Boundary Effects on the Thermodynamics of Quantum Fields Near a Black Hole

Levent Akant†, Emine Ertuğrul‡,
Department of Physics, Boğaziçi University
34342 Bebek, Istanbul, Turkey
†levent.akant@boun.edu.tr, ‡emine.ertugrul@boun.edu.tr,
April 30, 2015

Abstract

We study the thermodynamics of a quantum field in a spherical shell around a static black hole. We impose Dirichlet boundary conditions on the field and analyze their effects on the free energy and the entropy. We consider both bosonic and fermionic fields in Schwarzschild, Reissner-Nordström (RN) and dilatonic backgrounds. We show that the horizon divergencies get contributions from the boundary which, at the Hawking temperature are comparable to the bulk contributions. Moreover it is shown that the leading divergence is the same for all three geometries. Thermodynamics of the quantum fields are studied through the high temperature expansion. We give a derivation of the high temperature expansion in the presence of a chemical potential using Mellin transform and heat kernel methods.
1 Introduction

The analysis of thermodynamic properties of a quantum field around a black hole is an important step towards a deeper understanding the entanglement entropy of a black hole [1–3]. In this paper we consider a non interacting quantum field in a spherical shell around a static black hole, subject to Dirichlet boundary conditions, and investigate the effects of the boundaries of the shell on the thermodynamics of the quantum field. It is well known that the boundary conditions are necessary to define a sensible thermodynamics [4]; the inner wall regularizes the horizon divergencies and the outer wall provides an infrared cutoff. We will consider both bosonic and fermionic fields in Schwarzschild, Reissner-Nordström (RN) and dilatonic backgrounds.

Our main result is that at the Hawking temperature $T_H$ the contribution coming from boundaries to the entropy is comparable to the one coming from bulk contribution. Here the distinction between bulk and boundary contributions stems from the heat kernel expansion for the Dirichlet problem; the boundary terms are those in the heat kernel expansion which arise from the Dirichlet boundary condition on the Laplacian. It is well known that the leading bulk contribution to the entropy is proportional to the area $A_H$ of the horizon. We will show that at the Hawking temperature the leading boundary effect is also proportional to $A_H$ and of the same order of magnitude as the leading bulk term. More precisely we will show that the entropy of the quantum field at $T_H$ is

$$S = B \frac{A_H}{\delta^2} + C \log \delta^2.$$  \hspace{1cm} (1)

Here $\delta$ is a cutoff parameter regulating the horizon divergencies (brick wall cutoff [4]). Both coefficients $B$ and $C$ consist of a bulk plus a boundary contribution which are of the same order of magnitude. Moreover the coefficient $B$ will be shown to be the same for all three geometries considered.

Instead of the original metric we will work with the optical metric which is related to the former by a conformal scaling. We will assume that the mass parameter $m << T$ and employ a high temperature expansion to study the free energy and the entropy of the quantum field. We will also present an alternative derivation [7] of the high energy expansion via Mellin transform methods, without making use of the zeta function regularization. The resulting high $T$ expansion is in complete agreement with the one obtained by the latter method [8–15]. The leading order contribution to the entropy in the high temperature expansion was first calculated in [16] by a rather involved analysis. Later this result was reproduced in [17–19] by a simpler analysis based on the use of the optical metric [20]. Our leading order contri-
bution will be in complete agreement with the one given in the above papers. Higher order corrections in the high temperature expansion were obtained by [21, 22]. A WKB analysis of the problem was given in [23–25]. For an extensive review of the black hole entropy problem see [26].

In Sec. 2 we recall the optical metric construction and derive the high temperature expansion of the free energy for a neutral Bose field with non-vanishing chemical potential. We first express the free energy as a harmonic sum and then apply the Mellin transform method together with the heat kernel expansion. We also show how the resulting expansion is modified when one considers charged boson and fermions. In Sec. 3 we apply the results of Sec. 2 to static black hole backgrounds such as the Schwarzschild, the RN (including the extreme limit) and the dilaton backgrounds and derive the boundary effects on the horizon divergencies of the entropy, we evaluate the latter at the Hawking temperature and compare various bulk and boundary terms.

