THE NORM RESIDUE SYMBOL FOR HIGHER LOCAL FIELDS

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Abstract. In this paper we investigate the Kummer pairing associated to an arbitrary (one-dimensional) formal group. In particular, we obtain formulae describing the values of the pairing in terms of multidimensional $p$-adic differentiation, the logarithm of the formal group, the generalized trace and the norm on Milnor K-groups. The results are a generalization to higher-dimensional local fields of Kolyvagin’s explicit reciprocity laws. In particular, they constitute a generalization of Artin-Hasse, Iwasawa and Wiles reciprocity laws.

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1. INTRODUCTION

1.1. Background. The theory of finding explicit formulations for class field theory has a long and extensive history. Among the different formulations we highlight the reciprocity law of Artin-Hasse [1] for the Hilbert symbol $(, )_{p^n} : L^\times \times L^\times \rightarrow \langle \zeta_{p^n} \rangle$, for $L = \mathbb{Q}_p(\zeta_{p^n})$, i.e.,

$$ (u, \zeta_{p^n})_{p^n} = \zeta_{p^n}^{\text{Tr}_{L/\mathbb{Q}_p}(- \log u) / p^n}, $$

where $p > 2$ is a prime number, $\zeta_{p^n}$ is a $p^n$th primitive root of unity, and $u$ is any unit in $L$ such that $v_L(u - 1) > 2p^{n-1}$.

From this formula Iwasawa in [12] described the values $(u, w)_{p^n}$, for every principal unit $w$, in terms of $p$-adic differentiation:

$$ (u, w)_{p^n} = \zeta_{p^n}^{\text{Tr}_{L/\mathbb{Q}_p}(\psi(w) \log u) / p^n}, \quad \text{where} \quad \psi(w) = -\zeta_{p^n} w^{-1} \frac{dw}{d\pi_n}. $$

Here $dw/d\pi_n$ denotes the derivative of any power series $g(x) \in \mathbb{Z}_p[[X]]$, such that $w = g(x)$, evaluated at the uniformizer $\pi_n := \zeta_{p^n} - 1$; i.e., $g'(\pi_n)$. Also, it follows that the Hilbert symbol is characterized by the Artin-Hasse formula (1). In
other words: the symbol is characterized by its values on the torsion subgroup of \( \mathbb{Q}_p(\zeta_{p^n})^\times \).

Following the work of Iwasawa, Wiles [23] derived analogous formulae to describe the Kummer pairing associated to a Lubin-Tate formal group. This pairing is a generalization of the Hilbert symbol in which the multiplicative structure of the field is replaced by the formal group structure. Soon after, Kolyvagin [14] extended the formulae of Wiles to the Kummer pairing associated to an arbitrary formal group (of finite height.) The formulae of Kolyvagin describe the Kummer pairing in terms of \( p \)-adic derivations. Kolyvagin’s results also subsume those of de Shalit in [4].

In this article we generalize the formulae of Kolyvagin to arbitrary higher local fields (of mixed characteristic). Our formulae express the Kummer pairing associated to an arbitrary formal group with values in a higher local field—also called generalized Kummer pairing— in terms of multidimensional \( p \)-adic derivations, the logarithm of the formal group, the generalized trace and the norm of Milnor \( K \)-groups. Moreover, as in the work of Iwasawa, we show how to construct explicitly the multidimensional \( p \)-adic derivations from an Artin-Hasse type formula for the generalized Kummer pairing (cf. Equation (4)). This shows in particular that the generalized Kummer paring is characterized by its values on the torsion points associated to the formal group.

In a subsequent paper (cf. [8]) we provide a refinement of these formulae to the special case of a Lubin-Tate formal group. This has important consequences as it gives an exact generalization of Wiles’ reciprocity laws to higher local fields. As a byproduct we obtain exact generalizations of the formulae (1) (cf. [8] Corollary 5.3.1 ) and (2) (cf. [8] Theorem 5.5.1 ) to higher local fields. These formulae are described in more detail below.

Generalizations of (2) are also given by Kurihara (cf. [15] Theorem 4.4) and Zinoviev (cf. [24] Theorem 2.2 ) for the generalized Hilbert symbol associated to an arbitrary local field. [8] Theorem 5.5.1 further generalizes (2) to the Kummer pairing of an arbitrary Lubin-Tate formal group and an important family of higher local fields. In particular, when we take for the Lubin-Tate formal group the multiplicative formal group \( X + Y + XY \) and the higher local field to be \( \mathbb{Q}_p(\zeta_{p^n})\{\{T_1\}\cdots\{T_{d-1}\}\}, \) [8] Theorem 5.5.1 coincides with the results of Kurihara and Zoniviev.

Fukaya remarkably describes also in [10] similar formulae to those of [8] Theorem 5.5.1 for the Kummer pairing associated to an arbitrary \( p \)-divisible group \( G \). Furthermore, Fukaya’s formulae encompass also arbitrary higher local fields (containing the \( p^n \)th torsion group of \( G \)). However, [8] Theorem 5.5.1 in its specific conditions is sharper than the results in [10] for Lubin-Tate formal groups as we explain in more detail below.

We finally point out to a higher dimensional version of (1) that can be found in Zinoviev’s work ( [24] Corollary 2.1). The Corollary 5.3.1 in [8] further extends (1) to arbitrary Lubin-Tate formal groups and arbitrary higher local fields, in particular subsuming the formulae of Zinoviev. Moreover, we prove stronger results (cf. Proposition 5.3.3 and Equation (31) in [8]) which are not found, \textit{a priori}, in any of the formulae in literature.
It is worth mentioning that the techniques used here to obtain the explicit reciprocity laws were inspired by the work of Kolyvagin in [14]. This allows for a classical and conceptual approach to the higher-dimensional reciprocity laws.

1.2. Description of the formulae. Let $F$ be a formal group with coefficients in the ring of integers of the local field $K/\mathbb{Q}_p$ of finite height $h$. Let $S$ be a local field whose ring of integers $C$ is contained in the endomorphism ring of $F$; if $a \in C$, then $[a]_F(X) = aX + \cdots$ will denote the corresponding endomorphism. For a fixed uniformizer $\pi$ of $C$ we let $f := [\pi]_F$. Let $\kappa_n (\simeq (C/\pi^nC)^h)$ be the $\pi^n$th torsion group of $F$ and let $\kappa = \lim \kappa_n (\simeq C^h)$ be the Tate module. We will fix a basis \{c_i\}_{1 \leq i \leq h} for $\kappa$ and let \{e_n\}_{1 \leq i \leq h} be the corresponding reductions to the group $\kappa_n$.

In order to describe our formulae, let $L \supset K$ be a $d$-dimensional local field containing the torsion group $\kappa_n$, with ring of integers $\mathcal{O}_L$ and maximal ideal $\mu_L$. We will denote by $F(\mu_L)$ the set $\mu_L$ endowed with the group structure from $F$. For $m \geq 1$, we let $L_m = L(\kappa_m)$ and also fix a uniformizer $\gamma_m$ for $L_m$.

We define the Kummer pairing (cf. §2.2)

$$(\cdot)_{L,n} : K_d(L) \times F(\mu_L) \to \kappa_n$$

by $(\alpha, x) \mapsto (\alpha, x)_{L,n} = \Upsilon_L(\alpha)(z) \otimes F z$,

where $K_d(L)$ is the $d$th Milnor $K$-group of $L$ (cf. 2.1.1), $\Upsilon_L : K_d(L) \to G_L^b$ is Kato’s reciprocity map for $L$ (cf. §2.1.4), $f^{(n)}(z) = x$ and $\otimes F$ is the subtraction in the formal group $F$. Denote by $(\cdot)_{L,n}$ the $i$th coordinate of $(\cdot)_{L,n}$ with respect to the basis \{e_i\} of the group $\kappa_n$.

The main result in this paper is the following (cf. Theorem 5.3.1 for the precise formulation).

**Theorem** (Thm 5.3.1). Let $M = L_t$ for $t >> n$. Then there exists a $d$-dimensional derivation $\mathcal{D}^i_{M,m}$ (cf. Definition 4.2.1) such that

$$N_{M/L}(\alpha), x)_{L,n} = T_{M/S} \left( \frac{\mathcal{D}^i_{M,m}(\alpha)}{a_1 \cdots a_d} l_F(x) \right)$$

for all $\alpha = \{a_1, \ldots, a_d\} \in K_d(M)$ and all $x \in F(\mu_L)$. Here $l_F$ is the formal logarithm, $T_{M/S}$ is the generalized trace (cf. §3.1) and $N_{M/L}$ is the norm on Milnor $K$-groups.

Moreover, we can give an explicit description for $\mathcal{D}^i_{M,m}$ as follows. Let $\epsilon_i \in \kappa_t$ be any torsion point as in Remark 4.1.2, then

$$\mathcal{D}^i_{M,m}(\alpha) = -T_1 \cdots T_{d-1} \frac{\tau_{\beta,i}}{l(\epsilon_i) \frac{\partial}{\partial T_j}} \left[ \frac{\partial a_k}{\partial T_j} \right]_{1 \leq k, j \leq d}.$$

Here $\frac{\partial}{\partial T_j}$, $j = 1, \ldots, d$, denotes the partial derivatives of an element in the ring of integers of $M$ with respect to the local uniformizers $T_1, \ldots, T_{d-1}$, $T_d = \pi_M$ (cf. Section 4.1). The constant $\tau_{\beta,i}$ is an invariant of the formal group $F$ that is determined by the Artin-Hasse-type formula

$$\{T_1, \ldots, T_{d-1}, u\} \mathcal{D}^i_{M,m} = T_{M/S} (\log(u) \tau_{\beta,i}),$$

where $u \in V_{M,1} = \{u \in \mathcal{O}_M : v_M(u - 1) > v_M(p) - 1\}$ and log is the usual logarithm.
Additionally, we point to the following remarks. First, the bound on \( t \) is explicit as in [14] (cf. Theorem 5.3.1). Second, the constant \( \overline{\kappa}_{2,1} \) has a further interpretation as an invariant coming from the Galois representation associated to the Tate-module \( \kappa \simeq \lim_{\to} \kappa_{n} \) (cf. Section 5.2.1). Furthermore, we can give an explicit description of this invariant in the important case when \( F \) is a Lubin-Tate formal group as it is explained in the theorem below.

We highlight also that when \( F \) is a \( p \)-divisible group, Benois [2] and Fukaya [10] provide similar formulae for the Kummer pairing and remarkably give a further description of the invariant \( \overline{\kappa}_{2,1} \) in terms of differentials of the second kind associated to the \( p \)-divisible group and the theory of Fontaine [9]. On the other hand, since Fukaya’s formulae and (3) is the following: While Fukaya’s have no restrictions on \( \alpha \) for the symbol \( (\alpha, x)_{L,n} \), the formulae (3) have no restrictions on \( x \). However, for a Lubin-Tate formal group we may show that, under certain conditions, the theorem above can lead to sharper results than [10], as it is explained below.

Finally, we now describe in more detail the refinement of the formulae (3) in the case of a Lubin-Tate formal group \( F \) that it is addressed in the subsequent paper [8]. Let \( f \) be a power series in \( \mathcal{O}_{K}[[X]] \) such that \( f(X) \equiv \pi X \pmod{\deg 2} \) and \( f(X) \equiv X^{q} \pmod{\pi} \) for a uniformizer \( \pi \) of \( K \). Denote by \( F_{f} \) the Lubin-Tate formal group associated to \( f \). In this case we have that \( S = K, \mathcal{C} = \mathcal{O}_{K} \) and \( h = 1 \). We fix a generator \( e_{f} \) of \( \kappa \) and let \( e_{f,n} \) be the corresponding projection onto \( \kappa_{n} \), for all \( n \geq 1 \). Additionally, let \( \kappa_{\infty} = \bigcup_{n \geq 1} \kappa_{n} \). Denote by \( K \) the field \( K = \{ \{ T_{1} \} \} \cdots \{ \{ T_{d-1} \} \} \), and by \( K_{n}, K_{\infty}, K_{\kappa_{n}} \) and \( K_{\kappa_{\infty}} \), respectively, the fields \( K(\kappa_{n}), K(\kappa_{\infty}), K(\kappa_{n}), K(\kappa_{\infty}) \), respectively. In this context, we show in [8] the following refinement of the above theorem.

**Theorem** ([8]). Let \( L \) and \( \mathcal{L} \) be as above. Let \( r \) be maximal and \( r' \) minimal such that \( K_{r} \subset \mathcal{L} \cap K_{\infty} \subset K_{r'} \). Take \( s \geq \max\{r', n + r + \log_{q}(e(\mathcal{L}/K_{r})) \} \); here \( e(\mathcal{L}/K_{r}) \) is the ramification index of \( \mathcal{L}/K_{r} \). Then

\[
(N_{\mathcal{L}_{s}/\mathcal{L}(\alpha), x})_{\mathcal{L}, n} = \left[ \frac{\mathcal{T}_{\mathcal{L}_{s}/K}(QLS_{\alpha}(\alpha)I_{F_{f}}(x))}{F_{f}} \right] (e_{f,n}),
\]

where

\[
QLS_{\alpha}(\alpha) = \frac{T_{1} \cdots T_{d-1}}{\pi^{s} \prod(e_{f,s}) \frac{\partial e_{f,s}}{\partial x}} \det_{1 \leq i,j \leq d}^{a_{1} \cdots a_{d}}
\]

for all \( x \in F(\mu_{\ell}) \) and all \( \alpha = \{a_{1}, \ldots, a_{d} \} \in K_{d}(\mathcal{L}_{s})' := \cap_{i \geq s} N_{\mathcal{L}_{i}/\mathcal{L}_{s}}(K_{d}(\mathcal{L}_{i})). \)

Here \( T_{d} \) denotes the uniformizer \( \gamma_{s} \) of \( \mathcal{L}_{s} \).

The above theorem is an exact generalization of Wiles reciprocity laws to arbitrary higher local fields (cf. [23] Theorem 1.).

By studying how the theorem above is transformed when varying the uniformizer \( \pi \) of \( K \) and the power series \( f \in \Lambda_{\pi} \) we may prove, in [8] Theorem 5.5.1, the following higher dimensional version of Iwasawa’s reciprocity laws (2) for \( \mathcal{L} = K_{n}:

\[
(\alpha, x)_{\mathcal{L}, n} = \left[ \frac{\mathcal{T}_{\mathcal{L}/K}(QL_{n}(\alpha)I_{F}(x))}{F_{f}} \right] (e_{f,n})
\]

for all \( \alpha \in K_{d}(\mathcal{L}) \) and all \( x \in F_{f}(\mu_{\ell}) \) such that \( v_{\mathcal{L}}(x) \geq 2v_{\mathcal{L}}(p)/(q(q - 1)) \), where \( q \) denotes the ramification index \( e(K/Q_{p}) \). In particular, taking \( K = \mathbb{Q}_{p}, f(X) = (1 + X)^{p^{n}} - 1, F_{f}(X, Y) \) the multiplicative formal group \( X + Y + XY \) and
we review Kato’s higher di-

As we mentioned above, Fukaya [10] has similar formulae to (6) that, more remarkably, extend to arbitrary formal groups and arbitrary higher local fields. However, for Lubin-Tate formal groups formula (6) is sharper for $\mathcal{L} = \mathcal{K}_n$, as the condition on $x \in F(\mu_\mathcal{L})$ in [10] is $v_\mathcal{L}(x) > 2v_\mathcal{L}(p) (p - 1) + 1$.

In the deduction of (6) it is also shown, in [8] Corollary 5.3.1, the following Artin-Hasse formula for an arbitrary higher local field $\mathcal{M} \supset \mathcal{K}_n$:

$$(7) \quad \{T_1, \ldots, T_{d-1}, e_{g,n}\}, x\}_{\mathcal{M},n} = \left[ T_{\mathcal{M}/S} \left( \frac{1}{\xi^n t_g'(e_{g,n}) e_{g,n} l_{F_f}(x)} \right) \right]_{F_f} (e_{f,n})$$

for all $x \in F_f(\mu_\mathcal{M})$, where $g$ is a Lubin-Tate series in $\Lambda_n$ which is also a monic polynomial and $e_{g,n} = [1]_{f,g}(e_{f,n})$; here $[1]_{f,g}$ is the isomorphism of $F_f$ and $F_g$ congruent to $X$ (mod deg 2) and $l_g$ is the logarithm of $F_g$. By taking $K = \mathbb{Q}_p$, $f(X) = g(X) = (1 + X)^{p^n} - 1$, $F_f(X, Y)$ the multiplicative formal group $F_m(X, Y) = X + Y + XY$, $e_{f,n} = e_{g,n} = \zeta_{p^n} - 1$, and $\mathcal{M}$ the cyclotomic higher local field $\mathbb{Q}_p(\zeta_{p^n})\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, then (7) coincides with the Artin-Hasse formula of Zinoviev in [24] Corollary 2.1 (25).

Furthermore, (7) will be deduced as a consequence of the following stronger result (cf. [8] Proposition 5.3.3). Let $\mathcal{L} = \mathcal{K}_n$ and take $e_{g,n}$ and $l_g$ as above, then

$$(8) \quad \left\{\{u_1, \ldots, u_{d-1}, e_{g,n}\}, x\right\}_{\mathcal{L},n} = \left[ T_{\mathcal{L}/S} \left( \frac{\det \left[ \frac{\partial u_i}{\partial T_j} \right]}{u_1 \cdots u_{d-1} l_{F_f}(x)} \right) \right]_{F_f} (e_{f,n})$$

for all units $u_1, \ldots, u_{d-1}$ of $\mathcal{L}$ and all $x \in F_f(\mu_\mathcal{L})$.

Moreover, in the particular situation where $K = \mathbb{Q}_p$, $\pi = p$, $f(X) = (X + 1)^{p^n} - 1$, $F_f(X, Y) = F_m(X, Y)$, $l_f(X) = \log(X + 1)$, $\mathcal{L} = \mathbb{Q}_p(\zeta_{p^n})\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, we further have an additional formula (cf. [8] Equation (31)):

$$(9) \quad \left\{\{u_1, \ldots, u_{d-1}, \zeta_{p^n}\}, x\right\}_{\mathcal{L},n} = \left[ T_{\mathcal{L}/\mathbb{Q}_p} \left( \frac{1}{p^n \det \left[ \frac{\partial u_i}{\partial T_j} \right]} l_{F_f}(x) \right) \right]_{F_f} (e_{f,n})$$

for all units $u_1, \ldots, u_{d-1}$ of $\mathcal{L}$ and all $x \in F_f(\mu_\mathcal{L})$. For $u_1 = T_1, \ldots, u_{d-1} = T_{d-1}$ we obtain [24] Corollary 2.1 (24).

The sharper Artin-Hasse formulae (8) and (9) are not contained in any of the reciprocity laws in the literature.

This paper is organized as follows. In Section 2 we review Kato’s higher dimensional Local Class Field Theory and introduce the Kummer pairing along with its properties. In Section 3 we introduce the different components that appear in the formulae, namely the generalized trace ($\S 3.1$), the Iwasawa maps $\psi_{\mathcal{L},n}^L$ ($\S 3.3$) and the derivations $D_{\mathcal{L},n}^L$ ($\S 3.5$). In Section 4 we review the definitions and prove basic properties of multidimensional derivations. In Section 5 we finally deduce the formulae and show how to construct them explicitly from the Artin-Hasse-type formula (4). For convenience, we included an appendix with statements and proofs of several auxiliary results needed here.
The author would like to thank V. Kolyvagin for suggesting the problem treated in this article, for reading the manuscript and for providing valuable comments and improvements.

1.3. **Notation.** We will fix a prime number $p > 2$. If $x$ is a real number then $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.

For a complete discrete valuation field $L$ we define

$$L \{ \{T\} \} = \left\{ \sum_{-\infty}^{\infty} a_i T^i : a_i \in L, \inf_{-\infty} v_L(a_i) > -\infty, \lim_{i \to -\infty} v_L(a_i) = +\infty \right\}.$$  

This is a complete discrete valuation field with valuation $v_L(\sum a_i T^i) = \min_{i \in \mathbb{Z}} v_L(a_i)$, ring of integers $\mathcal{O}_L = \mathcal{O}_L(\{\{T\}\})$ and maximal ideal $\mu_L = \mu_L(\{\{T\}\})$; here $v_L$, $\mathcal{O}_L$ and $\mu_L$ denote the discrete valuation, ring of integers and maximal ideal of $L$, respectively. Observe that the residue field $k_L$ of $L$ is $k_L(\{\{T\}\})$, where $k_L$ is the residue field of $\mathcal{O}_L$. The field $L = (L \{ \{T_1\} \} \cdots \{\{T_{d-1}\}\})$ is defined inductively. In particular, if $L$ is a local field, then $L$ is a $d$-dimensional local field and we will endow it with the Parshin topology (see Chapter 1 of [7] or §6.1 of the Appendix).

For a $d$-dimensional local field $\mathcal{L} \supset K$ let $T_1, \ldots, T_{d-1}$ and $\pi_{\mathcal{L}}$ denote a system of uniformizers, and let $k_{\mathcal{L}} = F(\{T_1\} \cdots \{T_{d-1}\})$ be its residue field. Let $L_{(0)}$ be the standard field $L_{(0)}(\{T_1\} \cdots \{T_{d-1}\})$, where $L_{(0)}$ is a local field unramified over $K$ with residue field $F$. In particular, $\mathcal{L}/L_{(0)}$ is a finite totally ramified extension.

2. **The Kummer Pairing**

2.1. **Higher local class field theory.** We are now going to describe briefly the higher-dimensional class field theory from the point of view of Milnor $K$–groups. This theory parametrizes the abelian extensions of a higher local field in terms of norm subgroups of its Milnor $K$-group.

2.1.1. **Milnor-$K$-groups.** Let $R$ be a ring and $m \geq 0$. We denote by $K_m(R)$ the group

$$R^x \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} R^x / I$$

where $I$ is the subgroup of $(R^x) \otimes^m$ generated by

$$\{ a_1 \otimes \cdots \otimes a_m : a_1, \ldots, a_m \in R^x \text{ such that } a_i + a_j = 1 \text{ for some } i \neq j \}$$

$K_m(R)$ is called the $m^{th}$ Milnor-K-group of $R$. The element $a_1 \otimes \cdots \otimes a_m$ is denoted by $\{a_1, \ldots, a_m\}$.

The natural map

$$R^x \times \cdots \times R^x \to K_d(R) : (a_1, \ldots, a_d) \to \{a_1, \ldots, a_d\}$$

is called the symbol map and will be denoted by $\{ \}$. 

**Proposition 2.1.1.** The elements of the Milnor $K$-group satisfy the relations

1. $\{a_1, \ldots, a_i, \ldots, -a_i, \ldots, a_m\} = 1$
2. $\{a_1, \ldots, a_i, \ldots, a_j, \ldots, a_m\} = \{a_1, \ldots, a_j, \ldots, a_i, \ldots, a_m\}^{-1}$

**Proof.** See Appendix §6.3. □
From the definition we have $K_1(R) = R^\times$ and we define $K_0(R) := \mathbb{Z}$. We also have a product
\[ K_n(R) \times K_m(R) \to K_{n+m}(R), \]
where $\{a_1, \ldots, a_n\} \times \{a_{n+1}, \ldots, a_{n+m}\} \mapsto \{a_1, \ldots, a_{n+m}\}$.

