Quaternary Bicycle Matroids and the Penrose Polynomial for Delta-Matroids

Robert Brijder · Hendrik Jan Hoogeboom

Abstract In contrast to matroids, vf-safe delta-matroids have three kinds of minors and are closed under the operations of twist and loop complementation. We show that the delta-matroids representable over $GF(4)$ with respect to the nontrivial automorphism of $GF(4)$ form a subclass of the vf-safe delta-matroids closed under twist and loop complementation. In particular, quaternary matroids are vf-safe.

Using this result, we show that the matroid of a bicycle space of a quaternary matroid $M$ is obtained from $M$ by using loop complementation. As a consequence, the matroid of a bicycle space of a quaternary matroid $M$ is independent of the chosen representation. This also leads to, e.g., an extension of a known parity-type characterization of the bicycle dimension, a generalization of the tripartition of Rosenstiehl and Read [Ann. Disc. Math. (1978)], and a suitable generalization of the dual notions of bipartite and Eulerian binary matroids to a vf-safe delta-matroids.

Finally, we generalize a number of results concerning the Penrose polynomial from binary matroids to vf-safe delta-matroids. In this general setting the Penrose polynomial turns out to have a recursive relation much like the recursive relation of the Tutte polynomial.

Keywords bicycle matroid · quaternary matroid · Penrose polynomial · delta-matroid · graph tripartition

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R.B. is a postdoctoral fellow of the Research Foundation – Flanders (FWO).

R. Brijder
Hasselt University and Transnational University of Limburg, Belgium
E-mail: robert.brijder@uhasselt.be

H.J. Hoogeboom
Leiden Institute of Advanced Computer Science, Leiden University, The Netherlands
1 Introduction

It turns out that various results related to matroids are more generally and more efficiently obtained in the more general context of delta-matroids (or $\Delta$-matroids) defined by Bouchet [7]. While matroids are closed under taking its dual, $\Delta$-matroids are closed under the more general operation of twist which can be viewed as a “partial dual”. In [13] another operation for $\Delta$-matroids was defined, called loop complementation. Since $\Delta$-matroids in general are not closed under loop complementation, when studying loop complementation we often restrict to the (minor-closed) class of $\Delta$-matroids closed under both twist and loop complementation, called $vf$-safe $\Delta$-matroids. As twist and loop complementation form a group, $vf$-safe $\Delta$-matroids enjoy various interesting properties. In particular, $vf$-safe $\Delta$-matroids allow for three kinds of minors (a third in addition to the usual deletion and contraction). It was shown in [14] that binary $\Delta$-matroids (which includes all binary matroids) are $vf$-safe. In fact, the class of binary $\Delta$-matroids is closed under both twist and loop complementation. We generalize this result to $GF(4)$. First we consider an extension of the notion of representability of $\Delta$-matroids over some field $F$ using the notion of $\alpha$-symmetric matrices, where $\alpha$ is an automorphism of $F$. While quaternary $\Delta$-matroids are not always $vf$-safe, we show that the $\Delta$-matroids representable over $GF(4)$ with respect to the nontrivial automorphism $inv$ of $GF(4)$ form a subclass of the $vf$-safe $\Delta$-matroids closed under twist and loop complementation, cf. Theorem 3.6. As a consequence, every quaternary matroid is $vf$-safe, cf. Corollary 4.2.

Next we consider the effect of loop complementation on quaternary matroids (and binary matroids in particular). A “full” twist applied to a matroid obtains its dual matroid. We show in this paper that a “full” loop complementation applied to a quaternary matroid $M$ followed by taking all maximal sets results in a matroid representing any bicycle space corresponding to $M$, cf. Theorem 5.2. As a consequence, the matroids of two bicycle spaces corresponding to a quaternary matroid $M$ are equal. More generally, we show that loop complementation on a subset $Y$ of the ground set corresponds to the matroid of any bicycle space corresponding to $M$ relative to $Y$ (this extended notion of bicycle space is defined in [2] for binary matroids). The link between loop complementation and bicycle spaces has various consequences and explains why bicycle spaces appear often and in unexpected ways in the literature. We consider a number of results concerning the bicycle space of binary matroids and extend them to $vf$-safe $\Delta$-matroids. For example, we show that the well-known principal tripartition result of Rosenstiehl and Read [29] can be generalized to $vf$-safe $\Delta$-matroids. Also, the notions of Eulerian matroid and bipartite matroid are dual for binary matroids. We show that these notions can be linked to loop complementation, and this link suggests alternative definitions of Eulerian and bipartite matroid that coincide for binary matroids but are (unlike the usual definitions) dual for the larger class of $vf$-safe matroids (or, indeed, $vf$-safe $\Delta$-matroids).

The final application of the results concerning the link between loop complementation and bicycle spaces is the Penrose polynomial. The Penrose polynomial is introduced by Penrose [28] to study the four-color conjecture (it was not yet a theorem then), see [1] for a survey of the Penrose polynomial. The Penrose polynomial is defined in [2] for binary matroids $M$ in general and in terms of the dimensions
of the bicycle spaces of $M$ relative to the subsets $Y$ of the ground set of $M$. Using the obtained results concerning bicycle matroids, we straightforwardly generalize the Penrose polynomial to vf-safe $\Delta$-matroids, and then show that this polynomial allows for a recursive relation that characterizes the polynomial. For the case of binary $\Delta$-matroids, we also formulate this polynomial as a graph polynomial with a recursive relation. As the class of binary matroids is not closed under loop complementation (in contrast to the class of binary $\Delta$-matroids), we remark that this recursive relation is not valid when restricting to the narrow viewpoint of binary matroids. This provides a further example of why the more general viewpoint of (vf-safe) $\Delta$-matroids is often worthwhile to consider. In fact, another example is provided by Chun et al. [18], where it is shown, using a preprint of this paper on arXiv, that the Penrose polynomial for vf-safe $\Delta$-matroids and its recursive relation turn out to generalize the Penrose polynomial for graphs embedded in surfaces of [21] and its recursive relation. Finally, we consider evaluations of the Penrose polynomial for vf-safe $\Delta$-matroids inspired by the evaluations of the Penrose polynomial for binary matroids of [2].

2 Preliminaries

We first recall some basic notions and results.

Principal pivot transform.

Let $V$ and $W$ be finite sets. We consider $V \times W$-matrices $A$, i.e., matrices where the rows are indexed by $V$ and the columns by $W$. The rows and columns of $A$ are not ordered (note that matrix inversion, rank, etc. are defined for such matrices). For $X \subseteq V$ and $Y \subseteq W$, the $X \times Y$-submatrix of $A$ is denoted by $A[X,Y]$. We write simply $A[X]$ to denote $A[X,X]$. We define the deletion of $X$ in $A$ by $A \setminus X = A[V \setminus X,W \setminus X]$.

Let $A$ be a $V \times V$-matrix (over an arbitrary field $F$), and let $X \subseteq V$ be such that the principal submatrix $A[X]$ is nonsingular. The principal pivot transform (PPT) of $A$ on $X$, denoted by $A*X$, is defined as follows [33]. Let $A = X_{V \setminus X} (P \quad Q \quad \begin{bmatrix} S \end{bmatrix})$, then $A*X = X_{V \setminus X} \begin{pmatrix} P^{-1} & -P^{-1}Q \quad \begin{bmatrix} S - RP^{-1}Q \end{bmatrix} \end{pmatrix}$. PPT has many applications and is well motivated as it can be viewed as a partial matrix inversion (full matrix inversion corresponds to the case $X = V$) [32].

We now recall the following property of PPT.

Proposition 2.1 ([33][27]) Let $A$ be a $V \times V$-matrix, and let $X \subseteq V$ be such that $A[X]$ is nonsingular. Then, for all $Y \subseteq V$, $\det((A*X)[Y]) = \det(A[X\Delta Y]) / \det(A[X])$. In particular, $(A*X)[Y]$ is nonsingular iff $A[X\Delta Y]$ is nonsingular.

It is easy to verify (by the above definition of PPT) that $-(A*X)^T = (-A^T)*X$ for all $X \subseteq V$ with $A[X]$ nonsingular. As a consequence, if $A$ is skew-symmetric, i.e.,
\(-A^T = A\) (we allow nonzero diagonal entries in case \(F\) is of characteristic two), then \(A \ast X\) is skew-symmetric as well.

Hence, if \(A\) is a \(V \times V\)-symmetric matrix over \(GF(2)\), then so is \(A \ast X\). We identify \(V \times V\)-symmetric matrices \(A\) over \(GF(2)\) with (undirected) graphs \(G = (V, E)\) where \(\{x, y\} \in E\) iff \(A[\{x\}, \{y\}] = 1\) (we allow \(x = y\), i.e., loops); \(A\) is called the adjacency matrix of \(G\). Hence, we write, e.g., \(G[X], G \setminus X\), and \(A \ast X\) to denote \(A[X], A \setminus X\), and \(A \ast X\) respectively, where \(A\) is the adjacency matrix of \(G\). We will often use simply \(V\) to denote the vertex set of graph \(G\) under consideration.

**Twist and loop complementation on set systems.**

A set system (over \(V\)) is a tuple \(M = (V, D)\) with \(V\) a finite set called the ground set and \(D \subseteq 2^V\) a family of subsets of \(V\). Similar as with graphs, we will often use \(V\) to denote the ground set of set system \(M\) under consideration. We write simply \(X \in M\) to denote \(X \subseteq D\). Set system \(M\) is called proper if \(D \neq \emptyset\). We say that \(M\) is equicardinal if for all \(X, Y \in M\), \(|X| = |Y|\). A set system is called even if for all \(X, Y \in M\), \(|X|\) and \(|Y|\) have equal parity. We define, for \(X \subseteq V\), the restriction of \(X\) in \(M\) by \(M[X] = \{X, X\}\), where \(D' = \{Y \in D \mid Y \subseteq X\}\), and we define \(M \setminus X = M[V \setminus X]\). We define, for \(X \subseteq V\), the twist \([7]\) of \(M\) on \(X\), denoted by \(M \ast X\), as \((V, D \ast X)\), where \(D \ast X = \{Y \Delta X \mid Y \subseteq D\}\) and \(\Delta\) denotes symmetric difference. We say that \(u \in V\) is a loop in \(M\) if none of the \(X \in M\) contains \(u\). We say that \(u\) is a coloop in \(M\) if \(M \ast V\) is a loop in \(M\), i.e., all \(X \in M\) contain \(u\). Moreover, we define, for \(X \subseteq V\), loop complementation of \(M\) on \(X\), denoted by \(M + X\), as \((V, D')\), where \(Y \in D'\) iff \(\{Z \in M \mid Y \setminus X \subseteq \emptyset \}\) is odd \([13]\).