2 Thermodynamics Around the Black Hole

Consider the $d + 1$ dimensional static metric

\[ ds^2 = -F dt^2 + g_{ij} dx^i dx^j. \]  

The free energy of the Bose gas is given by

\[ F = \frac{1}{\beta} \sum_\sigma \log(1 - e^{-\beta(\epsilon_\sigma - \mu)}). \]  

Here the $\epsilon_\sigma$ ’s are the single particle energies determined by solving the Dirichlet problem for the Klein-Gordon equation

\[ \left[ -\Box + \xi R + m^2 \right] f_\sigma = 0 \]  

by the separation of variable $f_\sigma(t, x) = e^{-i\epsilon_\sigma t}\phi_\sigma(x)$ [27]. Under the scaling transformation

\[ \overline{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \overline{f}_\sigma = \Omega^{\frac{d-1}{2}} f_\sigma, \]  

the KG equation becomes

\[ \left\{ -\overline{\Box} + \frac{d-1}{4d} R + \Omega^{-2} \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right] \right\} \overline{f}_\sigma = 0. \]  

Here it is understood that the barred quantities are calculated with the new metric $\overline{g}_{\mu\nu}$. In particular if we choose $\Omega = F^{-1/2}$ we get the optical metric

\[ ds^2 = -dt^2 + \overline{g}_{ij} dx^i dx^j, \]  

3
where
\[ \overline{g}_{ij} = F^{-1} g_{ij}. \] (8)

Thus the KG equation in the new variables is
\[ \left\{ \partial_0^2 - \overline{\Delta} + \frac{d-1}{4d} \overline{R} + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right] \right\} \overline{f}_\sigma = 0. \] (9)

Since \( F \) does not depend on \( t \) we can still write the solution in the separated form \( \overline{f}_\sigma(t, x) = e^{-i\epsilon_\sigma t} \overline{\phi}_\sigma(x) \). Thus we get the eigenvalue problem for \( \epsilon_\sigma \)'s
\[ \left\{ -\overline{\Delta} + \frac{d-1}{4d} \overline{R} + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right] \right\} \overline{\phi}_\sigma = \epsilon_\sigma^2 \overline{\phi}_\sigma(x). \] (10)

Defining
\[ U_1 = \frac{d-1}{4d} \overline{R} + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right], \] (11)
we get
\[ H_1 \overline{\phi}_\sigma = \epsilon_\sigma^2 \overline{\phi}_\sigma(x), \] (12)
where
\[ H_1 = -\overline{\Delta} + U_1. \] (13)

Now coming back to the free energy, we write it as
\[ \mathcal{F} = -\frac{1}{\beta} \sum_\sigma \sum_{k=1}^{\infty} \frac{1}{k} \epsilon^{k\beta \mu} e^{-k\beta \epsilon_\sigma} \] (14)
\[ = -\sum_{k=1}^{\infty} \frac{1}{k\beta} \epsilon^{k\beta \mu} \sum_\sigma e^{-k\beta \epsilon_\sigma}. \] (15)

Using the subordination identity
\[ e^{-b \sqrt{x}} = \frac{b}{2\sqrt{\pi}} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{u^2}{4b}} e^{-ux}, \] (16)
we get
\[ \sum_\sigma e^{-k\beta \epsilon_\sigma} = \sum_\sigma e^{-k\beta m^{-1} \epsilon_\sigma} = \frac{k\beta m}{2\sqrt{\pi}} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{u(k\beta m)^2}{4b} - uTr} e^{-um^{-2}H}, \] (17)
where
\[ H = H_1 - m^2 = -\overline{\Delta} + U. \] (18)
with
\[
U = U_1 - m^2 = \frac{d-1}{4d}R + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right] - m^2. \tag{19}
\]

Defining \( y = \beta m \) let us write
\[
\mathcal{F}(y) = -m \sum_{k=1}^{\infty} e^{\beta y \pi} G(ky), \tag{20}
\]
with
\[
G(y) = \frac{e^{\beta y \pi_0}}{2\sqrt{2}} \int_{0}^{\infty} \frac{du}{u^{3/2}} e^{-\frac{u^2}{4\pi}T \xi - \frac{u^2}{4\pi}h}. \tag{21}
\]
and
\[
\mu_r = \mu - \pi_0, \quad \mu = \frac{\mu_r}{m}, \quad \pi_r = \frac{\pi}{m}, \quad \pi_0 = \frac{\pi_0}{m}. \tag{22}
\]
Now the sum in (20) is a harmonic sum whose small \( y = \beta m \) asymptotic is completely determined by the poles and the residues of the meromorphic extension of the Mellin transform of \( \mathcal{F} \) \cite{28}. Recall that the Mellin transform of a function \( f(y) \) is given by
\[
(\mathcal{M}f(y))(s) = \tilde{f}(s) = \int_{0}^{\infty} dy y^{s-1} f(y). \tag{23}
\]
Let us also recall the definition of the Laplace-Mellin transform of \( f \)
\[
(\mathcal{LM}f(y))(s, z) = \tilde{f}(s, z) = \int_{0}^{\infty} dy y^{s-1} e^{-z f(y)}. \tag{24}
\]
The asymptotic expansion of \( f(y) \) is then given according to the following dictionary. If the Mellin transform has the singular expansion \((\propto \text{ means the singular part of the expansion})
\[
\tilde{f}(s) \propto \sum_{w,k} A(w, k) \frac{(s-w)^{k+1}}{(s-w)^{k+1}}, \tag{25}
\]
then
\[
f(y) \sim \sum_{w,k} A(w, k) \frac{(-1)^k}{k!} y^{-w}(\log y)^k. \tag{26}
\]
If \( f(y) \) is given by a harmonic sum
\[
f(y) = \sum_{k=1}^{\infty} F(ky) \tag{27}
\]
then it is easy to see that
\[ \tilde{f}(s) = \zeta(s) \tilde{F}(s). \]  
(28)