2.1.2. Norm on Milnor-$K$-groups. Suppose $L/E$ is a finite extension of fields. Then the norm $N_{L/E} : L^* \to E^*$ induces a norm on the corresponding $K$-groups of $L$ and $E$, satisfying analogous properties to those of $N_{L/E}$. We recollect some of the properties in the following

**Proposition 2.1.2.** There is a group homomorphism
\[ N_{L/E} : K_m(L) \to K_m(E) \]
satisfying
1. When $m = 1$ this maps coincides with the usual norm.
2. For the tower $L/E_1/E_2$ of finite extensions we have $N_{L/E_1} = N_{E_1/E_2} \circ N_{L/E_1}$.
3. The composition $K_m(E) \to K_m(L) \xrightarrow{N_{L/E}} K_m(E)$ coincides with multiplication by $[L/E]$.
4. If $\{a_1, \ldots, a_m\} \in K_m(L)$, with $a_1, \ldots, a_i \in L^\times$ and $a_{i+1}, \ldots, a_m \in E^\times$, then
\[ N_{L/E}(\{a_1, \ldots, a_m\}) = N_{L/E}(\{a_1, \ldots, a_i\}) \cdot \{a_{i+1}, \ldots, a_m\} \in K_m(E). \]
The right hand side is the product of a norm in $K_i(L)$ and a symbol in $K_{m-i}(E)$.

**Proof.** See [6] Chapter IV and [11] Section 7.3. \qed

Note in particular that if $a_1 \in L^\times$ and $a_2, \ldots, a_m \in E^\times$, (1) and (4) imply
\[ N_{L/E}(\{a_1, \ldots, a_m\}) = \{N_{L/E}(a_1), a_2, \ldots, a_m\}. \]

2.1.3. Topological Milnor-$K$-groups. Suppose $\mathcal{L}$ is a $d$-dimensional local field. We endow $\mathcal{L}^*$ with the product Parshin topology (see Chapter 1 of [7] or §6.1.2 of the Appendix). This topology induces a topology on the Milnor-$K$-group as follows

**Definition 2.1.1.** We endow $K_d(\mathcal{L})$ with the finest topology $\lambda_d$ for which the map
\[ (\mathcal{L}^*)^\otimes d \to K_d(\mathcal{L}) : \ (a_1, \ldots, a_d) \mapsto \{a_1, \ldots, a_d\} \]
is sequentially continuous in each component with respect to the product topology on $\mathcal{L}^*$ and for which subtraction in $K_d(\mathcal{L})$ is sequentially continuous. Define
\[ K_d^{\top}(\mathcal{L}) = K_d(\mathcal{L})/\Lambda_d(\mathcal{L}), \]
with the quotient topology where $\Lambda_d(\mathcal{L})$ denotes the intersection of all neighborhoods of 0 with respect to $\lambda_d$ (and therefore it is a subgroup).

Fesenko proved, in [7] Chapter 6 Theorem 3, that
\[ \Lambda_d(\mathcal{L}) = \cap_{i \geq 1} K_d(\mathcal{L}), \]
and also the following
Proposition 2.1.3. Let $\mathcal{M}/\mathcal{L}$ be a finite extension of $d$-dimensional local fields, then the norm $N_{\mathcal{M}/\mathcal{L}} : K_d(\mathcal{M}) \to K_d(\mathcal{L})$ induces a norm

$$N_{\mathcal{M}/\mathcal{L}} : K_{d}^{\text{top}}(\mathcal{M}) \to K_{d}^{\text{top}}(\mathcal{L}).$$

For this norm we have $N_{\mathcal{M}/\mathcal{L}}(\text{open subgroup})$ is open in $K_{d}^{\text{top}}(\mathcal{L})$. In particular, $N_{\mathcal{M}/\mathcal{L}}(K_{d}^{\text{top}}(\mathcal{M}))$ is open in $K_{d}^{\text{top}}(\mathcal{L})$.

Proof. See Section 4.8, claims (1) and (2) of page 15 of [5].

2.1.4. The higher-dimensional reciprocity map. In the following theorem we recollect the main properties of Kato’s reciprocity map that will be needed in our formulations of the Kummer pairing. The reader is referred to [13] for a complete account on the topic.

Theorem 2.1.1 (A. Parshin, K. Kato). Let $\mathcal{L}$ be a $d$-dimensional local field. Then there exist a reciprocity map

$$\Upsilon_{\mathcal{L}} : K_d(\mathcal{L}) \to \text{Gal}(\mathcal{L}^{\text{ab}}/\mathcal{L}),$$

satisfying the properties

(1) If $\mathcal{M}/\mathcal{L}$ is a finite extension of $d$-dimensional local fields, then the following diagrams commute:

$$\begin{array}{ccc}
K_d(\mathcal{M}) & \xrightarrow{\Upsilon_{\mathcal{M}}} & \text{Gal}(\mathcal{M}^{\text{ab}}/\mathcal{M}) \\
N_{\mathcal{M}/\mathcal{L}} \downarrow & & \downarrow \text{restriction} \\
K_d(\mathcal{L}) & \xrightarrow{\Upsilon_{\mathcal{L}}} & \text{Gal}(\mathcal{L}^{\text{ab}}/\mathcal{L})
\end{array}$$

If moreover $\mathcal{M}/\mathcal{L}$ is abelian, then $\Upsilon_{\mathcal{L}}$ induces an isomorphism

$$K_d(\mathcal{L})/N_{\mathcal{M}/\mathcal{L}}(K_d(\mathcal{M})) \xrightarrow{\Upsilon_{\mathcal{L}}} \text{Gal}(\mathcal{M}/\mathcal{L})$$

(2) The reciprocity map $\Upsilon_{\mathcal{L}}$ is sequentially continuous if we endow $K_d(\mathcal{L})$ with the topology $\lambda_\mathcal{L}$ from definition 2.1.1.

Proof. The first assertion can be found in [13], Section 1, Theorem 2. The author has not found a formal proof of the second assertion, although it is a property that has been mentioned in other papers (cf. [24] Page 4809). We provide one in the Appendix §6.3. □

2.2. The Kummer pairing $(,)_n$. Let $\mathcal{L}$ be a $d$-dimensional local field containing $K$ and the group $\kappa_n$. The Kummer pairing

$$(,)_n : K_d(\mathcal{L}) \times F(\mu_\mathcal{L}) \to \kappa_n$$

is defined by $(\alpha, x)_{\mathcal{L}, n} = \Upsilon_{\mathcal{L}}(\alpha)(z) \ominus_F z$, where $f^{(n)}(z) = x$ and $\ominus_F$ is the subtraction in the formal group $F$.

Proposition 2.2.1. The pairing above satisfies the following:

(1) $(,)_n$ is bilinear and $C$-linear on the right.

(2) The kernel on the right is $f^{(n)}(F(\mu_\mathcal{L}))$.

(3) $(\alpha, x)_{\mathcal{L}, n} = 0$ if and only if $\alpha \in N_{\mathcal{L}/\mathcal{L}}(K_d(\mathcal{L}(z)))$, where $f^{(n)}(z) = x$.

(4) If $\mathcal{M}/\mathcal{L}$ is finite, $x \in F(\mu_\mathcal{L})$ and $\beta \in K_d(\mathcal{M})$. Then

$$(\beta, x)_{\mathcal{M}, n} = (N_{\mathcal{M}/\mathcal{L}}(\beta), x)_{\mathcal{L}, n}. $$
(5) Let $\mathcal{L} \supset \kappa_m$, $m \geq n$. Then
\[
(\alpha, x)_{\mathcal{L}, n} = f^{(m-n)}((\alpha, x)_{\mathcal{L}, m}) = (\alpha, f^{(m-n)}(x))_{\mathcal{L}, m}.
\]

(6) For a given $x \in K_d(\mathcal{L})$, the map $K_d(\mathcal{L}) \to \kappa_n : \alpha \mapsto (\alpha, x)_{\mathcal{L}, n}$ is sequentially continuous.

(7) Let $\mathcal{M}$ be a finite extension of $\mathcal{L}$, $\alpha \in K_d(\mathcal{L})$ and $y \in F(\mu_{\mathcal{M}})$. Then
\[
(\alpha, y)_{\mathcal{M}, n} = (\alpha, N_{\mathcal{M}/\mathcal{L}}^F(y))_{\mathcal{L}, n},
\]
where $N_{\mathcal{M}/\mathcal{L}}^F(y) = \oplus \sigma y^\sigma$, where $\sigma$ ranges over all embeddings of $\mathcal{M}$ in $\overline{\mathcal{L}}$ over $\mathcal{L}$.

(8) Let $t : F \to \overline{F}$ be a isomorphism. Then $(\alpha, t(x))_{\mathcal{L}, n} = t((\alpha, x)_{\mathcal{L}, n})$ for all $\alpha \in K_d(\mathcal{L})$, $x \in F(\mu_{\mathcal{L}})$.

Proof. We will only prove property 6. The proof of the other properties is the same as in [14] Section 3.3, or can be found in Section 6.3 of the appendix.

Property 6 follows from the fact that the reciprocity map $\Upsilon_{\mathcal{L}} : K_m(\mathcal{L}) \to \text{Gal}(\mathcal{L}^{ab}/\mathcal{L})$ is sequentially continuous (cf. Theorem 2.1.1 (2)). Indeed, for $z$ such that $f^{(n)}(z) = x$ consider the extension $\mathcal{L}(z)/\mathcal{L}$. The group $\text{Gal}(\mathcal{L}^{ab}/\mathcal{L}(z))$ is a neighborhood of $G_{\mathcal{L}}^{ab}$, so for any sequence $\{\alpha_m\}$ converging to zero in $K_d(\mathcal{L})$ we can take $m$ large enough such that $\Upsilon_{\mathcal{L}}(\alpha_m) \in \text{Gal}(\mathcal{L}^{ab}/\mathcal{L}(z))$, that is $\Upsilon_{\mathcal{L}}(\alpha_m)(z) = z$, so $(\alpha_m, x)_{\mathcal{L}, n} = 0$ for large enough $m$.

\[\square\]

2.3. Sequential continuity of the pairing. In this subsection we will show that the Kummer pairing is sequentially continuous in the second argument. This will play a vital role in showing the existence of the so-called Iwasawa map. This map allows us to express the Kummer pairing in terms of the generalized trace and the logarithm of the formal group (cf. §3.3).

Before we prove the sequential continuity of the pairing we need to introduce the following notation.

Definition 2.3.1. Let $\varrho$ denote the ramification index of $S$ over $\mathbb{Q}_p$. We say that a pair $(n, t)$ is admissible if there exist an integer $k$ such that $t - 1 - n \geq \varrho k \geq n$.

For example, the pair $(n, 2n+\varrho+1)$ is admissible with $k = [(n+\varrho)/\varrho]$. Moreover, in the special case where $\varrho = 1$, then the pair $(n, 2n+1)$ is admissible with $k = n$.

Let $\mathcal{L}$ be a local field. We will denote by $K_\mathcal{L}$ the field $K$ after adjoining the group $\kappa_\mathcal{L}$. For Sections 2 and 3 we will make the following assumptions

\[
(12) \quad \left\{\begin{array}{l}
(\alpha, x)_{\mathcal{L}, n} \text{ admissible pair}, \\
\mathcal{L} \supset K_\mathcal{L}.
\end{array}\right.
\]

We can now formulate the following proposition.

Proposition 2.3.1. Let $\mathcal{L}$ be as in (12). For a given $\alpha \in K_d(\mathcal{L})$, the map
\[
F(\mu_{\mathcal{L}, 1}) \to \kappa_n : x \mapsto (\alpha, x)_{\mathcal{L}, n},
\]
is sequentially continuous in the Parshin topology; if $x_j \to x$ then $(\alpha, x_j)_{\mathcal{L}, n} \to (\alpha, x)_{\mathcal{L}, n}$. Here $F(\mu_{\mathcal{L}, 1})$ is the set $\{x \in \mathcal{L} : v_{\mathcal{L}}(x) \geq |v_{\mathcal{L}}(p)/(p-1)| + 1\}$ considered with the operation induced by the formal group $F$.

Remark 2.3.1. We will make the following two assumptions during the proof of this proposition. First, assume that $\alpha = \{a_1, \ldots, a_d\} \in K_d(\mathcal{L})$ is such that $v_{\mathcal{L}}(a_1) = 1$. This will imply the result for $v_{\mathcal{L}}(a_1) = 0$ as well, by considering $a_1 = \pi_{\mathcal{L}}$ and
\[ a_1 = \pi_L u \text{ for any } u \in \mathcal{L}^* \text{ such that } v_\mathcal{L}(u) = 0; \text{ here } \pi_L \text{ denotes a uniformizer for } L. \]

Second, we will assume that the series \( r(X) = X \) is \( t \)-normalized (cf. \( \S 2.4 \)), otherwise we go to the isomorphic group law \( r(F(r^{-1}(X), r^{-1}(Y))) \). Thus for any \( m < t \) we will assume

\[
(\{a, a_2, \ldots, a_d\}, a)_{\mathcal{L}, m} = 0 \quad \forall a \neq 0 \in F(\mu_{\mathcal{L}}).
\]

**Proof.** We will drop the subscript \( \mathcal{L} \) from the pairing notation. Let \( x \in F(\mu_{\mathcal{L}, 1}) \) and \( \alpha = \{a_1, \ldots, a_d\} \in K_d(\mathcal{L}) \) with \( v_\mathcal{L}(a_1) = 1 \). Also, let \( \rho \) and \( k \) as in Definition 2.3.1. Let \( \tau = \rho k + 1 \), \( A(x) = a_1 + F f^{(\tau)}(x) \) and \( \alpha(x) = \{A(x), a_2, \ldots, a_d\} \in K_d(\mathcal{L}). \) Then

\[
(\alpha, x)_{n} = (\alpha \alpha(x)^{-1}, x)_n \oplus_F (\alpha(x), x)_n.
\]

We will show that the first term on the right-hand side is always zero, regardless of \( x \in F(\mu_{\mathcal{L}, 1}) \), and that the second term goes to zero when we take a sequence \( \{x_j\}_{j \geq 1} \) converging to zero. This completes the proof.

Let us start with the second term. Let \( m = n + \tau \). By (5) in the Proposition 2.2.1

\[
(\alpha(x), x)_n = (\alpha(x), f^{(m-n)}(x))_m = (\alpha(x), A(x) \oplus_F a_1)_m
\]

Here \( \oplus_F a \) denotes the inverse of \( a \) in the formal group law determined by \( F \). From (13) in the remark above, both \( (\alpha(x), A(x))_m \) and \( (\alpha, a_1)_m \) are equal to zero, so we may replace \( (\alpha(x), A(x))_m \) by \( (\alpha^{-1}, \oplus_F a_1)_m \) in (15) to obtain

\[
(\alpha(x), x)_n = (\alpha^{-1}, \oplus_F a_1)_m \oplus (\alpha(x), \oplus_F a_1)_m = (\alpha(x) \alpha^{-1}, \oplus_F a_1)_m.
\]

On the other hand, since \( F(X, Y) \equiv X + Y \pmod{XY} \), then \( A(x) \equiv a_1 + f^{(\tau)}(x) \) (mod \( a_1 f^{(\tau)}(x) \)) so dividing by \( a_1 \)

\[
A(x)/a_1 \equiv 1 + (f^{(\tau)}(x)/a_1) \quad \text{ (mod } f^{(\tau)}(x)).
\]

But \( v_\mathcal{L}(a_1) = 1 \), so \( A(x)/a_1 \) is a principal unit in \( \mathcal{L} \) for every \( x \in F(\mu_{\mathcal{L}, 1}) \). If we take a sequence \( \{x_j\}_{j \geq 1} \) converging to zero in the Parshin topology then, as \( f : \mu_{\mathcal{L}, 1} \to \mu_{\mathcal{L}, 1} \) is sequentially continuous in the Parshin topology by Lemma 6.2.3, we see that \( A(x_j) \) approaches to \( 1 \) as \( j \to \infty \). Hence \( \alpha(x_j) \alpha^{-1} \to \{1, a_2, \ldots, a_d \} \) as \( j \to \infty \), in the topology of \( K_d(\mathcal{L}) \). Notice that \( \{1, a_2, \ldots, a_d \} = 1 \) is the identity element in \( K_d(\mathcal{L}) \). Then by (6) in the Proposition 2.2.1

\[
(\alpha(x_j) \alpha^{-1}, \oplus_F a_1)_m \overset{j \to \infty}{\to} (1, \oplus_F a_1)_m = 0.
\]

Now we will show that first term on the right hand side of equation (14) is zero by showing that \( A(x)/a_1 \) is a \( p^k \)-th power in \( \mathcal{L}^* \) for \( x \in F(\mu_{\mathcal{L}, 1}) \). This is enough since it would imply that \( \alpha(x) \alpha^{-1} \) is \( p^k \)-divisible in \( K_d(\mathcal{L}) \), and from the fact that \( n \leq \rho k \), by Definition 2.3.1, we have that \( \pi^n \) divides \( p^k \). These two observations combined imply \( (\alpha \alpha(x)^{-1}, x)_n = 0. \)

To show that \( A(x)/a_1 \) is a \( p^k \)-th power, let us start by observing that from Proposition 6.2.3

\[
f^{(\tau)}(x) = l_F^{-1} \circ l_F(f^{(\tau)}(x)) = l_F^{-1}(\pi^k l_F(x)) = \pi^k w,
\]
for some $w \in \mu_L$. Then equation (16) implies $A(x)/a_1 = 1 + p^k w_2$ for some $w_2 \in \mu_L$, since $\pi^k = e^k$ for some unit $e$. Then $\log(A(x)/a_1) = \log(1 + p^k w_2) = p^k w_3$, where $w_3 \in \mu_L$. Again, by Proposition 6.2.3 (2), there exist a $w_4 \in \mu_L$ such that $\log(1 + w_4) = w_3$. Thus

$$A(x)/a_1 = (1 + w_4)p^k.$$  

\[\square\]

2.4. Norm Series. A power series $r(X) \in O_K[[X]]$ such that $r(0) = 0$ and $c(r) \in O_K^*$ is called $n$–normalized if for every $d$-dimensional field local $\mathcal{L}$ containing $\kappa_n$, it satisfies that 

$$(\alpha, x)_{\mathcal{L},n} = 0,$$

for all $x \in F(\mu_L)$ and all $\alpha = \{a_1, \ldots, a_d\} \in K_d(\mathcal{L})$ such that $a_i = r(x)$ for some $1 \leq i \leq d$. 

The following proposition will provide a way of constructing norm series.

**Proposition 2.4.1.** Let $g \in O_K[[X]]$, $g(0) = 0$ and $c(g) \in O_K^*$. The series $s = \prod_{v \in \kappa_n} g(F(\mathcal{X}, v))$ belongs to $O_K[[X]]$ and has the form $r_g(f^{(n)})$, where $r_g \in O_K[[X]]$. Then, the series $r_g$ is $n$–normalized and 

$$r_g'(0) = \prod_{v \neq 0 \in \kappa_n} \frac{g'(v)}{\pi^n}.$$

**Proof.** The proof of this Proposition is actually the same as in [14] Proposition 3.1. It can be found in Section 6.3 of the Appendix.

\[\square\]

3. The Iwasawa Map

In this section we introduce some basic properties of the generalized trace. We also introduce the modules $R_{\mathcal{L},1}$ and $R_{\mathcal{L}}$, necessary for the definition of the logarithmic derivatives. The main result in this section is Lemma 3.2.1 which guarantees the existence of the so called Iwasawa map, and thus giving a representation of the Kummer pairing in terms of the generalized trace and the logarithm.

3.1. The generalized trace. Let $E$ be a complete discrete valuation field. Following Kurihara [19], we define a map

$$c_{E\{\{T\}\}/E} : E\{\{T\}\} \to E \text{ by } c_{E\{\{T\}\}/E}(\sum_{i \in \mathbb{Z}} a_i T^i) = a_0.$$ 

Let $\mathcal{E} = E\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}$, we can define $c_{\mathcal{E}/E}$ by the composition

$$c_{E\{\{T_1\}\}/E \circ \cdots \circ c_{E\{\{T_{d-2}\}\}/E}}.$$ 

**Lemma 3.1.1.** This map satisfies the following properties

1. $c_{\mathcal{E}/E}$ is $E$-linear.
2. $c_{\mathcal{E}/E}(a) = a$, for all $a \in E$.
3. $c_{\mathcal{E}/E}$ is continuous with respect to the the Parshin topology on $\mathcal{E}$ and the discrete valuation topology on $E$.

**Proof.** See [24] Lemma 2.1.  

\[\square\]
Let \( \mathcal{L} \) be a \( d \)-dimensional local field and let \( \mathcal{L}_{(0)} \) be as in Section 1.3. We define generalized trace as the composition
\[
\mathcal{T}_{\mathcal{L}/S} := \text{Tr}_{\mathcal{L}_{(0)}/S} \circ c_{\mathcal{L}_{(0)}/\mathcal{L}_{(0)}} \circ \text{Tr}_{\mathcal{L}/\mathcal{L}_{(0)}}.
\]
Notice that if \( \mathcal{M}/\mathcal{L} \) is a finite extension of \( d \)-dimensional local fields then
\[
\mathcal{T}_{\mathcal{M}/S} = \mathcal{T}_{\mathcal{L}/S} \circ \text{Tr}_{\mathcal{M}/\mathcal{L}}.
\]

The generalized trace induces the pairing
\[
\langle , \rangle : \mathcal{L} \times \mathcal{L} \to S \quad \text{defined by} \quad \langle x, y \rangle = \mathcal{T}_{\mathcal{L}/S}(xy).
\]

We denote by \( \mathcal{D}(\mathcal{L}/S) \) the inverse of the dual of \( \mathcal{O}_{\mathcal{L}} \) with respect to this pairing. If \( \mathcal{L} \) is the standard higher local field \( \mathcal{L} \{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \), then \( \mathcal{D}(\mathcal{L}/S) = \mathcal{D}(\mathcal{L}/S)_{\mathcal{O}_{\mathcal{L}}} \).

Proposition 3.1.1. We have an isomorphism of \( C \)-modules
\[
\mathcal{L} \xrightarrow{\sim} \text{Hom}_{C}^{\text{seq}}(\mathcal{L}, S) : \alpha \mapsto (x \mapsto \mathcal{T}_{\mathcal{L}/S}(\alpha x)).
\]

In particular, \( \text{Hom}_{C}^{\text{seq}}(\mathcal{L}, S) = \text{Hom}_{C}(\mathcal{L}, S) \) since the generalized trace is continuous.

Proof. See Section 6.4 of Appendix. \( \square \)

3.2. Modules associated to the generalized trace.

3.2.1. The module \( \mathcal{R}_{\mathcal{L},1} \). Let \( \mathcal{L} \) be a \( d \)-dimensional local field and let \( v_{\mathcal{L}} \) denote the valuation \( \mathcal{L} \). Consider
\[
\mu_{\mathcal{L},1} := \{ x \in \mathcal{L} : v_{\mathcal{L}}(x) \geq \lfloor v_{\mathcal{L}}(p)/(p-1) \rfloor + 1 \}
\]
with the additive structure. Denote by \( \mathcal{R}_{\mathcal{L},1} \) the dual of \( \mu_{\mathcal{L},1} \) with respect to the pairing (18), i.e., \( \mathcal{R}_{\mathcal{L},1} := \{ x \in \mathcal{L} : \mathcal{T}_{\mathcal{L}/S}(x \mu_{\mathcal{L},1}) \subset C \} \). Then it can be shown (cf. §6.4 of the Appendix) that
\[
\mathcal{R}_{\mathcal{L},1} = \{ x \in \mathcal{L} : v_{\mathcal{L}}(x) \geq -v_{\mathcal{L}}(D(\mathcal{L}/S))/\lfloor v_{\mathcal{L}}(p)/(p-1) \rfloor - 1 \},
\]
where \( D(\mathcal{L}/S) \) is as in Section 3.1.

