THE CAUCHY–SZEGŐ PROJECTION AND ITS COMMUTATOR
FOR DOMAINS IN \( \mathbb{C}^n \) WITH MINIMAL SMOOTHNESS:
OPTIMAL BOUNDS

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Abstract. Let \( D \subset \mathbb{C}^n \) be a bounded, strongly pseudoconvex domain whose boundary \( bD \) satisfies the minimal regularity condition of class \( C^2 \). A 2017 result of Lanzani & Stein [16] states that the Cauchy–Szegő projection \( S_\omega \) maps \( L^p(bD, \omega) \) to \( L^p(bD, \omega) \) continuously for any \( 1 < p < \infty \) whenever the reference measure \( \omega \) is a bounded, positive continuous multiple of induced Lebesgue measure. Here we show that \( S_\omega \) (defined with respect to any measure \( \omega \) as above) satisfies explicit, optimal bounds in \( L^p(bD, \Omega_p) \), for any \( 1 < p < \infty \) and for any \( \Omega_p \) in the maximal class of \( A_p \)-measures, that is \( \Omega_p = \psi_p \sigma \) where \( \psi_p \) is a Muckenhoupt \( A_p \)-weight and \( \sigma \) is the induced Lebesgue measure. As an application, we characterize boundedness in \( L^p(bD, \Omega_p) \) with explicit bounds, and compactness, of the commutator \( [h, S_\omega] \) for any \( A_p \)-measure \( \Omega_p \), \( 1 < p < \infty \). We next introduce the notion of holomorphic Hardy spaces for \( A_p \)-measures, and we characterize boundedness and compactness in \( L^p(bD, \Omega_p) \) of the commutator \( [h, S_\omega] \) where \( S_\omega \) is the Cauchy–Szegő projection defined with respect to any given \( A_2 \)-measure \( \Omega_2 \). Earlier results rely upon an asymptotic expansion and subsequent pointwise estimates of the Cauchy–Szegő kernel, but these are unavailable in our setting of minimal regularity of \( bD \); at the same time, recent techniques [16] that allow to handle domains with minimal regularity, are not applicable to \( A_p \)-measures. It turns out that the method of quantitative extrapolation is an appropriate replacement for the missing tools.

1. Introduction

In this paper we present a new approach to the \( L^p \)-regularity problem for the Cauchy–Szegő projection \( S_\omega \) defined with respect to \( \omega = \Lambda \sigma \) (any) bounded, positive continuous multiple of the induced Lebesgue measure \( \sigma \) associated to a strongly pseudoconvex domain \( D \in \mathbb{C}^n \) that satisfies the minimal regularity condition of class \( C^2 \). As an application, we obtain the optimal quantitative bound

\[
\| S_\omega g \|_{L^p(bD, \Omega_p)} \lesssim [\Omega_p]_{A_p} \max \left\{ \frac{1}{\mu}, \frac{1}{\nu} \right\} \| g \|_{L^p(bD, \Omega_p)}, \quad 1 < p < \infty,
\]

for any \( \omega \in \{ \Lambda \sigma \}_\Lambda \) as above and for any \( \Omega_p \) in the class of Muckenhoupt measures \( A_p(bD) \), where \([\Omega_p]_{A_p}\) stands for the \( A_p \)-character of \( \Omega_p \) while the implicit constant depends solely on \( \omega \) and \( D \); see Theorem 1.1 for the precise statement. By contrast, the norm bounds obtained in the reference result [16]:

\[
\| S_\omega g \|_{L^p(bD, \omega)} \leq C(D, \omega, p) \| g \|_{L^p(bD, \omega)}, \quad 1 < p < \infty \quad \omega \in \{ \Lambda \sigma \}_\Lambda \text{ as above},
\]

and its recent improvement for \( \omega := \sigma \), [29, Theorem 1.1]:

\[
\| S_\sigma g \|_{L^p(bD, \Omega_p)} \leq \tilde{C}(D, [\Omega_p]_{A_p}) \| g \|_{L^p(bD, \Omega_p)}, \quad 1 < p < \infty, \quad \Omega_p \in A_p(bD)
\]

are unspecified functions of the stated variables.

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As is well known, the $A_p$-measures are the maximal (doubling) measure-theoretic framework for a great variety of singular integral operators. Any positive, continuous multiple of $\sigma$ is, of course, a member of $A_p(bD)$ for any $1 < p < \infty$, but $\{A_p(bD)\}_p$ far encompasses the family $\{\Lambda \sigma\}_\Lambda$. The classical methods for $S_\omega$ rely on the asymptotic expansion of the Cauchy–Szegő kernel, see e.g., the foundational works [3], [21] and [24], but this expansion is not available if $D$ is non-smooth; at the same time the Cauchy–Szegő projection $S_\omega$ may not be Calderón–Zygmund, see [19], thus none of (1.1)–(1.3) can follow by a direct application of the Calderón–Zygmund theory. The proof of (1.2) starts with the comparison of $S_\omega$, which we recall is the orthogonal projection of $L^2(bD, \omega)$ onto the holomorphic Hardy space $H^2(bD, \omega)$, with the members of an ad-hoc family $\{c_\epsilon\}_\epsilon$ of non-orthogonal projections (the so-called Cauchy–Leay integrals):

\[ c_\epsilon = S_\omega \circ \left[ I - (c_\epsilon^\dagger - c_\epsilon) \right] \text{ in } L^2(bD, \omega), \quad 0 < \epsilon < \epsilon(D), \]

where the upper-script $\dagger$ denotes the adjoint in $L^2(bD, \omega)$. The operators $\{c_\epsilon\}_\epsilon$ are bounded on $L^p(bD, \omega)$, $1 < p < \infty$, by an application of the $T(1)$ theorem. Furthermore, an elementary, Hilbert space-theoretic observation yields the factorization

\[ S_\omega = c_\epsilon \circ \left[ I - (c_\epsilon^\dagger - c_\epsilon) \right]^{-1} \text{ in } L^2(bD, \omega), \quad 0 < \epsilon < \epsilon(D), \]

along with its refinement

\[ S_\omega = \left( c_\epsilon + S_\omega \circ \left( (R_\epsilon^\dagger) - R_\epsilon^\dagger \right) \circ \left[ I - ((c_\epsilon^\dagger - c_\epsilon) \right]^{-1} \text{ in } L^2(bD, \omega), \quad 0 < \epsilon < \epsilon(D). \]

For the latter, the Cauchy–Leay integral $c_\epsilon$ is decomposed as the sum $c_\epsilon = c_\epsilon^\dagger + R_\epsilon^\dagger$ where the principal term $c_\epsilon^\dagger$ (bounded, by another application of the $T(1)$ theorem) enjoys the cancellation

\[ ||(c_\epsilon^\dagger) - c_\epsilon^\dagger||_{L^p(bD, \omega) \rightarrow L^p(bD, \omega)} \leq \epsilon^{1/2} C(p, D, \omega), \quad 1 < p < \infty, \quad 0 < \epsilon < \epsilon(D). \]

By contrast, the remainder $R_\epsilon$ has $L^p \rightarrow L^p$ norm that may blow up as $\epsilon \to 0$ but is weakly smoothing in the sense that

\[ R_\epsilon^\dagger \text{ and } (R_\epsilon^\dagger)^\dagger \text{ are bounded : } L^1(bD, \omega) \to L^\infty(bD, \omega) \text{ for any } 0 < \epsilon < \epsilon(D). \]

Combining (1.7) and (1.8) with the bounded inclusions

\[ L^2(bD, \omega) \hookrightarrow L^p(bD, \omega) \hookrightarrow L^1(bD, \omega), \quad 1 < p \leq 2, \]

one concludes that there is $\epsilon = \epsilon(p)$ such that the right hand side of (1.6), and therefore $S_\omega$, extends to a bounded operator on $L^p(bD, \omega)$ for any $\omega \in \{\Lambda \sigma\}_\Lambda$ whenever $1 < p < 2$; the proof for the full $p$-range then follows by duality.

Implementing this argument to all $A_p$-measures presents conceptual difficulties: for instance, there are no analogs of (1.8) nor of (1.9) that are valid with $\Omega_p$ in place of $\omega$ because $A_p$-measures may change with $p$. In [29, Lemma 3.3] it is shown that (1.7) is still valid for $p = 2$ if one replaces the measure $\omega$ in the $L^2(bD)$-space with an $A_2$-measure; the conclusion (1.3) then follows by Rubio de Francia’s original extrapolation theorem [26].

Here we obtain a new cancellation in $L^p(bD, \Omega_p)$ where the dependence of the norm-bound on the $A_p$-character of the measure is completely explicit, namely

\[ ||(c_\epsilon^\dagger) - c_\epsilon^\dagger||_{L^p(bD, \Omega_p) \rightarrow L^p(bD, \Omega_p)} \leq \epsilon^{1/2} |\Omega_p|_{A_p}^{\max\{1, \frac{1}{2-p}\}}, \quad 1 < p < \infty \]

where the implied constant depends on $p$, $D$ and $\omega$ but is independent of $\Omega_p$ and of $\epsilon$; see Proposition 3.3 for the precise statement. Combining (1.10) with the reverse Hölder inequality for $A_2(bD)$ we obtain

\[ ||S_\omega||_{L^2(bD, \Omega_1) \rightarrow L^2(bD, \Omega_2)} \leq |\Omega_2|_{A_2} \text{ for any } \Omega_2 \in A_2(bD) \text{ and any } \omega \in \{\Lambda \sigma\}_\Lambda. \]
The conclusion (1.1) then follows by quantitative extrapolation [6, Theorem 9.5.3], which serves as an appropriate replacement for (1.8) and (1.9). Incidentally, we are not aware of other applications of extrapolation to several complex variables; yet quantitative extrapolation seems to provide an especially well-suited approach to the analysis of orthogonal projections onto spaces of holomorphic functions* because orthogonal projections are naturally bounded, with minimal norm, on the Hilbert space where they are defined; thus there is always a baseline choice of \( \Omega_2 \) for which (1.11) holds trivially (that is, with \( [\Omega_2]_\Lambda_2 = 1 \)).

We anticipate that this approach will give new insight into other settings where this kind of \( L^p \)-regularity problems are unsolved, such as the strongly \( C \)-linearly convex domains of class \( C^{1,1} \) ([15]) and the \( C^{1,\alpha} \) model domains of [17].

It turns out that extrapolation is also an effective tool to study the commutator \([b, \mathcal{S}_\omega]\) although the latter is not a projection (never mind orthogonal!), thus there is no baseline \( L^2 \)-regularity satisfied by \([b, \mathcal{S}_\omega]\), nor is it possible to make a direct comparison à la (1.4) of \([b, \mathcal{S}_\omega]\) with the commutators \([b, \mathcal{C}_\omega]\) studied in [2]. Regardless, quantitative extrapolation can be employed to characterize boundedness in \( L^2(bD, \Omega_p) \) with explicit norm bounds, and compactness, of \([b, \mathcal{S}_\omega]\) under the minimal regularity of class \( C^2 \) and for all \( A_p \)-measures, in terms of appropriate BMO, or VMO conditions on the symbol \( b \); the complete statement is given in Theorem 1.2, which extends to the optimal setting seminal results of Coifman–Rochberg–Weiss [1] and Krantz–S.Y. Li [13].

We anticipate that extrapolation is also effective for characterizing finer properties, such as the Schatten-\( p \) norm of \([b, \mathcal{S}_\omega]\), and plan to address these questions in future work; see Feldman–Rochberg [4] for a related result.

Can the Cauchy–Szegő projection be defined for measures other than \( \omega \in \{ A \sigma \}_A \)? Orthogonal projections are highly dependent upon the choice of reference measure for the Hilbert space where they are defined: it is therefore natural to seek the maximal measure-theoretic framework for which the notion of Cauchy–Szegő projection is meaningful. The \( A_p \)-measures are an obvious choice in view of their historical relevance to the theory of singular integral operators. It turns out that in this context the Cauchy–Szegő projection is meaningful only if it is defined with respect to \( A_2 \)-measures that is, for \( p = 2 \): one may define \( \mathcal{S}_{\Omega_2} \) which is bounded: \( L^2(bD, \Omega_2) \to L^2(bD, \Omega_2) \) for any \( \Omega_2 \in A_2(bD) \) with the minimal operator norm: \( \| \mathcal{S}_{\Omega_2} \| = 1 \); on the other hand \( \mathcal{S}_{\Omega_p} \) may have no meaning if \( p \neq 2 \). Here we characterize boundedness and compactness in \( L^2(bD, \Omega_2) \) of the commutator \([b, \mathcal{S}_{\Omega_2}]\), see Theorem 1.5 below. The \( L^p(bD, \Omega_p) \)-regularity problem for \( \mathcal{S}_{\Omega_2} \), as well as the \( L^p(bD, \Omega_p) \)-regularity and -compactness problems for \([b, \mathcal{S}_{\Omega_2}]\), while meaningful, are, at present, unanswered for \( p \neq 2 \).

1.1. Statement of the main results. We let \( \sigma \) denote induced Lebesgue measure for \( bD \) and we henceforth refer to the family

\[
\{ A \sigma \}_A \equiv \{ \omega := A \sigma, \ A \in C(bD), \ 0 < c(D, A) \leq A(w) \leq C(D, A) < \infty \ \text{for any} \ w \in bD \}
\]

as the Leray Levi-like measures. This is because the Leray measure \( \Lambda \), which plays a distinguished role in the analysis [16] of the Cauchy–Leray integrals \( \{ \mathcal{C}_\epsilon \} \), and their truncations \( \{ \mathcal{C}_\epsilon^u \} \), is a member of this family on account of the identity

\[
d\Lambda(w) = \Lambda(w)d\sigma(w), \quad w \in bD,
\]

where \( \Lambda \in C(\overline{D}) \) satisfies the required bounds \( 0 < \epsilon(D) \leq \Lambda(w) \leq C(D) < \infty \) for any \( w \in bD \) as a consequence of the strong pseudoconvexity and \( C^2 \)-regularity and boundedness of \( D \). Hence we may equivalently express any Leray Levi-like measure \( \omega \) as

\[
\omega = \varphi \lambda
\]

*The Cauchy–Szegő and Bergman projections being two prime examples.
for some \( \varphi \in C(\partial D) \) such that \( 0 < m(D) \varphi(w) \leq M(D) < \infty \) for any \( w \in bD \). We refer to Section 2 for the precise definitions and to [25, Lemma VII.3.9] for the proof of (1.12) and a discussion of the geometric significance of \( \lambda \).

For any Leray Levi-like measure \( \omega \), the holomorphic Hardy space \( H^2(bD, \omega) \) is defined exactly as in the classical setting of \( H^2(bD, \sigma) \), simply by replacing \( \sigma \) with \( \omega \), see e.g., [18, (1.1) and (1.2)]. In particular \( H^2(bD, \omega) \) is a closed subspace of \( L^2(bD, \omega) \) and we let \( S_\omega \) denote the (unique) orthogonal projection of \( L^2(bD, \omega) \) onto \( H^2(bD, \omega) \).

Our goal is to understand the behavior of \( S_\omega \) on \( L^p(bD, \Omega_p) \) where \( \{ \Omega_p \} \) is any \( A_p \)-measure that is,

\[
\Omega_p = \phi_p \sigma
\]

where the density \( \phi_p \) is a Muckenhoupt \( A_p \)-weight. The precise definition is given in Section 2; here we just note that the Leray Levi-like measures are strictly contained in the class \( \{ \Omega_p \}_p \) in the sense that each \( \omega \in \{ \Lambda \sigma \}_\lambda \) is an \( A_p \)-measure for every \( 1 < p < \infty \). We may now state our first two main results.

**Theorem 1.1.** Let \( D \subset \mathbb{C}^n, n \geq 2, \) be a bounded, strongly pseudoconvex domain of class \( C^2 \); let \( \omega \) be any Leray Levi-like measure for \( bD \) and let \( S_\omega \) be the Cauchy–Szegő projection for \( \omega \). We have that

\[
\|S_\omega\|_{L^2(bD, \Omega_2) \to L^2(bD, \Omega_2)} \lesssim [\Omega_2]_{A_2} \text{ for any } \Omega_2 \in A_2(bD),
\]

where the implied constant depends solely on \( D \) and \( \omega \).

The \( L^p \)-estimate (1.1) follows from (1.15) by standard techniques, see [6, Theorem 9.5.3].

**Theorem 1.2.** Let \( D \subset \mathbb{C}^n, n \geq 2, \) be a bounded, strongly pseudoconvex domain of class \( C^2 \). The following hold for any \( b \in L^2(bD, \sigma) \) and for any Leray Levi-like measure \( \omega \):

1. if \( b \in \text{BMO}(bD, \sigma) \) then the commutator \( [b, S_\omega] \) is bounded on \( L^p(bD, \Omega_p) \) for any \( 1 < p < \infty \) and any \( A_p \)-measure \( \Omega_p \), with

\[
\|[b, S_\omega]\|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p)} \lesssim \|b\|_{\text{BMO}(bD, \sigma)} [\Omega_p]_{A_p}^{2 \max\{1, \frac{1}{p-1}\}}.
\]

Conversely, if \( [b, S_\omega] \) is bounded on \( L^p(bD, \Omega_p) \) for some \( p \in (1, \infty) \) and some \( A_p \)-measure \( \Omega_p \), then \( b \in \text{BMO}(bD, \sigma) \) with

\[
\|b\|_{\text{BMO}(bD, \sigma)} \lesssim [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} \|[b, S_\omega]\|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p)}.
\]

All implied constants depend on \( p, D \) and \( \omega \) but are independent of \( \Omega_p \).

2. if \( b \in \text{VMO}(bD, \sigma) \) then the commutator \( [b, S_\omega] \) is compact on \( L^p(bD, \Omega_p) \) for all \( 1 < p < \infty \) and all \( A_p \)-measures \( \Omega_p \). Conversely, if \( [b, S_\omega] \) is compact on \( L^p(bD, \Omega_p) \) for some \( p \in (1, \infty) \) and some \( A_p \)-measure \( \Omega_p \), then \( b \in \text{VMO}(bD, \sigma) \).

