UNIFORMIZATION OF HIGHER GENUS FINITE TYPE
LOG-RIEMANN SURFACES

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ABSTRACT. We consider a log-Riemann surface $S$ with a finite number of ramification points and finitely generated fundamental group. The log-Riemann surface is equipped with a local holomorphic diffeomorphism $\pi : S \rightarrow \mathbb{C}$. We prove that $S$ is biholomorphic to a compact Riemann surface with finitely many punctures $S$, and the pull-back of the 1-form $d\pi$ under the biholomorphic map $\phi : S \rightarrow S$ is a 1-form $\omega = \phi^*d\pi$ with isolated singularities at the punctures of exponential type, i.e. near each puncture $p$, $\omega = e^h \cdot \omega_0$ where $h$ is a function meromorphic near $p$ and $\omega_0$ a 1-form meromorphic near $p$.

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1. INTRODUCTION

In [BPM10a] we defined the notion of log-Riemann surface, as a Riemann surface $S$ equipped with a local diffeomorphism $\pi : S \rightarrow \mathbb{C}$ such that the set of points $\mathcal{R}$ added in the completion $S^\ast = S \sqcup \mathcal{R}$ of $S$ with respect to the flat metric on $S$ induced by $|d\pi|$ is discrete. It was shown in [BPM10a] that $\pi$ extends to the points $p \in \mathcal{R}$, and is a covering of a punctured neighbourhood of $p$ onto a punctured disk in $\mathbb{C}$; the point $p$ is called a ramification point of $S$ of order equal to the degree of the covering $\pi$ near $p$. The finite order ramification points may be added to $S$ to give a Riemann surface $S^\times$, called the finite completion of $S$. We call a log-Riemann surfaces of finite type if it has finitely many ramification points and finitely generated fundamental group.

In [BPM10b] we considered log-Riemann surfaces of finite type with simply connected finite completion and showed that these surfaces were biholomorphic to $\mathbb{C}$, with uniformization given by an entire function of the form $F(z) = \int Q(z)e^{P(z)}dz$.
for some polynomials $P, Q$, in the sense the entire function $F = \pi \circ \phi$, where $\phi : \mathbb{C} \to S^\times$ is a biholomorphic map, is of this form. It was also shown that conversely any such entire function arises as the uniformization of a simply connected finite type log-Riemann surface. The entire functions of the above form have also been studied by Nevanlinna [Nev32] and M. Taniguchi [Tan01].

The aim of this article is to generalize these uniformization theorems to arbitrary finite type log-Riemann surfaces. The role of the entire functions above is played by functions $f$ with finitely many critical points on punctured surfaces such that $df$ has exponential singularities at the punctures; here we say an isolated singularity $p$ of a holomorphic 1-form $\omega$ is an exponential singularity if locally $\omega = e^h \cdot \omega_0$ where $h$ is a function with a pole at $p$ and $\omega_0$ a 1-form meromorphic near $p$.

Ramification points of log-Riemann surfaces are defined in the next section; we define:

**Definition.** A log-Riemann surface $S$ is of finite type if it has finitely many ramification points and finitely generated fundamental group.

We prove the following:

**Theorem 1.1.** Let $S$ be a log-Riemann surface of finite type. Then there is a compact Riemann surface $C$, a finite set $A \subset C$ and a biholomorphic map $\phi : C - A \to S^\times$ such that the map $f = \pi \circ \phi$ has finitely many critical points, and $df$ has exponential singularities or poles at the points of $A$.

Conversely we have:

**Theorem 1.2.** Let $C$ be a compact Riemann surface, $A \subset C$ a finite set and $f$ a holomorphic function on $C - A$ with finitely many critical points such that $df$ has either exponential singularities or poles at the points of $A$. Then there is a finite type log-Riemann surface $S$ and a biholomorphic map $\phi : C - A \to S^\times$ such that $f = \pi \circ \phi$.

The proof of Theorem 1.1 proceeds in outline as follows: we first show that $S^\times$ has finitely many ends, each of which is homeomorphic to a punctured disk. We then show that each end can be isometrically embedded in a simply connected finite type log-Riemann surface. It follows from the uniformization theorem for simply connected log-Riemann surfaces of [BPM10b] that each end is biholomorphic to a punctured disk, and moreover the 1-form $d\pi$ has an exponential singularity at the pole. Since $S^\times$ has finitely many ends each of which is biholomorphic to a punctured disk, it follows that $S^\times$ is biholomorphic to a compact Riemann surface minus finitely many punctures, and $d\pi$ has exponential singularities at the punctures.

For the converse Theorem 1.2 since $f$ is a local diffeomorphism away from the finite set of critical points it suffices to show that there are only finitely many points added when $C$ is completed with respect to the flat path metric induced by $|df|$. Using the fact that $df$ has exponential singularities at the punctures, we show that a neighbourhood of each puncture is bi-Lipschitz equivalent to a neighbourhood of infinity in $\mathbb{C}$ under the path metric induced by $|Q(z)e^{P(z)}dz|$ for some polynomials $P, Q$. It follows from the converse of the uniformization theorem of [BPM10b] that
only finitely many points are added in the completion of a neighbourhood of each puncture of \( C \), and hence in the completion of \( C \).

In section 2 we study the topology of finite type log-Riemann surfaces, using the tools introduced in [BPM10b], and show that there are finitely many ends, which are all punctured disks. Then in section 3 we prove Theorems 1.1, 1.2 above.

2. Topology of finite type log-Riemann surfaces

We first recall some basic properties of log-Riemann surfaces from [BPM10a]. A log-Riemann surface \( S \) is equipped with a local holomorphic diffeomorphism \( \pi : S \to \mathbb{C} \) such that the following holds: the flat metric \( |d\pi| \) induces a path metric \( d \) on \( S \); letting \( \overline{S} = S \sqcup R \) be the metric completion of \( S \), the set \( R \) of points added is discrete.

Moreover for each \( w^* \in R \) the map \( \pi \) restricted to a sufficiently small punctured disk \( B(w^*, \epsilon) - \{ w^* \} \) is a connected covering of a punctured disk \( B(\pi(w^*), \epsilon) - \{ \pi(w^*) \} \) in \( \mathbb{C} \). The point \( w^* \) is called a ramification point of \( S \) of order equal to the degree \( 1 \leq n \leq +\infty \) of the covering.

A punctured disk neighbourhood of a finite order ramification point is biholomorphic to a punctured disk, hence the finite ramification points may be added to \( S \) to obtain a Riemann surface \( S^\infty \) called the finite completion of \( S \).

2.1. Decomposition into stars. The map \( \pi \) restricted to \( S \) is a local isometry. The locally Euclidean metric on \( S \) can be used to define a decomposition into simply connected open sets called stars defined as follows:

Let \( w_0 \in S \). Given an angle \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \), there is a \( 0 < \rho(w_0, \theta) \leq +\infty \) and a unique maximal unbroken geodesic segment \( \gamma(w_0, \theta) : [0, \rho(w_0, \theta)) \to S \) starting at \( w_0 \) which is the lift of the line segment \( \{ \pi(w_0) + te^{i\theta} : 0 \leq t < \rho(w_0, \theta) \} \), such that \( \gamma(w_0, \theta)(t) \to w^* \in R \) as \( t \to \rho(w_0, \theta) \) if \( \rho(w_0, \theta) < +\infty \).

