Some structures of Leibniz triple systems

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Abstract

In this paper, we investigate the Leibniz triple system $T$ and its universal Leibniz envelope $U(T)$. The involutive automorphism of $U(T)$ determining $T$ is introduced, which gives a characterization of the $\mathbb{Z}_2$-grading of $U(T)$. We give the relationship between the solvable radical $R(T)$ of $T$ and $\text{Rad}(U(T))$, the solvable radical of $U(T)$. Finally, Levi’s theorem for Leibniz triple systems is obtained.

Key words: Leibniz triple system, Lie triple system, Levi’s theorem, solvable radical, semisimplicity

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1 Introduction

The notion of Leibniz algebras was introduced by Loday \cite{13}, which is a “nonantisymmetric” generalization of Lie algebras. A Leibniz algebra is a vector space equipped with a bilinear bracket satisfying the (right) Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]. \quad (1.1)$$

A Leibniz algebra whose bracket is antisymmetric is a Lie algebra. In fact, such algebras were considered by Bloh in 1965, who called them $D$-algebras \cite{6}. Later, Loday introduced this class of algebras to search an “obstruction” to the periodicity in algebraic $K$-Theory. So far many results of Leibniz algebras have been studied, including extending important theorems in Lie algebras: there are analogs of Lie’s theorem, Engel’s theorem, Levi’s theorem and Cartan’s criterion for Leibniz algebras \cite{1, 5, 9, 10, 14}. 

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Leibniz triple systems were introduced by Bremner and Sánchez-Ortega [7]. Leibniz triple systems are defined in a functorial manner using the Kolesnikov-Pozhidaev algorithm, which takes the defining identities for a variety of algebras and produces the defining identities for the corresponding variety of dialgebras [11]. As an application of this algorithm, it is showed that associative dialgebras and Leibniz algebras can be obtained from associative and Lie algebras. In [7], Leibniz triple systems were obtained by applying the Kolesnikov-Pozhidaev algorithm to Lie triple systems. Furthermore, Leibniz triple systems are related to Leibniz algebras in the same way that Lie triple systems related to Lie algebras. So it is natural to prove analogs of results from the theory of Lie triple systems to Leibniz triple systems.

This paper proceeds as follows. Section 2 devotes to some basic facts about a Leibniz triple system \( T \). In Section 3, we give the involutive automorphism of \( U(T) \) determining \( T \), where \( U(T) \) is the universal Leibniz envelope of \( T \), using which to investigate the connection between automorphisms of \( T \) and those of \( U(T) \), and to describe the \( \mathbb{Z}_2 \)-graded subspace of \( U(T) \). In Section 4, the solvable radical of a Leibniz triple system and the definition of a semisimple Leibniz triple system are introduced. In Section 5, Levi’s theorem is extended to the case of Leibniz triple systems.

In the sequel, all vector spaces are finite-dimensional and defined over a fixed but arbitrarily chosen field \( \mathbb{K} \) of characteristic 0.

2 Preliminaries

**Definition 2.1.** [12, 15] A vector space \( T \) together with a trilinear map \((a, b, c) \mapsto [abc]\) is called a Lie triple system if

\[
[aac] = 0, \\
[abc] + [bca] + [cab] = 0, \\
[ab[cde]] = [[abc]de] + [c[abd]e] + [cd[abe]],
\]

for all \( a, b, c, d, e \in T \).

**Definition 2.2.** [7] A Leibniz triple system (LeibTS for short) is a vector space \( T \) with a trilinear operation \( \{\cdot, \cdot, \cdot\} : T \times T \times T \to T \) satisfying

\[
\{a\{bcd\}e\} = \{\{abc\}de\} - \{\{acb\}de\} - \{\{adb\}ce\} + \{\{adc\}be\}, \tag{2.2}
\]
\[
\{ab\{cde\}\} = \{\{abc\}de\} - \{\{abd\}ce\} - \{\{abe\}cd\} + \{\{abe\}dc\}. \tag{2.3}
\]

**Example 2.3.** A Lie triple system gives a LeibTS with the same ternary product. If \( L \) is a Leibniz algebra with product \([\cdot, \cdot]\), then \( L \) becomes a LeibTS by putting \( \{xyz\} = [[xy]z] \). More examples refer to [7].

**Proposition 2.4.** [7] Let \( T \) be a LeibTS. Then the following identities hold.

\[
\{a\{bcd\}e\} + \{a\{cbd\}e\} = 0, \quad \{ab\{cde\}\} + \{ab\{dce\}\} = 0, \tag{2.4}
\]
\{a\{bcd\}e\} + \{a\{cdb\}e\} + \{a\{dcb\}e\} = 0, \quad \{ab\{cde\}\} + \{ab\{dec\}\} + \{ab\{ecd\}\} = 0, \quad (2.5) \\
\{\{cde\}ba\} - \{\{cde\}ab\} - \{\{cba\}de\} + \{\{cab\}de\} - \{c\{abd\}e\} - \{cd\{abe\}\} = 0. \quad (2.6)

**Theorem 2.5.** \[7\] Suppose that \(T\) is a LeibTS. Define \(U(T) = T \oplus (T \otimes T)\) together with the product \([\cdot, \cdot]\) such that

\[
\begin{align*}
[a, b] &= a \otimes b, \quad [a \otimes b, c] = \{abc\}, \quad [a, b \otimes c] = \{abc\} - \{acb\}, \\
[a \otimes b, c \otimes d] &= \{abc\} \otimes d - \{abd\} \otimes c.
\end{align*}
\]

Then \(U(T)\) is a Leibniz algebra and \(T \otimes T\) is a subalgebra of \(U(T)\). \(U(T)\) is said to be the universal Leibniz envelope of \(T\).

Generally speaking, \(U(T)\) is not a Lie algebra even if \(T\) is a Lie triple system, so we try to add a condition.

**Proposition 2.6.** Let \(T\) be a Lie triple system and \(U(T)\) be defined as in Theorem 2.5. Denote by \(K(T) = \{\sum a \otimes b \in T \otimes T \mid \sum a \otimes b, T = [T, \sum a \otimes b] = 0\}\). If \(K(T) = 0\), then \(U(T)\) is a Lie algebra.

**Proof.** We need only to show that for all \(x = a + \sum b \otimes c \in U(T)\), \([x, x] = 0\), since \(U(T)\) is already a Leibniz algebra by Theorem 2.5.

