Resolve the multitude of microscale interactions to model stochastic partial differential equations

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Abstract

Constructing numerical models of noisy partial differential equations is very delicate. Our long term aim is to use modern dynamical systems theory to derive discretisations of dissipative stochastic partial differential equations. As a second step we here consider a small domain, representing a finite element, and apply stochastic centre manifold theory to derive a one degree of freedom model for the dynamics in the element. The approach automatically parametrises the microscale structures induced by spatially varying stochastic noise within the element. The crucial aspect of this work is that we explore how a multitude of noise processes may interact in nonlinear dynamics. We see that noise processes with coarse structure across a finite element are the significant noises for the modelling. Further, the nonlinear dynamics abstracts effectively new noise sources over the macroscopic time scales resolved by the model.

Contents

1 Introduction 2

2 Construct a memoryless normal form model 7

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1 Introduction

The ultimate aim is to accurately and efficiently model numerically the evolution of stochastic partial differential equations (SPDEs) as, for example, may be used to model pattern forming systems [14, 5]. Due to the forcing over many length and time scales, a SPDE typically has intricate spatio-temporal dynamics. Numerical methods to integrate stochastic ordinary differential equations are known to be delicate and subtle [20, e.g.]. We surely need to take considerable care for SPDEs as well [17, 40, e.g.].

An issue is that the stochastic forcing generates high wavenumber, steep variations, in spatial structures. Stable implicit integration in time generally damps far too fast such decaying modes, yet through stochastic resonance an accurate resolution of the life-time of these modes may be important on the large scale dynamics. We use the term “stochastic resonance” to include phenomena where stochastic fluctuations interact with each other and themselves through nonlinearity in the dynamical system to generate not only long time drifts but also potentially to change stability [21, 6, 14, 34, 39, e.g.] as seen here in equation (4). Thus we must reasonably resolve subgrid microscale structures so that numerical discretisation with large space-time grids achieve efficiency without sacrificing the subtle interactions that take place between the subgrid scale structures.

Centre manifold theory supports the macroscopic modelling of microscopic dynamics. For example, Knobloch & Wiesenfeld [21] and Boxler [6, 7]
1 Introduction

explicitly used centre manifold theory to support the modelling of SDEs and SPDEs. Indeed, Boxler [6] proves that “stochastic center manifolds, share all the nice properties of their deterministic counterparts.” Many others, such as Bergh und & Gentz [4], Blömker, Hairer & Pavliotis [5] and Kabanov & Pergamenshchikov [19], use the same separation of time scales that underlies centre manifold theory to form and support low-dimensional, long term models of SDEs and SPDEs that have both fast and slow modes. A centre manifold approach also illuminates the discretisation of deterministic partial differential equations [29, 31, 22, 30, 32, 23]. By merging these two applications of centre manifold theory we will model SPDEs with sound theoretical support; here we begin to develop good methods for the discretisation of SPDEs. Here we consider the case of just one finite element forming the domain. Later work will address how to couple many finite elements together to form a large scale discrete model of SPDEs. The crucial issue explored here is how to deal with noise that is distributed independently across space as well as time, that is, the noise is uncorrelated in space and time. We decompose the noise into its Fourier sine series and assume the infinite number of Fourier coefficients are an infinite number of independent noise sources. It eventuates that only a few combinations of these noise sources are important in the long term model. However, all do contribute in the infinite sums forming these few combinations.

The importance of this work is to establish a methodology to create accurate, finite dimensional, discrete models of the long term dynamics of SPDEs.

Analyse a prototype SPDE  Continuing earlier work [34], the simplest case, and that developed here, is the modelling of a SPDE on just one finite size element. As a prototype, let us consider the stochastically forced nonlinear partial differential equation

\[
\frac{\partial u}{\partial t} = -u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + u + \sigma \phi(x, t),
\]

such that \( u = 0 \) at \( x = 0, \pi \),

which involves the important generic physical processes of advection \( uu_x \), diffusion \( uu_{xx} \), some noise process \( \phi(x, t) \), and a linear reaction \( u \) that partially ameliorates diffusion to make the sin x mode dynamically neutral. Our primary aim is to work with the forcing by \( \phi(x, t) \), of strength \( \sigma \), being a white noise process that is delta correlated in both space and time. Here we express the additive noise in the orthogonal sine series

\[
\phi = \sum_{k=1}^{\infty} \phi_k(t) \sin kx,
\]
where the \( \phi_k(t) \) are independent white noises that are delta correlated in time.\(^1\) Our aim is to seek how the complex interactions, through the non-linearity of the prototype SPDE (1), of these spatially distributed noises affect the dynamics over the relatively large scale domain \([0, \pi]\). Analogously, Blömker et al. [5] rigorously modelled the stochastically forced Swift–Hohenberg equation by a stochastic Ginzburg–Landau equation as a prototype SPDE in a class of pattern forming stochastic systems.

Throughout the body of this paper, we interpret all noise processes and all stochastic differential equations in the Stratonovich sense so that the rules of traditional calculus apply. Thus the direct application of this modelling is to physical systems where the Stratonovich interpretation is the norm. However, the Appendix provides alternative proofs of some key properties of the nonlinear interaction of noise processes: these proofs use the Ito interpretation. Only in the Appendix is the Ito interpretation used, everywhere else the stochastic calculus is Stratonovich.

**Centre manifold theory supports the modelling**  We base the modelling upon the dynamics when the noise is absent, \( \sigma = 0 \). When \( \sigma = 0 \) the linear dynamics of the SPDE (1), described by

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \quad \text{such that} \quad u = 0 \text{ at } x = 0, \pi, \tag{3}
\]

are that modes \( u \propto \sin kx \exp \lambda t \) decay with rate \( \lambda_k = -(k^2 - 1) \) except for the \( k = 1 \) mode, \( u \propto \sin x \), which is linearly neutral, \( \lambda_1 = 0 \), and thus forms the basis of the long term model. The components of the forcing noise (2) with wavenumber \( k > 1 \) are orthogonal to this basic mode. Consequently, simple numerical methods, such as Galerkin projection onto the fundamental mode \( \sin x \), would ignore the “high wavenumber” modes, \( k > 1 \), of the noise (2) and hence completely miss subtle but important subgrid interactions. Instead, the systematic nature of centre manifold theory accounts for the subgrid scale interactions as a power series in the noise amplitude \( \sigma \) from the deterministic base (3).

Centre manifold theory applies to the nonlinear forced system (1) because in the linearised problem (3) there is some (here one) eigenvalue of 0 and all the other eigenvalues are negative (and bounded away from 0). After adjoining the trivial \( \frac{d\sigma}{dt} = 0 \), theory assures us that in some finite neighbourhood of \((u, \sigma) = (0, 0)\) there exists \([6, \text{Theorem 5.1 and 6.1}]\) a

\(^1\)The reason for expressing the noise in the sine expansion (2) is that the modes \( \sin kx \) are the eigenmodes of the linear dynamics and thus form a natural basis for analysis.
centre manifold \( u = v(a(t), x, t, \sigma) \) where the amplitude \( a \) of the neutral mode \( \sin x \) evolves according to \( \dot{a} = g(a, t, \sigma) \) for some function \( g \). Unfortunately, there is a caveat: Boxler’s [6] theory is as yet developed only for finite dimensional systems which satisfy a Lipschitz condition. Here, the SPDE (1) is infinite dimensional and the nonlinear advection \( u \frac{\partial u}{\partial x} \) involves the unbounded operator \( \frac{\partial}{\partial x} \). There is some infinite dimensional theory: Blömker et al. [5, Theorem 1.2] rigorously proved the existence and relevance of a stochastic Ginzburg–Landau model to the stochastic forced Swift–Hohenberg PDE; further, Caraballo, Langa & Robinson [8] and Duan, Lu & Schmalfuss [15] proved the existence of invariant manifolds for a wide class of reaction-diffusion SPDEs; they built on earlier work on inertial manifolds in SPDEs by Bensoussan & Flandoli [3]. I expect future theoretical developments to rigorously support the approach.

However, in the interim, a way to proceed is via a shadowing argument. The rapid dissipation of high wavenumber modes in (1), the spectrum \( \lambda_k \sim -k^2 \) for large wavenumber \( k \), ensures that the dynamics of the SPDE (1) is close enough to finite dimensional. By modifying the spatial derivatives in (1) to have a high wavenumber cutoff, the dynamics of the SPDE (1) is effectively that of a Lipschitz, finite dimensional system. The theorems of Boxler [6] then rigorously apply. For example, the modelling in Section 4 shows that just ten spatial modes in the noise \( \Phi(x, t) \) gives the coefficients in the model (24) correct to five decimal digits. Thus modifying \( \partial/\partial x \) to cutoff modes with wavenumber \( k > 20 \) is a nearby, Lipschitz, finite dimensional SDE system that is effectively indistinguishable from the original SPDE to five decimal digits and to the order of asymptotic expansion pursued here. Whenever theoretical support by Boxler [6] is invoked, I actually refer to this slightly modified system.