Definition 2.1. The star of \( w_0 \in S \) is the union of all maximal unbroken geodesics starting at \( w_0 \),

\[
V(w_0) := \bigcup_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \gamma(w_0, \theta)
\]

Similarly we also define for a ramification point \( w^* \) of order \( n \leq +\infty \) the star \( V(w^*) \) as the union of all maximal unbroken geodesics \( \gamma(w^*, \theta) \) starting from \( w^* \), where the angle \( \theta \in [-n\pi, n\pi] \):

\[
V(w^*) := \{ \gamma(w^*, \theta)(t) : 0 \leq t < \rho(w^*, \theta), -n\pi \leq \theta \leq n\pi \}
\]

The following Proposition is proved in [BPM10b]:

Proposition 2.2. For any \( w_0 \in S \) the star \( V(w_0) \) is a simply connected open subset of \( S \), and \( \pi \) maps \( V(w_0) \) biholomorphically onto a slit plane \( \mathbb{C} - L \), where \( L \) is a locally finite union of closed half-lines. The boundary \( \partial V(w_0) \subset S \) is a disjoint union of maximal unbroken geodesic segments \( \gamma : (0, +\infty) \to S \) such that \( \gamma(t) \to w^* \in R \) as \( t \to 0 \), \( \gamma(t) \to \infty \) as \( t \to \infty \).
The set of ramification points \( \mathcal{R} \) is discrete, hence countable. Let \( L \supset \pi(\mathcal{R}) \) be the union in \( \mathbb{C} \) of all straight lines joining points of \( \pi(\mathcal{R}) \). Then \( \mathbb{C} - L \) is dense in \( \mathbb{C} \). By a *generic fiber* we mean a fiber \( \pi^{-1}(z_0) = \{ w_i \} \) of \( \pi \) such that \( z_0 \in \mathbb{C} - L \).

From [BPM10] we have:

**Proposition 2.3.** Let \( \{ w_i \} \) be a generic fiber. Then:

1. The stars \( \{ V(w_i) \} \) are disjoint.
2. For \( w_i \neq w_j \), the components of \( \partial V(w_i), \partial V(w_j) \) are either disjoint or equal, and each component can belong to at most two such stars.
3. The union of the stars is dense in \( S \):
   \[ S = \bigcup_i V(w_i) = \bigcup_i V(w_i) \]

The above Proposition gives a cell decomposition of \( S \) into cells \( V(w_i) \) glued along boundary arcs \( \gamma \subset \partial V(w_i) \cap \partial V(w_j) \).

### 2.2. The skeleton and fundamental group.

Let \( \pi^{-1}(z_0) = \{ w_i \} \) be a generic fiber. The 1-skeleton of the cell decomposition into stars gives an associated graph:

**Definition 2.4.** The skeleton \( \Gamma(S, z_0) = (V, E) \) is the graph with vertices given by the stars \( V_i = V(w_i) \), and an edge between \( V(w_i) \) and \( V(w_j) \) for each connected component \( \gamma \) of \( \partial V(w_i) \cap \partial V(w_j) \). Each edge corresponds to a geodesic ray \( \gamma : (0, +\infty) \to S \) starting at a ramification point. This gives us a map from edges to ramification points, \( \text{foot} : \gamma \mapsto \text{foot}(\gamma) := \lim_{t \to 0} \gamma(t) \in \mathcal{R} \cap \partial V(w_i) \cap \partial V(w_j) \).

To each ramification point \( w^* \in \mathcal{R} \) we associate the subgraph \( \Gamma(w^*) \) with vertices and edges
\[ V(w^*) := \{ V_i : w^* \in V_i \}, E(w^*) := \{ \gamma : \text{foot}(\gamma) = w^* \} \]

We note that if \( \mathcal{R} \neq \emptyset \) then
\[ \Gamma(S, z_0) = \cup_{w^* \in \mathcal{R}} \Gamma(w^*) \]

From [BPM10] we have the following Propositions:

**Proposition 2.5.** If \( w^* \) is of finite order \( n \) then \( \Gamma(w^*) \) is a cycle in \( \Gamma(S, z_0) \) of length \( n \). If \( w^* \) is of infinite order then \( \Gamma(w^*) \) is a bi-infinite line in \( \Gamma(S, z_0) \).

**Proposition 2.6.** The log-Riemann surface \( S \) deformation retracts onto \( \Gamma(S, z_0) \). In particular \( \pi_1(S) = \pi_1(\Gamma(S, z_0)) \) is a free group.

The relation of \( \Gamma(S, z_0) \) to the finitely completed log-Riemann surface \( S^\times \) is as follows:

**Definition 2.7.** The finitely completed skeleton \( \Gamma^\times(S, z_0) \) is the graph obtained from \( \Gamma(S, z_0) \) as follows: for each finite order ramification point \( w^* \), add a vertex \( v = v(w^*) \) to \( \Gamma(S, z_0) \), remove all edges in the cycle \( \Gamma(w^*) \) and add an edge from \( v_i \) to \( v \) for each vertex \( v_i \) in the cycle \( \Gamma(w^*) \).

We have:
Proposition 2.8. The finitely completed log-Riemann surface $S^\times$ deformation retracts onto the finitely completed skeleton $\Gamma^\times(S, z_0)$. In particular $\pi_1(S^\times) = \pi_1(\Gamma^\times(S, z_0))$ is a free group.

Finally we have:

Proposition 2.9. If $\pi_1(S)$ is finitely generated then $\Gamma(S, z_0)$ is of the form $\Gamma_0 \cup T$ where $\Gamma_0$ is a finite connected subgraph and $T = \bigcup_{i \in I} T_i$ is a finite disjoint union of rooted trees $T_i$ each intersecting $\Gamma_0$ only at its root. A similar decomposition $\Gamma^\times(S, z_0) = \Gamma_0^\times \cup T^\times$ holds if $\pi_1(S^\times)$ is finitely generated.

Proof: Let $\Gamma_1 \subset \Gamma(S, z_0)$ be a maximal subtree. Then $\pi_1(\Gamma(S, z_0))$ is a free group with generators corresponding to edges of $\Gamma(S, z_0) - \Gamma_1$. It follows that $\Gamma(S, z_0)$ is given by adjoining finitely many edges $e_1, \ldots, e_n$ to a tree $\Gamma_1$. Let $\Gamma_1 \subset \Gamma_1$ be a finite subtree containing the vertices of $e_1, \ldots, e_n$ and $\Gamma_0$ the finite connected subgraph given by adjoining $e_1, \ldots, e_n$ to $\Gamma_1$. Then the closure $T$ of each connected component of $\Gamma(S, z_0) - \Gamma_0$ is contained in $\Gamma_1$, hence is a tree. Moreover since $\Gamma_1$ is connected, $T$ only intersects $\Gamma_0$ at a single vertex $v$ (otherwise $T \cup \Gamma_1 \subset \Gamma_1$ would contain a nontrivial loop). We let $T(v)$ be the union of all such trees $T'$ intersecting $\Gamma_0$ at $v$. Then $T(v)$ is a tree intersecting $\Gamma_0$ only at $v$, the trees $T(v), v$ a vertex of $\Gamma_0$, are disjoint and there are finitely many such. Letting $T = \bigcup_{v \in \Gamma_0} T(v)$, we have $\Gamma(S, z_0) = \Gamma_0 \cup T$ as required. The proof for $S^\times, \Gamma^\times(S, z_0)$ is similar. ∘

2.3. Ends of finite type log-Riemann surfaces. Let $S$ be a finite type log-Riemann surface, i.e. $S$ has finitely many ramification points and finitely generated fundamental group. We fix a generic fiber $\pi^{-1}(z_0)$ and the associated skeleton $\Gamma = \Gamma(S, z_0)$. Let $R_\infty \subset R$ be the set of infinite ramification points of $S$.

