I. INTRODUCTION

The asymptotic elimination of fast time scales from the Vlasov equation presents important analytical and computational advantages for its solution in complex plasma geometries \[1\]. Here, fast time scales in a collisionless plasma are either associated with particle orbital dynamics (e.g., the fast gyromotion of a charged particle about a magnetic field line) or wave-particle dynamics (e.g., the fast oscillatory motion of a charged particle in the presence of a high-frequency electromagnetic wave). In the former case, the elimination of a fast orbital time scale is often associated with the construction of an adiabatic (action) invariant (e.g., the magnetic moment of a charged particle in a strong magnetic field). Next, the construction of a reduced Vlasov equation, from which one or more fast time scales have been removed, can either be obtained through an iterative solution of a perturbatively-expanded Vlasov equation \[2\] or by performing one or more near-identity phase-space transformations resulting from applications of Hamiltonian perturbation theory \[3\]. The present paper focuses on applications of Lie-transform Hamiltonian perturbation theory.

The most general setting for carrying out Hamiltonian perturbation theory \[4\] on the Vlasov-Maxwell equations is to use an eight-dimensional extended phase space with coordinates \([z; w; t]\), where \(z\) denotes regular (six-dimensional) phase-space coordinates, \(w\) denotes the energy coordinate, and \(t\) denotes its canonically-conjugate coordinate. The extended Hamilton’s equations (\(\tau\) denotes the Hamiltonian-orbit parameter)

\[
\frac{dZ^a}{d\tau} = \{Z^a, H\} = J^{ab}(Z) \frac{\partial H(Z)}{\partial Z^b}, \tag{1}
\]

are expressed in extended phase space in terms of an extended Hamiltonian \(H(Z) \equiv H(z, t) - w\), where \(H(z, t)\) denotes the regular Hamiltonian, and the extended phase-space Lagrangian \(\Gamma \equiv \Gamma_z \ dZ^a\), which is a differential one-form in extended phase space (summation over repeated indices is, henceforth, implied). Note that the physical motion in extended phase space takes place on the subspace \(H \equiv 0\):

\[
w = H(z, t). \tag{2}
\]

The extended Poisson bracket \(\{\ , \\}\) is obtained from the extended phase-space Lagrangian \(\Gamma\), first, by constructing the Lagrange matrix \(\omega\) (with antisymmetric components \(\omega_{ab} \equiv \partial_a \Gamma_b - \partial_b \Gamma_a\) associated with the differential two-form \(\omega = d\Gamma = \frac{1}{2} \omega_{ab} dZ^a \wedge dZ^b\) and, second, by inverting the Lagrange matrix to obtain the Poisson matrix \(J \equiv \omega^{-1}\) with antisymmetric components \(J^{ab} \equiv \{Z^a, Z^b\}\). Hence, we obtain the extended Poisson bracket defined in terms of two arbitrary functions \(F\) and \(G\) as \(\{F, G\} \equiv \partial_a F J^{ab} \partial_b G\).

The extended Vlasov equation is expressed in terms of the extended Vlasov distribution \(F(Z)\) and the extended Hamilton’s equations \(\tag{1}\) as

\[
0 = \frac{dF}{dt} = \frac{dZ^a}{d\tau} \frac{\partial F(Z)}{\partial Z^a} \equiv \{F, H\}. \tag{3}
\]

In order to satisfy the physical constraint \(\tag{2}\), the extended Vlasov distribution is required to be of the form

\[
F(Z) \equiv c \delta([w - H(z, t)] f(z, t), \tag{4}
\]

where \(f(z, t)\) denotes the time-dependent Vlasov distribution on regular phase space. By integrating the extended Vlasov equation \(\tag{4}\) over the energy coordinate \(w\) (and using \(d\tau = dt\) ), we obtain the regular Vlasov equation

\[
0 = \frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \{f, H\}. \tag{5}
\]

Note that we use the same symbol \(\{\ , \\}\) for the Poisson bracket on regular phase space in Eq. \(\tag{5}\) since \(w\)-derivatives appearing in the extended Poisson bracket vanish identically on regular phase space.

Next, the extended Vlasov equation \(\tag{5}\) is coupled with Maxwell’s equations for the self-consistent electromagnetic fields

\[
\nabla \cdot E = 4\pi \rho \quad \text{and} \quad \nabla \times B - \frac{1}{c} \frac{\partial E}{\partial t} = \frac{4\pi}{c} J, \tag{6}
\]
where the charge-current densities
\[
\left( \begin{array}{c} e \, \mathbf{j} \\ \mathbf{c} \end{array} \right) = \sum e \int d^4 p \, \mathcal{F} \left( \begin{array}{c} e \\ \mathbf{v} \end{array} \right) \\
= \sum e \int d^4 p \, f \left( \begin{array}{c} e \\ \mathbf{v} \end{array} \right)
\]
are defined in terms of moments of the extended Vlasov distribution \( \mathcal{F} \) (with \( d^4 p = c^{-1} dw \, dp \)) and the electric and magnetic fields \( \mathbf{E} \equiv -\nabla \Phi - c^{-1} \partial \mathbf{A} / \partial t \) and \( \mathbf{B} \equiv \nabla \times \mathbf{A} \) satisfy
\[
\nabla \cdot \mathbf{B} = 0 \quad \text{and} \quad \nabla \times \mathbf{E} + c^{-1} \partial \mathbf{A} / \partial t = 0.
\]

The purpose of this paper is to show that: the asymptotic elimination of a fast time scale from the Vlasov equation \( \mathcal{F} \) introduces polarization and magnetization effects into the Maxwell equations \( \mathcal{A} \); and the reduced Vlasov-Maxwell equations possess exact conservation laws that can be derived from a variational principle by Noether method.

The remainder of this paper is organized as follows. In Sec. \( \text{II} \) we present a brief summary of the Lie-transform perturbation method used in the asymptotic elimination of a fast time scale from the Vlasov equation and its associated Hamiltonian dynamics. In Sec. \( \text{III} \) we present the reduced Vlasov-Maxwell equations introduced by the dynamical reduction associated with a near-identity phase-space transformation. Through the use of the push-forward representation of particle fluid moments, we present expressions for the charge-current densities involving momentum-space moments of the reduced Vlasov distribution and present explicit expressions for the reduced polarization charge-current densities and the divergenceless reduced magnetization current density. In Sec. \( \text{IV} \) we present the variational derivation of the reduced Vlasov-Maxwell equations and derive, through the Noether method, the exact reduced energy-momentum conservation laws (explicit proofs are presented in Appendix \( \text{A} \)). Lastly, in Sec. \( \text{V} \) we summarize the work present here and briefly discuss applications.

