Four-point boundary connectivities in critical two-dimensional percolation from conformal invariance

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ABSTRACT: We conjecture an exact form for an universal ratio of four-point cluster connectivities in the critical two-dimensional $Q$-color Potts model. We also provide analogous results for the limit $Q \to 1$ that corresponds to percolation where the observable has a logarithmic singularity. Our conjectures are tested against Monte Carlo simulations showing excellent agreement.
1 Introduction

The study of the geometry of random two-dimensional fractals has revealed the emergence of a profound mathematical connection between probability theory and stochastic processes [1, 2] on one hand and quantum field theory together with conformal symmetry on the other [3, 4]. Historically, a number of exact results were derived for the fractal dimensions of two-dimensional critical clusters in basic models of statistical mechanics such as percolation or Ising, built on the seminal contribution [5] and the so-called Coulomb Gas approach [6]. A deeper insight on how conformal invariance could be relevant to describe geometrical observables followed after [7] when J. Cardy derived, with methods borrowed from (boundary) conformal field theory [8, 9], an exact formula for the probability that at least a cluster should span the two horizontal sides of a rectangle in critical percolation [10]. The success of this approach suggested that a geometrical problem, such as critical percolation, could be solvable in two dimensions due to the infinite-dimensional nature of the conformal group [5].

However, at the same time, it was noticed how the conformal algebra associated to geometrical phase transitions could be more subtle [11]. In particular as a conformal field theory, critical percolation should have vanishing central charge (denoted by \(c\)) since its partition function does not depend on finite size effects [12]. However rightly at \(c = 0\) the stress-energy tensor is a null field and the field theory if not trivial, cannot be unitary. Absence of unitarity has serious consequences on the Operator Product Expansion (OPE) and ultimately produces logarithmic singularities in the four-point functions [13, 14]. Later,
it was conjectured in [15] that the OPE of two chiral fields with scaling dimension \( h \neq 0 \) at \( c = 0 \) should have the following expansion (\( z \in \mathbb{C} \))

\[
\lim_{z \to 0} \phi(z)\phi(0) = \frac{1}{2\pi i} \left[ 1 + \frac{2h}{b} z^2 (t(0) + \log(z) T(0)) + \ldots \right],
\]

(1.1)

where \( T(z) \) is the null stress energy tensor and \( t(z) \) was called its logarithmic partner. The parameter \( b \), termed the indecomposability parameter, is not fixed \( a \) priori by the algebra. Its name, in particular, stems from the fact that the fields \( T \) and \( t \) span a Jordan cell of dimension two which makes the conformal dilation operator non-diagonalizable. Conformal field theories that are built upon indecomposable representations of the Virasoro algebra are called logarithmic [16]. They are supposed to be ubiquitous in the study of random clusters and disordered two-dimensional systems [17, 18]. For detailed studies in higher dimensions, see also [19].

During the last decade a lot of effort has been put into the classification of logarithmic CFTs with special success on finite domains; see [20] and references therein. However not many exact correlation functions have been explicitly calculated and tested in statistical mechanics. Important exceptions are G. Watts result [21] and other generalizations of Cardy crossing formula on polygonal domains, such as hexagons or octagons [22, 23]. In particular, logarithmic singularities in crossing probabilities are hidden into higher-point correlation functions [23, 24]; for instance the six-point functions of the field \( \phi_{1,2} \) in the notations of Eq. (3.3). Such a field has vanishing scaling dimensions at \( c = 0 \) and its four-point function cannot be logarithmic [7], c.f. Eq. (1.1). We should also mention that the study of logarithmic conformal field theories in the bulk is considerably harder than on a finite geometry, due to the constraints of crossing symmetry. Recent developments for three [25–27] and four-point functions [28], are based on a conformal bootstrap approach to Liouville theory for \( c < 1 \).

In this paper we complement those existing results, introducing and exactly determining a geometrical observable that explicitly shows logarithmic behaviour at criticality. We focus on the \( Q \)-color Potts model [29] on a bounded domain and construct a ratio between four-point cluster connectivities, see Fig. 1. We then follow closely [7] and symmetry arguments to obtain a fully analytic expression for such a quantity in terms of Virasoro conformal blocks [5]. The main novelty of this approach is that the functions in Eq. (2.3) solve a third order differential equation that is associated to a null vector for a field with non-zero scaling dimension. They provide direct access at \( c = 0 \) to the OPE in Eq. (1.1), including then the parameter \( b \) for boundary percolation. Anticipating the content of Sec. 4, as expected, we recover \( b = -5/8 \), the same value argued in [15] and [30]. As noticed in [18, 31], our geometrical observable is also logarithmic at \( Q = 2 \), i.e. in the (extended [32]) Ising model. This case was already analyzed in [33] by the same authors but we recast it here in a more general context. We also provide numerical verifications of all our results through high-precision Monte Carlo simulations and further extend the analysis in [33] to the three-color and four-color Potts model.

The outline of the rest of the paper is as follows. In Sec. 2 we introduce the \( Q \)-
Four points \( x_1, x_2, x_3 \) and \( x_4 \) are marked on the boundary of a simply connected domain \( \mathcal{D} \) embedded into a regular two-dimensional lattice. The function \( P_{(12)(34)} \) is the probability that \( x_1 \) and \( x_2 \), and \( x_3 \) and \( x_4 \) are connected into the same FK cluster (in blue) but the two are different. Analogously \( P_{(14)(23)} \) is the probability that \( x_1 \) and \( x_4 \), and \( x_2 \) and \( x_3 \) are connected into the same FK cluster but the two are different. Finally \( P_{(1234)} \) is the probability that all the points belong to the same FK cluster. The non-normalized probability measure of any configurations (i.e. of any random graphs \( G \)) is given in Eq. (2.2).

Figure 1: Four points \( x_1, x_2, x_3 \) and \( x_4 \) are marked on the boundary of a simply connected domain \( \mathcal{D} \) embedded into a regular two-dimensional lattice. The function \( P_{(12)(34)} \) is the probability that \( x_1 \) and \( x_2 \), and \( x_3 \) and \( x_4 \) are connected into the same FK cluster (in blue) but the two are different. Analogously \( P_{(14)(23)} \) is the probability that \( x_1 \) and \( x_4 \), and \( x_2 \) and \( x_3 \) are connected into the same FK cluster but the two are different. Finally \( P_{(1234)} \) is the probability that all the points belong to the same FK cluster. The non-normalized probability measure of any configurations (i.e. of any random graphs \( G \)) is given in Eq. (2.2).

2 Four-point boundary connectivities in the \( Q \)-color Potts model

The \( Q \)-color Potts model [29] is defined by the Hamiltonian \((J > 0)\)

\[
H_Q = -J \sum_{\langle x,y \rangle} \delta_{s(x),s(y)}, \quad s(x) = 1, \ldots, Q
\]  

(2.1)

where the spin variable \( s(x) \) takes only positive integer values up to \( Q \), and the sum extends over next-neighbouring sites on a certain bounded domain \( \mathcal{D} \) embedded into a two-dimensional regular lattice. The boundary conditions for the spin are free. The Potts partition function \( Z(Q) = \sum_{\{s(x)\}} e^{-H_Q} \) admits a well known graph expansion, the so-called Fortuin and Kasteleyn [34] representation. Let \( p = 1 - e^{-J} \) the probability of drawing a bond between two next-neighbouring lattice sites in \( \mathcal{D} \), then it turns out, up to a multiplicative constant,

\[
Z(Q) = \sum_{G} p^{n_b}(1-p)^{\bar{n}_b} Q^{N_c}.
\]  

(2.2)

In Eq. (2.2) above, \( n_b \) (resp. \( \bar{n}_b \)) is the number of occupied (resp. empty) bonds in the domain \( \mathcal{D} \). Connected components, including isolated points, inside a graph \( G \) are called clusters (FK clusters). Each graph contains \( N_c \) clusters in which the Potts spins are forced to have the same color, hence the factor \( Q^{N_c} \). When \( Q = 2 \), Eq. (2.2) is the high-temperature expansion of the Ising model. Although the partition function \( Z(Q) \) can be defined for any complex \( Q \), in this paper we will consider only positive integer values including however
that corresponds to the percolation problem. In particular, we will be interested in determining boundary connectivities in the $Q$-color Potts model.

