Quantum moduli spaces from matrix models

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ABSTRACT: In this paper we show that the matrix model techniques developed by Dijkgraaf and Vafa can be extended to compute quantum deformed moduli spaces of vacua in four dimensional supersymmetric gauge theories. The examples studied give the moduli space of a bulk D-brane probe in geometrically engineered theories, in the presence of fractional branes at singularities.

KEYWORDS: Matrix models, Supersymmetric gauge theories

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1. Introduction

Supersymmetric gauge theories have many remarkable properties that have made them theoretically attractive through the years. Perhaps the most notable property of supersymmetric gauge theories is holomorphy [42]. In a nutshell, it is the ability to make exact predictions for the theories at strong coupling from (nonperturbative) weak coupling calculations. The data that is protected by supersymmetry is holomorphic, and usually it is the case that holomorphicity arguments alone are strong enough to solve the theory (meaning that we can extract all of the holomorphic information from an analysis of the symmetries of the theory, see for example [40, 44]).

It has been noted that the holomorphic information for field theories associated to D-branes can be encoded in a topological open string theory [45, 9]. These topological computations depend only on zero modes of the spacetime theory, so that the calculations are given by analyzing only the massless modes of a D-brane on a Calabi-Yau space. These are a finite number of degrees of freedom, and the calculations reduce to a matrix model whose fields are the chiral multiplets of the gauge theory. We are interested in the topological B-model, which in the most general setting is described in the work [25]. The tree level matrix model action is the classical superpotential of the theory. Already at this level one can predict Seiberg-like dualities [43] by changing the basis of fundamental branes in the B-model [3].

Recently, in a series of papers Dijkgraaf and Vafa [19, 20, 21] have argued that the matrix model can be used to compute not just the tree level superpotential, but all of the holomorphic information of the field theory. This has been argued only for situations when the classical supersymmetric gauge theory leads to a discrete set of vacua, as well as limits of this situation (for example extracting the Seiberg-Witten curve of the $\mathcal{N} = 2$ pure $U(N)$ gauge theory).

Each classical vacuum configuration can give rise to many quantum vacua. This follows because at the classical vacua the infrared physics is a pure gauge theory $\prod U(N_i)$, for
which the $SU(N_i)$ confine. The novel realization is that higher loops in the matrix model introduce powers of fields $S_i$ which in the weak coupling approximation can be understood as gaugino condensates for the $SU(N_i)$ fields. This weak coupling calculation can then be extrapolated to strong coupling, so long as we think of the $S_i$ as holomorphic coordinates describing the vacuum configurations. There is also a measure term for the $S_i$, so that when we vary the effective superpotential we get the exact vacua of the theory.

The solution via the matrix models [19] treats the variables $\mu_i = N_i/N$ as fixed filling fractions for a large $N$ saddle point calculation. The reason one can use the large $N$ saddle point is that we are only interested in the planar diagrams of the matrix model, and the $1/N$ expansion is an expansion in the genus of a Riemann surface. At higher genus string amplitudes will contribute to protected curvature and graviphoton couplings to the gauge theory, but not to the effective superpotential [3].

The recipe proceeds in the following steps: first we write the planar large $N$ expansion in terms of the appropriate 't Hooft couplings $g_i = gN_i$, $\lambda = gN$ so that $g_i = \mu_i \lambda$. After this is done we formally replace the variables $g_i$ by $S_i$, the associated gaugino condensate for the $SU(N_i)$ gauge field.

This has been verified in various cases getting results [19, 21, 22, 23, 24, 29, 30] from known matrix model solutions [35, 38] that can be matched to exact field theory calculations based on completely different techniques [2, 10, 22].

This paper will generalize the above idea to the case where the vacuum structure is not a set of isolated vacua, but instead it gives rise to a moduli space of vacua. We are particularly interested in situations where there are quantum corrections to the moduli space, and where the model is simple enough to be tractable at the matrix model level and the field theory as well.

Since the matrix model techniques are given by topological string amplitudes, it is natural to consider, as a starting point, D-brane probes in various geometries, and to engineer situations where we expect quantum corrections to the moduli space. The main tool of our geometrical understanding will be the brane engineering of geometric transitions. See [32, 36, 13, 12, 11, 15, 16, 39] for the description of the general setup.

Thus, we will be build theories by considering non compact Calabi-Yau spaces with isolated singularities, and placing a collection of fractional branes at the singularities 1. We will require that these fractional branes give rise to a consistent four dimensional gauge theory (the anomalies are canceled). In general, placing fractional branes at singularities leads to geometric transitions which deform the singularity structure. This deformed geometry is the geometry that should be seen by a probe brane in the bulk, see also [33, 34].

Because this new deformed geometry is not the original Calabi-Yau geometry (which corresponds to the classical moduli space of the probe brane), one can associate the geometric transition to nonperturbative effects of the gauge theory. These deformations can be computed exactly in the gauge theory in various cases [41].

