Quantum measurements need not conserve energy: relation to the Wigner-Araki-Yanase theorem

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Abstract

The paper focuses on the fact that quantum projective measurements do not necessarily conserve energy. On the other hand the Wigner-Araki-Yanase (WAY) theorem states that assuming a “standard” von Neumann measurement model and “additivity” of the total energy operator, projective measurements of a system must conserve energy as defined by the system’s energy operator. This paper explores the ideas behind the WAY theorem in hopes of uncovering the origin of the contradiction.

After Araki and Yanase published their proof of the WAY theorem, Yanase appended a new condition now known as the Yanase condition. Under the simplifying assumption that the observable being measured has discrete and non-degenerate eigenvalues, we prove that the Yanase condition actually follows from the hypotheses of the original WAY theorem.

The paper also proves that the hypotheses of the WAY theorem, together with the simplifying assumption, imply that the energy operator for the measuring apparatus must be a multiple of the identity, which seems physically unlikely. It seems probable that this surprising conclusion, along with the Yanase condition, also holds without the simplifying assumption.

1 Introduction

In orthodox quantum mechanics, there are two ways that a quantum system can change:

1. Continuous evolution as described by the Schroedinger equation;

2. A discrete change (“collapse of the wave function”) caused by a measurement of the system.

For continuous evolution, conservation of energy is automatically enforced by the mathematical structure of the theory. If the (mixed) state of the system at time $t = 0$ is $\rho(0)$ and the Hamiltonian (energy operator) is $H$, then the state $\rho(t)$ at time $t$ is

$$\rho(t) = e^{-iHt} \rho(0) e^{iHt}.$$
At a given time, the (average) energy of state $\sigma$ is $	ext{tr} [H\sigma] = \text{tr} [\sigma H]$, where tr denotes trace, so the energy of $\rho(t)$ is

\[
\text{energy of } \rho(t) = \text{tr} [e^{-iHt}\rho(0)e^{iHt}H] = \text{tr} [\rho(0)e^{iHt}He^{-iHt}] = \text{tr} [\rho(0)H] = \text{energy of } \rho(0).
\]

Since conservation of energy is one of the most universal and cherished principles of physics, one might expect a similar trivial calculation to establish conservation of energy for the discrete transition caused by a measurement. But such is not the case.

For our purposes, a measurement is described by a finite collection of “measurement operators” $M_1, M_2, \ldots, M_n$. If the state of the system before measurement is $\rho$, the state after the measurement (disregarding the measurement result, or assuming it unknown) is

\[
\sum_i M_i \rho M_i^\dagger.
\]

For the measurement to conserve (average) energy for all states $\rho$, it is necessary and sufficient that for all $\rho$,

\[
\sum_i \text{tr} [M_i \rho M_i^\dagger H] = \text{tr} [H\rho],
\]

which, using the cyclic property of the trace, is equivalent to

\[
\sum_i \text{tr} [(M_i^\dagger HM_i - H)\rho] = 0.
\]

It is plausible, and a simple exercise\(^4\) to prove, that this will hold for all states $\rho$ if and only if,

\[
\sum_i M_i^\dagger HM_i = H.
\]

In summary, only the special measurements satisfying (4) can conserve energy.

For projective measurements (i.e., the $M_i$ are orthogonal projectors which sum to the identity), readers familiar with operator theory may recognize that (4) is equivalent to requiring that the $M_i$ commute with $H$. This simpler condition underscores the very special nature of energy-conserving measurements.

**History**

It has been recognized at least since a seminal 1952 paper of Wigner \[^3\] that some quantum measurements do not conserve energy. Wigner’s observations

\[^2\] This may seem strange because normally, one might suppose that after a measurement, the result is known! See Subsection \[^3\] for an explanation.

\[^3\] This is the formulation of the standard text \[^1\], which is not the most general formulation. For a clear account of the general formulation, see Chapter 1 of Jacobs’ book \[^2\]. For projective measurements, which is all that we need consider, the two formulations are equivalent.

\[^4\] For $\rho = P_\phi$, the projector on a pure state $\phi$, and any operator $K$, tr $[KP_\phi] = \langle \phi, K\phi \rangle$, and it is a standard fact that $\langle \phi, K\phi \rangle = 0$ can hold for all $\phi$ only for $K = 0$. 

were generalized ten years later by Araki and Yanase [4] in what has become known as the WAY (Wigner-Araki-Yanase) theorem, on which there is an extensive modern literature.

I first learned of the possibility that quantum measurements might not conserve energy from the charismatically written [10], which presents a particular example of this phenomenon. It does not mention the WAY theorem, of which I assume its author was unaware, as was I. At that time, I formulated the core of the present work, but did not write it up because it is so mathematically trivial that I assumed that it must be known. On learning from references like [9, 11, 12] that the WAY theorem is of current interest, I thought that perhaps those observations might be of some interest, despite their mathematical simplicity.

The informal remarks of the Introduction essentially constitute a proof of a theorem stating that a quantum projective measurement conserves energy if and only if its projectors commute with the energy operator. This is formally stated as Theorem 1 in Section 2.

When I learned of the WAY theorem in the last few months, of course I wondered what might be its relation to the simple Theorem 1. The conclusion of the WAY theorem implies that all discrete quantum mechanical observables must commute with the energy observable! If physically correct, this would probably destroy much of the structure of quantum mechanics and its explanatory power. That makes it hard to believe.

But notice that I said the conclusion of the WAY theorem. Of course, the WAY theorem has hypotheses, which include acceptance of a so-called “standard” measurement model of von Neumann. Another important hypothesis assumes that the energy observable is of a particular “additive” form. If the WAY theorem’s conclusion is unbelievable, then chances are that one of its hypotheses is physically unrealistic.

In a search for the origin of the conundrum I studied a measurement model similar to the “standard” model, but algebraically more natural. It also led to WAY-type theorems with the same physical difficulties as the original WAY theorem. But the proofs seemed simpler and more transparent. The new WAY-type theorems will be presented in Section 4. The proof of the original WAY theorem (under the simplifying assumption that the system observable has non-degenerate eigenvalues) appears as a simple corollary in Section 5 which discusses the relations between the “standard” measurement model used by Araki and Yanase [4] and the more general model introduced in Section 4.

The reader will naturally wonder if the time invested in working through this material will be adequately repaid by the understanding gained. In honesty,

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5 Of which I have had time to read only a small fraction since I learned of the WAY theorem a few months ago. (See, for example, the bibliography of [12].) If I have overlooked some reference which should be included, that is the reason.

6 More precisely, all observables which can be “exactly” measured, which according to the usual textbook quantum mechanics, means all observables. The seminal papers on the WAY theorem also discuss a more general notion of “approximate measurability”. 
I feel compelled to let him know that it may not be, unless he is already a connoisseur of the ideas surrounding the WAY theorem. I have been led to regard von Neumann type measurement theory in general and the WAY theorem’s place in it as of questionable physical relevance.

On the other hand, the WAY theorem is characterized as “famous”, “remarkable”, and “important” in three recent papers by different authors, which makes me wonder if the difficulties which I have noticed are widely known. In retrospect, I regret the time I have invested in studying it, but having made the investment, it seemed worthwhile to write down what I have learned to save others the trouble.

Notation:

I hope that the notation informally introduced above will seem reasonably natural to most readers. We will not use the Dirac notation $|s\rangle$ for pure states, instead using the simpler $s$ which will be defined explicitly in the text instead of implicitly by Dirac notation.

The advantage of Dirac notation is that when one sees $|s\rangle$ (in browsing through a journal, say), one immediately knows what it represents. But Dirac notation is messy and sometimes hard to parse in more complicated expressions. Instead of the Dirac $|s\rangle\langle s|$ to represent the projector on the pure state $s$, we shall use the simpler $P_s$. In contexts in which the pure state $s$ is considered as the mixed state (positive operator of trace 1) which is $P_s$, we sometimes write $\tilde{s}$ instead of $P_s$.

We do not distinguish operators with carets, writing, for example, $M_i$ instead of $\hat{M}_i$. The identity operator will be denoted $I$, with the space on which it acts determined by the context.

