The Wess-Zumino-Witten term in non-commutative two-dimensional fermion models

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Abstract

We study the effective action associated to the Dirac operator in two dimensional non-commutative Field Theory. Starting from the axial anomaly, we compute the determinant of the Dirac operator and we find that even in the $U(1)$ theory, a Wess-Zumino-Witten like term arises.

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1 Introduction

Interest in non-commutative spaces has been renewed after the discovery that non-commutative gauge theories naturally arise when D-branes with constant B fields are considered [1]-[2]. These works as well as that in [3] prompted many investigations both in field theory and in string theory (see references in [3]). Concerning gauge field theories, recent results on chiral and gauge anomalies [4]-[6] have shown that well-known results on “ordinary” models extend naturally and interestingly to the case in which non-commutative spaces are considered. In this work we consider a problem which can be seen as closely related to that of anomalies, namely the evaluation of the two-dimensional fermion determinant in non-commutative space-time. This problem is of interest not only for the analysis of two-dimensional QED and QCD in non-commutative space, but also in connection with abelian and non-Abelian bosonization since, as it is well-known, the knowledge of the fermion determinant leads more or less directly to the bosonization rules.

We start by evaluating in Section II the chiral anomaly in two-dimensional non-commutative space-time in a way adapted to the calculation of fermion determinants through integration of the anomaly. This last is done in Section III where both the Abelian and \((U(N))\) non-Abelian fermion determinant is calculated exactly. In both cases we obtain for the determinant a Wess-Zumino-Witten term. Consequences of our results and possible extensions are discussed in section IV.

2 The chiral anomaly

Conventions

As usual, we define the \(*\)-product between a pair of functions \(\phi(x), \chi(x)\) as

\[
\phi * \chi(x) \equiv \exp \left( \frac{i}{2} \theta^\mu \partial_\mu \partial_{y^\nu} \right) \phi(x) \chi(y) |_{x=y}
\]

\[
= \phi(x) \chi(x) + \frac{i}{2} \theta_{\mu} \partial_{\mu} \phi \partial_{\nu} \chi(x) + O(\theta^2),
\]

(1)

and the (Moyal) bracket in the form

\[
\{\phi, \chi\}(x) \equiv \phi(x) * \chi(x) - \chi(x) * \phi(x),
\]

(2)
so that, when applied to (Euclidean) space-time coordinates $x^\mu, x^\nu$, one has

$$\{x^\mu, x^\nu\} = i\theta^{\mu\nu} \quad (3)$$

which is why one refers to non-commutative spaces. Here $\theta^{\mu\nu}$ is a real, antisymmetric constant tensor. Since we shall be interested in two dimensional space-time, one necessarily has $\theta^{\mu\nu} = \theta \varepsilon^{\mu\nu}$ with $\varepsilon^{\mu\nu}$ the completely antisymmetric tensor and $\theta$ a real constant. In the context of string theory, non-commutative spaces are believed to be relevant to the quantization of D-branes in background Neveu-Schwarz constant B-field $B_{\mu\nu}$ [1]-[3]. In this context $\theta^{\mu\nu}$ is related to the inverse of $B^{\mu\nu}$. Afterwards, this original interest was extended to the analysis of field theories in non-commutative space and then, as signaled in [3] it becomes relevant to know to what extent old problems and solutions in standard field theory fit in the new non-commutative framework.

A “non-commutative gauge theory” is defined just by using the $\ast$-product each time the gauge fields have to be multiplied. Then, even in the $U(1)$ Abelian case, the curvature $F_{\mu\nu}$ has a non-linear term (with the same origin as the usual commutator in non-Abelian gauge theories in ordinary space)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie (A_\mu \ast A_\nu - A_\nu \ast A_\mu)$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ie \{A_\mu, A_\nu\} \quad (4)$$

This field strength is gauge-covariant (not gauge-invariant, even in the Abelian case) under gauge transformations which should be represented by $U$ elements of the form

$$U(x) = \exp (i\lambda) \equiv 1 + i\lambda - \frac{1}{2} \lambda \ast \lambda + \ldots \quad (5)$$

