GEOMETRIC AND ANALYTIC STRUCTURES
ON THE HIGHER ADÈLES

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Abstract. The adèles of a scheme have local components — these are topological higher local fields. The topology plays a large role since Yekutieli showed in 1992 that there can be an abundance of inequivalent topologies on a higher local field and no canonical way to pick one. Using the datum of a topology, one can isolate a special class of continuous endomorphisms. Quite differently, one can bypass topology entirely and single out special endomorphisms (global Beilinson-Tate operators) from the geometry of the scheme. Yekutieli’s “Conjecture 0.12” proposes that these two notions agree. We prove this.

The adèles of a scheme $X$ generalize the classical adèles of Chevalley and Weil. The counterpart of a prime/finite place is a saturated flag of scheme points

$$\Delta := (\eta_0 > \cdots > \eta_n) \quad \eta_i \in X$$

with $\eta_{i+1} \in \{\eta_i\}$ a codimension one point. The counterpart of the local field at a prime becomes a higher local field $K$, see Theorem 2 below. Suppose $X$ is of finite type over a field $k$. In dimension one, the classical case, a local field has a canonical topology and thus comes with a canonical algebra of continuous $k$-linear endomorphisms, call it $E_K$. Sadly, this collapses dramatically for $\dim(X) \geq 2$: The adèles induce a topology on the higher local fields $K$. But as was discovered by Yekutieli [Yek92] in 1992, this topology is an additional datum. It cannot be recovered from knowing $K$ solely as a field. However, even if we know this topology, $K$ is no longer a topological field or ring. So it becomes quite unclear how to define the continuous endomorphism algebra $E_K$ for $\dim(X) \geq 2$. Approaches are:

1. (“Global BT operators”) Beilinson defines $E_{\Delta}^{\text{Beil}}$ using a flag $\Delta$ in the scheme.
2. (“Local BT operators”) Yekutieli defines $E_{\Delta}^{\text{Yek}}$ for a topological higher local field $K$.
3. (“$n$-Tate objects”) Adèles can be viewed as an $n$-Tate object [BGW15a], and let $E_{\Delta}^{\text{Tate}}$ be its endomorphism algebra in this category.

Yekutieli has shown that if $k$ is perfect, a flag $\Delta$ as in (1) also induces a topological higher local field structure, as in (2). So while a priori different, this suggests the

Conjecture (A. Yekutieli). Let $k$ be a perfect field. Suppose $X/k$ is a finite type $k$-scheme of pure dimension $n$ and $\Delta := (\eta_0 > \cdots > \eta_n)$ a saturated flag of points. Then there is a canonical isomorphism

$$E_{\Delta}^{\text{Yek}} \cong E_{\Delta}^{\text{Beil}}.$$ 

Theorem 1. The Conjecture is true. Even better,

$$E_{\Delta}^{\text{Yek}} \cong E_{\Delta}^{\text{Beil}} \cong E_{\Delta}^{\text{Tate}},$$

i.e. all three constructions of the endomorphism algebra agree.

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1.“Conjecture 0.12” of [Yek15]
See Theorem 12 for the precise result — the above statement is simplified since a careful formulation requires some preparations which we cannot supply in the introduction.

The theorem establishes a key merit of the \( n \)-Tate categories of [BGW14], namely that \( E_{Yek}, E_{Beil}, E_{Tate} \) all become “representable” in the sense that despite the original hand-made constructions of these algebras, they are nothing but genuine \( \text{End}(\cdot) \)-algebras of an exact category.

Our principal technical ingredient elaborates on the well-known structure theorem for the adèles. The original version is due to A. N. Parshin [Par76] (in dimension \( \leq 2 \)), A. A. Beilinson [Be˘ı80] (proof unpublished), and first published proof due to A. Yekutieli [Yek92]. The following version extends his result with regards to the ind-pro structure of the adèles [BGW14]. We write \( A_X(\Delta, -) \) to denote the adèles of the scheme \( X \) for a flag \( \Delta \). Notation is as in [Be˘ı80].

**Theorem 2.** Suppose \( X \) is a Noetherian reduced excellent scheme of pure dimension \( n \) and \( \Delta = \{(\eta_0 > \cdots > \eta_n)\} \) a saturated flag.

1. Then \( A_X(\Delta, \mathcal{O}_X) \) is a finite direct product of \( n \)-local fields \( \prod K_i \) such that each last residue field is a finite field extension of \( \kappa(\eta_n) \). Moreover,

\[
A_X(\Delta', \mathcal{O}_X) \overset{(*)}{\subseteq} \prod \mathcal{O}_i \subseteq \prod K_i = A_X(\Delta, \mathcal{O}_X),
\]

where \( \mathcal{O}_i \) denotes the first ring of integers of \( K_i \) and \( (*) \) is a finite ring extension.

2. These sit in a canonical staircase-shaped diagram

\[
\begin{array}{c}
A_{\{\eta_0\}}(\Delta, \mathcal{O}_X) \\
\downarrow \quad \quad \quad \quad \quad \quad \downarrow \\
A_{\{\eta_0\}}(\Delta', \mathcal{O}_X) \rightarrow A_{\{\eta_1\}}(\Delta', \mathcal{O}_X) \\
\quad \downarrow \quad \quad \quad \quad \downarrow \\
A_{\{\eta_1\}}(\Delta'', \mathcal{O}_X) \rightarrow A_{\{\eta_2\}}(\Delta'', \mathcal{O}_X) \\
\quad \quad \downarrow \quad \quad \quad \quad \downarrow \\
\quad \quad \quad \quad \quad \quad \quad \quad \cdot \\
\end{array}
\]

3. If \( X \) is finite type over a field \( k \), each field factor \( K := K_i \) in (1) is (non-canonically) isomorphic as rings\(^2\) to Laurent series,

\[
K \sim \kappa((t_1))((t_2)) \cdots ((t_n))
\]

for \( \kappa/k \) a finite field extension. This isomorphism can be chosen such that it is simultaneously an isomorphism

(a) of \( n \)-local fields,
(b) of \( n \)-Tate objects with values in abelian groups,
(c) (if \( k \) is perfect) of \( k \)-algebras,
(d) (if \( k \) is perfect) of \( n \)-Tate objects with values in finite-dimensional \( k \)-vector spaces,
(e) (if \( k \) is perfect) of topological \( n \)-local fields in the sense of Yekutieli.

4. After replacing each ring in (2), except the initial upper-left one, by a canonically defined finite ring extension, it splits canonically as a direct product of staircase-shaped

\(^2\text{but not necessarily } k \text{-algebras!}\)
diagrams of rings: Each factor has the shape
\[ \kappa((t_1)) \cdots ((t_n)) \]
\[ \kappa((t_1)) \cdots [ [t_n] ] \rightarrow \kappa((t_1)) \cdots ((t_{n-1})) \]
\[ \kappa((t_1)) \cdots [ [t_{n-1}] ] \rightarrow \kappa((t_1)) \cdots ((t_{n-2})) \]
\[ \vdots \]
\[ \kappa((t_1)) \cdots [ [t_1] ] \rightarrow \kappa((t_1)) \cdots ((t_2)) \]
\[ \uparrow \]

under any isomorphism as produced by (3).
(a) The upward arrows are going to the field of fractions,
(b) The rightward arrows correspond to passing to the residue field.

These are continuous/admissible epics resp. monics in both Yekutieli’s category of ST modules, as well as n-Tate objects.

(5) Suppose \( k \) is perfect. Then for each field factor \( K \), the notions of lattices (à la Beilinson, resp. Yekutieli, resp. Tate) need not agree, but are pairwise final and co-final (“Sandwich property”).

We refer to the main body of the text for notation and definitions. The reader will find these results in \[ \text{[Yek92, 3.3.2-3.3.6]} \] for Yekutieli’s result inspiring the above. Parts (3)-(5) appear to be new results.

These results focus on the case of schemes over a field, and as we shall explain below, are truly complicated only in the case of characteristic zero. Note also that, since we mostly work over a base field, our considerations are of a geometric/analytic nature, rather than an arithmetic one. Also, no thoughts on infinite places will appear here. See \[ \text{[Fes10]} \] for adèles directed towards arithmetic considerations.

Let us explain the relevance of (3): Yekutieli has already shown in \[ \text{[Yek92]} \] via an explicit example that in characteristic zero a random field automorphism of an \( n \)-local field \( K \) is frequently not continuous. Using Yekutieli’s technique in our context leads us to the following variation of his idea:

**Theorem 3.** Suppose an \( n \)-local field \( K \) is equipped with an \( n \)-Tate object structure in \( k \)-vector spaces. If \( \text{char}(k) = 0 \) and \( n \geq 2 \), then not every field automorphism of \( K \) will preserve the \( n \)-Tate object structure.

See Example \[ \text{[24]} \]. Jointly with Yekutieli’s original example, this shows that in (3), the validity of property (a) does not imply (b)-(e) being true as well.

The definitions of the endomorphism algebras \( E_{\text{Tate}}^{\Delta}, E_{\text{Yek}}^{\Delta}, E_{\text{Beil}}^{\Delta} \) all hinge on notions of lattices, whose definitions we shall address later in the paper. We shall show that the different notions of lattices used by Beilinson, Yekutieli or coming from Tate objects are all pairwise distinct. One might state the comparison as:

Beilinson lattices \( \subsetneq \) Yekutieli lattices \( \subsetneq \) Tate lattices,

modulo a slight abuse of language since these types of lattices live in different objects. See \[ \text{[4]} \] for an explicit counter-example. If these notions all agreed, this would have resulted in a particularly easy proof of Theorem 1. Even though this cannot work, the point of the proof

\[ \text{[4]} \] Moreover, these maps are induced from the corresponding upward and rightward arrows in (2), but due to the finite ring extensions interfering here, the precise nature of this is a little too subtle to make precise in the introduction.
of Theorem 1 turns out to be that Beilinson lattices determine Tate lattices, and moreover are final and co-final in the Grassmannian of all Tate lattices.

Let us survey the relation among the central players of this paper: Let us assume the base field $k$ is perfect.

The solid arrows refer to a canonical construction. Each dashed arrow expresses that a structure can non-cannotically be enriched with additional structure. By an “$n$-Tate object” we mean an $n$-Tate object in finite-dimensional $k$-vector spaces. By an “$n$-local field” we refer to an equicharacteristic $n$-local field with last residue field finite over $k$. By “flag in scheme” we refer to the adèle $A(\triangle, O_X)$, for a saturated flag $\triangle$ in a suitable scheme $X$ of finite type over $k$. By “TLF” we refer to a topological $n$-local field in the sense of Yekutieli. By “Laurent series” we refer to $k^\prime((t_1)) \cdots (t_n))$ with $k^\prime/k$ a finite field extension.

Arrow (1) refers to a certain construction $\#$, established in Theorem 14. Arrow (2) refers to the canonical $n$-Tate object structure of the adèles from [BGW14]. The downward solid arrows on the right, in particular Arrow (3), just refer to forgetting additional structure. Arrow (4) refers to Yekutieli’s construction of the TLF structure on the adèles [Yek92].

**Dangerous Bend.** It is a priori not clear that a TLF can be equipped with a system of liftings inverting Arrow (3) such that we would get a commutative diagram.

However, a different way to state the innovation in Theorem 2 is that it is possible to pick an isomorphism of $A(\triangle, O_X)$ with a Laurent series field such that we arrive at the same objects, no matter which path through Figure 0.1 we choose. That is, no matter through which arrows we produce an $n$-Tate object (resp. TLF), we get the same object.

The objects in Figure 0.1 come with three (a priori different) endomorphism algebras:

- $E_{\text{Beil}}$ of the flag of the scheme, global Beilinson-Tate operators.
- $E_{\sigma_{\text{Yek}}}$ of a TLF with a system of liftings $\sigma$, local Beilinson-Tate operators.
- $E_{\text{Tate}}$ the genuine endomorphisms in the category of $n$-Tate objects, i.e. really just a plain Hom-group. This, by the way, shows the conceptual advantage of working with $n$-Tate categories.

A deep result of Yekutieli, quoted below as Theorem 7, shows that $E_{\sigma_{\text{Yek}}}$ does not depend on $\sigma$, so we can speak of $E_{\text{Yek}}$ of a TLF. Our paper [BGW15a] shows that Arrow (2) induces an isomorphism $E_{\text{Beil}} \cong E_{\text{Tate}}$. Yekutieli’s Conjecture asks whether Arrow (4) induces an isomorphism $E_{\text{Beil}} \cong E_{\text{Yek}}$. We prove this in Theorem 12.
In [5] we prove that Arrow (1) induces an isomorphism $E_{Yek} \cong E_{Tate}$. This is a result of independent interest. It touches a slightly different aspect than Yekutieli’s Conjecture since it refers to the $n$-Tate structure produced by Arrow (1), while the conjecture is about the $n$-Tate structure of Arrow (2). Of course by Theorem [2] we know that we can find a system of liftings such that both $n$-Tate structures match, and then following this route gives a second proof of Yekutieli’s Conjecture.

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1. The topology problem for local fields

In this section we shall introduce the main players of the story. We will use this opportunity to give a survey over many (not even all) of the approaches to give higher local fields a topology or at least a structure replacing a topology. This issue is surprisingly subtle and many results are scattered over the literature.

1.1. Naïve topology. A complete discrete valuation field $K$ with the valuation $v$ comes with a canonical topology, which we shall call the naïve topology, namely: Take the sets $U_i := \{ x \in K \mid v(x) \geq i \}$ as an open neighbourhood basis of the identity. This topology is highly intrinsic to the algebraic structure:

We recall the crucial fact that a field cannot be a complete discrete valuation field with respect to several valuations:

Lemma 1 (F. K. Schmidt). If a field $K$ is complete with respect to a discrete valuation $v$,

(1) then every discrete valuation on $K$ is equivalent to $v$;

(2) any isomorphism of such fields stems from a unique isomorphism of their rings of integers;

(3) and is automatically continuous (in the naïve topology).

See Morrow [Mor12, §1], who has very clearly emphasized the importance of this uniqueness statement. A thorough study of such and related questions can be found in the original paper of Schmidt [Sch33].

Proof. For the sake of completeness, we give an argument, an alternative to the one in [Mor12].

(1) Let $w$ be a further discrete valuation, not equivalent to $v$, and $\pi_w$ a uniformizer for it. By the Approximation Theorem [FV02, Ch. I, (3.7) Prop.] one can pick an element $x \in K$ so that

$$w(x - \pi_w) \geq 1 \quad \text{and} \quad v(x - 1) \geq 1.$$  

By the first property, $w(x) \geq 1$. By the latter $x = 1 + a$ for some $a \in mO_K$ and if $l \geq 2$ is any integer (such that $l \nmid \text{char}(O_K/m)$ in case $O_K/m$ has positive characteristic), the series

$$\sum_{r=0}^{\infty} \left( \frac{1}{l^r} \right) a^r$$

converges, showing that $x$ is $l$-divisible. So $w(x) \in \mathbb{Z}$ is $l$-divisible, forcing $w(x) = 0$. This is a contradiction. (2) follows since the valuation determines the ring of integers, (3) follows since the naïve topology is defined solely in terms of the valuation. □

Definition 1 (Parshin [Par75], [Par78], Kato [Kat78]). For $n \geq 1$, an $n$-local field with last residue field $k$ is a complete discrete valuation field $K$ such that if $(O_1, m)$ denotes its ring of integers, $O_1/m$ is an $(n-1)$-local field with last residue field $k$. A 0-local field with last residue field $k$ is just $k$ itself.
Inductively unravelling this definition, every $n$-local field $K$ gives rise to the following staircase-shaped diagram

$$
\begin{array}{c}
K \\
\mathcal{O}_1 \rightarrow k_1 \\
\mathcal{O}_2 \rightarrow k_2 \\
\vdots
\end{array}
$$

where the $\mathcal{O}_i$ denote the respective rings of integers, and the $k_i$ the residue fields. We call the integers $(\text{char } \mathcal{O}_1, \ldots, \text{char } \mathcal{O}_n)$ the characteristic of $K$.

**Corollary 2.** Fix $n \geq 0$.

1. If a field $K$ possesses the structure of an $n$-local field at all, it is unique.
2. If $K \xrightarrow{\sim} K'$ is a field isomorphism of $n$-local fields, it is automatically continuous in the naive topology and induces isomorphisms of its residue fields,

$$
\begin{array}{c}
k_i \xrightarrow{\sim} k_i',
\end{array}
$$

each also continuous in the naive topology, as well as an isomorphism of last residue fields $k \xrightarrow{\sim} k'$.

**Proof.** This follows by induction from Lemma [1].

Note that the number $n$ is not uniquely determined. An $n$-local field is always also an $r$-local field for all $0 \leq r \leq n$.

**Example 3.** If $k$ is any field, the multiple Laurent series field $k((t_1)) \cdots ((t_n))$ is an example of an $n$-local field with last residue field $k$. It has characteristic $(0, \ldots, 0)$ or $(p, \ldots, p)$ depending on char($k$) = 0 or $p$. The field $\mathbb{Q}_p((t_1)) \cdots ((t_n))$ is an example of an $(n + 1)$-local field with last residue field $\mathbb{F}_p$. It has characteristic $(0, \ldots, 0, p)$. See [FK00] for many more examples.

Let $(R, \mathfrak{m})$ be a complete Noetherian local domain and $\mathfrak{m}$ its maximal ideal. A coefficient field is a sub-field $F$ so that the composition $F \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism of fields.

**Proposition 4** (Cohen’s Structure Theorem). Let $(R, \mathfrak{m})$ be a complete Noetherian local domain and $\mathfrak{m}$ its maximal ideal.

1. If $R$ contains a field (at all), a coefficient field exists.
2. If char($R$) = char($R/\mathfrak{m}$), a coefficient field exists.
3. If $F$ is any coefficient field and $x_1, \ldots, x_r \in \mathfrak{m}$ a system of parameters, $F[[x_1, \ldots, x_r]]$ is a regular local sub-ring inside $R$ so that $R$ is a finite module over it. If $R$ is regular, we even get an isomorphism of rings

$$
F[[x_1, \ldots, x_r]] \xrightarrow{\sim} R,
$$

where the map is evaluation of the series using convergence in the $\mathfrak{m}$-adic topology of $R$.
4. ([Yek15, Theorem 1.1]) Suppose $k$ is a perfect field and $R$ a $k$-algebra. Then one can find a coefficient field $F$ containing $k$ and such that $F \hookrightarrow R$ is a $k$-algebra morphism. If the residue field $R/\mathfrak{m}$ is finite over $k$, there is only one coefficient field having this additional property.
This stems from Cohen’s famous paper [Coh46]. Many more modern references exist, e.g. [HS06, Thm. 4.3.3] for an overview, [Mat80, Ch. 11] or [Mat89, §29 and §30] for the entire story. See Yekutieli’s paper [Yek15, §1] for (4).

An immediate consequence, modulo an easy induction, is the following (simple) excerpt of the classification theory for higher local fields:

**Proposition 5** (Classification). Let $K$ be an $n$-local field with last residue field $k$ such that all fields $K, k_i$ have the same characteristic. Then there exists a non-canonical isomorphism of fields

$$K \simeq k((t_1)) \cdots ((t_n)).$$

If the characteristic is allowed to change, the classification of $n$-local fields is significantly richer. We refer the reader to [FV02, Ch. II, §5] for the structure theory of complete discrete valuation fields, going well beyond the amount needed here. For the $n$-local field case, see [Zhu00], [Osi08], [MZ95, §0, Theorem] or [Mor12]. For our purposes here, the above version is sufficient.

1.2. Systems of liftings. Suppose $K$ is a complete discrete valuation field with ring of integers $\mathcal{O}$ and residue field $\kappa := \mathcal{O}/\mathfrak{m}$. By Cohen’s Structure Theorem, if $\text{char}(K) = \text{char}(\kappa)$, there exists a coefficient field $F \hookrightarrow \mathcal{O}$, in other words, the quotient map to the residue field $\mathcal{O} \twoheadrightarrow \kappa$ admits a section in the category of rings.

**Example 6.** Such a section is usually very far from unique. Consider $K = k(s)((t))$, a 1-local field with last residue field $k(s)$. Take any element in the maximal ideal $\alpha \in t \cdot k(s)[[t]]$. Then

$$k(s) \to k(s)((t)), \quad s \mapsto s + \alpha$$

defines a coefficient field. These are different whenever different $\alpha$ are chosen. Yekutieli has a much more elaborate version of this construction, producing an enormous amount of coefficient fields for the 2-local field $k((s))((t))$ with $\text{char}(k) = 0$. See Example [23] or [Yek92, Ex. 2.1.22].

**Example 7.** Suppose $K$ is an equicharacteristic complete discrete valuation field. If and only if the residue field is either (1) an algebraic extension of $\mathbb{Q}$, or (2) a perfect field of positive characteristic, then there is only one possible choice for the coefficient field [FV02, Ch. II §5.2-§5.4]. In all other cases there will be a multitude of coefficient fields.

There is a straight-forward extension of the concept of a coefficient field to $n$-local fields.

**Definition 2.** Let $K$ be an $n$-local field. An algebraic system of liftings $(\sigma_1, \ldots, \sigma_n)$ is a collection of ring homomorphisms

$$\sigma_i : k_i \to \mathcal{O}_i$$

which are sections to the residue field quotient maps $\mathcal{O}_i \twoheadrightarrow k_i$.

This concept appears for example in [Kat83 §1, p. 112], [MZ95], [Yek15].

**Example 8** (Madunts, Zhukov). By Example [23] an $n$-local field will surely have many systems of liftings if $n \geq 2$, and possibly as well if $n = 1$, depending on the last residue field. Still, if the last residue field is a finite field, and we choose uniformizers $t_1, \ldots, t_n$ for the rings of integers $\mathcal{O}_1, \ldots, \mathcal{O}_n$, Madunts and Zhukov [MZ95 §1] isolate a distinguished, canonical, system of liftings $h_{t_1,\ldots,t_n}$ for all $n$-local fields which are either (1) equicharacteristic $(p, \ldots, p)$ with $p > 0$ some prime, or (2) mixed characteristic $(0, p, \ldots, p)$ for some prime. This construction does not work for example for $k((t_1)) (\cdots ((t_n))$ with $\text{char}(k) = 0$, or the 2-local field $\mathbb{Q}_p((t))$ of characteristic $(0, 0, p)$. See [Zhu00 §1.3] for a survey. These liftings depend on the choice of $t_1, \ldots, t_n$. 


