Gravitational cubic interactions for a simple mixed-symmetry gauge field in AdS and flat backgrounds

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Abstract
Cubic interactions between the simplest mixed-symmetry gauge field and gravity are constructed in anti-de Sitter (AdS) and flat backgrounds. Non-Abelian cubic interactions are obtained in AdS following various perturbative methods including the Fradkin–Vasiliev construction, with and without Stückelberg fields. The action that features the maximal number of Stückelberg fields can be considered in the flat limit without loss of physical degrees of freedom. The resulting interactions in flat space are compared with a classification of vertices obtained via the antifield cohomological perturbative method. It is shown that the gauge algebra becomes Abelian in the flat limit, in contrast to what happens for totally symmetric gauge fields in AdS.

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1. Introduction

In the well-known papers [1, 2], the gravitational interaction problem (as well as self-interactions) for higher-spin gauge fields was solved at the first nontrivial order by going to a four-dimensional (anti-)de Sitter (AdS) background. This solved a longstanding problem and showed the importance of AdS backgrounds. Subsequently, these results led to the solution of the higher-spin interaction problem to all orders in interactions at the level of field equations in the seminal papers [3–5]. The results [1, 2] and [3–5] concern higher-spin gauge fields which, when described in the metric-like or Fronsdal formalism [6–9], are given by totally symmetric rank- tensors. For a review of the key mechanisms of higher-spin extensions of gravity, see [20] while various reviews on Vasiliev’s equations can be found in [21–23].

For recent works on cubic couplings among totally symmetric fields, see [10–19].
Mixed-symmetry gauge fields are neither totally symmetric nor totally antisymmetric ($p$-forms) and have first been described at the Lagrangian level around the flat background in [24–27]. For more recent works on mixed-symmetry fields in the constantly curved background, see [28–48] and references therein.

As far as the problem of finding consistent interactions for mixed-symmetry gauge fields is concerned, some analyses have been done in the flat background [49–52, 12, 14], but in the (A)dS$_d$ background almost nothing has been achieved apart from the very recent work [53] (see also the earlier works [54, 55]). In [56], the electromagnetic interactions of massive fields of the symmetry type studied in this paper have been studied in the St"uckelberg approach.

Generic irreducible mixed-symmetry gauge fields in AdS$_d$ are very different from their Minkowskian counterparts [57] in that they possess only one differential gauge symmetry associated with a single irreducible gauge parameter, not several at the same time like in flat spacetime. Gauge fields in AdS$_d$ can be described within the Alkalaev–Shaynkman–Vasiliev (ASV) approach [30] which is specific to AdS$_d$ and presents the advantage of being manifestly AdS$_d$-covariant with a minimal set of off-shell fields. However, the flat limit of the ASV formulation is not smooth in the sense of non-conservation of dynamical degrees of freedom.

There is always, however, the possibility of reinstalling all the differential gauge symmetries for a generic mixed-symmetry gauge field in AdS$_d$ at the price of adding extra fields, one for each supplementary differential gauge parameter, which can be shifted to zero at will provided the cosmological constant is nonvanishing [57], [41, 42, 45]. This is what we refer to as the St"uckelberg approach. Upon eliminating the extra fields of a St"uckelberg formulation, one arrives in AdS$_d$ at the ASV formulation.

It is the goal of this paper to study in details the cubic gravitational interaction problem in both flat and AdS$_d$ backgrounds for the simplest mixed-symmetry gauge field, i.e. one that is described in the metric-like formalism by a potential whose Young symmetry is $\Box$. Such a field will here be called hook, or [2, 1]-type, gauge field.

We will treat the gravitational interactions of the [2, 1]-type gauge field in AdS$_d$ using a modified $\frac{3}{2}$-order formalism and the Fradkin–Vasiliev approach. These techniques will be applied in turn to the St"uckelberg and ASV formulations. In flat spacetime, we will address the gravitational interaction problem in the metric-like formalism and using the cohomological reformulation [58] of the consistent deformation procedure [59].

The plan of the paper is as follows. After setting the notation and conventions, in section 2 we review the methods of investigating cubic interactions. We recall in section 3 some results about the free $[p, q]$-type gauge field in Minkowski and (A)dS$_d$ backgrounds and give an off-shell description of it in the frame-like St"uckelberg and ASV formalisms. Section 4 addresses the problem of cubic interactions in AdS$_d$. In section 4.1, we construct, from the St"uckelberg vantage point, consistent cubic gravitational interactions for the hook field in AdS$_d$ (where $d > 4$) which contain the usual Lorentz-covariant minimal coupling terms plus a finite sum of non-minimal terms, called ‘quasi-minimal’ [15, 20] in the context of totally symmetric gauge field in AdS$_d$. The St"uckelberg action obtained therein allows for a smooth flat limit. The full expressions for the gauge transformations of the fields at the first nontrivial order are explicitly given. In section 4.2, the St"uckelberg formulation is treated using the Fradkin–Vasiliev construction. Then, in section 4.3, we study the gravitational interactions within the ASV formulation. We show that it agrees with the results of section 4.2 upon partial gauge fixing of the St"uckelberg action. In section 5, using the cohomological reformulation of the N"other method [58, 60] and the results previously obtained in [50–52], we give the exhaustive list of cubic vertices in the flat background corresponding
to the set of fields used in sections 4.1 and 4.2. We make contact with the flat limit of the St"uckelberg action. In particular, we confirm that there are no possible non-Abelian vertices in flat space. In other words, switching on a cosmological constant enables one to deform an Abelian cubic action into a non-Abelian one, in sharp contrast to what happens for totally symmetric gauge fields [15] where the nature of the gauge algebra is not changed when going from AdS_d to the flat background. The conclusions are given in section 6. Finally, in the appendix we review the metric-like St"uckelberg formulation for the free hook field in AdS_d.

**Notation and conventions**

Base manifold indices, or world indices, are denoted by Greek letters $\mu, \nu, \ldots$, while Lorentz indices are denoted by lowercase Latin letters. The Lorentz algebra $\mathfrak{so}(d-1,1)$ is associated with the metric $\eta_{ab} = \text{diag}(+, -, \ldots, -)$ where the indices $a, b, \ldots$ run over the values $0, 1, \ldots, d-1$.

A group of $p$ totally antisymmetric Lorentz indices will be denoted by $[a_1 a_2 \ldots a_p] = a_1 a_2 \cdots a_p$. Moreover, square brackets indicate total antisymmetrization involving the minimal number of terms needed to achieve antisymmetrization. To further simplify the notation we will often use conventions whereby like letters imply complete antisymmetrization, e.g., for $\phi^{[2]}$.

