BECK-TYPE IDENTITIES: NEW COMBINATORIAL PROOFS
AND A THEOREM FOR PARTS CONGRUENT TO $t \mod r$

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ABSTRACT. Let $O_r(n)$ be the set of $r$-regular partitions of $n$, $D_r(n)$ the set of partitions of $n$ with parts repeated at most $r - 1$ times, $O_{1,r}(n)$ the set of partitions with exactly one part (possibly repeated) divisible by $r$, and let $D_{1,r}(n)$ be the set of partitions in which exactly one part appears at least $r$ times. If $E_{r,t}(n)$ is the excess in the number of parts congruent to $t \mod r$ in all partitions in $O_r(n)$ over the number of different parts appearing at least $t$ times in all partitions in $D_r(n)$, then $E_{r,t}(n) = |O_{1,r}(n)| = |D_{1,r}(n)|$. We prove this analytically and combinatorially using a bijection due to Xiong and Keith. As a corollary, we obtain the first Beck-type identity, i.e., the excess in the number of parts in all partitions in $O_r(n)$ over the number of parts in all partitions in $D_r(n)$ equals $(r - 1)|O_{1,r}(n)|$ and also $(r - 1)|D_{1,r}(n)|$. Our work provides a new combinatorial proof of this result that does not use Glaisher’s bijection. We also give a new combinatorial proof based of the Xiong-Keith bijection for a second Beck-Type identity that has been proved previously using Glaisher’s bijection.

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1. INTRODUCTION

Let $n$ be a non-negative integer. A partition $\lambda$ of $n$ is a non-increasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ that add up to $n$, i.e., $\sum_{i=1}^{\ell} \lambda_i = n$. The numbers $\lambda_i$ are called the parts of $\lambda$ and $n$ is called the size of $\lambda$. The number of parts of the partition is called the length of $\lambda$ and is denoted by $\ell(\lambda)$.

We will also use the exponential notation for parts in a partition. The exponent of a part is the multiplicity of the part in the partition. For example, $(5^2, 4^3, 3^1, 2^1)$ denotes the partition $(5, 5, 4, 3, 3, 3, 1, 1)$. Mostly, we will use the exponential notation when referring to rectangular partitions, i.e., partitions in which all parts are equal. Thus, we write $(m^i)$ for the partition consisting of $i$ parts equal to $m$.

The Ferrers diagram of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is an array of left justified boxes such that the $i$th row from to top contains $\lambda_i$ boxes. For example, the Ferrers diagram of the partition $(5, 5, 3, 3, 2, 1)$ is shown below.
We define several operations on partitions. Given partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$, we define partitions $\lambda \cup \mu$, $\lambda + \mu$, and $\lambda \mu$.

The partition $\lambda \cup \mu$ is the partition whose parts are precisely the parts of $\lambda$ and $\mu$, i.e., $\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$, arranged in non-increasing order.

The partition $\lambda + \mu$ is the partition $(\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots, \lambda_k + \mu_k)$, where $k = \max(\ell(\lambda), \ell(\mu))$ and, if $\ell(\lambda) < k$ or $\ell(\mu) < k$, the respective partition is padded with parts equal to 0.

If $\ell(\mu) \leq \ell(\lambda)$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell(\lambda)$, we define the partition $\lambda - \mu$ as the partition $(\lambda_1 - \mu_1, \lambda_2 - \mu_2, \ldots, \lambda_k - \mu_k)$, where, if $\ell(\mu) < \ell(\lambda)$, the partition $\mu$ is padded with parts equal to 0, i.e., $\mu_{\ell(\mu)+1} = \cdots = \mu_{\ell(\lambda)} = 0$.

For a non-negative integer $n$, a composition $\alpha$ of $n$ is a sequence of positive integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ that add up to $n$. Thus $(3, 2, 3, 1)$ and $(3, 1, 3, 2)$ are different compositions of 9. The sum of compositions is defined analogous to the sum of partitions.

Throughout the article, we make use of the following notation.

$O_r(n)$ is the set of $r$-regular partitions of $n$, i.e., partitions in which no part is divisible by $r$.

$D_r(n)$ is the set of partitions in which no part appears more than $r - 1$ times.

$F_r(n)$ is the set of $r$-flat partitions of $n$, i.e., partitions $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that for $1 \leq i \leq k$ we have $\lambda_i - \lambda_{i+1} \leq r - 1$. Here, we set $\lambda_{k+1} = 0$. We refer to $\lambda_i - \lambda_{i+1}$ as a difference of consecutive parts.

$O_{1,r}(n)$ is the set of partitions in which the set of parts divisible by $r$ has exactly one element (i.e., there is one part divisible by $r$, possibly repeated).

$D_{1,r}(n)$ is the set of partitions in which exactly one part appears at least $r$ times.

