Non-Renormalization Theorems for Operators with Arbitrary Numbers of Derivatives in $\mathcal{N} = 4$ Yang Mills Theory

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Abstract

We generalize the proof of the non-renormalization of the four derivative operators in $\mathcal{N} = 4$ Yang Mills theory with gauge group $SU(2)$ to show that certain terms with $2N$ derivatives are not renormalized in the theory with gauge group $SU(N)$. These terms may be determined exactly by a simple perturbative computation. Similar results hold for finite $\mathcal{N} = 2$ theories. We comment on the implications of these results.
1 Introduction

$\mathcal{N} = 4$ Yang-Mills theory is a remarkable theory in a number of ways. It is a finite and scale invariant. It is believed to exhibit an exact electric-magnetic duality. It also plays a crucial role in the Matrix model and in the AdS/CFT duality.

There is evidence that the quantum properties of this theory are even more remarkable than just finiteness. It has been shown, for example, that not only are the terms with two derivatives not renormalized, but the four derivative terms are not renormalized as well\footnote{\cite{1}}. The first suggestion of such a possibility was provided by the agreement of graviton-graviton scattering with the Matrix model\footnote{\cite{2}}. Subsequently, it has been observed that three graviton\footnote{\cite{3}} and even $N$-graviton scattering\footnote{\cite{4}} at tree level in supergravity agree with the predictions of the matrix model. While some of these analyses are specific to the case of the matrix model in $0+1$ dimensions (corresponding to non-compact eleven-dimensional space), some of these hold in higher dimensions, including $3+1$\footnote{\cite{4}}. This suggests that there should be non-renormalization theorems for $2N$ derivative terms in Yang-Mills theory with gauge group $SU(N)$ in various dimensions up to four. In $0+1$ dimensions, such theorems have been proven for terms with four\footnote{\cite{5}} and six derivatives\footnote{\cite{6}}. In four dimensions, only the four derivative terms have been shown to be unrenormalized.

In the present note, we generalize the arguments of \cite{1} to show that certain $2N$ derivative terms are not renormalized in 4-dimensional $\mathcal{N} = 4$ $SU(N)$ Yang Mills theory. The strategy is similar to that used to calculate multigraviton scattering amplitudes in the matrix model in \cite{4}. The theory has a large moduli space; at generic points, the gauge symmetry is $U(1)^{N-1}$. One can, however, consider regions of the moduli space in which there is a hierarchy of breakings; $SU(N)$ is broken to $SU(N-1) \times U(1)$, then to $SU(N-2) \times U(1) \times U(1)$, and so on. At each stage of this breaking, one can integrate out the most massive fields and obtain a suitable effective lagrangian. By focusing judiciously on certain terms in these effective lagrangians, one can make arguments similar in spirit to those of \cite{1}.

In the rest of the paper, we present the proof. In the next section, we review the analysis of \cite{4} with particular emphasis on the case of $3+1$ dimensions (corresponding to compactifying three of the $M$-theory dimensions). In section three, we generalize the argument of \cite{1} for non-renormalization of $F_{\mu
u}^4$ and other four derivative terms in $SU(2)$ to a statement about certain such operators in $SU(N)$ (while we suspect, as argued in \cite{7}, the statement holds in general, we will not attempt to prove it). In section 4, we show that certain six derivative terms in $SU(3)$...
are not renormalized. The strategy is to first look at an effective $SU(2)$ symmetric theory, and represent the effect of integrating out the heavy fields through a suitable spurion. The symmetries – scale invariance and $U(1)_R$ invariance, and an approximate shift symmetry for the background dilaton multiplet – are sufficient to completely determine certain six derivative terms in the theory. This argument can be generalized to $SU(N)$; this is presented in section 4. In section 5, we note that identical arguments and results hold for the finite $\mathcal{N} = 2$ theories. Section 6 contains some speculations.

2  Graviton scattering in 8 Dimensions and It’s Implications for $\mathcal{N} = 4$ Yang Mills Theory

The principle reason to suspect that there exists a large hierarchy of non-renormalization theorems comes from studies of multigraviton scattering in the matrix model. The agreement found in three graviton scattering in 11-dimensional Minkowski space [3] is impressive, and suggests that there should be non-renormalization theorems for some set of six derivative terms in the Matrix quantum mechanics. In [4], it was shown that there is actually agreement for certain terms in $N$-graviton scattering, for arbitrary $N$. Moreover, this agreement persists when the theory is compactified on tori of 1, 2 or 3 dimensions. As a result, one expects an infinite set of non-renormalization theorems in these theories. The strategy of the proof will be closely related to the approach of these earlier computations, so it is perhaps useful to review them here. We will consider specifically the case of compactification of 3 dimensions on a small torus, corresponding to $\mathcal{N} = 4$ Yang-Mills theory on a large torus[8, 9].