2 Energy-conserving projective measurements must commute with the energy operator

We start with equation (4) in the Introduction,

$$\sum_i M_i^\dagger H M_i = H,$$

which is equivalent to energy conservation for all states and show that when the $M_i$ are orthogonal projectors which sum to 1 (that is, the measurement is a projective measurement), they all must commute with $H$.

This holds for any operator $H$ representing a quantity conserved by the measurement, not just the energy operator. The Hilbert space on which $H$ operates can be finite or infinite dimensional. The collection $\{M_i\}$ can be finite or infinite.

\footnote{Or her, of course. I adhere to the long-standing and sensible grammatical convention that in contexts like this, “him”, “her”, “him or her”, and “her or him” carry identical meanings.}
To emphasize that the $M_i$ are assumed to be projectors (usually called “projection operators” or simply “projections” in the mathematical literature), we write $P_i$ instead of $M_i$. Thus the $P_i$ satisfy 

\[ P_i^\dagger = P_i, \quad P_i^2 = P_i \quad \text{for all } i, \quad P_i P_j = 0 = P_j P_i \quad \text{for } i \neq j, \quad \text{and } \sum_i P_i = I. \]

**Theorem 1** Let $H$ be a given operator, and \{P_i\} a collection of orthogonal projectors which sum to the identity. If the measurement defined by \{P_i\} conserves the observable quantity corresponding to $H$ (i.e., if $\sum_i P_i H P_i = H$) then all the $P_i$ commute with $H$: $P_i H = H P_i$ for all $i$.

**Proof:**

The Introduction explained what it means for a measurement to “conserve energy”, and the same definition is used for any observable $H$ instead of energy. Here it means that 

\[ \sum_i \text{tr} \left[ P_i H P_i \rho \right] = \text{tr} \left[ H \rho \right] \quad \text{for all mixed states } \rho, \]

which is equivalent to

\[ \sum_i P_i H P_i = H. \]

For any $k$,

\[ P_k H = \sum_i P_k P_i H P_i = P_k H P_k \]

because $P_k P_i = 0$ for $i \neq k$ and $P_k^2 = P_k$. Similarly,

\[ H P_k = \sum_i P_i H P_i P_k = P_k H P_k \quad \text{so} \]

\[ P_k H = P_k H P_k = H P_k. \]

Thus most projective measurements do not conserve energy; only the very special ones that commute with the energy operator $H$ can.

It is natural to wonder if the same or something similar is true for general, non-projective measurements. Again, it is trivial that if the measurement operators commute with $H$, then the measurement conserves the physical quantity corresponding to $H$. However, the converse is not so evident as for projective measurements, and may not be true.

Mathematicians may be interested in this problem. However its physical relevance is probably minor unless there turns out to be a simple, general condition that is equivalent to conservation of energy. The fact that typical projective measurements don’t conserve energy already poses a problem for the foundations of quantum mechanics.
3 Reviews

The reviews of this section are included to make the paper more nearly self-contained. Many will have no need for much of it. I suggest skimming and referring back to it when needed. But please do read the first paragraph of Subsection 3.1 for the definition of measurement which will be used throughout.

3.1 Review of measurement operators

A measurement in quantum mechanics is specified by a collection \( \{ M_i \} \) of measurement operators \( M_i \). For simplicity of language, we shall often refer to the collection \( \{ M_i \} \) of measurement operators as a measurement.

A collection \( \{ M_i \} \) of measurement operators is required to satisfy

\[
\sum_i M_i^\dagger M_i = I.
\]

The index \( i \) can run over any countable set, which when the set is finite is usually taken to be the set \( \{1, 2, \ldots, N\} \) of the first \( N \) integers, and this is the only situation that we shall consider.

The result of a measurement is one of the integers in this index set. For a quantum system in mixed state \( \rho \), the probability \( p(i) \) that the measurement result is \( i \) is

\[
p(i) = \text{tr} \left[ M_i \rho M_i^\dagger \right],
\]

and the measurement changes the premeasurement state \( \rho \) to the postmeasurement state

\[
\frac{M_i \rho M_i^\dagger}{\text{tr} [M_i \rho M_i^\dagger]}. \tag{5}
\]

The denominator, necessary to normalize the trace to 1, is just \( p(i) \). If we know that the measurement has been made but do not know the result, then the postmeasurement state is the mixed state which is the weighted average of (5) with weights the probabilities \( p(i) \):

\[
\text{postmeasurement state} = \sum_i p(i) \frac{M_i \rho M_i^\dagger}{p(i)} = \sum_i M_i \rho M_i^\dagger. \tag{6}
\]

If the mixed state \( \rho \) happens to be a pure state \( \rho = \hat{\phi} = P_\phi \), then (5) is always pure (it is the normalization of \( M_\phi \)), but (6) is rarely pure. Because measurement usually converts pure states into non-pure mixed states, the language of mixed states (positive operators of trace 1) is more natural than the language of pure states (unit vectors in a Hilbert space) to describe measurement operations.

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8 The rest of the paper will deal exclusively with projective measurements, so this section in unnecessarily general. This occurred because the first section was written before it became clear that the rest would not require the generality. However, specializing to projective measurements does not simplify anything, so I decided not to reset the type.
Many classical papers are formulated in terms of “measurement of observables”, which is a slightly different kind of measurement. An observable (like position, momentum, or energy) is mathematically represented by a Hermitian operator on a Hilbert space. For simplicity, we shall consider only observables whose spectrum consists only of eigenvalues. (The seminal papers which we shall discuss such as \cite{3, 4, 5} also make this assumption.) Let \( \{ \lambda_i \} \) be the collection of distinct eigenvalues of an observable \( H \), and let \( P_i \) denote the projector on the eigenspace for eigenvalue \( \lambda_i \). Then the \( P_i \) are orthogonal projectors (i.e., \( P_i P_j = 0 \) for \( i \neq j \)) which sum to the identity operator \( I \), and the collection \( \{ P_i \} \) constitutes a special kind of measurement operators. A measurement made with orthogonal projectors which sum to the identity is called a projective measurement.

Note that by definition, projectors are Hermitian operators. Also, the spectral theorem states that
\[
\sum_i \lambda_i P_i = H.
\]

Suppose we have a large collection of identical states \( \rho \) and perform a measurement on each one. Each measurement yields a result \( i \), with which is associated an eigenvalue \( \lambda_i \), physically interpreted as the measured value of the observable. The average of all these measured values, for a large enough sample, should be close to
\[
\sum_i p(i) \lambda_i = \sum_i \text{tr} \left[ P_i \rho P_i^\dagger \right] \lambda_i = \sum_i \text{tr} \left[ \rho P_i^2 \right] \lambda_i = \text{tr} \left[ \rho \sum_i P_i \lambda_i \right] = \text{tr} \left[ \rho H \right].
\] (7)

Thus \( \text{tr} \left[ \rho H \right] \) is the mathematical representation of the average value obtained by measuring the observable many times on identical systems in state \( \rho \).

Let \( H \) be an observable which we shall call the energy observable for ease of language, though what we say will apply to any observable such as spin, etc. To say that a measurement \( \{ M_i \} \) conserves energy \( H \) (or if \( H \) is an observable other than energy, whatever quantity it represents) means that for all mixed states \( \rho \), the average energy of the postmeasurement state is the same as the average energy of the premeasurement state, i.e., that
\[
\text{tr} \left[ \rho H \right] = \sum_i \text{tr} \left[ M_i \rho M_i^\dagger H \right].
\] (8)

As previously noted, this is equivalent to
\[
\sum_i M_i^\dagger H M_i = H
\].

### 3.2 Review of isometries

Let \( \mathcal{H} \) and \( \mathcal{K} \) be Hilbert spaces. A linear transformation \( U : \mathcal{H} \to \mathcal{K} \) is called an isometry if it preserves inner products (and consequently norms), that is, if
\[
\langle U \phi, U \psi \rangle = \langle \phi, \psi \rangle \quad \text{for all} \ \phi, \psi \in \mathcal{H}.
\]
Such isometries $U$ are often incorrectly called “unitary” operators in the physics literature. The difference between a unitary operator and an isometry is that a unitary operator is required to be surjective (“onto”); i.e., the range of a unitary operator must be all of $K$. This difference will be important to us, so it is worthwhile to note it explicitly.

