Some Properties of Macdonald Polynomials with Prescribed Symmetry.

Wendy Baratta

January 18, 2010

Abstract

The Macdonald polynomials with prescribed symmetry are obtained from the nonsymmetric Macdonald polynomials via the operations of $t$-symmetrisation, $t$-antisymmetrisation and normalisation. Motivated by corresponding results in Jack polynomial theory we proceed to derive an expansion formula and a related normalisation. Eigenoperator methods are used to relate the symmetric and antisymmetric Macdonald polynomials, and we discuss how these methods can be extended to special classes of the prescribed symmetry polynomials in terms of their symmetric counterpart. We compute the explicit form of the normalisation with respect to the constant term inner product. Surpassing our original motivation, this is used to provide a derivation of a special case of a conjectured $q$-constant term identity.

2000 Mathematics Subject Classification: Primary 81Q08, Secondary 81V08.

Keywords: Macdonald polynomials; prescribed symmetry; constant term identities.

1 Introduction

1.1 Background and overview

Nonsymmetric Macdonald polynomials were first introduced in 1994 [17, 7], six years after Macdonald’s paper [15] introducing what are now referred to as symmetric Macdonald polynomials $P_{\kappa}(z; q, t)$. The nonsymmetric Macdonald polynomials $E_{\eta}(z; q, t)$ can be regarded as building blocks of their symmetric counterparts, as $t$-symmetrisation of $E_{\eta}$ gives $P_{\eta^+}$. Generalising this action by applying a combination of $t$-symmetrising and $t$-antisymmetrising operators to $E_{\eta}$ generates the Macdonald polynomials with prescribed symmetry.

A polynomial is $t$-symmetric with respect to $z_i$ if $T_i f(z) = tf(z)$ and $t$-antisymmetric with respect to $z_i$ if $T_i f(z) = -f(z)$. The $t$-symmetrisation and $t$-antisymmetrisation operators are defined, respectively, by

$$ U^+ := \sum_{\sigma \in S_n} T_{\sigma}, \text{ and } U^- := \sum_{\sigma \in S_n} \left( -\frac{1}{t} \right)^{\ell(\sigma)} T_{\sigma}. \quad (1) $$

Here $S_n$ denotes the symmetric group on $n$ symbols. Also, with $s_i$ denoting the transposition operator with the action on functions

$$ s_i f (z_1, \ldots, z_n) = f (z_1, \ldots, z_{i+1}, z_i, \ldots, z_n), \quad (i = 1, \ldots, n - 1) $$. 


and \( \sigma := s_{i(n)} \ldots s_{i_1} \), the operator \( T_\sigma \) is specified by

\[
T_\sigma := T_{i(n)} \ldots T_{i_1},
\]

where

\[
T_i := t + \frac{t z_i - z_{i+1}}{z_i - z_{i+1}} (s_i - 1), \quad (i = 1, \ldots, n - 1).
\]

Note that when \( t = 1 \) the operators \( U^+ \) and \( U^- \) reduce to the standard symmetrising and antisymmetrising operators.

The study of Macdonald polynomials with prescribed symmetry began in 1999 [1,18] and was initially motivated by the analogous results in Jack polynomial theory [2,3]. Nonsymmetric Jack polynomials \( E_\eta(z; \alpha) \) are the limit \( q = t^\alpha, t \to 1 \) of Macdonald polynomials. Jack polynomials are eigenfunctions of the operator stemming from the type A Calogero-Sutherland quantum many body system. Cases where the system is multicomponent, containing both bosons and fermions, require eigenfunctions that are symmetric or antisymmetric, respectively, with respect to certain sets of variables. This requirement lead naturally to the introduction of Jack polynomials with prescribed symmetry [3,13].

Using the more extensively developed Jack theory as motivation we continue the study of prescribed symmetry Macdonald polynomials. Our first result is an expansion formula for the prescribed symmetry Macdonald polynomials in terms of the nonsymmetric Macdonald polynomials (Proposition 2). Following this we determine the normalisation required to obtain the prescribed symmetry Macdonald polynomial from the symmetrisation of the nonsymmetric polynomial (Proposition 3). In Section 3 eigenoperator methods are used to relate the symmetric and antisymmetric Macdonald polynomials, thus providing an alternate proof for a result of Marshall [19]. Our final investigation is of the inner product of prescribed symmetry Macdonald polynomials, giving general explicit formulas (Theorem 7) and then considering special cases where the antisymmetric components are of specific forms. Although originally motivated by the analogous Jack theory [1] these results have applications in \( q \)-constant terms identities, which are discussed in the second half of Section 4.

Before explicitly defining the Macdonald polynomials with prescribed symmetry, we give the required background on the nonsymmetric Macdonald polynomials.

### 1.2 Nonsymmetric Macdonald polynomials

The nonsymmetric Macdonald polynomials \( E_\eta := E_\eta(z; q, t) \) are polynomials of \( n \) variables \( z = (z_1, \ldots, z_n) \) having coefficients in the field \( \mathbb{Q}(q, t) \) of rational functions of the indeterminants \( q \) and \( t \). The compositions \( \eta := (\eta_1, \ldots, \eta_n) \) of non-negative integers components \( \eta_i \) label these polynomials. The nonsymmetric Macdonald polynomials can be defined, up to normalisation, as the unique simultaneous polynomials eigenfunctions of the commuting operators

\[
Y_i = t^{-n+1} T_1 \ldots T_{n-1} \omega T_1^{-1} \ldots T_{i-1}^{-1}, \quad (i = 1, \ldots, n)
\]

satisfying the eigenvalue equations

\[
Y_i E_\eta(z; q, t) = \eta_i E_\eta(z; q, t).
\]

In (3) \( \omega \) is given by \( \omega := s_{n-1} \ldots s_1 \tau_1 \), where the operator \( \tau_i \) has the action on functions

\[
(\tau_i f)(z_1, \ldots, z_n) := f(z_1, \ldots, q z_i, \ldots, z_n)
\]
and so corresponds to a $q$-shift of the variable $z_i$. The eigenvalue $\overline{\eta}_i$ in (11) is given by

$$\overline{\eta}_i := q^{n_i} t^{-l^0_i(i)},$$

where

$$l^0_\eta(i) := \# \{ j < i; \eta_j \geq \eta_i \} + \# \{ j > i; \eta_j > \eta_i \}.$$  

(6)

The nonsymmetric Macdonald polynomials are of the triangular form

$$E_\eta(z; q, t) := z^\eta + \sum_{\nu < \eta} b_{\eta\nu} z^\nu,$$

(7)

for coefficients $b_{\eta\nu} \in \mathbb{Q}(q, t)$. The coefficient of $z^\eta := z_1^{\eta_1} \ldots z_n^{\eta_n}$ is chosen to be unity as a normalisation. The ordering $\prec$ is a partial ordering on compositions having the same modulus, where $|\eta| := \sum_{i=1}^n \eta_i$ denotes the modulus of $\eta$. The partial ordering is defined by

$$\mu \prec \eta \text{ iff } \mu^+ < \eta^+ \text{ or in the case } \mu^+ = \eta^+ , \mu < \eta$$

where $\eta^+$ is the unique partition obtained by permuting the components of $\eta$, and $\mu < \eta$ iff $\mu \neq \eta$ and $\sum_{i=1}^n (\eta_i - \mu_i) \geq 0$ for all $1 \leq p \leq n$. The action of $T_i$ on $E_\eta$, for $1 \leq i \leq n - 1$, is given explicitly by (4)

$$T_i E_\eta(z) = \begin{cases} 
\frac{t-1}{1-\delta_{i,\eta}} E_\eta + t E_{s,\eta}, & \eta_i < \eta_{i+1} \\
 t E_\eta, & \eta_i = \eta_{i+1} \\
\frac{t-1}{1-\delta_{i,\eta}} E_\eta + \frac{(1-\delta_{i,\eta})(1-t^{-1}\delta_{i,\eta})}{(1-\delta_{i,\eta})^2} E_{s,\eta}, & \eta_i > \eta_{i+1},
\end{cases}$$

(8)

where $\delta_{i,\eta} = \overline{\eta}_i/\overline{\eta}_{i+1}$.

Alternatively, nonsymmetric Macdonald polynomials can be characterised as multivariate polynomials of the structure (7) orthogonal with respect to the inner product $\langle f, g \rangle_{q, t}$, defined by

$$\langle f, g \rangle_{q, t} := \text{CT} \left(f(z; q, t)g(z^{-1}, q^{-1}, t^{-1})W(z)\right).$$

(9)

In (4) $\text{CT}(f)$ denotes the constant term with respect to $z$ of any formal Laurent series $f$ and

$$W(z) := W(z; q, t) := \prod_{1 \leq i < j \leq n} \left(\frac{z_j}{z_i}; q\right)_\infty \left(\frac{z_i}{z_j}; q\right)_\infty \left(\frac{t_z^{-1}}{z_i}; q\right)_\infty \left(\frac{t_z}{z_i}; q\right)_\infty,$$

with the Pochhammer symbol defined by $(a; q)_\infty := \prod_{j=0}^{\infty} (1 - aq^j)$.