Now we can apply this method to \( F(y) \). The Mellin transform is given by
\[ \mathcal{M}(-\beta F(y))(s) = \zeta(s) \mathcal{M}(e^{\beta y}G(y))(s). \]  
(29)

But using the definition of the Laplace-Mellin transform \[29\] we can also write this as
\[ \mathcal{M}(-\beta F(y))(s) = \zeta(s) \tilde{G}(s, -\mu). \]  
(30)

Here
\[ \tilde{G}(s, -\mu) = \int_0^\infty dy y^{s-1}e^{\mu y}G(y). \]  
(31)

Now the singularities of this integral arise from the \( y \to 0 \) limit. First let us use the heat kernel expansion
\[ \text{Tr} e^{-\frac{t}{2m^2}} = \frac{1}{(4\pi u)^{d/2}} \left( a_0 + a_1 u^{1/2} + a_1 u + \ldots \right) \]  
(32)
to get
\[ \int_0^\infty du \frac{e^{-\frac{u}{2m^2}}}{u^{d/2}} \text{Tr} e^{-um^{-2}h} = \sum_{n=0}^\infty 2m^{d-n}a_{n/2} \left( \frac{y}{2} \right)^{\frac{n-d-1}{2}} K_{\frac{n+d+1}{2}}(y). \]  
(33)

Here \( K \)'s are the modified Bessel functions of second kind. Using their series expansions we get
\[ \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{du}{u^{3/2}} e^{-\frac{u}{m^2}} \text{Tr} e^{-um^{-2}h} \propto b_{-d-1} y^{-d-1} + b_{-d} y^{-d} + \ldots \]  
(34)

where \( b \) coefficients are
\[ b_{-d-1} = \frac{(2m)^d}{\sqrt{\pi}} \Gamma \left( \frac{d+1}{2} \right) a_0, \]
\[ b_{-d} = \frac{(2m)^{d-1}}{\sqrt{\pi}} \Gamma \left( \frac{d}{2} \right) a_{1/2}, \]
\[ b_{-d+1} = \frac{(2m)^{d-2}}{\sqrt{\pi}} \Gamma \left( \frac{d-1}{2} \right) (-m^2 a_0 + a_1) \]  
(35)
Here we listed only the first few coefficients that we will need in the following. Also note that for the Dirichlet problem the heat kernel coefficients are $[30,31]

\begin{align*}
a_0 &= \frac{1}{(4\pi)^{d/2}} \int_B dV, \\
a_{1/2} &= -\frac{1}{4(4\pi)^{d/2}} \int_{\partial B} dS, \\
a_1 &= \frac{1}{6(4\pi)^{d/2}} \left[ \int_B dV(-6U + \overline{R}) + 2 \int_{\partial B} dS K \right]. \tag{36}
\end{align*}

Here $K$ is the trace of the extrinsic curvature of $\partial B$ calculated with the inward normal.

Now plugging (33) into (21), expanding the exponential pre-factor, and multiplying the series we get for small $y$

\begin{equation*}
G(y) \asymp c_{-d-1} y^{-d-1} + c_{-d} y^{-d} + c_{-d+1} y^{-d+1} + \ldots, \tag{37}
\end{equation*}

where the coefficients of interest to us are

\begin{align*}
c_{-d-1} &= b_{-d-1}, \quad c_{-d} = b_{d} + \tau_0 b_{-d-1} \\
c_{-d+1} &= b_{-d+1} + \tau_0 b_{-d} + \frac{1}{2} \tau_0^2 b_{-d-1}. \tag{38}
\end{align*}

So

\begin{equation*}
\tilde{G}(s,-\mu) = \int_0^\infty dy y^{s-1} e^{-\mu y} G(y)
\end{equation*}

\begin{align*}
&\sim c_{-d-1} \frac{\Gamma(s-d-1)}{(-\mu)^{s-d-1}} + c_{-d} \frac{\Gamma(s-d)}{(-\mu)^{s-d}} + \\
&\quad + c_{-d+1} \frac{\Gamma(s-d+1)}{(-\mu)^{s-d+1}} + \ldots. \tag{39}
\end{align*}

