Two-Person Bargaining when the Disagreement Point is Private Information

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Abstract

We consider two-person bargaining problems in which (only) the disagreement outcome is private (and possibly correlated) information and it is common knowledge that disagreement is inefficient. We show that if the Pareto frontier is linear, the outcome of an ex-post efficient mechanism cannot depend on the disagreement payoffs. If the frontier is non-linear, the result continues to hold when the disagreement payoffs are independent or there is a player with at most two types. We discuss implications of these results for axiomatic bargaining theory and for full surplus extraction in mechanism design.

Keywords: bargaining problem, incomplete information, axiomatic method, efficiency, disagreement, correlation

JEL classification codes: C71, C78, D82.

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1 Introduction

In this paper, we analyze two-person bargaining problems in which the players have private information only about the disagreement outcome and in which it is common knowledge that Pareto efficiency requires agreement to be reached. Such a situation could, for example, arise when the players know each other very well, but each is lacking information about the other’s outside options. We study whether, in such situations, these outside options will influence the bargaining outcome.

As a simple example, suppose two players have to allocate an (indivisible) dollar to one of them, with each player only getting 25 cents if they fail to agree. Intuition suggests that the outcome will depend on the players’ risk attitudes and, indeed, under complete information, all standard bargaining solutions predict that a more risk averse player will get the dollar with a larger probability. However, risk attitudes typically are private information. Our results imply that, in this case, if it is common knowledge that both players are not too risk averse (and, for example, prefer the 50/50 lottery to disagreement), the degree of risk aversion is irrelevant: any symmetric and ex-post efficient mechanism has to prescribe the 50/50 lottery irrespective of the players’ risk attitudes. More generally, we show that, for bargaining problems with a linear Pareto frontier or independent disagreement payoffs, any ex-post efficient mechanism has to be ‘disagreement point independent’, i.e., the players’ (interim) expected utilities cannot depend on the values of their outside options.

The situation that we analyze differs from the classic bilateral trade model of Myerson and Satterthwaite (1983). In that model, disagreement (no trade) can be efficient, and there does not exist an ex-post efficient, incentive compatible and individually rational mechanism. In our context, disagreement is inefficient and -as in the example above- the players may even know a lottery that Pareto dominates disagreement for sure. Furthermore, there typically are many mechanisms that satisfy these three conditions; in the example, any (constant) mechanism that always selects the same lottery close to 50/50 satisfies them. The main result of the present paper is that only such ‘constant’ solutions can satisfy them: ‘disagreement point independence’ holds generically as long as the Pareto frontier is linear.

The model that we analyze amounts to a minimal change of the canonical (Nash) bargaining game with complete information. In that context, it is usually assumed
that the disagreement point is inefficient, and Pareto efficiency of the outcome is a standard axiom. The major bargaining solutions that have been proposed for this situation all satisfy Disagreement Point Monotonicity (DPM): if a player’s disagreement payoff increases, then the outcome (weakly) moves in her favor (Thomson, 1987).\(^1\) DPM is an intuitive idea and is a building block of the Harvard Negotiation Model (Fisher and Ury, 1981). The fifth principle of this negotiation method states that a bargainer should enhance her Best Alternative to Negotiated Agreement (BATNA): “The better your BATNA, the greater your power (...). In fact, the relative negotiating power of the two parties depends primarily upon how attractive to each is the option of not reaching an agreement.” (Getting to Yes, p. 52).\(^2\)

With complete information, efficiency and DPM, hence, are compatible. We only change the canonical model by making the players’ disagreement payoffs private information and study whether efficiency is compatible with the bargaining outcome being responsive to a player’s outside option. Although such responsiveness (called type-dependence below) is a weaker property than monotonicity, we show that it conflicts with efficiency. Hence, in the ‘allocate the dollar example’, when the players’ risk attitudes are private information, they cannot influence the lottery that the players will agree upon.\(^3\) Consequently, the incomplete information case is very different from the one with complete information. BATNA does not hold in this case. Although DPM is a very weak requirement in the latter domain, type-dependence is very demanding in the former.

We admit that type-dependence is somewhat less appealing than DPM is under complete information. If the disagreement payoff is private information, a player can claim that her outside option has improved, but, as the other party may not be able to verify this, why should he respond to it? The claim might be ignored, in particular, if the players’ outside options are independent or positively correlated. However, if they are negatively correlated, the second player can conclude that his partner’s

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\(^1\)DPM should be distinguished from Monotonicity (M), the main axiom that characterizes proportional solutions (Kalai, 1977) and which insists that, if the set of possible agreements is enlarged and the disagreement outcome does not change, none of the players becomes worse off.

\(^2\)Of course, BATNA can only be satisfied in a weak sense: a player’s bargaining power does not decrease, but will sometimes increase, if her outside option improves. The outside option principle (Binmore, Rubinstein and Wolinsky, 1986) holds that non-credible options are irrelevant.

\(^3\)In this example, type-independence corresponds to ordinality: the mechanism only depends on the players’ ordinal preferences (each prefers more money to less), but not on the intensity of the players’ preferences, i.e., how much each dislikes disagreement.
claim is likely to be true, and, equally important, that she knows that he knows this. Hence, under negative correlation, type-dependence has intuitive appeal: a player with a better option might be a tougher bargainer and this could translate into a different outcome. This certainly holds when the disagreement utilities are perfectly negatively correlated as this case is equivalent to having complete information. Hence, our intuition is that, with negative correlation, type-dependence and efficiency can be satisfied simultaneously. We, however, show that this intuition is wrong; our main result is that if the Pareto frontier is linear (and the players’ prior has full support), any feasible, efficient mechanism must be type-independent: how much a player dislikes disagreement cannot play any role. While this result may not be surprising when the players’ disagreement utilities are independent, it is far from obvious that it holds for all correlated priors.

Our paper aims to contribute to the literature on axiomatic bargaining with incomplete information. There is only a small literature on this topic. Two solution concepts have been proposed and axiomatized: the generalized Nash bargaining solution (Harsanyi and Selten, 1972; Myerson, 1979) and the neutral bargaining solution (Myerson, 1984). Occasionally also the ex-ante utilitarian solution (which maximizes the sum of ex-ante expected utilities of the players over the set of all feasible mechanisms) is used, as in Myerson and Satterthwaite (1983). It can be shown that, in the above ‘allocate the dollar’ example, when the players’ disagreement utilities are independent, the two former concepts are type-dependent (and, hence, predict that there will be some inefficiency), while the latter is efficient and, hence, type-independent. Our results imply that no solution can be both efficient and type-dependent.

This impossibility result raises the question of which of these properties, efficiency or type-dependence, should get priority. One might argue that, given that efficient solutions do not exist in general, efficiency should be given up. We, however, think that the issue is not that clear: if players cannot commit to a mechanism, then an inefficient mechanism is not renegotiation-proof. The players might not agree to a mechanism ex-ante if they know that they will renegotiate ex-post once they see that the mechanism’s outcome is inefficient.

The remainder of the paper is organized as follows. Section 2 introduces our main model. In Section 3 we show that, if the Pareto frontier is linear (and the prior has full support), an efficient mechanism must be type-independent. Section 4
shows that this result no longer holds when the frontier is non-linear. In Section 5, we adapt our model to the transferable utility case and show that the results from the Sections 3 and 4 continue to hold. Section 6 discusses three applications of our results for bargaining theory and Section 7 concludes. The more technical proofs are provided in the Appendix.

2 The model

We consider a two-person bargaining problem \((A, a_0)\), where \(A\) is a finite set of alternatives with \(|A| \geq 3\) and \(a_0 \in A\) is the disagreement (or status-quo) outcome. We assume that the players have Von Neumann Morgenstern utility functions \(u_1\) and \(u_2\), denote by \(u^a_i\) the utility that \(i\) attaches to \(a \in A\) and write \(u^a = (u^a_1, u^a_2)\). We write \(A^0 = A \setminus \{a_0\}\) for the set of (real) agreements and assume that the utilities of all agreements, \(\{u^a|a \in A^0\}\), are common knowledge, but that \(u^0_i := u^a_i\) is private information of player \(i\). Without loss of generality, we assume that different agreements have different utilities: if \(a, a' \in A^0\) and \(a \neq a'\), then \(u^a_i \neq u^{a'}_i\) for \(i = 1, 2\). Writing \(a_i\) for the best Pareto efficient alternative for player \(i\) from \(A^0\), we thus have \(a_1 \neq a_2\) and \(a_i\) also is the worst Pareto efficient alternative from \(A^0\) for player \(j\). We write \(a^1 := u^{a_1}\) and \(a^2 := u^{a_2}\). We denote by \(U^0 = \text{conv}\{u^a|a \in A^0\}\) the set of all utility pairs resulting from an agreement (where ‘\(\text{conv}\)’ denotes the convex hull) and by \(\partial U^0\) the Pareto boundary of \(U^0\); hence, \(\partial U^0\) is the set of all \(u \in U^0\) for which there does not exist \(v \in U^0\) with \(v_1 \geq u_1\) and \(v_2 \geq u_2\), with at least one of these inequalities being strict. (Note that, since there are no indifferences, weak and strong Pareto efficiency coincide.)

The model reflects a real-world scenario in which the payoffs from all possible agreements are common knowledge, but a player does not know how much the other one dislikes disagreement. Such a situation could, for example, arise when the players know each other very well, but each is lacking information about the other’s outside options.

