First Law, Counterterms and Kerr–AdS$_5$ Black Holes

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Abstract

We apply the counterterm subtraction technique to calculate the action and other quantities for the Kerr–AdS black hole in five dimensions using two boundary metrics; the Einstein universe and rotating Einstein universe with arbitrary angular velocity. In both cases, the resulting thermodynamic quantities satisfy the first law of thermodynamics. We point out that the reason for the violation of the first law in previous calculations is that the rotating Einstein universe, used as a boundary metric, was rotating with an angular velocity that depends on the black hole rotation parameter. Using a new coordinate system with a boundary metric that has an arbitrary angular velocity, one can show that the resulting physical quantities satisfy the first law.
1 Introduction

Since the discovery of the AdS/CFT correspondence \cite{1,2,3}, there has been considerable interest in Anti de–Sitter (AdS) spacetimes and their physical quantities. These quantities can reveal many important properties of the strongly coupled field theory on the boundary. In the last few years, there has been a debate concerning the thermodynamical quantities of Kerr–AdS black holes and the first law of black hole thermodynamics. This debate started with the work of Gibbons, Perry and Pope \cite{5} re-calculating the thermodynamical quantities of Kerr-AdS black holes in various dimensions using the background subtraction technique. Comparing their results with previous results \cite{12,4,10}, they showed that their quantities obey the first law of thermodynamics, while the quantities produced by some previous calculations, including those using counterterm method, do not obey the first law. This gave the impression that the counterterm technique did not produce the correct thermodynamical quantities for these Kerr–AdS solutions. In this article we show that using the standard counterterm calculation (i.e., without adding any new counterterms) for the Kerr–AdS case, one can produce physical quantities that satisfy the first law of thermodynamics. Here we take the boundary metric to be the non-rotating Einstein universe, similar to \cite{5}. One should notice that the boundary metric chosen here is not the one used in \cite{15,10}. In this case the thermodynamic quantities did not seem to satisfy the first law, i.e.,

\[ dE = TdS + \Omega_i dJ^i. \]  

These apparently different results of the counterterm method, naturally raise the question: Why do some choices of the boundary metric satisfy the first law and others do not? Let us remember that, according to the AdS/CFT duality, all boundary metrics in a given conformal class should produce the same quantities for a specific AdS solution. Of course, in certain cases, e.g., when the conformal symmetry is anomalous, some quantities such as energy
and action, depend on the chosen boundary metric. But, we know how these quantities change upon going from one boundary metric to another in the same conformal class. This should not affect the validity of the first law. In [9] Papadimitriou and Skenderis have formulated a variational problem for AdS gravity with Dirichlet boundary conditions. Their formulation naturally reproduces the known counterterms that leave the AdS action finite. Furthermore, they were able to show that all asymptotically locally AdS black holes satisfy the first law. Here, we discuss the particular case of Kerr–AdS$_5$ and ask the question; What went wrong in choosing the rotating Einstein universe (REU) as a boundary in [15, 10] calculations? We show that the reason for the violation of the first law is not that the REU was chosen as boundary metric but that it was rotating with an angular velocity, $\Omega_\infty = -a/l^2$, that depends on the black hole rotation parameter, $a$. The boundary angular velocity can be interpreted as that of an observer, or a rotating gas, at infinity which does not have to dependent on the black hole parameters. Working with a new coordinate system for Kerr–AdS$_5$ with arbitrary angular velocity at infinity, one can show that the relevant physical quantities satisfy the first law. Another interesting consequence of using the new coordinate system/boundary is that the first law is satisfied whether we used the energy associated with $\partial_t$ or $\partial_t + \Omega_\infty \partial_\phi$. This leads to the conclusion that in the counterterm method angular velocities, or other quantities, associated with a boundary metric should be independent from the black hole parameters, otherwise, the first law might be violated. It is interesting to notice that if we allow the angular velocity to vary this will lead to an additional term in the first law due to a surface tension on the boundary. The surface tension is nothing but the Casimir pressure in the boundary theory. We show that the existence of such a pressure will not affect the stability of the system since its compressibility is non-negative.
2 Counterterms and Gravitational Actions