Lemma 3.2.1. We have the isomorphism
\[
\mathcal{R}_{\mathcal{L},1}/\pi^n \mathcal{R}_{\mathcal{L},1} \xrightarrow{\sim} \text{Hom}_{C}^{\text{seq}}(\mu_{\mathcal{L},1}, C/\pi^n C),
\]
defined by
\[
\alpha \mapsto (x \mapsto \mathcal{T}_{\mathcal{L}/S}(\alpha x)).
\]

In particular, \( \text{Hom}_{C}^{\text{seq}}(\mu_{\mathcal{L},1}, C/\pi^n C) = \text{Hom}_{C}(\mu_{\mathcal{L},1}, C/\pi^n C) \).

Proof. Assume first that \( \mathcal{L} \) is the standard higher local field \( \mathcal{L} \{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). The proof is done by induction in \( d \). If \( d = 1 \) the result is known (cf. §3 and §4 of [14]). Suppose the result is true for \( d \geq 1 \) and let \( \mathcal{L} = E \{\{T_{d}\}\} \) where \( E = L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \).
Take $\Phi \in \text{Hom}_{\mathbb{C}}^{\times}\(\mu_{L,1}, C/\pi^nC\)$ and let $\Phi_i(x_i) = \Phi(x_iT_d^i)$ for all $x_i \in \mu_{L,1}$. Then $\Phi_i \in \text{Hom}_{\mathbb{C}}^{\times}(\mu_{L,1}, C/\pi^nC)$ and so by the induction hypothesis there exists $\overline{\pi}_{-i} \in R_{E,1}/\pi^nR_{E,1}$ such that

$$\Phi_i(x_i) = \mathbb{T}_{E/S}(\overline{\pi}_{-i}x_i).$$

Let $a_{-i} \in R_{E,1}$ be a representative of $\overline{\pi}_{-i}$. Thus for $x = \sum x_i T_d^i \in \mu_{L,1}$, the sequential continuity of $\Phi$ implies

$$\Phi(x) = \sum_{i \in \mathbb{Z}} \Phi(x_i T_d^i) = \sum_{i \in \mathbb{Z}} \mathbb{T}_{E/S}(a_{-i}x_i) \pmod{\pi^nC}.$$  

Let $\alpha = \sum a_i T_d^i$ and denote by $u_i$ the unit $a_i/\pi_{E,L}^{v_E(a_i)}$. We must show that

I. $\min\{v_E(a_i)\} > -\infty$,

II. $v_E(a_{-i}) \geq v_E(\pi^nR_{E,1})$ as $i \to \infty$ (i.e., conditions 1 and 2 imply that $\alpha \in R_{L,1}/\pi^nR_{L,1}$).

III. $\Phi(x) = \mathbb{T}_{L/S}(\alpha x) \pmod{\pi^nC}$, $\forall x \in \mu_{L,1}$.

Condition (I) follows immediately since $a_{-i} \in R_{E,1}$, i.e., by equation (19)

$$v_E(a_{-i}) \geq \left[ v_L(p)/(p - 1) \right] + 1 + v_L(D(L/S)) \quad \forall i \in \mathbb{Z}.$$ 

Suppose condition (II) was not true. Instead of passing to a subsequence we may assume for simplicity that $v_E(a_{-i}) < v_E(\pi^nR_{E,1})$ for all $i \geq 0$. For an arbitrary $y \in \mathcal{O}_L$, let

$$x = y(\pi_L^\delta u_0^{-1} + \pi_L^{\delta_1} u_1^{-1} T_d + \pi_L^{\delta_2} u_2^{-1} T_d^2 + \cdots),$$

where for any $i \geq 0$ we let $u_i = a_i/\pi_{E,L}^{v_E(a_i)}$ and

$$\delta_i = v_E(\pi^nR_{E,1}) - v_E(a_{-i}) + [v_L(p)/(p - 1)] \quad (\geq [v_L(p)/(p - 1)] + 1).$$

Then $x \in \mu_{L,1}$ and $a_{-i} x_i = \pi_{E,L}^{\delta_i} y$ for $i \geq 0$, where $w = v_E(\pi^nR_{E,1}) + [v_L(p)/(p - 1)]$.

The convergence of the right hand side of (20) and the fact that $\mathbb{T}_{E/S}(\pi_{E,L}^w y) = \mathbb{T}_{L/S}(\pi_{E,L}^w y)$ would imply that $\mathbb{T}_{L/S}(\pi_{E,L}^w y) \in \pi^nC$ for all $y \in \mathcal{O}_L$. Thus $\pi_{E,L}^n/\pi^n \in D(L/S)^{-1}$, which in turn implies $w \geq v_L(\pi^n) - v_L(D(L/S))$, that is,

$$v_E(R_{E,1}) \geq -v_L(D(L/S)) - [v_L(p)/(p - 1)] - 1 \quad \text{by (19).}$$

Finally, condition (III) immediately follows from equation (61).

Assume now that $L$ is an arbitrary $d$-dimensional local field $L$ and consider the finite extension $L/L(0)$, where $L(0) = L_0\{T_1\} \cdots \{T_{d-1}\}$ and $L_0/S$ is unramified; for example $L_0 = S$. In this case $R_{L,1} \cong \text{Hom}_{\mathcal{O}_L(0)}(\mu_{L,1}, \mathcal{O}_{L(0)})$ induces an isomorphism $R_{L,1}/\pi^nR_{L,1} \cong \text{Hom}_{\mathcal{O}_L(0)}(\mu_{L,1}, \mathcal{O}_{L(0)}/\pi^n\mathcal{O}_{L(0)})$. Now for a given $\phi \in \text{Hom}_{\mathcal{O}_L(0)}(\mu_{L,1}, C/\pi^nC)$ we fix an $x \in \mu_{L,1}$ and consider $\phi_x \in \text{Hom}_{\mathcal{O}_L(0)}^{\times}(\mu_{L,1}, C/\pi^nC)$ defined by

$$\phi_x(y) = \phi\left(\frac{x}{\pi_{L(0)}^a} y\right) \quad (y \in \mu_{L(0),1}),$$

where $a = v_{L(0)}(\mu_{L(0),1}) = [v_{L(0)}(p)/(p - 1)] + 1$. By the first part of the proof there exists an element $\psi(x) \in R_{L(0)}/\pi^nR_{L(0)}$ such that

$$\phi_x(y) = \mathbb{T}_{L(0)/S}(\psi(x) y) \quad (y \in \mu_{L(0),1}).$$
Thus $\psi$ induces an element in $\text{Hom}_{\mathcal{O}_{\ell(0)}}(\mu_{\ell,1}, R_{\ell(0)}/\pi^n R_{\ell(0)})$. Since

$$v_{\ell(0)}(D(\mathcal{L}(0)/S)) = V_{L_0}(D(L_0/S)) = 0,$$

then $v_{\ell(0)}(R_{\ell(0),1}) = -v_{\ell(0)}(\mu_{\ell(0),1}) = -a$ and so $\pi^a \psi$ can be considered as an element in $\text{Hom}_{\mathcal{O}_{\ell(0)}}(\mu_{\ell,1}, \mathcal{O}_{\ell(0)}/\pi^n \mathcal{O}_{\ell(0)})$. Thus there exists an element $\alpha \in R_{\ell}/\pi^n R_{\ell,1}$ such that $\pi^a \psi(x) = \text{Tr}_{\ell}/\ell(0)(\alpha x)$ for all $x \in \mu_{\ell,1}$. This together with (21) yields the desired result.

3.2.2. The module $R_{\ell}$. Let $T_{\ell}$ be the image of $F(\mu_{\ell})$ under the formal logarithm $l_F$. This is a $C$-submodule of $\mathcal{L}$ such that $T_{\ell} S = \mathcal{L}$. Indeed, let $x \in \mathcal{L}$ and take $n$ large enough such that $\pi^n x \in \mu_{\ell,1}$, then by Proposition 6.2.3 there exist a $y \in F(\mu_{\ell,1})$ such that $\pi^n x = l_F(y)$, thus $x \in T_{\ell} S$.

Let $\tilde{R}_{\ell}$ be the dual of $T_{\ell}$ with respect to the trace pairing $\mathbb{T}_{\ell}/S$, then by Proposition 3.1.1 and the fact that $\mathcal{L} = T_{\ell} S$, we have the isomorphism

$$R_{\ell} \simeq \text{Hom}^{\text{eq}}_{C}(T_{\ell}, C).$$

Let $\kappa_{\ell} = \kappa \cap \mathcal{L}$, i.e., the subgroup of torsion points contained in $\mathcal{L}$. Then $l_F$ induces a continuous isomorphism $l_F : F(\mu_{\ell})/\kappa_{\ell} \to T_{\ell}$. Thus we have the following result.

**Lemma 3.2.2.** The generalized trace induces an injective homomorphism

$$0 \to R_{\ell}/\pi^n R_{\ell} \to \text{Hom}^{\text{eq}}_{C}(T_{\ell}, C/\pi^n C) \to \text{Hom}^{\text{eq}}_{C}(F(\mu_{\ell})/\kappa_{\ell}, C/\pi^n C).$$

$$\alpha \mapsto \left( y \mapsto \mathbb{T}_{\ell}/S(\alpha y) \mapsto \left( x \mapsto \mathbb{T}_{\ell}/S(\alpha l_F(x)) \right) \right).$$

**Proof.** Immediate from the very definition of $R_{\ell}$. \hfill \Box

3.3. The map $\psi_{\ell,n}^i$. In this section we introduce the so-called Iwasawa map $\psi_{\ell,n}^i$. This map plays a vital role in the construction of the explicit reciprocity laws. The main goal in this paper is to show that, under certain conditions, the Iwasawa map is a logarithmic derivative. One of the key results to achieve this is contained in Proposition 3.3.2.

Recall that we denote by $(\ , \ )_{\ell,n}^i$ the $i$th coordinate of the paring $(\ , \ )_{\ell,n}$ with respect to the base $\{e_n^i\}$ of $\kappa_n$.

**Proposition 3.3.1.** Let $\mathcal{L}$ be as in (12). For a given $\alpha \in K_d(\mathcal{L})$ there exist a unique element $\psi_{\ell,n}^i(\alpha) \in R_{\ell,1}/\pi^n R_{\ell,1}$, such that

$$(\alpha, x)_{\ell,n}^i = \mathbb{T}_{\ell}/S\left( \psi_{\ell,n}^i(\alpha) l_F(x) \right) \qquad \forall x \in F(\mu_{\ell,1}),$$

and the map $\psi_{\ell,n}^i : K_d(\mathcal{L}) \to R_{\ell,1}/\pi^n R_{\ell,1}$ is a homomorphism.

**Proof.** Let us first take $\alpha$ to be an element of the form $\{a_1, \ldots, a_d\}$ and consider the map

$$\omega : \mu_{\ell,1} \to C/\pi^n C,$$

defined by

$$x \mapsto (\alpha, l_F^{-1}(x))_{\ell,n}^i.$$  

By Proposition 2.3.1 and Remark 6.2.2 this map is sequentially continuous and so by Lemma 3.2.1 there exist element $\psi_{\ell,n}^i(\alpha) \in R_{\ell,1}/\pi^n R_{\ell,1}$ satisfying (22). This defines a map $\psi_{\ell,n}^i : \mathbb{L}^{* \oplus d} \to R_{\ell,1}/\pi^n R_{\ell,1}$ satisfying the Steinberg relation, therefore it induces a map on $K_d(\mathcal{L})$.

\hfill \Box
The following are some basic properties of the Iwasawa map $\psi_{L,n}^i$.

**Proposition 3.3.2.** Consider the finite extension $\mathcal{M}/\mathcal{L}$ with $\mathcal{L}$ satisfying (12). Then

1. $\text{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}) \subset R_{\mathcal{L},1}$ and we have the commutative diagram

$$
\begin{array}{ccc}
K_d(\mathcal{M}) & \xrightarrow{\psi_{\mathcal{M},n}^i} & R_{\mathcal{M},1}/\pi^n R_{\mathcal{M},1} \\
N_{\mathcal{M}/\mathcal{L}} & & \downarrow \text{Tr}_{\mathcal{M}/\mathcal{L}} \\
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},n}^i} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}
\end{array}
$$

2. $R_{\mathcal{L},1} \subset R_{\mathcal{M},1}$ and we have the commutative diagram

$$
\begin{array}{ccc}
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},n}^i} & R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1} \\
\text{incl} & & \\
K_d(\mathcal{M}) & \xrightarrow{\psi_{\mathcal{M},n}^i} & R_{\mathcal{M},1}/\pi^n R_{\mathcal{M},1}
\end{array}
$$

The right-hand vertical map is induced by the embedding of $R_{\mathcal{L},1}$ in $R_{\mathcal{M},1}$.

3. Let $L \supset K_m$, $(m,t)$ admissible and $m \geq n$. Then for $\alpha \in K_d(\mathcal{L})$, $\psi_{\mathcal{L},n}^i(\alpha)$ is the reduction of $\psi_{\mathcal{L},m}^i(\alpha)$ from $R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1}$ to $R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}$, i.e., the diagram

$$
\begin{array}{ccc}
K_d(\mathcal{L}) & \xrightarrow{\psi_{\mathcal{L},m}^i} & R_{\mathcal{L},1}/\pi^m R_{\mathcal{L},1} \\
\psi_{\mathcal{L},n}^i & & \downarrow \text{reduction} \\
R_{\mathcal{L},1}/\pi^n R_{\mathcal{L},1}
\end{array}
$$

commutes.

4. If $t : (F,e_i) \to (\tilde{F},\tilde{e}_i)$ is an isomorphism, then

$$
\tilde{\psi}_{\mathcal{L},n}^i = \frac{1}{t'(0)} \psi_{\mathcal{L},n}^i.
$$

**Proof.** See Section 6.4 of Appendix.

We are ready to prove the following key result.

**Proposition 3.3.3.** Let $\mathcal{L}$ and $t$ as in (12). Let $r(X)$ be a $t$-normalized series. Then

$$
(\alpha,x)_{\mathcal{L},n}^i = T_{\mathcal{L}/S} \left( \log(a_1) \begin{bmatrix} -\psi_{\mathcal{L},n}^i(\alpha(r(x))) & r(x) & r'(x) \end{bmatrix} \right),
$$

where $\alpha$ and $\alpha(r(x))$ are elements in $K_d(\mathcal{L})$ defined, respectively, by

$\alpha = \{a_1, \ldots, a_d\}$ and $\alpha(r(x)) = \{r(x), a_2, \ldots, a_d\}$,

with $a_1 \in V_{\mathcal{L},1} = 1 + \mu_{\mathcal{L},1}$ and $x \in F(\mu_{\mathcal{L}})$. Here $\log$ is the usual logarithm.
Proof. We follow the same ideas as in the proof of Proposition 4.1 of [14]. To simplify the notation we will denote the normalized valuation \( v_\mathcal{L} / v_\mathcal{L}(p) \) by \( v \) and also omit the subscripts \( \mathcal{L} \) and \( F \) from the pairing notation, the formal sum \( \oplus_F \), and the formal logarithm \( l_F \). Furthermore, we are going to denote by \( \alpha(a) \) the element \( \{a, a_2, \ldots, a_d\} \) in \( K_d(\mathcal{L}) \).

Let \( \alpha = \alpha(a) \) with \( a_1 \in V_{\mathcal{L},1} \) and let \( x \in F(\mu_\mathcal{L}) \). Observe that we may assume that \( r(X) = X \) since we can go to the isomorphic formal group \( \tilde{F} = r(F(r^{-1}(X), r^{-1}(Y))) \) with torsion points \( e^i = r(e^i) \) through the isomorphism \( r : F \to \tilde{F} \). The series \( \tilde{r} = X \) is \( t \)-normalized for \( \tilde{F} \) and if the result were true for \( F \) and \( r \) then

\[
(\alpha, x)_{F,n} = (\alpha, r(x))_{\tilde{F},n} = \mathbb{T}_{\mathcal{L}/\mathcal{S}} \left( \log(u) \left[ -\psi_n^i(\alpha(r(x))) l'_F(r(x)) r(x) \right] \right).
\]

Since \( \psi_n^i(\alpha(r(x))) = r'(0)^{-1} \psi_n^i(\alpha(r(x))) \) and \( l'_F(r(X)) = l'_F(r(X)) = l'_F(r(X)) = r'(0) l'_F(r(x))/r(x)^2 \) the result follows.

We assume therefore that \( r(X) = X \). For the admissible pair \((n, t)\) let \( \varrho \) and \( k \) be as in Definition 2.3.1. Denote by \( \epsilon \) the unit \( \pi^{\varrho k}/p^k \). Let \( u \in V_{\mathcal{L},1} = \{ x \in \mathcal{L} : v(x-1) > 1/(p-1) \} \) and \( x \in \mu_\mathcal{L} \). By bilinearity of the pairing

\[
(\alpha, x)_n = (\alpha, x \circ (x a_1^{p^k}))_n \oplus (\alpha, x a_1^{p^k})_n.
\]

Now let \( m = n + \varrho k \) and \( y = x a_1^{p^k} \). Then by (5) of Proposition 2.2.1

\[
(\alpha, y)_n = f^{(m-n)}((\alpha, y)_m) = \pi^{\varrho k} (\alpha, y)_m = \epsilon(\alpha(a_1^{p^k}), y)_m.
\]

But \( r = X \) is \( t \)-normalized, hence we may replace \((\alpha(y), y)_m = 0 \) by the expression \((\alpha(x), x)_m = 0 \) and obtain

\[
(\alpha, y)_n = \epsilon(\alpha(x), x)_m \oplus \epsilon(\alpha(x), y)_m = \epsilon(\alpha(x), x \circ y)_m.
\]

By the properties of the logarithm in Proposition 6.2.3 we can express

\[
a_1^{p^k} = \exp \left( \log(a_1^{p^k}) \right) = \exp \left( p^k \log(a_1) \right) = 1 + p^k \log(a_1) + p^{2k} w,
\]

where \( w = \frac{z^2}{2} + p^k \frac{z^3}{3} + \cdots \), with \( z = \log(a_1) \). Since \( v(z) > 1/(p-1) \) then \( v(z^2) > 1/(p-1) \) and so \( v(w) > 1/(p-1) \). This follows, for example, by Proposition 2.4 of [14].

Since \( x \circ y \equiv x - y \) (mod \( xy \)) and \( y = x a_1^{p^k} \) with \( v(x) > 0 \), then \( v(x \circ y) = v(x - y) = v(x a_1^{p^k} - 1) > 1/(p-1) \). Thus, using the Taylor expansion of \( l = l_F \) around \( X = x \) we obtain

\[
l(y) = l(x + p^k z + p^{2k} w) = l(x) + l'(x)(xp^k z + xp^{2k} w) + p^{2k} w_1,
\]

where \( w_1 = l''(x) \frac{z^2}{2} + p^{k} l(3)(x) \frac{z^3}{3} + \cdots \) with \( \delta = xz + xp^k w \). Since \( v(\delta) > 1/(p-1) \) then \( v(w_1) > 1/(p-1) \). Moreover, \( l(y) = l(x) + l'(x)(xp^k z + p^{2k} w_2) \), with \( v(w_2) > 1/(p-1) \). Then

\[
l(x \circ y) = -l'(x)xp^k z - p^{2k} w_2.
\]

Observing that \( -l'(x)xz - p^k w_2 \in \mu_\mathcal{L} \) we have by the isomorphism given in 6.2.3 that there is an \( \eta \in F(\mu_\mathcal{L},1) \) such that \( l(x \circ y) = p^k l(\eta) = -l'(x)xp^k z - p^{2k} w_2 \). Thus

\[
x \circ y = [p^k](\eta) = [\pi^{\varrho k}](\eta) = f^{(\varrho k)}(\eta),
\]
for \( \tilde{\eta} = [e^{-1}]_F(\eta) \). Since \( n \leq \rho k \) then \( \pi^n \) divides \( \pi^m \) and we have that \( x \oplus y \in f^{(n)}(F(\muL)) \). Thus the first term on the right hand side of equation (24) is zero. By equations (25) and (26) and item (5) of Proposition 2.2.1 we have

\[
(27) \quad (\alpha, x)_n = \epsilon(\alpha(x), x \oplus y)_m = (\alpha(x), \eta)_n. 
\]

Since \( v(\eta) > 1/(p - 1) \) and \((n, t)\) is admissible we can use Proposition 3.3.1,

\[
(\alpha(x), \eta)_n = \mathbb{T}_{L/S}((\psi^i_n(\alpha(x)) \cdot \ell_p(\eta)) = \mathbb{T}_{L/S}(\psi^i_n(\alpha(x)) (-l'(x)zx - p^kw_2)). 
\]

Since \( \psi_{L,n}(\alpha(x)) \in R_{L,1}, w_2 \in T_{L,1}, \pi^n \mid p^k \) and \( \mathbb{T}_{L/S}(R_{L,1} / T_{L,1}) \subset C \), then we can write the last expression above as \( \mathbb{T}_{L/S}(\psi^i_n(\alpha(x)) (-l'(x)zx)) \). From Equation (27) we finally get

\[
(\alpha, x)_n = \mathbb{T}_{L/S}(\psi^i_n(\alpha(x))[ -l'(x)zx]).
\]

Keeping in mind that \( z = \log(a_1) \), the proposition follows. □

3.4. The map \( \rho_{L,n}^i \). We can define a \( C \)-linear structure on \( V_{L,1} = 1 + \muL \) by using the isomorphism \( \log : V_{L,1} \to T_{L,1} \), i.e., \( cu := \log^{-1}(c \log(u)) \) for \( c \in C \) and \( u \in V_{L,1} \). Let \( x \in F(\muL) \) and \( \alpha' = \{a_2, \ldots, a_d\} \in \mathbb{K}_{d-1}(L) \) be fixed. Consider the mapping

\[
V_{L,1} \to C/\pi^n C, \text{ defined by } a_1 \mapsto (\{a_1, \ldots, a_d\}, x)_{L,n}.
\]

According to Proposition 3.3.3 this is a continuous \( C \)-linear map and we have the following

**Proposition 3.4.1.** Let \( \alpha' \in \mathbb{K}_{d-1}(L) \) and \( x \in F(\muL) \). Consider the element \( \alpha = \{a_1\} \cdot \alpha' \in K_d(L) \) where \( a_1 \in V_{L,1} \). Then there exist a unique element \( \rho_{L,n}^i(\alpha', x) \in \mathbb{R}_{L,1}/\pi^n \mathbb{R}_{L,1} \) such that

\[
(28) \quad (\alpha, x)_{L,n} = \mathbb{T}_{L/S}(\log(a_1) \cdot \rho_{L,n}^i(\alpha', x)).
\]

Moreover, the map \( \rho_{L,n}^i : \mathbb{K}_{d-1}(L) \otimes F(\muL) \to \mathbb{R}_{L,1}/\pi^n \mathbb{R}_{L,1} \) is a homomorphism.

From Proposition 3.4.1 and Proposition 3.3.3 it follows the next proposition.

**Proposition 3.4.2.** Let \( L \) and \( t \) be as in (12) and let \( r(X) \) be a \( t \)-normalized series. Then for all \( x \in F(\muL) \) and all \( \alpha' = \{a_2, \ldots, a_d\} \in \mathbb{K}_{d-1}(L) \) we have

\[
\frac{\psi_{L,n}^i(\alpha(r(x)))}{r'(x)} = -\frac{\rho_{L,n}^i(\alpha', x)}{l_p'(x)},
\]

where \( \alpha(r(x)) = \{r(x), a_2, \ldots, a_d\} \in K_d(L) \).

