Collective 1/f fluctuation by pseudo-Casimir-invariants

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In this study, we propose a universal scenario explaining the 1/f fluctuation, including pink noises, in Hamiltonian dynamical systems with many degrees of freedom under long-range interaction. In the thermodynamic limit, the dynamics of such systems can be described by the Vlasov equation, which has an infinite number of Casimir invariants. In a finite system, they become pseudo-invariants, which yield quasistationary states. The dynamics then exhibit slow motion over them, up to the timescale where the pseudo-Casimir-invariants are effective. Such long-time correlation leads to 1/f fluctuations of collective variables, as is confirmed by direct numerical simulations. The universality of this collective 1/f fluctuation is demonstrated by taking a variety of Hamiltonians and changing the range of interaction and number of particles.

Introduction: The 1/f fluctuation is ubiquitous in nature: Certain quantities that fluctuate in time exhibit a power spectrum of the form $1/f^\nu$, where $f$ is the frequency and the exponent $\nu$ is typically in the range of $1/2 \lesssim \nu \lesssim 3/2$. The 1/f fluctuation, suggesting a long-time correlation, is observed in nature such as vacuum tubes [1], semiconductors [2], spin transport [3], oceans [4,5], quasars [6], solar wind [7,8], and proteins [9]. The 1/f fluctuation is also observed in model systems of water molecules [16,11], proteins [12], ferromagnetic bodies [13], and accretion disks [14]. (See, for instance, [15–19] for reviews.)

There have been multiple efforts to understand the mechanism of the 1/f fluctuation; however, a coherent explanation remains to be lacking. For instance, the superposition of Lorentzians (see [16,19], for instance) requires a certain distribution of multiple timescales, but the origin of multiple timescales must be explained. Our strategy here is to restrict our concern to Hamiltonian dynamical systems with many degrees of freedom and search for the possibility of 1/f spectra for collective variables, as observed in water molecules [10] and ferromagnetic bodies [13].

In Hamiltonian systems with a few degrees of freedom, the 1/f fluctuation has been observed and analyzed by the hierarchical structure of the phase space constructed by the KAM tori and chaotic sea [20,21]. The hierarchical structure is analyzed by perturbation theory in two-degrees-of-freedom systems [27] and also observed in a symplectic coupled map with 4 particles [28]. Nevertheless, such microscopic hierarchical structures exist only in a certain range for a system with a few degrees of freedom and is thus not generic in systems with many degrees of freedom. Hence, it remains to be elucidated how the 1/f fluctuation is generated in the collective motion of many-degree-of-freedom systems.

The aim of this article is to propose a universal scenario for the collective 1/f fluctuation in long-range Hamiltonian systems with many degrees of freedom. The target class of systems includes self-gravitating systems, plasmas, geophysical flows, trapped ions [29], among others [30,31]. In the thermodynamic limit with an infinite number of particles, the dynamics of such systems can be described by the Vlasov equation, the partial differential equation for the one-particle distribution function [33,34]. This equation is described by the distribution function on the one-particle phase space; therefore, the collective motion is naturally treated through the Vlasov equation.

An important feature of Vlasov dynamics is that they have an infinite number of Casimir invariants. The Casimir-invariants are exact in the thermodynamic limit, but in a system with a finite number of particles, they become pseudo-invariants and fluctuate slowly with time. These pseudo-Casimir-invariants play the role of constraints up to a certain timescale, but they break down in the long timescale, as has also been observed recently in the two-step relaxation of fluctuation amplitude in thermal equilibrium [13]. Such slow motion is owing to the pseudo-Casimir-invariants and one may expect a long time correlation in the dynamics of collective variables, i.e., the collective 1/f fluctuation. In the present article, we numerically demonstrate that this is indeed true and propose a general scenario for the collective 1/f fluctuation that persists up to the timescale where the constraint by pseudo-Casimir-invariants is effective.

