Semiparametric regression of mean residual life with censoring and covariate dimension reduction

Ge Zhao
Department of Mathematics and Statistics
Portland State University, Portland, OR 97201

Yanyuan Ma
Department of Statistics, Pennsylvania State University, State College, PA 16802

Huazhen Lin
Center of Statistical Research, School of Statistics
Southwestern University of Finance and Economics, Chengdu, China 611130

and Yi Li
Department of Biostatistics, University of Michigan, Ann Arbor, MI 48109

Abstract

We propose a new class of semiparametric regression models of mean residual life for censored outcome data. The models, which enable us to estimate the expected remaining survival time and generalize commonly used mean residual life models, also conduct covariate dimension reduction. Using the geometric approaches in semiparametrics literature and the martingale properties with survival data, we propose a flexible inference procedure that relaxes the parametric assumptions on the dependence of mean residual life on covariates and how long a patient has lived. We show that the estimators for the covariate effects are root-$n$ consistent, asymptotically normal, and semiparametrically efficient. With the unspecified mean residual life function, we provide a nonparametric estimator for predicting the residual life of a given subject, and establish the root-$n$ consistency and asymptotic normality for this estimator. Numerical experiments are conducted to illustrate the feasibility of the proposed estimators. We apply the method to analyze a national kidney transplantation dataset to further demonstrate the utility of the work.

Keywords: Mean residual life, kidney transplant, nonparametric estimation, semiparametric efficiency.
1 Introduction

End stage renal disease (ESRD) is one of the most lethal diseases in the US (Ferri, 2017). An ESRD patient loses kidney functions permanently and has to rely on renal replacement to survive. The common renal replacement therapies are dialysis and kidney transplant, where the latter is associated with better outcomes and quality of life. Due to severe shortages in kidney supplies, there are far more renal failure patients in need of kidney transplants than donors available, and the decision for organ allocation is often based on the Estimated Post-Transplant Survival (EPTS) score ([https://srtr.transplant.hrsa.gov/](https://srtr.transplant.hrsa.gov/)), which assesses a patient’s overall survival post transplantation.

EPTS is calculated from a Cox model which includes age, diabetes status, prior solid organ transplant and time on dialysis as covariates. With this model, however, candidates with diabetes or a prior solid organ transplant or being on dialysis treatment for many years will have a lower priority for transplantation. On the other hand, younger patients tend to have a higher priority for transplantation and age has effectively become the most dominant factor. A more efficient system should allocate organs to those who will benefit most among all the eligible patients (Bertsimas et al., 2013; Israni et al., 2014). One way to quantify the transplant efficiency is to compare the improvement of patients’ expected residual life with and without transplantation (Wolfe et al., 2008), where the expected residual life characterizes the remaining survival time given that a patient has survived up to a certain time. The residual life expectancy is advantageous to the overall life expectancy as the former takes into account the most updated information (Lin et al., 2016) once an
organ is available.

Since the seminal work of Oakes and Dasu (1990), much research has been conducted on mean residual life models. For example, Maguluri and Zhang (1994) proposed a univariate proportional mean residual life model and provided estimation and inference tools; Oakes and Dasu (2003) established the theoretical properties of the methods in Oakes and Dasu (1990); Chen and Cheng (2005) studied the proportional mean residual life model and proposed an alternative estimator for the covariate effect through a partial-score construction, similar to the partial likelihood approach; Chen et al. (2005) employed the inverse probability weighting approach to estimate the covariate effects; Müller and Zhang (2005) extended the mean residual life model to incorporate time-varying covariates; Chen and Cheng (2006) proposed a linear residual life model, and Chen (2007) proposed an additive mean residual life model. These models inspired quantile residual life model research (Jeong et al., 2008; Ma and Yin, 2010). However, all of these works assumed that the mean or quantile residual life depends on the covariates as well as how long a patient has lived through parametric functions. Violations of the model assumptions may lead to biased results (McLain and Ghosh, 2011; Liu et al., 2019), which motivates our work to relax these parametric assumptions.

Our work is also motivated by the kidney transplant data from the U.S. Scientific Registry of Transplant Recipients (Leppke et al., 2013; Israni et al., 2014), which feature, in addition to survival outcome data, rich information, such as treatment history, comorbidity conditions and demographic variables. To strike a balance between interpretation and flexibility when quantifying the covariates' impacts on the expected residual life so as to
identify patients who may benefit most from transplantation, we propose a semiparametric mean residual life regression model with covariate dimension reduction. Our model relaxes the parametric assumptions on the dependence of mean residual life on covariates and how long a patient has lived, as opposed to many existing residual life models, and in the meantime conducts covariate dimension reduction via a data driven fashion (Ma and Zhu, 2013). Using the geometric approaches in semiparametrics literature and the martingale properties with survival data, we obtain semiparametrically efficient estimators (Bickel et al., 1994; Tsiatis, 2006) for the covariate effects. With the unspecified mean residual life function, we provide a nonparametric estimator for predicting the residual life of a given subject, and establish the root-$n$ consistency and asymptotic normality for this estimator. We analyze the kidney transplant data and quantify the benefit with residual life expectancy.

The remainder is organized as follows. In Section 2 we propose the mean residual life model and introduce the notation. In Section 3 we derive an efficient estimator and discuss its properties. We also provide the estimation of the mean residual life function. The large sample properties of the estimator and the estimated mean residual life function are established in Section 4. We assess the finite sample properties of the methods using simulation studies in Section 5 and apply it to analyze the kidney transplant data in Section 6. We conclude the paper with a discussion in Section 7.
2 Semiparametric regression of mean residual life

Denote the failure time and the covariates by $T \in \mathbb{R}^+$ and $X \in \mathbb{R}^p$, respectively. For any given $t > 0$ (e.g. the time length a patient has lived so far), we propose a semiparametric regression model of mean residual life

$$E(T - t \mid T \geq t, X) = m(t, \beta^T X),$$

where $\beta \in \mathbb{R}^{p \times d}$ is the coefficient matrix with $d < p$, $m$ is an unspecified positive function of $t$ and $\beta^T X$. Here $d$ is the number of indices. When $d = 1$, the model reduces to the single index model; when $1 < d < p$, it corresponds to a dimension reduction structure; when $d = p$, the model is completely nonparametric. Our analysis first focuses on a fixed $d$, followed by selecting $d$ with a data driven fashion as discussed in Section 6. Assume further that $T$ is subject to random right censoring, i.e. $Z = \min(T, C)$ and $\Delta = I(T \leq C)$, where the censoring time $C$ satisfies $C \perp T \mid X$. Model (1) generalizes many existing mean residual life models. For example, it includes the proportional mean residual life model (Oakes and Dasu, 1990; Chen et al., 2005; Chen and Cheng, 2005) as a special case, by specifying $m(t, \beta^T X) = m_0(t) \exp(\beta^T X)$ and $d = 1$; it reduces to the additive model (Chen, 2007) by specifying $m(t, \beta^T X) = m_0(t) + \beta^T X$ and setting $d = 1$.

We assume that the observed data $(X_i, Z_i, \Delta_i)$, $i = 1, \ldots, n$, are independently and identically distributed realizations of $(X, Z, \Delta)$. To make (1) identifiable, we fix the upper $d \times d$ block of $\beta$ to be $I_d$. We aim to estimate the unspecified function $m(\cdot, \cdot, \cdot)$, as well as the column space of $\beta$, which is equivalent to estimating the lower $(p - d) \times d$ block of $\beta$. 
To associate the covariates with the upper and lower parts of $\beta$, we write $X = (X_u^T, X_l^T)^T$, where $X_u \in \mathbb{R}^d$ and $X_l \in \mathbb{R}^{p-d}$.

Model (1) leads to

$$
S(t \mid X) = S(t, \beta^T X) = \frac{m(0, \beta^T X)}{m(t, \beta^T X)} \exp \left\{ - \int_0^t \frac{1}{m(u, \beta^T X)} du \right\},
$$

$$
\lambda(t, \beta^T X) = \frac{m_1(t, \beta^T X) + 1}{m(t, \beta^T X)}, \tag{2}
$$

where $\lambda(t, \beta^T X)$ is the hazard function of $T$ conditional on $X$, and $m_1(t, \beta^T X) = \partial m(t, \beta^T X)/\partial t$. Likewise, we denote by $m_2(t, \beta^T X) = \partial m(t, \beta^T X)/\partial (\beta^T X)$. We can re-express (2) as

$$
m(t, \beta^T X) = e^{\Lambda(t, \beta^T X)} \int_t^\infty e^{-\Lambda(s, \beta^T X)} ds \tag{3}
$$

where $\Lambda(\cdot, \beta^T X)$ is the cumulative hazard function (Maguluri and Zhang, 1994).

## 3 A semiparametrically efficient estimator

### 3.1 Nuisance tangent spaces

Denote the conditional survival function, cumulative hazard function, hazard function and probability density function (pdf) of the censoring time $C$ by $S_c(z, X) = \operatorname{pr}(C \geq z \mid X)$, $\Lambda_c(z, X) = -\log S_c(z, X)$, $\lambda_c(z, X) = \partial \Lambda_c(z, X)/\partial z$ and $f_c(z, X) = -\partial S_c(z, X)/\partial z$ with $z < \tau$, where $\tau < \infty$ is the upper bound of the follow-up time. Let $p(X) \equiv \operatorname{pr}(C = \tau \mid X)$, $S_c(\tau, X) = f_c(\tau, X) = p(X)$, and $\lambda_c(\tau, X) = 1$. Here, $\lambda_c(z, X)$ and $f_c(z, X)$
are absolutely continuous on \((0, \tau)\), but with a discontinuity point at \(\tau\). For any \(\beta\), in addition to the survival function \(S(z, \beta^T X)\) and the hazard function \(\lambda(z, \beta^T X)\) in (2), we also define the pdf \(f(z, \beta^T X) = -\partial S(z, \beta^T X)/\partial z\) and the cumulative hazard function \(\Lambda(z, \beta^T X) = -\log S(z, \beta^T X)\). We write \(\lambda_2(s, \beta^T X) = \partial \lambda(s, \beta^T X)/\partial (\beta^T X)\) and \(\lambda_{20}(s, \beta^T X) = \partial \lambda_0(s, \beta^T X)/\partial (\beta^T X)\) as the derivatives of the hazard functions with respect to \(\beta^T X\), where \(\lambda_0(z, \beta^T X)\) is given in (2) when the mean residual function \(m(\cdot, \cdot) = m_0(\cdot, \cdot)\), where the subscript “0” indicates the truth.

The pdf of \((X, Z, \Delta)\) is

\[
f_{X,Z,\Delta}(x, z, \delta) = f_X(x)\lambda(z, \beta^T X)^\delta e^{-\int_0^\delta \lambda(s, \beta^T X)ds} \lambda_c(z, x)^{1-\delta} e^{-\int_0^\delta \lambda_c(s, x)ds}, \tag{4}
\]

where \(f_X(x)\) is the pdf of \(X\). We view the pdf in (4) as a semiparametric model where all unknown components, except for \(\beta\), are infinite dimensional nuisance parameters. The parameters \(\beta\) are parameters of interest with a finite dimension. We will estimate \(\beta\) by using a geometric approach, which avoids modeling the hazard function \(\lambda(z, \beta^T X)\) to be the product of a baseline function of \(z\) and a specific covariate function such as \(\exp(\beta^T X)\), as in a proportional hazards model. This entails more flexibility for the model.

Let \(Y(t) = I(Z \geq t)\) be the at risk process and \(N(t) = I(Z \leq t)\Delta\) be the counting process. Define the filtration \(\mathcal{F}_t\) to be \(\sigma\{N(u), Y(u), X, 0 \leq u < t\}\), and let \(M(t)\) be the martingale with respect to \(\mathcal{F}_t\), i.e. \(M(t, \beta^T X) = N(t) - \int_0^t Y(s)\lambda(s, \beta^T X)ds\). The nuisance tangent space, which will be utilized for deriving our estimator, is obtained as follows.

**Proposition 1.** The nuisance tangent space is \(T = T_1 \oplus T_2 \oplus T_3\), where each component
corresponds to \( f_X, m(\cdot, \cdot) \) and \( \lambda_c \), respectively. Specifically,

\[
\begin{align*}
\mathcal{T}_1 & = \{ a(X) : E\{a(X)\} = 0, a(X) \in \mathcal{R}^{(p-d)d}, \text{var}\{a(X)\} < \infty \}, \\
\mathcal{T}_2 & = \left[ \int_0^\infty \left\{ \frac{h_1(s, \beta^T X)}{m_1(s, \beta^T X) + 1} - \frac{h(s, \beta^T X)}{m(s, \beta^T X)} \right\} dM(s, \beta^T X) : \\
& \quad \forall h(z, \beta^T X) \in \mathcal{R}^{(p-d)d}, \text{var}\{h(z, \beta^T X)\} < \infty \}, \\
\mathcal{T}_3 & = \left[ \int_0^\infty h(s, X) dM_c(s, X) : \forall h(z, X) \in \mathcal{R}^{(p-d)d}, \text{var}\{h(z, X)\} < \infty \}.
\end{align*}
\]

The derivation of Proposition 1 is provided in Supplement S.1.

### 3.2 Derivation of an efficient score function

Taking the derivative of the logarithm of (4) with respect to \( \beta \), we obtain the score function

\[
S_\beta(\Delta, Z, X) = \int_0^\infty \left\{ \frac{m_{12}(s, \beta^T X)}{m_1(s, \beta^T X) + 1} - \frac{m_2(s, \beta^T X)}{m(s, \beta^T X)} \right\} \otimes X_t dM(s, \beta^T X),
\]

where \( m_{12}(t, \beta^T X) = \frac{\partial m_2(t, \beta^T X)}{\partial t} \) and \( X_t \) is the lower \( p - d \) components in \( X \). We can verify that, at \( \beta_0 \), \( S_\beta(\Delta, Z, X) \perp \mathcal{T}_1 \) and \( S_\beta(\Delta, Z, X) \perp \mathcal{T}_3 \) due to the martingale properties. To look for an efficient score by projecting \( S_\beta(\Delta, Z, X) \) at \( \beta_0 \) to \( \mathcal{T}_2 \), we search
for \( h^*(s, \beta_0^T X) \) such that

\[
S_{\text{eff}}(\Delta, Z, X) = S_{\beta}(\Delta, Z, X) - \int_0^\infty \left\{ \frac{h^*(s, \beta_0^T X)}{m(s, \beta_0^T X) + 1} - \frac{h^*(s, \beta_0^T X)}{m(s, \beta_0^T X)} \right\} dM(s, \beta_0^T X)
\]

\[
= \int_0^\infty \left\{ \frac{m_{12}(s, \beta_0^T X) \otimes X_t - h^*_1(s, \beta_0^T X)}{m(s, \beta_0^T X) + 1} - \frac{m_2(s, \beta_0^T X) \otimes X_t - h^*(s, \beta_0^T X)}{m(s, \beta_0^T X)} \right\} dM(s, \beta_0^T X)
\]

is orthogonal to \( \mathcal{T}_2 \), which implies that, for any \( h(s, \beta_0^T X) \), it must hold that

\[
0 = E \left[ \int_0^\infty a(s, \beta_0^T X)^T \left\{ \frac{h_1(s, \beta_0^T X)}{m(s, \beta_0^T X) + 1} - \frac{h(s, \beta_0^T X)}{m(s, \beta_0^T X)} \right\} ds \right],
\]

where

\[
a(s, \beta_0^T X) \equiv E \left[ \left\{ \frac{m_{12}(s, \beta_0^T X) \otimes X_t - h^*_1(s, \beta_0^T X)}{m(s, \beta_0^T X) + 1} - \frac{m_2(s, \beta_0^T X) \otimes X_t - h^*(s, \beta_0^T X)}{m(s, \beta_0^T X)} \right\} \right.
\]

\[
\times S_c(s, X) \mid \beta_0^T X \bigg] S(s, \beta_0^T X) \frac{m_1(s, \beta_0^T X) + 1}{m(s, \beta_0^T X)}.
\]

We can choose any \( h(s, \beta_0^T X) \) function. Specifically, by letting \( h(s, \beta_0^T X) = 0 \) for \( s < t \) and \( h(s, \beta_0^T X) = c(\beta_0^T X) \) for \( s \geq t \) with an arbitrary function \( c(\beta_0^T X) \), we obtain

\[
a(t, \beta_0^T X)/\{m_1(t, \beta_0^T X) + 1\} - \int_t^\infty a(s, \beta_0^T X)/m(s, \beta_0^T X)ds = 0.
\]

Solving this integral equation leads to

\[
a(t, \beta_0^T X) = \{m_1(t, \beta_0^T X) + 1\} \exp \left\{ - \int_0^t \frac{m_1(s, \beta_0^T X) + 1}{m(s, \beta_0^T X)} ds \right\} c(\beta_0^T X),
\]
for function $c(\cdot)$. Thus, reusing (5), we require that for all $h(t, \beta_0^T X)$,

$$
0 = E \left[ \int_0^\infty \{ m_1(t, \beta_0^T X) + 1 \} \exp \left\{ - \int_0^t \frac{m_1(s, \beta_0^T X) + 1}{m(s, \beta_0^T X)} ds \right\} c(\beta_0^T X)^T 
\times \left\{ \frac{h_1(t, \beta_0^T X)}{m_1(t, \beta_0^T X) + 1} - \frac{h(t, \beta_0^T X)}{m(t, \beta_0^T X)} \right\} dt \right] 
= -E \left\{ c(\beta_0^T X)^T h(0, \beta_0^T X) \right\}.
$$

Letting $h(0, \beta_0^T X) = c(\beta_0^T X)$ yields the only possibility of $c(\beta_0^T X) = 0$, hence $a(t, \beta_0^T X) = 0$. Inserting the expression of $a(t, \beta_0^T X)$ into (5), we have

$$
\frac{h_1^*(t, \beta_0^T X)}{m_1(t, \beta_0^T X) + 1} - \frac{h_1^*(t, \beta_0^T X)}{m(t, \beta_0^T X)} = \left\{ \frac{m_1(t, \beta_0^T X)}{m_1(t, \beta_0^T X) + 1} - \frac{m_2(t, \beta_0^T X)}{m(t, \beta_0^T X)} \right\} \otimes \frac{E \{ X_i S_c(t, X) \mid \beta_0^T X \}}{E \{ S_c(t, X) \mid \beta_0^T X \}}.
$$

Thus an efficient score is

$$
S_{eff}(\Delta, Z, X) = \int_0^\infty \left\{ \frac{m_1(s, \beta_0^T X)}{m(s, \beta_0^T X) + 1} - \frac{m_2(s, \beta_0^T X)}{m(s, \beta_0^T X)} \right\} \otimes \left[ X_t - \frac{E \{ X_i S_c(s, X) \mid \beta_0^T X \}}{E \{ S_c(s, X) \mid \beta_0^T X \}} \right] dM(s, \beta_0^T X).
$$

### 3.3 Construction of a semiparametrically efficient estimator

Based on (7), we construct a semiparametrically efficient estimator of $\beta$. First, a consistent estimating equation can be obtained from $E\{S_{eff}(\Delta, Z, X) \mid X \} = 0$, due to $E\{dM(t, \beta_0^T X) \mid X \} = 0$. Hence, to preserve the mean zero property and to simplify the computation, we can replace $m_1(s, \beta_0^T X)/m(s, \beta_0^T X) + 1 - m_2(s, \beta_0^T X)/m(s, \beta_0^T X)$
by an arbitrary function of $s$ and $\beta_0^T X$, say $g(s, \beta_0^T X)$, and still obtain

$$E \left( \int_0^\infty g(s, \beta_0^T X) \otimes \left[ X_t - \frac{E \{X_t S_c(s, X) \mid \beta_0^T X\}}{E \{S_c(s, X) \mid \beta_0^T X\}} \right] dM(s, \beta_0^T X) \right) = 0.$$ 

This provides a richer class of estimators than the estimator based on $S_{\text{eff}}$ alone.

