Abstract

Witten’s formulation of 2+1 gravity is investigated on the nonorientable three-manifold \( \mathbb{R} \times (\text{Klein bottle}) \). The gauge group is taken to consist of all four components of the 2+1 Poincaré group. We analyze in detail several components of the classical solution space, and we show that from four of the components one can recover nondegenerate spacetime metrics. In particular, from one component we recover metrics for which the Klein bottles are spacelike. An action principle is formulated for bundles satisfying a certain orientation compatibility property, and the corresponding components of the classical solution space are promoted into a phase space. Avenues towards quantization are briefly discussed.

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I. INTRODUCTION

The observation that vacuum Einstein gravity in 2+1 spacetime dimensions has no local dynamical degrees of freedom [1,2] has created interest in 2+1 gravity as an arena where quantum gravity can be investigated without many of the technical complications that are present in 3+1 spacetime dimensions. Several formulations of 2+1 dimensional gravity have been given, both classically and quantum mechanically, and it has become an issue to understand what the differences in these formulations are. For reviews, see Refs. [3,4].

In this paper we shall consider Witten’s connection formulation of 2+1 gravity [5,6]. For spacetimes with the topology $\mathbb{R} \times \Sigma$, where $\Sigma$ is a closed oriented surface, Witten’s formulation and its correspondence with the metric theory have been extensively discussed [6–16]; a comprehensive list of references can be found in Refs. [3,4]. The purpose of the present paper is to extend this discussion to the nonorientable spacetime $\mathbb{R} \times KB$, where $KB$ stands for the Klein bottle.

For a closed oriented surface $\Sigma$, a connection formulation that reproduces the standard metric solutions on $\mathbb{R} \times \Sigma$ can be defined as the theory of flat connections on principal $IO_0(2,1)$ bundles over $\mathbb{R} \times \Sigma$, where $IO_0(2,1)$ is the component of the identity of the 2+1 dimensional Poincare group $IO(2,1)$ [6,9]. For $\mathbb{R} \times KB$ a similar connection theory with the gauge group $IO_0(2,1)$ is well-defined, as we shall see, but it is not clear whether this theory in any sense describes nondegenerate spacetime metrics on $\mathbb{R} \times KB$. This motivates us to study the theory in which the gauge group consists of all of $IO(2,1)$. It will be shown that nondegenerate, flat metrics on $\mathbb{R} \times KB$ can be recovered from four connected components of the classical solution space of this theory, and from one of the components we recover metrics that admit spacelike Klein bottles. We shall also give an action principle for certain components of the theory, including these four. The action is sufficiently general to accommodate both the nonorientability of the manifold and the fact that the bundles are nontrivial in a manner involving the disconnected components of $IO(2,1)$.

We begin in Section II by describing the connection theory. This theory is a modest generalization of that given in Ref. [6], in that the three-manifold $M$ may be nonorientable and the gauge group is the full 2+1 Poincare group $IO(2,1)$. We first discuss the kinematics of connections on $IO(2,1)$ bundles over $M$, devoting special attention to gauge transformations and to the recovery of a spacetime metric on $M$. The equations of motion are then introduced by the requirement that the connection be flat. We next define a bundle to be orientation compatible iff the (potential) nonorientability of the base space intertwines with the (potential) nontriviality of the bundle in a certain way, and we construct for such bundles an action principle from which the equations of motion can be derived. $IO_0(2,1)$ bundles over oriented manifolds are orientation compatible, and for them our action reduces to that given in Refs. [3,13].

In Section III we specialize to $M = \mathbb{R} \times KB$, and we analyze in detail several of the connected components of the classical solution space. In Section IV we demonstrate that four of these components contain points from which one recovers nondegenerate metrics on $M$. The corresponding bundles are nontrivial and orientation compatible. In particular, from one component we recover metrics for which the induced metric on $KB$ is positive definite.

Section V offers some remarks on quantization. For the orientation compatible bundles,
the action principle of Section II endows the classical solution space, after excision of certain singular subsets, with a symplectic structure. This enables us to interpret these components of the solution space as cotangent bundles (again, after excision of certain singular subsets) and to apply methods of geometric quantization [17,18]. The results are qualitatively similar to those found for the $\text{IO}_0(2, 1)$ connection theory on the manifold $\mathbf{R} \times T^2$ in Ref. [10].

Section IV contains a brief summary and discussion. Some notation and facts about $\text{IO}(2, 1)$ and the Klein bottle have been collected into appendices A and B, and certain technical details are postponed to Appendix C. In Appendix D we discuss briefly the $\text{IO}(2, 1)$ connection theory on the manifold $\mathbf{R} \times T^2$.

II. 2+1 GRAVITY IN THE $\text{IO}(2, 1)$ CONNECTION FORMULATION

In this section we describe the theory of flat connections on principal $\text{IO}(2, 1)$ bundles over a three-manifold $M$, and its correspondence to 2+1 dimensional Einstein gravity. The notation and some facts about $\text{IO}(2, 1)$ have been collected into Appendix A.

A. Kinematics

The kinematical arena of the theory consists of a (connected, paracompact, Hausdorff, $C^\infty$) differentiable three-manifold $M$, a principal $\text{IO}(2, 1)$ bundle $P$ over $M$, and a connection $A$ in $P$. In a local chart $(U_\alpha, \varphi_\alpha)$, the representative of $A$ is a one-form $\alpha A_a$ on $U_\alpha$ taking values in the Lie algebra of $\text{IO}(2, 1)$. The lowercase Latin index is understood as an abstract tensor index on $M$. In terms of the basis $\{J_I, P_I\}$ of the Lie algebra of $\text{IO}(2, 1)$ (see Appendix A), $\alpha A_a$ can be expanded as

$$\alpha A_a = \alpha e^I_a P_I + \alpha A^I_a J_I ,$$

(2.1)

where $\alpha e^I_a$ and $\alpha A^I_a$ are one-forms on $U_\alpha$. The local representative of the curvature two-form of $A$ is

$$\alpha F_{ab} = 2 \alpha D_a \alpha e^I_b P_I + \alpha F^I_{ab} J_I ,$$

(2.2)

where

$$\alpha F^I_{ab} = 2 \partial_a \alpha A^I_b + \epsilon^I_{JK} \alpha A^J_a \alpha A^K_b ,$$

(2.3)

and $\alpha D_a$ is the $A$-dependent derivative operator in $U_\alpha$ defined by

$$\alpha D_a v^I = \partial_a v^I + \epsilon^I_{JK} \alpha A^J_a v^K .$$

(2.4)

Note that $\alpha D_a$ acts on the Lorentz indices but not on the tensor indices on $M$.

If $(U_\alpha, \varphi_\alpha)$ and $(U_\beta, \varphi_\beta)$ are two local charts with nonempty $U_\alpha \cap U_\beta$, the representatives of $A$ on $U_\alpha \cap U_\beta$ are related by the gauge transformation [19]

$$\beta A_a = \text{ad}(\psi^{-1}_{\alpha\beta}) \alpha A_a + \psi^{-1}_{\alpha\beta} d\psi_{\alpha\beta} ,$$

(2.5)
where \( \psi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1} \) is the transition function. Writing \( \psi_{\alpha\beta} = (R_{\alpha\beta}, w_{\alpha\beta}) \) in the notation of Appendix A, (2.5) gives the transformations

\[
\beta A^I_a K_I = R_{\alpha\beta}^{-1} (\alpha A^I_a K_I) R_{\alpha\beta} + R_{\alpha\beta}^{-1} \partial_a R_{\alpha\beta}
\]

\[
= \det (R_{\alpha\beta}) (R_{\alpha\beta}^{-1})^I_J \alpha A^I_a K_I + R_{\alpha\beta}^{-1} \partial_a R_{\alpha\beta} ,
\]

\[
\beta e^J_a = (R_{\alpha\beta}^{-1})^I_J (\alpha e^J_a + \alpha D_a w_{\alpha\beta}) .
\]

It follows that

\[
\beta F^I_{ab} = \det (R_{\alpha\beta}) (R_{\alpha\beta}^{-1})^I_J \alpha F^J_{ab} .
\]

\( \alpha e^J_a \) is interpreted as a triad on \( U_\alpha \), and the corresponding metric on \( U_\alpha \) is \( \alpha g_{ab} = \eta_{IJ} \alpha e^I_a \alpha e^J_b \). If \( \alpha e^I_a \) is nondegenerate, \( \alpha g_{ab} \) is nondegenerate with signature \((-+++)\). The collection of metrics \( \{\alpha g_{ab}\} \) does not in itself define a metric on \( M \), since in the overlaps \( U_\alpha \cap U_\beta \) the metrics \( \alpha g_{ab} \) and \( \beta g_{ab} \) need not coincide. To proceed, we use the fact that the bundle \( P \) is reducible to an \( O(2,1) \) bundle over \( M \). This means that there exists an open cover of \( M \) and an associated system of local charts such that the transition functions take values in \( O(2,1) \); conversely, any such system of local charts defines an \( O(2,1) \) reduction of \( P \) [14]. Now, in a system of local charts corresponding to an \( O(2,1) \) reduction of \( P \), (2.6[1]) shows that the metrics \( \alpha g_{ab} \) coincide in the overlaps, and the collection \( \{\alpha g_{ab}\} \) therefore defines a (possibly degenerate) metric \( g_{ab} \) on \( M \). If \( \alpha e^I_a \) are nondegenerate for every \( \alpha \), \( g_{ab} \) is nondegenerate with signature \((-+++)\). The quantities \( \alpha A^I_a J_I \) can be interpreted as the local representatives of a connection in the reduced \( O(2,1) \) bundle, and the local representatives of the curvature of this \( O(2,1) \) connection are \( \alpha F^I_{bc} J_I \). Similarly, \( \alpha F^I_{ab} \) can be interpreted as triad fields associated with this reduced bundle.

The metric \( g_{ab} \) obtained in this fashion depends on the \( O(2,1) \) reduction of \( P \). To investigate this dependence, we shall for the remainder of this subsection denote by \( \{(U_\alpha, \varphi_\alpha)\} \) and \( \{(U_\alpha, \bar\varphi_\alpha)\} \) two systems of local charts of \( P \) corresponding to two \( O(2,1) \) reductions of \( P \). The transition functions \( \psi_{\alpha\beta} = \varphi_\alpha \varphi_\beta^{-1} \) and \( \bar\psi_{\alpha\beta} = \bar\varphi_\alpha \bar\varphi_\beta^{-1} \) thus take values in \( O(2,1) \).

Let \( \theta_\alpha = \varphi_\alpha \bar\varphi_\alpha^{-1} \) be the transition function between the charts \( (U_\alpha, \varphi_\alpha) \) and \( (U_\alpha, \bar\varphi_\alpha) \). In a nonempty \( U_\alpha \cap U_\beta \), the consistency of the transition functions implies

\[
\bar\psi_{\alpha\beta} = \theta_\alpha^{-1} \psi_{\alpha\beta} \theta_\beta .
\]

In the notation of Appendix A, we write \( \psi_{\alpha\beta} = (S_{\alpha\beta}, 0) \), \( \theta_\alpha = (R_\alpha, w_\alpha) \). As \( \bar\psi_{\alpha\beta} \) takes values in \( O(2,1) \), (2.8) implies

\[
w_\beta = S_{\alpha\beta}^{-1} w_\alpha .
\]

Let now \( \alpha e^I_a \) and \( \bar\alpha e^I_a \) be the triads associated respectively with the charts \( (U_\alpha, \varphi_\alpha) \) and \( (U_\alpha, \bar\varphi_\alpha) \), and let us assume that \( \alpha e^I_a \) are nondegenerate for every \( \alpha \). Then there exists in each \( U_\alpha \) a unique vector field \( \alpha f^a \) such that

\[1\text{This follows from Refs. }[20,21]\text{ and the observation that the coset space } IO(2,1)/O(2,1) \text{ is homeomorphic to } \mathbb{R}^3. \text{ I thank Domenico Giulini for pointing this out.} \]
In a nonempty $U_\alpha \cap U_\beta$, equations $(2.6b)$, $(2.9)$, and $(2.10)$ show that $\alpha f^a$ and $\beta f^a$ coincide, and the collection $\{ \alpha f^a \}$ thus defines a vector field $f^a$ on $M$. From $(2.6b)$ one then obtains

$$w^I_\alpha = \alpha e^I_\alpha \alpha f^a .$$

(2.10)

In a system of local charts corresponding to an $O(2,1)$, we introduce a system of local charts $(\{U_\alpha, \varphi_\alpha\})$ corresponding to an $O(2,1)$ reduction of $P$. The index set in which $\alpha$ takes values is denoted by $I$. Without loss of generality we can assume this system to be chosen such that there is an atlas $\{(U_\alpha, \sigma_\alpha)\}$ of $M$ with the local manifold charts $\sigma_\alpha : U_\alpha \to \mathbb{R}^3$. 