To study $N$-graviton scattering (in the Discrete Light Cone (DLCQ) formulation of the theory) in the theory compactified to 8 dimensions, one considers the $\mathcal{N} = 4$ Yang-Mills theory with gauge group $SU(N)$. This theory has a moduli space; at generic points the symmetry $SU(N)$ is broken to $U(1)^{N-1}$, with the moduli of this breaking being identified with the coordinates of $N$ gravitons. If we write the $\mathcal{N} = 4$ theory in terms of six real (matrix-valued) scalar fields, $\phi^i$, then

$$\phi^i = \begin{pmatrix} v_1^i + \phi_1^i & 0 & 0 & \cdots \\ 0 & v_2^i + \phi_2^i & 0 & \cdots \\ 0 & 0 & v_3^i + \phi_3^i & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

(1)

It will be notationally convenient to extend the group to $U(N)$ so that the $v_i$’s are unconstrained.

The approach of [4] was to consider a hierarchy of expectation values, $v_N \gg v_{N-1} \gg \ldots$
...$v_2 \gg v_1$. One can then think of a sequence of breakings, first to $SU(N-1) \times U(1)$, then to $SU(N-2) \times U(1) \times U(1)$, and so on. To illustrate the procedure, consider first the case of $SU(3)$. In this case, the large expectation value of $\phi_1$ breaks the symmetry first to $SU(2) \times U(1)$ ($\times U(1)$, for $U(3)$). We can consider the effective lagrangian for the $SU(2)$ theory, obtained by integrating out the massive states associated with the first stage of breaking. At one loop, one can read off the result by a simple trick, generalizing the $SU(2)$ result (e.g. $[\Pi]$). This gives ($F_{12} = F_{11} - F_{22}$, etc., i.e. they are the differences of the diagonal matrix elements)

$$\mathcal{L} = \frac{1}{16\pi^2} \left( \frac{[(F^{\mu\nu}_{12})^4 + \ldots]}{|\phi_{12}|^4} + \frac{[(F^{\mu\nu}_{13})^4 + \ldots]}{|\phi_{13}|^4} + \frac{[(F^{\mu\nu}_{23})^4 + \ldots]}{|\phi_{23}|^4} \right).$$

(2)

Here the dots inside the braces denote

$$[(F^{\mu\nu})^4 + \ldots] = (F_{\mu\nu})^4 - \frac{1}{4}(F_{\mu\nu}^2)^2 + 2\partial^\mu \phi^i \partial_\mu \phi^j \partial^\nu \phi^i \partial_\nu \phi^j - \partial_\mu \phi^i \partial_\nu \phi^j \partial^\mu \phi^i \partial^\nu \phi^j.$$  

(3)

Now expand the denominators in small fluctuations about the expectation value, i.e. replace $\phi_1$ in the denominators by $v_3 \delta_{i3} + \phi_1$, and expand to second order in $\phi_1$. Identify $\phi_1 - \phi_2 = 2\phi^{[3]}$, where the label in braces denotes the Cartan generator. We can generalize this to an $SU(2)$-invariant expression by replacing $\phi^{[3]} \phi^{[3]}$ by $\phi^a \phi^a$. This yields:

$$\mathcal{L} = \ldots + \frac{9}{8\pi^2} [(F^{[8]}_{\mu\nu})^4 + \ldots] \phi^{[a]} \phi^{[b]} \left( \frac{\delta_{ij}}{|v_3|^6} - 6 \frac{v_i^3 v_j^3}{|v_3|^8} \right).$$

(4)

($F^{[8]}_{\mu\nu} \propto F^{\mu\nu}_{13} + F^{\mu\nu}_{23}$ corresponds to the generator conventionally called $T^8$ in $SU(3)$).

