The \emph{k}-metric dimension of a graph

Alejandro Estrada-Moreno\textsuperscript{(1)}, Juan A. Rodríguez-Velázquez\textsuperscript{(1)}, and Ismael G. Yero\textsuperscript{(2)}

\textsuperscript{(1)}Departament d’Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Av. Països Catalans 26, 43007 Tarragona, Spain. alejandro.estrada@urv.cat, juanalberto.rodriguez@urv.cat
\textsuperscript{(2)}Departamento de Matemáticas, Escuela Politécnica Superior de Algeciras Universidad de Cádiz, Av. Ramón Puyol s/n, 11202 Algeciras, Spain. ismael.gonzalez@uca.es

June 2, 2013

Abstract

As a generalization of the concept of metric basis, this article introduces the notion of \emph{k}-metric basis in graphs. Given a connected graph \( G = (V, E) \), a set \( S \subseteq V \) is said to be a \emph{k}-metric generator for \( G \) if the elements of any pair of vertices of \( G \) are distinguished by at least \( k \) elements of \( S \), \ie, for any two different vertices \( u, v \in V \), there exist at least \( k \) vertices \( w_1, w_2, \ldots, w_k \in S \) such that \( d_G(u, w_i) \neq d_G(v, w_i) \) for every \( i \in \{1, \ldots, k\} \). A metric generator of minimum cardinality is called a \emph{k}-metric basis and its cardinality the \emph{k}-metric dimension of \( G \). A connected graph \( G \) is \emph{k}-metric dimensional if \( k \) is the largest integer such that there exists a \emph{k}-metric basis for \( G \). We give a necessary and sufficient condition for a graph to be \emph{k}-metric dimensional and we obtain several results on the \emph{k}-metric dimension.

\textbf{Keywords:} \emph{k}-metric generator; \emph{k}-metric dimension; \emph{k}-metric dimensional graph; metric dimension; resolving set; locating set; metric basis.

\textbf{AMS Subject Classification Numbers:} 05C05; 05C12; 05C90.

1 Introduction

The problem of uniquely determining the location of an intruder in a network was the principal motivation of introducing the concept of metric dimension in graphs by Slater in \cite{19, 20}, where the metric generators were called \emph{locating sets}. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in \cite{9}, where metric generators were called \emph{resolving sets}.
Nevertheless, the concept of metric generator, in its primary version, has a weakness related with the possible uniqueness of the vertex identifying a pair of vertices of the graph. Consider, for instance, some robots which are navigating, moving from node to node of a network. On a graph, however, there is neither the concept of direction nor that of visibility. We assume that robots have communication with a set of landmarks \( S \) (a subset of nodes) which provide them the distance to the landmarks in order to facilitate the navigation. In this sense, one aim is that each robot is uniquely determined by the landmarks. Suppose that in a specific moment there are two robots \( x, y \) whose positions are only distinguished by one landmark \( s \in S \). If the communication between \( x \) and \( s \) is “unexpectedly blocked”, then the robot \( x \) will get “lost” in the sense that it can assume that it has the position of \( y \). So, for more realistic settings it could be desirable to consider a set of landmarks where each pair of nodes is distinguished by at least two landmarks.

A natural solution regarding that weakness is the location of one landmark in every node of the graph. But, such a solution, would have a very high cost. Thus, the choice of a correct set of landmarks is convenient for a satisfiable performance of the navigation system. That is, in order to achieve a reasonably efficiency, it would be convenient to have a set of landmarks, as “small” as possible, always having the guaranty that every object of the network will be properly distinguished.

From now on we consider a simple and connected graph \( G = (V, E) \). It is said that a vertex \( v \in V \) distinguish two different vertices \( x, y \in V \), if \( d_G(v, x) \neq d_G(v, y) \), where \( d_G(a, b) \) represents the length of a shortest \( a-b \) path. A set \( S \subseteq V \) is a metric generator for \( G \) if any pair of vertices of \( G \) is distinguished by some element of \( S \). Such a name for \( S \) raises from the concept of generator of metric spaces, that is, a set \( S \) of points in the space with the property that every point of the space is uniquely determined by its “distances” from the elements of \( S \). For our specific case, in a simple and connected graph \( G = (V, E) \), we consider the metric \( d_G : V \times V \rightarrow \mathbb{N} \), where \( d_G(x, y) \) is defined as mentioned above. With this metric, \( (V, d_G) \) is clearly a metric space. A metric generator of minimum cardinality is called a metric basis, and its cardinality the metric dimension of \( G \), denoted by \( \dim(G) \).

Other useful terminology to define the concept of metric generator in graphs is given at next. Given an ordered set \( S = \{s_1, s_2, \ldots, s_d\} \subset V(G) \), we refer to the \( d \)-vector (ordered \( d \)-tuple) \( r(u|S) = (d_G(u, s_1), d_G(u, s_2), \ldots, d_G(u, s_d)) \) as the metric representation of \( u \) with respect to \( S \). In this sense, \( S \) is a metric generator for \( G \) if and only if for every pair of different vertices \( u, v \) of \( G \), it follows \( r(u|S) \neq r(v|S) \).

In order to avoid the weakness of metric basis described above, from now on we consider an extension of the concept of metric generators in the following way. Given a simple and connected graph \( G = (V, E) \), a set \( S \subseteq V \) is said to be a \( k \)-metric generator for \( G \) if and only if any pair of vertices of \( G \) is distinguished by at least \( k \) elements of \( S \), i.e., for any pair of different vertices \( u, v \in V \), there exist at least \( k \) vertices \( w_1, w_2, \ldots, w_k \in S \) such that
\[
d_G(u, w_i) \neq d_G(v, w_i), \text{ for every } i \in \{1, \ldots, k\}. \tag{1}
\]
A \( k \)-metric generator of minimum cardinality in \( G \) will be called a \( k \)-metric basis and its cardinality the \( k \)-metric dimension of \( G \), which will be denoted by \( \dim_k(G) \).

As an example we take the cycle graph \( C_4 \) with vertex set \( V = \{x_1, x_2, x_3, x_4\} \) and edge set \( E = \{x_i x_j : j - i \equiv 1(2)\} \). We claim that \( \dim_2(C_4) = 4 \). That is, if we take the pair of
vertices \( x_1, x_3 \), then they are distinguished only by themselves. So, \( x_1, x_3 \) must belong to every 2-metric generator for \( C_4 \). Analogously, \( x_2, x_4 \) also must belong to every 2-metric generator for \( C_4 \). Other example is the graph \( G \) of Figure 1, for which \( \dim_2(G) = 4 \). To see this, note that \( v_3 \) does not distinguish any pair of vertices of \( V(G) − \{v_3\} \) and for each pair \( v_i, v_3 \), \( 1 \leq i \leq 5, i \neq 3 \), there exist two elements of \( V(G) − \{v_3\} \) that distinguish them. Hence, \( v_3 \) does not belong to any 2-metric basis for \( G \). To conclude that \( V(G) − \{v_3\} \) must be a 2-metric basis for \( G \) we proceed as in the case of \( C_4 \).

Note that every \( k \)-metric generator \( S \) satisfies that \( |S| \geq k \) and, if \( k > 1 \), then \( S \) is also a \((k−1)\)-metric generator. Moreover, 1-metric generators are the standard metric generators (resolving sets or locating sets as defined in [9] or [19], respectively). Notice that if \( k = 1 \), then the problem of checking if a set \( S \) is a metric generator reduces to check condition (1) only for those vertices \( u, v \in V - S \), as every vertex in \( S \) is distinguished at least by itself. Also, if \( k = 2 \), then condition (1) must be checked only for those pairs having at most one vertex in \( S \), since two vertices of \( S \) are distinguished at least by themselves. Nevertheless, if \( k \geq 3 \), then condition (1) must be checked for every pair of different vertices of the graph.

The literature about metric dimension in graphs shows several of its usefulness, for instance, applications to the navigation of robots in networks are discussed in [13] and applications to chemistry in [11, 12], among other ones. This invariant was studied further in a number of other papers including for example, [3, 4, 5, 7, 8, 10, 16, 17, 21, 22, 23]. Several variations of metric generators including resolving dominating sets [2], independent resolving sets [6], local metric sets [16], and strong resolving sets [14, 15, 18], etc. have been introduced and studied. For more information can be consulted the survey article [1] where is approached several metric dimension related topics. It is therefore our goal to introduce this extension of metric generators in graphs as a possible future tool for other possible more general variations of the applications described above.