1.3. Minimal higher topology. The naïve topology comes with a major drawback: Already for the multiple Laurent series field $k((t_1)) \cdots ((t_n))$ the formal series notation

$$
\sum a_{i_1,\ldots,i_n} t_1^{i_1} t_2^{i_2} \cdots t_n^{i_n}
$$

of an arbitrary element is usually not convergent in the topology once $n \geq 2$. The problem is that the topology is only made from the top valuation, sensitive to the exponent of $t_n$, but gives the first residue field — when viewed as a sub-field — the discrete topology. Also, the algebraic quotient maps $O_i \rightarrow k_i$ are not topological quotient maps, i.e. they do not induce the quotient topology on $k_i$. The Laurent polynomials $k[t^\pm 1, \ldots, t_n^\pm 1]$ are not dense for $n \geq 2$. This is a new phenomenon and complication in the case $n \geq 2$, which cannot be seen in the classical theory for $n = 1$. Dealing with this type of behaviour required some new ideas, and Parshin proposed to equip $n$-local fields with a different topology \cite{Par84} p. 145, bottom. 

**Example 9** (Parshin). There is a strong limitation to the properties a reasonable topology on $K := k((t_1))((t_2))$ can have, in the shape of the following obstruction: Assume $\mathcal{T}$ is any topology making the additive group $(K; +)$ a topological group and such that the quotient map

(1.2) \[ \mathcal{O}_1 \rightarrow \mathcal{O}_1/m, \text{ i.e. } k((t_1))[[t_2]] \rightarrow k((t_1)) \]

is continuous for $k((t_1))$ equipped with the naïve topology. Then $K$ is not a topological ring in this topology \cite{Par84} Remark 1 on p. 147: Suppose it were. The map in Equation 1.2 is continuous by assumption, so the subsets $U + t_2 k[[t_2]]$ are open in $k((t_1))[[t_2]]$ if and only if $U \subseteq k((t_1))$ is open. As multiplication with powers of $t_2$ would be continuous, this enforces the following: For $(U_i)_{i \in \mathbb{Z}}$ a sequence of open neighbourhoods of the identity in $k((t_1))$ such that $U_i = k((t_1))$ for all sufficiently large $i$, then the sets

(1.3) \[ V := \sum_i U_i t_2^i \subseteq K \]

must be open. These are finite sums of $t_2$-translates of sets we already know must be open. The following figure illustrates the nature of these open sets; the shaded range symbolizes those exponents $(i_1, i_2)$ whose monomials $t_1^{i_1} t_2^{i_2}$ are allowed to carry a non-zero coefficient:

![Diagram](https://via.placeholder.com/150)

This continues ad infinitum to the left; perhaps thinning out but never terminating. The dotted line makes the index such that $U_i = k((t_1))$ for all larger $i$. Now we observe that $V \cdot V = K$ is the entire field (under multiplication the condition $U_i = k((t_1))$ for large $i$ compensates that the open neighbourhoods may thin out to the left). Thus, if multiplication $K \times K \rightarrow K$ were continuous, the pre-image of some open $U \subseteq K$ would have to be open, thus contain some diagonal Cartesian open $V \times V$, but we just saw that multiplication maps this to all of $K$. See \cite{Zhu00}, \cite{Fes01} for a further analysis. For example, this observation extends to show that the multiplicative group $K^\times$ of an $n$-local field cannot be a topological group for $n \geq 3$ \cite{Kat00}.

It appears that the consensus of the practitioners in the field is that it is better to have a reasonable topology than insisting on working with topological rings, which carry an almost meaningless topology. Parshin \cite{Par84} then developed the theory by taking the open sets of the shape in Equation 1.3 as the general definition of a topology for the field $F((t))$: If the
additive group \((F; +)\) is equipped with a topological group structure, generate an additive group topology on \(F((t))\) from the sets \(V_{(U_i)}\) of the shape
\[
V_{(U_i)} := \sum_i U_i t^i \subseteq F((t))
\]
for \((U_i)_{i \in \mathbb{Z}}\) open neighbourhoods of the identity in \(F\) and \(U_i = F\) for \(i\) large enough. This is explained in more detail in [MZ95, §1], [Zhu00]. Giving \(k\) the discrete topology, this inductively equips \(k((t_1))\cdots((t_n))\) with a canonical topology. We call it Parshin's natural topology (there does not appear to be a standard name in the literature; e.g. Abrashkin and his students call it the ‘\(P\)-topology’ [Abr07, §1.2]). For \(n \geq 2\), the natural topology has quite different opens than the naïve topology.

If \(K\) is an equicharacteristic \(n\)-local field with last residue field \(k\), Prop. 5 provides an isomorphism \(\phi\) to such a multiple Laurent series field:
\[
K \xrightarrow{\phi} k((t_1))((t_2))\cdots((t_n)).
\]
Sadly, as was discovered by Yekutieli in 1992 (see Example 23 below), the induced topology usually depends on the choice of the isomorphism. That means, switching to a different \(\phi\) will frequently equip \(K\) with a truly different topology. We shall return to this crucial issue in §1.6.

Example 10 (Madunts, Zhukov). The situation is slightly better if we are in the situation of Example 8. If \(K\) is an \(n\)-local field, equicharacteristic \((p,\ldots,p)\) with \(p > 0\), and the last residue field is finite, Madunts and Zhukov define a topology (extending Parshin’s natural topology) based on their canonical lift \(h_{t_1,\ldots,t_n}\), cf. Example 8 and in a second step prove that the topology is independent of the choice of \(t_1,\ldots,t_n\). This also works for \(n\)-local fields of characteristic \((0, p, \ldots, p)\) and finite last residue field. Such a construction is not available for example for \(k((t_1))\cdots((t_n))\) with \(\text{char}(k) = 0\). In fact, Example 23 due to A. Yekutieli, shows that no such generalization can possibly exist.

Before we continue this line of thought, we discuss a further development of the natural topology:

1.4. Sequential spaces. Working with the natural topology, at least multiplication by a fixed element from the left or right are continuous, and one has
\[
x_n \rightarrow x, y_n \rightarrow y \quad \Rightarrow \quad x_n \cdot y_n \rightarrow x \cdot y,
\]
and the multiplication is continuous if one only tests it on sequences. Following this lead, Fesenko modified the natural topology into a new one in which continuity is detected by sequential continuity alone. We sketch the implications of this:

We recall that a subset \(Z \subset X\) of a topological space \(X\) is called sequentially closed if for every sequence \((x_n)\) with \(x_n \in Z\), convergent in \(X\), the limit \(\lim_n x_n\) also lies in \(Z\).

Definition 3 (Franklin). A topological space is called sequential if a subset is closed iff it is sequentially closed.

Franklin shows that equivalently sequential spaces are those spaces which arise as quotients of metric spaces [Fra65, (1.14) Corollary]. The inclusion admits a right adjoint, called sequential saturation,
\[
\text{Top}_{seq} \xrightarrow{sat} \text{Top}
\]
between the category \(\text{Top}\) (resp. \(\text{Top}_{seq}\)) of all (resp. sequential) topological spaces.

Definition 4 (Fesenko). The saturation topology on \(k((t_1))\cdots((t_n))\) is the sequential saturation of the natural topology. [Fes01].
This topology has many more open sets than the natural topology in general (see [Fes01, (2.2) Remark] for an explicit example), but a sequence is convergent in the saturation topology if and only if it converges in the natural topology. This is no contradiction since these topologies do not admit countable neighbourhood bases. Example 6 implies that we still cannot have a topological ring. However, we get something like a ‘sequential topological ring’. But this really is a completely different notion than a topological ring because ring objects in sequential spaces are not compatible with ring objects in topological spaces by the following example:

**Example 11** (Schwarz, Franklin). The categories $\text{Top}_{\text{seq}}$ and $\text{Top}$ have products, but they are not compatible, i.e.

$$(X_{\text{sat}} \times_{\text{Top}_{\text{seq}}} Y_{\text{sat}}) \neq (X \times_{\text{Top}} Y)_{\text{sat}}.$$ 

Explicit examples were given independently by Schwarz and Franklin. See [Fra65, Example 1.11] for the latter. Generally, colimits in $\text{Top}_{\text{seq}}$ and $\text{Top}$ are compatible, while limits, when they exist, tend to be different. We refer to [Fra65], [Fra67] for a detailed study.

**Remark 12.** For $n = 1$, the naïve, natural and saturation topology on $k((t))$ all agree.

**Remark 13.** Analogously to the case of higher local fields, the adèles of a scheme can also be equipped with sequential topologies. [Fes10], [Fes15]

**Remark 14.** A detailed exposition and elaboration on the notions of sequential groups and rings was given by A. Câmara [Câm15, §1]. He also studies a further topological approach. In [Câm13], [Câm14] he shows that $n$-local fields can also be viewed as locally convex topological vector spaces if one fixes a suitable embedding of a local field, serving as the ‘field of scalars’. The interested reader should consult A. Câmara for further information, much of which is not available in published form.

### 1.5. Kato’s ind-pro approach.

Kato [Kat83, §1] proposed that the concept of topology might in general not be the right framework to think about continuity in higher local fields. In the introduction to [Kat00] he proposes very clearly to abandon the idea of topology entirely, in favour of promoting the ind-pro structure of higher local fields, e.g. as in

$$k((t)) = \lim_{\substack{\longrightarrow \cr \longrightarrow}} \lim_{\substack{\longleftarrow \cr \longleftarrow}} t^{-i}k[[t]]/t^j,$$

to the essential datum. We note that this presentation of $k((t))$ is, in the category of linear topological vector spaces, inducing the naïve topology. Thus, the ind-pro perspective is another possible starting point to find a good generalization of it to higher local fields.

Instead of just working with vector spaces, such an ind-pro viewpoint makes sense for objects in almost any category. Let $C$ be an exact category, e.g. an abelian category. Then there is a category $\text{Ind}_h^\kappa(C)$ of admissible Ind-objects (of cardinality $\leq \kappa$), e.g. encoding objects defined by an inductive system

$$C_1 \hookrightarrow C_2 \hookrightarrow C_3 \hookrightarrow \cdots$$

with $C_i \in C$ and admissible monics as transition morphisms. Additionally, more complicated defining diagrams can be allowed. A precise definition and construction is given in Keller [Kel90] Appendix B] or in greater generality [BGW14, §3]. Following Keller’s ideas, $\text{Ind}_h^\kappa(C)$ is again an exact category and an analogous formalism exists for Pro-objects, $\text{Pro}_h^\kappa(C)$. See also Previdi [Pre11]. We shall frequently drop the cardinality $\kappa$ from the notation for the sake of
legibility. These categories sit in a commutative square of inclusion functors

\[
\begin{array}{ccc}
C & \longrightarrow & \text{Ind}^a C \\
\downarrow & & \downarrow \\
\text{Pro}^a C & \longrightarrow & \text{Ind}^a \text{Pro}^a C.
\end{array}
\]

One may now replace \(\text{Ind}^a \text{Pro}^a (C)\) by the smallest sub-category still containing \(\text{Ind}^a (C)\) and \(\text{Pro}^a (C)\), but also being closed under extensions. This is again an exact category, called the category of elementary Tate objects, \(\text{Tate}^{el} (C)\) \cite{Pre11}, \cite{BGW14}.

**Example 15.** Let \(\text{Ab}_{\text{fin}}\) be the abelian category of finite abelian groups. In the category of all abelian groups \(\text{Ab}\) we have

\[Q_p = \underleftarrow{\text{colim}}_{i} \underrightarrow{\text{lim}}_{j} \frac{1}{p^{i}} \mathbb{Z}/p^{j} \mathbb{Z}, \quad \text{where} \quad \frac{1}{p^{i}} \mathbb{Z}/p^{j} \mathbb{Z} \in \text{Ab}_{\text{fin}}.\]

Instead of regarding this colimit/limit inside the category \(\text{Ab}\), we could read the inner limit as a diagram \(J_{i} : \mathbb{N} \to \text{Ab}_{\text{fin}}, j \mapsto \frac{1}{p^{i}} \mathbb{Z}/p^{j} \mathbb{Z}\), defining an object in \(\text{Pro}^a (\text{Ab}_{\text{fin}})\), and using the dependency on \(i\) we get a diagram \(I : \mathbb{N} \to \text{Pro}^a (\text{Ab}_{\text{fin}}), i \mapsto [\{J_{i}\}]\) of Pro-objects. Considering the object defined by this diagram, we get an object \(I \in \text{Ind}^a \text{Pro}^a (\text{Ab}_{\text{fin}})\). One can easily check that it actually lies in \(\text{Tate}^{el} (\text{Ab}_{\text{fin}})\), see Definition 5 below. One can also define a functor \(\text{Tate}^{el} (\text{Ab}_{\text{fin}}) \to \text{Ab}\) which, using that \(\text{Ab}\) is complete and co-complete, evaluates the Ind-Pro-object described by these diagrams. This yields \(Q_p \in \text{Ab}\) as before. See \cite{BGW14} for more background. More examples along these lines can be found in \cite{BGW15a}.

Kapranov made the justification of Kato’s idea \cite{Kat00} very precise:

**Example 16** \((\text{Kapranov} \cite{Kap01b}, \cite{Kap01a})\). If \(C\) is the abelian category of finite-dimensional \(\mathbb{F}_q\)-vector spaces, \(q = p^n\), Kapranov proved that there is an equivalence of categories \(\text{Tate}^{el} (C) \cong \text{LT}\), where \(\text{LT}\) is the category of linearly locally compact topological \(\mathbb{F}_q\)-vector spaces \cite{Lef42}. Every equicharacteristic 1-local field with last residue field \(\mathbb{F}_q\) and equipped with the naive topology is an object of \(\text{LT}\). One can extend this example and interpret any 1-local field with last residue field \(\mathbb{F}_q\) as an object of \(\text{Tate}^{el} (C)\) for \(C\) the category of finite abelian \(p\)-groups, e.g. as in Example 15.

The category \(\text{Tate}^{el} (C)\) can be described as those objects \(V \in \text{Ind}^a \text{Pro}^a (C)\) which admit an exact sequence

\[(1.4) \quad L \hookrightarrow V \twoheadrightarrow V/L\]

so that \(L \in \text{Pro}^a (C)\) and \(V/L \in \text{Ind}^a (C)\).

**Definition 5.** Any \(L\) appearing in such an exact sequence will be called a (Tate) lattice in \(V\).

So Tate objects are those Ind-Pro-objects admitting a lattice. A category of this nature was first defined by Kato \cite{Kat00} in the 1980s (the manuscript was published only much later), but without an exact category structure, and independently by Beilinson \cite{Bel87} for a completely different purpose – Previdi proved the equivalence between Beilinson’s and Kato’s approaches \cite{Pre11}.

**Remark 17.** It is shown in \cite{BGW14} Thm. 6.7 that for idempotent complete \(C\), any finite set of lattices has a common sub-lattice and a common over-lattice. This can vaguely be interpreted as counterparts of the statement that finite unions and intersections of opens in a topological space should still be open.
Following Kato, this suggests to replace the topologically-minded category $\text{LT}$ (of Example 16) by $\text{Tate}^e(C)$, and for example a 2-local field over $F_q$ should be viewed as something like

\[(1.5) \quad F_q((t_1))((t_2)) \subset \text{Tate}^e(\text{Tate}^e(C)).\]

Instead of concatenating lengthy expressions, we shall call this a ‘2-Tate object’ and more generally define the following:

**Definition 6.** Define $1$-Tate$^e(C) := \text{Tate}^e(C)$, and $n$-Tate$^e(C) := \text{Tate}^e( (n-1)-\text{Tate}(C) )$ and $n$-Tate$(C)$ as the idempotent completion of the category $n$-Tate$^e(C)$. Objects in $n$-Tate$(C)$ will be called $n$-Tate objects. [BGW14, §7]

The slightly complicating presence of idempotent completions in this definition makes the categories substantially nicer to work with. See [BGW15a] for many instances of this effect.

**Example 18** (Kato). Kato [Kat83, §1] equips an n-local field $K$ along with a fixed algebraic system of liftings, Definition 2, with the structure of an n-Tate object in finite abelian groups. The definition depends on the system of liftings. See [Kat00] §1.2 for a detailed exposition. For multiple Laurent series we can use

\["k((t_1))((t_2)) \ldots ((t_n))" \]

\[= \text{colimit}_{i} \cdots \text{colimit}_{j} \text{colimit}_{n} \frac{1}{k[t_1, \ldots , t_n]/(t_1^{i_1}, \ldots , t_n^{i_n})}.\]

**Example 19** (Osipov). In the case of $C = \text{Vect}_F$ a closely related alternative model for $n$-Tate objects are the $C_n$-categories of Denis Osipov [Osi07]. There is also a variant for $C = \text{Ab}$ or including some abelian real Lie groups, the categories $C^n_{\text{el}}$ or $C^n_{\text{ar}}$ of [OP11].

Kato’s approach differs quite radically from the others. Since the concept of a topology is not used at all, it seems at first sight very unclear how one could even formulate any sort of ‘comparison’ between the ind-pro versus topological viewpoint.

### 1.6. **Yekutieli’s ST rings.**

Yekutieli’s approach, first introduced in [Yek92], uses topology again. However, instead of just looking at fields, he directly formulates an appropriate weakening of the concept of a topological ring for quite general (even non-commutative) rings.

For the moment, let $k$ be any ring and it will tacitly be understood as a topological ring with the discrete topology. Yekutieli works with his notion of semi-topological rings (ST rings): An ST ring is a $k$-algebra $R$ along with a $k$-linear topology on its underlying $k$-module such that for any given $r \in R$ both one-sided multiplication maps

\[(r \cdot -) : R \to R \quad \text{and} \quad (- \cdot r) : R \to R\]

are continuous. We follow his notation $\text{STRing}(k)$ for this category. Morphisms are continuous $k$-algebra homomorphisms. See [Yek15, §1] for a review of the theory. The material is developed in full detail in [Yek92, Chapter 1].

**Example 20** (Cámara). The left- and right-continuity is also a feature of both the natural and the saturation topology. In particular $k((t_1)) \cdots ((t_n))$ with the natural topology lies in $\text{STRing}(k)$. By a result of Cámara, this is no longer true for the saturation topology. In more detail: The topology on Yekutieli’s ST rings is always linear, i.e. admits an open neighbourhood basis made from additive sub-groups/or sub-modules. Cámara’s theorem [Câm15, Theorem 2.9 and Corollary] shows that the saturation topology from §1.4 is not a linear topology. For a 2-local field he shows that if one takes the topology generated only from those saturation topology opens which are simultaneously sub-groups, one recovers the natural topology.
Similarly, an ST module $M$ is an $R$-module along with a linear topology on its additive group such that for any given $r \in R$ and $m \in M$ the maps
\[(r \cdot -) : M \to M \quad \text{and} \quad (- \cdot m) : R \to M\]
are continuous. This additive $k$-linear category is denoted by $\text{STMod}(R)$. Yekutieli already points out that this category is not abelian. Although he does not phrase it this way, his results also imply that the situation is not too bad either:

**Proposition 21.** For any ST ring $R$, the category $\text{STMod}(R)$ is quasi-abelian in the sense of J.-P. Schneiders [Sch99].

**Proof.** Yekutieli already shows in [Yek92, Chapter 1] that the category is additive and has all kernels and cokernels. So one only has to check that pushouts preserve strict monics and pullbacks preserve strict epics. These verifications are immediate. □

We get a functor to ordinary modules by forgetting the topology and Yekutieli shows [Yek92, §1.2 and Prop. 1.2.4] that it has a left adjoint
\[\text{STMod}(R) \xrightarrow{\text{fine}} \text{Mod}(R),\]
where ‘fine’ equips an $R$-module $M$ with the so-called fine ST topology, the finest linear topology such that $M$ is an ST module at all (it exists by [Yek92, Lemma 1.1.1]). Being a left adjoint, ‘fine’ commutes with colimits.

**Example 22** (Yekutieli). Yekutieli defines an ST ring structure on multiple Laurent series:
\[(1.6) \quad k((t_1))(t_2) \cdots (t_n) \in \text{STRing}(k).\]
His construction is as follows: Write it as
\[\colim_i \lim_{j_i} \cdots \colim_i \lim_{j_n} \frac{1}{t_1^{j_1} \cdots t_n^{j_n}} k[t_1, \ldots, t_n]/(t_1^{i_1}, \ldots, t_n^{i_n})\]
and (1) equip the inner term with the fine ST $k$-module topology, (2) for the limits use that the inverse limit linear topology of ST topologies is again an ST topology [Yek92, Lemma 1.2.19], (3) the colimits are localizations, equip them with the fine topology over the ring we are localizing; this makes them ST rings again [Yek92, Prop. 1.2.9]. See [Yek15, Def. 1.17 and Def. 3.7] for the details.

Semi-topological rings ultimately remain a very subtle working ground. On the one hand, they behave very well with respect to many natural questions (e.g. Yekutieli develops inner Homs, shows a type of Matlis duality, etc., see [Yek92, Yek95]). On the other hand, just as for sequential spaces §1.4 harmless looking constructions can fail badly, e.g. [Yek15, Remark 1.29].

