AN ALGORITHM FOR COMPUTING THE CENTERED
HAUSDORFF MEASURES OF SELF-SIMILAR SETS

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ABSTRACT. We provide an algorithm for computing the centered
Hausdorff measures of self-similar sets satisfying the strong separa-
tion condition. We prove the convergence of the algorithm and
test its utility on some examples.

1. Introduction

We present an algorithm that takes as input a list \( \Psi = \{f_1, f_2, \ldots, f_m\} \)
of contracting similitudes in \( \mathbb{R}^n \) satisfying the strong separation condi-
tion (see section 2) and gives as output an estimate of the \( s \)-dimensional
centered Hausdorff measure, \( C_s(E) \), of the self-similar set \( E \) generated
by \( \Psi \). Here \( s \) is both the similarity and Hausdorff dimension of \( E \)
and it can be computed from the contracting factors of the similitudes
(see [2,1]). To our knowledge this is the first attempt at automatic
computation of the exact value of a metric measure (e.g. Hausdorff,
packing, spherical, centered, ...), a topic which has generated a consid-
erable quantity of research ([1], [4], [7], [8], [12], [17], [18], [19], [20], [21],
[22], [23], etc.)

The centered Hausdorff measure is a variant of the Hausdorff mea-
ure. These measures differ mainly in the natures of the coverings used
in their definitions. In the case of the centered Hausdorff measure we
consider only covers by closed balls \( B(x_i) \) centered at points \( x_i \) in the
given set.
The standard definition of \( C^s \) on subsets of \( \mathbb{R}^n \) consists of two steps. Given \( A \subset \mathbb{R}^n \), we first compute the premeasure \( C^s_0 \) as
\[
\begin{align*}
C^s_0 A &= \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \mathbb{N}} |B_i(x_i)|^s : A \subset \bigcup B_i \text{ and } |B_i| < \delta, x_i \in A \text{ for all } i \right\}.
\end{align*}
\]
Because the suppression of good candidates for the \( x_i \)'s may cause an increase of the infimum, this premeasure is not monotone, although it is \( \sigma \)-additive (see [15]). In order to avoid this difficulty we define
\[
\begin{align*}
C^s A &= \sup \{ C^s_0 B : B \subset A, B \text{ closed} \}.
\end{align*}
\]
The set function \( C^s \) so obtained is a metric measure. It turns out that the centered Hausdorff measure is bounded by constant multiples of the ordinary Hausdorff measure (see [14]). More precisely
\[
2^{-s} C^s(E) \leq H^s(E) \leq C^s(E)
\]
and so the centered Hausdorff dimension and the ordinary Hausdorff dimension coincide. In particular, for the self-similar set \( E \), we have that \( 0 < C^s(E) < \infty \). Thus \( C^s \) is a nice measure, but there remains the question of how to compute \( C^s A \) for some subset \( A \subset E \). A simplification comes from the following observation: for any Borel set \( A \subset \mathbb{R}^n \),
\[
\begin{align*}
C^s A &= C^s E \mu A,
\end{align*}
\]
where is \( \mu \) the so called natural or empirical probability measure on \( E \). Therefore, the problem of computing \( C^s \) on \( E \) reduces to computing \( C^s E \). Observe that the \( \mu \)-measure of any open subset \( A \subset E \) with boundary \( \partial A \) having null \( \mu \)-measure (and in particular of any open ball) can be easily obtained with arbitrary accuracy and, hence, the \( \mu \)-measures of compact subsets of \( E \) with \( \mu \)-null boundaries can also be computed through their complements. This gives a vast class \( \mathcal{C} \) of Borel sets with computable \( \mu \)-measure. In turn, the \( C^s \)-measure of any \( C^s \)-measurable set can be approximated with arbitrary accuracy by the \( C^s \)-measures of closed sets (see [6] Theorem 1.6 b).

How, then, to compute \( C^s E \)? Given the definitions, the task seems out of reach. The first obstruction comes from the need of the second step [12] in the definition of \( C^s \). However, in [10] it is proved that, for any subset \( A \) of a self-similar set \( E \) as above the measure and the premeasure coincide:

1.3. Theorem (Theorem 3 in [10]). Let \( A \) be either a closed or an open subset of a self-similar set \( E \) satisfying the open set condition. Then \( C^s_0 A = C^s A \).
With this result available, $C^s$ seems easier to compute than the Hausdorff or spherical Hausdorff measure. Namely, the differences between these three measures are that, for the Hausdorff measure, one optimizes among coverings by arbitrary convex sets; for the spherical Hausdorff measure one uses arbitrary balls; and for $C^s$ one uses coverings by balls with centers in $E$. For the Hausdorff and the spherical Hausdorff measures, the classes of available coverings are larger, and hence it is more difficult to find optimal coverings.

The second step which permits the computation of $C^s$ was also given in [10]. There it is proved that computing $C^s$ is equivalent to finding a centered ball with optimal inverse density:

1.4. Theorem (Theorem 5 in [10]). Suppose the invariant set $E$ of the system $\Psi$ satisfies the open set condition, with $\dim_H E = s$ and $|E| = R$, and let $\mu$ be the normalized Hausdorff measure on $E$. Then

$$C^s E = \inf \left\{ \frac{(2d)^s}{\mu(B(x,d))} : x \in E, d > 0 \right\} =: D_C^{-1}.$$  

Moreover, if $\Psi$ satisfies the SSC then

$$C^s E = \min \left\{ \frac{(2d)^s}{\mu(B(x,d))} : x \in E, \ c \leq d \leq R \right\},$$

where $c := \min_{i,j \in M, i \neq j} (f_i E, f_j E)$ and $R := |E|$.

From now on, $B(x,d)$ denotes the closed radius $d$ ball centered at $x \in \mathbb{R}^n$.

The statement (1.6) is crucial in our approach since it gives that if $B(x,d)$ is a ball of maximal inverse density, then $C^s(E) = \frac{(2d)^s}{\mu(B(x,d))}$. Therefore, to obtain the value of $C^s(E)$ we need only to find an optimal ball and compute its density. This is precisely how the algorithm proceeds. It searches for balls that maximize the density function $h_s(x,d) = \frac{(2d)^s}{\mu(B(x,d))}$. Actually, by the self-similar tiling principle (see [13]), we know that finding an optimal ball is equivalent to finding an optimal covering. This is so because we can get an optimal covering tiling the set $E$ with balls of optimal density.

$E$ may be tiled, without loss of $\mu$-measure, by tiles similar to a given tile $B$. By similar we mean that the tile is an image of $B$ under a composition of similitudes in $\Psi$. The only condition to be imposed on $B$ is that it be closed and have $\mu B \neq 0$.

Our algorithm computes $C^s E$ and provides an (approximate) ball of maximal density, together with an optimal covering of $E$ by balls centered at $E$. 
Now we describe the main steps of the algorithm. This is done rigorously in section 3. Recall that the goal is to find the maximal value of 
\[ h_s(x,d) = \frac{(2d)^s}{\mu(B(x,d))} \] for \( x \in E \) and \( d \in [c,R] \) (see (1.6)). At step \( k \), the set \( E \) is replaced by a finite set of points \( \{A_k\} \) such that \( \bigcup A_k = E \) and the measure \( \mu \) is replaced by a discrete probability measure \( \mu_k \) supported on \( A_k \) and converging weakly to \( \mu \) (see (3.4), (3.9) and Lemma 4.1(iv)). The objective now is to find the maximum of the discrete density function
\[ h_k(x,d(y)) = \frac{(2d(x,y))^s}{\mu_k(B(x,d(x,y)))} \] with \( x, y \in A_k \).

Here \( d(\cdot,\cdot) \) stands for the Euclidean distance. For each \( x \in A_k \) the algorithm searches for the maximal value of \( h_k(x,d(x,y)) \) for \( y \in A_k \). Once this has been found for every \( x \in A_k \), the algorithm finds the maximum of these values with respect to \( x \). Thus we only need to compute exactly \( \mu_k(B(x,d(x,y))) \) for every \( y \in A_k \). To this end, the points \( y \in A_k \) are listed in order of increasing distance to \( x \), and thus the points preceding a given point \( y \) in the list always belong to the ball \( B(x,d(x,y)) \).