We introduce now some other more necessary terminology for the article and the rest of necessary concepts will be introduced the first time they are mentioned in the work. If two vertices \( u, v \) are adjacent in \( G = (V, E) \), then we write \( u \sim v \) or we say that \( uv \in E(G) \). Given \( x \in V(G) \) we define \( N_G(x) \) to be the open neighborhood of \( x \) in \( G \). That is, \( N_G(x) = \{y \in V(G) : x \sim y\} \). The closed neighborhood, denoted by \( N_G[x] \), equals \( N_G(x) \cup \{x\} \). If there is no ambiguity, we will simply write \( N(x) \) or \( N[x] \). We also refer to the degree of \( v \) as \( \delta(v) = |N(v)| \). The minimum and maximum degrees of \( G \) are denoted by \( \delta \) and \( \Delta \), respectively. For a non-empty set \( S \subseteq V(G) \), and a vertex \( v \in V(G) \), \( N_S(v) \) denotes the set of neighbors that \( v \) has in \( S \), i.e., \( N_S(v) = S \cap N(v) \).
2 \textit{k}-metric dimensional graphs

It is clear that not for every integer \( k \) is possible to find a \( k \)-metric generator in a connected graph \( G \). That is, given a connected graph \( G \), there exists an integer \( t \) such that \( G \) does not contain any \( k \)-metric generator for every \( k > t \). According to that fact, a connected graph \( G \) is said to be a \textit{k}-metric dimensional graph, if \( k \) is the largest integer such that there exists a \( k \)-metric basis for \( G \). Notice that, if \( G \) is a \( k \)-metric dimensional graph, then for every positive integer \( k' \leq k \), \( G \) has at least a \( k' \)-metric basis. Since for every pair of vertices \( x, y \) of a graph \( G \) we have that they are distinguished at least by themselves, it follows that the whole vertex set \( V(G) \) is a \( 2 \)-metric generator for \( G \) and, as a consequence it follows that every graph \( G \) is \( k \)-metric dimensional for some \( k \geq 2 \). On the other hand, for any connected graph \( G \) of order \( n > 2 \) there exists at least one vertex \( v \in V(G) \) such that \( \delta(v) \geq 2 \). Since \( v \) does not distinguish any pair \( x, y \in N_G(v) \), there is no \( n \)-metric dimensional graph of order \( n > 2 \).

\textbf{Remark 1.} Let \( G \) be a \( k \)-metric dimensional graph of order \( n \). If \( n \geq 3 \), then \( 2 \leq k \leq n - 1 \). Moreover, \( G \) is \( n \)-metric dimensional if and only if \( G \cong K_2 \).

Next we give a characterization of \( k \)-metric dimensional graphs. To do so, we need some additional terminology. Given two vertices \( x, y \in V(G) \), we say that the set of \textit{distinctive vertices} of \( x, y \) is

\[ \mathcal{D}_G(x, y) = \{ z \in V(G) : d_G(x, z) \neq d_G(y, z) \} \]

and the set of \textit{non-trivial distinctive vertices} of \( x, y \) is

\[ \mathcal{D}_G^*(x, y) = \mathcal{D}_G(x, y) - \{ x, y \}. \]

\textbf{Theorem 2.} A connected graph \( G \) is \( k \)-metric dimensional if and only if \( k = \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)| \).

\textit{Proof.} (Necessity) If \( G \) is a \( k \)-metric dimensional graph, then for any \( k \)-metric basis \( B \) and any pair of vertices \( x, y \in V(G) \), we have \( |B \cap \mathcal{D}_G(x, y)| \geq k \). Thus, \( k \leq \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)| \).

Now we suppose that \( k < \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)| \). In such a case, for every \( x', y' \in V(G) \) such that \( |B \cap \mathcal{D}_G(x', y')| = k \), there exists a distinctive vertex \( z_{x'y'} \) of \( x', y' \) with \( z_{x'y'} \in \mathcal{D}_G(x', y') - B \). Hence, the set

\[ B \cup \left( \bigcup_{x', y' \in V(G) : |B \cap \mathcal{D}_G(x', y')| = k} \{ z_{x'y'} \} \right) \]

is a \((k + 1)\)-metric generator for \( G \), which is a contradiction. Therefore, \( k = \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)| \).

(Sufficiency) Let \( a, b \in V(G) \) such that \( \min_{x,y \in V(G)} |\mathcal{D}_G(x, y)| = |\mathcal{D}_G(a, b)| = k \). Since the set

\[ \bigcup_{x,y \in V(G)} \mathcal{D}_G(x, y) \]

is a \( k \)-metric generator for \( G \) and the pair \( a, b \) is not distinguished by \( k' > k \) vertices of \( G \), we conclude that \( G \) is a \( k \)-metric dimensional graph. \( \square \)
2.1 On some families of $k$-metric dimensional graphs for some specific values of $k$

The characterization proved in Theorem 2 gives a result on general graphs. Thus, next we particularize this for some specific classes of graphs or we bound its possible value in terms of other parameters of the graph. To this end, we need the following concepts. Two vertices $x, y$ are called false twins if $N(x) = N(y)$ and $x, y$ are called true twins if $N[x] = N[y]$. Two vertices $x, y$ are twins if they are false twin vertices or true twin vertices. A vertex $x$ is said to be a twin if there exists a vertex $y \in V(G)$ such that $x$ and $y$ are twin vertices in $G$. Notice that two vertices $x, y$ are twins if and only if $D^*_G(x, y) = \emptyset$.

**Corollary 3.** A connected graph $G$ of order $n \geq 2$ is 2-metric dimensional if and only if $G$ has twin vertices.

It is clear that $P_2$ and $P_3$ are 2-metric dimensional. Now, a specific characterization for 2-dimensional trees is obtained from Theorem 2 (or from Corollary 3). A leaf in a tree is a vertex of degree one, while a support vertex is a vertex adjacent to a leaf.

**Corollary 4.** A tree $T$ of order $n \geq 4$ is 2-metric dimensional if and only if $T$ contains a support vertex which is adjacent to at least two leaves.

An example of a 2-metric dimensional tree is the star graph $K_{1,n-1}$, whose 2-metric dimension is $\text{dim}_2(K_{1,n-1}) = n - 1$ (see Corollary 28). On the other side, an example of a tree $T$ which is not 2-metric dimensional is drawn in Figure 2. Notice that $S = \{v_1, v_3, v_5, v_6, v_7\}$ is a 3-metric basis of $T$.

![Figure 2: $S = \{v_1, v_3, v_5, v_6, v_7\}$ is a 3-metric basis of $T$.](image)

A cut vertex in a graph is a vertex whose removal increases the number of components of the graph and an extreme vertex is a vertex $v$ such that the subgraph induced by $N[v]$ is isomorphic to a complete graph. Also, a block is a maximal biconnected subgraph of the graph. Now, let $\mathcal{F}$ be the family of sequences of connected graphs $G_1, G_2, ..., G_k$, $k \geq 2$, such that $G_1$ is a complete graph $K_{n_1}$, $n_1 \geq 2$, and $G_i$, $i \geq 2$, is obtained recursively from $G_{i-1}$ by adding a complete graph $K_{n_i}$, $n_i \geq 2$, and identifying one vertex of $G_{i-1}$ with one vertex of $K_{n_i}$.

From this point we will say that a connected graph $G$ is a generalized tree\(^2\) if and only if there exists a sequence $\{G_1, G_2, ..., G_t\} \in \mathcal{F}$ such that $G_t = G$ for some $t \geq 2$. Notice that in these generalized trees every vertex is either, a cut vertex or an extreme vertex. Also, every complete graph used to obtain the generalized tree is a block of the graph. Note that if every $K_{n_i}$ is isomorphic to $K_2$, then $G_t$ is a tree, justifying the terminology used. With these concepts we give the following consequence of Theorem 2, which is a generalization of Corollary 4.

\(^1\)A biconnected graph is a connected graph such that if any vertex of graph were to be removed, the graph will remain connected.

\(^2\)In some works those graphs are called block graphs.
Corollary 5. A generalized tree $G$ is 2-metric dimensional if and only if $G$ contains at least two extreme vertices being adjacent to a common cut vertex.

The Cartesian product graph $G \square H$, of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, is the graph whose vertex set is $V(G \square H) = V_1 \times V_2$ and any two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \square H$ if and only if either:

(a) $x_1 = y_1$ and $x_2 \sim y_2$, or
(b) $x_1 \sim y_1$ and $x_2 = y_2$.

Proposition 6. Let $G$ and $H$ be two connected graphs of order greater than two. If $G \square H$ is $k$-metric dimensional, then $k \geq 3$.

Proof. Notice that for any vertex $(a, b) \in V(G \square H)$, $N_{G \square H}(a, b) = (N_G(a) \times \{b\}) \cup (\{a\} \times N_H(b))$. Now, for any two distinct vertices $(a, b), (c, d) \in V(G \square H)$ at least $a \neq c$ or $b \neq d$ and since $H$ is a connected graph of order greater than two, we have that at least $N_H(b) \neq \{d\}$ or $N_H(d) \neq \{b\}$. Thus, we obtain that $N_{G \square H}(a, b) \neq N_{G \square H}(c, d)$. Therefore, $G \square H$ does not contain any twin vertex and, by Remark 1 and Corollary 3, if $G \square H$ is $k$-metric dimensional, then $k \geq 3$. \hfill \Box

Proposition 7. Let $C_n$ be a cycle graph of order $n$. If $n$ is odd, then $C_n$ is $(n-1)$-metric dimensional and if $n$ is even, then $C_n$ is $(n-2)$-metric dimensional.

Proof. We consider two cases:

(1) $n$ is odd. For any pair of vertices $u, v \in V(C_n)$ there exist only one vertex $w \in V(C_n)$ such that $w$ does not distinguish $u$ and $v$. Therefore, by Theorem 2, $C_n$ is $(n-1)$-metric dimensional.

