Numerical range of weighted composition operators which contain zero

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Abstract

In this paper, we study when zero belongs to the numerical range of weighted composition operators $C_{ψ,ϕ}$ on the Fock space $F^2$, where $ϕ(z) = az + b$, $a, b \in \mathbb{C}$ and $|a| \leq 1$. In the case that $|a| < 1$, we obtain a set contained in the numerical range of $C_{ψ,ϕ}$ and find the conditions under which the numerical range of $C_{ψ,ϕ}$ contain zero. Then for $|a| = 1$, we precisely determine the numerical range of $C_{ψ,ϕ}$ and show that zero lies in its numerical range.

1 Introduction

The Fock space $F^2$ consists of all entire functions on the complex plane $\mathbb{C}$ which are square integrable with $dμ(z) = π^{-1}e^{-|z|^2}dA(z)$ that $dA$ is the Lebesgue measure on $\mathbb{C}$. For $f, g$ in $F^2$, the inner product on the Fock space is given by

$$\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)}dμ(z).$$

The set $\{e_m(z) = z^m/\sqrt{m!} : m \geq 0\}$ is an orthonormal basis for $F^2$. The reproducing kernel at $w$ in $\mathbb{C}$ for $F^2$ is given by $K_w(z) = e^{wz}$. Let $k_w$ denote the normalized reproducing kernel given by $k_w = K_w/\|K_w\|$, where $\|K_w\| = e^{\|w\|^2/2}$. Fock space is a very important tool for quantum stochastic calculus in the quantum probability. For more information about the Fock space, see [18].

Through this paper, for a bounded operator $T$ on $F^2$, the spectrum of $T$ and the point spectrum of $T$ are denoted by $σ(T)$ and $σ_p(T)$; respectively.

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For an entire function $\varphi$, the composition operator $C_\varphi$ on $\mathcal{F}^2$ is defined by the rule $C_\varphi(f) = f \circ \varphi$ for each $f \in \mathcal{F}^2$. For an entire function $\psi$, the weighted composition operator $C_{\psi,\varphi} : \mathcal{F}^2 \to \mathcal{F}^2$ is given by $C_{\psi,\varphi} h = \psi \cdot (h \circ \varphi)$. There is a vast literature on composition operators on the other spaces (see [5] and [13]). Moreover, recently many authors have worked on the weighted composition operators on the Fock spaces (see [3], [7], [11], [15], [16] and [17]). Bounded and compact composition operators on the Fock space over $\mathbb{C}^n$ were characterized in [3] by Carswell et al. They showed that $C_\varphi$ is bounded on the Fock space if and only if $\varphi(z) = az + b$, where $|a| \leq 1$ and if $|a| = 1$, then $b = 0$. In [14], Ueki found a necessary and sufficient condition for $C_{\psi,\varphi}$ to be bounded and compact. After that in [11], Le gave the easier characterizations for the boundedness and compactness of $C_{\psi,\varphi}$. Moreover, he found normal and isometric weighted composition operators on $\mathcal{F}^2$. Unitary weighted composition operators and their spectrum on the Fock space of $\mathbb{C}^n$ were characterized by Zhao in [15]. Note that there are some interesting papers [3], [15] and [17] which were written in another Fock space (see [18]), but their results hold for $\mathcal{F}^2$ by the same idea. Then we use them frequently in this paper.

For $T$ a bounded linear operator on a Hilbert space $H$, the numerical range of $T$ is denoted by $W(T)$ and is given by $W(T) = \{ \langle Tf, f \rangle : \|f\| = 1 \}$. The set $W(T)$ is convex, its closure contains $\sigma(T)$ and $\sigma_p(T) \subseteq W(T)$. There are some papers that the numerical range of composition operators and weighted composition operators on the Hardy space $H^2$ were investigated (see [1], [2], [9] and [12]).