This scalar product, introduced by Cherednik, is linear and positive definite. Macdonald showed that (17)

$$\langle E_\eta(q, t), E_\nu(q, t) \rangle_{q, t} = \delta_{\eta, \nu} N_\eta.$$

We will have future use for the explicit value of $N_\eta$. For this a number of quantities dependent on $\eta$ must be introduced. For each node $s = (i, j) \in \text{diag}(\eta)$ we define the arm length, $a_\eta(s) := \eta_i - j$, arm colength, $a'_\eta(s) := j - 1$, leg length, $l_\eta(s) := \# \{ k < i : j \leq \eta_k + 1 \leq \eta_i \} + \# \{ k < i :
\( j \leq \eta_k \leq \eta_i \) and leg colength \( l'_\eta(s) \), given by [9]. From these we define [21]

\[
\begin{align*}
d_\eta &:= d_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} \left( 1 - q^{a_\eta(s) + 1} t^{l'_\eta(s) + 1} \right), \\
d'_\eta &:= d'_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} \left( 1 - q^{a_\eta(s) + 1} t^{l'_\eta(s)} \right), \\
e_\eta &:= e_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} \left( 1 - q^{a_\eta(s) + 1} t^{n - l'_\eta(s)} \right), \\
e'_\eta &:= e'_\eta(q, t) = \prod_{s \in \text{diag}(\eta)} \left( 1 - q^{a_\eta(s) + 1} t^{n - 1 - l'_\eta(s)} \right).
\end{align*}
\]

In this notation the explicit formula for \( N_\eta \) is given by [see, e.g., 7]

\[
N_\eta = \frac{d'_\eta e_\eta}{d_\eta e'_\eta} \langle 1, 1 \rangle_{q, t}.
\]

In later sections we use the specialisation \( t = q^k \), \( k \in \mathbb{Z}^+ \). In this specialisation the weight function \( W(z) \) reduces to

\[
W(z; q, q^k) = \prod_{1 \leq i < j \leq n} \left( \frac{z_i}{z_j}; q \right)_k \left( \frac{q z_j}{z_i}; q \right)_k,
\]

where

\[
(a; q)_k := \Pi_{j=0}^{k-1} (1 - aq^j), \quad k > 0
\]

which is a Laurent polynomial. Furthermore [2]

\[
\langle 1, 1 \rangle_{q, q^k} = \frac{[nk]_q!}{[k]_q! m!},
\]

where the \( q \)-factorial is given in terms of \( q \)-numbers [13] \([m]_q := (1 - q^m)/(1 - q)\) by

\[
[k]_q! := [1]_q [2]_q \ldots [k]_q.
\]

Concluding the preliminary material, we now proceed to formally introduce the Macdonald polynomials with prescribed symmetry.

2 Macdonald Polynomials with Prescribed Symmetry and the Required Operator

2.1 The operator \( O_{I,J} \)

We begin our investigation into the Macdonald polynomials with prescribed symmetry by introducing a particular symmetrising operator \( O_{I,J} \). The sets \( I \) and \( J \) represent the variables which the operator \( O_{I,J} \) symmetrises and antisymmetrises with respect to. Explicitly

\[
T_i[O_{I,J}f(z)] = tO_{I,J}f(z) \quad \text{for } i \in I
\]

and

\[
T_j[O_{I,J}f(z)] = -O_{I,J}f(z) \quad \text{for } j \in J.
\]
For $O_{I,J}$ to be well defined $I$ and $J$ must be disjoint subsets of $\{1,...,n-1\}$, such that

$$i-1,i+1 \notin J \text{ for } i \in I \text{ and } j-1,j+1 \notin I \text{ for } j \in J.$$ 

In many cases we require the set $J$ to be decomposed into disjoint sets of consecutive integers, to be denoted $J_1,J_2,...,J_s$. For example, with $J = \{1,2,5,6,7\}$, $J_1 = \{1,2\}$ and $J_2 = \{5,6,7\}$.

Related to this we also require sets $\tilde{J}_j := J_j \cup \{\max(J_j) + 1\}$ and $\tilde{J} = \bigcup \tilde{J}_s$.

Since we are symmetrising with respect to a subset of variables, in contrast to the construction of $U^+$ and $U^-$ given in (11), we do not want to sum over all $\sigma \in S_n$. Instead we introduce $W_{I\cup J} := (s_k; \ k \in I \cup J)$, a subset of $S_n$ such that each $\omega \in W_{I\cup J}$,

$$\omega = \omega_I \omega_J, \text{ with } \omega_J \in W_J \text{ and } \omega_I \in W_I,$$ 

has the property that $\omega(i) = i$ if $i \notin \tilde{I} \cup \tilde{J}$.

The operator $O_{I,J}$ is then specified by

$$O_{I,J} := \sum_{\omega \in W_{I\cup J}} \left( -\frac{1}{t} \right)^{l(\omega_J)} T_\omega,$$

where $T_\omega$ is given by (2).

### 2.2 The polynomial $S^{(I,J)}_\eta(z)$

To motivate the introduction of the prescribed symmetry Macdonald polynomials we first consider the symmetric and antisymmetric Macdonald polynomials. These are denoted by $P_\kappa$ and $S_{\lambda+\delta}$, respectively, where $\kappa$ and $\lambda$ are partitions and $\delta := (n-1,\ldots,1,0)$. It is well known that these polynomials can be generated from nonsymmetric Macdonald polynomials via a process of symmetrisation and antisymmetrisation. Thus to generate $P_\kappa$ one would symmetrise any $E_\eta$ for which there exists a permutation $\sigma \in S_n$ such that $\sigma \eta = \kappa$. Similarly, to generate $S_{\lambda+\delta}$ one would antisymmetrise any $E_\mu$ such that there exists a permutation $\rho \in S_n$ where $\rho \mu = \lambda + \delta$.

Explicitly

$$U^+ E_\eta = b_\eta P_\kappa \quad \text{and} \quad U^- E_\mu = b'_\mu S_{\lambda+\delta},$$

for some non-zero $b_\eta, b'_\mu \in \mathbb{Q}(q,t)$. It follows quite naturally that the Macdonald polynomial with prescribed symmetry, denoted by $S^{(I,J)}_\eta(z;q,t)$, a polynomial $t$-symmetric with respect to the set $I$ and $t$-antisymmetric with respect to the set $J$, will be labeled by a composition $\eta^*$ such that

$$\eta^*_i \geq \eta^*_{i+1} \text{ for all } i \in I \text{ and } \eta^*_j > \eta^*_{j+1} \text{ for all } j \in J.$$ 

Such a polynomial can be generated by applying our prescribed symmetry operator $O_{I,J}$ to any $E_\eta$ such that there exists a $\sigma \in W_{I\cup J}$ with $\sigma \eta = \eta^*$. That is

$$O_{I,J} E_\eta(z; q,t) = a^{(I,J)}_\eta S^{(I,J)}_\eta(z),$$ 

for some non-zero $a^{(I,J)}_\eta$. This uniquely specifies $S^{(I,J)}_\eta(z)$ up to normalisation; for the latter we require that the coefficient of $z^{\eta^*}$ in the monomial expansion equals unity as in (10).

Our first task is to find the explicit formula for the proportionality $a^{(I,J)}_\eta$ in (16). We do this
by first computing the expansion formula of $S_{\eta'}(z^{-1}; q^{-1}, t^{-1})$ in terms of $E_{\eta}(z^{-1}; q^{-1}, t^{-1})$, a result which is of independent interest. We begin by deriving an explicit formula for the action of $T_i$ on $E_{\eta}(z^{-1}; q^{-1}, t^{-1})$ analogous to (8). This is done using the Cauchy formula for the nonsymmetric Macdonald polynomials, 20.

\[
\Omega(x, y; q, t) := \sum_{\eta} \frac{d\eta}{d\eta} E_{\eta}(x; q, t) E_{\eta}(y; q^{-1}, t^{-1}),
\]

the result [20]

\[
T_i^{(x)} \Omega(x, y^{-1}; q, t) = T_i^{(y)} \Omega(x, y^{-1}; q, t),
\]

and (8) itself. In (17) the superscripts denote which variables the respective operators act upon.

**Proposition 1** For $1 \leq i \leq n - 1$ we have

\[
T_i E_{\eta}(z^{-1}; q^{-1}, t^{-1}) = \begin{cases} 
\frac{t^{-1}}{1 - \delta_{i,\eta}} E_{\eta}(z^{-1}; q^{-1}, t^{-1}) + \frac{d\eta}{d\eta} \frac{(1 - \delta_{i,\eta})(1 - t^{-1}\delta_{i,\eta})}{(1 - \delta_{i,\eta})^2} E_{\eta}(z^{-1}; q^{-1}, t^{-1}) & \eta_i < \eta_{i+1} \\
\frac{t}{E_{\eta}(z^{-1}; q^{-1}, t^{-1})} & \eta_i = \eta_{i+1} \\
\frac{t^{-1}}{1 - \delta_{i,\eta}} E_{\eta}(z^{-1}; q^{-1}, t^{-1}) + t \frac{d\eta}{d\eta} E_{\eta}(z^{-1}; q^{-1}, t^{-1}) & \eta_i > \eta_{i+1}
\end{cases}
\]

**Proof** By (17) we have

\[
T_i^{(x)} \left( \frac{d\eta}{d\eta} E_{\eta}(x; q, t) E_{\eta}(y^{-1}; q^{-1}, t^{-1}) + \frac{d\eta}{d\eta} E_{\eta}(x; q, t) E_{\eta}(y^{-1}; q^{-1}, t^{-1}) \right) = T_i^{(y)} \left( \frac{d\eta}{d\eta} E_{\eta}(x; q, t) E_{\eta}(y^{-1}; q^{-1}, t^{-1}) + \frac{d\eta}{d\eta} E_{\eta}(x; q, t) E_{\eta}(y^{-1}; q^{-1}, t^{-1}) \right).
\]

Using (8) and equating coefficients of like terms gives (18).

