INVERSE NODAL PROBLEM FOR A CONFORMABLE FRACTIONAL DIFFUSION OPERATOR

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Abstract. In this paper, a diffusion operator including conformable fractional derivatives of order $\alpha$ $(0 < \alpha \leq 1)$ is considered. The asymptotics of the eigenvalues, eigenfunctions and nodal points of the operator are obtained. Furthermore, an effective procedure for solving the inverse nodal problem is given.

1. Introduction

The fractional derivative based on 1695 is widely used in applied mathematics and mathematical analysis. Since then, many researchers have developed different types of fractional derivative (see [1]-[4]). Unlike classical Newtonian derivatives, a fractional derivative is given via an integral form. For example, well-known Riemann-Liouville fractional derivative is one of them and is defined by

$$D_{a}^{\alpha}(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{dt^{n}} \int_{a}^{t} \frac{f(x)}{(t-x)^{\alpha-n+1}} dx,$$

for $\alpha \in [n-1, n)$.

In 2014, Khalil et al. introduced the definition of conformable fractional derivative [5]. In 2015, the basic properties and main results of this derivative was given by Abdeljawad and Atangana et al. ([6], [7]). The derivative arises in various fields such as quantum mechanics, dynamical systems, time scale problems, diffusions, conservation of mass, etc. (see [8]-[11]).

For about a century, inverse spectral theory for the different types of operators such as Sturm-Liouville, Dirac and diffusion has been investigated. The first and important result in this theory belongs to Ambarzumyan (see [12]). After this study, the theory has been developed by many authors. In recent years, the direct and inverse problems for the Sturm-Liouville and Dirac operators which include fractional derivative have been studied (see [13]-[20]). However, in current literature, there are no results in the inverse spectral theory for a diffusion operator which include conformable fractional derivative. These problems play an important role in mathematics and have many applications in natural sciences and engineering (see [21]-[25]).

The inverse nodal problems consist in recovering operators from given a dense set of zeros (nodes or nodal points) of eigenfunctions. In 1988, McLaughlin gave a solution of inverse nodal problem for the Sturm-Liouville operator (see [26]). Then, many important results for both the diffusion operators and the Sturm-Liouville

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operators have been studied by several researchers (see [27]-[41] and references therein).

In the present paper, we consider a diffusion operator with Dirichlet conditions which include conformable fractional derivatives of order \( \alpha \) \((0 < \alpha \leq 1)\) instead of the ordinary derivatives in a traditional diffusion operator. We reconstruct the potentials of the diffusion operator from nodes of its eigenfunctions and give an algorithm for solving the inverse nodal problem.

We note that the analogous results can be obtained also for other types of boundary conditions.

2. Preliminaries

In this section, Firstly, we recall known some concepts of the conformable fractional calculus. Then, we introduce a conformable fractional diffusion operator with Dirichlet boundary conditions on \([0, \pi]\).

**Definition 1.** Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be a given function. Then, the conformable fractional derivative of \( f \) of order \( \alpha \) with respect to \( x \) is defined by

\[
D_{x}^{\alpha} f(x) = \lim_{h \to 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}, \quad D_{x}^{\alpha} f(0) = \lim_{x \to 0^+} D_{x}^{\alpha} f(x),
\]

for all \( x > 0 \), \( \alpha \in (0, 1] \). If this limit exist and finite at \( x_0 \), we say \( f \) is \( \alpha \)-differentiable at \( x_0 \). Note that if \( f \) is differentiable, then,

\[
D_{x}^{\alpha} f(x) = x^{1-\alpha} f'(x).
\]

**Definition 2.** The conformable fractional Integral starting from \( 0 \) of order \( \alpha \) is defined by

\[
I_{x}^{\alpha} f(x) = \int_{0}^{x} f(t)d_{\alpha} t = \int_{0}^{x} t^{\alpha-1} f(t)dt, \quad \text{for all } x > 0.
\]

**Lemma 1.** Let \( f : [a, \infty) \rightarrow \mathbb{R} \) be any continuous function. Then, for all \( x > a \), we have

\[
D_{x}^{\alpha} I_{a} f(x) = f(x).
\]

**Lemma 2.** Let \( f : (a, b) \rightarrow \mathbb{R} \) be any differentiable function. Then, for all \( x > a \), we have

\[
I_{x}^{\alpha} D_{x}^{\alpha} f(x) = f(x) - f(a).
\]

**Definition 3.** (\(\alpha\)-integration by parts): Let \( f, g : [a, b] \rightarrow \mathbb{R} \) be two conformable fractional differentiable functions. Then,

\[
\int_{a}^{b} f(x)D_{x}^{\alpha} g(x)d_{\alpha} x = f(x)g(x)|_{a}^{b} - \int_{a}^{b} g(x)D_{x}^{\alpha} f(x)d_{\alpha} x.
\]

**Definition 4.** The space \( C_{\alpha}^{n}[a, b] \) consists of all functions defined on the interval \([a, b]\) which are continuously \( \alpha \)-differentiable up to order \( n \).