Note that \(T\) is a Lie triple system. Take \(a, b, c \in T\), then one has

\[
[a \otimes a, b] = \{aab\} = 0, \quad [b, a \otimes a] = \{baa\} - \{bba\} = 0,
\]

which imply \([a, a] = a \otimes a \in K(T) = 0\). It follows by (2.3) and the definition of Lie triple systems that

\[
\begin{align*}
[[a \otimes b, a \otimes b], c] &= \{(aba) \otimes b - \{abb\} \otimes a, c\} = \{(aba)bc\} - \{\{abb\}ac\} \\
&= \{ab\{abc\}\} + \{\{abc\}ab\} - \{(abc)ba\} = \{ab\{abc\}\} + \{\{abc\}ab\} + \{b\{abc\}a\} = 0,
\end{align*}
\]

and

\[
\begin{align*}
[c, [a \otimes b, a \otimes b]] &= [c, \{aba\} \otimes b - \{abb\} \otimes a] \\
&= \{c\{aba\}b\} - \{cb\{aba\}\} - \{c\{abb\}a\} + \{ca\{abb\}\} \\
&= \{b\{aba\}c\} + \{\{abb\}ac\} = \{ab\{bac\}\} - \{ba\{abc\}\} = 0,
\end{align*}
\]

that is, \([a \otimes b, a \otimes b] \in K(T) = 0\).

Now take \(a + \sum_{i=1}^{n} b_i \otimes c_i \in U(T)\), then

\[
\begin{align*}
[a + \sum_{i=1}^{n} b_i \otimes c_i, a + \sum_{i=1}^{n} b_i \otimes c_i] \\
&= a \otimes a + \sum_{i=1}^{n} \{b_i c_i a\} + \{ab_i c_i\} - \{ac_i b_i\} + \sum_{i,j=1}^{n} [b_i \otimes c_i, b_j \otimes c_j].
\end{align*}
\]
\[ \sum_{i<j} ([b_i \otimes c_i, b_j \otimes c_j] + [b_j \otimes c_j, b_i \otimes c_i]). \]

Note that for all \(d \in T\), \([d, [b_i \otimes c_i, b_j \otimes c_j] + [b_j \otimes c_j, b_i \otimes c_i]] = 0\) by using (1.1), and
\[
[b_i \otimes c_i, b_j \otimes c_j], d] = \{b_i c_i b_j c_j\} - \{b_i c_i b_j d\} - \{b_i c_i c_j d\}.
\]

Then \([b_i \otimes c_i, b_j \otimes c_j] + [b_j \otimes c_j, b_i \otimes c_i], d] = 0\), and so
\[
[b_i \otimes c_i, b_j \otimes c_j] + [b_j \otimes c_j, b_i \otimes c_i] \in K(T) = 0.
\]

Thus \([x, x] = 0\).

**Definition 2.7.** Let \(I\) be a subspace of a LeibTS \(T\). Then \(I\) is called a subsystem of \(T\), if \(\{III\} \subseteq I\); \(I\) is called an ideal of \(T\), if \(\{ITT\} + \{TIT\} + \{TTI\} \subseteq I\).

Let \(I\) be an ideal of a LeibTS \(T\) and define some notations as follows.

\[
i(I) = I \otimes T + T \otimes I,
\]
\[
j(I) = \left\{ \sum a \otimes b \in T \otimes T \mid \left[ \sum a \otimes b, T \right] + \left[T, \sum a \otimes b\right] \subseteq I \right\},
\]
\[
\mathcal{I}(I) = I + i(I), \quad \mathcal{J}(I) = I + j(I).
\]

It is easy to see that \(K(T) = j(0) = \mathcal{J}(0)\). The notion of \(K(T)\) will be used in Section 4, too.

**Theorem 2.8.** If \(I\) is an ideal of a LeibTS \(T\), then

1. \(i(I) \subseteq j(I), \mathcal{I}(I) \subseteq \mathcal{J}(I)\);
2. \(i(I)\) and \(j(I)\) are ideals of \(T \otimes T\);
3. \(\mathcal{I}(I)\) and \(\mathcal{J}(I)\) are ideals of \(U(T)\).

**Proof.** (1) follows from
\[
i(I), T] = [I \otimes T + T \otimes I, T] \subseteq \{ITT\} + \{TIT\} \subseteq I
\]
and
\[
[T, i(I)] = [T, I \otimes T + T \otimes I] \subseteq \{TIT\} + \{TTI\} \subseteq I.
\]

(2) \(i(I)\) is an ideal of \(T \otimes T\) since
\[
i(I), T \otimes T] = [I \otimes T + T \otimes I, T \otimes T] \subseteq \{ITT\} \otimes T + \{TIT\} \otimes T \subseteq I \otimes T \subseteq i(I)
\]
and

\[ [T \otimes T, i(I)] = [T \otimes T, I \otimes T + T \otimes I] \subseteq \{TTI\} \otimes T + \{TTT\} \otimes I \subseteq I \otimes T + T \otimes I = i(I). \]

To show that \( j(I) \) is an ideal of \( T \otimes T \), we take \( \sum t \otimes s \in T \otimes T \) and \( \sum a \otimes b \in j(I) \). It is sufficient to prove that for all \( c \in T \),

\[
\left[ \left[ \sum a \otimes b, \sum t \otimes s \right], c \right] \in I, \quad \left[ c, \left[ \sum a \otimes b, \sum t \otimes s \right] \right] \in I, \quad \left[ c, \left[ \sum t \otimes s, \sum a \otimes b \right] \right] \in I.
\]

In fact, it follows from (2.3) that

\[
\left[ \sum a \otimes b, \sum t \otimes s \right], c = \sum (\{abt\} \otimes s - \{abs\} \otimes t, c) = \sum (\{abt\} \otimes s) - \sum (\{abs\} \otimes t)\in I,
\]

\[
\left[ c, \left[ \sum a \otimes b, \sum t \otimes s \right] \right] = \sum [c, \{abt\} \otimes s - \{abs\} \otimes t]
\]

\[
\sum (\{c\{abt\} s - \{cs\{abt\}\} - \{c\{abs\} t\} + \{ct\{abs\}\})
\]

\[
= \sum (\{c[a \otimes b, t] s\} - \{ca \otimes b, t\}) - \{c[a \otimes b, s] t\} + \{ct[a \otimes b, s]\}) \in I,
\]

and

\[
\left[ \left[ \sum t \otimes s, \sum a \otimes b \right], c \right] = \sum ([\{tsa\} \otimes b - \{tsb\} \otimes a, c] = \sum \{tsa\} \otimes b - \{tsb\} \otimes a, c] = \sum \{tsc\} ab - \{tsc\} ba
\]

\[
= \sum (\{t[a \otimes b, c]\} + [\{tsc\}, a \otimes b]) \in I.
\]

Moreover, by (1.1), \( [c, \left[ \sum t \otimes s, \sum a \otimes b \right] = -[c, \left[ \sum a \otimes b, \sum t \otimes s \right]] \in I. \)

(3) It is routine to prove that \( I(I) \) is an ideal of \( U(T) \) by the facts of \( I \) being an ideal of \( T \) and \( i(I) \) being an ideal of \( T \otimes T \). \( J(I) \) is an ideal of \( U(T) \) since

\[
[J(I), U(T)] = [I + j(I), T + T \otimes T] = I \otimes T + [j(I), T] + [I, T \otimes T] + [j(I), T \otimes T]
\]

\[
\subseteq i(I) + I + j(I) = i + j(I) = J(I),
\]

and

\[
[U(T), J(I)] = \left[ T + T \otimes T, I + j(I) \right] = T \otimes I + [T \otimes T, I] + [T, j(I)] + [T \otimes T, j(I)]
\]

\[
\subseteq i(I) + I + j(I) = J(I).
\]

Thus the proof is completed. \( \square \)
3 The involutive automorphism of $U(T)$ determining $T$

Theorem 3.1 says that $U(T)$ is a $\mathbb{Z}_2$-graded Leibniz algebra with $U(T)_0 = T \otimes T$ and $U(T)_1 = T$. Then every Leibniz triple system $T$ is the $\bar{1}$ component of a $\mathbb{Z}_2$-graded Leibniz algebra. Moreover, for a $\mathbb{Z}_2$-graded Leibniz algebra $L = L_0 \oplus L_1$, $L_1$ has the structure of a Leibniz triple system by Example 2.3.

