Optimum design of complementary transient experiments for estimating thermal properties

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Abstract. In the design of transient thermal experiments for estimating the thermal conductivity, k, and volumetric heat capacity, C, the scaled sensitivity coefficients are utilized. These coefficients should be large and uncorrelated. In the current paper it is proven that they are the largest when the heated surface temperature of the sample is held at the maximum value. Hence, they can be assumed as ‘ideal reference sensitivities,’ which actual k- and C-sensitivity coefficients can refer to in order to evaluate their relative magnitude. In addition, by using two complementary experiments, the sensitivity coefficients can be made more uncorrelated than if a single experiment is used. The result of two combined experiments is that the above two parameters are found with the greatest accuracy. The parameters are also of equal accuracy, which is generally not the case. This problem is unique not only because of intrinsic solutions between the small and large time solutions but in the exchanging of the roles in the scaled sensitivity coefficients for k and C.

1. Introduction

Thermal properties need to be measured using transient methods for several reasons. One of which is to estimate different properties simultaneously [1]. Another is that properties of a medium (such as a biological tissue [2]) can change during heating.

One criterion for optimal experiments involves maximization of the confidence region. See [1, pp. 444 - 456]. One implication of this criterion is that the scaled sensitivity coefficients should be large and uncorrelated. Many geometries can be utilized, including one-dimensional and two-dimensional rectangular bodies and also 1D and 2D radial and axial ones. Various boundary conditions can also be imposed including temperature and heat flux, which are called the first and second kinds. Fourth-kind [3] accounting for thermal inertia of the heater as well as time-dependent boundary conditions are also possible. Another major consideration is the location of the sensors and the duration of the experiment.

In this work, in order to obtain the largest and the most uncorrelated sensitivity coefficients when simultaneously estimating thermal conductivity k and volumetric heat capacity C, two ideal transient experiments are considered. The former is the X21B(X11B10T0)0T0 problem, where “B(X11B10T0)” indicates that the time-dependent heat flux applied to the boundary x = 0 is the one coming from the surface heat flux of the X11B10T0 problem (which has prescribed temperature at x = 0, zero temperature at x = L and is initially at zero temperature). (See Refs. [1, 2] for the heat conduction numbering system devised by Cole et al.). The latter is a complementary problem denoted...
by \(X_{22B}(X_{12B10T0})0T0\), where the heat flux corresponding to the \(X_{12B10T0}\) problem (which has prescribed temperature at \(x = 0\) and zero heat flux at \(x = L\)) is applied to the sample of interest. It is proven that the heated surface scaled sensitivity coefficients for these two problems are the largest and, hence, they can be assumed as ‘ideal reference sensitivities,’ which actual \(k\)- and \(C\)-sensitivity coefficients can refer to. In addition, by using two combined complementary experiments as was done in Ref. [4], the sensitivity coefficients can be made more uncorrelated than if a single experiment is used. In other words, they allow us to minimize the area or hyper-volume of the confidence region [1, Chap. 8]. Concerning this, the confidence region is recalled to be at a minimum when the determinant of the related matrix \(X^T X\) for two parameters is a maximum, where \(X\) is the sensitivity matrix. When computing the above sensitivity functions, an exact solution for the \(X_{21B}(X_{11B10T0})0T0\) and \(X_{22B}(X_{12B10T0})0T0\) problems for the whole time region is derived using Green’s functions [5]. The small time solutions are derived too (still using GFs) and the relationship indicated above is demonstrated. Alternatively, the small time solutions can be derived using Laplace transforms [6]. An advantage of the early time solutions [7] is that they are mathematically simpler and the time range of validity is larger than one would expect. Both the early time and whole time solutions are helpful for several reasons. One is the computational advantage for early times of the one and the efficiency of another solution for moderate large time. Another reason is that having two different solutions for the “same” problem provides an indication of intrinsic verification [8,9]. Actually, this problem is unique not only because of intrinsic verification between the small and large time solutions but in the exchanging of the roles in the scaled sensitivity coefficients for \(k\) and \(C\).

2. \(X_{21B}(X_{11B10T0})0T0\) transient experiment

Consider the 1D transient experiment where the thin heater is located between two samples of the same thickness whose back surfaces are kept at zero temperature. A dimensional mathematical statement for this thermally-symmetric experiment denoted by \(X_{21B}(X_{11B10T0})0T0\) is

\[
\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L; \ t > 0) \quad (1a)
\]

\[-k \frac{\partial T}{\partial x}(0,t) = \phi_{X_{11B10T0}}(0,t) \quad (t > 0) \quad (1b)\]

\[T(L,t) = 0 \quad (t > 0) \quad (1c)\]

\[T(x,0) = 0 \quad (0 < x < L) \quad (1d)\]

where \(\phi_{X_{11B10T0}}(0,t)\) is the surface heat flux supplied by the thin heater during measurements. It is assumed that this heat flux is the same as the surface heat flux of the \(X_{11B10T0}\) case and, hence, can be fitted by using the following correlation in a series form

\[
\phi_{X_{11B10T0}}(0,t) = \frac{\phi_0}{\sqrt{\pi \phi t / L^2}} \sum_{n=0}^{\infty} \left\{ \exp\left(-\frac{n^2 L^2}{\phi t}\right) + \exp\left(-\frac{(n+1)^2 L^2}{\phi t}\right) \right\} \quad (2a)
\]

where \(\phi_0\) (units of \(W \cdot m^{-2}\)) and \(\phi\) (units of \(m^2 \cdot s^{-1}\)) are empirical coefficients known. The above heat flux tends to infinity for small times and goes to a constant for very large times. Alternatively, another series correlation is given by

\[
\phi_{X_{11B10T0}}(0,t) = \phi_0 \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\frac{n^2 \alpha}{L^2}} \right) \quad (2b)
\]
where \( \beta_m = m\pi \) are the “X11” dimensionless eigenvalues, as discussed in next subsection.