Second, we can obtain

$$\frac{E \{X_t Y(t) \mid \beta_0^T X\}}{E \{Y(t) \mid \beta_0^T X\}} = \frac{E \{X_t S_c(t, X) \mid \beta_0^T X\}}{E \{S_c(t, X) \mid \beta_0^T X\}},$$

where we define $S_{\text{eff}}$ to be $E \{X_t p(X) \mid \beta_0^T X\}/E \{p(X) \mid \beta_0^T X\}$ when $t > \tau$, with $p(X)$ defined in Section 3.1. We then verify that

$$E \left( \int_0^\infty g(s, \beta_0^T X) \otimes \left[ X_t - \frac{E \{X_t S_c(s, X) \mid \beta_0^T X\}}{E \{S_c(s, X) \mid \beta_0^T X\}} \right] dN(s) \right) = 0.$$ 

The proof of (8) and (9) is given in Supplement S.2. This implies that we can construct estimating equations of the form

$$\sum_{i=1}^n \Delta_i g(Z_i, \beta^T X_i) \otimes \left[ X_{ti} - \frac{E \{X_{ti} Y_i(Z_i) \mid \beta^T X_i\}}{E \{Y_i(Z_i) \mid \beta^T X_i\}} \right] = 0.$$

11
for any $g(\cdot, \cdot)$, with

$$
\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i) I(Z_j \geq Z_i)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i)}, \tag{11}
$$

$$
\hat{E} \{ X_i Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i) X_{ij} I(Z_j \geq Z_i)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i)}. \tag{12}
$$

Here, $\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \} = \hat{E} \{ Y_i(t) \mid \beta^T X_i \}_{t=Z_i}$ and similarly for other terms.

Third, when we choose to estimate the unknown components $m(\cdot, \cdot), m_1(\cdot, \cdot), m_2(\cdot, \cdot)$ and $m_{12}(\cdot, \cdot)$ nonparametrically, we then obtain the efficient estimator of $\beta$ by solving

$$
\sum_{i=1}^{n} \Delta_i \left\{ \frac{\hat{m}_{12}(Z_i, \beta^T X_i)}{\hat{m}_1(Z_i, \beta^T X_i)} + 1 - \frac{\hat{m}_2(Z_i, \beta^T X_i)}{\hat{m}(Z_i, \beta^T X_i)} \right\} \otimes \left[ X_{hi} - \frac{\hat{E} \{ X_h Y_i(Z_i) \mid \beta^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] = 0. \tag{13}
$$

We estimate $m(t, \beta^T X)$ nonparametrically via estimating $\Lambda(t, \beta^T X)$ by

$$
\hat{\Lambda}(t, \beta^T X) = \sum_{i=1}^{n} \int_{0}^{t} \frac{K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} dN_i(s), \tag{14}
$$

at any $t$, and using (3) to obtain

$$
\hat{m}(t, \beta^T X) = e^{\hat{\Lambda}(t, \beta^T X)} \int_{t}^{\infty} e^{-\hat{\Lambda}(s, \beta^T X)} ds, \tag{15}
$$

where $\hat{\Lambda}(t, \beta^T X)$ is a kernel smoothed version of the Nelson-Aalen estimator (Ramlau-Hansen, 1983; Andersen et al., 1993). The estimators of the derivatives of $\Lambda(t, \beta^T X)$ and $m(t, \beta^T X)$ are given in Supplement S.3.1. To bypass the zero-denominator issue when applying the
nonparametric estimator to a finite sample case, we propose a trimmed version in Supplement S.3.2 and show that it retains the same asymptotic properties proved in Section 4.

4 Asymptotic properties and semiparametric efficiency

We list the regularity conditions for the results of root-$n$ consistency and asymptotic normality of the estimators proposed in Section 3. We also establish semiparametric efficiency of the estimator obtained by solving (13).

C1 (kernel function) The kernel function $K_h(\cdot) = h^{-d}K(\cdot/h)$ where $K(a) = \prod_{j=1}^d K(a_j)$ for $a = (a_1, ..., a_d)^T$ is symmetric on each individual entry and $K(a_j)$ is differentiable, decreasing when $x \geq 0$, and $\int K(x)dx = 1, \int x^j K(x)dx = 0$, for $1 \leq j < \nu$, $0 < \int x^\nu K(x)dx < \infty$, and $\int K^2(x)dx, \int x^2 K^2(x)dx, \int K^{\nu 2}(x)dx, \int x K^{\nu 2}(x)dx, \int x^{2\nu} K^{\nu 2}(x)dx$, $\int x^2 K^{\nu 2}(x)dx$ are all bounded. When there is no confusion, we use the same $K$ for both univariate and multivariate kernel functions for simplicity.

C2 (bandwidths) The bandwidths $h$ and $b$ satisfy $h \to 0, nh^{2\nu} \to 0, b \to 0$ and $nh^{d+2}b \to \infty$, where $2\nu > d + 1$.

C3 (density functions of covariates) For all $\beta \in \mathcal{B}$, the parameter space, the probability density function of $\beta^T X$, $f_{\beta^T X}(\beta^T x)$, has a compact support and is bounded away from zero and $\infty$. Further $f_{\beta^T X}(\beta^T x)$ has a derivative, up to the fourth order, that is bounded uniformly on the support.
C4 (smoothness) For all $\beta \in \mathcal{B}$ and $t > 0$, $E\{X_j I(Z_j \geq t) \mid \beta^T X_j = \beta^T x\}$, and its derivatives, up to the fourth order, are bounded uniformly as functions of $\beta^T x$; $E\{X_j X_j^T I(Z_j \geq t) \mid \beta^T X_j = \beta^T x\}$ and its first and second order derivatives are bounded uniformly as functions of $\beta^T x$.

C5 (survival function) For all $\beta \in \mathcal{B}$, it holds that, at any $t$, $\partial^{i+j}S(t, \beta^T x)/\partial t^i \partial (\beta^T x)^j$, $\partial^{i+j}E\{S_c(t, X) \mid \beta^T x\}/\partial t^i \partial (\beta^T x)^j$ and $\partial^{i+j}f(t, \beta^T x)/\partial t^i \partial (\beta^T x)^j$ exist and are bounded and bounded away from zero, for all $i \geq 0, j \geq 0, i + j \leq 4$. In addition, $S_c(\tau, X)$ is bounded way from zero.

C6 (boundedness) The true parameter $\beta_0$ is an interior point in $\mathcal{B}$ and $\mathcal{B}$ is bounded.

C7 (uniqueness) The equation

$$E \left( \Delta \left\{ \frac{m_{12}(Z, \beta^{\top} X)}{m_1(Z, \beta^{\top} X)} + 1 \right\} \otimes \left[ X_t - E \left\{ X_t Y(Z) \mid \beta^{\top} X \right\} \right] \right) = 0$$

has a unique solution in $\mathcal{B}$.

Conditions C1 and C2 are commonly assumed in kernel regression analysis (Silverman, 1986; Ma and Zhu, 2013). Conditions C3–C5 assume boundedness of event time, censoring time, covariates and their expectations, which hold for real datasets. The smoothness of several functions is imposed by constraining their derivatives, which are common conditions (Silverman, 1978). It is natural to make a boundedness assumption on the parameter space $\mathcal{B}$ as in Condition C6 in practical problems (Härdle et al., 1997). Condition C7 precludes that the estimating equation is degenerate.
Condition $\text{C3}$ can be slightly modified for the trimmed estimators.

$\text{C3'}$ (The density of index, relaxed.) Uniformly for any $\beta$ in a local neighborhood of $\beta_0$, the density function of $\beta^T X$, i.e. $f_{\beta^T X}(v)$, is bounded, and there exists a constant $\epsilon > 0$ such that $\int_{\{v: f_{\beta^T X}(v) \leq d_n\}} f_{\beta^T X}(v) dv < n^{-\epsilon}$ for sufficiently large $n$. Here $d_n \to 0$ as $n \to \infty$, and $n^{-\epsilon} = O(h^2 + n^{-1/2}h^{-1/2})$, where $h$ satisfies Condition $\text{C2}$. In addition, the derivatives of $f_{\beta^T X}(\cdot)$, up to the fourth order, are bounded.

Condition $\text{C3'}$, weaker than Condition $\text{C3}$, requires the tail of $f_{\beta^T X}$ to be sufficiently thin to ensure the near zero values of $f_{\beta^T X}(\cdot)$ not affect the overall performance of our estimator. It guarantees that the trimmed nonparametric estimators will retain the same asymptotic properties discussed below.

Theorems 1 and 2 demonstrate the root-$n$ consistency and asymptotical normality of the profile parameter estimator $\hat{\beta}$. The proofs are given in Supplement S.5 and S.6.

**Theorem 1.** Under Conditions $\text{C1-C7}$, the estimator, $\hat{\beta}$, obtained by solving (10) or (13) is consistent, i.e. $\hat{\beta} - \beta \to 0$ in probability when $n \to \infty$.

**Theorem 2.** Under Conditions $\text{C1-C7}$, the estimator, $\hat{\beta}$, obtained by solving (10) or (13) satisfies $\sqrt{n}(\hat{\beta} - \beta) \to N(0, A^{-1}BA^{-1}\beta^T)$ in distribution when $n \to \infty$, where

$$A = E \left\{ \frac{\partial}{\partial \text{vec}(\beta)^T} \text{vec} \left( \Delta g(Z, \beta^T X) \otimes \left[ a(X_i) - \frac{E \{ a(X_i)Y(Z) | \beta^T X \}}{E \{ Y(Z) | \beta^T X \}} \right] \right) \right\},$$

$$B = E \left\{ \text{vec} \left( \Delta g(Z, \beta^T X) \otimes \left[ a(X_i) - \frac{E \{ a(X_i)Y(Z) | \beta^T X \}}{E \{ Y(Z) | \beta^T X \}} \right] \right)^{\otimes 2} \right\}.$$
Here $\text{vecl}(A)$ represents the vectorization of the lower $(p-d) \times d$ block of a generic matrix $A$ and $A^{\otimes 2} = AA^T$ for any matrix or vector $A$. Note that in (10), $a(X_t) = X_t$ and in (13), $a(X_t) = X_t$, $g(Z, \beta^T X) = \hat{m}_{12}(Z, \beta^T X)/\{\hat{m}_1(Z, \beta^T X) + 1\} - \hat{m}_2(Z, \beta^T X)/\hat{m}(Z, \beta^T X)$.

Further, the estimator, $\hat{\beta}$, obtained from solving (13) is semiparametrically efficient and satisfies

$$\sqrt{n}(\hat{\beta} - \beta) \rightarrow N(0, [E\{S_{\text{eff}}^{\otimes 2}(\Delta, Z, X)\}]^{-1})$$

in distribution, where $S_{\text{eff}}(\Delta, Z, X)$ is given in (7).

With $S_{\text{eff}}$ being a martingale,

$$E\{S_{\text{eff}}^{\otimes 2}(\Delta, Z, X)\} = E\left[ \int_0^\infty \left( \frac{m_{12}(s, \beta^T X)}{m_1(s, \beta^T X)} + 1 - \frac{m_2(s, \beta^T X)}{m(s, \beta^T X)} \right) \otimes \left[ X_t \frac{E\{X_t S_c(s, X) \mid \beta^T X\}}{E\{S_c(s, X) \mid \beta^T X\}} \right] dN(s) \right],$$

which leads to a consistent estimator of $E\{S_{\text{eff}}^{\otimes 2}(\Delta, Z, X)\}$ as follows

$$\frac{1}{n} \sum_{i=1}^n \frac{\lambda_2(z_i, \hat{\beta}^T x_i)}{\hat{\lambda}(z_i, \hat{\beta}^T x_i)} \otimes \left[ x_{il} \frac{E\{X_i Y(z_i) \mid \beta^T x_i\}}{E\{Y(z_i) \mid \beta^T x_i\}} \right]^{\otimes 2}.$$

Here $E\{Y(z_i) \mid \beta^T x_i\}$, $E\{X_i Y(z_i) \mid \beta^T x_i\}$, $\lambda(z_i, \beta^T x_i)$ and $\hat{\lambda}_2(z_i, \beta^T x_i)$ are given in (11), (12), (S.3) and (S.4) respectively.
Theorem 3. Under Conditions C1-C7, the nonparametric estimator \( \hat{m}(t, \hat{\beta}^T x) \) satisfies

\[
\sqrt{n}h \left\{ \hat{m}(t, \hat{\beta}^T x) - m(t, \beta^T x) \right\} \to N\{0, \sigma^2_m(t, \beta^T x)\}
\]

in distribution, where

\[
\sigma^2_m(t, \beta^T x) = e^{2\Lambda(t, \beta^T x)} \int K^2(u)du \frac{\lambda(r, \beta^T x)}{\int E[I(Z \geq r) \mid \beta^T x]} \left\{ I(r < t) \int_t^\infty e^{-\Lambda(s, \beta^T x)}ds 
+ \int_{\max(r, t)}^\infty e^{-\Lambda(s, \beta^T x)}ds \right\}^2 dr.
\]

The proof is provided in Supplement S.7. Theorem 3 implies that we can even estimate the variance \( \sigma^2_m(t, \beta^T x) \), without estimating \( \lambda \) or \( f_{\beta^T x}(\beta^T x) \), by using

\[
\hat{\sigma}^2_m(t, \hat{\beta}^T x) = e^{2\Lambda(t, \hat{\beta}^T x)} \int K^2(u)du \sum_{i=1}^{nt} \frac{\hat{\Lambda}(\tau(i), \hat{\beta}^T x) - \hat{\Lambda}(\tau(i-1), \hat{\beta}^T x)}{1/n \sum_{j=1}^{n} Y(j)Y(j-1)}K_h(\beta^T x_j - \hat{\beta}^T x_j) \left[ I(\tau(i-1) < t) \sum_{j=1}^{nt} I(\tau(j) > t)e^{-\hat{\Lambda}(\tau(j-1), \hat{\beta}^T x)}(\tau(j) - \max(t, \tau(j-1))) 
+ \sum_{j=1}^{nt} I(\tau(j) > \max(t, \tau(i-1)))e^{-\hat{\Lambda}(\tau(j-1), \hat{\beta}^T x)}(\tau(j) - \max(r, \tau(i-1), \tau(j-1))) \right]^2
\]

where \( nt \) is the total number of the observed events and \( n_t \) is the number of the observed events up to time \( t \).
5 Simulation

The section features three simulation studies for evaluating the finite sample performance of our method in estimating $\beta$ and $m(t, \beta^T X)$. For comparisons, we also implement three estimators, two for the proportional mean residual life model, named as “PM1” and “PM2” for Chen and Cheng (2005) and Chen et al. (2005) respectively, and one for the additive mean residual life model (Chen, 2007), named as “additive”. All of these competing methods implicitly require $d = 1$.

**Study 1:** We generate event times with a hazard function $\lambda(t, \beta^T X) = 2te^{\beta^T X}/\{1 + (1 + t^2)e^{\beta^T X}\}$ such that the true mean residual life is

$$m(t, \beta^T X) = \frac{1 + e^{\beta^T X}}{1 + t^2 e^{\beta^T X}} \left[ \frac{\pi}{2} - \tan^{-1} \left\{ \frac{t}{1 + e^{\beta^T X}} \right\} \right].$$

Each component in $X$ is generated independently from the standard normal distribution. We consider $d = 1, p = 10$ and set the true parameters to be $\beta = (1, -0.75, 0.4, -0.2, 0.15, 0, -0.15, 0.2, -0.4, 0.75)^T$.

**Study 2:** We generate the event times from a log-logistic distribution with shape parameter 8.0 and scale parameter $\exp(\beta^T X)$. The corresponding mean residual life function is

$$m(t, \beta^T X) = \frac{\exp(\beta^T X)}{8} \left[ 1 + \left\{ \frac{t}{\exp(\beta^T X)} \right\}^8 \right]^{\frac{1}{8}} \int_{z(t)}^{1} (1 - u)^{7/8} u^{1/8} du,$$

where $z(t) = 1/[1 + \{t \exp(-\beta^T X)\}^8]$. Each component in $X$ is generated independently from the uniform distribution over $[0, 1]$. We consider $d = 1, p = 7$ and set the true
parameter $\beta = (1, 1.3, -1.3, 1, -0.5, 0.5, -0.5)^T$.

**Study 3:** We consider $d = 2$ and $p = 6$ and generate the event times with a hazard function $\lambda(t, \beta^T X) = t \{e^{(\beta^T x)_1} + e^{(\beta^T x)_2}\}$ where $(\beta^T X)_k$ denotes the $k$th entry of $\beta^T X$, and the true parameters are $\beta = \{(1, 0, 2.75, -0.75, -1, 2)^T, (0, 1, -3.125, -1.125, 1, -2)^T\}$. The mean residual life time function is

$$m(t, \beta^T X) = e^{\frac{1}{2}t^2 \{e^{(\beta^T x)_1} + e^{(\beta^T x)_2}\}} \int_t^\infty e^{-\frac{1}{2}s^2 \{e^{(\beta^T x)_1} + e^{(\beta^T x)_2}\}} ds.$$

For each simulation configuration, we generate 1,000 data sets with $n = 500$, and generate the censoring times from a gamma distribution with different parameter values to achieve various censoring rates.

The results for the estimation of $\beta$ under Study 1 are given in Table \ref{tab:study1} with three censoring rates, 0%, 20% and 40%. The proposed method has much smaller biases and standard deviations, whereas “PM1”, “PM2” and “additive” are biased with larger standard deviations. The performances of all of the estimators deteriorate when the censoring rate increases, though our method still outperforms the others. We also demonstrate the true and estimated mean residual life functions in Figure \ref{fig:study1}. The error plots, shown in the last row of Figure \ref{fig:study1} demonstrate that our method fare well for estimating $m(t, \beta^T x)$ when $t$ is not too large. The contour plots reveal that bias increases as censoring rate increases and the estimation deteriorates when $t$ is large. We further illustrate the dependence of $\hat{m}(t, \beta^T x)$ on $t$ at fixed $\beta^T x$ and the dependence of $\hat{m}(t, \beta^T x)$ on $\beta^T x$ at fixed $t$ values in Figure \ref{fig:study1}. These results show an overall satisfactory performance of our semiparametric
method. As Figure 1 reveals, the performance of our method is better when $t$ is in the interior of the range because more observations are available for the local estimation, as opposed to a larger $t$ with fewer observations still available. In contrast, regardless of the magnitude of $t$, the mean residual life function estimated by “PM1”, “PM2” and “additive” is severely biased, as shown in the last three rows from Figure S.1. This is because these models assume a pre-determined functional form of the mean residual life, which in this case is misspecified.

Tables S.1 and S.2 report the results of Studies 2 and 3 in relation to $\hat{\beta}$, respectively. We also provide the average estimation result of $m(t, \beta^T x)$ using a contour plot in Figure S.2 and Figure S.4. We further plot $\hat{m}(t, \beta^T x)$ as a function of $\beta^T x$ at fixed $t$ and as a function of $t$ at fixed $\beta^T x$ respectively in Figure S.3 and Figure S.5. Similar to the conclusion in the first simulation study, the performance of estimating $\beta$ by our proposed estimator is satisfactory. The performance of the mean residual life estimation is better when $t$ is smaller, deteriorates when $t$ becomes larger, and is better for smaller censoring rates. When $t$ is fixed, $\hat{m}(t, \beta^T x)$ from the proposed method has a stable performance in the entire range of $\beta^T x$ for all of the censoring rates. We also show $\hat{m}(t, \beta^T x)$ as a function of $t$ when $\beta^T x$ is fixed, in which the proposed estimator is consistent, while “PM1”, “PM2” and “additive” deviate from the true curve.

To recap, for estimating $\beta$, the proposed method yields much smaller biases than “PM1”, “PM2” and “additive”; for estimating the mean residual life function, “PM1”, “PM2” and “additive” estimators deviate much from the truth when the model is misspecified, but the proposed estimator recovers the truth well.
6 Analysis of the Kidney Transplant Data

We apply the proposed method to analyze a kidney transplant data set from the U.S. Scientific Registry of Transplant Recipients. The registry is maintained by the United Network for Organ Sharing and Organ Procurement and Transplantation Network (UNOS/OPTN) and includes all waitlisted kidney transplant candidates and transplant recipients in the United States (https://unos.org/). To evaluate the possible benefit of transplantation, we use the residual life to estimate how much longer a patient can survive if he receives a transplant than otherwise.

To avoid the confounding cohort effect, we consider the patients who were waitlisted in the same year of 2011. There were 42,217 patients in this cohort with an average followup of 907 days after waitlisting. During the followup, a total of 22,295 patients received kidney transplants. The response variable is the survival time in days ($T_i$) starting from waitlisting. In the transplant group, 5.82% of the observations were censored, while in the non-transplant group, the censoring rate was 26.27%. The covariates $X_{app}$ included in our analysis are gender ($X_1$), race ($X_2$), cold ischemia time ($X_3$), insurance coverage ($X_4$), body mass index ($X_5$), diagnosis type ($X_6$), peak PRA/CPRA ($X_7$), previous malignancy status ($X_8$) and diabetes indicator ($X_9$). In the transplant group of patients, we further included the waiting time $X_w$, calculated as the difference between operation and waitlisting. The hypothesis to test is that a patient with more prompt transplant operation may get more benefit. The goal of this study is to quantify the potential residual life increment if a patient receives a kidney transplant given the covariate profile.
As the survival trajectories for patients who received transplants might differ from those for patients who did not, we analyze the associations between the residual survival time and the covariates using model (1) for the transplant and non-transplant groups, separately. To proceed, we first determine the number of indices \( d \) through Validated Information Criterion (VIC) \( \text{Ma et al., 2015} \), where the smallest VIC value corresponds to the selected \( d \) value. In our analysis, \( d = 1 \) is chosen based on this criterion for both models and we fix \( d = 1 \) for these two models. The results are reported in Table 2. The index vector is normalized by fixing the first component (gender) at 1, hence only 9 coefficient estimates are reported in the transplant group and 8 in the non-transplant group.

