Contracting asymptotics of the lapse-scalar field 
sub-system of the Einstein-scalar field equations

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Abstract

We prove an asymptotic stability result for a linear coupled hyperbolic–elliptic system on a large class of singular background spacetimes in CMC gauge on the $n$-torus. At each spatial point these background spacetimes are perturbations of Kasner-like solutions of the Einstein-scalar field equations which are not required to be close to the homogeneous and isotropic case. We establish the existence of a homeomorphism between Cauchy data for this system and a set of functions naturally associated with the asymptotics in the contracting direction, which we refer to as asymptotic data. This yields a complete characterization of the degrees of freedom of all solutions of this system in terms of their asymptotics. Spatial derivative terms can in general not be fully neglected which yields a clarification of the notion of asymptotic velocity term dominance (AVTD).

1 Introduction

Do cosmological solutions of Einstein’s equations exhibit any generic coherent behavior in the neighborhood of the Big Bang singularity? Mathematically, a cosmological solution is a globally hyperbolic Lorentzian manifold with closed Cauchy hypersurfaces. Certain symmetry-defined classes of vacuum cosmological spacetimes have infinite dimensional families of solutions which exhibit asymptotically velocity term dominated (AVTD) type behavior near the singularity [19, 21]. Such solutions are modeled in the limit towards the Big Bang singularity (i.e., in the contracting time direction) by solutions to an asymptotic model system – the so-called VTD system – in which spatial derivative terms are dropped from the equations. The Gowdy vacuum solutions in areal gauge [14, 24, 27] and wave gauge [3], the polarized and half polarized $T^2$-symmetric vacuum solutions in areal gauge

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[1, 20], and the polarized and half polarized U(1)-symmetric vacuum solutions in wave-
type gauges [16, 17, 22] all have infinite dimensional families of solutions which exhibit
AVTD type behavior near the singularity.

Heuristic and numerical studies suggest that the general set of cosmological solutions
do not exhibit AVTD behavior in any gauge near the singularity [9–13, 32]. Rather, it
is conjectured that general solutions have oscillatory behavior, with an infinite sequence
of AVTD-like epochs. While this so-called “BKL conjecture” (named after the authors of
[7, 8]) is believed to be very difficult to check rigorously, it is also known to fail in cases
where weak null singularities form [25].

In contrast to the general cosmological setting, there are reasons to suspect that
generic solutions of the Einstein system coupled to the scalar field equations may exhibit
AVTD behavior near the singularity. Several authors [5, 6, 19] noted that if the matter
was modeled by a scalar field (or a fluid with a stiff equation of state) the singular dy-
namics would be monotonic rather than oscillatory, greatly simplifying the mathematical
analysis. This simplification has since been exploited, first, in work by Andersson and
Rendall [4], who prove, using Fuchsian methods (see below), the existence of an infinite
dimensional family of solutions to the Einstein–scalar field system with AVTD asymp-
totics. These solutions are not limited to being close to the FLRW solution. While the
family of solutions obtained is general in the sense of function counting, the question
of whether the VTD asymptotic data yield solutions corresponding to an open set of
Cauchy data is not addressed by the Fuchsian methods, particularly when restricted to
the analytic category.

In remarkable more recent work, Rodnianski and Speck [29, 30] show that this AVTD
behavior is nonlinearly stable under perturbations of the homogeneous and isotropic
(FLRW) solution of the Einstein-scalar field system on the n-torus. It is shown in [29]
for the linearized problem for example that for each solution of the linearized Einstein-
scalar field system there is a function \( \Psi_{Bang}(x) \) such that

\[
\| t \partial_t \varphi(t, \cdot) - \Psi_{Bang} \| \leq C t^{2/3 - \epsilon}, \quad \| \partial_i \varphi(t, \cdot) - \log t \partial_i \Psi_{Bang} \| \leq C,
\]

in terms of some smallness parameter \( \epsilon > 0 \) where \( \varphi \) is the linearized scalar field. The
scalar field \( \varphi \) satisfies a second-order wave equation, however, and its solution set there-
fore has two functional degrees of freedom. In order to study how these two degrees of
freedom are reflected in the asymptotics, one therefore needs to go beyond the single
asymptotic datum \( \Psi_{Bang} \) which was identified in the estimates by Rodnianski and Speck.
In fact, as we show the function \( \Psi_{Bang} \) is completely determined by the asymptotics of
the other fields, and, the estimates above therefore do not reflect any of the actual de-
grees of freedom of the scalar field – this statement also holds for the fully nonlinear
Einstein-scalar field system [30].

The Fuchsian method [1, 2, 14, 18, 23, 24, 27] is a common method for identifying asymptotic
degrees of freedom – asymptotic data – (often heuristically by formal expansions
at the singularity) and then for rigorously validating these (by proving the existence of

\footnote{\textsuperscript{1.1}See [29] for more information regarding the norms and the constants \( C, c > 0 \).}
a solution of the equation which realizes each choice of asymptotic data in some well-defined sense). While this typically yields infinite dimensional families of solutions with well-understood asymptotics (as in most of the works listed above), this method alone does not necessarily imply that the constructed family is open, or even generic within the set of all solutions. In order to address particularly this last issue, Ringström [28] has initiated a line of research which is concerned with the question whether the degrees of freedom of the solution space are equivalent in a certain sense to the asymptotic degrees of freedom. Ringström proves the existence (and also the non-existence) of homeomorphisms between Cauchy data and asymptotic data for a large class of linear wave equations with spatially homogeneous coefficients.

The main outstanding questions which motivate our work are as follows. How are the full degrees of freedom, which parametrize the space of all solutions of the Einstein-scalar field system, reflected in the asymptotic behavior of solutions near the singularity? Can we derive formulas for the asymptotics of generic solutions to the Einstein–scalar fields system which contain all these degrees of freedom? If yes, do the VTD equations, that is, the asymptotic model equations where spatial derivatives are dropped, imply the same asymptotic representation of the full degrees of freedom?

Rather than investigate these questions for the full Einstein–scalar field system, which, in light of the technical arguments required for the results of [29, 30] appears out of reach, we focus on a coupled sub-system in this paper that is obtained from the Einstein-scalar system. As in the Rodnianski-Speck work we choose CMC gauge with zero shift. Our sub-system is then formed by the hyperbolic scalar field equation and the elliptic type equation for the lapse treating the spatial metric $\gamma^a_b$ and the trace-free part of the second fundamental form $\chi^a_b$ as given functions (that is, the remaining Einstein–scalar field equations are not imposed on these). We shall defer the discussion of the derivation of this sub-system to Section 2.1. It turns out that our unknowns are the scalar field $\phi$, its conjugate momentum $\pi = t \partial_t \phi/\alpha$, and the lapse $\alpha$, in terms of which this system takes the form

\begin{align*}
-t\partial_t \pi + (1 - \alpha)\pi + \alpha t^2 \gamma^a_b D_a D_b \phi + t^2 \gamma^a_b D_a \alpha D_b \phi &= 0, \\
t^2 \gamma^a_b D_a D_b \alpha - \left(t^2 \chi^a_b \chi^b_a + \frac{1}{n} + \pi^2\right)\alpha + 1 &= 0, \\
-t\partial_t \phi + \alpha \pi &= 0.
\end{align*}

(1.1) – (1.3)

Notice that $D_a$ is the covariant derivative associated with $\gamma$. We shall refer to Eqs. (1.1) – (1.3) as the lapse-scalar field system.

A particular goal for our work is therefore to characterize the map $\Psi$ between Cauchy data and asymptotic data for the lapse-scalar field system under as general conditions as possible, in particular not restricting to near-FLRW backgrounds, and to derive as much detail regarding the asymptotics as possible. Since detail is the paramount aim for us here, we, in fact, focus on linearizations of the lapse-scalar field system only; see Section 2.1. We conjecture that the amount of detail in controlling the asymptotics of linearized solutions, which we establish, suffices to conclude essentially the same state-
ments for solutions to the nonlinear system. We will address nonlinearities in future work exploiting recent results regarding a large class of Fuchsian-type equations in [26].

Roughly stated, our main result is as follows. For $\gamma^{ab}$ and $\chi^a_b$, which at each spatial point decay to a Kasner–scalar field solution, we consider the linearization of the system Eqs. (1.1) – (1.3) on the $n$-torus, cf. Section 2. We prove that there is a homeomorphism with respect to a natural topology between Cauchy initial data (prescribed at a regular time $T > 0$) and asymptotic data (associated with the limit at the singular time $t = 0$). The existence of such a homeomorphism does not only yield that the full degrees of freedom of the solution set are equivalently characterized by the Cauchy data or asymptotic data. It also follows that Cauchy data depend continuously on the asymptotic data and vice versa which precludes chaotic behavior. Most importantly, however, this map and its inverse are open maps which implies that any perturbation of Cauchy data (or asymptotic data), i.e., any change within an open set, guarantees that the corresponding asymptotic data (or Cauchy data) also change within some open set, i.e., are actual perturbations as well. All of this implies asymptotic stability. The Fuchsian methods in [1,2,14,18,27], which typically yield continuous maps from asymptotic data to Cauchy data, have often been criticized for the potential lack of openness of these maps. It is indeed not clear whether the constructed families of solutions are open (let alone dense) subsets of the solution space. The precise version of our main theorem is Theorem 3.2 which we shall complement with more detailed discussions there.

Remark 1.1. Theorem 3.2 provides a characterization of the asymptotics of solutions to the linearized lapse-scalar field system that contains the full set of degrees of freedom for solutions of these equations. A surprising result is that while spatial derivatives do not contribute to the very leading-order terms (consistent with AVTD behavior), the full set of degrees of freedom can only be described in terms of the asymptotics by incorporating spatial derivatives of sufficiently significant order. This shows that at least for the linearized lapse-scalar field system the VTD-equations (where these spatial derivatives are dropped) do not suffice to describe the full asymptotic behavior of generic solutions.

Remark 1.2. The method we use in this paper extends some of the general theory of [28] in important ways. First, the system of equations considered here have spatially non-homogeneous coefficients; indeed, the background metric is allowed to approach (with suitable decay estimates) a different solution of the Kasner–scalar field system at each spatial point. Second, Ringström studies the case in which the asymptotics are determined by the velocity terms alone (that is the case in which spatial derivatives are fully negligible in the sense above). As mentioned in Remark 1.1, to fully characterize solutions of the linearized lapse-scalar field system by their asymptotics and obtain a homeomorphism between Cauchy data and asymptotic data, certain spatial derivative terms must be accounted for and included in the asymptotic model equation. Finally our results are not restricted to wave equations (as opposed to [28]) as we take the elliptic lapse equation into account as well.

The outline of the paper is as follows. In Section 2 we provide some necessary preliminaries. In particular we derive the lapse-scalar field sub-system from the ADM formu-
lation of the Einstein-scalar field equations in CMC gauge with zero shift in Section 2.1. Section 2.2 is devoted to Kasner-scalar field solutions and their properties which play a essential role for our analysis. In Section 3 we present and discuss our main result. Before we discuss the proof in several sub-steps in Section 5, we introduce a new methodology in Section 4, i.e., a systematic way to asymptotically match solutions of in principle any system of equations of interest with certain asymptotic model equations, essentially following the paradigm used to define the AVTD property in [21]. This approach is the key to our proof of our main result in Section 5. We believe that it is a robust approach which has the potential to apply to more general classes of problems. In particular it allows us to reach a satisfactory level of detail in describing the asymptotics of the solutions of our system of equations.

2 Preliminaries

2.1 The lapse-scalar field system

In this section we briefly discuss the origin of the lapse-scalar field system Eqs. (1.1) – (1.3). The starting point is the standard ADM formulation of the Einstein-(minimally coupled) scalar field system (with zero potential) under the CMC-zero shift gauge condition

\[ K = -1/t, \quad \beta^a = 0. \]  

(2.1)

Here, and throughout the paper Latin indices \( a, b, \ldots \) take values 1, 2, \ldots, \( n \) (and therefore represent purely spatial tensors), while Greek indices \( \mu, \nu, \ldots \) take values 0, 1, \ldots, \( n \) (and therefore represent spacetime tensors). With the above gauge choice, the Einstein-scalar field system can be written as in [29]:

1. Constraint equations:

\[
\text{Scal} [\gamma] - \chi_a^b \chi_b^a + \frac{n-1}{nt^2} = \frac{1}{\alpha^2} \dot{\phi}^2 + \gamma_{ab} \dot{D}_a \phi \dot{D}_b \phi, \quad (2.2)
\]

\[
D_b \chi_a^b = -\frac{1}{\alpha} \phi \dot{D}_a \phi, \quad \chi_a^a = 0. \quad (2.3)
\]

2. Evolution equations for \( \gamma_{ab} \) and \( \chi_a^b \):

\[
\dot{\gamma}_{ab} = 2\alpha \chi_c^a \gamma_{cb} - \frac{2\alpha}{nt} \gamma_{ab}, \quad (2.4)
\]

\[
\dot{\chi}_a^b = -\gamma_{bc} D_a D_c \alpha - \alpha t \chi_a^b + \frac{\alpha - 1}{nt^2} \delta_a^b + \alpha \text{Ric}[\gamma]_a^b - \alpha \gamma_{bc} D_a \phi D_c \phi. \quad (2.5)
\]

3. Scalar field equation:

\[
- \frac{1}{\alpha} \frac{d}{dt} \left( \frac{1}{\alpha} \dot{\phi} \right) - \frac{1}{\alpha t} \dot{\phi} + \gamma_{ab} D_a D_b \phi + \frac{1}{\alpha} \gamma_{ab} D_a \alpha D_b \phi = 0. \quad (2.6)
\]
1. Lapse equation:

\[ \gamma^{ab} D_a D_b \alpha = \frac{\alpha - 1}{f^2} + \alpha \text{Scal}[\gamma] - \alpha \gamma^{ab} D_a \phi D_b \phi. \]  

(2.7)

Here, \( \gamma^{ab} \) is the 3-metric, \( \chi^a_b \) the tracefree part of the second fundamental form, \( \alpha \) is the lapse and \( \phi \) is the scalar field. A dot refers to the time derivative \( \partial_t \), and \( D_a \) is the Levi-Civita connection compatible with \( \gamma \). In order to extract the lapse-scalar field system Eqs. (1.1) – (1.3), now we first eliminate \( \text{Scal}[\gamma] \) from Eq. (2.7) using Eq. (2.2). This equation together with Eq. (2.6) then takes the form

\[ -t \partial_t (t \partial_t \phi) + (1 - \alpha) t \partial_t \phi + \frac{t \partial_t \phi}{\alpha} + \alpha^2 t^2 \gamma^{ab} D_a D_b \phi + \alpha t^2 \gamma^{ab} D_a \alpha D_b \phi = 0, \]  

(2.8)

\[ t^2 \gamma^{ab} D_a D_b \alpha - \alpha^2 \alpha \chi^a_b \chi^b_a - \frac{\alpha - n}{n} - \frac{1}{\alpha} (t \partial_t \phi)^2 = 0. \]  

(2.9)

Eqs. (1.1) – (1.3) are obtained by defining \( \pi = t \partial_t \phi / \alpha \) and replacing \( t \partial_t \phi \) by \( \pi \alpha \). If we used \( t \partial_t \phi \) instead of \( \pi \) as a variable in Eqs. (1.1) – (1.3), then time-derivative of \( \alpha \) would be needed to take care of when the background spacetime is not spatially homogeneous. Notice carefully that in working with Eqs. (1.1) – (1.3), we do not impose any of the remaining equations above. In fact, we shall consider \( \gamma^{ab} \) as any given time-dependent Riemannian metric and \( \chi^a_b \) as any given time-dependent tracefree (1,1)-tensor field. Only later we shall demand these fields to have certain asymptotics in the limit \( t \searrow 0 \).

Consider now arbitrary smooth fields \( \phi(t, x), \alpha(t, x), \pi(t, x) \), which represent a background solution. Rather than being exact solutions of the lapse-scalar field system, it is sufficient to only require \( \hat{\phi}, \hat{\alpha}, \hat{\pi} \) to satisfy Eqs. (1.1) – (1.3) up to residuals \( f^{(1)}(t, x), f^{(2)}(t, x) \) and \( f^{(3)}(t, x) \):

\[ -t \partial_t \hat{\pi} + (1 - \hat{\alpha}) \hat{\pi} + \hat{\alpha} t^2 \gamma^{ab} D_a D_b \hat{\phi} + t^2 \gamma^{ab} D_a \hat{\alpha} D_b \hat{\phi} = -f^{(1)}, \]  

(2.10)

\[ t^2 \gamma^{ab} D_a D_b \hat{\alpha} - t^2 \chi^a_b \chi^b_a \hat{\alpha} - \hat{\alpha} \hat{\pi}^2 + \chi^a_b D_a \hat{\pi} = -f^{(2)}, \]  

(2.11)

\[ -t \partial_t \hat{\phi} + \hat{\alpha} \hat{\pi} = -f^{(3)}. \]  

(2.12)

Specific conditions on the \( t \searrow 0 \) asymptotics of the residuals \( f^{(1)}, f^{(2)}, f^{(3)} \) are contained in Theorem 3.2 later. The fields

\[ u = \pi - \hat{\pi}, \quad \nu = \alpha - \hat{\alpha}, \quad \varphi = \phi - \hat{\phi}, \]  

(2.13)

therefore satisfy the system

\[ -t \partial_t u - \hat{\pi} \nu + (1 - \hat{\alpha}) u + \hat{\alpha} t^2 \gamma^{ab} D_a D_b \varphi \]  

\[ + \nu t^2 \gamma^{ab} D_a D_b \varphi + t^2 \gamma^{ab} D_a \hat{\alpha} D_b \varphi + t^2 \gamma^{ab} D_a \nu D_b \varphi \]  

\[ + u \nu + \nu t^2 \gamma^{ab} D_a D_b \varphi + t^2 \gamma^{ab} D_a \nu D_b \varphi = f^{(1)}, \]  

(2.14)

\[ t^2 \gamma^{ab} D_a D_b \nu - \left( t^2 \chi^a_b \chi^b_a + \frac{1}{n} + \hat{\pi}^2 \right) \nu - 2 \hat{\pi} \hat{\alpha} u - \nu u^2 - 2 \hat{\pi} \nu u - \hat{\alpha} u^2 = f^{(2)}, \]  

(2.15)

\[ -t \partial_t \varphi + \hat{\alpha} u + \hat{\pi} \nu + \nu u = f^{(3)}. \]  

(2.16)
The linearized lapse-scalar field equations are obtained by deleting all nonlinear terms from Eq. (2.14) – (2.16). The idea is that $\phi(t, x), \dot{\alpha}(t, x), \dot{\pi}(t, x)$ serve as a background-(almost)-solution of the lapse-scalar field equations with respect to which two kinds of problems may be of interest: First, the dynamics of its linear perturbations $(u, \nu, \varphi)$, and second, nonlinear stability (or instability). As mentioned before this paper focuses exclusively on the linear case.

2.2 Kasner-scalar field spacetimes

The background solutions of interest to this paper are related to a simple family of solutions: The *Kasner-scalar field spacetimes* (referred to as (generalised) Kasner spacetimes in [29]). In this section we briefly summarise their most important properties.

The Kasner-scalar field spacetimes are spatially homogeneous (but in general very anisotropic) solutions $(\gamma^{ab}, \chi^a_b, \alpha, \phi)$ of the Einstein-scalar field system in zero shift CMC gauge where the spatial manifold is the $n$-torus $M = T^n$. The spatial metric takes the form

$$
\gamma^{ab} = \text{diag} \left( t^{-2q_1}, \ldots, t^{-2q_n} \right) \tag{2.17}
$$

written in terms of standard Cartesian coordinates on $M$ where the Kasner exponents $q_1, \ldots, q_n$ are real numbers subject to

$$
\sum_{i=1}^n q_i = 1, \quad \sum_{i=1}^n q_i^2 = 1 - A^2, \tag{2.18}
$$

for any \(^2\) \(A \in [0, a_+], \quad a_+ := \sqrt{1 - \frac{1}{n}}. \tag{2.19}\)

The scalar field is given by

$$
\tilde{\phi} = A \log t + B \tag{2.20}
$$

for any real constant $B$. Furthermore we have

$$
\bar{\alpha} = 1, \tag{2.21}
$$

and

$$
\chi^a_b = \frac{1}{t} \text{diag} \left( \frac{1}{n} - q_1, \ldots, \frac{1}{n} - q_n \right). \tag{2.22}
$$

Notice that this implies the useful formula

$$
t^2 \chi^a_b \chi^b_a = 1 - \frac{1}{n} - A^2. \tag{2.23}
$$

\(^2\)Without loss of generality we assume that $A$ is non-negative throughout this whole paper. If $A$ is negative we can replace $\tilde{\phi}$ by $-\tilde{\phi}$ since the Einstein-scalar field system with zero potential is invariant under the transformation $\phi \mapsto -\phi$. 

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Two well-known special cases are (1) the isotropic FLRW-case given by \( A = A_+ \) and \( q_1 = \ldots = q_n = 1/n \), and (2) the vacuum Kasner solutions characterized by \( A = 0 \).  

For the later discussion, it is useful to establish certain bounds on the set of admissible Kasner exponents for any given \( A \in [0, A_+] \). Setting \( p_i = q_i - 1/n \), Eq. (2.18) transforms into

\[
\sum_{i=1}^{n} p_i = 0, \quad \sum_{i=1}^{n} p_i^2 = 1 - A^2 - 1/n = A_+^2 - A^2. \tag{2.24}
\]

All vectors \( (p_1, \ldots, p_n) \in \mathbb{R}^n \) consistent with this are on the intersection of the plane through the origin perpendicular to the vector \( n_1 = (1, \ldots, 1)/\sqrt{n} \) and the \( (n-1) \)-sphere around the origin with radius \( \sqrt{A_+^2 - A^2} \). Combining the square of the first identity in Eq. (2.24)

\[
p_1^2 = \sum_{i,j=2}^{n} p_ip_j = -\frac{1}{2} \sum_{i,j=2}^{n} (p_i - p_j)^2 + (n - 1) \sum_{i=2}^{n} p_i^2
\]

with the second identity eventually yields

\[
p_1^2 = \frac{n-1}{n} (A_+^2 - A^2) - \frac{1}{2n} \sum_{i,j=2}^{n} (p_i - p_j)^2.
\]

The largest possible value

\[
|p_1| = \sqrt{\frac{n-1}{n}} \sqrt{A_+^2 - A^2} = A_+ \sqrt{A_+^2 - A^2}
\]

is therefore obtained when

\[
p_i = -p_1/(n - 1)
\]

for all \( i = 2, \ldots, n \). We conclude that for any given \( A \in [0, A_+] \) the largest and the smallest possible Kasner exponents are

\[
q_{\text{upper}} = 1 - A_+^2 + A_+ \sqrt{A_+^2 - A^2}, \quad q_{\text{lower}} = 1 - A_+^2 - A_+ \sqrt{A_+^2 - A^2}. \tag{2.25}
\]

Given any collection \( (q_1, \ldots, q_n) \) of Kasner exponents, we write

\[
q_{\max} = \max\{q_1, \ldots, q_n\}, \quad q_{\min} = \min\{q_1, \ldots, q_n\}. \tag{2.26}
\]

Given \( n \) and \( A \) satisfying Eq. (2.19), which Kasner-scalar field solution, cf. Eq. (2.18), has the smallest value of \( q_{\max} \)? The above considerations imply that this is achieved when one of the Kasner exponents takes the value of \( q_{\text{lower}} \) and all other Kasner exponents take the value

\[
q_{\max,0} = \frac{1}{n} + \frac{A_+}{n-1} \sqrt{A_+^2 - A^2} = \left(1 - A_+^2\right) \left(1 + \sqrt{1 - A^2/A_+^2}\right). \tag{2.27}
\]

\( ^{22} \)Note that in the vacuum case \( \phi = B = \text{const} \) has no dynamics and no gravitational interaction.
3 Main result

Before stating our main theorem Theorem 3.2 below we introduce further notation and conventions. As explained above we focus on the linearization of the lapse-scalar field system Eq. (2.14) – (2.16) which we write now as follows. First, in order to emphasize that \( \gamma^{ab} \) and \( \chi^a_b \) are given fields just as \( \hat{\alpha} \), \( \hat{\phi} \) and \( \hat{\pi} \), we write \( \hat{\gamma}^{ab} \) and \( \hat{\chi}^b_a \) in all of what follows now. We shall formulate our precise assumptions on these background fields in Definition 3.1 below. Second, we introduce the switching parameter \( \sigma \), which multiplies certain terms. The parameter \( \sigma \) is used to switch between the full system (corresponding to \( \sigma = 1 \)) and an asymptotic model system (corresponding to \( \sigma = 0 \)), but any value in between may be of interest as well. Note that \( \sigma \) multiplies the spatial derivative terms as well as other terms that turn out to be of higher order. In a first study of this section and Theorem 3.2, it is sufficient to just think of the case \( \sigma = 1 \). Different values of the switching parameter become relevant in our discussion of asymptotic matching problems in Section 4 and, in particular, for our proof of Theorem 3.2 in Section 5. With all this our system takes the form

\[
-t \partial_t u - \sigma (\hat{\pi} - A) \nu + \sigma (1 - \hat{\alpha}) u + \sigma \hat{\alpha} t^2 \hat{\gamma}^{ab} D_a D_b \varphi
+ \sigma \nu t^2 \hat{\gamma}^{ab} D_a D_b \hat{\phi} + \sigma t^2 \hat{\gamma}^{ab} D_a \hat{\alpha} D_b \varphi + \sigma t^2 \hat{\gamma}^{ab} D_a \nu D_b \hat{\phi} = f^{(1)},
\]

\[
\sigma t^2 \hat{\gamma}^{ab} D_a D_b \nu - \nu - 2 A u - \sigma \left( \left( t^2 \hat{\chi}^b_a \hat{\chi}_a^b + \frac{1}{n} + A^2 - 1 \right) + (\hat{\pi}^2 - A^2) \right) \nu
- 2 \sigma (\hat{\pi} \hat{\alpha} - A) u = f^{(2)},
\]

\[
-t \partial_t \varphi + u + A \nu + \sigma (\hat{\alpha} - 1) u + \sigma (\hat{\pi} - A) \nu = f^{(3)}.
\]

In order to write our main result now let us first equip \( M = \mathbb{T}^n \) with the time-independent flat reference metric \( \delta_{ab} \). In terms of Cartesian coordinates on \( M \) we assume that \( \delta_{ab} \) takes the form

\[
\delta_{ab} = \text{diag} (1, \ldots, 1).
\]

Below, all spatial index operations are performed with the metric \( \delta_{ab} \). Let \( \partial_a \) be the (time-independent) covariant derivative associated with \( \delta_{ab} \). For any two smooth time-dependent tensor fields\(^3\) \( S \) and \( \tilde{S} \) on \( M \) of the same arbitrary rank \((0,r)\) we write

\[
(S, \tilde{S})_{\delta} = S_{a_1 \ldots a_r} \tilde{S}^{a_1 \ldots a_r}, \quad |S|_{\delta}^2 = (S, S)_{\delta},
\]

and

\[
S = \tilde{S} + O(f)
\]

for some smooth function \( f(t,x) \) provided that for each non-negative integers \( k \) and \( \ell \)

\(^3\)Arbitrary smooth (time-dependent or time-independent) tensor fields \( S \) of rank \((0,r)\) on \( M \) are written in either the index-free form as \( S \), or, if we want to emphasize the rank as \( S_{[r]} \), or, using abstract indices as \( S_{a_1 \ldots a_r} \).
there is a constant $C > 0$ such that\footnote{This notion of the $O$-symbol can be relaxed. In fact, some of the quantities below do not require control over any time-derivatives and/or only over a finite number of spatial derivatives. Nevertheless, we use this strict version of the $O$-symbol in order to simplify the statement of the main theorem.} \footnote{Recall that spatial index manipulations are performed with the flat reference metric $\delta_{ab}$. In our notation, $\gamma_{ab}^{-1}$ (the inverse of $\gamma^{ab}$) does therefore in general differ from $\gamma_{ab}$.}  

$$\left|(t\partial_t)^k \partial^k(S(t,x) - \tilde{S}(t,x))\right|_\delta \leq C f(t,x),$$

for all $x \in M$ and all $t \in (0,T]$ where $T > 0$ is some final time. Here, $\partial^k(S - \tilde{S})$ is the short hand notation for the tensor field $\partial_{\alpha_1} \cdots \partial_{\alpha_k} (S_{\alpha_1 \cdots \alpha_k} - \tilde{S}_{\alpha_1 \cdots \alpha_k})$ of rank $(0,k+r)$.

