Photon-added coherent states as nonlinear coherent states

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Abstract

The states $|\alpha, m\rangle$, defined as $\hat{a}^m|\alpha\rangle$ up to a normalization constant and $m$ is a nonnegative integer, are shown to be the eigenstates of $f(\hat{n}, m)\hat{a}$ where $f(\hat{n}, m)$ is a nonlinear function of the number operator $\hat{n}$. The explicit form of $f(\hat{n}, m)$ is constructed. The eigenstates of this operator for negative values of $m$ are introduced. The properties of these states are discussed and compared with those of the state $|\alpha, m\rangle$.

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I. INTRODUCTION

Coherent states are important in many fields of physics \[1,2\]. Coherent states \(|\alpha\rangle\), defined as the eigenstates of the harmonic oscillator annihilation operator \(\hat{a}\), \(\hat{a}|\alpha\rangle = \alpha|\alpha\rangle\) \[3\], have properties like the classical radiation field. There exist states of the electromagnetic field whose properties, like squeezing, higher order squeezing, antibunching and sub-Poissonian statistics \[4,5\], are strictly quantum mechanical in nature. These states are called as nonclassical states. The coherent states define the limit between the classical and nonclassical behaviour of the radiation field as far as the nonclassical effects are considered. A generalization of the coherent states was done by q-deforming the basic commutation relation \([\hat{a}, \hat{a}^\dagger] = 1\) \[6,7\]. A further generalization is to define states that are eigenstates of the operator \(f(\hat{n})\hat{a}\),

\[
f(\hat{n})\hat{f}|f, \alpha\rangle = \alpha|f, \alpha\rangle,
\]

(1)

where \(f(\hat{n})\) is a operator valued function of the number operator \(\hat{n} = \hat{a}^\dagger\hat{a}\). These eigenstates are called as nonlinear coherent states and they are nonclassical. In the linear limit, \(f(\hat{n}) = 1\), the nonlinear coherent states become the usual coherent states \(|\alpha\rangle\). The nonlinear coherent states were introduced, as f-coherent states, in connection with the study of the oscillator whose frequency depends on its energy \[8\]. A class of nonlinear coherent states can be realized physically as the stationary states of the center-of-mass motion of a trapped ion \[9\]. These nonlinear coherent states exhibit nonclassical features like squeezing and self-splitting.

The photon-added coherent states \(|\alpha, m\rangle\) \[10\] are defined as

\[
|\alpha, m\rangle = \hat{a}^m|\alpha\rangle \sqrt{\langle \alpha | \hat{a}^m \hat{a}^m | \alpha \rangle},
\]

(2)

where \(m\) is a nonegative integer. The states \(|\alpha, m\rangle\) exhibit nonclassical features like phase squeezing and sub-Poissonian statistics. These states are produced in the interaction of a two-level atom with a cavity field initially prepared in the coherent state \(|\alpha\rangle\) \[10\]. In the present contribution it is shown that the photon-added coherent states can be interpreted as nonlinear coherent states. This is done by showing that the states \(|\alpha, m\rangle\) obey the equation
with a suitable choice for the function $f(\hat{n}, m)$. The operator $f(\hat{n}, m)$ is a valid operator for negative values of $m$ also. The eigenstates of $f(\hat{n}, m)\hat{a}$ with negative values of $m$ are constructed and their properties compared with those of $|\alpha, m\rangle$.

II. CONSTRUCTION OF $f(\hat{n}, m)$

In this section we construct the explicit form of the operator valued function $f(\hat{n}, m)$. The coherent states $|\alpha\rangle$ satisfy, by definition,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle,$$  \hspace{1cm} (4)

where $\alpha$ is a complex number. Premultiplying both the sides of this equation by $\hat{a}^\dagger m$ yields

$$\hat{a}^\dagger m \hat{a}|\alpha\rangle = \alpha \hat{a}^\dagger m |\alpha\rangle,$$  \hspace{1cm} (5)

where $m$ is a nonnegative integer. Using the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$, the above equation is written as