We will show that these results can be reproduced using large $N$ matrix models associated to the superpotential of the gauge theory. In the matrix model approach, one

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1These are branes wrapped on blown-down cycles with a $B$ field through them.
also needs to treat the probe branes on a different footing than the branes at the singularities. The reason the probe branes have to be analyzed separately is that they have massless modes (the moduli) that one is not allowed to integrate out. Understanding how these zero modes get affected by performing the matrix integral will give us the expected deformed moduli space.

The reader interested in learning more about matrix models should read the excellent reviews [18, 31].

The body of the paper is organized in three main sections, each covering a different example of computations. First we study the examples considered by Dijkgraaf and Vafa [19] with a brane in the bulk, a particular case of which is the brane setup of Klebanov and Strassler [36], which is reviewed briefly. Then we study the $\mathcal{N} = 4$ gauge theory and we show that the moduli space of vacua does not get quantum corrected. Thirdly we study a field theory which results from deforming the theory of branes at the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ singularity with discrete torsion. We close the paper with a conclusion and discussion.

2. Branes at the conifold singularity and the canonical example

Let us consider a theory where we place $M$ fractional branes at the conifold singularity, and add a single probe brane to the system. The geometry of the conifold is described by a non-degenerate quadratic equation in four variables being set equal to zero, for example

$$w^2 - uv = z^2$$  \hspace{1cm} (2.1)

and the conformal field theory associated to the conifold was first discussed in [17].

The field theory associated to this set of branes in the geometry has been studied in the work of Klebanov and Strassler [36]. The effect of placing the fractional branes at the singularity makes the probe moduli space become the deformed conifold, due to the Affleck-Dine-Seiberg superpotential [1]. This effect has been generalized to many probes in [3]. We will make a brief review of the calculation done by Klebanov and Strassler, and we will then proceed to calculate the same effect with the associated matrix model.

We will consider a $SU(M+S) \times SU(S)$ gauge theory with four matter fields $A_{1,2}$ and $B_{1,2}$ transforming in the $(M+S,\bar{S})$ and $(\bar{M}+\bar{S},S)$. We will further specialize to $S = 1$ for simplicity, which indicates that we have only one bulk probe in the brane configuration. The theory has a classical superpotential given by

$$W = \lambda \text{tr}(\epsilon^{ij} \epsilon^{kl} A_i B_k A_j B_l)$$

The superpotential above can be written as

$$W = \lambda (\epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl})$$

where $N_{ij} = A_i B_j$ with the $SU(M+S)$ indices contracted. These are gauge invariant meson superfields with respect to the confining gauge theory, and are the moduli of the theory.
It is easy to show that solving for the classical moduli space of vacua we have
\[ \epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl} = 0 \] (2.2)
which is the conifold geometry.

Now, the SU(\(M + S\)) theory for \(M\) large and fixed \(S = 1\) is confining, and by dimensional transmutation it has a dynamical scale \(\Lambda\) replacing the coupling constant.

Strong coupling effects generate an effective superpotential
\[ W = \lambda (\text{tr} (\epsilon^{ij} \epsilon^{kl} A_i B_k A_j B_l)) + (M - 1) \left( \frac{\Lambda^{3M+1}}{\epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl}} \right)^{1/M-1} \] (2.3)
And after minimizing the superpotential one finds that
\[ \epsilon^{ij} \epsilon^{kl} N_{ik} N_{jl} = (2\Lambda^{3M+1}/\lambda^M)^{1\over M-1} \] (2.4)
which is the geometry of the deformed conifold, namely, we are getting the following equation
\[ uv + w^2 = z^2 + \epsilon \]
Notice that the phase of \(\epsilon\) is determined by a choice of vacuum for the \(U(M)\) pure \(N = 1\) gauge theory associated to the infrared of the fractional branes at the conifold singularity. The size of \(\epsilon\) is determined by the dynamical scale of the theory.

This is a particular example in a general class of field theories first studied by Cachazo et al \[13, 12, 11\], and which has been labeled the canonical example in [21]. The classical geometry associated to the field theory is given by the following equation in four variables
\[ w^2 - uv = P(z)^2 \] (2.5)
where \(P(z)\) is a polynomial of degree \(n\). The above geometry has singularities of the conifold type at the roots of \(P(z)\), and one can place fractional branes at each of these singularities. Each of these sets of fractional branes via strong coupling effects deforms the above geometry so that in the end one finds that one should arrive at the following type of geometry [11]
\[ uv + w^2 = P(z)^2 + f(z) \] (2.6)
where \(f(z)\) is given by a polynomial in \(z\) of degree \(n - 1\). In the particular case of the conifold we have that \(P\) is of degree one, and the deformation \(f(z)\) is a polynomial of degree zero (a constant).

These theories are also of the form \(U(M) \times U(N)\), and besides the fields \(A_i, B_i\) these theories have an additional pair of adjoint fields \(z\) and \(\tilde{z}\). The classical superpotential for the theory is given by
\[ W = \text{tr} \left( z(A_1 B_1 - A_2 B_2) - (B_1 A_1 - B_2 A_2)\tilde{z} \right) + \text{tr} [V(z) - V(\tilde{z})] \] (2.7)
and these are a deformation of an \(\mathcal{N} = 2\) SYM theory by the potential term \(\text{tr} (V(z) - V(\tilde{z}))\).