The following definition introduces the relevant class of background fields used in the main result.

**Definition 3.1** (Asymptotically point-wise Kasner–scalar field background). Let $\hat{\gamma}^{ab}$ be a smooth time-dependent Riemannian metric, $\hat{\chi}^b_a$ a smooth time-dependent symmetric $(1,1)$-tensor field, and $\hat{\alpha}, \hat{\pi}, \hat{\phi}$ be smooth time-dependent scalar fields on $M$. We say that the collection of fields $(\gamma^{ab}, \chi^b_a, \alpha, \pi, \phi)$ on $M = \mathbb{T}^n$ is an asymptotically point-wise Kasner–scalar field background with decay $\beta$ for some smooth time-independent positive function $\beta$ on $M$ provided:

1. There are smooth functions $q_1(x), \ldots, q_n(x)$ and $A(x)$ on $M$ such that Eqs. (2.18) and (2.19) hold pointwise at each $x \in M$, and smooth time-dependent fields $\hat{h}^{ab}$, $\tilde{h}_{ab}$ on $M$ with\footnote{Equations (2.25) and (2.26) hold pointwise at each $x \in M$.}  

$$\hat{h}^{ab}\gamma^{bc} = O(t^\beta), \quad \tilde{h}_{ab}\gamma_{bc} = O(t^\beta),$$

such that

$$\hat{\gamma}^{ab}(t,x) = \gamma^{ab}(t,x) + \hat{h}^{ab}, \quad \hat{\gamma}^{-1}_{ab}(t,x) = \gamma^{-1}_{ab}(t,x) + \tilde{h}_{ab}. \quad (3.7)$$

Here $\gamma^{ab}(t,x)$ is defined by Eq. (2.17) pointwise at each $x \in M$.

2. We have

$$t\chi^b_a = \gamma^b_a + O(t^\beta),$$

where $\gamma^b_a$ is defined by Eq. (2.22) pointwise at each $x \in M$.

3. The functions $\hat{\alpha}$, $\hat{\pi}$ and $\hat{\phi}$ on $M$ satisfy

$$\hat{\alpha} = 1 + O(t^\beta), \quad \hat{\pi} = A + O(t^\beta), \quad \hat{\phi} = (A + O(t^\beta)) \log t. \quad (3.8)$$

Given any asymptotically point-wise Kasner–scalar field background we refer to the functions $q_1, \ldots, q_n$ as the Kasner exponents and $A$ as the scalar field strength. We refer to $q_{\text{max}}$, $q_{\text{min}}$, $q_{\text{upper}}$ and $q_{\text{lower}}$ by the formulas Eq. (2.26) and (2.25) pointwise at each $x \in M$.

Notice that the exact Kasner-scalar field solutions in Section 2.2 are special examples of asymptotically point-wise Kasner–scalar field background. It is important to point
out however that although the asymptotically point-wise Kasner–scalar field background fields asymptote at each point to a homogeneous solution of the Einstein–scalar field equations, the fields themselves are not required in the present work to be solutions to these equations. The findings in [29, 30] demonstrate that this class of background fields is sufficiently general and in fact goes far beyond special backgrounds with large degrees of symmetry (in particular the FLRW Kasner scalar field background). We remark that it is in fact possible to generalise our results here to Kasner footprint maps introduced in [29, 30]. We formulate our theorem below for the slightly more specialised class of backgrounds above, however, just to reduce the otherwise heavy notation (but without sacrificing the essential features of our results).

In order to facilitate the following now, we introduce a smooth function $\xi(x)$ on $M$ with range in $[0, 1]$ by (recall Eq. (2.19))

$$A = A_+ \sqrt{1 - \xi^2},$$

and then define

$$\lambda_c = \begin{cases} 2A^2 = 2(1 - \xi^2)A_+^2 & \text{for } \xi \in [0, 1/3], \\ \frac{(1+\xi)^3}{\xi}A_+^2 & \text{for } \xi \in [1/3, 1]. \end{cases}$$

In order to simplify some of the following formulas, we also define

$$S^{(1)}(t, x) = \int_t^T \left( f^{(1)}(s, x) - A(x)f^{(2)}(s, x) \right) \left( \frac{t}{s} \right)^{2A^2(s)} s^{-1} ds$$

$$S^{(2)}(t, x) = - (2A^2(x) - 1) \int_0^t S^{(1)}(s, x)s^{-1} ds$$

$$- \int_0^t \left( f^{(3)}(s, x) + A(x)f^{(2)}(s, x) \right) s^{-1} ds,$$

and

$$\mathcal{L}[g] = \sigma \left( \dot{\alpha}t^{2} \dot{\gamma}^{ab} D_{a}D_{b}g + t^{2} \dot{\gamma}^{ab} D_{b} \dot{\alpha} D_{a}g \right),$$

for any smooth functions $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ and $g(t, x)$ for which these integrals are finite. Given this we are now in the position for stating our main result.

**Theorem 3.2.** Pick $T$ to be a sufficiently small constant in $(0, 1]$ and an integer $n \geq 3$. Consider an arbitrary asymptotically point-wise Kasner–scalar field background $\Gamma = (\dot{\gamma}^{ab}, \dot{\chi}^{a}, \dot{\alpha}, \dot{\pi}, \dot{\phi})$ with decay $\beta$, where $\beta$ is smooth positive time-independent function on $M = \mathbb{T}^n$ with

$$\beta > \lambda_c - \min\{2A^2, 2(1 - q_{\max})\},$$

such that

$$\lambda_c < \min\{4(1 - q_{\max}), 2A^2 + 2(1 - q_{\max})\}$$

holds where $\lambda_c$ is defined in Eq. (3.10). Suppose the source terms $f^{(1)}(t, x)$, $f^{(2)}(t, x)$ and $f^{(3)}(t, x)$ in Eqs. (3.1) – (3.3) are smooth functions on $(0, T] \times M$ such that

$$\int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|_{\delta, H^{k+4}(M)} + \|s^{-\lambda_1} f^{(2)}\|_{\delta, H^{k+4}(M)} + \|s^{-\lambda_1 + 2\kappa} f^{(3)}\|_{\delta, H^{k+5}(M)} \right) s^{-1} ds < \infty,$$

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for all integers \( k \geq 0 \) and for some smooth function \( \lambda_1(x) \) on \( M \) with

\[
\lambda_1 > \max\{\lambda_c - 2(1 - q_{\max}), \lambda_c - \beta, 0\}.
\] (3.16)

Then, for each constant \( \sigma \in [0, 1] \), there is a homeomorphism in the \( C^\infty \)-topology

\[
\Psi : (C^\infty(M))^2 \to (C^\infty(M))^2, \quad (\varphi(0), \varphi(1)) \mapsto (u_*, \varphi_*)
\]

with the following property. For each integer \( k \geq 0 \) and smooth function \( \lambda(x) \) with

\[
\lambda < \min\{\lambda_1 + 2(1 - q_{\max}), 4(1 - q_{\max}), 2A^2 + 2(1 - q_{\max}), \beta + 2(1 - q_{\max}), 2A^2 + \beta\},
\] (3.17)

there is a constant \( C > 0 \), such that for any \((\varphi(0), \varphi(1)) \in (C^\infty(M))^2\), the solution \((u, \nu, \varphi)\) of Eqs. (3.1) – (3.3) determined by Cauchy data \( \varphi(T, \cdot) = \varphi_* \) and \( u(T, \cdot) = u_* \) with \((u_*, \varphi_*) = \Psi(\varphi(0), \varphi(1))\) satisfies

\[
\begin{align*}
& \left\| t^{-\lambda} \left( u(t, \cdot) + 2A^2 s^{-1} ds - S(1)(t, \cdot) \right) \right\|^2_{H^k(M)} \\
& + \left\| t^{-\lambda} \left( \varphi(t, \cdot) - \varphi(0) - 2A^2 - 1\right)^{2A^2} \varphi(1) \\
& \quad + (2A^2 - 1) \int_0^t \int_{\tau}^T \mathcal{L}[\varphi(0)](s, \cdot) \left( \frac{t}{s} \right)^{2A^2} s^{-1} ds \tau^{-1} d\tau - S(2)(t, \cdot) \right\|^2_{H^k(M)} \leq C
\end{align*}
\] (3.18)

for all \( t \in (0, T] \) and all integers \( k \geq 0 \). The constant \( C > 0 \) may depend on \( k, \lambda, T, f^{(1)}, f^{(2)}, f^{(3)}, \varphi(0), \varphi(1) \) and \( \Gamma \).

The proof of Theorem 3.2 is discussed in detail in Section 5. In the remainder of this section we discuss consequences of Theorem 3.2. In particular we clarifying the role of spatial derivatives for the asymptotic description of the degrees of freedom of solutions of the lapse-scalar field system. First let us just mention that the estimate (3.18) for \( \nu \) and \( \varphi \) can be complemented by a corresponding estimate for \( \nu \); in order to keep the statement of the theorem short we have not incorporated this estimate.

On the one hand, this theorem states that any solution of Eqs. (3.1) – (3.3) launched by Cauchy data \((u_*, \varphi_*)\) is described asymptotically by Eqs. (3.18) and (3.17) for asymptotic data\(^{3,4}\) \((\varphi(0), \varphi(1)) = \Psi^{-1}(u_*, \varphi_*)\). On the other hand, it states that for any choice of asymptotic data \((\varphi(0), \varphi(1))\) there exists a solution of the Cauchy problem for Eqs. (3.1) – (3.3) with Cauchy data \((u_*, \varphi_*) = \Psi(\varphi(0), \varphi(1))\) which realizes these asymptotic data via Eqs. (3.18). The full degrees of freedom of the entire solution set are therefore parametrized by either Cauchy data (the state of the “system” at the “regular time” \( t = T > 0 \)), or equivalently, by asymptotic data (the “state of the system at the singular time” \( t = 0 \)). This map between asymptotic data and Cauchy data \( \Psi \) is invertible, and, both directions are open maps with respect to the \( C^\infty \)-topology. As discussed before,
this implies that the solutions are *asymptotically stable* under perturbations of either the asymptotic data or the Cauchy data.

Eq. (3.18) yields the asymptotic behavior of arbitrary solutions up to $O(t^\lambda)$ (in the limit $t \to 0$). Indeed, $\lambda$ can always be chosen a little larger than $2A^2$ since the upper bound in Eq. (3.17) is larger than $2A^2$:

- $\lambda_1 + 2(1 - q_{\text{max}}) > 2A^2$ because of Eqs. (3.16) and (3.10).
- $4(1 - q_{\text{max}}) > 2A^2$ because
  
  \[
  4(1 - q_{\text{max}}) - 2A^2 \geq 4(1 - q_{\text{upper}}) - 2A^2 = 4A^2_+ (1 - \xi) - 2A^2_+ (1 - \xi^2) \\
  = 2A^2_+ (1 - \xi)(2 - (1 + \xi)) = 2A^2_+ (1 - \xi)^2 > 0,
  \]
  
  for all $\xi \in [0, 1)$ using Eqs. (3.9) and (2.25).
- $2A^2 + 2(1 - q_{\text{max}}) > 2A^2$ because $q_{\text{max}} < 1$ for all $\xi < 1$.
- $\beta + 2(1 - q_{\text{max}}) > 2A^2$ because
  
  \[
  \beta + 2(1 - q_{\text{max}}) - 2A^2 \\
  \geq \begin{cases} 
  \lambda_c - 2A^2 + 2(1 - q_{\text{max}}) - 2A^2 & \text{if } 2(1 - q_{\text{max}}) \geq 2A^2, \\
  \lambda_c - 2(1 - q_{\text{max}}) + 2(1 - q_{\text{max}}) - 2A^2 & \text{if } 2(1 - q_{\text{max}}) \leq 2A^2
  \end{cases}
  \geq 0,
  \]
  
  as a consequence of Eq. (3.14).
- $2A^2 + \beta > 2A^2$ because $\beta > 0$.

Because of this, Eq. (3.18) is “correct up to $O(t^{2A^2+\epsilon})$” for any sufficiently small $\epsilon > 0$. It can thus be interpreted as the asymptotic representation of the full degrees of freedom of the solution set as it contains both the first asymptotic datum, $\varphi(0)$, associated with the $t$-power 0 and the second asymptotic datum, $\varphi(1)$, with the $t$-power $2A^2$. Hence it does make sense to call $(\varphi(0), \varphi(1))$ *asymptotic data*. Observe that these are associated with the following limits of an arbitrary solution:

\[
\varphi(0)(x) = \lim_{t \to 0} \varphi(t, x),
\]

\[
-2A^2(x)\varphi(1)(x) = \lim_{t \to 0} \left( u(t, x) - \int_t^T L[\varphi(0)](s, x) \left( \frac{t - s}{s} \right)^{2A^2(x)} s^{-1} ds - S(1)(t, x) \right) t^{-2A^2(x)}.
\]

Let us stress that the limit of the first integral on the right-hand side by itself multiplied with $t^{-2A^2(x)}$ would in general not be finite (see below).

One of the most interesting conclusions which we can draw from Eq. (3.18) is that any of its terms whose $t$-power is smaller than $2A^2$ is *significant for the asymptotic characterization of the full degrees of freedom*. Obviously it does not only contain the
“pure asymptotic data terms”, i.e., the term associated with the $t$-powers $0$ and $2A^2$. Of particular interest to us are the terms involving $L[\varphi(0)]$ which originate in the spatial derivatives in the equation. The terms $S^{(1)}$ and $S^{(2)}$ are generated by the source terms $f^{(1)}, f^{(2)}, f^{(3)}$ of the equation. In principle, there might have been additional terms generated by the fields in $\Gamma$; thanks to our minimal assumptions regarding $\Gamma$, in particular Eq. (3.14), such contributions do however not show up in Eq. (3.18) at orders compatible with Eq. (3.17). We justify now that the above terms generated by spatial derivatives are significant for the asymptotic characterization of the full degrees of freedom in general – a statement which demonstrates that the notion of AVTD cannot be applied naively. We see this as follows. The $t$-powers associated with those spatial derivative terms in Eq. (3.18) are all of the form $2(1-q)$ where $q$ is one of the Kasner exponents. The most dominant term among these is therefore the one associated with the $t$-power $2(1-q_{\text{max}})$. If $2(1-q_{\text{max}}) > 2A^2$ at all points $x \in M$, then all spatial derivative terms would be negligible. If $2A^2 > 2(1-q_{\text{max}})$ at some point $x \in M$, however, then some terms generated by spatial derivatives are significant for the asymptotic characterization of the full degrees of freedom. We can shed light on this by considering the following sharp bounds (recall Eqs. (2.25) and (2.27))

$$2(1-q_{\text{max}}) \geq 2(1-q_{\text{upper}}) = 2A^2(1-\xi),$$

and

$$2(1-q_{\text{max}}) \leq 2(1-q_{\text{max},0}) = 2(A^2_+ - (1-A^2_+) \xi).$$

The possible range for $2(1-q_{\text{max}})$ therefore depends on $\xi$, but so does $2A^2 = 2A^2_+(1-\xi^2)$. Whether spatial derivatives are hence significant in the sense above or not depends on the following bounds

$$\frac{2(1-q_{\text{max}})}{2A^2} \geq \frac{2A^2_+(1-\xi)}{2A^2_+(1-\xi^2)} = \frac{1}{1+\xi} > \frac{1}{2}, \quad \text{(3.19)}$$

if $\xi \in [0, 1)$, and

$$\frac{2(1-q_{\text{max}})}{2A^2} \leq \frac{A^2_+ - (1-A^2_+) \xi}{A^2_+(1-\xi^2)} = 1 - \frac{1-(n-1)\xi}{(n-1)(1-\xi^2)} \xi.$$
terms are significant in this sense for all choices of Kasner exponents consistent with the fundamental constraints when \( \xi \in [0, 1/(n - 1)) \). Only if \( \xi > 1/(n - 1) \), there exist certain choices of Kasner exponents for which spatial derivative terms are completely insignificant; this is the case when the Kasner exponents are such that \( q_{\text{max}} \) is sufficiently close to the smallest possible value given by Eq. (2.27).

This discussion therefore suggests that Eq. (3.18) is the correct asymptotic formula for all solutions of our equations and represents the full degrees of freedom for any \( \xi \in [0, 1) \).

What is therefore the origin of the restriction Eq. (3.15) which prevents us from going too close to the vacuum case \( \xi = 1 \)?

- Given arbitrary \( n \geq 3 \) (and hence \( A_+ \) by Eq. (2.19)), then Eq. (3.15) holds for all \( \xi \in [0, \Xi) \) and all admissible choices of Kasner exponents provided \( \Xi \approx 0.58 \) (the smallest real root of \( \xi^3 + 19\xi^2 - 13\xi + 1 \) in the interval \((1/3, 1))\).

- Given arbitrary \( n \geq 3 \), then Eq. (3.15) holds for all \( \xi \in [0, \Xi) \) and all admissible choices of Kasner exponents with \( q_{\text{max}} \) sufficiently close to Eq. (2.27) provided \( \Xi \approx 0.82 \) (the smallest real root of \( \xi^3 + 19\xi^2 - 13\xi + 1 \) in the interval \((1/3, 1))\).

- Eq. (3.15) holds for all \( \xi \in [0, 1) \), all sufficiently large \( n \geq 3 \) and all admissible choices of Kasner exponents with \( q_{\text{max}} \) sufficiently close to Eq. (2.27).

Hence, only if \( n \) is sufficiently large and \( q_{\text{max}} \) is sufficiently small, Eq. (3.15) yields no restriction on \( \xi \). Recall that the above arguments were based on the idea that, in order to describe the full degrees of freedom in terms of the asymptotics, we need to accurately describe the solutions only up to \( O(t^\lambda) \) with \( \lambda \) a little larger than \( 2A^2 \). As we discuss in detail in Section 5, our estimates however demand the asymptotic formula Eq. (3.18) to be correct up to order \( O(t^\lambda) \) with \( \lambda \) a little larger than \( \lambda_c \) (which can be larger than \( 2A^2 \); see Eq. (3.10)). Now replacing \( 2A^2 \) by \( \lambda_c \) in Eq. (3.19) and considering \( \xi > 1/3 \) (which implies that \( \lambda_c > 2A^2 \)), we lose the lower bound of \( 1/2 \):

\[
\frac{2(1 - q_{\text{max}})}{\lambda_c} \geq \frac{8\xi(1 - \xi)}{(1 + \xi)^3}.
\]

This is the reason why Eq. (3.18) is not accurate enough when \( \xi \) is too close to \( \xi = 1 \), i.e., when Eq. (3.15) is violated.

We believe however that this just a technical problem which can be overcome (even though we have not looked at the details yet) as follows. As we will see, the quantity \( \lambda_c \) shows up for the first time in one of the “rough estimates” in Proposition 5.1 below. The same arguments which we then employ to “improve” a different one of these rough estimates in Proposition 5.2 should “similarly improve” the particular estimates which gives rise to \( \lambda_c \) as well. In this way it should be possible to “push \( \lambda_c \) arbitrary close to \( 2A^2 \) iteratively on any fixed sub-interval for \( \xi \) in \((0, 1)\). The larger this sub-interval is, the more differentiability we however expect to lose.

We want to close this section with more remarks anticipating some motivation in Section 4. Recall that we had introduced the switching parameter \( \sigma \) in Eqs. (3.1) – (3.3) and we argued that one can think of the case \( \sigma = 1 \) for the time being. It turns
out that this parameter plays an important role for our proof. In fact, the asymptotic formula provided by Eq. (3.18) essentially turns out to be the general solution of the equations obtained by setting \( \sigma = 0 \) in Eqs. (3.1) – (3.3). In a sense we make precise in the following, the homeomorphism constructed in Theorem 3.2 can be interpreted as a (invertible) map between solutions of the equations with \( \sigma = 0 \) and solutions of the equations with an arbitrary value of \( \sigma \in [0, 1] \) which “match” as in Eq. (3.18) (or vice versa). Given this homeomorphism for two different values \( \sigma_1 \) and \( \sigma_2 \) in [0, 1] (but everything else is the same), composing the inverse of the latter with the first clearly yields a homeomorphism which “matches” any solution of the \( \sigma_1 \)-version of the equation with a solution of the \( \sigma_2 \)-version. Asymptotic matching problems are discussed in the next section. These provide the basis of the proof of the main theorem in Section 5.

4 Matching problems

4.1 Asymptotic and finite matching problems

In this section we present a framework for characterizing the asymptotics of solutions of, in principle, any given (system of) evolution PDE. This framework assumes that the sought asymptotics are themselves solutions of a, in general, simpler “effective” version of the original equation – a (asymptotic) model equation. In general relativity, the classical problem prototype of this perspective is that of finding AVTD solutions as defined rigorously in [21]. In order to introduce our framework consider a PDE of the schematic form

\[
A[v](t, x) + \sigma L[v](t, x) = 0,
\]

where \( v(t, x) \) is the unknown defined on some time interval \( t \in (0, T], T > 0 \), and for all \( x \in M \). In general, \( v \) may be a time-dependent section in some vector bundle \( E \) over \( M \). At this stage, we only assume that \( A \) and \( L \) in Eq. (4.1) are linear differential operators with smooth coefficients. The quantity \( \sigma \) is a real number restricted to a bounded closed interval \( I \) of \( \mathbb{R} \) (which in general needs be chosen such that the character of the equation is not changed).

The main idea of the splitting manifested by the two terms in Eq. (4.1) is that the operator \( A \) contains all those terms which are significant asymptotically (like the “velocity terms” in the AVTD setting above), while \( L \) contains all terms which are negligible (for example spatial derivatives in the context of AVTD). Of particular interest are the cases \( \sigma = 1 \) – the full equation – and \( \sigma = 0 \) – the “model equation” (or effective equation). Notice that going from the full equation to the model equation means that one neglects all terms given by the operator \( L \). Since the terms in \( L \) are effectively “switched off” when we set \( \sigma = 0 \) we call \( \sigma \) the switching parameter as in Eq. (3). For different choices of the parameter \( \sigma \) we often speak of different versions of the equation. The hope is that the model equation describes the asymptotics of the full equation sufficiently accurately (in the precise sense below) in the limit \( t \searrow 0 \). Below we present a method to verify this.