As recalled below, loop complementation for set systems turns out to generalize loop complementation for graphs.

We assume left associativity of set system operations. Therefore, e.g., \(M + X \setminus Y\) denotes \((M + X) \setminus Y\). Twist and loop complementation are involutions and they commute on distinct elements. Hence for \(X, Y \subseteq V\) we have, e.g., \(M \ast X \ast Y = M \ast (X \Delta Y), M + X + Y = M + (X \Delta Y)\), and if \(X \setminus Y = \emptyset\), \(M + X + Y = M + Y \ast X\). It turns out that \(\ast X\) and \(+ X\) (for any \(X \subseteq V\)) generate the group \(S_3\) of permutations on 3 elements \([13]\). We denote by \(\ast X\) the third element \(\ast X + X \ast X = + X + X \ast X\) of order 2, called the dual pivot on \(X\). We have that \(M \ast X\) is equal to \((V, D')\), where \(Y \in D'\) iff \(\{Z \in M \mid Y \setminus Z \subseteq Y \cup X\}\) is odd. Let \(\min(M) = (V, \min(D))\) and \(\max(M) = (V, \max(D))\), where \(\min(D)\) (\(\max(D)\), resp.) are the sets in \(D\) which are minimal (maximal, resp.) with respect to set inclusion. We denote by \(d_M = \min_{Y \in M}(|Y|)\) the smallest cardinality among the sets in \(M\). It is observed in \([13]\) that \(\min(M) = \min(M + X)\), thus \(d_M = d_{M \ast X}\). Since \(\min(M) = \max(M \ast V) \ast V\), we have similarly \(\min(M) = \max(M + X)\).

In case of singletons \(\{u\}\), we also write \(M \setminus u, M \ast u, \text{ etc.}\) to denote \(M \setminus \{u\}, M \ast \{u\}, \text{ etc.}\)

**Matroids and \(\Delta\)-matroids.**

We assume the reader is familiar with the basic notions concerning matroids, which can be found, e.g., in \([36] [26]\). We recall here \(\Delta\)-matroids and their link with matroids.
A proper set system $M$ is a $\Delta$-matroid \cite{7} if for all $X, Y \in M$ and for all $x \in X \Delta Y$, there is a $y \in X \Delta Y$ ($x = y$ is allowed), such that $X \Delta \{x, y\} \in M$. If $M$ is a $\Delta$-matroid, then so is $M \ast X$ for all $X \subseteq V$.

A set system $M$ is an equicardinal $\Delta$-matroid iff $M$ is a matroid described by its bases \cite{8}. Also, if $M$ is a $\Delta$-matroid, then $\min(M)$ and $\max(M)$ are matroids \cite{12}.

In this paper, we assume, unless stated otherwise, that matroids are described by their bases. Hence, if $M$ is a matroid, then $M \ast V = M^*$ is the dual matroid of $M$. The nullity and rank of a matroid $M$ are denoted by $\nu(M)$ and $\rho(M)$, respectively. Note that $\rho(M) = d_M$ and $\nu(M) = d_{M^*}$ for matroids $M$, and $\rho(\min(M)) = d_M$ and $\nu(\max(M)) = d_{M^*}$ for $\Delta$-matroids $M$. Also note that the notions of loop and coloop for set systems coincide with the eponymous notions for matroids.

A minor of a $\Delta$-matroid is a $\Delta$-matroid obtained by applying a (possibly empty) sequence of operations of the form $\setminus u$ and $\ast u \setminus u$ with $u \in V$. Note that the usual (matroid-theoretical) notion of deletion of $u$ for a matroid $M$ is equal to $M \setminus u$ if $u$ is not a coloop and equal to $M \ast u \setminus u$ otherwise. Similarly, contraction of $u$ for a matroid $M$ is equal to $M \ast u \setminus u$ if $u$ is not a loop and equal to $M \setminus u$ otherwise. Since, in this paper, we never apply $\setminus u$ to a coloop $u$ or $\ast u \setminus u$ to a loop $u$, the reader may think of $\setminus u$ and $\ast u \setminus u$ as the usual matroid-theoretical notions of deletion and contraction, respectively. A $\Delta$-matroid $M$ is called $vf$-safe if any set system in the orbit of $M$ under $+$ and $\ast$ is a $\Delta$-matroid. The class of $vf$-safe $\Delta$-matroids is minor closed \cite{14}. There are (delta-)matroids that are not $vf$-safe, such as the 6-point line $U_{2,6}$, $P_6$, and the non-Fano matroid $F_7^*$, see \cite{26} for a description of these matroids. In fact, they are excluded minors for the class of $vf$-safe $\Delta$-matroids \cite{14}.

### 3 $\alpha$-symmetry and delta-matroids

Let $\alpha$ be an automorphism of a field $\mathbb{F}$. By abuse of notation, we extend $\alpha$ point-wise to vectors, matrices, and subspaces over $\mathbb{F}$. Hence for a $V \times V$-matrix $A = (a_{i,j})_{i,j \in V}$, we let $\alpha(A) = (\alpha(a_{i,j}))_{i,j \in V}$. Moreover, for subspace $L \subseteq \mathbb{F}^V$, we let $\alpha(L) = \{\alpha(v) \mid v \in L\}$.

Let $A$ be a $V \times V$-matrix over some field $\mathbb{F}$, and let $\alpha$ be an involutive automorphism of $\mathbb{F}$, i.e., $\alpha^2(x) = x$ for all $x$ of $\mathbb{F}$ (the identity automorphism is considered an involutive automorphism here). Then $A$ is called $\alpha$-symmetric if $\alpha(-A^T) = A$. Note that if $A$ is $\alpha$-symmetric, then $A^T$ is $\alpha$-symmetric. Also note that $A$ is id-symmetric with id the identity automorphism iff $A$ is skew-symmetric.

**Lemma 3.1** Let $A$ be a $V \times V$-matrix over some field $\mathbb{F}$, and let $\alpha$ be an automorphism of $\mathbb{F}$. If $X \subseteq V$ is such that $A[X]$ is nonsingular, then $\alpha(A \ast X) = \alpha(A) \ast X$.

**Proof** For any nonsingular matrix $P$, we have $\alpha(P^{-1}) = (\alpha(P))^{-1}$. Thus, if $A = \begin{pmatrix} x & \nu \setminus x \\ \nu \setminus x & P & Q & R & S \end{pmatrix}$, then in both cases we obtain:

$$
\begin{pmatrix}
\alpha(P)^{-1} & -\alpha(P)^{-1}\alpha(Q) \\
\alpha(R)\alpha(P)^{-1} & \alpha(S) - \alpha(R)\alpha(P)^{-1}\alpha(Q)
\end{pmatrix}.
$$
If $A$ is $\alpha$-symmetric and $X \subseteq V$ is such that $A[X]$ is nonsingular, then $A*X$ is $\alpha$-symmetric. Indeed, $\alpha(-(A*X)^T) = (A^T)*X = A*X$, where in the second equality we use Lemma 3.1.

Let $A$ be a $V \times V$ matrix. We define the set system $\mathcal{M}_A = (V,D)$ where $X \in D$ iff $A[X]$ is nonsingular. By convention, $A[\emptyset]$ is nonsingular. The next result is a straightforward extension of a result of [7] (the original formulation restricts to the case $\alpha = \text{id}$).

Lemma 3.2 ([7]) Let $\alpha$ be an involutive automorphism of some field $F$, and let $A$ be a $\alpha$-symmetric $V \times V$-matrix over $F$. Then $\mathcal{M}_A$ is a $\Delta$-matroid.

Proof Let $X, Y \in \mathcal{M}_A$ and $x \in X \Delta Y$. If entry $A*X[\{x\}]$ is nonzero, then by Proposition 2.1 $X\Delta\{x\} \in \mathcal{M}_A$ and we are done. Thus assume that $A*X[\{x\}]$ is zero. Since $A[Y]$ is nonsingular, $A*X[X\Delta Y]$ is nonsingular by Proposition 2.1. Hence there is a $\alpha \in X\Delta Y$ with entry $A*X[\{x\}, \{y\}]$ nonzero (note that $x \neq y$). Since $A*X$ is $\alpha$-symmetric, $A*X[\{x,y\}]$ is of the form

$$\begin{pmatrix}
  x & y \\
  \alpha(-t_1) & t_2
\end{pmatrix}
$$

for some $t_1 \in F \setminus \{0\}$ and $t_2 \in F$. Thus $A*X[\{x,y\}]$ is nonsingular and $X\Delta\{x,y\} \in \mathcal{M}_A$.

We say that a $\Delta$-matroid $M$ is $\alpha$-representable over $F$, if $M = \mathcal{M}_A*X$ for some $\alpha$-symmetric $V \times V$-matrix $A$ and $X \subseteq V$. In this way, the notion of representable from [7] coincides with $\alpha$-representable.

For a graph $G$, $\mathcal{M}_G$ is even iff $G$ has no loops. It is easy to verify that for graphs $G$ and $G'$, $\mathcal{M}_G = \mathcal{M}_{G'}$ iff $G = G'$. In fact, $G$ is uniquely determined by $V$ and the sets of cardinality 1 and 2 of $\mathcal{M}_G$, see [12, Property 3.1].

A $V \times V$-matrix $A$ over $F$ is called principally unimodular (PU, for short) if for all $Y \subseteq V$, $\det(A[Y]) \in \{0, 1, -1\}$. Note that any $V \times V$-matrix over $GF(2)$ or $GF(3)$ is principally unimodular.

We now consider the field $GF(4)$. Let us denote the unique nontrivial automorphism of $GF(4)$ by inv. Note that inv($x$) = $x^{-1}$ for all $x \in GF(4) \setminus \{0\}$, and thus inv is an involutive automorphism.

Theorem 3.3 Let $A$ be a inv-symmetric $V \times V$-matrix over $GF(4)$. Then $A$ is a principally unimodular.

Proof Recall that $1 = -1$ in $GF(4)$. We have $\det(A) = \det(\text{inv}(A^T)) = \text{inv}(\det(A)) = \text{inv}(\det(A))$. Thus $\det(A) \in \{0, 1\}$.

Remark 3.4 The proof of Theorem 3.3 essentially uses that the field $F$ under consideration is of characteristic 2, i.e., $F = GF(2^k)$ for some $k \geq 1$, and that $F$ has an involutive automorphism $\alpha$ with only trivial fixed points (the set of fixed points form
The automorphisms of $GF(2^k)$ are of the form $x \mapsto x^\ell$, with $1 \leq \ell \leq k$, and $\alpha$ is an involution when either $k = 1$ (and thus $\ell = 1$) or both $k$ is even and $\ell = k/2$. Moreover, for $\ell = k/2$ and $k$ even, the corresponding automorphism $\alpha$ has only trivial fixed points if $\ell = 1$. Consequently, the proof of Theorem 3.5 only works for $\alpha = \text{inv}$ and $F = GF(4)$ (and, of course, $\alpha = \text{id}$ and $F = GF(2)$).