**Stochastic induced drift affects stability** Previously [34], I discussed that when the \( \sin 2x \) component of the noise (2) is large enough, and in the absence of any other noise component, then stochastic resonance may make a qualitative change in the nature of the solutions in that it restabilises the zero equilibrium. The model described the evolution of the amplitude \( a(t) \) of the \( \sin x \) mode as

\[
\dot{a} \approx -\frac{\sigma^2}{38} a - \frac{1}{12} a^3 + \frac{1}{6} \sigma a \Phi_2 + \frac{\sqrt{315}}{1936 \sqrt{3}} \sigma^2 a \Phi,
\]

for some white noise \( \Phi(t) \) independent of \( \Phi_2 \) over long times. The second key theorem of centre manifolds is that models such as (11) do capture the long term dynamics of the original stochastic system (1). For example,
1 Introduction

Theorem 7.1(i) [6] assures us that all nearby solutions of the SPDE (1) exponentially quickly in time approach a solution of the model (4) embedded on the centre manifold \( u = v(a(t), x, t, \sigma) \). This property is sometimes called “asymptotic completeness” [36]. It assures us that apart from exponentially decaying transients, models such as (4) potentially describe all the long term dynamics of the original system.

Although the nonlinearity induced stochastic resonance generates the effectively new multiplicative noise, \( \propto \sigma^2 a \Phi \), its most significant effect is the enhancement of the stability of the equilibrium \( a = 0 \) through the \( \sigma^2 a / 88 \) term. For examples, Boxler [6, p.544], Drolet & Vinals [13, 14] and Knobloch & Weisenfeld [21] and Vanden–Eijnden [39, p.68] found the same sort of stability modifying linear term in their analyses of stochastically perturbed bifurcations and systems. Boxler [6, Theorem 7.3(a)] proves that the stability of an original SDE is the same as the model SDE on the stochastic centre manifold. Analogously to these effects of microscale noise on the macroscale dynamics, Just et al. [18] sought to determine how microtime deterministic chaos, not noise, translates into a new effective stochastic noise in the slow modes of a deterministic dynamical system. Here we explore further the modelling of such induced changes to the stability of the system (1) through the transformation by nonlinearity of microscale noise into macroscale drift and noise. Indeed, our more complete analysis here shows that noise in all other spatial modes contribute to destabilise the equilibrium, see [24].

The approach For the first part of the analysis, Sections 2 and 3, the requirement of white, delta correlated noise is irrelevant; the results are valid for quite general time dependent, additive forcing. In these two sections we show how to remove “memory” integrals over the past history of the noise, Section 2. However, in a nonlinear system there are effects quadratic in the noise processes; in Section 3 the techniques reduce the number of memory integrals but cannot eliminate them all. In the second part of the analysis, Sections 4 and 5, the critical assumption of white, delta correlated noise enables analysis of the nonlinear interactions. Modelling the Fokker–Planck equations of the irreducible quadratic noises shows that their probability density functions (PDFs) approximately factor into a multivariate Gaussian and a slowly evolving conditional probability, such a factorisation is also the key to the modelling by Just et al. [18] of fast deterministic chaos as noise on the slow modes. This factorisation abstracts effectively new noise processes over the long time scales of interest in the model. Section 4 discusses the specifics, such as the appropriate version of (4), for the SPDE (1) with delta
correlated noise in space and time; whereas Section 3 presents generic transformations of the irreducible quadratic noises for use in analysing general SDEs. Appendix A provides alternate proofs, using Ito calculus, of some of the key results in the modelling of nonlinear interactions among the noise components.

2 Construct a memoryless normal form model

The centre manifold approach identifies that the long term dynamics of a SPDE such as (1) is parametrised by the amplitude \( a(t) \) of the neutral mode \( \sin x \). Arnold et al. [11] investigated stochastic Hopf bifurcations this way, and the approach is equivalent to the slaving principle for SDEs used by Schoner & Haken [37]. However, most researchers generate models with convolutions over fast time scales of the noise [34, §2, e.g.]. Here we keep the model tremendously simpler by removing these ‘memory’ convolutions. This removal of convolutions was originally developed for SDEs by Coullet, Elphick & Tirapegui [12], Sri Namachchivaya & Lin [26], and Roberts & Chao [9, 34].

As discussed, centre manifold theory supports the models we consider herein. However, crucial features of the model reflect the steps taken to construct the model; thus the next two sections discuss the iterative construction. However, note that centre manifold theorems only depend upon certain basic properties and that the residuals of the governing equations are of some specified order of smallness. For example, Boxler [6, Theorem 8.1] assure us that if we satisfy the SPDE (1) to some residual \( \mathcal{O}(\|a,\sigma\|^4) \), then the stochastic centre manifold and the evolution thereon have the same order of error. Note that the amplitude \( a \) and the noise intensity \( \sigma \) are the small parameters in the asymptotic expansions forming the model. The support such theorems give to our modelling is independent of the details of construction.

One complication is that I construct models to residuals of \( \mathcal{O}(a^4 + \sigma^2) \), for example. Theory covers this when we simply define a new small parameter \( \epsilon = \sigma^{1/2} \), then a residual of \( \mathcal{O}(a^4 + \sigma^2) = \mathcal{O}(\|(a, \epsilon)\|^4) \), for example; hence the theorem applies to assure us the errors are of \( \mathcal{O}(\|(a, \epsilon)\|^4) = \mathcal{O}(a^4 + \sigma^2) \), for example. I use this latter form to report the residuals and errors. Because the critical aspect of constructing the centre manifold model is simply the ultimate order of the residual of the SPDE (1), the specific details of the computation are not recorded here. Instead computer algebra [35, §1] performs all the details. Here I just report on critical steps in the method.
Consider the task of iteratively constructing a stochastic model for the SPDE (1) using iteration [28]. We seek solutions in the form \( u = v(a, x, t, \sigma) = a \sin x + \cdots \) such that the amplitude \( a \) evolves according to some prescription \( \dot{a} = g(a, t, \sigma) \), such as (1). The steps in the construction proceed iteratively. Suppose that at some stage we have an asymptotic approximation to the model, then the next iteration is to seek small corrections, denoted \( v' \) and \( g' \), to improve the asymptotic approximation. As the iterations proceed, the small corrections \( v' \) and \( g' \) get systematically smaller, that is, of higher order in the small parameters \( a \) and \( \sigma \) of the asymptotic expansion. As explained in [28], substitute \( u = v + v' \) and \( \dot{a} = g + g' \) into the SPDE (1), linearise the problem for \( v' \) and \( g' \) by dropping products of small corrections, and obtain that the corrections should satisfy

\[
\frac{\partial v'}{\partial t} - \frac{\partial^2 v'}{\partial x^2} - v' + g' \sin x = \text{residual}.
\]

Here the “residual” is the residual of the SPDE (1) evaluated for the currently known asymptotic approximation. For example, if at some stage we had determined the deterministic part of the model was

\[
u = a \sin x - \frac{1}{6} a^2 \sin 2x + \frac{1}{32} a^3 \sin 3x + \mathcal{O}(a^4, \sigma)
\]

such that

\[
\dot{a} = -\frac{1}{12} a^3 + \mathcal{O}(a^4, \sigma),
\]

then the residual of the SPDE (1) for the next iteration would be simply the stochastic forcing,

\[
\text{residual} = \sigma \sum_{k=1}^{\infty} \phi_k \sin kx + \mathcal{O}(a^4).
\]

The terms in the residual split into two categories, as is standard in singular perturbations:

- the components in \( \sin kx \) for \( k \geq 2 \) cause no great difficulties, we include a corresponding component in the correction \( v' \) to the field in proportion to \( \sin kx \) —when the coefficient of \( \sin kx \) in the residual is time dependent the component in the correction \( v' \) is \( \mathcal{H}_k \phi_k(t) \sin kx \) in which the operator \( \mathcal{H}_k \) denotes convolution over past history with \( \exp[-(k^2 - 1)t] \);\(^2\)

- but any component in \( \sin x \), such as \( \phi_1 \) in this iteration with this residual, must cause a contribution to the evolution correction \( g' \), here simply \( g' = \phi_1 \), as no uniformly bounded component in \( v' \) of \( \sin x \) can

\(^2\text{Namely } \mathcal{H}_k \phi = \exp[-(k^2 - 1)t] * \phi(t) = \int_{-\infty}^{t} \exp[-(k^2 - 1)(t - \tau)] \phi(\tau) \, d\tau.\)
match a \( \sin x \) component of the residual—this is the standard solvability condition for singular perturbations.

However, a more delicate issue arises for the next corrections. In the next iteration the next

\[
\text{residual} = a\sigma \left[ \frac{1}{2} H_2 \phi_2 \sin x + \left( \frac{1}{3} \phi_1 + H_3 \phi_3 \right) \sin 2x \right] + \sum_{k=3}^{\infty} \frac{k}{2} (H_{k+1} \phi_{k+1} - H_{k-1} \phi_{k-1}) \sin kx + O(a^4 + \sigma^2). \tag{5}
\]

Many are tempted to simply use the solvability condition and match the \( \sin x \) component in this residual directly by the correction \( a\sigma \frac{1}{2} H_2 \phi_2 \) to the evolution \( g' \). But this choice introduces incongruous short time scale convolutions of the forcing into the model (11) of the long time evolution. The appropriate alternative \[12, 26, 9, 34\] is to recognise that part of the convolution can be integrated: since for any \( \Phi(t) \), \( \frac{d}{dt} H_k \Phi = -(k^2 - 1) H_k \Phi + \Phi \), thus

\[
H_k \Phi = \frac{1}{k^2 - 1} \left[ -\frac{d}{dt} H_k \Phi + \Phi \right], \tag{6}
\]

and so split such a convolution in the residual, when multiplied by the neutral mode \( \sin x \), into:

- the first part, \(-\frac{d}{dt} H_k \Phi/(k^2 - 1)\), that is integrated into the next update \( v' \) for the subgrid field;
- and the second part, \( \Phi/(k^2 - 1) \), that updates \( \dot{a} \) in the evolution.