As \(u^0_i\) (the utility of disagreement) is private information of player \(i\), the possible values of \(u^0_i\) constitute the types of this player. For each player \(i\), let \(T_i\) denote the (finite) set of his possible types and let \(T = T_1 \times T_2\) denote the product type set with generic element \(t = (t_1, t_2)\). Denote \(\bar{t}_i := \max T_i\) and \(\underline{t}_i := \min T_i\). The players have a common prior \(f\) on \(T\), with \(f_i(t_i) = \sum_{t_j \in T_j} f(t_i, t_j) > 0\) for \(i = 1, 2\) and all \(t_i \in T_i\).
We write \( f_i(t_j|t_i) = f(t)/f_i(t_i) \) for the associated conditional probability. We say that \( f \) has full support if \( f(t) > 0 \) for all \( t \in T \). Note that we allow the players’ types to be correlated. We write \( \Gamma = (A, a_0, T_1, T_2, u_1, u_2, f) \) for this Bayesian bargaining problem. As is standard in the literature (Myerson, 1979, 1984), we assume that each player already knows his type when the bargaining starts.

As far as disagreement is concerned, we make two assumptions. First, \( u^j_i < t_i < \bar{t}_i < u^i_i \) for each player \( i \). This implies that player \( j \)'s best outcome is unacceptable to player \( i \) (\( i \neq j \)) so that the players need to compromise. Second, we assume that it is common knowledge that \( a_0 \) is Pareto inefficient, i.e., for every \( t \in T \), there exists a lottery \( \alpha(t) \in \Delta(A^0) \) that Pareto dominates \( a_0 \). Hence, if we write \( U = \text{conv}[U^0 \cup \{t|t \in T\}] \) and \( \partial U \) for its Pareto frontier, we have \( \partial U = \partial U^0 \).

Note that, if \( f \) has full support, it is common knowledge that the lottery \( \alpha(\bar{t}_1, \bar{t}_2) \) is better than disagreement. When \( f \) does not have full support, the set of dominating lotteries can be contingent on \( t \), with no lottery dominating all \( t \in T \), as illustrated by case (b) in Figure 1 below. This figure depicts situations in which the players’ utility functions have been normalized such that \( u^1 = (1,0) \) and \( u^2 = (0,1) \). Such normalization is without loss of generality and, whenever convenient, we will use it. In a normalized problem, \( \partial U \) is a piece-wise linear curve that connects \((0,1)\) to \((1,0)\). Figure 1 shows three cases of our model: (a) a linear frontier with full support; (b) a linear frontier with a triangular support; and (c) a non-linear frontier with full support. The disagreement payoffs are distributed in the dotted regions.

![Figure 1](image_url)

Our model amounts to a minimal change of the canonical (Nash) bargaining game with complete information; the only difference is that in our model the dis-
agreement utilities are private information. Our model is inspired by Börgers and Postl (2009), in which the case with $A = \{a_0, a_1, a_2\}$ and independent, symmetric beliefs is investigated. That paper differs in its assumptions and emphasis. It allows $a_0$ to be efficient (and, hence, to be a real compromise), shows that this implies that no efficient mechanism can be incentive compatible and derives some numerical properties of \textit{ex-ante} efficient incentive compatible rules. Our assumption that $a_0$ is inefficient makes the model more tractable. On the other hand, our model is more general since we allow correlated (and asymmetric) beliefs and take into account individual rationality constraints.

2.1 Mechanisms and desirable properties

Let a Bayesian bargaining game $\Gamma$ be given. Because of the revelation principle (Myerson, 1979), we can restrict ourselves to (incentive compatible) direct mechanisms in which the message spaces are the type spaces. A mechanism $\mu : T \rightarrow \Delta(A)$ assigns to each type profile $t = (t_1, t_2)$ a probability distribution $\mu(t)$ over the set of alternatives. We write $\mu^a$ (resp. $\mu^0$) for the probability that alternative $a$ (resp. $a_0$) is chosen. For each player $i$, the interim expected probability that alternative $a \in A$ is chosen if she has type $t_i$ but reports $\hat{t}_i$ while player $j$ reports honestly is given by

$$\mu_i^a(\hat{t}_i|t_i) = \sum_{t_j \in T_j} \mu^a(\hat{t}_i, t_j)f_i^c(t_j|t_i),$$

and the corresponding interim expected utility is given by

$$U_i^\mu(\hat{t}_i|t_i) = \mu_i^0(\hat{t}_i|t_i)t_i + \sum_{a \in A^0} \mu_i^a(\hat{t}_i|t_i)u_i^a.$$

Mechanism $\mu$ is incentive compatible (IC) if reporting truthfully is a Bayesian Nash equilibrium. If $\mu$ is incentive compatible, we denote the \textit{ex-post} equilibrium utility of player $i$ by $u_i^\mu(t)$; the interim probability of $a \in A$ given $t_i$ and truthtelling by $\mu_i^a(t_i)$, and the interim utility from truthtelling by $U_i^\mu(t_i)$. From now on, when we refer to a mechanism, we always mean that it is physically feasible ($\mu(t) \in \Delta(A)$ for all $t \in T$) and incentive compatible.

We note that, since the players already know their types at the start of the interaction and each one can unilaterally enforce disagreement, only mechanisms
that are individually rational (IR), i.e., that satisfy $U^t_i(t_i) \geq t_i$ for all $t_i \in T_i$ and 
$i = 1, 2$, can actually be implemented. Although IR can, therefore, be seen as 
a necessary requirement for feasibility, some of our results do not depend on this 
assumption and we will mention this property only when we actually need it.

We will mainly be interested in efficient mechanisms. An efficient mechanism 
generates a Pareto efficient outcome with probability 1. Efficiency is a very desirable 
property: if $\mu$ is inefficient, then players know from the start that, if they accept 
$\mu$, with positive probability, they will have an incentive to renegotiate; foreseeing 
this they may reject this mechanism. This argument has special force if $f$ has full 
support, since in that case, the players even know a range of outcomes to which 
they could renegotiate.

**Definition 1.** Mechanism $\mu$ is efficient if $\Pr_f \{ t : u^\mu(t) \in \partial U \} = 1$. It is strongly 
efficient if $u^\mu(t) \in \partial U$ for all $t \in T$.

Our concept of strong efficiency coincides with the classical ex-post efficiency 
concept from Holmström and Myerson (1983). That paper discusses six efficiency 
concepts, which differ depending on whether or not incentive constraints are taken 
into account (classical or incentive efficiency) and on the point in time (ex-ante, 
interim or ex-post) at which the evaluation is taking place. The next Proposition 
links efficiency to ex-ante incentive efficiency; together with Proposition 2, it gives 
the reader a better idea of our model. The proof is given in the Appendix.4

**Proposition 1.** Let $\Gamma$ be a bargaining game with $f$ having full support.

1. If $\partial U$ is linear, an efficient mechanism is ex-ante incentive efficient.
2. If $\partial U$ is non-linear, this need not hold.
3. There always exists a mechanism that is individually rational, efficient and 
ex-ante incentive efficient.

In a bargaining game $\Gamma$ in which $f$ has full support, there exist many individually 
rational and efficient mechanisms; any constant mechanism that Pareto dominates

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4A mechanism $\mu$ is ex-ante incentive efficient if there does not exist another mechanism $\nu$ which, 
when types are not yet known, both players consider at least as good as $\mu$, with at least one player 
viewing it as strictly better. A mechanism $\mu$ is interim incentive efficient if there does not exist 
another mechanism that all types of both players (weakly) prefer to $\mu$, with the preference being 
strict for at least one type. Our proof shows that Proposition 1(2) also holds if ex-ante is replaced 
by interim.
\( a_0 \) satisfies these conditions. On the other hand, if \( f \) does not have full support, no such mechanism may exist.

**Proposition 2** (Existence and multiplicity). Let \( \Gamma \) be a bargaining game.

1. If \( f \) has full support, there exist multiple individually rational and strongly efficient mechanisms.
2. If \( f \) does not have full support, an individually rational and strongly efficient mechanism may not exist.

The proof of the second part of this Proposition is deferred to Section 3, where we will show that it follows almost immediately from Theorem 1. Because of Proposition 2(2), we will focus mainly on the case with full support. Note that, when the prior \( f \) is independent, our assumption that \( f_i(t_i) > 0 \) for all \( t_i \) implies that \( f \) has full support.

In this paper, we investigate whether there exist efficient mechanisms that depend non-trivially on the players’ types, i.e., that take into account some of the private information that the players have. As the game theoretic approach to bargaining is welfaristic (Sen, 1979), real outcomes only matter through their utilities; hence, we focus on the interim utilities associated with a mechanism. We say that a mechanism is *type-dependent* if for at least one of the players, the interim utility function is not constant.

**Definition 2.** Mechanism \( \mu \) is type-dependent if, for at least one player \( i \), the interim utility function \( t_i \rightarrow U_\mu^i(t_i) \) is not a constant function; it is type-independent if both \( U_1^\mu \) and \( U_2^\mu \) are constant functions.

We note that type-independence is a much weaker property than the mechanism prescribing the same lottery for each \( t \). Type-independence relates to the interim stage and there can be much variation ex-post without this influencing a player’s interim utility.\(^5\)

It is easily seen that, if the players’ disagreement payoffs are statistically independent, an efficient mechanism must be type-independent. Efficiency requires that \( \mu^0(t) = 0 \) for all \( t \) and if \( \mu \) prescribes a more attractive lottery on \( \partial U \) for type \( t_i \) than for some other type \( t'_i \), then clearly it is not incentive compatible. Hence, we have the following result, which is independent of the shape of the Pareto frontier,

\(^5\)An explicit example to illustrate this fact is available from the authors upon request.
also holds if there are more than two players, and also holds when \( T_i \) is an interval. This result provides a benchmark; from now on we will assume that \( f \) is not independent.