As mentioned earlier, extending the notion of Cauchy–Szegő projection to the realm of \( A_p \)-measures requires, first of all, a meaningful notion of holomorphic Hardy space for such measures, but this does not immediately arise from the classical theory [28]; here we adopt the approach of [18, (1.1)] and give the following

**Definition 1.3.** Suppose \( 1 \leq p < \infty \) and let \( \Omega_p \) be an \( A_p \)-measure. We define \( H^p(bD, \Omega_p) \) to be the space of functions \( F \) that are holomorphic in \( D \) with \( \mathcal{N}(F) \in L^p(bD, \Omega_p) \), and set

\[
\|F\|_{H^p(bD, \Omega_p)} := \|\mathcal{N}(F)\|_{L^p(bD, \Omega_p)}.
\]

Here \( \mathcal{N}(F) \) denotes the non-tangential maximal function of \( F \), that is

\[
\mathcal{N}(F)(\xi) := \sup_{z \in \Gamma_\alpha(\xi)} |F(z)|, \quad \xi \in bD,
\]
where $\Gamma_\alpha(\xi) = \{z \in D : |(z - \xi) \cdot \nu_\xi| < (1 + \alpha)\delta_\xi(z), |z - \xi|^2 < \alpha \delta_\xi(z)\}$, with $\nu_\xi$ is the (complex conjugate of) the outer unit normal vector to $\xi \in bD$, and $\delta_\xi(z) =$ the minimum distance of $z$ to $bD$ and the distance of $z$ to the tangent space at $\xi$.

For the class of domains under consideration it is known that if $\Omega_p$ is Leray Levi-like, such definition agrees with the classical formulation [18, (1.2)]; see also [28].

**Proposition 1.4.** Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, strongly pseudoconvex domain of class $C^2$. Then, for any $1 \leq p < \infty$ and any $A_p$-measure $\Omega_p$ we have that $H^p(bD, \Omega_p)$ is a closed subspace of $L^p(bD, \Omega_p)$. More precisely, suppose that $\{F_n\}_n$ is a sequence of holomorphic functions in $D$ such that $\|N(F_n) - f\|_{L^p(bD, \Omega_p)} \to 0$ as $n \to \infty$. Then, there is an $F$ holomorphic in $D$ such that $N(F)(w) = f(w)\quad \Omega_p$-a.e. $w \in bD$.

The proof relies on the following observation, which is of independent interest: for any $1 \leq p < \infty$ and any $A_p$-measure $\Omega_p$ with density $\psi_p$, there is $p_0 = p_0(\Omega_p) \in (1, p)$ such that

\begin{equation}
H^p(bD, \Omega_p) \subset H^{p_0}(bD, \sigma) \quad \text{with} \quad \|F\|_{H^{p_0}(bD, \sigma)} \leq C_{\Omega_p, D} \|F\|_{H^p(bD, \Omega_p)},
\end{equation}

where

\[ C_{\Omega_p, D} = \left( \int_{bD} \psi_p(w)^{\frac{p_0}{p_0 - p_1}} \sigma(w) \right)^{\frac{p_1}{p_0 - p_1}}. \]

On the other hand in the context of $A_p$-measures the notion of Hilbert space is meaningful only in relation to $A_2$-measures (that is for $p = 2$), hence Cauchy–Szegő projections can be associated only to such measures: on account of Proposition 1.4, for any $A_2$-measure $\Omega_2$ there exists a unique, orthogonal projection:

\[ S_{\Omega_2} : L^2(bD, \Omega_2) \to H^2(bD, \Omega_2) \]

which is naturally bounded on $L^2(bD, \Omega_2)$ with minimal norm $\|S_{\Omega_2}\| = 1$. We may now state our final main result.

**Theorem 1.5.** Let $D \subset \mathbb{C}^n$, $n \geq 2$, be a bounded, strongly pseudoconvex domain of class $C^2$. The following hold for any $b \in L^2(bD, \Omega_2)$:

1. if $b \in \text{BMO}(bD, \sigma)$ then the commutator $[b, S_{\Omega_2}]$ is bounded on $L^2(bD, \Omega_2)$ for any $A_2$-measure $\Omega_2$. Conversely, if $[b, S_{\Omega_2}]$ is bounded on $L^2(bD, \Omega_2)$ for some $A_2$-measure $\Omega_2$, then $b \in \text{BMO}(bD, \sigma)$;

2. if $b \in \text{VMO}(bD, \sigma)$ then the commutator $[b, S_{\Omega_2}]$ is compact on $L^2(bD, \Omega_2)$ for any $A_2$-measure $\Omega_2$. Conversely, if $[b, S_{\Omega_2}]$ is compact on $L^2(bD, \Omega_2)$ for some $A_2$-measure $\Omega_2$, then $b \in \text{VMO}(bD, \sigma)$.

All the implied constants depend solely on $D$ and $\Omega_2$.

We conclude with a few remarks concerning the statements and proofs of Theorems 1.2 and 1.5.

- Since commutators are not projection operators, there can be no perfect counterpart to the identities (1.5) and (1.6) that have $[b, S_{\omega}]$ (resp. $[b, \mathcal{E}_\omega]$) in place of $S_{\omega}$ (resp. $\mathcal{E}_\omega$). Instead, the proofs of Theorem 1.5 and Theorem 1.2 make use of the following new analogs of (1.5) and (1.6):

\begin{equation}
[b, S_{\Omega_2}] = \left( [b, \mathcal{E}_\omega] + S_{\Omega_2} \circ [b, I - (\mathcal{E}_\omega^i - \mathcal{E}_\omega)] \right) \circ \left( I - (\mathcal{E}_\omega^i - \mathcal{E}_\omega) \right)^{-1} \quad \text{in} \quad L^2(bD, \Omega_2)
\end{equation}
for any $A_2$-like measure $\Omega_2$ and for any $0 < \epsilon < \epsilon(D)$ as above; and
\begin{equation}
[b, S_{\omega}] = \left[ [b, \mathcal{E}_z] + S_{\omega} \circ \left[ b, I - (\mathcal{E}_z^\dagger - \mathcal{E}_z) \right] \right] - [b, S_{\omega}] \circ \left( I - (\mathcal{E}_z^\dagger - \mathcal{E}_z) \right)^{-1}
\end{equation}
in $L^2(bD, \omega)$
for any Leray Levi-like measure $\omega$ and for any $0 < \epsilon < \epsilon(D)$. These identities are proved in Section 4.

- In the statement of Theorem 1.5 we assume that the symbol $b$ is in $L^2(bD, \Omega_2)$ because the latter is the natural function space where the Cauchy–Szegő projection $S_{\Omega_2}$ is defined. However such an assumption is not restrictive because $D$ is bounded and of class $C^2$, and $A_p$-measures for such domains are absolutely continuous with respect to the Leray Levi-like measures, hence $\Omega_2(bD) < \infty$ for any such measure, and for any $1 < p < \infty$. It follows that $\text{BMO}(bD, \sigma) \subset L^2(bD, \Omega_2)$ for any $\Omega_2 \in A_2(bD)$, see (4.26).

- The space $\text{BMOA}(bD, \sigma)$ (resp. $\text{VMOA}(bD, \sigma)$) is the proper subspace of $\text{BMO}(bD, \sigma)$ (resp. $\text{VMOA}(bD, \sigma)$) obtained by changing the a-priori condition that $b \in L^1(bD, \sigma)$ with the stricter requirement that $b$ is in the holomorphic Hardy space $H^1(bD, \sigma)$, see [23]; by the above argument, $\text{BMOA}(bD, \sigma) \subset H^2(bD, \Omega_2)$ for any $\Omega_2 \in A_2(bD)$. Changing the a-priori condition that $b \in L^2(bD, \Omega_2)$ to $b \in H^2(bD, \Omega_2)$ in Theorem 1.5 produces new statements that are true for $b \in \text{BMOA}(bD, \sigma)$ (resp. $b \in \text{VMOA}(bD, \sigma)$), with the same proof.

1.2. Further results. It is clear from (1.20) and (1.21) that one also needs to prove quantitative results for the Cauchy–Leray integrals $\{\mathcal{E}_z\}_z$ that extend the scope of the earlier works [16] and [2] from Leray Levi-like measures, to $A_p$-measures: these are stated in Theorems 3.1 and 3.2, and Proposition 3.3.

1.3. Organization of this paper. In the next section we recall the necessary background and give the proof of Proposition 1.4. All the quantitative results pertaining to the Cauchy–Leray integral are collected in in Section 3. The main results of this paper: Theorem 1.1; Theorem 1.2, and Theorem 1.5, are proved in Section 4.

2. Preliminaries and proof of Proposition 1.4

In this section we introduce notations and recall certain results from [2, 16] that will be used throughout this paper. We will henceforth assume that $D \subset \mathbb{C}^n$ is a bounded, strongly pseudoconvex domain of class $C^2$; that is, there is $\rho \in C^2(\mathbb{C}^n, \mathbb{R})$ which is strictly plurisubharmonic and such that $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ and $bD = \{w \in \mathbb{C}^n : \rho(w) = 0\}$ with $\nabla \rho(w) \neq 0$ for all $w \in bD$. (We refer to such $\rho$ as a defining function for $D$; see e.g., [25] for the basic properties of defining functions. Here we assume that one such $\rho$ has been fixed once and for all.) We will throughout make use of the following abbreviated notations:

$$
\|T\|_p = \|T\|_{L^p(bD, d\mu) \to L^p(bD, d\mu)}, \quad \text{and} \quad \|T\|_{p,q} = \|T\|_{L^p(bD, d\mu) \to L^q(bD, d\mu)}
$$

where the operator $T$ and the measure $\mu$ will be clear from context.

- The Levi polynomial and its variants. Define

$$
\mathcal{L}_0(w, z) := (\partial \rho(w), w - z) - \frac{1}{2} \sum_{j,k} \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} (w_j - z_j)(w_k - z_k),
$$

\footnote{Indeed, if $f \in \text{BMOA}(bD, \sigma)$ then $f \in L^2(bD, \sigma)$ by the above argument. Hence $f \in H^1(bD, \sigma)$ \cap \L^2(\mathbb{C}, \omega)$ and this implies that $f \in H^2(bD, \sigma)$, see [18, Corollary 2].}
where $\partial \rho(w) = (\frac{\partial \rho}{\partial w_1}(w), \ldots, \frac{\partial \rho}{\partial w_n}(w))$ and we have used the notation $\langle \eta, \zeta \rangle = \sum_{j=1}^{n} \eta_j \zeta_j$ for $\eta = (\eta_1, \ldots, \eta_n), \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$. The strict plurisubharmonicity of $\rho$ implies that

$$2 \text{Re} \, \mathcal{L}_0(w, z) \geq -\rho(z) + c|w - z|^2,$$

for some $c > 0$, whenever $w \in bD$ and $z \in \overline{D}$ is sufficiently close to $w$. We next define

$$(2.1) \quad g_0(w, z) := \chi \, \mathcal{L}_0 + (1 - \chi)|w - z|^2$$

where $\chi = \chi(w, z)$ is a $C^\infty$-smooth cutoff function with $\chi = 1$ when $|w - z| \leq \mu/2$ and $\chi = 0$ if $|w - z| \geq \mu$. Then for $\mu$ chosen sufficiently small (and then kept fixed throughout), we have that

$$(2.2) \quad \text{Re} \, g_0(w, z) \geq c(-\rho(z) + |w - z|^2)$$

for $z$ in $\overline{D}$ and $w$ in $bD$, with $c$ a positive constant; we will refer to $g_0(w, z)$ as the modified Levi polynomial. Note that $g_0(w, z)$ is polynomial in the variable $z$, whereas in the variable $w$ it has no smoothness beyond mere continuity. To amend for this lack of regularity, for each $\epsilon > 0$ one considers a variant $g_\epsilon$ defined as follows. Let $\{\tau_{jk}^\epsilon(w)\}$ be an $n \times n$-matrix of $C^1$ functions such that

$$\sup_{w \in bD} \left| \frac{\partial^2 \rho(w)}{\partial w_j \partial w_k} - \tau_{jk}^\epsilon(w) \right| \leq \epsilon, \quad 1 \leq j, k \leq n.$$ 

Set

$$\mathcal{L}_\epsilon(w, z) = \langle \partial \rho(w), w - z \rangle - \frac{1}{2} \sum_{j,k} \tau_{jk}^\epsilon(w)(w_j - z_j)(w_k - z_k),$$

and define

$$g_\epsilon(w, z) = \chi \mathcal{L}_\epsilon + (1 - \chi)|w - z|^2, \quad z, w \in \mathbb{C}^n.$$ 

Now $g_\epsilon$ is of class $C^1$ in the variable $w$, and

$$|g_0(w, z) - g_\epsilon(w, z)| \lesssim \epsilon|w - z|^2, \quad w \in bD, z \in \overline{D}.$$ 

We assume that $\epsilon$ is sufficiently small (relative to the constant $c$ in (2.2)), and this gives that

$$(2.3) \quad |g_\epsilon(w, z)| \leq |g_\epsilon(w, z)| \leq \tilde{C} |g_0(w, z)|, \quad w, z \in bD$$ 

where the constants $C$ and $\tilde{C}$ are independent of $\epsilon$; see [16, Section 2.1].

- **The Leray–Levi measure for $bD$.** Let $j^*$ denote the pullback under the inclusion $j : bD \hookrightarrow \mathbb{C}^n$.

Then the linear functional

$$(2.4) \quad f \mapsto \frac{1}{(2\pi i)^n} \int_{bD} f(w) j^* (\partial \rho \wedge (\bar{\partial} \rho)^{n-1})(w) =: \int_{bD} f(w) d\lambda(w)$$

where $f \in C(bD)$, defines a measure $\lambda$ with positive density given by

$$d\lambda(w) = \frac{1}{(2\pi i)^n} j^* (\partial \rho \wedge (\bar{\partial} \rho)^{n-1})(w).$$

We point out that the definition of $\lambda$ depends upon the choice of defining function for $D$, which here has been fixed once and for all; hence we refer to $\lambda$ as "the" Leray–Levi measure.

- **A space of homogeneous type.** Consider the function

$$(2.5) \quad d(w, z) := |g_0(w, z)|^{1/2}, \quad w, z \in bD.$$ 

It is known [16, (2.14)] that

$$|w - z| \lesssim d(w, z) \lesssim |w - z|^{1/2}, \quad w, z \in bD.$$
and from this it follows that the space of Hölder-type functions [16, (3.5)]:
\[(2.6) \quad |f(w) - f(z)| \lesssim d(w, z)^\alpha \quad \text{for some } 0 < \alpha \leq 1 \quad \text{and for all } w, z \in bD\]
is dense in \(L^p(bD, \omega), 1 < p < \infty\) for any Leray Levi-like measure see [16, Theorem 7].

It follows from (2.3) that
\[(2.7) \quad \mathcal{C}d(w, z)^2 \leq |g_r(w, z)| \leq \mathcal{C}d(w, z)^2, \quad w, z \in bD\]
for any \(\epsilon\) sufficiently small. It is shown in [16, Proposition 3] that \(d(w, z)\) is a quasi-distance: there exist constants \(A_0 > 0\) and \(\mathcal{C}_d > 1\) such that for all \(w, z, z' \in bD\),
\[
\begin{align*}
1) & \quad d(w, z) = 0 \quad \text{iff} \quad w = z; \\
2) & \quad A_0^{-1}d(z, w) \leq d(w, z) \leq A_0 d(z, w); \\
3) & \quad d(w, z) \leq \mathcal{C}_d(d(w, z') + d(z', z)).
\end{align*}
\]

Letting \(B_r(w)\) denote the boundary balls determined via the quasi-distance \(d\),
\[(2.9) \quad B_r(w) := \{z \in bD : d(z, w) < r\}, \quad \text{where } w \in bD,\]
we have that
\[(2.10) \quad c_\omega^{-1} r^{2n} \leq \omega(B_r(w)) \leq c_\omega r^{2n}, \quad 0 < r \leq 1,\]
for some \(c_\omega > 1\), see [16, p. 139]. It follows that the triples \(\{bD, d, \omega\}\), for any Leray Levi-like measure \(\omega\), are spaces of homogeneous type, where the measures \(\omega\) have the doubling property:

Lemma 2.1. The Leray Levi-like measures \(\omega\) on \(bD\) are doubling, i.e., there is a positive constant \(C_\omega\) such that for all \(x \in bD\) and \(0 < r \leq 1\),
\[
0 < \omega(B_{2r}(w)) \leq C_\omega \omega(B_r(w)) < \infty.
\]
Furthermore, there exist constants \(c_\omega \in (0, 1)\) and \(C_\omega > 0\) such that
\[
\omega(B_r(w) \setminus B_r(z)) + \omega(B_r(z) \setminus B_r(w)) \leq C_\omega \left(\frac{d(w, z)}{r}\right)^{c_\omega}
\]
for all \(w, z \in bD\) such that \(d(w, z) \leq r \leq 1\).

Proof. The proof is an immediate consequence of (2.10). \(\square\)

- A family of Cauchy-like integrals. In [16, Sections 3 and 4] an ad-hoc family \(\{C_\epsilon\}_\epsilon\) of Cauchy-Fantappiè integrals is introduced (each determined by the aforementioned denominators \(g_r(w, z)\)) whose corresponding boundary operators \(\{\mathcal{C}_\epsilon\}_\epsilon\) play a crucial role in the analysis of \(L^p(bD, \lambda)\)-regularity of the Cauchy-Szegő projection. We henceforth refer to \(\{\mathcal{C}_\epsilon\}_\epsilon\) as the Cauchy-Leray integrals; we record here a few relevant points for later reference.

