Unified Universal Seesaw Models*

Peter Cho

California Institute of Technology, Pasadena, CA 91125

Abstract

A set of Grand Unified Theories based upon the gauge groups $SU(5)_L \times SU(5)_R$, $SO(10)_L \times SO(10)_R$ and $SU(4)_C \times SU(4)_L \times SU(4)_R$ is explored. Several novel features distinguish these theories from the well-known $SU(5)$, $SO(10)$ and $SU(4)_C \times SU(2)_L \times SU(2)_R$ models which they generalize. Firstly, Standard Model quarks and leptons are accompanied by and mix with heavy $SU(2)_L \times SU(2)_R$ singlet partners. The resulting fermion mass matrices are seesaw in form. Discrete parity symmetries render the determinants of these mass matrices real and eliminate CP violating gauge terms. The unified seesaw models consequently provide a possible resolution to the strong CP problem. Secondly, $\sin^2 \theta_W$ at the unification scale is numerically smaller than the experimentally measured $Z$ scale value. The weak angle must therefore increase as it evolves down in energy. Finally, proton decay is suppressed by small seesaw mixing factors in all these theories.

*Work supported in part by the U.S. Dept. of Energy under Contract no. DEAC-03-81ER40050.
1. Introduction

Among the many questions left unanswered by the Standard Model of particle physics, the origin of fermion masses ranks as one of the most intriguing and important. Details of the fermion mass spectrum remain a perplexing mystery, and even its gross features are not understood. One general characteristic which remains unexplained within the context of the minimal Standard Model is the disparity between the electroweak scale and quark and lepton masses. In the most extreme case, the mass of the electron is roughly a million times smaller than the weak scale. This dichotomy can of course be accommodated in the Standard Model by tuning certain Yukawa couplings to be sufficiently small. However, a more natural explanation for this mass gap would be preferable.

In the past few years, such an explanation has been offered in which the familiar neutrino seesaw mechanism [1] is applied to charged fermions as well [2,3]. This universal seesaw proposal necessitates the introduction of new heavy partners for each of the known Standard Model fermions with which they mix. The lightness of observed quarks and leptons then results as a natural consequence of the seesaw mechanism. This scheme obviously works best for the first generation of fermions and worst for the third. In particular, achieving the anomalously large mass for the top quark is problematic. Nonetheless, the basic idea of a universal seesaw mechanism is appealing and sheds some light on the fermion mass puzzle.

A second fundamental question left unaddressed by the Standard Model but which theories with a universal seesaw mechanism can resolve is the strong CP problem. Such theories generally possess a parity symmetry which prohibits a CP violating $\theta_{QCD}$ term from appearing in the QCD Lagrangian and renders Yukawa coupling matrices hermitian. So while the fermion mass matrix can be complex and generate weak CP violation, the argument $\theta_{QFD}$ of its determinant is zero. The physically observable parameter $\theta = \theta_{QCD} + \theta_{QFD}$ consequently vanishes at tree order. Universal seesaw models thus offer a solution to the strong CP problem which does not involve axions [4,5,6].

The universal seesaw mechanism has been studied in the past mainly within the context of the left-right symmetric $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)$ model. In this paper, we explore a number of possibilities for embedding this mechanism within a unified theory. In particular, we investigate models based upon the gauge groups $SU(5)_L \times SU(5)_R$, $SO(10)_L \times SO(10)_R$ and $SU(4)_C \times SU(4)_L \times SU(4)_R$. As we shall see, such unified theories provide a rationale for the seemingly ad hoc introduction of heavy $SU(2)_L \times SU(2)_R$ singlet
fermions in their un-unified counterparts. Moreover, these particular models generalize the
well-known $SU(5)$ [7] and $SO(10)$ [8] Grand Unified Theories and the $SU(4)_C \times SU(2)_L \times
SU(2)_R$ Pati-Salam model [9]. So they are of interest in their own right.

To help guide our exploration, we will adopt the following set of unified seesaw model
building rules:

I. The model must reproduce the measured $Z$ scale values for the Standard Model cou-
plings [10,11,12]

$$\sin^2 \theta_W(M_Z) = 0.2325 \pm 0.0008$$

$$\alpha_{EM}^{-1}(M_Z) = 127.8 \pm 0.2$$

$$\alpha_s(M_Z) = 0.118 \pm 0.007.$$ (1.1a, 1.1b, 1.1c)

II. The model must satisfy other phenomenological constraints such as limits on new
particle masses and bounds on proton decay.

III. The model should incorporate heavy $SU(2)_L \times SU(2)_R$ singlet fermions which mix
with Standard Model quarks and leptons to allow for a seesaw mass matrix whose
determinant is real.

IV. The model should contain fermions in anomaly free but complex representations in
accordance with the “survival hypothesis” [13].

V. The model preferably maintains left-right symmetry from the unification scale down
to the Standard Model subgroup level.

These requirements are listed in approximate order of importance. The first two experi-
mental constraints are binding and must be satisfied by any realistic Grand Unified Theory.
The remaining theoretical guidelines are more negotiable. In particular, the last item is
included only to help restrict the large number of possible symmetry breaking patterns
in the models we shall explore. So we may relax this final aesthetic condition in order to
fulfill the other more stringent requirements in this list.

The remainder of our paper is organized as follows. We present the $SU(5) \times SU(5)$ and
$SO(10) \times SO(10)$ models in sections 2 and 3. These theories illustrate the basic features of
all unified universal seesaw models. They also serve as warmups for the $SU(4) \times SU(4) \times
SU(4)$ model which is discussed in greater detail in section 4. Finally, we close in section 5
with some indications for possible further investigation of this new class of Grand Unified
Theories.
2. The prototype $SU(5) \times SU(5)$ model

The first model that we shall explore is based upon the gauge group $G = SU(5)_L \times SU(5)_R$. This theory represents an obvious generalization of the Georgi-Glashow $SU(5)$ model and shares many of its attractive features. It is also the simplest unified seesaw model and has been analyzed in the past. While this theory ultimately turns out not to be phenomenologically viable, it is worth reviewing since many of its basic characteristics are common to all unified universal seesaw models.

To begin, we impose a $Z_2$ symmetry on the chiral theory which combines a spatial inversion with interchanging the $SU(5)$ factors in the product group $G$. Such a discrete symmetry is needed to ensure the equality of the $SU(5)_L$ and $SU(5)_R$ couplings constants above the unification scale. In its absence, the couplings would run differently and diverge even if they were set equal at one particular renormalization point. The generalized parity operation enforces a left-right symmetry on the Lagrangian which may be violated only softly by superrenormalizable interactions. It also dictates a one-to-one correspondence among matter field representations of $SU(5)_L$ and $SU(5)_R$. The spectrum of this theory consequently exhibits an explicit parity doubling.