In Section 2, we investigate $W(C_{\psi,\varphi})$, where $\varphi(z) = az + b$ with $0 < |a| < 1$. In Proposition 2.1, we find a subset contained in $W(C_{\psi,\varphi})$, where $\psi(\frac{b}{1-\alpha}) \neq 0$. In Theorem 2.2, we show that if $C_{\psi,az+b}$ is compact, where $\psi(\frac{b}{1-\alpha}) \neq 0$ and $a$ is not a positive real number, then $W(C_{\psi,\varphi})$ contains zero. Then in Theorem 2.3, for $C_{\psi,\varphi}$ with $\psi(\frac{b}{1-\alpha}) = 0$, we show that $W(C_{\psi,\varphi})$ contains a closed disk with center at 0. Moreover, in Remark 2.4, for a constant function $\varphi$, we show that $W(C_{\psi,\varphi})$ contains zero.

In Section 3, for $\varphi(z) = az + b$, with $|a| = 1$, we find the numerical range of $C_{\psi,\varphi}$ and see that $W(C_{\psi,\varphi})$ contains zero.

2 $\varphi(z) = az + b$, whith $|a| < 1$

Suppose that $\psi$ is an entire function and $\varphi(z) = az + b$, where $|a| < 1$. If $C_{\psi,\varphi}$ is a bounded operator on $\mathcal{F}^2$, then by [16] Theorem 1, $0 \in \sigma(C_{\psi,\varphi})$. 

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Hence, \( 0 \in W(C_{\psi, \varphi}) \). In this section, we study when \( 0 \) belongs to \( W(C_{\psi, \varphi}) \) and we work on the numerical range of bounded weighted composition operator \( C_{\psi, \varphi} \), where \( \varphi(z) = az + b \) with \( |a| < 1 \). In this section, we assume that \( q(z) = e^{\overline{\varphi}(a-1)z} \psi(z + p) \), where \( p = \frac{b}{1-a} \) is the fixed point of \( \varphi \). In the proof of Proposition 2.1, we will see that \( q \) belongs to \( \mathcal{F}^2 \) and we assume that \( \sum_{j=0}^{\infty} \hat{q}_j \frac{z^j}{\sqrt{j!}} \) is the representation series of \( q \) in \( \mathcal{F}^2 \).

**Proposition 2.1.** Suppose that \( \psi \) is an entire function and \( \varphi(z) = az + b \), where \( 0 < |a| < 1 \). Let \( C_{\psi, \varphi} \) be bounded on \( \mathcal{F}^2 \). Suppose that \( \psi(p) \neq 0 \), where \( p = \frac{b}{1-a} \) is the fixed point of \( \varphi \). Let \( n \) be a non-negative integer and \( m \) be a positive integer. Then \( W(C_{\psi, \varphi}) \) contains the ellipse with foci at \( a^n \) and \( a^{n+m} \) and a major axis

\[
\sqrt{|a^n - a^{n+m}|^2 + \frac{|\hat{q}_n a^n \sqrt{(m+n)!}|^2}{m!n!}}
\]

and a minor axis

\[
\frac{|\hat{q}_n a^n \sqrt{(m+n)!}|}{\sqrt{m!n!}}.
\]

**Proof.** By [15, Corollary 1.2], \( C_{k_p, z-p} \) is unitary and [11, Proposition 3.1] implies that \( C_{k_p, z-p} = C_{k_{z-p}, z+p} \). Since \( \varphi(z+p) - p = a(z+p) + b - p = az \) and

\[
k_{z-p}(z)k_p(\varphi(z+p))\psi(z+p) = e^{\overline{\varphi}(a-1)z} \psi(z+p),
\]

we obtain that

\[
C_{k_p, z-p}C_{\psi, \varphi}C_{k_p, z-p} = C_{k_{z-p}, z+p}C_{\psi, \varphi}C_{k_p, z-p} = C_{q, az},
\]

where \( q = e^{\overline{\varphi}(a-1)z} \psi(z+p) \) (since \( C_{q, az} \) is a bounded operator on \( \mathcal{F}^2 \), \( q = C_{q, az}(1) \) belongs to \( \mathcal{F}^2 \)). It shows that \( C_{\psi, \varphi} \) is unitary equivalent to \( C_{q, az} \). Thus, \( W(C_{\psi, \varphi}) = W(C_{q, az}) \) and so we investigate the numerical range of \( C_{q, az} \). Let \( M = \text{span}\{e_1, e_2\} \), when \( e_1(z) = \frac{z^n}{\sqrt{n!}} \) and \( e_2(z) = \frac{z^{n+m}}{\sqrt{(n+m)!}} \). We can see that