Now, one can contract $\phi^{[a]} \phi^{[b]}$. The propagator, in the presence of a background $\phi^{[3]}$, is given by

$$\langle \phi^{[i]} \phi^{[j]} \rangle = \frac{\delta_{ij}}{k^2 + M^2} + \frac{4\partial^\mu \phi^{[3]} \partial_\mu \phi^{[3]} j + \delta_{ij}(\ldots)}{(k^2 + M^2)^3}.$$  

(5)

where $M^2 = 2g^2 |v_{12}|^2$. Integrating over $k$ yields six-derivative terms in the low energy effective theory which agree precisely with tree level calculations in supergravity (similar statements hold in 11, 10 and 9 dimensions). As a result, we expect that, in $SU(N)$, non-renormalization theorems hold for certain terms with $2N$ derivatives. In the following, we will show that this is the case, in regions of the moduli space where the expectation values are hierarchically ordered, for operators of the form

$$\frac{[(F^{[N-1]}_{\mu\nu})^4 + \ldots]}{|v_N|^6} \left( \frac{\partial_\mu \phi^{[N-2]} \partial^\mu \phi^{[N-2]}}{|v_{N-1}|^4} \right) \left( \frac{\partial_\mu \phi^{[N-3]} \partial^\mu \phi^{[N-3]}}{|v_{N-2}|^4} \right) \ldots \left( \frac{\partial_\mu \phi^{[1]} \partial^\mu \phi^{[1]}}{|v_{12}|^2} \right).$$

(6)

Here, in $SU(N)$ ($N > 3$), we are labeling the elements of the Cartan subalgebra, $1, \ldots N - 1$. 
3 The Non-Renormalization Theorem for the four derivative terms: Extension to $SU(N)$

Let us review, first, the proof of the theorem for the group $SU(2)$. Presumably, it would be easy to prove the theorem if one had a convenient superspace formulation for theories with sixteen supersymmetries. Lacking this, it was noted in [1] that one can exploit an $\mathcal{N} = 2$ subgroup of the full supersymmetry, for which a full off-shell superspace formulation is available. It is useful to first describe the theory in an $\mathcal{N} = 1$ language. The theory consists of three chiral multiplets, $\phi_i$, and a gauge multiplet, $W_\alpha$, all in the adjoint representation of the group. An $SU(3) \times U(1)_R$ subgroup of the full $SU(4)$ R-symmetry is manifest in this description. The $\phi_i$’s transform as a triplet, each with charge $+2/3$ under the $U(1)_R$.

In an $\mathcal{N} = 2$ description, the theory consists of a vector multiplet and a hypermultiplet. The vector multiplet consists of the $\mathcal{N} = 1$ vector multiplet and one of the chiral fields, say $\phi_1$. One can write

$$\psi = \phi + \bar{\theta}W + \bar{\theta}^2G$$

(we have dropped the subscript 1 on $\phi$). The kinetic term for the $\psi$’s is (explicitly indicating the adjoint $SU(2)$ index)

$$\mathcal{L}_{vee} = \int d^2 \theta \int d^2 \bar{\theta}\psi^\alpha \psi^\alpha.$$  

Our focus in the following will be on the vector multiplets. We will study flat directions corresponding to expectation values of the scalar components of these multiplets (only). Note that when these fields get expectation values, the $U(1)_R$ symmetry which we described above is broken, but another $U(1)$, which we will call $X$, survives, which is a linear combination of the original $U(1)$ and an $SU(3)$ generator. The hypermultiplets have charge $+4/3$ under this symmetry.

The fact that the kinetic term is an integral over half of superspace allows one to prove many remarkable properties of the theory. Perhaps somewhat more remarkable is that in the $\mathcal{N} = 4$ case, one can prove statements that involve integrals over the full $\mathcal{N} = 2$ superspace. Consider the flat direction in which $SU(2)$ is broken to $U(1)$ by the scalar field in the vector multiplet. Call the light vector multiplet in this direction

$$\psi \sim \psi^\alpha \psi^\alpha$$  

(9)
We can ask what sorts of terms one can write involving an integral over the full superspace, which respect the symmetries of the theory. Such an integral has the form \[11\]:

\[
\mathcal{L}_\theta = \int d^8 \theta \mathcal{H}(\psi, \psi^\dagger).
\] (10)

The theory is conformally invariant and so \(\mathcal{H}\) must be dimensionless. It must respect the \(U(1)_R\) symmetry, under which \(\psi\) transforms by a phase. These conditions restrict \(\mathcal{H}\) to the form \[11, 1\]:

\[
\mathcal{H} = \frac{1}{16 \pi^2} \ln(\psi) \ln(\psi^\dagger).
\] (11)

No scale is necessary in the logarithm, since the dependence on the scale would vanish after integration over \(\theta\)'s. In other words, this expression is scale invariant. Related to this, the integral of \(\mathcal{H}\) vanishes under an \(R\) transformation, since the integral over a chiral superfield over the full superspace is zero. If one now includes a background dilaton field in a vector multiplet, it is easy to see that this cannot appear in \(\mathcal{H}\) without spoiling both the scale and \(R\) symmetries. As a result, the one loop expression for the four derivative terms in the effective lagrangian is exact.

