Abstract. We give a simple proof of the fact that the classical Ornstein-Uhlenbeck operator $L$ is R-sectorial of angle $\arcsin|1 - \frac{2}{p}|$ on $L^p(\mathbb{R}^n, \exp(-|x|^2/2)dx)$ (for $1 < p < \infty$). Applying the abstract holomorphic functional calculus theory of Kalton and Weis, this immediately gives a new proof of the fact that $L$ has a bounded $H^{\infty}$ functional calculus with this optimal angle.

1. Introduction

The Ornstein-Uhlenbeck operator appears in many areas of mathematics: as the number operator of quantum field theory, the analogue of the Laplacian in the Malliavin calculus, the generator of the transition semigroup associated with the simplest mean-reverting stochastic process (the Ornstein-Uhlenbeck process), or as the operator associated with the classical Dirichlet form on $\mathbb{R}^d$ equipped with the Gaussian measure $d\mu = e^{-|x|^2/2}dx$. For the sake of this paper, the Ornstein-Uhlenbeck operator will be defined via the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t>0}$ given by

**Definition 1.** For $t > 0$ and $f \in L^p(\mu)$, define $T_t f : \mathbb{R}^d \to \mathbb{C}$ as

$$x \mapsto \int_{\mathbb{R}^d} M_t(x, y) f(y) dy,$$

where $M_t : \mathbb{R}^{2d} \to \mathbb{R}$ is given by

$$(x, y) \mapsto \frac{1}{(2\pi)^d} \left( \frac{1}{1 - e^{-2t}} \right)^d \exp \left( -\frac{1}{2} \frac{|e^{-t}x - y|^2}{1 - e^{-2t}} \right),$$

the Mehler kernel.

Let us recall the basic properties of the Ornstein-Uhlenbeck semigroup used in this article.

**Theorem A.** For each $p \in [1, \infty]$ and each $t > 0$, the map $f \mapsto T_t f$ is bounded $L^p(\mu) \to L^p(\mu)$, with operator norm at most 1, and is a positive operator. For $p \in [1, \infty)$, $T_t : L^p(\mu) \to L^p(\mu)$ is a $C_0$ semigroup, i.e. as $t \to 0$, $T_t \to I$ strongly and $T_tT_s = T_{t+s}$ for all $t, s > 0$. 

\vspace{1cm}

**Date:** December 21, 2018.

1991 Mathematics Subject Classification. Primary: 47A60; Secondary: 35K08, 47F05.

Key words and phrases. Ornstein-Uhlenbeck operator, Mehler kernel, Gaussian harmonic analysis, Holomorphic functional calculus, R-sectorial.

The author gratefully acknowledges financial support by the discovery Grant DP160100941 of the Australian Research Council. This research is also supported by an Australian Government Research Training Program (RTP) Scholarship.
For a proof of these preliminary facts, see for example Theorem 2.5 of [6]. It should be noted that although the Ornstein-Uhlenbeck semigroup arises from many different areas of mathematics, these basic properties can be proven solely with use of the explicit kernel and elementary techniques. It is a simple calculation to show that \( T_t \) is bounded with norm 1 on both \( L^\infty(\mu) \) and \( L^1(\mu) \), from which interpolation can be used to deduce boundedness with norm 1 on \( L^p(\mu) \) for \( p \in [1, \infty] \). Positivity follows from non-negativity of the Mehler kernel. The \( C_0 \) nature follows as in typical proofs of the strong continuity of the classical heat semigroup, and the semigroup property follows from a tedious exercise in integrating Gaussian functions. Due to Theorem A, we can talk about the generator of the Ornstein-Uhlenbeck semigroup on \( L^p(\mu), p \in [1, \infty) \), whose negative we shall call the Ornstein-Uhlenbeck operator and denote by \( L \). Theorem 1.4 of [2], with the \( C_0 \) nature of the Ornstein-Uhlenbeck semigroup, implies that \( L \) is a closed densely-defined unbounded operator on \( L^p(\mu), p \in [1, \infty) \), which uniquely determines \( T_t \). Thus from here on, we will use the notation \( \exp(-tL) \) for the operator \( T_t \), on any of its possible domains (in arguments, \( p \) will have already been fixed so that there will be no confusion).

