Maximum Sum Rate of Aloha with Capture

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Abstract

The sum rate performance of random-access networks crucially depends on the access protocol and receiver structure. Despite extensive studies, how to characterize the maximum sum rate of the simplest version of random access, Aloha, remains an open question. In this paper, a comprehensive study of the sum rate performance of Aloha networks is presented. By extending the unified analytical framework proposed in [20], [21] from the classical collision model to the capture model, the network steady-state point in saturated conditions is derived as a function of the signal-to-interference-plus-noise ratio (SINR) threshold which determines a fundamental tradeoff between the information encoding rate and the network throughput. To maximize the sum rate, both the SINR threshold and backoff parameters of nodes should be properly selected. Explicit expressions of the maximum sum rate and the optimal setting are obtained under various assumptions on the receiver model and channel conditions. The analysis reveals that similar to the sum capacity of the multiple access channel, the maximum sum rate of Aloha also logarithmically increases with the mean received signal-to-noise ratio (SNR), but the high-SNR slope is only $e^{-1}$. Effects of backoff, multipacket reception, channel fading and power control on the sum rate performance of Aloha networks are further discussed, which shed important light on the practical network design.

Index Terms

Random access, Aloha, sum rate, network throughput, backoff, capture model, collision model

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I. INTRODUCTION

Random access provides a simple and elegant solution for multiple users to share a common channel. Studies on random-access protocols date back to 1970s [1]. After decades of extensive research, random access has found wide applications to Ethernet, IEEE 802.11 networks and wireless ad-hoc networks [2]. The minimum coordination and distributed control make it highly appealing for low-cost data networks.

In sharp contrast to the simplicity in concept, the performance analysis of random-access networks has been known as notoriously difficult, which is mainly due to the lack of a coherent analytical framework. Numerous models have been proposed based on distinct assumptions. According to the receiver structure, they can be broadly divided into three categories:

1) Collision model: In the classical collision model, when more than one node transmit their packets simultaneously, a collision occurs and none of them can be successfully decoded. A packet transmission is successful only if there are no concurrent transmissions. The collision model was first proposed by Abramson in [1], and has been widely used in most of random-access studies [3]–[21].

2) Capture model: Though an elegant and useful simplification of receivers, the collision model could be overly pessimistic if there exists a large difference of received power. It was first pointed out by Roberts in [22] that even with multiple concurrent transmissions, the strongest signal could be successfully detected as long as the signal-to-interference ratio (SIR) is high enough. It was referred to as the “capture effect”, which has been extensively studied in [23]–[36]. With the capture model, each node’s packet is decoded independently by treating others’ as background noise. A packet can be successfully decoded as long as its received signal-to-interference-plus-noise ratio (SINR) is above a certain threshold. It is clear that multiple packets can be decoded simultaneously if the SINR threshold is sufficiently low.

3) Joint-decoding model: The capture model is essentially a single-user detector. Multiuser detectors, such as Minimum Mean Square Error (MMSE) and Successive Interference Cancelation (SIC), have been also applied to random-access networks [37]–[43]. By jointly decoding multiple nodes’ packets, the efficiency can be greatly improved, though at the cost of receiver complexity.

Note that the capture model and the joint-decoding model both have the so-called “multipacket-reception (MPR)” capability [44], [45], and have been referred to as the MPR model in many references [32], [36], [37], [39]–[41], [43]. Here we distinguish them apart because they assume different receiver structures. Compared to the capture model, although a sophisticated multiuser detector is much more efficient, it
may not be affordable to low-cost data networks. Therefore, in this paper, we specifically focus on the performance analysis based on the capture model.

A. Maximum Network Throughput of Aloha

In random-access networks, due to the uncoordinated nature of transmitters, the number of successfully decoded packets in each time slot varies from time to time. In the literature, the time average of the number of successfully decoded packets per time slot is usually adopted as an important performance metric, which is referred to as the network throughput.

The network throughput performance depends on a series of key factors including the receiver model and protocol design. With the classical collision model, for instance, at most one packet can be successfully decoded at each time slot. Therefore, the network throughput, which is also the fraction of time that an effective output is produced in this case, cannot exceed 1. The maximum network throughput of Aloha was shown to be only $e^{-1}$ with the collision model [3], which indicates that over 60% of the time is wasted when the network is either in collision or idle states. To improve the efficiency, Carrier Sense Multiple Access (CSMA) was further introduced in [4], with which the network throughput can approach 1 by reducing the sensing time. On the other hand, significant improvement in network throughput was also observed when the capture model is adopted [23]–[27], [29], [30], [32], [33], [36]. Intuitively, with the capture model, more packets can be successfully decoded by reducing the SINR threshold. The network throughput is thus greatly improved, and may exceed 1 if the SINR threshold is sufficiently small.

Despite extensive studies, how to maximize the network throughput has been an open question for a long time. In Abramson’s landmark paper [3], by modeling the aggregate traffic as a Poisson random variable with parameter $G$, the network throughput of Aloha with the collision model can be easily obtained as $Ge^{-G}$, which is maximized at $e^{-1}$ when $G = 1$. To enable the network to operate at the optimum point, nevertheless, it requires the connection between the mean traffic rate $G$ and key system parameters such as transmission probabilities of nodes, which turns out to be a challenging issue. Various retransmission strategies were developed to adjust the transmission probability of each node according to the number of backlogged nodes to stabilize the network [5]–[8]. Yet most of them were based on the realtime feedback information on the backlog size, which may not be available in a distributed network. Decentralized retransmission control was further studied in [6], [10]–[12], where algorithms were proposed to either estimate and feed back the backlog size [10], [12], or update the transmission probability of each node recursively according to the channel output [6], [11].
The above analytical approaches were also applied to the capture model. By assuming Poisson distributed aggregate traffic, for instance, the network throughput was derived as a function of the mean traffic rate $G$ and the SIR threshold in [24], [25], [27] under distinct assumptions on channel conditions. Similar to the case of collision model, the maximum network throughput can be obtained by optimizing $G$, yet how to properly tune the system parameters to achieve the maximum network throughput remains unknown. Retransmission control strategies developed in [5], [10] and [11] were further extended to the capture model in [28], [29] and [30], respectively. To evaluate the network throughput performance for given transmission probabilities of nodes, various Markov chains were also established in [23], [26], [33], [36] to model the state transition of each individual user. The computational complexity, nevertheless, sharply increases when sophisticated backoff strategies are further involved, which renders it extremely difficult, if not impossible, to search for the optimal configuration to maximize the network throughput.

The difficulty originates from the modeling of random-access networks. As demonstrated in [21], the modeling approaches in the literature can be roughly divided into two categories: channel-centric [3]–[8], [10]–[12] and node-centric [9], [13]–[18]. By focusing on the state transition process of the aggregate traffic, the channel-centric approaches capture the essence of contention among nodes, which, nevertheless, ignore the behavior of each node’s queue and thus shed little light on the effect of backoff parameters on the performance of each single node. With the node-centric approaches, on the other hand, the modeling complexity becomes prohibitively high if interactions among nodes’ queues are further taken into consideration. To simplify the analysis, a key approximation, which has been widely adopted and shown to be accurate for performance evaluation of large multi-queue systems [46], is to treat each node’s queue as an independent queueing system with identically distributed service time. The service time distribution is still crucially determined by the aggregate activities of head-of-line (HOL) packets of all the nodes, which requires proper modeling of HOL packets’ behavior.

In our recent work [20], [21], a unified analytical framework for two representative random-access protocols, Aloha and CSMA, was established, where the network steady-state operating points were characterized based on the fixed-point equations of the limiting probability of successful transmission of HOL packets by assuming the classical collision model. Both steady-state points were derived as explicit functions of key system parameters including the aggregate input rate, the number of nodes and the transmission probabilities of nodes, which enable the characterization of stable regions and performance optimization. In this paper, the proposed analytical framework is further extended to incorporate the
capture model. Specifically, we consider an $n$-node Aloha network, where each node always has packets to transmit, and the received signal-to-noise ratios (SNRs) of nodes’ packets are exponentially distributed with the same mean received SNR $\rho$. The network steady-state point, which is characterized as the single non-zero root of the fixed-point equation of the limiting probability of successful transmission of HOL packets, is found to be closely dependent on transmission probabilities of nodes, the SINR threshold $\mu$ and the mean received SNR $\rho$. The maximum network throughput $\hat{\lambda}_{\text{max}}$ is obtained as an explicit function of the SINR threshold $\mu$ and the mean received SNR $\rho$, which is shown to be monotonically increasing as $\mu$ decreases or $\rho$ increases. The optimal transmission probabilities of nodes to achieve the maximum network throughput are also derived, and verified by simulation results.

B. Maximum Sum Rate of Aloha

From the information-theoretic perspective, random access can be regarded as a multiple access channel with a random number of active transmitters. It is well known that the sum capacity of an $n$-user Additive-White-Gaussian-Noise (AWGN) multiple access channel is determined by the received SNRs, i.e.,

$$C_{\text{sum}} = \log_2(1 + \sum_{i=1}^{n} \text{SNR}_i).$$

With random access, however, the number of active transmitters is a random variable whose distribution is determined by the protocol and parameter setting. Moreover, to achieve the sum capacity, a joint decoding of all transmitted codewords should be performed at the receiver side, which might be unaffordable for random-access networks. Therefore, the sum rate performance of random access becomes closely dependent on assumptions on the access protocol and receiver design.

There has been a great deal of effort to explore the information-theoretic limit of random-access networks. For instance, the concept of rate splitting \cite{47} was first introduced to Aloha networks in \cite{38}, where a joint coding scheme was developed for the two-node case. If each node independently encodes its information, \cite{42} showed that the sum rate performance of Aloha networks can be improved by adaptively adjusting the encoding rate according to the number of nodes and the transmission probability of each node. \cite{38} and \cite{42} are based on the assumption of joint decoding of multiple nodes’ packets at the receiver side. With the capture model, the effects of power allocation and modulation on the sum rate of Aloha in AWGN channels were analyzed in \cite{31} and \cite{34}, respectively. Queueing stability and channel fading were further considered in \cite{35}, where the sum rates with various cross-layer approaches were derived. In \cite{19}, by assuming that the channel state information (CSI) is available at the transmitter

\footnote{Note that different terminologies were used in these studies. In \cite{31}, for instance, “average spectral efficiency” was used to denote the sum rate of Aloha. In \cite{19, 34, 35, 42}, it was referred to as “throughput”.}
side and the collision model is adopted at the receiver side, the scaling behavior of the sum rate of Aloha as the number of nodes $n$ goes to infinity was characterized, and shown to be identical to that of the sum capacity of the multiple access channel.

Although various analytical models were developed in the above studies, many of them rely on numerical methods to calculate the sum rate, and the high computational complexity makes it hard to further optimize the sum rate. As we will demonstrate in this paper, the sum rate optimization of Aloha networks can be decomposed into two parts: 1) For given information encoding rate $R$, or equivalently, SINR threshold $\mu$, the network throughput can be maximized by properly choosing the backoff parameters, i.e., the transmission probabilities of each node. 2) As the information encoding rate and the maximum network throughput are both functions of the SINR threshold $\mu$, the sum rate can be further optimized by tuning $\mu$.

Specifically, we first focus on fading channels over which the received SNRs of nodes’ packets are exponentially distributed with the same mean received SNR $\rho$. By assuming that all the nodes encode their information independently at the same rate and each codeword lasts for one time slot, the maximum sum rate of Aloha with the capture model is derived as a function of the mean received SNR $\rho$. It is shown that similar to the sum capacity of the multiple access channel, the maximum sum rate of Aloha also logarithmically increases with $\rho$, but the high-SNR slope is only $e^{-1}$. At the low SNR region, the maximum sum rate is a monotonic increasing function of the number of nodes $n$, and approaches $e^{-1} \log_2 e \approx 0.5307$ as $n \to \infty$. To achieve the maximum sum rate, both the SINR threshold and the backoff parameters should be carefully selected according to the mean received SNR $\rho$. Explicit expressions of the optimal SINR threshold at different SNR regions are derived, and verified by simulations.

As the sum rate performance is critically determined by the assumptions on the receiver and channel conditions, the analysis is further applied to the collision model and AWGN channels. Specifically, the maximum sum rate of Aloha with the collision model over fading channels is further characterized, and shown to be comparable to that with the capture model. It indicates that despite an overly pessimistic estimation on the network throughput, the collision model serves as a good approximation for the capture model when analyzing the sum rate performance of Aloha networks. To demonstrate the effect of fading, we also derive the maximum sum rate of Aloha over AWGN channels, and the comparison corroborates that fading always hurts if no CSI is available at the transmitter side. Finally, the analysis is extended to incorporate distinct mean received SNRs of nodes.
The remainder of this paper is organized as follows. Section II presents the system model. Section III focuses on the network throughput analysis, where the maximum network throughput and the optimal backoff parameters are obtained as functions of the SINR threshold and the mean received SNR. The maximum sum rate is derived in Section IV, and the effects of key factors, including backoff, multipacket-reception capability, channel fading and power control, are discussed in Section V. Conclusions are summarized in Section VI.