II. DYNAMICAL REDUCTION BY NEAR-IDENTITY PHASE-SPACE TRANSFORMATION

A. Near-identity Phase-space Transformation

The process by which a fast time scale is removed from Hamilton’s equations \( \mathcal{F} \) involves a near-identity transformation \( T_\epsilon : \mathcal{Z} \rightarrow \mathcal{Z}(\mathcal{Z}; \epsilon) \equiv T_\epsilon \mathcal{Z} \) on extended particle phase space, where
\[
\mathcal{Z}^\epsilon(\mathcal{Z}; \epsilon) = \mathcal{Z}^a + \epsilon G_1^a + \epsilon^2 \left( G_2^a - \frac{G_1^b \partial G_1^b}{2} \right) + \cdots.
\]
and its inverse near-identity transformation \( T_{\epsilon}^{-1} : \mathcal{Z} \rightarrow \mathcal{Z}(\mathcal{Z}; \epsilon) \equiv T_{\epsilon}^{-1} \mathcal{Z} \), where
\[
\mathcal{Z}^a(\mathcal{Z}, \epsilon) = \mathcal{Z}^a - \epsilon G_1^a - \epsilon^2 \left( G_2^a - \frac{G_1^b \partial G_1^b}{2} \right) + \cdots.
\]

In Eqs. (9)-(10), the dimensionless ordering parameter \( \epsilon \ll 1 \) is defined as the ratio of the fast time scale over a slow time scale of interest, and the \( n \)-th order generating vector field \( G_n \) is chosen to remove the fast time scale at order \( \epsilon^n \) from the perturbed Hamiltonian dynamics. Examples of asymptotic elimination of fast time scales by Lie-transform perturbation method include guiding-center Hamiltonian theory \( \mathcal{H} \), gyrocenter Hamiltonian theory \( \mathcal{H} \), and oscillation-center Hamiltonian theory \( \mathcal{H} \).

B. Pull-back and Push-forward Operators

Next, we define the pull-back operator on scalar fields induced by the near-identity transformation \( \mathcal{F} \):
\[
T_\epsilon : \mathcal{F} \rightarrow \mathcal{F} \equiv T_\epsilon \mathcal{F},
\]
i.e., the pull-back operator \( T_\epsilon \) transforms a scalar field \( \mathcal{F} \) on the phase space with coordinates \( \mathcal{Z} \) into a scalar field \( \mathcal{F} \) on the phase space with coordinates \( \mathcal{Z} \):
\[
\mathcal{F}(\mathcal{Z}) = T_\epsilon \mathcal{F}(\mathcal{Z}) = \mathcal{F}(T_\epsilon \mathcal{Z}) = \mathcal{F}(\mathcal{Z}).
\]
Using the inverse transformation \( \mathcal{F} \), we also define the push-forward operator:
\[
T_{\epsilon}^{-1} : \mathcal{F} \rightarrow \mathcal{F} \equiv T_{\epsilon}^{-1} \mathcal{F},
\]
i.e., the push-forward operator \( T_{\epsilon}^{-1} \) transforms a scalar field \( \mathcal{F} \) on the phase space with coordinates \( \mathcal{Z} \) into a scalar field \( \mathcal{F} \) on the phase space with coordinates \( \mathcal{Z} \):
\[
\mathcal{F}(\mathcal{Z}) = T_{\epsilon}^{-1} \mathcal{F}(\mathcal{Z}) = \mathcal{F}(T_{\epsilon}^{-1} \mathcal{Z}) = \mathcal{F}(\mathcal{Z}).
\]
Note that both induced transformations \( \mathcal{F} \) and \( \mathcal{F} \) satisfy the scalar-invariance property \( \mathcal{F}(\mathcal{Z}) = \mathcal{F}(\mathcal{Z}) \).

In Lie-transform perturbation theory \( \mathcal{F} \), the pull-back and push-forward operators \( \mathcal{F} \) and \( \mathcal{F} \) are expressed as Lie transforms: \( \mathcal{F}^{\pm 1} \equiv \exp(\pm \sum_{n=1}^\infty \epsilon^n L_n) \) defined in terms of the Lie derivative \( L_n \) generated by the \( n \)-th-order vector field \( G_n \), which appear in the \( n \)-th order terms found in the near-identities \( \mathcal{F} \) and \( \mathcal{F} \). A Lie derivative is a special differential operator that preserves the tensorial nature of the object it operates on \( \mathcal{F} \). For example, the Lie derivative \( L_n \) of the scalar field \( \mathcal{H} \) is defined as the scalar field \( L_n \mathcal{H} \equiv \nabla_b \partial_n \mathcal{H} \), while the Lie derivative \( L_n \) of a one-form \( \Gamma \equiv \Gamma_a \, d^2 \) is defined as the one-form
\[
L_n \Gamma \equiv G_n^{a} \omega_{ab} \, d^2 b + d(G_n^{a} \Gamma_a),
\]
where \( \omega_{ab} \equiv \partial_a \Gamma_b - \partial_b \Gamma_a \) are the components of the two-form \( \omega \equiv d \Gamma \).

The pull-back and push-forward operators \( \mathcal{F} \) can now be used to transform an arbitrary operator
The new extended phase-space coordinates are chosen are specified at each order $n$ are independent of the fast time scale.

$$C : F(Z) \rightarrow C[F](Z)$$ acting on the extended Vlasov distribution function $F$. First, since $C[F](Z)$ is a scalar field, it transforms to $T^{-1}_n[C[F]](Z)$ with the help of the push-forward operator $\mathcal{F}$. Next, we replace the extended Vlasov distribution function $F$ with its pull-back representation $F = T_e F$ and, thus, define the transformed operator

$$C_e[F] \equiv T^{-1}_e[C[T_e F]].$$

(13)

By applying the induced transformation on the extended Vlasov operator $d/d\tau$ defined in Eq. 3, we obtain

$$d_e F = T^{-1}_e \left( \frac{d}{d\tau} T_e F \right) = \{F, \mathcal{C}_e\},$$

(14)

where the total derivative $d_e/d\tau$ along the transformed particle orbit is defined in terms of the transformed Hamiltonian

$$\mathcal{H}_e \equiv \Gamma^{-1}_e \mathcal{H},$$

(15)

and the transformed Poisson bracket

$$\{F, C\}_e = T^{-1}_e(\{T_e F, T_e C\}).$$

(16)