**Definition 5.** Let \( 1 \leq p < \infty \), \( a > 0 \). The space \( L_{\alpha}^{p}(0, a) \) consists of all functions \( f : [0, a] \rightarrow \mathbb{R} \) satisfying the condition

\[
\left( \int_{0}^{a} |f(x)|^{p} d_{\alpha} x \right)^{1/p} < \infty.
\]
Lemma 3. [43], The space $L^p_\alpha (0,a)$ associated with the norm function
\[ \|f\|_{p,\alpha} := \left( \int_0^a |f(x)|^p \, d_\alpha x \right)^{1/p} \]
is a Banach space. Moreover if $p = 2$ then $L^2_\alpha (0,a)$ associated with the inner product for $f, g \in L^2_\alpha (0,a)$
\[ \langle f, g \rangle := \int_0^a f(x)g(x) \, d_\alpha x \]
is a Hilbert space.

Definition 6. [43], Let $p \in \mathbb{R}$ be such that $p \geq 1$. The Sobolev space $W^p_\alpha (0,a)$ consists of all functions on the interval $[0,a]$, such that $f(x)$ is absolutely continuous and $D^\alpha_x f(x) \in L^p_\alpha (0,a)$.

More detail knowledge about the conformable fractional calculus can be seen in [5] and [6].

Now, let us consider the boundary value problem $L_\alpha = L_\alpha (p(x), q(x))$ of the form
\[
\begin{align*}
\ell_\alpha y & := -D^\alpha_x D^\alpha_x y + [2\lambda p(x) + q(x)] y = \lambda^2 y, \quad 0 < x < \pi \\
U(y) & := y(0) = 0 \\
V(y) & := y(\pi) = 0
\end{align*}
\]
where $\lambda$ is the spectral parameter, $p(x), D^\alpha_x p(x), q(x) \in W^2_\alpha (0,\pi)$ are real valued functions, $p(x) \neq \text{const.}$ and
\[ \int_0^\pi p(x) \, d_\alpha x = 0. \]

The operator $L_\alpha$ is called as conformable fractional diffusion operator (CFDO).

Let the functions $S(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of the equation (1) satisfying the initial conditions
\[ S(0,\lambda) = 0, \quad D^\alpha_x S(0,\lambda) = 1 \quad \text{and} \quad \psi(\pi,\lambda) = 0, \quad D^\alpha_x \psi(\pi,\lambda) = 1 \]
respectively.

Denote
\[ \Delta (\lambda) = W_\alpha [ S(x,\lambda), \psi(x,\lambda) ] = S(x,\lambda) D^\alpha_x \psi(x,\lambda) - \psi(x,\lambda) D^\alpha_x S(x,\lambda). \]

Where, the function $W_\alpha [ \varphi(x,\lambda), \psi(x,\lambda) ]$ is called the fractional Wronskian of the functions $S(x,\lambda)$ and $\psi(x,\lambda)$. It is proven in [17] that $W_\alpha$ does not depend on $x$ and putting $x = 0$ and $x = \pi$ in (6) it can be written as
\[ \Delta (\lambda) = V(S) = -U(\psi). \]

Definition 7. The function $\Delta (\lambda)$ is called the characteristic function of the problem $L_\alpha$.

Let us calculate an asymptotic of the eigenvalues of the problem $L_\alpha$. Firstly, we rewritten equation (1) as
\[
D^\alpha_x D^\alpha_x y + \frac{D^2_x p(x)}{\lambda - p(x)} D^\alpha_x y + (\lambda - p(x))^2 y = (q(x) + p^2(x)) y + \frac{D^2_x p(x)}{\lambda - p(x)} D^\alpha_x y.
\]
It is easily shown that the system of functions \( \{ \cos (\frac{\lambda}{x} x^\alpha - Q(x)) , \sin (\frac{\lambda}{x} x^\alpha - Q(x)) \} \) is a fundamental system for the differential equation

\[
D_x^\alpha D_x^\alpha y + \frac{D_x^\alpha p(x)}{\lambda - p(x)} D_x^\alpha y + (\lambda - p(x))^2 y = 0
\]

where

\[
Q(x) := \int_0^x p(t) dt.
\]