$$(\hat{a} \hat{a}^\dagger m - m \hat{a}^\dagger (m-1))|\alpha\rangle = \alpha \hat{a}^\dagger m |\alpha\rangle,$$  \hspace{1cm} (6)

which, making use of the identity $\frac{1}{1 + \hat{a}^\dagger \hat{a}} \hat{a} \hat{a}^\dagger = 1$, leads to

$$(\hat{a} - \frac{m}{1 + \hat{a}^\dagger \hat{a}} \hat{a}) \hat{a}^\dagger m |\alpha\rangle = \alpha \hat{a}^\dagger m |\alpha\rangle.$$  \hspace{1cm} (7)

Comparing Eq. (7) and Eq. (3) gives the expression for $f(\hat{n}, m)$ as

$$f(\hat{n}, m) = 1 - \frac{m}{1 + \hat{a}^\dagger \hat{a}}.$$

This shows that the photon-added coherent states can be interpreted as nonlinear coherent states. These states are a class nonlinear coherent states than can be physically realized in the interaction of a two- level atom with a cavity field that is initially prepared in the coherent state $|\alpha\rangle$. 

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III. \(|\alpha, m\rangle\) AS DEFORMED m-PHOTON NUMBER STATE \(|m\rangle\)

In this section it is shown that the photon-added coherent states can be written as the nonunitarily deformed number state \(|\alpha, m\rangle\). This is achieved by the method given by Shanta, et al \[11\]. Firstly, a brief review of the method is given. Consider an ”annihilation operator” \(\hat{A}\) which annihilates a set of number states \(|n_i\rangle, i = 1, 2, ... k\). Then we can construct a sector \(S_i\) by repeatedly applying \(\hat{A}^\dagger\), the adjoint of \(\hat{A}\), on the number state \(|n_i\rangle\). Thus we have \(k\) sectors corresponding to the states that are annihilated by \(\hat{A}\). A given sector may turnout to be either finite or infinite dimensional. If a sector, say \(S_j\), is of infinite dimension then we construct an operator \(\hat{G}_j^\dagger\) such that the commutator \([\hat{A}, \hat{G}_j^\dagger] = 1\) holds in the sector. Then the eigenstates of \(\hat{A}\) can be written as \(e^{\alpha\hat{G}_j^\dagger} |n_j\rangle\). If the operator \(\hat{A}\) is of the form \(f(\hat{n})\hat{a}^p\), where \(p\) is a nonnegative integer and \(f(\hat{n})\) is a operator valued function of the number operator \(\hat{a}^\dagger\hat{a}\), such that it annihilates the number state \(|j\rangle\) then \(\hat{G}_j^\dagger\) is constructed as

\[
\hat{G}_j^\dagger = \frac{1}{p} \hat{A}^\dagger \frac{1}{\hat{A}\hat{A}^\dagger} (\hat{a}^\dagger\hat{a} + p - j).
\] (9)

The photon-added coherent states are the eigenstates of \(f(\hat{n}, m)\hat{a}\) with \(f(\hat{n}, m)\) given by Eq.(8). This operator annihilates the vacuum state \(|0\rangle\) and the \(m\)-photon state \(|m\rangle\). The states in between the vacuum state and the \(m\)-photon state are not annihilated. In this sense it is different from the \(m\)-photon annihilation operator \(\hat{a}^m\) which annihilates all the states \(|i\rangle, i = 0, 1, 2, ... m\). To discuss the case of \(|\alpha, m\rangle\) let

\[
\hat{A} = (1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}})\hat{a}.
\] (10)

The adjoint \(\hat{A}^\dagger\) is given by

\[
\hat{A}^\dagger = \hat{a}^\dagger (1 - \frac{m}{1 + \hat{a}^\dagger\hat{a}}),
\] (11)