2.1 Heat flux supplied by the heater. The X11B10T0 case

The temperature to the X11B10T0 case, valid for all \( t > 0 \), is given by Carslaw and Jaeger [10] as

- short-time form [10, p. 310, equation (6)]

\[
T_{x11b10t0}(x,t) = T_0 \sum_{n=0}^{\infty} \left\{ \text{erfc} \left( \frac{2nL+x}{\sqrt{4\alpha t}} \right) - \text{erfc} \left( \frac{2(n+1)L-x}{\sqrt{4\alpha t}} \right) \right\}
\]  

(3a)

- large-time form [10, p. 313, equation (10)]

\[
T_{x11b10t0}(x,t) = T_0 \left[ 1 - \frac{x}{L} - 2 \sum_{n=1}^{\infty} \sin \left( \frac{\beta_n x}{L} \right) \frac{\beta_n e^{-\beta_n^2 t}}{\beta_n^2} \right]
\]  

(3b)

The solution for the heat flux at \( x = 0 \) is, respectively,

\[
q_{x11b10t0}(0,t) = \frac{kT_0}{\sqrt{\pi \alpha t}} \sum_{n=0}^{\infty} \left[ \exp \left( -\frac{n^2 L^2}{\alpha t} \right) + \exp \left( -\frac{(n+1)^2 L^2}{\alpha t} \right) \right]
\]  

(4a)

\[
q_{x11b10t0}(0,t) = \frac{kT_0}{L} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\beta_n^2 t / \tau} \right)
\]  

(4b)

where the group \( kT_0 / L \) (units of \( \text{W} \cdot \text{m}^{-2} \)) corresponds to the coefficient \( \phi_0 \) of Eqs. (2a) and (2b) as well as \( \alpha \) (units of \( \text{m}^2 \cdot \text{s}^{-1} \)) corresponds to \( \Phi \) of the same correlations.

The temperature solution to the X21B(X11B10T0)0T0 experiment at \( x = 0 \) (where the sensor is located) is discussed in next subsections for the whole time region and for early times.

2.2 Exact solution for the whole time region

Using the heat flux defined by Eq. (2b) as being known during measurements, the surface temperature of the specimen can be computed using large-cotive Green’s functions [5] as

\[
T_{x21b(x11b10t0)0t0}(0,t) = \frac{1}{C} \int_{\tau=0}^{t} \varphi_{x11b10t0}(0,\tau) G_{x21}(0,0,0,\tau) d\tau
\]

\[
= \frac{\phi_0}{C} \int_{\tau=0}^{t} \left( 1 + 2 \sum_{n=1}^{\infty} e^{-\beta_n^2 \tau / \tau} \right) \frac{2}{L} \sum_{n=1}^{\infty} e^{-\eta_n^2 \Omega(\tau - \tau)} d\tau, \quad \eta_n = (n-1/2)\pi
\]  

(5)

By some algebra given in Appendix A, Eq. (5) becomes

\[
T_{x21b(x11b10t0)0t0}(0,t,\phi/kC) = \frac{\phi_0 L}{k} \left\{ 1 - 2 \sum_{n=1}^{\infty} e^{-\eta_n^2 \Omega(1-\phi/kC) / \eta_n^2} + 2 \sum_{n=1}^{\infty} e^{-\eta_n^2 \Omega(1-\phi/kC) / \eta_n^2} \right\}
\]  

(6)
where, for an accuracy of about $10^{-4}$ (with $A = 2, 3, \ldots, 15$), the required number terms for either summation is [11]

$$
N = \text{ceil} \left( \frac{1}{\pi} \sqrt{A \ln(10) \frac{1}{\tilde{t}}} + \frac{1}{2} \right), \quad M = \text{ceil} \left( \frac{1}{\pi \sqrt{C \phi / k}} \sqrt{A \ln(10) \frac{1}{\tilde{t}}} \right) \quad (7)
$$

where $\tilde{t} = \alpha t / L^2 = kt / (CL^2)$. For the extremely accurate value of $A = 15$ (errors less than $10^{-15}$), the required numbers of terms for the $n$-summation for times $\tilde{t} = 0.01, 0.1$ and 0.5 are respectively, 20, 7 and 4. Even the number of 20 terms is not large. Therefore, the solution defined by Eq. (6) converges nicely with a relatively small numbers of terms. Furthermore, a more efficient solution is available for small dimensionless times, as discussed in the next subsections.