Proposition 2.10. The skeleton $\Gamma$ is of the form

$$\Gamma = \Gamma_0 \cup \left( \bigcup_{w^* \in R_\infty} L^+(w^*) \cup L^-(w^-) \right)$$

where $\Gamma_0$ is a finite connected subgraph such that $(\Gamma(w^*) - \Gamma_0)$ is a disjoint union of two open half-lines $L^+(w^*), L^-(w^-)$ for all $w^* \in R_\infty$.

Proof: By Proposition 2.9 there is a finite connected subgraph $\Gamma_0$ such that $\Gamma = \Gamma_0 \cup T$ where $T$ is a finite disjoint union of trees $T = \bigcup_{i \in I} T_i$ each meeting $\Gamma_0$ in a single vertex. In particular, all cycles of $\Gamma$ are contained in $\Gamma_0$. Hence the subgraphs $\Gamma(w^*)$ associated to the finite ramification points $w^*$ of $S$ are contained in $\Gamma_0$, and recalling that $\Gamma = \bigcup_{w^* \in R} \Gamma(w^*)$, it follows that $\Gamma = \Gamma_0 \cup \bigcup_{w^* \in R_\infty} \Gamma(w^*)$.

Each intersection $\Gamma(w^*) \cap \Gamma_0, w^* \in R_\infty$, if nonempty, is connected, since otherwise there would be a finite subinterval of $\Gamma(w^*)$ contained in $T$ meeting $\Gamma_0$ in exactly two vertices, which is not possible. Hence either $\Gamma(w^*)$ is disjoint from $\Gamma_0$ or $\Gamma(w^*) \cap \Gamma_0$ is connected and $(\Gamma(w^*) - \Gamma_0)$ is a disjoint union of two open half-lines $L^+(w^*), L^-(w^-)$.

If $\Gamma(w^*)$ is disjoint from $\Gamma_0$ then $\Gamma(w^*)$ is contained in a tree $T$ meeting $\Gamma_0$ in a single vertex $v$ say, and there is a simple arc $\gamma$ in $T$ starting at $v$ which meets $\Gamma(w^*)$ in a single vertex. It follows that by adding such paths $\gamma$ to $\Gamma_0$, we can ensure that $\Gamma_0$ meets each $\Gamma(w^*), w^* \in R_\infty$ and each $(\Gamma(w^*) - \Gamma_0)$ is a disjoint union of two open half-lines $L^+(w^*), L^-(w^-)$ as required.
It remains to check that for $w_1^* \neq w_2^*$, any two half-lines $L^+(w_1^*), L^-(w_2^*)$ are disjoint; this follows from observing that any intersection would give rise to a cycle of $\Gamma$ not contained in $\Gamma_0$. ◢

We make the convention of labelling the half-lines $L^+(w^*), L^-(w^*)$ above such that any branch of $\arg(w - \pi(w^*))$ defined in a punctured neighbourhood of $w^*$ is bounded below in the stars of $S$ corresponding to $L^+(w^*)$ and is bounded above in those corresponding to $L^-(w^*)$. We let $l(w^*) \subset \mathbb{C}$ be the closed half-line in the direction $(\pi(w^*) - z_0)$ starting from $\pi(w^*)$. An immediate corollary of the above Proposition is the following:

**Lemma 2.11.** (’Clean sheets’) For any star $V \in L^\pm(w^*), w^* \in \mathcal{R}_\infty$, we have $\overline{V} \cap \mathcal{R} = \{w^*\}$ (closure taken in $\overline{S}$) and $\pi$ maps $V$ biholomorphically onto the slit plane $\mathbb{C} - l(w^*)$.

**Proof:** The half-lines $L^+(w^*), L^-(w_1^*)$ are disjoint for $w_1^* \neq w^*$, so $V \notin \Gamma(w_1^*)$ for $w_1^* \neq w^*$ and $\overline{V} \cap \mathcal{R} = \{w^*\}$, from which the second conclusion follows. ◢

We now describe a neighbourhood of infinity in $S^\times$:

Fix $\epsilon > 0$ small enough so that the punctured neighbourhoods $U(w^*) := \{0 < d(w, w^*) \leq \epsilon\}, w^* \in \mathcal{R}_\infty$ are disjoint, and $\pi$ restricted to each $U(w^*)$ is a universal covering of $\pi(U(w^*))$. Fix $R > |z_0|, |\pi(w^*)| + \epsilon, w^* \in \mathcal{R}_\infty$, and let $U(\infty) := \pi^{-1}(\{|z| \geq R\})$. Then on each component of $U(\infty)$, $\pi$ is a connected covering of $\pi(U(\infty))$. For $w^* \in \mathcal{R}_\infty$, let $S^+(w^*)$ (respectively $S^-(w^*)$) be the closure (in $S^\times$) of the union of the stars in $L^+(w^*)$ (respectively $L^-(w^*)$). Let $S_0$ be the closure of the finite union of the stars in $\Gamma_0$. Then by Proposition 2.11 we have

$$S^\times = S_0 \cup \bigcup_{w^* \in \mathcal{R}_\infty} S^+(w^*) \cup S^-(w^*)$$

Moreover, since $L^+(w^*), L^-(w^*)$ and $\Gamma_0$ are connected, the sets $S^+(w^*), S^-(w^*)$ and $S_0$ are connected.

**Proposition 2.12.** The set

$$E = U(\infty) \cup \bigcup_{w^* \in \mathcal{R}_\infty} U(w^*) \cup S^+(w^*) \cup S^-(w^*)$$

is a neighbourhood of infinity in $S^\times$, i.e. $S^\times - E$ is pre-compact.

**Proof:** Let $(z_k)$ be a sequence in $S^\times - E$. Now for any star $V$ it is easy to see from Propositions 2.2, 2.3 that the set $V - (U(\infty) \cup w^* \in \mathcal{R}_\infty U(w^*))$ is precompact in $S^\times$. As $S_0$ is the closure of a finite union of stars, $S_0 - (U(\infty) \cup w^* \in \mathcal{R}_\infty U(w^*))$ is precompact. Since $z_k$ does not lie in any $S^\pm(w^*), (z_k)$ must be contained in $S_0$, and $z_k \notin (U(\infty) \cup w^* \in \mathcal{R}_\infty U(w^*)), \text{ hence the preceding remark implies that } (z_k) \text{ has a convergent subsequence. ◢}$

**Lemma 2.13.** The set $E$ has finitely many connected components.

**Proof:** Since $S^+(w^*), S^-(w^*), S_0$ are connected unions of closures of stars, it is easy to see that the intersections $U(\infty) \cap S^+(w^*), U(\infty) \cap S^-(w^*), U(\infty) \cap S_0$ are connected. As these intersections cover $U(\infty)$, any component of $U(\infty)$ must be a finite union of these sets, therefore $U(\infty)$ has finitely many components. Similarly, the sets $U(w^*), S^\pm(w^*), w^* \in \mathcal{R}_\infty$ are connected, so any component of $E$ is a finite
Lemma 2.14. Let $U$ be a connected component of $U(\infty)$. Let $\gamma : \mathbb{R} \to \mathbb{C}$ be the curve $\gamma(t) = \text{Re}^{it}$. Then either:

(1) $U$ does not meet $\bigcup_{w^* \in \mathcal{R}_\infty} (S^+(w^*) \cup S^-(w^*))$ and $\pi : U \to \{|z| \geq R\}$ is a finite sheeted covering.

or:

(2) $U$ meets $\bigcup_{w^* \in \mathcal{R}_\infty} (S^+(w^*) \cup S^-(w^*))$ and $\pi : U \to \{|z| \geq R\}$ is a universal covering. Moreover there are unique $w^*_+ = \pi^*_+(U), w^*_- = \pi^*_-(U) \in \mathcal{R}_\infty$ such that for any lift $\tilde{\gamma}$ of $\gamma$ to $U$, $\tilde{\gamma}(t) \to \infty$ through $S^+(w^*_+)$ as $t \to +\infty$ and $\tilde{\gamma}(t) \to \infty$ through $S^-(w^*_-)$ as $t \to -\infty$.