\[
C_{q, az}(e_1)(z) = (1 + \hat{q}_1 z + \frac{\hat{q}_2 z^2}{\sqrt{2!}} + \cdots) \frac{a^n z^n}{\sqrt{n!}}
\]

\[
= \left( \frac{a^n z^n}{\sqrt{n!}} + \hat{q}_1 a^n \frac{z^{n+1}}{\sqrt{2!}} + \frac{\hat{q}_2 a^n z^{n+2}}{\sqrt{2!n!}} + \cdots + \frac{\hat{q}_m a^n z^{n+m}}{\sqrt{m!n!}} + \cdots \right)
\]

\[
= \left( \frac{a^n z^n}{\sqrt{n!}} + \frac{\hat{q}_1 a^n z^{n+1}}{\sqrt{2!n!}} + \frac{\hat{q}_2 a^n z^{n+2}}{\sqrt{2!n!}} + \cdots + \frac{\hat{q}_m a^n z^{n+m}}{\sqrt{m!n!}} + \cdots \right)
\]

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and

\[ C_{q,az}(e^2)(z) = (1 + \hat{q}_1 z + \hat{q}_2 \frac{z^2}{\sqrt{2!}} + \cdots) \frac{a^{n+m} z^{n+m}}{\sqrt{(n+m)!}} \]

\[ = a^{n+m} \frac{z^{n+m}}{\sqrt{(n+m)!}} + \hat{q}_1 a^{n+m} \frac{z^{n+m+1}}{\sqrt{(n+m)!}} + \cdots. \]

Let \( T \) be the compression of \( C_{q,az} \) to \( M \). Then the matrix representation of \( T \) is

\[ \begin{bmatrix} a^n \sqrt{n!} & 0 \\ \hat{q}_m a^n \sqrt{(n+m)!} \cdot n! \end{bmatrix}. \]

By \([10, p.3-4]\), \( W(T) \) is an ellipse with foci at \( a^n, a^{n+m} \) and the minor axis \( |a^n \hat{q}_m| \frac{\sqrt{(n+m)!}}{\sqrt{m!n!}} \) and the major axis \( \sqrt{|a^n - a^{n+m}|^2 + |\hat{q}_m a^n \sqrt{(n+m)!}|^2}. \) Since \( W(T) \subset W(C_{\psi,\varphi}) \), the result follows. \( \Box \)

Let \( \varphi(z) = az \), where \( 0 < a < 1 \). Since \( C_{az} \) is normal (see \([11, Theorem 3.3]\), \([3, Theorem 2]\), \([17, Proposition 2.6]\) and \([10, Theorem 1.4-4, p.16]\) state that \( W(C_{az}) = [0,1] \). By the Open Mapping Theorem, 0 is not an eigenvalue for \( C_{az} \). Invoking \([10, Theorem 1.5-5, p.20]\), \( W(C_{az}) = (0,1] \) and so 0 dose not belong to \( W(C_{az}) \). In the next theorem, we prove that 0 belongs to the numerical range of compact weighted composition operator \( C_{\psi,az+b} \), where \( a \) is not a positive real number. In the proof of Theorem 2.2, we use the notation \( \mathbb{D} \) which is the open unit disk in the complex plane \( \mathbb{C} \). Moreover, some ideas of the proof of the next theorem is similar to \([12, Proposition 2.1]\).

**Theorem 2.2.** Suppose that \( \psi \) is an entire function and \( \varphi(z) = az + b \), where \( 0 < |a| < 1 \). Let \( \psi(\frac{b}{1-a}) \neq 0 \). Assume that \( C_{\psi,\varphi} \) is compact on \( \mathbb{F}^2 \). If \( a \) is not a positive real number, then \( W(C_{\psi,\varphi}) \) contains zero and it is closed.

