SPACES OF GEODESIC TRIANGULATIONS ARE CELLS

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ABSTRACT. It has been shown that spaces of geodesic triangulations of surfaces with negative curvature are contractible. Here we propose an approach aiming to prove that the spaces of geodesic triangulations of a surface with negative curvature are homeomorphic to Euclidean spaces \( \mathbb{R}^n \).

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1. INTRODUCTION

This work is a continuation of [10], where we proved that the spaces of geodesic triangulations of a surface with negative curvature are contractible. We promote this result in this paper to

**Theorem 1.1.** The spaces of geodesic triangulations of a surface with negative curvature are homeomorphic to Euclidean spaces \( \mathbb{R}^n \).

One motivation to study the spaces of geodesic triangulations of surfaces is that they are natural discretization of spaces of diffeomorphism groups of smooth surfaces. Bloch-Connelly-Henderson [1] proved that the spaces of geodesic triangulations of a convex polygon in \( \mathbb{R}^2 \) are homeomorphic to \( \mathbb{R}^n \) for some \( n > 0 \). This result is regarded as a discrete version of Smale’s theorem stating that the group of diffeomorphisms of the 2-disk fix the boundary pointwise is contractible. The goal for this paper is to establish a complete analogy for the spaces of geodesic triangulations of surfaces of negative curvature and those of convex polygons.
Spaces of geodesic triangulations have been studied since Cairns connected it with the smoothing problem of triangulated 4-manifolds. Recent works by Luo-Wu-Zhu [11] and Erickson-Lin [3] identified the homotopy types of spaces of geodesic triangulations of flat tori and surfaces of negative curvature. Luo-Wu-Zhu [12] also showed that the spaces of Delaunay triangulations of the unit 2-sphere is homotopy equivalent to $SO(3)$, if not empty. However, homotopy types of spaces of geodesic triangulations of the unit 2-sphere remain unknown.

Open contractible manifolds are not necessarily homeomorphic to Euclidean spaces. The typical examples are given by Whitehead manifolds. This paper shows that this pathological behavior will not occur for spaces of geodesic triangulations of surfaces of negative curvature.

1.1. The Main Theorem. Recall the following notations from [10]. Let $M$ be a connected closed orientable smooth surface with a smooth Riemannian metric $g$ of negative Gaussian curvature. A triangulation of $M$ is a homeomorphism $\psi$ from $|T|$ to $M$, where $|T|$ is the carrier of a 2-dimensional simplicial complex $T = (V, E, F)$ with the vertex set $V$, the edge set $E$, and the face set $F$. We label vertices of $T$ as $1, 2, \ldots, n$, where $n = |V|$ is the number of vertices. The edge in $E$ determined by vertices $i$ and $j$ is denoted as $ij$. Each edge is identified with the closed interval $[0, 1]$.

Let $T^{(1)}$ be the 1-skeleton of $T$. A geodesic triangulation in the homotopy class of $\psi$ is an embedding $\phi : T^{(1)} \rightarrow M$ satisfying that

1. The restriction $\phi_{ij}$ of $\phi$ on the edge $ij$ is a geodesic parameterized with constant speed, and

2. $\phi$ is homotopic to $\psi|_{T^{(1)}}$.

Denote $X = X(M, T, \psi)$ as the space of geodesic triangulations homotopic to $\psi$. Then $X$ is a metric space, with the distance function

$$d_X(\phi, \psi) = \max_x d_g(\phi(x), \psi(x)).$$

**Theorem 1.2 ([10]).** The space $X(M, T, \psi)$ is contractible.

The main result of this paper is

**Theorem 1.3.** The space $X(M, T, \psi)$ is homeomorphic to $\mathbb{R}^{2n}$, where $n = |V|$.

We will use the following notations in the rest of this paper. Let $\widetilde{X} = \widetilde{X}(M, T, \psi)$ be the super space of $X$, containing all the continuous maps $\phi : T^{(1)} \rightarrow M$ satisfying that

1. The restriction $\phi_{ij}$ of $\phi$ on the edge $ij$ is geodesic parameterized with constant speed, and

2. $\phi$ is homotopic to $\psi|_{T^{(1)}}$.