The covariant derivative implementing infinitesimal gauge transformations takes the form

$$D_\mu [A] \lambda = \partial_\mu \lambda + i e (\lambda \ast A_\mu - A_\mu \ast \lambda) \quad (6)$$

so that an infinitesimal gauge transformation on $A_\mu$ reads as usual

$$\delta A_\mu = \frac{1}{e} D_\mu \lambda \quad (7)$$

Concerning finite gauge transformations, one has

$$A_\mu^U = \frac{i}{e} U(x) \ast \partial_\mu U^{-1}(x) + U(x) \ast A_\mu \ast U^{-1}(x) \quad (8)$$
Given a fermion field $\psi$, one can easily see that the combination
\[
\gamma^\mu D_\mu[A]\psi = \gamma^\mu \partial_\mu \psi - ie\gamma^\mu A_\mu \psi
\] (9)
transforms covariantly under gauge transformations (8),
\[
\gamma^\mu D_\mu[A^U]\psi^U = U \gamma^\mu D_\mu[A]\psi
\] (10)
with
\[
\psi^U = U(x) \star \psi
\] (11)
and
\[
U(x) \star U^{-1}(x) = U^{-1} \star U(x) = 1
\] (12)
A gauge invariant Dirac action can be defined in the form
\[
S_f = \int d^d x \bar{\psi}(x) \star i\gamma^\mu D_\mu[A]\psi(x)
\] (13)

**The Anomaly**

Chiral transformations will be written as
\[
\psi'(x) = U_5(x) \star \psi
\] (14)
with
\[
U_5(x) = \exp_i(\gamma_5 \alpha(x)) = 1 + \gamma_5 \alpha + \frac{1}{2} \alpha(x) \star \alpha(x) + \ldots
\] (15)

The chiral anomaly $A_d$ in $d$-dimensional space can be calculated from the formula
\[
\log J_d[\alpha] = -2A_d ,
\] (16)
\[
A_d = \text{Tr} \ \gamma_5 \delta \alpha(x)|_{\text{reg}}
\] (17)
here $J_d[\alpha]$ is the Fujikawa Jacobian associated with an infinitesimal chiral transformation $U = 1 + \gamma_5 \delta \alpha$ and $\text{Tr}$ includes a matrix and functional space trace.

Let us specialize to the two dimensional case. We shall use the heat-kernel regularization so that (17) will be understood as
\[
A_2 = \int d^2 x \ A_2(x) \star \delta \alpha(x) ,
\] (18)
$$A_2(x) = \lim_{M \to \infty} \text{Tr} \gamma_5 \exp_* \left( \frac{D^* D}{M^2} \right). \tag{19}$$

After some standard manipulations, (19) takes the form

$$A_2(x) = \frac{1}{4\pi} \text{tr} \gamma_5 D^* D = \frac{1}{4\pi} \text{tr} (\gamma_5 \gamma^\mu \gamma^\nu) D_\mu \ast D_\nu. \tag{20}$$

Here tr is just the matrix trace. Using $\text{tr}(\gamma_5 \gamma^\mu \gamma^\nu) = 2i \varepsilon^{\mu\nu}$, eq. (20) can be written as

$$A_2(x) = \frac{e}{2\pi} \varepsilon^{\mu\nu} (\partial_\mu A_\nu - ie A_\mu \ast A_\nu) = \frac{e}{4\pi} \varepsilon^{\mu\nu} F_{\mu\nu}. \tag{21}$$

This result coincides with that first obtained in [4].

