ALGEBRAIC INFINITE DELOOPING AND DERIVED DESTABILIZATION

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Abstract. Working over the prime field of characteristic two, consequences of the Koszul duality between the Steenrod algebra $A$ and the (big) Dyer-Lashof algebra are studied, with an emphasis on the interplay between instability for the Steenrod algebra action and that for the Dyer-Lashof operations. The central algebraic framework is the category $\mathcal{QM}^{gr}$ of length-graded modules over the Steenrod algebra equipped with an unstable action of the Dyer-Lashof algebra, with compatibility via the Nishida relations.

A first ingredient is a functor from modules over the Steenrod algebra, $M$, to $\mathcal{QM}^{gr}$, that arose in the work of Kuhn and McCarty on the homology of infinite loop spaces. This functor is given in terms of derived functors of destabilization from the category $M$ of modules over the Steenrod algebra to unstable modules, enriched by taking into account the action of Dyer-Lashof operations.

A second ingredient is the derived functors of the Dyer-Lashof indecomposables functor $q: \mathcal{QM}^{gr} \to M^{gr}$ to length-graded modules over the Steenrod algebra. These are related to functors used by Miller in his study of a spectral sequence to calculate the homology of an infinite delooping. An important fact is that these functors can be calculated as the homology of an explicit Koszul complex with terms expressed as certain Steinberg functors. The latter are quadratic dual to the more familiar Singer functors.

By exploiting the explicit complex built from the Singer functors which calculates the derived functors of destabilization, Koszul duality leads to an algebraic infinite delooping spectral sequence. This is conceptually similar to Miller’s spectral sequence, but there seems to be no direct relationship.

The spectral sequence sheds light on the relationship between unstable modules over the Steenrod algebra and all $A$-modules.

1. Introduction

To motivate the study of the rich algebraic structures which intervene, consider a spectrum $X$ and its associated infinite loop space $\Omega^\infty X$ and their respective mod-2 homologies (denoted here simply by $H_*(X)$ and $H_*(\Omega^\infty X)$). The homology of the spectrum, $H_*(X)$, is an $A$-module, where the mod 2 Steenrod algebra $A$ acts on the right. However, unlike the homology of a space, it is not an unstable $A$-module in general. On the other hand, $H_*(\Omega^\infty X)$ is an unstable $A$-module and there is much more structure: it is a bicommutative Hopf algebra and is equipped with an action of the Dyer-Lashof algebra and all these structures are compatible. Here the focus is upon the residual additive structure after passage to the algebra indecomposables $QH_*(\Omega^\infty X)$.

Given $H_*(X)$ as an $A$-module, there are various strategies available for calculating $H_*(\Omega^\infty X)$, at least when $X$ is connected. For instance, Kuhn and McCarty [KM13] use the spectral sequence of the associated Goodwillie-Arone tower;

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Haugseng and Miller [HM16] use the spectral sequence associated to a cosimplicial resolution constructed from Eilenberg-MacLane spectra.

In their work, Kuhn and McCarty give an algebraic approximation to their spectral sequence with $E^\infty$-page expressed in terms of the derived functors of destabilization; an important fact is that the Dyer-Lashof action is visible on the $E^\infty$-page. (In this homological setting, the destabilization functor $\Omega_k : M \to \mathcal{M}$ is the right adjoint to the inclusion of the category $\mathcal{M}$ of unstable modules in $M$, the category of $\mathcal{A}$-modules. This functor is left exact and has non-trivial right derived functors $\Omega^i_k$.)

From a purely algebraic viewpoint, the fundamental algebraic object appearing in [KM13] is, for $M \in M$, the length-graded $\mathcal{A}$-module $H^0 R M := \bigoplus \Sigma^{-1} \Omega^1 \Sigma^{1-i} M$, equipped with an unstable action of the (big) Dyer-Lashof algebra that is compatible with the $\mathcal{A}$-action via the Nishida relations. The length grading reflects the fact that the (big) Dyer-Lashof algebra is a homogeneous quadratic algebra, hence has a length grading.

Such objects form a category $QM^{gr}$ and the above can be considered as a functor from $M$ to $QM^{gr}$. (Forgetting the length grading gives the category $QM$ used by Kuhn and McCarty.) As the notation suggests, $H^0 R M$ is the zeroth homology of a functorial chain complex $R M$ taking values in $QM^{gr}$. A secondary purpose of this paper is to underline the importance of the functor $H^0 R : M \to QM^{gr}$ and give its fundamental properties (see Sections 7 and 8).

The object appearing in Kuhn and McCarty’s algebraic approximation is not $H^0 R M$ (for $M = H_* X$) but $\Sigma H^0 R \Sigma^{-1} M$, where $\Sigma : QM^{gr} \to QM^{gr}$ is the left adjoint to the desuspension functor $\Sigma^{-1} : QM^{gr} \to QM^{gr}$. This exhibits the interplay between the notions of instability with respect to the $\mathcal{A}$-module structure and with respect to the Dyer-Lashof action. For instance, $H^0 R M$ is not necessarily $\mathcal{A}$-unstable but is Dyer-Lashof unstable. The suspension $\Sigma H^0 R M$ is $\mathcal{A}$-unstable, but need not be Dyer-Lashof unstable; the functor $\Sigma$ corrects this. It is also interesting to understand the relationship between $H^0 R M$ and $H^0 R \Sigma M$; this is explained in Corollary 8.3, generalizing a classical result for unstable modules over the Steenrod algebra.

To introduce the second ingredient, let us return to topology and the attempt to recover $H_*(X)$ from $H_*(\Omega^\infty X)$ by killing the extraneous structure. This strategy was carried out by Miller, who constructed an infinite delooping spectral sequence [Mil78]. The first step passes to the algebra indecomposables $QH_* (\Omega^\infty X)$ and the second to the indecomposables for the residual Dyer-Lashof action; in the spectral sequence, the higher derived functors of these appear. The restriction axiom for the action of Dyer-Lashof operations $Q_0 x = x^2$ implies that $QH_* (\Omega^\infty X)$ is the desuspension of an object of $QM$, which leads Miller to work with a slightly different category.

From the current viewpoint, the basic functor to consider is the Dyer-Lashof indecomposables functor

$$q : QM^{gr} \to M^{gr}$$

that is left adjoint to the trivial Dyer-Lashof action functor (here $M^{gr}$ is the category of length-graded $\mathcal{A}$-modules). This has left derived functors (in the sense of relative homological algebra), $L_n q$, which are of independent interest. These are related to the functors $\text{Untor}^A_* (\mathbb{F}, -)$ used by Miller, who identified some key properties.
Restricted to $\mathcal{M}$, the functors $\mathbb{L}_n q$ ($n \in \mathbb{N}$) turns out to be non-trivial only in length grading $n$; this is a manifestation of the Koszul property, à la Priddy [P5770]. Moreover, the associated functor $\mathbb{L}_n : \mathcal{M} \to \mathcal{M}$ (termed here the Steinberg functor), is exact. This functor is quadratic dual to the important Singer functor $\mathcal{A}_n : \mathcal{M} \to \mathcal{M}$ which arises as a component of the left adjoint to the forgetful functor $\mathcal{Q} \mathcal{M} \to \mathcal{M}$. The Steinberg functors do not seem to have been studied in full generality, but arise for example implicitly in the work of Kuhn on the Whitehead conjecture [Kuh15].

For $N \in \mathcal{Q} \mathcal{M}^{\mathbb{Z}}$, with length-gradings components $N[i]$ ($i \in \mathbb{N}$), in fixed length grading $n \in \mathbb{N}$, there is a Koszul complex $\mathbb{L}_n N$ in $\mathcal{M}$:

$$
\mathcal{L}_n N[0] \xrightarrow{d_n} \mathcal{L}_{n-1} N[1] \xrightarrow{d_{n-1}} \mathcal{L}_{n-2} N[2] \to \ldots \to \mathcal{L}_0 N[n] = N[n],
$$

where $\mathcal{L}_i N[i]$ is placed in homological degree $i$. The differential is defined using the action of Dyer-Lashof operations on $N$. A related complex occurs in the work of Miller [Mil78].

By Corollary 5.35 the $i$th homology of $\mathcal{L}_n N$ is $(\mathbb{L}_n q N)[n]$. Since the Steinberg functors are relatively well understood as functors on $\mathcal{M}$, this leads to a good understanding of the derived functors $\mathbb{L}_n q$.

Combining the Kuhn-McCarty algebraic approximation with the existence of Miller’s infinite delooping spectral sequence leads to the following question: is there a natural spectral sequence to recover $M \in \mathcal{M}$ from the input $H_0 \mathcal{Q} M$? For convergence reasons, it is only reasonable to ask this question when one does not necessarily want to insist that $M$ be $(-1)$-connected. In this generality the answer is no, as can be seen by considering the module $M = \Sigma^{-2} \mathcal{A}^*$ (the dual Steenrod algebra), since $H_0 \mathcal{Q} \Sigma^{-2} \mathcal{A}^*$ is zero.

This does not preclude the existence of such a spectral sequence to recover $M$ from $H_0 \mathcal{Q} M$ when $M$ is 0-connected. For example, one could use the filtration of an $\mathcal{A}$-module given by Miller in [Mil78]. However, the author knows of no approach which leads to such a spectral sequence with an explicitly identified $E^2$-page.

On the other hand, the Koszul duality between Steenrod and Dyer-Lashof actions does lead to a spectral sequence that gives important information on the structure of a bounded-below $\mathcal{A}$-module. This is the subject of Section 5 which contains the following result:

**Theorem 1.** Let $M \in \mathcal{M}$ be bounded-below.

1. There is a natural second quadrant homological spectral sequence $(E^r_{s,t}, d^r)$ with $d_r$ of $(s,t)$-bidegree $(r-1, -r)$ and $E^2$-page:

$$
E^2_{s,t} = \mathbb{L}_s q \left( H_0 \mathcal{Q} \Sigma^{-s} M \right)^{[-s]}.
$$

2. The spectral sequence converges strongly to $M$ with associated increasing filtration $\mathfrak{v}_t M \subset M$, $t \in \mathbb{N}$.

3. There is a natural isomorphism

$$
E^2_{-t,t} \cong \ker \left\{ \mathcal{L}_t \Sigma^{-1} \Omega^\infty \Sigma^t M \to \mathcal{L}_{t-1} \Sigma^{-1} \Omega^\infty \Sigma^t M \right\}.
$$

4. The $t = 0$ edge morphism identifies with the natural inclusion

$$
\mathfrak{v}_0 M \cong \Sigma^{-1} \Omega^\infty \Sigma M \hookrightarrow M
$$

in $(s,t)$-degree $(0, 0)$ and is zero elsewhere.

5. $E^2_{s,t} = 0$ for $t > -s$, so that the spectral sequence is concentrated below the anti-diagonal and there is a second edge homomorphism:

$$
E^\infty_{s,t} = \mathfrak{v}_t M / \mathfrak{v}_{t-1} M \hookrightarrow E^2_{-t,t}.
$$

6. $E^\infty_{s,t} = 0$ if $s \neq -t$ and $E^\infty_{-t,t} \cong \mathfrak{v}_t M / \mathfrak{v}_{t-1} M$. 

Some comments are in order. The length zero part of $H_0\mathcal{R}\Sigma^l M$ is $\Sigma^{-1}\Omega^{\infty}\Sigma^{1+l} M$. Since any bounded-below $\mathcal{A}$-module $M$ can be recovered as

$$M \cong \text{colim}_d \Sigma^{-d}\Omega^{\infty}\Sigma^d M$$

one must check that the spectral sequence really does give an effective means of recovering $M$. This is the case, since the contribution from $H_0\mathcal{R}\Sigma^l M$ has connectivity which increases exponentially with $l$ (for $l$ sufficiently large).

Some of the consequences, including connectivity results, are summarized in the following:

**Corollary 2.** Let $M$ be a bounded-below $\mathcal{A}$-module. Then $M$ admits a natural, exhaustive filtration in $M$:

$$0 \subset v_0 M \subset v_1 M \subset \ldots \subset v_t M \subset \ldots \subset M$$

such that

1. $v_0 M = \Sigma^{-1}\Omega^{\infty}\Sigma M$;
2. for $t \in \mathbb{N}$,
   $$v_t M/v_{t-1} M \subset \ker \{ \mathcal{L}_t\Sigma^{-1}\Omega^{\infty}\Sigma^{1+t} M \to \mathcal{L}_{t-1}\Sigma^{-1}\Omega^{\infty}_1\Sigma^t M \};$$
3. if $M$ is $c$-connected, then $v_t M/v_{t-1} M$ is at least $(2^t(d(M) + 1) - (t + 2))$-connected, where $d(M) = \sup\{ (c + t + 1), -1 \}$;
4. the module $v_t M \subset M$ admits a finite filtration such that the associated graded satisfies:
   $$\Sigma^{2^t} \text{gr}(v_t M) \in \mathcal{U}$$

is unstable.

Moreover, there is a natural morphism of filtered objects relating the respective filtrations for $M$ and $\Sigma M$:

$$v_t M \hookrightarrow \Sigma^{-1}v_t \Sigma M$$

which, for $t = 0$, is the natural inclusion:

$$\Sigma^{-1}\Omega^{\infty}\Sigma M \hookrightarrow \Sigma^{-2}\Omega^{\infty}\Sigma^2 M. \tag{10.2}$$

In the case $M = \Sigma^n\mathcal{A}^*$, for $n \in \mathbb{Z}$, one obtains a skewed length filtration of the (dual) Steenrod algebra (see Section 10.2). This should be compared with Miller’s results, which exhibits the length filtration of $\mathcal{A}/\mathcal{A}\text{Sq}^1$ arising from his spectral sequence. It is of interest that an analogous calculation occurs in the work of Haugseng and Miller [HM16], which is ‘dual’, being related to the Dyer-Lashof algebra.

Clearly, if $M = \Sigma^{-1}N$ where $N$ is unstable, one has $v_0 M = M$ and the filtration is constant. The case $M = \Sigma^{-2}N$ ($N$ as above), already shows up non-trivial behaviour and passage to further desuspensions increases complexity (see Section 10.1).

2. Preliminaries

### 2.1. Recollections on $\mathcal{A}$-modules

Throughout $\mathbb{F}$ denotes the prime field of characteristic two and $\mathcal{A}$ the mod-2 Steenrod algebra. Unless otherwise stated, all $\mathcal{A}$-modules are homological, so that operations act on the right and $\text{Sq}^i$ has homological degree $-i$.

Here and in Section 3 the conventions of Kuhn and McCarty [KMc13] are followed; for unstable modules in the cohomological framework (ie. with $\mathcal{A}$ acting on the left), see [Sch94].

Duality permits the passage between the homological and cohomological settings, restricting to an equivalence of categories on objects of finite type. In particular, if $M$ is a cohomological $\mathcal{A}$-module, $M^*$ denotes the dual homological $\mathcal{A}$-module.
Notation 2.1. (Cf. [KM13].) Denote by

1. $\mathcal{M}$ the category of locally-finite $\mathcal{A}$-modules and $\mathcal{M}_\mathcal{A} \subset \mathcal{M}$ the full subcategory of modules that are bounded below;
2. $\Sigma : \mathcal{M} \to \mathcal{M}$ the suspension functor;
3. $\Phi : \mathcal{M} \to \mathcal{M}$ the Frobenius functor which doubles degrees;
4. $\mathcal{U} \subset \mathcal{M} \subset \mathcal{M}$ the full subcategory of unstable modules;
5. $\Omega^\infty : \mathcal{M} \to \mathcal{U}$ the right adjoint to the inclusion $\mathcal{U} \hookrightarrow \mathcal{M}$, with right derived functors $\Omega_i^\infty$, $i \geq 0$.

Notation 2.2. For $M$ a homologically $\mathbb{Z}$-graded object and $n \in \mathbb{Z}$, let $M_n$ denote the degree $n$ component of $M$.

Definition 2.3. A $\mathbb{Z}$-graded object $M$ is $c$-connected if $M_n = 0$ for $n \leq c$. In particular $M$ is $(-1)$-connected if and only if it is concentrated in non-negative degrees.

Example 2.4. The dual Steenrod algebra $\mathcal{A}^*$ is an $(-1)$-connected object of $\mathcal{M}$ hence, for any $n \in \mathbb{Z}$, $\Sigma^n \mathcal{A}^*$ lies in $\mathcal{M}$. The higher derived functors of $\Omega^\infty$ vanish on these objects:

$$\Omega_i^\infty \Sigma^n \mathcal{A}^* = 0$$

for $i > 0$. For $i = 0$, one has $\Omega^\infty \Sigma^n \mathcal{A}^* = F(n)^* \in \mathcal{U}$, where $F(n)$ is the free (cohomological) unstable module on a generator of degree $n$, in particular is zero for $n < 0$.

The categories $\mathcal{M}$, $\mathcal{M}_\mathcal{A}$ and $\mathcal{U}$ are abelian and the functors $\Sigma$ and $\Phi$ are exact. The following Proposition is the homological version of a standard result for cohomological unstable modules:

Proposition 2.5. The functors $\Sigma$, $\Phi$ restrict to endofunctors of $\mathcal{U}$ and the top Steenrod operation induces a natural transformation $Sq^0 : 1 \to \Phi$.