Before proceeding to calculate the sensitivity coefficients, it may be noted that equation (6) produces $\phi_s L / k = (T_{\text{rise}})_m$ as time goes to infinity, which represents the sample temperature rise for $t \to \infty$. Actually, the same result is obtained for all times equal to or greater than zero for $\phi = k / C$ since $\cot(\eta_p) = 0$ and $\tan(\beta_p) = 0$ in Eq. (6).

The sensitivity coefficients for $k$ and $C$ can be obtained from equation (6) by direct differentiation. However, it is more complicated than utilizing finite differences, which is possible because the computed temperatures can be calculated so precisely. A central finite difference scheme for computing the dimensionless scaled $k$- and $C$-sensitivities $X_{k,21}$ and $X_{C,21}$ is

$$
X_{k,21} = \frac{k}{(T_{\text{rise}})_m} \left( \frac{\partial T_{X21}(0,\tilde{t})}{\partial k} \right) - \frac{T_{X21}(0,\tilde{t},k(1+\varepsilon),C) - T_{X21}(0,\tilde{t},k(1-\varepsilon),C)}{2\varepsilon (T_{\text{rise}})_m} \quad (8a)
$$

$$
X_{C,21} = \frac{C}{(T_{\text{rise}})_m} \left( \frac{\partial T_{X21}(0,\tilde{t})}{\partial C} \right) - \frac{T_{X21}(0,\tilde{t},k,C(1+\varepsilon)) - T_{X21}(0,\tilde{t},k,C(1-\varepsilon))}{2\varepsilon (T_{\text{rise}})_m} \quad (8b)
$$

where the subscript “$X21$” can be used as a short notation. The errors using this equation are about $\varepsilon^2$ so that if $\varepsilon = 0.00001 = 10^{-5}$, the errors would be about $10^{-10}$. Numerical experiments with $\varepsilon$ values from 0.01 to 0.00000001 have shown that both small values, such as 0.01 or 0.001, can cause significant errors; but extremely small values such as 0.00000001 also can. To get an accuracy of about 8 decimal places when compared with the sensitivity coefficients for $k$ and $C$ obtained from equation (6) by direct differentiation, an $\varepsilon$ value of 0.00001 is recommended.

2.3 One-term approximate solution for very short times

For very early times, less than the so-called deviation time defined as [7, 8]

$$
\tilde{t}^{(1)} = \frac{0.1}{A \alpha} (2L - x)^3 \quad (A = 2, 3, \ldots, 15), \quad (9)
$$

the temperature and heat flux of the X11B10T0 finite problem treated in Subsection 2.1 are the same (with errors less than $10^{-4}$) as the ones of the X10B1T0 semi-infinite case. The solution is [5, 10]

$$
q_{X11B10T0}(x,t) = q_{X10B1T0}(x,t) = \frac{T_h k}{\sqrt{\pi \alpha t}} e^{-x^2 / \alpha t} \quad (t \leq \tilde{t}^{(1)}) \quad (10)
$$

At the boundary surface $x = 0$, the heat flux is
\[ q_{X_{11}B_{10}T_0}(0,t) = q_{X_{10}B_{10}T_0}(0,t) = \frac{T_i k}{\sqrt{\pi \alpha t}} \] 

\((t \leq 0.4L^2 / (A\alpha))\quad (11)\)

Therefore, the heat flux applied to the sample during measurements defined by Eqs. (2a) or (2b) may be taken at very early times as

\[ \varphi_{X_{11}B_{10}T_0}(0,t) = \varphi_{X_{10}B_{10}T_0}(0,t) = \frac{\varphi_0}{\sqrt{\pi \alpha t / L^2}} \] 

\((t \leq 0.4L^2 / (A\phi))\quad (12)\)

The surface temperature for the X21B(X11B10T0)T0 problem given by Eq. (6) for the whole time region can accurately be replaced at early times (less than the deviation time) with the surface temperature for the X21B(X10B1T0)T0 case. This can be calculated using Green’s functions [5] where a further approximation with errors less than \(10^{-4}\) can be performed for co-times \((t - \tau) \leq 0.1 \cdot (2L - x)^2 / (A\alpha)\). In fact,

\[ G_{X_{21}}(x,0,t-\tau) = G_{X_{20}}(x,0,t-\tau) = \frac{1}{\sqrt{\pi \alpha (t-\tau)}} \exp\left[-\frac{x^2}{4\alpha(t-\tau)}\right] \] 

\((13)\)

where \(\alpha = k / C\) is the unknown thermal diffusivity of the sample. Therefore,

\[ T_{X_{21}B(X_{11}B_{10}T_0)T_0}(0,t,k) = T_{X_{20}B(X_{10}B_{10}T_0)T_0}(0,t,k) = \frac{1}{C} \int_{t=0}^{\infty} \varphi_{X_{10}B_{10}T_0}(0,T)G_{X_{20}}(0,0,t-\tau)d\tau \]

\[ = \frac{\varphi_0 L}{\pi \sqrt{(kC)\phi}} \frac{d\tau}{\sqrt{\tau(t-\tau)}} = \frac{\varphi_0 L}{\sqrt{(kC)\phi}} = \frac{(T_{rise})}{\sqrt{C\phi / k}} = (T_{rise})_0 \]

\((14)\)

that is valid with errors less than \(2 \cdot 10^{-4}\) for \(t \leq \min\{0.4L^2 / (A\alpha), 0.4L^2 / (A\phi)\}\). Also, \((T_{rise})_0\) is the temperature rise at \(t = 0\); while \((T_{rise}) = \varphi_0 L / k\) is the temperature rise for \(t \to \infty\). Equation (14) states that the surface temperature is constant for a while when the sample heating starts. In particular, for \(\phi = k / C\), this constant is equal to \((T_{rise})_\alpha\).