This is a twisted superpotential deformation of the \(\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2\) orbifold. One can show
\[The U(N) \times U(0) theory gives rise to a standard one-matrix model\]
easily that this is the geometry of the moduli space of vacua by using non-commutative geometry techniques \[8, 4\]. Because these theories can be considered as softly broken $N = 2$ theories, one can analyze them via the Seiberg-Witten solution of the undeformed theory \[44, 10\]. The curve where $u = v = 0$, $w^2 = P(z)^2 + f(z)$ is the Seiberg-Witten curve of the associated $\mathcal{N} = 2$ gauge theory when one goes to a generic point in moduli space for $U(N)$, where the gauge group is broken to $U(N) \rightarrow U(1)^N$, and for which $P$ is of degree $N$.

When $M = N = 1$ we have a single probe brane, and the variables, $u, v, w, z$ can be identified with

$$u = (A_1 B_2), v = (A_2 B_1), w = \frac{(A_1 B_1 + A_2 B_2)}{2}, z = \tilde{z}$$

and $2V'(z) = P(z)$.

There are also vacua where two fractional branes are located at different singularities, and these lead to additional isolated vacua. The condition $\tilde{z} = z$ is the one that ensures that the $A_i, B_i$ have massless modes.

When we have a $U(N) \times U(M)$ theory, we need to think in terms of the eigenvalues of $z$ and $\tilde{z}$ as gauge invariant quantities. In the case where $V(z) = mz^2$, the fields $z$ and $\tilde{z}$ can be integrated out, and one recovers the conifold field theory as described above.

We will be interested again in the $U(N + 1) \times U(1)$ field theory: one probe in the presence of $N$ fractional branes. This will simplify the analysis that would be required to take care of the $U(N + 1) \times U(M + 1)$ theory where the fractional branes between the two types of branes are split in different singularities, plus a single probe.

The field theories associated to these examples have been studied by matrix model techniques in the work of Dijkgraaf and Vafa in the particular case where there are no moduli, for the $U(M) \times U(0)$ theory. Here we will show that the deformed moduli space of vacua from a matrix model computation will lead exactly to the geometry given by (2.6).

Now, we need to write the associated matrix model to the above situation. For the case of the $U(N) \times U(0)$ theory, the solution of the vacuum of the theory is described by solving the matrix model

$$\int [dz] \exp(N' \mu^{-1}tr(V(z)))$$

about a saddle point of the large $N'$ limit of $N' \times N'$ matrices with some quantum distribution of eigenvalues. Each classical saddle point of $V$ gives a location where one can place classical eigenvalues. On the eigenvalue plane, the quantum eigenvalues will produce cuts on this plane, and the theory is solved by a Riemann surface which is a double cover of the eigenvalue plane, the spectral curve.

$N'$, although it begins its life as $N$, in the end should not be identified with $N$. Instead each variable $N_i$ (the number of eigenvalues in a cut) in the expansion should be replaced by an associated gaugino condensate $S_i$. The $S_i$ is obtained later from the periods of a differential of the spectral curve of the solution of the matrix model, which is given by

$$y^2 = (V')^2(z) + f(z)$$
This effectively gives us a map from the $f$ to the $S_i$.

Now, we will show that this same technology can be used to derive the quantum deformed moduli space of the $U(N + 1) \times U(1)$ theory.

The idea is to look at a $U(N' + 1) \times U(1)$ matrix model, where again $N'$ begins it’s life as $N$, but we treat the $U(N')$ diagrams in the planar limit, while we keep the probe separate from the $U(N')$ theory, and take $N' \to \infty$.

We should consider writing a multi-matrix model with all the fields that enter in the superpotential

$$\int \!(dz|d\tilde{z}|dA|dB) \exp(W)$$

but where the integral is over all massive modes of the theory, and we keep the massless information. This is indicated with a prime on the integration measure, which indicates that we should remove the set of variables which are massless.

It is easy to show that the classical moduli space of the probe is given by

$$w^2 - uv = P(\tilde{z})^2,$$

with $u, v, w, \tilde{z}$ defined as above, making sure that the $U(N' + 1)$ indices are contracted completely, see for example [3]. One can also check that for these solutions one of the eigenvalues of $z$ is equal to $\tilde{z}$, while all the other eigenvalues sit classically at critical points of $V(z)$.

To include the effects of confinement, one should solve the matrix model in the large $N'$ limit. By standard methods one can turn it into an integral over the eigenvalues of $z$ times the Vandermonde determinant. This is the gauge fixing procedure for $z$. We will be careful to distinguish one of the eigenvalues $z_0$ which is equal to $\tilde{z}$. This ensures that some of the components of $A, B$ are massless, and that we are allowed to have a moduli space.

From a more invariant point of view, one notices that the equations of motion for the $B$ imply that $zA = A\tilde{z}$ with $\tilde{z}$ a number and $A$ a column vector. If $A$ is not zero, then this implies that $\tilde{z}$ is equal to one of the eigenvalues of the matrix $z$. This is exactly what we need to explore the moduli space of vacua.