$F_{1,r}(n)$ is the set of partition in which exactly one difference of consecutive parts is at least $r$ and all other differences of consecutive parts are at most $r - 1$.

The notation is meant to remind the reader that the partitions in a set with subscript 1, $r$ have a single violation of the rule describing the partitions in the corresponding set with subscript $r$.

For $1 \leq t \leq r - 1$, we denote by $E_{r,t}(n)$ the excess in the number of parts congruent to $t \pmod{r}$ in all partitions in $O_r(n)$ over the number of different parts that appear at least $t$ times in a partition, counted in all partitions in $D_r(n)$.

Given a partition $\lambda$, let $\ell_t(\lambda)$ be the number of parts congruent to $t \pmod{r}$ in $\lambda$, and let $\overline{\ell}_t(\lambda)$ be the number of different parts that appear at least $t$ times in $\lambda$ (each counted with multiplicity 1). Then

$$E_{r,t}(n) = \sum_{\lambda \in O_r(n)} \ell_t(\lambda) - \sum_{\lambda \in D_r(n)} \overline{\ell}_t(\lambda).$$

In [5], George Beck conjectured a companion identity to Euler’s partition identity. Recall that Euler’s partition identity states that

$$|O_2(n)| = |D_2(n)|.$$

Beck conjectured that

$$|O_{1,2}(n)| = |D_{1,2}(n)| = b(n),$$

where $b(n)$ is the difference between the number of parts in all partitions in $O_2(n)$ and the number of parts in all partitions in $D_2(n)$. Andrews proved these identities in [1] using generating functions. Since then, in a fairly short time, many articles appeared giving generalizations of this result as well as combinatorial proofs in
A THIRD BECK-TYPE IDENTITY 3

many cases. See for example [7, 13, 3, 8, 9, 10, 2, 4]. Some authors have started referring to these companion identities as Beck-type identities.

Some of the earlier generalizations [13] gave companion identities to Glaisher’s identity

\[(2) \quad |O_r(n)| = |D_r(n)|.\]

The Beck-type identity is

\[(3) \quad |O_{1,r}(n)| = |D_{1,r}(n)| = \frac{1}{r-1} b_r(n),\]

where \(b_r(n)\) is the difference between the number of parts in all partitions in \(O_r(n)\) and the number of parts in all partitions in \(D_r(n)\), i.e.,

\[b_r(n) = \sum_{\lambda \in O_r(n)} \ell(\lambda) - \sum_{\lambda \in D_r(n)} \ell(\lambda).\]

We refer to these identities as first Beck-type identities.

In [7], Fu and Tang gave two generalizations of (1). For one of the generalizations, Fu and Tang gave a combinatorial proof and, as a particular case, they obtained a combinatorial proof for

\[|O_{1,r}(n)| = |D_{1,r}(n)|.\]

So far, all combinatorial proofs of Beck-type identities rely on variations of Glaisher’s bijection used to prove (2).

The second generalization of (1) provided in [7], for which the authors give a proof using generating functions, is the following theorem.

**Theorem 1.1** (Fu-Tang). For all \(n \geq 0\) and \(r \geq 2\),

\[|O_{1,r}(n)| = |D_{1,r}(n)| = E_{r,1}(n).\]

In this article we give a more general theorem of which Theorem 1.1 is a particular first case. Our main theorem is given below. If \(t = 1\) we obtain the statement of Theorem 1.1.

**Theorem 1.2.** For all integers \(n, r, t\) with \(n \geq 0\), \(r \geq 2\) and \(1 \leq t \leq r-1\), we have

\[(4) \quad |O_{1,r}(n)| = |D_{1,r}(n)| = E_{r,t}(n).\]

We refer to (4) as a third Beck-type identity. We provide analytic and combinatorial proofs of the theorem. Our combinatorial proof uses a recent bijection of Xiong and Keith [12] for Glaisher’s identity (2). Their proof is a variant of a bijection due to Stockhofe [11].

Importantly, the first Beck-type identity (3) follows directly from Theorem 1.2. Thus, the work of this article provides a new combinatorial proof for (3) that does not use Glaisher’s bijection.