The components of an irreducible $\mathfrak{gl}_p$ tensor whose symmetry type consists of a Young diagram with two columns, the first of length $p$ and the second of length $q$, will be denoted $\phi^{[p][q]}$. The Young symmetry described above is abbreviated by $\phi \sim [p, q]$. The components of a Lorentz tensor of type $[p, q]$ are denoted by $\phi^{[p, q]}$. Note that, by abuse of notation, we do not consider (anti) self-duality constraints on what we call Lorentz (or AdS) tensors in $d = 2n$, so that a Lorentz tensor, in our conventions, only obeys over-(anti)symmetrization and trace constraints. The torsion-free, Lorentz connection on the base manifold is denoted by $\nabla$. The background vielbein one-forms. It is useful to set $\lambda = \sqrt{|\Lambda|}$.

2. Cubic interactions

**Generalities.** Having a quadratic action $S_0[\Phi]$ that is invariant under some Abelian gauge transformations $\delta_0 \Phi$ as an input, one may look for interaction vertices by expanding the nonlinear corrections in powers of some formal coupling constant $g$:

$$S = S_0 + g S_1 + \cdots$$

$$\delta \Phi = \delta_0 \Phi + g \delta_1 \Phi + O(g^2).$$

The consistency condition $\delta S = 0$ at the leading nontrivial order, which corresponds to cubic vertices, implies that

$$\frac{\delta S_1}{\delta \Phi} \delta_0 \Phi + \frac{\delta S_0}{\delta \Phi} \delta_1 \Phi = 0.$$  \hspace{1cm} (2.2)

Noting that $\frac{\delta S_1}{\delta \Phi}$ is the left-hand side of the linear equations of motion, one can rewrite (2.2) as

$$\frac{\delta S_1}{\delta \Phi} \bigg|_{\delta S_0 = 0} \delta_0 \Phi = 0$$

$$\delta_1 \Phi = 0$$

\hspace{1cm} (2.3)

which is much easier to solve in practice. Given some solution $S_1$, the expression for the gauge transformations $\delta_1 \Phi$ can be extracted from (2.2). In general, $\delta_1 \Phi$ has a complicated form and is not needed for most purposes.
Auxiliary fields, 1st-, 3/2-formalisms. In the first-order formalism which is widely used in higher-spin theory, there are auxiliary fields which we denote collectively by $\Omega_1$. The auxiliary fields $\Omega_1$ can be expressed, modulo gauge transformations, in terms of physical fields $\Phi_1$. Splitting (2.2) in terms of $\Phi_1$ and $\Omega_1$ gives

$$\frac{\delta S_1}{\delta \Phi_1} \frac{\delta S_0}{\delta \Omega_1} + \frac{\delta S_0}{\delta \Phi_1} \frac{\delta S_1}{\delta \Omega_1} = 0.$$  

(2.4)

On the one hand, one can use the first-order formalism treating both $\Phi_1$ and $\Omega_1$ as independent fields. But this requires a lot of calculations including corrections to gauge transformations of the auxiliary field $\Omega_1$, which often turn out to be the most complicated ones. Alternatively, in the frame-like formulation of gravity and supergravity, there is the well-known 3/2-order formalism where one takes into account the variations of physical fields $\Phi_1$ only, all the calculations being done on the solutions of the complete algebraic equations for the auxiliary field $\Omega_1$.

$$\frac{\delta S_1}{\delta \Phi_1} \frac{\delta S_0}{\delta \Omega_1} + \frac{\delta S_0}{\delta \Phi_1} \frac{\delta S_1}{\delta \Omega_1} = 0.$$  

(2.5)

The advantage is that there is no need to consider the corrections $\delta \Omega_1$ to the $\Omega_1$ gauge transformations. However, one has to solve nonlinear equations for $\Omega_1$ and this can be highly nontrivial. In section 4.1, we use a modified 3/2-order formalism very well suited for the investigations of cubic vertices

$$\frac{\delta S_1}{\delta \Phi_1} \frac{\delta S_0}{\delta \Omega_1} + \frac{\delta S_0}{\delta \Phi_1} \frac{\delta S_1}{\delta \Omega_1} = 0.$$  

(2.6)

Here also there is no need to consider $\delta \Omega_1$ and we have to make all calculations on the solutions of free $\Omega_1$ field equations only. If one is not interested in $\delta \Phi_1$, then the equation (2.3) can be used to find $S_1$.

Fradkin–Vasiliev cubic interactions [2]. It is very convenient for the purpose of finding interactions to reformulate a field theory in the unfolded form [61]:

$$R^A = 0, \quad R^A = dW^A - F^A(W), \quad F^B \frac{\partial F^A}{\partial W^B} \equiv 0,$$  

(2.7)

so that (i) the fields $W^A$ form a set of differential forms of some degrees; (ii) all exterior derivatives of fields are expressed in terms of fields themselves; moreover, $F^A(W)$ is a function built with the use of the exterior product of fields only; (iii) $d^2 \equiv 0$ implies an integrability condition for $F^A(W)$. In other words the set $W^A$ is closed under the de Rham derivative and there are no other relations between $dW^A$ than those given by $F^A$. Then, (2.7) ensures the gauge symmetry

$$\delta W^A = d\xi^A + \xi^B \frac{\partial F^A}{\partial W^B},$$  

(2.8)

where the first term is absent if the form degree of $W^A$ is zero.

The anti-de Sitter background itself can be thought of as a part of the unfolded system of equations:

$$dh^a + \omega^a_{\ c} \wedge h^c = 0,$$  

(2.9)

$$d\sigma^a + \omega^a_{\ c} \wedge \sigma^c + \omega^c_{\ b} = \Lambda h^a \wedge h^b.$$  

(2.10)