II. SYSTEM MODEL

Consider a slotted Aloha network where \( n \) nodes transmit to a single receiver. All the nodes are synchronized and can start a transmission only at the beginning of a time slot. For each node, assume that it always has packets in its queue and each packet transmission lasts for one time slot. We ignore the subtleties of the physical layer such as the switching time from receiving mode to transmitting mode and the delay required for information exchange.

Let \( g_k \) denote the channel gain from node \( k \) to the receiver, which can be further written as

\[
g_k = \gamma_k \cdot h_k. \tag{1}\]

\( h_k \) is the small-scale fading coefficient of node \( k \) which varies from time slot to time slot and is modeled as a complex Gaussian random variable with zero mean and unit variance. The large-scale fading coefficient \( \gamma_k \) characterizes the long-term channel effect such as path loss and shadowing. Due to the slow-varying nature, the large-scale fading coefficients are usually available at the transmitter side through channel measurement and feedback.\(^2\) Let us first assume that uplink power control is performed to overcome the effect of large-scale fading. Specifically, denote the transmission power of node \( k \) as \( \bar{P}_k \). Then we have

\[
\bar{P}_k \cdot |\gamma_k|^2 = P_0. \tag{2}\]

In this case, each node has the same mean received signal-to-noise ratio (SNR) \( \rho = P_0/\sigma^2 \). The assumption of uplink power control will be relaxed in Section \( \text{V-D} \) where the analysis is extended to incorporate distinct mean received SNRs.

Throughout the paper, we assume that the receiver always has perfect channel state information but

\(^2\)More specifically, we assume that the time slot length is equal to the channel coherence time.

\(^3\)In this paper, we assume perfect and instant feedback. The effect of feedback error and delay on the network performance of Aloha is an interesting topic that deserves attention in the future study.
the transmitters are unaware of the instantaneous realizations of the small-scale fading coefficients. As a result, each node independently encodes its information at a given rate $R$ bit/s/Hz. Assume that each codeword lasts for one time slot. At the receiver side, no joint decoding is performed among nodes’ packets or with previously received packets. Instead, each node’s packet is decoded independently by treating others’ as background noise at each time slot. Such a simplified receiver has been widely adopted in the literature [22]–[36] and is referred to as the “capture model”.

Let
\[
\mu = 2^R - 1
\]  
(3)
denote the signal-to-interference-plus-noise ratio (SINR) threshold at the receiver. For each node’s packet, if its received SINR exceeds the threshold $\mu$, it can be successfully decoded and rate $R$ can be supported for reliable communications.\(^4\) Note that when the SINR threshold $\mu$ is sufficiently small, more than one packets could be successfully decoded at each time slot. It is clear that the number of successfully decoded packets in time slot $t$, denoted by $N_t$, is a time-varying variable. As a result, the total received information rate, i.e., $R \cdot N_t$ bit/s/Hz, also varies with time. In this paper, we focus on the long-term system behavior and define the sum rate as the time average of the received information rate:
\[
R_s = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} R \cdot N_t = R \cdot \hat{\lambda}_{out},
\]  
(4)
where
\[
\hat{\lambda}_{out} = \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} N_t
\]  
(5)
is the average number of successfully decoded packets per time slot, which is referred to as the network throughput.

Both the information encoding rate $R$ and the network throughput $\hat{\lambda}_{out}$ depend on the SINR threshold $\mu$. Intuitively, by reducing $\mu$, more packets can be successfully decoded at each time slot, yet the information encoding rate becomes smaller. Therefore, the SINR threshold $\mu$ should be carefully chosen to maximize the sum rate. Note that the network throughput $\hat{\lambda}_{out}$ is also crucially determined by the protocol design and

\(^4\)Note that here we assume that each codeword only covers one channel coherence time period. Without coding over fading states, the decoding delay is greatly reduced, but a certain rate loss is also caused, as we will show in Section IV-B and Section V-A.

\(^5\)More specifically, denote the received SINR of node $k$ as $SINR_k$. If $\log_2(1 + SINR_k) > R$, then by random coding the error probability of node $k$’s packet is exponentially reduced to zero as the block length goes to infinity. Here we assume that the block length is sufficiently large such that node $k$’s packet can be successfully decoded as long as $SINR_k \geq \mu$. 


backoff parameters. In the next section, we will specifically focus on the network throughput performance of Aloha networks.

III. NETWORK THROUGHPUT

An $n$-node buffered Aloha network is essentially an $n$-queue-single-server system whose performance is determined by the aggregate activities of the head-of-line (HOL) packets. Let us first characterize the state transition process of HOL packets.

A. State Characterization of HOL Packets

The behavior of each HOL packet can be modeled as a discrete-time Markov process. As Fig. 1 shows, a fresh HOL packet is initially in State 0, and moves to State D if it does not transmit. Define the phase of a HOL packet as the number of collisions it experiences. A phase-$i$ HOL packet moves to State 0 if its transmission is successful, and otherwise shifts to State $\min(K, i + 1)$, where $K$ denotes the cutoff phase. Intuitively, to alleviate the contention, nodes should reduce their transmission probabilities as they experience more collisions. Therefore, we assume that the transmission probabilities $\{q_i\}_{i=0,\ldots,K}$ form a monotonic non-increasing sequence.

![State transition diagram of an individual HOL packet in Aloha networks.](image)

In Fig. 1, $p_t$ denotes the probability of successful transmission of HOL packets at time slot $t$. It can

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6Note that a similar Markov chain of the HOL packet was established in [20] where the transmission probability of each fresh HOL packet was assumed to be 1. Here a new state, i.e., State D, is introduced to incorporate a general transmission probability of $0 < q_0 \leq 1$ for each fresh HOL packet.

7Note that in Fig. 1, the probability of successful transmission of HOL packets at time slot $t$, $p_t$, is assumed to be state independent. Intuitively, given that a HOL packet is attempting to transmit, the probability that its transmission is successful is determined by the overall activities of all the other HOL packets, rather than its own state. Therefore, no matter which state the HOL packet is currently staying at, its probability of successful transmission depends on the attempt rate of other HOL packets at the moment, which is denoted as $p_t$ in Fig. 1.
be easily shown that the Markov chain is uniformly strongly ergodic if and only if the limit

$$\lim_{t \to \infty} p_t = p$$

exists [48]. The steady-state probability distribution of the Markov chain in Fig. 1 can be further obtained as

$$\pi_0 = \frac{1}{\sum_{i=0}^{K-1} \frac{(1-p)^i}{q_i} + \frac{(1-p)^K}{pq_K}}$$

and

$$\pi_D = \frac{1 - q_0}{q_0} \pi_0, \quad \pi_i = \frac{(1-p)^i}{q_i} \pi_0, \quad i = 1, \ldots, K-1, \quad \pi_K = \frac{(1-p)^K}{pq_K} \pi_0.$$  

Note that \(\pi_0\) is the service rate of each node’s queue as the queue has a successful output if and only if the HOL packet is in State 0.

**B. Steady-state Point in Saturated Conditions**

If we regard an \(n\)-node buffered Aloha network as an \(n\)-queue-single-server system, the network throughput \(\hat{\lambda}_{out}\) is indeed the system output rate, which is equal to the aggregate input rate \(\hat{\lambda}\) if each node’s buffer has a non-zero probability of being empty. As \(\hat{\lambda}\) increases, the network will eventually become saturated where each node is busy with a non-empty queue. In this case, the network throughput is determined by the aggregate service rate, i.e.,

$$\hat{\lambda}_{out} = n\pi_0,$$

which, as (7) shows, depends on the steady-state probability of successful transmission of HOL packets \(p\). In this section, we will characterize the network steady-state operating point in saturated conditions based on the fixed-point equation of \(p\).

Specifically, for HOL packet \(j\), let \(S_j\) denote the set of nodes which have concurrent transmissions. It can be successfully decoded at the receiver side if and only if its received SINR is above the threshold \(\mu\), i.e.,

$$\frac{\sum_{k \in S_j} P_k |g_k|^2}{\sum_{k \in S_j} P_k |g_k|^2 + \sigma^2} \geq \mu,$$  

where \(P_k = P_0 |h_k|^2\) denotes the received power, which is given by \(P_k = P_0 |h_k|^2\) by combining (1) and (2). Suppose that \(|S_j| = i\). The steady-state probability of successful transmission
of HOL packet \( j \) given that there are \( i \) concurrent transmissions, \( r_i^j \), can be then written as
\[
\begin{align*}
  r_i^j = \Pr \left\{ \frac{|h_j|^2}{\sum_{k \in S_j} |h_k|^2 + 1/\rho} \geq \mu \right\}, \\
\end{align*}
\tag{10}
\]
where \( \rho = P_0/\sigma^2 \) is the mean received SNR. With \( h_k \sim \mathcal{CN}(0, 1) \), \( r_i^j \) can be easily obtained as \cite{24, 36}
\[
\begin{align*}
  r_i^j &= \exp \left( -\frac{\mu}{\rho} \right) \left( \frac{\mu}{\mu + 1} \right)^i. \\
\end{align*}
\tag{11}
\]
The right-hand side of (11) is independent of \( j \), indicating that all the HOL packets have the same conditional probability of successful transmission. Therefore, we drop the superscript \( j \), and write the steady-state probability of successful transmission of HOL packets \( p \) as
\[
\begin{align*}
  p &= \sum_{i=0}^{\infty} r_i \cdot \Pr\{ \text{\( i \) concurrent transmissions} \}. \\
\end{align*}
\tag{12}
\]
In saturated conditions, all the nodes have non-empty queues. According to the Markov chain shown in Fig. 1, the probability that the HOL packet is requesting transmission is given by \( (\pi_D + \pi_0)q_0 + \sum_{i=1}^{K} \pi_i q_i \), which is equal to \( \pi_0/\rho \) according to (8). Therefore, the probability that there are \( i \) concurrent transmissions can be obtained as
\[
\begin{align*}
  \Pr\{ \text{\( i \) concurrent transmissions} \} &= \binom{n-1}{i} (1 - \pi_0/\rho)^{n-1-i} \cdot (\pi_0/\rho)^i. \\
\end{align*}
\tag{13}
\]
By substituting (11) and (13) into (12), the steady-state probability of successful transmission of HOL packets \( p \) can be obtained as
\[
\begin{align*}
  p &= \exp \left( -\frac{\mu}{\rho} \right) \cdot \left( 1 - \frac{\pi_0}{\mu + 1} \cdot \frac{\pi_0}{\rho} \right)^{n-1} \text{ for large } n \approx \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \cdot \frac{\pi_0}{\rho} \right\}, \\
\end{align*}
\tag{14}
\]
where the approximation is obtained by applying \( (1 - x)^n \approx \exp(-nx) \) for \( 0 < x < 1 \). Finally, by substituting (7) into (14), we have
\[
\begin{align*}
  p &= \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \cdot \frac{1}{\sum_{i=0}^{\infty} \pi_i^K \frac{(1-p)^i}{q_i} + \frac{(1-p)^K}{qK}} \right\}. \\
\end{align*}
\tag{15}
\]
The following theorem states the existence and uniqueness of the root of the fixed-point equation (15).

\(^8\) Note that with a small network size, i.e., \( n \leq 5 \) for instance, the approximation error may become noticeable. It, nevertheless, rapidly declines as the number of nodes \( n \) increases.
Theorem 1. The fixed-point equation (15) has one single non-zero root \( p_A \) if \( \{q_i\}_{i=0,\ldots,K} \) is a monotonic non-increasing sequence.

Proof: See Appendix A.

As we can see from (15), the non-zero root \( p_A \) is closely dependent on backoff parameters \( \{q_i\}_{i=0,\ldots,K} \).