The article is organized as follows. In section 2, we use generating functions to prove Theorem 1.2. In section 3 we introduce Xiong and Keith’s bijection and give a combinatorial proof of Theorem 1.2. We also show combinatorially how (3) follows from our main theorem. Finally, in section 4, we give a new combinatorial proof of a second conjecture of George Beck [6] which was proved analytically in [1] and generalized in [13].
2. Analytic Proof of Theorem 1.2

The generating functions for $|\mathcal{O}_r(n)|$ and $|D_r(n)|$ are

$$
\sum_{n=0}^{\infty} |\mathcal{O}_r(n)|q^n = \prod_{n=0}^{\infty} \frac{1}{(1-q^n+1)(1-q^n+2)\ldots(1-q^n+r-1)} = \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n};
$$

$$
\sum_{n=0}^{\infty} |\mathcal{D}_r(n)|q^n = \prod_{n=1}^{\infty} (1+q^n+q^{2n}+\ldots+q^{(r-1)n}) = \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n}.
$$

The generating functions for $|\mathcal{O}_{1,r}(n)|$ and $|\mathcal{D}_{1,r}(n)|$ are

$$
\sum_{n=0}^{\infty} |\mathcal{O}_{1,r}(n)|q^n = \sum_{n=0}^{\infty} |\mathcal{D}_{1,r}(n)|q^n = \sum_{m=1}^{\infty} \frac{q^{mr}}{1-q^{mr}} \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n}.
$$

The generating function for the number of parts congruent to $t \pmod{r}$ in all partitions in $\mathcal{O}_r(n)$ is

$$
\frac{\partial}{\partial z} \bigg|_{z=1} \prod_{n=0}^{\infty} \frac{1}{(1-q^n+1)(1-q^n+2)\ldots(1-q^n+r-1)}.
$$

The generating function for the number of different parts that appear at least $t$ times in all partitions in $\mathcal{D}_r(n)$ is

$$
\frac{\partial}{\partial z} \bigg|_{z=1} \prod_{n=1}^{\infty} (1+q^n+q^{2n}+\ldots+q^{(t-1)n}+q^{(t-1)n}+\ldots+q^{(r-1)n}).
$$

Then

$$
\sum_{n=0}^{\infty} E_{r,t}(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n} \sum_{n=0}^{\infty} \frac{q^{rn+t}}{1-q^{rn+t}} - \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n} \sum_{n=1}^{\infty} \frac{q^t}{1-q^t} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.
$$

We have

$$
\sum_{n=0}^{\infty} q^{rn+t} = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} q^{rn+t} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} q^{rn+t} = \sum_{m=1}^{\infty} \frac{q^m}{1-q^m};
$$

therefore,

$$
\sum_{n=0}^{\infty} E_{r,t}(n)q^n = \prod_{n=1}^{\infty} \frac{1-q^n}{1-q^n} \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}.
$$

3. Combinatorial Proof of Theorem 1.2

Recall that the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is called $r$-flat if $\lambda_i - \lambda_{i+1} \leq r - 1$ for all $1 \leq i \leq \ell(\lambda) - 1$ and $\lambda_{\ell(\lambda)} \leq r - 1$. I.e., in an $r$-flat partition differences of consecutive parts as well as the smallest part are strictly less than $r$. To make explanations less cumbersome, we set $\lambda_{\ell(\lambda)+1} = 0$. As mentioned in the introduction, $\mathcal{F}_r(n)$ is the set of all $r$-flat partitions of $n$. Conjugation gives a bijection (and, in fact, an involution) from $\mathcal{F}_r(n)$ to $\mathcal{D}_r(n)$. 
Next, we introduce a beautiful bijection between the set of $r$-flat partitions and the set of $r$-regular partitions given by Xiong and Keith in [12]. We denote this transformation by $\xi : \mathcal{F}_r(n) \rightarrow \mathcal{O}_r(n)$ and for the remainder of the article we refer to $\xi$ as the Xiong-Keith bijection. This bijection will be an important building block in the combinatorial proof of Theorem 1.2.

Step 1. Let $(\mu, \nu)$ be a pair of partitions such that $\lambda = \mu \cup \nu$, $\nu = r\eta$ for some partition $\eta$, $\mu$ is $r$-flat, and removing any part of $\mu$ congruent to $0 \pmod{r}$ leaves a partition that is not $r$-flat. If $\mu$ is $r$-regular, let $\beta^* = \emptyset$ and go to step 3.

Step 2. Let $(\alpha, \beta)$ be a pair of partitions such that $\mu = \alpha \cup \beta$, $\alpha$ is $r$-regular and $\beta = r\gamma$ for some partition $\gamma$. For $1 \leq i \leq \ell(\alpha)$, let $u_i$ be the number of parts in $\beta$ that are less than $\alpha_i$. For $1 \leq i \leq \ell(\beta)$, let $v_i$ be the number of parts in $\alpha$ that are greater than $\beta_i$. Consider the partition $u = (u_1, u_2, \ldots, u_{\ell(\alpha)})$ and the composition $v = (v_1, v_2, \ldots, v_{\ell(\beta)})$. Let $\alpha^* = \alpha - ru$ and $\beta^* = \beta + rv$.

Step 3. Write the partition $\nu \cup \beta^*$ as $ra$ and define $\xi(\lambda) = \alpha^* + ra^* \in \mathcal{O}_r(n)$.