**Example 23** (Yekutieli). In [Yek92, Ex. 2.1.22] Yekutieli exhibited an example greatly clarifying the problem underlying the search for a canonical topology on $n$-local fields. A detailed exposition is given in [Yek15, Ex. 3.13]. We sketch the construction since we shall need to refer to some of its ingredients later: Suppose $\text{char}(k) = 0$ and let \(\{b_i\}_{i \in I}\) be a transcendence basis for $k((t_1))/k$ with $b_i \in k[[t_1]]$; such exists since if $b_i \notin k[[t_1]]$, replace it by $b_i^{-1}$. The index set $I$ will necessarily be infinite. Then for any choice of elements $c_i \in k((t_1))[[t_2]]$, $i \in I$, Yekutieli constructs a map
\[(1.7) \quad \sigma : k((t_1)) \to k((t_1))[[t_2]]
\quad b_i \mapsto b_i + c_i t_2,\]
which is a particular choice of a coefficient field (On the purely $k$-transcendental sub-field
\[ k\{\{b_1\}_{i \in I}\} \rightarrow k\{(t_1)\}\])
the existence of this map is clear right away. Lifting this morphism along the algebraic extension
\[ k\{(t_1)\}/k\{\{b_1\}_{i \in I}\} \] is the subtle point and hinges on \text{char}(k) = 0 \; \text{[Yek15]}. We may assume
\[ b_0 = t_1 \text{ and } c_0 = 0 \text{ for some index } 0 \in I, \] so that \( \sigma \) maps \( t_1 \) to itself. Yekutieli shows that \( \sigma \)
lifts to a field automorphism \( \tilde{\sigma} \) of \( k\{(t_1)\}\) sending one such coefficient field to another and
\( t_2 \) to itself. Since the sub-field \( k\{t_1, t_2\} \) is element-wise fixed by \( \tilde{\sigma} \), but is dense in the natural
topology, Fesenko’s saturation topology and Yekutieli’s ST topology, \( \tilde{\sigma} \) will not be continuous
unless all \( c_i \) are zero. It follows that if \( K \) is an \( n \)-local field and
\[ \phi : K \simeq k\{(t_1)\} \cdots \{(t_n)\} \]
some field isomorphism \( \phi \) from Prop. 6 the topology pulled back from the right-hand side to
\( K \) depends on the choice of \( \phi \), because we could twist this map with arbitrary discontinuous
automorphisms \( \tilde{\sigma} \).

**Example 24.** We use this paper as an opportunity to strengthen Yekutieli’s example in order
to show that Kato’s ind-pro structure, as explained in Example 15 will also not be preserved by
a random field automorphism. We assume at least a passing familiarity with [BGW14]. Recall
that \( \text{Vect}_f \) denotes the abelian category of finite-dimensional $k$-vector spaces. Again, suppose
\text{char}(k) = 0. Consider Yekutieli’s map \( \sigma \), as in Equation 1.7 and recall that we can choose the
\( c_i \) quite arbitrarily. We will use this now: Pick any surjective set-theoretic map \( Q : I \twoheadrightarrow \Z \).
Such a map exists since the indexing set \( I \) is infinite. We take
\[ c_i := t_1^{Q(i)} \in k\{(t_1)\}\]\nWe write either side as a 2-Tate object in finite-dimensional \( k \)-vector spaces 2-Tate\( (\text{Vect}_f) \),
as in Example 13. If \( \tilde{\sigma} \) is induced from a morphism of 2-Tate objects, it is in particular an
automorphism of a 1-Tate object (namely, a 1-Tate object with values in 1-Tate objects, see
Equation 1.10), namely of
\[ \lim \cdots \lim k\{(t_1)\}\][\( t_2 \)] for every pair \( (i_2, j_2) \) there exists a pair \( (i_2', j_2') \) so that \( \tilde{\sigma} \)
restricts to
\[ \tilde{\sigma}_{(i_2, j_2)} : V_{i_2, j_2} \rightarrow V_{i_2', j_2'} \].
If this is the case, the converse translation is as follows: these \( \tilde{\sigma}_{(i_2, j_2)} \), for each \( i_2 \) fixed
and varying over \( j_2 \), induce a morphism of Pro-diagrams (see [BGW14, §4.1, Def. 4.1] for a
definition), and then varying over \( i_2 \) they induce a morphism of Tate diagrams, made from
these Pro-diagrams (see [BGW14] Def. 5.2] for a definition). This in turn gives the desired
morphism of Tate objects. Unravel Equation 1.10 in the case \( i_2 := 0 \) and take any \( j_2' \) (we may
imagine taking this arbitrarily large, if we want) so that we have the existence of indices \( i_2' \) and
\( j_2 \) with
\[ \tilde{\sigma}_{(0, j_2)} : k\{(t_1)\}\][\( t_2 \)] \rightarrow \[ \frac{1}{t_2} k\{(t_1)\}\][\( t_2 \)]
\[ b_i \mapsto b_i + t_1^{Q(i)} \cdot t_2. \]
The restriction of this morphism in the category $\text{Tate}(\text{Vect})$ to the lattice $k[[t_1]]$ becomes
\begin{equation}
\tilde{\sigma} \mid_{(0,j_2)}: \frac{k[[t_1]][[t_2]]/((t_2^j))}{(t_2^j)} \rightarrow \frac{1}{t_2^j} k((t_1))[[t_2]]/((t_2^j)).
\end{equation}

But lattices are Pro-objects. Thus, by [BGW14, Prop. 5.8] the morphism $\tilde{\sigma} \mid_{(0,j_2)}$ factors through a Pro-subobject $L$ of the right-hand side
\[
\begin{array}{ccc}
k[[t_1]] & \xrightarrow{\tilde{\sigma}} & k((t_1))[[t_2]]/((t_2^j)) \\
\downarrow & & \downarrow \\
L & & k((t_1))[[t_2]]/((t_2^j))
\end{array}
\]

Alternatively one could use the following stronger fact: For a morphism of Tate objects, morphisms originating from a lattice factor through a lattice in the target [BGW15b, Prop. 2.7 (1)]. Now, the Pro-system
\begin{equation}
\left( m \mapsto \frac{1}{t_2^j} \cdot \frac{1}{t_2^m} \right) \text{ in } \text{Pro}^a(\text{Vect})
\end{equation}

is a co-final system of lattices in the target, so in particular the image of $\tilde{\sigma} \mid_{(0,j_2)} k[[t_1]]$ as in Equation 1.11 would have to factor over some object in this system. As we could assume
\[b_i \in k[[t_1]]\] for all $i \in I$ in Example 23 and $Q$ is surjective, Equation 1.10
\[
k[[t_1]][[t_2]]/((t_2^j)) \xrightarrow{\frac{1}{t_2^j}} k((t_1))[[t_2]]/((t_2^j))
\]
\[b_i \mapsto b_i + t_1^{Q(i)} \cdot t_2
\]

produces a contradiction since arbitrarily negative powers of $t_1$ lie in the image of this map, but each of the lattices in the system in Equation 1.12 only has $t_1$ powers with an overall lower bound on the exponent. In other words: Even though $\tilde{\sigma}$ exists as a field automorphism, there is no automorphism of 2-Tate objects inducing it.

We summarize: A general field automorphism of the 2-local field $k((t_1))((t_2))$ for $\text{char}(k) = 0$ need not preserve (1) the natural or saturation topologies, (2) Yekutieli’s ST topology, (3) or Kato’s 2-Tate object structure.

We thank Denis Osipov for pointing out to us that those automorphisms which preserve the $n$-Tate structure of Laurent series $k((t_1)) \cdots ((t_n))$ are also automatically continuous in all of the aforementioned topologies [Osi07, Prop. 2.3, (i)]. See also Example 43.

Remark 25 (Characteristic $p > 0$). Contrary to the usual intuition, the situation is much simpler in positive characteristic $p > 0$:

(1) (Kato) Kato produces a canonical ind-pro structure. See [Kat00, §1.1, Prop. 2 & Example].
(2) (Madunts, Zhukov) The paper [MZ95] constructs a canonical topology, following Parshin.
(3) (Yekutieli) Yekutieli proves that all field isomorphisms between equicharacteristic $n$-local fields of positive characteristic $p > 0$ must automatically be continuous, i.e. isomorphisms in $\text{STRing}(k)$ [Yek92, Prop. 2.1.21]. This is based on a surprising idea using differential operators. See [Yek92, Thm. 2.1.14 and Prop. 2.1.21].

Despite these positive results, it still seems reasonable to approach the uniqueness problem for the topology for arbitrary $n$-local fields without using this work-around in positive characteristic.
Example 23 and Example 24 suggest that looking at \( n \)-local fields per se, there are too many automorphisms to make reasonable and especially canonical use of topological concepts. As a result, Yekutieli proposes to rigidify the category of \( n \)-local fields by choosing and fixing a topology on them. This will be an extra datum. Working in this context, one can restrict one’s attention to those field automorphisms which are also continuous. This greatly cuts down the size of the automorphism group: For an \( n \)-local field, we define the ring

\[
O(K) := \mathcal{O}_1 \times_{k_1} \mathcal{O}_2 \times_{k_2} \cdots \times_{k_{n-1}} \mathcal{O}_n \subset K.
\]

It consists of those elements in \( \mathcal{O}_1 \) whose residual image lies in \( \mathcal{O}_2 \) such that their residual image lies in \( \mathcal{O}_3 \) and so forth.

**Definition 7** (Yekutieli). Let \( k \) be a perfect field. A topological \( n \)-local field (TLF) consists of the following data:

1. an \( n \)-local field \( K \) as in Definition 2,
2. a topology \( T \) on \( K \), which makes it an ST ring,
3. a ring homomorphism \( k \to O(K) \) such that the composition \( k \to O(K) \to k_n \) is a finite extension of fields;

and we assume there exists a (non-canonical, not part of the datum) field isomorphism \( \phi : k((t_1))(t_2)) \cdots \to K \), which is also an isomorphism in \( \text{STRing}(k) \), where the left-hand side is equipped with the standard ST ring structure, as explained in Example 22.

A morphism of TLFs is a field morphism, which is simultaneously an ST ring morphism and preserves the \( k \)-algebra structure given by (3).

Any such isomorphism \( \phi \) will be called a parametrization. We wish to stress that the parametrization is not part of the data. We only demand that an isomorphism exists at all. See [Yek15, §3], [Yek95] for a detailed discussion of TLFs.

**Dangerous Bend.** Despite the name, a ‘topological \( n \)-local field’ is not a field object (or even ring object) in the category \( \text{Top} \).

**Example 26.** Since Yekutieli’s Example 23 shows that a general field automorphism \( \phi \) will not be continuous in the ST ring topology, it implies that it will not be a TLF automorphism.

**Remark 27.** All of these approaches to topologization not only apply to higher local fields, but are also natural techniques to equip similar algebraic structures with a topology, e.g. double loop Lie algebras \( g((t_1))(t_2)) \). [Fes06]

### 2. Adèles of schemes

In [11] we have introduced higher local fields and their topologies. In the present section we shall recall one of the most natural sources producing these structures: the adèles of a scheme. Mimicking the classical one-dimensional theory of Chevalley and Weil, this construction is due to Parshin in dimension two [Par76], and then was extended to arbitrary dimension by Beilinson [Bei80].

#### 2.1. Definition of Parshin-Beilinson adèles

We follow the notation of the original paper by Beilinson [Bei80]. We assume that \( X \) is a Noetherian scheme. For us, any closed subset of \( X \) tacitly also denotes the corresponding closed sub-scheme with the reduced sub-scheme structure, e.g. for a point \( \eta \in X \) we write \( \overline{\{\eta\}} \) to denote the reduced closed sub-scheme whose generic point is \( \eta \). For points \( \eta_0, \eta_1 \in X \), we write \( \eta_0 > \eta_1 \) if \( \overline{\{\eta_0\}} \ni \eta_1 \neq \eta_1 \). Denote
by \( S(X)_n := \{(\eta_0 > \cdots > \eta_n), \eta_i \in X\} \) the set of nondegenerate chains of length \( n + 1 \). Let \( K_n \subseteq S(X)_n \) be an arbitrary subset.

We will allow ourselves to denote the ideal sheaf of the reduced closed sub-scheme \( \overline{\{\eta\}} \) by \( \eta \) as well. This allows a slightly more lightweight notation and is particularly appropriate for affine schemes, where the \( \eta \) are essentially just prime ideals.

For any point \( \eta \in X \), define \( \eta K := \{(\eta_1 > \cdots > \eta_n) \text{ s.t. } (\eta > \eta_1 > \cdots > \eta_n) \in K_n\} \), a subset of \( S(X)_{n-1} \). Let \( F \) be a \emph{coherent} sheaf on \( X \). For \( n = 0 \) and \( n \geq 1 \) respectively, we define inductively

\[
A(K_0, F) := \prod_{\eta \in K_0} \lim_{\to} F \otimes_{O_X} O_{X, \eta}/\eta^i
\]

\[
A(K_n, F) := \prod_{\eta \in X} \lim_{\to} A(\eta K_n, F \otimes_{O_X} O_{X, \eta}/\eta^i).
\]

For a \emph{quasi-coherent} sheaf \( F \), we define

\[
A(K_n, F) := \colim F_j A(K_n, F_j),
\]

where \( F_j \) runs through all coherent sub-sheaves of \( F \). As it is built successively from ind-limits and countable Mittag-Leffler pro-limits, \( A(K_n, \cdot) \) is an exact functor from the category of quasi-coherent sheaves to the category of \( O_X \)-module sheaves. We state the following fact in order to provide some background, but it will not play a big role in this paper:

\[\text{Theorem 4 (Beilinson [Beï80 §2])}. \] For a Noetherian scheme \( X \), and a quasi-coherent sheaf \( F \) on \( X \), there is a functorial resolution

\[
0 \rightarrow F \rightarrow \mathcal{A}^0 \rightarrow \mathcal{A}^1 \rightarrow \mathcal{A}^2 \rightarrow \cdots
\]

in the category of \( O_X \)-module sheaves, made from the flasque sheaves defined by \( \mathcal{A}^i(U) := A(S(U)_i, F) \).

We will not go into further detail. See Huber [Hub91a, Hub91b] for a detailed proof (the only proof available in print, as far as we know) as well as further background.

\[\text{Example 28.} \] If \( X/k \) is an integral proper curve, the complex \( 2.3 \) for \( F := O_X \) becomes

\[
0 \rightarrow O_X \rightarrow k(X) \oplus \prod_{x \in U_0} \hat{O}_x \rightarrow \prod_{x \in U_0} \hat{O}_x \rightarrow 0,
\]

where \( k(X) \) is the sheaf of rational functions, \( U_0 \) is the set of closed points in any open \( U \) (read these terms as sheaves in \( U \)), \( \hat{O}_x := \text{Frac} \hat{O}_x \). In particular, we obtain \( H^1(X, O_X) \) as the cohomology of the global sections. Note that the global sections of the right-most term just correspond to the classical adèles of the curve. Hence, the Parshin-Beilinson adèles really extend the classical framework. As discussed in \( \ref{28} \) the fields \( \hat{O}_x \) have a well-defined intrinsic topology, just because they are 1-local fields. For \( \dim X \geq 2 \), we would get higher local fields and the question of a topology begins to play a significant role.

\[\text{Remark 29 (Other adèle theories).} \] In this paper, whenever we speak of “adèles”, we will refer to the Parshin-Beilinson adèles as described in this section, or the papers [Beï80, Hub91b]. There are other notions of adèles as well: First of all, the Parshin-Beilinson adèles truly generalize the classical adèles only in the function field case: the adèles of a number field feature the infinite places as a very important ingredient, and these are not covered by the Parshin-Beilinson formalism. In a different direction, for us a higher local field has a ring of integers in each of its residue fields, corresponding to a valuation taking values in the integers. However, one can also look at this story from the perspective of higher-rank valuations, i.e. taking values in \( \mathbb{Z}^r \) with a lexicographic ordering. This yields further, more complicated, rings of integers, along with corresponding notions of adèles. See Fesenko [Fes03, Fes10]. Finally, instead of allowing...
just quasi-coherent sheaves as coefficients, one may also allow other sheaves as coefficients. See for example [Gor08, CPT15].

2.2. Local endomorphism algebras. We axiomatize the basic algebraic structure describing well-behaved endomorphisms, for example of $n$-local fields, or vector spaces over $n$-local fields. In particular, this will apply to $n$-local fields built from the ad` eles.

Definition 8. A Beilinson $n$-fold cubical algebra is

1. an associative unital $k$-algebra $A$;
2. two-sided ideals $I^+_i, I^-_i$ such that we have $I^+_i + I^-_i = A$ for $i = 1, \ldots, n$.

This structure appears in [Be˘ı80], but does not carry a name in loc. cit. In all examples of relevance to us, $A$ will be non-commutative. The rest of this section will be devoted to three rather different ways to produce examples of this type of algebra.

2.3. Tate categories/ind-pro approach.

Theorem 5 (BGW15a, Theorem 1). Let $\mathcal{C}$ be an idempotent complete and split exact category. For every object $X \in \mathcal{N}$, its endomorphism algebra carries the structure of a Beilinson $n$-fold cubical algebra, we call it

$$E^{\text{Tate}}(X) := \text{End}_{\mathcal{N}}(X).$$

In particular, we can look at finite-dimensional $k$-vector spaces, i.e. $\mathcal{C} := \text{Vect}_f$, and then the Tate objects à la $k((t_1)) \cdots ((t_n))$ in [15] automatically carry a cubical endomorphism algebra. See [BGW15a] for the construction of the algebra structure and for further background. The above result is not given in the broadest possible formulation, e.g. even if $\mathcal{C}$ is not split exact, the ideals $I^+_i, I^-_i$ can be defined. Moreover, they even make sense in arbitrary Hom-groups and not just endomorphisms. Without split exactness, one then has to be careful with the property $I^+_i + I^-_i = A$ however, which may fail in general.

The introduction of [BGW15a] provides a reasonably short survey to what extent the above theorem can be stretched, and which seemingly plausible generalizations turn out to be problematic.

2.4. Yekutieli’s TLF approach. Yekutieli also constructs such an algebra, but taking a topological local field as its input.

Theorem 6 (A. Yekutieli). Let $k$ be a perfect field. Let $K$ be an $n$-dimensional TLF over $k$. Then there is a canonically defined Beilinson $n$-fold cubical $k$-algebra

$$E^{\text{Yek}}(K) \subseteq \text{End}_k(K),$$

contained in the algebra of all $k$-linear endomorphisms.

This is [Yek15, Theorem 0.4]. We briefly summarize what lies behind this: Firstly, Yekutieli introduces the notion of topological systems of liftings $\sigma$ for TLFs [Yek15, Def. 3.17] (actually it is easy to define: this is an algebraic system of liftings, as in our Definition 2, where the sections $\sigma_i$ have to be ST morphisms. We have already seen in Example 23 that this truly cuts down the possible choices). Then he gives a very explicit definition of a Beilinson $n$-fold cubical algebra called $E^K$ in loc. cit., depending on this choice of liftings. The precise definition is [Yek15, Def. 4.5 and 4.14], and we refer the reader to this paper for a less dense presentation and many more details:

Definition 9 (Yekutieli). Let $k$ be a perfect field and $K$ an $n$-dimensional TLF over $k$.

---

4For some applications it can be sensible to allow non-unital $A$ as well, but we would not have a use for this level of generality here.
(1) If \( M \) is a finite \( K \)-module, a Yekutieli lattice \( L \) is a finite \( \mathcal{O}_1 \)-submodule of \( M \) such that \( K \cdot L = M \).

(2) Fix any system of liftings \( \sigma = (\sigma_1, \ldots, \sigma_n) \) in the sense of Yekutieli [Yek15] Def. 3.17. For finite \( K \)-modules \( M_1, M_2 \), define

\[
E^{Yek}_\sigma(M_1, M_2) \subseteq \text{Hom}_k(M_1, M_2)
\]

to be those \( k \)-linear maps so that

(a) for \( n = 0 \) there is no further restriction, all \( k \)-linear maps are allowed;

(b) for \( n \geq 1 \) and all Yekutieli lattices \( L_1 \subset M_1, L_2 \subset M_2 \), there have to exist Yekutieli lattices \( L'_1 \subset M_1, L'_2 \subset M_2 \) such that

\[
L'_1 \subseteq L_1, \quad L'_2 \subseteq L_2, \quad f(L'_1) \subseteq L_2, \quad f(L'_2) \subseteq L_2
\]

and for all such choices \( L_1, L'_1, L_2, L'_2 \) the induced \( k \)-linear homomorphism

\[
\mathcal{T} : L_1/L'_1 \to L'_2/L_2
\]

must lie in \( E^{Yek}_\sigma(\sigma_2, \ldots, \sigma_n)(L_1/L'_1, L'_2/L_2) \). For this read \( L_1/L'_1 \) and \( L'_2/L_2 \) as \( k_1 \)-modules via the lifting \( \sigma_1 : k_1 \to \mathcal{O}_1 \). Yekutieli calls any such pair \( (L'_1, L'_2) \) an \( f \)-refinement of \( (L_1, L_2) \).

(3) Define \( I_{i,\sigma}^+(M_1, M_2) \) to be those \( f \in E^{Yek}_\sigma(M_1, M_2) \) such that there exists a Yekutieli lattice \( L \subset M_2 \) with \( f(M_1) \subseteq L \). Dually, \( I_{i,\sigma}^-(M_1, M_2) \) is made of those such that there exists a lattice \( L \subset M_1 \) with the property \( f(L) = 0 \).

(4) For \( i = 2, \ldots, n \), and both “+/−”, we let \( I_{i,\sigma}^\pm(M_1, M_2) \) consist of those \( f \in E^{Yek}_\sigma(M_1, M_2) \) such that for all lattices \( L_1, L'_1, L_2, L'_2 \) as in part (2), Equation \( \mathcal{O} \), the condition

\[
\mathcal{T} \in I_{(i-1), (\sigma_2, \ldots, \sigma_n)}(L_1/L'_1, L'_2/L_2)
\]

holds.

(5) For any finite \( K \)-module \( M \), these ideals equip \( (E^{Yek}_\sigma(M, M), I_{i,\sigma}^\pm(M, M)) \) with the structure of a Beilinson \( n \)-fold cubical algebra. Yekutieli calls elements of \( E^{Yek}_\sigma \) a local Beilinson-Tate operator.

The verification that this is indeed a cubical algebra is essentially [Yek15] Lemma 4.17 and 4.19.