**Definition 4.1** (Asymptotic matching problem). Pick \( \sigma_1, \sigma_2 \in I = [0, 1] \). Given any solution \( v_1 \) (of some prescribed regularity) of the version of Eq. (4.1) given by \( \sigma = \sigma_1 \), find
a solution \( v_2 \) (of some prescribed regularity) of the version of Eq. (4.1) given by \( \sigma = \sigma_2 \) such that
\[
\|v_1(t, \cdot) - v_2(t, \cdot)\| \to 0, \quad \text{as } t \downarrow 0,
\]
for some (possibly time-dependent) norm \( \|\cdot\| \). If this can be done for each \( v_1 \) in some set \( \Omega \) and if the corresponding \( v_2 \) is always uniquely determined, we say that this matching problem is well-posed. In this case it gives rise to a map \( \Psi_{\sigma_1 \to \sigma_2} \) – the (asymptotic) matching map – with domain \( \Omega \) of the schematic form
\[
\Psi_{\sigma_1 \to \sigma_2} : v_1 \mapsto v_2.
\]

In the sense of the map in Definition 4.1, we call the \( \sigma = \sigma_1 \)-version of Eq. (4.1) the source equation and the \( \sigma = \sigma_2 \)-version the target equation. We have intentionally not yet assumed any particular PDE type for Eq. (4.1) at this point. It is clear that the question whether a matching problem is well-posed or not depends strongly on the PDE type as well as the choice of norm and regularity requirements in the definition.

In agreement with the above comments, the maps \( \Psi_{1 \to 0} \) and \( \Psi_{0 \to 1} \) are of particular interest. When the model equation \( (\sigma = 0) \) is sufficiently simple and its solution can be parametrized by “data” we shall call these “asymptotic data”. The former is therefore essentially the map from Cauchy data (for the full equation) to asymptotic data. Provided the equation has a well-posed initial value problem, the latter is the map from asymptotic data to Cauchy data and is strongly related to the the Fuchsian method mentioned earlier. We remark that, indeed, it turns out to be useful to write Eq. (4.1) and Definition 4.1 in terms of a general family of switching parameters \( \sigma, \sigma_1 \) and \( \sigma_2 \) (as opposed to restricting to special values 1 or 0). In doing this one is able to essentially treat all specific cases in a uniform manner.

Suppose a matching problem in Definition 4.1 is well-posed and the map \( \Psi_{\sigma_1 \to \sigma_2} \) has been constructed for some \( \sigma_1, \sigma_2 \in I \). A natural question is: Is this map invertible, maybe even a homeomorphism in some sense? If this is the case, then the set of all solutions of the \( \sigma = \sigma_1 \)-version of the equation is in one-to-one correspondence with the set of all solutions of \( \sigma = \sigma_2 \)-version of the equation and the correspondence is described in terms of the asymptotics. This yields a strong characterization of the asymptotics.

How does Theorem 3.2 now fit into this framework? In order to set up an asymptotic matching problem, and therefore to choose \( A, L \) and \( F \) in Eq. (4.8), one often starts off with some heuristic considerations. First we hope to be able to choose the background fields \( \phi(t, x), \bar{\alpha}(t, x), \bar{\pi}(t, x) \) in Eqs. (2.10) – (2.12) to carry the main singular behavior of solutions of the lapse-scalar field system, in the sense that the functions \( f^{(1)}(t, x), f^{(2)}(t, x) \) and \( f^{(3)}(t, x) \) determined by Eqs. (2.10) – (2.12) behave in some less singular manner in the limit \( t \downarrow 0 \). Since the fields \( u, \nu \) and \( \varphi \) defined by Eq. (2.13) are therefore interpreted as small perturbations, it is generally expected that solutions of Eqs. (2.14) – (2.16) are predominantly driven by linear terms, and, that nonlinearities contribute only at higher order in the limit \( t \downarrow 0 \). In fact, the study of the role of nonlinearities could
be phrased as the following asymptotic matching problem with switching parameter $\sigma$:

$$
-t \partial_t u - \ddot{\pi} \nu + (1 - \dot{\alpha}) u + \dot{\alpha} t^2 \dot{\gamma}_{ab} D_a D_b \varphi
+ \nu t^2 \dot{\gamma}_{ab} D_a \dot{D}_b \varphi + t^2 \ddot{\gamma}_{ab} D_a \nu D_b \varphi
+ \sigma \left( \nu t^2 \dot{\gamma}_{ab} D_a \dot{D}_b \varphi + t^2 \ddot{\gamma}_{ab} D_a \nu D_b \varphi \right) = f^{(1)},
$$

$$
t^2 \dot{\gamma}_{ab} D_a D_b \nu - \left( \frac{t^2}{2} \chi_{a}^{b} \chi_{b}^{a} + \frac{1}{n} + \nu \pi^2 \right) \nu - 2 \pi \dot{\alpha} u - \sigma \left( \nu u^2 + 2 \pi \nu u + \dot{\alpha} u^2 \right) = f^{(2)},
$$

$$
-t \partial_t \varphi + \dot{\alpha} u + \ddot{\pi} \nu + \sigma \nu u = f^{(3)}.
$$

If this asymptotic matching problem is well-posed in some suitable sense, this would give a clear and precise justification for considering “the dynamics to be driven by the linear system $\sigma = 0$”. As mentioned earlier, we do not analyze the problem in this paper. Instead, the major task for this paper here is to characterize the asymptotics of solutions of this linear system $\sigma = 0$ as precisely as possible and under conditions as general as possible. Now, which terms of the linear system are expected to drive the asymptotics? If AVTD holds and if the decay parameter $\beta$ is sufficiently large, Eqs. (3.1) – (3.3) with the switching parameter $\sigma$ as introduced there should be the correct asymptotic matching problem.

### 4.2 A strategy to analyze matching problems

How can the asymptotic matching map in Definition 4.1 be constructed rigorously (if it exists)? In this paper, we provide a pathway which applies to equations Eq. (4.1) whose Cauchy problem is well-posed in the sense that for each smooth Cauchy data imposed at each initial time $t_\star \in (0, T]$ there is a unique smooth solution defined on $(0, T] \times M$. A prominent class of evolution equations ruled out by this are parabolic problems. In fact we restrict to evolution equations which are essentially hyperbolic. In this paper, we denote Cauchy data for solutions $v$ of Eq. (4.1) as $v_{\tau}$ if they are imposed at initial time $\tau \in (0, T]$. In the special case $\tau = T$, we often write $v_*$ instead of $v_{\tau}$. Notice that if Eq. (4.1) is first-order in time, as we will usually assume later, then $v_{\tau} = v(\tau, \cdot)$ and $v_* = v(T, \cdot)$. It is clear that if the Cauchy problem is well-posed in this sense, the asymptotic matching map $\Psi_{\sigma_1 \to \sigma_2}$ (if it exists) may be understood as a map between Cauchy data imposed at $t = T$, i.e., we may think of the map

$$
\Psi_{\sigma_1 \to \sigma_2} : v_{1_*} \mapsto v_{2_*},
$$

instead of Eq. (4.3).

With the Cauchy problem at our disposal, the idea is now to approximate an asymptotic matching map by a sequence of finite matching maps.

**Definition 4.2** (Finite matching problem). Suppose that the Cauchy problem of Eq. (4.1) is well-posed in the sense above. Pick $\sigma_1, \sigma_2 \in I$. For any $\tau \in (0, T]$ and any smooth...
solution $v_1$ of the version of Eq. (4.1) given by $\sigma = \sigma_1$, find the uniquely determined smooth solution $v_2$ of the version of Eq. (4.1) given by $\sigma = \sigma_2$ such that

$$v_1[\tau] = v_2[\tau].$$

(4.5)

The map of the schematic form

$$\psi_{\sigma_1 \to \sigma_2} : (\tau, v_{1*}) \mapsto v_{2*}$$

(4.6)

is called the finite matching map.

While the finite matching map can always be found under these assumptions, the hope is that the asymptotic matching map $\Psi_{\sigma_1 \to \sigma_2}$ can be constructed as the limit

$$\Psi_{\sigma_1 \to \sigma_2} = \lim_{\tau \searrow 0} \psi_{\sigma_1 \to \sigma_2}(\tau, \cdot).$$

(4.7)

We point out that a very similar idea was originally put forward in [1, 2, 14] to solve singular initial value problems underlying the Fuchsian method.

In order to step through this program for solving asymptotic matching problems now, it turns out that we must solve a hierarchy of Cauchy problems. Let us generalise Eq. (4.1) slightly by adding a so far arbitrary smooth source term $F(t, x)$ to the right-hand side

$$A[v](t, x) + \sigma L[v](t, x) = F(t, x).$$

(4.8)

As in Definition 4.1 and Definition 4.2, we pick $\sigma_1, \sigma_2 \in I$ and consider any two smooth solutions $v_1$ and $v_2$ of

$$A[v_1] + \sigma_1 L[v_1] = F_1, \quad v_1(T, \cdot) = v_{1*},$$

(4.9)

$$A[v_2] + \sigma_2 L[v_2] = F_2, \quad v_2(T, \cdot) = v_{2*},$$

(4.10)

where the two smooth source term functions $F_1(t, x)$ and $F_2(t, x)$ are at this stage allowed to be different (in fact, we shall see that it is crucial to allow them to be different in order to “modify” an asymptotic matching problem suitably; see below). Given the solution $v_1$ of the Cauchy problem Eq. (4.9), the quantity

$$w = v_1 - v_2$$

(4.11)

describes the difference of $v_1$ and the unknown solution $v_2$ of Eq. (4.10) whose Cauchy data $v_{2*}$ we aim to find to solve the finite matching problem with matching time $\tau$. This is therefore a solution of

$$A[w] + \sigma_2 L[w] = (\sigma_2 - \sigma_1)L[v_1] + F_1 - F_2 =: F_3, \quad w(\tau, \cdot) = 0.$$

(4.12)

Solving the Cauchy problem Eq. (4.12) clearly requires that the Cauchy problem Eq. (4.9) has been solved first. In principle this allows us to construct the finite matching map. In order to derive continuity estimates which may eventually allow us to take the limit Eq. (4.7) we must consider two $w$ and $\tilde{w}$ defined as above for any two smooth solutions
v_1 \text{ and } \tilde{v}_1 \text{ of Eq. (4.9)} \text{ with two (possibly different) sets of smooth source terms } F_1, F_2, 
\bar{F}_1 \text{ and } \bar{F}_2 \text{ and two (possibly different) matching times}^{4.1} \tau, \tilde{\tau} \in (0,T]. \text{ This leads to the problem}
\begin{align}
A[w - \tilde{w}] + \sigma_2 L[w - \tilde{w}] = (\sigma_2 - \sigma_1)L[v_1 - \tilde{v}_1] + (F_1 - \bar{F}_1) - (F_2 - \bar{F}_2) &=: F_4,
(w - \tilde{w})(\tilde{\tau}, \cdot) = w(\tilde{\tau}, \cdot). \tag{4.13}
\end{align}

This Cauchy problem can only be solved once the Cauchy problem
\begin{align}
A[v_1 - \tilde{v}_1] + \sigma_1 L[v_1 - \tilde{v}_1] &= F_1 - \bar{F}_1 =: F_5,
(v_1 - \tilde{v}_1)(T, \cdot) &= v_{1*} - \tilde{v}_{1*} \tag{4.14}
\end{align}

has been solved.

It is a very useful observation now that all of these Cauchy problems Eqs. (4.9), (4.10), (4.12), (4.13) and (4.14) are essentially of the same form Eq. (4.8) with the same A and L, with either \( \sigma = \sigma_1 \) or \( \sigma = \sigma_2 \), with various source terms \( F_1, F_2, F_3, F_4 \) or \( F_5 \) and various Cauchy data imposed at various times. In all of what follows it is therefore key to find estimates that are uniform in \( \sigma \) and Cauchy data appears in some particular way. We require these estimates to hold whether we evolve to the future or to the past within the time-interval \((0, T]\). In our particular application here we find two different types of estimates, one where the initial condition is imposed at the “end” of the time interval \( t = T \) (as for Eqs. (4.9), (4.10) and (4.14)), and second, where the initial condition is imposed at the “beginning” of the time interval \( t = \tau \) (as for Eqs. (4.12) and (4.13)).

### 4.3 Heuristics of an example problem

In order to develop some intuition for the issues we encounter let us consider the following example. A \( 1+1\)-Euler-Poisson-Darboux equation takes the form
\begin{align}
t \partial_t (t \partial_t v) - a t \partial_t v - t^{2(1-p)} \partial_x^2 v &= g \tag{4.15}
\end{align}

We note that \( g(t, x) \) in Eq. (4.15) could be any smooth source term function. We shall assume here that \( a \) and \( p \) are constants with \( a \in (0, \infty) \) and \( p \in (-\infty, 1) \). In consistency with the AVTD phenomenology, we expect that solutions of Eq. (4.15) should be dominated by the first two terms (possibly including the source term \( g \)) in the limit \( t \searrow 0 \), while the spatial derivative term should be negligible. The choice
\begin{align}
A[v] = t \partial_t (t \partial_t v) - a t \partial_t v - g, \quad L[v] = -t^{2(1-p)} \partial_x^2 v \tag{4.16}
\end{align}

puts Eq. (4.15) into the form Eq. (4.1). In order to preserve the wave-character of the equation and to avoid arbitrary large wave speeds, we restrict \( \sigma \) to \( I := [0,1] \).

For any value of \( \sigma \in I \) the formal expansions of solutions are of the form\(^4.2\)
\begin{align}
v(t, x) &= v_0(x) + t^av_1(x) + \sigma \frac{\partial_x^2 v_0(x)}{2(1-p)(2(1-p) - a)} t^{2(1-p)} + O(t^{2(1-p)+\epsilon}) + O(t^{4(1-p)}), \tag{4.17}
\end{align}

\(^4.1\)Without loss of generality we always assume \( \tau \leq \tilde{\tau} \).

\(^4.2\)The special case \( a = 2(1-p) \) is ignored in all of what follows.
as long as \( g \) decays sufficiently fast\textsuperscript{13}. The quantities \( v(0)_1(x) \) and \( v(1)_1(x) \) are smooth functions. Let us now pick any \( \sigma_1, \sigma_2 \in I \) and consider the asymptotic matching problem as in Definition 4.1. Let us first pick any constant

\[
\lambda < \min\{a, 2(1 - p)\}
\]

and set \( || \cdot || = ||t^{-\lambda} \cdot ||_{L^2(M)} \) for the norm in Eq. (4.2). Then, for any given solution \( v_1 \) of the \( \sigma_1 \)-version of Eq. (4.16), we can find a solution \( v_2 \) of the \( \sigma_2 \)-version such that Eq. (4.2) holds. This function \( v_2 \) has to have the same coefficient \( v(0) \) in Eq. (4.17) as \( v_1 \). However, this does not fix \( v_2 \) uniquely because Eq. (4.2) with the choice Eq. (4.18) does not impose any restrictions on the coefficient \( v(1) \) in Eq. (4.17). Indeed it can be proved rigorously that this asymptotic matching problem is therefore ill-posed. In rough terms, the problem is that \( \lambda \) is too small to cover both asymptotic degrees of freedom represented by the first two terms in the formal expansion Eq. (4.17).

Let us therefore attempt to pick \( \lambda \) larger than \( a \). Let us first assume that \( a < 2(1 - p) \) and pick

\[
\lambda \in (a, 2(1 - p)).
\]

In this case, the issue above is resolved: Given any solution \( v_1 \) of the \( \sigma_1 \)-version of Eq. (4.16), we can find a unique solution \( v_2 \) of the \( \sigma_2 \)-version such that Eq. (4.2) holds by matching both \( v(0)_1 \) and \( v(1)_1 \) in Eq. (4.17) (one can prove that this can be done in this case). Indeed if \( a < 2(1 - p) \), the asymptotic matching problem for the choice (4.19) is well-posed for any pair of switch parameters \( \sigma_1, \sigma_2 \in I \). In addition, the asymptotic matching map Eq. (4.3) turns out to be bijective, i.e., \( (\Psi_{\sigma_1 \rightarrow \sigma_2})^{-1} = \Psi_{\sigma_2 \rightarrow \sigma_1} \), and in fact a homeomorphism in the \( C^\infty \)-topology.

Eq. (4.17) however also suggests that spatial derivatives may cause trouble if \( a > 2(1 - p) \). In this case, even if we choose \( \lambda > a \), the third term in Eq. (4.17) is not “negligible” in comparison to the second term. Moreover, this third term depends explicitly on the choice of \( \sigma \). It is therefore impossible to fully match two versions of Eq. (4.16) given by two different switch parameters.

As we show later, this is precisely the situation we encounter in the proof of our main theorem. Is there a way to resolve this? Start again from any solution \( v_1 \) of the \( \sigma_1 \)-version of Eq. (4.16) and its formal expansion Eq. (4.17). Suppose that we can determine the leading coefficient \( v(0) \) somehow. Then define

\[
v_1^{\text{mod}} = (\sigma_1 - K) \frac{\partial^2_x v(0)}{2(1 - p)(2(1 - p) - a)}
\]

for (so far) any \( K \in \mathbb{R} \) and calculate

\[
t\partial_t(t \partial_t v_1^{\text{mod}}) - a t \partial_t v_1^{\text{mod}} - \sigma_1 t^{2(1-p)} \partial^2_x v_1^{\text{mod}} = (\sigma_1 - K) \partial^2_x v(0)(x) t^{2(1-p)} - \sigma_1 t^{2(1-p)} \partial^2_x v_1^{\text{mod}} =: \sigma_1(\sigma_1 - K) g_1^{\text{mod}}(x) = O(t^{(1-p)})
\]

\textsuperscript{13}For the heuristic discussion in this section we shall not formulate specific decay conditions for \( g \) now.
Setting now
\[ \tilde{v}_1 = v_1 - v_{1 \text{mod}} \]  
(4.22)

implies that
\[ t \partial_t (t \partial_t \tilde{v}_1) - a t \partial_t \tilde{v}_1 - \sigma_1 t^{2(1-p)} \partial_x^2 \tilde{v}_1 = g + \sigma_1 (\sigma_1 - K) g_1^{(\text{mod})} - (\sigma_1 - K) \partial_x^2 v_{1(0)}(x) t^{2(1-p)}, \]
(4.23)

We claim now that any solution of this equation with leading term \( v_{(0)} \), in particular, \( \tilde{v}_1 \) given by Eq. (4.22), can be fully matched to some solution of
\[ t \partial_t (t \partial_t \tilde{v}_2) - a t \partial_t \tilde{v}_2 - \sigma_2 t^{2(1-p)} \partial_x^2 \tilde{v}_2 = g + \sigma_2 (\sigma_2 - K) g_2^{(\text{mod})} - (\sigma_2 - K) \partial_x^2 v_{2(0)}(x) t^{2(1-p)} \]
(4.24)

for any sufficiently fast decaying \( g_2^{(\text{mod})}(t, x) \). This is so because any such solution \( \tilde{v}_1 \) has the formal expansion
\[ \tilde{v}_1(t, x) = v_{(0)}(x) + t^a v_{(1)}(x) + K \frac{\partial_x^2 v_{(0)}(x)}{2(1-p)(2(1-p) - a)} t^{2(1-p)} + O(t^{2(1-p)+a}) + O(t^{4(1-p)}); \]
(4.25)
i.e., the spatial derivative term does not depend on the choice of switch parameter anymore. Indeed, for any such \( \tilde{v}_1 \) we can therefore find \( \tilde{v}_2 \) as a solution of Eq. (4.24) such that
\[ \| t^{-\lambda} (\tilde{v}_1(t, \cdot) - \tilde{v}_2(t, \cdot)) \|_{L^2(M)} \to 0 \]
for any \( \lambda \in (a, \min\{2(1-p) + a, 4(1-p)\}) \) as required by Definition 4.1.

The choice of \( K \) in this argument is free. So how do we pick \( K \), and, also \( g_1^{(\text{mod})} \) and \( g_2^{(\text{mod})} \)? This depends on the actual problem of interest. If the source equation of the matching problem is the full equation \( \sigma = \sigma_1 = 1 \), then we would typically choose \( K = \sigma_1 \) so that both additional terms in Eq. (4.23) disappear (irrespective of the choice of \( g_1^{(\text{mod})} \)) and we recover the original full source equation. The additional terms in the target equation (4.24) do in general however not disappear. Only the term involving \( g_2^{(\text{mod})} \) is identically zero in the special case \( \sigma = \sigma_2 = 0 \), i.e., when the target equation is the model equation. Vice versa, if the source equation is the model equation \( \sigma = \sigma_1 = 0 \), then we would typically pick \( K = \sigma_2 \). Again, \( g_1^{(\text{mod})} \) drops out there, and, in the target equation all additional terms disappear (and we recover the original equations). Since \( g_1^{(\text{mod})} \) and \( g_2^{(\text{mod})} \) disappear from the problem for all of these cases we will choose them to be zero in all of what follows. So, intuitively because spatial derivatives are not fully negligible when \( a > 2(1-p) \), it is necessary to modify the equations of interest in the way above in order to obtain a well-posed matching problem. This idea of “modifying” an asymptotic matching problem to “make it well-posed” shall be applied later in the proof of our main theorem.
5 Proof of the main result for the lapse-scalar field system

5.1 Main steps of the proof

The purpose of this whole section now is to prove Theorem 3.2. Our proof exploits the ideas in Section 4.1 and follows the strategy in Section 4.2 incorporating the insights from Section 4.3. Eqs. (3.1) – (3.3) are of the form Eq. (4.8) with

\[
A[(u, \nu, \varphi)] = - \left( t \partial_t u + A \nu, \nu + 2Au, t \partial_t \varphi - u - A \nu \right), \tag{5.1}
\]

\[
L[(u, \nu, \varphi)] = - \left( (\tilde{\pi} - A) \nu - (1 - \tilde{\alpha}) \right. \\
\left. - \tilde{\alpha} t^2 \hat{\gamma}^{ab} D_a D_b \varphi - \nu t^2 \hat{\gamma}^{ab} D_a D_b \hat{\phi} - t^2 \hat{\gamma}^{ab} D_a \hat{\alpha} D_b \varphi - t^2 \hat{\gamma}^{ab} D_a \nu D_b \nu \right) \\
+ \frac{1}{\nu} \left( \frac{\nu}{n} + A^2 - 1 \right) + t^2 \hat{\chi}^a b \hat{\lambda}_b a + (\tilde{\pi}^2 - A^2) \nu + 2(\tilde{\pi} \hat{\alpha} - A) u \right), \tag{5.2}
\]

\[
F = (f(1), f(2), f(3)). \tag{5.3}
\]

The plan of attack is to analyze the hierarchy of Cauchy problems Eqs. (4.9), (4.14), (4.12) and (4.13) as outlined in Section 4.2.