This paper presents a new proof of the following theorem

**Theorem 2.** For \( p \in (1, \infty) \), the Ornstein-Uhlenbeck operator has a bounded \( H^\infty(\Sigma_{\theta_p}) \) functional calculus on \( L^p(\mu) \), where \( \sin(\theta_p) = \left| 1 - \frac{2}{p} \right| \).

See [5] for the theory of \( H^\infty \) functional calculus, and note that the difficulty here is to prove the boundedness of the calculus with precisely the angle \( \theta_p \) (which is known to be best possible).

This result was originally proven by García-Cuerva, Mauceri, Meda, Sjögren and Torrea in [4]. They use Mauceri’s abstract multiplier theorem to reduce the problem to precisely estimating \( u \mapsto ||L^u|| \). To do so, they express \( L^u \) as an integral of the semigroup, using a carefully chosen contour of integration. They then consider the kernels of operators corresponding to different parts of the contour, and decompose them into a local and global part. To treat the global parts they then use a range of subtle kernel estimates.

In [1], Carbonaro and Dragicaevic reproved and extended this result to treat arbitrary generators of symmetric contraction semigroups on an \( L^p \) space. To prove this striking result, they first reduce the problem to proving a bilinear embedding for the semigroup, with constants depending optimally on the angle \( \theta_p \). They then use the Bellman function method, controlling the bilinear form by an optimally (depending on \( p \)) chosen function. This function turns out to be a known Bellman function introduced by Nazarov and Treil, but just proving that it has the right properties is a highly non-trivial task.

In contrast, the proof presented in this paper is mostly self-contained and completely transparent, requiring only simple manipulations of the kernel of the Ornstein-Uhlenbeck semigroup. It is based on an approach designed by van Neerven and Portal in [7], where they also recover classical results about the Ornstein-Uhlenbeck semigroup in a very direct manner. Their idea is to separate algebraic difficulties from analytic difficulties by considering a non-commutative functional calculus of the Gaussian position and momentum operators (the Weyl calculus). Using this calculus, one sees how to modify the kernels in a way that make their analysis straightforward. A posteriori, the use of
Optimal Holomorphic Functional Calculus for the Ornstein-Uhlenbeck Operator

The Weyl calculus can be removed, and the proof can be read as a simple computation exploiting the change of time parameter \( t \mapsto \frac{1 + e^{-t}}{1 - e^{-t}} \) (which has been used by many authors before).

We shall use the following abstract result on the \( H^\infty \) functional calculus (Theorem 10.7.13 of [5])

**Theorem B.** Let \((\Omega, m)\) be a measure space (\(\sigma\)-algebra omitted) and fix \( p \in (1, \infty) \). If an unbounded operator \( T \) on \( L^p(\Omega, m) \) generates an analytic semigroup which is a positive contraction semigroup for real time, then \( T \) is R-sectorial and \( T \) has a bounded \( H^\infty \) functional calculus of the same angle as the angle of R-sectoriality.

See [5] for the theory of R-sectoriality. Theorem 2 then follows once we have proven that \( L \) generates an analytic semigroup and is R-sectorial of angle \( \theta_p \), which we do in Theorem 5.

Throughout the paper, we make use of the following notation. The function \( \phi : \mathbb{R}^d \to \mathbb{R} \) will have action \( x \mapsto \frac{x^2}{2} \). The Borel measure \( \mu \) on \( \mathbb{R}^d \) will have density \( d\mu = e^{-\phi(x)}dx \). The Lebesgue measure on \( \mathbb{R}^d \) will be denoted by \( \lambda \). As we only ever work over \( \mathbb{R}^d \) with Borel \( \sigma \)-algebra, the measurable space over which we consider Lebesgue spaces will be dropped from notation. For \( \theta \in [0, \pi] \), we will write \( \Sigma_\theta \) for the sector \( \Sigma_\theta = \{ z \in \mathbb{C}\setminus\{0\}; \Re(z) > \cos(\theta)|z| \} \). We make use of alphabetical indexing for other’s theorems, and numerical indexing for new or re-proven results.