In what follows, we will not use the full system of unfolded equations to describe a dynamical field of some spin, but only several Yang–Mills-like curvatures $R^A$ that are relevant for the cubic action as they contribute to the quadratic action. The strategy is as follows:
(1) for a required multiplet of gauge fields, for which we would like to investigate cubic interactions, one has to give unfolded curvatures $R_0^A$ that are linear in gauge fields,

$$R_0^A = DW^A + F_B^A(h)WB$$

(2.11)

but could be nonlinear in the background fields as manifested in $F_B^A(h)$. Note that $\sigma$ can appear only as a part of the Lorentz covariant derivative $D = d + \sigma$. The curvature $R_0^A$ can be read off from [10, 30, 39, 41, 42, 45, 62–66]. The indices $A, B, \ldots$ run over certain set of irreducible Lorentz tensors. As the number of gauge forms $W^A$ for some particular field is finite $R_0^A = 0$ cannot describe propagating fields. The way out is that not all of the curvatures $DW^A + F_B^A(h)WB$ are zero, some being proportional to zero-forms $C^a$, called generalized Weyl tensors:

$$R_0^A = F_A^a(h)C^a,$$

(2.12)

which are consistent unfolded equations provided that $DC^a$ satisfy their own equations. In what follows, the curvatures for $C^a$ are not needed;

(2) find a quadratic action of the form

$$S = \frac{1}{2} \sum_{A, B} \int I_{ABu \ldots} R_0^A \wedge R_0^B \wedge h^u \wedge \cdots \wedge h^u,$$

(2.13)

which is gauge invariant by construction. The $I_{ABu \ldots}$ are some invariant tensors built out of $\epsilon_{abu \ldots}$ and $\eta_{ab}$. The action contains some free coefficients, which are generally responsible for normalization of actions for individual constituents of the multiplet and for a freedom to add boundary terms;

(3) to extend the curvatures $R_0^A$,

$$R^A = DW^A + F_B^A(h)WB + \frac{1}{2} gF_B^A(h)WB \wedge WC$$

(2.14)

with terms quadratic in the fields while maintaining the integrability condition (2.7) to the order $g$, which implies

$$\delta R^A = \xi^C R_0^D \frac{\partial F^A}{\partial W^C \partial W^D} + O(g^2) = g \xi^C R_0^D F_{CD}^A + O(g^2),$$

(2.15)

where we have replaced $R^D$ with $R_0^D$ on the right-hand side as $F_{CD}^A$ has already brought in one power of $g$. The coefficients $F_{CD}^A$ are in fact the structure constants of some higher-spin algebra [67, 68], so that $R = dW + W \star W$. In this paper, we do not consider the full higher-spin algebra as we look for some particular cubic vertices and do not know $F_{CD}^A$ for all generators;

(4) to insert the corrected curvatures $R^A$ into the action instead of the linearized $R_0^A$ and to adjust free coefficients such that the action is gauge invariant to the order $g$:

$$\delta S = \sum_{A, B} \int I_{ABu \ldots} \delta R^A \wedge R^B \wedge h^u \wedge \cdots \wedge h^u$$

(2.16)

$$= \sum_{A, B} \int I_{ABu \ldots} \xi^C \wedge R_0^D F_{CD}^A \wedge R_0^B \wedge h^u \wedge \cdots \wedge h^u + O(g^2).$$

(2.17)

At this stage one may use a modified $\frac{1}{2}$-order formalism (2.6) or, if there is no need in finding $\delta_1 \Phi$, one can apply a modified (2.3),

$$\left[ \frac{\delta S_0}{\delta \Phi} + \frac{\delta S_1}{\delta \Phi} \right]_{\phi = 0} = 0,$$

(2.18)
where $\delta_1 = \delta_1^a + \delta_1^b$ and $F_{5C}^A$ are responsible for $\delta_1^b$, which appears naturally and makes the expression in brackets more simple (2.15). In this case, there are additional simplifications due to the fact that most of $R_{5A}$ are zero on-shell, a few being equal to Weyl tensors (2.12). Applying (2.12) one reduces (2.16) to
\[
\delta S = g \sum_{A,B} \int I_{AB;\ldots;}C^aC^bF_{5C}^A\varepsilon^C \wedge F_{5D}^B (h) \wedge F_{5E}^C (h) \wedge h^\mu \wedge \cdots \wedge h^n + O(g^2) = 0, \tag{2.19}
\]
which is a purely algebraic problem of adjusting free coefficients in order for various combinations of Weyl tensors to cancel each other.

Actually, in the original paper [2]5, the coefficients $F_{5C}^A$ were completely known for the multiplet of totally symmetric fields of spins $s = 0, 1, \ldots$ in contrast to this paper where we just probe some of $F_{5C}^A$ for the $\mathfrak{osp}(1|2)$ higher-spin algebra of [68] describing certain mixed-symmetry fields in addition to totally symmetric ones.

3. Type-$[p, q]$ gauge fields

In this section, we present a frame-like formulation for type-$[p, q]$ fields. The cases [1, 1] and [2, 1] will be treated in details in the rest of the paper.

**Minkowski space.** According to [38, 63], the unfolded formulation for the type-$[p, q]$ field in Minkowski space starts with two fields
type $[p, q]$: $e^a_p[q], \omega^a_{q[p+1]}$ \hspace{1cm} (3.1)
as a vielbein and spin-connection spin-$[p, q]$ field, where the subscripts on the above fields indicate their respective differential form degrees. One can construct linearized unfolded curvatures for these fields as
\[
R_{p+1}^{[q]} = \mathcal{D}_p e^a_p[q] - h_{[p-q+1]}^{a[p+1]} \omega^a_{q[p+1]}, \tag{3.2}
\]
\[
R_{q+1}^{[p+1]} = d\omega^a_{q[p+1]}, \tag{3.3}
\]
on-mass-shell the curvatures obey
\[
R_{p+1}^{[q]} = 0, \tag{3.4}
\]
\[
R_{q+1}^{[p+1]} = h_{m_{[q+1]}}^{a[p+1],m_{[q+1]}}, \tag{3.5}
\]
where $C_0^{a[p+1],m_{[q+1]}}$ is the Weyl tensor. The quadratic action has the form
\[
S_{p,q} = \frac{1}{2} \int \left( R_{p+1}^{[q]} \wedge \omega^{a[p+1]}_q + (-1)^{p+1} e^a_p [q] \wedge R_{q+1}^{[p+1]} h_{a[p+1]+1} \right), \tag{3.6}
\]
where we have used volume forms
\[
H_{a[k]} = \epsilon_{a_{1}a_{2}a_{3}\cdots a_{k}} h_{a_{k+1}} \wedge \cdots \wedge h^{b_{n}},
\]
which form a basis set of $(d - k)$ forms, resulting in the identity
\[
h^c \wedge H_{a_{1}a_{2}\cdots a_{k}} = \frac{(-1)^{c}}{(d - k + 1)} \sum_{i=1}^{k} (-1)^{c} \delta^{c}_{i} h_{a_{1}a_{2}\cdots a_{k}}. \tag{3.7}
\]

However, it turns out that this set of fields/curvatures does not admit a straightforward deformation to the anti-de Sitter space.