**Proof.** We know that \( W(C_{\psi,\varphi}) = \psi(p)W(C_{\psi,\varphi}) \), where \( p = \frac{b}{1-a} \) is the fixed point of \( \varphi \). By \([17, Proposition 2.6]\), \( \sigma(C_{\psi,\varphi}) = \{0, 1, a, a^2, \ldots\} \) and by \([4, Theorem 7.1, p. 214]\), \( \sigma_p(C_{\psi,\varphi}) = \{1, a, a^2, \ldots\} \). Since \( \sigma_p(C_{\psi,\varphi}) \subset W(C_{\psi,\varphi}) \), the convex hull of some arbitrary elements of \( \sigma_p(C_{\psi,\varphi}) \) is a subset of \( W(C_{\psi,\varphi}) \). We claim that there is a set \( M \) that \( M \subset W(C_{\psi,\varphi}) \) and \( 0 \in M \). We break the problem into three cases.

(a) Assume \( a = |a|e^{i\theta} \) and \( e^{i\theta} \) is not a root of 1. Then \( \{e^{in\theta} : n \geq 0\} \) is dense.
in $\partial \mathbb{D}$. We can find $n_1, n_2, n_3, n_4$ such that $|a|^{n_1} e^{i\theta_1}, |a|^{n_2} e^{i\theta_2}, |a|^{n_3} e^{i\theta_3}, |a|^{n_4} e^{i\theta_4}$ lie in the quadrants I, II, III, IV; respectively. It is not hard to see that 0 is contained in the interior of the polygonal region $P$ whose vertices are $|a|^{n_1} e^{i\theta_1}, |a|^{n_2} e^{i\theta_2}, |a|^{n_3} e^{i\theta_3}, |a|^{n_4} e^{i\theta_4}$. Let $M$ be the union of $P$ and its interior region.

(b) Assume that $a = |a| e^{i\theta}$ that $e^{i\theta}$ is a primitive root of 1 of order $n > 2$. Let $P$ be the polygonal region whose vertices are $1, |a| e^{i\theta}, |a|^2 e^{2i\theta}, ..., |a|^{n-1} e^{(n-1)i\theta}$ (note that $e^{i\theta}, e^{2i\theta}, ..., e^{(n-1)i\theta}$ are the $n$th root of 1). Since $n > 2$, the argument of $a$ is not 0 or $\pi$ and so there are at least three vertices which are non-colinear points. It is not hard to see that none of sides of $P$ contains zero and so 0 belongs to the interior of the polygonal region $P$. Again let $M$ be the union of $P$ and its interior region.

(c) Assume that $a = -|a|$. As we know, the convex hull of $\sigma_p(C_{\psi,\varphi})$ is a subset of $W(C_{\psi,\varphi})$. Then $[a, 1] \subseteq W(C_{\psi,\varphi})$. Let $M$ be the closed line segment with end points $a$ and 1. Since in these three cases, $M \subseteq W(C_{\psi,\varphi})$ and 0 $\in M$, 0 $\in W(C_{\psi,\varphi})$. Moreover, invoking [6, Theorem 1], $W(C_{\psi,\varphi})$ is closed. \hfill $\Box$

Note that if $\varphi$ and $\psi$ satisfy the hypotheses of Theorem 2.2 and the argument of $a$ is not 0 or $\pi$, then by the proof of Theorem 2.2, 0 lies in the interior of $W(C_{\psi,\varphi})$. In the next theorem, we show that 0 belongs to the interior of $W(C_{\psi,az+b})$, where $0 < |a| < 1$ and $\psi(\frac{b}{1-a}) = 0$.

**Theorem 2.3.** Suppose that $\psi$ is an entire function and $\varphi(z) = az + b$, where $0 < |a| < 1$. Let $n$ be a non-negative integer and $m$ be a positive integer. Assume that $C_{\psi,\varphi}$ is bounded on $\mathbb{F}^2$. Suppose that $\psi(p) = 0$, where $p = \frac{b}{1-a}$ is the fixed point of $\varphi$. Then $W(C_{\psi,\varphi})$ contains a closed disk with center at 0 and radius $|\frac{a^n a^{n+m} \sqrt{n!}}{m!}|/2$.