The Poisson-bracket transformation $\{ , \} \rightarrow \{ , \}_e$ can also be performed through the transformation of the extended phase-space Lagrangian, $T_e \equiv \Gamma^{-1}_e \Gamma + dS$, is expressed as $\mathcal{F}$

$$T_e = \Gamma_0 + \epsilon \left( \Gamma_1 + dS_1 - L_1 \Gamma_0 \right) + \epsilon^2 \left( \Gamma_2 + dS_2 - L_2 \Gamma_0 - L_1 \Gamma_1 + \frac{1}{2} L_1^2 \Gamma_0 \right) + \cdots,$$

where $S \equiv \epsilon S_1 + \epsilon^2 S_2 + \cdots$ denotes a (canonical) scalar field used to simplify the transformed phase-space Lagrangian $T_e$ at each order $\epsilon^n$ in the perturbation analysis. Note that the choice of $S$ has no impact on the new Poisson-bracket structure

$$\varpi_e = \mathcal{D}_e = d(\Gamma^{-1}_e \Gamma) = \Gamma^{-1}_e d\Gamma \equiv \Gamma^{-1}_e \omega,$$

since $d^2 S = 0$ (i.e., $\partial_{ab}^2 S - \partial_{ab} S = 0$) and $\Gamma^{-1}$ commutes with $d$. By inverting the reduced Lagrange matrix $\varpi_e = \Gamma^{-1}_e \omega \rightarrow \varpi_e \equiv \varpi^{-1}_e$, we thus, obtain the reduced Poisson matrix $\Gamma_e$, with antisymmetric components $\mathcal{J}_e \equiv \{z^a, \mathcal{Z}\}_e$, and define the reduced Poisson bracket $\{F, C\}_e \equiv \partial_a F \mathcal{J}_e^{ab} \partial_b C$. Lastly, we note that the extended-Hamiltonian transformation may be re-expressed in terms of the regular Hamiltonians $H$ and $\mathcal{H} \equiv \Gamma^{-1}_e H - \partial S/\partial t$ as $\mathcal{F}$

$$\mathcal{H} = H_0 + \epsilon \left( H_1 - L_1 H_0 - \frac{\partial S_1}{\partial t} \right) + \epsilon^2 \left( H_2 - L_2 H_0 - L_1 H_1 + \frac{1}{2} L_1^2 H_0 - \frac{\partial S_2}{\partial t} \right) + \cdots.$$

The new extended phase-space coordinates are chosen (i.e., the generating vector field $G_n$ and the scalar field $S_n$ are specified at each order $n = 1, 2, \ldots$ in the perturbation analysis) so that $d_\tau \mathcal{Z} / d\tau = \{\mathcal{Z}, \mathcal{H}\}_e$ are independent of the fast time scale.

## III. REDUCED VLASOV-MAXWELL EQUATIONS

### A. Reduced Vlasov Equation

The push-forward transformation of the extended Vlasov distribution $\mathcal{F}$ yields the reduced extended Vlasov distribution

$$\mathcal{F}(Z) \equiv \mathcal{C}(\varpi - \mathcal{H}(Z, t)) \mathcal{J}(Z, t),$$

(17)

where the reduced extended Hamiltonian $\mathcal{H} \equiv \mathcal{H}(Z, t) - \varpi$ is defined in Eq. 15. The extended reduced Vlasov equation

$$\frac{d_e F}{d\tau} = \{F, \mathcal{H}\}_e = 0$$

(18)

can be converted into the regular reduced Vlasov equation by integrating it over the reduced energy coordinate $\varpi$, which yields the reduced Vlasov equation

$$0 = \frac{d_e \mathcal{F}}{d\tau} = \frac{\partial d_e \mathcal{F}}{\partial \varpi} + \{F, \mathcal{H}\}_e,$$

(19)

where $\mathcal{F}(Z, t)$ denotes the time-dependent reduced Vlasov distribution on the new reduced phase space. Hence, we see that the pull-back and push-forward operators play a fundamental role in the transformation of the Vlasov equation to the reduced Vlasov equation.

### B. Reduced Maxwell Equations

We now investigate how the pull-back and push-forward operators are used in the transformation of Maxwell’s equations. The charge-current densities can be expressed in terms of the general expression (where time dependence is omitted for clarity)

$$J^\mu(r) = \sum e \int d^3 x \int d^4 p \, v^\mu \delta^3(\mathbf{x} - \mathbf{r}) \mathcal{F},$$

(20)

where the delta function $\delta^3(\mathbf{x} - \mathbf{r})$ means that only particles whose positions $\mathbf{x}$ coincide with the field position $\mathbf{r}$ contribute to the moment $J^\mu(r)$. By applying the extended (time-dependent) phase-space transformation $T_e : Z \rightarrow \mathcal{Z}$ (where time $t$ itself is unaffected) on the right side of Eq. 20, we obtain the push-forward representation for $J^\mu$:

$$J^\mu(r) = \sum e \int d^3 \mathbf{x} d^4 p \, (\mathcal{T}_e^{-1} \mathcal{V}^\mu) \delta^3(\mathbf{x} + \mathbf{r} - \mathbf{r}) \mathcal{F}$$

$$= \sum e \int d^4 \mathbf{p} \left[ (\mathcal{T}_e^{-1} \mathcal{V}^\mu) \mathcal{F} - \nabla \cdot (\mathbf{p} \mathcal{T}_e^{-1} \mathcal{V}^\mu \mathcal{F}) + \cdots \right],$$

(21)

where $\mathcal{T}_e^{-1} \mathcal{V}^\mu = (e, \mathcal{T}_e^{-1} \mathbf{v})$ denotes the push-forward of the particle four-velocity $\mathbf{v}^\mu$ and the displacement $\mathbf{\rho}_e \equiv \mathcal{T}_e^{-1} \mathbf{x}$. 

The total derivative $d_e / d\tau$ along the transformed particle orbit is defined in terms of the transformed Hamiltonian $\mathcal{H}_e \equiv \Gamma^{-1}_e \mathcal{H}$, with antisymmetric components $\mathcal{J}_e \equiv \{z^a, \mathcal{Z}\}_e$, and define the reduced Poisson bracket $\{F, C\}_e \equiv \partial_a F \mathcal{J}_e^{ab} \partial_b C$. Lastly, we note that the extended-Hamiltonian transformation may be re-expressed in terms of the regular Hamiltonians $H$ and $\mathcal{H} \equiv \Gamma^{-1}_e H - \partial S/\partial t$ as $\mathcal{F}$

$$\mathcal{H} = H_0 + \epsilon \left( H_1 - L_1 H_0 - \frac{\partial S_1}{\partial t} \right) + \epsilon^2 \left( H_2 - L_2 H_0 - L_1 H_1 + \frac{1}{2} L_1^2 H_0 - \frac{\partial S_2}{\partial t} \right) + \cdots.$$
\[ T^{-1}_c \mathbf{x} - \bar{\mathbf{x}} \] between the push-forward \( T^{-1}_c \mathbf{x} \) of the particle position \( \mathbf{x} \) and the (new) reduced position \( \bar{\mathbf{x}} \) is expressed as

\[ \rho_e = - \varepsilon G_1^x - \varepsilon^2 \left( G_2^x - \frac{1}{2} G_1 \cdot dG_1^x \right) + \cdots \]  

(22)

in terms of the generating vector fields \( (G_1, G_2, \cdots) \) associated with the near-identity transformation.