Connectivities in the $Q$-color Potts model are probabilities that a certain set of points marked by $x_1, \ldots, x_n$ are partitioned into FK clusters. A non-normalized probability measure for the allowed graph configurations is given by $m(\mathcal{G}) = p^n_b (1 - p)^{n_b} Q^{N_c}$ according to Eq. (2.2). The normalized probability measure for the graphs would be of course $Z^{-1} m(\mathcal{G})$ as $Z(Q) = 1$ only at $Q = 1$; the normalization factor is however not essential here since only ratios of connectivities will be considered. Moreover, if we focus on configurations in which $n$ points are on the boundary, it can be shown [35] that the number of linearly independent connectivities is also the number of non-crossing non-singleton partitions of a set of $n$ elements (also known as Riordan numbers [36]). In particular if $n = 4$ there are only three linearly independent four-point boundary connectivities that are schematically represented in Fig. 1. With obvious notations such four-point functions will be denoted by $P_{(12)(34)}$, $P_{(14)(23)}$ and $P_{(1234)}$, see again Fig. 1.

For $1 \leq Q \leq 4$, the Potts model undergoes a second order ferromagnetic phase transition for a critical reduced inverse temperature $J = J_c$. The ferromagnetic phase transition is instead of the first order for $Q > 4$. In geometrical terms, at $J < J_c$ there is a finite probability that any bulk point will be connected to the boundary of $\mathcal{D}$. Such a probability vanishes as a power law as $J \to J_c$ with a critical exponent that coincides with the one of the one-point function of the order parameter; for instance [37].

At $J = J_c$, and in the scaling limit when the mesh of the lattice is sent to zero, connectivities are conjectured to be conformally covariant [10]. We can then define the dimensionless conformal invariant quantity

$$R = \frac{P_{(14)(23)}}{P_{(14)(23)} + P_{(12)(34)} + P_{(1234)}}. \quad (2.3)$$

Let us now clarify the statement that $R$ is a conformal invariant quantity. If $x_j = (s_j, t_j)$ belongs to $\mathcal{D}$ we introduce complex coordinates $w_j = s_j + i t_j \in \mathbb{C}$. Then the bounded domain $\mathcal{D}$ (for instance the unit disk) can be mapped conformally through the mapping $z = z(w)$ into the upper half plane. When the points $w_j$ are on the boundary of $\mathcal{D}$, they are mapped on the real axis and we can always choose $z_1 < z_2 < z_3 < z_4$. Conformal invariance implies that $R$ calculated on the upper half plane is only a function of the anharmonic ratio

$$\eta = \frac{z_{21} z_{43}}{z_{31} z_{42}} \quad (2.4)$$

where $z_{ij} = z_i - z_j$. We can then determine $R$ on the original domain $\mathcal{D}$ simply replacing $z_j = z(w_j)$ ($j = 1, \ldots, 4$) into the expression for $R(\eta)$. In the next sections, we will obtain exactly $R$. The conjecture for Eq. (2.3) is valid for all the integer values $Q = 1, 2, 3, 4$ and will be eventually tested against Monte Carlo simulations. Anticipating the content of the remaining sections, the reader will find a plot of our theoretical predictions for the universal ratio in Eq. (2.3) in Fig. 5.

It is important to observe that, although their leading short-distance singularities are
the same, four-point boundary connectivities cannot be obtained from the knowledge of the boundary correlation functions of the Potts order parameter. A paradigmatic example [35] is $Q = 2$, where the unique boundary four-point function of the spin is a linear combination of the three connectivities in Fig. 1.

3 Duality and conformal symmetry

Boundary-condition-changing operators and duality—The quantum field theory that describes the critical large-distance fluctuations of the two-dimensional $Q$-color Potts model is a conformal field theory (CFT) [38] with central charge

$$c = 1 - \frac{6}{p(p + 1)}$$

(3.1)

and the parameter $Q = 4 \cos^2[\pi/(p + 1)]$. The central charge in Eq. (3.1) enters the commutation relations of the Virasoro algebra generators

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}\delta_{n+m,0}n(n^2 - 1).$$

(3.2)

In principle $p \in \mathbb{R}$, however since $Q$ is integer and $1 \leq Q \leq 4$, we only consider the cases $p = 2$ (percolation), $p = 3$ (Ising), $p = 5$ (three-color Potts model) and $p \to \infty$ (four-color Potts model). Notice that at $Q = 1$ the central charge in Eq. (3.1) is zero, a well-known circumstance of critical percolation that in particular implies that in such a case the CFT is non-unitary (the only unitary CFT with zero central charge is indeed trivial). This lack of unitary reflects itself into the presence, as we will discuss, of logarithmic singularities [11, 13–15] in the four-point connectivities. Let us now briefly review some basics of (boundary) CFTs in two dimensions [8, 9]. When a scaling field is inserted at the boundary of a planar domain $D$, its scaling dimensions are eigenvalues of the operator $L_0$ [8]. The scaling dimensions of a primary field (for a definition see [39]) $\phi_{r,s}$ sitting at the boundary are then given by

$$h_{r,s} = \left[\frac{p+1}{4p} - \frac{sp}{p(p+1)}\right]^2 - 1;$$

(3.3)

and it turns out that for our purposes $r, s$ are positive integers. In such a case the fields $\phi_{r,s}$ are also dubbed degenerate [5] and their correlation functions satisfy partial differential equations of degree $rs$. The operator content of the CFT depends on the boundary conditions. For the $Q$-color Potts model natural boundary conditions for the spin variable on $D$ are either free or fixed to a definite color $\alpha = 1, \ldots, Q$. Remarkably, conformal symmetry is however also compatible with inhomogeneous boundary conditions that are associated to the insertion of scaling fields at the boundary called boundary-condition-changing (bcc) operators [9]. For instance if the value of the spin on the boundary switches from $\alpha$ to $\beta \neq \alpha$ nearby $x$, such a discontinuity in the boundary conditions is realized, in the scaling limit, by the insertion of a scaling field $\phi_{\alpha\beta}(x)$. In particular, the bcc operator $\phi_{\alpha\beta}$ can be identified with a field $\phi_{r,s}$, whose scaling dimensions are given in Eq. (3.3); we will dis-
cuss which field in the next subsection. Before we remind, as pointed out first in [7], how correlation functions of bcc operators can be related to connectivities in the $Q$-color Potts model. The argument exploits the duality transformation of the Potts partition function on a planar domain $\mathcal{D}$; in particular the mapping also requires a transformation of the lattice. A duality transformation [40] relates partition functions on the dual lattice with fixed boundary conditions and in the ordered phase to connectivities calculated with free boundary conditions on the original lattice in the disordered phase. In particular if we denote by $Z^\ast_{\alpha\beta\alpha\beta}$ the dual partition function (calculated on the dual lattice) with boundary conditions as in Fig. 2a ($\alpha \neq \beta$), we have [35, 40]

$$Z^\ast_{\alpha\beta\alpha\beta}(x_1, x_2, x_3, x_4) = A'(P_{(12)(34)} + P_{(14)(23)} + P_{(1234)}),$$

(3.4)

where $A'$ is a normalization constant that does not depend on $x_1, \ldots, x_4$. The duality relation Eq. (3.4) can be understood as follows. Perform an FK graph expansion of the partition function $Z^\ast_{\alpha\beta\alpha\beta}$, then the dual FK clusters cannot connect regions where the spins are fixed to have different colors at the boundary. We can distinguish three cases:

- Dual graph configurations contain at least a dual cluster connecting the two regions with boundary conditions $\beta$ (there is an horizontal dual crossing). Applying a duality transformation these configurations are in one-to-one correspondence with the ones that contribute to $P_{(12)(34)}$.