(2) $n$ is even. In this case, $C_n$ is 2-antipodal. For any pair of vertices $u, v \in V(C_n)$, such that $d(u, v) = 2l$, we can take a vertex $x$ such that $d(u, x) = d(v, x) = l$. So, $\mathcal{D}_G(u, v) = V(C_n) - \{x, y\}$, where $y$ is antipodal to $x$. On the other hand, if $d(u, v)$ is odd, then $\mathcal{D}_G(u, v) = V(C_n)$. Therefore, by Theorem 2, the graph $C_n$ is $(n-2)$-metric dimensional. \hfill \Box

Now, according to Remark 1 we have that every graph of order $n$, different from $K_2$, is $k$-metric dimensional for some $k \leq n-1$. Next we characterize those graphs being $(n-1)$-metric dimensional.

Theorem 8. A graph $G$ of order $n \geq 3$ is $(n-1)$-metric dimensional if and only if $G$ is a path or $G$ is an odd cycle.

\footnote{The diameter of $G = (V, E)$ is defined as $D(G) = \max_{u,v \in V(G)}\{d_G(u, v)\}$. We say that $u$ and $v$ are antipodal vertices or mutually antipodal if $d_G(u, v) = D(G)$. We recall that $G = (V, E)$ is 2-antipodal if for each vertex $x \in V$ there exists exactly one vertex $y \in V$ such that $d_G(x, y) = D(G)$.}
Proof. Since \( n \geq 3 \), by Remark 1, \( G \) is \( k \)-metric dimensional for some \( k \in \{2, \ldots, n - 1\} \). By Proposition 26, if \( G = P_n \), then \( G \) is \((n - 1)\)-metric dimensional. By Proposition 7, we have that if \( G \) is an odd cycle, then \( G \) is \((n - 1)\)-metric dimensional.

On the contrary, let \( G \) be a \((n - 1)\)-metric dimensional graph. Hence, for every pair of vertices \( x, y \in V(G) \) there exists at most one vertex which does not distinguish \( x, y \). Suppose \( \Delta(G) > 2 \) and let \( v \in V(G) \) such that \( \{u_1, u_2, u_3\} \subset N(v) \). Figure 3 shows all the possibilities for the links between these four vertices. Figures 3a, 3b and 3d show that \( v, u_1 \) do not distinguish \( u_2, u_3 \). Figure 3c shows that \( u_1, u_2 \) do not distinguish \( v, u_3 \). Thus, from the cases above we deduce that there is a pair of vertices which is not distinguished by at least two other different vertices. Thus \( G \) is not a \((n - 1)\)-metric dimensional graph, which is a contradiction. As a consequence, \( \Delta(G) \leq 2 \) and we have that \( G \) is either a path or a cycle graph. Finally, by Proposition 7, we have that if \( G \) is a cycle, then \( G \) has odd order.

2.2 Bounding the value \( k \) for which a graph is \( k \)-metric dimensional

In order to continue presenting our results, we need to introduce some definitions. A vertex of degree at least three in a graph \( G \) will be called a major vertex of \( G \). Any end-vertex (a vertex of degree one) \( u \) of \( G \) is said to be a terminal vertex of a major vertex \( v \) of \( G \) if \( d_G(u, v) < d_G(u, w) \) for every other major vertex \( w \) of \( G \). The terminal degree \( \text{ter}(v) \) of a major vertex \( v \) is the number of terminal vertices of \( v \). A major vertex \( v \) of \( G \) is an exterior major vertex of \( G \) if it has positive terminal degree. Let \( \mathcal{M}(G) \) be the set of exterior major vertices of \( G \) having terminal degree greater than one.  

Given \( w_i \in \mathcal{M}(G) \) and a terminal vertex \( u_{ij} \) of \( w_i \), we denote by \( P(u_{ij}, w_i) \) the shortest path that starts at \( u_{ij} \) and ends at \( w_i \). Let \( l(u_{ij}, w_i) \) be the length of \( P(u_{ij}, w_i) \). Now, given \( w_i \in \mathcal{M}(G) \) and two terminal vertices \( u_{ij}, u_{ir} \) of \( w_i \) we denote by \( P(u_{ij}, w_i, u_{ir}) = u_{ij} - w_i - u_{ir} \)

\[ P(u_{ij}, w_i, u_{ir}) = u_{ij} - w_i - u_{ir} \]
the shortest path from \( u_{ij} \) to \( u_{ir} \) containing \( w_i \) and by \( \varsigma(u_{ij}, u_{ir}) \) the length of \( P(u_{ij}, w_i, u_{ir}) \). Finally, given \( w_i \in \mathcal{M}(G) \) and the set of terminal vertices \( U_i = \{u_{i1}, u_{i2}, \ldots, u_{ik}\} \) of \( w_i \), for \( j \neq r \) we define \( \varsigma(w_i) = \min_{u_{ij}, u_{ir} \in U_i} \{\varsigma(u_{ij}, u_{ir})\} \) and \( l(w_i) = \min_{u_{ij} \in U_i} \{l(u_{ij}, w_i)\} \). From the local parameters above we define the following global parameter

\[
\varsigma(G) = \min_{w_i \in \mathcal{M}(G)} \{\varsigma(w_i)\}.
\]

An example which helps to understand the notation above is given in Figure 5. According to this notation we present the following result.

**Theorem 9.** Let \( G \) be a connected graph such that \( \mathcal{M}(G) \neq \emptyset \). If \( G \) is \( k \)-metric dimensional, then \( k \leq \varsigma(G) \).

**Proof.** We claim that there exist at least one pair of vertices \( x, y \in V(G) \) such that \( |\mathcal{D}_G(x, y)| = \varsigma(G) \). To see this, let \( w_1 \in \mathcal{M}(G) \) and let \( u_{11}, u_{12} \) be two terminal vertices of \( w_1 \) such that \( \varsigma(G) = \varsigma(w_1) = \varsigma(u_{11}, u_{12}) \). Let \( u_{11}' \) and \( u_{12}' \) be the vertices adjacent to \( w_1 \) in the shortest paths \( P(u_{11}, w_1) \) and \( P(u_{12}, w_1) \), respectively. Notice that it could happen \( u_{11}' = u_{11} \) or \( u_{12}' = u_{12} \). Since every vertex \( v \not\in V(P(u_{11}, w_1, u_{12})) - \{w_1\} \) satisfies that \( d_G(u_{11}', v) = d_G(u_{12}', v) \), and the only distinctive vertices of \( u_{11}', u_{12}' \) are those ones belonging to \( P(u_{11}', u_{11}) \) and \( P(u_{12}', u_{12}) \), we have that \( |\mathcal{D}_G(u_{11}', u_{12}')| = \varsigma(G) \). Therefore, by Theorem 2, if \( G \) is \( k \)-metric dimensional, then \( k \leq \varsigma(G) \). \( \square \)

The upper bound of Theorem 9 is tight. For instance, it is achieved for every tree different from a path as it is proved further in Section 4, where is studied the \( k \)-metric dimension of tree graphs.

A *clique* in a graph \( G \) is a set of vertices \( S \) such that the subgraph induced by \( S \), denoted by \( \langle S \rangle \), is isomorphic to a complete graph. The maximum cardinality of a clique in a graph \( G \) is the *clique number* and it is denoted by \( \omega(G) \). We will say that \( S \) is an \( \omega(G) \)-clique if \( |S| = \omega(G) \).

**Theorem 10.** Let \( G \) be a graph of order \( n \) different from the complete graph. If \( G \) is \( k \)-metric dimensional, then \( k \leq n - \omega(G) + 1 \).

**Proof.** Let \( S \) be an \( \omega(G) \)-clique. Since \( G \) is not complete, there exists a vertex \( v \not\in S \) such that \( N_S(v) \subset S \). Let \( u \in S \) with \( v \not\sim u \). If \( N_S(v) = S - \{u\} \), then \( d(u, x) = d(v, x) = 1 \) for every \( x \in S - \{u\} \). Thus, \( |\mathcal{D}_G(u, v)| \leq n - \omega(G) + 1 \). On the other hand, if \( N_S(v) \neq S - \{u\} \), then there exists \( u' \in S - \{u\} \) such that \( u' \not\sim v \). Thus, \( d(u, v) = d(u', v) = 2 \) and for every \( x \in S - \{u, u'\} \), \( d(u, x) = d(u', x) = 1 \). So, \( |\mathcal{D}_G(u, u')| \leq n - \omega(G) + 1 \). Therefore, Theorem 2 leads to \( k \leq n - \omega(G) + 1 \). \( \square \)

Examples where the previous bound is achieved are those connected graphs \( G \) of order \( n \) and \( \omega(G) = n - 1 \), so \( n - \omega(G) + 1 = 2 \). Notice that so there exists at least two twin vertices. Hence, by Corollary 3 these graphs are 2-metric dimensional.