The coefficients in the expansion of $S_{\eta'}^{(I,J)}(z^{-1}; q^{-1}, t^{-1})$ in terms of $\{ E_{\mu}(z^{-1}; q^{-1}, t^{-1}) \}$ can be computed explicitly in terms of the quantities $d\eta$ and $d\eta'$. The derivation makes use of the the fact that for $\eta_i < \eta_{i+1}$ we have [21]

\[
\frac{d\eta}{d\eta} = \frac{1 - \delta_{i,\eta}}{t - \delta_{i,\eta}} \quad \text{and} \quad \frac{d\eta'}{d\eta'} = \frac{t^{-1} - \delta_{i,\eta}}{1 - \delta_{i,\eta}}.
\]

**Proposition 2** Let $\omega \in W_{I\cup J}$ be decomposed as in (15). Let $\omega \eta^* = \mu$ and $\omega I \eta^* = \mu_I$. The coefficients in

\[
S_{\eta'}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}) = \sum_{\mu \in W_{I\cup J}(\eta^*)} \hat{b}_{\eta'}^{\mu} E_{\mu}(z^{-1}; q^{-1}, t^{-1}), \quad \hat{b}_{\eta'}^{\rho^*} = 1,
\]

are specified by

\[
\hat{b}_{\eta'}^{\mu} = (-1)^{l(\omega_I)} l^{(\omega)} d\eta' d\mu / d\mu_i d\mu_i.
\]

Similarly, the coefficients in

\[
S_{\eta'}^{(I,J)}(z) = \sum_{\mu \in W_{I\cup J}(\eta^*)} \hat{c}_{\eta'}^{\mu} E_{\mu}(z), \quad \hat{c}_{\eta'}^{\rho^*} = 1,
\]

are specified by

\[
\hat{c}_{\eta'}^{\mu} = (-1)^{l(\omega_J)} d\eta' d\mu / d\mu_i d\mu_i.
\]
Proof We write
\[
\sum_{\mu \in W_{I \cup J}(\eta^*)} \hat{b}_{\eta^* \mu} E_{\mu}(z^{-1}; q^{-1}, t^{-1}) = \sum_{\mu \in W_{I \cup J}(\eta^*)} \chi_{i,i+1} \left( \hat{b}_{\eta^* \mu} E_{\mu}(z^{-1}; q^{-1}, t^{-1}) + \hat{b}_{\eta^* s_i \mu} E_{s_i \mu}(z^{-1}; q^{-1}, t^{-1}) \right)
\]
where \( \chi_{i,i+1} = 1/2 \) if \( \mu = s_i \mu \) and 1 otherwise. For \( i \in I \) we require
\[
T_i S_{\eta^*}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}) = t S_{\eta^*}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}).
\]
If \( \mu_i = \mu_{i+1} \) holds due to the relation in (18). Hence we consider the case where \( \mu_i < \mu_{i+1} \). Expanding the left hand side of (24) using (18) gives simultaneous equations and solving these show
\[
\frac{\hat{b}_{\eta^* \mu}}{\hat{b}_{\eta^* s_i \mu}} = \frac{1 - t \delta_{i,\mu}}{1 - \delta_{i,\mu}} \quad \text{for all } i \in I.
\]
Since \( \mu_i < \mu_{i+1} \) can be used to rewrite (25) as
\[
\frac{\hat{b}_{\eta^* \mu}}{\hat{b}_{\eta^* \eta^*}} = t \frac{d_{s_i \mu}}{d_{\mu}}.
\]
By noting \( \eta^* = \omega^{-1} I \mu_I = s_i \ldots i(\omega^{-1}) \mu_I \) where each \( s_i \) interchanges components we can apply (26) repeatedly to obtain
\[
\frac{\hat{b}_{\eta^* \mu}}{\hat{b}_{\eta^* \eta^*}} = \hat{b}_{\eta^* \mu} = t^{(\omega)} \frac{d_{s_i \mu}}{d_{\mu}},
\]
where the first equality follows from the normalisation \( \hat{b}_{\eta^* \eta^*} = 1 \).

To complete the derivation we require a formula for the ratio \( \hat{b}_{\eta^* \mu}/\hat{b}_{\eta^* \mu} \). Since for all \( j \in J \) we have
\[
T_j S_{\eta^*}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}) = - S_{\eta^*}^{(I,J)}(z^{-1}; q^{-1}, t^{-1}),
\]
applying the above methods gives \( \hat{b}_{\eta^* \mu}/\hat{b}_{\eta^* s_j \mu} = -d_{\mu}/d_{s_j \mu} \), and consequently
\[
\frac{\hat{b}_{\eta^* \mu}}{\hat{b}_{\eta^* \mu}} = (-1)^{(\omega)} \frac{d_{\mu}}{d_{\mu}}.
\]
Combining (28) with (27) we obtain (21).

The derivation of (26) is as above, only replacing (18) with (8).

We now use Proposition 2 to determine \( \omega_{\eta}^{(I,J)} \). To present the result requires some notation. Write \( \eta^{(\epsilon_1, \epsilon_2)} \), where \( \epsilon_1, \epsilon_2 \in \{+, 0, -\} \), to denote the element of \( W_{I \cup J}(\eta) \) with the properties that \( \eta^{(+,+)}(\eta^{(0,0)}) \) has \( \eta_i^{(+,+)} \geq \eta_i^{(+,+)} \) for all \( i \in I \) \( \eta_j^{(0,+)} > \eta_j^{(0,+)} \) for all \( j \in J \), \( \eta^{(0,-)}(\eta^{(-,-)}) \) has \( \eta_i^{(-,-)} \leq \eta_i^{(-,-)} \) for all \( i \in I \) \( \eta_j^{(0,-)} < \eta_j^{(0,-)} \) for all \( j \in J \), and \( \eta^{(0,0)} \) has \( \eta_i^{(0,0)} = \eta_{i+1}^{(0,0)} \) for all \( i \in I \). For example for \( \mu = \omega_{I \omega J} \eta^* \) we have \( \omega_{I \eta} = \mu^{(+,0)} \) and \( \omega_{J \eta} = \mu^{(-,0)} \). Also introduce
\[
M_{I, \eta} := \sum_{\sigma' \in W_I} \mu^{(\sigma')}.
\]
Proposition 3 The proportionality constant \(a_{\eta}^{(I,J)}\) in (10) is specified by

\[
a_{\eta}^{(I,J)} = (-1)^{l(\omega_J)} M_{I,\eta} \frac{d_{\bar{\eta}}' d_{\bar{\eta}}(-\omega_J) d_{\bar{\eta}}(-\omega_J)}{d_{\bar{\eta}}'(0,+)}
\]

where \(\omega_J\) is such that \(\omega_J \eta^{(+,+)} = \eta^{(+,0)}\).

Proof Let \(G(x,y)\) be defined by

\[
G(x,y) = \sum_{\eta \in W_{I,J}(\eta^*)} \frac{d_{\bar{\eta}} E_{\eta}(x; q, t) E_{\eta}(y^{-1}; q^{-1}, t^{-1})}{d_{\bar{\eta}}(\eta^*)}.
\] (29)

It follows from (8) and (18) that \(O_{I,J}^{(x)} G(x,y) = O_{I,J}^{(y)} G(x,y)\) for \(i \in I \cup J\), and hence

\[
O_{I,J}^{(x)} G(x,y) = O_{I,J}^{(y)} G(x,y).
\] (30)

By (13) and (14) we have \(O_{I,J}^{(\eta)} E_{\eta}(y^{-1}; q^{-1}, t^{-1}) = b_{\eta}^{(I,J)} S_{\eta}^{(I,J)}(\eta)\) for some \(b_{\eta}^{(I,J)} \in \mathbb{Q}(q,t)\). Hence substituting (29) into (30) and recalling (16) shows

\[
S_{\eta^*}^{(I,J)}(x) \sum_{\eta \in W_{I,J}(\eta^*)} \frac{d_{\bar{\eta}}}{d_{\bar{\eta}}'} a_{\eta}^{(I,J)} E_{\eta}(y^{-1}) = S_{\eta^*}^{(I,J)}(y^{-1}) \sum_{\eta \in W_{I,J}(\eta^*)} \frac{d_{\bar{\eta}}}{d_{\bar{\eta}}'} b_{\eta}^{(I,J)} E_{\eta}(x).
\]