By the method of variation of parameters general solution of equation (8) or (1) is

\[
y(x, \lambda) = c_1 \cos (\frac{\lambda}{x} x^\alpha - Q(x)) + c_2 \sin (\frac{\lambda}{x} x^\alpha - Q(x))
\]

\[
\quad + \int_0^x \frac{\sin (\frac{\lambda}{x} x^\alpha - Q(x))}{\lambda - p(t)} \left[ (q(t) + p^2(t)) y(t, \lambda) + \frac{D_x^\alpha p(t)}{\lambda - p(t)} D_x^\alpha y(t, \lambda) \right] dt.
\]

Since \( S(x, \lambda) \) is the solution of equation (1) satisfying the initial conditions (5).

From (11), we get

\[
S(x, \lambda) = \frac{1}{\lambda - p(0)} \sin (\frac{\lambda}{x} x^\alpha - Q(x))
\]

\[
\quad + \int_0^x \frac{\cos (\frac{\lambda}{x} x^\alpha - Q(0))}{\lambda - p(t)} \left[ (q(t) + p^2(t)) S(t, \lambda) + \frac{D_x^\alpha p(t)}{\lambda - p(t)} D_x^\alpha S(t, \lambda) \right] dt.
\]

**Theorem 1.** For \( |\lambda| \to \infty \), the following asymptotic formula is valid:

\[
S(x, \lambda) = \frac{1}{\lambda} \sin (\frac{\lambda}{x} x^\alpha - Q(x))
\]

\[
\quad + \frac{1}{2x^\alpha} \left\{ (p(x) + p(0)) \sin (\frac{\lambda}{x} x^\alpha - Q(x))
\quad - \left( \int_0^x (q(t) + p^2(t)) dt \right) \cos (\frac{\lambda}{x} x^\alpha - Q(x))
\quad + \int_0^x (q(t) + p^2(t)) \cos \left[ \frac{\lambda}{x} (x^\alpha - 2t^\alpha) - Q(x) + 2Q(t) \right] dt \right\}
\]

\[
\quad + \int_0^x D_x^\alpha p(t) \sin \left[ \frac{\lambda}{x} (x^\alpha - 2t^\alpha) - Q(x) + 2Q(t) \right] dt
\]

\[
\quad + \frac{1}{4x^\alpha} \left\{ \left[ 4p^2(0) + \frac{2(p(x) + p(0))}{1 + \alpha} - 2^{1 + \alpha} x^{1 + \alpha}(0)(p(x) - p(0))1 + \alpha \right]
\quad - \frac{\pi}{2} \left( \int_0^x (q(t) + p^2(t)) dt \right)^2 \sin (\frac{\lambda}{x} x^\alpha - Q(x))
\quad - \left( \int_0^x (q(t) + p^2(t)) (p(x) + p(0) + 2p(t)) dt \right) \cos (\frac{\lambda}{x} x^\alpha - Q(x)) \right\}
\]

\[
\quad + o \left( \frac{1}{x^\alpha} \exp \left( \frac{\lambda}{x} x^\alpha \right) \right)
\]

uniformly with respect to \( x \in [0, \pi] \), where \( \tau = \text{Im} \lambda \).
Proof. We denote
\[
S_0(x, \lambda) = \frac{\sin \left( \frac{\lambda}{n} x^\alpha - Q(x) \right)}{\lambda - p(0)}
\]
and
\[
S_n(x, \lambda) = \frac{\pi}{\lambda - p(t)} \left[ (q(t) + p^2(t)) S_{n-1}(t, \lambda) + \frac{D^2 p(t)}{\lambda - p(t)} D_t^2 S_{n-1}(t, \lambda) \right] d_\alpha t,
\]
for \( n = 1, 2, \ldots \)

Applying successive approximations method to the equations (12) and taking into account Taylor’s expansion formula for the function \( \frac{1}{1-x} \), \( u \to 0 \), we arrive at the estimates (14).