It is routine to verify that $U$ is an isometry if and only if
\[ U^\dagger U = I. \]  
(9)
One also easily checks that $UU^\dagger$ is the projector on the range of $U$. Thus a unitary operator satisfies in addition to (9),
\[ UU^\dagger = I. \]

For later reference, it is worth noting that, from (9), the adjoint $U^\dagger$ of an isometry $U$ acts as an inverse for $U$ on the range of $U$: If $U\phi = \psi$, then $U^\dagger \psi = \phi$.

Also, we shall use the following simple fact. Let $R$ denote the projector on the range of $U$. Then obviously,
\[ RU = U, \]
and taking adjoints shows that also
\[ U^\dagger R = U^\dagger. \]

### 3.3 Review of a von Neumann type measurement model

This subsection sets up the measurement model in which the WAY theorem is formulated. The model is generally attributed to von Neumann. Busch and Lahti \[6\] call it the “Standard Model of Quantum Measurement Theory”. Readers already familiar with the von Neumann model may only need to skim this section.

In the early days of quantum theory, much attention was given to the transition between the classical world governed by everyday Newtonian physics and the much stranger quantum world seemingly governed entirely differently. Exactly how does the quantum world become classical?

It seemed that light might be shed on this problem by examining in detail the process of measuring an observable like the spin of a spin-1/2 quantum particle in a given direction. According to quantum theory, the measurement is a projective measurement implemented by two projectors $\{ P_+, P_- \}$ corresponding to “up” and “down” spins. Some feel that this measurement occurs on a quantum level which has to be somehow amplified to be classically observable. For example, spin can be observed with a macroscopic Stern-Gerlach apparatus which seems to obey the laws of Newtonian physics.

I am trying to explain a point of view with which I have never been comfortable. I don’t see why the Stern-Gerlach apparatus could not be regarded as a physical implementation of the measurement operators $\{ P_+, P_- \}$.

However, suppose we accept the interpretation that the $\{ P_+, P_- \}$ measurement has to somehow be amplified to be observable on the classical level. The following mechanism, usually attributed to von Neumann, has been proposed.
Let $s_+$ and $s_-$ be the (pure) quantum states which are eigenvectors of $P_+$ and $P_-:
\begin{align*}
P_+ s_+ &= (1/2)s_+, \\
P_- s_- &= (-1/2)s_- .
\end{align*}
We imagine that a Stern-Gerlach apparatus also has a complete set of two orthogonal pure states $a_+, a_-$. Consider the association
\[ s_+ \mapsto s_+ \otimes a_+, \quad s_- \mapsto s_+ \otimes a_- . \]

Here $s_+ \otimes a_\pm$ are states in a Hilbert space $S \otimes A$ which is the tensor product of the original state space $S$ for the particle whose spin is being measured and a Hilbert space $A$ for the apparatus. Then we perform the measurement with projection operators $I \otimes P_{a_\pm}$, with $\{P_{a_+}, P_{a_-}\}$ a projective measurement in the apparatus space. The postmeasurement state is then either $s_+ \otimes a_+$ or $s_- \otimes a_-$ (which after tracing out the state space becomes $a_+$ or $a_-$).

This process is supposed to somehow explain how quantum measurements get converted to classical ones which we can perform in the laboratory. To me, it seems rather silly, no more explanatory than simply imagining the quantum measurement with $\{P_\pm\}$ as implemented in some way which we choose not to (or cannot) describe in detail.

Anyway, the WAY theorem is formulated in terms of this measurement model, so we have to tentatively accept it to continue. The observable $S$ which we want to measure will be arbitrary, not necessarily spin. It operates on a Hilbert space $S$. We shall call $S$ the system observable and $S$ the system space.

The measuring apparatus (assumed to obey the laws of quantum mechanics despite the above motivating remarks) is an observable on a Hilbert space $A$. The system together with apparatus operates on $S \otimes A$.

## 4 WAY-type theorems in a new framework

### 4.0 Warning

Subsection 4.1 presents a measurement theory which is similar to the “standard” von Neumann theory, but not quite equivalent. It is a little more general, and, I think, algebraically more natural. Section 5 makes contact with the traditional von Neumann measurement theory in which Araki and Yanase’s original proof [4] of the WAY theorem is formulated.

### 4.1 Interpretations of Araki and Yanase’s setup

Following Araki and Yanase [4], we assume that the system observable $S$ has discrete spectrum (possibly degenerate) with distinct eigenvalues $\{\lambda_i\}$. Let $Q_i$ denote the projector on the eigenspace for eigenvalue $\lambda_i$. Then the spectral theorem states that $Q_i Q_j = 0$ for $i \neq j$ and
\[
S = \sum_i \lambda_i Q_i .
\]
We shall actually be concerned with the projective measurement with measurement operators \( \{Q_i\} \) rather than measuring \( S \) itself; that is, the eigenvalues of \( S \) will not enter into our considerations beyond the definition (10). Also, little insight will be lost by assuming that each \( Q_i \) has one-dimensional range spanned by a unit vector \( \phi_i \), and we shall first consider this case when it simplifies the exposition. Our generalizations to \( Q_i \) of arbitrary dimension (finite or infinite) will be routine.

Let \( \phi_i \) be an orthonormal basis for the system space \( S \) such that the range of each \( Q_i \) is spanned by some subcollection of \( \{\phi_i\} \), where \( i \) runs over some index set. Corresponding to this orthonormal basis let \( \{X_i\} \) be an orthonormal basis , for the apparatus space \( A \), where \( i \) runs over the same index set. Let \( U \) be the unique isometry \( U : S \to S \otimes A \) satisfying

\[
U\phi_i = \phi_i \otimes X_i \quad \text{for all } i .
\]

(11)

Here we depart from the “standard” von Neumann measurement model, and from Araki and Yanase \([4]\) in particular. The differences may seem small, but the setups are not equivalent as one might imagine.\(^9\) Ours is more general, as will become apparent in Section 5.

The more usual notation is to choose a fixed unit vector \( \xi \) in \( A \), and define

\[
U(\phi_i \otimes \xi) = \phi_i \otimes X_i .
\]

(12)

That may give the impression that \( U \) might later be defined on all of \( S \otimes A \). We emphasize that it is only defined on the subspace spanned by all \( \phi_i \otimes \xi \), and an extension to all of \( S \otimes A \) is typically not considered in the literature. Araki and Yanase \([4]\) then consider an operator \( L : S \otimes A \to S \otimes A \) and state as a hypothesis that \( L \) commutes with \( U \):

\[
UL = LU .
\]

(13)

But this makes no sense unless \( U \) is defined on the range of \( L \), which, for arbitrary \( L \), might be expected to contain vectors not of the form \( \phi \otimes \xi \). There are various ways to circumvent this, but in general confusion seems inevitable because the reader has no way to know which circumvention the authors intended. We present one such circumvention below.

Our first task is to fix on an interpretation for (13). The starting point will be to consider \( U \) as a map from \( S \) into \( S \otimes A \):

\[
U : S \to S \otimes A .
\]

(This accounts for the difference between the formulation below and the “standard” von Neumann model.)

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\(^9\)It might seem perverse to use Greek \( \phi \) to denote vectors in \( S \) and Roman \( X \) for \( A \), but this is the notation of Araki and Yanase \([4]\). We use it both for comparison and also because it seems to make it easier to sort out at a glance which vector is in which space.

\(^{10}\)When this subsection was written, I thought that they would turn out to be equivalent, and was surprised to find out that they are not.
If $U : S \rightarrow S \otimes A$, then the $L$’s on the left and right of (13) must be different. The domain of the right side is the domain of $U$, namely $S$. Hence the domain of the left side, which is the domain of the left-side $L$, must also be $S$. But the domain of the right-side $L$ has to include the range of $U$, which is a subspace of $S \otimes A$. The point is that the domains of the left-side and right-side $L$’s are in different Hilbert spaces, so the left-side $L$ cannot be the same as the right-side $L$. Therefore, we should use different symbols for the two.