B. Dynamics

The dynamics of the theory consists of the statement that the connection $A$ is flat. In a local chart this amounts by $(2.2)$ to the equations of motion

$$\alpha F^I_{ab} = 0 ,$$

(2.13a)
$$\alpha D_{[a} \alpha e^I_{b]} = 0 .$$

(2.13b)

In a system of local charts corresponding to an $O(2,1)$ reduction of $P$, $(2.13a)$ says that the connection on the reduced bundle is flat, and $(2.13b)$ can then be understood as a compatibility condition for the $O(2,1)$ connection and the triad.

In the theory where $\text{IO}(2,1)$ is replaced by $\text{IO}_0(2,1)$, an $O_0(2,1)$ reduction of $P$ defines on $M$ the three-form $e^I_{[a} F^I_{bc]}$. If $M$ is orientable, integrating this three-form over $M$ yields then an action functional from which the equations of motion $(2.13)$ can be derived. We shall now show that this action can be generalized to a class of bundles within our $\text{IO}(2,1)$ theory, including certain bundles for nonorientable $M$. The idea is to define on $M$ a density analogous to $e^I_{[a} F^I_{bc]}$ by taking advantage of the factor $\det(R_{\alpha \beta})$ in $(2.7)$.

Let $P$ be a principal $\text{IO}(2,1)$ bundle over $M$. Let $P'$ be the principal $\mathbb{Z}_2$ bundle over $M$ that is obtained by collapsing the fibers in $P$ into $\mathbb{Z}_2$ with the homomorphism $\text{IO}(2,1) \to \mathbb{Z}_2; (R, w) \mapsto \det(R)$. Let $x \in M$, and let $\pi_1(M, x)$ be the homotopy group of $M$ with the base point $x$. As $\mathbb{Z}_2$ is discrete and Abelian, lifting closed paths in $M$ with the base point $x$ into paths in $P'$ defines the holonomy map $h_x : \pi_1(M, x) \to \mathbb{Z}_2$ $[19]$. By construction $h_x$ is a group homomorphism. We say that $P'$ is orientation compatible iff for every $x \in M$, $h_x$ takes the orientation preserving homotopy classes in $\pi_1(M, x)$ to 1 and the orientation reversing homotopy classes (if any) to $-1$. It is clear that an equivalent definition is to require $h_x$ to have this property for some $x \in M$. We say that $P$ is orientation compatible iff $P'$ is orientation compatible.

For the rest of this subsection we assume $P$ to be orientation compatible. As in subsection [1A], we introduce a system of local charts $(\{U_\alpha, \varphi_\alpha\})$ corresponding to an $O(2,1)$ reduction of $P$. The index set in which $\alpha$ takes values is denoted by $I$. Without loss of generality we can assume this system to be chosen such that there is an atlas $\{(U_\alpha, \sigma_\alpha)\}$ of $M$ with the local manifold charts $\sigma_\alpha : U_\alpha \to \mathbb{R}^3$. 

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Let $P'$ be the principal $\mathbb{Z}_2$ bundle over $M$ that is obtained from $P$ by collapsing the fibers into $\mathbb{Z}_2$ as above. Let $\{(U_\alpha, \varphi_\alpha')\}$ be the system of local charts of $P'$ induced by the system $\{(U_\alpha, \varphi_\alpha)\}$. We now single out one value of $\alpha$, say $\alpha = 0$, and we shall use the chart $(U_0, \varphi_0')$ and the orientation of $U_0$ induced by the manifold chart $\sigma_0$ as a reference in terms of which the action will be defined. This choice is analogous to choosing the orientation on $M$ in the theory considered in Refs. [6,13].

For every $\alpha$, choose a point $x_\alpha \in U_\alpha$. Let $p'_\alpha \in P'$ be the point in the fiber over $x_\alpha$ such that $\varphi_\alpha'(p'_\alpha) = 1$. Let $\gamma_\alpha$ be a path in $M$ starting from $x_\alpha$ and ending at $x_0$. We now define the numbers $\epsilon_\gamma$ and $\eta_\gamma$ in $\mathbb{Z}_2$ in the following way. Let $\tilde{\gamma}_\alpha$ be the lift of $\gamma_\alpha$ to $P'$ that starts from $p'_\alpha$. We set $\epsilon_{\gamma_\alpha} = 1$ if the end point of $\tilde{\gamma}_\alpha$ is $p'_0$, and $\epsilon_{\gamma_\alpha} = -1$ otherwise. We set $\eta_{\gamma_\alpha} = 1$ if $\gamma_\alpha$ takes the orientation of $U_\alpha$ induced by the manifold chart $\sigma_\alpha$ to the orientation of $U_0$ induced by the manifold chart $\sigma_0$, and $\eta_{\gamma_\alpha} = -1$ otherwise. The orientation compatibility of $P'$ guarantees that for a fixed $\alpha$, the product $\epsilon_{\gamma_\alpha} \eta_{\gamma_\alpha}$ is independent of the choice of the path $\gamma_\alpha$ and the choice of the point $x_\alpha \in U_\alpha$. The product $\epsilon_{\gamma_\alpha} \eta_{\gamma_\alpha}$ therefore defines a function $E: I \to \mathbb{Z}_2$. In particular one has $E(0) = 1$.

Let now $\mathcal{A}$ be a connection in $P$. In each $U_\alpha$, use the local chart $(U_\alpha, \varphi_\alpha)$ as in subsection 1A to define the three-form

$$\alpha \mathcal{E}_{abc} = E(\alpha) \alpha e_I [a \alpha F_{bc}^I].$$

(2.14)

Let $\alpha \mathcal{E}_{abc}$ be the components of $\alpha \mathcal{E}_{abc}$ in the manifold chart $(U_\alpha, \sigma_\alpha)$; the overlined lowercase Latin indices are concrete indices in this chart. In a nonempty $U_\alpha \cap U_\beta$, it now follows from (2.6) and (2.7) that $\alpha \mathcal{E}_{abc}$ and $\beta \mathcal{E}_{abc}$ are related by the absolute value of the Jacobian of the manifold transition function $\sigma_\alpha \sigma_\beta^{-1}$. This means that the collection $\{\alpha \mathcal{E}_{abc}\}$ defines on $M$ a density $\bar{\mathcal{E}}_{abc}$ [28]. We can thus integrate $\bar{\mathcal{E}}_{abc}$ over $M$ to obtain the action

$$S = \frac{1}{2} \int_M \bar{\mathcal{E}}.$$

(2.15)

If $M$ is noncompact, suitable fall-off conditions may need to imposed, and suitable boundary terms may need to be added to (2.15). Using the local representatives of $\bar{\mathcal{E}}_{abc}$, it is straightforward to verify that the variation of (2.14) gives, under suitable boundary conditions in the noncompact case, the equations of motion (2.13). It is also clear how this discussion generalizes if $M$ is replaced by a manifold with a boundary.

Our definition of the action (2.15) relied on the system $\{(U_\alpha, \varphi_\alpha)\}$ of local charts corresponding to an O(2,1) reduction of $P$. To see how the action depends on the choice of this system, let $\{(U_\alpha, \varphi_\alpha)\}$ be another such system as in subsection 1A. Let us first suppose that the transition function $\varphi_0'(\varphi_0')^{-1}$ is the identity. The difference of $\bar{\mathcal{E}}_{abc}$ and $\bar{\mathcal{E}}_{abc}$ is then a total divergence whose components in the manifold chart $(U_\alpha, \sigma_\alpha)$ are $\bar{E}(\alpha) \partial_{[a} (\alpha F_{bc]} w_\alpha)$; this divergence is a well-defined density by virtue of (2.7) and (2.8). Given suitable fall-off/boundary conditions, the action defined in terms of the new system of local charts is therefore equal to the old action. Finally, if $\varphi_0'(\varphi_0')^{-1}$ is not the identity, the two actions differ only by an overall sign.

If $M$ is orientable, the construction of the action simplifies considerably. In this case orientation compatibility is equivalent to the triviality of $P'$, and one can choose the local charts of $P$ so as to induce a global chart of $P'$. One can also choose $\{(U_\alpha, \sigma_\alpha)\}$ to be an
oriented atlas. The factor $E(\alpha)$ in (2.14) is then equal to 1 for all $\alpha$ and can be omitted: $\mathcal{E}_{abc}$ can be thought of just as a three-form on $M$ [23].

Finally, note that when $M = \mathbb{R} \times \Sigma$, where $\Sigma$ is a closed two-manifold, one can perform a Hamiltonian decomposition of the action (2.13), in close analogy with the decomposition in the IO$\theta(2, 1)$ theory on orientable manifolds [21, 22]. The Hamiltonian form of the action defines on the fields a symplectic structure, which can be pulled back into a symplectic structure on (smooth subsets of) the solution space. We shall discuss this explicitly in Section 3 in the case where $\Sigma$ is the Klein bottle.

C. Solutions

We shall now recall how to describe the solution space of the connection theory in terms of the fundamental group of $M$.

Let for the moment $G$ be a general Lie group, let $P$ be a principal $G$ bundle over $M$, and let $\mathcal{A}$ be a flat connection on $P$. Choose a point $x \in M$, and choose a point $p \in P$ in the fiber over $x$. As $\mathcal{A}$ is flat, lifting closed paths in $M$ with the base point $x$ into horizontal paths in $P$ starting at $p$ defines the holonomy map $\Phi_p: \pi_1(M, x) \to G$ [13], which is by construction a group homomorphism. If $q \in P$ is another point in the fiber over $x$, one can write $q = pg$ for some $g \in G$, and the holonomy maps are related by $\Phi_p g = g \Phi_p g^{-1}$. Thus, $\mathcal{A}$ defines a $G$ conjugacy class of elements in $\text{Hom}(\pi_1(M, x), G)$, that is, a point in the quotient space $\text{Hom}(\pi_1(M, x), G)/G$. Conversely, given $\Phi \in \text{Hom}(\pi_1(M, x), G)$, there exists a principal $G$ bundle $P$ over $M$ and a flat connection $\mathcal{A}$ on $P$ such that $\Phi/G$ is the conjugacy class of homomorphisms defined by the holonomy maps of $\mathcal{A}$, and the reconstruction of $P$ and $\mathcal{A}$ from $\Phi$ is unique up to isomorphisms [24–26]. Finally, given another point $x' \in M$, the holonomy maps $\Phi_{p'} \in \text{Hom}(\pi_1(M, x'), G)$ can be related to those in $\text{Hom}(\pi_1(M, x), G)$ by introducing paths $\gamma$ in $M$ from $x$ to $x'$: the conjugacy classes are related just by $\Phi'/G = (\Phi/G) \circ \Gamma_\gamma$, where $\Gamma_\gamma: \pi_1(M, x') \to \pi_1(M, x)$ is the isomorphism induced by $\gamma$.

We therefore have the following statement: The space of flat connections modulo bundle isomorphisms is parametrized by $\text{Hom}(\pi_1(M), G)/G$, where $\pi_1(M)$ is the (base point independent) fundamental group of $M$.

There are four issues that deserve a comment. Firstly, in the present case of flat connections the reconstruction [24, 25] of $P$ and $\mathcal{A}$ from $\Phi$ takes the following simple form. Let $\tilde{M}$ be the universal covering space of $M$, and let $\pi_1(M)$ denote the base point independent fundamental group of $M$. There exists an action $\rho$ of $\pi_1(M)$ on $\tilde{M}$ such that the quotient space $\tilde{M}/\rho$ is homeomorphic to $M$ [27]. We denote this action by $\tilde{x} \mapsto \rho_\alpha(\tilde{x})$, $\tilde{x} \in \tilde{M}$, $\alpha \in \pi_1(M)$. Introduce the product bundle $\tilde{P} = \tilde{M} \times G$, and let $\tilde{\mathcal{A}}$ be the flat connection on $\tilde{P}$ induced by the product structure. Given $\Phi \in \text{Hom}(\pi_1(M), G)$, define the action $\tilde{\rho}$ of $\pi_1(M)$ on $\tilde{P}$ by $(\tilde{x}, g) \mapsto (\rho_\alpha(\tilde{x}), (\Phi(\alpha))^{-1}g)$ for $\alpha \in \pi_1(M)$. It is straightforward to show that the quotient space $\tilde{P}/\tilde{\rho}$ is a principal $G$ bundle $P$ over $M$, and that $\tilde{\mathcal{A}}$ induces a flat connection $\mathcal{A}$ on $P$. To study the holonomy map of $\mathcal{A}$ in $P$, let $x \in M$, and let $\tilde{x} \in \tilde{M}$ be a

2I thank Alan Rendall, Domenico Giulini, and Don Marolf for pointing out this construction.
point in the $\rho$-equivalence class that defines $x$. The quotient construction $M = \tilde{M}/\rho$ defines an isomorphism $f_\tilde{x}: \pi_1(M, x) \to \pi_1(M)$. Let $p \in P$ be the $\tilde{\rho}$-equivalence class of $(\tilde{x}, e) \in \tilde{P}$, where $e$ is the identity element in $G$. Clearly $p$ is in the fiber over $x$. The holonomy map $\Phi_p \in \text{Hom}(\pi_1(M, x), G)$ of $A$ can then be verified to be the composition $\Phi \circ f_\tilde{x}$.

