A LEHTO–VIRTANEN-TYPE THEOREM AND A RESCALING PRINCIPLE FOR AN ISOLATED ESSENTIAL SINGULARITY OF A HOLOMORPHIC CURVE IN A COMPLEX SPACE

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To the memory of Professor Shoshichi Kobayashi

Abstract. We establish a Lehto–Virtanen-type theorem and a rescaling principle for an isolated essential singularity of a holomorphic curve in a complex space, which are useful for establishing a big Picard-type theorem and a big Brody-type one for holomorphic curves.

1. Introduction

Let $V$ be a complex space. For a holomorphic mapping $f : \mathbb{D} \setminus \{0\} \to V$, we say that $f$ has an isolated essential singularity at the origin if $f$ does not extend holomorphically to $\mathbb{D}$. Our aim is to contribute to the study of the Kobayashi hyperbolicity and the Brody one of a complex space by establishing a Lehto–Virtanen-type theorem and a rescaling principle for an isolated essential singularity of a holomorphic curve in a complex space.

Notation 1.1. Set $\mathbb{D}(a, r) := \{z \in \mathbb{C} : |z| < r\}$ for every $a \in \mathbb{C}$ and every $r > 0$, and set $\mathbb{D}(r) := \mathbb{D}(0, r)$ for every $r > 0$. Then $\mathbb{D}(1) = \mathbb{D}$. For every metric $\delta$ on a complex space $V$, set $\text{diam}_\delta(S) := \sup\{\delta(a, a') : a, a' \in S\}$ for a non-empty subset $S$ in $V$. Finally, for a complex space $V$, let $d_V$ be the Kobayashi pseudometric on $V$.

For the foundation of hyperbolic complex spaces, see the books [4, 5].

1.1. A Lehto–Virtanen-type theorem and a big Picard-type theorem. The following is a generalization of Lehto–Virtanen [7].

Theorem 1 (a Lehto–Virtanen-type theorem). Let $V$ be a complex space equipped with a metric $\delta$ inducing the topology of $V$, and $f : \mathbb{D} \setminus \{0\} \to V$ be a holomorphic mapping having an isolated essential singularity at the origin. If $\bigcap_{r > 0} f(\mathbb{D}(r) \setminus \{0\}) \neq \emptyset$, then there exists a sequence $(z_n)$ in $\mathbb{D} \setminus \{0\}$ tending to 0 as $n \to \infty$ such that $\lim_{n \to \infty} f(z_n)$ exists in $V$ and that $\liminf_{n \to 0} \text{diam}_\delta(f(\partial \mathbb{D}(|z_n|))) > 0$.

There always exists a metric $\delta$ on $V$ inducing the topology of $V$. When $V$ is Kobayashi hyperbolic, we can set $\delta = d_V$ in Theorem 1 and the following
is an immediate consequence of Theorem 1 (see also Example 2.2 and Facts 2.3 and 2.5).

**Corollary 1.2** (a big Picard-type theorem, Kwack [6]). Let $V$ be a complex space and $f : \mathbb{D} \setminus \{0\} \to V$ be a holomorphic mapping having an isolated essential singularity at the origin. If $\bigcap_{r>0} f(\mathbb{D}(r) \setminus \{0\}) \neq \emptyset$, then $V$ is not Kobayashi hyperbolic.

**Remark 1.3.** The corresponding little Picard-type theorem follows from the definition of the Kobayashi hyperbolicity: see Fact 2.7.

1.2. A rescaling principle and a big Brody-type theorem. When a complex space is compact, it admits a nice metric (see Theorem 2.8).

**Theorem 2** (a rescaling principle). Let $V$ be a compact complex space equipped with a metric satisfying the conditions in Theorem 2.8, and $f : \mathbb{D} \setminus \{0\} \to V$ be a holomorphic mapping having an isolated essential singularity at the origin. Then there are sequences $(z_k)$ and $(\rho_k)$ in $\mathbb{C}$ and $(0, \infty)$, respectively, and a non-constant holomorphic mapping $g : X \to V$, where $X$ is either $\mathbb{C}$ or $\mathbb{C} \setminus \{0\}$, such that $\lim_{k \to \infty} z_k = 0$ on $\mathbb{C}$, that $\lim_{k \to \infty} \rho_k = 0$ in $\mathbb{R}$, and that $\lim_{k \to \infty} f(z_k + \rho_k v) = g(v)$ locally uniformly on $X$.