It is not hard to see that the exact value of \( \mu_k(B(x,d(x,y))) \) is obtained from the place of \( y \) in this list (see (3.5) for the homogeneous case and (3.11) for the general case). It remains to show that in the limit \( k \to \infty \) this process converges to \( C^s(E) \). This is done in section 4.

The convergence is shown in two steps. First, by means of the Markov operator \( M \) associated to the set \( E \) (see section 2 for notation and definitions), the measures \( \mu_k \) are shown to converge weakly to the invariant measure \( \mu \). The basic properties of these measures yield a sequence of pairs of points \( (x_k,y_k) \in A_k \times A_k \) such that \( h_k(x_k,d(x_k,y_k)) \to C^s(E) \). However, there is no reason why these \( (x_k,y_k) \) should optimize \( h_k \). Nonetheless, we are able to show that this holds asymptotically, which is enough for our purposes. An interesting technical point is that in the proofs an essential role is played by a result of Mattila [11] implying that \( \mu(\partial B(x,d)) = 0 \).

In section 5 we apply the algorithm to treat several sets whose centered Hausdorff measures were available in the literature. It is remarkable that in all these cases the optimal value (and also optimal ball and covering) is attained at an early iteration. The algorithm also yields conjectural values (which, in many cases, are upper bounds) for sets whose measure is unknown. Research is in progress to explore the rate of convergence and show that the method yields the precise values of \( C^s(E) \). Preliminary results seem to indicate that, for self-similar sets
with less than four similarities, four decimal digits of accuracy are attainable by personal computers without any serious effort to optimize the code’s design.

2. Preliminaries

Let Ψ = \{f_1, f_2, ..., f_m\} be a list of contracting similitudes on \( \mathbb{R}^n \), \(|f_i(x) - f_i(y)| = r_i|x - y|\) where \(0 < r_i < 1\). The unique non-empty compact set satisfying

\[ E = S\Psi(E), \]

where \( S\Psi(X) = \bigcup_{i \in M} f_i(X) \), is called the self-similar set associated to \( \Psi \). Sometimes we shall refer to \( E \) as the attractor or invariant set of the iterated function system (IFS) \( \Psi = \{f_1, f_2, ..., f_m\} \).

We shall use the following notation. Let \( M := \{1, ..., m\} \) and

\[ M^k = \{i_k = (i_1, ..., i_k) : i_j \in M \quad \forall j = 1, ..., m\}. \]

For \( i_k = i_1i_2..i_k \in M^k \), we write

\[ f_{i_k} = f_{i_1} \circ f_{i_2} \circ ... \circ f_{i_k}, \]
\[ r_{i_k} = r_{i_1}r_{i_2}...r_{i_k}, \]

and for \( A \subset \mathbb{R}^n \), we write

\[ A_{i_k} = f_{i_k}(A). \]

We shall refer to the sets \( E_{i_k} = f_{i_k}(E) \) as the cylinder sets of generation \( k \).

Throughout the paper we shall assume that the system \( \Psi \) satisfies the strong separation condition (SSC). That is, \( f_i(E) \cap f_j(E) = \emptyset \) for all \( i \neq j \), where \( i, j \in \{1, ..., m\} \). The Hausdorff dimension of \( E \), \( \dim_H E \), is given by the unique real number \( s \) such that

\[ (2.1) \quad \sum_{i=1}^{m} r_i^s = 1. \]

Moreover, the Hausdorff measure of \( E \) is finite and positive.

We shall denote by \( \mu \) the natural probability measure, or normalized Hausdorff measure, defined on the ring of cylinder sets by

\[ \mu(E_i) = r_i^s, \]

and then extended to Borel subsets of \( E \). The measure \( \mu \) is regular in the sense that there are positive numbers \( a \) and \( b \) such that

\[ (2.3) \quad ar^s \leq \mu(B(x, r)) \leq br^s \quad \text{for} \ x \in E \text{ and } 0 < r \leq 1. \]

Let \( \mathcal{P}(\mathbb{R}^n) \) be the space of probability measures on \( \mathbb{R}^n \). The well known fact that the measure \( \mu \) is the unique invariant measure for the
Markov operator (see, for example, [2]), plays an important role in our proofs. Let \( M : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n) \) be the Markov operator associated with the IFS \( \Psi \) with probabilities \( \{r_1^s, \ldots, r_m^s\} \),

\[
M(\nu) = r_1^s \nu \circ f_1^{-1} + \ldots + r_m^s \nu \circ f_m^{-1} \quad \forall \nu \in \mathcal{P}(\mathbb{R}^n).
\]

Then \( \mu \) is the unique probability measure satisfying

\[
M(\mu) = \mu.
\]

3. Description of the algorithm

In this section we introduce an algorithm to compute the centered Hausdorff measure for self-similar sets satisfying the SSC.

Given \( A \subset \mathbb{R}^n \), \( |A| \) stands for the diameter of \( A \).

In [10] it is proved that if \( E \) is a self-similar set satisfying the SSC, then

\[
C^s E = \min \left\{ \frac{(2d)^s}{\mu(B(x,d))} : x \in E \text{ and } c \leq d \leq R \right\}
\]

where \( c := \min_{i,j \in M, i \neq j} \text{dist}(f_i E, f_j E) \) and \( R := |E| \) (see Theorem 1.4).

Our method depends strongly on (3.1) as, to find the value of \( C^s E \), we construct an algorithm for minimizing the value of

\[
h_k(x, d) := \frac{(2d)^s}{\mu(B(x,d))}
\]

when \( x \in E \) and \( c \leq d \leq R \).

The idea is to construct a sequence \( \{A_k\} \) of countable sets and a sequence \( \{\mu_k\} \) of discrete measures supported on the \( A_k \) such that the \( \mu_k \) converge weakly to \( \mu \) and \( \bigcup_{k=1}^{\infty} A_k = E \), where \( A \) stands for the closure of \( A \). This allows us to construct another sequence \( \{\tilde{m}_k\} \) converging to \( C^s E \) by choosing on the \( k \)th step a pair \( (\tilde{x}_k, \tilde{y}_k) \in A_k \times A_k \) satisfying

\[
\tilde{m}_k := h_k(\tilde{x}_k, \text{dist}(\tilde{x}_k, \tilde{y}_k)) = \min \{h_k(x_k, \text{dist}(x_k, y_k)) : (x_k, y_k) \in A_k \times A_k\},
\]

where \( h_k(x, d) := \frac{(2d)^s}{\mu_k(B(x,d))} \).

We describe first the algorithm for self-similar sets where all the contraction ratios coincide, as this case illustrates better the central idea of the construction. After this we shall explain the modifications needed to treat the case of unequal contraction ratios. Observe that if \( r_i = r_j := r \) for all \( i \neq j \), the invariant measure \( \mu \) satisfies that

\[
\mu(E_{i_k}) = r^{k_i} = \frac{1}{m^k} \quad \forall i_k \in M^k.
\]

3.3. Algorithm. (Homogeneous case: \( r_i = r := r \) \forall i \neq j, i, j = 1, \ldots, m)
1. **Construction of \(A_k\).** Let \(A_1 = \{x_i \in \mathbb{R}^n : f_i(x_i) = x_i, \ i = 1, ..., m\}\) be the set of the fixed points of the similitudes in \(\Psi\). For \(k \in \mathbb{N}^+\), let \(A_k = S^k(\Psi)\) be the set of \(m^k\) points obtained by applying \(S^k(\Psi)(x) = \bigcup_{i \in M} f_i(x)\) to each of the \(m^{k-1}\) points of \(A_{k-1}\).