For $X \subseteq V$, we define $A + X$ to be the $V \times V$ matrix with $A + X[[x], [y]] = A[[x], [y]] + 1$ if $x = y \in X$ and $A + X[[x], [y]] = A[[x], [y]]$ otherwise. The following result is a straightforward generalization of a result of [13] formulated for the case $F = GF(2)$.

**Proposition 3.5 (Theorem 8 of [13])** Let $A$ be a principally unimodular $V \times V$-matrix over a field $F$ of characteristic 2. Then, for all $X \subseteq V$, $\mathcal{M}_{A+X} = \mathcal{M}_A + X$.

**Proof** It suffices to show the result for $X = \{u\}$ with $u \in V$. By the definition of loop complementation, we need to show that, for $Z \subseteq V$, $Z \in \mathcal{M}_{A+u}$ iff (1) $Z \in \mathcal{M}_A$ when $u \notin Z$ and (2) exactly one of $Z, Z \setminus \{u\}$ is in $\mathcal{M}_A$ when $u \in Z$.

Let $Z \subseteq V$. First assume that $u \notin Z$. Then $A[Z] = (A + u)[Z]$, thus $\det A[Z] = \det (A + u)[Z]$ and so $Z \in \mathcal{M}_{A+u}$ iff $Z \in \mathcal{M}_A$. Now assume that $u \in Z$, which implies that $A[Z]$ and $(A + u)[Z]$ differ in exactly one position: $(u, u)$. We may compute determinants by Laplace expansion over the $u$-column, and summing minors. As $A[Z]$ and $(A + u)[Z]$ differ at only the matrix-element $(u, u)$, these expansions differ only by the minor $\det A[Z \setminus \{u\}]$. Thus $\det (A + u)[Z] = \det A[Z] + \det A[Z \setminus \{u\}]$, and this computation is in $GF(2)$ as $A$ is PU and $F$ of characteristic 2. Hence $Z \in \mathcal{M}_{A+u}$ iff exactly one of $Z, Z \setminus \{u\}$ is in $\mathcal{M}_A$. \qed

Note that, for a graph $G$ (i.e., a symmetric matrix over $GF(2)$), $G + X$ is obtained from $G$ by complementing the existence of loops for the vertices in $X$, hence the name loop complementation for the set systems operation $+ X$.

A $\Delta$-matroid $M$ is said to be representable over $F$, if $M = \mathcal{M}_A + X$ for some skew-symmetric $V \times V$-matrix $A$ and some $X \subseteq V$. A $\Delta$-matroid is said to be binary if it is representable over $GF(2)$. The following result is an adaption of the proof of [14] Theorem 8.2 where it is shown that the class of binary $\Delta$-matroids is closed under twist and loop complementation.

**Theorem 3.6** The class of $\Delta$-matroids inv-representable over $GF(4)$ is closed under twist and loop complementation.

**Proof** Let $M$ be a $\Delta$-matroid inv-representable over $GF(4)$. Then $M = \mathcal{M}_A \ast X$ for some inv-symmetric $V \times V$-matrix $A$ over $GF(4)$ and $X \subseteq V$. Let $\phi$ be a sequence of twist and loop complementations over $V$. Let $W \in \mathcal{M}_A \ast X \phi$, and consider now $\phi' = \ast X \phi \ast W$. By the $S_3^2$ group structure of $\ast$ and $+$, $\phi'$ can be put in the following normal form: $\mathcal{M}_A \phi' = \mathcal{M}_A + Z_1 + Z_2 + Z_3$ for some $Z_1, Z_2, Z_3 \subseteq V$ with $Z_1 \subseteq Z_2$. By Theorem 3.5, $A$ is PU. By Proposition 3.5, $\mathcal{M}_A + Z_1 = \mathcal{M}_A + Z_1$. Thus $\mathcal{M}_A + Z_1 + Z_2 + Z_3 = \mathcal{M}_A + Z_1 + Z_3$. By construction $\emptyset \in \mathcal{M}_A \phi'$. Hence we have $\emptyset \in \mathcal{M}_A + Z_1 \ast Z_2$. Therefore $Z_2 \in \mathcal{M}_A + Z_1$ and so $A + Z_1[Z_2]$ is nonsingular. Thus, $A + Z_1 \ast Z_2$ is defined. Consequently, $A' = A + Z_1 \ast Z_2 + Z_3$ is defined and $\mathcal{M}_A \phi' = \mathcal{M}_A$. Hence $M \phi = \mathcal{M}_A \ast X \phi = \mathcal{M}_A \ast W$ and thus inv-symmetric matrix $A'$ represents $M \phi$. Therefore, $M \phi$ a $\Delta$-matroid inv-representable over $GF(4)$. \qed
Consequently, the class of vf-safe $\Delta$-matroids contains the class of inv-representable $\Delta$-matroids over $GF(4)$ (and therefore also the class of binary $\Delta$-matroids). In contrast with Theorem 3.6 it is shown in [14] that there are $\Delta$-matroids id-representable over $GF(4)$ that are not vf-safe.

4 Quaternary matroids

Let $M = (V, \mathcal{B})$ be a matroid representable over $F$, and described by its bases. Let $B$ be a standard representation of $M$ over $F$. Then $B$ is equal to

$$
x \begin{array}{ccc} v \setminus x & \quad \scriptstyle v \setminus x & \quad \scriptstyle v \setminus x \\ x & \begin{pmatrix} I & S \end{pmatrix} & \end{array}
$$

for some $X \in \mathcal{B}$, where $I$ is the identity matrix of suitable size. Let $\alpha$ be an involutive automorphism of $F$. We define $R(B, \alpha)$ to be the $\alpha$-symmetric $V \times V$-matrix

$$
x \begin{array}{ccc} x & \quad \scriptstyle v \setminus x & \quad \scriptstyle v \setminus x \\ v \setminus x & \begin{pmatrix} 0 & S \end{pmatrix} & \end{array}
$$

$$
\alpha(-S^T) 0
$$

We now recall the following result of de Frayseix [20] and Bouchet [7] which states that a matroid is representable in the classical matroid sense iff it is representable in the $\Delta$-matroid sense. Therefore, the class of matroids representable over some field $F$ is a subclass of the class of $\Delta$-matroids representable over $F$.

**Proposition 4.1 (Theorem 4.4 of [7])** Let $M$ be a matroid representable over $F$, let $\alpha$ be an involutive automorphism of $F$, and let $B$ be a $X \times V$-matrix over $F$ that is a standard representation of $M$. Then $M = M_X \alpha = M^*$ with $A_R(B, \alpha)$.

The formulation of Proposition 4.1 is slightly more general than the original formulation in [7], which assumes $\alpha = \text{id}$. However, note that if $A = R(B, \alpha)$ and $A' = R(B, \text{id})$, then $A[Y]$ is nonsingular iff $A'[Y]$ is nonsingular for all $Y \subseteq V$. Thus $M_X \alpha = M^*$.

In case $F = GF(2)$, we can view $A$ from Proposition 4.1 as (an adjacency matrix representation of) a $(X, V \setminus X)$-bipartite graph $G$. Graph $G$ is often called the fundamental graph of $M$ with respect to the basis $X \in M$, consisting of all edges $\{u, v\}$ such that $X \Delta \{u, v\}$ is a basis. If $Y \in M$, then $M \ast Y = M \ast G \ast X \ast Y = M \ast (X \Delta Y) = M_{G \ast (X \Delta Y)}$ where in the last equality we use that $X \Delta Y \in M \ast X = M^*$ since $Y \in M$. Therefore, every fundamental graph of $M$ can be obtained from $G$ by applying PPT [11 Section 2].

Hence, by Proposition 4.1 a matroid $M$ is representable over $F$ in the usual (matroid) sense iff $M$ is $\alpha$-representable for some involutive automorphism $\alpha$ of $F$ (recall that we allow $\alpha = \text{id}$) iff $M$ is $\alpha$-representable for all involutive automorphisms $\alpha$ of $F$. Therefore, choosing $\alpha = \text{id}$ may not necessarily be the most convenient extension of the matroid notion of representability to $\Delta$-matroids. Indeed, in view of Theorem 3.6 and the remark below it, we argue that over $GF(4)$, inv-representability is a more natural extension of the matroid notion of representability to $\Delta$-matroids.
In particular, every quaternary matroid is a \( \Delta \)-matroid inv-representable over \( GF(4) \). Hence by Theorem 3.6 we have the following result, which was conjectured in [14].

**Corollary 4.2** Every quaternary matroid is sf-safe.

## 5 Bicycle matroids

Let \( v \in \mathbb{F}^V \) be a vector. The *support* of \( v \) is the set \( X \subseteq V \) such that the entries of \( X \) in \( v \) are nonzero and entries of \( V \setminus X \) in \( v \) are zero. Let \( L \) be a subspace of \( \mathbb{F}^V \). We denote by \( M(L) \) the matroid with ground set \( V \) such that for all \( C \subseteq V \), \( C \) is a circuit of \( M(L) \) iff there is a \( v \in L \) with support \( C \) and \( C \) is minimal with this property among the nonempty subsets of \( V \). Note that for a \( X \times V \)-matrix \( A \), the matroid \( M(\ker(A)) \) equals the column matroid of \( A \), denoted by \( M(A) \). The orthogonal complement of \( L \), denoted by \( L^\perp \), is \( \{ v \in \mathbb{F}^V \mid \langle u, v \rangle = 0 \text{ for all } u \in L \} \) where \( \langle u, v \rangle = \sum_{x \in V} u(x)v(x) \) for all \( u, v \in \mathbb{F}^V \).

Consider now the case \( \mathbb{F} = GF(4) \). Inspired by terminology from [34], we call \( L \cap inv(L^\perp) \) the *bicycle space of \( L \)* and denote it by \( BC_L \). We have \( BC_{L^\perp} = inv(BC_L) \).

The dimension of \( BC_L \) is called the *bicycle dimension* of \( L \). Moreover, for all \( Y \subseteq V \) and vector \( v \) over \( V \), denote by \( \pi_Y(v) \) the vector obtained from \( v \) by setting all entries of \( V \setminus Y \) to 0. Then we call \( \{ v \in L \mid \pi_Y(v) \in inv(L^\perp) \} \) the *bicycle space of \( L \) relative to \( Y \)* and we denote it by \( BC_L(Y) \). Note that \( BC_L(\emptyset) = L \) and \( BC_L(V) = BC_L \).