The AdS/CFT duality states that

$$<e^{\int \phi_0(x)O(x)}>_{CFT} = Z_{AdS}(\phi)$$

where $\phi$ is a bulk field and $\phi_0$ is its value on the boundary. If $\phi = g$ is the metric on AdS and $\phi_0 = \gamma$ is its value on the boundary, then $O = T_{\mu\nu}$ is the energy momentum tensor of the boundary field theory. In the low energy limit, we have

$$Z_{CFT}(\gamma) \simeq e^{-I_{AdS}(g)}$$

i.e., AdS gravitational action acts as the effective CFT action. The gravity action for asymptotically anti-de-Sitter space $\mathcal{M}$, with boundary $\partial \mathcal{M}$, is given by,

$$I_{bulk} + I_{surf} = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^{n+1}x \sqrt{-g} \left( R + \frac{n(n-1)}{l^2} \right) - \frac{1}{8\pi G} \int_{\partial \mathcal{M}} d^n x \sqrt{-h} K.$$ \hspace{1cm} (4)

Where, $\Lambda = -n(n-1)/2l^2$ is the cosmological constant and the second term is the Gibbons–Hawking boundary term. $h_{ab}$ is the induced metric on the boundary and $K$ is the trace of the extrinsic curvature $K_{ab}$ of the boundary. Since asymptotically AdS spacetimes have infinite volumes, this action diverges unless one uses some regularization method. The most commonly used regularization techniques are: i) the background subtraction technique and ii) the counterterm subtraction technique. The background subtraction technique utilizes the fact that divergent contributions in the AAdS space action is due to the asymptotic region (i.e., where $r \to \infty$). Therefore, one can obtain a finite action by subtracting the AdS space action from the AAdS action. The main problem with such a technique is that any physics common between the two manifolds cancels out and will not be carried by the resulting finite action. For example, physical quantities on the gravity side dual to Casimir energy and conformal anomaly vanish on the gravity side upon using the background subtraction method. On the other hand if one
calculates such quantities on the field theory side, one obtains non-vanishing expressions. This creates a clear mismatch between the two sides of the duality, since this piece of action carries important information about the strongly coupled CFT on the boundary. The counterterm subtraction technique uses the fact that divergent contributions to the AAdS gravitational actions can be written as surface terms that depend on the metric \( h \) and its covariant derivatives \(^2\). By calculating these expressions and using them as counterterms one can define a finite gravitational action\(^{[21]}\).

\[
I_{\text{ct}} = \frac{1}{8\pi G} \int_{\partial M} d^n x \sqrt{-h} \left[ \frac{(n-1)}{l} - \frac{l R}{2(n-2)} \right]. \tag{5}
\]

Here \( R \) and \( R_{ab} \) are the Ricci scalar and tensor for \( h \). Using these counterterms one can construct a divergence-free stress–energy tensor al’ a Brown and York from the finite action \( I = I_{\text{bulk}} + I_{\text{surr}} + I_{\text{ct}} \) by defining (see Ref. \(^{[26]}\) for more details):

\[
T^{ab} = 2 \sqrt{-h} \frac{\delta I}{\delta h^{ab}}. \tag{6}
\]

We will be interested in using this stress tensor to calculate conserved quantities for AdS solutions, specifically the total energy and angular momentum of kerr-AdS\(_5\) solution. The Brown-York conserved charge is given by \(^{[26]}\):

\[
Q_\xi = \int_\Sigma d^{D-2} x \sqrt{\sigma} u^\mu T_{\mu\nu} \xi^\nu. \tag{7}
\]

where \( \xi \) is a Killing vector and \( u_\mu = -N t_\mu \), while \( N \) and \( \sigma \) are the lapse function and the space-like metric which appear in the ADM–like decomposition of the boundary metric

\[
ds^2 = -N^2 dt^2 + \sigma_{ab}(dx^a + N^a dt)(dx^b + N^b dt). \tag{8}
\]

