Diresonance in production and scattering of heavy mesons

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Abstract

We consider the production and scattering amplitudes of heavy mesons in a situation, where there are two closely spaced narrow resonances, which structure we refer to as a diresonance. Assuming strong overlapping of the resonances coupled to common channels, it is found, using the unitarity and analyticity constraints, that the production amplitudes by a weak source should have similar behavior with energy in different channels. In particular the ratio of the coefficients for each pole contribution to the production amplitude is fixed at $-1$. 
The spectroscopy of resonances near the charm threshold attracts a considerable renewed interest. Recent experimental data not only provide evidence of new states but also suggest that the nature of well known resonances merits a reexamination at the level of fine details. The new exotic charmonium-like states\cite{1} considerably expand the standard spectrum of charmonium, and challenge us for a better understanding of the strong dynamics. Furthermore the apparent significance of multiquark states at the onset of open charm threshold may also impact the properties of the known resonances and may be instrumental in resolving some long-standing puzzles. One such puzzle is related to an inconsistency between BES and CLEO results for the production and decays of $\psi(3770)$ resonance, in particular the fraction of its decays into non-$D\bar{D}$ states\cite{2,3}. Recently BES Collaboration has reanalyzed their data on $e^+e^-$ annihilation in the energy region between 3.700 and 3.872 GeV \cite{4}. They reported observation of an anomalous line-shape behavior of the cross section which is inconsistent with the presence of only one simple $\psi(3770)$ resonance in this energy region. It is claimed that this anomalous behavior could be better understood in terms of two resonances near the c.m. energies of 3.764 GeV and 3.779 GeV. This result could violate the conventional interpretation of $\psi(3770)$ as being a dominantly $1^3D_1$ charmonium state with an admixture of $2^3S_1$, and clearly suggests a more complicated structure, possibly including strong dynamics of the $D$ meson pairs near the threshold. Namely, a diresonance structure may arise from existence of both a charmonium state and a ’molecular’ $D\bar{D}$ threshold resonance.

Apriori one would expect that the nature of each of the two individual peaks could be studied by further exploring their relative coupling to various channels. However the purpose of the present paper is to argue that this standard method, applicable to sufficiently widely separated resonances, is unlikely to be applicable for a strongly overlapping pair of resonances, such as the one indicated by the BES data, i.e. when the splitting between the positions of the resonances is comparable with their widths, and all of these parameters are small in a typical energy scale for the process. (It is natural to call such a structure as diresonance.) Using unitarity and analyticity constraints, we find, under the simplest assumptions, similar to those involved in the standard Breit-Wigner treatment of a single resonance, that the two states are necessarily strongly mixed, and that various final states produced e.g. in the $e^+e^-$ annihilation have the behavior of the production cross section in the diresonance region proportional to one another. Furthermore, if a diresonant production amplitude is written as a linear combination of two poles, then in the limit where the diresonance parameters, the two widths and the splitting between the poles, can be considered

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as small, the relative factor between the two poles is necessarily equal to $-1$.

In practical terms this universal behavior implies that it would be problematic to disentangle experimentally the underlying origin of the states in a diresonance complex. In particular, even if the suggested by the experiment structure around 3.77 GeV, originates from an overlap of charmonium and molecular states, the mixing between them effectively erases any difference in their experimental signature.

We start our discussion of a diresonance structure with a simple case of just one scattering channel. The radial part of the wave function of the particle with mass $m$ with arbitrary complex energy $E$ and orbital momentum $l = 0$ (here we consider $S$-wave motion for simplicity) has the following form at large distances $r$:

$$ R = \frac{1}{r} \left[ B^*(E) e^{i k r} + B(E) e^{-i k r} \right], \quad k = \sqrt{2 m E}, \quad (1) $$

where the coefficients of the incoming and outgoing waves are related by complex conjugation due to the requirement that the wave function is real at real negative $E$. In the familiar case of a single resonance at $E_r = E_0 - i \Gamma/2$, with $E_0$ and $\Gamma$ being the position and the width of the resonance, one has $B(E_r) = 0$, which ensures that the wave function of the resonant state vanishes at spatial infinity. One can then expand the function $B(E)$ near the position of the resonant level $E_r$ as

$$ B(E) = (E - E_0 + \frac{i}{2} \Gamma) b, \quad (2) $$

with $b$ being a smooth function of energy, i.e. $b$ changes on a scale much larger than the resonance width $\Gamma$. The scattering $S$-matrix element is then found as follows

$$ S = \exp(2i \delta) = \frac{B^*(E)}{B(E)} = \frac{E - E_0 - i \Gamma/2}{E - E_0 + i \Gamma/2} \exp(2i \delta^{(0)}), \quad (3) $$

where $\delta^{(0)}$ is a nonresonant phase which is a smooth function of energy defined as $\exp(2i \delta^{(0)}) = b^*/b$.