Proof: We know $\pi : U \to \{|z| \geq R\}$ is a covering. Let $\tilde{\gamma}$ be a lift of $\gamma$ to $U$. As $\{|z| \geq R\}$ is a punctured disk, clearly $\pi$ restricted to $U$ is a finite sheeted covering if and only if $\tilde{\gamma}$ is periodic.

Suppose $\tilde{\gamma}$ does not meet $\bigcup_{w^* \in \mathcal{R}_\infty} (S^+(w^*) \cup S^-(w^*))$. Then $\tilde{\gamma}$ is contained in $\{|\pi(w)| = R\} \cap S_0$ which is compact (since $S_0$ is the closure of a finite union of stars), so it follows that $\tilde{\gamma}$ is periodic and $\pi$ is a finite sheeted covering. This proves (1).

Now suppose $\tilde{\gamma}$ meets some $S^\pm(w^*)$, say $S^+(w^*)$. Then it follows from Lemma 2.11 that $\tilde{\gamma}(t) \to \infty$ through $S^+(w^*)$ as $t \to +\infty$, so $\pi$ is a universal covering. Therefore $\tilde{\gamma}(t) \to \infty$ as $t \to -\infty$ as well. Since $\{|\pi(w)| = R\} \cap S_0$ is compact, it follows that as $t \to -\infty$, $\tilde{\gamma}(t) \to \infty$ through $\bigcup_{w^* \in \mathcal{R}_\infty} (S^+(w^*) \cup S^-(w^*))$. As these sets are disjoint, $\tilde{\gamma}(t) \to \infty$ as $t \to -\infty$ through one of these sets, say $S^+(w^*)$. As any branch of $\arg(w)$ is bounded below on $S^+(\tilde{w}^*)$, we must have $\tilde{\gamma}(t) \to \infty$ through $S^-(\tilde{w}^*)$ as $t \to -\infty$. Letting $w^*_+ = w^*, w^*_- = \tilde{w}^*$, this proves (2). \(\diamondsuit\)

It follows from the Lemma that each $S^+(w^*), S^-(w^*)$, meets a unique component of $U(\infty)$ which is necessarily as in case (2) above; denote these components by $U^+(\infty, w^*)$ and $U^-(\infty, w^*)$ respectively.

We define maps $u, d : \mathcal{R}_\infty \to \mathcal{R}_\infty$ (for ‘up’, ‘down’ respectively) by $u(w^*) := w^*_+ (U^-(\infty, w^*)), d(w^*) := w^*_-(U^+(\infty, w^*))$. Since $w^* = \pi^*_+(U) = \pi^*_+(U')$, the maps $u, d$ are mutual inverses, and so $\mathcal{R}_\infty$ splits into disjoint cycles $\mathcal{R}_\infty = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_m$ invariant under $u, d$.

Lemma 2.15. For any cycle $\mathcal{R}_i$, the set

$$W(\mathcal{R}_i) := \bigcup_{w^* \in \mathcal{R}_i} \left( S^-(w^*) \cup U(w^*) \cup S^+(w^*) \cup U^-(\infty, w^*) \right)$$

is connected. Moreover for distinct cycles $\mathcal{R}_i, \mathcal{R}_j$ the sets $W(\mathcal{R}_i), W(\mathcal{R}_j)$ are disjoint.

Proof: Since, for any $w^*, S^-(w^*)$ and $S^+(u(w^*))$ meet $U^-(\infty, w^*)$, it follows easily that $W(\mathcal{R}_j)$ is connected.

If for two cycles $\mathcal{R}_i, \mathcal{R}_j$ the sets $W(\mathcal{R}_i), W(\mathcal{R}_j)$ meet, then, for some $w^*_k \in \mathcal{R}_i, w^*_l \in \mathcal{R}_j$, the sets $A_k = (S^-(w^*_k) \cup U(w^*_k) \cup S^+(w^*_k) \cup U^-(\infty, w^*_k)), k = 1, 2$, must meet. Then either $w^*_1 = w^*_2$, or $w^*_1 \neq w^*_2$, in which case one of the two
intersections $S^+(w_1^*) \cap U^-(\infty, w_2^*), S^+(w_2^*) \cap U^-(\infty, w_1^*)$ must be nonempty (all other intersections between the constituents of $A_1, A_2$ are empty), and consequently either $w_1^* = u(w_2^*)$ or $w_2^* = u(w_1^*)$. Thus $\mathcal{R}_i = \mathcal{R}_j$. \(\diamondsuit\)

**Lemma 2.16.** Let $C$ be a component of $E$. Then either:

1. $C = U$ for some component $U$ of $U(\infty)$ such that $\pi : U \to \{ |z| \geq R \}$ is a finite-sheeted covering.

or:

2. $C = W(\mathcal{R}_i)$ for some cycle of infinite ramification points $\mathcal{R}_i \subset \mathcal{R}_\infty$.

**Proof:** We first observe that any component $U$ of $U(\infty)$ as in case (1) of Lemma 2.14 does not meet $\bigcup_{w^* \in \mathcal{R}_\infty} (S^-(w^*) \cup U(w^*) \cup S^+(w^*))$, and is hence a connected component of $E$, while any component of $U(\infty)$ as in case (2) of Lemma 2.14 is contained in a unique $W(\mathcal{R}_i)$ by Lemma 2.15. We know $C$ is a finite union of components of $U(\infty)$ and of the sets $(U(w^*) \cup S^+(w^*) \cup S^-(w^*))$. In particular $C$ must meet $U(\infty)$.

If $C$ contains a component $U$ of $U(\infty)$ as in case (1) of Lemma 2.14 then by the previous observation $C = U$ and we are done.