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5 For some reviews, see e.g. [21, 69, 22].
Stückelberg formulation. The BMV conjecture [57], proved in [41, 42, 47], states that an irreducible $\text{AdS}_d$ massless field decomposes into a direct sum of irreducible Minkowski massless fields in the flat limit $\Lambda \to 0$. For a unitary spin-[$p, q$] field the decomposition is

$$\text{AdS: } \text{unitary spin-}[p, q]|_{\Lambda \to 0} \cong \text{spin-}[p, q] \oplus \text{spin-}[p - 1, q] : \text{Minkowski.} \quad (3.8)$$

This can be understood as follows. A spin-[$p, q$] massless field in Minkowski space possesses two independent differential gauge symmetries with parameters having the types [[$p - 1, q$]] and [[$p, q - 1$]]. The first gauge symmetry gets broken in $\text{AdS}_d$; its role was to remove, from the components of the spin-[$p, q$] field, the spin-[$p - 1, q$] polarization. The spin-[$p, q - 1$] gauge parameter is still activated in $\text{AdS}_d$. Thus, if one wants the total number of physical degrees of freedom to be a conserved quantity in the limit $\Lambda \to 0$, then the spin-[$p - 1, q$] polarization has to decouple and will become an independent massless field. The above scenario is what one expects from representation theory: in the flat limit an irreducible representation of $\mathfrak{so}(d - 1, 2)$ becomes an $\mathfrak{so}(d - 1, 1)$-reducible one.

That $\mathfrak{so}(d - 1, 1)$ is not semisimple indeed results in a drastic difference between formulations in $\text{AdS}_d$ and Minkowski backgrounds. However, from a given field-theoretical formulation in a flat space in one higher dimension, one can obtain all the information one wishes about the corresponding massive or massless field in $\text{AdS}_d$ or flat backgrounds. This is a powerful technique, systematically applied in various cases by one of the authors [70, 71, 39, 66, 45, 72] and that can be used in order to derive the Lagrangian formulation of any given field (massive or massless, in $\text{AdS}_d$ or Minkowski): starting from the maximal Stückelberg description of a generically massive field in the $\text{AdS}_d$ background, one has at one’s disposal all the fields needed to describe its massless $m^2 \to 0$ and/or flat $\Lambda \to 0$ limit without losing any degrees of freedom. For example, given a totally symmetric spin-$s$ massive field, its massless limit produces a set of massless fields whose spectrum of spins is obtained by a dimensional reduction of length-$s$ one-row Young’s diagram [1, 1, ..., 1] of $\mathfrak{so}(d - 1)$ to various one-row diagrams of $\mathfrak{so}(d - 2)$ with fewer and fewer cells. Hence, in the simplest case of a spin-$s$ massive field, one may take a set of fields that is used to describe massless fields of spins 0, 1, ..., $s$. The massive Lagrangian is then a sum of Lagrangians of massless fields supplemented by various mixing terms with 1 and 0 derivatives. The procedure is quite natural and has already been tested both in Minkowski and $\text{AdS}_d$ for all fields whose spin is given by an arbitrary two-row Young diagram [45].

This idea was also mentioned in [73] and has been used [41, 42] in order to describe in a geometric way massive and massless $\text{AdS}_d$ fields of arbitrary symmetry type, starting from the dimensional reduction of the Minkowskian $(d + 1)$-dimensional geometric formulation [38, 63] developed by one of the authors. In that way, it was possible to reproduce and understand the Brink–Metsaev–Vasiliev (BMV) pattern of Stückelberg fields and their possible decouplings, at any given value of the mass parameter. For a very recent and interesting work related to [73], see e.g. [74, 75] and references therein.

The advantage (and definition) of the maximal Stückelberg formulation is that all limits (massless and/or flat) are smooth and all the fields have a gauge transformation. For a massive field in $\text{AdS}_d$, at critical values of $m^2$ certain mixing coefficients in the Lagrangian go to zero and the latter splits into two pieces, one describing a massless (or partially massless) field in $\text{AdS}_d$ which still has more degrees of freedom than the Minkowski massless field with the same spin. If, instead of choosing a critical value for $m^2$ in the maximal Stückelberg Lagrangian, one takes the limit $\Lambda \to 0$, then all the mixing coefficients go to zero and one obtains a direct sum of massless Lagrangians, one for each of the fields appearing in the initial Lagrangian.
Stückelberg formulation for type-[p, q] fields. According to (3.8) in order to construct Stückelberg formulation for the type-[p, q] field, one may take

\[
\text{type } [p-1, q] : \quad e^{a[q]}_{p-1} \quad \alpha q^{a[p]},
\]

(3.9)

\[
\text{type } [p, q] : \quad e^{a[q]}_{p} \quad \alpha q^{a[p+1]},
\]

(3.10)

The ansatz for curvatures contains all mixing terms and reads

\[
R^{a[q]}_{p} = D e^{a[q]}_{p-1} - h_{m[p-q]} \alpha q^{a[q]m[p-q]} - \alpha e^{a[q]}_{p},
\]

(3.11)

\[
R^{a[p]}_{q} = D e^{a[p]}_{q} - \beta h_{m} \alpha q^{a[p]m},
\]

(3.12)

\[
R^{a[q]}_{q+1} = D e^{a[q]}_{q+1} - \gamma p^2 h_{m} \alpha q^{a[q+1]},
\]

(3.13)

\[
R^{a[p+1]}_{q+1} = D e^{a[p+1]}_{q+1} - \delta h^a \omega a^n.
\]

(3.14)

The integrability condition (2.7) implies that

\[
\alpha = \Lambda \delta (1)^q, \quad \gamma = -\delta, \quad \beta = \Lambda \delta (1)^{p-1}.
\]

(3.15)

Choosing \( \delta \sim \sqrt{|\Lambda|} \equiv \lambda \), one can have a smooth flat limit \( \Lambda \to 0 \), with the four curvatures decoupling into two independent sets of (3.2)–(3.3). On-mass-shell we have

\[
R^{a[q]}_{p} = 0, \quad R^{a[p]}_{q} = h_{m[p-q]} C^{a[p]m[q]} + 1,
\]

(3.16)

\[
R^{a[q]}_{q+1} = 0, \quad R^{a[p+1]}_{q+1} = h_{m[p+1]} C^{a[p+1]} + 1,
\]

(3.17)

featuring two Weyl tensors in accordance with the BMV conjecture [57].