For the example residual (5), the term \( \frac{1}{2} a\sigma H_2 \phi_2 \sin x \) in the residual thus forces a term \(-\frac{1}{6} a\sigma H_2 \phi_2 \sin x \) into the subgrid field, and a term \( \frac{1}{6} a\sigma \phi_2 \) into the evolution \( \dot{a} \). When the residual component has many convolutions, then apply this separation recursively. Consequently, all fast time convolutions linear in the forcing are removed from the evolution equation for the amplitude \( a(t) \).

Continuing this iterative construction gives more and more accurate models. The iteration terminates when the residuals are zero to some specified order. Then the Approximation Theorem of centre manifold theory \[6\] assures us that the model has the same order of error as the residual.

For example, terminating the iterative construction so that residual \( \| = O(a^4 + \sigma^2) \), we find the field

\[
\mathbf{u} = a \sin x - \frac{1}{6} a^2 \sin 2x + \frac{1}{12} a^3 \sin 3x + \sigma \sum_{k=2}^{\infty} H_k \phi_k \sin kx
\]

Tony Roberts, March 29, 2022
3 Quadratic noise has irreducible interactions

\[ a \sigma \left[ -\frac{1}{6} \mathcal{H}_2 \phi_2 \sin x + \frac{1}{3} \mathcal{H}_2 \phi_1 + \mathcal{H}_3 \phi_3 \right] \sin 2x + \sum_{k=3}^{\infty} \frac{k}{2} \mathcal{H}_k (\mathcal{H}_{k+1} \phi_{k+1} - \mathcal{H}_{k-1} \phi_{k-1}) \sin kx \]  
\[ + \mathcal{O}(a^4 + \sigma^2). \]  

(7)

The corresponding model for the evolution,

\[ \dot{a} = -\frac{1}{12} a^3 + \sigma \phi_1 + \frac{1}{6} a \sigma \phi_2 + a^2 \sigma (\frac{1}{18} \phi_1 + \frac{1}{96} \phi_3) + \mathcal{O}(a^5 + \sigma^2), \]  

(8)

has no fast time convolutions, only the direct influence of the forcing. This is the normal form for a noisy model.

Note the generic feature that the originally linear noise, through the non-linearities in the system, appears as a multiplicative noise in the model. But it is only the coarse structure of the noise that appears in the model: all noise with wavenumber \( k > 3 \) is ineffective in these the most important terms in a model.

3 Quadratic noise has irreducible interactions

Continue the iterative construction of the stochastic centre manifold model to effects quadratic in the magnitude \( \sigma \) of the noise. We seek terms in \( \sigma^2 \) as these generate mean drift terms, and also seek terms in \( a \sigma^2 \) as these affect the linear stability of the SPDE \[ \text{[34, Figure 2]} \] and \[ \text{[31, p.544]}. \]  

Computer algebra \[ \text{[33, §1.1–4]} \] determines the stochastic model evolution

\[ \dot{a} = -\frac{1}{12} a^3 - \frac{7}{5400} a^5 \]  
\[ + \sigma \phi_1 + \frac{1}{6} a \sigma \phi_2 + a^2 \sigma (\frac{1}{18} \phi_1 + \frac{1}{96} \phi_3) + a^3 \sigma (\frac{1}{36} \phi_2 + \frac{1}{4320} \phi_4) \]  
\[ + \sigma^2 \left[ \frac{1}{6} \phi_1 \mathcal{H}_2 \phi_2 + \sum_{k=2}^{\infty} \frac{\phi_k \mathcal{H}_{k+1} \phi_{k+1} + \phi_{k+1} \mathcal{H}_k \phi_k}{2(2k^2 + 2k - 1)} \right] \]  
\[ + a \sigma^2 \left[ \frac{1}{18} \phi_1 \mathcal{H}_2 \phi_1 + \frac{17}{528} \phi_1 \mathcal{H}_3 \phi_3 + \frac{1}{72} \phi_1 \mathcal{H}_2 \mathcal{H}_3 \phi_3 \right. \]  
\[ - \frac{1}{44} \phi_2 \mathcal{H}_2 \phi_2 + \frac{1}{66} \phi_3 \mathcal{H}_2 \phi_1 + \frac{1}{22} \phi_3 \mathcal{H}_2 \mathcal{H}_2 \phi_3 \]  
\[ + \sum_{k=3}^{\infty} c_k^0 \phi_k \mathcal{H}_k \phi_k + \sum_{k=3}^{\infty} c_k^*(\phi_{k+1} \mathcal{H}_{k-1} \phi_{k-1} + \phi_{k-1} \mathcal{H}_{k+1} \phi_{k+1}) \]  
\[ + \sum_{k=2}^{\infty} c_k^+ \phi_k \mathcal{H}_{k+1} (\mathcal{H}_{k+2} \phi_{k+2} - \mathcal{H}_k \phi_k) \]

Tony Roberts, March 29, 2022
\[ + \sum_{k=4}^{\infty} c_k^\alpha \mathcal{H}_{k-1}(\mathcal{H}_k \phi_k - \mathcal{H}_{k-2} \phi_{k-2}) \bigg] + \mathcal{O}(a^6 + \sigma^3), \tag{9} \]

where constants
\[
\begin{align*}
c_k^0 &= \frac{1}{2(k^2 - 1)(2k^2 - 2k - 1)(2k^2 + 2k - 1)}, \\
c_k^* &= \frac{4k^4 - 2k^2 + 1}{12k^2(2k^2 - 2k - 1)(2k^2 + 2k - 1)}, \\
c_k^\pm &= \frac{k \pm 1}{4(2k^2 \pm 2k - 1)}. 
\end{align*}
\]

The model (9) provides accurate simulations of the original SPDE (1), as the model is obtained through solving the SPDE. This accuracy will hold whether the forcing noise \(\phi(x, t)\) is deterministic or stochastic, space-time correlated or independent at each point in space and time. In deterministic cases, Chicone & Latushkin’s [11] theory of infinite dimensional centre manifolds supports (9) as a model of the deterministic but nonautonomous PDE (1). However, we proceed to exclusively consider the case when the applied forcing \(\phi(x, t)\) is stochastic.

The outstanding challenge with effects quadratic in the noise is that we apparently cannot directly eliminate history integrals of the noise, such as \(\phi_1 \mathcal{H}_2 \phi_2\), from the model.

4 Stochastic resonance affects deterministic terms

Chao & Roberts [9, 34] argued that quadratic terms involving memory integrals of the noise were effectively new drift and new noise terms when viewed over the long time scales of the relatively slow evolution of the model (9). Analogously, Just et al. [18] argued that fast time deterministic chaos appears as noise when viewed over long time scales. The arguments of Chao & Roberts [9, 34] rely upon the noise being stochastic white noise. In previous sections, the model was a strong model in that (9) could faithfully track given realisations of the original SPDE [3, Theorem 7.1(i), e.g.]; however, now we derive a weak model, such as (4), which in a weak sense maintains fidelity to solutions of the original SPDE, but we cannot know which realisation because of the effectively new noises on the macroscale.
**Abandon fast time convolutions**  The undesirable feature of the large time model is the inescapable appearance in the model of fast time convolutions in the quadratic noise term, namely $H_2 \phi_1 = e^{-3t} \ast \phi_1$ and $H_2 H_3 \phi_3 = e^{-3t} \ast e^{-8t} \ast \phi_3$. These require resolution of the fast time response of the system to these fast time dynamics in order to maintain fidelity with the original SPDE and so incongruously require small time steps for a supposedly slowly evolving model. However, maintaining fidelity with the full details of a white noise source is a pyrrhic victory when all we are interested in is the relatively slow long term dynamics. Instead we need only those parts of the quadratic noise factors, such as $\phi_1 H_2 \phi_1$ and $\phi_1 H_2 H_3 \phi_3$, that over long time scales are firstly correlated with the other processes that appear and secondly independent of the other processes: these not only introduce factors in new independent noises into the model but also introduce a deterministic drift due to stochastic resonance (as also noted by Drolet & Vinal [13]).

In this problem, and to this order of accuracy, we need to understand the long term effects of quadratic noise effects taking the form $\phi_1 H_p \phi_i$ and $\phi_1 H_q H_p \phi_i$. These terms appear in the right-hand side of the evolution equation [9] in the form $\dot{a} = \cdots \sigma^2 c \phi_j H_p \phi_i \cdots$, for example. Equivalently rewrite this form as $\dot{a} = \cdots \sigma^2 c \phi_j H_p \phi_i dt \cdots$. In this latter form we aim to replace such a noise term by a corresponding stochastic differential so that $\dot{a} = \cdots \sigma^2 c dy_1 \cdots$ for some stochastic process $y_1$ with some drift and volatility: $dy_1 = () dt + () dW$ for a Wiener process $W$. Thus we must understand the long term dynamics of stochastic processes $y_1$ and $y_2$ defined via the nonlinear SDES

$$ \frac{dy_1}{dt} = \phi_j H_p \phi_i \quad \text{and} \quad \frac{dy_2}{dt} = \phi_j H_q H_p \phi_i. $$

**Canonical quadratic noise interactions**  To proceed following the argument by Chao & Roberts [9 §4.1] we name the two coloured noises that appear in the nonlinear terms. Define $z_1 = H_p \phi_1$ and $z_2 = H_q H_p \phi_1$. From [10] they satisfy the SDES

$$ \frac{dz_1}{dt} = -\beta_1 z_1 + \phi_i \quad \text{and} \quad \frac{dz_2}{dt} = -\beta_2 z_2 + z_1, $$

where for this SPDE the rates of decay $\beta_1 = p^2 - 1$ and $\beta_2 = q^2 - 1$. Now put the SDES and together: we must understand the long term properties of $y_1$ and $y_2$ governed by the coupled system

$$ \dot{y}_1 = z_1 \phi_1, \quad \dot{z}_1 = -\beta_1 z_1 + \phi_i, $$

Tony Roberts, March 29, 2022
4 Stochastic resonance affects deterministic terms

\[
\dot{y}_2 = z_2 \phi_1, \quad \dot{z}_2 = -\beta_2 z_2 + z_1.
\]  

(12)

There are two cases to consider: when \( i = j \) the two source noises \( \phi_i \) and \( \phi_j \) are identical; but when \( i \neq j \) the two noise sources are independent.