The suspension $\Sigma$ has right adjoint $\Omega : \mathcal{U} \to \mathcal{U}$ which fits into the natural exact sequence

$$0 \to \Sigma\Omega M \to M \xrightarrow{Sq^0} \Phi M \to \Sigma\Omega_1 M \to 0,$$

for $M \in \mathcal{U}$, where $\Omega_1$ is the first right derived functor of $\Omega$ and the higher derived functors vanish.

Definition 2.6. For $M \in \mathcal{U}$, let $q_0 : \Phi \Sigma \Omega M \to \Sigma \Omega_1 M$ denote the natural transformation given by the composite

$$\Phi \Sigma \Omega M \hookrightarrow \Phi M \to \Sigma \Omega_1 M$$

of the morphisms from the exact sequence (1) of Proposition 2.5.

Remark 2.7. The notation $q_0 : \Phi \Sigma \Omega M \to \Sigma \Omega_1 M$ indicates a relationship with the Dyer-Lashof operation $Q_0$ (cf. Section 3).

The derived functors $\Omega_s^\infty$ ($s \in \mathbb{N}$) and $\Omega_i$ ($i \in \{0, 1\}$) are related by a short exact sequence derived from the Grothendieck spectral sequence for the composite of functors. Short exact sequences of this form go back to the work of Singer (cf. [Sin80] for example).

Proposition 2.8. For $M \in \mathcal{M}$ and $s \in \mathbb{N}$, there is a natural short exact sequence in $\mathcal{M}$:

$$0 \to \Omega_1 \Omega_s^\infty M \to \Omega_{s+1}^\infty \Sigma^{-1} M \to \Omega_{s+1}^\infty \Omega_1 M \to 0.$$
2.2. A delooping spectral sequence. As a warm-up to considering the passage between unstable \(\mathcal{A}\)-modules and \(\mathcal{A}\)-modules, consider delooping an unstable module of the form \(\Omega M\).

**Definition 2.9.** For \(M \in \mathcal{U}\), let \(\mathfrak{f}_i M\) denote the natural increasing filtration of \(M\) defined by \(\mathfrak{f}_{-1} M = 0\) and

\[
\mathfrak{f}_i M := \ker\{M \xrightarrow{\Phi^i} \Phi^{i+1} M\}
\]

for \(i \in \mathbb{N}\).

**Remark 2.10.** By definition, \(\mathfrak{f}_{-1} M = 0\) and \(\mathfrak{f}_0 M = \Sigma \Omega M\). Moreover, \(\bigcup_i \mathfrak{f}_i M = M\), the submodule of elements in positive degree. Hence the filtration is exhaustive if and only if \(M\) is 0-connected.

The Hilbert series of an \(\mathbb{N}\)-graded \(F\)-vector space \(V\) is defined by \(H_V(t) := \sum_{n \geq 0} \dim V_n t^n\). One has \(H_{\Phi V}(t) = H_V(t^2)\).

**Remark 2.11.** If \(M\) is 0-connected, then the Hilbert series of \(M\) is determined recursively by those of \(\Omega M\) and \(\Omega_1 M\) by using the Euler characteristic of the exact sequence \([1]\) of Proposition 2.5 which gives the identity:

\[
H_M(t) - H_M(t^2) = t(H_{\Omega M}(t) - H_{\Omega_1 M}(t)).
\]

This observation can be refined using the spectral sequence associated to the filtration of Definition 2.9.

The filtration \(\mathfrak{f}_i M\) is based upon the action of \(S_{\mathbb{Q}}\) upon \(M\), that is \(M\) considered as an \(\mathbb{N}\)-graded vector space equipped with a *Verschiebung*. These objects form an abelian category \(\mathcal{U}^F\) (using the notation of [GLM92 Section 1.1.1]); the full subcategory of 0-connected objects is denoted \(\mathcal{U}^F_0\). The category \(\mathcal{U}^F_0\) splits as a product of categories indexed by the prime to 2 part of the degree; each of the factors is equivalent to the category of graded modules concentrated in negative degree over the graded polynomial ring \(F[x]\) with \(x\) in degree one, hence the homological algebra of these categories is well understood.

Consider \(M \in \mathcal{U}^F_0\) of finite type. There is a minimal injective resolution in \(\mathcal{U}^F_0\) of the form

\[
0 \to M \to I_0 \to I_1 \to 0
\]

and, borrowing the notation used for \(\mathcal{U}\), the socle of \(I_0\) is isomorphic to \(\Sigma \Omega M\) and the socle to \(I_1\) to \(\Sigma \Omega_1 M\).

The natural filtration of Definition 2.9 provides a filtered complex \(\mathfrak{f}_i I\), and hence an associated spectral sequence, which has a particularly simple form. The \(E^0\)-page is concentrated on the diagonals of total (cohomological) degree 0 and 1, the differential \(d_0\) is trivial (due to the hypothesis of minimality) and the differential \(d_1\) is induced by the natural transformation \(q_0\) of Definition 2.6.

The hypothesis that the resolution be minimal served only to deduce that the differential \(d_0\) is trivial; in the general case, the spectral sequence identifies as above from the \(E^1\)-page. This allows the result to be made more precise when working with \(\mathcal{U}\). In the following, homological indexing is used, so that \(I_1\) is placed in degree \(-1\).

**Theorem 2.12.** Let \(M \in \mathcal{U}\) be 0-connected and of finite type. Then there is a strongly convergent homological spectral sequence of unstable modules that is concentrated in the fourth quadrant and satisfies the following properties.

1. The non-zero terms of the \(E^1\)-page are:

\[
E^1_{p,-p} \cong \Phi^p \Sigma \Omega M
\]

\[
E^1_{p,-p-1} \cong \Phi^p \Sigma \Omega_1 M
\]
for \( p \geq 0 \), with \( d_1 : E^1_{p,-p} \to E^1_{p-1,-p} \), zero for \( p = 0 \) and \( \Phi^{-1}q_0 \), for \( p > 0 \).

(2) The \( E^2 \)-page is given by

\[
E^2_{p,0} \cong \Sigma \Omega M \\
E^2_{p,-p} \cong \Phi^{-1}\ker q_0 \\
E^2_{p,-p-1} \cong \Phi^p\coker q_0
\]

for \( p > 0 \).

(3) For \( r \geq 1 \), the differential \( d^r \) is determined by \( d^r : E^r_{r,-r} \to E^r_{0,0} \) for \( n \in \mathbb{N} \) via:

\[
E^r_{r+n,-r+n} \to E^r_{n,n-1} = \Phi^n(E^r_{r,-r} \to E^r_{0,-1}).
\]

(4) The \( E^\infty \)-page is concentrated on the anti-diagonal, with \( E^\infty_{p,-p} \cong f_p M/f_{p-1}M \).

Proof. (Indications.) Because of the finite-type hypothesis, one can dualize and work with cohomological unstable modules, where the argument may be more familiar. In this case, \( M \) admits a projective resolution by 0-connected projectives of finite type. Since unstable projectives are reduced, this is also a projective resolution in vector spaces with a Frobenius. The spectral sequence is then constructed as indicated in the preceding discussion. \( \square \)

Remark 2.13.

(1) Theorem 2.12 provides a spectral sequence calculating \( M \) with \( E^2 \)-page expressed functorially in terms of \((\Sigma \Omega M, \Sigma \Omega_1 M, q_0)\). Clearly convergence requires that \( M \) be 0-connected.

(2) The spectral sequence gives a systematic way of encoding the information contained in the exact sequence (1) of Proposition 2.5.

3. Dyer-Lashof actions

This section introduces the algebraic categories which underlie the constructions of the paper, in particular the category \( \mathbb{Q}M \) of modules equipped with actions of both the Steenrod algebra and Dyer-Lashof operations (satisfying additional conditions).

3.1. Introducing the Dyer-Lashof operations. As in [KM13], Dyer-Lashof operations \( Q^i \) are indexed by \( i \in \mathbb{Z} \), with \( |Q^i| = i \). (This should be contrasted with [Mil78], where no negative Dyer-Lashof operations are required.) The lower indexing of Dyer-Lashof operations is useful:

Notation 3.1. Suppose that the operations \( Q^i \) act on the \( \mathbb{Z} \)-graded module \( M \); for \( a \in \mathbb{Z} \) define \( Q_a : \Sigma^a \Phi M \to M \) by \( Q_ax := Q^{a+i}x \) (so that \( Q_a \) is a linear map of degree zero).

The \( \mathcal{A} \)-modules with an action of Dyer-Lashof operations considered here satisfy:

(1) Adem relations:

\[
Q^rQ^s = \sum_i \left( \frac{i-s-1}{2i-r} \right) Q^{r+s-i}Q^i;
\]

(2) Nishida relations:

\[
(Q^r x)Sq^s = \sum_i \left( \frac{s-r}{r-2i} \right) Q^{s-r+i}(xSq^i);
\]

(3) Dyer-Lashof instability: \( Q_a \) acts trivially for \( a < 0 \).

Remark 3.2.
The instability condition implies that negative Dyer-Lashof operations act trivially on elements of non-negative degree.

The Dyer-Lashof operations generate the big Dyer-Lashof algebra, namely the homogeneous quadratic algebra generated by \( \{Q_i \mid i \in \mathbb{Z}\} \) subject to the Adem relations. The usual Dyer-Lashof algebra (as used for example in [Mil78]) is the quotient by the ideal generated by terms of negative excess.

The big Dyer-Lashof algebra is bigraded, with internal grading and length grading. The length grading plays a fundamental role here.

Notation 3.3. Let

(1) \( \widehat{QM} \) denote the category of modules over the big Dyer-Lashof algebra in \( \mathcal{M} \) which satisfy the Nishida relations;

(2) let \( QM \subset \widehat{QM} \) denote the full subcategory of objects satisfying the Dyer-Lashof instability condition.

Remark 3.4. The category \( QM \) is of principal interest here. The suspension \( \Sigma : \mathcal{M} \to \mathcal{M} \) induces a functor \( \Sigma : \widehat{QM} \to \widehat{QM} \) (as for Steenrod operations, Dyer-Lashof operations commute with the suspension). However, this does not preserve \( QM \), due to the relation

\[ Q_a \Sigma = \Sigma Q_{a+1} \]

for \( a \in \mathbb{Z} \), which allows \( Q_{-1} \) to act non-trivially on the suspension of an object of \( QM \).

Here objects \( M \in \widehat{QM} \) such that \( \Sigma^{-t} M \in QM \) (for \( t \in \mathbb{N} \)) arise occasionally; the full subcategory of such objects is denoted \( \Sigma^t QM \subset \widehat{QM} \). There is an increasing filtration

\[ QM \subset \Sigma QM \subset \Sigma^2 QM \subset \ldots \subset \widehat{QM} \]

(Cf. Proposition 3.10 below).

Lemma 3.5. The category \( QM \) is abelian and the forgetful functor \( QM \to \mathcal{M} \) is exact. Equipping an object of \( \mathcal{M} \) with trivial action by Dyer-Lashof operations defines an exact functor \( \text{triv} : \mathcal{M} \to QM \).

Proof. Straightforward. □

Proposition 3.6. [KM13] The forgetful functor \( QM \to \mathcal{M} \) has an exact left adjoint \( R : \mathcal{M} \to QM \). Moreover, \( R \) takes values in the category of bigraded modules over the big Dyer-Lashof algebra, so that for \( M \in \mathcal{M} \)

\[ RM \cong \bigoplus_{s \geq 0} R_s M \]

where \( s \) denotes the length grading and each \( R_s M \) belongs to \( \mathcal{M} \), in particular \( R_0 M = M \).

If \( M \in \mathcal{M} \) is \((d-1)\)-connected, then \( R \) is \((2^d - 1)\)-connected. In particular, if \( M \) is 0-connected, then so is \( RM \) and, if moreover \( M \) is of finite type, then so is \( RM \).

Proof. (Indications.) The functor \( R \) is defined by forming the free module on the big Dyer-Lashof algebra and imposing the instability condition. The \( A \)-action is recovered from the Nishida relations. As in the proof of [KM13] Lemma 4.19, an explicit basis of \( R_s M \) can be given in terms of allowable monomials, which shows that \( R \) is exact. The final connectivity and finite-type statements follow easily from this (compare [KM13] Lemma 4.10). □

Notation 3.7. Let \( q : QM \to \mathcal{M} \) denote the Dyer-Lashof indecomposables functor, namely the left adjoint to \( \text{triv} : \mathcal{M} \to QM \).
Lemma 3.8.

1. The functor \( q : \mathcal{QM} \to \mathcal{M} \) is right exact.
2. The composites of \( q \) with \( R : \mathcal{M} \to \mathcal{QM} \) and \( \text{triv} : \mathcal{M} \to \mathcal{QM} \) are both naturally equivalent to the identity functor of \( \mathcal{M} \).
3. For \( N \in \mathcal{QM} \) and the adjunction counit \( \omega_N : R_N \to qN \) is the natural projection (adjunction counit for \( q \)). In particular, for \( M \in \mathcal{M} \) and \( \mu_M : R\mathcal{R}M \to \mathcal{R}M \) the morphism \( q\mu_M : R\mathcal{R}M \to M \) is the projection \( \pi_M \) onto \( M = R_0M \).

Proof. Straightforward. \( \square \)

Proposition 3.9. The Frobenius functor \( \Phi : \mathcal{M} \to \mathcal{M} \) extends to an exact functor \( \Phi : \mathcal{QM} \to \mathcal{QM} \).

Restricted to \( \mathcal{QM} \), the operation \( Q_0 \) induces a natural transformation \( Q_0 : \Phi \to 1\mathcal{QM} \).

Proof. The fact that \( \Phi \) defines a functor \( \Phi : \mathcal{QM} \to \mathcal{QM} \) is a straightforward calculation with the Adem and Nishida relations (analogous to the result for \( Sq_0 \) stated in Proposition 2.5). In particular, the compatibility with the Adem relations follows from the congruence

\[
\binom{\alpha - 1}{\beta} \equiv \binom{2\alpha - 1}{2\beta} \mod 2
\]

and the compatibility with the Nishida relations from the congruence

\[
\binom{\alpha}{\beta} \equiv \binom{2\alpha}{2\beta} \mod 2.
\]

Restricting to \( \mathcal{QM} \), consider the Adem relation for \( Q^rQ_0x \), where \( Q_0 \) acts via \( Q^{[x]} \):

\[
Q^rQ_0x = \sum_i \binom{i - |x| - 1}{2i - r} Q^{r+|x|-i}Q_i^i x.
\]

For the binomial coefficient to be non-trivial, we require \( 2i \geq r \), whereas instability for the Dyer-Lashof action implies that \( Q^{r+|x|-i}Q_0 = 0 \) unless \( r + |x| - i \geq |x| + i \), namely \( r \geq 2i \). It follows that the Adem relation reduces to \( Q^rQ_0x = 0 \) if \( r \) is odd and

\[
Q^{2i}Q_0x = Q_0Q_i^i x,
\]

as required. \( \square \)

Proposition 3.10. The desuspension \( \Sigma^{-1} : \mathcal{M} \to \mathcal{M} \) extends to an exact functor \( \Sigma^{-1} : \mathcal{QM} \to \mathcal{QM} \).

This admits a left adjoint \( \Sigma : \mathcal{QM} \to \mathcal{QM} \) that fits into a natural exact sequence

\[
0 \to \Sigma^{-1}\Sigma N \to \Phi N \xrightarrow{Q_0} N \to \Sigma^{-1}\Sigma N \to 0
\]

for \( N \in \mathcal{QM} \).

Proof. Analogous to the construction of the exact sequence of Proposition 2.5. \( \square \)

Remark 3.11. The functor \( \Sigma_1 \) is the first left derived functor of \( \Sigma \) in the sense of the relative homological algebra of Section 4.

Notation 3.12. For \( M \in \mathcal{M} \), let \( \varepsilon_M : R\mathcal{R}M \to \Sigma^{-1}R\Sigma M \) denote the natural transformation in \( \mathcal{QM} \) induced by the natural inclusion \( \Sigma M \to R\Sigma M \).
Corollary 3.13. For $M \in \mathcal{M}$ there is a natural isomorphism $\Sigma \mathcal{A}M \cong \mathcal{A} \Sigma M$ and the exact sequence of Proposition 3.10 induces a natural short exact sequence

$$0 \to \Phi \mathcal{A}M \xrightarrow{Q_0} \mathcal{A}M \xrightarrow{\Sigma \mathcal{A}M} 0$$

in $\mathcal{Q}M$. In particular, $Q_0$ induces a natural inclusion in $\mathcal{M}$:

$$\Phi M \hookrightarrow \mathcal{A}_1 M.$$  

Remark 3.14. There is a natural transformation $\mathcal{A} \Phi M \to \Phi \mathcal{A}M$ for $M \in \mathcal{M}$, which is induced by the natural inclusion $\mathcal{M} \to \mathcal{A}M$ in $\mathcal{M}$. This is not injective in general, since the left hand side contains elements in odd degree.

In a similar vein to Proposition 3.10 is the following:

Proposition 3.15. For $N \in \mathcal{Q}M \cap \mathcal{Z}$, the linear map

$$S_{Q_0} : N \to \Phi N$$

is a morphism of $\mathcal{Q}M \cap \mathcal{Z}$.