We first focus on the model for the transplant group. Our analysis finds that several covariates, such as the body mass index \( (X_5) \), have no significant effects on the mean residual life, which agrees with the previous studies \( \text{Friedman et al., 2003} \). The waiting time \( X_w \) turns out not significant in the model for the transplant group. To confirm this, we investigate the possible confounding between \( X_w \) and other covariates. Specifically, we exclude \( X_w \) and perform the same analysis for the operation group using the remaining 9 covariates. The VIC results in dimension \( d = 1 \) as well and we report the results in Table 2. We then examine the dependence between \( X_w \) and the index \( \hat{\beta}^T X_{app} \), where \( \hat{\beta} \) contains the values reported in the middle part of Table 2 via the distance correlation test \( \text{Szekely et al., 2014} \). The \( p \)-value is less than 0.001, indicating a significant confounding effect. That is, the correlation between \( X_w \) and the index \( \hat{\beta}^T X_{app} \) masks the direct link between the waiting time and expected residual life. Indeed, the waiting time affects survival benefits in a very complex way \( \text{Meier-Kriesche et al., 2000; Gill et al., 2005} \) and
hence might infer complicated strategies for organ allocation (Meier-Kriesche and Kaplan, 2002).

We also estimate the mean residual function, i.e. \(\hat{m}_{\text{treat}}(t, \hat{\beta}_{\text{treat}}^T X_{\text{app}})\), where the subscript “treat” indicates the transplant group. We estimate \(\hat{m}_{\text{treat}}(t, \hat{\beta}_{\text{treat}}^T X_{\text{app}})\) with and without including \(X_w\) as a covariate. As the numerical results are almost identical, we only focus on the result without including \(X_w\) as a covariate in our later discussion.

We also present the model results for the patients that did not receive kidney transplantation during the followup in Table 2, which reveals that the sets of significant variables differ across the transplant and non-transplant groups. For example, BMI (\(X_5\)), which is not significant in the transplant group, has a positive effect on the mean residual life in the non-transplant group, which has been noted in the literature (Kalantar-Zadeh et al., 2005). Another example is diagnosis type (\(X_6\)), though not significant in the non-transplant group, has a significantly negative effect in the transplant group. This can be easily understood because hypertension (Frei et al., 1995), polycystic kidney disease (Kasiske et al., 2003) and vascular disease (Grimm et al., 1997) are high-risk factors for the post-transplant survival benefits. The last example is the previous malignancy status (\(X_8\)), which has opposite effects in the non-transplant and transplant groups. We believe this is caused by the correlation between covariates. Specifically, the type of malignancies (Brattström et al., 2013), the waiting time between malignancy diagnosed and transplantation (Penn, 1997) and the malignancy treatment before transplantation (Oechslin et al., 1996), which all affect patients’ survival, are highly correlated with \(X_8\).

Denote the estimated mean residual life time by \(\hat{m}_{\text{wait}}(t, \hat{\beta}_{\text{wait}}^T X_{\text{app}})\), where the subscript
“wait” indicates the waiting (or the non-transplant) group. The contour plot of both \( \hat{m}_{\text{treat}}(t, \beta_{\text{treat}}^T \text{X}_{\text{app}}) \) and \( \hat{m}_{\text{wait}}(t, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) are given in Figure 2.

Given a patient with \( \text{X} \) and alive at time \( t \), \( \hat{m}_{\text{treat}}(t, \beta_{\text{treat}}^T \text{X}_{\text{app}}) - \hat{m}_{\text{wait}}(t, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) provides an estimate of his/her mean residual life time difference between receiving and not receiving kidney transplant. Because the difference is a function of three variables, namely \( t, \beta_{\text{treat}}^T \text{X}_{\text{app}} \) and \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \), we present the difference using various plots. Figure 3 plots curves that change with \( t \) at several fixed \( (\beta_{\text{treat}}^T \text{X}_{\text{app}}, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) values, curves that change with \( \beta_{\text{treat}}^T \text{X}_{\text{app}} \) at several fixed \( (t, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) values, and curves that change with \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \) at several fixed \( (t, \beta_{\text{treat}}^T \text{X}_{\text{app}}) \) values. Further, Figure S.6 plots the contour of the mean residual life time as a function of \( (\beta_{\text{treat}}^T \text{X}_{\text{app}}, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) at a fixed time \( t \), as a function of \( (t, \beta_{\text{wait}}^T \text{X}_{\text{app}}) \) at a fixed index value \( \beta_{\text{treat}}^T \text{X}_{\text{app}} \) and as a function of \( (t, \beta_{\text{treat}}^T \text{X}_{\text{app}}) \) at a fixed index value \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \).

In summary, we find that when the waiting time is less than 500 days, regardless of the index values \( \beta_{\text{treat}}^T \text{X}_{\text{app}} \) and \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \), 150 to 300 more days on average can be gained if a patient receives transplantation; with the waiting time between 500 and 1000 days, kidney transplants can still lead to a reasonably large improvement if the patient’s index value \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \) is positive (indicating overall good health), regardless of his/her \( \beta_{\text{treat}}^T \text{X}_{\text{app}} \) value. However, if the waiting time is more than 1000 days or the waiting time is more than 500 days but with a negative index \( \beta_{\text{wait}}^T \text{X}_{\text{app}} \) (indicating poor health conditions), the benefit of kidney transplant is the least.
7 Discussion

The work is to address a severe shortage of organs that are needed to sustain ESRD patients’ life and aims to design a feasible strategy to increase the potential efficiency brought by each available kidney. Instead of evaluating the patients’ expected survival time, as is done in current strategy, we propose to consider the potential residual life prolonged by kidney transplant. We compare the patients’ expected residual life with and without transplant and use the difference to gauge the potential benefit gained from the transplant. Patients with larger differences may have a higher priority for organ allocation than those with smaller incremental values.

We have proposed a flexible semiparametric regression model of mean residual life, which relaxes the parametric assumptions on the dependence of mean residual life on covariates and how long a patient has lived. To strike a balance between interpretation and flexibility, our procedure also enables reduce the covariate dimension $p$ to $d$: when $d = 1$, the model falls to the single index model, while $d = p$ corresponds to a completely nonparametric model. We suggest to use the Validated Information Criterion (Ma et al., 2015) to choose $d$, which seems to fare well in practice.
Table 1: Results of study 1, based on 1000 simulations with sample size 500. “Prop.” is the semiparametric method, “PM1” and “PM2” are the proportional mean residual life methods, “additive” is the additive method. “bias” is the average absolute bias of each component in $\hat{\beta}$, “sd” is the sample standard deviation of the corresponding estimators, “MSE” is the mean squared error.

|                | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ | $\beta_8$ | $\beta_9$ | $\beta_{10}$ |
|----------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-------------|
| true           | -0.75     | 0.4       | -0.2      | 0.15      | 0.0       | -0.15     | 0.2       | -0.4      | 0.75        |
| No censoring   |           |           |           |           |           |           |           |           |             |
| Prop. bias     | 0.002     | 0.004     | 0.001     | 0.000     | 0.002     | 0.005     | 0.003     | 0.002     | 0.003       |
| sd             | 0.091     | 0.090     | 0.092     | 0.091     | 0.088     | 0.083     | 0.098     | 0.081     | 0.092       |
| MSE            | 0.008     | 0.008     | 0.008     | 0.008     | 0.008     | 0.007     | 0.01      | 0.006     | 0.008       |
| PM1 bias       | 0.057     | 0.027     | 0.014     | 0.018     | 0.003     | 0.025     | 0.019     | 0.004     | 0.059       |
| sd             | 0.383     | 0.319     | 0.299     | 0.300     | 0.307     | 0.307     | 0.296     | 0.323     | 0.382       |
| MSE            | 0.152     | 0.105     | 0.091     | 0.114     | 0.103     | 0.097     | 0.088     | 0.105     | 0.151       |
| PM2 bias       | 0.129     | 0.069     | 0.004     | 0.043     | 0.017     | 0.029     | 0.014     | 0.046     | 0.153       |
| sd             | 0.824     | 0.736     | 0.695     | 0.665     | 0.649     | 0.626     | 0.631     | 0.687     | 0.761       |
| MSE            | 0.532     | 0.392     | 0.431     | 0.377     | 0.422     | 0.358     | 0.311     | 0.397     | 0.513       |
| additive bias  | 0.066     | 0.023     | 0.017     | 0.017     | 0.004     | 0.026     | 0.021     | 0.012     | 0.060       |
| sd             | 0.396     | 0.339     | 0.322     | 0.316     | 0.346     | 0.337     | 0.335     | 0.340     | 0.395       |
| MSE            | 0.184     | 0.127     | 0.114     | 0.124     | 0.137     | 0.114     | 0.115     | 0.119     | 0.184       |
| 20% censoring  |           |           |           |           |           |           |           |           |             |
| Prop. bias     | 0.002     | 0.016     | 0.000     | 0.001     | 0.003     | 0.003     | 0.001     | 0.006     | 0.001       |
| sd             | 0.122     | 0.127     | 0.124     | 0.123     | 0.116     | 0.118     | 0.116     | 0.124     | 0.120       |
|          | MSE       | 0.015 | 0.016 | 0.015 | 0.015 | 0.013 | 0.014 | 0.014 | 0.013 | 0.015 | 0.014 |
|----------|-----------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| PM1 bias |           | 0.098 | 0.054 | 0.029 | 0.020 | 0.006 | 0.022 | 0.022 | 0.065 | 0.032 | 0.087 |
| sd       |           | 0.569 | 0.444 | 0.413 | 0.419 | 0.411 | 0.430 | 0.435 | 0.442 | 0.500 |
| MSE      |           | 0.333 | 0.200 | 0.172 | 0.176 | 0.168 | 0.185 | 0.193 | 0.197 | 0.257 |
| PM2 bias |           | 0.136 | 0.071 | 0.009 | 0.076 | 0.025 | 0.058 | 0.058 | 0.054 | 0.097 | 0.131 |
| sd       |           | 1.548 | 1.132 | 1.113 | 1.147 | 1.171 | 1.194 | 1.110 | 1.333 | 1.545 |
| MSE      |           | 2.413 | 1.284 | 1.237 | 1.320 | 1.371 | 1.427 | 1.233 | 1.784 | 2.401 |
| additive bias | | 0.094 | 0.052 | 0.033 | 0.011 | 0.007 | 0.024 | 0.056 | 0.038 | 0.070 |
| sd       |           | 0.512 | 0.403 | 0.413 | 0.372 | 0.388 | 0.409 | 0.386 | 0.435 | 0.461 |
| MSE      |           | 0.271 | 0.165 | 0.171 | 0.139 | 0.151 | 0.168 | 0.152 | 0.191 | 0.217 |

|          | Prop. bias | 0.003 | 0.016 | 0.005 | 0.004 | 0.010 | 0.010 | 0.007 | 0.000 | 0.005 |
| sd       |           | 0.175 | 0.190 | 0.155 | 0.169 | 0.204 | 0.181 | 0.161 | 0.140 | 0.188 |
| MSE      |           | 0.030 | 0.036 | 0.024 | 0.029 | 0.042 | 0.033 | 0.026 | 0.02   | 0.035 |
| PM1 bias |           | 0.121 | 0.067 | 0.049 | 0.031 | 0.000 | 0.026 | 0.046 | 0.037 | 0.106 |
| sd       |           | 0.734 | 0.482 | 0.541 | 0.394 | 0.410 | 0.429 | 0.456 | 0.447 | 0.606 |
| MSE      |           | 0.553 | 0.237 | 0.295 | 0.156 | 0.168 | 0.184 | 0.209 | 0.201 | 0.378 |
| PM2 bias |           | 0.232 | 0.068 | 0.063 | 0.129 | 0.066 | 0.242 | 0.047 | 0.200 | 0.186 |
| sd       |           | 5.141 | 3.178 | 4.173 | 3.268 | 4.487 | 4.783 | 3.618 | 4.341 | 5.914 |
| MSE      |           | 26.45 | 10.09 | 17.39 | 10.68 | 20.11 | 22.91 | 13.08 | 18.86 | 34.97 |
| additive bias | | 0.107 | 0.063 | 0.042 | 0.036 | 0.017 | 0.027 | 0.033 | 0.041 | 0.101 |
| sd       |           | 0.566 | 0.462 | 0.475 | 0.398 | 0.376 | 0.427 | 0.420 | 0.420 | 0.625 |
| MSE      |           | 0.332 | 0.217 | 0.227 | 0.160 | 0.141 | 0.183 | 0.178 | 0.178 | 0.401 |

40% censoring
Table 2: Analysis of the kidney transplant data, where “SE” is the estimated standard error of $\hat{\beta}$

|                  | $\hat{\beta}_2$ | $\hat{\beta}_3$ | $\hat{\beta}_4$ | $\hat{\beta}_5$ | $\hat{\beta}_6$ | $\hat{\beta}_7$ | $\hat{\beta}_8$ | $\hat{\beta}_9$ | $\hat{\beta}_w$ |
|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| **Transplant including waiting time $X_w$** |                  |                  |                  |                  |                  |                  |                  |                  |                  |
| **Estimate**     | -0.928           | 0.488            | 1.045            | 0.016            | -1.057           | 1.006            | 0.475            | -1.041           | -0.053           |
| **SE**           | 0.293            | 0.254            | 0.315            | 0.231            | 0.277            | 0.294            | 0.258            | 0.338            | 0.194            |
| **p-value**      | $< 0.001$        | 0.027            | $< 0.001$        | 0.471            | $< 0.001$        | $< 0.001$        | 0.033            | 0.001            | 0.392            |
| **Transplant excluding waiting time $X_w$** |                  |                  |                  |                  |                  |                  |                  |                  |                  |
| **Estimate**     | -0.875           | 0.477            | 0.912            | 0.061            | -0.779           | 0.966            | 0.379            | -0.958           |
| **SE**           | 0.280            | 0.109            | 0.218            | 0.155            | 0.360            | 0.263            | 0.162            | 0.269            |
| **p-value**      | 0.002            | $< 0.001$        | $< 0.001$        | 0.694            | 0.030            | $< 0.001$        | 0.019            | $< 0.001$        |
| **Non-transplant** |                  |                  |                  |                  |                  |                  |                  |                  |                  |
| **Estimate**     | -0.875           | 1.016            | 1.116            | 1.061            | -0.033           | 0.995            | -1.106           | -1.123           |
| **SE**           | 0.221            | 0.263            | 0.363            | 0.338            | 0.158            | 0.218            | 0.432            | 0.332            |
| **p-value**      | $< 0.001$        | $< 0.001$        | 0.002            | 0.002            | 0.834            | $< 0.001$        | 0.011            | $< 0.001$        |
Figure 1: Performance of the semiparametric method on mean residual life function of study 1. First row: contour plot of true $m(t, \beta^T X)$; Second row: contour plot of $\hat{m}(t, \beta^T X)$; Third row: contour plot of $|\hat{m}(t, \beta^T X) - m(t, \beta^T X)|$. Left to right columns: no censoring; 20% censoring rate; 40% censoring rate.
Figure 2: Mean residual life function in kidney transplant application study. Left panel: Patient receive kidney transplant. Right panel: Patients did not received kidney transplant.

Figure 3: Mean residual life difference between receiving and not receiving transplant. Left panel: time is the variable at the following setting of \([X_3, X_5, X_6, X_8]\), red: \([100, 50, 2, 3]\), blue: \([20, 45, 9, 1]\), orange: \([80, 30, 4, 1]\), yellow: \([50, 30, 2, 1]\), green: \([50, 15, 7, 1]\). The rest variables remain the same at \([X_1, X_2, X_4, X_7, X_9]\) = \([1, 1, 1, 50, 0]\). Middle panel: \(\beta^T_{\text{treat}} X\) is the variable with the following setting, blue: \(\beta^T_{\text{wait}} X = -40\), green: \(\beta^T_{\text{wait}} X = 50\), red: \(\beta^T_{\text{wait}} X = 100\), solid: \(t = 100\), dashed: \(t = 500\), dotted: \(t = 1000\). Right panel: \(\beta^T_{\text{wait}} X\) is the variable with the following setting, blue: \(\beta^T_{\text{treat}} X = -40\), green: \(\beta^T_{\text{treat}} X = 50\), red: \(\beta^T_{\text{treat}} X = 100\), solid: \(t = 100\), dashed: \(t = 500\), dotted: \(t = 1000\).
Supplement to “Semiparametric regression of mean residual life with censoring and covariate dimension reduction”

S.1 Proof of Proposition 1

Proof: Let $T_1$, $T_2$ and $T_3$ be the nuisance tangent spaces corresponding to $f_X$, $m(\cdot, \cdot)$ and $\lambda_c$ respectively. The result of $T_1$ follows obviously. To obtain $T_2$, let $m(z, \beta^T X) + \gamma \mathbf{h}(z, \beta^T X)$ be a sub model of $m(z, \beta^T X)$, where $\mathbf{h}(z, \beta^T X) \in \mathcal{R}^{(p-d)d}$ with $\text{var}\{\mathbf{h}(z, \beta^T X)\} < \infty$, differentiate the log of (4) with respect to $\gamma$ and evaluate it at $\gamma = 0$. Then, $T_2$ is

$$
\Delta \left\{ \frac{h_1(z, \beta^T X)}{m_1(z, \beta^T X) + 1} - \frac{h(z, \beta^T X)}{m(z, \beta^T X)} \right\} \left[ 1 \right] - \int_0^z \frac{h_1(s, \beta^T X)m(s, \beta^T X) - \{m_1(s, \beta^T X) + 1\}h(s, \beta^T X)}{m^2(s, \beta^T X)} dt
$$

$$
= \Delta \left\{ \frac{h_1(z, \beta^T X)}{m_1(z, \beta^T X) + 1} - \frac{h(z, \beta^T X)}{m(z, \beta^T X)} \right\} - \int_0^z \left\{ \frac{h_1(s, \beta^T X)}{m_1(s, \beta^T X) + 1} - \frac{h(s, \beta^T X)}{m(s, \beta^T X)} \right\} \lambda(s, \beta^T X) ds
$$

$$
= \int_0^\infty \left\{ \frac{h_1(s, \beta^T X)}{m_1(s, \beta^T X) + 1} - \frac{h(s, \beta^T X)}{m(s, \beta^T X)} \right\} dM(s, \beta^T X)
$$

where $h_1(z, \beta^T X) = \partial h(z, \beta^T X)/\partial z$.

To obtain $T_3$, let $\lambda_c(t, X)\{1 + \gamma \mathbf{h}(t, X)\}$ be a submodel of $\lambda_c(t, X)$, where $\mathbf{h}(z, X) \in \mathcal{R}^{(p-d)d}$, with $\text{var}\{\mathbf{h}(z, X)\} < \infty$. We then obtain $T_3$ as follows

$$
\frac{\partial \log f(X, Z, \Delta)}{\partial \gamma} |_{\gamma = 0} = (1 - \Delta)h(Z, X) - \int_0^Z h(s, X)\lambda_c(s, X) ds
$$

$$
= \int_0^\infty h(s, X) dM_c(s, X),
$$

31
where $M_c(t, X) = N_c(t) - \int_0^t I(Z \geq s) \lambda_c(s, X) ds$ is a martingale process (See Theorem 1.3.2 in Fleming and Harrington (1991)). A similar result was also established by Prentice and Kalbfleisch (2003) for a mixed discrete and continuous Cox regression model. Because $\lambda_c(t, X)$ can be any positive function, $h(s, X)$ can be any function. This leads to the form of $T_3$.

By taking conditional expectations given $X$, it follows that $T_1 \perp T_2$ and $T_1 \perp T_3$. Further, $T_2 \perp T_3$ because the martingale integrals associated with $M(s, \beta^T X)$ and $M_c(s, X)$ are independent conditional on $X$ due to $T \parallel C \mid X$. 

