On the number of even roots of permutations

Lev Glebsky, Melany Licón, Luis Manuel Rivera

Abstract

Let $\sigma$ be a permutation on $n$ letters. We say that a permutation $\tau$ is an even (resp. odd) $k$th root of $\sigma$ if $\tau^k = \sigma$ and $\tau$ is an even (resp. odd) permutation. In this article, we obtain generating functions for the number of even and odd $k$th roots of a permutation, in terms of its cycle type. Our result implies known generating functions of Moser and Wyman and also some generating functions for sequences in The On-line Encyclopedia of Integer Sequences (OEIS).

Keywords: Roots of permutations; even permutations; generating functions.

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1 Introduction

A classical problem in group theory and combinatorics is the study of problems related to the solution of the equation $x^k = a$ over groups, where $k$ is a fixed positive integer (see, e.g., [6, 11, 12, 13, 20, 21, 23]). One of the most studied situations is the case of the symmetric group $S_n$. For example, there is a characterization that determines when a given permutation has a $k$th root in $S_n$ (see, e.g., [1, 3, 7]) and there are several results about the probability that a randomly selected permutation of length $n$ has a $k$th root (see, e.g., [2, 5, 13, 14, 18]). In addition, Pavlov [10] gave an explicit formula for the number of solutions in $S_n$ of the equation $x^k = \sigma$, and Leaños et al., [10] gave a multivariable exponential generating function. Finally, Roichman [19] gave a formula for such a number expressed as an alternating sum of $\mu$-unimodal $k$th roots of the identity permutation.

In this article, we are interested in the number of even permutations as the $k$th roots of a given permutation. To our knowledge, there are only few results in this direction and only for the case of the identity permutation. Moser and Wyman [13] studied the case of $k = 2$. In OEIS [15] there are only a few sequences for the number of even $k$th roots of identity permutation: A000704 ($k = 2$), A061129 ($k = 4$), A061130 ($k = 6$), A061131 ($k = 8$) and A061132 ($k = 10$). For the odd $k$th roots of the identity permutation, in OEIS we find sequences A001465 ($k = 2$), A061136 ($k = 4$) and A061137 ($k = 6$).

1After finishing this work, we were aware of the existence of a generating function, given in terms of the cycle index of the symmetric group $S_n$ due to Chernoff [4].
1.1 Basic definitions and main result

In order to formulate our main result, we need some definitions and notation. The cycle type of an \( n \)-permutation is a vector \( c = (c_1, \ldots, c_n) \), which means that for every \( i \in [n] \), the permutation has \( c_i \) cycles of length \( i \). We say that a permutation \( \sigma \) is of cycle type \((\ell_1)^{a_1} \cdots (\ell_m)^{a_m}\), with \( a_i > 0 \), if \( \sigma \) has exactly \( a_i \) cycles of length \( \ell_i \) in its disjoint cycle factorization and does not have any cycles of any other length. We use \( \mathbb{N} \) (respectively \( \mathbb{N}_0 \)) to denote the set of positive (respectively, non-negative) integers. Let \( k, \ell \in \mathbb{N} \). Let

\[
G_k(\ell) = \{ g \in \mathbb{N} : \gcd(g\ell, k) = g \}.
\]

It is easy to see that if \( k = p_1^{a_1} \cdots p_j^{a_j} \), where \( p_1, \ldots, p_j \) are distinct primes and \( a_i > 0 \) for \( i \in \{1, \ldots, j\} \), then

\[
G_k(\ell) = \left\{ p_1^{b_1} \cdots p_j^{b_j} : b_i = a_i \text{ if } p_i | \ell \text{ and } b_i \in \{0, 1, \ldots, a_i\} \text{ if } p_i \nmid \ell \right\}.
\]

The main result of this paper is the following.

