Thermoelasticplastic deformation of a functionally graded spherical layer

E. T. Akinlabi1*, E. P. Dats2**, E. V. Murashkin3***

1University of Johannesburg, Johannesburg, South Africa;
2Institute of Applied Mathematics FEB RAS, Vladivostok, Russia;
3Ishlinsky Institute for Problems in Mechanics of RAS, Moscow, Russia.
E-mail: *etakinlabi@uj.ac.za, **dats@dvo.ru, ***murashkin@ipmanet.ru

Abstract. The paper deals with the problem of calculation of residual stresses and displacements in functionally graded material under thermal treatment. The investigated multilayered object has a spherical form. The model of thermoelastic plastic deformation is used. The method of construction of analytical solution is proposed and discussed. The distributions of stresses and displacements in the one-dimensional problem of heating a multilayer sphere are obtained and graphically analysed.

1. Introduction

There are a lot of indepth studies on functionally graded materials, structures, and coatings devoted to their chemo-mechanical properties and manufacturing technologies [1, 2, 3]. The functionally graded material can be produced by additive manufacturing technologies and manifest time dependence of its elasto-viscous and plastic properties.

There are several ways to mathematically describe the gradient distribution of mechanical characteristics inside a thermoelastoplastic material. One of the simplest approaches in setting the gradient for physical parameters is to represent the material in the form of a multilayered structure, where in each layer different values of Lame’s moduli, thermal expansion coefficient, yield strength, etc. are constant. Application of a discrete distribution of such characteristics with the subsequent description of multilayered deformation material allows us to determine the ability of this approach to qualitatively describe the stress-strain state in the framework of continuous distribution models (specified by means of continuous functions of the coordinates) in the functionally graded media. The undoubted advantage of the discrete gradient specification is the ability to construct analytical solutions for the thermoelasticplastic problems, taking into account nonlinear effects.

The analytical solutions to a number of boundary value problems of thermoelasticplastic deformation were obtained under conditions of thermo-mechanical treatments, showing the properties of axial symmetry. For example, in the studies [4, 5, 6, 7, 8, 9, 10], solutions to the boundary value problems of residual deformations and stresses calculations under conditions of central (spherical) symmetry for an elastoplastic material were considered, the features of the solution under non-stationary thermal gradient were determined. The studies [11, 12, 13, 14, 15, 16] are devoted to problems of constructing solutions for stresses and displacements in an elastoplastic material under conditions of axial symmetry in cylindrical
coordinates. The features of calculating the stress state in the case of a plane stress and a plane
strain state of a material were revealed taking account of the dependence of the yield strength
on temperature.

2. Constitutive and governing equations for a thermoelasticplastic continuum
Throughout the paper we will use the model of Prandtl–Reuss infinitesimal elasto-plastic
continuum generalised on thermal expansion effects. The total strain tensor \( d_{ij} \) divides on
an thermoelastic (reversible) \( e_{ij} \) and a plastic (irreversible) \( p_{ij} \) component by equation

\[
d_{ij} = e_{ij} + p_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j),
\]

where \( u_i \) are the components of the displacement vector, \( \partial_j \) is the operator of partial
differentiation with respect to the spatial coordinate.

Constitutive equation between the elastic components of the strain tensor and the components
of the Cauchy stress tensor can be assumed in the form similar to the thermoelastic Duhamel-
Neumann law:

\[
\sigma_{ij}^{(v)} = 2\mu_v e_{ij}^{(v)} + (\lambda_v e_{kk}^{(v)} - (3\lambda_v + 2\mu_v)\alpha_v \Delta)\delta_{ij}.
\]

Hereinafter, the upper and/or lower index \( v = 1 \ldots n \) defines the serial number of the material
layer in the range from 0 to \( n \), counting from the origin; \( \lambda_v, \mu_v \) are the Lame’s parameters, \( \alpha_v \)
is the coefficient of linear thermal expansion in each layer, \( \Delta \) denotes difference between initial
(room) and current (increased) temperature determining the degree of heating. In this case, the
thermal expansion deformation \( e_T^{(v)} \) of the material is determined by the dependence

\[
e_T^{(v)} = \alpha_v \Delta = \alpha_v (T - T_0).
\]

The beginning of the process of irreversible deformation (plastic flow) is connected with the
fulfilment of the yield criterion in the heating domain

\[
f^{(v)}(\sigma_{ij}) = 0.
\]

The surface defined by equation (3) will a plastic potential in the frameworks of the von Mises
maximum principle. The associated law of plastic flow for this assumption is written as

\[
\partial t \varepsilon_{ij}^{(v)} = \xi^{(v)} \frac{\partial f^{(v)}}{\partial \sigma_{ij}}, \quad \xi^{(v)} = \sqrt{\partial t \varepsilon_{kl}^{(v)} \partial_t \varepsilon_{ik}^{(v)} \left( \frac{\partial f^{(v)}}{\partial \sigma_{mn}} \frac{\partial f^{(v)}}{\partial \sigma_{nm}} \right)^{-\frac{1}{2}}}, \quad \partial_t = \frac{\partial}{\partial t}.
\]

In the regions of reversible and irreversible deformation, equilibrium equations are valid, which in the absence of mass forces can be written as

\[
\partial_j \sigma_{ji} = 0.
\]

The system of equations (1)–(5) should be supplemented by boundary conditions for the
particular boundary value problem.

