FROM ANDERSON TO ZETA

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Abstract. For an irreducible crystallographic root system $\Phi$ and a positive integer $t$ relatively prime to the Coxeter number $h$ of $\Phi$, we give a natural bijection $A$ from the set $\tilde{W}_t$ of affine Weyl group elements with no inversions of height $t$ to the finite torus $\tilde{\mathcal{Q}}/t\tilde{\mathcal{Q}}$. Here $\tilde{\mathcal{Q}}$ is the coroot lattice of $\Phi$. This bijection is defined uniformly for all irreducible crystallographic root systems $\Phi$ and is equivalent to the Anderson map $\mathcal{A}_{GMV}$ defined by Gorsky, Mazin and Vazirani when $\Phi$ is of type $A_{n-1}$.

Specialising to $t = mh + 1$, we use $A$ to define a uniform $W$-set isomorphism $\zeta$ from the finite torus $\tilde{\mathcal{Q}}/(mh + 1)\tilde{\mathcal{Q}}$ to the set of $m$-nonnesting parking functions $\text{Park}^{(m)}(\Phi)$ of $\Phi$. The map $\zeta$ is equivalent to the zeta map $\zeta_{HL}$ of Haglund and Loehr when $m = 1$ and $\Phi$ is of type $A_{n-1}$.

1. Introduction

The aim of this article is to describe uniform generalisations to all irreducible crystallographic root systems $\Phi$ of two bijections that arose from the study of the Hilbert series $\mathcal{D}\mathcal{H}(n; q, t)$ of the space of diagonal harmonics of the symmetric group $S_n$. These bijections are the Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani [GMV14, Section 3.1] and the zeta map $\zeta_{HL}$ of Haglund and Loehr [Hag08, Theorem 5.6].

1.1. The Hilbert series of the space of diagonal harmonics. The space of diagonal harmonics of the symmetric group $S_n$ is a well-studied object in algebraic combinatorics. Its Hilbert series has two (conjectural) combinatorial interpretations [Hag08, Conjecture 5.2]:

$$\mathcal{D}\mathcal{H}(n; q, t) = \sum_{(P, \sigma) \in \mathcal{P}\mathcal{F}_n} q^{\text{dinv}'(P, \sigma)}t^{\text{area}(P, \sigma)} = \sum_{(w, D) \in \mathcal{D}_n} q^{\text{area}'(w, D)}t^{\text{bounce}(w, D)},$$

where $\mathcal{P}\mathcal{F}_n$ is the set of parking functions of length $n$ viewed as vertically labelled Dyck paths and $\mathcal{D}_n$ is the set of diagonally labelled Dyck paths of length $n$. There is a bijection $\zeta_{HL}$ due to Haglund and Loehr [Hag08, Theorem 5.6] that maps $\mathcal{P}\mathcal{F}_n$ to $\mathcal{D}_n$ and sends the bistatistic $(\text{dinv}', \text{area})$ to $(\text{area}', \text{bounce})$, demonstrating the second equality.

In their study of rational parking functions, Gorsky, Mazin and Vazirani introduced the Anderson map $\mathcal{A}_{GMV}$ as a bijection from the set of $t$-stable affine permutations $\tilde{S}_n^t$ to the set of $t/n$-parking functions $\mathcal{P}\mathcal{F}_{t/n}$. Here $t$ is any positive integer relatively prime to $n$ and $\mathcal{P}\mathcal{F}_{n+1/n} = \mathcal{P}\mathcal{F}_n$. They used it to define a combinatorial Hilbert series for the set of $t$-stable affine permutations that generalises the conjectured formulas for $\mathcal{D}\mathcal{H}(n; q, t)$ and related $\mathcal{A}_{GMV}$ to $\zeta_{HL}$ [GMV13, Definition 3.26, Theorem 5.3].
1.2. Beyond type $A_{n-1}$. One can view all the objects $\mathcal{P}_n$, $\mathcal{D}_n$, $\tilde{S}_n$ and $\mathcal{P}_{t/n}$ as well as the maps $\zeta_{HL}$ and $\mathcal{A}_{GMV}$ as being associated with the root system of type $A_{n-1}$. We will generalise both the zeta map $\zeta_{HL}$ of Haglund and Loehr and the Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani to all irreducible crystallographic root systems $\Phi$. We work at three different levels of generality, in order from most general to least general these are the rational level, the Fuß-Catalan level, and the Coxeter-Catalan level. All the objects we now mention will be defined in later sections.

1.3. The rational level. We fix a positive integer $t$ that is relatively prime to the Coxeter number $h$ of $\Phi$. We define the Anderson map $\mathcal{A}$ as a bijection from the set $\tilde{\mathcal{W}}_t$ of affine Weyl group elements with no inversions of height $t$ to the finite torus $\tilde{Q}/t\tilde{Q}$. Here $\tilde{Q}$ is the coroot lattice of $\Phi$. If $\Phi$ is of type $A_{n-1}$ it reduces to the Anderson map $\mathcal{A}_{GMV}$ of Gorsky, Mazin and Vazirani that maps the set $\tilde{S}_n$ of $t$-stable affine permutations to the set of rational parking functions $\mathcal{P}_{t/n}$.

1.4. The Fuß-Catalan level. At the Fuß-Catalan level, we specialise to $t = mh + 1$ for some positive integer $m$. We consider an affine hyperplane arrangement in the ambient space of $\Phi$ called the $m$-extended Shi arrangement. Every region $R$ of that arrangement has a unique minimal alcove $w_RA_0$, and the set $\text{Alc}_m^m$ of minimal alcoves of regions of the $m$-extended Shi arrangement corresponds to $\tilde{\mathcal{W}}_{mh+1}$. Thus the Anderson map $\mathcal{A}$ gives a bijection from $\text{Alc}_m^m$ to the finite torus $\tilde{Q}/(mh+1)\tilde{Q}$.

The set $\text{Park}^{(m)}(\Phi)$ of $m$-nonnesting parking functions was defined by Rhoades [Rho14] as a model for the set of regions of the $m$-extended Shi arrangement that carries an action of the Weyl group $W$. The finite torus $\tilde{Q}/(mh+1)\tilde{Q}$ also has a natural $W$-action. Using the inverse $\mathcal{A}^{-1}$ of the Anderson map we define a $W$-set isomorphism $\zeta$ from $\tilde{Q}/(mh+1)\tilde{Q}$ to $\text{Park}^{(m)}(\Phi)$. We call this the zeta map.

1.5. The Coxeter-Catalan level. At the Coxeter-Catalan level, we specialise further to $m = 1$. Thus we have $t = h + 1$. In the case where $\Phi$ is of type $A_{n-1}$, we identify the combinatorial objects $\mathcal{P}_n$ and $\mathcal{D}_n$ with the finite torus $\tilde{Q}/(h+1)\tilde{Q}$ and the set $\text{Park}(\Phi)$ of nonnesting parking functions of $\Phi$ respectively. With these identifications, our zeta map $\zeta$ coincides with the zeta map $\zeta_{HL}$ of Haglund and Loehr.

1.6. Structure of the article. This paper is structured as follows. Section 2 gives some basic definitions about irreducible crystallographic root systems and various hyperplane arrangements associated with them. Sections 3 to 6 are devoted to defining the uniform bijection $\mathcal{A}$ at the Fuß-Catalan level. We use a chain of various maps that already occur in the literature in some form. The intermediate objects are introduced when they are needed. Sections 3 to 6 can be summarised in the following chain of bijections.

$$ \mathcal{A} $$

Shi regions $\rightarrow$ Shi alcoves $\rightarrow$ Sommers region $\rightarrow (mh+1)A_0 \rightarrow \tilde{Q}/(mh+1)\tilde{Q}$

$R \rightarrow w_RA_0 \rightarrow w^{-1}_RA_0 \rightarrow w_fw^{-1}_RA_0 \rightarrow w_Rw^{-1}_f \cdot 0$
Section 7 defines the uniform bijection $\zeta$ at the Fuß-Catalan level. It can be summarised in the following commutative diagram of bijections. Here $\Theta$ is a natural bijection defined in Theorem 7.1.

In Section 8 we generalise $A$ to the rational level, as a bijection from $\tilde{W}^+$ to $\tilde{Q}/t\tilde{Q}$. After Sections 9 to 11 have provided the necessary combinatorial setup, Section 12 shows that our Anderson map $A$ is equivalent to the Anderson map $A_{GMV}$ of Gorsky, Mazin and Vazirani in the case where $\Phi$ is of type $A_{n-1}$. In Section 13 we specialise to the Coxeter-Catalan level and show that in the case where $m = 1$ and $\Phi$ is of type $A_{n-1}$, our zeta map $\zeta$ is equivalent to the zeta map $\zeta_{HL}$ of Haglund and Loehr. Its content can be summarised in the following commutative diagram of bijections. Here $\epsilon$, $\delta$ and $\chi$ are natural bijections defined in Section 13.

2. Root systems and hyperplane arrangements

Let $\Phi$ be an irreducible crystallographic root system of rank $r$ with ambient space $V$. For background on root systems see [Hum90]. Choose a set of simple roots $\Delta$ for $\Phi$ and let $\Phi^+$ be the corresponding set of positive roots. Let $W$ be the Weyl group of $\Phi$ and let $S$ be the set of simple reflections corresponding to $\Delta$. Then $S$ generates $W$ and $(W, S)$ is a Coxeter system.

**Example.** The root system of type $A_{n-1}$ is $\Phi = \{e_i - e_j : i, j \in [n], i \neq j\}$. It has rank $n - 1$ and ambient space $V = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0\}$. We let $\alpha_i = e_i - e_{i+1}$ for $i \in [n-1]$ and choose $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$. The Weyl group $W$ of $\Phi$ is the symmetric group $S_n$ on $n$ letters that acts on $V$ by permuting coordinates. It is generated by $S = \{s_1, s_2, \ldots, s_{n-1}\}$, where $s_i = (i \ i+1)$ is the $i$-th adjacent transposition, corresponding to the reflection through the linear hyperplane orthogonal to $\alpha_i$.

For $d \in \mathbb{Z}$ and $\alpha \in \Phi$, define the affine hyperplane

$$H_\alpha^d := \{x \in V : \langle x, \alpha \rangle = d\}.$$ 

2.1. The Coxeter arrangement. The Coxeter arrangement is the central hyperplane arrangement in $V$ given by all the linear hyperplanes $H_\alpha^0 = H_\alpha$ for $\alpha \in \Phi^+$. The complement of this arrangement falls apart into connected components which we call chambers. The Weyl group $W$ acts simply transitively on the chambers. Thus we define the dominant chamber by

$$C := \{x \in V : \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta\}.$$
and write any chamber \( wC \) for a unique \( w \in W \).

2.2. The affine Coxeter arrangement and the affine Weyl group. The root order on \( \Phi^+ \) is the partial order defined by \( \alpha \leq \beta \) if and only if \( \beta - \alpha \) can be written as a sum of positive roots. The set of positive roots \( \Phi^+ \) with this partial order is called the root poset of \( \Phi \). It has a unique maximal element, the highest root \( \tilde{\alpha} \). Write \( \tilde{\alpha} = \sum_{\alpha \in \Delta} c_\alpha \alpha \) as a linear combination of the simple roots and define the Coxeter number of \( \Phi \) as \( h = 1 + \sum_{\alpha \in \Delta} c_\alpha \).

The affine Coxeter arrangement is the affine hyperplane arrangement in \( V \) given by all the affine hyperplanes \( H^d_\alpha \) for \( \alpha \in \Phi \) and \( d \in \mathbb{Z} \). The complement of this arrangement falls apart into connected components which are called alcoves. We call an alcove dominant if it is contained in the dominant chamber. Define \( s^d_\alpha \) as the reflection through the affine hyperplane \( H^d_\alpha \). That is, \( s^d_\alpha(x) := x - 2\frac{\langle x, \alpha \rangle - d}{\langle \alpha, \alpha \rangle} \alpha \).

We will also write \( s_\alpha \) for the linear reflection \( s^0_\alpha \).

Let the affine Weyl group \( \tilde{W} \) be the group of affine automorphisms of \( V \) generated by all the reflections through hyperplanes in the affine Coxeter arrangement, that is \( \tilde{W} := \langle s^d_\alpha : \alpha \in \Phi \text{ and } d \in \mathbb{Z} \rangle \).

The affine Weyl group \( \tilde{W} \) acts simply transitively on the alcoves of the affine Coxeter arrangement. Thus we define the fundamental alcove by

\[
A_0 := \{ x \in V : \langle x, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha} \rangle < 1 \}
\]

and write any alcove of the affine Coxeter arrangement as \( w_\alpha A_0 \) for a unique \( w_\alpha \in \tilde{W} \).

If we define \( \tilde{S} := S \cup \{ s^1_\alpha \} \), then \( \tilde{S} \) generates \( \tilde{W} \) and \( (\tilde{W}, \tilde{S}) \) is a Coxeter system.

For a root \( \alpha \in \Phi \), its coroot is defined as \( \alpha^\vee = 2\frac{\alpha}{\langle \alpha, \alpha \rangle} \). The set \( \Phi^\vee = \{ \alpha^\vee : \alpha \in \Phi \} \) is itself an irreducible crystallographic root system, called the dual root system of \( \Phi \). Clearly \( \Phi^{\vee\vee} = \Phi \).