We construct the sector \(S_0\) by repeatedly applying \(\hat{A}^\dagger\) on the vacuum state \(|0\rangle\). \(S_0\) is the set \(|i\rangle, i = 0, 1, 2, ... m - 1\) and it is finite dimensional. The sector \(S_m\), built by the repeated application of \(\hat{A}^\dagger\) on \(|m\rangle\), is the set \(|i\rangle, i = m, m + 1, ...\) and it is of infinite dimension. Hence we can construct an operator \(\hat{G}^\dagger\) such that \([\hat{A}, \hat{G}^\dagger] = 1\) holds in \(S_m\). To construct \(\hat{G}^\dagger\),
corresponding to the operator \( \hat{A} \) given by Eq. (10), we set \( p = 1 \) and \( j = m \) in Eq. (9) and this yields

\[
\hat{G}^\dagger = \hat{a}^\dagger. \tag{12}
\]

Thus on the sector \( S_m \) we have \( [\hat{A}, \hat{a}^\dagger] = 1 \). Thus the photon-added coherent states, which are the eigenstates of \( \hat{A} \), can be written as \( e^{\alpha \hat{a}^\dagger} |m\rangle \). However, this is not a unitary deformation.

**IV. EIGENSTATES OF \( f(\hat{n}, m)\hat{a} \) WITH NEGATIVE \( m \)**

The form of \( f(\hat{n}, m) \), given by Eq. (8), suggests that it is a well defined operator valued function, on the harmonic oscillator Hilbert space, for negative integer values of \( m \) also. In this section the nonlinear coherent states, with negative \( m \) in the expression for \( f(\hat{n}, m) \), are constructed. Denoting the eigenstates by \( |\alpha, -m\rangle \), the equation to determine them is

\[
(1 + \frac{m}{1 + \hat{a}^\dagger \hat{a}})\hat{a}|\alpha, -m\rangle = \alpha|\alpha, -m\rangle. \tag{13}
\]

Expanding \( |\alpha, -m\rangle \) in terms of the number states \( |n\rangle \) as

\[
|\alpha, -m\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{14}
\]

where \( c_n \)'s are the expansion coefficients and substituting the expansion in Eq. (13) leads to the recursion relation

\[
c_n = \frac{m! \sqrt{n!} \alpha^n}{(n + m)!} c_0. \tag{15}
\]

The constant \( c_0 \) is determined by normalization. The normalized \( |\alpha, -m\rangle \) is given by

\[
|\alpha, -m\rangle = N \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!(n + 1)(n + 2)...(n + m)}} |n\rangle; \tag{16}
\]

\[
N^{-1} = \sqrt{\sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2}} = \frac{1}{m! \sqrt{2 F_2(1, 1, m + 1, m + 1, |\alpha|^2)}},
\]

where \( 2 F_2(1, 1, m + 1, m + 1, |\alpha|^2) \) is the generalized Hypergeometric function \([12] \). The number state expansion for the state \( |\alpha, m\rangle \) is \([10] \).
\[ |\alpha, m\rangle = \frac{\exp(-|\alpha|^2/2)}{\sqrt{L_m(-|\alpha|^2)m!}} \sum_{n=0}^{\infty} \frac{\alpha^n \sqrt{(m+n)!}}{n!} |n + m\rangle, \quad (17) \]

where \( L_m(x) \) is the Laguerre polynomial of order \( m \) defined by \[ L_m(x) = \sum_{n=0}^{m} \frac{(-x)^n m!}{n!(m-n)!}. \quad (18) \]

The state \( |\alpha, -m\rangle \) involves a superposition of all the Fock states starting with the vacuum state \( |0\rangle \). But in the state \( |\alpha, m\rangle \) the Fock states \( |0\rangle, |1\rangle, \ldots |m-1\rangle \) are not present. This important difference leads to different limiting cases of the states \( |\alpha, m\rangle \) and \( |\alpha, -m\rangle \) when \( \alpha \to 0 \). In the limit \( \alpha \to 0 \) the state \( |\alpha, -m\rangle \) becomes the vacuum state \( |0\rangle \), which has properties like a classical radiation field, irrespective of the value of \( m \) and the state \( |\alpha, m\rangle \) becomes the Fock state \( |m\rangle \) which is nonclassical. In the limit \( m \to 0 \) the states \( |\alpha, m\rangle \) and \( |\alpha, -m\rangle \) become the coherent state \( |\alpha\rangle \). Thus, \( |\alpha, -m\rangle(|\alpha, m\rangle) \) is a state that is intermediate between the vacuum state (the number state \( |m\rangle \)) and the coherent state.