In [12], the authors prove that that $\sigma_1 \leq \ell(\alpha^*)$ and they show that $\xi$ is a bijection. Moreover, $\lambda$ and $\xi(\lambda)$ have the same number of parts congruent to $t \pmod{r}$.

In view of this discussion, $E_{r,t}(n)$ equals the number of parts congruent to $t \pmod{r}$ in all partitions in $\mathcal{F}_r(n)$ minus the number of differences of consecutive parts that are at least $t$ in all partitions in $\mathcal{F}_r(n)$. Given a partition $\lambda$, denote by $d_t(\lambda)$ the number of differences of consecutive parts of $\lambda$ that are at least $t$. Then

$$E_{r,t}(n) = \sum_{\lambda \in \mathcal{F}_r(n)} (\ell_t(\lambda) - d_t(\lambda)).$$

Note that it is possible for $\lambda \in \mathcal{F}_r(n)$ to have $\ell_t(\lambda) - d_t(\lambda) < 0$.

For example, if $r = 4$, $t = 2$ and $\lambda = (10, 7, 7, 5, 4, 3) \vDash 36$, we have $\ell_2(\lambda) = 1$ and $d_2(\lambda) = 3$ and thus $\ell_2(\lambda) - d_2(\lambda) = -2$.

When considering examples for fairly large $n$ and $r$, it is often easier to work with $r$-modular Ferrers diagrams.

**Definition 1.** The $r$-modular Ferrers diagram of a partition $\lambda$ is a diagram in which, if $\lambda_i = q_i r + s_i$ with $1 \leq s_i \leq r$, then the $i$th row has $q_i$ boxes filled with $r$ and the last box is filled with $s_i$. Note that, if $\lambda_i$ is not divisible by $r$, then $s_i$ is the remainder of $\lambda_i$ upon division by $r$. If $\lambda_i$ is divisible by $r$, then $s_i = r$.

**Example 1.** The 4-modular diagram of $\lambda = (10, 7, 7, 5, 4, 3)$ is

```
4 1 2
4 3
4 1
4
3
```

Before proving Theorem 1.2, we show combinatorially that the sets of partitions involved in the theorem are equinumerous with the partitions in $\mathcal{F}_{1,r}(n)$.

**Theorem 3.1.** For all $n \geq 0$, we have $|\mathcal{D}_{1,r}(n)| = |\mathcal{F}_{1,r}(n)|$ and $|\mathcal{F}_{1,r}(n)| = |\mathcal{O}_{1,r}(n)|$.

**Corollary 3.2.** For all $n \geq 0$, we have $|\mathcal{D}_{1,r}(n)| = |\mathcal{O}_{1,r}(n)|$.

**Proof of Theorem 3.1.** Conjugation is a bijection between $\mathcal{D}_{1,r}(n)$ and $\mathcal{F}_{1,r}(n)$. Thus $|\mathcal{D}_{1,r}(n)| = |\mathcal{F}_{1,r}(n)|$. 
Next, we adapt the Xiong-Keith bijection to obtain a bijection \( \varphi : \mathcal{F}_{1,r}(n) \rightarrow \mathcal{O}_{1,r}(n) \).

Begin with a partition \( \lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l) \) in \( \mathcal{F}_{1,r}(n) \). Then there is exactly one consecutive difference in \( \lambda \) that is greater than or equal to \( r \), say \( \lambda_i - \lambda_{i+1} \geq r \). Write \( \lambda_i - \lambda_{i+1} = rk + d \) where \( 0 \leq d < r \) and let \( \tilde{\lambda} = \lambda - ((rk)^i) \). Then \( \tilde{\lambda} \in \mathcal{F}_r(n - irk) \).

The partition \( \tilde{\lambda} \) is \( r \)-flat because all of the consecutive differences in \( \tilde{\lambda} \) are equal to the corresponding consecutive differences in \( \lambda \) except \( \tilde{\lambda}_j - \tilde{\lambda}_{j+1} = d < r \).

Using the Xiong-Keith bijection, we map \( \tilde{\lambda} \in \mathcal{F}_r(n - irk) \) to \( \tilde{\mu} = \xi(\tilde{\lambda}) \in \mathcal{O}_r(n - irk) \). Finally, let \( \mu = \tilde{\mu} \cup ((rk)^i) \), i.e., insert \( i \) parts equal to \( rk \) into \( \tilde{\mu} \). Set \( \varphi(\lambda) = \mu \).

Then \( \varphi(\lambda) \in \mathcal{O}_{1,r}(n) \). We illustrate the mapping \( \varphi \) in Example 2 below.