3.5. The maps \( D_{L,n}^i \). Let \( L \) be as in (12). Define \( D_{L,n}^i : \mathbb{O}_{L}^d \to \mathbb{R}_{L,1}/\pi^n \mathbb{R}_{L,1} \) by

\[
(29) \quad D_{L,n}^i(a_1, \ldots, a_d) = \psi_{L,n}^i(\{a_1, \ldots, a_d\})a_1 \cdots a_d,
\]

and consider \( \mathbb{R}_{L,1}/\pi^n \mathbb{R}_{L,1} \) as an \( \mathbb{O}_{L} \)-module.

**Proposition 3.5.1.** The map \( D_{L,n}^i \) satisfies

1. **Leibniz Rule:**

\[
D_{L,n}^i(a_1, \ldots, a_i a_j', \ldots, a_d) = a_j D_{L,n}^i(a_1, \ldots, a_j', \ldots, a_d) + a_i' D_{L,n}^i(a_1, \ldots, a_j', \ldots, a_d).
\]

2. **Steinberg relation:** \( D_{L,n}^i(a_1, \ldots, a_d) = 0 \) if \( a_j + a_k = 1 \) for some \( j \neq k \).
(3) Skew-symmetric:
\[ D^i_{\mathcal{L}, n}(a_1, a_2, \ldots, a_k, \ldots, a_d) = -D^i_{\mathcal{L}, n}(a_1, a_2, \ldots, a_k, \ldots, a_d). \]

(4) \[ D^i_{\mathcal{L}, n}(a_1, a_d) = 0 \] if some \( a_j \) is a \( p^k \)-th power in \( \mathcal{O}_\mathcal{L} \) with \( \pi^n|p^k \).

Proof. Property (1) follows from the fact that \( \tilde{\psi}_{\mathcal{L}, n} \) is a homomorphism. Property (2) follows from the Steinberg relation \( \{a_1, \ldots, a_j, \ldots, 1 - a_j, \ldots, a_d\} = 1 \) for elements in the Milnor K-group \( K_d(\mathcal{L}) \) and property (3) follows from the fact that \( \{a_1, \ldots, a_j, \ldots, a_d\} = \{a_1, \ldots, a_k, \ldots, a_j, \ldots, a_d\}^{-1} \), in \( K_d(\mathcal{L}) \) (cf. Proposition 3.2.1).

Proposition 3.5.2. Let \( \mathcal{L} \) and \( t \) be as in (12) and \( r(X) \) a \( t \)-normalized series. Then for all \( x, y \in F(\mu_\mathcal{L}) \) and all \( a_2, \ldots, a_d \in \mathcal{L}^* \) we have
\[
\frac{D^i_{\mathcal{L}, n}(\alpha(r(x \oplus y))}{r'(x \oplus y)} = F_X(x, y) \frac{D^i_{\mathcal{L}, n}(\alpha(r(x))}{r'(x)} + F_Y(x, y) \frac{D^i_{\mathcal{L}, n}(\alpha(r(y))}{r'(y)},
\]
where \( \alpha(z) \) denotes \( \{z, a_2, \ldots, a_d\} \in \mathcal{O}_\mathcal{L}^t \).

Proof. This follows from the fact that
\[
\rho^i_{\mathcal{L}, n}(\alpha', x \oplus y) = \rho^i_{\mathcal{L}, n}(\alpha', x) + \rho^i_{\mathcal{L}, n}(\alpha', y),
\]
for \( \alpha' = \{a_2, \ldots, a_d\} \in K_{d-1}(\mathcal{L}) \) and from differentiating \( l_F(F(X, Y)) = l_F(X) + l_F(Y) \) with respect to \( X \) and \( Y \). \( \square \)

Let \( \hat{F} \) be the formal group \( r(F(r^{-1}(X), r^{-1}(Y))) \). The series \( r(X) = X \) is \( t \)-normalized for \( \hat{F} \). Denote by \( \hat{\oplus} \) the sum according to this formal group and \( \hat{D}^i_{\mathcal{L}, n}, \hat{\psi}_{\mathcal{L}, n} \) the corresponding maps. According to Proposition 3.3.2 (4) we have that
\[
\hat{D}^i_{\mathcal{L}, n} = \frac{1}{r'(0)} D^i_{\mathcal{L}, n}.
\]

Therefore, Proposition 3.5.2 in terms of \( \hat{\oplus} \) and \( \hat{D}^i_{\mathcal{L}, n} \) reads as

Corollary 3.5.1. Let \( \mathcal{L} \) be as in (12), then
\[
D^i_{\mathcal{L}, n}(\alpha(x \oplus y)) = \hat{F}_X(x, y) D^i_{\mathcal{L}, n}(\alpha(x)) + \hat{F}_Y(x, y) D^i_{\mathcal{L}, n}(\alpha(y)).
\]

In order to take advantage of this differentiability property of the map \( D^i_{\mathcal{L}, n} \) with respect to formal group law \( \hat{F} \), we will show that any element in the maximal ideal \( \mu_\mathcal{L} \) can be expressed as an infinite sum with respect to the formal group law \( \hat{F} \). This is accomplished in the following subsection.

3.5.1. Representations of elements as formal group series. In order to simplify the notation, we introduce the following

Definition 3.5.1. Let \( J_n = \{ \bar{i} = (i_1, \ldots, i_n) : 0 \leq i_1, \ldots, i_{d-1} < p^n \} \). Let \( \mathfrak{A} \) be the set of all series in \( X_1 \mathcal{O}_\mathcal{L}[[X_1, \ldots, X_d]] \) of the form
\[
\bigoplus_{k=1}^{\infty} \left( \bigoplus_{\gamma_{i,k} \in J_n} \gamma_{i,k}^{p^n} X_1^{i_1} \cdots X_{d-1}^{i_{d-1}} X_d^k \right),
\]
where \( \gamma_{i,k} \in \mathcal{O}_\mathcal{L} \). Here \( \oplus \) denotes addition in the formal group law \( F \).
The following lemma will be used in Proposition 3.5.3 to express every element in the maximal ideal as an infinite formal group sum.

**Lemma 3.5.1.** For $x \in \mathcal{O}_L$, there exist elements $\gamma_{\overline{t}} \in \mathcal{O}_L$, with $\overline{t} \in J_n$, such that

$$x \equiv \sum_{\overline{t} \in J_n} \gamma_{\overline{t}}^p \cdot T_{i_1} \cdots T_{i_{d-1}} \pmod{\pi_L}.$$

**Proof.** For a direct proof using induction see §6.4 of the Appendix. Alternatively, the proposition is equivalent to the following two facts:

1. $k_L((T_1)) \cdots ((T_{d-1}))$ is a finite extension of $k_L((T_1^{p^n})) \cdots ((T_{d-1}^{p^n}))$ of degree $p^{n(d-1)}$ and generated by the elements $T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}$ for $0 \leq i_1, \ldots, i_{d-1} < p^n$.

2. $k_L((T_1^{p^n})) \cdots ((T_{d-1}^{p^n}))$ is the image of $k_L((T_1)) \cdots ((T_{d-1}))$ under the Frobenius homomorphism

$$\sigma_p : k_L((T_1)) \cdots ((T_{d-1})) \to k_L((T_1)) \cdots ((T_{d-1})),$$

i.e., every element of $k_L((T_1^{p^n})) \cdots ((T_{d-1}^{p^n}))$ is a $p^n$th power of an element in $k_L((T_1)) \cdots ((T_{d-1}))$.

Both facts are easily proven by induction from the fact that, for a field $k$ of characteristic $p$, the extension $[k((T) : k((T^p))]$ has degree $p$ and $\sigma_p(k)((T^p))$ is the image of $k((T))$ under the Frobenius homomorphism $\sigma_p : k((T)) \to k((T))$.

**Proposition 3.5.3.** For $y \in \mu_L$, there exist a power series $\eta \in \mathfrak{A}$ such that $y = \eta(T_1, \ldots, \pi_L)$.

**Proof.** Fix $y \in \mu_L$. Denote by $T^\overline{t}$ the product $T_1^{i_1} \cdots T_{d-1}^{i_{d-1}}$ for $\overline{t} \in J_n$ and by $\oplus$ the formal sum $\oplus_F$. Then Lemma 3.5.1 applied to $y/\pi_L$ gives as elements $\gamma_{\overline{t},1} \in \mathcal{O}_L$ such that

$$\frac{y}{\pi_L} \equiv \sum_{\overline{t} \in J_n} \gamma_{\overline{t},1}^p \cdot T^\overline{t} \equiv \frac{1}{\pi_L} \oplus_{\overline{t} \in J_n} \gamma_{\overline{t},1}^p \cdot T^\overline{t} \cdot \pi_L \pmod{\pi_L}.$$

In other words, $y \equiv \oplus_{\overline{t} \in J_n} \gamma_{\overline{t},1}^p \cdot T^\overline{t} \cdot \pi_L \pmod{\pi_L^2}$. Denote by $y_1$ the formal sum $\oplus_{\gamma_{\overline{t},1} \in J_n} \gamma_{\overline{t},1}^p \cdot T^\overline{t} \cdot \pi_L$. Suppose we have defined, for $1 \leq k \leq m-1$, elements

$$y_k = \bigoplus_{\overline{t} \in J_n} \gamma_{\overline{t},k}^p \cdot T^\overline{t} \cdot \pi_L^k, \gamma_{\overline{t},k} \in \mathcal{O}_L,$$

such that $y \equiv \bigoplus_{k=1}^{m-1} y_k \equiv 0 \pmod{\pi_L^m}$.

Then, again by Lemma 3.5.1, there exist elements $\gamma_{\overline{t},m} \in \mathcal{O}_L$ such that

$$\frac{1}{\pi_L^m} \left( y \oplus \bigoplus_{k=1}^{m-1} y_k \right) \equiv \sum_{\overline{t} \in J_n} \gamma_{\overline{t},m}^p \cdot T^\overline{t} \equiv \frac{1}{\pi_L^m} \bigoplus_{\overline{t} \in J_n} \gamma_{\overline{t},m}^p \cdot T^\overline{t} \cdot \pi_L^m \pmod{\pi_L}.$$

Denote by $y_m$ the formal sum $\bigoplus_{\overline{t} \in J_n} \gamma_{\overline{t},m}^p \cdot T^\overline{t} \cdot \pi_L^m$. Then

$$y \equiv \bigoplus_{k=1}^{m-1} y_k \equiv y_m \pmod{\pi_L^m} \quad \text{or} \quad y \equiv \bigoplus_{k=1}^{m} y_k \equiv 0 \pmod{\pi_L^{m+1}}.$$

Therefore $y = \bigoplus_{k=1}^{\infty} y_k$, which is what we wanted to prove. \(\square\)
Corollary 3.5.2. For every $x \in \mathcal{O}_\mathcal{L}$, there exists $\gamma_{\mathcal{L},k} \in \mathcal{O}_\mathcal{L}$ such that

$$\sum_{k=0}^{\infty} \sum_{T \in J_n} \gamma_{\mathcal{L},k} T_1^{i_1} \cdots T_d^{i_d-1} \pi_k^d.$$

Proof. Take $F$ to be the additive formal group $X + Y$.

3.5.2. Differentiability properties of the map $\mathcal{D}_{\mathcal{L},n}$. Let $r$ be a $t$-normalized series for $F$. Let $\hat{F}$ be the formal group $r(F(r^{-1}(X), r^{-1}(Y)))$. For $y \in \mu_\mathcal{L}$ we will denote by $\hat{\eta}_y(X_1, \ldots, X_d)$ the multivariable series

$$\bigoplus_{k=1}^{\infty} \left( \bigoplus_{T \in J_n} \gamma_{\mathcal{L},k} T_1^{i_1} \cdots T_d^{i_d-1} \pi_k^d \right), \quad (\gamma_{\mathcal{L},k} \in \mathcal{O}_\mathcal{L})$$

with respect to $\hat{F}$, such that $y = \hat{\eta}_y(T_1, \ldots, \pi_\mathcal{L})$, whose existence is guaranteed by Proposition 3.5.3.

Proposition 3.5.4. Let $\mathcal{L}$ be as in (12). For $y = \tilde{\eta}_y(T_1, \ldots, \pi_\mathcal{L}) \in \mu_\mathcal{L}$, $\tilde{\eta}_y$ as in (34), we have

$$D^{\pi}_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, y) = \sum_{i=1}^{d} \frac{\partial \tilde{\eta}_y}{\partial X_i} \bigg|_{X_i = T_k, k=1, \ldots, d} D^{\pi}_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, T_i),$$

for all $a_1, \ldots, a_{d-1} \in \mathcal{O}_\mathcal{L}$, where $T_d = \pi_\mathcal{L}$.

Proof. Let $y = \hat{\sum}_{k=1}^{\infty} y_k$, where

$$y_k = \bigoplus_{T \in J_n} \gamma_{\mathcal{L},k} T_1^{i_1} \cdots T_d^{i_d-1} \pi_k^d, \quad (\gamma_{\mathcal{L},k} \in \mathcal{O}_\mathcal{L})$$

Thus, $\tilde{\eta}_y = \hat{\sum}_{k=1}^{\infty} \eta_m$, where

$$\eta_m(X_1, \ldots, X_d) = \bigoplus_{T \in J_n} \gamma_{\mathcal{L},k} T_1^{i_1} \cdots T_d^{i_d-1} X_d^m.$$

Let us fix $a_1, \ldots, a_{d-1} \in \mathcal{L}^*$ and denote by $D(x)$ the element $D^{\pi}_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x)$ to simplify notation.

First notice that

$$v_{\mathcal{L}}(y_k) \xrightarrow{k \to \infty} \infty \implies D(y_k) = 0 \text{ for } k \text{ large enough.}$$

This follows from $D^{\pi}_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x) = \psi^{\pi}_{\mathcal{L},n}(a_1, \ldots, a_{d-1}, x) a_1 \cdots a_{d-1} x$ and the fact that $\psi^{\pi}_{\mathcal{L},n}$ has values in $R_\mathcal{L}/\pi^{n}R_\mathcal{L}$, i.e., $\pi^{n}_{\mathcal{L}}\psi^{\pi}_{\mathcal{L},n} = 0$.

Thus, from $\tilde{\eta}_y \equiv \hat{\sum}_{m=1}^{k-1} \eta_m \pmod{\pi^k}$ and equation (35), it is enough to consider the finite formal sum $\hat{\sum}_{m=1}^{k-1} \eta_m$. The proposition follows now from the fact that $D(\gamma^{\pi^k}) = p^n \gamma^{p^n-1} D(\gamma) = 0 \pmod{p^n}$, $D(xy) = yD(x) + xD(y)$, and corollary 3.5.1.

□

Corollary 3.5.3. Let $\mathcal{L}$ be as in (12). Let $y_i = \tilde{\eta}_y(T_1, \ldots, \pi_\mathcal{L}) \in \mu_\mathcal{L}$, for $1 \leq i \leq d$, where $\eta_{y_i}$ is a multivariable series of the form (34). Then

$$D^{\pi}_{\mathcal{L},n}(y_1, \ldots, y_d) = \det \left[ \frac{\partial \tilde{\eta}_y}{\partial X_j} \right]_{i,j} \bigg|_{X_i = T_k, k=1, \ldots, d} D^{\pi}_{\mathcal{L},n}(T_1, \ldots, T_d),$$

where $T_d = \pi_\mathcal{L}$.

□
Proof. The proof is immediate, see Section 6.4 of the Appendix.

From the above corollary we see that the map \( D_{L,n}^i \) behaves like a multidimensional derivation. Our goal in the following sections is to give conditions that will guarantee this. In particular, it will follow that (36) not only holds for elements in the maximal ideal \( \mu_L \) but in the full ring of integers \( \mathcal{O}_L \) and, moreover, it is independent of the power series representing the elements \( y_k, k = 1 \ldots d \).

4. Multidimensional derivations

In this section we recall the main properties of multidimensional derivations and set them in the context needed to deduce our formulae. We start by introducing one dimensional derivations.

We will use the following notation and assumptions. Let \( L \) be a \( d \)-dimensional local field with local uniformizers \( T_1, \ldots, T_{d-1} \) and \( \pi_L \). Let \( W \) be an \( \mathcal{O}_L \)-module that is \( p \)-adically complete, i.e.,

\[ W \cong \lim_{\leftarrow} W/p^nW. \]

For example, if \( p^nW = 0 \) for some \( n \), then \( W \) is \( p \)-adically complete. Actually, this is going to be our situation, since \( W \) will be the \( \mathcal{O}_L \)-module \( R_{L,1}/\pi^nR_{L,1} \).

4.1. Derivations and the module of differentials.

**Definition 4.1.1.** A derivation of \( \mathcal{O}_L \) into \( W \) over \( \mathcal{O}_K \) is a map \( D : \mathcal{O}_L \to W \) such that for all \( a, b \in \mathcal{O}_L \) we have

\[
\begin{align*}
(1) & \quad D(ab) = aD(b) + bD(a), \\
(2) & \quad D(a + b) = D(a) + D(b), \\
(3) & \quad D(a) = 0 \text{ if } a \in \mathcal{O}_K.
\end{align*}
\]

We denote by \( D_{\mathcal{O}_K}(\mathcal{O}_L, W) \) the \( \mathcal{O}_L \)-module of all derivations \( D : \mathcal{O}_L \to W \). The universal object in the category of derivations of \( \mathcal{O}_L \) over \( \mathcal{O}_K \) is the \( \mathcal{O}_L \)-module of Khaler differentials, denoted by \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \). This is the \( \mathcal{O}_L \)-module generated by finite linear combinations of the symbols \( da \), for all \( a \in \mathcal{O}_L \), divided out by the submodule generated by all the expressions of the form \( dab - adb - bda \) and \( d(a + b) - da - db \) for all \( a, b \in \mathcal{O}_L \) and \( da \) for all \( a \in \mathcal{O}_K \). The derivation \( \mathfrak{d} : \mathcal{O}_L \to \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \) is defined by sending \( a \) to \( da \).

If \( D : \mathcal{O}_L \to W \) is a derivation, then \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \) is universal in the following way. There exist a unique homomorphism \( \xi : \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to W \) of \( \mathcal{O}_L \)-modules such that the diagram

\[
\begin{array}{ccc}
\mathcal{O}_L & \xrightarrow{\mathfrak{d}} & \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \\
D \downarrow & & \downarrow \xi \\
W
\end{array}
\]

is commutative.

Let \( \tilde{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) \) be the \( p \)-adic completion of \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \), i.e.,

\[ \tilde{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) = \lim_{\leftarrow} \Omega_{\mathcal{O}_K}(\mathcal{O}_L)/p^n\Omega_{\mathcal{O}_K}(\mathcal{O}_L). \]

Since we are assuming that \( W \) is \( p \)-adically complete, the homomorphism \( \beta \) induces the homomorphism

\[ \xi : \tilde{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L) \to W. \]
Denote by \( \mathcal{D}(L/K) \subset \mathcal{O}_L \) the annihilator ideal of the torsion part of \( \hat{\Omega}_{O_K}(\mathcal{O}_L) \). Then we have the following proposition.

**Proposition 4.1.1.** The module \( \hat{\Omega}_{O_K}(\mathcal{O}_L) \) is generated by the elements \( dT_1, \ldots, dT_{d-1} \), \( d\pi_L \), and there is an isomorphism of \( \mathcal{O}_L \)-modules

\[
\hat{\Omega}_{O_K}(\mathcal{O}_L) \cong \mathcal{O}_L^{(d-1)} \oplus (\mathcal{O}_L/\mathcal{D}(L/K)).
\]

Moreover, if \( D(L/K) \) is as in Section 3.1, then

\[
\mathcal{D}(L/K) \mid D(L/K).
\]

In particular, if \( L \) is the standard higher local field \( L\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \) we have the isomorphism of \( \mathcal{O}_L \)-modules

\[
\hat{\Omega}_{O_K}(\mathcal{O}_L) \cong \mathcal{O}_LdT_1 \oplus \cdots \oplus \mathcal{O}_LdT_{d-1} \oplus (\mathcal{O}_L/D(L/K)\mathcal{O}_L)d\pi_L,
\]

where \( \pi_L \) is a uniformizer for \( L \) and \( D(L/K) \) is the different of the extension \( L/K \). Thus in this case

\[
\mathcal{D}(L/K) = D(L/K)\mathcal{O}_L.
\]

**Proof.** See [16] Section 12.0 (b). For an alternative proof see Section 6.5 of the Appendix.

\[\square\]

4.1.1. **Canonical derivations.** We will now define what it means to differentiate an element in \( \mathcal{O}_L \) with respect to the uniformizers \( T_1, \ldots, T_{d-1} \) and \( T_d = \pi_L \).

First we will assume \( L \) is the standard higher local field \( L\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \). Then we will define derivations over \( \mathcal{O}_K \)

\[
D_k : \mathcal{O}_L \to \mathcal{O}_L \quad (1 \leq k \leq d - 1)
\]
as follows. Let \( L_0 = L \) and \( L_k = L_{k-1}\{\{T_k\}\}, k = 1, \ldots, d - 1 \). For \( 1 \leq k \leq d - 1 \), we define the derivation of \( \mathcal{O}_{L_k} \) over \( \mathcal{O}_K \)

\[
D_k : \mathcal{O}_{L_k} \to \mathcal{O}_{L_k} \quad \text{by} \quad D_k\left( \sum a_i T_k^i \right) = \sum a_i i T_k^{i-1} \quad (a_i \in \mathcal{O}_{L_{k-1}}).
\]

We now lift these derivations to derivations of \( \mathcal{O}_L \) over \( \mathcal{O}_K \), by induction, in the following way. Suppose \( D : \mathcal{O}_{L_{k-1}} \to \mathcal{O}_{L_{k-1}} \) is a derivation of \( \mathcal{O}_{L_{k-1}} \) over \( \mathcal{O}_K \). Then \( D \) extends to a derivation of \( \mathcal{O}_{L_k} \) over \( \mathcal{O}_K \)

\[
D : \mathcal{O}_{L_k} \to \mathcal{O}_{L_k} \quad \text{by} \quad D\left( \sum a_i T_k^i \right) := \sum_i D(a_i) T_k^i.
\]

This derivation is well-defined since \( D \) is continuous with respect to the valuation topology of \( \mathcal{O}_{L_{k-1}} \). Thus the derivations (37) are well-defined and we can now introduce the following definition.

**Definition 4.1.2.** The derivations \( D_k : \mathcal{O}_L \to \mathcal{O}_L, 1 \leq k \leq d - 1 \) from Equation (37), will be denoted by \( \frac{\partial}{\partial T_k} \) for \( 1 \leq k \leq d - 1 \).

We now assume \( L \) is any \( d \)-dimensional local field not necessarily standard. Let \( \mathcal{L}_0 \subset \mathcal{L} \) be the standard local field defined in Section 1.3. For \( g(x) = a_0 + \cdots + a_k x^k \in \mathcal{O}_{\mathcal{L}_0}[X] \) we denote by \( \frac{\partial}{\partial T_i}g(x), i = 1, \ldots, d \), the polynomial

\[
\frac{\partial a_k}{\partial T_i} x^k + \cdots + \frac{\partial a_0}{\partial T_i} \in \mathcal{O}_{\mathcal{L}_0}[X] \quad (i = 1, \ldots, d - 1)
\]
and
\[ \frac{\partial g}{\partial T^i}(X) = g'(x) \quad (i = d) \]
If \( a \in \mathcal{O}_L \), let \( g(x) \in \mathcal{O}_L[X] \) such that \( a = g(\pi_L) \). Then we denote by \( \frac{\partial a}{\partial T^i} \) the element \( \frac{\partial g}{\partial T^i}(\pi_L), \ i = 1, \ldots, d \). Even though this definition depends on the choice of \( g(X) \), for the purpose of Proposition 4.1.2 below, this choice is immaterial.