- Dual graph configurations contain at least a dual cluster connecting the two regions with boundary conditions $\alpha$ (there is a vertical dual crossing). Applying a duality transformation these configurations are in one-to-one correspondence with the ones that contribute to $P_{(14)(23)}$, see Fig. 2a.

- Dual graph configurations do not contain any cluster that connects regions on the boundary with the same color (there are no dual crossings). Applying a duality transformation these configurations are in one-to-one correspondence with $P_{(1234)}$.

Notice that cannot be simultaneous horizontal and vertical crossings; this possibility was instead investigated in [21]. Summing over the three possibilities we obtain Eq. (3.4). The partition function $Z^\ast_{\alpha\beta\alpha\beta}(x_1, x_2, x_3, x_4)$ is in turn proportional [7, 9] in the scaling limit to the four-point function $\langle \phi_{\alpha\beta}(x_1)\phi_{\beta\alpha}(x_2)\phi_{\alpha\beta}(x_3)\phi_{\beta\alpha}(x_4) \rangle$. At the critical self-dual point we then conclude that

$$\langle \phi_{\alpha\beta}(x_1)\phi_{\beta\alpha}(x_2)\phi_{\alpha\beta}(x_3)\phi_{\beta\alpha}(x_4) \rangle = A'' P_1(x_1, x_2, x_3, x_4),$$

(3.5)

with $A''$ constant and $P_1 \equiv P_{(12)(34)} + P_{(14)(23)} + P_{(1234)}$. Similar consideration can be applied formally to all the correlation functions $\langle \phi_{\alpha_1\alpha_2}(x_1)\phi_{\alpha_2\alpha_3}(x_2)\phi_{\alpha_3\alpha_4}(x_3)\phi_{\alpha_4\alpha_1}(x_4) \rangle$, for an arbitrary choice of the color $\alpha_i$. For instance if $\alpha_i \neq \alpha_j$ for any $i$ and $j$, the four-point function above is proportional to $P_{(1234)}$ in the scaling limit.

*Conformal blocks and the universal ratio $R$—* Following [7], we identify the bcc operator
Figure 2: (a) Partition function \( Z^*_{\alpha\beta\alpha\beta} \). It is drawn a dual FK cluster (dashed line) leading to a vertical crossing. These cluster configurations are dual to the ones contributing to \( P_{(14)(23)} \). In particular it is not possible to connect \( x_1 \) with \( x_3 \) or \( x_2 \) with \( x_4 \) without crossing the dual dashed cluster. (b) The limit \( x_1 \to x_2 \) in the function \( P_{(14)(23)} \) produces configurations where two distinct FK clusters meet at \( x_2 \). Field theoretically this is interpreted as the insertion of the field \( \phi_{1,5} \) at the boundary point \( x_2 \).

\( \phi_{\alpha\beta} \) with the field \( \phi_{1,3} \), whose scaling dimensions are given (cft. Eq. (3.3)) by

\[
h(p) = \frac{p - 1}{p + 1}. \tag{3.6}
\]

The identification holds for any integer \( 1 \leq Q \leq 4 \). Let us now consider the boundary four-point function of the field \( \phi_{1,3} \); as discussed in Sec. 3 we work on the upper half plane, denoted hereafter by \( \mathbb{H} \). The four points are then ordered on the real axis and chosen such that \( z_1 < z_2 < z_3 < z_4 \). Exploiting global conformal symmetry on the upper half plane (i.e. \( SL(2, \mathbb{R}) \) Möbius transformations), the four-point function of \( \phi_{1,3} \) can be written as

\[
\langle \phi_{1,3}(z_1)\phi_{1,3}(z_2)\phi_{1,3}(z_3)\phi_{1,3}(z_4) \rangle \mathbb{H} = \frac{1}{(z_{12}z_{34})^{2h}(1 - \eta)^{2h}} G(\eta). \tag{3.7}
\]

The function \( G(\eta) \) solves [5] an Ordinary Differential Equation (ODE) of degree 3 that can be obtained by the condition of decoupling of the null-vector at level three in the Verma module of \( \phi_{1,3} \). Deriving such a differential equation is standard, the reader can consult for instance [41]. It turns out

\[
6(1 - h)\bar{h}^2(1 - 2\eta)G(\eta) + \left[ 2(-2 + \eta)\eta - 3h \left( 1 - 5\eta + 5\eta^2 \right) + \bar{h}^2 \left( 3 - 19\eta + 19\eta^2 \right) \right] G'(\eta) + (-1 + \eta)^2 \left[ (-2 + 4h + 4\eta - 8h)G''(\eta) + (-1 + \eta)\eta G'''(\eta) \right] = 0, \tag{3.8}
\]

and \( h \) depends on \( p \) as in Eq. (3.6) whereas \( p \) is related to the central charge and \( Q \) by Eq. (3.1). The equation above is of Fuchsian type with regular singular points in \( \eta = 0, 1 \) and \( \infty \), therefore we can write its Frobenius series near \( \eta = 0 \) as \( G_p(\eta) = \eta^p \sum_{k=0}^{\infty} a_k \eta^k \) where conventionally we set \( a_0 = 1 \). The series has radius of convergence \( |\eta| < 1 \) however this is enough for our purposes since by a global conformal transformation we can set \( z_1 = 0 \),
\[ z_2 = \eta, \; z_3 = 1 \text{ and } z_4 = \infty, \] and therefore recalling that the four points are ordered on the boundary \( \eta \in [0, 1] \). Notice also that Eq. (3.8) is symmetric under the transformation \( \eta \rightarrow (1 - \eta) \). The exponent \( \rho \) solves the indicial equation

\[ \rho(\rho - h)(\rho - 3h - 1) = 0 \quad (3.9) \]

and the three roots coincide with the scaling dimensions \( h_{1,1}, h_{1,3} \) and \( h_{1,5} \) given in Eq. (3.3). This is of course expected since [5] the roots of Eq. (3.9) are the scaling dimensions of the leading singularities produced in the OPE \( \lim_{z_1 \to z_2} \phi_{1,3}(z_1)\phi_{1,3}(z_2) \) as it can be seen from Eq. (3.7); in particular \( h_{1,5} = 3h + 1 \). When the differences between the roots of the indicial equation are not integer and always for the largest root, the Frobenius series are directly the Virasoro conformal blocks of the CFT. Denoting by \( F^c_{\rho} \) the four-point conformal block where we fixed the external legs to be the field \( \phi_{1,3} \) (of dimension \( h(c) \)) and the internal field has scaling dimension \( \rho \), we have

\[ F^c_{\rho}(\eta) = \frac{G_{\rho}(\eta)}{(1 - \eta)^{2h}} = \eta^\rho(1 + a_1(\rho, c, h)\eta + a_2(\rho, c, h)\eta^2 + \ldots). \quad (3.10) \]

To lighten the notation we did not show the dependence of \( G_{\rho} \) from \( c \) (alias \( h \)). The coefficients in the series expansion in Eq. (3.10) can be calculated from Zamolodchikov recursive formula [42], see Appendix B, and provide a further verification of the solution of Eq. (3.8). When the indicial equation have roots \( \rho_1 \) and \( \rho_2 \), such that \( \rho_1 - \rho_2 = \nu \) is a positive integer but no solution is logarithmic the Frobenius series may not necessarily coincide with the conformal blocks. The two power series can indeed mix at order \( \nu \).