Otherwise all components of $U(\infty)$ contained in $C$ are as in case (2) of Lemma 2.14 and are contained in the union of the sets $W(\mathcal{R}_i)$. As the sets $(U(w^*) \cup S^+(w^*) \cup S^-(w^*))$ are also contained in the union of the $W(\mathcal{R}_i)$, it follows that $C$ is contained in the union of the $W(\mathcal{R}_i)$. Since the $W(\mathcal{R}_i)$ are pairwise disjoint, $C = W(\mathcal{R}_i)$ for some $i$. \(\diamondsuit\)

**Lemma 2.17.** Each component of $E$ is homeomorphic to a closed disk with a puncture, $\{ 0 < |z| \leq 1 \}$.

**Proof:** This is clear for the components $U$ of $U(\infty)$ which are finite sheeted coverings of $\{ |z| \geq R \}$. For a component $W(\mathcal{R}_i)$ with $\mathcal{R}_i = \{ w^*_0, \ldots, w^*_{n-1} \}$, where $w^*_{i+1} = u(w^*_i)$, we can decompose $W(\mathcal{R}_i)$ as follows:

Define $\tilde{U}(w^*_i) = \frac{U(w^*_i) - (S^-(w^*_i) \cup S^+(w^*_i))}{\pi(w^*_i) - \pi(w^*_j)}$ and $\tilde{U}^-(\infty, w^*_i) = U^-(\infty, w^*_i) - (S^-(w^*_i) \cup S^+(w^*_i+1))$. Then branches of $\log(\pi(w) - \pi(w^*_i))$, $\log(\pi(w))$ map $\tilde{U}(w^*_i), \tilde{U}^-(\infty, w^*_i)$ univalently to semi-infinite horizontal strips $V_j, W_j$ of the form $\{ \Re z \leq A_j, B_j \leq \Im z \leq C_j \}$ and $\{ \Re z \geq A'_j, B_j \leq \Im z \leq C_j \}$ respectively. Similarly appropriate logarithms map $S^-(w^*_i), S^+(w^*_j)$ to lower and upper half-planes $L_j, H_j$ of the form $\{ \Im z \leq B_j \}$ and $\{ \Im z \geq C_j \}$ respectively.

Clearly $W(\mathcal{R}_i)$ is homeomorphic to the disjoint union of the $V_j, W_j, L_j, H_j$ glued together as follows: each strip $V_j$ is glued to the half-planes $L_j, H_j$ along the boundary arcs $\{ \Re z \leq A_j, \Im z = B_j \}$ and $\{ \Re z \leq A_j, \Im z = C_j \}$ respectively, while each strip $W_j$ is glued to the half-planes $L_j, H_{j+1}$ along the boundary arcs $\{ \Re z \geq A'_j, \Im z = B_j \}$ and $\{ \Re z \geq A_j, \Im z = C_{j+1} \}$ respectively. It is easy to see that the resulting quotient is homeomorphic to a closed disk with a puncture $\{ 0 < |z| \leq 1 \}$. \(\diamondsuit\)

**Proposition 2.18.** The Riemann surface $S^X$ is homeomorphic to a closed surface with finitely many punctures (one for each component of $E$).
Proof: It follows from Lemmas $2.12$, $2.13$, and $2.17$ above that $S^x - Z$ is a compact surface with finitely many boundary components which are Jordan curves, and $S^x$ is given by attaching a closed disk with a puncture to each boundary component, hence $S^x$ is a closed surface with finitely many punctures. ∘

3. Uniformization theorems

We have seen above that a finite type log-Riemann surface has finitely many ends, each homeomorphic to a punctured disk (henceforth by an ‘end’ we will mean a component of a neighbourhood of infinity). We shall show that each is indeed biholomorphic to a punctured disk, by showing that each end can be isometrically embedded in a log-Riemann surface whose finite completion is simply connected, and then using the uniformization theorem of $[BPM10b]$ to conclude that the end is biholomorphic to a neighbourhood of infinity in $\mathbb{C}$, hence is a punctured disk.

It will turn out that to each end is associated an integer $K$, the index of the end, which corresponds to the index of a holomorphic vector field with an exponential singularity associated to the end. We define below a countable family of log-Riemann surfaces $S(w_0, \ldots, w_{n-1}, w, K)$ indexed by an integer $K$ such that $S(w_0, \ldots, w_{n-1}, w, K)$ has an end with index $K$; any end of a finite type log-Riemann surface with index $K$ will then be embeddable into $S(w_0, \ldots, w_{n-1}, w, K)$.

3.1. A family of finite type log-Riemann surfaces. Given $n$ points $w_0, \ldots, w_{n-1} \in \mathbb{C}$ (not necessarily distinct), a point $w$ distinct from $w_0, \ldots, w_{n-1}$ and an integer $K \in \mathbb{Z}$ we define a finite type log-Riemann surface $S(w_0, \ldots, w_{n-1}, w, K)$ as follows:

Choose a point $z_0$ not lying on any of the lines passing through $w_i, w_j, 0 \leq i < j \leq n-1$, and a point $w$ not lying on any of the lines passing through $z_0$ and $w_i, j = 0, \ldots, n-1$. Let $l_1, l_0, \ldots, l_{n-1} \subset \mathbb{C}$ be the closed half-lines starting at the points $w, w_0, \ldots, w_{n-1}$ in the directions $(w-z_0), (w_0-z_0), \ldots, (w_{n-1}-z_0)$ respectively. Consider the slit planes $C := \mathbb{C} - l, C_j := \mathbb{C} - l_j, C_j^* := \mathbb{C} - (l \cup l_j), j = 0, \ldots, n-1$. Completing these slit planes with respect to the path-metric induced from $\mathbb{C}$ gives metric spaces $\overline{C} = \mathbb{C} \cup (l^+ \cup l^-), \overline{C_j} = C_j \cup (l_j^- \cup l_j^+)$. $\overline{C_j^*} = C_j^* \cup (l^- \cup l^+) \cup (l_j^- \cup l_j^+)$ where $l^+, l_j^+$ are isometric copies of $l, l_j$ respectively (representing the ‘top’ and ‘bottom’ sides of the lines). We construct the log-Riemann surface $S(w_0, \ldots, w_{n-1}, w, K)$ by pasting together copies of $\overline{C}, \overline{C_j}, \overline{C_j^*}$ along the ‘slits’ $l^+, l_j^+$ as follows:

For $j = 0, \ldots, n-1$ we take a copy of $\overline{C_j}$ and a family $(\overline{C_j}^{(k)})_{k \in \mathbb{Z}}$ of copies $\overline{C_j}^{(k)}$ of $\overline{C_j}$. We treat the cases $K = 0, K > 0, K < 0$ separately:

1. $K = 0$ : For $j = 0, \ldots, n-1$, we identify isometrically the following lines: $l_j^- \subset \overline{C_j}$ is identified with $l_j^+ \subset \overline{C_j}^{(0)}$, $l_j^- \subset \overline{C_j}$ with $l_j^+ \subset \overline{C_j}^{(0)}$ and $l_j^- \subset \overline{C_j}^{(k)}$, with $l_j^+ \subset \overline{C_j}^{(1)}$, and $l_j^- \subset \overline{C_j}^{(k)}$ with $l_j^+ \subset \overline{C_j}^{(k+1)}$ for $k \neq 0$.

It is not hard to see that we obtain a log-Riemann surface $S = S(w_0, \ldots, w_{n-1}, w, 0)$ with $n$ ramification points of infinite order projecting onto the points $w_0, \ldots, w_{n-1}$.
and one of order \(n\) projecting onto \(w\), such that \(S^\times\) is simply connected. The log-Riemann surface \(S\) has a skeleton with stars \(C_j^\times\) forming a cycle \(C\) of length \(n\), each being attached to two half-line segments formed by the stars \((C_j^\times)_{k=0}^2\) and \((C_j^\times)_{k=2}\).