Use the Fokker–Planck equation Following Chao \& Roberts [9, 34] and analogous to Just et al. [18], we explore the long term dynamics of the canonical quadratic system (12) via the Fokker–Planck equation for the PDF \( P(y, z, t) \). Vanden–Eijnden [39] similarly uses the Kolmogorov forward equation to model the slow modes in SDEs. See in the canonical system (12) that, upon neglecting the forcing, the \( z \) variables naturally decay exponentially whereas the \( y \) variables would be naturally constant. Consequently, in the long-term we expect the \( z \) variables to settle onto some more-or-less definite stationary probability distribution, whereas the \( y \) variables would evolve slowly. Thus we proceed to approximately factor the joint PDF into

\[
P(y, z, t) \approx p(y, t) G_0(z),
\]  

(13)

where \( G_0(z) \) is some distribution to be determined, depending upon the coefficients \( \beta \), and the quasi-PDF \( p(y, t) \) evolves slowly in time according to a PDE we interpret as a Fokker–Planck equation for the long term evolution.

Begin by analysing the Fokker–Planck equation for the PDF \( P(y, z, t) \) of the canonical system (12). Recall that throughout we adopt the Stratonovich interpretation of SDEs; thus the Fokker–Planck equation of (12) is

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial z_1} (\beta_1 z_1 P) + \frac{\partial}{\partial z_2} [(\beta_2 z_2 - z_1) P] + \frac{1}{2} \frac{\partial^2 P}{\partial z_1^2} + \frac{1}{2} s \sum_{k=1}^{2} \frac{\partial}{\partial y_k} \left( z_k \frac{\partial P}{\partial z_1} \right) + \frac{1}{2} \sum_{k,l=1}^{2} \frac{\partial}{\partial y_k} \left( z_k z_l \frac{\partial P}{\partial y_l} \right),
\]  

(14)

where the parameter \( s = 1 \) for the identical noise case \( i = j \), whereas \( s = 0 \) for the independent noise case \( i \neq j \).

A deterministic centre manifold captures the long term dynamics

The first line in the Fokker–Planck equation (14) represents all the rapidly dissipative processes: the terms \( \partial_{z_k} [\beta_k z_k P] \) move probability density \( P \) to the vicinity of \( z_k = 0 \); which is only balanced by the spread induced through the stochastic noise term \( \frac{1}{2} P_{z_1 z_1} \) and the forcing term \( \partial_{z_2} [-z_1 P] \). In contrast, the terms in the second line of the Fokker–Planck equation (14) describes that the PDF \( P \) will slowly spread in the \( y_k \) directions over long times. This

Tony Roberts, March 29, 2022
strong disparity in time scales leads many [13, 37, 18, e.g.] to the conditional factorisation (13). However, we go further systematically by appealing to deterministic centre manifold theory [9, 33, 16]. Consider the terms in the second line of the Fokker–Planck equation (14) to be small perturbation terms through assuming that the structures in the $y_k$ variables are slowly varying, that is, treat $\partial / \partial y_k$ as asymptotically “small” parameters [27, 33], as is appropriate over long times. Then “linearly”, that is, upon ignoring the “small” terms in the second line, the dynamics of the Fokker–Planck equation (14) are that of exponential attraction to a manifold of equilibria $P \propto G_0(z)$ at each $y$, say the constant of proportionality is $p(y)$. Theory for slow variations in space [27, 33] then assures us that a centre manifold exists for the Fokker–Planck equation (14), and that all dynamics (as it is a linear pde) are exponentially quickly attracted to the dynamics on the centre manifold. The approximation theorem then assures us that the long term dynamics of the PDF $P$, when the small terms in the second line of (14) are accounted for, may be expressed as a series in gradients in $y$ of the slowly evolving $p(y, t)$: using $\nabla$ for the vector gradient $\partial / \partial y$, the PDF $P(y, z, t) = G_0(z)p + G_1(z) \cdot \nabla p + G_2(z) : \nabla \nabla p + \cdots$, \hspace{1cm} (15)

where instead of being constant the quasi-conditional probability $p$ evolves slowly in time according to a series in gradients of $p$ in $y$ of the Kramers–Moyal form [25, 24, 38, e.g.]

$$\frac{\partial p}{\partial t} = -U \cdot \nabla p + D : \nabla \nabla p + \cdots.$$ \hspace{1cm} (16)

In practice we truncate this Kramers–Moyal expansion to include up to the second order gradients in $y$ for two reasons: firstly, Pawula’s theorem implies any higher order truncation may lead to negative probabilities; and secondly, we interpret the second order truncation of the Kramers–Moyal expansion (16) as a Fokker–Planck equation for the long term evolution of the interesting $y$ processes. Just et al. [18] in their equation (11) similarly truncate to second order. Deterministic centre manifold theory assures us that all solutions are attracted to this model [33, §2.2.2, e.g.].

**Construct the long term model**  The approximation theorem of centre manifolds [33, §2.2.3, e.g.] asserts that we simply substitute the ansatz (15–16) into the governing Fokker–Planck equation (14) and solve to reduce the residuals to some order of asymptotic error, then the centre manifold model is constructed to the same order of accuracy. Here the order of accuracy is measured by the number of $y$ gradients, $\nabla$, as this is the small perturbation...
in this problem. Consequently, for example, an error denoted as $O(\nabla^q p)$ denotes all terms of the form $a^{q_1+q_2} \frac{\partial}{\partial y_1}^{q_1} \frac{\partial}{\partial y_2}^{q_2}$ for $q_1 + q_2 \geq q$. See in the ansatz (15), anticipating the relevant parts of the model, I already truncated the expansions at the second order in such gradients, that is, the displayed part of these expansions have errors $O(\nabla^3 p)$. Computer algebra machinations \[35, §2\] driven by the residuals of the Fokker–Planck equation (14) readily find the coefficients of the centre manifold model (15–16). The computer algebra \[35, §2\] determines that large time solutions of the processes (12) have the pdf

$$P = A \exp \left\{ - (\beta_1 + \beta_2) \left[ z_1^2 - 2 \beta_2 z_1 z_2 + \beta_2 (\beta_1 + \beta_2) z_2^2 \right] \right\}$$

for any normalisation constants $A$, $B_1$ and $B_2$. Simultaneously with finding the next order corrections to this, we find the relatively slowly varying, quasi-conditional probability density $p$ evolves according to the Kramers–Moyal expansion

$$\frac{\partial p}{\partial t} = - \frac{1}{2} s \frac{\partial p}{\partial y_1} + \nabla \nabla p + O(\nabla^3 p), \quad (17)$$

where the constant diffusion matrix

$$\mathbb{D} = \frac{1}{4\beta_1} \begin{bmatrix} \frac{1}{\beta_1 + \beta_2} & \frac{1}{\beta_1 + \beta_2} \\ \frac{1}{\beta_1 + \beta_2} & \frac{1}{\beta_2 (\beta_1 + \beta_2)} \end{bmatrix}. \quad (18)$$

**Translate to a corresponding SDE** Interpret (17) as a Fokker–Planck equation and see it corresponds to the SDEs

$$\dot{y}_1 = \frac{1}{2} s + \frac{\psi_1(t)}{\sqrt{2\beta_1}} \quad \text{and} \quad \dot{y}_2 = \frac{1}{\beta_1 (\beta_1 + \beta_2)} \left( \frac{\psi_1(t)}{\sqrt{2\beta_1}} + \frac{\psi_2(t)}{\sqrt{2\beta_2}} \right). \quad (19)$$

Of course there are many coupled SDEs whose Fokker–Planck equation is (17): the reason is there are many $2 \times 2$ volatility matrices $S$ of coupled SDEs that give the same diffusivity matrix $\mathbb{D} = \frac{1}{2} S S^T$; for example, Just et al. \[18\] choose $S$ to be the positive definite, symmetric square root of the diffusivity matrix $2 \mathbb{D}$. For our purposes any of the possible volatility matrices would suffice: in resorting to the Fokker–Planck equations we necessarily lose fidelity of paths, and now only require fidelity of distributions and correlations; as
commented earlier, the results is a weak model, not a strong model. For simplicity, obtain the form of the noise terms in (19) by the unique Cholesky factorisation of the diffusion matrix

\[ \mathbb{D} = \frac{1}{2} \mathbb{L} \mathbb{L}^\top \quad \text{for matrix} \quad \mathbb{L} = \begin{bmatrix} \frac{1}{\sqrt{2\beta_1}} & 0 \\ \frac{1}{\sqrt{2\beta_1(\beta_1 + \beta_2)}} & \frac{1}{\sqrt{2\beta_2(\beta_1 + \beta_2)}} \end{bmatrix}. \quad (20) \]

Choosing the volatility matrix in the SDEs to be the lower triangular Cholesky matrix \( \mathbb{L} \) ensures that nearly half the terms in the volatility matrix are zero, and it also ensures that when we go to higher order convolutions of noise in Section 5 this \( 2 \times 2 \) factorisation remains in the higher order factorisations, see (28). As argued by Chao & Roberts [9, 34] and proved in Appendix A, \( \phi_1(t) \) are new noises independent of \( \phi_i \) and \( \phi_j \) over long time scales. The remarkable feature to see in the SDEs (19) is that for the case of identical noise, \( \phi_i = \phi_j \), that is the case \( s = 1 \), there is a mean drift \( \frac{1}{2} \) in the stochastic process \( y_1 \); there is no mean drift in any other process nor in the other case, \( s = 0 \).