[i.] Each \(\mathcal{C}_\epsilon\) admits a primary decomposition in terms of an “essential part” \(\mathcal{C}_\epsilon^e\) and a “remainder” \(\mathcal{R}_\epsilon\), which are used in the proof of the \(L^2(bD, \omega)\)-regularity of \(\mathcal{C}_\epsilon\). However, at this stage the magnitude of the parameter \(\epsilon\) plays no role (this is because of the “uniform” estimates (2.7)) and we temporarily drop reference to \(\epsilon\) and simply write \(\mathcal{C}\) in lieu of \(\mathcal{C}_\epsilon; C(w, z)\) for \(C_\epsilon(w, z)\), etc.. Thus
\[
(2.11) \quad \mathcal{C} = \mathcal{C}^e + \mathcal{R},
\]
with a corresponding decomposition for the integration kernels:
\[
(2.12) \quad C(w, z) = C^e(w, z) + R(w, z).
\]
The “essential” kernel $C'(w,z)$ satisfies standard size and smoothness conditions that ensure the boundedness of $C'$ in $L^2(bD,\omega)$ by a $T(1)$-theorem for the space of homogeneous type $\{bD,d,\omega\}$. On the other hand, the “remainder” kernel $R(w,z)$ satisfies improved size and smoothness conditions granting that the corresponding operator $\mathcal{R}$ is bounded in $L^2(bD,\omega)$ by elementary considerations; see [16, Section 4].

[i.] One then turns to the Cauchy–Szegő projection, for which $L^2(bD,\omega)$-regularity is trivial but $L^p(bD,\omega)$-regularity, for $p \neq 2$, is not. It is in this stage that the size of $\epsilon$ in the definition of the Cauchy-type boundary operators of item [i.] is relevant. It turns out that each $C^\epsilon$ admits a further, “finer” decomposition into (another) “essential” part and (another) “reminder”, which are obtained by truncating the integration kernel $C^\epsilon(w,z)$ by a smooth cutoff function $\chi_{s}^\epsilon(w,z)$ that equals 1 when $d(w,z) < s = s(\epsilon)$. One has:

\begin{equation}
C^\epsilon = C^\epsilon_s + R^\epsilon_s
\end{equation}

where

\begin{equation}
\| (C^\epsilon_s)^\dagger - C^\epsilon_s \|_p \lesssim \epsilon^{1/2} M_p
\end{equation}

for any $1 < p < \infty$, where $M_p = \frac{p}{p-1} + p$. Here and henceforth, the upper-script “$*$” denotes adjoint in $L^2(bD,\omega)$ (hence $(C^\epsilon_s)^\dagger$ is the adjoint of $C^\epsilon_s$ in $L^2(bD,\omega)$); see [16, Proposition 18]. Furthermore $R^\epsilon_s$ and $(R^\epsilon_s)^\dagger$ are controlled by $d(w,z)^{-2n+1}$ and therefore are easily seen to be bounded

\begin{equation}
R^\epsilon_s, \ (R^\epsilon_s)^\dagger : L^1(bD,\omega) \to L^\infty(bD,\omega),
\end{equation}

see [16, (5.2) and comments thereafter].

• **Bounded mean oscillation on $bD$.** The space BMO($bD,\lambda$) is defined as the collection of all $b \in L^1(bD,\lambda)$ such that

\[
\|b\|_s := \sup_{z \in bD, r > 0, B_r(z) \subset bD} \frac{1}{\lambda(B_r(z))} \int_{B_r(z)} |b(w) - b_B|d\lambda(w) < \infty,
\]

with the balls $B_r(z)$ as in (2.9) and where

\begin{equation}
b_B = \frac{1}{\lambda(B)} \int_B b(z)d\lambda(z).
\end{equation}

BMO($bD,\lambda$) is a normed space with $\|b\|_{\text{BMO}(bD,\lambda)} := \|b\|_s + \|b\|_{L^1(bD,\lambda)}$. We note the inclusion

\begin{equation}
\text{BMO}(bD,\lambda) \subset L^p(bD,\lambda), \quad 1 \leq p < \infty,
\end{equation}

which is a consequence of the John–Nirenberg inequality [27, Corollary p. 144] and of the compactness of $bD$. On account of (1.12), it is clear that

\[
\text{BMO}(bD,\lambda) = \text{BMO}(bD,\lambda) \quad \text{with} \quad \|b\|_{\text{BMO}(bD,\lambda)} \approx \|b\|_{\text{BMO}(bD,\lambda)},
\]

where BMO($bD,\lambda$) is the classical BMO space (where the reference measure is induced Lebesgue).

• **Vanishing mean oscillation on $bD$.** The space VMO($bD,\lambda$) is the subspace of BMO($bD,\lambda$) whose members satisfy the further requirement that

\begin{equation}
\lim_{a \to 0} \sup_{B \subset bD: r_B = a} \frac{1}{\lambda(B)} \int_B |f(z) - f_B|d\lambda(z) = 0,
\end{equation}

where $r_B$ is the radius of $B$. As before, it is clear that VMO($bD,\lambda$) = VMO($bD,\lambda$).
- **Muckenhoupt weights on bD.** Let \( p \in (1, \infty) \). A non-negative locally integrable function \( \psi \) is called an \( A_p(bD, \sigma) \)-weight, if

\[
[\psi]_{A_p(bD, \sigma)} := \sup_B \langle \psi \rangle_B \langle \psi^{1-p'} \rangle_B^{p-1} < \infty,
\]

where the supremum is taken over all balls \( B \) in \( bD \), and \( \langle \phi \rangle_B := \frac{1}{\sigma(B)} \int_B \phi(z) d\sigma(z) \). Moreover, \( \psi \) is called an \( A_1(bD, \sigma) \)-weight if \( [\psi]_{A_1(bD, \sigma)} := \inf \{ C \geq 0 : \langle \psi \rangle_B \leq C \psi(x), \forall x \in B, \forall \text{ balls } B \in bD \} < \infty \).

Similarly, one can define the \( A_p(bD, \lambda) \)-weight for \( 1 \leq p < \infty \).

As before, the identity (1.12) grants that \( A_p(bD, \sigma) = A_p(bD, \lambda) \) with \( [\psi]_{A_p(bD, \sigma)} \approx [\psi]_{A_p(bD, \lambda)} \), thus we will henceforth simply write \( A_p(bD) \) and \( [\psi]_{A_p(bD)} \). At times it will be more convenient to work with \( A_p(bD, \lambda) \), and in this case we will refer to its members as \( A_p \)-like weights.

**Proof of Proposition 1.4.** To streamline notations, we write \( \Omega \) for \( \Omega_p \), and \( \psi \) for \( \psi_p \). It is clear that \( H^p(bD, \Omega) \) is a subspace of \( L^p(bD, \Omega) \), where the density function \( \psi \) of \( \Omega \) is in \( A_p \).

Our first claim is that for every \( F \in H^p(bD, \Omega) \), the non-tangential (also known as admissible) limit \( F^b \) exists \( \Omega \)-a.e. \( w \in bD \). In fact, note that for \( \psi \in A_p \), there exists \( 1 < p_1 < p \) such that \( \psi \) is in \( A_{p_1} \). Set \( p_0 = p/p_1 \). Then it is clear that \( 1 < p_0 < p \). Let \( P = p/p_0 \) and \( P' \) be the conjugate of \( P \), which is \( P' = p/(p - p_0) \). So we get \( \psi^{1-P'} = \psi^{-p_0/(p-p_0)} \in A_{P'} = A_p/(p-p_0) \).

Hence, for \( F \in H^p(bD, \Omega) \),

\[
\|F\|_{H^0(bD, \lambda)} = \left( \int_{bD} [N(F)(w)]^{p_0 \psi(w)} \frac{\psi(w)}{\psi(w)^{-p_0} d\lambda(w)} \right)^{1/p_0} \leq \left( \int_{bD} [N(F)(w)]^{P' \psi(w)} d\lambda(w) \right)^{1/P'} = \left( \int_{bD} \psi(w)^{-p_0/(p-p_0)} d\lambda(w) \right)^{p_0/(p-p_0)} = \|F\|_{H^p(bD, \Omega)} \left( \psi^{-p_0/(p-p_0)}(bD) \right)^{p_0/(p-p_0)}.
\]

Since \( \psi^{-p_0/(p-p_0)} \in A_p/(p-p_0) \) and \( bD \) is compact we have that \( \psi^{-p_0/(p-p_0)}(bD) \) is finite.

Hence, we see that \( H^p(bD, \Omega) \subset H^0(bD, \lambda) \). Thus \( F \) has admissible limit \( F^b \) for \( \lambda \)-a.e. \( z \in bD \) ([28, Theorem 10]), and hence \( F \) has admissible limit \( F^b \) for \( \Omega \)-a.e. \( z \in bD \) since the measure \( \Omega \) (with the density function \( \psi \) : \( d\Omega(z) = \psi(z) d\lambda(z) \)) is absolutely continuous. So the boundary function \( F^b \) exists.

Next, and from the definition of the non-tangential maximal function \( N(F) \), we have that

\[
|F^b(z)| \leq N(F)(z) \quad \text{for } \Omega \text{ - a.e. } z \in bD.
\]

Thus, \( F^b \in L^p(bD, \Omega) \). Also note that with the same methods of [28] one can show that

\[
N(F)(z) \lesssim M(F^b)(z) \quad \text{for } \Omega \text{ - a.e. } z \in bD,
\]

where \( M(F^b) \) is the Hardy–Littlewood maximal function on the boundary \( bD \). Since the maximal function is bounded on \( L^p(bD, \Omega) \), we obtain that

\[
\|F\|_{H^p(bD, \Omega)} = \|N(F)\|_{L^p(bD, \Omega)} \lesssim \|M(F^b)\|_{L^p(bD, \Omega)} \lesssim C[\psi]_{A_p} \|F^b\|_{L^p(bD, \Omega)}.
\]

Suppose now that \( \{F_n\} \) is a sequence in \( H^p(bD, \Omega) \) and \( f \in L^p(bD, \Omega) \) such that

\[
\|N(F_n) - f\|_{L^p(bD, \Omega)} \to 0.
\]
Then, it is clear that
\[
\|\mathcal{N}(F_n) - f\|_{L^p(bD,\lambda)} \leq \|\mathcal{N}(F_n) - f\|_{L^p(bD,\Omega)} \left(\psi^{-\frac{m}{p-m}}(bD)\right)^{\frac{p-m}{p}} \to 0.
\]
Since \(H^{p_0}(D,\lambda)\) is a proper subspace of \(L^{p_0}(bD,\lambda)\), then in particular \(\{F_n\}_n\) is in \(H^{p_0}(D,\lambda)\) and \(f\) is in \(L^{p_0}(bD,\lambda)\), we see that there is \(F\) that is holomorphic in \(D\) and such that
\[
F^b(w) = f(w) \quad \lambda - \text{a.e. } w \in bD.
\]
Again, this implies that
\[
F^b(w) = f(w) \quad \Omega - \text{a.e. } w \in bD.
\]
Moreover, invoking the weighted boundedness of the Hardy–Littlewood maximal function, we see that
\[
\|\mathcal{N}(F)\|_{L^p(bD,\Omega)} \lesssim \|f\|_{L^p(bD,\Omega)}.
\]
The proof of Proposition 1.4 is complete. \(\square\)

3. Quantitative estimates for the Cauchy–Leray integral and its commutator

As before, in the proofs of all statements in this section we adopt the shorthand \(\Omega\) for \(\Omega_p\) and \(\psi\) for \(\psi_p\).

**Theorem 3.1.** Let \(D \subset \mathbb{C}^n\), \(n \geq 2\), be a bounded, strongly pseudoconvex domain of class \(C^2\). Then the Cauchy-type integral \(\mathcal{C}_\epsilon\) is bounded on \(L^p(bD,\Omega_p)\) for any \(0 < \epsilon < \epsilon(D)\), any \(1 < p < \infty\) and any \(A_p\)-measure \(\Omega_p\), with
\[
\|\mathcal{C}_\epsilon\|_{L^p(bD,\Omega_p)\to L^p(bD,\Omega_p)} \lesssim [\Omega_p]_{A_p}^{\max\{1,\frac{1}{p-1}\}},
\]
where the implied constant depends on \(p\), \(D\), and \(\epsilon\) but is independent of \(\Omega_p\).

It follows that for any \(A_2\)-measure \(\Omega_2\), the \(L^2(bD,\Omega_2)\)-adjoint \(\mathcal{C}_\epsilon^\dagger\) is also bounded on \(L^p(bD,\Omega_p)\) with same bound.

**Proof.** To begin with, we first recall that the Cauchy integral operator \(\mathcal{C}_\epsilon\) can be split into the essential part and remainder, that is, \(\mathcal{C}_\epsilon = \mathcal{C}_\epsilon^e + \mathcal{R}_\epsilon\). Denote by \(\mathcal{C}_\epsilon^e(w,z)\) and \(\mathcal{R}_\epsilon(w,z)\) the kernels of \(\mathcal{C}_\epsilon^e\) and \(\mathcal{R}_\epsilon\), respectively.

Recall from [16] that \(\mathcal{C}_\epsilon^e(w,z)\) is a standard Calderón–Zygmund kernel, i.e. there exists a positive constant \(A_1\) such that for every \(w, z \in bD\) with \(w \neq z\),
\[
\begin{align*}
\left| \mathcal{C}_\epsilon^e(w,z) \right| & \leq A_1 \frac{1}{d(w,z)^{2n}}; \\
\left| \mathcal{C}_\epsilon^e(w,z) - \mathcal{C}_\epsilon^e(w',z) \right| & \leq A_1 \frac{d(w,w')}{d(w,z)^{2n+1}}, \quad \text{if } d(w,z) \geq cd(w,w'); \\
\left| \mathcal{C}_\epsilon^e(w,z) - \mathcal{C}_\epsilon^e(w,z') \right| & \leq A_1 \frac{d(z,z')}{d(w,z)^{2n+1}}, \quad \text{if } d(w,z) \geq cd(z,z')
\end{align*}
\]
for an appropriate constant \(c > 0\) and where \(d(z,w)\) is a quasi-distance suitably adapted to \(bD\). And hence, the \(L^p(bD)\) boundedness \((1 < p < \infty)\) of \(\mathcal{C}_\epsilon^e\) follows from a version of the \(T(1)\) Theorem. Moreover, we also get that there exists a positive constant \(A_2\) such that for every \(w, z \in bD\) with \(w \neq z\),
\[
\left| \mathcal{C}_\epsilon^e(w,z) \right| \geq A_2 \frac{1}{d(w,z)^{2n}}.
\]
However, the kernel $R_\epsilon(w, z)$ of $\mathcal{R}_\epsilon$ satisfies a size condition and a smoothness condition for only one of the variables as follows: there exists a positive constant $C_R$ such that for every $w, z \in bD$ with $w \neq z$,

\[
\begin{aligned}
&d) \quad |R_\epsilon(w, z)| \leq C_R \frac{1}{d(w, z)^{2n-1}}, \\
&c) \quad |R_\epsilon(w, z) - R_\epsilon(w, z')| \leq C_R \frac{d(z, z')}{d(w, z)^{2n}}, \quad \text{if } d(w, z) \geq c_R d(z, z')
\end{aligned}
\]

for an appropriate large constant $C_R$.

Since the kernel of $\mathcal{C}_\epsilon^2$ is a standard Calderón–Zygmund kernel on $bD \times bD$, according to [9] (see also [14]), we can obtain that $\mathcal{C}_\epsilon^2$ is bounded on $L^p(bD, \Omega)$ with

\[
\|\mathcal{C}_\epsilon^2\|_{L^p(bD, \Omega)\rightarrow L^p(bD, \Omega)} \lesssim [\psi]_{A_p}^{\max\{1, \frac{1}{2p}\}}.
\]

Thus, it suffices to show that $\mathcal{R}_\epsilon$ is bounded on $L^p(bD, \Omega)$ with the appropriate quantitative estimate.

To see this, we claim that for every $f \in L^p(bD, \Omega), \tilde{z} \in bD$, there exists a $q \in (1, p)$ such that

\[
\big| (\mathcal{R}_\epsilon(f))^{\#}(\tilde{z}) \big| \lesssim \left( M(|f|^q)(\tilde{z}) \right)^{1/q},
\]

where $F^\#$ is the sharp maximal function of $F$ as recalled in Section 2.

We now show this claim. Since $bD$ is bounded, there exists $\mathcal{C} > 0$ such that for any $B_r(z) \subset bD$ we have $r < \mathcal{C}$. For any $\tilde{z} \in bD$, let us fix a ball $B_r = B_r(z_0) \subset bD$ containing $\tilde{z}$, and let $z$ be any point of $B_r$. Now take $j_0 = \lfloor \log_2 \mathcal{C} \rfloor + 1$. Since $d$ is a quasi-distance, there exists $i_0 \in \mathbb{Z}^+$, independent of $z, r$, such that $d(w, z) > c_R r$ whenever $w \in bD \setminus B_{2^{i_0} r}$, where $c_R$ is in (3.3). We then write

\[
\mathcal{R}_\epsilon(f)(z) = (\mathcal{R}_\epsilon(f\chi_{bD \cap B_{2^{2j_0}}}) (z) - \mathcal{R}_\epsilon(f\chi_{bD \cap B_{2^{j_0}}}) (z)) = I(z) + II(z).
\]

For the term $I$, by using Hölder’s inequality and the fact that $\mathcal{R}_\epsilon$ is bounded on $L^q(bD, \lambda)$, $1 < q < \infty$, we have

\[
\frac{1}{\lambda(B_r)} \int_{B_r} |I(z) - I_{B_r}| d\lambda(z) \leq 2 \left( \frac{1}{\lambda(B_r)} \int_{B_r} \left| \mathcal{R}_\epsilon(f\chi_{bD \cap B_{2^{j_0}}}) (z) \right|^q d\lambda(z) \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( \frac{1}{\lambda(B_r)} \int_{bD \cap B_{2^{j_0}}} |f(z)|^q d\lambda(z) \right)^{\frac{1}{q}}
\]

\[
\lesssim \left( M(|f|^q)(\tilde{z}) \right)^{\frac{1}{q}}.
\]

To estimate $II$, observe that if $i_0 \geq j_0$, then $bD \setminus B_{2^{i_0} r} = \emptyset$ and $|II(z) - II(z_0)| = 0$. If $i_0 < j_0$, then we have

\[
|II(z) - II(z_0)| = |\mathcal{R}_\epsilon(f\chi_{bD \cap B_{2^{i_0}}}) (z) - \mathcal{R}_\epsilon(f\chi_{bD \cap B_{2^{j_0}}}) (z_0)|
\]

\[
\leq \int_{bD \setminus B_{2^{i_0} r}} |R_\epsilon(w, z) - R_\epsilon(w, z_0)| |f(w)| d\lambda(w)
\]

\[
\leq d(z, z_0) \int_{bD \setminus B_{2^{i_0} r}} \frac{1}{d(w, z_0)^{2n}} |f(w)| d\lambda(w)
\]
In fact, by Hölder’s inequality and that (3.6)

\[
\int_{bD \setminus B_{2^j r}} \frac{1}{d(w, z_0)^{2n}} |f(w)|^q d\lambda(w) \leq \sum_{j=0}^{j_0} \frac{1}{(2j)^{2n}} \int_{d(w, z_0) \leq 2^{j+1}r} |f(w)|^q d\lambda(w)
\]

\[
\leq \sum_{j=0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} |f(w)|^q d\lambda(w)
\]

\[
\lesssim j_0 M(||f||^q)(\tilde{z}).
\]

Similarly, we have

\[
\int_{bD \setminus B_{2^j r}} \frac{1}{d(w, z_0)^{2n}} d\lambda(w) \leq \sum_{j=0}^{j_0} \frac{1}{\lambda(B_{2^{j+1}r})} \int_{B_{2^{j+1}r}} d\lambda(w) \lesssim j_0.
\]

Thus, we get that \(|II(z) - II(z_0)| \lesssim j_0 \left( M(||f||^q)(\tilde{z}) \right)^{\frac{1}{q}}
\]

Therefore,

\[
\frac{1}{\lambda(B_r)} \int_{B_r} |II(z) - II_{B_r}| d\lambda(z) \leq \frac{2}{\lambda(B_r)} \int_{B_r} |II(z) - II(z_0)| d\lambda(z) \lesssim j_0 \left( M(||f||^q)(\tilde{z}) \right)^{\frac{1}{q}}
\]

\[
\lesssim r \left( \log_2 \left( \frac{C}{r} \right) + 1 \right) \left( M(||f||^q)(\tilde{z}) \right)^{\frac{1}{q}} \lesssim (M(||f||^q)(\tilde{z}))^{\frac{1}{q}},
\]

where the last inequality comes from the fact that \( r \log_2 \left( \frac{C}{r} \right) \) is uniformly bounded. Combining the estimates on \( I \) and \( II \), we see that the claim (3.4) holds.