**Theorem 1.1.** Let \( k, n \) be positive integer. Let \( c_1, \ldots, c_n \) be non-negative integers such that \( n = c_1 + 2c_2 + \cdots + nc_n \). Then the coefficient of \( \frac{t^{c_1}}{c_1!} \cdots \frac{t^{c_n}}{c_n!} \) in the expansion of

\[
\frac{1}{2} \exp \left( \sum_{\ell \geq 1} \sum_{g \in G_k(\ell)} \frac{\ell^g - 1}{g} \frac{t^g}{\ell^g} \right) + \frac{1}{2} \exp \left( \sum_{\ell \geq 1} \sum_{g \in G_k(\ell)} (-1)^{\ell+1} \frac{\ell^g - 1}{g} \frac{t^g}{\ell^g} \right)
\]

is the number of even \( k \)th roots of a permutation of cycle type \( c = (c_1, \ldots, c_n) \), and in the expansion of

\[
\frac{1}{2} \exp \left( \sum_{\ell \geq 1} \sum_{g \in G_k(\ell)} \frac{\ell^g - 1}{g} \frac{t^g}{\ell^g} \right) - \frac{1}{2} \exp \left( \sum_{\ell \geq 1} \sum_{g \in G_k(\ell)} (-1)^{\ell+1} \frac{\ell^g - 1}{g} \frac{t^g}{\ell^g} \right)
\]

is the number of odd \( k \)th roots of a permutation of cycle type \( c \).

The known result of the identity permutation is a consequence of this theorem. The outline of this paper is as follows. In Section 2 we will prove several propositions and lemmas that we use in the proof of our main result. The proof of Theorem 1.1 is at the end of this section. In Section 3 we show a few special cases of Theorem 1.1 which allow some nice simplifications.

## 2 Auxiliary results and proof of Theorem 1.1

First, we present two known results, which will be used in the proof or our main result.
Proposition 2.1 ([10 Proposition 5]). A permutation of cycle type \((\ell)^c\) has a kth root if and only if the equation

\[ g_1x_1 + \cdots + g_hx_h = c \]

has non-negative integer solutions, where \(G_k(\ell) = \{g_1, \ldots, g_h\}\).

The following result shows a generating function for the number of kth roots of a permutation.

Theorem 2.2 ([10 Theorem 2]). Let \(k, n\) be positive integer. Let \(c_1, \ldots, c_n\) be non-negative integers such that \(n = c_1 + 2c_2 + \cdots + nc_n\). Then the coefficient of \(t^{c_1} \cdots t^{c_n} c_1! \cdots c_n!\) in the expansion of

\[ \exp \left( \sum_{\ell \geq 1} \sum_{g \in G_m(\ell)} \frac{g^{\ell - 1}}{g} t^g \right) \]

is the number of kth roots of a permutation of cycle type \(c = (c_1, \ldots, c_n)\).

The outline of the proof is as follows. First, we work with the difference between the number of even kth roots and the number of odd kth roots of a permutation (Lemma 2.5). The next step was to obtain a multivariable exponential generating function for such a difference (Lemma 2.10). In order to do this, first we assign a sign to the number of kth roots, of certain type, of permutations with all its cycles of the same length (Proposition 2.7). Using this, we obtain an exponential generating function for the difference between the number of even kth roots and odd kth roots of permutations with all its cycles of the same length (Lemma 2.9). Finally, the proof of Theorem 1.1 is obtained as a consequence of Theorem 2.2 and Lemma 2.10.

We need the following easy proposition about groups in general.

Proposition 2.3. Let \(G\) be a group and \(K\) be a field. Let \(\phi: G \rightarrow (K, \cdot) (g \mapsto \phi^g)\) be a homomorphism to the multiplicative group of \(K\) and \(X, Y \subseteq G\) be finite. Then

\[
\left( \sum_{g \in X} \phi^g \right) \left( \sum_{h \in Y} \phi^h \right) = \sum_{g \in X} \sum_{h \in Y} \phi^{gh}.
\]

Let \(re_k(\sigma)\) (resp. \(ro_k(\sigma)\)) denote the number of even (resp. odd) kth roots of permutation \(\sigma\). The support of an n-permutation \(\sigma\) is defined as \(\text{supp}(\sigma) = \{a \in \{1, \ldots, n\} : \sigma(a) \neq a\}\).