Araki and Yanase [4] refer to $L$ as a conserved quantity, which they define as one which satisfies (13) (their equation (2.6)). For ease of language, we are going to call this “conserved” quantity “energy”, with the understanding that it could well be something else such as spin. Instead of $L$, we shall use the symbol $H$ for energy, with an appropriate subscript to indicate to which system the $H$ refers, $H_S$ for the energy operator on $S$, $H_{S \otimes A}$ for the energy operator on $S \otimes A$ and $H_A$ for the energy operator on $H_A$. Then equation (13) reads:

$$UH_S = H_{S \otimes A}U \quad .$$

(14)

In the mathematical literature, equation (14) would be verbalized by saying that $U$ intertwines $H_S$ and $H_{S \otimes A}$ instead of saying that $H$ commutes with $U$.

This implies that the range of $H_S$ is contained in the domain of $U$, namely $S$, so

$$H_S : S \rightarrow S \quad .$$

Establishing the domain and codomain of $H_{S \otimes A}$ is a bit trickier. From the right side of (13), the domain of $H_{S \otimes A}$ must contain $US$, which is the span of all $\phi_i \otimes X_i$. Again from (13), this span is invariant under $H_{S \otimes A}$, so there seems no harm in taking the codomain of $H_{S \otimes A}$ as $US$, or as $S \otimes A$ when convenient:

$$H_{S \otimes A} : US \rightarrow US \quad \text{or} \quad H_{S \otimes A} : US \rightarrow S \otimes A \quad .$$

The question of fixing the domain and codomain of $H_{S \otimes A}$ might seem nit-picking, but it arises in the following way in the context of the WAY theorem. Araki and Yanase [4] consider an $H_{S \otimes A}$ assumed to be of the form

$$H_{S \otimes A} = H_1 \otimes I + I \otimes H_2 \quad .$$

(15)

Since the right side need not lie in $US$, to even consider such an $H_{S \otimes A}$ for arbitrary $H_1$ and $H_2$, one needs to enlarge its codomain beyond $US$. However, such an $H_{S \otimes A}$ can satisfy (14) only if $H_1$ and $H_2$ are such that the range of $H_{S \otimes A}$ actually does lie in $US$. To consider arbitrary $H_{S \otimes A}$ satisfying (15), it seems reasonable to take the codomain as $S \otimes A$:

$$H_{S \otimes A} : US \rightarrow S \otimes A \quad .$$

11 This is a different usage than our usage of “conserved” to refer exclusively to a quantity which is not changed by a measurement. Beyond this paragraph, we shall never use “conserved” in the Araki-Yanase sense.
But when we want to emphasize that the range of $H_{S\otimes A}$ must actually lie in $U_S$ when (14) holds, we will write $H_{S\otimes A} : U_S \rightarrow U_S$.

Under the additivity assumption (15), the WAY theorem of Araki and Yanase [4] concludes that the system observable $S$ must commute with the system energy operator $H_1$, $H_1 S = S H_1$, but that assumes their “standard” von Neumann measurement model. Within the present setup this conclusion would seem something of a red herring for the following reason.

It would be natural to imagine that $H_1$ would be the energy operator on $H_S$ and $H_2$ the energy operator on $H_A$. If that were the case, then commutation of $H_1$ with $S$ would be equivalent to commutation of $H_1$ with the projectors of the measurement $\{Q_i\}$, which by Theorem 4 is equivalent to conservation of energy in the system space $H_S$. This would indeed seem an interesting conclusion.

But it is rarely the case that $H_1 = H_S$. At this point, there is no substitute for an explicit calculation to convince the reader of this. Also, this simple calculation contains the essence of our proof of WAY-type theorems, and illustrates their essential simplicity.

Consider a two-dimensional system $S$ with orthonormal bases $\{\phi_1, \phi_2\}$ for $S$, $\{X_1, X_2\}$ for $A$, and $\{\phi_1 \otimes X_1, \phi_2 \otimes X_2\}$ for $U_S \subset S \otimes A$, and a system observable $S = \sum \lambda_i P_{\phi_i}$. All matrices will be written with respect to whichever of these bases is relevant.

Consider arbitrary $H_1$ and $H_2$ with matrices:

$$H_1 = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \quad \text{and} \quad H_2 = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$  

We are going to observe that the necessity for $H_{S\otimes A} = H_1 \otimes I + I \otimes H_2$ to hold invariant the range of $U$, which is the span of $\{\phi_1 \otimes X_1, \phi_2 \otimes X_2\}$, constrains (the matrices of) $H_1$ and $H_2$ to be diagonal. We have

$$H(\phi_1 \otimes X_1) = h_{11} \phi_1 \otimes X_1 + h_{21} \phi_2 \otimes X_1 + k_{11} \phi_1 \otimes X_1 + k_{21} \phi_1 \otimes X_2$$

Projecting onto the span of $\{\phi_i \otimes X_1\}_{i=1}^2$ shows that

$$h_{21} = 0 = k_{21}.$$  

Considering similarly $H(\phi_2 \otimes X_2)$ yields $h_{12} = 0 = k_{12}$, so $H_1$ and $H_2$ must be diagonal, which implies that $H_1$ commutes with $S$. Now

$$H_{S\otimes A} = H_1 \otimes I + I \otimes H_2 = \begin{bmatrix} h_{11} + k_{11} & 0 \\ 0 & h_{22} + k_{22} \end{bmatrix},$$

where the matrix is with respect to the basis $\{\phi_i \otimes X_1\}$. The relation $UH_S = H_{S\otimes A}U$ implies that the matrix of $H_S$ with respect to $\{\phi_1, \phi_2\}$ is the same as the matrix of $H_{S\otimes A}$ with respect to $\{\phi_1 \otimes X_1, \phi_2 \otimes X_2\}$:

$$H_S = \begin{bmatrix} h_{11} + k_{11} & 0 \\ 0 & h_{22} + k_{22} \end{bmatrix} \neq H_1.$$
This shows explicitly that $H_S$ is unitarily equivalent to $H_{S \otimes A}|US$, not to $H_1$ as one might imagine. By the symmetry of the situation, all three of $H_A, H_{S \otimes A}|US, H_A$ are unitarily equivalent.

Before continuing, we point out how the essence of the WAY-type theorems, both ours and that of Araki and Yanase [4], is revealed by the above calculation. Our assumed intertwining relation (14) requires that $H$ hold invariant the range of $U$, which is the span of $\{\phi_1 \otimes X_1, \phi_2 \otimes X_2\}$. But only very special $H$ of the form $H = H_1 \otimes I + I \otimes H_2$ can hold invariant this span.

The system energy $H_S$ has physical meaning, but the physical meaning of $H_1$, if any, seems unclear. And if the physical meaning of $H_1$ is unclear, the import of the conclusion that $H_1$ commutes with the system observable $S$, seems even more obscure. (That conclusion would be the conclusion of the WAY theorem if the Araki/Yanase setup were the same as ours.) The conclusion that we want is that $S$ commutes with $H_S$, not $H_1$. Fortunately, we shall see that this desired conclusion does hold, not just for this example but in general, along with the conclusion that $S$ commutes with $H_1$.

The reader may wonder if we are merely playing with words in calling $H_S$ the system energy operator instead of $H_1$, but a little reflection will dispel this worry. We started with the system $S$ with energy operator named $H_S$. The system $S \otimes A$ with corresponding energy operator was just a mathematical construction.

\section*{4.2 WAY-type theorems in our setup}

This subsection will give simple proofs of variants of the WAY theorem in our setup. The results will be stated in the generality in which they have been proved, but to simplify the notation, the proofs will assume that the eigenspaces of the system observable are one-dimensional. All the important ideas of the proof are present in this case. The notationally complicated proofs for eigenspaces of possibly greater dimension are relegated to an appendix.

For the convenience of skimming readers, we first summarize the notation of the preceding sections and introduce two new notations. The WAY-type theorems and related results will be mainly consequences of the general setup developed in Subsection 4.1. To specify it completely in a theorem’s hypotheses would result in an excessively cumbersome statement.