(2) \(K > 0\) : In this case we take \(K\) copies \((C_j^\times)_{1 \leq i \leq K}\) of \(C\) as well. For \(i = 0, \ldots, n-2\), we make the same isometric identifications as above: \(l^- \subset C_j^\times\) is identified with \(l^0 \subset C_j^\times\), \(l^0 \subset C_j^\times\), \(l^1 \subset C_j^\times\), \(l^1 \subset C_j^\times\), and \(l^2 \subset C_j^\times\) with \(l^2 \subset C_j^\times\) for \(k \neq 0\). We make the following further isometric identifications: \(l^- \subset C_{n-1}^\times\) is identified with \(l^0 \subset C_{n-1}^\times\), \(l^- \subset C_{(K)}^\times\) with \(l^0 \subset C_{(K)}^\times\), and if \(K > 1\) then \(l^- \subset C_{(i)}^\times\) is identified with \(l^0 \subset C_{(i+1)}^\times\) for \(i = 1, \ldots, K - 1\).

In this case we obtain a log-Riemann surface \(S(w_0, \ldots, w_{n-1}, w, K)\) with ramification points of infinite order projecting onto the points \(w_0, \ldots, w_{n-1}\) and one of order \(n + K\) projecting onto \(w\), such that \(S^\times(w_0, \ldots, w_{n-1}, w, K)\) is simply connected. The log-Riemann surface has a skeleton with the stars \(C_0^\times, \ldots, C_{n-1}^\times, C^{(1)}, \ldots, C^{(K)}\) forming a cycle of length \(n + K\), with each \(C_j^\times\) being attached to two half-line segments formed by the stars \((C_j^\times)_{k=0}^2\) and \((C_j^\times)_{k=2}\).

From the uniformization theorem of \([BPM10b]\) we have:

**Proposition 3.1.** The finite completion \(S^\times(w_0, \ldots, w_{n-1}, w, K)\) for \(K \geq 0\) is biholomorphic to \(C\). There is a polynomial \(P\) of degree \(n\) and a uniformization \(F : C \to S^\times(w_0, \ldots, w_{n-1}, w, K)\) satisfying \((\pi \circ F)'(z) = z^{n+K}e^{P(z)}\).

We remark that the \(n\) infinite ramification points of \(S(w_0, \ldots, w_{n-1}, w, K)\) all belong to one cycle. The corresponding end \(W(\mathbb{R}_\infty)\) is the only end of \(S^\times(w_0, \ldots, w_{n-1}, w, K)\). Under the uniformization \(F\) above, the end \(W(\mathbb{R}_\infty)\) is biholomorphic to a punctured disk neighbourhood of \(\infty\) in \(C\).

(3) \(K < 0\) : In this case we replace the two copies \((C_j^\times)_{1}^{(1)}, (C_j^\times)_{1+(K)}\) of \(C_j^\times\) with two additional copies \((C_j^\times)_0^{(1)}, (C_j^\times)_0^{(2)}\) of \(C_j^\times\) instead. For \(j = 0, \ldots, n-1\), we make the following isometric identifications: \(l^- \subset C_j^\times\) is identified with \(l^0 \subset C_j^\times\), \(l^- \subset C_j^\times\), \(l^1 \subset C_j^\times\), and \(l^2 \subset C_j^\times\) with \(l^2 \subset C_j^\times\) for \(k \neq 0\). We identify \(l^0 \subset C_0^\times\) with \(l^0 \subset C_0^\times\), \(l^0 \subset C_0^\times\), \(l^1 \subset C_0^\times\), \(l^1 \subset C_0^\times\), \(l^2 \subset C_0^\times\), and \(l^2 \subset C_0^\times\) for \(k \neq 0\). If \(K < -1\) then we identify \(l^0 \subset C_0^\times\) with \(l^0 \subset C_0^\times\), \(l^0 \subset C_0^\times\), \(l^1 \subset C_0^\times\), \(l^1 \subset C_0^\times\), \(l^2 \subset C_0^\times\), and \(l^2 \subset C_0^\times\) for \(k \neq 0\). Finally we identify \(l^0 \subset C_0^\times\) with \(l^0 \subset C_0^\times\) for \(k \leq -1\) and \(k \geq 2 + (K)\).

In this case the log-Riemann surface \(S(w_0, \ldots, w_{n-1}, w, K)\) we obtain is of finite type, has again \(n\) infinite order ramification points projecting onto \(w_0, \ldots, w_{n-1}\), but now two finite order ramification points projecting onto \(w\), one of order \(n\) and one of order two. Moreover there is an end \(U\) of the finite completion which is a finite-sheeted covering of \(|z| > R\) of degree \((-K)\). The finite completion is not simply connected in this case, but we still have:
Proposition 3.2. The finite completion \( S^\times(w_0, \ldots, w_{n-1}, w, K) \) for \( K < 0 \) is biholomorphic to the punctured plane \( \mathbb{C}^* \). There is a uniformization \( F : \mathbb{C}^* \to S^\times(w_0, \ldots, w_{n-1}, w, K) \) satisfying \((\pi \circ F)'(z) = z^K(z - p_1)^n(z - p_2)e^{P(z)} \) for some polynomial \( P \) of degree \( n \) and two distinct non-zero \( p_1, p_2 \in \mathbb{C} \).

Proof: It is straightforward to see that the finitely completed skeleton \( \Gamma^\times \) of \( S = S(w_0, \ldots, w_{n-1}, K) \) has only one cycle corresponding to the end \( U \) which is a degree \((-K)\) covering of \( \{|z| > R\} \). The end \( U \) is biholomorphic to a punctured disk, hence we may add a point \( q \) to \( S^\times \) to obtain a Riemann surface \( S^* \) such that \( U \cup \{q\} \) is biholomorphic to a disk, and \( \pi \) extends to have a pole of order \((-K)\) at \( q \). Moreover the surface \( S^* \) is simply connected. Since it has only finitely many ramification points, using the Kobayashi-Nevanlinna parabolicity criterion of [BPM10b] it follows that \( S^* \) is biholomorphic to \( \mathbb{C} \).

We now argue as in the proof of Theorem 1.1 of [BPM10b]. Let \( F : \mathbb{C} \to S^* \) be a uniformization such that \( F(0) = q \), and choose a basepoint \( q' \neq q \) in \( S \). The approximation Theorem 2.10 of [BPM10b] gives a sequence of pointed finite-sheeted log-Riemann surfaces \((S_k, q_k')\) converging to \((S, q')\) in the sense of Caratheodory (see [BPM10b]). For \( k \) large the surfaces \( S_k^\times \) also have one end which is a degree \((-K)\) covering of \( \{|z| > R\} \), and hence as above we can add a point \( q_k \) to obtain a Riemann surface \( S_k^* \) which is biholomorphic to \( \mathbb{C} \). Let \( F_k : \mathbb{C} \to S_k^* \) be a uniformization such that \( F_k(0) = q_k \). Since \( \pi_k \circ F_k \) is finite-to-one in a neighbourhood of infinity, it follows that \( \pi_k \circ F_k \) is a rational function, which moreover has only one finite pole of order \((-K)\) at \( 0 \), two critical points of orders \( n - 1 \) and \( 1 \) corresponding to the finite ramification points of \( S_k \) projecting onto \( w \), and \( n \) critical points of increasing orders (as \( k \to \infty \)) corresponding to the finite ramification points projecting onto \( w_0, \ldots, w_{n-1} \). By the Caratheodory convergence theorem of [BPM10b], normalizing the \( F_k \)'s appropriately, we have \( \pi_k \circ F_k \to \pi \circ F \) uniformly on compacts of \( \mathbb{C}^* \) as \( k \to \infty \).