2. **Construction of \(\mu_k\).** For all \(k \in \mathbb{N}^+\), set

\[
(3.4) \quad \mu_k(x) = \frac{1}{m^k} \quad \forall x \in A_k.
\]

Thus,

\[
\mu_k = \frac{1}{m^k} \sum_{i=1}^{m^k} \delta_{x_i}
\]

is a probability measure with \(\text{spt}(\mu_k) = A_k = \{x_1, ..., x_{m^k}\}\).

3. **Construction of \(\tilde{\mu}_k\).**

3.1 Given \(x_l \in A_k\), compute the \(m^k\) distances \(\text{dist}(x_l, x)\) for every \(x \in A_k\).

3.2 Arrange the distances in increasing order.

3.3 Let \(d_1 \leq d_2 \leq ... \leq d_{m^k}\) be the list of ordered distances.

Since for each \(j = 1, ..., m^k\), \(B(x_l, d_j)\) contains \(j + t\) points of \(A_k\), where \(t\) is the cardinality of \(A_k \cap \partial B(x_l, d_j)\), i.e. \(d_l = d_{l+1} = ... = d_{l+t} \neq d_{l+t+1}\), there holds

\[
(3.5) \quad \mu_k(B(x_l, d_j)) = \frac{j + t}{m^k}.
\]

3.4 Let \(i_k(l) = (i_1(l), ..., i_k(l)) \in M^k\) be the unique sequence of length \(k\) such that \(x_l = f_{i_k(l)}(x)\) for some \(x \in A_1\). We shall use the notation \(x_{i_k(l)} := f_{i_k(l)}(x)\). Compute

\[
(3.6) \quad \frac{(2d_j)^s}{\mu_k(B(x_l, d_j))} = \frac{(2d_j)^s}{\frac{j+t}{m^k}} = \frac{m^k (2d_j)^s}{l+t}
\]

only for those \(m^{k-1}(m-1)\) distances \(d_j\) satisfying the following condition.

**Condition:** If

\[
d_j = \text{dist}(x_l, y)
\]

for some \(y \in A_k\) such that \(y = f_{i_k(z)}(z) = f_{j_1...j_k}(z)\) for some \(z \in A_1\), then

\[
(3.7) \quad i_1(l) \neq j_1.
\]

3.5 Find the minimum of the \(m^{k-1}(m-1)\) values computed in step 3.4.

3.6 Repeat steps 3.1-3.5 for each \(x_l \in A_k, \ l = 1, ..., m^k\).
3.7 Take the minimum of the $m^k$ values computed in step 3.6.

3.8. Algorithm (General case). The main difference between this case and the previous one is that the values of the measures $\mu_k$ are different. The structure of the algorithm is the same in both cases. Consequently, we shall list only the changes needed to find the measure when the contraction ratios are unequal.

1. In step 2, replace (3.4) with

$$\mu_k(x) = r_{i_k}^s \quad \forall x \in A_k,$$

where $i_k = (i_1, ..., i_k) \in M^k$ is the unique sequence of length $k$ such that $x = f(i_k(y))$ for some $y \in A_1$. Thus, $\mu_k$ is a probability measure with $\text{spt}(\mu_k) = A_k = \{x_1, ..., x_{m^k}\}$. If, for every $j = 1, ..., m^k$, we denote by $i_k(j) \in M^k$ the unique sequence of length $k$ such that $x_j = f_{i_k(j)}(y)$ for some $y \in A_1$, then we can write

$$\mu_k = \sum_{j=1}^{m^k} r_{i_k(j)}^s \delta_{x_j}.$$  

2. In step 3.3, replace (3.5) with

$$\mu_k(\mathcal{B}(x_l, d_j)) = \sum_{q=1}^{j+t} r_{i_k(q)}^s,$$

where $i_k(q) \in M^k$ is such that $d_q = d(x_l, x_{i_q})$ for all $q = 1, ..., j + t$.

3. In step 3.4, replace (3.6) with

$$\frac{(2d_j)^s}{\mu_k(\mathcal{B}(x_l, d_j))} = \frac{(2d_j)^s}{\sum_{q=1}^{j+t} r_{i_k(q)}^s}.$$  

3.13. Remark. Note that, in (3.6) and (3.12), we only compute the values of the inverse density function for those distances between points that belong to different basic cylinder sets. Namely, if $d_j = \text{dist}(x_l, x_p)$ for some $x_p \in A_k$, then $\mu_k(\mathcal{B}(x_l, d_j))$ is only computed by the algorithm when $x_l \in E_i$ and $x_p \in E_j$ with $i \neq j$.

3.14. Notation. In the rest of the paper we shall use the following notation. Let $A_k = S\Psi(A_{k-1})$ be the set of $m^k$ points obtained after $k$ iterations with $A_1 = \{x \in \mathbb{R}^n : f_i(x) = x, i = 1, ..., m\}$. We write

$$A := \bigcup_{k=1}^{\infty} A_k.$$
For $k \in \mathbb{N}^+$ and for each $x_l \in A_k$, let $D_k^l$ be the set of $m^{k-1}(m - 1)$ distances satisfying condition (3.7) in the construction of the algorithm, denote by

$$D_k := \bigcup_{l=1}^{m^k} D_k^l$$

the set of the $m^{2k-1}(m - 1)$ distances that appear in step 3.5, and write

$$D := \bigcup_{k=0}^{\infty} D_k.$$

Observe that $D$ only takes values in the interval $[c, R]$ (see (3.1)).

From now on, we shall assume, without lost of generality, that $R := |E| = 1$.

4. Convergence of the algorithm

4.1. Preliminary results. The next two lemmas collect some basic results needed in the proofs of our theorems. We shall prove only those statements that do not follow directly from the construction of the algorithm.

4.1. Lemma.

(i) For every $x \in E$ there exists a sequence $\{x_k\} \subset A_k$ such that $\lim_{k \to \infty} x_k = x$.

(ii) For every $k \in \mathbb{N}^+$,

$$A_k \subset A_{k+1}.$$

(iii) Let $k \in \mathbb{N}^+$, $x \in A_k$, and $i_k \in M^k$ be such that $f_{i_k}(y) = x$ for some $y \in A_1$. Then

$$\mu_k(x) = \mu_k(E_{i_k}) = \mu(E_{i_k}).$$

(iv) For every $k \in \mathbb{N}^+$,

$$\mu_{k+1} = M(\mu_k).$$

Moreover, $\{\mu_k\}_{k \in \mathbb{N}^+}$ converges weakly to $\mu$ and thus

$$\lim_{k \to \infty} \mu_k(A) = \mu(A)$$

for every set $A$ satisfying $\mu(\partial A) = 0$.

Proof. (i) and (ii) follow directly from the construction of the algorithm.

(iii) By (2.2), (3.2), (3.4), and (3.9), it suffices to notice that the SSC ensures the existence, for any cylinder set $E_{i_k}$ of generation $k$, of a unique point $x \in A_k \cap E_{i_k}$. The existence is clear by construction and the uniqueness holds because the cardinality of $A_k$ is equal to the number of cylinder sets $E_{i_k}$.
(iv) Let $x \in A_{k+1}$. By the SSC, there exists a unique $j \in M$ such that $f_j^{-1}(x) \in A_k$. Moreover, if $i_{k+1}(x) = (i_1(x), \ldots, i_{k+1}(x)) \in M^{k+1}$ is the unique sequence of length $k+1$ such that $x = f_{i_{k+1}(x)}(y)$ for some $y \in A_1$, then $i_1(x) = j$ and thus, by (2.4) and (3.10),
\[
M(\mu_k)(x) = r^s_j \mu_k \circ f_j^{-1}(x) = r^s_j r^s_{i_2(x)} \ldots r^s_{i_{k+1}(x)} = r^s_{i_{k+1}(x)} = \mu_{k+1}(x).
\]
This proves (4.3).

The weak convergence holds because the sequence $\{M^k(\nu)\}$ converges weakly to the invariant measure $\mu$ for every compactly supported probability measure $\nu$ (see [3]). In particular, $\{\mu_k\} = \{M^k(\mu_1)\}$ converges weakly to $\mu$. Finally, this is equivalent to (4.4) (see, for example, [5] Theorem 2.5.11).