We next embed the Standard Model within the Grand Unified Theory following the Georgi-Glashow model blueprint. Color $SU(3)$ and weak $SU(2)$ are identified with the diagonal $SU(3)_{L+R}$ subgroup of $G$ and the $SU(2)_L$ subgroup of $SU(5)_L$ respectively. $U(1)_{EM}$ is generated by the diagonal sum of the familiar $SU(5)_L$ and $SU(5)_R$ electric charge generators. The $SU(3) \times SU(2) \times U(1)$ content of a single fermion family representation

$$\mathcal{F} \sim (\overline{5} + 10, 1) + (1, \overline{5} + 10)$$

is then readily established. The fermions’ colors, flavors and electric charges are indicated by conventional letter names in the matrices below:

$$\begin{pmatrix} D_1^c \\ D_2^c \\ D_3^c \\ e \\ -\nu \end{pmatrix}_L \sim (\overline{5}, 1) \quad \begin{pmatrix} 0 & U_3^c & -U_2^c & -u_1 & -d_1 \\ -U_3^c & 0 & U_1^c & -u_2 & -d_2 \\ U_2^c & -U_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -E^c \\ d_1 & d_2 & d_3 & E^c & 0 \end{pmatrix}_L^{ij} \sim (10, 1)$$

(2.1)

$$\begin{pmatrix} D_1^c \\ D_2^c \\ D_3^c \\ e \\ -\nu \end{pmatrix}_R \sim (1, \overline{5}) \quad \begin{pmatrix} 0 & U_3^c & -U_2^c & -u_1 & -d_1 \\ -U_3^c & 0 & U_1^c & -u_2 & -d_2 \\ U_2^c & -U_1^c & 0 & -u_3 & -d_3 \\ u_1 & u_2 & u_3 & 0 & -E^c \\ d_1 & d_2 & d_3 & E^c & 0 \end{pmatrix}_R^{i'j'} \sim (1, 10).$$

(2.2)
Three generations of families are assumed as in the Standard Model and assigned to three copies of $F$.

We now specify a simple symmetry breaking pattern that starts with the unified chiral gauge group and cascades down to unbroken color and electromagnetism:

$$SU(5)_L \times SU(5)_R$$
$$\downarrow M_{GUT}$$
$$SU(3)_L \times SU(2)_L \times U(1)_L \times SU(3)_R \times SU(2)_R \times U(1)_R$$
$$\downarrow \Lambda_{LR}$$
$$SU(3)_{L+R} \times SU(2)_L \times SU(2)_R \times U(1)_{L+R}$$
$$\downarrow v_R$$
$$SU(3)_{L+R} \times SU(2)_L \times U(1)_Y$$
$$\downarrow v_L$$
$$SU(3)_{L+R} \times U(1)_{EM}.$$  \hspace{1cm} (2.3)

A minimal number of fundamental Higgs fields is introduced into the theory to achieve this pattern. As in the Georgi-Glashow model, $SU(5)_L$ and $SU(5)_R$ are broken with scalars $\Phi_{L} \sim (24, 1)$ and $\Phi_{R} \sim (1, 24)$ that transform in their adjoint representations. The fermion families decompose under the resulting $(SU(3) \times SU(2) \times U(1))^2$ subgroup as

$$F \sim \left[(\overline{3}, 1, 1, 1)^{\frac{4}{2}, 0} + (1, \overline{3}, 1, 1)^{-\frac{4}{2}, 0} + (3, 2, 1, 1)^{\frac{4}{2}, 0} + (1, 1, 1, 1)^{1, 0}\right]_L$$
$$+ \left[(1, 1, \overline{3}, 1)^{0, \frac{1}{2}} + (1, 1, 1, \overline{3})^{0, -\frac{1}{2}} + (1, 1, \overline{3}, 1)^{0, -\frac{3}{2}} + (1, 1, 3, 2)^{0, \frac{3}{2}} + (1, 1, 1, 1)^{0, 1}\right]_R.$$  \hspace{1cm} (2.4)

The subsequent breaking of chiral color and chiral hypercharge to their diagonal subgroups is performed at the $\Lambda_{LR}$ scale by Higgs fields $\omega \sim (\overline{5}, 5)$ and $\Omega \sim (10, \overline{10})$. If these scalars develop the vacuum expectation values

$$\langle \omega \rangle_1' = \langle \omega \rangle_2' = \langle \omega \rangle_3' = \langle \Omega \rangle_{[12]}^{[12]} = \langle \Omega \rangle_{[23]}^{[23]} = \langle \Omega \rangle_{[31]}^{[31]} = \langle \Omega \rangle_{[45]}^{[45]} = \Lambda_{LR},$$ \hspace{1cm} (2.5)

the chirally colored $(\overline{3}, 1, 1, 1)$ and $(1, 1, \overline{3}, 1)$ and chirally hypercharged $(1, 1, 1, 1)^{1, 0}$ and $(1, 1, 1, 1)^{0, 1}$ fields in (2.4) marry together and acquire Dirac masses through the Yukawa interactions

$$L_{\text{Yukawa}}(\omega, \Omega) = -f_{\omega}(\overline{\psi}_L)^{\prime i}(\omega)^{\prime i}(\psi_R)_{i'} + \frac{f_{\Omega}}{2}(\overline{\psi}_L)_{ij}(\Omega)^{ij}_{i'j'}(\psi_R)_{i'j'} + \text{h.c.} \hspace{1cm} (2.6a)$$
These fourteen $SU(2)_L \times SU(2)_R$ singlet fermions automatically emerge in the unified theory as the heavy seesaw partners that are added by hand in un-unified seesaw models. They are denoted by capital letters in (2.2). The remaining sixteen fields in (2.4) reside within $SU(2)$ doublets and stay massless at the $\Lambda_{LR}$ scale. They essentially correspond to the known Standard Model fermions plus a right handed neutrino and are represented by the lower case letters in (2.2).

The last two steps in pattern (2.3) are accomplished by scalars $\phi_L \sim (5, 1)$ and $\phi_R \sim (1, 5)$ which break $SU(2)_L$ and $SU(2)_R$ via the VEV’s $\langle \phi_{L,R} \rangle = (0, 0, 0, v_{L,R}/\sqrt{2})^T$. Masses connecting heavy and light fermions are then generated by the Yukawa terms

$$\mathcal{L}_{\text{Yukawa}}(\phi) = f_\phi \left[ (\Psi_L^T)^i C(\Psi_L)^{ij} (\phi_L^\dagger) j + (\Psi_R^T)^i C(\Psi_R)^{ij'} (\phi_R^\dagger) j' \right] + f'_\phi \left[ \epsilon_{ijklm} (\Psi_L^T)^{ij} C(\Psi_L)^{kl} (\phi_L)^m + \epsilon_{i'j'k'l'} (\Psi_R^T)^{ij'} C(\Psi_R)^{k'l'} (\phi_R)^{m'} \right] + \text{h.c.} \quad (2.6b)$$