Proposition 2.4. Let \(\sigma\) be a permutation such that \(\sigma = \sigma_1\sigma_2\) and \(\text{supp}(\sigma_1) \cap \text{supp}(\sigma_2) = \emptyset\). Let \(re_k(\sigma)\) (resp. \(ro_k(\sigma)\)) be the number of even (resp. odd) kth roots \(\tau\) of \(\sigma\) such that \(\tau = \tau_1\tau_2\) with \(\tau_1^k = \sigma_1\) and \(\tau_2^k = \sigma_2\). Then

\[
re_k(\sigma) - ro_k(\sigma) = (re_k(\sigma_1) - ro_k(\sigma_1)) (re_k(\sigma_2) - ro_k(\sigma_2)).
\]
Proof. Consider the parity of permutations as a homomorphism \( \phi : S_n \to \{-1,1\} \). Let \( X = \{ \tau_1 \in S_n : \tau_1^k = \sigma_1 \} \) and \( Y = \{ \tau_2 \in S_n : \tau_2^k = \sigma_2 \} \). Then \( \sum_{\tau_1 \in X} \phi^{\tau_1} = \text{re}_k(\sigma_1) - \text{ro}_k(\sigma_1) \) and \( \sum_{\tau_2 \in Y} \phi^{\tau_2} = \text{re}_k(\sigma_2) - \text{ro}_k(\sigma_2) \). Therefore, by Proposition 2.3 we have that

\[
\sum_{\tau_1 \in X} \tau_2 \in Y \phi^{\tau_1} = \text{re}'_k(\sigma_1) - \text{ro}'_k(\sigma_1).
\]

The following result shows that for a given permutation \( \sigma \) we can obtain the difference \( \text{re}_k(\sigma) - \text{ro}_k(\sigma) \) by working with the different lengths in the cycles of \( \sigma \) separately.

Lemma 2.5. Let \( \sigma \) be an \( n \)-permutation that has \( k \)th roots. Suppose that the disjoint cycle factorization of \( \sigma \) can be expressed as the product \( \sigma_1 \sigma_2 \cdots \sigma_m \) where \( \sigma_i \) is the product of all the disjoint cycles of length \( \ell_i \) in \( \sigma \), for every \( i \), with \( \ell_i \neq \ell_j \), for \( i \neq j \). Then

\[
\text{re}_k(\sigma) - \text{ro}_k(\sigma) = \prod_{i=1}^m (\text{re}_k(\sigma_i) - \text{ro}_k(\sigma_i)).
\]

Proof. It is well-known that every \( k \)th root of \( \sigma \) can be written as \( \tau_1 \cdots \tau_m \) with \( \tau_i^k = \sigma_i \), for every \( i \) (see, e.g., [10, §3]). The result follows by Proposition 2.4 and induction.

Sometimes, we use the following fact: if \( \alpha \) is an \( \ell \)-cycle, then \( \alpha^m \) is a product of exactly \( \gcd(m, \ell) \) disjoint \( \ell/\gcd(m, \ell) \)-cycles. Let \( g, k, \ell \) be fixed positive integers and \( p \) be a fixed non-negative integer. We use \( f_{k,\ell,g,p}(c) \) to denote the number of permutations of cycle type \( (g\ell)^p \) that are \( k \)th roots of a permutation of cycle type \( (\ell)^c \), \( c \in \mathbb{N}_0 \). The following proposition has been proven, in essence, by Moreno et al. [10].

Proposition 2.6. Let \( g, k, \ell \) be fixed positive integers and \( p \) be a fixed non-negative integer. Let \( c \in \mathbb{N}_0 \). If \( g \in G_k(\ell) \) and \( c = gp \), then

\[
f_{k,\ell,g,p}(c) = \frac{(gp)!\ell^{p(g-1)}}{g^p p!},
\]

and \( f_{k,\ell,g,p}(c) = 0 \) in any other case.

In view of previous proposition, for \( g \in G_k(\ell) \) we define

\[
f_{k,\ell,g}(c) = \begin{cases} f_{k,\ell,g,p}(c) & c = gp \\ 0 & \text{other case} \end{cases}
\]

Now, we assign a sign to the number \( f_{k,\ell,g}(c) \), which helps to know whether the roots of cycle type \( (g\ell)^p \) of a permutation of cycle type \( (\ell)^c \) are even.
Proposition 2.7. Let $k, \ell$ be fixed positive integers. Let $g \in G_k(\ell), c \in \mathbb{N}_0$ and 

\[ a(c) = (-1)^{c(\ell g + 1)/g} f_{k, \ell, g}(c). \]

If $\sigma$ is a permutation of cycle type $(\ell)^c$ and $c = gp$, then $a(c) \neq 0$. In addition, the $k$th roots of cycle type $(g\ell)^p$ of $\sigma$ are even permutations if and only if $a(c) > 0$.