3 The two-dimensional fermion determinant

Let us write the gauge field in the two-dimensional case in the form

$$A = \frac{1}{e} (i \theta \exp_* (\gamma_5 \phi + i\eta)) \ast \exp_* (\gamma_5 \phi - i\eta) \tag{22}$$

Note that in the $\theta_{\mu\nu} \to 0$ limit, eq. (22) reduces to the usual decomposition of a two-dimensional gauge field in the form

$$eA_\mu = \varepsilon_{\mu\nu} \partial_\nu \phi + \partial_\mu \eta \tag{23}$$

which allows to decouple fermions from the gauge-field and then obtain the fermion determinant as the Jacobian associated to this decoupling [8]. Now, the form (22) was precisely proposed in [9] to achieve the decoupling in the case of non-Abelian gauge field backgrounds, this leading to the calculation of the $QCD_2$ fermion determinant in a closed form. Afterwards [10], it was shown that writing a two dimensional gauge field as in eq. (22) (without the $\ast$-product but in the $U(N)$ case) does correspond to the choice of a gauge condition. Eq. (22) is then the extension of this approach for a case in which non-commutativity arises from the use of the $\ast$-product.

At the classical level, the change of fermionic variables

$$\psi = \exp_* (\gamma_5 \phi + i\eta) \ast \chi$$
$$\bar{\psi} = \bar{\chi} \ast \exp_* (\gamma_5 \phi - i\eta) \tag{24}$$
completely decouples the gauge field, written as in (22), leading to an action of free massless fermions,

\[ S_f = \int d^2x \, \bar{\chi} \star i\partial \chi \]  

(25)

Of course, this is not the whole story: at the quantum level there is a Fujikawa Jacobian \( J \) associated to change (24). In order to compute this Jacobian, we follow the method introduced in [8]-[9]. Consider then the change of variables

\[ \psi = U_t \star \chi_t , \]
\[ \bar{\psi} = \bar{\chi}_t \star U_t^\dagger \]  

(26)

where

\[ U_t = \exp_\star (t (\gamma_5 \phi + i\eta)) \]  

(27)

and \( t \) is a real parameter, \( 0 \leq t \leq 1 \). Given the fermion determinant defined as

\[ \det(\partial - ie A) = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp \left(-S_f[\bar{\psi}, \psi] \right) \]  

(28)

we proceed to the change of variables (26) which leads to

\[ \det(\partial - ie A) = J[\phi, \eta; t] \int \mathcal{D}\bar{\chi}_t\mathcal{D}\chi_t \exp \left(-S_f[\bar{\chi}_t, \chi_t] \right) \]

(29)

where \( J[\phi, \eta; t] \) stands for the Jacobian

\[ \mathcal{D}\bar{\psi}\mathcal{D}\psi = J[\phi, \eta; t] \mathcal{D}\bar{\chi}_t\mathcal{D}\chi_t \]  

(30)

and we have defined

\[ D_t = U_t^\dagger \star (\partial - ie A\star) \, U_t \]  

(31)

Now, since the l.h.s. in (29) does not depend on \( t \) we get, after differentiation,

\[ \frac{d}{dt} \log \det D_t = - \frac{d}{dt} \log J[\phi, \eta; t] \]  

(32)

or, after integrating on \( t \) and using that \( D_0 = \partial - ie A \) and \( D_1 = \partial \)

\[ \det(\partial - ie A) = \det \partial \exp \left(-2 \int_0^1 dt A_2(t) \right) \]  

(33)
where we have used
\[ A_2(t) = \frac{d}{dt} \log J[\phi, \eta; t] \tag{34} \]

Now, it is trivial to identify \( A_2(t) \) with the two-dimensional chiral anomaly as defined in eq.(17), just by writing \( \delta \alpha = \phi dt \),
\[ A_2(t) = \delta \alpha \tag{35} \]