The photon-added coherent states are obtained by the action of \( \hat{a}^{\dagger m} \) on the coherent state. The states \( |\alpha, -m\rangle \) can be written in a similar form using the inverse operators \( \hat{a}^{-1} \) and \( \hat{a}^{\dagger -1} \). These operators are defined in terms of their actions on the number states \( |n\rangle \) as follows:

\[ \hat{a}^{-1}|n\rangle = \frac{1}{\sqrt{n+1}} |n + 1\rangle, \quad (19) \]
\[ \hat{a}^{\dagger -1}|n\rangle = \frac{1}{\sqrt{n}} |n - 1\rangle \quad \text{for } n \neq 0, \quad (20) \]
\[ \hat{a}^{\dagger -1}|0\rangle = 0. \quad (21) \]

Using these inverse operators and Eqn.(16) the state \( |\alpha, -m\rangle \) can be written as

\[ |\alpha, -m\rangle = N \hat{a}^{\dagger -m} \hat{a}^{-m} |\alpha\rangle. \quad (22) \]

The states \( |\alpha, -m\rangle \) correspond to the nonlinear coherent states with \( -m \) replacing \( m \) in \( f(\hat{n}, m) \). However, they are obtained by the action of \( \hat{a}^{\dagger -m} \hat{a}^{-m} \) on \( |\alpha\rangle \) and not \( \hat{a}^{\dagger -m} \) on \( |\alpha\rangle \). Using the method reviewed in Section III we can show that

\[ |\alpha, -m\rangle = e^{\alpha \hat{G}^{\dagger}} |0\rangle, \quad (23) \]

where \( \hat{G}^{\dagger} = \hat{a}^{\dagger} \frac{1+\hat{a}^{\dagger} \hat{a}}{1+m+\hat{a}^{\dagger} \hat{a}}. \)
V. SQUEEZING IN $|\alpha, -m\rangle$

The state $|\alpha, -m\rangle$ exhibits squeezing in both x- and p-quadratures. The x- and p-quadratures are given in terms of $\hat{a}$ and $\hat{a}^\dagger$ by

$$x = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \quad p = \frac{\hat{a}^\dagger - \hat{a}}{i\sqrt{2}}. \quad (24)$$

The mean values of the relevant operators in the state $|\alpha, -m\rangle$ are

$$\langle \hat{a} \rangle = \alpha N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2 (n + m + 1)}, \quad (25)$$

$$\langle \hat{a}^2 \rangle = \alpha^2 N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2 (n + m + 1)(n + m + 2)}, \quad (26)$$

and

$$\langle \hat{a}^\dagger \hat{a} \rangle = N^2 \sum_{n=1}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2 n}. \quad (27)$$

The mean values of $\hat{a}^\dagger$ and $\hat{a}^{12}$ are obtained by taking the complex conjugates of $\langle \hat{a} \rangle$ and $\langle \hat{a}^2 \rangle$ respectively. The uncertainty in x is

$$\langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{2} \left[ 1 + 2 \langle \hat{a}^\dagger \hat{a} \rangle + \langle \hat{a}^{12} \rangle - \langle \hat{a} \rangle^2 - \langle \hat{a}^\dagger \rangle^2 - 2 \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle \right], \quad (28)$$

and that in p is

$$\langle p^2 \rangle - \langle p \rangle^2 = \frac{1}{2} \left[ 1 + 2 \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^2 \rangle - \langle \hat{a}^{12} \rangle + \langle \hat{a} \rangle^2 + \langle \hat{a}^\dagger \rangle^2 - 2 \langle \hat{a} \rangle \langle \hat{a}^\dagger \rangle \right]. \quad (29)$$