The $\mathbb{Z}_2$-grading of $U(T)$ can be characterized by an involutive automorphism. In what follows we will use this involutive automorphism of $U(T)$ to investigate the connection between automorphisms of $T$ and those of $U(T)$, and describe the $\mathbb{Z}_2$-graded subspace of $U(T)$.

**Theorem 3.1.** Let $T$ be a LeibTS and $U(T)$ the universal Leibniz envelope of $T$. Then there is a unique involutive automorphism $\theta : U(T) \to U(T)$ such that

$$T = \{ x \in U(T) \mid \theta(x) = -x \}.$$

**Proof.** For any $a + \sum b \otimes c \in U(T)$, define $\theta(a + \sum b \otimes c) = -a + \sum b \otimes c$. It is clear that $\theta$ is linear and $\theta^2 = \text{id}$. Note that

$$[\theta (a + \sum b \otimes c), \theta (a' + \sum b' \otimes c')] = [-a + \sum b \otimes c, -a' + \sum b' \otimes c']$$

$$= a \otimes a' - \sum \{ bca' \} - \sum \{ abc' \} + \sum \{ ac'b' \} + \sum \{ bc'b' \} \otimes c' - \sum \{ bcc' \} \otimes b',$$

and

$$\theta \left( a + \sum b \otimes c, a' + \sum b' \otimes c' \right)$$

$$= \theta \left( a \otimes a' + \sum \{ bca' \} + \sum \{ abc' \} - \sum \{ ac'b' \} + \sum \{ bc'b' \} \otimes c' - \sum \{ bcc' \} \otimes b' \right)$$

$$= a \otimes a' - \sum \{ bca' \} - \sum \{ abc' \} + \sum \{ ac'b' \} + \sum \{ bc'b' \} \otimes c' - \sum \{ bcc' \} \otimes b'.$$

Then $\theta$ is an involutive automorphism of $U(T)$ and $T = \{ x \in U(T) \mid \theta(x) = -x \}$.

If $\phi$ is another involutive automorphism of $U(T)$ with $T = \{ x \in U(T) \mid \phi(x) = -x \}$, then $\theta|_T = \phi|_T = -\text{id}_T$ and

$$\theta \left( a + \sum b \otimes c \right)$$

$$= \theta \left( a + \sum [b, c] \right) = \theta(a) + \sum [\theta(b), \theta(c)] = \phi(a) + \sum [\phi(b), \phi(c)] = \phi \left( a + \sum [b, c] \right)$$

$$= \phi \left( a + \sum b \otimes c \right),$$

which implies $\theta = \phi$, so $\theta$ is unique. $\theta$ is called the **involutive automorphism of $U(T)$ determining $T$**.

\[\square\]
Proposition 3.2. Suppose that $T$ is a LeibTS and $f$ is an automorphism of $T$. Then there is a unique automorphism $\tilde{f}$ of $U(T)$ such that $f = \tilde{f}|_T$ and $\tilde{f}\theta = \theta f$, where $\theta$ is the involutive automorphism of $U(T)$ determining $T$.

Proof. Define $\tilde{f} : U(T) \to U(T)$ by $\tilde{f}(a + \sum b \otimes c) = f(a) + \sum f(b) \otimes f(c)$. It is clear that $\tilde{f}$ is well defined, $\tilde{f}|_T = f$ and $\tilde{f}$ is a linear isomorphism. Moreover, we have

$$\tilde{f}\left(\left[a + \sum b \otimes c, a' + \sum b' \otimes c'\right]\right)$$

$$= \tilde{f}\left(a \otimes a' + \sum \{bca'\} + \sum \{ab'c'\} - \sum \{ac'b'\} + \sum \{bcb'\} \otimes c' - \sum \{bcc'\} \otimes b'\right)$$

$$= f(a) \otimes f(a') + \sum \{f(b)f(c)f(a')\} + \sum \{f(a)f(b')f(c')\} - \sum \{f(a)f(c')f(b')\}$$

$$+ \sum \{f(b)f(c)f(b')\} \otimes f(c') - \sum \{f(b)f(c)f(b')\} \otimes f(b')$$

$$= \left[f(a) + \sum f(b) \otimes f(c), f(a') + \sum f(b') \otimes f(c')\right]$$

$$= \left[\tilde{f}\left(a + \sum b \otimes c\right), \tilde{f}\left(a' + \sum b' \otimes c'\right)\right],$$

and

$$\tilde{f}\theta\left(a + \sum b \otimes c\right) = \tilde{f}\left(-a + \sum b \otimes c\right) = -f(a) + \sum f(b) \otimes f(c)$$

$$= \theta\left(f(a) + \sum f(b) \otimes f(c)\right) = \theta \tilde{f}\left(a + \sum b \otimes c\right).$$

Hence $\tilde{f}$ is an automorphism and $\tilde{f}\theta = \theta \tilde{f}$. $\tilde{f}$ is unique since $U(T)$ is generated by $T$. $\square$

Proposition 3.3. If $f$ is an automorphism of $U(T)$ satisfying $f\theta = \theta f$, then $f(T) \subseteq T$ and $f|_T$ is an automorphism of $T$.

Proof. For all $a \in T$, $f\theta(a) = f(-a) = -f(a)$. Since $f\theta = \theta f$, it follows that $\theta f(a) = -f(a)$, which implies $f(a) \in T$ by Theorem 3.1. It is clear that $f|_T$ is an injective endomorphism of $T$. Note that

$$f(T \otimes T) = f([T, T]) = [f(T), f(T)] = f(T) \otimes f(T) \subseteq T \otimes T,$$

and $f$ is surjective, so $f|_T$ is surjective. Thus $f|_T$ is an automorphism of $T$. $\square$

Note that $U(T)$ is a $\mathbb{Z}_2$-graded Leibniz algebra with $U(T)_0 = T \otimes T$ and $U(T)_1 = T$. Then a subspace $V$ of $U(T)$ is $\mathbb{Z}_2$-graded if and only if $V = V \cap T + V \cap (T \otimes T)$. Suppose that $V$ is a $\theta$-invariant subspace of $U(T)$. Then for $a + \sum b \otimes c \in V$, $\theta (a + \sum b \otimes c) = -a + \sum b \otimes c \in V$. Since $\text{ch} \mathbb{K} = 0$, it follows $a \in V \cap T$ and $\sum b \otimes c \in V \cap (T \otimes T)$, that is, $V = V \cap T + V \cap (T \otimes T)$. Conversely, if $V$ is a subspace of $U(T)$ such that $V = V \cap T + V \cap (T \otimes T)$, then $a + \sum b \otimes c \in V$ implies that $a \in V$ and $\sum b \otimes c \in V \cap (T \otimes T)$, so $\theta (a + \sum b \otimes c) = -a + \sum b \otimes c \in V$ and $V$ is a $\theta$-invariant subspace of $U(T)$.