Without loss of generality, let \( q_i = q_0 \cdot Q_i \) where \( q_0 \) is the initial transmission probability and \( Q_i \) is an arbitrary monotonic non-increasing function of \( i \) with \( Q_0 = 1 \) and \( Q_i \leq Q_{i-1}, i = 1, \ldots, K \). With the cutoff phase \( K = 0 \), or the backoff function \( Q_i = 1, i = 0, \ldots, K \), for instance, \( p_A \) can be explicitly written as

\[
p_A = \exp \left( -\frac{\mu}{\rho} - \frac{n\mu}{\mu+1}q_0 \right).
\]

(16)

C. Maximum Network Throughput for Given \( \mu \) and \( \rho \)

It has been shown in Section III-B that the network operates at the steady-state point \( p_A \) in saturated conditions. By combining (9) and (14), the network throughput at \( p_A \) can be written as

\[
\hat{\lambda}_{out} = (\mu + 1) \cdot \left( -\frac{p_A \ln p_A}{\mu} - \frac{p_A}{\rho} \right),
\]

(17)

where \( p_A \) is an implicit function of the transmission probabilities \( q_i, i = 0, \ldots, K \), which is given in (15). It can be seen from (17) and (15) that the network throughput is crucially determined by the backoff parameters \( \{q_i\} \). In this section, we focus on the maximum network throughput \( \hat{\lambda}_{max} = \max_{\{q_i\}} \hat{\lambda}_{out} \). The following theorem presents the maximum network throughput \( \hat{\lambda}_{max} \) and the corresponding optimal backoff parameters \( \{q^*_i\} \).

Theorem 2. For given SINR threshold \( \mu \in (0, \infty) \) and mean received SNR \( \rho \in (0, \infty) \), the maximum network throughput is given by

\[
\hat{\lambda}_{max} = \begin{cases} 
\frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) & \text{if } \mu \geq \frac{1}{n-1} \\
n \exp \left\{ \frac{n\mu}{\mu+1} - \frac{\mu}{\rho} \right\} & \text{otherwise},
\end{cases}
\]

(18)

which is achieved at

\[
q^*_i = \begin{cases} 
q_m Q_i & \text{if } \mu \geq \frac{1}{n-1} \\
1 & \text{otherwise},
\end{cases}
\]

(19)
\[ q_m = \frac{\mu + 1}{n\mu} \left( \sum_{i=0}^{K-1} \frac{\exp\left( -1 - \frac{\mu}{\rho} \right) \left[ 1 - \exp\left( -1 - \frac{\mu}{\rho} \right) \right]^i}{Q_i} + \frac{\left[ 1 - \exp\left( -1 - \frac{\mu}{\rho} \right) \right]^K}{Q_K} \right). \]  

(20)

**Proof:** See Appendix B

(18) shows that for given SINR threshold \( \mu \), the maximum network throughput \( \hat{\lambda}_{\text{max}} \) is a monotonic increasing function of the mean received SNR \( \rho \). As \( \rho \to \infty \), we have

\[
\lim_{\rho \to \infty} \hat{\lambda}_{\text{max}} = \left\{ \begin{array}{ll}
\frac{\mu + 1}{\mu} - 1 & \text{if } \mu \geq \frac{1}{n-1} \\
\frac{\mu}{n} \exp\left\{-\frac{n\mu}{\mu + 1}\right\} & \text{otherwise,}
\end{array} \right.
\]

which approaches \( e^{-1} \) when \( \mu \gg 1 \). On the other hand, for given \( \rho \), \( \hat{\lambda}_{\text{max}} \) monotonically decreases as the SINR threshold \( \mu \) increases, as Fig. 2a illustrates. With a lower \( \mu \), the receiver can decode more packets among multiple concurrent transmissions, and thus better throughput performance can be achieved.

Corollary 1 shows that multipacket reception is possible if the SINR threshold \( \mu \) is sufficiently small.

**Corollary 1.**

1) For \( \mu \geq \frac{1}{n-1} \), \( \hat{\lambda}_{\text{max}} > 1 \) if and only if \( \frac{1}{n-1} \leq \mu < \frac{1}{e-1} \) and \( \rho > \frac{\mu}{\ln\frac{\mu + 1}{\mu} - 1} \).

2) For \( \mu < \frac{1}{n-1} \), \( \hat{\lambda}_{\text{max}} > 1 \) if and only if \( \rho > \frac{\mu}{\ln n - \frac{n\mu}{\mu + 1}} \).

**Proof:** See Appendix C

As Fig. 2b illustrates, with \( n = 50 \), if the SINR threshold \( \mu = 0.01 < \frac{1}{n-1} \), \( \hat{\lambda}_{\text{max}} > 1 \) when the mean received SNR \( \rho > -25.3 \text{dB} \) according to Corollary 1. On the other hand, if \( \mu = 0.5 \), we have \( \frac{1}{n-1} < \mu < \frac{1}{e-1} \approx 0.582 \). In this case, \( \hat{\lambda}_{\text{max}} > 1 \) when the mean received SNR \( \rho > 7 \text{dB} \).

**D. Simulation Results**

In this section, simulation results are presented to verify the preceding analysis. In particular, we consider a saturated Aloha network with BEB, i.e., each node always has a packet to transmit, and the transmission probabilities of each HOL packet are given by \( q_i = q_0 \cdot \frac{1}{i} \), \( i = 0, \ldots, K \). Section III-B has shown that it operates at the steady-state point \( p_A \), which is closely determined by the number of nodes \( n \) and the backoff parameters \( \{q_i\} \). The expression of \( p_A \) is given in (15) and verified by simulation results.
Fig. 2. (a) Maximum network throughput $\hat{\lambda}_{\text{max}}$ versus SINR threshold $\mu$. (b) Maximum network throughput $\hat{\lambda}_{\text{max}}$ versus mean received SNR $\rho$. $n = 50$.

presented in Fig. 3.

Fig. 4 illustrates the corresponding network throughput performance. The network throughput $\hat{\lambda}_{\text{out}}$ has been derived as a function of $p_A$ in (17), which varies with the backoff parameters. As we can see from Fig. 4, the network throughput performance is sensitive to the setting of the initial transmission probability $q_0$. According to Theorem 2 when the SINR threshold $\mu \geq \frac{1}{n-1}$, the maximum network throughput $\hat{\lambda}_{\text{max}}$ is achieved when $q_0$ is set to be $q_m$. Otherwise, $\hat{\lambda}_{\text{max}}$ is achieved with $q_i = 1, i = 0, \ldots, K$. The expressions of $\hat{\lambda}_{\text{max}}$ and the corresponding optimal backoff parameters $q_i^*$ are given in (18) and (19), respectively, and verified by simulation results presented in Fig. 4.

It can be also observed from Fig. 4 that the maximum network throughput $\hat{\lambda}_{\text{max}}$ closely depends on the SINR threshold $\mu$ and the mean received SNR $\rho$. Fig. 5 presents the simulation results of $\hat{\lambda}_{\text{max}}$ under various values of $\mu$ and $\rho$. We can clearly see from Fig. 5 that $\hat{\lambda}_{\text{max}}$ increases as the mean received SNR $\rho$ grows or the SINR threshold $\mu$ decreases.

Note that in spite of the improvement on the network throughput performance by reducing the SINR threshold $\mu$, the information encoding rate that can be supported for reliable communications, i.e., $R = \log_2(1 + \mu)$, is quite low when $\mu$ is small. It is clear that the SINR threshold $\mu$ determines a tradeoff between the network throughput and the information encoding rate. In the next section, we will further study how to maximize the sum rate by properly choosing the SINR threshold $\mu$.

*In simulations, the steady-state probability of successful transmission of HOL packets $p_A$ is obtained by calculating the ratio of the number of successful transmissions to the total number of attempts of HOL packets over a long time period, i.e., $10^8$ time slots.*
IV. MAXIMUM SUM RATE

It has been demonstrated in Section II that the sum rate of Aloha networks is determined by the information encoding rate $R$ and the network throughput $\hat{\lambda}_{out}$. By combining (3) and (4), the maximum sum rate can be written as

$$C = \max_{\mu, \{q_i\}} \hat{\lambda}_{out} \log_2(1 + \mu) = \max_{\mu} \log_2(1 + \mu) \max_{\{q_i\}} \hat{\lambda}_{out}. \quad (22)$$

Section III further shows that if backoff parameters $\{q_i\}$ are properly selected, the network throughput is maximized at $\hat{\lambda}_{max}$, which is a function of the SINR threshold $\mu$ and the mean received SNR $\rho$. By
combining (22) and Theorem 2, the maximum sum rate can be further written as

\[ C = \max_{\mu > 0} f(\mu), \] (23)

where the objective function \( f(\mu) \) is given by

\[ f(\mu) = \begin{cases} 
\frac{\mu + 1}{\mu} \exp\left(-1 - \frac{\mu}{\rho}\right) \log_2(1 + \mu) & \text{if } \mu \geq \frac{1}{n-1} \\
 n \exp\left\{-n\frac{\mu}{\mu + 1} - \frac{\mu}{\rho}\right\} \log_2(1 + \mu) & \text{otherwise}.
\end{cases} \] (24)

The following theorem presents the maximum sum rate and the optimal SINR threshold \( \mu^* \).

**Theorem 3.** For given mean received SNR \( \rho \in (0, \infty) \), the maximum sum rate is

\[ C = \begin{cases} 
\frac{\mu^*_h + 1}{\mu^*_h} \exp\left(-1 - \frac{\mu^*_h}{\rho}\right) \log_2(1 + \mu^*_h) & \text{if } \rho \geq \rho_0 \\
n \exp\left\{-n\frac{\mu^*_l}{\mu^*_l + 1} - \frac{\mu^*_l}{\rho}\right\} \log_2(1 + \mu^*_l) & \text{otherwise},
\end{cases} \] (25)

which is achieved at

\[ \mu^* = \begin{cases} 
\mu^*_h & \text{if } \rho \geq \rho_0 \\
\mu^*_l & \text{otherwise},
\end{cases} \] (26)

where \( \mu^*_h \) and \( \mu^*_l \) are the roots of the following equations:

\[ \frac{\mu + 1}{\rho} + \frac{1}{\mu} = e, \] (27)
and
\[
\left(\mu + 1\right) \frac{n + 1}{\rho} + \frac{\mu + 1}{\mu + 1} = e,
\]
respectively, and
\[
\rho_0 = \frac{n - 1}{1 - (n - 1) \ln \frac{n}{n - 1}},
\]

**Proof:** See Appendix D. \(\blacksquare\)

Note that \(\rho_0\) is a monotonic decreasing function of \(n \in [2, \infty)\), and \(\lim_{n \to \infty} \rho_0 = 2\). When the number of nodes \(n\) is large, \(\rho_0\) is close to 3dB.

**A. Optimal SINR Threshold \(\mu^*\)**

Theorem 3 shows that to achieve the maximum sum rate, the SINR threshold \(\mu\) should be carefully selected. Fig. 6a illustrates how the optimal SINR threshold \(\mu^*\) varies with the mean received SNR \(\rho\). At the low SNR region, i.e., \(\rho < \rho_0\), for instance, we can obtain from (26) and (28) that \(\mu^*_{\rho < \rho_0} = \mu^*_l \approx e^{-\mathbb{W}_0(-\frac{1}{n})} - \frac{1}{n}\) for large \(n\), where \(\mathbb{W}_0(z)\) is the principal branch of the Lambert W function [49]. With a large number of nodes \(n\), \(\mu^*_p < \rho_0 \ll 1\), implying that multiple packets can be successfully decoded. At the high SNR region, we can obtain from (26-27) that \(\mu^*_{\rho \geq \rho_0} = \mu^*_h \approx e^{\mathbb{W}_0(\rho)}\) for large \(\rho\). As we can see from Fig. 6a, with \(\rho \gg 1\), the optimal SINR threshold monotonically increases with the mean received SNR \(\rho\).

By combining (26) with Theorem 2, we can also obtain the maximum network throughput with \(\mu = \mu^*\) as
\[
\hat{\lambda}_{\text{max}}^{\mu = \mu^*} = \begin{cases} \frac{\mu^*_h + 1}{\mu^*_h} \exp \left( -1 - \frac{\mu^*_h}{\rho} \right) & \text{if } \rho \geq \rho_0 \\ n \exp \left( -n \frac{\mu^*_l}{\mu^*_l + 1} - \frac{\mu^*_l}{\rho} \right) & \text{otherwise.} \end{cases}
\]

As we can see from Fig. 6b, at the low SNR region, i.e., \(\rho < \rho_0\), \(\hat{\lambda}_{\text{max}}^{\mu = \mu^*}\) linearly increases with the number of nodes \(n\). In this case, the optimal SINR threshold \(\mu^*_{\rho < \rho_0} = \mu^*_l\) is a monotonic decreasing function of \(n\), and thus more packets can be successfully decoded as \(n\) grows. For large \(n\), we have \(\hat{\lambda}_{\text{max}, \rho < \rho_0} \approx ne^{-1}\) according to (30). At the high SNR region, Fig. 6a has shown that the optimal SINR threshold \(\mu^*_{\rho \geq \rho_0} = \mu^*_h\) is much larger than 1, with which at most one packet can be successfully decoded at each time slot. Therefore, the maximum network throughput quickly drops below 1, and eventually approaches \(e^{-1}\) as \(\rho \to \infty\).
(b) Maximum network throughput $\hat{\lambda}_{\text{max}}^{\mu^*}$ versus mean received SNR $\rho$.