The \(k\)- and \(C\)-scaled sensitivity coefficients at the heated surface \(x = 0\) (where the temperature sensor is located) can be obtained by direct differentiation as

\[ k \frac{\partial T_{X_{21}}}{\partial k}(0,t) = C \frac{\partial T_{X_{21}}}{\partial C}(0,t) = \frac{1}{2} \frac{\varphi_0 L}{\sqrt{(kC)\phi}} \right) = \frac{1}{2} \frac{(T_{rise})_0}{\sqrt{C\phi / k}} = \frac{1}{2} (T_{rise})_0 \]

\((15)\)

Notice that these two sensitivity coefficients are equal and functions of the product \(kC\). In particular, they are both equal to the negative of the temperature rise divided by 2. Therefore, their sum is equal to the negative of the temperature rise at the heated surface according to the sensitivity equation [3]

\[ k \frac{\partial T_{X_{21}}}{\partial k}(0,t) + C \frac{\partial T_{X_{21}}}{\partial C}(0,t) = -T_{X_{21}B(X_{11}B_{10}T_0)T_0}(0,t) \]

\((16)\)

In a dimensionless form, with errors less than \(2 \cdot 10^{-4}\), Eq. (15) becomes

\[ X_{s,21} = \frac{k}{(T_{rise})_\alpha} \frac{\partial T_{X_{21}}}{\partial k}(0,t) = X_{C,21} = \frac{C}{(T_{rise})_\alpha} \frac{\partial T_{X_{21}}}{\partial C}(0,t) = -\frac{1}{2\sqrt{C\phi / k}} \]

\((17)\)
where \( \tilde{t} = \alpha t / L^2 = kt / (CL^2) \). Also, for \( \phi = k / C \), it results in \( X_{1X21} = X_{C,X21} = -1 / 2 \).

### 2.4 Three-terms approximate solution for short times

For times less than a certain time given by

\[
t_d^{(2)} = \frac{0.1}{A\alpha} (2L + x)^2 \quad (A = 2,3,...,15),
\]

which is still called deviation time but is longer than the previous one defined by equation (9), the exact solutions defined by Eqs. (3a) and (4a) can be computed with only two terms \( (n = 0 \text{ and } 1) \) with errors less than \( 10^{-4} \). However, as for the location \( x = 0 \) (which is here of interest) it results in \( t_d^{(2)} = t_d^{(3)} \), another and longer deviation time has to be considered. It is defined as

\[
t_d^{(3)} = \frac{0.1}{A\alpha} (4L - x)^2 \quad (A = 2,3,...,15),
\]

and the exact solutions stated before can be computed with only three terms \( (n = 0, 1, 2) \) with errors less than \( 10^{-4} \) for \( t \leq t_d^{(3)} \). The superscripts “(1),” “(2)” and “(3)” appearing in the symbol of the deviation time indicate the number of terms required, which are one, two and three, respectively. In particular, the first three terms for the surface heat flux defined by equation (4a) are

\[
q_{X11\theta10T0}(0,t) = \frac{kT_0}{\sqrt{\pi\alpha t}} \left( 1 + e^{-\frac{x^2}{\alpha t}} + e^{-\frac{L^2}{\alpha t}} \right) \quad (t \leq 1.6L^2 / (A\alpha))
\]

Therefore, the heat flux applied to the sample during measurements defined by Eqs. (2a) or (2b) may be taken at early times as

\[
\phi_{X11\theta10T0}(0,t) = \frac{\phi_0}{\sqrt{\pi\phi t} / L^2} \left( 1 + 2e^{-\frac{L^2}{\phi t}} \right) \quad (t \leq 1.6L^2 / (A\phi))
\]

Then, the surface temperature of the sample can still be derived using the Green’s function solution equation. Also, the Green’s function for the X21 case at \( x = x' = 0 \) can be replaced with the three-term approximate GF with errors less than \( 10^{-4} \) for co-times \( (t - \tau) \leq 1.6L^2 / (A\alpha) \). It is [5]

\[
G_{X21}(0,0,t - \tau) = \frac{1}{\sqrt{\pi\alpha(t - \tau)}} \left[ 1 - 2e^{\frac{L^2}{\alpha(t - \tau)}} \right]
\]