The root lattice \( Q \) of \( \Phi \) is the lattice in \( V \) spanned by all the roots in \( \Phi \). The coroot lattice \( \hat{Q} \) of \( \Phi \) is the lattice in \( V \) spanned by all the coroots in \( \Phi^\vee \). It is not hard to see that \( \tilde{W} \) acts on the coroot lattice. To any \( \mu \in \hat{Q} \), there corresponds the translation

\[
t_\mu : V \to V \\
x \mapsto x + \mu.
\]

If we identify \( \hat{Q} \) with the corresponding group of translations acting on the affine space \( V \) then we may write \( \tilde{W} = W \ltimes \hat{Q} \) as a semidirect product.

For an alcove \( w_\alpha A_0 \) and a root \( \alpha \in \Phi \) there is a unique integer \( k \) such that \( k < \langle x, \alpha \rangle < k + 1 \) for all \( x \in w_\alpha A_0 \). We denote this integer by \( k(w_\alpha, \alpha) \). We call the tuple \( (k(w_\alpha, \alpha))_{\alpha \in \Phi^+} \) the address of the alcove \( w_\alpha A_0 \).

Notice that we have \( k(w_\alpha, -\alpha) = -k(w_\alpha, \alpha) - 1 \) and \( k(w w_\alpha, w(\alpha)) = k(w_\alpha, \alpha) \).
for all $\alpha \in \Phi$ and $w \in W$. Also note that if $k(w_\alpha, \alpha) = k(w_b, \alpha)$ for all $\alpha \in \Phi^+$, then $w_\alpha = w_b$.

2.2.1. The Shi arrangement. The Shi arrangement is the affine hyperplane arrangement given by all the hyperplanes $H^d_\alpha$ for $\alpha \in \Phi^+$ and $d = 0, 1$. It was first introduced in [Shi87] and arose from the study of the Kazhdan-Lusztig cells of the affine Weyl group of type $A_{n-1}$. The complement of these hyperplanes, we call them Shi hyperplanes, falls apart into connected components which we call the regions of the Shi arrangement, or Shi regions for short. We call a Shi region dominant if it is contained in the dominant chamber.

An ideal in the root poset is a subset $I \subseteq \Phi^+$ such that whenever $\alpha \in I$ and $\beta \leq \alpha$, then $\beta \in I$. Dually, we define an order filter as a subset $J \subseteq \Phi^+$ such that whenever $\alpha \in J$ and $\alpha \leq \beta$ then $\beta \in J$. For a dominant Shi region $R$ define

$$\phi(R) := \{\alpha \in \Phi^+: (x, \alpha) > 1 \text{ for all } x \in R\}.$$  

It is easy to see that $\phi(R)$ is an order filter in the root poset of $\Phi$. In fact, $\phi$ even defines a bijection between the set of dominant Shi regions and the set of order filters in the root poset [Shi97, Theorem 1.4].

The minimal elements of an order filter $J$ are pairwise incomparable, so they form an antichain $A$. One can recover the order filter $J$ from the antichain $A$ since

$$J = \{\beta \in \Phi^+: \beta \geq \alpha \text{ for some } \alpha \in A\}.$$  

This gives a simple bijection between the order filters and the antichains of the root poset. Putting it together with $\phi$ gives a bijection from dominant Shi regions to antichains in the root poset.

2.3. The m-extended Shi arrangement. For a positive integer $m$, the $m$-extended Shi arrangement, or simply $m$-Shi arrangement, is the affine hyperplane arrangement given by all the hyperplanes $H^d_\alpha$ for $\alpha \in \Phi^+$ and $-m < d \leq m$. We call them $m$-Shi hyperplanes. The complement of these hyperplanes falls apart into connected components, which we call the regions of the $m$-Shi arrangement, or $m$-Shi regions for short. Notice that the 1-Shi arrangement is exactly the Shi arrangement introduced in Section 2.2.1.

Following [Ath05], we will encode dominant $m$-Shi regions by geometric chains of ideals or equivalently geometric chains of order filters. Suppose $\mathcal{I} = (I_1, I_2, \ldots, I_m)$ is an ascending (multi)chain of $m$ ideals in the root poset of $\Phi$, that is $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_m$. Setting $J_i := \Phi^+ \setminus I_i$ for $i \in [m]$ and $\mathcal{J} := (J_1, J_2, \ldots, J_m)$ gives us the corresponding descending chain of order filters. That is, we have $J_1 \supseteq J_2 \supseteq \ldots \supseteq J_m$. The ascending chain of ideals $\mathcal{I}$ and the corresponding descending chain of order filters $\mathcal{J}$ are both called geometric if the following conditions are satisfied simultaneously.

1. $(I_i + I_j) \cap \Phi^+ \subseteq I_{i+j}$ for all $i, j \in \{0, 1, \ldots, m\}$ with $i + j \leq m$, and
2. $(J_i + J_j) \cap \Phi^+ \subseteq J_{i+j}$ for all $i, j \in \{0, 1, \ldots, m\}$.

Here we set $I_0 := \emptyset$, $J_0 := \Phi^+$ and $J_i := J_m$ for $i > m$. We call $\mathcal{I}$ and $\mathcal{J}$ positive if $\Delta \subseteq I_m$, or equivalently $\Delta \cap J_m = \emptyset$.

If $R$ is a dominant $m$-Shi region define $\theta(R) := (I_1, I_2, \ldots, I_m)$ and $\phi(R) := (J_1, J_2, \ldots, J_m)$, where

$$I_i := \{\alpha \in \Phi^+: \langle x, \alpha \rangle < i \text{ for all } x \in R\} \text{ and } J_i := \{\alpha \in \Phi^+: \langle x, \alpha \rangle > i \text{ for all } x \in R\}$$
for $i \in \{0, 1, \ldots, m\}$. It is not difficult to verify that $\theta(R)$ is a geometric chain of ideals and that $\phi(R)$ is the corresponding geometric chain of order filters.

In fact $\theta$ is a bijection from dominant $m$-Shi regions to geometric chains of ideals. Equivalently $\phi$ is a bijection from dominant $m$-Shi regions to geometric chains of order filters [Ath05, Theorem 3.6].

3. Minimal alcoves of $m$-Shi regions

Any alcove of the affine Coxeter arrangement is contained in a unique $m$-Shi region. We will soon see that for any $m$-Shi region $R$ there is a unique alcove $w_R A_0 \subseteq R$ such that for all $w_a A_0 \subseteq R$ and all $\alpha \in \Phi^+$ we have

$$|k(w_R, \alpha)| \leq |k(w_a, \alpha)|.$$  

We call $w_R A_0$ the minimal alcove of $R$. We say that an alcove $w_a A_0$ is an $m$-Shi alcove if it is the minimal alcove of the $m$-Shi region containing it. We define $\text{Alc}_m = \{w_R A_0 : R$ is an $m$-Shi region\} to be the set of $m$-Shi alcoves.

3.1. The address of a dominant $m$-Shi alcove. We first concentrate on dominant $m$-Shi regions and their minimal alcoves. The following lemma from [Shi87a, Theorem 5.2] gives necessary and sufficient conditions for a tuple $(k_\alpha)_{\alpha \in \Phi^+}$ to be the address of some alcove $w_a A_0$.

**Lemma 3.1** ([Ath05, Lemma 2.3]). Suppose that for each $\alpha \in \Phi^+$ we are given some integer $k_\alpha$. Then there exists $w_a \in \tilde{W}$ with $k(w_a, \alpha) = k_\alpha$ for all $\alpha \in \Phi^+$ if and only if

$$k_\alpha + k_\beta \leq k_{\alpha + \beta} \leq k_\alpha + k_\beta + 1$$

for all $\alpha, \beta \in \Phi^+$ with $\alpha + \beta \in \Phi^+$.

For a geometric chain of order filters $\mathcal{J} = (J_1, J_2, \ldots, J_m)$ and $\alpha \in \Phi^+$, define $k_\alpha(\mathcal{J}) = \max\{k_1 + k_2 + \ldots + k_l : \alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_l \text{ and } \alpha_i \in J_{k_i} \text{ for all } i \in [l]\}$ where $k_i \in \{0, 1, \ldots, m\}$ for all $i \in [l]$.  

![Figure 1. The 49 minimal alcoves of the 2-Shi arrangement of type $A_2$.](image)
It turns out that the integer tuple \((k_\alpha(J))_{\alpha \in \Phi^+}\) satisfies the conditions of Lemma 3.1 [Ath05 Corollary 3.4], so there is a unique \(w_a \in \tilde{W}\) with 
\[k(w_a, \alpha) = k_\alpha(J)\] for all \(\alpha \in \Phi^+\).

The alcove \(w_a A_\circ\) is exactly the minimal alcove \(w_R A_\circ\) of the dominant \(m\)-Shi region \(R := \phi^{-1}(J)\) corresponding to \(J\) [Ath05 Proposition 3.7].

3.2. **Floors of dominant \(m\)-Shi regions and alcoves.** A wall of an \(m\)-Shi region \(R\) is a hyperplane supporting a facet of \(R\). It is called a **floor** of \(R\) if it does not contain the origin and separates \(R\) from the origin. Define in a similar fashion the walls and floors of an alcove, or more generally of any open or closed polyhedron in \(V\). So any wall of an alcove is a hyperplane of the affine Coxeter arrangement, and any wall of an \(m\)-Shi region is an \(m\)-Shi hyperplane.

The floors of a dominant \(m\)-Shi region \(R\) can be seen in the corresponding geometric chain of order filters \(J := \phi(R)\) as follows. If \(k\) is a positive integer, a root \(\alpha \in \Phi^+\) is called a **rank \(k\) indecomposable element** of a geometric chain of order filters \(J = (J_1, J_2, \ldots, J_m)\) if the following hold:

1. \(k_\alpha(J) = k\),
2. \(\alpha \notin J_i + J_j\) for any \(i, j \in \{0, 1, \ldots, m\}\) with \(i + j = k\) and
3. if \(\alpha + \beta \in J_i\) and \(k_{\alpha + \beta}(J) = t \leq m\) for some \(\beta \in \Phi^+\), then \(\beta \in J_{i-k}\).

The following theorem relates the indecomposable elements of \(J\) to the floors of \(R\) and \(w_R A_\circ\).

**Theorem 3.2 ([Ath05 Theorem 3.11]).** If \(R\) is a dominant \(m\)-Shi region, \(J = \phi(R)\) is the corresponding geometric chain of order filters and \(\alpha \in \Phi^+\), then the following are equivalent:

1. \(\alpha\) is a rank \(k\) indecomposable element of \(J\),
2. \(H^k_\alpha\) is a floor of \(R\), and
3. \(H^k_\alpha\) is a floor of \(w_R A_\circ\).

3.3. **\(m\)-Shi regions and alcoves in other chambers.** The following easy lemma, generalising [ARR15 Lemma 10.2], describes what the \(m\)-Shi arrangement looks like in each chamber.

**Lemma 3.3.** For \(w \in W\), the \(m\)-Shi hyperplanes that intersect the chamber \(wC\) are exactly those of the form \(H^k_{w(\alpha)}\) where \(\alpha \in \Phi^+\) and either \(1 \leq k < m\) or \(k = m\) and \(w(\alpha) \in \Phi^+\).

**Proof.** If an \(m\)-Shi hyperplane \(H^k_{\beta}\) with \(\beta \in \Phi\) and \(1 \leq k \leq m\) intersects \(wC\), then there is some \(x \in wC\) with \(\langle x, \beta \rangle = k\). So \(w^{-1}(x) \in C\) and \(\langle w^{-1}(x), w^{-1}(\beta) \rangle = k > 0\), thus \(\alpha := w^{-1}(\beta) \in \Phi^+\). If \(k = m\), then \(\beta = w(\alpha) \in \Phi^+\) since otherwise \(H^k_{\beta}\) is not an \(m\)-Shi hyperplane.

Conversely, if \(\alpha \in \Phi^+\) and either \(1 \leq k < m\) or \(k = m\) and \(w(\alpha) \in \Phi^+\), then \(H^k_{w(\alpha)}\) is an \(m\)-Shi hyperplane. Take \(x \in C\) with \(\langle x, \alpha \rangle = k\). Then \(w(x) \in wC\) and \(\langle w(x), w(\alpha) \rangle = k\), so \(H^k_{w(\alpha)}\) intersects \(wC\).

We are now ready for our first main theorem about minimal alcoves of \(m\)-Shi regions, which we will use frequently and without mention. It is already known for dominant regions [Ath05 Proposition 3.7, Theorem 3.11].

**Theorem 3.4.** Every region \(R\) of the \(m\)-Shi arrangement contains a unique minimal alcove \(w_R A_\circ\). That is, for any \(\alpha \in \Phi^+\) and \(w_a \in \tilde{W}\) such that \(w_a A_\circ \subseteq R\), we have \(|k(w_R, \alpha)| \leq |k(w_a, \alpha)|\). The floors of \(w_R A_\circ\) are exactly the floors of \(R\).
The concept of the proof is as follows. Start with an $m$-Shi region $R$ contained in the chamber $wC$. Consider $R_{\text{dom}} := w^{-1}R \subseteq C$. This is not in general an $m$-Shi region, but it contains a unique $m$-Shi region $R_{\text{min}}$ that is closest to the origin. We take its minimal alcove $w_{\text{min}}A_0$ and find that $ww_{\text{min}}A_0$ is the minimal alcove of $R$.