Therefore, $V$ being a $\theta$-invariant subspace of $U(T)$ is equivalent to that $V$ is a $\mathbb{Z}_2$-graded subspace of $U(T)$.
Theorem 3.4. Let $V$ be a $\theta$-invariant subspace of $U(T)$. If $V$ is an ideal of $U(T)$, then $V \cap T$ is an ideal of $T$ and $V \cap (T \otimes T)$ is an ideal of $T \otimes T$ such that
\[i(V \cap T) \subseteq V \cap (T \otimes T) \subseteq j(V \cap T), \quad \mathcal{I}(V \cap T) \subseteq V \subseteq \mathcal{J}(V \cap T).\]

Proof. Denote $I = V \cap T$ and $\mathfrak{M} = V \cap (T \otimes T)$. Then $V = I + \mathfrak{M}$. Using the condition that $V$ is an ideal of $U(T)$, one has
\[\{V, U(T)\} = [I + \mathfrak{M}, T + T \otimes T] = I \otimes T + [\mathfrak{M}, T] + [I, [T, T]] + [\mathfrak{M}, T \otimes T] \subseteq I + \mathfrak{M},\]
\[\{U(T), V\} = [T + T \otimes T, I + \mathfrak{M}] = T \otimes I + [[T, T], I] + [T, \mathfrak{M}] + [T \otimes T, \mathfrak{M}] \subseteq I + \mathfrak{M}.
It follows that
\[I \otimes T + T \otimes I + [\mathfrak{M}, T \otimes T] + [T \otimes T, \mathfrak{M}] \subseteq \mathfrak{M},\]
\[[\mathfrak{M}, T] + [I, [T, T]] + [[T, T], I] + [T, \mathfrak{M}] \subseteq I.
Then $i(I) \subseteq \mathfrak{M} \subseteq j(I)$ and $\mathfrak{M}$ is an ideal of $T \otimes T$. Note that
\[\{TTI\} = [[T, T], I] \subseteq I,\]
\[\{TIT\} = [[T, I], T] = [T \otimes I, T] \subseteq [\mathfrak{M}, T] \subseteq I,\]
\[\{ITT\} = [[I, T], T] = [I \otimes T, T] \subseteq [\mathfrak{M}, T] \subseteq I.
Thus $I$ is an ideal of $T$ and $\mathcal{I}(I) \subseteq I + \mathfrak{M} \subseteq \mathcal{J}(I)$. \qed

4 The solvable radical of a Leibniz triple system

Given an arbitrary Leibniz algebra $L$, the central lower sequence of an ideal $I$ of $L$ is defined as
\[I^{(0)} = I, I^{(n+1)} = [I^{(n)}, I^{(n)}], \quad \forall n \geq 1.\]
Suppose that $I$ is an ideal of a LeibTS $T$. Let $I^{[0]} = I$ and for $n \geq 1$,
\[I^{[n+1]} = \{TTI^{[n]}\} + \{I^{[n]}T I\} + \{I^{[n]}I^{[n]}\}.\]

Proposition 4.1. If $I$ is an ideal of a LeibTS $T$, then $I^{[n]}$ is also an ideal of $T$ and $I \supseteq I^{[1]} \supseteq \cdots \supseteq I^{[n]} \supseteq \cdots$, $\forall n \in \mathbb{N}$.

Proof. First we show $\{I^{[1]}T T\} \subseteq I^{[1]}$. By (2.2) and (2.3), it follows that
\[\{TTI\} \subseteq \{T I \{TT\}\} + \{T I \{IT\}\} + \{T I \{TI\}\} \subseteq \{TII\} + \{TIT\} + \{ITT\} \subseteq I^{[1]},\]
\[\{I IT\} \subseteq \{IT \{IT\}\} + \{IT \{TT\}\} \subseteq \{ITI\} + \{IT\} \subseteq I^{[1]},\]
\[\{IT T\} \subseteq \{(IT)T\} + \{(IT)I\} + \{(IT)I\} \subseteq \{ITT\} \subseteq I^{[1]} + \{IT\} = I^{[1]},\]
which imply $\{I^{[1]}T T\} \subseteq I^{[1]}$. Similarly, one can prove $\{TTI^{[1]}T\} \subseteq I^{[1]}$ and $\{TTI^{[1]}I\} \subseteq I^{[1]}$. So $I^{[1]}$ is an ideal of $T$. Suppose that $I^{[k]}$ is an ideal of $T$. Then $I^{[k+1]} = (I^{[k]}T I^{[k]})I^{[k]}$ is an ideal of $T$. Hence $I^{[n]}$ is an ideal of $T$ for all $n \in \mathbb{N}$. It is clear that $I \supseteq I^{[1]} \supseteq \cdots \supseteq I^{[n]} \supseteq \cdots$, $\forall n \in \mathbb{N}$. \qed
Definition 4.2. An ideal $I$ of a LeibTS $T$ is called \textit{solvable}, if there is a positive integer $k$ such that $I^{[k]} = 0$.

Theorem 4.3. Let $I$ be an ideal of a LeibTS $T$. Then the following statements are equivalent.

1. $I$ is a solvable ideal of $T$.
2. $\mathcal{I}(I)$ is a solvable ideal of $U(T)$.
3. $\mathcal{J}(I)$ is a solvable ideal of $U(T)$.

Proof. (1)$\Rightarrow$(2). We will use the induction to show

$$\mathcal{I}(I)^{(2n)} \subseteq I^{[n]} + I^{[n]} \otimes T.$$ 

Consider the base step $n = 1$. We obtain

$$\mathcal{I}(I)^{(1)} = [\mathcal{I}(I), \mathcal{I}(I)] = [I + i(I), I + i(I)] = I \otimes I + [i(I), I] + [I, i(I)] + [i(I), i(I)].$$

Note that

$$[i(I), I] = \{ITI\} + \{TII\} \subseteq I^{[1]}, \quad [I, i(I)] \subseteq \{IIT\} + \{ITI\} \subseteq I^{[1]},$$

$$[i(I), i(I)] = [I \otimes T + T \otimes I, I \otimes T + T \otimes I] \subseteq \{ITI\} \otimes T + \{ITT\} \otimes I + \{TII\} \otimes T + \{TIT\} \otimes I \subseteq I^{[1]} \otimes T + I \otimes I.$$ 