We now prove that \( \mathcal{R}_e \) is bounded on \( L^p(bD, \Omega) \). In fact, for \( f \in L^p(bD, \Omega) \)

\[
(3.5) \quad \| \mathcal{R}_e(f) \|_{L^p(bD, \Omega)} \leq C \left( \Omega(bD)(\mathcal{R}_e(f)_{bD})^p + \| \mathcal{R}_e(f) \|^p_{L^p(bD, \Omega)} \right)
\]

\[
\leq C \left( \Omega(bD)(\mathcal{R}_e(f)_{bD})^p + \| (M(||f||^q))^{\frac{1}{q}} \|^p_{L^p(bD, \Omega)} \right)
\]

\[
\lesssim \Omega(bD)(\mathcal{R}_e(f)_{bD})^p + \| f \|_{L^p(bD, \Omega)}^{p \max\{1, \frac{1}{q} \}}
\]

where the second inequality follows from (3.4) and the last inequality follows from the fact that the Hardy–Littlewood function is bounded on \( L^p(bD, \Omega) \). We point out that

\[
(3.6) \quad \Omega(bD)(\mathcal{R}_e(f)_{bD})^p \lesssim ||f||_{L^p(bD, \Omega)}^p.
\]

In fact, by Hölder’s inequality and that \( \mathcal{R}_e \) is bounded on \( L^q(bD, \lambda) \), \( 1 < q < \infty \), we have

\[
\Omega(bD)(\mathcal{R}_e(f)_{bD})^p \leq \Omega(bD) \left( \frac{1}{\lambda(bD)} \int |\mathcal{R}_e(f)(z)|^q d\lambda(z) \right)^{\frac{p}{q}}
\]
Following the standard approach (see for example [A]), we now prove that the commutator
\[(i): \text{We begin with proving the sufficiency. Suppose}
\]
\[
\text{Proof. Proof of Part (i):}
\]
\[
\text{Thus, it suffices to verify that}
\]
\[
\text{Note that}
\]
\[
\text{Moreover, for any}
\]
\[
\text{and with}
\]
\[
\text{we have that}
\]
\[
\text{Therefore, (3.6) holds, which, together with (3.5), implies that}
\]
\[
\text{Theorem 3.2. Let}
\]
\[
\text{and let}
\]
\[
\text{be the Leray Levi measure for}
\]
\[
\text{Theorem 3.2.}
\]
\[
\text{Let}
\]
\[
\text{implies that}
\]
\[
\text{The implied constants depend solely on}
\]
\[
\text{Moreover, for any}
\]
\[
\text{and with}
\]
\[
\text{denoting the adjoint of}
\]
\[
\text{we have that (i) and (ii) above also hold with}
\]
\[
\text{Proof. Proof of Part (i):}
\]
\[
\text{we begin with proving the sufficiency. Suppose}
\]
\[
\text{we now prove that the commutator}
\]
\[
\text{Note that}
\]
\[
\text{and}
\]
\[
\text{and}
\]
\[
\text{where}
\]
\[
\text{In fact, employing the same decomposition as in the proof of Theorem 3.1, we obtain that}
\]
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\]
\[
\text{where}
\]
\[
\text{Hence, we have}
\]
\[
\text{We have}
\]
\[
\text{Therefore,}
\]
\[
\text{with}
\]
\[
\text{with the correct quantitative bounds. The proof of Theorem 3.1 is complete.}
\]
Then, combining the size condition \((\text{for some } R \text{ with } \lambda R \not\equiv 0)\) and the smoothness conditions in \((3.1)\), we get that for every ball \(B = B_r(z_0) \subset bD\) with \(r \leq r_0\), there exists another ball \(B' = B_{r'}(w_0) \subset bD\) with \(A_4 r' \leq d(w_0, z_0) \leq (A_3 + 1)r\) such that at least one of the following properties holds:

a) For every \(z \in B\) and \(w \in B', C^x_1(z, w)\) does not change sign and \(|C^x_1(z, w)| \geq \frac{A_4}{d(w, z)^{2n}}\);

b) For every \(z \in B\) and \(w \in B'\), \(C^x_2(z, w)\) does not change sign and \(|C^x_2(z, w)| \geq \frac{A_5}{d(w, z)^{2n}}\).

Then, without loss of generality, we assume that the property a) holds. Then combining with the size estimate of \(R(z, w)\) in \((3.3)\), we obtain that there exists a positive constant \(A_6\) such that for every \(z \in B\) and \(w \in B\), \(C^x_1(w, z) + R_1(w, z)\) does not change sign and that

\[
|C^x_1(w, z) + R_1(w, z)| \geq \frac{A_6}{d(w, z)^{2n}}.
\]

We test the \(\text{BMO}(bD, \lambda)\) condition on the case of balls with big radius and small radius.

Case 1: In this case we work with balls with a large radius, \(r \geq r_0\).

By \((2.10)\) and by the fact that \(\lambda(B) \geq \lambda(B_{r_0}(z_0)) \approx \gamma_0^{2n}\), we obtain that

\[
\frac{1}{\lambda(B)} \int_B |b(z) - b_B| d\lambda(z) \lesssim \frac{1}{\lambda(B_{r_0}(z_0))} ||b||_{L^1(bD, \lambda)} \lesssim \gamma_0^{-2n} ||b||_{L^1(bD, \lambda)}.
\]

Case 2: In this case we work with balls with a small radius, \(0 < r < r_0\).
We aim to prove that for every fixed ball $B = B_r(z_0) \subset bD$ with radius $r < \gamma_0$,

\[(3.11) \quad \frac{1}{\lambda(B)} \int_B |b(z) - b_B|d\lambda(z) \lesssim \|b, C_\varepsilon\|_p,\]

which, combining with (4.36), finishes the proof of the necessity part.

Now let $\tilde{B} = B_r(w_0)$ be the ball chosen as above, and let $m_b(\tilde{B})$ be the median value of $b$ on the ball $\tilde{B}$ with respect to the measure $\lambda$ defined as follows: $m_b(\tilde{B})$ is a real number that satisfies simultaneously

$$\lambda(\{w \in \tilde{B} : b(w) > m_b(\tilde{B})\}) \leq \frac{1}{2}\lambda(\tilde{B}) \quad \text{and} \quad \lambda(\{w \in \tilde{B} : b(w) < m_b(\tilde{B})\}) \leq \frac{1}{2}\lambda(\tilde{B}).$$

Then, following the idea in [20, Proposition 3.1] by the definition of median value, we choose $F_1 := \{w \in \tilde{B} : b(w) \leq m_b(\tilde{B})\}$ and $F_2 := \{w \in \tilde{B} : b(w) \geq m_b(\tilde{B})\}$. Then it is direct that $\tilde{B} = F_1 \cup F_2$, and moreover, from the definition of $m_b(\tilde{B})$, we see that

\[(3.12) \quad \lambda(F_i) \geq \frac{1}{2}\lambda(\tilde{B}), \quad i = 1, 2.\]

Next we define $E_1 = \{z \in B : b(z) \geq m_b(\tilde{B})\}$ and $E_2 = \{z \in B : b(z) < m_b(\tilde{B})\}$. Then $B = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$. Then it is clear that $b(z) - b(w)$ is non-negative for any $(z, w) \in E_1 \times F_1$, and is negative for any $(z, w) \in E_2 \times F_2$. Moreover, for $(z, w)$ in $(E_1 \times F_1) \cup (E_2 \times F_2)$, we have

\[(3.13) \quad |b(z) - b(w)| \geq |b(z) - m_b(\tilde{B})|.

Therefore, from (3.10), (3.12), and (3.13) we obtain that

\[
\begin{align*}
\frac{1}{\lambda(B)} \int_{E_1} |b(z) - m_b(\tilde{B})|d\lambda(z) \\
\quad \lesssim \frac{1}{\lambda(B)} \frac{\lambda(F_1)}{\lambda(\tilde{B})} \int_{E_1} |b(z) - m_b(\tilde{B})|d\lambda(z) \\
\quad \lesssim \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} \frac{1}{d(w, z)^{2n}} |b(z) - b(w)|d\lambda(w)d\lambda(z) \\
\quad \lesssim \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} \left| C_{\varepsilon, 1}(w, z) + R_{1, \varepsilon}(w, z) \right| (b(z) - b(w))d\lambda(w)d\lambda(z) \\
\quad \lesssim \frac{1}{\lambda(B)} \int_{E_1} \int_{F_1} \left| C_{\varepsilon}(w, z) (b(z) - b(w)) \right| d\lambda(w)d\lambda(z) \\
\quad \lesssim \frac{1}{\lambda(B)} \int_{E_1} \left| b, C_\varepsilon \right| (\chi_{F_1})(z) d\lambda(z),
\end{align*}
\]

(3.14)

where the last but second inequality follows from the fact that $C_{\varepsilon, 1}(w, z) + R_{1, \varepsilon}(w, z)$ is the real part of $C_{\varepsilon}(w, z)$.

Then, by using Hölder’s inequality and (4.36), $\Omega_p \in A_p$ with the density function $\psi$, we further obtain that the right-hand side of (3.14) is bounded by

$$\frac{1}{\lambda(B)} \left( \int_{E_1} \psi^{-\frac{p}{p'}}(z)d\lambda(z) \right)^{\frac{1}{p'}} \left( \int_{E_1} \left| b, C_\varepsilon \right| (\chi_{F_1})(z) \psi(z)d\lambda(z) \right)^{\frac{1}{p}}.$$
For the sake of simplicity we drop the subscript for a nonzero \( \phi \). Similarly, we can obtain that

\[
\frac{1}{\lambda(B)} \int_{E_2} \left| b(z) - m_b(\tilde{B}) \right| d\lambda(z) \lesssim \| b, \mathcal{C}_e \|_{L^p(bD, \Omega_p)}.
\]

As a consequence, we get that

\[
\frac{1}{\lambda(B)} \int_B \left| b(z) - m_b(\tilde{B}) \right| d\lambda(z) \lesssim \frac{1}{\lambda(B)} \int_{E_1} \left| b(z) - m_b(\tilde{B}) \right| d\lambda(z)
\]

\[
\lesssim \| b, \mathcal{C}_e \|_{L^p(bD, \Omega_p)}.
\]

Therefore,

\[
\frac{1}{\lambda(B)} \int_B \left| b(z) - b_B \right| d\lambda(z) \leq \frac{2}{\lambda(B)} \int_B \left| b(z) - m_b(\tilde{B}) \right| d\lambda(z) \lesssim \| b, \mathcal{C}_e \|_{L^p(bD, \Omega_p)}
\]

which gives (3.11). Combining the estimates in Case 1 and Case 2 above, we see that \( b \) is in \( \text{BMO}(bD, \lambda) \). The proof of Part (1) is concluded.

Proof of Part (ii): We begin by showing the sufficiency. Suppose \( b \in \text{VMO}(bD, \lambda) \). Note that \( [b, \mathcal{C}_e] = [b, \mathcal{C}_e^j] + [b, \mathcal{R}_e] \), and that \( [b, \mathcal{C}_e^j] \) is a compact operator on \( L^p(bD, \omega) \) (following the standard argument in [13]), it suffices to verify that \( [b, \mathcal{R}_e] \) is compact on \( L^p(bD, \Omega_p) \). However, this follows from the approach in the proof of (ii) of Theorem D ([2, Theorem 1.1]).

We now prove the necessity. Suppose that \( b \in \text{BMO}(bD, \lambda) \) and that \( [b, \mathcal{C}_e] \) is compact on \( L^p(bD, \Omega) \) for some \( 1 < p < \infty \). Without loss of generality, we assume that \( \| b \|_{\text{BMO}(bD, \lambda)} = 1 \).

We now use the idea from [2]. To show \( b \in \text{VMO}(bD, \lambda) \), we seek the contradiction: there is no bounded operator \( T : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N}) \) with \( T e_j = T e_k \neq 0 \) for all \( j, k \in \mathbb{N} \). Here, \( e_j \) is the standard basis for \( \ell^p(\mathbb{N}) \). Thus, it suffices to construct the approximates to a standard basis in \( \ell^p \), namely a sequence of functions \( \{g_j\} \) such that \( \|g_j\|_{L^p(bD, \Omega_p)} \approx 1 \), and for a nonzero \( \phi \), we have \( \| \phi - [b, \mathcal{C}_e] g_j \|_{L^p(bD, \Omega_p)} < 2^{-j} \).

Suppose that \( b \notin \text{VMO}(bD, \lambda) \), then there exist \( \delta_0 > 0 \) and a sequence \( \{B_j\}_{j=1}^\infty := \{B_{r_j}(z_j)\}_{j=1}^\infty \) of balls such that

\[
(3.15) \quad \frac{1}{\lambda(B_j)} \int_{B_j} \left| b(z) - b_{B_j} \right| d\lambda(z) \geq \delta_0.
\]

Without loss of generality, we assume that for all \( j \), \( r_j < \gamma_0 \), where \( \gamma_0 \) is the fixed constant in the argument for (3.10).

Now choose a subsequence \( \{B_{j_i}\} \) of \( \{B_j\} \) such that

\[
(3.16) \quad r_{j_{i+1}} \leq \frac{1}{4 c_\omega} r_{j_i},
\]

where \( c_\omega \) is the constant such that

\[
(3.17) \quad c_\omega^{-1} r^{2n} \leq \lambda(B_r(w)) \leq c_\omega r^{2n}, \quad 0 < r \leq 1.
\]

For the sake of simplicity we drop the subscript \( i \), i.e., we still denote \( \{B_{j_i}\} \) by \( \{B_j\} \).
Thus, combining with (3.18) we can find disjoint subsets \( F_{j,1}, F_{j,2} \subset \tilde{B}_j \) such that
\[
F_{j,1} \subset \{ w \in \tilde{B}_j : b(w) \leq m_b(\tilde{B}_j) \}, \quad F_{j,2} \subset \{ w \in \tilde{B}_j : b(w) \geq m_b(\tilde{B}_j) \},
\]
and
\[
\lambda(F_{j,1}) = \lambda(F_{j,2}) = \frac{\lambda(\tilde{B}_j)}{2}.
\]

Next we define \( E_{j,1} = \{ z \in B : b(z) \geq m_b(\tilde{B}_j) \}, \ E_{j,2} = \{ z \in B : b(z) < m_b(\tilde{B}_j) \}, \) then \( B_j = E_{j,1} \cup E_{j,2} \) and \( E_{j,1} \cap E_{j,2} = \emptyset \). Then it is clear that \( b(z) - b(w) \geq 0 \) for \((z, w) \in E_{j,1} \times F_{j,1}\) and \( b(z) - b(w) < 0 \) for \((z, w) \in E_{j,2} \times F_{j,2}\). And for \((z, w)\) in \((E_{j,1} \times F_{j,1}) \cup (E_{j,2} \times F_{j,2})\), we have
\[
|b(z) - b(w)| \geq |b(z) - m_b(\tilde{B}_j)|.
\]

We now consider
\[
\tilde{F}_{j,1} := F_{j,1} \bigcup_{\ell = j+1}^{\infty} \tilde{B}_\ell \quad \text{and} \quad \tilde{F}_{j,2} := F_{j,2} \bigcup_{\ell = j+1}^{\infty} \tilde{B}_\ell, \quad \text{for } j = 1, 2, \ldots.
\]

Then, based on the decay condition of the radius \( \{ r_j \} \), we obtain that for each \( j \),
\[
\lambda(\tilde{F}_{j,1}) \geq \lambda(F_{j,1}) - \lambda\left( \bigcup_{\ell = j+1}^{\infty} \tilde{B}_\ell \right) \geq \frac{1}{2} \lambda(\tilde{B}_j) - \sum_{\ell = j+1}^{\infty} \lambda(\tilde{B}_\ell) \geq \frac{1}{4} \lambda(\tilde{B}_j).
\]

Now for each \( j \), we have that
\[
\frac{1}{\lambda(\tilde{B}_j)} \int_{\tilde{B}_j} |b(z) - b_{B_j}| d\lambda(z) \leq \frac{2}{\lambda(\tilde{B}_j)} \int_{\tilde{B}_j} |b(z) - m_b(\tilde{B}_j)| d\lambda(z)
\]
\[
= \frac{2}{\lambda(\tilde{B}_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) + \frac{2}{\lambda(\tilde{B}_j)} \int_{E_{j,2}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z).
\]

Thus, combining with (3.15) and the above inequalities, we obtain that as least one of the following inequalities holds:
\[
\frac{2}{\lambda(\tilde{B}_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}, \quad \frac{2}{\lambda(\tilde{B}_j)} \int_{E_{j,2}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}.
\]

We may assume that the first one holds, i.e.,
\[
\frac{2}{\lambda(\tilde{B}_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z) \geq \frac{\delta_0}{2}.
\]

Therefore, for each \( j \), from (3.18) and (3.19) and by using (3.14), we obtain that
\[
\frac{\delta_0}{4} \leq \frac{1}{\lambda(\tilde{B}_j)} \int_{E_{j,1}} |b(z) - m_b(\tilde{B}_j)| d\lambda(z)
\]
This contradiction shows that

\[ \text{We denote the limit function by } g. \]

Moreover, since \( \psi \in A_p \), it follows that there exist positive constants \( C_1, C_2 \) and \( \sigma \in (0,1) \) such that for any measurable set \( E \subset B \),

\[ \left( \frac{\lambda(E)}{\lambda(B)} \right)^\sigma \leq C_1 \frac{\Omega_p(E)}{\Omega_p(B)} \leq C_2 \left( \frac{\lambda(E)}{\lambda(B)} \right)^\sigma. \]

Hence, from (3.20), we obtain that \( 4^{-\frac{\sigma}{p}} \leq \| f_j \|_{L^p(bD, \Omega_p)} \leq 1 \). Thus, it is direct to see that \( \{ f_j \}_j \) is a bounded sequence in \( L^p(bD, \Omega_p) \) with a uniform \( L^p(bD, \Omega_p) \)-lower bound away from zero.