Around $-m$ ($m = 0, 1, 2, \ldots$),

\begin{equation*}
\Gamma(s-n) \sim (-1)^m \frac{1}{m!} \frac{1}{s-n+m}. \tag{40}
\end{equation*}

and we also have the simple pole of $\zeta$ at $s = 1$

\begin{equation*}
\zeta(s) \sim \frac{1}{s-1} + \gamma. \tag{41}
\end{equation*}
Here $\gamma$ is the Euler-Mascheroni constant. Thus the poles of $\zeta(s)\tilde{G}(s, -\mu)$ are seen to be the integers $d, d-1, d-2, \ldots$. Here all the poles are simple except the one at $s = 1$ which is a double pole. Thus we have the following residues:

Around $s = d + 1$:

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d + 1)c_{-d-1}\frac{1}{s - d - 1}.$$  \hfill (42)

around $s = d$:

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d)(c_{-d-1}\mu_r + c_{-d})\frac{1}{s - d}.$$  \hfill (43)

around $s = d - 1$:

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d - 1)(\frac{1}{2}c_{-d-1}\mu_r^2 + c_{-d}\mu_r + c_{-d+1})\frac{1}{s - d + 1}.$$  \hfill (44)

So the small $y$ asymptotic of $\mathcal{F}$ is

$$\mathcal{F} \sim -m\zeta(d + 1)c_{d-1}\left(\frac{T}{m}\right)^{d+1} - m\zeta(d)(c_{d-1}\mu_r + c_{-d})\left(\frac{T}{m}\right)^d$$

$$-m\zeta(d - 1)\left(\frac{1}{2}c_{-d-1}\mu_r^2 + c_{-d}\mu_r + c_{-d+1}\right)\left(\frac{T}{m}\right)^{d-1} + \ldots.$$  \hfill (45)

Here using (38) we get

$$c_{d-1} = b_{d-1} = \frac{(2m)^d}{\sqrt{\pi}} \Gamma\left(\frac{d + 1}{2}\right) a_0$$

$$c_{d-1}\mu_r + c_{-d} = \mu b_{d-1} + b_{-d} = \mu \left(\frac{2m)^d}{\sqrt{\pi}}\right) \Gamma\left(\frac{d + 1}{2}\right) a_0 + \left(\frac{2m)^d-1}{\sqrt{\pi}}\right) \Gamma\left(\frac{d}{2}\right) a_{1/2}$$

$$\frac{1}{2}c_{d-1}\mu_r^2 + c_{-d}\mu_r + c_{-d+1} = \frac{1}{2}\mu^2 b_{d-1} + \mu b_{-d} + b_{d-1}$$

$$= \frac{1}{2}\mu^2 \frac{(2m)^d}{\sqrt{\pi}}\Gamma\left(\frac{d + 1}{2}\right) a_0 + \mu \left(\frac{2m)^d-1}{\sqrt{\pi}}\right) \Gamma\left(\frac{d}{2}\right) a_{1/2}$$

$$+ \frac{(2m)^d-2}{\sqrt{\pi}}\Gamma\left(\frac{d - 1}{2}\right) (-m^2 a_0 + a_1).$$  \hfill (46)
Specializing to $d = 3$ we get
\[
\mathcal{F} = -\frac{\zeta(4)}{\sqrt{\pi}} 8 a_0 T^4 + -\frac{\zeta(3)}{\sqrt{\pi}} [8\mu a_0 + 2 \sqrt{\pi} a_{1/2}] T^3
- \frac{\zeta(2)}{\sqrt{\pi}} [(4\mu^2 - 2m^2)a_0 + 2\sqrt{\pi}\mu a_{1/2} + 2a_1] T^2
+ \frac{1}{2\sqrt{\pi}} \left[ \left( \frac{8}{3}\mu^3 - 4m^2\mu \right) a_0 + 2\sqrt{\pi}(\mu^2 - m^2)a_{1/2} + 4\mu a_1 + 2\sqrt{\pi} a_{3/2} \right]
\times T \log(T/m) + \ldots \tag{47}
\]

In the case of charged bosons we have
\[
\mathcal{F}_{\text{charged}} = \mathcal{F}(\mu) + \mathcal{F}(-\mu) = -\frac{\zeta(4)}{\sqrt{\pi}} 16 a_0 T^4 - \frac{\zeta(3)}{\sqrt{\pi}} 4a_{1/2} T^3
- \frac{\zeta(2)}{\sqrt{\pi}} 4 \left[ (2\mu^2 - m^2)a_0 + a_1 \right] T^2
+ \left[ 2(\mu^2 - m^2)a_{1/2} + \frac{2}{\pi} a_{3/2} \right] \times T \log(T/m) + \ldots \tag{48}
\]