It is worth mentioning that the metric restricted to the boundary, \( h_{ab} \), diverges due to the infinite conformal factor that depends on a radial coordinate that we might call \( r \). One can have a well defined boundary metric \( \gamma \) as
\[ \gamma_{ab} = \lim_{r \to \infty} \Omega^2 h_{ab} \]  \hspace{1cm} (9)

where \( \Omega \) is some positive function with first order pole in \( r \). This defines a conformal structure on the boundary \([2]\) rather than a specific boundary metric \( i.e. \), a class of boundary metrics for a specific AdS solution which are related by conformal transformations. As we stressed in the introduction, this puts all possible metrics in a given conformal class on equal footing. In principle, one can use any of them to calculate the action and conserved quantities of a given AdS solutions up to pieces dual to CFT conformal anomalies and Casimir energies which should not affect the thermodynamic properties of such a solution.

As a consequence of the counterterm subtraction technique one can relate the field theory’s energy momentum tensor predicted by the duality \( \hat{T}^{ab} \) and the CFT energy momentum tensor \([27]\):

\[ \sqrt{-\gamma} \gamma_{ab} \hat{T}^{bc} = \lim_{r \to \infty} \sqrt{-h} h_{ab} T^{bc} \]  \hspace{1cm} (10)

3 The General Five-Dimensional Kerr–AdS Solution

The five-dimensional Kerr-AdS\(_5\) solution was first introduced by Hawking, Hunter and Taylor-Robinson \([4]\), where they discussed its relevance to the AdS/CFT correspondence. In addition to mass parameter \( M \) and AdS radius \( l \), this solution has two rotation parameters \( (a, b) \). The metric in Boyer-Lindquest-type coordinates has the following form

\[ ds^2 = -\frac{\Delta_r}{\rho^2} \left( dt - a \frac{\sin^2 \theta}{\Xi_a} d\phi - b \frac{\cos^2 \theta}{\Xi_b} d\psi \right)^2 + \frac{\Delta_\theta}{\rho^2} \left( adt - \frac{(r^2 + a^2)}{\Xi_a} d\phi \right)^2 + \frac{(1 + r^2/l^2)}{r^2 \rho^2} \left( abdt - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\psi \right)^2 + \rho^2 \frac{\Delta_r}{\Delta_\theta} dr^2 + \rho^2 \frac{\Delta_\theta}{\Delta_r} (bdt - \frac{(r^2 + b^2)}{\Xi_b} d\psi)^2 + \rho^2 d\theta^2 , \]  \hspace{1cm} (11)
\[
\begin{align*}
\rho &= r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \\
\Xi_a &= 1 - a^2 / l^2, \quad \Xi_b = 1 - b^2 / l^2, \\
\Delta_r &= \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2)(1 + r^2 / l^2) - 2MG, \\
\Delta_\theta &= 1 - a^2 / l^2 \cos^2 \theta - b^2 / l^2 \sin^2 \theta.
\end{align*}
\]

The inverse temperature, computed by requiring regularity of the Euclidean section, is given by:
\[
\beta = \frac{1}{T} = \frac{2\pi r_+ (r_+^2 + a^2)(r_+^2 + b^2)}{2r_+^6 + r_+^4 (l^2 + b^2 + a^2) - a^2 b^2 l^2}.
\]

while the area of the horizon is
\[
A = \frac{2\pi^2 (r_+^2 + a^2)(r_+^2 + b^2)}{r_+ \Xi_a \Xi_b}.
\]

In these coordinates the angular velocities on the horizon have the form:
\[
\Omega^a_H = a \frac{\Xi_a}{r_+^2 + a^2}, \quad \Omega^b_H = b \frac{\Xi_b}{r_+^2 + b^2}.
\]