**Remark 1.4.** Originally, similar results to Theorems 1 and 2 have been established in [9, Lemma 3.1, Theorem 1] for quasiregular mappings from a punctured ball to a (compact) Riemannian manifold having an isolated essential singularity at the puncture, and are applied not only to deduce “big” results from their corresponding “little” ones (e.g., Holopainen–Rickman’s big Picard-type theorem [2] from Holopainen–Rickman’s little one [3] for quasiregular mappings when the target is compact) but also to establish the density of repelling periodic points in the Julia set for local uniformly quasiregular dynamics including complex dynamics [5].

**Remark 1.5** (a holomorphic mapping exceptional in Julia’s sense). In Theorem 2 when $V$ is a compact Hermitian manifold equipped with an Hermitian metric $\delta_V$, the sequence $(z_k)$ can identically equal 0 (and then $X = \mathbb{C} \setminus \{0\}$) if the mapping $f$ is exceptional in Julia’s sense in that

$$\limsup_{z \to 0} |z| f^\#(z) < \infty,$$

where we set $f^\#(z) := \lim_{w \to z} \delta_V(f(z), f(w))/|z - w|$ on $\mathbb{D} \setminus \{0\}$. Conversely, if $\limsup_{z \to 0} |z| f^\#(z) = \infty$, then the case $X = \mathbb{C}$ can occur. Some examples of both non-exceptional $f$ and exceptional $f$ in Julia’s sense have been known and studied in the Nevanlinna theory: see Lehto–Virtanen [7].

When $V$ is compact, the following improvement of Corollary 1.2 immediately follows from Theorem 2 (see also Remark 2.7).

**Corollary 1.6** (a big Brody-type theorem). If there is a holomorphic mapping from $\mathbb{D} \setminus \{0\}$ to a compact complex space $V$ having an isolated essential singularity at the origin, then $V$ is not Brody hyperbolic.
Remark 1.7. Corollary 1.6 is also a consequence of Corollary 1.2 and the equivalence between the Kobayashi hyperbolicity and the Brody one for compact complex spaces, the latter of which is known as Brody’s theorem [1]. See Remark 2.7 below.

1.3. Organization of this article. We gather some background materials in Section 2, and show Theorems 1 and 2 in Sections 3 and 4 respectively. Section 5 is devoted to some details on Remark 1.5.

2. Background

For the definition of a complex space and its Kobayashi pseudometric, see [5, §VII] or [10, §2].

Recall that \(d_{\mathbb{D}}\) is the Kobayashi pseudometric on a complex space \(\mathbb{D}\).

Example 2.1. The Kobayashi pseudometric \(d_{\mathbb{D}}\) on \(\mathbb{D}\) coincides with the Poincaré (or hyperbolic) metric on \(\mathbb{D}\), which is a Kähler metric given by \(d_{\mathbb{D}} = |dz|/(1 - |z|^2)\) on \(\mathbb{D}\).

Example 2.2 (cf. [5, Propositions IV.1.1 and VI.2.1]). The Kobayashi pseudometric \(d_{\mathbb{D}\setminus\{0\}}\) on \(\mathbb{D}\setminus\{0\}\) coincides with the hyperbolic metric on \(\mathbb{D}\setminus\{0\}\), which is a Kähler metric given by \(d_{\mathbb{D}\setminus\{0\}} = |dz|/(-|z|\log|z|)\) on \(\mathbb{D}\setminus\{0\}\). In particular, the arc length of the circle \(\partial \mathbb{D}(r)\) with respect to \(d_{\mathbb{D}\setminus\{0\}}\) is \(O(1/(-\log r))\) as \(r \to 0\).

Fact 2.3. The Kobayashi pseudometrics on complex spaces enjoy the non-increasing property under holomorphisms in that for complex spaces \(X, Y\), a holomorphic mapping \(f : X \to Y\), and points \(x, x' \in X\),

\[d_Y(f(x), f(x')) \leq d_X(x, x')\]

In particular, the Kobayashi pseudometrics are invariant under biholomorphisms between complex spaces.