In this expansion one eventually encounters the logarithmic terms arising from the double pole at $s = 1$. For example, at $d = 3$ the fourth term in the expansion is logarithmic.
\[
\mathcal{F} = -m\zeta(3)_{c_{-3}} \left( \frac{T}{m} \right)^4 - m\zeta(2)_{(c_{-3}\mu + c_{-2})} \left( \frac{T}{m} \right)^3
+ m \left( \frac{1}{2} c_{-3\mu}^2 + c_{-2\mu} + c_{-1} \right) \left( \frac{T}{m} \right)^2 \log \left( \frac{T}{m} \right) + \ldots \tag{49}
\]

Finally, for a Fermi gas we have
\[
\mathcal{F}_{\text{fermion}} = -\frac{2}{\beta} \sum_{\sigma} \left[ \log(1 + e^{-\beta(\epsilon_{\sigma} - \mu)}) + \log(1 + e^{-\beta(\epsilon_{\sigma} + \mu)}) \right] \tag{50}
\]

Again expanding the logarithms we get
\[
\mathcal{F} = -4 \sum_{\sigma} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\beta} \cosh(k\beta\mu) e^{-k\beta\epsilon_{\sigma}} \tag{51}
\]
The rest of the analysis parallels the bosonic case. Because of the alternating nature of the series expansion the Mellin transform now generates the Dirichlet $\eta$ function instead of the Riemann $\zeta$.

$$F_{\text{fermion}} = -2\eta(4)\sqrt{\pi} 16a_0T^4 - 2\eta(3)\sqrt{\pi} 4a_{1/2}T^3 - 2\eta(2)\sqrt{\pi} 4\left[(2\mu^2 - m^2)a_0 + a_1\right]T^2 + \ldots$$

Here

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}.$$  

Unlike the bosonic case we do not have a logarithmic term in this expansion since the $\eta$ function is entire and consequently the poles of the Mellin transform are all simple.

### 3 Boundary Effects in Black Hole Backgrounds

We will consider Schwarzschild, Reissner-Nordström (RN) and dilaton black holes in $1+3$ dimensions and calculate the free energy and the entropy for a neutral Bose field. For charged bosons and charged fermions the results can be written down at once with the minor modifications explained at the end of Sec 2. In all three geometries the metric has the general form

$$ds^2 = F(r)dt^2 - F^{-1}(r)dr^2 + r(r - a)[d\theta^2 + \sin^2 \theta d\phi^2],$$

where $a = 0$ for Schwarzschild and RN space-times. We will take $B$ as the "spherical" shell defined by $r_1 \leq r \leq r_2$ with $r_1 = r_H + \epsilon$, where $r_H$ is the radial coordinate of the horizon and $\epsilon$ is the brick wall cutoff.

The optical metric is

$$ds^2 = F^{-2}(r)dr^2 + r(r - a)F^{-1}(r)[d\theta^2 + \sin^2 \theta d\phi^2].$$

The quantities needed for the calculation of the heat kernel coefficients (36) are then as follows.

The volume element is given as

$$dV = F^{-2}(r - a)\sin \theta dr \wedge d\theta \wedge d\phi.$$  

The area element of a constant $r$ surface is given as

$$dS_r = F^{-1}(r - a)\sin \theta d\theta \wedge d\phi,$$
On the other hand,

\[-6U + \overline{R} = -6F\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] + 6m^2. \tag{58}\]

The inward normal to the inner wall is given by the value at \(r = r_1\) of the vector field

\[N = F\partial_r, \tag{59}\]

and the trace of the corresponding intrinsic curvature is

\[K = -\nabla \cdot N = F'(r) - \frac{(2r - a)F(r)}{r(r - a)}. \tag{60}\]

Setting \(\mu = 0\), we get

\[F = -\zeta(4)\frac{1}{\pi^2}VT^4 - \zeta(3)\left(\frac{1}{8\pi}A\right)T^3 \]

\[-\zeta(2)\left\{-\frac{1}{4\pi^2} \left[\int_B dV F\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] - \int_{\partial B} dS \frac{K}{3}\right]\right\}T^2 \]

\[+ \frac{1}{16\pi} \int_{\partial B} dS \left\{-F m^2 + \left(\xi - \frac{1}{6}\right)R\right\} + \] ...