The action, which is valid both in Minkowski and AdS, is simply a sum of (3.6), where the curvatures are to be replaced with (3.11)–(3.14).

\[
S = S_{p-1,q} + \frac{-\Lambda}{(p+1)^2 \delta} S_{p,q},
\]

(3.18)

where the relative coefficient is fixed by gauge invariance. Note that the action does not have a manifestly gauge invariant form. This can be cured in AdS.

In anti-de Sitter space, one can indeed cast the action into a manifestly gauge invariant form

\[
S = a_1 \int R_{q+1}^{a[q+1]} d^{p-q-1} \wedge R_{q+1}^{a[p]} d^{a[p-q]} \wedge H_{a}^{2q+2} + a_2 \int R_{q+1}^{a[q+1]} d^{p-q} \wedge R_{q+1}^{a[p+1]} d^{a[p-q]} \wedge H_{a}^{2q+2} + a_3 \int R_{q+1}^{a[q]} \wedge R_{q+1}^{a[p]} d^{p+q+1} \wedge H_{a}^{2q+1} + a_4 \int D(R_{q+1}^{a[q+1]} d^{p-q-1} \wedge R_{q+1}^{a[p+1]} \wedge H_{a}^{2q+3}),
\]

(3.19)

where the last term is a boundary term and allows to set \( a_1/a_2 \) at will; then, we may put \( a_4 = 0 \). The action is manifestly gauge invariant. However, if one expands the first three terms, one finds contributions of the form \( \omega a^{a[q]d[p-q]} \wedge D \omega q^{a[p+1]} d_{(p-q)} H_{a}^{2q+1} \) that are of...
the third order in derivatives upon solving equations for \( \omega \) in terms of tetrad-like fields \( e \). The requirement for higher-derivative terms to vanish gives one constraint on \( a_1, a_2, a_3 \):

\[
- \frac{2(q + 1)}{d - 2q - 1} \left( a_1 \frac{\Lambda}{\delta} + a_2 (d - p - q - 1) \delta \right)
+ a_3 (-)^{p+q+1} \frac{(p + 1)! (d - p - q - 1)!}{(q + 1)! (d - 2q - 1)!} = 0.
\]

(3.20)

**Interlude.** One \( [76, 10] \) can collect \( \omega_{a,b} \) and \( h^a \) as different components of a single \( \mathfrak{s}(d - 1, 2) \)-connection \( \Omega^{AB} \equiv - \Omega^{B,A} \equiv \Omega_{\mu}^{A,B} dx^\mu, A, B, \ldots = 0 \ldots d \), where the decomposition of \( \Omega^{AB} \) into a \( d \times d \) antisymmetric matrix \( \sigma^{a,b} \) and a Lorentz vector \( h^a \) can be performed in a \( \mathfrak{s}(d - 1, 2) \)-covariant way by introducing a normalized vector field \( V^A, V^C V_C = 1 \), called compensator [10],

\[
H^A \lambda \equiv D_0 V^A, \quad \Omega_L^{AB} = \Omega^{AB} + \lambda (V^A H^B - H^A V^B),
\]

(3.21)

where \( D_0 = d + \Omega, (D_0)^2 = 0 \), which is equivalent to (2.9)–(2.10). The \( \mathfrak{s}(d - 1, 2) \)-covariant definitions for \( h^a \) and \( \omega_{a,b} \) are given by \( H^A \) and \( \Omega_L^{AB} \), with the Lorentz covariant derivative \( D = d + \Omega_L \), which is manifested in

\[
H^A \lambda = 0, \quad DV^A = 0, \quad DH^A = 0
\]

(3.22)

and the fact that, choosing \( V^A = \delta^A_{d} \) (\( d \) means the value of the index rather than an index), one recovers

\[
H^a_{\mu} = \Omega_{\mu}^{a,d}, \quad H^d = 0, \quad \Omega_L^{a,b} = \Omega^{a,b}.
\]

(3.23)

**Manifestly AdS-covariant formulation.** One can go further and construct a formulation for type-[\( p, q \)] fields in addition to being manifestly gauge invariant is also manifestly covariant under global symmetries of AdS [10, 30, 77, 32, 33].

One first notes that with the help of the gauge parameter \( \xi_{p-1}^{a[q]} \) of \( e_{[p]} \)

\[
\delta_{p-1}^{a[q]} = D e_{p-2}^{a[q]} + (-1)^{p-q} h_{[p-q]} e_{p}^{a[q][p-q]} + \alpha \xi_{p-1}^{a[q]}
\]

(3.24)

one can gauge away \( e_{p-1}^{a[q]} \) completely. It is obvious that \( a_{q-1}^{a[p]} \) and \( e_{p-1}^{a[q]} \) have the same number of components. In the gauge where \( e_{p-1}^{a[q]} = 0 \), the zero-curvature condition on (3.11) implies that

\[
h_{[p-q]} a_{q}^{[m][p-q]} - \alpha e_{p-1}^{a[q]} = 0,
\]

(3.25)

i.e. \( e_{p}^{a[q]} \) is just an avatar for \( a_{q}^{a[p]} \). The curvature (3.13) then does not carry any new information and can be abandoned to the benefit of (3.12). The resulting formulation is based on two fields \( a_{q}^{a[p]} \) and \( a_{q}^{a[p+1]} \) with the unfolded curvatures of the form

\[
R_{q+1}^{a[p]} = D a_{q}^{a[p]} + \lambda h_{[p]} a_{q}^{a[p]},
\]

(3.26)

\[
R_{q+1}^{a[p+1]} = D a_{q}^{a[p+1]} - \lambda h_{a}^{a} a_{q}^{a[p]},
\]

(3.27)

where we have made the choice \( \delta = \sqrt{| \Lambda |} \equiv \lambda \). The action now reduces to three terms

\[
S = a_1 \int R_{q+1}^{a[p+1][p-q-1]} R_{q+1}^{a[p+1]} H_{a[2q+2]}
+ a_2 \int R_{q+1}^{a[p+1][p-q-1]} R_{q+1}^{a[p-q]} H_{a[2q+2]}
+ a_3 \int D (R_{q+1}^{a[p+1][p-q-1]} R_{q+1}^{a[p+2]} H_{a[2q+3]})
\]