The push-forward representation for the charge-current densities, therefore, naturally introduces polarization and magnetization effects into the Maxwell equations. Hence, the microscopic Maxwell’s equations (6) are transformed into the macroscopic (reduced) Maxwell equations:

\[ \nabla \cdot \mathbf{D} = 4\pi \bar{\rho}, \quad (23) \]

\[ \nabla \times \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c} \mathbf{J}, \quad (24) \]

where the reduced charge-current densities \( \mathcal{J}^e = (c \bar{\rho}, \bar{\mathbf{J}}) \) are defined as moments of the reduced Vlasov distribution \( \mathcal{F} \):

\[ (c \bar{\rho}, \bar{\mathbf{J}}) = \sum e \int d^4p \mathcal{F} \left( c, \frac{d \bar{\mathbf{x}}}{dt} \right), \quad (25) \]

where \( d \bar{\mathbf{x}}/dt \) denotes the reduced (e.g., guiding-center) velocity. The microscopic electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) are, thus, replaced by the macroscopic fields

\[ \mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}_e, \quad \mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}_e, \quad (26) \]

where \( \mathbf{P}_e \) and \( \mathbf{M}_e \) denote the polarization and magnetization vectors associated with the dynamical reduction introduced by the phase-space transformation \( \mathcal{L} \).

C. Push-forward Representation of Charge-Current Densities

We now derive explicit expressions for the reduced polarization \( \mathbf{P}_e \) and the reduced magnetization \( \mathbf{M}_e \) by using the push-forward representation method. First, we derive the push-forward representation \( \mathcal{L} \) for the charge density \( (\mathcal{J}^0 = c \rho) \):

\[ \rho = \sum e \int d^4p \mathcal{F} - \nabla \cdot \left( \sum e \int d^4p \rho_e \mathcal{F} + \cdots \right) = \bar{\rho} - \nabla \cdot \mathbf{P}_e, \quad (27) \]

where \( \bar{\rho} = \sum e \int d^4p \mathcal{F} \) denotes the reduced charge density that appears in Eq. (28) and the polarization vector is defined as

\[ \mathbf{P}_e = \sum e \int d^4p \left[ \rho_e \mathcal{F} - \frac{1}{2} \nabla \cdot (\rho_e \mathcal{F}) + \cdots \right], \quad (28) \]

where the quadrupole contribution \( \frac{1}{4} \nabla \cdot (\rho_e \mathcal{F}) \) (which will be useful in what follows) is retained in Eq. (28), while the reduced electric-dipole moment (for each particle species)

\[ \pi_e = e \bar{\rho}_e \quad (29) \]

is associated with the fast-time-averaged charge separation induced by the near-identity phase-space transformation.

Secondly, we derive the push-forward expression for the current density \( \mathbf{J} \), where the push-forward of the particle velocity \( \mathbf{v} = d \mathbf{x}/dt \) (using the Lagrangian representation)

\[ \mathbf{T}_c^{-1} \dot{\mathbf{x}} = \mathbf{T}_c^{-1} \frac{d \mathbf{x}}{dt} = \left[ \mathbf{T}_c^{-1} \frac{d \mathbf{x}}{dt} \right] \mathbf{T}_c^{-1} \mathbf{x} \]

\[ \equiv \frac{d \mathbf{x}}{dt} + \frac{d \mathbf{\rho}^e}{dt} \quad (30) \]

is expressed in terms of the reduced velocity \( d \bar{\mathbf{x}}/dt \), which is independent of the fast time scale, and the particle polarization velocity \( d \mathbf{\rho}_e/dt \), which has both fast and slow time dependence. Note that the fast-time-average particle polarization velocity \( d \mathbf{\rho}_e/dt \), which is nonvanishing under certain conditions, represents additional reduced dynamical effects (e.g., the standard polarization drift in guiding-center theory \[11,12\]) not included in \( d \mathbf{\bar{m}}/dt \). Hence, the push-forward expression \( \mathcal{L} \) for the current density \( \mathbf{J} \) is

\[ \mathbf{J} = \sum e \int d^4p \left[ \frac{d \mathbf{\bar{x}}}{dt} + \frac{d \mathbf{\rho}_e}{dt} \right] \mathcal{F} - \nabla \cdot \left[ \sum e \int d^4p \rho_e \left( \frac{d \mathbf{\bar{x}}}{dt} + \frac{d \mathbf{\rho}_e}{dt} \right) \mathcal{F} + \cdots \right] \quad (31) \]

We may now replace the polarization velocity \( d \rho_e/dt \) in Eq. (31) by using the following identity based on the reduced polarization vector \( \mathbf{P}_e \):

\[ \frac{d \mathbf{P}_e}{dt} = \sum e \int d^4p \left( \frac{d \mathbf{\rho}_e}{dt} \right) \mathcal{F} - \nabla \cdot \left[ \sum e \int d^4p \rho_e \left( \frac{d \mathbf{\bar{x}}}{dt} \right) \mathcal{F} \right] + \frac{1}{2} \frac{d \mathbf{\bar{x}}}{dt} \left( \rho_e, \rho_e \right) + \cdots \quad (32) \]

where the reduced Vlasov equation \( \mathcal{L} \) was used and integration by parts was performed. Using the vector identity \( \nabla \cdot (\mathbf{BA} - \mathbf{AB}) = \nabla \times (\mathbf{A} \times \mathbf{B}) \), the push-forward representation for the current density is, therefore, expressed as

\[ \mathbf{J} \equiv \mathcal{J} + \frac{\partial \mathbf{P}_e}{\partial t} + e \nabla \times \mathbf{M}_e, \quad (33) \]

where \( \mathcal{J} \equiv \sum e \int d^4p (d \mathbf{\bar{x}}/dt) \mathcal{F} \) denotes the reduced current density appearing in Eq. (28), \( \mathbf{J}_{\text{pol}} \equiv \partial \mathbf{P}_e/\partial t \) denotes the reduced polarization current, and the divergenceless reduced magnetization current \( \mathbf{J}_{\text{mag}} \equiv \nabla \times \mathbf{M}_e \).
\( c \nabla \times \mathbf{M}_e \) is expressed in terms of the reduced magnetization vector