If \( p(X) \) denotes the minimal polynomial of \( \pi_L \) over the extension \( L/L_0 \), then from Proposition 4.1.1 the equation \( p(\pi_L) = 0 \) implies that the elements \( dT_1, \ldots, dT_{d-1} \) and \( d\pi_L \) satisfy the relation
\[ \frac{\partial p}{\partial T_1}(\pi_L) dT_1 + \cdots + \frac{\partial p}{\partial T_{d-1}}(\pi_L) dT_{d-1} + p'(\pi_L) d\pi_L = 0, \]
and
\[ v_L(\mathcal{D}(L/K)) = \min \left\{ v_L \left( \frac{\partial p}{\partial T^1}(\pi_L) \right), \ldots, v_L \left( \frac{\partial p}{\partial T^{d-1}}(\pi_L) \right), v_L(p'(\pi_L)) \right\}. \]

We now deduce the following proposition.

**Proposition 4.1.2.** Let \( D : \mathcal{O}_L \to W \) be a derivation over \( \mathcal{O}_L \). Then the following relation holds
\[ \frac{\partial p}{\partial T_1}(\pi_L) D(T_1) + \cdots + \frac{\partial p}{\partial T_{d-1}}(\pi_L) D(T_{d-1}) + p'(\pi_L) D(\pi_L) = 0 \]
and for all \( a \in \mathcal{O}_L \) we have
\[ D(a) = \frac{\partial a}{\partial T_1} D(T_1) + \cdots + \frac{\partial a}{\partial T_{d-1}} D(T_{d-1}) + \frac{\partial a}{\partial T_d} D(\pi_L). \]

Conversely, let \( w_1, \ldots, w_d \in W \) such that
\[ \frac{\partial p}{\partial T_1}(\pi_L) w_1 + \cdots + \frac{\partial p}{\partial T_{d-1}}(\pi_L) w_{d-1} + p'(\pi_L) w_d = 0. \]

Then the map \( D : \mathcal{O}_L \to W \) defined by
\[ (39) \quad D(a) := \sum_{k=1}^d \frac{\partial a}{\partial T_k} w_k, \]
is a well-defined derivation from \( \mathcal{O}_L \) into \( W \) over \( \mathcal{O}_K \).

As a particular case, if \( w_1, \ldots, w_d \in W \) are annihilated by \( \mathcal{D}(L/K) \), then (39) defines a derivation over \( \mathcal{O}_K \).

**Proof.** The proof follows from the proof of Proposition 4.1.1 and the fact that \( D_{\mathcal{O}_K}(\mathcal{O}_L,W) \cong \text{Hom}_{\mathcal{O}_L}(\hat{\Omega}_{\mathcal{O}_K}(\mathcal{O}_L),W) \).

4.1.2. **Derivations on one-dimensional local fields.** The following three propositions are taken from [14] and will be needed in the deduction of our formulae. We put them here for convenience.

If \( L \) is a finite extension of the local field \( K \), then we denote by \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K) \) the \( \mathcal{O}_K \)-module of differentials of \( \mathcal{O}_L \) over \( \mathcal{O}_K \).

**Proposition 4.1.3.**

1. \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \cong \mathcal{O}_L/D(L/K) \) as \( \mathcal{O}_L \)-modules. Moreover, the element \( d\pi_L \) generates \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \).
2. If \( M \) is a finite extension of \( L \), the homomorphism \( \Omega_{\mathcal{O}_K}(\mathcal{O}_L) \to \Omega_{\mathcal{O}_K}(\mathcal{O}_M) \) is an embedding.
Proof. cf. [14] Proposition 5.1.

We will denote by $K_m$ (resp. $L_m$) the field obtained by adjoining the $m$-th torsion points to $K$ (resp. $L$), i.e., $K_m = K(\kappa_m)$. Let $v$ denote the normalized valuation $v_M/v_M(p)$, for every finite extension $M$ of $\mathbb{Q}_p$.

**Proposition 4.1.4.** There are explicit positive constants $c_1$, $c_2 \in \mathbb{R}$, depending only on $(F, \pi)$, such that

1. $v(D(L_m/L)) \leq m/\varrho + \log_2(m)/(p-1) + c_2$ and $v(D(K_m/K)) \geq m/\varrho - c_1$.
   Where $\varrho$ is the ramification index of $S$ over $\mathbb{Q}_p$.

2. Let $p_m$ be the period (i.e., the generator of the annihilator ideal) of the $\mathcal{O}_{K_m}$-submodule of $\Omega\mathcal{O}_{K_m}(\mathcal{O}_{K_m})$ generated by differentials $de^j_m$, $j = 1, \ldots, h$. Then $v(p_m) \geq m/\varrho - c_1$.

Proof. cf. [14] Proposition 5.3.

**Remark 4.1.1.** According to Kolyvagin (cf. [14] page 325) we may take for $c_2$ the constant

$$c_2 = \frac{2}{p-1} + \frac{2v(\pi)p}{(p-1)^2} + v(D(K_{2^e-1}/K)).$$

As for the constant $c_1$, if $K/S$ is an unramified extension we may take

$$c_1 = \frac{1}{\varrho(q_S^h - 1)},$$

where $q_S$ is the cardinality of the residue field of $S$.

**Remark 4.1.2.** By Proposition 4.1.4 (2), there exists a torsion point $\epsilon^i_m$, with $1 \leq j \leq h$, which we will denote by $\epsilon_m$, such that $v(p_m) \geq m/\varrho - c_1$. Here $p_m$ denoted the period of the $\mathcal{O}_{K_m}$-submodule of $\Omega\mathcal{O}_{K_m}(\mathcal{O}_{K_m})$ generated by $de^i_m$.

**Remark 4.1.3.** If $L \supset \kappa_n$ is a $d$-dimensional local field then the inequality Proposition 4.1.4(2) also holds, namely $v(D(L_m/L)) \leq m/\varrho + \log_2(m)/(p-1) + c_2$. This follows from the fact that $v(D(L_m/L)) \leq v(D(K_m/K))$.

**Proposition 4.1.5.** Suppose $K/S$ is an unramified extension and let $q = |k_S|$. Let $h$ be the height of $F$ with respect to $C = \mathcal{O}_S$, cf. Proposition 6.2.2. Then

1. $v(D(K_m/K)) \geq m/\varrho - 1/q(q^h - 1)$.

2. $K(\epsilon^i_m)$ is totally unramified over $K$ and $\epsilon^i_m$ is a uniformizer for $K(\epsilon^i_m)$.

Moreover

$v($ period of $de^i_m$ $) = m/\varrho - 1/q(q^h - 1)$.

In particular, if $h = 1$ then $K_m = K(\epsilon^i_m)$ and $D(K_m/K) = \pi^m/\pi_1\mathcal{O}_{K_m}$, where $\pi_1$ is a uniformizer for $K_1$.

Proof. cf. [14] Proposition 5.6.

4.2. Multidimensional derivations. Let $L$, $\tilde{L}$, $L$, $\tilde{L}$, $K$ and $W$ be as in the beginning of the previous section.

**Definition 4.2.1.** A $d$-dimensional derivation of $\mathcal{O}^d_L$ into $W$ over $\mathcal{O}_K$ is map $D: \mathcal{O}^d_L \to W$ such that for all $a_1, \ldots, a_d$ and $a'_1, \ldots, a'_d$ in $\mathcal{O}_L$, and any $1 \geq i \geq d$, it satisfies

1. Leibniz rule: $D(a_1, \ldots, a_i a'_i, \ldots, a_d) = a'_i D(a_1, \ldots, a_i, \ldots, a_d) + a_i D(a_1, \ldots, a'_i, \ldots, a_d)$.

2. Linearity: $D(a_1, \ldots, a_i + a'_i, \ldots, a_d) = D(a_1, \ldots, a_i, \ldots, a_d) + D(a_1, \ldots, a'_i, \ldots, a_d)$.
(3) Alternating: \( D(a_1, \ldots, a_d) = 0 \) if \( a_i = a_j \) for \( i \neq j \).

(4) Constants: \( D(a_1, \ldots, a_d) = 0 \) if \( a_i \in O_K \) for some \( 1 \leq i \leq d \).

We denote by \( D_d^{d}(O_L, W) \) the \( O_L \)-module of all \( d \)-dimensional derivations \( D: O_d^L \to W \).

Consider the \( d \)th exterior product \( \Omega_d^{d}(O_L) := \bigwedge^d O_L \Omega_{O_K}(O_L) \) (cf. [17] Chapter 19 §1 ). This is the \( O_L \)-module \( \Omega_{O_K}(O_L) \otimes \cdots \otimes \Omega_{O_K}(O_L) \) divided out by the \( O_L \)-submodule generated by the elements

\[ x_1 \otimes \cdots \otimes x_d, \]

where \( x_i = x_j \) for some \( i \neq j \). The symbols \( x_1 \otimes \cdots \otimes x_d \) will be denoted by

\[ x_1 \wedge \cdots \wedge x_d, \]

instead. For \( \Omega_{O_K}^d(O_L) \) we consider the \( d \)-dimensional derivation

\[ \mathfrak{d}: O_L^d \to \Omega_{O_K}^d(O_L) \]

that sends \( (a_1, \ldots, a_d) \) to the wedge product \( a_1 \wedge \cdots \wedge a_d \). This \( O_L \)-module is the universal object in the category of \( d \)-dimensional derivations of \( O_L \) over \( O_K \), i.e,

**Proposition 4.2.1.** If \( D: O_L^d \to W \) is a \( d \)-dimensional derivation over \( O_K \) then there exist a homomorphism \( \beta: \Omega_{O_K}^d(O_L) \to W \) of \( O_L \)-modules such that the diagram

\[ \begin{array}{ccc}
O_L^d & \xrightarrow{\mathfrak{d}} & \Omega_{O_K}^d(O_L) \\
D \downarrow & & \downarrow \lambda \\
& W & 
\end{array} \]

is commutative.

**Proof.** It is clear that \( \lambda \) is the homomorphism defined by \( \lambda(da_1 \wedge \cdots \wedge da_d) = D(a_1, \ldots, a_d) \). \( \square \)

**Proposition 4.2.2.** The \( O_L \)-module \( \hat{\Omega}_{O_K}^d(O_L) \) is generated by \( dT_1 \wedge \cdots \wedge dT_{d-1} \wedge d\pi_L \) and we have an isomorphism of \( O_L \)-modules

\[ \hat{\Omega}_{O_K}^d(O_L) \cong \Omega_{O_L}(O_L) \] as \( O_L \)-modules, where \( \pi_L \) is a uniformizer for \( L \) and \( D(L/K) \) is the different of the extension \( L/K \). Thus in this particular case

\[ \mathfrak{D}(L/K) = D(L/K)O_L. \]

**Proof.** This follows immediately from Lemma 4.1.1. \( \square \)

The above proposition implies

\[ \mathfrak{D}(L/K) \mid D(L/K). \]

We will show in the proposition below how every multidimensional derivation is characterized by the derivations in Definition 4.1.2.
Proposition 4.2.3. Let $D \in D^d_K(\mathcal{O}_L^d, W)$. Then $D(\mathcal{L}/K)$ annihilates $D(T_1, \ldots, \pi_L)$, and for $a_1, \ldots, a_d \in \mathcal{O}_L$ we have

\begin{equation}
D(a_1, \ldots, a_d) = \text{det} \left[ \frac{\partial a_i}{\partial T_j} \right]_{1 \leq i, j \leq d} D(T_1, \ldots, \pi_L).
\end{equation}

Conversely, we can construct a multidimensional derivation in the following way. Let $w \in W$ be annihilated by $D(\mathcal{L}/K)$, then the map $D(a_1, \ldots, a_d) := \text{det} \left[ \frac{\partial a_i}{\partial T_j} \right]_{1 \leq i, j \leq d} w$, is well-defined and belongs to $D^d_K(\mathcal{O}_L^d, W)$.

In other words, the map $D \mapsto D(T_1, \ldots, \pi_L)$, defines an isomorphism from $D^d_K(\mathcal{O}_L^d, W)$ to the $D(\mathcal{L}/K)$-torsion subgroup of $W$.

Proof. This follows from Proposition 4.2.2 and the fact that $D^d_K(\mathcal{O}_L^d, W) \cong \text{Hom}_{K^e} (\hat{\Omega}_K^d(\mathcal{O}_L), W)$.

\[ \square \]

5. Deduction of the Formulae

We finally deduce the main formulae to describe the Kummer pairing for the field $\mathcal{L}$ and the level $n$, i.e., $\langle \cdot, \cdot \rangle_{\mathcal{L}, n}$. The strategy of the proof is the following. We start by introducing an auxiliary finite field extension $\mathcal{M}$ of $\mathcal{L}$, containing sufficiently many torsion points, and also introduce higher levels $k$ and $m$ with $k \geq m \geq n$. In Section 5.1, we show that, under certain conditions, the map $D^k_{\mathcal{M}, k}$ is a derivation. In Section 5.2, we introduce an Artin-Hasse type formula for the generalized Kummer pairing of level $k$. This formula is characterized by an invariant attached to the Tate representation. With this invariant, we descend to the level $m$ and manufacture an explicit derivation $\mathcal{D}^k_{\mathcal{M}, m}$ in Definition 5.2.2 and then show that it coincides with $D^k_{\mathcal{M}, m}$. The final descend back to the field $\mathcal{L}$ and level $n$ will be guaranteed by Proposition 5.1.2 and accomplished in full detail in Section 5.3.

For the rest of this section we will use the following notation and assumptions:

\begin{equation}
\begin{cases}
\beta = (k, t) \text{ admissible pair}, \\
\pi^k | D(K_t/K), \\
\mathcal{M} \supset K_t.
\end{cases}
\end{equation}

Additionally, $\pi_\mathcal{M}$ will denote a uniformizer for $\mathcal{M}$ and $\pi_t$ a uniformizer for $K_t$. We will also denote by $\mathcal{K}_t$ the field $K_t \{\{T_1\} \ldots \{T_{d-1}\}\}$. The above conditions, and the fact that $D(K_t/K) = \mathcal{D}(\mathcal{K}_t/K)$ (cf. Proposition 4.1.1), imply

$\pi^k | \mathcal{D}(\mathcal{K}_t/K) | \mathcal{D}(\mathcal{M}/K)$

Note that the assumption of (42) on the different $D(K_t/K)$ is satisfied, for example, when $(t - k)/\varrho \geq c_1$; here $c_1$ is the constant from Proposition 4.1.4.
5.1. The reduction of the map $D^i_{M,k}$.

Proposition 5.1.1. Let $M$ be as in (42). Then the reduction

$$D^i_{M,k} : \mathcal{O}_M^i \to R_{M,1}/(\langle \pi^k/\pi_M \rangle R_{M,1})$$

of $D^i_{M,k}$ to $R_{M,1}/(\langle \pi^k/\pi_M \rangle R_{M,1})$ is a $d$-dimensional derivation over $\mathcal{O}_K$.

Proof. Let us fix $a_1, \ldots, a_{d-1} \in \mathcal{O}_M$. From Proposition 4.1.2 and the fact that

$$\mathcal{D}(M/K)D^i_{M,k}(a_1, \ldots, a_{d-1}, T_j) = 0 \pmod{\pi^k R_{M,1}}, \quad j = 1, \ldots, d - 1,$$

$$\mathcal{D}(M/K)D^i_{M,k}(a_1, \ldots, a_{d-1}, \pi_M) = 0 \pmod{\pi^k R_{M,1}},$$

we can construct a derivation

$$D : \mathcal{O}_M \to R_{M,1}/\pi^k R_{M,1},$$

such that $D(\pi_M) = D^i_{M,k}(a_1, \ldots, a_{d-1}, \pi_M)$ and $D(T_k) = D^i_{M,k}(a_1, \ldots, a_{d-1}, T_j)$,

$j = 1, \ldots, d - 1$, in the following way

$$D(a) = \sum_{j=1}^{d-1} \frac{\partial a}{\partial T_j} D(T_j) + \frac{\partial a}{\partial T_d} D(\pi_M),$$

where $a \in \mathcal{O}_M$.

According to Proposition 3.5.4 both $D$ and $D^i_{M,k}(a_1, \ldots, a_{d-1}, \cdot)$ coincide in $\mu_M$. But from the Leibniz rule it follows, by comparing $D(\pi_M x)$ and $D^i_{M,k}(a_1, \ldots, a_{d-1}, \pi_M x)$ when $x \in \mathcal{O}_M$, that they coincide $\pmod{\langle \pi^k/\pi_M \rangle R_{M,1}}$ in all of $\mathcal{O}_M$.

It follows now that

$$D^i_{M,k} : \mathcal{O}_M^i \to R_{M,1}/(\langle \pi^k/\pi_M \rangle R_{M,1})$$

satisfies all conditions from definition 4.2.1 and so by Proposition 4.2.3 we have that it is a $d$-dimensional derivation such that

$$D^i_{M,k}(a_1, \ldots, a_d) = \det \begin{bmatrix} \frac{\partial a_i}{\partial T_j} \end{bmatrix}_{1 \leq i, j \leq d} D_{M,k}(T_1, \ldots, T_{d-1}, \pi_M),$$

where $a_1, \ldots, a_d \in \mathcal{O}_M$. \hfill $\square$

5.1.1. Description of the map $\psi^i_{M,m}$ in terms of $D^i_{M,m}$. Observe that for any $m \leq k$ the pair $(m, t)$ is also admissible, so Proposition 5.1.1 also holds for this pair. Thus, $D^i_{M,m} : \mathcal{O}_M^i \to R_{M,1}/(\langle \pi^m/\pi_M \rangle R_{M,1})$ is also a derivation as well and, moreover, we can express the map $\psi^i_{M,m}$ out of $D^i_{M,m}$ as follows. For $u_1, \ldots, u_d$ in $\mathcal{O}_M^* = \{ x \in \mathcal{O}_M : v_M(x) = 0 \}$ we let

$$\begin{align*}
\psi^i_{M,m}(u_1, \ldots, u_{d-1}, \pi_M) &= D^i_{M,m}(u_1, \ldots, u_{d-1}, \pi_M) \pmod{\pi^m R_{M,1}}
\psi^i_{M,m}(u_1, \ldots, u_d) &= D^i_{M,m}(u_1, \ldots, u_d) \pmod{\pi^m R_{M,1}}
\psi^i_{M,m}(u_1, \ldots, \pi_M u_d) &= k \psi^i_{M,m}(u_1, \ldots, \pi_M) + \psi^i_{M,m}(u_1, \ldots, u_d), \quad k \in \mathbb{Z}
\psi^i_{M,m}(a_1, \ldots, a_d) &= 0, \text{ whenever } a_i = a_j \text{ for } i \neq j, \quad a_1, \ldots, a_d \in M^*
\end{align*}$$

(43)
It is clear from the definition that this is independent from the choice of a uniformizer $\pi_\mathcal{M}$ of $\mathcal{M}$. Notice that the fourth property says that $\psi_{\mathcal{M},m}^i$ is alternate, in particular it is skew-symmetric, i.e.,

$$\psi_{\mathcal{M},m}^i(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_d) = -\psi_{\mathcal{M},m}^j(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_d).$$

whenever $i \neq j$.

5.1.2. Descending from $\mathcal{M}$ to $\mathcal{L}$ and from level $m$ to level $n$. The following proposition will be used in the main result (Theorem 5.3.1) to descend from the auxiliary field $\mathcal{M}$ to the ground field $\mathcal{L}$ and from the level $m$ to the level $n$.

Let $\mathcal{L} \supset \kappa_n$ be a $d$-dimensional local field and $v$ denote the normalized valuation $v_\mathcal{L}/v_\mathcal{L}(p)$.

**Proposition 5.1.2.** Let $m, n$ be integers such that $v(f^{(m-n)}(x)) > 1/(p-1)$ for all $x \in F(\mu_\mathcal{L})$ (cf. Remark 5.1.1). Let $(m, t)$ be admissible and put $\mathcal{M} = \mathcal{L}_t$. Then $\text{Tr}_{\mathcal{M}/\mathcal{L}}$ induces a homomorphism from $R_{\mathcal{M},1}/\pi^m R_{\mathcal{M},1}$ to $\mathcal{L}/\pi^n R_\mathcal{L}$ and we have the representation

$$\psi_{\mathcal{M},m}^i(x) = \text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(x)) l_F(x),$$

for all $x \in K_d(\mathcal{M}$) and $x \in F(\mu_\mathcal{L})$. In particular, $\text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(x))$ belongs to $\mathcal{L}/\pi^n R_\mathcal{L}$ and it is the unique element satisfying (44).

**Proof.** This proof was inspired by Proposition 6.1 of [14]. Since $e_n^i = f^{(m-n)}(e_n^i)$ and $f^{(m-n)}(x) \in \mu_{\mathcal{L},1} \subset \mu_{\mathcal{M},1}$, then by Proposition 2.2.1 (4) and (5) we have

$$\begin{align*}
(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^{i, m}_\mathcal{L} &= \frac{1}{\pi^{m-n}}(\alpha, f^{(m-n)}(x))^{i, m}_{\mathcal{M}, m} \\
&= \frac{1}{\pi^{m-n}} \text{T}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha) l(f^{(m-n)}(x)) = \text{T}_{\mathcal{L}/\mathcal{S}}(\text{T}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) l(x).
\end{align*}$$

From the condition on $m$ we have that $\pi^{m-n} \mathcal{T}_\mathcal{L} \subset \mathcal{T}_{\mathcal{L},1}$. Thus, after taking the dual with respect to $\text{T}_{\mathcal{L}/\mathcal{S}}$ we obtain

$$\frac{1}{\pi^{m-n}} R_\mathcal{L} \supset R_{\mathcal{L},1}, \text{ or, } \pi^n R_\mathcal{L} \supset \pi^m R_{\mathcal{L},1}.$$ 

Then from $\text{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M},1}) \subset R_{\mathcal{L},1}$, cf. Proposition 3.3.2 (1), it follows that

$$\text{Tr}_{\mathcal{M}/\mathcal{L}}(\pi^m R_{\mathcal{M},1}) \subset \pi^n R_{\mathcal{L},1} \subset \pi^n R_\mathcal{L}.$$ 

(45)

It then follows that $\text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) \in \mathcal{L}/\pi^n R_\mathcal{L}$, and moreover since $(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)^{i, m}_\mathcal{L} \subset C/\pi^n C$ then $\text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M},m}^i(\alpha)) \in R_\mathcal{L}/\pi^n R_\mathcal{L}$ and the uniqueness follows from Lemma 3.2.2.

**Remark 5.1.1.** For $m$ in Proposition 5.1.2 we can take any

$$m > n + \log_p (v_\mathcal{L}(p)/(p-1)) + g/(p-1).$$

The proof of this claim can be found in Proposition 6.2 of [14]. In Section §6.6 we reproduce the proof.
5.2. Artin-Hasse-type formulae for the Kummer pairing. Having shown that the reduction of $D_{M,m}$ is a multidimensional derivation we know then, after Proposition 4.2.3, that this map is completely characterized by its value at the local uniformizers $T_1, \ldots, T_{d-1}, \pi_M$. In this section, we will normalize this description using torsion points of the formal group $F$ instead. Concretely, we will show that there exist a torsion element $\epsilon_t \in \kappa_t$ (cf. Definition 5.2.1) such that the derivation $D_{M,m}$ can be described by its values at $T_1, \ldots, T_{d-1}, \epsilon_t$. This description is done via an invariant associated to an Artin-Hasse type formula for the Kummer pairing of level $k$.

5.2.1. The invariant $\overline{\tau}_{\beta,i,j}$. Recall that we have fixed a basis $\{e^i\}_{i=1}^h$ for $\kappa = \lim_{\leftarrow} \kappa_n$. We also denoted by $e^i_t$ the reduction of $e^i$ to $\kappa_n$. Clearly $\{e^i_t\}$ is a basis for $\kappa_n$.