Proof. This is proved by using the Nishida relations, using $\mathcal{A}$-instability (without the hypothesis of Dyer-Lashof instability). The morphism $S_{Q_0}$ is trivial on elements of odd degree, hence consider $(Q^n x) S_{Q_0}$ where $s + |x| = 2j$, so that $S_{Q_0}$ acts via $S_{Q^n}$. The Nishida relation gives

$$(Q^n x) S_{Q_0} = \sum_{i} \left( \binom{s - j}{j - 2i} Q^{s - j + i} (x S_{Q^n}) \right)$$

where $2i \leq j$. By $\mathcal{A}$-instability, the terms on the right hand side are trivial if $2i > |x|$, hence we may assume that $2i \leq |x|$. The binomial coefficient is non-trivial only if $(s - j) \geq (j - 2i)$, equivalently if $2j - s \leq 2i$. Now $|x| = 2j - s$, hence non-trivial terms occur only when both $2i \leq |x|$ and $|x| \leq 2i$, hence only when $|x| = 2i$, so that $x S_{Q^n} = x S_{Q_0}$.

For $|x| = 2i$, the relation $s + |x| = 2j$ then implies $s - j = j - 2i$ and $s = 2n$ for some $n = j - i$ and one has

$$(Q^{2n} x) S_{Q_0} = Q^n (x S_{Q_0}).$$

For $|x|$ odd, one has $(Q^n x) S_{Q_0} = 0$. □

3.2. Length grading. As observed in Remark 3.12, the big Dyer-Lashof algebra is bigraded when equipped with the length grading. Moreover, the functor $\mathcal{A} \to \mathcal{Q}M$ takes values in the category of bigraded objects of $\mathcal{Q}M$, namely those modules $N \in \mathcal{Q}M$ equipped with a length decomposition:

$$N \cong \bigoplus_{s \in \mathbb{Z}} N^{|s|}$$

such that $N^{|s|} \in \mathcal{M}$ and $Q_a : \Sigma^s \Phi N^{|s|} \to N^{|s+1|}$ (graded linear map, not $\mathcal{A}$-linear in general).

Notation 3.16.

1. Denote by $\mathcal{Q}M_{gr}$ the category of length-graded objects of $\mathcal{Q}M$ and length-grading preserving morphisms, equipped with the forgetful functor $\mathcal{Q}M_{gr} \to \mathcal{Q}M$, $N \mapsto \bigoplus_{s \in \mathbb{Z}} N^{|s|}$.
2. Denote by $\mathcal{M}_{gr}$ the category of length-graded objects of $\mathcal{M}$ and length-grading preserving morphisms, equipped with the exact forgetful functor $\mathcal{M}_{gr} \to \mathcal{M}$.
3. For $l \in \mathbb{Z}$, let $\cdot (l) : \mathcal{Q}M_{gr} \to \mathcal{Q}M_{gr}$ (respectively $\cdot (l) : \mathcal{M}_{gr} \to \mathcal{M}_{gr}$) denote the exact functor which increases length grading by $l$, so that $N(l)^{|s|} = N^{|s+l|}$. 

Remark 3.17. In applications here, the length grading is always bounded below, that is $N^{[s]} = 0$ for $s \ll 0$. Frequently the length grading will be defined only for $s \in \mathbb{N}$, in which case it is extended by zero to negative degrees.

Proposition 3.18.

1. The categories $QM^{gr}$ and $M^{gr}$ are abelian and the forgetful functor $QM^{gr} \to M^{gr}$ is exact.
2. The trivial action functor induces an exact functor $\text{triv} : M^{gr} \to QM^{gr}$.
3. For $l \in \mathbb{Z}$, the functor $\cdot (l) : QM^{gr} \to QM^{gr}$ is an equivalence of categories (respectively for $M^{gr}$) and these equivalences are compatible via the forgetful and trivial action functors.
4. The functor $q$ induces a functor $q : QM^{gr} \to M^{gr}$.
5. The functor $R$ factorizes across a functor $R : M \to QM^{gr}$ which extends to a functor $R : M^{gr} \to QM^{gr}$.
6. The functors $\Sigma^{-1}, \Sigma, \Phi$ extend to functors on $QM^{gr}$.
7. For $N \in QM^{gr}$, the morphism $Q_0$ defines a natural transformation in $QM^{gr}$:

$$\Phi N(1) \xrightarrow{Q_0} N.$$

Proof. Straightforward.

The following is the length-graded version of Proposition 3.10:

Proposition 3.19. The exact functor $\Sigma^{-1} : QM^{gr} \to QM^{gr}$ admits a left adjoint $\Sigma : QM^{gr} \to QM^{gr}$ which fits into a natural exact sequence

$$0 \to \Sigma^{-1} \Sigma N \to (\Phi N)(1) \xrightarrow{Q_0} N \to \Sigma^{-1} \Sigma N \to 0$$

for $N \in QM^{gr}$.

3.3. Length truncations.

Definition 3.20. For $l \in \mathbb{Z}$, let $\tau^{[\leq l]} : QM^{gr} \to QM^{gr}$ denote the length truncation functor defined by

$$(\tau^{[\leq l]} N)^{[s]} = \begin{cases} N^{[s]} & s \leq l \\ 0 & s > l, \end{cases}$$

for $N \in QM^{gr}$.

Clearly one has the following:

Proposition 3.21. For $l \in \mathbb{Z}$, the functor $\tau^{[\leq l]} : QM^{gr} \to QM^{gr}$ is exact and there is a commutative diagram of natural surjections:

$$\xymatrix{ N \ar[r]^{\tau^{[\leq l+1]} N} \ar[d] & \tau^{[\leq l]} N, }$$

for $N \in QM^{gr}$. 

3.4. **Connectivity estimates for** $\Sigma_1$. The following elementary result is the basis for the stable range which appears in many situations:

**Lemma 3.22.** For $M \in \mathcal{M}$ of connectivity $d - 1$, the module $\Phi M$ is $(2d - 1)$-connected.

**Proposition 3.23.** For $N \in \mathcal{QM}^{gr}$ such that $\tau_{[\leq -1]} N = 0$ and, for $i \geq 0$, $N^{[i]}$ is $d_i - 1$ connected for $d_i \in \mathbb{Z}$,

1. $\left(\Sigma_1 N\right)^{[0]} = 0$;
2. $\left(\Sigma_1 N\right)^{[i]}$ is at least $(2d_{i-1})$-connected, for $i > 0$.

**Proof.** A straightforward consequence of the exact sequence of Proposition 3.19 together with Lemma 3.22. □

4. **Relative left derived functors of** $q$

The derived functors of the Dyer-Lashof indecomposables that are introduced in this section play a central rôle in the paper.

4.1. **The class of relative projectives and relative left derived functors.**

The adjunction $\mathcal{R} : \mathcal{M} \rightleftarrows \mathcal{QM} : \text{Forget}$ defines a projective class in $\mathcal{QM}$, in the sense of relative homological algebra. The class of projective objects is $\{\mathcal{R}M | M \in \mathcal{M}\}$.

The comonad associated to the adjunction will be denoted $\mathcal{R} : \mathcal{QM} \to \mathcal{QM}$ (omitting the forgetful functor from the notation), equipped with the counit $\mu : \mathcal{R} \to 1_{\mathcal{QM}}$ (corresponding to the action map) and $\Delta : \mathcal{R} \to \mathcal{R} \mathcal{R}$, induced by the adjunction unit $1_{\mathcal{M}} \to \mathcal{R}$. For $N \in \mathcal{QM}$, this induces a simplicial object with $n$th term $\mathcal{R}^{n+1} N$ and equipped with the augmentation $\mathcal{R} N \to N$.

The associated chain complex is acyclic (as is seen, as usual, by applying the forgetful functor and applying the contracting homotopy for the augmented chain complex which is provided by the adjunction unit). Thus $\mathcal{R}^{\bullet+1} N$ provides a functorial (relative) projective resolution of $N$.

**Definition 4.1.** For $F : \mathcal{QM} \to \mathcal{C}$ a right exact additive functor to an abelian category $\mathcal{C}$, the (relative) left derived functors $L_i F : \mathcal{QM} \to \mathcal{C}$ are defined by

$$L_i F(N) := H_i(F(\mathcal{R}^{\ast+1} N)),$$

so that $L_0 F = F$.

**Remark 4.2.**

1. These left derived functors can be calculated with respect to any relative projective resolution.
2. For current purposes, one could simply define these as cotriple derived functors.

**Example 4.3.** The functor $\Sigma : \mathcal{QM} \to \mathcal{QM}$ is a left adjoint, hence is right exact. The left derived functors $L_i \Sigma$ are trivial for $i > 1$ and $L_1 \Sigma \cong \Sigma_1$, the functor appearing in Proposition 3.10.

**Example 4.4.** The indecomposables functor $q : \mathcal{QM} \to \mathcal{M}$ is right exact, hence there are derived functors

$$L_i q : \mathcal{QM} \to \mathcal{M}.$$ 

For $N \in \mathcal{QM}$, since $q \mathcal{R}$ is the identity functor on $\mathcal{M}$ (by Lemma 3.3), the complex $q(\mathcal{R}^{\ast+1} N)$ has the form

$$\ldots \to \mathcal{R}^2 N \to \mathcal{R} N \to N$$
in \( M \), equipped with the augmentation \( N \to qN \). The morphism \( \mathcal{R}N \to N \) is the difference between the \( \mathcal{R} \)-action structure morphism and the projection \( \mathcal{R}N \to q\mathcal{R}N \to N \).

In particular, this complex gives an exact functor from \( QM \) to the category \( \text{Ch}_M \) of chain complexes in \( M \).

**Definition 4.5.** Let \( c_q : QM \to \text{Ch}_M \) be the exact functor \( q\mathcal{R}^{\bullet+1} \) of Example 4.4.

**Notation 4.6.** For \( M \in M^{gr} \), let \( \mathcal{R}M \subset \mathcal{R}M \) denote the kernel of the natural projection \( \mathcal{R}M \to M \).

The resolution \( \mathcal{R}^{\bullet+1}N \), for \( N \in QM \), has a reduced subobject, with \( i \)th term \( \mathcal{R}(\mathcal{R})^iN \) (the analogue of the reduced bar construction), which is again a resolution.

**Definition 4.7.** Let \( C_q : QM \to \text{Ch}_M \) denote the sub-complex of \( C_q \) given by applying \( q \) to the reduced resolution, so that \( (C_qN)_i = (\mathcal{R})^iN \).

**Proposition 4.8.** For \( N \in QM \), the inclusion \( C_qN \hookrightarrow C_qN \) is a quasi-isomorphism.

**Proof.** Standard. □

**4.2. First properties of \( L_q \).**

**Proposition 4.9.** Objects of the projective class of \( QM \) are acyclic, namely for \( N = \mathcal{R}M \) (where \( M \in M \)), the augmented chain complex \( \mathcal{R}^{\bullet+1}N \to N \) is acyclic. In particular, \( L_iF(\mathcal{R}M) = 0 \) for \( i > 0 \) and \( L_0F(\mathcal{R}M) = F(\mathcal{R}M) \).

**Proof.** Standard: the adjunction provides an extra degeneracy, hence a contracting homotopy in \( QM \). □

**Example 4.10.** For \( M \in M \), \( L_iq(\mathcal{R}M) = 0 \) for \( i > 0 \) and \( L_0q(\mathcal{R}M) = M \).

**Proposition 4.11.** For \( 0 \to N_1 \to N_2 \to N_3 \to 0 \) a short exact sequence in \( QM \), there is a natural long exact sequence of derived functors of \( q \):

\[ \ldots \to L_iqN_1 \to L_iqN_2 \to L_iqN_3 \to L_{i-1}qN_1 \to \ldots . \]

**Proof.** Applying the exact functor \( C_q : QM \to \text{Ch}_M \) to the short exact sequence gives a short exact sequence of chain complexes

\[ 0 \to C_qN_1 \to C_qN_2 \to C_qN_3 \to 0. \]

The long exact sequence is given by passage to homology. □

The previous results can be made more precise by using the length grading, since the functor \( \mathcal{R} : M \to QM \) extends to \( \mathcal{R} : M^{gr} \to QM^{gr} \) and the forgetful functor respects length grading.

**Proposition 4.12.**

1. For \( N \in QM^{gr} \), the augmented chain complex \( \mathcal{R}^{\bullet+1}N \to N \) is defined in \( QM^{gr} \).
2. If \( N = \mathcal{R}M \in QM^{gr} \), for \( M \in M^{gr} \), then this augmented chain complex is acyclic.
3. The functor \( c_q \) extends to an exact functor \( c_q : QM^{gr} \to \text{Ch}_M^{gr} \).
4. The functor \( c_q \) commutes with the length grading shift functor \( \cdot(l) \) for \( l \in \mathbb{Z} \); namely there is a natural isomorphism \( c_q(N(l)) \cong (c_qN)(l) \).
(5) The derived functors $L_\ast q$ induce functors

$$L_\ast q : \mathcal{Q} \mathcal{M}^{\geq 0} \to \mathcal{M}^{\geq 0}.$$ 

(6) For $M \in \mathcal{M}$, $L_\ast q \mathcal{Q} \mathcal{Q} M$ is zero for $i > 0$ and $L_0 q \mathcal{Q} \mathcal{Q} M \cong M$, considered as concentrated in length 0.

Proof. Straightforward. □

The following observation is fundamental (and is analogous to a result proved in [Mil78, Section 3]).

Proposition 4.13. For $s \in \mathbb{N}$, the composite functor

$$\mathcal{M} \xrightarrow{\text{triv}} \mathcal{Q} \mathcal{M}^{\geq 0} \xrightarrow{L_{s-1}} \mathcal{M}^{\geq 0}$$

is exact.

Proof. Since the Dyer-Lashof action is trivial, the underlying bigraded vector space of $L_\ast q M$ only depends upon the underlying graded vector space $M$, whence the result follows by semisimplicity of the category of graded vector spaces and Proposition 4.12. (This may also be seen by inspection of the complex.) □

The functor $L_\ast q : \mathcal{M} \to \mathcal{M}^{\geq 0}$, when restricted to the full subcategory $\mathcal{M}_{\geq 1}$ of $(-2)$-connected objects, is identified explicitly in Theorem 5.3.1 which leads to a Koszul complex calculating these derived functors (see Corollary 5.3.2).

4.3. Relating $L_\ast q$ to Miller’s $\text{Untor}_s^R(\mathbb{F}, -)$. Miller [Mil78 Section 2.2] introduced the category $a_0R$-mod of non-negatively graded $n$-allowable modules over the Dyer-Lashof algebra. This can be enriched (as in [Mil78 Section 4], but not requiring instability of the $\mathcal{A}$-module structure) to take into account the Steenrod action (which is compatible with the Dyer-Lashof action via the Nishida relations). This gives the category $a_0\mathcal{A}^0 R$-mod of non-negatively graded $n$-allowable modules over the Dyer-Lashof and Steenrod algebras.

Remark 4.14. The restriction to non-negatively graded objects is implicit in [Mil78].

The category $a_0\mathcal{A}^0 R$-mod is $\mathcal{Q} \mathcal{M}^{\geq 0}$, the full subcategory of $\mathcal{Q} \mathcal{M}$ of non-negatively graded objects. The category $a_1\mathcal{A}^0 R$-mod is the full subcategory of $a_0\mathcal{A}^0 R$-mod of modules such that $\Sigma N \in \mathcal{Q} \mathcal{M}$ (equivalently, $Q_0$ acts trivially on $N$).

Forgetting the Dyer-Lashof instability condition, an object of $a_1 R$-mod is an $R(-\infty)$-module, where $R(-\infty)$ is the quotient of the big Dyer-Lashof algebra by the two-sided ideal generated by the operations $Q_i$ with $i < 0$. This inclusion fits into an adjunction

$$R(-\infty) \text{-mod} \xrightarrow{\cong} a_1 R \text{-mod}$$

which Miller uses to define a projective class. The analogous construction works for $a_1\mathcal{A}^0 R$-mod: the projectives are of the form $\Sigma^{-1} \mathcal{A} \Sigma M$, where $M \in \mathcal{M}$ is non-negatively graded.

Remark 4.15. The category $\mathcal{Q} \mathcal{M}$ is equivalent to the category of $\mathcal{A}$-modules in $\mathcal{M}$. The above shows that $a_1\mathcal{A}^0 R$-mod is the category of $\Sigma^{-1} \mathcal{A} \Sigma$-modules in $\mathcal{M}_{\geq 0}$, where $\mathcal{M}_{\geq 0} \subset \mathcal{M}$ is the full subcategory of $(-1)$-connected objects. Relative projective resolutions are associated as before to the functor $\Sigma^{-1} \mathcal{A} \Sigma$.

This can be seen explicitly by considering the reduced bar resolution used in [Mil78 Section 3], where the functor $U : \mathbb{F}_2 \text{-mod} \to a_1 R \text{-mod}$ extends to $\mathcal{M}_{\geq 0} \to a_1 \mathcal{A}^0 R$-mod and the latter is the restriction of $\Sigma^{-1} \mathcal{A} \Sigma$ to $\mathcal{M}_{\geq 0}$.

Definition 4.16. (Cf. [Mil78 Section 2.2] and [Mil78 Section 4].) Let $\text{Untor}_s^R(\mathbb{F}, -)$ be the left derived functors of the composite $a_1\mathcal{A}^0 R$-mod $\to \mathcal{Q} \mathcal{M} \to \mathcal{M}$ with respect to the projective class defined by $\Sigma^{-1} \mathcal{A} \Sigma$. 