\[
\mathbf{M}_e = \sum_e \frac{e}{c} \int d^4 \mathbf{p}_e \mathbf{p}_e \times \left( \frac{1}{2} \frac{d_e \mathbf{p}_e}{dt} + \frac{d_e \mathbf{x}_e}{dt} \right) \mathcal{F},
\]

which represents the sum (for each particle species) of the intrinsic (fast-time-averaged) magnetic-dipole moment

\[
\mu_e = \frac{e}{2c} \left( \mathbf{p}_e \times \frac{d_e \mathbf{x}_e}{dt} \right),
\]

and a moving electric-dipole contribution \( \pi_e \times \frac{d_e \mathbf{x}_e}{dt} \), as suggested by classical electromagnetic theory. Here, we present a variational principle for reduced \( \mathbf{M}_e \), previously by Pfirsch \([13]\) and Pfirsch and Morrison \([14]\) using the Hamilton-Jacobi formulation and by Kaufman et al. \([10, 16, 17, 18, 19]\) using the Low-Lagrangian formalism. Here, we present a variational principle for reduced Vlasov-Maxwell equations based on the reduced Lagrangian density \([20]\)

\[
\mathcal{L} = \frac{\mathbf{F} : \mathbf{F}}{16\pi} - \sum \int d^4 \mathbf{p} \mathcal{F} \widetilde{\mathbf{H}},
\]

where \( \mathcal{F} \) and \( \widetilde{\mathbf{H}} \) denote the reduced extended Vlasov distribution and the reduced extended Hamiltonian, respectively. In this Section, we use the convenient spacetime metric \( g^{\mu\nu} = \text{diag}(-1, 1, 1, 1) \), so that the electromagnetic field tensor \( F_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) is defined in terms of the four-potential \( A_{\mu} = (-\Phi, \mathbf{A}) \) and, thus, \( F_{0} = E_t \) and \( F_{ij} = \epsilon_{ijk} B^k \). In order to simplify our presentation, we use canonical four-momentum coordinates \( p_\mu = (-\mathbf{p}/c, \mathbf{p}) \), so that the reduced extended Hamiltonian \( \widetilde{\mathbf{H}} \) is required to be invariant under the gauge transformation

\[
A_{\mu} \rightarrow A_{\mu} + \partial_{\mu} \chi \quad \text{and} \quad p_\mu \rightarrow p_\mu + (e/c) \partial_{\mu} \chi,
\]

where \( \chi \) denotes the electromagnetic gauge field.

Note that, as a result of the dynamical reduction of the Vlasov equation, the reduced Hamiltonian \( \widetilde{\mathbf{H}} \) is not only a function of the four-potential \( A_{\mu} \) but also of the field tensor \( F_{\mu\nu} \). From these dependences, we express the reduced four-current density \([20]\) as

\[
\mathcal{J}^\mu = (c \mathcal{F}, \mathbf{J}) \equiv -c \sum \int d^4 \mathbf{p} \mathcal{F} \frac{\partial \widetilde{\mathbf{H}}}{\partial A_{\mu}},
\]

and we introduce the reduced antisymmetric polarization-magnetization tensor \([17]\)

\[
K^{\mu\nu} \equiv - \sum \int d^4 \mathbf{p} \mathcal{F} \frac{\partial \widetilde{\mathbf{H}}}{\partial F_{\mu\nu}},
\]

where the reduced polarization and magnetization vectors \( \mathbf{K}^{\Phi} = P_\mu^\Phi \) and \( \mathbf{K}^{\mathbf{B}} = \epsilon_{ijk} M_{\mu k} \) are defined as

\[
(P_\mu^\Phi, M_\mu^\Phi) \equiv - \sum \int d^4 \mathbf{p} \mathcal{F} \left( \frac{\partial \widetilde{\mathbf{H}}}{\partial \mathcal{E}}, \frac{\partial \widetilde{\mathbf{H}}}{\partial \mathbf{B}} \right).
\]

We begin with an expression for the variation of the reduced Lagrangian density

\[
\delta \mathcal{L} = \frac{\partial \mathcal{J}^\mu}{\partial x_\nu} \equiv - \sum \int d^4 \mathbf{p} \mathcal{F} \left\{ \mathcal{F}, \mathcal{H} \right\}_\mu^\nu
\]

\[+ \frac{\delta A_\nu}{4\pi} \left( \frac{\partial}{\partial x_\mu} \left( F^{\mu\nu} - 4\pi K^{\mu\nu} \right) + \frac{4\pi}{c} \mathcal{J}^\nu \right),
\]

which is generated by the four-potential variation \( \delta A_\nu \) and the Eulerian variation \([20]\) for the reduced Vlasov distribution \( \delta \mathcal{F} \equiv - \delta \mathcal{Z} \frac{\partial \mathcal{F}}{\partial \mathcal{E}} = \{ \mathcal{S}, \mathcal{F} \}_\nu^\mu \), where \( \mathcal{S} \) is the generating scalar field for a virtual displacement on reduced phase space, \( \delta \mathcal{Z} \equiv \{ \mathcal{Z}, \mathcal{S} \}_\nu \). Note that the divergence term \( \delta \mu \mathcal{J}^\mu \), where the reduced Noether four-density is

\[
\mathcal{J}^\mu = \sum \int d^4 \mathbf{p} \mathcal{F} \frac{\partial \mathcal{F}}{\partial x_\nu} + \frac{\delta A_\nu}{4\pi} \left( F^{\mu\nu} - 4\pi K^{\mu\nu} \right),
\]

does not contribute to the variational principle \( \int d^4 x \delta \mathcal{L} = 0 \) but instead is used to derive exact conservation laws by applications of the Noether method.