Since the conformal blocks are directly the contribution in Eq. (3.7) of the OPE channels, we will propose an identification of the universal ratio \( R \) in Eq. (2.3) in terms of them. The identification is based on the following observations.

**Obs. 1:** The three linearly independent connectivities \( P_{(12)(34)}, P_{(14)(23)} \) and \( P_{(1234)} \) are in the scaling limit proportional to the four-point function of \( \phi_{1,3} \). Therefore, once the common prefactor in Eq. (3.7) has been factor out, they are linear combinations of the three conformal blocks.

**Obs. 2:** The field \( \phi_{1,2k+1} \) when inserted at the boundary point \( x \) anchors \( k \) FK-clusters [43, 44]. In particular, if we consider the limit \( x_1 \to x_2 \) in \( P_{(14)(23)} \) we generate configurations where two distinct FK clusters meet at \( x_2 \) and are separated by a dual FK cluster (see the previous subsection and Fig. 2b). Such a cluster configuration is associated to the insertion at \( x_2 \) of the field \( \phi_{1,5} \) and therefore the leading singularity for \( \eta \to 0 \) of \( P_{(14)(23)} \) has to be the same as the one of the conformal block \( F^c_{3h+1} \). However \( 3h+1 > h > 0 \) and we conclude then that no other conformal blocks can enter \( P_{(14)(23)} \) except the one of \( \phi_{1,5} \). This observation identifies (apart from an overall constant) the numerator in Eq. (2.3) as \( (1 - \eta)^{-2h}G_{3h+1} \) through Eq. (3.10).

**Obs. 3:** Consider the OPE of two bcc operators \( \phi_{\alpha\beta} \) as they appear in Eq. (3.5), it has the structure

\[ \phi_{\alpha\beta} \cdot \phi_{\beta\alpha} = 1 + X + \ldots, \quad (3.11) \]
where \( 1 \) is the identity field and \( X \) denotes a field with scaling dimension larger than zero that is compatible with the boundary conditions. Certainly such a field cannot be \( \phi_{\alpha\beta} \), since this would imply a discontinuity of the boundary conditions that is not allowed by the OPE in Eq. (3.11), see also [35] for analogous arguments for kink fields in the bulk. Therefore we are led to the conclusion that the conformal blocks that enter the denominator in Eq. (2.3) can only be \( F^c_0 \) (i.e. the identity conformal block) and \( F^c_{3h+1} \) (i.e. the \( \phi_{1,5} \) conformal block).

**Obs. 4:** The denominator in Eq. (2.3) is obviously symmetric under the exchange \( x_1 \rightarrow x_3 \). Such a symmetry corresponds to the transformation \( \eta \rightarrow (1 - \eta) \). Imagine now, according to the previous observation, to have expressed the denominator in Eq. (2.3) as a linear combination \( \alpha F^c_0(\eta) + \beta F^c_{3h+1}(\eta) \equiv S(\eta) \), then it is easy to verify, substituting into Eq. (3.7) that the symmetry under the exchange \( x_1 \rightarrow x_3 \) requires the function \( S(\eta) \) to satisfy the functional equation \( S(\eta) = S(1 - \eta) \). Obviously \( S \) is a solution of Eq. (3.8) and from the fusion matrix given in [38] (see Eq. (5.11) there) it can be moreover verified that it exists only a linear independent function \( S \), constructed from the conformal blocks of the identity and \( \phi_{1,5} \) that satisfies such a property (the other can be chosen a linear combination of the conformal blocks of \( \phi_{1,3} \) and \( \phi_{1,5} \)).

In summary the universal ratio \( R \) in Eq. (2.3) will be

\[
R(\eta) = A_Q \frac{G_{3h+1}(\eta)}{S(\eta)},
\]

where \( A_Q \) imposes \( R(1) = 1 \) that should be clear from the geometrical interpretation in Fig. 1.

### 4 Analytic expressions for the connectivities

We are ready to present explicit results for the ratio \( R \) in Eq. (2.3) for any integer \( 1 \leq Q \leq 4 \). When needed to distinguish among different \( Q \)’s in Eq. (2.3) we will introduce an extra index \( Q \) (see for instance Eq. (4.5) or Eq. (4.10)). Finally, for more technical details, we invite to read the three final Appendices.

**Q=1: Percolation** — We have (c.f. Eq. (3.1)) \( p = 2 \) and \( h_{1,3} = h = 1/3, h_{1,5} = 3h + 1 = 2 \). The solutions \( G_{1/3}(\eta) \) and \( G_2(\eta) \) are free of logarithms and can be easily determined using the method described in Appendix A. It turns out

\[
G_{1/3}(\eta) = \eta^{1/3} \left( 1 - \frac{1}{2} \eta - \frac{2}{7} \eta^2 - \frac{1}{7} \eta^3 - \frac{58}{637} \eta^4 - \frac{83}{1274} \eta^5 + o(\eta^5) \right), \quad (4.1)
\]

\[
G_2(\eta) = \eta^2 \left( 1 + \frac{1}{3} \eta + \frac{37}{198} \eta^2 + \frac{112}{891} \eta^3 + \frac{469}{5049} \eta^4 + \frac{3304}{45441} \eta^5 + o(\eta^5) \right); \quad (4.2)
\]
and we actually generated $O(10^5)$ terms in all the power series. The conformal blocks $F_{1/3}^0(\eta)$ and $F_{2}^0(\eta)$ are obtained through Eq. (3.10) as it can be verified from Appendix B. The conformal block of the identity field is singular at $c = 0$ \cite{Gurarie-Ludwig} and needs to be regularized. In particular, Eq. (3.12) does not hold \textit{verbatim} at $c = 0$; although it might be still true in the limit $Q \to 1$. Here, we proceed pragmatically finding the Frobenius solution with $\rho = 0$ of Eq. (3.8), that is symmetric under the exchange $\eta \to (1 - \eta)$; such a solution is logarithmic. Using the method described in Appendix A and in particular the normalization $b_0(\sigma) = \sigma$ we obtain

$$
\tilde{G}_0(\eta) = -\frac{8}{45} \log(\eta) G_2(\eta) + \left(1 - \frac{2}{3} \eta^2 + \frac{119}{225} \eta^4 + \frac{152}{2025} \eta^6 + \frac{18947}{735075} \eta^8 + \frac{27058}{2205225} \eta^{10} + o(\eta^{12})\right).
$$

(4.3)

Remarkably, the three power series in Eqs. (4.1-4.3) can be expressed through combinations of hypergeometric functions $3 F_2$; see Eqs. (C.7-C.8) or Eqs. (C.10-C.12). In particular we can construct directly a symmetric solution of Eq. (3.8), whose series expansion near $\eta = 0$ coincides with Eq. (4.3), thus proving that $\tilde{G}_0(\eta) = \tilde{G}_0(1 - \eta)$.

Notice also that the second linearly independent solution symmetric under the transformation $\eta \to (1 - \eta)$ can be chosen $G_{1/3}(\eta) + G_{1/3}(1 - \eta)$. However such a function cannot enter into $S(\eta)$ since it contains a subleading singularity $\eta^{1/3}$. In conclusion, we conjecture that at $Q = 1$, $S(\eta) = \tilde{G}_0(\eta)$, given in Eq. (4.3).