Suppose $R$ is an $m$-Shi region contained in the chamber $wC$. Let $R_{\text{dom}} := w^{-1}R \subseteq C$. Notice that $R_{\text{dom}}$ need not itself be an $m$-Shi region. By Lemma [5.3] the walls of $R$ are of the form $H_{\alpha}^k$ where $\alpha \in \Phi^+$ and either $0 \leq k < m$ or $m = k$ and $w(\alpha) \in \Phi^+$. Thus the walls of $R_{\text{dom}} = w^{-1}R$ are of the form $H_{\alpha}^k$ with $\alpha \in \Phi^+$ and either $0 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$. In particular, they are $m$-Shi hyperplanes. The only $m$-Shi hyperplanes $H$ that may intersect $R_{\text{dom}}$ are those such that $w(H)$ is not an $m$-Shi hyperplane, that is those of the form $H_{\alpha}^m$ with $w(\alpha) \in -\Phi^+$.

Now suppose $R'$ is a dominant $m$-Shi region and $J' = \phi(R')$ is the corresponding geometric chain of order filters. Then $R' \subseteq R_{\text{dom}}$ if and only if for every $m$-Shi hyperplane $H_{\alpha}^k$, whenever all of $R_{\text{dom}}$ is on one side of $H_{\alpha}^k$, then all of $R'$ is on the same side of it. Equivalently, $R' \subseteq R_{\text{dom}}$ precisely when for all $1 \leq k \leq m$ and $\alpha \in \Phi^+$ we have $\alpha \in J'_k$ if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, and $\alpha \in I'_k$ if $\langle x, \alpha \rangle < k$ for all $x \in R_{\text{dom}}$.

Let $J = (J_1, J_2, \ldots, J_m)$ be the chain of order filters with $\alpha \in J_k$ if and only if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$. To see that $J$ is geometric, first note that if $\alpha \in J_i$, $\beta \in J_j$ and $\alpha + \beta \in \Phi^+$, then $\langle x, \alpha + \beta \rangle = \langle x, \alpha \rangle + \langle x, \beta \rangle > i + j$ for all $x \in R_{\text{dom}}$, so $\alpha + \beta \in J_{i+j}$. Let $R' \subseteq R_{\text{dom}}$ be some $m$-Shi region contained in $R_{\text{dom}}$ and let $J' = \phi(R')$ be the corresponding geometric chain of order filters. Then $R'$ and $R_{\text{dom}}$ are on the same side of every $m$-Shi hyperplane that does not intersect $R_{\text{dom}}$, so in particular $J_k = J'_k$ for $1 \leq k < m$. Whenever $\alpha \in J_m$, then $\langle x, \alpha \rangle > m$ for all $x \in R_{\text{dom}}$, so $\alpha \in J'_m$. Thus $J_m \subseteq J'_m$. If $i + j < m$, assume without loss of generality that $i, j > 0$, so that $i, j < m$ and

$$(I_i + I_j) \cap \Phi^+ = (I'_i + I'_j) \cap \Phi^+ \subseteq I'_{i+j} \subseteq I_{i+j},$$

since $J'$ is geometric. This shows that $J$ is geometric. Thus there is a dominant region $R_{\text{min}} = \phi^{-1}(J)$. We clearly have $\alpha \in J_k$ if $\langle x, \alpha \rangle > k$ for all $x \in R_{\text{dom}}$, and whenever $\langle x, \alpha \rangle < k$ for all $x \in R_{\text{dom}}$, then $\alpha \in I'_k \subseteq I_k$. Thus $R_{\text{min}} \subseteq R_{\text{dom}}$. Observe that $k_\alpha(J) \leq k_\alpha(J')$ for all $\alpha \in \Phi^+$. Also note that $\langle x, \alpha \rangle > k_\alpha(J)$ for all $x \in R_{\text{dom}}$.

Let $w_{\text{min}}A_0$ be the minimal alcove of $R_{\text{min}}$ [Ath05]. Thus we have $k(w_{\text{min}}, \alpha) = k_\alpha(J)$ for all $\alpha \in \Phi^+$. So if $w_\alpha A_0$ is any alcove contained in $R_{\text{dom}}$, say $w_\alpha A_0 \subseteq R'$ for some $m$-Shi region $R' \subseteq R_{\text{dom}}$, then if $J' = \phi(R')$ we have $k(w_{\text{min}}, \alpha) = k_\alpha(J) \leq k_\alpha(J') \leq k(w_\alpha, \alpha)$ for all $\alpha \in \Phi^+$.

So if we define $w_R := ww_{\text{min}}$, then $w_RA_0 \subseteq R$ and $k(w_R, \alpha) = k(w_{\text{min}}, w^{-1}(\alpha))$ for all $\alpha \in \Phi$. If $w_\alpha A_0$ is any alcove contained in $R$, $\alpha \in \Phi^+$ and $w^{-1}(\alpha) \in \Phi^+$, then

$$k(w_\alpha, \alpha) = k(w^{-1}w_\alpha, w^{-1}(\alpha)) \geq k(w_{\text{min}}, w^{-1}(\alpha)) = k(w_R, \alpha),$$
since $w^{-1}w_{a}A_{0} \subseteq R_{\text{dom}}$. Note that in this case $k(w_{R},\alpha) = k(w_{\text{min}},w^{-1}(\alpha)) \geq 0$, since $w^{-1}(\alpha) \in \Phi^{+}$ and $w_{\text{min}}A_{0}$ is dominant. If instead $w^{-1}(\alpha) \in -\Phi^{+}$, then

$$k(w_{a},\alpha) = k(w^{-1}w_{a},w^{-1}(\alpha)) = -k(w^{-1}w_{a},-w^{-1}(\alpha)) - 1 \leq -k(w_{\text{min}},-w^{-1}(\alpha)) - 1 = k(w_{\text{min}},w^{-1}(\alpha)) = k(w_{R},\alpha).$$

Note that in this case, $k(w_{R},\alpha) = -k(w_{\text{min}},-w^{-1}(\alpha)) - 1 < 0$. So either way we have $|k(w_{R},\alpha)| \leq |k(w_{a},\alpha)|$.

Suppose $H_{\alpha}^{k}$ is a floor of $w_{R}A_{0}$. Then it is the only hyperplane separating $s_{\alpha}^{k}w_{R}A_{0}$ from $w_{R}A_{0}$. Thus $k(s_{\alpha}^{k}w_{R},\beta) = k(w_{R},\beta)$ for all $\beta \neq \pm\alpha$ and $|k(s_{\alpha}^{k}w_{R},\alpha)| = |k(w_{R},\alpha)| - 1$. Since $w_{R}A_{0}$ is the minimal alcove of $R$ this implies that $s_{\alpha}^{k}w_{R}A_{0}$ is not contained in $R$. Thus $H_{\alpha}^{k}$ must be an $m$-Shi hyperplane, and therefore a floor of $R$.

Suppose $H_{\alpha}^{k}$ is a floor of $R_{\text{dom}}$, where $\alpha \in \Phi^{+}$. Then we claim that $\alpha$ is a rank $k$ indecomposable element of $J = \phi(R_{\text{min}})$. To see this, first note that $\langle x,\alpha \rangle > k$ for all $x \in R_{\text{dom}}$, so $\alpha \in J_{k}$. Also, $\langle x,\alpha \rangle < k + 1$ for some $x \in R_{\text{dom}}$, so $k_{\alpha}(J) = k$.

Suppose $\alpha = \beta + \gamma$ with $\beta \in J_{i}$ and $\gamma \in J_{j}$ and $i + j = k$. Then $\langle x,\beta \rangle > i$ and $\langle x,\gamma \rangle > j$ imply that $\langle x,\alpha \rangle > k$ for $x \in R_{\text{dom}}$, so $H_{\alpha}^{k}$ does not support a facet of $R$, a contradiction. If $\alpha + \beta \in J_{i}$ and $k_{\alpha+\beta}(J) = t \leq m$ for some $\beta \in \Phi^{+}$ then we have $\langle x,\alpha + \beta \rangle > t$ for all $x \in R_{\text{dom}}$ so we cannot have $\langle x,\alpha \rangle < t - k$ for all $x \in R_{\text{dom}}$, since together they would imply that $\langle x,\alpha \rangle > k$ for all $x \in R_{\text{dom}}$, so $H_{\alpha}^{k}$ would not support a facet of $R$. Since $t - k < m$, the hyperplane $H_{\beta}^{k} \cap \Phi$ does not intersect $R_{\text{dom}}$, so this implies that $\langle x,\beta \rangle > t - k$ for all $x \in R_{\text{dom}}$, so $\beta \in J_{t-k}$. This verifies the claim. From the fact that $\alpha$ is a rank $k$ indecomposable element of $J$ it follows that $H_{\alpha}^{k}$ is a floor of $w_{\text{min}}A_{0}$ by Theorem 3.2.

Now suppose that $H_{\alpha}^{k}$ is a floor of $R$. Then $w_{\text{min}}^{-1}(\alpha)$ is a floor of $R_{\text{dom}}$ and thus a floor of $w_{\text{min}}A_{0}$. So $H_{\alpha}^{k}$ is a floor of $w_{R}A_{0} = w_{\text{min}}A_{0}$. \hfill \Box

The following lemma characterises the $m$-Shi alcoves. It is a straightforward generalisation of [Shi87, Proposition 7.3].

Lemma 3.5. An alcove $w_{a}A_{0}$ is an $m$-Shi alcove if and only if all its floors are $m$-Shi hyperplanes.

Proof. The forward implication is immediate from Theorem 3.4. For the backward implication, we prove the contrapositive: we show that every alcove that is not an $m$-Shi alcove has a floor that is not an $m$-Shi hyperplane. So suppose $w_{a}A_{0}$ is an alcove contained in an $m$-Shi region $R$, and $w_{a} \neq w_{R}$. Consider the set

$$K = \{ x \in V \mid k(w_{R},\alpha) < \langle x,\alpha \rangle < k(w_{a},\alpha) + 1 \text{ for all } \alpha \in \Phi \text{ with } k(w_{R},\alpha) \geq 0 \text{ and } k(w_{a},\alpha) < \langle x,\alpha \rangle < k(w_{R},\alpha) + 1 \text{ for all } \alpha \in \Phi \text{ with } k(w_{R},\alpha) < 0 \}. $$

Then any alcove $w_{R}A_{0}$ has either $w_{R}A_{0} \subseteq K$ or $w_{R}A_{0} \cap K = \emptyset$. For $\alpha \in \Phi$, we have $k(w_{R},\alpha) \leq k(w_{a},\alpha) \leq k(w_{R},\alpha)$ whenever $w_{R}A_{0} \subseteq K$ and $k(w_{R},\alpha) \geq 0$. Similarly $k(w_{a},\alpha) \leq k(w_{R},\alpha)$ whenever $w_{R}A_{0} \subseteq K$ and $k(w_{R},\alpha) < 0$. Thus any hyperplane of the affine Coxeter arrangement that separates two alcoves contained in $K$ also separates $w_{R}A_{0}$ and $w_{a}A_{0}$. Since no $m$-Shi hyperplane separates $w_{R}A_{0}$ and $w_{a}A_{0}$, no $m$-Shi hyperplane separates two alcoves contained in $K$. Since $K$ is convex, there exists a sequence $(w_{1},w_{2},\ldots,w_{l})$ with $w_{1} = w_{a}$, $w_{l} = w_{R}$, and $w_{i}A_{0} \subseteq K$ for all $i \in [l]$, such that $w_{i}A_{0}$ shares a facet with $w_{i+1}A_{0}$ for all $i \in [l-1]$. 


So the supporting hyperplane of the common facet of $w_1 A_o = w_a A_o$ and $w_2 A_o$ is a floor of $w_a A_o$ which is not an $m$-Shi hyperplane. 

4. AFFINE ROOTS

One may understand the affine Weyl group $\tilde{W}$ in terms of its action on the set of affine roots $\Phi$. The affine roots are the vectors in $\tilde{V} := V \oplus \mathbb{R}\delta$ of the form $\alpha + k\delta$, where $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Here $\delta$ is just a formal variable. Since $\tilde{W} = W \ltimes \tilde{Q}$, we can write any $w_a \in \tilde{W}$ as $w_a = wt_\mu$ with $w \in W$ and $t_\mu$ the translation by $\mu \in \tilde{Q}$. Then $w_a$ acts on the affine roots by

$$w_a(\alpha + k\delta) = w(\alpha) + (k + (\mu, \alpha))\delta.$$  

This action imitates the action of $\tilde{W}$ on the half-spaces defined by hyperplanes of the affine Coxeter arrangement. To see this, define

$$\mathcal{H}_\alpha^k := \{x \in V : (x, \alpha) < k\}$$

for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. Then $w_a(\mathcal{H}_\alpha^k) = \mathcal{H}_\alpha^k$ if and only if $w_a(\alpha + k\delta) = \beta + l\delta$. The set of positive affine roots is

$$\tilde{\Phi}^+ := \{\alpha + k\delta : \alpha \in \Phi^+ \text{ and } k > 0\} \cup \{\alpha + k\delta : \alpha \in -\tilde{\Phi}^+ \text{ and } k \geq 0\}.$$  

So an affine root $\alpha + k\delta$ is positive if and only if the corresponding half-space $\mathcal{H}_\alpha^k$ contains the fundamental alcove $A_o$ and we may write $\tilde{\Phi}$ as the disjoint union of $\tilde{\Phi}^+$ and $-\tilde{\Phi}^+$. The set of affine simple roots is $\tilde{\Delta} := -\Delta \cup \{\tilde{\alpha} + \delta\}$. The affine simple roots correspond to the half-spaces that contain $A_o$ and share a wall with it.