Secondly, we are here adopting the viewpoint that connections are identified only under those bundle isomorphisms whose projection to the diffeomorphism group $\text{Diff}(M)$ is in the component of the identity. In physical terms, this means treating the diffeomorphisms connected to the identity as “gauge” but the diffeomorphisms disconnected from the identity as symmetries. If one wanted to identify connections also under the bundle isomorphisms that project into the disconnected components of $\text{Diff}(M)$, one would need to take the quotient of $\text{Hom}(\pi_1(M), G)/G$ with respect to the full automorphism group of $\pi_1(M)$. By construction, $\text{Hom}(\pi_1(M), G)/G$ is already invariant under the inner automorphisms of $\pi_1(M)$ [28,29].

Thirdly, $\text{Hom}(\pi_1(M), G)/G$ consists of points arising from all $G$ bundles over $M$ that admit flat connections. For a given $M$, there may be several such bundles, in which case any single bundle only yields a subset of $\text{Hom}(\pi_1(M), G)/G$.

Fourthly, on some bundles it is natural to introduce additional structure that is not invariant under all bundle isomorphisms. To consider the case of interest for us, suppose that $G$ is not connected, and let $G_0$ be the component of the identity in $G$. Suppose now that one introduces some structure that is not invariant under those bundle isomorphisms that permute the components in the fibers. For example, one could introduce a base point $\ast$ in $M$, and fix one component in the fiber over $\ast$ to be associated with $G_0$. One can then require that solutions related by a bundle isomorphism should be regarded as equivalent only if the isomorphism leaves this structure invariant. The solution space is then $\text{Hom}(\pi_1(M), G)/G_0$. The space $\text{Hom}(\pi_1(M), G)/G$ is recovered as the quotient of $\text{Hom}(\pi_1(M), G)/G_0$ with respect to conjugation by $\pi_0(G) = G/G_0$. In physical language, adopting $\text{Hom}(\pi_1(M), G)/G_0$ as the solution space means that the gauge transformations that are disconnected from the identity are treated as symmetries rather than as gauge. The above discussion specializes readily to our version of Witten’s 2+1 gravity. Choosing to treat both the large (that is, disconnected from the identity) gauge transformations and the large diffeomorphisms of $M$ as symmetries rather than as gauge, we see that the solution space is $\text{Hom}(\pi_1(M), \text{IO}(2, 1))/\text{IO}_0(2, 1)$, where $\pi_1(M)$ is the (base point independent) fundamental group of $M$.

D. Spacetime metrics

Given a solution $\mathcal{A}$ to the connection theory on the bundle $P$, and given a local chart $(U_\alpha, \varphi_\alpha)$ such that the triad $\varphi_\alpha^I$ is nondegenerate, equations (2.13) imply that the metric $\alpha_{g_{ab}} = \eta_{IJ} \varphi_\alpha^I \varphi_\alpha^J$ on $U_\alpha$ is flat, i.e., satisfies the 2+1 dimensional vacuum Einstein equations [11,22]. If there exists a system of local charts corresponding to an $\text{O}(2, 1)$ reduction of $P$ such that $\varphi_\alpha^I$ is nondegenerate for every $\alpha$, the collection $\{\varphi_\alpha^I\}$ then defines a flat Lorentz-metric $g_{ab}$ on $M$.

By (2.12) and (2.13), an infinitesimal change in the system of local charts changes $g_{ab}$ by a Lie derivative with respect to a vector field on $M$. Conversely, it is seen from (2.14) that for any vector field $f^a$ on $M$ there exists an infinitesimal change in the $\text{O}(2, 1)$ reduction such
that the infinitesimal change in \( g_{ab} \) is just \( \mathcal{L}_f g_{ab} \). Thus, at the infinitesimal level, a solution of the connection theory that yields a spacetime metric on \( M \) yields a whole diffeomorphism equivalence class of such metrics.

In order to consider the effect of finite changes in the system of local charts on the metric, more precise assumptions are required. Progress in this direction has been made in the case \( M = \mathbb{R} \times \Sigma \), where \( \Sigma \) is a closed oriented two-manifold, under the assumption that the induced metric on \( \Sigma \) is spacelike \([6–9,12,13]\). In particular, Ref. \[9\] gives a set of assumptions that allows a precise control of the diffeomorphism equivalence classes of metrics when \((M, g_{ab})\) is assumed to be a domain of dependence of \( \Sigma \). If \( \Sigma \) may be non-spacelike, the situation appears more open. Some examples and discussion in the special case \( \Sigma = T^2 \) can be found in Ref. \[16\].

III. CONNECTION SOLUTIONS ON \( \mathbb{R} \times (\text{KLEIN BOTTLE}) \)

In this section we shall analyze the solution space of the connection theory on \( \mathbb{R} \times KB \). Some facts about \( KB \) and its fundamental group \( \pi_1(KB) := \pi \) are collected in Appendix B.

We choose to treat both the large gauge transformations and the large diffeomorphisms of \( M \) as symmetries, in the way explained in subsection II C. Since \( \pi_1(\mathbb{R} \times KB) \simeq \pi_1(KB) = \pi \), the solution space \( \mathcal{M} \) is

\[
\mathcal{M} = \text{Hom}(\pi, \text{IO}(2,1))/\text{IO}_0(2,1) \ .
\]

(3.1)

To obtain a convenient description of \( \text{Hom}(\pi, \text{IO}(2,1)) \), recall from Appendix B that \( \pi \) is generated by a pair of elements \((a,b)\) with the single relation \((B2)\). A point in \( \text{Hom}(\pi, \text{IO}(2,1)) \) is therefore uniquely specified by the images of the two generators, and the images must satisfy the relation arising from \((B2)\). Denoting the images of \( a \) and \( b \) respectively by \( A \) and \( B \), we thus obtain

\[
\text{Hom}(\pi, \text{IO}(2,1)) = \left\{ (A, B) \in \text{IO}(2,1) \times \text{IO}(2,1) \mid BABA^{-1} = E \right\} ,
\]

(3.2)

where \( E \) stands for the identity element in \( \text{IO}(2,1) \). \( \text{Hom}(\pi, \text{IO}(2,1)) \) and \( \mathcal{M} \) clearly inherit a topology and (where smooth) a differentiable structure from those of \( \text{IO}(2,1) \times \text{IO}(2,1) \).

For reasons to be explored in Section IV, we shall be mainly interested in those components of \( \text{Hom}(\pi, \text{IO}(2,1)) \) where \( A \) is either in \( \text{IO}_P(2,1) \) or \( \text{IO}_T(2,1) \), and \( B \) is either in \( \text{IO}_0(2,1) \) or \( \text{IO}_{TP}(2,1) \) (see Appendix A for the notation). As \( a \) and \( b \) are respectively orientation reversing and orientation preserving, these components are precisely the ones that arise from orientation compatible bundles over \( \mathbb{R} \times KB \). We shall therefore examine in detail only these four cases and, for the sake of contrast, the case where both \( A \) and \( B \) are in \( \text{IO}_0(2,1) \). We devote a separate subsection to each case.

A. \( \mathcal{M}_{0,0} \)

We begin by taking \( A \in \text{IO}_0(2,1) \) and \( B \in \text{IO}_0(2,1) \). This part of \( \mathcal{M} \) is

\[
\mathcal{M}_{0,0} = \text{Hom}(\pi, \text{IO}_0(2,1))/\text{IO}_0(2,1) \ ,
\]

(3.3)
where
\[ \text{Hom}(\pi, \text{IO}_0(2,1)) = \{ (A, B) \in \text{IO}_0(2,1) \times \text{IO}_0(2,1) \mid BABA^{-1} = E \} \] (3.4)

Analyzing \( \mathcal{M}_{0,0} \) is straightforward. In the notation of Appendix [A], we write \( A = (R_A, w_A) \) and \( B = (R_B, w_B) \). One way to proceed is to use the parametrization \( R_B = \exp(v^I K_I) \), in terms of which the \( \text{O}_0(2,1) \) component of the relation \( BABA^{-1} = E \) takes the more transparent form \( \exp(v^I K_I) \exp[(R_A v)^I K_I] = \mathbb{1} \). We shall now list the points in \( \mathcal{M}_{0,0} \) by giving for each point a unique representative in \( \text{Hom}(\pi, \text{IO}_0(2,1)) \) (3.4). The parameters take arbitrary values except when otherwise stated. There are seven different subsets, denoted by \( A_1 \) to \( A_7 \).

For \( A_1 \),
\[ R_A = \exp(\mu K_2) \quad w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \quad B = (\mathbb{1}, 0) \] (3.5)
where \( \mu > 0 \).

For \( A_2 \),
\[ R_A = \exp(\pm K_0 + K_2) \quad w_A = \frac{1}{2} \mu \begin{pmatrix} \mp 1 \\ 0 \\ 1 \end{pmatrix}, \quad B = (\mathbb{1}, 0) \] (3.6)
Here, and from now on everywhere with \( \pm \) signs, upper and lower signs give distinct sets of points.

For \( A_3 \),
\[ R_A = \mathbb{1} \quad B = (\mathbb{1}, 0) \]
\[ w_A = \begin{pmatrix} \tilde{b} \\ 0 \\ 0 \end{pmatrix} \text{ or } w_A = \begin{pmatrix} \pm 1 \\ 0 \\ 1 \end{pmatrix} \text{ or } w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \] (3.7)
where \( b > 0 \). The three possibilities for \( w_A \) label the \( \text{O}_0(2,1) \) conjugacy classes of timelike or zero, null nonzero, and spacelike vectors in \( M^{2+1} \).

For \( A_4 \),
\[ R_A = \exp(\tilde{\mu} K_0) \quad w_A = \begin{pmatrix} \tilde{b} \\ 0 \\ 0 \end{pmatrix}, \quad B = (\mathbb{1}, 0) \] (3.8)
where \( 0 < \tilde{\mu} < \pi \) or \( \pi < \tilde{\mu} < 2\pi \).

For \( A_5 \),
\[ R_A = \exp(\pi K_0) \quad w_A = \begin{pmatrix} \tilde{b} \\ 0 \\ 0 \end{pmatrix}, \quad R_B = \mathbb{1} \quad w_B = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix}, \] (3.9)
where \( a \geq 0 \).

For \( A_6 \),

\[
R_A = \exp(\pi K_0) , \quad w_A = 0 , \quad R_B = \exp(\lambda K_2) , \quad w_B = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} ,
\]

and

\[
(3.10)
\]

where \( \lambda > 0 \).

For \( A_7 \),

\[
R_A = \exp(\hat{\mu} K_0) , \quad w_A = \begin{pmatrix} -\hat{b} \\ 0 \\ 0 \end{pmatrix} , \quad R_B = \exp(\pi K_0) , \quad w_B = 0 ,
\]

where \( 0 \leq \hat{\mu} < 2\pi \).

We see that \( \mathcal{M}_{0,0} \) consists of two connected components, one given by \( \bigcup_{i=1}^{6} A_i \) and the other by \( A_7 \). The latter component is clearly a manifold with topology \( S^1 \times \mathbb{R} \). The former component is close to being a two-dimensional manifold but contains certain singular subsets. The projection of \( \bigcup_{i=1}^{6} A_i \) into \( \text{Hom}(\pi, O_0(2, 1))/O_0(2, 1) \) is shown in Figure 1.

To examine connections that give rise to the points in \( \mathcal{M}_{0,0} \), we envisage \( KB \) as the closed square \( Q = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1 \} \) with the boundaries identified in the manner explained in Appendix \( B \). In the coordinate patch consisting of the open square \( Q = \{(x, y) \mid 0 < x < 1, 0 < y < 1 \} \), define the one-forms

\[
A^0 = \tilde{\mu} dy , \quad e^0 = -\tilde{b} dy ,
\]

where \( \tilde{\mu} \) and \( \tilde{b} \) are constants. It is clear that these one-forms uniquely continue into smooth one-forms on \( KB \). The continued one-forms define a flat connection on the trivial principal \( IO_0(2, 1) \) bundle over \( \mathbb{R} \times KB \), and the holonomies of this connection are given by \( (3.8) \). It is straightforward to find one-forms on \( KB \) with similar properties also for the holonomies \( (3.5)–(3.7) \).

To understand the holonomies \( (3.9) \) and \( (3.10) \), it is easiest to introduce a local chart \( (\mathbb{R} \times Q) \times IO_0(2, 1) \) in which the transition function upon exiting and re-entering across the vertical boundaries \( x = 0 \) and \( x = 1 \) is still the identity, but the transition function upon exiting and re-entering across the horizontal boundaries \( y = 0 \) and \( y = 1 \) is \( (\exp(\pi K_0), 0) \). It is clear that this is a local chart in the trivial principal \( IO_0(2, 1) \) bundle over \( \mathbb{R} \times KB \). In this chart, consider the one-forms

\[
A^2 = \lambda dx , \quad e^2 = bdx ,
\]

where \( \lambda \) and \( b \) are constants. These one-forms define a flat connection on the trivial principal \( IO_0(2, 1) \) bundle over \( \mathbb{R} \times KB \), and the holonomies of this connection are given by \( (3.10) \).