Proposition 4.17. For \( N \in \alpha_1 \mathcal{A}^0 R\text{-mod} \), there is a natural isomorphism
\[
\Sigma \text{Untor}_R^\mathbb{L} (F, N) \cong \mathbb{L}_* q(\Sigma N).
\]

Proof. Straightforward (using the fact that \( q \) commutes with \( \Sigma^{-1} \)). \( \square \)

Example 4.18. Consider \( N := \Sigma^{-1} \mathcal{R} \Sigma M \), for \( M \in \mathcal{M}^{\geq 0} \). Observe that \( N \) is the quotient of \( R^\mathbb{L} M \) by the image of \( Q_0 \), by Proposition 3.10.

Then \[
\text{Untor}_R^\mathbb{L} (F, \Sigma^{-1} \mathcal{R} \Sigma M) \cong \begin{cases} 0 & * > 0 \\ M & * = 0. \end{cases}
\]

5. The Steinberg functors and the Koszul complex for \( \mathbb{L}_* q \)

The Steinberg functors \( \mathcal{L}_s \) that are introduced in this section are of independent interest. Their importance here is through the rôle that they play in the Koszul complex (see Definition 5.37) with homology the calculating the derived functors \( L_* q \) (see Corollary 5.40).

5.1. The Steinberg functor. The functor \( \mathcal{R} : \mathcal{M} \to \mathcal{Q} \mathcal{M}^{\mathbb{G}} \) has length components \( \mathcal{R}_s : \mathcal{M} \to \mathcal{M} \) for \( s \in \mathbb{N} \), where \( \mathcal{R}_s \) is the \( s \)th Singer functor (this should be taken as the definition of the Singer functors here). The adjunction unit induces natural transformations \( \mathcal{R}_s \mathcal{R}_t \to \mathcal{R}_{s+t} \) and, in particular, there is a natural surjection \( (\mathcal{R}_1)^s \to \mathcal{R}_s \).

The Singer functors have a quadratic (co)presentation, and are thus determined by the surjection \( (\mathcal{R}_1)^2 \to \mathcal{R}_2 \). The Steinberg functors are defined by the quadratic dual construction.

Definition 5.1. For \( s \in \mathbb{N} \), define the Steinberg functor \( \mathcal{L}_s : \mathcal{M} \to \mathcal{M} \) on \( M \in \mathcal{M} \) by:
\[
\mathcal{L}_s M := \bigcap_{i=0}^{s-2} \ker \{ (\mathcal{R}_1)^s M \to (\mathcal{R}_1)^i \mathcal{R}_2(\mathcal{R}_1)^{s-(i+2)} M \}
\]

for the natural transformations induced by \( (\mathcal{R}_1)^2 \to \mathcal{R}_2 \).

The following records properties of the Steinberg functors.

Proposition 5.2. For \( s \in \mathbb{N} \), the following properties hold.

1. The functor \( \mathcal{L}_s \) is exact and there is a natural inclusion \( \mathcal{L}_s \hookrightarrow (\mathcal{R}_1)^s \) that is an isomorphism for \( s \in \{0, 1\} \).

2. For \( s = 2 \), there is a natural short exact sequence
\[
0 \to \mathcal{L}_2 \to (\mathcal{R}_1)^2 \to \mathcal{R}_2 \to 0.
\]

3. For \( s_1, s_2 \in \mathbb{N} \), there is a natural inclusion
\[
\mathcal{L}_{s_1 + s_2} \hookrightarrow \mathcal{L}_{s_1} \mathcal{L}_{s_2}
\]
and, for varying \( s_1, s_2 \), these are coassociative. In particular, for \( s > 0 \), there is a natural inclusion \( \mathcal{L}_s \hookrightarrow \mathcal{L}_{s-1} \mathcal{L}_1 \).

Proof. The exactness of \( \mathcal{L}_s \) can be checked as for \( \mathcal{R}_s \). Indeed, as for the proof of Proposition 4.13, the underlying graded vector space of \( \mathcal{L}_s M \) depends only upon that of \( M \). The remaining statements are clear. \( \square \)
Corollary 5.3. For $2 \leq s \in \mathbb{N}$ and $M \in \mathcal{M}$, there is a natural isomorphism:

$$\mathcal{L}_s M := \bigcap_{i=0}^{s-2} (\mathcal{L}_1)^i \mathcal{L}_2 (\mathcal{L}_1)^{s-(i+2)} M,$$

where the intersection is formed within $(\mathcal{L}_1)^s M$.

Proof. Follows from the exactness of $\mathcal{L}_1$ and the short exact sequence identifying $\mathcal{L}_2$. □

This constructs the Steinberg functors $\mathcal{L}_s$ as the quadratic duals to the Singer functors $\mathcal{H}_s$. An alternative approach, again explaining the dual nature of the construction, uses the Hecke algebra, as in [Kuh15, Section 4], as sketched below.

Notation 5.4. For $s \in \mathbb{N}$, let

1. $\mathcal{H}_s$ denote the Hecke algebra of type $A_{s-1}$ defined by

   $$\mathcal{H}_s := \text{End}_{\mathcal{L}_1}(F[\mathbb{F}_s \backslash \mathbb{F}])$$

   where $B_s < GL_s$ is the Borel subgroup of upper triangular matrices (for $s = 0$, take $\mathcal{H}_0 = F$);

2. $D(s)$ denote the Dickson algebra $H^*(BV_s^{GL_s})$ (a cohomological unstable algebra), where $V_s$ is an elementary abelian 2-group of rank $s$;

3. $D(s)\mathcal{H}$, the category of $D(s)$-modules in cohomological unstable modules.

Proposition 5.5. For $s \in \mathbb{N}$, the Hecke algebra $\mathcal{H}_s$ acts by natural transformations upon the functor $(\mathcal{H}_1)^*: \mathcal{M} \to \mathcal{M}$.

Proof. It is more transparent to present this proof for the cohomological Singer functors. For unstable modules these are considered by Lannes and Zarati [LZ87] and they extend to all $\mathcal{A}$-modules as in [Pow14] (which is written for the odd primary case). To avoid confusion, denote the cohomological Singer functors by $R_s$.

For $s \in \mathbb{N}$, $(R_1)^s F$ is isomorphic to the unstable algebra $H^*(BV_s)^{B_s}$ and $R_s F$ is the Dickson algebra $D(s)$. The Hecke algebra $\mathcal{H}_s$ acts via morphisms of $D(s)\mathcal{H}$:

$$\mathcal{H}_s \to \text{End}_{D(s)\mathcal{H}} (H^*(BV_s)^{B_s}).$$

Now, for a (cohomological) $\mathcal{A}$-module $M$, $R_s M$ is a $D(s)$-module in the category of $\mathcal{A}$-modules and there is a natural isomorphism of $\mathcal{A}$-modules:

$$(R_1)^s M \cong H^*(BV_s^{B_s}) \otimes_{D(s)} R_s M.$$

The action of $\mathcal{H}_s$ on $H^*(BV_s)^{B_s}$ therefore induces a natural action on $(R_1)^s M$. (Moreover, this action is $D(s)$-linear.) □

The Hecke algebra $\mathcal{H}_s$ is equipped with the involution $\sim$: $\mathcal{H}_s \to \mathcal{H}_s$ (see [Kuh15 Proposition 4.4]) and contains the idempotent $e_s \in \mathcal{H}_s$ of [Kuh15 Definition/Proposition 4.6] and the corresponding idempotent $\hat{e}_s$.

We record the following generalization of the results of [Kuh15]:

Proposition 5.6. For $s \in \mathbb{N}$ and $M \in \mathcal{M}$, there are natural isomorphisms:

$$\mathcal{B}_s M \cong \hat{e}_s (\mathcal{H}_1)^s M$$

$$\mathcal{L}_s M \cong e_s (\mathcal{H}_1)^s M.$$

Proof. The result follows from [Kuh15 Corollary 4.9] and [Kuh15 Corollary 4.10]. □

Notation 5.7. [HSN10] Proposition 2.2] For $s \in \mathbb{N}$, denote by
(1) \(M_s\) the cohomological unstable module defined using the Steinberg idempotent \(\text{St}_s \in F[GL_s]\):
\[
M_s := \text{St}_s H^*(BV_s)
\]
for \(V_s\) an elementary abelian 2-group of rank \(s\);
(2) \(L_s\) the cohomological unstable module defined by the canonical decomposition
\[
M_s \cong L_s \oplus L_{s-1}.
\]

Remark 5.8. The notation \(L_s\) must not be confused with that used in [Kuh15], where a suspension is introduced.

Notation 5.9. Write the top Dickson invariant as \(\omega_s \in D(s)\) (this is the product of the non-zero classes of \(H^1(BV_s)\) so that \(|\omega_s| = 2^s - 1\); see [HSN10], for example).

Remark 5.10. By construction, \(M_s\) belongs to \(D(s)\)-\(\mathcal{W}\) and, by [HSN10] Proposition 2.3, \(L_s = \omega_s M_s\), in particular \(L_s \subset M_s\) is a sub-object in \(D(s)\)-\(\mathcal{W}\).

In the cohomological setting, it follows from [Kuh15] Section 4.2 that the Steinberg functor corresponds to the functor
\[
M \mapsto M_s \otimes_{D(s)} R_s M.
\]

Corollary 5.11. For \(s \in \mathbb{N}\), there is a natural isomorphism in \(\mathcal{M}\):
\[
\mathcal{L}_s F \cong M_s^*.
\]

Proof. This follows from Proposition [5.6] \(\square\)

Remark 5.12. Using [Kuh15] Theorem 6.3, Kuhn deduces that \(\mathcal{L}_s \Sigma F \cong \Sigma L_s^*\) (using the notation adopted here, rather than Kuhn’s), so that the right hand side should be understood as \(\Sigma(\omega_s M_s)^*\).

5.2. The Steinberg functor and suspension.

Proposition 5.13. The natural surjection \(\Sigma \mathcal{A}_1 \twoheadrightarrow \mathcal{A}_1 \Sigma\), given by the restriction of \(\varepsilon : \Sigma \mathcal{A} \twoheadrightarrow \mathcal{A} \Sigma\) to length one, induces a natural surjection for \(s \in \mathbb{N}\)
\[
\Sigma \mathcal{L}_s \twoheadrightarrow \mathcal{L}_s \Sigma.
\]

Proof. Clearly \(\varepsilon\) induces a natural surjection \(\Sigma(\mathcal{A}_1)^* \twoheadrightarrow (\mathcal{A}_1)^* \Sigma\). The result follows by applying the idempotent \(e_s\). \(\square\)

The above suspension morphism fits into a short exact sequence analogous to that of Corollary 5.11.

Proposition 5.14. For \(M \in \mathcal{A}\) that is of finite type and \(s \in \mathbb{N}\), the suspension morphism fits into a natural short exact sequence
\[
0 \rightarrow \Sigma^{-1} \mathcal{L}_{s-1} \Sigma \Phi M \rightarrow \mathcal{L}_s M \rightarrow \Sigma^{-1} \mathcal{L}_s \Sigma M \rightarrow 0.
\]

Proof. It is convenient to give the proof working in cohomological \(\mathcal{A}\)-modules, using the hypothesis that \(M\) is of finite type and bounded below to translate to this setting.

Using the cohomological Singer functors \(R_s\), there is a natural short exact sequence
\[
0 \rightarrow \Sigma^{-1} R_s \Sigma N \rightarrow R_s N \rightarrow \Phi R_{s-1} N \rightarrow 0
\]
of \(D(s)\)-modules in \(\mathcal{M}\), where \(D(s)\) acts upon \(\Phi R_{s-1} N\) via the natural surjection \(D(s) \twoheadrightarrow \Phi D(s - 1)\) which has kernel \(\omega_s D(s)\).

The underlying \(D(s)\)-module of \(M_s\) is free (cf. for example the 2-primary case of [Mit85] Corollary 3.11), hence applying the functor \(M_s \otimes_{D(s)} \_\) yields an exact sequence
\[
0 \rightarrow \Sigma^{-1} L_s' \Sigma N \rightarrow L_s' N \rightarrow M_s \otimes_{D(s)} \Phi R_{s-1} N \rightarrow 0
\]
L isomorphism of $H$ hence there is a natural isomorphism

$$D(s) \otimes_{D(s)} R_s(-) \cong \Phi D(s-1) \Phi R_{s-1} N \cong R_{s-1} \Phi N.$$ 

Hence, there is a natural isomorphism

$$M_s \otimes_{D(s)} \Phi R_{s-1} N \cong L_{s-1} \otimes_{D(s-1)} R_{s-1} \Phi N,$$

using the identification $M_s/\omega_s M_s \cong L_{s-1}$ as $D(s)$-modules, where $D(s)$ acts on $L_{s-1}$ via the composite $D(s) \to \Phi D(s-1) \to D(s-1)$ and the $D(s-1)$-module structure of $L_{s-1}$.

Now, $L_{s-1}$ identifies as $\omega_{s-1} M_{s-1}$, hence there is a natural isomorphism

$$L_{s-1} \otimes_{D(s-1)} R_{s-1} \Phi N \cong \Sigma^{-1} M_{s-1} \otimes_{D(s-1)} R_{s-1} \Sigma \Phi N$$

and, by definition, the latter is $\Sigma^{-1} L_{s-1} \Sigma \Phi N$. This provides the required short exact sequence.

\[ \square \]

**Remark 5.15.** For $s = 1$, the short exact sequence is

$$0 \to \Phi M \to \mathcal{L}_1 M \to \Sigma^{-1} \mathcal{L}_2 \Sigma M \to 0$$

(where $\mathcal{L}_1$ identifies with $R_1$).

For $s > 1$, by composing with the natural surjection $\mathcal{L}_{s-1} \Phi M \to \Sigma^{-1} \mathcal{L}_{s-1} \Sigma \Phi M$, the inclusion of the kernel induces a natural transformation

$$\mathcal{L}_{s-1} \Phi M \to \mathcal{L}_s M.$$ 

Upon composing with the natural inclusion $\mathcal{L}_s M \hookrightarrow \mathcal{L}_{s-1} \mathcal{L}_1 M$, one obtains a natural transformation $\mathcal{L}_{s-1} \Phi M \to \mathcal{L}_{s-1} \mathcal{L}_1 M$. This is not the natural inclusion induced by $\Phi M \hookrightarrow \mathcal{L}_1 M$.

Rather, the natural transformation (2) is given as the composite

$$\mathcal{L}_{s-1} \Phi M \to \mathcal{L}_{s-1} \mathcal{L}_1 M \to \mathcal{L}_s M$$

decreasing this inclusion with the projection induced by the Steinberg idempotent. The fact that this factorizes across $\Sigma^{-1} \mathcal{L}_{s-1} \Sigma \Phi M$ follows, since the Steinberg relation imposes 'complete unallowability' (as in [Kuh13]).

**5.3. The Steinberg functor and instability.** It is important to understand the behaviour of the functors $R_s$, $(R_1)^s$ and $\mathcal{L}_s$ when applied to modules which are iterated suspensions of unstable modules. It is a fundamental fact that these functors preserve instability.

**Proposition 5.16.** For $0 < s \in \mathbb{N}$ and $M \in \mathcal{W}$,

1. $(R_1)^s M$ is unstable;
2. for $0 < d \in \mathbb{N}$, $(R_1)^s \Sigma^{-d} M$ admits a finite filtration with associated graded $\text{gr}((R_1)^s \Sigma^{-d} M)$ such that

$$\Sigma^{2d} \text{gr}((R_1)^s \Sigma^{-d} M)$$

is unstable.

**Proof.** Since $R_1$ is an exact functor, by induction upon $s$ it suffices to consider the case $s = 1$. This case is treated by induction upon $d$, the case $d = 0$ following from the fact that $R_1$ restricts to a functor $R_1 : \mathcal{W} \to \mathcal{W}$.

For the inductive step, consider the exact sequence derived from Proposition 3.18,

$$0 \to \Phi \Sigma^{-d} M \to R_1 \Sigma^{-d} M \to \Sigma^{-1} \mathcal{R}_1 \Sigma^{-(d-1)} M \to 0.$$
The right hand term is treated by the inductive hypothesis and the left hand term by using the natural isomorphism $\Phi \Sigma^{-d} M \cong \Sigma^{-2d} \Phi M$, where $\Phi M$ is unstable. □

**Corollary 5.17.** For $0 < s \in \mathbb{N}$ and $M \in \mathcal{W}$,

1. $\mathcal{R}_s M$ and $\mathcal{L}_s M$ are unstable;
2. for $0 < d \in \mathbb{N}$, $\mathcal{R}_s \Sigma^{-d} M$ (respectively $\mathcal{L}_s \Sigma^{-d} M$) admit finite filtrations with associated graded $\text{gr}(\mathcal{R}_s \Sigma^{-d} M)$ (respectively $\text{gr}(\mathcal{L}_s \Sigma^{-d} M)$) such that

$$\Sigma^{2d} \text{gr}(\mathcal{R}_s \Sigma^{-d} M)$$

$$\Sigma^{2d} \text{gr}(\mathcal{L}_s \Sigma^{-d} M)$$

are unstable.