Using separation of variables it follows that

\[
S_{\eta^*}^{(I,J)}(y^{-1}) = a_{\eta^*} \sum_{\eta \in W_{I,J}(\eta^*)} \frac{d_{\bar{\eta}}}{d_{\bar{\eta}}'} a_{\eta}^{(I,J)} E_{\eta}(y^{-1})
\] (31)

for some constant \(a_{\eta^*}\). Equating coefficients for \(\eta = \omega_I \omega_J \eta^*\) in (31) and (20) shows

\[
a_{\eta^*} \frac{d_{\bar{\eta}}}{d_{\bar{\eta}}'} a_{\eta}^{(I,J)} = (-1)^{l(\omega_J)} l(\omega_I) \frac{d_{\bar{\eta}}'}{d_{\bar{\eta}}'(0,+)}.
\] (32)

The identity (32) must hold for all \(\eta \in W_{I,J}(\eta^*)\), and in particular for \(\eta = \eta^{(-,-)}\). For such a composition we can use (5) to show

\[
O_{I,J} E_{\eta^{(-,-)}}(x) = (-1)^{l(\omega_J)} M_{I,\eta} S_{\eta^{(-,-)}}^{(I,J)}(x),
\]

where \(\omega_J^{-1} \eta^{(-,-)} = \eta^{(-,-)}\). Consequently

\[
a_{\eta^{(-,-)}}^{(I,J)} = (-1)^{l(\omega_J)} M_{I,\eta}.
\] (33)

Substituting (33) into (32) with \(\eta = \eta^{(-,-)}\) implies

\[
a_{\eta^{(-,-)}^*} = \frac{l(\omega_I)}{M_{I,\eta}} \frac{d_{\bar{\eta}}'}{d_{\bar{\eta}}'(0,+)}.
\] (34)

Substituting (34) in (32) gives the desired result. \(\square\)

Corollary 1 We have the evaluation formula

\[
S_{\eta^{(-,0)}}^{(I,\emptyset)}(\tilde{J}) = \frac{n_I}{a_{\eta^{(-,0)}}^{(I,0)}} E_{\eta^{(-,0)}}(\tilde{J}) = \frac{n_I l(\eta)}{M_{I,\eta^*} d_{\eta^{(-,0)}}'}. \]
(35)

where \(n_I := \sum_{\sigma \in W_I} l(\sigma) = \Pi_s([I_s])!\).
Proof Using \( T_i f (t^J) = tf (t^J) \) the first equality of (35) can be derived immediately from (16).

With \( J = \emptyset \) and \( \eta = \eta^* \) Proposition 3 gives

\[
E_{\eta^*}(t^J ; q, t) = E_{\eta^*}(t^J ; q, t) = t^{J(\eta)} \frac{d_{\eta^*}}{d_{\eta^*}}.
\]

Substituting this and the well known result (see e.g. [16])

\[
E_\eta(t^J ; q, t) = t^{J(\eta)} \frac{d_{\eta}}{d_{\eta}}
\]
gives the final equality.

We now move on to our first related result, deducing the form of Macdonald polynomials with prescribed symmetry in specific cases.

3 Special Forms of the Prescribed Symmetry Polynomials

3.1 The main result

We begin by introducing some notation to simplify the labeling of the \( S_{\eta^*}^{(I, J)} \) of interest. With \( N_p := \{n_1, n_2, \ldots, n_p\} \), let

\[
(k_{n_0}, \delta_{N_p}) := (\kappa_1, \ldots, \kappa_{n_0}, n_1 - 1, n_1 - 2, \ldots, 1, 0, \ldots, n_p - 1, \ldots, 1, 0)
\]

and

\[
I^{n_0} := \{1, \ldots, n_0 - 1\}
\]

and

\[
J^{n_0, N_p} := \bigcup_{i=1}^{p} \left\{ \sum_{j=1}^{\eta} n_j - 1 + 1, \ldots, \sum_{j=1}^{\eta+1} n_j - 1 \right\}
\]

(36)

For example with \( \kappa_3 = (3, 3, 2) \) and \( N_p = \{4, 2\} \), \((\kappa_3, \delta_{\{4, 2\}}) = (3, 3, 2, 3, 2, 1, 0, 1, 0)\), \( I^3 = \{1, 2\} \)

and \( J^{3,\{4,2\}} = \{4, 5, 6, 8\} \).

Related to the set \( J^{n_0, N_p} = J \) are the generalised Vandermonde products \( \Delta_{n_0, N_p} (z) \) and \( \Delta_{I, N_p} (z) \), defined by

\[
\Delta_{n_0, N_p} (z) := \prod_{\beta=1}^{p} \prod_{\substack{\eta \leq \min(J_{\beta}) \leq \iota \leq \min(J_{\beta}) \leq i \leq \max(J_{\beta})}} (z_i - z_j)
\]

and

\[
\Delta_{I, N_p} (z) := \prod_{\beta=1}^{p} \prod_{\substack{\eta \leq \min(J_{\beta}) \leq \iota \leq \min(J_{\beta}) \leq i \leq \max(J_{\beta})}} (z_i - t^{-1} z_j).
\]

In the theory of Jack polynomials with prescribed symmetry, these being denoted by \( S_{\eta^*}^{(I, J)} (z; \alpha) \), using the properties of the eigenoperators it was found that [3] with \( \eta^* = (k_{n_0}, \delta_{N_p}) \), \((I, J) = (I^{n_0}, J^{n_0, N_p}) \) and \( \kappa \) a partition such that \( \kappa_1 < \min(n_1, \ldots, n_p) \)

\[
S_{\eta^*}^{(I, J)} (z; \alpha) = \Delta_{n_0, N_p} (z) J_{\alpha}^{(p+\alpha)}(z_1, \ldots, z_{n_0}),
\]

(37)

where \( J_{\kappa}(z; \alpha) \) is a symmetric Jack polynomial. It was shown in [1], using different eigenoperator properties to the Jack case, that the Macdonald analogue of (37), with the ordering of the \( t \)-symmetric and the \( t \)-antisymmetric variables and partitions switched, holds when \( p = 1 \). Our interest is in the Macdonald analogue of (37) for \( p \geq 1 \). The stated related results, along with computational evidence, leads us to conjecture the following.
Conjecture 1 Let \( \eta^* = (\kappa_{n_0}, \delta_{N_p}) \), \((I, J) = (I^{n_0}, J^{n_0, N_p})\) and \( \kappa \) a partition such that \( \kappa_1 < \min(n_1, \ldots, n_p) \), then for \( p \geq 1 \)
\[
S_{\eta^*}^{(I, J)}(z; q, t) = \Delta_t^{n_0, N_p}(z) P_\kappa(z_1, \ldots, z_{n_0}; qt^p, t). \tag{38}
\]

We first prove a special case of the conjecture then consider the general case.

Theorem 4 With \( \eta^* = (0^{n_0}, \delta_{N_p}) \) and \((I, J) = (I^{n_0}, J^{n_0, N_p})\) we have
\[
S_{\eta^*}^{(I, J)}(z; q, t) = \Delta_t^{n_0, N_p}(z). \tag{39}
\]

Proof The result follows from \( S_{\eta^*}^{(I, J)} \) having leading term \( z^{(0^{n_0}, \delta_{N_p})} \) and the requirement that \( S_{\eta^*}^{(I, J)} \) be \( t \)-antisymmetric with respect to \( J^{n_0, N_p} \).

Due to the structure of the eigenoperator for the Macdonald polynomials the methods used in the Jack theory cannot be generalised to prove Conjecture 1 similarly the proof in 4 only works for the one-block case with the antisymmetric variables before the symmetric. However, within 3 a brief note is made on how one may show the following result
\[
S_{\rho+\delta}(z; \alpha) = \Delta(z)J_\alpha(\lambda/(\lambda+\alpha))(z; \alpha), \tag{40}
\]
where \( S_{\rho+\delta} \) is the antisymmetric Jack polynomial, using the fact that
\[
\tilde{H}_H^{(C, Ex)} \Delta f = \Delta \tilde{H}_H^{(C, Ex)}(\alpha/(\lambda+\alpha)) f,
\]
We refer the reader to 3 for the definition of the Jack polynomial eigenoperator \( \tilde{H}_H^{(C, Ex)} \) and further details of the suggested method. Low order cases have indicated that this method can be generalised to prove 4. Therefore, although the Macdonald analogue of (39), was found by Marshall in 19 using the orthogonality properties of the Macdonald polynomials we give the alternative derivation as suggested by 3 and give suggestions as to how it could be generalised to prove 4. Before stating the theorem we introduce the eigenoperator for the symmetric Macdonald polynomials 4
\[
D_\kappa^1(q, t) := t^{n-1} \sum_{i=1}^n Y_i,
\]
explicitly,
\[
D_\kappa^1(q, t) P_\kappa(z) = c_{\eta^*} P_\kappa(z), \quad c_{\eta^*} \in \mathbb{Q}(q, t).
\]

Theorem 5 We have
\[
S_{\kappa+\delta}(z; q, t) = \Delta_t(z) P_\kappa(z; qt). \tag{41}
\]