In a global chart, one-forms gauge equivalent to \( (3.13) \) are given by

\[
A^0 = \pi dy , \quad A^1 = \sin(\pi y) \lambda dx , \quad A^2 = \cos(\pi y) \lambda dx ,
\]

\[
e^1 = \sin(\pi y) bdx , \quad e^2 = \cos(\pi y) bdx .
\]

\[(3.14)\]
Finding one-forms with the analogous properties for the holonomies \((3.9)\) is straightforward.

To understand the remaining component \(A_7\), consider again the one-forms on \(KB\) defined by \((3.12)\). Consider the IO\(_0(2, 1)\) bundle over \(R \times KB\) such that there exists a local chart \((R \times Q) \times \text{IO}_0(2, 1)\) in which the transition function across the horizontal boundaries \(y = 0\) and \(y = 1\) is the identity, but the transition function across the vertical boundaries \(x = 0\) and \(x = 1\) is \((\exp(\pi K_0), 0)\). It is straightforward to verify that this defines a nontrivial bundle whose Euler characteristic \(\chi\) is equal to 1. The local expressions \((3.12)\) define by continuity a flat connection on this bundle, and the holonomies of this flat connection are given by \((3.11)\).

\[\mathcal{B}. \quad \mathcal{M}_{T,0}\]

We next take \(A \in \text{IO}_T(2, 1)\) and \(B \in \text{IO}_0(2, 1)\). This part of \(\mathcal{M}\) is

\[\mathcal{M}_{T,0} = \{ (A, B) \in \text{IO}_T(2, 1) \times \text{IO}_0(2, 1) \mid BABA^{-1} = E \} / \text{IO}_0(2, 1) \quad . \quad (3.15)\]

We can proceed as with \(\mathcal{M}_{0,0}\). Writing \(R_A = -R\), where \(R \in \text{O}_0(2, 1)\), one immediately sees that the projection of \(\mathcal{M}_{T,0}\) into \(\text{Hom}(\pi, \text{O}(2, 1))/\text{O}_0(2, 1)\) is homeomorphic to the corresponding projection of \(\mathcal{M}_{0,0}\). We shall therefore divide \(\mathcal{M}_{T,0}\) into seven different subsets, denoted by \(B_1\) to \(B_7\), whose projections into \(\text{Hom}(\pi, \text{O}(2, 1))/\text{O}_0(2, 1)\) are homeomorphic to the corresponding projections of the sets \(A_1\) to \(A_7\). As before, the parameters take arbitrary values except when otherwise stated, and the parametrization is unique.

For \(B_1\),

\[R_A = -\exp(\mu K_2) \quad , \quad w_A = 0 \quad , \quad R_B = \mathbb{I} \quad , \quad w_B = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad . \quad (3.16)\]

where \(\mu > 0\).

For \(B_2\),

\[R_A = -\exp(\pm K_0 + K_2) \quad , \quad w_A = 0 \quad , \quad R_B = \mathbb{I} \quad , \quad w_B = \frac{1}{2\pi} \begin{pmatrix} \pm 1 \\ 0 \\ 1 \end{pmatrix} \quad . \quad (3.17)\]

For \(B_3\),

\[A = (-\mathbb{I}, 0) \quad , \quad R_B = \mathbb{I} \quad , \]

\[w_B = \begin{pmatrix} \tilde{a} \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad w_B = \begin{pmatrix} \pm 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad w_B = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad , \quad (3.18)\]

where \(a > 0\).

For \(B_4\),
\[ R_A = -\exp(\tilde{\mu}K_0) \ , \ w_A = 0 \ , \ R_B = \mathbb{1} \ , \ w_B = \begin{pmatrix} -\tilde{a} \\ 0 \\ 0 \end{pmatrix}, \] (3.19)

where \( 0 < \tilde{\mu} < \pi \) or \( \pi < \tilde{\mu} < 2\pi \).

For \( B_5 \),

\[ R_A = -\exp(\pi K_0) \ , \ w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \ , \ R_B = \mathbb{1} \ , \ w_B = \begin{pmatrix} -\tilde{a} \\ 0 \\ 0 \end{pmatrix}, \] (3.20)

where \( b \geq 0 \).

For \( B_6 \),

\[ R_A = -\exp(\pi K_0) \ , \ w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \ , \ R_B = \exp(\lambda K_2) \ , \ w_B = 0, \] (3.21)

where \( \lambda > 0 \).

For \( B_7 \),

\[ R_A = -\exp(\tilde{\mu}K_0) \ , \ w_A = 0 \ , \ R_B = \exp(\pi K_0) \ , \ w_B = \begin{pmatrix} -\tilde{a} \\ 0 \\ 0 \end{pmatrix}, \] (3.22)

where \( 0 \leq \tilde{\mu} < 2\pi \).

We see that \( M_{T,0} \) consists of two connected components, one given by \( \bigcup_{i=1}^{6} B_i \) and the other by \( B_7 \). They are closely analogous to the corresponding components of \( M_{0,0} \).

Connections that yield the points in \( M_{T,0} \) can again be investigated by envisaging \( KB \) as the closed square \( \overline{Q} \) and constructing the bundle in terms of a local chart associated with the open square \( Q \). The component \( \bigcup_{i=1}^{6} B_i \) comes from a bundle that admits charts in which the transition function across the vertical boundaries \( x = 0 \) and \( x = 1 \) is the identity, but the transition function across the horizontal boundaries \( y = 0 \) and \( y = 1 \) is \((\exp(\pi K_0),0)\). The component \( B_7 \) comes from a bundle in which the transition function across the horizontal boundaries is as above but the transition function across the vertical boundaries is \((\exp(\pi K_0),0)\). Expressions for the components of the connection one-form in these local charts are easily written down; for example, the one-form

\[ A^0 = \tilde{\mu}dy \ , \ e^0 = -\tilde{a}dx \] (3.23)

gives the holonomies \( B_4 \) (3.19) and \( B_7 \) (3.22) in the appropriate bundles.

C. \( M_{P,0} \)

We next take \( A \in IO_P(2,1) \) and \( B \in IO_0(2,1) \). This part of \( M \) is

\[ M_{P,0} = \{ (A,B) \in IO_P(2,1) \times IO_0(2,1) \mid BAB^{-1} = E \} / IO_0(2,1), \] (3.24)
We can proceed as above. Analyzing the conjugacy classes is now more involved than in the previous cases, and we outline in Appendix C this analysis for the $O(2,1)$ part of $\mathcal{M}_{P,0}$. $\mathcal{M}_{P,0}$ consists of five different subsets, denoted by $C_1$ to $C_5$. As before, the parameters take arbitrary values except when otherwise stated, and the parametrization is unique. We write $P = \text{diag}(1, -1, 1)$.

For $C_1$,

$$R_A = P, \ w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}, \ R_B = \exp(\lambda K_2), \ w_B = 0,$$

where $\lambda > 0$.

For $C_2$,

$$R_A = P, \ w_A = \frac{1}{2}\bar{p} \begin{pmatrix} \pm 1 \\ 0 \\ 1 \end{pmatrix}, \ R_B = \exp(\pm K_0 + K_2), \ w_B = 0.$$

For $C_3$,

$$R_A = P, \ R_B = \mathbb{I}, \ w_B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix},$$

$$w_A = \begin{pmatrix} \tilde{b} \\ 0 \\ 0 \end{pmatrix} \text{ or } w_A = \begin{pmatrix} \pm 1 \\ 0 \\ 1 \end{pmatrix} \text{ or } w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix},$$

where $b > 0$, and in addition $a \geq 0$ when $w_A$ is given by the first of the three alternatives.

For $C_4$,

$$R_A = P, \ w_A = \begin{pmatrix} -\tilde{b} \\ 0 \\ 0 \end{pmatrix}, \ R_B = \exp(\tilde{\lambda} K_0), \ w_B = 0,$$

where $0 < \tilde{\lambda} < 2\pi$.

For $C_5$,

$$R_A = P \exp(\mu K_1), \ w_A = 0, \ R_B = \mathbb{I}, \ w_B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix},$$

where $\mu > 0$.

We see that $\mathcal{M}_{P,0}$ is connected, and close to being a two dimensional manifold. The projection of $\mathcal{M}_{P,0}$ into $\text{Hom}(\pi, O(2,1))/O_0(2,1)$ is shown in Figure 2. In terms of a local chart associated with the square $Q$ as before, the bundle admits charts in which the transition function across the vertical boundaries $x = 0$ and $x = 1$ is the identity, but the transition function across the horizontal boundaries $y = 0$ and $y = 1$ is $(P,0)$. Expressions for the
components of the connection one-form in such a chart are again easily found. For future reference, we list them all here:

\( C_1 : \ A^2 = \lambda dx \ , \ e^2 = bdy \); (3.30a)
\( C_2 : \ \pm A^0 = A^2 = dx \ , \ \pm e^0 = e^2 = \frac{1}{2} pdy \); (3.30b)
\( C_3 : \ A^I = 0 \ , \ e^I = adx \)
\( e^0 = \tilde{b}dy \) or \( \pm e^0 = e^2 = dy \) or \( e^2 = bdy \); (3.30c)
\( C_4 : \ A^0 = \tilde{\lambda} dx \ , \ e^0 = -\tilde{b}dy \); (3.30d)
\( C_5 : \ A^1 = \mu dy \ , \ e^1 = adx \). (3.30e)

D. \( \mathcal{M}_{P,TP} \)

We next take \( A \in IO_P(2,1) \) and \( B \in IO_{TP}(2,1) \). This part of \( \mathcal{M} \) is

\[ \mathcal{M}_{P,TP} = \{ (A, B) \in IO_P(2,1) \times IO_{TP}(2,1) \mid BABA^{-1} = E \}/IO_0(2,1) \ . \] (3.31)

We can again proceed as above. Analyzing the conjugacy classes is now even more involved; one way to proceed is to use the results given at the end of Appendix C about the \( O_0(2,1) \) conjugacy classes in \( O_P(2,1) \). One finds that \( \mathcal{M}_{P,TP} \) consists of two subsets, which we denote by \( D_1 \) and \( D_2 \). Representatives for points in these subsets are given respectively by

\[ R_A = P \ , \ w_A = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \ , \ R_B = -P \exp(\pi K_0) \exp(\lambda K_2) \ , \ w_B = 0 \ , \] (3.32)
and

\[ R_A = P \exp(\mu K_1) \ , \ w_A = 0 \ , \ R_B = -P \ , \ w_B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \ , \] (3.33)

and the parametrizations become unique after the identifications \((\lambda, b) \sim (-\lambda, -b)\) and \((\mu, a) \sim (-\mu, -a)\).

The connected components of \( \mathcal{M}_{P,TP} \) are just \( D_1 \) and \( D_2 \). Each becomes a two-dimensional manifold if the points \( \lambda = b = 0 \) and \( \mu = a = 0 \) are respectively excised. In terms of a local chart associated with the square \( Q \) as before, the bundle corresponding to \( D_1 \) admits charts in which the transition function across the vertical boundaries \( x = 0 \) and \( x = 1 \) is \((-P \exp(\pi K_0), 0)\), whereas the transition function across the horizontal boundaries \( y = 0 \) and \( y = 1 \) is \((P, 0)\). In the bundle corresponding to \( D_2 \) the transition function across the horizontal boundaries is as above but the transition function across the vertical boundaries is \((-P, 0)\). Expressions for the components of the connection one-form in such charts are easily found: for \( D_1 \) and \( D_2 \) one has respectively

\[ A^2 = \lambda dx \ , \ e^2 = bdy \ , \] (3.34)
and

\[ A^1 = \mu dy \ , \ e^1 = adx \ . \] (3.35)
E. $\mathcal{M}_{T,TP}$

We finally take $A \in \text{IO}_T(2,1)$ and $B \in \text{IO}_{TP}(2,1)$. This part of $\mathcal{M}$ is

$$\mathcal{M}_{T,TP} = \{(A,B) \in \text{IO}_T(2,1) \times \text{IO}_{TP}(2,1) \mid BABA^{-1} = E\}/\text{IO}_0(2,1).$$

(3.36)

$\mathcal{M}_{T,TP}$ is isomorphic to $\mathcal{M}_{P,TP}$ via the map $(A,B) \mapsto (AB,B)$. In the notation of Appendix B, this isomorphism is generated by the element $(1,-1)$ of $\text{Out}(\pi) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

IV. SPACETIME METRICS ON $\mathbb{R} \times (\text{KLEIN BOTTLE})$

In this section we shall examine nondegenerate metrics that are recovered from the connection solutions. We shall show that four of the components of $\mathcal{M}$ contain points that yield such metrics.

A. $\mathcal{M}_{P,0}$

We begin with the component $\mathcal{M}_{P,0}$.