It is also important to mention that the coefficient of the logarithm in Eq. (4.3) is related to the Gurarie-Ludwig \cite{Gurarie-Ludwig} indecomposability parameter $b$. It was indeed argued that CFTs at $c = 0$ could be characterized by a universal number $b$ appearing in the regularized OPE of a chiral field with itself, see Eq. (1.1). In particular if $h$ is the (chiral) scaling dimension of such a field, the leading small $\eta$ behaviour of the function $\tilde{G}_0$ (notice the definition of

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\textbf{Figure 3:} Percolation (left) and Ising (right). The main plots show the results of Monte Carlo simulations for sizes $L = 33, 65, 129, 257$ together with the theoretical predictions derived from CFT. The insets show the deviations of the numerical data from the CFT predictions.
the prefactor in Eq. (3.7)) has to be \[15\]
\[\tilde{G}_0(\eta) = 1 - 2h\eta + \frac{h^2}{b}\eta^2 \log(\eta) + \ldots, \] (4.4)
Comparing with Eq. (4.3) for \(h = 1/3\) we obtain \(b = -5/8\). This is indeed the same value of the indecomposability parameter that was argued to describe critical boundary percolation \[30\]. However no logarithmic observable in critical two-dimensional percolation was fully calculated so-far. Summarizing we have (c.f. Eq. (2.3))
\[R_{Q=1} = A_1 \frac{G_2(\eta)}{G_0(\eta)}, \] (4.5)
and \(A_1\) ensures \(R_{Q=1}(\eta = 1) = 1\); see Appendix C for an explicit expression for Eq. (4.5), including the constant \(A_1\). Comparison with the numerics is presented in Fig. 3.
\[Q=2;\ \text{Ising model}—\] Here \(p = 3\) and \(h_{1,3} = h = 1/2, h_{1,5} = 3h + 1 = 5/2\). This case was solved in \[33\], however we report it for completeness. The power series solution \(G_0(\eta)\) and \(G_{5/2}\) are
\[G_0(\eta) = G_0(1 - \eta) = 1 - \eta + \eta^2, \] (4.6)
\[G_{5/2}(\eta) = \eta^{5/2} \left( 1 + \frac{1}{4} \eta + \frac{49}{384} \eta^2 + \frac{125}{1536} \eta^3 + \frac{37025}{638976} \eta^4 + \frac{37547}{851968} \eta^5 + o(\eta^5) \right). \] (4.7)
and are related to the conformal blocks \(F_{0}^{1/2}(\eta)\) and \(F_{5/2}^{1/2}(\eta)\) through Eq. (3.10). Since the function \(G_0(\eta)\) is symmetric, it follows from Eq. (3.12) \(G_0(\eta) = S(\eta)\); i.e. at \(c = 1/2\), the coefficient \(\beta\) in the linear combination in Eq. (3.12) is zero which simplified the discussion in \[33\]. It can also be proven that the power series for \(G_{5/2}\) can be re-summed as follows
\[G_{5/2}(\eta) = G_0(\eta) \int_0^\eta d\eta' g(\eta'), \] (4.8)
being \[33, 45\]
\[g(\eta) = \frac{16}{21\pi} \frac{(2 - \eta)(1 + \eta)(-1 + 2\eta)E(\eta) + (2 + \eta(-4 + \eta + \eta^2))K(\eta)}{\sqrt{(1 - \eta^2)\eta}(1 + (-1 + \eta)\eta^2)}. \] (4.9)
In Eq. (4.9) above \(E(\eta)\) and \(K(\eta)\) are the complete elliptic integrals of first and second kind (with Mathematica convention for the modulus). In conclusion \[33\] the ratio \(R\) at \(Q = 2\) is given by
\[R_{Q=2} = A_2 \frac{G_{5/2}(\eta)}{G_0(\eta)} = A_2 \int_0^\eta d\eta' g(\eta'); \] (4.10)
where \(A_2\) ensures \(R_{Q=2}(1) = 1\). The logarithmic behaviour emerges in Eq. (4.10) in the limit \(\eta \to 1\) and algebraically is understood by the collision of the primary field \(\phi_{1,5}\) with the descendant \(L_{-2}\phi_{1,3}\) \[46, 47\]. Comparison with the numerics is presented in Fig. 3.
\[Q=3;\ \text{Three-color Potts model}—\] The model corresponds to \(p = 5\) and \(h_{1,3} = h = 2/3, h_{1,5} = 3h + 1 = 3\). There are no logarithmic solutions; using the method of Appendix A we
obtain the power series

\[ G_0(\eta) = G_0(1 - \eta) = 1 - \frac{4}{3} \eta + \frac{4}{3} \eta^2, \quad (4.11) \]

\[ G_{2/3}(\eta) = \eta^{2/3}(1 - \eta + \frac{3}{4} \eta^2), \quad (4.12) \]

\[ G_3(\eta) = \frac{81}{52} \left[ G_0(\eta) - \frac{4}{3} G_2(1 - \eta) \right]. \quad (4.13) \]

The conformal blocks \( F_{2/3}^{4/5}(\eta) \) and \( F_3^{4/5}(\eta) \) are obtained from Eq. (3.10). Using the recursive formula in Appendix B we can also verify directly that \( G_0(\eta)(1 - \eta)^{-4/3} = F_0^{4/5}(\eta) + \frac{20}{81} F_3^{4/5}(\eta) \), thus proving that \( G_0(\eta) \) in Eq. (4.11) is actually \( S(\eta) \); cf Eq. (3.12). We therefore have at \( Q = 3 \)

\[ R_{Q=3} = A_3 \frac{G_3(\eta)}{G_0(\eta)} = 1 - \frac{(1 - \eta)^{2/3} \left( 1 - \frac{2\eta}{3} + \eta^2 \right)}{1 - \frac{4\eta}{3} + \frac{4\eta^2}{3}}. \quad (4.14) \]

The analytic prediction in Eq. (4.14) is compared against numerical simulations in Fig. 4.

\[ Q=4; \text{ Four-color Potts model} \]—Finally we consider the four-color Potts model for which \( p \to \infty \) and \( h_{1,3} = h = 1, h_{1,5} = 3h + 1 = 4 \). All the Frobenius power series reduce to polynomials; Appendix A produces the following basis of solutions: \( G_0(\eta) = 1, G_1(\eta) = \eta - \frac{3}{2} \eta^2 + \eta^3, G_4(\eta) = \eta^4 \). However, as we remarked in Sec. 4 only the conformal block \( F_4 \) is obtained from \( G_4 \) using Eq. (3.10). The conformal blocks of the identity and the field \( \phi_{1,3} \) are derived from the recursive formula in Appendix B (with the limit \( c \to 1 \) taken first)
and are
\[
F_0^1(\eta) = \frac{G_0(\eta) - 2G_1(\eta) + \frac{3}{2}G_4(\eta)}{(1-\eta)^2},
\]
(4.15)
\[
F_1^1(\eta) = \frac{G_1(\eta) - \frac{1}{4}G_4(\eta)}{(1-\eta)^2}.
\]
(4.16)

According to Eq. (3.12) there exists only a linear combination of \(F_0^1\) and \(F_1^1\) that leads to a function \(S\) symmetric under the transformation \(\eta \rightarrow (1-\eta)\). We indeed find \(F_0^1 + \frac{5}{8}F_1^1 = (1-\eta)^2(1-\eta + \eta^2)^2\) and therefore \(S(\eta) = (1-\eta + \eta^2)^2\). Consistently with the discussion below Eq. (3.12) the other linear combination leading to a symmetric function can be chosen \(F_1^1 - \frac{1}{4}F_1^1\). Summarizing at \(Q = 4\)
\[
R_{Q=4} = A_4 \left( \frac{\eta^2}{1-\eta+\eta^2} \right)^2 = \left( \frac{\eta^2}{1-\eta+\eta^2} \right)^2.
\]
(4.17)

Comparison of Eq. (4.17) with the numerical simulations is given in Fig. 4.