To obtain the inverse map, we simply reverse the process. Begin with \( \mu \in \mathcal{O}_{1,r}(n) \). Then there is one part of \( \mu \) that is divisible by \( r \). Suppose the part divisible by \( r \) is \( rk \) with \( k > 0 \) and it occurs \( j > 0 \) times in \( \mu \). Let \( \tilde{\mu} \) be the partition obtained from \( \mu \) by removing all \( j \) parts equal to \( rk \). Then \( \tilde{\mu} \in \mathcal{O}_r(n-jrk) \).

Using the inverse of the Xiong-Keith bijection, we map \( \tilde{\mu} \in \mathcal{O}_r(n-jrk) \) to \( \tilde{\lambda} = \xi^{-1}(\tilde{\mu}) \in \mathcal{F}_r(n - ijk) \). Finally, let \( \lambda = \tilde{\lambda} + ((rk)^i) \), i.e., add \( rk \) to each of the first \( j \) parts of \( \tilde{\lambda} \). Since \( \lambda_j - \lambda_{j+1} \geq r \), we have \( \lambda \in \mathcal{F}_{1,r}(n) \). Then \( \varphi^{-1}(\mu) = \lambda \). □

**Example 2.** Consider \( \lambda = (27, 24, 20, 15, 13, 10, 6, 5, 2) \in \mathcal{F}_{1,5}(122) \) with \( i = 3 \). We show the 5-modular diagram of \( \lambda \) below along with the highlighted cells that will be removed to obtain \( \tilde{\lambda} \).

\[
\lambda =
\begin{array}{ccccccccc}
5 & 5 & 5 & 5 & 2 \\
5 & 5 & 5 & 4 &   \\
5 & 5 & 5 & 3 &   \\
5 & 5 & 5 &   \\
5 & 1 &   \\
5 &   \\
5 &   \\
5 &   \\
\end{array}
\]

Then \( \lambda \) maps to \( \tilde{\lambda} = (22, 19, 15, 15, 13, 10, 6, 5, 2) \in \mathcal{F}_5(107) \) after the block removal.

As can be seen in [12, pg. 562-563], under the Xiong-Keith bijection, \( \tilde{\lambda} \) maps to \( \tilde{\mu} = \xi(\tilde{\lambda}) = (32, 24, 23, 16, 12) \in \mathcal{O}_5(107) \). Finally, add 3 parts of size 5 to \( \tilde{\mu} \) to obtain \( \mu \in \mathcal{O}_{1,5}(122) \).

\[
\mu =
\begin{array}{ccccccccc}
5 & 5 & 5 & 5 & 5 & 2 \\
5 & 5 & 5 & 4 &   \\
5 & 5 & 5 & 3 &   \\
5 & 5 & 5 & 1 &   \\
5 & 5 & 2 &   \\
5 &   &   \\
5 &   &   \\
\end{array}
\]

We are now ready to complete the combinatorial proof of Theorem 1.2.
Combinatorial Proof of Theorem 1.2. We prove that $E_{r,t}(n) = |F_{1,r}(n)|$. Then, Theorem 3.1 implies that $E_{r,t}(n) = |D_{1,r}(n)| = |O_{1,r}(n)|$.

Recall that $E_{r,t}(n)$ is the excess in the number of parts congruent to $t \pmod{r}$ in all partitions in $O_r(n)$ over the number of different parts that appear at least $t$ times in a partition, counted in all partitions in $D_r(n)$.

Denote by $O_{r,t}^*(n)$ the set of partitions in $O_r(n)$ with exactly one part congruent to $t \pmod{r}$ marked. Note that if $\lambda \in O_r(n)$ with $\lambda_i = \lambda_j \equiv t \pmod{r}$ and $i \neq j$, then the partition with with the part $\lambda_i$ marked is different from the partition with part $\lambda_j$ marked.

Denote by $\mathcal{F}_{r,t}(n)$ the set of partition in $F_r(n)$ with exactly one part overlined and part $\lambda_i$ may be overlined only if $\lambda_i - \lambda_{i+1} \geq t$ (where $\lambda_{i+1} = 0$ if $\lambda_i$ is the last part). Note that the overlining marks a consecutive difference greater than or equal to $t$. Via conjugation, the overlining marks the last occurrence of a part that is repeated at least $t$ times in the corresponding partition in $D_r(n)$.

Then

$$|O_{r,t}^*(n)| = \sum_{\lambda \in O_r(n)} \ell_t(\lambda) \quad \text{and} \quad |\mathcal{F}_{r,t}(n)| = \sum_{\lambda \in D_r(n)} \ell_t(\lambda).$$

To prove that $E_{r,t}(n) = |F_{1,r}(n)|$, we create a bijection between $O_{r,t}^*(n)$ and $\mathcal{F}_{r,t}(n) \cup F_{1,r}(n)$. We achieve this by creating bijections

$$\psi_1 : \mathcal{F}_{r,t}(n) \cup F_{1,r}(n) \rightarrow P_{r,t}(n)$$

and

$$\psi_2 : O_{r,t}^* \rightarrow P_{r,t}(n),$$

where

$$P_{r,t}(n) = \{(\mu, ((ar + t)i)) \mid \mu \in F_r(n - i(ar + t)), a \geq 0, i > 0\}.$$

To define $\psi_1$, start with $\nu \in \mathcal{F}_{r,t}(n) \cup F_{1,r}(n)$. Then we have two cases.