\[= S = -\frac{\partial F}{\partial T} = 4\zeta(4)\frac{1}{\pi^2}VT^3 + 3\zeta(3)\left(\frac{1}{8\pi}A\right)T^2 \]

\[+ 2\zeta(2)\left\{-\frac{1}{4\pi^2} \left[\int_B dV F\left[m^2 + \left(\xi - \frac{1}{6}\right)R\right] - \int_{\partial B} dS \frac{K}{3}\right]\right\}T \]

\[+ \frac{1}{16\pi} \int_{\partial B} dS \left\{-F m^2 + \left(\xi - \frac{1}{6}\right)R\right\} + \] ...

\[\tag{62}\]

### 3.1 Schwarzschild Black Hole

We will now specialize to 1 + 3 Schwarzschild geometry for which

\[F = 1 - \frac{2M}{r}, \quad R = 0, \tag{63}\]
and

\[ K = 2 \frac{3M - r}{r^2}, \tag{64} \]

Now using (56) we get the volume of \( B \) as

\[ V = 4\pi \left[ \frac{r^3}{3} + 2Mr^2 + 12M^2r + 32M^3\log(r - 2M) - \frac{16M^4}{r - 2M} \right]_{r_1}^{r_2}. \tag{65} \]

Therefore

\[ V \approx 64\pi M^3 \left( \frac{M}{\epsilon} - 2\log(\epsilon) \right). \tag{66} \]

Similarly using (57) the area of \( \partial B \) is given as

\[ A = 4\pi \left( \frac{r_1^3}{r_1 - 2M} + \frac{r_2^3}{r_2 - 2M} \right). \tag{67} \]

So \( A \) diverges as

\[ A \approx \frac{32\pi M^3}{\epsilon}. \tag{68} \]

At \( O(T) \) the bulk contribution is

\[ \int_B dV F m^2 = 4\pi m^2 \int_{r_1}^{r_2} dr \frac{r^2}{F} \]

\[ = 4\pi m^2 \left[ (4M^2r + Mr^2 + \frac{r^3}{3} + 8M^3\log(r - 2M)) \right]_{r_1}^{r_2} \]

\[ \approx -32\pi M^3m^2 \log \epsilon. \tag{69} \]

At the same order the boundary contribution to the horizon divergence comes only from the integral over \( r = r_1 \) and is seen to be

\[ \int_{r=r_1} dS_{r_1} \frac{1}{3} K = \frac{8\pi}{3} \left[ \frac{r_1(3M - r_1)}{r_1 - 2M} \right] \]

\[ \approx \frac{16\pi M^2 1}{3\epsilon}. \tag{70} \]

If we compare various bulk and boundary terms appearing in the coefficients of the expansion (62) we get

\[ \frac{A}{V} = O(M^{-1}), \tag{71} \]
\[ \int_{\partial B} \frac{1}{3} K dS = O(M^{-2}), \]  
\[ \int_B F m^2 dV \to 0. \]  
(72)  
(73)

Thus for generic values of \( T \) we see that the boundary contributions are suppressed against the bulk contributions by inverse powers of \( M \). However we must evaluate the entropy at the Hawking temperature \( T_H \) which is \( O(M^{-1}) \). In this case we have for example

\[ \frac{AT_H^2}{VT_H^3} = O(M^0), \]  
(74)

and the boundary contributions become as important as the bulk terms.

At this point we evaluate the entropy (62) at the Hawking temperature

\[ T_H = \frac{1}{8\pi M}. \]  
(75)

We also trade the cutoff \( \epsilon \) with the proper length cutoff \( \delta \)

\[ \epsilon = T_H \pi \delta^2 = \frac{\delta^2}{8M}. \]  
(76)

The result is

\[
S = \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8} \zeta(3) - \frac{2}{3} \zeta(2) \right] \frac{1}{\delta^2} \\
+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}.
\]  
(77)

Here \( L \) is an infrared cutoff which may be taken as the radial coordinate of the outer wall of \( B \). The terms proportional to \( \zeta(4) \) originate from the leading \( O(T^3) \) term of the high temperature expansion which is proportional to the optical volume and are in complete agreement with the results of [16] and [17]. Moreover the dependence of \( S \) on the cutoff \( \delta \) is of the right form for the cancellation against the divergencies in the reciprocal of the bare gravitational constant [5,6].