Monte Carlo simulations — We give some details about the numerical experiments. Simulation have been carried on triangular lattices on triangles of side \(L\) where \(L = 33, 65, 129, 257\). The value of the reduced inverse temperature has been set to the exactly known \([29]\) critical value in the thermodynamic limit \(e^{2J_c} = 2 \cos \left[ \frac{2}{3} \arccos \left( \frac{\sqrt{Q}}{2} \right) \right]\).

As done in \([33]\), we map the four points \(z_1 = 0, z_2 = \eta, z_3 = 1\) and \(z_4 = \infty\) on the boundary of an equilateral triangle by a Schwartz-Christoffel transformation
\[
w(z) = \frac{6z \Gamma \left( \frac{2}{3} \right)}{\sqrt{\pi} \Gamma \left( \frac{1}{3} \right) } F_1(1/2, 2/3; 3/2, 9z^2).
\]
(4.19)

In particular, the points \(z_1 = 0, z_3 = 1\) are mapped through Eq. (4.19) into the midpoints of an equilateral triangle with length-side two and vertices at \(w(-1/3) = -1, w(1/3) = 1\) and \(w(\infty) = -i\sqrt{3}\). The image of the point \(z_2\) moves therefore along the triangle between \(w(z_1) = 0\) and \(w(z_2) = e^{-i\pi/3}\). Symmetries of the triangle are also taken into account in order to enhance the statistics. The algorithm employed is the Swendsen-Wang cluster algorithm \([48]\) giving direct access to the FK clusters. The random number generator is given in \([49]\) and the number of samples collected is up to \(10^{10}\) for the largest sizes considered. As the size is increased all crossing events become rarer and this happens in a more severe way for higher values of the parameter \(Q\). This can obviously be traced back to the leading scaling dimension \(h_{1,3}\) setting the dimensions of the numerator and denominator in Eq. (2.3). Its value gets bigger as \(Q\) is increased. This has limited the maximal size of the triangular lattice for \(Q = 3, 4\) to \(L = 129\). The results of the simulations for the universal ratio \(R\) as a function of \(\eta\) are shown together with the CFT predictions in the already referred to Figs. 3 and 4.
Figure 5: The universal ratio $R$ in Eq. (2.3) in the critical $Q$-color Potts model as a function of the anharmonic ratio $\eta$.

5 Conclusions

In this paper we constructed an universal ratio $R$, see Eq. (2.3), that involves four-point boundary connectivities of FK clusters in the two-dimensional $Q$-color Potts model. Exploiting lattice duality and conformal symmetry we conjectured an exact expression for $R$ at criticality for any integer values $1 \leq Q \leq 4$. In particular we considerably expanded the study in [33], to the three-color and four-color Potts model. Remarkably we also provided a conjecture for $R$ in the percolation problem that corresponds to the limit $Q \to 1$. Our theoretical results are plotted in Fig. 5.

The percolation case is particularly interesting since critical properties are described by a non-unitary CFT with vanishing central charge. Non-unitary extensions of minimal conformal models [5] are notoriously hard to address theoretically and few exact correlation functions have been obtained during the years. In particular earlier studies focused on generalizations of Cardy formula [7].

We calculated exactly four-point functions at $c = 0$ of an operator with non-vanishing scaling dimension and interpreted them as cluster connectivities in critical percolation. In particular, the sum of connectivities in Fig. 1 furnishes a fully explicit example of a logarithmic singularity at $c = 0$. Consistently with previous analysis [30] and the original proposal [15] we also derived the value $b = -5/8$ for the indecomposability parameter of boundary percolation. This is another direct, although not easily accessible numerically, verification of the indecomposability parameter $b$ in boundary percolation. We checked extensively our conjectures with high-precision Monte Carlo simulations on a triangular lattice, confirming both universality of the ratio in Eq. (2.3) and its remarkable agreement with the predictions of conformal invariance for any integer $1 \leq Q \leq 4$.

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A Frobenius series

For mathematical details we refer to the classic volume [53]. Given the ODE in Eq. (3.8) we write its truncated Frobenius series as

\[
G_\rho(\eta) = \sum_{k=0}^{N} a_k \eta^k.
\]

We denote by \(L\) the action of the differential operator in Eq. (3.8) then it turns out

\[
L(G_\rho) = \sum_{k=0}^{N} f_k(\rho) \eta^{\rho+k-1} = 0. \tag{A.1}
\]

The \(k = 0\) coefficient in Eq. (A.1) fixes \(\rho\) throughout

\[
f_0(\rho) = a_0 \rho(\rho - h)(\rho - 3h - 1) = 0. \tag{A.2}
\]

Once we fixed the normalization \(a_0 = 1\), the remaining coefficients \(a_1, \ldots, a_N\) in the power series are obtained solving a linear lower triangular system of equations. The solution is required in symbolic form to avoid numerical truncation errors when summing the series and can be obtained efficiently up to \(N = O(10^5)\) with Mathematica. In practice, the \(N\) equations for the unknowns \(a_1, \ldots, a_N\) are obtained from (A.1) as

\[
\frac{1}{l!} \left. \frac{d^l}{d\eta^l} [\eta^{-\rho+1} L(G_\rho)] \right|_{\eta=0} = f_l(\rho) = 0, \quad l = 1, \ldots, N. \tag{A.3}
\]

The constant vector of the linear system is fixed by the normalization condition \(a_0 = 1\). However, when there are roots in Eq. (A.2) that differ by integers, the linear system (A.3) might be inconsistent [53]. In particular let \(\rho_1 > \rho_0\) two roots of Eq. (A.2) such that \(\rho_1 - \rho_0 = \nu\) is a positive integer and denote by \(M^{(\rho_0)}\) the \(N \times N\) lower triangular matrix associated to Eq. (A.3) for \(\rho = \rho_0\). For simplicity we assume that no other pair of roots are separated by integers. In this case the coefficient \(a_{\nu}\) will appear as a free variable (i.e. the element \([M^{(\rho_0)}]_{\nu\nu} = 0\)). Furthermore let \(M^{(\rho_0)}_{\nu}\) the matrix obtained replacing the \(\nu\)-th column of \(M^{(\rho_0)}\) with the constant vector of the linear system. If the linear system is consistent (i.e. \(\text{rk}(M^{(\rho_0)}_{\nu}) = \text{rk}(M^{(\rho_0)})\)) then the particular solution will give the coefficients of the power series of \(G_{\rho_0}\), normalized by \(a_0 = 1\). The kernel of \(M^{(\rho_0)}\) will be spanned by the coefficients of the power series \(G_{\rho_1}\) and necessarily \(a_0 = \cdots = a_{\nu-1} = 0\); conventionally we choose \(a_{\nu} = 1\).

In this way we generated all the linearly independent power series at \(Q = 3\) (where \(a_3\) is a free variable, \(G_0\) is the particular solution and \(G_3\) spans the kernel of \(M^{(0)}\)) and \(Q = 4\) (where actually \(a_1\) and \(a_4\) are both free variables, \(G_0\) is the particular solution and the kernel of \(M^{(0)}\) is spanned by \(G_1\) and \(G_4\)). At \(Q = 1\) and \(Q = 2\) the linear system associated to Eq. (A.3) can be inconsistent. For percolation this happens when \(\rho_0 = 0, \rho_1 = 3h + 1 = 2\) and for Ising when \(\rho_0 = h = 1/2\) and \(\rho_1 = 3h + 1 = 5/2\); in both cases \(\nu = 2\). When the
linear system is inconsistent the particular solution does not exist and it will be replaced by a Frobenius power series with a logarithmic singularity; \( G_{\rho_1} \) instead continues to span the kernel of \( M^{(\rho_0)} \) and is free of logarithms.