Case 1: $\nu \in \mathcal{F}_{r,t}(n)$. Suppose the overlined part is $\nu_i$. Then $\nu_i - \nu_{i+1} \geq t$. Let $\mu = \nu - (t^i)$. Note that $\mu$ is $r$-flat and $\mu_i - \mu_{i+1} < r - t$. Define $\psi_1(\nu) = (\mu, (t^i))$.

For example, if $\nu = (4,3,1) \in \mathcal{F}_{3,2}(8)$, then $\mu = (2,1,1)$ and $(t^i) = (2^i)$. We show the mapping below, highlighting the removed cells.

![Mapping](image1)

Case 2: $\nu \in F_{1,r}(n)$. Then there is a single consecutive difference $\nu_i - \nu_{i+1}$ that is greater than or equal to $r$. Write $\nu_i - \nu_{i+1} - t$ as $ar + d$ where $a \geq 0$ and $0 \leq d < r$. Then, $\nu_i - \nu_{i+1} = ar + t + d$. Let $\mu = \nu - ((ar + t)i)$. Note that $\mu$ is $r$-flat, and if $a = 0$, then $\mu_i - \mu_{i+1} \geq r - t$. Define $\psi_1(\nu) = (\mu, (ar + t)i)$.

For example, if $\nu = (5,2,1) \in F_{1,3}(8)$ and $t = 2$, then $\mu = (3,2,1)$ and $(t^i) = (2^i)$. We show the mapping below, highlighting the removed cells.

![Mapping](image2)
Since in case 1 we have $\mu_i - \mu_{i+1} < r - t$ and in case 2, if $a = 0$, $\mu_i - \mu_{i+1} \geq r - t$, it follows that $\psi_1(F_{r, t}(n)) \cap \psi_1(F_{1, r}(n)) = \emptyset$.

The inverse of $\psi_1$ maps $(\mu, (ar + t)) \in P_{r, t}(n)$ to $\nu = \mu + ((ar + t))$. If $a \neq 0$, then $\nu_i - \nu_{i+1} \geq r$ and $\nu \in F_{1, r}(n)$. If $a = 0$, then either $t \leq \nu_i - \nu_{i+1} < r$ and we overline $\nu_i$ to obtain $\nu \in F_{r, t}(n)$, or $\nu_i - \nu_{i+1} \geq r$ and $\nu \in F_{1, r}(n)$.

To define $\psi_2$, start with $\lambda \in O_{r, t}^*(n)$. Then there is one marked part of size $ar + t$ with $a \geq 0$. Suppose the marked part is the $i$th part of size $ar + t$. Let $\eta$ be the partition obtained from $\lambda$ by removing $i$ parts equal to $ar + t$ (including the marking). Then $\eta \in O_r(n - i(ar + t))$. Let $\mu = \xi^{-1}(\eta)$ be the image of $\eta$ under the inverse of the Xiong-Keith bijection. Then $\mu \in F_r(n - i(ar + t))$ and $(\mu, ((ar + t)^i)) \in P_{r, t}(n)$.

**Example 3.** Consider $\lambda = (32, 24, 23, 16, 12, 7, 7^*) \in O_{5, 2}^*(121)$.

\[
\begin{align*}
\lambda &= \begin{array}{cccccccc}
5 & 5 & 5 & 5 & 5 & 5 & 2 \\
5 & 5 & 5 & 5 & 4 \\
5 & 5 & 5 & 5 & 3 \\
5 & 5 & 5 & 1 \\
5 & 5 & 2 \\
5 & 2 \\
5^* & 2^* \\
\end{array} \\
\eta &= \begin{array}{cccccccc}
5 & 5 & 5 & 5 & 5 & 5 & 2 \\
5 & 5 & 5 & 5 & 4 \\
5 & 5 & 5 & 5 & 3 \\
5 & 5 & 5 & 1 \\
5 & 5 & 2 \\
\end{array} \\
\mu &= \begin{array}{ccccccc}
5 & 5 & 5 & 5 & 5 & 2 \\
5 & 5 & 5 & 4 \\
5 & 5 & 5 \\
5 & 5 & 3 \\
5 & 5 \\
5 & 1 \\
5 \\
2 \\
\end{array}
\end{align*}
\]

Then $\lambda \mapsto \eta = (32, 24, 23, 16, 12) \in O_5(107)$.