\section*{S.2 Proof of Equations (8) and (9)}

Proof: First, we note that, when $t \leq \tau$,

\begin{align*}
E \{ X_i Y(t) \mid \beta^T X \} &= E[E \{ X_i I(T \geq t) I(C \geq t) \mid \beta^T X, X \} \mid \beta^T X] \\
&= E[E \{ X_i I(T \geq t) I(C \geq t) \mid X \} \mid \beta^T X] \\
&= E \{ X_i S(t, \beta^T X) S_c(t, X) \mid \beta^T X \} \\
&= S(t, \beta^T X) E \{ X_i S_c(t, X) \mid \beta^T X \},
\end{align*}

where the second to last equality holds because of $T \parallel C \mid X$ and that $\text{pr}(T \geq t \mid X)$ is a function of $\beta^T X$ only. Similarly, for $t \leq \tau$, we obtain that

\begin{equation*}
E \{ Y(t) \mid \beta^T X \} = S(t, \beta^T X) E \{ S_c(t, X) \mid \beta^T X \}.
\end{equation*}

Hence, when $t \leq \tau$,

\begin{equation*}
\frac{E \{ X_i Y(t) \mid \beta^T X \}}{E \{ Y(t) \mid \beta^T X \}} = \frac{E \{ X_i S_c(t, X) \mid \beta^T X \}}{E \{ S_c(t, X) \mid \beta^T X \}}. \tag{S.1}
\end{equation*}

Second, when $t > \tau$, $Y(t) = 0$ and $S_c(t, X) = 0$. Hence, we have a 0/0 scenario in (S.1).
in which case, we define (S.1) to be

\[
\frac{E \{ X_lS_c(\tau, X) \mid \beta^T X \}}{E \{ S_c(\tau, X) \mid \beta^T X \}} = \frac{E \{ X_lp(X) \mid \beta^T X \}}{E \{ p(X) \mid \beta^T X \}},
\]

a time-invariant constant. Here, \( p(X) \) is defined in Section 3.1. Hence, (S) holds over \([0, \infty)\), with the truth of \( \beta \) being \( \beta_0 \).

With a generic \( \beta \), (S.1) leads to

\[
\begin{align*}
E \left( \int_0^\infty g(s, \beta^T X) \otimes \left[ X_l - \frac{E \{ X_lS_c(s, X) \mid \beta^T X \}}{E \{ S_c(s, X) \mid \beta^T X \}} \right] Y(s)\lambda_0(s, \beta^T X)ds \right) \\
= E \left( \int_0^\infty g(s, \beta^T X) \otimes \left[ E \{ X_lY(s) \mid \beta^T X \} - \frac{E \{ X_lS_c(s, X) \mid \beta^T X \}}{E \{ S_c(s, X) \mid \beta^T X \}}E \{ Y(s) \mid \beta^T X \} \right] \lambda_0(s, \beta^T X)ds \right) \\
= 0,
\end{align*}
\]

as the quantity inside the square bracket is zero. In addition,

\[
E \left( \int_0^\infty g(s, \beta^T X) \otimes \left[ X_l - \frac{E \{ X_lS_c(s, X) \mid \beta^T X \}}{E \{ S_c(s, X) \mid \beta^T X \}} \right] dM(s, \beta^T X) \right) = 0,
\]

because \( dM(s, \beta^T X) = dN(s) - Y(s)\lambda_0(s, \beta^T X)ds \) is a martingale. Therefore, we have

\[
E \left( \int_0^\infty g(s, \beta^T X) \otimes \left[ X_l - \frac{E \{ X_lS_c(s, X) \mid \beta^T X \}}{E \{ S_c(s, X) \mid \beta^T X \}} \right] dN(s) \right) = 0.
\]

Hence, (S) holds with the truth of \( \beta \) being \( \beta_0 \).
S.3 Nonparametric Estimators

S.3.1 Nonparametric Estimators of Hazard and Mean Residual Life Functions and Their Derivatives

The nonparametric estimators of $\Lambda_2(t, \beta^T X)$, $\lambda(t, \beta^T X)$, $\lambda_2(t, \beta^T X)$, $m_1(t, \beta^T X)$, $m_2(t, \beta^T X)$, and $m_{12}(t, \beta^T X)$ are

\[
\hat{\Lambda}_2(t, \beta^T X) = -\sum_{i=1}^{n} \frac{I(Z_i \leq t) \Delta_i K_h'(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)} + \sum_{i=1}^{n} I(Z_i \leq t) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}^2, \tag{S.2}
\]

\[
\hat{\lambda}(t, \beta^T X) = -\sum_{i=1}^{n} \frac{K_h(Z_i - t) \Delta_i K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}, \tag{S.3}
\]

\[
\hat{\lambda}_2(t, \beta^T X) = -\sum_{i=1}^{n} \frac{K_h(Z_i - t) \Delta_i K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)} + \sum_{i=1}^{n} K_h(Z_i - t) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X)}^2, \tag{S.4}
\]

\[
\hat{m}_1(t, \beta^T X) = \hat{\lambda}(t, \beta^T X)e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} e^{-\hat{\lambda}(s, \beta^T X)} ds - 1, \tag{S.5}
\]

\[
\hat{m}_2(t, \beta^T X) = \hat{\Lambda}_2(t, \beta^T X)e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} e^{-\hat{\lambda}(s, \beta^T X)} ds - e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} \hat{\Lambda}_2(s, \beta^T X) e^{-\hat{\lambda}(s, \beta^T X)} ds, \tag{S.6}
\]

\[
\hat{m}_{12}(t, \beta^T X) = \hat{\Lambda}_2(t, \beta^T X)e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} e^{-\hat{\lambda}(s, \beta^T X)} ds + \hat{\lambda}(t, \beta^T X)\hat{\Lambda}_2(t, \beta^T X)e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} e^{-\hat{\lambda}(s, \beta^T X)} ds - \hat{\lambda}(t, \beta^T X)e^{\hat{\lambda}(t, \beta^T X)} \int_{t}^{\infty} \hat{\Lambda}_2(s, \beta^T X) e^{-\hat{\lambda}(s, \beta^T X)} ds. \tag{S.6}
\]
The trimmed estimators of (S.11), (S.12), (S.13), and (S.2) - (S.4) are

\[
\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i) I(Z_j \geq Z_i)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i)} \left\{ \frac{1}{n} \sum_{k=1}^{n} K_h(\beta^T X_k - \beta^T X_i) > d_n \right\}, \tag{S.7}
\]

\[
\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \} = \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i) X_i I(Z_j \geq Z_i)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X_i)} \left\{ \frac{1}{n} \sum_{k=1}^{n} K_h(\beta^T X_k - \beta^T X_i) > d_n \right\}. \tag{S.8}
\]

\[
\hat{\lambda}(t, \beta^T X) = \sum_{j=1}^{n} \int_{0}^{t} \frac{K_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X)} I \left\{ \frac{1}{n} \sum_{k=1}^{n} Y_j(s) K_h(\beta^T X_k - \beta^T X) > d_n \right\} dN_i(s), \tag{S.9}
\]

\[
\hat{\lambda}_2(t, \beta^T X) = -\sum_{j=1}^{n} \frac{K_h(Z_j - t) \Delta_j K_h'(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X)} I \left\{ \frac{1}{n} \sum_{k=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X) > d_n \right\} \left\{ \frac{1}{n} \sum_{k=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X) > d_n \right\}, \tag{S.10}
\]

\[
\hat{\lambda}(t, \beta^T X) = \sum_{j=1}^{n} \frac{K_h(Z_j - t) \Delta_j K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X)} I \left\{ \frac{1}{n} \sum_{k=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X) > d_n \right\}, \tag{S.11}
\]

\[
\hat{\lambda}_2(t, \beta^T X) = -\sum_{j=1}^{n} \frac{K_h(Z_j - t) \Delta_j K_h(\beta^T X_j - \beta^T X)}{\sum_{j=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X)} I \left\{ \frac{1}{n} \sum_{k=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X) > d_n \right\} \left\{ \frac{1}{n} \sum_{k=1}^{n} I(Z_k \geq Z_j) K_h(\beta^T X_k - \beta^T X) > d_n \right\}. \tag{S.12}
\]
S.4 Two useful lemmas

S.4.1 Lemma 1

**Lemma 1.** Under the regularity conditions C1-C5 listed above,

\[
\hat{E} \{Y(Z) \mid \beta^T X\} = E\{Y(Z) \mid \beta^T X\} + O_p((nh)^{-1/2} + h^2), \quad (S.13)
\]

\[
\hat{E} \{XY(Z) \mid \beta^T X\} = E\{XY(Z) \mid \beta^T X\} + O_p((nh)^{-1/2} + h^2), \quad (S.14)
\]

\[
\hat{\lambda}(z, \beta^T X) = \lambda(z, \beta^T X) + O_p((nhb)^{-1/2} + h^2 + b^2) \quad (S.15)
\]

\[
\hat{\lambda}_2(z, \beta^T X) = \lambda_2(z, \beta^T X) + O_p((nhb^3)^{-1/2} + h^2 + b^2) \quad (S.16)
\]

\[
\hat{\Lambda}(z, \beta^T X) = \Lambda(z, \beta^T X) + O_p((nh)^{-1/2} + h^2) \quad (S.17)
\]

\[
\hat{\Lambda}_2(z, \beta^T X) = \Lambda_2(z, \beta^T X) + O_p((nh^3)^{-1/2} + h^2) \quad (S.18)
\]

uniformly for all \(z, \beta^T X\).

Proof: For notation convenience, we prove the results for \(d = 1\). We prove

\[
\hat{E} \{XY(Z) \mid \beta^T X\} = E\{XY(Z) \mid \beta^T X\} + O_p((nh)^{-1/2} + h^2)
\]

and

\[
\hat{\Lambda}_2(z, \beta^T X) = \Lambda_2(z, \beta^T X) + O_p((nh^3)^{-1/2} + h^2).
\]

and skip the remaining results because of the similar arguments.

First, for any \(X\) and \(\beta\) in a local neighborhood of \(\beta_0\),

\[
\frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) = f_{\beta^T X}(\beta^T X) + O_p(n^{-1/2} h^{-1/2} + h^2), \quad (S.19)
\]
To see this, the absolute bias of the left hand side of (5.19) is

\[
\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} K_h (\beta^T x_j - \beta^T X) \right\} - f_{\beta^T X}(\beta^T X) \right|
\]

\[
= \left| \int \frac{1}{h} K \left( \frac{\beta^T x_j - \beta^T X}{h} \right) f_{\beta^T X}(\beta^T x_j) d\beta^T x_j - f_{\beta^T X}(\beta^T X) \right|
\]

\[
= \left| \int K(u) f_{\beta^T X}(\beta^T X + hu) du - f_{\beta^T X}(\beta^T X) \right|
\]

\[
= \left| \int K(u) \left\{ f_{\beta^T X}(\beta^T X) + f'_{\beta^T X}(\beta^T X) hu + \frac{1}{2} f''_{\beta^T X}(\xi) h^2 u^2 \right\} du - f_{\beta^T X}(\beta^T X) \right|
\]

\[
\leq \frac{h^2}{2} \sup_{\beta^T X} |f''_{\beta^T X}(\beta^T X)| \int u^2 K(u) du,
\]

where throughout the text, \(\xi\) is between \(\beta^T X\) and \(\beta^T X + hu\). The variance is

\[
\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} K_h (\beta^T x_j - \beta^T X) \right\}
\]

\[
= \frac{1}{n} \text{var} K_h (\beta^T X_j - \beta^T X)
\]

\[
= \frac{1}{n} \left[ E K_h^2 (\beta^T X_j - \beta^T X) - \left\{ E K_h (\beta^T X_j - \beta^T X) \right\}^2 \right]
\]

\[
= \frac{1}{n} \left[ \int \frac{1}{h^2} K^2 (\beta^T x_j - \beta^T X) / h \right] f_{\beta^T X}(\beta^T x_j) d\beta^T x_j - f_{\beta^T X}(\beta^T X) + O(h^2)
\]

\[
= \frac{1}{nh} \int K^2(u) f_{\beta^T X}(\beta^T X + hu) du - \frac{1}{n} f_{\beta^T X}(\beta^T X) + O(h^2/n)
\]

\[
\leq \frac{1}{nh} f_{\beta^T X}(\beta^T X) \int K^2(u) du + \frac{h}{2n} \sup_{\beta^T X} |f''_{\beta^T X}(\beta^T X)| \int u^2 K^2(u) du + \frac{1}{n} f_{\beta^T X}(\beta^T X) + O(h^2/n).
\]
Therefore, applying the central limit theorem, we have that

$$\frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) = f_{\beta^T X}(\beta^T X) + O_p(n^{-1/2}h^{-1/2} + h^2)$$

for all $\beta$ under conditions [C1][C3].Condition [C3] also holds for any $\beta$ in a local neighborhood of $\beta_0$ due to the continuity. Similarly, We have

$$-\frac{1}{n} \sum_{j=1}^{n} K_h'(\beta^T X_j - \beta^T X) = f_{\beta^T X}'(\beta^T X) + O_p(n^{-1/2}h^{-3/2} + h^2). \quad (S.20)$$

To show (S.14), the absolute bias and variance are

$$\left| E \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq z) \right\} K_h(\beta^T X_j - \beta^T X) \right| - f_{\beta^T X}(\beta^T X) E\{X_j(Z_j \geq z) \mid \beta^T X\}$$

$$= \frac{h^2}{2} \int \frac{\partial^2}{\partial (\beta^T X)^2} f_{\beta^T X}(\xi) E\{X_j(Z_j \geq z) \mid \xi\} u^2 K(u) du$$

$$\leq \frac{h^2}{2} \sup_{\beta^T X} \left| \frac{\partial^2}{\partial (\beta^T X)^2} f_{\beta^T X}(\beta^T X) E\{X_j(Z_j \geq z) \mid \beta^T X\} \right| \left\{ \int u^2 K(u) du \right\},$$

and

$$\text{var} \left\{ \frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq z) K_h(\beta^T X_j - \beta^T X) \right\}$$

$$\leq \frac{1}{nh} \sup_{\beta^T X} \left| f_{\beta^T X}(\beta^T X) E\{X_j X_j^T I(Z_j \geq z) \mid \beta^T X\} \right| \int K^2(u) du$$

$$+ \frac{h}{2n} \sup_{\xi} \left| \frac{\partial^2}{\partial (\beta^T X)^2} f_{\beta^T X}(\xi) E\{X_j X_j^T I(Z_j \geq z) \mid \xi\} \right| \int u^2 K^2(u) du + O(1/n)$$
under conditions \([C_1, C_3]\). So

\[
\frac{1}{n} \sum_{j=1}^{n} X_j I(Z_j \geq z)K_h(\beta^T X_j - \beta^T X) = f_{\beta^T X}(\beta^T X)E\{X_j I(Z_j \geq z) \mid \beta^T X\} + O_p(n^{-1/2}h^{-1/2} + h^2)
\]  

(S.21)

under conditions \([C_1, C_3]\).

To show (S.18), let

\[
\hat{\Lambda}_{21}(z, \beta^T X) = -\sum_{i=1}^{n} \frac{I(Z_i \leq z)\Delta_i K'_h(\beta^T X_i - \beta^T X)}{\sum_{j=1}^{n} I(Z_j \geq Z_i)K_h(\beta^T X_j - \beta^T X)}
\]

\[
\hat{\Lambda}_{22}(z, \beta^T X) = \sum_{i=1}^{n} I(Z_i \leq z)\Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\sum_{j=1}^{n} I(Z_j \geq Z_i)K_h(\beta^T X_j - \beta^T X)}{\left\{\sum_{j=1}^{n} I(Z_j \geq Z_i)K_h(\beta^T X_j - \beta^T X)\right\}^2}.
\]

Then \(\hat{\Lambda}_2(z, \beta^T X) = \hat{\Lambda}_{21}(z, \beta^T X) + \hat{\Lambda}_{22}(z, \beta^T X)\). To analyze \(\hat{\Lambda}_{21}\),

\[
E\hat{\Lambda}_{21}(z, \beta^T X) = E \left[ \frac{-I(Z_i \leq z)\Delta_i K'_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X)S(Z_i, \beta^T X)E\{S_c(Z_i, X_j) \mid \beta^T X_j = \beta^T X, Z_i\}} \right] + E \left[ \frac{1}{n} \sum_{i=1}^{n} \frac{-I(Z_i \leq z)\Delta_i K'_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X)S(Z_i, \beta^T X)E\{S_c(Z_i, X_j) \mid \beta^T X_j = \beta^T X, Z_i\}} O_p(A) \right].
\]
The first term is

$$
E \left[ \frac{-I(Z_i \leq z) \Delta_i K'_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_j = \beta^T X, Z_i\}} \right]
$$

$$
= \int I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \frac{f_{\beta^T X}(\beta^T X) E\{S_c(z, X_i) \mid \beta^T X, z\} f_{\beta^T X}(\beta^T X)}{f_{\beta^T X}(\beta^T X) S(z, \beta^T X) E\{S_c(z, X_j) \mid \beta^T X, z\}} dz_i
$$

$$
- \frac{h^2 \partial^3}{6 \partial(\beta^T X)^3} \int I(z_i \leq z) \frac{f_{\beta^T X}(\beta^T X) (\xi) f(z_i, \xi) E\{S_c(z, X_i) \mid \xi, z_i\} f_{\beta^T X}(\beta^T X)}{f_{\beta^T X}(\beta^T X) S(z, \beta^T X) E\{S_c(z, X_j) \mid \beta^T X, z_i\}} d\xi d\beta^T X d\beta^T X
$$

$$
\leq h^2 \sup_{z, \beta^T X, \xi} \left| \frac{\partial^3}{6 \partial(\beta^T X)^3} \int I(z_i \leq z) f_{\beta^T X}(\beta^T X) (\xi) f(z_i, \xi) E\{S_c(z, X_i) \mid \xi, z_i\} f_{\beta^T X}(\beta^T X)}{f_{\beta^T X}(\beta^T X) S(z, \beta^T X) E\{S_c(z, X_j) \mid \beta^T X, z_i\}} d\xi d\beta^T X d\beta^T X
$$

$$
\times \int |u^3 K'(u)| du
$$

$$
= O(h^2)
$$

under Condition C1-C5. Similarly, we conclude that

$$
\frac{1}{n} \sum_{i=1}^{n} \frac{-I(Z_i \leq z) \Delta_i K'_h(\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_j = \beta^T X, Z_i\}} O_p(A) = O_p(h^2 + (nh)^{-1/2})
$$

41
under conditions $[C1-C5]$ due to $A = O_p\{h^2 + (nh)^{-1/2}\}$. Therefore

\[
E\tilde{\Lambda}_{21}(z, \beta^T X) = \int I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \left[ f(z_i, \beta^T X) E\{S_c(z_i, X_i) | \beta^T X, z_i\} f_{\beta^T X}(\beta^T X) \right] / \partial \beta^T X \, dz_i + O((nh)^{-1/2} + b^2 + h^2).
\]

For $\tilde{\Lambda}_{22}$, let $B = -1/n \sum_{i=1}^{n} I(Z_j \geq Z_i) K_h(\beta^T X_j - \beta^T X) - \partial f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq Z_i) | \beta^T X\} / \partial \beta^T X$, then

\[
\tilde{\Lambda}_{22}(z, \beta^T X) = -\frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{S_c(z_i, X_j) | \beta^T X, z_i\} \right] / \partial \beta^T X \times \{1 + O_p(B) + O_p(A)\}.
\]

We have

\[
E\left[ -\frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq Z_i) | \beta^T X, Z_i\} \right] / \partial \beta^T X \right] = -\int I(z_i \leq z) K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{S_c(z_i, X_j) | \beta^T X, z_i\} \right] / \partial \beta^T X \times f(z_i, \beta^T X) E\{S_c(z_i, X_i) | \beta^T X_i, z_i\} f_{\beta^T X}(\beta^T X_i) \, dz_i \, d\beta^T X_i
\]

\[
= -\int I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{S_c(z_i, X_j) | \beta^T X, z_i\} \right] / \partial \beta^T X \times f(z_i, \beta^T X) E\{S_c(z_i, X_i) | \beta^T X_i, z_i\} \, dz_i \frac{f^2_{\beta^T X}(\beta^T X S^2(z_i, \beta^T X) E^2\{S_c(z_i, X_j) | \beta^T X, z_i\})}{f^2_{\beta^T X}(\beta^T X S^2(z_i, \beta^T X) E^2\{S_c(z_i, X_j) | \beta^T X, z_i\})}
\]

\[
-\frac{h^2}{2} \int I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{S_c(z_i, X_j) | \beta^T X, z_i\} \right] / \partial \beta^T X \times \frac{\partial^2}{2 \partial (\beta^T X)^2} E\{S_c(z_i, X_i) | \xi\} f(z_i, \xi) f_{\beta^T X}(\xi) u^2 K(u) \, dz_i \, du,
\]

(S.23)
therefore

\[
E \left[ -\frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T z) E \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \} \right]}{f_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \}} \right] \\
+ \int I(z_i \leq z) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) S(z_i, \beta^T X) E \{ S_c(z_i, X_j) \mid \beta^T X, z_i \} \right]}{f_{\beta^T X}(\beta^T X) S^2(z_i, \beta^T X) E^2 \{ S_c(z_i, X_j) \mid \beta^T X, z_i \}} \frac{\partial \beta^T X}{f(z, \beta^T X) d z_i} \\
\leq h^2 \sup_{z, z_i, \beta^T X, \xi} \left| \frac{\partial \left[ f_{\beta^T X}(\beta^T X) S(z_i, \beta^T X) E \{ S_c(z_i, X_j) \mid \beta^T X, z_i \} \right]}{f_{\beta^T X}(\beta^T X) S^2(z_i, \beta^T X) E^2 \{ S_c(z_i, X_j) \mid \beta^T X, z_i \}} \frac{\partial \beta^T X}{2 \beta^T X} \right| \left| \int u^2 K(u) d u \right| \\
= O(h^2)
\]

under conditions C1-C5. Recall \( B = O_p(n^{-1/2} h^{-3/2} + h^2) \), then

\[
-\frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \} \right]}{f_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \}} O_p(B) \\
= O_p(n^{-1/2} h^{-3/2} + h^2), \\
-\frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \} \right]}{f_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq Z_i) \mid \beta^T X, Z_i \}} O_p(A) \\
= O_p(n^{-1/2} h^{-1/2} + h^2)
\]

under condition C1-C5. Therefore

\[
E \hat{A}_{22} = -\int I(z_i \leq z) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) S(z_i, \beta^T X) E \{ S_c(z_i, X_j) \mid \beta^T X, z_i \} \right]}{f_{\beta^T X}(\beta^T X) S^2(z_i, \beta^T X) E^2 \{ S_c(z_i, X_j) \mid \beta^T X, z_i \}} \frac{\partial \beta^T X}{f(z_i, \beta^T X) d z_i} \\
+ O(n^{-1/2} h^{-3/2} + h^2)
\]

43
In addition
\[
\int \left( I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \left[ f(z_i, \beta^T X) E\{S_c(z_i, X_j) \mid \beta^T X, z_i\} f_{\beta^T X}(\beta^T X) \right] / \partial \beta^T X \right) dz_i \\
- I(z_i \leq z) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) S(z_i, \beta^T X) E\{S_c(z_i, X_j) \mid \beta^T X, z_i\} \right] / \partial \beta^T X
\]
\[
\int f(z_i, \beta^T X) \right) dz_i
\]
= \Lambda_2(z, \beta^T X).