Since \( [b, \mathcal{E}_\gamma] \) is compact, we obtain that the sequence \( \{ [b, \mathcal{E}_\gamma](f_j) \}_j \) has a convergent subsequence, denoted by

\[ \{ [b, \mathcal{E}_\gamma](f_j) \}_j. \]

We denote the limit function by \( g_0 \), i.e.,

\[ [b, \mathcal{E}_\gamma](f_j) \to g_0 \quad \text{in } L^p(bD, \Omega_p), \quad \text{as } i \to \infty. \]

Moreover, \( g_0 \neq 0 \).

After taking a further subsequence, labeled \( \{ g_{j_i} \}_{i=1}^\infty \), we have

- \( \| g_{j_i} \|_{L^p(bD, \Omega_p)} \approx 1 \);
- \( g_{j_i} \) are disjointly supported;
- and \( \| g_0 - [b, \mathcal{E}_\gamma]g_{j_i} \|_{L^p(bD, \Omega_p)} < 2^{-i} \).

Take \( a_j = 2^{-j} \), so that \( \{ a_j \}_{j=1}^\infty \subset \ell^\infty \setminus \ell^1 \). It is immediate that \( g = \sum_j a_j g_j \in L^p(bD, \Omega_p) \), hence \( [b, \mathcal{E}_\gamma] g \in L^p(bD, \Omega_p) \). But, \( g_0 \sum_j a_j \equiv \infty \), and yet

\[ \left\| g_0 \sum_j a_j \right\|_{L^p(bD, \Omega_p)} \leq \| [b, \mathcal{E}_\gamma] g \|_{L^p(bD, \Omega_p)} + \sum_j\| g_0 - [b, \mathcal{E}_\gamma] g_j \|_{L^p(bD, \Omega_p)} < \infty. \]

This contradiction shows that \( b \in \text{VMO}(bD, \lambda) \).

The proof of Theorem 3.2 is complete. \( \square \)

We now turn to the proof of the new cancellation (1.10).

**Proposition 3.3.** For any fixed \( 0 < \epsilon < \epsilon(D) \) as in [16], there exists \( s = s(\epsilon) > 0 \) such that

\[ \| (\mathcal{E}_\epsilon^*)^{\dagger} - \mathcal{E}_\epsilon^{\dagger} \|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p)} \lesssim \epsilon^{1/2} \Omega_p^{\max\{1, \frac{1}{p-1}\}} \]

for any \( 1 < p < \infty \) and for any \( A_p \)-measure, \( \Omega_p \) where the implied constant depends on \( D \) and \( p \) but is independent of \( \Omega_p \) and of \( \epsilon \). As before, here \( (\mathcal{E}_\epsilon^*)^{\dagger} \) denotes the adjoint in \( L^2(bD, \omega) \).
Proof. Recall from [16, (5.7)], that $\mathcal{C}_e^s$ is given by
\[
\mathcal{C}_e^s(f)(z) = \mathcal{C}_e(f(\cdot)\chi_s(\cdot, z))(z), \quad z \in bD,
\]
where $\chi_s(w, z)$ is the cutoff function given by $\chi_s(w, z) = \bar{\chi}_{s,w}(z)\bar{\chi}_{s,z}(w)$ with
\[
\bar{\chi}_{s,w}(z) = \chi\left(\frac{\text{Im}\langle \partial \rho(w), w - z \rangle}{cs^2} + i\frac{|w - z|^2}{cs^2}\right).
\]
Here $\chi$ is a non-negative $C^1$-smooth function on $\mathbb{C}$ such that $\chi(u + iv) = 1$ if $|u + iv| \leq 1/2$, $\chi(u + iv) = 0$ if $|u + iv| \geq 1$, and furthermore $|\nabla \chi(u + iv)| \lesssim 1$.

Then we also have the essential part and the remainder of $\mathcal{C}_e^s$, which are given by
\[
(3.24)
\]
and hence we have
\[
\|\mathcal{L}_bD\|\langle 3.22 \rangle
\]
where the constant $M$ is given by
\[
(3.24)
\]
and hence we have
\[
\mathcal{C}_e^s = \mathcal{C}_e^s + \mathcal{R}_e^s, \quad \text{where } \mathcal{R}_e^s \text{ is the remainder. Further, } (\mathcal{C}_e^s)^* = (\mathcal{C}_e^s)^* + (\mathcal{R}_e^s)^*.
\]
To prove (3.21), we note that $(\mathcal{C}_e^s)^* = \varphi^{-1}(\mathcal{C}_e^s)^* \varphi$, where $\varphi$ is the density function of $\omega$ satisfying (1.13). Thus, $(\mathcal{C}_e^s)^* - \mathcal{C}_e^s = (\mathcal{C}_e^s)^* - (\mathcal{C}_e^s)^* - \varphi^{-1}[\varphi, (\mathcal{C}_e^s)^*]$, which gives
\[
\|((\mathcal{C}_e^s)^* - \mathcal{C}_e^s)\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} \leq \|\varphi^{-1}[\varphi, (\mathcal{C}_e^s)^*]\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} + \|\varphi^{-1}[\varphi, (\mathcal{C}_e^s)^*]\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)}.
\]
Thus, we claim that
\[
\|((\mathcal{C}_e^s)^* - \mathcal{C}_e^s)\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} \leq c_{\mathcal{C}_e}^s \|\psi\|_{A_p}^{\max(1, \frac{1}{r-1})} M(p, D, \omega)
\]
and that
\[
\|\varphi^{-1}[\varphi, (\mathcal{C}_e^s)^*]\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} \leq c_{\mathcal{C}_e}^s \|\psi\|_{A_p}^{\max(1, \frac{1}{r-1})} M(p, D, \omega).
\]
We first prove (3.22). To begin with, we write $\mathcal{C}_e^s - (\mathcal{C}_e^s)^* = \mathcal{A}_e^s + \mathcal{B}_e^s$, where
\[
\mathcal{A}_e^s = \mathcal{C}_e^s - (\mathcal{C}_e^s)^* \quad \text{and} \quad \mathcal{B}_e^s = (\mathcal{C}_e^s) - (\mathcal{C}_e^s)^*.
\]
Let $A_e^s(w, z)$ be the kernel of $\mathcal{A}_e^s$. Then, from [16, (5.10) and (5.9)], we see that
\[
1) \quad |A_e^s(w, z)| \lesssim \epsilon \frac{1}{d(w, z)^{2n+1/2}} \quad \text{for any } s \leq s(\epsilon);
\]
and
\[
2) \quad |A_e^s(w, z) - A_e^s(w', z)| \lesssim \epsilon \frac{1}{d(w, z)^{2n+1/2}} \quad \text{for any } s \leq s(\epsilon);
\]
and
\[
3) \quad |A_e^s(w, z) - A_e^s(w', z')| \lesssim \epsilon \frac{1}{d(w, z)^{2n+1/2}} \quad \text{for any } s \leq s(\epsilon).
\]
From the kernel estimates, we see that $\epsilon^{-1/2}A_e^s, s \leq s(\epsilon), is a standard Calderón–Zygmund operator on $bD \times bD$, according to [9] (see also [14]), we can obtain that $\epsilon^{-1/2}A_e^s$ is bounded on $L^p(bD, \Omega)$ with
\[
\|\epsilon^{-1/2}A_e^s\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} \leq \|\psi\|_{A_p}^{\max(1, \frac{1}{r-1})} M(p, D, \omega),
\]
where the constant $M(p, D, \omega)$ depends only on $p$, $D$ and $\omega$. Hence, we obtain that
\[
\|A_e^s\|_{L^p(bD, \Omega)\to L^p(bD, \Omega)} \leq \epsilon^{1/2}\|\psi\|_{A_p}^{\max(1, \frac{1}{r-1})} M(p, D, \omega) \quad \text{for any } s \leq s(\epsilon).
\]
Thus, it suffices to consider $\mathcal{B}_e^s$. Note that now the kernel $R_e^s(w, z)$ of $\mathcal{R}_e^s$ is given by
\[
R_e^s(w, z) = R_e(w, z)\chi_s(w, z),
\]
where $R_\epsilon(w, z)$ is the kernel of $\mathcal{R}_\epsilon$ satisfying (3.3). Thus, it is easy to see that $R_\epsilon^s(w, z)$ satisfies the following size estimate

$$4) \quad |R_\epsilon^s(w, z)| \leq \tilde{c}_\epsilon \frac{\chi_s(w, z)}{d(w, z)^{2n-1}},$$

where the constant $\tilde{c}_\epsilon$ is large, depending on $\epsilon$. For the regularity estimate of $R_\epsilon^s(w, z)$ on $z$, we get that for $d(w, z) \geq c_R d(z, z')$,

$$|R_\epsilon^s(w, z) - R_\epsilon^s(w, z')| \leq \tilde{c}_\epsilon \frac{d(z, z') \cdot \chi_s(w, z) + \frac{1}{d(w, z)^{2n-1}} \cdot |\chi_s(w, z) - \chi_s(w, z')|}{d(w, z) |\chi_s(w, z) - \chi_s(w, z')|},$$

where the last inequality follows from (3.3). Note that from the definition of $\chi_s(w, z)$, we get that $|\chi_s(w, z) - \chi_s(w, z')|$ vanishes unless $d(w, z) \approx d(w, z') \approx s$. Moreover, under this condition, we further have

$$|\chi_s(w, z) - \chi_s(w, z')| \approx \frac{d(z, z')}{d(w, z)}.$$

Combining these estimates, we obtain that

$$5) \quad |R_\epsilon^s(w, z) - R_\epsilon^s(w, z')| \leq \frac{\tilde{c}_\epsilon d(z, z') \chi_s(w, z)}{d(w, z)^{2n}}, \quad \text{if } d(w, z) \geq c_R d(z, z').$$

We now consider the operator $\mathcal{T} = s^{-1} R_\epsilon^s$. Then we claim that

$$\|\mathcal{T}(f)\|_{L^p(bD, \Omega)} \leq \tilde{c}_\epsilon \max \{|\psi| \chi_s, M(p, D, \omega)\} \|f\|_{L^p(bD, \Omega)}.$$ (3.25)

To prove (3.25), following the proof of Theorem 3.1, we first show that for every $f \in L^p(bD, \Omega), \tilde{z} \in bD$, there exists a $q \in (1, p)$ such that

$$\left| (\mathcal{T}(f))^{\#}(\tilde{z}) \right| \leq \tilde{c}_\epsilon \left( M(|f|^q)(\tilde{z}) \right)^{1/q},$$ (3.26)

where $F^{\#}$ is the sharp maximal function of $F$ as recalled in Section 2.

We now show (3.26). Since $bD$ is bounded, there exists $C > 0$ such that for any $B_r(z) \subset bD$ we have $r < C$. For any $\tilde{z} \in bD$, let us fix a ball $B_r = B_r(z_0) \subset bD$ containing $\tilde{z}$, and let $z$ be any point of $B_r$. Since $d$ is a quasi-distance, there exists $t_0 \in \mathbb{Z}^+$, independent of $z, r$, such that $d(w, z) > c_R r$ whenever $w \in bD \setminus B_{2t_0r}$, where $c_R$ is in (3.3). We then write

$$\mathcal{T}(f)(z) = \mathcal{T}(f \chi_{bD \cap B_{2t_0r}})(z) - (f \chi_{bD \cap B_{2t_0r}})(z) =: I(z) + II(z).$$

For the term $I$, by using Hölder’s inequality we have

$$\frac{1}{\lambda(B_r)} \int_{B_r} |I(z) - I_{B_r}| d\lambda(z) \leq 2 \left( \frac{1}{\lambda(B_r)} \int_{B_r} |\mathcal{T}(f \chi_{bD \cap B_{2t_0r}})(z)|^q d\lambda(z) \right)^{1/q}$$

$$\lesssim \tilde{c}_\epsilon \left( \frac{1}{\lambda(B_r)} \int_{bD \cap B_{2t_0r}} |f(z)|^q d\lambda(z) \right)^{1/q}$$

$$\lesssim \tilde{c}_\epsilon \left( M(|f|^q)(\tilde{z}) \right)^{1/q},$$

where the second inequality follows from the boundedness in [16, (4.22)] since the kernel of $\mathcal{T}$ satisfies the conditions in [16, (4.22)].
Thus, we get that $T_A$ is bounded on $L^p(\Omega \backslash B_r)$. In this case, we have

$$|II(z) - II(z_0)| = |T(f \chi_{B_r\backslash B_{2r}})(z) - T(f \chi_{B_r\backslash B_{2r}})(z_0)|$$

$$\leq \int_{B_r\backslash B_{2r}} \frac{c_r}{s} \frac{d(z, z_0)}{d(w, z_0)^{2n}} |f(w)| d\lambda(w)$$

$$\leq \int_{(B_r\backslash B_{2r}) \cap B_s} \frac{c_r}{s} \frac{d(z, z_0)}{d(w, z_0)^{2n}} |f(w)| d\lambda(w),$$

where $B_s := B_s(z_0)$ and the last inequality follows from the property of the function $\chi_s(w, z_0)$. Thus, we see that if $2^s r \geq s$, then the last term is zero. So we just need to consider $2^s r < s$. In this case, we have

$$|II(z) - II(z_0)| \leq \frac{c_r}{s} \int_{B_r\backslash B_{2r}} \frac{1}{d(w, z_0)^{2n}} |f(w)| d\lambda(w)$$

$$\leq \frac{c_r}{s} \left( \int_{B_r\backslash B_{2r}} \frac{1}{d(w, z_0)^{2n}} d\lambda(w) \right)^{\frac{1}{q'}} \left( \int_{B_r\backslash B_{2r}} \frac{1}{d(w, z_0)^{2n}} |f(w)|^q d\lambda(w) \right)^{\frac{1}{q}}.$$

We take $j_0 = \lfloor \log_2 \frac{s}{T} \rfloor + 1$. Continuing the estimate:

$$\int_{B_r\backslash B_{2r}} \frac{1}{d(w, z_0)^{2n}} |f(w)|^q d\lambda(w) \leq \sum_{j=0}^{j_0} \int_{2^{j+1} r \leq d(w, z_0) \leq 2^j r} \frac{1}{d(w, z_0)^{2n}} |f(w)|^q d\lambda(w)$$

$$\leq \sum_{j=0}^{j_0} \frac{1}{(2r)^{2n}} \int_{d(w, z_0) \leq 2^j r} |f(w)|^q d\lambda(w)$$

$$\leq \sum_{j=0}^{j_0} \frac{1}{\lambda(B_{2^{j+1} r})} \int_{B_{2^{j+1} r}} |f(w)|^q d\lambda(w) \lesssim j_0 M(|f|^q)(\bar{z}).$$

Similarly, we have

$$\int_{B_r\backslash B_{2r}} \frac{1}{d(w, z_0)^{2n}} d\lambda(w) \lesssim \sum_{j=0}^{j_0} \frac{1}{\lambda(B_{2^{j+1} r})} \int_{B_{2^{j+1} r}} d\lambda(w) \lesssim j_0.$$

Thus, we get that $|II(z) - II(z_0)| \lesssim c_r j_0 (M(|f|^q)(\bar{z}))^{\frac{1}{q}}$. Therefore,

$$\frac{1}{\lambda(B_r)} \int_{B_r} |II(z) - II(z_0)| d\lambda(z) \leq \frac{2}{\lambda(B_r)} \int_{B_r} |II(z) - II(z_0)| d\lambda(z) \lesssim c_r \frac{r}{s} j_0 (M(|f|^q)(\bar{z}))^{\frac{1}{q}}$$

$$\lesssim c_r \left( \log_2 \left( \frac{s}{T} \right) + 1 \right) (M(|f|^q)(\bar{z}))^{\frac{1}{q}} \lesssim c_r (M(|f|^q)(\bar{z}))^{\frac{1}{q}},$$

where the last inequality comes from the fact that $A (\log_2 \left( \frac{1}{A} \right) + 1)$ is uniformly bounded for $A \in (0, 1]$.