The solution for the surface temperature is mathematically given by

\[
T_{X21X11\theta10T0T0}(0,t) = \frac{1}{C} \int_{t=0}^{t} \phi_{X11\theta10T0}(0,\tau) G_{X21}(0,0,t - \tau) d\tau
\]

\[
= \frac{\phi_0L}{\pi\sqrt{(k\alpha\phi)}} \int_{t=0}^{t} \frac{1}{\sqrt{\tau(t - \tau)}} \left[ 1 - 2e^{\frac{cL^2}{k(t-\tau)}} + e^{\frac{L^2}{\phi \tau}} \right] d\tau
\]

where the cross term was dropped as negligible. Some needed integrals are given in Mathematica. For convenience, one of these integrals is
\[
\int_{t-0}^{t} e^{-\frac{\tau^2}{2\sigma t}}d\tau = \pi \text{erfc}\left(\sqrt{\frac{\tau^2}{\sigma t}}\right)
\]

\[\quad (\sigma = k / C \quad \text{or} \quad \phi) \quad (23)\]

Using these integrals, equation (22) produces

\[
T_{x_{21}|x_{11}|0|0|0}(0, t, \phi) = \frac{\phi_0 L}{(kC)\phi} \left[ 1 - 2\text{erfc}\left(\sqrt{\frac{CL^2}{kt}}\right) + 2\text{erfc}\left(\sqrt{\frac{E}{\phi t}}\right) \right]
\]

(24a)

where

\[
\frac{\phi_0 L}{(kC)\phi} = (T_{rise})_0 = \frac{(T_{rise})_0}{\sqrt{C\phi / k}} = \frac{\phi_0 L / k}{\sqrt{C\phi / k}}
\]

(24b)

The above expression is valid with errors less than \(2 \times 10^{-4}\) for \(t \leq \min\{1.6L^2 / (A\alpha), 1.6L^2 / (A\phi)\}\).

Also, it states that the surface temperature is time-independent for \(\phi = k / C\) (as expected) and equal to \(\phi_0 L / k = (T_{rise})_0\), which represents the sample temperature rise for \(t \rightarrow \infty\).

The thermal conductivity scaled sensitivity coefficient obtained by differentiating equation (24a) is

\[
k \frac{\partial T_{x_{21}}}{\partial k}(0, t, \phi) = \frac{1}{2} T_{x_{21}|x_{11}|0|0|0}(0, t) - \frac{2\phi_0 L / k}{\sqrt{\pi k t / L^2}} e^{\frac{CL^2}{t}}
\]

\[\quad = \frac{1}{2} \left(\frac{T_{rise}}{\sqrt{C\phi / k}}\right) - \frac{4}{\sqrt{\pi k t / (CL^2)}} e^{\frac{CL^2}{t}} - 2\text{erfc}\left(\sqrt{\frac{CL^2}{kt}}\right) + 2\text{erfc}\left(\sqrt{\frac{E}{\phi t}}\right)\]

(25)

Similarly, the volumetric heat capacity scaled sensitivity coefficient is

\[
C \frac{\partial T_{x_{21}}}{\partial C}(0, t, \phi) = \frac{1}{2} T_{x_{21}|x_{11}|0|0|0}(0, t) + \frac{2\phi_0 L / k}{\sqrt{\pi k t / L^2}} e^{\frac{CL^2}{t}}
\]

\[\quad = \frac{1}{2} \left(\frac{T_{rise}}{\sqrt{C\phi / k}}\right) - \frac{4}{\sqrt{\pi k t / (CL^2)}} e^{\frac{CL^2}{t}} - 2\text{erfc}\left(\sqrt{\frac{CL^2}{kt}}\right) + 2\text{erfc}\left(\sqrt{\frac{E}{\phi t}}\right)\]

(26)

The sum of these two scaled sensitivity coefficients is equal to the negative of the temperature rise at the heated surface according to the sensitivity equation (16). In a dimensionless form, they are

\[
X_{,x_21} = \frac{k}{(T_{rise})_0} \frac{\partial T_{x_{21}}}{\partial k}(0, t, \phi) = -\frac{1}{2\sqrt{C\phi / k}} \left[ 1 + \frac{4e^{-\frac{CL^2}{t}}}{\sqrt{\pi t}} - 2\text{erfc}\left(\frac{1}{\sqrt{t}}\right) + 2\text{erfc}\left(\frac{1}{\sqrt{t}(C\phi / k)}\right) \right] \quad (27a)
\]

\[
X_{,c_{x21}} = \frac{C}{(T_{rise})_0} \frac{\partial T_{x_{21}}}{\partial C}(0, t, \phi) = -\frac{1}{2\sqrt{C\phi / k}} \left[ 1 - \frac{4e^{-\frac{CL^2}{t}}}{\sqrt{\pi t}} - 2\text{erfc}\left(\frac{1}{\sqrt{t}}\right) + 2\text{erfc}\left(\frac{1}{\sqrt{t}(C\phi / k)}\right) \right] \quad (27b)
\]

where \(\tilde{t} = \alpha t / L^2 = kt / (CL^2)\). At very early times, it is convenient to use equations (17) based on one-term approximation. As regards Eqs. (8a) and (8b) related to Eq. (6) with the number of terms defined by Eq. (7), they are computationally efficient for \(\tilde{t} \geq \min\{1.6 / A, 1.6 / [A(C\phi / k)]\}\) though they are
valid for the whole time region. Substituting $\tilde{t} = 1.6 / A$ in the first of the two Eqs. (7) gives $N = \text{ceil}(0.4A + 0.5)$, that is, 7 terms for an accuracy of $A = 15$ (errors less than $10^{-15}$). For $\phi = k / C$, 

$$X_{i,X21} = -\frac{1}{2} \left( 1 + \frac{4e^{-2i}}{\sqrt{\pi t}} \right) \quad X_{c,X21} = -\frac{1}{2} \left( 1 - \frac{4e^{-2i}}{\sqrt{\pi t}} \right)$$