(3.28)
with no restriction on $a_1, a_2, a_3$. The boundary term again serves [30, 77] as a tool to adjust $a_1/a_2$ at will. The on-mass-shell condition (3.16)–(3.17) reduces to

$$R_{q+1}^{[p]} = h_{m[q+1]}C^{[p],[m][q+1]},$$

(3.29)

$$R_{q+1}^{[p+1]} = h_{m[q+1]}C^{[p+1],[m][q+1]}.$$  

(3.30)

Now one can realize (3.26)–(3.27) as two projections $h_{q+1}^{[p+1]}$ and $h_{q+1}^{[p+1]}$ of a single curvature $R_{q+1}^{[p+1]} = D_0 W_{q}^{[p+1]}$ for a generalized $\mathfrak{so}(d - 1, 1)$-connection $W_{q}^{[p+1]}$, which has an analogous decomposition into two generalized Lorentz connections to be identified with $\omega_{q}^{[p]}$ and $\omega_{q}^{[p+1]}$. The action can also be rewritten in a $\mathfrak{so}(d - 1, 2)$-covariant form [77, 30]

$$S = a_1 \int R_{q+1}^{[p+1]} U^{[p+1]} M_{V_M} \wedge R_{q+1}^{[p+1]} U^{[p+1]} A_{[p+1]} \wedge H_{U[2q+2]}$$

$$+ a_2 \int U^{[p]} M_{V_M} \wedge R_{q+1}^{[p+1]} U^{[p]} A_{[p]} \wedge H_{U[2q+2]}$$

$$+ a_3 \int D_0 (R_{q+1}^{[p]}) U^{[p]} M_{V_M} \wedge R_{q+1}^{[p]} U^{[p]} A_{[p-1]} \wedge H_{U[2q+3]}$$

(3.31)

with

$$H_{U[k]} = e_{U_1...U_{k-1}B_{12}...B_{k}} H_{B_{12}} \wedge \cdots \wedge H_{B_{k}W}.$$  

Such formulation in terms of generalized connections of the anti-de Sitter algebra is referred to as the ASV formulation due to [30], where it was introduced first, see [32, 33, 41, 42, 64, 46] for developments and generalizations. Within the ASV formulation a set of frame-like fields is organized in a compact way as various projections of a single generalized $\mathfrak{so}(d - 1, 2)$-connection. However, this is achieved at the price of losing Stückelberg symmetries and associated fields that make the flat limit smooth. Therefore, ASV formulation is restricted to AdS and has a singular flat limit, with the Van Dam–Veltman–Zakharov-like discontinuity in the number of physical degrees of freedom.

4. Gravitational interactions of type-\([2, 1]\) fields in AdS

In this section, we will present three different ways of constructing gravitational interactions for the \([2, 1]\)-type gauge fields in AdS$_d$.

4.1. Stückelberg formulation and $\frac{3}{2}$-approach

Firstly we apply a modified $\frac{3}{2}$-formalism in the presence of a maximal set of Stückelberg fields.

Kinematics. In accordance with section 3, we will use the following fields for the description of hook: two-form $\Phi^a$ and one-forms $\Omega^{a[3]}$, $\Omega^{a[2]}$ and $f^a$, leaving notations $e^a$ and $\omega^{a[2]}$ for the description of the graviton. In this notation, the free Lagrangian for a hook in AdS can be written as follows:

$$\mathcal{L}_0 = \left[ -\frac{1}{2} \Omega^{a[3]} h_{m[n]} \Omega^{[m[2]} + \Omega^{a[3]} D \Phi^a \right] H_{d[4]}$$

$$+ \left[ \frac{1}{4} \Omega^{a[2]} h_{m[n]} \Omega^{[m]} - \Omega^{a[2]} D f^a \right] H_{d[3]} + m [h_{m[n]} \Omega^{a[2]} f^a + \Omega^{a[2]} \Phi^a] H_{d[3]},$$

(4.1)

where $m^2 = 3\lambda^2$. It is invariant under the following set of gauge transformations:

$$\delta_0 \Phi^a = D e^a + h_{m[n]} \Phi^a m^m + \frac{m}{3} h^m h_m^m \xi^m,$$

(4.2)
\[ \delta_0 \Omega^{[3]} = D \chi^{[3]} + \frac{m}{3} h^a \chi^{[2]}, \quad (4.3) \]
\[ \delta_0 f^a = D \xi^a + h_m \chi^{ma} + m z^a, \quad (4.4) \]
\[ \delta_0 \Omega^{[2]} = D \chi^{[2]} - m h_m \chi^{ma[2]} . \quad (4.5) \]

Correspondingly, we can construct four gauge invariant objects (Yang–Mills-like curvatures)\(^6\):

\[ R^a = D \Phi^a - h_m h_m \Omega^{[2]a} - \frac{m}{3} h^a h_m f^m , \quad (4.6) \]
\[ R^{[3]} = D \Omega^{[3]} + \frac{m}{3} h^a \Omega^{[2]a} , \quad (4.7) \]
\[ K^a = D f^a + h_m \Omega^{ma} - m \Phi^a , \quad (4.8) \]
\[ K^{[2]} = D \Omega^{[2]} - m h_m \Omega^{ma[2]} . \quad (4.9) \]

They satisfy the following differential identities:

\[ D R^{[3]} = - \frac{m}{3} h^a K^{[2]a} , \quad DR^a = - h_m h_m R^{[2]a} - \frac{m}{3} h^a h_m K^m , \quad (4.10) \]
\[ D K^{[2]} = m h_m R^{[2]a} , \quad DK^a = - h_m K^{ma} - m R^a . \quad (4.11) \]

Note here that on the solutions of the equations for the auxiliary fields \( \Omega^{[3]} \) and \( \Omega^{[2]} \), we have

\[ R^a = 0 , \quad K^a = 0 \implies h_m h_m R^{[2]a} = 0 , \quad h_m K^{ma} = 0 . \quad (4.12) \]

**Minimal interactions.** To illustrate how our modified formalism work let us begin with the free Lagrangian for a massless hook in a flat spacetime and corresponding initial gauge transformations, where now \( D^2 = 0 \),

\[ \mathcal{L}_0 = \frac{1}{2} \left[ - h_m h_m \Omega^{[2]a} \right] \Omega^{[3]} + 2 \Omega^{[3]} D \Phi^a \mathcal{H}^{[4]} . \quad (4.13) \]

\[ \delta_0 \Omega^{[3]} = D \chi^{[3]} , \quad \delta_0 \Phi^a = D z^a + h_m h_m \chi^{ma[2]} . \quad (4.14) \]