We first notice that the Cauchy problem is well-posed. This can be proved using the information provided in [29, 30] and with standard arguments incorporating Proposition 5.1. The first step is now to obtain “rough (a-priori) estimates” for solutions of the Cauchy problem which are independent of the value of \( \sigma \) and the particular choice of \( F \). These are summarised in Proposition 5.1 below. Before we can state them we introduce the following norms and energies. First recall the notation Eq. (3.5) for any smooth (time-dependent or time-independent) \((0, r)\)-tensor fields \( S \) and \( \tilde{S} \) on \( M \). Recall also that all index operations are performed with the reference metric \( \delta_{ab} \) in Eq. (3.4), that \( \partial_a \) is the covariant derivative associated with \( \delta_{ab} \), and, that \( \partial^a S \) represents the \((0, r + k)\) tensor field \( \partial_{a_1} \cdots \partial_{a_k} S_{b_1 \cdots b_r} \). Based on this we define

\[
\|S\|_\delta^2 = \int_M |S|^2_\delta \ dx, \quad \|S\|_{\delta, H^k(M)}^2 = \sum_{l=0}^k \|\partial^l S\|_\delta^2,
\]

where we integrate with respect to the volume element defined by \( \delta_{ab} \). Given any asymptotically point-wise Kasner–scalar field background \( \Gamma = (\hat{\gamma}^{ab}, \hat{\chi}^a b, \hat{\lambda}_b a, \hat{\alpha}, \hat{\pi}, \hat{\phi}) \) and the corresponding metric \( \tilde{\gamma}^{ab} \) (see Definition 3.1), we also write

\[
\|t^{-\lambda} \partial S\|_{\delta, t^2 \tilde{\gamma}}^2 = \int_M t^{2\tilde{\gamma}^{cd}} \left( t^{-\lambda} \partial_c S, t^{-\lambda} \partial_d S \right)_{\delta} \ dx, \tag{5.5}
\]
for all \( t \in (0, T] \). Moreover, we define the following energies as
\[
e_{\sigma, \lambda}[S, \tilde{S}](t) := \frac{1}{2} \int_M \left( \left| t^{-\lambda}S_\delta \right|^2 + \sigma t^2 \partial^d \left( t^{-\lambda} \partial_\delta \tilde{S} \right) \right) \, dx, \\ e_{\sigma, \lambda}[S](t) := e_{\sigma, \lambda}[S, \tilde{S}](t), \quad e_\lambda[S](t) := e_{\sigma, \lambda}[S, 0](t),
\]
any smooth function \( \lambda(x) \). Given any smooth time-dependent functions \( u, \varphi, \nu \) on \( M \) (not necessarily solutions of Eqs. (3.1) – (3.3)), we also set
\[
E_{k, \sigma, \lambda}[u, \varphi](t) = \sum_{l=0}^k e_{\sigma, \lambda}[\partial^l u, \partial^l \varphi](t), \quad E_{k, \sigma, \lambda}[\varphi](t) = \sum_{l=0}^k e_{\sigma, \lambda}[\partial^l \varphi](t), \\
E_{k, \lambda}[u](t) = \sum_{l=0}^k e_\lambda[\partial^l u](t).
\]

**Proposition 5.1** (Rough a-priori estimates for solutions of the lapse-scalar field system). Consider an arbitrary asymptotically point-wise Kasner–scalar field background \( \Gamma = (\tilde{\gamma}^{ab}, \tilde{\chi}_a, \tilde{\alpha}, \tilde{\pi}, \tilde{\phi}) \) with positive decay \( \beta \). Let \( \nu = (u, \varphi) \) be any smooth solution of Eq. (4.8) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2), and,
\[
F = (F^{(1)}(t, x), F^{(2)}(t, x), F^{(3)}(t, x))
\]
for arbitrary smooth \( F^{(1)}(t, x), F^{(2)}(t, x) \) and \( F^{(3)}(t, x) \) and \( \sigma \in I = [0, 1] \). Pick any sufficiently small \( \epsilon > 0 \) and any integer \( k \geq 0 \). Then there is a constant \( C > 0 \) such that for any \( t_0 \in (0, T] \) the following estimates hold:

1. For all \( t \in [t_0, T] \) and for any smooth function \( \lambda > \lambda_c \) (see Eq. (3.10)), we have\(^{5,1}\)
\[
E_{k, \sigma, \lambda}[u, \varphi](t) \leq C \left( E_{k, \sigma, \lambda}[u, \varphi](t_0) \right. \\
+ \int_{t_0}^t \left( \| s^{-\lambda} F^{(1)} \|_{\tilde{\delta}_H^k(M)}^2 + \| s^{-\lambda} F^{(2)} \|_{\tilde{\delta}_H^k(M)}^2 + \| s^{-\lambda} F^{(3)} \|_{\tilde{\delta}_H^{k+1}(M)}^2 \right) \delta^{-1+2\epsilon} \, ds.
\]

2. For all \( t \in [t_0, T] \) and for any smooth function \( \lambda \geq 0 \) on \( M \), we have
\[
\sqrt{E_{k, \lambda}[\varphi](t)} \leq \sqrt{E_{k, \lambda}[\varphi](t_0)} \\
+ C \int_{t_0}^t \left( \sqrt{E_{k, \lambda}[u](s)} + \| s^{-\lambda} F^{(2)} \|_{\tilde{\delta}_H^k(M)} \right) \delta^{-1} \, ds.
\]

\(^{5,1}\)In this whole paper we use the following standard convention for constants \( C \) in estimates. First \( C \) may be different in each step of any derivation. Second, \( C \) is uniform in all quantities which appear explicitly in the corresponding estimate except for those quantities listed explicitly in corresponding text. It may however depend on all quantities which do not appear explicitly in the corresponding estimate except for those quantities listed explicitly in corresponding text.
3. For all $t \in (0, t_\ast)$ and for any smooth function $\lambda < 0$ on $M$, we have
\[
E_{k,\sigma,\lambda}[u, \varphi](t) \leq C \left( E_{k,\sigma,\lambda+\epsilon}[u, \varphi](t_\ast) \right.
+ \left. \int_t^{t_\ast} \left( \|s^{-\lambda}F^{(1)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda}F^{(2)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda}F^{(3)}\|_{\delta,H^{k+1}(M)}^2 \right)s^{-1-2\epsilon} ds \right). \tag{5.13}
\]

4. For all $t \in (0, t_\ast)$ and for any smooth function $\lambda \leq 0$ on $M$, we have
\[
\sqrt{E_{k,\lambda}^1}[\varphi](t) \leq \sqrt{E_{k,\lambda}^1}[\varphi](t_\ast)
+ C \int_t^{t_\ast} \left( \sqrt{E_{k,\lambda}[u]}(s) + \|s^{-\lambda}F^{(2)}\|_{\delta,H^k(M)} + \|s^{-\lambda}F^{(3)}\|_{\delta,H^{k+1}(M)} \right)s^{-1} ds. \tag{5.14}
\]

The constants $C$ may depend on $T$, $k$, $\lambda$, $\epsilon$ and $\Gamma$. Finally, for all $t \in (0, T]$ and for any smooth $\lambda$ on $M$, we have
\[
E_{k,\lambda}[\nu](t) \leq C \left( E_{k,\lambda}[u](t) + \|t^{-\lambda}F^{(2)}\|_{\delta,H^k(M)} \right). \tag{5.15}
\]

where the constant $C > 0$ may depend on $T$, $k$ and $\Gamma$.

The proof can be found in Section 5.2. We point out that the restrictions for $\lambda$ in both Eqs. (5.11) and (5.13) are disappointing. In particular the latter suggests that $u$ might not even be bounded at $t = 0$ even though we expect from Eq. (3.18) that $u$ should decay with some specific positive power. By sacrificing some differentiability, it turns out that we can indeed recover the optimal decay for $u$. This step turns out to be crucial in the proof of Theorem 3.2. This result is summarised in the following proposition.

**Proposition 5.2** (Improved decay estimates for solutions of the lapse-scalar field system).
Consider an arbitrary asymptotically point-wise Kasner–scalar field background $\Gamma = (\delta^{ab}, \chi^b, \alpha, \pi, \phi)$ with positive decay $\beta$. Given any $u_\ast, \varphi_\ast \in C^\infty(M)$, let $v = (u, \nu, \varphi)$ be the smooth solution of Eq. (4.8) with $A$ and $L$ given by Eqs. (5.1) and (5.2), and, $F$ given by Eq. (5.10), satisfying the Cauchy condition $u(T, x) = u_\ast(x)$, $\varphi(T, x) = \varphi_\ast(x)$. Pick any sufficiently small $\epsilon > 0$ and any integer $k \geq 0$. Then there is a constant $C > 0$, which may depend on $T$, $k$, $\lambda$, $\epsilon$ and $\Gamma$, such that\textsuperscript{5.2}
\[
E_{k,\lambda}[u](t) \leq C \left( \|u_\ast\|_{\delta,H^{k+2}(M)}^2 + \sigma \|\varphi_\ast\|_{\delta,H^{k+3}(M)}^2 + \|\varphi_\ast\|_{\delta,H^{k+2}(M)}^2 \right) \tag{5.16}
+ \int_t^T \left( \|s^{-\lambda}F^{(1)}\|_{\delta,H^{k+2}(M)}^2 + \|s^{-\lambda}F^{(2)}\|_{\delta,H^{k+2}(M)}^2 + \|s^{-\lambda-2\epsilon}F^{(3)}\|_{\delta,H^{k+3}(M)}^2 \right)s^{-1-2\epsilon} ds
\]
for all $t \in (0, T]$ for any smooth function $\lambda$ on $M$ with $\lambda < \min\{2(1 - q_{\max}), 2A^2\}$.\textsuperscript{5.2}

\textsuperscript{5.2} We define $\kappa$ as any smooth strictly positive function strictly smaller than $1 - q_{\max}(x)$ at each $x \in M$; $q_{\max}$ is defined in Eq. (2.26).
The proof is discussed in Section 5.3. Thanks to the above estimates, we can now understand the issue which was heuristically motivated in Section 4.3. In order to solve the Cauchy problem Eq. (4.12) we need to estimate the source term $F_3$ there. When we do this for the case here we encounter integrals of the form

$$\int_{t_*}^{t} \|s^{-\lambda} \partial s^2 \partial^a \partial^b \partial||^2_{H^k(M)} s^{-1} ds$$

for $t \geq t_*$ which are required to be bounded in the limit $t_* \rightarrow 0$ for $\lambda > \lambda_c$. We have discussed before that this is only possible in the exceptional case that $2(1 - q_{max}) > \lambda_c$. In fact, the asymptotic matching problem is therefore in general ill-posed and we need to consider suitable “modifications”. In order to work out these modifications as in Section 4.3 we require a map which yields the asymptotic datum $\varphi(0)$ for any given solution. The process of finding this is referred to as pre-matching; the motivation for this name is given in its proof discussed in Section 5.4.

**Proposition 5.3** (Pre-matching for the lapse-scalar field system). Consider an arbitrary asymptotically point-wise Kasner–scalar field background $\Gamma = (\hat{\gamma}^{ab}, \hat{\chi}^a, \hat{\alpha}, \hat{\pi}, \hat{\phi})$ with positive decay $\beta$. Suppose the source terms $f^{(1)}(t, x)$, $f^{(2)}(t, x)$ and $f^{(3)}(t, x)$ in Eqs. (3.1) – (3.3) are smooth functions on $(0, T] \times M$ such that

$$\int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda_2} f^{(2)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda_3} f^{(3)}\|_{\delta, H^{k+3}(M)} \right) s^{-1} ds < \infty,$$

holds for some integer $k \geq 0$ and for some smooth function $\lambda_1(x)$ on $M$ with $\lambda_1 > 0$. Pick $\sigma \in I = [0, 1]$. Then there is map

$$\Psi^{(\text{pre})} : (C^\infty(M))^2 \rightarrow C^\infty(M),$$

and a constant $C > 0$ such that, given any $u_*, \varphi_* \in C^\infty(M)$ and the corresponding smooth solution $(u, \nu, \varphi)$ of the Cauchy problem of Eqs. (3.1) – (3.3) with Cauchy data with $\varphi(T, \cdot) = \varphi_*$ and $u(T, \cdot) = u_*$,

$$\left\| t^{-\lambda} \left( \varphi(t, \cdot) - \Psi^{(\text{pre})}(u_*, \varphi_*) \right) \right\|^2_{\delta, H^k(M)} \leq C \left( \|u_*\|^2_{\delta, H^{k+2}(M)} + \|\varphi_*\|^2_{\delta, H^{k+3}(M)} \right)$$

$$+ \int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|^2_{\delta, H^{k+2}(M)} + \|s^{-\lambda_2} f^{(2)}\|^2_{\delta, H^{k+2}(M)} + \|s^{-\lambda_3} f^{(3)}\|^2_{\delta, H^{k+3}(M)} \right) s^{-1} ds, \tag{5.17}$$

for all $t \in (0, T]$, for any smooth $0 < \lambda_1 \leq \lambda_1$ and $\lambda < \min\{\tilde{\lambda}, 2(1 - q_{max}), 2A^2\}$. In fact, there is only one map with this property. Moreover, for any smooth $\lambda < \min\{2(1 - q_{max}), 2A^2\}$, we have

$$\left\| t^{-\lambda} \left[ \left( \varphi(t, \cdot) - \Psi^{(\text{pre})}(u_*, \varphi_*) \right) - \left( \varphi(t, \cdot) - \Psi^{(\text{pre})}(\tilde{u}_*, \varphi_*) \right) \right] \right\|_{\delta, H^k(M)} \leq C \left( \|u_* - \tilde{u}_*\|_{\delta, H^{k+2}(M)} + \|\varphi_* - \tilde{\varphi}_*\|_{\delta, H^{k+3}(M)} \right), \tag{5.18}$$

for any $u_*, \varphi_*, \tilde{u}_*, \varphi_* \in C^\infty(M)$ and all $t \in (0, T]$. The constants $C$ above may depend on $k$, $T$, $\lambda$, $\lambda$ and $\Gamma$. The function $\kappa$ is defined in footnote 5.2.
Once we have obtained the pre-matching map as in the previous proposition, we are now in the position to formulate (and solve) suitably modified asymptotic matching problems for the lapse-scalar field system.

**Proposition 5.4** (Modified asymptotic matching problem for the lapse-scalar field system). Consider an arbitrary asymptotically point-wise Kasner–scalar field background \( \Gamma = (\gamma_{ab}, \chi_a^b, \alpha, \pi, \phi) \) with positive decay \( \beta \) satisfying Eq. (3.14) such that Eq. (3.15) holds where \( \lambda_c \) is defined in Eq. (3.10). Suppose the source terms \( f^{(1)}(t, x) \), \( f^{(2)}(t, x) \) and \( f^{(3)}(t, x) \) in Eqs. (3.1) – (3.3) are smooth functions on \((0, T] \times M\) such that

\[
\int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|_{\delta, H^{k+4}(M)} + \|s^{-\lambda_1} f^{(2)}\|_{\delta, H^{k+4}(M)} + \|s^{-\lambda_1+2\kappa} f^{(3)}\|_{\delta, H^{k+5}(M)} \right) s^{-1} ds < \infty,
\]

holds for some integer \( k \geq 0 \) and for some smooth function \( \lambda_1(x) \) on \( M \) with Eq. (3.16). Pick \( \sigma_1, \sigma_2 \in I = [0, 1], K \in \mathbb{R} \) and any sufficiently small \( \epsilon > 0 \). Suppose that for some integers \( k_0, k_1 \geq 0 \), there is a continuous map

\[
\Phi : H^{k+4+k_0}(M) \times H^{k+4+k_1}(M) \to H^{k+4}(M) \times H^{k+5}(M), \quad (u_{1x}, \varphi_{1x}) \mapsto (u_{1s}, \varphi_{1s})
\]

and a map\(^5,3\)

\[
\chi : (C^\infty(M))^2 \to C^\infty(M), \quad (u_{1x}, \varphi_{1x}) \mapsto \varphi_{1(0)}
\]

with the property that there is a constant \( C > 0 \), which may depend on \( k, T, \lambda \) and \( \Gamma \), such that for every \( (u_{1x}, \varphi_{1x}) \in (C^\infty(M))^2 \),

\[
\left\| t^{-\lambda} \left( \varphi_{1(t, \cdot)} - \varphi_{1(0)} \right) \right\|^2_{\delta, H^{k+2}(M)} \leq C \left( \|u_{1x}\|_{\delta, H^{k+4+k_0}(M)}^2 + \|\varphi_{1x}\|_{\delta, H^{k+4+k_1}(M)}^2 \right)
\]

\[
+ \int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|_{\delta, H^{k+4}(M)}^2 + \|s^{-\lambda_1} f^{(2)}\|_{\delta, H^{k+4}(M)}^2 + \|s^{-\lambda_1+2\kappa} f^{(3)}\|_{\delta, H^{k+5}(M)}^2 s^{-1} ds \right),
\]

(5.19)

\[
E_{k+2,\lambda}[u_{1}] (t) \leq C \left( \|u_{1x}\|_{\delta, H^{k+4+k_0}(M)}^2 + \|\varphi_{1(1)}\|_{\delta, H^{k+4+k_1}(M)}^2 \right)
\]

\[
+ \int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|_{\delta, H^{k+4}(M)}^2 + \|s^{-\lambda_1} f^{(2)}\|_{\delta, H^{k+4}(M)}^2 + \|s^{-\lambda_1+2\kappa} f^{(3)}\|_{\delta, H^{k+5}(M)}^2 s^{-1} ds \right),
\]

(5.20)

for all \( t \in (0, T] \), and for any smooth function \( \lambda < \min\{\lambda_1, 2(1 - q_{max}), 2A^2\} \) on \( M \). Here \( v_1 = (u_{1(t, x)}, \nu_{1(t, x)}, \varphi_{1(t, x)}) \) is the solution of Eq. (4.8) with \( A \) and \( L \) given by

\(^5,3\) Just to avoid any confusions, we stress that this map \( \chi \) is completely unrelated to the trace-free part of the second fundamental form \( \chi_a^b \).
Eqs. (5.1) and (5.2) and
\[
F^{(1)} = f^{(1)} + (\sigma_1 - K) \left( \partial_t^2 \gamma^{ab} D_a D_b \varphi_1(0) + t^2 \gamma^{ab} D_a \hbar D_b \varphi_1(0) \right),
\]
\[
F^{(2)} = f^{(2)}, \quad F^{(3)} = f^{(3)}, \quad \sigma = \sigma_1,
\]
satisfying the Cauchy condition
\[
\varphi_1(t, \cdot) = \varphi_{1*}, \quad u_1(t, \cdot) = u_{1*}.
\]
Then there is a map
\[
\Psi_{\sigma_1 \to \sigma_2} : (C^\infty(M))^2 \to (C^\infty(M))^2
\]
with the following properties:

1. Pick any \((u_{1*}, \varphi_{1*}) \in (C^\infty(M))^2\) as above and consider the corresponding solution 
\((u_1(t, x), \nu_1(t, x), \varphi_1(t, x))\) as above. The solution \(v_2 = (u_2, \nu_2, \varphi_2)\) of Eq. (4.8) with 
\(A\) and \(L\) given by Eqs. (5.1) and (5.2) and
\[
F^{(1)} = f^{(1)} + (\sigma_2 - K) \left( \partial_t^2 \gamma^{ab} D_a D_b \varphi_1(0) + t^2 \gamma^{ab} D_a \hbar D_b \varphi_1(0) \right),
\]
\[
F^{(2)} = f^{(2)}, \quad F^{(3)} = f^{(3)}, \quad \sigma = \sigma_2,
\]
determined by Cauchy data
\[
(u_{2*}, \varphi_{2*}) = \Psi_{\sigma_1 \to \sigma_2}(u_{1*}, \varphi_{1*})
\]
with \(\varphi(T, \cdot) = \varphi_*\) and \(u(T, \cdot) = u_*\) satisfies
\[
\left\| t^{-\lambda} (u_1(t, \cdot) - u_2(t, \cdot)) \right\|_{H^k(M)}^2 + \left\| t^{-\lambda} (\varphi_1(t, \cdot) - \varphi_2(t, \cdot)) \right\|_{H^k(M)}^2 \leq C \left( \left\| u_{1*} \right\|_{H^k(M)}^2 + \left\| \varphi_{1*} \right\|_{H^k(M)}^2 + \int_0^T \left( \left\| s^{-\lambda_1} f^{(1)} \right\|_{H^{k+4}(M)}^2 + \left\| s^{-\lambda_1} f^{(2)} \right\|_{H^{k+4}(M)}^2 + \left\| s^{-\lambda_1 + 2\sigma} f^{(3)} \right\|_{H^{k+5}(M)}^2 \right) s^{-1} ds \right)
\]
for all \(t \in (0, T]\) and any smooth function \(\lambda\) on \(M\) with Eq. (3.17). The constant \(C\) here may depend on \(k, T, \lambda\) and \(\Gamma\).

2. The map \(\Psi_{\sigma_1 \to \sigma_2}\) is continuous in the sense that
\[
\left\| \Psi_{\sigma_1 \to \sigma_2}(u_{1*}, \varphi_{1*}) - \Psi_{\sigma_1 \to \sigma_2}(\tilde{u}_{1*}, \tilde{\varphi}_{1*}) \right\|_{H^k(M)} \leq C \left( \left\| u_{1*} - \tilde{u}_{1*} \right\|_{H^{k+4+k_0}(M)} + \left\| \varphi_{1*} - \tilde{\varphi}_{1*} \right\|_{H^{k+4+k_1}(M)} \right)
\]
for any smooth \(u_{1*}, \varphi_{1*}, \tilde{u}_{1*}, \tilde{\varphi}_{1*}\). The constant \(C\) here may depend on \(k, T, \lambda, \Gamma\) and \(\Phi\).
3. Given any smooth \((u_1, \varphi_1) \in (C(\infty)(M))^2\) and the corresponding solution \(v_1 = (u_1, \nu_1, \varphi_1)\) as above, suppose there is a smooth solution \((\tilde{u}_2, \tilde{\nu}_2, \tilde{\varphi}_2)\) of Eq. (4.8) with \(A\) and \(L\) given by Eqs. (5.1) and (5.2) and (5.23) such that
\[
\left\| t^{-\lambda} (t \partial_t \varphi_1(t, \cdot) - t \partial_t \tilde{\varphi}_2(t, \cdot)) \right\|_{L^2(M)} + \left\| t^{-\lambda} (\varphi_1(t, \cdot) - \tilde{\varphi}_2(t, \cdot)) \right\|_{H^1(M)} \leq C \tag{5.27}
\]
for all \(t \in (0, T]\) and some \(\lambda\) consistent with Eq. (3.17) where the constant \(C\) here may depend on \(k, T, \lambda, \Gamma, (u_1, \nu_1, \varphi_1)\) and \((\tilde{u}_2, \tilde{\nu}_2, \tilde{\varphi}_2)\). Then \((\tilde{u}_2, \tilde{\nu}_2, \tilde{\varphi}_2)\) agrees identically with the solution \(v_2 = (u_2, \nu_2, \varphi_2)\) of Eq. (4.8) with \(A\) and \(L\) given by Eqs. (5.1), (5.2) and (5.23) determined by the Cauchy data Eq. (5.24).

4. Pick any two smooth \((u_1, \varphi_1)\) and \((\tilde{u}_1, \tilde{\varphi}_1)\). Supposing that \(\Psi_{\sigma_1 \rightarrow \sigma_2}(u_1, \varphi_1) = \Psi_{\sigma_1 \rightarrow \sigma_2}(\tilde{u}_1, \tilde{\varphi}_1)\), it follows that \(\varphi_1(0) = \tilde{\varphi}_1(0)\).

The function \(\kappa\) is defined in footnote 5.2.

The proof is discussed in Section 5.4. It is not always clear whether maps \(\Phi\) and \(\chi\) assumed in this proposition can always be found given \(\sigma_1, \sigma_2\) and \(K\). In the proof of Theorem 3.2 (which we can finally discuss now) we show how one can pick these quantities in the most important cases of interest. Before we do this however let us remark that this proposition here restricts to asymptotic matching problems with modifications based on time-independent “leading order terms” \(\varphi_1(0)(x)\). While this turns out to be sufficient for our purposes here, more general problems may require time-dependent choices of \(\varphi_1(0)\), which take into account expansions of higher orders of the solution at \(t = 0\).

Proof of Theorem 3.2. For the following proof we assume \(\sigma \neq 0\). The case \(\sigma = 0\) is trivial and we pick \(\Psi = \text{id}\).

We start off with the following considerations. Given any smooth \((\varphi(0), \varphi(1)) \in (C(\infty)(M))^2\),

\[
u_1(t, x) = -2A^2(x)t^2A^2(x)\varphi(1)(x) + \int_0^T \mathcal{L}[\varphi(0)](s, x)(t/s)^{2A^2(x)}s^{-1}ds + S^{(1)}(t, x) \\
\varphi_1(t, x) = \varphi(0)(x) + (2A^2(x) - 1)t^2A^2(x)\varphi(1)(x) - (2A^2(x) - 1)\int_0^t \int_{\tau}^T \mathcal{L}[\varphi(0)](s, x)(\tau/s)^{2A^2(x)}s^{-1}d\tau d\tau + S^{(2)}(t, x) \\
\nu_1(t, x) = -2A(x)u_1(t, x) - f^{(2)}(t, x)
\]

represents the general solution of Eq. (4.8) with Eqs. (5.1), (5.2) and (5.21), \(\sigma_1 = 0\), \(\sigma_2 = \sigma\) and \(K = \sigma_2\) for \(\varphi_1(0) = \varphi(0)\). Recall the definitions of \(\mathcal{L}[\cdot]\), \(S^{(1)}\) and \(S^{(2)}\) in Eqs. (3.13), (3.11) and (3.12). Recall here that all integrals are well-defined and finite. The corresponding Cauchy data \((u_1, \varphi_1)\) are found by evaluating Eq. (5.28) at \(t = T\),
and, the corresponding map $\Phi : (\varphi(0), \varphi(1)) \mapsto (u_{1+}, \varphi_{1+})$ is therefore well-defined. We can write

$$\mathcal{E}[\varphi(0)] := - \int_0^T \mathcal{L}[\varphi(0)] s^{-1} ds$$

$$= - \frac{2A^2}{2A^2 - 1} \varphi(0) + \frac{2A^2}{2A^2 - 1} (\varphi_{1+} - S^{(2)}(T, \cdot)) + u_{1+} - S^{(1)}(T, \cdot)$$

$$\varphi(1) = - \frac{1}{2A^2 T^2 A^2} \left( u_{1+} - S^{(1)}(T, \cdot) \right).$$

(5.29)

Observe carefully that the time integral here acts only on the time-dependent coefficients of $\mathcal{L}[]$ and not on $\varphi(0)$ (which does not depend on time by assumption). In fact, $\mathcal{E}$ is therefore a standard linear differential operator acting on $\varphi(0)$ by spatial derivatives only. Following Appendix II of [15], it is easy to show that $\mathcal{E}$ is elliptic. Since it is therefore a continuous map $H^{k+2}(M) \to H^k(M)$ for any $k > n/2 - 2$ (see Corollary 2.2 there), it follows that

$$\Phi : H^{k+2}(M) \times H^k(M) \to H^k(M) \times H^k(M), \quad (\varphi(0), \varphi(1)) \mapsto (u_{1+}, \varphi_{1+})$$

is a continuous map.