Notice that elements in $\widetilde{X}$, which are called geodesic mappings, may not be embeddings of $T^{(1)}$ to $M$. The space $\widetilde{X}$ is also a metric space, with the same distance function as that of $X$. Denote $\widetilde{M}$ as the universal covering of $M$ and $p : \widetilde{M} \rightarrow M$ is a covering map. Then we have a parametrization of $\widetilde{X}$ by $\widetilde{M}^n$.

**Proposition 1.4 ([10]).** There exists a natural homeomorphism from every geodesic mappings $\widetilde{X}$ to $\widetilde{M}^n$, denoted as $\phi \mapsto \widetilde{\phi}$, such that $p(\widetilde{\phi}(i)) = \phi(i)$ for any $i \in V$.

Intuitively, for each vertex in $V$, we can pick one fixed lift of it to $\widetilde{M}$ by $p^{-1}$. Then $\widetilde{\phi}$ is determined by the positions of these vertices in $\widetilde{M}$. Hence, $X$ is a $2n$-dimensional open manifold.
1.2. Organization. This paper is organized as follows. In Section 2, we will summarize prerequisites to prove Theorem 1.3, including a generalized Tutte’s embedding theorem established in [10], the mean value coordinates introduced by Floater in [4], and the construction of an exhaustion of $X$ by compact subsets. In Section 3, we will prove Theorem 1.3 assuming a key lemma on the existence of special homotopy in $X$. In Section 4, we will describe the detailed construction of the homotopy in $X$ to prove the key lemma.

2. PREPARATIONS FOR THE MAIN THEOREM

This section consists of three known results, including a generalized Tutte’s embedding theorem, the mean value coordinates, and a characterization of subsets of $\mathbb{R}^k$ homeomorphic to $\mathbb{R}^k$.

2.1. A generalized Tutte’s embedding theorem. Tutte’s embedding theorem is a fundamental result on the construction of straight-line embeddings of a 3-connected planar graph in the plane, which has profound applications in computational geometry. The connection between Tutte’s embedding theorem and spaces of geodesic triangulations of surfaces was first established in [9] if the surface is a convex polygon in the plane. This connection is further generalized to the cases of surfaces of negative curvature in [10] to prove Theorem 1.2. We summarize this result briefly as follows.

Let $(i, j)$ be the directed edge starting from the vertex $i$ and ending at the vertex $j$. Denote $\bar{E} = \{(i, j) : ij \in E\}$ as the set of directed edges of $T$. A positive vector $w \in \mathbb{R}_{>0}^{\bar{E}}$ is called a weight of $T$. For any weight $w$ and any geodesic mapping $\varphi \in X$, we call $\varphi$ is $w$-balanced if for any $i \in V$,

$$\sum_{j : ij \in E} w_{ij}v_{ij} = 0.$$  

Here $v_{ij} \in T_{q_i}M$ is defined with the exponential map $\exp : TM \to M$ such that $\exp_{q_i}(tv_{ij}) = \varphi_{ij}(t)$ for $t \in [0, 1]$, where $q_i$ is a point on $M$.

**Theorem 2.1.** Assume $(M, g)$ has strictly negative Gaussian curvature. For any weight $w$, there exists a unique geodesic triangulation $\varphi \in X(M, T, \psi)$ that is $w$-balanced. Such induced map $\Phi(w) = \varphi$ is continuous from $\mathbb{R}_{>0}^{\bar{E}}$ to $X$.

Theorem 2.1 is a generalization of the embedding theorems by Colin de Verdière (see Theorem 2 in [2]) and Hass-Scott (see Lemma 10.12 in [7]) from the cases of symmetric weights to non-symmetric weights.

The space of weights can be normalized as follows. Define a map $\alpha : \mathbb{R}_{>0}^{\bar{E}} \to W$, where

$$W := \{w \in \mathbb{R}_{>0}^{\bar{E}} : \sum_{j : j \sim i} w_{ij} = 1 \text{ for any } i \in V\},$$

by the formula

$$\alpha(w)_{ij} = \frac{w_{ij}}{\sum_{k : k \sim i} w_{ik}}.$$  

Then a geodesic triangulation $\phi$ is $w$-balanced if and only if it is $\alpha(w)$-balanced, i.e.,

$$\Phi = \Phi \circ \alpha.$$
2.2. Mean value coordinates. The mean value coordinate was introduced by Floater \cite{Floater2005} as a generalization of barycentric coordinates to produce parametrization of surfaces in the plane. The same construction works for geodesic triangulations of surfaces.