The quark and lepton mass matrices thus assume the seesaw forms

$$\mathcal{L}_{\text{mass}} = \begin{pmatrix} w_L & U_L \end{pmatrix} \begin{pmatrix} 0 & \sqrt{2} f_\phi^\dagger v_L \\ \sqrt{2} f_\phi v_R & f_\phi^\dagger \Lambda_{LR} \end{pmatrix} \begin{pmatrix} u_R \\ U_R \end{pmatrix}$$

$$+ \begin{pmatrix} d_L & D_L \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} f_\phi^\dagger v_L \\ \frac{1}{2} f_\phi v_R & f_\phi^\dagger \Lambda_{LR} \end{pmatrix} \begin{pmatrix} d_R \\ D_R \end{pmatrix} \quad (2.7)$$

$$+ \begin{pmatrix} e_R^\dagger & E_R^\dagger \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} f_\phi v_L \\ \frac{1}{2} f_\phi^\dagger v_R & f_\phi \Lambda_{LR} \end{pmatrix} \begin{pmatrix} e_L^\dagger \\ E_L^\dagger \end{pmatrix} + \text{h.c.}$$

It is important to recall that the fermion fields are $N_F = 3$ dimensional vectors in family space. The Yukawa couplings in eqns. (2.6a,b) are consequently $N_F \times N_F$ matrices with generation indices that have been suppressed. Parity constrains $f_\omega$ and $f_\Omega$ to be hermitian, while the form of the second term in (2.6b) automatically renders $f'_\phi$ symmetric. If these Yukawa couplings are approximately comparable in magnitude, then the mass matrices have the well-known seesaw eigenvalues

$$m \simeq -O\left( f \frac{v_L v_R}{\Lambda_{LR}} \right)$$

$$M \simeq O\left( f \Lambda_{LR} \right) \quad (2.8)$$
and corresponding eigenvectors

\[
\begin{pmatrix}
q' \\
Q'
\end{pmatrix} = \begin{pmatrix}
1 & O(v_L/\Lambda_{LR}) \\
O(v_R/\Lambda_{LR}) & 1
\end{pmatrix}
\begin{pmatrix}
q \\
Q
\end{pmatrix}
\]

provided \( v_L v_R \ll \Lambda_{LR}^2 \). We thus recover the universal seesaw mechanism in this \( SU(5) \times SU(5) \) theory.

The fermion mass matrices in (2.7) are generally complex and induce weak CP violation as in the Standard Model. But their determinants are real. This can be simply verified by rewriting the down-type quark matrix for example as

\[
M_{dD} = \begin{pmatrix}
1 & 0 \\
0 & v_R/\Lambda_{LR}
\end{pmatrix}
\begin{pmatrix}
0 & \frac{1}{2} f_\phi^\dagger \Lambda_{LR} \\
\frac{1}{2} f_\phi^\dagger \Lambda_{LR} & f_\omega^\dagger \Lambda_{LR}^3 / v_L v_R
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & v_L/\Lambda_{LR}
\end{pmatrix}.
\]

(2.10)

Since the diagonal matrices are real while the middle matrix is hermitian, we conclude that \( \arg(\det M_{dD}) = 0 \). So as a result of the generalized parity symmetry in the \( SU(5) \times SU(5) \) model, the complex argument \( \theta_{QFD} \) of the total mass matrix as well as the \( \theta_{QCD} \) term in the QCD Lagrangian vanish at tree order. The seesaw GUT therefore provides a possible solution to the strong CP problem.

Unfortunately, the symmetry breaking pattern in (2.3) is not phenomenologically viable. Recall that once the embedding of the electroweak subgroup inside the gauge group \( G \) is specified, the value of \( \sin^2 \theta_W \) at the unification scale is fixed:

\[
\sin^2 \theta_W(M_{GUT}) = \frac{\Tr (T^3_L)^2}{\Tr Q^2} = \frac{3}{16} = 0.1875.
\]

(2.11)

In this \( SU(5) \times SU(5) \) model, there are twice as many electrically charged fermions as in the \( SU(5) \) theory but precisely the same number of weak \( SU(2)_L \) doublets. So \( \sin^2 \theta_W(M_{GUT}) \) is half as large as in the Georgi-Glashow model \[7\] and starts out numerically smaller than \( \sin^2 \theta_W(M_Z) = 0.2325 \). Moreover, renormalization effects decrease the value of \( \sin^2 \theta_W(\mu) \) for \( \mu < M_{GUT} \) in the \( SU(5) \times SU(5) \) theory just as in the \( SU(5) \) model \[10\]. Therefore, pattern (2.3) cannot duplicate the \( Z \) scale measurement and must be rejected.

One can try to search for alternate breaking patterns in which \( \sin^2 \theta_W \) increases as it evolves down in energy from the GUT scale. Maximal enhancement is achieved if the first stage of symmetry breaking is taken to be \( SU(5)_L \times SU(5)_R \to SU(3)_L \times SU(2)_L \times U(1)_L \times SU(5)_R \) \[14,15\]. This clearly leads to trouble with proton decay. Moreover, detailed calculation demonstrates that this asymmetrical breaking pattern still cannot yield the values for the Standard Model couplings in \[11,13\]. We therefore conclude that an \( SU(5) \times SU(5) \) seesaw theory is ruled out.
3. The $SO(10) \times SO(10)$ model

The GUT scale value for $\sin^2 \theta_W$ tends to be small in all unified universal seesaw models as we have seen in the particular case of $SU(5) \times SU(5)$. So in order for these theories to have any chance of being phenomenologically viable, we must find some mechanism for enhancing $\sin^2 \theta_W$ as it evolves down in energy from the unification scale. We will illustrate a general strategy for overcoming this problem in the context of an $SO(10) \times SO(10)$ model.

This second theory represents an obvious generalization of the first considered in the preceding section, and a number of parallel features can immediately be established. For instance, a discrete interchange symmetry must again be imposed on the separate factors in the gauge group $G = SO(10)_L \times SO(10)_R$. As a result, particle representations occur in pairs, and fermion families in particular transform as

$$\mathcal{F} \sim (16, 1) + (1, 16)$$

which generalizes the $SU(5) \times SU(5)$ assignments in (2.1). There are however some significant differences between the two models. Most importantly, the larger size of $SO(10) \times SO(10)$ allows several new possibilities for electroweak subgroup embedding and symmetry breaking. As we shall see, this greater flexibility provides the key to increasing $\sin^2 \theta_W$ at the $Z$ scale.

Among the many different potential breaking schemes, we focus upon the following pattern which maintains explicit left-right symmetry down to the Standard Model:
\[ SO(10)_L \times SO(10)_R \]
\[ L^\alpha \quad R^\alpha \]
\[ g_{10L} \quad g_{10R} \]
\[ \downarrow M_{\text{GUT}} \]
\[ SU(4)_L \times SU(2)_L \times SU(2)'_L \times SU(4)_R \times SU(2)_R \times SU(2)'_R \]
\[ U^A_L \quad T^i_L \quad T^{i'}_L \quad U^A_R \quad T^i_R \quad T^{i'}_R \]
\[ g_{4L} \quad g_{2L} \quad g^{i'}_{2L} \quad g_{4R} \quad g_{2R} \quad g^{i'}_{2R} \]
\[ \downarrow \Lambda_{LR} \]
\[ SU(4)_{L+R} \times SU(2)_L \times SU(2)_R \times SU(2)'_{L+R} \]
\[ U^A = U^A_L + U^A_R \quad T^i_L \quad T^i_R \quad T^{i'}_L = T^{i'}_L + T^{i'}_R \]
\[ g_3 \quad g_{2L} \quad g_{2R} \quad g^{i'}_2 \]
\[ \downarrow \Lambda_C \]
\[ SU(3)_{L+R} \times SU(2)_L \times SU(2)_R \times U(1)_{L+R} \]
\[ U^a \quad T^i_L \quad T^i_R \quad S = T^{3'} + \sqrt{\frac{2}{3}} U^{15} \]
\[ g_3 \quad g_{2L} \quad g_{2R} \quad g_1 \]
\[ \downarrow v_R \]
\[ SU(3)_{L+R} \times SU(2)_L \times U(1)_Y \]
\[ U^a \quad T^i_L \quad \frac{Y}{2} = T^3_R + S \]
\[ g_3 \quad g_{2L} \quad g^i \]
\[ \downarrow v_L \]
\[ SU(3)_{L+R} \times U(1)_{EM} \]
\[ U^a \quad Q = T^3_L + \frac{Y}{2} \]
\[ g_3 \quad e \]