Combining the estimates for $I$ and $II$, we see that the claim (3.26) holds. We now prove that $\mathcal{T}$ is bounded on $L^p(bD, \Omega)$. In fact, for $f \in L^p(bD, \Omega)$

$$(3.27) \quad \|\mathcal{T}(f)\|_{L^p(bD, \Omega)} \leq \left( \Omega(bD)(\mathcal{T}(f)_{bD})^p + \|\mathcal{T}(f)^p\|_{L^p(bD, \Omega)} \right)$$
\[
\Omega(bD)(\mathcal{T}(f)_{bD})^p \leq \Omega(bD)(\mathcal{T}(f)_{bD})^p + c_p^\varepsilon M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}^p
\]

where the second inequality follows from (3.26) and the last inequality follows from the fact that the Hardy–Littlewood function is bounded on \(L^p(bD, \Omega)\) with the constant \(M(p, D, \omega)\) depending only on \(p, D\) and \(\omega\). We point out that

\[
\Omega(bD)(\mathcal{T}(f)_{bD})^p \leq \Omega(bD)(\mathcal{T}(f)_{bD})^p + c_p^\varepsilon [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}^p.
\]

In fact, by Hölder’s inequality and the fact that \(\mathcal{T}\) is bounded on \(L^q(bD, \lambda)\) for all \(1 < q < \infty\) (from [16, (4.22)]), we have

\[
\Omega(bD)(\mathcal{T}(f)_{bD})^p \leq \Omega(bD) \left( \frac{1}{\lambda(bD)} \int_{bD} |\mathcal{T}(f)(z)|^q d\lambda(z) \right)^{\frac{p}{q}}
\]

\[
\leq c_p^\varepsilon \Omega(bD) \left( \frac{1}{\lambda(bD)} \int_{bD} |f(z)|^q d\lambda(z) \right)^{\frac{p}{q}}
\]

\[
\leq c_p^\varepsilon \Omega(bD) \inf_{z \in bD} \left( M(|f|^q)(z) \right)^{\frac{p}{q}}
\]

\[
\leq c_p^\varepsilon \int_{bD} \left( M(|f|^q)(z) \right)^{\frac{p}{q}} d\Omega(z)
\]

\[
\leq c_p^\varepsilon [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}^p.
\]

Therefore, (3.28) holds, which, together with (3.27), implies that the claim (3.25) holds.

As a consequence, we obtain that for every \(1 < p < \infty\),

\[
\|R_\varepsilon^{s, \ast} f\|_{L^p(bD, \Omega)} \leq s \cdot \tilde{c}_\varepsilon \cdot [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}.
\]

By using the fact that \(\Omega \frac{1}{p-1} \) is in \(A_p'\) when \(\Omega\) is in \(A_p\) (\(p'\) is the conjugate index of \(p\)), and by using duality, we obtain that

\[
\|R_\varepsilon^{s, \ast} f\|_{L^p(bD, \Omega)} \leq s \cdot \tilde{c}_\varepsilon \cdot [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}.
\]

Hence, we have

\[
\|B_\varepsilon^{s} f\|_{L^p(bD, \Omega)} \leq s \cdot \tilde{c}_\varepsilon \cdot [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}.
\]

Since \(\varepsilon\) is a fixed small positive constant, we see that

\[
s \cdot \tilde{c}_\varepsilon \approx \varepsilon^{1/2}
\]

if \(s\) is sufficiently small. Thus, we obtain that

\[
\|B_\varepsilon^{s} f\|_{L^p(bD, \Omega)} \leq \varepsilon^{1/2} \cdot [\Omega_p]_{A_p}^{\max\{1, \frac{1}{p-1}\}} M(p, D, \omega) \|f\|_{L^p(bD, \Omega)}.
\]

Combining the estimates of (3.24) and (3.32), we obtain that the claim (3.22) holds.

We now prove (3.23). To begin with, following [16, Section 6.2], we consider a partition of \(\mathbb{C}^n\) into disjoint cubes of side-length \(\gamma\), given by \(\mathbb{C}^n = \bigcup_{k \in \mathbb{Z}^n} Q_k^\gamma\). Then we revert to our
domain $D$. For a fixed $\gamma > 0$, we write $1_k$ for the characteristic function of $Q_k^\gamma \cap bD$. Then we consider $1_k(z)(C^*_x)^*(1_j f)(z)$, we note that if $z$ is not in the support of $f$, then

$$
1_k(z)(C^*_x)^*(1_j f)(z) = 1_k(z) \int_{bD} \frac{C_z(z, w) \chi_s(w, z)}{C_z(z, w)} f(w) 1_j(w) d\lambda(w).
$$

Thus, by choosing $s$ small enough and $\gamma = cs$, where $c$ is a fixed positive constant, we see that $1_k(z)(C^*_x)^*(1_j f)(z)$ vanishes if $Q^\gamma_j$ and $Q_k^\gamma$ do not touch. Next, following the proof in [16, Section 6.3], and combining our result Theorem 3.1 we see that

$$
\|1_k(C^*_x)^*(1_j f)\|_{L^p(bD, \Omega)} \lesssim \epsilon \|C^*_x\|_{L^p(bD, \Omega)} \|f\|_{L^p(bD, \Omega)}
$$

where

$$
\psi \text{ is the density of } \Omega, \text{ and that there exists an absolute positive constant } C.
$$

Hence for $\epsilon = \epsilon(\Omega_2)$ we write

$$
\epsilon^{1/2}[\Omega_2]_{A_2} M(D, \omega) \leq \frac{1}{2},
$$

where $\psi_2$ is the density of $\Omega_2$. Thus, Proposition 3.3 grants

$$
\|((C^*_x)^\dagger - C^*_x)^j g\|_{L^2(bD, \Omega_2)} \leq \frac{1}{2j} \|g\|_{L^2(bD, \Omega_2)}, \quad j = 0, 1, 2, \ldots.
$$

Hence for $\epsilon = \epsilon(\Omega_2)$ as in (4.1) we have that

$$
\|(C^*_x)^\dagger - C^*_x)^j g\|_{L^2(bD, \Omega_2)} \leq 2 \|g\|_{L^2(bD, \Omega_2)},
$$

and by Theorem 3.1 (for $p = 2$) we conclude that

$$
\|A_\epsilon g\|_{L^2(bD, \Omega_2)} \leq C_1(D, \omega) [\Omega_2]_{A_2} \|g\|_{L^2(bD, \Omega_2)}.
$$

We next proceed to bound the norm of $B_\epsilon g$; to this end we recall that the reverse H"older inequality for $\Omega_2 = \psi_2 d\lambda$ ([6, Theorem 9.2.2])

$$
\left( \frac{1}{\lambda(bD)} \int_{bD} \psi_2^{1+\gamma}(z) d\lambda(z) \right)^{\frac{1}{1+\gamma}} \leq C(\Omega_2) \frac{1}{\lambda(bD)} \int_{bD} \psi_2(z) d\lambda(z)
$$

is true for some $\gamma = \gamma(\Omega_2) > 0$.

Recall that by keeping track of the constants (see [6, (9.2.6) and (9.2.7)]), we have that

$$
\sup_{\Omega_2 \in A_2} \gamma(\Omega_2) \leq 1
$$

and that there exists an absolute positive constant $C_2$ such that

$$
\sup_{\Omega_2 \in A_2} C(\Omega_2) \leq C_2^2 < \infty.
$$
Hence, we obtain that

$$
(4.4) \quad \left( \int_{bD} \psi_2^{1+\gamma}(z) \, d\lambda(z) \right)^{\frac{1}{\gamma+1}} \leq C_2^2 \frac{1}{\lambda(bD)^{\frac{1}{\gamma+1}}} \int_{bD} \psi_2(z) \, d\lambda(z).
$$

Recall that, for Leray Levi-like measures we have \(d\omega(z) = \varphi(z) \, d\lambda(z)\), where \(0 < m_\omega \leq \varphi(z) \leq M_\omega < \infty\) for any \(z \in bD\).

Hence, using the shorthand \(\|\|\) we have

$$
\|B_\epsilon g\|_{L^2(bD, \Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} \left( \int_{bD} |S_\omega H|^2(z) \, d\omega(z) \right)^{\frac{\gamma}{2(\gamma+1)}} \left( \int_{bD} \psi_2^{1+\gamma}(z) \, d\lambda(z) \right)^{\frac{1}{2}}.
$$

Hölder’s inequality for \(q := \frac{\gamma + 1}{\gamma}\), where \(\gamma = \gamma(\Omega_2)\) is as in (4.3), now gives that

$$
\|B_\epsilon g\|_{L^2(bD, \Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} \left( \int_{bD} |S_\omega H|^2(z) \, d\omega(z) \right)^{\frac{\gamma}{2(\gamma+1)}} \left( \int_{bD} \psi_2^{1+\gamma}(z) \, d\lambda(z) \right)^{\frac{1}{2}}.
$$

Comparing the above with (4.4) we obtain

$$
\|B_\epsilon g\|_{L^2(bD, \Omega_2)} \leq m_\omega^{-\frac{\gamma}{\gamma+1}} C_2 \|S_\omega(H)\|_{L^p(bD, \omega)} \frac{1}{\lambda(bD)^{\frac{1}{2(1+\gamma)}}} \left( \int_{bD} \psi_2(z) \, d\lambda(z) \right)^{\frac{1}{2}},
$$

where \(p := \frac{2(1+\gamma)}{\gamma}\). But

$$
\|S_\omega(H)\|_{L^p(bD, \omega)} \leq C(\omega, D) \|H\|_{L^p(bD, \omega)}
$$

by the \(L^p(bD, \omega)\)-regularity of \(S_\omega\) [16], since \(H\) is shorthand for \((\mathcal{R}_\epsilon^s)^\dagger - \mathcal{R}_\epsilon^s) h\), and each of \((\mathcal{R}_\epsilon^s)^\dagger\) and \((\mathcal{R}_\epsilon^s)^\dagger\) takes \(L^1(bD, \omega)\) to \(L^\infty(bD, \omega)\) (again by [16]). We obtain

$$
\|S_\omega(H)\|_{L^p(bD, \omega)} \leq C(\omega, D) \omega(bD)^\frac{1}{2} \|h\|_{L^1(bD, \omega)} \leq C(\omega, D) \omega(bD)^\frac{1}{2} m_\omega \|h\|_{L^2(bD, \Omega_2)} \left( \int_{bD} \psi_2^{-1}(z) \, d\lambda(z) \right)^{\frac{1}{2}}.
$$

But \(h := (\mathcal{R}_\epsilon^s)^{-1} g\) and choosing \(\epsilon = \epsilon(\Omega_2)\) as in (4.1) we have that

$$
\|h\|_{L^2(bD, \Omega_2)} \leq 2 \|g\|_{L^2(bD, \Omega_2)} \quad \text{by (4.2)}.
$$

Combining all the pieces we obtain

$$
\|B_\epsilon g\|_{L^2(bD, \Omega_2)} \leq 2 C_2 C(\omega, D) M_\omega \left( m_\omega^{-1} \left( \frac{\omega(bD)}{\lambda(bD)} \right) \right)^{\frac{1}{2}} \left[ \Omega_2 \right]^{\frac{1}{2}} \|g\|_{L^2(bD, \Omega_2)}.
$$

Hence the desired bound for \(\|B_\epsilon g\|_{L^2(bD, \Omega_2)}\) holds true because

$$
\left( m_\omega^{-1} \left( \frac{\omega(bD)}{\lambda(bD)} \right) \right)^{-\frac{1}{2}} \leq C_3(\omega, D) < \infty \quad \text{for any } 0 \leq \gamma < \infty.
$$

The proof of Theorem 1.1 is complete once we recall that \(\left[ \Omega_2 \right]_{A_2} \geq 1\). \(\Box\)
4.2. Proof of Theorem 1.2: a preliminary result. We begin with the following specialized statement, which will be used in the proof of Theorem 1.2.

**Theorem 4.1.** Let \( D \subset \mathbb{C}^n, \ n \geq 2, \) be a bounded, strongly pseudoconvex domain of class \( C^2 \) and let \( \lambda \) be the Leray Levi measure for \( bD \). The following hold for any \( b \in L^2(bD, \lambda) \) and any \( 1 < p < \infty \):

1. If \( b \in \text{BMO}(bD, \lambda) \) then the commutator \([b, S_\omega]\) is bounded on \( L^p(bD, \omega) \) for any Leray Levi-like measure \( \omega \) with

\[
\|[b, S_\omega]\|_{L^p(bD, \omega) \to L^p(bD, \omega)} \lesssim \|b\|_{\text{BMO}(bD, \lambda)}.
\]

Conversely, if \([b, S_\omega]\) is bounded on \( L^p(bD, \omega) \) for some Leray Levi-like measure \( \omega \), then \( b \in \text{BMO}(bD, \lambda) \) with

\[
\|[b]\|_{\text{BMO}(bD, \lambda)} \lesssim \|[b, S_\omega]\|_p.
\]

Here the implicit constants depend only on \( p, D \) and \( \omega \).

2. If \( b \in \text{VMO}(bD, \lambda) \) then the commutator \([b, S_\omega]\) is compact on \( L^p(bD, \omega) \) for any Leray Levi-like measure \( \omega \). Conversely, if \([b, S_\omega]\) is compact on \( L^p(bD, \omega) \) for some Leray Levi-like measure \( \omega \), then \( b \in \text{VMO}(bD, \lambda) \).

The implied constants in (1) and (2) depend solely on \( p, \omega \) and \( D \).

**Proof of Part (1).** We first prove the sufficiency: we suppose that \( b \in \text{BMO}(bD, \lambda) \) and show that \([b, S_\omega]: L^p(bD, \omega) \to L^p(bD, \omega)\) is bounded for all \( 1 < p < \infty \). Note that by duality it suffices to show that \([b, S_\omega]: L^p(bD, \omega) \to L^p(bD, \omega)\) is bounded for \( 1 < p \leq 2 \).

We first establish boundedness in \( L^2(bD, \omega) \). The starting point are the following basic identities for any fixed \( 0 < \epsilon < \epsilon(D) \):

\[
S_\omega \xi_\epsilon^\dagger f = (\xi_\epsilon, S_\omega)^\dagger f = (\xi_\epsilon, S_\omega^\dagger)f = (S_\omega^\dagger)f = S_\omega f,
\]

which are valid for any \( f \in L^2(bD, \omega) \) and for any \( \epsilon \) (whose value is of no import here). We recall that the upper-script "\( \dagger \)" denotes the adjoint in \( L^2(bD, \omega) \).

A computation that uses (4.5) gives that

\[
-S_\omega[b, T_\epsilon] f + S_\omega bT_\epsilon f = \xi_\epsilon(bf)
\]

is true with

\[
T_\epsilon := I - (\xi_\epsilon^\dagger - \xi_\epsilon)
\]

whenever \( f \) is taken in the Hölder-like subspace (2.6) – the latter ensuring that all terms in (4.6) are meaningful; more precisely for such functions \( f \) we have that \( bf \in L^2(bD, \omega), \) since \( b \in \text{BMO}(bD, \lambda) \subset L^2(bD, \lambda) \) on account of (2.17), and \( L^2(bD, \lambda) = L^2(bD, \omega) \) by (1.12). We also have that \( bT_\epsilon f \in L^2(bD, \omega) \) because \( T_\epsilon f \in C(bD) \) by [16, Proposition 6 and (4.1)]. On the other hand, the classical Kerzman–Stein identity [11]

\[
S_\omega T_\epsilon f = \xi_\epsilon f , \quad f \in L^2(bD, \omega),
\]

gives that

\[
bS_\omega T_\epsilon f = b\xi_\epsilon f , \quad f \in L^2(bD, \omega).
\]

Combining (4.6) and (4.8) we obtain

\[
[b, S_\omega] T_\epsilon f = ([b, \xi_\epsilon] - S_\omega[b, T_\epsilon]) f
\]

whenever \( f \) is in the Hölder-like space (2.6). However the righthand side of (4.9) is meaningful and indeed bounded in \( L^2(bD, \omega) \) by Theorem 3.2 (which applies to Leray Levi-like measures); thus (4.9) extends to an identity on \( L^2(bD, \omega) \). Furthermore, we have that \( T_\epsilon \) is invertible in \( L^2(bD, \omega) \) as a consequence of the following two facts (1.), \( \xi_\epsilon \) and \( (\xi_\epsilon)^\dagger \) are
bounded in $L^2(bD, \omega)$ and (2.), $T_\epsilon$ is skew adjoint (that is, $(T_\epsilon)^\dagger = -T_\epsilon$); see the proof in [19, p. 68] which applies verbatim here. We conclude that

\begin{equation}
[b, S_\omega]g = ([b, \mathcal{C}_\epsilon] - S_\omega[b, T_\epsilon]) \circ T_\epsilon^{-1}g, \quad g \in L^2(bD, \omega).
\end{equation}

But the righthand side of (4.10) is bounded in $L^2(bD, \omega)$ by what has just been said. Thus $[b, S_\omega]$ is also bounded, with

\begin{equation}
\| [b, S_\omega] \|_2 \lesssim \| T_\epsilon^{-1} \|_2 \| b \|_{BMO(bD, \lambda)} \lesssim \| b \|_{BMO(bD, \lambda)}.
\end{equation}

We next prove boundedness on $L^p(bD, \omega)$ for $1 < p < 2$ (as we will see in (4.15) below, it is at this stage that the choice of $\epsilon$ is relevant). We start by combining the “finer” decomposition of $\mathcal{C}_\epsilon$, see (2.13), with the classical Kerzman–Stein identity (4.7), which gives us

\begin{equation}
\mathcal{C}_\epsilon = S_\omega(T_\epsilon^* + R_\epsilon^*) \text{ in } L^2(bD, \omega),
\end{equation}

where

$$T_\epsilon^* := I - ((\mathcal{C}_\epsilon^*)^\dagger - \mathcal{C}_\epsilon^*) \equiv I - \mathcal{C}_\epsilon^*$$

see (2.14), and

$$R_\epsilon^* := R_\epsilon^* - (R_\epsilon^*)^\dagger$$

see (2.15). Plugging (4.12) in (4.9) gives us

\begin{equation}
[b, S_\omega]T_\epsilon^* f = ([b, \mathcal{C}_\epsilon] - S_\omega[b, T_\epsilon] - [b, S_\omega]R_\epsilon^*) f
\end{equation}

whenever $f$ is in the Hölder-like space (2.6). We claim that all three terms in the righthand side of (4.13) are in fact meaningful in $L^p(bD, \omega)$: the first two terms are so by the results of [2] and [16]; on the other hand, the boundedness of the third term is a consequence of the boundedness of $[b, S_\omega]$ in $L^2(bD, \omega)$ that was just proved, and of the mapping properties (2.15), giving us:

\begin{equation}
[b, S_\omega]R_\epsilon^* : L^p(bD, \omega) \rightarrow L^1(bD, \omega) \rightarrow L^\infty(bD, \omega)
\rightarrow L^2(bD, \omega) \rightarrow L^2(bD, \omega) \rightarrow L^p(bD, \omega).
\end{equation}