The eigenvalues of \( L_\alpha \) coincide with the zeros of its characteristic function \( \Delta(\lambda) = S(\pi, \lambda) \). Hence, using the formulae (4) and (14) one can establish the following asymptotic
\[
\Delta(\lambda) = \frac{1}{\lambda} \sin \left( \frac{\lambda}{n} \pi^\alpha \right)
+ \frac{1}{2 \lambda} \left\{ (p(\pi) + p(0)) \sin \left( \frac{\lambda}{n} \pi^\alpha \right) \right.
- \left( \int_0^\pi (q(t) + p^2(t)) d_\alpha t \right) \cos \left( \frac{\lambda}{n} \pi^\alpha \right)
+ \left. \int_0^\pi (q(t) + p^2(t)) \cos \left( \frac{\lambda}{n} (\pi^\alpha - 2t^\alpha) + 2Q(t) \right) d_\alpha t \right\}
+ \frac{1}{2n} \left\{ 4p^2(0) + \frac{2(p(\pi) + p(0))^2 + 2^{2+\alpha} p^{2+\alpha}(0) + (p(\pi) - p(0))^{2+\alpha}}{1 + \alpha} \right.
- \frac{1}{\pi} \left( \int_0^\pi (q(t) + p^2(t)) d_\alpha t \right)^2 \sin \left( \frac{\lambda}{n} \pi^\alpha \right)
- \left( \int_0^\pi (q(t) + p^2(t)) (p(\pi) + p(0) + 2p(t)) d_\alpha t \right) \cos \left( \frac{\lambda}{n} \pi^\alpha \right)
\]
+ \left. o \left( \frac{1}{\lambda} \exp \left( \frac{1}{\alpha} \pi^\alpha \right) \right) \right|, |\lambda| \to \infty.
\]
By the standard method using (15) and Rouche’s theorem (see [42]) and taking \( \Delta(\lambda_n) = 0 \) one can prove that eigenvalues \( \lambda_n \) have the form
\[
\lambda_n = \frac{n \alpha}{\pi^{n-1}} + \frac{a_1 - A_n}{2n \pi} + \frac{(p(\pi) + p(0)) a_1 + 2a_2}{4n^2 \pi^{2-\alpha} \alpha} + o \left( \frac{1}{n^2} \right), |n| \to \infty,
\]
where \( n \in \mathbb{Z}\setminus\{0\} \), \( x_n^0 = 0, x_n^j = \pi, j \in \mathbb{Z} \),
\[
a_1 = \int_0^\pi (q(t) + p^2(t)) d_\alpha t, \quad a_2 = \int_0^\pi (q(t) + p^2(t)) p(t) d_\alpha t,
\]
\[
A_n^j = \int_0^{x_n^j} (q(t) + p^2(t)) \cos \left( \frac{2\pi t^\alpha}{\alpha} - 2Q(t) \right) d_\alpha t - \int_0^{x_n^j} D_t^2 p(t) \sin \left( \frac{2\pi t^\alpha}{\alpha} - 2Q(t) \right) d_\alpha t.
\]

Corollary 1. According to (16) for sufficiently large \( |n| \) the eigenvalues \( \lambda_n \) are real and simple.
3. Main Results

In this section, under condition (4) we obtain the asymptotics for the nodal points of \( L_\alpha \) and prove a constructive procedure for solving the inverse nodal problem.

**Theorem 2.** For sufficiently large \(|n|\), the eigenfunction \( S(x, \lambda_n) \) has exactly \(|n| - 1\) nodes \( x_n^j \) in \((0, \pi)\):

\[
0 < x_1^1 < x_1^2 < \ldots < x_n^{n-1} < \pi \quad \text{for} \quad n > 0
\]

and

\[
0 < x_1^{-1} < x_1^{-2} < \ldots < x_n^{n+1} < \pi \quad \text{for} \quad n < 0.
\]

Moreover, the numbers \( x_n^j \) satisfy the following asymptotic formula:

\[
(x_n^j)^\alpha = \frac{j\pi^\alpha}{n} + \frac{Q(x_n^j)}{n\pi^{1-\alpha}} + \frac{1}{2n\pi^{2-2\alpha}} \left[ \int_0^{x_n^j} (q(t) + p^2(t)) d_\alpha t - \frac{a_1}{\pi^\alpha} (x_n^j)^\alpha - \left( A_n^\alpha - \frac{A_n^\alpha}{\pi^\alpha} (x_n^j)^\alpha \right) \right] + \frac{1}{2n\pi^{3-3\alpha} x_n^j} \left[ \int_0^{x} (q(t) + p^2(t)) p(t) d_\alpha t - \left( a_2 + \frac{\alpha p(\pi) + p(0)}{2} \right) (x_n^j)^\alpha \right] + o \left( \frac{1}{n^\alpha} \right),
\]

uniformly with respect to \( j \).