Definition 2.4 (Kobayashi hyperbolicity and the Brody one). A complex space \(V\) is said to be Kobayashi hyperbolic (resp. Brody hyperbolic) if the Kobayashi pseudometric \(d_V\) is a metric on \(V\) (resp. if there is no non-constant holomorphic mapping from \(\mathbb{C}\) to \(V\)).

Fact 2.5. If a complex space \(V\) is Kobayashi hyperbolic, then \(d_V\) induces the topology of \(V\).

Remark 2.6. For a non-constant holomorphic mapping \(g : \mathbb{C}\setminus\{0\} \to V\), \(g \circ \exp : \mathbb{C} \to V\) is non-constant and holomorphic. Conversely, for a non-constant holomorphic mapping \(g : \mathbb{C} \to V\), \(g|((\mathbb{C}\setminus\{0\})) : \mathbb{C}\setminus\{0\} \to V\) is non-constant and holomorphic.

Hence, a complex space \(V\) is Brody hyperbolic if and only if there is no non-constant holomorphic mapping from \(X\) to \(V\), where \(X\) is either \(\mathbb{C}\) or \(\mathbb{C}\setminus\{0\}\).

Fact 2.7 (a little Picard-type theorem and Brody’s theorem). If a complex space \(V\) is Kobayashi hyperbolic, then it is also Brody hyperbolic; this is almost by the definition of the Kobayashi hyperbolicity and that of the Brody one.
Brody’s theorem asserts that the converse is also true if in addition $V$ is compact, that is, a Brody hyperbolic compact complex space is Kobayashi hyperbolic.

When a complex space is compact, it admits a nice metric.

**Theorem 2.8** (cf. [10] Subsection 4.1). For every compact complex space $V$, there is a metric $\delta$ on $V$ satisfying that

1. The distance $\delta$ induces the (equipped) topology of $V$, and that
2. There is an open covering $\{U_x : x \in V\}$ of $V$ such that for every $x \in V$, $U_x$ is a Kobayashi hyperbolic subdomain in $V$ containing $x$ and satisfies $\delta \leq d_{U_x}$ on $U_x$.

The following local Lipschitz continuity of holomorphic curves into compact complex spaces is derived from the properties of the metric $\delta$ in Theorem 2.8 and the non-increasing property of the Kobayashi pseudometrics, and plays a key role in the proof of Theorem 2.

**Theorem 2.9** (cf. [10] Subsection 2.3). Let $V$ be a compact complex space equipped with a metric $\delta$ satisfying the conditions in Theorem 2.8. Then for every open disk $D(a, r)$ and every holomorphic mapping $f$ from an open neighborhood of $D(a, r)$ in $\mathbb{C}$ to $V$, we have

$$
L_{f, D(a, r)} := \sup_{w, w' \in D(a, r), w \neq w'} \frac{\delta(f(w), f(w'))}{d_{D(a, r)}(w, w')} < \infty,
$$

which satisfies the invariance

$$
L_{f \circ \phi, D(b, s)} = L_{f, D(a, r)}
$$

for every biholomorphism $\phi : D(b, s) \to D(a, r) = \phi(D(a, r))$.

**Definition 2.10.** For complex spaces $X, Y$, let $O(X, Y)$ be the set of all holomorphic mappings from $X$ to $Y$.

We conclude this section with the following generalization of Zalcman’s lemma [11].

**Theorem 2.11.** Let $D$ be a domain in $\mathbb{C}$ and $V$ a compact complex space equipped with a metric satisfying the conditions in Theorem 2.8. If a family $\mathcal{F}$ in $O(D, V)$ is not normal at a point $a \in D$, then there are sequences $(f_k)$, $(z_k)$, and $(\rho_k)$ in $\mathcal{F}$, $D$, and $(0, \infty)$, respectively, and a non-constant holomorphic mapping $g : \mathbb{C} \to V$ such that $\lim_{k \to \infty} z_k = a$, that $\lim_{k \to \infty} \rho_k = 0$, and that $\lim_{k \to \infty} f_k(z_k + \rho_kv) = g(v)$ locally uniformly on $\mathbb{C}$.