Hence $\mathcal{I}(I)^{(1)} \subseteq I^{[1]} + I^{[1]} \otimes T + I \otimes I$. So

$$\mathcal{I}(I)^{(2)} = [\mathcal{I}(I)^{(1)}, \mathcal{I}(I)^{(1})] \subseteq [I^{[1]} + I^{[1]} \otimes T + I \otimes I, I^{[1]} + I^{[1]} \otimes T + I \otimes I]$$

$$\subseteq I^{[1]} \otimes I^{[1]} + \{I^{[1]}T I^{[1]}\} + \{III^{[1]}\} + \{I^{[1]}I^{[1]}T\} + \{I^{[1]}T I^{[1]}\} \otimes T + \{I^{[1]}TT\} \otimes I^{[1]}$$

$$+ \{III^{[1]}\} \otimes T + \{IIT\} \otimes I^{[1]} + \{I^{[1]}II\} + \{I^{[1]}TI\} \otimes I + \{III\} \otimes I$$

$$\subseteq I^{[1]} + I^{[1]} \otimes T.$$

Suppose $\mathcal{I}(I)^{(2k)} \subseteq I^{[k]} + I^{[k]} \otimes T$. Then

$$\mathcal{I}(I)^{(2k+1)} = [\mathcal{I}(I)^{(2k)}, \mathcal{I}(I)^{(2k)}] \subseteq [I^{[k]} + I^{[k]} \otimes T, I^{[k]} + I^{[k]} \otimes T]$$

$$\subseteq I^{[k]} \otimes I^{[k]} + \{I^{[k]}T I^{[k]}\} + \{I^{[k]}I^{[k]}T\} + \{I^{[k]}T I^{[k]}\} \otimes T + \{I^{[k]}TT\} \otimes I^{[k]}$$

$$\subseteq I^{[k+1]} + I^{[k]} \otimes I^{[k]} + I^{[k+1]} \otimes T.$$

Hence

$$\mathcal{I}(I)^{(2k+2)} = [\mathcal{I}(I)^{(2k+1)}, \mathcal{I}(I)^{(2k+1)}]$$

$$\subseteq [I^{[k+1]} + I^{[k]} \otimes I^{[k]} + I^{[k+1]} \otimes T, I^{[k+1]} + I^{[k]} \otimes I^{[k]} + I^{[k+1]} \otimes T]$$

$$\subseteq I^{[k+1]} \otimes I^{[k+1]} + \{I^{[k]}I^{[k]}I^{[k+1]}\} + \{I^{[k+1]}T I^{[k+1]}\} + \{I^{[k+1]}I^{[k]}I^{[k]}\} + \{I^{[k]}I^{[k]}I^{[k]}\} \otimes I^{[k]}.$$
Thus $\mathcal{I}(I)^{(2n)} \subseteq I^{[n]} + I^{[n]} \otimes T$ by induction, which implies that $\mathcal{I}(I)$ is solvable.

(2) $\Rightarrow$ (1). It is sufficient to prove $I^{[n]} \subseteq \mathcal{I}(I)^{(n)}$ for all $n \in \mathbb{N}$. In fact,

$I^{[1]} = \{TTI\} + \{ITI\} + \{IT\} = [[T, I], I] + [[I, T], I] + [[I, I], T]
\subseteq [[T, I], I] + [[I, T], I] + [I, [I, T]] \subseteq [i(I), I] + [I, i(I)] \subseteq [\mathcal{I}(I), \mathcal{I}(I)] = \mathcal{I}(I)^{(1)}$.

Suppose $I^{[k]} \subseteq \mathcal{I}(I)^{(k)}$. Then

$I^{[k+1]} = \{TTI^{[k]}\} + \{I^{[k]}TI^{[k]}\} + \{I^{[k]}I^{[k]}T\}
\subseteq [[T, I^{[k]}], I^{[k]}] + [[I^{[k]}, T], I^{[k]}] + [I^{[k]}, [I^{[k]}, T]]
\subseteq [[T, \mathcal{I}(I)^{(k)}], \mathcal{I}(I)^{(k)}] + [[\mathcal{I}(I)^{(k)}, T], \mathcal{I}(I)^{(k)}] + [\mathcal{I}(I)^{(k)}, [\mathcal{I}(I)^{(k)}, T]]
\subseteq [\mathcal{I}(I)^{(k)}, \mathcal{I}(I)^{(k)}] = \mathcal{I}(I)^{(k+1)}$.

Hence $I^{[n]} \subseteq \mathcal{I}(I)^{(n)}$ follows from the induction. So (1) $\iff$ (2).

(3) $\Rightarrow$ (2). It is clear since $\mathcal{I}(I) \subseteq \mathcal{J}(I)$.

(2) $\Rightarrow$ (3). Note that

$[\mathcal{J}(I), U(T)] = [I + j(I), T + T \otimes T]
\subseteq I \otimes T + [j(I), T] + \{ITT\} + [j(I), T] \otimes T
\subseteq I \otimes T + I \subseteq \mathcal{I}(I)$.

Then $\mathcal{J}(I)^{(1)} = [\mathcal{J}(I), \mathcal{J}(I)] \subseteq [\mathcal{J}(I), U(T)] \subseteq \mathcal{I}(I)$. Suppose $\mathcal{J}(I)^{(k)} \subseteq \mathcal{I}(I)^{(k-1)}$. Then

$\mathcal{J}(I)^{(k+1)} = [\mathcal{J}(I)^{(k)}, \mathcal{J}(I)^{(k)}] \subseteq [\mathcal{I}(I)^{(k-1)}, \mathcal{I}(I)^{(k-1)}] = \mathcal{I}(I)^{(k)}$.

It follows that $\mathcal{J}(I)^{(n+1)} \subseteq \mathcal{I}(I)^{(n)}$ by induction and the solvability of $\mathcal{I}(I)$ implies the solvability of $\mathcal{J}(I)$. Thus (2) $\iff$ (3).

Therefore, the above three items are equivalent and the proof is completed.

Corollary 4.4. Let $V = I + \mathfrak{M}$ be a $\theta$-invariant ideal of $U(T)$. Then $I$ is a solvable ideal of $T$ if and only if $V$ is a solvable ideal of $U(T)$. In particular, $T$ is a solvable LeibTS if and only if $U(T)$ is a solvable Leibniz algebra.

Proposition 4.5. Suppose that $I$ and $J$ are solvable ideals of a LeibTS $T$. Then $I + J$ is a solvable ideal of $T$.

Proof. It is clear that $I + J$ is an ideal of $T$. Now we will use the induction to prove $(I + J)^{[n]} \subseteq I^{[n]} + J^{[n]} + I \cap J$. The base step $n = 1$ is true, for

$(I + J)^{[1]} = \{T, I + J, I + J\} + \{I + J, T, I + J\} + \{I + J, I + J, T\} \subseteq I^{[1]} + J^{[1]} + I \cap J$.  