You might query the role of the neglected terms in the Kramers–Moyal expansions of the PDF (15) and the supposed Fokker–Planck equation (17). In the PDF (15) the neglected \( \mathcal{O}(\nabla^3 p) \) terms provide more details of the non-Gaussian structure of the PDF in the slowly evolving long time dynamics. The effects of the neglected \( \mathcal{O}(\nabla^3 p) \) terms in (17) correspond to algebraically decaying departures from the second order truncation that we interpret as a Fokker–Planck equation; Chao & Roberts [9] demonstrated this algebraic decay to normality in some numerical simulations (Chatwin [10] discussed this algebraic approach to normality in detail in the simpler situation of dispersion in a channel). Such algebraically decaying transients may represent slow decay of non-Markovian effects among the \( y \) variables. However, the truncation (17) that we interpret as a Fokker–Planck equation is the lowest order structurally stable model and so will adequately model the dynamics over the longest time scales.

**Temporarily truncate the noise to simplify discussion** We want to simplify the detailed model (9) further by eliminating the nonlinear fast time convolutions to deduce a model nearly as simple as (4). However, dealing with the infinite sums in the model (9) is confusing when the focus is on transforming the nonlinear fast time convolutions. Thus temporarily we discuss the case when the applied spatio-temporal noise (2) is truncated to the first three modes: \( \phi = \sum_{k=1}^3 \phi_k(t) \sin kx \). Just these three noise components have a range of interactions that are representative of the noise interactions.
appearing in the model \[ \text{(9)} \] to the order of accuracy reported here and for the nonlinearity of this problem. Thus to focus on the transformations of the noise, temporarily consider the model \[ \text{(9)} \] with the truncated noise, that is,

\[
\dot{a} = -\frac{1}{12}a^3 - \frac{7}{3456}a^5 + \sigma \phi_1 + \frac{1}{6}a\sigma \phi_2 + a^2\sigma(\frac{1}{18}\phi_1 + \frac{1}{96}\phi_3) + a^3\sigma\frac{1}{54}\phi_2
\]

\[
+ \sigma^2(\frac{1}{6}\phi_1 \mathcal{H}_2 \phi_2 + \frac{1}{12}\phi_3 \mathcal{H}_2 \phi_2 + \frac{1}{22}\phi_2 \mathcal{H}_3 \phi_3)
\]

\[
+ a\sigma^2\left[\frac{1}{18}\phi_1 \mathcal{H}_2 \phi_1 - \frac{1}{44}\phi_2 \mathcal{H}_2 \phi_2 + \frac{1}{4048}\phi_3 \mathcal{H}_3 \phi_3 + \frac{10}{528}\phi_1 \mathcal{H}_3 \phi_3
\]

\[
+ \frac{1}{66}\phi_3 \mathcal{H}_2 \phi_1 - \frac{3}{44}\phi_2 \mathcal{H}_3 \mathcal{H}_2 \phi_2 + \frac{1}{6}\phi_1 \mathcal{H}_2 \mathcal{H}_3 \phi_3 + \frac{1}{22}\phi_3 \mathcal{H}_2 \mathcal{H}_3 \phi_3
\]

\[
- \frac{1}{23}\phi_3 \mathcal{H}_4 \mathcal{H}_3 \phi_3\right] + \mathcal{O}(a^6 + \sigma^3).
\]

\[
(21)
\]

**Transform the strong model \[ (21) \] to be usefully weak.** The quadratic noises in \[ (21) \] involve the convolutions \( \mathcal{H}_2, \mathcal{H}_3 \) and \( \mathcal{H}_4 \) which have corresponding decay rates \( \beta \) of 3, 8 and 15 respectively. Thus from the various instances of \[ (19) \], to obtain a model for long time scales we replace the quadratic noises as follows:

\[
\phi_1 \mathcal{H}_2 \phi_2 \mapsto \frac{\psi_1}{\sqrt{6}},
\]

\[
\phi_3 \mathcal{H}_2 \phi_2 \mapsto \frac{\psi_2}{\sqrt{6}},
\]

\[
\phi_2 \mathcal{H}_3 \phi_3 \mapsto \frac{\psi_3}{4},
\]

\[
\phi_1 \mathcal{H}_2 \phi_1 \mapsto \frac{1}{2} + \frac{\psi_4}{\sqrt{6}},
\]

\[
\phi_2 \mathcal{H}_2 \phi_2 \mapsto \frac{1}{2} + \frac{\psi_5}{\sqrt{6}},
\]

\[
\phi_3 \mathcal{H}_3 \phi_3 \mapsto \frac{1}{2} + \frac{\psi_6}{4},
\]

\[
\phi_1 \mathcal{H}_3 \phi_3 \mapsto \frac{\psi_7}{4},
\]

\[
\phi_3 \mathcal{H}_2 \phi_1 \mapsto \frac{\psi_8}{\sqrt{6}},
\]

\[
\phi_2 \mathcal{H}_3 \mathcal{H}_2 \phi_2 \mapsto \frac{\psi_5}{11\sqrt{6}} + \frac{\psi_9}{44},
\]

\[
\phi_1 \mathcal{H}_2 \mathcal{H}_3 \phi_3 \mapsto \frac{\psi_7}{44} + \frac{\psi_{10}}{11\sqrt{6}},
\]

Tony Roberts, March 29, 2022
Stochastic resonance affects deterministic terms

\[ \phi_3\mathcal{H}_2\mathcal{H}_3\Phi_3 \mapsto \psi_6 + \frac{\psi_{11}}{11\sqrt{6}}, \]

\[ \phi_3\mathcal{H}_4\mathcal{H}_3\Phi_3 \mapsto \psi_6 + \frac{\psi_{12}}{23\sqrt{30}}, \]

where \( \psi_1, \ldots, \psi_{12} \) are independent white noises, that is, derivatives of independent Wiener processes. Thus transform the strong model (21) to the weak model

\[ \dot{a} = -\frac{1}{12}a^3 - \frac{7}{3456}a^5 \]

\[ + \sigma \phi_1 + \frac{1}{6}a^2\Phi_2 + a^2\sigma(\frac{1}{18}\phi_1 + \frac{1}{90}\phi_3) + a^3\sigma\frac{1}{36}\Phi_2 \]

\[ + \sigma^2 \left( \frac{\psi_1}{6\sqrt{6}} + \frac{\psi_2}{22\sqrt{6}} + \frac{\psi_3}{88} \right) + \frac{1}{2}a\sigma^2 \left[ \frac{1}{18} - \frac{1}{44} + \frac{1}{4048} \right] \]

\[ + \sigma^2 \left[ \frac{\psi_4}{18\sqrt{6}} - \frac{7\psi_5}{242\sqrt{6}} + \frac{2549\psi_6}{4096576} + \frac{9\psi_7}{704} + \frac{\psi_8}{66\sqrt{6}} - \frac{3\psi_9}{1936} \right] \]

\[ + \frac{\psi_{10}}{66\sqrt{6}} + \frac{\psi_{11}}{242\sqrt{6}} - \frac{\psi_{12}}{529\sqrt{30}} \right] + \mathcal{O}(a^6 + \sigma^3). \] (22)

Here the new noises \( \psi_k \) only appear in two different combinations. Thus we do not need to use them individually, only their combined effect. Combining the new noises into two effective new noise processes \( \Phi_1 \) and \( \Phi_2 \) \[ (23) \]

The Stratonovich model (23) is a weak model of the original Stratonovich SPDE (11) because we have replaced detailed knowledge of the interactions of rapid fluctuations, seen in the convolutions of (21), by their long time scale statistics; similarly Just et al. (18) replaced detailed knowledge of rapid chaos by its long time scale statistics. Vanden–Eijnden (39) comments that stronger results can be obtained. However, resolving rapid fluctuations seems futile when they are stochastic, as required for this section, because describing them as stochastic admits we do not know their detail anyway. The model (23) is useful because it only resolves long time scale dynamics and hence, for example, we are empowered to efficiently simulate it numerically using large time steps.

---

3The combinations \( \sigma \phi_1 + 0.07144 \sigma^2 \Phi_1 \) and \( \frac{1}{6}a^2\Phi_2 + 0.02999 \sigma^2 a\Phi_2 \) in (23) could be combined, but then one must be careful with the correlations with the other noise terms on the second line of (23).
But furthermore, we readily discover crucial stability information in the weak model (23). See that the quadratic interactions of noise processes, through stochastic resonance, generate the mean effect term $0.01654 \sigma^2 a$. As it is a term linear in $a$ with positive coefficient $0.01654 \sigma^2$, this term destabilises the origin. Roberts [34] demonstrated in numerical simulations how the same term in $\sigma^2 a$, but with a negative coefficient, stabilises the origin as expected. Thus here we are empowered by our analysis to predict instead that the stochastic solutions of the SPDE (1) will linger about and switch between two fixed points obtained from the deterministic part of (23), namely $u \approx a \sin x$ for amplitudes $a \approx \pm 0.45 \sigma$.