Proof. As $c = gp$, we have that $a(c) = (-1)^p(\ell g + 1) f_{k, \ell, g,p}(c)$, and Proposition 2.6 implies that $a(c) \neq 0$. The result follows because the sign of a $q$-cycle is $(-1)^{q+1}$ and hence the sign of the product of $p$ cycles of length $(\ell g)$ is $(-1)^{p(\ell g + 1)}$. \hfill \Box

The exponential generating function, in the variable $t_\ell$, for the number $a(c)$ in the previous proposition is given in the following result.

Proposition 2.8. Let $\ell, k \in \mathbb{N}$. Let $g \in G_k(\ell)$ fixed. Then

\[ \sum_{c \geq 0} (-1)^{c/g(\ell g + 1)} f_{k, \ell, g}(c) \frac{t_\ell^c}{c!} = \exp \left( (-1)^{\ell g + 1} \ell \frac{g^{g-1}}{g} t_\ell^g \right). \]

Proof. From Proposition 2.6, we have that $f_{k, \ell, g}(c) \neq 0$ if and only if $c = gp$, for some $p \in \mathbb{N}_0$. Therefore

\[ \sum_{c \geq 0} (-1)^{c/g(\ell g + 1)} f_{k, \ell, g}(c) \frac{t_\ell^c}{c!} = \sum_{p \geq 0} (-1)^p(\ell g + 1) \frac{(gp)! p^{(g-1)}}{g^p p!} \frac{t_\ell^p}{(gp)!} \]

\[ = \sum_{p \geq 0} \left( (-1)^{\ell g + 1} \ell \frac{g^{g-1}}{g} t_\ell^g \right)^p \frac{1}{(p)!} \]

\[ = \exp \left( (-1)^{\ell g + 1} \ell \frac{g^{g-1}}{g} t_\ell^g \right). \ \Box \]

Let $re_k(\ell, c)$ (resp. $ro_k(\ell, c)$) denote the number of even (resp. odd) $k$th roots of any permutation of cycle type $(\ell)^c$.

Lemma 2.9. Let $\ell \in \mathbb{N}$. Then

\[ \sum_{c \geq 0} (re_k(\ell, c) - ro_k(\ell, c)) \frac{t_\ell^c}{c!} = \exp \left( \sum_{g \in G_k(\ell)} (-1)^{g \ell + 1} \ell \frac{g^{g-1}}{g} t_\ell^g \right). \]

Proof. Let $\sigma$ be any permutation of cycle type $(\ell)^c$ and let $S$ be the set of all disjoint cycles in $\sigma$. Let $G_k(\ell) = \{g_1, \ldots, g_m\}$, with $g_1 < \cdots < g_m$. By Proposition 2.1, $\sigma$ has $k$th roots if and only if the equation

\[ g_1 x_1 + \cdots + g_m x_m = c \]

...
has non-negative integer solutions, where a solution \((p_1, \ldots, p_m)\) of previous equation means that \(\sigma\) has \(k\)th roots of cycle type \((g_1\ell)^{p_1} \cdots (g_m\ell)^{p_m}\). We can obtain all these roots by running over all the weak ordered partitions \((A_1, \ldots, A_m)\) of \(S\). Indeed, if \((A_1, \ldots, A_m)\) is such a partition, the number of \(k\)th roots associated to this partition is given by \(f_{k,\ell, g_1}(|A_1|) \cdots f_{k,\ell, g_m}(|A_m|)\), where this product is different from 0 if \(|A_i|\) is a multiple of \(g_i\), for every \(i\). Let \(A\) be the set of all weak ordered partitions of \(S\) into \(m\) blocks. The number of \(k\)th roots of \(\sigma\) is equal to
\[
\sum_{(A_1, \ldots, A_m) \in A} f_{k,\ell, g_1}(|A_1|) \cdots f_{k,\ell, g_m}(|A_m|).
\]