**Proof.** A straightforward consequence of Proposition 5.16 □

**Remark 5.18.** The analysis of $\mathcal{L}_s \Sigma^{-d}$ can be refined by using induction upon $s$ and $d$ together with the short exact sequence provided by Proposition 5.14.

5.4. **Identifying $\mathcal{L}_s \Sigma^d\mathcal{F}$**, Corollary 5.11 identifies $\mathcal{L}_s \mathcal{F}$, for $s \in \mathbb{N}$; this can be generalized to consider $\mathcal{L}_s \Sigma^d\mathcal{F}$ for all $d \in \mathbb{Z}$. It is conceptually clearer to present this in the cohomological setting.

The localized algebra $D(s)[\omega_s^{-1}]$ is an algebra in cohomological $\mathcal{A}$-modules and the construction of the Singer functors shows that

$$\Sigma^{-d} R_s \Sigma^d \mathcal{F} = \omega_d^d D(s) \subset D(s)[\omega_s^{-1}],$$

for any $d \in \mathbb{Z}$, as $D(s)$-modules in $\mathcal{A}$-modules.

As in Remark 5.10, it follows that there is an isomorphism

$$(\Sigma^{-d} \mathcal{L}_s \Sigma^d \mathcal{F})^* \cong \omega_d^d M_s$$

where the right hand side is understood as a sub-object of $M_s[\omega_s^{-1}] := M_s \otimes_{D(s)} D(s)[\omega_s^{-1}]$. (Here, $\omega_d^d M_s$ is of finite type, so the duality causes no difficulty.)

In particular, for $d \geq 0$, this gives a decreasing filtration with

$$\omega_d^d M_s = \omega_d^{d-1} L_s \subseteq L_s, \text{ for } d > 0.$$

For current purposes, it is sufficient to describe bases of the underlying graded vector spaces. Recall the following standard definition:

**Definition 5.19.**

1. A sequence of length $s$ (for $s \in \mathbb{N}$) is an ordered sequence $\{i_1, \ldots, i_s\} \in \mathbb{Z}^s$ (empty if $s = 0$). The degree of $I$ is $d(I) := \sum_j i_j$ and the length $s$ is denoted by $l(I)$.
2. The sequence $I$ is admissible if $i_j \geq 2i_{j+1}$ for all $0 \leq j < s$; the excess of an admissible sequence $I$ is $e(I) = \sum_j (i_j - 2i_{j+1}).$

**Notation 5.20.** For applications here, sequences will usually lie in $\mathbb{N}^s$ and, often, will be sequences of positive integers (that is lying in $\mathbb{N}^*_s$).

However, much of this material should be considered in the context of the big Steenrod algebra (cf. [BCL05] and [Pow11]), where the indexing is by $\mathbb{Z}$.

**Proposition 5.21.** For $0 < s \in \mathbb{N}$ and $d \in \mathbb{Z}$, the graded vector space $\mathcal{L}_s \Sigma^d \mathcal{F}$ has a basis indexed by admissible sequences of length $s$:

$$\{\sigma_{l,d-s}|\text{admissible, } i_s > d\}$$

where $\sigma_{l,d-s}$ has degree $d(I) + d - s$.

**Proof.** (Indications.) This is best understood in the cohomological setting, so that the above refers to a dual basis. For $d = 0$, this is standard. The effect of multiplying by $\omega_s$ is to shift an admissible sequence by the excess zero sequence $(2^{s-1}, 2^{s-2}, \ldots, 1)$ of length $s$. □
Remark 5.22. For \( d \geq 1 \), this result can be deduced from [Mi78, Proposition 3.3.11] by using Proposition 4.17.

For completeness, the underlying relationship between Dyer-Lashof operations and admissible sequences is explained below. Proposition 5.21 can also be deduced from Lemmas 5.24, 5.27 and 5.29.

Notation 5.23. For \( 0 < s \in \mathbb{N} \) and \( J \in \mathbb{N}^s \) an ordered sequence of non-negative integers, write \( Q_J \) for the iterated operation \( Q_J := Q_{j_1}Q_{j_2} \cdots Q_{j_s} \), to be interpreted either as a formal composition (as in \( (\mathcal{Z})^s = (\mathcal{A}^1)^s \)) or as the quotient in \( \mathcal{B}^s \) (respectively \( \mathcal{Z}^s \)).

Lemma 5.24. For \( \alpha \in \mathbb{Z}^s \), write
\[
(\mathcal{A}^1)^s \oplus \mathcal{P}_d \text{ has a basis } \{Q_{j_1}Q_{j_2} \cdots Q_{j_s} | J \in \mathbb{N}^s \}.
\]

Lemma 5.25. For \( \alpha \in \mathbb{Z}^s \), \( \mathcal{Z}^s \oplus \mathcal{P}_d \) has a basis
\[
\{Q_{j_1}Q_{j_2} \cdots Q_{j_s} | J \in \mathbb{N}^s, j_i > j_{i+1} \text{ for } 1 \leq i < s\}.
\]

Proof. This follows from the proof of [KM13, Lemma 4.19].

Lemma 5.26. For \( \alpha \in \mathbb{Z}^s \), \( \mathcal{Z}^s \oplus \mathcal{P}_d \) has a basis
\[
\{Q_{j_1}Q_{j_2} \cdots Q_{j_s} | J \in \mathbb{N}^s, j_i > j_{i+1} \text{ for } 1 \leq i < s\}.
\]

Proof. (Sketch.) The case \( s = 2 \) follows from Lemma 5.24 and then the general case from the construction of \( \mathcal{Z}^s \), which corresponds to the assertion that there is a PBW-basis of this form. This is equivalent to the Koszul property à la Priddy [Pr70]; see Section 5.6 below.

Remark 5.26. The previous result should be compared with Kuhn’s comments following [Kuh15, Theorem 6.3], where \( e_s \) is described as rewriting terms in Dyer-Lashof allowable form and \( e_s \) as rewriting in Dyer-Lashof ‘completely unallowable’ form.

To proceed, Dyer-Lashof operations must be rewritten with upper indexing:

Lemma 5.27. For \( \alpha \in \mathbb{Z}^s \),
\[
Q_{\alpha + t \cdot d} = Q_{\alpha + t \cdot d} \cdots Q_{\alpha + t \cdot d},
\]
where \( \alpha = \alpha(j, d) \in \mathbb{Z}^s \) is the ordered sequence of integers given by
\[
\alpha_t := \alpha + \sum_{i > t} 2^i - (t + 1) j_i + 2^{s-t} d.
\]

Proof. Induction upon \( s \).

Notation 5.28. For \( \alpha \in \mathbb{Z}^s \), write \( \alpha + 1 \) for the sequence \( (\alpha_1 + 1, \ldots, \alpha_s + 1) \).

Lemma 5.29. For \( d \in \mathbb{Z} \), the association \( J \mapsto \alpha(j, d) + 1 \) induces a bijection between \( \mathbb{N}^s \) and
\[
\{ I \in \mathbb{N}^s | \text{admissible, } i_s > d \}.
\]
If \( d \geq 0 \), then \( \alpha(j, d) + 1 \in \mathbb{N}^s; \alpha(j, -1) + 1 \in \mathbb{N}^s \).

Proof. It is straightforward to check that \( J \mapsto \alpha(j, d) + 1 \) is one to one, hence it suffices to identify the image. Since \( \alpha_s = j_s + d \) and \( j_s \geq 0 \), the condition on \( i_s \) follows immediately. Now, for \( 0 \leq t < s \), \( \alpha_t - 2 \alpha_{t+1} = j_t - j_{t+1} \); it follows that \( \alpha(j, d) + 1 \) is admissible if and only if \( j_t > j_{t+1} \) for all \( t \). This establishes the bijection; the final statement is clear.

Remark 5.30. The above re-indexing is intimately related to the quadratic duality of [Pow11].
5.5. Connectivity. Proposition 5.21 leads to an understanding of the connectivity of objects of the form \( L^t M \), for \( M \in \mathcal{M} \), by applying the following result:

**Proposition 5.31.** For \( t \in \mathbb{N} \) and \( d \in \mathbb{Z} \), \( L^t \Sigma^d F \) is precisely \( 2^t (d + 1) - (t + 2) \)-connected (that is, the lowest degree class is in degree \( 2^t (d + 1) - (t + 1) \)).

In particular, the lowest degree class of \( L^t \Sigma^{-1} F \) is in degree \( -(t + 1) \).

**Proof.** This result can be proved either by using Proposition 5.21 or directly from Lemma 5.25 (The reader is encouraged to use the Lemma in the case \( d = -1 \) and then deduce the general case.) \( \square \)

**Corollary 5.32.** For \( M \in \mathcal{M} \) which is \((d - 1)\)-connected and \( t \in \mathbb{N} \), \( L^t M \) is \((2^t (d + 1) - (t + 2))\)-connected hence is at least \((2^t d - 1)\)-connected.

**Proof.** The first statement is an immediate consequence of Proposition 5.31. The second follows from \( 2^t - (t + 2) \geq -1 \), for \( t \in \mathbb{N} \), where the inequality is strict for \( t \geq 2 \). \( \square \)

**Remark 5.33.** Clearly the first statement gives a much better bound for connectivity for large \( t \). The weaker bound is sometimes more convenient for describing generic behaviour.

5.6. The Koszul property. In [Mil78, Section 3], Miller observes that Priddy’s results on Koszul duality [Pri70] carry over to \( \text{Untor}_R^R(\mathbb{F}, -) \). This is also true for \( L^* q \).

**Theorem 5.34.** For \( M \in \mathcal{M}_{\geq -1} \), considered as an object of \( QM^g \) via \( \text{triv} \), \((L^s q M)^{[i]} = 0 \) if \( i \neq s \) and there is a natural isomorphism in \( \mathcal{M} \):

\[
(L^s q M)^{[s]} \cong \mathcal{L}^s M.
\]

**Proof.** For the first statement, by exactness of \( L^s q \) restricted to \( \mathcal{M} \) (see Proposition 4.13), it suffices to consider the case \( M = \Sigma^d \mathbb{F} \). Here the result follows as for [Mil78, Proposition 3.1.2]; indeed, for \( d \geq 1 \), the statement can be deduced from this result by using Proposition 4.17 (The cases \( d \in \{-1, 0\} \) can then be deduced from this by using the spectral sequence of Section 9).

The identification of \((L^s q M)^{[s]} \) follows by considering the reduced complex \( \overline{C}_q M \) in length degree \( s \) (see Proposition 4.18). In homological degree \( s \) this identifies the cycles as the kernel of the map defining \( \mathcal{L}^s M \) and there are no non-trivial boundaries. \( \square \)

**Remark 5.35.** The hypothesis \( d \geq -1 \) for the vanishing of \((L^s q M)^{[i]} \) (for \( i \neq s \)) can be relaxed by using the Koszul duality result of [BCL05] (cf. also [Pow11]).

If \( N \in QM^g \), then the Dyer-Lashof action induces natural transformations:

\[
\mathcal{R}_1 N^{[i]} \rightarrow N^{[i+1]}
\]

for \( i \in \mathbb{Z} \). Since \( \mathcal{L}_1 = \mathcal{R}_1 \), composing with the natural inclusion \( \mathcal{L}_s \rightarrow \mathcal{L}_{s-1} \mathcal{L}_1 \) (for \( s = 0 \) this is taken to be zero) leads to the natural ‘Koszul differential’:

\[
d^Z_s : \mathcal{L}_s N^{[i]} \rightarrow \mathcal{L}_{s-1} N^{[i+1]}
\]

which can be considered as a natural transformation in \( M^g \):

\[
\mathcal{L}_s N \rightarrow \mathcal{L}_{s-1} (N(-1)).
\]
Lemma 5.36. For $N \in \mathcal{QM}^{gr}$ and $s \in \mathbb{N}$, the composite

$$\mathcal{L}_s N[i] \xrightarrow{d^s} \mathcal{L}_{s-1} N[i+1] \xrightarrow{d^{s-1}} \mathcal{L}_{s-2} N[i+2]$$

is trivial.

Proof. This is the standard Koszul complex argument: since the 'coproducts' $\mathcal{L}_{s+t} \to \mathcal{L}_s \mathcal{L}_t$ are coassociative (by Proposition 5.2), that $d^2 = 0$ follows from the fact that $\mathcal{L}_2$ is the kernel of $(\mathcal{R})^2 \to \mathcal{R}_2$. □

Definition 5.37. For $N \in \mathcal{QM}^{gr}$ and $n \in \mathbb{N}$, let $\mathcal{L}_n N$ denote the complex in $\mathcal{M}$:

$$\mathcal{L}_n N[0] \xrightarrow{d^0} \mathcal{L}_{n-1} N[1] \xrightarrow{d^1} \mathcal{L}_{n-2} N[2] \to \ldots \to \mathcal{L}_0 N[n] = N[n],$$

where $\mathcal{L}_n N[n-i]$ is placed in homological degree $i$.

Corollary 5.38. (Cf. [Mil78, Theorem 3.3.16].) Let $N \in \mathcal{QM}^{gr}$ such that $\tau^{[\leq -1]} N = 0$ and $N[i] \in \mathcal{M}_{\geq -1}$ for each $i$. For $s, t \in \mathbb{N}$, there is a natural isomorphism

$$(\mathcal{L}_t \mathcal{q} N)^{[s]} \cong H_t (\mathcal{L}_s N).$$

In particular, $(\mathcal{L}_t \mathcal{q} N)^{[s]} = 0$ for integers $t > s \geq 0$ and

$$(\mathcal{L}_s \mathcal{q} N)^{[s]} \cong \ker \{ \mathcal{L}_s N[0] \xrightarrow{d^s} \mathcal{L}_{s-1} N[1] \}. $$

Proof. The spectral sequence associated to the length filtration of $N$ degenerates to the Koszul complexes, by Theorem 5.34 (which uses the hypothesis $N[i] \in \mathcal{M}_{\geq -1}$). □

Example 5.39. For $N \in \mathcal{QM}^{gr}$ such that $\tau^{[\leq -1]} N = 0$ and $N[i] \in \mathcal{M}_{\geq -1}$ for each $i$, there are natural isomorphisms:

$$(\mathcal{L}_t \mathcal{q} N)^{[s]} \cong \ker \{ \mathcal{R}_1 N^{[s-1]} \to N^{[i]} \}$$

$$(\mathcal{L}_t \mathcal{q} N)^{[s]} \cong \coker \{ \mathcal{R}_1 N^{[s-1]} \to N^{[i]} \},$$

where the morphism is given by the Dyer-Lashof action, since $\mathcal{L}_0$ is the identity functor and $\mathcal{R}_1 = \mathcal{R}_1$. In particular, $(\mathcal{L}_t \mathcal{q} N)^{[0]} = 0$ and $(\mathcal{L}_t \mathcal{q} N)^{[0]} = N[0]$.

Corollary 5.40. Let $N \in \mathcal{QM}^{gr}$ such that $\tau^{[\leq -1]} N = 0$ and $N[i]$ is $(d_i - 1 \geq -2)$-connected for each $i$. For $s, t \in \mathbb{N}$, $(\mathcal{L}_t \mathcal{q} N)^{[s]}$ is at least $(2^t (d_{s-t} + 1) - (t + 2))$-connected, hence is at least $(2^t d_{s-t} - 1)$-connected.

Proof. A consequence of Corollaries 5.39 and 5.38. □

6. Derived functors of $\mathcal{q}$ and desuspension

In this section, the behaviour of the functors $\mathcal{L}_* \mathcal{q}$ on a (de) suspension is considered.

6.1. Relating $\mathcal{L}_* \mathcal{q} (\Sigma^{-1} M)$ and $\Sigma^{-1} \mathcal{L}_* \mathcal{q} M$. Recall from Section 2 that the suspension morphism

$$\varepsilon_{\Sigma^{-1} M} : \mathcal{R} \Sigma^{-1} \to \Sigma^{-1} \mathcal{R} M$$

is a natural transformation in $\mathcal{QM}^{gr}$ for $M \in \mathcal{M}^{gr}$. This is compatible with the comonad structure of $\mathcal{R}$.

Lemma 6.1.
(1) For $N \in \mathcal{QM}^{gr}$ there is a natural commutative diagram in $\mathcal{QM}^{gr}$:

$$
\begin{array}{c}
\mathcal{R} \Sigma^{-1}N \xrightarrow{\Sigma^{-1}(\epsilon)} \Sigma^{-1}\mathcal{R} N \\
\mu_{\Sigma^{-1}N} \\
\Sigma^{-1}N \xrightarrow{\epsilon} \Sigma^{-1}N,
\end{array}
$$

where $\mu$ denotes the adjunction counit.

(2) For $M \in \mathcal{M}^{gr}$, the adjunction unit $\eta$ fits into a commutative diagram:

$$
\begin{array}{c}
\Sigma^{-1}M \xrightarrow{\Sigma^{-1}(\eta)} \Sigma^{-1}\mathcal{R} M \\
\eta_{\Sigma^{-1}M} \\
\mathcal{R} \Sigma^{-1}M \xrightarrow{\epsilon} \Sigma^{-1}\mathcal{R} M.
\end{array}
$$

(3) For $N \in \mathcal{QM}^{gr}$, there is a natural commutative diagram in $\mathcal{QM}^{gr}$:

$$
\begin{array}{c}
\mathcal{R} \Sigma^{-1}N \xrightarrow{\Sigma^{-1}(\epsilon)} \Sigma^{-1}\mathcal{R} N \\
\Delta_{\Sigma^{-1}N} \\
\mathcal{R} \mathcal{R} \Sigma^{-1}N \xrightarrow{\epsilon} \Sigma^{-1}\mathcal{R} \mathcal{R} N,
\end{array}
$$

in which $\Delta : \mathcal{R} \to \mathcal{R} \mathcal{R}$ is the comonad structure map.