The *girth* of a graph \( G \) is the length of a shortest cycle in \( G \).
Theorem 11. Let $G$ be a graph of minimum degree $\delta \geq 2$, maximum degree $\Delta \geq 3$ and girth $g \geq 4$. If $G$ is $k$-metric dimensional, then

$$k \leq n - 1 - (\Delta - 2) \sum_{i=0}^{\left\lceil \frac{g}{2} \right\rceil - 2} (\delta - 1)^i.$$ 

Proof. Let $v \in V$ be a vertex of maximum degree in $G$. Since $\Delta \geq 3$ and $g \geq 4$, there are at least three different vertices adjacent to $v$ and $N(v)$ is an independent set\(^4\). Given $u_1, u_2 \in N(v)$ and $i \in \{0, \ldots, \left\lceil \frac{g}{2} \right\rceil - 2\}$ we define the following sets.

$$A_0 = N(v) - \{u_1, u_2\}.$$  
$$A_1 = \bigcup_{x \in A_0} N(x) - \{v\}.$$  
$$A_2 = \bigcup_{x \in A_1} N(x) - A_0.$$  

$$A_{\left\lceil \frac{g}{2} \right\rceil - 2} = \bigcup_{x \in A_{\left\lceil \frac{g}{2} \right\rceil - 3}} N(x) - A_{\left\lceil \frac{g}{2} \right\rceil - 4}.$$  

Now, let $A = \{v\} \cup \left( \bigcup_{i=0}^{\left\lceil \frac{g}{2} \right\rceil - 2} A_i \right)$. Since $\delta \geq 2$, we have that $|A| \geq 1 + (\Delta - 2) \sum_{i=0}^{\left\lceil \frac{g}{2} \right\rceil - 2} (\delta - 1)^i$. Also, notice that for every vertex $x \in A$, $d(u_1, x) = d(u_2, x)$. Thus, $u_1, u_2$ can be only distinguished by themselves and at most $n - |A| - 2$ other vertices. Therefore, $|D_G(u_1, u_2)| \leq n - |A|$ and the result follows by Theorem 2. \hfill \square

The bound of Theorem 11 is sharp. For instance, it is attained for the graph of Figure 4. Since in this case $n = 8$, $\delta = 2$, $\Delta = 3$ and $g = 5$, we have that $k \leq n - 1 - (\Delta - 2) \sum_{i=0}^{\left\lceil \frac{5}{2} \right\rceil - 2} (\delta - 1)^i = 6$. Table 1 shows every pair of vertices of this graph and their corresponding non-trivial distinctive vertices. Notice that by Theorem 2 the graph is 6-metric dimensional.

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{The graph satisfies the equality in the upper bound of Theorem 11.}
\end{figure}
\end{center}

\(^4\)An independent set or stable set is a set of vertices in a graph, no two of which are adjacent.
and let $G$ be a connected graph of order $n ≥ 2$. Then $\dim_2(G) = 2$ if and only if $G ≅ P_n$. 

| $x, y$ | $D^*_G(x, y)$ |
|-------|----------------|
| $v_1, v_3$ | $\{v_4, v_5, v_6, v_7 \}$ |
| $v_1, v_5$ | $\{v_2, v_3, v_5, v_8 \}$ |
| $v_1, v_6$ | $\{v_1, v_5, v_6, v_7 \}$ |
| $v_1, v_7$ | $\{v_2, v_3, v_5, v_6 \}$ |
| $v_1, v_8$ | $\{v_2, v_3, v_4, v_7 \}$ |
| $v_2, v_5$ | $\{v_1, v_3, v_4, v_8 \}$ |
| $v_2, v_6$ | $\{v_1, v_3, v_5, v_7 \}$ |
| $v_2, v_7$ | $\{v_1, v_3, v_4, v_8 \}$ |
| $v_3, v_4$ | $\{v_1, v_2, v_5, v_8 \}$ |
| $v_3, v_5$ | $\{v_1, v_2, v_6, v_7 \}$ |
| $v_3, v_6$ | $\{v_4, v_5, v_7, v_8 \}$ |
| $v_3, v_7$ | $\{v_2, v_4, v_6, v_8 \}$ |
| $v_4, v_5$ | $\{v_3, v_6, v_7, v_8 \}$ |
| $v_4, v_8$ | $\{v_1, v_3, v_5, v_7 \}$ |
| $v_5, v_7$ | $\{v_1, v_3, v_4, v_8 \}$ |
| $v_7, v_8$ | $\{v_1, v_4, v_5, v_6 \}$ |

Table 1: Pairs of vertices of the graph of Figure 4 and their non-trivial distinctive vertices.

3 The $k$-metric dimension of graphs

In this section we present some results that allow to compute the $k$-metric dimension of several families of graphs. We also give some tight bounds on the $k$-metric dimension of a graph.

**Theorem 12** (Monotony of the $k$-metric dimension). Let $G$ be a $k$-metric dimensional graph and let $k_1, k_2$ be two integers. If $1 ≤ k_1 < k_2 ≤ k$, then $\dim_{k_1}(G) < \dim_{k_2}(G)$.

**Proof.** Let $B$ be a $k$-metric basis of $G$. Let $x ∈ B$. Since all pairs of vertices in $V(G)$ are distinguished by at least $k$ vertices of $B$, we have that $B - \{x\}$ is a $(k - 1)$-metric generator for $G$ and, as a consequence, $\dim_{k-1}(G) ≤ |B - \{x\}| < |B| = \dim_k(G)$. Proceeding analogously, we obtain that $\dim_{k-1}(G) > \dim_{k-2}(G)$ and, by a finite repetition of the process we obtain the result. □

**Corollary 13.** Let $G$ be a $k$-metric dimensional graph of order $n$.

(i) For every $r ∈ \{1, ..., k\}$, $\dim_r(G) ≥ \dim(G) + (r - 1)$.

(ii) For every $r ∈ \{1, ..., k - 1\}$, $\dim_r(G) < n$.

3.1 The $k$-metric dimension of some specific families of graphs

**Theorem 14.** Let $G$ be a connected graph of order $n ≥ 2$. Then $\dim_2(G) = 2$ if and only if $G ≅ P_n$. 

10
Proof. In [5] was shown that $\dim(G) = 1$ if and only if $G \cong P_n$.

(Necessity) If $\dim_2(G) = 2$, then by Corollary 13 (i) we have that $\dim(G) = 1$, i.e.,

$$2 = \dim_2(G) \geq \dim(G) + 1 \geq 2.$$ 

Hence, $G$ must be isomorphic to a path graph.

(Sufficiency) By Corollary 13 (i) we have $\dim_2(P_n) \geq \dim(P_n) + 1 = 2$ and, since the leaves of $P_n$ distinguish every pair of vertices of $P_n$, we conclude $\dim_2(P_n) = 2$. \qed

Corollary 13 (i) and the fact that $\dim(G) = 1$ if and only if $G \cong P_n$ immediately lead to the following result.

**Theorem 15.** Let $G$ be a $k$-metric dimensional graph different from a path. Then for any $r \in \{2, \ldots, k\}$

$$\dim_r(G) \geq r + 1.$$ 

Let $D_k(G)$ be the set obtained as the union of the sets of distinctive vertices $D_G(x, y)$ whenever $|D_G(x, y)| = k$, i.e.,

$$D_k(G) = \bigcup_{|D_G(x, y)|=k} D_G(x, y).$$

**Remark 16.** If $G$ is a $k$-metric dimensional graph, then $\dim_k(G) \geq |D_k(G)|$.

*Proof.* Since every pair of vertices $x, y$ is distinguished only by the elements of $D_G(x, y)$, if $|D_G(x, y)| = k$, then for any $k$-metric basis $B$ we have $D_G(x, y) \subseteq B$ and, as a consequence, $D_k(G) \subseteq B$. \qed

The bound given in Remark 16 is tight. For instance, Proposition 30 shows that there exists a family of trees attaining this bound for every $k$. Other examples can be derived from the following result.

**Theorem 17.** Let $G$ be a $k$-metric dimensional graph of order $n$. Then $\dim_k(G) = n$ if and only if $V(G) = D_k(G)$.

*Proof.* Suppose that $V(G) = D_k(G)$. Now, since every $k$-metric dimensional graph $G$ satisfies that $\dim_k(G) \leq n$, by Remark 16 we obtain that $\dim_k(G) = n$.

On the contrary, let $\dim_k(G) = n$. Note that for every $a, b \in V(G)$, we have $|D_G(a, b)| \geq k$. If there exists at least one vertex $x \in V(G)$ such that $x \notin D_k(G)$, then for every $a, b \in V(G)$, we have $|D_G(a, b) - \{x\}| \geq k$ and, as a consequence, $V(G) - \{x\}$ is a $k$-metric generator for $G$, which is a contradiction. Therefore, $V(G) = D_k(G)$. \qed

It is interesting the particular case $k = 2$ of Theorem 17.

**Corollary 18.** Let $G$ be a connected graph of order $n \geq 2$. Then $\dim_2(G) = n$ if and only if every vertex is a twin.
We show other examples of graphs that satisfy Theorem 17 for \( k \geq 3 \). To this end, we recall that the join graph \( G + H \) of the graphs \( G = (V_1, E_1) \) and \( H = (V_2, E_2) \) is the graph with vertex set \( V(G + H) = V_1 \cup V_2 \) and edge set \( E(G + H) = E_1 \cup E_2 \cup \{uv : u \in V_1, v \in V_2\} \). We give now some examples of graphs satisfying the assumptions of Theorem 17. Let \( W_{1,n} = C_n + K_1 \) be the wheel graph and \( F_{1,n} = P_n + K_1 \) be the fan graph. The vertex of \( K_1 \) is called the central vertex of the wheel or the fan, respectively. Since \( V(F_{1,4}) = D_3(F_{1,4}) \) and \( V(W_{1,5}) = D_4(W_{1,5}) \), by Theorem 17 we have that \( \dim_3(F_{1,4}) = 5 \) and \( \dim_4(W_{1,5}) = 6 \) respectively.

Given two non-trivial graphs \( G \) and \( H \), it holds that any pair of twin vertices \( x, y \in V(G) \) or \( x, y \in V(H) \) are also twin vertices in \( G + H \). As a direct consequence of Corollary 18, the next result holds.