### 3.2 Reissner-Nordst"orm Black Hole

For the RN background we have

\[ F(r) = \left( 1 - \frac{r_-}{r} \right) \left( 1 - \frac{r_+}{r} \right), \quad R = \frac{2r_+ r_-}{r^4}. \]  
(78)
Thus using (56) we get the volume of the spherical shell as

\[
V = 4\pi \left[ \frac{r_-^6}{(r_--r_+)^2(r_- - r)} + \frac{r_+^6}{(r_--r_+)^2(r_+ - r)} + \left(3r_-^2 + 4r_- r_+ + 3r_+^2\right) r \right.
+ \left. (r_+ + r_-) r^2 + \frac{r^3}{3} + \frac{2r^3 (2r_--3r_+)}{(r_--r_+)^3} \log(r-r_-) \right.
+ \left. \frac{2(3r_- - 2r_+ r_+^4 \log(r-r_+))}{(r_--r_+)^3} \right].
\]

(79)

thus

\[
V \asymp 4\pi \frac{r_+^5}{(r_--r_+)^2} \left[ \frac{r_+}{\epsilon} + \frac{3r_- - 2r_+}{r_+ - r_-} 2 \log \epsilon \right].
\]

(80)

Similarly using (57) we get

\[
A \asymp \frac{4\pi r_+^4}{r_+ - r_-} \frac{1}{\epsilon}.
\]

(81)

We use (58) to get

\[
\int_B F \left[ m^2 - \left(\xi - \frac{1}{6}\right) R \right] = 4\pi m^2 \int_{r_1}^{r_2} dr \frac{r^2}{F} + 4\pi \left(\xi - \frac{1}{6}\right) \int_{r_1}^{r_2} dr \frac{R}{r^2} F^2
\]

\[
= 4\pi m^2 \left[ (r_+^2 + r_-^2 + r_+ r_-) r + \frac{1}{2} (r_+ + r_-) r^2 + \frac{r^3}{3} \right.
+ \left. \frac{r^4}{r_+ - r_-} \log(r-r_-) + \frac{r^4}{r_+ - r_+} \log(r-r_+ \right]_{r_1}^{r_2}
+ \left. 4\pi \left(\xi - \frac{1}{6}\right) \frac{2r_- r_+}{r_+ - r_-} [\log(r-r_+) - \log(r-r_-)]_{r_1}^{r_2} \right)
\]

\[
 \asymp -4\pi \frac{r_+^4 - r_-^4}{r_+ - r_-} \frac{2(\xi - \frac{1}{6}) r_- r_+}{r_+ - r_-} \log \epsilon.
\]

(82)

General expression for the extrinsic curvature over a \( r = \text{constant} \) surface is

\[
K = -\frac{2r^2 + 3(r_+ + r_-) r - 4r_+ r_-}{r^3}.
\]

(83)

Finally, using (60), the surface integral of extrinsic curvature over the inner wall of \( B \) is

\[
\int_{r=r_1} dS_r K = -4\pi \frac{2r_+^3 - 3r_1^2(r_+ + r_-) + 4r_1^2 r_+ r_-}{(r_1 - r_-)(r_1 - r_+)}.
\]

(84)
Thus

\[ \int_{\partial B} dS \frac{1}{3} K \approx \frac{4\pi r_+^2}{3} \epsilon. \]  
(85)

Now evaluating \( S \) at the Hawking temperature

\[ T_H = \frac{r_+ - r_-}{4\pi r_+^2} \]  
(86)

\[ A_H = 4\pi r_+^2 \]  
(87)

with

\[ \epsilon = T_H \pi \delta^2 = \frac{(r_+ - r_-)}{4r_+^2} \delta^2, \]  
(88)

we get

\[ S = \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8} \zeta(3) - \frac{2}{3} \zeta(2) \right] \frac{1}{\delta^2} \]
\[ + \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{2r_+ - 3r_-}{2r_+} + \frac{\zeta(2)}{8\pi} m^2 A_H + \zeta(2) \left( \xi - \frac{1}{6} \right) \frac{r_-}{r_+} \right] \log \left( \frac{\delta^2}{L^2} \right). \]

(89)

For the extreme Reissner-Nordstörm Black Hole case i.e. \( \lim r_- \to r_+ \) we have

\[ S = \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8} \zeta(3) - \frac{2}{3} \zeta(2) \right] \frac{1}{\delta^2} \]
\[ + \frac{1}{\pi^2} \left[ \frac{\zeta(4)}{\pi^2} \frac{1}{2} + \frac{\zeta(2)}{8\pi} m^2 A_H + \zeta(2) \left( \xi - \frac{1}{6} \right) \right] \log \left( \frac{\delta^2}{L^2} \right) \]
\[ + \ldots \]  
(90)

Note that the coefficient of the \( \delta^{-2} \) term is the same as in the Schwarzschild case.