The coefficients of the logarithmic solution are determined as follows (again we refer to \([53]\) for a comprehensive discussion that includes the case of repeated roots in Eq. (A.2)). We introduce a formal power series

\[ G_{\sigma}(\eta) = \eta^\sigma \sum_{k=0}^{\infty} b_k(\sigma) \eta^k, \quad (A.4) \]

and solve the linear system in (A.3) choosing \( b_0(\sigma) = (\sigma - \rho_0) \). In this way all the coefficients \( g_k(\sigma) \) are analytic in the limit \( \sigma \to \rho_0 \). However since \([53]\) \( b_k(\rho_0) = 0 \) for \( k < \nu \), it follows that \( G_{\rho_0} \) determined through (A.4) is \( b_\nu(\rho_0) G_{\rho_1} \) (recall that we chose \( a_\nu = 1 \) for \( G_{\rho_1} \)). To obtain the linear independent solution associated to \( \rho_0 \) we observe that from the analyticity of the \( b_k \)’s the differential operator \( L \) commutes with the derivative with respect to \( \sigma \) and it turns out

\[ L (\partial_\sigma|_{\sigma=\rho_0} G_{\sigma}) = \partial_\sigma|_{\sigma=\rho_0} L(G_{\sigma}) = \partial_\sigma|_{\sigma=\rho_0}(\eta^{\sigma-1} f_0(\sigma)(\sigma - \rho_0)) = 0. \quad (A.5) \]

The linear independent solution associated to the root \( \rho_0 \) is then

\[ \tilde{G}_{\rho_0}(\eta) = \eta^{\rho_0} \left( \log(\eta) \sum_{k=\nu}^{\infty} b_k(\rho_0) \eta^k + \sum_{k=0}^{\infty} \left. \frac{db_k}{d\sigma} \right|_{\sigma=\rho_0} \eta^k \right) \]

\[ \equiv \beta \log(\eta) G_{\rho_1}(\eta) + \sum_{k=0}^{\infty} c_k \eta^{k+\rho_0}, \quad (A.6) \]

where we defined \( \beta \equiv b_\nu(\rho_0) \) and \( c_k = \left. \frac{db_k}{d\sigma} \right|_{\sigma=\rho_0} \). It should be noticed that multiplying \( b_0(\sigma) \) by any analytic function \( F(\sigma) \) that is \( O(1) \) at \( \sigma = \rho_0 \), the above procedure produces an equally valid solution of Eq. (3.8) that corresponds to a linear combination of Eq. (A.6) and \( G_{\rho_1} \); in particular \( c_2 \) can be arbitrarily redefined. To efficiently generate the power series (A.6) we can first determine \( G_{\rho_1} \) then fix \( c_0 = 1 \) (that in turns fixes \( \beta \)) and choose \( c_2 = \left. \frac{db_2}{d\sigma} \right|_{\sigma=\rho_0} \) with the choice \( F(\sigma) = 1 \). Finally we set up a recursion for the \( c_k \) with \( k > 2 \) requiring Eq. (A.6) to be a solution of Eq. (3.8).

B Recursive formula for the Virasoro conformal blocks

The Virasoro conformal blocks \( F^\nu_\rho(\eta) \) can be obtained directly using Al. Zamolodchikov recursive formula \([42]\). The formula is conveniently written in terms of the elliptic nome \( q = e^{i\tau} \), where the the modulus \( \tau \) is related to the anharmonic ratio by

\[ \tau = i \frac{K(1 - \eta)}{K(\eta)}, \quad (B.1) \]
where $K$ as in Eq. (4.9) is the complete elliptic integral of the first kind. The Virasoro conformal blocks for an internal field with dimension $\rho$ and external legs with dimensions $\{h_i\}$ ($i = 1, \ldots, 4$) can be explicitly calculated as

$$F(\eta, c, \rho, \{h_i\}) = (16q)^{\rho-\frac{c}{24}} \eta^{\frac{c}{24}} (1-\eta)^{-\frac{c}{24} - h_2 - h_3} \vartheta_3(q)^{-\frac{c}{24} - 4 \sum h_i} H(q, c, \rho, \{h_i\}),$$

(B.2)

and $\vartheta_3$ is a Jacobi theta function. The function $H$ in (B.2) satisfies the recursion

$$H(q, c, \rho, \{h_i\}) = 1 + \sum_{r,s} (16q)^{r+s} R_{r,s}(c, \{h_i\}) H(c, h_{r,s} + rs, \{h_i\}, q).$$

(B.3)

Explicit expressions for $R_{r,s}$ and $h_{r,s}(c)$ (cft Eq. (3.1) and Eq. (3.6)) can be found in [42] and we will not repeat them here. Since one is interested in generating a series expansion in $q$ (and ultimately in $\eta$) of Eq. (B.3) up to order $N$, the level of recursion is fixed by $rs = N$. Notice that in principle when the internal field is degenerate, i.e. $\rho = h_{r,s}$, the conformal block might be singular. For all the non-logarithmic cases examined in this paper, this does not happen as the corresponding factor $R_{r,s}$ in the numerator of Eq. (B.3) vanishes as well. Since $F^\rho(\eta) = F(\eta, c, \rho, \{h, h, h, h\})$, we used Eqs. (B.2) and (B.3) to verify (to order $N = 10$ in the recursion) all the identities quoted in Sec. 4.

C Solutions at $Q = 1$ in terms of hypergeometric functions $3F_2$

The ODE that is associated to the percolation problem is obtained replacing $h = 1/3$ in Eq. (3.8) and reads

$$4(2\eta - 1)G(\eta) + (-6 + 8\eta - 8\eta^2)G(\eta) + 3(\eta - 1)\eta(2(2\eta - 1)G''(\eta) + 3(\eta - 1)\eta G''(\eta)) = 0.$$  (C.1)

Explicit form for functions entering the ratio $R_{Q=1}$ — Remarkably, one can formally find two linearly independent hypergeometric solutions [54] (one long and the other short)

$$F_L(\eta) = (\eta(1-\eta))^{4/9} 3F_2 \left( \frac{2}{9}, \frac{1}{18}; \frac{7}{9}; \frac{1}{3}; \frac{4}{27} (1-\eta)^2 \eta^2 \right),$$

(C.2)

$$F_S(\eta) = (1-\eta)^2 \eta^2 3F_2 \left( \frac{4}{3}, \frac{3}{2}; \frac{7}{3}; \frac{8}{3}; 4\eta(1-\eta) \right).$$

(C.3)