We have listed underneath each of the subgroup factors in this pattern our nomenclature conventions for the associated generators and coupling constants.  

\[ ^1 \text{The ranges of the } SO(10), \text{ } SU(4), \text{ } SU(3) \text{ and } SU(2) \text{ generator labels } \alpha, \ A, \ a \text{ and } i \text{ are respectively 1-45, 1-15, 1-8 and 1-3.} \]
The generators at each level in (3.2) are linear combinations $H = \sum_i c_i G_i$ of those at the previous level, and the corresponding couplings are related as $h^{-2} = \sum_i (g_i/c_i)^{-2}$. In particular, the electric charge generator

$$Q = T_L^3 + T_R^3 + T_L^{3'} + T_R^{3'} + \sqrt{\frac{2}{3}} U_L^{15} + \sqrt{\frac{2}{3}} U_R^{15}$$

(3.3)
of the final unbroken $U(1)_{EM}$ subgroup is a combination of elements in the Cartan subalgebras of $SU(2)_{L,R}$, $SU(2)'_{L,R}$ and $SU(4)_{L,R}$. The corresponding relation among these groups’ coupling constants

$$e(\mu)^{-2} = g_{2L}(\mu)^{-2} + g_{2R}(\mu)^{-2} + g_1'(\mu)^{-2} + g_2'(\mu)^{-2} + \frac{2}{3} g_{4L}(\mu)^{-2} + \frac{2}{3} g_{4R}(\mu)^{-2}$$

(3.4)
fixes the GUT scale value of the weak mixing angle:

$$\sin^2 \theta_W(M_{GUT}) = \frac{e(M_{GUT})^2}{g_{2L}(M_{GUT})^2} = \frac{3}{16} = 0.1875.$$

(3.5)

Since the $SU(3) \times SU(2) \times U(1)$ content of the fermion representation (3.1) in the $SO(10) \times SO(10)$ model is identical to that of (2.1) in the $SU(5) \times SU(5)$ theory except for an additional electrically neutral $SU(2)_L \times SU(2)_R$ singlet field, we have again found a value for $\sin^2 \theta_W(M_{GUT})$ which is precisely half as large as in the Georgi-Glashow model. But the behavior of $\sin^2 \theta_W$ below the unification scale is qualitatively different:

$$\sin^2 \theta_W(\mu) = \begin{cases} 
1 & \Lambda_L \leq \mu \leq M_{GUT} \\
1 + \left( \frac{g_{2L}}{g_{2R}} \right)^2 + \left( \frac{g_{2L}}{g_2} \right)^2 + \frac{2}{3} \left( \frac{g_{4L}}{g_4} \right)^2 + \frac{2}{3} \left( \frac{g_{4L}}{g_4} \right)^2 & \Lambda_C \leq \mu \leq \Lambda_{LR} \\
1 + \left( \frac{g_{2L}}{g_{2R}} \right)^2 + \left( \frac{g_{2L}}{g_2} \right)^2 + \frac{2}{3} \left( \frac{g_{4L}}{g_4} \right)^2 & v_R \leq \mu \leq \Lambda_C \\
1 + \left( \frac{g_{2L}}{g_{2R}} \right)^2 + \left( \frac{g_{2L}}{g_2} \right)^2 & v_L \leq \mu \leq v_R \\
1 & \Lambda_{LR} \leq \mu \leq M_{GUT} 
\end{cases}$$

(3.6a, 3.6b, 3.6c, 3.6d)

In (3.6a), the $SU(2)$ couplings $g_{2L}$, $g_{2R}$, $g'_2L$ and $g'_2R$ are all asymptotically free and increase as they run down in energy. However, the $SU(4)$ couplings $g_{4L}$ and $g_{4R}$ increase even faster.

9
So the denominator in (3.6a) decreases and the total fraction grows larger for \( \mu < M_{GUT} \). This rising trend continues until the \( \Lambda_C \) scale is reached. At that point, \( \sin^2 \theta_W(\mu) \) begins to decrease and continues downward all the way to \( \mu = M_Z \). The final sign and magnitude of the net change in \( \sin^2 \theta_W \) depend in detail upon the numerical values of the various intermediate scales and beta functions of the couplings appearing within the multilevel pattern (3.2). But we at least see how an enhancement of the weak mixing angle may be achieved in principle [17,18].

Unification by itself cannot uniquely determine all the symmetry breaking scales in (3.2). However, a number of phenomenological considerations restrict their values. Firstly, \( K \bar{K} \) mixing places lower mass limits of 1.6-2.5 TeV on \( W_R^\pm \) gauge bosons in manifestly left-right symmetric \( SU(2)_L \times SU(2)_R \times U(1) \) theories [19,20]. Therefore, \( v_R \) must lie at least in the multi-TeV region. Secondly, limits on the lepton family number violating decay \( K_L \to \mu^\pm e^\mp \) provide a bound on the \( \Lambda_C \) scale, for it is mediated by \( SU(4)_{L+R} \) gauge boson exchange. Including renormalization effects [21], we estimate that the branching fraction limit \[ \frac{\Gamma(K_L \to \mu^\pm e^\mp)}{\Gamma(K^+ \to \mu^+ \nu_\mu)} < 3.54 \times 10^{-11} \] (3.7)
restricts \( \Lambda_C \gtrsim 10^6 \) GeV. Finally, the unification scale \( M_{GUT} \) must be sufficiently large to allow for an acceptable proton lifetime.