**Observation 1.** If \( f \) is independent \((f(t) = f_1(t_1)f_2(t_2)\) for all \( t \)), then any efficient mechanism is type-independent.\(^6\)

### 3 The linear case

Theorem 1 below is one of the main results of the paper. It states that the above observation continues to hold when beliefs are correlated, provided that \( f \) has full support and the Pareto frontier \( \partial U \) is linear. Hence, even when type-dependence appears natural and appealing, such as when types are strongly negatively correlated, it is incompatible with efficiency. Although the case of a linear Pareto frontier is somewhat special, it is interesting and relevant. It arises, for example, when there is pure conflict, i.e., there are just three alternatives, \( A = \{a_0, a_1, a_2\} \) with \( a_i \) defined as above. Linearity also results when the players can use monetary transfers to reach a compromise, a case which we will discuss in more detail in Section 5.

**Theorem 1.** If the Pareto frontier is linear and \( f \) has full support, then any efficient mechanism is type-independent.\(^7\)

**Proof of Theorem 1.** Let \( \Gamma \) be a bargaining game as defined in Section 2 and assume \( \partial U \) is linear. Without loss of generality, assume \( \Gamma \) is normalized such that \( u_1 + u_2 = 1 \) for each \( u \in \partial U \). Let \( \mu \) be an efficient mechanism. Given the probability distribution \( f \) on \( T \), define \( G^\mu = \langle T_1, T_2, u^\mu_{1}, u^\mu_{2} \rangle \) as the game in which the set of pure strategies of player \( i \) is \( T_i \) \((i = 1, 2)\) and in which \( i \) gets payoff \( u^\mu_i(t) \) if \( t \in T \) is played. It is easy to see that \( \mu \) is incentive compatible if and only if \( f \) is a correlated equilibrium of \( G^\mu \). Note that, since \( f \) has full support and \( \mu \) is efficient, \( G^\mu \) is a constant-sum game:

\(^6\)Miralles (2012) has derived a somewhat similar result in a different context. He studies the allocation of two (identical) objects to two (ex-ante identical) players when monetary transfers are impossible and each player must get an object. He shows that when the players’ values are independent, one cannot do better than random allocation; hence, the intensity of the players’ preferences cannot play a role.

\(^7\)We conjecture that Theorem 1 continues to hold when each \( T_i \) is an interval in \( \mathbb{R} \) and \( f \) admits a continuous density. As there is only a small literature on correlated equilibria in non-finite games, following Hart and Schmeidler (1989), and there are some subtle issues, we did not pursue this direction.
\( u_i^1(t) + u_i^2(t) = 1 \) for all \( t \in T \). Let \( v_i \in R \) be the value of this game for player \( i \); hence, \( v_1 + v_2 = 1 \). For general (finite) zero-sum games Forges (1990) already observed that the beliefs associated with pure strategies that are played with positive probability in a correlated equilibrium are optimal strategies, i.e., they yield at least the value against any strategy of the opponent. The next lines of our proof build on (or follow from) this insight. Note that, since \( f \) has full support, all strategies \( t_i \) are played with positive probability in the correlated equilibrium \( f \). Furthermore, as a player can always play a maximin strategy (rather than follow a recommendation to play \( t_i \)), we must have \( u_i^\mu(t_i, f_i^c(t_i)) \geq v_i \) for each \( i \) and each \( t_i \). None of these inequalities can be strict, as in that case we would have \( u_i(f) > v_i \) for the ex-ante expected payoff of player \( i \) in \( f \) and, hence, \( u_j(f) < v_j \) for \( j \neq i \), which is impossible since player \( j \) can guarantee himself \( v_j \). Hence, we have \( U_i^\mu(t_i) = u_i^\mu(t_i, f_i^c(t_i)) = v_i \) for all \( i \) and \( t_i \). Consequently, the mechanism \( \mu \) is type-independent. \( \Box \)

Following Crémer and McLean (1988), we say that \( f \) satisfies the full rank condition if, for \( i = 1, 2 \), the set \( \{ f_i^c(t_i) | t_i \in T_i \} \) of posterior beliefs of player \( i \) is linearly independent. Note that this condition is generically satisfied if \( |T_1| = |T_2| \). The next result shows that, if the beliefs \( f \) satisfy this condition and have full support, any efficient mechanism \( \mu \) must yield constant ex-post utility, i.e., \( u_i^\mu(t) = u_i^\mu(t') \) for all \( t, t' \) and \( i \). The intuition behind this result is that, if each \( t_i \) appears with positive probability in the correlated equilibrium, all incentive constraints must hold with equality. Writing \( n = |T_1| = |T_2| \), this gives \( 2n(n - 1) \) equations in \( u^\mu \). On top of this, we have the \( 2n \) interim utility constraints, \( U_i^\mu(t_i) = v_i \), encountered in the proof of Theorem 1, which together yields \( 2n^2 \) equations, just as much as there are variables in \( u^\mu \). The CM-condition guarantees that the equations are independent; hence, there is a unique solution, which must be constant.

**Proposition 3.** If the Pareto frontier is linear, \( f \) has full support and the Crémer-McLean full rank condition holds, then any efficient mechanism \( \mu \) is constant, i.e., if \( t, t' \in T \), then \( u_i^\mu(t) = u_i^\mu(t') \) for \( i = 1, 2 \).

In the remainder of this Section we make four remarks that serve to put Theorem 1 in perspective.

**Remark 1:** \( T \) must be a product set.
The assumption that $T$ is a product set, $T = T_1 \times T_2$ is routinely made in the literature (see e.g., Börgers (2015), p.174). Theorem 1, however, does not hold if this assumption is not satisfied. For example, efficient type-dependent solutions can exist if the disagreement points lie on a curve $C = \{t|t_2 = \varphi(t_1)\}$, i.e., the case of perfect correlation in which Spearman’s rank correlation coefficient $\rho = \pm 1$. A simple example is as follows. Assume that each player can be of 2 types, $l$ or $h$, with $l < h$ and $f(l,h) = f(h,l) = \frac{1}{2}$. Consider the symmetric mechanism $\mu$ that prescribes the lottery $p \ast a_1 + (1 - p) \ast a_2$ (with $p > \frac{1}{2}$) if $(h,l)$ is reported and disagreement if $(l,l)$ or $(h,h)$ is reported. If $h < p < 1 - l$, then $\mu$ is incentive compatible and efficient, with $U^\mu(l) < U^\mu(h)$; hence, $\mu$ is type-dependent. A similar example can be constructed if the disagreement outcomes are known to be very bad, i.e., if for each $t$ with $f(t) > 0$, we have $t_1 + t_2 < \frac{1}{2}$.

The above example relies on the mechanism prescribing an inefficient outcome if the players report a type combination that has prior probability 0. This implies that the game $G^\mu$, defined in the proof of Theorem 1, is no longer constant-sum. If we insist on having efficient outcomes for all type combinations, $G^\mu$ is again zero-sum and the proof remains valid; hence, we have:

**Corollary 1.** If the Pareto frontier is linear, any strongly efficient mechanism is type-independent.

This corollary allows us to prove the following Proposition, which in turn proves the second part of Proposition 2 from Section 2.

**Proposition 4.** If the Pareto frontier $\partial U$ is linear and there does not exist a lottery on $\partial U$ that Pareto dominates all $t$ in the support of $f$, then there does not exist a strongly efficient and individually rational mechanism.

**Proof of Proposition 4.** Let the conditions be satisfied and assume that $\mu$ is a strongly efficient mechanism. Then $\mu$ is type-independent, hence, it produces a constant interim utility pair $u = (u_1, u_2) \in \partial U$. By the conditions of the Proposition, there exist some $t \in T$ that is not Pareto dominated by $u$. Hence, we must have $t_1 \geq u_1$ or $t_2 \geq u_2$. Since $t$ is not Pareto efficient, $t \neq u$; hence, one of these inequalities must be strict. But if $t_i > u_i$, then $\mu$ is not individually rational. \(\square\)

**Remark 2:** The result only holds for two players.
Theorem 1 no longer holds if there are more than two players (unless \( f \) is independent). In the Appendix we construct a symmetric, three-player example with correlated beliefs in which an efficient and type-dependent mechanism exists.

**Remark 3: Epsilon-efficiency**

With a weaker efficiency concept, type-dependent solutions become possible. For example, define mechanism \( \mu \) to be \( \varepsilon \)-efficient if \( \Pr_{f \{t|\mu^0(t) \leq \varepsilon} = 1 \); hence, whatever the state of the world, disagreement occurs at most with probability \( \varepsilon \). It is easily seen that such \( \varepsilon \)-efficiency is compatible with type-dependence. We do not consider \( \varepsilon \)-efficiency to be an attractive concept. In the Introduction and in Section 2 we argued that efficiency is a very desirable property. That argument implies that an \( \varepsilon \)-efficient mechanism is just as unsatisfactory as any other mechanism that is not fully efficient.

**Remark 4: Indirect mechanisms**

Theorem 1 implies that any efficient incentive compatible outcome can be implemented by a constant lottery, the most simple indirect mechanism. Hence, players do not need to reveal any information about their disagreement payoff. In contexts in which such information is sensitive, this can be a desirable property.

### 4 The non-linear case

In this Section, we investigate to what extent Theorem 1 can be generalized if the Pareto frontier is non-linear (i.e., if it is only piece-wise linear). We show that the result continues to hold if at least one player has at most two types. We also provide an example in which each player has three types and in which an efficient, individually rational, type-dependent solution exists.

**Theorem 2.** Suppose the Pareto frontier is non-linear and \( f \) has full support. If \( \min(|T_1|,|T_2|) = 2 \), an efficient mechanism is type-independent. If \( \min(|T_1|,|T_2|) > 2 \), an efficient, individually rational and type-dependent mechanism can exist.

