Mott Transition for Strongly-Interacting 1D Bosons in a Shallow Periodic Potential

G. Boéris,1 L. Gori,2 M. D. Hoogerland,3 A. Kumar,2 E. Lucioni,2 L. Tanzi,2 M. Inguscio,2,4 T. Giamarchi,5 C. D’Errico,2 G. Carleo,1 G. Modugno,2 and L. Sanchez-Palencia1

1Laboratoire Charles Fabry, Institut d’Optique, CNRS, Univ Paris Sud 11, 2 avenue Augustin Fresnel, F-91127 Palaiseau cedex, France
2LENS and Dipartimento di Fisica e Astronomia, Università di Firenze, and CNR-INFM 50019 Sesto Fiorentino Italy
3Department of Physics, University of Auckland Private Bag 92019, Auckland, New Zealand
4INRIM, 10135 Torino, Italy
5Department of Quantum Matter Physics, University of Geneva, 24 quai Ernest-Ansermet, 1211 Geneva, Switzerland

(Dated: October 6, 2015)

We investigate the superfluid-insulator transition of one-dimensional interacting Bosons in both deep and shallow periodic potentials. We compare a theoretical analysis based on Monte-Carlo simulations in continuum space and Luttinger liquid approach with experiments on ultracold atoms with tunable interactions and optical lattice depth. Experiments and theory are in excellent agreement. It provides a quantitative determination of the critical parameter for the Mott transition and defines the regime of validity of widely used approximate models, namely the Bose-Hubbard and sine-Gordon models.

Introduction.— The interplay of repulsive interactions and a periodic potential in a quantum fluid triggers a superfluid-insulator transition known as the Mott transition, provided the potential period is commensurate to the inverse fluid density. The most familiar notion of Mott transition takes place in the limit of a deep periodic potential. In this case the lattice Hubbard model microscopically captures the dominant interaction and hopping processes, the strengths of which, \( U \) and \( J \), strongly depend on the periodic potential amplitude \( V \). Then, the Mott transition is driven by the competition of these sole two parameters at \( J \sim U \) [1, 2]. Quite strikingly in one dimension (1D) a Mott transition can exist even for a vanishingly small periodic potential provided the repulsive interactions are strong enough [3–6]. In the limit of a shallow potential, its amplitude \( V \) becomes sub-relevant and the transition is mostly controlled by the interaction strength \( g \) alone [6].

Ultracold atoms provide a remarkable laboratory to study this physics [7, 8]. So far the Mott transition has been observed in both deep [9–13] and shallow [14] optical lattices. However, the characterization of the Mott transition in shallow potentials remains a formidable challenge for both theory and experiments, with direct consequences not only in the ultracold atom realm but also in condensed matter for problems such as spin chains for instance [6, 15, 16].

On the theoretical side, while the Hubbard limit is now well documented [7, 8, 17] and its Mott transition has been extensively studied [18–24] its regime of validity beyond the deep-lattice limit is still unknown in 1D, with ab-initio results having been reported so-far only for 3 dimensions (3D) [25]. In the limit of a vanishing potential, an estimate of the transition values may be found in the sine-Gordon model whose coefficients are determined perturbatively [14, 26]. This, however, ignores the unavoidable renormalization of the field-theoretic coupling parameters by the potential, which may significantly affect the transition. On the experimental side, the Mott transition has been clearly observed in the shallow lattice limit using modulation spectroscopy and transport measurements [14]. However, the experimental uncertainties did not allow to a precise determination of the phase diagram.

In this Letter, we report the first quantitative joint theoretical and experimental investigation of the Mott transition for strongly-interacting 1D Bosons in a shallow periodic potential. Using continuous-space quantum Monte Carlo calculations, we determine the exact quantum phase diagram. Our calculations confirm the field-theoretical universal predictions and provide, in addition, an accurate quantitative values of the critical parameters of the Mott transition. Experimentally, we perform transport measurements on a Bose gas with tunable interactions down to the limit of very shallow lattices and we analyze them with a phase slip based model to determine accurately the Mott transition. The numerical and experimental results are in excellent agreement and show significant deviation from the perturbative sine-Gordon theory using bare Luttinger parameters.