Given \( \varphi \in X \), the mean value coordinates are defined to be

\[
\omega_{ij} = \frac{\tan(\alpha_{ij}/2) + \tan(\beta_{ij}/2)}{|v_{ij}|},
\]

where \( |v_{ij}| \) equals to the geodesic length of \( \varphi_{ij}([0, 1]) \), and \( \alpha_{ij} \) and \( \beta_{ij} \) are the two inner angles in \( \varphi(T^{(1)}) \) at the vertex \( \varphi(i) \) sharing the edge \( \varphi_{ij}([0, 1]) \). This produces a continuous map \( \Psi \) from \( X \) to \( \mathbb{R}^E \). Furthermore, \( \Phi \circ \Psi = \text{id}_X \) by Floater’s mean value theorem (see Proposition 1 in \cite{Floater2005}). Hence, the mean value coordinates introduce a section from \( X \) to \( \mathbb{R}^E \).

2.3. Simple connectivity at infinity. There exist open contractible manifold in \( \mathbb{R}^k \) which are not homeomorphic to \( \mathbb{R}^k \) such as Whitehead manifolds. The following theorem characterizes when a homeomorphism exists. Recall that a topological space \( S \) is simply connected at infinity if for any compact subset \( K \) of \( S \), there is a compact set \( D \) in \( S \) containing \( K \) so that the induced map

\[
\pi_1(S - D) \to \pi_1(S - K)
\]

is trivial.

**Theorem 2.2** (\cite{Bauer2007}). A contractible open subset of \( \mathbb{R}^k \) with \( k \geq 3 \) which is simply connected at infinity is homeomorphic to \( \mathbb{R}^k \).

3. PROOF OF THE MAIN THEOREM

In this section, we will prove Theorem 1.3 by constructing an exhaustion of \( X \) by compact subsets and verifying Theorem 2.2.

3.1. Exhaustion by Compact Sets of \( X \). For any \( \phi \in X \), denote \( \theta^i_{jk}(\phi) \) as the inner angle at vertex \( i \) in the triangle \( \phi(\triangle ijk) \), and

\[
\theta_m(\phi) = \min_{i,j,k \in F} \theta^i_{jk}(\phi).
\]

For any \( \epsilon > 0 \), denote \( K_\epsilon = \{ \phi \in X : \theta_m(\phi) \geq \epsilon \} \). The following proposition shows that \( \{K_{1/n}\}_n \) form an exhaustion by compact sets of \( X \). For any \( \phi \in \bar{X} \), we denote \( l_{ij} = l_{ij}(\phi) \) as the length of the geodesic arc \( \phi_{ij}([0, 1]) \) in \( M \).

**Proposition 3.1.** For any \( \epsilon > 0 \), \( K_\epsilon \) is compact.

**Proof.** We will show that any sequence \( \phi^{(n)} \) in \( K_\epsilon \) has a subsequence converging to some point in \( K_\epsilon \). By picking a subsequence, we may assume that \( l_{ij}(\phi^{(n)}) \) converge to some \( \tilde{l}_{ij} \) in \( [0, \infty] \) for any \( ij \in E \).

If \( \tilde{l}_{ij} = \infty \) for some \( ij \in E \), then by the hyperbolic law of cosines and the comparison theorems of geodesic triangles, \( \theta^i_{jk}(\phi^{(n)}) \to 0 \) for any \( \Delta ijk \in F \). This is impossible because \( \phi^{(n)} \in K_\epsilon \). Therefore, \( \tilde{l}_{ij} < \infty \) for any \( ij \in E \). By a standard compactness argument, we assume that \( \phi^{(n)} \) converge to \( \check{\phi} \in \bar{X} \), with \( \tilde{l}_{ij} = l_{ij}(\check{\phi}) \).

Assume \( \tilde{l}_{ij} = 0 \) for some \( ij \in E \), \( \tilde{l}_{ij} > 0 \) for some \( ij \in E \), since \( \check{\phi} \) is not null-homotopic,. Then there exists \( \Delta ijk \in F \) such that \( l_{ij} = 0 \) and \( \tilde{l}_{ij} > 0 \). Then by the comparison theorems of geodesic triangles and hyperbolic law of sines, \( \theta^i_{jk}(\phi^{(n)}) \to 0 \). This is contradictory to the fact that \( \phi^{(n)} \in K_\epsilon \) for any \( n \). \( \square \)
The following proposition gives a useful estimate to control the minimum angle in the triangulation by conditions on the weights.