It is useful to imagine constructing a low energy effective field theory at each symmetry breaking stage in pattern (3.2) in order to simplify the renormalization group analysis of coupling constant evolution. Particles that can acquire masses at a certain scale are integrated out together and do not contribute to subsequent renormalization group running. We thus find the following one-loop gauge boson and fermion contributions to the \( U(1) \) and \( SU(n) \) beta functions \( \beta(g_n) = b_n g_n^2 / 16 \pi^2 \): 

\[
\begin{align*}
    b_Y &= \frac{20}{3} N_F \\
    b_1 &= \frac{8}{9} N_F \\
    b'_2 &= -\frac{22}{3} \\
    b_n &= -\left( \frac{11n}{3} - \frac{4}{3} N_F \right).
\end{align*}
\] (3.8)

\footnote{2 Ordinary quarks and leptons are singlets under \( SU(2)'_{L+R} \), while their seesaw partners acquire heavy masses and decouple at the \( \Lambda_{LR} \) scale in (3.2). There is consequently no fermion contribution to the \( SU(2)'_{L+R} \) beta function coefficient \( b'_2 \).}
It is then straightforward to integrate the renormalization group equations to obtain a linear system of equations that relates the three high energy quantities $\alpha_{10}(M_{GUT}) = g_{10}(M_{GUT})^2/4\pi$, $\log(M_{GUT}/\Lambda_{LR})$ and $\log(\Lambda_{LR}/\Lambda_C)$ to the three low energy parameters $\sin^2 \theta_W(M_Z)$, $\alpha_{EM}(M_Z)$ and $\alpha_s(M_Z)$:

$$\begin{pmatrix}
\frac{13}{3} & 3b_{2L} + \frac{4}{3}b_{4L} & b_{2L} + b'_L + \frac{2}{3}b_4 \\
1 & b_{2L} & b_{2L} \\
2 & 2b_{4L} & b_4
\end{pmatrix}
\begin{pmatrix}
2\pi/\alpha_{10}(M_{GUT}) \\
\log M_{GUT}/\Lambda_{LR} \\
\log \Lambda_{LR}/\Lambda_C
\end{pmatrix}
= \begin{pmatrix}
2\pi \cos^2 \theta_W(M_Z)/\alpha_{EM}(M_Z) - (b_1 + b_2) \log \Lambda_C/v_R - b_V \log v_R/M_Z \\
2\pi \sin^2 \theta_W(M_Z)/\alpha_{EM}(M_Z) - b_{2L} \log \Lambda_{LR}/M_Z \\
2\pi/\alpha_s(M_Z) - b_3 \log \Lambda_C/M_Z
\end{pmatrix}.$$ (3.9)

Unfortunately, no consistent solution to this matrix equation exists which satisfies the phenomenological restrictions on the intermediate scales and reproduces the high precision numbers in (1.1). A fit for the GUT scale parameters based upon the inputs $v_R = 5$ TeV, $\Lambda_C = 1000$ TeV and $\Lambda_{LR} = 100,000$ TeV yields the results

$$\alpha_{10}(M_{GUT}) = 0.025$$ (3.10)

$$M_{GUT} = 1.3 \times 10^{15} \text{ GeV}$$

which imply the $Z$ scale values

$$\sin^2 \theta_W(M_Z) = 0.197$$

$$\alpha_{EM}^{-1}(M_Z) = 124.3$$ (3.11)

$$\alpha_s(M_Z) = 0.137.$$ 

The match between these theoretical numbers and the experimental measurements in (1.1) is obviously poor. Nonetheless, we see that the basic strategy of embedding part of the hypercharge generator within an asymptotically free subgroup has led to an increase in $\sin^2 \theta_W(M_Z)$ over its unification value [18]. This trick must generally be employed in any unified universal seesaw model.

At this point, we could explore other symmetry breaking schemes for the $SO(10) \times SO(10)$ theory in which manifest left-right symmetry is broken at an earlier stage than in pattern (3.2) so as to further enhance the value for $\sin^2 \theta_W(M_Z)$. Alternatively, we could continue to search for a phenomenologically viable chiral $G_L \times G_R$ model based upon an even larger group such as $E_6 \times E_6$. But we turn instead to explore a somewhat different theory with the gauge structure $G_C \times G_L \times G_R$ in the following section.
4. The $SU(4) \times SU(4) \times SU(4)$ model

The prototypical example of a unified $G_C \times G_L \times G_R$ theory is the $SU(3)_C \times SU(3)_L \times SU(3)_R$ model \cite{22}. This amusing “trinification” theory has been studied in the past as an alternative to $SU(5)$ and $SO(10)$ unification. The $SU(3)^3$ model however cannot accommodate a heavy $SU(2)_L \times SU(2)_R$ partner for each Standard Model fermion. So we are led to consider the next simplest possibility based upon the gauge group

$$G = SU(4)_C \times SU(4)_L \times SU(4)_R$$ (4.1)

which is supplemented with a cyclic $Z_3$ symmetry to ensure equality among the separate $SU(4)$ coupling constants. This theory represents an obvious generalization of $SU(3)^3$ trinification as well as the Pati-Salam model \cite{9}. Indeed, Pati and Salam originally proposed $G$ as a possible global symmetry of nature in which lepton number plays the role of a fourth color. The similarities and differences between our model and these others that have been studied in the past will become evident as we proceed.

Embedding the Standard Model subgroup within $SU(4)^3$ is straightforward. We take a generalized set of Gell-Mann matrices as generators of $SU(4)_C$. The first eight members of this set are associated with color $SU(3)$, while the fifteenth matrix

$$U^{15}_C = \sqrt{\frac{3}{2}} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}$$ (4.2)

generates a commuting $U(1)_C$ factor. For $SU(4)_{L,R}$, we use the set of $4 \times 4$ Pauli matrices

$$\frac{\sigma^i}{2\sqrt{2}}, \quad \frac{\tau^j}{2\sqrt{2}}, \quad \frac{\sigma^i\tau^j}{2\sqrt{2}} \quad i, j = 1, 2, 3$$ (4.3)

as normalized generators. The linear combinations

$$T^{i}_{L,R} = \frac{\sigma_i(1 + \tau^3)}{4} = \frac{1}{2} \begin{pmatrix} \sigma^i \\ 0 \end{pmatrix}$$
$$T^{i}_{L,R}' = \frac{\sigma_i(1 - \tau^3)}{4} = \frac{1}{2} \begin{pmatrix} 0 \\ \sigma^i \end{pmatrix}$$
$$S_{L,R} = \frac{\tau^3}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ (4.4)
belong to an $SU(2)_{L,R} \times SU(2)'_{L,R} \times U(1)_{L,R}$ subalgebra of $SU(4)_{L,R}$. Weak $SU(2)$ and its right handed analog are identified as $SU(2)_L$ and $SU(2)_R$. Finally, we choose
\[ Q = T^3_L + T^3'_L + T^3_R + T^3'_R + \sqrt{\frac{2}{3}} U^{15}_C \] (4.5)
as the generator of electromagnetism. This definition implies $\sin^2 \theta_W(M_{GUT}) = 3/14 = 0.2143$. While this GUT scale value is still below the $Z$ scale measurement $\sin^2 \theta_W(M_Z) = 0.2325$, it is certainly closer than the corresponding $\sin^2 \theta_W(M_{GUT}) = 3/16 = 0.1875$ that we found in the $SU(5) \times SU(5)$ and $SO(10) \times SO(10)$ models. So we already see one clear advantage of the $SU(4)^3$ theory over its predecessors.