This standard Breit-Wigner treatment of a single resonance can be readily extended to the case of a diresonance, i.e. in the situation when the scattering amplitude has two closely separated poles at $E_1 - i \Gamma_1/2$ and $E_2 - i \Gamma_2/2$ with both widths $\Gamma_1$ and $\Gamma_2$ and the difference $E_1 - E_2$ being considered as ‘small’. The expansion of the coefficient $B(E)$ having two zeros in the diresonance region is obviously given by

$$ B(E) = (\Delta_1 + i \gamma_1)(\Delta_2 + i \gamma_2) b, \quad (4) $$
where the notation is introduced $\Delta_a = E - E_a, \gamma_a = \Gamma_a / 2$ ($a = 1, 2$), and, similarly to Eq.(2), the coefficient $b$ is a slowly varying function of the energy, which can be approximated by a constant on the energy scale of the diresonance region. The corresponding expression for the $S$ matrix element then takes the form

$$S = \frac{(\Delta_1 - i \gamma_1)(\Delta_2 - i \gamma_2)}{(\Delta_1 + i \gamma_1)(\Delta_2 + i \gamma_2)} \exp(2i \delta(0)),$$

(5)

Let us consider now the amplitude $A(E)$ for production of the scattering state in the diresonance region by a point-like source. The production process is assumed to be weak, so that it is sufficient to consider only the lowest order in the coupling to the source. The energy dependence of such amplitude is proportional to the inverse of the coefficient of the incoming wave in the wave function (1): $A(E) = g/B(E)$ with $g$ being a real (at real $E$) smooth function of energy. Indeed, according to the familiar “$\psi(0)$ rule” the absolute value of the amplitude is proportional to $|\psi(0)|$, provided that the wave function is normalized to a fixed amplitude at infinity, $R = (1/r) \sin(kr + \delta)$, which implies the relation $|A(E)| \propto 1/|B(E)|$. On the other hand, according to the Watson’s theorem, the phase of $A$ is given by $\delta$, i.e. the phase is that of $1/B(E)$. Using this relation we readily find an analytical formula for the production amplitude in the diresonance energy region

$$A(E) = \frac{(g/b)}{(\Delta_1 + i \gamma_1)(\Delta_2 + i \gamma_2)}.$$

(6)

The latter expression for the diresonance production amplitude when written as a sum over two resonances:

$$A(E) = \frac{(g/b)}{E_1 - E_2 + i\gamma_2 - i\gamma_1} \left( \frac{1}{\Delta_1 + i \gamma_1} - \frac{1}{\Delta_2 + i \gamma_2} \right),$$

(7)

tells us that the relative phase between the two resonance factors has to be equal to $\pi$ and the coefficients of the pole factors should be the same.

As a simple cross check we considered a toy model with the scattering of a particle with mass $m$ in a central potential with two Gaussian barriers:

$$V(r) = \frac{1}{2 m r_0^2} x \left\{ h_1 \exp \left[ -\frac{(x - x_1)^2}{w_1} \right] + h_2 \exp \left[ -\frac{(x - x_2)^2}{w_2} \right] \right\},$$

(8)

where $x = r/r_0$ is a dimensionless ratio of the distance $r$ to an arbitrary scale $r_0$ and the parameters $h_i, x_i, w_i$ are also dimensionless. The distance scale $r_0$ also sets the scale $E_0 = (2mr_0^2)^{-1}$ for the energy of the particle. We calculated numerically the dimensionless
“production amplitude” as the value of the wave function at the origin, $\psi(0)$, at energy $E$, provided that at large $r$ the wave function is normalized to a wave with unit amplitude. We found that every time the parameters of the potential $h_i, x_i, w_i$ are tuned in such a way that a diresonance structure appears, the production amplitude is closely approximated by the expression (7). We illustrate this behavior in Fig. 1 for one specific set of parameters $(h_1 = 5.5, h_2 = 0.94, x_1 = 1.75, x_2 = 6.92, w_1 = 0.86, w_2 = 0.45)$. The fit curve shown in the plot corresponds to a constant ratio $g/b$ in Eq.(7), so that the relative factor between the two poles exactly equals $-1$. If the fit is relaxed and this relative factor is also treated as a fit parameter, we find the best approximation for it as $-0.90 + 0.02i$.

![Figure 1](image)

Figure 1: The dimensionless “production cross section” $|A(E)|^2$ vs the energy $E$ (in units of $E_0$) in the diresonance region in a toy model with potential scattering. The circles are the numerical data and the curve is the fit by the formula in Eq.(7).