Now, for a given partition \((A_1, \ldots, A_m)\) with
\[
f_{k,\ell, g_1}(|A_1|) \cdots f_{k,\ell, g_m}(|A_m|) \neq 0,
\]
the sign of
\[
(-1)^{|A_1|/g_1}(\ell g_1+1) f_{k,\ell, g_1}(|A_1|) \cdots (-1)^{|A_m|/g_m}(\ell g_m+1) f_{k,\ell, g_m}(|A_m|),
\]
determine the parity of the \(k\)th roots of \(\sigma\) of cycle type \((g_1\ell)^{p_1} \cdots (g_m\ell)^{p_m}\), where \(p_i = |A_i|/g_i\). Therefore, the number \(r_{k}(\ell, c) - r_{k}(\ell, c)\) is equal to
\[
\sum_{(A_1, \ldots, A_m) \in A} (-1)^{|A_1|/g_1}(\ell g_1+1) f_{k,\ell, g_1}(|A_1|) \cdots (-1)^{|A_m|/g_m}(\ell g_m+1) f_{k,\ell, g_m}(|A_m|),
\]
and the desired exponential generating function is obtained by Proposition 5.1.3 in Stanley’s book [24] and Proposition 2.8.

Let \(r_{k}(c)\) (resp. \(r_{k}(c)\)) denote the number of even (resp. odd) \(k\)th roots of a permutation of cycle type \(c\). The following multivariable exponential generating function, in the variables \(t_1, t_2, \ldots\), for the difference between the number of even \(k\)th roots and the number of odd \(k\)th roots of permutations of any cycle type follows from Lemmas 2.6 and 2.9.

**Lemma 2.10.** Let \(n, k\) be a positive integers and let \(c_1, \ldots, c_n\) be non-negative integers. For \(n = c_1 + 2c_2 + \cdots + nc_n\), the coefficient of \(\frac{t_1^{c_1} \cdots t_n^{c_n}}{c_1! \cdots c_n!}\) in the expansion of
\[
\exp \left( \sum_{\ell \geq 1} \sum_{g \in G_k(\ell)} (-1)^{g+1} \frac{\ell^{g-1}}{g} t_{\ell}^g \right)
\]
is equal to the number \(r_{k}(c) - r_{k}(c)\), with \(c = (c_1, \ldots, c_n)\).

**Proof of Theorem 2.7** Let \(r_{k}(\sigma)\) denote the number of \(k\)th roots of permutation \(\sigma\). We have that
\[
2r_{k}(\sigma) = r_{k}(\sigma) + r_{k}(\sigma) + r_{k}(\sigma) - r_{k}(\sigma) = r_{k}(\sigma) + (r_{k}(\sigma) - r_{k}(\sigma)).
\]
Similarly \(2r_{k}(\sigma) = r_{k}(\sigma) - (r_{k}(\sigma) - r_{k}(\sigma))\). Therefore, the result follows immediately from Theorem 2.2 and Lemma 2.10.
3 Particular cases

If \( k \) is odd, then any solution of the equation \( x^k = \sigma \) should have the same parity as \( \sigma \), so the generating function is the same as the one given in Theorem 2.2. Therefore, in this section \( k \) is a fixed even integer.

In some examples, we will use, without an explicit mention, the following observation.

**Observation 3.1.** Let \( k \) be an even integer. If \( \ell \) is even, then \( G_k(\ell) \) is a set of even integers.

**Permutations of cycle type** \((\ell)^c\)

For a fixed positive integer \( \ell \), we have that

\[
\sum_{c \geq 0} \text{re}_k(\ell, c) \frac{t^c}{c!} = \frac{1}{2} \exp \left( \sum_{g \in G_k(\ell)} \frac{\ell g - 1}{g} t^g \right) + \frac{1}{2} \exp \left( \sum_{g \in G_k(\ell)} (-1)^g \frac{\ell g - 1}{g} t^g \right),
\]

and

\[
\sum_{c \geq 0} \text{ro}_k(\ell, c) \frac{t^c}{c!} = \frac{1}{2} \exp \left( \sum_{g \in G_k(\ell)} \frac{\ell g - 1}{g} t^g \right) - \frac{1}{2} \exp \left( \sum_{g \in G_k(\ell)} (-1)^g \frac{\ell g - 1}{g} t^g \right).
\]