**Remark 2.12.** A proof of Zalcman’s lemma begins as follows: by the non-normality of $\mathcal{F}$ (in $O(D, V)$) at $a$ and the Arzelà–Ascoli theorem, we can choose an $r > 0$ small enough and a sequence $(f_k)$ in $\mathcal{F}$ such that

$$
\lim_{k \to \infty} L_{f_k, D(a, r)} = \infty.
$$

Such a sequence $(f_k)$ in $\mathcal{F} = O(D, V)$ for $D = \mathbb{D}$ and an open disk $D(a, r)$ in $D = \mathbb{D}$ satisfying 2.24 can also be chosen if $V$ is not Kobayashi hyperbolic.

Now Zalcman’s lemma and Brody’s theorem (see Fact 2.7) are shown simultaneously by rescaling $(f_k)$ appropriately: for the details, we recommend [10] Proof of Theorem 2.6], where a compact complex space which is not necessarily an Hermitian manifold is carefully treated.
Let $V$ be a complex space equipped with a metric $\delta$ inducing the topology of $V$, and $f : \mathbb{D} \setminus \{0\} \to V$ a holomorphic mapping having an isolated essential singularity at the origin. Suppose that $\bigcap_{r>0} f(\mathbb{D}(r) \setminus \{0\}) \neq \emptyset$. Then we can fix a sequence $(z_n)$ in $\mathbb{D} \setminus \{0\}$ tending to 0 as $n \to \infty$ such that $a := \lim_{n \to \infty} f(z_n)$ exists in $V$. Fix an open neighborhood $W$ of $a$ in $V$ equivalent to an analytic subset in an open subset $\Omega$ in $\mathbb{C}^d$ for some $d \in \mathbb{N}$, and fix a subdomain $W' \subset W$ containing $a$.

If $\lim_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z_n|)) > 0$, then we are done. So, suppose that $\lim_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z_n|)) = 0$. Taking a subsequence if necessary, we can even assume that
\begin{equation}
\lim_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z_n|)) = 0.
\end{equation}

Then for every $n \in \mathbb{N}$ large enough, $f(\partial \mathbb{D}(|z_n|)) \subset W'$. For every $n \in \mathbb{N}$ large enough, since the origin is an isolated essential singularity of $f$, by Riemann’s extension theorem, the following minimum
\begin{equation}
r'_n := \min\{r \in (0, |z_n|) : f(\partial \mathbb{D}(r)) \not\subset W'\} > 0
\end{equation}
exists, and then $f(\overline{\mathbb{D}(|z_n|)} \setminus \mathbb{D}(r'_n)) \subset W'$ by the continuity of $f$. Fix a sequence $(z'_n)$ in $\mathbb{D} \setminus \{0\}$ tending to 0 as $n \to \infty$ such that for every $n \in \mathbb{N}$ large enough, $z'_n \in \partial \mathbb{D}(r'_n) \cap f^{-1}(W' \setminus W')$. By the compactness of $\overline{W'} \setminus W'$, we can assume that the limit $b := \lim_{n \to \infty} f(z'_n)$ exists in $\overline{W'} \setminus W'$. It remains to show that $\lim \inf_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z'_n|)) > 0$.

Suppose contrary that $\lim \inf_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z'_n|)) = 0$. Taking a subsequence if necessary, we can even assume that
\begin{equation}
\lim_{n \to \infty} \diam_\delta f(\partial \mathbb{D}(|z'_n|)) = 0.
\end{equation}

Since $a$ and $b$ are distinct points in $W$, which we identify with an analytic subset in an open subset $\Omega$ in $\mathbb{C}^d$, there exists an affine coordinate system $w = (w_1, \ldots, w_d)$ on $\Omega$ such that $w(a) = 0$ and that $w_1(b) \neq 0$. Set $w \circ f = (f_1, \ldots, f_d) : f^{-1}(W) \to w(W)$. Then, for every $n \in \mathbb{N}$ large enough, under the assumptions (3.1) and (3.11), we have both
\begin{equation}
f_1(\partial \mathbb{D}(|z_n|)) \subset \mathbb{D}(|w_1(b)|/3) \quad \text{and} \quad f_1(\partial \mathbb{D}(|z'_n|)) \subset \mathbb{D}(w_1(b), |w_1(b)|/3).
\end{equation}