Combining \( E\hat{\Lambda}_{21}(z, \beta^T X) \) and \( E\hat{\Lambda}_{22}(z, \beta^T X) \) gives
\[
\left| E\hat{\Lambda}_2(z, \beta^T X) - \Lambda_2(z, \beta^T X) \right| = O(n^{-1/2}h^{-3/2} + h^2)
\]
under conditions C1-C5.

The variance of \( \hat{\Lambda}_2(z, \beta^T X) \) is
\[
\text{var}\{\hat{\Lambda}_2(z, \beta^T X)\} = \text{var}\{\hat{\Lambda}_{21}(z, \beta^T X) + \hat{\Lambda}_{22}(z, \beta^T X)\}
\]
\[
\leq 2\text{var}\{\hat{\Lambda}_{21}(z, \beta^T X)\} + 2\text{var}\{\hat{\Lambda}_{22}(z, \beta^T X)\}.
\]

The first term
\[
\text{var}\{\hat{\Lambda}_{21}(z, \beta^T X)\}
\]
\[
\leq \frac{2}{n} \text{var} \left[ -I(Z_i \leq z) \Delta_i K_h'(\beta^T X_i - \beta^T X) \right] \\
\]
\[
+ 2\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} -I(Z_i \leq z) \Delta_i K_h'(\beta^T X_i - \beta^T X) \right] O_p(A)
\]

44
The first part is
\[
\frac{2}{n} \text{var} \left[ \frac{-I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right] = \\
\frac{2}{n} E \left[ \frac{-I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right]^2 \\
\approx \\
\frac{2}{n h^3} \int f_{\beta^T X}(\beta^T X) S^2(z_i, \beta^T X) E\{S_c(z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\} \left\{ \int K^2(u) du \right\} \\
+ \frac{1}{n h^3} \sup_{z_i, z_i, \beta^T X, \xi} \left| \frac{\partial^2}{\partial z_i^2} f_{\beta^T X}(\beta^T X) S^2(z_i, \beta^T X) E\{S_c(z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\} \right| \\
\times \left\{ \int u^2 K^2(u) du \right\} + O(1/n) \\
= O\{1/(nh^3)\} \quad (S.24)
\]

under condition C1-C5 The second part is
\[
2 \text{var} \left[ \frac{\sum_{i=1}^{n} -I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right] = \\
2 E \left[ \frac{\sum_{i=1}^{n} -I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right]^2 \\
= \left\{ \left( E \left[ \frac{-I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right] \right)^2 + \frac{1}{n} \text{var} \left[ \frac{-I(Z_i \leq z) \Delta_i K_h' (\beta^T X_i - \beta^T X)}{f_{\beta^T X}(\beta^T X) S(Z_i, \beta^T X) E\{S_c(Z_i, X_j) \mid \beta^T X_i = \beta^T X, Z_i\}^2} \right] \right\} O\{1/(nh) + h^4\} \\
= O\{1/(nh) + h^4\}
\]
under conditions \( C_1 \) to \( C_5 \), where the second last equation is because of (S.22) and (S.24).

Therefore \( \text{var}\{\hat{\Lambda}_{21}(z, \beta^T X)\} = O\{1/(nh^3)\} \) under conditions \( C_1 \) to \( C_5 \).

For \( \hat{\Lambda}_{22}(z, \beta^T X) \),

\[
\text{var}\{\hat{\Lambda}_{22}(z, \beta^T X)\} \\
\leq \frac{2}{n} \text{var} \left[ I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq Z_i) \mid \beta^T X \} \right] \right] \\
+ 4\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq Z_i) \mid \beta^T X \} \right] \right] \\
+ 4\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq Z_i) \mid \beta^T X \} \right] \right] \\
\times \left\{ \int K^2(u)du \right\} \\
+ \frac{h}{n} \sup_{z_i, \beta^T X} \left| \frac{\partial}{\partial \beta^T X} \left[ f_{\beta^T X}(\beta^T X) S(z_i, \beta^T X) E\{S_c(z_i, X_j) \mid \beta^T X \} \right] \right|^2 \\
\times \frac{\partial^2}{\partial \beta^T X^2} E\{S_c(z_i, X_j) \mid \beta^T X \} \left\{ \int u^2 K^2(u)du \right\} + O(1/n) \\
= O\{1/(nh)\} \quad (S.25)
\]
under conditions C1-C5. The second part is

\[
4\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{I(Z_j \geq Z_i \mid \beta^T X)\} / \partial \beta^T X \right]}{f_{\beta^T X}(\beta^T X) E^2 \{I(Z_j \geq Z_i \mid \beta^T X, Z_i)\}} O_p(B) \right] \\
\leq 4E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{I(Z_j \geq Z_i \mid \beta^T X)\} / \partial \beta^T X \right]}{f_{\beta^T X}(\beta^T X) E^2 \{I(Z_j \geq Z_i \mid \beta^T X, Z_i)\}} \right)^2 \right] \\
\times O \{1/(nh^3) + h^4\} \\
= O \{1/(nh^3) + h^4\}
\]

under conditions C1-C5, where the second last equation is because of (S.23) and (S.25). The last part is

\[
4\text{var} \left[ \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{I(Z_j \geq Z_i \mid \beta^T X)\} / \partial \beta^T X \right]}{f_{\beta^T X}(\beta^T X) E^2 \{I(Z_j \geq Z_i \mid \beta^T X, Z_i)\}} O_p(A) \right] \\
\leq 4E \left[ \left( \frac{1}{n} \sum_{i=1}^{n} I(Z_i \leq z) \Delta_i K_h(\beta^T X_i - \beta^T X) \frac{\partial \left[ f_{\beta^T X}(\beta^T X) E \{I(Z_j \geq Z_i \mid \beta^T X)\} / \partial \beta^T X \right]}{f_{\beta^T X}(\beta^T X) E^2 \{I(Z_j \geq Z_i \mid \beta^T X, Z_i)\}} \right)^2 \right] \\
\times O \{(nh)^{-1} + h^4\} \\
= O \{(nh)^{-1} + h^4\}
\]

under conditions C1-C5, where the second last equation is because of (S.23) and (S.25). Therefore \(\text{var}\{\hat{\Lambda}_{22}(z, \beta^T X)\} = O\{1/(nh^3)\}\) under conditions C1-C5.

Summarizing the results above, \(\text{var}\{\hat{\Lambda}_2(z, \beta^T X)\} = O\{1/(nh^3)\}\). Hence the estimator \(\hat{\Lambda}_2(z, \beta^T X)\) satisfies

\[
\hat{\Lambda}_2(z, \beta^T X) = \Lambda_2(z, \beta^T X) + O_p\{(nh^3)^{-1/2} + h^2\}
\]

under conditions C1-C5.
To show that the trimmed estimators have the same asymptotic results, we prove (S.7) and skip the others. For further reading about the trimmed kernel estimators, please see Appendix A.2 of Härdle and Stoker (1989). For notational simplicity, let \( \hat{f}(\beta^T X) \equiv n^{-1} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) \). The absolute bias of the trimmed estimator is

\[
\begin{align*}
\leq & \left| E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \left\{ \hat{f}(\beta^T X) > d_n \right\} \right] - E\{I(Z_j \geq Z) \mid \beta^T X\} \right| \\
= & \left| E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \left\{ \hat{f}(\beta^T X) > d_n \right\} \right] \\
- & E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
+ & E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
- & E \left[ \frac{n^{-1} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
+ & E \left[ \frac{n^{-1} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)I(Z_j \geq Z)}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] - E\{I(Z_j \geq Z) \mid \beta^T X\} \right|.
\end{align*}
\]
When Condition $C_3$ is replaced by Condition $C_3'$ the first term satisfies

\[
\begin{align*}
&\left| E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \{ \hat{f}(\beta^T X) > d_n \} \right] 
- E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \{ f_{\beta^T X}(\beta^T X) > d_n \} \right] \right| \\
&\quad \leq \left| E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \{ \hat{f}(\beta^T X) > d_n, f_{\beta^T X}(\beta^T X) \leq d_n \} \right] \right| \\
&\quad \quad + \left| E \left[ \frac{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \{ \hat{f}(\beta^T X) \leq d_n, f_{\beta^T X}(\beta^T X) > d_n \} \right] \right| \\
&\quad \leq \left| E \left[ I \{ \hat{f}(\beta^T X) > d_n, f_{\beta^T X}(\beta^T X) \leq d_n \} \right] \right| + \left| E \left[ I \{ \hat{f}(\beta^T X) \leq d_n, f_{\beta^T X}(\beta^T X) > d_n \} \right] \right| \\
&\quad \leq O_p \left\{ n^{-\epsilon} + h^2 + (nh)^{-1/2} \right\} + O_p \left\{ h^2 + (nh)^{-1/2} \right\}.
\end{align*}
\]
The second term is
\[
E \left[ \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) \frac{1}{\sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
- E \left[ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
\leq E \left[ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) \frac{1}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
- E \left[ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) \frac{1}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] \\
\leq E \left[ \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) \frac{1}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right] O_p(h^2 + (nh)^{-1/2}) \\
= O_p(h^2 + (nh)^{-1/2}).
\]

The third term is
\[
E \left[ \left| \frac{1}{n} \sum_{j=1}^{n} K_h(\beta^T X_j - \beta^T X) I(Z_j \geq Z) \frac{1}{f_{\beta^T X}(\beta^T X)} I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right| - E \left( I(Z_j \geq Z) \right) \left| \beta^T X \right| \\
= E \left| I(Z_j \geq Z) I \left\{ f_{\beta^T X}(\beta^T X) > d_n \right\} \right| - E \left( I(Z_j \geq Z) \right) \left| \beta^T X \right| + O_p(h^2) \\
= E \left( I(Z_j \geq Z) \right) I \left\{ f_{\beta^T X}(\beta^T X) \leq d_n \right\} \right] + O_p(h^2) \\
\leq E \left( I \left\{ f_{\beta^T X}(\beta^T X) \leq d_n \right\} \right) + O_p(h^2) \\
= O_p(n^{-\epsilon} + h^2 + (nh)^{-1/2}) \\
= O_p(h^2 + (nh)^{-1/2}).
\]

50
It is easy to see the variance of this trimmed estimator,

\[
\text{var} \left[ \frac{\sum_{j=1}^{n} K_h (\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h (\beta^T X_j - \beta^T X)} I \left\{ \hat{f}(\beta^T X) > d_n \right\} \right] = O_p\{(nh)^{-1/2}\}.
\]

Therefore

\[
\frac{\sum_{j=1}^{n} K_h (\beta^T X_j - \beta^T X) I(Z_j \geq Z)}{\sum_{j=1}^{n} K_h (\beta^T X_j - \beta^T X)} I \left\{ \hat{f}(\beta^T X) > d_n \right\} = E\{I(Z_j \geq Z) \mid \beta^T X\} + O_p\{h^2 + (nh)^{-1/2}\}.
\]

We give detailed proof for uniform result of (S.14) only. Because the domain of \( \beta^T X \) is compact, we divide it into rectangular regions. In each region, the distance between a point \( \beta^T x \) in this region and the nearest grid point is less than \( n^{-2} \). We need only \( N \leq Cn^2 \) grid points, where \( C \) is a constant. Let the grid points be \( \kappa_1, \ldots, \kappa_N \). Let \( \hat{\rho}(\beta^T X) = \hat{E}\{XY(Z) \mid \beta^T X\} \) and \( \rho(\beta^T X) = E\{XY(Z) \mid \beta^T X\} \). Then for any \( (\beta^T X) \), there exists a \( \kappa_i, 1 \leq i \leq N \), such that

\[
|\hat{\rho}(\beta^T X) - \rho(\beta^T X)| \leq |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| + |\hat{\rho}(\beta^T X) - \hat{\rho}(\kappa_i)| + |\rho(\beta^T X) - \rho(\kappa_i)|
\]

for an absolute constant \( D_1 \) under Conditions [C1] and [C5]. Thus, for any \( D \geq D_1 \),

\[
\text{pr}(\sup_{\beta^T X} |\hat{\rho}(\beta^T X) - \rho(\beta^T X)| > 2D[h^2 + \{\log(nh)^{-1}\}^{1/2}])
\]

\[
\leq \text{pr}(\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| > 2D[h^2 + \{\log(nh)^{-1}\}^{1/2}] - D_1n^{-2})
\]

\[
\leq \text{pr}(\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| > D[h^2 + \{\log(nh)^{-1}\}^{1/2}]\}
\]

under Condition [C2]. Using Bernstein’s inequality on \( \hat{\rho}(\kappa_i) \), under Conditions [C1]-[C5] we
have

\[
\Pr[|\hat{\rho}(\kappa_i) - \rho(\kappa_i)| \geq A\{\log n/(nh)\}^{1/2}] \leq 2 \exp \left\{ \frac{-nA^2 \log n/(nh)}{2D_2h^{-1} + 2/3AD_3(\log n)^{1/2}(nh)^{-1/2}} \right\}
\]

\[
= 2 \exp \left\{ \frac{-A^2 \log n}{2D_2 + 2/3AD_3(\log n/h)^{1/2}} \right\}
\]

\[
\leq 2 \exp \left( \frac{-A^2 \log n}{2D_2 + AD_3} \right)
\]

for all \( A > D_3 + \sqrt{D_3^2 + 4D_2} \), where \( D_2 \) and \( D_3 \) are constants satisfying

\[
\var \{\hat{\rho}(\beta^T X) - \rho(\beta^T X)\} \leq \frac{D_2}{nh},
\]

\[
\left| \frac{K_h(\beta^T X_i - \beta^T X)X_iI(Z_i \geq Z)}{1/n \sum_{j=1}^n K_h(\beta^T X_j - \beta^T X)} - \rho(\beta^T X) \right| \leq D_3 \text{ with probability 1.}
\]

This leads to

\[
\Pr[\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - E\hat{\rho}(\kappa_i)| \geq A\{\log n/(nh)\}^{1/2}] \leq 2Cn^2 \exp \left( \frac{-A^2 \log n}{2D_2 + AD_3} \right)
\]

\[
= 2C \exp \left( \left\{2 - \frac{A^2}{(2D_2 + AD_3)}\right\} \log n \right) \to 0
\]

because \( A > D_3 + \sqrt{D_3^2 + 4D_2} \). Combining the above results, for \( A_1 = \max(A, D) \),

\[
\Pr(\sup_{\beta^T X} |\hat{\rho}(\beta^T X, \beta) - \rho(\beta^T X)| > 2A_1[h^2 + \{\log n(hn)^{-1}\}^{1/2}])
\]

\[
\leq \Pr(\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| > A_1[h^2 + \{\log n(hn)^{-1}\}^{1/2}])
\]

\[
\leq \Pr(\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| > A_1h^2) + \Pr(\sup_{\kappa_i} |\hat{\rho}(\kappa_i) - \rho(\kappa_i)| > A_1\{\log n(hn)^{-1}\}^{1/2})
\]

\[
\to 0.
\]

The uniform convergence results concerning (S.15)-(S.18) are slightly different because
these functions contain the additional component $Z$. Nevertheless, under Condition $C_5$ the support of $(\beta^T X_i, Z_i)$ or $(\beta^T X_j, Z_j)$ is also bounded so we can similarly divide the region using $N \leq C n^{2+2}$ grid points while the distance of a point to the nearest grid point is less than $n^{-2}$. The rest of the analysis can then be similarly carried out as above, then the uniform convergence is established.  