Consider first the set $C_1$ (3.25). In a local chart over the set $\mathbb{R} \times Q \subset \mathbb{R} \times KB$ defined as in subsection III C, a connection one-form giving the holonomies (3.25) is given by (3.30a). Changing the local chart by the transition function $(1, (t,0,0)^T)$, where $t$ is the coordinate on $\mathbb{R}$, $A^I$ remains unchanged but the new triad is

$$e^0 = dt, \quad e^1 = t\lambda dx, \quad e^2 = bdy.$$  

(4.1)

By assumption $\lambda > 0$. If now $b \neq 0$, squaring (4.1) gives on $(0, \infty) \times Q$ the nondegenerate metric

$$ds^2 = -dt^2 + t^2\lambda^2 dx^2 + b^2 dy^2.$$  

(4.2)

and adding charts with $O(2,1)$ transition functions to cover $(0, \infty) \times KB$ as explained in subsection III A clearly gives on $(0, \infty) \times KB$ the metric obtained by continuation of (4.2).

From the identifications (B1) in Appendix B one sees that this spacetime is constructed by taking the quotient of the region $T > |X|$ in $M^{2+1}$ with respect to the two holonomies (3.25). This spacetime is a modest generalization of one causal region of Misner space [30].

Consider then the set $C_5$ (3.29). Proceeding as above, we start from the connection one-form (3.30c), and change the local chart by $(1, (t,0,0)^T)$ to obtain the new triad

$$e^0 = dt, \quad e^1 = adx, \quad e^2 = -t\mu dy.$$  

(4.3)

By assumption $\mu > 0$. If now $a \neq 0$, one obtains as above on $(0, \infty) \times KB$ the nondegenerate metric

$$ds^2 = -dt^2 + t^2\mu^2 dy^2 + a^2 dx^2.$$  

(4.4)
The spacetime is constructed by taking the quotient of the region $T > |Y|$ in $M^{2+1}$ with respect to the two holonomies (3.29). This spacetime is therefore another generalization of one causal region of Misner space [30]. Note that it is not isometric to (4.2).

Consider next the set $C_3$, with $w_A$ given by the last alternative in (3.27). By assumption $b > 0$. If $a \neq 0$, one proceeds as above from (3.30c) to obtain on $(-\infty, \infty) \times KB$ the nondegenerate metric

$$ds^2 = -dt^2 + a^2 dx^2 + b^2 dy^2 .$$

(4.5)

The spacetime is constructed by taking the quotient of $M^{2+1}$ with respect to the two holonomies (3.27).

All the three classes of spacetimes (4.2), (4.4), and (4.5) are globally hyperbolic, with KB spacelike surfaces, and (4.5) is in addition geodesically complete. Further, these spacetimes are domains of dependence in the sense of Ref. [4]. It appears likely that the methods of Refs. [7,9,31] could be adapted to show that these three classes of spacetimes exhaust, up to isometries, all flat 2+1 spacetimes that are domains of dependence of a spacelike Klein bottle [32].

The spacetimes (4.2), (4.4), and (4.5) were obtained by taking the quotient of $M^{2+1}$ or some subset of it with respect the two IO(2,1) holonomies. This is similar to what happens in the IO(2,1) connection formulation of 2+1 gravity on $\mathbb{R} \times T^2$ [7,2,10]. We shall now show that the similarity extends further, in that all points of $M_{P,0}$, except a set of measure zero, yield nondegenerate spacetime metrics.

Let us first reconsider the set $C_5$ with $a \neq 0$. A gauge transformation of (3.30a) with the transition function $(1, (1/2(t+1), 0, 1/2(t-1))^T)$ leads to the triad

$$e^0 = \frac{1}{2} dt - \frac{1}{2} (t-1) \mu dy$$

$$e^1 = adx$$

$$e^2 = \frac{1}{2} dt - \frac{1}{2} (t+1) \mu dy$$

(4.6)

which is nondegenerate for $t > 0$. It is straightforward to verify, using the coordinate transformations given in Ref. [30] in the context of Misner space, that the metric arising from (4.6) describes the spacetime that is obtained as the quotient of the region $T > -Y$ in $M^{2+1}$ with respect to the two holonomies (3.29). The $t=constant$ Klein bottles go from timelike, for $t < 0$, via null, at $t = 0$, to spacelike, for $t > 0$.

For $C_3$ the situation is straightforward. Whenever $w_A$ and $w_B$ in (3.27) are linearly independent, one easily finds a nondegenerate co-triad such that the resulting spacetime is simply the quotient of $M^{2+1}$ with respect to the two holonomies (3.27). The Klein bottles are spacelike only in the case (4.3).

Consider then the set $C_2$. Starting from the connection one-form (3.30d) and performing a gauge transformation by the transition function $(1, (\pm 1/2 t, 0, -\pm 1/2 t)^T)$ leads to the triad

$$e^0 = \pm \frac{1}{2} p dy \pm \frac{1}{2} dt$$

$$e^1 = \pm t dx$$

$$e^2 = \frac{1}{2} p dy - \frac{1}{2} dt$$

(4.7)

If $p \neq 0$, this triad is is nondegenerate for $t > 0$. The spacetime is obtained as the quotient of the region $T > \pm Y$ in $M^{2+1}$ with respect to the holonomies (3.27).
Finally, consider the set $C_4$. Performing on (3.30d) a gauge transformation by the transition function $(1, (0, 0, t)^T)$ leads to the triad
\[
e^0 = -\tilde{b}dy \ , \ e^1 = -t\tilde{\lambda}dx \ , \ e^2 = dt . \tag{4.8}
\]
By assumption $\tilde{\lambda} > 0$. If $\tilde{b} \neq 0$, the triad (4.8) is nondegenerate for $t > 0$, and the metric is
\[
ds^2 = -\tilde{b}^2dy^2 + dt^2 + t^2\tilde{\lambda}^2dx^2 . \tag{4.9}
\]
This spacetime can be obtained in the following way. One first removes from $M^{2+1}$ the axis $X = Y = 0$ and passes to the angular coordinates $(T, \rho, \varphi)$ by
\[
X = \rho \sin \varphi \ , \ Y = -\rho \cos \varphi , \tag{4.10}
\]
where $\rho > 0$ and $\varphi$ is identified with period $2\pi$. One then passes to the universal covering space, which means dropping the periodicity of $\varphi$. Finally, one takes the quotient of this covering space with respect to the two isometries
\[
(T, \rho, \varphi) \mapsto (T - \tilde{b}, \rho, -\varphi) , \tag{4.11a}
\]
\[
(T, \rho, \varphi) \mapsto (T, \rho, \varphi + \tilde{\lambda}) . \tag{4.11b}
\]
The metric (4.9) is obtained through the coordinate transformation
\[
T = \tilde{\lambda}y , \quad \rho = t , \quad \varphi = \tilde{\lambda}x . \tag{4.12}
\]
If $\varphi$ were periodic with period $2\pi$, the isometries (4.11) would be precisely the holonomies (3.28). However, when $\varphi$ is not periodic, the holonomies (3.28) do not have a natural (associative) action for generic values of $\tilde{\lambda}$. This means that the holonomies (3.28) represent the isometries (4.11) of the covering space only in a local sense.\footnote{This point is missed in the corresponding discussion for $\mathbb{R} \times T^2$ in Ref. [16]. The statement therein that the IO$_O(2,1)$ holonomies can be used for quotienting the covering space is incorrect.}

Triads gauge-equivalent to (4.8) are obtained by adding multiples of $2\pi$ to $\tilde{\lambda}$. For generic values of $\tilde{\lambda}$, the resulting metrics are not diffeomorphic to (4.9). A similar phenomenon was noted for $\mathbb{R} \times T^2$ in Ref. [16].

Finally, note that in all the new local charts that we obtained through the gauge transformations in this subsection, the transition functions across the boundaries of $Q$ are the same as in the original local chart. Had we for example attempted to adapt the construction of the triad (1.6) to the set $C_1$, with the transition function $(1, (\frac{1}{2}(t+1), \frac{1}{2}(t-1))^T, 0)$, this would no longer have been true: the new transition function across the horizontal boundaries of $Q$ would not have been in $O(2,1)$, and the metrics across the boundary would not have agreed. This is related to the fact that the holonomies (3.29) preserve the domain $T > -Y$ in $M^{2+1}$, but the holonomies (3.28) do not preserve the domain $T > -X$.\footnote{This point is missed in the corresponding discussion for $\mathbb{R} \times T^2$ in Ref. [16]. The statement therein that the IO$_O(2,1)$ holonomies can be used for quotienting the covering space is incorrect.}
We now turn to $\mathcal{M}_{T,0}$.

Let us begin with the sets $B_4$, $B_5$, and $B_6$. It is most convenient not to use the local chart mentioned in subsection [Ill] but instead a chart in which the transition function across the vertical boundaries $x = 0$ and $x = 1$ is the identity, and the transition function across the horizontal boundaries $y = 0$ and $y = 1$ is $(-\exp(\pi K_0), 0)$. Connection one-forms for $B_4$, $B_5$, and $B_6$ in this chart are given respectively by

\begin{align*}
A^0 &= (\tilde{\mu} + \pi)dy , \quad e^0 = -\tilde{a}dx , \quad e^1 = -(\tilde{\mu} + \pi)tdy , \quad e^2 = dt , \\
A^I &= 0 , \quad e^0 = -\tilde{a}dx , \quad e^1 = dt , \quad e^2 = bdy ,
\end{align*}

(4.13)

and

\begin{align*}
A^2 &= \lambda dx , \quad e^0 = \lambda tdx , \quad e^1 = dt , \quad e^2 = bdy .
\end{align*}

(4.14)

The metric coming from (4.14) describes, for $\tilde{\alpha} \neq 0 \neq b$, the spacetime obtained as the quotient of $M^{2+1}$ with respect to the holonomies (3.20). Similarly, the metric coming from (4.13) describes, for $\lambda \neq 0 \neq b$, the spacetime obtained by taking the quotient of the Rindler wedge $X > |T|$ with respect to the holonomies (3.21). The metric coming from (4.13) describes, for $\mu + \pi \neq 0 \neq \tilde{a}$, the spacetime obtained by first cutting out the $T$-axis from $M^{2+1}$, then going to the universal covering space, and finally taking a quotient with respect to two isometries that are locally represented by the holonomies (3.19). All these spacetimes are space orientable but time-nonorientable. Again, connection one-forms gauge-equivalent to (4.13) are obtained by adding multiples of $2\pi$ to $\lambda$.

Let us next consider $B_1$. In the local chart mentioned in subsection [Ill], a connection one-form yielding the holonomies (3.16) is

\begin{align*}
A^2 &= \mu dy , \quad e^2 = adx .
\end{align*}

(4.16)

We now change the chart by the transition function $(t, (t \cos(\pi y), t \sin(\pi y), 0)^T)$. The new triad is

\begin{align*}
e^0 &= \cos(\pi y)dt + t(\mu - \pi)\sin(\pi y)dy , \\
e^1 &= \sin(\pi y)dt + t(\mu + \pi)\cos(\pi y)dy , \\
e^2 &= adx ,
\end{align*}

(4.17)

and in the new chart the transition functions across the boundaries are the same as in the old chart. Squaring the triad (4.17) gives the metric

\begin{align*}
\int s^2 &= -\cos(2\pi y)dt^2 + t^2 \left[(\mu^2 + \pi^2)\cos(2\pi y) + 2\pi \mu\right]dy^2 \\
&\quad + 2\pi \sin(2\pi y)ttdy + a^2 dx^2 .
\end{align*}

(4.18)

Recall that $\mu > 0$. From now on we assume $\mu < \pi$ and $a \neq 0$. The metric (4.18) is then nondegenerate for $t > 0$, and it clearly defines a nondegenerate metric on $\mathbb{R} \times KB$. 

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To understand (4.18), consider for the moment the metric
\[ ds^2 = -\cos(2\pi \theta)dt^2 + t^2 \left[ (\mu^2 + \pi^2) \cos(2\pi \theta) + 2\pi \mu \right] d\theta^2 
+ 2\pi \sin(2\pi \theta)tdt + a^2 d\varphi^2 , \]
where \( t > 0 \), and \( \theta \) and \( \varphi \) both take all real values. Define a transformation to the new coordinates \((U, V, Y)\) by
\[
U = 2^{1/2} t \exp(\mu \theta) \sin(\pi \theta + \pi/4) ,
V = 2^{1/2} t \exp(-\mu \theta) \cos(\pi \theta + \pi/4) ,
Y = a\varphi .
\]
(4.20)
The transformation is clearly not globally one-to-one, but in any sufficiently small interval in \( \theta \) it is one-to-one to its image. In any such interval, the metric (4.19) takes the form
\[ ds^2 = -dUdV + dY^2 , \]
(4.21) which is the 2+1 Minkowski metric in double null coordinates. Patching together the intervals in \( \theta \), we thus see that the metric (4.19) describes the space \( \tilde{B} \) that is obtained by removing from \( M_{2+1} \) a spacelike geodesic and then passing to the universal covering space. We can think of the system \((U, V, Y)\) as a set of coordinates on a single sheet of \( \tilde{B} \), with the removed geodesic being at \( U = V = 0 \). With an appropriately placed cut in the system \((U, V, Y)\), the isometry
\[
(t, \varphi, \theta) \mapsto (t, -\varphi, \theta + 1) \] (4.22a)
has the effect \((U, V, Y) \mapsto (-e^\mu U, -e^{-\mu} V, -Y)\): this is a boost in the constant \( Y \) planes with rapidity \( \mu \), followed by the inversion \((U, V, Y) \mapsto (-U, -V, -Y)\). The isometry
\[
(t, \varphi, \theta) \mapsto (t, \varphi + 1, \theta) \] (4.22b)
is a translation in \( Y \) with magnitude \( a \). Comparing (4.18) to (4.19), and recalling the identifications of the coordinates \((x, y)\), it becomes clear that (4.18) is obtained by taking the quotient of \( \tilde{B} \) with respect to the two isometries (4.22), which are locally represented by the holonomies (3.16).