One of the features of the Kerr-AdS solution in Boyer-Lindquest coordinates is the non-vanishing angular velocities \(\Omega^a_\infty = -a / l^2\), \(\Omega^b_\infty = -b / l^2\), in the \(\phi\) and \(\psi\) directions at spatial infinity. This is in contrast to the asymptotically flat Kerr solutions case which has a vanishing \(\Omega_\infty\). It implies that observers at spatial infinity associated with this coordinate system are not co-rotating with the freely falling gas at infinity as in the asymptotically flat Kerr case. Notice the dependence of the angular velocities at infinity on the angular parameters of the black holes. In principle, an observer or a gas at infinity can have any angular velocity, it does not have to be related to the rotation parameters of the black hole. We are going to realize the importance of such a simple observation when we discuss the first law.
3.1 Previous Calculations

In previous calculations [15, 10] the counterterm method has been used to calculate the action, stress tensor and conserved charges of Kerr-AdS$_5$ (for a similar calculation but using different time-like killing vector please see [9]). In this calculation the induced metric on the boundary is defined as the hypersurface at $r \rightarrow \infty$, where $r$ is the radial coordinate in the above Boyer-Lindquest-type form of the Kerr-AdS$_5$ solution. Therefore, it was natural to choose the boundary on which the dual field lives to be

$$ds^2 = -dt^2 + \frac{2a \sin^2 \theta}{\Xi_a} dtd\phi + \frac{2b \cos^2 \theta}{\Xi_b} dtd\psi + l^2 \left[ \frac{d\theta^2}{\Delta_a} + \frac{\sin^2 \theta}{\Xi_a} d\phi^2 + \frac{\cos^2 \Pi}{\Xi_b} d\psi^2 \right].$$

(16)

We are going to refer to this boundary as the rotating Einstein universe (REU). Calculating the total energy and angular momentum one obtains the following expressions:

$$\mathcal{M} = \frac{\pi l^2}{96G\Xi_a \Xi_b} \left[ 7\Xi_a \Xi_b + \Xi_a^2 + \Xi_b^2 + 72GM/l^2 \right],$$

(17)

and

$$\mathcal{J}_a = \frac{\pi Ma}{2\Xi_a^2 \Xi_b}, \quad \mathcal{J}_b = \frac{\pi Mb}{2\Xi_b^2 \Xi_a}.$$  

(18)

The action is given by

$$I_5 = -\frac{\pi \beta l^2}{96\Xi_a \Xi_b G} \left[ 12(r_+^2/l^2)(1 - \Xi_a - \Xi_b) + \Xi_a^2 + \Xi_b^2 + \Xi_b \Xi_a \\
+ 12r_+^4/l^4 - 2(a^4 + b^4)/l^4 - 12(a^2b^2/l^4)(r_+^2l^{-2} - 1/3) - 12 \right].$$

(19)

The above physical quantities satisfy the following thermodynamic relation

$$S = \beta \left( \mathcal{M} - \Omega_H^a \mathcal{J}_a + \Omega_H^b \mathcal{J}_b \right) - I_5 = \frac{A}{4G},$$

(20)

The general variation of the total energy expressions can not be put in the form of the first law

$$d\mathcal{M} \neq TdS + \Omega_H^a d\mathcal{J}_a + \Omega_H^b d\mathcal{J}_b.$$  

(21)
3.2 Kerr-AdS$_5$ Revisited

Another natural conformal boundary for the Kerr-AdS solution is the Einstein universe (EU)

\[ ds^2 = -dT^2 + l^2 \left[ d\Theta^2 + \sin^2 \Theta d\Phi^2 + \cos^2 \Theta d\psi^2 \right] . \]  

(22)

This metric is the hypersurface at \( y \to \infty \) for any asymptotically AdS solution in global coordinates, with \( y \) as a radial coordinate. Performing the following coordinate transformations \[ \Xi_a y^2 \sin^2 \Theta = (r^2 + a^2) \sin^2 \theta \quad \Phi = \phi + a t/l^2 \quad T = t \]
\[ \Xi_b y^2 \cos^2 \Theta = (r^2 + b^2) \cos^2 \theta \quad \Psi = \psi + b t/l^2 \]  