This effective lagrangian includes terms with four powers of \(F_{\mu\nu}\), as well as terms with derivatives of scalars and fermions. If one compares with component field computations, one finds complete agreement up to terms which vanish if one uses the lowest order equations of motion \[12\]. It is also not hard to guess a generalization to \(SU(N)\). There are now \(N-1\) massless fields; one can write them as differences of diagonal entries of an \(N \times N\) matrix, \(\psi_{ij} = \psi_i - \psi_j\). Then a guess for a generalization of the \(SU(2)\) result, which is symmetric under permutations as well as scale invariance and \(R\) symmetries, is \[7, 13\]:

\[
\mathcal{H} = \frac{1}{16 \pi^2} \sum_{i<j} \ln(\psi_{ij}) \ln(\psi_{ij}^\dagger).
\] (12)

This expression respects all of the symmetries. It agrees with an explicit one loop computation. If this term were unique, one could again immediately prove a non-renormalization theorem. However, the symmetries we have used up to now do not suffice to uniquely determine \(\mathcal{H}\). Ratios of different \(\psi_{ij}\)'s are both scale invariant and \(U(1)_R\) invariant. In other words, functions such as

\[
f(\tau, \tau^\dagger) \frac{\psi_{ij} \psi_{mn}^\dagger}{\psi_{kl} \psi_{op}^\dagger}
\] (13)

for various choices of \(i, j\ldots\) are invariant under the \(U(1)_R\) invariance and scale invariance for any choice of \(f\). It is possible that one can still constrain the function completely using the full
$SU(4)$ R-symmetry, which is not manifest in the $\mathcal{N} = 2$ setup. We will not attempt this here. Instead, we will content ourselves with a more limited statement about four derivative terms in these theories.

Consider, first, the case of $SU(3)$. Suppose that one eigenvalue of $\phi$, say $\phi_3$ is much larger than the others; more precisely, $\phi_{13} \approx \phi_{23} \gg \phi_{12}$. In this limit, we can integrate out the fields with mass of order $\phi_{13}$ to obtain an $SU(2)$-symmetric (Wilsonian) effective action. This action, again, can be written as an integral of a function over the whole superspace. It must be scale invariant and $R$-invariant. In general, again, it can involve ratios of the $\psi_{ij}$'s. But certain operators cannot be generated by such ratios. In particular, consider those terms which involve four factors of $F_{\mu\nu}^{[8]}$. $F^{[8]}$ couples only to heavy fields. On dimensional grounds, these terms are suppressed by at least four factors of the expectation value $v_{13}$ or $v_{23}$. Restricting our attention to terms with precisely four such factors limits the possible dimensionless ratios which can be relevant. In general, $H$ could involve $\frac{\psi_{12}}{\psi_{13}}$, for example. But $\psi_{12}$ would contribute either a factor of $F_{12}$, which is not relevant here, or a factor of $v_{12}$, which would imply a suppression by a power of $v_{13}$. Similarly, $\frac{\psi_{13}}{\psi_{23}} = 1 + \frac{\psi_{12}}{\psi_{23}}$ is irrelevant. As a result, $H$ must take the form

$$H = \frac{1}{16\pi^2} (\ln(\psi_{13}) \ln(\psi_{13}^\dagger) + \ln(\psi_{23}) \ln(\psi_{23}^\dagger)).$$

Again, introducing a background dilaton in the theory, one sees that this coupling is not renormalized.

In this way, we have established that in $SU(3)$, the terms in the effective action proportional to

$$\frac{[(F_{\mu\nu}^{[N-1]})^4 + \ldots]}{|v_{13}|^4} + \frac{[(F_{\mu\nu}^{[23]})^4 + \ldots]}{|v_{23}|^4}$$

are not renormalized. This result clearly generalizes to the case where $SU(N)$ is broken to $SU(N - 1)$, to terms involving

$$\sum_{i=1}^{N-1} \frac{[(F_{\mu\nu}^{[N_i]})^4 + \ldots]}{|v_{N_i}|^4}.$$  

As always, we are assuming the existence of a suitable Wilsonian effective action.

In support of this argument, one can consider the two loop corrections to the four derivative terms. If one examines the various two loop diagrams, it is easy to see that, in $SU(N)$, in all of the diagrams, the term proportional to $\frac{(F_{\mu\nu}^{[N-1]})^4}{(v_N)^4}$ is proportional to $N(N + 1)$. As a result, the cancellation in the case of $SU(2)$ (guaranteed by the theorem of [1]) insures the cancellation to this order in $SU(N)$. 