Recall that the system observable $S$ has distinct eigenvalues $\lambda_i$ and that the spectral decomposition of $S$ is

$$S = \sum_k \lambda_k Q_k$$

where the $Q_k$ are orthogonal projectors (so that $\{Q_k\}$ is a measurement).

Let $\{\phi_i\}_{i \in I}$ be an orthonormal basis for $S$, where $I$ is some index set, and such that for each $k$, the range of $Q_k$ is spanned by some collection of the $\phi_i$. Thus when $S$ has non-degenerate eigenvalues (i.e., all the $Q_k$ are one-dimensional), the notation can be chosen so that $Q_k = P_{\phi_k}$, and we assume this choice. Let $\{X_i\}_{i \in I}$ be an orthonormal basis for $A$. 

Denote by $U$ the unique isometry $U : S \to S \otimes A$ satisfying 
\begin{equation}
U \phi_i = \phi_i \otimes X_i \quad \text{for all } i.
\end{equation}
Similarly define an isometry $V : A \to S \otimes A$ by $V X_i = \phi_i \otimes X_i$. Define an 
"apparatus observable" 
\[ A := \sum_k \lambda_k P_k, \]
where $P_k$ is the projector on $A$ with range spanned by the $X_i$ for which $\phi_i \in$ Range $Q_k$. Under the simplifying assumption that $Q_k = P_{\phi_k}$, we have $P_k = P_{X_k}$.

Let 
\[ H_S : S \to S, \quad H_A : A \to A, \quad H_{S \otimes A} : U S \to S \otimes A. \]

We call these “energy operators” on their respective spaces, but they could be arbitrary Hermitian operators. We say that the measurement \{\$Q_i\$\} conserves energy on $S$ if its operators commute with $H_S : Q_i H_S = H_S Q_i$ for all $i$. The same language applies to $S \otimes A$ and $A$ with \{\$Q_i\$\}, $H_S$ replaced by \{\$U Q_i U^\dagger$\}, $H_{S \otimes A}$ for $S \otimes A$ and \{\$P_i\$\}, $H_A$ for $A$.

The following proposition is little more than a tautology which systemizes the facts which we will need to prove the conclusion of the WAY theorem for the setup of Subsection 4.1 (which is similar but not identical to the setup of Araki and Yanase [4]).

**Proposition 2** Assume the general setup just described and that $U H_S = H_{S \otimes A} U$. Then the following are equivalent:

(i) $H_S$ commutes with $S$: $S H_S = H_S S$;

(i)$'$ The measurement \{\$Q_i\$\} conserves energy on $S$;

(ii) $H_{S \otimes A}$ commutes with $U S U^\dagger$: $(U S U^\dagger) H_{S \otimes A} = H_{S \otimes A} (U S U^\dagger)$;

(ii)$'$ The measurement \{\$U Q_i U^\dagger$\} on the range of $U$ conserves energy on $U S$;

(iii) $H_A$ commutes with $A$: $A H_A = H_A A$;

(iii)$'$ The measurement \{\$P_i\$\} conserves energy on $A$.

**Proof:** This proof does not require the simplifying assumption that $Q_i = P_{\phi_i}$. The equivalence of the various items and their primed versions (e.g., (i) and (i)$'$) is immediate from the spectral theorem, part of which states that an operator commutes with $S$ if and only if it commutes with all of the spectral projectors $Q_i$ for $S$.

Since the domain of $H_{S \otimes A}$ is $U S$, to show that $U Q_i U^\dagger$ commutes with $H_{S \otimes A}$ when $H_S$ commutes with $Q_i$, it is sufficient to show that $(H_{S \otimes A} U Q_i U^\dagger) U = U H_S U Q_i U^\dagger$. 

\[\]
\((UQ_iU^\dagger H_{S\otimes A})U\). We have, using \(U^\dagger U = I\) and \(H_SQ_i = Q_iH_S\),

\[
[H_{S\otimes A}(UQ_iU^\dagger)]U = UH_SQ_i = UQ_iH_S = (UQ_iU^\dagger)(UH_S)
\]

\[
= [(UQ_iU^\dagger)H_{S\otimes A}]U.
\]

For the converse, that \(Q_iH_S = H_SQ_i\) when \((UQ_iU^\dagger)H_{S\otimes A} = H_{S\otimes A}(UQ_iU^\dagger)\), first note that using \(U^\dagger U = I\), \(UH_S = H_{S\otimes A}U\) implies that \(H_S = U^\dagger H_{S\otimes A}U\). Hence,

\[
H_SQ_i = U^\dagger H_{S\otimes A}UQ_i = U^\dagger H_{S\otimes A}(UQ_iU^\dagger)U = U^\dagger(UQ_iU^\dagger)H_{S\otimes A}U = Q_iU^\dagger H_{S\otimes A}U = Q_iH_S.
\]

We have shown that (i), (i)’, (ii), and (ii)’ are equivalent. The equivalence of (iii), and (iii)’ with the rest follows similarly from the symmetry of the setup.

Next we obtain WAY-type theorems from the Proposition. Various hypotheses on the form of \(H_{S\otimes A}\), such as the Araki/Yanase assumption that it is of the form \(H_{S\otimes A} = H_1 \otimes I + I \otimes H_2\), are easily seen to imply that \(H_{S\otimes A}\) commutes with the \(UQ_iU^\dagger\), which from the proposition implies that energy is conserved not only in \(S \otimes A\), but also in \(S\) and \(A\).

In my view, this is the physically relevant conclusion that one wants, as discussed in Subsection 4.1. However, it is not the form of the conclusion of Araki/Yanase’s WAY theorem \[4\], that \(S\) commutes with \(H_1\). For comparison and completeness, we obtain the latter also.

Say that \(H_{S\otimes A} : S \otimes A \rightarrow S \otimes A\) is diagonal with respect to the projectors \(\{UQ_iU^\dagger\}\) if \((UQ_iU^\dagger)H_{S\otimes A}(UQ_jU^\dagger) = 0\) for all \(i\) and all \(j \neq i\), with a similar meaning for “\(H_S\) is diagonal with respect to \(\{Q_i\}\)” etc. When \(H_{S\otimes A}\) holds invariant the range of \(U\), (as it does when \(H_{S\otimes A}U = UH_S\)) this implies that \(H_{S\otimes A}\) commutes with all \(UQ_iU^\dagger\) and conversely.

The next result looks like the WAY theorem expressed in our setup, though (in the context of non-degenerate eigenvalues for the system observable) it is actually more general, as shown in Section 5. Its statement assumes the notation of the preceding discussion. However, we summarize it first for the benefit of skimming readers who may want to get a feel for the result in order to decide whether to read further.

The object of interest is a quantum (“system”) observable \(S\) on a Hilbert space \(S\) which is to be measured. The measurement apparatus is an observable \(A\) on a Hilbert space \(A\), and the total system-apparatus Hilbert space is \(S \otimes A\). On each of these three Hilbert spaces is defined an energy operator denoted \(H_S\) for \(S\), \(H_A\) for \(A\), and \(H_{S\otimes A}\) for \(S \otimes A\). These are related by an isometry \(U\).
which intertwines $H_S$ and $H_{S\otimes A}$: $UH_S = H_{S\otimes A}U$ and embeds $S$ into $S \otimes A$. The measurement is carried out in $S \otimes A$, but the result is translated back into $S$ via the identification furnished by $U$.

**Theorem 3 (WAY-type theorem with Yanase-type condition proved.)**

Suppose that $H_{S\otimes A}$ is of the form

$$H_{S\otimes A} = H_1 \otimes I + I \otimes H_2$$

and that $UH_S = H_{S\otimes A}U$. Then the system observable $S$ commutes with both the system energy operator $H_S$ and $H_1$, and the apparatus observable $A$ commutes with both the apparatus energy operator $H_A$ and $H_2$.

**Proof** (for $S$ with one-dimensional eigenspaces): Though stated in general, for simplicity and clarity this proof will be given under the simplifying assumption that $Q_k = P_{\phi_k}$ and $P_k = P_{X_k}$. The general proof is relegated to the Appendix.

