Conditional tail risk expectations for location-scale mixture of elliptical distributions

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July 21, 2020

Abstract We present general results on the univariate tail conditional expectation (TCE) and multivariate tail conditional expectation for location-scale mixture of elliptical distributions. Examples include the location-scale mixture of normal distributions, location-scale mixture of Student-\(t\) distributions, location-scale mixture of Logistic distributions and location-scale mixture of Laplace distributions. We also consider portfolio risk decomposition with TCE for location-scale mixture of elliptical distributions.

Keyword Tail conditional expectations; Portfolio allocations; Multivariate risk measures; Location-scale mixture; Elliptical distributions

1 Introduction and Motivation

Tail conditional expectation (TCE), one of important risk measures, is common and practical. TCE of a random variable \(X\) is defined as \(TCE_X(x_q) = E(X|X > x_q)\), where \(x_q\) is a particular value, generally referred to as the \(q\)-th quantile with \(F_X(x_q) = 1 - q\). Here \(F_X(x) = 1 - F_X(x)\) is tail distribution function of \(X\). The TCE has been discussed in many literatures (see Landsman and Valdez (2003), Ignatieva and Landsman (2015, 2019)). Recently, a new type of multivariate tail conditional expectation (MTCE) was defined by Landsman et al. (2016). It is the following special case when \(q = (q, q, \cdots, q)\).

\[
MTCE_q(X) = E[X|X > VaR_q(X)]
\]

\[= E[X|X_1 > VaR_{q_1}(X_1), \cdots, X_n > VaR_{q_n}(X_n)],\; q = (q_1, \cdots, q_n) \in (0, 1)^n,\]
where $\mathbf{X} = (X_1, X_2, \cdots, X_n)^T$ is an $n \times 1$ vector of risks with cumulative distribution function $F_{\mathbf{X}}(x)$ and tail function $T_{\mathbf{X}}(x)$,

$$VaR_q(\mathbf{X}) = (VaR_{q_1}(X_1), VaR_{q_2}(X_2), \cdots, VaR_{q_n}(X_n))^T,$$

and $VaR_{q_k}(X_k)$, $k = 1, 2, \cdots, n$ is the value at risk (VaR) measure of the random variable $X_k$, being the $q_k$-th quantile of $X_k$ (see Landsman et al. (2016)). On the basis of it, Mousavi et al. (2019) study multivariate tail conditional expectation for scale mixtures of skew-normal distribution.

Closely related to tail conditional expectation is portfolio risk decomposition with TCE, it’s research has experienced a rapid growth in the literature. Portfolio risk decomposition based on TCE for the elliptical distribution was studied in Landsman and Valdez (2003) and extended to the multivariate skew-normal distribution in Vernic (2006). The phase-type distributions and multivariate Gamma distribution were researched in Cai and Li (2005) and Furman and Landsman (2007), respectively. Furthermore, Hashorva and Ratovomirija (2015) considered the capital allocation with TCE for mixed Erlang distributed risks joined by the Sarmanov distribution, and Ignatieva and Landsman (2019) has given the expression of TCE-based allocation for the generalised hyperbolic distribution. Recently, Zuo and Yin (2020) deals with the tail conditional expectation for univariate generalized skew-elliptical distributions and multivariate tail conditional expectation for generalized skew-elliptical distributions.

The rest of the paper is organized as follows. In Section 2 we define the location-scale mixture of elliptical distributions and establish some properties of it. Furthermore, we give several examples as special cases of it. In Section 3 we derive TCE for univariate cases of mixture of elliptical distributions, and in Section 4, we provide expression of MTCE for mixture of elliptical distributions. Section 5 offers expression of portfolio risk decomposition with TCE for mixture of elliptical distributions. Section 6 gives concluding remarks.

2 Mixture of elliptical distributions

In this section we introduce the location-scale mixture of elliptical (LSME) distributions and some its properties. Let $\mathbf{Y} \sim LSEM_n(\mu, \Sigma, \Theta, g_n)$ be an $n$-dimensional LSME distribution with location parameter $\mu$ and positive definite scale matrix $\Sigma = (\sigma_{i,j})^n_{i,j=1}$, if

$$\mathbf{Y} = \mu + \Theta \beta + \Theta^{1/2} \Sigma^{1/2} \mathbf{X},$$

(2.1)
where $\beta \in \mathbb{R}^n$, and $X \sim E_n(0, I_n, g_n)$. Assume that $X$ is independent of non-negative scalar random variable $\Theta$. We have

$$Y|\Theta = \theta \sim E_n(\mu + \theta \beta, \theta \Sigma, g_n).$$

(2.2)

Here $X$ is an $n$-dimensional elliptical random vector, and denoted by $X \sim E_n(\mu, \Sigma, g_n)$. If it’s probability density function exists, the form will be

$$f_X(x) := \frac{1}{\sqrt{\mid \Sigma \mid}} g_n \left\{ \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \ x \in \mathbb{R}^n,$$

(2.3)

where $\mu$ is an $n \times 1$ location vector, $\Sigma$ is an $n \times n$ scale matrix and $g_n(u), u \geq 0$, is the density generator of $X$. This density generator satisfies condition: (see Fang et al. (1990))

$$\int_0^\infty u^{\frac{n}{2} - 1} g_n(u)du < \infty.$$ 

(2.4)