**S.4.2 Lemma 2**

**Lemma 2.** The estimator $\hat{\Lambda}(t, \hat{\beta}^T X)$ has the expansion

$$\sqrt{n}h \left\{ \hat{\Lambda}(t, \hat{\beta}^T X) - \Lambda(t, \beta^T X) \right\} = \sqrt{\frac{h}{n}} \sum_{i=1}^{n} \int_{0}^{t} \frac{I \left\{ \sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X) > 0 \right\}}{f_{\beta^T X}(\beta^T X) E \{ I(z > s) \mid \beta^T X \}} K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) + o_p(1),$$

and satisfies

$$\sqrt{n}h \left\{ \hat{\Lambda}(t, \hat{\beta}^T X) - \Lambda(t, \beta^T X) \right\} \rightarrow N\{0, \sigma^2(t, \beta^T X)\}$$

in distribution when $n \rightarrow \infty$ for all $t, \beta^T X$ under Conditions $C_1, C_5$ where

$$\sigma^2(t, \beta^T X) = \int K^2(u) du \int_{0}^{t} \frac{\lambda(s, \beta^T X)}{f_{\beta^T X}(\beta^T X) E \{ I(Z > s) \mid \beta^T X \}} ds.$$
Proof: For notational convenience, let $d = 1$ and $\nu = 2$. For any $t$ and $\beta^T X$, define

$$
\phi_n(s, \beta^T X) = \sum_{j=1}^{n} Y_j(s)K_h(\beta^T X_j - \beta^T X),
$$

$$
Q_n(t, \beta^T X) = \hat{\Lambda}(t, \beta^T X) - \int_0^t I \{ \phi_n(s, \beta^T X) > 0 \} \lambda(s, \beta^T X) ds,
$$

$$
D_n(t, \beta^T X) = \int_0^t \lambda(s, \beta^T X) \left[ 1 - I \{ \phi_n(s, \beta^T X) > 0 \} \right] ds.
$$

Then

$$
\sqrt{n}h \left\{ \hat{\Lambda}(t, \beta^T X) - \Lambda(t, \beta^T X) \right\} = \sqrt{n}h Q_n(t, \beta^T X) - \sqrt{n}h D_n(t, \beta^T X).
$$

We first show that $\sqrt{n}h D_n(t, \beta^T X) \to 0$ in probability uniformly. It suffices to show that

$$
\sqrt{n}h \left[ 1 - I \{ \phi_n(s, \beta^T X) > 0 \} \right] \xrightarrow{p} 0,
$$

which is equivalent to show that for any $\epsilon > 0$,

$$
\Pr \left( \sqrt{n}h \left[ 1 - I \{ \phi_n(s, \beta^T X) > 0 \} \right] > \epsilon \right) \to 0.
$$

Now for $\epsilon \geq \sqrt{n}h$, the above automatically holds. For $\epsilon < \sqrt{n}h$, this is equivalent to show

$$
\Pr \left\{ n^{-1} \phi_n(s, \beta^T X) \leq 0 \right\} \to 0. \quad (S.26)
$$

Because $n^{-1} \phi_n(s, \beta^T X) = f_{\beta^T X}(\beta^T X)E \{ I(Z_j \geq s) \mid \beta^T X \} + O_p \{ h^2 + (nh)^{-1/2} \}$, (S.26) is equivalent to

$$
\Pr \left[ f_{\beta^T X}(\beta^T X)E \{ I(Z_j \geq s) \mid \beta^T X \} + O_p \{ h^2 + (nh)^{-1/2} \} \leq 0 \right] \to 0,
$$

54
which automatically holds under Condition $C_3$ and $C_5$. Hence $\sqrt{nh}D_n(t, \beta^T X) \to 0$ in probability uniformly. Second we inspect the asymptotic property of $\sqrt{nh}Q_n(t, \beta^T X)$. Recall $M_i(s, \beta^T X)$ is the martingale corresponding to the counting process $N_i(s)$ and satisfies $dM_i(s, \beta^T X) = dN_i(s) - Y_i(s)\lambda(s, \beta^T X)ds$

\[
\sqrt{nh}Q_n(t, \beta^T X) = \int_0^t \sqrt{nh} \left\{ \frac{1}{\sum_{j=1}^n Y_j(s)K_h(\beta^T X_j - \beta^T X)} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)dN_i(s) \right\} \lambda(s, \beta^T X)ds
\]

\[
= \int_0^t \sqrt{nh}I\{\phi_n(s, \beta^T X) < 0\} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)dN_i(s)
\]

\[
+ \int_0^t \sqrt{nh}I\{\phi_n(s, \beta^T X) > 0\} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)dN_i(s)
\]

\[
- \int_0^t \sqrt{nh}I\{\phi_n(s, \beta^T X) > 0\} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)Y_i(s)\lambda(s, \beta^T X)ds
\]

\[
= \int_0^t \sqrt{nh}I\{\phi_n(s, \beta^T X) > 0\} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)dM_i(s, \beta^T X) \quad (S.27)
\]

\[
+ \int_0^t \sqrt{nh}I\{\phi_n(s, \beta^T X) < 0\} \sum_{i=1}^n K_h(\beta^T X_i - \beta^T X)dN_i(s). \quad (S.28)
\]
We decompose (S.27) as

\[
\int_0^t \sqrt{n} h \frac{1}{\sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X)} \sum_{i=1}^{n} K_h(\beta^T X_i - \beta^T X) dt = \sqrt{n} h \frac{n}{\sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X) dt} K_h(\beta^T X_i - \beta^T X) dt + O_p\{h^4 + (nh)^{-1}\}
\]

\[
= Q_{n1} - Q_{n2} + o_p(1),
\]

(S.29)

where

\[
Q_{n1} = \sqrt{n} \sum_{n=1}^{t} \int_0^t \frac{1}{f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq s) | \beta^T X\}} K_h(\beta^T X_i - \beta^T X) dt M_i(s, \beta^T X),
\]

\[
Q_{n2} = \sqrt{n} \sum_{n=1}^{t} \int_0^t \frac{1}{f_{\beta^T X}(\beta^T X) E^2\{I(Z_j \geq s) | \beta^T X\}} K_h(\beta^T X_i - \beta^T X) dt M_i(s, \beta^T X)
\]

\[
\times \left[ \frac{1}{n} \sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq s) | \beta^T X\} \right]
\]

\[
\times K_h(\beta^T X_i - \beta^T X) dt M_i(s, \beta^T X)
\]

\[
= \sqrt{n} h \frac{1}{\sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X) dt} \sum_{i=1}^{n} K_h(\beta^T X_i - \beta^T X) dt M_i(s, \beta^T X)
\]

\[
\times \left[ \frac{1}{n} \sum_{j=1}^{n} Y_j(s) K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X) E\{I(Z_j \geq s) | \beta^T X\} \right]
\]

\[
\times K_h(\beta^T X_i - \beta^T X) dt M_i(s, \beta^T X),
\]

and the remaining term in (S.29) is \(o_p(1)\) because \(\sqrt{n/h} O_p(h^4 + (nh)^{-1}) = O_p(n^{1/2} h^{7/2} + \)
\((nh^3)^{-1/2}\) = \(o_p(1)\) by Condition \(\mathbb{C}_2\).

Using the U-statistic property, \(Q_{n2}\) has leading order terms \(Q_{n21} + Q_{n22} - Q_{n23}\), where

\[
Q_{n21} = \sqrt{\frac{h}{n}} E \left( \sum_{i=1}^{n} \int_{0}^{t} \frac{I \{ \phi_n(s, \beta^T X) > 0 \}}{f^2_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq s) \mid \beta^T X \}} \times \left[ Y_j(s) K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X) E \{ I(Z \geq s) \mid \beta^T X \} \right] \times K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) \mid \Delta_i, \beta^T X_i, Z_i \right).
\]

\[
Q_{n22} = \sqrt{\frac{h}{n}} E \left( \sum_{j=1}^{n} \int_{0}^{t} \frac{I \{ \phi_n(s, \beta^T X) > 0 \}}{f^2_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq s) \mid \beta^T X \}} \times \left[ Y_j(s) K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X) E \{ I(Z \geq s) \mid \beta^T X \} \right] \times K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) \mid \Delta_j, \beta^T X_j, Z_j \right).
\]

\[
Q_{n23} = \sqrt{nh} E \left( \int_{0}^{t} \frac{I \{ \phi_n(s, \beta^T X) > 0 \}}{f^2_{\beta^T X}(\beta^T X) E^2 \{ I(Z_j \geq s) \mid \beta^T X \}} \times \left[ Y_j(s) K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X) E \{ I(Z \geq s) \mid \beta^T X \} \right] \times E \{ K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) \} \right).
\]

\[
I \{ \phi_n(s, \beta^T X) > 0 \} = I \left[ f_{\beta^T X}(\beta^T X) E \{ I(Z_j \geq s) \mid \beta^T X \} + o_p(h^2 + (nh)^{-1}) > 0 \right] = 1 \text{ al-}
\]

57
most surely. Thus, almost surely,

\[ Q_{n21} = \sqrt{\frac{h}{n}} \sum_{i=1}^{n} \left( \int_{0}^{t} \frac{1}{f_{\beta^T X}(\beta^T X)E\{I(Z_j \geq s) | \beta^T X\}} \times E\left[ Y_j(s)K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X)E\{I(Z \geq s) | \beta^T X\} \right] \times K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) \right) \]

\[ = \sqrt{\frac{h}{n}} \sum_{i=1}^{n} \int_{0}^{t} \frac{O(h^2)}{f_{\beta^T X}(\beta^T X)E\{I(Z_j \geq s) | \beta^T X\}} K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X) \rightarrow 0. \]

uniformly as \( h \rightarrow 0 \). Similarly, almost surely,

\[ Q_{n22} = \sqrt{\frac{h}{n}} \sum_{j=1}^{n} \int_{0}^{t} \frac{1}{f_{\beta^T X}(\beta^T X)E^2\{I(Z_j \geq s) | \beta^T X\}} \times [Y_j(s)K_h(\beta^T X_j - \beta^T X) - f_{\beta^T X}(\beta^T X)E\{I(Z \geq s) | \beta^T X\}] \times E\{K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X)\} \]

\[ = 0. \]

Obviously, \( Q_{n23} = E(Q_{n22}) = 0 \), hence \( Q_{n2} \rightarrow 0 \) in probability as \( n \rightarrow \infty \).

For (S.28)

\[ \int_{0}^{t} \sqrt{nh} \frac{I\{\phi_n(s, \beta^T X) \leq 0\}}{\sum_{j=1}^{n} Y_j(s)K_h(\beta^T X_j - \beta^T X)} \sum_{i=1}^{n} K_h(\beta^T X_i - \beta^T X) dN_i(s) \rightarrow 0 \]
in probability uniformly. We have obtained

$$\sqrt{n h} Q_n(t, \beta^T X) = \sqrt{\frac{h}{n}} \sum_{i=1}^{n} \int_0^t I\{\phi_n(s, \beta^T X) > 0\} \frac{f_{\beta \gamma}(\beta^T X) I\{Z \geq s\} | \beta^T X\} K_h(\beta^T X_i - \beta^T X) dM_i(s, \beta^T X)$$

+ \text{o}_p(1).$$

Applying martingale central limit theorem on (S.30), we have

$$\frac{h}{n} \sum_{i=1}^{n} \int_0^t \frac{\lambda(s, \beta^T X) I\{\phi_n(s, \beta^T X) > 0\}}{[f_{\beta \gamma}(\beta^T X) E\{I(Z \geq s) | \beta^T X\}]^2} K_h^2(\beta^T X_i - \beta^T X) Y_i(s) ds$$

$$= \int_0^t \frac{\lambda(s, \beta^T X) I\{\phi_n(s, \beta^T X) > 0\}}{[f_{\beta \gamma}(\beta^T X) E\{I(Z \geq s) | \beta^T X\}]^2} \frac{1}{n} \sum_{i=1}^{n} h K_h^2(\beta^T X_i - \beta^T X) Y_i(s) ds$$

$$= \int_0^t \frac{\lambda(s, \beta^T X) I\{\phi_n(s, \beta^T X) > 0\}}{[f_{\beta \gamma}(\beta^T X) E\{I(Z \geq s) | \beta^T X\}]^2}$$

$$\times \left[ f_{\beta \gamma}(\beta^T X) E\{I(Z_i \geq s) | \beta^T X\} \int K^2(u) du + O_p(n^{-1/2} h^{-1/2} + h^2) \right] ds$$

$$\overset{p}{\to} \int K^2(u) du \int_0^t \frac{\lambda(s, \beta^T X)}{f_{\beta \gamma}(\beta^T X) E\{I(Z_i \geq s) | \beta^T X\}} ds$$

$$= \sigma^2(t, \beta^T X).$$

(S.31)
Next we inspect the following integration for any $\epsilon > 0$.

\[
\sum_{i=1}^{n} \int_{0}^{t} \frac{h}{n} \frac{I\{\phi_n(s, \beta^T X) > 0\}}{I\{\beta^T X\} E\{I(Z \geq s) \mid \beta^T X\}} K_h^2(\beta^T X_i - \beta^T X) Y_i(s) \times I\left[ \frac{h}{n} \frac{I\{\phi_n(s, \beta^T X) > 0\}}{I\{\beta^T X\} E\{I(Z \geq s) \mid \beta^T X\}} K_h(\beta^T X_i - \beta^T X) > \epsilon \right] \lambda(s, \beta^T X) ds
\]

\[
= \int_{0}^{t} \frac{I\{\phi_n(s, \beta^T X) > 0\}}{I\{\beta^T X\} E\{I(Z \geq s) \mid \beta^T X\}} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{h K_h^2(\beta^T X_i - \beta^T X) Y_i(s)}{\lambda(s, \beta^T X)} \right) \lambda(s, \beta^T X) ds
\]

\[
\leq \int_{0}^{t} \sup_{1 \leq i \leq n} \left[ \frac{I\{\phi_n(s, \beta^T X) > 0\}}{I\{\beta^T X\} E\{I(Z \geq s) \mid \beta^T X\}} \frac{\frac{1}{n} \sum_{i=1}^{n} h K_h^2(\beta^T X_i - \beta^T X) Y_i(s)}{\epsilon \sqrt{n h}} \right] ds,
\]

where $m_i = |I\{\phi_n(s, \beta^T X) > 0\} K\{((\beta^T X_i - \beta^T X)/h)\}$, which is bounded following Condition $C_1$. In the above display,

\[
\sup_{1 \leq i \leq n} \left[ \frac{I\{\phi_n(s, \beta^T X) E\{I(Z \geq s) \mid \beta^T X\}}{\epsilon \sqrt{n h}} < \frac{m_i}{\epsilon \sqrt{n h}} \right] = 0
\]

as long as $n$ is large enough because the right hand side converges to 0 by Condition $C_2$ but the left hand side will be always larger than 0 by conditions $C_3$ and $C_5$. On the other hand,

\[
\frac{1}{n} \sum_{i=1}^{n} h K_h^2(\beta^T X_i - \beta^T X) Y_i(s)
\]

\[
= f_{\beta^T X}(\beta^T X) E\{I(Z_i \geq s) \mid \beta^T X\} \int K^2(u) du + O_p(n^{-1/2} h^{-1/2} + h^2)
\]

\[
\rightarrow 0
\]

60
in probability uniformly. Hence

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \int_{0}^{t} \frac{h}{n} \frac{I\{\phi_n(s, \beta^T X) > 0\}}{f_{\beta^T X}(\beta^T X) E\{I(Z \geq s) \mid \beta^T X\}} K_h^2(\beta^T X_i - \beta^T X) Y_i(s) \\
\times I \left( \sqrt{\frac{h}{n}} \frac{I\{\phi_n(s, \beta^T X) > 0\}}{f_{\beta^T X}(\beta^T X) E\{I(Z \geq s) \mid \beta^T X\}} K_h|\beta^T X_i - \beta^T X| > \epsilon \right) \lambda(s, \beta^T X) ds = 0
\]

(S.32)

with probability 1 uniformly for any \( \epsilon > 0 \).

In summary

\[
\sqrt{nh} \left\{ \hat{\Lambda}(t, \beta^T X) - \Lambda(t, \beta^T X) \right\} \to N(0, \sigma^2(t, \beta^T X))
\]

uniformly.

\[\square\]

S.5 Proof of Theorem 1

Because the result regarding (13) is the most difficult to establish, we provide only the proof concerning (13), the result concerning (10) is based on a similar proof.

For each \( n \), let \( \hat{\beta}_n \) satisfy

\[
\frac{1}{n} \sum_{i=1}^{n} \frac{\Delta_i}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \otimes \left[ X_{ii} - \hat{E} \left\{ X_{ii} Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\} \right] = 0.
\]

Under condition C6, there exists a subsequence of \( \hat{\beta}_n, n = 1, 2, \ldots \), that converges. For notational simplicity, we still write \( \hat{\beta}_n, n = 1, 2, \ldots \), as the subsequence that converges and let the limit be \( \beta^* \).
From the uniform convergence in (S.13), (S.14), (S.15), (S.16) given in Lemma 1,

\[
\frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \hat{\lambda}_i(Z_i, \hat{\beta}_n^T X_i) \right] \circ \left[ X_{ii} - \frac{\hat{E} \left\{ X_{ii}Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \frac{\lambda_i(Z_i, \hat{\beta}_n^T X_i) + O_p \{(nh)^{-1/2} + h^2 + b^2\}}{\lambda(Z_i, \hat{\beta}_n^T X_i) + O_p \{(nh)^{-1/2} + h^2 + b^2\}} \right] \circ \left[ X_{ii} - \frac{E \left\{ X_{ii}Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\} + O_p \{(nh)^{-1/2} + h^2 + b^2\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\} + O_p \{(nh)^{-1/2} + h^2 + b^2\}} \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \frac{\lambda_i(Z_i, \hat{\beta}_n^T X_i) + O_p \{(nh)^{-1/2} + h^2 + b^2\}}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] \circ \left[ X_{ii} - \frac{E \left\{ X_{ii}Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] + o_p(1).
\]

Thus, for sufficiently large \( n \),

\[
\frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \frac{\lambda_i(Z_i, \hat{\beta}_n^T X_i)}{\lambda(Z_i, \hat{\beta}_n^T X_i)} \right] \circ \left[ X_{ii} - \frac{E \left\{ X_{ii}Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^T X_i \right\}} \right] = \frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \frac{\lambda_i(Z_i, \beta^T X_i)}{\lambda(Z_i, \beta^T X_i)} \right] \circ \left[ X_{ii} - \frac{E \left\{ X_{ii}Y_i(Z_i) \mid \beta^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^T X_i \right\}} \right] + O_p(\hat{\beta}_n - \beta^*)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \Delta_i \left[ \frac{\lambda_i(Z_i, \beta^T X_i)}{\lambda(Z_i, \beta^T X_i)} \right] \circ \left[ X_{ii} - \frac{E \left\{ X_{ii}Y_i(Z_i) \mid \beta^T X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^T X_i \right\}} \right] + o_p(1),
\]
under Condition [C1][C2] where the last equality is because \( \hat{\beta}_n \) converges to \( \beta^* \). In addition,

\[
\frac{1}{n} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta^{*T}X_i)}{\lambda(Z_i, \beta^{*T}X_i)} \otimes \left[ X_{li} - \frac{E \left\{ X_i Y_i(Z_i) \mid \beta^{*T}X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^{*T}X_i \right\}} \right]
\]

\[
= E \left( \frac{\Delta \lambda_i(Z, \beta^{*T}X)}{\lambda(Z, \beta^{*T}X)} \otimes \left[ X_l - \frac{E \left\{ X_i Y(Z) \mid \beta^{*T}X_i \right\}}{E \left\{ Y(Z) \mid \beta^{*T}X_i \right\}} \right] \right) + o_p(1)
\]

under Condition [C1][C2]. Thus, for sufficient large \( n \)

\[
0 = \frac{1}{n} \sum_{i=1}^{n} \Delta_i \frac{\hat{\lambda}_i(Z_i, \hat{\beta}_n^{T}X_i)}{\hat{\lambda}(Z_i, \hat{\beta}_n^{T}X_i)} \otimes \left[ X_{li} - \frac{\hat{E} \left\{ X_i Y_i(Z_i) \mid \hat{\beta}_n^{T}X_i \right\}}{\hat{E} \left\{ Y_i(Z_i) \mid \hat{\beta}_n^{T}X_i \right\}} \right]
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \hat{\beta}_n^{T}X_i)}{\lambda(Z_i, \hat{\beta}_n^{T}X_i)} \otimes \left[ X_{li} - \frac{E \left\{ X_i Y_i(Z_i) \mid \hat{\beta}_n^{T}X_i \right\}}{E \left\{ Y_i(Z_i) \mid \hat{\beta}_n^{T}X_i \right\}} \right] + o_p(1)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta^{*T}X_i)}{\lambda(Z_i, \beta^{*T}X_i)} \otimes \left[ X_{li} - \frac{E \left\{ X_i Y_i(Z_i) \mid \beta^{*T}X_i \right\}}{E \left\{ Y_i(Z_i) \mid \beta^{*T}X_i \right\}} \right] + o_p(1)
\]

\[
= E \left( \frac{\Delta \lambda_i(Z, \beta^{*T}X)}{\lambda(Z, \beta^{*T}X)} \otimes \left[ X_l - \frac{E \left\{ X_i Y(Z) \mid \beta^{*T}X_i \right\}}{E \left\{ Y(Z) \mid \beta^{*T}X_i \right\}} \right] \right) + o_p(1)
\]

under conditions [C1][C2] and [C6]. Note that

\[
E \left( \frac{\Delta \lambda_i(Z, \beta^{*T}X)}{\lambda(Z, \beta^{*T}X)} \otimes \left[ X_l - \frac{E \left\{ X_i Y(Z) \mid \beta^{*T}X_i \right\}}{E \left\{ Y(Z) \mid \beta^{*T}X_i \right\}} \right] \right)
\]

is a nonrandom quantity that does not depend on \( n \), hence it is zero. Thus the uniqueness requirement in Condition [C7] ensures that \( \beta^* = \beta_0 \).

63
We now show that the subsequence that converges includes all but a finite number of \( n \)'s. Assume this is not the case, then we can obtain an infinite sequence of \( \hat{\beta}_n \)'s that do not converge to \( \beta^* \). As an infinite sequence in a compact set \( \mathcal{B} \), we can thus obtain another subsequence that converges, say to \( \beta^{**} \neq \beta^* \). Identical derivation as before then leads to \( \beta^{**} = \beta_0 \), which is a contradiction to \( \beta^{**} \neq \beta^* \). Thus we conclude \( \hat{\beta} - \beta_0 \to 0 \) in probability when \( n \to \infty \) under condition C1-C6.

\[ \square \]

S.6 Proof of Theorem 2

We only provide the proof concerning (13); the result concerning (10) follows by using a similar and simpler proof.