(23)

the Kerr-AdS$_5$ solution (11) take the following form, which is manifestly asymptotic to AdS spacetime \[30];

\[ ds^2 = - (1 + y^2/l^2)dT^2 + \frac{dy^2}{1 + y^2/l^2 - \frac{2M}{\Delta y^2}} + y^2 d\Omega_3^2 \]
\[ + \frac{2M}{\Delta \Theta^3 y^2} (dT - a \sin^2 \Theta d\Phi - b \cos^2 \Theta d\Psi)^2 + \ldots \]  

(24)

where

\[ \Delta \Theta = 1 - a^2/l^2 \sin^2 \Theta - b^2/l^2 \cos^2 \Theta \quad d\Omega_3^2 = d\Theta^2 + \sin^2 \Theta d\Phi^2 + \cos^2 \Theta d\psi^2 \]  

(25)

In this coordinate system and other coordinate systems used in this paper, we are going to keep the thermodynamic quantities as a function of the outer horizon radius, \( r_+ \), in the Boyer-Lindquest-type coordinates in order to compare different expressions obtained using different boundary metrics. Using counterterms to calculate the action and total energy for Kerr-AdS$_5$ in these coordinates with the Einstein universe as our boundary metric, one gets the following

\[ \mathcal{M}' = \frac{\pi}{32 \Xi_a \Xi_b l^2} \left[ MG(16 \Xi_a + 16 \Xi_b - 8 \Xi_a \Xi_b) + 3 \frac{\Xi_a^2 \Xi_b^2 l^2}{G} \right] . \]  

(26)
\[ I'_5 = \frac{\pi \beta}{32 G l^2 \Xi_a \Xi_b} \left[ 4 \left( r_+^2 + a^2 \right) \left( r_+^2 + a^2 \right) \left( l^2 / r_+^2 - 1 \right) + 3 \Xi_a \Xi_b l^4 \right] \]  

(27)

The angular momenta are the same as in (18). The above quantities satisfy the following thermodynamic relation

\[ S = \beta (\mathcal{M}' - \Omega \mathcal{J}) - I'_5 = \frac{A}{4G}, \]  

(28)

Also, they satisfy the first law

\[ d \mathcal{M}' = T dS + \Omega dJ. \]  

(29)

It is worth mentioning that the same coordinate system has been considered in a background method calculation used by Gibbons, Perry and Pope [5] to produce the action and other physical quantities for Kerr-AdS$_D$. Their expressions satisfy the above statistical relation (20) and the first law (1). In a more recent work [7] the same authors considered the vacuum energy of a Kerr-AdS$_5$ black hole and argued that it is the same as that of AdS space (i.e., \( E_c = \frac{3 \pi l^2}{32G} \)). As we have discussed in section 2 counterterms can be used to obtain the same consistent results produced by the background method. Furthermore, it produces the correct quantities dual to the Casimir energy or the conformal anomaly on the field theory side. As we have mentioned earlier, the boundary field theory lives on Einstein Universe (22). Using results of field theory on the Einstein universe (See for example [29]), one can check that the Casimir energy and conformal anomaly for the boundary field theory match that calculated using the counterterm method. The Casimir energy is given by

\[ E_{\text{casimir}} = \frac{3 N^2}{16l}, \]  

(30)

and the trace anomaly vanishes on both sides

\[ T^\mu_\mu = 0. \]  

(31)
4 The First Law, Counter-terms and Kerr-AdS$_5$

We would like to discuss the first law for Kerr-AdS$_5$ upon using the REU in (16) as a boundary metric and write an expression for the variation of the total energy in terms of the relevant thermodynamic parameters. It is important to remind the reader that the simple form of the first law is due to thermodynamic quantities that were measured by an observer at rest relative to a free thermal gas at infinity. For example, the total energy of the Schwarzschild or Kerr black hole calculated using the ADM mass is the energy measured by an observer at rest relative to the hole at infinity. Obviously, a non-inertial observer measures different energy due to non-inertial forces that might appear in his frame. Concerning rotation, there are two types of non-inertial forces that appear in a rotating frame; centripetal force and Coriolis force. Coriolis force does not depend on the size of the system, therefore, it would not contribute to the thermodynamic energy of the system. As we have seen in the previous section, and as pointed out in [5], the variation of the total energy, obtained using (16) as a boundary, can not be put in the form of the first law. But, it can be written as