Fig. 6. (a) Optimal SINR threshold $\mu^*$ versus mean received SNR $\rho$. (b) Maximum network throughput $\hat{\lambda}_{\text{max}}^{\mu^*}$ versus mean received SNR $\rho$.

Fig. 7. (a) Maximum sum rate $C$ at the high SNR region. (b) Maximum sum rate $C$ at the low SNR region.

B. Maximum Sum Rate $C$

Similar to Section IV-A, let us take a closer look at the maximum sum rate $C$ at different SNR regions.

1) $\rho \geq \rho_0$: With $\rho \geq \rho_0$, it has been shown in Section IV-A that the optimal SINR threshold $\mu^*_{\rho \geq \rho_0} = \mu^*_h \approx e^{\mathcal{W}_0(\rho)}$ for large $\rho$. The maximum sum rate in this case can be then approximated by

$$C_{\rho \geq \rho_0} \approx (1 + e^{\mathcal{W}_0(\rho)}) \exp\left(-1 + \frac{e^{\mathcal{W}_0(\rho)}}{\rho}\right) \log_2(1 + e^{\mathcal{W}_0(\rho)}), \quad (31)$$
for $\rho \gg 1$. As Fig. 7a shows, the approximation (31) works well when the mean received SNR $\rho$ is large, i.e., $\rho \geq 15$dB. Moreover, a logarithmic increase of the maximum sum rate $C$ can be observed at the high SNR region. The following corollary presents the high-SNR slope of $C$.

**Corollary 2.** $\lim_{\rho \to \infty} \frac{C}{\log_2 \rho} = e^{-1}$.

*Proof: See Appendix [E]*

Recall that the high-SNR slope of the ergodic sum capacity of multiple access channels is 1 when single-antenna is employed at both the transmitters and the receiver. To achieve the ergodic sum capacity, however, a joint decoding of all received signals is required and the codewords should span multiple fading states. With the capture model, in contrast, each node’s packet is decoded independently by treating others’ as background noise at each time slot. When the mean received SNR is high, at most one packet can be successfully decoded each time due to a large SINR threshold $\mu^* \gg 1$. Corollary [2] shows that with the simplified receiver, the high-SNR slope of the maximum sum rate of Aloha networks is significantly lower than that of the sum capacity.

2) $\rho < \rho_0$: For $\rho < \rho_0$, it has been shown in Section [IV-A] that the optimal SINR threshold $\mu_{\rho<\rho_0}^* = \mu_l^* \approx e^{-W_0(-\frac{1}{n})} - 1$ for large $n$. The corresponding maximum sum rate can be then approximated by

$$C_{\rho<\rho_0} \approx -nW_0\left(-\frac{1}{n}\right) \exp\left(-n\left(1 - e^{W_0(-\frac{1}{n})}\right) - \frac{e^{-W_0(-\frac{1}{n})} - 1}{\rho}\right) \log_2 e,$$

for $n \gg 1$. As we can see from Fig. 7b, the approximation (32) works well when the number of nodes $n$ is large. The following corollary further presents the limiting maximum sum rate as $n \to \infty$ at the low SNR region.

**Corollary 3.** $\lim_{n \to \infty} C_{\rho<\rho_0} = e^{-1}\log_2 e$.

*Proof: See Appendix [E]*

Note that it has been shown in Section [IV-A] that with $\rho < \rho_0$, the maximum network throughput $\dot{\lambda}_{\mu=\mu^*} \approx ne^{-1}$, which grows with the number of nodes $n$ unboundedly. Although more packets can be successfully decoded as $n$ increases, the information carried by each packet decreases due to a diminishing information encoding rate, i.e., $R = \log_2(1 + \mu_{\rho<\rho_0}^*) \approx \frac{1}{n} \log_2 e$ for large $n$. Therefore, as the number of nodes $n \to \infty$, the maximum sum rate reaches a limit that is independent of the mean received SNR, as Corollary [3] indicates. It is in sharp contrast to the ergodic sum capacity of multiple access channels which linearly increases with $n$ and $\rho$ at the low SNR region.
Fig. 8. Sum rate $R_s$ versus SINR threshold $\mu$ under different values of mean received SNR $\rho$. $n = 50$. $K = 0$ and $q_0 = q_0^*$.  

**C. Simulation Results**

In this section, simulation results are presented to verify the preceding analysis. Again we consider a saturated Aloha network with the cutoff phase $K = 0$. Section III has shown that for given SINR threshold $\mu$ and mean received SNR $\rho$, the initial transmission probability $q_0$ should be set as $q_0^*$, which is given in Theorem 2, to maximize the network throughput. Fig. 8 illustrates the corresponding sum rate. We can clearly observe from Fig. 8 that the sum rate performance is sensitive to the SINR threshold $\mu$ especially when the mean received SNR $\rho$ is small. To achieve the maximum sum rate, $\mu$ should be properly set according to the mean received SNR $\rho$. The expressions of the optimal SINR threshold $\mu^*$ and the maximum sum rate $C$ are given in Theorem 3 and verified by simulation results presented in Fig. 8.

**V. Discussions**

So far we have shown that to optimize the sum rate performance of Aloha networks, the SINR threshold $\mu$ and backoff parameters $\{q_i\}$ should be properly set according to the mean received SNR $\rho$, and the maximum sum rate logarithmically increases with $\rho$ with the high-SNR slope of $e^{-1}$. In this section, we will further discuss how the performance is affected by key factors such as backoff, multipacket-reception capability, channel fading and power control.

**A. Effect of Adaptive Backoff**

Backoff is a key component of random-access networks. It has been shown in Sections III and IV that to achieve the maximum sum rate, backoff parameters, i.e., the transmission probabilities $\{q_i\}$ of nodes,
should be adaptively selected according to the number of nodes \( n \) and the mean received SNR \( \rho \). In many studies, however, nodes are supposed to transmit their packets with a fixed probability \([23], [25], [26], [33], [36]\). To see how the rate performance of Aloha deteriorates without adaptive backoff, let us assume that each node transmits its packet with a constant probability \( q \) at each time slot, i.e., \( q_i = q \), \( i = 0, \ldots, K \). In this case, the network steady-state point in saturated conditions can be obtained from \([15]\) as

\[
p^{q_i=q}_A = \exp \left( -\frac{\mu}{\rho} - \frac{nq\mu}{\mu + 1} \right),
\]

and the corresponding network throughput is

\[
\hat{\lambda}^{q_i=q}_{out} = nq \exp \left( -\frac{\mu}{\rho} - \frac{nq\mu}{\mu + 1} \right),
\]

according to \([17]\). The sum rate can be then written as

\[
R^{q_i=q}_s = nq \exp \left( -\frac{\mu}{\rho} - \frac{nq\mu}{\mu + 1} \right) \cdot \log_2(1 + \mu),
\]

which is an increasing function of the mean received SNR \( \rho \).

As \( \rho \to \infty \), it can be easily obtained from \([35]\) that \( R^{q_i=q}_s = \lim_{\rho \to \infty} R^{q_i=q}_s = nq \exp \left( -\frac{nq\mu}{\mu + 1} \right) \cdot \log_2(1 + \mu) \), with the maximum

\[
\max_{\mu} R^{q_i=q}_s = nq \exp \left( -nq \left( 1 - e^{\frac{1}{\rho} \left( -\frac{1}{nq} \right)} \right) \right) \cdot \log_2 e^{\frac{1}{\rho} \left( -\frac{1}{nq} \right)},
\]

which is achieved at

\[
\mu^{*,q_i=q} = e^{-\frac{1}{\rho}} - 1.
\]

\([37]\) shows that the optimal SINR threshold \( \mu^{*,q_i=q} \) monotonically decreases as the number of nodes \( n \) grows. For large \( n \gg 1 \), it can be easily obtained from \([36,37]\) that \( \mu^{*,q_i=q} \approx \frac{1}{nq} \), and

\[
\max_{\mu} R^{q_i=q}_s \approx nq \exp \left( -\frac{1}{\rho} \right) \cdot \log_2 e.
\]

Recall that it has been shown in Section \([IV-B]\) that the maximum sum rate increases with the mean received SNR \( \rho \) unboundedly. Here \([38]\) indicates that with a constant transmission probability, the sum rate converges to a limit that is much lower than 1 as \( \rho \to \infty \). It corroborates that adaptive backoff is indispensable for random-access networks.
It is interesting to note that when \( q = 1 \), all the nodes persistently transmit their packets, and the Aloha network reduces to a typical multiple access channel. It is well known that for an \( n \)-user AWGN multiple access channel, if the capture model is adopted at the receiver side\(^{10}\) and all the users have equal received power, the sum rate approaches \( \log_2 e \) as \( n \to \infty \) \(^{50}\). Here we can see from \(^{38}\) that an additional factor of \( e^{-1} \) is introduced, which is mainly attributed to the effect of channel fading\(^{11}\).

### B. Effect of Multipacket Reception

In this paper, we focus on the capture model where a packet can be successfully decoded at the receiver side if and only if the received SINR is above the threshold \( \mu \). If \( \mu \) is small enough, multiple packets can be successfully decoded each time, and thus the network throughput performance is substantially enhanced compared to the classical collision model where at most one packet can be successfully decoded at each time slot. Such an improvement has been widely observed in the literature \(^{23}–^{27},^{29},^{30},^{32},^{33},^{36}\). As we will demonstrate in this section, despite a prominent throughput increase brought by multipacket reception, the maximum sum rate with the capture model is indeed comparable to that with the collision model.

Specifically, let us consider the classical collision model where a packet transmission is successful only if there are no concurrent transmissions. To support an information encoding rate of \( R \) bit/s/Hz, the SNR threshold at the receiver side should be set to \( \mu = 2^R - 1 \). Appendix \( \Gamma \) shows that in this case, the maximum sum rate is given by

\[
C_{\text{collision}} = \exp\left(-1 - \frac{e^{W_0(\rho)} - 1}{\rho}\right) \cdot \log_2(e^{W_0(\rho)}),
\]

which is achieved when the SNR threshold is set to be \( \mu^{*,\text{collision}} = e^{W_0(\rho)} - 1 \). The corresponding maximum network throughput is given by

\[
\hat{\lambda}_{\text{max},\mu = \mu^{*,\text{collision}}} = \exp\left(-1 - \frac{e^{W_0(\rho)} - 1}{\rho}\right).
\]

We can see from \(^{40}\) that the maximum network throughput with the collision model is always smaller than \( e^{-1} \), and the factor \( \exp\left(-\frac{e^{W_0(\rho)} - 1}{\rho}\right) \) describes the throughput loss due to channel fading. As Fig. \( 9a \) shows, at the low SNR region, it is much lower than the maximum network throughput with the

\(^{10}\) It is also referred to as the “conventional CDMA receiver” in the literature.

\(^{11}\) Note that in this paper, each codeword is assumed to last for one channel coherence time period. Without coding over different fading states, the channel fluctuations cannot be averaged out, thus leading to a significant rate loss compared to the AWGN case.
capture model that can be much higher than 1 thanks to multipacket reception. The gap is, nevertheless, diminished as the mean received SNR $\rho$ increases. Both of them approach $e^{-1}$ as $\rho \to \infty$.

In spite of the significant throughput improvement, only marginal gains on the sum rate performance can be observed at the low SNR region. As Fig. 9b illustrates, the maximum sum rate with the collision model is close to that with the capture model even when the mean received SNR $\rho$ is small. As we have demonstrated in Section IV-B, the maximum sum rate with the capture model is limited by $e^{-1} \log_2 e \approx 0.5307$ with $\rho < \rho_0$. The rate difference rapidly decreases as the mean received SNR $\rho$ increases, indicating that the collision model serves as a good approximation for the capture model when analyzing the sum rate performance of Aloha networks.