Let $\mathcal{M}$, $\kappa_t$ and $\beta = (k,l)$ be as in (42). Let $\mathcal{M}_n = \mathcal{M}(\kappa_n)$. The action of $G_M = \text{Gal}(\mathcal{M}/\mathcal{M})$ on $\kappa$ defines a continuous representation
\[ \tau : G_M \to GL_n(C). \]

The reduction of $\tau$ to $GL_h(C/\pi^nC)$ is the analogous representation of $G_M$ on $\kappa_n$ and will be denoted by $\tau_n$. This clearly induces an embedding $\tau_n : G(\mathcal{M}_n/\mathcal{M}) \to GL_h(C/\pi^nC)$.

If $a \in \mathcal{M}^*$, then
\[ \tau_{k+t}(\mathbf{Y}_M(\{T_1, \ldots, T_{d-1}, a\})) \]
is congruent to the identity matrix $I$ (mod $\pi^t$) because the Galois group $G(\mathcal{M}_n/\mathcal{M})$ fixes $\kappa_t$ and so correspond in $GL_h(C/\pi^k+tC)$ to matrices $\equiv I$ (mod $\pi^t$), i.e., there exist characters $\chi_{M,\beta;i,j} : \mathcal{M}^* \to C/\pi^kC$ such that
\[ \tau_{k+t}(\mathbf{Y}_M(\{T_1, \ldots, T_{d-1}, a\})) = I + \pi^t(\chi_{M,\beta;i,j}(a)) \in GL_h(C/\pi^{k+t}C). \]

For $\mathcal{M} = \kappa_t$ we simply write $\chi_{\beta;i,j}$. By Proposition 2.1.2 (4) we have that
\[ N_{\mathcal{M}/\mathcal{K}_t}(\{T_1, \ldots, T_{d-1}, a\}) = \{ T_1, \ldots, T_{d-1}, N_{\mathcal{M}/\mathcal{K}_t}(a) \}, \]
where $\mathcal{K}_t = K_t \{T_1\} \cdots \{T_{d-1}\}$, and Proposition 2.1.2 (4) implies
\[ \chi_{M,\beta;i,j}(a) = \chi_{\beta;i,j}(N_{\mathcal{M}/\mathcal{K}_t}(a)) \]

The definition of the pairing $(,)_t$ implies, for $v \in \kappa_t$, that
\[ \left( \{T_1, \ldots, T_{d-1}, a\}, v \right)_t = \left( \chi_{M,\beta;i,j}(a) \right)_{i,j} (v^j), \]
as an identity of column vectors, where the right hand side is the product of the matrix $\left( \chi_{M,\beta;i,j}(a) \right)_{i,j}$ with the column vector $(v^j)$ formed by the coordinates of $v$ with respect to to $\{e^i_t\}$. In particular, for $v = e^i_t$ we have
\[ \left( \{T_1, \ldots, T_{d-1}, a\}, e^i_t \right)_t = \chi_{M,\beta;i,j}(a). \]

According to Proposition 3.4.1 we see that $\chi_{M;i,j}$ uniquely determines a constant $c_{M,\beta;i,j} \in R_{M,1}/\pi^k R_{M,1}$ such that
\[ \chi_{M,\beta;i,j}(u) = T_M/S(\log(u) c_{M,\beta;i,j}) \quad \forall u \in \mathcal{V}_{M,1}. \]
Namely, $R_{M,k}(T_1, \ldots, T_{d-1}, e^i_t) = c_{M,\beta;i,j}$.

Observe that $c_{M,\beta;i,j}$ is, by Equation (47), the image of $c_{\beta,i,j} \in c_{\mathcal{K}_t,\beta;i,j}$ under the map
\[ R_{\mathcal{K}_t,1}/\pi^k R_{\mathcal{K}_t,1} \to R_{M,1}/\pi^k R_{M,1} \]
$(R_{\mathcal{K}_t,1} \subset R_{M,1})$. So we will denote $c_{M,\beta;i,j}$ by $\overline{\tau}_{\beta;i,j}$. 

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Definition 5.2.1. For the torsion element $\epsilon_t$ in Remark 4.1.2 we will denote the constants $c_{\beta,i,j}$ and $\tau_{\beta,i,j}$ by $c_{\beta,j}$ and $\tau_{\beta,i}$, respectively, so that
\begin{equation}
\{ T_1, \ldots, T_{d-1}, u \}, \epsilon_t \}_{l,M,k} = T_{M/S}( \log(u) \tau_{\beta,i} ) \quad \forall u \in V_{M,1}.
\end{equation}
The choice of this $\epsilon_t$ is independent of $i$.

Note that Equation (49) defines an Artin-Hasse-type formula for the Kummer pairing, in which the constant $\tau_{\beta,i}$ is characterized by the value of the Kummer pairing at the torsion point $\epsilon_t$. We will see in the next section how to construct the $D_{l,M,m}$ from the value $\tau_{\beta,i}$.

Finally, observe that $c_{\beta,i,j}$ is an invariant of the isomorphism class of $(F,e_j)$ because if $g : (F,e_j) \to (\tilde{F},\tilde{e}_j)$ is such isomorphism then $\rho^i_{M,k}(T_1, \ldots, T_{d-1}, g(x)) = \rho^i_{M,k}(T_1, \ldots, T_{d-1}, x)$.

From Proposition 3.4.2 we conclude that

Proposition 5.2.1. Let $M$ and $\beta = (k, l)$ be as in (42). If $r(X)$ is a $t$-normalized series for $F$, then
\begin{equation}
D_{l,M,k}(T_1, \ldots, T_{d-1}, r(e^i_t)) = -r'(e^i_t) T_1 \cdots T_{d-1} \tau_{\beta,i}^{l/M}.
\end{equation}
In particular, this holds for the torsion element $\epsilon_t$ and the invariant $\tau_{\beta,i}$.

5.2.2. Explicit description of $D_{l,M,m}$ using the invariant $\tau_{\beta,i}$. Let $L$ be a $d$-dimensional local field. Define
\begin{equation}
R'_{L,1} := \{ x \in L : v_L(x) \geq -v_L(D(L/S)) - \lfloor v_L(p)/(p-1) \rfloor \}.
\end{equation}
Note that
\[ v_L(x) \geq -v_L(D(L/S)) - \left\lfloor \frac{v_L(p)}{p-1} \right\rfloor \quad \text{iff} \quad v_L(x) \geq -v_L(D(L/S)) - \frac{v_L(p)}{p-1}. \]
This holds since $v_L(x)$ and $v_L(D(L/S))$ are integers, therefore the conditions
\[ \frac{v_L(p)}{p-1} \geq -v_L(D(L/S)) - v_L(x) \quad \text{and} \quad \frac{v_L(p)}{p-1} \geq -v_L(D(L/S)) - v_L(x), \]
are equivalent by the very definition of the integral part of a real number. Comparing with Equations (51) and (19) we see that $R'_{L,1} = \pi_L R_{L,1}$.

If $M/L$ is a finite extension of $d$-dimensional local fields, then clearly
\begin{equation}
R'_{M,1} = (1/D(M/L)) R'_{L,1}.
\end{equation}

Proposition 5.2.2. Let $\beta = (k, t)$, $K_t$ and $\mathcal{K}_t$ be as in (42). Let $a_j \in \mathcal{O}_K$, such that $d\epsilon_t^j = a_j d\pi_1$ in $\Omega_{\mathcal{O}_K}(\mathcal{O}_K)$. Then
\begin{equation}
c_{\beta,i,j} = \frac{a_j R'_{\mathcal{K}_t,1} + \tau^{k}_{\mathcal{K}_t} R'_{\mathcal{K}_t,1} + \pi^{k} R_{\mathcal{K}_t,1}}{\pi^{k} R_{\mathcal{K}_t,1}}.
\end{equation}

Proof. The proof is the same as of Proposition 6.5 of [14]. We will reproduce it again in Section 6.6 of the Appendix.

Lemma 5.2.1. Let $M$ be as in (42). Let $\epsilon_t$ and $\tau_{\beta,i}$ be as in Definition 5.2.1. Let $b \in \mathcal{O}_M$ such that $dT_1 \wedge \cdots \wedge d\epsilon_t = b dT_1 \wedge \cdots \wedge d\pi_M$ in $\Omega_{\mathcal{O}_M}(\mathcal{O}_M)$; this $b$ exists by Proposition 4.2.2. Then there exists $a \gamma_i \in R_{M,1}/(\pi^{k}/\pi_M) R_{M,1}$ such that
\begin{equation}
-\tau_{\beta,i}/l'(\epsilon_t) = b \gamma_i.
\end{equation}
Moreover, all such $\gamma_i$’s coincide when reduced to
\[ R_{M,1}/(\pi^m/\pi_M)R_{M,1}, \]
for $m \leq k$ satisfying $m/\varrho \leq k/\varrho - v(D(M/K)) + t/\varrho - c_1$; $c_1$ is the constant from Proposition 4.1.4.

**Remark 5.2.1.** We can interpret the element $\gamma_i$ (mod $(\pi^k/\pi_M)R_{M,1}$) as follows. Choose a polynomial $g(X)$ with coefficients in the ring of integers of the maximal subextension of $M$ unramified over $K$, such that $\epsilon_t = g(\pi_M)$. Then
\[ \gamma_i = -\frac{\tau_{\beta,i}}{\ell'(\epsilon_t) g'(\pi_M)}. \]
Moreover, modulo $(\pi^m/\pi_M)R_{M,1}$, this is independent of the choice of $g(X)$.

**Proof of Lemma 5.2.1.** Let $\lambda_i$ be a representative for $c_{\beta,i}$ in $R_{K,i,1}$. By Proposition 5.2.2, $\lambda_i \in aR_{K,i,1} + (\pi^k/\pi_M)R_{K,i,1}$ where $\delta_t = a\delta_t$. Let $b \in O_M$ such that $dT_1 \wedge \cdots \wedge d\pi_t = b dT_1 \wedge \cdots \wedge d\pi_M$, in particular $D(M/K_i) = bO_M$, and set $b = a b$. We have, by 52, that
\[ \frac{1}{b}R_{K,i,1} = (1/D(M/K_i))R_{K,i,1} = R_{M,1} \subseteq R_{M,1}, \]
\[ \frac{\pi^k}{\pi_t} R_{K,i,1} \subseteq \frac{\pi^k}{\pi_M} D(M/K_i) R_{M,1} \subseteq \frac{\pi^k}{\pi_M} R_{M,1}, \]
where the last inclusion follows since $D(M/K_i)$ is divisible by $\pi_t/\pi_M$; this follows from the general inequality $v_M(D(M/K_i)) \geq c(M/K_i) - 1$ (cf. [3] Chapter 1 Proposition 5.4). It thus follows
\[ -\frac{\tau_{\beta,i}}{\ell'(\epsilon_t)} = -\frac{\lambda_i}{\ell'(\epsilon_t) g'(\pi_M)} \in b \frac{R_{M,1}}{(\pi^k/\pi_M)R_{M,1}}. \]

Now let us prove the second assertion. Since $-\tau_{\beta,i}/\ell'(\epsilon_i) = b \gamma_i$, then $\gamma_i$ is uniquely defined (mod $(\pi^k/\pi_M)bR_{M,1}$). Let $m \leq k$, thus $\gamma_i$ is uniquely defined (mod $(\pi^m/\pi_M)R_{M,1}$) as long as $\pi^m|(\pi^k/b)$. But this condition is fulfilled when $m/\varrho \leq k/\varrho - v(D(M/K)) + t/\varrho - c_1$. Indeed, $dT_1 \wedge \cdots \wedge d\delta_t = b dT_1 \wedge \cdots \wedge d\pi_M$ implies that $v(\epsilon_t) + v(b) = v(D(M/K)) (\leq v(D(M/K)))$, with $\epsilon_t$ as in Remark 4.1.2, therefore by the same remark
\[ m/\varrho \leq k/\varrho - v(D(M/K)) + t/\varrho - c_1 \leq k/\varrho - v(D(M/K)) + v(\epsilon_t) = k/\varrho - v(b). \]

**Definition 5.2.2.** Using the same notation and assumptions of Lemma 5.2.1, we construct the explicit $d$-dimensional derivation $\mathcal{D}_{M,m}^i : O_M^d \to R_{M,1}/(\pi^m/\pi_M)R_{M,1}$, using Proposition 4.2.3, in the following way
\[ \mathcal{D}_{M,m}(a_1, \ldots, a_d) := \det \left[ \frac{\partial a_i}{\partial T_j} \right]_{1 \leq i, j \leq d} T_1 \cdots T_{d-1} \gamma_i, \]
where \( a_1, \ldots, a_d \in \mathcal{O}_M \). Moreover, from \( \mathcal{D}^i_{M,m} \) we can construct an explicit logarithmic derivative \( \mathcal{D} \log^i_{M,m} : K_d(M) \to R_{M,1}/(\pi^m/\pi_M^2)R_{M,1} \), by setting

\[
\begin{aligned}
\mathcal{D} \log^i_{M,m}(u_1, \ldots, u_d, \pi_M) &= \mathcal{D}^i_{M,m}(u_1, \ldots, u_{d-1}, \pi_M) (\text{mod } \pi_M^m R_{M,1}) \\
\mathcal{D} \log^i_{M,m}(u_1, \ldots, u_d) &= \mathcal{D}^i_{M,m}(u_1, \ldots, u_d) (\text{mod } \pi_M^m R_{M,1}) \\
\mathcal{D} \log^i_{M,m}(u_1, \ldots, \pi_M^k u_d) &= k \mathcal{D} \log^i_{M,m}(u_1, \ldots, \pi_M) + \mathcal{D} \log^i_{M,m}(u_1, \ldots, u_d), \quad k \in \mathbb{Z} \\
\mathcal{D} \log^i_{M,m}(a_1, \ldots, a_d) &= 0, \text{ whenever } a_j = a_k \text{ for } j \neq k, \quad a_1, \ldots, a_d \in \mathcal{M}^*
\end{aligned}
\]

where \( u_1, \ldots, u_d \) are in \( \mathcal{O}_M = \{ u \in \mathcal{O}_M : \psi_M(u) = 0 \} \).

We now give conditions on when the map \( \mathcal{D}^i_{M,m} \) (respectively \( \psi^i_{M,m} \)) and the explicit \( \mathcal{D} \log^i_{M,m} \) (respectively \( \mathcal{D} \log^i_{M,m} \)) coincide.

**Proposition 5.2.3.** Let \( M \) be as in (42). Let \( m \leq k \) such that \( m/\varrho \leq k/\varrho + t/\varrho - v(D(M/K)) - c_1; c_1 \) is the constant from Proposition 4.1.4. Then the reduction of

\[ D^i_{M,m} : \mathcal{O}_M^d \to R_{M,1}/\pi^m R_{M,1} \]

to \( R_{M,1}/(\pi^m/\pi_M^2)R_{M,1} \) is a \( d \)-dimensional derivation over \( \mathcal{O}_K \). Moreover, this \( d \)-dimensional derivation coincides with the explicit \( d \)-dimensional derivation \( \mathcal{D} \log^i_{M,m} \) from Definition 5.2.2.

**Remark 5.2.2.** In particular,

\[ D^i_{M,m}(T_1, \ldots, T_{d-1}, e^j_1) = -T_1 \cdots T_{d-1} \gamma_{i,j} \theta_{\beta;i,j}/\psi'(e^j_1) (\text{mod } \pi^m/\pi_M)R_{M,1} \]

for all \( 1 \leq j \leq h \).

**Proof.** First notice that, by Proposition 3.3.2 (3),

\[ D^i_{M,m} : \mathcal{O}_M^d \to R_{M,1}/(\pi^m/\pi_M)R_{M,1} \]

is the reduction of

\[ D^i_{M,k} : \mathcal{O}_M^d \to R_{M,1}/(\pi^k/\pi_M)R_{M,1} \]

and the latter is a \( d \)-dimensional derivation over \( \mathcal{O}_K \) according to Proposition 5.1.1. Also, according Proposition 5.2.1 we have

\[ (54) \quad D^i_{M,k}(T_1, \ldots, T_{d-1}, e_t) = -T_1 \cdots T_{d-1} \gamma_{i,j} \theta_{\beta;i,j}/\psi'(e_t). \]

Moreover, for \( b \) as in Lemma 5.2.1, we have that

\[ D^i_{M,k}(T_1, \ldots, T_{d-1}, e_t) = b D^i_{M,k}(T_1, \ldots, T_{d-1}, \pi_M). \]

This together with (54) imply that \( D^i_{M,k}(T_1, \ldots, T_{d-1}, \pi_M)/T_1 \cdots T_{d-1} \) is one of the \( \gamma_i \)'s that satisfies (53). Therefore, by the second assertion of Lemma 5.2.1 and by the way \( \mathcal{D} \log^i_{M,m} \) was constructed in Definition 5.2.2, we conclude that the three maps \( D^i_{M,k}, D^i_{M,m} \) and \( \mathcal{D} \log^i_{M,m} \) coincide \( \text{mod } (\pi^m/\pi_M)R_{M,1} \). This concludes the proof.

The statement in the remark holds because \( D^i_{M,m} \) is a multidimensional derivation and from the formula for \( D^i_{M,m}(T_1, \ldots, T_{d-1}, r(e^j_1)) \) in equation (50). \( \square \)
5.3. Main formulae. Let $\mathcal{L} \supset K_n$ be a $d$-dimensional local field. Let $\varrho$ be the ramification index of $S/\mathbb{Q}_p$, and

\begin{equation}
4.1.3
m = n + 2 + \lfloor \varrho \log_p \left( v_L(p)/(p - 1) \right) + \varrho/(p - 1) \rfloor.
\end{equation}

Take $k$ an integer large enough such that

\begin{equation}
5.1.1
\frac{k}{\varrho} + \frac{k}{\varrho} \geq m/\varrho + c_1 + v(D(\mathcal{M}/K)) \quad \text{and}
\end{equation}

\begin{equation}
5.2.2
k + \varrho + 1 \geq c_1 \varrho,
\end{equation}

where $t = 2k + \varrho + 1$, $\mathcal{M} = \mathcal{L}_t$, and $c_1$ is the constant from Proposition 4.1.4. Condition (56) holds, for example, when

\begin{equation}
5.2.3\frac{k}{\varrho} \geq \frac{m}{\varrho} + \frac{\log_2(2k + \varrho + 1)}{p - 1} + c_1 + c_2 + v(D(\mathcal{L}/K)),
\end{equation}

where $c_2$ is the constant from Proposition 4.1.4; see the proof of Theorem 5.3.1 for a deduction of (56) from (58). We now formulate the main result.

**Theorem 5.3.1.** Let $\mathcal{L}$, $\mathcal{M}$ and $m$ be as above. Then

\begin{equation}
(N_{\mathcal{M}/\mathcal{L}}(\alpha), x)_{\mathcal{L}, n} = \mathbb{T}_{\mathcal{L}/S}(\text{Tr}_{\mathcal{M}/\mathcal{L}}(\mathfrak{D}\log_{\mathcal{M}, m}^i(\alpha) l(x))) = \mathbb{T}_{\mathcal{M}/S}(\mathfrak{D}\log_{\mathcal{M}, m}^i(\alpha) l(x)),
\end{equation}

for all $\alpha \in K_d(\mathcal{M})$ and all $x \in F(\mu_L)$. Here $\mathfrak{D}\log_{\mathcal{M}, m}^i$ is the explicit logarithmic derivative constructed in Definition 5.2.2.

**Proof.** By considering the tower $\mathcal{L}_t \supset \mathcal{L} \supset K$ and the upper bound in Proposition 4.1.4 (1) and Remark 4.1.3, we get

\[
v(D(\mathcal{M}/K)) = v(D(\mathcal{L}_t/\mathcal{L}))+v(D(\mathcal{L}/K)) \leq \frac{t}{\varrho} + \log_2(t)/(p - 1) + c_2 + v(D(\mathcal{L}/K)).
\]

Adding $m/\varrho + c_1$ we obtain, by (58), the inequality (56). The definition of $t$ clearly implies that $(k, t)$ is admissible and condition (57) implies $(t - k)/\varrho \geq c_1$. Thus $\mathcal{M}$, $t$, $k$ and $m$ defined as above, satisfy the hypothesis of Proposition 5.3.1 and of Definition (5.2.2). The result now follows from Proposition 5.1.2, the Remark 5.1.1 and Equation (43), i.e., the map $\psi_{\mathcal{M}, m}$ and $\mathfrak{D}\log_{\mathcal{M}, m}^i$ coincide. It remains only to check that $\text{Tr}_{\mathcal{M}/\mathcal{L}}((\pi^m/\pi_\mathcal{M}) R_{M, 1}) \subset \pi^n R_L$, so that $\text{Tr}_{\mathcal{M}/\mathcal{L}}(\mathfrak{D}\log_{\mathcal{M}, m}^i(\alpha))$ is well defined in $R_L/\pi^n R_L$. To do this we notice that condition (55) implies

\[
m - 1 > n + \varrho \log_p \left( v_L(p)/(p - 1) \right) + \varrho/(p - 1),
\]

and we can apply Remark 5.1.1 to $m - 1$ and get, by Equation (45), that

\[
\text{Tr}_{\mathcal{M}/\mathcal{L}} \left( \pi^{m-1} \frac{\pi}{\pi_\mathcal{M}} R_{M, 1} \right) \subset \text{Tr}_{\mathcal{M}/\mathcal{L}}(\pi^{m-1} R_{M, 1}) \subset \pi^n R_L.
\]

Bearing in mind that $\pi_\mathcal{M}^{k/\varrho} \pi$, since $\pi^k D(\mathcal{M}/K)$ (and $D(\mathcal{M}/K) = D(\mathcal{M}/K)$ implies $e(\mathcal{M}/K) > 1$ (i.e., $\mathcal{M}/K$ is not unramified). □

6. Appendix

6.1. Higher-dimensional local fields. The reader is welcome to visit the nice paper [7] for an account on higher local fields and all the results not proven in this section. In the rest of this section $E$ will denote a local field, $k_E$ its residue field and $\pi_E$ a uniformizer for $E$. 


Definition 6.1.1. \( \mathcal{K} \) is an \( d \)-dimensional local field, i.e., a field for which there is a chain of fields \( \mathcal{K}_d = \mathcal{K}, \mathcal{K}_{d-1}, \ldots, \mathcal{K}_0 \) such that \( \mathcal{K}_{i+1} \) is a complete discrete valuation ring with residue field \( \mathcal{K}_i \), \( 0 \leq i \leq d - 1 \), and \( \mathcal{K}_0 \) is a finite field of characteristic \( p \).

If \( k \) is a finite field then \( \mathcal{K} = k((T_1)) \ldots ((T_d)) \) is a \( d \)-dimensional local field with \( \mathcal{K}_i = k((T_1)) \ldots ((T_i)) \), \( 1 \leq i \leq d \).

If \( E \) is a local field, then \( \mathcal{K} = E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \) is defined inductively as \( E_{d-1}\{\{T_{d-1}\}\} \) where \( E_{d-1} = E\{\{T_1\}\} \ldots \{\{T_{d-2}\}\} \). We have that \( \mathcal{K} \) is a \( d \)-dimensional local field with residue field \( \mathcal{K}_{d-1} = k_{E_{d-1}}((T_{d-1})) \), and by induction

\[ k_{E_{d-1}} = k_E((T_1)) \ldots ((T_{d-2})). \]

Therefore \( \mathcal{K}_{d-1} = k_E((T_1)) \ldots ((T_{d-1})) \). These fields are called the standard fields.