Now the same argument as in [BPM10b] shows that the nonlinearity of \( \pi \circ F \) is a rational function of the form

\[
\frac{(\pi \circ F)''(z)}{(\pi \circ F)'(z)} = \frac{K - 1}{z} + \frac{n - 1}{z - p_1} + \frac{1}{z - p_2} + P'(z)
\]

for two distinct non-zero \( p_1, p_2 \in \mathbb{C} \) and some polynomial \( P \) of degree \( n \), and the result follows upon integration of the above. \( \diamond \)

We note that as before the \( n \) infinite ramification points all lie in the same cycle. However in this case \((K < 0)\), the surface \( S^\times(w_0, \ldots, w_{n-1}, w, K) \) has two ends, one of the form \( W(\mathcal{R}_\infty) \) corresponding to the cycle of infinite ramification points, and the other the degree \((-K)\) covering \( U \) of \( \{|z| \geq R\} \). Under the uniformization \( F : \mathbb{C}^* \to S^\times(w_0, \ldots, w_{n-1}, w, K) \) these are biholomorphic to punctured disks in \( \mathbb{C} \).

3.2. Isometric embedding of ends. Let \( S \) be a log-Riemann surface of finite type and consider the decomposition of the skeleton

\[
\Gamma = \Gamma_0 \sqcup (\sqcup_{w^* \in \mathcal{R}_\infty} L^+(w^*) \sqcup L^-(w^-))
\]

We observe that adding to \( \Gamma_0 \) finite initial segments of each half-line \( L^+(w^*), w^* \in \mathcal{R}_\infty \) gives a similar decomposition for which all properties proved in the previous
We first show that (in the notation of Lemma 2.17) the union of all but 'piece' a family \((\pi(w))\) corresponding to the cycle of infinite ramification points of \(U\). The indices \(\Gamma(\pi(w))\) in the skeleton, \(\Gamma(\pi(w))\) (say) respectively. We have \(\pi(w)\) has been chosen so that all these segments have lengths lying in an interval \([N - c_1, 2N + c_1]\) where \(N, c_1, c_2 \geq 2\) are integers such that \(N \geq 8(\#R_\infty + 1)(c_1 + c_2)\).

Now let \(R_\pi = \{w_0^\pi, \ldots, w_{n-1}^\pi\} \subset R_\infty\) be a cycle of infinite ramification points (with \(w_{n+1}^\pi = u(w_1^\pi)\)), and \(W(R_\pi)\) the corresponding component of \(E\). Let \(w_j = \pi(w_j^\pi)\). We have:

**Theorem 3.3.** There exists \(K \in \mathbb{Z}\) and an isometric embedding \(\iota : W(R_\pi) \to S(w_0, \ldots, w_{n-1}, w, K)\) such that \(\pi \circ \iota = \pi\) and \(\iota(W(R_\pi))\) is the end of \(S^\times(w_0, \ldots, w_{n-1}, w, K)\) corresponding to the cycle of infinite ramification points of \(S^\times(w_0, \ldots, w_{n-1}, w, K)\) (here \(\pi\) is the projection mapping of \(S(w_0, \ldots, w_{n-1}, w, K)\)).

**Proof:** We first show that (in the notation of Lemma 2.17) the union of all but one of the sets making up \(W(R_\pi)\), namely \(S^\times(w_0^\pi), \bar{U}^\times(\infty, w_0^\pi), S^+(w_1^\pi), \bar{U}(w_1^\pi), S^-(w_1^\pi), \ldots, S^-(w_{n-1}^\pi), \bar{U}^\times(\infty, w_{n-1}^\pi), S^+(w_0^\pi)\), can be isometrically embedded into \(S(w_0, \ldots, w_{n-1}, K)\) for \(|K| \leq N\), and the embedding then extends to the remaining 'piece' \(\bar{U}(w_0^\pi)\) if \(K\) is chosen appropriately.

Let \(|K| \leq N/2\) and let \(S_K = S(w_0, \ldots, w_{n-1}, w, K)\). We recall that \(S_K\) contains a family \((C_0(-k))_{k \geq 1}\) of copies of \(C_0\). Given \(k_0 \geq 1\), clearly there is a unique isometry \(\iota\) of \(S^-(w_0^\pi)\) onto \(\cup_{k \geq k_0} C_0(-k)\) such that \(\pi \circ \iota = \pi\). We first show that \(k_0\) can be chosen so that \(\iota\) extends to an isometric embedding of the union of \(S^-(w_0^\pi), \bar{U}^\times(\infty, w_0^\pi), S^+(w_1^\pi), \bar{U}(w_1^\pi), S^-(w_1^\pi), \ldots, S^-(w_{n-1}^\pi), \bar{U}^\times(\infty, w_{n-1}^\pi), S^+(w_0^\pi)\). The indices \(j, j + 1\) will be taken modulo \(n\) throughout.

Suppose for some \(0 \leq j \leq n - 2\) we are given \(k_j \geq 1\) and an isometric embedding \(\iota\) of \(S^-(w_j^\pi)\) into \(S_K\) with image \(\cup_{k \geq k_j} C_j(-k)\). If \(1 \leq k_j \leq 2N - c_1 - 2\) then there is a unique isometric extension of \(\iota\) to \(\bar{U}^\times(\infty, w_j^\pi)\), with image passing through the stars \(C_j(-k_j+1), \ldots, C_j(0), C_j^*(1), C_{j+1}^{(1)}, \ldots, C_j^{(k_j+1)}\), where \(k_j = a_j' - (k_j + 1) \geq 1, a_j'\) being the length of the segment \(\Gamma(\pi(w_j^\pi)) - (L^-(w_j^\pi) \cup L^+(w_{j+1}^\pi))\) (in the skeleton \(\Gamma\)). This determines a further isometric extension to \(S^+(w_{j+1}^\pi)\) with image in the stars \(C_j^{(k)}, k \geq k_j + 1\).

There is also an isometric extension to \(\bar{U}(w_{j+1}^\pi)\) which, if \(k_{j+1} \geq 2N - c_1 - 1\), has image in the stars \(C_j^{(k)}(-k), \ldots, C_j^{(1)}, C_j^{(1)}, C_{j+1}^{(0)}, \ldots, C_j^{(k_j-1)}\) where \(k_{j+1} = a_j + 1 - k_{j+1} \geq 1, a_j + 1\) being the length of the segment \(\Gamma(w_{j+1}^\pi) - (L^-(w_{j+1}^\pi) \cup L^+(w_{j+1}^\pi))\) again, this gives a unique isometric extension to \(S^-(w_{j+1}^\pi)\) with image in the stars \(C_j^{(k)}, k \geq k_{j+1} + 1\).

By hypothesis \(a_j' \in [2N - c_1, 2N + c_1]\) so if \(k_j \in [N - j(c_1 - c_2), N + j(c_1 + c_2)]\) then \(k_{j+1} = a_j' - k_j \in [N - j + 1)c_1 - c_2 - 1, N + (j + 1)c_1 + c_2 + 1]\) \(\subset [1, 2N - c_1 - 2]\), and \(k_{j+1} = a_j + 1 - k_{j+1} \in [N - j + 1)c_1 - c_2, N + (j + 1)c_1 + c_2] \subset [1, \infty)\) (using
the hypotheses on \( N, c_1, c_2 \). It follows that choosing \( k_0 \in [N - c_2, N + c_2] \) we can extend \( \iota \) inductively to an isometry defined on the union of \( S^-(w_0^*), \hat{U}^-(\infty, w_0^*), S^+(w_1^*), \hat{U}(w_1^*), S^-(w_n^*), \ldots, S^-(w_n^*) \).