The characteristic function of $X$ takes the form $\varphi_X(t) = \exp \left\{ it^T \mu \right\} \psi \left( \frac{1}{2} t^T \Sigma t \right), \ t \in \mathbb{R}^n$, with function $\psi(t) : [0, \infty) \rightarrow \mathbb{R}$, called the characteristic generator. Furthermore, the condition

$$|\psi'(0)| < \infty,$$

(2.5)

guarantees the existence of the covariance matrix of $X$ (see Fang et al. (1990)). Suppose $A$ is a $k \times n$ matrix, and $b$ is a $k \times 1$ vector. Then

$$AX + b \sim E_k \left( A \mu + b, A \Sigma A^T, g_k \right).$$

(2.6)

To express conditional tail risk measures for $n$-dimensional mixture of elliptical distributions we introduce the cumulative generator $\overline{G}_n(u)$ (see Landsman et al.(2018)):

$$\overline{G}_n(u) = \int_u^\infty g_n(v)dv.$$ 

(2.7)

Let $X^* \sim E_n(\mu, \Sigma, \overline{G}_n)$ be an elliptical random vector with generator $\overline{G}_n(u)$, whose the density function (if it exists)

$$f_{X^*}(x) = \frac{-1}{\psi'(0) \sqrt{\mid \Sigma \mid}} \overline{G}_n \left\{ \frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \ x \in \mathbb{R}^n.$$

(2.8)

We list some examples of the mixture of elliptical family, including location-scale mixture of normal (LSMN) distributions, location-scale mixture of Student-\(t\) (LSMSt) distributions, location-scale mixture of Logistic (LSMLo) distributions and location-scale mixture of Laplace (LSMLa) distributions.
Example 2.1 (Mixture of normal distribution). An $n$-dimensional normal random vector $\mathbf{X}$ with location parameter $\mu$ and scale matrix $\Sigma$ has density function
\[ f_{\mathbf{X}}(x) = \frac{(2\pi)^{-\frac{n}{2}}}{\sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\}, \quad x \in \mathbb{R}^n, \]
and denoted by $\mathbf{X} \sim N_n(\mu, \Sigma)$. Therefore, the location-scale mixture of normal random vector $\mathbf{Y} \sim LSMN_n(\mu, \Sigma, \beta, \Theta)$ is defined as
\[ \mathbf{Y} = \mu + \Theta \beta + \Theta^\frac{1}{2} \Sigma^\frac{1}{2} \mathbf{X}, \quad (2.9) \]
and $\mu$, $\Sigma$, $\Theta$ and $\beta$ are the same as in (2.1).

Example 2.2 (Mixture of student-t distribution). An $n$-dimensional student-$t$ random vector $\mathbf{X}$ with location parameter $\mu$, scale matrix $\Sigma$ and $m > 0$ degrees of freedom has density function
\[ f_{\mathbf{X}}(x) = \frac{c_n}{\sqrt{|\Sigma|}} \left[ 1 + \frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{m} \right]^{-\frac{m+n}{2}}, \quad x \in \mathbb{R}^n, \]
where $c_n = \frac{\Gamma\left( \frac{m+n}{2} \right)}{\Gamma\left( \frac{m}{2} \right) \Gamma\left( \frac{n}{2} \right)}$, and denoted by $\mathbf{X} \sim St_n(\mu, \Sigma, m)$. So that the location-scale mixture of student-$t$ random vector $\mathbf{Y} \sim LSMSt_n(\mu, \Sigma, \beta, \Theta)$ satisfies
\[ \mathbf{Y} = \mu + \Theta \beta + \Theta^\frac{1}{2} \Sigma^\frac{1}{2} \mathbf{X}, \quad (2.10) \]
and $\mu$, $\Sigma$, $\Theta$ and $\beta$ are the same as in (2.1).

Example 2.3 (Mixture of Logistic distribution). Density function of an $n$-dimension Logistic random vector $\mathbf{X}$ with location parameter $\mu$ and scale matrix $\Sigma$ can be expressed as
\[ f_{\mathbf{X}}(x) = \frac{c_n}{\sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \left[ 1 + \exp \left\{ -\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\} \right]^{-\frac{m+n}{2}}, \quad x \in \mathbb{R}^n, \]
where $c_n = (2\pi)^{-n/2} \left[ \sum_{i=0}^{\infty} (-1)^i i^{-1} i^{-n/2} \right]^{-1}$, and denoted by $\mathbf{X} \sim Lo_n(\mu, \Sigma)$. The location-scale mixture of Logistic random vector $\mathbf{Y} \sim LSMLo_n(\mu, \Sigma, \beta, \Theta)$ satisfies
\[ \mathbf{Y} = \mu + \Theta \beta + \Theta^\frac{1}{2} \Sigma^\frac{1}{2} \mathbf{X}, \quad (2.11) \]
and $\mu$, $\Sigma$, $\Theta$ and $\beta$ are the same as in (2.1).

Example 2.4. (Mixture of Laplace distribution). Density of Laplace random vector $\mathbf{X}$ with location parameter $\mu$ and scale matrix $\Sigma$ is given by
\[ f_{\mathbf{X}}(x) = \frac{c_n}{\sqrt{|\Sigma|}} \exp \left\{ -\frac{1}{2} |(x - \mu)^T \Sigma^{-1} (x - \mu)|^{1/2} \right\}, \quad x \in \mathbb{R}^n, \]
where $c_n = \frac{\Gamma\left( \frac{n}{2} \right)}{2 \pi^{n/2} \Gamma\left( \frac{1}{2} \right)}$, and denoted by $\mathbf{X} \sim La_n(\mu, \Sigma)$. Hence, the location-scale mixture of Laplace random vector $\mathbf{Y} \sim LSMLa_n(\mu, \Sigma, \beta, \Theta)$ is defined as
\[ \mathbf{Y} = \mu + \Theta \beta + \Theta^\frac{1}{2} \Sigma^\frac{1}{2} \mathbf{X}, \quad (2.12) \]
and $\mu$, $\Sigma$, $\Theta$ and $\beta$ are the same as in (2.1).
3 Univariate cases