2. R-Sectoriality of \( L \)

To simplify things, for the rest of the article we will assume that \( p \in (1, \infty) \) is fixed. Similarly, all concepts of boundedness and R-boundedness will be on either \( L^p(\mu) \) or \( L^p(\lambda) \) without explicit mention of the space, the measure being clear from context.

**Lemma 3.** \( M_t \) has the alternate form for \( t > 0 \) and \( x, y \in \mathbb{R}^d \),

\[
M_t(x, y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{1}{1 - e^{-2t}} \right)^{\frac{d}{2}} \exp \left( -s_t \left( \frac{x + y}{\sqrt{2}} \right)^2 - \frac{1}{4s_t} \left( \frac{x - y}{\sqrt{2}} \right)^2 \right) \exp \left( \frac{1}{2} (\phi(x) - \phi(y)) \right),
\]

where \( s_t = \frac{1 - e^{-t}}{1 + e^{-t}} \).

**Proof.** We will just rearrange the exponent from Definition 1 and show that it is equal to the exponent given above for all \( x, y \in \mathbb{R}^d \) and \( t > 0 \), as that is all that has changed between the two representations. For each \( t > 0, x, y \in \mathbb{R}^d \) we have
Moreover, the supremum of the angles of sectors for which

Define the isometry

\[ R \text{-sectorial of the known optimal angle, we shall apply the following Proposition 10.3.3 of [5] } \]

Hence we need only show that the Ornstein-Uhlenbeck operator has an analytic extension to a

\[ \frac{1}{2}(x^2 - y^2) + \frac{1}{2}(\phi(x) - \phi(y)) \]

\[ = -\frac{1}{2}(1 - e^{-2t})(e^{-2t}x^2 - y^2 + \frac{1}{2}(1 - e^{-2t}) (x^2 - y^2)) + \frac{1}{2}(\phi(x) - \phi(y)) \]

\[ = -\frac{1}{8(1 - e^{-2t})}((1 - e^{-t})(x + y)^2 + (1 + e^{-t})^2(x - y)^2) + \frac{1}{2}(\phi(x) - \phi(y)) \]

\[ = -\left( \frac{1 - e^{-t}}{1 + e^{-t}} \right)^2 + \frac{1}{4} \frac{1 + e^{-t}}{1 - e^{-t}} \left( \frac{x - y}{\sqrt{2}} \right)^2 + \frac{1}{2}(\phi(x) - \phi(y)) \]

\[ = -\left( \frac{x + y}{2\sqrt{2}} \right)^2 + \frac{1}{4} \frac{x - y}{\sqrt{2}} \left( \frac{x - y}{\sqrt{2}} \right)^2 + \frac{1}{2}(\phi(x) - \phi(y)) . \]

\[ \square \]

The next definition; albeit a simple one, forms the backbone of the rest of our arguments.

**Definition 4.** Define the isometry \( U_p : L^p(\mu) \to L^p(\lambda) \) by

\[ U_p f = \left( x \mapsto f(x) \exp \left( -\frac{\phi(x)}{p} \right) \right) . \]

To get to the proof of the critical result of this paper, that the Ornstein Uhlenbeck operator is R-sectorial of the known optimal angle, we shall apply the following Proposition 10.3.3 of [5].

**Theorem C.** Let \( A \) be a linear operator on a Banach space \( X \). Then the following are equivalent.

1. \( A \) is R-sectorial of some angle \( \theta < \frac{\pi}{2} \).

2. \(-A\) is the generator of an R-bounded analytic semigroup.

Moreover, the supremum of the angles of sectors for which \( \exp(-zA) \) is R-bounded is \( \frac{\pi}{2} - \theta \).