7
4 Six Derivative Terms in SU(3)

First focus on the case of SU(3). In the flat direction, \( \phi \) is a diagonal matrix. As in our discussion of the previous section, we can take it to be principally in the \( T^8 \) direction, with a small component in the \( T^3 \) direction, i.e. we can write (for simplicity, writing as a \( U(3) \) matrix)

\[
\phi = \begin{pmatrix}
    v_3 & 0 & 0 \\
    0 & \phi[3] & 0 \\
    0 & 0 & -\phi[3]
\end{pmatrix}
\]

(17)

\( v_3 \) and \( \phi[3] \) are complex.

To study whether the six-derivative operator implied by equation 6 is renormalized, we might try to use the \( N = 2 \) setup of the previous section in the full theory. Six derivative terms would then correspond to integrals over the full super space of terms with four covariant derivatives. The particular operator would be generated by terms such as

\[
\int d^8 \theta f(\tau, \tau^\dagger)(D_\alpha \psi_{12})(D^\alpha \psi_{12})(\bar{D}_{\dot{\alpha}} \psi_{31})(\bar{D}^{\dot{\alpha}} \psi_{31}) \frac{1}{|\psi_{31}|^4 |\psi_{12}|^2}.
\]

(18)

This term, one can check, is consistent with all of the symmetries. However, it is not easy to show that this is not renormalized, as for the four derivative terms, since the integral is scale and \( R \)-invariant for any choice of the function \( f(\tau, \tau^\dagger) \) (in particular, it is invariant for functions \( f(\tau - \tau^\dagger) \), corresponding to possible perturbative corrections).

Instead, we resort to a different strategy, which closely parallels the calculation of section 2. We note that for small \( \phi_3 \), the low energy theory is approximately an SU(2) \( \times U(1) \) \( \mathcal{N} = 4 \) supersymmetry gauge theory. This theory, for constant background \( \phi[8] \) and \( F[8]_{\mu\nu} \) possesses not only unbroken supersymmetry but unbroken \( R \) symmetry. For slowly varying \( F[8]_{\mu\nu} \) and \( \phi[8] \), supersymmetry is broken, as is the \( U(1)_R \) symmetry. This breaking is described by operators which couple these fields to the SU(2) degrees of freedom. The leading such operator is obtained from

\[
\mathcal{H} = \frac{1}{16\pi^2} \sum_{i=1}^{2} \ln(\psi_{3i}) \ln(\psi_{3i}^\dagger).
\]

(19)

As before, one expands for small \( \phi_1, \phi_2 \), and obtains the same SU(2) expression as we did earlier:

\[
\mathcal{L}_{eff} = \frac{9}{8\pi^2} \left[ \frac{(F[8]_{\mu\nu})^4}{|v_3|^4} + \ldots \right] \left[ \frac{|\phi^a|^2}{|v_3|^2} + \frac{\phi^a \phi^a}{(v_3)^2} + \frac{\phi^{a\dagger} \phi^{a\dagger}}{(v_3^*)^2} \right]
\]

(20)
The braces now denote
\[
[(F^{\mu\nu})^4 + \ldots] = (F_{\mu\nu})^4 - \frac{1}{4}(F_{\mu\nu}^2)^2 + \partial_\mu \phi \partial^\mu \phi \partial_\nu \phi^\dagger \partial^\nu \phi^\dagger
\]
and \(\phi\) is a complex field, \(\phi = \phi^1 + i\phi^2\).

This lagrangian can be viewed as a perturbation of the low energy, SU(2) theory. For non-vanishing background \(F^{[8]}\) and \(\partial \phi^{[8]}\) it breaks the supersymmetries. The last two terms in eqn. 20 also violate the \(U(1)\) R-symmetry, and we will focus on these. The first point to note about these terms is that they are not renormalized. This is established by our earlier proof of the non-renormalization of the four derivative terms \(\Phi\).

Treating \(L_{\text{eff}}\) of eqn. 20 as a perturbation, we want to consider flat directions of the low energy SU(2) theory, and construct the effective lagrangian in these flat directions to first order in the perturbation. We can do this in a manner similar to the treatment of the \(F^4_{\mu\nu}\) terms if we treat the supersymmetry breaking terms as a spurion, as follows. Describe the gauge coupling by a chiral field, \(\tau\) (chiral with respect to both \(\theta\) and \(\bar{\theta}\)). Take the highest component of \(\tau\) to be proportional to \(F^4_{\mu\nu}\), i.e.
\[
\tau = a + \frac{i}{g^2} + \ldots \theta^2 \bar{\theta}^2 m^2,
\]
with
\[
m^2 = \frac{9}{8\pi^2} \left[\frac{[(F^{[8]}_{\mu\nu})^4 + \ldots]}{|v_3|^4 (v_3)^2}\right].
\]
Then the \(R\)-symmetry violating \(\phi^a \phi^a\) correction to the effective action arises from the coupling of \(\tau\):
\[
L_{\text{eff}} = \int d^2\theta d^2\bar{\theta} \psi^a \psi^a \phi^a \tau.
\]