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Suppose \((I + J)^{[k]} \subseteq I^{[k]} + J^{[k]} + I \cap J\). Then
\[(I + J)^{[k+1]} = \{T(I + J)^{[k]}(I + J)^{[k]}\} + \{(I + J)^{[k]}T(I + J)^{[k]}\} + \{(I + J)^{[k]}(I + J)^{[k]}T\}\]
\[\subseteq \{T, I^{[k]} + J^{[k]} + I \cap J, T[I^{[k]} + J^{[k]} + I \cap J]\} + \{I^{[k]} + J^{[k]} + I \cap J, T[I^{[k]} + J^{[k]} + I \cap J, T]\} + \{I^{[k]} + J^{[k]} + I \cap J, I^{[k]} + J^{[k]} + I \cap J, T\}\]
\[\subseteq I^{[k+1]} + J^{[k+1]} + I \cap J.\]

So there is \(n_1 \in \mathbb{N}\) such that \((I + J)^{[n_1]} \subseteq I \cap J\). Note that \(I \cap J\) is a solvable ideal of \(T\), which implies \(I + J\) is solvable.

\[\square\]

**Definition 4.6.** Let \(R(T)\) denote the sum of all solvable ideals in \(T\). Then \(R(T)\) is the unique maximal solvable ideal by Proposition 4.5, called the **solvable radical** of \(T\).

For a Leibniz algebra \(L\), there is a corresponding Lie algebra \(L/Ker(L)\), where \(Ker(L)\) is a two-side ideal of \(L\) spanned by \(\{[x, x] \mid x \in L\}\). Similarly, for a LeibTS \(T\), we define \(Ker(T)\) to be the subspace of \(T\) spanned by all elements in \(T\) of the form \(\{abc\} − \{acb\} + \{bca\}\), for all \(a, b, c \in T\).

**Theorem 4.7.** Let \(T\) be a LeibTS and \(U(T)\) be its universal Leibniz envelope. Then

1. \(Ker(T)\) is an ideal of \(T\) satisfying \(TTKer(T) = TKer(T)T\) = 0.
2. \(T/Ker(T)\) is a Lie triple system. Moreover, \(T\) is a Lie triple system if and only if \(Ker(T) = 0\).
3. \(Ker(T) \subseteq R(T)\).
4. \(Ker(U(T)) \cap T = Ker(T)\).
5. \(\mathcal{I}(Ker(T)) \subseteq Ker(U(T)) \subseteq J(Ker(T))\).

**Proof.** (1) For \(\{abc\} − \{acb\} + \{bca\} \in Ker(T)\) and \(d, e \in T\), by (2.4) and (2.5), we have
\[
\{d, e, \{abc\} − \{acb\} + \{bca\}\} = \{d, e, \{abc\} + \{cab\} + \{bca\}\} = 0,
\]
\[
\{d, \{abc\} − \{acb\} + \{bca\}, e\} = \{d, \{abc\} + \{cab\} + \{bca\}, e\} = 0,
\]
which imply \(TTKer(T) = TKer(T)T = 0\). Using (2.2) and (2.3), we have
\[
\{\{abc\} − \{acb\}, d, e\} = \{\{abc\}de\} = \{a\{bcd\}e\} + \{\{adb\}ce\} − \{\{ade\}be\}
\]
\[
= a\{bcd\}e + \{ad\{bce\}\} + \{\{ade\}bc\} − \{\{ade\}cb\},
\]
and
\[
\{\{bca\}de\} = bc\{ade\} + \{bcd\}ae + \{bce\}ad − \{bce\}da.
\]
Note that
\[\{\{ade\}bc\} - \{\{ade\}cb\} + \{bc\{ade\}\} \in \text{Ker}(T),\]
\[\{\{bce\}ad\} - \{\{bce\}da\} + \{ad\{bce\}\} \in \text{Ker}(T),\]
and
\[\{a\{bcd\}e\} + \{\{bcd\}ae\} = (\{a\{bcd\}e\} - \{ae\{bcd\}\}) + (\{\{bcd\}ae\} - \{\{bcd\}ae\} + \{ae\{bcd\}\}) \in \text{Ker}(T).\]

Therefore, \(\{\{abc\} - \{acb\} + \{bca\}, d, e\} \in \text{Ker}(T)\) and \(\text{Ker}(T)\) is an ideal of \(T\).

(2) Since \(\text{Ker}(T)\) is an ideal of \(T\), \(T/\text{Ker}(T)\) is also a LeibTS with the product \([abc] = \{abc\}\). \([aa\overline{b}] = 0\) comes from \(\{aab\} = \{aab\} - \{aba\} + \{aba\} \in \text{Ker}(T)\), and then

\[\overline{[abc]} + [\overline{bca}] + [\overline{c\overline{a}b}] = [\overline{abc}] + [\overline{bca}] - [\overline{a\overline{c}b}] = (\{abc\} - \{acb\} + \{bca\}) = 0.\]

Moreover,
\[\overline{[\overline{abc}]d\overline{e}] = [ab\{cde\}] = \{\{abc\}c\} - \{\{abc\}d\}c = \{\{abc\}d\} + \{\{abc\}c\}\]
\[= \{\{abc\}d\} + \{c\{ab\}d\} + \{cd\{abc\}\} = ([\overline{abc}]\overline{d}e) + [c\{ab\}d\] + [\overline{cd}\{abc\}]].

Hence \(T/\text{Ker}(T)\) is a Lie triple system. It is clear \(T\) is a Lie triple system if and only if \(\text{Ker}(T) = 0\).

(3) Since
\[\text{Ker}(T)[1] = \{TKer(T)\text{Ker}(T)\} + \{\text{Ker}(T)TKer(T)\} + \{\text{Ker}(T)\text{Ker}(T)T\} = 0,\]
\(\text{Ker}(T)\) is a solvable ideal of \(T\) and so \(\text{Ker}(T) \subseteq R(T)\).

(4) \(\text{Ker}(T) \subseteq \text{Ker}(U(T)) \cap T\) follows from \(\{abc\} - \{acb\} + \{bca\} = [a, b \otimes c] + [b \otimes c, a] = [a + b \otimes c, a + b \otimes c] - [a, a] - [b \otimes c, b \otimes c] \in \text{Ker}(U(T))\).

Conversely, for all \(x = [a + \sum b_i \otimes c_i, a + \sum b_i \otimes c_i] \in \text{Ker}(U(T))\), suppose \(x = d + m\) with \(d \in T, m \in T \otimes T\). Then \(d = \sum (\{b_i c_i a\} + \{a b_i c_i\} - \{a c_i b_i\}) \in \text{Ker}(T)\), that is, \(\text{Ker}(U(T)) \cap T \subseteq \text{Ker}(T)\).

(5) Note that \(\text{Ker}(U(T))\) is an ideal of \(U(T)\) and \(\text{Ker}(U(T)) \cap T = \text{Ker}(T)\). It follows from Theorem 3.4 that \(\mathcal{I}(\text{Ker}(T)) \subseteq \text{Ker}(U(T)) \subseteq \mathcal{J}(\text{Ker}(T)).\)

Let \(L\) be a Leibniz algebra and \(\text{Rad}(L)\) the solvable radical of \(L\). \(L\) is called semisimple if \(\text{Rad}(L) = \text{Ker}(L)\). Likewise, we introduce the definition of a semisimple LeibTS.