We now recall the notion of bicycle space of a binary matroid \( M \). Let \( M \) be a binary matroid over \( V \) and let \( \mathcal{C} \mathcal{F}_M \) be the cycle space of \( M \), i.e., the subspace of \( GF(2)^V \) generated by the circuits of \( M \). The *bicycle space of \( M \) relative to \( Y \subseteq V \)* is defined as \( BC_M(Y) = \{ C \in \mathcal{C} \mathcal{F}_M \mid C \cap Y \in \mathcal{C} \mathcal{F}_M \} \) (where vectors over \( GF(2) \) are identified by their support), see [23]. We (may) consider \( GF(2)^V \) as a subspace of \( GF(4)^V \). Observe that if \( L \subseteq GF(2)^V \), then \( BC_L = L \cap L^\perp \). It is well known that \( \mathcal{C} \mathcal{F}_M \) is equal to the null space \( \ker(A) \) of any binary representation \( A \) of \( M \), see [26] Proposition 9.2.2 (in particular, dim(\( \mathcal{C} \mathcal{F}_M \)) = \( \nu(M) \)). Thus \( BC_{\ker(A)}(Y) = BC_M(Y) \) for all \( Y \subseteq V \) and so the definition of \( BC_L(Y) \) is consistent with the definition of \( BC_M(Y) \). The *bicycle matroid* \( \mathcal{B}M_M(Y) \) of binary matroid \( M \) relative to \( Y \subseteq V \) is the (unique) binary matroid with ground set \( V \) and cycle space \( BC_M(Y) \). The notion of bicycle matroid for the case \( Y = V \) was introduced in [22].

In contrast to the binary case, \( BC_{\ker(B)} \) and \( BC_{\ker(B')} \) may differ when \( B \) and \( B' \) are different representations over \( GF(4) \) of a quaternary matroid \( M \). However, we know from [34] that, for all \( L \subseteq GF(4)^V \), the *dimensions of \( BC_{\ker(B)} \) and \( BC_{\ker(B')} \) are equal*. We now extend this result by showing that the *matroids of \( BC_{\ker(B)} \) and \( BC_{\ker(B')} \) are equal*. In fact, we show \( M(BC_{\ker(B)}(Y)) = M(BC_{\ker(B')}(Y)) \) for all \( Y \subseteq V \). As a consequence we may speak of the bicycle matroid of a quaternary matroid \( M \) relative to \( Y \subseteq V \). Moreover we give an explicit formula for this bicycle matroid in terms of \( M \) (independent of representation). Also, the proof of this result below is direct, and therefore not obtained as a consequence of an evaluation of the Tutte polynomial as in [34].

First we prove a technical lemma.
Lemma 5.1 Let $B$ be a $X \times V$-matrix over some $\mathbb{F}$ with characteristic 2 such that $X \subseteq V$ and $B[X]$ the identity matrix of suitable size. Let $\alpha$ be an involutive automorphism of $\mathbb{F}$ and let $A = R(B, \alpha)$. Then for all $Y \subseteq V$, the null space of the $\alpha$-symmetric matrix $A + (X \cup Y) \ast (X \setminus Y)$ is equal to $\{v \in \ker(B) \mid \pi_Y(v) \in \alpha(\ker(B)^\perp)\}$.

Proof Recall from, e.g., [26, Proposition 2.2.23], that the null space of

$$
B = \begin{pmatrix} X & V \setminus X \end{pmatrix} \begin{pmatrix} 1 & S \end{pmatrix}
$$

is the orthogonal complement of the null space of

$$
B' = \begin{pmatrix} X & V \setminus X \end{pmatrix} \begin{pmatrix} -S^T & 1 \end{pmatrix}.
$$

Although signs are irrelevant over fields with characteristic 2, we leave them for didactical purposes. We observe that $B = (A + V)[X, V]$ and $\alpha(B') = (A + V)[V \setminus X, V]$. Thus, $\ker(A + V) = \ker(B) \cap \ker(\alpha(B')) = \ker(B) \cap \alpha(\ker(B)^\perp)$ (which proves the case $Y = V$). Let

$$
S = \begin{pmatrix} X \setminus Y & V \setminus (Y \setminus X) \end{pmatrix} \begin{pmatrix} S_1 & S_2 \ S_3 & S_4 \end{pmatrix}.
$$

Then the null space of

$$
B'' = \begin{pmatrix} X \setminus Y & X \setminus Y & Y \setminus X & V \setminus (Y \setminus X) \end{pmatrix} \begin{pmatrix} I & 0 & S_1 & S_2 \ 0 & I & S_3 & S_4 \ 0 & \alpha(-S_1^2) & I & 0 \ 0 & \alpha(-S_2^2) & 0 & 0 \end{pmatrix}.
$$

is equal to $\{v \in \ker(B) \mid \pi_Y(v) \in \alpha(\ker(B)^\perp)\}$. We show that the null space of $B''$ is equal to the null space of $A' = A + (X \cup Y) \ast (X \setminus Y)$. We have

$$
A' = \begin{pmatrix} X \setminus Y & X \setminus Y & Y \setminus X & V \setminus (Y \setminus X) \end{pmatrix} \begin{pmatrix} I & 0 & -S_1 & -S_2 \ 0 & I & S_3 & S_4 \ \alpha(-S_1^2) & \alpha(-S_2^2) & I + \alpha(S_1^2)S_1 & \alpha(S_2^2)S_2 \ \alpha(-S_1^2) & \alpha(-S_2^2) & \alpha(S_1^2)S_1 & \alpha(S_2^2)S_2 \end{pmatrix}.
$$

Now consider the following nonsingular matrix

$$
A'' = \begin{pmatrix} X \setminus Y & X \setminus Y & Y \setminus X & V \setminus (Y \setminus X) \end{pmatrix} \begin{pmatrix} I & 0 & 0 & 0 \ 0 & I & 0 & 0 \ \alpha(S_1^2) & 0 & I & 0 \ \alpha(S_2^2) & 0 & 0 & I \end{pmatrix}.
$$

Observe that $A''A' = B''$ (here we use that $\mathbb{F}$ has characteristic 2 to remove the “incorrect” signs of $(A''A')[X \setminus Y, V \setminus X]$). Hence $\ker(A + (X \cup Y) \ast (X \setminus Y)) = \ker(A') = \ker(B'')$. \qed
We note that (the proof of) Lemma 5.1 holds also when automorphism α is not involutive. Of course, in that case, $A + (X \cup Y) \ast (X \setminus Y)$ need not be $\alpha$-symmetric.

It follows from the strong principal minor theorem [23] Theorem 2.9 that, if $A$ is $\alpha$-symmetric matrix, then matroid $\max(\mathcal{A}_A)$ is equal to $M(A)$ (in fact, the strong principal minor theorem holds for so-called "quasi-symmetric matrices", but it is straightforward to verify that $\alpha$-symmetric matrices are quasi-symmetric). We use this observation in the next result.

**Theorem 5.2.** Let $M$ be a quaternary matroid, and let $B$ be a representation of $M$ over $GF(4)$. For all $Y \subseteq V$, $M(\BC_{\ker(B)}(Y))$ is equal to the matroid $\max(M + Y)$.

**Proof.** Without loss of generality we assume that $B$ is a standard representation of $M$. Say, $B$ is a $X \times V$-matrix. Let $A = R(B, \text{inv})$ and let $A' = A + (X \cup Y) \ast (X \setminus Y)$.

By Lemma 5.1, $M(A') = M(\BC_{\ker(B)}(Y))$ and thus it suffices to show that $M(A') = \max(M + Y)$.

Recall that (through the strong principal minor theorem) $M(A') = \max(\mathcal{A}_A)$ since $A'$ is an inv-symmetric matrix. By Theorem 3.3, $A$ is PU and by Proposition 3.5, $\mathcal{A}_A = \mathcal{A}_A + (X \cup Y) \ast (X \setminus Y)$. By Proposition 4.1, $\mathcal{A}_A \ast X = M$ and thus $\mathcal{A}_A + (X \cup Y) \ast (X \setminus Y) = M \ast X + (X \cup Y) \ast (X \setminus Y)$.

Recall that $\ast$ and $+$ commute on distinct elements and that they form $S_3$ on common elements. Hence, $M \ast X + (X \cup Y) \ast (X \setminus Y) = M \ast (X \setminus Y) \ast (X \setminus Y) + Y + (X \setminus Y) \ast (X \setminus Y) = M \ast (X \setminus Y) + Y \ast (X \setminus Y) = M + Y \ast X$. We obtain $M(A') = \max(M + Y \ast X)$. Since $\max(N \ast Z) = \max(N)$ for all set systems $N$ and all subsets $Z$ of the ground set, we have $\max(M + Y \ast X) = \max(M + Y)$ and the result follows. \(\square\)

Theorem 5.2 suggests the following definition. For a vf-safe matroid $M$, we call matroid $\max(M + Y)$ the **bicycle matroid relative to $Y \subseteq V$** and denote it by $BM_M(Y)$. Note that this definition is consistent with (and therefore generalizes) the above definition of bicycle matroid for binary matroids.

Note that $\max(M' + V) = \max(M \ast V + V) = \max(M + V \ast V) = \max(M + V)$, so the bicycle matroid of $M$ is invariant under taking the dual matroid.

We call the nullity of $BM_M(Y)$ the **bicycle dimension of $M$ relative to $Y$**. In case $Y = V$, we simply speak of the bicycle dimension of $M$. We have $\nu(\max(M + Y)) = d_{M + Y \ast V} = d_{M + V \ast Y}$, and $d_{M \ast V \ast Y} = d_{M \ast V + V} = d_{M \ast Y}$, and so we obtain the following corollary to Theorem 5.2.

**Corollary 5.3.** Let $M$ be a quaternary matroid and $Y \subseteq V$. The bicycle dimension of $M$ relative to $Y$ is $d_{M \ast V \ast Y}$. In particular, the bicycle dimension of $M$ is $d_{M \ast V}$.