4.5. Lemma. If $(x_0, d) \in E \times [c, 1]$ is such that
\[
(4.6) \quad C^s E = \frac{(2d)^s}{\mu(B(x_0, d))},
\]
then $\partial B(x_0, d) \cap E \neq \emptyset$.

Proof. Suppose that this is not the case. As both $\partial B(x_0, d)$ and $E$ are compact sets, there exists $0 < \epsilon < \text{dist}(E, \partial B(x_0, d))$ such that
\[
(B(x_0, d) \setminus B(x_0, d - \epsilon)) \cap E = \emptyset.
\]
Thus, $\mu(B(x_0, d)) = \mu(B(x_0, d - \epsilon))$, contradicting the minimality of $(x_0, d)$. \qed

Observe that for any pair $(x_0, d) \in E \times [c, 1]$ satisfying (4.6), Lemma 4.5 guarantees the existence of a point $y \in E$ such that
\[
(4.7) \quad C^s E = \frac{(2d)^s}{\mu(B(x_0, d))} = \frac{(2\text{dist}(x_0, y))^s}{\mu(B(x_0, \text{dist}(x_0, y))}.
\]

4.2. Convergence. In this section we show the convergence of the algorithm described in section 3. We do it in two steps. First, given $x_0$, $r$, and $y$ as in (4.7), we prove, in Theorem 4.9, the existence of a sequence $\{(x_k, d_k)\}_{k=1}^\infty$ in $A_k \times D_k$ such that
\[
m_k := h_k(x_k, d_k) := \frac{(2d_k)^s}{\mu_k(B(x_k, d_k))} \rightarrow \frac{(2d)^s}{\mu(B(x_0, d))} = h_s(x_0, d) = C^s(E).
\]
However, the algorithm’s sequence \( \{(\tilde{x}_k, \tilde{d}_k)\}_{k=1}^{\infty} \) is constructed by choosing on the \( k \)th step a pair \((\tilde{x}_k, \tilde{d}_k) \in A_k \times D_k \) such that
\[
\tilde{m}_k := h_k(\tilde{x}_k, \tilde{d}_k) := \frac{(2\tilde{d}_k)^s}{\mu_k(B(\tilde{x}_k, \tilde{d}_k))} = \min\{h_k(x_k, d_k) : (x_k, d_k) \in A_k \times D_k\}.
\]

Actually, for each \( k \in \mathbb{N}^+ \) there might be more than one pair \((\tilde{x}_k, \tilde{d}_k)\) satisfying (4.8). Thus, if we chose a sequence \( \{(\tilde{x}_k, \tilde{d}_k)\}_{k=1}^{\infty} \) from the set of pairs selected by the algorithm, it does not need to coincide with the sequence \( \{(x_k, d_k)\}_{k=1}^{\infty} \) given in Theorem 4.9. Therefore, one needs to show that the sequence of minimal values \( \tilde{m}_k \) converges to the minimum \( C^s(E) \). In computer experiments we have observed that in many cases the sequence \( h_k(\tilde{x}_k, \tilde{d}_k) \) is not monotone. Nonetheless, in Theorem 4.13 we prove the desired convergence.

From now on, we use the notation given in (4.8) and write \( r_{\max} := \max\{r_1, \ldots, r_m\} \). We refer to the sequence \( \tilde{m}_k \) as the sequence chosen by the algorithm.

4.9. Theorem. Let \( x_0, r, \) and \( y \) be as in (4.7). Then, there exist sequences \( \{x_k\} \) and \( \{y_k\} \) such that \( x_k, y_k \in A_k \) for all \( k \in \mathbb{N}^+ \), \( x_k \to x_0 \), \( y_k \to y \), \( d_k := \text{dist}(x_k, y_k) \to d = \text{dist}(x_0, y) \), and
\[
(4.10) \quad \frac{(2d_k)^s}{\mu_k(B(x_k, d_k))} \to \frac{(2d)^s}{\mu(B(x_0, d))}.
\]

Proof. The existence of the convergent sequences \( \{x_k\}, \{y_k\}, \) and \( \{d_k\} \) is given by Lemma 4.1 (i).

As \( d_k \to d \) and, by condition (3.7), \( \mu_k(B(x_k, d_k)) \) is bounded away from zero for all \( k \in \mathbb{N}^+ \), we need only to show that
\[
(4.11) \quad |\mu_k(B(x_k, d_k)) - \mu(B(x_0, d))| \to 0 \text{ as } k \to \infty.
\]

Let \( \delta > 0 \) and let \( \phi_\delta \) be a compactly supported continuous function on \( \mathbb{R}^n \) such that \( 0 \leq \phi_\delta \leq 1 \) everywhere, \( \phi_\delta \equiv 1 \) on \( B(x_0, d + \delta) \), and \( \phi_\delta \equiv 0 \) off of \( B(x_0, d + 2\delta) \). By the weak convergence of \( \mu_k \) to \( \mu \) (Lemma 4.1 (iv)), we have that
\[
\int \phi_\delta d\mu = \lim_{k \to \infty} \int \phi_\delta d\mu_k.
\]

Moreover, as \( \mu(\partial B(x_0, d)) = 0 \),
\[
|\mu(B(x_0, d)) - \int \phi_\delta d\mu|
\]
is small when $\delta$ is small enough. So (4.11) holds if
\[ |\mu_k(B(x_k, d_k)) - \int \phi d\mu_k| \]
goes to zero as $\delta \to 0$ and $k \to \infty$. For each $\delta > 0$ take $k_0 = k_0(\delta) \in \mathbb{N}^+$ such that for all $k \geq k_0$, $B(x_0, d - \delta) \subset B(x_k, d_k) \subset B(x_0, d + \delta)$, and $r_{\max}^k \leq \delta$. Hence
\begin{equation}
(4.12) \quad \int \phi d\mu_k \geq \mu_k(B(x_k, d_k)),
\end{equation}
and so
\[ |\mu_k(B(x_k, d_k)) - \int \phi d\mu_k| = \int \phi d\mu_k - \mu_k(B(x_k, d_k)) \leq \mu_k(B(x_0, d + 2\delta) \setminus B(x_k, d_k)) \leq \mu_k(R(2\delta)), \]
where $R(\delta) := B(x_0, d + \delta) \setminus B(x_0, d - \delta)$.

There remains only to show the convergence of $\mu_k(R(2\delta))$. Let \{x_1, ..., x_J\} be the set of points in $A_k \cap R(2\delta)$ and, for $j = 1, ..., J$, denote by $i_k(j) \in M^k$ the unique sequence of length $k$ satisfying
\[ f_{i_k(j)}(y) = x_j \quad \text{for some} \quad y \in A_1, \]
and by $E_{i_k(j)}$ the unique cylinder set of generation $k$ such that $x_j \in E_{i_k(j)}$. Then (3.10) and (4.2) imply
\[ \mu_k(R(2\delta)) = \sum_{j=1}^{J} \mu_k(x_j) = \sum_{j=1}^{J} \mu(E_{i_k(j)}) \leq \mu(\{E_{i_k} \subset E \text{ such that } E_{i_k} \cap R_{2\delta} \neq \emptyset \text{ and } i_k \in M^k\}) \leq \mu(R_{3\delta}). \]
The last inequality holds because $|E_{i_k}| \leq r_{\max}^k \leq \delta$ for all $i_k \in M^k$ with $k \geq k_0$. This completes the proof of (4.11) as $\mu(\partial B(x_0, d)) = 0$ (see [11]). \hfill \qed

4.13. Theorem. The sequence \{m_k\}_{k \in \mathbb{N}} chosen by the algorithm, given in (4.8), converges to $C^s(E)$. Moreover, for any $k \in \mathbb{N}$ such that $\mu_k(B(\bar{x}_k, \bar{d}_k)) = \mu(B(\bar{x}_k, \bar{d}_k))$, there holds
\[ C^s(E) \leq m_k. \]

Proof. The second statement in the theorem is immediate, as $\mu_k(B(\bar{x}_k, \bar{d}_k)) = \mu(B(\bar{x}_k, \bar{d}_k))$ and (1.6) together imply
\[ C^s(E) \leq h_s(\bar{x}_k, \bar{d}_k) = h_k(\bar{x}_k, \bar{d}_k) = m_k. \]
Thus, we need only to prove the convergence.