Proof Since the unique symmetric eigenfunction of \( D_\kappa^1(q, qt) \) with leading term \( m_\kappa \) (the monomial symmetric polynomial indexed by \( \kappa \)) is \( P_\kappa(z; q, qt) \), (41) will hold if for any symmetric function \( f(z) \)
\[
D_\kappa^1(q, t) \Delta_t(z) f(z) = \Delta_t(z) D_\kappa^1(q, qt) f(z). \tag{42}
\]
Hence, our task will be to prove (42). We begin by deriving a more explicit form for the left hand side of (42). Since \( \Delta_t(z) f(z) \) is \( t \)-antisymmetric the left hand side can be rewritten as
\[
\left(T_1 T_{n-1} \omega - t T_2 T_{n-1} \omega + \ldots + (-t)^{n-1} \omega\right) \Delta_t(z) f(z)
\]
which, by the definition of \( \omega \) is equal to
\[
\left( T_1...T_{n-1} - tT_2...T_{n-1} + ... + (-t)^{n-1} \right) \Delta_t \left( qz_n, z_1, ..., z_{n-1} \right) f \left( qz_n, z_1, ..., z_{n-1} \right).
\]

For simplicity we let \( \Theta_m = T_m...T_{n-1}, g_k = \Delta_t \left( qz_k, z_1, ..., z_n \right) f \left( qz_k, z_1, ..., z_n \right) \) and
\[
T_i = \frac{(1 - t) z_{i+1}}{z_i - z_{i+1}} + \frac{tz_i - z_{i+1}}{z_i - z_{i+1}} s_i.
\]

We begin by deducing the coefficient, \( c[k, m] \) say, of each \( g_k \) after being operated on by \( \Theta_m \). For this to be non-zero we require \( m \leq k \). For \( qz_k \) to appear in the first position of \( f \) we must take the term that has had \( s_i \) act on it for each \( i = n - 1, ..., k + 1 \) and therefore
\[
c[k, k] = \prod_{i=k+1}^{n} \frac{tz_k - z_i}{z_k - z_i}.
\]

It can be shown by (backward) induction on \( m \) that for \( m < k \)
\[
c[k, m] = (-1)^{k-m} \frac{(t - 1) z_k}{z_m - z_k} \prod_{i=m+1}^{k-1} \frac{tz_i - z_k}{z_i - z_k} \prod_{i=k+1}^{n} \frac{tz_k - z_i}{z_k - z_i}.
\]

We note that an important part of the inductive proof is to keep \( \Delta \) and \( f \) of the form \( \Delta_t \left( qz_k, z_1, ..., z_n \right) f \left( qz_k, z_1, ..., z_n \right) \). This is done by observing that \( s_i f(z) = f(z) \) and
\[
s_i \Delta_t(z) = \frac{tz_{i+1} - z_i}{tz_i - z_{i+1}} \Delta_t(z).
\]

To derive the coefficient of \( \Delta_t \left( qz_k, z_1, ..., z_n \right) f \left( qz_k, z_1, ..., z_n \right) \) in the overall operator we must evaluate \( \sum_{m=1}^{k} (-t)^{m-1} c[k, m] \). This is done by proving
\[
\sum_{m=j}^{k} (-t)^{m-1} c[k, m] = (-1)^{j-1} \prod_{i=j}^{k-1} \frac{z_k - tz_i}{z_k - z_i} \prod_{i=k+1}^{n} \frac{tz_k - z_i}{z_k - z_i} \times \Delta_t(z).
\]

Inductively with a base case of \( j = k - 1 \). It follows that the coefficient of \( f \left( qz_k, z_1, ..., z_n \right) \) in \( D_n^1(q, t) \Delta_t(z) f(z) \) is
\[
\prod_{i=j}^{k-1} \frac{z_k - tz_i}{z_k - z_i} \prod_{i=k+1}^{n} \frac{tz_k - z_i}{z_k - z_i} \times \Delta_t \left( qz_k, z_1, ..., z_n \right).
\]

By noting
\[
\Delta_t \left( qz_k, z_1, ..., z_n \right) = \prod_{i=j}^{k-1} \frac{qz_k - t^{-1}z_i}{z_k - t^{-1}z_k} \prod_{i=k+1}^{n} \frac{qz_k - t^{-1}z_i}{z_k - t^{-1}z_i} \Delta_t(z)
\]
we simplify [42] to
\[
\Delta_t(z) \prod_{i=1}^{n} \frac{qz_k - z_i}{z_k - z_i},
\]
and hence
\[
D_n^1(q, t) \Delta_t(z) f(z) = \Delta_t(z) \sum_{k=1}^{n} \prod_{i=1}^{n} \frac{qz_k - z_i}{z_k - z_i} f \left( qz_k, z_1, ..., z_n \right).
\]
We now simplify the right hand side of (41). We have

\[ D_n^1 (q, qt) f (z) = (qt)^{n-1} \sum_{i=1}^n Y_i^{(q)} f (z) \]

\[ = (qt)^{n-1} \left( (qt)^{-1} T_i^{(q)} ... T_{n-1}^{(q)} \omega + ... + \omega T_i^{-1(q)} ... T_{n-1}^{-1(q)} \right) f (z) \]

(43)

where \( Y_i^{(q)} \), \( T_i^{(q)} \), \( T_i^{-1(q)} \) are the operators \( Y_i \), \( T_i \), \( T_i^{-1} \) with \( t \) replaced by \( qt \). Since \( f (z) \) is symmetric we have \( T_i^{-1(q)} f (z) = (qt)^{-1} f (z) \). Using this and the action of \( \omega \) (43) simplifies to

\[ (T_1^{(q)} ... T_{n-1}^{(q)} + T_1^{(q)} + 1) f (qz_n, z_1, ..., z_{n-1}) \]

We let \( \Theta_m^{(q)} = T_m^{(q)} ... T_{n-1}^{(q)} \) and denote the coefficient of each \( f (qz_k, z_1, ..., z_n) \) by \( \hat{c}_q[k, m] \) for each \( m \leq k \). Similarly to before

\[ \hat{c}_q[k, k] = \prod_{i=k+1}^n \frac{qtz_k - z_i}{z_k - z_i} \]

and, by induction,

\[ \hat{c}_q[k, m] = (-1)^{k-m} \frac{(qt - 1) z_k}{zm - z_k} \prod_{i=m+1}^{k-1} \frac{qtz_k - z_i}{z_i - z_k} \prod_{i=k+1}^n \frac{qtz_k - z_i}{z_k - z_i} \]

For use in (43) we require \( \sum_{m=1}^k \hat{c}_q[k, m] \). This is found by induction on

\[ \sum_{m=1}^k \hat{c}_q[k, m] = \prod_{i=j}^n \frac{qtz_k - z_i}{z_k - z_i} \]

Therefore

\[ \Delta_z (x) D_n^1 (q, qt) f (z) = \Delta_z (x) \sum_{k=1}^n \prod_{i=1}^n \frac{qtz_k - z_i}{z_k - z_i} f (qz_k, z_1, ..., z_n), \]

which shows (41), and consequently (40), to be true.

\[ \square \]

Trial cases suggest that for a symmetric function \( f (z_1, ..., z_n) \) with leading term \( m, \kappa \) and \( \kappa_1 < \text{min} (n_1, ..., n_p) \) one has

\[ D_n^1 (q, t) \Delta_t^{n_0, N_p} (z) f (z_1, ..., z_n) = \Delta_t^{n_0, N_p} (z) D_n^1 (qt^p, t) f (z_1, ..., z_n) \] 

(44)

We believe it to be possible to prove Conjecture 1 by first proving (44). At this stage however it is not clear how one would keep track of the blocks of variables within the antisymmetrising set, making a strategy used to prove Theorem 5 problematic.

### 3.2 A consequence of the conjecture

A major result in the theory of Jack polynomials with prescribed symmetry is the evaluation \( U_{q^{(I,J)}} (z; \alpha) \) [8], where \( U_{q^{*}} (z; \alpha) \) is defined by

\[ U_{q^{(I,J)}} (z; \alpha) := \frac{S_{q^{(I,J)}} (z; \alpha)}{\Delta_{q^{(I,J)}} (z)} \]
It does not appear possible to generalise the Jack method to find the analogous Macdonald evaluation, \( U^{(I,J)}_{\eta^*}(z; q, t) \), where

\[
U^{(I,J)}_{\eta^*}(z; q, t) := \frac{S^{(I,J)}_{\eta^*}(z; q, t)}{\Delta_{n, N_p}(z)}.
\]

by (38) use of the evaluation formula for the symmetric Macdonald polynomials \([16]\)

\[
P^\kappa(t_\delta; q, t) = \prod_{s \in \text{diag}(\kappa)} \frac{1 - q^{a'_s(\kappa)} t^l - l(\eta)}{1 - q^{a'_s(\kappa)} t^{l(\eta)} + 1},
\]

gives the following as a corollary to Conjecture 1

\[
U^{(I,J)}_{\eta^*}(z; q, t) = \prod_{s \in \text{diag}(\eta^*)} \frac{1 - (qt^p a'_s(\eta))^l - l(\eta)}{1 - (qt^p a'_s(\eta)) t^{l(\eta)} + 1}.
\]

4 The Inner Product of Prescribed Symmetry Polynomials and Constant Term Identities

4.1 The inner product of prescribed symmetry polynomials

We begin this section by finding the explicit formulas for the inner product of the prescribed symmetry polynomials

\[
\left< S^{(I,J)}_{\eta^*}(z), S^{(I,J)}_{\eta^*}(z) \right>_{q,t},
\]

in terms of nonsymmetric Macdonald polynomials. We then proceed to show how these formulas can be used to prove specialisations of certain constant term conjectures. We first consider the inner product of \( O_{I,J} E_\eta \).