Finally, let us consider \( B_3 \). When \( w_A \) is given by the last of the three alternatives in (3.18), a nondegenerate metric is obtained as the \( \mu \to 0 \) limit of (4.18). To understand this spacetime as a quotient space, it is now not necessary to pass to the covering space after removing the spacelike geodesic from \( M_{2+1} \): the spacetime is simply obtained by removing from \( M_{2+1} \) the \( Y \) axis and taking the quotient with respect to the two holonomies (3.18). It is clear how to adapt the construction to the case where \( \tilde{a} \neq 0 \) of the three alternatives in (3.18).

C. \( \mathcal{M}_{P,TP} \) and \( \mathcal{M}_{T,TP} \)

We finally turn to \( \mathcal{M}_{P,TP} \) and \( \mathcal{M}_{T,TP} \). We shall demonstrate that in the component \( D_1 \subset \mathcal{M}_{P,TP} \) (3.32) there are points that yield nondegenerate metrics. The same then also holds for the component of \( \mathcal{M}_{T,TP} \) that is isomorphic to \( D_1 \) via an outer automorphism of \( \pi \).
We start from the connection \[(3.34)\] in the local chart mentioned in subsection \[\text{III D}\], and we change the chart by the transition function \((1, (t \cos(\pi x), t \sin(\pi x), 0)^T)\). The new triad is
\[
e^0 = \cos(\pi x) dt + t(\lambda - \pi) \sin(\pi x) dx ,
\]
\[
e^1 = \sin(\pi x) dt + t(\lambda + \pi) \cos(\pi x) dx ,
\]
\[
e^2 = b dy ,
\]
and in the new chart the transition functions across the boundaries are the same as in the old chart. Squaring the triad \[(4.23)\] gives the metric
\[
ds^2 = -\cos(2\pi x) dt^2 + t^2 \left[(\lambda^2 + \pi^2) \cos(2\pi x) + 2\pi \lambda \right] dx^2
\]
\[+2\pi \sin(2\pi x) tdtdx + b^2 dy^2 .\]

For \(|\lambda| < \pi\) and \(b \neq 0\), the metric \[(4.24)\] is nondegenerate for \(t > 0\), and it clearly defines a nondegenerate metric on \(\mathbb{R} \times KB\). Comparing \[(4.24)\] with \[(4.19)\], it is seen as in subsection \[\text{IV B}\] that the spacetime described by \[(1.24)\] is obtained by taking the quotient of the space \(\tilde{B}\) with respect to two isometries. In the local null coordinates \((U,V,Y)\) on \(\tilde{B}\), one of the isometries is \((U,V,Y) \mapsto (-e^\lambda U, -e^{-\lambda} V, Y)\): this is a boost in the constant \(Y\) planes with rapidity \(\lambda\), followed by the space and time inversion in the constant \(Y\) planes. The other isometry is \((U,V,Y) \mapsto (V, U, Y + b)\), which is a space inversion in the constant \(Y\) planes followed by a translation in \(Y\). In local coordinate patches these isometries are represented by the holonomies \[(3.32)\], but globally this is true only when \(\lambda = 0\).

\section*{V. SYMPLECTIC STRUCTURE AND QUANTIZATION}

In the previous two sections we have investigated several components of the solution space \(\mathcal{M}\). In this section we shall briefly discuss the possibilities for quantizing the theory.

As was mentioned in Section \[\text{III}\], the spaces \(\mathcal{M}_{T,0}, \mathcal{M}_{P,0}, \mathcal{M}_{T,TP}\), and \(\mathcal{M}_{P,TP}\) come from orientation compatible bundles over \(\mathbb{R} \times KB\). This raises the possibility of endowing (smooth subsets of) these spaces with a symplectic structure, by first performing a Hamiltonian decomposition of the action \[(2.15)\] as in the orientable case \[(3.11)\] and then pulling the resulting symplectic structure on the fields back to a symplectic structure \(\Omega\) on the solution space. Using our explicit parametrizations, it is straightforward to verify that \(\Omega\) is nondegenerate, with the exception of certain subsets whose projections into \(\text{Hom}(\pi, O(2,1))/O_0(2,1)\) have measure zero. Avenues towards quantization can thus be explored for example via the geometric quantization techniques of Refs. \[\text{[17]}\] and \[\text{[18]}\].

For brevity, we shall concentrate on \(\mathcal{M}_{P,0}\). Analogous considerations hold for \(\mathcal{M}_{T,0}, \mathcal{M}_{T,TP}\), and \(\mathcal{M}_{P,TP}\).

The symplectic structure \(\Omega\) on (smooth subsets of) \(\mathcal{M}_{P,0}\) is readily read off by substituting the connection one-forms \[(3.30)\] into the Hamiltonian decomposition of the action. The result is:
\[
C_1 : \quad \Omega = db \wedge d\lambda ; \quad (5.1a)
\]
\[
C_2 : \quad \Omega = 0 ; \quad (5.1b)
\]
\[ C_3 : \quad \Omega = 0 \quad ; \quad (5.1c) \]
\[ C_4 : \quad \Omega = \tilde{b} \wedge d\tilde{\lambda} \quad ; \quad (5.1d) \]
\[ C_5 : \quad \Omega = -da \wedge d\mu . \quad (5.1e) \]

The Poisson brackets are thus \( \{ b,\lambda \} = \{ \tilde{b},\tilde{\lambda} \} = -\{ a,\mu \} = 1. \)

Experience with the orientable case [6,16] suggests that one could use \( \Omega \) to interpret most of \( \mathcal{M}_{P,0} \) as a cotangent bundle over the subspace where the holonomies are in \( O(2,1) \). It is immediately seen from (5.1) that this is possible for the disjoint open sets \( C_1, C_4, \) and \( C_5. \)

One can therefore quantize \( C_1, C_4, \) and \( C_5 \) in the geometric quantization framework of Refs. [17,18], directly following the treatment of the \( IO_0(2,1) \) torus theory in Ref. [16]. Borrowing terminology that has been invoked for the torus [11], one might refer to the quantum theories arising from \( C_1 \) and \( C_5 \) as two distinct “spacelike sector” theories, and to the quantum theory arising from \( C_4 \) as a “timelike sector” theory.

As \( \mathcal{M}_{P,0} \) is connected, it is natural to ask whether there exist larger quantum theories that would in some sense connect the individual quantum theories built from \( C_1, C_4, \) and \( C_5. \)

As \( C_5 \) is classically connected to \( C_1 \) and \( C_4 \) in \( \mathcal{M}_{P,0} \), only in a non-smooth manner through \( C_3 \), it is not obvious whether there is a natural larger quantum theory that would connect the quantization of \( C_5 \) to the quantizations of \( C_1 \) or \( C_4. \) However, we shall now show that there is a larger quantum theory that connects the quantizations of \( C_1 \) and \( C_4. \)

Let us start from \( C_1 \) in the parametrization (3.25), with \( \lambda > 0. \) We introduce a new parametrization by
\[
\begin{align*}
\lambda &= (2 \sinh 2r)^{1/2} , \\
b &= (2 \sinh 2r)^{1/2} \frac{p_r}{2 \cosh 2r} , 
\end{align*}
\]
where \( r > 0 \) and \( p_r \) is arbitrary. The symplectic structure (5.1a) becomes
\[ \Omega = dp_r \wedge dr . \quad (5.3) \]

Conjugating the holonomies (3.25) by the boost \( \left( \exp \left( \pm \frac{1}{2} \ln(\tanh r)K_1 \right), 0 \right) \) yields
\[
R_A = P , \quad w_A = \frac{p_r}{2 \cosh 2r} \begin{pmatrix} \pm e^{-r} & 0 \\ 0 & e^r \end{pmatrix} , \quad R_B = \exp(\pm e^{-r}K_0 + e^rK_2) , \quad w_B = 0 . \quad (5.4)
\]

Now, the holonomies (5.4) continue smoothly to \( r \leq 0. \) At \( r = 0, \) (5.4) coincides with (3.26), with the upper and lower signs matching, provided we set \( p = p_r. \) For \( r < 0, \) conjugating (5.4) by \( \left( \exp \left( \mp \frac{1}{2} \ln(\tanh r)K_1 \right), 0 \right) \) yields the holonomies (3.28), provided we set
\[
\begin{align*}
\tilde{\lambda} &= \pm(-2 \sinh 2r)^{1/2} , \\
\tilde{b} &= \mp \frac{(-2 \sinh 2r)^{1/2} p_r}{2 \cosh 2r} .
\end{align*}
\]

Because \( \tilde{\lambda} \) in (3.28) should be understood as an angular parameter, equations (5.3) need to be interpreted appropriately; for us it is sufficient to note that they have a unambiguous
meaning for $-\frac{1}{2}\text{arsinh}(\pi^2/2) < r < 0$, separately for the upper and lower signs. The expression (5.3) for the symplectic structure is valid for $-\frac{1}{2}\text{arsinh}(\pi^2/2) < r < \infty$ for both upper and lower signs, and clearly coincides with (5.1d) for $r < 0$. We have therefore constructed a set of local coordinate systems from which it is seen that the set $C_1 \cup C_2 \cup C_4$ is a smooth non-Hausdorff manifold. It can be viewed as the cotangent bundle $T^*\mathcal{B}$ over a tadpole-like non-Hausdorff base manifold $\mathcal{B}$ (Figure 3). $\mathcal{B}$ consists of the base space of $C_1$ (open half-line) and the base space of $C_4$ (open interval) glued together in a non-Hausdorff fashion by the two points that constitute the $O(2, 1)$ projection of $C_2$.

One can now quantize the cotangent bundle $T^*\mathcal{B}$ in the geometric quantization framework of Refs. [17,18], again closely following the treatment of the torus theory in Ref. [16]. In particular, the subtleties arising from the non-Hausdorff property of $\mathcal{B}$ can be handled as in Ref. [16]. The resulting quantum theory contains operators that induce transitions between the quantum theories built from $C_1$ and $C_4$.

**VI. CONCLUSIONS AND DISCUSSION**

In this paper we have investigated a connection formulation of 2+1 gravity that generalizes Witten’s formulation [6] to nonorientable three-manifolds and to the full four-component 2+1 Poincare group IO(2, 1). We first defined the theory, for a general three-manifold $M$, as a theory of flat connections in IO(2, 1) bundles over $M$, and we discussed in some detail the notion of gauge transformations and the recovery of spacetime metrics. We then defined a class of bundles as orientation compatible iff the (potential) nonorientability of $M$ intertwines with the (potential) nontriviality of the bundle in a certain way. It was shown that for orientation compatible bundles the theory has a natural action principle, which reduces to that given in Ref. [6] for IO$^0(2, 1)$ bundles over oriented manifolds.

We next specialized to $M = \mathbb{R} \times KB$, where $KB$ is the Klein bottle. We analyzed in detail several of the connected components of the solution space $\mathcal{M}$, including all the seven components that arise from orientation compatible bundles. We demonstrated that four of these seven components contain points from which one can recover nondegenerate spacetime metrics on $\mathbb{R} \times KB$. Some of these spacetimes are obtained by taking the quotient of the 2+1 Minkowski space $M^{2+1}$ under the action of the holonomies of the connection; to obtain the others, one first removes from $M^{2+1}$ a geodesic, passes to the universal covering space, and then takes the quotient with respect to certain isometries that can be associated with IO(2, 1) elements only in a local sense. In particular, from one of the connected components of $\mathcal{M}$ we recovered spacetime metrics in which the induced metric on $KB$ is positive definite.