**Proof.** It is obvious that according to (16) for sufficiently large \(|n|\) in the domain \( \Gamma_n = \{ \lambda \mid | \lambda - \frac{a_1}{\pi^\alpha} | \leq 1 \} \) there is exactly one eigenvalue \( \lambda_n \). Taking into account the real-valuedness of \( p(x), q(x) \) we say that is also an eigenvalue \( \lambda_n = \overline{\lambda_n} \). Therefore, the functions \( S(x, \lambda_n) \) are real-valued for sufficiently large \(|n|\).

Substituting (16) in (14) we get

\[
\lambda_n S(x, \lambda_n) = \sin \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + \frac{1}{2n\pi^{2-2\alpha} x_n^j} \left[ (a_1 - A_n^\alpha) \frac{x_n^\alpha}{\pi^\alpha} - \int_0^x (q(t) + p^2(t)) d_\alpha t \right] \cos \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + \left( p(x) + p(0) \right) \sin \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + \left( \int_0^x (q(t) + p^2(t)) \cos \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + 2Q(t) \right) d_\alpha t + \int_0^x D^\alpha p(t) \sin \left( \frac{n(x^\alpha - 2x_n^\alpha)}{\pi^\alpha} - Q(x) + 2Q(t) \right) d_\alpha t + \frac{1}{4n \pi^{2-2\alpha} x_n^j} \left[ \left( (p(\pi) + p(0)) a_1 + 2a_2 \right) \frac{x_n^\alpha}{\pi^\alpha} + (p(x) + p(0)) a_1 \frac{x_n^\alpha}{\pi^\alpha} - \int_0^x (q(t) + p^2(t)) \left( p(x) + p(0) + 2p(t) \right) d_\alpha t \right] \cos \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + \left[ 4p^2(0) + 2(p(x)+p(0))^{1+\alpha} - 22^{+\alpha} p^{1+\alpha}(0) + (p(x)+p(0))^{1+\alpha} \right] \frac{1}{1+\alpha} + a_1 \frac{x_n^\alpha}{\pi^\alpha} \int_0^x (q(t) + p^2(t)) d_\alpha t - \left( a_1 \frac{x_n^\alpha}{\pi^\alpha} \right)^2 + \frac{1}{2} \left( \int_0^x (q(t) + p^2(t)) d_\alpha t \right)^2 \sin \left( \frac{nx_n^\alpha}{\pi^\alpha} - Q(x) \right) + o \left( \frac{1}{n^\alpha} \right), \quad |n| \to \infty,
\]

uniformly in \( x \in [0, \pi] \). From \( S(x_n^j, \lambda_n) = 0 \), we get
\[
\sin \left( \frac{n(x^j_n)^{\alpha}}{\pi^{n-1}} - Q(x^j_n) \right) \\
+ \frac{1}{2n\pi^{1-\alpha}} \left\{ \left( a_1 - A^i_n \right) \frac{(x^j_n)^{\alpha}}{\pi^{\alpha}} - \frac{x^j_n}{\pi} \left( q(t) + p^2(t) \right) d_\alpha t \right\} \cos \left( \frac{n(x^j_n)^{\alpha}}{\pi^{n-1}} - Q(x^j_n) \right) \\
+ (p(x^j_n) + p(0)) \sin \left( -\frac{n(x^j_n)^{\alpha}}{\pi^{n-1}} + Q(x^j_n) \right) \\
+ \int_0^{x^j_n} (q(t) + p^2(t)) \cos \left( \frac{n((x^j_n)^{\alpha} - 2t^{\alpha})}{\pi^{n-1}} - Q(x^j_n) + 2Q(t) \right) d_\alpha t \\
+ \int_0^{x^j_n} D_\alpha^p p(t) \sin \left( \frac{n((x^j_n)^{\alpha} - 2t^{\alpha})}{\pi^{n-1}} - Q(x^j_n) + 2Q(t) \right) d_\alpha t \left\{ \left( (p(\pi) + p(0)) a_1 + 2a_2 \right) \frac{(x^j_n)^{\alpha}}{\pi^{\alpha}} + (p(x^j_n) + p(0)) a_1 \frac{(x^j_n)^{\alpha}}{\pi^{\alpha}} \\
- \frac{1}{4n^2 \pi^{2-\alpha}} \int_0^{x^j_n} (q(t) + p^2(t)) d_\alpha t - a_1 \left( \frac{x^j_n}{\pi^{\alpha}} \right)^2 \\
- \frac{1}{2} \left( \int_0^{x^j_n} (q(t) + p^2(t)) d_\alpha t \right)^2 \right\} \sin \left( \frac{n((x^j_n)^{\alpha} - 2t^{\alpha})}{\pi^{n-1}} - Q(x^j_n) \right) \right\} + o \left( \frac{1}{n^{\alpha}} \right) = 0, \ |n| \to \infty.
\]