**Proof.** As we saw in the proof of Proposition 2.1, $W(C_{\psi,\varphi}) = W(C_{\psi,az})$, so we investigate the numerical range of $C_{\psi,az}$. We assume that $M =$ span\{$e_1, e_2$}, where $e_1(z) = \frac{z^n}{\sqrt{n!}}$ and $e_2(z) = \frac{z^{n+m}}{\sqrt{(n+m)!}}$. We have

\[
C_{\psi,az}(e_1) = (\hat{q}_1 z + \hat{q}_2 \frac{z^2}{\sqrt{2!}} + \cdots + \hat{q}_n \frac{z^n}{\sqrt{n!}})
\]

\[
= \hat{q}_1 \frac{a^n z^{n+1}}{\sqrt{n!}} + \hat{q}_2 a^n \frac{z^{n+2}}{\sqrt{2!n!}} + \cdots + \hat{q}_m a^n \frac{z^{n+m}}{\sqrt{n!m!}} + \cdots
\]
and
\[ C_{q,az}(e_2) = (\tilde{q}_1 z + \tilde{q}_2 \frac{z^2}{\sqrt{2!}} + \cdots) a^{n+m} \frac{z^{n+m}}{\sqrt{(n+m)!}} \]
\[ = \tilde{q}_1 a^{n+m} \frac{z^{n+m+1}}{\sqrt{(n+m)!}} + \cdots. \]

Let \( T \) be the compression of \( C_{\psi,\varphi} \) to \( M \). Then the matrix representation of \( T \) is
\[
\begin{bmatrix}
0 & 0 \\
\tilde{q}_m a^n \sqrt{(n+m)!} & 0
\end{bmatrix}.
\]

By [10, Example 1, p. 1],
\[ W(T) = \left\{ z : |z| \leq \frac{|\tilde{q}_m a^n \sqrt{(n+m)!}|}{2\sqrt{n!}m!} \right\}. \]

Since \( W(T) \subseteq W(C_{\psi,\varphi}) \), \( W(C_{\psi,\varphi}) \) contains a closed disk with center at 0 and radius \( \frac{|\tilde{q}_m a^n \sqrt{(n+m)!}|}{2\sqrt{n!}m!} \). \qed

Remark 2.4. Suppose that for some complex number \( b \), \( \varphi \equiv b \) and \( \psi \) is a non-zero entire function. Then \( C_{\psi,\varphi} f = f(b) \psi = \langle f, \|\psi\|K_b \rangle \frac{\psi}{\|\psi\|} \). By [2, Proposition 2.5], we can find \( W(C_{\psi,\varphi}) \) as follows.
(a) If \( K_b = \frac{c}{\|\psi\|} \psi \) for some non-zero complex number \( c \), then \( W(C_{\psi,\varphi}) \) is the closed line segment from 0 to \( \overline{c} \).
(b) If \( K_b \perp \psi \), then \( W(C_{\psi,\varphi}) \) is the closed disk centered at the origin with radius \( \frac{\|\psi\| \|b\|^2}{2} \).
(c) Otherwise \( W(C_{\psi,\varphi}) \) is a closed ellipse with foci at 0 and \( \psi(b) \).
Then we can see that in the case that \( \varphi \) is constant, \( W(C_{\psi,\varphi}) \) contains zero.

In the first part of the following example, we give a compact weighted composition operator \( C_{\psi,az+b} \), where \( a \) is a positive real number and \( 0 \in W(C_{\psi,az+b}) \) (see Theorem 2.2). Also in the second part, we give an example which satisfy the conditions of Theorem 2.3.

Example 2.5. (a) Suppose that \( \varphi(z) = \frac{1}{2} z - \frac{1}{2} \) and \( \psi(z) = e^z \). By [17, Corollary 2.4], \( C_{\psi,\varphi} \) is compact. It is easy to see that 1 is the fixed point of
\( \varphi \) and \( q(z) = e^{K_{1/2}(z)} \). The representation series of \( q \) in \( \mathcal{F}^2 \) is

\[
\sum_{j=0}^{\infty} \frac{e^{2j}}{2^j \sqrt{j!} \sqrt{j!}} z^j.
\]

Let \( n = m = 1 \). By Proposition 2.1, \( W(C_{\psi,\varphi}) \) contains the ellipse with foci at 1/2 and 1/4 and the major axis \( \sqrt{\frac{1}{16} + \frac{e^2}{8}} \). It states that 0 belongs to \( W(C_{\psi,\varphi}) \).