**Remark 19.** Let \( G \) and \( H \) be two nontrivial graphs of order \( n_1 \) and \( n_2 \), respectively. If all the vertices of \( G \) and \( H \) are twin vertices, then \( G + H \) is 2-metric dimensional and

\[
\dim_2(G + H) = n_1 + n_2.
\]

Note that in Remark 19, the graphs \( G \) and \( H \) could be non connected. Moreover, \( G \) and \( H \) could be nontrivial empty graphs. For instance, \( N_r + N_s \), where \( N_r, N_s, r, s > 1 \), are empty graphs, is the complete bipartite graph \( K_{r,s} \) which satisfies that \( \dim_2(K_{r,s}) = r + s \).

### 3.2 Bounding the \( k \)-metric dimension of graphs

We begin this subsection with a necessary definition of the twin equivalence relation \( \mathcal{R} \) on \( V(G) \) as follows:

\[
x \mathcal{R} y \iff N_G[x] = N_G[y] \text{ or } N_G(x) = N_G(y).
\]

We have three possibilities for each twin equivalence class \( U \):

(a) \( U \) is singleton, or

(b) \( N_G(x) = N_G(y) \), for any \( x, y \in U \), or

(c) \( N_G[x] = N_G[y] \), for any \( x, y \in U \).

We will refer to the type (c) classes as the true twin equivalence classes i.e., \( U \) is a true twin equivalence class if and only if \( U \) is not singleton and \( N_G[x] = N_G[y] \), for any \( x, y \in U \).

Let us see three different examples where every vertex is a twin. An example of graph where every equivalence class is a true twin equivalence class is \( K_r + (K_s \cup K_t) \), \( r, s, t \geq 2 \). In this case, there are three equivalence classes composed by \( r, s \) and \( t \) true twin vertices, respectively. As an example where no class is composed by true twin vertices we take the complete bipartite graph \( K_{r,s} \), \( r, s \geq 2 \). Finally, the graph \( K_r + N_s \), \( r, s \geq 2 \), has two equivalence classes and one of them is composed by \( r \) true twin vertices. On the other hand, \( K_1 + (K_r \cup N_s) \), \( r, s \geq 2 \), is an example where one class is singleton, one class is composed by true twin vertices and the other one is composed by false twin vertices.

In general, we can state the following result.
**Remark 20.** Let $G$ be a connected graph and let $U_1, U_2, ..., U_t$ be the non-singleton twin equivalence classes of $G$. Then

$$\dim_2(G) \geq \sum_{i=1}^{t} |U_i|.$$  

**Proof.** Since for two different vertices $x, y \in V(G)$ we have that $D_2(x, y) = \{x, y\}$ if and only if there exists an equivalence class $U_i$ such that $x, y \in U_i$, we deduce

$$D_2(G) = \bigcup_{i=1}^{t} U_i.$$  

Therefore, by Remark 16 we conclude the proof.

Notice that the result above leads to Corollary 18, so this bound is tight. Now we consider the connected graph $G$ of order $r + s$ obtained from a null graph $N_r$ of order $r \geq 2$ and a path $P_s$ of order $s \geq 1$ by connecting every vertex of $N_r$ to a given extreme vertex of $P_s$. In this case, there are $s$ singleton classes and one class, say $U_1$, of cardinality $r$. By the previous result we have $\dim_2(G) \geq |U_1| = r$ and, since $U_1$ is a 2-metric generator for $G$, we conclude that $\dim_2(G) = r$.

We recall that the strong product graph $G \boxtimes H$ of two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph with vertex set $V(G \boxtimes H) = V_1 \times V_2$, where two distinct vertices $(x_1, x_2), (y_1, y_2) \in V_1 \times V_2$ are adjacent in $G \boxtimes H$ if and only if one of the following holds.

- $x_1 = y_1$ and $x_2 \sim y_2$, or
- $x_1 \sim y_1$ and $x_2 = y_2$, or
- $x_1 \sim y_1$ and $x_2 \sim y_2$.

**Theorem 21.** Let $G$ and $H$ be two nontrivial connected graphs of order $n$ and $n'$, respectively. Let $U_1, U_2, ..., U_t$ be the true twin equivalence classes of $G$. Then

$$\dim_2(G \boxtimes H) \geq n' \sum_{i=1}^{t} |U_i|.$$  

Moreover, if every vertex of $G$ is a true twin, then

$$\dim_2(G \boxtimes H) = nn'.$$

**Proof.** For any two vertices $a, c \in U_i$ and $b \in V(H)$,

$$N_{G \boxtimes H}[a, b] = N_G[a] \times N_H[b]$$

$$= N_G[c] \times N_H[b]$$

$$= N_{G \boxtimes H}[(c, b)].$$

13
Thus, \((a, b)\) and \((c, b)\) are true twin vertices. Hence,

\[
\mathcal{D}_2(G \boxtimes H) \supseteq \bigcup_{i=1}^{t} U_i \times V(H).
\]

Therefore, by Remark 16 we conclude \(\dim_2(G \boxtimes H) \geq n' \sum_{i=1}^{t} |U_i|\).

Finally, if every vertex of \(G\) is a true twin, then \(\bigcup_{i=1}^{t} U_i = V(G)\) and, as a consequence, we obtain \(\dim_2(G \boxtimes H) = nn'\).

Now we present a lower bound for the \(k\)-metric dimension of a \(k'\)-metric dimensional graph \(G\) with \(k' \geq k\). To this end, we require the use of the following function for any exterior major vertex \(w_i \in V(G)\) having terminal degree greater than one, \(i.e.,\) \(w_i \in \mathcal{M}(G)\). Notice that this function uses the concepts already defined in Page 8. Given an integer \(r \leq k'\),

\[
I_r(w_i) = \begin{cases} 
(\text{ter}(w_i) - 1) (r - l(w_i)) + l(w_i), & \text{if } l(w_i) \leq \left\lfloor \frac{r}{2} \right\rfloor, \\
(\text{ter}(w_i) - 1) \left\lceil \frac{r}{2} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor, & \text{otherwise.}
\end{cases}
\]

In Figure 5 we give an example of a graph \(G^*\), which helps to clarify the notation above. In such a case we have \(\mathcal{M}(G^*) = \{v_3, v_5, v_{15}\}\) and, for instance, \(\{v_1, v_8, v_{12}\}\) are terminal vertices of \(v_3\). So, \(v_3\) has terminal degree three \((\text{ter}(v_3) = 3)\) and it follows that

\[
l(v_3) = \min\{l(v_{12}, v_3), l(v_8, v_3), l(v_1, v_3)\} = \min\{1, 2, 2\} = 1,
\]

and

\[
\varsigma(v_3) = \min\{\varsigma(v_{12}, v_1), \varsigma(v_{12}, v_8), \varsigma(v_8, v_1)\} = \min\{3, 3, 4\} = 3.
\]

Similarly, it is possible to observe that \(\text{ter}(v_5) = 2, l(v_5) = 1, \varsigma(v_5) = 3, \text{ter}(v_{15}) = 2, l(v_{15}) = 2\) and \(\varsigma(v_{15}) = 4\). Thus, \(\varsigma(G^*) = 3\), and by Theorem 9, we have that \(G^*\) is \(k\)-metric dimensional for some \(k \leq 3\). Since every graph is at least 2-metric dimensional, we can consider the integer \(r = 2\) and we have the following.

- Since \(l(v_3) = 1 \leq \left\lfloor \frac{r}{2} \right\rfloor\), it follows that \(I_r(v_3) = (\text{ter}(v_3) - 1) (r - l(v_3)) + l(v_3) = (3 - 1)(2 - 1) + 1 = 3\).

- Since \(l(v_5) = 1 \leq \left\lfloor \frac{r}{2} \right\rfloor\), it follows that \(I_r(v_5) = (\text{ter}(v_5) - 1) (r - l(v_5)) + l(v_5) = (2 - 1)(2 - 1) + 1 = 2\).

- Since \(l(v_{15}) = 2 > \left\lfloor \frac{r}{2} \right\rfloor\), it follows that \(I_r(v_{15}) = (\text{ter}(v_{15}) - 1) \left\lceil \frac{r}{2} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor = (2 - 1) \left\lceil \frac{2}{2} \right\rceil + \left\lfloor \frac{2}{2} \right\rfloor = 2\).

Therefore, according to the result below, \(\dim_2(G^*) \geq 3 + 2 + 2 = 7\).
Theorem 22. If $G$ is a $k$-metric dimensional graph such that $|\mathcal{M}(G)| \geq 1$, then for every $r \in \{1, \ldots, k\}$,

$$\dim_r(G) \geq \sum_{w_i \in \mathcal{M}(G)} I_r(w_i).$$

Proof. Let $S$ be a $r$-metric basis of $G$. Let $w_i \in \mathcal{M}(G)$ and let $u_{ij}, u_{is}$ be two different terminal vertices of $w_i$. Let $u_{ij}', u_{is}'$ be the vertices adjacent to $w_i$ in the paths $P(u_{ij}, w_i)$ and $P(u_{is}, w_i)$, respectively. Notice that $\mathcal{D}_G(u_{ij}', u_{is}') = V(P(u_{ij}, w_i)) - \{w_i\}$ and, as a consequence, it follows that $|S \cap (V(P(u_{ij}, w_i)) - \{w_i\})| \geq r$. Now, if $\text{ter}(w_i) = 2$, then we have $|S \cap (V(P(u_{ij}, w_i)) - \{w_i\})| \geq r = I_r(w_i)$.