$$d\mathcal{M} = TdS + \Omega_H^a d\mathcal{I}_a + \Omega_H^b d\mathcal{I}_b + J_a d\Omega_{\infty}^a + J_b d\Omega_{\infty}^b + d\mathcal{M}_{\text{cas}}. \quad (32)$$

As one can see, the additional terms depend on the $\Omega_{\infty}$'s variations, this is why the first law is satisfied upon choosing Eu as a boundary, since it has a vanishing $\Omega_{\infty} = 0$. Let us ignore the last term for a moment. The energy not only depends on the usual extensive variables ($S, J$), but also on the intensive variables ($\Omega_{\infty}^a, \Omega_{\infty}^b$). This indicates that this expression is not a well defined thermodynamic energy and we better define another energy function which depend on extensive variables only;

$$\mathcal{M} = \mathcal{M}' + \Omega_{\infty}^a J^a + \Omega_{\infty}^b J^b, \quad (33)$$
therefore,

\[ d\mathcal{M}' = TdS + (\Omega^a_H - \Omega^a_\infty) dJ^a + (\Omega^b_H - \Omega^b_\infty) dJ^b \quad (34) \]

The meaning of this new energy function \( \mathcal{M}' \) is simple, it is the energy measured by an observer co-rotating with free gas of particles at infinity. The time-like killing vectors of these two energies are related by

\[ \partial_{\nu'} = \partial_t + \Omega^a_\infty \partial_\phi + \Omega^b_\infty \partial_\psi \quad (35) \]

As a result one has to use the time frame of the rotating free gas at infinity to get a meaningful thermodynamic expression for the energy of the system. This relation has been noticed in \([5, 6, 9]\), and we stress on its importance from a thermodynamic point of view. As one can see \( d\mathcal{M}' \) can be put in the following form \([9]\)

\[ d\mathcal{M}' = TdS + (\Omega^a_H - \Omega^a_\infty) dJ^a + (\Omega^b_H - \Omega^b_\infty) dJ^b + d\mathcal{M}_c \quad (36) \]

where \( \mathcal{M}_c = \mathcal{M}_c(a, b) \) is the vacuum part of the energy. Notice that \((a, b, r_+)\) can be considered functions of \((J_a, J_b, S)\) regarding equation (14), and (18), therefore, the last term violates the first law. The first law is apparently violated because two independent physical quantities, namely; \( \Omega_H \) and \( \Omega_\infty \), are related through their dependence on the same parameter \( a \). One should regard \( \Omega_\infty \) as a boundary property (i.e., of an observer, or a gas at infinity) which does not have to dependent on the black hole parameters. Notice that if \( \Omega_\infty \) depends on \( r_+ \) instead of \( a \) the first law will be again violated.

5 Another Coordinate System for Kerr-AdS\(_5\)

In this section we present a different coordinate system for Kerr-AdS\(_5\) black hole with one rotation parameter\(^1\). This coordinate system can describe Kerr-AdS\(_5\) from the point of view of an observer rotating with respect to a freely

\(^1\)We choose for simplicity, one rotation parameter, but it can be easily generalized to two parameters and other dimensions as well.
falling gas at infinity. The observer’s angular velocity $\Omega_\infty = c/l^2$ and that at the horizon are independent in contrast to that of Boyer-Linquest-type coordinate. This coordinate system can be obtained through the following coordinate transformation resulting in new coordinates $(t', r', \theta', \phi', \psi')$

$$
\Xi_c y' \sin^2 \Theta = (r'^2 + c^2) \sin^2 \theta', \quad \Phi = \phi' + c t'/l^2 \quad y'^2 \cos^2 \Theta = (r'^2) \cos^2 \theta', \quad \Psi = \psi', \quad T = t' \quad (37)
$$