### 3.3 Dilaton Black Hole

For the dilaton metric

\[ F(r) = \left( 1 - \frac{2M}{r} \right), \quad R = \frac{a^2(r - 2M)}{2(r - a)^2 r^3}, \]  
(91)
and consequently we have

\[ V \propto 4\pi \left( 8M^3(2M - a) \frac{1}{\epsilon} + 4M^2(3a - 8M) \log \epsilon \right) \]  

(92)

\[ A \propto 4\pi \left( 4M^2(2M - a) \frac{1}{\epsilon} \right) \]  

(93)

\[ \int_B F \left[ m^2 + \left( \xi - \frac{1}{6} \right) R \right] \propto \frac{8\pi}{3} M^2(a - 2M) \log \epsilon \]  

(94)

General expression for the extrinsic curvature over a \( r = \) constant surface is

\[ K = \frac{-2r^2 + (6M + a)r - 4aM}{r^2(r - a)} \]  

(95)

Integrating over the inner wall of \( B \) gives

\[ \int_{\partial B} dS \frac{1}{3} K \propto \frac{4\pi}{3} \left( 2M(2M - a) \frac{1}{\epsilon} \right) \]  

(96)

After setting \( A_H = 4\pi 2M(2M - a) \), \( T_H = 1/(8\pi M) \) for the entropy we have;

\[ S = \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8} \zeta(3) - \frac{2}{3} \zeta(2) \right] \frac{1}{\delta^2} \]

\[ + \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{(8M - 3a)}{8M} + \frac{\zeta(2)}{48\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}. \]  

(97)

Note that the coefficient of the \( \delta^{-2} \) term is the same as in the Schwarzschild and RN cases.

**Acknowledgement**

This work is supported by Boğaziçi University BAP Project No. 6942. The authors would like to thank O. T. Turgut for useful conversations.

**References**

[1] S. Hawking, Comm. Math. Phys. **43**, 199 (1975).

[2] J. B. Hartle, S. Hawking, Phys. Rev. **D13**, 2188 (1976).

[3] J. Beckenstein, Phys. Rev. **D7**, 2333 (1974).
[4] G. ’t Hooft, Nucl. Phys. B256, 727 (1985).
[5] N. D. Birrell, P. C. W. Davies, Quantum Fields in Curved Space, Cambridge University Press (1984).
[6] L. Susskind, J. Uglum, Phys. Rev. D50, 2700 (1994).
[7] L. Akant, E. Ertugrul, Y. Gul, O. T. Turgut, arXiv:1501.02743.
[8] J. S. Dowker, G. Kennedy, J. Phys. A: Math. Gen. 11, 895 (1978).
[9] A. Actor, J. Phys. A20, 5351 (1987).
[10] J. S. Dowker, J. P. Schofield, Phys. Rev. D38, 3327 (1988).
[11] J. S. Dowker, J. P. Schofield, Nucl. Phys. B327, 267 (1989).
[12] K. Kirsten, J. Phys. A24, 3281 (1991).
[13] K. Kirsten, Class. Quantum Grav. 8, 2239 (1991).
[14] D. J. Toms, Phys. Rev. Lett. 69, 1152 (1992).
[15] D. J. Toms, Phys. Rev. D47, 2483 (1993).
[16] S. N. Solodukhin, Phys. Rev. D51, 618 (1995).
[17] S. P. de Alwis, N. Ohta, Phys. Rev. D52, 3529 (1995).
[18] A. D. Barvinsky, V. P. Frolov, A I. Zelnikov, Phys. Rev. D51, 1741 (1995).
[19] J. L. F. Barbon, Phys. Rev. D50, 2712 (1994).
[20] G. W. Gibbons, M. J. Perry, Proc. Roy. Soc. London A358, 467 (1978).
[21] G. Cognola, L. Vanzo, S. Zerbini, Class. Quant. Grav. 12, 1927 (1995).
[22] G. Cognola, L. Vanzo, S. Zerbini, Phys. Rev. D52, 4548 (1995).
[23] A. Ghosh, P. Mitra, Phys. Rev. Lett. 73, 2521 (1994).
[24] A. Ghosh, P. Mitra, Phys. Lett. B357, 295 (1995).
[25] A. Ghosh, Nucl. Phys. B814, 212 (2009).
[26] S. N. Solodukhin, Living Rev. Rel. 14, 8 (2011).
[27] K. S. Thorne, D. A. MacDonald, R. H. Price, Black Holes: The Membrane Paradigm, Yale University Press (1986).

[28] P. Flajolet, X. Gourdon, P. Dumas, Theoretical Computer Science 144, 3-58 (1995).

[29] J. Jorgenson, S Lang, Basic Analysis of Regularized Series and Products, Springer-Verlag (1993).

[30] T. P. Branson, P. B. Gilkey, Comm. Partial Differential Equations 15 245 (1990).

[31] D. V. Vassilevich, Phys. Rept. 388 279 (2003).