For \{m_k\}_{k \in \mathbb{N}^+} as in (4.8), let \{m_{k_i}\}_{i \in \mathbb{N}^+} be a convergent subsequence of \{m_k\}_{k \in \mathbb{N}^+} and $\tilde{m} := \lim_{i \to \infty} m_{k_i}$. Since the algorithm only takes
values in the set \( A \times (D \cap [c, 1]) \), the sequence \( \{ \tilde{m}_k \}_{k \in \mathbb{N}^+} \) is contained in a compact set and it is enough to show that \( \tilde{m} = C^s(E) \).

Let \( \lambda = \tilde{m} - C^s(E) \) and for \( \{ \tilde{m}_{k_i} \}_{i \in \mathbb{N}^+} \), let \( \{ m_{k_i} \}_{i \in \mathbb{N}^+} \) be the corresponding subsequence of the convergent sequence \( \{ m_k \}_{k \in \mathbb{N}^+} \). Then, given \( 0 < \epsilon < |\lambda|/2 \), there exists \( i_0 \in \mathbb{N}^+ \) such that for any \( i \geq i_0 \),

\[
\begin{align*}
|h_{k_i}(x_{k_i}, d_{k_i}) - C^s(E)| &\leq \epsilon/2, \\
|h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}) - \tilde{m}| &\leq \epsilon/2.
\end{align*}
\]

First we want to show that \( \tilde{m} \leq C^s(E) \). If this is not the case, then \( \lambda > \epsilon > 0 \). Together with (4.14) and (4.15) this implies

\[
 h_{k_i}(x_{k_i}, d_{k_i}) \leq C^s(E) + \epsilon/2 = \tilde{m} - \lambda + \epsilon/2 \leq h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}) - \lambda + \epsilon < h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}),
\]

contradicting the minimality of \( h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}) \). Therefore, \( \tilde{m} \leq C^s(E) \).

With the aim of showing the reverse inequality, suppose that there exists \( i_1 \in \mathbb{N}^+ \) such that for any \( i \geq i_1 \),

\[
(4.16) \quad h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}) \geq h_s(\tilde{x}_{k_i}, \tilde{d}_{k_i}) - \epsilon/2,
\]

and assume that \( \tilde{m} > C^s(E) \). Then \( \lambda < -\epsilon < 0 \) and thus, (4.14), (4.15), and (4.16) give

\[
C^s(E) = \tilde{m} - \lambda \geq h_{k_i}(\tilde{x}_{k_i}, \tilde{d}_{k_i}) - \epsilon/2 - \lambda \geq h_s(\tilde{x}_{k_i}, \tilde{d}_{k_i}) - \epsilon - \lambda > h_s(\tilde{x}_{k_i}, \tilde{d}_{k_i}),
\]

contradicting the minimality of \( h_s(x_0, d) = C^s(E) \). There remains only to prove (4.16). Notice that, as \( \tilde{d}_{k_i} \in [c, 1] \), (2.3) implies that it is enough to show that for any sequence \( \{(x_k, d_k)\}_{k=1}^\infty \) in \( A \times (D \cap [c, 1]) \) and \( \delta > 0 \), there exists \( k_0 \in \mathbb{N}^+ \) such that for any \( k \geq k_0 \),

\[
(4.17) \quad |\mu_k(B(x_k, d_k)) - \mu(B(x_k, d_k))| < \delta.
\]

For every \( k \in \mathbb{N}^+ \) satisfying that \( r_k \max < c \), let

\[
\begin{align*}
G_k &= \{ E_{ik} : i_k \in M^k \text{ and } E_{ik} \subset B(x_k, d_k) \}, \\
P_k &= \{ E_{ik} : i_k \in M^k, E_{ik} \cap B(x_k, d_k) \neq \emptyset \text{ and } E_{ik} \cap E \setminus B(x_k, d_k) \neq \emptyset \}, \\
R_k &= \{ E_{ik} \in P_k : A_k \cap E_{ik} \cap B(x_k, d_k) \neq \emptyset \}.
\end{align*}
\]

Then,

\[
\begin{align*}
R_k &\subset P_k, \\
E \cap B(x_k, d_k) &= G_k \cup (B(x_k, d_k) \cap P_k), \\
\mu_k(B(x_k, d_k)) &= \mu_k(G_k) + \mu_k(R_k), \quad \text{and} \\
\mu(B(x_k, d_k)) &= \mu(G_k) + \mu(B(x_k, d_k) \cap P_k).
\end{align*}
\]
Moreover, by (4.2), \( \mu(G_k) = \mu_k(G_k) \) and \( \mu(R_k) = \mu_k(R_k) \). All this together with the triangle inequality, gives

\[
| \mu_k(B(x_k,d_k)) - \mu(B(x_k,d_k)) | = | \mu_k(G_k) + \mu_k(R_k) - \mu(G_k) - \mu(B(x_k,d_k) \cap P_k) |
\]

\[
= | \mu_k(R_k) - \mu(B(x_k,d_k) \cap P_k) | \leq \mu(R_k) - \mu(B(x_k,d_k) \cap P_k) + \mu((B(x_k,d_k) \cap P_k) \setminus R_k) \leq \mu(R_k) + \mu((B(x_k,d_k) \cap P_k) \setminus R_k) \leq \mu(P_k) \leq \mu(B(x_k,d_k + r^k_{\max}) \setminus B(x_k,d_k - r^k_{\max})),
\]

where the last inequality holds because \( P_k \subseteq B(x_k,d_k + r^k_{\max}) \setminus B(x_k,d_k - r^k_{\max}) \).

Thus, we want to show that \( \mu(B(x_k,d_k + r^k_{\max}) \setminus B(x_k,d_k - r^k_{\max})) < \delta \) if \( k \) is big enough. This is true, since \( E \times [c,1] \) is a compact set and the continuity of \( \mu \), together with the fact that \( \mu(\partial B(x,d)) = 0 \) (see [11]), implies that \( \mu(B(x,d + r^k_{\max}) \setminus B(x,d - r^k_{\max})) < \delta \) for any \( (x,d) \in E \times [c,1] \), if \( k \) is big enough. This proves (4.17) and concludes the proof of the theorem.

4.18. Remark. Note that the second statement in Theorem 4.13 provides an upper bound for \( C^s(E) \). If on the \( k \)th step the algorithm selects a ball containing all the \( k \)th generation cylinder sets that it intersects, \( \tilde{m}_k \) provides us with an upper bound for \( C^s(E) \). In general, the existence of such a ball at step \( k \) is not guaranteed as it depends on the geometry of the set \( E \). However, \( \tilde{m}_1 \) will always provide an upper bound for \( C^s(E) \) as the optimal ball selected on the first step already contains the whole set. Observe that this value is also an upper bound for the spherical Hausdorff measure. Even when we do not have such a ball, we can obtain natural upper bounds by estimating the value of \( \mu(B(\tilde{x}_k,\tilde{d}_k)) \), as, for every \( k \in \mathbb{N} \) there holds

\[
C^s(E) \leq \frac{(2\tilde{d}_k)^s}{\mu(B(\tilde{x}_k,\tilde{d}_k))}
\]

(see section 5 for further discussion and examples).