Proof. The first two points are straightforward, using the fact that $\epsilon$ is constructed via the adjunction from $\eta$. The third then follows. □

Proposition 6.2. For $N \in \mathcal{QM}^{gr}$, $\epsilon$ induces natural transformations:

1. $\mathcal{R}^{+1}\Sigma^{-1}N \to \Sigma^{-1}\mathcal{R}^{+1}N$ in $\text{ChQM}^{gr}$;
2. $\mathcal{C}q\Sigma^{-1}N \to \Sigma^{-1}\mathcal{C}qN$ in $\text{ChM}^{gr}$;
3. $\mathcal{L}i\Sigma^{-1}N \to \Sigma^{-1}\mathcal{L}iN$ in $\mathcal{M}^{gr}$, for $i \in \mathbb{N}$, which is an isomorphism for $i = 0$.

Proof. The first statement follows from Lemma 6.1 and the remaining statements follow since $q$ commutes with $\Sigma^{-1}$. □

Remark 6.3. Analogous results hold in the non-length-graded case, replacing $\mathcal{M}^{gr}$ and $\mathcal{QM}^{gr}$ respectively by $\mathcal{M}$ and $\mathcal{QM}$.

The morphism appearing in Proposition 6.2 (3) fits into a long exact sequence associated to a Grothendieck composite functor spectral sequence.

Proposition 6.4. There is a natural isomorphism of functors $q\Sigma \cong \Sigma q : \mathcal{QM} \to \mathcal{M}$ and the associated Grothendieck spectral sequence

$$(L_{-1}q\Sigma) \Rightarrow \Sigma L_{-1}q$$

degenerates to a long exact sequence of the form

$$\ldots \Rightarrow (L_{n-1}q\Sigma) \Rightarrow \Sigma L_n q \Rightarrow (L_n q) \Sigma \Rightarrow (L_{n-1}q) \Sigma \Rightarrow \ldots$$

The natural morphisms $\Sigma L_{n-1}q \to (L_{n-1}q) \Sigma$ evaluated on $\Sigma^{-1}N$ yields the natural transformation of Proposition 6.2 (3).

Proof. For the existence of the spectral sequence, it suffices to use the fact that $\Sigma$ carries relative projectives to relative projectives, which holds by Corollary 3.13. The remainder is straightforward. □
Remark 6.5. This spectral sequence is analogous to that introduced by Miller (see [Mil78, Equation 2.3.3] and following). This is defined for functors on a certain category of allowable Hopf algebras and is the composite of the indecomposables functors $Q$ and the functor $\text{Untor}_R^R(\mathbb{F}, -)$. The functor $Q$ has $\mathbb{L}_nQ = 0$ for $n > 1$ so the spectral sequence is of similar form.

6.2. The suspension and the Koszul complex $\mathcal{L}_s$.

Proposition 6.6. The suspension morphism $\mathcal{L}_s \Sigma^{-1} \rightarrow \Sigma^{-1} \mathcal{L}_s$, of Proposition 5.13 induces a surjective morphism of Koszul complexes for $N \in \mathcal{Q}\mathcal{M}^{gr}$:

$$\mathcal{L}_s \Sigma^{-1} N \rightarrow \Sigma^{-1} \mathcal{L}_s N,$$

for $s \in \mathbb{N}$.

The induced morphism in homology,

$$H_t \mathcal{L}_s \Sigma^{-1} N \rightarrow H_t \Sigma^{-1} \mathcal{L}_s N$$

identifies via the isomorphism of Corollary 5.38 with the natural transformation

$$\left(\mathbb{L}_s q \Sigma^{-1} N\right)[s] \rightarrow \left(\Sigma^{-1} \mathbb{L}_t q N\right)[s]$$

of Proposition 6.3.

Proof. For the first statement, it suffices to check that the differential of the Koszul complexes is compatible with the suspension morphisms $\mathcal{L}_s \Sigma^{-1} \rightarrow \Sigma^{-1} \mathcal{L}_s$. This follows from the commutativity of the following diagram, in which the horizontal morphisms are the coproducts of Proposition 5.2 and the vertical morphisms the suspension morphisms of Proposition 5.13:

$$\xymatrix{ \mathcal{L}_{s+t} \Sigma^{-1} \ar[r] & \mathcal{L}_s \mathcal{L}_{t} \ar[d] \ar[r] & \mathcal{L}_s \Sigma^{-1} \mathcal{L}_t \ar[d] \\
\Sigma^{-1} \mathcal{L}_{s+t} \ar[r] & \Sigma^{-1} \mathcal{L}_s \mathcal{L}_t }$$

where $s, t \in \mathbb{N}$.

The identification of the induced morphism in homology follows from unravelling the definitions. □

Remark 6.7. By Proposition 5.13, the kernel of the surjective morphism of complexes $\mathcal{L}_s \Sigma^{-1} N \rightarrow \Sigma^{-1} \mathcal{L}_s N$ is a complex with $i$th term

$$\Sigma^{-1} \mathcal{L}_{-1} \Sigma^{-1} \Phi N$$

and with differential induced from $\mathcal{L}_s \Sigma^{-1} N$.

7. The functor $H_0\mathcal{R}$ and destabilization

The relationship with instability of $\mathcal{R}$-modules is established by the construction of a chain complex that calculates the derived functors of destabilization. Such results go back to Singer (see [Sin80] for example), the work of Lannes and Zarati [LZ87] on derived functors of destabilization and more recent treatments such as [Pow14] (working with cohomology and for odd primes) and [KML13]. The latter reference is followed here, since the relationship with the action of Dyer-Lashof operations is fundamental.
7.1. The chain complex. This section presents the construction of the chain complex of [KM13] in a form suitable for current purposes.

**Notation 7.1.** For $M \in \mathcal{M}$, let $d_M : M \to \mathcal{R}_1 \Sigma M \subset \mathcal{R} \Sigma M$ denote the Singer differential (see [KM13, Definition 4.11]) and also its extension to a morphism of $\mathcal{Q} \mathcal{M}$:

$$d_M : \mathcal{R} M \to \mathcal{R} \Sigma M.$$

The following resumes some of the fundamental properties of $d_M$:

**Proposition 7.2.** [KM13] For $M \in \mathcal{M}$,

1. the kernel of $\Sigma d_{-1} : M \to \Sigma \mathcal{R}_1 M$ is $\Omega^\infty M \subset M$;
2. the composite $d_{\Sigma M} d_M : \mathcal{R} M \to \mathcal{R} \Sigma^2 M$ is trivial;
3. $d$ commutes with the suspension $\varepsilon : \mathcal{R} \to \Sigma^{-1} \mathcal{R} \Sigma$.

The above statements can be made more precise by observing that the differential respects the length grading, increasing it by one, hence can be written as the natural transformation of $\mathcal{R} \mathcal{M}^{gr}$:

$$d_M : \mathcal{R} M \to (\mathcal{R} \Sigma M)(-1),$$

for $M \in \mathcal{M}$, using the notation of Section 3.2.

**Definition 7.3.** For $M \in \mathcal{M}$, let $\mathcal{R}_s M$ denote the homological chain complex in $\mathcal{R} \mathcal{M}^{gr}$ with

$$\mathcal{R}_s M := (\mathcal{R} \Sigma^{-s} M)(s)$$

and differential $d_{\Sigma^{-1}M}(s) : \mathcal{R}_s M \to \mathcal{R}_{s-1} M$.

**Notation 7.4.**

1. Let $Ch \mathcal{R} \mathcal{M}^{gr}$ denote the category of $\mathbb{Z}$-graded, homological chain complexes in $\mathcal{R} \mathcal{M}^{gr}$.
2. For $d \in \mathbb{Z}$, let $[d] : Ch \mathcal{R} \mathcal{M}^{gr} \to Ch \mathcal{R} \mathcal{M}^{gr}$ denote the $d$-fold homological shift of complexes.

**Proposition 7.5.** The construction $\mathcal{R}$ defines an exact functor $\mathcal{R} : \mathcal{M} \to Ch \mathcal{R} \mathcal{M}^{gr}$. This satisfies the following properties for $M \in \mathcal{M}$:

1. $H_0 \mathcal{R} M \cong \bigoplus_{i \geq 0} \Sigma^{-1} \Omega_i^\infty \Sigma^{1-i} M \in \mathcal{Q} \mathcal{M}^{gr}$;
2. for $d \in \mathbb{Z}$, there is a natural isomorphism $\mathcal{R} \Sigma^d M \cong (\mathcal{R} M)[d](d)$ and $H_0 \mathcal{R} M \cong H_0 \mathcal{R} \Sigma^{-d} M[d]$;
3. there is an exact sequence of complexes in $\mathcal{Q} \mathcal{M}^{gr}$:

$$0 \to (\Phi \mathcal{R} M)(1) \xrightarrow{Q_0} \mathcal{R} M \xrightarrow{\varepsilon} \Sigma^{-1} \mathcal{R} \Sigma M \to 0.$$

**Proof.** The first statement is a consequence of [KM13 Theorem 4.22] and the second is clear from the definition of the chain complex; the final statement is a chain complex version of Corollary 3.13 (cf. [KM13 Proposition 4.26]). □

**Proposition 7.6.** For $M \in \mathcal{U}$ unstable, there is a natural isomorphism $H_0 \mathcal{R} M \cong \mathcal{R} M$ in $\mathcal{Q} \mathcal{M}^{gr}$.

**Proof.** This fundamental fact relies on the calculation of certain derived functors of destabilization by Lannes and Zarati [LZ87], which can be recovered from the chain complex viewpoint as in [KM13 Theorem 4.34] (see also [Pow14]). Namely, for $M$ unstable, $\Omega_i^\infty \Sigma^{1-i} M \cong \Sigma \mathcal{R}_1 M$ and the proof shows that $\bigoplus_{i \geq 0} \Sigma^{-1} \Omega_i^\infty \Sigma^{1-i} M \cong \mathcal{R} M$ in $\mathcal{Q} \mathcal{M}^{gr}$ (i.e. the Dyer-Lashof action is the canonical one). □

The following connectivity result is useful:
Corollary 7.7. If $M \in \mathcal{M}$ is $c$-connected, then for $i \in \mathbb{N}$, $(H_0 \mathcal{R}M)^{[i]}$ is at least $(2^i(c + 1) - 1)$-connected.

If $c \geq -1$ and for $l \in \mathbb{N}$, $(\mathbb{L}_* q H_0 \mathcal{R}M)^{[l]}$ is at least $(2^l(c + 1) - 1)$-connected.

Proof. Using the identification of $H_0 \mathcal{R}M \cong \bigoplus_{i \geq 0} \Sigma^{-i}\Omega^\infty \Sigma^1 \mathcal{M}$ from Proposition 7.5 this result follows from [KM13 Corollary 4.31] (and can be proved directly from the connectivity statement in Proposition 5.40).

The final statement then follows by applying Corollary 5.40.

\[\Box\]

7.2. The long exact sequence for $H_0 \mathcal{R}$. The exactness of $\mathcal{R} : \mathcal{M} \to \text{Ch}_\mathcal{QM}^{\mathcal{M}}$ leads to the following:

Proposition 7.8. For $0 \to M_1 \to M_2 \to M_3 \to 0$ a short exact sequence in $\mathcal{M}$, there is a natural long exact sequence in $\text{Ch}_\mathcal{QM}^{\mathcal{M}}$:

$$\ldots \to H_0 \mathcal{R} \Sigma^{-1} M_3(1) \to H_0 \mathcal{R} M_1 \to H_0 \mathcal{R} M_2 \to H_0 \mathcal{R} M_3 \to H_0 \mathcal{R} \Sigma M_1(-1) \to \ldots$$

Proof. This is the long exact sequence in homology associated to the short exact sequence in $\text{Ch}_\mathcal{QM}^{\mathcal{M}}$:

$$0 \to \mathcal{R} M_1 \to \mathcal{R} M_2 \to \mathcal{R} M_3 \to 0,$$

by using the natural isomorphisms $H_1 \mathcal{R} M_3 \cong H_0 \mathcal{R} \Sigma^{-1} M_3(1)$ and $H_{-c} \mathcal{R} M_1 \cong H_0 \mathcal{R} \Sigma M_1(-1)$ provided by Proposition 7.5 (the non-displayed terms are treated similarly).

\[\Box\]

7.3. Connectivity estimates. For $M \in \mathcal{M}$ and $c \in \mathbb{Z}$, there is a natural truncation short exact sequence

$$0 \to M_{\leq c} \to M \to M_{\geq c+1} \to 0$$

in which $M_{\leq c}$ is the subobject of $M$ formed by elements of degree at most $c$ so that $M_{\geq c+1}$ is at least $c$-connected.

Proposition 7.9. For $M \in \mathcal{M}$, $c \in \mathbb{Z}$ and $i \in \mathbb{N}$, the natural morphism

$$(H_0 \mathcal{R}(M_{\leq c}))^{[i]} \to (H_0 \mathcal{R}M)^{[i]}$$

is

1. surjective in degrees $\leq 2^i(c + 1) - 1$;
2. injective if $i = 0$ and, if $i > 0$, in degrees $\leq 2^{i-1} c - 1$.

In particular, for $c \in \mathbb{N}$, it is an isomorphism in degrees

1. $\leq c$ for $i = 0$;
2. $\leq 2^{i-1} c - 1$ for $i > 0$.

Proof. The result follows by applying the connectivity result of Corollary 7.7 to the exact sequence

$$(H_0 \mathcal{R} \Sigma^{-1}(M_{\geq c+1}))^{[i-1]} \to (H_0 \mathcal{R}(M_{\leq c}))^{[i]} \to (H_0 \mathcal{R}M)^{[i]} \to (H_0 \mathcal{R}(M_{\geq c+1}))^{[i]}$$

given by Proposition 7.8.

\[\Box\]

Corollary 7.10. For $M \in \mathcal{M}$ that is $(-1)$-connected and $0 < c, l \in \mathbb{N}$, the morphism

$$(\mathbb{L}_* q H_0 \mathcal{R} M_{\leq c})^{[l]} \to (\mathbb{L}_* q H_0 \mathcal{R} M)^{[l]}$$

induced by the canonical inclusion $M_{\leq c} \hookrightarrow M$ is an isomorphism in degrees $\leq 2^{l-1} c - 1$.

Proof. The result follows by combining Corollary 5.40 with Proposition 7.9.

\[\Box\]
8. The suspension morphism for $H_0\mathcal{R}$

The behaviour of $H_0\mathcal{R}$ with respect to the suspension morphism is of independent interest; the material of this section sheds light on results of Kuhn and McCarty [KM13].

8.1. The suspension morphism $\varepsilon$. Recall from Proposition 7.5 that, for $M \in \mathcal{M}$, the suspension morphism induces a morphism of complexes in $\mathcal{QM}^\gr$

$$\varepsilon : \mathcal{R}M \rightarrow \Sigma^{-1}\mathcal{R}\Sigma M.$$  

The morphism $H_0\varepsilon$ has adjoint $\sum H_0\mathcal{RM} \rightarrow H_0\mathcal{R}\Sigma M$.

**Theorem 8.1.** Let $M \in \mathcal{M}$.

1. There is a natural short exact sequence in $\mathcal{QM}^\gr$:

$$0 \rightarrow \sum H_0\mathcal{RM} \rightarrow H_0\mathcal{R}\Sigma M \rightarrow \sum_1 H_0\mathcal{R}\Sigma M(-1) \rightarrow 0.$$  

2. There are natural isomorphisms in $\mathcal{QM}^\gr$:

$$\sum H_0\mathcal{RM} \cong \text{image}(H_0\mathcal{RM} \xrightarrow{H_0\varepsilon} \Sigma^{-1}H_0\mathcal{R}\Sigma M) \cong \bigoplus_{i \geq 0} \Omega_i \Sigma^{1-i}(\Sigma M).$$

3. Writing $\text{im}(Q_0)_M$ for the image of $\Phi(H_0\mathcal{RM})(1)$ $\xrightarrow{Q_0} H_0\mathcal{RM}$, there are natural short exact sequences in $\mathcal{QM}^\gr$:

$$0 \rightarrow \text{im}(Q_0)_M \rightarrow H_0\mathcal{RM} \rightarrow \Sigma^{-1}\sum H_0\mathcal{RM} \rightarrow 0$$

$$0 \rightarrow \Sigma^{-1}\sum_1 H_0\mathcal{RM} \rightarrow \Phi(H_0\mathcal{RM})(1) \rightarrow \text{im}(Q_0)_M \rightarrow 0.$$  

4. There is a natural isomorphism in $\mathcal{QM}^\gr$

$$\text{im}(Q_0)_M \cong \Sigma^{-1} \bigoplus_{i \geq 1} \Omega_i \Sigma^{1-i}(\Sigma M).$$

**Proof.** The short exact sequence of complexes in $\mathcal{QM}^\gr$ given by Proposition 7.5

$$0 \rightarrow (\Phi\mathcal{RM})(1) \xrightarrow{Q_0} \mathcal{RM} \rightarrow \Sigma^{-1}\mathcal{R}\Sigma M \rightarrow 0,$$

induces a long exact sequence in homology, occurring as the top row of the following commutative diagram:

$$
\begin{array}{cccccc}
\ldots & \rightarrow & \Phi(H_0\mathcal{RM})(1) & \xrightarrow{Q_0} & H_0\mathcal{RM} & \xrightarrow{H_0\varepsilon} & \Sigma^{-1}H_0\mathcal{R}\Sigma M \\
& & & & \sum H_0\mathcal{RM} & \xrightarrow{\text{im}(Q_0)_M} & \Sigma^{-1}\sum H_0\mathcal{RM} \\
& & & & \sum_1 H_0\mathcal{RM} & \rightarrow & \Phi(H_{-1}\mathcal{RM})(1) \\
\end{array}
$$

where the exactness of the functors $\Sigma^{-1}$, $\Phi$ and $(1)$ provides the commutation with formation of homology and the identification of the kernel and cokernel of $Q_0$ comes from Proposition 8.19.