**Proof of Theorem 2.** The proof of the first part of the theorem is given in the Appendix. To prove the second part, we give a symmetric example with \( |T_1| = |T_2| = 3 \). Let \( A = \{a_0, a_1, a_2, a_3\} \) with \( a_0, a_1 \) and \( a_2 \) as in Section 2, let the utility functions be normalized as in Section 2, and assume \( u_i(a_3) = \frac{7}{10} \) for \( i = 1, 2 \). Furthermore, assume that each player has three types \( s_1, s_2 \) and \( w \), with \( u_i^0(t) \leq \frac{1}{2} \) for all \( t \). The
prior probability distribution over the players’ types is given in Table 1 where \( \varepsilon > 0 \) is assumed to be small.

| \( f \) | \( s_1 \) | \( s_2 \) | \( w \) |
|---|---|---|---|
| \( s_1 \) | \( \varepsilon \) | \( \varepsilon \) | \( \frac{1}{4} - 2\varepsilon \) |
| \( s_2 \) | \( \varepsilon \) | \( \varepsilon \) | \( \frac{1}{4} - 2\varepsilon \) |
| \( w \) | \( \frac{1}{4} - 2\varepsilon \) | \( \frac{1}{4} - 2\varepsilon \) | \( 4\varepsilon \) |

Table 1

Note that the situation is fully symmetric and that the players’ beliefs are almost perfectly negatively correlated:

\[
f_i^c(w|s_1) = f_i^c(w|s_2) \approx 1, \quad f_i^c(w|w) \approx 0.
\]

Consider the following symmetric mechanism \( \mu \). If each player reports a type in \( \{s_1, s_2\} \) and the reports differ, then \( a_1 \) is selected; on the other hand, \( a_2 \) is selected if the reports are the same. If player 1 reports a type in \( \{s_1, s_2\} \) and player 2 reports type \( w \), then the lottery \( (\frac{1}{6}, \frac{5}{6}) \) over \( a_1 \) and \( a_3 \) is selected. Symmetry dictates the lottery \( (\frac{1}{6}, \frac{5}{6}) \) over \( a_2 \) and \( a_3 \) if the roles of the players are reversed. Finally, if each player reports \( w \), then \( a_3 \) is selected for sure. Clearly, \( \mu \) is efficient. Then the payoffs associated with this mechanism are as in Table 2:

| \( u^\mu \) | \( s_1 \) | \( s_2 \) | \( w \) |
|---|---|---|---|
| \( s_1 \) | \( (0, 1) \) | \( (1, 0) \) | \( (\frac{3}{4}, \frac{7}{12}) \) |
| \( s_2 \) | \( (1, 0) \) | \( (0, 1) \) | \( (\frac{3}{4}, \frac{7}{12}) \) |
| \( w \) | \( (\frac{7}{12}, \frac{3}{4}) \) | \( (\frac{7}{12}, \frac{3}{4}) \) | \( (\frac{7}{10}, \frac{7}{10}) \) |

Table 2

Assume \( \varepsilon \) to be small. If both players report honestly then \( U_i^\mu(s_1) = U_i^\mu(s_2) \approx \frac{3}{4} \) and \( U_i^\mu(w) \approx \frac{7}{12} \); hence \( \mu \) is type-dependent. Clearly, \( \mu \) is also individually rational. If the other player tells the truth, then reporting \( s_i \) instead of \( s_j \) results in the same payoff, while misreporting as \( w \) yields \( \approx \frac{7}{10} < \frac{3}{4} \); if type \( w \) reports \( s_i \), then his expected payoff is \( \approx \frac{1}{2} < \frac{7}{12} \); hence, \( \mu \) is incentive compatible.

\( \square \)

Note that, in this example, we did not have to specify the disagreement utilities of the types \( s_1, s_2 \) and \( w \); the example works as long as \( t_i = u_i^0(t) \leq \frac{1}{2} \) for all \( i \)
and \( t \). Hence, it is allowed that \( u_0^i(s_1,t_2) = u_0^i(s_2,t_2) \) for all \( t_2 \). In this case, \( s_1 \) and \( s_2 \) only differ in name: these types have the same preferences and beliefs, and they can be thought of as a type \( s \) having been split into two. One of the axioms Harsanyi and Selten (1972) use to characterize their generalized Nash solution is “player splitting”; it insists that such splitting (keeping utilities and beliefs the same) should not change the solution; see also Weidner (1992). Our example shows that “player splitting” may not be as innocent as it first might look: as soon as a player is split into two different decision making agents, there can be coordination problems between them, and this can change the outcome.

5 Transferable utility

In the model from Section 2, the players only have lotteries at their disposal to reach a compromise. In this Section, we investigate the same problem, but now with the assumption that compromises can also be reached by making monetary transfers. Specifically, as in Myerson and Satterthwaite (1983) we assume that the individuals are risk neutral and have additively separable utility for money and the alternatives from \( A \). Hence, we assume that each player \( i \)’s utility function is given by \( u_i(a,m_i) = v_i(a) + m_i \), where \( v_i(a) \) denotes the VNM-utility that player \( i \) attaches to the alternative \( a \in A \) and \( m_i \) is the amount of money that player \( i \) holds \textit{ex-post}. In this specification, utility is transferable: if player \( i \) gives one unit of money to player \( j \), \( u_i \) decreases with one unit while \( u_j \) increases by 1. Note that the assumption that money enters each player’s utility function with coefficient 1 implies that utility becomes interpersonally comparable. More precisely, utility gains (and losses) can be compared as each such difference is equivalent to a certain amount of money, which both players value in the same way.

In the transferable utility literature, it is usually assumed that constraints on money holdings are not binding\(^8\) which, among others, implies that Pareto efficiency

\(^8\)For example, Myerson (1991, p. 422) defines transferable utility as there being “a commodity-called money-that players can \textit{freely} transfer among themselves, such that any player’s utility payoff increases by one unit for every unit of money that he gets.” (Emphasis added.) In cooperative game theory, TU is simply defined as, for each coalition, there being a surplus that can be arbitrarily divided, while in the matching literature it has been directly defined as the Pareto frontier being a straight line with slope -1 (Chiappori, 2017, p.6, 27). The mechanism design literature usually also assumes that transfers can be unbounded, although it has been noted that this may influence the results (Crémé and McLean, 1988, p. 1255).
is equivalent to maximizing the sum of the players’ utilities, \( u_1 + u_2 \). In this case, the Pareto frontier is linear and Theorem 1 applies. In this Section, we investigate the more general case in which budgets can be limited. This not only is more realistic, it is also interesting since the Pareto frontier then can be non-linear. We will see that a result as in Theorem 1 holds given some conditions on the players’ budgets, but certainly not in general.

5.1 Model

We first describe how the model from Section 2 is changed to fit the TU-context. Let \( A, a_0, T \) and \( f \) be as in Section 2. For each player \( i \), let \( u_i \) be as above and assume that this player has an initial endowment (budget) \( b_i \geq 0 \) of money. A bargaining problem with transferable utility is a tuple \( \Gamma = (A, a_0, T, f, v, b) \). Note that when player \( i \) has budget \( b_i \), the utility that his type \( t_i \) assigns to disagreement is \( t_i + b_i \). An (ex-post) allocation for \( \Gamma \) is a triple \((a, m_1, m_2)\) with \( a \in A, m_i \geq 0 \) (for \( i = 1,2 \)) and \( m_1 + m_2 = b_1 + b_2 \). As an allocation is fully determined by the pair \((a, m_1)\), we simplify notation by writing \((a, m_1)\) instead of \((a, m_1, m_2)\). We write \( u(a, m_1) \) for the utility pair associated with the allocation \((a, m_1)\); hence \( u(a, m_1) = (v_1(a) + m_1, v_2(a) + b_1 + b_2 - m_1) \). We say an allocation is utilitarian if it maximizes the sum of the players’ utilities, while \( a \in A \) is called utilitarian if \((a, b_1)\) is utilitarian.

The players’ ideal points \( a_1 \) and \( a_2 \) are defined as in Section 2, and we continue to assume that \( t_i > v_i(a_j) \) for each player \( i \) and \( j \neq i \) (so that player \( i \) will accept \( a_j \) only when he gets sufficient financial compensation). Given that utility now is interpersonally comparable, we can no longer normalize \( v_1 \) and \( v_2 \) as in Section 2, as this would imply that \( a_1 \) and \( a_2 \) are equally good from the social point of view. However, without loss of generality, we can still normalize \( v_1 \) and \( v_2 \) such that \( v_1(a_2) = v_2(a_1) = 0 \) and \( v_1(a_1) = 1 \). The former just pins down the level of zero utility while the latter simply amounts to choosing a convenient unit of account. With respect to disagreement, we assume that there exists \( a^* \in A \) such that

\[
t_1 + t_2 < v_1(a^*) + v_2(a^*) \quad \text{for all } t \in T.
\]

Note that this assumption is slightly weaker than the one we made in Section 2, as this inequality still allows \( t = (t_1, t_2) \) to be Pareto efficient when transfers are
Clearly, all allocations with \( a = a^* \) are Pareto efficient. On top of this, there can be other efficient allocations, which are not utilitarian. For example, if \( v_1(a_2) + v_2(a_2) < v_1(a_1) + v_2(a_1) \), the allocation \((a_2, 0)\) still is Pareto efficient: it is the best possible outcome for player 2 and player 1 does not have any money to induce player 2 to accept \( a_1 \). In this case, if \( A = \{a_0, a_1, a_2\} \) and \( a_1 \) is utilitarian, all lotteries on the set \( \{(a_1, 0), (a_2, 0)\} \) are Pareto efficient. Below, we write \( \partial U(b) \) for the Pareto frontier generated by \( \Gamma \), \( U(b) \) for the utilitarian part of the frontier and \( NU(b) \) for the non-utilitarian part.