The most general ansatz for a cubic vertex with two derivatives has the form (here we put the gravitational coupling constant to 1)

\[ \mathcal{L}_1 = \left[ \alpha_1 \epsilon_m h_m \Omega^{[2]a[2]} \Omega^{[3]} + \alpha_2 \epsilon^a h_m \Omega^{[2]a[2]} \Omega_m^{[2]} - \Omega^{[3]} \omega^a \Phi^m \mathcal{H}^{[4]} \right] . \quad (4.15) \]

Let us consider gauge transformations for the graviton, first,

\[ \delta \epsilon^a = D \eta^{a} + h_m \eta^{ma} , \quad \delta \omega^{[2]} = D \eta^{[2]} . \quad (4.16) \]

Direct calculations show that the variation of \( \mathcal{L}_1 \) under the above transformations cancels provided that \( \alpha_1 = - \frac{1}{2} \) and \( \alpha_2 = - \frac{1}{2} \), together with appropriate corrections to gauge transformations

\[ \delta \Phi^a = \eta^{ma} \Phi^m + \eta_m h_m \Omega^{ma[2]} . \quad (4.17) \]

\(^6\) To make connection with (3.15), \( \alpha = m, \beta = m, \gamma = m/3, \delta = -m/3, p = 2, q = 1.\)
As we see, this vertex does not deform the gauge algebra. It may seem that we took too many derivatives, but we were not able to avoid this four-derivative vertex.

A few comments are in order.

Cubic vertices with four derivatives: vertex $\Omega^{[3]}\Omega^{[3]}R$. The most general ansatz for this vertex has the form

$$\mathcal{L}_{41} = [a_1 \Omega^{[2]}_m \Omega^{[2]}_n R^{[2]}_{mn} + a_2 \Omega^{[3]}_m \Omega^{[3]}_{mn} R^{[2]}_{n} + a_4 \Omega^{[3]}_m \Omega^{[3]}_{mn} R^{[2]}_{n} R^{[2]}_{m} + a_4 \Omega^{[2]}_m \Omega^{[2]}_{mn} R^{[2]}_{n} R^{[2]}_{m}] H_{[4]}.$$  (4.22)

However, due to the identity

$$0 = \Omega^{[3]}_m \Omega^{[3]}_{mn} R^{[2]}_{n} R^{[2]}_{m} H_{[4]} \sim [-3 \Omega^{[2]}_m \Omega^{[2]}_{mn} R^{[2]}_{n} + 2 \Omega^{[3]}_m \Omega^{[3]}_{mn} R^{[2]}_{n} R^{[2]}_{m}] H_{[4]},$$

the first and second terms are not independent. Let us put $a_2 = 0$. Then, using the on-shell identities $h_m h_m R^{[2]}_{mn} = 0$ and $h_m R^{[2]}_{mn} = 0$, the $\chi^{[3]}$ variation of the action can be cast into the form

$$\delta_1 \mathcal{L}_{41} = \left[ 2 \Omega^{[2]}_m \Omega^{[2]}_{mn} R^{[2]}_{n} R^{[2]}_{m} H_{[4]} \right],$$

This enforces $a_3 = 2a_4$, while the first term can be compensated by

$$\delta_1 \Phi^a = -\frac{2a_4}{3} \chi^{[2]}_m R^{[2]}_{nm}.$$  (4.23)

A few comments are in order.

- As we see, this vertex does not deform the gauge algebra. It may seem that we took too many derivatives, but we were not able to avoid this four-derivative vertex.

- In all subsequent calculations it is crucial that the combination $(2a_1 + a_4)$ is non zero. Maybe the relation between these two parameters becomes clear in the first-order formalism, but for simplicity in what follows we put $a_4 = 0$.

At this stage we have to consider the AdS$_d$-covariantization of this vertex, taking into account that now $D^2 \neq 0$. There are two sources for non-invariance of cubic vertices in this case: terms proportional to $m$ in the definition of the curvature tensors and in the gauge transformations

$$\Delta R^{[3]} = \frac{m}{3} R^{[2]} \Omega^{[2]}, \quad \delta_1 \Omega^{[3]} = \frac{m}{3} R^{[2]} \chi^{[2]}$$

and this produces

$$\text{res}_{11} = \frac{2ma_1}{3} [-\chi^{[2]}_m (h_m \Omega^{[2]}_n + 2h^m \Omega^{[2]}_{nm}) + \Omega^{[2]}_m (h_m \chi^{[2]}_n + 2h^m \chi^{[2]}_{nm})] H_{[4]}.$$  (4.24)
Cubic vertices with four derivatives: vertex $\Omega^{[2]} \Omega^{[2]} R$. As far as we know, the only possible vertex with four derivatives and bilinear in the hook sector looks like
\[ \mathcal{L} = f^{a} K^{[2]} \Omega^{[2]} R^{[2]} H_{d[5]}. \]  
(4.25)

It is similar to the first nontrivial term in the decomposition of the Gauss–Bonnet invariant but now for two different spin-2 fields. By construction, such a vertex exists in $d \geq 5$ only. For the modified $\frac{1}{d}$-formalism, it is convenient to rewrite this vertex in the form $\Omega \Omega R$. So in what follows, we will use
\[ \mathcal{L}_{42} = a_{2} [\Omega^{am} \Omega^{am} R^{[2]} - \Omega^{[2]} \Omega^{am} R^{am} ] H_{d[4]}. \]  
(4.26)

As for the AdS-covariantization of this vertex, we again have two sources for non-invariance—terms proportional to $m$ in the curvature and gauge transformations
\[ \Delta K^{[2]} = -m h_{m} \Omega^{am} [2], \quad \delta \Omega^{[2]} = -m h_{m} \chi^{am} [2]. \]

This produces
\[ \text{res}_{12} = -ma_{2} [2 \Omega^{am} h_{m} \chi^{[2]} R^{[2]} + \Omega^{[2]} h_{m} \chi^{am} [2] R^{m} - h_{m} \chi^{am} [2] \Omega^{am} R^{m} ] H_{d[4]} + ma_{2} [2 \chi^{am} h_{m} \Omega^{am} [2] R^{[2]} + \chi^{am} h_{m} \Omega^{am} [2] R^{m} - h_{m} \Omega^{am} [2] \chi^{am} R^{m} ] H_{d[4]}. \]  
(4.27)