**Theorem 3.1.** Let $Y \sim \text{LSME}_1(\mu, \sigma^2, \beta, \Theta, g_1)$ be an univariate location-scale mixture of elliptical random variable defined as (2.1). We suppose

$$\int_0^\infty g_1(u)du < \infty. \tag{3.13}$$

Then

$$TCE_Y(y_q) = \mu + E_{\Theta}[\theta \beta + \delta_\theta(\sqrt{\theta}\sigma)^2], \tag{3.14}$$

where

$$\delta_\theta = \frac{1}{\sqrt{\sigma^2}G_1(\frac{1}{2}z^2_q)}1 - \phi_1(z_q),$$

with $Z \sim E_1(0, 1, g_1)$ and $z_q = \frac{y_q - \mu - \theta \beta}{\sqrt{\sigma^2}}$. 

Proof. Using definition and tower property of expectations, we obtain

$$TCE_Y(y_q) = E[Y|Y > y_q] = E_{\Theta}[E(Y|Y > y_q, \Theta)].$$

Since

$$E[Y|Y > y_q, \Theta = \theta] = E[(Y|\Theta = \theta)(Y|\Theta = \theta) > y_q] = E[M|M > y_q] = TCE_M(y_q),$$

where $M \sim E_1(\mu + \theta \beta, \theta\sigma^2, g_1)$, and the second equality we have used (3.1).

Using Theorem 1 in Landsman and Valdez (2003), we obtain (3.14), which completes the proof of Theorem 3.1.

**Remark 3.1.** We find that $TCE_{Y|\Theta}(y_q)$ is a special case of Theorem 1 in Landsman and Valdez (2003).

**Corollary 3.1.** Let $Y \sim \text{LSMN}_1(\mu, \sigma^2, \beta, \Theta)$ be an univariate location-scale mixture of normal random variable defined as (2.9). Under conditions in Theorem 3.1, we obtain the TCE for location-scale mixture of normal distributions. Its’ form is the same as (3.14), where

$$\delta_\theta = \frac{1}{\sqrt{\sigma^2}\phi_1(\frac{1}{2}z^2_q)}1 - \Phi_1(z_q).$$

Additionally, $\phi_1(\cdot)$ and $\Phi_1(\cdot)$ denote the density and distribution functions of normal distributions, respectively.
Proof. Letting the density generator $\mathcal{G}_1(u) = g_1(u) = \phi_1(u) = (2\pi)^{-\frac{1}{2}}e^{-u}$ in Theorem 3.1, we directly obtain our result. This completes the proof of Corollary 3.1.

**Corollary 3.2.** Let $Y \sim \text{LSTMSt}_1(\mu, \sigma^2, \beta, \Theta)$ be an univariate location-scale mixture of student-t random variable defined as (2.10). Under conditions in Theorem 3.1, we obtain the TCE for location-scale mixture of Student-t distributions. Its’ form is the same as (3.1), where

$$
\delta_g = \frac{\frac{1}{\sqrt{\theta}}\mathcal{G}_1\left(\frac{\sqrt{z_q}}{\theta}\right)}{F_Z(z_q)} = \frac{\frac{1}{\sqrt{\theta}}c_1 \frac{m(1 + \frac{z_q^2}{m})^{-(m-1)/2}}{F_Z(z_q)}}{\frac{1}{\sqrt{\theta}}t_{m,1}(z_q; 0, 1)} = \frac{1}{T_{m,1}(z_q; 0, 1)}.
$$

In addition, $t_{m,1}(z_q; 0, 1)$ and $T_{m,1}(z_q; 0, 1)$ are the density and distribution functions of Student-t distributions, respectively (see Landsman et al. (2016)).

Proof. Letting $g_1(u) = c_1(1 + \frac{2u}{m})^{-(m+1)/2}$, $\mathcal{G}_1(u) = c_1 \frac{m(1 + \frac{2u}{m})^{-(m-1)/2}}{F_Z(z_q)}$ and $c_1 = \frac{\Gamma((m+1)/2)}{\Gamma(m/2)(mn)^{1/2}}$ (see Landsman et al. (2016)) in Theorem 3.1, we immediately obtain our result. This completes the proof of Corollary 3.2.

**Corollary 3.3.** Let $Y \sim \text{LSMLo}_1(\mu, \sigma^2, \beta, \Theta)$ be an univariate location-scale mixture of Logistic random variable defined as (2.11). Under conditions in Theorem 3.1, we obtain the TCE for location-scale mixture of Logistic distributions. Its’ form is the same as (3.1), where

$$
\delta_g = \frac{\frac{1}{\sqrt{\theta}}\mathcal{G}_1\left(\frac{\sqrt{z_q}}{\theta}\right)}{F_Z(z_q)} = \frac{\frac{1}{\sqrt{\theta}}c_1 \frac{\exp\left(-\frac{\sqrt{z_q}}{\theta}\right)}{1+\exp\left(-\frac{\sqrt{z_q}}{\theta}\right)}}{F_Z(z_q)} = \left[\frac{1}{2(\sqrt{2\pi})^{-1} + \phi(z_q)}\right] \frac{\frac{1}{\sqrt{\theta}}\phi(z_q)}{F_Z(z_q)}.
$$

In addition, $\phi(\cdot)$ is the density functions of normal distributions (see Landsman and Valdez (2003)).