\(^1\)One might object that in our earlier non-renormalization argument, it was crucial that we considered operators which are suppressed only by \(v_3^4\), yet here we are dealing with operators suppressed by \(v_3^3\). In general, this would be a valid objection, but the terms which interest us here violate the \(U(1)_{R}\)-symmetry of the low energy SU(2) theory, and a more careful examination of possible contributions to \(H\) indicates that other possible corrections of this \(R\)-symmetry breaking type are suppressed by further powers of \(v_3\). In particular, these could arise from contributions to \(H\) of the form
\[
\frac{\psi_{12} \psi_{12}^\dagger \psi_{31} \psi_{31}^\dagger}{v_3^3 v_3^3}.
\]
However, such terms do not respect the \(U(1)_{R}\) symmetry of the full theory. The integral over \(\frac{\psi_{12} \psi_{12}^\dagger}{v_3^3 v_3^3}\) vanishes. Note that there are possible corrections to the terms involving \(\phi^a \phi^a\), coming from operators such as \(\frac{\psi_{12} \psi_{12}^\dagger}{v_3^3 v_3^3}\) and our arguments are not powerful enough to determine if these are or are not renormalized.
In order to determine the terms in the effective action linear in \( m^2 \) which violate the \( R \)-symmetry, we just need to analyze the possible \( \tau \)-dependence of the effective action. The \( \tau \)-independent terms are just those of the usual \( SU(2) \)-theory. By the same arguments as in [1], there are no possible \( \tau \)-dependent terms one can add to the lagrangian (without covariant derivatives, i.e. involving less than six derivatives), except for one which vanishes in the case of constant (lowest component) \( \tau \). This term is:

\[
\int d^8 \theta \ln(\psi) \tau^\dagger + c.c. \tag{26}
\]

The term is scale invariant. It is invariant under the \( R \) symmetry because under the symmetry \( \ln \psi \) shifts by a constant, and the remaining integral gives zero since \( \tau^\dagger \) is antichiral. It is the unique term involving an integral over the full superspace with a non-trivial \( \tau \)-dependence. However, it has the wrong \( g^2 \)-dependence to correspond to Feynman diagrams – it has too few powers of \( g^2 \). So it is not generated in the theory.

So, in fact, it would seem that there are no terms in the effective theory linear in the symmetry breaking. This is a non-renormalization theorem, but it seems too strong. How are we to account for the explicit loop corrections in [4]? Here, one must be careful about the use of the equations of motion. In the absence of the symmetry-breaking term, \( \partial^2 \phi = 0 \). As pointed out by [12], the \( \mathcal{N} = 2 \) action differs from that computed by the (component) background field method by terms which vanish by the tree level equations of motion. Including the quantum corrections, these terms are sixth order in derivatives. In the presence of the perturbation, however, \( \partial^2 \phi = 2m^2 \phi \). In this case, there are additional terms in the effective action. These can be worked out using formulas which are conveniently collected in [14]. These authors work out the lagrangian of eqn. 11 in components. Examining their results (eqns. B.1-B.9 of that paper), there is one term bilinear in the \( \phi \)'s:

\[
\frac{1}{2} \mathcal{H}_{\phi, \phi} \nabla^2 \phi \nabla^2 \phi^\dagger \tag{27}
\]

Using the equations of motion and the actual form of \( \mathcal{H} \) this yields:

\[
\frac{1}{2\pi^2} m^2 \beta m^2 \phi^\dagger. \tag{28}
\]

This is to be compared to the computation of [4], where one studied

\[
m^2 \langle \phi \phi \rangle + c.c. = \frac{1}{8\pi^2} \frac{m^2}{|v_{12}|^2} \Box \phi \Box \phi + c.c. \tag{29}
\]

which, by the equations of motion, is equal to the expression, eqn. 28 above, after an integration by parts.
This argument establishes that there are no further renormalizations of any \( R \)-symmetry violating terms in \( SU(3) \), with six derivatives. The two loop terms in \([4]\) are generated by a combination of the four derivative terms from integrating out the most massive fields, plus the four derivative \( SU(2) \) terms. Neither of these are renormalized.