For the orientation compatible bundles, we used the Hamiltonian decomposition of the action to define a symplectic structure on the fields as in the orientable case [11], and we then pulled this symplectic structure back to a symplectic structure on the associated components of $\mathcal{M}$. This symplectic structure allows one to interpret these components of $\mathcal{M}$, after excision of certain singular subsets, as cotangent bundles over the “base spaces” where the IO(2, 1) holonomies lie in O(2, 1). One can thus approach quantization of these components of $\mathcal{M}$ via the geometric quantization methods of Refs. [17,18]. For the component of $\mathcal{M}$ that was found to yield metrics with spacelike Klein bottles, the resulting quantum theories are closely analogous to those that were constructed for the IO$^0(2, 1)$ con-
nection theory on $\mathbb{R} \times T^2$ in Ref. [16]. Extending the terminology of Ref. [11], one recovers two distinct “spacelike sector” quantum theories and one “timelike sector” quantum theory. Further, there exists a natural larger quantum theory that incorporates both the “timelike sector” theory and one of the “spacelike sector” theories, and contains operators that induce transitions between these two smaller theories.

When defining the solution space $\mathcal{M}$, we chose to treat both the large gauge transformations and the large diffeomorphisms as symmetries rather than as gauge. Treating the large diffeomorphisms as gauge would mean taking the quotient of $\mathcal{M}$ with respect to the action of the outer automorphisms of the fundamental group of the Klein bottle; treating the large gauge transformations as gauge would mean taking the quotient of $\mathcal{M}$ with respect to conjugation by $\text{IO}(2, 1)/\text{IO}_0(2, 1)$. Either option would result into discrete identifications on $\mathcal{M}$. In the parametrizations of Section III, the identifications are straightforward to implement by restricting the ranges of the parameters. For example, in $C_1$ (3.25) either option would result into the restriction $b \geq 0$. However, these identifications are not compatible with the interpretation of certain components of $\mathcal{M}$ (including $C_1$) as cotangent bundles in the manner of Section V, since the prospective momenta then no longer take values in all of $\mathbb{R}$. Handling these identifications in the quantum theory would therefore require new input.

As the torus is a double cover of the Klein bottle (see Appendix B), there exists a natural map $\mathcal{P}: \mathcal{M} \to \mathcal{M}_{\text{torus}}$, where $\mathcal{M}_{\text{torus}}$ is the solution space of the $\text{IO}(2, 1)$ connection theory on $\mathbb{R} \times T^2$. In terms of pairs of $\text{IO}(2, 1)$ elements, in the notation of Section III, this map is given by $(A, B) \mapsto (A^2, B)$. The images of $\mathcal{M}_{0,0}$, $\mathcal{M}_{P,0}$, and $\mathcal{M}_{T,0}$ under $\mathcal{P}$ lie in the component $\mathcal{M}_{0,0}^{\text{torus}}$ that comes from the $\text{IO}_0(2, 1)$ connection theory [16]. The symplectic form on $\mathcal{M}_{0,0}^{\text{torus}}$ can therefore be pulled back with $\mathcal{P}^*$ to two-forms on $\mathcal{M}_{0,0}$, $\mathcal{M}_{P,0}$, and $\mathcal{M}_{T,0}$. For $\mathcal{M}_{P,0}$ and $\mathcal{M}_{T,0}$, this yields symplectic forms that agree, up to an overall numerical factor, with the symplectic forms obtained from our action (2.15). For $\mathcal{M}_{0,0}$, on the other hand, the resulting two-form is identically vanishing. We also see that $\mathcal{P}$ gives rise to a map that takes all our nondegenerate metrics on $\mathbb{R} \times KB$ to nondegenerate metrics on $\mathbb{R} \times T^2$. In terms of our local coordinates $(x, y)$, this map means that the coordinates become identified according to $(x, y) \sim (x + 1, y) \sim (x, y + 2)$.

In this paper we have not attempted to investigate directly a metric formulation of Einstein gravity on $\mathbb{R} \times KB$. If the induced metric on the Klein bottle is assumed to be spacelike, a metric formulation could presumably be analyzed by the methods of Refs. [7,9,31], and it appears likely that all the classical solutions would be isometric to the spacetimes (4.2), (4.4), and (4.5) [32]. If this is true, one could eliminate the supermomentum constraints by adopting a spatially locally homogeneous slicing, and one would then be led to the “minisuperspace” metric theory for $\mathbb{R} \times KB$ discussed in Refs. [33,34]. For the nonstatic solutions (4.2) and (4.4), one could further solve the remaining super-Hamiltonian constraint by adopting the York time gauge, arriving at unconstrained “square root Hamiltonian” theories as in Ref. [7]. The correspondence between metric quantization and connection quantization could then be investigated by methods that have been utilized for $\mathbb{R} \times T^2$ in Refs. [12,14,35,36].

Our method to demonstrate the existence of nondegenerate spacetime metrics was to explicitly construct such metrics from points in the solution space of the connection theory. It would be interesting to understand whether our collection of nondegenerate metrics is in
any sense an exhaustive one, and whether there is some easily characterizable property of
the connection theory that determines which connected components of the solution space
yield nondegenerate metrics. For the $\text{IO}_0(2,1)$ connection theory on $\mathbb{R} \times \Sigma$, where $\Sigma$ is
a closed orientable two-surface of genus $g > 1$, such a property is known: nondegenerate
metrics with spacelike $\Sigma$ are obtained precisely when the bundle has maximal (or minimal)
Euler class $\{1,2,3\}$.

One is prompted to ask whether, for a general three-manifold $M$, orientation compat-
ibility of the bundle might be a necessary condition for recovering nondegenerate metrics
from the connection theory. An intuitive idea suggesting an affirmative answer is that one
might expect the spacetime to be always the quotient of some domain in $M^{2+1}$ with respect
to the action of the holonomy group; this is known to be the case for $\mathbb{R} \times \Sigma$ with $\Sigma$ closed,
orientable, and spacelike $\{4\}$. However, we have found the $\text{IO}(2,1)$ connection theory on
$\mathbb{R} \times KB$ to yield certain nondegenerate metrics that come from quotienting the covering
space of a multiply connected subset of $M^{2+1}$ with respect to isometries that cannot be
globally interpreted as elements of $\text{IO}(2,1)$. In Appendix 4 we shall show that a similar
observation holds for a family of nondegenerate metrics arising from the $\text{IO}(2,1)$ theory
on $\mathbb{R} \times T^2$, such that the holonomy group is in $\text{IO}_0(2,1) \cup \text{IO}_{TP}(2,1)$ but not in $\text{IO}_0(2,1)$.
A similar observation holds already in the $\text{IO}_0(2,1)$ theory on $\mathbb{R} \times T^2$ when the tori are
not required to be spacelike $\{4\}$. What would be needed is a better understanding of the
relation between the holonomy group of the connection and the isometry groups that are
employed in the quotient constructions.

In this paper we have given an action principle only for orientation compatible bundles.
This meant that we were able to introduce a symplectic structure only on those components
of the solution space that came from orientation compatible bundles. It would be important
to understand whether this limitation could be removed. One possibility for examining this
issue might be to focus not on an action functional but directly on the classical solution
space. This avenue has been developed in Refs. $\{28,37,38\}$, where a symplectic structure
was constructed on the spaces Hom($\Pi,G)/G$, where $\Pi$ is a discrete group satisfying certain
conditions, and $G$ is a Lie group whose Lie algebra admits a symmetric nondegenerate
bilinear form that is invariant under the adjoint action of $G$. For $G = \text{IO}_0(2,1)$, a bilinear
form satisfying the hypotheses is $P^I J_I$, in the notation of Appendix $\{3\}$. When $\Pi = \pi_1(T^2) \simeq
\mathbb{Z} \times \mathbb{Z}$, it is straightforward to verify that the approach of Refs. $\{28,37,38\}$ with $P^I J_I$ yields
a symplectic structure that agrees with the one obtained from the action functional of the
$\text{IO}_0(2,1)$ theory on $\mathbb{R} \times T^2$ $\{12,16\}$. This raises the hope that the construction of Refs.
$\{28,37,38\}$ could be generalized so as to apply to the $\text{IO}(2,1)$ theory on $\mathbb{R} \times KB$. One
potential difficulty here is, however, that the although $P^I J_I$ is invariant under the adjoint
action of $\text{IO}_0(2,1)$ and $\text{IO}_{TP}(2,1)$, it changes sign under the adjoint action of $\text{IO}_{P}(2,1)$ and $\text{IO}_{TP}(2,1)$.

In conclusion, we have seen that one connected component of the $\text{IO}(2,1)$ connection
formulation of 2+1 gravity on $\mathbb{R} \times KB$ is closely analogous to the single connected com-
ponent of the $\text{IO}_0(2,1)$ connection formulation on $\mathbb{R} \times T^2$, both classically and quantum
mechanically. This connected component is arguably the most interesting one from the
viewpoint of spacetime metrics. Although the classical $\text{IO}(2,1)$ theory on $\mathbb{R} \times KB$ is in its
own right solvable in explicit form, it remains a subject to further work to fully examine the
possibilities for quantizing all the components of this theory, as well as to fully assess the
relevance of all these components for spacetime metrics.

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APPENDIX A: $\text{IO}(2,1)$

In this appendix we collect some properties of $\text{IO}(2,1)$ and establish our notation.

The 2+1 dimensional Poincare group $\text{IO}(2,1)$ can be defined as the group of pairs $(R, w)$, where $R$ is an $\text{O}(2,1)$ matrix and $w$ is a column vector with three real entries. The group multiplication law is

$$ (R_2, w_2) \cdot (R_1, w_1) = (R_2 R_1, R_2 w_1 + w_2) \ . $$

(A1)

When points in the 2+1 dimensional Minkowski space $M^{2+1}$ are represented by column vectors as $v = (T, X, Y)^T$ and the entries are the usual Minkowski coordinates associated with the line element $ds^2 = -dT^2 + dX^2 + dY^2$, the action of a group element $(R, w)$ on $M^{2+1}$ is

$$ (R, w): v \mapsto Rv + w \ . $$

(A2)

$\text{IO}(2,1)$ is the semidirect product of the Lorentz subgroup $\text{O}(2,1)$, in which the elements are of the form $(R, 0)$, and the translational subgroup, in which the elements are of the form $(I, w)$, where $I$ stands for the $3 \times 3$ identity matrix. $\text{IO}(2,1)$ consists of the component of the identity $\text{IO}_0(2,1)$, which in (A2) preserves both space and time orientation, and the disconnected components $\text{IO}_P(2,1)$, $\text{IO}_T(2,1)$, and $\text{IO}_{TP}(2,1)$, which reverse respectively the spatial orientation, time orientation, and both space and time orientation. The corresponding four components of $\text{O}(2,1)$ are denoted by $\text{O}_0(2,1)$, $\text{O}_P(2,1)$, $\text{O}_T(2,1)$, and $\text{O}_{TP}(2,1)$.

A basis for the Lie algebra of $\text{IO}(2,1)$ is provided by the elements $J_I$ and $P_I$ that can be respectively identified as the standard bases of the Lorentz and translational subalgebras. The uppercase Latin index takes values in $\{0, 1, 2\}$, and the Lie brackets are

$$ [J_I, J_J] = \epsilon^K_{I,J} J_K $$

$$ [J_I, P_J] = \epsilon^K_{I,J} P_K $$

$$ [P_I, P_J] = 0 \ , $$

(A3)

where $\epsilon^I_{J,K}$ is obtained from the totally antisymmetric symbol $\epsilon_{IJK}$ by raising the index with the 2+1 Minkowski metric $\eta_{IJ} = \text{diag}(-1, 1, 1)$. Our convention is $\epsilon_{012} = 1$. The Lorentz indices are raised and lowered with $\eta_{IJ}$ throughout the paper.
The adjoint representation of the Lie algebra of $O_0(2,1)$ is spanned by the matrices $K_j$ whose components are $(K_j)^T_L = \epsilon^T J_L$. Using the SU(1,1) parametrization of $O_0(2,1)$ $[10,11]$, it is straightforward to verify that every matrix in $O_0(2,1)$ can be written as $\exp(v' K_I)$, where $v'^I v_I \geq -\pi^2$. The only redundancy in this parametrization is that when $v'^I v_I = -\pi^2$, $v^I$ and $-v^I$ give the same element. For the use of Section $[11]$ we note the parametrizations of a rotation, a boost and a null rotation respectively as

$$\exp(v K_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos v & -\sin v \\ 0 & \sin v & \cos v \end{pmatrix}, \quad \exp(v K_2) = \begin{pmatrix} \cosh v & \sinh v & 0 \\ \sinh v & \cosh v & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\exp(v(\pm K_0 + K_2)) = \begin{pmatrix} 1 + \frac{1}{2}v^2 & v & \mp \frac{1}{2}v^2 \\ v & 1 & \mp v \\ \pm \frac{1}{2}v^2 & \pm v & 1 - \frac{1}{2}v^2 \end{pmatrix}.$$  

(A4)

APPENDIX B: KLEIN BOTTLE

The Klein bottle $KB$ can be constructed as the quotient manifold $R^2/H$, where $H$ is the group of diffeomorphisms of $R^2 = \{(x, y)\}$ generated by the two elements

\begin{align*}
a: (x, y) &\mapsto (-x, y + 1), \\
b: (x, y) &\mapsto (x + 1, y).
\end{align*}

(B1a)  

(B1b)

As $a$ reverses the orientation of $R^2$, $KB$ is nonorientable. Mapping $R^2$ into a fundamental domain, $KB$ can be visualized as the closed square $\overline{Q} = \{(x, y) \in R^2 | 0 \leq x \leq 1, 0 \leq y \leq 1\}$, with the vertical boundaries identified parallelly as $(0, y) \sim (1, y)$ and the horizontal boundaries identified antiparallelly as $(x, 0) \sim (1 - x, 1)$.