**Lemma 6** With \( K_{I,J}(t) := \Pi_{t=1}^{\eta^*} ||J||_t \Pi_{i=1}^{\eta^*} ||I_i||_{t-1}! \) we have

\[
\left< O_{I,J} E_\eta(z), O_{I,J} E_\eta(z) \right>_{q,t} = K_{I,J}(t) \left< O_{I,J} E_\eta(z), E_\eta(z) \right>_{q,t},
\]

where \( I_i \) and \( J_j \) denote the decomposition of \( I \) and \( J \) as a union of sets of consecutive integers.

**Proof** We begin by rewriting the left hand side of (6) as

\[
\left< O_{I,J} E_\eta(z), \sum_{\omega \in W_{I,J}} \left( -\frac{1}{t} \right)^{l(\omega)} T_\omega E_\eta(z) \right>_{q,t}
= \sum_{\omega \in W_{I,J}} \left< O_{I,J} E_\eta(z), \left( -\frac{1}{t} \right)^{l(\omega)} T_\omega E_\eta(z) \right>_{q,t}.
\]

Since \( T_i^{-1} \) is the adjoint operator of \( T_i \), that is \( \left< f, T_i g \right>_{q,t} = \left< T_i^{-1} f, g \right>_{q,t} \), it follows that (10) is equal to

\[
\sum_{\omega \in W_{I,J}} \left< T_{i \omega}^{-1} O_{I,J} E_\eta(z), \left( -\frac{1}{t} \right)^{l(\omega)} E_\eta(z) \right>_{q,t}.
\]
Using (13) and (14) we rewrite (47) as
\[
\sum_{\omega \in W_{I,J}} \left( \frac{(-1)^{l(\omega)}}{l(\omega)} \right) O_{I,J} E_{\eta} (z) \left( \frac{1}{t} \right) E_{\eta} (z) \right)_{q,t}.
\]
By definition of the inner product \( \langle \cdot, t^{-1} \rangle_{q,t} = \langle t, \cdot \rangle_{q,t} \) and therefore we have
\[
\sum_{\omega \in W_{I,J}} \left( \frac{1}{l(\omega)} \right) O_{I,J} E_{\eta} (z) E_{\eta} (z) \right)_{q,t} = \sum_{\omega \in W_{I,J}} \frac{1}{l(\omega)} \langle O_{I,J} E_{\eta} (z) , E_{\eta} (z) \rangle_{q,t}.
\]
The final result is obtained using the identity \( \sigma \in S_n q^l(\sigma) = [n]_q! \).

We now present the main theorem of this subsection.

**Theorem 7** We have
\[
\langle S^{(I,J)}_{\eta}\rangle (z) , S^{(I,J)}_{\eta}\rangle (z) \rangle_{q,t} = K_{I,J} (t) \frac{a_{\eta^*} (z; q, t)}{a_{\eta^*} (q^{-1}, t^{-1})} \langle E_{\eta} (z) , E_{\eta} (z) \rangle_{q,t}.
\]

**Proof** Using (16) we are able to rewrite (45) as
\[
\left( \frac{1}{a_{\eta^*} (I,J)} (q,t) \frac{1}{a_{\eta^*} (I,J)} (q,t) \right) O_{I,J} E_{\eta} (z) , E_{\eta} (z; q, t) \right)_{q,t}
\]
by the definition of the inner product and Lemma 6 we write this as
\[
\frac{K_{I,J} (t)}{a_{\eta^*} (I,J) (q,t)} \frac{O_{I,J} E_{\eta} (z; q, t)}{a_{\eta^*} (q^{-1}, t^{-1})} \langle E_{\eta} (z) , E_{\eta} (z; q, t) \rangle_{q,t}.
\]
Again using (16) we write (49)
\[
\frac{K_{I,J} (t)}{a_{\eta^*} (I,J) (q^{-1}, t^{-1})} \langle S^{(I,J)}_{\eta}\rangle (z) , E_{\eta} (z; q, t) \rangle_{q,t}
\]
By (22) and the orthogonality of the Macdonald polynomials we get the desired result.

**4.2 Special cases of the prescribed symmetry inner product**

Following the theory of Jack polynomials [1] we were lead to finding explicit formulas for (45) in the special case where \( \eta^* \) is of the form
\[
\eta^*_{(n_0:n_1)} := (0_{n_0}, \delta_{n_1}) = (0, \ldots, 0, n_1 - 1, \ldots, 1, 0),
\]
\( I = \emptyset \) and \( J = J_{n_0,n_1} \). Upon further inspection of this formula it was observed that the result could be used to provide an alternative derivation of a specialisation of a constant term conjecture from [2].

Whilst working with the constant term identities it became apparent that a further conjecture in [2] was related to the more general inner product formula where \( \eta^* \) was given by
\[
\eta^*_{(n_0:N_p)} := (0_{n_0}; \delta_{N_p}) = (0, \ldots, 0, n_1 - 1, \ldots, 1, 0, n_2 - 1, \ldots, 1, 0, \ldots, n_p - 1, \ldots, 1, 0),
\]
I = ∅ and J = J_{n_0,N_p}.

Using the theory developed in the previous section we give an explicit formula for

\[ \langle S_{\eta^{*}(n_0;N_p)}^{(I,J)}(z), S_{\eta^{*}(n_0;N_p)}^{(I,J)}(z) \rangle_{q,q^k}, \]

deriving the more specific formula for \( \eta^{*} = \eta^{*}_{(n_0;n_1)} \) as a corollary.

**Theorem 8** With \( \eta^{*} = \eta^{*}_{(n_0;N_p)} \) as defined by (51), I = ∅ and J = J_{n_0,n_1} we have

\[ \langle S_{\eta^{*}}^{(I,J)}, S_{\eta^{*}}^{(I,J)} \rangle_{q,q^k} = \prod_{i=1}^{p} [n_i]_q q - k q^{kq_i - 1} \frac{n_i(qi - 1)}{2} [k]_q! \left(1 - q^k\right)^{\max(N_p)} \frac{\prod_{j=1}^{\max(N_p)} (1 - q^j k(q(n-m(j))))}{[k]_q!n}. \]  

(52)

where \( m(j) := \sum_{k=i_j}^{p} (n_k^{+} - j) \), \( i_j := \#\{n_k^{+} \in N_p^{+}, n_k < j\} + 1 \), where \( N_p^{+} := \sigma(N_p) \) such that \( n_1^{+} \geq \ldots \geq n_p^{+} \).

**Proof** By (48) our task is to simplify

\[ K_{0,J}(q^k) \frac{\hat{c}_{\eta^{*}}^{\eta^{*}}}{a_{\eta^{*}}(q^{-1},q^{-k})} \langle E_{\eta^{*}}(z), E_{\eta^{*}}(z) \rangle_{q,q^k}. \]

For simplicity we take \( \eta = \eta^{*} \), and hence \( \hat{c}_{\eta^{*}}^{\eta^{*}} = 1 \). The \( p \) disjoint sets in \( J \) indicate

\[ K_{I,J}(q^k) = \prod_{i=1}^{p} [n_i]_q. \]

and with I = ∅, \( a_{\eta^{*}}(q^{-1},q^{-k}) \) simplifies to

\[ a_{\eta^{*}}(q^{-1},q^{-k}) = \frac{d'_{\eta^{*}}(q^{-1},q^{-k})}{d'_{\eta^{*}}(q^{-1},q^{-k})}. \]

Lastly, by (10) and (12), we have

\[ \langle E_{\eta^{*}}(q,q^k), E_{\eta^{*}}(q,q^k) \rangle_{q,q^k} = \frac{d'_{\eta^{*}}(q,q^k) e_{\eta^{*}}(q,q^k)}{d'_{\eta^{*}}(q,q^k) e'_{\eta^{*}}(q,q^k)} \frac{[nk]_q!}{[k]_q!n}. \]

Putting this all together allows us to rewrite (52) as

\[ \prod_{i=1}^{p} [n_i]_q q^{-1} q^{-k} \frac{d'_{\eta^{*}}(q^{-1},q^{-k}) d'_{\eta^{*}}(q,q^k) e_{\eta^{*}}(q,q^k)}{d'_{\eta^{*}}(q^{-1},q^{-k}) d_{\eta^{*}}(q,q^k) e'_{\eta^{*}}(q,q^k)} \frac{[nk]_q!}{[k]_q!n}. \]

(53)

We begin by simplifying

\[ \frac{d'_{\eta^{*}}(q^{-1},q^{-k})}{d'_{\eta^{*}}(q^{-1},q^{-k})} \frac{d'_{\eta^{*}}(q,q^k)}{d_{\eta^{*}}(q,q^k)} \]

(54)

In comparison with \( \eta^{*} \), \( \eta^{*}_{(-)} \) has one additional empty box above each row. Hence the leg length of each \( s \in \text{diag}(\eta^{*}_{(-)}) \) is one greater than it’s corresponding box in \( \text{diag}(\eta^{*}) \). It follows from this and the definition of \( d_q \) and \( d'_q \) that \( a_{\eta^{*}_{(-)}}(q^{-1},q^{-k}) = d_q \). Hence we can rewrite (54) as

\[ \frac{d_{\eta^{*}}(q^{-1},q^{-k}) d'_{\eta^{*}}(q,q^k)}{d'_{\eta^{*}}(q^{-1},q^{-k}) d_{\eta^{*}}(q,q^k)} = q^{k q^{-1} n_i (n_i - 1)/2}. \]