At this point the integral to do is

$$\int \prod_{i \neq 0}(d\lambda_i)[dA|dB) \Delta^2 \exp\left\{N\mu^{-1} \sum_i (\lambda_i - \tilde{z})(A_i \cdot B_i) + N\mu^{-1} \sum_i V(\lambda_i) - N\mu^{-1} V(\tilde{z}) + N\mu^{-1}(z_0 - \tilde{z})(A_1^0(B_1)_0 - (A_2)^0(B_2)_0)\right\}$$

For us, the moduli will be given by $z = z_0$ and the zero components of $A, B$. The notation above has the symbol $A \cdot B$ which indicates the combination $A_1B_1 - A_2B_2$, and we have made the color indices explicit with respect to the $U(N + 1)$ gauge group because we want to keep track of the eigenvalue we singled out.

Now we can integrate out $A_i, B_j$. This being a Gaussian integral over four coordinates of mass $\lambda_i - z$ gives us the result $\delta = \prod_{i \neq 0}(\lambda_i - z)^2$ up to a multiplicative constant $^3$.

The logarithm of the Vandermonde determinant and $\delta$ is then

$$\log(\Delta^2 \delta) = \sum_{i \neq j} 2\log(\lambda_i - \lambda_j) - 2 \sum_{i \neq 0} \log(\lambda_i - \tilde{z})$$

$^3$This constant is irrelevant for our discussion, but plays a role when we consider the vev of the superpotential of the theory at a given vacuum.
Now we want to solve for the saddle point of this setup in the large $N'$ limit. Notice that the eigenvalues $\tilde{z}$ and $z_0$ do not have an interaction between them, because we have not integrated out the massless modes of $A, B$. The saddle point equation for the zero components of $A, B$ generic make $\tilde{z} = z_0$ in this situation.

One sees that the saddle point equations for the eigenvalues $\lambda_i$ are the same as when we have the theory $U(N') \times U(0)$, because the contribution from $z_0$ cancels the contribution from $z$ when they are equal. The saddle point equation for the eigenvalues $\lambda_i$ is

$$N' \mu^{-1} V'(\lambda_i) - 2 \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j} = 0 \quad (2.11)$$

Now let $w(\lambda) = \frac{1}{N'} \sum_{i \neq 0} \frac{1}{\lambda - \lambda_i}$ be the resolvent of the matrix model.

From here it follows that in the large $N'$ limit one has

$$w(\lambda)^2 + w(\lambda) \mu^{-1} V'(\lambda) = f(\lambda) \quad (2.12)$$

where $f(\lambda)$ is a polynomial of degree $n - 1$.

The saddle point equations for $\tilde{z} = z_0$ are identically equal to

$$\mu^{-1} V'(z_0) + 2w(z_0) + \mu^{-1}(A_1)^0(B_1)_0 - (A_2)^0(B_2)_0 = 0$$

From here we get the quantum corrected relation in the variables from equation (2.8)

$$w^2 - uv = \frac{1}{4}(V'(z_0) + 2\mu w(z_0))^2$$

and it follows from equation (2.12) that

$$w^2 - uv = \frac{1}{4}((V')^2(z) - f(z)) \quad (2.13)$$

which up to a normalization factor on the polynomial $f$ is the same result as the deformed geometry of Cachazo et al [11].

One can repeat the calculation with many probes. The results are then grouped in block diagonal sets of matrices. The results for each such probe brane are the same as above. Thus one obtains that the moduli space is a symmetric product of the deformed geometry. This is also expected from the calculations performed in [3]. This happens because the probe branes do not affect the eigenvalue distribution of the fractional branes. After all, there is an equality between the eigenvalues of $z$ and some eigenvalues of $\tilde{z}$ which cancels terms in the saddle point for the fractional brane eigenvalues, and the probes do not seem to interact with each other. Also the interactions between the eigenvalues corresponding to different probes cancel each other for the same reason. We will see that this is exactly what happens in the $\mathcal{N} = 4$ SYM; the lesson is that the branes in the bulk will behave locally like the maximally supersymmetric brane, except when they are at a singularity.
3. $\mathcal{N} = 4$ SYM

Let us discuss a case where there are no quantum correction, namely, $\mathcal{N} = 4$ gauge theory. Again, we should do the integral

$$\int [dX][dY][dZ]' \exp(-N\mu^{-1} tr(X[Y,Z]))$$  \hspace{1cm} (3.1)

We can go to the eigenvalue basis for $X$, and we obtain the integral

$$\int \prod d\lambda_i [dY][dZ]' \Delta^2 \exp(-N\mu^{-1} \sum_i \lambda_i[Y,Z]_{ii})$$  \hspace{1cm} (3.2)

Now, we integrate the massive modes of $Y, Z$. These are the off-diagonal components of $Y, Z$, and the mass matrix for $Y_{ij}, Z_{ji}$ is $N\mu^{-1}(\lambda_{ii} - \lambda_{jj})$. In total there are four scalars with this mass term, so they give rise to the following

$$\int [\prod d\lambda_i dY_{ii} dZ_{ii}'][\Delta^2 \prod_{i<j}(\lambda_i - \lambda_j)^{-2}] = \int [\prod d\lambda_i dY_{ii} dZ_{ii}']$$  \hspace{1cm} (3.3)

The result follows from the identity $\Delta = \prod_{i<j}(\lambda_i - \lambda_j)$.