We may rephrase Corollary 5.3 in terms of subspaces of $GF(4)^V$. Let $L$ be a subspace of $GF(4)^V$. Then, by Corollary 5.3, $\dim(BL(Y)) = d_{M(L) \ast V \ast Y}$. In particular, $\dim(L \cap \text{inv}(L^\perp)) = d_{M(L) \ast Y}$. The equality of the bicycle dimension and the value $d_{M \ast V}$ was already shown for the case of binary matroids $M$ in [15] as a consequence of calculating the Tutte polynomial at $(-1, -1)$ in an alternative way. The present paper explains this equality for quaternary matroids in general in a direct way (without considering the Tutte polynomial) as a corollary to Theorem 5.2.
Remark 5.4 Theorem 5.2 identifies, for quaternary matroids \( M \), a relationship between the matroids \( M = \max(M), M^\ast = M \ast V = \max(M \ast V) \), and \( \max(M + V) \). It turns out that the matroids \( \max(M), \max(M \ast V), \) and \( \max(M + V) \) are also in some weak sense related for \( \Delta \)-matroids \( M \) in general. Matroids \( M_1 \) and \( M_2 \) are said to be orthogonal if for all circuits \( C \) of \( M_1 \) and \( C_2 \) of \( M_2 \), \( |C \cap C_2| \neq 1 \). It is well known that any matroid \( M \) is orthogonal to its dual \( M^\ast \). In [15], vf-safe \( \Delta \)-matroids are shown to be “essentially” equivalent to a particular class of multimatroids [9] called tight 3-matroids [11] (we will not recall multimatroids in this paper). Theorem 3.2 of [10] shows that the matroids corresponding to disjoint transversals of a multimatroid are orthogonal when projecting the ground sets onto a common ground set \( V \).

This translates to vf-safe \( \Delta \)-matroids as follows: for any vf-safe \( \Delta \)-matroid \( M \), the matroids \( \max(M), \max(M \ast V), \) and \( \max(M + V) \) are mutually orthogonal. In case \( M \) is a vf-safe matroid, we have that \( M \), its dual \( M^\ast \), and \( \max(M + V) \) are mutually orthogonal.

By definition, the \( \Delta \)-matroid \( M \ast V \) is constructed from \( M \) by adding the sets that are included in an odd number of bases. Corollary 5.3 opens the possibility of parity-type characterizations of the bicycle dimension. Indeed, quaternary matroid \( M \) has an odd number of bases iff the bicycle dimension of \( M \) is zero, a result shown by Chen [17] for the case where \( M \) is a graphic matroid (and later realized to hold for binary matroids in general). Moreover, by the definition of dual pivot, \( d_M \ast V > 1 \) iff the number of bases of \( M \) is even and for all \( v \in V \), \( v \) is in an even number of bases of \( M \) iff for all \( v \in V \), \( v \) is in an even number of bases and in an even number of cobases of \( M \), the latter of which is the \( q \geq 1 \) characterization (\( q \) being equal to the bicycle dimension of \( M \)) of de Fraysseix [19] Théorème 1).

Remark 5.5 Unfortunately, the other two characterizations for \( q = 0 \) and \( q = 1 \) stated in [19] Théorème 1 do not hold. These characterizations are formulated in terms of the principal tripartition which we recall in Subsection 6.2. We give a counterexample for each characterization. The cycle matroid \( M \) of \( K_4 \), the complete graph on four vertices, has 16 bases, while every element of the ground set occurs in 8 bases. Every element is part of a 4-cycle, which is a cocycle as well. Hence, the tripartition equals \( (P, Q, R) = (\emptyset, \emptyset, V) \). The first characterization of [19] Théorème 1 predicts \( q = 0 \), while actually \( q = 2 \). The uniform matroid \( U_{2,3} \) is the cycle matroid of \( K_3 \). It has three bases, and each element in the ground set occurs in two bases. Moreover the ground set forms a cycle that becomes a cocycle when any element is removed. Thus the tripartition for \( U_{2,3} \) equals \( (P, Q, R) = (V, \emptyset, \emptyset) \). Now \( R \) is empty, as is the set of elements occurring in an odd number of bases. The second characterization of [19] Théorème 1) predicts \( q = 1 \), while actually \( q = 0 \). In fact, the two characterizations for \( q = 0 \) and \( q = 1 \) are not disjoint, as in the latter example also \( Q = \emptyset \), predicting also \( q = 0 \).

6 Consequences

In this section we discuss a number of consequences for binary matroids of Theorem 5.2 and we give an example. In Section 7 we use the result to generalize the Penrose polynomial to \( \Delta \)-matroids.
6.1 Fundamental graph of a matroid

We now consider fundamental graphs of binary matroids.

**Corollary 6.1** Let $G$ be a fundamental graph of a binary matroid $M$. Then the column matroid of $G + V$ is equal to the bicycle matroid of $M$.

**Proof** Let $\mathcal{M}_G = M \star Z$ for some $Z \in M$. Then by Theorem 5.2, the bicycle matroid of $M$ relative to $V$ is $\max(\mathcal{M}_G + V) = \max(\mathcal{M}_G + V + Z) = \max(\mathcal{M}_G + V)$ which in turn is equal to the column matroid of $G + V$.

Consequently, if $G_1$ and $G_2$ are fundamental graphs of some binary matroid $M$, then the column matroids of $G_1 + V$ and $G_2 + V$ are equal.

Since every bipartite graph is the fundamental graph of some matroid $M$, we obtain the following result stated (without proof) in [22].

**Corollary 6.2 (Proposition 3 of [22])** Let $M$ be a binary matroid. Then $M$ is the column matroid of a graph $G$ such that $G + V$ is bipartite iff $M$ is the bicycle matroid of a binary matroid.

**Remark 6.3** The Tanner graph [30] is a popular notion within coding theory. A (linear) code $\mathcal{C}$ is a subspace of $GF(2)^V$ (for some finite set $V$), a parity-check matrix $H$ for $\mathcal{C}$ is a matrix with $\ker(H) = \mathcal{C}$. Matrix $H$ is said to be in a standard form if $H = (I \mid B)$ where $I$ is an identity matrix. If $H$ is a $m \times n$-parity-check matrix in standard form, then the Tanner graph $T$ of $H$ is a $(U, V)$-bipartite graph with $U = \{u_1, \ldots, u_m\}$ and $V = \{v_1, \ldots, v_n\}$ such that $\{u_i, v_j\}$ is an edge of $T$ iff entry $H_{i,j}$ is equal to 1. The elements of $U$ and $V$ are called check nodes and bit nodes, respectively. The vertices $v_1, \ldots, v_m$ and their edges are often ignored, see, e.g., Fig. 2 in [24]. The obtained bipartite graph $G$ is therefore exactly a fundamental graph of the binary matroid $M$ with cycle space $\mathcal{C}$. Hence it seems fruitful to consider Tanner graphs from a matroid perspective (indeed, e.g., edge local complementation is applied to the Tanner graph, see [24], which corresponds to taking a different fundamental graph of $M$). However, surprisingly, considering the extensive literature on the notion of Tanner graph, this perspective seems to have not yet been taken.

In the next subsections we meet the fundamental graph again.

6.2 Principal tripartition

The well-known principal tripartition for binary matroids from Rosenstiehl and Read [29, Theorem 2.1] is as follows.

**Proposition 6.4 (Principal tripartition [29])** Let $M$ be a binary matroid. Then every $u \in V$ belongs to exactly one of the following sets: $P = \{v \in V \mid \exists X \in \mathcal{C}_M, v \in X, X \setminus \{v\} \in \mathcal{C}_M^\perp\}$, $Q = \{v \in V \mid \exists X \in \mathcal{C}_M^\perp, v \in X, X \setminus \{v\} \in \mathcal{C}_M\}$, and $R = \{v \in V \mid \exists X \in \mathcal{C}_M \cap \mathcal{C}_M^\perp, v \in X\}$.
We now generalize the principal tripartition result from binary matroids to vf-safe matroids. The characterization of the previous section allow us to use a formulation that avoids (bi)cycles.

**Theorem 6.5** Let $M$ be a vf-safe $\Delta$-matroid. Then every element of $V$ belongs to exactly one of the following sets: $P = \{v \in V \mid v$ is not a coloop of $\operatorname{max}(M + V + v)\}$, $Q = \{v \in V \mid v$ is not a coloop of $\operatorname{max}(M + V + v + v)\}$, and $R = \{v \in V \mid v$ is not a coloop of $\operatorname{max}(M + V + v + v + v)\}$. Moreover, this tripartition coincides with the tripartition of Proposition 6.4 when $M$ is a binary matroid.

Note that, in particular, quaternary matroids satisfy the tripartition result of Theorem 6.5.

The proof of Theorem 6.5 uses the following result from [14].

**Proposition 6.6** (Theorem 14 in [14]) Let $M$ be a vf-safe $\Delta$-matroid and let $v \in V$. Then the matroids $\operatorname{max}(M)$, $\operatorname{max}(M + v)$, and $\operatorname{max}(M + v + v)$ are such that precisely two of the three are equal, to say $M_1$. Moreover, the rank of the third $M_2$ is one smaller than the rank of $M_1$, and $M_3$ is the direct sum of $M_2 \setminus v$ and the matroid consisting of the coloop $v$.

**Proof** (of Theorem 6.5) By applying Proposition 6.6 to $M + V$, we have that two of three matroids $\operatorname{max}(M + V + v)$, $\operatorname{max}(M + V + v + v)$, $\operatorname{max}(M + V + v + v + v)$ are equal and have $v$ as a coloop and the third does not have $v$ as a coloop. Hence $v$ belongs to precisely one of $P$, $Q$, and $R$.

Assume now that $M$ is a binary matroid. We show that the tri partitions of Proposition 6.4 and Theorem 6.5 coincide. Now, there exists a $X \in \mathcal{M}_M$ with $X \setminus \{v\} \in \mathcal{M}_M$ and $v \in X$ iff there exists a $X \in \mathcal{M}_{\operatorname{BM}_M(v \setminus \{v\})}$ with $v \in X$ iff $v$ is not a coloop of $\operatorname{BM}_M(V \setminus \{v\}) = \operatorname{max}(M + (V \setminus \{v\})) = \operatorname{max}(M + V + v)$ (where the first equality is by Theorem 5.2). Similarly, there exists a $X \in \mathcal{M}_M = \mathcal{M}_M^+$ with $X \setminus \{v\} \in \mathcal{M}_M^+$ and $v \in X$ iff there exists a $X \in \mathcal{M}_{\operatorname{BM}_M(v \setminus \{v\})}$ with $v \in X$ iff $v$ is not a coloop of $\operatorname{BM}_M(V \setminus \{v\}) = \operatorname{max}(M + V + (V \setminus \{v\})) = \operatorname{max}(M + (V \setminus \{v\} + v \setminus v)) = \operatorname{max}(M + V + v)$. Finally, there exists a $X \in \mathcal{M}_M \setminus \mathcal{M}_M^+$ with $v \in X$ iff $v$ is not a coloop of $\operatorname{BM}_M(V) = \operatorname{max}(M + V)$. $\square$

Theorem 6.5 and its relation to Proposition 6.6 allows one to generalize results associated to the principal tripartition result (and, moreover, allows for easier proofs of these results). For example Table 3 in [29], which states how the tripartition changes for a cycle matroid of a graph $G$ when applying various operations on $G$, is readily obtained as a consequence of Proposition 6.6.