**Proposition 3.2.** For any \( \epsilon > 0 \), there exists \( \delta = \delta(M, T, \psi, \epsilon) > 0 \), such that for any weight \( w \in W \) and \( \triangle ijk \in F \), if \( w_{ij} + w_{ik} \geq 1 - \delta \), then \( \Phi(w) \notin K_{\epsilon} \).

**Proof.** If not, we can find a sequence of weights \( w^{(n)} \) in \( W \), and a sequence of triangles \( \triangle i_{n}j_{n}k_{n} \) such that \( w_{i_{n}j_{n}}^{(n)} + w_{i_{n}k_{n}}^{(n)} \to 1 \) and \( \Phi(w^{(n)}) \in K_{\epsilon} \) for any \( n \in \mathbb{N}^+ \). By picking a subsequence, we may assume that \( \triangle i_{n}j_{n}k_{n} \) is a constant triangle \( \triangle ijk \), and \( w^{(n)} \) converge to some \( \tilde{w} \) in the closure of \( W \) with \( \tilde{w}_{ij} + \tilde{w}_{ik} = 1 \), and \( \Phi(w^{(n)}) \) converge to some \( \tilde{\phi} \in K_{\epsilon} \). Then by the continuity \( \tilde{\phi} \) is \( \tilde{w} \)-balanced, and thus \( \tilde{\phi}_{ij}, \tilde{\phi}_{ik} \) are on the same geodesic. This is contradictory to the assumption that \( \tilde{\phi} \) is in \( X \). \( \square \)

### 3.2. Contraction of loops at infinity

Since \( X \) is already known to be a contractible manifold, we only need to prove the following proposition to complete the proof of Theorem 1.3.

**Proposition 3.3.** For any \( \epsilon > 0 \), there exists a positive constant \( \delta = \delta(M, T, \psi, \epsilon) < \epsilon \) such that any loop \( \gamma \) in \( X - K_\delta \) is contractible in \( X - K_\epsilon \).

Denote \( \partial X \) as the boundary of \( X \) in \( \tilde{X} \), and fix a vertex \( v \in V \), and we will prove Proposition 3.3 assuming the following key homotopy lemma.

**Lemma 3.4.** For any \( \epsilon > 0 \) there exists a positive constant \( \delta = \delta(M, T, \psi, \epsilon) < \epsilon \) such that for any loop \( \gamma \) in \( X - K_\delta \), there exists a continuous map

\[
H : S^1 \times [0, 1] \to \tilde{X}
\]

such that

1. \( H(s, 0) = \gamma(s) \) for any \( s \in S^1 \),
2. \( H(s, 1) \in \partial X \) for any \( s \in S^1 \),
3. \( H(s, t) \in X - K_\epsilon \) for any \( s \in S^1 \) and \( t \in [0, 1] \),
4. for any \( (s, t) \in S^1 \times [0, 1] \) and \( i \in V - \{v\} \),

\[
(H(s, t))(i) = (H(s, 0))(i) = (\gamma(s))(i).
\]

The proof of Lemma 3.4 is postponed to Section 4. Intuitively, the homotopy \( H \) above moves only one vertex \( v \) in the family of triangulations in \( \gamma \) so that all the triangulations stay in \( X - K_\epsilon \) and become degenerate in the end.

**Proof of Proposition 3.3.** Recall that \( \Phi \) is the map given by the generalized Tutte’s embedding theorem, and \( \Psi \) is the map given by mean value coordinates. Let \( \delta \) and \( H \) be as in Lemma 3.4. It is straightforward to check that \( \alpha \circ \Psi \circ H \) is a continuous map on \( S^1 \times [0, 1] \), and \( \theta_m \circ H(s, t) \to 0 \) as \( t \to 1 \).