Given this, the next step of this proof is now to apply Proposition 5.4 with $\sigma_1 = 0$, $\sigma_2 = \sigma$ and $K = \sigma_2$ for this choice of $\Phi$ and for map $\chi$ being the projection on the first component. Notice that we write $(\varphi(0), \varphi(1))$ instead of $(u_{1+}, \varphi_{1+})$ and $\varphi(0)$ instead of $\varphi_{1+}$. In order to match the property of $\Phi$ above with that in Proposition 5.4, we replace $k$ in Eq. (5.31) (which holds for any $k > n/2 - 2$) by $k + 5$ and set $k_0 = 3$ and $k_1 = 1$. Now in order to check Eqs. (5.19) and (5.20), we exploit Proposition 5.3 to the $\sigma_1$-system where $f^{(1)}$, $f^{(2)}$, $f^{(3)}$ have to be replaced by the components of the source term in (5.21). We conclude that $\varphi(0) = \Psi^{(pre)}(u_0, v_0)$ and that for any $\lambda < \min\{\lambda_1, 2(1 - q_{\max}), 2A^2\}$

$$\|e^{-\lambda} (\varphi(t, \cdot) - \Psi^{(pre)}(u_0, v_0))\|^2_{d, H^{-k+\theta_0}(M)} \leq C \left( \|\varphi(0)\|_{d, H^{k+\theta_0}(M)}^2 + \|\varphi(1)\|_{d, H^{k+\theta_0}(M)}^2 \right)$$

$$+ \int_0^T \left( \|s^{-\lambda_1} f^{(1)}\|^2_{d, H^{k+\theta_0}(M)} + \|s^{-\lambda_1} f^{(2)}\|^2_{d, H^{k+\theta_0}(M)} + \|s^{-\lambda_1 + 2\sigma_2} f^{(3)}\|^2_{d, H^{k+\theta_0}(M)} \right) s^{-1} ds$$

$$+ \int_0^T \left( \|s^{-2\sigma_2} \mathcal{L}[\varphi(0)]\|_{d, H^{k+\theta_0}(M)}^2 s^{-1} ds \right),$$

for all $t \in (0, T]$, where the last term can also be estimated by $\|\varphi(0)\|_{d, H^{k+\theta_0}(M)}^2$. The same arguments apply to Eqs. (5.20) and (5.16). Proposition 5.4 therefore yields the asymptotic matching map $(u_{2+}, \varphi_{2+}) = \Psi_{0 \to \sigma}(\varphi(0), \varphi(1))$ which is $C^\infty$-continuous. It yields smooth $(u_{2+}, \varphi_{2+}) = \Psi_{0 \to \sigma}(\varphi(0), \varphi(1))$ such that the solution $v_2 = (u_2, v_2, \varphi_2)$ of Eq. (4.8) with Eqs. (5.1), (5.2) and (5.23) determined by Cauchy data $\varphi_2(T, \cdot) = \varphi_{2+}$ and $u_2(T, \cdot) = u_{2+}$ satisfies the estimate Eq. (5.25) for all $t \in (0, T]$ and for any $\lambda$ in the range (3.17). Replacing the right-hand side in Eq. (5.25) by a generic constant $C > 0$, Eq. (3.18) follows from Eq. (5.28). Since Eq. (4.8) with Eqs. (5.1), (5.2) and (5.23) agrees
with Eqs. (3.1) – (3.3) (because $\sigma = \sigma_2$ and $K = \sigma_2$), the map $\Psi$ asserted in Theorem 3.2 is given by

$$\Psi = \Psi_{\sigma_1 \rightarrow \sigma_2}.$$

This map must be injective as Eq. (3.18) could not be satisfied for the same solution $(u, \nu, \varphi)$ of Eqs. (3.1) – (3.3) given two different choices of $(\varphi_{(0)}, \varphi_{(1)})$

Let us now address the surjectivity of $\Psi$. The main idea for this part of the proof is to apply Proposition 5.4 with $\sigma_2 = 0$, $\sigma_1 = \sigma$ and $K = \sigma_1$, $\Phi = \text{id}$ (which is why we simply write $(u_{1*}, \varphi_{1*})$ instead of $(u_{1X}, \varphi_{1X})$) and $\chi = \Psi^{(\text{pre})}$ given by Proposition 5.3 for $\sigma = \sigma_1$ in order to construct a candidate for the inverse of $\Psi$. Observe here that due $K = \sigma_1$, it is not important which map $\Phi$ we choose and we therefore pick the identity for simplicity. As before we write

$$\varphi_{1(0)} = \Psi^{(\text{pre})}(u_{1*}, \varphi_{1*}).$$

The solution $(u_1(t, x), \nu_1(t, x), \varphi_1(t, x))$ is uniquely determined by solving the well-posed Cauchy problem of Eq. (4.8) with Eqs. (5.1), (5.2) and (5.21) with arbitrary smooth Cauchy data $(u_{1*}, \varphi_{1*})$. We conclude from Proposition 5.2 and Proposition 5.3 that the hypothesis of Proposition 5.4 holds under the same restrictions as before. The $C^\infty$-continuous map $\Psi_{\sigma_1 \rightarrow \sigma_2} = \Psi_{\sigma \rightarrow 0}$ asserted by Proposition 5.4 therefore yields smooth $(u_{2*}, \varphi_{2*}) = \Psi_{\sigma \rightarrow 0}(u_{1*}, \varphi_{1*})$ such that the solution $v_2 = (u_2, \nu_2, \varphi_2)$ of Eq. (4.8) with Eqs. (5.1), (5.2) and (5.23) determined by Cauchy data $\varphi_2(T, \cdot) = \varphi_{2*}$ and $u_2(T, \cdot) = u_{2*}$ satisfies the estimate Eq. (5.25) for all $t \in (0, T]$ and any smooth $\lambda$ in the range (3.17). Now, this solution $v_2$ must be of the form Eq. (5.28) for some smooth $(\varphi_{(0)}, \varphi_{(1)})$. Eq. (5.25) implies that $\varphi_{(0)}$ equals Eq. (5.32), and, $\varphi_{(1)}$ is given by Eq. (5.30) (with $u_{1*}$ replaced by $u_{2*}$). It is straightforward to conclude from the information given in the proof of Proposition 5.3 that $\chi = \Psi^{(\text{pre})}$ is $C^\infty$-continuous. We have therefore constructed a $C^\infty$-continuous map from the set of smooth $(u_{1*}, \varphi_{1*})$ to the set of smooth $(\varphi_{(0)}, \varphi_{(1)})$ with the above properties. Now pick any smooth $(u_{1*}, \varphi_{1*})$ and let $(\varphi_{(0)}, \varphi_{(1)})$ be determined by this map. Let $(\tilde{u}_{1*}, \tilde{\varphi}_{1*}) = \Psi(\varphi_{(0)}, \varphi_{(1)})$. The uniqueness property of $\Psi$ asserted by Proposition 5.4 implies that $(\tilde{u}_{1*}, \tilde{\varphi}_{1*}) = (u_{1*}, \varphi_{1*})$. We conclude that $\Psi$ is therefore surjective, and, in fact, that both $\Psi$ and inverse $\Psi^{-1}$ are $C^\infty$-continuous. □

5.2 Proof of Proposition 5.1: Rough a-priori estimates for solutions of the lapse-scalar field system

This subsection is concerned with the proof of Proposition 5.1. Let us start by picking arbitrary $\sigma \in [0, 1]$ and arbitrary asymptotically point-wise Kasner–scalar field background $\Gamma = (\dot{\gamma}^{ab}, \dot{x}_a^b, \dot{\alpha}, \dot{\pi}, \dot{\phi})$ with positive decay $\beta$. Given any smooth solution $v = (u, \nu, \varphi)$ of Eq. (4.8) with $A$ and $L$ given by Eqs. (5.1) and (5.2), and, $F$ given by (5.10). Interpreting $A$ and $L$ as operators on arbitrary smooth time-dependent $(0, r)$-tensor fields, the fields

$$u_{a_1...a_k} = \partial_{a_1} \cdots \partial_{a_k} u, \quad \nu_{a_1...a_k} = \partial_{a_1} \cdots \partial_{a_k} \nu, \quad \varphi_{a_1...a_k} = \partial_{a_1} \cdots \partial_{a_k} \varphi,$$

satisfy

$$(A + \sigma L) \left[ (u_{a_1...a_k}, \nu_{a_1...a_k}, \varphi_{a_1...a_k}) \right] = (F^{(1)}_{a_1...a_k}, F^{(2)}_{a_1...a_k}, F^{(3)}_{a_1...a_k}).$$

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for the same $A$ and $L$, but where the source terms are given as
\[
(F^{(1)}_{a_1...a_k}, F^{(2)}_{a_1...a_k}, F^{(3)}_{a_1...a_k}) = \left( \partial_{a_1} \cdots \partial_{a_k} F^{(1)}, \partial_{a_1} \cdots \partial_{a_k} F^{(2)}, \partial_{a_1} \cdots \partial_{a_k} F^{(3)} \right) \\
+ \left[ A + \sigma L, \partial^k \right] \left( [u_{a_1 \cdots a_k}, \nu_{a_1 \cdots a_k}, \varphi_{a_1 \cdots a_k}] \right),
\]  
(5.35)

with $[\cdot, \cdot]$ being the commutator.

The first step of the proof of Proposition 5.1 is now to estimate the smooth tensorial solutions $(u_{a...b}, \nu_{a...b}, \varphi_{a...b})$ of Eq. (5.34) for any smooth tensorial source term fields $F^{(1)}_{a...b}$, $F^{(2)}_{a...b}$ and $F^{(3)}_{a...b}$. Since in this first step we therefore do not impose Eq. (5.35), these solutions do in general not satisfy Eq. (5.33). Only in the second step we consider Eq. (5.35) which yields Lemma 5.6. The proof of Proposition 5.1 is then essentially a combination of these two lemmas. Recall the definitions in Eqs. (5.4), (5.5), (5.6) and (5.7).

**Lemma 5.5** (Estimates for smooth solutions of the tensorial equations). Consider an arbitrary asymptotically point-wise Kasner–scalar field background $\Gamma = (\tilde{\gamma}^{ab}, \tilde{\chi}^a, \tilde{\alpha}, \tilde{\pi}, \tilde{\phi})$ with positive decay $\beta$. Let $(u, \nu, \varphi)$ be any smooth $(0, r)$-tensorial solution of Eq. (5.34) with $A$ and $L$ given by Eqs. (5.1) and (5.2) for arbitrary smooth time-dependent $(0, r)$-tensor fields $F^{(1)}(t, x)$, $F^{(2)}(t, x)$ and $F^{(3)}(t, x)$ on $M$ and $\sigma \in I = [0, 1]$. Pick any sufficiently small $\epsilon > 0$. Then there is a constant $C > 0$ such that for any $t \in [0, T]$ the following estimates hold:

1. For all $t \in [t_*, T]$ and for any smooth function $\lambda > \lambda_c$ (see Eq. (3.10)), we have
\[
ed_{\sigma, \lambda}[u, \varphi](t) \leq C \left( \ned_{\sigma, \lambda}[u, \varphi](t_*) \\
+ \int_{t_*}^t \left( \| s^{-\lambda} F^{(1)} \|_\delta^2 + \| s^{-\lambda} F^{(2)} \|_\delta^2 + \sigma^2 \| s^{-\lambda} \partial F^{(3)} \|_{\delta, t^2, \Gamma}^2 \right) s^{-1+2\epsilon} ds \right).
\]
(5.36)

2. For all $t \in [t_*, T]$ and for any smooth function $\lambda \geq 0$ on $M$, we have
\[
\sqrt{\ned_{\lambda}[\varphi](t)} \leq \sqrt{\ned_{\lambda}[\varphi](t_*)} \\
+ C \int_{t_*}^t \left( \sqrt{\ned_{\lambda}[u](s)} + \| s^{-\lambda} F^{(2)} \|_\delta + \| s^{-\lambda} F^{(3)} \|_\delta \right) s^{-1} ds.
\]
(5.37)

3. For all $t \in (0, t_*]$ and for any smooth function $\lambda < 0$ on $M$, we have
\[
ed_{\sigma, \lambda}[u, \varphi](t) \leq C \left( \ned_{\sigma, \lambda}[u, \varphi](t_*) \\
+ \int_t^{t_*} \left( \| s^{-\lambda} F^{(1)} \|_\delta^2 + \| s^{-\lambda} F^{(2)} \|_\delta^2 + \sigma^2 \| s^{-\lambda} \partial F^{(3)} \|_{\delta, t^2, \Gamma}^2 \right) s^{-1-2\epsilon} ds \right).
\]
(5.38)
4. For all \( t \in (0, t_\ast] \) and for any smooth function \( \lambda \leq 0 \) on \( M \), we have
\[
\sqrt{e_\lambda[\varphi]}(t) \leq \sqrt{e_\lambda[\varphi]}(t_\ast) + C \int_t^{t_\ast} \left( \sqrt{e_\lambda[u]}(s) + \|s^{-\lambda}F^{(2)}\|_\delta + \|s^{-\lambda}F^{(3)}\|_\delta \right)s^{-1}ds.
\] (5.39)

The constants \( C \) may depend on \( T, \lambda, \epsilon \) and \( \Gamma \). Moreover, for all \( t \in (0, T] \) and for any smooth function \( \lambda \) on \( M \), we have
\[
\sqrt{e_\sigma,\lambda[\nu]}(t) \leq C \left( \sqrt{e_\lambda[u]}(t) + \|t^{-\lambda}F^{(2)}\|_\delta \right),
\] (5.40)
where the constant \( C > 0 \) may depend on \( T \) and \( \Gamma \).

The proof of this lemma is discussed towards the end of this subsection. We remark that large parts of this proof consist in standard wave-type energy arguments using integration by parts. However, the presence of the elliptic variable \( \nu \) leads to a few peculiarities which we wish to point out in the proof. We also remark that in contrast to the proof of Proposition 5.2 in [29] (which restricts to estimates when the Cauchy data is imposed at the final time \( T \)), we find it useful here to write the estimates in terms of the variable \( u \), instead for \( \partial_t \varphi \), and to keep the estimates for \( u \) and \( \varphi \) separate from those for \( \nu \). This is crucial in order to avoid all terms involving time-derivatives of \( \nu \). We also remark that the function \( \lambda(x) \) acts as the exponent of a time-weight. Such time weights become crucial when we need to distinguish the two different kinds of Cauchy problems later on: (1), where the Cauchy condition is imposed at the final time, and, (2), where the Cauchy condition is imposed at the matching time.

The next step is to estimate the source terms.

**Lemma 5.6** (Estimates for the tensor source terms given by Eq. (5.35)). Consider an arbitrary asymptotically point-wise Kasner–scalar field background \( \Gamma = (\gamma^{ab}, \chi^a_b, \dot{\alpha}, \dot{\pi}, \dot{\phi}) \) with positive decay \( \beta \). Pick any smooth (scalar) solution \( v = (u, \nu, \varphi) \) of Eq. (4.8) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2) for arbitrary smooth scalar fields \( F^{(1)}(t, x) \), \( F^{(2)}(t, x) \) and \( F^{(3)}(t, x) \) on \( M \) and \( \sigma \in I = [0, 1] \). Let the \((0, k)\)-tensor fields \( u[k], \varphi[k], \nu[k] \) be defined by Eq. (5.33) for arbitrary integers \( k \geq 1 \). Then there is a constant \( C > 0 \), which may depend on \( k, T \) and \( \Gamma \), such that the following estimates hold for the fields defined in
Equation (5.35):

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} F_i^{(1)} \right\|_{\delta}^2 \leq C \left( \left\| t^{-\lambda} F^{(1)} \right\|_{\delta,H^k(M)}^2 + \left\| t^{-\lambda} F^{(2)} \right\|_{\delta,H^{k-1}(M)}^2 \right) \\
+ E_{k-2\epsilon}[u,\varphi](t) + E_{k-1,\lambda}[u,\varphi](t),
\]

(5.41)

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} F_i^{(2)} \right\|_{\delta}^2 \leq C \left( \left\| t^{-\lambda} F^{(2)} \right\|_{\delta,H^k(M)}^2 + E_{k-1,\lambda}[u](t) + E_{k,\lambda-2\epsilon}[u](t) \right),
\]

(5.42)

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} \partial F_i^{(3)} \right\|_{\delta,t^2\gamma}^2 \leq C \left( \left\| t^{-\lambda} F^{(3)} \right\|_{\delta,H^{k+1}(M)}^2 + \left\| t^{-\lambda} F^{(2)} \right\|_{\delta,H^{k-1}(M)}^2 \right) \\
+ E_{k-1,\lambda}[u,\varphi](t) + E_{k,\lambda-\epsilon,\lambda-2\epsilon}[u,\varphi](t).
\]

(5.44)

The function \( \kappa \) here is defined in footnote 5.2.

Before we provide the proof of this lemma, we now prove first the main result of this subsection, Proposition 5.1.

**Proof of Proposition 5.1.** Pick any smooth (scalar) solution \( v = (u, \nu, \varphi) \) of Eq. (4.8) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2) for arbitrary smooth scalar fields \( F^{(1)}(t, x) \), \( F^{(2)}(t, x) \) and \( F^{(3)}(t, x) \) on \( M \) and \( \sigma \in I = [0, 1] \). Let the \((0, k)\)-tensor fields \( u_{[k]}, \varphi_{[k]}, \psi_{[k]} \) be defined by Eq. (5.33) for arbitrary integers \( k \geq 1 \), and \( F_{[k]}^{(1)}, F_{[k]}^{(2)}, F_{[k]}^{(3)} \) by Eq. (5.35).

Consider first the case that \( \lambda > \lambda_c \) (see Eq. (3.10)). Also suppose that \( k \geq 1 \) first. Then combine the sum Eq. (5.36) from \( r = 0 \) to \( k \) with Eqs. (5.41), (5.42) and (5.44) as follows:

\[
E_{k,\sigma,\lambda}[u,\varphi](t) \leq C \left( \int_{t_s}^{t} E_{k,\lambda}[u,\varphi](s)s^{-1+2\epsilon}ds \right) \\
+ \int_{t_s}^{t} \left( \left\| s^{-\lambda} F^{(1)} \right\|_{\delta,H^{k+1}(M)}^2 + \left\| s^{-\lambda} F^{(2)} \right\|_{\delta,H^{k-1}(M)}^2 + \left\| s^{-\lambda} F^{(3)} \right\|_{\delta,H^{k+1}(M)}^2 \right) s^{-1+2\epsilon}ds,
\]

for all \( t \in [t_s, T] \) where we exploit that the function \( \kappa \) is strictly positive. Notice that we get the same estimate for \( k = 0 \) without the the second term on the right-hand side. In this case Eq. (5.11) follows directly. If \( k \geq 1 \), however, Eq. (5.11) follows after we apply Grönwall’s lemma; notice that the constant \( C > 0 \), which may depend on \( k \) in addition now, may blow up in the limit \( \epsilon \to 0 \).

Suppose next that \( \lambda < 0 \). If \( k = 0 \), Eq. (5.13) follows directly from Eq. (5.38). For
For any $k \geq 1$, a similar procedure as above yields

$$E_{k,\sigma,\lambda}[u, \varphi](t) \leq C \left( E_{k,\sigma,\lambda}[u, \varphi](t_*) + \int_t^{t_*} E_{k-1,\lambda}[u, \varphi](s) s^{-1-2\epsilon} ds + \int_t^{t_*} E_{k,\lambda-2\epsilon}[u, \varphi](s) s^{-1-2\epsilon} ds \right)$$

$$+ \int_t^{t_*} \left( \|s^{-\lambda} F^{(1)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(2)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(3)}\|_{\delta,H^{k+1}(M)}^2 \right) s^{-1-2\epsilon} ds,$$

for all $t \in (0, t_*]$. Notice carefully that we have not estimated the energy of order $k-1$ by the energy of order $k$ in contrast to the above. Supposing that $\epsilon > 0$ is sufficiently small there is a constant $\eta > 0$ (recall that $\kappa > 0$) such that Grönwall’s lemma implies

$$E_{k,\sigma,\lambda}[u, \varphi](t) \leq C \left( E_{k,\sigma,\lambda}[u, \varphi](t_*) + \frac{1}{2\epsilon} \left( t_*^{2\epsilon} - t^{2\epsilon} \right) \sup_{s \in [t, t_*)} E_{k-1,\lambda+2\epsilon}[u, \varphi](s) \right.$$\n
$$+ \int_t^{t_*} \left( \|s^{-\lambda} F^{(1)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(2)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(3)}\|_{\delta,H^{k+1}(M)}^2 \right) s^{-1-2\epsilon} ds,$$

for a constant $C > 0$ which may depend on $\eta$ in addition. If $k = 1$ we can estimate the second term on the right-hand side by the $k = 0$-estimate above which yields Eq. (5.13). For any $k \geq 2$, the same estimate holds with $k$ replaced by $k-1$ and $\lambda$ by $\lambda+2\epsilon$ (provided $\epsilon$ is sufficiently small). This can be used to estimate the second term on the right-hand above. The same argument applied repeatedly yields for any $k \geq 0$

$$E_{k,\sigma,\lambda}[u, \varphi](t) \leq C \left( E_{k,\sigma,\lambda+2\epsilon}[u, \varphi](t_*) \right.$$\n
$$+ \int_t^{t_*} \left( \|s^{-\lambda} F^{(1)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(2)}\|_{\delta,H^k(M)}^2 + \|s^{-\lambda} F^{(3)}\|_{\delta,H^{k+1}(M)}^2 \right) s^{-1-2(2k+1)\epsilon} ds,$$

For any fixed value of $k$ and $\epsilon$ we may therefore write $\epsilon$ instead of $(2k+1)\epsilon$ which establishes Eq. (5.13). The constants $C > 0$ here may depend on $T$, $\sigma$, $\lambda$, $\epsilon$, and $\Gamma$.

Applying Eqs. (5.42) and (5.43) to Eq. (5.37) for any $\lambda \geq 0$ yields Eq. (5.12). Applying Eqs. (5.42) and (5.43) to Eq. (5.39) for any $\lambda \leq 0$ yields Eq. (5.14). Finally, Eq. (5.15) is established as part of the proof of Lemma 5.6 in Eq. (5.76).

Before we discuss the proofs of the two lemmas of this subsection next, let us introduce
some more notation:

\[ L^{[1,1]} = \sigma (1 - \dot{\alpha}) = \sigma L_1^{[1,1]}, \]
\[ L^{[1,2]} = -A - \sigma (\dot{\pi} - A) + \sigma t^2 \chi_{ab} D_a D_b \dot{\phi} = L_0^{[1,2]} + \sigma L_1^{[1,2]}, \]
\[ L^{[1,3],b} = \sigma t^2 \chi_{ab} D_a \dot{\alpha} = \sigma L_1^{[1,3],b}, \]
\[ L^{[1,4],b} = \sigma t^2 \chi_{ab} D_a \dot{\phi} = \sigma L_1^{[1,4],b}, \]
\[ L^{[2,1]} = -1 - \sigma \left( t^2 \dot{\chi}_{a}^{\ b} \dot{\chi}_{b}^{\ a} + \frac{1}{n} + A^2 - 1 \right) + (\dot{\pi}^2 - A^2) = L_0^{[2,1]} + \sigma L_1^{[2,1]}, \]
\[ L^{[2,2]} = -2A - 2\sigma (\dot{\pi} \dot{\alpha} - A) = L_0^{[2,2]} + \sigma L_1^{[2,2]}, \]
\[ L^{[3,1]} = A + \sigma (\dot{\pi} - A) = L_0^{[3,1]} + \sigma L_1^{[3,1]}, \]
\[ L^{[3,2]} = 1 + \sigma (\dot{\alpha} - 1) = L_0^{[3,2]} + \sigma L_1^{[3,2]}. \]

For some of our discussion it is also useful to express the covariant derivative \( D_a \) by \( \partial_a \) using the tensor field (recall the index conventions introduced in Section 3 and footnote 3.3)

\[
C_{c b}^{a} = \frac{1}{2} \gamma^{ad} \left( \partial_c \dot{\chi}_{bd}^{\ 1} + \partial_b \dot{\chi}_{cd}^{\ 1} - \partial_d \dot{\chi}_{bc}^{\ 1} \right), \tag{5.53}
\]
\[
C_{c}^{a} = \dot{\gamma}_{cb} C_{c b}^{a} = -\partial_c \dot{\gamma}^{ac} + \frac{1}{2} \dot{\gamma}_{bc} \partial_d \dot{\gamma}^{bc}, \tag{5.54}
\]

Given this Eq. (5.34) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2) takes the form\(^{5,4}\)

\[
-t \partial_t u + \sigma \dot{\alpha} t^2 \chi_{ab} \partial_a \partial_b \varphi - \sigma \left( t^2 \dot{\chi}_{a}^{\ b} C_{b}^{a} - L_1^{[1,3],a} \right) \partial_a \varphi
+ L_0^{[1,1]} u + L_1^{[1,2]} u + \sigma L_1^{[1,4],b} \partial_b \nu = F^{(1)}, \tag{5.54}
\]
\[
\sigma t^2 \gamma_{ab} \partial_a \partial_b \nu - \sigma t^2 C_a^{a} \partial_a \nu + L_0^{[2,1]} u + L_1^{[2,2]} u = F^{(2)}, \tag{5.55}
\]
\[
-t \partial_t \varphi + L_0^{[3,1]} \nu + L_1^{[3,2]} u = F^{(3)}. \tag{5.56}
\]

**Proof of Lemma 5.5.** Consider any smooth time-dependent \((0, r)\)-tensorial solution \((u, \nu, \varphi)\) of Eqs. (5.54) – (5.56). We start by projecting Eq. (5.54) on \( u^{a...b}, \) extracting a total \( \partial \)-derivative term (“integration by parts”) and expressing the resulting factor \( \partial_c u^{a...b} \) by Eq. (5.56). Then we multiply the result by \( t^{-2\lambda} \) for a so far arbitrary smooth function

\(^{5,4}\)Observe carefully that we are using the index-free notation for tensorial quantities here. These equations therefore cover, but do not restrict to, scalars as a particular case.
\[0 = -\frac{1}{2} t \partial_t \left( t^{-\lambda} u \right)_\delta^2 + \sigma \frac{\dot{a} t^2}{L^{[3,2]}} \gamma^{cd} \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_\delta + \left( L^{[1,1]} - \lambda \right) \left( t^{-\lambda} u \right)_\delta^2 + L^{[1,2]} \left( t^{-\lambda} u, t^{-\lambda} \nu \right)_\delta + \sigma \left( -t^2 \partial_c \dot{\alpha} \dot{c}^{cd} - \dot{a} t^2 \left( \partial_c \gamma^{cd} \right) + \frac{\dot{\alpha} t^2}{L^{[3,2]}} \partial_c L^{[3,2]} \gamma^{cd} \right. \]
\[\left. - \dot{a} t^2 C^{cd} + L_1^{[1,3]} \gamma^{cd} + 2 \dot{a} t^2 \log \partial_\epsilon \lambda \gamma^{cd} \right) \left( t^{-\lambda} u, t^{-\lambda} \partial_d \varphi \right)_\delta + \sigma \left( \frac{\dot{\alpha} t^2}{2L^{[3,2]}} \partial_c L^{[3,1]} \gamma^{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \varphi \right)_\delta + \partial_c \left( \partial_\epsilon \lambda \gamma^{cd} \right) \left( t^{-\lambda} u, t^{-\lambda} \partial_d \varphi \right)_\delta \right) .
\]