5 Generalization to \( SU(N) \)

In the case of \( SU(N) \), we can repeat these arguments. First, just as for the \( F_4^{\mu\nu} \) terms, we can show that certain \( F_6^{\mu\nu} \) terms are not renormalized. In particular, consider first breaking \( SU(N) \) to \( SU(N-1) \) by an expectation value, \( v_N \). We have seen that the four derivative terms in the one loop effective action involving \( \frac{(F_2^{[N-1]})^4}{(v_N)^4} \) are not renormalized. This yields the obvious generalization of the effective action of eqn. \([20]\) where the sum now runs over the generators of the adjoint representation of \( SU(N-1) \):

\[
L_{\text{eff}} = \frac{1}{2\pi^2} \frac{N^2}{(N-1)^2} \frac{[(F_2^{[N-1]})^4 + \ldots]}{|v_N|^4} \left( \frac{\phi^a|^2}{|v_N|^2} + \frac{\phi^a \phi^a}{(v_N)^2} + \frac{\phi^a \phi^a}{(v_N)^2} \right). \tag{30}
\]

(30)

We can again describe the \( R \)-symmetry violating part of this perturbation by treating the highest component of \( \tau \), \( m^2 \), as a spurion:

\[
m^2 = \frac{1}{2\pi^2} \frac{N^2}{(N-1)^2} \frac{[(F_2^{[N-1]})^4 + \ldots]}{|v_N|^4} \]

(31)

We have already established that this term is not renormalized. Now we can consider the \( SU(N-1) \) theory, with this interaction as a perturbation. We consider the breaking of this symmetry to \( SU(N-2) \). The four derivative terms are described by

\[
\mathcal{H} = \frac{1}{16\pi^2} \sum_{i=1}^{N-2} \ln(\psi_{N-1,i}) \ln(\psi_{N-1,i}^\dagger). \tag{32}
\]

(32)

Again, there are no \( \tau \)-dependent corrections. Since the perturbation can be described in terms of a background \( \tau \), we see, again, that up to terms related to the equations of motion, there are no \( \tau \)-dependent corrections to the four derivative terms in the low energy theory, corresponding to the absence of corrections to six derivative symmetry violating terms in the full theory.

Now consider the (\( \tau \)-independent) terms. We want to consider these as perturbations in the lower energy \( SU(N-2) \) theory. Rewriting this expression in terms of the Cartan generators, we obtain:

\[
1 \frac{(N-1)^2}{2\pi^2 (N-2)^2} \frac{\partial^2 (F_2^{[N-2]})}{|v_{N-1}|^2} \sum_{i=1}^{N-3} (\phi^i \phi^i + \text{c.c.} + \ldots) \tag{33}
\]
Here we have kept only terms which are relevant to our analysis, i.e. those for which the equations of motion will yield factors of $m^2$. Again, this can be generalized to an expression invariant under $SU(N-2)$. Using the equations of motion, it reduces to the expression obtained in the (component) background field method.

Further operators can now be obtained by iteration. Finally, we are left with the operator:

$$
\frac{1}{g^2}\left(\frac{g^2}{2\pi^2}\right)^{N-1} \prod_{n=2}^{N} \frac{n^2}{(n-1)^2} \frac{\left[(F_{\mu\nu}^{[N-1]})^4 + \ldots \right]}{v_N^4} \frac{\partial_\mu \phi^{[N-2]} \partial^\mu \phi^{[N-2]}}{u_N^2|v_{N-1}|^2} + c.c. \right) + c.c.
$$

In sum, we see that a set of non-zero terms with a particular symmetry structure are obtained by symmetry arguments (up to one overall coefficient). They are generated at precisely the expected order in the coupling, with precisely the values obtained from explicit component field computations. Because they are generated from structures which are not renormalized, these operators are themselves not renormalized. So we have exhibited what we promised: a set of operators with up to $2N$ derivatives which are not renormalized.

6 Finite $\mathcal{N} = 2$ Theories

In [1], it was noted that not only are the $F_{\mu\nu}^{4}$ terms note renormalized in $\mathcal{N} = 4$ theories, but identical arguments imply that they are not renormalized in $\mathcal{N} = 2$ theories. The same applies to the $2N$ derivative terms we have considered here; the scale invariance and $R$ symmetries which were necessary in the $\mathcal{N} = 4$ case also hold in these theories. All of the arguments we have given above go through word for word. The extra matter multiplets in these theories play a similar role to that of the hypermultiplets in $\mathcal{N} = 4$ theories. Like those fields, they carry charge $+4/3$ under the $U(1)_R$ symmetry.