C. Effect of Fading

Section IV has shown that with the capture model, the network throughput of Aloha over fading channels can be much higher than $e^{-1}$, the maximum network throughput with the collision model in ideal channel conditions. In the literature, it is sometimes mistaken as an improvement brought by fading channels \cite{24, 26, 27, 30}: the variation of the received power of nodes is enlarged due to fading, and thus the chance that one of them can be captured, i.e., with much higher received power than others, is increased. As we will demonstrate in this section, the gain comes from the receiver rather than the channel fading. If the receiver model is fixed, both the maximum network throughput and the maximum sum rate of Aloha over AWGN channels are always higher than that over fading channels.
Let us first assume that the capture model is adopted. Specifically, by setting the small-scale fading coefficients \( |h_k| = 1, k = 1, \ldots, n \), the steady-state probability of successful transmission of a HOL packet given that there are \( i \) concurrent transmissions over AWGN channels can be easily obtained from (10) as

\[
r^i_{AWGN} = \Pr \left\{ \frac{1}{i + 1/\rho} \geq \mu \right\} = \begin{cases} 1 & \text{if } i \leq \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor \\ 0 & \text{otherwise.} \end{cases} \tag{41}
\]

By substituting (41) and (13) into (12), the steady-state probability of successful transmission of HOL packets can be written as

\[
p = \sum_{i=0}^{\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor} \binom{n-1}{i} (1-\pi_0/p)^{n-i} \cdot (\pi_0/p)^i \begin{cases} 0 & \text{if } \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor < 0 \\ 1 & \text{if } \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor \geq n-1 \\ I_{1-\pi_0/p} \left( n-1-\left\lceil \frac{1}{\mu} - \frac{1}{\rho} \right\rceil, \left\lceil \frac{1}{\mu} - \frac{1}{\rho} \right\rceil + 1 \right) & \text{otherwise,} \end{cases} \tag{42}
\]

where \( I_x(a, b) \) is the regularized incomplete beta function. Appendix \( \text{H} \) shows that with \( K = 0 \), the maximum sum rate is given by

\[
C^{AWGN} = \begin{cases} e^{-1} \log_2(1 + \rho) & \text{if } \rho \geq \rho_1 \\ n \log_2 \left( 1 + \frac{1}{n-1+\rho} \right) & \text{otherwise,} \end{cases} \tag{43}
\]

which is achieved when the SINR threshold is set to be

\[
\mu^{*,AWGN} = \begin{cases} \rho & \text{if } \rho \geq \rho_1 \\ \frac{1}{n-1+\rho} & \text{otherwise,} \end{cases} \tag{44}
\]

and the corresponding maximum network throughput is given by

\[
\hat{\lambda}_{\text{max}}^{\mu=\mu^{*,AWGN}} = \begin{cases} e^{-1} & \text{if } \rho \geq \rho_1 \\ n & \text{otherwise,} \end{cases} \tag{45}
\]

where \( \rho_1 \) is the root of the following equation:

\[
n \log_2 \left( 1 + \frac{1}{n-1+\rho} \right) = e^{-1} \log_2(1 + \rho). \tag{46}
\]

For large number of nodes \( n \), \( \rho_1 \approx e^e - 1 \).
We can see from (44) that at the low SNR region, by setting the SINR threshold \( \mu = \mu^\ast_{p<\rho_1} = \frac{1}{n-1+\rho} \),

we have \( \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor = n - 1 \), with which all the packets can be successfully decoded according to (42).

To maximize the network throughput, all the nodes should transmit with probability 1, and thus the maximum network throughput is equal to the number of nodes \( n \). As \( n \to \infty \), the maximum sum rate \( C_{p<\rho_1}^{AWGN} \to \log_2 e \) according to (43), which is consistent to the asymptotic sum rate of an \( n \)-user AWGN multiple access channel with the capture model [50]. Recall that it has been shown in Section IV that in the fading case, the maximum sum rate and the corresponding maximum network throughput at the low SNR region are approximately given by \( e^{-1} \log_2 e \) and \( ne^{-1} \), respectively, for large \( n \). Both of them are significantly lower than that over AWGN channels, as Fig. 10 illustrates.

At the high SNR region, by setting the SINR threshold \( \mu = \mu^\ast_{p\geq\rho_1} = \rho \), we have \( \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor = 0 \). According to (42), with \( \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor = 0 \), a packet can be successfully decoded if and only if there are no concurrent transmissions. In this case, the receiver reduces to the collision model. The maximum network throughput is then \( e^{-1} \), and the maximum sum rate logarithmically increases with the mean received SNR \( \rho \) with the high-SNR slope \( e^{-1} \). As we can see from Fig. 10b, although the sum rates over AWGN and fading channels have the same high-SNR slope, substantial gains are observed in the AWGN case.

If the collision model is adopted, the maximum network throughput of Aloha over perfect channel conditions is well known to be \( e^{-1} \). The maximum sum rate of Aloha over AWGN channels can be then easily obtained as \( e^{-1} \log_2 (1 + \rho) \). We can clearly see from (39) that over fading channels, the
maximum sum rate is upper-bounded by 
\[ e^{-1} \log_2 \left( e^{W_0(\rho)} \right) < e^{-1} \log_2 (1 + \rho). \] 
\[ (40) \] also shows that the corresponding maximum throughput is always smaller than 
\[ e^{-1} \log_2 (1 + \rho). \] It corroborates that if CSI is not available at the transmitter side, fading always hurts \[ (50). \]

\[ D. \text{ Effect of Power Control} \]

So far we have focused on a homogeneous Aloha network where all the nodes have the same mean received SNR \( \rho \). In this section, the analysis will be extended to the heterogeneous case, where nodes in the same group have identical mean received SNRs but SNRs differ from group to group.

Specifically, assume that \( n \) nodes are divided into \( M \) groups. Group \( i \) has \( n_i \) nodes, and each node in Group \( i \) has the mean received SNR \( \rho_i \), \( i = 1, \ldots, M \). For HOL packet \( j \), let \( S_j \) denote the set of nodes that have concurrent transmissions. It can be successfully decoded at the receiver if and only if its received SINR is above the SINR threshold \( \mu \), i.e.,
\[ \frac{P_j}{\sum_{k \in S_j} P_k + \sigma^2} \geq \mu, \] where \( P_k \) denotes the received power of node \( k \)’s packet. Suppose that 
\[ S_j = \bigcup_{m=1}^{M} S_j^m, \] where \( S_j^m \) denotes the set of nodes which have concurrent transmissions in Group \( m \), and \( |S_j^m| = i_m, m = 1, \ldots, M \). The steady-state probability of successful transmission of HOL packet \( j \) given that there are \( \{i_m\}_{m=1,\ldots,M} \) concurrent transmissions, \( r_{\{i_m\}}^j \), can be then written as
\[ r_{\{i_m\}}^j = \Pr \left\{ \frac{|h_j|^2}{\sum_{m=1}^{M} \sum_{k \in S_j^m} |h_k|^2 \cdot \frac{\rho_m}{\rho_j} + \frac{1}{\rho_j}} \geq \mu \right\}. \] (47)

Appendix I shows that with \( h_k \sim \mathcal{CN}(0, 1) \), \( r_{\{i_m\}}^j \) is given by
\[ r_{\{i_m\}}^j = \frac{\exp \left( -\frac{\mu}{\rho_j} \right)}{\prod_{m=1}^{M} (1 + \frac{\rho_m}{\rho_j})^{i_m}}. \] (48)

The steady-state probability of successful transmission of HOL packet \( j \), \( p^{(j)} \), can be written as
\[ p^{(j)} = \sum_{i_1=0}^{n_1} \cdots \sum_{i_j=0}^{n_j-1} \cdots \sum_{i_M=0}^{n_M} r_{\{i_m\}}^j \cdot \prod_{m=1}^{M} \Pr \{ i_m \text{ concurrent transmissions in Group } m \}. \] (49)
For each node in Group \( m \), the probability that it is busy with the HOL packet requesting transmission in saturated conditions is given by \( (\pi_D^{(m)} + \pi_0^{(m)})q_0 + \sum_{i=1}^{K} \pi_i^{(m)} q_i \), which is equal to \( \frac{\pi_0^{(m)}}{p^{(m)}} \) according
to (8). Therefore, we have

\[
\Pr\{i_m \text{ concurrent transmissions in Group } m\} = \begin{cases} 
{n_m \choose i_m} \left(1 - \frac{\pi_0^{(m)}}{p^{(m)}}\right)^{n_m - i_m} \cdot \left(\frac{\pi_0^{(m)}}{p^{(m)}}\right)^{i_m} & m \neq j \\
{n_{j-1} \choose i_j} \left(1 - \frac{\pi_0^{(j)}}{p^{(j)}}\right)^{n_j - i_j} \cdot \left(\frac{\pi_0^{(j)}}{p^{(j)}}\right)^{i_j} & m = j.
\end{cases}
\]

By combining (48-50), the steady-state probability of successful transmission of HOL packet \(j\) can be obtained as

\[
p^{(j)} = \exp \left( -\frac{\mu}{\rho_j} \right) \cdot \left(1 - \frac{\mu}{\mu + 1} \cdot \frac{\pi_0^{(j)}}{p^{(j)}}\right)^{n_j - 1} \prod_{m=1, m\neq j}^{M} \left(1 - \frac{\mu}{\mu + \rho_j/\rho_m} \cdot \frac{\pi_0^{(m)}}{p^{(m)}}\right)^{n_m}
\]

for large \(n_1, \ldots, n_M\).

\[
\exp \left( -\frac{\mu}{\rho_j} - \sum_{m=1}^{M} \frac{n_m \mu}{\mu + \rho_j/\rho_m} \cdot \frac{\pi_0^{(m)}}{p^{(m)}} \right).
\]

(51)

Finally, by substituting (7) into (51), we have

\[
p^{(j)} = \exp \left( -\frac{\mu}{\rho_j} - \sum_{m=1}^{M} \frac{n_m \mu}{\mu + \rho_j/\rho_m} \cdot \frac{1}{\sum_{i=0}^{K-1} \frac{p^{(m)}}{q_i} \left(1 - \frac{\pi_0^{(m)}}{p^{(m)}}\right)^{i}} \right). 
\]

(52)

We can see from (52) that in the heterogeneous case, HOL packets in different groups have distinct steady-state probabilities of successful transmission. With \(M\) groups, \(M\) non-zero roots \(\{p_A^{(m)}\}_{m=1, \ldots, M}\) can be obtained by jointly solving \(M\) fixed-point equations given in (52). Note that nodes in the same group have the same steady-state probability of successful transmission and thus the same throughput performance. For each node in Group \(m\), \(m = 1, \ldots, M\), the node throughput can be obtained from (7) as

\[
\lambda^{(m)}_{out} = \frac{1}{\sum_{i=0}^{K-1} \frac{1 - p^{(m)}_A}{q_i} + \frac{1 - p^{(m)}_A}{p^{(m)}_A q_K}}
\]

and the network throughput is

\[
\hat{\lambda}_{out} = \sum_{m=1}^{M} n_m \lambda^{(m)}_{out}.
\]

To illustrate the above results, let us focus on the two-group case and assume that the cutoff phase \(K = 0\). The steady-state probabilities of successful transmission of HOL packets in Group 1 and Group 2 can be obtained from (52) as

\[
p_A^{(1)} = \exp \left( -\frac{\mu}{\rho_1} - \frac{n_1 \mu q_0}{\mu + 1} - \frac{n_2 \mu q_0}{\mu + \rho_1/\rho_2} \right)
\]

(54)
and

\[ P_A^{(2)} = \exp \left( -\frac{\mu}{\rho_2} - \frac{n_1 \mu q_0}{\mu + \rho_2/\rho_1} - \frac{n_2 \mu q_0}{\mu + 1} \right), \]  

(55)

respectively. By combining (54-55) with (53), the node throughput can be obtained as

\[ \lambda^{(1)}_{\text{out}} = q_0 \exp \left( -\frac{\mu}{\rho_1} - \frac{n_1 \mu q_0}{\mu + 1} - \frac{n_2 \mu q_0}{\mu + \rho_1/\rho_2} \right), \]  

(56)

and

\[ \lambda^{(2)}_{\text{out}} = q_0 \exp \left( -\frac{\mu}{\rho_2} - \frac{n_1 \mu q_0}{\mu + \rho_2/\rho_1} - \frac{n_2 \mu q_0}{\mu + 1} \right). \]  

(57)

(56-57) shows that the throughput performance is closely determined by the mean received SNRs. If the two groups have equal mean received SNRs \( \rho_1 = \rho_2 = \rho \), for instance, we can see from (54-55) that all the HOL packets have the same steady-state probability of successful transmission, i.e., \( P_A^{(1)} = P_A^{(2)} \). The node throughput can be obtained from (56-57) as

\[ \hat{\lambda}_{\text{out}}(1) = \hat{\lambda}_{\text{out}}^{(2)} = q_0 \exp \left( -\frac{\mu}{\rho} - \frac{(n_1 + n_2) \mu q_0}{\mu + 1} \right). \]

In this case, each node has an equal probability of accessing the channel, thus achieving the same throughput performance.