Model and theoretical approach.— We consider zero-temperature interacting 1D Bosons of mass \( m \) with a contact interaction of strength \( g \), subjected to a periodic potential \( V(x) = V \sin^2(kx) \) of spacing \( a = \pi/k \) and amplitude \( V \). Both the large \( V \) and small \( V \) limits have the possibility of a Mott transition when the interactions are increased [17]. In spite of their qualitative different natures the two limiting cases are, however, expected to belong to the same universality class for they both lead to the same low-energy sine-Gordon model [4, 6, 17, 27]. Within the Tomonaga-Luttinger liquid (TLL) approach, the homogeneous superfluid is parameterized by the Luttinger parameter \( K \), which characterizes the interaction strength. For weak interactions, the periodic potential is essentially irrelevant, except in renormalizing the effective value of the Luttinger parameter. For strong interactions, the TLL may be unstable upon introducing a periodic potential, which signals the Mott insulator phase. More precisely, the Mott transition may be triggered either by changing the fluid density to commensurability at sufficiently strong interactions (Mott-\( \delta \) transition) or by increasing the interactions at commensurability (Mott-\( U \) transition). The TLL theory predicts the universal critical values \( K_c = 1/p^2 \) and \( K_c = 2/p^2 \)
for the Mott-$\delta$ and Mott-$U$ transitions respectively, where $p$ is the commensurability order [5, 6, 27].

However the TLL theory involves effective parameters that are not easily related to the microscopic Hamiltonian parameters and the critical curve $g_r(V)$ is presently not quantitatively known. To precisely determine the Mott transitions, we use quantum Monte-Carlo (QMC) simulations. This allows us (i) to determine quantitatively the phase diagram in terms of the microscopic parameters and (ii) to compute explicitly the Luttinger parameter $K$ as a function of the microscopic ones and make the link with field theory. We use the same implementation of the continuous-space worm algorithm [28, 29] in the grand-canonical ensemble as used in Ref. [30], which is numerically exact for all the physical quantities we study in the following [31].

**Incommensurate transition.**— We start with the incommensurate (Mott-$\delta$) transition, which may be triggered by changing the chemical potential $\mu$. In order to accurately determine the critical point, several quantities are examined. The particle density $n$ is computed directly in QMC and the compressibility $\kappa = \partial^2 n/\partial \mu^2$ is computed independently from particle number fluctuations. The hydrodynamic superfluid density $n_s$ is found from the superfluid stiffness $\Gamma_s$, defined as the response of the system to a twist of the boundary conditions, which is computed using the winding-number estimator [31]. We then deduce the superfluid fraction $f_s = n_s/n$ and the Luttinger parameter $K = \pi \sqrt{n_s \kappa}$.

The QMC results are shown versus the chemical potential in Fig. 1 for $V = 2E$, where $E = \hbar^2 k^2/2m$ is the recoil energy, $g = \hbar^2/ma$, and various system sizes. The density [Fig. 1(a1)] increases monotonically with the chemical potential $\mu$ and exhibits a plateau at commensurability, $na = 1$, where the superfluid density drops to zero. This is the signature of the Mott-$\delta$ transition. The critical chemical potentials $\mu_c^{\pm}$ corresponding to the two edges of the plateau are accurately determined from the crossing points of the compressibility for different system sizes [Fig. 1(a2)]. They can also be found from the drop of the superfluid fraction $f_s$, which yields similar values for $\mu_c^{\pm}$ [Fig. 1(a3)]. At the Mott-$\delta$ transition, the Luttinger parameter is expected to exhibit the universal discontinuity from $K = 1$ to $K = 0$. Our data [Fig. 1(a4)] are perfectly compatible with this prediction of the TLL theory.

Repeating the same calculations for various values of the interaction strength, we find the Mott lobe in the $g-\mu$ plane shown in Fig. 1(b). The black points and joining lines are determined from the $K = 1$ criterion while the colored points are extracted from the crossing point of the compressibility (red), the cusp of the compressibility (green), and the crossing point of the Luttinger parameter (blue). The different methods yield results in excellent agreement all along the lobe within a few percents. The Mott gap $\Delta = \mu_c^+ - \mu_c^-$ is shown in the Inset of Fig. 1 versus the interaction strength. The black dashed line is a fit with the exponential closing $\Delta(g) \propto \exp(-b/\sqrt{g-g_c})$ predicted by the Berezinskii-Kosterlitz-Thouless (BKT) theory [32–34], with $b$ and $g_c$ as fitting parameters. We find that the fit is good for all values of $g$ except very close to the hard-core limit, $g \to +\infty$. The fit also yields a fair estimate of the critical value at commensurability, $g_c/(\hbar^2/ma) = 2.2 \pm 0.1$, which corresponds to the tip of the lobe.