**Lemma 4.1.** If $\alpha + k\delta \in \tilde{\Phi}^+$ and $w_a \in \tilde{W}$, then $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$ if and only if $\mathcal{H}_\alpha^k$ separates $w_a A_o$ from $A_o$.

**Proof.** For the forward implication suppose $\alpha + k\delta \in \tilde{\Phi}^+$, $w_a \in \tilde{W}$ and $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$. Then $A_o \subseteq \mathcal{H}_\alpha^k$ but $A_o \nsubseteq w_a^{-1}(\mathcal{H}_\alpha^k)$. So $w_a A_o \nsubseteq \mathcal{H}_\alpha^k$. Thus the hyperplane $\mathcal{H}_\alpha^k$ separates $w_a A_o$ from $A_o$.

Conversely, if $\alpha + k\delta \in \tilde{\Phi}^+$ and $\mathcal{H}_\alpha^k$ separates $w_a A_o$ from $A_o$, then $A_o \subseteq \mathcal{H}_\alpha^k$ but $w_a A_o \nsubseteq \mathcal{H}_\alpha^k$. Thus $\mathcal{H}_\alpha^k \subseteq w_a^{-1}(\mathcal{H}_\alpha^k)$ and $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$. \hfill $\square$

**Lemma 4.2.** If $\alpha + k\delta \in \tilde{\Phi}^+$, $k > 0$ and $w_a \in \tilde{W}$, then $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Delta}$ if and only if $\mathcal{H}_\alpha^k$ is a floor of $w_a A_o$.

**Proof.** For the forward implication, suppose $\alpha + k\delta \in \tilde{\Phi}^+$, $k > 0$, $w_a \in \tilde{W}$ and $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Delta}$. Then by Lemma 4.1 the hyperplane $\mathcal{H}_\alpha^k$ separates $w_a A_o$ from $A_o$. But we also have that $w_a^{-1}(\mathcal{H}_\alpha^k)$ shares a wall with $A_o$, so $\mathcal{H}_\alpha^k$ is a floor of $w_a A_o$.

Conversely, if $\alpha + k\delta \in \tilde{\Phi}^+$ and $\mathcal{H}_\alpha^k$ is a floor of $w_a A_o$, then $k > 0$. By Lemma 4.1 $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Phi}^+$, so $A_o \nsubseteq w_a^{-1}(\mathcal{H}_\alpha^k)$. But since $\mathcal{H}_\alpha^k$ shares a wall with $w_a A_o$, $w_a^{-1}(\mathcal{H}_\alpha^k)$ shares a wall with $A_o$. So $w_a^{-1}(\alpha + k\delta) \in -\tilde{\Delta}$. \hfill $\square$

5. THE SOMMERS REGION

We define the Sommers region [Som05 Section 5] as

$$D_{\Phi}^{m+1} = \{x \in V \mid (x, \alpha) > -m \text{ for all } \alpha \in \Delta \text{ and } (x, \tilde{\alpha}) < m + 1\}.$$  

The following theorem gives a simple bijection from minimal alcoves of regions of the $m$-Shi arrangement to alcoves contained in the Sommers region.
Theorem 5.1 ([FY10, Theorem 7.1]). The map \( w_R A_o \mapsto w_R^{-1} A_o \) is a bijection from \( \text{Alc}_m^0 \) to the set of alcoves contained in the Sommers region \( D_{m}^{h+1} \).

Proof. Suppose \( w_R A_o \in \text{Alc}_m^0 \) is an \( m \)-Shi alcove. Then by Lemma 3.5 all floors of \( w_R A_o \) are \( m \)-Shi hyperplanes. Let \( \alpha_s \in -\Delta = \Delta \cup \{ -\tilde{\alpha} - m \delta \} \) and let \( w_R(\alpha_s) = \alpha + k \delta \). If \( k > 0 \) then \( \alpha + k \delta \in \tilde{\Phi}^+ \) so since \( w_R^{-1}(\alpha + k \delta) = \alpha_s \in -\Delta \) we have \( H^m_{\alpha} \) is a floor of \( w_R A_o \) by Lemma 4.2. Thus we have \( k \leq m \) and if \( \alpha \in -\Phi \) then also \( k < m \).

So \( w_R(\alpha_s - m \delta) = w_R(\alpha_s) - m \delta = \alpha + (k - m) \delta \in -\tilde{\Phi}^+ \). If instead \( k \leq 0 \), then we also have \( w_R(\alpha_s - m \delta) \in -\tilde{\Phi}^+ \). Thus \( w_R(\alpha_s - m \delta) \in -\tilde{\Phi}^+ \).

Since \( -\alpha_s + m \delta \in \tilde{\Phi}^+ \), Lemma 4.1 implies that the hyperplane corresponding to \( -\alpha_s + m \delta \) does not separate \( w_R^{-1} A_o \) from \( A_o \). This hyperplane is \( H^m_{-\alpha} = H^m_{\alpha} \) for \( \alpha_s = \alpha \in \Delta \), and \( H^{m+1}_{\tilde{\alpha}} \) for \( \alpha_s = -\tilde{\alpha} - \delta \). Thus \( w_R^{-1} A_o \) is contained in the Sommers region \( D_{m}^{h+1} \). Reversing the argument shows that if \( w_R A_o \subseteq D_{m}^{h+1} \) then \( w_R^{-1} A_o \in \text{Alc}_m^0 \).

Consider the dominant \( m \)-Shi region furthest away from the origin

\[
R_f := \{ x \in V : \langle x, \alpha \rangle > m \text{ for all } \alpha \in \Delta \}
\]

and its minimal alcove \( w_f A_o := w_R A_o \) [Ath05, Section 4]. Then the definition of the Coxeter number \( h \) implies that \( w_f A_o = \{ x \in V : \langle x, \alpha \rangle > m \text{ for all } \alpha \in \Delta \text{ and } \langle x, \tilde{\alpha} \rangle < m(h-1)+1 \} \).

Lemma 5.2. The affine transformation \( w_f \) maps the Sommers region \( D_{m}^{h+1} \) to \( (mh + 1)A_o \), the \( (mh + 1) \)-fold dilation of the fundamental alcove.

Proof. The affine transformation \( w_f \) maps the affine half-spaces that contain \( A_o \) and share a wall with it to the affine half-spaces that contain \( w_f A_o \) and share a wall with it. In terms of affine roots this means that

\[
w_f(-\Delta \cup \{ \alpha + \delta \}) = w_f(\tilde{\Delta}) = \{-\alpha - m \delta : \alpha \in \Delta \} \cup \{ \tilde{\alpha} + (m(h-1)+1) \delta \}.
\]
Thus
\[ w_f\{-\alpha + m\delta : \alpha \in \Delta\} \cup \{\tilde{\alpha} + (m+1)\delta\} = w_f(\tilde{\Delta} + m\delta) = w_f(\tilde{\Delta}) + m\delta \]
\[ = \{-\alpha : \alpha \in \Delta\} \cup \{\tilde{\alpha} + (mh+1)\delta\}. \]
So \(w_f\) maps the walls of \(D^m_{\Phi} \) to the walls of \((mh+1)A_\circ\) as required. \(\square\)

Using Theorem 5.1 we deduce the following corollary.

**Corollary 5.3.** The map \(w_RW_\circ \mapsto w_FW_{R_\circ}^{-1}A_\circ\) is a bijection from \(A_{\circ}^m\) to the set of alcoves contained in \((mh+1)A_\circ\).

Since the volume of \((mh+1)A_\circ\) is \((mh+1)^r\) times that of \(A_\circ\), it contains \((mh+1)^r\) alcoves. So we deduce the following corollary.

**Corollary 5.4.** The number of \(m\)-Shi alcoves equals \((mh+1)^r\).

Thus using Theorem 3.4 we also recover the following result, which was originally proven by Yoshinaga using the theory of free arrangements [Yos04, Theorem 1.2].

**Corollary 5.5.** The number of \(m\)-Shi regions equals \((mh+1)^r\).

6. The finite torus

Here we follow a remark in [Soh09, Section 6]. The fundamental alcove \(A_\circ\) is a fundamental domain for the action of \(\tilde{W} = W \ltimes \hat{Q}\) on \(V\). Thus its dilation \((mh+1)A_\circ\) is a fundamental domain for the action of the subgroup \(\tilde{W}_{mh+1} := W \ltimes (mh+1)\hat{Q}\subseteq \tilde{W}\) on \(V\). So the set \(\{w_a : w_a A_\circ \in (mh+1)A_\circ\}\) is a right transversal for \(\tilde{W}_{mh+1}\), that is it contains exactly one element of each right coset of \(\tilde{W}_{mh+1}\) in \(\tilde{W}\).

Notice that any set of (the translations by) representatives of the finite torus \(\hat{Q}/(mh+1)\hat{Q}\) is also a right transversal of \(\tilde{W}_{mh+1} = W \ltimes (mh+1)\hat{Q}\). If \(w_a = wt_\mu \in \tilde{W}\) then \(t_\mu = t_{-w_\mu^{-1}0}\) is in the same right coset of \(\tilde{W}_{mh+1}\) as \(w_a\). Thus the map \(w_a \mapsto -w_a^{-1} \cdot 0\) is a bijection from \(\{w_a : w_a A_\circ \in (mh+1)A_\circ\}\) to some set of representatives of \(\hat{Q}/(mh+1)\hat{Q}\). Using Corollary 5.3 we deduce the following theorem.

**Theorem 6.1.** The map \(\rho : w_RW_\circ \mapsto -w_RW_{F_\circ}^{-1} \cdot 0 + (mh+1)\hat{Q}\) is a bijection from \(A_{\circ}^m\) to \(\hat{Q}/(mh+1)\hat{Q}\).

We define the Anderson map \(A\) as a sign-changed version of \(\rho\).

**Theorem 6.2.** The map \(A : w_RW_\circ \mapsto w_RW_{F_\circ}^{-1} \cdot (mh+1)\hat{Q}\) is a bijection from \(A_{\circ}^m\) to \(\hat{Q}/(mh+1)\hat{Q}\).

Later, we will justify this terminology by relating it to the Anderson map \(A_{GMV}\) defined by Gorsky, Mazin and Vazirani [GMV14, Section 3.1].

7. The \(m\)-nonnesting parking functions

For an \(m\)-Shi region \(R\) and a positive integer \(k\), define
\[ FL_k(R) := \{\alpha \in \Phi : H^k_\alpha \text{ is a floor of } R\} \]
\[ = \{\alpha \in \Phi : H^k_\alpha \text{ is a floor of } w_RW_\circ\}. \]
Notice that \(FL_m(R) \subseteq \Phi^+\) and \(FL_k(R) = \emptyset\) for \(k > m\). Define \(W_R = \langle s_\alpha : \alpha \in FL_m(R)\rangle \subseteq W\).
The set $\text{Park}^{(m)}(\Phi)$ of $m$-nonnesting parking functions, first introduced in [Rho14], is the set of equivalence classes of pairs $(w, R)$ with $w \in W$ and $R$ a dominant $m$-Shi region under the equivalence relation

$$(w_1, R_1) \sim (w_2, R_2) \text{ if and only if } R_1 = R_2 \text{ and } w_1 W_{R_1} = w_2 W_{R_1}.$$ 

$\text{Park}^{(m)}(\Phi)$ is endowed with a left action of $W$ defined by

$$u \cdot [w, R] := [uw, R]$$

for $u \in W$. Thus we say that $\text{Park}^{(m)}(\Phi)$ is a $W$-set.

If $R$ is a dominant $m$-Shi region, $\alpha \in FL_m(R)$ and $\beta > \alpha$, then the inequality $\langle x, \beta \rangle > m$ for all $x \in R$ is implied by the inequality $\langle x, \alpha \rangle > m$ for all $x \in R$, so $H^\alpha_m$ is not a floor of $R$ and $\beta \notin FL_m(R)$. Thus $FL_m(R)$ is an antichain in the root poset. So there is some $u \in W$ with $I := u(FL_m(R)) \subseteq \Delta$ by [Som05, Theorem 6.4]. In particular, $W_R$ is a parabolic subgroup of $W$ (a subgroup conjugate to some subgroup generated by a subset of $S$) and any left coset $wW_R$ of $W_R$ in $W$ has a unique representative $w'$ such that $w'(FL_m(R)) \subseteq \Phi^+$.

### 7.1. $m$-nonnesting parking functions and $m$-Shi alcoves

**Theorem 7.1.** The map

$$\Theta : \text{Park}^{(m)}(\Phi) \to \text{Alc}^m_{\Phi},$$

$$[w, R] \mapsto w' w_R A_\circ$$

is a well-defined bijection. Here $w'$ is the unique representative of $w W_R$ with $w'(FL_m(R)) \subseteq \Phi^+$.