(55)
where the equality follows upon use of the definition of $d^*$, given above (10), and the simple identity

\[
\frac{1 - x}{1 - x^{-1}} = -x.
\]

We now consider the simplification of the ratio $e/e'$. Explicitly we have

\[
\frac{e_{\eta^*} (q, q^k)}{e'_{\eta^*} (q, q^k)} = \frac{\prod_s \left( 1 - q^{a^s (s)+1} q^{k(n-l^s (s))} \right)}{\prod_r \left( 1 - q^{a^r (r)+1} q^{k(n-1-l^r (r))} \right)}.
\]

(56)

In [21] Sahi showed $e_{\eta} = e_{s, \eta}$ and $e'_{\eta} = e'_{s, \eta}$, hence, the products in (56) are independent of the row order. For simplicity we take $\eta^* = \eta^{*+}$. In such a composition $n - l^s_{\eta^{*+}}((i, j)) = n - 1 - l^s_{\eta^{*+}}((i + 1, j))$ and consequently most terms in (56) cancel. The terms unique to the numerator correspond to the boxes in the top row of $\eta^{*+}$, and therefore the terms remaining in the denominator will correspond to the bottom box of each column. The leg column lengths of the latter set are given by $m(j) - 1$, (for $j = 1 \ldots \max(N_p) - 1$). Hence the ratio of the $e'$’s in our expansion is given by

\[
\frac{e_{\eta^*} (q, q^k)}{e'_{\eta^*} (q, q^k)} = \frac{(1 - q^{kn+1}) (1 - q^{kn+2}) \ldots (1 - q^{kn+\max(N_p)-1})}{\prod_{j=1}^{\max(N_p)-1} (1 - q^j q^{k(m(j))})}
\]

\[
\times \frac{(1 - q)^{\max(N_p)} [kn + \max(N_p)] q^k}{\prod_{j=1}^{\max(N_p)-1} (1 - q^j q^{k(n-m(j))}) (1 - q^{\max(N_p)} q^{kn}) [kn] q^k}.
\]

The last simplification was made by noting $m(\max(N_p)) = 0$. Substituting each simplification into (53) gives the required result.

We now give the analogous result for $\eta^* = \eta^*_{(n_0,n_1)}$.

**Corollary 2** With $\eta^*_{(n_0,n_1)}$ given by (50) we have

\[
\left< S_{\eta^*_{(n_0,n_1)}}^{(I,J)} (z) , S_{\eta^*_{(n_0,n_1)}}^{(I,J)} (z) \right>_{q,q^k} = \frac{[n_1] q^k [n_1 + nk] q^k (1 - q)^{n_1}}{(kq^k) q^{n_1(k+1)+nk; q^{-(k+1)}} q^{(kn_1-1)n_1}}.
\]

(57)

**Proof** The result follows immediately from Theorem 8 by substituting $p = 1$ into the right hand side of (52) and simplifying using (11).

We now present the two conjectures put forward by Baker and Forrester [2]. We conclude the paper by showing how our results can be used to prove the special case of these conjectures when $a = b = 0$. 

**4.3 The constant term identities**
Conjecture 2 [2, Conj 2.1] We have

\[
D_1(n_0; n_1; a, b; q) = \text{CT} \left( \prod_{n_0+1 \leq i < j \leq n} (z_i - q^{k+1}z_j)(z_i^{-1} - q^{k}z_j^{-1}) \prod_{i < j} \left( \frac{z_i}{z_j} \right)_q \left( \frac{q z_j}{z_i} \right)_q \right) 
\times \prod_{i=1}^{n} \left( \frac{z_i}{z_i} \right)_a \left( \frac{q z_i}{z_i} \right)_b 
= \frac{\Gamma_q(1) \Gamma_q(a + b + 1) \Gamma_q(1 + k(l + 1))}{(\Gamma_q(1 + k))^n} \prod_{l=0}^{n_1-1} \frac{\Gamma_q((k + 1)j + a + b + kn_0 + 1) \Gamma_q((k + 1)j + k + kn_0)}{\Gamma_q((k + 1)j + 1) \Gamma_q((k + 1)j + b + kn_0 + 1)}
\]

The \(n_0 = 0\) case of Conjecture 2 reduces to the \(q\)-Morris constant term identity, well known in the theory of Selberg integrals (see, e.g., [9]). Within [2] Baker and Forrester were able to prove Conjecture 2 for the cases \(a = k\) and \(n_1 = 2\). In a related work [5] they also proved the case where \(a = b = 0\). In both cases a combinatorial identity of Bressaud and Goulden [6] is used. Following this Hamada [11] confirmed the general cases \(n_1 = 2\) and \(n_1 = 3\) using a \(q\)-integration formula of Macdonald polynomials and Gessel [10] showed the conjecture to be true for \(n_1 = 2, n - 1, 3\) and also for the cases where \(n \leq 5\).

Conjecture 3 [2, Conj 2.2] Define \(D_p(n_1, \ldots, n_p; n_0; a, b, k; q)\) by

\[
D_p(n_1, \ldots, n_p; n_0; a, b, k; q) := \text{CT} \left( \prod_{\alpha = 1}^{p} \prod_{\min(\tilde{J}_\alpha) \leq i < j} \left( \frac{z_i}{z_j} \right)_q \left( \frac{q z_j}{z_i} \right)_q \prod_{i=1}^{n} \left( \frac{z_i}{z_i} \right)_a \left( \frac{q z_i}{z_i} \right)_b \right)
\]

where \(\tilde{J}_\alpha\) is given by [70]. Then for \(n_p > n_j\) \((j = 1, \ldots, p - 1)\) we have

\[
\frac{D_p(n_1, \ldots, n_p; n_0; n_p + 1; 0, 0, 0; k; q)}{D_p(n_1, \ldots, n_p; n_0; 0, 0, 0; k; q)} = \frac{[n_p + 1]_{q^{k+1}}}{[k]_q!} \frac{\Gamma_q((k + 1) + n_p + 1 + k \sum_{j=0}^{p-1} n_j)}{\Gamma_q((k + 1) n_p + k \sum_{j=0}^{p-1} n_j)} \frac{[n_p + 1]_{q^{k+1}} [k + n_p]_{q}!}{[k]_q! [k n + n_p]_{q}!} \quad (58)
\]

Using the following Lemma reclaim the result for the \(a = b = 0\) case of Conjecture 2 proved by Baker and Forrester in [5]. Although the result is already known to be true, the following highlights the strong connection between the conjectured constant terms and Macdonald polynomial theory. On this point the special case of Conjecture 2 corresponding to the \(q\)-Morris identity is well known to relate to Macdonald polynomial theory and furthermore has generalisations involving the Macdonald polynomial in an identity due to Kaneko [12].
Lemma 9 Let \( J = \{r, \ldots, r + s\} \subseteq \{1, \ldots, n\} \) and \( h(z) \) be antisymmetric with respect to \( z_j, j \in J \) then
\[
\text{CT}
\left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z)\right) = \frac{[s]_a!}{s!} \text{CT}
\left(\prod_{r \leq i < j \leq r+s} (z_i - z_j) h(z)\right).
\]

Proof Let \( S_J = \{s_j; j = r, \ldots, r + s - 1\} \). For any permutation \( \sigma \in S_J \), the operation \( z \rightarrow \sigma z \) leaves the constant term unchanged. Hence
\[
\text{CT}
\left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z)\right) = \text{CT}
\left(\prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) h(\sigma z)\right).
\]

Since \( h(x) \) is antisymmetric \( h(\sigma x) = (-1)^{l(\sigma)} h(x) \), summing over all permutations gives
\[
\text{CT}
\left(\prod_{r \leq i < j \leq r+s} (z_i - az_j) h(z)\right) = \frac{1}{s!} \text{CT}
\left(\sum_{\sigma \in S_J} (-1)^{l(\sigma)} \prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) h(\sigma z)\right).
\]

Since
\[
\sum_{\sigma \in S_J} (-1)^{l(\sigma)} \prod_{r \leq i < j \leq r+s} (z_{\sigma(i)} - az_{\sigma(j)}) = \sum_{\omega \in S_J} (-1)^{l(\omega)} \sum_{\sigma \in S_J} (-a)^{l(\sigma)} z^{\sigma \delta}.
\]

Letting \( \omega = \sigma^{-1} \) we see that the constant term of (59) is \( \sum_{\sigma \in S_J} a^{l(\sigma)} \). The result follows from the identity \( \sum_{\sigma \in S_J} a^{l(\sigma)} = [s]_a! \).