The image of \( S^-(w_{n-1}^*) \) lies in the stars \( C_{n-1}^{(-k)} \), \( k \geq k_{n-1} + 1 \), and \( k_{n-1} \leq N + (n - 1)c_1 + c_2 \). As \( \hat{U}^-(\infty, w_{n-1}^*) \) corresponds to a segment in the skeleton of length \( a_{n-1}' \geq 2N - c_1 \) and \( |K| \leq N/2 \), it is not hard to see that \( \iota \) has a unique isometric extension to \( \hat{U}^-(\infty, w_{n-1}^*) \) with image starting in the star \( C_{n-1}^{(-k_{n-1})} \) and ending in a star \( C_0^{(k_0')} \), where \( k_0' \geq 1 \) is given by \( k_0' = a_{n-1}' - (k_{n-1} + 1 + K) \). Then \( \iota \) has a unique isometric extension to \( S^+(w_0^*) \), with image lying in the stars \( C^{(k)}, k \geq k_0' + 1 \).

It remains to define \( \iota \) on \( \hat{U}(w_0^*) \). As \( \iota \) is given on \( \hat{U}(w_0^*) \cap S^+(w_0^*) \) and \( S^-(w_0^*) \), there is an extension to \( \hat{U}(w_0^*) \) if and only if the number of stars from \( C_0^{(k_0')} \) to \( C^{(-k_0)} \) is equal to the length of the segment \( \Gamma(w_0^*) - (L^-(w_0^*) \cup L^+(w_0^*)) \), or in other words if \( k_0' + k_0 + 1 = a_0 \).

Using the recursion formulae for \( k_j, k_j' \), this reduces to

\[
K = \sum_{j=0}^{n-1} (a_j' - a_j) - (n - 1)
\]

and by the hypothesis on the lengths \( a_j', a_j \) the right-hand side above is bounded above by \( 2nc_1 + (n - 1) \leq N/2 \). Hence \( K \) can be chosen as required so that \( \iota \) extends to an isometric embedding of \( W(R_i) \) into \( S_K \).

**Corollary 3.4.** Each end of a finite type log-Riemann surface is biholomorphic to a punctured disk.

**Proof:** Each end is either a component \( U \) of \( U(\infty) \) such that \( \pi : U \to \{ |z| \geq R \} \) is a finite-sheeted covering, in which case the interior of \( U \) is biholomorphic to a punctured disk, or is a component of \( E \) of the form \( W(R_i) \). It follows from the previous Theorem and the remarks following Propositions 3.1, 3.2 that the end \( W(R_i) \) is biholomorphic to a punctured disk.

### 3.3. Proofs of main theorems.

We are now in a position to prove the main Theorems [1, 12]

**Proof of Theorem 1.1:** It follows immediately from Proposition 2.18 and Corollary 3.4 that the finite completion \( S^\infty \) of a finite type log-Riemann surface is biholomorphic to a closed Riemann surface with finitely many punctures corresponding to the ends of \( S^\infty \). For an end which is a finite-sheeted degree \( d \) covering of a neighbourhood of \( \infty \) in \( \hat{C} \), it follows that \( \pi \) extends meromorphically to have a pole of order \( d \) at the corresponding puncture.

For an end \( W(R_i) \) corresponding to a cycle of infinite ramification points, by Theorem 3.3 and Propositions 3.1, 3.2 we obtain a biholomorphic map \( F \) from a punctured disk neighbourhood of \( \infty \) in \( \hat{C} \) to \( W(R_i) \) such that in terms of the local coordinate \( z = F^{-1} \), \( d\tau = R(z)e^{P(z)}dz \) for some polynomial \( P \) and some rational function \( R \). It follows that \( d\tau \) has an exponential singularity at the corresponding puncture.
\*\*\*  

**Proof of Theorem 1.2.** Given a non-constant meromorphic map \( f \) on a closed Riemann surface with punctures \( S \) such that \( df \) has exponential singularities at the punctures, we note that \( df \) has no poles or zeroes in a neighbourhood of each puncture, hence the set \( A \) of poles and critical points of \( f \) is finite. Let \( S' = S - A \). Then \( f : S' \to \mathbb{C} \) is a local diffeomorphism, and it suffices to show that finitely many points are added in the completion \( \overline{S} \) of \( S' \) with respect to the path metric induced by \( \|df\| \). If \( p \in S' \) tends to a puncture which is a critical point of \( f \), then \( p \) tends to a unique limit in \( \overline{S} \), which is a finite ramification point of \( S' \), while if \( p \) tends to a pole of \( f \) then \( p \) tends to infinity in \( \overline{S} \).

It remains to show that as \( p \) tends to a puncture \( p_0 \) which is an exponential singularity of \( df \), then \( p \) can only accumulate in \( \overline{S} \) on a finite set of points (only depending on \( p_0 \)). Let \( df = e^h \omega \) near \( p_0 \), where \( h \) is a meromorphic function with a pole of order \( n \) say at \( p_0 \) and \( \omega \) is a 1-form meromorphic near \( p_0 \). We can choose a local coordinate \( z \) such that \( z(p) = \infty \), \( h(z) = z^n \), and \( df = g(z)e^{z^n} dz \) where \( g \) is a function meromorphic near \( z = \infty \). Then for some integer \( k \) and some \( C > 0 \) we have

\[
\frac{1}{C}\|z^ke^{z^n}\|\|dz\| \leq |df| \leq C\|z^ke^{z^n}\|\|dz\|
\]

It follows that the path-metrics \( d, d' \) induced by \( |df| \) and \( |z^ke^{z^n}|\|dz\| \) on a punctured neighbourhood \( D \) of \( p_0 \) are bi-Lipschitz equivalent, hence so are the completions of \( D \) with respect to \( d, d' \).

If \( k \geq 0 \) then by Theorem 1.2 of \([BPM10a]\) the function \( \int z^ke^{z^n} dz \) defines a log-Riemann surface with \( n \) infinite ramification points and we are done.

If \( k = -m < 0 \) then the metric induced by \( |z^ke^{z^n}dz| \) is bi-Lipschitz equivalent to that induced by \( |\eta| = |(1/z^m - C/z^{m+n})e^{z^n} dz| \) where the constant \( C \) is chosen so that the residue of the 1-form \( \eta \) at \( \infty \) vanishes, so \( \eta \) has a primitive \( F = \int \eta \) on \( \mathbb{C}^* \). We can approximate \( F \) by rational functions

\[
R_N(z) = \int \left( \frac{1}{z^m - \frac{C_N}{z^{m+n}}} \right) \left( 1 + \frac{z^n}{N} \right)^N dz
\]

where the constants \( C_N \to C \) are chosen so that the residue at \( \infty \) of the 1-form in the integral vanishes. Then each \( R_N \) defines a log-Riemann surface structure \( S_N \) on \( \mathbb{C}^* \) with \( n \) finite order ramification points each of order \( N + 1 \). By the compactness Theorem 2.11 of \([BPM10a]\), there is a subsequence of \( S_N \) which converges in the sense of Caratheodory to a log-Riemann surface structure \( S \) on \( \mathbb{C}^* \) with at most \( n \) ramification points. By the Caratheodory convergence Theorem 1.2 of \([BPM10a]\), the maps \( R_N \) converge along the same subsequence uniformly on compacts to the projection mapping of \( S \), which is hence given by \( F \). It follows that there are at most \( n \) points added in the completion of a neighbourhood of \( \infty \) with respect to the metric induced by \( |\eta| \).

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