**Dangerous Bend.** Something is very important to stress in this context: The system of liftings plays an absolutely crucial role here. The quotients

\[
L_1/L'_1 \quad \text{and} \quad L'_2/L_2
\]

in Equation \( \mathcal{X} \) carry a canonical structure as torsion \( \mathcal{O}_1 \)-modules. There is no canonical way to turn them into modules over the residue field \( k_1 \); the residue map

\[
\mathcal{O}_1 \twoheadrightarrow k_1
\]

goes in the wrong direction. So we really need a section to this map, i.e. a system of liftings. As we have seen in the Example 23 (due to Yekutieli), there can be very different sections, so a priori there is a critical dependence of \( E^{Yek}_\sigma \) on \( \sigma \).

The key technical input then becomes a rather surprising observation originating from Yekutieli [Yek92]: Every change between Yekutieli’s systems of liftings must essentially come from a continuous differential operator, see [Yek15] §2, especially Theorem 2.8 for \( M_1 = M_2 \) for a precise statement, and these in turn lie in \( E^{K}_\sigma \) regardless the \( \sigma \). This establishes the independence of the system of liftings chosen.

**Theorem 7** (Yekutieli [Yek15]). The sub-algebra \( E^{Yek}_\sigma(M_1, M_2) \subseteq \text{Hom}_k(M_1, M_2) \) is independent of the choice of \( \sigma \), and a choice of \( \sigma \) always exists.
In order to distinguish his algebra, called \(E^\text{End}_K\) in loc. cit., from the other variants appearing in this paper, we shall call it \(E^\text{Yek}\) in this paper. By the above theorem, a reference to \(\sigma\) is no longer needed at all.

**Remark 30.** If one looks at the \(n\)-dimensional TLF \(K := k((t_1)) \cdots (t_n))\) over \(k\), then a precursor of Yekutieli’s algebra is Osipov’s algebra “\(\text{End}_K\)” of his 2007 paper [Osi07], §2.3. As an associative algebra, it agrees with \(E^\text{Yek}(K, K)\) and \(\sigma\) the standard lifting. However, Osipov’s definition really uses the concrete presentation of \(K\) as Laurent series, so (a priori) it does not suffice to know \(K\) as a plain TLF or \(n\)-local field.

2.5. **Beilinson’s global approach.** Now suppose \(X/k\) is a reduced scheme of finite type and pure dimension \(n\). We use the notation of [23].

**Definition 10.** Let \(\Delta = \{ (\eta_0 > \cdots > \eta_i) \} \subseteq S(X)_i\) (for some \(i\)) be a singleton set with \(\text{codim}_X \{ \eta_i \} = r\).

1. Define \(\Delta' := \{ (\eta_i > \cdots > \eta_n) \} \subseteq S(X)_{i-1}\), removing the initial entry.
2. Write \(F_\Delta := A(\Delta, F)\) for \(F\) a quasi-coherent sheaf on \(X\).

The notation \(M_\Delta\) also makes sense if \(M\) is an \(\mathcal{O}_{y_0}\)-module since any such defines a quasi-coherent sheaf.

**Definition 11 (Beilinson [Beil80]).** Suppose \(\Delta = \{ (\eta_0 > \cdots > \eta_i) \}\) is given.

1. If \(M\) is a finitely generated \(\mathcal{O}_{y_0}\)-module, a Beilinson lattice in \(M\) is a finitely generated \(\mathcal{O}_{y_1}\)-module \(L \subseteq M\) such that \(\mathcal{O}_{y_1} \cdot L = M\).
2. Let \(M_1\) and \(M_2\) be both finitely generated \(\mathcal{O}_{y_0}\)-modules. Define \(\text{Hom}_\mathcal{O}(M_1, M_2) := \text{Hom}_k(M_1, M_2)\) as all \(k\)-linear maps. Define \(\text{Hom}_\Delta(M_1, M_2)\) to be the \(k\)-submodule of all those maps \(f \in \text{Hom}_\mathcal{O}(M_1, M_2)\) such that for all Beilinson lattices \(L_1 \subseteq M_1, L_2 \subseteq M_2\) there exist lattices \(L'_1 \subseteq M_1, L'_2 \subseteq M_2\) such that

\[
L'_1 \subseteq L_1, \quad L'_2 \subseteq L_2, \quad f(L'_1) \subseteq L_2, \quad f(L_1) \subseteq L_2.
\]

and for all such choices \(L_1, L'_1, L_2, L'_2\) the induced \(k\)-linear morphism

\[
\overline{f} : (L_1/L'_1) \to (L_2/L'_2)
\]

lies in \(\text{Hom}_\Delta(L_1/L'_1, L_2/L'_2)\).

3. Define \(I^\pm_{\Delta}(M_1, M_2)\) to be those \(f \in \text{Hom}_\mathcal{O}(M_1, M_2)\) such that there exists a lattice \(L \subseteq M_2\) with \(f(M_1) \subseteq L\). Dually, \(I^\pm_{\Delta}(M_1, M_2)\) is made of those such that there exists a lattice \(L \subseteq M_1\) with the property \(f(L) = 0\).

4. For \(i = 2, \ldots, n\), and both “\(+/−\)”, we let \(I^\pm_{\Delta}(M_1, M_2)\) consist of those \(f \in \text{Hom}_\mathcal{O}(M_1, M_2)\) such that for all lattices \(L_1, L'_1, L_2, L'_2\) as in part (3) the condition

\[
\overline{f} \in I^\pm_{\Delta}(L_1/L'_1, L_2/L'_2)
\]

holds.

With these definitions in place we are ready to formulate another principal source of algebras as in Definition 8.

**Theorem 8 (Beilinson, [Beil80], §3).** Suppose \(X/k\) is a reduced finite type scheme of pure dimension \(n\). Let \(\eta_0 > \cdots > \eta_n \in S(X)_n\) be a flag with \(\text{codim}_X \{ \eta_i \} = i\). Then

\[
E^\text{Beil}_{\Delta} := \text{Hom}_\Delta(\mathcal{O}_{y_0}, \mathcal{O}_{y_0})
\]

is an associative sub-algebra of all \(k\)-linear maps from \(\mathcal{O}_X\) to itself. For \(i = 1, 2, \ldots, n\), define \(E^\pm_{\Delta} = I^\pm_{\Delta}(\mathcal{O}_{y_0}, \mathcal{O}_{y_0})\). Then \((E^\pm_{\Delta}, (I^\pm_{\Delta}))\) is a Beilinson \(n\)-fold cubical algebra. We shall call its elements global Beilinson-Tate operators.
The structure of this definition is very close to the variant of Yekutieli. However, some essential ingredients differ significantly: On the one hand, no system of liftings is used, so there is no counterpart of the Dangerous Bend in §2.4 and no need for a result like Yekutieli’s Theorem 7. On the other hand, we pay the price of using the $r$-dimensional local rings $O_{\eta^r}$ of $X$. Thus, we really use some data of the scheme $X$ which a stand-alone TLF cannot provide.

3. Structure theorems

In order to proceed, we shall need a few structural results about the structure of the local adèles. The following result
- is classical (and nearly trivial) in dimension one,
- is due to Parshin in dimension two [Par76],
- is due to Beilinson in general [Bei80], but the proof remained unpublished,
- and the first proof in print is due to Yekutieli [Yek92, §3.3.2-3.3.6].

We shall give a self-contained proof in this paper—needless to say, following similar ideas than those used by Yekutieli—but a number of steps are done a bit differently and we strengthen parts of the results, especially in view of Kato’s ind-pro perspective (§1.5).

The following section relies on a number of standard facts from commutative algebra. For the convenience of the reader, we will cite them from the Appendix §A, where we have collected the relevant material.

Definition 12. A saturated flag $\triangle$ in $X$ is a singleton set $\triangle = \{(\eta_0 > \cdots > \eta_r)\} \subseteq S(X)_r$ such that $\text{codim}_X \{\eta_i\} = i$.

Whenever we need to relate adèles between different schemes, in order to be sure what we mean, we write $A_X(-, -)$ to denote adèles of a scheme $X$. Note that flags $\eta_0 > \cdots > \eta_r$ in $X$ also make sense as flags for closed sub-schemes if all their entries are contained in them.

Theorem 9 (Structure Theorem). Suppose $X$ is a Noetherian reduced excellent scheme of pure dimension $n$ and $\triangle = \{(\eta_0 > \cdots > \eta_r)\}$ a saturated flag for some $r$.

1. Then $A_X(\triangle, O_X)$ is a finite direct product of $r$-local fields $\prod K_i$ such that each last residue field is a finite field extension of $\kappa(\eta_r)$, the rational function field of $\{\eta_r\} \subseteq X$.

Moreover,

$$A_X(\triangle', O_X) \subseteq (\star) \subseteq \prod K_i = A_X(\triangle, O_X),$$

where $O_i$ denotes the first ring of integers of $K_i$ and $(\star)$ is the normalization, a finite ring extension.

2. If we regard $\triangle'$ as a flag in the closed sub-scheme $\overline{\{\eta_1\}}$ instead, the corresponding decomposition of Equation (3.1) exists for $A_{\overline{\{\eta_1\}}}(\triangle', O_{\overline{\{\eta_1\}}})$ as well, say

$$\prod k_j = A_{\overline{\{\eta_1\}}}(\triangle', O_{\overline{\{\eta_1\}}})$$

(with a possibly different number of factors), and the residue fields of the $O_i$ in Equation (3.1) are finite extensions of these field factors. Here to each $k_j$ correspond $\geq 1$ factors in Equation (3.1).

3. If $X$ is of finite type over a field $k$, then each $K_i$ is non-canonically ring isomorphic to $k'(t_1) \cdots (t_r)$ for $k'/k(\eta_r)$ a finite field extension. If $k$ is perfect, it can be promoted to a $k$-algebra isomorphism.

4. For a quasi-coherent sheaf $F$, $A(\triangle, F) \cong F \otimes_{O_X} A(\triangle, O_X)$.
In claim (2) we state that for each field factor $k_j$ in Equation \ref{e:3.1} there may be several field factors $K_i$ in Equation \ref{e:3.1} but at least one, corresponding to it. In a concrete case such a branching pattern may for example look like

\[
\begin{array}{cccc}
\text{(3.3)}
\end{array}
\]

where the dots in the bottom row represent the field factors $k_j$, and the dots of the top row the higher local fields $K_i$ corresponding the them, that is: for each such factor the top ring of integers $O_i \subseteq K_i$ has a finite field extension of $k_j$ as its respective residue field.

We devote the entire section to the proof, split up into several pieces.

Unravelling the inductive definition from Equation \ref{e:2.1} yields the formula

\[
A_X(\Delta, \mathcal{F}) = \lim_{i_0} \cdots \lim_{i_r} \text{colim}_{\eta_0} \cdots \text{colim}_{\eta_r} \mathcal{F} \otimes O_{(\eta_r)} / \eta_r^{1r} \otimes \cdots \otimes O_{(\eta_0)} / \eta_0^{10},
\]

where we have allowed ourselves the use of the following viewpoint/shorthands:

- As already the inner-most colimit corresponds to the localization at $\eta_r$ (i.e. taking the stalk), we can henceforth work with rings and modules instead of the scheme and its coherent sheaves. More precisely, we can do this computation in $O_{\eta_r}$-modules.

- We (temporarily) use the notation

$$O_{X, \eta_i} = \text{colim}_{\eta_i} O_{(\eta_i)}$$

for the system of finitely generated $O_{\eta_r}$-submodules $O_{(\eta_i)} \subseteq O_{\eta_i}$.

- We write $\eta_i$ not just for the scheme point $\eta_i$, but also for its radical ideal — under the transition to look at the stalk rather than working with sheaves, the ideal sheaf of the reduced closed sub-scheme $\{\eta_i\}$ corresponds to a radical ideal.

Equation \ref{e:3.3} the commutativity of tensor products with colimits, and Lemma \ref{lem:A.1} settles Theorem \ref{thm:B} (4).

To proceed, let us consider generally objects of the shape

\[
\lim_{i_0} \cdots \lim_{i_j} A_j \otimes O_{(\eta_{j-1})} / \eta_{j-1}^{1r} \otimes \cdots \otimes O_{(\eta_0)} / \eta_0^{10},
\]

where the objects $A_j$ are yet to be defined. We had just seen that $A(\Delta, \mathcal{F})$ is of this shape for $j := r$ and $A_r := \mathcal{F} \otimes O_{\eta_r}$. Now suppose we consider an arbitrary object of the shape as in Equation \ref{e:3.5}. As colimits commute with tensor products, we get

\[
\begin{align*}
&\cong \lim_{i_0} \cdots \lim_{i_j} A_j \otimes (\text{colim}_{\eta_{j-1}} O_{(\eta_{j-1})} / \eta_{j-1}^{1r} \otimes \cdots \otimes O_{(\eta_0)} / \eta_0^{10}) \\
&\cong \cdots \text{colim} \left( \lim_{\eta_{j-2}} \cdots \lim_{\eta_{j-1}} (A_j ((O_{\eta_j} - \eta_{j-1})^{-1}) / \eta_{j-1}^{1j} \otimes \cdots \otimes O_{(\eta_0)} / \eta_0^{1i}) \right)
\end{align*}
\]

(as the colimit is just the localization $O_{\eta_{j-1}}$ and then use Lemma \ref{lem:A.2}). Then we have recovered the shape of Equation \ref{e:3.5} for $j - 1$. Hence, inductively, $A_X(\Delta, \mathcal{F}) = A_0$. Thus, Theorem \ref{thm:B} is essentially a result on the structure of $A_0$ for the special case $\mathcal{F} := O_X$.

**Definition 13.** For the sake of an induction, we shall give the following auxiliary rings a name:

\[
A_{j-1} := \lim_{i_{j-1}} A_j ((O_{\eta_j} - \eta_{j-1})^{-1}) / \eta_{j-1}^{1j}.
\]
Equivalently, $A_j := A(\eta_j > \cdots > \eta_r, \mathcal{O}_X)$ for $0 \leq j \leq r$.

We now argue inductively along $j$:

**Lemma 31.** Assume for some $j$ we have shown the following:

1. $A_j$ is a faithfully flat Noetherian $\mathcal{O}_{\eta_j}$-algebra of dimension $j$.
2. The maximal ideals of $A_j$ are precisely the primes minimal over $\eta_j A_j$.
3. $A_j$ is a finite product of reduced $j$-dimensional local rings, each complete with respect to its maximal ideal.

Then the analogous statements for $j-1$ are true.

(We apologize to the reader for this slightly redundant formulation, but we also intend the numbering as a guide along the steps in the proof.)

Beginning with $j := r$ we had set $A_r := \widehat{O}_{r}$. It is clear that all properties are satisfied since

$$\dim \widehat{O}_{r} = \dim O_{r} = \text{codim}_X \eta_r = r.$$ 

**Proof.** (*Step 1*) By construction $A_{j-1}$ is an $\eta_j - 1 A_j - 1$-adically complete Noetherian ring. $A_j$ is an $\mathcal{O}_{\eta_j}$-algebra (property 1 for $A_j$), so by the universal property of localization we have

$$A_j \rightarrow A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]$$

$$\mathcal{O}_{\eta_j} \rightarrow (\mathcal{O}_{\eta_j})_{\eta_j - 1},$$

but $(\mathcal{O}_{\eta_j})_{\eta_j - 1} = \mathcal{O}_{\eta_j - 1}$. So $A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]$ and its $\eta_j - 1$-adic completion are $\mathcal{O}_{\eta_j - 1}$-algebras. (*Step 2: Maximal ideals under localization*) Next, we determine the maximal ideals $m_i$ of $A_{j-1}$:

By Lemma A.13

$$\eta_j - 1 A_{j-1} \subseteq \mathfrak{m}_j := \bigcap m_i,$$

i.e. they are in bijective correspondence with the maximal ideals of $A_{j-1}/\eta_j - 1 A_{j-1} \cong A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]/\eta_j - 1$. The primes of the localization $A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]$ correspond bijectively to those primes $P \subset A_j$ such that $P \cap (\mathcal{O}_{\eta_j} - \eta_j - 1) = \emptyset$. By induction (properties 1 & 2 for $A_j$) we know that the maximal ideals in $A_j$ are the (finitely many) primes which are minimal over $\eta_j A_j$. Moreover, $A_j$ is faithfully flat over $\mathcal{O}_{\eta_j}$, so by Lemma A.10 the primes $P$ minimal over $\eta_j A_j$ are those minimal with the property $P \cap \mathcal{O}_{\eta_j} = \eta_j$. Hence, for them $P \cap (\mathcal{O}_{\eta_j} - \eta_j - 1) = \eta_j - \eta_j - 1 \neq \emptyset$; they all disappear in the localization. Thus, the maximal ideals of $A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]$ correspond to primes in $A_j$ having at least dimension 1. This enforces that $A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]/\eta_j - 1$ is zero-dimensional. Hence, the maximal ideals $P$ of

$$A_j[[\mathcal{O}_{\eta_j} - \eta_j - 1]]/\eta_j - 1 \cong A_{j-1}/\eta_j - 1 A_{j-1}\,$$

are exactly the minimal primes of it, i.e. they are primes minimal over $\eta_j - 1 A_{j-1}$ in $A_{j-1}$ (proving property 2 for $A_{j-1}$). (*Step 3: Faithful flatness*) $A_{j-1}$ is clearly flat over $\mathcal{O}_{\eta_j}$ since it arises from repeated localization and completion from $\mathcal{O}_{\eta_j}$, and both operations are flat. Moreover, again by faithful flatness of $A_j$ over $\mathcal{O}_{\eta_j}$, $\eta_{j-1} - 1 A_j \cap \mathcal{O}_{\eta_j} = \eta_j - 1$, hence $\eta_{j-1} - 1 A_j \cap (\mathcal{O}_{\eta_j} - \eta_{j-1}) = \eta_j - 1 - \eta_{j-1} = \emptyset$; so the ring in Equation 3.7 is not the zero ring. By Lemma A.10 this shows that $A_{j-1}$ is even a faithfully flat $\mathcal{O}_{\eta_j - 1}$-algebra (proving property 1 for $A_{j-1}$). (*Step 4: Reducedness*) Next, we claim that $A_{j-1}$ is reduced. To see this, recall that quotient rings and localization commute in the sense that $(R/I)_{S} \cong R_S/I_R S$ if $S$ is a multiplicative subset of $R$. 


and \( \overline{S} \) denotes its image in \( R/I \). Using this fact repeatedly, we rewrite

\[
A_j^{-1} = \lim_{r \to j} A_j[(O_{\eta_j} - \eta_j^{-1})]/\eta_j^{r-1}
\]

\[
= \lim_{r \to j} \left( \lim_{s \to j} A_{j+1}[(O_{\eta_{j+1}} - \eta_j^{-1})]/\eta_j^{s}ight) [(O_{\eta_j} - \eta_j^{-1})]/\eta_j^{r-1}
\]

\[
= \cdots = \lim_{r \to j} \cdots \lim_{r \to j} A_r[S^{-1}]/\eta_r^{-1} \cdots \eta_j^{r-1},
\]

but \( A_r := \overline{O}_{\eta_j} \) so \( A_r \) arises as a completion of a localization of \( X \). Both localization and completion (with respect to arbitrary ideals) are regular morphisms by Lemma [A14]. Thus, the composition is regular, it is also faithfully flat, so by faithfully flat ascent, Lemma [A15], \( A_j^{-1} \) is reduced. With the same argument \( A_j^{-1}/\eta_j^{-1}A_j^{-1} \cong \lim_{m \to j} \cdots \lim_{m \to j} (A_r/\eta_j^{-1})[S^{-1}]/\eta_r^{-1} \cdots \eta_j^{r-1} \),

but \( A_r/\eta_j^{-1} = \overline{O}_{\eta_j}/\eta_j^{-1} \). As \( \eta_j^{-1} \) is prime, \( \overline{O}_{\eta_j}/\eta_j^{-1} \) is a domain and thus \( \overline{O}_{\eta_j}/\eta_j^{-1} \) is at least reduced. Hence, \( A_j^{-1}/\eta_j^{-1}A_j^{-1} \) is reduced and zero-dimensional, so by Lemma [A14] we have

\[
(3.8) \quad A_j^{-1}/\eta_j^{-1}A_j^{-1} \cong \prod_m [A_j[(O_{\eta_j} - \eta_j^{-1})]/\eta_j^{-1}],
\]

where \( m \) runs through the finitely many (automatically minimal) primes in \( A_j^{-1}/\eta_j^{-1}A_j^{-1} \). The localizations of the right-hand side are reduced zero-dimensional local rings, i.e. by Lemma [A16] they must be fields. We obtain a complete system of pairwise orthogonal idempotents \( e_1, \ldots, e_\ell \in A_j^{-1}/\eta_j^{-1}A_j^{-1} \) giving the decomposition of Equation (3.8). Using Lemma [A17] these idempotents lift uniquely to a complete system of pairwise orthogonal idempotents \( e_1, \ldots, e_\ell \) in \( A_j^{-1} \).

Hence, \( A_j^{-1} \) is a finite product of reduced \((j - 1)\)-dimensional local rings (proving property 3 for \( A_j^{-1} \)).

After this preparation we are ready to establish the rest of Theorem 9.