In order to have a gauge-invariant regularization ensuring that the \( \eta \) part of the transformation does not generate a Jacobian, we adopt, in agreement with (18) and (19),
\[ A_2(t) = \lim_{M \to \infty} \text{Tr} \left( \gamma_5 \exp \left( \frac{\not{D}_t \not{D}_t}{M^2} \right) \phi \right) \tag{36} \]

so that finally one has
\[ A_2(t) = \frac{e}{2\pi} \int d^2 x \varepsilon^{\mu \nu} \left( \partial_\mu A_\nu^t - ie A_\mu^t * A_\nu^t \right) * \phi = \frac{e}{4\pi} \int d^2 x \varepsilon^{\mu \nu} F_{\mu \nu}^t * \phi \tag{37} \]

where we have introduced
\[ \gamma_\mu A_\mu^t = - \frac{1}{e} (i \not{\partial} U_t) * U_t^{-1} \tag{38} \]

and analogously for \( F_{\mu \nu}^t \). In summary, we can write for the \( U(1) \) fermion determinant
\[ \det(\not{D} - ie A) = \exp \left( - \frac{e}{2\pi} \int d^2 x \int_0^1 \text{d} t \varepsilon^{\mu \nu} F_{\mu \nu}^t * \phi \right) \det \not{D} \tag{39} \]

It will be convenient to use the relation
\[ \gamma^\mu \gamma_5 = -ie \varepsilon^{\mu \nu} \gamma_\nu \tag{40} \]

to rewrite (39) in the form
\[ \det(\not{D} - ie A) = \exp \left( \frac{ie}{2\pi} \text{tr} \int d^2 x \int_0^1 \text{d} t \gamma_5 \phi * \left( \not{D} A^t - ie A^t * A^t \right) \right) \det \not{D} \tag{41} \]

Then, one can exploit the identity
\[ \text{tr} \int d^2 x \frac{1}{2} \frac{d}{dt} A^t * A^t = \frac{1}{e} \text{tr} \int d^2 x \gamma_5 i \not{\partial} \phi * A^t + 2 \text{tr} \int d^2 x \gamma_5 A^t * \phi * A^t + \frac{1}{e} \text{tr} \int d^2 x (\not{D} \phi) * A \tag{42} \]

\[ 7 \]
and find for \((41)\)

\[
\log \det (\slashed{\partial} - ie \slashed{A}) = -\frac{e^2}{4\pi} \text{tr} \int d^2x \ A^* A + \frac{e^2}{2\pi} \text{tr} \int dt \int d^2x \ \gamma_5 \phi^* A^* A
\]

\[
+ \frac{e}{2\pi} \int dt \int d^2x \ (\slashed{\partial} \eta)^* A + \log \det \slashed{\partial}
\]

(43)

This is the final form for the fermion determinant in a \(U(1)\) gauge theory. In order to write it in a more suggestive way connecting it with the Wess-Zumino-Witten term, let us consider the light-cone gauge \(A_+ = 0\), then one can see after some algebra that [11]

\[
\log \left( \frac{\det (\slashed{\partial} - ie \slashed{A})}{\det \slashed{\partial}} \right) = -\frac{1}{8\pi} \int d^2x \left( \partial_\mu g(x)^{-1} \right) \ast (\partial_\mu g(x))
\]

\[
+ \frac{i}{12\pi} \epsilon_{ijk} \int_B d^3y g(x,t)^{-1} \ast (\partial_i g(x,t)) \ast g(x,t)^{-1} \ast (\partial_j g(x,t)) g^{-1} \ast (\partial_k g(x,t))
\]

(44)

here we have written \(A_- = (i/e)g(x) \ast \partial_- g^{-1}(x)\) with \(g(x) = \exp(2\phi(x))\), \(g(x,t) = \exp(2\phi(x)t)\) and \(d^3y = d^2x dt\) so that the integral in the second line of eq.(44) runs over the three dimensional manifold \(B\), which in compactified Euclidean space can be identified with a ball with boundary \(S^2\). Index \(i\) runs from 1 to 3. As in the ordinary commutative case, because the determinant was computed in Euclidean space, elements \(g\) should be considered as belonging to \(U(1)\) \(\mathbb{C}\) (the complexified \(U(1)\)) [11]-[12].