To cover bilateral trade as in Myerson and Satterthwaite (1983), we also allow \( |A| = 2 \); hence \( A = \{a_0, a^*\} \), with \( a^* \) as in the inequality above. In this case, without loss of generality, we can assume that player 1 prefers \( a^* \) to \( a_0 \), which implies that \( a^* = a_1 \). To see that this case indeed covers bilateral trade, view player 1 as the buyer and player 2 as the seller, and interpret \( a_0 \) and \( a^* \) as no trade and trade, respectively. Then the only difference with Myerson and Satterthwaite (1983) is that we assume it is common knowledge that trade is necessary to maximize surplus. Note that, in this case, the allocation \((a_0, 0)\) in which the seller keeps the object and gets all the money, although not utilitarian, is still Pareto efficient: \( U(b) \) is the line segment between \( u(a^*, 0) \) and \( u(a^*, b_1 + b_2) \), while \( NU(b) \) is the line segment between \( u(a_0, 0) \) and \( u(a^*, 0) \).

In the remainder of this Section, we distinguish two cases: (1) At least one of the \( a_i \) is utilitarian (which, without loss of generality we assume to be \( a_1 \)); (2) None of the \( a_i \) is utilitarian. We will see that, in the first case, a result as in Theorem 1 continues to hold provided that player 1 has high enough budget as compared to player 2, and that the second case is very much like the non-linear case discussed in Section 4.

### 5.2 At least one ideal point is utilitarian

Let \( \Gamma = (A, a_0, T, f, v, b) \) be a bargaining problem with transferable utility and, without loss of generality, assume that \( a_1 \) is utilitarian. If \( a_2 \) is also utilitarian, then the Pareto frontier is linear and Theorem 1 applies. We will refer to this case as a surplus division problem. There are two other cases: \( |A| \geq 3 \) and \( a_2 \) is not utilitarian, or \( |A| = 2 \) and \( A = \{a_0, a_1\} \) (bilateral trade). In both of these cases, Theorem 1 cannot be directly applied as the Pareto frontier is non-linear. Nevertheless, we will
show that, if the budget of player 1 is sufficiently large relative to that of player 2, an efficient and individually rational mechanism must exclusively select utilitarian allocations. This implies that the proof of Theorem 1 can be applied and that the result continues to hold. Hence, we have the following theorem.

**Theorem 3.** Let $\Gamma$ be a bargaining problem with transferable utility and a prior $f$ with full support.

(1) If both $a_1$ and $a_2$ are utilitarian (hence, $\Gamma$ is a surplus division problem), then any efficient mechanism is type-independent.

(2) If only $a_1$ is utilitarian, then, if $b_1 > \frac{v_1(a_1) + b_2}{\min_{t_2} f_1^c(t_2|t_1)} - b_2$, any individually rational and efficient mechanism is type-independent.

**Proof of Theorem 3.** It suffices to prove the second statement. First assume $|A| \geq 3$ and $a_2$ is not utilitarian. Let $a^m$ be the alternative from $A$ that is the least preferred by player 1 from the set of utilitarian alternatives. The utilitarian part of the frontier, $U(b)$, then is the line segment between $u(a^m, 0)$ and $u(a_1, b_1 + b_2)$. Furthermore, it is easily seen that if $a$ is not utilitarian, $(a, m_1)$ can only be Pareto efficient if $m_1 = 0$. Let $\mu$ be an efficient mechanism and suppose that there exists some $t = (t_1, t_2)$ for which $\mu$ selects a point from $NU(b)$ with positive probability. For this specific $t_1$, write $\Pi$ for the set of $t_2$ for which $\mu$ selects a non-utilitarian lottery. Then $\Pi \neq \emptyset$ and for all such lotteries we have that the expected payoff of player 1 is less than $u_1(a^m, 0) = v_1(a^m)$. Simplifying notation by writing $p(t_2) = f_1^c(t_2|t_1)$ and $\pi = \sum_{t_2 \in \Pi} p(t_2)$, we have:

$$U_1(t_1) < \sum_{t_2 \in \Pi} p(t_2)v_1(a^m) + \sum_{t_2 \in T_2 \setminus \Pi} p(t_2)(v_1(a_1) + b_1 + b_2)$$

$$= \pi v_1(a^m) + (1 - \pi)(v_1(a_1) + b_1 + b_2)$$

$$\leq v_1(a_1) + (1 - \pi)(b_1 + b_2).$$

For $\mu$ to be individually rational, we must have $U_1^\mu(t_1) \geq t_1 + b_1 \geq b_1$; hence

$$v_1(a_1) + (1 - \pi)(b_1 + b_2) \geq b_1,$$

or

$$v_1(a_1) + b_2 \geq \pi(b_1 + b_2).$$
Since $\pi \geq \min_{t_2} f^*_1(t_2|t_1) \geq \min_{t} f^*_1(t_2|t_1)$, this condition can only be satisfied if

$$\min_{t} f^*_1(t_2|t_1)(b_1 + b_2) \leq v_1(a_1) + b_2 \text{ or } b_1 \leq \frac{v_1(a_1) + b_2}{\min_{t} f^*_1(t_2|t_1)} - b_2.$$  

But the condition from the Theorem states that this inequality does not hold. This implies that, under the conditions of the Theorem, a mechanism that is efficient and individually rational must select utilitarian allocations with probability 1. But then the assumption that $f$ has full support together with the argument from the proof of Theorem 1 implies that $\mu$ must be type-independent.

Next, assume $|A| = 2$ and $A = \{a_0, a_1\}$. Then $a_1$ is the unique utilitarian alternative, hence, $a^m = a_1$ and the proof proceeds exactly as in the case $|A| \geq 3$. \Box

As mentioned, case (2) of this theorem covers situations of bilateral trade in which it is common knowledge that maximizing surplus requires trade. For this case, the theorem says that, if the buyer’s budget is sufficiently large compared to the seller’s, efficiency and individual rationality require trade with probability 1, and a constant payment from the buyer to the seller. It is interesting that the lower bound on the buyer’s budget depends on her beliefs: if she attaches small probability to some seller types, this bound may be quite large.

The reader may be surprised about the large bound on $b_1$, and also that $b_1$ must be large relative to $b_2$. Isn’t it possible to specify a simple bound for $b_1$ that is independent of $b_2$? For example, in a normalized problem (with $v_i(a) \in [0, 1]$ for all $i$ and $a$) isn’t it sufficient that $b_1 \geq 1$? The answer is “no” as the following example illustrates. This example modifies the one with almost perfectly negatively correlated beliefs constructed in Theorem 2 to fit the current context. The mechanism $\mu$ selects the non-linear part of the frontier (specifically $(a_2, 0)$) only when both players are of type $s$ and miscoordinate. As this happens with small probability, the selected outcome usually is on the linear part of the frontier, and by choosing the transfer payments appropriately, we can make sure that all incentive constraints and all individual rationality constraints are satisfied.

**Example 1.** If $b_2 = 0$ and $b_1 = 1$, there exists a bargaining problem with transferable
utility $\Gamma$ for which there exists an individually rational, efficient and type-dependent mechanism.

To construct an example, we add monetary transfers to the game used in the proof of Theorem 2. Specifically, let $\Gamma$ be given by: $A = \{a_0, a_1, a_2\}$ with $v_1(a_1) = 1$, $v_2(a_2) = V < 1$, $v_1(a_2) = v_2(a_1) = 0$; $T_i = \{s_1, s_2, w\}$ and $f$ as in Table 1, where $\varepsilon > 0$ is assumed to be small; $b_2 = 0$ and $b_1 = 1$. We claim that, for appropriate values of the transfers $x$ and $y$ and disagreement utilities $u_i(s_1) = u_i(s_2) = s > u_i(w) = w$, the mechanism $\mu$ described in Table 3 is individually rational, efficient and type-dependent.

| $\mu$ | $s_1$ | $s_2$ | $w$ |
|-------|-------|-------|-----|
| $s_1$ | $(a_2, 0)$ | $(a_1, 1)$ | $(a_1, x)$ |
| $s_2$ | $(a_1, 1)$ | $(a_2, 0)$ | $(a_1, x)$ |
| $w$   | $(a_1, y)$ | $(a_1, y)$ | $(a_1, \frac{x+y}{2})$ |

Table 3

It is easy to see that $\mu$ is efficient and type-dependent; hence, we only have to check the incentive constraints and the individual rationality constraints. Note that for $\varepsilon$ small, we have $U_1^\mu(s) \approx 1+x$, $U_1^\mu(w) \approx 1+y$, $U_2^\mu(s) \approx 1-y$ and $U_2^\mu(w) \approx 1-x$. Hence, the IR constraints are satisfied for player 1 if $x, y > s$. They are satisfied for player 2 if $x, y < 1-s$. These conditions hold for intermediate values of $x$ and $y$. If type $s_k$ of player 1 reports $w$, he expects $\approx 1+y$; hence, he does not deviate if $x > y$. His type $w$ will not deviate if $y > 0$. Finally, a type $s_k$ of player 2 will not deviate if $y < x$, while type $w$ of this player will not deviate if $1-x > \frac{V+1}{2}$. It is easy to find values of the parameters for which the system has a solution. For example, if $V = 0.4$, $s = 0.2$ and $w = 0.1$, then any $x, y$ with $0.2 < x < 0.3$ and $0.2 < y < x$ satisfies all inequalities.