(27c)

2.5 Numerical results

Some numerical values (to 8 digits, i.e. $A = 8$) for the scaled sensitivity coefficients are given in Table 1 for $\phi = k / C$. They were obtained using equation (8), with equation (6), Eq. (17) for $\phi = k / C$ and Eq. (27c), each in its own time range of validity. In detail, equations (8) and (6) were used for $i \geq 0.3$, equation (17) for $\tilde{t} = 0.01$ and equation (27c) for $\tilde{t} = 0.1$ and 0.2. In particular, when dealing with equations (8) and (6), the maximum number of terms defined by equation (6) for $\tilde{t} = 0.3$ was only of $N = M = 3$. A plot is shown in figure 1 for visual purposes and for $\phi = k / C$. This figure indicates that, in the current theoretical experiment with $\phi = k / C$, the $k$-sensitivity is the largest and can be assumed as an ‘ideal reference sensitivity’ which $k$-sensitivity coefficients of real experiments can refer to in order to evaluate their relative magnitude.

**Table 1.** Dimensionless $k$- and $C$-scaled sensitivity coefficients for the heated surface of the X21B(X11B10T0)0T0 and X22B(X12B10T0)0T0 (discussed ahead) problems when $\phi = k / C$.

| $\tilde{t}$ | $X_{i,X21} = X_{c,X22}$ | $X_{c,X21} = X_{c,X22}$ | $X_i + X_c$ |
|-------------|-----------------|-----------------|-------------|
| 0.01        | -0.50000000     | -0.50000000     | -1.00000000 |
| 0.10        | -0.50016200     | -0.49983800     | -1.00000000 |
| 0.20        | -0.51700073     | -0.48299927     | -1.00000000 |
| 0.30        | -0.57349303     | -0.42650697     | -1.00000000 |
| 0.40        | -0.64644983     | -0.3535017      | -1.00000000 |
| 0.50        | -0.71596389     | -0.28403611     | -1.00000000 |
| 0.60        | -0.77514144     | -0.22485865     | -1.00000000 |
| 0.70        | -0.82321393     | -0.17678605     | -1.00000000 |
| 0.80        | -0.86146119     | -0.13853881     | -1.00000000 |
| 0.90        | -0.89160143     | -0.10839857     | -1.00000000 |
| 1.00        | -0.91524675     | -0.08475325     | -1.00000000 |
| 1.25        | -0.95424011     | -0.04575989     | -1.00000000 |
| 1.50        | -0.97530407     | -0.02469593     | -1.00000000 |
| 1.75        | -0.98667288     | -0.01332712     | -1.00000000 |
| 2.00        | -0.99280812     | -0.00719188     | -1.00000000 |
| 2.25        | -0.99611896     | -0.00388104     | -1.00000000 |
| 2.50        | -0.99790563     | -0.00209437     | -1.00000000 |
| 2.75        | -0.99886979     | -0.00113021     | -1.00000000 |
| 3.00        | -0.99939009     | -0.00060991     | -1.00000000 |
| 8.00        | -1.00000000     | -0.00000000     | -1.00000000 |

a Numerical values obtained using Eq. (17) for both X21 and X22 cases.
b Numerical values obtained using Eq. (27c) for X21 and Eq. (32) with $\phi = k / C$ for X22
c Numerical values obtained using Eqs. (6) and (8) for X21 and Eqs. (30) and (31) for X22.
Figure 1. Dimensionless scaled sensitivity coefficients for $k$ and $C$ for the heated surface location of the X21B(X11B10T0)0T0 and X22B(X12B10T0)0T0 (discussed ahead) problems for $\phi = k/C$.

3. X22B(X12B10T0)0T0 transient experiment
Consider another 1D transient experiment similar to the one of Section 2 where, however, the back surfaces of the two samples are thermally insulated. A dimensional mathematical statement for this thermally-symmetric experiment denoted by X22B(X12B10T0)0T0 is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (0 < x < L; \ t > 0) \quad (28a)$$

$$-k \frac{\partial T}{\partial x}(0,t) = \phi_{X22B10T0}(0,t) \quad (t > 0) \quad (28b)$$

$$-k \frac{\partial T}{\partial x}(L,t) = 0 \quad (t > 0) \quad (28c)$$

$$T(x,0) = 0 \quad (0 < x < L) \quad (28d)$$

where $\phi_{X22B10T0}(0,t)$ is the surface heat flux of the X12B10T0 case. It may be taken as

$$\phi_{X12B10T0}(0,t) = \frac{\phi_0}{\sqrt{\pi} \phi t / L^2} \sum_{n=0}^{\infty} (-1)^n \left\{ \exp \left( -\frac{n^2 L^2}{\phi t} \right) - \exp \left( \frac{(n+1)^2 L^2}{\phi t} \right) \right\} \quad (29a)$$

or, alternatively, as

$$\phi_{X12B10T0}(0,t) = 2\phi_0 \sum_{n=1}^{\infty} e^{-\eta_n^2 \phi t / L^2} \quad (29b)$$

where $\eta_n = (n-1/2)\pi$ are the “X12” eigenvalues. Following the methodology proposed in Subsections 2.2-2.4, the surface temperature solution and, hence, the sensitivity coefficients may be
derived for the whole time region and for short times. However, for the sake of brevity, the solution derivation will not be given here.