Proof. Letting $g_1(u) = c_1 \frac{\exp(-u)}{[1+\exp(-u)]^2}$, $\mathcal{G}_1(u) = c_1 \frac{\exp(-u)}{1+\exp(-u)}$ and $c_1 = \frac{1}{2}$ (see Landsman and Valdez (2003)) in Theorem 3.1, we directly obtain our result. This completes the proof of Corollary 3.3.

**Corollary 3.4.** Let $Y \sim \text{LSMLa}_1(\mu, \sigma^2, \beta, \Theta)$ be an univariate location-scale mixture of Laplace random variable defined as (2.11). Under conditions in Theorem 3.1, we obtain the TCE for location-scale mixture of Laplace distributions. Its’ form is the same as (3.1), where

$$
\delta_g = \frac{\frac{1}{\sqrt{\theta}}\mathcal{G}_1\left(\frac{\sqrt{z_q}}{\theta}\right)}{F_Z(z_q)} = \frac{\frac{1}{\sqrt{\theta}}c_1 (1 + \sqrt{\frac{z_q^2}{\theta}}) \exp(-\sqrt{\frac{z_q^2}{\theta}})}{F_Z(z_q)} = \sqrt{2} \left[\frac{1}{\sqrt{\theta}}\frac{e^{\frac{z_q^2}{\theta}}}{F_Z(z_q)}\right].
$$

Additionally, $e(\cdot)$ is the density functions of exponential power distributions with a density generator of the form $g_1(u) = c_1 \exp(-\sqrt{u})$ and $c_1 = \frac{1}{2\sqrt{2}}$ (see Landsman and Valdez (2003)).

Proof. Letting $g_1(u) = c_1 \exp(-\sqrt{2u})$, $\mathcal{G}_1(u) = c_1 (1 + \sqrt{2u}) \exp(-\sqrt{2u})$ and $c_1 = \frac{1}{2}$ (see Landsman et al. (2016)) in Theorem 3.1, we immediately obtain our result. This completes the proof of Corollary 3.4.
4 Multivariate cases

In this section, we consider the multivariate TCE for mixture of elliptic distributions. To calculate it we definite shifted cumulative generator (see Landsman et al. (2016))

\[
\overline{G}_{n-1}^*(u) = \int_u^\infty g_n(v + a)dv, \quad a \geq 0, \quad n > 1, \quad (4.15)
\]

with

\[
\overline{G}_{n-1}^*(u) < \infty. \quad (4.16)
\]

Here we consider \( g_{n-1}^*(u) = g_n(u + a) \) as a density generator, if it satisfies the condition:

\[
\int_0^\infty u^{\frac{n}{2}-1} g_n(u + a)du < \infty, \quad \forall a \geq 0. \quad (4.17)
\]

Let \( Y \sim \text{LSME}_n(\mu, \Sigma, \beta, \Theta, g_n) \) and \( M = Y|\Theta = \theta \sim E_n(\mu + \theta\beta, \theta \Sigma, g_n) \). Then

\[
Z = (\theta \Sigma)^{-\frac{1}{2}}(M - \mu - \theta\beta) \sim E_n(0, I_n, g_n).
\]

Writing

\[
\xi_q = (\xi_{q,1}, \xi_{q,2}, \ldots, \xi_{q,n})^T = (\theta \Sigma)^{-\frac{1}{2}}(y_q - \mu - \theta\beta),
\]

where \( y_q = \text{VaR}_q(Y) \), and \( \xi_{q,-k} = (\xi_{q,1}, \xi_{q,2}, \ldots, \xi_{q,k-1}, \xi_{q,k+1}, \ldots, \xi_{q,n})^T \).

To derive formula for MTCE we introduce tail function \( F_{Z-k}(t) \) of \((n-1)\)-dimensional random vector \( Z-k = (Z_1, Z_2, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_n)^T \),

\[
F_{Z-k}(t) = \int_t^\infty f_{Z-k}(v)dv, \quad v, t \in \mathbb{R}^{n-1}, \quad dv = dv_1dv_2 \cdots dv_n,
\]

with the pdf

\[
f_{Z-k}(z-k) = -\frac{1}{\psi^*(0)} \overline{G}_{n-1}^* \left\{ \frac{1}{2} z_{-k}^TZ_{-k} \right\} = -\frac{1}{\psi^*(0)} \overline{G}_{n} \left\{ \frac{1}{2} z_{-k}^T z_{-k} + \frac{1}{2} \xi_{q,k}^2 \right\}, \quad k = 1, 2, \ldots, n,
\]

where \( \psi^*(\cdot) \) is the characteristic generator corresponding to \( \overline{G}_{n-1}^* \), and \( \overline{G}_{n-1}^* \) as formula \( (4.15) \).

**Theorem 4.1.** Let \( Y \sim \text{LSME}_n(\mu, \Sigma, \beta, \Theta, g_n) \) be an \( n \)-dimensional location-scale mixture of elliptical random variable defined as \( (2.4) \). We suppose satisfy conditions \( (2.5), (4.16) \) and \( (4.17) \).