When we consider the right hand side of the equation, all of the elements appearing in the integral are moduli, so we should not integrate over them at all. From here, it is clear that the quantum moduli space is given exactly by the classical moduli space, namely, three matrices that commute. This is a well known result, and the matrix model result is consistent with this fact.

We should mention that it is important to notice that the eigenvalue measure was canceled exactly by the integration of the massive fields, so it performs a non-trivial test of the matrix model technology.

4. A third example

So far we have studied examples which are essentially $N = 2$ or $N = 4$ supersymmetric, and could be argued to be derived from the Seiberg-Witten curve with a small superpotential.

Now, we will describe a matrix model which is obtained from a different type of geometry, and is more naturally thought of as an $\mathcal{N} = 1$ gauge theory. The theory is pure $U(N)$ gauge field theory with three adjoints $X, Y, Z$, and the following superpotential

$$W = g(tr(XYZ + XZY) - 2tr(M_1^2 X + M_2^2 Y + M_3^2 Z))$$  \hspace{1cm} (4.1)

We can always rescale the fields so that $M_1^2 = M_2^2 = M_3^2 = 1$. However, it is helpful to consider the dependence on the holomorphic couplings $M_i^2$ when we describe the quantum moduli space from the field theory point of view. In the UV, if one arrives at a fixed point of the renormalization group, then $g$ can be identified with the gauge coupling of the theory, after we normalize $X, Y, Z$ properly in their kinetic term.

This theory has been analyzed previously in various places, and it corresponds to a deformation of the $\mathbb{C}^3/\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with discrete torsion $[20, 6]$. A bulk probe brane
has worldvolume gauge theory $U(2)$, and one can show that the following matrices are proportional to the identity on the moduli space $u = X^2$, $v = Y^2$, $t = Z^2$, $\gamma = \{[X,Y],Z\}/4$.

We can choose an open set of the moduli space to be parametrized as

$$X = \alpha \sigma_3$$
$$Y = \alpha^{-1} \sigma_3 + \beta \sigma_1$$
$$Z = \alpha^{-1} \sigma_3 + \beta^{-1}(1 - \alpha^{-2})\sigma_1 + \delta \sigma_3$$

from here $X^2 = x^2$, $Y^2 = x^{-2} + \beta^2$, $Z^2 = x^{-2} + \beta^{-2}x^{-4} + \delta^2$, $\gamma = \alpha \beta \delta$.

These are easily seen to satisfy the following constraint

$$uvw - u - v - t + 2 = \gamma^2$$

and there is a conifold type singularity at $u = v = t = 1$, and $\gamma = 0$. At this location there are two types of fractional brane solutions, with $X = Y = Z = \pm 1$. The field theory in the vicinity of the singularity is in the same universality class as the conifold of Klebanov and Witten. In particular, we can add a large number of fractional branes of the same type, and it is expected that the geometry will be deformed by a geometric transition. Reintroducing the couplings $M_i^2$, one finds that

$$uvw - M_1^4u - M_2^4v - M_3^4t + 2M_1^2M_2^2M_3^2 = \gamma^2$$

If we place fractional branes at the tip of the singularity, one expects to have a geometric transition for the conifold, and the singularity should be deformed to

$$uvw - M_1^4u - M_2^4v - M_3^4t + (2 + \epsilon)M_1^2M_2^2M_3^2 = \gamma^2$$

where $\epsilon$ is a function of $g$ and $\tau$ (the gauge coupling of the associated gauge theory). The form of this dependence is guaranteed by the symmetries of the theory. When $M_i^2 = 0$ the theory has a $U(1)^3$ symmetry which rotates $X$, $Y$, $Z$ independently (we can choose $g$ to transform to cancel the above rotation), and then the $M_i^2$ transform under these charges. The equation of the moduli space then has definite quantum numbers under these charges.

It is easy to see that one can not generate terms that are quadratic on the variables $u, v, t$ and have holomorphic behavior when we take $M_i^2 \to 0$. Determining this function $\epsilon$ is a very interesting problem in quantum field theory, as in principle it is an arbitrary function of a particular combination of $g$ and $\tau$ determined by anomalies. We will not pursue this direction here however.

We will calculate the geometry of a probe in the presence of these fractional branes by the matrix model technique, and we will compute the deformed moduli space. We will show that it agrees with the above shape for the deformation. This is, we will consider a $U(N + 2)$ theory.