This has intentionally not been integrated in space yet. Similarly, projecting Eq. (5.55) on \( \nu^{a...b} \) yields:
\[0 = L^{[2,1]} \left| t^{-\lambda} \nu \right|_\delta^2 + \sigma t^2 \left( \partial_c \gamma^{cd} + C^{cd} - 2 \log t \partial_\epsilon \lambda \gamma^{cd} \right) \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \nu \right)_\delta - \sigma \gamma^{cd} \left( t^{-\lambda} \partial_c \nu, t^{-\lambda} \partial_d \nu \right)_\delta + \sigma L_1^{[1,4]} \gamma^{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \nu \right)_\delta \left( t^{-\lambda} \nu, t^{-\lambda} \nu \right)_\delta - \sigma \gamma^{cd} \left( t^{-\lambda} \partial_c \nu, t^{-\lambda} \partial_d \nu \right)_\delta + \partial_c \left( \sigma \dot{\alpha} t^2 \gamma^{cd} \right) \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \nu \right)_\delta .
\]

The next step is now to incorporate the asymptotics of the coefficients in Eqs. (5.54) – (5.56). It follows from the hypothesis that
\[L^{[1,1]} = \sigma O(t^3), \quad L^{[1,2]} = -A + \sigma \left( O(t^3) + O(t^2 \epsilon) \right), \quad L_1^{[1,3]} = \sigma \alpha t^2 \partial_\epsilon, \quad L_1^{[1,4]} = \sigma t^2 \alpha \partial_\epsilon \dot{\alpha}. \quad L^{[2,1]} = -1 - \sigma O(t^3), \quad L^{[2,2]} = -2A - \sigma O(t^3), \quad L^{[3,1]} = A + \sigma O(t^3), \quad L^{[3,2]} = 1 + \sigma (\dot{\alpha} - 1), \quad (5.57)\]

where \( \kappa \) is defined in footnote 5.2. A straightforward calculation involving Eqs. (3.6) and (3.7) shows that
\[-t^2 \partial_\epsilon \dot{\alpha} \gamma^{cd} - \dot{a} t^2 \left( \partial_c \gamma^{cd} \right) + \frac{\dot{\alpha} t^2}{L^{[3,2]}} \partial_c L^{[3,2]} \gamma^{cd} - \dot{a} t^2 C^{cd} + L_1^{[1,3]} \gamma^{cd} + 2 \dot{a} t^2 \log \partial_\epsilon \lambda \gamma^{cd} \]
\[= \left( -\frac{1}{2} \sigma \gamma^{bc} \partial_\epsilon \gamma^{bc} + 2 \log t \partial_\epsilon \lambda + O(t^{3 - \epsilon}) \right) \gamma^{cd} ,
\]
which follows from Eqs. (5.53), (5.45) – (5.52) and from a calculation of the form
\[
\tilde{\gamma}_{bc}^{-1} \partial_t \tilde{\gamma}^{bc} = \tilde{\gamma}_{bc}^{-1} \partial_t \gamma_{bc} = \tilde{\gamma}_{bc}^{-1} \partial_t \gamma_{bc} + \sigma \gamma_{bc}^{-1} \partial_t \gamma_{bc} + \sigma \gamma_{bc}^{-1} \partial_t h^{bc} = \tilde{\gamma}_{bc}^{-1} \partial_t \tilde{\gamma}^{bc} + \sigma \gamma_{bc}^{-1} \partial_t \gamma_{bc} + \sigma \gamma_{bc}^{-1} \partial_t h^{bc} = \tilde{\gamma}_{bc}^{-1} \partial_t \tilde{\gamma}^{bc} + \sigma O(t^{\beta - \epsilon}),
\]
where \(\epsilon > 0\) is any constant. A similar calculation involving Eq. (2.22) yields that
\[
t\partial_t \left( \frac{\tilde{\alpha}t^2}{2L|3,2|} \right) \tilde{\gamma}^{cd} + \frac{\tilde{\alpha}t^2}{2L|3,2|} (t\partial_t \tilde{\gamma}^{cd}) - \lambda \frac{\tilde{\alpha}t^2}{L|3,2|} \tilde{\gamma}^{cd} = (t\tilde{\gamma}^{ac} + \left( 1 - \frac{1}{n} - \lambda \right) \delta_a^c + O(t^3)) (\delta_f^a + O(t^3)) t^2 \tilde{\gamma}^{fd}.
\]
Thanks to all this our first identity above becomes
\[
0 = -\frac{1}{2} t\partial_t \left( t^{-\lambda} u^2 \right) + \sigma (1 + O(t^3)) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \left( t^{-\lambda} u \right)^2_{\delta} - \left( A + O(t^3) \right) \left( t^{-\lambda} u \right)_{\delta} + \sigma \left( \frac{1}{2} \gamma_{bc}^{-1} \partial_t \gamma^{bc} + 2 \log t \partial_t A + O(t^{\beta - \epsilon}) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} u, t^{-\lambda} \partial \varphi \right)_{\delta} + \sigma \left( t \tilde{\gamma}^{ac} + \left( 1 - \frac{1}{n} - \lambda \right) \delta_a^c + O(t^3) \right) t^2 \tilde{\gamma}^{fd} \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( \partial_c A + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( A + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c \nu, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \partial_a \tilde{\gamma}^{2,ac} \left( t^{-\lambda} u, t^{-\lambda} \partial_c \nu \right)_{\delta} - \left( t^{-\lambda} u, t^{-\lambda} F^{(1)} \right)_{\delta} + \sigma \left( 1 + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c F^{(3)}, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \partial_c \left( \sigma \tilde{\gamma}^{2,cd} \left( t^{-\lambda} u, t^{-\lambda} \partial_d \varphi \right)_{\delta} \right).
\]
For the second identity above we find
\[
0 = -\left( 1 + O(t^3) \right) t^{-\lambda} \nu^2_{\delta} - \left( A + O(t^3) \right) \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( \frac{1}{2} \gamma_{bc}^{-1} \partial_t \gamma^{bc} + 2 \log t \partial_t A + O(t^{\beta - \epsilon}) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( t \tilde{\gamma}^{ac} + \left( 1 - \frac{1}{n} - \lambda \right) \delta_a^c + O(t^3) \right) t^2 \tilde{\gamma}^{fd} \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( \partial_c A + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \left( A + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c \nu, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \sigma \partial_a \tilde{\gamma}^{2,ac} \left( t^{-\lambda} u, t^{-\lambda} \partial_c \nu \right)_{\delta} - \left( t^{-\lambda} u, t^{-\lambda} F^{(1)} \right)_{\delta} - \sigma \left( 1 + O(t^3) \right) t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c F^{(3)}, t^{-\lambda} \partial_d \varphi \right)_{\delta} + \partial_c \left( \sigma \tilde{\gamma}^{2,cd} \left( t^{-\lambda} u, t^{-\lambda} \partial_d \varphi \right)_{\delta} \right).
\]
Defining
\[
\tilde{c} = \frac{1}{2} \int_M \left( (1 + O(t^3)) t^{-\lambda} \nu^2_{\delta} + \sigma t^2 \tilde{\gamma}^{cd} \left( t^{-\lambda} \partial_c \nu, t^{-\lambda} \partial_d \nu \right)_{\delta} \right) dx,
\]
38
Eq. (5.62) yields for all $t \in (0, T]$ and for any smooth $\lambda(x)$:

$$
\hat{c}(t) \leq C \left( \sqrt{e_\lambda[u](t)} \sqrt{e_{\sigma,\lambda}[\nu](t)} + \sqrt{e_{\sigma,\lambda}[\nu](t)} \|t^{-\lambda} F^{(2)}\|_\delta \right),
$$

where $C$ may depend on $T$ and $\Gamma$. Comparing the definition of $\hat{c}$ and Eq. (5.7) implies Eq. (5.40) for some constant $C > 0$ with the same dependencies.

Setting

$$
U = (t^{-\lambda}u \quad t^{-\lambda}v \quad \sqrt{\sigma} t^{-\lambda} \partial_c \varphi \quad \sqrt{\sigma} t^{-\lambda} \partial_d \varphi)^T,
$$

we can cast Eq. (5.63) into the form

$$
\frac{1}{2} t \partial_t \left( \left| t^{-\lambda} u \right|^2_\delta + \sigma \left( 1 + O(t^\beta) \right) t^2 \gamma^{cd} \left( t^{-\lambda} \partial_c \varphi, t^{-\lambda} \partial_d \varphi \right)_\delta \right)
= U^T \cdot \dot{M}_1 \cdot U
- \left( t^{-\lambda}u, t^{-\lambda} F^{(1)} \right)_\delta + \sigma(1 + O(t^\beta)) t^2 \gamma^{ad} \left( t^{-\lambda} \partial_c F^{(3)}, t^{-\lambda} \partial_d \varphi \right)_\delta + \partial_c (\ldots),
$$

with

$$
\dot{M}_1 = \begin{pmatrix}
\dot{M}_{1,11} & \dot{M}_{1,12} \\
\dot{M}_{1,21} & \dot{M}_{1,22}
\end{pmatrix}
$$

and

$$
\dot{M}_{1,11} = \begin{pmatrix}
-\lambda & -A/2 \\
-A/2 & 0
\end{pmatrix} + O(t^\beta) + O(t^{2\kappa}),
$$

$$
\dot{M}_{1,12} = \dot{M}_{1,21} = \begin{pmatrix}
-\sqrt{\sigma} \frac{1}{2} \left( \frac{1}{2} \gamma_{bc} \partial_a \gamma^{bc} - 2 \log t \partial_a \lambda + O(t^{\beta-\epsilon}) \right) t^\beta \gamma^{ad} \\
\frac{1}{2} \sqrt{\sigma} (\partial_a A + O(t^\beta)) t^\beta \gamma^{ad}
\end{pmatrix},
$$

$$
\dot{M}_{1,22} = \begin{pmatrix}
(\mathfrak{T} a)^c + \left( 1 - \frac{1}{n} - \lambda \right) \delta^{c}_a + O(t^\beta) \gamma^{ad} \\
\frac{1}{2} (A + O(t^\beta)) \gamma^{cd}
\end{pmatrix}.
$$

In the same way we find for Eq. (5.62)

$$
0 = -U^T \cdot \dot{M}_2 \cdot U - \left( t^{-\lambda}u, t^{-\lambda} F^{(2)} \right)_\delta + \partial_c \left( \sigma t^2 \gamma^{cd} \left( t^{-\lambda}u, t^{-\lambda} \partial_c \nu \right)_\delta \right),
$$

with

$$
\dot{M}_2 = \begin{pmatrix}
\dot{M}_{2,11} & \dot{M}_{2,12} \\
\dot{M}_{2,21} & \dot{M}_{2,22}
\end{pmatrix}
$$

and

$$
\dot{M}_{2,11} = \begin{pmatrix}
0 & A \\
A & 1
\end{pmatrix} + O(t^\beta),
$$

$$
\dot{M}_{2,12} = \dot{M}_{2,21} = \begin{pmatrix}
0^d \\
0^d \frac{1}{2} \sqrt{\sigma} \left( \frac{1}{2} \gamma_{bc} \partial_a \gamma^{bc} - 2 \log t \partial_a \lambda + O(t^{\beta-\epsilon}) \right) t^\beta \gamma^{ad}
\end{pmatrix},
$$

$$
\dot{M}_{2,22} = \begin{pmatrix}
0^d & 0^d \\
0^d & \gamma^{cd}
\end{pmatrix}.
$$
For the following it is useful to add a multiple of Eq. (5.65) to (5.64) using a so far unspecified smooth function $\mu(x)$ as follows:

$$\frac{1}{2} t \partial_t \left( \left| t^{-\lambda} u \right|_\delta^2 + \sigma(1 + O(t^3)) t^2 \xi_{cd}(t^{-\lambda} \partial_d \varphi, t^{-\lambda} \partial_d \varphi) \right)$$

$$=: U^T \cdot (\hat{M}_1 - (\lambda + \mu) \hat{M}_2) \cdot U - \sigma \partial_c (\lambda + \mu) t^2 \xi_{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \nu \right)_\delta$$

$$- \left( t^{-\lambda} u, t^{-\lambda} F^{(1)} \right)_\delta - \sigma(1 + O(t^3)) t^2 \xi_{cd} \left( t^{-\lambda} \partial_c F^{(2)}, t^{-\lambda} \partial_d \varphi \right)_\delta$$

$$- (\lambda + \mu) \left( t^{-\lambda} \nu, t^{-\lambda} F^{(2)} \right)_\delta + \partial_c \left( (\lambda + \mu) \sigma t^2 \xi_{cd} \left( t^{-\lambda} \nu, t^{-\lambda} \partial_d \nu \right)_\delta \right)$$

with

$$\hat{M} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{12} \\ \hat{M}_{21} & \hat{M}_{22} \end{pmatrix}$$

and

$$\hat{M}_{11} = \begin{pmatrix} \lambda & -A/2 - (\lambda + \mu)A \\ -A/2 - (\lambda + \mu)A & (\lambda + \mu) \end{pmatrix} + O(t^3) + O(t^2 \sigma),$$

$$\hat{M}_{21} = \hat{M}_{22} = \begin{pmatrix} \xi_{cd} \nu & (1 - \frac{1}{2} - \lambda) \delta_{a}^c + O(t^{2}) \xi_{cd} \nu & \frac{1}{2} (A + O(t^{2})) \xi_{cd} \nu & - (\lambda + \mu) \xi_{cd} \nu \end{pmatrix}.$$

Let us now define

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad \Lambda_1 = \begin{pmatrix} 1 & A \\ A & 1 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} \xi_{cd} \nu & 0 \\ 0 & \xi_{cd} \nu \end{pmatrix}.$$
two matrices at each spatial point:

\[ M_{11,0} = \Lambda_1^{-1}(\dot{M}_{11} + \lambda \Lambda_1) = \frac{1}{1 - A^2} \begin{pmatrix} 1 & -A \\ -A & 1 \end{pmatrix} \left( \begin{pmatrix} 0 & \frac{1}{2}A \delta_a^c \\ \frac{1}{2}A \delta_a^c & -\mu \delta_a^c \end{pmatrix} \right) \]  

(5.67)

\[ M_{22,0} = \Lambda_2^{-1}(\dot{M}_{22} + \lambda \Lambda_2) = \begin{pmatrix} \text{diag} (1 - q_1, \ldots, 1 - q_n) & \frac{1}{2}A \delta_a^c \\ \frac{1}{2}A \delta_a^c & -\mu \delta_a^c \end{pmatrix}, \]  

(5.68)

using Eq. (2.22). Notice that the eigenvalues of both matrices clearly depend on the choice of the function \( \mu(x) \) and on \( x \), and, that the second matrix can clearly be decomposed into separate \( 2 \times 2 \)-blocks for each Kasner exponent \( q_1, \ldots, q_n \). In order to facilitate the subsequent analysis, the task is now to vary \( \mu \) in order (1) to maximize the smallest of all eigenvalues of Eqs. (5.67) and (5.68), or, (2) to minimize the largest of all eigenvalues. With straightforward arguments exploiting the bounds Eq. (2.25) on the Kasner exponents, we find for each \( x \in M \):

(1) The smallest of all eigenvalues of Eqs. (5.67) and (5.68) is maximized for \( \mu = -1/2 \). This maximal value is zero. We therefore conclude that \( \dot{M} \) is uniformly positive definite for all sufficiently small \( t > 0 \) and all \( x \in M \) provided \( \lambda(x) < 0 \) and \( \mu(x) = -1/2 \) for all \( x \in M \).

(2) Pick \( A \) and \( q_1, \ldots, q_n \) as before, and \( \xi \in [0,1] \) (see Eq. (3.9)). The largest of all eigenvalues of Eqs. (5.67) and (5.68) is minimized for \(^{5.5} \)

\[ \mu = \mu_c = \begin{cases} \frac{1}{2}(1 - 4A^2) = \frac{1}{2}(1 - 4(1 - \eta^2)A^2) & \text{for } \xi \in [0,1/3], \\ \frac{4\varepsilon^2 - A^2(1-\xi)(1+\xi)\gamma}{4(1-\xi)^2} & \text{for } \xi \in [1/3,1]. \end{cases} \]  

(5.69)

This minimal value is \( \lambda_c \) (see Eq. (3.10)). We therefore conclude that \( \dot{M} \) is uniformly negative definite for all sufficiently small \( t > 0 \) and all \( x \in M \) provided \( \lambda(x) > \lambda_c(x) \) and \( \mu(x) \) as above for all \( x \in M \).

Let us now first assume the conditions where \( \dot{M} \) is positive definite as above. Integrating Eq. (5.66) for any fixed \( t \in (0,T) \) with respect to the volume element associated with \( \delta_{ab} \) on \( M \) yields, for any constant \( \eta > 0 \) (recall that \( \mu = -1/2 \)),

\[ t \partial_t e \geq -2\eta \varepsilon - \frac{C}{2\eta} \left( \| t^{-\lambda} F(1) \|_\delta^2 + \sigma^2 \| t^{-\lambda} \partial F(3) \|_\delta^2 t^{-\lambda} \|_{\delta, t=\Gamma} \right) - \int_M (\lambda - 1/2 \left( t^{-\lambda} \nu_v, t^{-\lambda} F(2) \right)_{\delta} \right) dx, \]

where \( C \) is a constant that may depend on \( T \) and \( \Gamma \) and where we have set

\[ \varepsilon = \frac{1}{2} \int_M \left( \| t^{-\lambda} u \|_\delta^2 + \sigma (1 + O(t^2)) t^{2\sigma_{cd}} \left( t^{-\lambda} \partial_v \varphi, t^{-\lambda} \partial_d \varphi \right)_{\delta} \right) dx. \]

\(^{5.5}\)Since \( \xi \) can have different values at each \( x \in M \), we think of \( \mu_c \) as being any smooth function arbitrarily close to Eq. (5.69).
For any \( t_* \in (0, T] \), we divide by \( t \) and integrate this over \([t, t_*]\) for \( t \in (0, t_*] \):

\[
t^{2\eta}e(t) \leq t^{2\eta}e(t_*) + \frac{C}{2\eta} \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|^2_\delta + \sigma^2\|s^{-\lambda}\partial F^{(3)}\|^2_{\delta,t,\varphi}\right) s^{-1+2\eta} ds \\
+ \int_t^{t_*} \left( \int_M (\lambda - 1/2) \left( s^{-\lambda}\nu, s^{-\lambda}F^{(2)} \right) dx \right) s^{-1+2\eta} ds.
\]

Comparing the definition of \( e \) to that of \( e_{\sigma,\lambda}[u, \varphi] \) in Eq. (5.6) allows us to conclude that

\[
e_{\sigma,\lambda-\eta}[u, \varphi](t) \leq C \left( e_{\sigma,\lambda-\eta}[u, \varphi](t_*) + \int_t^{t_*} \left( \|s^{-\lambda-\eta}F^{(1)}\|^2_\delta + \sigma^2\|s^{-\lambda-\eta}\partial F^{(3)}\|^2_{\delta,t,\varphi}\right) s^{-1} ds \right) \\
+ \int_t^{t_*} \left( \int_M (\lambda - 1/2) \left( s^{-\lambda-\eta}\nu, s^{-\lambda-\eta}F^{(2)} \right) dx \right) s^{-1} ds,
\]

for a constant \( C > 0 \) which may depend on \( T, \eta \) and \( \Gamma \). This inequality holds for any smooth \( \lambda < 0 \). Since \( \eta > 0 \) is arbitrary, it also holds for \( \lambda - \eta \) replaced by \( \lambda \):

\[
e_{\sigma,\lambda}[u, \varphi](t) \leq C \left( e_{\sigma,\lambda}[u, \varphi](t_*) + \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|^2_\delta + \sigma^2\|s^{-\lambda}\partial F^{(3)}\|^2_{\delta,t,\varphi}\right) s^{-1} ds \right) \\
+ \int_t^{t_*} \left( \int_M (\lambda - 1/2) \left( s^{-\lambda}\nu, s^{-\lambda}F^{(2)} \right) dx \right) s^{-1} ds.
\]

(5.70)

For any constant \( \epsilon > 0 \), Eq. (5.40) and the Grönwall lemma imply

\[
e_{\sigma,\lambda-\epsilon}[u, \varphi](t) \leq C \left( e_{\sigma,\lambda-\epsilon}[u, \varphi](t_*) + t^{2\epsilon} \int_t^{t_*} \left( \|s^{-\lambda-\epsilon}F^{(1)}\|^2_\delta + \|s^{-\lambda-\epsilon}F^{(2)}\|^2_\delta + \sigma^2\|s^{-\lambda}\partial F^{(3)}\|^2_{\delta,t,\varphi}\right) s^{-1} ds \right).
\]

If \( \epsilon > 0 \) is sufficiently small, this estimate must also hold when \( \lambda - \epsilon \) is replaced by \( \lambda \) under the same conditions for \( \lambda \), which yields Eq. (5.38).

In the case that \( \mathcal{M} \) is negative definite on the other hand (see above), the same arguments lead to the estimate

\[
e_{\sigma,\lambda}[u, \varphi](t) \leq C \left( e_{\sigma,\lambda}[u, \varphi](t_*) + \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|^2_\delta + \sigma^2\|s^{-\lambda}\partial F^{(3)}\|^2_{\delta,t,\varphi}\right) s^{-1} ds \right) \\
+ \int_t^{t_*} \left( \int_M (\lambda + \mu_c) \left( s^{-\lambda}\nu, s^{-\lambda}F^{(2)} \right) dx \right) s^{-1} ds,
\]

(5.71)

for any \( \lambda > \lambda_c \) in Eq. (3.10) and for all \( t \in [t_*, T] \) where \( \mu_c \) is given by Eq. (5.69). The same line of arguments applied to Eq. (5.71) yields Eq. (5.36).

For any sufficiently small \( t \in (0, T] \) we find easily, applying similar arguments as before to Eq. (5.56), that

\[
t^\delta \partial_t \sqrt{e_{\lambda}[\varphi](t)} \leq C \left( \sqrt{e_{\lambda}[u](t)} + \sqrt{e_{\sigma,\lambda}[\nu](t)} + \|t^{-\lambda}F^{(3)}\|_\delta \right),
\]

so long as \( \lambda(x) \geq 0 \). Eq. (5.40) therefore implies Eq. (5.37). With the same arguments we find Eq. (5.39) for any smooth \( \lambda(x) \leq 0 \).
Proof of Lemma 5.6. In order to establish this, we apply the product estimates of the form of Prop. 3.7 in Chapter 13 of [31] together with suitable estimates for the coefficients as follows. For example, we encounter expressions like this

\[
\left(\sigma\partial_{e\cdots f}(\tilde{\alpha}t^2\gamma_{cd})\partial_{c\cdots d}b\right) \left(\sigma\partial_{e\cdots f}(\tilde{\alpha}t^2\gamma_{cd})\partial_{c\cdots d}b\right) \\
= \partial_{c\cdots d}b \left(\sigma^2\partial_{e\cdots f}(\tilde{\alpha}t^2\gamma_{cd})\partial_{e\cdots f}(\tilde{\alpha}t^2\gamma_{cd})\delta^{a_1\cdots a_m} \cdots \delta^{b_1\cdots b_n}\right) \partial_{c'\cdots d'}b'.
\]

Recall Eq. (3.7), it suffices to establish that \(\sigma^2 A^{cc'd'd'} \leq C t^{2c} \sigma t^{2c} \gamma^{cc' dd'}\) for each \(x\) and all sufficiently small \(t > 0\) for some uniform constant \(C\) (which may depend on \(T\) and \(\Gamma\)) where the tensor \(A^{cc'd'd'}(t, x)\) is interpreted as a symmetric bilinear form acting on the space of \((0, 2)\) tensors at \((t, x)\). Recalling that \(\gamma^{cc' dd'} = 1\) is bounded by Eqs. (3.6) and (3.7), that

\[
t^2 \gamma^c \delta_{de} = t^2 \gamma^c \left(\delta_{d'} + O(t^3)\right) \delta_{de},
\]

that \(\tilde{\alpha}\) satisfies Eq. (3.8) and that very similar arguments imply that terms like \(B^c = t^2 \alpha C^c - L^{1,3, c}\) have the bound \(\sigma^2 \partial_{e\cdots f}B^c \partial_{e\cdots f}B^d \leq C t^{2c} \gamma^{cc' dd'}\), it follows that

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} F^{(l)}_{[l]} \right\|_d^2 \leq C \left( \left\| t^{-\lambda} F^{(l)}_{[l]} \right\|_{H^{k}(M)}^2 + E_{k, \lambda - 2\kappa}[u, \varphi](t) + E_{k - 1, \lambda}[u, \varphi](t) + E_{k - 1, \lambda}[v](t) \right)
\]

for all \(t \in (0, T]\) where \(\kappa\) is defined in footnote 5.2. The constant \(C > 0\) may depend on \(k, T\) and \(\Gamma\). The same arguments as above lead to the estimates

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} \partial F^{(l)}_{[l]} \right\|_d^2 \leq C \left( \left\| t^{-\lambda} F^{(l)}_{[l]} \right\|_{H^{k}(M)}^2 + E_{k - 1, \lambda}[u](t) + E_{k - 1, \lambda}[v](t) + E_{k, \lambda - 2\kappa}[v](t) \right),
\]

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} \partial F^{(l)}_{[l]} \right\|_d^2 \leq C \left( \left\| t^{-\lambda} F^{(l)}_{[l]} \right\|_{H^{k}(M)}^2 + E_{k - 1, \lambda - \nu}[u](t) + E_{k - 1, \lambda}[v](t) \right),
\]

\[
\sum_{l=0}^{k} \left\| t^{-\lambda} \partial F^{(l)}_{[l]} \right\|_d^2 \leq C \left( \left\| t^{-\lambda} F^{(l)}_{[l]} \right\|_{H^{k+1}(M)}^2 + E_{k - 1, \lambda}[u](t) + E_{k - 1, \lambda}[v](t) + E_{k, \lambda - \nu - 2\kappa}[u](t) \right).
\]

This last estimate crucially depends on the fact that \(L^{3,2} - 1 = O(t^3)\), see Eqs. (3.8) and Eq. (5.52).
Now let us apply Eq. (5.40) to Eq. (5.73) using Eq. (5.8):

\[
\sum_{t=0}^{k} \left\| t^{-\lambda} F_\delta^{(2)} \right\|_\delta^2 \leq C \left( \left\| t^{-\lambda} F_\delta^{(2)} \right\|_\delta^{2,\kappa(M)} + E_{k-1,\lambda}[u](t) + E_{k,\lambda-2\kappa}[u](t) \\
+ \sum_{t=0}^{k-1} \left\| t^{-\lambda} F_\delta^{(2)} \right\|_\delta^2 + \sum_{t=0}^{k} \left\| t^{-(\lambda-2\kappa)} F_\delta^{(2)} \right\|_\delta^2 \right).
\]

Hence, provided \( t \in (0, T) \) for a sufficiently small \( T \), we can redefine the constants to find Eq. (5.42) (recall that \( \kappa \) is strictly positive by definition). The resulting constant \( C \) may depend on \( T, k \) and \( \Gamma \) as before. Plugging now Eq. (5.42) into Eqs. (5.40) yields

\[
E_{k,\lambda}[\nu](t) \leq C \left( E_{k,\lambda}[u](t) + \left\| t^{-\lambda} F_\delta^{(2)} \right\|_\delta^{2,\kappa(M)} \right).
\]

(5.76)

This can be used to rewrite Eq. (5.72) as Eq. (5.41), Eq. (5.74) as Eq. (5.43) and Eq. (5.75) as Eq. (5.44).