Then

\[
MTCE_q(Y) = \mu + E_0 \left[ \theta \beta + \sqrt{\theta \Sigma} \frac{1}{2} \xi_q \right], \quad (4.18)
\]
where
\[ \delta_q = (\delta_{1q}, \delta_{2q}, \ldots, \delta_{nq})^T, \]
with \( \delta_{kq} = -c_n \psi^{\prime}(0) \frac{F_{z-k}(\xi_{q-k})}{F_{z}(\xi_q)} \) and \( c_n = \frac{\Gamma(n/2)}{(2\pi)^{n/2}} \left[ \int_0^\infty u^{n-2} g_n(u) du \right]^{-1}. \)

Proof. Using the tower property of expectations, we obtain
\[ \text{MTCE}_q(Y) = E[Y|Y > y_q] = E[\Theta(E(Y|Y > y_q, \Theta))]. \]

Since
\[ E[Y|Y > y_q, \Theta = \theta] = E[(Y|\Theta = \theta)|(Y|\Theta = \theta) > y_q] = E[M|M > y_q] = \text{MTCE}_q(M), \]
where \( M \sim E_n(\mu + \theta\beta, \theta\Sigma, g_n) \), and the second equality we have used (2.2). Using Theorem 1 in Landsman et al. (2016), we obtain (4.19), which completes the proof of Theorem 4.1.

Remark 4.1. We remark that Theorem 1 in Landsman et al. (2016), which corresponding the result of \( \text{MTCE}_q(Y|\Theta) \) with \( q = (q, q, \ldots, q)^T \), is a special case of our Theorem 4.1.

Corollary 4.1. Suppose \( Y \sim LSMN_n(\mu, \Sigma, \beta, \Theta) \) is an \( n \)-variate location-scale mixture of normal random variable defined as (2.9). Under conditions in Theorem 4.1, we obtain the MTCE for location-scale mixture of normal distributions. Its’ form is the same as (4.18), where
\[ \delta_{kq} = -c_n \psi^{\prime}(0) \frac{F_{z-k}(\xi_{q-k})}{F_{z}(\xi_q)} = \phi_1(\xi_{k,q}) \frac{\Phi_{z-k}(\xi_{q-k})}{\Phi_z(\xi_q)}. \]

Additionally, \( \phi_n(\cdot) \) and \( \Phi_n(\cdot) \) denote the density and distribution functions of normal distributions, respectively.

Proof. Letting the density generator \( G_n(u) = g_n(u) = \phi_n(u) = c_n e^{-u}, c_n = (2\pi)^{-\frac{n}{2}} \) and
\[ \psi^{\prime}(0) = -(2\pi)^{\frac{n}{2}} \phi_1(\xi_{q,k}) \]
in Theorem 4.1 we directly obtain our result. This completes the proof of Corollary 4.1.

Corollary 4.2. Suppose that \( Y \sim LMSSt_n(\mu, \Sigma, \beta, \Theta) \) is an \( n \)-variate location-scale mixture of student-t random vector defined as (2.10). Under conditions in Theorem 4.1, we obtain the MTCE for
Proof. Letting $m$-variate location-scale mixture of Student-$t$ distributions. Its’ form is the same as (4.18), where

$$
\delta_{k,q} = -c_n \psi^*(0) \frac{F_{Z_{-k}}(\xi_{q,-k})}{F_z(\xi_q)}
$$

$$
= \frac{\Gamma(m/2)}{2^{(m-1)/2} \sqrt{\pi (m-1)}} \left( \frac{m-1}{m} \right)^{\frac{m}{2}} \left( 1 + \frac{\zeta^2_{q,k}}{m} \right)^{-\frac{(m+n-2)}{2}} \frac{F_{m-1,n-1}(\xi_{q,-k};0, \Delta_k)}{F_{m,n}(\xi_q;0, I_n)},
$$

and

$$
\Delta_k = \left( \frac{m(1+\zeta^2_{q,k})}{m-1} \right) I_{m-1},
$$

$I_k (k = n - 1 \text{ or } n)$ is a $k$-dimensional identity matrix. In addition, $T_{m-1,n-1}(\xi_{q,-k};0, \Delta_k)$ and $T_{m,n}(\xi_q;0, I_n)$ are distribution functions of Student-$t$ distributions (see Landsman et al. 2016).

Proof. Letting $g_n(u) = c_n(1 + \frac{2u}{m})^{-(m+n)/2}$, $g_n(u) = c_m(\frac{m}{m+n})^{-(m+n)/2}$, $c_n = \frac{\Gamma((m+n)/2)}{\Gamma(m/2)(m\pi)^{m/2}}$ (see Landsman et al. (2016)) and

$$
\psi^*(0) = -\frac{\Gamma(m/2)}{\Gamma(m+n/2)} \pi(\frac{(m-1)/2}{m+n/2}) \left( 1 + \frac{\zeta^2_{q,k}}{m} \right)^{-\frac{(m+n-2)/2}{2}}
$$
in Theorem 4.1, we immediately obtain our result. This completes the proof of Corollary 4.2.

Corollary 4.3. Assume $Y \sim LSMLo_n(\mu, \Sigma, \beta, \Theta)$ is an $n$-variate location-scale mixture of Logistic random vector defined as (2.11). Under conditions in Theorem 4.1, we obtain the MTCE for location-scale mixture of Logistic distributions. Its’ form is the same as (4.18), where

$$
\delta_{k,q} = -c_n \psi^*(0) \frac{F_{Z_{-k}}(\xi_{q,-k})}{F_z(\xi_q)}
$$

$$
= \frac{L(- \exp(-\frac{\xi^2_{q,k}}{2}), \frac{n-1}{2}, 1) \exp(-\frac{\xi^2_{q,k}}{2}) F_{Z_{-k}}(\xi_{q,-k})}{\sqrt{2\pi} \left[ \sum_{i=0}^{\infty} (-1)^i i^{-n/2} \right] F_z(\xi_q)},
$$