(b) Let \( \varphi(z) = \frac{1}{2} z + \frac{1}{2} \) and \( \psi(z) = K_{1}(z) - e^{-1} \). Note that \( C_{\psi,\varphi} = C_{K_{1},\varphi} - e^{-1} C_{\varphi} \) and so by [11] Theorem 2 and [13] Proposition 2.2, \( C_{\psi,\varphi} \) is bounded. We have \( \psi(-1) = 0 \) and \( q(z) = e^{z/2}(e^{z-1} - e^{-1}) = e^{-1}(e^{z-1} - e^{z}) \).

It is not hard to see that the representation series of \( q \) in \( \mathcal{F}^2 \) is

\[
\sum_{j=0}^{\infty} \frac{3^j - 1}{2^j \sqrt{j!} \sqrt{j!}} z^j.
\]

Let \( n = 0 \) and \( m = 1 \). By Theorem 2.3, \( W(C_{\psi,\varphi}) \) contains the closed disk with center at 0 and radius \( \frac{1}{2e} \).

### 3 \( \varphi(z) = az + b, \text{ whit } |a| = 1 \)

In this section, we completely find the numerical range of \( C_{\psi,az+b} \), where \( |a| = 1 \). Let \( S \) be a subset of complex plane \( \mathbb{C} \). For \( a \in \mathbb{C} \), we define \( aS = \{as : s \in S\} \); we use this definition in the next theorem.

**Theorem 3.1.** Suppose that \( \varphi(z) = az + b \), where \( |a| = 1 \). Let \( C_{\psi,\varphi} \) be a bounded weighted composition operator on \( \mathcal{F}^2 \). Then

(a) If \( a \neq 1 \) and \( a \) is a primitive root of 1 of order \( n \), then \( W(C_{\psi,\varphi}) = \psi(0)e^{a|b|^2} P \), where \( P \) is the union of the polygon with \( n \) sides and vertices at \( 1, a, ..., a^{n-1} \) and its interior region.

(b) If \( a \) is not a root of 1, then \( W(C_{\psi,\varphi}) = \psi(0)e^{a|b|^2} \mathbb{D} \cup \{\psi(0)e^{a|b|^2}a^m : m \geq 0\} \).

(c) If \( a = 1 \), then \( W(C_{\psi,\varphi}) = \psi(0)e^{|b|^2} \mathbb{D} \).

**Proof.** Suppose that \( a \neq 1 \). By [10], Proposition 2.1, \( \psi(z) = \psi(0)e^{-abz} = \psi(0)K_{-\overline{b}}(z) \). Let \( u = \overline{\psi(0)} \). We have
\[
C_{ka,z-u}C_{\psi,\varphi}C_{k-u,z+u} = \frac{1}{\|K_u\|^2}C_{e^{\pi z-u}C_{\psi,\varphi}C_{e^{-\pi z+u}}} \\
= \frac{1}{\|K_u\|^2}e^{\pi z} \cdot \psi(z-u) \cdot e^{(-\pi(az+b))(z-u)}C_{(z+u)(az+b)(z-u)} \\
= \frac{1}{\|K_u\|^2}e^{\pi z} \cdot \psi(z-u) \cdot e^{-\pi(az-au+b)}C_{az+u(1-a)+b} \\
= C_{\tilde{\varphi},\tilde{\psi}},
\]
where
\[
\tilde{\varphi}(z) = az + \frac{-ab}{a-1}(1-a) + b = az
\]
and
\[
\tilde{\psi}(z) = e^{-|u|^2}e^{\pi z} \cdot \psi(z-u) \cdot e^{-\pi(az-au+b)} = \psi(0)e^{\frac{u|b|^2}{a-1}}.
\]