It may appear that the rigid constraint on the relative contribution of the two single-resonance factors in a diresonance complex is a limitation of the considered situation with one scattering channel. For this reason we proceed to discussing scattering in two coupled channels, where we find that the same constraint on the relative strength of the two pole factors applies in each of the channels, so that in fact the production of the final states in the two channels is described by the same energy dependence with the only free factor being
the ratio of the overall yield of each of the final channels.

In this discussion we consider two channels $a$ and $b$ and consider the $S$ matrix for the scattering and the production by a localized source:

$$S = \begin{pmatrix} S_{aa} & S_{ab} & iA_a \\ S_{ba} & S_{bb} & iA_b \\ iA_a & iA_b & 1 \end{pmatrix},$$  \tag{9}

where $A_{a,b}$ are the production amplitudes for each of the channels by a weak localized source, so that the quadratic in the strength effect of the source in the diagonal (33) element of the $S$ matrix can be neglected. In the zeroth order in the production amplitudes we find that the general solution to the unitarity conditions for the two-channel scattering matrix with a diresonance singularity has the form

$$S_{mn} = \left[ \delta_{mn} - \frac{2i \left( \Delta_1 \gamma_2 + \Delta_2 \gamma_1 \right) \eta_m \eta_n}{(\Delta_1 + i \gamma_1)(\Delta_2 + i \gamma_2)} \right] \exp[i\left(\delta^{(0)}_m + \delta^{(0)}_n\right)]; \quad m, n = a, b, \tag{10}$$

where $\delta_{a,b}^{(0)}$ is the nonresonant scattering phase in the corresponding channel and the real factors $\eta_a$ and $\eta_b$ satisfy the condition $\eta_a^2 + \eta_b^2 = 1$. It can be noted that in the single resonance case these factors are determined by the corresponding branching fractions for the resonance: $\eta_n^2 = \Gamma_n / \Gamma$. In the diresonance case we do not find a simple direct relation of these factors to the individual width parameters $\Gamma_1$ and $\Gamma_2$. However these factors can still be interpreted in terms of the branching ratios (for the diresonance complex) in the sense that $\eta_n^2$ gives the probability of the branching of the scattering into the corresponding channel. It is also quite clear that setting $\eta$ to zero in one channel and $\eta^2 = 1$ in the other, returns us to the previously discussed case of a single channel.

The unitarity relation for the matrix (9) in the first order in the production amplitudes $A_{a,b}$ then yields these amplitudes in the form

$$A_n = \frac{\mu \eta_n}{(\Delta_1 + i \gamma_1)(\Delta_2 + i \gamma_2)} \exp(i\delta^{(0)}_n); \quad n = a, b, \tag{11}$$

where the real smooth factor $\mu$ characterizes the strength of the coupling of the source.

The expression (11) implies that the single-channel behavior of a diresonance production amplitude also holds for multiple channels. Namely, when expressed as a linear combination of two poles the relative coefficient between the two pole terms is necessarily equal to $-1$ as shown in Eq. (7), and the relative yield in each channel is determined by the branching factor $\eta^2$. 
One can readily notice that the relative factor $-1$ is a trivial consequence of the production amplitude behaving as $(\Delta E)^{-2}$ away from the diresonance region. Clearly, in order to invalidate such a behavior, one would have to introduce in the dependence of the coefficient $g/b$ an energy scale comparable to the parameters $(E_1 - E_2), \Gamma_1$ and $\Gamma_2$. Thus our conclusions are applicable in the situation, which we refer to as a diresonance, where these parameters are small in the typical scale in the problem. In this respect the assumption is quite similar to the familiar Breit-Wigner approximation for a single resonance, where the resonance has to be considered as narrow, i.e. with a small width. In the diresonance case it is also the splitting between the poles, which has to be "narrow" in addition to the width parameters, for our approximation to be valid.

We believe that our consideration of a diresonance structure is quite generic and may be applicable to the suggested by experiment\cite{4} structure in the $e^+e^-$ cross section near 3.77 GeV, and possibly to other similar structures. A specific detailed application of the discussed diresonance properties to the production of $D$ meson pairs in the $\psi(3770)$ region should also include the $P$-wave kinematics with different thresholds for the pairs of neutral and charged mesons, as well as the Coulomb effects. Such analysis can be done along the lines presented in Ref.\cite{6}, if more detailed data become available. At this point we can only remark that the two-pole fit to the data\cite{4} does not contradict the expression\cite{7} for the production amplitude. However the error range is still too large for any further conclusions to be drawn.

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