5. Examples

The first part of this section is devoted to showing that the algorithm recovers easily examples from the literature in which the centered Hausdorff measure has been computed. We have checked how it performs for the Cantor type sets analyzed in [4], [21], [22], and [23]. Here we will illustrate the accuracy of the algorithm by testing it on three of these examples. We will also see that the algorithm suggests the correct minimizing balls for \( h_s(x,d) \). In the cases where \( C^s(E) \) remains unknown the algorithm still provides candidates for minimizing balls (and hence
for optimal coverings, by [9] and [10]). These potentially minimizing balls yields conjectural values for \( C^s(E) \) that, in the cases considered here, are rigorous upper bounds (see Theorem 4.13, Remark 4.18, Example 5.9, and Conjecture 5.14). In Remark 5.7 we point out that, in some cases, we can use the balls selected by the algorithm to obtain also the corresponding lower bounds, although the proofs we have so far are case specific and are not simple. In the last part of this section we present other examples not included in the literature together with some conjectures on the precise values of \( C^s(E) \) for these cases.

Recall that, by Theorem 1.4, our goal is to find the minimal value of
\[
h_s(x,r) := \frac{(2d)^s}{\mu(B(x,r))}
\]
when \( x \in E \) and \( c \leq r \leq R \). For this the algorithm selects, at the \( k \)th step, some of the pairs \( (\tilde{x}_k, \tilde{d}_k) \in A_k \times A_k \) satisfying
\[
\tilde{m}_k = h_k(\tilde{x}_k, \tilde{d}_k) = \frac{(2\tilde{d}_k)^s}{\mu_k(B(\tilde{x}_k, \tilde{d}_k))} = \min \{ h_k(x_k, d_k) : (x_k, d_k) \in A_k \times D_k \},
\]
where \( \tilde{d}_k = |\tilde{x}_k - \tilde{y}_k| \), \( A_0 = \{ x_i \in \mathbb{R}^n : f_i(x_i) = x_i, i = 1, \ldots, m \} \) is the set of fixed points for the similitudes in the IFS, and \( A_k = S\Psi(A_{k-1}) \) (see section 3). We shall refer to \( B(\tilde{x}_k, \tilde{d}_k) \) as the minimizing ball (or interval).

1. \( \lambda \)-Cantor sets in the real line

Let \( K(\lambda) \) be the attractor of the iterated function system
\[
\{ f_0(x) = \lambda x, f_1(x) = 1 - \lambda + \lambda x \}, \quad x \in [0, 1].
\]
In [21] it is proved that if \( 0 < \lambda \leq \frac{1}{3} \), then
\[
C^s(K(\lambda)) = 2^s(1 - \lambda)^s,
\]
where \( s = \frac{\log 2}{\log \lambda} \) is the Hausdorff dimension of \( K(\lambda) \).

Let \( \lambda = \frac{1}{3} \). Then \( K(\frac{1}{3}) \) is the middle-third Cantor set. The following table gives the results of the algorithm for \( K(\frac{1}{3}) \).
after 14 iterations.

| $k$  | $\tilde{m}_k$ | $(\tilde{x}_k, \tilde{y}_k)$ | $d_k$ | $B(\tilde{x}_k, d_k)$ |
|------|---------------|-----------------|------|------------------|
| $k = 0$ | 1.54856 | (0, 1) | 1 | $[-1, 1]$ |
|       |               | (1, 0) |      |                  |
| $k = 1$ | 1.03238 | $(\frac{2}{3}, \frac{1}{3})$ | 0.333333 | $[\frac{1}{3}, 1]$ |
|       |               | $(\frac{1}{3}, \frac{2}{3})$ |      | $[0, \frac{2}{3}]$ |
| $2 \leq k \leq 13$ | 1.19902 | $(\frac{2}{3}, 0)$ | 0.666667 | $[0, \frac{4}{3}]$ |
|       |               | $(\frac{1}{3}, 1)$ |      | $[-\frac{1}{3}, 1]$ |

Note that, from (5.1) with $\lambda = \frac{1}{3}$, it follows that

$$C^s(K(\frac{1}{3})) = \frac{\log 2}{\log 3} \approx 1.199023,$$

so the algorithm has found the exact value of the Hausdorff centered measure already at the third iteration! A simple computation shows that, for any $k \geq 2$,

$$h_k(\frac{1}{3}, \frac{2}{3}) = h_k(\frac{2}{3}, \frac{2}{3}) = (\frac{2}{3})^s = \frac{4^s}{2} = C^s(K(\frac{1}{3})).$$

Notice also that $h_k(\frac{1}{3}, \frac{2}{3})$ does not depend on $k$. In fact, one checks easily that

$$h_s(\frac{1}{3}, \frac{2}{3}) = h_s(\frac{2}{3}, \frac{2}{3}) = C^s(K(\frac{1}{3})).$$

Thus, $B(\frac{1}{3}, \frac{2}{3}) = [-\frac{1}{3}, 1]$ and $B(\frac{2}{3}, \frac{2}{3}) = [0, \frac{4}{3}]$ are the actual minimizing intervals for $h_s(x, r)$ and the algorithm has found them already on the second step.

2. $(\lambda_1, \lambda_2)$-symmetry Cantor sets in the real line

Let $K(\lambda_1, \lambda_2)$ be the symmetric Cantor set defined as the attractor of the iterated function system \(\{f_0(x) = \lambda_1 x, f_1(x) = \lambda_2 x + \frac{1-\lambda_2}{2}, f_2(x) = 1 - \lambda_1 + \lambda_1 x, x \in [0, 1]\}\). By [4] we know that if

(5.2) \[ \frac{1 - 2\lambda_1 - \lambda_2}{2} \geq \lambda, \]

then

(5.3) \[ C^s(K(\lambda_1, \lambda_2)) = 1, \]
where $\lambda \equiv \max\{\lambda_1, \lambda_2\}$.

Here is the output of the algorithm when $\lambda_1 = \frac{1}{8}$ and $\lambda_2 = \frac{1}{5}$:

| $k$ | $\bar{m}_k$ | $\bar{x}_k, \bar{y}_k$ | $d_k$ | $B(\bar{x}_k, d_k)$ |
|-----|-------------|----------------------|------|-------------------|
| 0   | 1           | $\left(\frac{1}{2}, 1\right)$ | 0.5  | [0, 1]            |
| 1   | 1           | $\left(\frac{1}{2}, 1\right)$ | 0.5  | [0, 1]            |
| 2   | 1           | $\left(\frac{1}{2}, 1\right)$ | 0.5  | [0, 1]            |
| 3   | 1           | $\left(\frac{1}{2}, 1\right)$ | 0.5  | [0, 1]            |

There is no need to iterate more as this stability is empirical evidence that the algorithm has found the optimal ball. Indeed, we have tested the algorithm with this kind of symmetric Cantor sets for different values of $\lambda_1$ and $\lambda_2$, and in all our trials, $[0, 1]$ appears as the minimizing interval for every $k \geq 0$. Thus, $[0, 1]$ is the natural candidate to be the minimizing ball for $h_s$ as well.

In fact, it is easy to use the exact value of $C_s(K(\lambda_1, \lambda_2))$ given in [4] (see (5.3)) to check that this is the case. To see this, notice that for any $\lambda_1, \lambda_2$ as in (5.2),

$$A_0 = \left\{0, \frac{1}{2}, 1\right\}$$

and $h_k\left(\frac{1}{2}, \frac{1}{2}\right) = h_s\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{(2\frac{1}{2})^s}{1} = 1 = C_s(K(\lambda_1, \lambda_2))$.