We first expand (13) as

\[
0 = n^{-1/2} \sum_{i=1}^{n} \frac{\lambda_i(Z_i, \beta^T X_i)}{\lambda(Z_i, \beta^T X_i)} \otimes \left[ X_{ti} - \frac{\hat{E}\{X_{ti}Y_i(Z_i) | \beta^T X_i\}}{\hat{E}\{Y_i(Z_i) | \beta^T X_i\}} \right]
\]

\[
= n^{-1/2} \sum_{i=1}^{n} \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{ti} - \frac{\hat{E}\{X_{ti}Y_i(Z_i) | \beta_0^T X_i\}}{\hat{E}\{Y_i(Z_i) | \beta_0^T X_i\}} \right] + \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_i^T \beta)} \left[ \frac{\lambda_i(Z_i, \beta^T X_i)}{\lambda(Z_i, \beta^T X_i)} \otimes \left[ X_{ti} - \frac{\hat{E}\{X_{ti}Y_i(Z_i) | \beta^T X_i\}}{\hat{E}\{Y_i(Z_i) | \beta^T X_i\}} \right] \right] \otimes X_i^T \right\}_{\beta = \hat{\beta}}
\]

\times \sqrt{n}(\hat{\beta} - \beta_0),
\]

where \( \hat{\beta} \) is on the line connecting \( \beta_0 \) and \( \beta \).
We first consider (S.34). Because of Theorem 1 and Lemma 1,

\[
\frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_i^T \beta)} \left( \Delta_i \frac{\hat{l}_i(Z_i, \beta^T X_i)}{\lambda(Z_i, \beta^T X_i)} \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta^T X_i \}} \right] \right) \right\}_{\beta = \beta} \]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial (X_i^T \beta_0)} \left( \Delta_i \frac{\hat{l}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \right) \right\}_{\beta = \beta} + o_p(1)
\]

\[
= -\frac{1}{n} \sum_{i=1}^{n} \left( \frac{\lambda_{i}^{\otimes 2}(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{li} - \frac{\hat{E} \{ X_{li} Y_i(Z_i) \mid \beta_0^T X_i \}}{\hat{E} \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \right] \otimes X_{li}^T \right) + o_p(1)
\]

(S.35)

Because of Lemma 1, (S.35) converges uniformly in probability to

\[
-\hat{E} \left( \int_{0}^{\infty} \frac{\lambda_{i}^{\otimes 2}(s, \beta_0^T X)}{\lambda(s, \beta_0^T X)} \otimes \left[ X_{li} - \frac{E \{ X_{li} Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes X_{li}^T dN(s) \right)
\]

\[
= -\hat{E} \left( \int_{0}^{\infty} \frac{\lambda_{i}^{\otimes 2}(s, \beta_0^T X)}{\lambda(s, \beta_0^T X)} \otimes \left[ X_{li} - \frac{E \{ X_{li} Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes X_{li}^T Y(s) \lambda(s, \beta_0^T X) ds \right)
\]

\[
= -\hat{E} \left( \int_{0}^{\infty} \frac{\lambda_{i}^{\otimes 2}(s, \beta_0^T X)}{\lambda(s, \beta_0^T X)} \otimes \left[ X_{li} - \frac{E \{ X_{li} Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes X_{li}^T \right) Y(s) ds
\]

\[
= -\hat{E} \left( \int_{0}^{\infty} \frac{\lambda_{i}^{\otimes 2}(s, \beta_0^T X)}{\lambda(s, \beta_0^T X)} \otimes \left[ X_{li} - \frac{E \{ X_{li} Y(s) \mid \beta_0^T X \}}{E \{ Y(s) \mid \beta_0^T X \}} \right] \otimes E \{ X_{li} Y(s) \mid \beta_0^T X \} \right) \frac{Y(s) ds}{E \{ Y(s) \mid \beta_0^T X \}}
\]

\[
= -\hat{E} \{ S_{\text{eff}}(\Delta, Z, X) \}^{\otimes 2},
\]

where the last equality is because the second term above is zero by first taking expectation.
conditional on $\beta_0^T X$.

Similarly, from Lemma 11 the term in (S.36) converges uniformly in probability to the limit of

$$
E \left\{ \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_1(Z_i, \beta_0^T X_i) \otimes X_i \right) \right\}.
$$

Now let $\hat{\lambda}_{1,-i}(Z, \beta_0^T X)$ be the leave-one-out version of $\hat{\lambda}_1(Z, \beta_0^T X)$, i.e. it is constructed the same as $\hat{\lambda}_1(Z, \beta_0^T X)$ except that the $i$th observation is not used. Obviously,

$$
\frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1,-i}(Z_i, \beta_0^T X_i) \otimes X_i \right)
$$

Let $E_i$ mean taking expectation with respect to the $i$th observation conditional on all other observations, then

$$
E_i \left\{ \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{1,-i}(Z_i, \beta_0^T X_i) \otimes X_i \right) \right\} \otimes X_i
$$

$$
= E_i \left\{ \frac{\partial}{\partial \beta_0} \int \hat{\lambda}_{1,-i}(s, \beta_0^T X_i) \otimes X_i \right\} \otimes X_i
$$

Here, the last equality is because the integrand has expectation zero conditional on $\beta_0^T X_i$ and all other observations, and the third to last equality is because the expectation is
with respect to $X_i$ and does not involve $\beta_0$. Therefore, the term in (S.36) converges in probability uniformly to

$$E \left\{ \frac{\Delta_i}{\lambda(Z_i, \beta_0^T X_i)} \frac{\partial}{\partial (X_i^T \beta_0)} \left( \hat{\lambda}_{i,-i}(Z_i, \beta_0^T X_i) \otimes \left[ X_{ii} - \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right] \right) \otimes X_{ii}^T \right\} = 0$$

Combining the results concerning (S.35) and (S.36), thus the expression in (S.34) is $-E\{S_{\text{eff}}(\Delta, Z, X)^{\otimes 2}\} + o_p(1)$.

Next we decompose (S.33) into

$$n^{-1/2} \sum_{i=1}^n \Delta_i \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{ii} - \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right] = T_1 + T_2 + T_3 + T_4,$$

where

$$T_1 = n^{-1/2} \sum_{i=1}^n \Delta_i \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ X_{ii} - \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right],$$

$$T_2 = n^{-1/2} \sum_{i=1}^n \Delta_i \left( \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} - \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \right) \otimes \left[ X_{ii} - \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right],$$

$$T_3 = n^{-1/2} \sum_{i=1}^n \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} - \frac{\hat{E} \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{\hat{E} \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right],$$

$$T_4 = n^{-1/2} \sum_{i=1}^n \Delta_i \left( \frac{\hat{\lambda}_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} - \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \right) \otimes \left[ \frac{E \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{E \{Y_i(Z_i) \mid \beta_0^T X_i\}} - \frac{\hat{E} \{X_{ii}Y_i(Z_i) \mid \beta_0^T X_i\}}{\hat{E} \{Y_i(Z_i) \mid \beta_0^T X_i\}} \right].$$

67
First,

\[ T_2 = n^{-1/2} \sum_{i=1}^{n} \int \left\{ \frac{\hat{\lambda}_i(s, \beta_0^T X_i)}{\lambda(s, \beta_0^T X_i)} - \frac{\lambda_i(s, \beta_0^T X_i)}{\lambda(s, \beta_0^T X_i)} \right\} \otimes \left[ X_{li} - \frac{E \{ X_{li} Y_i(s) \mid \beta_0^T X_i \}}{E \{ Y_i(s) \mid \beta_0^T X_i \}} \right] dN_i(s) \]

\[ = o_p \left( n^{-1/2} \sum_{i=1}^{n} \int \left[ X_{li} - \frac{E \{ X_{li} Y_i(s) \mid \beta_0^T X_i \}}{E \{ Y_i(s) \mid \beta_0^T X_i \}} \right] Y_i(s) \lambda(s, \beta_0^T X_i) ds \right) \]

\[ = o_p(1), \]

where the last equality above is because the quantity inside the parentheses is a mean zero
normal random quantity of order $O_p(1)$. Further,

\[
T_3 = n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( -\tilde{E} \left\{ X_i Y_i(Z_i) \mid \beta_0^T X_i \right\} \frac{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}}{E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} + \frac{\tilde{E} \{ Y_i(Z_i) \mid \beta_0^T X_i \} E \{ X_i Y_i(Z_i) \mid \beta_0^T X_i \}}{[E \{ Y_i(Z_i) \mid \beta_0^T X_i \}]^2} \right) + o_p(1)
\]

\[
= n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left( -\frac{1}{n} \sum_{j=1}^{n} K_h(\beta_0^T X_j - \beta_0^T X_i) X_{i,j} I(Z_j \geq Z_i) \frac{f_{\beta_0^T X}(\beta_0^T X_i) E \{ Y_i(Z_i) \mid \beta_0^T X_i \}}{f_{\beta_0^T X}(\beta_0^T X_i)} \right) \frac{E \{ X_i Y_i(Z_i) \mid \beta_0^T X_i \}}{[E \{ Y_i(Z_i) \mid \beta_0^T X_i \}]^2} \left(n^{-1} \sum_{j=1}^{n} K_h(\beta_0^T X_j - \beta_0^T X_i) - f_{\beta_0^T X}(\beta_0^T X_i) \right) \right) + o_p(1)
\]

\[
= n^{-3/2} \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ -\frac{K_h(\beta_0^T X_j - \beta_0^T X_i) X_{i,j} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i) E \{ Y_i(Z_i) \mid \beta_0^T X_i \}} \frac{f_{\beta_0^T X}(\beta_0^T X_i)}{f_{\beta_0^T X}(\beta_0^T X_i)} \frac{E \{ X_i Y_i(Z_i) \mid \beta_0^T X_i \}}{[E \{ Y_i(Z_i) \mid \beta_0^T X_i \}]^2} \right] + o_p(1)
\]

\[
= T_{31} + T_{32} + T_{33} + o_p(1),
\]
where

\[
\begin{align*}
T_{31} &= n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes E \left[ - \frac{K_h(\beta_0^T X_j - \beta_0^T X_i)X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)E \{Y_i(Z_i) | \beta_0^T X_i\}} \right. \\
& \quad \left. + \frac{E \{X_{ij} Y_i(Z_i) | \beta_0^T X_i\} K_h(\beta_0^T X_j - \beta_0^T X_i)I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)[E \{Y_i(Z_i) | \beta_0^T X_i\}]^2} \right] \Delta_i, Z_i, X_i \\
T_{32} &= n^{-1/2} \sum_{j=1}^{n} E \left( \Delta_j \frac{\lambda_j(Z_j, \beta_0^T X_j)}{\lambda(Z_j, \beta_0^T X_j)} \otimes \left[ - \frac{K_h(\beta_0^T X_j - \beta_0^T X_i)X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)E \{Y_i(Z_i) | \beta_0^T X_i\}} \right. \\
& \quad \left. + \frac{E \{X_{ij} Y_i(Z_i) | \beta_0^T X_i\} K_h(\beta_0^T X_j - \beta_0^T X_i)I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)[E \{Y_i(Z_i) | \beta_0^T X_i\}]^2} \right] \Delta_j, Z_j, X_j \\
T_{33} &= -n^{1/2} E \left( \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ - \frac{K_h(\beta_0^T X_j - \beta_0^T X_i)X_{ij} I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)E \{Y_i(Z_i) | \beta_0^T X_i\}} \right. \\
& \quad \left. + \frac{E \{X_{ij} Y_i(Z_i) | \beta_0^T X_i\} K_h(\beta_0^T X_j - \beta_0^T X_i)I(Z_j \geq Z_i)}{f_{\beta_0^T X}(\beta_0^T X_i)[E \{Y_i(Z_i) | \beta_0^T X_i\}]^2} \right] \right)
\end{align*}
\]

Here we used U-statistic property in the last equality above. Now when \( nh^4 \to 0 \),

\[
\begin{align*}
T_{31} &= n^{-1/2} \sum_{i=1}^{n} \Delta_i \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ - \frac{E\{X_{ii} Y_i(Z_i) | \beta_0^T X_i\}}{E \{Y_i(Z_i) | \beta_0^T X_i\}} \right. \\
& \quad \left. + \frac{E \{X_{ii} Y_i(Z_i) | \beta_0^T X_i\} E \{Y_i(Z_i) | \beta_0^T X_i\}}{[E \{Y_i(Z_i) | \beta_0^T X_i\}]^2} \right] + O(n^{1/2}h^2) \\
= & \quad o_p(1).
\end{align*}
\]
Thus, $T_{33} = o_p(1)$ as well. To analyze $T_{32}$,

$$
T_{32} = n^{-1/2} \sum_{j=1}^{n} E \left( \frac{\lambda_i(Z_i, \beta_0^T X_i)}{\lambda(Z_i, \beta_0^T X_i)} \otimes \left[ - \frac{K_h(\beta_0^T X_j - \beta_0^T X_i)X_jI(Z_j \geq Z_i)}{f_{\theta_0^T X}(\beta_0^T X_i)I(Z_i \geq Z_j)} \right] \right) + \frac{E \{X_i I(Z \geq Z_i) | \beta_0^T X = \beta_0^T X_i, Z_i\}}{f_{\theta_0^T X}(\beta_0^T X_i)} I(z_j \geq Z_i) K_h(\beta_0^T X_j - \beta_0^T X_i) \right) \right)

$$

$$
= n^{-1/2} \sum_{j=1}^{n} \left[ \int_0^{z_j} E \left( \frac{\lambda_i(s, \beta_0^T X_j)}{\lambda(Z_i, \beta_0^T X_i)} \right) \otimes \left[ \frac{E \{X_i S_c(s, X) | \beta_0^T X = \beta_0^T X_j\}}{E [S_c(s, X) | \beta_0^T X = \beta_0^T X_j]} - x_{ij} \right] S_c(s, X_i)ds | \beta_0^T X_i = \beta_0^T x_j \right) + O_p(n^{1/2}h^2)

$$

$$
= n^{-1/2} \sum_{j=1}^{n} \int Y_j(s) \lambda(s, \beta_0^T X_j) \otimes \left[ E \{X_i Y_j(s) | \beta_0^T X_j\} \right] ds + O_p(n^{1/2}h^2).

$$

When $nh^4 \to 0$, plugging the results of $T_1$ and $T_{32}$ to (S.37), the expression in (S.33) is

$$
n^{-1/2} \sum_{i=1}^{n} \Delta_i \otimes \left[ X_i - \frac{\hat{E} \{X_i Y_i(Z_i) | \beta_0^T X_i\}}{E \{Y_i(Z) | \beta_0^T X_i\}} \right] \right]

$$

$$
= n^{-1/2} \sum_{i=1}^{n} \int \lambda_i(t) \otimes \left[ X_i - \frac{E \{X_i Y_i(t) | \beta_0^T X_i\}}{E \{Y_i(t) | \beta_0^T X_i\}} \right] dM_i(t) + o_p(1)

$$

$$
= n^{-1/2} \sum_{i=1}^{n} S_{ef}(\Delta_i, Z_i, X_i) + o_p(1).

$$
Finally,

\[
\mathbf{T}_4 = n^{-1/2} \sum_{i=1}^{n} \Delta_i \left\{ \frac{\hat{\lambda}_i(Z_i, \beta_0^T \mathbf{x}_i)}{\hat{\lambda}(Z_i, \beta_0^T \mathbf{x}_i)} - \frac{\lambda_i(Z_i, \beta_0^T \mathbf{x}_i)}{\lambda(Z_i, \beta_0^T \mathbf{x}_i)} \right\} \\
\times \left[ \frac{E \{ \mathbf{X}_i \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}}{E \{ \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}} - \frac{\hat{E} \{ \mathbf{X}_i \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}}{\hat{E} \{ \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}} \right]
\]

= \text{op}(n^{-1/2} \sum_{i=1}^{n} \Delta_i \left[ \frac{E \{ \mathbf{X}_i \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}}{E \{ \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}} - \frac{\hat{E} \{ \mathbf{X}_i \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}}{\hat{E} \{ \mathbf{Y}_i(Z_i) \mid \beta_0^T \mathbf{x}_i \}} \right])

= \text{op}(n^{-1/2} \sum_{i=1}^{n} \int Y_i(s) \lambda(s, \beta_0^T \mathbf{x}_i) \left[ \frac{E \{ \mathbf{X}_i \mathbf{Y}_i(s) \mid \beta_0^T \mathbf{x}_i \}}{E \{ \mathbf{Y}_i(s) \mid \beta_0^T \mathbf{x}_i \}} - \mathbf{x}_i \right] ds + \text{op}(n^{1/2} h^2) + \text{op}(1),
\]

where the last equality is because the integrand has mean zero conditional on \( \beta_0^T \mathbf{x} \), and the second to last equality is obtained following the same derivation of \( \mathbf{T}_3 \). Using these results in (S.33), combined with the results on (S.34), it is clear that the theorem holds. \( \square \)

### S.7 Proof of Theorem 3

We expand \( \sqrt{n} h \{ \hat{m}(t, \beta^T \mathbf{x}) - m(t, \beta^T \mathbf{x}) \} \) as

\[
\sqrt{n} h \{ \hat{m}(t, \beta^T \mathbf{x}) - m(t, \beta^T \mathbf{x}) \} = \sqrt{n} h \left\{ e^{\hat{\Lambda}(t, \beta^T \mathbf{x})} - e^{\Lambda(t, \beta^T \mathbf{x})} \right\} \int_t^\infty e^{-\Lambda(s, \beta^T \mathbf{x})} ds \tag{S.38}
\]

\[
+ \sqrt{n} h e^{\hat{\Lambda}(t, \beta^T \mathbf{x})} \int_t^\infty \left\{ e^{-\hat{\Lambda}(s, \beta^T \mathbf{x})} - e^{-\Lambda(s, \beta^T \mathbf{x})} \right\} ds. \tag{S.39}
\]

\[
+ \sqrt{n} h \left\{ e^{\hat{\Lambda}(t, \beta^T \mathbf{x})} - e^{\Lambda(t, \beta^T \mathbf{x})} \right\} \int_t^\infty \left\{ e^{-\hat{\Lambda}(s, \beta^T \mathbf{x})} - e^{-\Lambda(s, \beta^T \mathbf{x})} \right\} ds. \tag{S.40}
\]
It is easy to see that the term in (S.40) satisfies

\[
\sqrt{nh} \left\{ e^{\hat{\Lambda}(t, \beta^T x)} - e^{\Lambda(t, \beta^T x)} \right\} \int_t^\infty \left\{ e^{-\Lambda(s, \beta^T x)} - e^{-\hat{\Lambda}(s, \beta^T x)} \right\} ds
\]

\[
= \sqrt{nh} O_p\left\{ \hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x) \right\} \int_t^\infty e^{-\Lambda(s, \beta^T x)} O_p\left\{ \hat{\Lambda}(s, \beta^T x) - \Lambda(s, \beta^T x) \right\} ds
\]

\[
= O_p(\sqrt{nh}) O_p\{ h^4 + (nh)^{-1} \}
\]

\[
= o_p(1)
\]

by Condition C2.

We inspect the terms in (S.38) and (S.39). For (S.38), based on Lemma 1,

\[
\sqrt{nh} \left\{ e^{\hat{\Lambda}(t, \beta^T x)} - e^{\Lambda(t, \beta^T x)} \right\} \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds
\]

\[
= \sqrt{nh} e^{\Lambda(t, \beta^T x)} \left( \hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x) + O_p\{ (\hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x))^2 \} \right) \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds
\]

\[
= \sqrt{nh} e^{\Lambda(t, \beta^T x)} \left\{ \hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x) \right\} \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds + o_p(1),
\]

where the last step uses Condition C2.