Return to the full spectrum of noise (9) Now we deal with the full complexity of the infinite sums of nonlinear noise interactions in the strong model (9). First, see that we obtain the exact numerical coefficient for the stochastic resonance term $\sigma^2 a$ for the full spectrum of noise through the infinite sum $\sum_{k=3}^{\infty} c_k \phi_k \mathcal{H}_k \phi_k$. Terms of this form are the only ones contributing to this stochastic resonance. The exact numerical coefficient is thus $(1/18 - 1/44 + \sum_{k=3}^{\infty} c_k^3)/2 = 0.016563$ to five significant digits. Curiously, in this problem, it is only the $\phi_2 \sin 2x$ component of the noise that acts to stabilise $u = 0$ through its negative contribution to this sum, as explored in [34]; all other noise components act to destabilise $u = 0$ through their positive contribution.

Second, and similarly, the other infinite sums over the noise components in (9) modify the coefficients in the weak model (23). But, as for the stochastic resonance term, the modification to the coefficients are not large: the plain $\sigma^2$ term from the third line of (23) has coefficients $\sim 1/k^2$ but $\phi_k \mathcal{H}_k \phi_k \sim 1/k$ (from (19) and that $\beta \sim k^2$), so that the terms in the sum are $\sim 1/k^3$; similarly the infinite sums in lines 6–8 of (10) have terms $\sim 1/k^4$ or smaller. Further, when combining the infinitude of new noise terms in the analogue of (22) to find the exact version of the weak model (23), the coefficients are the root-sum-squares of the coefficients of the new noise processes in the infinite sums; thus terms $O(1/k^3)$ and $O(1/k^4)$ in the sums are effectively terms $O(1/k^6)$ and $O(1/k^8)$. Indeed, computer algebra [35, §1.5] demonstrates that at most ten terms in these sums determine the coefficients of the weak model correct to five significant digits, namely

$$
\dot{a} = 0.016563 \sigma^2 a - \frac{1}{12} a^3 - \frac{7}{3456} a^5 \\
+ \sigma \phi_1 + \frac{1}{\sigma} a \sigma \phi_2 + a^2 \sigma (\frac{1}{18} \phi_1 + \frac{1}{56} \phi_3) + a^3 \sigma \frac{1}{54} \phi_2 \\
+ 0.071843 \sigma^2 \phi_1 + 0.030368 \sigma^2 a \Phi_2 + O(a^6 + \sigma^3). 
$$

(24)
The weak model (23) with just the three main noise processes has coefficients correct to about 1% when compared to this model for the full spectrum of noise.

5 Higher order analysis requires more convolutions

In the strong model (9) we only seek to resolve the quadratic noise terms in \( \sigma^2 \) and \( \sigma^2 a \). If we seek quadratic noise terms of higher order in the amplitude \( a \), such as terms in \( \sigma^2 a^2 \) and \( \sigma^2 a^3 \), then we would face more convolutions of the noise, such as \( \Phi \mathcal{H}_s \mathcal{H}_t \mathcal{H}_q \mathcal{H}_p \Phi \), for example. At higher orders the infinite sums over the noise modes would have been considerably more complicated. Such complication may be difficult to handle, but the techniques are routine; whereas here the techniques have to be extended to handle more convolutions of the noise processes.

To handle more noise convolutions and thus be more complete we have to extend the canonical system of noise interactions (12). For possibly \( n \) convolutions of noise processes, extend the system (12) to discuss

\[
\begin{align*}
\dot{y}_1 &= z_1 \Phi_j, \\
\dot{z}_1 &= -\beta_1 z_1 + \Phi_i, \\
\dot{y}_2 &= z_2 \Phi_j, \\
\dot{z}_2 &= -\beta_2 z_2 + z_1, \\
\vdots & \quad \vdots \\
\dot{y}_n &= z_n \Phi_j, \\
\dot{z}_n &= -\beta_n z_n + z_{n-1}.
\end{align*}
\]

(25)

Recall that the constants \( \beta_k \) appearing here are just the decay rates of various of the fundamental modes of the linearised SPDE. Thus the results of this section from analysing this canonical hierarchy of quadratic noise effects apply to general dynamical systems.

Consider the Fokker–Planck equation for the PDF \( P(\mathbf{y}, \mathbf{z}, t) \) of the canonical system (25), it is a straightforward extension of the Fokker–Planck equation (14), again recall that we adopt the Stratonovich interpretation of SDEs:

\[
\frac{\partial P}{\partial t} = \frac{\partial}{\partial z_1}(\beta_1 z_1 P) + \sum_{k=2}^{n} \frac{\partial}{\partial z_k}[(\beta_k z_k - z_{k-1})P] + \frac{1}{2} \frac{\partial^2 P}{\partial z_1^2} \\
+ \frac{1}{2} \sum_{k=1}^{n} \frac{\partial}{\partial y_k} \left( z_k \frac{\partial P}{\partial z_1} \right) + \frac{1}{2} \sum_{k,l=1}^{n} \frac{\partial}{\partial y_k} \left( z_k z_l \frac{\partial P}{\partial y_l} \right). \tag{26}
\]

Tony Roberts, March 29, 2022
Higher order analysis requires more convolutions

Using the same arguments as in Section 4, treating \( y_k \) derivatives as asymptotically small parameters, this Fokker-Planck equation has a centre manifold, that is exponentially quickly attractive, and may be constructed by making the residual of the Fokker-Planck equation (20) zero to some order. For a given number of convolutions \( n \), computer algebra \([35, \S 2]\) readily derives the terms in the centre manifold model (15–16). For example, it appears that the leading order Gaussian can be written in terms of a sum of squares as

\[
G_0 = \mathcal{A} \exp(-\sum_{k=1}^{n} \beta_k \zeta_k^2)
\]

where

\[
\zeta_1 = z_1, \\
\zeta_2 = z_1 - (\beta_1 + \beta_2)z_2, \\
\zeta_3 = z_1 - (\beta_1 + 2\beta_2 + \beta_3)z_2 + (\beta_1\beta_2 + (\beta_1 + \beta_2 + \beta_3)\beta_3)z_3, \\
\zeta_4 = z_1 - (\beta_1 + 2\beta_2 + 2\beta_3 + \beta_4)z_2 + (\beta_1\beta_2 + (\beta_1 + \beta_2 + \beta_3 + \beta_4)\beta_4)z_3 - (\beta_1\beta_2\beta_3 + (\beta_1\beta_2 + \beta_1\beta_3 + \beta_2\beta_3)\beta_4 + (\beta_1 + \beta_2 + \beta_3 + \beta_4)\beta_3)z_4.
\]

However, using this algorithm, determining terms in \( \nabla p \) and \( \nabla \nabla p \) requires more computer memory and time than I currently have available for anything more than the case of \( n = 3 \) general noise convolutions with \( \beta_k \) as variable parameters.\(^4\) For the accessible \( n = 3 \) case, we find the relatively slowly varying, quasi-conditional probability density \( p \) evolves according to the Fokker–Planck like PDE (17) but now the \( 3 \times 3 \) diffusion matrix has entries

\[
D_{11} = \frac{1}{4\beta_1}, \\
D_{12} = D_{21} = \frac{1}{4\beta_1(\beta_1 + \beta_2)}, \\
D_{22} = \frac{1}{4\beta_1\beta_2(\beta_1 + \beta_2)}, \\
D_{13} = D_{31} = \frac{1}{4\beta_1(\beta_1 + \beta_2)(\beta_1 + \beta_3)}, \\
D_{23} = D_{32} = \frac{\beta_1 + \beta_2 + \beta_3}{4\beta_1\beta_2(\beta_1 + \beta_2)(\beta_1 + \beta_3)(\beta_2 + \beta_3)}, \\
D_{33} = \frac{\beta_1 + \beta_2 + \beta_3}{4\beta_1\beta_2\beta_3(\beta_1 + \beta_2)(\beta_1 + \beta_3)(\beta_2 + \beta_3)}.
\]

(27)

The \( 2 \times 2 \) upper-left block is reassuringly identical to the earlier diffusion matrix (18).

\(^4\)For any specific convolution of noises, when the coefficients \( \beta_1 \) are all specified numbers, the computer algebra of \([35, \S 2]\) analyses the case of \( n = 4 \) convolutions within 20 seconds CPU on my current desktop computer.
Fortunately, the alternative derivation in Appendix A of the diffusion matrix $D$ is significantly more efficient. Computing the $4 \times 4$ diffusion matrix I find the expressions extremely complicated and apparently not worth recording. But the Cholesky factorisation is accessible.