Fix such $n \in \mathbb{N}$ as satisfies (3.2). Let $\ell$ be a line segment in the ring domain $\mathbb{D}(|z_n|) \setminus \overline{\mathbb{D}(|z'_n|)} \subset f^{-1}(W)$ having one end point in $\partial \mathbb{D}(|z_n|)$ and the other in $\partial \mathbb{D}(|z'_n|)$. Then the path $f_1(\ell)$ in $w_1(W)$ joins the closed curves $f_1(\partial \mathbb{D}(|z_n|))$ and $f_1(\partial \mathbb{D}(|z'_n|))$, so by (3.2), we may fix $y_0 \in \ell$ such that
\begin{equation}
f_1(y_0) \not\in \mathbb{D}(|w_1(b)|/3) \cup \overline{\mathbb{D}(w_1(b), |w_1(b)|/3)}.
\end{equation}

Since $f_1$ is a holomorphic function on $f^{-1}(W)$ and takes the value $f_1(y_0)$ at least at $y_0 \in \mathbb{D}(|z_n|) \setminus \overline{\mathbb{D}(|z'_n|)}$, by the argument principle,
\begin{equation}
1 \leq \int_{\partial(\mathbb{D}(|z_n|) \setminus \overline{\mathbb{D}(|z'_n|)})} \frac{f'_1(z)dz}{f_1(z) - f(y_0)} = \int_{(f_1)_{+}(\partial(\mathbb{D}(|z_n|)))} \frac{dw_1}{w_1 - f(y_0)} - \int_{(f_1)_{+}(\partial(\mathbb{D}(|z'_n|)))} \frac{dw_1}{w_1 - f(y_0)}.
\end{equation}
where the boundary $\partial(\mathbb{D}(|z_n|) \setminus \overline{\mathbb{D}(|z'_n|)})$ is canonically oriented. On the other hand, by (2.2) and (3.3), the residue theorem yields

$$\int_{(f_1)_* \partial(\mathbb{D}(|z_n|))} \frac{dw_1}{w_1 - f(y_0)} = \int_{(f_1)_* \partial(\mathbb{D}(|z'_n|))} \frac{dw_1}{w_1 - f(y_0)} = 0,$$

which contradicts (3.4).

Hence $\liminf_{n \to \infty} \operatorname{diam} f(\partial \mathbb{D}(|z'_n|)) > 0$, and the proof is complete. \(\square\)

Remark 3.1. The final residue theoretic argument applied to $f_1$ can be replaced by a more topological argument (for $f_1$) as in [9, Proof of Lemma 3.1]. In [9, Lemma 3.1], the target Riemannian $n$-manifold $M$ of a quasiregular mapping $f : \mathbb{B}^n \setminus \{0\} \to M$ was assumed to be compact, but this assumption can be relaxed as $\bigcap_{r>0} f(\mathbb{B}^n(r) \setminus \{0\}) \neq \emptyset$ as in Theorem 1. Moreover, in [9, Lemma 3.1], we only claimed that $\limsup_{r \to 0} \operatorname{diam}(f(\partial \mathbb{B}^n(r))) > 0$, but this assertion can be strengthened that there exists a sequence $(x_j)$ in $\mathbb{B}^n \setminus \{0\}$ tending to 0 as $j \to \infty$ such that $\lim_{j \to \infty} f(x_j)$ exists in $M$ and that $\lim\inf_{j \to \infty} \operatorname{diam}(f(\partial \mathbb{B}^n(|x_j|))) > 0$, as in Theorem 1.

4. Proof of Theorem 2

Let $V$ be a compact complex space and $f : \mathbb{D} \setminus \{0\} \to V$ be a holomorphic mapping having an isolated essential singularity at the origin, and fix a metric $\delta$ on $V$ satisfying the conditions in Theorem 2.8. Define a function $Q_f : \mathbb{D}(2/3) \setminus \{0\} \to \mathbb{R}_{\geq 0}$ by

$$Q_f(z) := L_{f, \mathbb{D}(z, |z|/2)},$$

where the right hand side is defined in (2.4). We study the cases that $\limsup_{z \to 0} Q_f(z) = \infty$ and that $\limsup_{z \to 0} Q_f(z) < \infty$, separately.