In [16], the graph counterpart of Proposition 6.6 (i.e., the case $M = \mathcal{M}_G$ for some graph $G$) was explicitly seen as a tripartition result with the property of “being a coloop” that decides to which of the three classes of the tripartition a particular vertex belongs (similar as in Theorem 6.5). However, no concrete link with the result of [29] was established in [16].

Let, for a vf-safe $\Delta$-matroid $M$ and $v \in V$, $\operatorname{nmax}_M(v) = \{v(\operatorname{max}(M)), v(\operatorname{max}(M + v)), v(\operatorname{max}(M + v + v))\}$. Note that by using Proposition 6.6 the definitions of $P$, $Q$, and $R$ of Theorem 6.5 can be rephrased as follows: $v \in P$ iff $v(\operatorname{max}(M + V + v)) = \{v(\operatorname{max}(M)), v(\operatorname{max}(M + v)), v(\operatorname{max}(M + v + v))\}$.
nmax_{M+V}(v), v \in Q \text{ iff } v(\max(M + V + v)) = nmax_{M+V}(v), \text{ and } v \in R \text{ iff } v(\max(M + V)) = nmax_{M+V}(v).

One may again rephrase this in terms of rank instead of nullity. However, we choose nullity due to the following result.

We let v(G) be the nullity of the adjacency matrix of a graph G. Moreover, let for all v \in V, nmax_{G}(v) = \max\{v(G), v(G \setminus v), v(G + v)\}. We reformulate the tripartition for binary matroids M in terms of a fundamental graph of M.

**Corollary 6.7** Let P, Q, R be the tripartition associated with a binary matroid M. Let G be a fundamental graph of M = \mathcal{M}_G * Z. If v \in V \setminus Z, then

1. v \in P \text{ iff } nmax_{G+V}(v) = v(G + V + v),
2. v \in Q \text{ iff } nmax_{G+V}(v) = v(G + V \setminus v), \text{ and }
3. v \in R \text{ iff } nmax_{G+V}(v) = v(G + V).

If v \in Z, then the roles of P and Q are reversed.

**Proof** Let v \in V \setminus Z. Then max(M + V + v) = \max(\mathcal{M}_G * Z + V + v) = \max(\mathcal{M}_G + V + v) = \max(\mathcal{M}_G + V, v)\}. Hence v(\max(M + V + v)) = v(G + V \setminus \{v\}). Similarly, v(\max(M + V + v)) = v(G + V + v) and v(\max(M + V)) = v(G + V). It is a well-known property of the Schur complement \mathcal{F}*v\setminus\{v\} for a matrix \mathcal{F} that its nullity is equal to the nullity of \mathcal{F}, see, e.g., [17]. Hence v(G + V + v) = v(G + V \setminus v).

Finally, let v \in Z. Then max(M + V + v) = \max(\mathcal{M}_G * Z + V + v) = \max(\mathcal{M}_G + V * v + v) = \max(\mathcal{M}_G + V + v) \text{ and similarly max}(M + V * v) = \max(\mathcal{M}_G + V + v). Hence the roles of P and Q are reversed with respect to the case of v \in V \setminus Z. \qed

The fact that the three values v(G + v), v(G \setminus v), v(G) in Corollary 6.7 are of the form m, m, m + 1 (in some order) has been shown in [6] Lemma 2], see also [13]. As a consequence, it suffices to know two of these three values to be able to determine the third.

### 6.3 Eulerian and bipartite binary matroids

Matroid M is said to be **bipartite** when every circuit of M is of even cardinality, and M is said to be **Eulerian** when there are disjoint circuits of M whose union is equal to V. If M is binary, then M is Eulerian iff V \in \mathcal{G}_\mathcal{M}. It is shown in [15] that a binary matroid M is Eulerian iff its dual M^* is bipartite. We obtain the following two dual characterizations.

**Theorem 6.8** Let M be a binary matroid.

1. M is bipartite iff M + V is an even \Delta-matroid.
2. M is Eulerian iff M + V is an even \Delta-matroid.

**Proof** (1) By the proof of [22] Proposition 2], M is bipartite iff each diagonal entry of G + X \setminus X is 1 where G is the fundamental graph of M with respect to some X \in M. Since \mathcal{M}_G = M \setminus X, M is bipartite iff M + X + V = M + V + X is an even \Delta-matroid. The latter is in turn equivalent to M + V being an even \Delta-matroid.

(2) Note M^* + V = M + V + V = M + V * V is even iff M + V is even. \qed
Remark 6.9 Theorem 6.8 suggests an alternative extension of the notions of bipartite and Eulerian from binary matroids to vf-safe (delta-)matroids. This alternative extension is natural as the two notions remain each others dual notions—the (original) notions of bipartite and Eulerian are known to not be dual for (nonbinary) matroids in general. For example, the vf-safe (in fact, quaternary) uniform matroid $U_{3,6}$ is bipartite, but $U_{3,6}^* = U_{3,6}$ is not Eulerian.

Remark 6.10 Proposition 1 of [22] (which is used in the proof of [22, Proposition 2]) states that a matroid $M$ is binary iff $M = \max\mathcal{M}_F$ for some graph $F$. In fact, the proof of this result implicitly takes $F = G + X \ast X$ where $G$ is the fundamental graph of $M$ with respect to $X$. Indeed, $M = \max\mathcal{M} = \max(M \ast X) = \max(\mathcal{M}_{G + X \ast X})$ where $G$ is such that $M_G = M \ast X$.

6.4 Example

Let us consider the graph $F$ of Figure 1 (left-hand side) with six (labeled) edges, and consider the cycle matroid $M$ of $F$ over $V = \{1, \ldots, 6\}$, see Figure 1 (right-hand side). To avoid notational clutter, we often denote sets within sets by juxtaposition in this example. The bases of $M$ are the six spanning trees of $F$, thus $M = (V, \{235, 236, 245, 246, 345, 346\})$.

The cycle space of $M$ has dimension 3, and is generated by $\{1, 234, 56\}$. Its cocycle space is also of dimension 3, and is generated by $\{23, 24, 56\}$.

The empty set $\emptyset$ is not a set in $M \ast V$, as $M$ has an even number of bases. However, $M \ast V$ contains $\{5\}$ and $\{6\}$, as both are contained in three bases. Thus the bicycle dimension of $M$ is $d_{M \ast V} = 1$.

From $M$ one constructs $M + V$ by adding the sets that contain an odd number of bases, these are eight 4-element sets and the 5-element sets $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 6\}$. Thus matroid $\max(M + V)$ equals $(V, \{12345, 12346\})$. Hence, the only nontrivial cycle of $\max(M + V)$ is $\{5, 6\}$.

Consider basis $Z = \{2, 4, 6\}$ of $M$. The standard representation of $M$ with respect to $Z$ is given in Figure 1 Using that representation we deduce the (bipartite) fundamental graph $G$ of $M$ with respect to $Z$ (the construction is described in [7]).

The $\Delta$-matroid $\mathcal{M}_G$ is equal to $M \ast Z = (V, \{3456, 34, 56, \emptyset, 2356, 23\})$.

Fig. 1 A graph $F$ and a binary representation of its cycle matroid $M$. 
The adjacency matrix of $G + V$ is as follows. Due to the simple block structure of the matrix, nullities are easily computed. From that we infer the tripartition using Corollary 6.7.

$$A(G + V) =
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 \\
3 & 0 & 1 & 1 & 0 & 0 \\
4 & 0 & 0 & 1 & 1 & 0 \\
5 & 0 & 0 & 0 & 1 & 1 \\
6 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The latter formulation allows us to consider $P_M(y)$ for arbitrary vf-safe $\triangle$-matroids $M$ (or, indeed, set systems) instead of binary matroids. We first consider the transition polynomial for set systems, which has the generalized Penrose polynomial as a specialization.

In this section all set systems are assumed to be proper. Note that $M_{\emptyset} = (\emptyset, \{\emptyset\})$ is proper.

7.1 Transition Polynomials

In a 4-regular graph we can partition the edges into a set of circuits (called a circuit partition). The number of resulting circuits depends on the choices (or transitions)
made at each vertex. Jaeger shows how several classic graph polynomials arise by counting circuits and applying particular weights for the three possible transitions at each vertex \[23\]. Here we follow this approach, but in the more abstract fashion given in \[15\].

Let \( V \) be a finite set. We define \( \mathcal{P}_3(V) \) to be the set of triples \((V_1, V_2, V_3)\) where \( V_1, V_2, \) and \( V_3 \) are pairwise disjoint subsets of \( V \) such that \( V_1 \cup V_2 \cup V_3 = V \). Therefore \( V_1, V_3, \) and \( V_3 \) form an “ordered partition” of \( V \) where \( V_i = \emptyset \) for some \( i \in \{1, 2, 3\} \) is allowed.

We now recall the transition polynomial for set systems from \[15\].

**Definition 7.1** Let \( M \) be a proper set system. We define the transition polynomial of \( M \) (weighted by \([a, b, c]\)) as follows:

\[
Q_{[a,b,c]}(M)(y) = \sum_{(A,B,C) \in \mathcal{P}_3(V)} d_A a |b| c |c| y^{d_A + b + c},
\]

The next lemma shows that \( P_M(y) \) is a specialization of \( Q_{[a,b,c]}(M)(y) \).

**Lemma 7.2** Let \( M \) be a proper set system. Then \( P_M(y) = Q_{[0,1,-1]}(M)(y) \).

**Proof** Indeed, we have \( Q_{[0,1,-1]}(M)(y) = \sum_{(A,B,C) \in \mathcal{P}_3(V)} (-1)^{|C|} y^{d_M + b + c} = \sum_{C \subseteq V} (-1)^{|C|} y^{d_M(V \setminus C) + c} \) and \( d_M(V \setminus C) + c = d_M(V \setminus C) + C + C \), where in the last equality we use that \( d_M + x = d_N \) for all set systems \( N \) over \( V \) and \( X \subseteq V \). Now, \( d_M(V \setminus C) + C + C = d_M(V \setminus C) + C \) and the result follows.

Several other specializations of the transition polynomial are well-known polynomials. It is shown in \[15\] that the two-variable interlace polynomial \[5\] of a graph \( G \) is equal to \( q(G)(x,y) = Q_{[1, x^{-1}, 0]}(\mathcal{M}_G)((y - 1)/(x - 1)) \) (the single-variable case is the case \( x = 2 \), see \[4\]). Moreover, \( Q_{[0, b, c]}(\mathcal{M}_G)(y) \) equals the bracket polynomial of \( G \) as studied in \[31\]. Furthermore, we recall from \[15\] that the Tutte polynomial \( t_M(x,y) \) for a matroid \( M \) is closely related to \( Q_{[a, b, 0]}(M)(y) \).