On the contrary, we assume $\text{ter}(w_i) > 2$. Let $W_i$ be the set of terminal vertices of $w_i$, and let $u_{ij}'$ be the vertex adjacent to $w_i$ in the path $P(u_{ij}, w_i)$ for every $u_{ij} \in W_i$. Let $U(w_i) = \bigcup_{u_{ij} \in W_i} V(P(u_{ij}, w_i)) - \{w_i\}$ and let $x_i = \min \{ |S \cap V(P(u_{ij}, w_i))| \}$. Since $S$ is a $r$-metric generator of minimum cardinality (it is a $r$-metric basis of $G$), it is satisfied that $0 \leq x_i \leq \min\{ l(w_i), \lceil \frac{r}{2} \rceil \}$. Let $u_{ia}$ be a terminal vertex such that $|S \cap (V(P(u_{ia}, w_i)) - \{w_i\})| = x_i$. Since for every terminal vertex $u_{i\beta} \in W_i - \{u_{ia}\}$ we have that $|S \cap \mathcal{D}_G(u_{i\beta}', u_{i\alpha}')| \geq r$, it follows that $|S \cap (V(P(u_{i\beta}, w_i)) - \{w_i\})| \geq r - x_i$. Thus,

$$|S \cap U(w_i)| = |S \cap (V(P(u_{ia}, w_i)) - \{w_i\})| + \sum_{\beta=1, \beta \neq \alpha}^\text{ter}(w_i) |S \cap (V(P(u_{i\beta}, w_i)) - \{w_i\})| \geq (\text{ter}(w_i) - 1) (r - x_i) + x_i.
$$

Now, if $x_i = 0$, then $|S \cap U(w_i)| \geq (\text{ter}(w_i) - 1) r > I_r(w_i).$ On the contrary, if $x_i > 0$, then the function $(\text{ter}(w_i) - 1) (r - x_i) + x_i$ is decreasing with respect to $x_i$. So, the minimum value of the function is achieved in the highest possible value of $x_i$. Thus, $|S \cap U(w_i)| \geq I_r(w_i)$. Since $\bigcap_{w_i \in \mathcal{M}(G)} U(w_i) = \emptyset$, it follows that

$$\dim_r(G) \geq \sum_{w_i \in \mathcal{M}(G)} |S \cap U(w_i)| \geq \sum_{w_i \in \mathcal{M}(G)} I_r(w_i).$$

\qed
Now, in order to give some consequences of the bound above we will use some notation defined on Page 8 to introduce the following parameter, which we need at next.

\[ \mu(G) = \sum_{v \in \mathcal{M}(G)} \text{ter}(v). \]

Notice that for \( k = 1 \) Theorem 22 leads to the bound on the metric dimension of a graph, established by Chartrand et al. in [5]. In such a case, \( I_1(w_i) = \text{ter}(w_i) - 1 \) for all \( w_i \in \mathcal{M}(G) \) and thus,

\[ \dim(G) \geq \sum_{w_i \in \mathcal{M}(G)} (\text{ter}(w_i) - 1) = \mu(G) - |\mathcal{M}(G)|. \]

Next we give the particular cases of Theorem 22 for \( r = 2 \) and \( r = 3 \).

**Corollary 23.** If \( G \) is a connected graph, then

\[ \dim_2(G) \geq \mu(G). \]

**Proof.** If \( \mathcal{M}(G) = \emptyset \), then \( \mu(G) = 0 \) and the result is direct. Suppose that \( \mathcal{M}(G) \neq \emptyset \). Since \( G \) is \( k \)-metric dimensional for some \( k \geq 2 \), \( I_2(w_i) = \text{ter}(w_i) \) for all \( w_i \in \mathcal{M}(G) \). Therefore, \( \dim_2(G) \geq \sum_{w_i \in \mathcal{M}(G)} \text{ter}(w_i) = \mu(G) \).

**Corollary 24.** If \( G \) is \( k \)-metric dimensional for some \( k \geq 3 \), then

\[ \dim_3(G) \geq 2\mu(G) - |\mathcal{M}(G)|. \]

**Proof.** If \( \mathcal{M}(G) = \emptyset \), then the result is direct. Suppose that \( \mathcal{M}(G) \neq \emptyset \). Since \( G \) is \( k \)-metric dimensional for some \( k \geq 3 \), \( I_3(w_i) = 2\text{ter}(w_i) - 1 \) for all \( w_i \in \mathcal{M}(G) \). Therefore, \( \dim_3(G) \geq \sum_{w_i \in \mathcal{M}(G)} (2\text{ter}(w_i) - 1) = 2\mu(G) - |\mathcal{M}(G)| \).

In next section we give some results on trees which show that the bounds proved in Theorem 22 and Corollaries 23 and 24 are tight. Specifically those results are Theorem 27 and Corollaries 28 and 29, respectively.

### 4 The particular case of trees

To study the \( k \)-metric dimension of a tree it is of course necessary to know first the value \( k \) for which a given tree is \( k \)-metric dimensional. That is what we do at next. In this sense, now on we need the terminology and notation already described in Page 8 and also the following one. Given an exterior major vertex \( v \) in a tree \( T \) and the set of its terminal vertices \( v_1, ..., v_\alpha \), the set \( \mathcal{B}(v) = \bigcup_{i=1}^{\alpha} V(P(v, v_i)) \) is called a branch of \( T \) at \( v \) (a \( v \)-branch for short).

**Theorem 25.** If \( T \) is a \( k \)-metric dimensional tree different from a path, then \( k = \varsigma(T) \).
Proof. Since $T$ is not a path, $\mathcal{M}(T) \neq \emptyset$. Let $\mathcal{M}(T) = \{w_1, w_2, \ldots, w_r\}$ and let $T_i = (V_i, E_i)$ be the subgraphs induced by the sets of its $w_i$-branches with $1 \leq i \leq r$. Also we consider the set of vertices $V' = V(T) - \bigcup_{i=1}^r V_i$. Without loss of generality let $w_1 \in \mathcal{M}(G)$ and let $u_{11}, u_{12}$ be two terminal vertices of $w_1$ such that $\varsigma(G) = \varsigma(w_1) = \varsigma(u_{11}, u_{12})$. Notice that, for instance, the two neighbors of $w_1$ belonging to the paths $P(w_1, u_{11})$ and $P(w_1, u_{12})$, say $u_{11}'$ and $u_{12}'$ satisfy $|D_G(u_{11}', u_{12}')| = \varsigma(T)$.

It only remains to prove that for every $x, y \in V(T)$ it holds that $|D_G(x, y)| \geq \varsigma(T)$. Note that $|V_i| \geq \varsigma(T) + 1$ for $1 \leq i \leq r$. With this fact in mind, we consider three cases.

Case 1: $x \in V_i$ and $y \in V_j$ for some $i, j \in \{1, \ldots, r\}$, $i \neq j$. In this case, $x, y$ are distinguished by $w_i$ or by $w_j$. Now, if $w_i$ distinguishes the pair $x, y$, then at most one element of $V_i$ does not distinguish $x, y$ (see Figure 6). So, $x$ and $y$ are distinguished by at least $|V_i| - 1$ vertices of $T$ or by at least $|V_j| - 1$ vertices of $T$.

![Figure 6](image)

Figure 6: In this example, $w_i$ distinguishes the pair $x, y$, and $z$ is the only vertex in $V_i$ that does not distinguish $x, y$.

Case 2: $x \in V'$ or $y \in V'$. Thus, $V' \neq \emptyset$ and, as a consequence, $|\mathcal{M}(T)| \geq 2$. Hence, we have one of the following situations.

- There exist two vertices $w_i, w_j \in \mathcal{M}(T)$, $i \neq j$, such that the shortest path from $x$ to $w_i$ and the shortest path from $y$ to $w_j$ have empty intersection, or
- for every vertex $w_l \in \mathcal{M}(T)$, it follows that either $y$ belongs to the shortest path from $x$ to $w_l$ or $x$ belongs to the shortest path from $y$ to $w_l$.

In the first case, $x, y$ are distinguished by the elements of $V_i$ or by the elements of $V_j$ and in the second one, $x, y$ are distinguished by the elements of $V_i$.

Case 3: $x, y \in V_j$ for some $w_j \in \mathcal{M}(T)$. If $x, y \in V(P(u_{jl}, w_j))$ for some $l \in \{1, \ldots, \text{ter}(w_j)\}$, then there exists at most one vertex of $V(P(u_{jl}, w_j))$ which does not distinguish $x, y$. Since $\text{ter}(w_j) \geq 2$, the vertex $w_j$ has at least other terminal vertex $u_{jq}$ with $q \neq l$. So, $x, y$ are distinguished by $|V(P(u_{jl}, w_j, u_{jq}))| - 1$, and since $|V(P(u_{jl}, w_j, u_{jq}))| \geq \varsigma(T) + 1$, we are done. If $x \in V(P(u_{jl}, w_j))$ and $y \in V(P(u_{lj}, w_j))$ for some $l, q \in \{1, \ldots, \text{ter}(w_j)\}$, $l \neq q$, then there exists at most one vertex of $V(P(u_{jl}, w_j, u_{jq}))$ which does not distinguish $x, y$. Since $|V(P(u_{jl}, w_j, u_{jq}))| \geq \varsigma(T) + 1$, the result follows.