This provides the short exact sequence of part 1, by using the isomorphism $H_{-1}\mathcal{RM} \cong H_0\mathcal{R}\Sigma M(-1)$ given by Proposition 7.5. The identification of $\sum H_0\mathcal{RM}$ follows using the argument employed in the proof of [KM13 Proposition 1.11].

The two short exact sequences of (3) are given by dévissage of the long exact sequence, as indicated by the dotted arrows in the diagram.

Finally, in length grading $i$, the short exact sequence

$$0 \rightarrow \text{im}(Q_0)_M[i] \rightarrow H_0\mathcal{RM}[i] \rightarrow \Sigma^{-1}\sum H_0\mathcal{RM}[i] \rightarrow 0$$

identifies as

$$0 \rightarrow (\text{im}(Q_0)_M)[i] \rightarrow \Sigma^{-1} \bigoplus \Omega_i \Sigma^{1-i} M \rightarrow \Sigma^{-1} \bigoplus \Omega_i \Sigma^{1-i} \Sigma M \rightarrow 0,$$
where the surjection is the desuspension of the surjection derived from the short exact sequence of Proposition 2.8
\[ 0 \to \Omega_1 \Omega_i ^{\infty} \Sigma \to \Omega_i ^{\infty} \to \Omega_i ^{\infty} \Sigma \to 0 \]
applied to \( \Sigma ^{1} - M \). This yields the identification of \( \text{im}(Q_0)_M \). \( \square \)

**Example 8.2.** In [KM13] Proposition 1.11, for \( M \in \mathcal{M} \), Kuhn and McCarty introduce an object \( L_n M \) that lies in \( \mathcal{QM} \) and such that each component is unstable. This is used in their description of the \( E^\infty \)-page of the algebraic approximation to the infinite looping spectral sequence (see [KM13] Theorem 1.12).

This object is naturally isomorphic to \( \Sigma H_0 \Sigma - M \), where the passage to \( \Sigma \) serves to ensure \( \mathcal{A} \)-instability. (The notation \( L_n M \) is not used here, since it could lead to confusion with the Steinberg functors.)

**Corollary 8.3.** For \( M \in \mathcal{M} \), there is a natural exact sequence in length-graded objects of \( \Sigma \mathcal{QM} \cap \mathcal{U} \):
\[ 0 \to \Sigma (\Sigma H_0 \mathcal{R}M) \to (\Sigma H_0 \mathcal{R} \Sigma M) \to \Sigma^2 \text{im}(Q_0)_{\Sigma M}(-1) \to 0 \]
that identifies in each length grading with the exact sequence of Proposition 8.2.

**Proof.** The suspensions ensure that each term lies in \( \mathcal{U} \), hence \( S_{Q_0} \) is a morphism of \( \Sigma \mathcal{QM} \) by Proposition 8.15. The terms are identified by Theorem 8.1 (by reassembling two of the short exact sequences). \( \square \)

**Remark 8.4.**

1. The previous results should be compared with [KM13] Corollary 4.36.
2. Corollary 8.3 shows what is required to pass from \( H_0 \mathcal{R}M \) to \( H_0 \mathcal{R} \Sigma M \); whereas \( \Sigma H_0 \mathcal{R}M \) is calculated as a functor of \( H_0 \mathcal{R}M \), one also requires the input of \( \text{im}(Q_0)_{\Sigma M} \). With these in hand, the delooping spectral sequence of Theorem 2.12 can be used, enhanced so as to take into account the Dyer-Lashof action. (Compare Example 8.6 below.)

**Corollary 8.5.** For \( M \in \mathcal{M} \) which is \( c \)-connected and \( i \in \mathbb{N} \), the following statements hold.

1. The natural inclusion
\[ (\Sigma H_0 \mathcal{R}M)^{(i)} \hookrightarrow (H_0 \mathcal{R} \Sigma M)^{(i)} \]
is an isomorphism in degrees \( \leq 2i + 1(c + 2) \).
2. The natural surjection
\[ (H_0 \mathcal{R}M)^{(i)} \twoheadrightarrow (\Sigma^{-1} \Sigma H_0 \mathcal{R}M)^{(i)} \]
is an isomorphism for \( i = 0 \) and, for \( i > 0 \), in degrees \( \leq 2^i (c + 1) - 2 \).

In particular, if \( c \geq -1 \), the suspension morphism
\[ (H_0 \mathcal{R}M)^{(i)} \xrightarrow{H_0 \varepsilon} (\Sigma^{-1} H_0 \mathcal{R} \Sigma M)^{(i)} \]
is an isomorphism in degrees \( \leq 2^i (c + 1) - 2 \).

**Proof.** For the first statement, consider the short exact sequence
\[ 0 \to (\Sigma H_0 \mathcal{R}M)^{(i)} \to (H_0 \mathcal{R} \Sigma M)^{(i)} \to (\Sigma_1 H_0 \mathcal{R} \Sigma M)^{(i+1)} \to 0 \]
of Theorem 8.1. Now \( (H_0 \mathcal{R} \Sigma M)^{(i)} \) is at least \( (2^i (c + 2) - 1)\)-connected, by Corollary 7.7; hence Proposition 3.23 implies that \( (\Sigma_1 H_0 \mathcal{R} \Sigma M)^{(i+1)} \) is at least \( 2^{i+1} (c + 2) \)-connected, for \( i \in \mathbb{N} \).
The second point is proved by a similar argument, based upon the connectivity of $\Phi H_0\mathcal{R}M(1)$ and hence of the image of $\text{im}(Q_0)M$. The conclusion for $H_0\varepsilon$ follows by putting these statements together. \hfill \square

Example 8.6. Suppose that $\text{im}(Q_0)M = 0$, then one has
\[ \Sigma_1 H_0\mathcal{R}M \cong \Sigma \Phi H_0\mathcal{R}M(1). \]
In this case, the short exact sequence of Theorem 8.1 part (1) for $\Sigma^{-1}M$ becomes:
\[ 0 \to \Sigma H_0\mathcal{R}\Sigma^{-1}M \to H_0\mathcal{R}M \to \Sigma \Phi H_0\mathcal{R}M \to 0 \]
in $QM^{gr}$.

If, moreover, $Q_0$ is trivial on $H_0\mathcal{R}\Sigma^{-1}M$, then $\Sigma H_0\mathcal{R}\Sigma^{-1}M \cong H_0\mathcal{R}\Sigma^{-1}M$. After suspension, the short exact sequence then becomes:
\[ 0 \to \Sigma(\Sigma H_0\mathcal{R}\Sigma^{-1}M) \to (\Sigma H_0\mathcal{R}M) \to \Phi(\Sigma H_0\mathcal{R}M) \to 0. \]
Here $(\Sigma H_0\mathcal{R}\Sigma^{-1}M)$ and $(\Sigma H_0\mathcal{R}M)$ are length-graded unstable modules and the surjection $(\Sigma H_0\mathcal{R}M) \to \Phi(\Sigma H_0\mathcal{R}M)$ is induced by $S_{Q_0}$, with surjectivity a consequence of the vanishing of $Q_0$.

In this case, it follows that there is an identification
\[ \Omega(\Sigma H_0\mathcal{R}M) \cong (\Sigma H_0\mathcal{R}\Sigma^{-1}M). \]

This situation arises for example when considering the case $M = \Sigma^n \AAA^*$ for $n > 0$ an integer (cf. Example 2.4).

9. An infinite delooping spectral sequence

The spectral sequence constructed in this section is derived from a double complex. The construction is analogous to that of the Grothendieck spectral sequence associated to a composite of functors: one of the spectral sequences associated to the double complex degenerates and the second gives the required spectral sequence.

9.1. Construction of the spectral sequence. In Section 7 introduced the exact functors $\mathcal{R} : M \to \text{Ch} QM^{gr}$ and $\mathcal{C}_q : QM^{gr} \to \text{Ch} M^{gr}$. The composite $\mathcal{C}_q \mathcal{R}$ defines an exact functor to homological bicomplexes in $M^{gr}$, where
\[ (\mathcal{C}_q \mathcal{R}M)_{s,t} = (\mathcal{C}_q)_{t}(\mathcal{R}_s M) \]
so that the bicomplex is concentrated in bidegrees $(s, t) \in \mathbb{Z} \times \mathbb{N}$ and the terms of the bicomplex belong to $M^{gr}$ (i.e. are length-graded $\AAA$-modules). In particular, this can be considered as a length-graded bicomplex in $M$. Because of the shifting properties given in Proposition 7.5, without loss of generality, we may restrict to length grading zero:
\[ (\mathcal{C}_q \mathcal{R}M)^{[0]}. \]

Lemma 9.1. For $M \in M$, $(\mathcal{C}_q \mathcal{R}M)^{[0]}$ is a second quadrant, homological bicomplex (concentrated in the quadrant $s \leq 0$, $t \geq 0$).

Proof. An immediate consequence of the fact that $\mathcal{R}_s M$ for $s > 0$ is concentrated in positive length degree. \hfill \square

Remark 9.2. If $M \in M$, Proposition 3.6 provides a local finiteness property of this bicomplex: for $n, d \in \mathbb{Z}$,
\[ ((\mathcal{C}_q)_{n-s}(\mathcal{R}_s M))_d = 0 \]
for $s < 0$. This is sufficient to ensure strong convergence of the spectral sequences considered below.
There are two spectral sequences associated to the horizontal and vertical filtrations of the bicomplex. The horizontal filtration is the decreasing filtration defined by
\[(C_q R M)^{[0]}_{s \leq -n}\]
for \(n \in \mathbb{N}\), so that \((C_q R M)^{[0]}_{s \leq 0} = (C_q R M)^{[0]}\).

Since \(R s M = \Sigma^{-s} M(s)\), which is acyclic for \(q\), the associated spectral sequence has
\[E^1_{s,t} = \begin{cases} M & (s, t) = (0, 0) \\ 0 & \text{otherwise} \end{cases}\]
and degenerates at \(E^1\).

The vertical filtration is the increasing filtration defined by
\[(C_q R M)^{[0]}_{t \leq n}\]
for \(n \in \mathbb{N}\), so that \((C_q R M)^{[0]}_{t \leq 0} = (R M)^{[0]}\).

**Notation 9.3.** For \(n \in \mathbb{N}\), denote by \(v_n M \subset M\) the image in homology of the morphism of the total complexes associated to \((C_q R M)^{[0]}_{t \leq n} \hookrightarrow (C_q R M)^{[0]}\).

**Theorem 9.4.** Let \(M \in \mathcal{M}\).

1. There is a natural second quadrant homological spectral sequence \((E^r_{s,t}, d^r)\) with \(d^r\) of \((s,t)\)-bidegree \((r-1, -r)\) and \(E^2\)-page:
   \[E^2_{s,t} = \mathbb{L}_t \Sigma^{-s} M \{[-s]\}.\]

2. There is a natural isomorphism
   \[E^2_{t,0} = \ker \left\{ \mathcal{L}_t \Sigma^{-1} \Omega \Sigma^{-1+t} M \rightarrow \mathcal{L}_{t-1} \Sigma^{-1} \Omega \Sigma^t M \right\}.\]

3. The \(t = 0\) edge morphism identifies with the natural inclusion
   \[v_0 M \cong \Sigma^{-1} \Omega \Sigma M \hookrightarrow M\]
in \((s,t)\)-degree \((0,0)\) and is zero elsewhere.

4. \(E^2_{s,t} = 0\) for \(t > -s\), so that the spectral sequence is concentrated below the anti-diagonal and there is a second edge homomorphism:
   \[E^\infty_{s,t} \cong v_t M/v_{t-1} M \hookrightarrow E^2_{s,t}.\]

5. The spectral sequence converges strongly to \(M\); \(E^\infty_{s,t} = 0\) if \(s \neq -t\) and \(E^\infty_{-s,t} \cong v_t M/v_{t-1} M\).

**Proof.** This is the second spectral sequence associated to the bicomplex \((C_q R M)^{[0]}\). Since the functor \(C_q\) is exact,
\[E^1_{s,t} \cong (C_q R M)^{[0]}_{s \leq -t}\]
and the differential \(d^1\) is that of \(C_q\). This identifies the \(E^2\)-page of the spectral sequence, by using the isomorphism \(H_0 R \Sigma^{-s} M \cong H_0 R \Sigma^{-s} M(s)\) provided by Proposition 7.3. Identification of the edge morphism is straightforward.

The vanishing above the anti-diagonal for \(s \leq -2\) follows from Corollary 5.38 which also gives the identification of \(E^2_{s,t}\). Since the spectral sequence is homological, the anti-diagonal provides a second edge homomorphism.

Strong convergence, under the hypothesis that \(M \in \mathcal{M}\), is a consequence of Remark 9.2 and the associated filtration of \(M\) is given by the construction of the spectral sequence. \(\square\)
Corollary 9.5. For $M \in \mathcal{M}$, the edge morphism $E^\infty_{2,1} \rightarrow E^2_{2,1}$ is an isomorphism, hence
\[
v_1 M / v_0 M \cong \ker \{ Z \Sigma^{-1} \Omega^{\infty} \Sigma^2 M \rightarrow \Sigma^{-1} \Omega^{\infty} \Sigma^1 M \}.
\]
Proof. The terms of $E^2_{-1,1}$ are permanent cycles and $E^2_{-1,1}$ is given by Example 5.39. □

Corollary 9.6. For $M \in \mathcal{M}$ and $t \in \mathbb{N}$, the module $v_t M \subset M$ admits a finite filtration such that the associated graded satisfies:
\[
\Sigma^2 \text{gr}(v_t M) \in \mathcal{M}
\]
is unstable.

Proof. For $n \in \mathbb{N}$, the subquotient $v_n M / v_{n-1} M$ is a submodule of $Z \Sigma^{-1} \Omega^{\infty} \Sigma^{1+n} M$
by Theorem 9.4, hence the result follows by applying Corollary 5.17. □

Remark 9.7. The attentive reader will observe that the $E^2$-page of the spectral sequence of Theorem 9.4 depends upon $\Sigma^{-1} \Omega^{\infty} \Sigma^{t+1} M$ for all $t \in \mathbb{N}$. Since any bounded-below $\mathcal{M}$-module $M$ can be recovered as
\[
M \cong \text{colim}_n \Sigma^{-d} \Omega^{\infty} \Sigma^d M,
\]
the input would appear to contain the information to be calculated. That the spectral sequence does provide an effective tool for calculations is explained by the connectivity results of the following section, in particular Theorem 9.12.

9.2. Connectivity results.

Proposition 9.8. Let $M \in \mathcal{M}$ be $c$-connected and consider the spectral sequence of Theorem 9.4. Then $E^2_{2,t}$ has connectivity at least
\[
d(M) = \sup \{(c + t + 1), -1\}.
\]
If $c + t + 1 \leq -1$, then $E^2_{2,t}$, for $0 \leq i \leq t$ has connectivity at least $-(i + 2)$.

Proof. For the first statement, by Corollary 5.38 it suffices to consider the connectivity of $Z \Sigma^{-1} \Omega^{\infty} \Sigma^{1+t} M$. The module $\Sigma^{-1} \Omega^{\infty} \Sigma^{1+t} M$ has lowest class in degree at least $d(M)$, using the fact that $\Omega^{\infty} \Sigma^{1+t} M$ is unstable, hence is always $(-1)$-connected. The result then follows from Proposition 5.31. The second statement is proved by a similar argument. □

Remark 9.9. This leads to an explicit form of the convergence stated in Theorem 9.4, by replacing Remark 9.2 by the result of Proposition 9.8. In particular, as soon as $c + t > 0$, this ensures exponential growth of the connectivity with $t$.

Example 9.10. Consider the fundamental example $M = \Sigma^n \mathcal{A}^*$, with $n \in \mathbb{Z}$ (cf. Example 2.4). Then $\Omega^i \mathcal{A}^* = 0$ for $i > 0$ and $\Omega^{\infty} \Sigma^{t+1} M = \Omega^{\infty} \Sigma^{n+t+1} \mathcal{A}$ (for $t \in \mathbb{N}$) is 0-connected unless $1 + t + n = 0$. In particular, for $E^2_{2,t}$ to be non-zero in negative degree, this requires $n < 0$.