By Proposition 3.2, there exists \( \kappa > 0 \) such that for any weight \( w \) and \( \triangle ijk \in F \), if \( w_{ij} + w_{ik} \geq 1 - \kappa \), then

\[
\theta_m(\Phi(w)) \leq \epsilon.
\]
Claim: There exists $t_0 \in (0, 1)$ such that if $w = \alpha \circ \Psi \circ H(s, t_0)$ for some $s \in S^1$, then there exists $\triangle ijk \in \text{star}(v) := \{\sigma \in F : v \in \sigma\}$ such that
\[ w_{ij} + w_{ik} \geq 1 - \kappa. \]

Let us first finish the proof assuming this claim. By Lemma 3.4, $\gamma$ is homotopic to $\beta := H(\cdot, t_0)$ in $X - K_\epsilon$. Let
\[ \widetilde{\beta} = \alpha \circ \Psi \circ \beta : S^1 \to W_0, \]
where
\[ W_0 := \{w \in W : w_{ij} + w_{ik} \geq 1 - \kappa \text{ for some } \triangle ijk \in \text{star}(v)\}. \]
Since $\Phi \circ \widetilde{\beta} = \beta$, it suffices to show that $\widetilde{\beta}$ is contractible in $W_0$. Fix a vertex $u$ in the neighbor of $v$ denoted as $N(v)$. Define
\[ \widetilde{\pi} : S^1 \to W_0 \]
to be a loop in $W_0$ such that
\[ \widetilde{\pi}_{ij}(s) = \widetilde{\beta}_{ij}(s) \quad \text{if } i \neq u, \]
\[ \widetilde{\pi}_{uv}(s) = 1 - \kappa, \]
and
\[ \widetilde{\pi}_{ui}(s) = \frac{\kappa}{|N_u| - 1} \quad \text{if } i \in N_u - \{v\}. \]
Then $\widetilde{\beta}$ is linearly homotopic to $\widetilde{\pi}$ in $W_0$. Moreover, $\widetilde{\pi}$ lies in an affine subset of $W_0$ defined by
\[ W_u := \{w \in \mathbb{R}^E_{>0} : \sum_{j : j \sim i} w_{ij} = 1 \text{ for } i \in V, \text{ and } w_{ui} = \widetilde{\pi}_{ui}(s) \text{ for any } i \in N_u\}. \]
Hence, $\widetilde{\pi}$ is null-homotopic in $W_0$. This completes the proof of Proposition 3.3.

3.3. Proof of the Claim. If it is not true, we can find a sequence $t_n \to 1$, and a sequence $s_n \in S^1$, such that $w^{(n)} = \alpha \circ \Psi \circ H(s_n, t_n)$ satisfies that
\[ w^{(n)}_{ij} + w^{(n)}_{ik} \leq 1 - \kappa \]
for any $\triangle ijk \in \text{star}(v)$. Let $\phi^{(n)} = H(s_n, t_n)$. By picking a subsequence we may assume that $s_n$ converge to some $\bar{s} \in S^1$. Then $\phi^{(n)}$ converge to some $\bar{\phi} := H(\bar{s}, 1) \in \partial X$, and all the edge length $l^{(n)}_{ij} = l_{ij}(\phi^{(n)})$ converge to $\bar{l}_{ij} := l_{ij}(\bar{\phi})$, where
\[ \bar{l}_{ij} = l_{ij}(H(\bar{s}, 1)) = l_{ij}(\gamma(s)) > 0 \]
for any edge $ij$ not incident vertex $v$. Denote $\bar{w}^{(n)} = \Psi \circ H(s_n, t_n)$ as the mean value coordinates of $\phi^{(n)}$ defined by equation.