3. Deformation of the multilayered functionally graded material increasing
temperature
Let us consider a thermoelasticplastic hollow spherical structure consisting of \( n \) layers having
different thermo-mechanical characteristics. We assume that the main mechanism of deformation
of the material of the sphere is the temperature deformations due to uniform heating of the
material of the spherical layer. Obviously, if we take different values of the coefficient of thermal expansion for each layer, then the resulting strain gradient will lead to the appearance of temperature stresses.

Integration of the equilibrium equations (5) after substituting the constitutive equations (1)–(4) leads to the formulas for the components of the stress tensor and the displacement vector valid in each layer under conditions of central symmetry in terms of spherical coordinate system

\[
\sigma_{rr}^v = A_v + \frac{1}{B^3_v}, \quad \sigma_{\theta\theta}^v = \sigma_{\phi\phi}^v = A_v - \frac{1}{2B^3_v}, \quad u_r^{(v)} = r\alpha_v \Delta + \frac{A_v r}{3\lambda_v + 2\mu_v} - \frac{B_v}{4\mu_v r^2}. \tag{6}
\]

Here \(A_v, B_v\) are unknown integration constants determined from the boundary conditions of the problem. In general, the size of each layer can be different, and the radial coordinate of the layer with the number \(v\) varies in the range \(R_{v-1} \leq r \leq R_v\), where \(R_{v-1}, R_v\) is the inner and outer surface of the \(v\)th-layer.

To determine the integration constants in (6), we use the boundary conditions, which on contact surfaces and on the outer surface of the sphere can be taken in the form:

\[
\sigma_{rr}^v(R_v) = \sigma_{rr}^{v+1}(R_v), \quad u_r^v(R_v) = u_r^{v+1}(R_v), \quad \sigma_r^{(v)}(R_n) = 0 \quad (v = 1 \ldots n - 1). \tag{7}
\]

The stress gradient arising during non-uniform thermal expansion can cause the process of irreversible deformation accumulation in the material. In this case, under conditions of spherical symmetry and the yield criterion \((\sigma_{rr} - \sigma_{\theta\theta})^2 = 4k^2\) (where \(k\) is the yield strength under a pure shear), areas of irreversible deformation (plastic flow) are formed in the material. The stress strain state in such areas should be described by the relations:

\[
\sigma_{rr}^{(v)} = F_v - 4s_v k_v \ln(r), \quad \sigma_{\theta\theta}^{(v)} = \sigma_{\phi\phi}^{(v)} = F_v - 4s_v k_v \ln(r) - 2s_v k_v, \\
u_r^{(v)} = \frac{G_v}{r^2} - \frac{4s_v k_v}{\lambda_v + 2\mu_v} r \ln(r) + r \left(\frac{F_v}{3\lambda_v + 2\mu_v} + \alpha_v \Delta\right). \tag{8}
\]

This function means that it belongs to the region of plastic flow. The integration constants \(F_v, G_v\) along with the constants (6) are found in the system of boundary conditions (7) taking account of the conditions of continuous stress and displacements in elastoplastic boundaries \(a_v\):

\[
\sigma_{rr}^{(v)}(a_v) = \sigma_{rr}^{(v)}(a_v), \quad \sigma_{rr}^{(v)}(a_v) = u_r^{(v)}(a_v), \quad v = 0 \ldots m, \tag{9}
\]

where \(m\) is the number of areas of irreversible deformation. The values of the elastoplastic boundaries \(a_v\) are determined from a numerical solution of a system of equations that specify the continuity of the circumferential stresses on them:

\[
\sigma_{\theta\theta}(a_v) = \sigma_{\theta\theta}(a_v), \quad v = 0 \ldots m. \tag{10}
\]

Thus, for \(n\) layers with \(m\) plastic flow regions, we have a system of \(n + m + 1\) equations for determining unknown parameters \(A_v, B_v, F_v, G_v, a_v\).