3.1 Exact solution for the whole time region

The surface temperature solution is

\[
T_{X22B(X12B10T0T0)}(0,\bar{t}, \frac{\phi}{kC}) = \frac{\phi L}{C k} \left\{ 1 + 2 \sum_{m=1}^{\infty} e^{-\frac{\beta_m}{C T} k} \frac{\tan \left[ \beta_m \sqrt{k / (C \phi)} \right]}{\beta_m \sqrt{k / (C \phi)}} \right\} \left( 1 - 2 \sum_{n=1}^{\infty} e^{-\frac{\eta_n}{C T} k} \frac{\cot \left[ \eta_n \sqrt{k / (C \phi)} \right]}{\eta_n \sqrt{k / (C \phi)}} \right)
\]

(30)

where \( \beta_m = m\pi \) are the “X22” dimensionless eigenvalues and \( \frac{\phi L}{(C \phi)} = (T_{rise})_\infty \) is the sample temperature rise for \( t \to \infty \). For the case of \( \phi = \alpha = k / C \), the term inside the braces is equal to 1 since \( \tan(\beta_m) = 0 \) and \( \cot(\eta_n) = 0 \). It is significant to notice that the two expressions defined by equations (6) and (30) are mirror images in the sense that \( k / (C \phi) \) by \( (C \phi) / k \) can be interchanged to get identical expressions if also \( \phi \) in the exponentials is replaced by \( k / C \). This mirror image is rather amazing. It has application to the case of optimal experiments for estimating \( k \) and \( C \).

The maximum number of terms for either summation appearing in equation (30) is still given by equation (7). Sensitivity coefficient for the \( k \) and \( C \) properties can be found using differences as

\[
X_{k,X22} = \frac{k}{(T_{rise})_\infty} \frac{\partial T_{X22}}{\partial k}(0,\bar{t}) = \frac{T_{X22B(X12B10T0T0)}[0,\bar{t}, k(1+\epsilon), C] - T_{X22B(X12B10T0T0)}[0,\bar{t}, k(1-\epsilon), C]}{2\epsilon(T_{rise})_\infty}
\]

(31a)

\[
X_{C,X22} = \frac{C}{(T_{rise})_\infty} \frac{\partial T_{X22}}{\partial C}(0,\bar{t}) = \frac{T_{X22B(X12B10T0T0)}[0,\bar{t}, k, C(1+\epsilon)] - T_{X22B(X12B10T0T0)}[0,\bar{t}, k, C(1-\epsilon)]}{2\epsilon(T_{rise})_\infty}
\]

(31b)

where the subscript “X22” can be used as a short notation. They are efficient and accurate since the temperature is known very accurately.

3.2 One-term approximate solution for very short times

For early times, less than the so-called deviation time \( t_d^{(1)} \) defined by equation (9), the surface temperature for the X22B(X12B10T0)0T0 problem can accurately be replaced with the surface temperature for the X22B(X10B10T0)0T0 case. The latter can be calculated using Green’s functions where a further approximation (with errors less than \( 10^{-4} \)) can be performed. In fact, \( G_{X22}(x,0,t-\tau) = G_{X20}(x,0,t-\tau) \) defined by equation (13) for co-times \( (t-\tau) \leq 0.1 \cdot (2L-x)^2 / (A \alpha) \). Therefore, \( T_{X22B(X12B10T0T0)}(0,\bar{t}) = T_{X20B(X10B10T0)}(0,\bar{t}) \) defined by equation (14a).

It follows that, for short times, the sensitivity coefficients are still given by equation (15) and, in dimensionless form, by equation (17), where the subscript “X21” has to be replaced with “X22.”

3.3 Three-terms approximate solution for short times

The scaled sensitivity coefficients in dimensionless form for \( \bar{t} \leq \min\{1.6 / A,1.6 /[A(C\phi / k)]\} \) are
\[ X_{1X2} = \frac{k}{(T_{\text{ref}})_e} \frac{\partial T_{X2}}{\partial k} \left( 0, t, \frac{\phi}{kC} \right) = -\frac{1}{2\sqrt{C\phi / k}} \left[ 1 - 4e^{-\beta t} \sqrt{\pi t} + 2\text{erfc} \left( \frac{1}{\sqrt{\pi t}} \right) \right] \] (32a)

\[ X_{C22} = \frac{C}{(T_{\text{ref}})_e} \frac{\partial T_{X2}}{\partial C} \left( 0, t, \frac{\phi}{kC} \right) = -\frac{1}{2\sqrt{C\phi / k}} \left[ 1 + 4e^{-\beta t} \sqrt{\pi t} - 2\text{erfc} \left( \frac{1}{\sqrt{\pi t}} \right) + 2\text{erfc} \left( \frac{1}{\sqrt{\pi t} (C\phi / k)} \right) \right] \] (32b)

which are the same as equations (27a) and (27b) except for the + and – signs inside the brackets. For \( \phi = k / C \), we obtain Eq. (27c) except for the + and – signs inside the brackets. Therefore, the \( k \)-scaled sensitivity coefficient of the X21B(X11B10T0) case is equal to the \( C \)-scaled sensitivity function for the X22B(X12B10T0) case. This same reversal occurs for the \( C \) sensitivities.