**Commensurate transition.**— In order to characterize the commensurate (Mott-$U$) transition more precisely, we now vary the interaction strength $\gamma = mg/\hbar^2 n$ at commensura-
the integrable Lieb-Liniger model [35]. However, in the inter-

cal value \((J/U)_{\text{c}} = 0.297 \pm 0.01\) (the shaded area corresponds to this 

even relatively weak interactions is significant and cannot be

even relatively weak interactions is significant and cannot be

observed a clear change of behavior of the \(g_1\) function from algebraic for \(\gamma < \gamma_c\) to exponential for

\(\gamma > \gamma_c\) (see Insets of Fig. 2). This is the signature of the Mott-

U transition. The finite correlation length \(l_c\) in the insulating

phase is shown in the left panel of Fig. 2. It is of only a few

lattice sites long for strong interactions and increases up to a

value comparable to the system size for \(\gamma > \gamma_c\). This is compati-

ble with the expected divergence of the correlation length at

the transition. In the superfluid phase, the algebraic decay of

the correlation function is compatible with the TLL theory

prediction \(g_1(x) \sim 1/x^{1/2K}\). The two values of the Luttinger

parameter found from a fit to this prediction and from the ther-

dydynamic prediction \(K = \pi \sqrt{\gamma_c / \kappa}\) are in good agreement (see

right panel of Fig. 2). When increasing the interaction towards

the insulating phase, the Luttinger parameter decreases down

to \(K \simeq 2\) as predicted by the TLL theory.

To locate the Mott-U transition point accurately, we resort to the

BKT renormalization group equations to perform the finite size scaling of the Luttinger parameter [31]. The results are shown in Fig. 3 (black points). In the strong potential limit, the results are compatible with the prediction of the BH model with the critical value \((J/U)_{\text{c}} = 0.297 \pm 0.01\) [21, 22] and the hopping \(J\) and interaction strength \(U\) calculated from the exact Wannier functions [20]. In the vanishing potential limit, the results converge to the critical value \(\gamma(V = 0) \simeq 3.5\) (red cross in Fig. 3) found from the exact relation \(K(g)\) for the integrable Lieb-Liniger model [35]. However, in the inter-

mediate regime, we find a strong deviation from the pinning transition line (red dashed line) computed in Ref. [26] from the perturbative sine-Gordon theory using the bare Luttinger parameters. This shows that to quantitatively obtain the phase diagram the renormalization of the Luttinger parameters by even relatively weak interactions is significant and cannot be ignored in the perturbative field theory.

**Experiment.**— Having quantitatively characterized the Mott transition from theory, we now turn to the experimental investigation. The experiment starts with a Bose-Einstein condensate of 35 000 \(^{39}\text{K}\) atoms with tunable scattering length at a broad Feshbach resonance [36]. The BEC is split into about 1000 vertical 1D tubes by adiabatically loading a strong horizontal 2D optical lattice. Each tube contains on average 36 atoms and the transverse trapping frequency, \(\omega_\perp = 2\pi \times 40\)

kHz corresponds to an energy higher than all other energy scales, realizing an effective 1D geometry. In the longitudinal direction we then adiabatically rise a weak optical lattice with spacing \(a = \lambda / 2 = 532 \text{ nm}\) and normalized amplitude \(V/E\), ranging from 1.0(1) to 4.0(4). The temperature of the system, \(T \simeq 30\) nK, is below the 1D degeneracy temperature \(T_0 \simeq 50\) nK [37]. A magnetic field along the vertical direction holds the system against gravity and a residual harmonic trap potential, with frequency \(\omega_\parallel = 2\pi \times 160\) Hz, makes it inhomogeneous. By varying the 3D scattering length \(a_{\text{pp}}\), we can tune the Luttinger parameter \(\gamma\) in the range 0.07 – 7.4. The system parameters are chosen to obtain a mean atom number per potential period \(\langle na\rangle = 1\). This implies that in most of the tubes there are one or two regions with local commensurate density, \(na=1\), which can undergo a Mott-U transition. There is however a fraction of the tubes for which \(na < 1\) which can-