If last equality is divided by \( \cos \left( \frac{n(x^j_n)^{\alpha}}{\pi^{n-1}} - Q(x^j_n) \right) \) and necessary arrangements are made, we obtain

\[
\tan \left( \frac{n(x^j_n)^{\alpha}}{\pi^{n-1}} - Q(x^j_n) \right) = \\
\left\{ 1 + \frac{1}{2n\pi^{1-\alpha}} \left[ p(x^j_n) + p(0) + \int_0^{x^j_n} (q(t) + p^2(t)) \sin \left( \frac{2nt^{\alpha}}{\pi^{n-1}} - 2Q(t) \right) d_\alpha t \right. \\
+ \int_0^{x^j_n} D_\alpha^p p(t) \cos \left( \frac{2nt^{\alpha}}{\pi^{n-1}} - 2Q(t) \right) d_\alpha t \right\} \\
+ \frac{1}{4n^2 \pi^{2-\alpha}} \left[ 4p^2(0) + 2p(x^j_n) + p(0) + \int_0^{x^j_n} p(x^j_n) + p(0) \right] \\
+ a_1 \left( \frac{x^j_n}{\pi^{\alpha}} \right)^2 \int_0^{x^j_n} (q(t) + p^2(t)) d_\alpha t - a_1 \left( \frac{x^j_n}{\pi^{\alpha}} \right)^2 \left( \int_0^{x^j_n} (q(t) + p^2(t)) d_\alpha t \right)^2 \right\}^{-1} \times \\
\times \left\{ \frac{1}{2n\pi^{1-\alpha}} \left[ A^i_n - a_1 \frac{(x^j_n)^{\alpha}}{\pi^{\alpha}} - A^i_n + \int_0^{x^j_n} (q(t) + p^2(t)) d_\alpha t \right] \right\}
\]
Corollary 2. From (17) it is clear that the set $X$ of all nodal points is dense in the interval $[0, \pi]$.

For each fixed $x \in [0, \pi]$ and $\alpha \in (0, 1]$. We can choose a sequence $\{j_n\} \subset X$ so that $\lim_{|n| \to \infty} x_{j_n} = x$. Then, there exist finite limits and corresponding equalities hold:

$$Q(x) = \pi^{1-\alpha} \lim_{|n| \to \infty} \left( n \left( x_{j_n} \right)^{\alpha} - j_n \pi^\alpha \right),$$

$$f(x) := 2\pi^{1-\alpha} \lim_{|n| \to \infty} \left[ \pi^{1-\alpha} \left( n \left( x_{j_n} \right)^{\alpha} - j_n \pi^\alpha \right) - Q(x_{j_n}) \right],$$

$$g(x) := \pi^{1-\alpha} \lim_{|n| \to \infty} \left\{ 2\pi^{1-\alpha} \left[ n\pi^{1-\alpha} \left( n \left( x_{j_n} \right)^{\alpha} - j_n \pi^\alpha \right) - Q(x_{j_n}) \right] \right. - f(x_{j_n}) + A_n x_{j_n} - A_n \left( x_{j_n} \right)^{\alpha} \right\}$$

and

$$f(x) = \int_0^x \left( g(t) + p^2(t) \right) d_\alpha t - \frac{x^\alpha}{\pi^\alpha} \int_0^\pi \left( q(t) + p^2(t) \right) d_\alpha t,$$

$$g(x) = \int_0^x \left( q(t) + p^2(t) \right) p(t) d_\alpha t - \frac{x^\alpha}{\pi^\alpha} \int_0^\pi \left( q(t) + p^2(t) \right) p(t) d_\alpha t - x^\alpha \frac{p(\pi) + p(0)}{\pi^\alpha} \int_0^\pi \left( g(t) + p^2(t) \right) d_\alpha t.$$
Therefore we can prove the following theorem for the solution of the inverse nodal problem.

**Theorem 3.** Given any dense subset of nodal points $X_0 \subset X$ uniquely determines the functions $p(x)$ and $q(x)$ a.e. on $[0, \pi]$. Moreover, these functions can be found by the following algorithm.