The quotient construction implies that $R^2$ is the universal covering space of $KB$, and that the fundamental group $\pi_1(KB) := \pi$ is isomorphic to $H$. We denote by $(a, b)$ a pair of generators of $\pi$ that corresponds to the pair $(\overline{a}, \overline{b})$. From (B1) we have the relation

$$aba^{-1} = e,$$

(B2)

where $e$ stands for the identity. Conversely, $\pi$ can be defined as the discrete group generated by two elements with the single relation $[\overline{a}, \overline{b}].$ $[27]$.

The elements of the automorphism group of $\pi$, $\text{Aut}(\pi)$, are labeled by the triplets $(\epsilon, \eta, n)$, where $\epsilon$ and $\eta$ take values in $Z_2$ and $n$ takes values in $Z$. The automorphisms act according to

$$(\epsilon, \eta, n): a \mapsto a'^\epsilon b^n, \quad b \mapsto b^n,$$

(B3)

and the multiplication law is $(\epsilon', \eta', n') \cdot (\epsilon, \eta, n) = (\epsilon\epsilon', \eta\eta', n + \eta' \cdot n)$. We therefore have $\text{Aut}(\pi) \simeq Z_2 \times (Z_2 \times Z)$, where $\times S$ stands for a semidirect product. The subgroup of inner automorphisms, $\text{Inn}(\pi)$, consists of the elements for which $\epsilon = 1$ and $n$ is even. The quotient group $\text{Out}(\pi) := \text{Aut}(\pi)/\text{Inn}(\pi)$ is thus isomorphic to $Z_2 \times Z_2$, and the homomorphism from $\text{Out}(\pi)$ to $\pi$ is $(\epsilon, \eta, n) \mapsto (\epsilon, \epsilon^\eta n) \in Z_2 \times Z_2$.

Taking the quotient of $R^2$ with respect to the discrete group generated by the two diffeomorphisms $\overline{a}^2$ and $\overline{b}$ gives the torus $T^2$. It follows that $T^2$ is a double cover of $KB$. 

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APPENDIX C: LORENTZ SUBSPACE OF $\mathcal{M}_{P,0}$

In this appendix we consider the subspace $m \subset \mathcal{M}_{P,0}$ in which the holonomies are in $O(2,1)$. We have

$$m = \tilde{m}/O_0(2,1) \ ,$$

where

$$\tilde{m} = \{ (R_A, R_B) \in O_P(2,1) \times O_0(2,1) \mid R_B R_A R_B^{-1} = I \} \ .$$

(C1)

(C2)

We shall give for each point in $m$ a unique representative in $\tilde{m}$. The extension of this analysis to $\mathcal{M}_{P,0}$ is straightforward.

We write $R_A = P F$, where $F \in O_0(2,1)$ and $P = \text{diag}(1,-1,1)$. We parametrize $R_B$ as $R_B = \exp(v^t I_k)$, where $v^t$ is interpreted as a Lorentz-vector. The relation $R_B R_A R_B^{-1} = I$ then takes the form

$$\exp[(Fv)^t I_k] = \exp[(Pv)^t I_k] \ .$$

(C3)

Suppose first that $v$ is spacelike, $v^t v_I > 0$. By $O_0(2,1)$ conjugation we can uniquely set $v = (0,0,\lambda)^T$, where $\lambda > 0$. Then $Pv = v$, and (C3) implies $Fv = v$. This means that $F$ is either the identity or a boost that fixes $v$. If $R$ is a boost that fixes $v$, conjugation by $R$ leaves $R_B$ invariant but sends $P F$ to $R F R^{-1} = P (PRP) FR^{-1} = PR^{-1} FR^{-1} = P F R^{-2}$. Choosing $R$ suitably one can thus set $F = I$.

Suppose next that $v$ is nonzero null, $v^t v_I = 0$ but $v \neq 0$. The situation is analogous to the previous one. By $O_0(2,1)$ conjugation we can set $v = (\pm 1, 0, 1)^T$, after which (C3) implies $Fv = v$. $F$ can then be conjugated to $I$ by a null rotation that fixes $v$.

Suppose next that $v$ is timelike, $v^t v_I < 0$. The situation is analogous to those above. By $O_0(2,1)$ conjugation we can uniquely set $v = (\tilde{\lambda}, 0, 0)^T$, where $-\pi < \tilde{\lambda} < 0$ or $0 < \tilde{\lambda} \leq \pi$. Then $Pv = v$, (C3) implies $Fv = v$, and $F$ can be conjugated to $I$ by a rotation that fixes $v$.

What remains is the case $v = 0$. The relation (C3) is now an identity. Conjugation by $R = \exp(-u^t I_k) \in O_0(2,1)$ sends $P F$ to $R F R^{-1} = P (PRP) FR^{-1} = P \exp(\mu I_k) F \exp(\mu I_k)$. This conjugation is perhaps most easily analyzed in the SU(1,1) parametrization of $O_0(2,1)$, building $R$ from infinitesimal conjugations by integration. One finds that $F$ can be uniquely conjugated to $\exp(\mu K_1)$ with $\mu \geq 0$.

APPENDIX D: IO(2,1) CONNECTION THEORY ON $\mathbf{R} \times T^2$

In this appendix we discuss briefly the connection theory with the full gauge group IO(2,1) on the manifold $\mathbf{R} \times T^2$.

As $\pi_1(T^2) \cong \mathbf{Z} \times \mathbf{Z}$, the classical solution space is parametrized by pairs of commuting IO(2,1) elements modulo overall IO$_0(2,1)$ conjugation. The component where all the holonomies are in IO$_0(2,1)$ reduces to the theory considered in Ref. [10]. Here we wish to investigate the components where the holonomies are in IO$_{TP}(2,1)$ and IO$_0(2,1)$. It is sufficient to consider the space
\[ \mathcal{M}^{\text{forms}}_{0,TP} = \{ (A, B) \in \text{IO}_0(2,1) \times \text{IO}_{TP}(2,1) \mid ABA^{-1}B^{-1} = E \} / \text{IO}_0(2,1) \]. \quad (D1) 

The space \( \mathcal{M}^{\text{forms}}_{TP,TP} \), where \( (A, B) \in \text{IO}_{TP}(2,1) \times \text{IO}_{TP}(2,1) \), is isomorphic to \( \mathcal{M}^{\text{forms}}_{0,TP} \) via a large diffeomorphism of the torus.

With the help of the results in Appendix \( \square \) about the \( O_0(2,1) \) conjugacy classes in \( O_P(2,1) \), analyzing \( \mathcal{M}^{\text{forms}}_{0,TP} \) is straightforward. \( \mathcal{M}^{\text{forms}}_{0,TP} \) consists of two connected components, which we denote by \( T_1 \) and \( T_2 \). Representatives for the points in \( T_1 \) are given by

\[ R_A = \exp(\mu K_1) \quad , \quad w_A = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \quad , \quad R_B = -P \exp(\lambda K_1) \quad , \quad w_B = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix} \] \quad (D2)

and the parametrization becomes unique after the identification \( (\lambda, \mu, a, b) \sim (\lambda, -\mu, -a, -b) \). Representatives for the points in \( T_2 \) are given by

\[ R_A = \exp(\pi K_0) \quad , \quad w_A = \begin{pmatrix} 0 \\ b \\ 0 \end{pmatrix} \quad , \quad R_B = -P \exp(\lambda K_1) \quad , \quad w_B = 0 \] \quad (D3)

and the parametrization becomes unique after the identification \( (\lambda, b) \sim (-\lambda, -b) \). The action (2.15) endows \( T_1 \) and \( T_2 \) with the symplectic structures

\[ T_1 : \quad \Omega = db \wedge d\lambda - da \wedge d\mu \] \quad (D4a)
\[ T_2 : \quad \Omega = db \wedge d\lambda \] \quad (D4b)

\( T_1 \) and \( T_2 \) can therefore each be regarded as a cotangent bundle, if the points with \( \lambda = \mu = 0 \) and \( \lambda = 0 \) are respectively excised. Methods of geometric quantization \( \square \) can thus be applied as in Ref. \[16].

We now concentrate on \( T_1 \). We envisage the torus as the closed square \( \overline{Q} = \{(u, v) \in \mathbb{R}^2 \mid 0 \leq u \leq 1, 0 \leq v \leq 1\} \), with the horizontal boundaries identified as \((u, 0) \sim (u, 1)\) and the vertical boundaries identified as \((0, v) \sim (1, v)\). In the pairs \((A, B) \in \text{IO}(2,1) \times \text{IO}(2,1)\) used in \( \square \), we identify the first member as coming from a closed loop at constant \( u \) and the second member as coming from a closed loop at constant \( v \). Consider now a bundle that admits a local chart \((\mathbb{R} \times \overline{Q}) \times \text{IO}(2,1)\) such that the transition function across the horizontal boundaries \( v = 0 \) and \( v = 1 \) is the identity but the transition function across the vertical boundaries \( u = 0 \) and \( u = 1 \) is \((-P, 0)\). In this chart, consider the connection one-form

\[ A^1 = \lambda du + \mu dv \quad , \quad e^1 = adu + bdv \] \quad (D5)

\( \square \) clearly defines on the bundle a connection whose holonomies are \( \square \).

To obtain a nondegenerate triad, we change the chart by the transition function \((\mathbb{I}, (t \cos(\pi u), -t \sin(\pi u), 0)^T)\). The new triad is

\[ e^0 = \cos(\pi u) dt + t \sin(\pi u)[(\lambda - \pi) du + \mu dv] \quad , \quad e^1 = adu + bdv \] \quad , \quad (D6)
\[ e^2 = -\sin(\pi u) dt - t \cos(\pi u)[(\lambda + \pi) du + \mu dv] \]
and in the new chart the transition functions across the boundaries are the same as in the old chart. Squaring the triad (D6) gives the metric

$$ds^2 = -\cos(2\pi u)dt^2 + 2\pi \sin(2\pi u)tdtdv + (adu + bdv)^2 + t^2 \left\{ \cos(2\pi u) \left[ (\lambda du + \mu dv)^2 + \pi^2 du^2 \right] + 2\pi (\lambda du + \mu dv)du \right\}.$$  \hfill (D7)

For $|b\lambda - a\mu| < \pi|b|$, the metric (D7) is nondegenerate for $t > 0$, and it clearly defines a nondegenerate metric on $\mathbb{R} \times T^2$. The local coordinate transformation

$$U = 2^{1/2}t \exp(\lambda u + \mu v) \sin(\pi u + \pi/4) ,$$
$$V = 2^{1/2}t \exp(-\lambda u - \mu v) \cos(\pi u + \pi/4) ,$$
$$Y = au + bv ,$$  \hfill (D8)

brings the metric (D7) to the explicitly flat double null form (4.21). It is then seen as in subsection IV B that the spacetime (D7) is obtained by cutting a spacelike geodesic from Minkowski space, going to the universal covering space, and taking the quotient with respect to two isometries that can be locally represented by the two holonomies (D2).
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* Electronic address: louko@csd.uwm.edu. Address after September 1, 1995: Department of Physics, University of Maryland, College Park, MD 20742-4111, USA.

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FIGURES

FIG. 1. The projection of the component \( \bigcup_{i=1}^{6} A_i \subseteq \mathcal{M}_{0,0} \) into \( \text{Hom}(\pi, O_0(2,1))/O_0(2,1) \). We denote the projections of \( A_i \) respectively by \( A_i' \). \( A_4' \cong \mathbb{R} \cup \mathbb{R} \) is completed into a smooth circle by the two points \( A_3' \) and \( A_5' \), at which also the two lines \( A_1' \) and \( A_6' \) join the circle. The two points that constitute \( A_2' \) are close to \( A_1' \) and \( A_3' \) in a non-Hausdorff way.

FIG. 2. The projection of \( \mathcal{M}_{P,0} \) into \( \text{Hom}(\pi, O(2,1))/O_0(2,1) \). The projections of the sets \( C_i \) are denoted respectively by \( C_i' \). \( C_1' \), \( C_4' \), and \( C_5' \) all meet at the point \( C_3' \).

FIG. 3. The tadpole-like non-Hausdorff manifold \( \mathcal{B} \), consisting of the sets \( C_1' \), \( C_2' \), and \( C_4' \) of Figure 2. One end of the open half-line \( C_1' \) is glued to both ends of the open interval \( C_1' \) by the two points that constitute \( C_2' \). \( \mathcal{B} \) is a manifold, but the two points constituting \( C_2' \) do not have disjoint neighborhoods.