### 3.4 Numerical results

Numerical values are given in table 1. They were obtained using equations (17), (32) for \( \phi = k / C \) and equation (30) with Eq. (31), each in its own time range of validity. A plot is shown in figure 1 for visual proposes. This figure indicates that the scaled sensitivity coefficients are large in magnitude as well as in the previous experiment. In particular, in the current experiment the \( C \)-sensitivity is the largest and can be assumed as an ‘ideal reference sensitivity’ which real \( C \)-sensitivity coefficients can refer to in order to evaluate their relative magnitude.

**Conclusions**

It was shown that the scaled sensitivity coefficients for \( k \) and \( C \) can be made largest keeping the heated surface temperature at the maximum value during the transient experiment. This allows us to obtain ideal reference sensitivities, which actual sensitivities can refer to in order to evaluate their relative magnitude. In addition, the sensitivity coefficients can be made more uncorrelated by using two complementary experiments rather than a single one experiment. Also, when computing the sensitivity functions for the heated surface of the X21B(X11B10T0) and X22B(X12B10T0) experiments, it was noted that the roles in the scaled sensitivity coefficients for \( k \) and \( C \) can be exchanged. In addition, the small and large time solutions for their calculation exhibit intrinsic verification.

**Appendix A. Solution derivation for the whole time region**

The first part of the integral in Eq. (5) is

\[ \int \sum_{m=1}^{\infty} e^{-\eta_m^2 \frac{t-c(t)}{T}} d\tau = \frac{L^2}{\alpha} \left( \sum_{m=1}^{\infty} \frac{1}{\eta_m^2} - \sum_{m=1}^{\infty} e^{-\eta_m^2 \frac{t-c(t)}{T}} \right) \] (A.1)

Solving the second integral in the same equation yields

\[ \int \sum_{m=1}^{\infty} e^{-\beta m^2} \sum_{n=1}^{\infty} e^{-\eta_n^2 \frac{t-c(t)}{T}} d\tau = \sum_{m=1}^{\infty} e^{-\beta m^2} \int_{t=0}^{\infty} e^{-\eta_n^2 \frac{t-c(t)}{T}} d\tau = L^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\eta_n^2 \frac{t-c(t)}{T}} - e^{-\beta m^2 \frac{t-c(t)}{T}} \] (A.2)

Combining the above two integrals in equation (5) gives the surface temperature as

\[ T_{X21B(X11B10T0)}(0,t) = 2\phi L \left[ \frac{1}{k} \left( \frac{1}{2} - \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\eta_n^2 \frac{t-c(t)}{T}} - e^{-\beta m^2 \frac{t-c(t)}{T}} \right) + 2\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\eta_n^2 \frac{t-c(t)}{T}} - e^{-\beta m^2 \frac{t-c(t)}{T}} \right] \] (A.3)

where the double summation can be expanded to
The two single summations without the exponentials converge slowly. However, the following relations (found using Mathematica) can be employed to remove this convergence problem,

\[ \sum_{m=1}^{\infty} \frac{1}{\beta_m^2 - \eta_n^2/\phi} = \sum_{n=1}^{\infty} \frac{1}{(m\pi)^2 - \eta_n^2/\phi} = \frac{1}{2\eta_n\sqrt{\alpha/\phi}} \]  
(A.5a)

\[ \sum_{n=1}^{\infty} \frac{1}{\beta_m^2/\alpha - \eta_n^2} = \sum_{n=1}^{\infty} \frac{1}{\beta_m^2/\phi - [(n-1/2)\pi]^2} = -\frac{\tan(\beta_m\sqrt{\phi/\alpha})}{2\beta_m^2\sqrt{\phi/\alpha}} \]  
(A.5b)

Introducing these algebraic identities in equation (A.4) gives

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} e^{-\eta_n^2/\phi} - \frac{e^{-\eta_n^2/\phi}}{\beta_m^2C\phi - \eta_n^2k} = \sum_{m=1}^{\infty} \frac{1}{2k} \sum_{n=1}^{\infty} e^{-\eta_n^2/\phi} \frac{1}{\eta_n^2} \cot\left(\eta_n\sqrt{\alpha/\phi}\right) \]  
\[ + \frac{1}{2k} \sum_{m=1}^{\infty} e^{-\eta_n^2/\phi} \frac{\tan(\beta_m\sqrt{\phi/\alpha})}{\beta_m^2\sqrt{\phi/\alpha}} \]  
(A.6)

Substituting this result into equation (A.3) yields the surface temperature defined by Eq. (6). Unlike the double-summation form defined by equation (A.3), Eq. (6) converges nicely with a relatively small numbers of terms, as shown in Subsection 2.2.

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