**Step-1:** For each fixed $x \in [0, \pi]$ and $\alpha \in (0, 1]$, choose a sequence $(x_n^\alpha) \subset X_0$ such that $\lim_{|n| \to \infty} x_n^\alpha = x$.

**Step-2:** Find the function $Q(x)$ from (19) and calculate
\[
p(x) = D_x^\alpha Q(x),
\]

**Step-3:** Find the function $f(x)$ from (20) and determine
\[
q(x) - \frac{1}{\pi} \int_0^{\pi} q(t) d\alpha t := r(x) = D_x^\alpha f(x) - p^2(x) + \frac{1}{\pi} \int_0^{\pi} p^2(t) d\alpha t,
\]

**Step-4:** For each fixed $x \in [0, \pi]$ and $\alpha \in (0, 1]$, $\alpha Q(x) - x^\alpha (p(\pi) + p(0)) \neq 0$, find $g(x)$ from (21) and calculate
\[
\frac{1}{\pi} \int_0^{\pi} q(t) d\alpha t = \frac{1}{\alpha Q(x) - x^\alpha (p(\pi) + p(0))} \left[ g(x) - \int_0^{\pi} (r(t) + p^2(t)) p(t) d\alpha t + \frac{x^\alpha (p(\pi) + p(0))}{\pi} \int_0^{\pi} (r(t) + p^2(t)) d\alpha t \right],
\]

**Step-5:** Calculate the function $q(x)$ via the formula
\[
q(x) = r(x) + \frac{1}{\pi} \int_0^{\pi} q(t) d\alpha t.
\]

**Proof.** Formula (24) it is obvious from (10).

Differentiating (22) we get
\[
D_x^\alpha f(x) = q(x) + p^2(x) - \frac{1}{\pi} \int_0^{\pi} \left( q(t) + p^2(t) \right) d\alpha t.
\]

Denote $r(x) := q(x) - \frac{1}{\pi} \int_0^{\pi} q(t) d\alpha t$. We obtain immediately formula (25).

Substituting the function $q(x) = r(x) - \frac{1}{\pi} \int_0^{\pi} q(t) d\alpha t$ in (23) and taking (4) into account we get formula (26).

Finally, from (25) and (26) we arrive at (27).

\[\square\]

**References**

[1] Miller KS. An Introduction to fractional calculus and fractional differential equations. J Wiley and Sons New York: 1993.

[2] Oldham K, Spanier J. The fractional calculus. Theory and Applications of Differentiation and Integration of Arbitrary Order: (USA) Academic Press; 1974.

[3] Kilbas A, Srivastava H, Trujillo J. Theory and applications of fractional differential equations. Math Studies. North-Holland New York; 2006.

[4] Podlubny I. Fractional differential equations. USA: Academic Press; 1999.

[5] Khalil R, Al Horania M, Yousefa A, et al. A new definition of fractional derivative. J Comput Appl Math. 2014;279:65–70.

[6] Abdeljawad T, On conformable fractional calculus. J Comput Appl Math. 2015;279:57-66.
Atangana A, Baleanu D, Alsaedi A. New properties of conformable derivative. Open Math. 2015;13:889–898.

Chung WS. Fractional Newton mechanics with conformable fractional derivative. J Comput Appl Math. 2015;290:150–158. doi: 10.1016/j.cam.2015.04.049.

Anderson DR, Ullness DJ. Properties of the Katugampola fractional derivative with potential application in quantum mechanics. J Math Phys. 2015;56:063502. doi: 10.1063/1.4922018.

Benkhettou N, Hassani S, Torres DFM. A conformable fractional calculus on arbitrary time scales. J King Saud Univ Sci. 2016;28:93–98. doi: 10.1016/j.jksus.2015.05.003.

Zhou HW, Yang S, Zhang SQ. Conformable derivative approach to anomalous diffusion. Phys A. 2018;491:1001–1013. doi: 10.1016/j.physa.2017.09.101.

Ambarzumyan VA, Über eine Frage der eigenwerttheorie. Z Phys. 1929;53:690–695.

Rivero M, Trujillo JJ, Velasco MP. A fractional approach to the Sturm-Liouville problem. Centr Eur J Phys. 2013;11(10):1246–1254.

Klimek M, Agrawal OP. Fractional Sturm-Liouville problem. Comput Math Appl. 2013;6:795–812.