3. **Cantor type sets in the plane**

Let $S$ be the attractor of the iterated function system \( \{f_1, f_2, f_3, f_4\} \) where \( f_i(x) = \lambda_i x + b_i, \ i = 1, 2, 3, 4, \ x = (x_1, x_2) \in \mathbb{R}^2, \ 0 < \lambda_i \leq \frac{1}{2 + \sqrt{2}}, \ b_1 = (0, 0), \ b_2 = (1 - \lambda_2, 0), \ b_3 = (1 - \lambda_3, 1 - \

\[
C_s(S) = D^{-1}_{\max},
\]

\[
(5.4)
\]
where
\[
D_{\text{max}} = \max_{1 \leq t \leq 4} \left\{ \frac{1}{(2\sqrt{2}(1 - \lambda_t))^s} \right\}
\]
\[
d^k_{\text{min}} = \min \left\{ \frac{\sqrt{2}\lambda^s_k}{(\min\{1 - \lambda_i, 1 - \lambda_j\})^s}, \frac{\lambda^s_k + \lambda^s_i + \lambda^s_j}{(\sqrt{2}(1 - \lambda_p))^s} \right\}
\]
\[(k, i, j, p) \in \{(1, 2, 3, 4), (2, 1, 3, 4), (3, 2, 4, 1), (4, 1, 3, 2)\}\]
and
\[0 < \dim_H(S) = s < 1.\]

Next, we test the algorithm for the case \(\lambda_1 = \lambda_3 = \frac{1}{400}\) and \(\lambda_2 = \lambda_4 = \frac{1}{20}\). In this case we have that 
\[s = \log_{20}(\sqrt{3} + 1) \approx 0.335495\] and
\[C^s(S) = \left(\frac{19\sqrt{2}}{10}\right)^s \approx 1.3932\]
see [23].

The next table shows the results given by the algorithm after six iterations.

**Table 1.**

| \(k\) | \(\tilde{m}_k\) | \((\tilde{x}_k, \tilde{y}_k)\) | \(d_k\) | \(B(\tilde{x}_k, d_k)\) |
|------|----------------|----------------------------|--------|---------------------|
| \(k = 0\) | 1.4174 | \((0, 0), (1, 1)\) | \(\sqrt{2} = 1.41421\) | \(B((0, 0), \sqrt{2})\) |
| | | \((1, 0), (0, 1)\) | | \((1, 0), (0, 1)\) |
| \(1 \leq k \leq 5\) | 1.39321 | \((\frac{1}{20}, \frac{19}{20}), (1, 0)\) | \(\frac{19\sqrt{2}}{20} = 1.3435\) | \(B((\frac{1}{20}, \frac{19}{20}), \frac{19\sqrt{2}}{20})\) |
| | | \((\frac{19}{20}, \frac{1}{20}), (0, 1)\) | | \(B((\frac{19}{20}, \frac{1}{20}), \frac{19\sqrt{2}}{20})\) |

We note that the pattern of the previous examples is repeated here. There exists \(k_0\) such that, for all \(k \geq k_0\), the same balls are selected. The results in Table 1 show that for all \(k \geq 1\), the optimal balls for the discrete density function \(h_k\) are \(B((\frac{1}{20}, \frac{19}{20}), \frac{19\sqrt{2}}{20})\) and \(B((\frac{19}{20}, \frac{1}{20}), \frac{19\sqrt{2}}{20})\), so these should be considered as the natural candidates to minimize \(h_s\). As before, by
(5.4), a simple computation shows that these candidates are in fact the minimizing balls:

\[
\begin{align*}
    h_s((\frac{1}{20}, \frac{19}{20}), \frac{19\sqrt{2}}{20}) &= h_s((\frac{19}{20}, \frac{1}{20}), \frac{19\sqrt{2}}{20}) = \frac{(2\frac{19\sqrt{2}}{20})^s}{1} = \\
    h_k((\frac{1}{20}, \frac{19}{20}), \frac{19\sqrt{2}}{20}) &= h_k((\frac{19}{20}, \frac{1}{20}), \frac{19\sqrt{2}}{20}) = \left(\frac{19\sqrt{2}}{10}\right)^s = C^s(S).
\end{align*}
\]

The third equality in (5.6) holds because \(\mu_k(B((\frac{1}{20}, \frac{19}{20}), \frac{19\sqrt{2}}{20})) = \mu_k(B((\frac{19}{20}, \frac{1}{20}), \frac{19\sqrt{2}}{20})) = 1\) for any \(k \geq 1\).

5.7. Remark. As we have seen in the preceding examples the algorithm is also helpful for finding natural candidates for optimal balls. For instance, an output repeated for all iterations \(k \geq k_0\) is a natural candidate. Once the algorithm has suggested such an optimal ball, this can be used to provide new proofs of (5.1), (5.3), and (5.5). We explain the main idea for the first case. The other cases can be treated in a similar fashion.

\(\lambda\)-Cantor sets in the real line. We noticed that when we applied the algorithm to different \(K(\lambda)\) we obtained two values for \((\tilde{x}_k, \tilde{y}_k)\), namely \((\lambda, 1 - \lambda)\) and \((1 - \lambda, 1 - \lambda)\). Once we have a candidate for the optimal ball, we can recover (5.1) by showing that the minimum of \(h_s(x, r)\) is attained for the pairs \((\lambda, 1 - \lambda)\) and \((1 - \lambda, 1 - \lambda)\), i.e.

\[
C^s E = \min \left\{ \frac{(2r)^s}{\mu(B(x, r))} : x \in E \text{ and } c \leq r \leq R \right\} = \frac{(2(1 - \lambda))^s}{\mu(B(\lambda, 1 - \lambda))}
\]

\[
= \frac{(2(1 - \lambda))^s}{\mu(B(1 - \lambda, 1 - \lambda))} = 2^s(1 - \lambda)^s.
\]

The proof reduces to showing that

\[
\frac{(2r)^s}{\mu(B(\lambda, r))} \geq 2^s(1 - \lambda)^s \quad \text{for any } r \in [1 - 2\lambda, 1 - \lambda],
\]

since the other cases are trivial. The inequality (5.8) can be proved by hand using the geometry of the set.

For the \((\lambda_1, \lambda_2)\)-symmetry Cantor sets in the real line the same method can be used to show that the optimal interval is actually the one chosen by the algorithm, namely the interval \([0, 1]\). Finally, for the attractor \(S\) of the third example, the candidate for an optimal ball is

\[
B((\lambda_1, \lambda_1), \sqrt{2}(1 - \lambda_1)) \text{ whenever } \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4.
\]
We note that the results in [4], [21], [22], and [23] are obtained using the relation between the upper spherical density of the invariant measure and the centered Hausdorff measure. More precisely, in these references it is shown, that from the fact that, if $C_s(E) < \infty$, there holds

$$d_s^\ast(C^s_E, x) = \limsup_{r \to 0} \frac{C_s(B(x, r) \cap E)}{(2r)^s} = 1,$$

for almost all $x \in E$ (see [16]), there follows

$$C^s(E) = (d_s^\ast(\mu, x))^{-1},$$

for almost all $x \in E$, where $\mu = \frac{C^s_E}{C^s(E)}$ is the invariant measure. So they proved that

$$d_s^\ast(\mu, x) = \limsup_{r \to 0} \frac{1}{h_s(x, r)} = \frac{1}{2^s(1 - \lambda)^s},$$

for every point $x$ in a certain set of full $\mu$ measure. Notice that the method suggested by the algorithm is different since it is not necessary to pass to the limit.

Next, we present the results obtained by the algorithm for a family of Sierpinski gaskets whose centered Hausdorff measure is not known.