This represents, then, another large class of theories for which the coefficients of terms with arbitrarily large numbers of derivatives can be calculated exactly.

7 Conclusions

We have established that in $\mathcal{N} = 4$ Yang-Mills theories, there is a large set of non-renormalization theorems. This property, already guessed from the behavior of the matrix model, is quite remarkable, and one might wonder both about extensions and possible applications. Given the
complete agreement of the three graviton scattering amplitude in the case of the matrix model, we might expect complete agreement in the field theory for amplitudes which scale correctly with separation. Note also that our techniques do not permit study of terms with more than $2N$ derivatives, which can be generated by operators with covariant derivatives.

Another question is: while non-renormalization theorems plus dualities account for the agreement of the matrix model and supergravity\cite{15, 16, 17}, in what sense do the agreement of each of the coefficients of the $2N$ derivative terms between the matrix model and supergravity constitute independent tests of the dualities? Given that the results follow from the structure of iterations of the one loop action, one suspects that the answer is that they do not.

These observations should also have implications for the understanding of the AdS/CFT correspondence. It should be possible to generalize the analyses of\cite{18, 19} for scattering of two D3-branes to N D3-branes in an AdS background. This is currently under investigation. Finally, one can ask what sorts of non-perturbative information can be extracted from the theory using these results, and whether they are applicable to other non-trivial field theories. What other interesting facts may be gleaned about these theories, as well as more complete answers to the questions raised above, are all subjects worthy of further study.

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References

[1] M. Dine and N. Seiberg, “Comments on Higher Derivative Operators in Some SUSY Field Theories,” Phys. Lett. B409 (1997) 239, \[hep-th/9705057\].

[2] T. Banks, W. Fischler, S. Shenker and L. Susskind, “M theory as a Matrix Model: A Conjecture,” Phys. Rev. D55 (1997) 5112; \[hep-th/9610043\].

[3] Y. Okawa and T. Yoneya, “Multibody Interactions of D Particles in Supergravity and Matrix Theory,” Nucl. Phys. B538 (1999) 67; \[hep-th/9806108\]. Y. Okawa and T. Yoneya, “Equations of Motion and Galilei Invariance in D Particle Dynamics,” Nucl. Phys. B541 (1999) 163.
[4] M. Dine, R. Echols and J.P. Gray, “Tree Level Supergravity and the Matrix Model,” hep-th/9810021.

[5] S. Paban, S. Sethi and M. Stern, “Constraints from Extended Supersymmetry in Quantum Mechanics,” Nucl. Phys. B534 (1998) 137, hep-th/9805018.

[6] S. Sethi and M. Stern, “Supersymmetry and the Yang-Mills Effeaction Action at Finite N,” JHEP 9906:004 (1999), hep-ph/9903049.

[7] D.A. Lowe and R. von Unge, “Constraints on Higher Derivative Operators in Maximally Supersymmetric Gauge Theory,” hep-th/9811017.

[8] W. Taylor, “D-Brane Field Theory on Compact Spaces,” Phys. Lett. B394 (1997) 283.

[9] L. Susskind, “T Duality in Matrix Theory and S Duality in Field Theory,” hep-th/9611164.

[10] D. Berenstein and R. Corrado, “Matrix-Theory in Various Dimensions,” Phys. Lett. B406 (1997) 37, hep-th/9702108.

[11] F. Gonzalez-Rey, U. Lindstrom, M. Rocek and R. von Unge, “On $\mathcal{N}=2$ Low Energy Effective Actions,” Phys. Lett. B388 (1996) 581, hep-th/9607089.

[12] V. Periwal and R. von Unge, “Accelerating D-Branes,” Physics Letters B430 (1998) 71, hep-th/9801121.

[13] M. Dine and N. Seiberg, unpublished.

[14] G. Chalmers, M. Rocek, R. von Unge, “Monopoles in Quantum-Corrected N=2 Super Yang-Mills Theory,” hep-th/9612195.

[15] N. Seiberg, “Why is the Matrix Model Correct,” Phys. Rev. Lett. 79 (1997) 3577, hep-th/9710009.

[16] A. Sen, “D0 Branes on $T^n$ and Matrix Theory,” hep-th/9709220, Adv. Theor. Math. Phys. 2 (1998) 51, hep-th/9709220.

[17] J. Polchinski, “M-Theory and the Light Cone,” hep-th/9903165.

[18] M.R. Douglas and W. Taylor, “Branes in the Bulk of Anti-De Sitter Space,” hep-ph/9807225.

[19] S.R. Das, “Brane Waves, Yang-Mills Theories, and Causality,” JHEP 9902:012 (1999), hep-th/9901004.