Model: We consider the $\alpha$-Hamiltonian mean-field ($\alpha$-HMF) model [36], which is described by the Hamiltonian

$$H_{\alpha}(q,p) = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \frac{1}{2N_\alpha} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \frac{1 - \cos(q_j - q_k)}{r_{jk}^\alpha}, \quad (1)$$

where $\alpha \geq 0$. This system represents XY spins, each of which is located at a one-dimensional lattice point with a unit lattice spacing. The variable $q_j$ denotes the phase of the $j$th particle, and $p_j$ is the conjugate momentum. A quantum version of this system can be experimentally realized through trapped ions [29]. Here, the spatial boundary condition is set to be periodic, and accordingly, the distance between the $j$th and $k$th particles is defined as $r_{jk} = \min(|j - k|, N - |j - k|)$ for $k \neq j$ and $r_{jk} = 1$ for $k = j$. The normalization factor $N_\alpha$ is introduced to ensure the extensivity of energy, as is defined by $N_\alpha = \sum_{k=0}^{N-1} 1/r_{jk}^\alpha$. By taking $\alpha = 0$ with $N_0 = N$, the $\alpha$-HMF model is reduced to the

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Hamiltonian mean-field (HMF) model\textsuperscript{37, 38}. In the opposite limit, \( \alpha \to \infty \), it can be reduced to the model with the nearest-neighbor couplings with \( N_{\infty} = 3 \). Hence, the dependence on the coupling ranging from long to short is investigated by varying the value of \( \alpha \).

Here, we investigate the order parameter defined by \( M = \sum_{j=0}^{N-1} (\cos q_j, \sin q_j) / N \). The \( \alpha \)-HMF model shows the second-order phase transition at the specific energy \( E_c = 3/4 \), which corresponds to the critical temperature \( T_c = 1/2 \), irrespective of the value of \( \alpha \) for \( 0 \leq \alpha < 1 \)\textsuperscript{39–41}. The boundary between the long- and short-range is given by \( \alpha = 1 \), beyond which the mean-field description is not valid. We introduce the scaling \( x = j / N \) such that the domain of \( x \) is restricted to \( x \in [-1/2, 1/2] \) because of the periodic boundary condition.

In the limit \( N \to \infty \), the dynamics of the \( \alpha \)-HMF model are described by the Vlasov equation\textsuperscript{42}:

\[
\frac{\partial F}{\partial t} + \frac{\partial \mathcal{H}[F]}{\partial p} \frac{\partial F}{\partial q} - \frac{\partial \mathcal{V}[F]}{\partial q} = 0, \tag{2}
\]

where \( F(q, p, x, t) \) is the one-particle distribution function, \( \mathcal{H}[F](q, p, x, t) \) is the one-particle Hamiltonian functional defined by

\[
\mathcal{H}[F] = \frac{p^2}{2} + \mathcal{V}[F](q, x, t),
\]

\[
\mathcal{V}[F] = -\frac{1}{\kappa_\alpha} \int_{-1/2}^{1/2} dx' \int_{-\pi}^{\pi} dq' \int_{-\infty}^{\infty} dp' \frac{\cos(q - q')}{|x - x'|^\alpha} F(q', p', x', t), \tag{3}
\]

and \( \kappa_\alpha = \int_{-1/2}^{1/2} (1/|x|^\alpha) dx \). From the Poisson structure of the Vlasov equation, it is easy to show that a Casimir functional,

\[
\mathcal{C}[F](t) = \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp c(F(q, p, x, t)) \tag{4}
\]

is a constant of motion for any differentiable function \( c \) if \( c(F) \to 0 \) in \( |p| \to \infty \) (see the Appendix\textsuperscript{43}). We stress that the validity of the Vlasov description is guaranteed for \( \alpha < 1 \), as the integral of \( \kappa_\alpha \) does not converge for \( \alpha \geq 1 \)\textsuperscript{44}.

**Numerical tests and results:** The initial values of \( (|q_k|, p_k) )_{k=0}^{N-1} \) are randomly chosen from the one-particle distribution function in thermal equilibrium:

\[
F_{eq}(q, p; T, M) = A \exp \left[ -\left( \frac{p^2}{2} - M \cos q \right) / T \right], \tag{5}
\]

where \( A \) is the normalization factor, and \( T \) is temperature. From the rotational symmetry of the system, the direction of the order parameter is set to \( M = (M, 0) \). The magnetization \( M \) is determined for a given \( T \) by solving the self-consistent equation: \( M = \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp F_{eq}(q, p; T, M) \cos q \).

The temperature \( T \) is introduced to parameterize the set of thermal equilibria. All numerical simulations are performed by integrating the canonical equations of motion associated with the \( N \)-body Hamiltonian\textsuperscript{40} without thermal noise by using the fourth-order sympletic integrator\textsuperscript{46} with the time step \( \Delta t = 0.1 \).