Then \(C_{\psi,\varphi}\) is unitary equivalent to \(\psi(0)e^{\frac{u|b|^2}{a-1}}C_{az}\) (see [15, Corollary 1.2]). We try to find \(W(C_{az})\). We prove that \(\sigma_p(C_{az}) = \{1, a, a^2, \ldots\}\). It is easy to see that \(C_{az}(z^j) = a^jz^j\) for each non-negative integer \(j\). Then \(\{1, a, \ldots\} \subseteq \sigma_p(C_{az})\). Since by [3, Lemma 2], \(C_{az}^* = C_{az}\), \(C_{az}\) is an isometry. We infer from [4, Exercise 7, p.213] that \(\sigma_p(C_{az}) \subseteq \partial \mathbb{D}\) (note that \(C_{az}\) is invertible). Assume that there is \(\lambda \in \sigma_p(C_{az})\) such that \(|\lambda| = 1\) and \(\lambda\) does not belong to \(\{a^m : m \geq 0\}\). Thus, there exists a non-zero function \(f \in F^2\) that
\[
C_{az}(f) = \lambda f.\tag{2}
\]
It shows that \(f(0) = \lambda f(0)\). Hence \(f(0) = 0\). Assume that for each \(j < k\), \(f^{(j)}(0) = 0\). We prove that \(f^{(k)}(0) = 0\). Taking \(k\)th derivatives on the both sides of Equation (2) yields \(a^k f^{(k)}(0) = \lambda f^{(k)}(0)\). Then \(f^{(k)}(0) = 0\). Thus, \(f \equiv 0\) which is a contradiction. It states that \(\sigma_p(C_{az}) = \{1, a, a^2, \ldots\}\). Moreover, by [15, Corollary 1.4], \(\sigma(C_{az}) = \{a^m : m = 0\}^\infty\).

(a) Suppose that \(a \neq 1\) and \(a\) is a primitive root of \(1\) of order \(n\). If \(n = 2\), then \(\sigma(C_{az}) = \{-1, 1\}\). Invoking [10, Theorem 1.4-4, p.16], \(W(C_{az})\) is the convex hull of \(\sigma(C_{az})\) which is equal to \([-1, 1]\). Since \(-1, 1 \in \sigma_p(C_{az})\), we conclude that \(W(C_{az}) = [-1, 1]\). Now assume that \(n > 2\). Let \(P\) be the convex hull of \(\{1, a, \ldots, a^{n-1}\}\) which is the union of polygon with \(n\) sides and vertices at \(1, a, \ldots, a^{n-1}\) and its interior region. We can see that \(P\) is a subset of \(W(C_{az})\) (note that \(\sigma_p(C_{\psi,\varphi}) = \{1, a, \ldots, a^{n-1}\}\) and \(\sigma_p(C_{az}) \subseteq W(C_{az})\)). We show that \(W(C_{az}) = P\). By [10, Theorem 1.4-4, p.16], \(W(C_{az}) = P\). Since all vertices of \(P\) belong to \(\sigma_p(C_{az})\), \(W(C_{az}) = P\) (see also [10, Corollary 1.5-7, p.20]). It shows that \(W(C_{\psi,\varphi}) = \psi(0)e^{\frac{u|b|^2}{a-1}}P\).
(b) Assume that $a$ is not a root of 1. Since $\sigma_p(C_{az}) = \{a^m\}_{m=0}^\infty$ and 
$\{a^m : m \geq 0\}$ is a dense subset of the unit circle, we get 
$\{a^m : m \geq 0\} \cup \mathbb{D}$ is a subset of $W(C_{az})$. Moreover, since $C_{az}$ is an isometry, 
$\|C_{az}\| = 1$ and so $W(C_{az}) \subseteq \mathbb{D}$. Now we show that for each $\lambda \in \partial \mathbb{D}$ that $\lambda \notin \{a^m : m \geq 0\}$, $\lambda \notin W(C_{az})$. Suppose that there is $\lambda \in \partial \mathbb{D}$ which does not belong to
$\{a^m : m \geq 0\}$ and $\lambda \in W(C_{az})$. By [10] Theorem 1.3-3, p.10], $\lambda \in \sigma_p(C_{az})$ 
which is a contradiction. Then $W(C_{az}) = \{a^m : m \geq 0\} \cup \mathbb{D}$ and so
$W(C_{\psi,\varphi}) = \psi(0)e^{\frac{\alpha |h|^2}{2}} \mathbb{D} \cup \{\psi(0)e^{\frac{\alpha |h|^2}{2}} a^m : m \geq 0\}$.