In Fig. 1 the variance in p is shown for real $\alpha$ for various values of m. As expected the uncertainty in p is close to $\frac{1}{2}$, the uncertainty in p in the vacuum state, when $\alpha$ is close to zero. In the case of the state $|\alpha, m\rangle$ the variance is close to $m + \frac{1}{2}$ when $\alpha$ is close to zero. For real values of $\alpha$ the p-quadrature is always squeezed, $\langle p^2 \rangle - \langle p \rangle^2 < \frac{1}{2}$, for the state $|\alpha, -m\rangle$. For large values of $\alpha$ the variance in p approaches that of the coherent state. If $\alpha$ becomes $i\alpha$ the expression for variance in p becomes the expression for variance in x. Since p shows squeezing for real $\alpha$ the x-quadrature exhibits squeezing for imaginary $\alpha$. 


VI. PHOTON STATISTICS OF $|\alpha, -m\rangle$

The photon number distribution $p(n)$ for the state $|\alpha, -m\rangle$ is

$$p(n) = |\langle n | \alpha, -m \rangle|^2,$$

$$= N \frac{|\alpha|^{2n} n!}{(n + m)!^2}. \quad (30)$$

When $m = 0$ the distribution becomes the Poissonian distribution whose mean value is $|\alpha|^2$.

A measure of the variance of the photon number distribution is given by the Mandel’s $q$ parameter [14],

$$q = \frac{\langle (\Delta \hat{n})^2 \rangle - \langle \hat{n} \rangle}{\langle \hat{n} \rangle}. \quad (31)$$

The coherent states have $q$ as zero. Negative value of $q$ indicates that the distribution $p(n)$ is sub-Poissonian and it is a nonclassical feature. The photon-added coherent states $|\alpha, m\rangle$ are always sub-Poissonian for all values of $m$. For the state $|\alpha, -m\rangle$ the mean values of $\hat{n}$ and $\hat{n}^2$ are given by

$$\langle \hat{n} \rangle = N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2} n, \quad (32)$$

$$\langle \hat{n}^2 \rangle = N^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n} n!}{(n + m)!^2} n^2. \quad (33)$$

In Fig. 2 the $q$-parameter, calculated from Eqs. (31)-(33), for the state $|\alpha, -m\rangle$ is shown as a function of $|\alpha|$. The states $|\alpha, -m\rangle$ have $q$ always greater than zero indicating that they are super-Poissonian. For small values of $\alpha$ the $q$-parameter is close to zero and for large values of $\alpha$ it approaches zero.

VII. SUMMARY

The photon-added coherent states are nonlinear coherent states. They are the eigenstates of the operator $(1 - \frac{m}{1+\alpha^2})\hat{a}$ when $m$ takes positive integer values. This operator is a meaningful operator when $m$ takes negative integer values also. The corresponding eigenstates $|\alpha, -m\rangle$ are nonclassical. The photon-added coherent state $|\alpha, m\rangle$ results from the
action of $\hat{a}^m$ on the coherent state $|\alpha\rangle$ while the state $|\alpha, -m\rangle$ comes from the action of $\hat{a}^\dagger - m \hat{a} - m$ on the coherent state $|\alpha\rangle$. Both $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ show squeezing. While $|\alpha, m\rangle$ is sub-Poissonian the state $|\alpha, -m\rangle$ is not. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ become $|m\rangle$ and $|0\rangle$ respectively in the limit $\alpha \to 0$ but they become the coherent state $|\alpha\rangle$ when $m \to 0$. The states $|\alpha, m\rangle$ and $|\alpha, -m\rangle$ are the result of deforming the number states $|m\rangle$ and $|0\rangle$ respectively.

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FIGURES

FIG. 1. Uncertainty $S$, $\langle p^2 \rangle - \langle p \rangle^2$, in $p$ as a function of $\alpha$ for $m=1$, $m=5$ and $m=10$ for the state $|\alpha, -m\rangle$. The real $\alpha$ is denoted as $r$.

FIG. 2. Mandel’s $q$ parameter as a function of $|\alpha|$ for $m = 1$, $m = 5$ and $m = 10$. $|\alpha|$ is denoted as $r$.