From now on we will assume \( \mathcal{K} \) has mixed characteristic, i.e., \( \text{char}(\mathcal{K}) = 0 \) and \( \text{char}(\mathcal{K}_{d-1}) = p \). The following theorem classifies all such fields.

Theorem 6.1.1 (Classification Theorem). Let \( \mathcal{K} \) be an \( d \)-dimensional local field of mixed characteristic. Then \( \mathcal{K} \) is a finite extension of a standard field

\[ E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\}, \]

where \( E \) is a local field, and there is a finite extension of \( \mathcal{K} \) which is a standard field.

Proof. cf. [7] § 1.1 Classification Theorem. \( \square \)

Definition 6.1.2. An \( d \)-tuple of elements \( t_1, \ldots, t_d \in \mathcal{K} \) is called a system of local parameters of \( \mathcal{K} \), if \( t_d \) is a prime in \( \mathcal{K}_d \), \( t_{d-1} \) is a unit in \( \mathcal{O}_\mathcal{K} \) but its residue in \( \mathcal{K}_{d-1} \) is a prime element of \( \mathcal{K}_{d-1} \), and so on.

For the standard field \( E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \) we can take as a system of local parameters \( t_d = \pi_E, t_{d-1} = T_{d-1}, \ldots, t_1 = T_1 \).

Definition 6.1.3. We define a discrete valuation of rank \( d \) to be the map \( v = (v_1, \ldots, v_d) : \mathcal{K}^* \to \mathbb{Z}^d, v_d = v_{\mathcal{K}_d^*}; v_{d-1}(x) = v_{\mathcal{K}_{d-1}}(x_{d-1}) \) where \( x_{d-1} \) is the residue in \( \mathcal{K}_{d-1} \) of \( xt_{d-1}v_n(x) \), and so on.

Although the valuation depends, for \( n > 1 \), on the choice of \( t_2, \ldots, t_d \), it is independent in the class of equivalent valuations.

6.1.1. Topology on \( \mathcal{K} \). We define the topology on \( E\{\{T_1\}\} \ldots \{\{T_{d-1}\}\} \) by induction on \( d \). For \( d = 1 \) we define the topology to be the topology of a one-dimensional local field. Suppose we have defined the topology on a standard \( d \)-dimensional local field \( E_d \) and let \( \mathcal{K} = E_d\{\{T\}\} \). Denote by \( P_{E_d}(c) \) the set \( \{x \in E_d : v_{E_d}(x) \geq c\} \). Let \( \{V_i\}_{i \in \mathbb{Z}} \) be a sequence of neighborhoods of zero in \( E_d \) such that

\[
\begin{align*}
1. & \text{ there is a } c \in \mathbb{Z} \text{ such that } P_{E_d}(c) \subset V_i \text{ for all } i \in \mathbb{Z}. \\
2. & \text{ for every } l \in \mathbb{Z} \text{ we have } P_{E_d}(l) \subset V_i \text{ for all sufficiently large } i.
\end{align*}
\]

and put \( V_{|V_i|} = \{ \sum b_i T^i : b_i \in V_i \} \). These sets form a basis of neighborhoods of 0 for a topology on \( \mathcal{K} \). For an arbitrary \( d \)-dimensional local field \( L \) of mixed characteristic we can find, by the Classification Theorem, a standard field that is a finite extension of \( L \) and we can give \( L \) the topology induced by the standard field.
Proposition 6.1.1. Let $L$ be a $d$-dimensional local field of mixed characteristic with the topology defined above.

1. $L$ is complete with this topology. Addition is a continuous operation and multiplication by a fixed $a \in L$ is a continuous map.
2. Multiplication is a sequentially continuous map, i.e., if $x, y_k \in L$ and $y_k \to y$ in $L$ then $xy_k \to xy$.
3. This topology is independent of the choice of the standard field above $L$.
4. If $K$ is a standard field and $L/K$ is finite, then the topology above coincides with the natural vector space topology as a vector space over $K$.
5. The reduction map $O_L \to k_L = L^{d-1}$ is continuous and open (where $O_L$ is given the subspace topology from $L$, and $k_L = L^{d-1}$ the $(d-1)$-dimensional topology).

Proof. All the proofs can be found in [19] Theorem 4.10.

6.1.2. Topology on $K^*$. Let $R \subset K = K_d$ be a set of representatives of the last residue field $K_0$. Let $t_1, \ldots, t_d$ be a fixed system of local parameters for $K$, i.e., $t_d$ is a uniformizer for $K$, $t_d^{-1}$ is a unit in $O_K$ but its residue in $K_{d-1}$ is a uniformizer element of $K_{d-1}$, and so on. Then

$$K^* = V_K \times \langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times R^*,$$

where the group of principal units $V_K = 1 + M_K$ and $R^* = R - \{0\}$. From this observation we have the following,

Proposition 6.1.2. We can endow $K^*$ with the product of the induced topology from $K$ on the group $V_K^*$ and the discrete topology on $\langle t_1 \rangle \times \cdots \times \langle t_d \rangle \times R^*$. In this topology we have,

1. Multiplication is sequentially continuous, i.e., if $a_n \to a$ and $b_n \to b$ then $a_nb_n \to ab$.
2. Every Cauchy sequence with respect to this topology converges in $K^*$.

Proof. See [7] Chapter 1 §1.4.2.

6.2. Formal groups. In this section we will state some useful results in the theory of formal groups.

6.2.1. The Weierstrass lemma. Let $E$ be a discrete valuation field of zero characteristic with integer ring $O_E$ and maximal ideal $\mu_E$.

Lemma 6.2.1 (Weierstrass lemma). Let $g = a_0 + a_1 X + \cdots \in O_E[[X]]$ be such that $a_0, \ldots, a_{n-1} \in \mu_E$, $n \geq 1$, and $a_n \notin \mu_E$. Then there exist a unique monic polynomial $c_0 + \cdots + X^n$ with coefficients in $\mu_E$ and a series $b_0 + b_1 X \cdots$ with coefficients in $O_E$ and $b_0$ a unit, i.e., $b_0 \neq \mu_E$, such that

$$g = (c_0 + \cdots + X^n)(b_0 + b_1 X \cdots).$$

Proof. See [17] IV. §9 Theorem 9.2.
6.2.2. The group $F(\mu_M)$. Let $K$ be a $d$-dimensional local field containing the local field $K$, say, $K = K\{\{T_i\}\} \cdots \{\{T_d\}\}$. Denote by $F(\mu_K)$ the group with underlying set $\mu_K$ and operation defined by the formal group $F$. More generally, if $M$ is an algebraic extension of $K$ we define

$$F(\mu_M) := \bigcup_{\mathcal{M} \supset \mathcal{L} \supset K, \mathcal{L} / \mathcal{K} < \infty} F(\mu_\mathcal{L}).$$

An element $f \in \text{End}(F)$ is said to be an isogeny if the map $f : F(\mu_K) \to F(\mu_K)$ induced by it is surjective with finite kernel.

If the reduction of $f$ in $k_K[[X]]$, $k_K$ the residue field of $K$, is not zero then it is of the form $f_1(X^p^c)$ with $f_1(0) \in \mathcal{O}_K^*$, cf. [14] Proposition 1.1. In this case we say that $f$ has finite height. If on the other hand the reduction of $f$ is zero we say it has infinite height.

**Proposition 6.2.1.** $f$ is an isogeny if and only if $f$ has finite height. Moreover, in this situation $|\ker f| = p^h$.

**Proof.** If the height is infinite, the coefficients of $f$ are divisible by a uniformizer of the local field $K$, so $f$ cannot be surjective. Let $h < \infty$ and $x \in \mu_\mathcal{L}$ where $\mathcal{L}$ is a finite extension of $K$. Consider the series $f - x$ and apply Lemma 6.2.1 with $E = \mathcal{L}$, i.e.,

$$f - x = (c_0 + \cdots + X^{p^c^0})(b_0 + b_1X + \cdots),$$

where $c_i \in \mu_\mathcal{L}$, $b_i \in \mathcal{O}_\mathcal{L}$ and $b_0 \in \mathcal{O}_\mathcal{L}^*$. Therefore the equation $f(X) = x$ is equivalent to the equation $c_0 + \cdots + X^{p^c} = 0$ and since the $c_i \in \mu_\mathcal{L}$ every root belongs to $\mu_\mathcal{L}$.

Moreover, the polynomial $P(X) = c_0 + \cdots + X^{p^c}$ is separable because $f'(X) = \pi t(X)$, $t(X) = 1 + \cdots$ is an invertible series and $f' = P'(b_0 + b_1X + \cdots) + P(b_0 + b_1X + \cdots)$ so $P'$ can not vanish at a zero of $P$. We conclude that $P$ has $p^h$ roots, i.e., $|\ker f| = p^h$.

**Proposition 6.2.2.** Denote by $j$ the degree of inertia of $S/\mathbb{Q}_p$ and by $h_1$ the height of $f = [\pi]_F$. Then $j$ divides $h_1$, namely $h_1 = jh$. Let $\kappa_n$ be the kernel of $f^{(n)}$. Then

$$\kappa_n \simeq (C/\pi^n C)^h \quad \text{and} \quad \lim \kappa_n \simeq C^h,$$

as $C$-modules. This $h$ is called the height of the formal group with respect to $C = \mathcal{O}_S$.

**Proof.** cf. [14] Proposition 2.3.

**Remark 6.2.1.** Notice that since the coefficients of $F$ are in the local field $K$ then $\kappa_n \subset K$ for all $n \geq 1$.

6.2.3. The logarithm of the formal group. We define the logarithm of the formal group $F$ to be the series

$$l_F = \int_0^X \frac{dX}{F_X(0, X)}$$

Observe that $F_X(0, X) = 1 + \cdots \in \mathcal{O}_K[[X]]^*$ then $l_F$ has the form...
where $a_i \in \mathcal{O}_K$.

**Proposition 6.2.3.** Let $E$ be a field of characteristic 0 that is complete with respect to a discrete valuation, $\mathcal{O}_E$ the valuation ring of $E$ with maximal ideal $\mu_E$ and valuation $v_E$. Consider a formal group $F$ over $\mathcal{O}_E$, then

1. The formal logarithm induces a homomorphism
   
   $$l_F : F(\mu_E) \to E$$

   with the additive group law on $E$.

2. The formal logarithm induces the isomorphism
   
   $$l_F : F(\mu_E) \cong \mu_E$$

   for all $r \geq [v_E(p)/(p-1)] + 1$ and
   
   $$v_E(l(x)) = v(x) \quad (\forall x \in \mu_E).$$

   In particular, this holds for $\mu_{E,1} = \{ x \in E : v_E(x) > v_E(p)/(p-1) + 1 \}$. 

**Proof.** [22] IV Theorem 6.4 and Lemma 6.3.

**Lemma 6.2.2.** Let $E$ and $v_E$ as in the previous proposition. Then

$$v_E(n!) \leq \frac{(n-1)v_E(p)}{p-1},$$

and $v_E(x^n/n!) \to \infty$ as $n \to \infty$ for $x \in \mu_{E,1}$.

**Proof.** The first assertion can be found in [22] IV. Lemma 6.2. For the second one notice that

$$v_E(x^n/n!) \geq nv_E(x) - v(n!) \geq nv_E(x) - (n-1)\frac{v_E(p)}{p-1}$$

$$= v_E(x) + (n-1)\left(v_E(x) - \frac{v_E(p)}{p-1}\right).$$

Since we are assuming that $x \in \mu_{E,1}$, i.e., $v_E(x) > v_E(p)/(p-1)$, then $v_E(x^n/n!) \to \infty$ as $n \to \infty$. 

**Lemma 6.2.3.** Let $L$ be a $d$-dimensional local field containing the local field $K$, $g(X) = a_1X + \frac{a_2}{2}X^2 + \cdots + \frac{a_n}{n}X^n + \cdots$ and $h(X) = a_1X + \frac{a_2}{2}X^2 + \cdots + \frac{a_n}{n}X^n + \cdots$ with $a_i \in \mathcal{O}_K$. Then $g$ and $h$ define, respectively, maps $g : \mu_L \to \mu_L$ and $h : \mu_{L,1} \to \mu_{L,1}$ that are sequentially continuous in the Parshin topology.

**Proof.** We may assume $L$ is a standard $d$-dimensional local field. Let $\mathcal{V}_{\{V_i\}}$ be a basic neighborhood of zero that we can consider to be a subgroup of $L$, and let $c > 0$ such that $P_L(c) \subset \mathcal{V}_{\{V_i\}}$. If $x_n \in \mu_L$ for all $n$, then there exists an $N_1 > 0$ such that $v_L(x_n/i)$, $v_L(x'/i) > c$ for all $i > N_1$ and all $n$; because $iv_L(x_n) - v(i) \geq iv_L(x_n) - \log_p(i) \geq i - \log_p(i) \to \infty$ as $i \to \infty$. On the other hand, if $x_n \in \mu_{L,1}$ for all $n$, then there exists an $N_2 > 0$ such that $v_L(x_n/i!)$, $v_L(x'/i!)$, $v_L(x'/i!)$ for all $i > N_2$ and all $n$ by Lemma 6.2.2. Then, for $N = \max\{N_1, N_2\}$, we have

$$\sum_{i=N+1}^{\infty} a_i x_n^i - x^i \in \mathcal{V}_{\{V_i\}}.$$

$$\sum_{i=N+1}^{\infty} a_i x_n^i - x^i \in P_L(c) \subset \mathcal{V}_{\{V_i\}}.$$
Now, since multiplication is sequentially continuous and \( x_n \rightarrow x \) then

\[
\sum_{i=1}^{N} a_i \frac{x_i^n - x^n}{i} \rightarrow 0, \quad \sum_{i=1}^{N} a_i \frac{x_i^n - x^n}{i!} \rightarrow 0 \quad \text{as } n \rightarrow \infty
\]

Thus for \( n \) large enough we have that

\[
g(x_n) - g(x), \quad h(x_n) - h(x) = \sum_{i=1}^{N} + \sum_{i=N+1}^{\infty} x_i \in \mathcal{V}_{\{V_i\}}.
\]

\[\square\]

**Remark 6.2.2.** In particular, \( \log : \mu_\mathcal{L} \rightarrow \mu_\mathcal{L}, \ l_F : \mu_\mathcal{L} \rightarrow \mu_\mathcal{L} \) and \( \exp_F = l_F^{-1} : \mu_{\mathcal{L},1} \rightarrow \mu_{\mathcal{L},1} \) are sequentially continuous.

### 6.3. Proofs of propositions and lemmas in section 2.

**Proof of Proposition 2.1.1.** To simplify the notation we will assume \( m = 2 \).

1. Noticing that \((1 - a)/(1 - 1/a) = -a\) it follows that
   \[
   \{a, -a\} = \{a, 1 - a\} \{a, 1 - 1/a\}^{-1} = \{a, 1 - 1/a\}^{-1} = \{1/a, 1 - 1/a\} = 1
   \]

2. This follows immediately from the previous item
   \[
   \{a, b\} \{b, a\} = \{-b, b\} \{b, a\} = \{-a, a\} = \{-ab, b\} \{-ab, a\} = \{-ab, ab\} = 1
   \]

\[\square\]

**Proof of Theorem 2.1.1 (2).** Let \( \mathcal{M} \) be a finite abelian extension of \( \mathcal{L} \), thus \( \text{Gal}(\mathcal{L}^a/\mathcal{M}) \) is an open neighborhood of \( G^a \). Let \( x_n \) be a convergent sequence to the zero element of \( K_d(\mathcal{L}) \). Since \( N_{\mathcal{M}/\mathcal{L}}(K^d_\text{top}(\mathcal{M})) \) is a open subgroup of \( K^d_\text{top}(\mathcal{L}) \) by Proposition 2.1.3, then

\[
\overline{x_n} \in N_{\mathcal{M}/\mathcal{L}}(K^d_\text{top}(\mathcal{M})) \quad (n >> 0),
\]

where \( \overline{x_n} \) is the image of \( x_n \) in \( K^d_\text{top}(\mathcal{L}) \). Thus, there exist \( y_n \in K_d(\mathcal{M}) \) and \( \beta_n \in \Lambda_m(\mathcal{L}) \) such that

\[
x_n = \beta_n N_{\mathcal{M}/\mathcal{L}}(y_n) \quad (n >> 0).
\]

From equation (11) we have that \( \beta_n \in \cap_{i \geq 1} K_d(\mathcal{L}) \) which implies that \( \Upsilon_\mathcal{L}(\beta_n) \) is the identity element in \( G^a_\mathcal{L} \) (because \( G^a_\mathcal{L} \) is a profinite group). Therefore

\[
\Upsilon_\mathcal{L}(x_n) = \Upsilon_\mathcal{L}(N_{\mathcal{M}/\mathcal{L}}(y_n)) \quad (n >> 0),
\]

but the element on the right hand side of this equality is the identity on \( \text{Gal}(\mathcal{L}^a/\mathcal{M}) \) by the second item of this Theorem. It follows that \( \Upsilon_\mathcal{L}(x_n) \) converges to the identity element of \( G^a_\mathcal{L} \).

\[\square\]

**Proof of Proposition 2.2.1.** The first 5 properties follow from the definition of the pairing and Theorem 2.1.1.

Let us prove property 7. Let \( f^{(a)}(z) = y \) an take a finite Galois extension \( N \supset \mathcal{M}(z) \) over \( \mathcal{L} \). Let \( G = G(N/\mathcal{L}) \) and \( H = G(N/\mathcal{M}), w = [G : H], \) and \( V : G/G' \rightarrow H/H' \) the transfer homomorphism. Let \( g = \Upsilon_\mathcal{L}(a) \), then by Theorem 2.1.1 we have \( V(\Upsilon_\mathcal{L}(a)) = \Upsilon_\mathcal{M}(a) \). The explicit computation of \( V \) at \( g \in G \) proceeds as follows (cf. [20] § 3.5). Let \( \{c_i\} \) be a set of representatives of the right cosets of \( H \) in \( G \), i.e., \( G = \sqcup Hc_i \). Then for each \( c_i, i = 1, \ldots w \) there exist a \( c_j \) such that
\(c_j g c_j^{-1} = h_i \in H\) and no two \(c_j\)'s are equal; this is because \(c_j g\) belongs to one and only one of the right cosets \(Hc_j\). Then

\[V(g) = \prod_{i=1}^{w} h_i.\]

Also, notice that since \(g c_j^{-1} = c_j^{-1} h_i\)

\[h_i(z) \otimes z = c_j^{-1} (h_i(z) \otimes z) = g(c_j^{-1}(z)) \otimes c_j^{-1}(z).\]

So we have

\[(a, y)_{\mathcal{M}, n} = \sum_{\mathcal{M}} (a(z) \otimes z = V(g)(z) \otimes z = (\prod_{i=1}^{w} h_i) z \otimes z = \sum_{i=1}^{w} (h_i(z) \otimes z)\]

\[= \sum_{i=1}^{w} (g(c_j^{-1}(z)) \otimes c_j^{-1}(z)) = g(\sum_{i=1}^{w} c_j^{-1}(z)) \otimes (\sum_{i=1}^{w} c_j^{-1}(z)) = (a, N_Y^{\mathcal{M}/\mathcal{L}}(y))_{\mathcal{L}, n}\]

the last equality being true since \(g = Y_{\mathcal{L}}(a)\) and

\[f^{(n)}(\sum_{i=1}^{w} c_j^{-1}(z)) = \sum_{i=1}^{w} c_j^{-1}(y) = N_Y^{\mathcal{M}/\mathcal{L}}(y).\]

\[\Box\]

**Proof of Proposition 2.4.1.** First, the coefficients of \(s\) are in \(\mathcal{O}_K\) because \(s^\sigma = s\) for every \(\sigma \in G_K = \text{Gal}(\overline{K}/K)\). Now, applying Lemma 6.2.1 to \(s\) and \(f^{(n)}\), we get \(s = P s_1\) and \(f^{(n)} = Q f_1\), where \(P\) and \(Q\) is a monic polynomials and \(s_1, f_1 \in \mathcal{O}_K[[X]]^*\). Since \(s(F(X, v)) = s(X)\) for all \(v \in \kappa_n\), then \(P(v) = 0\) for all \(v \in \kappa_n\) and so \(Q = \prod_{v \in \kappa_n} (X - v)\) divides \(P\). This implies that \(s\) is divisible by \(f^{(n)}\), i.e., \(s = f^{(n)}(a_0 + a_1 X + \cdots)\). In particular,

\[s - f^{(n)}(0) = f^{(n)}(a_1 X + \cdots).\]

But from \(s(F(X, v)) = s(X)\) we see that \(a_1 X + \cdots\) must satisfy the same property and so \(a_1 v + \cdots = 0\) for all \(v \in \kappa_n\). Therefore this series is also divisible by \(f^{(n)}\) and repeating the process we get \(s = r_g(f^{(n)})\). Let us compute now \(c(r_g)\). Taking the logarithmic derivative on \(s\) and then multiplying by \(X\) we get

\[s'(X) s(X) = \sum_{v \in \kappa_n} \frac{g(F(X, v)) F_X(X, v) X}{g(F(X, v))},\]

which implies

\[\frac{s'(0)}{\prod_{0 \neq v \in \kappa_n} g(v)} = g'(0),\]

From \(s' = r_g'(f^{(n)}f^{(n)})'\) we obtain

\[r_g'(0) = \frac{s'(0)}{f^{(n)}(0)} = \frac{c(g) \prod_{0 \neq v \in \kappa_n} g(v)}{\pi^n}.\]

Each \(g(v)\) is associated to \(v\), \(0 \neq v \in \kappa_n\), then \(\prod_{0 \neq v \in \kappa_n} g(v)\) is associated to \(\prod_{0 \neq v \in \kappa_n} v\), but the latter is associated to \(\pi^n\) from the equation \(f = P f_1\). Then \(c(r_g) \in \mathcal{O}_K\). Finally, we will show that \((\{a_1, \ldots, a_{i-1}, r_g(x), a_i, \ldots, a_{d-1}\}, x) = 0\). Let \(L\) be a local field containing \(\kappa_n\), \(L = L \{T_1\} \cdots \{T_{d-1}\}, x \in F(\mu_L)\) and \(z\) such that \(f^{(n)}(z) = x\). Then

\[r_g(x) = \prod_{v \in \kappa_n} g(z \otimes F v) = \prod_{i} N_{\mathcal{L}/\mathcal{L}}(z_i),\]
where the $z_i$ are pairwise non-conjugate over $L$ distinct roots of $f^{(n)}(X) = x$, so

$$\{a_1, \ldots, a_{i-1}, r_g(x), a_{i+1}, \ldots, a_d\} = \{a_1, \ldots, a_{i-1}, N_{L(z)/L}(\prod_i g(z_i)), a_{i+1}, \ldots, a_d\}$$

$$= N_{L(z)/L}(\{a_1, \ldots, a_{i-1}, \prod_i g(z_i), a_{i+1}, \ldots, a_d\}).$$

The last equality follows from Proposition 2.1.2 (1) and (4). The result now follows from Proposition 2.2.1. \qed

6.4. Proofs of Section 3.

Proof of Proposition 3.1.1. Assume first that $L$ is the standard higher local field $L\{\{T_1\} \cdots \{T_{d-1}\}\}$. The proof is done by induction in $d$. If $d = 1$ the result is known. Suppose the result is true for $d \geq 1$ and let $L = E\{\{T_d\}\}$ where $E = L\{\{T_1\} \cdots \{T_{d-1}\}\}$.