We recall that the Casimir-invariants hold under conditions of (i) thermodynamic limit \( N \to \infty \) and (ii) long-range interaction \( (\alpha < 1) \). For large but finite \( N \), the Casimir-invariants are no longer invariants; they fluctuate with time, thus becoming pseudoinvariants. To unveil the possible relationship between the pseudo-Casimir-invariants and \( 1/f \) fluctuation, we first numerically examine the \( N \) dependence and \( \alpha \) dependence in the low- and high-energy sides of the \( \alpha \)-HMF model, respectively.

In the low-energy ordered phase (\( T < T_c \)), the relationship is examined through the \( N \) dependence by fixing the parameter \( \alpha = 0 \) (the HMF model) for simplicity. The power spectra of \( M^2(t) \) are presented in Fig. 1(a) for several values of \( N \), which exhibits \( 1/f \) fluctuations down to a certain frequency \( \tau^{-1} \). From the figure, this maximum timescale is estimated as \( \tau \sim 200N \), suggesting that the \( 1/f \) fluctuation persists to small frequencies, in the larger \( N \). The timescale \( \tau \sim N \) is consistent with the fact that the collision term of the product of two \( O(1/\sqrt{N}) \) terms is added to the Vlasov equation\textsuperscript{40} for finite \( N \), to break the Casimir
invariants. The power spectra of individual $\cos q_j(t)$ do not exhibit $1/f$ spectra (see the Appendix B), and the observed $1/f$ spectra are caused by collective motion. For a crossover of $1/f$ fluctuation between large and small $N$, see the Appendix C.

To quantitatively reveal the timescale where the constraint by the pseudo-Casimir-invariants is effective, we also compute the time-dependent variance $V(t)$ defined by

$$V(t) = \sum_{n=0}^{n_0-1} \frac{1}{n_0} \left[ \frac{1}{\ell} \sum_{k=1}^{\ell} \|M(nt+k)\|^2 - \left( \frac{1}{\ell} \sum_{k=1}^{\ell} \|M(nt+k)\| \right)^2 \right]$$  \hspace{1cm} (6)

where $\ell = 2, 4, 8 \cdots$ and $n_0$ is a sufficiently large number. The variance $V$ is scaled as $NV/T$, which represents the susceptibility at the thermal equilibrium level, according to the fluctuation-response relation [13]. The temporal evolution of the scaled variance is shown in Fig. 1(b) with indications of the timescale $\tau = 200N$ around which the variance reaches the asymptotic level.

From Fig. 1(b), we can understand that the constraint by the pseudo-Casimir-invariants is effective up to the timescale $\tau$. The small collision term is negligible in the short-time regime and the fluctuation is restricted in the initial Casimir level set, which is an iso-Casimir surface in function space, up to the end of the plateau [47]. The plateau level is theoretically predicted using linear response theory for the Vlasov dynamics [48, 49] under the conservation of the Casimir invariants [50]. With time, the collision term becomes non-negligible, altering the Casimir invariants and moving to another level set. The state is trapped on the new level set (although the next plateau is invisible in the figure because of the logarithmic axis). The change in level sets continues and the cumulative variance [6] thus slowly increases, whereas the suppression by the pseudo-Casimir-invariants remains effective. We underline that the plateau is not perfectly flat, where a perfect plateau suggests that the Casimir invariants are exact as in the limit $N \rightarrow \infty$. The arrival to the asymptotic level at $\tau$ suggests that the state has traveled over the possible level sets and the constraint by the pseudo-Casimir-invariants is no longer effective for timescales larger than $\tau$ [51].

In the high-energy disordered phase ($T > T_c$), linear response theory predicts that the variance on a Casimir level set agrees with the thermal equilibrium level [48, 52], and, accordingly, no two-step relaxation of $V(t)$ appears (see the Appendix D). In contrast, the disordered phase exists for any value of $\alpha$. We, therefore, change the strategy and investigate the dependence of the range of interaction $\alpha$ by choosing the initial condition from thermal equilibrium [5] by setting $M = 0$ for $T = 0.6 (> T_c)$. The power spectra of $M^2(t)$ for $T = 0.6$ are presented in Fig. 2 for $N = 32, 1024$, and 8192. As we expected, the $1/f$ spectrum tends to disappear as $\alpha$ increases, i.e., as the interaction range becomes shorter. Moreover, this tendency is enhanced by increasing $N$ and the exponent $\nu$ goes to 0 in $\alpha > 1$ as reported in Fig. 2(d). Note that the collective motion is again necessary for observing the $1/f$ spectrum (see the Appendix E). At the low-energy end, a similar figure with Fig. 2 has the same tendency but does not coincide exactly (see the Appendix H).