and pdf of $Z_{-k}$:

$$
f_{Z_{-k}}(t) = -\frac{1}{\psi^*(0)} \frac{\exp(-\frac{\xi^2_{q,k}}{2})}{1 + \exp(-\frac{\xi^2_{q,k}}{2})}, \quad k = 1, 2, \cdots, n, \ t \in \mathbb{R}^{n-1},
$$

and

$$
\psi^*(0) = \frac{(2\pi)^{(n-1)/2}}{\Gamma((n-1)/2)} \left[ \int_0^\infty \frac{t^{(n-3)/2} \exp(-t - \frac{\xi^2_{q,k}}{2}) dt}{1 + \exp(-t - \frac{\xi^2_{q,k}}{2})} \right]
$$

$$
= \frac{(2\pi)^{(n-1)/2} L(- \exp(-\frac{\xi^2_{q,k}}{2}), \frac{n-1}{2}, 1)}{\exp(\frac{\xi^2_{q,k}}{2})}.
$$

Additionally, $L(\cdot)$ is the well known Lerch zeta function (see Lin and Srivastava (2004)).
Proof. Letting $g_n(u) = c_n \frac{\exp(-u)}{[1+\exp(-u)]^2}$, $G_n(u) = c_n \frac{\exp(-u)}{1+\exp(-u)}$,

$$c_n = (2\pi)^{-n/2} \left[ \sum_{i=0}^{\infty} (-1)^{i-1} i^{1-n/2} \right]^{-1}$$

and formula (4.19) in Theorem 4.1, we directly obtain our result. This completes the proof of Corollary 4.3.

**Corollary 4.4.** Assume that $Y \sim LSLM_n(\mu, \Sigma, \beta, \Theta)$ be an $n$-variate location-scale mixture of Laplace random vector defined as (2.11). Under conditions in Theorem 4.1, we obtain the MTCE for location-scale mixture of Laplace distributions. Its’ form is the same as (4.18), where

$$\delta_{k,q} = -c_n \psi'_{\theta} (0) \frac{\mathcal{F}_{\gamma_k}(\xi_{q,-k})}{\mathcal{F}_{\gamma}(\xi_{q})} = -\frac{\Gamma(n/2)\psi'_{\theta} (0) \mathcal{F}_{\gamma_k}(\xi_{q,-k})}{2\pi^{n/2}\Gamma(n)} \mathcal{F}_{\gamma}(\xi_{q}),$$

and pdf of $Z_{-k}$:

$$f_{Z_{-k}}(t) = \frac{1}{\psi'_{\theta} (0)} \left( 1 + \sqrt{t^T t + \xi_{q,k}^2} \right) \exp \left\{ -\sqrt{t^T t + \xi_{q,k}^2} \right\}, \ k = 1, 2, \ldots, n, \ t \in \mathbb{R}^{n-1},$$

and

$$\psi'_{\theta} (0) = -\frac{(2\pi)^{(n-1)/2}}{\Gamma(n/2)} \left[ \int_0^{\infty} t^{n-3} \left( 1 + \sqrt{t + \xi_{q,k}^2} \right) \exp \left\{ -\sqrt{t + \xi_{q,k}^2} \right\} d\xi \right]. \ (4.20)$$

Proof. Letting $g_n(u) = c_n \exp(-\sqrt{2u})$, $G_n(u) = c_n (1 + \sqrt{2u}) \exp(-\sqrt{2u})$, $c_n = \frac{\Gamma(n/2)}{2\pi^{n/2}\Gamma(n)}$ (see Landsman et al. 2016) and formula (4.20) in Theorem 4.1 we immediately obtain our result. This completes the proof of Corollary 4.4.

5 **Portfolio risk decomposition with TCE**

Let $Y = (Y_1, Y_2, \ldots, Y_n)^T \sim LSEM_n(\mu, \Sigma, \beta, \Theta, g_n)$, $e = (1, 1, \ldots, 1)^T$ is an $n \times 1$ vector whose elements are all equal to 1. We define

$$S = \sum_{j=1}^{n} Y_j = e^T Y = e^T \mu + \Theta e^T \beta + \Theta \hat{\Sigma} e^T X,$$ (5.21)

which is the sum of mixture of elliptical risks.

**Proposition 5.1.** Under the conditions (5.15) and (5.21), the TCE of $S$ can be expressed as

$$TCE_S(s_q) = \mu_S + E_0 \left[ \theta \beta_S + \delta_S (\sqrt{\sigma_S})^2 \right],$$ (5.22)
where
\[ \delta_S = \frac{1}{\sqrt{\delta_S}} \gamma'_1 \left( \frac{1}{2} z_q^2 \right), \]
with \( Z \sim E_1(0, 1, g_1) \) and \( z_q = \frac{k_S - \mu_S - \theta \beta_S}{\sqrt{\delta_S}} \).

Proof. Let \( L = e^T \Sigma^* X \). Due to (2.4), we get \( L \sim E_1(0, \sigma^2_S, g_1) \) with \( \sigma^2_S = e^T \Sigma e \). So that \( L' \sim E_1(0, 1, g_1) \) with \( L' = \frac{L}{\sigma_S} \). Therefore,
\[ S = \mu_S + \Theta \beta_S + \Theta \frac{1}{2} \sigma_L L' \sim LSE_1(\mu_S, \sigma^2_S, \beta_S, \Theta, g_1), \tag{5.23} \]
with \( \mu_S = e^T \mu \) and \( \beta_S = e^T \beta \).

By using Theorem 3.1, we obtain (5.23), which completes the proof of Proposition 5.1.