Dropping the primes from the new coordinates, leaves the metric component in the following form

$$
g_{tt} = - \frac{r^2}{l^2} - D_\theta + \frac{2 m \Xi_c \Delta^2}{r^2 \Delta^3} + O(\frac{1}{r^8})
$$

$$
g_{t\phi} = \frac{c}{l^2} \left( \frac{r^2 + c^2}{\Xi_c} \right) \sin^2 \theta - \frac{2 m a \sin^2 \theta \Xi_\Delta}{r^2 \Delta^3} + O(\frac{1}{r^8})
$$

$$
g_{rr} = \frac{l^2}{r^2} - \frac{l^4 D_\theta}{r^4} + 2 m l^4 \frac{\Xi_c}{r^6 \Delta^2} + \frac{l^6}{r^6} \left[ D_\theta + \frac{b^4}{l^4} \sin^2 \theta \right] + O(\frac{1}{r^8})
$$

$$
g_{\theta\theta} = \frac{r^2 + c^2 \cos^2 \theta}{1 - \frac{c^2}{r^2} \cos^2 \theta} + O(\frac{1}{r^8})
$$

$$
g_{\phi\phi} = \frac{(r^2 + c^2) \sin^2 \theta}{\Xi_c} - \frac{2 m a^2 \Xi_c \sin^4 \theta}{r^2 \Delta^3} + O(\frac{1}{r^8})
$$

$$
g_{\psi\psi} = r^2 \cos^2 \theta, \quad (38)
$$

where

$$
D_\theta = 1 + c^2/l^2 \sin^2 \theta, \quad \Delta_\theta = 1 - c^2/l^2 \cos^2 \theta - (a c)/l^2 \sin^2 \theta \quad (39)
$$

$$
\Xi_c = 1 - c^2/l^2, \quad \Xi_a = 1 - a^2/l^2, \quad \Delta = 1 - c^2/l^2 \cos^2 \theta - a^2/l^2 \sin^2 \theta \quad (40)
$$

The inverse temperature, $\beta$ and the area of the horizon $A$ are the same as in (13) and (14), but the angular velocities at the horizon have the form:

$$
\hat{\Omega}_H = a \frac{(r^2/l^2 + 1)}{r^2 + a^2} + c/l^2. \quad (41)
$$

Notice that, the previous two coordinate systems are special cases of the coordinate system presented here, corresponding to $c = a$ and $c = 0$. The
hypersurface as $r \to \infty$ is chosen to be our boundary metric in the counterterm calculation. The action is given by

$$\hat{I}_5 = \frac{\pi \beta}{96 G l^2 \Xi_a} \left[ 12 \left( r_+^2 + a^2 \right) (l^2 - r_+^2) + \frac{l^4 \Xi_a}{\Xi_c} (9 \Xi_c + c^4/l^4) \right]$$

The energy associated with the killing vector $\partial_t$ is given by

$$\hat{M} = \frac{\pi}{4 \Xi_a^2} \left[ M(3 - a^2/l^2 + 2 a c/l^2) \right] + \frac{\pi l^2}{96 \Xi_c G} (9 \Xi_c + c^4/l^4).$$

Notice the dependence of the energy on $c$. The angular momentum is

$$J = \frac{\pi M a}{2 \Xi_a}.$$

All the above quantities satisfy the statistical relation (28). The Casimir energy and conformal anomaly of the boundary field theory predicted from geometry side are given by

$$E_c = \frac{\pi l^2}{96 \Xi_c} \left[ 9 \Xi_c + c^4/l^4 \right]$$

and

$$T_a^a = -\frac{c^2 N^2}{4 \pi l^6} \left[ c^2/l^2 \cos^2 \theta (3 \cos^2 \theta - 2) - \cos 2\theta \right],$$

which match exactly the expressions of the Casimir energy and trace anomaly for $D = 4$ $N = 4$ SYM theory on the rotating Einstein universe with angular velocity $\Omega_\infty = c/l^2$. Notice here that the conformal invariance is broken because of the non-vanishing angular velocity (i.e. $\Omega_\infty = c/l^2$) at infinity not the black hole rotation parameter $a$.