Next, as a result of the variational principle, where the variations \( \mathcal{S} \) and \( \delta A_\nu \) are arbitrary (but are required to vanish on the integration boundaries), we obtain the reduced Vlasov equation \([13]\), \( \{ \mathcal{F}, \mathcal{H} \}_\nu^\mu = 0 \), and the reduced (macroscopic) Maxwell equations

\[
\frac{\partial}{\partial x_\mu} \left( F^{\mu\nu} - 4\pi K^{\mu\nu} \right) = -\frac{4\pi}{c} \mathcal{J}^\nu,
\]

from which we recover the reduced Maxwell equations \([20]\) and \([21]\). Here, the polarization-magnetization four-current is expressed in terms of the tensor \([33]\) as

\[
- \partial_\mu K^{\mu\nu} = \left( \nabla \cdot \mathbf{P}_e - e^{-1} \partial_t \mathbf{P}_e + \nabla \times \mathbf{M}_e \right)
\equiv \left( \rho_{\text{pol}}, e^{-1} J_{\text{pol}} + c^{-1} J_{\text{mag}} \right).
\]

Note that the electromagnetic field tensor also satisfies \( \partial_\mu F_{\mu\nu} + \partial_\nu F_{\mu\nu} + \partial_\nu F_{\mu\nu} = 0 \). In addition, we note that the reduced electric-dipole and reduced magnetic-dipole moments \([20]\) and \([33]\) are also expressed in terms of derivatives of the reduced Hamiltonian \( \widetilde{\mathbf{H}} \) as

\[
\pi_e \equiv - \frac{\partial \widetilde{\mathbf{H}}}{\partial \mathbf{E}} \quad \text{and} \quad \mu_e \equiv - \left( \frac{\partial \widetilde{\mathbf{H}}}{\partial \mathbf{B}} + \frac{1}{c} \frac{d \mathbf{x}}{dt} \times \frac{\partial \widetilde{\mathbf{H}}}{\partial \mathbf{E}} \right),
\]
which provides a useful consistency check on the reduced Vlasov-Maxwell equations. Lastly, the reduced charge conservation law

\[ 0 = \frac{\partial J^{\nu}}{\partial x^{\nu}} \equiv \frac{\partial \pi}{\partial t} + \nabla \cdot J \]  

(44)

follows immediately from the reduced Maxwell equations \[\text{(43)}\] as a result of the antisymmetry of \(F^{\mu\nu}\) and \(K^{\mu\nu}\) (i.e., \(\partial^2_{\mu\nu} F^{\mu\nu} \equiv 0 \equiv \partial^2_{\mu\nu} K^{\mu\nu}\)).

B. Reduced Energy-momentum Conservation Laws

We now derive the energy-momentum conservation law from the reduced Noether equation \(\delta \overline{L} \equiv \partial_{\mu} \overline{J}^\mu\) associated with space-time translations generated by \(\delta x^\sigma \equiv (c \delta t, \delta x)\), where the variations \((\overline{S}, \delta A_\nu, \delta \overline{L})\) are

\[
\begin{align*}
\overline{S} &= \overline{\pi}_\sigma \delta x^\sigma \\
\delta A_\nu &= F_{\nu\sigma} \delta x^\sigma - \partial_\nu (A_\sigma \delta x^\sigma) \\
\delta \overline{L} &= -\partial_\sigma (\overline{L} \delta x^\sigma)
\end{align*}
\]

(45)

Here, we note that \(\delta x^\mu = \{x^\nu, \overline{S}\}_\epsilon\), while the variations \(\delta A_\nu\) and \(\delta \overline{L}\) are expressed in terms of the Lie derivative \(\mathcal{L}_{\delta x}\) generated by \(\delta x^\nu\) as \(\delta A_\nu \equiv \mathcal{L}_{\delta x}(A_\nu \delta x^\nu)\) and \(\delta L \overline{d} \overline{L} \equiv \mathcal{L}_{\delta x}(\overline{L} \delta x^\nu)\).

After some cancellations introduced through the use of the reduced Maxwell equations \[\text{(43)}\], we obtain the reduced energy-momentum conservation law \(\partial_{\mu} T^{\mu\nu} \equiv 0\), where the reduced energy-momentum tensor is defined as

\[
T^{\mu\nu} = \frac{g^{\mu\nu}}{16\pi} F : F - \frac{1}{4\pi} (F^{\mu\sigma} - 4\pi K^{\mu\sigma}) F_{\sigma\nu} + \sum \int d^3 \pi \mathcal{F} \left( \frac{d \mathcal{A}^\mu}{dt} \right) - \frac{1}{c} \overline{J}^\mu \mathcal{A}^\nu.
\]

(46)

Here, the antisymmetric tensor \(K^{\mu\nu}\), defined in Eq. (39), represents the effects of reduced polarization and magnetization. Note that, while the last two terms are individually gauge-dependent, their sum is invariant under the gauge transformation \(\overline{J}^\mu \rightarrow \overline{J}^\mu + \mathcal{A}_{\nu} \partial_{\mu} \mathcal{A}^\nu\). Explicit proofs of energy-momentum conservation for the reduced Vlasov-Maxwell equations based on the reduced energy-momentum tensor \[\text{(40)}\] are presented in Appendix A. Lastly, additional angular-momentum conservation laws can be derived from the reduced Noether equation \(\delta \overline{L} \equiv \partial_{\mu} \overline{J}^\mu\) by considering invariance of the reduced Lagrangian density \(\overline{L}\) with respect to arbitrary rotations in space.

V. SUMMARY

In this paper, the general theory for the reduced Vlasov-Maxwell equations was presented based on the asymptotic elimination of fast time scales by Lie-transform Hamiltonian perturbation method. This dynamical reduction is based on a near-identity transformation on extended phase space, which induces transformations on the Vlasov distribution and the Vlasov operator, as well as introducing a natural (push-forward) representation of charge-current densities in terms of reduced charge-current densities and their associated reduced polarization and magnetization effects. The variational formulation of the reduced Vlasov-Maxwell equations allows the derivation of exact energy-momentum conservation laws by Noether method.

The Table shown below summarizes the polarization and magnetization effects observed in reduced Vlasov-Maxwell equations that have important applications in plasma physics.

| Reduced Dynamics | \(\pi_\epsilon\) | \(\mu_\epsilon\) |
|------------------|-----------------|-----------------|
| Guiding-center   | \((mc^2/B^2) \mathbf{E}_{\perp} + \epsilon \mathbf{b} \times \mathbf{v}_B / \Omega\) | \(-\mathbf{\pi} \mathbf{b}\) |
| Gyrocenter       | \(\epsilon (c \mathbf{B}_0 / B_0) \times \left( \epsilon \mathbf{A}_1 / c + m \mathbf{u}_{E1} + \mathbf{p}_\parallel \mathbf{B}_1 / B_0 \right)\) | \(-\mathbf{\pi} (\mathbf{B}_0 + \epsilon \mathbf{B}_1 / B_0)\) |
| Oscillation-center | \(\epsilon^2 \mathbf{e} \mathbf{k} \times (-i \hat{\xi}^* \times \hat{\xi})\) | \(\epsilon^2 \mathbf{e} \omega / c (-i \hat{\xi}^* \times \hat{\xi})\) |