With these expressions, we can obtain the generating functions of the following sequences in OEIS: A000704, A061129, A061130, A061131, A061132, A001465, A061136 and A061137. For example, sequence A061131 corresponds to the number of even 8th roots of the identity permutation. In this case \( \ell = 1 \) and \( G_8(1) = \{1, 2, 4, 8\} \). Therefore

\[
\sum_{c \geq 0} \text{re}_8(1, c) \frac{t^c}{c!} = \frac{1}{2} \exp \left( t + \frac{1}{2} t^2 + \frac{1}{4} t^4 + \frac{1}{8} t^8 \right) + \frac{1}{2} \exp \left( t - \frac{1}{2} t^2 - \frac{1}{4} t^4 - \frac{1}{8} t^8 \right)
= \exp(t) \cosh \left( \frac{1}{2} t^2 + \frac{1}{4} t^4 + \frac{1}{8} t^8 \right).
\]

We can make further simplifications of equations (1) and (2). First, we consider the case when \( \ell \) is even. By Observation 3.1 we have that

\[
\sum_{c \geq 0} \text{re}_k(\ell, c) \frac{t^c}{c!} = \cosh \left( \sum_{g \in G_k(\ell)} \frac{\ell g - 1}{g} t^g \right)
\]

and

\[
\sum_{c \geq 0} \text{ro}_k(\ell, c) \frac{t^c}{c!} = \sinh \left( \sum_{g \in G_k(\ell)} \frac{\ell g - 1}{g} t^g \right).
\]
For $\ell$ odd, let $G_{O_k}(\ell) = \{g \in G_k(\ell) : g \text{ is odd}\}$ and let $G_{E_k}(\ell) = G_k(\ell) - G_{O_k}(\ell)$. Then
\[
\sum_{c \geq 0} \text{re}_k(\ell, c) \frac{t^c}{c!} = \frac{1}{2} \exp \left( \sum_{g \in G_k(\ell)} \frac{\ell^g - 1}{g} t^g \right) + \frac{1}{2} \exp \left( \sum_{g \in G_{O_k}(\ell)} \frac{\ell^g - 1}{g} t^g - \sum_{g \in G_{E_k}(\ell)} \frac{\ell^g - 1}{g} t^g \right),
\]

Similarly, for the case of odd $k$th roots we have
\[
\sum_{c \geq 0} \text{ro}_k(\ell, c) \frac{t^c}{c!} = \exp \left( \sum_{g \in G_{O_k}(\ell)} \frac{\ell^g - 1}{g} t^g \right) \sinh \left( \sum_{\substack{g \in G_{E_k}(\ell) \mid g \text{ even} \}} \frac{\ell^g - 1}{g} t^g \right).
\]

For the case of the identity permutation ($\ell = 1$) we have that $G_k(1) = \{m : m|k\}$. Therefore,
\[
\sum_{c \geq 0} \text{re}_k(1, c) \frac{t^c}{c!} = \exp \left( \sum_{\substack{g \mid k \text{ odd} \}} \frac{1}{g} t^g \right) \cosh \left( \sum_{\substack{g \mid k \text{ even} \}} \frac{1}{g} t^g \right),
\]
and
\[
\sum_{c \geq 0} \text{ro}_k(1, c) \frac{t^c}{c!} = \exp \left( \sum_{\substack{g \mid k \text{ odd} \}} \frac{1}{g} t^g \right) \sinh \left( \sum_{\substack{g \mid k \text{ even} \}} \frac{1}{g} t^g \right).
\]

In particular, for the case $k = 2^m$, we have
\[
\sum_{c \geq 0} \text{re}_{2^m}(1, c) \frac{x^c}{c!} = \frac{1}{2} \exp \left( \sum_{i=0}^{m} \frac{1}{2^i} x^{2^i} \right) + \frac{1}{2} \exp \left( x - \sum_{i=1}^{m} \frac{1}{2^i} x^{2^i} \right) = \exp(x) \cosh \left( \frac{1}{2} x^2 + \cdots + \frac{1}{2^m} x^{2^m} \right).
\]

This generating function was used in the work of Koda, Sato and Tskeghara [9]. For the case of odd roots we have
\[
\sum_{c \geq 0} \text{ro}_{2^m}(1, c) \frac{x^c}{c!} = \exp(x) \sinh \left( \frac{1}{2} x^2 + \cdots + \frac{1}{2^m} x^{2^m} \right).
\]
3.1 Square roots of permutations

For the case of even square roots we have the following consequence of Theorem 1.1.