To show that $H_1$ commutes with $S$, we exploit the fact that because of $H_{S\otimes A}U = UH_S$, $H_{S\otimes A}$ holds invariant the range of $U$, which is spanned by the $\phi_i \otimes X_i$. Suppose for some $k \neq j$, $\langle \phi_k, H_1 \phi_j \rangle \neq 0$. Then

$$\langle \phi_k \otimes X_j, H_{S\otimes A}(\phi_j \otimes X_j) \rangle = \langle \phi_k, H_1 \phi_j \rangle \cdot 1 + 0 \cdot \langle X_j, H_2 X_j \rangle \neq 0$$

shows that $H_{S\otimes A}$ does not hold invariant the span of all $\phi_i \otimes X_i$. If it did, we would have $H_{S\otimes A}(\phi_j \otimes X_j) = \sum_i c_i \phi_i \otimes X_i$ for some scalars $c_i$, and consequently,

$$\langle \phi_k \otimes X_j, H_{S\otimes A}(\phi_j \otimes X_j) \rangle = 0.$$  

The proof that $H_2$ commutes with the apparatus observable $A$ is the same, using the system-apparatus symmetry of the setup.

Thus

$$H_1 \phi_i = d_i \phi_i \quad \text{and} \quad H_2 X_i = b_i X_i$$

for some scalars $d_i, b_i$ and all $i$, and consequently

$$H(\phi_i \otimes X_i) = (d_i + b_i) \phi_i \otimes X_i.$$  

The relation $UH_S = H_{S\otimes A}U$ shows that $U$ implements a unitary equivalence between $H_S$ and $H_{S\otimes A}|\text{Range } U$. Under this equivalence, the basis $\{\phi_i\}$ for $S$ goes over into the basis $\{\phi_i \otimes X_i\}$ for $S \otimes A$. Hence the matrix of $H_S$ with respect to $\{\phi_i \otimes X_i\}$ is the same as the matrix of $H_{S\otimes A}$ with respect to $\{\phi_i \otimes X_i\}$, namely the diagonal matrix $\text{diag}(d_i)$. The matrix of $S$ with respect to $\{\phi_i\}$ is also diagonal, namely $\lambda_i$. Hence $H_S$ commutes with $S$ Similarly, $A$ commutes with $H_2$, which is called the “Yanase condition” (see below), and also with $H_A$.

The so-called “Yanase condition” requires some explanation. A year after the Araki/Yanase WAY theorem [4] was published, Yanase published [5] which seems to adjoin in some way to the WAY theorem the condition that $H_2$ commute with the apparatus observable $A$. This has become known as the “Yanase
condition”. Yanase’s language is obscure to me, and seemingly to other authors. There may be various interpretations, but there seems substantial agreement that the Yanase condition is physically desirable.

The above proofs are valid under hypotheses on the form of $H_{S\otimes A}$ considerably more general than the Araki/Yanase hypothesis $H_{S\otimes A} = H_1 \otimes I + I \otimes H_2$ stated, but more general formulations make the hypotheses too cumbersome and obscure the simplicity of the proofs. We indicate here some typical generalizations.

If $D_1$ and $D_2$ are diagonal Hermitian operators (with respect to $\{Q_i\}$ and $\{P_i\}$ respectively, then the proof of Theorem 3 goes through with hypothesis

$$H_{S\otimes A} = H_1 \otimes D_2 + D_1 \otimes H_2.$$  \hfill (17)

If $D_1$ is a Hermitian operator which is diagonal with respect to $\{Q_i\}$ and

$$H_{S\otimes A} = D_1 \otimes (V^\dagger U)D_1(V^\dagger U)^\dagger,$$  \hfill (18)

the easy part of the proof of Theorem 3 establishes its conclusion. (Again, $U$ implements a unitary equivalence between $H_S$ and the operator $H_{S\otimes A}$, which is obviously diagonal with respect to $\{\phi_i \otimes X_i\}$.) The above statements involving forms (17) and (18) assume non-degenerate eigenvalues (i.e., the $\{Q_i\}$ are one-dimensional). I have not examined more general cases.

5 Comparison with the traditional approach

Again, we denote the system Hilbert space by $S$ and the apparatus space by $A$, with $S \otimes A$ the system-apparatus space. For simplicity, the discussion will assume that the spectral projections of the system observable have one-dimensional range. Let $\{\phi_i\}$ be an orthonormal basis for $S$ and $\{X_i\}$ an orthonormal basis for $A$. Assume a system observable $S$ of the form

$$S = \sum_i \lambda_i P_{\phi_i} \text{ with distinct } \lambda_i.$$  

Let $\xi$ denote some distinguished unit vector in $A$ and $[\xi]$ the one-dimensional subspace that it spans. Define

$$U : S \otimes [\xi] \to S \otimes A$$  \hfill (19)

to be the unique isometry which satisfies

$$U(\phi_i \otimes \xi) = \phi_i \otimes X_i \text{ for all } i.$$  \hfill (20)

Since $S \otimes [\xi]$ is naturally identified with $S$ via the map $\phi \otimes \xi \mapsto \phi$, I formulated the approach of Section 3 in the expectation that the results would be equivalent to those of the traditional approach. I was surprised and initially puzzled when they turned out to differ significantly. This will be discussed in more detail.
below. We shall also derive the original WAY theorem (for the special case of non-degenerate system observable eigenvalues) as formulated by Araki and Yanase from the Section 4 results.

Let $H$ denote the energy operator on $S \otimes A$, previously called $H_{S \otimes A}$. We change the name to avoid confusion with the previous approach and for easier comparison with the approach and notation of Araki/Yanase. The traditional approach allows us to write the equation

$$HU = UH$$

with some hope of giving it meaning, since all operators in it are defined on some subspace of $S \otimes A$. However, since $U : S \otimes [\xi] \to S \otimes A$ is never extended to $S \otimes A$, the traditional approach seems to me algebraically unnatural. The approach of Subsection 4.1 turns out to be actually, not just cosmetically, more general.

For (21) to be meaningful, $U$, which so far is only defined on the subspace $S \otimes [\xi]$, must hold invariant the range of $H$. In particular, the range of $H$ must be in the domain of $U$, which is $S \otimes [\xi]$.

Araki and Yanase assume that $H$ is of the form

$$H = H_1 \otimes I + I \otimes H_2$$

with $H_1$ the energy operator on $S$ and $H_2$ the energy operator on $A$. We shall see that this that forces $H_2$ to be a multiple of the identity operator. We leave it to the reader to decide whether this is physically reasonable.

We have

$$H(\phi \otimes \xi) = (H_1 \phi) \otimes \xi + \phi \otimes (H_2 \xi)$$

For this to be in the domain of $U$ (so that we can write $UH =HU$), we must have

$$H_2 \xi = a \xi \quad \text{for some scalar } a.$$  

We do not exclude the case $a = 0$. Finally, we have

$$H(\phi \otimes \xi) = (H_1 \phi) \otimes \xi + a \phi \otimes \xi$$

Before continuing, we must deal with a notational problem. In Section 4, we started with an observable $S$ on an abstract Hilbert space $S$ and an energy operator $H_S : S \to S$ on the same space.

In the traditional model which we are analyzing, $S$ is effectively replaced by $S' := S \otimes [\xi] \subset S \otimes A$ by identifying $S$ with $S'$ via the unitary map

$$W : S \to S', \quad \phi \mapsto \phi \otimes \xi.$$

In our discussion of the traditional model, $S'$ will take the place of the $S$ in the Section 4 discussions.

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13 However, Araki and Yanase use $L$ for what they call the "conserved" quantity instead of our $H$. 

Under the unitary identification $W$, an energy operator $H_S$ on $S$ corresponds to
\[ WHSW^\dagger : S' \to S' \] (24)
on $S'$. So, $WHSW^\dagger(\phi \otimes \xi) = (H_S\phi) \otimes \xi$. However, the traditional approach does not explicitly specify an energy operator on $S$ or $S'$. To continue to compare the approaches, we must decide which operator on $S'$ to take as the analog of the Section 4 energy operator $H_S$. A natural assumption is that $H_1$ is to be taken as $H_S$, the energy operator on the present $S$, and we shall make that assumption:
\[ H_S := H_1. \]

But now we have two “natural” candidates for an energy operator on $S'$, $WHSW^\dagger = WH_1W^\dagger$, and the restriction of $H$ to the invariant subspace $S'$, denoted $H|S'$. Equation (23) shows that these are not necessarily the same:
\[ H|S' = WH_1W^\dagger + aI. \]

However, the difference is physically insignificant because an energy operator is defined only up to an arbitrary additive constant multiple of the identity. It also turns out to be mathematically insignificant: both “natural” definitions lead to the conclusion of the WAY theorem.