We need to consider the following matrix integral in the large $N$ limit

$$\int [dX][dY][dZ]^n \exp(-N\mu^{-1}W)$$

These are elements of the center of the associated quiver algebra.
Again, we diagonalize $X$, but we need to keep two eigenvalues singled out for the probe brane, let them be $\lambda^{0,1}$. Let the other eigenvalues be $x_i$ For these special $\lambda$ eigenvalues we will not integrate the associated block of $2 \times 2$ matrices of $X, Y, Z$, but we will look instead at the quantum corrected moduli space. Hence, we will have

$$
\int d\lambda^0 d\lambda^1 \prod_i dx_i [dY][dZ] \Delta^2 \exp(-N\mu^{-1} \sum_i x_i \{Y, Z\}_{ii} - 2x_i - 2Y_{ii} - 2Z_{ii})$$

(4.7)

$$
+ \lambda^0(Y_{0i}Z_{i0} + Y_{0i}Z_{0i} + \lambda^1(Y_{1i}Z_{i1} + Y_{i1}Z_{i1}) + W_{red})
$$

(4.8)

$$
\int d\lambda^0 d\lambda^1 \prod_i dx_i \prod_{i<j}(x_i + x_j)^{-2} \prod_i (x_i + \lambda_0)^{-2} (x_i + \lambda_1)^{-2}
$$

(4.9)

Where $W_{red}$ is the superpotential associated to the $2 \times 2$ block of matrices that are associated to the eigenvalues $\lambda^{0,1}$. Again we have made the gauge theory indices explicit in the above formula. Here, the Vandermonde determinant is over the $(N+2) \times (N+2)$ matrix $X$. We have separated the action in terms of the probe terms, and those of the large $N$ condensate. We will now integrate the $Y, Z$ terms that are not appearing in $W_{red}$, to get an effective action for $2 \times 2$ block matrices associated to $\lambda^{0,1}$. We will also ignore the contribution to the measure of the diagonal components $Y_{ii}, Z_{ii}$ because it is subleading in the large $N$ limit. However, because there is a linear term in the action, we will take care of the appropriate shift.

Indeed

$$x_i(2Z_{ii}Y_{ii} - 2Z_{ii} - 2Y_{ii}) = 2x_i(Z_{ii} - x_i^{-1})(Y_{ii} - x_i^{-1}) - 2x_i^{-1}
$$

This is, the effective classical potential for the eigenvalue $x_i$ is

$$V(x_i) = -2(x_i + x_i^{-1})
$$

(4.10)

Now, we can integrate out the off diagonal components $Y_{ij}$ and $Z_{ij}$, and we are left with the measure for these terms, which is equal to a constant times

$$
\prod_{i<j}(x_i + x_j)^{-2} \prod_i (x_i + \lambda_0)^{-2} (x_i + \lambda_1)^{-2}
$$

Now, let us evaluate the saddle point equations for the $x_i$.

We find that

$$
\frac{N'}{\mu}(2 - \frac{2}{x_i^2}) + 2 \sum_{j \neq i} \left(\frac{1}{x_i - x_j} - \frac{1}{x_i + x_j}\right) + \sum_{a=0,1} \frac{2}{x_i - \lambda_a} - \frac{2}{x_i + \lambda_a}
$$

(4.11)

One can show that $Y_{0,1}$ can be considered a moduli only if it is massless, and this happens when $\lambda_0 = -\lambda_1$. If we substitute this into the equation above, then we see that the effect of the probe on the condensate cancels when we sum over the eigenvalues, similar to the first example we studied in this paper.

We therefore only need to consider the saddle point for the fractional branes on their own

$$
\frac{N'}{\mu}(2 - \frac{2}{x_i^2}) + 2 \sum_{j \neq i} \left(\frac{1}{x_i - x_j} - \frac{1}{x_i + x_j}\right)
$$

(4.12)
Now let
\[ w(\lambda^2) = \frac{1}{N'} \sum_i \frac{x_i}{\lambda^2 - x_i^2} \] (4.13)

Take equation (4.12), multiply it by \( x_i(\lambda^2 - x_i^2)^{-1} \) and sum over \( i \). We arrive at the following equation
\[ w(\lambda^2)^2 - \frac{1}{N'} w'(\lambda^2) + \mu^{-1} w(\lambda^2)(1 - \frac{1}{\lambda^2}) + \frac{A}{\lambda^2} = 0 \] (4.14)
with \( A \) a number. Again, in the large \( N' \) limit, we drop the term in \( w' \), and we obtain an algebraic equation for \( w(\lambda^2) \).

Now, we can go back to the saddle point for the block matrix associated to the eigenvalues \( \lambda_{0,1} \). For this reduced set, we obtain the following saddle point equations, which will describe the quantum corrected moduli space
\[
\begin{align*}
\{X, Z\} &= 2 \\
\{X, Y\} &= 2 \\
\{Y, Z\} &= 2 + 4\mu^{-1} w(X^2)
\end{align*}
\] (4.15-4.17)

These are solved by the following
\[
\begin{align*}
X &= \lambda \sigma_3 \\
Y &= \lambda^{-1} \sigma_3 + \beta \sigma_1 \\
Z &= \lambda^{-1} \sigma_3 + \beta' \sigma_1 + \delta \sigma_2
\end{align*}
\] (4.18-4.20)

With the constraint
\[ \beta \beta' = 1 - \frac{1}{\lambda^2} + 2\mu w(\lambda^2) \]

Now we can again evaluate the “gauge invariant coordinates” for this solution, and we obtain
\[ u = \lambda^2, \quad v = \lambda^{-2} + \beta^2, \quad t = \lambda^{-2} + (\beta')^2 + \delta^2, \quad \gamma = \lambda \beta \delta \]