**Corollary 3.2.** The coefficient of \( t_1^{c_1} \cdots t_n^{c_n} / (c_1! \cdots c_n!) \) in the expansion of

\[
\prod_{j \geq 1} \exp \left( t_{2j-1} \cosh \left( \sum_{j \geq 1} \left( \frac{2j-1}{2} t_{2j-1}^2 + jt_{2j}^2 \right) \right) \right)
\]

is the number of even square roots of a permutation of cycle type \( c = (c_1, \ldots, c_n) \), and in the expansion of

\[
\prod_{j \geq 1} \exp \left( t_{2j-1} \sinh \left( \sum_{j \geq 1} \left( \frac{2j-1}{2} t_{2j-1}^2 + jt_{2j}^2 \right) \right) \right)
\]

is the number of odd square roots of a permutation of cycle type \( c \).

**Proof.** We rewrite Theorem 1.1 for the case of even square roots. When \( k = 2 \), \( G_2(\ell) \subseteq \{1, 2\} \). We have two cases depending on the parity of \( \ell \). If \( \ell = 2j - 1 \), with \( j \in \mathbb{N} \), then \( G_2(2j - 1) = \{1, 2\} \). Thus

\[
\sum_{g \in G_2(\ell)} (-1)^{\ell g} \ell g^{-1} t_{2j-1}^g = t_{2j-1} - \frac{2j-1}{2} t_{2j-1}^2
\]

and

\[
\sum_{g \in G_2(\ell)} \ell g^{-1} t_{2j-1}^g = t_{2j-1} + \frac{2j-1}{2} t_{2j-1}^2.
\]

If \( \ell = 2j \), with \( j \in \mathbb{N} \), then \( G_2(2j) = \{2\} \). Therefore

\[
\sum_{g \in G_2(\ell)} (-1)^{\ell g} \ell g^{-1} t_{2j}^g = -jt_{2j}^2
\]

and

\[
\sum_{g \in G_2(\ell)} \ell g^{-1} t_{2j}^g = jt_{2j}^2.
\]

Therefore, the exponential generating in Theorem 1.1 becomes

\[
\frac{1}{2} \left( \exp \left( \sum_{j \geq 1} \left( t_{2j-1} + \frac{2j-1}{2} t_{2j-1}^2 + jt_{2j}^2 \right) \right) + \exp \left( \sum_{j \geq 1} \left( t_{2j-1} - \frac{2j-1}{2} t_{2j-1}^2 - jt_{2j}^2 \right) \right) \right).
\]

From which we obtain

\[
\frac{1}{2} \prod_{j \geq 1} \exp \left( t_{2j-1} \right) \left( \prod_{j \geq 1} \exp \left( \frac{2j-1}{2} t_{2j-1}^2 + jt_{2j}^2 \right) + \prod_{j \geq 1} \exp \left( -\frac{2j-1}{2} t_{2j-1}^2 - jt_{2j}^2 \right) \right),
\]

9
that is equal to
\[
\prod_{j \geq 1} \exp(t\bar{j}-1) \cosh \left( \sum_{j \geq 1} \left( \frac{2j-1}{2} t_{2j-1}^2 + j t_{2j}^2 \right) \right).
\]

The proof for the case of odd square roots is similar.

\[\Box\]

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L. G., UNIVERSIDAD AUTÓNOMA DE SAN LUIS POTOSÍ, MEXICO.
E-mail: glebsky@cactus.iico.uaslp.mx.

M. L. and L. M. R., UNIVERSIDAD AUTÓNOMA DE ZACATECAS, MEXICO.
E-mails: mliiqon@gmail.com and luismanuel.rivera@gmail.com.