For $n < 0$, the only terms in the spectral sequence of negative degree arise from
\[
E^2_{n+1,-(n+1)} = \mathcal{L}_{-(n+1)} \Sigma^{-1} \mathcal{F}.
\]
In particular, these are permanent cycles in the spectral sequence and contribute to the negative degree part of $\Sigma^n \mathcal{A}$. By Proposition 5.31, the lowest degree non-trivial class in $\mathcal{L}_{-(n+1)} \Sigma^{-1} \mathcal{F}$ is in degree $-(n+1) + 1 = n$, as expected. This is again a manifestation of Koszul duality (cf. Section 5.6).

Proposition 9.8 gives a generalized stable range for the increasing filtration $v_n M$ of $M$. 

Corollary 9.11. Let \( M \in \mathcal{M} \) be c-connected and \( n \in \mathbb{N} \). Then
\[
v_n M \mapsto M
\]
is an isomorphism in degrees \( \leq 2^{n+1}(d(M) + 1) - (n + 3) \), where \( d(M) = \sup\{c + n + 2\}, -1\} \).

Forgetting the \( \mathcal{A} \)-action, the spectral sequence splits into degree summands (as for the length grading), in particular one can truncate and consider the behaviour in degrees \( \leq D \), for an integer \( D \). Then, combining Corollary 9.11 with the connectivity result Corollary 7.10 leads to the following:

Theorem 9.12. Let \( M \in \mathcal{M} \) be \( (-1) \)-connected and let \( n \in \mathbb{N} \). The spectral sequence of Theorem 9.4 to calculate \( M \) in degrees \( \leq 2^{n+1}(n + 1) - 1 \) depends only upon
\[
\tau^{\leq t} H_0 R \Sigma^t (M_{\leq \ell_t})
\]
for \( 0 \leq t \leq n \), where
\[
c_0 = 2^{n+1}(n + 1) - 1
\]
\[
c_t = 2^{n-t+2}(n + 1) \text{ for } t > 0.
\]

9.3. The suspension morphism and the spectral sequence. The natural suspension morphism of complexes, \( \varepsilon : \mathcal{R}M \to \Sigma^{-1}\Sigma M \) induces a morphism of bicomplexes
\[
\mathcal{C}_q \mathcal{R}M \to \mathcal{C}_q \Sigma^{-1}\Sigma M
\]
and hence, by composing with the natural transformation \( \mathcal{C}_q \Sigma^{-1} \to \Sigma^{-1} \mathcal{C}_q \) of Proposition 6.2 and restricting to length grading zero, a morphism of bicomplexes
\[
(\mathcal{C}_q \mathcal{R}M)^{[0]} \to \Sigma^{-1}(\mathcal{C}_q \Sigma \mathcal{R}M)^{[0]}.
\]

Theorem 9.13. For \( M \in \mathcal{M} \),

1. the suspension morphism induces a morphism of spectral sequences
\[
E^r_{s,t}(M) \to \Sigma^{-1}E^r_{s,t}(\Sigma M);
\]
2. given on the \( E^2 \)-page by the natural transformation
\[
(1 + qH_0 R \Sigma^{-s} M)^{[-s]} \to \Sigma^{-1}(1 + qH_0 R \Sigma^{-1-s} M)^{[-s]}.
\]
provided by Proposition 6.2,
3. on \( E^2_{-t,t} \), the morphism is induced by the natural transformation
\[
L_t \Sigma^{-1} \Omega^\infty \Sigma^{1+t} M \to \Sigma^{-1} L_t \Sigma^{-1} \Omega^\infty \Sigma^{2+t} M
\]
induced by the natural transformation \( L_t \Sigma^{-1} \to \Sigma^{-1} L_t \) of Proposition 5.13 together with the natural transformation induced by \( \Omega^\infty \to \Sigma^{-1} \Omega^\infty \Sigma \);
4. the morphism induces a natural monomorphism of filtered objects
\[
v_t M \mapsto \Sigma^{-1} v_t \Sigma M
\]
which, for \( t = 0 \), is the natural transformation
\[
\Sigma^{-1} \Omega^\infty \Sigma M \mapsto \Sigma^{-2} \Omega^\infty \Sigma^2 M.
\]

Proof. Straightforward. \( \square \)
9.4. A characterization of instability. As an immediate application of the spectral sequence of Theorem 9.4 one has the following characterization of instability in terms of the $E^2$-page of the spectral sequence.

**Theorem 9.14.** For $M \in \mathcal{M}$, the following conditions are equivalent:

(1) $\Sigma M \in \mathcal{W}$.

(2) For all $t > 0$ and $i > 0$, $(\mathbb{L}_t q H_0 \mathcal{R} \Sigma^t M)^{[1]} = 0$.

(3) For all $t > 0$, $(\mathbb{L}_t q H_0 \mathcal{R} \Sigma^t M)^{[0]} = 0$.

(4) For all $t > 0$, the morphism from the Koszul complex

$$\mathcal{L}_t \Sigma^{-1} \Omega^\infty \Sigma^{1+t} M \to \mathcal{L}_{t-1} \Sigma^{-1} \Omega^\infty \Sigma^t M$$

is injective.

**Proof.** The equivalence of (3) and (4) follows from Corollary 5.38.

The implication (1) $\Rightarrow$ (2) follows from Propositions 7.6 and 4.9 and (2) $\Rightarrow$ (3) is clear.

The spectral sequence of Theorem 9.4 provides the implication (3) $\Rightarrow$ (1), since it implies that the $t = 0$ edge morphism is an isomorphism in total $(s,t)$-degree zero. \qed

10. Examples

Some insight into the behaviour of the spectral sequence of Theorem 9.4 is gained by considering basic examples such as the case $M = \Sigma^{-2} N$, with $N \in \mathcal{W}$ and the cases $M = \Sigma^n q \mathcal{R}^*$, for $n \in \mathbb{Z}$.

10.1. The spectral sequence for desuspensions of unstable modules. Consider the spectral sequence of Theorem 9.4 for $M = \Sigma^{-2} N$, where $N \in \mathcal{W}$. This is not covered by Theorem 9.14. Then $E^2_{p,0} = \Sigma^{-1} \Omega^\infty \Sigma^{-1} N = \Sigma^{-1} \Omega N$. To calculate the $E^2_{1,-} \Sigma$ column it suffices to consider $\tau^{[\leq 1]} H_0 \mathcal{R} \Sigma^{-1} N$, this identifies as

$$\Sigma^{-1} \Omega^\infty N = \Sigma^{-1} N \quad \Sigma^{-1} \Omega^\infty \Sigma^{-1} N,$$

where the dotted arrow indicates the action of Dyer-Lashof operations.

The short exact sequence given by Proposition 2.8 for $\Omega^\infty \Sigma^{-1} N$ identifies as

$$0 \to \Omega_1 N \to \Omega_1 \Sigma^{-1} N \to \Omega_1 \Omega^\infty N \to 0$$

and $\Omega_1 \Omega^\infty N \cong \mathcal{R}_1 N$, since $N$ is unstable.

The inclusion $\Sigma^{-1} \Omega_1 N \to \Sigma^{-1} \Omega_1 \Sigma^{-1} N$ induces a natural short exact sequence in $\mathcal{Q} \mathcal{M}^\mathbb{Z}$:

$$0 \to \Sigma^{-1} \Omega_1 N(1) \to \tau^{[\leq 1]} H_0 \mathcal{R} \Sigma^{-1} N \to \tau^{[\leq 1]} \Sigma^{-1} \mathcal{R} N \to 0.$$

Now, by Corollary 5.38 $(\mathbb{L}_t q \Sigma^{-1} \mathcal{R} N)^{[1]}$ is calculated as the homology of the complex:

$$\mathcal{R}_1 \Sigma^{-1} N \to \Sigma^{-1} \mathcal{R}_1 N,$$

where the morphism is surjective, hence the non-zero homology corresponds to $(\mathbb{L}_t q \Sigma^{-1} \mathcal{R} N)^{[1]}$ given by the kernel, which is isomorphic to $\Phi \mathcal{R} \Sigma^{-1} N$ by the short exact sequence of Proposition 5.18.

The long exact sequence for $\mathbb{L}_* q$ associated to the short exact sequence \[3\] induces an exact sequence:

$$(\mathbb{L}_1 q \Sigma^{-1} \Omega_1 N(1))^{[1]} \to (\mathbb{L}_1 q H_0 \mathcal{R} \Sigma^{-1} N)^{[1]} \to (\mathbb{L}_1 q \Sigma^{-1} \mathcal{R} N)^{[1]} \to \Sigma^{-1} \Omega_1 N(1),$$

where the left hand term is zero for length degree reasons.

Hence this identifies with the exact sequence

$$0 \to (\mathbb{L}_1 q H_0 \mathcal{R} \Sigma^{-1} N)^{[1]} \to \Phi \Sigma^{-1} N(1) \to \Sigma^{-1} \Omega_1 N(1)$$
and the right hand morphism is induced by the canonical surjection $\Phi N \to \Sigma \Omega_1 N$ (recalling that $\Phi \Sigma^{-1} \cong \Sigma^{-2}\Phi$).

Thus the $E^2$-page of the spectral sequence is closely related to the exact sequence of Proposition 2.5

$$0 \to \Sigma \Omega N \to N \to \Phi N \to \Sigma \Omega_1 N \to 0$$

(after applying $\Sigma^{-2}$). In particular

$$v_0(\Sigma^{-2} N) \cong \Sigma^{-1} \Omega N$$

$$v_1(\Sigma^{-2} N)/v_0(\Sigma^{-2} N) \cong \Sigma^{-2} \ker\{\Phi N \to \Sigma \Omega_1 N\}$$

$$v_1(\Sigma^{-2} N) \cong \Sigma^{-2} N.$$

The suspension morphism of spectral sequences can also be described explicitly in this case, via the natural transformations

$$v_t \Sigma^{-2} N \to \Sigma^{-1} v_t \Sigma^{-1} N.$$

Now, $M = \Sigma^{-1} N$ falls within the case of Theorem 9.14, in particular $v_t \Sigma^{-1} N = \Sigma^{-1} N$ for all $t \geq 0$. By inspection

$$v_0 \Sigma^{-2} N = \Sigma^{-1} \Omega N \to \Sigma^{-1} v_0 \Sigma^{-1} N = \Sigma^{-2} N$$

is the natural inclusion and on $v_1$ one has the identity.

Remark 10.1. The above approach may be applied to consider modules of the form $M = \Sigma^{-d} N$ for $N \in \mathcal{U}$ and any $d \in \mathbb{N}$. The case $d = 3$ already shows how the complexity in calculating the $E^2$-page increases significantly with $d$.

10.2. The spectral sequence for $\Sigma^n \mathcal{A}^*$, with $n \in \mathbb{N}$. In [Mil78], Miller explains (working cohomologically) how to recover $\mathcal{A}/\mathcal{A} Sq^1$ from $\text{Untor}^R_{\mathcal{A}}(F, F)$, the homological filtration corresponding to the length filtration of $\mathcal{A}/\mathcal{A} Sq^1$ (see [Mil78, Example 3.3.12]).

Similarly, the spectral sequence of Theorem 9.4 can be used to calculate $\Sigma^n \mathcal{A}^*$ (now working in homology). The $E^2$-page is concentrated on the anti-diagonal, hence the spectral sequence degenerates and

$$E_{\infty}^{s,t} = \mathcal{L}^{s-t} \Sigma^{-1} F(n + t + 1).$$

Here, as in [Mil78], the calculation is only carried out at the level of graded vector spaces, although this can be made much more precise.

Recall the definition of an admissible sequence from Definition 5.19 and the fact that the excess of an admissible sequence identifies as

$$e(I) = i_1 - \sum_{j>1} i_j.$$ 

By convention, the excess of $\emptyset$ is taken to be $\infty$.

Notation 10.2. For $I$ a sequence of length $s$ and $0 \leq j \leq s$, write

$$\omega_j I := I \setminus \{i_1, \ldots, i_j\},$$

so that $\omega_0 I = I$, $\omega_s I = \emptyset$ and, for $0 < j < s$, $\omega_{j-1} I = \{i_j\} \cup \omega_j I$.

Clearly if $I$ is admissible, so is $\omega_j I$ and $e(\omega_j I) \leq e(I)$ for $j < s$.

Proposition 10.3. Let $d \in \mathbb{Z}$. For $I$ an admissible sequence of length $s$, there is a unique integer $j \in \{0, \ldots, s\}$ such that both the following conditions hold

$$e(\omega_j I) \leq d + j \quad \text{if } j < s$$

$$e(\omega_{j-1} I) \geq d + j \quad \text{if } j > 0.$$
Proof. The definition of excess implies that, for \( I \) admissible of length \( s > 0 \),
\[
e(\omega(I)) \geq e(\omega_1 I) \geq \cdots \geq e(\omega_{s-1} I) > 0.
\]
Choose \( j \in \{0, \ldots, s+1\} \) maximal such that the conditions of the Proposition hold. If \( j = 0 \), there is nothing to prove. Otherwise, by the above, for \( 0 \leq k < j \) one has
\[
e(\omega_k I) \geq e(\omega_{j-1} I) \geq d + j > d + k.
\]
Hence there is no solution to the given inequalities for \( k < j \). \( \Box \)

Remark 10.4. For \( j = 0 \), the condition of Proposition 10.3 reduces to \( e(I) \leq d \); for \( j = s \) it reduces to \( i_s \geq d + s \).

Corollary 10.5. For \( d \in \mathbb{Z} \) and \( I \neq \emptyset \) an admissible sequence of length \( s \), there is a unique \( i \in \{0, \ldots, s\} \) such that
\[
e(\omega_I) \leq d + j \quad \text{if } j < s
\]
\[
i_j \geq d + d(\omega_I) + j \quad \text{if } j > 0.
\]
Proof. Suppose that \( j > 0 \). Since \( \omega_{j-1} I = \{i_j\} \cup \omega_j(I) \), one has \( e(\omega_{j-1} I) = i_j - d(\omega_I) \), where \( d(\omega_I) = 0 \), by convention. Hence the condition \( i_j \geq d + d(\omega_I) + j \) is equivalent to \( e(\omega_{j-1} I) \geq d + j \). The result therefore follows from Proposition 10.3 \( \Box \)

Corollary 10.6 is the combinatorial input to the spectral sequence calculating \( \Sigma^n \mathcal{A}' \), together with the calculation of \( 
L^* \Sigma^n F \) given in Proposition 5.21 and the following classical fact (see [Sch94] for example):

Proposition 10.6. The free (cohomological) unstable module \( F(n) \) on a generator of degree \( n \) has basis \( Sq^t \iota_n \), where \( |\iota_n| = n \) and \( I \) is an admissible sequence with \( e(I) \leq n \).

Proposition 10.7. For \( 0 < n \in \mathbb{N} \) and \( t \in \mathbb{N} \), as graded vector spaces
\[

\mathcal{L}^* \Sigma^{-1} F(n + t + 1) \cong \langle \sigma_t St^{l''} \iota_n \rangle
\]
where the sum ranges over admissible sequences \( I', I'' \) such that
\begin{enumerate}
\item \( I'(I') = t \);
\item \( e(I'') \leq (n + 1) + t \);
\item \( i'_t \geq (n + 1) + d(I'') + t \) if \( t > 0 \).
\end{enumerate}
The element \( \sigma_t Lt^{l''} \iota_n \) has degree \( d(I') + d(I'') + n \).

Proof. A basis of \( \Sigma^{-1} F(n + t + 1) \) is indexed by elements \( \Sigma^{-1} St^{l''} \iota_{n+t+1} \), where \( I'' \) is admissible with \( e(I'') \leq n + 1 + t \) and the element has degree \( d(I'') + n + t \). Proposition 5.21 then gives the stated basis for \( \mathcal{L}^* \Sigma^{-1} F(n + t + 1) \). \( \Box \)

Combining Corollary 10.5 with Proposition 10.7 shows that the spectral sequence calculates \( \Sigma^n \mathcal{A}' \), as expected.

Remark 10.8.
\begin{enumerate}
\item It is instructive to consider the suspension morphism of Theorem 9.13 in this case. Details are left to the reader.
\item This analysis extends to all integers \( n \in \mathbb{Z} \). (Cf. Example 9.10.)
\end{enumerate}

Remark 10.9. The infinite delooping spectral sequence of \([Mil78]\) can also be used to recover \( \Sigma^n \mathcal{A}' \), for \( n \in \mathbb{N} \); this relies upon an analysis of \( \text{Unitor}_{\mathbb{F}}(\mathbb{F}, \Sigma F(n - 1)) \).

It is interesting to compare this with the above calculation. For this, Corollary 10.5 is replaced by the following observation for a fixed natural number \( n \in \mathbb{N} \):
For an admissible sequence $I \neq \emptyset$ of length $s$, there is a unique integer $j \in \{0, \ldots, s\}$ such that the following conditions are satisfied:

\[
\begin{align*}
eq (\omega_j I) & \leq n & & \text{if } j < s \\
i_j & > d(\omega_j I) + n & & \text{if } j > 0,
\end{align*}
\]

where $d(\omega_j I) = 0$, by convention.

There is a degree shift which occurs in the calculation, due to the intervention of the homological degree of $\text{Un}^R_{\mathfrak{g}}$.

Remark 10.10. The above arguments based upon decompositions of admissible sequences are related to the calculations occurring in [HM16, Section 5], which involve allowable monomial bases of the Dyer-Lashof algebra.

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