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**FIG. 2.** (color online) Power spectra in the $\alpha$-HMF model. $T = 0.6$. (a) $N = 32$ (100). (b) $N = 1024$ (100). (c) $N = 8192$ (60). The number in parentheses represents the number of sample orbits. $\alpha = 0, 0.5, 0.8, 1, 1.5, 2$, and 5 from top to bottom. For graphical reasons, the vertical scale has been modified suitably. (d) $\alpha$ dependence of the exponent $\nu$ (minus of the slope), which is computed by the least-square method between the two vertical lines of the panels (a)-(c).
Finally, we investigate whether the existence of phase transition in the \( \alpha \)-HMF model is essential to the 1/\( \nu \) fluctuation. To this end, we introduce a globally coupled Fermi-Pasta-Ulam-Tsingou (FPUT) system [53]:

\[
H_{\text{FPUT}} = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \varphi(q_j - q_k), \quad \varphi(q) = \frac{a^2}{2} + \frac{a^4}{4},
\]

which does not exhibit the phase transition. The collective observables that are naïvely expected, such as the variance of \( q \) or the potential energy, do not exhibit the 1/\( \nu \) fluctuation (see the Appendix H). In contrast, the power spectrum of the collective variable \( \Phi^2(t) \) exhibits 1/\( \nu \) fluctuation as shown in Fig. 3 where \( \Phi = \sum_j (\cos \phi_j, \sin \phi_j) / N \) and \( \phi \) is defined on the \((q,p)\) plane as \((q,p) = (r \cos \phi, r \sin \phi)\) with \( r = \sqrt{q^2 + p^2} \). To understand this result, we note that on one Casimir level set, fluctuations should occur on each iso-action curve, whereas the angle fluctuation will be experimentally verifiable. In real systems, the Poisson structure providing the Casimir invariants will be disturbed by dissipation and randomness. It will be important to examine the robustness of our scenario under such perturbations. Finally, the theory to predict the exponent \( \nu \) in the 1/\( \nu \) fluctuation may be hidden and is extracted by setting suitable observables with the aid of the proposed scenario.

To confirm the importance of the Casimir invariants (that are based on the Poisson structure of the Vlasov equation), we also examined the Kuramoto model [57], as a non-Hamiltonian and dissipative version of the HMF model, and confirmed that the system does not exhibit 1/\( \nu \) fluctuation (see the Appendix G) [58]. The superposition of Lorentzian spectra is also unsuitable to explain the observed 1/\( \nu \) fluctuation by considering the timescales determined by the Landau damping modes (see the Appendix H).

It is important to examine the universality of our result. Although we considered the thermal equilibrium states of simple models here, the existence of Casimir invariants does not depend on consideration of reference states or the details of interaction potentials in long-range Hamiltonian systems, and the slow dynamics by pseudo-Casimir-invariants will generally be applied to the so-called quasistationary states [34] for instance. As there exists a variety of examples that can be effectively described by long-range Hamiltonian systems, such as plasmas, free electron laser [30, 51–52], water molecules [63], and trapped ions [23, 64–67], collective 1/\( \nu \) fluctuation will be experimentally verifiable. In real systems, the Poisson structure providing the Casimir invariants will be disturbed by dissipation and randomness. It will be important to examine the robustness of our scenario under such perturbations. Finally, the theory to predict the exponent \( \nu \) in the 1/\( \nu \) spectrum must be formulated based on the slow motion of pseudo-Casimir-invariants.

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Appendix A: Casimir invariants

Let us consider the Vlasov equation

$$\frac{\partial F}{\partial t} + \frac{\partial \mathcal{H}[F]}{\partial p} \frac{\partial F}{\partial q} - \frac{\partial \mathcal{H}[F]}{\partial q} \frac{\partial F}{\partial p} = 0$$  \hspace{1cm} (A1)

for the one-particle distribution function $F(q, p, x, t)$, where $q$ and $p$ are the conjugate position and momentum, and $x$ represents the spatial position. We prove that a Casimir functional

$$\mathcal{C}[F](t) = \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp \ c(F(q, p, x, t))$$  \hspace{1cm} (A2)

is a constant of motion if the one-particle Hamiltonian functional $\mathcal{H}[F](q, p, x, t)$ belongs to the $C^2$-class with respect to $q$ and $p$, and if $|c(F(q, p, x, t))|$ rapidly decreases in $|p| \to \infty$. Differentiating with respect to time $t$ and using the Vlasov equation, we have