Proof of Thm. 9 Recall that \( A_X(\triangle, O_X) = A_0 \). From Lemma [31] property 3, for \( A_0 \) it follows that \( A_X(\triangle, O_X) \) is a finite product of fields. We may unwind \( A_X(\triangle', O_X) \) entirely analogously as in Equation (3.4) and obtain \( A_X(\triangle', O_X) = A_1 \) and thus (by the very definition of \( A_0 \), Equation (3.0))

\[
A_X(\triangle, O_X) = A_0 = \lim_{i \to 0} A_1[(O_{\eta_i} - \eta_i^{-1})]/\eta_i^{j_0}
\]

\[
= \lim_{i \to 0} A_X(\triangle', O_X)[(O_{\eta_i} - \eta_i^{-1})]/\eta_i^{j_0}.
\]

By Lemma [31] the ring \( A_1 \) is a finite product of one-dimensional reduced complete local rings. Denote by \( Q_i \) the minimal primes of \( A_1 \). Being reduced, the first arrow in

\[
A_X(\triangle', O_X) = A_1 \hookrightarrow \prod_i A_1/Q_i
\]

\[
\hookrightarrow \prod_i A_1/Q_i[(O_{\eta_i} - \eta_i^{-1})]
\]

\[
\hookrightarrow \lim_{i \to 0} \prod_i A_1/Q_i[(O_{\eta_i} - \eta_i^{-1})]/\eta_i^{j_0} = \prod_i A_X(\triangle, O_X)/Q_i
\]

is injective. The injectivity of the third follows from being Noetherian. Consider the normalization of \( A_X(\triangle', O_X) \) in \( A_X(\triangle, O_X) \), this is a finite extension since \( X \) is excellent and the rings we deal with are formed by constructions preserving excellence. By Lemma [A3] the normalization arises as the product of the integral closures \( N_i \) of each \( A_X(\triangle', O_X)/Q_i \) in the respective field of fractions \( A_X(\triangle, O_X)/Q_i \):

\[
A_X(\triangle', O_X) \hookrightarrow \prod_i A_1/Q_i \hookrightarrow \prod_i N_i \hookrightarrow \prod_i A_X(\triangle, O_X)/Q_i = A_X(\triangle, O_X).
\]
Clearly each $N_i$ is a one-dimensional normal complete semi-local ring. Such a local ring is a discrete valuation ring by Lemma [A112]. Hence, $A_X(\triangle, \mathcal{O}_X)$ is a finite product of complete discrete valuation fields, $N_i$ are their respective rings of integers. Under the normalization each local ring of $A_X(\triangle', \mathcal{O}_X)$ gets extended to a semi-local ring, leading to a branching into some $g \geq 1$ maximal ideals over it, and thus to a branching like (for example)

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}
\]

once we look at all local rings together: dots in the upper row represent maximal ideals of the normalizations, i.e. factors $N_i$. Dots in the lower row represent maximal ideals of $A_X(\triangle', \mathcal{O}_X)$, so by Lemma [B] equivalently minimal primes of $A_X(\triangle', \mathcal{O}_X)/\eta_1$. The respective residue fields $\kappa_i := N_i/m_i$ also follow to be finite ring extensions of $(A_X(\triangle', \mathcal{O}_X)/Q_1)/\eta_1$. By direct inspection one sees that $A_X(\triangle, \mathcal{O}_X)/\eta_1$ can be identified with $A_{(\eta_1)}(\triangle', \mathcal{O}_{\eta_1})$, i.e. identified with $A(\triangle', \mathcal{O}_X)$, but taking $X := \{\eta_1\}$ as the scheme and reading $\triangle'$ as an element of $S(\{\eta_1\})_{r-1}$ instead of $S(X)_{r-1}$. Therefore, by induction on the dimension of $X$, in the Figure above the lower row dots equivalently correspond canonically to the factors $k_j$; and the upper row dots to the $\kappa_i$. Moreover, again by induction, the ring $A_X(\triangle', \mathcal{O}_X)/\eta_1$ is a finite product of $(r-1)$-local fields in a canonical fashion, and the $\kappa_i$ finite field extensions thereof. Going all the way down, by induction on $r$, this shows that the last residue fields are finite extensions of

\[
A_{(\eta_r)}(\{\eta_r\}, \mathcal{O}_{\{\eta_r\}}) = \lim_{\rightarrow} O_{(\eta_{r-1}), \eta_r}/\eta_r^i = \kappa(\eta_r).
\]

directly from the definition of the adèles, Equation [2.1]. This establishes part (2) of the claim. Each $\kappa_i$ is (a finite extension of $-$ and thus itself) a complete discrete valuation field whose residue field is $(r-1)$-local. Thus, each $F_i$ is an $r$-local field. This establishes part (1) of the theorem. Finally, if all the fields in this induction are $k$-algebras, each complete discrete valuation ring $R_i$ is equicharacteristic, so by Cohen’s Structure Theorem, Prop. [4] there is a non-canonical isomorphism $\simeq \kappa_i[[t]]$. Hence, $F_i \simeq \kappa_i((t))$ and inductively this shows that $r$-local fields are multiple Laurent series fields, proving part (3) of the theorem. If $k$ is perfect, pick each coefficient field such that it is additionally a sub-$k$-algebra. Part (4) is just the sheaf version of Lemma [A11] □

We can easily extract the higher local field structure of the local adèles from the previous result. Recall that we write $A_Z(-, -)$ to denote adèles of a scheme $Z$.

**Theorem 10** (Structure Theorem II). Suppose $X$ is a purely $n$-dimensional reduced Noetherian excellent scheme and $\triangle = \{\eta_0 > \cdots > \eta_r\}$ a saturated flag. Then we get a diagram

\[
\begin{array}{ccc}
A_{(\eta_0)}(\triangle, \mathcal{O}_X) & \longrightarrow & A_{(\eta_1)}(\triangle', \mathcal{O}_X) \\
\| & & \downarrow \\
A_{(\eta_0)}(\triangle', \mathcal{O}_X) & \longrightarrow & A_{(\eta_1)}(\triangle'', \mathcal{O}_X)
\end{array}
\]

(3.9)

\[
\begin{array}{ccc}
A_{(\eta_0)}(\triangle''), \mathcal{O}_X & \longrightarrow & A_{(\eta_2)}(\triangle''', \mathcal{O}_X) \\
\| & & \downarrow \\
& \cdots & \\
A_{(\eta_r)}(\triangle''', \mathcal{O}_X) & \longrightarrow & \cdots
\end{array}
\]
where

1. the upward arrows are precisely the inclusions of Theorem 2 \(\text{(part 1)}\), Equation 3.4;
2. the rightward arrows are taking the quotient of \(A_{(\eta_1)}(\Delta', \mathcal{O}_X)\) by \(\eta_{i+1}\);
3. After replacing each ring in Diagram 3.9 except the initial upper-left one, by a canonically defined finite ring extension, it splits canonically as a direct product of staircase-shaped diagrams of rings: Each factor has the shape

\[
\kappa((t_1)) \cdots ((t_n))
\]

\[
\kappa((t_1)) \cdots [(t_n)] \longrightarrow \kappa((t_1)) \cdots ((t_{n-1}))
\]

\[
\kappa((t_1)) \cdots [(t_{n-1})] \rightarrow \kappa((t_1)) \cdots ((t_{n-2}))
\]

\[
\vdots
\]

In particular, each object in it is a direct factor of a finite extension of the corresponding entry in Diagram 3.9

(a) The upward arrows are going to the field of fractions,
(b) The rightward arrows correspond to passing to the residue field.

4. These factors are indexed uniquely by the field factors of the upper-left entry \(A_X(\Delta, \mathcal{O}_X) = \prod K_i\). Each field factor \(k_j\) of \(A_{(\eta_1)}(\Delta', \mathcal{O}_X)\) in any row of Diagram 3.9 corresponds to \(\geq 1\) field factors in the row above, such that the respective residue field is a finite field extension of the chosen \(k_j\).

An elaboration: As we already know, each \(A_{(\eta_1)}(\Delta', \mathcal{O}_X)\) decomposes as a finite direct product of fields. In particular, in Diagram 3.9 we get such a decomposition in every single row (and of the two terms in each row, we refer to the one following after \(\rightarrow\)), and there is a matching between the field factors of the individual rows. For each field factor \(k_j\) of a row, there are \(\geq 1\) field factors in the row above it, such that the respective residue field is finite over the given \(k_j\). If we follow the graphical representation of this branching behaviour as in Diagram 3.3, we get a simple description of the entire branching behaviour from the top row all to the bottom row: If we begin with the field factors of the upper-left entry \(A_X(\Delta, \mathcal{O}_X) = \prod K_i\), the matching to the indexing of the field factors of \(A_{(\eta_1)}(\Delta', \mathcal{O}_X)\) in the rows below is obtained by following the downward paths top-to-bottom in the tree graph obtained by concatenating the branching diagrams (like Diagram 3.3) on each level, e.g. as in the figure below.
Proof. The first step (both logically as well as visually in the diagram)

\[ A_{\{\eta_0\}}(\Delta, \mathcal{O}_X) \xrightarrow{\eta_0} A_{\{\eta_1\}}(\Delta', \mathcal{O}_X) \]

is literally just Theorem 9 applied to the scheme \( X := \{\eta_0\} \) and the flag \( \Delta \). To continue to the next step, just inductively apply Theorem 9 to \( X := \{\eta_1\} \) instead and note that the \( i \)-fold truncated flag of sub-schemes can be viewed as a flag of sub-schemes in this smaller scheme as well. □

Definition 14. For a point (or ideal) \( \eta \), we shall write

\[ \operatorname{colim}_{f \in \eta} \mathcal{O}(f^{-\infty}) \]

to denote the colimit over all coherent sub-sheaves (or finitely generated sub-modules) of the localization \( \mathcal{O}_\eta \).

Lemma 32. Suppose \( X \) is a purely \( n \)-dimensional reduced Noetherian excellent scheme and \( \Delta = \{\eta_0 > \cdots > \eta_r\} \) a saturated flag. Suppose \( \mathcal{F} \) is a coherent sheaf. Then the following \( \mathcal{O}_{\eta_i} \)-modules are pairwise canonically isomorphic for all \( j = 1, \ldots, r \):

1. \( \mathcal{F}_\Delta = A(\eta_0 > \cdots > \eta_r, \mathcal{F}) \). (this intentionally does not depend on \( j \))
2. \( \operatorname{colim}_{f \not\in \eta_i} \lim_{j_i \geq 1} \lim_{j_{i+1} \geq 1} \cdots \lim_{j_r \geq 1} A \left( \eta_{j_1} > \cdots > \eta_{j_r}, \frac{\mathcal{F}_{\eta_j} \otimes \mathcal{O}(f^{-\infty}) \otimes \cdots \otimes \mathcal{O}(f^{-\infty})}{\eta_{j_1} + \cdots + \eta_{j_r}} \right) \),
   where the denominator tacitly is to be understood as \((\eta_{j_1} + \cdots + \eta_{j_r}) \cdot (\text{numerator})\).
3. \( \operatorname{colim}_{f \not\in \eta_i} \lim_{j_i \geq 1} \lim_{j_{i+1} \geq 1} \cdots \lim_{j_r \not\in \eta_j} A \left( \eta_{j_1} > \cdots > \eta_{j_r}, \frac{\mathcal{F} \otimes \mathcal{O}(f^{-\infty}) \otimes \cdots \otimes \mathcal{O}(f^{-\infty})}{\eta_{j_1} + \cdots + \eta_{j_r}} \right) \),
   where the denominator tacitly is to be understood as \((\eta_{j_1} + \cdots + \eta_{j_r}) \cdot (\text{numerator})\).
4. \( \operatorname{colin}_{\ell=1} \operatorname{colin}_{\ell=1} \cdots \operatorname{colin}_{\ell=1} A \left( \eta_{j_1} > \cdots > \eta_{j_r}, \frac{L_{\ell-1}}{L_{\ell-1}} \right) \),
   where for all \( \ell = 1, \ldots, j \) the \( L_{\ell} \) run through all finitely generated \( \mathcal{O}_{\eta_i} \)-submodules of \( \frac{L_{\ell-1}}{L_{\ell-1}} \) (in case \( \ell > 1 \)) or \( \mathcal{F}_{\eta_i} \) (in case \( \ell = 1 \)) in ascending order; and the \( L_{\ell} \) run through all full rank finitely generated \( \mathcal{O}_{\eta_i} \)-submodules of \( L_{\ell} \) in descending order.

Statement (1) intentionally does not depend on the choice of \( j \). We merely use the numbering of the above statement as a guideline through the steps of the proof. Overall, we are just collecting a large number of different ways to express the same object.

Proof. Compare with [Bra14b]. First of all, recall that

\[ A(\eta_0 > \cdots > \eta_r, \mathcal{F}) = A(\eta_0 > \cdots > \eta_r, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F}, \]

and we see that it suffices to prove the claim for \( \mathcal{F} := \mathcal{O}_X \). The isomorphism of the objects in (2) and (3) is clear from the definition since \( \mathcal{F}_{\eta_i} \) will generally only be a quasi-coherent sheaf, see Equation 2.2. Next, we demonstrate the isomorphism between (2) and (4) for any fixed \( j \):
Suppose we are given $\ell \geq 1$. Define for any $O\langle f_{\ell-1}^{-\infty} \rangle$ in the $\ell$-th colimit and $i_{\ell} \geq 1$ in the $\ell$-th limit
\begin{equation}
L_{\ell} := O_{n_{\ell}}\text{-span of } O\langle f_{0}^{-\infty}, \ldots, f_{\ell-1}^{-\infty} \rangle \subseteq \frac{L_{\ell-1}}{L_{\ell-1}'} (\text{if } \ell > 1) \text{ or } O_{n_0} (\text{if } \ell = 1)
\end{equation}
\begin{equation}
L_{\ell}' := O_{n_{\ell}}\text{-span of } \eta_{1_{\ell}}^{i_{\ell}} + \cdots + \eta_{n_{\ell}}^{i_{\ell}} \subseteq \frac{L_{\ell-1}}{L_{\ell-1}'} (\text{if } \ell > 1) \text{ or } O_{n_0} (\text{if } \ell = 1).
\end{equation}
As $O\langle f_{\ell-1}^{-\infty} \rangle$ is a coherent sheaf by construction, cf. Definition 14, $L_{\ell}$ is a finitely generated $O_{n_{\ell}}$-module. The same is true for $L_{\ell}'$ and we clearly have $L_{\ell}' \subseteq L_{\ell}$. This shows that there is a morphism between the indexing sets of the limits/collimits in (2) to the indexing sets of the $L_{\ell}, L_{\ell}'$ in (4). Moreover, we unravel by induction
\begin{equation}
\frac{L_{\ell}}{L_{\ell}'} = \frac{O_{n_{\ell}} \cdot O\langle f_{0}^{-\infty} \rangle \otimes \cdots \otimes O\langle f_{\ell-1}^{-\infty} \rangle}{\eta_{1_{\ell}}^{i_{\ell}} + \cdots + \eta_{n_{\ell}}^{i_{\ell}}}
\end{equation}
(a quotient of sub-spaces of $\frac{L_{\ell-1}}{L_{\ell-1}'}$ for $\ell > 1$, or $O_{n_0}$ if $\ell = 1$).

We see that $A(\eta_{j+1} > \cdots > \eta_r, L_{\ell}/L_{\ell}')$ agrees with the $A(\cdot, \cdot)$ appearing in formulation (2). Summarized, the ind-pro limits of (2) define a sub-system of the ind-pro limits in (4), running over the same objects as in (2). Next, note that for all finitely generated $O_{n_{\ell}}$-submodules of $\frac{L_{\ell-1}}{L_{\ell-1}'}$ or $L_{\ell}$ we can lift generators from sub-quotients to rational functions, allowing us to form a co-final system within the ind-pro limits of (2). This implies that (4) is canonically isomorphic to (2). Now, prove the full claim by induction on $j$: We verify (1) $\equiv$ (2) in the special case $j = 1$ by hand. Now assume (3) for any given $j$. Then by unwinding the definition of $A(\eta_{j+1} > \cdots > \eta_r, -)$ we literally obtain (2) for $j + 1$. Since we already have proven (3) $\equiv$ (2) for all $j$, this sets up the entire induction along $j$, establishing our claim.

This result has a particularly nice consequence for flags of the maximal possible length:

**Corollary 33.** We keep the assumptions and notation as before, but additionally demand that $X$ is of finite type over a field $k$ and $r = n$. Then
\begin{equation}
A(\eta_0 > \cdots > \eta_n, F) = \colim_{L_{\ell} \in \{ L_{\ell-1} \}} \cdots \colim_{L_n \in \{ L_n \}} \frac{L_n}{L_{\ell-1}},
\end{equation}
where for all $\ell = 1, \ldots, n$ the $L_{\ell}$ run through all Beilinson lattices (for the flag $\eta_{\ell-1} > \cdots > \eta_n$) in
\begin{equation}
\frac{L_{\ell-1}}{L_{\ell-1}'} (\text{in case } \ell > 1) \quad \text{or} \quad F_{\eta_0} (\text{in case } \ell = 1)
\end{equation}
in ascending order; and the $L_{\ell}' \subseteq L_{\ell}$ run through all contained Beilinson lattices in descending order.

**Proof.** Just apply Lemma 52 in the special case $r = n$. \hfill \Box

In the formulation of the following lemma we shall employ the notation $\hat{(-)}$, which refers to omission here and not to completion or the like.

**Lemma 34.** We keep the assumptions and notation as before, still finite type over a field $k$.

1. Assume we are given finitely generated $O_{n_0}$-modules $M_1, M_2$. Then a $k$-vector space morphism $f \in \Hom_k(M_1, M_2)$ is an element of $\Hom_\Delta(M_1, M_2)$ if and only if
(a) one can provide a final and co-final collection of Beilinson lattices $L'_\ell \subseteq L_\ell$ of $M_1$, and $N_\ell \subseteq N'_\ell$ of $M_2$ (in either case for $\ell = 1, \ldots, n$) as in Corollary \ref{cor:beilinson-lattices}, such that

$$\frac{L_n}{L'_n} \to \frac{N_n}{N'_n}$$

inducing the map $f$ in the iterated Ind- and Pro-diagrams

$$f : M_1 \to M_2$$

$$\colim \cdots \colim \frac{L_n}{L'_n} \to \colim \cdots \colim \frac{N_n}{N'_n}.$$

(2) Suppose $f \in \text{Hom}_\Delta(M_1, M_2)$. Then $f \in I^+_\Delta(M_1, M_2)$ if and only if $f$ admits a factorization of the shape

$$\colim \cdots \colim \frac{L_n}{L'_n} \to \colim \cdots \colim \frac{N_n}{N'_n},$$

i.e. instead of a colimit running over all $N_i$, it factors through a fixed $N_i$ (depending only on $N_1, N'_1, \ldots, N_{i-1}, N'_{i-1}$).

(3) Similarly, $f \in I^-_\Delta(M_1, M_2)$ holds if and only if $f$ admits a factorization of the shape

$$\colim \cdots \colim \frac{L_n}{L'_n} \to \colim \cdots \colim \frac{N_n}{N'_n},$$

i.e. instead of having the limit run over all $L_i$, it vanishes on a fixed $L_i$ (depending only on $L_1, L'_1, \ldots, L_{i-1}, L'_{i-1}$).

**Proof.** In view of Cor. \ref{cor:beilinson-lattices} this follows rather straight-forwardly from Beilinson’s Definition \ref{def:beilinson-lattices}. For (1): Once $f \in \text{Hom}_\Delta(M_1, M_2)$ holds true for a $k$-linear map $f$, Definition \ref{def:beilinson-lattices} allows us to produce many such factorizations; firstly over

$$\frac{L_1}{L'_1} \to \frac{N_1}{N'_1},$$

(for any prescribed $L_1$ and $N'_1$) and then inductively further down the flag $\Delta$. Conversely, given such factorizations, they clearly define a $k$-linear map and the condition of Definition \ref{def:beilinson-lattices} follows from the map being of this shape. (2) and (3) follow just from unravelling Beilinson’s definition in view of Cor. \ref{cor:beilinson-lattices} and the fact that all $L_\ell, L'_\ell$ (for all $\ell = 1, \ldots, n$) are Beilinson lattices. □

**Proposition 35.** For $\Delta = \{(\eta_0 > \cdots > \eta_n)\}$ and $\mathcal{F}$ a coherent sheaf, the presentation of Corollary \ref{cor:beilinson-lattices}

$$\mathcal{F}_\Delta = \colim \cdots \colim \frac{L_n}{L'_n},$$

also equips $\mathcal{F}_\Delta$ with a canonical structure as an $n$-Tate object in $ST$ $k$-modules (with their exact structure, Prop. \ref{prop:tate-structures}). Or, executing the colimits and limits, as an $ST$ $k$-module itself.

**Proof.** We only need to know that the transition maps of the Ind- and Pro-diagrams are admissible monics and epics. This was already shown by Yekutieli, albeit in a slightly different language \cite{yekutieli2015} Lemma 4.3, (2) and (4)]. For the second claim, we only need to know that the respective limits and colimits exist in ST modules; this is \cite{yekutieli2015} Lemma 4.3, (3) and (6). □
Theorem 11 (Structure Theorem III). Suppose $X$ is a purely $n$-dimensional reduced scheme of finite type over a field $k$ and $\Delta = \{(\eta_0 > \cdots > \eta_n)\}$ a saturated flag. Then each direct summand of the upper-left object in Diagram 3.9 of Theorem 11 carries a canonical structure

1. of $n$-local fields,
2. of Tate objects in $n$-Tate($\text{Ab}$), i.e. with values in abelian groups,
3. of Tate objects in $n$-Tate($\text{Vect}_f$), i.e. with values in finite-dimensional $k$-vector spaces,
4. of $k$-algebras,
5. (if $k$ is perfect) of topological $n$-local fields in the sense of Yekutieli,

and one can find (non-canonically) a simultaneous field and $n$-Tate($\text{Ab}$) isomorphism to a multiple Laurent series field

$$\kappa((t_1)) \cdots ((t_n))$$

$$\kappa((t_1)) \cdots [t_n] \xrightarrow{\kappa((t_1)) \cdots ((t_{n-1})} \kappa((t_1)) \cdots ((t_{n-2}))$$

with its standard field and $n$-Tate($\text{Ab}$) structure. Here $\kappa/k$ is a finite field extension. Moreover,

1. Beilinson’s notion of lattices, and
2. the notion of Tate lattices with respect to the $n$-Tate structure in $n$-Tate($\text{Ab}$),

sandwich each other (but usually do not agree).

If $k$ is perfect, one can find such an isomorphism which is additionally also an isomorphism of $k$-algebras, objects in $n$-Tate($\text{Vect}_f$), $n$-dimensional TLFs. In this case, even

1. Beilinson’s notion of lattices,
2. Yekutieli’s notion of lattices with respect to the TLF structure,
3. Tate lattices with respect to the $n$-Tate structure in $n$-Tate($\text{Vect}_f$),

all pairwise sandwich each other (but usually do not agree).