Gauge bosons in this model transform according to the 45-dimensional representation
\[ G \sim (15, 1, 1) + (1, 15, 1) + (1, 1, 15) \] (4.6)
which automatically remains invariant under cyclic $Z_3$ permutations. In the fermion sector, a single family of left handed fields is assigned to the anomaly free but complex representation
\[ F \sim (4, \overline{1}, 1) + (1, 4, \overline{1}) + (\overline{4}, 1, 4). \] (4.7)
One generation of left handed quarks and leptons along with their seesaw partners fit smugly inside $(4, \overline{1}, 1)$, while conjugate fields appear in $(\overline{4}, 1, 4)$. The remaining $(1, 4, \overline{4})$ contains a new set of leptons. All these particles’ colors, flavors and electric charges are indicated in the matrices below:

\begin{align*}
\Psi_{CL}(4, \overline{1}, 1) &= \begin{pmatrix}
d_1 & u_1 & D_1 & U_1 \\
d_2 & u_2 & D_2 & U_2 \\
d_3 & u_3 & D_3 & U_3 \\
e & \nu & E & N
\end{pmatrix}_L \tag{4.8a} \\
\Psi_{LR}(1, 4, \overline{4}) &= \begin{pmatrix}
I^0 & I^+ & J^0 & J^+ \\
I^- & I^{0c} & J^- & J^{0c} \\
K^0 & K^+ & L^0 & L^+ \\
K^- & K^{0c} & L^- & L^{0c}
\end{pmatrix}_L \tag{4.8b} \\
\Psi_{RC}(\overline{4}, 1, 4) &= \begin{pmatrix}
d^c_1 & d^c_2 & d^c_3 & e^c \\
u^c_1 & \nu^c_2 & \nu^c_3 & \nu^c \\
D^c_1 & D^c_2 & D^c_3 & E^c \\
U^c_1 & U^c_2 & U^c_3 & N^c
\end{pmatrix}_L \tag{4.8c}
\end{align*}

There exist a large number of potential symmetry breaking chains that start from the GUT group and end with the Standard Model. The simplest schemes which retain manifest
left-right symmetry down to the $SU(3) \times SU(2) \times U(1)$ subgroup do not sufficiently enhance $\sin^2 \theta_W$ as it runs down in energy to reproduce the measured $Z$ scale value. However, if left-right symmetry is broken either spontaneously or softly at the first stage, then we can find viable breaking patterns that lead to phenomenologically interesting results. One such possibility is the following:

$$SU(4)_C \times SU(4)_L \times SU(4)_R$$

$$\downarrow \Lambda_L = M_{GUT}$$

$$SU(4)_C \times SU(2)_L \times SU(2)'_L \times U(1)_L \times SU(4)_R$$

$$\downarrow \Lambda_R$$

$$SU(4)_C \times SU(2)_L \times SU(2)'_L \times U(1)_L \times SU(2)_R \times SU(2)'_R \times U(1)_R$$

$$\downarrow \Lambda_C$$

$$SU(3)_C \times U(1)_C \times SU(2)_L \times SU(2)'_L \times U(1)_L \times SU(2)_R \times SU(2)'_R \times U(1)_R$$

$$\downarrow \Lambda_{LR}$$

$$SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)'_{LR} \times U(1)_C$$

$$\downarrow v_R$$

$$SU(3)_{L+R} \times SU(2)_L \times U(1)_Y$$

$$\downarrow v_L$$

$$SU(3)_{L+R} \times U(1)_{EM}.$$ (4.9)

The renormalization group analysis of coupling constant running in this pattern is similar to that described in the preceding section for the $SO(10) \times SO(10)$ model. The only qualitatively new feature that we include in the $SU(4)^3$ analysis is scalar contributions to beta functions. These come from the Higgs sector of the theory which we will discuss in detail shortly. The results of the renormalization group analysis yield a wide range of values for the symmetry breaking scales in (4.9) that reproduce the Standard Model parameters in (1.14) and satisfy all other phenomenological constraints. For simplicity, we merge the intermediate $\Lambda_{LR}$ and $\Lambda_C$ thresholds together and quote one set of representative values for these scales:

$$\Lambda_L = M_{GUT} = 6.47 \times 10^{11} \text{ GeV}$$

$$\Lambda_R = \Lambda_C = 2.07 \times 10^7 \text{ GeV}$$

$$\Lambda_{LR} = 1.0 \times 10^5 \text{ GeV}$$

$$v_R = 5.0 \times 10^3 \text{ GeV}$$

$$v_L = 2.46 \times 10^2 \text{ GeV}.$$ (4.10)
The evolution of $\sin^2 \theta_W$ for this choice of scales is illustrated in fig. 1.

We now consider the minimal Higgs content of the $SU(4)^3$ model needed to perform the several stages of symmetry breaking in (4.9) and to provide fermion masses. The first three steps result from vacuum expectation values of the adjoint fields in

$$\Phi = \Phi_C (15, 1, 1) + \Phi_L (1, 15, 1) + \Phi_R (1, 1, 15).$$

These scalars’ VEV’s

$$\langle \Phi_{L,R} \rangle = \Lambda_{L,R} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & \end{pmatrix} \quad \langle \Phi_C \rangle = \Lambda_C \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

break the separate $SU(4)$ factors in $G$ as

$$SU(4)_{L,R} \xrightarrow{\langle \Phi_{L,R} \rangle} SU(2)_{L,R} \times SU(2)_{L,R} \times U(1)_{L,R}$$

$$SU(4)_C \xrightarrow{\langle \Phi_C \rangle} SU(3)_C \times U(1)_C.$$ (4.13a)

(4.13b)

The $SU(2)'_L$ and $SU(2)'_R$ subgroups under which the seesaw fermions transform are subsequently reduced at the $\Lambda_{L,R}$ scale to the diagonal $U(1)'_{L,R}$ generated by $S'_{L,R} = T^3_L + T^3_R$. We introduce two sets of scalars

$$\phi^I = \phi^I_{CL} (4, \overline{4}, 1) + \phi^I_{LR} (1, 4, \overline{4}) + \phi^I_{RC} (\overline{4}, 1, 4)$$

labeled by the flavor index $I = u, d$ to accomplish this breaking. The $\phi^u_{LR}$ and $\phi^d_{LR}$ fields are presumed to acquire the distinct vacuum expectation values

$$\langle \phi^u_{LR} \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_L \\ 0 & 0 & 0 & 0 \\ 0 & v_R & 0 & \Lambda_{L,R} \end{pmatrix} \quad \langle \phi^d_{LR} \rangle = \begin{pmatrix} 0 & 0 & v_L & 0 \\ 0 & 0 & 0 & 0 \\ v_R & 0 & \Lambda_{L,R} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ (4.15)