**Proposition 7.3 (Theorem 24 of \[15\])** Let \( M \) be a matroid. Then \( Q_{[a,b,0]}(M)(y) = a^y b^0 t_M(1 + \frac{y}{a}, 1 + \frac{b}{a}), \) where \( t_M \) is the Tutte polynomial.

We quote several results from \[15\] that will be useful in the present paper. The following result illustrates the effect on the transition polynomials \( Q_{[a,b,c]}(M) \) of the application of the operations \(+V, \ast V, \) and \( \ast V \).

**Proposition 7.4** (\[15\]) Let \( M \) be a proper set system over \( V \). Then \( Q_{[a,b,c]}(M)(y) = Q_{[a,b,c]}(M + V)(y) = Q_{[a,b,c]}(M \ast V)(y) = Q_{[a,b,c]}(M \ast V)(y) \).

We say that \( u \in V \) is singular in a proper set system \( M \) if \( u \) is a loop or a coloop of \( M \).

The transition polynomial satisfies a recursive formulation \[15\] Theorem 28]. Here we state the case where the \( c = 0 \).
Proposition 7.5 ([15]) Let $M$ be a $\Delta$-matroid, and let $Q(M)(y) = Q_{[a,b,0]}(M)(y)$. 
(0) For $M = M_S = (\emptyset, \emptyset)$ we have $Q(M)(y) = 1$.

Now let $u \in V$.
(1) If $u$ is nonsingular in $M$, then $Q(M)(y) = aQ(M \setminus u)(y) + bQ(M * u \setminus u)(y)$.
(2) If $u$ is a loop of $M$, then $Q(M)(y) = (a + by)Q(M \setminus u)(y)$.
(3) If $u$ is a coloop of $M$, then $Q(M)(y) = (b + ay)Q(M * u \setminus u)(y)$.

Note that the recursive relation of Proposition 7.5 characterizes $Q_{[a,b,0]}(M)(y)$.

Let us write for graphs $G$, $Q_{[a,b,c]}(G)(y) = Q_{[a,b,c]}(\mathcal{M}_G)(y)$.

Remark 7.6 Theorem 5.6 directly implies that the graph polynomials $Q_{[a,b,c]}(G)$ can be straightforwardly defined for inv-symmetric $V \times V$-matrices $A$ over $GF(4)$ in general instead of graphs $G$ i.e., symmetric $V \times V$-matrices over $GF(2)$, while maintaining their recursive relations and some of its evaluations. As an example, consider the interface polynomial $Q(A)(y) = \sum_{X \subset V} \sum_{Z \in \mathcal{X}} (y - 2)^{|X + Z|}$ of [3] for inv-symmetric $V \times V$-matrices $A$. Following the exact same reasoning as done in [15] for inv-symmetric $V \times V$-matrices $A$ instead of graphs $G$, we have that $Q(A)(y) = Q_{[1,1,1]}(\mathcal{X} - \mathcal{X})(y) = 2$. By Theorem 5.6, $\mathcal{X}$ is vf-safe. Again following the reasoning of [15] by using the recursive relation of $Q_{[a,b,c]}(M)(y)$ (of which Proposition 7.5 above is a special case), we have that (1) if $\{u,v\} \subseteq V$ with $u \neq v$, $A[[u]] = A[[v]] = 0$, and $A[[u,v]] \neq 0$, then $Q(A)(y) = Q(A \setminus u)(y) + Q(A \setminus v)(y) + Q(A \setminus \{u,v\})(y)$, and (2) if $A[V, [u]]$ is a zero vector, then $Q(A)(y) = yQ(A \setminus u)(y)$. Also, we have the evaluation $Q(A)(0) = 0$ if $|V| > 0$.

We now consider the case where $M$ is even.

Lemma 7.7 Let $M$ be a proper even set system. Then $Q_{[a,b,0]}(M)(y) = (-1)^{d_B - V} Q_{[-a,b,0]}(M)(y)$.

Proof We have $Q_{[a,b,0]}(M)(y) = \sum_{x \in \mathcal{X}} a^{[A]} b^{[B]} y^{d_{M,B}}$. Since $M$ is even, the parity of $|B| + d_{M,B}$ does not depend on $B$. Hence, $Q_{[a,-b,0]}(M)(y) = \sum_{x \in \mathcal{X}} a^{[A]} (-b)^{|B|} y^{d_{M,B}} = (-1)^{d_B - V} Q_{[a,b,0]}(M)(y)$. Similarly, since $M$ is even and $B = V \setminus A$, the parity of $|A| + d_{M,V,A}$ is $|V| - |B| + d_{M,B}$ does not depend on $B$ and thus does not depend on $B$. Hence, $Q_{[-a,b,0]}(M)(y) = \sum_{x \in \mathcal{X}} (-a)^{|A|} (-b)^{|B|} y^{d_{M,V,A}} = (-1)^{d_{M,V}} Q_{[a,b,0]}(M)(y)$.

7.2 The specialization $Q_{[1,-1,0]}(M)(y)$

Instead of directly studying the Penrose polynomial $P_M(y)$, we consider first $P_M + V * V$ which by Proposition 7.3 is equal to $Q_{[0,1,1]}(M + V * V)(y) = Q_{[1,0,-1]}(M + V)(y) = Q_{[1,-1,0]}(M)(y) = \sum_{x \in \mathcal{X}} (-1)^{|X|} y^{d_{M,x}}$. We denote $P_M + V * V(y) = Q_{[1,-1,0]}(M)(y)$ by $p_1(M)(y)$. The advantage of $p_1(M)(y)$ over $P_M(y)$ is that $p_1(M)(y)$ is defined in terms of twist of instead of the more elaborate loop complementation. As a result, it is easier to prove results for $p_1(M)(y)$ and then translate them to $P_M(y)$ instead of working directly with $P_M(y)$. Also, for some results concerning $p_1(M)(y)$, $M$ needs only to be $\Delta$-matroid, where the corresponding result for $P_M(y)$ requires $M$ to be a vf-safe $\Delta$-matroid.
Let $M$ be a proper set system. We obviously have, $p_1(M)(1) = 0$ if $|V| > 0$. Also, $p_1(M \ast X)(y) = (-1)^{|X|} p_1(M)(y)$ for all $X \subseteq V$. As a consequence, if $M$ is such that $M \ast X = M$ for some $X \subseteq V$ with $|X|$ odd, then $p_1(M)(y) = 0$. Note that, in this case, $M$ is not even. Finally note that by Proposition 7.3, $p_1(M)(y) = (-1)^{p(M)} t_M(1 - y, 1 - y)$ for the case where $M$ is a matroid.

Example 7.8 Let $F_7$ be the Fano matroid, see [26] for a description of this matroid. We compute $p_1(F_7)$ by case analysis on the cardinality of $X \subseteq V$, from 0 to 7, $p_1(F_7)(y) = y^3 - 7y^2 + 21y - (28y^0 + 7y^2) + 35y^3 - 21y^2 + 7y^3 - y^4 = -y^4 + 8y^3 - 35y^2 + 56y - 28$.

We claim that $F_7 \# V = F_7$. Clearly the triplets (three-element subsets) of $F_7$ and $F_7 \# V$ coincide. It is easy to verify that every smaller set belongs to an even number of bases, and does not belong to $F_7 \# V$. Each two-element subset belongs to five triplets, of which one is a line. Thus it is contained in four bases. Each single-element subset belongs to 15 triplets, including three lines. Thus it is contained in twelve bases. The empty set is contained in all 28 bases.

As $F_7 \# V = F_7$ we know that $F_7 + V \# V = F_7 + V$, thus $p_1(F_7 + V) = 0$. We also conclude that $F_7 + V$ consists of the 28 bases of $F_7$ and the 28 bases of $F_7^*$. By Proposition 7.4 we have the following recursive relation for $p_1(M)$.

Corollary 7.9 Let $M$ be a $\Delta$-matroid, and $u \in V$. If $u$ is nonsingular in $M$, then

$$p_1(M)(y) = p_1(M \setminus u)(y) - p_1(M \ast u \setminus u)(y).$$

If $u$ is a loop of $M$, then $p_1(M)(y) = (1 - y)p_1(M \setminus u)(y)$, and if $u$ is a coloop of $M$, then $p_1(M)(y) = (y - 1)p_1(M \ast u \setminus u)(y)$.

Remark 7.10 It may strike the reader that the recursive relations of Corollary 7.9 are very similar to the recursive relations of the characteristic polynomial $c_M(y)$ of a matroid $M$. Indeed, for matroids $M$, the recursive relations coincide when either (1) $u$ is nonsingular in $M$ or (2) $u$ is a coloop of $M$. However, when $u$ is a loop of $M$, then $c_M(y)$ differs as it is equal to 0.

Let us write for graphs $G$, $p_1(G)(y) = p_1(\mathcal{H}_G)(y)$. The next lemma shows that the graph polynomial $p_1(G)(y)$ may (in a way similar to the interface polynomial [24]) be recursively computed. We show in Theorem 7.15 below that $p_1(G)(y)$ computes (up to a sign) the Penrose polynomial for a binary matroid $M$ where $G$ is a particular graph depending on $M$. Again note that it is straightforward to consider more generally $p_1(A)(y)$ for inv-symmetric $V \times V$ matrices $A$ instead of graphs $G$, and results such as Lemma 7.11 can be formulated for $p_1(A)(y)$ as well. However, for convenience we choose to restrict to graphs $G$.

Lemma 7.11 Let $G$ be a graph. Then

$$p_1(G)(y) = \sum_{X \subseteq V} (-1)^{|X|} y^{v(G[X])}.$$ 

Moreover, $p_1(G)(y)$ satisfies the following characterizing recursive relation. If $u$ is a looped vertex, then

$$p_1(G)(y) = p_1(G \setminus u)(y) - p_1(G \ast u \setminus u)(y).$$
If \{u,v\} is an edge where both \( u \) and \( v \) are not looped, then
\[
p_1(G)(y) = p_1(G \setminus u)(y) + p_1(G \ast \{u,v\} \setminus u)(y).
\]

If \( u \) is an isolated vertex (i.e., no edge is adjacent to \( u \)) of \( G \), then
\[
p_1(G)(y) = (1 - y)p_1(G \setminus u)(y).
\]

Finally, if \( G \) is the empty graph, then \( p_1(G)(y) = 1 \).