5.3 Proof of Proposition 5.2: Improved decay estimates

Suppose that \((u, \nu, \varphi)\) is given as in the hypothesis of Proposition 5.2. The \((0, k)\)-tensor fields associated with spatial derivatives of order \( k \) therefore satisfy Eqs. (5.54) – (5.56) with Eq. (5.55) and Eqs. (5.45) – (5.52). Plugging (5.55) into (5.54) yields

\[
-t \partial_t u + \left( L^{[1,1]} - L^{[2,2]} \frac{L^{[1,2]}_{[2,1]}}{L^{[2,1]}} \right) u \\
= F^{(1)} - \frac{L^{[1,2]}_{[2,1]}}{L^{[2,1]}} F^{(2)} - \sigma \alpha t^2 \gamma_{ab} \partial_a \partial_b \nu + \sigma (t^2 \hat{\alpha} C^a - L^{[1,3],a}_{1}) \partial_a \varphi \\
+ \sigma L^{[1,2]}_{[2,1]} t^2 \gamma_{ab} \partial_a \partial_b \nu - \sigma \left( L^{[1,2]}_{[2,1]} t^2 C^a + L^{[1,4],a}_{1} \right) \partial_a \nu.
\]

Similarly to the first part of the proof of Lemma 5.5 we project this equation onto \( t^{-2\lambda} u_{a...b} \) for any smooth function \( \lambda(x) \). This yields the identity

\[
\frac{1}{2} \partial_t [t^{-\lambda} u_{\delta}]^2 - \lambda - \left( L^{[1,1]} - L^{[2,2]} \frac{L^{[1,2]}_{[2,1]}}{L^{[2,1]}} \right) [t^{-\lambda} u_{\delta}]^2 \\
- \left( t^{-\lambda} u, t^{-\lambda} F^{(1)}_{[k]} \right)_\delta + \frac{L^{[1,2]}_{[2,1]}}{L^{[2,1]}} \left( t^{-\lambda} u, t^{-\lambda} F^{(2)}_{[k]} \right)_\delta \\
+ \sigma \alpha t^2 \gamma_{ab} \left( t^{-\lambda} u, t^{-\lambda} \partial_a \partial_b \varphi \right)_\delta - \sigma (t^2 \hat{\alpha} C^c \gamma_{cb}^{-1} - t^2 \partial_b \hat{\alpha}) \gamma_{ab} \left( t^{-\lambda} u, t^{-\lambda} \partial_a \varphi \right)_\delta \\
- \sigma L^{[1,2]}_{[2,1]} t^2 \gamma_{ab} \left( t^{-\lambda} u, t^{-\lambda} \partial_a \partial_b \nu \right)_\delta + \sigma \left( L^{[1,2]}_{[2,1]} t^2 C^c \gamma_{cb}^{-1} + t^2 \partial_b \hat{\phi} \right) \gamma_{ab} \left( t^{-\lambda} u, t^{-\lambda} \partial_a \nu \right)_\delta.
\]
From Eqs. (5.77) – (5.60) we get

\[
L^{[1,1]} - L^{[2,2]} \frac{L^{[1,2]}}{L^{[2,1]}} = 2A^2 + O(t^\beta) + O(t^{2\kappa}), \quad \frac{L^{[1,2]}}{L^{[2,1]}} = -A + O(t^\beta) + O(t^{2\kappa}),
\]

where \( \kappa \) is defined in footnote 5.2. If \( \lambda < 2A^2 \), this implies that

\[
t\partial_t \sqrt{e^\delta|u|_k}(t) \leq -C \left( \|t^{-\lambda}F^{(1)}_k\|_\delta + \|t^{-\lambda}F^{(2)}_k\|_\delta + \sqrt{e\lambda - 2\kappa [\varphi_{k+2}](t)} + \sqrt{e\lambda - 2\kappa [\varphi_{k+1}](t)} \right)
\]

where \( C > 0 \) may depend on \( T \) and \( \Gamma \). We replace \( k \) by \( l \) and sum from this estimate from \( 0 \) to \( k \) and restrict to \( k \geq 1 \) (the case \( k = 0 \) follows with similar, but simpler arguments). The first term on the right-hand side can then be estimated using the following alternative to Eq. (5.77), which provides sharper decay control at the cost of weaker regularity control:

\[
\sum_{l=0}^{k} \|t^{-\lambda}F^{(1)}_l\|_\delta^2 \leq C \left( \sigma^2 \|t^{-\lambda}F^{(1)}\|\|M_{k+1} - 2\kappa [\varphi](t) + E_{k-1,\lambda - \beta}[u](t) + E_{k-1,\lambda}[\nu](t) \right).
\]

The second term on the right-hand side above can be estimated directly with Eqs. (5.42) and all energies involving \( \nu \) with Eq. (5.15). If we now tighten our assumption on \( \lambda \) so that \( \lambda < \min\{2\kappa, 2A^2\} \) and pick any sufficiently small \( \varepsilon > 0 \), we can use Eq. (5.14) to show that

\[
t\partial_t \sqrt{E_{k,\lambda}[u]}(t) \geq -C \left( \|t^{-\lambda}F^{(1)}\|\|M_{h^k}(M) + \|t^{-\lambda}F^{(2)}\|\|M_{h^k}(M) + \|t^{-\lambda+2\kappa}F^{(2)}\|\|M_{h^{k+2}}(M) \right.
\]

\[
+ t^\varepsilon \int_{t}^{t^*} \left( \|s^{-\lambda+2\kappa}F^{(2)}\|\|M_{h^{k+2}}(M) + \|s^{-\lambda+2\kappa}F^{(3)}\|\|M_{h^{k+3}}(M) \right)s^{-1-\varepsilon} ds
\]

\[
+ t^\varepsilon \sqrt{E_{k+2,\lambda-2\kappa}[\varphi](t)} + \sqrt{E_{k-1,\lambda}[u](t)} + \sqrt{E_{k,\lambda}[\nu](t)} \right)
\]

\[
\leq Ct^{2\kappa} \sup_{s \in [t, t^*]} \sqrt{E_{k+2,\sigma,\lambda - 2\kappa + 2}[u, \varphi](s)}
\]

\[
+ t^\varepsilon \int_{t}^{t^*} \sqrt{E_{k+2,\lambda-2\kappa+2}[u](s)} s^{-1+\varepsilon} ds \leq Ct^{\varepsilon} \sup_{s \in [t, t^*]} \sqrt{E_{k+2,\sigma,\lambda - 2\kappa + 2}[u, \varphi](s)}
\]

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for all \( t \in (0, t_*) \). The constant \( C > 0 \) here may depend on \( T, k, \lambda \) and \( \Gamma \). Hence
\[
\sqrt{E_{k, \lambda}[u](t)} \leq C \left( \sqrt{E_{k, \lambda}[u](t_*)} + \sqrt{E_{k+2, \lambda-2\kappa+\epsilon}[\varphi](t_*)} + \sup_{s \in (0, T)} \sqrt{E_{k+2, \sigma, \lambda-2\kappa+2\epsilon}[u, \varphi](s)} \right. \\
\left. + \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|_{\delta, H^{k}(M)} + \|s^{-\lambda}F^{(2)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda+2\kappa-\epsilon}F^{(3)}\|_{\delta, H^{k+3}(M)} \right) s^{-1} ds \right) + \sup_{s \in [t, t_*)} \sqrt{E_{k-1, \lambda+\epsilon}[u](s)}.
\]

The constant \( C > 0 \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \). Notice that the last term is not present if \( k = 0 \). If \( k = 1 \), we can combine this estimate with the \( k = 0 \) estimate to establish the result. If \( k \geq 2 \), we proceed inductively in a manner very similar to proof of Proposition 5.1 and find
\[
\sqrt{E_{k, \lambda}[u](t)} \leq C \left( \sqrt{E_{k, \lambda+\epsilon}[u](t_*)} + \sqrt{E_{k+2, \lambda-2\kappa+\epsilon}[\varphi](t_*)} + \sup_{s \in (0, T)} \sqrt{E_{k+2, \sigma, \lambda-2\kappa+2\epsilon}[u, \varphi](s)} \right. \\
\left. + \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|_{\delta, H^{k}(M)} + \|s^{-\lambda}F^{(2)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda+2\kappa-\epsilon}F^{(3)}\|_{\delta, H^{k+3}(M)} \right) s^{-1} ds \right).
\]

The constant \( C > 0 \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \). In order to be able to apply Eq. (5.13) we first need to square this estimate. In general, the Hölder inequality implies the existence of a constant \( C > 0 \), which depends on \( T \) and \( 2\eta_1 - \eta_2 \) such that for any smooth function \( f(t) \) and any constant \( \eta_1 \in \mathbb{R} \)
\[
\left( \int_t^{t_*} f(s)s^{-1+\eta_1} ds \right)^2 \leq C \int_t^{t_*} f^2(s)s^{-1+\eta_2} ds
\]
provided \( \eta_2 < 2\eta_1 \). Here we pick \( \eta_1 = -\epsilon \) and \( \eta_2 = -3\epsilon \) and apply Eq. (5.13) to eventually find:
\[
E_{k, \lambda}[u](t) \leq C \left( E_{k, \lambda+\epsilon}[u](t_*) + E_{k+2, \lambda-2\kappa+\epsilon}[\varphi](t_*) + E_{k+2, \sigma, \lambda-2\kappa+3\epsilon}[u, \varphi](t_*) \right. \\
\left. + \int_t^{t_*} \left( \|s^{-\lambda}F^{(1)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda}F^{(2)}\|_{\delta, H^{k+2}(M)} + \|s^{-\lambda+2\kappa-\epsilon}F^{(3)}\|_{\delta, H^{k+3}(M)} \right) s^{-1-3\epsilon} ds \right).
\]

The constant \( C > 0 \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \) as before. We can now choose \( t_* = T \), replace the energies on the right-hand side by norms of Cauchy data imposed at \( t = T \), and, replace \( 3\epsilon \) by \( 2\epsilon \). This yields Eq. (5.16) and completes the proof of Proposition 5.2.

5.4 Proofs of Proposition 5.3 and 5.4: The asymptotic matching problem of the lapse-scalar field system

Proof of Proposition 5.3. Considering \( \sigma \in I \), \( f^{(1)} \), \( f^{(2)} \) and \( f^{(3)} \) as fixed, it is useful to rephrase the problem addressed by this proposition as the following asymptotic matching
problem in the sense of Definition 4.1:

1. Let \((u, \nu, \varphi)\) be the smooth solution of the Cauchy problem of Eqs. (3.1) – (3.3) determined by smooth Cauchy data \((u_*, \varphi_*)\) imposed at \(t = T\).

2. The functions \(u\) and \(\nu\) found in step 1 then determine the equation

\[
t \partial_t \varphi^{(0)} = \sigma^{(0)}(u + A\nu)
\]

derived from Eq. (3.3) for a “new” switch parameter \(\sigma^{(0)} \in [0,1]\) and a “new” unknown \(\varphi^{(0)}\) (which does in general not equal the given function \(\varphi\) found in step 1).

Each choice of smooth \((u_*, \varphi_*)\) therefore determines the asymptotic matching problem for Eq. (5.79). Here we are clearly only interested in the case \(\sigma^{(0)}_1 = 1\) and \(\sigma^{(0)}_2 = 0\). We notice that (5.79) is an equation of the form Eq. (4.8) with

\[
A = t \partial_t, \quad L = 0, \quad F = \sigma^{(0)}(u + A\nu).
\]

In Eqs. (4.9) – (4.14) we then assume

\[
F_1 = u + A\nu, \quad F_2 = 0, \quad \tilde{F}_1 = \tilde{u} + A\tilde{\nu}, \quad \tilde{F}_2 = 0,
\]

where \((\tilde{u}, \tilde{\nu}, \tilde{\varphi})\) is the smooth solution of the Cauchy problem of Eqs. (3.1) – (3.3) determined by any (possibly different) Cauchy data \((\tilde{u}_*, \tilde{\varphi}_*)\).

Let us suppose for the moment that we can indeed establish the well-posedness of this asymptotic matching problem in the case \(\sigma^{(0)}_1 = 1\) and \(\sigma^{(0)}_2 = 0\). The map \(\Psi^{(\text{pre})}\), whose existence is asserted in Proposition 5.3, is then related to the asymptotic matching map associated with Eq. (5.79), which we refer to as \(\Psi^{(0)}_{1\to 0}\), by

\[
\Psi^{(\text{pre})}(u_*, \varphi_*) = \Psi^{(0)}_{1\to 0}(\varphi^{(0)}_*, u_*, \varphi_*)\big|_{\varphi^{(0)}_* = \varphi_*}.
\]

Notice that here and in all of what follows we make the implicit dependence of \(\Psi^{(0)}_{1\to 0}\) on \(u_*\) and \(\varphi_*\) via \(u\) and \(\nu\) in Eq. (5.79) explicit by adding these as arguments to the map.

Given \((u_*, \varphi_*)\) as above now, the first step is to consider the corresponding finite matching problem with the corresponding finite matching map

\[
\psi^{(0)}_{1\to 0} : (0, T] \times (C^\infty(M))^3 \to C^\infty(M), \quad (\tau, \varphi^{(0)}_*, u_*, \varphi_*) \mapsto \psi^{(0)}_{1\to 0}(\tau, \varphi^{(0)}_*, u_*, \varphi_*).
\]

According to the general discussion above, the plan is to find estimates for the hierarchy of Cauchy problems Eqs. (4.9), (4.14), (4.12) and (4.13) for Eqs. (5.80) and (5.81) and arbitrary \(\tau, \varphi^{(0)}_*, u_*\) and \(\varphi_*\) as above. As before, we first require estimates for two the smooth solutions \((u, \nu, \varphi)\) and \((\tilde{u}, \tilde{\nu}, \tilde{\varphi})\) of the Cauchy problem of Eqs. (4.9) and (4.14) for Eq. (5.1) – (5.3) with the source term \(F = \tilde{F} = (f^{(1)}, f^{(2)}, f^{(3)})\) and the switch parameter \(\sigma \in I\) now writing

\[
w^{(0)} = \varphi^{(1)}_0 - \varphi^{(2)}_0.
\]
Picking any smooth \( \lambda < \min \{2\kappa, 2A^2\} \) and sufficiently small constant \( \epsilon > 0 \), Eq. (5.16) is the required estimate for the function \( u \), and yields the required estimate for \( u - \tilde{u} \):

\[
E_{k,\lambda}[u - \tilde{u}](t) \leq C \left( \|u_s - \tilde{u}_s\|^2_{\delta,H^{k+2} (M)} + \|\varphi_s - \tilde{\varphi}_s\|^2_{\delta,H^{k+3} (M)} \right)
\]  

(5.84)

for all \( t \in (0, T) \). The constants \( C \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \).

Now, first we observe that Eq. (5.12) allows us to estimate the Cauchy problems Eqs. (4.12) and (4.13) for Eqs. (5.80) and (5.81) for any smooth \( \lambda > 0 \) when, respectively, (1), \( \varphi \) is replaced by \( u^{(0)} \), \( u \) by \( \sigma^{(0)}u \), and \( F^{(2)} = \sigma f^{(2)} \) and \( F^{(3)} = 0 \), and, (2), \( \varphi \) is replaced by \( u^{(0)} - \tilde{u}^{(0)} \), \( u \) by \( \sigma^{(0)}(u - \tilde{u}) \) and \( F^{(2)} = F^{(3)} = 0 \). For \( t_* = \tau \) and \( t_* = \tilde{\tau} \) (respectively) we find imposing the initial conditions in Eqs. (4.12) and (4.13) (and assuming without loss of generality \( \tau \leq \tilde{\tau} \)):

\[
\sqrt{E_{k,\lambda}\left[u^{(0)}\right](t)} \leq C \int_{\tau}^{t} \left( \sqrt{E_{k,\lambda}\left[\tilde{u}\right](s)} + \|s^{-\lambda} f^{(2)}\|_{\delta,H^{k} (M)} \right) s^{-1} ds
\]

(5.85)

and

\[
\sqrt{E_{k,\lambda}\left[u^{(0)} - \tilde{u}^{(0)}\right](t)} \leq C \left( \sqrt{E_{k,\lambda}\left[u^{(0)}\right](\tilde{\tau})} + \int_{\tau}^{t} \sqrt{E_{k,\lambda}\left[u - \tilde{u}\right](s)} s^{-1} ds \right)
\]

(5.86)

for all \( t \in [\tau, T] \) and \( t \in [\tilde{\tau}, T] \), respectively. The constants \( C \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \). Plugging Eq. (5.85) into the last estimate with \( t = \tilde{\tau} \) and applying Eq. (5.78) with \( \eta_1 = 0 \) and \( \eta_2 = -2\epsilon \) generates the following result

\[
E_{k,\lambda}\left[u^{(0)} - \tilde{u}^{(0)}\right](t) \leq C \left( (\tilde{\tau}^{2\kappa} - \tau^{2\kappa}) \sup_{s \in [\tau, \tilde{\tau}]} E_{k,\lambda+2\kappa}\left[u\right](s) + \sup_{s \in [\tilde{\tau}, T]} E_{k,\lambda+2\kappa}\left[u - \tilde{u}\right](s) \right.
\]

\[
+ \left. \int_{\tau}^{\tilde{\tau}} \|s^{-\lambda} f^{(2)}\|^2_{\delta,H^{k} (M)} \right) s^{-1-2\epsilon} ds \right).
\]

The constants \( C \) here may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \). Using now Eq. (5.13) and (5.84) assuming that \( 0 < \lambda < \min \{\lambda_1, 2\kappa, 2A^2\} \) with \( \lambda_1 \) as given in the hypothesis:

\[
E_{k,\lambda}\left[u^{(0)} - \tilde{u}^{(0)}\right](t) \leq C \left( (\tilde{\tau}^{2\kappa} - \tau^{2\kappa}) \left( \|u_s\|^2_{\delta,H^{k+2} (M)} + \|\varphi_s\|^2_{\delta,H^{k+3} (M)} \right) \right.
\]

\[
+ \left. \int_{0}^{T} \left( \|s^{-\lambda} f^{(1)}\|^2_{\delta,H^{k+2} (M)} + \|s^{-\lambda} f^{(2)}\|^2_{\delta,H^{k+2} (M)} + \|s^{-\lambda+2\kappa-\epsilon} f^{(3)}\|^2_{\delta,H^{k+3} (M)} \right) s^{-1-6\epsilon} ds \right)
\]

\[
+ \|u_s - \tilde{u}_s\|^2_{\delta,H^{k+2} (M)} + \|\varphi_s - \tilde{\varphi}_s\|^2_{\delta,H^{k+3} (M)} + \int_{\tau}^{\tilde{\tau}} \|s^{-\lambda} f^{(2)}\|^2_{\delta,H^{k} (M)} s^{-1-2\epsilon} ds \right)
\]

provided \( \epsilon > 0 \) is sufficiently small.
When an energy needs to be replaced by a norm, the following estimate becomes handy. For any smooth function $u(t,x)$ and any smooth function $\lambda(x)$ we have

$$
\|t^{-\lambda} u(t,\cdot)\|_{H^k(M)}^2 = \sum_{l=0}^k \|\partial^l (t^{-\lambda} u(t,\cdot))\|_{S}^2
$$

$$
\leq \sum_{l=0}^k \sum_{m=0}^l C_{k,l,m} \|t^{-\lambda} \partial^l u(t,\cdot)\|_{S}^2 \|t^\lambda \partial^{l-m}\|_{S} \leq C t^{-2\epsilon} \sum_{l=0}^k \|t^{-\lambda} \partial^l u(t,\cdot)\|_{S}^2
$$

$$
= CE_{k,\lambda,\epsilon}[u](t),
$$

(5.86)

for all $t \in (0,T)$ for any $\epsilon > 0$, where the constant $C > 0$ may depend on $k$ and $\lambda$.

Applying this to the previous estimate and using the fact that all solutions $\varphi_2^{(0)}$ of the $\sigma^{(0)} = \sigma^{(0)}_2$ is the $0$-version of Eq. (5.79) are constant in time we have found

$$
\left\| t^{-\lambda} \left[ (\varphi_1^{(0)}(t,\cdot) - \psi_1^{(0)}(\tau, \varphi_1^{(0)}(s), u_s, \varphi_s)) - (\tilde{\varphi}_1^{(0)}(t,\cdot) - \psi_1^{(0)}(\tilde{\tau}, \tilde{\varphi}_1^{(0)}, \tilde{u}_s, \tilde{\varphi}_s)) \right]\right\|^2_{H^k(M)}
$$

$$
\leq C \left( \tau^{2\epsilon} - \tau^{2\epsilon} \right) \left( \|u_s\|^2_{H^{k+2}(M)} + \|\varphi_s\|^2_{H^{k+3}(M)} \right) + \int_0^T \left( \|s^{-\lambda} f^{(1)}\|^2_{H^{k+2}(M)} + \|s^{-\lambda} f^{(2)}\|^2_{H^{k+3}(M)} + \|s^{-\lambda+2\epsilon-\epsilon f^{(3)}}\|^2_{H^{k+3}(M)} \right) s^{-1-8\epsilon} ds
$$

$$
+ \|u_s - \tilde{u}_s\|^2_{H^{k+2}(M)} + \|\varphi_s - \tilde{\varphi}_s\|^2_{H^{k+3}(M)} + \int_\tau^\tau \|s^{-\lambda} f^{(2)}\|^2_{H^{k+3}(M)} s^{-1-4\epsilon} ds,
$$

(5.87)

where $C$ here may depend on $T$, $k$, $\lambda$, $\epsilon$ and $\Gamma$. A uniform continuity estimate for $\psi_1^{(0)}(\tau, \varphi_1^{(0)}(s), u_s, \varphi_s) - \psi_1^{(0)}(\tilde{\tau}, \tilde{\varphi}_1^{(0)}, \tilde{u}_s, \tilde{\varphi}_s)$ follows directly by setting $t = T$ and rearranging.

The finite matching map Eq. (5.83) thus has a unique continuous extension to the domain $[0,T] \times H^k(M) \times H^{k+2}(M) \times H^{k+3}(M)$ and co-domain $H^k(M)$. When this extended map is evaluated at $\tau = 0$ we obtain the map $\Psi_{k,0}^{(0)} : H^k(M) \times H^{k+2}(M) \times H^{k+3}(M) \rightarrow H^k(M)$, which we shall claim to be the asymptotic matching map. The continuity property of this map follows from Eq. (5.87) by setting $\tau = \tilde{\tau} = 0$. In general this map clearly depends on $k$. However, a standard argument shows that the restriction of this map to the dense sub-domain $(C^\infty(M))^3$ does not depend on the choice of $k$.