### 5.1 First Law

Now following the discussion on the previous section, the energy associated with the killing vector is given by $\partial_t + \Omega_\infty \partial_\phi$,

$$\mathcal{M} = \frac{\pi}{4 \Xi_a^2} \left[ M(3 - a^2/l^2) \right] + \frac{\pi l^2}{96 \Xi_c G} (9 \Xi_c + c^4/l^4).$$
The energy and angular velocity

\[ \Omega_H = \hat{\Omega}_H - \Omega_\infty = a \frac{(r_2^2/l^2 + 1)}{r_+^2 + a^2}, \]  

(48)

satisfy both (28) and the first law as well;

\[ d\hat{M} = TdS + \hat{\Omega}_H d\mathcal{J}. \]  

(49)

This is true as long as we think of \( c \) as a fixed input parameter. \( c \) can be thought as a fixed parameter as a consequence of fixing the boundary metric. This serves as a boundary condition on the metric in the AdS/CFT set up, for more details please see [9]. The first law is directly satisfied in agreement with the general results of [9].

It is worth mentioning that the energy \( \hat{M} \) associated with the killing vector \( \partial_t \) and the angular velocity \( \hat{\Omega}_H \) satisfy both (28) and the first law;

\[ d\hat{M} = TdS + \hat{\Omega}_H d\mathcal{J}. \]  

(50)

It is intriguing to notice that if one allows \( c \) to vary, it will lead to an additional term in the first law proportional to \( \frac{dM_{\text{Cas}}}{dA} \). Varying \( c \) is the same as varying the area \( A \) of the spatial part of the boundary metric. It allows the existence of external forces that act on the thermal gas at infinity. This term can be interpreted as the work done by surface tension, since the system has a curved boundary and the energy depends on the boundary surface area \( A \). From the boundary theory point of view this surface tension is nothing but the Casimir pressure, in addition to the usual conformal pressure (i.e., which is proportional to 1/3 of the energy density). Calculating the compressibility of the Casimir pressure, one find that it is non-negative for \( 0 \leq c \leq l \). Instead of writing the expression for compressibility, which is rather long, we draw the compressibility as a function of \( c \) in Figure 1.

This is a sign of thermodynamic stability of the system against small changes in the volume of the the boundary. Notice here that the range \( 0 \leq c \leq l \) contains the values of \( c \) that do not change the metric signature.
and keep the velocity of any object rotating with an angular velocity $\Omega_\infty$ less than that of light.

6 Concluding Remarks

We use the standard counterterm method for the Kerr–AdS$_5$ case to produce physical quantities that satisfy the first law of thermodynamics. Here we choose the boundary metric to be the non-rotating Einstein universe, similar to [5]. In this work we point out the reason for the apparent violation of the first law in some previous calculations [15, 10]. We show that the reason for the violation of the first law is not that REU was chosen as the boundary metric but that it was rotating with an angular velocity $\Omega_\infty = -a/l^2$ that depends on the black hole rotation parameter, $a$. This boundary angular velocity is that of an observer, or a thermal gas at infinity and does not have to depend on the black hole parameters. Choosing to work with a new coordinate system for Kerr–AdS$_5$ with arbitrary angular velocity at infinity, one can show that the relevant physical quantities satisfy the first law. This leads to the conclusion that, in the counterterm method, angular velocities
or other quantities associated with a boundary metric should be independent from the black hole parameters, otherwise, the first law might be violated. It is interesting to notice that if we allow the angular velocity to vary it will lead to an additional term in the first law due to a surface tension on the boundary. The surface tension is nothing but the Casimir pressure in the boundary theory. We show that the existence of such pressure will not affect the stability of the system since its compressibility is non-negative.

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