As the difference between \( \rho_1 \) and \( \rho_2 \) grows, nevertheless, the node throughput performance becomes increasingly polarized. We can see from (54-55) that with \( \rho_1 \gg \rho_2 \), \( P_A^{(1)} \gg P_A^{(2)} \), which indicates that much more packets from Group 1 can be successfully received than Group 2. The throughput performance of nodes in Group 1 is then much better than that in Group 2, i.e., \( \lambda^{(1)}_{\text{out}} \gg \lambda^{(2)}_{\text{out}} \) according to (56-57), implying serious unfairness among nodes.

As the maximum network throughput \( \hat{\lambda}_{\max} = \max_{q_0} \hat{\lambda}_{\text{out}} \) does not have an explicit expression in general, we can only numerically calculate the maximum sum rate \( C = \max_{\mu} \hat{\lambda}_{\max} \cdot \log_2(1 + \mu) \). Fig. 11 illustrates how the maximum sum rate \( C \) varies with the ratio of \( \rho_1 \) and \( \rho_2 \) by fixing the mean SNR of nodes \( \bar{\rho} = \frac{\sum_{m=1}^{2} n_m \rho_m}{\sum_{m=1}^{2} n_m} \) to 0dB, 10dB, 15dB and 20dB. It is interesting to note from Fig. 11 that with a large SNR ratio \( \rho_1/\rho_2 \gg 1 \), the maximum sum rate is higher than that with \( \rho_1/\rho_2 = 1 \), which suggests that despite serious unfairness, the sum rate performance may be improved by introducing a large SNR difference among nodes.

To see why, let us take a closer look at the network performance with \( \rho_1/\rho_2 \to \infty \). In this case, the network throughput can be obtained from (56-57) as \( \hat{\lambda}_{\text{out}}^{1/2} \to \infty = n_1 q_0 \exp \left\{ -\frac{\mu}{n_1 + n_2} \rho - \frac{n_1 \mu q_0}{\mu + 1} \right\} \), which reduces to the throughput of Group 1. For large \( \mu \), it can be obtained that \( \hat{\lambda}_{\max}^{1/2} \to \infty = \frac{\mu + 1}{\mu} \exp \left\{ -1 - \frac{\mu}{\rho} \right\} \) and \( \hat{\lambda}_{\max}^{1/2} = \frac{\mu + 1}{\mu} \exp \left\{ -1 - \frac{\mu}{\rho} \right\} \) according to (18), which can be further written as \( \hat{\lambda}_{\max}^{1/2} \to \infty (\bar{\rho}) = \)
Fig. 11. Maximum sum rate versus $\rho_1/\rho_2$ for a two-group Aloha network. $n_1 = n_2 = 25$. $K = 0$.

\[ \hat{\lambda}_{\text{max}}^{\rho_1/\rho_2 = 1}(\frac{n_1 + n_2}{n_1} \bar{\rho}) > \hat{\lambda}_{\text{max}}^{\rho_1/\rho_2 = 1}(\bar{\rho}). \] As a result, for $\bar{\rho} \gg 1$, we have $C_{\rho_1/\rho_2 \to \infty}(\rho) = C_{\rho_1/\rho_2 = 1}(\frac{n_1 + n_2}{n_1} \bar{\rho}) > C_{\rho_1/\rho_2 = 1}(\bar{\rho})$. Intuitively, the channel efficiency is maximized by allocating all the resources to the strongest node(s). Here we can see that even without a central controller for resource allocation, the fundamental tradeoff between efficiency and fairness still holds true for random-access networks.

The tradeoff nevertheless becomes less significant when the network operates at the low SNR region. It can be observed from Fig. 11 that with $\bar{\rho} = 0$ dB, the maximum sum rate is insensitive to the SNR ratio. It indicates that power control is desirable in this case, with which the fairness performance can be improved without sacrificing the sum rate.

VI. CONCLUSION

In this paper, the unified analytical framework proposed in [20], [21] is extended to incorporate the capture model. By assuming that the received SNRs of nodes’ packets are exponentially distributed with the same mean received SNR $\rho$ in a saturated Aloha network, explicit expressions of the maximum network throughput and the corresponding optimal backoff parameters are obtained, based on which the maximum sum rate is derived by optimizing the SINR threshold $\mu$. The analysis shows that with a low SNR, the maximum sum rate linearly increases with the number of nodes $n$, and approaches $e^{-1} \log_2 e$ as $n \to \infty$. At the high SNR region, a logarithmic growth of the maximum sum rate is observed as $\rho$ increases, with the high-SNR slope of $e^{-1}$. Effects of key factors, including backoff, receiver model, channel fading and power control, on the sum rate performance are also studied.

The analysis sheds important light on the practical network design. For instance, it is demonstrated that
to achieve the maximum sum rate, the transmission probabilities of nodes should be adaptively selected according to the network size and the mean received SNR $\rho$. With a fixed transmission probability, the sum rate may significantly deteriorate, and converges to a limit that is much lower than 1 as $\rho \to \infty$. Moreover, the throughput performance of each node is found to be closely dependent on its mean received SNR. Although a large SNR difference among nodes may be beneficial to the sum rate performance, it introduces serious unfairness. A uniform mean received SNR is shown to be crucial for achieving a good balance between fairness and sum rate when the network operates at the low SNR region.

Note that the analysis is based on the capture model, which is essentially a single-user detector. Performance gains on the maximum sum rate and network throughput can be expected if multiuser detectors, such as SIC, are adopted. It is therefore important to further extend the analysis to incorporate more advanced receiver structures. Moreover, this paper considers a saturated network, where the network throughput is pushed to the limit, yet the mean queueing delay is infinite and the network could be unstable. It is of great practical significance to further study the maximum sum rate of Aloha under certain system constraints, such as stability or delay requirements. Finally, a key assumption throughout the paper is that the nodes are unaware of the instantaneous realizations of the small-scale fading, and they encode their packets independently at the same rate. It has been shown in the literature that if CSI is available at the transmitter side, the network performance of Aloha can be significantly improved by adaptively adjusting the transmission probability according to CSI [19], [39]. How to characterize the maximum sum rate with CSI at the transmitter side is another interesting and challenging issue, which deserves much attention in the future study.

**APPENDIX A**

**PROOF OF THEOREM 1**

*Proof*: The right-hand side of (15) can be written as

$$h(p) = \exp \left( -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \cdot \frac{1}{g(p)} \right),$$

where $g(p) = \sum_{i=0}^{K-1} \frac{p(1-p)^i}{q_i} + \frac{(1-p)^K}{q_K}$. Define $\tilde{q}_i = 1/q_i$, for $0 \leq i \leq K - 1$, and $\tilde{q}_i = 1/q_K$ for $i \geq K$. $g(p)$ can be then written as

$$g(p) = \sum_{i=0}^{\infty} p(1-p)^i \tilde{q}_i = E_X[\tilde{q}_X],$$

(59)
where $X$ is a geometric random variable with parameter $p$.

Suppose that $0 < p_1 < p_2 \leq 1$. Let $X_1$ and $X_2$ denote geometric random variables with parameters $p_1$ and $p_2$, respectively. Then we have $X_1 \geq_{st} X_2$ \footnote{$X_1 \geq_{st} X_2$ denotes that a random variable $X_1$ is larger than a random variable $X_2$ in the usual stochastic order, i.e., $\Pr(X_1 > x) \geq \Pr(X_2 > x)$ for all $x \in (-\infty, \infty)$.}. As $\{q_i\}$ is a monotonic non-increasing sequence, we have $\tilde{q}_{X_1} \geq_{st} \tilde{q}_{X_2}$. We can then conclude from (59) that $g(p_1) \geq g(p_2)$. Therefore, $g(p)$ is a monotonic non-increasing function with respect to $p$, which indicates that $h(p)$ is a monotonic non-increasing function according to (58).

Moreover, as $\lim_{p \to 0} h(p) = \exp(\mu \rho - \mu + 1) > 0$ and $\lim_{p \to 1} h(p) = \exp(\mu \rho - \mu + 1 \cdot q_0) < 1$, we can then conclude that (15) has a single non-zero root if $\{q_i\}_{i=0,\ldots,K}$ is a monotonic non-increasing sequence.

**APPENDIX B**

**PROOF OF THEOREM 2**

**Proof:** It is shown in (17) that the network throughput can be obtained as an explicit function of $p_A$. The following lemma first presents $\hat{\lambda}_{\max}^p = \max_{p_A \in (0,1]} \hat{\lambda}_{\text{out}}$ and the corresponding optimal steady-state point $p_A^*$.

**Lemma 1.** For given SINR threshold $\mu \in (0, \infty)$ and mean received SNR $\rho \in (0, \infty)$, $\hat{\lambda}_{\max}^p$ is given by

$$\hat{\lambda}_{\max}^p = \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right),$$

(60)

which is achieved at

$$p_A^* = \exp \left( -1 - \frac{\mu}{\rho} \right).$$

(61)

**Proof:** According to (17), the second-order derivative of $\hat{\lambda}_{\text{out}}$ with respect to $p_A$ is given by $-\frac{\mu + 1}{\mu p_A} < 0$, for $p_A \in (0, \infty)$. Therefore, we can conclude that $\hat{\lambda}_{\text{out}}$ is a strictly concave function of $p_A \in (0, \infty)$ with one global maximum at $p_A^*$, where $p_A^*$ is the root of $\frac{d\hat{\lambda}_{\text{out}}}{dp_A} = 0$, i.e.,

$$\frac{\mu}{\mu + 1} \cdot \left( -\frac{\ln p_A - 1}{\mu} - \frac{1}{\rho} \right) = 0,$$

(62)

which is given by (61). (60) can be obtained by substituting (61) into (17).

We can see from Lemma 1 and (15) that to achieve $\lambda_{\max}^p$, the backoff parameters $\{q_i\}_{i=0,\ldots,K}$ should be carefully selected such that $p_A = p_A^*$. For given backoff function $Q_i$, the optimal initial transmission
probability $q_m$ for achieving $\hat{\lambda}_{\text{max}}^p$ can be easily obtained by combining (15) and (61) as (20).

Note that $q_m$ should not exceed 1. Lemma 2 shows that $\hat{\lambda}_{\text{max}}^p$ is achievable for $q_i \in (0, 1], i = 0, \ldots, K$, if and only if the SINR threshold $\mu \geq \frac{1}{n-1}$.

**Lemma 2.** $\hat{\lambda}_{\text{max}}^p$ is achievable if and only if $\mu \geq \frac{1}{n-1}$.

**Proof:** Define $\tilde{Q}_i = 1/Q_i$, for $0 \leq i \leq K-1$, and $\tilde{Q}_i = 1/Q_K$ for $i \geq K$. Let $Y$ denote a geometric random variable with parameter $\exp \left(-1 - \frac{\mu}{\rho}\right)$. (20) can be then written as

$$q_m = \frac{\mu + 1}{n\mu} \cdot E_Y[\tilde{Q}_Y].$$

As $Q_i \leq 1$ for $i = 0, 1, \ldots, K$, we have $E_Y[\tilde{Q}_Y] \geq 1$.

1) if: if $\mu \geq \frac{1}{n-1}$, with $Q_i = 1$ for $i = 0, 1, \ldots, K$, we have $E_Y[\tilde{Q}_Y] = 1$ and $q_m = \frac{\mu + 1}{n\mu} \leq 1$. In this case, $\hat{\lambda}_{\text{max}}^p$ can be achieved by setting $q_0 = q_m$.

2) only if: if $\mu < \frac{1}{n-1}$, we have $q_m > 1$ according to (63), which indicates that $\hat{\lambda}_{\text{max}}^p$ is not achievable.

For $\mu < \frac{1}{n-1}$, $\hat{\lambda}_{\text{max}}^p$ is not achievable for $q_i \in (0, 1], i = 0, \ldots, K$. The following lemma shows that in this case, the maximum network throughput $\hat{\lambda}_{\text{max}}$ is always smaller than $\hat{\lambda}_{\text{max}}^p$, which is achieved by setting $q_i = 1, i = 0, \ldots, K$.

**Lemma 3.** For given SINR threshold $\mu < \frac{1}{n-1}$, the maximum network throughput $\hat{\lambda}_{\text{max}}$ is given by

$$\hat{\lambda}_{\text{max}}^{\mu < \frac{1}{n-1}} = n \exp \left(-\frac{n\mu}{\mu + 1} - \frac{\mu}{\rho}\right),$$

which is achieved at $q_i^* = 1, i = 0, \ldots, K$.