For (S.40), using Condition C2 as well, we get

\[
\sqrt{nh} e^{\Lambda(t, \beta^T x)} \int_t^\infty \left\{ e^{-\hat{\Lambda}(s, \beta^T x)} - e^{-\Lambda(s, \beta^T x)} \right\} ds
\]

\[
= \sqrt{nh} e^{\Lambda(t, \beta^T x)} \int_t^\infty \left[ \hat{\Lambda}(s, \beta^T x) - \Lambda(s, \beta^T x) + O_p\{ (nh)^{-1} + h^4 \} \right] e^{-\Lambda(s, \beta^T x)} ds
\]

\[
= \sqrt{nh} e^{\Lambda(t, \beta^T x)} \int_t^\infty \left\{ \hat{\Lambda}(s, \beta^T x) - \Lambda(s, \beta^T x) \right\} e^{-\Lambda(s, \beta^T x)} ds + o_p(1).
\]

Now combine the leading terms in (S.38) and (S.39) and use the expansion of \( \hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x) \) —
$\Lambda(t, \beta^T x)$ in Lemma 2:

$$\sqrt{nh} e^{\Lambda(t, \beta^T x)} \left\{ \hat{\Lambda}(t, \beta^T x) - \Lambda(t, \beta^T x) \right\} \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds$$

$$+ \sqrt{nh} e^{\Lambda(t, \beta^T x)} \int_t^\infty \left\{ \hat{\Lambda}(s, \beta^T x) - \Lambda(s, \beta^T x) \right\} e^{-\Lambda(s, \beta^T x)} ds$$

$$= e^{\Lambda(t, \beta^T x)} \sum_{i=1}^n \int_0^\infty \sqrt{\frac{n}{h}} I \left\{ \phi_n(r, \beta^T x) > 0 \right\} K_h (\beta^T X_i - \beta^T x)$$

$$\times \left\{ I(r < t) \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds + \int_{\max(r,t)}^\infty e^{-\Lambda(s, \beta^T x)} ds \right\} dM_i(r, \beta^T x) + o_p(1). \tag{S.41}$$

Note that $I \left\{ \phi_n(r, \beta^T x) > 0 \right\} = 1$ almost surely and according to Lemma 1:

$$\frac{1}{n} \sum_{i=1}^n hK_h^2(\beta^T X_i - \beta^T x) Y_i(r) = f_{\beta^T X}(\beta^T x)E\{I(Z \geq r) \mid \beta^T x\} \int K^2(u) du + o_p(1).$$

The leading term in (S.41) converges to $N\{0, \sigma_m^2(t, \beta^T x)\}$ uniformly by martingale central limit theorem, where

$$\sigma_m^2(t, \beta^T x)$$

$$= e^{2\Lambda(t, \beta^T x)} \int_0^\infty \frac{\lambda(r, \beta^T x)}{f_{\beta^T X}(\beta^T x)} \int_0^\infty \frac{\lambda(s, \beta^T x)}{f_{\beta^T X}(\beta^T x)} ds$$

$$\times \left\{ I(r < t) \int_t^\infty e^{-\Lambda(s, \beta^T x)} ds + \int_{\max(r,t)}^\infty e^{-\Lambda(s, \beta^T x)} ds \right\} dr.$$

Therefore $\sqrt{nh} \left\{ \hat{m}(t, \beta^T x) - m(t, \beta^T x) \right\} \rightarrow N\{0, \sigma_m^2(t, \beta^T x)\}$ uniformly for all $t$ and $\beta^T x$. \hfill \square
Table S.1: Results of study 2, based on 1000 simulations with sample size 500. “Prop.” is the semiparametric method, “PM1” and “PM2” are the proportional mean residual life methods, “additive” is the additive method. “bias” is the average absolute bias of each component in $\hat{\beta}$, “sd” is the sample standard deviation of the corresponding estimators. “MSE” is the mean squared error.

|               | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $\beta_6$ | $\beta_7$ |
|---------------|-----------|-----------|-----------|-----------|-----------|-----------|
| true          | 1.3       | -1.3      | 1         | -0.5      | 0.5       | -0.5      |
| No censoring  |           |           |           |           |           |           |
| Prop. bias    | 0.001     | 0.002     | 0.000     | 0.000     | 0.001     | 0.000     |
| Prop. sd      | 0.030     | 0.030     | 0.026     | 0.026     | 0.022     | 0.023     |
| Prop. MSE     | 0.001     | 0.001     | 0.001     | 0.001     | 0.000     | 0.001     |
| PM1 bias      | 0.003     | 0.003     | 0.003     | 0.002     | 0.002     | 0.001     |
| PM1 sd        | 0.061     | 0.061     | 0.051     | 0.040     | 0.040     | 0.042     |
| PM1 MSE       | 0.004     | 0.004     | 0.003     | 0.002     | 0.002     | 0.002     |
| PM2 bias      | 0.005     | 0.005     | 0.004     | 0.003     | 0.002     | 0.001     |
| PM2 sd        | 0.073     | 0.071     | 0.059     | 0.046     | 0.046     | 0.049     |
| PM2 MSE       | 0.005     | 0.005     | 0.004     | 0.002     | 0.002     | 0.002     |
| additive bias | 0.011     | 0.011     | 0.004     | 0.008     | 0.009     | 0.010     |
| additive sd   | 0.100     | 0.101     | 0.086     | 0.077     | 0.080     | 0.079     |
| additive MSE  | 0.010     | 0.010     | 0.007     | 0.006     | 0.006     | 0.006     |
| 20% censoring |           |           |           |           |           |           |
| Prop. bias    | 0.001     | 0.001     | 0.001     | 0.001     | 0.002     | 0.002     |
| Prop. sd      | 0.032     | 0.040     | 0.035     | 0.031     | 0.040     | 0.032     |

75
|        | MSE   | 0.002 | 0.002 | 0.001 | 0.001 | 0.002 | 0.001 |
|--------|-------|-------|-------|-------|-------|-------|-------|
| PM1    | bias  | 0.008 | 0.014 | 0.003 | 0.005 | 0.000 | 0.001 |
|        | sd    | 0.065 | 0.066 | 0.054 | 0.047 | 0.046 | 0.047 |
| MSE    |       | 0.004 | 0.004 | 0.003 | 0.002 | 0.002 | 0.002 |
| PM2    | bias  | 0.034 | 0.274 | 0.003 | 0.187 | 0.064 | 0.186 |
|        | sd    | 0.108 | 0.192 | 0.087 | 0.133 | 0.084 | 0.134 |
| MSE    |       | 0.012 | 0.112 | 0.007 | 0.053 | 0.011 | 0.052 |
| additive bias | 0.001 | 0.014 | 0.005 | 0.006 | 0.005 | 0.007 |
|        | sd    | 0.127 | 0.136 | 0.115 | 0.089 | 0.099 | 0.096 |
| MSE    |       | 0.016 | 0.018 | 0.013 | 0.008 | 0.009 | 0.009 |
|        |       |       |       |       |       |       |       |
|        | 40% censoring |       |       |       |       |       |       |
|        | MSE   | 0.002 | 0.001 | 0.001 | 0.001 | 0.001 | 0.002 |
|        | sd    | 0.043 | 0.054 | 0.044 | 0.038 | 0.039 | 0.044 |
| MSE    |       | 0.002 | 0.002 | 0.002 | 0.001 | 0.001 | 0.001 |
| PM1    | bias  | 0.012 | 0.017 | 0.005 | 0.006 | 0.001 | 0.002 |
|        | sd    | 0.077 | 0.078 | 0.069 | 0.056 | 0.056 | 0.059 |
| MSE    |       | 0.006 | 0.006 | 0.005 | 0.003 | 0.003 | 0.003 |
| PM2    | bias  | 0.078 | 0.625 | 0.013 | 0.423 | 0.156 | 0.428 |
|        | sd    | 0.160 | 0.307 | 0.140 | 0.214 | 0.134 | 0.218 |
| MSE    |       | 0.030 | 0.486 | 0.019 | 0.226 | 0.042 | 0.231 |
| additive bias | 0.021 | 0.032 | 0.006 | 0.001 | 0.004 | 0.003 |
|        | sd    | 0.158 | 0.152 | 0.132 | 0.107 | 0.109 | 0.103 |
| MSE    |       | 0.023 | 0.023 | 0.017 | 0.011 | 0.012 | 0.011 |

76
Table S.2: Results of study 3, based on 1000 simulations with sample size 500. “mean” is the average absolute bias of \((\hat{\beta})\) of each component in \(\beta\), “sd” and “MSE” are the sample standard deviation and mean squared error of the corresponding estimations.

|            | \(\beta_{31}\) | \(\beta_{41}\) | \(\beta_{51}\) | \(\beta_{61}\) | \(\beta_{32}\) | \(\beta_{42}\) | \(\beta_{52}\) | \(\beta_{62}\) |
|------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| true       | 2.75           | -0.75          | -1.0           | 2.0            | -3.125         | -1.125         | 1.0            | -2.0          |
| Prop. mean |                |                |                |                |                |                |                |                |
| No censoring | 0.067          | 0.047          | 0.025          | 0.056          | 0.083          | 0.052          | 0.023          | 0.051          |
| sd         | 0.441          | 0.300          | 0.304          | 0.383          | 0.553          | 0.363          | 0.368          | 0.423          |
| MSE        | 0.199          | 0.092          | 0.093          | 0.149          | 0.312          | 0.134          | 0.136          | 0.182          |
| 20% censoring | 0.085          | 0.036          | 0.032          | 0.056          | 0.063          | 0.044          | 0.033          | 0.052          |
| Prop. mean |                |                |                |                |                |                |                |                |
| sd         | 0.511          | 0.404          | 0.453          | 0.490          | 0.719          | 0.472          | 0.457          | 0.503          |
| MSE        | 0.268          | 0.164          | 0.206          | 0.243          | 0.520          | 0.224          | 0.210          | 0.255          |
| 40% censoring | 0.083          | 0.034          | 0.008          | 0.075          | 0.109          | 0.050          | 0.019          | 0.040          |
| Prop. mean |                |                |                |                |                |                |                |                |
| sd         | 0.583          | 0.469          | 0.474          | 0.519          | 0.676          | 0.525          | 0.540          | 0.565          |
| MSE        | 0.347          | 0.221          | 0.225          | 0.275          | 0.468          | 0.278          | 0.292          | 0.321          |
Figure S.1: Mean residual life function estimation in Study 1. Row 1: \( m(t, \beta^T x) \) as a function of \( \beta^T x \) at \( t = 1 \). Row 2 to Row 5: \( m(t, \beta^T x) \) as a function of \( t \) at \( \beta^T x = 1.5 \) from method "semiparametric", "PM1", "PM2", "additive". Left to right columns: no censoring; 20% censoring rate; 40% censoring rate. Black line: True \( m(t, \beta^T x) \); Blue line: Median of \( \hat{m}(t, \beta^T x) \); Blue dashed line: 2.5% empirical percentile curve; Blue dotted line: 97.5% empirical percentile curve.
Figure S.2: Performance of the semiparametric method on mean residual life function of study 2. First row: contour plot of true $m(t, \beta^T X)$; Second row: contour plot of $\tilde{m}(t, \beta^T X)$; Third row: contour plot of $|\tilde{m}(t, \beta^T X) - m(t, \beta^T X)|$. Left to right columns: no censoring; 20% censoring rate; 40% censoring rate.
Figure S.3: Mean residual life function estimation in Study 2. Row 1: $m(t, \beta^T x)$ as a function of $\beta^T x$ at $t = 0.7$. Row 2 to Row 5: $m(t, \beta^T x)$ as a function of $t$ at $\beta^T x = -1$ from method “semiparametric”, “PM1”, “PM2”, “additive”. Left to right columns: no censoring; 20% censoring rate; 40% censoring rate. Black line: True $m(t, \beta^T x)$; Blue line: Median of $\hat{m}(t, \beta^T x)$; Blue dashed line: 2.5% empirical percentile curve; Blue dotted line: 97.5% empirical percentile curve.
Figure S.4: Performance of the semiparametric method on mean residual life function at fixed $t$ of study 3. First row: contour plot of true $m(t = 1, \beta^T X)$; Second row: contour plot of averaged $\overline{m}(t = 1, \beta^T X)$ over 1000 simulations; Third row: contour plot of $|\overline{m}(t = 1, \beta^T X) - m(t = 1, \beta^T X)|$. Left to right columns: no censoring; 20% censoring rate; 40% censoring rate.
Figure S.5: Mean residual life function estimation in Study 3. Row 1: \( m(t, \beta^T x) \) as a function of \( t \) at \( \beta^T x = [1, 1]' \). Row 2: \( m(t, \beta^T x) \) as a function of \( (\beta^T x)_1 \) at \( t = 1 \) and \( (\beta^T x)_2 = 1 \). Row 3: \( m(t, \beta^T x) \) as a function of \( (\beta^T x)_2 \) at \( t = 1 \) and \( (\beta^T x)_1 = 1 \). Left to right columns: no censoring; 20% censoring rate; 40% censoring rate. Black line: True \( m(t, \beta^T x) \); Blue line: Median of \( \hat{m}(t, \beta^T x) \); Blue dashed line: 2.5% empirical percentile curve; Blue dotted line: 97.5% empirical percentile curve.
Figure S.6: Contour plot of mean residual life difference $\hat{m}_{treat}(t, \beta^T_{treat}X) - \hat{m}_{wait}(t, \beta^T_{wait}X)$. First row: fix $t$ at 100, 500 and 1000 from left to right. Second row: fix $\beta^T_{treat}X$ at -40, 50 and 100 from left to right. Third row: fix $\beta^T_{wait}X$ at -40, 50 and 100 from left to right.
References

Andersen, P., Borgan, Ø., Gill, R., and Keiding, N. (1993), *Statistical Models Based on Counting Processes*, New York: Springer.

Bertsimas, D., Farias, V. F., and Trichakis, N. (2013), “Fairness, Efficiency, and Flexibility in Organ Allocation for Kidney Transplantation,” *Operations Research*, 61, 73–87.

Bickel, P., Klaassen, C., Ritov, Y., and Wellner, J. (1994), *Efficient and Adaptive Estimation for Semiparametric Models*, New York: Springer.

Brattström, C., Granath, F., Edgren, G., Smedby, K. E., and Wilczek, H. E. (2013), “Overall and Cause-Specific Mortality in Transplant Recipients with a Pretransplantation Cancer History,” *Transplantation*, 96, 297–305.

Chen, Y. Q. (2007), “Additive Expectancy Regression,” *Journal of the American Statistical Association*, 102, 153–166.

Chen, Y. Q., and Cheng, S. (2005), “Semiparametric Regression Analysis of Mean Residual Life with Censored Survival Data,” *Biometrika*, 92, 19–29.

Chen, Y. Q., and Cheng, S. (2006), “Linear Life Expectancy Regression with Censored Data,” *Biometrika*, 93, 303–313.

Chen, Y. Q., Jewell, N. P., Lei, X., and Cheng, S. C. (2005), “Semiparametric Estimation of Proportional Mean Residual Life Model in Presence of Censoring,” *Biometrics*, 61, 170–178.

Ferri, F. F. (2017), *Ferri’s Clinical Advisor 2018 E-Book: 5 Books in 1*, Missouri: Elsevier Health Sciences.
Fleming, T. R., and Harrington, D. P. (1991), *Counting Processes and Survival Analysis*, New York: Wiley.

Frei, U., Schindler, R., Wieters, D., Grouven, U., Brunkhorst, R., and Koch, K. M. (1995), “Pre-transplant Hypertension: a Major Risk Factor for Chronic Progressive Renal Allograft Dysfunction?,” *Nephrology Dialysis Transplantation*, 10, 1206–1211.

Friedman, A. N., Miskulin, D. C., Rosenberg, I. H., and Levey, A. S. (2003), “Demographics and Trends in Overweight and Obesity in Patients at Time of Kidney Transplantation,” *American Journal of Kidney Diseases*, 41, 480–487.

Gill, J. S., Tonelli, M., Johnson, N., Kiberd, B., Landsberg, D., and Pereira, B. J. (2005), “The Impact of Waiting Time and Comorbid Conditions on the Survival Benefit of Kidney Transplantation,” *Kidney International*, 68, 2345–2351.

Grimm, Jr, R. H., Svendsen, K. H., Kasiske, B., Keane, W. F., and Wahl, M. M. (1997), “Proteinuria is a Risk Factor for Mortality Over 10 Years of Follow-up,” *Kidney International Supplement*, 63, 10–14.

Härdle, W., Spokoiny, V., Sperlich, S. et al. (1997), “Semiparametric Single Index Versus Fixed Link Function Modelling,” *The Annals of Statistics*, 25, 212–243.

Härdle, W., and Stoker, T. M. (1989), “Investigating Smooth Multiple Regression by the Method of Average Derivatives,” *Journal of the American statistical Association*, 84, 986–995.

Israni, A. K., Salkowski, N., Gustafson, S., Snyder, J. J., Friedewald, J. J., Formica, R. N., Wang, X., Shteyn, E., Cherikh, W., Stewart, D. et al. (2014), “New National Allocation
Policy for Deceased Donor Kidneys in the United States and Possible Effect on Patient Outcomes,” *Journal of the American Society of Nephrology*, 25, 1842–1848.

Jeong, J.-H., Jung, S.-H., and Costantino, J. P. (2008), “Nonparametric Inference on Median Residual Life Function,” *Biometrics*, 64, 157–163.

Kalantar-Zadeh, K., Abbott, K. C., Salahudeen, A. K., Kilpatrick, R. D., and Horwich, T. B. (2005), “Survival Advantages of Obesity in Dialysis Patients,” *The American Journal of Clinical Nutrition*, 81, 543–554.

Kasiske, B. L., Snyder, J. J., Gilbertson, D., and Matas, A. J. (2003), “Diabetes Mellitus After Kidney Transplantation in the United States,” *American Journal of Transplantation*, 3, 178–185.

Leppke, S., Leighton, T., Zaun, D., Chen, S.-C., Skeans, M., Israni, A. K., Snyder, J. J., and Kasiske, B. L. (2013), “Scientific Registry of Transplant Recipients: Collecting, Analyzing, and Reporting Data on Transplantation in the United States,” *Transplantation Reviews*, 27, 50–56.

Lin, H., Fei, Z., and Li, Y. (2016), “A Semiparametrically Efficient Estimator of the Time-Varying Effects for Survival Data with Time-Dependent Treatment,” *Scandinavian Journal of Statistics*, 43, 649–663.

Liu, Y., Lin, C., and Zhou, Y. (2019), “Nonparametric Estimate of Conditional Quantile Residual Lifetime for Right Censored Data,” *Statistics and Its Interface*, 12, 61–70.

Ma, Y., and Yin, G. (2010), “Semiparametric Median Residual Life Model and Inference,” *Canadian Journal of Statistics*, 38, 665–679.
Ma, Y., Zhang, X. et al. (2015), “A Validated Information Criterion to Determine the Structural Dimension in Dimension Reduction Models,” *Biometrika*, 102, 409–420.

Ma, Y., and Zhu, L. (2013), “Efficient Estimation in Sufficient Dimension Reduction,” *Annals of Statistics*, 41, 250–268.

Maguluri, G., and Zhang, C.-H. (1994), “Estimation in the Mean Residual Life Regression Model,” *Journal of the Royal Statistical Society, Series B*, 56, 477–489.

McLain, A. C., and Ghosh, S. K. (2011), “Nonparametric Estimation of the Conditional Mean Residual Life Function with Censored Data,” *Lifetime Data Analysis*, 17, 514–532.

Meier-Kriesche, H.-U., and Kaplan, B. (2002), “Waiting Time on Dialysis as the Strongest Modifiable Risk Factor for Renal Transplant Outcomes: A Paired Donor Kidney Analysis,” *Transplantation*, 74, 1377–1381.

Meier-Kriesche, H.-U., Port, F. K., Ojo, Akinlolu, O., Rudich, S. M., Hanson, J. A., Cibrik, D. M., Leichtman, A. B., and Kaplan, B. (2000), “Effect of Waiting Time on Renal Transplant Outcome,” *Kidney International*, 58, 1311–1317.

Müller, H.-G., and Zhang, Y. (2005), “Time-Varying Functional Regression for Predicting Remaining Lifetime Distributions from Longitudinal Trajectories,” *Biometrics*, 61, 1064–1075.

Oakes, D., and Dasu, T. (1990), “A Note on Residual Life,” *Biometrika*, 77, 409–410.

Oakes, D., and Dasu, T. (2003), “Inference for the Proportional Mean Residual Life Model,” *Lecture Notes-Monograph Series*, 43, 105–116.
Oechslin, E., Kiowski, W., Schneider, J., Follath, F., Turina, M., and Gallino, A. (1996), “Pretransplant Malignancy in Candidates and Posttransplant Malignancy in Recipients of Cardiac Transplantation,” *Annals of Oncology*, 7, 1059–1063.

Penn.I (1997), “Evaluation of Transplant Candidates with Pre-existing Malignancies,” *Annals of Transplantation*, 2, 14–17.

Prentice, R. L., and Kalbfleisch, J. D. (2003), “Mixed Discrete and Continuous Cox Regression Model,” *Lifetime Data Analysis*, 9, 195–210.

Ramlau-Hansen, H. (1983), “The Choice of a Kernel Function in the Graduation of Counting Process Intensities,” *Scandinavian Actuarial Journal*, 1983, 165–182.

Silverman, B. W. (1978), “Weak and Strong Uniform Consistency of the Kernel Estimate of a Density and its Derivatives,” *The Annals of Statistics*, pp. 177–184.

Silverman, B. W. (1986), *Density Estimation for Statistics and Data Analysis*, Vol. 26, Boca Raton, Florida: CRC Press.

Szekely, G. J., Rizzo, M. L. et al. (2014), “Partial Distance Correlation with Methods for Dissimilarities,” *The Annals of Statistics*, 42, 2382–2412.

Tsiatis, A. (2006), *Semiparametric Theory and Missing Data*, New York: Springer.

Wolfe, R., McCullough, K. P., Schaubel, D., Kalbfleisch, J., Murray, S., Stegall, M. D., and Leichtman, A. (2008), “Calculating Life Years from Transplant (LYFT): Methods for Kidney and Kidney-pancreas Candidates,” *American Journal of Transplantation*, 8, 997–1011.