**Lemma 5.1.** Let \( Y = (Y_1, Y_2, \cdots, Y_n)^T \sim LSE_n(\mu, \Sigma, \beta, \Theta, g_n) \) as (2.1). Then the vector \( Y_{k,S} = (Y_k, S)^T \), \( 1 \leq k \leq n \) has a mixture of elliptical distribution, namely,
\[ Y_{k,S} \sim LSE_2(\mu_{k,S}, \Sigma_{k,S}, \beta_{k,S}, \Theta, g_2), \]
where \( \mu_{k,S} = (\mu_k, e^T \mu)^T = (\mu_k, \sum_{i=1}^{n} \mu_i)^T \),
\[ \Sigma_{k,S} = \begin{pmatrix} \sigma^2_k & \sigma_{k,S} \\ \sigma_{k,S} & \sigma^2_S \end{pmatrix} \]
and \( \beta_{k,S} = (\beta_k, \beta_S)^T \), and \( \sigma^2_k = \sigma_{k,k}, \sigma_{k,S} = \sum_{i=1}^{n} \sigma_{k,i}, \beta_S = e^T \beta = \sum_{i=1}^{n} \beta_i \),
\[ \sigma^2_S = e^T \Sigma e = \sum_{i,j=1}^{n} \sigma_{i,j}. \]

Proof. Write \( P = (P_1, P_2, \cdots, P_n)^T = \Sigma^* X \). Due to (2.4), we know \( P \sim E_n(0, \Sigma, g_n) \).

From (2.1), we get \( Y_k = \mu_k + \Theta \beta_k + \Theta \frac{1}{2} \sigma_L P_k, 1 \leq k \leq n \). According to (5.23), we know \( S = \mu_S + \Theta \beta_S + \Theta \frac{1}{2} L \) with \( L \sim E_1(0, \sigma^2_S, g_1) \). So that \( Y_{k,S} = \mu_{k,S} + \Theta \beta_{k,S} + \Theta \frac{1}{2} (P_k, L)^T \).

By Lemma 1 in Landsman and Valdez (2003), we obtain \( (P_k, L)^T \sim E_2(\mu_{k,S}, \Sigma_{k,S}, g_2) \). Therefore, \( Y_{k,S} \sim LSE_2(\mu_{k,S}, \Sigma_{k,S}, \beta_{k,S}, \Theta, g_2) \). This completes the proof of Lemma 5.1

**Lemma 5.2.** Let \( Y = (Y_1, Y_2)^T \sim LSE_2(\mu, \Sigma, \beta, \Theta, g_2) \). Assume that condition (5.13) holds. Then
\[ TCE_{Y_1|Y_2}(y_q) = \mu_1 + E_0[\theta \beta_1 + \delta_2 \theta \sigma_1 \sigma_2 \rho_{1,2}], \tag{5.24} \]
where
\[ \delta_2 = \frac{1}{\sqrt{\delta_2}} \gamma'_2 \left( \frac{1}{2} z_{2,q}^2 \right), \]
\[ \rho_{1,2} = \frac{\sigma_{1,2}}{\sigma_1 \sigma_2}, \sigma_1 = \sqrt{\sigma_{1,1}}, \sigma_2 = \sqrt{\sigma_{2,2}} \text{ and } z_{2,q} = \frac{y_q - \mu_2 - \theta \beta_2}{\sqrt{\delta_2}}. \]
Proof. Using the tower property of expectations, we have

\[ TCE_{Y_1|Y_2}(y_q) = E[Y_1|Y_2 > y_q] = E_\theta[E[Y_1|Y_2 > y_q, \Theta = \theta]] = E_\theta[E[Q_1|Q_2 > y_q]], \]

where \((Q_1, Q_2)^T = Y|\Theta = \theta \sim E_2(\mu + \theta \beta, \theta \Sigma, g_2)\), and the third equality we have used (2.2).

By Lemma 2 in Landsman and Valdez (2003), we can obtain (5.24). This completes the proof of Lemma 5.2.

We use the above two Lemmas to give portfolio risk decomposition with TCE as follows.

**Theorem 5.1.** Let \( Y = (Y_1, Y_2, \cdots, Y_n)^T \sim LSM_E n(\mu, \Sigma, \beta, \Theta, g_n) \) be an \( n \)-dimensional location-scale mixture of elliptical random vector defined as (2.1). We suppose condition (3.4) holds, and let \( S = \sum_{i=1}^n Y_i. \) Then the contribution of risk \( Y_k, 1 \leq k \leq n, \) to the total TCE can be given by

\[ TCE_{Y_k|S}(s_q) = \mu_k + E_\theta[\beta_k + \delta_S \theta \sigma_k \sigma_S \rho_k, S], \tag{5.25} \]

where \( \rho_{k,S} = \frac{\sigma_{k,S}}{\sigma_k \sigma_S} \) and \( \delta_S \) is the same as in Proposition 5.1.

Proof. By Lemma 5.1 we know \( Y_{k,S} = (Y_k, S)^T \sim LSM E_2(\mu_{k,S}, \Sigma_{k,S}, \beta_{k,S}, \Theta, g_2), (1 \leq k \leq n). \) Let \( Y \) subject to \( Y_{k,S} \) in Lemma 5.2, we can immediately obtain (5.25). This completes the proof of Theorem 5.1.