This restriction is referred to as $\Psi_{k,0}^{(0)} : (C^\infty(M))^3 \rightarrow C^\infty(M)$. In order to prove that this is is asymptotic matching map of interest we need to establish Eq. (4.2) for some choice of norm. Here, however, we are rather interested in the related map $\Psi^{(pre)}$ given by Eq. (5.82). The analogue of Eq. (4.2) is Eq. (5.17) which we establish now by reconsidering Eq. (5.85) with $\tau = 0$ and Eq. (5.78) with $\eta_1 = 0$ and $\eta_2 = -2\epsilon$ and
combining this with Eq. (5.16):

\[ E_k,\lambda[w^{(0)}](t) \leq C \left( \|u_*\|^2_{L^2(M)} + \|\varphi_*\|^2_{H^1(M)} \right) \]

\[ + \int_0^T \left( \|s^{-\lambda}f^{(1)}\|^2_{L^2(M)} + \|s^{-\lambda}f^{(2)}\|^2_{L^2(M)} + \|s^{-\lambda+2\kappa-\epsilon}f^{(3)}\|^2_{L^2(M)} \right) s^{1-6\epsilon} ds, \]

provided \( 0 < \lambda < \min\{\lambda_1, 2\kappa, 2A^2\} \). Having replaced the energy on the left-hand side by a norm following Eq. (5.86), we can we rephrase the conditions for \( \lambda \) in a more useful way exploiting that \( T \in (0,1] \). It is clear that this inequality holds also when \( \lambda \) on the right-hand side is replaced by any smooth exponent \( \tilde{\lambda} > 0 \); in order to guarantee that the right-hand side is finite we should demand that \( 0 < \tilde{\lambda} < \lambda_1 \). The exponent \( \lambda \) on the left-hand side then only needs to satisfy the upper bound \( \lambda < \min\{\tilde{\lambda}, 2\kappa, 2A^2\} \). This leads to Eq. (5.17). Eq. (5.18) follows directly from Eq. (5.87) by setting \( \tau = \tilde{\tau} \) noticing that that inequality holds for any smooth \( \lambda < \min\{2\kappa, 2A^2\} \).

Last but not least we notice that the resulting map is clearly uniquely determined by Eq. (5.17) given that all solutions of the \( \sigma^{(0)} = 0 \)-version of Eq. (5.79) are constant. □

**Proof of Proposition 5.4.** We notice that Eq. (4.8) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2), and, Eq. (5.21) and Eq. (5.23), respectively, are of the form Eqs. (4.9) and (4.10) with

\[
F_1 = \left( f^{(1)} + (\sigma_1 - K) \left( \partial t^2 \alpha \partial_a \partial_b \varphi_1(0) - \left( t^2 \partial t^2 - t^2 \alpha \partial_t \partial_a \partial_b \right) \partial_a \varphi_1(0) \right), f^{(2)}, f^{(3)} \right),
\]

\[
F_2 = \left( f^{(1)} + (\sigma_2 - K) \left( \partial t^2 \alpha \partial_a \partial_b \varphi_1(0) - \left( t^2 \partial t^2 - t^2 \alpha \partial_t \partial_a \partial_b \right) \partial_a \varphi_1(0) \right), f^{(2)}, f^{(3)} \right).
\]

In Eqs. (4.9) – (4.14) we then assume that

\[
\tilde{F}_1 = \left( f^{(1)} + (\sigma_1 - K) \left( \partial t^2 \alpha \partial_a \partial_b \tilde{\varphi}_1(0) - \left( t^2 \partial t^2 - t^2 \alpha \partial_t \partial_a \partial_b \right) \partial_a \tilde{\varphi}_1(0) \right), f^{(2)}, f^{(3)} \right),
\]

\[
\tilde{F}_2 = \left( f^{(1)} + (\sigma_2 - K) \left( \partial t^2 \alpha \partial_a \partial_b \tilde{\varphi}_1(0) - \left( t^2 \partial t^2 - t^2 \alpha \partial_t \partial_a \partial_b \right) \partial_a \tilde{\varphi}_1(0) \right), f^{(2)}, f^{(3)} \right),
\]

where \( \varphi_1(0) \) and \( \tilde{\varphi}_1(0) \) are defined as in the proposition. It is useful to write \( L \) in terms of the coefficients Eqs. (5.45) – (5.51), see Eqs. (5.54) – (5.56).

Under the given hypothesis the finite matching map (analogous to the one defined in Definition 4.2) is taken to be of the type

\[ \psi_{\sigma_1 \to \sigma_2} : (0,T] \times (C^\infty(M))^2 \to (C^\infty(M))^2, \quad (\tau, u_{1x}, \varphi_{1x}) \mapsto (u_{2x}, \varphi_{2x}), \]

which is certainly well-defined. According to the general strategy, the plan is now to analyze the hierarchy of Cauchy problems Eqs. (4.9), (4.14), (4.12) and (4.13). We write

\[ w^{(1)} = u_1 - u_2, \quad w^{(2)} = \varphi_1 - \varphi_2, \quad w^{(3)} = \nu_1 - \nu_2. \]

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Given Eqs. (5.19) and (5.20), which can be interpreted as estimates for the Cauchy problems (4.9) and (4.14), let us proceed with Eqs. (4.12) and (4.13). The source terms of Eqs. (4.12) and Eq. (4.13) take the form

\[ F_3 = (\sigma_2 - \sigma_1) \left( \partial_t \gamma^{ab} \partial_a \partial_b (\varphi_1 - \varphi_1(0)) + \left( t^2 \partial_t C^a - t^2 \gamma^{ab} \partial_b \alpha \right) \partial_a (\varphi_1 - \varphi_1(0)) \right. \\
+ L_{\delta}^{1,1} u_1 + L_{\delta}^{1,2} \nu_1 + L_{\delta}^{1,4,b} \partial_b \nu_1, \\
\left. - t^2 \gamma^{ab} \partial_a \partial_b \nu_1 - t^2 C^a \partial_a \nu_1 + L_{\delta}^{2,1} \nu_1 + L_{\delta}^{2,2} u_1, \quad L_{\delta}^{3,1} \nu_1 + L_{\delta}^{3,2} u_1 \right), \]

\[ F_4 = (\sigma_2 - \sigma_1) \left( \partial_t \gamma^{ab} \partial_a \partial_b ((\varphi_1 - \tilde{\varphi}_1) + (\varphi_1(0) - \tilde{\varphi}_1(0))) \right. \\
+ \left( t^2 \partial_t C^a - t^2 \gamma^{ab} \partial_b \alpha \right) \partial_a ((\varphi_1 - \tilde{\varphi}_1) + (\varphi_1(0) - \tilde{\varphi}_1(0))) \\
+ L_{\delta}^{1,1} (u_1 - \tilde{u}_1) + L_{\delta}^{1,2} (\nu_1 - \tilde{\nu}_1) + L_{\delta}^{1,4,b} \partial_b (\nu_1 - \tilde{\nu}_1), \\
\left. - t^2 \gamma^{ab} \partial_a \partial_b (\nu_1 - \tilde{\nu}_1) - t^2 C^a \partial_a (\nu_1 - \tilde{\nu}_1) + L_{\delta}^{2,1} (\nu_1 - \tilde{\nu}_1) + L_{\delta}^{2,2} (u_1 - \tilde{u}_1), \\
L_{\delta}^{3,1} (\nu_1 - \tilde{\nu}_1) + L_{\delta}^{3,2} (u_1 - \tilde{u}_1) \right). \]

The first aim is now to estimate these \( F_3 \) and \( F_4 \). This is straightforward. For example,

\[
\left\| t^{-\lambda} \partial_t^2 \gamma^{ab} \partial_a \partial_b (\varphi_1 - \varphi_1(0)) \right\|^2_{\delta, H^k (M)} \leq C \left( \left\| u_1 \right\|^2_{\delta, H^{k+\kappa}(M)} + \left\| \varphi_1 \right\|^2_{\delta, H^{k+\kappa}(M)} \\
+ \int_0^T \left( \left\| s^{-\lambda_1} f^{(1)} \right\|^2_{\delta, H^{k+\kappa}(M)} + \left\| s^{-\lambda_1} f^{(2)} \right\|^2_{\delta, H^{k+\kappa}(M)} + \left\| s^{-\lambda_1 + 2\kappa} f^{(3)} \right\|^2_{\delta, H^{k+\kappa}(M)} \right) s^{-1} \, ds \right)
\]

using the multiplication property of Sobolev-regular functions (see for example Proposition 2.3 in Appendix I of [15]), and, using Eq. (5.19) for any smooth

\[
\lambda < 2\kappa + \min\{\lambda_1, 2(1 - q_{\text{max}}), 2A^2\}, \tag{5.89}
\]

where \( \kappa \) is defined in footnote 5.2. Under the same assumptions we find

\[
\left\| t^{-\lambda} \partial_t C^a \partial_a (\varphi_1 - \varphi_1(0)) \right\|^2_{\delta, H^k (M)} \leq C \left\| t^{-\lambda + 2\kappa - \epsilon} (\varphi_1 - \varphi_1(0)) \right\|^2_{\delta, H^{k+\kappa}(M)},
\]

using Eq. (5.53) which can therefore be estimated by the same expression as above. The same expression above also bounds the term \( t^2 \gamma^{ab} \partial_a \alpha \partial_b (\varphi_1 - \varphi_1(0)) \) (we could give a better bound for this term taking into account that \( \partial_a \alpha = O(t^3) \)). Exploiting the known asymptotics of the coefficients, Eq. (5.86) and Eq. (5.15) with \( F_1^{(2)} = f^{(2)} \), we find in the same way that

\[
\left\| t^{-\lambda} \left( L_{\delta}^{1,1} u_1 + L_{\delta}^{1,2} \nu_1 + L_{\delta}^{1,4,b} \partial_b \nu_1 \right) \right\|^2_{\delta, H^k (M)} \\
\leq C \left( E_{k+1, \lambda-\mu+2\kappa} [u_1(t)] + \left\| t^{-\lambda + \mu - 2\kappa} f^{(2)} \right\|^2_{\delta, H^{k+1}(M)} \right).
\]

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for any smooth \( \lambda(x) \) where \( \mu(x) \) is any smooth function\(^5\) on \( M \) that satisfies

\[
\mu \in (0, \min \{\beta, 2\kappa\}).
\] (5.90)

We also find for any smooth \( \lambda \)

\[
\left\| t^{-\lambda} \left( -t^{2}\gamma^{ab} \partial_{a} \partial_{b} \nu_{1} + t^{2} C^{a} \partial_{a} \nu_{1} - L_{1}^{[2,1]} \nu_{1} - L_{1}^{[2,2]} u_{1} \right) \right\|_{\delta,H^{k}(M)}^{2}
\leq C \left( E_{k+2,\lambda-\mu}[u_{1}](t) + \|s^{-\lambda+\mu} f^{(2)}\|_{\delta,H^{k+2}(M)} \right),
\]

and

\[
\left\| t^{-\lambda} \left( L_{1}^{[3,1]} \nu_{1} + L_{1}^{[3,2]} u_{1} \right) \right\|_{\delta,H^{k}(M)}^{2} \leq C \left( E_{k,\lambda-\mu}[u_{1}](t) + \|s^{-\lambda+\mu} f^{(2)}\|_{\delta,H^{k}(M)} \right).
\]

All the constants \( C > 0 \) in these estimates may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \). Putting all of these together yields

\[
\left\| t^{-\lambda} F_{3}^{(1)} \right\|_{\delta,H^{k}(M)}^{2} + \left\| t^{-\lambda} F_{3}^{(2)} \right\|_{\delta,H^{k}(M)}^{2} + \left\| t^{-\lambda} F_{3}^{(3)} \right\|_{\delta,H^{k+1}(M)}^{2}
\leq C \left( E_{k+2,\lambda-\mu+2\epsilon}[u_{1}](t) + \|u_{1} \|_{\delta,H^{k+4+4\epsilon}(M)} + \|\varphi_{1} \|_{\delta,H^{k+4+4\epsilon}(M)}
+ \int_{0}^{T} \left( \|s^{-\lambda_{1}} f^{(1)}\|_{\delta,H^{k+4}(M)}^{2} + \|s^{-\lambda_{1}} f^{(2)}\|_{\delta,H^{k+4}(M)}^{2} + \|s^{-\lambda_{1}+2\epsilon} f^{(3)}\|_{\delta,H^{k+5}(M)}^{2} \right) s^{-1} ds \right),
\] (5.91)

and

\[
\left\| t^{-\lambda} F_{4}^{(1)} \right\|_{\delta,H^{k}(M)}^{2} + \left\| t^{-\lambda} F_{4}^{(2)} \right\|_{\delta,H^{k}(M)}^{2} + \left\| t^{-\lambda} F_{4}^{(3)} \right\|_{\delta,H^{k+1}(M)}^{2}
\leq C \left( E_{k+2,\lambda-\mu+2\epsilon}[u_{1} - \bar{u}_{1}](t) + \|u_{1} \|_{\delta,H^{k+4+4\epsilon}(M)} + \|\varphi_{1} - \varphi_{1} \|_{\delta,H^{k+4+4\epsilon}(M)} \right). \] (5.92)

Both these estimates require Eqs. (5.89) and (5.90). The constants \( C > 0 \) in these estimates may depend on \( T, k, \lambda, \epsilon \) and \( \Gamma \).

Before we proceed now, let us note the following useful estimate obtained by combining Eq. (5.12) with Eqs. (5.66) and (5.78) for \( \eta_{1} = 0 \) and \( \eta_{2} = -2\epsilon \). Given any smooth solution of Eq. (4.8) with \( A \) and \( L \) given by Eqs. (5.1) and (5.2) for an arbitrary \( \sigma \in [0,1] \) and source term (i.e., this is not restricted to the special setting of this proof), an arbitrary smooth \( \lambda > 0 \) and an arbitrary integer \( k \geq 0 \), then

\[
\left\| t^{-\lambda} u(t, \cdot) \right\|_{\delta,H^{k}(M)}^{2} + \left\| t^{-\lambda} \varphi(t, \cdot) \right\|_{\delta,H^{k}(M)}^{2} \leq C \left( E_{k,\lambda+\epsilon}[\varphi](t_{*}) + \sup_{s \in [t_{*}, t]} E_{k,\lambda+3\epsilon}[u](s)
+ \int_{t_{*}}^{t} \left( \|s^{-\lambda} F^{(2)}\|_{\delta,H^{k}(M)}^{2} + \|s^{-\lambda} F^{(3)}\|_{\delta,H^{k}(M)}^{2} \right) s^{-1-4\epsilon} ds \right),
\] (5.93)

\(^{5,6}\)The function \( \mu \) here is unrelated to the function \( \mu \) introduced in Eq. (5.66).
for all $t \in [t_*, T]$. The constant $C$ here may depend on $T$, $k$, $\lambda$, $\epsilon$ and $\Gamma$.

Now, picking any smooth
\[
\lambda > \lambda_c,
\]
where $\lambda_c$ is defined in Eq. (3.10), and setting $t_* = \tau$ and $t_* = \hat{\tau}$ (respectively), Eq. (5.11) applied to the two Cauchy problems Eqs. (4.12) and (4.13) yields (assuming $\tau \leq \hat{\tau}$)

\[
E_{k, \sigma_2, \lambda}[w^{(1)}, w^{(2)}](t)
\leq C \int_\tau^T \left( \| s^{-\lambda} F_3^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_3^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1+2\epsilon} ds,
\]

and

\[
E_{k, \sigma_2, \lambda}[w^{(1)} - \tilde{w}^{(1)}, w^{(2)} - \tilde{w}^{(2)}](t)
\leq C \left( \int_\tau^{\hat{\tau}} \left( \| s^{-\lambda} F_3^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_3^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1+2\epsilon} ds
+ \int_\tau^T \left( \| s^{-\lambda} F_4^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1+2\epsilon} ds \right),
\]

for all $t \in [\tau, T]$ and $t \in [\hat{\tau}, T]$. Combining these now with Eq. (5.93) yields

\[
\left\| t^{-\lambda} w^{(1)}(t, \cdot) \right\|^2_{\delta, H^k(M)} + \left\| t^{-\lambda} w^{(2)}(t, \cdot) \right\|^2_{\delta, H^k(M)}
\leq C \int_\tau^T \left( \| s^{-\lambda} F_3^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_3^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1-4\epsilon} ds,
\]

provided Eq. (5.94) holds (notice that $\lambda_c > 0$; see Eq. (3.10)), and

\[
\left\| t^{-\lambda} (w^{(1)}(t, \cdot) - \tilde{w}^{(1)}(t, \cdot)) \right\|^2_{\delta, H^k(M)} + \left\| t^{-\lambda} (w^{(2)}(t, \cdot) - \tilde{w}^{(2)}(t, \cdot)) \right\|^2_{\delta, H^k(M)}
\leq C \left( \left( \tau^{2\epsilon} - \hat{\tau}^{2\epsilon} \right) \int_\tau^{\hat{\tau}} \left( \| s^{-\lambda} F_3^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_3^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1+4\epsilon} ds
+ \int_\tau^{\hat{\tau}} \left( \| s^{-\lambda} F_4^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1-4\epsilon} ds
+ \int_\tau^T \left( \| s^{-\lambda} F_4^{(1)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(2)} \|^2_{\delta, H^k(M)} + \| s^{-\lambda} F_4^{(3)} \|^2_{\delta, H^{k+1}(M)} \right) s^{-1+4\epsilon} ds \right).
\]

In both cases, the constant $C$ may depend on $T$, $k$, $\lambda$, $\epsilon$ and $\Gamma$. We wish to combine these estimates with Eqs. (5.91) and (5.92) next which leads to the restrictions Eq. (5.90) and

\[
\lambda_c < \lambda < 2\kappa + \min\{\lambda_1, 2(1 - q_{max}), 2A^2\}.
\]

Moreover, we wish to use Eq. (5.20) to estimate the remaining energies under the condition that

\[
\lambda < \mu + \min\{\lambda_1, 2\kappa, 2A^2\}.
\]

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This implies that Eqs. (3.14), (3.16) and (3.15) must hold. Continuing to assume that \( \epsilon > 0 \) is sufficiently small (and replacing multiples of \( \epsilon \) by smaller multiples if convenient) and applying Eq. (5.20) generates the conclusion

\[
\left\| t^{-\lambda}(w(1)(t, \cdot) - \tilde{w}(1)(t, \cdot)) \right\|_{\delta, H^k(M)}^2 + \left\| t^{-\lambda}(w(2)(t, \cdot) - \tilde{w}(2)(t, \cdot)) \right\|_{\delta, H^k(M)}^2 
\leq C \left( (\tau^{2\lambda} - \tau^{2\epsilon}) \left( \| u_1 \times \|_{\delta, H^{k+k_0}} + \| \varphi_1 \times \|_{\delta, H^{k+k_1}} \right) 
+ \int_0^T \left( \left\| s^{-\lambda_1} f(1) \right\|_{\delta, H^{k+1}(M)}^2 - \| s^{-\lambda_1} f(2) \right\|_{\delta, H^{k+1}(M)}^2 + \| s^{-\lambda_1+2\epsilon} f(3) \right\|_{\delta, H^{k+1}(M)}^2 \right) ds 
+ \int_0^T \left( \left\| s^{-\lambda_1} f(1) \right\|_{\delta, H^{k+1}(M)}^2 - \| s^{-\lambda_1} f(2) \right\|_{\delta, H^{k+1}(M)}^2 + \| s^{-\lambda_1+2\epsilon} f(3) \right\|_{\delta, H^{k+1}(M)}^2 \right) ds 
+ \left\| u_1 \times - \tilde{u}_1 \times \right\|_{\delta, H^{k+k_0}} + \left\| \varphi_1 \times - \tilde{\varphi}_1 \times \right\|_{\delta, H^{k+k_1}} \right).
\]

This holds for any smooth \( \lambda \) satisfying Eq. (3.17) provided Eq. (3.15) holds. Evaluating this at \( t = T \), transferring all norms of \( u_1, \tilde{u}_1, \varphi_1, \tilde{\varphi}_1 \) to the right-hand side and then estimating them in terms of \( u_1, \tilde{u}_1, \varphi_1, \tilde{\varphi}_1 \) exploiting the continuity of \( \Phi \), leads to our uniform continuity estimate for the finite matching map. This map Eq. (5.88) thus has a unique continuous extension to the domain \([0, T] \times H^{k+k_0} \times H^{k+k_1}(M)\) and co-domain \((H^k(M))^2\). The extended map evaluated at \( \tau = 0 \) is referred to as \( \Psi_{\sigma_1 \rightarrow \sigma_2, k} : H^{k+k_0}(M) \times H^{k+k_1}(M) \to (H^k(M))^2\) which we claim to be the asymptotic matching map asserted in the proposition. It is therefore continuous in the sense

\[
\left\| \Psi_{\sigma_1 \rightarrow \sigma_2, k}(u_1, \varphi_1) - \Psi_{\sigma_1 \rightarrow \sigma_2, k}(\tilde{u}_1, \tilde{\varphi}_1) \right\|_{H^k(M)} \leq C \left( \left\| u_1 \times - \tilde{u}_1 \times \right\|_{\delta, H^{k+k_0}} + \left\| \varphi_1 \times - \tilde{\varphi}_1 \times \right\|_{\delta, H^{k+k_1}} \right),
\]

where the constant \( C > 0 \) may depend on \( T, k, \lambda, \Phi \) and \( \Gamma \). Even though \( \Psi_{\sigma_1 \rightarrow \sigma_2, k} \) clearly depends on \( k \), standard arguments imply that its restriction to the dense sub-domain \((C^{\infty}(M))^2\) does not depend on the choice of \( k \) and therefore yields the map \( \Psi_{\sigma_1 \rightarrow \sigma_2} : (C^{\infty}(M))^2 \to (C^{\infty}(M))^2\), which satisfies the continuity estimate Eq. (5.26). Eq. (5.25) follows when apply the same chain of arguments to Eq. (5.97) which we had applied to the estimate after Eq. (5.97) above and then take the limit \( \tau \to 0 \).

Let us next investigate the uniqueness statement Eq. (5.27) of Proposition 5.4. Pick any smooth Cauchy data \((u_1, \varphi_1)\) and let \((u_1, \varphi_1)\) be the corresponding solution, and the solution \((u_2, \varphi_2)\) determined by the Cauchy data \((u_2, \varphi_2) = \Psi_{\sigma_1 \rightarrow \sigma_2}(u_1, \varphi_1)\) as before. Then given any other (possibly different) smooth solution \((\tilde{u}_2, \tilde{\varphi}_2)\) of the same target equation we define

\[
\omega = (\omega(1), \omega(2), \omega(3)) = (u_2, \varphi_2, \nu_2) - (\tilde{u}_2, \tilde{\varphi}_2, \tilde{\nu}_2).
\]

This is therefore a smooth solution of the \( \sigma = \sigma_2 \)-version of the equation with zero source term. Given any \( \lambda > \lambda_c \) and sufficiently small \( \epsilon > 0 \), Eqs. (5.93) together with Eq. (5.11)
yields
\[
\left\| t^{-\lambda+5\epsilon} \omega^{(1)}(t, \cdot) \right\|^2_{\delta H^0(M)} + \left\| t^{-\lambda+5\epsilon} \omega^{(2)}(t, \cdot) \right\|^2_{\delta H^0(M)} \\
\leq C \left\{ t_{*}^{-\lambda} \left\| \omega^{(2)}(t_{*}, \cdot) \right\|^2_{L^2(M)} + t_{*}^2 \left( \left\| t_{*}^{-\lambda+\epsilon} \omega^{(1)}(t_{*}, \cdot) \right\|^2_{L^2(M)} + \left\| t_{*}^{-\lambda} \omega^{(2)}(t_{*}, \cdot) \right\|^2_{H^1(M)} \right) \right\}
\]
for any \( t_{*} \in (0, T] \) and for all \( t \in [t_{*}, T] \). Assuming that \( \lambda \) is now in the range Eq. (3.17) it follows that the right-hand side vanishes in the limit \( t_{*} \to 0 \). This implies uniqueness.

Finally pick any two smooth \((u_{1x}, \varphi_{1x})\) and \((\tilde{u}_{1x}, \tilde{\varphi}_{1x})\). Supposing that
\[
\Psi_{\sigma_1 \to \sigma_2}(u_{1x}, \varphi_{1x}) = \Psi_{\sigma_1 \to \sigma_2}(\tilde{u}_{1x}, \tilde{\varphi}_{1x}) = (u_{2x}, \varphi_{2x}),
\]
we need to establish that \( \varphi_{1(0)} = \tilde{\varphi}_{1(0)} \). Let \((u_1, \nu_1, \varphi_1), (\tilde{u}_1, \nu_1, \tilde{\varphi}_1)\) and \((u_2, \nu_2, \varphi_2)\) be the corresponding solutions. For any smooth \( \lambda < \min\{\lambda_1, 2(1 - q_{\text{max}}), 2A^2\} \) and all \( t \in (0, T] \) we have
\[
\left\| t^{-\lambda} \left( \varphi_{1(0)} - \tilde{\varphi}_{1(0)} \right) \right\|_{H^k(M)} \\
\leq \left\| t^{-\lambda} \left( \varphi_{1(t, \cdot)} - \varphi_{1(0)} \right) \right\|_{H^k(M)} + \left\| t^{-\lambda} \left( \tilde{\varphi}_{1(t, \cdot)} - \tilde{\varphi}_{1(0)} \right) \right\|_{H^k(M)} \\
+ \left\| t^{-\lambda} \left( \varphi_{1(t, \cdot)} - \varphi_{2(t, \cdot)} \right) \right\|_{H^k(M)} + \left\| t^{-\lambda} \left( \tilde{\varphi}_{1(t, \cdot)} - \tilde{\varphi}_{2(t, \cdot)} \right) \right\|_{H^k(M)}.
\]
It is a consequence of Eq. (5.19) and of Eq. (5.25) that \( \lambda \) can be chosen such that the right-hand side approach zero in the limit \( t = 0 \). This completes the proof of Proposition 5.4.

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