As can be seen in [12, pg. 562-563], under the Xiong-Keith bijection, $\eta \mapsto \mu = \xi^{-1}(\eta) = (22, 19, 15, 15, 13, 10, 6, 5, 2) \in F_5(107)$.

So $\lambda \mapsto (\mu, (7^2)) \in P_{5, 2}(121)$.
The inverse of $\psi_2$ maps $(\mu, ((ar + t)^i)) \in \mathcal{P}_{r,t}(n)$ to $\nu = \mu \cup ((ar + t)^i)$. Then, the partition obtained by marking the $i$th part equal to $ar + t$ in $\nu$ is in $\mathcal{O}_{r,t}(n)$.

Therefore, $|\mathcal{O}_{r,t}(n)| = |\mathcal{P}_{r,t}(n)| = |\mathcal{F}_{r,t}(n)|$, which finishes the combinatorial proof of the theorem.

Next, we show combinatorially that the first Beck-type identity (3) follows from Theorem 1.2. Therefore, we obtain a new combinatorial proof of (3).

**Corollary 3.3.** For all $n \geq 0$ and $r \geq 2$ we have

$$|\mathcal{O}_{1,r}(n)| = |\mathcal{D}_{1,r}(n)| = \frac{1}{r - 1} b_r(n).$$

**Proof.** We have

$$r^{-1} \sum_{t=1}^{r-1} E_{r,t}(n) = r^{-1} \left( \sum_{\lambda \in \mathcal{O}_r(n)} \ell_t(\lambda) - \sum_{\lambda \in \mathcal{D}_r(n)} \ell_t(\lambda) \right) = \sum_{\lambda \in \mathcal{O}_r(n)} \ell(\lambda) - \sum_{t=1}^{r-1} \sum_{\lambda \in \mathcal{D}_r(n)} \ell_t(\lambda).$$

Given a partition $\lambda \in \mathcal{D}_r(n)$, each part of $\lambda$ is counted in $\sum_{t=1}^{r-1} \sum_{\lambda \in \mathcal{D}_r(n)} \ell_t(\lambda)$ as many times as its multiplicity. Thus $\sum_{t=1}^{r-1} \sum_{\lambda \in \mathcal{D}_r(n)} \ell_t(\lambda) = \sum_{\lambda \in \mathcal{D}_r(n)} \ell(\lambda)$ and

$$r^{-1} \sum_{t=1}^{r-1} E_{r,t}(n) = b_r(n).$$

On the other hand, from Theorem 1.2,

$$r^{-1} \sum_{t=1}^{r-1} E_{r,t}(n) = (r - 1)|\mathcal{O}_{1,r}(n)| = (r - 1)|\mathcal{D}_{1,r}(n)|.$$
Proof. Denote by $O_r(n)$, respectively $D_r(n)$, the set of partitions in $O_r(n)$, respectively $D_r(n)$, with exactly one part overlined. Only the last occurrence of a part may be overlined. Then

$$|O_r(n)| = \sum_{\lambda \in O_r(n)} t(\lambda)$$

and

$$|D_r(n)| = \sum_{\lambda \in D_r(n)} t(\lambda).$$

Next, we create bijections between $O_r(n)$, $D_r(n)$ and $T_r(n)$ respectively, and certain sets of pairs of partitions $(\mu, (1^i))$, where $\mu$ is an $r$-flat partition.

Start with $\lambda \in O_r(n)$ and suppose the overlined part is equal to $i \not\equiv 0 \pmod{r}$. Let $\nu$ be the partition obtained from $\lambda$ by removing the overlined part. Define $\mu = \xi^{-1}(\nu) \in F_r(n-i)$. Set $\psi_o(\lambda) = (\mu, (1^i))$. This gives a bijection

$$\psi_o : O_r(n) \to A_o(n) := \{ (\mu, (1^i)) \mid \mu \in F_r(n-i), i \not\equiv 0 \pmod{r} \}.$$ 

Example 4. Consider $\lambda = (32, 24, 23, 16, 16, 12) \in O_5(123)$.

Then $i = 16$ and $\nu = (32, 24, 23, 16, 12) \in O_5(107)$.

As can be seen in [12, pg. 562-563], under the Xiong-Keith bijection, $\nu$ maps to $\mu = \xi^{-1}(\nu) = (22, 19, 15, 13, 10, 6, 5, 2) \in F_5(107)$. So $\lambda \mapsto (\mu, (1^{16})) \in A_o(123)$.