Definition 4.8. A LeibTS \(T\) is said to be semisimple, if \(R(T) = \text{Ker}(T)\).

Proposition 4.9. Let \(T\) be a LeibTS. Then the following statements hold.

(1) \(T/R(T)\) is a semisimple Lie triple system.
(2) Let $I$ be an ideal of $T$. If $T/I$ is a semisimple LeibTS, then $R(T) \subseteq \text{Ker}(T) + I$.

(3) If $I$ is an ideal of $T$ and $T/I$ is a semisimple Lie triple system, then $R(T) \subseteq I$.

**Proof.** (1) $T/R(T)$ is clearly a Lie triple system since $\text{Ker}(T) \subseteq R(T)$. If $I/R(T)$ is a solvable ideal of $T/R(T)$, then $I^{[k]}/R(T) = (I/R(T))^{[k]} = 0$, for some $k \in \mathbb{N}$. Then $I^{[k]} \subseteq R(T)$, it follows that $I$ is a solvable ideal of $T$, so $I = R(T)$. Hence the only solvable ideal of $T/R(T)$ is 0 and $T/R(T)$ is semisimple.

(2) Since $T/I$ is semisimple, $R(T/I) = \text{Ker}(T/I)$. Note that $R(T) + I/I$ is solvable in $T/I$ and $\text{Ker}(T/I) = \text{Ker}(T) + I/I$. Then

$$R(T) + I/I \subseteq R(T/I) = \text{Ker}(T/I) = \text{Ker}(T) + I/I,$$

which implies $R(T) \subseteq \text{Ker}(T) + I$.

(3) If $T/I$ is a Lie triple system, then $\text{Ker}(T/I) = 0$, so $\text{Ker}(T) \subseteq I$, and hence $R(T) \subseteq I$ by using (2). \qed

**Theorem 4.10.** $R(T) = \text{Rad}(U(T)) \cap T$ and $\text{Rad}(U(T)) = \mathcal{I}(R(T)) + K(T) = \mathcal{J}(R(T))$.

**Proof.** Suppose that $\theta$ is the involutive automorphism of $U(T)$ determining $T$. Since $\text{Rad}(U(T))$ is solvable, its homomorphic image $\theta(\text{Rad}(U(T)))$ is solvable, then $\theta(\text{Rad}(U(T))) \subseteq \text{Rad}(U(T))$ and $\text{Rad}(U(T))$ is $\theta$-invariant. So

$$\text{Rad}(U(T)) = \text{Rad}(U(T)) \cap T + \text{Rad}(U(T)) \cap (T \otimes T),$$

$$\mathcal{I}(\text{Rad}(U(T)) \cap T) \subseteq \text{Rad}(U(T)) \cap T + \mathcal{J}(\text{Rad}(U(T)) \cap T).$$

Then $\mathcal{I}(\text{Rad}(U(T)) \cap T)$ is solvable. By Theorem 1.5, $\text{Rad}(U(T)) \cap T$ and $\mathcal{J}(\text{Rad}(U(T)) \cap T)$ is solvable, so $\mathcal{J}(\text{Rad}(U(T)) \cap T) = \text{Rad}(U(T))$.

Since $\text{Rad}(U(T)) \cap T$ is solvable, $\text{Rad}(U(T)) \cap T \subseteq R(T)$, which implies $\text{Rad}(U(T)) = \mathcal{J}(\text{Rad}(U(T)) \cap T) \subseteq \mathcal{J}(R(T))$. But $\mathcal{J}(R(T))$ is solvable by the solvability of $R(T)$, then $\mathcal{J}(R(T)) \subseteq \text{Rad}(U(T))$, hence $\mathcal{J}(R(T)) = \text{Rad}(U(T)) = \mathcal{J}(\text{Rad}(U(T)) \cap T)$. So $R(T) = \text{Rad}(U(T)) \cap T$ and $\mathcal{J}(R(T)) = \mathcal{J}(R(T))$.

Consider the factor Leibniz algebra $\overline{U(T)} = U(T)/\mathcal{I}(R(T))$. Since $\overline{U(T)} = T + T \otimes T = \overline{T} + \overline{T} \otimes \overline{T} = U(\overline{T})$ and $\overline{T} = T + \mathcal{I}(R(T))/\mathcal{I}(R(T)) \cong T/(T \cap \mathcal{I}(R(T))) = T/R(T)$, it follows that $\overline{T}$ is a semisimple Lie triple system, so $R(\overline{T}) = 0$. Hence

$$\text{Rad}(U(T)) \subseteq \text{Rad}(U(\overline{T})) = \text{Rad}(U(\overline{T})) = \mathcal{J}(R(\overline{T})) + \overline{\theta(0)} = K(T),$$

which implies $\text{Rad}(U(T)) \subseteq K(T) + \mathcal{I}(R(T))$. On the other hand, note that $K(T) = \mathcal{J}(0) \subseteq \mathcal{J}(R(T))$ and $\mathcal{I}(R(T)) \subseteq \mathcal{J}(R(T))$, so $\text{Rad}(U(T)) = \mathcal{I}(R(T)) + K(T) = \mathcal{J}(R(T))$. \qed

**Theorem 4.11.** If $U(T)$ is a semisimple Leibniz algebra, then $T$ is a semisimple LeibTS; if $T$ is a semisimple LeibTS with $K(T) = 0$, then $U(T)$ is a semisimple Leibniz algebra.
Proof. If \( U(T) \) is semisimple, then \( \text{Rad}(U(T)) = \text{Ker}(U(T)) \), so
\[
R(T) = \text{Rad}(U(T)) \cap T = \text{Ker}(U(T)) \cap T = \text{Ker}(T),
\]
that is, \( T \) is semisimple.

If \( T \) is semisimple, then \( R(T) = \text{Ker}(T) \), which implies \( \text{Rad}(U(T)) = \mathcal{I}(\text{Ker}(T)) = \mathcal{J}(\text{Ker}(T)) = \text{Ker}(U(T)) \). Hence \( U(T) \) is semisimple. \( \square \)

5 Levi’s theorem for Leibniz triple systems

Definition 5.1. [8] A vector space \( V \) is called a **module** for a Lie triple system \( T \) if the vector space direct sum \( T + V \) is itself a Lie triple system such that (1) \( T \) is a subsystem, (2) \( [abc] \) lies in \( V \) if any one of \( a, b, c \) is in \( V \), (3) \( [abc] = 0 \) if any two of \( a, b, c \) are in \( V \).