In the next section, we consider the problem of calculating the stress strain state of a three-layered elastoplastic sphere subjected to unsteady thermal action. The problem statement can be carried out with various boundary conditions that determine the features of the deformation of an inhomogeneous layer with different properties: 1) the free external and internal surface of the sphere; 2) a fixed inner and free outer surface of the sphere. Various boundary conditions on the inner surface make it possible to evaluate the effect of non-uniform thermal expansion and mechanical characteristics of the material on its stress strain state.
4. Deformation under conditions of free thermal expansion of the inner and outer surfaces of the three-layered spherical structure

In this case, the solution of the thermoelastic problem is described by the relations (6), (7). Note that, under the conditions of the same thermal expansion ($\alpha_i = \alpha_{i+1}$), the integration constants in the relations (6) take zero values, and therefore stresses in the multilayered material do not arise during heating, despite differences in llama parameters for each layer. Thus, the gradient level of the coefficient of linear thermal expansion during the transition from layer to layer in this formulation of the problem has a decisive influence.

Figures 1–3 show the characteristic distributions of the fields of increasing temperature stresses at elevated temperatures.
Figure 3. Stresses. \( R_0 = 0.05 \), \( R_1/R_0 = 2 \), \( R_2/R_0 = 3 \), \( R_3/R_0 = 4 \), \( \alpha_1 = 3\alpha_0 \), \( \alpha_2 = 2\alpha_0 \), \( \alpha_3 = \alpha_0 \), \( \mu_1 = 3\mu_0 \), \( \mu_2 = 2\mu_0 \), \( \mu_3 = \mu_0 \), \( \Delta T = 520^\circ C \).

Acknowledgments
The paper is financially supported by joint science and technology collaboration South Africa (National Research Foundation) and Russia (Russian Foundation for Basic Research) (project No. 19-51-60001 / RUSA180527335500) and by the Ministry of Science and Higher Education of the Russian Federation.

References
[1] Akinlabi E T and Akinlabi S A 2012 Effect of heat input on the properties of dissimilar friction stir welds of aluminium and copper American Journal of Materials Science 2 147–152
[2] Mahamood R M, Akinlabi E T, Shukla M and Pityana S 2013 Scanning velocity influence on microstructure, microhardness and wear resistance performance of laser deposited Ti6Al4V/TiC composite Materials & design, 50 656–666
[3] Mahamood R M and Akinlabi E T 2017 Functionally graded materials (Springer)
[4] Dats E P, Murashkin E V and Velmurugan R 2015 Calculation of Irreversible Deformations in a Hollow Elastic-Plastic Sphere under Nonsteady Thermal Action Bulletin of Yakovlev Chuvash State Pedagogical University. Series: Mechanics of Limit State 25 3 168–175
[5] Dats E P, Mokrin S N and Murashkin E V 2012 Calculation of accumulated residual deformation in the process of "heating-cooling" of an elastoplastic ball Bulletin of the I. Yakovlev Chuvash State Pedagogical University. Series: Limit State Mechanics 14 4 250–264
[6] Burenin A, Murashkin E and Dats E 2018 Residual stresses in AM fabricated ball during a heating process AIP Conference Proceedings 1959 070008
[7] Murashkin E and Dats E 2018 Thermal Residual Stresses Computing in Elastic-Plastic Ball with Rigid Inclusion under Heat Treatments Lecture Notes in Engineering and Computer Science 2235 811–814
[8] Murashkin E and Dats E 2018 Applications of Multi-Physics Modelling for Simulations of Thermo-Elastic-Plastic Materials In Fourth International Conference on Mathematics and Computers in Sciences and in Industry, Corfu Island, Greece, 24 - 27 Aug, 2017 (Conference Publishing Services of IEEE) 76–80
[9] Murashkin E and Dats E 2017 Thermoelastoplastic Deformation of a Multilayer Ball Mechanics of Solids 52 5 30–36
[10] Murashkin E V, Dats E P and Khliukhov V V 2017 Numerical Analysis of the Elastic-Plastic Boundaries in the Thermal Stresses Theory Frameworks Journal of Physics: Conf. Series 937 012060
[11] Burenin A A, Dats E P and Murashkin E V 2014 Formation of the residual stress field under local thermal actions Mechanics of Solids 49 2 218–224
[12] Dats E P, Murashkin E V and Gupta N K 2017 On Yield Criterion Choice in Thermoelastoplastic Problems Procedia IUTAM 23 187–200
[13] Dats E, Stadnik N and Murashkin E 2017 On a Multi-Physics Modelling Framework for Thermo-elastic-plastic Materials Processing Procedia Manufacturing 7 427–434

[14] Dats E, Stadnik N and Murashkin E 2017 On Heating of Thin Circular Elastic-plastic Plate with the Yield Stress Depending on Temperature Procedia Engineering 173 891–896

[15] Murashkin E and Dats E Coupled thermal stresses analysis in the composite elastic-plastic cylinder Journal of Physics: Conf. Series 991 012060

[16] Murashkin E and Dats E 2017 Piecewise Linear Yield Criteria in the Problems of Tnermoplasticity IAENG International Journal of Applied Mathematics 47 261 – 264