5.3 None of the ideal points is utilitarian

We now consider the case in which neither $a_1$ or $a_2$ is utilitarian. Let $a^*$ be a utilitarian alternative; hence, $a^* \neq a_1, a_2$. In this case, the Pareto frontier contains at least three linear segments, one with slope $-1$, another with slope $< -1$ and one with slope $> -1$. The condition from Theorem 3 already suggests that this case is qualitatively different from the one in the previous subsection. On the one hand, we must have $b_1 > b_2$, to ensure that non-utilitarian Pareto allocations between
$a_2$ and $a^*$ are not selected, but a symmetric argument requires $b_2 > b_1$. Clearly, these conditions cannot be satisfied simultaneously. We can use the example from the proof of Theorem 2 (Section 4) to construct an efficient, individually rational and type-dependent mechanism: for arbitrary $(b_1, b_2)$, change the mechanism $\mu$ from that example by never letting players exchange any money. Formally, consider the mechanism $\nu$ defined by $\nu_1(t) = (\mu_1(t), b_1)$ and $\nu_2(t) = (\mu_2(t), b_2)$; then $\nu$ trivially satisfies the three desired properties.

**Corollary 2.** Let $\Gamma$ be a bargaining problem with transferable utility in which neither $a_1$ nor $a_2$ is utilitarian. For every $b_1, b_2 \geq 0$ and $f$ with full support, there can exist a mechanism that is efficient, individually rational and type-dependent.

## 6 Applications to Bargaining Theory

In this Section, we discuss three implications of our results for the theory of bargaining under incomplete information. In our first application, we consider non-transferable utility and two players with equal bargaining power. In the second application, utility is transferable and, as in the mechanism design literature, one of the players (the informed principal) has all the bargaining power so that she can dictate the mechanism. We show that in both cases, *ex-post* efficiency results in a very small class of implementable bargaining solutions. In the third application, rather than focusing on efficiency, we put emphasis on fairness, and on proportional solutions (for the NTU-case) and the egalitarian solution (in the TU-case). Among others, we show that, if the beliefs are independent, an *interim* egalitarian solution almost always selects disagreement. This is in sharp contrast to the case with complete information in which there always is an outcome that is efficient (all surplus is divided) as well as egalitarian (the players get an equal share of the surplus).

### 6.1 Symmetric players

Thus far, we focused on a fixed bargaining problem: all elements of $\Gamma$ were assumed given. The axiomatic approach to bargaining theory derives its strength from insisting that the solutions of different problems should be related to (or consistent with) each other. In such axiomatizations, the symmetry axiom plays an important role. For example, in Nash’s (1950) axiomatization of his complete information solution, the Independence of Irrelevant Alternatives Axiom allows one to transform any
problem in an equivalent linear and symmetric one. The symmetry axiom, which merely states that it does not matter which player is called player 1, is undisputed and can be easily extended to games with incomplete information. We introduce the following notion of symmetry.

**Definition 3.** A bargaining problem $\Gamma$ (as in Section 2) is symmetric if $\{u(a) | a \in A^0\}$ is symmetric, $T_1 = T_2$ and $f$ is symmetric, $f(t_1, t_2) = f(t_2, t_1)$ for all $t_1, t_2 \in T_1$. If $\Gamma$ is symmetric, then a mechanism $\mu$ for $\Gamma$ is symmetric if $u^\mu_1(t_1, t_2) = u^\mu_2(t_2, t_1)$ for all $t_1, t_2 \in T_1$.

The symmetry axiom requires that the solution $\mu(\Gamma)$ of a symmetric bargaining problem $\Gamma$ is symmetric (see for example, Harsanyi and Selten, 1972; Weidner, 1992).

Suppose that two players have equal bargaining power and that $\Gamma$ is a symmetric bargaining problem with $f$ having full support. Theorem 1 implies that, if $\partial U$ is linear, symmetry and efficiency lead to an (essentially) unique solution. For example, if $A = \{a_0, a_1, a_2\}$, the solution must be $\mu(t) = \frac{1}{2}a_1 + \frac{1}{2}a_2$ for all $t$. (Note that this symmetric solution is individually rational). Since symmetric problems play a crucial role in axiomatizations, we believe that Corollary 3 will also prove to be relevant for asymmetric problems with incomplete information.

**Corollary 3.** Let $\Gamma$ be a symmetric, linear bargaining problem with $f$ having full support. If $\mu$ is a symmetric and efficient mechanism, then $\mu$ is interim equivalent to a 50/50 lottery over the players’ most preferred alternatives.

### 6.2 Monopoly and surplus extraction

Let $\Gamma$ be a surplus division problem with incomplete information ($|T_1| > 1$ or $|T_2| > 1$). Suppose player 1 has all the bargaining power and can dictate the mechanism; hence, she is an informed principal. As in Crémer and McLean (1988) and Severinov (2008), say that player 1 can extract all surplus if there exists a mechanism that is efficient and that always gives player 2 his disagreement payoff.

It has been shown that, if types are independent, an informed principal cannot benefit from her private information at all; hence, in this situation, she cannot extract full surplus (Mylovanov and Tröger, 2014). In contrast, Severinov (2008) has shown that the informed principal can generically extract all surplus if there are at least three players and the players’ types are correlated. Specifically, full extraction
of the surplus is possible if two conditions are satisfied: the convex independence condition (CI) and the identifiability condition (KS). When there are only two players, these conditions cannot be satisfied simultaneously: CI cannot hold if the types are independent, while KS can only hold in that case (Kosenok and Severinov, 2008, p. 135). The literature has left open the question of what an informed principal can achieve in the 2-person case. Corollary 4 fills this gap, it shows that, generically, not all surplus can be extracted, not even if types are strongly correlated. In other words, the result of Severinov (2008) is sharp.

**Corollary 4.** Let $\Gamma$ be a surplus division problem in which player 1 has all the bargaining power and $|T_2| > 1$. If $f$ has full support, then player 1 cannot extract all surplus.

The result is easy to see. If $\mu$ is a mechanism that allows player 1 to extract all surplus, then player 2’s interim utility must be constant because of Theorem 1, while on the other hand it should be given by $U^\mu_2(t_2) = t_2 + b_2$. If $|T_2| > 1$, these conditions cannot be satisfied simultaneously.

### 6.3 Egalitarianism implies inefficiency

Throughout this paper, we focused on efficiency and did not discuss the second important dimension of bargaining: the requirement that the outcome be fair. Of course, fairness can be defined in various ways; however, when there is transferable utility, equal division of the gains from cooperation is unambiguous and natural. In the case of complete information, such equal division $(u_1 - t_1 = u_2 - t_2)$ of the maximal surplus $(u_1 + u_2 - t_1 - t_2)$ is known as the egalitarian solution. If information is complete, there always is a unique egalitarian outcome. When the disagreement outcome is private information, it is natural to look at equality at the *interim* stage. If the players’ budgets are $b_1$ and $b_2$, this amounts to the requirement $U^\mu_1(t_1) - t_1 - b_1 = U^\mu_2(t_2) - t_2 - b_2$ for all $t_1$ and $t_2$. This equality is trivially satisfied

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9Condition CI (convex independence) was introduced in Crémer and McLean (1988). It requires that, for $i = 1, 2$, there does not exist $t_i'$ such that $f_i'(t_i')$ belongs to the convex hull of the set $\{f_i(t_i) : t_i \in T_i \setminus \{t_i'\}\}$, where $f_i(t_i)$ denotes the conditional distribution of $f$ on $T_i$ given $t_i$.

The identifiability condition (KS) was introduced in Kosenok and Severinov (2008). A distribution $f$ satisfies KS if, for all full support distributions $g \neq f$, there is at least one $i$ and one $t_i' \in T_i$ such that $g_i'(t_i')$ does not belong to the convex hull of the set $\{f_i(t_i) : t_i \in T_i \setminus \{t_i'\}\}$. Severinov (2008) focus on the case with more than 2 players. In the 2-person case, one of their requirements is equivalent to $f$ having full support.
by the mechanism that always selects $a_0$, which also satisfies IR. However, since egalitarianism requires type-dependence (in fact, type-monotonicity), Theorem 1 immediately implies that egalitarian mechanisms must be inefficient: in surplus division problems with incomplete information and $f$ having full support, efficiency requires $U_i^\mu(t_i)$ to be independent of $t_i$. When the prior $f$ is independent, egalitarianism leads to a very inefficient outcome, as the next Proposition shows: The outcome will be disagreement, except possibly in the lowest state $\underline{t} = (\underline{t}_1, \underline{t}_2)$; when there is no budget to redistribute ($b_1 = b_2 = 0$), egalitarianism even leads to disagreement with probability 1.

**Proposition 5.** Let $\Gamma$ be a surplus division problem with budgets $b_1$ and $b_2$ and incomplete information ($|T_1| > 1$ or $|T_2| > 1$).

1. If the prior $f$ has full support, there does not exist an efficient and egalitarian mechanism.

2. If $f$ is independent and $b_1, b_2 \geq 0$, then an egalitarian mechanism $\mu$ selects the disagreement outcome for all $t \neq \underline{t}$.

3. If in (2) $b_1 = b_2 = 0$, then $U_i^\mu(t_i) = t_i$ for all $t_i \in T_i$ and $i = 1, 2$.\(^{10}\)

**Remark 5: The value of money.** We note that in case (2) of the Proposition, although $\mu^0(t) = 1$ for all $t \neq \underline{t}$ (hence, there is disagreement, unless each player happens to be of the lowest type), we can still have $U_i^\mu(t_i) > t_i + b_i$ for all $t_i$ and $i = 1, 2$, provided that $b_1, b_2 > 0$. The reason is that, in an egalitarian mechanism $\mu$, we can have $\mu^0(t) < 1$, so that the lowest types, when they meet, can possibly generate a surplus. Of course, if only these types would get more than their disagreement payoff, $\mu$ would not be egalitarian; however, if each player has some money, then this can be used to share this surplus with the other types, so as to get an egalitarian outcome. If $b_1 = b_2 = 0$, such redistribution is not possible, and for this reason we get a stronger result in case (3). The following is an explicit (symmetric) example. Let each player have two types, $w = 0.1$ and $s = 0.4$ which are equally likely. Assume that $A$ contains an agreement $\hat{a}$ that gives each player utility 0.3 and that each player has a budget $b \geq 0.1$. Let $\mu$ be given by $\mu(w, w) = \hat{a}$ and disagreement otherwise;\(^{10}\)

\(^{10}\)de Clippel (2010) has already shown that, for social choice problems with incomplete information and with transfers, if types are independent, then interim incentive efficiency and monotonicity (as in Kalai (1977)) are incompatible. We obtain a similar result, using a different monotonicity concept.
furthermore, there are only transfers when the players have different types, in which case \( w \) gives 0.1 to \( s \). This mechanism is egalitarian, satisfies IR and IC, and yields utilities \( U^\mu_i(w) = 0.15 + b \) and \( U^\mu_i(s) = 0.45 + b \).