$$\frac{d \mathcal{C}[F]}{dt} = \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp \ c'(F) \left( -\frac{\partial \mathcal{H}[F]}{\partial p} \frac{\partial F}{\partial q} \frac{\partial \mathcal{H}[F]}{\partial q} \frac{\partial F}{\partial p} \right)$$

$$= \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp \ \left( -\frac{\partial \mathcal{H}[F]}{\partial p} \frac{\partial c(F)}{\partial q} + \frac{\partial \mathcal{H}[F]}{\partial q} \frac{\partial c(F)}{\partial p} \right).$$  \hspace{1cm} (A3)

Performing the integration by parts and noting the rapid decrease in $c(F)$ for $|p| \to \infty$, we have

$$\frac{d \mathcal{C}[F]}{dt} = \int_{-1/2}^{1/2} dx \int_{-\pi}^{\pi} dq \int_{-\infty}^{\infty} dp \ c(F) \left( \frac{\partial}{\partial q} \frac{\partial \mathcal{H}[F]}{\partial p} - \frac{\partial}{\partial p} \frac{\partial \mathcal{H}[F]}{\partial q} \right) = 0,$$  \hspace{1cm} (A4)

for $\mathcal{H}[F]$ in the $C^2$-class. The decreasing speed of $c(F)$ is sufficient to cause the term $c(F)\partial_q \mathcal{H}[F]$ to vanish in the limit $|p| \to \infty$, and the condition is reduced to $c(F) \to 0$ for $|p| \to \infty$ if $\mathcal{H}[F]$ consists of the $p$-dependent kinetic and $q$-dependent potential parts.

Appendix B: Power spectra of individual particles

In the $\alpha$-Hamiltonian mean-field model, no $1/f$ fluctuation occurs in the time series concerning the individual particle. Figures 4 and 5 respectively report flat power spectra for $N$ dependence with $\alpha = 0$ and $\alpha$ dependence with $N = 1024$. We thus conclude that the observed $1/f$ fluctuation comes from collective motion.

**FIG. 4.** (color online) (a) Power spectra of $\cos q_0(t)$ and (b) average over $N$ of power spectra of $\cos q_j(t)$ in the $\alpha$-HMF model. $\alpha = 0$, $T = 0.45$, $N = 128$ (red), 256 (orange), 512 (blue), and 1024 (magenta) from top to bottom, where the horizontal scales have been multiplied by $10^{-2}$, $10^{-4}$, and $10^{-6}$ in the last three curves, respectively, for graphical reasons.
FIG. 5. (color online) (a) Power spectra of $\text{cos} \varphi(t)$ and (b) average over $N$ of power spectra of $\text{cos} \varphi(t)$ in the $\alpha$-HMF model. $N = 1024$. $T = 0.6$. $\alpha = 0$ (red), 0.5 (orange), 1 (blue), and 5 (magenta) from top to bottom, where the horizontal scales have been multiplied by $10^{-2}$, $10^{-4}$, and $10^{-6}$ in the last three curves, respectively, for graphical reasons.

FIG. 6. (color online) Power spectra of $M^2(t)$ in the HMF model ($\alpha = 0$). $T = 0.45(< T_c)$. $N = 3, 4, \cdots, 10, 20$, and 50 from bottom to top. For graphical reasons, the vertical scale has been multiplied by 10 for $N = 4$, $10^2$ for $N = 5$, etc. Each curve is the average over 100 samples. The gray line segments guide the eyes for the slopes $-0.62$ (lower) and $-1.18$ (upper) obtained by the least-square method in the intervals of the segments. The $1/f$ fluctuation due to pseudo-Casimir-invariants is proposed as the enclosed domain by the two black dashed lines.

Appendix C: $N$ dependence in low-energy ordered phase with small $N$

By further investigating the $N$ dependence of the power spectrum, we note the existence of two distinct types of $1/f$ fluctuation as in Fig. 5. For larger $N(\gtrsim 9)$, the collective $1/f$ fluctuation is observed. For smaller $N(3 \lesssim N \lesssim 8)$, however, the $1/f$ fluctuation exhibits slopes clearly different from the previous type. One possible explanation of the $1/f$ fluctuation with small $N$ might be the hierarchical structure of phase space [28], whereas we note that 10 particles are sufficiently large to exhibit the collective $1/f$ fluctuation based on the pseudo-Casimir-invariants.