Therefore, $\varsigma(T) = \min_{x, y \in V(T)} |D_G(x, y)|$ and by Theorem 2 we have the result. \qed

Since any path is a particular case of a tree and its behavior with respect to the $k$-metric dimension is relative different, here we analyze them in first instance. In Theorem 14 we noticed
that the 2-metric dimension of a path $P_n(n \geq 2)$ is two. Here we give a formula for the $k$-metric dimension of any path graph for $k \geq 3$.

**Proposition 26.** Let $k \geq 3$ be an integer. For any path graph $P_n$ of order $n \geq k + 1$,

$$\dim_k(P_n) = k + 1.$$  

**Proof.** Let $v_1$ and $v_n$ be the leaves of $P_n$ and let $S$ be a $k$-metric basis of $P_n$. Since $|S| \geq k \geq 3$, there exists at least one vertex $w \in S \cap (V(P_n) - \{v_1, v_n\})$. For any vertex $w \in V(P_n) - \{v_1, v_n\}$ there exist at least two vertices $u, v \in V(P_n)$ such that $w$ does not distinguish $u$ and $v$. Hence, $|S| = \dim_k(P_n) \geq k + 1$.

Now, notice that for any pair of vertices $u, v \in V(P_n)$ there exists at most one vertex $w \in V(P_n) - \{v_1, v_n\}$ such that $w$ does not distinguish $u$ and $v$. Thus, we have that for every $S \subseteq V(P_n)$ such that $|S| = k + 1$ and every pair of vertices $x, y \in V(P_n)$, there exists at least $k$ vertices of $S$ such that they distinguish $x, y$. So $S$ is a $k$-metric generator for $P_n$. Therefore, $\dim_k(P_n) \leq |S| = k + 1$ and, consequently, the result follows. 

Once studied the path graphs, we are now able to give a formula for the $r$-metric dimension of any $k$-metric dimensional tree different from a path which, among other usefulness, shows that Theorem 22 is tight.

**Theorem 27.** If $T$ is a tree which is not a path, then for any $r \in \{1, ..., \varsigma(T)\}$,

$$\dim_r(T) = \sum_{w_i \in \mathcal{M}(T)} I_r(w_i).$$

**Proof.** Since $T$ is not a path, $T$ contains at least one vertex belonging to $\mathcal{M}(T)$. Hence, let $\mathcal{M}(T) = \{w_1, w_2, \ldots, w_t\}$. Let $T_i = (V_i, E_i)$ be the subgraphs induced by the $w_i$-branch, $1 \leq i \leq t$. Also we consider the set $V' = V(T) - \bigcup_{i=1}^t V_i$. For every $w_i \in \mathcal{M}(T)$, we suppose $u_i$ is a terminal vertex of $w_i$ such that $l(u_{ij}, w_i) = l(w_i)$. Let $U(w_i) = \{u_{i1}, u_{i2}, \ldots, u_{is}\}$ be the set of terminal vertices of $w_i$. Now, for every $u_{ij} \in U(w_i)$, let the path $P(u_{ij}, w_i) = u_{ij}u_{ij}^1u_{ij}^2\ldots u_{ij}^{l(w_i)-1}w_i$ and we consider the set $S(u_{ij}, w_i) \subseteq V(P(u_{ij}, w_i)) - \{w_i\}$ given by:

$$S(u_{ij}, w_i) = \begin{cases} 
\{u_{ij}, u_{ij}^1, \ldots, u_{ij}^{l(w_i)-1}\}, & \text{if } l(w_i) \leq \left[\frac{r}{2}\right] \text{ and } u_{ij} = u_{i1}, \\
\{u_{ij}, u_{ij}^1, \ldots, u_{ij}^{l(w_i)-1}\}, & \text{if } l(w_i) \leq \left[\frac{r}{2}\right] \text{ and } u_{ij} \neq u_{i1}, \\
\{u_{ij}, u_{ij}^1, \ldots, u_{ij}^{l(w_i)-1}\}, & \text{if } l(w_i) > \left[\frac{r}{2}\right] \text{ and } u_{ij} = u_{i1}, \\
\{u_{ij}, u_{ij}^1, \ldots, u_{ij}^{l(w_i)-1}\}, & \text{if } l(w_i) > \left[\frac{r}{2}\right] \text{ and } u_{ij} \neq u_{i1}.
\end{cases}$$

Also, notice that

$$|S(u_{ij}, w_i)| = \begin{cases} 
l(w_i), & \text{if } l(w_i) \leq \left[\frac{r}{2}\right] \text{ and } u_{ij} = u_{i1}, \\
r - l(w_i), & \text{if } l(w_i) \leq \left[\frac{r}{2}\right] \text{ and } u_{ij} \neq u_{i1}, \\
\left[\frac{r}{2}\right], & \text{if } l(w_i) > \left[\frac{r}{2}\right] \text{ and } u_{ij} = u_{i1}, \\
\left[\frac{r}{2}\right], & \text{if } l(w_i) > \left[\frac{r}{2}\right] \text{ and } u_{ij} \neq u_{i1}.
\end{cases}$$
Let $S(w_i) = \bigcup_{u_{ij} \in U(w_i)} S(u_{ij}, w_i)$ and $S = \bigcup_{w_i \in \mathcal{M}(T)} S(w_i)$. Since for every $w_i \in \mathcal{M}(T)$ it follows that $\bigcap_{u_{ij} \in U(w_i)} S(u_{ij}, w_i) = \emptyset$ and $\bigcap_{w_i \in \mathcal{M}(T)} S(w_i) = \emptyset$, we obtain that $|S| = \sum_{w_i \in \mathcal{M}(T)} I_r(w_i)$.

Also notice that for every $w_i \in \mathcal{M}(T)$, such that $\text{ter}(w_i) = 2$ we have $|S(w_i)| = r$ and, if $\text{ter}(w_i) > 2$, then we have $|S(w_i)| \geq r + 1$. We claim that $S$ is an $r$-metric generator for $T$. Let $u, v$ be two distinct vertices of $T$. We consider the following cases.

**Case 1:** $u, v \in V_i$ for some $i \in \{1, ..., t\}$. We have the following subcases.

**Subcase 1.1:** $u, v \in V(P(u_{ij}, w_i))$ for some $j \in \{1, ..., \text{ter}(w_i)\}$. Hence there exists at most one vertex of $S(w_i) \cap V(P(u_{ij}, w_i))$ which does not distinguish $u, v$. If $\text{ter}(w_i) = 2$, then there exists at least one more exterior major vertex $w_j \in \mathcal{M}(T)$ with $i \neq j$. So, the elements of $S(w_j)$ distinguish $u, v$. Since $|S(w_j)| \geq r$, we deduce that at least $r$ elements of $S$ distinguish $u, v$. On the other hand, if $\text{ter}(w_i) > 2$, then since $|S(w_i)| \geq r + 1$, we obtain that at least $r$ elements of $S(w_i)$ distinguish $u, v$.

**Subcase 1.2:** $u \in V(P(u_{ij}, w_i))$ and $v \in V(P(u_{ij}, w_i))$ for some $j, l \in \{1, ..., \text{ter}(w_i)\}$, $j \neq l$. According to the construction of the set $S(w_i)$, there exists at most one vertex of $(S(w_i) \cap V(P(u_{ij}, w_i)))$ which does not distinguish $u, v$.

Now, if $\text{ter}(w_i) = 2$, then there exists $w_j \in \mathcal{M}(T) \setminus \{w_i\}$. If $d(u, w_i) = d(v, w_i)$, then the $r$ elements of $S(w_i)$ distinguish $u, v$ and, if $d(u, w_i) \neq d(v, w_i)$, then the elements of $S(w_j)$ distinguish $u, v$.

On the other hand, if $\text{ter}(w_i) > 2$, then since $|S(w_i)| \geq r + 1$, we deduce that at least $r$ elements of $S(w_i)$ distinguish $u, v$.

**Case 2:** $u \in V_i, v \in V_j$, for some $i, j \in \{1, ..., t\}$ with $i \neq j$. In this case, either the vertices in $S(w_i)$ or the vertices in $S(w_j)$ distinguish $u, v$. Since $|S(w_i)| \geq r$ and $|S(w_j)| \geq r$ we have that $u, v$ are distinguished by at least $r$ elements of $S$.

**Case 3:** $u \in V'$ or $v \in V'$. Without loss of generality we assume $u \in V'$. Since $V' \neq \emptyset$, we have that there exist at least two different vertices in $\mathcal{M}(T)$. Hence, we have either one of the following situations.

- There exist two vertices $w_i, w_j \in \mathcal{M}(T)$, $i \neq j$, such that the shortest path from $u$ to $w_i$ and the shortest path from $v$ to $w_j$ have empty intersection, or

- for every vertex $w_l \in \mathcal{M}(T)$, it follows that either $v$ belongs to every shortest path from $u$ to $w_l$ or $u$ belongs to every shortest path from $v$ to $w_l$.

Notice that in both situations, since $|S(w_i)| \geq r$, for every $w_i \in \mathcal{M}(T)$, we have that $u, v$ are distinguished by at least $r$ elements of $S$. In the first case, $u$ and $v$ are distinguished by the elements of $S(w_i)$ or by the elements of $S(w_j)$ and, in the second one, $u$ and $v$ are distinguished by the elements of $S(w_i)$.