3.3.1. If $\bar{l}_{vi} > 0$ for any $i \in N_v$. Then the inner angles $\bar{\theta}_{jk}^{i} := \theta_{jk}^{i}(\bar{\phi})$ are well-defined for any $\triangle ijk \in F$, and there exists some inner angle $\bar{\theta}_{jk}^{i} = \pi$ since $\bar{\phi} \in \partial X$. Notice that $\bar{\theta}_{jk}^{i}$ is the only inner angle at $i$ in $\bar{\phi}$ equal to $\pi$. Then $\bar{w}^{(n)}_{ij} \to \infty, \bar{w}^{(n)}_{ik} \to \infty$, and $\bar{w}^{(n)}_{im}$ converge to some real number for any $m \in N_i - \{j, k\}$. Therefore, $w^{(n)}_{ij} + w^{(n)}_{ik} \to 1$, which contradicts to the assumption. See Figure 1.
3.3.2. If \( \bar{l}_{iv} = 0 \) for some \( i \in N_v \). Since \( w_{iv}^{(n)} \leq 1 - \kappa \) for any \( n \), by picking a subsequence we may assume that there exists some \( m \in N_i \) such that for any \( n \),
\[
\frac{\bar{w}_{iv}^{(n)}}{w_{iv}^{(n)}} \geq \frac{\kappa /(|N_i| - 1)}{1 - \kappa}.
\]
Assume \( \triangle ijv, \triangle ikv \in F \), then equation (2) implies
\[
\frac{\bar{w}_{im}^{(n)}}{w_{im}^{(n)}} = \frac{\kappa}{w_{iv}^{(n)}} \geq \frac{\kappa /(|N_i| - 1)}{1 - \kappa}.
\]
Therefore,
\[
\lim_{n \to \infty} \bar{w}_{im}^{(n)} \to \infty.
\]
\[\] If \( m \notin N_v, \bar{w}_{im}^{(n)} \to (\Psi \circ H(\bar{s},1))_{im} = (\Psi \circ \gamma(\bar{s}))_{im} < \infty \). Hence, \( m \in N_v \cap N_i = \{j, k\} \) and we may assume that \( m = j \). See Figure 2. Let \( \triangle ijh \) be the triangle in \( F \) with \( h \neq v \). Notice that by the last property of the homotopy in Lemma 3.4,
\[
\lim_{n \to \infty} l_{ij}^{(n)} = \bar{l}_{ij} = l_{ij}(\gamma(\bar{s}))(i), (\gamma(\bar{s}))(j)) > 0,
\]
and
\[
\lim_{n \to \infty} \theta_{ijh}^{(n)}(\phi^{(n)}) = \lim_{n \to \infty} \theta_{ijh}^{(n)}(\gamma(s_n)) \to \theta_{ijh}^{(n)}(\gamma(\bar{s})) \in (0, \pi).
\]
Then
\[
\lim_{n \to \infty} \frac{\bar{w}_{ij}^{(n)}}{w_{ij}^{(n)}} = \lim_{n \to \infty} \frac{1}{l_{ij}^{(n)}} \cdot \left( \frac{\tan \theta_{ij}^{(n)}}{2} + \frac{\tan \theta_{ijh}^{(n)}}{2} \right) = \lim_{n \to \infty} \frac{1}{l_{ij}^{(n)}} \cdot \tan \frac{\theta_{ij}^{(n)}}{2}
\]
\[
\leq \lim_{n \to \infty} \frac{1}{l_{ij}^{(n)}} \cdot \frac{\theta_{ij}^{(n)}}{2} \leq \frac{1}{l_{ij}^{(n)}} \cdot \frac{\theta_{ij}^{(n)}}{2} = \lim_{n \to \infty} \frac{l_{ij}^{(n)}}{l_{ij}^{(n)}} = 0.
\]
This contradicts to the equation (3), which completes the argument.

4. CONSTRUCTION OF THE HOMOTOPY IN LEMMA 3.4

We first introduce a few more notions and auxiliary propositions in Section 4.1, then prove Lemma 3.4 in Section 4.2. Proofs of the auxiliary propositions will be provided in Section 4.3.
4.1. Convex Disks and Kernels. Fix a vertex $v \in V$, and denote $\text{star}(v)$ as the closed 1-ring neighborhood of $a$ in $T$. For any $\phi \in X$, $\phi(A)$ denotes the closed (polygonal) disk in $M$ bounded by the union of the geodesic arcs $\phi_{ij}([0, 1])$ with $\triangle ivj \in F$.

Recall the following definitions. A subset $D$ of $M$ is a disk if it is homeomorphic to a unit closed disk. A disk $D \subset M$ is (strongly) convex, if for any $x, y \in D$, there exists a unique geodesic arc in $D$ connecting $x$ and $y$. Such geodesic is denoted as $g = g[D, x, y] : [0, 1] \to M$, with constant speed and $g(0) = x$ and $g(1) = y$. Assume $D$ is a subset of $M$, then the kernel $\ker(D)$ of $D$ is a subset of $D$ containing all the points $x \in D$ such that, for any $y \in D$ there exists a geodesic arc $\gamma$ in $D$ connecting $x$ and $y$. Intuitively, $\ker(D)$ contains all the points who can “see” the whole space $D$. It has been shown that $\ker(D)$ is convex. See [R] for example. Assume $\phi \in X$, and $x \in \ker(\phi(A))$, and $y \in \phi(A)$, then there exists a unique geodesic arc in $\phi(A)$ connecting $x$ and $y$. Such geodesic is still denoted as $g = g[D, x, y] : [0, 1] \to M$, with constant speed and $g(0) = x$ and $g(1) = y$. The uniqueness follows from the fact that $\phi(A)$ is simply connected with a metric of negative curvature.