Appendix D: $N$ dependence in high-energy disordered phase

Next, we study the power spectra of the square order parameter and temporal evolution of the variance in the high-energy disordered phase of the Hamiltonian mean-field model ($\alpha = 0$), by setting the temperature to $T = 0.6$ in Fig. 7 and $T = 0.8$ in Fig. 8 where the critical temperature is $T_c = 0.5$. At both temperatures, $1/f$ power spectra are observed, but the slopes $-0.70$ and $-0.64$ in the long-time region are quite close to that observed with small $N$ in the low-energy ordered phase. In the high-energy side, the largest Lyapunov exponent tends to vanish as $N$ increases [68–70]. Accordingly, one possible understanding of the small exponent is that the system is nearly integrable and the hierarchical structure appears together with the constraint by the pseudo-Casimir-invariants. The comparison between $T = 0.6$ and $T = 0.8$ supports this hypothesis as the small exponent becomes significant as $T$ increases, where the dynamics are more regular.
At the high-energy side, we define the variance as

\[
V(t) = \left[ \sum_{n=0}^{n_0-1} \frac{1}{n_0} \left( \frac{1}{t} \sum_{k=1}^{t} \|M(nt + k)\|^2 - \left( \frac{1}{t} \sum_{k=1}^{t} M(nt + k) \right)^2 \right) \right]
\]

(D1)
because the order parameter vector \( M \) fluctuates on the two-dimensional plane around \( M = 0 \). Both with and without the Casimir constraints, the scaled variance \( NV/T \) is theoretically predicted as \( NV/T = 2T_c/(T - T_c) \) [13], which is twice the susceptibility. Accordingly, there is no two-step relaxation of variance in Figs. 7(b) and 8(b). The arrival to the thermal equilibrium level, therefore, does not imply the end of effectivity of the pseudo-Casimir-invariants, because the arrival is possible on one Casimir level set.

![Graphs showing power spectra and temporal evolution of scaled variance](image)

FIG. 7. (color online) \( N \) dependence in the HMF model (\( \alpha = 0 \)). \( T = 0.60 (> T_c) \). \( N = 20 \) (black starts), 50 (red triangles), 100 (orange inverse triangles), 200 (green diamonds), 500 (blue circles), and 1000 (magenta squares). (a) Power spectra of \( M^2(t) \). \( N \) increases from bottom to top. For graphical reasons, the vertical scale has been multiplied by 10\(^1\) for \( N = 50 \), 10\(^2\) for \( N = 100 \), etc. The gray line segments guide the eyes for the slopes \(-0.70\) (left upper), \(-1.07\) (right upper), and \(-1.02\) (right lower) obtained by the least-square method in the intervals of segments. (b) Temporal evolution of the scaled variance of magnetization \( M \). \( N \) has the same values as those in (a). The black horizontal line represents the thermal equilibrium level, 10. In both panels, each curve is the average over 100 samples.

![Graphs showing power spectra and temporal evolution of scaled variance](image)

FIG. 8. (color online) Similar to Fig. 7 but with \( T = 0.8 \). In panel (a), the vertical scales are suitably modified. The gray line segments guide the eyes for the slopes \(-0.64\) (upper left) and \(-0.99\) (lower right) obtained by the least-square method in the intervals of segments. In panel (b), the black horizontal line represents the thermal equilibrium level, 10/3.

**Appendix E: \( \alpha \) dependence in low-energy ordered phase**

Fixing \( T = 0.45 (< T_c) \), we computed power spectra of \( M^2 \) by varying the value of \( \alpha \) (Fig. 9). Note that, for \( \alpha > 1 \), the initial condition randomly chosen from the distribution function \( A \exp \left[ -(p^2/2 - M \cos q)/T \right] \) is not guaranteed to be in thermal equilibrium. As in the high-energy disordered phase, the slope of power spectrum goes to zero as \( \alpha \) increases, but \( \alpha = 1 \)
seems not the threshold. Instead, the slope goes to zero at $\alpha$ beyond which the ordered phase disappears. We may imagine that, in the low-energy side, wrecks of the Casimir invariants remain for $\alpha \gtrsim 1$. This expectation does not contradict with the fact that the Casimir invariants exist if the interaction is long-range, since validity of the converse is not mentioned. One difficulty studying around $\alpha = 1$ is that $N$ dependence becomes weak as shown in Fig. 10. An in-depth investigation in the middle range of $\alpha \gtrsim 1$ needs to be performed.