**Remark 5.1.** Letting the density generator \( \overline{g}_1(u) = g_1(u) = \phi_1(u) = (2\pi)^{-\frac{1}{2}} e^{-u} \) in Theorem 5.1 we obtain the portfolio risk decomposition with TCE for location-scale mixture of normal distributions. Its’ form is the same as (5.25), where

\[ \delta_S = \frac{1}{\sqrt{\sigma_S}} \phi_1(\frac{1}{2} z_q) \]

Additionally, \( \phi_1(\cdot) \) and \( \Phi_1(\cdot) \) denote the density and distribution functions of normal distributions.

**Remark 5.2.** Letting \( g_1(u) = c_1(1 + \frac{2u}{m})^{-\frac{(m+1)/2}{2}}, \overline{g}_1(u) = c_1 \frac{m}{m-1} (1 + \frac{2u}{m})^{-\frac{(m-1)/2}{2}} \) and \( c_1 = \frac{\Gamma((m+1)/2)}{\Gamma(m/2) (m\pi)^{\frac{m}{4}}} \) (see Landsman et al. (2016)) in Theorem 5.1 we obtain the portfolio risk decomposition with TCE for location-scale mixture of Student-t distributions. Its’ form is the same as (5.25), where

\[ \delta_S = \frac{\sqrt{\sigma_S} \overline{g}_1(\frac{1}{2} z_q)}{\overline{F}_z(z_q)} = \frac{1}{\sqrt{\sigma_S}} c_1 \frac{m}{m-1} (1 + \frac{z_q^2}{m})^{-\frac{(m-1)/2}{2}} \overline{F}_z(z_q) = \frac{1}{\sqrt{\sigma_S}} t_{m,1}(z_q; 0, 1). \]

In addition, \( t_{m,1}(z_q; 0, 1) \) and \( T_{m,1}(z_q; 0, 1) \) are the density and distribution functions of Student-t distributions, respectively (see Landsman et al. (2016)).
Remark 5.3. Letting $g_1(u) = c_1 \frac{\exp(-u)}{1+\exp(-u)}$, $G_1(u) = c_1 \frac{\exp(-u)}{1+\exp(-u)}$ and $c_1 = \frac{1}{2}$ (see Landsman and Valdez (2003)) in Theorem [5.1] we obtain the portfolio risk decomposition with TCE for location-scale mixture of Logistic distributions. Its’ form is the same as (5.25), where

$$\delta_S = \frac{1}{\sqrt{\delta_0}} G_1(\frac{1}{2} z^2_q) = \frac{1}{\sqrt{\delta_0}} \frac{c_1 \exp(-\frac{1}{2} z^2_q)}{F_Z(z_q)} = \frac{1}{\sqrt{\delta_0}} \frac{\phi(z_q)}{F_Z(z_q)}.$$

In addition, $\phi(\cdot)$ is the density functions of normal distributions (see Landsman and Valdez (2003)).

Remark 5.4. Letting $g_1(u) = c_1 \exp(-\sqrt{2}u)$, $G_1(u) = c_1(1+\sqrt{2}u)\exp(-\sqrt{2}u)$ and $c_1 = \frac{1}{2}$ (see Landsman et al. (2016)) in Theorem [5.1] we obtain the portfolio risk decomposition with TCE for location-scale mixture of Laplace distributions. Its’ form is the same as (5.25), where

$$\delta_S = \frac{1}{\sqrt{\delta_0}} G_1(\frac{1}{2} z^2_q) = \frac{1}{\sqrt{\delta_0}} c_1 (1+\sqrt{2}z^2_q)\exp(-\sqrt{2}z^2_q) = \frac{1}{\sqrt{\delta_0}} \frac{\phi(z_q)}{F_Z(z_q)}.$$

Additionally, $\phi(\cdot)$ is the density functions of exponential power distributions with a density generator of the form $g_1(u) = c_1 \exp(-\sqrt{u})$ and $c_1 = \frac{1}{2} \sqrt{2}$ (see Landsman and Valdez (2003)).

6 Concluding remarks

In this paper we consider the univariate and multivariate location-scale mixture of elliptical distribution, which is ($A = \Sigma^+$) generalization of normal mean-variance mixture distribution in Kim and Kim (2019). It has received much attention in finance and insurance applications, since this distribution not only include location-scale mixture of normal (LSMN) distributions, location-scale mixture of Student-t (LSMSt) distributions, location-scale mixture of Logistic (LSMLo) distributions and location-scale mixture of Laplace (LSMLa) distributions, but also include the generalized hyperbolic distribution (GHD) and the slash distribution. The GHD is a special case of this mixture random variable with $X \sim N_n(0, I_n)$ and the distribution of $\Theta$ given by a generalized inverse gaussian $N^{-1}(\lambda, \chi, \psi)$ (see Kim and Kim (2019) for details). The GHD is an important distribution, and has a lot of applications (see Kim (2010) and Ignatieva and Landsman (2015)). Slash distribution also is a special case of this mixture random variable with $X \sim N_n(0, I_n)$ and $\Theta \sim BP(\eta = 1, \alpha = 1, \beta = q/2)$. Here $BP(\cdot)$ is the 3-parameter beta prime (BP) or inverted beta distribution (see Kim and Kim (2019) for details). This distribution has been discussed in many literatures (see Gneiting (1997), Genç (2007) and Wang and Genton (2006)). We also consider univariate TCE, multivariate TCE and portfolio risk decomposition with TCE for location-scale mixture of elliptical distribution. As special cases, we provided univariate TCE, multivariate TCE and portfolio risk decomposition with TCE for LSMN, LSMSt, LSMLo and LSMLa distributions.
Acknowledgments

The research was supported by the National Natural Science Foundation of China (No.11571198, 11701319)

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