If one is happy with plain field isomorphisms without extra structure, this is of course part of the original results of Parshin and Beilinson. The construction and very definition of the canonical TLF structure/ST module structure is due to Yekutieli [Yek92, Yek15]. However, we know from Example 23 going back to Yekutieli’s work, that a general field isomorphism will not preserve this structure, and from its variation Example 24 that it would also not preserve the $n$-Tate structure.

For the sake of clarifying the implications of this, let us state a particular consequence separately:

**Corollary 36.** We keep the assumptions as before, and suppose that $k$ is perfect. For each field factor $K_i$ of $A(\Delta, \mathcal{O}_X) = \prod K_i$, if $e$ denotes the idempotent cutting it out, we can find a $k$-algebra isomorphism of fields

$$e \colim_{L_i} \cdots \colim_{L_n} L_i \rightarrow \prod_{L_i} L_n (\text{cf. Cor. 33})$$

$$= \colim_{t_n} \cdots \colim_{t_1} t_n \cdots t_1 k[t_1, \ldots, t_n]/(t_1^j, \ldots, t_n^j),$$
so that for each colimit index \( L_s \) (resp. limit \( L'_s \)) in the first row, it can be sandwiched between \( i_s \) (resp. \( j_s \)) in the second row, and reversely for the second row.

**Proof of Theorem 11.** We prove this by several separate inductions. Firstly, some general remarks: The argument will deal with objects which are simultaneously objects in a Tate category, ST module category and rings. Here the word 'simultaneously' refers to the fact that we deal with objects in these different categories which, under the natural forgetful functors or evaluation-of-limits functors agree.

- If \( k \) is perfect, we work in the categories of \( k \)-algebras, ST modules and Tate objects of finite-dimensional \( k \)-vector spaces, as a shorthand
  \[
  n\text{-Tate} := n\text{-Tate}(\text{Vect}_f).
  \]

- If \( k \) is not perfect, we work in the categories of rings and Tate objects of all abelian groups, let
  \[
  n\text{-Tate} := n\text{-Tate}(\text{Ab}).
  \]
  In this case, simply ignore all statements about \( k \)-algebra structures, \( k \)-vector space structures or ST module structures in the proof below.

(Step 1) In the first step, we work by induction along \( j \), starting from \( j = n \) and going downwards. Firstly, observe that

\[
C_j := \lim_{\longrightarrow} \lim_{i > j} \lim_{i \geq 1} \eta_{j-1} + \eta_j + \cdots + \eta_n
\]

is

1. (as a ring, \( k \)-algebra) an \( \eta_j \)-adically complete semi-local ring with Jacobson radical \( \eta_j \) and minimal primes all lying over \( \eta_j \). It follows that \( C_j \) is one-dimensional. The morphism

\[
q : C_j \to C_j/\eta_j
\]

is the quotient map to \( C_j/\eta_j \), a reduced Artinian ring.

2. (as a Tate object) an object in \( \text{Pro}^n((n-j)\text{-Tate}) \). In this category \( q \) is an admissible epic, since it is the natural mapping from a Pro-diagram to one of its entries.

3. (as ST modules) defining an object in Yekutieli’s category of ST modules. Equip the inner term with the fine ST module structure. (Much like in Example 22) each limit is equipped with its limit topology, resulting again in an ST module [Yek92, Lemma 1.2.19], and equip the colimits, which are localizations, with the fine topology over the ring we are localizing (or equivalently with the colimit topology [Yek92, Cor. 1.2.6]); this makes them ST rings again. Then \( q \) is an admissible epic in ST modules and (equivalently) induces the quotient topology by [Yek15, Lemma 4.3].

Next, by comparison of definitions, we literally find

\[
C_j = A_X(\eta_j \cdots \eta_j, O_X)/\eta_j - 1
\]

so that these rings agree with those of the lower downward diagonal in Theorem 10. In order to work with the rings of integers of their total rings of quotients, we replace \( C_j \) by its normalization \( C'_j \); as also described in Theorem 9 (1). This is a finite ring extension/k-algebra extension (since \( C_j \) is excellent). It is a finite product of complete discrete valuation rings, say

\[
C'_j = \prod O_t
\]

with residue fields \( \kappa_t := O_t/m_t \).

and by the finiteness of normalization each \( \kappa_t \) is finite over \( C_j/\eta_j \). A direct computation reveals that \( C_j/\eta_j = \text{Quot}(C_{j+1}) \). This is true as rings, as \( k \)-algebras, and it also holds as Tate objects
or ST modules — literally from the definition of the Tate object structure/topology on $C_j$. By the direction of our induction in $j$, we have already performed the entire above analysis for $C_{j+1}$. Assembling the pieces of the induction steps, we arrive at diagrams like

(1) (as rings, $k$-algebras) the upward dotted arrows are always the inclusion into the total ring of quotients; these maps are injective. In the case of the unbent dotted arrow it is additionally a product of the inclusions of the discrete valuation rings $\mathcal{O}$ into their field of fractions. The maps denoted by $\gamma$ are normalizations; the integral closure in the total ring of quotients. The dashed upward arrows are products of finite field extensions. Each quotient $C_j/\eta_j$ is itself a product of fields.

(2) (as Tate objects, ST modules) the upward bent arrows are admissible monics in Tate objects since they are the inclusion of an entry of an admissible Ind-diagram into the Ind-object defined by this diagram. Analogously, an admissible monic in ST modules for essentially the same reason, just with the colimit carried out.

(Step 2) Now we again work by induction, but this time starting on the upper left and going downwards. Consider the left-most upward arrow $\prod \mathcal{O} \to \prod \kappa$ in the above Figure. By the dashed arrow, a product of finite field extensions, these maps will usually not be the inclusion of rings of integers into their field of fractions. However, we may replace the $\mathcal{O}$ by the integral closure, call it $\mathcal{O}^*$, in these bigger fields (as before, we use the same notation for perhaps different factors to avoid an index overkill). Since the $\mathcal{O}$ are complete discrete valuation rings, the $\mathcal{O}^*$ are also complete discrete valuation rings, cf. Lemma A.11 (there can only be one factor
since we are inside a field). We write $\kappa^*$ for their residue fields so that $\kappa^*/\kappa$ is a finite field extension. This leads to

\[
\prod \mathcal{O}^* \to \prod \kappa^*
\]

\[
\prod \mathcal{O} \to \prod \kappa
\]

\[
C_j \to C_j/\eta_j
\]

\[
C_{j+1} \to C_{j+1}/\eta_{j+1}
\]

Now proceed by induction to the right. Instead of looking at the upward arrow $\prod \mathcal{O} \to \prod \kappa$ in the second column, consider the composition all up along $\prod \mathcal{O} \to \prod \kappa^*$. Since both dashed arrows are products of finite field extensions, this is the same situation as for the previous step. Proceed this way all down the lower right.

\[
\prod \mathcal{O} \to \prod \kappa
\]

\[
\gamma
\]

\[
C_j \to C_j/\eta_j
\]

\[
C_{j+1} \to C_{j+1}/\eta_{j+1}
\]

\[
(\text{Step 3}) \text{ Now delete all objects from the diagram apart from the } \prod \mathcal{O}^* \cdot \cdot \cdot \text{ and } \prod \kappa^* \cdot \cdot \cdot \text{'s. We arrive at a finite product of staircase diagrams of genuine } n\text{-local fields, and moreover we see that we have finite ring extensions } C_j \to \prod \mathcal{O}^* \cdot \cdot \cdot \text{. We will focus on only one factor of this product, leaving only the staircase diagram of a single } n\text{-local field along with the finite ring extensions from the } C_j. \text{ After re-naming the objects following our convention for higher local}
\]
fields, cf. Equation 3.11 we arrive at the diagram

\[
\begin{array}{c}
K \\
O_1 \to k_1 \\
\Downarrow \\
C_1 \\
\Downarrow \\
O_2 \to k_2 \\
\Downarrow \\
C_2 \\
\end{array}
\]

While it is outside the general pattern, it can easily be shown that we also have a finite ring map \( C_0 \to K \); in fact this is the projection on a direct factor of the ring \( C_0 \). Since \( K \) is an \( n \)-local field, the \( k_j \) are \((n-j)\)-local fields and \( O_{i+1} \) their first rings of integers.

(1) (as Tate objects) Now \( k_0 := K \), as a factor of \( C_0 \), is an \( n \)-Tate object and inductively \( O_{j+1} \) and its maximal ideal \( m \subset O_{j+1} \) are Tate lattices in \( k_j \), and the quotient \( O_{j+1}/m = k_{j+1} \) is an \((n-j)\)-Tate object. So all the \( O_j \) are objects in \( \text{Pro}^{(n-j)\text{-Tate}} \), and by sandwiching

\[
(3.12) \quad m^N O_j \subseteq \eta_j O_j \subseteq m O_j
\]

the morphism \( C_j \to O_j \) turns out to come from a morphism of Pro-diagrams and thus the \( C_j \to O_j \) are all morphisms in \( \text{Pro}^{(n-j)\text{-Tate}} \) as well.

(2) (ST modules) Moreover, if \( k \) is perfect, \( k_0 = K \), as a factor of \( C_0 \), is an ST \( k \)-module. This ST module structure on \( C_0 \) is precisely the one employed by Yekutieli, see [Yek15, §6] for a survey, or [Yek92, Definition 3.2.1] and [Yek92, Prop. 3.2.4] for details. This renders all \( k_j \) and \( O_j \) ST modules by the sub-space and quotient topologies. By Equation 3.12 and [Yek92, Prop. 1.2.20] the morphism \( C_j \to O_j \) is a morphism of ST modules.

\textbf{(Step 4)} Next, we work by induction, starting from \( j = n \) again and working downward. Assume we have constructed and fixed an isomorphism

\[
\phi_j : k'((t_n)((t_{n-1})) \cdots ((t_{j+1})) \sim k_j
\]

for some field \( k' \), simultaneously in the categories of rings, \( k \)-algebras, \((n-j)\)-Tate objects, ST modules. Since \( O_j \) is an equicharacteristic complete discrete valuation ring with residue field \( k_j \), Cohen’s Structure Theorem allows us to pick a coefficient field isomorphic to \( k_j \) in \( O_j \), write \([-]_* : k_j \to O_j \), and thus get an isomorphism of rings

\[
\psi : k_j[[t_j]] \to O_j
\]

\[
\sum_s a_s t_j^s \mapsto \text{evaluate } \sum_s [a_s]_* t_j^s
\]

with \( a_s \in k_j \) and \( t_j \) some (arbitrary) uniformizer of \( O_j \). If \( k \) is perfect, we can assume to have picked the coefficient field as a sub-\( k \)-algebra and so that \( \psi \) is a \( k \)-algebra isomorphism. Otherwise, we must content ourselves with a ring isomorphism in general. Rewrite this morphism as

\[
\psi : \varprojlim_{i, j} k_j[[t_j]]/(t_j^{i+j}) \to \varprojlim_{i, j} O_j/m^{i+j}
\]

\[
\sum a_s t_j^s \mapsto \text{evaluate } \sum_s [a_s]_* t_j^s.
\]
Now, if we can produce an entry-wise isomorphism between the Pro-diagrams defined by either side of the morphism, and these are objects in a category $\mathcal{C}$, this defines an isomorphism in $\text{Pro}^a(\mathcal{C})$. However, via $\phi_j$ this can be achieved:
\[
k'(t_n)\cdots((t_j+1)) (t_j/((t_j)^j) \rightarrow O_j/m^j
\]
sending
\[
\sum_s a_s t^s_j \mapsto \text{evaluate } \sum_s [\phi_j(a_s)] t^s_j.
\]
Since each $\phi_j$ induces an isomorphism of $(n-j)$-Tate objects, this defines a straight morphism of Pro-diagrams and thus an isomorphism of objects in $\text{Pro}^a((n-j)\text{-Tate})$, and by an analogous procedure for the colimits, an isomorphism in $(n-j+1)$-Tate. In rings or $k$-algebras, this colimit is just the localization, so we get the field of fractions $k_{j-1}$ of $O_j$. Thus, we have constructed
\[
\phi_{j-1} : k'(t_n)\cdots((t_j+1)) (t_j) \sim k_{j-1}
\]
both in rings/$k$-algebras as well as Tate objects. Carrying out the Pro-limit and equipping it with the limit topology, respectively the colimit topology for the colimit, we also obtain that $\phi_{j-1}$ is a morphism of ST modules. Finally, once the entire induction is done, we obtain a Tate object and ST module isomorphism between $K$ and the multiple Laurent series. If $K$ is perfect, this produces a parametrization of the $n$-local field and thus gives an alternative proof that $K$ is a TLF (see Definition 7). Moreover, this turns $\phi_0$ into an isomorphism of TLFs. Finally, since
\[
C_0 = A(\triangle, O_X),
\]
this proves all our claims except the assertions about lattices, which we prove in the following propositions. □

**Proposition 37.** Below, statements about Yekutieli lattices need the assumption that $k$ be perfect. For $\triangle = \{\eta_0 > \cdots > \eta_n\}$ and $\mathcal{F}$ a coherent sheaf,

1. For $\triangle' := \{\eta_1 > \cdots > \eta_n\}$ defined as usual, and $L \subseteq F_m$ a Beilinson lattice,
   
   a) $L_{\triangle'} \subseteq F_{\triangle}$ is a Tate lattice (Def. 4) if we read $F_{\triangle}$ as its underlying $n$-Tate object;
   
   b) each $O_i, L_{\triangle'} \subseteq K_i \subseteq F_{\triangle}$ is a Yekutieli lattice (Def. 9) if we read $F_{\triangle}$ with the limits/colimits carried out. Here $O_i$ and $K_i$ refer to the factors of the normalization of $O_{\triangle'}$, as in Theorem 9.

2. ("Beilinson lattices are final and co-final") For every Tate (resp. Yekutieli) lattice $T$, there exist Beilinson lattices $L_1 \subseteq L_2$ such that
\[
L_{1\triangle'} \subseteq T \subseteq L_{2\triangle'}.
\]

**Proof.** (1.a) Note that by Corollary 33 the initial colimit is over all Beilinson lattices, so our $L$ can be regarded as an object in this outer-most Ind-diagram. Thus, it is itself of the shape
\[
\lim_{L'_1} \cdots \text{colim} \lim_{L_n} L_n \xrightarrow{L_n} L_n
\]
and thus a Pro-subobject. The quotient by it explicitly has the shape
\[
\text{colim} \lim_{L_1} \cdots \text{colim} \lim_{L_n} L_n \xrightarrow{L_n} L_n
\]
and this is visibly an Ind-quotient in the outer-most Tate category. Thus, \( L_{\Delta'} \) is a Tate lattice.

(1.b) We have

\[ L_{\Delta'} \subseteq \mathcal{F}_{\Delta}. \]

Moreover, the \( \mathcal{O}_{\eta_1} \)-module structure of the Beilinson lattice induces

\[ \mathcal{O}_{\eta_1} \otimes L \rightarrow L \]

(3.13)

\[ (\mathcal{O}_{\eta_1})_{\Delta'} \otimes L_{\Delta'} \rightarrow L_{\Delta'}. \]

Since the maximal ideals of \( \mathcal{O}_{\Delta'} \) lie over \( \eta_1 \), we have \( (\mathcal{O}_{\eta_1})_{\Delta'} = \mathcal{O}_{\Delta'}. \) This makes \( L_{\Delta'} \), a finitely generated \( \mathcal{O}_{\Delta'} \)-module. By Theorem [3] the normalization \((-)'\) of \( \mathcal{O}_{\Delta'} \) satisfies \( (\mathcal{O}_{\Delta'})' = \prod \mathcal{O}_i \subseteq \prod K_i = \mathcal{O}_{\Delta}. \) Let \( e \) be the idempotent cutting out \( K_i \) from \( \mathcal{O}_{\Delta} \), and then also \( \mathcal{O}_i \) from \( (\mathcal{O}_{\Delta'})' \). Inside \( \mathcal{F}_{\Delta} \), we can take \( \mathcal{O}_{\Delta} \)-spans of elements; in particular, \( e(\mathcal{O}_i \cdot L_{\Delta'}) \) defines a finitely generated \( \mathcal{O}_i \)-submodule of \( K_i \). As \( L \) was a Beilinson lattice, we have

\[ \mathcal{O}_{\eta_0} \cdot L = \mathcal{F}_{\eta_0} \]

and as in Equation [3.13] this implies

\[ (\mathcal{O}_{\eta_0})_{\Delta'} \cdot L_{\Delta'} = (\mathcal{F}_{\eta_0})_{\Delta'}, \]

but the maximal ideals of the \( (\mathcal{O}_{\eta_1})_{\Delta'} \)-module structure of \( L_{\Delta'} \) all lie over \( \eta_1 \), so as the localization at \( \eta_0 \) inverts this, it follows that \( (\mathcal{O}_{\eta_0})_{\Delta'} = \mathcal{O}_{\Delta} \) and \( (\mathcal{F}_{\eta_0})_{\Delta'} = \mathcal{F}_{\Delta} \). Thus, \( \mathcal{O}_{\Delta} \cdot L_{\Delta'} = \mathcal{F}_{\Delta} \) and therefore

\[ e\mathcal{F}_{\Delta} = e(\mathcal{O}_{\Delta} \cdot L_{\Delta'}) \subseteq e(\mathcal{O}_i \cdot L_{\Delta'}) \subseteq e\mathcal{F}_{\Delta}. \]

It follows that \( \mathcal{O}_i \cdot L_{\Delta'} \subseteq K_i \) is a Yekutieli lattice.

(2) For Yekutieli lattices, the claim follows from Cor. [36]. Each entry in

\[ \text{colim}_{i_n} \lim_{j_n} \cdots \lim_{j_1} \lim_{i_1} \frac{1}{t_1^{i_1} \cdots t_n^{i_n}} k[t_1, \ldots, t_n]/(t_1^{i_1}, \ldots, t_n^{i_n}) \otimes \mathcal{F} \]

(3.14)

for fixed \( i_n \) defines a Yekutieli lattice under evaluation, namely

\[ \frac{1}{t_1^{i_1} \cdots t_n^{i_n}} k((t_1)) \cdots ((t_{n-1}))[[t_n]] \otimes \mathcal{F}. \]

Now use that lattices of this shape are final and co-final in all Yekutieli lattices (Yekutieli calls these lattices ‘standard’, see [Yek15, Example 4.2]). Thus, it suffices to show that the lattices in Equation [3.14] are final and co-final in all Beilinson lattices, but this is precisely Cor. [36].

For Tate lattices, the corresponding claim is proven in [BGW15a, Lemma 23].

**Theorem 12.** Suppose \( X \) is a purely \( n \)-dimensional reduced scheme of finite type over a field \( k \) and \( \Delta = \{(\eta_0 > \cdots > \eta_n)\} \) a saturated flag.

1. ([BGW15a, Theorem 5]) There is a canonical isomorphism of \( n \)-fold cubical algebras

\[ E^{Tate}(\mathcal{O}_{X\Delta}) \sim E^{Beil}_{\Delta}. \]

2. Suppose \( k \) is perfect. Then for each field factor \( K \) in \( \mathcal{O}_{X\Delta} = \prod K \), cut out by the idempotent \( e \in E^{Beil}_{\Delta} \), there are canonical isomorphisms of \( n \)-fold cubical algebras

\[ E^{Tate}(K) \sim e E^{Beil}_{\Delta} \sim e E^{Yek}(K). \]

**Proof.** (1) See [BGW15a, Theorem 5].

(2) Let us compare Beilinson’s and Yekutieli’s approaches. Take any system of liftings in Yekutieli’s sense, so that for finitely generated \( \mathcal{O}_{\eta_1} \)-modules \( M_1, M_2 \), the adèle completions, cut down to the factor \( K \) of \( \mathcal{O}_{X\Delta} \), i.e. \( eM_{1\Delta} \) and \( eM_{2\Delta} \) are finitely generated \( e\mathcal{O}_{X\Delta} \)-modules, i.e. finitely generated \( K \)-modules. Then

\[ E^{Yek}_e(eM_{1\Delta}, eM_{2\Delta}) \subseteq \text{Hom}_k(eM_{1\Delta}, eM_{2\Delta}). \]
By Yekutieli’s Theorem\footnote{Yekutieli,蹉跎.} the choice of \( \sigma \) does not matter. In Beilinson’s approach

\[
E_{\Delta}^{\text{Beil}} = \text{Hom}_{\Delta}(M_1, M_2) \subseteq \text{Hom}_k(M_1, M_2).
\]

We leave it to the reader to check that multiplication with \( e \in \mathcal{O}_{X\Delta} \) also defines an endomorphism \( e \in E_{\Delta}^{\text{Beil}} \); one just has to confirm that \( e \) behaves according to the axioms with respect to Beilinson lattices. Then

\[
e \text{Hom}_k(M_1, M_2)e = \text{Hom}_k(eM_1, eM_2),
\]

so that \( eE_{\Delta}^{\text{Beil}}e \) and \( E_{\Delta}^{\text{Yek}}(eM_1, eM_2) \) are both \( k \)-subspaces of \( \text{Hom}_k(eM_1, eM_2) \), so we can produce an isomorphism by just showing that they are equal. However, they are both defined by (exactly the same) properties with respect to Beilinson resp. Yekutieli lattices and by Prop. \ref{prop:beilinson-lattices} these lattices sandwich each other. The defining properties just demand the existence of lattices smaller or larger than others, so that the truth of these conditions is unaffected by the sandwiching procedure. We give an example verification: Suppose some \( \varphi \) is bounded in Beilinson’s sense, i.e.

\[
\varphi(-) \subseteq L_{\Delta^i}
\]

for some Beilinson lattice. By Prop. \ref{prop:beilinson-lattices} \( \mathcal{O}_i \cdot L_{\Delta^i} \) is actually a Yekutieli lattice, so it is also Yekutieli bounded. Conversely, if \( \varphi(-) \subseteq \hat{L} \) for a Yekutieli lattice \( \hat{L} \), then by co-finality there exists a Beilinson lattice \( L \) so that \( \hat{L} \subseteq L_{\Delta^i} \) and the latter verifies that \( \varphi \) is also bounded in Beilinson’s sense. This argument works for all \( I^i_n \), \( i = 1, \ldots, n \). The ideals \( I^i_n \) can be dealt with analogously; this time using Prop. \ref{prop:beilinson-lattices} to find Beilinson lattices \( L \) with \( L_{\Delta^i} \subseteq \hat{L} \). The algebra multiplication is composition in either case, so it also agrees trivially. \( \square \)

We can use Theorem\ref{thm:geometric-analytic} to obtain a formulation ‘in coordinates’:

**Definition 15.** Suppose \( (A, \{I^i_n\}_{i=1,\ldots,n}) \) is a Beilinson \( n \)-fold cubical algebra. A system of good idempotents consists of elements \( P^+_i \in A \) with \( i = 1, \ldots, n \) such that the following conditions are met:

\begin{itemize}
  \item \( [P^+_i, P^+_j] = 0 \), \quad \text{(pairwise commutativity)}
  \item \( P^+_i P^+_j = P^+_i \)
  \item \( P^+_i A \subseteq I^+_i \)
  \item \( P^+_i A \subseteq I^+_i \) \quad \text{(and we define } P^-_i := 1_A - P^+_i \).
\end{itemize}

This definition originates from \cite[Def. 14]{Bra14a}.