Let $\phi : L \to S$ be a sequentially continuous $C$-linear map and define, for each $i \in \mathbb{Z}$, the sequentially continuous map $\phi_i(x) = \phi(xT_d^i)$ for all $x \in E$. Then clearly $\phi_i \in \text{Hom}_C(E, S)$ and by the induction hypothesis we know that there exists an $a_{-i} \in E$ such that $\phi(xT_d^i) = T_{E/S}(a_{-i}x)$ for all $x \in E$. Let $\alpha = \sum a_i T_d^i$, we must show that

I. $\min \{v_E(a_i)\} > -\infty$.
II. $v_E(a_{-i}) \to \infty$ as $i \to \infty$ (i.e., conditions (I) and (II) imply that $\alpha \in L$).
III. $\phi(x) = T_{E/S}(\alpha x)$, $\forall x \in L$.

For any $x = \sum x_i T^i \in L$ we have, by the sequential continuity of $\phi$ that

$$\phi(x) = \sum_{i \in \mathbb{Z}} \phi(x_i T_d^i) = \sum_{i \in \mathbb{Z}} T_{E/S}(a_{-i} x_i).$$

(61)

Suppose (I) was not true, then there exist a subsequence $\{a_{n_k}\}$ such that $v_E(a_{n_k}) \to -\infty$ as $n_k \to \infty$ or as $n_k \to -\infty$. In the first case we take an $x = \sum x_i T_d^i \in L$ such that $x_i$ is equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i} x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$. Then the sum on the right of (61) would not converge. In the second case we take $x_i$ to be equal to $1/a_{n_k}$ if $i = -n_k$ and 0 if $i \neq -n_k$. So $a_{-i} x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and again the sum on the right would not converge.

Suppose (II) was not true. Then $v_L(a_{n_k}) < M$ for some positive integer $M$ and a of negative integers $n_k \to -\infty$. Then take $x = \sum x_i T_d^i \in L$ such that $x_i$ is equal to $1/a_{n_k}$ for $i = -n_k$ and 0 for $i \neq -n_k$. So $a_{-i} x_i = 1$ if $i = -n_k$ and 0 if $i \neq -n_k$ and the sum on the right of (61) would not converge.

Finally, (III) follows by noticing that by (I) and (II) the sum $\sum_{i \in \mathbb{Z}} a_{-i} x_i$ converges and

$$\sum_{i \in \mathbb{Z}} \text{Tr}_{L/S}(a_{-i} x_i) = \text{Tr}_{L/S}(\sum_{i \in \mathbb{Z}} a_{-i} x_i) = T_{E/S}(\alpha).$$

Assume now that $L$ is an arbitrary $d$-dimensional local field. Then let $L_{(0)}$ be the standard local field from Section 1.3. Since $L/L_{(0)}$ is a finite extension, then $\text{Tr}_{L/L_{(0)}}$ induces a pairing $L \times L \to L_{(0)}$. From this and the first part of the proof the result now follows. \qed

Proof of equation (19). By induction on $d$. For $d = 1$ this is proven in [14] §4.1. Suppose it is true for $d \geq 1$, and let $L = E_d\{\{T_d\}\}$, where $E_d = L\{\{T_1\} \cdots \{T_{d-1}\}\}$. \qed
If $x = \sum_{i \in \mathbb{Z}} x_i T_i \in R_{\mathcal{L}, 1}$, then $\mu_{E_d, 1} \subset \mu_{\mathcal{L}, 1}$ we have that also $\mu_{E_d, 1} T_d^{-1} \subset \mu_{\mathcal{L}, 1}$
\[ T_{\mathcal{L}/S}(x \mu_{E_d, 1} T_d^{-1}) \subset C, \]
which implies $T_{E_d/S}(x \mu_{E_d, 1}) \subset C$, since $T_{\mathcal{L}/S} = T_{E_d/S} \circ c_{\mathcal{L}/E_d}$. By induction hypothesis we have $v_{E_d}(x_i) \geq -v_L(D(L/S)) - |v_L(p)/(p-1)| - 1$ for all $i \in \mathbb{Z}$, therefore
\[ v_{\mathcal{L}}(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - |v_L(p)/(p-1)| - 1. \]
Conversely, if $v_{\mathcal{L}}(x) = \min v_{E_d}(x_i) \geq -v_L(D(L/S)) - |v_L(p)/(p-1)| - 1$, then $v_{E_d}(x_i) \geq -v_L(D(L/S)) - |v_L(p)/(p-1)| - 1$ for all $i \in \mathbb{Z}$. Then, by the induction hypothesis $T_{E_d/S}(x \mu_{E_d, 1}) \subset C$ for all $i \in \mathbb{Z}$, and therefore
\[ T_{\mathcal{L}/S}(x \mu_{\mathcal{L}, 1}) = \sum_{i \in \mathbb{Z}} T_{\mathcal{L}/S}(x_i T_i) \mu_{\mathcal{L}, 1} = \sum_{i \in \mathbb{Z}} T_{E_d/S}(x_i \mu_{E_d, 1}) \subset C. \]
Thus, identity (19) holds for standard local fields $L\{\{T_1\} \cdots \{T_{d-1}\}\}$. In the general case of an arbitrary $d$-dimensional local field $\mathcal{L}$, it is enough to consider the finite extension $\mathcal{L}(0)$ from Section 1.3.

Proof of Proposition 3.3.2. (1) From the identity $T_{\mathcal{M}/S} = T_{\mathcal{L}/S} \circ \text{Tr}_{\mathcal{M}/\mathcal{L}}$ and the fact that $\mu_{\mathcal{L}, 1} \subset \mu_{\mathcal{M}, 1}$ we obtain
\[ T_{\mathcal{L}/S}(\text{Tr}_{\mathcal{M}/S}(R_{\mathcal{M}, 1} \mu_{\mathcal{L}, 1})) = T_{\mathcal{L}/S}(\text{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M}, 1} \mu_{\mathcal{L}, 1})) \subset T_{\mathcal{M}/S}(R_{\mathcal{M}, 1} \mu_{\mathcal{L}, 1}) \subset C, \]
from which follows that $\text{Tr}_{\mathcal{M}/\mathcal{L}}(R_{\mathcal{M}, 1}) \subset R_{\mathcal{L}, 1}$. Now by Proposition 2.2.1 (4) we have, for $b \in K_d(\mathcal{M})$ and $x \in f(\mu_{\mathcal{L}, 1})$, that
\[ (N_{\mathcal{M}/\mathcal{L}}(b), x)_{\mathcal{L}, n} = (b, x)_{\mathcal{M}, n} = T_{\mathcal{M}/S}(\psi_{\mathcal{M}, m}(b))l(x) = T_{\mathcal{L}/S}(\psi_{\mathcal{L}, m}(b))l(x). \]
It follows from Proposition 3.3.1 that
\[ \psi_{\mathcal{L}, m}(N_{\mathcal{M}/\mathcal{L}}(b)) = \text{Tr}_{\mathcal{M}/\mathcal{L}}(\psi_{\mathcal{M}, m}(b)). \]
(2) This is proved in a similar fashion to the previous property but this time using Proposition 2.2.1 (7).
(3) This follows from Proposition 3.3.1 and Proposition 2.2.1 (5). Indeed, since $\epsilon_n^i = f(m-n)(\epsilon_n^i)$ and $a_n = (a, f(m-n)(x))_m$ we get
\[ \pi^{m-n}(a, x)^i_n = (a, f(m-n)(x))^i_m \pmod{\pi^m C} \]
That is
\[ \pi^{m-n} T_{\mathcal{L}/S}(\psi_{\mathcal{L}, n}(a)) l_F(x) = T_{\mathcal{L}/S}(\psi_{\mathcal{L}, m}(a)) l_F(f(m-n)(x)) \]
\[ = \pi^{m-n} T_{\mathcal{L}/S}(\psi_{\mathcal{L}, m}(a)) l_F(x) \pmod{\pi^m C} \]
Upon dividing by $\pi^{m-n}$ the result follows.
(4) This property follows from Proposition 2.2.1 (8) and $l_F(0) = t(0)l_F$.  

Proof of Lemma 3.5.1. For a $d$-dimensional local field $\mathcal{L}$, let $\mathcal{L}(0)$ be as in Section 1.3. Since $\mathcal{L}$ and $\mathcal{L}(0)$ have the same residue field, it is enough to prove the result for a standard higher local field. Thus we assume $\mathcal{L}$ is standard and proceed by induction on $d$. For $d = 1$, the result follows since $k_L$ is perfect. Suppose it is proved
for \( R = \mathcal{O}_{\mathcal{L}_d} \), where \( \mathcal{L}_d = \mathcal{L}_d \{ \{ T_1 \} \} \ldots \{ \{ T_{d-1} \} \} \). Let \( \mathcal{L} = \mathcal{L}_d \{ \{ T_d \} \} \) and \( x \in \mathcal{O}_{\mathcal{L}} \). Then \( x \equiv \sum_{j \geq m} a_j T_d^j \pmod{\pi_\mathcal{L}} \), \( a_j \in R \). Thus, by the induction hypotheses

\[
x \equiv \sum_{0 \leq u < p^n} T_u^2 \left( \sum_{m \leq i \leq k p^n} a_{i+k p^n} D_{i+k p^n} \right) \pmod{\pi_\mathcal{L}}
\]

\[
\equiv \sum_{0 \leq u < p^n} T_u^2 \left( \sum_{m \leq i \leq k p^n} \sum_{0 \leq i_1, \ldots, i_{d-1} < p^n} \gamma_{i_1, \ldots, i_{d-1}, i} T_{i_1} \cdots T_{i_{d-1}} \right) T_{d+1} \pmod{\pi_\mathcal{L}}
\]

\[
\equiv \sum_{0 \leq u < p^n} \sum_{0 \leq i_1, \ldots, i_{d-1} < p^n} \gamma_{i_1, \ldots, i_{d-1}, i} T_{i_1} \cdots T_{i_{d-1}} \pmod{\pi_\mathcal{L}}
\]

where \( \gamma_{i_1, \ldots, i_{d-1}} = \sum_k \gamma_{i_1, \ldots, i_{d-1}, i_1, \ldots, j_k} T_{d+1}^{i_1} \) and regrouping terms is valid since the series are absolutely convergent in the Parshin topology. Also by noticing that the congruence

\[
\sum_{c \leq k} b_k \equiv \left( \sum_{c \leq k} b_k \right) \pmod{\pi_\mathcal{L}},
\]

holds in \( k_{\mathcal{L}_{d-1}} (\{ T_d \}) \), where \( k_{\mathcal{L}_{d-1}} \) is the residue field of \( \mathcal{L}_{d-1} \).

**Proof of Corollary 3.5.2.** This follows from Proposition 3.5.4 and Proposition 3.5.1 (3). Indeed, let us illustrate the proof in the case \( d = 2 \), i.e., \( \mathcal{L} \) is a 2-dimensional local field with a systems of local uniformizers \( T_1 \) and \( T_2 = \pi_\mathcal{L} \). To simplify the notation we will denote \( D_{\mathcal{L}_n} \) by \( D \). From Proposition 3.5.4 we have

\[
D(\eta_1 (T_1, T_2), \eta_2 (T_1, T_2)) = \left. \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_2} \right|_{X_i = T_i, i = 1, 2} D(T_1, T_1) + \left. \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_2} \right|_{X_i = T_i, i = 1, 2} D(T_2, T_2)
\]

But \( D(T_1, T_1) = D(T_2, T_2) = 0 \), \( D(T_2, T_1) = -D(T_1, T_2) \) from Proposition 3.5.1 (3), therefore

\[
D(\eta_1 (T_1, T_2), \eta_2 (T_1, T_2)) = \left. \left( \frac{\partial \eta_1}{\partial X_1} \frac{\partial \eta_2}{\partial X_2} - \frac{\partial \eta_1}{\partial X_2} \frac{\partial \eta_2}{\partial X_1} \right) \right|_{X_i = T_i, i = 1, 2} D(T_2, T_1).
\]

The corollary follows.

**6.5. Proofs of Section 4.**

**Proof of Proposition 4.1.1.** To simplify the notation let us denote \( \hat{\Omega}_{\mathcal{O}_{\mathcal{L}}} (\mathcal{O}_{\mathcal{L}}) \) by \( \hat{\Omega} \), where \( \Omega = \Omega_{\mathcal{O}_{\mathcal{L}}} (\mathcal{O}_{\mathcal{L}}) \). We will start by showing that \( \Omega / \pi_n \Omega \) is generated by \( d \pi_\mathcal{L} \) and \( dT_1, \ldots, dT_{d-1} \) for all \( n \).

Let \( x \in \mathcal{O}_{\mathcal{L}} \), then by corollary 3.5.2, we have that

\[
x = \sum_{k=0}^{\infty} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} < p^n} \gamma_{i_1, \ldots, i_{d-1}, i} T_{i_1} \cdots T_{i_{d-1}} \right) \pi_k.
\]
Therefore, in $\Omega/p^n\Omega$, we can consider the truncated sum
\[
\sum_{k=0}^{m} \left( \sum_{0 \leq i_1, \ldots, i_{d-1} < p^n} \gamma^{p^n}_{i_k} T_1^{i_1} \cdots T_{d-1}^{i_{d-1}} \pi^k_L \right),
\]
where $m$ is such that $p^n | \pi^{m+1}_L$. Thus, $dx$ is generated by $d\pi_L$ and $dT_i$, $i = 1, \ldots, d-1$ in $\Omega/p^n\Omega$.

We will assume the notation of Section 4.1.1 and let $T_d = \pi_L$. Let $b_i = \frac{\partial g_i}{\partial T_i}(\pi_L)$, $i = 1, \ldots, d$, and let $O$ be the ideal of $\Omega_L$ such that
\[
(64) \quad \omega(\Omega) = \min_{1 \leq i \leq d} \{\omega(\pi_L)\}.
\]
Without loss of generality we may assume that $\omega(\pi_L) = \min_{1 \leq i \leq d} \{\omega(\pi_L)\}$, and then we define
\[
w = b_1 dT_1 + \cdots + b_{d-1} dT_{d-1} + dT_d \in \Omega.
\]
It is clear that $\partial w = 0$ and also that $dT_1, \ldots, dT_{d-1}, w$ generate $\Omega/p^n\Omega$ for all $n$.

We will show now that
\[
\frac{\Omega}{\pi^n\Omega} \simeq \frac{\Omega_L}{p^n\pi^n\Omega_L} \oplus \cdots \oplus \frac{\Omega_L}{p^n\pi^n\Omega_L + \mathfrak{D}O_L}
\]
for all $n \geq 1$. These isomorphisms are compatible: $\simeq_{n+1} \equiv \simeq_n \ (\text{mod } p^n)$, then we can take the projective limit $\lim_{\rightarrow}$ to obtain the result. This will imply in particular that $\mathfrak{D}$ is the annihilator ideal of the torsion part of $\Omega_{O_K}(\Omega_L)$, i.e., $\mathfrak{D}(\Omega_L/K) = \mathfrak{D}$.

In order to construct the isomorphism (65) we consider the derivations $D_k : \Omega_L \to \Omega_L$ for $k = 1, \ldots, d-1$ and $D_d : \Omega_L \to \Omega_L/\mathfrak{D}$ as follows
\[
D_i(g(\pi_L)) = \frac{\partial g}{\partial T_i}(\pi_L) - \frac{b_i}{b_d} \frac{\partial g}{\partial T_d}(\pi_L) \quad i = 1, \ldots, d-1
\]
and
\[
D_d(g(\pi_L)) = \frac{\partial g}{\partial T_d}(\pi_L)
\]
for $g(x) \in \Omega_L^{(0)}[X]$. It is clear from the very definition that these are well-defined derivations which are independent of the choice of $g(x)$. Define the map
\[
\partial : \Omega_L \longrightarrow \frac{\Omega_L}{p^n\pi^n\Omega_L} \oplus \frac{\Omega_L}{p^n\pi^n\Omega_L} \oplus \cdots \oplus \frac{\Omega_L}{p^n\pi^n\Omega_L + \mathfrak{D}O_L}
\]
by
\[
a \rightarrow (D_1(a), \ldots, D_d(a))
\]
where $D_k$ is the reduction of $D_k$. This is a well-defined derivation of $\Omega_L$ over $\Omega_K$ and by the universality of $\Omega$, this induces a homomorphism of $\Omega_L$-modules
\[
\partial : \frac{\Omega}{p^n\Omega} \longrightarrow \frac{\Omega_L}{p^n\pi^n\Omega_L} \oplus \cdots \oplus \frac{\Omega_L}{p^n\pi^n\Omega_L + \mathfrak{D}O_L}.
\]
Let us show that $\partial$ is an isomorphism. Indeed, it is clearly surjective since for $(a_0, \ldots, a_{d-1}) \in \Omega_L \oplus \cdots \oplus \Omega_L \oplus (\Omega_L/D(L/K)\Omega_L)$ we have that
\[
\partial(a_1dT_1 + \cdots + a_{d-1}dT_{d-1} + a_d w) = (a_1, \ldots, a_d).
\]
since
\[
\overline{D_k}(w) = \begin{cases} 1, & k = d, \\ 0, & 1 \leq k \leq d-1, \end{cases} \quad \overline{D_k}(dT_i) = \begin{cases} 0, & k \neq i, \\ 1, & k = i \neq d, \end{cases}
\]

Also, \( \partial \) is injective for if \( a = a_1 dT_1 + \cdots + a_{d-1} dT_{d-1} + a_d w \in \Omega/p^n \Omega \) is such that \( \partial(a) = 0 \), then \( D_k(a) = a_k = 0 \) in \( \mathcal{O}_L/p^n \mathcal{O}_L \), for \( 1 \leq k \leq d-1 \), and \( D_d(a) = a_d = 0 \) (mod \( p^n \mathcal{O}_L + \mathfrak{D} \mathcal{O}_L \)). But then \( a_d w = 0 \), since \( \mathfrak{D} w = 0 \), and therefore \( a = 0 \) mod \( p^n \Omega \). This concludes the proof.

Notice that if \( \mathcal{L} \) is the standard higher local field \( L\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \), then we take as a system of uniformizers \( T_1, \ldots, T_{d-1} \) and \( \pi_L \), and in this case \( b_i = 0 \) for \( i = 1, \ldots, d-1 \), form which the second claim in the statement of the proposition follows.

\[ \square \]

### 6.6. Proofs of Section 5.

**Proof of Remark 5.1.1.** Let \( k = m - n \). Then \( f^{(k)}r \) is divisible by \( \pi^k \), which implies that every term \( a_i X^i \) of the series \( f^{(k)} \) satisfies \( v(a_i) + v(i) \geq k/\rho \). If \( v(a_i) > 1/(\rho - 1) \) then there is nothing to prove. If on the other hand \( v(a_i) \leq 1/(\rho - 1) \) then \( v(i) \geq k/\rho - 1/(\rho - 1) \). In this case

\[
v(x^i) \geq \frac{i}{v(c_i(p))} \geq \frac{p^k}{v(c_i(p))} > \frac{1}{\rho - 1}
\]

for all \( x \in \mu_L \), since \( k/\rho - 1/(\rho - 1) > \log_p(v(c_i(p))/(\rho - 1)) \). Then

\[
v(f^{(k)}(x)) > \frac{1}{\rho - 1}
\]

for all \( x \in \mu_L \).

\[ \square \]

**Proof of Proposition 5.2.2.** We begin by taking a representative \( \lambda_{i,j} \) of \( c_{\beta;i,j} \) in \( R_{K_{\xi_1},1} \). We have to show that

\[
\lambda_{i,j} \in \beta_j R_{K_{\xi_1},1} + \left( \frac{\pi^k}{\pi_M} \right) R_{K_{\xi_1},1}.
\]

Let \( M \supset K_i, \pi_M \) and \( \pi_t \) uniformizers for \( M \) and \( K_i \), respectively, and \( \mathcal{M} = M\{\{T_1\}\} \cdots \{\{T_{d-1}\}\}, K_i = K_i\{\{T_1\}\} \cdots \{\{T_{d-1}\}\} \). Let \( \eta \in \mathcal{O}_M \) such that \( d \pi_t = \beta d \pi_M \); this exist by Proposition 4.1.3. Then \( D(M/K_i) = \beta \mathcal{O}_M \). Set \( \beta_j = \beta a_j \in \mathcal{O}_M \). Clearly, \( d e^j_i = \beta_j d \pi_M \). By Proposition 5.1.1,

\[
D_{\mathcal{M},k}^j : \mathcal{O}_M^d \to R_{M,1}/(\pi^k/\pi_M)R_{M,1}
\]

is a \( d \)-dimensional derivation over \( \mathcal{O}_R \), which together with Proposition 5.2.1 implies

\[
r'(e^j_i) \beta_j D_{\mathcal{M},k}(T_1, \ldots, T_{d-1}, \pi_M) = D_{\mathcal{M},k}^j(T_1, \ldots, T_{d-1}, r(e^j_i)) = -r'(e^j_i) T_1 \cdots T_{d-1} \sum_{i\in I} \frac{\varphi_{\beta;i,j}}{l'(e^j_i)} \pmod{(\pi^k/\pi_M)R_{M,1}}.
\]

Recall that \( \varphi_{\beta;i,j} \) is the image of \( c_{\beta;i,j} \) under the map \( R_{K_{\xi_1},1}/\pi^k R_{K_{\xi_1},1} \to R_{M,1}/\pi^k R_{M,1}; R_{K_{\xi_1},1} \subset R_{M,1} \). This identity implies

\[
\lambda_{i,j} \in \beta_j R_{M,1} + \left( \frac{\pi^k}{\pi_M} \right) R_{M,1}.
\]

(67)
Then
\[ v_M(\lambda_{i,j}) \geq \min \left\{ v_M(\beta_j R_{M,1}), v_M\left( \frac{\pi^k}{\pi_M} R_{M,1} \right) \right\} \]
\[ \geq \min \left\{ v_M(\beta_j) - v_M(D(M/S)) - \frac{e(M)}{p-1} - 1, \right. \]
\[ \left. v_M\left( \frac{\pi^k}{\pi_M} \right) - v_M(D(M/S)) - \frac{e(M)}{p-1} - 1 \right\} \]

We will further assume that \( M \) is the local field obtained by adjoining to \( K_t \) the roots of the Eisenstein polynomial \( X^n - \pi_t, (n, p) = 1 \). Then \( e(M/\mathbb{Q}_p) = n e(K_t/\mathbb{Q}_p) \) and \( D(M/K_t) = n \pi_M^{n-1} = \pi_t/\pi_M \).
\[ v_M(\lambda_{i,j}) \geq \min \left\{ v_M(\alpha_j) - v_M(D(K_t/S)) - \frac{e(M)}{p-1} - 1, \right. \]
\[ \left. v_M(\frac{\pi^k}{\pi_{K_t}}) - v_M(D(K_t/S)) - \frac{e(M)}{p-1} - 1 \right\} \]

Dividing everything by \( e(M/K_t) = n \) and noticing that \( v_M(x) = e(M/K_t)v_{K_t}(x) \) for \( x \in K_t \) we obtain
\[ v_{K_t}(\lambda_{i,j}) \geq \min \left\{ v_{K_t}(\alpha_j) - v_{K_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n}, \right. \]
\[ \left. v_{K_t}\left( \frac{\pi^k}{\pi_{K_t}} \right) - v_{K_t}(D(K_t/S)) - \frac{e(K_t)}{p-1} - \frac{1}{n} \right\} \]

Letting \( n \to \infty \) we obtain
\[ v_{K_t}(\lambda_{i,j}) \geq \min \left\{ v_{K_t}(\alpha_j) - v_{K_t}(D(K_t/S)) - e(K_t)/(p-1), \right. \]
\[ \left. v_{K_t}(\frac{\pi^k}{\pi_{K_t}}) - v_{K_t}(D(K_t/S)) - e(K_t)/(p-1) \right\} \]
which implies (66). \( \square \)

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