**Fig. 9.** (color online) Power spectra in the $\alpha$-HMF model. $T = 0.45$. (a) $N = 32$ (100). (b) $N = 1024$ (100). (c) $N = 8192$ (20). The number in parentheses represents the number of sample orbits. $\alpha = 0$ (red), 0.2 (orange), 0.4 (green), 0.6 (blue), 0.8 (magenta), 0.9 (gray), 0.95 (black), 1.2 (red), and 1.5 (orange) from top to bottom. For graphical reasons, the vertical scale has been modified suitably. (d) $\alpha$ dependence of the exponent $\nu$ (minus of the slope), which is computed by the least-square method between the two vertical lines of the panels (a)-(c). $N = 32$ (orange triangles), 1024 (red circles), and 8192 (blue squares). (inset) $\alpha$ dependence of time average of $M$.

**Appendix F: Dependence of observables**

In the globally coupled Fermi-Pasta-Ulam-Tsingou system

$$H_{\text{FPUT}} = \sum_{j=0}^{N-1} \frac{p_j^2}{2} + \frac{1}{2N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \varphi(q_j - q_k), \quad \varphi(q) = \frac{q^2}{2} + \frac{q^4}{4},$$

we compute the other power spectra of the variance of $q$,

$$\Delta = X_2 - X_1^2, \quad X_n = \frac{1}{N} \sum_{j=0}^{N-1} q_j^n,$$

the potential energy $U$, and $\Theta^2$, where the order parameter $\Theta$ is defined by

$$\Theta = \frac{1}{N} \sum_{j=0}^{N-1} (\cos \theta_j, \sin \theta_j),$$

and $\theta$ is the angle variable, which is conjugate to the action variable associated with the one-particle Hamiltonian

$$\mathcal{H}_{\text{FPUT}} = \frac{p^2}{2} + \frac{1}{2N} \sum_{j=0}^{N-1} X^c_j - \frac{3}{2} q^2 + \frac{1}{4} q^4.$$
FIG. 10. (color online) Temporal evolution of $V(t)$ in the $\alpha$-HMF model. $T = 0.45$. (a) $N = 32$ (100). (b) $N = 1024$ (100). (c) $N = 8192$ (20). The number in parentheses represents the number of sample orbits. $\alpha = 0$ (red triangles), 0.2 (orange inverse triangles), 0.4 (green diamonds), 0.6 (blue circles), 0.8 (magenta squares), 0.9 (gray crosses), 0.95 (black stars), 1.2 (red open triangles), and 1.5 (orange open inverse triangles) from bottom to top. The horizontal scales of $\alpha = 1.2$ and 1.5 are out of range for $t > 100$ in (b) and (c). (d) The horizontal axis is scaled as $t/N^\beta$. (inset) $\alpha$ dependence of $\beta$, which is obtained by fixing $\beta = 1$ for $(N, \alpha) = (8192, 0)$). The lower red and upper blue lines are for $N = 1024$ and $N = 8192$ respectively. The black solid line represents $\beta = 1 - \alpha^2$.

The value of $X_2^C$ is computed to satisfy the self-consistent equation from the canonical thermal equilibrium distribution, which is proportional to exp($-\mathcal{H}_{\text{PUT}}/T$). For $T = 1$, we have $X_2^C \approx 0.34252$.

The power spectra are presented in Fig. 11. The variables $\Delta$ and $U$ have no $1/f$ fluctuation, but $\Theta^2$ is another suitable observable to find the hidden $1/f$ fluctuation as $\Phi^2$.

Appendix G: Kuramoto model

The Kuramoto model is expressed as

$$\frac{d\theta_j}{dt} = \omega_j - \frac{K}{N} \sum_{k=1}^{N} \sin(\theta_j - \theta_k),$$

where $\theta_j$ represents the phase of the $j$th oscillator, $\omega_j$ is the natural frequency depending on the number of oscillators, and $K > 0$ is the coupling constant. The natural frequency $\omega_j$ is independently and randomly drawn from a probability distribution function $g(\omega)$. The Kuramoto model describes the synchronization transition from the nonsynchronized state to the partially synchronized state. The extent of synchrony is measured by the order parameter

$$Z = \frac{1}{N} \sum_{k=1}^{N} e^{i\theta_k}.$$ 

If $g(\omega)$ is even and unimodal, then we find the critical coupling constant $K_c = 2/|\pi g(0)|$, where the nonsynchronized state is stable (unstable) for $K < K_c$ ($K > K_c$).