**Proposition 38.** Let \( X/k \) be a reduced finite type scheme of pure dimension \( n \) over a perfect field \( k \). If \( \Delta \) is a saturated flag of points and \( K \) a field factor in

\[
\mathcal{O}_{X\Delta} = \prod_m K_m,
\]

then an isomorphism

\[
K \simeq (t_n)((t_n-1)) \cdots ((t_1)) \quad \text{with} \quad [\kappa : k] < \infty
\]

as in Theorem\ref{thm:geometric-analytic} can be chosen so that for \( f \in E^{\text{Beil}}(K) \) we have the following characterization of the ideals:

\begin{enumerate}
  \item \( f \in I^+_i \) holds iff for all choices of \( e_1, \ldots, e_{i-1} \in \mathbb{Z} \) there exists some \( e_i \in \mathbb{Z} \) such that instead of needing to run over the \( i \)-th colimit in

\[
\text{im}(f) \subseteq \left\{ \lim\colim_{e_j} \cdots \lim\colim_{e_1} \right\}.
\]

\end{enumerate}

it can, as indicated by the omission symbol \( (\widetilde{-}) \), be replaced by this index \( e_i \).
(2) $f \in I_i^+$ holds iff for all $e_1, \ldots, e_{i-1} \in \mathbb{Z}$ there exists $e_i \in \mathbb{Z}$ so that the $i$-th colimit can be replaced, as in

$$\left\{ \text{colim}_{e_1} \cdots \text{colim}_{j_1} \cdots \text{colim}_{e_n} \sum_{j_1 = e_1, \ldots, e_n = e_n}^{j_1 - 1, \ldots, j_n - 1} a_{e_1 \ldots e_n} t_{j_1}^{a_1} \cdots t_{j_n}^{a_n} \right\} \subseteq \ker(f),$$

by the index $e_i$.

(3) Fix such isomorphisms for all field factors $K_m$ in Equation (3.17). Denote by $\kappa_m$ the last residue field of $K_m$. If we define the $\kappa_m$-linear maps

$$m P_i^+ \sum_{a_{t_1} \ldots a_n} t_{j_1}^{a_1} \cdots t_{j_n}^{a_n} = \sum_{a_i \geq 0} a_{t_1} \cdots a_n t_{j_1}^{a_1} \cdots t_{j_n}^{a_n} \quad (1 \leq i \leq n)$$

on the right-hand side in Equation (3.17) for each field factor $K_m$, then the aforementioned isomorphisms equip $O_{X_r}$ with a system of good idempotents.

$$P_i^+: O_{X_r} \rightarrow O_{X_r}$$

$$\prod_{m=1}^m K_m \rightarrow \prod_{m=1}^m K_m$$

$$(x_1, \ldots, x_w) \mapsto (1 P_i^+ x_1, \ldots, w P_i^+ x_w).$$

We stress that (3) would not be true for a randomly chosen field isomorphism in Equation (3.16).

**Proof.** (1) + (2): This is just unravelling properties that we have already established by now. By Lemma 3.2 we know that $f \in I_i^+(K, K)$ holds if and only if $f$ admits a factorization

$$\text{colim}_{e_1} \cdots \text{colim}_{j_1} \text{colim}_{e_n} \sum_{j_1 = e_1, \ldots, e_n = e_n}^{j_1 - 1, \ldots, j_n - 1} \frac{1}{t_{j_1}^{a_1} \cdots t_{j_n}^{a_n}} \kappa(t_1, \ldots, t_n)/(t_{j_1}^{a_1} \cdots t_{j_n}^{a_n})$$

Now, as the image factors through some fixed $N_i$ in Equation (3.17), this is equivalent to factoring over some fixed $e_i \in \mathbb{Z}$. Stated along with its dependencies on the other indices this becomes: For all $e_1, \ldots, e_{i-1} \in \mathbb{Z}$, there exists $e_i \in \mathbb{Z}$, so that

$$a_{e_1 \ldots e_n} t_{j_1}^{a_1} \cdots t_{j_n}^{a_n}.$$
4. Different types of lattices

We remain in the situation of the preceding section. Suppose we look at some flag of points \( \Delta = \{(\eta_0 > \cdots > \eta_n)\} \) in a scheme \( X \). We have seen in Prop. 37 that for any Beilinson lattice \( L \) in \( M \) we can produce a Tate lattice \( L_{\Delta'} \) in \( M_{\Delta} \). Does the converse hold?

It does not, and it is indeed very easy to find counterexamples. For example, if \( L \) is a Beilinson lattice, it is by definition an \( O_{\eta_1} \)-module. As a result,

\[
O_{\Delta'} \otimes L_{\Delta'} \longrightarrow L_{\Delta'}
\]
defines an \( O_{\Delta'} \)-module structure on \( L_{\Delta'} \). But Tate lattices have no reason to carry any module structure at all. For example, let \( x_1, \ldots, x_r \) an arbitrary family of elements in \( O_{\Delta} \), some ‘noise’. Then if \( L \subseteq O_{\Delta} \) is a Tate lattice, so is \( L + k \langle x_1, \ldots, x_r \rangle \subseteq O_{\Delta} \). This is true for the simple reason that adding or quotienting out some finite-dimensional vector space will not affect being a Pro- or Ind-object inside \( \text{Tate}^f(\text{Vect}_f) \). This shows that a general Tate lattice need not come from a Beilinson or Yekutieli lattice. The rest of this section will be devoted to discussing a more sophisticated example, where a Tate and Yekutieli lattice does carry (the natural!) module structure, but still does not come from a Beilinson lattice.

Consider the affine 2-space \( A^2 = \text{Spec} k[s, t] \) and the singleton flag \( \Delta := \{((0) > (s^2 - t^3)) > (s, t))\} \). For the sake of brevity, we employ the shorthand

\[
A_j := A(\eta_j > \cdots > \eta_2, O_{A^2}) \in \text{Rings},
\]

(we had already used this notation earlier; cf. Definition 13) and we regard these only as commutative rings for the moment. We compute

\[
A_2 = k[[s, t]] \\
A_1 = \lim_{j} k[[s, t]] \left[ (k[s, t]_{(s,t)} - (s^2 - t^3))^{-1} \right] / (s^2 - t^3)^j.
\]

To understand \( A_1 \) as a ring, note that \( k[[s, t]] \) is a 2-dimensional regular local domain. Already inverting only \( t \) removes the maximal ideal, so that \( k[[s, t]][t^{-1}] \) is a 1-dimensional regular domain — since \( k[[s, t]] \) is regular, it is factorial, and so all height one primes are principal. Therefore, \( k[[s, t]][t^{-1}] \) is actually a principal ideal domain. Hence, \( k[[s, t]] \left[ (k[s, t]_{(s,t)} - (s^2 - t^3))^{-1} \right] \) is a localization thereof, and thus itself a principal ideal domain. The ideal \( (s^2 - t^3) \) is then necessarily maximal, thus completing at this ideal yields a regular complete local ring of dimension one, i.e. a discrete valuation ring. Hence, by Cohen’s Structure Theorem (cf. Proposition 4) there exists an isomorphism \( A_1 \simeq \mathcal{X}[[w]] \) with

\[
\mathcal{X} := A_1/(s^2 - t^3) = k[[s, t]]/(s^2 - t^3) \left[ (k[s, t]_{(s^2-t^3)} - (0))^{-1} \right] / (0)^j,
\]

where the overline denotes that we refer to the images of these elements after taking the quotient by \( (s^2 - t^3) \). Thus, \( \mathcal{X} = \text{Frac} k[[s, t]]/(s^2 - t^3) \). Next,

\[
A_0 = \lim_{j} A_1 \left[ (k[s, t]_{(s^2-t^3)} - (0))^{-1} \right] / (0)^j,
\]
so this is just the field of fractions of $A_1$. We therefore could draw a diagram (except for the $k[[u]]$ entry, which will be constructed only below)

$$
\begin{array}{c}
A_0 \\
\uparrow \\
A_1 \longrightarrow A_1/(s^2-t^3) \\
\uparrow \\
k[[u]] \\
\uparrow \\
A_2/(s^2-t^3) \longrightarrow A_2/(s,t).
\end{array}
$$

The upper-right diagonal entries are fields, the lower-left diagonal entries are one-dimensional local domains, the upward arrows are localizations, and the rightward arrows quotients by the respective maximal ideals. Note that $A_2/(s^2-t^3)$ is the completed local ring of the standard cusp singularity. In particular, it is not a normal ring. The famous integral closure inside the field of fractions is $k[[u]]$ via the inclusion $t \mapsto u^2$, $s \mapsto u^3$. In particular, $\mathcal{L} := A_1/(s^2-t^3) \simeq k((u))$ since $\frac{1}{u} = \frac{w}{w^3} = u$ and $t$ is already a unit in $A_1$ as we had discussed above. In particular, after these isomorphisms we may rephrase the previous diagram in the shape

$$
\begin{array}{c}
k((u))((w)) \\
\uparrow \\
k((u))[[w]] \longrightarrow k((u)) \\
\uparrow \\
k[[u]] \\
\uparrow \\
k[[s,t]]/(s^2-t^3) \longrightarrow k.
\end{array}
$$

If we follow Beilinson’s definition of a lattice, Definition 11, the lattices in $\mathcal{O}_{(0)} = k(s,t)$ are finitely generated $k[s,t]_{(s^2-t^3)}$-submodules $L \subseteq k(s,t)$ so that $k(s,t) \cdot L = k(s,t)$. A quotient of such, say $L'_1 \subseteq L_1$, would be, for example,

$$
\frac{L_1}{L'_1} = \frac{k[s,t]_{(s^2-t^3)}}{(s^2-t^3)^N \cdot k[s,t]_{(s^2-t^3)}} = \left( \frac{L_1}{L'_1} \right)_{\triangle'} = \frac{k((u))[[w]]}{w^N \cdot k((u))[[w]]},
$$

where $\triangle' = (s^2-t^3) > (s,t)$ and $N \geq 0$ some integer. Now, the Beilinson lattices inside $L_1/L'_1$ are $k[s,t]_{(s,t)}$-modules, for example,

$$
(t^p \cdot k[s,t]_{(s,t)})_{\triangle''} \equiv (u^{2p} \cdot k[u^2,u^3]_{(u)})_{\triangle''} \equiv u^{2p} \cdot k[[u,w]] \subset \left( \frac{L_1}{L'_1} \right)_{\triangle'}.
$$

Any Beilinson lattice $\mathcal{L} \subseteq L_1/L'_1$ is generated by polynomials in the variables $s,t$, and thus after applying $(\cdot)_{\triangle''}$ is generated from elements of the shape $\sum_{i,j \geq 0} a_{ij} u^{2^i + 3^j}$ only. So we see that for $N = 1$, there exists no Beilinson lattice $\mathcal{L} \subseteq L_1/L'_1$ so that $\mathcal{L}_{\triangle''} \equiv u \cdot k[[u,w]] \equiv u \cdot k[[u]] \langle 1, w, \ldots, w^{N-1} \rangle$ (these agree in $(L_1/L'_1)_{\triangle'}$, since $w^N \equiv 0$). In particular, $u \cdot k[[u,w]]$ is a Tate lattice, an $(\mathcal{O}_{A^1})_{\triangle''}$-module, yet cannot be of the shape $\mathcal{L}_{\triangle''}$ for a Beilinson lattice.

In summary, we have strict inclusions

Beilinson lattices $\not\subseteq$ Yekutieli lattices $\not\subseteq$ Tate lattices,

with a slight abuse of language since they each live in different categories and objects. Follow Prop. 57 to make it precise.
5. Stand-alone higher local fields

In this section we shall address a different question, closely related to what we have done in the preceding sections, but still essentially independent. We will now forget about schemes. Instead, let $k$ be a perfect field and $K$ an $n$-dimensional TLF over $k$. Then for finite $K$-modules $V_1, V_2$ we have Yekutieli’s cubical algebra, Definition 9, $E_{Yek}(V_1, V_2)$. However, we could try to interpret $K$ as an $n$-Tate object in finite-dimensional $k$-vector spaces (in some way still to discuss...) so that we also have the corresponding cubical algebra as $n$-Tate objects, Theorem 6. Clearly our Theorem 12 strongly suggests that these two algebras might be isomorphic. However, such a result would not follow readily from Theorem 12 itself since the latter really uses the adèle theory of a surrounding scheme $X$ and the notion of Beilinson lattices.

This sets the stage for the present section: We shall prove such a comparison result by a direct method.

There will be two variations: (1) We consider the multiple Laurent series field $K = k((t_1))((t_2))\cdots((t_n))$. This is canonically a TLF, Example 22, and simultaneously canonically an $n$-Tate object, Example 18. In this case both cubical algebras are defined and we shall show that they are canonically isomorphic.

(2) We shall consider a general TLF. In this case one has to choose a presentation as an $n$-Tate object. This makes the comparison a little more involved, but thanks to the results of Yekutieli’s paper [Yek15], one still arrives at an isomorphism.

5.1. Variant 1: Multiple Laurent series fields. Let $k$ be a field. Recall the following:

1. $k[[t]]$ is a principal ideal domain,
2. every non-zero ideal is of the form $(t^n)$ for $n \geq 0$,
3. every finitely generated module is of the form $k[[t]]^\oplus \bigoplus_{i=1}^m k[[t]]/t^{n_i}$.
4. the forgetful functor $\text{Mod}_f(k[[t]]) \to \text{Vect}(k)$ is exact and factors through an exact functor $\text{Mod}_f(k[[t]]) \to \text{Pro}^\otimes_{\aleph_0}(k)$,
5. the forgetful functor $\text{Vect}_f(k((t))) \to \text{Vect}(k)$ is exact and factors through an exact functor $\text{Vect}_f(k((t))) \to \text{Tate}_{\aleph_0}^\otimes(k)$.

Define $K := k((t_1))\cdots((t_n))$.

Lemma 39. The forgetful functor $\text{Vect}_f(K) \to \text{Vect}(k)$ is exact and factors through an exact functor $T : \text{Vect}_f(K) \to n\text{-Tate}_{\aleph_0}^\otimes(k)$.

Proof. This follows from property 5 and induction on $n$. \qed

Denote by $V_k(n)$ the $n$-Tate space associated to $k((t_1))\cdots((t_n))$ as explained in Example 18. Recall from Definition 9 that a Yekutieli lattice in $V_k(n)$ is a $k((t_1))\cdots((t_{n-1}))/[[t_n]]$-submodule $L \subset V_k(n)$ such that $k((t_1))\cdots((t_n)) \cdot L = V_k(n)$.

Lemma 40. Every Yekutieli lattice of $V_k(n)$ is of the form $t_n^i k((t_1))\cdots((t_{n-1}))/[[t_n]]$. In particular, it is a free $k((t_1))\cdots((t_{n-1}))/[[t_n]]$-module of rank 1.
By the universal property of kernels, we conclude that the Yekutieli lattice Lemma 42.

Remark 43. A key fact used in the statement and proof of this theorem is that the forgetful functor \( n \)-Tate\(_{\mathcal{R}_0}^e \) to \( \mathcal{V}ect(k) \) is injective on Hom-sets. This is immediate for \( n = 1 \), and for \( n > 1 \), it follows by induction.

Proof. It suffices to assume \( n = 1 \). For the general case, just replace the field \( k \) by the field \( k((t_1)) \cdots (t_{n-1}) \) and replace the \( k \)-algebra \( k[[t]] \), by the \( k((t_1)) \cdots (t_{n-1}) \)-algebra \( k((t_1)) \cdots (t_{n-1})[[t_n]] \).

Now, let \( M \subset k((t)) \) be a finitely generated \( k[[t]] \)-sub-module such that \( k((t)) \cdot M = k((t)) \).

Let \( \{f_1, \ldots, f_m\} \) be a set of generators for \( M \) over \( k[[t]] \). Re-ordering as necessary, we can assume that \( \text{ord}_t = 0 f_i \leq \text{ord}_t = 0 f_{i+1} \) for all \( i \). Define \( \ell := \text{ord}_t = 0 f_1 \). By definition, we have \( M \subset t^\ell k[[t]] \subset k((t)) \).

Conversely, because \( k \) is a field, there exists a unit in \( g \in k[[t]]^\times \) such that \( f_1 g = t^\ell \). Because \( t^\ell k[[t]] \) is a cyclic \( k[[t]] \)-module generated by \( t^\ell \), we conclude that \( M \supset t^\ell k[[t]] \) as well.

The following result is the analogue of Prop. 37, but we are not working with more input than the TLF and \( n \)-Tate object \( K \). In particular, the notion of a Beilinson lattice as in Prop. 37 would not even make sense in the present context.

Lemma 41. Denote by \( \text{Gr}^\text{Yek}(k) \) the partially ordered set of Yekutieli lattices. There is a final and co-final inclusion of partially ordered sets \( \text{Gr}^\text{Yek}(k) \subset \text{Gr}(V_k(n)) \), where the latter denotes the Grassmannian of Tate lattices (i.e. the Sato Grassmannian as defined in \([BGW14]\)).

Proof. The \( n \)-Tate object \( V_k(n) \) is represented by the admissible Ind-diagram

\[
\cdots \rightarrow t_n^1 V_k(n-1)[[t_n]] \rightarrow t_n^{-1} V_k(n-1)[[t_n]] \rightarrow \cdots
\]

We see that every Yekutieli lattice arises in this diagram. Therefore every Yekutieli lattice is a Tate lattice of \( V_k(n) \), i.e. \( \text{Gr}^\text{Yek}(k) \subset \text{Gr}(V_k(n)) \). Further, by the definition of Hom-sets in \( n \)-Tate\(_{\mathcal{R}_0}^e(k) \) (which implies that the sub-category \( \text{Pro}^a((n-1)\text{-Tate}^e_{\mathcal{R}_0}(k)) \) is left filtering), we see that every Tate lattice in \( V_k(n) \) factors through a Yekutieli lattice in the above diagram. Therefore the sub-poset of Yekutieli lattices is final. It remains to show that every Tate lattice \( L \) of \( V_k(n) \) contains a Yekutieli lattice. This will follow from the same argument by which one shows that \( \text{Ind}^a(C) \) is right filtering in \( \text{Tate}^e(C) \) (cf. \([BGW14]\) Proposition 5.8)). Denote by \( \mathcal{O}_1(0) \) the Yekutieli lattice \( V_k(n-1)[[t_n]] \subset V_k(n) \). Consider the map

\[
\mathcal{O}_1(0) \hookrightarrow V_k(n) \twoheadrightarrow V_k(n)/L.
\]

Because \( \text{Pro}^a((n-1)\text{-Tate}^e_{\mathcal{R}_0}(k)) \) is left filtering in \( n \)-Tate\(_{\mathcal{R}_0}^e(k) \), there exists an \((n-1)\)-Tate object \( P \) such that the above map factors as

\[
\mathcal{O}_1(0) \rightarrow P \twoheadrightarrow V_k(n)/L.
\]

Further, \( \mathcal{O}_1(0) \) is represented by the admissible Pro-diagram

\[
\cdots \mathcal{O}_1(0)/t_n^i \rightarrow \mathcal{O}_1(0)/t_n^{-i-1} \rightarrow \cdots \rightarrow V_k(n-1).
\]

Therefore, by the definition of Hom-sets in \( \text{Pro}^a((n-1)\text{-Tate}^e_{\mathcal{R}_0}(k)) \) (which implies that the sub-category \((n-1)\)-Tate\(_{\mathcal{R}_0}^e(k) \) is right filtering), we see that there exists \( i \) such that the map \( \mathcal{O}_1(0) \rightarrow P \) factors as

\[
\mathcal{O}_1(0) \rightarrow \mathcal{O}_1(0)/t_n^i \rightarrow P.
\]

By the universal property of kernels, we conclude that the Yekutieli lattice \( t_n^i \mathcal{O}_1(0) \) is a common Tate sub-lattice of \( \mathcal{O}_1(0) \) and \( L \). \( \square \)