**Proof.** This is well-defined, since if $[w_1, R_1] = [w_2, R_2]$ then $R_1 = R_2$ and $w_1 W_{R_1} = w_2 W_{R_1}$, so $w'_1 = w'_2$. To see that $w' w_R A_\circ \in \text{Alc}^m_{\Phi}$, first note that, since $w_R A_\circ$ is a dominant $m$-Shi alcove, by Lemma 3.5 all its floors are of the form $H^k_\alpha$ with $1 \leq k \leq m$ and $\alpha \in \Phi^+$. From the fact that $w'(FL_m(R)) \subseteq \Phi^+$ we deduce that the floors of $w' w_R A_\circ$ are all either of the form $H^k_\alpha$ with $1 \leq k < m$ and $\beta \in \Phi$ or of the form $H^m_\alpha$ with $\beta \in \Phi^+$. Thus they are $m$-Shi hyperplanes, so $w' w_R A_\circ \in \text{Alc}^m_{\Phi}$ by Lemma 3.5.

To see that $\Theta$ is injective, suppose that $\Theta([w_1, R_1]) = w'_1 w_R A_\circ = w'_2 w_R A_\circ = \Theta([w_2, R_2])$. Now $w'_1 w_{R_1} A_\circ \subseteq w'_1 C$ and $w'_2 w_{R_2} A_\circ \subseteq w'_2 C$, so $w'_1 = w'_2$. Thus $w_{R_1} A_\circ = w_{R_2} A_\circ$ and therefore $R_1 = R_2$. We also get that $w_1 W_{R_1} = w'_1 W_{R_1} = w'_2 W_{R_1} = w_2 W_{R_1}$, so $[w_1, R_1] = [w_2, R_2]$.

To see that $\Theta$ is surjective, note that if $w_R A_\circ \in \text{Alc}^m_{\Phi}$, say with $w_R A_\circ \subseteq wC$, then by Lemma 3.5 and Lemma 3.3 all of its floors are of the form $H^k_{w(\alpha)}$ where $\alpha \in \Phi^+$ and either $1 \leq k < m$ or $k = m$ and $w(\alpha) \in \Phi^+$. Thus all floors of $w^{-1} w_R A_\circ$ are $m$-Shi hyperplanes, so by Lemma 3.5 it is the minimal alcove of a dominant $m$-Shi region $R_{\text{dom}}$. Furthermore $w(FL_m(R_{\text{dom}})) = FL_m(R) \subseteq \Phi^+$. Thus $\Theta([w, R_{\text{dom}}]) = w w_{R_{\text{dom}}} A_\circ = w_R A_\circ$. \qed

A similar bijection using ceilings instead of floors was given for the special case where $m = 1$ in [ARR15, Proposition 10.3]. Note that the proof furnishes a description of $\Theta^{-1}$: we have $\Theta^{-1}(w_R A_\circ) = [w, R_{\text{dom}}]$ where $w_R A_\circ \in wC$ and $R_{\text{dom}}$ is the $m$-Shi region containing $w^{-1} w_R A_\circ$. 


7.2. \(m\)-nonnesting parking functions and the finite torus. The Weyl group \(W\) acts on the coroot lattice \(\tilde{Q}\) and thus also on the finite torus \(\tilde{Q}/(mh+1)\tilde{Q}\). In [Rho14, Proposition 9.9] it is shown that there is a \(W\)-set isomorphism\(^1\) from \(\text{Park}^{(m)}(\Phi)\) to \(\tilde{Q}/(mh+1)\tilde{Q}\). That is, there is a bijection from \(\text{Park}^{(m)}(\Phi)\) to \(\tilde{Q}/(mh+1)\tilde{Q}\) that commutes with the action of \(W\) on both sets. The following theorem makes this isomorphism explicit.

**Theorem 7.2.** The map

\[
\Gamma : \text{Park}^{(m)}(\Phi) \rightarrow \tilde{Q}/(mh+1)\tilde{Q}
\]

\[
[w, R] \mapsto \omega w R w_f^{-1} \cdot 0 + (mh+1)\tilde{Q}
\]

is a \(W\)-set isomorphism. In addition, we have \(\Gamma = A \circ \Theta\).

To prove this theorem, we will need the following result due to Haiman.

**Lemma 7.3** ([Hai94, Lemma 7.4.1]). The set \((mh+1)A \cap \tilde{Q}\) is a system of representatives for the orbits of the \(W\)-action on \(\tilde{Q}/(mh+1)\tilde{Q}\). The stabilizer of an element of \(\tilde{Q}/(mh+1)\tilde{Q}\) represented by \(\mu \in (mh+1)A \cap \tilde{Q}\) is generated by the reflections through the linear hyperplanes parallel to the walls of \((mh+1)A \cap \tilde{Q}\) that contain \(\mu\).

**Proof of Theorem 7.2.** We will first show that \(\Gamma = A \circ \Theta\). Suppose \(R\) is a dominant \(m\)-Shi region and \(w R A_\circ\) its minimal alcove. Write \(w R w_f^{-1} = ut - \mu\). By Corollary 5.3 we have \(w_f R^{-1} A_\circ \subseteq (mh+1)A_\circ\), so \(\mu = w_f w_R^{-1} \cdot 0 \in (mh+1)A_\circ \cap \tilde{Q}\). We claim that the stabilizer of

\[
w R w_f^{-1} \cdot 0 + (mh+1)\tilde{Q} = -u(\mu) + (mh+1)\tilde{Q}
\]

in \(\tilde{Q}/(mh+1)\tilde{Q}\) is \(W_R = (s_\beta : \beta \in FL_m(R))\).

To see this, we calculate that for \(\beta \in \Phi^+\) we have the following equivalences.

\[
\beta \in FL_m(R)
\]
\[
\Rightarrow H_\beta^m \text{ is a floor of } R
\]
\[
\Rightarrow H_\beta^m \text{ is a floor of } w R A_\circ
\]
\[
\Rightarrow w_R^{-1}(\beta + m\delta) \in -\tilde{\Delta}
\]
\[
\Rightarrow w_R^{-1}(\beta) \in -\Delta - m\delta
\]
\[
\Rightarrow w_f w_R^{-1}(\beta) \in w_f(-\Delta - m\delta) = \Delta \cup \{-\tilde{\alpha} - (mh+1)\delta\}
\]
\[
\Rightarrow \beta = w_f w_R^{-1}(\alpha) \text{ for some } \alpha \in \Delta \text{ or } \beta = w_f w_R^{-1}(-\tilde{\alpha} - (mh+1)\delta)
\]
\[
\Rightarrow \beta = u(\alpha) \text{ and } \langle \mu, \alpha \rangle = 0 \text{ for some } \alpha \in \Delta \text{ or } \beta = u(-\tilde{\alpha}) \text{ and } \langle \mu, \tilde{\alpha} \rangle = mh + 1.
\]

\(^1\)Rhoades mistakenly writes the root lattice \(Q\) in place of the coroot lattice \(\tilde{Q}\). However, his result still stands as written: it turns out that \(Q/(mh+1)Q\) and \(\tilde{Q}/(mh+1)\tilde{Q}\) are isomorphic as \(W\)-sets.
Here we used Lemma 4.2, Lemma 5.2, $w_R w_f^{-1} = ut_{-\mu}$ and the definition of the action of $\bar{W}$ on $\tilde{\Phi}$. Combining this with Lemma 7.3 we get
\[
\text{Stab}(w_R w_f^{-1} \cdot 0 + (mh + 1)\check{Q}) = \text{Stab}(-u(\mu) + (mh + 1)\check{Q}) = u\text{Stab}(\mu + (mh + 1)\check{Q})u^{-1} = u\langle s_\alpha : \mu \text{ lies in a wall of } (mh + 1)A^\circ \text{ orthogonal to } \alpha \rangle u^{-1} = u\langle s_\beta : \beta \in FL_m(R) \rangle u^{-1} = \langle s_\beta : \beta \in FL_m(R) \rangle u^{-1} = \langle s_\beta : \beta \in FL_m(R) \rangle \check{W}.
\]
Let $w'$ be the unique element of $wW_R$ with $w'FL_m(R) \subseteq \Phi^+$. We calculate that
\[
\Gamma([w,R]) = www_R w_f^{-1} \cdot 0 + (mh + 1)\check{Q} = w'w w_f^{-1} \cdot 0 + (mh + 1)\check{Q} = A(w'w R A_\circ) = A(\Theta([w,R])),
\]
using that $w^{-1}w' \in W_R = \text{Stab}(w_R w_f^{-1} \cdot 0 + (mh + 1)\check{Q})$. So $\Gamma = A \circ \Theta$ is a well-defined bijection. Since for $u \in W$ we have
\[
\Gamma(u \cdot [w,R]) = \Gamma([uw,R]) = uww_R w_f^{-1} \cdot 0 + (mh + 1)\check{Q} = u \cdot \Gamma([w,R])
\]
we see that $\Gamma$ is a $W$-set isomorphism.

We define the zeta map as $\zeta := \Gamma^{-1} = \Theta^{-1} \circ A^{-1}$. Later, we will justify this terminology by relating it to the zeta map $\zeta_{HL}$ of Haglund and Loehr [Hag08, Theorem 5.6].

**Theorem 7.4.** The map $\zeta$ is a $W$-set isomorphism from $\tilde{Q}/(mh+1)\check{Q}$ to $\text{Park}^{(m)}(\Phi)$.

## 8. A rational generalisation

In this section, we generalise the Anderson map $A$.

For a root $\alpha \in \Phi$ we write it as a linear combination of the simple roots and define the height $\text{ht}(\alpha)$ as the sum of the coefficients. For an affine root $\alpha + k\delta \in \tilde{\Phi}$, we define its height as $\text{ht}(\alpha + k\delta) := kh - \text{ht}(\alpha)$. In particular, $\text{ht}(\alpha + k\delta) > 0$ if and only if $\alpha + k\delta \in \check{\Phi}^+$ and $\text{ht}(\alpha + k\delta) = 1$ if and only if $\alpha + k\delta \in \check{\Delta}$. We define the height of an affine hyperplane $H^k_\alpha$ as the height of the corresponding positive affine root.

We say that a positive affine root $\alpha + k\delta \in \check{\Phi}^+$ is an inversion of $w_\alpha \in \tilde{W}$ if $w_\alpha(\alpha + k\delta) \in -\Phi^+$. By Lemma 4.1, this is the case if and only if $H^k_\alpha$ separates $w_\alpha^{-1}A_\circ$ from $A_\circ$.

Let $t$ be any positive integer relatively prime to the Coxeter number $h$. Define $\tilde{W}^t$ to be the set of $w_\alpha \in \tilde{W}$ that have no inversions of height $t$. So $w_\alpha \in \tilde{W}^t$ if
and only if \( w^{-1}_a A_\circ \subseteq D^t_{\Phi} \), where \( D^t_{\Phi} \) is the region in \( V \) bounded by all the affine hyperplanes of height \( t \). Explicitly, if we write \( t = ah + b \) for \( 1 \leq b < h \), then
\[
D^t_{\Phi} = \{ x \in V : (x, \alpha) > -a \text{ for all } \alpha \in \Phi^+ \text{ with } \text{ht}(\alpha) = b \\
\text{and } (x, \alpha) < a + 1 \text{ for all } \alpha \in \Phi^+ \text{ with } \text{ht}(\alpha) = h - b \}.
\]

We call \( D^t_{\Phi} \) the Sommers region. For \( t = mh + 1 \), it coincides with the Sommers region \( D^{mh+1}_{\Phi} \) defined in Section 5. In particular, we see that \( w_a \in \tilde{W}^{mh+1} \) if and only if \( w^{-1}_a A_\circ \subseteq D^{mh+1}_{\Phi} \) if and only if \( w^a A_\circ \in \text{Alc}_{\Phi}^m \) by Theorem 6.3. The following theorem summarises our considerations.

**Theorem 8.1.** The map \( w_a \mapsto w^{-1}_a A_\circ \) is a bijection from \( \tilde{W}^t \) to the set of alcoves contained in the Sommers region \( D^t_{\Phi} \).

It was shown in Som05, Theorem 5.7 that there is a \( w_f \in \tilde{W} \) with \( w_f(D^t_{\Phi}) = t A_\circ \). The following theorem demonstrates that this is unique.

**Theorem 8.2.** There is a unique \( w_f \in \tilde{W} \) with \( w_f(D^t_{\Phi}) = t A_\circ \).

**Proof.** It remains to show that if \( w_\circ \in \tilde{W} \) and \( w_a(t A_\circ) = t A_\circ \), then \( w_a = e \) is the identity. Index the set of simple roots as \( \Delta = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \} \) and write the highest root as \( \hat{\alpha} = \sum_{i=1}^r c_i \alpha_i \). Define the coweight lattice as
\[
\hat{P} := \{ x \in V : (x, \alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Phi \}.
\]

It is generated by the set of fundamental coweights \( \{ \hat{\omega}_1, \hat{\omega}_2, \ldots, \hat{\omega}_r \} \), the dual basis of \( \Delta \). It contains the coroot lattice \( \tilde{Q} \) and the index \( f := [\hat{P} : \tilde{Q}] \) is called the index of connection.