Khosravian-Arab H, Dehghan M, Eslahchi MR. Fractional Sturm-Liouville boundary value problems in unbounded domains, theory and applications. J Comput Phys. 2015;299:526–560.

Al-Towailb M A. A q-fractalional approach to the regular Sturm-Liouville problems. Electron J Differ Equ. 2017;88:1–13.

Mortazaasl H, Jodayree Akbarfam A. Trace formula and inverse nodal problem for a conformable fractional Sturm-Liouville problem. Inverse Problems in Science and Engineering. 201; doi: 10.1080/17415977.2019.1615909.

Allahverdiev BP, Tuna H, Yalçinkaya Y. Conformable fractional Sturm-Liouville equation. Mathematical Methods in the Applied Sciences. 2019;42(10):3508-3526.

Keskin B. Inverse problems for one dimensional conformable fractional Dirac type integro differential system. Inverse Problems. 2020; https://doi.org/10.1088/1361-6420/ab7e03.

Adalar I, Ozkan AS. Inverse problems for a conformable fractional Sturm-Liouville operators. Journal of Inverse and Ill-posed Problems. to appear.

Monje CA, Chen Y, Vinagre BM, et al. Fractional-order systems and controls: fundamentals and applications. London (UK): Springer-Verlag; 2010.

Baleanu D, Guvenc ZB, Machado JT. New trends in nanotechnology and fractional calculus applications. New York (US): Springer; 2010.

Mainardi F. Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models. London, UK: Imperial College Press; 2010.

Silva MF, Machado JAT. Fractional order PD α joint control of legged robots. J Vib Control. 2006;12:1483–1501. doi: 10.1177/1077546306070608.

Pfalzler A. Efficient solution of a vibration equation involving fractional derivatives. Int J Nonlin Mech. 2010;45:169–175. doi: 10.1016/j.ijnonlinmech.2009.10.006.

McLaughlin JR. Inverse spectral theory using nodal points as data—a uniqueness result. J Differential Equations. 1988;73(2):354-362.

Buterin SA, Shieh CT. Inverse nodal problem for differential pencils. Applied Mathematics Letters. 2009;22(8):1240-1247.

Buterin SA, Shieh CT. Incomplete inverse spectral and nodal problems for differential pencils. Results Math. 2012;62:167-179.

Hald OH, McLaughlin JR. Inverse problems: recovery of BV coefficients from nodes. Inverse Problems. 1998;14:245-273.

Shieh CT, Yurko VA. Inverse nodal and inverse spectral problems for discontinuous boundary value problems. J Math Anal Appl. 2008;347:266-272.

Yang CF. Reconstruction of the diffusion operator with nodal data. Z Naturforsch A. 2010;65:100-106.

Yang XF. A new inverse nodal problem. J Differential Equations. 2001;169:633-653.

Yang XF. A solution of the inverse nodal problem. Inverse Problems. 1997;13:203-213.

Law CK, Yang CF. Reconstructing the potential function and its derivatives using nodal data. Inverse Problems. 1998:14:299-312.

Browne PJ, Sleeman BD. Inverse nodal problems for Sturm–Liouville equations with eigenparameter dependent boundary conditions. Inverse Problems. 1996;12:377–381.

Shen CL, Shieh CT. An inverse nodal problem for vectorial Sturm–Liouville equation. Inverse Problems. 2000;16:349–356.
[37] Buterin SA, Yurko VA. Inverse spectral problem for pencils of differential operators on a finite interval. Vestnik Bashkir Univ. 2006;4:8-12.

[38] Yurko VA. An inverse problem for pencils of differential operators. Matem Sbornik. 2000; 191(10):137-160 (in Russian), English transl. in Sbornik Mathematics. 2000;191(10):1561–1586.

[39] Hald OH, McLaughlin JR. Solutions of inverse nodal problems. Inverse Problems. 1989;5:307-347.

[40] Law CK, Shen CL, Yang CF. The inverse nodal problem on the smoothness of the potential function. Inverse Problems. 1999;15:253-263. Errata: Inverse Problems. 2001;17(2):361–363.

[41] Koyunbakan H. A new inverse problem for the diffusion operator. Appl Math Lett. 2006;19:995-999.

[42] Freiling G, Yurko VA. Inverse Sturm–Liouville problems and their applications. New York: Nova Science Publishers; 2001.

[43] Wang Y, Zhou J, Li Y. Fractional sobolev’s spaces on time scales via conformable fractional calculus and their application to a fractional differential equation on time scales. Adv Math Phys. 2016;2016:9636491:21.