**Proof:** According to (15), the initial transmission probability $q_0$ can be written as

$$q_0 = \frac{\mu + 1}{n\mu} \left(-\ln p_A - \frac{\mu}{\rho}\right) \cdot z(p_A).$$

where $z(p_A) = \sum_{i=0}^{K-1} \frac{p_A(1-p_A)^i}{Q_i} + \frac{(1-p_A)^K}{Q_K^K}$. Similar to $g(p)$ in Appendix A, it can be proved that $z(p_A)$ is a monotonic non-increasing function of $p_A \in (0, 1]$. Note that $-\ln p_A$ is also a monotonic non-increasing function of $p_A \in (0, 1]$. Therefore, we can conclude from (65) that $p_A$ is a monotonic
non-increasing function of \( q_0 \). With 0 < \( q_0 \) ≤ 1, we can obtain from (15) that

\[
p_A \geq \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \cdot \frac{1}{\sum_{i=0}^{K-1} p_A (1 - p_A)^i Q_i + (1 - p_A)^K Q_K} \right\}, \tag{66}
\]

where “=” holds when \( q_0 = 1 \). Note that the backoff function \( Q_i \leq 1, i = 0, \ldots, K \). We can further obtain from (66) that

\[
p_A \geq \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \right\}, \tag{67}
\]

where “=” holds when \( q_0 = 1 \) and \( Q_i = 1, i = 0, \ldots, K \). When \( \mu < \frac{1}{n-1} \), we can see from (67) that \( p_A > p_A^* \). According to the proof of Lemma 1, the network throughput \( \hat{\lambda}_{\text{out}} \) is a monotonic decreasing function of \( p_A \) when \( p_A > p_A^* \). Therefore, in this case, \( \hat{\lambda}_{\text{out}} \) is maximized when \( p_A \) is minimized, i.e., \( p_A = \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \right\} \) according to (67), which is achieved at \( q_i^* = 1 \). (64) can be then obtained by substituting \( p_A = \exp \left\{ -\frac{\mu}{\rho} - \frac{n\mu}{\mu + 1} \right\} \) into (17).

Finally, (18) and (19) can be obtained by combining Lemma 1, Lemma 2 and Lemma 3.

\section*{Appendix C}
\textbf{Proof of Corollary 1}

\textit{Proof:} 1) When \( \mu \geq \frac{1}{n-1} \), we have \( \hat{\lambda}_{\text{max}} = \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) \) according to (18).

i) if: if \( \frac{1}{n-1} \leq \mu < \frac{1}{e-1} \), we have \( \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) > 1 \) if \( \rho > \frac{\mu}{\ln \left( \frac{\mu + 1}{\mu} - 1 \right)} > 0 \).

ii) only if: if \( \mu \geq \frac{1}{e-1} \), we have \( \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) \leq \exp \left( -\frac{\mu}{\rho} \right) < 1 \) for \( \rho > 0 \). On the other hand, if \( \rho \leq \frac{\mu}{\ln \left( \frac{\mu + 1}{\mu} - 1 \right)} \), we have \( \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) \leq 1 \).

2) When \( \mu < \frac{1}{n-1} \), we have \( \hat{\lambda}_{\text{max}} = n \exp \left( -\frac{n\mu}{\mu + 1} - \frac{\mu}{\rho} \right) \) according to (18).

i) if: if \( \rho > \frac{\mu}{\ln n - \frac{\mu}{\mu + 1}} \), we have \( n \exp \left\{ -\frac{n\mu}{\mu + 1} - \frac{\mu}{\rho} \right\} > 1 \).

ii) only if: if \( \rho \leq \frac{\mu}{\ln n - \frac{\mu}{\mu + 1}} \), we have \( n \exp \left\{ -\frac{n\mu}{\mu + 1} - \frac{\mu}{\rho} \right\} \leq 1 \).

\section*{Appendix D}
\textbf{Proof of Theorem 3}

\textit{Proof:} According to (23-24), we can rewrite the maximum sum rate as \( C = \max (C_1, C_2) \), where

\[
C_1 = \max_{\mu \geq \frac{1}{n-1}} \frac{\mu + 1}{\mu} \exp \left( -1 - \frac{\mu}{\rho} \right) \cdot \log_2(1 + \mu), \tag{68}
\]
and
\[
C_2 = \max_{0 < \mu \leq \frac{1}{n-1}} n \exp \left( -\frac{n \mu}{\mu + 1} - \frac{\mu}{\rho} \right) \cdot \log_2(1 + \mu).
\]

Let us first focus on \( C_1 \).

1) Denote the objective function of (68) as \( f_1(\mu) \) and let us first prove the following lemma.

**Lemma 4.** \( f_1(\mu) \) is a monotonic decreasing function of \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \) if \( \rho < \rho_0 \). Otherwise, it has one global maximum at \( \mu_h^* \), where \( \mu_h^* \) is the root of (27).

**Proof:** \( f_1(\mu) \) is a continuously differentiable function of \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \). The first-order derivative of \( f(\mu) \) can be written as
\[
f_1'(\mu) = \exp \left( -1 - \frac{\mu}{\rho} \right) \log_2 e \cdot G_1(\mu),
\]
where
\[
G_1(\mu) = \frac{1}{\mu} - \frac{1}{\mu^2} \ln(1 + \mu) - \frac{1}{\rho} \cdot \frac{1 + \mu}{\mu} \ln(1 + \mu).
\]

It can be easily obtained from (71) that
\[
\lim_{\mu \to \frac{1}{n-1}} G_1(\mu) = (n - 1) - (n - 1)^2 \ln \frac{n}{n - 1} - \frac{n}{\rho} \ln \frac{n}{n - 1},
\]
and
\[
\lim_{\mu \to \infty} G_1(\mu) = -\infty.
\]

Moreover, the first-order derivative of \( G_1(\mu) \) can be obtained from (71) as
\[
G_1'(\mu) = -\frac{1}{\mu^2} \left( \frac{2 + \mu}{1 + \mu} - \frac{2}{\mu} \ln(1 + \mu) \right) - \frac{1}{\rho} \left( \frac{1}{\mu} - \frac{\ln(1 + \mu)}{\mu^2} \right) < 0,
\]
for \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \), which indicates that \( G_1(\mu) \) is a monotonic decreasing function of \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \). 

i) If \( \rho \geq \rho_0 \), we can obtain from (72)(73) that \( \lim_{\mu \to \frac{1}{n-1}} G_1(\mu) \geq 0 \) and \( \lim_{\mu \to \infty} G_1(\mu) < 0 \). As \( G_1(\mu) \) is a monotonic decreasing function of \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \), there must exist \( \mu_h^* \in \left[ \frac{1}{n-1}, \infty \right) \), such that \( G_1(\mu) > 0 \) for \( \mu \in \left[ \frac{1}{n-1}, \mu_h^* \right) \) and \( G_1(\mu) < 0 \) for \( \mu \in (\mu_h^*, \infty) \), where \( \mu_h^* \) is the root of \( G_1(\mu) = 0 \), which is given in (27). We can then obtain from (70) that \( f_1'(\mu) > 0 \) for \( \mu \in \left[ \frac{1}{n-1}, \mu_h^* \right) \) and \( f_1'(\mu) < 0 \) for \( \mu \in (\mu_h^*, \infty) \), which indicates that \( f_1(\mu) \) has one global maximum at \( \mu_h^* \).

ii) If \( \rho < \rho_0 \), we can obtain from (72) that \( \lim_{\mu \to \frac{1}{n-1}} G_1(\mu) < 0 \). As \( G_1(\mu) \) is a monotonic decreasing function of \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \), we have \( G_1(\mu) < 0 \) for \( \mu \in \left[ \frac{1}{n-1}, \infty \right) \). According to (70), we can conclude
that in this case $f_1(\mu)$ is a monotonic decreasing function as $f_1'(\mu) < 0$ for $\mu \in \left[ \frac{1}{n-1}, \infty \right)$. □

According to Lemma 4, we can conclude that the optimal SINR threshold for $C_1$ is

$$
\mu_1^* = \begin{cases} 
\mu_h^* & \text{if } \rho \geq \rho_0 \\
\frac{1}{n-1} & \text{otherwise.}
\end{cases}
$$

(75)

2) For $C_2$, denote the objective function of (69) as $f_2(\mu)$ and let us first prove the following lemma.

**Lemma 5.** $f_2(\mu)$ is a monotonic non-decreasing function of $\mu \in \left( 0, \frac{1}{n-1} \right]$ if $\rho \geq \rho_0$. Otherwise, it has one global maximum at $\mu_1^*$, where $\mu_1^*$ is the root of (28).

**Proof:** $f_2(\mu)$ is a continuously differentiable function of $\mu \in \left( 0, \frac{1}{n-1} \right]$. The first-order derivative of $f_2(\mu)$ can be written as

$$
f_2'(\mu) = \frac{n}{(1 + \mu)^2} \exp \left( -\frac{n\mu}{\mu + 1} - \frac{\mu}{\rho} \right) \log_2 e \cdot G_2(\mu),
$$

(76)

where

$$
G_2(\mu) = (1 + \mu) - \left( \frac{(1 + \mu)^2}{\rho} + n \right) \ln(1 + \mu).
$$

(77)

It can be easily obtained from (77) that

$$
\lim_{\mu \to 0} G_2(\mu) = 1,
$$

(78)

and

$$
\lim_{\mu \to \frac{1}{n-1}} G_2(\mu) = \frac{n}{n-1} - n \ln \frac{n}{n-1} - \frac{1}{\rho} \cdot \left( \frac{n}{n-1} \right)^2 \ln \frac{n}{n-1}.
$$

(79)

Moreover, the first-order derivative of $G_2(\mu)$ can be obtained from (77) as

$$
G_2'(\mu) = 1 - \frac{n}{1 + \mu} - \frac{1 + \mu}{\rho} (1 + 2 \ln(1 + \mu)) < 0
$$

(80)

for $\mu \in \left( 0, \frac{1}{n-1} \right]$, which indicates that $G_2(\mu)$ is a monotonic decreasing function of $\mu \in \left( 0, \frac{1}{n-1} \right]$. 

i) If $\rho \geq \rho_0$, we can obtain from (79) that $\lim_{\mu \to \frac{1}{n-1}} G_2(\mu) \geq 0$. As $G_2(\mu)$ is a monotonic decreasing function of $\mu \in \left( 0, \frac{1}{n-1} \right]$, we have $G_2(\mu) \geq 0$ for $\mu \in \left( 0, \frac{1}{n-1} \right]$. According to (76), we can conclude that in this case $f_2(\mu)$ is a monotonic non-decreasing function as $f_2'(\mu) \geq 0$ for $\mu \in \left( 0, \frac{1}{n-1} \right]$. 

ii) If $\rho < \rho_0$, we can obtain from (78-79) that $\lim_{\mu \to 0} G_2(\mu) > 0$ and $\lim_{\mu \to \frac{1}{n-1}} G_2(\mu) < 0$. As $G_2(\mu)$ is a monotonic decreasing function of $\mu \in \left( 0, \frac{1}{n-1} \right]$, there must exist $\mu_1^* \in \left( 0, \frac{1}{n-1} \right]$, such that $G_2(\mu) > 0$ for
\( \mu \in (0, \mu^*_l) \) and \( G_2(\mu) < 0 \) for \( \mu \in (\mu^*_l, \frac{1}{n-1}] \), where \( \mu^*_l \) is the root of \( G_2(\mu) = 0 \), which is given in (28).

We can then obtain from (76) that \( f'_2(\mu) > 0 \) for \( \mu \in (0, \mu^*_l) \) and \( f'_2(\mu) < 0 \) for \( \mu \in (\mu^*_l, \frac{1}{n-1}] \), which indicates that \( f_2(\mu) \) has one global maximum at \( \mu^*_l \).

According to Lemma 5, we can conclude that the optimal SINR threshold for \( C_2 \) is

\[
\mu^*_2 = \begin{cases} 
\frac{1}{n-1} & \text{if } \rho \geq \rho_0 \\
\mu^*_l & \text{otherwise.} 
\end{cases} 
\tag{81}
\]

3) By combining (75) and (81), we can see that if \( \rho \geq \rho_0 \), \( C_1 = f_1(\mu^*_h) \) and \( C_2 = f_2 \left( \frac{1}{n-1} \right) \). As \( f_2 \left( \frac{1}{n-1} \right) = f_1 \left( \frac{1}{n-1} \right) \) and \( f_1 \left( \frac{1}{n-1} \right) \leq C_1 \), we have \( C_1 \geq C_2 \). Therefore, we can conclude that in this case the maximum sum rate \( C = C_1 \) and the optimal SINR threshold \( \mu^* = \mu^*_h \).