The fundamental alcove \( A_\circ \) has a vertex at 0, and its other vertices are \( \frac{1}{c_i} \hat{\omega}_i \) for \( i \in [r] \). Define \( L \) as the lattice generated by \( \{ \frac{1}{c_i} \hat{\omega}_i : i \in [r] \} \). Then
\[
[L : \tilde{Q}] = [\hat{P} : \tilde{Q}] = c_1 c_2 \cdots c_r f.
\]

Now a case-by-case check using the classification of irreducible crystallographic root systems reveals that every prime that divides either \( f \) or some \( c_i \) also divides the Coxeter number \( h \). So since \( t \) is relatively prime to \( h \) it is also relatively prime to \( [L : \tilde{Q}] \) Som97, Remark 3.6. This implies that the map
\[
\frac{L}{\tilde{Q}} \rightarrow \frac{L}{\tilde{Q}}
\]
\[
x + \tilde{Q} \mapsto tx + \tilde{Q}
\]
is invertible. Since 0 is the only vertex of \( A_\circ \) in the coroot lattice, \( \frac{1}{c_i} \hat{\omega}_i \notin \tilde{Q} \) for all \( i \in [r] \), so we also have \( t \frac{1}{c_i} \hat{\omega}_i \notin \tilde{Q} \) for all \( i \in [r] \). Thus 0 is the only vertex of \( t A_\circ \) that is in \( \tilde{Q} \).

Now if \( w_a(t A_\circ) = t A_\circ \), then \( w_a \cdot 0 \in \tilde{Q} \) must be a vertex of \( t A_\circ \). Thus \( w_a \cdot 0 = 0 \). So \( w_a \in W \). Since \( w_a(t A_\circ) = t A_\circ \) we must have \( w_aC = C \), therefore \( w_a = e \) as required.

Using Theorem 8.1 we deduce the following.

**Corollary 8.3.** The map \( w_a \mapsto w_f w^{-1}_a A_\circ \) is a bijection from \( \tilde{W}^t \) to the set of alcoves contained in \( t A_\circ \).

Replacing \( mh + 1 \) by \( t \) in Section 6 we deduce the following theorem, generalising Theorem 6.2.

**Theorem 8.4.** The map \( A : w_a \mapsto w_a w^{-1}_f \cdot 0 + t \tilde{Q} \) is a bijection from \( \tilde{W}^t \) to \( \tilde{Q}/t \tilde{Q} \).
As we noted above, \( w_R A_0 \in \text{Alc}_{n}^{\oplus} \) if and only if \( w_R \in \tilde{W}^{m_h+1} \), so this restricts to the Anderson map of Theorem 6.2 up to the abuse of notation of not distinguishing between \( w_R \) and \( w_R A_0 \).

9. Affine permutations

We now restrict to the special case where \( \Phi \) is of type \( A_{n-1} \) in order to relate our Anderson map \( A \) from Theorem 8.4 to the Anderson map \( A_{GMV} \) of [GMV14, Section 3.1]. We have

\[
\begin{align*}
\Phi &= \{ e_i - e_j : i, j \in [n], i \neq j \}, \\
\Phi^+ &= \{ e_i - e_j : i, j \in [n], i < j \}, \\
\Delta &= \{ e_i - e_{i+1} : i \in [n-1] \}, \\
V &= \{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i = 0 \}, \text{ and} \\
\hat{Q} &= \{ (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^{n} x_i = 0 \}.
\end{align*}
\]

The Weyl group \( W \) is the symmetric group \( S_n \) acting on \( V \) by permuting coordinates, the rank of \( \Phi \) is \( r = n - 1 \) and the Coxeter number is \( h = n \).

The affine Weyl group \( \tilde{W} \) also has a combinatorial model as \( \tilde{S}_n \), the set of affine permutations of period \( n \). These are the bijections \( \tilde{\sigma} : \mathbb{Z} \to \mathbb{Z} \) with

\[
\tilde{\sigma}(l + n) = \tilde{\sigma}(l) + n \text{ for all } l \in \mathbb{Z} \text{ and}
\]

\[
\sum_{i=1}^{n} \tilde{\sigma}(i) = \binom{n + 1}{2}.
\]

To identify the affine Weyl group \( \tilde{W} \) with \( \tilde{S}_n \), we index its generating set as \( \tilde{S} = \{ s_0, s_1, \ldots, s_{n-1} \} \), where \( s_i = s_{e_i - e_{i+1}} \) for \( i \in [n-1] \) and \( s_0 = s_{\tilde{e}_1 - \tilde{e}_n} \). Here \( \tilde{e}_1 - \tilde{e}_n = \tilde{\alpha} \) is the highest root of \( \Phi \). Then we let the generators \( s_0, s_1, \ldots, s_{n-1} \) of \( \tilde{W} \) act on \( \mathbb{Z} \) by

\[
\begin{align*}
\sigma(l) &= l + 1 \text{ for } l \equiv j \pmod{n}, \\
\sigma(l) &= l - 1 \text{ for } l \equiv j + 1 \pmod{n}, \text{ and} \\
\sigma(l) &= l \text{ otherwise.}
\end{align*}
\]

We use this identification extensively and do not distinguish between elements of the affine Weyl group and the corresponding affine permutations. Since \( w_a \in \tilde{W} \) is uniquely defined by its values on \( [n] \), we sometimes write it in window notation as \( w_a = [w_a(1), w_a(2), \ldots, w_a(n)] \).

For \( w_a \in \tilde{W} \), write \( w_a(i) = w(i) + n \mu_i \) with \( w(i) \in [n] \) for all \( i \in [n] \). Then \( w \in S_n, \mu := (\mu_1, \mu_2, \ldots, \mu_n) \in \hat{Q} \) and \( w_a = w t_\mu \).

It turns out that the set \( \tilde{W}^t \) of affine Weyl group elements with no inversions of height \( t \) is the set \( \tilde{S}_n^t \) of \( t \)-stable affine permutations [GMV14, Definition 2.13]:

\[
\tilde{S}_n^t := \{ w_a \in \tilde{S}_n : w_a(i + t) > w_a(i) \text{ for all } i \in \mathbb{Z} \}.
\]
Figure 3. The balanced 4-flush abacus \( A(\Delta_{w_a}) \) for \( w_a = [-3, 10, 4, -1] \). Note that the values in the window of \( w_a^{-1} = [5, -6, 8, 3] \) are the lowest gaps of \( A(\Delta_{w_a}) \) in each runner. The levels of the runners of \( A(\Delta_{w_a}) \) are 1, -2, 0, 1. Thus \( w_a^{-1} \cdot 0 = (1, -2, 0, 1) \).

10. **Abaci**

For any affine permutation \( w_a \), we consider the set

\[
\Delta_{w_a} := \{ l \in \mathbb{Z} : w_R(l) > 0 \} = w_R^{-1}(\mathbb{Z}_{>0}).
\]

We define its abacus diagram \( A(\Delta_{w_a}) \) as follows: draw \( n \) runners, labelled 1, 2, \ldots, \( n \) from left to right, with runner \( i \) containing all the integers congruent to \( i \) modulo \( n \) arranged in increasing order from top to bottom. We say that the \( k \)-th level of the abacus contains the integers \( (k-1)n+i \) for \( i \in [n] \) and arrange the runners in such a way that the integers of the same level are on the same horizontal line. We circle the elements of \( \mathbb{Z} \setminus \Delta_{w_a} \) and call them beads, whereas we call the elements of \( \Delta_{w_a} \) gaps. Notice that the fact that \( w_a(l+n) = w_a(l) + n > w_a(l) \) for all \( l \in \mathbb{Z} \) implies that whenever \( l \in \Delta_{w_a} \), then also \( l+n \in \Delta_{w_a} \). We say that \( \Delta_{w_a} \) is \( n \)-invariant. Thus the abacus \( A(\Delta_{w_a}) \) is \( n \)-flush, that is whenever \( l \) is a gap then all the \( l+kn \) for \( k \in \mathbb{Z}_{>0} \) below it are also gaps. Or equivalently whenever \( l \) is a bead then so are all the \( l-\) kn for \( k \in \mathbb{Z}_{>0} \) above it.

If \( w_a \in \tilde{S}_n \) then \( \Delta_{w_a} \) is also \( t \)-invariant so the abacus \( A(\Delta_{w_a}) \) is also \( t \)-flush.

For an \( n \)-flush abacus \( A \) define \( \text{level}_i(A) \) to be the highest level of a bead on runner \( i \) in \( A \) for \( i \in [n] \). Define the integer tuple

\[
\text{levels}(A) = (\text{level}_1(A), \text{level}_2(A), \ldots, \text{level}_n(A)).
\]

The following theorem is well-known.

**Theorem 10.1.** For \( w_a \in \tilde{S}_n \), we have \( \text{levels}(A(\Delta_{w_a})) = w_a^{-1} \cdot 0 \).

**Proof.** This follows from the observations that \( \text{levels}(A(\Delta_{s_j})) = 0 \) and

\[
\text{levels}(A(\Delta_{w_a s_j})) = s_j \cdot \text{levels}(A(\Delta_{w_a}))
\]

for \( w_a \in \tilde{S}_n \) and \( j = 0, 1, \ldots, n \). \( \square \)

In particular \( \text{levels}(A(\Delta_{w_a})) \in \tilde{Q} \), so the sum of the levels of \( A(\Delta_{w_a}) \) is zero. We call such an abacus balanced.

Let \( M_{w_a} \) be the minimal element of \( \Delta_{w_a} \) (that is, the smallest gap of \( A(\Delta_{w_a}) \))
and define \( \tilde{\Delta}_{wa} = \Delta_{wa} - M_{wa} \). This is also an \( n \)-invariant set, so we form its \( n \)-flush abacus \( A(\tilde{\Delta}_{wa}) \). This is a normalized abacus, that is its smallest gap is 0.

**Remark 10.2.** It is easy to see that for an \( n \)-invariant set \( \Delta \) with levels \( x_1, x_2, ..., x_n \) the levels of \( \Delta + 1 \) are \( x_n + 1, x_1, x_2, ..., x_{n-1} \). Thus we define the bijection

\[
g : \mathbb{Z}^n \to \mathbb{Z}^n \quad (x_1, x_2, ..., x_n) \mapsto (x_n + 1, x_1, x_2, ..., x_{n-1})
\]

and get that

\[
\text{levels}(A(\tilde{\Delta}_{wa})) = g^{-M_{wa}} \cdot \text{levels}(A(\Delta_{wa})).
\]

In particular \( \sum_{i=1}^n \text{level}_i(A(\tilde{\Delta}_{wa})) = -M_{wa} \).

### 11. Rational parking functions and the finite torus

Let \( t \) be a positive integer relatively prime to \( n \). A **rational \( t/n \)-parking function** is a tuple \( (f_1, f_2, ..., f_n) \) of \( n \) nonnegative integers such that

\[
|\{i \in [n] : f_i \leq \frac{t}{n}(k - 1)\}| \geq k \text{ for all } k \in [n].
\]

This is equivalent to the condition that the increasing rearrangement \( (g_1, g_2, ..., g_n) \) of \( (f_1, f_2, ..., f_n) \) satisfies \( g_i \leq \frac{t}{n}(i - 1) \) for all \( i \in [n] \). Denote the set of \( t/n \)-parking functions of length \( n \) by \( P_{\mathcal{F}_{t/n}} \). Any rearrangement of a \( t/n \)-parking function is still a \( t/n \)-parking function, so the symmetric group \( S_n \) acts on \( P_{\mathcal{F}_{t/n}} \) by permuting positions.

A standard combinatorial model for rational \( t/n \)-parking functions is **vertically labelled \( t/n \)-Dyck paths**. A **\( t/n \)-Dyck path** is a lattice path in \( \mathbb{Z}^2 \) consisting of North and East steps that starts at \((0, 0)\) and ends at \((t, n)\) and never goes below the diagonal \( y = \frac{t}{n}x \). A **vertical labelling** of a \( t/n \)-Dyck path \( P \) is a permutation \( \sigma \in S_n \) with \( \sigma(i) < \sigma(i + 1) \) whenever the \( i \)-th North step of \( P \) is followed by another North step. In that case we say that \( i \) is a **rise** of \( P \) with label \( (\sigma(i), \sigma(i + 1)) \). We think of \( \sigma \) as labelling the \( n \) North steps of \( P \) from bottom to top. So the condition is that labels in the same column should increase from bottom to top. A vertically labelled \( t/n \)-Dyck path is a pair \( (P, \sigma) \) of a \( t/n \)-Dyck path \( P \) and a vertical labelling \( \sigma \) of \( P \). The following lemma is well-known.

**Lemma 11.1** ([GMV13, Lemma 2.5]). For a vertically labelled \( t/n \)-Dyck path \( (P, \sigma) \), let \( P_i \) be the \( x \)-coordinate of the \( i \)-th North step in \( P \) for all \( i \in [n] \). Then
the map \((P, \sigma) \mapsto \sigma \cdot (P_1, P_2, \ldots, P_n)\) is a bijection from the set of vertically labelled \(t/n\)-Dyck paths to \(\mathcal{PF}_{t/n}\).