**Proof** We have \( p_1(G)(y) = p_1(\mathcal{M}_G)(y) = \sum_{X \subseteq V} (-1)^{|X|}d_{\mathcal{M}_G}^X \). It is shown in [14] that \( d_{\mathcal{A}*X} = \nu(A[X]) \) for any symmetric or skew-symmetric matrix \( A \). Hence \( d_{\mathcal{M}*X} = \nu(G[X]) \).

If \( u \) is a looped vertex, then \( p_1(G)(y) = p_1(G \setminus u)(y) - p_1(G \ast u \setminus u)(y) \) follows from Corollary 7.9 and the fact that \( G \ast u \) is defined when \( u \) is looped.

If \( \{u,v\} \) is an edge where both \( u \) and \( v \) are not looped, then by Corollary 7.9
\[
p_1(G)(y) = p_1(\mathcal{M}_G)(y) = p_1(\mathcal{M}_G \setminus u)(y) - p_1(\mathcal{M}_G \ast u \setminus u)(y) = p_1(\mathcal{M}_G \setminus u)(y) + p_1(\mathcal{M}_G \ast u \setminus u)(y).
\]

An example of the recursive computation is given in Figure 4 (ignore the caption of the figure for now). The graph operations \( \ast u \) on a looped vertex and \( \ast \{u,v\} \) on an unlooped edge are known as local complementation and edge local complementation, respectively. Local complementation “complements” the edges in the neighbourhood \( N_G(u) = \{v \in V \mid \{u,v\} \in E(G), u \neq v\} \) of \( u \) in \( G \); for each \( v, w \in N_G(u) \), \( \{v, w\} \in E(G) \) iff \( \{v, w\} \notin E(G \ast \{u\}) \), and \( \{v\} \in E(G) \) iff \( \{v\} \notin E(G \ast \{u\}) \) (the case \( v = w \)).

The other edges are left unchanged. We will not recall the explicit graph theoretical definition of edge local complementation in this paper. It can be found in, e.g., [13].

7.3 The Penrose Polynomial as a specialization of \( Q(M) \)

Results for \( p_1(M)(y) = P_{M*V \ast V}(y) \) can be straightforwardly translated to \( P_M(y) \). Similar as for \( p_1 \), we have \( P_M(1) = 0 \) if \( |V| > 0 \). Since \( p_1(M \ast Z)(y) = (-1)^{|Z|}p_1(M)(y) \), we have \( P_{M+Z} = (-1)^{|Z|}P_M \) (this is also be easily verified from the definition of \( P_M \)).

**Example 7.12** For the Fano matroid \( F_7 \) and its dual \( F_7^\perp \) we compute the Penrose polynomial, cf. [2]. Thus, \( P_{F_7}(y) = p_1(F_7 \ast V \setminus V)(y) = p_1(F_7 \setminus V)(y) = y^4 - 8y^3 + 35y^2 - 56y + 28 \), see Example 7.8. Note that \( F_7 \setminus V = F_7 \), so \( F_7 \setminus V = F_7 \) and \( P_{F_7} = 0 \).

If \( M \) is an equicardinal set system, then \( \max(M \ast V) = M \ast V \) and thus, for every \( X \subseteq V \), the sets of \( M \ast V \) are also present in \( M \ast V \ast X \). Consequently, \( d_{M*V}*X \leq d_{M*V} \).

Therefore, the degree of \( P_M(y) \) can be at most \( d_{M*V} \). In fact, it cannot be less than that value, without becoming nontrivial, generalizing [2] Proposition 2.
Let $M$ be a vf-safe recursive relations do not exist for 4-regular graphs or matroids—one needs to step We find recursive relations that characterize the Penrose polynomial. It seems such

Note that Theorem 7.14 holds for every set system

For notational convenience we assume that

Moreover, if $u$ is a coloop of $M$ and let $u \in V$. If $u$ is nonsingular in

Finally, if $V = \varnothing$, then $P_M(y) = 1$.

Note that Theorem 7.14 holds for every set system $M$ such that $M \neq V$ and is a $\Delta$-matroid. For notational convenience we assume that $M$ is vf-safe, but the reader may easily recover the loss of generality.

We now show that, for a binary matroid $M$, $P_M(y)$ may also be viewed (up to a sign) as the graph polynomial $p_1(G)(y)$ where $G$ is a fundamental graph of $M$.

Let $M$ be a binary matroid and $Z$ a basis of $M$. Then $M \neq V \ast Z = G_{\bar{Z}}$ for some graph $G$ and $P_M(y) = (-1)^{|v(M)|} p_1(G)(y)$.

Since $Z \in M = \max(M)$, we have $Z \in M \neq V$, and so $\varnothing \in M \neq V \ast Z$. Hence $M \neq V \ast Z = G_{\bar{Z}}$ for some graph $G$. Moreover, $p_M(y) = P_M(y) = P_{G_{\bar{Z}}}(y) = p_1(G_{\bar{Z}})(y) = p_1(G_{\bar{Z}})(y) = (-1)^{|V \setminus Z|} p_1(G)(y)$. As $|V \setminus Z| = \nu(M)$, the result follows.

Example 7.15 Consider the cycle matroid $M_D$ of the diamond graph $D$ with edge set $V = \{1, 2, \ldots, 5\}$, cf. Figure 1. Its eight bases, the spanning trees of $D$, are all subsets of $V$ of cardinality $3$ except for the forbidden triangles $\{1, 4, 5\}$ and $\{2, 3, 5\}$. We
Fig. 3 Diamond graph $D$.

$$p_1 = 4(1-y)(2-y)$$

$$p_1 = 2(1-y)(2-y)$$

$$p_1 = (1-y)(2-y)$$

$$p_1 = (y-1)(2-y)$$

$$p_1 = (1-y)^2$$

$$p_1 = 1 - y$$

compute the Penrose polynomial $P_{M_D}(y)$ recursively with the help of Theorem 7.15 and Lemma 7.11.

We have that $M_D \# V$ is obtained from $M_D$ by adding the family of sets $\{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}$. Now, binary $\Delta$-matroid $M_D \# V \ast \{1,2,3\}$ contains $\emptyset$, which hence represents a graph $G$. The sets in $M_D \# V \ast \{1,2,3\}$ of cardinality one or two (which uniquely determine $G$) are $\{\{2\}, \{3\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}\}$.
By applying the graph operations of local complementation and edge local complementation, we determine the polynomial $p_1(G)(y) = 4(1 - y)(2 - y)$, see Figure 2. Thus $P_M(y) = 4(y - 1)(2 - y)$.

We now extend the result from [2] Proposition 1] that for an Eulerian matroid $M$ the value $P_M(2)$ of the Penrose polynomial of a binary matroid equals the size of its cocycle space. We also consider $P_M(-1)$.

**Theorem 7.17** Let $M$ be a set system such that $M \uparrow V$ is even. Then $P_M(y) = (-1)^{d_{M \uparrow V}} Q_{\{1,0\}}(M \uparrow V)(y)$. In particular, $P_M(-1) = (-1)^{d_{M \uparrow V}} 2^{|V|}$. If $M$ is moreover a vf-safe $\Delta$-matroid, then $P_M(2) = (-1)^{d_{M \uparrow V}} 2^{|V|} d_M$.

**Proof** Note that $M \uparrow V$ is an even set system iff $M \ast V + V$ has that property. Thus, we have by Lemma 7.17 $P_M(y) = p_1(M \ast V + V)(y) = (-1)^{d_{M \uparrow V}} Q_{\{1,1,0\}}(M \ast V + V)(y)$, where in the last equality we use that $d_{M \uparrow V} = d_N$ and $Q_{\{1,1,0\}}(N \ast V)(y) = Q_{\{1,0\}}(N)(y)$ for every set system $N$ over $V$. Note also that $Q_{\{1,0\}}(N)(1) = 2^{|V|}$ for every set system $N$ over $V$.

By [15] we have $Q_{\{1,0\}}(N)(-2) = (-1)^{|V|}(-2)^{d_{M \uparrow V}}$ for any vf-safe $\Delta$-matroid $N$. Thus, $P_M(2) = (-1)^{d_{M \uparrow V}} Q_{\{1,1,0\}}(M \uparrow V)(-2) = (-1)^{d_{M \uparrow V}} (-2)^{d_M}$. \[\square\]

As another example, for the nonbinary matroid $U_{2,5}$ we have $U_{2,5} \uparrow V = U_{2,5}$, so $U_{2,5} \uparrow V$ is even. Then $P_{U_{2,5}}(y) = p_1(U_{2,5} \ast V + V)(y) = p_1(U_{2,5} \uparrow V)(y) = p_1(U_{2,5})(y) = y^3 - 5y^2 + 10y - 10 + 5y - y^2 = y^3 - 6y^2 + 15y - 10$. Thus $P_{U_{2,5}}(2) = 4 = 2^{p(U_{2,5})}$, as predicted by Theorem 7.17.

We now turn to binary matroids. By Theorem 7.17 and Theorem 6.8(2) we have the following result (we also use that $Q_{\{1,1,0\}}(N)(1) = 2^{|V|}$ for every set system $N$).

**Corollary 7.18** Let $M$ be an Eulerian binary matroid. Then we have $P_M(y) = (-1)^{|V|M} Q_{\{1,1,0\}}(M \uparrow V)(y)$, $P_M(-1) = (-1)^{|V|M} 2^{|V|}$, and $P_M(2) = 2^{p(M)}$.

We now generalize the equality between the Penrose polynomial at $-2$ and the Tutte polynomial at $(0, -3)$, given in [2] Theorem 2, from binary matroids to vf-safe matroids. It is well known that $|t_M(0, -3)|$ is equal to the number of nowhere-zero 4-flows of a binary matroid $M$.

**Corollary 7.19** Let $M$ be a vf-safe matroid. Then $P_M(-2) = 2^{p(M)} t_M(0, -3)$ where $t_M(x,y)$ is the Tutte polynomial.

**Proof** First we need the following auxiliary result. Let $a, b, c, d$ be arbitrary values. Then $Q_{\{a,b,c,d\}}(M)(-2) = Q_{\{a+b+d+c\}}(M)(-2)$ for vf-safe $\Delta$-matroids $M$. This equality is a special case of [15] Theorem 7, which is more generally stated there in terms of tight matroids (the conversion to vf-safe $\Delta$-matroids is similar as in the proof of Theorem 38.2 in [15]).

By the above auxiliary result, we have $P_M(-2) = Q_{\{0,1,0\}}(M)(-2) = Q_{\{1,2,0\}}(M)(-2)$. Now by Proposition 7.3 we obtain $Q_{\{1,2,0\}}(M)(-2) = 2^{p(M)} t_M(0, -3)$. \[\square\]
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