(c) Assume that $a = 1$. By [11] Proposition 2.1], $\psi(z) = \psi(0)K_{-b}(z)$. We 
know that $C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}$ is unitary (see [15] Corollary 1.2). Then $W(C_{\psi, z+b}) = 
\psi(0)\|K_{-b}\|W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$. We try to find $W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$. Since $\sigma(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}) = 
\partial \mathbb{D}$ (see [15] Corollary 1.4]), [10] Theorem 1.4-4, p. 16] implies that $W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}) = \mathbb{D}$. Since $W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$ is convex, it is not hard to see that $\mathbb{D} \subseteq W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$.

Because $C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}$ is unitary, $\|C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}\| = 1$. Hence by [10] Theorem 
1.3-3, p.10], if $\lambda \in \partial \mathbb{D}$ is an element of $W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$, then $\lambda$ must 
belong to $\sigma_p(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$. We claim that $\sigma_p(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}) = \emptyset$. Assume that 
$\lambda \in \sigma_p(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b})$. Let $\mu$ be an arbitrary unimodular number. It is not 
hard to see that there exists $u \in \mathbb{C}$ such that $e^{2\text{Im}(u\lambda)} = \mu$. By Equation 
(1), we get

$$C_{k_u, z-u} C_{\frac{\kappa - b}{\|K_{-b}\|} z + b} C_{k_u, z + u} = C_{\tilde{\psi}, z + b},$$

where $\tilde{\psi}(z) = \mu \psi(z)$. Then $C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}$ is unitary equivalent to $\mu C_{\psi, \varphi}$. It
states that $\lambda \mu^{-1}$ is an eigenvalue of $C_{\psi, \varphi}$. Thus, $\sigma_p(C_{\psi, \varphi}) = \partial \mathbb{D}$. Since $C_{\psi, \varphi}$ is normal (see [11] Theorem 3.3]) and the Fock space is sparable, by [4] Proposition 
5.7, p.47], $C_{\psi, \varphi}$ cannot have an uncountable collection of eigenvalues which is a contradiction. Therefore, $\sigma_p(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}) = \emptyset$ and it shows that

$W(C_{\frac{\kappa - b}{\|K_{-b}\|} z + b}) = \mathbb{D}$. Thus, $W(C_{\psi, \varphi}) = \psi(0)e^{\frac{\alpha |h|^2}{2}} \mathbb{D}$. \hfill \Box

In Theorem 3.1, we saw that 0 lies in the interior of the numerical range of 
$C_{\psi, az+b}$, where $|a| = 1$. In the next example, we compute the numerical 
range of some weighted composition operators using Theorem 3.1.

Example 3.2. (a) Let $\varphi(z) = iz + 3$ and $\psi(z) = K_{3i}(z)$. By Theorem
3.1(a), \( W(C_{\psi, \varphi}) = e^{\frac{9}{i} \pi} P \), where \( P \) is the union of the polygon with \( n \) sides and vertices at \( 1, i, -1, -i \) and its interior region.

(b) Let \( \varphi(z) = e^{\frac{\sqrt{3}}{i}} z + 2 \) and \( \psi(z) = K_{-2e^{-\sqrt{3}}}(z) \). Theorem 3.1(b) implies that \( W(C_{\psi, \varphi}) = t\mathbb{D} \cup \{t(e^{\sqrt{3}i})^m : m \geq 0\} \), where
\[
t = e^{4e^{\sqrt{3}} - 1}.
\]

(c) Let \( \varphi(z) = z + 2 \) and \( \psi(z) = K_{-2}(z) \). We infer from Theorem 3.1(c) that \( W(C_{\psi, \varphi}) = e^{2\mathbb{D}} \).

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