The egalitarian solution insists that the two players gain equally from cooperation, hence, it is applicable only in situations in which the players’ utility levels can be measured on a common scale, i.e., if utility differences are interpersonally comparable. This implies that the egalitarian solution cannot be applied in the model of Section 2 as there we assumed that the two scales can be varied independently. In that context, proportional solutions are still meaningful, see (Kalai, 1977, Myerson, 1977). For the model of Section 2, we define a mechanism \( \mu \) to be \((\text{interim})\) proportional, if there exists a pair of weights \( \lambda_1, \lambda_2 > 0 \) such that \( \lambda_1(U^\mu_i(t_1) - t_1) = \lambda_2(U^\mu_j(t_2) - t_2) \) for all \( t_1 \) and \( t_2 \). If there is incomplete information, i.e., \( |T_1| > 1 \) or \( |T_2| > 1 \), the Pareto frontier is linear and \( f \) has full support, Theorem 1 implies that, if \( \mu \) is efficient, \( U^\mu_i(t_i) \) does not depend on \( t_i \). Hence, we have a similar impossibility result as in Proposition 5(1).

**Corollary 5.** If \( \Gamma \) is a bargaining game with \( |T_1| > 1 \) or \( |T_2| > 1 \), a linear Pareto frontier \( \partial U \) and a prior \( f \) with full support, there does not exist an efficient and proportional mechanism.

7 Conclusion

We made a minimal change to the canonical two-person (Nash) bargaining model by allowing the players’ disagreement payoffs to be private information, while maintaining the assumption that it is common knowledge that disagreement is inefficient. In the resulting model, multiple \( \text{ex-post} \) efficient (and individually rational) mechanisms exist, provided that the players’ prior over the possible disagreement points (type-pairs) has full support. In Nash’s complete information model, all standard solution concepts insist on efficiency and satisfy Disagreement Point Monotonicity (DPM): if one player’s disagreement utility increases, then this player’s bargaining outcome (weakly) improves. We investigated whether, if the disagreement payoffs are private information, efficiency is compatible with a weaker version of DPM, type-dependence, and showed that, when the Pareto frontier is linear or the players’ disagreement payoffs are independent, these properties cannot be satisfied simultaneously. This holds both in a non-transferable utility context as well as in a TU-
context. When the Pareto frontier is non-linear, however, type-dependent solutions become possible.

We attach more weight to our impossibility results for the linear case than to the possibility results of the non-linear one. The reason is that linear problems play an important role in axiomatizations of bargaining solutions. For example, in the complete information case, Nash’s IIA axiom allows to transform a general problem into a linear one. It seems likely that linear problems will also play an important role in axiomatizations when there is incomplete information.

The incompatibility of the two properties forces one to make a choice between efficiency and type-dependence: which of these principles should an impartial mediator use? The answer can depend on the context and on which commitment possibilities the players (and the mediator) have. In our view, when renegotiation is possible, \textit{ex-post} efficiency is indispensable as an axiom: rational players will not agree to a mechanism of which they know \textit{ex-ante} that they might renegotiate away from its recommendation once it has been received. Hence, the bargaining solutions proposed by Harsanyi-Selten and Myerson, which generally select inefficient outcomes, do not seem viable in this context. On the other hand, as argued in Holmström and Myerson (1983) and Myerson (1984), the \textit{ex-ante} utilitarian solution is not appealing either since it does not adequately deal with intra-player fairness issues.¹¹ Since these are, essentially, the only general solutions that have been proposed in the literature on bargaining games with incomplete information, we can conclude that, at this stage, we do not yet have a good solution concept for bargaining problems with incomplete information in which the players have limited commitment possibilities.

In the applied literature (for example, Fisher and Ury, 1981) it has been stressed that a bargainer foremost should aim to improve his disagreement outcome as this yields a better final outcome. This is a direct application of Disagreement Point Monotonicity. Although, in our model, the set of disagreement payoffs is exogenous and not the subject of choice and, hence, the model cannot directly be used to evaluate this recommendation when there is incomplete information, our paper sug-

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¹¹Also see Kim (2017). This paper uses Myerson’s neutral solution to study the question which type of mediator two bargainers will choose in a situation of incomplete information. The model is symmetric, with each player being either $s$ or $w$, and these types having different preferences. The paper shows that, to avoid information leakage, each type will propose the mediator that is preferred by type $s$, but this mediator is \textit{ex-ante} inefficient.
gests that this recommendation is incomplete: when player $i$’s disagreement payoff is private information, the bargaining outcome may be independent of how good or how bad it is.

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Appendix: Proofs not given in the main text

Proof of Proposition 1. Consider a normalized bargaining problem $\Gamma$. (1) If $\partial U$ is linear and $\mu$ is an efficient mechanism, then, since $f$ has full support, $u^i(t) + u^j(t) = 1$ for all $t \in T$. As there cannot exist a mechanism with a total *ex-ante* sum of payoffs larger than 1, $\mu$ is *ex-ante* incentive-efficient.

(2) Assume $\Gamma$ has as set of alternatives $A = \{a_0, a_1, a_2, a_3\}$ with $u_i(a_3) = 0.7$, for $i = 1, 2$. Assume that each player has two types, $s$ and $w$ that are equally likely and independent. Let $\mu$ be the mechanism that selects $a_1$ when the players report the same type and $a_2$ when the players report different types. Then $\mu$ is efficient, with *ex-ante* utility 0.5 for each player. Hence, $\mu$ is *ex-ante* dominated by the mechanism $\nu$ that always selects $a_3$. (Note that $\nu$ also dominates $\mu$ at the *interim* stage.)

(3) Our assumptions imply that the intersection of $\partial U$ and the individually rational region for $\bar{t}$, denoted by $\partial U^*$, is non-empty. Pick any point $u \in \partial U^*$. Since $\partial U$ is the boundary of the convex set $U$, there exists a supporting hyperplane at $u$; hence, there exists $\lambda = (\lambda_1, \lambda_2) \in R^2_+$ such that $\lambda \cdot u \geq \lambda \cdot v$ for all $v \in U$. Let $L \in \Delta(A^0)$ be a lottery that generates the payoff $u$, and let $\mu$ be the constant mechanism given by $\mu(t) = L$ for all $t$. Then $\mu$ is efficient and individually rational. Furthermore, if $\nu$ is a mechanism with *ex-ante* payoff $U^\nu$, then $\lambda \cdot u \geq \lambda \cdot U^\nu$; hence, $\mu$ is *ex-ante* efficient.

Proof of Proposition 2. (1) Let $f$ have full support and write $\bar{t} = (\bar{t}_1, \bar{t}_2)$. The situation is as in Figure 1(c). The lottery $\alpha(\bar{t})$ on $A^0$ Pareto dominates $t$ for all $t \in T$ and the same holds for all lotteries that are close to $\alpha(\bar{t})$, and for all efficient lotteries that Pareto dominate these (if they exist). Any mechanism $\mu$ that always selects the same efficient lottery from this (infinite) set satisfies the conditions from the Proposition.

(2) As Figure 1(b) shows, if $f$ does not have full support, there may not be a lottery that Pareto dominates all $t \in T$. Proposition 4 in Section 3 shows that in such a case an individually rational and strongly efficient mechanism may not exist.

Proof of Observation 1. Suppose $\mu$ is efficient. For a given strategy of player $j$ and each message $t_i$ of player $i$, $\mu$ induces a lottery $\mu(t_i) \in \Delta(A^0)$. All types of player $i$ have the same preference relation over this set of lotteries, hence IC implies that $U^\mu_i(t_i)$ must be independent of $t_i$. Hence, $\mu$ is type-independent.
Proof of Proposition 3. The proof of Theorem 1 established that \( u^c_i(t_i, f^c_i(t_i)) = v_i \) for all \( i \) and \( t_i \). In fact, since each \( t_i \) appears with positive probability (since \( f \) has full support) in the correlated equilibrium, all such incentive inequalities must hold with equality. If \( v_i > u^c_i(t'_i, f^c_i(t_i)) \) for at least one \( t'_i \neq t_i \), then \( f^c_i(t_i) \) yields player \( j \) more than her value \( v_j \), contradiction.