Suppose first that $\limsup_{z \to 0} Q_f(z) = \infty$. Then there exists a sequence $(y_k)$ in $\mathbb{D} \setminus \{0\}$ such that $\lim_{k \to \infty} y_k = 0$ and $\lim_{k \to \infty} Q_f(y_k) = \infty$. Fix $\epsilon \in (0, 1)$. Then for every $k \in \mathbb{N}$ large enough, a holomorphic mapping $g_k : \mathbb{D}(1 + \epsilon) \to V$ is defined by

$$g_k(w) := f \left( y_k + \frac{|y_k|}{2}w \right).$$

Then for every $k \in \mathbb{N}$ large enough, there exist distinct $w_k, w'_k \in \mathbb{D}$ such that

$$\frac{\delta(g_k(w_k), g_k(w'_k))}{d_{\mathbb{D}}(w_k, w'_k)} \geq \frac{1}{2} \frac{L_{g_k, \mathbb{D}}}{L_{g_k, \mathbb{D}}} = \frac{1}{2} Q_f(y_k),$$

where the final equality is by (2.2). We claim that that the family $\{g_k : k \in \mathbb{N}\}$ is not normal at a point $a \in \mathbb{D}$; otherwise, decreasing $\epsilon > 0$ if necessary, $\{g_k : k \in \mathbb{N}\}$ is normal on $\mathbb{D}(1 + \epsilon)$, so there is a locally uniform limit point $g$ of $\{g_k : k \in \mathbb{N}\}$ on $\mathbb{D}(1 + \epsilon)$, which is in $\mathcal{O}(\mathbb{D}(1 + \epsilon), V)$. Then by (4.1), we have

$$\infty = \limsup_{k \to \infty} \frac{1}{2} Q_f(y_k) \leq L_{g, \mathbb{D}},$$

which contradicts (2.1), so the claim holds.
By this claim, Theorem 2.1 yields sequences \((z_j), (\rho_j), \text{ and } (k_j)\) in \(\mathbb{C}, (0, \infty), \) and \(N, \) respectively, and a non-constant \(g \in \mathcal{O}(\mathbb{C}, V)\) such that \(\lim_{j \to \infty} z_j = a, \) \(\lim_{j \to \infty} \rho_j = 0, \) \(\lim_{j \to \infty} k_j = \infty, \) and
\[
\lim_{j \to \infty} g_{k_j}(z_j + \rho_j v) = g(v)
\]
locally uniformly on \(\mathbb{C}.\) Since
\[
g_{k_j}(z_j + \rho_j v) = f((y_{k_j} + (|y_{k_j}|/2)z_j) + ((|y_{k_j}|/2)\rho_j)v)
\]
on \(\mathbb{C}, \) \(\lim_{j \to \infty}(y_{k_j} + (|y_{k_j}|/2)z_j) = 0 \) in \(\mathbb{C},\) and \(\lim_{j \to \infty}(|y_{k_j}|/2)\rho_j = 0 \) in \(\mathbb{R},\) we are done in the case that \(\limsup_{z \to 0} Q_f(z) = \infty.\)

Suppose next that \(\limsup_{z \to 0} Q_f(z) < \infty.\) For every \(k \in \mathbb{N},\) define a holomorphic mapping \(g_k : \mathbb{D}(e^k) \setminus \{0\} \to V\) by
\[
g_k(v) := f(0 + e^{-k}v).
\]
Then for every \(v \in \mathbb{C} \setminus \{0\},\)
\[
\limsup_{k \to \infty} L_{g_k, D(v, |v|/2)} = \limsup_{k \to \infty} Q_f(e^{-k}v) \leq \limsup_{z \to 0} Q_f(z) < \infty,
\]
where the first equality is by \((2.2).\) Hence the family \(\{g_k : k \in \mathbb{N}\}\) is locally equicontinuous on \(\mathbb{C} \setminus \{0\}.\) By the Arzelà–Ascoli theorem, taking a subsequence if necessary, the locally uniform limit \(g := \lim_{k \to \infty} g_k\) exists on \(\mathbb{C} \setminus \{0\},\) which is in \(\mathcal{O}(\mathbb{C} \setminus \{0\}, V).\) It remains to show that \(g\) is non-constant.