Therefore, $S$ is an $r$-metric generator for $T$ and, by Theorem 22, the proof is complete. \(\square\)

In the case $r = 1$, the formula of Theorem 27 leads to
\[
\dim(T) = \mu(T) - |\mathcal{M}(T)|,
\]
which is the result obtained in [5]. Other interesting particular cases are the following ones for $r = 2$ and $r = 3$, respectively. That is, by Theorem 27 we have the next results.
Corollary 28. If $T$ is a tree different from a path, then

$$\dim_2(T) = \mu(T).$$

Corollary 29. If $T$ is a tree different from a path with $\varsigma(T) \geq 3$, then

$$\dim_3(T) = 2\mu(T) - |\mathcal{M}(T)|.$$

As mentioned before, the two corollaries above show that the bounds given in Corollaries 23 and 24 are achieved. We finish our exposition with a formula for the $k$-metric dimension of a $k$-metric dimensional tree with some specific structure, also showing that the inequality $\dim_k(T) \geq |\mathcal{D}_k(T)|$, given in Remark 16, can be reached.

Proposition 30. Let $T$ be a tree different from a path and let $k \geq 2$ be an integer. If $\text{ter}(w_i) = 2$ and $\varsigma(w_i) = k$ for every $w_i \in \mathcal{M}(T)$, then $\dim_k(T) = |\mathcal{D}_k(T)|$.

Proof. Since every vertex $w_i \in \mathcal{M}(T)$ satisfies that $\text{ter}(w_i) = 2$ and $\varsigma(w_i) = k$, we have that $\varsigma(T) = k$. Thus, by Theorem 25, $T$ is $k$-metric dimensional tree. Since $I_k(w_i) = k$ for every $w_i \in \mathcal{M}(T)$, by Theorem 27 we have that $\dim_k(T) = k|\mathcal{M}(T)|$. Let $u_{i,r}, u_{i,s}$ be the terminal vertices of $w_i$. As we have shown in the proof of Theorem 25, for every pair $x, y \in V(T)$ such that $x \notin V (P(u_{i,r}, w_i, u_{i,s})) - \{w_i\}$ or $y \notin V (P(u_{i,r}, w_i, u_{i,s})) - \{w_i\}$, $x, y$ are distinguished by at least by $k + 1$ vertices of $T$. Hence, $|\mathcal{D}_G(x, y)| \neq k - 2$. So, if $|\mathcal{D}_G(x, y)| = k - 2$, then $x, y \in V (P(u_{i,r}, w_i, u_{i,s})) - \{w_i\}$ for some $w_i \in \mathcal{M}(T)$. If $d(x, w_i) \neq d(y, w_i)$, then $x, y$ are distinguished by more than $k$ vertices (those vertices not in $V (P(u_{i,r}, w_i, u_{i,s})) - \{w_i\}$). Thus, $d(x, w_i) = d(y, w_i)$ and $V (P(u_{i,r}, w_i, u_{i,s})) - \{x, y, w_i\}$ are distinctive vertices of $x, y$. Since $|V (P(u_{i,r}, w_i, u_{i,s})) - \{w_i\}| = k$ and $\bigcap_{w_i \in \mathcal{M}(T)} V (P(u_{i,r}, w_i, u_{i,s})) = \emptyset$, we have that $|\mathcal{D}_k(T)| = k|\mathcal{M}(T)|$. Therefore, $\dim_k(T) = |\mathcal{D}_k(T)|$.

\[ \text{Figure 7: Example of a 3-metric dimensional tree that satisfies the equality of Remark 16. In this case } \mathcal{M}(T) = \{w_1, w_2\}, \text{ ter}(w_i) = 2 \text{ and } \varsigma(w_i) = 3 \text{ for every } w_i \in \mathcal{M}(T). \text{ Then Proposition 30 leads to } \dim_3(T) = |\mathcal{D}_3(T)|=6. \]

References

[1] R. F. Bailey, P. J. Cameron, Base size, metric dimension and other invariants of groups and graphs, Bulletin of the London Mathematical Society 43 (2) (2011) 209–242.

URL http://blms.oxfordjournals.org/content/43/2/209.1.abstract
[2] R. C. Brigham, G. Chartrand, R. D. Dutton, P. Zhang, Resolving domination in graphs, Mathematica Bohemica 128 (1) (2003) 25–36. URL http://mb.math.cas.cz/mb128-1/3.html

[3] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas, C. Seara, D. R. Wood, On the metric dimension of cartesian product of graphs, SIAM Journal on Discrete Mathematics 21 (2) (2007) 423–441. URL http://epubs.siam.org/doi/abs/10.1137/050641867

[4] G. G. Chappell, J. Gimbel, C. Hartman, Bounds on the metric and partition dimensions of a graph, Ars Combinatoria 88 (2008) 349–366. URL http://www.cs.uaf.edu/~chappell/papers/metric/metric.pdf

[5] G. Chartrand, L. Eroh, M. A. Johnson, O. R. Oellermann, Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 (1-3) (2000) 99–113. URL http://dx.doi.org/10.1016/S0166-218X(00)00198-0

[6] G. Chartrand, C. Poisson, P. Zhang, Resolvability and the upper dimension of graphs, Computers & Mathematics with Applications 39 (12) (2000) 19–28. URL http://dx.doi.org/10.1016/S0898-1221(00)00126-7

[7] G. Chartrand, E. Salehi, P. Zhang, The partition dimension of a graph, Aequationes Mathematicae 59 (1-2) (2000) 45–54. URL http://dx.doi.org/10.1007/PL00000127

[8] M. Fehr, S. Gosselin, O. R. Oellermann, The partition dimension of cayley digraphs, Aequationes Mathematicae 71 (1-2) (2006) 1–18. URL http://link.springer.com/article/10.1007%2Fs00010-005-2800-z

[9] F. Harary, R. A. Melter, On the metric dimension of a graph, Ars Combinatoria 2 (1976) 191–195. URL http://www.ams.org/mathscinet-getitem?mr=0457289

[10] T. W. Haynes, M. A. Henning, J. Howard, Locating and total dominating sets in trees, Discrete Applied Mathematics 154 (8) (2006) 1293–1300. URL http://www.sciencedirect.com/science/article/pii/S0166218X06000035

[11] M. Johnson, Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 (2) (1993) 203–236, pMID: 8220404. URL http://www.tandfonline.com/doi/abs/10.1080/10543409308835060

[12] M. A. Johnson, Browsable structure-activity datasets, in: R. Carbó-Dorca, P. Mezey (eds.), Advances in Molecular Similarity, chap. 8, JAI Press Inc, Stamford, Connecticut, 1998, pp. 153–170. URL http://books.google.es/books?id=1vvMsHXd2AsC
[13] S. Khuller, B. Raghavachari, A. Rosenfeld, Landmarks in graphs, Discrete Applied Mathematics 70 (3) (1996) 217–229. 
URL http://www.sciencedirect.com/science/article/pii/0166218X95001062

[14] D. Kuziak, I. G. Yero, J. A. Rodríguez-Velázquez, On the strong metric dimension of corona product graphs and join graphs, Discrete Applied Mathematics 161 (7–8) (2013) 1022–1027. 
URL http://www.sciencedirect.com/science/article/pii/S0166218X12003897

[15] O. R. Oellermann, J. Peters-Fransen, The strong metric dimension of graphs and digraphs, Discrete Applied Mathematics 155 (3) (2007) 356–364. 
URL http://www.sciencedirect.com/science/article/pii/S0166218X06003015

[16] F. Okamoto, B. Phinezy, P. Zhang, The local metric dimension of a graph, Mathematica Bohemica 135 (3) (2010) 239–255. 
URL http://dml.cz/dmlcz/140702

[17] V. Saenpholphat, P. Zhang, Conditional resolvability in graphs: a survey, International Journal of Mathematics and Mathematical Sciences 2004 (38) (2004) 1997–2017. 
URL http://www.hindawi.com/journals/ijmms/2004/247096/abs/

[18] A. Sebö, E. Tannier, On metric generators of graphs, Mathematics of Operations Research 29 (2) (2004) 383–393. 
URL http://dx.doi.org/10.1287/moor.1030.0070

[19] P. J. Slater, Leaves of trees, Congressus Numerantium 14 (1975) 549–559.

[20] P. J. Slater, Dominating and reference sets in a graph, Journal of Mathematical and Physical Sciences 22 (4) (1988) 445–455. 
URL http://www.ams.org/mathscinet-getitem?mr=0966610

[21] I. Tomescu, Discrepancies between metric dimension and partition dimension of a connected graph, Discrete Applied Mathematics 308 (22) (2008) 5026–5031. 
URL http://www.sciencedirect.com/science/article/pii/S0012365X07007200

[22] I. G. Yero, D. Kuziak, J. A. Rodríguez-Velázquez, On the metric dimension of corona product graphs, Computers & Mathematics with Applications 61 (9) (2011) 2793–2798. 
URL http://www.sciencedirect.com/science/article/pii/S0898122111002094

[23] I. G. Yero, J. A. Rodríguez-Velázquez, A note on the partition dimension of cartesian product graphs, Applied Mathematics and Computation 217 (7) (2010) 3571–3574. 
URL http://www.sciencedirect.com/science/article/pii/S0096300310008921