It is an easy algebraic manipulation to determine that
\[ \gamma^2 = uvt - u - v - t - t((\beta \beta')^2 - 1 - \frac{1}{\lambda^4}) \] (4.21)

We now obtain from equation (4.14) that
\[ (\beta \beta')^2 = (1 - \frac{2}{\lambda^2} + \frac{1}{\lambda^4}) + \frac{4A\mu^2}{\lambda^2} \]

So that the quantum deformed moduli space is given by the equation
\[ \gamma^2 = uvt - u - v - t + 2 - 4A\mu^2 \] (4.22)

which is a rather simple modification of the above geometry. One can verify that this is just as expected, the quantum corrected moduli space in equation (4.5).
5. Conclusion and outlook

In this paper we have seen various examples of matrix models that we can solve and then obtain the quantum deformed moduli space of certain supersymmetric gauge theories. The results match exact field theoretical results, at least at the level of the shape of the deformation. It is clear that at least in some cases the matrix models are an effective way to compute quantum effects in supersymmetric gauge theories.

We noticed that in the examples studied, bulk branes played a particularly simple role in that they did not affect the fractional brane condensates, but they felt the deformations in the geometry caused by the fractional branes. It would be very interesting if this is found to happen in every situation.

The models studied in this paper are naturally associated with some geometric construction. One can argue that these systems are simpler because the classical geometry gives us the constraints and natural variables to analyze the classical moduli space, and the quantum effects (both in the field theory and the matrix model) modify these classical constraints leading to the same deformed moduli space. Perhaps the fact mentioned above that bulk branes are sufficiently simple is the key to solving these problems.

In general, there is no known answer as to how one can get a quiver theory from a given geometry. Even the inverse problem of finding a Calabi-Yau geometry which corresponds to a given quiver theory with superpotential can be quite involved.

Conservatively, one might assume that it is exactly in these situations in which one has a Calabi-Yau threefold that is tractable that one might be able to solve the associated matrix model. The classical geometry provides the right loop variables to understand the matrix model. If this is true then one should be able to solve geometrically the associated matrix models to theories that have appeared in \cite{24,1}, and deformations of models of \cite{6} which in spirit are not too different from the third example studied in this paper. However, the matrix models in question might be a lot harder to analyze \cite{28,21}. Also, one might be able to solve examples associated to fractional branes at isolated toric singularities \cite{28} and theories with $Sp$ and $SO$ gauge groups \cite{15}.

On a more optimistic scenario, one could do better than the above and address 'all' of the possible quiver theories. For example, dualities often require the knowledge of deformed moduli spaces to show that they describe the same universality class \cite{13}. There is a classical version of Seiberg duality for quiver diagrams as equivalences of derived categories \cite{5}. This suggests an equivalence of the full topological string theories associated to the dual theories. One might hope to be able to produce a proof that takes care of all of the quantum aspects of the duality as well by using these techniques.

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References

[1] I. Affleck, M. Dine and N. Seiberg, “Dynamical Supersymmetry Breaking In Supersymmetric QCD,” Nucl. Phys. B 241, 493 (1984).

[2] O. Aharony, N. Dorey and S. P. Kumar, “New modular invariance in the N = 1* theory, operator mixings and supergravity singularities,” JHEP 0006, 026 (2000) [arXiv:hep-th/0006008].

[3] D. Berenstein, “On the universality class of the conifold,” JHEP 0111, 060 (2001) [arXiv:hep-th/0110184].

[4] D. Berenstein, “Reverse geometric engineering of singularities,” JHEP 0204, 052 (2002) [arXiv:hep-th/0201093].

[5] D. Berenstein and M. R. Douglas, “Seiberg duality for quiver gauge theories,” arXiv:hep-th/0207027.

[6] D. Berenstein, V. Jejjala and R. G. Leigh, “Marginal and relevant deformations of N = 4 field theories and non-commutative moduli spaces of vacua,” Nucl. Phys. B 589, 196 (2000) [arXiv:hep-th/0005087].

[7] D. Berenstein and R. G. Leigh, “Discrete torsion, AdS/CFT and duality,” JHEP 0001, 038 (2000) [arXiv:hep-th/0001055].

[8] D. Berenstein and R. G. Leigh, “Resolution of stringy singularities by non-commutative algebras,” JHEP 0106, 030 (2001) [arXiv:hep-th/0105229].

[9] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, “Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes,” Commun. Math. Phys. 165, 311 (1994) [arXiv:hep-th/9309140].

[10] F. Cachazo and C. Vafa, “N = 1 and N = 2 geometry from fluxes,” arXiv:hep-th/0206017.

[11] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, “A geometric unification of dualities,” Nucl. Phys. B 628, 3 (2002) [arXiv:hep-th/0110028].

[12] F. Cachazo, S. Katz and C. Vafa, “Geometric transitions and N = 1 quiver theories,” arXiv:hep-th/0108120.

[13] F. Cachazo, K. A. Intriligator and C. Vafa, “A large N duality via a geometric transition,” Nucl. Phys. B 603, 3 (2001) [arXiv:hep-th/0103067].

[14] L. Chekhov and A. Mironov, “Matrix models vs. Seiberg-Witten/Whitham theories,” arXiv:hep-th/0209085.