First, in guiding-center Hamiltonian theory \[\text{[5, 6, 21]}\] for a strongly magnetized plasma in the presence of a background electric field \(\mathbf{E} \equiv -\nabla \Phi\), the fast time scale is associated with the rapid gyromotion of a charged par-
ticle about a magnetic field line and fast-time-averaging is carried out by averaging with respect to the gyroangle. The fast-time-averaged reduced electric-dipole moment \( \mathbf{\Pi}_{gc} \equiv c \mathbf{\Pi}_{gc} \) includes effects due to the background electric field \( (\mathbf{v}_E = c \mathbf{E} \times \mathbf{B}/B^2) \) as well as the magnetic \( (\nabla \mathbf{B} \text{ and curvature}) \) drift velocity \( \mathbf{v}_B \). The fast-time-averaged particle polarization velocity \( d_{gc} \mathbf{P}_{gc}/dt \) includes the standard polarization drift velocity \( (c/\mathbf{B} \mathbf{\Omega}) \partial \mathbf{E}_L / \partial t \), which is not included in the guiding-center drift velocity \( d_{gc} \mathbf{\Pi}_{gc}/dt \equiv \mathbf{v}_{gc} = \mathbf{v}_E + \mathbf{v}_B \). On the other hand, the reduced magnetic-dipole moment yields the classical parallel magnetization term \( \mathbf{\Pi}_{gc} \equiv - \mathbf{\Pi} \hat{b} \) (where \( \mathbf{\Pi} \) denotes the guiding-center magnetic-moment adiabatic invariant and \( \hat{b} \equiv \mathbf{B}/B \) denotes the unit vector along a magnetic-field line), which enables the reconciliation of the particle current \( \mathbf{J} \) with the guiding-center current \( \mathbf{J}_{gc} \) through the relation \( \mathbf{J} \equiv \mathbf{J}_{gc} + \mathbf{J}_{mag} \) (valid in a static magnetized plasma).

Next, gyrocenter Hamiltonian theory \( \mathcal{H} \) describes the reduced (gyroangle-independent) perturbed guiding-center Hamiltonian dynamics associated with low-frequency, electric and magnetic fluctuations \( (\epsilon \mathbf{E}_1, \epsilon \mathbf{B}_1) \) in a strongly magnetized plasma (with static magnetic field \( \mathbf{B}_0 = B_0 \hat{b}_0 \) and \( \mathbf{E}_0 = 0 \)). Note that the results shown here are valid only in the limit of zero Larmor radius \( \mathcal{L} \). The reduced electric-dipole moment includes not only the perturbed polarization-drift term \( (mc \hat{b}_0/\mathbf{B}_0 \times \mathbf{u}_{E_1}) \), but also the effects due to magnetic flutter \( (c \mathbf{\Pi} \hat{b}_0 \times \mathbf{b}_1/\mathbf{B}_0) \) and the inductive part of the perturbed \( \mathbf{E} \times \mathbf{B} \) velocity (i.e., the polarization drift velocity includes the higher-order correction \( -\partial_0 \mathbf{A}_1 \times \hat{b}_0/\mathbf{B}_0 \) to the perturbed \( \mathbf{E} \times \mathbf{B} \) velocity \( -\nabla \Phi_1 \times c \hat{b}_0/\mathbf{B}_0 \)). On the other hand, the reduced magnetic-dipole moment includes a correction to the classical parallel magnetization term due to the perturbed magnetic field: \( \mathbf{\Pi}_{sv} \equiv - \mathbf{\Pi} (\hat{b}_0 + c \mathbf{B}_1/\mathbf{B}_0) \), where \( \mathbf{\Pi} \) now denotes the gyrocenter magnetic-moment adiabatic invariant.

Lastly, oscillation-center Hamiltonian theory \( \mathcal{S} \) describes the reduced dynamics of charged particles interacting with a high-frequency electromagnetic wave in a weakly-inhomogeneous plasma for which the eikonal approximation is valid. The eikonal representation for the wave fields is \( (\mathbf{E}_1, \mathbf{B}_1) \equiv (\mathbf{E}_1, \mathbf{B}_1) \exp(i\epsilon_0^{-1} \Theta) + c.c. \), where \( \epsilon_0 \ll 1 \) denotes the eikonal small parameter while the eikonal phase \( \Theta(\epsilon_0 \mathbf{r}, \epsilon_0 t) \) is used to define \( \omega \equiv -\epsilon_0^{-1} \partial_t \Theta \) and \( \mathbf{k} \equiv \epsilon_0^{-1} \nabla \Theta \), with \( \omega' \equiv \omega - \mathbf{k} \cdot \mathbf{v} \) denoting the Doppler-shifted wave frequency. Note that fast-time-averaging, here, is carried out by averaging with respect to the eikonal phase \( \Theta \). By considering the simplest case of an unmagnetized plasma \( \mathcal{S} \), the first-order term for the displacement \( \mathbf{\rho}_c = \epsilon \mathbf{x} + \cdots \) has the eikonal amplitude

\[
\tilde{\mathbf{x}} = -\frac{\epsilon}{m \omega'^2} \left( \mathbf{E}_1 + \frac{\mathbf{v}}{c} \times \mathbf{B}_1 \right),
\]

where \( \epsilon \ll 1 \) denotes the amplitude of the electromagnetic wave represented by the first-order fields \( \mathbf{E}_1 \) and \( \mathbf{B}_1 \). Note that both reduced oscillation-center electric-dipole and magnetic-dipole moments are quadratic functions of the wave fields. An additional wave-action conservation law results from the invariance of the reduced Lagrangian density on the eikonal phase but its derivation is outside the scope of this work.

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APPENDIX A: EXPLICIT PROOFS OF ENERGY-MOMENTUM CONSERVATION FOR THE REDUCED VLASOV-MAXWELL EQUATIONS