We can make any definition that we want for the energy operator $H_S'$ on $S'$, but it is comforting that for our purposes, it will be irrelevant which of the two “natural” definitions we use. We shall define
\[ H_S' := H|S'. \] (25)

We also define a new system operator $S'$ on $S'$ to be the old $S$ transferred to $S'$ via the identification $W$:
\[ S' := WSW^\dagger . \]

Now we are in precisely the situation considered in Section 4 with its system space $S$ replaced by $S'$, its system observable $S$ by $S'$, its system energy operator $H_S$ by $H_S'$, and its $U$ by the $U$ defined above by (20), so its analysis and the theorems proved there apply directly. We do not rename $U$ because whether the $U$ of Section 4 or the $U$ of (20) is meant will always be clear from the context. (It will always be the $U$ defined by (20).)

The hypothesis $UH_S = H_S\otimes AU$ of Theorem 3 here reads
\[ UH_S = HU. \]

Since the domain of both sides is $S'$, that is tautologically equivalent to
\[ UH_S|S' = HU|S' , \] which does hold under the hypothesis $UH = HU$ of the Araki/Yanase WAY theorem [4] because
\[ UH_S|S' = UH|S' = HU|S' . \]
Now Theorem 3 implies that the new system observable $S' := WSW^\dagger$ commutes with $WH_1W^\dagger$ ($H_1$ transferred to $S'$), and hence $S$ commutes with $H_1$. This proves the WAY theorem of Araki and Yanase [4] for the special case of nondegenerate eigenvalues, but we can say more.

We shall next observe that necessarily, $H_2 = aI$. We proved above that $H_1$ commutes with the system observable $S$, for which $\phi_i$ are eigenvectors corresponding to distinct eigenvalues. That implies that

$$H_1\phi_i = d_i\phi_i \quad \text{for some scalars } d_i \text{ and all } i.$$  

(26)

From $UH = HU$, it follows that $H = H_1 \otimes I + I \otimes H_2$ holds invariant the range of $U$, which is the span of $\{\phi_i \otimes X_i\}$. From $H_1\phi_i = d_i\phi_i$, it follows that $H_1 \otimes I$ holds invariant this range. Hence $I \otimes H_2$ must also hold invariant the span of $\{\phi_i \otimes X_i\}$. But from the Parseval equality,

$$(I \otimes H_2)(\phi_i \otimes X_i) = \phi_i \otimes H_2X_i = \sum_j \langle X_j, H_2X_i \rangle \phi_i \otimes X_j.$$  

For $j \neq i$, $\phi_i \otimes X_j$ is orthogonal to all $\phi_k \otimes X_k$, as well as all $\phi_i \otimes X_m$ for $m \neq j$, so $\langle X_j, H_2X_i \rangle = 0$ for $j \neq i$, which shows (again by Parseval) that

$$H_2X_i = b_iX_i \quad \text{with } b_i = \langle X_i, H_2X_i \rangle.$$  

Finally, $U$ implements a unitary equivalence between $H[S'] = H_1 \otimes I + I \otimes aI$ and $H[\text{Range } U]$, so the eigenvalues of the former, namely $d_i + a$, must equal the eigenvalues of the latter, namely $d_i + b_i$. Hence $b_i = a$ for all $i$. In more detail,

$$H(\phi_i \otimes \xi) = H_1\phi_i \otimes \xi + \phi \otimes H_2\xi = (d_i + a)\phi_i \otimes \xi,$$

and

$$H(\phi_i \otimes X_i) = H_1\phi_i \otimes X_i + \phi_i \otimes H_2X_i = (d_i + b_i)\phi_i \otimes X_i.$$  

Since $UH = HU$,

$$(d_i + a)\phi_i \otimes X_i = (d_i + a)U(\phi_i \otimes \xi) = U((d_i + a_i)(\phi_i \otimes \xi)) = UH(\phi_i \otimes \xi) = H(\phi_i \otimes X_i) = (d_i + b_i)\phi_i \otimes X_i,$$

whence $d_i + b_i = d_i + a$ for all $k$, so $b_i = a$ and

$$H_2 = aI \quad .$$  

(27)

Of course, this implies the Yanase condition (that the apparatus observable with eigenvectors $\{X_i\}$ commutes with $H_2$).

I was surprised by the conclusion $H_2 = aI$ because it seems so unphysical. How are we to understand this? Why did $H_2 = aI$ not already appear in the Section 4 analysis? This bothered me because I was worried that there might be an error, and I wanted to understand $H_2 = aI$ independently of the detailed analysis. The way of looking at the situation that convinced me that there probably is no error appears just below for the benefit of the reader who may be similarly uneasy.
Once perceived, the reason for the difference is easy to understand. There was no $a$ in the Section 4 analysis. It entered the present analysis at

$$H_{S}(\phi \otimes \xi) := H(\phi \otimes \xi) = (H_{1}\phi) \otimes \xi + \phi \otimes H_{2}\xi = (H_{1}\phi) \otimes \xi + \phi \otimes a\xi,$$

which arose because $H_{2}\xi = a\xi$ was forced in order to make $UH = HU$ well defined. This makes the restriction of $I \otimes H_{2}$ to $S'$ equal to $aI$: $(I \otimes H_{2})|S' = aI$.

Independently of our analysis, the Araki/Yanase WAY theorem implies that $H_{1}\phi_{i} = d_{i}\phi_{i}$ for some scalars $d_{i}$. It follows that $U$ intertwines $(H_{1} \otimes I)|S'$ and $(H_{1} \otimes I)|\text{Range } U$:

$$U(H_{1} \otimes I)(\phi_{i} \otimes \xi) = (d_{i}\phi_{i}) \otimes X_{i} = (H_{1} \otimes I)U(\phi_{i} \otimes \xi).$$

Also, from $UH = HU$, $U$ intertwines $H|S'$ and $H|\text{Range } U$. Hence, from $H = H_{1} \otimes I + I \otimes H_{2}$, $U$ also intertwines $(I \otimes H_{2})|S'$ and $(I \otimes H_{2})|\text{Range } U$. Since $(I \otimes H_{2})$ is $aI$ on $S'$, it must also be $aI$ on $\text{Range } U$, i.e., $H_{2} = aI$.

The following theorem summarizes. The first paragraph of hypotheses merely summarizes the setup just described.

**Theorem 4 (Extension of WAY theorem for non-degenerate eigenvalues)**

Let $S = \sum \lambda_{i}P_{\phi_{i}}$ be an observable on a Hilbert space $S$, where $\{\phi_{i}\}$ is an orthonormal basis for $S$, and the $\lambda_{i}$ are real scalars. Let $X_{i}$ be an orthonormal basis for a Hilbert space $A$, where $i$ runs over the same index set as for $\phi_{i}$. Let $\xi$ be a unit vector in $A$. Define an isometry $U : S \otimes [\xi] \to S \otimes A$ as the unique isometry satisfying $U(\phi_{i} \otimes \xi) := \phi_{i} \otimes X_{i}$.

Let $H$ be an operator on $S \otimes A$ of the form $H = H_{1} \otimes I + I \otimes H_{2}$, where $H_{1} : S \to S$ and $H_{2} : A \to A$. Assume that $HU$ commutes with $UH$ in the sense that for any $\phi_{i} \otimes \xi$, $HU(\phi_{i} \otimes \xi) = UH(\phi_{i} \otimes \xi)$.

Then

(i) $H_{1}$ commutes with $S$; equivalently, $H_{1}\phi_{i} = d_{i}\phi_{i}$ for some constants $d_{i}$;

(ii) The measurement $\{P_{\phi_{i}}\}$ defined by the spectral projectors $P_{\phi_{i}}$ for $S$ conserves energy, as defined by the energy operator $H_{1}$ on $S$;

(iii) Necessarily, $H_{2}\xi = a\xi$ for some constant $a$, and $H_{2} = aI$.