Heavy Dirac masses for the $U, N, D$ and $E$ fermions are then generated via the Yukawa interaction

$$\mathcal{L}_{\text{Yukawa}} (\phi^I) = f^I \text{Tr} \left[ (\Psi^T_{RC} (4, \overline{4}, 1) C \Psi_{CL} (4, \overline{4}, 1) \phi^I_{LR} (1, 4, \overline{4}) \right.$$  

$$+ (\Psi^T_{LR} (1, 4, \overline{4}) C \Psi_{RC} (\overline{4}, 1, 4) \phi^I_{CL} (4, \overline{4}, 1) \right] + \text{h.c.}.$$ (4.16a)
We also give $O(\Lambda_{LR})$ masses to all the new exotic leptons in (4.8) through a second Yukawa term

$$L_{\text{Yukawa}}(\chi) = \frac{g}{2} \text{Tr} [(\Psi_{\text{CL}}^T)(4, \mathbf{4}, 1) C \Psi_{\text{CL}}(4, \mathbf{4}, 1) \chi_{\text{CL}}(6, 6, 1) + (\Psi_{\text{LR}}^T)(1, 4, \mathbf{4}) C \Psi_{\text{LR}}(1, 4, \mathbf{4}) \chi_{\text{LR}}(1, 6, 6) + (\Psi_{\text{RC}}^T)(\mathbf{4}, 1, 4) C \Psi_{\text{RC}}(\mathbf{4}, 1, 4) \chi_{\text{RC}}(6, 1, 6)] + \text{h.c.}$$

(4.16b)

which antisymmetrically couples fermions to the additional Higgs field

$$X = \chi_{\text{CL}}(6, 6, 1) + \chi_{\text{LR}}(1, 6, 6) + \chi_{\text{RC}}(6, 1, 6).$$

(4.17)

The only components of $X$ that may develop nonvanishing vacuum expectation values which do not break color and electromagnetism but do violate $U(1)_{LR}$ are $(\chi_{\text{LR}})^{[12]}$, $(\chi_{\text{LR}})^{[34]}$, $(\chi_{\text{LR}})^{[12]}$, and $(\chi_{\text{LR}})^{[34]}$. We choose these VEV’s to all equal $\Lambda_{LR}$.

The final two symmetry breaking steps in (4.9) result from the $v_R$ and $v_L$ entries in (4.15) and

$$\langle \phi_{\text{CL}}^f \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & v_L & 0 & 0 \end{pmatrix} \quad \langle \phi_{\text{RC}}^f \rangle = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_R \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (4.18)$$

The Yukawa Lagrangian induces mixing between the heavy seesaw fermions and their light Standard Model partners. The final forms of the quark and charged lepton mass matrices appear as

$$M_{uU} = u_L^c U_L \begin{pmatrix} 0 & f^u v_R \\ f^u v_L & f^u \Lambda_{LR} \end{pmatrix} \quad M_{dD} = d_L^c D_L \begin{pmatrix} 0 & f^d v_R \\ f^d v_L & f^d \Lambda_{LR} \end{pmatrix}, \quad (4.19a)$$

and

$$M_{\text{charged leptons}} = \begin{pmatrix} e_L^c & E_L & I^-_L & J^-_L & K^-_L & L^-_L \\ E_L^c & f^d v_R & (f^u + f^d) v_L & 0 & 0 & 0 \\ I^+_L & (f^u + f^d) v_R & 0 & -g \Lambda_{LR} & 0 & 0 \\ J^+_L & 0 & 0 & 0 & -g \Lambda_{LR} & 0 \\ K^+_L & 0 & (f^u + f^d) v_R & 0 & 0 & -g \Lambda_{LR} \\ L^+_L & 0 & 0 & 0 & 0 & -g \Lambda_{LR} \end{pmatrix}. \quad (4.19b)$$
We refrain from explicitly writing down the neutral lepton matrix since it is larger and more complicated than those exhibited above.

We should recall that the $f^u$, $f^d$ and $g$ Yukawa couplings are $N_F \times N_F$ matrices in fermion family space. As we saw before in the $SU(5) \times SU(5)$ theory, it is useful to invoke a parity symmetry $P$ to constrain the forms of these Yukawa matrices. We therefore follow ref. [23] and promote the discrete $Z_3$ symmetry in our $SU(4)^3$ model to $S_3$ through the addition of a parity operation and its two cyclic partners. $P$ performs a conventional spatial inversion and swaps the $SU(4)_L$ and $SU(4)_R$ factors in the gauge group. Its action upon the $SU(4)_C \times SU(4)_L \times SU(4)_R$ gauge fields

$$C^\mu(\vec{x}, t) \rightarrow C^\mu(-\vec{x}, t) \quad L^\mu(\vec{x}, t) \rightarrow R^\mu(-\vec{x}, t) \quad R^\mu(\vec{x}, t) \rightarrow L^\mu(-\vec{x}, t) \quad (4.20a)$$

forbids a CP violating topological term from appearing in the gauge part of the Lagrangian.

In the fermion sector, parity maps left handed fields into their right handed analogs which we express as left handed conjugates:

$$\Psi_{LR}(\vec{x}, t) \rightarrow -C(\Psi_{LR}^c)^*(-\vec{x}, t)$$
$$\Psi_{CL}(\vec{x}, t) \rightarrow -C(\Psi_{CL}^c)^*(-\vec{x}, t) = -C(\Psi_{RC})^\dagger(-\vec{x}, t) \quad (4.20b)$$
$$\Psi_{RC}(\vec{x}, t) \rightarrow -C(\Psi_{RC}^c)^*(-\vec{x}, t) = -C(\Psi_{CL})^\dagger(-\vec{x}, t).$$

Finally, the scalars transform under $P$ as

$$\Phi_C(\vec{x}, t) \rightarrow \Phi_C^\dagger(-\vec{x}, t) \quad \chi_{LR}(\vec{x}, t) \rightarrow \chi_{LR}^\dagger(-\vec{x}, t) \quad \phi_{LR}^I(\vec{x}, t) \rightarrow (\phi_{LR}^I)^\dagger(-\vec{x}, t)$$
$$\Phi_L(\vec{x}, t) \rightarrow \Phi_L^\dagger(-\vec{x}, t) \quad \chi_{CL}(\vec{x}, t) \rightarrow \chi_{RC}^\dagger(-\vec{x}, t) \quad \phi_{CL}^I(\vec{x}, t) \rightarrow (\phi_{RC}^I)^\dagger(-\vec{x}, t)$$
$$\Phi_R(\vec{x}, t) \rightarrow \Phi_R^\dagger(-\vec{x}, t) \quad \chi_{RC}(\vec{x}, t) \rightarrow \chi_{CL}^\dagger(-\vec{x}, t) \quad \phi_{RC}^I(\vec{x}, t) \rightarrow (\phi_{CL}^I)^\dagger(-\vec{x}, t). \quad (4.20c)$$

It is straightforward to check that the Yukawa interactions in $(4.16a, b)$ remain invariant under parity only if the $f^I$ and $g$ coupling matrices are hermitian. The fermion mass matrices can thus be complex, but their determinants are real. So $\bar{\theta} = \theta_{QCD} + \theta_{QFD}$ vanishes at tree level, and the $SU(4)^3$ model provides a possible solution to the strong CP problem.
We next diagonalize the fermion mass matrices in (4.19a, b) neglecting small inter-generational mixing between families. The masses of Standard Model quarks and charged leptons fix the diagonal elements in $f^u$, $f^d$ and $g$:

$$
\begin{align*}
&f^d \approx \frac{\Lambda_{LR}}{v_L v_R} \begin{pmatrix} m_d = 0.009 \\ m_s = 0.181 \\ m_b = 4.5 \end{pmatrix} = \begin{pmatrix} 0.0007 \\ 0.0147 \\ 0.3654 \end{pmatrix} \\
&f^u \approx \frac{\Lambda_{LR}}{v_L v_R} \begin{pmatrix} m_u = 0.005 \\ m_c = 1.5 \\ m_t = 130 \end{pmatrix} = \begin{pmatrix} 0.0004 \\ 0.1218 \\ 10.556 \end{pmatrix} \\
g \approx \frac{\Lambda_{LR}}{v_L v_R} \begin{pmatrix} (m_u+m_d)^2 \\ (m_c+m_d)^2 \\ (m_c+m_b)^2 \\ (m_t+m_b)^2 \\ m_t+m_b \end{pmatrix} = \begin{pmatrix} 0.0017 \\ 0.7998 \\ 233.9 \end{pmatrix}.
\end{align*}
$$