Similarly, start with $\lambda \in D_r(n)$ and suppose the overlined part is equal to $i$. Let $\nu$ be the partition obtained from $\lambda$ by removing the overlined part. Define $\mu = \nu'$, the conjugate of $\nu$. It follows that $\mu \in F_r(n-i)$ and $\mu_i - \mu_{i+1} < r - 1$. Set $\psi_d(\lambda) = (\mu, (1^i))$. This gives a bijection

$$\psi_d : D_r(n) \to A_d(n) := \{ (\mu, (1^i)) \mid \mu \in F_r(n-i), \mu_i - \mu_{i+1} < r - 1 \}.$$ 

Example 5. Consider $\lambda = (20, 20, 20, 17, 13, 10, 10, 3) \in D_5(123)$. 

Then $\lambda = \begin{array}{cccccccc} 5 & 5 & 5 & 5 & 5 & 2 \\ 5 & 5 & 5 & 5 & 4 \\ 5 & 5 & 5 & 5 & 3 \\ 5 & 5 & 5 & 5 & 1 \\ 5 & 5 & 5 & 5 & 1 \\ 5 & 5 & 2 \end{array}$
Then \( i = 20 \) and \( \nu = (20, 20, 17, 13, 10, 10, 10, 3) \in D_5(103) \). Under conjugation, \( \nu \) maps to \( \mu = \nu' = (8^3, 7^7, 4^3, 3^4, 2^3) \in F_5(103) \). So \( \lambda \mapsto (\mu, (1^{20})) \in A_d(123) \).

Finally, start with \( \lambda \in T_r(n) \) and suppose that the part occurring more than \( r \) times but less than \( 2r \) times. Let \( \nu \) be the partition obtained from \( \lambda \) by removing \( r \) parts equal to \( j \). Let \( i = rj \). Define \( \mu = \nu' \), the conjugate of \( \nu \). It follows that \( \mu \in F_r(n - i) \) and \( 0 < \mu_j - \mu_{j+1} \). Set \( \psi_\ell(\lambda) = (\mu, (1^i)) \). This gives a bijection \( \psi_\ell : T_r(n) \to A_\ell(n) := \{(\mu, (1^i)) \mid \mu \in F_r(n - i), i \equiv 0 \pmod{r}, 0 < \mu_{i/r} - \mu_{(i/r)+1}\} \).

**Example 6.** Consider \( \lambda = (20, 17, 13, 10, 10, 10, 10, 10, 3) \in T_5(123) \).

\[
\lambda = \begin{array}{cccc}
5 & 5 & 5 & 5 \\
5 & 5 & 5 & 2 \\
5 & 5 & 3 \\
5 & 5 \\
5 & 5 \\
5 & 5 \\
5 & 5 \\
5 & 5 \\
3 \\
\end{array}
\]

Then \( j = 10, i = 5(10) = 50 \), and \( \nu = (20, 17, 13, 10, 10, 3) \in D_5(73) \). Under conjugation, \( \nu \) maps to \( \mu = \nu' = (6^3, 5^7, 3^4, 2^3, 1^3) \in F_5(73) \). So \( \lambda \mapsto (\mu, (1^{50})) \in A_\ell(123) \).

Our goal is to show that \( |A_\ell(n)| + |A_o(n)| = |A_d(n)| \). Notice that \( A_\ell(n) \cap A_o(n) = \emptyset \) and

\[
A_\ell(n) \cup A_o(n) = \{(\mu, (1^i)) \mid \mu \in F_r(n - i)\} \setminus \{(\mu, (1^i)) \mid \mu \in F_r(n - i), i \equiv 0 \pmod{r}, \mu_{i/r} - \mu_{(i/r)+1} = 0\}.
\]

Thus, to show that \( |A_\ell(n)| + |A_o(n)| = |A_d(n)| \), we need to show that the sets

\[
A := \{(\mu, (1^i)) \mid \mu \in F_r(n - i), \mu_i - \mu_{i+1} = r - 1\}
\]

and

\[
B := \{(\mu, (1^i)) \mid \mu \in F_r(n - i), i \equiv 0 \pmod{r}, \mu_{i/r} - \mu_{(i/r)+1} = 0\}
\]

are equinumerous.

We create a bijection \( \zeta : A \to B \) as follows. Start with \( (\mu, (1^i)) \in A \). Then \( \mu_j - \mu_{j+1} = r - 1 \). Let \( \nu = \mu - ((r - 1)^2) \). We have \( \nu_j - \nu_{j+1} = 0 \). Let \( \zeta((\mu, (1^i))) = (\nu, (1^{i/r})) \in B \).

Conversely, if \( (\nu, (1^i)) \in B \), then \( i = rj \) for some \( j > 0 \) and \( \nu_j - \nu_{j+1} = 0 \). Let \( \mu = \nu + ((r - 1)^2) \in F_r(n - j) \) and \( \mu_j - \mu_{j+1} = r - 1 \). Then \( \zeta^{-1}(\nu, (1^i)) = (\mu, (1^i)) \in A \).

This completes the combinatorial proof of Theorem 4.1.
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