Recall that to interpret the PDE (17) as a Fokker–Planck equation of some SDEs, we desire the Cholesky factorisation of the diffusion matrix. The Cholesky factorisation here is $D = \frac{1}{2} L L^T$ for lower triangular matrix $L$ with non-zero entries:

\[
\begin{align*}
L_{11} &= \frac{1}{\sqrt{2}\beta_1}, \\
L_{21} &= \frac{1}{\sqrt{2}\beta_1(\beta_1 + \beta_2)}, \\
L_{22} &= \frac{1}{\sqrt{2}\beta_2(\beta_1 + \beta_2)}, \\
L_{31} &= \frac{1}{\sqrt{2}\beta_1(\beta_1 + \beta_2)(\beta_1 + \beta_3)}, \\
L_{32} &= \frac{1}{\sqrt{2}\beta_2(\beta_1 + \beta_3)} \left[ \frac{1}{\beta_1 + \beta_2} + \frac{1}{\beta_2 + \beta_3} \right], \\
L_{33} &= \frac{1}{\sqrt{2}\beta_3(\beta_2 + \beta_3)(\beta_1 + \beta_3)}, \\
L_{41} &= \frac{1}{\sqrt{2}\beta_1(\beta_1 + \beta_2)(\beta_1 + \beta_3)(\beta_1 + \beta_4)}, \\
L_{42} &= \frac{1}{\sqrt{2}\beta_2(\beta_1 + \beta_3)} \left[ \frac{1}{(\beta_2 + \beta_3)(\beta_2 + \beta_4)} \\
&\quad + \frac{1}{(\beta_1 + \beta_4)(\beta_2 + \beta_4)} + \frac{1}{(\beta_1 + \beta_2)(\beta_1 + \beta_4)} \right], \\
L_{43} &= \frac{1}{\sqrt{2}\beta_3(\beta_2 + \beta_4)} \left[ \frac{1}{(\beta_1 + \beta_3)(\beta_2 + \beta_3)} \\
&\quad + \frac{1}{(\beta_1 + \beta_4)(\beta_3 + \beta_4)} + \frac{1}{(\beta_1 + \beta_3)(\beta_1 + \beta_4)} \right], \\
L_{44} &= \frac{1}{\sqrt{2}\beta_4(\beta_1 + \beta_4)(\beta_2 + \beta_4)(\beta_3 + \beta_4)}. \tag{28}
\end{align*}
\]

The upper-left entries are also reassuringly identical to the earlier $2 \times 2$ case (20). These formulae empower us to transform general quadratic non-

\[ \text{There are some intriguing hints of relatively simple patterns developing in the entries of } L. \text{ Maybe an even more direct derivation via a change in measure for the hierarchy (25) could be exploited to derive general formulae for more convolutions of noise.} \]

Tony Roberts, March 29, 2022
linear combinations of noise processes into effectively new and independent noise processes for the long time dynamics of quite general SDEs and SPDEs.

6 Conclusion

The crucial virtue of the weak models (1) and (24), as also recognised by Just et al. [18], is that we may accurately take large time steps as all the fast dynamics have been eliminated. The critical innovation here is we have demonstrated, via the particular example SPDE (1), how it is feasible to analyse the net effect of many independent subgrid stochastic effects. We see three important results: we can remove all memory integrals (convolutions) from the model; nonlinear effects quadratic in the noise processes effectively generate a mean drift; and nonlinear effects quadratic in the noise processes effectively generate abstract new noises. The general formulae in Section 5, together with the iterative construction of centre manifold models [28], empower us to model quite generic SPDEs.

My aim is to construct sound, discrete models of SPDEs. Here we have treated the whole domain as one element. The next step in the development of this approach to creating good discretisations of SPDEs is to divide the spatial domain into finite sized elements and then systematically analyse their subgrid processes together with the appropriate physical coupling between the elements, as we have instigated for deterministic PDEs [20, 31, 23, e.g.].

A Ito proves quadratic stochastic resonance

In this Appendix we resort to Ito interpretation of SDEs rather than the Stratonovich interpretation used throughout the body of this work. This Appendix considers some of the properties of noise quadratically interacting with itself that were established through Fokker–Planck equations in Sections 4 and 5. Here we provide alternate more direct proofs of some of these properties.

A.1 Noise interacting with itself over long times

This subsection analyses the simplest case of one noise quadratically interacting with itself, that is, $\phi_i = \phi_j$. Thus we explore the large time dynamics of
the first pair of Stratonovich sdes in \ref{12}. The equivalent Ito sdes, written in the more usual capital letters, is for some Wiener process $W$

$$dY = \frac{1}{2} dt + Z dW \quad \text{and} \quad dZ = -\beta Z dt + dW,$$

(29)

where all subscripts are omitted for simplicity, $Y = y_1$, $Z = z_1$ and $dW = \phi_j dt = \phi_i dt$.

Consider the dynamics over any time interval $[a, b]$, provided times $a$ and $b$ are large enough for initial transients to have decayed, and for simplicity use just $\int$ to denote $\int_a^b$ and just $\Delta$ to denote the difference $[t]_{t=a}^b$.

**Theorem 1** The process $Y$ has drift $\frac{1}{2}$ and variance growing linearly at a rate $1/(2\beta)$.

**Proof:** Integrate the $Y$ equation to $\Delta Y = \frac{1}{2} \Delta t + \int Z dW$ and take expectations:

$$E[\Delta Y] = \frac{1}{2} \Delta t + E\left[ \int Z dW \right] = \frac{1}{2} \Delta t,$$

by the martingale property of Ito integrals. Hence $Y$ has drift $\frac{1}{2}$.

Now consider

$$\text{Var}\left[ \Delta (Y - \frac{1}{2} t) \right] = \text{Var}\left[ \int Z dW \right]$$

$$= \int E[Z^2] dt \quad \text{by Ito isometry}$$

$$= \frac{\Delta t}{2\beta},$$

as $Z$ is a well known Ornstein–Uhlenbeck process. Hence the variance of $Y$ grows linearly at rate $1/(2\beta)$.

Rather than appeal to $Z$ being an Ornstein–Uhlenbeck process we could instead recognise $Z = \int_{-\infty}^t \exp\{-\beta(t-s)\} dW_s$, from the defining convolution; then

$$E[Z^2] = \text{Var}\left[ \int_{-\infty}^t \exp\{-\beta(t-s)\} dW_s \right]$$

which by the Ito isometry

$$= \int_{-\infty}^t E[\exp\{-\beta(t-s)\}^2] \ ds$$
A Ito proves quadratic stochastic resonance

\[ \int_{-\infty}^{t} \exp\left\{ -2\beta(t-s) \right\} ds = \frac{1}{2\beta}, \]

as before. The next subsection uses this route to find covariances with any number of convolutions.

Given that \( \Delta Y \) approaches as Gaussian over long time scales, as established in the Fokker–Planck analysis leading to (17–18) and shown in some numerical simulations by Chao & Roberts [9], the process \( Y \) may be thus modelled over long time scales by the SDE

\[ dY = \frac{1}{2} dt + \frac{1}{\sqrt{2\beta}} dW_1 \]

for some Wiener process \( W_1 \), as analogously derived in (19). But before this corollary is of any use, we need to establish that the Wiener process \( W_1 \) is effectively independent of the original Weiner process \( W \) when viewed over large time scales. The next theorem shows the correlation

\[ \mathbb{E} [\Delta W \cdot \Delta W_1] = 0. \]

**Theorem 2** For the processes \( Y \) and \( Z \) with Ito SDE (29), the correlation

\[ \mathbb{E} [\Delta W \cdot \Delta (Y - \frac{1}{2} t)] = 0, \]

and hence the increments \( \Delta W \) and \( \Delta (Y - \frac{1}{2} t) \) are independent.

**Proof:** Trivially \( \mathbb{E} [\Delta W \cdot \Delta t] = 0 \), so we need only consider \( \mathbb{E} [\Delta W \cdot \Delta Y] \).

Since \( (W - W_a)(Y - Y_a) = 0 \) at \( t = a \), it follows that

\[ \Delta W \cdot \Delta Y = \Delta [(W - W_a)(Y - Y_a)]. \]

Hence

\[ \mathbb{E} [\Delta W \cdot \Delta Y] = \mathbb{E} [\Delta [(W - W_a)(Y - Y_a)]] \]

\[ = \mathbb{E} \left[ \int [(W - W_a)(Y - Y_a)] \right] \]

which by Ito’s formula [3 p.62, e.g.]

\[ = \mathbb{E} \left[ \int Y - Y_a + Z(W - W_a) \, dW + \int Z + \frac{1}{2}(W - W_a) \, dt \right] \]

\[ = \mathbb{E} \left[ \int Y - Y_a + Z(W - W_a) \, dW \right] + \int \mathbb{E} [Z] + \frac{1}{2} \mathbb{E} [W - W_a] \, dt \]

\[ = 0, \]

by the martingale property of Ito integrals, by the fact that \( Z \) is an Ornstein–Uhlenbeck process and hence has zero expectation after any initial transients, and since Wiener increments have zero expectation. Consequently, the increments \( \Delta W \) and \( \Delta (Y - \frac{1}{2} t) \) are independent.

Tony Roberts, March 29, 2022
Two distinct and interacting noises  Now turn to the case of one noise interacting with another, that is, when $\phi_i \neq \phi_j$. Thus explore the large time dynamics of the first pair of Stratonovich SDEs in (12). Now the equivalent Ito SDEs for some independent Wiener processes $\mathcal{W}$ and $\hat{\mathcal{W}}$ are

$$
\text{d}Y = Z \, \text{d}W \quad \text{and} \quad \text{d}Z = -\beta Z \, \text{d}t + \text{d}\hat{\mathcal{W}},
$$

where $\text{d}W = \phi_j \, \text{d}t$ and $\text{d}\hat{\mathcal{W}} = \phi_i \, \text{d}t$.