Lemma 42. For any \( V_1, V_2 \in \mathcal{V}ect_f(K) \), there is an equality of subsets of \( \text{Hom}_k(V_1, V_2) \)

\[
E^\text{Yek}(V_1, V_2) = \text{Hom}_{n\text{-Tate}^e_{\mathcal{R}_0}(k)}(T(V_1), T(V_2)),
\]

where \( T \) denotes the functor of Lemma 39.
Proof. We prove this by induction on $n$. For $n = 0$, there is nothing to show. For the induction step, by the universal properties of direct sums, it suffices to show the equality for $V_1 = V_2 = k((t_1)) \cdots ((t_n))$; note that for this $V$, $T(V) := V_k(n)$. We begin by showing that $\text{End}_{n, \text{Tate}^d_{\text{el}}(k)}(V_k(n)) \subset E^\text{Yek}(V)$. Let $\varphi$ be an endomorphism of $V_k(n)$. Let $L_1 = t_n^i V_k(n-1)[[t_n]]$ and $L_2 = t_n^i V_k(n-1)[[t_n]]$ be a pair of Yekutieli lattices of $k((t_1)) \cdots ((t_n))$. We begin by showing that this pair admits a $\varphi$-refinement (see Definition 9). By the standard Ind-diagram for $V_k(n)$, and the definition of Hom-sets in $n, \text{Tate}^d_{\text{el}}(k)$, there exists a Yekutieli lattice $N = t_0^i V_k(n-1)[[t_n]]$ such that $\varphi(L_1) \subset N$. Let $i^0_2 = \min(j, i_2)$, and define $L'_2 = t_0^i V_k(n-1)[[t_n]]$. Next, consider the map $L_1 \to L'_2 / L_2$. The quotient $L'_2 / L_2 \cong V_k(n-1)[[t_n]] / (t_2^i - t_n^0)$ is an elementary $(n-1)$-Tate space. By the definition of Hom-sets in $\text{Pro}^\alpha((n-1), \text{Tate}^d_{\text{el}}(k))$ (which implies that the sub-category of $(n-1)$-Tate spaces is right filtering), the map above factors through an admissible epic in $\text{Pro}^\alpha((n-1), \text{Tate}^d_{\text{el}}(k))$

$$L_1 \to L_1 / t_n^i L_1 \to L'_2 / L_2.$$  

We define $L'_1 = t_n^i + t_0^i V_k(n-1)[[t_n]]$, and observe that $(L'_1, L'_2) \varphi$-refines $(L_1, L_2)$. Furthermore, because $(n-1)$-Tate spaces is a full sub-category of $\text{Pro}^\alpha((n-1), \text{Tate}^d_{\text{el}}(k))$, the map $\overline{\varphi}$ is a map of $(n-1)$-Tate spaces. By the inductive hypothesis, this map is an element in $E^\text{Yek}(L_1 / t_n^i L_1, L_2 / L_2)$. We conclude that

$$\text{End}_{n, \text{Tate}^d_{\text{el}}(k)}(V_k(n)) \subset E^\text{Yek}(k((t_1)) \cdots ((t_n))).$$

To complete the induction step, it remains to show the reverse inclusion. Let $\varphi \in E^\text{Yek}(K)$. We begin by showing that, given any two Yekutieli lattices $L_1$ and $L_2$ such that $\varphi(L_1) \subset L_2$, then the map $L_1 \to L_2$ is a map of admissible Pro-objects (in $(n-1)$-Tate spaces). By Lemma 10, $L_0 \cong t_0^i V_k(n-1)[[t_n]]$ for $a = 1, 2$. By the definition of Yekutieli’s $E^\text{Yek}$ Definition 9 for each $\ell > 0$, there exists a $\varphi$-refinement $(L_1^\ell, L_2^\ell)$ of the pair $(L_1, t_n^i L_2)$. Without loss of generality, we can take $L_2^\ell = L_2$, and we therefore obtain a square

$$\begin{array}{ccc}
L_1 \ & \ & \varphi \ & \ & \varphi \\
\downarrow & & & & & \downarrow \\
L_1^\ell \ & \ & t_n^i L_2^\ell \\
\end{array}$$

By the definition of local $BT$-operators, for all $\ell \geq 0$, the induced map $L_1 / L_1^\ell \to L_2 / t_n^i L_2$ is a local $BT$-operator, and thus, by induction hypothesis, a map of $(n-1)$-Tate spaces. Because an inclusion of Yekutieli lattices is an admissible monic of admissible Pro-objects (e.g. by Lemma 10), for all $\ell \geq 0$, the map

$$L_1 \to L_1 / L_1^\ell \to L_2 / t_n^i L_2$$

is a map of admissible Pro-objects. Taking the limit over all $n$ (in $\text{Pro}^\alpha((n-1), \text{Tate}^d_{\text{el}}(k))$), we obtain a map of admissible Pro-objects $L_1 \to \lim_{\ell} L_2 / t_n^i L_2 \cong L_2$. The forgetful functor

$$\text{Pro}^\alpha((n-1), \text{Tate}^d_{\text{el}}(k)) \to \text{Vect}(k)$$

preserves limits (by construction). Therefore, we conclude that the map of $k$-vector spaces underlying the map of admissible Pro-objects is equal to the limit of the maps

$$L_1 \to L_1 / L_1^\ell \to L_2 / t_n^i L_2$$

but this is just $\varphi$. We have shown that $\varphi$ restricts to a map of admissible Pro-objects on any pair of lattices $L_1$ and $L_2$ such that $\varphi(L_1) \subset L_2$. It remains to show that $\varphi$ is a map of $n$-Tate spaces. Let $L_\ell = t_n^i V_k(n-1)[[t_n]]$. Then $\ell \mapsto L_\ell$ is an admissible Ind-diagram (in
we have constructed an ascending chain of inclusions of Yekutieli lattices \( \varphi \). There exists a pair of Yekutieli lattices \((L_0, L'_0)\) which \(\varphi\)-refine \((L_0, L_0)\). In particular, \(\varphi(L_0) \subseteq L'_0\). For the induction step, suppose we have constructed an ascending chain of inclusions of Yekutieli lattices

\[
L'_0 \hookrightarrow \ldots \hookrightarrow L'_n
\]

such that \(\varphi(L_i), L_i \subseteq L'_i\) for \(i \leq n\). Consider the pair of Yekutieli lattices \((L_{n+1}, L'_{n+1})\). Then there exists a pair of Yekutieli lattices \((L_n, L_k)\) which \(\varphi\)-refines this pair. Further (e.g. by Lemma [10]), there exists a Yekutieli lattice \(L'_{n+1}\) which contains both \(L_b\) and \(L_{n+1}\). This completes the induction step. Above we have shown that the maps \(L_\ell \rightarrow L'_\ell\) are maps of admissible \(\text{Pro}\)-objects (in \((n - 1)\)-Tate spaces) for each \(\ell\). Therefore, we conclude that \(\varphi\) lifts to a map of admissible \(\text{Ind}\)-diagrams. By construction, the ascending chain of lattices

\[
L'_0 \hookrightarrow \ldots \hookrightarrow L'_\ell \hookrightarrow \ldots
\]

is final in the Grassmannian of Tate lattices \(Gr(V_k(n))\) (because the chain \(L_0 \hookrightarrow \cdots \hookrightarrow L_\ell \hookrightarrow \cdots\) is). We conclude that \(V_k(n)\) is the colimit of this ascending chain, and that the map of colimits

\[
V_k(n) \cong \colim_\ell L_\ell \rightarrow \colim_\ell L'_\ell \cong V_k(n)
\]

is a map of \(n\)-Tate spaces. But, this map is equal to \(\varphi\) (e.g. because the forgetful map \(n\)-Tate \(\text{End}_{\text{Tate}}(k) \rightarrow \text{Vect}(k)\) preserves colimits, by construction). We conclude that

\[
E_{\text{Yek}}^k((t_1)) \cdots ((t_n)) \subseteq \text{End}_{n\text{-Tate}^\ell}(k)(V_k(n)).
\]

This finishes the proof. \(\Box\)

**Lemma 44.** For any \(V_1, V_2 \in \text{Vect}_{\ell}(K)\), the equality

\[
E_{\text{Yek}}(V_1, V_2) = \text{Hom}_{n\text{-Tate}^\ell}(k)(T(V_1), T(V_2))
\]

of Lemma [12] restricts to an equality of two-sided ideals

\[
I_{i, \text{Yek}}(V_1, V_2) = I_{i, \text{Tate}}(T(V_1), T(V_2))
\]

for \(1 \leq i \leq n\).

**Proof.** We prove this by induction on \(n\). For \(n = 0\), there is nothing to show. Because every Yekutieli lattice of \(V\) induces a Tate lattice of \(V_k(n)\), knowing any conditions defining \(I_\ell^\pm\) for all Tate lattices, implies it for all Yekutieli lattices. Thus, we immediately get

\[
I_{i, \text{Yek}}(V_1, V_2) \supseteq I_{i, \text{Tate}}(T(V_1), T(V_2))
\]

The converse direction is a bit more involved. Not every Tate lattice is a Yekutieli lattice, but with the help of Lemma [11] we shall reduce checking conditions for Tate lattices to Yekutieli lattices. Suppose we want to check whether \(\varphi \in I_{i, \text{Tate}}(T(V_1), T(V_2))\) holds. For \(i = 1\), Lemma [11] implies that having image contained in a Yekutieli lattice is the same as having image contained in a Tate lattice, and analogously for kernels. Thus, to deal with \(i = 2, \ldots, n\) we only need to confirm that this argument survives refinements: We know that if \(L_1 \subseteq T(V_1), L_2 \subseteq T(V_2)\) are Tate lattices and we pick Tate lattices \(L'_1 \subseteq T(V_1), L'_2 \subseteq T(V_2)\) such that

\[
L'_1 \subseteq L_1, \quad L_2 \subseteq L'_2, \quad f(L'_1) \subseteq L_2, \quad f(L_1) \subseteq L'_2,
\]

we have the \(f\)-refinement

\[
\bar{f}: L_1/L'_1 \rightarrow L_2/L'_2.
\]
We need to show that \( \mathcal{T} \in I_{-1}^{\pm, \text{Tate}}(L_1/L_1', L_2'/L_2) \), just assuming this holds whenever all of the above lattices are also Yekutieli lattices. So let \( L_1, L_2 \) be Tate lattices for which we want to check the defining property. By Lemma 41 there exist Yekutieli lattices \( N_{2,a} \subset L_2 \) and \( N_{1,a} \supset L_1 \). Also, by Lemma 41 we can choose a \( \varphi \)-refinement \((N_{1,a}, L_2')\) of \((L_1', N_{2,a})\) with \( N_{1,a} \) a Yekutieli lattice, and we can also choose a \( \varphi \)-refinement \((L_1, N_{2,b})\) of \((N_{1,b}, L_2')\) with \( N_{2,b} \) a Yekutieli lattice. These refinements define a commuting diagram

\[
\begin{array}{ccc}
N_{1,b}/N_{1,a} & \longrightarrow & N_{2,b}/N_{2,a} \\
\downarrow & & \downarrow \\
L_1/N_{2,a} & \longrightarrow & L_2'/N_{2,a} \\
\downarrow & & \downarrow \\
L_1/L_1' & \longrightarrow & L_2'/L_2.
\end{array}
\]

By assumption, the top horizontal map is in \( I_{-1, \text{Yek}}^{\pm}(N_{1,b}/N_{1,a}, N_{2,b}/N_{2,a}) \). Further, the upper vertical arrows are admissible monics, while the lower vertical arrows are admissible epics. In particular, all the vertical maps split, so we have a commuting diagram

\[
\begin{array}{ccc}
N_{1,b}/N_{1,a} & \longrightarrow & N_{2,b}/N_{2,a} \\
\downarrow & & \downarrow \\
L_1/L_1' & \longrightarrow & L_2'/L_2.
\end{array}
\]

in which the top map is in \( I_{-1, \text{Yek}}^{\pm}(N_{1,b}/N_{1,a}, N_{2,b}/N_{2,a}) \). Because this is a categorical ideal \([\text{Yek15, Lemma 4.16 (2)}]\), we conclude that the bottom map is in \( I_{-1, \text{Tate}}^{\pm}(L_1/L_1', L_2'/L_2) \) as claimed.

Of course combining Lemma 42 with Lemma 44 implies:

**Theorem 13.** The functor

\[
\text{Vect}_f(K) \rightarrow T^{n-\text{Tate}_{\aleph_0}}(k)
\]

induces canonical isomorphisms

\[
E^{\text{Yek}}(V_1, V_2) \cong \text{Hom}_{n-\text{Tate}_{\aleph_0}}(k) (T(V_1), T(V_2))
\]

so that for \( V_1 = V_2 \) this becomes an isomorphism of Beilinson cubical algebras.

This finishes the comparison.

**Example 45** (Osipov, Yekutieli). Yekutieli has shown that elements in \( E^{\text{Yek}}(V_1, V_2) \) are morphisms of ST modules, i.e., they are continuous in the ST topology \([\text{Yek15, Thm. 4.24}]\). However, he also proved that \( E^{\text{Yek}}(V_1, V_2) \) is strictly smaller than the algebra of all ST module homomorphisms for \( n \geq 2 \) \([\text{Yek15, Example 4.12 and following}]\). This generalizes an observation due to Osipov, who had established the corresponding statements for Laurent series with Parshin’s natural topology \([\text{Osi07, §2.3}]\).

5.2. **Variant:** TLFs. Instead of working with an explicit model like \( k((t_1)) \cdots ((t_n)) \) we can also work with a general TLF. Firstly, recall that this forces us to assume that the base field \( k \) is perfect. Even though we cannot associate an \( n \)-Tate vector space over \( k \) to a TLF directly, we can do so using Yekutieli’s concept of a system of liftings:
Definition 16. Let $k$ be a perfect field. Moreover, let $K$ be an $n$-dimensional TLF over $k$ and $\sigma = (\sigma_1, \ldots, \sigma_n)$ a system of liftings in the sense of Yekutieli. Suppose $V$ is a finite-dimensional $K$-vector space.

1. If $n = 0$, $K = k$ and every finite-dimensional $k$-vector space is literally a 0-Tate object over $\text{Vect}_f(k)$.

2. If $n \geq 1$ the ring of integers $\mathcal{O}_1 := \mathcal{O}_1(K)$ is a (not finitely generated) $k_1(K)$-module. Let $b_1, \ldots, b_r$ be any $K$-basis of $V$ and $\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\}$ its $\mathcal{O}_1$-span inside $V$. We can partially order all such bases by the inclusion relation among their $\mathcal{O}_1$-spans. Note that each

$$(\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\})/m_1^m$$

is a finite torsion $\mathcal{O}_1$-module and thus a finite-dimensional $k_1(K)$-vector space by the lifting $\sigma_1$.

3. Thus, if we assume that for each finite-dimensional vector space $V$ over the $(n-1)$-dimensional TLF $k_1(K)$ along with the system of liftings $(\sigma_2, \ldots, \sigma_n)$ comes with a fixed model, denoted $V^\flat$, as an $(n-1)$-Tate object in $k$-vector spaces,

$$\lim_{\text{colim}} \lim_{\text{over all bases}} ((\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\})/m_1^m)^3$$

defines an $n$-Tate object in $k$-vector spaces.

4. Inductively, this associates a canonical $n$-Tate object to each finite-dimensional $K$-vector space (but depending on the chosen system of liftings).

It is easy to check that the colimit over the bases $b_1, \ldots, b_r$ is filtering.

The technical result as well as the key idea underlying the proof of the following is entirely due to Yekutieli:

Theorem 14. Let $k$ be a perfect field and $K$ an $n$-dimensional TLF over $k$.

1. For any system of liftings $\sigma$, the construction in Definition 16 gives rise to a functor $\mathcal{W}_\sigma$:

$$\text{Vect}_f(K) \xrightarrow{\mathcal{W}_\sigma} \text{n-Tate}^d(\text{Vect}_f(k)) \xrightarrow{\text{eval}} \text{Vect}(k)$$

so that the composition agrees with the forgetful functor to $k$-vector spaces as in Lemma 77.

2. For any $V_1, V_2 \in \text{Vect}_f(K)$, the functor $\mathcal{W}_\sigma$ induces an isomorphism

$$E^{\text{Yek}}(V_1, V_2) \xrightarrow{\sim} \text{Hom}_{n, \text{Tate}^d}(\mathcal{W}_\sigma V_1, \mathcal{W}_\sigma V_2).$$

3. For any two systems of liftings $\sigma, \sigma'$, there exists a canonical $n$-Tate automorphism $e_{\sigma, \sigma'}$ so that $\mathcal{W}_{\sigma'} = e_{\sigma, \sigma'} \circ \mathcal{W}_\sigma$.

4. For any $V_1, V_2$, the image of $\text{Hom}_{n, \text{Tate}^d}(\mathcal{W}_\sigma V_1, \mathcal{W}_\sigma V_2)$ under ‘eval’ is independent of the choice of $\sigma$, and agrees with $E^{\text{Yek}}(V_1, V_2)$.

The interesting aspect of (3) is the existence of a canonical isomorphism. The existence of an abundance of rather random isomorphisms is clear from the outset.

Proof. (1) and (2): The proof is basically a repetition of everything we have done with $k((t_1)) \cdots ((t_n))$ in this section. The argument works basically verbatim. Replace each $k((t_1)) \cdots ((t_n))$ by $K$, each $(-)[[t]]$ by the respective ring of integers $\mathcal{O}$, and each power $t^i$ by $m^i$ with $m$ the respective maximal ideal. The only slight change is that in Equation 5.1 we take the colimit over all bases $b_1, \ldots, b_r$ in Lemma 11. Part (3) is deep in principle, but easy for us since we can rely on the theory set up in Yek15. In Definition 16 part (2), we can read the finite $\mathcal{O}_1$-module

$$(\mathcal{O}_1 \otimes \{b_1, \ldots, b_r\})/m_1^m.$$
as a $k_1(K)$-vector space either by the lifting $\sigma$ or $\sigma'$. The assumptions of [Yek15] Theorem 2.8, (2) are satisfied; the above is a finite $O_1$-module and it is a precise Artinian local ring by [Yek15] Lemma 3.14. By Yekutieli’s theorem, loc. cit., the identity automorphism on the module transforms the two $k_1(K)$-vector space structures of $\sigma$ and $\sigma'$ via $\text{GL}_{(\tau)}(D^\text{cont}_{K/k})$ and by [Yek15] Lemma 4.11 this lies in Yekutieli’s $E^K = E^K_{\sigma}$, i.e. our $E^\text{Yek}(K)$ (at this point in Yekutieli’s paper it has not yet been proven that this is independent of $\sigma$, but of course we may already use this here). Finally, by part (2) this is nothing but an automorphism as an $n$-Tate object, giving the desired $e_{\sigma,\sigma'}$. Part (4) follows from (3): The images just differ by an inner automorphism, but that means that they are the same. □

APPENDIX A. RESULTS FROM COMMUTATIVE ALGEBRA

For the convenience of the reader, we list various facts from commutative algebra which we need in various proofs:

**Lemma.** Suppose $R$ is a Noetherian ring.

1. ([Eis95] Thm. 7.2 (a))] For an ideal $I$, and a finitely generated $R$-module $M$, $\widehat{M}_I \cong M \otimes_R \widehat{R}_I$.
2. ([Eis95] Lemma 2.4] For a multiplicative set $S$ and $M$ an $R$-module, we have $M[S^{-1}] \cong M \otimes_R S^{-1}$.
3. ([AM69] Prop. 10.15 (iv))] For an ideal $I$ and $\widehat{R}_I$ the $I$-adic completion. Then $I\widehat{R}_I$ is contained in the Jacobson radical of $\widehat{R}_I$.
4. ([Eis95] Cor. 2.16)] Every Artinian ring $R$ is isomorphic to $\prod P_i$ (i.e. a product of Artinian local rings), where $P$ runs through the finitely many maximal primes of $R$.
5. ([Mat89] Thm. 7.2(3))] Let $R$ be a ring and $M$ an $R$-module. Then $M$ is faithfully flat over $R$ iff $M \neq mM$ for every maximal ideal of $R$.
6. ([Mat89] Thm. 7.5(ii))] If $M$ is a faithfully flat $R$-algebra and $I$ an ideal of $R$, $IM \cap R = I$.
7. ([Eis95] Cor. 7.5)] (Lifting of Idempotents) Suppose $R$ is a Noetherian ring which is complete with respect to an ideal $I$. Then any system of pairwise orthogonal idempotents $e_1, \ldots, e_r \in R/I$ lifts uniquely to pairwise orthogonal idempotents $e_1, \ldots, e_r \in R$.
8. ([HS06] Cor. 2.1.13) Suppose $R$ is a reduced ring, $Q_1, \ldots, Q_r$ its minimal primes, and $R'$ the integral closure in its total ring of quotients $\text{Quot}(R)$. Then $R' \cong \prod_{i=1}^r (R/Q_i)'$, where $(R/Q_i)'$ denotes the integral closure in the field of fractions of $R/Q_i$.
9. ([Lam99] Thm. 21.10) Let $R$ be a ring and $e$ an idempotent. Then $\text{rad}(eR) = e \text{ rad } R$, where $\text{ rad } R$ denotes the Jacobson radical of $R$. Thus, $eR/e \text{ rad } R \cong \overline{\pi}(R/e \text{ rad } R)$, where $\overline{\pi}$ denotes the image of $e$ in $R/\text{ rad } R$.
10. Suppose $R \to S$ is a faithfully flat morphism. Let $P$ be a prime in $R$. Then $P$ is of the shape $Q \cap S$ for a prime ideal $Q$ in $S$ minimal over $PS$. Conversely, for every prime ideal $Q$ minimal over $PS$ we have $Q \cap S = P$.
11. Let $(R, m)$ be a Noetherian complete local ring and $R \to S$ a finite extension. Then $S$ is semi-local and decomposes as a finite product of complete local rings, $S \cong \prod S_m'$, where $m'$ runs through the finitely many maximal ideals of $S$.
12. ([Eis95] Prop. 11.1] Suppose $(R, m)$ is a 1-dimensional regular local ring. Then it is a discrete valuation ring.
13. Let $R$ be a reduced excellent ring and $I$ an ideal. Then $\widehat{R}_I$ is also reduced.
Let $R$ be an excellent ring, $I$ an ideal. Then the canonical map $R \to \hat{R}_I$ is regular.

Let $R \to S$ be a regular, faithfully flat ring homomorphism. Then $R$ is reduced iff $S$ is reduced.

Proof. All given references also give a full proof. As additional remarks: For [11] we refer to [HS06, Prop. 4.3.2]: Use that $S/mS$ has finitely many minimal primes $m'$ and therefore by the Chinese Remainder Theorem $S/mS \cong \prod (S/mS)_{m'}$. Then use lifting of idempotents. □

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