![Figure 2](image_url) ![Figure 3](image_url)
We have studied, both theoretically and experimentally, the Mott transition of strongly-interacting 1D Bosons in a periodic potential from deep to shallow potentials. Our ab-initio QMC calculations validate the field-theoretic universal predictions and in addition provide a quantitative determination of the phase diagram. It shows that the renormalization of the Luttinger parameter is significant even for weak periodic potentials. The numerical analysis give excellent agreement with experiments for the Mott-U transition. The experimental observation of our numerical phase diagram for the Mott-δ transition is still beyond reach for ultracold atomic systems due to the requirement of a fine control of atom number in box-shaped potentials. In spite of recent progress in that direction [42, 43], it remains a great challenge for future studies.

This research was supported by the EU FET-Proactive QUIC (H2020 grant No. 641122), the ERC ALoGlaDis (FP7/2007-2013 grant No. 256294), Marie Curie IEF (FP7/2007-2013 grant No. 327143), MIUR (grant No. RBFR12NLNA), and the Swiss NSF under Division II. Numerical calculations were performed using HPC resources from GENCI-CCRT/CINES (grant No. c2015056853) and the GMPCS cluster of the LUMAT federation (FR LUMAT).
2764), and make use of the ALPS scheduler library and statistical analysis tools [44–46].

Note added.— During the completion of this manuscript, a preprint appeared reporting the numerical study of the Mott–U transition with results consistent with ours [47].
In this Supplemental Material we give additional details on the methods and the analysis we have performed in the main Letter. In particular we provide a detailed description of i) the quantum Monte Carlo calculations, ii) the determination of the superfluid density, iii) the finite-size scaling analysis, and iv) the experimental methods.

Quantum Monte Carlo calculations

We consider 1D interacting Bosons at zero temperature governed by the continuous-space Hamiltonian,

\[ \hat{H} = \int \! dx \left[ \frac{\hbar^2}{2m} \nabla \bar{\psi} \nabla \psi + g \frac{1}{2} \bar{\psi} \psi \psi \right] + V(x) \psi \psi, \]

(S1)

where \( \bar{\psi}(x) \) is the Bose field, \( m \) is the particle mass, \( g \) is the contact interaction strength, and \( V(x) = V \sin^2(kx) \) is a periodic potential of spacing \( a = \pi/k \) and amplitude \( V \).

We study the interacting quantum system from first principles, by means of quantum Monte Carlo (QMC) numerical simulations. Path-integral QMC approaches allow us to compute the exact thermodynamic properties of a generic, interacting bosonic quantum system in any dimensions. This is achieved by means of the path-integral representation of the grand-canonical partition function \( \hat{Z} \) governing the continous-space Hamiltonian, \( \hat{H} \), by an angle \( \theta \). It is directly computed by QMC using the usual winding number estimator [1]. However, this quantity strongly depends on the ratio \( \beta \) [5]. The relevant quantity to study is the superfluid density \( n \), appearing in the hydrodynamic action

\[ S[\theta] = \int_0^L \! dx \int_0^{\beta} \! d\tau \left[ \frac{\hbar^2 n_2}{2m} (\partial_\tau \theta)^2 + \frac{\hbar^2 K}{2} (\partial_\theta \theta)^2 \right]. \]

(S3)

Computing the partition function \( Z = \int \mathcal{D} \theta e^{-S[\theta]} \), correctly taking into account winding configurations of the form \( \theta(x, \tau) = 2\pi k x / L \), one deduces the relation

\[ \Upsilon = n \left( 1 - 4 \pi^2 n \frac{\hbar^2 \beta}{mL} \sum_{k = -\infty}^{\infty} e^{-2\pi^2 n \hbar^2 k^2 \beta / mL} \right). \]

(S4)

In our calculations, we invert this relation and extract \( n \) from the stiffness \( \Upsilon \), computed by QMC.