**Proposition 4.1.** Given a convex disk $D \in M$, and two distinct points $x, y \in D$ with $x \in \text{int}(D)$, then there exists unique point $z = z(D, x, y) \in \partial D$ such that $y$ is on the geodesic arc $g(D, x, z)$.

Intuitively the map $y \mapsto z(D, x, y)$ is a geodesic radial projection to $\partial D$ with center at $x$.

**Proposition 4.2.** If $D \subset M$ is a convex disk, then there exists a unique minimizer $b(D)$ in $D$ for the functional

$$E(x) = \int_D d_D(x, y)^2 dA(y)$$

where $A(y)$ is the area form on $M$. Further $b(D) \in \text{int}(D)$.

Intuitively, such minimizer is the barycentre of the domain $D$.

Let $v$ be a vertex in $V$. Given $\phi \in X$ and $x \in \ker(\phi(A))$, define $\phi_x : T^{(1)} \to M$ as $(\phi_x)_{ij} = \phi_{ij}$ if $v \notin \{i, j\}$ and $(\phi_x)_{iv} = g[\phi(A), x, \phi(i)]$ if $iv \in E$. Intuitively, $\phi_x$ is constructed by moving the vertex $\phi(v)$ to $x$.

Let $K = K(M)$ denote the set of nonempty compact subsets of $M$, and it is naturally a metric space under the Haussdorff distance. Denote

$$K_c = \{D \in K(M) : D \text{ is a convex disk}\},$$

$$P = \{(D, x, y) \in K(M) \times M \times M : D \in K_c(M), x \in D, y \in D\},$$

and

$$P_0 = \{(D, x, y) \in P : x \neq y\}.$$

**Proposition 4.3.** The following properties hold.

(1) $D \mapsto \partial D$ is continuous map from $K_c$ to $K$,

(2) $D \mapsto b(D)$ is a continuous map from $K_c$ to $M$,

(3) $(D, x, y, t) \mapsto g[D, x, y](t)$ is a continuous map from $P \times [0, 1]$ to $M$,

(4) $(D, x, y) \mapsto z(D, x, y)$ is a continuous map from $P_0$ to $M$,

(5) $\phi \mapsto \ker(\phi(A))$ is a continuous map from $X$ to $K$, and

(6) $\phi \mapsto \phi_x$ is a continuous map from $\{(\phi, x) \in X \times M : x \in \ker(\phi(A))\}$ to $\tilde{X}$, and $\phi_x \in X$ if and only if $x \in \text{int}(\ker(\phi(A)))$. 

4.2. Proof of Lemma 3.4

Proof of Lemma 3.4. For any $\epsilon > 0$, denote

$$\bar{K}_\epsilon = \{ \phi \in X : D = \ker(\phi(A)) \},$$

$$ \exists x \in \partial D, y \in g[D, \phi(v), x]([0, 1]) \text{ s.t. } \phi(v) \in g[D, b(D), x]([0, 1]) \text{ and } \phi_y \in K_\epsilon \}.$$ 

It is obvious that $\bar{K}_\epsilon \subset \bar{K}_\epsilon$. Assuming $\gamma$ is a loop in $X - K_\delta$, we will construct a desired homotopy $H$ in Lemma 3.4.

Claim 1: $\bar{K}_\epsilon$ is compact for any $\epsilon > 0$.

Then for any $\epsilon > 0$, we can pick a $\delta > 0$ such that $\bar{K}_\epsilon \subset K_\delta$. Assuming $\gamma$ is a loop in $X - K_\delta$, we will construct a desired homotopy $H$ in Lemma 3.4.

Claim 2: There exists a small perturbation, which is a homotopy, from $\gamma$ to $\gamma_0$ in $X - K_\delta$ such that

$$(\gamma_0(s))(v) \neq b((\gamma_0(s))(A))$$

for any $s \in S^1$.

Assuming the above two claims and $\gamma = \gamma_0$ without loss of generality, we can construct a homotopy $H(s, t)$ as following. Define

1. $D_s = \ker((\gamma(s))(A))$,
2. $b_s = b(D_s)$,
3. $v_s = (\gamma(s))(v)$,
4. $z_s = z(D_s, b_s, v_s)$,
5. $y(s, t) = g[D_s, v_s, z_s](t)$, and
6. $H(s, t) = (\gamma(s))_{y(s,t)}$.