Hence we need only show that the Ornstein-Uhlenbeck operator has an analytic extension to a sector of the correct angle, and that it is R-bounded on each smaller sector. We will in fact show a lot
more with no more effort. We shall work with the reparametrisation of the kernel of the semigroup in terms of $s_t$ from Lemma 3. The function $t \mapsto s_t$ is analytic and can clearly be extended to $\mathbb{C}\setminus i\pi(2\mathbb{Z} + 1)$. We will consider the analytic extension $z \mapsto s_z$ on domains of the form

$$E := \{z \in \mathbb{C}; s_z \in \Sigma_{\frac{1}{2} - \theta_p}; z \notin i\pi\mathbb{Z}\}$$

where $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|^2$. We will show the Ornstein-Uhlenbeck semigroup extends to an analytic semigroup on the domain $E$. Moreover, we will simultaneously show that the Ornstein-Uhlenbeck semigroup is R-bounded on sets of the form

$$E_{\epsilon, \delta} = \{z \in \mathbb{C}; |\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2 + \epsilon; \text{dist} (z, i\pi(2\mathbb{Z} + 1)) > \delta; z \notin 2i\pi\mathbb{Z}\}$$

for all $\epsilon, \delta > 0$. Note that, in terms of the reparametrisation $s_z$, these sets are just open sectors of angle $\frac{1}{2} - \theta_p$ or less, with certain points removed. We claim that $\Sigma_{\frac{1}{2} - \theta_p} \subset E$, and that for all $\epsilon' > 0$ there exists $\epsilon, \delta > 0$ such that $\Sigma_{\frac{1}{2} - \theta_p - \epsilon'} \subset E_{\epsilon, \delta}$ (see [7] for details of this calculation). These results combined will imply that the maximal domain of analyticity of the Ornstein-Uhlenbeck semigroup contains the sector $\Sigma_{\frac{1}{2} - \theta_p}$, and that it is R-bounded on each smaller sector, which combined with the quoted Theorem C will show at least that the Ornstein-Uhlenbeck operator is R-sectorial of the desired angle.

**Theorem 5.** For $p \in (1, \infty)$, the Ornstein-Uhlenbeck operator on $L^p(\mu)$ is R-sectorial of angle $\theta_p$, where $\sin(\theta_p) = M_p := \left|1 - \frac{2}{p}\right|^2$.

**Proof.** To determine (R-)boundedness of the analytic extension of $\exp(-tL)$ on $L^p(\mu)$ we conjugate by the isometry $U_p : L^p(\mu) \rightarrow L^p(\lambda)$, and work with $U_p \exp(-tL)U_p^{-1}$ on $L^p(\lambda)$. As isometries preserve (R-)boundedness, $\exp(-tL)$ has an analytic extension to $z \in \mathbb{C}$ if and only if $U_p \exp(-tL)U_p^{-1}$ does, and both families of operators will be R-bounded on the same subdomains of the domain of analyticity.

Using the integral kernel of Lemma 3 and the explicit form of the isometry $U_p$ from Definition 4, we find the integral representation for $f \in L^p(\lambda)$:

$$U_p \exp(-tL)U_p^{-1}f = \left(x \mapsto \int_{\mathbb{R}^d} k_t(x, y)f(y)dy\right),$$

with

$$k_t(x, y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1 - e^{-2t}}\right)^{\frac{d}{4}} \exp \left(-s_t \left(\frac{x + y}{2\sqrt{2}}\right)^2 - \frac{1}{4s_t} \left(\frac{x - y}{\sqrt{2}}\right)^2\right) \exp \left(\left(\frac{1}{2} - \frac{1}{p}\right)(\phi(x) - \phi(y))\right)$$

and $s_t = \frac{1}{1 - e^{-2t}}$. If $U_p \exp(-tL)U_p^{-1}$ were to have an analytic extension $U_p \exp(-zL)U_p^{-1}$ for $z$ in some domains containing $[0, \infty)$, uniqueness theory of analytic functions implies that $U_p \exp(-zL)U_p^{-1}$ would also have an integral representation, with kernel

$$k_z(x, y) = \frac{1}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{1 - e^{-2z}}\right)^{\frac{d}{4}} \exp \left(-s_z \left(\frac{x + y}{2\sqrt{2}}\right)^2 - \frac{1}{4s_z} \left(\frac{x - y}{\sqrt{2}}\right)^2\right) \exp \left(\left(\frac{1}{2} - \frac{1}{p}\right)(\phi(x) - \phi(y))\right),$$
where \( s_z = \frac{1}{\pi e^{-z}} \). We will now work on bounding this kernel. We start by assuming that \( z \in E \) (see Equation (11)). Note that this implies \( \Re(s_z) > 0 \) and \( 1 - e^{-2z} \neq 0 \). Then we have:

\[
\begin{align*}
|k_z(x, y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( -\Re(s_z) \left( \frac{x+y}{2\sqrt{2}} \right)^2 - \frac{1}{4} \Re \left( \frac{1}{s_z} \right) \left( \frac{x-y}{\sqrt{2}} \right)^2 \right) \exp \left( \left( \frac{1}{2} - \frac{1}{p} \right) (\phi(x) - \phi(y)) \right) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( -\Re(s_z) \left( \frac{x+y}{2\sqrt{2}} \right)^2 + M_p \frac{1}{4} (x^2 - y^2) - \frac{1}{4} \Re \left( \frac{1}{s_z} \right) \left( \frac{x-y}{\sqrt{2}} \right)^2 \right) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( -\Re(s_z) \left( \frac{x+y}{2\sqrt{2}} \right)^2 + M_p \frac{1}{4} \left( \frac{x+y}{2\sqrt{2}} \right) \left( \frac{x-y}{\sqrt{2}} \right) - \frac{1}{4} \Re \left( \frac{1}{s_z} \right) \left( \frac{x-y}{\sqrt{2}} \right)^2 \right)
\end{align*}
\]

For notational simplicity, let \( u = \frac{x+y}{2\sqrt{2}} \) and \( k = \frac{x-y}{\sqrt{2}} \). Then rewriting in terms of \( u \) and \( k \) and completing the square in \( u \) gives

\[
\begin{align*}
|k_z(x, y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( -\Re(s_z) u^2 + M_p u k - \frac{1}{4} \Re \left( \frac{1}{s_z} \right) k^2 \right) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( - \left( \sqrt{\Re(s_z)} u - \frac{M_p}{2\sqrt{\Re(s_z)}} k \right)^2 - \frac{1}{4} \left( \Re \left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) k^2 \right) .
\end{align*}
\]

So

\[
\begin{align*}
|k_z(x, y)| &\leq \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( - \frac{1}{4} \left( \Re \left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) k^2 \right) \\
&= \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( - \frac{1}{4} \left( \Re \left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) \left( \frac{x-y}{\sqrt{2}} \right)^2 \right) .
\end{align*}
\]

Let \( g_z : \mathbb{R}^d \to \mathbb{R} \) be the mapping

\[
\begin{align*}
x &\mapsto \frac{1}{(2\pi)^{\frac{d}{2}}} \left| \frac{1}{1 - e^{-2z}} \right| \frac{d}{4} \exp \left( - \frac{1}{8} \left( \Re \left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} \right) x^2 \right) .
\end{align*}
\]

Then we have that for all \( z \in E, f \in L^p(\lambda) \) and a.e. \( x \in \mathbb{R}^d \)

\[
\left| (U_p \exp(-tL)U_p^{-1} f) (x) \right| \leq (g_z * |f|)(x)
\]