This was proved above. Condition (iii) implies the Yanase condition. The proof of Theorem 4 does not require that the $\lambda_{i}$ be distinct. I believe that the generalization to degenerate eigenvalues should be routine, but I have not worked out the details.

6 Return to the original conundrum

6.1 The conundrum

The original motivation for studying the WAY theorem was to figure out which hypothesis might be responsible for its hard-to-believe conclusion that all (discrete) observables (which can be exactly measured) commute with the energy
operator. The condition that the observable be discrete is probably not the origin of the problem; quite likely a similar theorem could be proved for observables like position and momentum with a “continuous” spectrum. Or, since real measurements cannot be made with arbitrary precision, one could probably construct a quantum mechanics for which all observables are discrete.

The condition that the discrete observables be “exactly” measurable (in a sense defined in [3, 4] but not discussed in the present paper) also seems relatively harmless. Doesn’t a Stern-Gerlach apparatus “exactly” measure whether a particle’s spin in a given direction is “up” or “down”? The particle leaves the apparatus going in just one of two possible directions which are easily distinguishable.

The other two hypotheses of the WAY theorem are the assumption of a von Neumann type measurement model throughout, and the assumption that the energy observable $H$ is “additive”: $H = H_1 \otimes I + I \otimes H_2$. The assumption that $H$ is additive can certainly can be questioned. For example, suppose the quantum system is an electron, and the apparatus a proton. We would certainly not expect the energy operator for a hydrogen atom to be the sum of the energy operators for a free electron and for a free proton. (Of course, this is intended as a metaphor, not as a mathematically meaningful objection.)

However, even if additivity seems unlikely in all situations, it seems that it could be possible in some situations. And in those situations, it would seem strange if no observable which did not commute with the energy operator could be (exactly) measured, as the WAY theorem implies. That focuses attention to the von Neumann model as possibly the physically unrealistic assumption.

6.2 Can the von Neumann model be generalized to imply conservation of energy in measurements?

Let us return to the observation of the Introduction that quantum projective measurements, need not conserve energy. Since there is arguably no principle more pervasive in physics than conservation of energy, this is certainly unsettling. A natural way to save the principle is to imagine that every measurement involves an interaction of the measured system with a measuring apparatus, and that energy gained or lost by the system would be lost or gained by the apparatus.

The WAY-type theorems show that under the hypotheses of Araki and Yanase [4], or those of Section 4, such a resolution is impossible (or unnecessary) because under those hypotheses, the system energy is always conserved by a measurement. Measurements which should violate conservation of energy according to textbook quantum mechanics are simply impossible under the WAY hypotheses. If a von Neumann-type measurement model is to be retained, then it seems that the hypothesis of additivity of the energy observable, $H = H_1 + H_2$, should go.
7 Appendix

This appendix indicates how the simplifying assumption of the proofs of the WAY-type theorems in Subsection 4.2 that the system observable has non-degenerate eigenvalues, can be removed. This is just for the reader’s convenience because the extensions will be more or less routine, though annoyingly complicated.

Proof of Theorem 3 We use the notation of Subsection 4.2 Let
\[ \phi_{k,1}, \phi_{k,2}, \ldots, \phi_{k,n_k} \]
be an orthonormal basis for the spectral subspace of the system observable \( S \) which is the range of its spectral projection \( Q_k \) for eigenvalue \( \lambda_k \). In other words, \( \{ \phi_{k,j} \}_{j=1}^{n_k} \) spans \( \text{Range } Q_k \). For convenience, the notation assumes that this range is finite dimensional, but that is unnecessary. Let \( \{ X_{k,j} \} \) be a similarly indexed orthonormal basis for the spectral subspace of the apparatus observable \( A \), so that \( \{ X_{k,j} \}_{j=1}^{n_k} \) spans the range of the spectral projection \( P_k \) for eigenvalue \( \lambda_k \) of the apparatus observable \( A \).

Define \( U : S \rightarrow S \otimes A \) as the unique isometry satisfying
\[ U \phi_{k,j} := \phi_{k,j} \otimes X_{k,j} \quad \text{for all } k \text{ and } 1 \leq j \leq n_k. \]

Similarly define \( V : A \rightarrow S \otimes A \) by
\[ VX_{k,j} := \phi_{k,j} \otimes X_{k,j} \quad \text{for all } k \text{ and } 1 \leq j \leq n_k, \]

To show that \( H_1 \) commutes with \( S \), suppose for some \( \phi_{k,j} \) and some \( \phi_{m,p} \) with \( m \neq k \), \( \langle \phi_{m,p}, H_1 \phi_{k,j} \rangle \neq 0 \). Then
\[ \langle \phi_{m,p} \otimes X_{k,j}, H_{S \otimes A} (\phi_{k,j} \otimes X_{k,j}) \rangle = \langle \phi_{m,p}, H_1 \phi_{k,j} \rangle \cdot 1 + 0 \cdot \langle X_{k,j}, H_2 X_{k,j} \rangle \neq 0 \]
shows that \( H_{S \otimes A} (\phi_{k,j} \otimes X_{k,j}) \) is not contained in the range of \( U \), which is \( \text{Span } \{ \phi_{s,t} \otimes X_{s,t} \} \), contrary to the assumed \( H_{S \otimes A} U = U H_S \). Hence for all \( k \), \( H_1 \) holds invariant \( \text{Range } Q_k = \text{Span } \{ Q_{k,j} \}_{j=1}^{n_k} \), which for Hermitian \( H_1 \) is equivalent to \( H_1 Q_k = Q_k H_1 \), and to \( H_1 S = S H_1 \). By the system-apparatus symmetry, the same argument shows that \( H_2 \) commutes with the apparatus observable \( A \) and all its spectral projections \( P_k \).

Next we show that \( S \) commutes with \( H_S \). Let \( R_k \) denote the span of all \( \phi_{k,j} \otimes X_{k,j} \) for \( 1 \leq j \leq n_j \). Now \( U H_S = H_{S \otimes A} U \) together with the form of \( U \) shows that \( U \) implements a unitary equivalence of \( H_S \) with \( H_{S \otimes A} \)\( | \text{Range } U \), which sends \( \text{Range } Q_k \) onto \( R_k \). Let \( \phi_k \in \text{Range } Q_k \), say \( \phi_k = \sum_j c_{k,j} \phi_{k,j} \). Let \( X_k := \sum_j c_{k,j} X_{k,j} \), where the \( c_{k,j} \) are the same as in the expansion for \( \phi_k \). Then \( U \phi_k = \phi_k \otimes X_k \).

We have
\[ U H_S \phi_k = H_{S \otimes A} U \phi_k = H_{S \otimes A} (\phi_k \otimes X_k) = (H_1 \phi_k) \otimes X_k + \phi_k \otimes H_2 X_k. \quad (28) \]

We want to show that \( H_S \) holds \( \text{Range } Q_k \) invariant. Since \( U \) is an isometry, this is the same as showing that for \( s \neq k \), \( (28) \) is orthogonal to \( U (\text{Range } Q_s) \).
That this is so is seen by taking the inner product of the right side of (28) with a typical vector spanning $U(\text{Range } Q_s)$ for $s \neq k$, say $\phi_{s,t} \otimes X_{s,t}$. Each of the terms on the right side will have a factor $\langle X_{s,t}, X_k \rangle = 0$ or $\langle \phi_{s,t}, \phi_k \rangle = 0$, so the total result is zero.

Finally, the invariance of Range $Q_k$ under $H_S$ shows that $H_S$ commutes with all $Q_k$ and with $S$. Similarly, $H_A$ commutes with all $P_k$ and $A$. This completes the proof of the stated version of Theorem 3 without the simplifying assumption that the $Q_i$ have one-dimensional ranges.

It should be noted that the statement of the WAY theorem by Araki and Yanase in [4] is even more general than what we have just proved. For that reason, we continue to call Theorem 3 a “WAY-type” theorem. I know of no obstacle to the proof of the full Araki/Yanase statement [4], but also I have not thought through what might be involved.

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