Finite size scaling

To precisely determine the Mott-\( U \) critical interaction strength \( U_c \), we study the evolution of the Luttinger parameter \( K \) for the increasing system sizes \( L/a = 30, 50, \) and \( 100 \). When \( L \) is increased, the temperature \( T \) is lowered by keeping the ratio \( \beta / L \) constant in order to consistently compute ground-state properties. We resort to the known Berezinskii-Kosterlitz-Thouless (BKT) renormalization group (RG) equations to perform the finite size scaling and determine the critical interaction strength in the thermodynamic limit. In particular, the finite-size behaviour of the Luttinger parameter \( K \) and of renormalized potential strength \( V \) are governed by the renormalization equations [6–10].

\[ \frac{dK}{dt} = -\frac{\pi^6}{16} \left( \frac{V}{E} \right)^2 K^2 ; \quad \frac{dV}{dt} = -(2 - K)V. \]

(S5)

Dividing these two equations one by another, one finds that the quantity

\[ \zeta = \frac{2}{K} + \ln \left( \frac{K}{2} - \frac{\pi^2}{16} \left( \frac{V}{E} \right)^2 \right) \]

(S6)

is conserved along the RG trajectories. The separatrix between the superfluid and the insulating phases then corresponds to \( K = 2 \) and \( V = 0 \), i.e. \( \zeta = 1 \). In practice, we use

\[ \zeta = \frac{2}{K} + \ln \left( \frac{K}{2} - \frac{\pi^2}{16} \left( \frac{V}{E} \right)^2 \right). \]

(S7)
the values the Luttinger parameter $K_1$ and $K_2$ for two system sizes $L_1$ and $L_2$, and compute $\zeta$ using the integral relation

$$\int_{K_1}^{K_2} \frac{dK}{K^2(\ln K/2 - \zeta) + 2K} = \ln(L_2/L_1).$$

(S7)

The Mott-$U$ critical point is then determined using the criterion $\zeta = 1$.

**Experimental methods**

In the experiment, a Bose-Einstein condensate of $^{39}$K atoms is split in about $10^3$ 1D subsystems (potential tubes) by means of a 2D optical lattice. The overall Thomas-Fermi distribution of atoms in the tube labelled by $i$ and $j$ along the two horizontal directions is given by $N_{ij} = N_0 (1 - 2\pi N_0 (l^2 + j^2)/SN_{ij}^{1/2})$, where $N_0 = 3.5 \times 10^4$ is the total atom number and $N_{ij} \approx 50$ is the atom number in the central tube. On $N_i$ there is a 6% statistical error while a 30% systematic uncertainty is due to the imaging calibration. Along the vertical direction of the tube, where we load the weak periodic potential, the trapping frequency is $\omega_z = 2\pi \times 160$ Hz. A good estimate of the mean atomic density for each tube is provided by the largest of the Thomas-Fermi and the Tonks value [11].

The mean site occupation $\langle n a \rangle$ is then calculated by averaging over all the tubes. During the lattice loading we employ an optimal value of the 3D scattering length to get commensurability on average, i.e. $\langle n a \rangle = 1$. Given $\omega_z$ and $N_i$, the optimal value is $a_{3D} = 220 a_0$. After the lattice loading, by varying the 3D scattering length $a_{3D}$, we tune the 1D scattering length $a_{1D} = \ell_1^2 (1 - 1.03a_{3D}/\ell_1)/2a_{3D}$, and thus the Lieb-Liniger parameter $\gamma$, where $\ell_1 = \sqrt{\hbar/m\omega_z}$ is the harmonic transverse size of a tube, fixed by the 2D lattice. The value of $\gamma$ is estimated by averaging over all the tubes $1/(n_{max} a_{1D})$, where the peak occupation $n_{max} a$ is the largest of the Thomas-Fermi and the Tonks value. The mean peak occupation, i.e. the peak occupation averaged over all the tubes, is $\bar{n}_{max} a \simeq 1.2$ and the peak density in the central tube is less than two, thus preventing localization mechanisms at occupations other than one. Despite the inhomogeneity of our system, at sufficiently strong interactions we clearly observe a suppression of the system dynamics. We interpret the latter as due to the fact that within each tube, a part of the atoms reaches the localization condition $na = 1$ stopping also the remaining adjacent parts with different occupation. We estimate that about one quarter of the atoms resides in tubes where the occupation is always $na < 1$, justifying the plateau in $p$, shown in Fig. 4 of the main paper.