Therefore, provided the family of convolution operators \( f \in L^p(\lambda) \mapsto g_z * f \) is (R-)bounded for \( z \) in (a subset of) \( E \), we will have proven, by domination and isometry, that \( \exp(-zL) \) is (R-)bounded on (the same subset of) \( E \) (to see that domination implies R-boundedness, see Proposition 8.1.10)
of \([5]\), and note that in the proof of said proposition the fixed positive operator can be replaced by an R-bounded family of positive operators. For \(z \in E\), we find
\[
\Re \left( \frac{1}{s_z} \right) - \frac{M_p^2}{\Re(s_z)} = \frac{\Re(s_z)}{|s_z|^2} - \frac{M_p^2}{\Re(s_z)} = \frac{1}{\Re(s_z)} \left( \frac{\Re(s_z)^2}{|s_z|^2} - M_p^2 \right)
\]
thus, since \(\Re(s_z) > 0\) and \(|\Re(s_z)|^2/|s_z|^2 = \cos^2(\arg(s_z)) > M_p^2\) by definition of \(E\) (since \(\cos \left( \frac{z}{T} - \theta_p \right) = \sin(\theta_p) = M_p\)). So for \(z \in E\), \(g_z \in L^1(\lambda)\) and so by Young’s convolution inequality, convolution by \(g_z\) is a bounded operator on \(L^p(\lambda)\) with operator norm at most \(||g_z||_{L^1(\lambda)}\). Now we will focus on sets of the form \(E_{\epsilon, \delta}\) for some fixed \(\epsilon, \delta > 0\) (see Equation (2)). We will show that
\[
\sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} \sup_{|y| > |x|} |g_z(y)| \, dx < \infty,
\]
from which we can apply Proposition 8.2.3 of \([5]\) to find that the family of convolution operators \(\{g_z\}_{z \in E_{\epsilon, \delta}}\) is R-bounded on \(L^p(\lambda)\). Noting that each \(g_z\) is radially decaying and positive, the quantity to bound is
\[
\sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} \sup_{|y| > |x|} |g_z(y)| \, dx = \sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} g_z(x) \, dx
\]
\[
= \sup_{z \in E_{\epsilon, \delta}} \int_{\mathbb{R}^d} \left( \frac{1}{2\pi} \right)^\frac{d}{2} \frac{1}{1 - e^{-2z_0}} \left[ \exp \left( -\frac{1}{8} \left( \frac{\Re(1/s_z)}{\Re(s_z)} - \frac{M_p^2}{\Re(s_z)} \right)^2 x^2 \right) \right] \, dx
\]
\[
\leq \sup_{z \in E_{\epsilon, \delta}} 2^d \left( \frac{1}{1 - e^{-2z_0}} \right) \left( \frac{\epsilon}{\Re(s_z)} \right)^{-\frac{d}{2}}
\]
\[
\leq \sup_{z \in E_{\epsilon, \delta}} e^{-\frac{d}{2} 2^d} \left( \frac{|s_z|}{1 - e^{-2z_0}} \right)^{\frac{d}{2}}
\]
\[
\leq \sup_{z \in E_{\epsilon, \delta}} e^{-\frac{d}{2} 2^d} \left( \frac{1 - e^{-z}}{1 + e^{-z}} \right)^{\frac{d}{2}}
\]
\[
= \sup_{z \in E_{\epsilon, \delta}} e^{-\frac{d}{2} 2^d} \left( \frac{1}{1 + e^{-z}} \right)^d
\]
\[
< \infty
\]
since \(z\) is bounded away from \((2\pi + 1)i\pi\). So the family of convolution operators \(\{g_z\}_{z \in E_{\epsilon, \delta}}\) is R-bounded. By pointwise domination, \(U_p \exp(-zL)U_p^{-1}\) is bounded for \(z \in E\), and is R-bounded.
on subsets $E_{\epsilon,\delta} \subset E$ of the form \([2]\). Hence by isometric equivalence, $\exp(-zL)$ shares the same properties. Hence the claim follows from the discussion precluding this proof. \(\square\)

References

[1] Andrea Carbonaro and Oliver Dragičević. Functional calculus for generators of symmetric contraction semigroups. *Duke Math. J.*, 166(5):937–974, 2017.

[2] Klaus-Jochen Engel and Rainer Nagel. *One-parameter Semigroups for Linear Evolution Equations*. Springer-Verlag, New York, 2000.

[3] Jay B. Epperson. The hypercontractive approach to exactly bounding an operator with complex Gaussian kernel. *J. Funct. Anal.*, 87(1):1–30, 1989.

[4] José García-Cuerva, Giancarlo Mauceri, Stefano Meda, Peter Sjögren, and José Luis Torrea. Functional calculus for the Ornstein-Uhlenbeck operator. *J. Funct. Anal.*, 183(2):413–450, 2001.

[5] Tuomas Hytönen, Jan van Neerven, Mark Veraar, and Lutz Weis. *Analysis in Banach Spaces Volume II: Probabilistic Methods and Operator Theory*. Springer International Publishing, 2017.

[6] Wilfredo Urbina-Romero. *Gaussian Harmonic Analysis*. Springer International Publishing, 2019.

[7] Jan van Neerven and Pierre Portal. Weyl calculus with respect to the Gaussian measure and restricted $L^p - L^q$ boundedness of the Ornstein-Uhlenbeck semigroup in complex time. *arXiv:1702.03602*, 2018.

Hanna Neumann Building #145, Science Road The Australian National University Canberra ACT 2601.

E-mail address: Sean.Harris@anu.edu.au