We will often use this lemma implicitly and fail to distinguish between \(\mathcal{PF}_{t/n}\) and its combinatorial realization in terms of vertically labelled \(t/n\)-Dyck paths. In this model, the natural \(S_n\)-action on \(\mathcal{PF}_{t/n}\) is realized by defining for \(\tau \in S_n\)

\[
\tau \cdot (P, \sigma) = (P, (\tau \sigma)')
\]

where \((P, (\tau \sigma)')\) comes from labelling the North steps of \(P\) with \(\tau \sigma\) and then sorting the labels in each column increasingly from bottom to top.

The number of full lattice squares (boxes) between the \(i\)-th North step of a \(t/n\)-Dyck path \(P\) and the diagonal \(y = \frac{n}{t}x\) is \(a_i := \lfloor \frac{n}{t}(i-1) \rfloor - P_i\) for all \(i \in [n]\). We call the tuple \((a_1, a_2, \ldots, a_n)\) the \textit{area vector} of \(P\) and say that row \(i\) has area \(a_i\) for \(i \in [n]\).

Now we make the connection between rational \(t/n\)-parking functions and the finite torus \(\hat{Q}/t\hat{Q}\) of type \(A_{n-1}\), following [Ath05, Section 5.1]. First recall that

\[
\hat{Q} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n : \sum_{i=1}^n x_i = 0 \}.
\]

The natural projection

\[
\text{mod } t : \hat{Q} \to \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_t^n : \sum_{i=1}^n x_i = 0 \}
\]

has kernel \(t\hat{Q}\). Furthermore, since \(n\) and \(t\) are relatively prime, the natural projection

\[
\text{mod } (1, 1, \ldots, 1) : \{(x_1, x_2, \ldots, x_n) \in \mathbb{Z}_t^n : \sum_{i=1}^n x_i = 0 \} \to \mathbb{Z}_t^n/(1, 1, \ldots, 1)
\]

is a bijection to the set \(\mathbb{Z}_t^n/(1, 1, \ldots, 1)\) of cosets of the cyclic subgroup of \(\mathbb{Z}_t^n\) generated by \((1, 1, \ldots, 1)\). Thus if \(\pi_{\hat{Q}} := \text{mod } (1, 1, \ldots, 1) \circ \text{mod } t\), then

\[
\pi_{\hat{Q}} : \hat{Q}/t\hat{Q} \to \mathbb{Z}_t^n/(1, 1, \ldots, 1)
\]

is a well-defined bijection.

It is well-known that the set of rational \(t/n\)-parking functions is a set of representatives for \(\mathbb{Z}_t^n/(1, 1, \ldots, 1)\), so the natural projection

\[
\pi_{\mathcal{PF}} : \mathcal{PF}_{t/n} \to \mathbb{Z}_t^n/(1, 1, \ldots, 1)
\]

is a bijection.
Note that \( W = S_n \) naturally acts on \( Q/\tilde{Q}, \mathcal{PF}_{t/n} \) and \( \mathbb{Z}^n/(1,1,...,1) \) and that both \( \pi_Q \) and \( \pi_{PF} \) are isomorphisms with respect to these actions.

12. The Anderson map

Now we show that in the case where \( \Phi \) is of type \( A_{n-1} \) our Anderson map \( A \) from Theorem 8.4 is equivalent to the Anderson map \( A_{GMV} \) of Gorsky, Mazin and Vazirani [GMV14, Section 3.1]. The following theorem makes this precise.

**Theorem 12.1.** Suppose \( \Phi \) is of type \( A_{n-1} \) and \( t \) is a positive integer relatively prime to \( n \). Then

\[
\pi_{\tilde{Q}} \circ A = \pi_{PF} \circ A_{GMV}.
\]

**Proof.** We have already observed that \( \tilde{W}^t = \tilde{S}^n_t \), so the domains of both maps agree. Let us recall the description of \( A_{GMV} \) from [GMV14, Section 3.1]. We use English notation, preferring top-justified over bottom-justified Young diagrams.

Take \( w_a \in \tilde{S}^n_t \). As in Section 10 we consider the set

\[
\Delta_{w_a} := \{ i \in \mathbb{Z} : w_a(i) > 0 \}
\]

and let \( M_{w_a} \) be its minimal element. Let \( \Delta_{w_a} := \Delta_{w_a} - M_{w_a} \). In contrast to [GMV14], we shall use \( \Delta_{w_a} \) in place of \( \Delta_{w_a} \) and therefore also have a different labelling of \( \mathbb{Z}^2 \).

View the integer lattice \( \mathbb{Z}^2 \) as the set of square boxes. Define the rectangle

\[
R_{t,n} := \{(x,y) \in \mathbb{Z}^2 : 0 \leq x < t, 0 \leq y < n\}
\]

and label \( \mathbb{Z}^2 \) by the linear function

\[
l(x,y) := -n - nx + ty.
\]

Define the top-justified Young diagram

\[
D_{w_a} := \{(x,y) \in R_{t,n} : l(x,y) \in \Delta_{w_a}\}
\]

and let \( P_{w_a} \) be the \( t/n \)-Dyck path that defines its lower boundary. Label its \( i \)-th North step by \( \sigma(i) := w_a(l_i + M_{w_a}) \), where \( l_i \) is the label of the rightmost box of \( D_{w_a} \) in the \( i \)-th row from the bottom (or the label of \((-1,i-1)\) if its \( i \)-th row is empty). Then we have \( \sigma \in S_n \) and define \( A_{GMV}(w_a) \) as the vertically labelled \( t/n \)-Dyck path \( (P_{w_a}, \sigma) \).

![Figure 6. The vertically labelled 9/4-Dyck path \( A_{GMV}(w_a) \) for \( w_a = [-3,10,4,-1] \). It has area vector \((0,2,3,2)\) and labelling \( \sigma = 2431 \). It corresponds to the 9/4-parking function \((4,0,1,0)\). The positive beads of the normalized abacus \( A(\Delta_{w_a}) \) are shaded in gray.](image-url)
Note that the labels of boxes in the $i$-th row of $D_{w_n}$ from the bottom (those with $y$-coordinate $i - 1$) are those congruent to $t(i - 1)$ modulo $n$. Thus we define the permutation $\tau \in S_n$ by
\[
\tau(i) \equiv t(i - 1) \mod n
\]
for all $i \in [n]$. The fact that $t$ is relatively prime to $n$ implies that this indeed gives a permutation of $n$.

We now turn our attention to $w_f$. From \cite[Lemma 2.16]{GMV} we get that its inverse is
\[
w_f^{-1} = [t - c, 2t - c, \ldots, nt - c]
\]
where $c = \frac{(t-1)(n+1)}{2}$. Since $\Delta_{w_f} = w_f^{-1}(\mathbb{Z}_{>0})$ the set of lowest gaps of the runners of the balanced abacus $A(\Delta_{w_f})$ is
\[
\{w_f^{-1}(1), w_f^{-1}(2) \ldots w_f^{-1}(n)\} = [t - c, 2t - c, \ldots, nt - c].
\]
Thus the set of lowest gaps of the runners of the normalized abacus $A(\tilde{\Delta}_{w_f})$ is $\{0, t, 2t, \ldots, (n - 1)t\}$. This is exactly the set of labels of $(-1, i - 1)$ for $i \in [n]$. Thus all the labels in $R_{c,n}$ are beads in $A(\tilde{\Delta}_{w_f})$. Therefore $D_{w_f}$ is empty and $A_{GMV}(w_f) = (\mu_f, \sigma) = (0, 0, \ldots, 0)$.

For $x, y \in \mathbb{Z}^n$ write $x \equiv y$ if the projections of $x$ and $y$ into $\mathbb{Z}^n/(1, 1, \ldots, 1)$ agree. Then $\equiv$ is compatible both with addition and with the $S_n$-action on $\mathbb{Z}^n$. The set of lowest gaps of the runners of the abacus $A(\tilde{\Delta}_{w_f} + t)$ is $\{t, 2t, \ldots, nt\}$. Thus
\[
\text{levels}(A(\tilde{\Delta}_{w_f} + t)) = \text{levels}(A(\tilde{\Delta}_{w_f})) + (0, 0, \ldots, 0, t) \equiv \text{levels}(A(\tilde{\Delta}_{w_f})).
\]
We also have
\[
\text{levels}(A(\tilde{\Delta}_{w_f} + n)) = \text{levels}(A(\tilde{\Delta}_{w_f})) + (1, 1, \ldots, 1) \equiv \text{levels}(A(\tilde{\Delta}_{w_f})).
\]
In terms of the bijection $g$ from Remark 10.2, this means that
\[
g' \cdot \text{levels}(A(\tilde{\Delta}_{w_f})) \equiv \text{levels}(A(\tilde{\Delta}_{w_f})),
\]
and
\[
g^n \cdot \text{levels}(A(\tilde{\Delta}_{w_f})) \equiv \text{levels}(A(\tilde{\Delta}_{w_f})).
\]
Since $t$ and $n$ are coprime, this implies that
\begin{equation}
(1) \quad g \cdot \text{levels}(A(\tilde{\Delta}_{w_f})) \equiv \text{levels}(A(\tilde{\Delta}_{w_f})).
\end{equation}

Now take $w_n \in \tilde{S}_n^t$. Let $P_t$ be the number of boxes on the $i$-th row of $D_{w_n}$ from the bottom. This is the number of gaps of $A(\tilde{\Delta}_{w_n})$ on runner $\tau(i)$ that are in $R_{c,n}$. Equivalently, it is the number of gaps of $A(\tilde{\Delta}_{w_n})$ on runner $\tau(i)$ that are smaller than the smallest gap on runner $\tau(i)$ of $A(\tilde{\Delta}_{w_f})$. Thus
\[
P_t = \text{level}_{\tau(i)}(A(\tilde{\Delta}_{w_f})) - \text{level}_{\tau(i)}(A(\tilde{\Delta}_{w_n})),
\]
that is
\begin{equation}
(2) \quad (P_t, P_2, \ldots, P_n) = \tau^{-1} \cdot \left[\text{levels}(A(\tilde{\Delta}_{w_f})) - \text{levels}(A(\tilde{\Delta}_{w_n}))\right].
\end{equation}

Now we start looking at the labelling $\sigma$ of the $t/n$-Dyck path $P_{w_n}$. We have for $i \in [n]$
\[
\sigma(i) := w_n(l_i + M_{w_n}) \equiv w_n(\tau(i) + M_{w_n}) \mod n.
\]
Define $r \in S_n$ by $r(i) \equiv i + 1 \mod n$. Write $w_n = wt - \mu$ with $w \in W = S_n$ and $\mu \in \tilde{Q}$, simultaneously viewing $w$ as an affine permutation in $S_n^\tau$ also. Then
\[
w(r^{M_{w_n}}(\tau(i))) \equiv w(\tau(i) + M_{w_n}) \equiv w_n(\tau(i) + M_{w_n}) \mod n.
\]
Since $\sigma(i)$ and $w(r^{M_w}(\tau(i)))$ are congruent modulo $n$ and both in $[n]$, they are equal. Thus

$$\sigma = w \circ r^{M_w} \circ \tau.$$  

Now we calculate

$$\mathcal{A}_{GMV}(w_a) = (P_{w_a}, \sigma)$$
$$= (w \circ r^{M_w} \circ \tau) \cdot \tau^{-1} \cdot [\text{levels}(A(\Delta_{w_f}) - \text{levels}(A(\Delta_{w_a})))$$
$$\equiv (w \circ r^{M_w}) \cdot [g^{M_w} \cdot \text{levels}(A(\Delta_{w_f})) - \text{levels}(A(\Delta_{w_a})))$$
$$= (w \circ r^{M_w}) \cdot [g^{-M_w} \cdot \text{levels}(A(\Delta_{w_f})) - g^{-M_w} \cdot \text{levels}(A(\Delta_{w_a})))$$
$$= (w \circ r^{M_w}) \cdot [r^{-M_w} \cdot \text{levels}(A(\Delta_{w_f})) - \text{levels}(A(\Delta_{w_a})))$$
$$= w \cdot (w_f^{-1} \cdot 0 - w_a^{-1} \cdot 0)$$
$$= w \cdot (w_f^{-1} \cdot 0 - \mu)$$
$$= wt_{-\mu}w_f^{-1} \cdot 0$$
$$= w_a w_f^{-1} \cdot 0$$
$$= \mathcal{A}(w_a).$$

Here we used Equation (3), Equation (2), Equation (1), Remark 10.2 and Theorem 10.1 in that order.

\[\square\]

13. The zeta map

In this section, take $\Phi$ be the root system of type $A_{n-1}$ and set $m = 1$. Thus we write $\mathcal{A}_{C_1}$ for $\mathcal{A}_{C_1}(\Phi)$, $\text{Park}(\Phi)$ for $\text{Park}^{(1)}(\Phi)$, and so on. Our aim is to relate our zeta map $\zeta$ from Theorem 7.4 to the zeta map $\zeta_{HL}$ of Haglund and Loehr [Hag08, Theorem 5.6]. To do this, we give a combinatorial model for $\text{Park}(\Phi)$ in terms of diagonally labelled Dyck paths.

Recall that we can bijectively map a Shi region $R$ to the order filter $J = \phi(R)$ in the root poset and then to the antichain $A$ of minimal elements of $J$. Then $A = FL(R)$, the set of $\alpha \in \Phi^+$ such that $H^1_\alpha$ is a floor of $R$ [Ath05].