The real and imaginary part of $F_L$, denoted by $F_L^{(R)}$ and $F_L^{(I)}$ respectively, and $F_S$, which is real, constitute an independent basis of the solutions of (C.1) in the range $0 \leq \eta \leq 1/2$. Although $F_S(\eta)$ relies upon evaluation of a $3F_2$ hypergeometric function with argument in the range $[1, \infty)$ it proves stable for numerical evaluation (in its Mathematica implementation) so it will be preferred to explicit series expression of $G_0(\eta)$ and $G_2(\eta)$ (and $G_{1/3}(\eta)$ too) that will be also given. In order to retrieve the functions $G_0(\eta)$, and $G_2(\eta)$ in Sec. 4 from the above $F_L(\eta)$, $F_S(\eta)$ within the range $0 \leq \eta \leq 1$ we have to proceed in the following way: we use linear combinations of $F_L^{(R)}(\eta)$, $F_L^{(I)}(\eta)$, and $F_S(\eta)$ for $0 \leq \eta \leq 1/2$ and of $F_L^{(R)}(1-\eta)$, $F_L^{(I)}(1-\eta)$, and $F_S(1-\eta)$ for $1/2 \leq \eta \leq 1$ and impose the suitable matching
condition at $\eta = 1/2$ and normalization. Define the constants

$$\alpha_0 = -\frac{3^{7/6} \Gamma \left( -\frac{1}{18} \right) \Gamma \left( \frac{7}{9} \right) \Gamma \left( \frac{7}{6} \right)}{2^{13/9} \pi \Gamma \left( \frac{1}{6} \right)}$$  \hspace{1cm} (C.4)$$

$$\beta_0 = \frac{1}{45} \left( 35 - 12 \log(3) + 2\pi \left( -\sqrt{3} + \cot \left( \frac{\pi}{9} \right) \right) + \cot \left( \frac{2\pi}{9} \right) \right)$$  \hspace{1cm} (C.5)$$

$$\alpha_2 = -\frac{\Gamma \left( -\frac{2}{9} \right) \Gamma \left( -\frac{1}{18} \right) \Gamma \left( \frac{7}{9} \right) \Gamma \left( \frac{8}{9} \right) \pi \Gamma \left( -\frac{2}{9} \right) \Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{11}{6} \right)}{2^{1/9} \sqrt{3} \pi \Gamma \left( \frac{1}{3} \right) \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{4}{3} \right) \Gamma \left( \frac{7}{6} \right) \Gamma \left( \frac{11}{6} \right)}.$$  \hspace{1cm} (C.6)$$

The function $\tilde{G}_0(\eta)$ is obtained by imposing the function to be continuous, having vanishing first derivatives and continuous second derivative in $\eta = 1/2$ and setting the function to be one for $\eta = 0$. This yields the following expression

$$\tilde{G}_0(\eta) = \alpha_0 \left[ \sqrt{3} \sin \left( \frac{2\pi}{9} \right) - \cos \left( \frac{2\pi}{9} \right) \right] F_L^{(R)}(\zeta) +$$

$$+ \alpha_0 \left[ -\sin \left( \frac{2\pi}{9} \right) - \sqrt{3} \cos \left( \frac{2\pi}{9} \right) \right] F_L^{(I)}(\zeta) + \beta_0 F_S(\zeta)$$  \hspace{1cm} (C.7)$$

where $\zeta = \min(\eta, 1-\eta)$.

In order to reproduce $G_2(\eta)$ we have to impose the function to be equal to $F_S(\eta)$ for $0 \leq \eta \leq 1/2$ and impose continuity of the function and the first two derivatives at $\eta = 1/2$. The outcome is

$$G_2(\eta) = \begin{cases} 
F_S(\eta) & \text{if } 0 \leq \eta \leq 1/2, \\
\alpha_2 F_L^{(I)}(1-\eta) + F_S(1-\eta) & \text{if } 1/2 \leq \eta \leq 1.
\end{cases}$$  \hspace{1cm} (C.8)$$

The power series expansions of Eqs. (C.7-C.8) coincide with Eq. (4.3) and Eq. (4.2) respectively. Moreover since we can also calculate explicitly the value of this function in $\eta = 1$ we can fix the normalization constant $A_1$ for the ratio $R_{Q=1}$ given in (4.5)

$$A_1 = \frac{3^{7/6} \pi \Gamma \left( \frac{5}{9} \right) \Gamma \left( \frac{5}{6} \right) \Gamma \left( \frac{7}{6} \right)}{4 \cos(13\pi/18) \Gamma \left( -\frac{2}{9} \right) \Gamma \left( \frac{1}{6} \right) \Gamma \left( \frac{11}{6} \right)}.$$  \hspace{1cm} (C.9)$$

To have an idea of the power of the derived expression we compare the value of this expression when $\eta = 1/2$ with the series expansion and the numerics. The truncated series expansion (with 10^5 term) provides the value $R_{Q=1}^{\text{(series)}}(1/2) = 0.119993 \ldots$ while the exact expression gives $R_{Q=1}(1/2) = 0.117680185 \ldots$. If we extrapolate to the thermodynamic limit the numerically obtained values for $\eta = 1/2$ by fitting them with a power law function we get 0.11766(4) fully consistent with our conjectured exact expression.

Explicit series expression around $\eta = 0$—We provide also explicit series expressions derived by working out the functions $F_L$ and $F_S$ where the argument of the $3F_2$ functions is...
making them more useful for studying the $\eta \approx 0$ region:

$$
\tilde{G}_0(\eta) = (1 - \eta + \eta^2)^{2/3} g_0 \left( \frac{27}{4} \left( 1 - \eta \right)^2 \eta^2 \right) + 
\frac{1}{45} \left( 35 - 4i\pi + 4\pi \csc \left( \frac{\pi}{9} \right) \right) G_2(\eta) \quad (C.10)
$$

$$
G_{1/3}(\eta) = ((1 - \eta)\eta)^{1/3} (1 - \eta + \eta^2)^{1/6} 3F_2 \left( \frac{1}{18}, \frac{5}{18}, \frac{11}{18}; \frac{1}{6}, \frac{7}{6}; \frac{27}{4} \left( 1 - \eta \right)^2 \eta^2 \right) \quad (C.11)
$$

$$
G_2(\eta) = ((1 - \eta)\eta)^2 3F_2 \left( \frac{4}{3}, \frac{3}{2}, \frac{7}{3}, \frac{8}{3}; 3; 4\eta(1 - \eta) \right) = ((1 - \eta)\eta)^2 (1 - \eta + \eta^2)^{-7/3} 3F_2 \left( \frac{7}{9}, \frac{10}{9}, \frac{13}{9}; \frac{11}{6}, 2; \frac{27}{4} \left( 1 - \eta \right)^2 \eta^2 \right) \quad (C.12)
$$

where we used the additional function

$$
g_0(\zeta) = 1 + \frac{\Gamma \left( \frac{5}{9} \right) \Gamma \left( \frac{8}{9} \right)}{\Gamma \left( -\frac{2}{9} \right) \Gamma \left( \frac{1}{9} \right)} \sum_{k=1}^{\infty} (-\zeta)^k \frac{\Gamma \left( \frac{1}{9} - k \right) \Gamma \left( -\frac{2}{9} + k \right)}{(k-1)!k!} \Gamma \left( \frac{8}{9} - k \right) 
\times \left( \log(\zeta) - i\pi + \psi_0 \left( \frac{5}{9} - k \right) + \psi_0 \left( \frac{8}{9} - k \right) + \psi_0 \left( \frac{2}{9} + k \right) - \psi_0(k) - \psi_0(1+k) - \psi_0 \left( \frac{1}{6} - k \right) \right)
$$

and $\psi_0$ is the digamma function. The presence of digamma functions comes not as a surprise and is analogous to the ones encountered when dealing with the logarithmic companion solution to $2F_1(a, b; c; z)$ of the simple hypergeometric differential equation when $c = 1, 2, 3 \ldots$ and $a, b \neq 1, 2, \ldots, n - 1$. A full understanding of the relation between the various functions presented here would entail a better knowledge of the connection formulas for $3F_2$. Unfortunately this theory is not as developed as the one for the $2F_1$.

All of these functions indeed reproduce exactly the series expansions given in the main text in Eq. (4.3), Eq. (4.1), and Eq. (4.2) but as already noted the series expression for $\tilde{G}_0$ is not efficient an the form given in Equation (C.7) should be preferred.

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