In this Appendix, we present explicit proofs of energy-momentum conservation based on the reduced energy-momentum stress tensor \( \mathbf{T} \). We begin with the reduced energy conservation law \( \partial \mathcal{E}/\partial t + \nabla \cdot \mathbf{S} = 0 \), where the reduced energy density \( \mathcal{E} \equiv T^{00} \) is

\[
\mathcal{E} = -\frac{1}{8\pi} \left( |\mathbf{E}|^2 - |\mathbf{B}|^2 \right) + \frac{\mathbf{D} \cdot \mathbf{E}}{4\pi} + \sum d^4 \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial t} - \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial \mathbf{E}} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t},
\]

where the \( \mathbf{\Pi} \)-integration was performed. First, using the reduced charge conservation law \( \mathbf{J} \) and the identity

\[
\sum d^4 \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial t} = \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial \mathbf{E}} - \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial \mathbf{B}} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t},
\]

we obtain

\[
\frac{\partial \mathcal{E}}{\partial t} = \frac{\mathbf{D} \cdot \partial \mathbf{D}}{4\pi} + \frac{\mathbf{H} \cdot \partial \mathbf{B}}{4\pi} + \mathbf{E} \cdot \mathbf{J} + \nabla \cdot (\mathbf{J} \Phi) - \sum d^4 \mathbf{\Pi} \left( \nabla \cdot \left( \frac{\partial \mathbf{\Pi}}{\partial t} \right) \right),
\]

where we used the reduced Vlasov equation \( \mathbf{J} \) to obtain the last term. Lastly, using the reduced Maxwell equations \( \mathbf{E} \) and \( \mathbf{B} \) and the identity

\[
\int d^3 \mathbf{\Pi} \left( \nabla \cdot \left( \frac{\partial \mathbf{\Pi}}{\partial t} \right) \right) = \int d^4 \mathbf{\Pi} \frac{\partial \mathbf{\Pi}}{\partial t} \frac{d^4 \mathbf{\Pi}}{dt},
\]

we finally obtain

\[
\frac{\partial \mathcal{E}}{\partial t} = -\nabla \cdot \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} - \mathbf{J} \Phi \right) + \sum d^4 \mathbf{\Pi} \frac{\partial \mathcal{H}}{\partial \mathbf{E}} \frac{d^4 \mathbf{\Pi}}{dt} = -\nabla \cdot \mathbf{S},
\]
where the energy-density flux $S^i \equiv c T^{i0}$.

Next, we consider the reduced momentum conservation law $\partial \Pi / \partial t + \nabla \cdot \mathbf{T} = 0$, where the reduced momentum density $\Pi^i \equiv \Pi^0 \delta_{ij} / c$ is

$$\Pi = \frac{\mathbf{D} \times \mathbf{B}}{4\pi c} - \frac{n}{c} \mathbf{A} + \sum \int d^3 \mathbf{p} \mathbf{j} \mathbf{p},$$

which has the Minkowski form\(^\text{[10]}\). By substituting the reduced Vlasov-Maxwell equations\(^\text{[8]}\) and\(^\text{[23]-[24]}\) and the reduced charge conservation law\(^\text{[44]}\), we obtain

$$\frac{\partial \Pi}{\partial t} = \nabla \cdot \left[ \frac{1}{4\pi} (\mathbf{B} \cdot \mathbf{H} + \mathbf{D} \cdot \mathbf{E}) + \frac{1}{c} \mathbf{J} \mathbf{A} \right]$$

$$- \sum \int d^3 \mathbf{p} \mathbf{j} \left( \frac{d \mathbf{r}}{dt} \mathbf{p} \right)$$

$$- \sum \int d^3 \mathbf{p} \nabla \mathbf{H} \mathbf{p} - \nabla \mathbf{A} \cdot \frac{\mathbf{j}}{c}$$

Lastly, using the identities

$$\sum \int d^3 \mathbf{p} \mathbf{H} \nabla \mathbf{H} \equiv \nabla \cdot \mathbf{H} - \nabla \mathbf{A} \cdot \frac{\mathbf{j}}{c}$$

and

$$\nabla \mathbf{H} \cdot \mathbf{B} + \nabla \mathbf{E} \cdot \mathbf{D} = \nabla \left( \frac{|\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2} - 4\pi \mathbf{M}_e \cdot \mathbf{B} \right)$$

$$+ 4\pi (\nabla \mathbf{E} \cdot \mathbf{P}_e + \nabla \mathbf{B} \cdot \mathbf{M}_e),$$

we finally obtain

$$\frac{\partial \Pi}{\partial t} = \nabla \cdot \left[ \frac{1}{4\pi} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) - \sum \int d^3 \mathbf{p} \mathbf{j} \left( \frac{d \mathbf{r}}{dt} \mathbf{p} \right) \right]$$

$$+ \frac{1}{c} \mathbf{J} \mathbf{A} - \frac{1}{4\pi} \left( \frac{|\mathbf{E}|^2}{2} + \frac{|\mathbf{B}|^2}{2} - 4\pi \mathbf{M}_e \cdot \mathbf{B} \right) \equiv - \nabla \cdot \mathbf{T}.$$

[1] For a recent review, see W. M. Tang and V. S. Chan, Plasma Phys. Contr. Fusion 47, R1-R34 (2005).
[2] R. J. Hastie, J. B. Taylor, and F. A. Haas, Ann. Phys. 41, 302-338 (1967).
[3] R. G. Littlejohn, J. Math. Phys. 23, 742-747 (1982).
[4] A. J. Brizard, Phys. Lett. A 291, 146-149 (2001).
[5] R. G. Littlejohn, J. Plasma Phys. 29, 111-125 (1983).
[6] A. J. Brizard, Phys. Plasmas 2, 459-471 (1995).
[7] For a recent review, see A. J. Brizard and T. S. Hahn, Foundations of Nonlinear Gyrokinetic Theory, PPPL Report 4133 (2006).
[8] J. R. Cary and A. N. Kaufman, Phys. Fluids 24, 1238-1250 (1981).
[9] R. Abraham and J. E. Marsden, Foundations of Mechanics, 2nd ed. (Benjamin/Cummings, Reading, MA, 1978).
[10] J. D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York, 1975), sec. 6.7.
[11] P. P. Sosenko, J. Plasmas Phys. 53, 223-234 (1995).
[12] P. P. Sosenko, P. Bertrand, and V. K. Decyk, Phys. Scr. 64, 264-272 (2001).
[13] D. Pfirsch, Z. Naturforsch. 39a, 1-8 (1984).
[14] D. Pfirsch and P. J. Morrison, Phys. Rev. A 32, 1714-1721 (1985).
[15] A. N. Kaufman and B. M. Boghosian, Contemp. Math. 28, 169-176 (1984).
[16] A. N. Kaufman and D. D. Holm, Phys. Lett. A 105, 277-279 (1984).
[17] P. L. Similon, Phys. Lett. A 112, 33-37 (1985).
[18] B. M. Boghosian, Covariant Lagrangian methods of relativistic plasma theory, Ph. D. thesis, University of California, Davis (1987); see also arXiv:physics/0307148.
[19] H. Ye and A. N. Kaufman, Phys. Fluids B 4, 1735-1753 (1992).
[20] A. J. Brizard, Phys. Rev. Lett. 84, 5768-5771 (2000).
[21] A. N. Kaufman, Phys. Fluids 29, 1736-1737 (1986).