Definition 5.2. [16] Let \( T \) be a Lie triple system and \( V \) a vector space. \( V \) is called a **\( T \)-module** if there exists a bilinear map \( \delta : T \times T \to \text{End}(V) \) such that for all \( a, b, c, d \in T \),
\[
\delta(c,d)\delta(a,b) - \delta(b,d)\delta(a,c) - \delta(a,[bcd]) + D(b,c)\delta(a,d) = 0,
\]
\[
\delta(c,d)D(a,b) - D(a,b)\delta(c,d) + \delta([abc],d) + \delta(c,[abd]) = 0,
\]
where \( D(a,b) = \delta(b,a) - \delta(a,b) \).

It is not difficult to prove that the above two definitions about the module for Lie triple systems are equivalent.

Theorem 5.3. [8] If \( T \) is a finite-dimensional semisimple Lie triple system over a field of characteristic 0, then every finite-dimensional module \( V \) is completely reducible.

Theorem 5.4 (Levi’s theorem for Lie triple systems). [12] If \( T \) is a Lie triple system, then there is a semisimple subsystem \( S \) such that \( T = R(T) \perp S \), where \( R(T) \) is the solvable radical of \( T \).

Theorem 5.5 (Levi’s theorem for Leibniz triple systems). Let \( T \) be a LeibTS and \( R(T) \) its solvable radical. Then there is a semisimple subsystem \( S \) of \( T \) such that \( T = S \perp R(T) \). In particular, \( S \) is a semisimple Lie triple system.

Proof. By Theorem 4.7, \( \overline{T} = T/\text{Ker}(T) \) is a Lie triple system. Then there are a semisimple subsystem \( \overline{S} \) and the solvable radical \( \overline{R} \) of \( \overline{T} \) such that \( \overline{T} = \overline{S} \perp \overline{R} \) from Levi’s theorem for Lie triple systems. Let \( \pi : T \to \overline{T} \) be the canonical projection. Then \( \pi(R(T)) \) is solvable since \( R(T) \) is solvable, hence \( \pi(R(T)) \subseteq \overline{R} \).

Conversely, denote by \( \pi^{-1}(\overline{R}) \) the inverse image of \( \overline{R} \). Then \( \pi^{-1}(\overline{R}) \) is a subsystem of \( T \). Since \( \text{Ker}\pi = \text{Ker}(T) \), \( \pi^{-1}(\overline{R})/\text{Ker}(T) \cong \overline{R} \). Note that \( \text{Ker}(T) \) and \( \overline{R} \) are both solvable, which implies that \( \pi^{-1}(\overline{R}) \) is solvable, then \( \pi^{-1}(\overline{R}) \subseteq R(T) \), and so \( \overline{R} \subseteq \pi(R(T)) \). Thus \( \pi(R(T)) = \overline{R} \) and \( R(T) = \pi^{-1}(\overline{R}) \).
Denote by $F = \pi^{-1}(\overline{S})$ the inverse image of $\overline{S}$. Then $F$ is a subsystem of $T$ and $\overline{S} = F/Ker(T)$. Since $\overline{T} = T/Ker(T)$ and $\overline{R} = R(T)/Ker(T)$, it follows that

$$T/Ker(T) = F/Ker(T) + R(T)/Ker(T).$$

If $a \in F \cap R(T)$, then $\bar{a} \in (F/Ker(T)) \cap (R(T)/Ker(T)) = \overline{S}$, which implies $a \in Ker(T)$, hence $F \cap R(T) = Ker(T)$. So

$$T = F + R(T), \quad F \cap R(T) = Ker(T).$$

Define a bilinear map $\delta : \overline{S} \times \overline{S} \to \text{End}(F)$ by $\delta(\bar{s}_1, \bar{s}_2)(a) = \{as_1s_2\}$. Then $\{as_1s_2\} \in F$ since $\overline{S} = F/Ker(T)$ and $F$ is a subsystem of $T$. If $\bar{s}_1 = \bar{s}_1'$ and $\bar{s}_2 = \bar{s}_2'$, then $s_1 - s_1', s_2 - s_2' \in Ker(T)$, it follows from $\{TTKer(T)\} = \{TKer(T)T\} = 0$ that

$$\{as_1s_2\} - \{as_1's_2'\} = \{as_1s_2\} - \{as_1's_2\} + \{as_1's_2'\} - \{as_1's_2'\} = 0.$$

Hence $\delta$ is well defined. Let $D(\bar{s}_1, \bar{s}_2)(a) = \delta(\bar{s}_2, \bar{s}_1)(a) - \delta(\bar{s}_1, \bar{s}_2)(a) = \{as_2s_1\} - \{as_1s_2\}$. Then by (2.3) and (2.6),

$$\begin{align*}
\delta(s_3, s_4)\delta(s_1, s_2)(a) - \delta(s_2, s_4)\delta(s_1, s_3)(a) - \delta(s_2, \{s_2s_3s_4\})(a) + D(\bar{s}_2, \bar{s}_3)\delta(\bar{s}_1, \bar{s}_4)(a) &= \{as_1s_2\} - \{as_1s_3s_4\} - \{as_1s_2s_3s_4\} + \{as_1s_4s_3s_2\} - \{as_1s_4s_2s_3\} \\
&= 0, \\
\delta(s_3, s_4)D(\bar{s}_1, \bar{s}_2)(a) - D(\bar{s}_1, \bar{s}_2)\delta(s_3, s_4)(a) + \delta(\{s_1s_2s_3\}, s_4)(a) + \delta(s_3, \{s_1s_2s_4\})(a) &= \{as_2s_1\} - \{as_1s_2s_3s_4\} - \{as_1s_2s_3s_4\} + \{as_1s_3s_4s_2\} \\
&\quad + \{as_1s_2s_3s_4\} + \{as_1s_2s_3s_4\} - \{as_1s_2s_3s_4\} + \{as_1s_2s_3s_4\} \\
&= 0.
\end{align*}$$

Thus $F$ is an $\overline{S}$-module. By Theorem 5.3, $F$ is a completely reducible $\overline{S}$-module since $\overline{S}$ is a semisimple Lie triple system. Note that $Ker(T)$ is an ideal of $F$. Then $Ker(T)$ is a submodule of $F$. Hence by the completely reducibility of $F$, there is a complementary submodule $S$ of $F$ such that $F = S + Ker(T)$. Let $\sigma : F \to S$ be the canonical projection. Then $Ker(\sigma) = Ker(T)$ and $S \cong F/Ker(T)$ is a semisimple Lie triple system such that

$$T = F + R(T) = S + Ker(T) + R(T) = S + R(T).$$

Similarly, $R(T)$ is an $\overline{S}$-module and there is a complementary submodule $R'$ of $R(T)$ such that $R(T) = R' + Ker(T)$. If $a \in S \cap R(T)$, then $a = a' + t$, for some $a' \in R'$ and $t \in Ker(T)$. Then $a' \in S \subseteq F$, which implies $a' \in R(T) \cap F = Ker(T)$. Moreover, $a' \in R' \cap Ker(T) = 0$. Hence $a = t \in Ker(T) \cap S = 0$, i.e., $R(T) \cap S = 0$. Therefore, $T = S + R(T)$ with $S$ both a Lie triple system and a semisimple subsystem of $T$. \qed
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