5.9. Example (Sierpinski gasket $S(r)$). Let $S(r)$ be the attractor of the system $\Psi = \{f_0, f_1, f_2\}$, where

\begin{align*}
(5.10) \quad f_0(x, y) &= r(x, y), \\
(5.11) \quad f_1(x, y) &= r(x, y) + (1 - r, 0), \quad \text{and} \\
(5.12) \quad f_2(x, y) &= r(x, y) + \left(\frac{1}{2}(1 - r), (1 - r)\sqrt{3}\right).
\end{align*}

After testing many examples, we have discovered that for $r < 0.25$ the algorithm quickly stabilizes at $[2(1 - r)(r^2 + r + 1)^{1/2}]^s$, where $s = \frac{-\log 3}{\log r}$ is the Hausdorff dimension of $S(r)$. We conjecture that, in these cases,

$$C^s(S(r)) = [2(1 - r)(r^2 + r + 1)^{1/2}]^s.$$ 

We illustrate this with Table 5.9, which shows the output of the algorithm for $r = 0.2$. 
Output of the algorithm for the case $r = 0.2$.

| $k$   | $\tilde{m}_k$ | $(\tilde{x}_k, \tilde{y}_k)$ | $d_k$ | $B(\tilde{x}_k, d_k)$               |
|-------|---------------|-------------------------------|-------|-------------------------------------|
| $k = 0$ | 1.60504       | $\left((\frac{5}{10}, \frac{\sqrt{3}}{2}), (1, 0)\right)$, $\left((\frac{5}{10}, \frac{\sqrt{3}}{2}), (0, 0)\right)$ | 1     | $B(\left((\frac{5}{10}, \frac{\sqrt{3}}{2}), 1\right)$, $B(\left((\frac{5}{10}, \frac{\sqrt{3}}{2}), 1\right)$ |
| $k = 1$ | 1.51231       | $\left((\frac{2}{10}, 0), (\frac{5}{10}, \frac{\sqrt{3}}{2})\right)$, $\left((\frac{8}{10}, 0), (\frac{5}{10}, \frac{\sqrt{3}}{2})\right)$ | 0.91615 | $B(\left((\frac{2}{10}, 0), \frac{\sqrt{31}}{5}\right)$, $B(\left((\frac{8}{10}, 0), \frac{\sqrt{31}}{5}\right)$ |
| $2 \leq k \leq 8$ | 1.48326       | $\left((\frac{9}{50}, \frac{\sqrt{3}}{50}), (\frac{5}{10}, \frac{\sqrt{3}}{2})\right)$ | 0.890842 | $B(\left((\frac{9}{50}, \frac{\sqrt{3}}{50}), \frac{4\sqrt{31}}{25}\right)$ |

The pattern observed in our trials for these Sierpinski gaskets is the following. For any $k \geq 2$ the optimal ball chosen by the algorithm is $B(\bar{x}, d(\bar{x}, \bar{x}_2)) = B(\bar{x}, (1 - r)\sqrt{r^2 + r + 1})$, where $\bar{x} = f_0(f_1(\bar{x}_2))$ and $\bar{x}_2 = f(\bar{x}_2) = f(x_2, y_2) \in \mathbb{R}^2$ is the fixed point of $f_2(x, y)$ (see Figure 1). Observe that $S(r) \subset B(\bar{x}, d(\bar{x}, \bar{x}_2))$. Hence, for any $k \in \mathbb{N}$,

$$\mu_k(B(\bar{x}, d(\bar{x}, \bar{x}_2))) = \mu(B(\bar{x}, d(\bar{x}, \bar{x}_2))) = 1.$$  

Then, by Theorem 4.18 we have a rigorous upper bound for $C^\alpha(S(r))$, namely

$$C^\alpha(S(r)) \leq [2(1 - r)(r^2 + r + 1)^{\frac{1}{2}}]^s \quad \forall \ r < 0.25.$$  

5.13. Remark. Note that in all previous examples the minimizing ball (or interval) is big enough to cover the whole set. That is, if $B$ is the optimal ball, then $\mu(B) = 1$. The reason for this lies in the fact that the contraction factors are very small compared with the distances that separate the cylinder sets.

In the case of the Sierpinski gaskets $S(r)$ (Example 5.9), if $r < 0.25$, then the candidate for an optimal ball is selected already at the second step because $r$ is still quite small in comparison with $c$ (see Theorem 1.4 for notation). With larger values for $r$, the centers of the selected balls are the same after the second step, but the end point varies slightly. Therefore, the value of $\tilde{m}_k$ changes slightly after the second step, but in a few steps the output is stable to three decimal places. This leads, for example, to the following conjectures for the Sierpinski gasket $S(\frac{1}{3})$.  


Figure 1:
Minimizing ball selected by the algorithm for $S(r)$ with $r < 0.25$.

and the well-known planar $\frac{1}{4}$-Cantor set, $C(\frac{1}{4})$, both self-similar sets with dimension 1.
5.14. **Conjecture.** Let \( S(\frac{1}{3}) \) be the attractor of the system \( \Psi = \{f_0, f_1, f_2\} \) given in (5.10) with \( r = \frac{1}{3} \) and let \( C(\frac{1}{4}) \) be the attractor of the system \( \tilde{\Psi} = \{g_0, g_1, g_2, g_3\} \) where

\[
\begin{align*}
g_0(x, y) &= \frac{1}{4}(x, y), \\
g_1(x, y) &= \frac{1}{4}(x, y) + \left(\frac{3}{4}, 0\right), \\
g_2(x, y) &= \frac{1}{4}(x, y) + \left(0, \frac{3}{4}\right), \\
g_3(x, y) &= \frac{1}{4}(x, y) + \left(\frac{3}{4}, \frac{3}{4}\right).
\end{align*}
\]

The conjectural values given by the algorithm for these self-similar sets are

1. \( C^1(S(1/3)) \simeq 1.537. \)
2. \( C^1(C(1/4)) \simeq 1.95. \)

We will analyze the planar \( \frac{1}{4} \)-Cantor set to show how to obtain a rigorous upper bound from the results given by the algorithm. The Sierpinski gasket \( S(\frac{1}{3}) \) can be treated in a similar manner.

First we present a table showing the results of seven iterations of the algorithm (see Table 3). For simplicity we focus on only one of the minimizing balls obtained. In this case we believe that the description of the pairs \( (\tilde{x}_k, \tilde{y}_k) \) by their corresponding codes is more illustrative than the description by their coordinates, so we will use the following standard notation: for every \( k \in \mathbb{N} \) and \( i_1i_2...i_k \in M^k \),

\[
x_{i_1i_2...i_k} = g_{i_1i_2...i_{k-1}}(x_{i_k}) = g_{i_1} \circ g_{i_2} \circ ... \circ g_{i_{k-1}}(x_{i_k}),
\]

where \( x_{i_k} = g_{i_k}(x_{i_k}) \), \( M = \{0, 1, 2, 3\} \), and \( M^k = \{i_1i_2...i_k : i_j \in M \land j = 1, ..., k\} \). Observe that if \( i_t = i_{t+1} = ... = i_k \) for some \( 0 < t \leq k \), then \( x_{i_1i_2...i_k} = x_{i_1i_2...i_t} \). We shall always use the shorter notation.
Table 3.

Output of the algorithm for $C\left(\frac{1}{4}\right)$.

| $k$  | $\tilde{m}_k$ | $(\tilde{x}_k, \tilde{y}_k)$ | $\tilde{d}_k$ |
|------|---------------|-----------------------------|-------------|
| $k = 0$ | 2.66667       | $(x_0, x_3)$                | 1           |
| $k = 1$ | 1.92296       | $(x_{20}, x_{03})$          | 0.901388    |
| $k = 2$ | 1.95814       | $(x_{20}, x_{020})$         | 0.795495    |
| $k = 3$ | 1.95542       | $(x_{20}, x_{3})$           | 0.790569    |
| $k = 4$ | 1.95306       | $(x_{20}, x_{3})$           | 0.790569    |
| $k = 5$ | 1.95388       | $(x_{20}, x_{020023})$      | 0.790662    |
| $k = 6$ | 1.95417       | $(x_{20}, x_{020021})$      | 0.790662    |

For simplicity, we are going to use the value of $\tilde{m}_3$ in Theorem 4.13 to obtain the upper bound for $C^1(C\left(\frac{1}{4}\right))$. It easily seen that $\mu_3(B(\tilde{x}_3, \tilde{d}_3)) = \mu(B(\tilde{x}_3, \tilde{d}_3)) = \frac{3}{4} + \frac{3}{4} + \frac{3}{4}$ (see Figure 2), so by Theorem 4.13 we have that

$$C^1(C\left(\frac{1}{4}\right)) \leq f_3(\tilde{x}_3, \tilde{d}_3) = 1.95542.$$
Figure 2: 
$B(\tilde{x}_4, \tilde{d}_4)$: Minimizing ball selected by the algorithm at step $k = 4$ for $C(\frac{1}{4})$.

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