\square

Theorem 10 We have
\[
D_1 \left(n_1; n_0; 0, 0, k; q \right) = \frac{\Gamma_q (n_1 + 1)}{\Gamma_q ((1 + k)n_0 + (k + 1)j + 1)} \prod_{j=0}^{n_1-1} \frac{\Gamma_q ((k + 1)j + 1) + kn_0 + 1)}{\Gamma_q ((k + 1)j + 1)}
\times \prod_{l=0}^{n_0-1} \frac{\Gamma_q (1 + k(l + 1))}{\Gamma_q (1 + kl)}.
\]

Proof With \( n^*_{(n_0, n_1)} \) defined by (50), Theorem 4 gives
\[
S_{n^*_{(n_0, n_1)}}(z) = \Delta_{n_0, \{n_1\}}(z)
\]

and hence
\[
(S_{n^*_{(n_0, n_1)}})_q(z) = \text{CT}
\left(\prod_{n_0+1 \leq i < j \leq n} (z_i - q^{-k}z_j)(z_{j-1} - q^{k}z_{j-1}) \prod_{1 \leq i < j \leq n} \left(\frac{z_i}{z_j}; k\right) \left(\frac{q z_j}{z_i}; k\right)\right).
\]

To be able to apply Lemma 9 we view (61) as
\[
\text{CT}
\left(\prod_{n_0+1 \leq i < j \leq n} (z_i - q^{-k}z_j) h(z)\right)
\]

where \( h(z) \) is antisymmetric with respect to \( z_{n_0+1}, \ldots, z_n \). Using Lemma 9 twice we see that
\[
D_1 \left(n_1; n_0; 0, 0, k; q \right) = \frac{[n_1]_{q+k-1}!}{[n_1]_{q-k}!} (S_{n^*_{(n_0, n_1)}})_q(z).
\]
Using (57) and
\[
\frac{[n_1]q^k!}{[n_1]q^{-k}!} = q^{k(n_1-1)n_1}
\]
we obtain
\[
D_1 (n_1; n_0; 0, 0, k; q) = \frac{[n_1]q^{k+1}![n_1 + nk]q^l (1 - q)^{n_1}}{([k]q^l)^n (q^{n_1(k+1)+nk}; q^{-k+1})_{n_1}}. \tag{62}
\]
To show (62) is equivalent to (60) we firstly use the identity [14] \( \Gamma_q (1 + n) = [n]q! \) to show
\[
\frac{[n_1]q^{k+1}!}{([k]q^l)^n} = \frac{\Gamma_q (n_1 + 1)}{(\Gamma_q (1 + k))^n}
\]
and
\[
\prod_{l=0}^{n_0-1} \frac{\Gamma_q (1 + k(l + 1))}{\Gamma_q (1 + kl)} = [n_0k]q!.
\]
Lastly, to show
\[
\frac{[n_1 + nk]q^l (1 - q)^{n_1}}{(q^{n_1(k+1)+nk}; q^{-k+1})_{n_1}} = [n_0]q!] \prod_{j=0}^{n_1-1} \frac{[(k + 1) j + km + k]q^l}{[(k + 1) j + km]q^l},
\]
we expand both sides and compare terms.

Our final result in this section is proving the specialisation \( a = b = 0 \) of Conjecture 3, which until now has not been done. We do this by finding the analogue of Theorem 10 for \( \eta^{*}_{(n_0, N_p)} \) and stating the result as a corollary. We begin with a generalisation of Lemma 9.

Lemma 11 Let \( J \subseteq \{1, \ldots, n\} = J_1 \cup \ldots \cup J_s \) and \( h(z) \) be antisymmetric with respect to \( z_j, \ j \in J \) then
\[
\text{CT} \left( \prod_{\alpha=1}^{s} \prod_{\min(J_\alpha) \leq i < j \leq \max(J_\alpha)} (z_i - a z_j) h(z) \right) = \prod_{\alpha=1}^{s} \frac{[J_\alpha]q!}{|J_\alpha|!} \text{CT} \left( \prod_{\alpha=1}^{s} \prod_{\min(J_\alpha) \leq i < j \leq \max(J_\alpha)} (z_i - z_j) h(z) \right).
\]

Theorem 12 We have
\[
D_p (N_p; n_0; 0, 0, k; q) = \prod_{\alpha=1}^{p} \left( [n_\alpha]q^{k+1}! \right) \frac{(kn + \max(N_p))q^l}{([k]q^l)^n} \times \frac{(1 - q)^{\max(N_p)}}{\prod_{i=1}^{\max(N_p)} (1 - q^i q^{kn_{(n-i)}})}.
\tag{63}
\]

Proof With \( \eta^{*}_{(n_0, N_p)} \) defined by (51), Theorem 4 implies
\[
S_{\eta^{*}_{(n_0, N_p)}} (z) = \Delta_{n_0, N_p} (z)
\]
and hence the inner product \( \langle S_{\eta^{*}}, S_{\eta'} \rangle_{q, k} \) can be written as
\[
\langle S_{\eta^{*}}, S_{\eta'} \rangle_{q, k} = \text{CT} \left( \prod_{\alpha=1}^{p} \prod_{\min(J_\alpha) \leq i < j \leq \max(J_\alpha)} (z_i - q^{-k} z_j) (z_i^{-1} - q^{k} z_j^{-1}) \prod_{1 \leq i < j \leq N} \left( \frac{z_i}{z_j}; q \right)_k \left( \frac{z_j}{z_i}; q \right)_k \right).
\]

19
By apply Lemma [11] twice to (52) and noting that
\[ \frac{[n_i]_{q^k}!}{[n_i]_{q^{-k}}!} = q^{\frac{k(n_i-1)n_i}{2}} , \]
the result is obtained using the methods of the derivation in Theorem [10].

Conjecture 3 is verified by substituting (63) into the left hand side of (58) and making the obvious simplifications.

Acknowledgement.

I would like to thank my supervisor Peter Forrester for his valuable advice and support and also Ole Warnaar for his comments on an earlier draft. This work was supported by an APA scholarship, the ARC and the University of Melbourne.

References

[1] T. H. Baker, C. F. Dunkl, and P. J. Forrester. Polynomial eigenfunctions of the Calogero-Sutherland-Moser models with exchange terms. CRM Ser. Math. Phys., pages 37–51. Springer, New York, 2000.

[2] T. H. Baker and P. J. Forrester. Generalized weight functions and the Macdonald polynomials. 1996. q-alg/9603005

[3] T. H. Baker and P. J. Forrester. The Calogero-Sutherland model and polynomials with prescribed symmetry. Nuclear Phys. B, 492, 1997.

[4] T. H. Baker and P. J. Forrester. A q-analogue of the type A Dunkl operator and integral kernel. Internat. Math. Res. Notices, (14):667–686, 1997.

[5] T. H. Baker and P. J. Forrester. Generalizations of the q-Morris constant term identity. J. Combin. Theory Ser. A, 81(1):69–87, 1998.

[6] D. M. Bressoud and I. P. Goulden. Constant term identities extending the q-Dyson theorem. Trans. Amer. Math. Soc., 291(1):203–228, 1985.

[7] I. Cherednik. Nonsymmetric Macdonald polynomials. Internat. Math. Res. Notices, (10):483–515, 1995.

[8] C. F. Dunkl. Orthogonal polynomials of types A and B and related Calogero models. Comm. Math. Phys., 197(2):451–487, 1998.

[9] P. J. Forrester and S. O. Warnaar. The importance of the Selberg integral. Bull. Amer. Math. Soc. (N.S.), 45(4):489–534, 2008.

[10] I. M. Gessel, L. Lv, G. Xin, and Y. Zhou. A proof of Askey’s conjectured q-analogue of Selberg’s integral and a conjecture of Morris. SIAM J. Math. Anal., 19(4):969–986, 1988.

[11] S. Hamada. Proof of Baker-Forrester’s constant term conjecture for the cases N_1 = 2,3. Kyushu J. Math., 56(2):243–266, 2002.
[12] Kaneko. J. Constant term identities of Forrester-Zeilberger-Cooper. *Discrete Math.*, 173(1-3):79–90, 1997.

[13] Y. Kato and T. Yamamoto. Jack polynomials with prescribed symmetry and hole propagator of spin Calogero-Sutherland model. *J. Phys. A*, 31(46):9171–9184, 1998.

[14] T. H. Koornwinder. *q*-special functions, an overview, 2005. [arXiv:math/0511148v1](https://arxiv.org/abs/math/0511148v1).

[15] I. G. Macdonald. A new class of symmetric functions. *I.R.M.A*, 20:131–171, 1988.

[16] I. G. Macdonald. *Symmetric functions and Hall polynomials*. The Clarendon Press Oxford University Press, New York, second edition, 1995.

[17] I. G. Macdonald. *Affine Hecke algebras and orthogonal polynomials*, volume 157 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2003.

[18] D. Marshall. *Macdonald polynomials*. Masters thesis, the University of Melbourne, 1999.

[19] D. Marshall. Symmetric and nonsymmetric Macdonald polynomials. *Ann. Comb.*, 3(2-4):385–415, 1999. On combinatorics and statistical mechanics.

[20] K. Mimachi and M. Noumi. A reproducing kernel for nonsymmetric Macdonald polynomials. *Duke Math. J.*, 91(3):621–634, 1998.

[21] S. Sahi. A new scalar product for nonsymmetric Jack polynomials. *Internat. Math. Res. Notices*, (20):997–1004, 1996.

[22] R. P. Stanley. *Enumerative combinatorics. Vol. 1*. Cambridge University Press, New York/-Cambridge, 1997.

---

Wendy Baratta  
Department of Mathematics  
University of Melbourne  
Australia  
(E-mail: w.baratta@ms.unimelb.edu.au)