On the other hand, if \( \rho < \rho_0 \), \( C_1 = f_1 \left( \frac{1}{n-1} \right) \) and \( C_2 = f_2(\mu^*_l) \). As \( f_1 \left( \frac{1}{n-1} \right) = f_2 \left( \frac{1}{n-1} \right) \) and \( f_2 \left( \frac{1}{n-1} \right) \leq f_2(\mu^*_l) \), we have \( C_2 \geq C_1 \). Therefore, we can conclude that in this case the maximum sum rate \( C = C_2 \) and the optimal SINR threshold \( \mu^* = \mu^*_l \).

**APPENDIX E**

**PROOF OF COROLLARY 2**

**Proof:** We can easily obtain from (27) that \( \lim_{\rho \to \infty} \mu^*_h = \infty \) and

\[
\lim_{\rho \to \infty} \frac{\mu^*_h}{\rho} \ln \mu^*_h = \lim_{\rho \to \infty} \left( \frac{1}{\mu^*_h} + \frac{1 + \mu^*_h}{\rho} \right) \ln(1 + \mu^*_h) = 1. \tag{82}
\]

By combining (26), we have

\[
\lim_{\rho \to \infty} \mu^* = \lim_{\rho \to \infty} \mu^*_h = \infty, \tag{83}
\]

\[
\lim_{\rho \to \infty} \frac{\mu^*}{\rho} = \lim_{\rho \to \infty} \frac{\mu^*_h}{\rho} = \lim_{\rho \to \infty} \frac{1}{\ln \mu^*_h} = 0. \tag{84}
\]

Moreover, by applying L’Hôpital’s rule on the left-hand side of (82), we have \( \lim_{\rho \to \infty} \frac{d\mu^*_h}{d\rho} (1 + \ln \mu^*_h) = 1 \), which indicates that

\[
\lim_{\rho \to \infty} \frac{d\mu^*}{d\rho} (1 + \ln \mu^*) = 1. \tag{85}
\]

Therefore, by combining (25) with (83, 85), we can obtain that

\[
\lim_{\rho \to \infty} \frac{C}{\log_2 \rho} = e^{-1} \cdot \lim_{\rho \to \infty} \frac{\log_2 \mu^*}{\log_2 \rho} = e^{-1} \cdot \lim_{\rho \to \infty} \frac{\rho}{\mu^*(1 + \ln \mu^*)} = e^{-1}. \tag{86}
\]
APPENDIX F

PROOF OF COROLLARY 3

Proof: When $\rho < \rho_0$, the optimal SINR threshold $\mu^* = \mu^*_l$ according to (26). We can easily obtain from (28) that

$$\lim_{n \to \infty} \mu^*_l = 0.$$  \hfill (87)

By combining (28) and (87), we can further obtain that

$$\lim_{n \to \infty} n \log_2 (1 + \mu^*_l) = \lim_{n \to \infty} \frac{\log_2 e}{\mu^*_l + 1 + \frac{\mu^*_l + 1}{n\rho}} = \log_2 e,$$  \hfill (88)

and

$$\lim_{n \to \infty} \frac{n\mu^*_l}{\mu^*_l + 1} = \lim_{n \to \infty} \frac{\mu^*_l}{\ln(1 + \mu^*_l) - \frac{\mu^*_l(\mu^*_l + 1)}{\rho}} = 1.$$  \hfill (89)

Therefore, by combining (25) with (87,89), we can obtain that

$$\lim_{n \to \infty} C_{\rho < \rho_0} = \lim_{n \to \infty} n \exp \left\{ - \frac{n\mu^*_l}{\mu^*_l + 1} - \frac{\mu^*_l}{\rho} \right\} \log_2 (1 + \mu^*_l) = e^{-1} \log_2 e.$$  \hfill (90)

APPENDIX G

DERIVATION OF (39-40)

Based on the collision model, a packet transmission is successful if and only if there are no concurrent transmissions and its received SNR is above the threshold $\mu$. The steady-state probability of successful transmission of HOL packets, $p$, can be then written as

$$p = \Pr\{\text{no concurrent transmissions}\} \cdot \Pr\{\text{received SNR is above the threshold } \mu\}.\quad (91)$$

It has been shown in Section III-B that in saturated conditions, the probability of a node being busy with the HOL packet requesting transmission is equal to $\pi_0/p$. The probability that there are no concurrent transmissions can be then obtained as

$$\Pr\{\text{no concurrent transmissions}\} = (1 - \pi_0/p)^{n-1}.\quad (92)$$
Since the received SNR is exponentially distributed with mean $\rho$, the probability that the received SNR is above the threshold $\mu$ is given by

$$\Pr\{\text{received SNR is above the threshold } \mu\} = \exp\left(-\frac{\mu}{\rho}\right). \tag{93}$$

By substituting (92) and (93) into (91), we have

$$p = (1 - \pi_0/p)^{n-1} \cdot \exp\left(-\frac{\mu}{\rho}\right) \approx \exp\left(-\frac{\mu}{\rho} - \frac{n\pi_0}{p}\right), \tag{94}$$

which can be further written as

$$p = \exp\left(-\frac{\mu}{\rho} - \frac{n}{\sum_{i=0}^{K-1} \frac{p (1-p)^i}{q_i} + \frac{(1-p)^K}{q_K}}\right), \tag{95}$$

according to (7).

By following a similar derivation in Appendix A, it can be proved that (95) has a single non-zero root if $\{q_i\}_{i=0,...,K}$ is a monotonic non-increasing sequence. Denote the non-zero root of (95) as $p^*_A$. The network throughput can then be obtained as

$$\hat{\lambda}_{\text{collision}} = -p^*_A \ln p^*_A - \frac{p^*_A \mu}{\rho} \text{ by combining (9) and (94).} \tag{96}$$

As the second-order derivative of $\hat{\lambda}_{\text{collision}}$ with respect to $p^*_A$ is $-\frac{1}{p^*_A} < 0$ for $p^*_A \in (0, \infty)$, $\hat{\lambda}_{\text{collision}}$ is a strictly concave function of $p^*_A \in (0, \infty)$ with one global maximum at $p^*_A$. It can be easily obtained that

$$\max_{p^*_A} \hat{\lambda}_{\text{collision}} = \exp\left(-1 - \frac{\mu}{\rho}\right),$$

which is achieved at $p^*_A = \exp\left(-1 - \frac{\mu}{\rho}\right)$, and is achievable for $\mu \in (0, \infty)$. Therefore, we have

$$C_{\text{collision}} = \max_{\mu > 0} \exp\left(-1 - \frac{\mu}{\rho}\right) \cdot \log_2(1 + \mu). \tag{97}$$

Let $f(\mu)$ denote the objective function of (97). It can be easily shown that $f'(\mu) \geq 0$ for $\mu \in (0, \mu^{*,\text{collision}}]$ and $f'(\mu) < 0$ for $\mu \in (\mu^{*,\text{collision}}, \infty)$, indicating that $f(\mu)$ has one global maximum at $\mu^{*,\text{collision}}$, where $\mu^{*,\text{collision}}$ is the root of $(\mu + 1)(\mu + 1)/\rho = e$, which is given by

$$\mu^{*,\text{collision}} = e^W_0(\rho) - 1. \tag{98}$$

(39) and (40) can be then obtained by substituting (98) into (97) and (96), respectively.
According to (42), the network steady-state point in saturated conditions $p_A$ crucially depends on $\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor$. Let us specifically consider the following cases:

1) $\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor < 0$: It can be seen from (42) that $p_A = 0$. In this case, no packet can pass through due to an excessively high SINR threshold. Both the network throughput and the sum rate are 0.

2) $\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor \geq n - 1$: It can be seen from (42) that $p_A = 1$. In this case, all the packets can be successfully decoded, and the network throughput is $\hat{\lambda}_{out} = nq_0$ by combining (7) and (9). To maximize the network throughput, all the nodes should transmit with probability $q_0 = 1$, and the maximum network throughput is $\lambda_{\max} = n$. The corresponding sum rate is then $n \log_2(1+\mu)$, which is a monotonic increasing function of the SINR threshold $\mu$, and is maximized when $\mu = \frac{1}{n-1+\frac{1}{\rho}}$.

3) $0 \leq \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor < n - 1$: It can be obtained from (42) and (7) that in this case, $p_A$ with $K = 0$ can be written as

$$p_A = I_{1-q_0} \left( n - 1 - \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor , \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor + 1 \right). \quad (99)$$

The network throughput can be then obtained by combining (7), (9) and (99) as

$$\hat{\lambda}_{out} = nq_0 I_{1-q_0} \left( n - 1 - \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor , \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor + 1 \right), \quad (100)$$

which has one global maximum at $q_0^*$, where $q_0^*$ is the root of the following equation:

$$\int_0^{1-q_0} t^{n-2-\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor } \frac{1}{1-t} \left( 1-t \right)^{\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor } dt = (1-q_0)^{n-2-\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor } q_0^{\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor + 1}. \quad (101)$$

The corresponding sum rate is then given by

$$R_s = nq_0^* I_{1-q_0^*} \left( n - 1 - \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor , \left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor + 1 \right) \log_2(1+\mu). \quad (102)$$

Note that when $\frac{1}{1+\frac{1}{\rho}} < \mu \leq \rho$, $\left\lfloor \frac{1}{\mu} - \frac{1}{\rho} \right\rfloor = 0$. In this case, the optimal transmission probability can be obtained from (101) as $q_0^* = \frac{1}{n}$, and the maximum network throughput can be obtained from (100) as $\lambda_{\max} = (1-\frac{1}{n})^{n-1} \approx e^{-1}$ for large number of nodes $n$. The corresponding sum rate is then $e^{-1} \log_2(1+\mu)$, which is a monotonic increasing function of the SINR threshold $\mu$, and is maximized when $\mu = \rho$. 
Fig. 12 illustrates how the sum rate varies with the SINR threshold $\mu$. It can be clearly observed from Fig. 12 that there are two local maximum points at $\mu = \frac{1}{n-1+\rho/n}$ and $\mu = \rho$, respectively. Which one is the global maximum point is determined by the mean received SNR $\rho$. Let $\rho_1$ denote the root of (46). If $\rho < \rho_1$, we have $n \log_2 \left( 1 + \frac{1}{n-1+\rho} \right) > e^{-1} \log_2 (1 + \rho)$. In this case, the maximum sum rate is $C = n \log_2 \left( 1 + \frac{1}{n-1+\rho} \right)$, which is achieved when the SINR threshold $\mu = \frac{1}{n-1+\rho}$. Otherwise, $\mu = \rho$ is the global maximum point, and the maximum sum rate is given by $C = e^{-1} \log_2 (1 + \rho)$. It can be clearly seen from Fig. 12 that for $\rho = 5\text{dB} < \rho_1 \approx 11.5\text{dB}$, $\mu = \frac{1}{n-1+\rho}$ is the optimal SINR threshold. When $\rho$ increases to 15dB, the maximum sum rate is achieved at $\mu = \rho$.

![Graph](image)

Fig. 12. Sum rate of Aloha over AWGN channels versus SINR threshold. $n = 50$. $K = 0$ and $q_0 = q_0^\ast$. (a) $\rho = 5\text{dB}$. (b) $\rho = 15\text{dB}$.

**APPENDIX I**

**DERIVATION OF (48)**

Let $Z = \sum_{m=1}^{M} \sum_{k \in S^m_j} |h_k|^2 \cdot \frac{\rho_m}{\rho_j} - |h_j|^2 / \mu$. According to (47), we have

$$r_{\{i_m\}}^j = \Pr \{ Z \leq -1/\rho_j \}. \quad (103)$$

With $h_k \sim \mathcal{CN}(0, 1)$, the Laplace transform of $Z$ can be written as

$$\mathcal{L}_Z(s) = \frac{\mu}{\mu - s} \cdot \prod_{m=1}^{M} \left( \frac{1}{1 + \frac{\rho_m}{\rho_j} s} \right)^{i_m}. \quad (104)$$
By applying the partial fraction expansion and the inverse Laplace transform for \( s \leq 0 \), we have

\[
f_Z(z) = \mu \exp(\mu z) \cdot \prod_{m=1}^{M} \left( \frac{1}{1 + \frac{\rho_m}{\rho_j} \mu} \right)^{i_m}.
\]

(105)

Finally, by combining (105) with (103), we have

\[
\{i_m \} \sim \frac{\exp\left(-\frac{\mu}{\rho_j}\right)}{\prod_{m=1}^{M} \left(1 + \frac{\rho_m}{\rho_j} \mu\right)^{i_m}}.
\]

(106)

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