It is at this point that it is necessary to make a specific choice of $\epsilon$. Given $1 < p < 2$ we pick $\epsilon$ (hence $s = s(\epsilon)$) sufficiently small so that the operator $T_\epsilon^*$ is invertible on $L^p(bD, \omega)$ (with bounded inverse) on account of (2.14). That is:

\begin{equation}
\epsilon^{1/2}M_p := \epsilon^{1/2} \left( \frac{p}{p-1} + p \right) < 1.
\end{equation}

Combining (4.13) with the above considerations we obtain

\begin{equation}
[b, S_\omega]g = ([b, \mathcal{C}_\epsilon] - S_\omega[b, T_\epsilon] - [b, S_\omega]R_\epsilon^*) \circ (T_\epsilon^*)^{-1}g, \quad g \in L^p(bD, \omega).
\end{equation}

We conclude that $[b, S_\omega]$ is bounded on $L^p(bD, \omega)$ with

$$\| [b, S_\omega] \|_p \lesssim \left( 1 + \| S_\omega \|_p + \| T_\epsilon^{-1} \|_2 \| R_\epsilon^* \|_{1, \infty} \right) \| (T_\epsilon^*)^{-1} \|_p \| b \|_{BMO(bD, \lambda)}.$$

We next prove the necessity. Suppose that $b \in L^2(bD, \lambda)$ and that the commutator $[b, S_\omega] : L^p(bD, \omega) \rightarrow L^p(bD, \omega)$ is bounded for some $1 < p < \infty$. We aim to show that $b \in BMO(bD, \lambda)$. We fix $0 < \epsilon < \epsilon(D)$ arbitrarily and observe that the basic identity

$$\langle S_\omega \rangle f = \langle \mathcal{C}_\epsilon S_\omega \rangle f \quad \text{for any } f \in L^2(bD, \omega)$$

grants that the following equality

\begin{equation}
[b, \mathcal{C}_\epsilon]S_\omega f = (I - \mathcal{C}_\epsilon)[b, S_\omega] f
\end{equation}
is valid whenever $f$ is in the Hölder-like space (2.6), which is a dense subspace of $L^q(bD, \omega)$, $1 < q < \infty$. Now the righthand side of (4.17) extends to a bounded operator on $L^p(bD, \omega)$ by the main result of [2] along with our assumption on $[b, S_\omega]$. Thus (4.17) extends to $L^p(bD, \omega)$.

Next, we apply (4.17) to $f \in H^p(bD, \omega) \cap H^p(bD, \omega)$, which is dense in $H^p(bD, \omega)$, and obtain that

$$
\| [b, \epsilon_c](f) \|_{L^p(bD, \omega)} = \| [b, \epsilon_c](S_\omega f) \|_{L^p(bD, \omega)} = \| (I - \epsilon_c) [b, S_\omega](f) \|_{L^p(bD, \omega)}
$$

$$
\leq \| (I - \epsilon_c) \|_p \| [b, S_\omega] \|_p \| f \|_{L^p(bD, \omega)}
$$

$$
\leq \| (I - \epsilon_c) \|_p \| [b, S_\omega] \|_p \| f \|_{H^p(bD, \omega)},
$$

which gives

$$
\| [b, \epsilon_c] \|_{H^p(bD, \omega) \to L^p(bD, \omega)} \leq \| (I - \epsilon_c) \|_p \| [b, S_\omega] \|_p.
$$

Thus $[b, \epsilon_c]$ extends to a bounded operator: $H^p(bD, \omega) \to L^p(bD, \omega)$ with

$$
\| [b, \epsilon_c](f) \|_{L^p(bD, \omega)} \lesssim \| (1 - \epsilon_c) \|_p \| [b, S_\omega] \|_p \| f \|_{H^p(bD, \omega)}.
$$

It follows by duality that

$$
[b, \epsilon_c]^*: L^{p'}(bD, \omega) \to H^{p'}(bD, \omega) \to L^{p'}(bD, \omega)
$$

is bounded, where $1/p + 1/p' = 1$, and that $\| [b, \epsilon_c]^* \|_p \lesssim \| (I - \epsilon_c) \|_p \| [b, S_\omega] \|_p$.

Recall that $[b, \epsilon_c]^*$ is the adjoint of $[b, \epsilon_c]$ in $L^2(bD, \omega)$. Hence, we obtain that $[b, \epsilon_c]^* = \varphi^{-1} [b, \epsilon_c]^* \varphi$, where $\varphi$ are continuous functions on $bD$ such that (1.12) holds, and $[b, \epsilon_c]^*$ is the adjoint of $[b, \epsilon_c]$ in $L^2(bD, \lambda)$.

Since $[b, \epsilon_c]^*$ is bounded on $L^{p'}(bD, \omega)$ and $\varphi$ has uniform positive upper and lower bounds, we obtain that $[b, \epsilon_c]^*$ is also bounded on $L^{p'}(bD, \omega)$ and moreover,

$$
\| [b, \epsilon_c]^* \|_{p'} \lesssim \| [b, \epsilon_c]^* \|_{p'}.
$$

Next, we note that $[b, \epsilon_c]^* = [b, \epsilon_c]^*$. Hence, we further have $[b, \epsilon_c]^* = [b, (\epsilon_c)^*] + [b, (\epsilon_c)^*]$. Since $\epsilon_c^*$ is a standard Calderón–Zygmund operator (see [16]) with the kernel lower bound

$$
|C_{\epsilon_c}^*(w, z)| \geq A_2 \frac{1}{d(w, z)^{2n}},
$$

we also have that $(\epsilon_c^*)^*$ a standard Calderón–Zygmund operator, and that the same lower bound as above holds for the kernel of $(\epsilon_c^*)^*$. Moreover the remainder $R_c$ has the kernel upper bound $|R_c(w, z)| \leq C_R \frac{1}{d(w, z)^{2n-1}}$ (see [16]), which implies that $(\epsilon_c)^*$ has the same kernel upper bound.

As a consequence, we see that by applying Theorem 3.2 to $[b, \epsilon_c]^*$, we obtain that $b \in BMO(bD, \lambda)$ with $\| b \|_{BMO(bD, \lambda)} \lesssim \| [b, \epsilon_c]^* \|_{p'}$, which further implies that

$$
\| b \|_{BMO(bD, \lambda)} \lesssim \| (I - \epsilon_c) \|_p \| [b, S_\omega] \|_p \lesssim \| [b, S_\omega] \|_p,
$$

where the implicit constant is independent of those $\epsilon$ in $(0, \epsilon(D))$ (see Theorem 3.1). The proof of Part (1) is concluded. \hfill \Box

**Proof of Part (2).** Suppose that $b$ is in VMO$(bD, \lambda)$. We claim that $[b, S_\omega]$ is compact on $L^2(bD, \omega)$. This is immediate from (4.10) which shows that $[b, S_\omega]$ is the composition of compact operators (namely $[b, \epsilon_c]$ and $[b, T_\epsilon]$), by Theorem 3.2 with the operators $T_\epsilon^{-1}$ (which is bounded by the results of [16]) and $S_\omega$ (trivially bounded in $L^2(bD, \omega)$). The compactness in $L^p(bD, \omega)$ for $1 < p < 2$ follows by applying this same argument to the identity (4.16), once we point out that the extra term $[b, S_\omega] R_\epsilon$ which occurs in the righthand side of (4.16) is compact in $L^p(bD, \omega)$ on account of the compactness, just proved, of $[b, S_\omega]$ in $L^2(bD, \omega)$, and the chain of bounded inclusions (4.14); the compactness in the range $2 < p < \infty$ now follows by duality. This concludes the proof of sufficiency.
To prove the necessity, we suppose that \( b \in \text{BMO}(bD, \lambda) \) and that \([b, S_\omega]\) is compact on \(L^p(bD, \omega)\) for some \(1 < p < \infty\), and we prove that \( b \in \text{VMO}(bD, \lambda)\). To this end, we note that (4.17) (however with \( f \) taken in \(H^2(bD, \omega) \cap H^p(bD, \omega)\)) shows that \([b, C_\epsilon]\) is compact as an operator from \(H^p(bD, \omega) \rightarrow L^p(bD, \omega)\) since it is the composition of a compact operator (namely \([b, S_\omega]\), which is assumed to be compact on \(L^p\) and hence on its subspace \(H^p\)) with the bounded operator \(I - C_\epsilon\) (by the results of [16]). Thus

\[
[b, C_\epsilon]^*: L^p(bD, \omega) \rightarrow H^{p'}(bD, \omega) \hookrightarrow L^p(bD, \omega)
\]
is compact by duality, where \(1/p + 1/p' = 1\).

Following the argument at the end of the proof of Part (1), we see that this implies that \( b \in \text{VMO}(bD, \lambda) \) by Theorem 3.2.

The proof of Theorem 4.1 is concluded. \(\square\)

### 4.3. Proof of Theorem 1.2.

We prove each part separately.

**Proof of Part (1).** We first show the sufficiency. To this end, it suffices to prove that

\[
[b, S_\omega]g \lesssim [\Omega_2]_A^2 \|b\|_{\text{BMO}(bD, \lambda)} \|g\|_{L^2(bD, \Omega_2)}
\]

holds for any \(g \in C(bD)\) and for any \(A_2\)-like measure \(\Omega_2\), where the implied constant depends only on \(\omega\) and \(D\), because the \(L^p\)-estimate (1.16) will then follow by extrapolation [5, Section 9.5.2]. To prove (4.20), for any \(\epsilon > 0\) we write

\[
[b, S_\omega]g = \tilde{A}_\epsilon g + \tilde{B}_\epsilon g + C_\epsilon g \quad \text{where}
\]

\[
\tilde{A}_\epsilon g := [b, C_\epsilon] \circ (T_\epsilon^*)^{-1} g; \quad \tilde{B}_\epsilon g := -[b, S_\omega] \circ ((\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*) \circ (T_\epsilon^*)^{-1} g,
\]

and

\[
C_\epsilon g := S_\omega \circ [b, I - ((\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*)] \circ (T_\epsilon^*)^{-1} g
\]

where again

\[
T_\epsilon^* h := (I - ((\mathcal{C}_\omega^*)^\dagger - \mathcal{C}_\omega^*))h.
\]

We first consider \(\tilde{A}_\epsilon\). By choosing \(\epsilon = \epsilon(\Omega_2)\) as in (4.1), we see that \((T_\epsilon^*)^{-1}\) is bounded on \(L^2(bD, \Omega_2)\) with \(|(T_\epsilon^*)^{-1}|_{L^2(bD, \Omega_2)} \leq 2\). Hence Theorem 3.2 grants

\[
\|\tilde{A}_\epsilon g\|_{L^1(bD, \Omega_2)} \lesssim \|b\|_{\text{BMO}(bD, \lambda)} \|\Omega_2\|_A^2 \|g\|_{L^2(bD, \Omega_2)}.
\]

To control the operator \(\tilde{B}_\epsilon\), with same \(\epsilon\) as above, it suffices to prove the boundedness of \([b, S_\omega] \circ ((\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*)\). To this end, we combine the mapping properties (2.15) with Part (1) of Theorem 4.1 and the reverse Hölder’s inequality, and obtain that

\[
[b, S_\omega] \circ ((\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*) : L^2(bD, \Omega_2) \hookrightarrow L^1(bD, \omega) \rightarrow L^\infty(bD, \omega)
\]

\[
\rightarrow L^{p_0}(bD, \omega) \rightarrow L^{p_0}(bD, \omega) \hookrightarrow L^2(bD, \Omega_2);
\]

here \(p_0 > 2\) has been chosen so that its Hölder conjugate \(p_0^*\) satisfies

\[
\left( \int_{bD} \Omega_2^{p_0}(z)d\omega(z) \right)^{\frac{1}{p_0^*}} \leq M(D, \omega) \int_{bD} \Omega_2(z)d\omega(z) = M(D, \omega) \Omega_2(bD),
\]

where the constant \(M(D, \omega)\) is independent of \(\Omega_2\). Moreover, by writing \(h := (T_\epsilon^*)^{-1} g, \ H = ((\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*) h\) and \(\tilde{B}_\epsilon g = -[b, S_\omega](H)\), we have that

\[
\|H\|_{L^{p_0}(bD, \omega)} \lesssim \omega(bD)^{\frac{1}{p_0^*}} \|(\mathcal{R}_\epsilon^*)^\dagger - \mathcal{R}_\epsilon^*) h\|_{L^\infty(bD, \omega)} \lesssim \omega(bD)^{\frac{1}{p_0^*}} \Omega_2^{1}(bD)^{\frac{1}{2}} \|h\|_{L^2(bD, \Omega_2)}
\]

\[
\|H\|_{L^{p_0}(bD, \omega)} \lesssim \omega(bD)^{\frac{1}{p_0^*}} \Omega_2^{1}(bD)^{\frac{1}{2}} \|h\|_{L^2(bD, \Omega_2)}
\]

\[
\|H\|_{L^{p_0}(bD, \omega)} \lesssim \omega(bD)^{\frac{1}{p_0^*}} \Omega_2^{1}(bD)^{\frac{1}{2}} \|h\|_{L^2(bD, \Omega_2)}
\]
and that
\[ \|b, S_\omega (\overline{\mathbf{T}})\|_{L^2(bD, \Omega_2)} \leq \|b, \overline{S_\omega (\overline{\mathbf{T}})}\|_{L^2(bD, \omega)} \|\Omega_2\|^{\frac{1}{2}}_{L^2(bD, \omega)} \leq \|\overline{\mathbf{T}}\|_{L^2(bD, \omega)} M(D, \omega) \|\Omega_2\|_{bD}^{\frac{1}{2}}. \]

Hence, we have the norm
\[ \|\tilde{B}_t g\|_{L^2(bD, \Omega_2)} \lesssim M(D, \omega) \|\Omega_2\|_{bD}^{\frac{1}{2}} \|\overline{\mathbf{BMO}}(bD, \lambda)\|_{\Omega_2}^{\frac{1}{2}} \|\overline{\mathbf{BMO}}(bD, \lambda)\| \|g\|_{L^2(bD, \Omega_2)}, \]

where the last inequality follows from the definition of the \(A_2\) constant.

To bound the norm of \(C_\epsilon g\), we start by writing
\[ C_\epsilon g = S_\omega (\tilde{H}); \quad \tilde{H} := [b, I - ((\ell^\epsilon)^{-1})] h; \quad h := (T_\epsilon)^{-1} g, \]

hence (1.15) grants
\[ \|C_\epsilon g\|_{L^2(bD, \Omega_2)} \lesssim \|\Omega_2\|_{A_2} \|\tilde{H}\|_{L^2(bD, \Omega_2)}. \]

Furthermore,
\[ \|\tilde{H}\|_{L^2(bD, \Omega_2)} \leq \|b, \ell^\epsilon\|_{L^2(bD, \Omega_2)} + \|b, (\ell^\epsilon)^{-1}\|_{L^2(bD, \Omega_2)}. \]

Now Theorem 3.2 (for \(p = 2\)) gives that
\[ \|b, \ell^\epsilon\|_{L^2(bD, \Omega_2)} \leq C(\omega, D) \|b\|_{\overline{\mathbf{BMO}}(bD, \lambda)} \|\Omega_2\|_{A_2} \|h\|_{L^2(bD, \Omega_2)}, \]

and choosing \(\epsilon = \epsilon(\Omega_2)\) as in (4.1) we conclude that
\[ \|h\|_{L^2(bD, \Omega_2)} \leq 2 \|g\|_{L^2(bD, \Omega_2)}. \]

Combining all of the above we obtain
\[ \|b, \ell^\epsilon\|_{L^2(bD, \Omega_2)} \leq 2C(\omega, D) \|b\|_{\overline{\mathbf{BMO}}(bD, \lambda)} \|\Omega_2\|_{A_2} \|g\|_{L^2(bD, \Omega_2)}. \]

It now suffices to show that
\[ (4.22) \quad \|b, (\ell^\epsilon)^{-1}\|_{L^2(bD, \Omega_2)} \leq C(\omega, D) \|b\|_{\overline{\mathbf{BMO}}(bD, \lambda)} \|\Omega_2\|_{A_2} \|h\|_{L^2(bD, \Omega_2)}. \]

To see this, we first recall that from [16, (5.7)], \(\ell^\epsilon\) is given by
\[ \ell^\epsilon (f)(z) = \ell_\epsilon (f \cdot \chi_s (\cdot, z))(z), \quad z \in bD \]

(see the proof of Proposition 3.3). Recall also that \((\ell^\epsilon)^{-1} = \varphi^{-1}(\ell^\epsilon)^* \varphi\), where \(\varphi\) is the density function of \(\omega\) satisfying (1.13). Next, we observe that
\[ [b, (\ell^\epsilon)^{-1}](h)(x) = b(x) \varphi^{-1}(x)(\ell^\epsilon)^* (\varphi \cdot h)(x) - \varphi^{-1}(x)(\ell^\epsilon)^* (b \cdot \varphi \cdot h)(x) \]
\[ = \varphi^{-1}(x) [b, (\ell^\epsilon)^*] (\varphi \cdot h)(x). \]