In the limit $N \to \infty$, the Kuramoto model is described by the equation of continuity

$$\frac{\partial f}{\partial t} + \frac{\partial }{\partial \theta} (\nu f) = 0,$$
FIG. 11. (color online) Power spectra of (a) $\Delta$, (b) potential energy, and (c) $\Theta^2$ in the globally coupled FPUT model. $T = 1$. $N = 100$ (top orange), 1000 (middle red), and 10000 (bottom blue), where the vertical scales of the last two lines in (c) have been multiplied by 10 and 100, respectively, for graphical reasons. The curves are averages over 20 samples obtained with the time step of $\Delta t = 0.01$. The gray line segments in (c) are obtained by the least square method in the intervals of the segments, and their slopes are $-0.61$, $-0.72$, and $-0.72$ for $N = 100$, 1000, and 10000 respectively.

where $f(\theta, \omega, t)$ is the probability distribution function satisfying $\int_0^{2\pi} f(\theta, \omega, t) d\theta = g(\omega)$, and

$$v = \omega - K \int_{-\infty}^{\infty} d\omega \int_0^{2\pi} d\theta' \sin(\theta - \theta') f(\theta', \omega, t).$$  \hfill (G4)

The equation of continuity (G3) corresponds to the Vlasov equation (A1). This correspondence becomes clear by rewriting the Vlasov equation in the form

$$\frac{\partial F}{\partial t} + \nabla_{(q, p)} \cdot (XF) = 0,$$  \hfill (G5)

where

$$\nabla_{(q, p)} = \left( \frac{\partial}{\partial q}, \frac{\partial}{\partial p} \right)$$  \hfill (G6)

and $X$ is the Hamiltonian vector field

$$X = \left( \frac{\partial \mathcal{H}[f]}{\partial p}, -\frac{\partial \mathcal{H}[f]}{\partial q} \right).$$  \hfill (G7)

Nevertheless, the equation of continuity has no Poisson structure and no Casimir invariants accordingly. We can not expect $1/f$ fluctuation of collective variables originated from the Casimir mechanism.

To confirm absence of $1/f$ fluctuation in the order parameter, we set $g(\omega)$ as the normal distribution

$$g(\omega) = \frac{1}{\sqrt{2\pi T}} \exp(-\omega^2 / 2T)$$  \hfill (G8)

with “temperature” $T = 0.6$, which gives $K_c = 1.236...$. We numerically integrate the equation of motion (G1) by using the fourth-order Runge-Kutta method with the time step $\Delta t = 0.1$. The power spectrum of $|Z(t)|^2$ is reported in Fig. 12. The observable is a collective variable, but no $1/f$ fluctuation appears irrespective of values of the coupling constant $K$. 

\medskip

\begin{figure}[h] 
\centering 
\includegraphics[width=\textwidth]{fig11.png} 
\caption{(color online) Power spectra of (a) $\Delta$, (b) potential energy, and (c) $\Theta^2$ in the globally coupled FPUT model. $T = 1$. $N = 100$ (top orange), 1000 (middle red), and 10000 (bottom blue), where the vertical scales of the last two lines in (c) have been multiplied by 10 and 100, respectively, for graphical reasons. The curves are averages over 20 samples obtained with the time step of $\Delta t = 0.01$. The gray line segments in (c) are obtained by the least square method in the intervals of the segments, and their slopes are $-0.61$, $-0.72$, and $-0.72$ for $N = 100$, 1000, and 10000 respectively.} 
\end{figure}
FIG. 12. (color online) Power spectra of $|Z(t)|^2$ in the Kuramot model. The coupling constant is $K = 1.0$ (blue lower), $K = K_c$ (red middle), and $K = 1.5$ (orange upper). The vertical scale for $K = 1.5$ is multiplied by 10 for graphical reasons. Each curve is the average over 20 samples. The upper gray line segment guides the eyes for the slope $\sim 2$.

Appendix H: Superposition of Lorentzian spectra

One traditional explanation of $1/f$ fluctuations is a superposition of Lorentzian spectra originating from the exponentially damping correlation (see [18,19], for instance). A long-range system has several exponential Landau damping modes in general, but the $1/f$ spectra reported in this article cannot be explained by this superposition: The second slowest damping rate is of $O(1)$ around $T = T_c$ in the $\alpha$-HMF model, and no accumulation of Landau poles appears in the slow timescale of $f \sim 10^{-3}$.

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