We have numerically evaluated these matrices using the indicated GeV quark masses and the scale values in (4.10). The resulting Yukawa couplings for the first and second families are reasonable in size. We thus see the seesaw mechanism at work generating small quark and lepton masses without an excessive fine tuning of Yukawa couplings. Unfortunately, the results for the third family are corrupted by the huge top quark mass. The large value for $m_t$ can of course be offset by the inverted seesaw prefactor in (4.21). But then we are left with very small Yukawas for the lightest quarks and leptons as in the Standard Model.

Finally, we investigate proton decay in the $SU(4)^3$ theory. Recall that left handed Standard Model fermions and antifermions appear in separate multiplets in (I.8). Therefore, gauge boson exchange cannot mediate fermion number violating transitions such as $P \rightarrow \pi^0 e^+$. Proton decay only proceeds through $\chi$ scalar exchange graphs like the one illustrated in fig. 2. We expect the mass of the $\chi^4$ scalar shown in the figure to be on the order of the unification scale $M_{GUT} = 6.47 \times 10^{11}$ GeV. This mass seems much too light to yield a proton lifetime consistent with the experimental lower limit $[10]$

$$\tau_P > 5 \times 10^{32} \text{ yrs.} \quad (4.22)$$

However, the diagram in fig. 2 is further suppressed by $O(v_R/\Lambda_{LR})^2$ as a result of seesaw mixing between fermion gauge and mass eigenstates. Such seesaw suppression of proton decay is generic in all unified seesaw models. Naive dimensional analysis yields the proton lifetime estimate

$$\tau_P \approx \frac{16\pi}{(g_{11})^4} \left( \frac{\Lambda_{LR}}{v_R} \right)^4 \frac{M_{\chi}^4}{m_P^5} \quad (4.23)$$
where $g_{11}$ is the Yukawa coupling for the first family in (4,16) while $16\pi$ represents a two body phase space factor. Inserting numerical values, we find $\tau_P \simeq 4.6 \times 10^{33}$ yrs which is consistent with the bound in (4.22).

5. Conclusion

The $SU(5) \times SU(5)$, $SO(10) \times SO(10)$ and $SU(4) \times SU(4) \times SU(4)$ models that we have investigated in this paper illustrate the basic features of unified universal seesaw theories. They also generalize several well-known models that have been studied in the past. Many possible extensions of this work would be interesting to pursue. Gauge boson mixing, neutrino masses and loop contributions to $\theta$ should all be further analyzed in these models. Moreover, a number of alternatives to the symmetry breaking patterns that we have considered here remain to be examined in the $SO(10) \times SO(10)$ and $SU(4)^3$ theories. Generalizations to $E_6 \times E_6$ and $SU(5)^3$ which may maintain left-right symmetry down to the Standard Model subgroup could also be constructed. In short, unified universal seesaw models represent a new class of Grand Unified Theories in which there is much room for further exploration.

Acknowledgements

It is a pleasure to thank Howard Georgi and Sheldon Glashow for numerous discussions in the past on many issues related to this work.
References

[1] M. Gell-Mann, P. Ramond and R. Slansky, in *Supergravity*, edited by F. van Nieuwenhuizen and D. Freedman, (North-Holland, Amsterdam, 1979) p. 315;
   T. Yanagida, Prog. Th. Physics B135, 66 (1978).
[2] S. Rajpoot, Phys. Lett. 191, 122 (1987).
[3] A. Davidson and K.C. Wali, Phys. Rev. Lett. 59, 393 (1987);
   A. Davidson and K.C. Wali, Phys. Rev. Lett. 60, 1813 (1988).
[4] K.S. Babu and R.N. Mohapatra, Phys. Rev. Lett. 62, 1079 (1989).
[5] K.S. Babu and R.N. Mohapatra, Phys. Rev. D41, 1286 (1990).
[6] S.M. Barr, D. Chang and G. Senjanovic, Phys. Rev. Lett. 67, 2765 (1991).
[7] H. Georgi and S. L. Glashow, Phys. Rev. Lett. 32, 438 (1974).
[8] H. Georgi, in *Particles and Fields*, edited by C.E. Carlson, (A.I.P., New York, 1975) p. 575;
   H. Fritzsch and P. Minkowski, Ann. Phys. 93, 193 (1975).
[9] J.C. Pati and A. Salam, Phys. Rev. D8, 1240 (1973);
   J.C. Pati and A. Salam, Phys. Rev. D10, 275 (1974).
[10] Review of Particle Properties, Phys. Rev. D45, Part 2 (1992).
[11] A. Giveon, L.J. Hall and U. Sarid, Phys. Lett. B271, 138 (1991).
[12] S. Bethke, Talk presented at XXVI Int. Conf. High Energy Physics, Dallas, Aug. 6-12, 1992.
[13] H. Georgi, Nucl. Phys. B156, 126 (1979).
[14] A. Davidson and K.C. Wali, Phys. Rev. Lett. 58, 2623 (1987).
[15] P. Cho and S. Glashow, unpublished.
[16] H. Georgi, H.R. Quinn and S. Weinberg, Phys. Rev. Lett. 33, 451 (1974).
[17] S. Dawson and H. Georgi, Phys. Rev. Lett. 43, 821 (1979).
[18] H. Georgi, Private communication.
[19] G. Beall, M. Bander and A. Soni, Phys. Rev. Lett. 48, 848 (1982).
[20] G. Ecker and W. Grimus, Nucl. Phys. B258, 328 (1985).
[21] N.G. Deshpande and R.J. Johnson, Phys. Rev. D27, 1193 (1983).
[22] A. DeRujula, H. Georgi and S.L. Glashow, in *Fifth Workshop on Grand Unification*,
    edited by K. Kung, H. Fried and P. Frampton (World Scientific, Singapore, 1984) p. 88.
[23] E.D. Carlson and M.Y. Wang, HUTP-92/A057 (1992).
Figure Captions

Fig. 1. Evolution of $\sin^2 \theta_W(\mu)$ over the range $M_Z \leq \mu \leq M_{GUT}$ in the $SU(4)^3$ model. Dashed lines mark the locations of the intermediate $v_R$, $\Lambda_{LR}$ and $\Lambda_C = \Lambda_R$ scales.

Fig. 2. Dominant contribution to proton decay from $\chi$ scalar exchange in the $SU(4)^3$ model. Primed and unprimed fields denote mass and gauge eigenstates respectively.