The CM-condition implies that both players have the same number of types. Denote this number by \( n \). First focus on player 1. For \( t_1 \in T_1 \), write \( f^c_1(t_1) \) for the vector of conditional probabilities \( f^c_1(t_2|t_1)_{t_2 \in T_2} \), and denote by \( u^\mu_1(t_1) \) the vector \( u^\mu_1(t_1, t_2)_{t_2 \in T_2} \) of payoffs of player 1. Note that with this notation, we have the incentive inequalities are given by

\[
\begin{align*}
f^c_1(t_1) \cdot u^\mu_1(t_1) &= v_1, & \text{for all } t_1, \quad (3) \\
f^c_1(t_1) \cdot u^\mu_1(t_1) - f^c_1(t_1) \cdot u^\mu_1(t'_1) &= 0, & \text{for all } t_1 \text{ and } t'_1 \neq t_1. \quad (4)
\end{align*}
\]

The system (3) and (4) has \( n^2 \) equations and \( n^2 \) variables. This system can be written as

\[
f^c_1(t_1) \cdot u^\mu_1(t'_1) = v_1, \text{ for all } t_1 \text{ and } t'_1 \neq t_1. \quad (5)
\]

The constraint matrix of (5) is given by

\[
M = \begin{bmatrix}
f^c_1(t_1^1) & 0 & \cdots & 0 \\
0 & f^c_1(t_1^1) & \cdots & 0 \\
f^c_1(t_1^n) & 0 & \cdots & 0 \\
0 & f^c_1(t_1^n)
\end{bmatrix}
\]

where the elements in \( T_1 \) are indexed by \( t_1^1, \ldots, t_1^n \) and \( 0 \) denotes the row vector with \( n(n-1) \) zeros. Notice that \( M \) is an \( n^2 \times n^2 \) matrix, and that \( f \) having full rank implies that \( M \) has linearly independent rows and hence full rank. Consequently, by the invertible matrix theorem, system (5) has a unique solution given by \( u^\mu_1(t) = v_1 \) for all \( t \). Similarly we can solve \( u^\mu_2(t) = v_2 \) for all \( t \). It is easily seen that the set of mechanisms that implement this utility vector is non-empty. \( \square \)
Proof of Remark 2: A three-player example. Let \( A = \{a_0, a_1, a_2, a_3\} \) where each player \( i \in \{1, 2, 3\} \) prefers \( a_i \) most and \( a_j \) least (for all \( j \neq i \)), i.e., \( u_i(a_i) = 1 \) and \( u_i(a_j) = 0 \). Each player has two types, 1 and 2, with disagreement values \( u^1 = 0 \) and \( 0 < u^2 < 1/3 \). Furthermore, for \( t = (k, l, m) \), write \( f(t) = f_{klm} > 0 \). Suppose that \( f \) is symmetric (\( f_{121} = f_{112} = f_{211} = f_{221} = f_{212} \)) with, for some \( \varepsilon \in (0, \frac{1}{12}] \),

\[
    f_{111} = \frac{1}{2} - 4\varepsilon, \quad f_{222} = \frac{1}{2} - 5\varepsilon, \quad f_{211} = \varepsilon, \quad f_{221} = 2\varepsilon.
\]

Consider the following mechanism \( \mu : \{1, 2\}^3 \rightarrow \Delta(A) \): If all players report the same type, then an equal randomization over \( A \setminus \{a_0\} \) is implemented. If only two players \( i \) and \( j \) report the same type, then these players (the majority) are favored and a fair lottery over \( \{a_i, a_j\} \) is implemented. This mechanism is efficient and type-dependent. \( \square \)

Proof of Theorem 2. In the main text, we already proved the second part. Here we show that the first part holds: if \( |T_1| = 2 \), \( |T_2| = l \geq 2 \) and \( f(t) > 0 \) for all \( t \), then an efficient mechanism is type-independent. The proof is by contradiction. Assume that \( \mu \) is a type-dependent and efficient mechanism. Denote a type of player 1 (resp. 2) by \( i \) (resp. \( j \)) and simplify notation by writing \( a_{ij} = u^i_{1}(i, j) \) and \( b_{ij} = u^j_{2}(i, j) \) for the players’ expected utilities in the truth-telling equilibrium when the types are \( i \) and \( j \). We first consider the case in which all entries in the first row of player 2’s payoff matrix \( B \) are different, \( b_{ij} \neq b_{ik} \) for all \( j \neq k \). Without loss of generality, we can assume that the columns are ordered such that \( b_{11} > b_{12} > \cdots > b_{1l} \). Since \( f(t) > 0 \) for all \( t \), incentive compatibility implies that we must have \( b_{21} < b_{22} < \cdots < b_{2l} \), as otherwise some column \( b_j \) would be strictly dominated for player 2. Furthermore, since \( \mu \) is efficient, we must have that the entries in the first row of player 1’s payoff matrix \( A \) are increasing \( (a_{11} < a_{12} < \cdots < a_{1l}) \), while those in the second row are decreasing \( (a_{21} > a_{22} > \cdots > a_{2l}) \). Incentive compatibility for player 1 implies that the payoffs of one row cannot all be larger than the payoffs in the other row; hence, if \( \alpha_j = a_{ij} - a_{2j} \), then \( \alpha_j \) is increasing in \( j \) with \( \alpha_1 < 0 \) and \( \alpha_l > 0 \). For \( i = 1, 2 \), write \( f^i_\varepsilon \) for the conditional distribution of \( j \) given that player 1 has type \( i \). The incentive constraints of player 1 are:

\[
    E(\alpha|f^i_\varepsilon) \geq 0 \quad \text{and} \quad E(\alpha|f^2_\varepsilon) \leq 0. \quad (6)
\]
Writing $p_j$ for the conditional probability that player 1 is of type 1, if player 2 has type $j$, the incentive constraints for any pair of types $j < k$ of player 2 are given by

$$b_{1j}p_j + b_{2j}(1 - p_j) \geq b_{1k}p_j + b_{2k}(1 - p_j),$$

$$b_{1j}p_k + b_{2j}(1 - p_k) \leq b_{1k}p_k + b_{2k}(1 - p_k).$$

Subtracting the second inequality from the first, we get

$$(b_{1j} - b_{1k} + b_{2j} - b_{2k})(p_j - p_k) \geq 0.$$ 

Hence, $p_j \geq p_k$ since the first term is positive. Hence, the odds ratio is declining. Consequently, if player 1 is of type 2 he is assigning higher probabilities to the higher values of $j$ as when he is of type 1. As $\alpha_j$ is increasing in $j$, this implies

$$E(\alpha|f^c_1) \leq E(\alpha|f^c_2).$$

Combining this with (6) yields

$$E(\alpha|f^c_1) = E(\alpha|f^c_2) = 0.$$ 

Since $\alpha_j$ is increasing in $j$, this is possible only if $f^c_{1j} = f^c_{2j}$ for all $j$. Hence, $f$ is independent. But then, by Observation 1, $\mu$ must be type-independent. This contradicts the assumption that we started with. Consequently, if all entries in the first row of the matrix $B$ are different, then a type-dependent, efficient mechanism cannot exist.

Now assume that $b_{1j} = b_{1k}$ for some $j, k$. As above, we then must have $b_{2j} = b_{2k}$ as otherwise either column $j$ or $k$ would be strictly dominated for player 2. Furthermore, since all payoff vectors are Pareto efficient and $\mu$ is also incentive compatible for player 1, we must have $a_{1j} = a_{1k}$ and $a_{2j} = a_{2k}$. Hence the columns $j$ and $k$ of the matrices $A$ and $B$ are identical. Now, consider the reduced problem in which each set of identical columns is merged, say into a set $J$ and with $f_{iJ} = \sum_{j \in J} f_{ij}$ being the probability that state $(i, J)$ is realized. The mechanism $\mu$ naturally induces a mechanism $\mu'$ on this reduced problem, which is efficient. The first part of the proof applies to this reduced problem so that $\mu'$ is type-independent. This implies that all entries in the reduced matrices $A'$ and $B'$ must be the same. But then there can
only be one equivalence class to start with, hence, \( \mu \) is type-independent.

\[ \square \]

**Proof of Proposition 5.** It suffices to prove the second and the third statements. First assume that there is 2-sided incomplete information: \(|T_1| > 1 \) and \(|T_2| > 1\).

(2) Let \( f \) be independent. Assume \( \mu \) is egalitarian and individually rational. Let \( \mu^0_i(t_i) \) be the probability that \( a_0 \) is chosen when player \( j \) is truthful and player \( i \) has type \( t_i \), and write \( U^\mu_i(t_i) = \mu^0_i(t_i)t_i + r_i(t_i) \). Consider two types \( x \) and \( y \) of player \( i \) with \( x < y \). Two incentive constraints for these types are given by

\[
\mu^0_i(x)x + r_i(x) \geq \mu^0_i(y)x + r_i(y)
\]

and

\[
\mu^0_i(y)y + r_i(y) \geq \mu^0_i(x)y + r_i(x),
\]

while egalitarianism requires that

\[
\mu^0_i(x)x + r_i(x) - x = \mu^0_i(y)y + r_i(y) - y.
\]

Substituting this expression in the incentive constraints and simplifying, we obtain

\[
\mu^0_i(y)(y - x) \geq y - x.
\]

Since \( y > x \), this implies \( \mu^0_i(y) \geq 1 \). Clearly, as a probability, \( \mu^0_i(y) \leq 1 \); hence \( \mu^0_i(y) = 1 \). This establishes that \( \mu^0_i(t_i) = 1 \) for all \( t_i \neq t \), which implies that \( \mu^0(t_i, t_j) = 1 \) for all \( t_j \), all \( t_i \neq t \) and \( i = 1, 2 \). This, in turn, implies \( \mu^0(t) = 1 \) for all \( t \neq 1 \).

(3) On top of the assumptions from (2), assume \( b_1 = b_2 = 0 \). Let \( t_i \neq t \). From (2), we know \( \mu^0_i(t_i) = 1 \), hence elements from \( A^0 \) are selected with probability 0, and since no money is available, we have \( r_i(t_i) = 0 \). Consequently, \( U^\mu_i(t_i) = t_i \) for all \( t_i \neq t \). Egalitarianism then implies that also \( U^\mu_2(t_2) = t_2 \). Hence \( U^\mu_i(t_i) = t_i \) for all \( t_i \) and \( i = 1, 2 \).

If there is 1-sided incomplete information, say \(|T_2| = 1 \), then all previous statements hold for \( i = 1 \). In (3), egalitarianism implies that \( U^\mu_2(t_2) = t_2 \). \[ \square \]