The complete model we use to fit the temperature dependence of the momentum, $p(t)$, as shown in Fig. 4 of the main paper, is

$$p(t) = p_{max} e^{-Gt} \sin(\omega't)$$

with amplitude $p_{max} = m^* \omega'^2 z_0/\omega'$, frequency $\omega' = \sqrt{\omega^2 - G^2}$ and damping rate $G$. Here $m^*$ is the effective mass due to the shallow lattice, $z_0 \simeq 3 \mu$m is the trap displacement, and $\omega' = \omega_c \sqrt{m/m^*}$ is the lattice renormalized frequency.

In the absence of an exact theoretical model for the critical momentum $p$, to enter the dynamical instability regime at finite interaction, we use a quantum phase slips based model to predict the interaction dependence of $p$. Along the lines of Ref. [12] we use the equation for the quantum phase slip nucleation rate [13, 14]

$$\Gamma = B L(U) \sqrt{n a J U/\pi} = \sqrt{\frac{7.1(\pi/2 - p \lambda/2\hbar)^{5/2}}{2\pi \sqrt{U/na}}} \exp\left[-7.1 \sqrt{naJ/(\pi/2 - p \lambda/2\hbar)^{5/2}}\right].$$

Here $L(U) \simeq 2U^{1/3}$ is the mean length of the tubes and $B$ is a phenomenological constant. Assuming that the system enters the strongly dissipative regime when the nucleation rate $\Gamma$ exceeds a constant critical value, from Eq. (S8) we can obtain $p$ for each given value of the interaction $U$. Since this relation is valid in the Bose-Hubbard regime, we can employ it to estimate the critical momentum $p(U)$ for the experimental case with $V/E_n = 4$. The phenomenological parameter $B$ is arbitrarily adjusted to reproduce the measured $p$, at one interaction value, $U = 2.4J$. In Fig. S1 we compare the measured and the predicted $p$, as a function of the scattering length $a_{3D} \propto U$. The quantum phase slip-based model well reproduces the experimental behaviour of $p$, with the interaction, showing a quadratic dependence of $p$ on $a_{3D}$. This result justifies the choice of a quadratic polynomial fit in Fig. 4 of the main paper, at least for the measurements with larger $V$. For shallower lattices the Tomonaga-Luttinger Liquid model for the nucleation rate could in principle be used [13], but this would require the Luttinger parameter $K$ as a function of the interaction $\gamma$ in the presence of the lattice, which is still unknown.

---

[1] D. M. Ceperley, Rev. Mod. Phys. 67, 279 (1995).
[2] M. Boninsegni, N. Prokof’ev, and B. Svistunov, Phys. Rev. Lett. 96, 070601 (2006).
[3] M. Boninsegni, N. V. Prokof’ev, and B. V. Svistunov, Phys. Rev. E 74, 036701 (2006).
[4] B. Gaveau and L. S. Schulman, J. Phys. A 19, 1833 (1986).
[5] N. V. Prokof’ev and B. V. Svistunov, Phys. Rev. B 61, 11282 (2000).
[6] V. L. Berezinskii, Sov. Phys. JETP 34, 610 (1972).
[7] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. 6, 1181 (1973).
[8] N. Prokof’ev, O. Ruebenacker, and B. Svistunov, Phys. Rev. Lett. 87, 270402 (2001).
[9] N. Prokof’ev and B. Svistunov, Phys. Rev. A 66, 043608 (2002).
[10] T. Giamarchi, *Quantum Physics in One Dimension* (Carendon press, Oxford, 2004).
[11] V. Dunjko, V. Lorent, and M. Olshanii, Phys. Rev. Lett. 86, 5413 (2001).
[12] L. Tanzi, E. Lucioni, S. Chaudhuri, L. Gori, A. Kumar,
Figure S1. For $V/E_R = 4$ the experimental critical momentum (blue dots) is compared to the one predicted by the quantum phase slip based model, which applies up to the critical scattering length $a_c$ to enter the MI regime (gray dashed line). For $a > a_c$, $p_c$ has instead a plateau due to tubes with occupation $na < 1$ (black dotted line).

C. D’Errico, M. Inguscio, and G. Modugno, Phys. Rev. Lett. 111, 115301 (2013).

[13] A. Polkovnikov, E. Altman, E. Demler, B. Halperin, and M. D. Lukin, Phys. Rev. A 71, 063613 (2005).

[14] I. Danshita and A. Polkovnikov, Phys. Rev. A 85, 023638 (2012).