[15] K. Dasgupta, K. Oh and R. Tatar, “Geometric transition, large N dualities and MQCD dynamics,” Nucl. Phys. B 610, 331 (2001) [arXiv:hep-th/0105066].

[16] K. Dasgupta, K. Oh and R. Tatar, “Open/closed string dualities and Seiberg duality from geometric transitions in M-theory,” JHEP 0208, 026 (2002) [arXiv:hep-th/0106040].

[17] K. Dasgupta, K. h. Oh, J. Park and R. Tatar, “Geometric transition versus cascading solution,” JHEP 0201, 031 (2002) [arXiv:hep-th/0110050].

[18] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254, 1 (1995) [arXiv:hep-th/9306153].

– 13 –
[19] R. Dijkgraaf and C. Vafa, “Matrix models, topological strings, and supersymmetric gauge theories,” arXiv:hep-th/0206255.

[20] R. Dijkgraaf and C. Vafa, “On geometry and matrix models,” arXiv:hep-th/0207106.

[21] R. Dijkgraaf and C. Vafa, “A perturbative window into non-perturbative physics,” arXiv:hep-th/0208048.

[22] N. Dorey, T. J. Hollowood and S. Prem Kumar, “An exact elliptic superpotential for $N = 1^*$ deformations of finite $N = 2$ gauge theories,” Nucl. Phys. B 624, 95 (2002) [arXiv:hep-th/0108221].

[23] N. Dorey, T. J. Hollowood, S. Prem Kumar and A. Sinkovics, “Exact superpotentials from matrix models,” arXiv:hep-th/0209089.

[24] N. Dorey, T. J. Hollowood, S. P. Kumar and A. Sinkovics, “Massive vacua of $N = 1^*$ theory and S-duality from matrix models,” arXiv:hep-th/0209099.

[25] M. R. Douglas, “D-branes, categories and $N = 1$ supersymmetry,” J. Math. Phys. 42, 2818 (2001) [arXiv:hep-th/0011017].

[26] M. R. Douglas, “D-branes and discrete torsion,” arXiv:hep-th/9807235.

[27] M. R. Douglas and B. Fiol, “D-branes and discrete torsion. II,” arXiv:hep-th/9903031.

[28] B. Feng, A. Hanany and Y. H. He, “Phase structure of D-brane gauge theories and toric duality,” JHEP 0108, 040 (2001) [arXiv:hep-th/0104259].

[29] F. Ferrari, “On exact superpotentials in confining vacua,” arXiv:hep-th/0210135.

[30] H. Fuji, Y. Ookouchi, “Comments on Effective Superpotentials via Matrix Models,” arXiv:hep-th/0210148.

[31] P. Ginsparg and G. W. Moore, “Lectures On 2-D Gravity And 2-D String Theory,” arXiv:hep-th/9304011.

[32] R. Gopakumar and C. Vafa, “M-theory and topological strings. I,” arXiv:hep-th/9809187.

[33] S. Gukov and D. Tong, “D-brane probes of G(2) holonomy manifolds,” Phys. Rev. D 66, 087901 (2002) [arXiv:hep-th/0202125].

[34] S. Gukov and D. Tong, “D-brane probes of special holonomy manifolds, and dynamics of $N = 1$ three-dimensional gauge theories,” JHEP 0204, 050 (2002) [arXiv:hep-th/0202126].

[35] V. A. Kazakov, I. K. Kostov and N. A. Nekrasov, “D-particles, matrix integrals and KP hierarchy,” Nucl. Phys. B 575, 413 (1999) [arXiv:hep-th/9810035].

[36] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities,” JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].

[37] I. R. Klebanov and E. Witten, “Superconformal field theory on threebranes at a Calabi-Yau singularity,” Nucl. Phys. B 536, 199 (1998) [arXiv:hep-th/9807080].

[38] I. K. Kostov, “Exact solution of the six-vertex model on a random lattice,” Nucl. Phys. B 575, 513 (2000) [arXiv:hep-th/9911023].

[39] K. h. Oh and R. Tatar, “Duality and confinement in $N = 1$ supersymmetric theories from geometric transitions,” arXiv:hep-th/0112040.
[40] N. Seiberg, “Naturalness versus supersymmetric nonrenormalization theorems,” Phys. Lett. B 318, 469 (1993) [arXiv:hep-ph/9309335].

[41] N. Seiberg, “Exact results on the space of vacua of four-dimensional SUSY gauge theories,” Phys. Rev. D 49, 6857 (1994) [arXiv:hep-th/9402044].

[42] N. Seiberg, “The Power of holomorphy: Exact results in 4-D SUSY field theories,” arXiv:hep-th/9408013.

[43] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) [arXiv:hep-th/9411149].

[44] N. Seiberg and E. Witten, “Electric - magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory,” Nucl. Phys. B 426, 19 (1994) [Erratum-ibid. B 430, 485 (1994)] [arXiv:hep-th/9407087].

[45] E. Witten, “Chern-Simons gauge theory as a string theory,” Prog. Math. 133, 637 (1995) [arXiv:hep-th/9207094].