As in the proof of Theorem 1, write the increments $\Delta Y = \int Z \, \text{d}W$ and then the martingale property and Ito isometry assure us that $E[\Delta Y] = 0$ and $\text{Var}[\Delta Y] = \Delta t/(2\beta)$. Similarly to the proof of Theorem 2, the increment $\Delta Y$ is uncorrelated with both $\Delta W$ and $\Delta \hat{\mathcal{W}}$—use Ito’s formula for products of processes that depend upon multiple noises [2, p.185, e.g.]:

$$
E[\Delta W \cdot \Delta Y] = E\left[\int (W - W_a) (Y - Y_a) \, \text{d}W\right] = E\left[\int (W - W_a) Z + (Y - Y_a) \, \text{d}W + \int Z \, \text{d}t\right] = E\left[\int (W - W_a) Z + (Y - Y_a) \, \text{d}W\right] + E[Z] \, \text{d}t = 0;
$$

$$
E[\Delta \hat{\mathcal{W}} \cdot \Delta Y] = E\left[\int (\hat{\mathcal{W}} - \hat{\mathcal{W}}_a) (Y - Y_a) \, \text{d}\hat{\mathcal{W}}\right] = E\left[\int (\hat{\mathcal{W}} - \hat{\mathcal{W}}_a) Z \, \text{d}W + \int Y - Y_a \, \text{d}\hat{\mathcal{W}}\right] = E\left[\int (W - W_a) Z \, \text{d}W\right] + E\left[\int Y - Y_a \, \text{d}\hat{\mathcal{W}}\right] = 0.
$$

Consequently, given that $\Delta Y$ approaches as Gaussian over long time scales, we may model the process $Y$ by the SDE $\text{d}Y = \frac{1}{\sqrt{2\beta}} \text{d}W_1$ for some effectively independent Wiener process $W_1$, as analogously derived in (19).

This subsection gives alternative and more direct proofs of some of the Fokker–Planck analysis of Section 4 on the most elementary canonical noise interactions. However, this subsection does not establish the key property that the increments $\Delta Y$ approaches a Gaussian for long times. Instead the Relevance Theorem of centre manifolds together with the structural stability of the Fokker–Planck equation (17) assure us of this key property.
A.2 Multiple convolutions of quadratic noises

To complete the analysis we here explore noise processes interacting with multiple convolutions of their past history. Thus consider the Ito version of the Stratonovich hierarchy (25):

\[
\begin{align*}
\text{d}Y_1 &= \frac{1}{2} \text{d}s \text{d}t + Z_1 \text{d}W, & \text{d}Z_1 &= -\beta_1 Z_1 \text{d}t + \text{d}\hat{W}, \\
\text{d}Y_2 &= \quad Z_2 \text{d}W, & \text{d}Z_2 &= ( -\beta_2 Z_2 + Z_1) \text{d}t, \\
& \quad \vdots \\
\text{d}Y_n &= \quad Z_n \text{d}W, & \text{d}Z_n &= ( -\beta_n Z_n + Z_{n-1}) \text{d}t,
\end{align*}
\]

(31)

where \( W = \hat{W} \) in the case of a noise interacting with itself, \( s = 1 \), otherwise they are independent, \( s = 0 \). Ito calculus provides alternate confirmation, to that derived in Section 4 of the effective large time dynamics of the processes \( Y_m \).

The processes \( Y_m \) have zero drift except for the case \( W = \hat{W} \) when instead process \( Y_1 \) has drift \( \frac{1}{2} \). We need the covariances of the fluctuations in these processes in order to establish that the correlations among the fluctuations is determined by the lower triangular matrix \( \mathbb{L} \) in (28). For conciseness define the fluctuation process \( \mathcal{Y}_m = Y_m - \delta_{m1} \frac{1}{2} \text{st} \).

Theorem 3 The expectation \( E [\Delta \mathcal{Y}_m] = 0 \) for all \( m \), and the covariances \( E [\Delta \mathcal{Y}_k \Delta \mathcal{Y}_m] \) are given by the corresponding elements in \( \Delta t \mathbb{L} \mathbb{L}^T \) for the lower triangular matrix \( \mathbb{L} \) in (28).

Proof: Firstly, immediately from definition of \( \mathcal{Y}_m \) and the Ito sdes (31), 
\[
\text{d}\mathcal{Y}_m = Z_m \text{d}W.
\]
Recall that an unadorned \( \int \) denotes \( \int_a^b \) and \( \Delta \) denotes the difference \( [ \; ]_{t=a}^{t=b} \). Thus \( \Delta \mathcal{Y}_m = \int Z_m \text{d}W \), then by the Ito isometry 
\[
E [\Delta \mathcal{Y}_m] = 0.
\]

Secondly, consider the covariances

\[
E [\Delta \mathcal{Y}_k \Delta \mathcal{Y}_m] = E \left[ \int Z_k \text{d}W \int Z_m \text{d}W \right] = \int E [Z_k Z_m] \text{d}t,
\]

(32)

by an extension of the Ito isometry.

Find these covariances by observing, and this is actually the definition from convolutions of the right-hand column in the hierarchy (31),

\[
Z_1 = \int_{-\infty}^{t} e^{-\beta_1 (t-s)} \text{d}\hat{W}_s \quad \text{and} \quad Z_m = \int_{-\infty}^{t} e^{-\beta_m (t-s)} Z_{m-1}(s) \text{d}s.
\]
The first is an Ito integral. Turn the others into Ito integrals by defining
\[ h_1(t) = e^{-\beta_1 t} \] and
\[ h_m(t) = e^{-\beta_m t} \ast h_{m-1}(t) = \int_0^t e^{-\beta_m (t-s)} h_{m-1}(s) \, ds; \]
for example, when the decay rates \( \beta_m \) differ
\[ h_2(t) = \frac{e^{-\beta_2 t} - e^{-\beta_1 t}}{\beta_1 - \beta_2}, \]
\[ h_3(t) = \frac{e^{-\beta_1 t}}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)} + \frac{e^{-\beta_2 t}}{(\beta_2 - \beta_3)(\beta_2 - \beta_1)} + \frac{e^{-\beta_3 t}}{(\beta_3 - \beta_1)(\beta_3 - \beta_2)}. \]
Then inductively
\[
Z_m = \int_{-\infty}^t e^{-\beta_m (t-\tau)} \int_{-\infty}^\tau h_{m-1}(\tau - s) \, d\hat{W}_s \, d\tau
\]
\[
= \int_{-\infty}^t \int_s^t e^{-\beta_m (t-\tau)} h_{m-1}(\tau - s) \, d\tau \, d\hat{W}_s
\]
\[
= \int_{-\infty}^t \int_0^{t-s} e^{-\beta_m (t-s-\tau)} h_{m-1}(\tau) \, d\tau \, d\hat{W}_s
\]
\[
= \int_{-\infty}^t h_m(t-s) \, d\hat{W}_s.
\]
Consequently, by an extension of the Ito isometry
\[
E[Z_m Z_k] = E\left[ \int_{-\infty}^t h_m(t-s) \, d\hat{W}_s \int_{-\infty}^t h_k(t-s) \, d\hat{W}_s \right]
\]
\[
= \int_{-\infty}^t E[h_m(t-s) h_k(t-s)] \, ds
\]
\[
= \int_0^\infty h_m(t) h_k(t) \, dt. \tag{34}
\]
Computer algebra \cite{35} §3 readily computes the convolutions and integrals in equations \( \text{(33)} \) and \( \text{(34)} \). The resultant covariances \( E[Z_k Z_m] \) are correctly twice the corresponding elements in the diffusion matrices \( \mathbb{D} \) in \( \text{(18)} \) and \( \text{(27)} \).

The computer algebra \cite{35} §3 easily computes higher order covariance matrices. But the expressions for order \( m \geq 4 \) are too hideous to record in detail here. However, \( \text{(28)} \) records the expressions computed for the fourth order Cholesky factorisation.

The factorisation \( \text{(28)} \) is needed to weakly model convolutions of noise by effectively new and independent noises as discussed in Section \( \text{4} \). But again, we need to be sure that these effectively new noise processes are independent of the original processes \( \mathcal{W} \) and \( \hat{\mathcal{W}} \).
Theorem 4  For the processes $Y_m$ and $Z_m$ with Ito SDE (31), the correlation
$E[\Delta W \cdot \Delta Y_m] = E[\Delta \hat{W} \cdot \Delta Y_m] = 0$, and hence the increments $\Delta W$, $\Delta \hat{W}$
and $\Delta Y_m$ are independent.

Proof:  As in Theorem 2, since $(W - W_a)(Y_m - Y_{ma}) = 0$ at $t = a$, it
follows that
$\Delta W \cdot \Delta Y_{ma} = \Delta((W - W_a)(Y_m - Y_{ma}))$.
Hence, using Ito’s formula for products of processes that depend upon mul-
tiple noises [2, p.185, e.g.]

$$E[\Delta W \cdot \Delta Y_m] = E[\Delta((W - W_a)(Y_m - Y_{ma}))]$$
$$= E\left[\int (W - W_a)(Y_m - Y_{ma}) \, dW + \int Z_m \, dt\right]$$
$$= E\left[\int Y_m - Y_{ma} d\hat{W} + \int (W - W_a) Z_m \, dW\right]$$
$$= 0,$$
by the martingale property of Ito integrals, including $Z_m = \int_{-\infty}^{t} h_m(t - s) \, d\hat{W}_s$ as deduced above. Consequently, the increments $\Delta W$ and $\Delta Y_m$ are independent.

Similarly,

$$E[\Delta \hat{W} \cdot \Delta Y_m] = E[\Delta((\hat{W} - \hat{W}_a)(Y_m - Y_{ma}))]$$
$$= E\left[\int (\hat{W} - \hat{W}_a)(Y_m - Y_{ma}) \, d\hat{W} + \int (\hat{W} - \hat{W}_a) Z_m \, dW\right]$$
$$= E\left[\int Y_m - Y_{ma} d\hat{W}\right] + E\left[\int (\hat{W} - \hat{W}_a) Z_m \, dW\right]$$
$$= 0,$$
by the martingale property of Ito integrals. Consequently, the increments $\Delta \hat{W}$ and $\Delta Y_m$ are independent.

Tony Roberts, March 29, 2022
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