A Dyck path is a lattice path from $(0,0)$ to $(n,n)$ consisting of North and East steps that never goes below the diagonal $y = x$. The set of Dyck paths is exactly the set of $n + 1/n$-Dyck paths with their last East step removed. Thus we will identify $n + 1/n$-Dyck paths with their corresponding Dyck path.

We identify antichains in the root poset of $\Phi$ with Dyck paths in the usual way: say that $(i,j)$ is a valley of the Dyck path $D$ if the $i$-th East step of $D$ is followed immediately by its $j$-th North step. Then the antichain $A$ is identified with the Dyck path $D$ which has a valley at $(i,j)$ if and only if $e_i - e_j \in A$ [Arm13, Section 3.1].

For $w \in W = S_n$, we call $w$ a diagonal labelling of $D$ if $w(i) < w(j)$ whenever $(i,j)$ is a valley of $D$. In that case the pair $(w, D)$ is a diagonally labelled Dyck path and we say that the valley $(i,j)$ has label $(w(i), w(j))$. We think of $w$ as labelling the boxes of the diagonal from $(0,0)$ to $(n,n)$ from bottom to top in such a way that the label below any valley is less than the label to the right of it. We have
$w(A) \subseteq \Phi^+$ if and only if $w$ is a diagonal labelling of the Dyck path $D$ corresponding to $A$. Thus we get the following lemma.

Lemma 13.1. The map

$$\epsilon : \text{Park}(\Phi) \to D_n$$

$$[w, R] \mapsto (w', D),$$

where $D$ is the Dyck path corresponding to the antichain $FL(R)$ and $w' \in wW_R$ is the unique representative with $w'(FL(R)) \subseteq \Phi^+$, is a bijection.

Proof. An easy verification, which is left to the interested reader. □

Since $W_R$ is generated by the transpositions $(ij)$ such that $(i, j)$ is a valley of $D$ and the condition $w'(FL(R)) \subseteq \Phi^+$ is equivalent to $w'(i) < w'(j)$ whenever $(i, j)$ is a valley of $D$ we can get $w'$ from $w$ with a simple sorting procedure: for all maximal chains of indices $i_1 < i_2 < \ldots < i_l$ such that $(i_j, i_{j+1})$ is a valley of $D$ for all $j \in [l-1]$ sort the values of $w$ on positions $i_1, i_2, \ldots, i_l$ increasingly. The result is $w'$. From this we also get the $S_n$-action on $D_n$ that turns $\epsilon$ into an $S_n$-isomorphism: for $u \in S_n$ define

$$u \cdot (w, D) := ((uw)', D)$$

where $(uw)'$ arises from $uw$ through the sorting procedure described above. Note the analogy between this action and the $S_n$-action on parking functions realized as vertically labelled Dyck paths.

One may also view diagonally labelled Dyck paths as a combinatorial model for Shi alcoves. The following lemma makes this explicit.

Lemma 13.2. The map

$$\delta : \text{Alc}_\Phi \to D_n$$

$$w_R A_\circ \mapsto (w, D),$$

where $w_R A_\circ \in wC$ and $D$ is the Dyck path corresponding to the antichain $FL(R_{dom})$, where $R_{dom}$ is the dominant Shi region containing $w^{-1}w_R A_\circ$, is a bijection. Furthermore $\delta = \epsilon \circ \Theta^{-1}$.

Proof. An immediate check from the definitions. □

The zeta map $\zeta_{HL}$ of Haglund and Loehr is a bijection from $\mathcal{P}F_n$, viewed as the set of vertically labelled Dyck paths of length $n$, to the set $D_n$ of diagonally labelled Dyck paths of length $n$ [Hag08, Theorem 5.6]. It has many different descriptions, and our favorite is the one given by Armstrong, Loehr and Warrington [ALW14, Section 5.2], which we now recall.
Given a vertically labelled Dyck path \((P, \sigma)\), form its diagonal reading word \(w\) by reading the labels of rows of area 0 from bottom to top, then the labels of rows of area 1 from bottom to top, \ldots, then the labels of rows of area \(n-1\) from bottom to top. Write \(w\) as a labelling of the boxes of the diagonal from \((0,0)\) to \((n,n)\) from bottom to top and draw the Dyck path \(D\) such that \((w,D)\) has a valley with label \((a,b)\) if and only if \((P, \sigma)\) has a rise with label \((a,b)\). Then define \(\zeta_{HL}(P, \sigma) := (w,D)\).

The following theorem relates our zeta map \(\zeta\) from Theorem 7.4 to the zeta map \(\zeta_{HL}\) of Haglund and Loehr. Let \(\chi := \pi_{PF} \circ \pi_{\tilde{Q}}\) be the natural \(S_n\)-isomorphism from \(\tilde{Q}/(h+1)\tilde{Q}\) to \(PF_n\).

**Theorem 13.3.** If \(\Phi\) is of type \(A_{n-1}\) and \(m = 1\), then
\[
\zeta_{HL} = \epsilon \circ \zeta \circ \chi^{-1} = \delta \circ \mathcal{A}_{GMV}^{-1}.
\]

**Proof.** We have
\[
\epsilon \circ \zeta \circ \chi^{-1} = \epsilon \circ \Theta^{-1} \circ \mathcal{A}^{-1} \circ \chi^{-1} = \delta \circ \mathcal{A}_{GMV}^{-1}
\]
using the definition of \(\zeta\), Lemma 13.2 and Theorem 12.1, committing the mild abuse of notation of identifying \(w_R\) with \(w_{R}\).

First we need to check that if \((\delta \circ \mathcal{A}_{GMV}^{-1})(P, \sigma) = (w,D)\), then \(w\) is the diagonal reading word of \((P, \sigma)\). Equivalently we need to verify that if \(w_{RA}_o \subseteq wC\) and \(\mathcal{A}_{GMV}(wR) = (P_{wR}, \sigma)\) then the diagonal reading word of \((P_{wR}, \sigma)\) is \(w\).

First suppose that \(w = e\) is the identity. That is, \(w_{RA}_o \subseteq C\). Equivalently \(w_{R}^{-1}\) is affine Grassmanian, that is \(w_{R}^{-1}(1) < w_{R}^{-1}(2) < \ldots < w_{R}^{-1}(n)\). The set of lowest gaps on the runners of the balanced abacus \(\mathcal{A}(\Delta_{wR})\) is
\[
\{w_{R}^{-1}(1), w_{R}^{-1}(2), \ldots, w_{R}^{-1}(n)\}.
\]
Thus the set of lowest gaps of the normalized abacus \(\mathcal{A}(\tilde{\Delta}_{wR})\) is
\[
\{w_{R}^{-1}(1) - M_{wR}, w_{R}^{-1}(2) - M_{wR}, \ldots, w_{R}^{-1}(n) - M_{wR}\},
\]
where \(M_{wR}\) is the minimal element of \(\Delta_{wR}\). This equals the set \(\{l_1, l_2, \ldots, l_n\}\) of labels of the boxes to the left of the North steps of the Dyck path \(P_{wR}\).

Let \((a_1, a_2, \ldots, a_n)\) be the area vector of \(P_{wR}\). Then we have \(l_i = na_i + i - 1\). Thus \(l_i < l_j\) if and only if either \(a_i < a_j\) or \(a_i = a_j\) and \(i < j\). Furthermore, the label of the \(i\)-th North step of \(P_{wR}\) is \(\sigma(i) = w_R(l_i + M_{wR})\). So the \(j\)-th label being
read in the diagonal reading word is \( d(j) = w_R(l_j + M_{wR}) \), where \( l_j \) is the \( j \)-th smallest element of \( \{l_1, l_2, \ldots, l_n\} \). But the \( j \)-th smallest element of
\[ \{l_1, l_2, \ldots, l_n\} = \{w^{-1}_R(1) - M_{wR}, w^{-1}_R(2) - M_{wR}, \ldots, w^{-1}_R(n) - M_{wR}\} \]
is just \( w^{-1}_R(j) - M_{wR} \), so \( d(j) = w_R(w^{-1}_R(j) - M_{wR} + M_{wR}) = j \). Thus the diagonal reading word of \((P_{wR}, \sigma)\) is \( d = e = w \), as required.

In general if \( w_RA_0 \subseteq wC \) then \( w_R = w_{wD} \), where \( w_D \in S_n^{wR} \) and \( wDA_0 \subseteq C \). We have \( \Delta_{wD} = \Delta_{wR} \) and thus also \( M_{wD} = M_{wR} \) and \( \tilde{\Delta}_{wD} = \tilde{\Delta}_{wR} \). Therefore \( P_{wR} = P_{wD} \) and the tuple \((l_1, l_2, \ldots, l_n)\) is also the same for \( w_R \) and \( w_D \). Thus the \( j \)-th label being read in the diagonal reading word of \( A_{GMV}(w_R) = (P_{wR}, \sigma) \) is
\[ d(j) = w_R(w^{-1}_R(j) - M_{wR} + M_{wR}) = w(j) \].
So the diagonal reading word of \((P_{wR}, \sigma)\) is \( d = w \), as required.

The second property we need to check is that if \( (\epsilon \circ \zeta \circ \chi^{-1})(P, \sigma) = (w, D) \) then \((w, D)\) has a valley labelled \((a, b)\) if and only if \((P, \sigma)\) has a rise labelled \((a, b)\). But this follows from general considerations: since \( \epsilon \circ \zeta \circ \chi^{-1} \) is a composition of \( S_n \)-isomorphisms it is itself an \( S_n \)-isomorphism. In particular, the \( S_n \)-stabilizers of \((P, \sigma)\) and \((w, D)\) must agree. But \((P, \sigma)\) has a rise labelled \((a, b)\) if and only if \( b \) is the smallest integer with \( a < b \leq n \) such that the transposition \((ab)\) fixes \((P, \sigma)\), and similarly \((w, D)\) has a valley labelled \((a, b)\) if and only if \( b \) is the smallest integer with \( a < b \leq n \) such that the transposition \((ab)\) fixes \((w, D)\). Thus \((w, D)\) has a valley labelled \((a, b)\) if and only if \((P, \sigma)\) has a rise labelled \((a, b)\). This concludes the proof. 

\[ \square \]

14. Outlook

Given the algebraically defined uniform zeta map \( \zeta \) that has the combinatorial interpretation \( \zeta_{HL} \) if \( \Phi \) is of type \( A_{m-1} \) and \( m = 1 \) it is natural to ask for combinatorial interpretations in other types and for larger \( m \). The extended abstract \[ST14\] supplies an answer for type \( C_n \) and \( m = 1 \). For types \( B_n \) and \( D_n \) similar combinatorial zeta maps can be defined. This will be the subject of future work.

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References

[ALW14] Drew Armstrong, Nicholas A. Loehr, and Gregory S. Warrington. Rational parking functions and Catalan numbers. 2014. arXiv:1403.1845.
[Arm13] Drew Armstrong. Hyperplane arrangements and diagonal harmonics. Journal of Combinatorics, 4:157–190, 2013.
[ARR15] Drew Armstrong, Victor Reiner, and Brendon Rhoades. Parking Spaces. Advances in Mathematics, 269:647–706, 2015.
[Ath05] Christos A. Athanasiadis. On a Refinement of the Generalized Catalan Numbers for Weyl Groups. Transactions of the American Mathematical Society, 357:179–196, 2005.
[FV10] Susanna Fishel and Monica Vazirani. A bijection between dominant Shi regions and core partitions. European Journal of Combinatorics, 31:2087–2101, 2010.
[GMV14] Eugene Gorsky, Mikhail Mazin, and Monica Vazirani. Affine permutations and rational slope parking functions. 2014. arXiv:1403.0303.
[Hag08] Jim Haglund. The \( q,t \)-Catalan numbers and the space of diagonal harmonics. American Mathematical Society, Providence, RI, 2008.
[Hai94] Mark D. Haiman. Conjectures on the quotient ring by diagonal invariants. Journal of Algebraic Combinatorics, 133:17–76, 1994.
[Hum90] James E. Humphreys. *Reflection Groups and Coxeter Groups*. Cambridge University Press, Cambridge, 1990.

[Rho14] Brendon Rhoades. Parking Spaces: Fuss analogs. *Journal of Algebraic Combinatorics*, 40:417–473, 2014.

[Shi87a] Jian-yi Shi. Alcoves corresponding to an affine Weyl group. *Journal of the London Mathematical Society*, 35:42–55, 1987.

[Shi87b] Jian-yi Shi. Sign types corresponding to an affine Weyl group. *Journal of the London Mathematical Society*, 35:56–74, 1987.

[Shi97] Jian-yi Shi. The number of ⊕-sign types. *Quarterly Journal of Mathematics*, 48:93–105, 1997.

[Som97] Eric N. Sommers. A family of affine Weyl group representations. *Transformation Groups*, 2:375–390, 1997.

[Som05] Eric N. Sommers. b-Stable Ideals in the Nilradical of a Borel Subalgebra. *Canadian Mathematical Bulletin*, 48:460–472, 2005.

[ST14] Robin Sulzgruber and Marko Thiel. Type C parking functions and a zeta map. 2014. arXiv:1411.3885.

[Yos04] Masahiko Yoshinaga. Characterization of a free arrangement and conjecture of Edelman and Reiner. *Inventiones Mathematicae*, 157:449–454, 2004.

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