Thus, it suffices to show that \([b, (\ell^\epsilon)^*]\) is bounded on \(L^2(bD, \Omega_2)\). Assume that this is the case, then based on the fact that \(\varphi\) is the density function of \(\omega\) satisfying (1.13), we obtain that
\[ \|b, (\ell^\epsilon)^{-1}\|_{L^2(bD, \Omega_2)} = \|\varphi^{-1}[b, (\ell^\epsilon)^*] (\varphi \cdot h)(x)\|_{L^2(bD, \Omega_2)} \]
\[ \leq \frac{M(D, \varphi)}{m(D, \varphi)} \|b, (\ell^\epsilon)^*\|_{L^2(bD, \Omega_2)} \|h\|_{L^2(bD, \Omega_2)}. \]

Next, by noting that for any \(\Omega_2 \in A_2\), \(f_1 \in L^2(bD, \Omega_2)\) and \(f_2 \in L^2(bD, \Omega_2^{-1})\) (recall that \(\Omega_2^{-1}\) is also an \(A_2\) weight), we have
\[ \langle [b, (\ell^\epsilon)^*] f_1, f_2 \rangle = \int_{bD} [b, (\ell^\epsilon)^*] f_1(x) f_2(x) d\lambda(x) \]
\[ = \int_{bD} f_1(x) [b, (\ell^\epsilon)^*] f_2(x) d\lambda(x) \]
\[= \int_{bD} f_1(x) \psi_\epsilon^2(x) \left[ b, \mathcal{C}_\epsilon^* \right](f_2)(x) \psi_\epsilon^2(x) \, d\lambda(x), \]

which gives that \(|\langle (b, \mathcal{C}_\epsilon^*) \rangle(f_1, f_2)\| \leq \|f_1\|_{L^2(bD, \Omega_2)} \|\left[ b, \mathcal{C}_\epsilon^* \right](f_2)\|_{L^2(bD, \Omega_2^{-1})}, \]

and therefore

\[\|\left[ b, \mathcal{C}_\epsilon^* \right]\|_{L^2(bD, \Omega_2) \to L^2(bD, \Omega_2^{-1})} \leq \|\left(\left[ b, \mathcal{C}_\epsilon^* \right]\right)^*\|_{L^2(bD, \Omega_2^{1/2}) \to L^2(bD, \Omega_2^{-1})} \cdot \]

Now Theorem 3.2 gives that the right-hand side in the above inequality is bounded by

\[C(\omega, D)\|b\|_{\text{BMO}(bD, \Omega_2^{-1/2})} \Omega_2^{-1/2} \|f\|_{A_2}, \]

which, together with the fact that \(\Omega_2^{-1} = [\Omega_2], \)

leads to

\[\|\left[ b, \mathcal{C}_\epsilon^* \right] h \|_{L^2(bD, \Omega_2)} \leq \frac{M(b, \omega)}{m(b, \omega)} C(\omega, D)\|b\|_{\text{BMO}(bD, \Omega_2)} \|f\|_{A_2}, \]

We next prove the necessity. Suppose that \(b \in L^2(bD, \Lambda) \) and that the commutator \([b, S_\omega] : L^p(bD, \Omega_p) \to L^p(bD, \Omega_p)\) is bounded for some \(1 < p < \infty\) and for some \(A_{\epsilon}\)-measure \(\Omega_p\). We aim to show that \(b \in \text{BMO}(bD, \Lambda)\); we will do so by proving that (for any arbitrarily fixed \(0 < \epsilon < c(D)\)) the commutator \([b, \mathcal{C}_\epsilon^*]\) is bounded on \(L^p(bD, \Omega_p')\) where \(1/p + 1/p' = 1; \)

\(\Omega_p'= \Omega_p \downarrow \), and \(\mathcal{C}_\epsilon^*\) is the \(L^2(bD, \sigma)\)-adjoint of \(\mathcal{C}_\epsilon\); the desired conclusion will then follow by Theorem 3.2.

To this end, we proceed as in the proof of Theorem 4.1: we first note that the basic identity

\[S_\omega = \mathcal{C}_\epsilon S_\omega \quad \text{in} \quad L^2(bD, \omega) \]

grants the validity of

\[(4.23) \quad [b, \mathcal{C}_\epsilon] S_\omega f = (I - \mathcal{C}_\epsilon) [b, S_\omega] f \]

whenever \(f\) is in the Hölder-like space \((2.6), \) which is a dense subspace of \(L^q(bD, \Omega_q)\) for any \(1 < q < \infty\) and any \(A_{\epsilon}\)-measure \(\Omega_q\). But the right-hand side of (4.23) extends to a bounded operator on \(L^p(bD, \Omega_p)\) by Theorem 3.1 and by our assumption on \([b, S_\omega]\). Thus

(4.23) extends to \(L^p(bD, \Omega_p).\)

Next, we apply (4.23) to \(f \in H^2(bD, \omega) \cap H^p(bD, \Omega_p),\) which is dense in \(H^p(bD, \Omega_p),\) and obtain that

\[\|\left[ b, \mathcal{C}_\epsilon \right] f \|_{L^p(bD, \Omega_p)} = \|\left[ b, \mathcal{C}_\epsilon \right] (S_\omega f) \|_{L^p(bD, \Omega_p)} = \|\left( I - \mathcal{C}_\epsilon \right) [b, S_\omega] f \|_{L^p(bD, \Omega_p)} \]

\[\leq \|\left( I - \mathcal{C}_\epsilon \right) [b, S_\omega] f \|_{L^p(bD, \Omega_p)} \to L^p(bD, \Omega_p) \|f\|_{L^p(bD, \Omega_p)} \]

\[\leq \|\mathcal{C}_\epsilon \|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p)} \|f\|_{L^p(bD, \Omega_p)}, \]

where the last inequality follows from Theorem 3.1, and the implicit constant depends on \(p, D\) and \(\epsilon\) (which we recall has been fixed once and for all). Thus \([b, \mathcal{C}_\epsilon] \) extends to a bounded operator:

\[H^p(bD, \Omega_p) \to L^p(bD, \Omega_p).\]

It follows by duality that

\[(4.24) \quad [b, \mathcal{C}_\epsilon] : L^p(bD, \Omega_p') \to H^p(bD, \Omega_p') \to L^p(bD, \Omega_p') \]

is bounded, where \(1/p + 1/p' = 1, \) and that

\[(4.25) \quad \|\left[ b, \mathcal{C}_\epsilon \right] \|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p')} \leq \|\mathcal{C}_\epsilon \|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p')}, \]

Now the adjoints of \([b, \mathcal{C}_\epsilon] \) in \(L^2(bD, \omega)\) and in \(L^2(bD, \sigma)\) (respectively denoted by upper-indexes \(\dagger\) and \(*\) ) are related to one another via the identity \([b, \mathcal{C}_\epsilon] \dagger = \varphi^{-1}[b, \mathcal{C}_\epsilon]^* \varphi, \) where \(\varphi\) and its reciprocal \(\varphi^{-1}\) satisfy (1.12). Since \([b, \mathcal{C}_\epsilon]^\dagger\) is bounded on \(L^p(bD, \Omega_p')\) and \(\varphi\) has positive and finite upper and lower bounds on \(bD,\) we obtain that \([b, \mathcal{C}_\epsilon]^*\) is also bounded on \(L^p(bD, \Omega_p')\) and moreover,

\[\|\left[ b, \mathcal{C}_\epsilon \right]^* \|_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p')} \leq \|\left[ b, \mathcal{C}_\epsilon \right] \|^\dagger_{L^p(bD, \Omega_p) \to L^p(bD, \Omega_p')} \cdot \]

But \([b, \mathcal{C}_\epsilon] = [b, \mathcal{C}'_\epsilon]\), hence the conclusion \(b \in \text{BMO}(bD, \lambda)\) and the desired bound:

\[
\|b\|_{\text{BMO}(bD, \lambda)} \lesssim [\Omega_{p, A_p}^{\text{max}}(1, \frac{1}{p})] \|b, S_\omega\|_{L_p(bD, \Omega_p)} \to L^p(bD, \Omega_p)
\]

follow from Theorem 3.2 and (4.25). The proof of Part (1) is concluded.

**Proof of Part (2).** This follows a similar approach to the proof of (2) of Theorem 4.1 with standard modifications which can be seen from the proof of (1) above and the extrapolation on weighted Lebesgue spaces [10]; we omit the details.

The proof of Theorem 1.2 is complete. \(\square\)

### 4.4. Proof of Theorem 1.5.

As before, the superscript \(\bullet\) designates the adjoint with respect to the inner product \((\cdot, \cdot)_{\Omega_2}\) of \(L^2(bD, \Omega_2)\). Thus, \(S_{\Omega_2}\) is the orthogonal projection of \(L^2(bD, \Omega_2)\) onto \(H^2(bD, \Omega_2)\) in the sense that

\[
S_{\Omega_2}^\bullet = S_{\Omega_2},
\]

where \(H^2(bD, \Omega_2)\) is the holomorphic Hardy space given in Definition 1.3 and the \(S_{\Omega_2}^\bullet\) denotes the adjoint of \(S_{\Omega_2}\) in \(L^2(bD, \Omega_2)\).

To begin with, we first point out that if \(b\) is in \(\text{BMO}(bD, \lambda)\), then \(b\) is in \(L^2(bD, \Omega_2)\), where \(\Omega_2\) has the density function \(\psi \in A_2\). Then, following the result in [7, Section 5.2] (see also [8, Theorem 3.1] in \(\mathbb{R}^n\)), we see that

\[
\text{BMO}(bD, \lambda) = \text{BMO}_{L^p_{\Omega_2}}(bD, \lambda)
\]

for all \(1 \leq p < \infty\) and the norms are mutually equivalent, where \(\text{BMO}_{L^p_{\Omega_2}}(bD, \lambda)\) is the space of all \(b \in L^1(bD, \lambda)\) such that

\[
\|b\|_{*_{\Omega_2}} := \sup_B \left( \frac{1}{\Omega_2(B)} \int_B |b(z) - b_B|^p d\Omega_2(z) \right)^{\frac{1}{p}} < \infty, \quad b_B = \frac{1}{\lambda(B)} \int_B b(w) d\lambda(w),
\]

and

\[
\|b\|_{\text{BMO}_{L^p_{\Omega_2}}(bD, \lambda)} = \|b\|_{*_{\Omega_2}} + \|b\|_{L^1(bD, \lambda)}.
\]

Since \(bD\) is compact, we see that \(b \in L^p(bD, \Omega_2)\) for \(1 \leq p < \infty\), that is

\[
(4.26) \quad \text{BMO}(bD, \lambda) \subset L^p(bD, \Omega_2) \quad \text{for any} \quad 1 \leq p < \infty.
\]

We split the proof into two parts.

**Proof of Part (1).** We first prove the sufficiency. We suppose that \(b\) is in \(\text{BMO}(bD, \lambda)\) and show that \([b, S_{\Omega_2}] : L^2(bD, \Omega_2) \to L^2(bD, \Omega_2)\) for every \(\varphi \in A_2\) with

\[
(4.27) \quad \|\|b, S_{\Omega_2}\|\|_2 \lesssim N([\psi]_{A_2}),
\]

where \(N(s)\) is a positive increasing function on \([1, \infty)\).

We start with the following basic identity

\[
(4.28) \quad S_{\Omega_2} \mathcal{C}_\epsilon (f) = (\mathcal{C}_\epsilon S_{\Omega_2})^\bullet (f) = (\mathcal{C}_\epsilon^\bullet S_{\Omega_2})^\bullet (f) = (S_{\Omega_2})^\bullet (f) = S_{\Omega_2} (f),
\]

which is valid for any \(f \in L^2(bD, \Omega_2)\) and for any \(\epsilon\) (whose value is not important here).

It follows from (4.28) that

\[
(4.29) \quad S_{\Omega_2} [b, T_{\epsilon, \Omega_2}] (f) = S_\omega b T_{\epsilon, \Omega_2} (f) = \mathcal{C}_\epsilon (bf),
\]

where

\[
T_{\epsilon, \Omega_2} := I - (\mathcal{C}_\epsilon^\bullet - \mathcal{C}_\epsilon)
\]

and \(f\) is any function taken in the H"older-like space (2.6). On the other hand, the classical Kerzman–Stein identity [11]

\[
(4.30) \quad S_{\Omega_2} T_{\epsilon, \Omega_2} f = \mathcal{C}_\epsilon f, \quad f \in L^2(bD, \Omega_2),
\]

But \([b, \mathcal{C}_\epsilon]^\bullet = [b, \mathcal{C}'_\epsilon]\), hence the conclusion \(b \in \text{BMO}(bD, \lambda)\) and the desired bound:

\[
\|b\|_{\text{BMO}(bD, \lambda)} \lesssim [\Omega_{p, A_p}^{\text{max}}(1, \frac{1}{p})] \|b, S_\omega\|_{L_p(bD, \Omega_p)} \to L^p(bD, \Omega_p)
\]

follow from Theorem 3.2 and (4.25). The proof of Part (1) is concluded.

**Proof of Part (2).** This follows a similar approach to the proof of (2) of Theorem 4.1 with standard modifications which can be seen from the proof of (1) above and the extrapolation on weighted Lebesgue spaces [10]; we omit the details.

The proof of Theorem 1.2 is complete. \(\square\)
gives that
\begin{equation}
(4.31) \quad bS_{Ω_2}T_{ε,Ω_2}f = bC_ε f, \quad f \in L^2(bD, Ω_2).
\end{equation}
Combining (4.29) and (4.31) we obtain
\begin{equation}
(4.32) \quad [b, S_{Ω_2}]T_{ε,Ω_2}f = ([b, C_ε] + S_{Ω_2}[b, T_{ε,Ω_2}])f
\end{equation}
whenever \( f \) is in the Hölder-like space (2.6). We now point out that the righthand side of (4.32) is meaningful in \( L^2(bD, Ω_2) \) by the same argument as before. We observe here that
\begin{equation}
(4.33) \quad [b, T_{ε,Ω_2}] = [b, I] - [b, C_ε] + [b, C_ε] = -[b, C_ε]\}
\end{equation}
and by (i) in Theorem 3.2, we get that \([b, T_{ε,Ω_2}]\) is also bounded on \( L^2(bD, Ω_2) \).

Furthermore, we have that \( T_{ε,Ω_2} \) is invertible in \( L^2(bD, Ω_2) \) by the analogous two facts as in the proof of Theorem 4.1: (1), \( C_ε \) and \( C_ε^* \) are bounded in \( L^2(bD, Ω_2) \) and (2), \( T_{ε,Ω_2} \) is skew adjoint (that is, \( (T_{ε,Ω_2}^*)^* = -T_{ε,Ω_2}^* \)). We conclude that
\begin{equation}
(4.34) \quad [b, S_{Ω_2}]g = ([b, C_ε] + S_{Ω_2}[b, T_{ε,Ω_2}]) \circ T_{ε,Ω_2}^{-1}g, \quad g \in L^2(bD, Ω_2).
\end{equation}
But the righthand side of (4.34) is bounded in \( L^2(bD, Ω_2) \) and
\begin{equation}
(4.35) \quad \|b, S_{Ω_2}\|_2 \lesssim \|T_{ε,Ω_2}^{-1}\|_2 \|b, C_ε\|_2 (1 + \|S_{Ω_2}\|_2) \lesssim \|T_{ε,Ω_2}^{-1}\|_2 \|Ω_2\|_{A_2}\|b\|_{BMO(bD, λ)},
\end{equation}
where the last inequality follows from (i) in Theorem 3.2 and the fact that \( \|S_{Ω_2}\|_2 = 1 \) by the definition of \( S_{Ω_2} \).

Hence we see that (4.27) holds with \( N(s) := C_2s^2 \) and \( C := \|T_{ε,Ω_2}^{-1}\|_2\|b\|_{BMO(bD, λ)} \).

We next prove the necessity. Suppose that \( b \) is in \( L^2(bD, λ) \) and that the commutator \( [b, S_{Ω_2}] : L^2(bD, Ω_2) \to L^2(bD, Ω_2) \) is bounded.

Repeating the same steps in the proof of the necessity part in Theorem 4.1, we see that \( [b, C_ε] \) is bounded from \( L^2(bD, Ω_2) \) to \( L^2(bD, Ω_2) \) with
\begin{equation}
(4.36) \quad \|b, C_ε\|_2 \lesssim \|I - C_ε\|_2\|b, S_{Ω_2}\|_2,
\end{equation}
where \( \|I - C_ε\|_2 < \infty \) follows from Theorem 3.1.

Then, by using (i) in Theorem 3.2 (simply noting that \( b \in L^2(bD, Ω_2) \) implies that \( b \in L^1(bD, λ) \) since \( Ω_2^{-1}(bD) < \infty \)), we obtain that \( b \) is in \( \text{BMO}(bD, λ) \) with \( \|b\|_{\text{BMO}(bD, λ)} \lesssim \|[b, C_ε]\|_2 \), which, together with (4.36), gives
\begin{equation}
\|b\|_{\text{BMO}(bD, λ)} \lesssim \|I - C_ε\|_2\|b, S_{Ω_2}\|_2.
\end{equation}

Proof of Part (2). To prove the sufficiency, we assume that \( b \) is in \( \text{VMO}(bD, λ) \) and we aim to prove that \( [b, S_{Ω_2}] \) is compact on \( L^2(bD, Ω_2) \).

In fact, the argument that \( [b, S_{Ω_2}] \) is compact on \( L^2(bD, Ω_2) \) is immediate from (4.34), which shows that \( [b, S_{Ω_2}] \) is the composition of compact operators (namely \( [b, C_ε] \) and \( [b, T_{ε,Ω_2}] \), by (ii) of Theorem 3.2) with the bounded operators \( T_{ε,Ω_2}^{-1} \) (by the results of [16]) and \( S_{Ω_2} \).

To prove the necessity, we suppose that \( b \in \text{BMO}(bD, λ) \) and that \( [b, S_{Ω_2}] \) is compact on \( L^2(bD, Ω_2) \), and we show that \( b \) is in \( \text{VMO}(bD, λ) \). To this end, we note that (4.36) shows that
\begin{equation}
[b, C_ε] : L^2(bD, Ω_2) \to L^2(bD, Ω_2)
\end{equation}
is compact. But this implies that \( b \in \text{VMO}(bD, λ) \) by (ii) of Theorem 3.2.

The proof of Theorem 1.5 is concluded. \( \square \)
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