Then the properties (1), (2), and (4) in Lemma 3.4 are satisfied by $H(s, t)$ directly from the definition. Notice that $\gamma(s) \subset X - K_\delta \subset X - K_\epsilon$, then when we deform the vertex $v$ in the homotopy $H(s, t)$, the geodesic triangulation $H(s, t) = (\gamma(s))_{y(s,t)}$ will never lie in $K_\epsilon$ by the definition of $\bar{K}_\epsilon$. This shows that property (3) holds for the homotopy.

4.2.1. Proof of Claim 1. Assume $\phi^{(n)}$ is sequence in $\bar{K}_\epsilon$ and $D_n = \ker(\phi^{(n)}(A))$, then there exist $x_n \in \partial D_n$, $t_n \in [0, 1]$, $y_n = g[D_n, \phi^{(n)}(v), x_n](t_n)$, such that $\phi^{(n)}(v) = g[D_n, b(D_n), x_n](t'_n)$ for some $t'_n \in [0, 1]$ and $\phi_{y_n}^{(n)} \in K_\epsilon$. By picking a subsequence we may assume that $\phi^{(n)} \to \phi \in \bar{X}$, $y_n \to y \in M$, and $\phi_{y_n}^{(n)} \to \phi_y \in \bar{K}_\epsilon$. Then by Proposition 4.3, we have

$$D_n = \ker(\phi^{(n)}(A)) = \ker(\phi_{y_n}^{(n)}(A)) \to \ker(\phi_y(A)) =: \mathcal{D},$$
and \( \bar{D} \) is a convex disk. Moreover, \( b(D_n) \to b(\bar{D}) \).

By picking a subsequence we may assume that \( x_n \to \bar{x} \in \partial \bar{D}, t_n \to \bar{t} \in [0, 1], \) and \( t'_n \to \bar{t}' \in [0, 1] \). Then \( \bar{y} = g(D, \phi(v), \bar{x})[\bar{t}] \) and \( \phi(v) = g(D, b(D), \bar{x})[\bar{t}'] \). It remains to prove that \( \bar{\phi} \in X \). Assume the opposite, we have \( \bar{\phi}(v) \in \partial \bar{D} \). Then \( \bar{\phi}(v) = \bar{x} \) and \( \bar{y} = \bar{x} = \bar{\phi}(v) \in \partial \bar{D} \). This contradicts to the assumption that \( \bar{\phi}_y \in K_{\epsilon} \).

4.2.2. Proof of Claim 2. Recall that by Proposition 1.4, \( \bar{X} \) is homeomorphic to \( \tilde{M^n} \), and \( p : \tilde{M} \to M \) is a covering map. We can parameterize \( \tilde{M} \) by \( \mathbb{R}^2 \), then by a routine compactness argument, there exists some small \( \eta > 0 \) such that if \( a \in \mathbb{R}^2 \) and \( |a|_2 < \eta \), then \( \gamma(s)_{p(\gamma(s)(v) + a)} \in X - K_\delta \) for any \( s \in S^1 \). Geometrically, this means that if the local perturbation \( a \) of the image \( \gamma(s)(v) \) of the vertex \( v \) lifted to \( \bar{X} \) is small enough, then the geodesic triangulations stay in \( X - K_\delta \).

It suffices to construct a continuous function \( g : S^1 \to \mathbb{R}^2 \) such that for any \( s \in S^1 \), \( |g(s)|_2 < \eta \) and

\[
g(s) \neq (\gamma(s)_{b_a})'(v) - \gamma(s)(v) := g_0(s).
\]

Then \( \gamma(s)_{p(\gamma(s)(v) + t-g(s))} \) is a desired homotopy satisfying condition [3] by the definition of \( g(s) \).

There exists some smooth \( g_1 : S^1 \to \mathbb{R}^2 \) such that \( |g_0(s) - g_1(s)|_2 < \eta/2 \) for any \( s \in S^1 \). By Lebesgue measure theory, there exists some \( a \in \mathbb{R}^2 \) with \( |a| < \eta/2 \) such that \( g_1(s) \neq a \) for any \( s \in S^1 \). So \( g(s) = g_0(s) - g_1(s) + a \) is a desired function. \( \square \)

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