So, we have found for the two-dimensional non-commutative fermion determinant that, even for a \(U(1)\) gauge field background, a Wess-Zumino-Witten term arises due to non-commutativity of the \(*\)-product. Of course, in the \(\theta^{\mu\nu} \to 0\) limit in which the \(*\)-product becomes the ordinary one, the \(U(1)\) fermion determinant contribution to the gauge field effective action reduces to \((-1/2\pi) \int d^2x \phi \phi \partial^\mu \partial_\mu \phi\) which is nothing but the Schwinger determinant result expressed in a gauge-invariant way.

The method we have employed has the advantage that it can be trivially generalized to the case of a \(U(N)\) gauge group. One has just to take into account that in (22) one has

\[
\phi = \phi^a t^a , \quad \eta = \eta^a t^a
\]

(45)

with \(t^a\) the \(U(N)\) generators. Then, as originally shown in [9] for the commutative case, the fermion determinant can be seen to be given by

\[
\det(\slashed{\partial} - ie \slashed{A}) = \exp \left( -\frac{e}{4\pi} \text{tr} e \int d^2x \int_0^1 dt \epsilon^{\mu\nu} F_{\mu\nu}^t \ast \phi \right) \det \slashed{\partial}
\]

(46)
where $\text{tr}^c$ is a trace over the $U(N)$ algebra. Then, following the same steps leading to (44), one gets, in the $U(N)$ case

$$\log \left( \frac{\det(\partial - ieA)}{\det \phi} \right) = -\frac{1}{8\pi} \text{tr}^c \int d^2 x \left( \partial_\mu g^{-1} \right) \ast (\partial_\mu g)$$

$$+ \frac{i}{12\pi} \epsilon_{ijk} \text{tr}^c \int_B d^3 y g^{-1} \ast (\partial_i g) \ast g^{-1} \ast (\partial_j g) g^{-1} \ast (\partial_k g)$$

(47)

where again, in the light-cone gauge we have written

$$A_- = -\frac{i}{e} g \ast \partial_- g^{-1}, \quad A_+ = 0$$

(48)

$$g = \exp_s(2\phi^a t^a)$$

(49)

Eq. (47) is the generalization of the expression given in [13] for the two-dimensional non-Abelian fermion determinant to the case of non-commutative space-time.

4 Conclusion

We studied in this article the effective action of the gauge degrees of freedom in a two-dimensional non-commutative Field Theory of fermions coupled to a gauge field. Using Fujikawa’s approach, we computed the chiral anomaly and, from it, the fermionic determinant of the non-commutative Dirac operator.

As it was to be expected, the result for the fermion determinant corresponds to the ∗-deformation of the standard result. Now, the fact that a Moyal bracket enters in the field strength curvature even in the Abelian case, has important consequences, some of which have already been signaled in [4]-[6] where chiral and gauge anomalies in non-commutative spaces have been analyzed.

In our framework, where the anomaly was integrated in order to obtain the fermion determinant, this reflects in the fact that a Wess-Zumino-Witten like term arises both in the Abelian and in the non-Abelian cases (eqs.(44) and (47) respectively). This should have, necessarily, implications in relevant aspects of two-dimensional theories since, as it is well-known, bosonization is closely related to the form of the fermion determinant [13]. Indeed, the bosonization rules for fermion currents as well as the resulting current algebra
can be easily derived by differentiation of the Dirac operator determinant \( \det(\slashed{d} - i \beta) \) with respect to the source \( s_\mu \) (see [10] for a review). Now, as one learns from ordinary non-Abelian bosonization, where the Polyakov-Wiegmann identity plays a central rôle in the bosonization recipe, here one should have an analogous identity which will lead to non-trivial changes at the level of currents and, a fortiori, for the current algebra. In view of the relevance of these objects in connection with two-dimensional bosonic and fermionic models, it will be worthwhile to pursue the investigation initiated here in this direction.

Acknowledgements

F.A.S. would like to thank Claude Viallet for discussions on anomalies in non-commutative spaces and LPTHE, Paris U. VI-VII for hospitality. This work is supported in part by grants from CICBA, CONICET (PIP 4330/96), and ANPCYT (PICT 97/2285) Argentina.

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