By the compactness of \(V,\) we have \(\bigcap_{r>0} f(\mathbb{D}(r) \setminus \{0\}) \neq \emptyset,\) so that by Theorem 1 there is a sequence \((z_j)\) in \(\mathbb{D} \setminus \{0\}\) tending to 0 as \(j \to \infty\) such that \(a := \lim_{j \to \infty} f(z_j)\) exists in \(V\) and \(\liminf_{j \to \infty} \text{diam}_A f(\partial \mathbb{D}(|z_j|)) > 0.\) Then there is a sequence \((k_j)\) in \(\mathbb{N}\) tending to \(\infty\) as \(j \to \infty\) such that for every \(j \in \mathbb{N},\) \(\partial \mathbb{D}(|z_j|) \subset \mathbb{D}(e^{-k_j}) \setminus \mathbb{D}(e^{-k_j-1}).\)

If \(g\) is constant, then, since
\[
g_{k_j}(\mathbb{D} \setminus \mathbb{D}(e^{-1})) = f(\mathbb{D}(e^{-k_j}) \setminus \mathbb{D}(e^{-k_j-1})) \supset f(\partial \mathbb{D}(|z_j|)) \ni f(z_j)
\]
for every \(j \in \mathbb{N},\) we must have not only \(g \equiv a = \lim_{j \to \infty} f(z_j)\) on \(\mathbb{C} \setminus \{0\}\) but also
\[
0 = \text{diam}_A(a) = \limsup_{j \to \infty} \text{diam}_A(g_{k_j}(\mathbb{D} \setminus \mathbb{D}(e^{-1})))
\]
\[
\geq \liminf_{j \to \infty} \text{diam}_A f(\partial \mathbb{D}(|z_j|)),
\]
which contradicts that \(\liminf_{j \to \infty} \text{diam}_A f(\partial \mathbb{D}(|z_j|)) > 0.\) Hence \(g\) is non-constant, and we are done in the case that \(\limsup_{z \to 0} Q_f(z) < \infty.\)

Now the proof of Theorem 2 is complete.

**Remark 4.1.** The proof of Theorem 2 is similar to [9, Proof of Theorem 1] for quasiregular mappings. In the holomorphic curve case, however, the locally *Lipschitz continuity* (2.1) of holomorphic curves and the invariance of Kobayashi pseudometrics under biholomorphisms between complex spaces make the argument much simpler than that in the quasiregular case.
Let \( V \) be a compact Hermitian manifold equipped with an Hermitian metric \( \delta_V \). Then \( \delta_V \) satisfies the properties in Theorem 2.8 (cf. [10, §2.3]). Let \( f : \mathbb{D} \setminus \{0\} \to V \) be a holomorphic curve having an isolated essential singularity at the origin, and recall that \( f^\#(z) := \lim_{w \to z} \delta_V(f(z), f(w))/|z - w| \) on \( \mathbb{D} \setminus \{0\} \).

For every \( z \in \mathbb{D}(2/3) \setminus \{0\} \), the Kobayashi (pseudo)metric \( d_{\mathbb{D}(z,|z|/2)} \) on \( \mathbb{D}(z, |z|/2) \) is given by the Kähler metric \( d\mathbb{D}(z,|z|/2) = (|z|/2)|dw|/((|z|/2)^2 - |w - z|^2) \) (cf. Example 2.1 and Fact 2.3) and we have

\[
\lim_{w \to z} \frac{\delta_V(f(w), f(w'))}{d\mathbb{D}(z,|z|/2)(w, w')} = \left(\frac{|z|/2 - |w - z|^2}{|z|/2}\right) f^\#(w) \leq \frac{|z|}{2} f^\#(w) \leq |w| f^\#(w).
\]

For every \( z \in \mathbb{D}(2/3) \setminus \{0\} \), setting \( w = z \) in the first equality in (5.1), we also have

\[
\lim_{w' \to z} \frac{\delta_V(f(z), f(w'))}{d\mathbb{D}(z,|z|/2)(z, w')} = \frac{|z|}{2} f^\#(z).
\]

These computations conclude that the case that \( \limsup_{z \to 0} Q_f(z) < \infty \) in the proof of Theorem 2 occurs if and only if \( f \) is exceptional in Julia’s sense.

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