Vertices with three derivatives. As we have seen from the previous subsection, the four-derivative vertices produce contributions to the $\chi^{[3]}$, and $\chi^{[2]}$-variations only. It means that any variation under the $\zeta^{a}$- and $\xi^{a}$-transformations for the three-derivative vertex has to be compensated by corrections to gauge transformations only. This put severe restrictions on such vertices. The only one we have managed to find is
\[ \mathcal{L}_{3} = b_{0} \Omega^{[3]} f_{m} R^{am} H_{d[4]}, \quad \delta \Phi^{a} = -b_{0} R^{am} \xi_{m}. \]  
(4.28)

So we have only one new parameter $b_{0}$ to compensate for the non-invariance coming from the minimal interactions (4.19) and (4.21) on the one hand and the non-invariance coming from four-derivative vertices (4.24) on the other hand. Happily, with a heavy use of on-shell relations $h_{m} h_{m} R^{am} [2] = 0$ and $h_{m} K^{am} = 0$, one can show that all residual variations can be canceled provided we set
\[ a_{1} = -\frac{3}{4(d - 3)m^{2}}, \quad a_{2} = -\frac{3}{4m^{2}}, \quad b_{0} = \frac{1}{m}. \]  
(4.29)

and introduce important corrections to the gauge transformations
\[ \delta_{1} \eta^{a} = \frac{3}{2(d - 3)m} [ \chi^{am} [2] \Omega_{m[2]} - \Omega^{am} [2] \chi_{m[2]} ]. \]  
(4.30)

Thus, we finally obtain a non-trivial deformation of the gauge algebra. To summarize, we found the following cubic vertex and corresponding gauge transformations:
\[ \mathcal{L}_{1} = \left[ \frac{3}{2} c_{m} h_{m} \Omega^{am} [2] \Omega_{m[2]} - \frac{3}{2} c^{a} h_{m} \Omega^{am} [2] \Omega_{m[2]} - \Omega^{am} [2] \omega^{a} \Phi^{m} \right] H_{d[4]} + \frac{1}{m} \Omega^{am} [2] f_{m} R^{am} H_{d[4]} \]
\[ - \frac{3}{4 (d - 3) m^{2}} \Omega^{[2]} \Omega^{am} [2] H_{d[4]} \]
\[ - \frac{3}{4 m^{2}} [ \Omega^{am} \Omega^{am} R^{[2]} - \Omega^{[2]} \Omega^{am} R^{am} ] H_{d[4]} \]  
(4.31)

\[ \delta_{1} \Phi^{a} = \eta^{am} \Phi_{m} + \eta_{m} h_{m} \Omega^{am} [2] - c_{m} h_{m} \chi^{am} [2] - \omega^{am} z_{m} + \frac{1}{2 (d - 3) m^{2}} \chi^{am} R^{am} [2], \]  
(4.32)
\[ \delta_1 f^a = -\omega^a_{m} \xi^m + \eta^a_{m} f^m - e_m \chi^m + \eta_m \Omega^m, \quad (4.33) \]
\[ \delta_1 e^a = \frac{3}{2(d-3)m} [\chi^{am(2)} \Omega_{m(2)} - \Omega^{am(2)} \chi_{m(2)}]. \quad (4.34) \]

### 4.2. Stückelberg formulation and Fradkin–Vasiliev approach

In this section, we consider the application of the Fradkin–Vasiliev approach to the Stückelberg description of the hook field. First of all we have to rewrite the free Lagrangian in terms of gauge invariant curvatures. The result reads
\[ \mathcal{L}_0 = [a_1 R^{am(2)} R^{am(2)} m + a_2 K^{am(2)} K^{am(2)} + a_3 R^{am(3)} K^{am(3)}] H_{d(4)}, \quad (4.35) \]
where
\[ \frac{(d-4)a_1}{3} + a_2 = -\frac{3}{4m^2}, \quad a_3 = -\frac{1}{m}. \quad (4.36) \]
Again we see that there is an ambiguity in the choice of coefficients but the choice will be fixed after switching on interactions.

Now we have to construct deformed curvatures both for the hook field and for the graviton, so that the variations will be proportional to the free curvatures. For the graviton, the result is easy to find
\[ \hat{R}^{am(2)} = R^{am(2)} + ma_0 \left[ \Omega^{am(2)} \Omega_{m(2)} - \frac{2}{3} \Omega^{am(2)} \Omega_{m} \right], \quad (4.37) \]
\[ \hat{T}^a = T^a + a_0 \Omega^{am(2)} \Omega_{m(2)}, \quad (4.38) \]
with the corresponding variations having the form
\[ \delta \hat{R}^{am(2)} = ma_0 \left[ K^{am(2)} \chi_{m(2)} - \frac{2}{3} K^{am} \chi_m \right], \quad (4.39) \]
\[ \delta \hat{T}^a = a_0 \left[ -\chi^{am(2)} K_{m(2)} + R^{am(2)} \chi_{m(2)} \right]. \quad (4.40) \]

The deformations for the hook’s curvatures simply correspond to the standard Lorentz minimal coupling
\[ \hat{R}^a = R^a + \omega^{am} \Phi_m - h_m e_m \Omega^m - \frac{m}{3} (e^a h_m + h^a e_m) f^m, \quad (4.41) \]
\[ \hat{R}^{am(3)} = R^{am(3)} + \omega^{am} \Omega^{am(3)} + \frac{m}{3} \omega^{am} \Omega^{2}, \quad (4.42) \]
\[ \hat{K}^a = K^a + \omega^{am} f_m + e_m \Omega^m, \quad (4.43) \]
\[ \hat{K}^{am(2)} = K^{am(2)} + \omega^{am} \Omega^{am(2)} - m e_m \Omega^{am(2)}, \quad (4.44) \]

In what follows, we will only need the part of the variation that does not vanish on-shell. It has a simple form
\[ \delta \hat{R}^{am(3)} = R^a \chi_{m(2)}, \quad \delta \hat{K}^{am(2)} = R^a \chi_{m}, \quad \delta \hat{K}^a = R^{am} \xi_m. \quad (4.45) \]
Now, following the general procedure, we consider the interacting Lagrangian
\[ L_0 + L_1 = \left[ a_1 \tilde{R}^{[2]} m \tilde{R}^{[2]}_m + a_2 \tilde{K}^{[2]} m + a_3 \tilde{R}^{[3]} + \frac{1}{4 \lambda^2} \tilde{R}^{[2]} \tilde{R}^{[2]} \right] H_{d[4]} . \] (4.46)

The next problem is to adjust the coefficients so that all variations vanish on-shell. For the \( \chi^{[3]} \)-transformations, we obtain
\[ \delta_1 L_0 + \delta_0 L_1 = -4a_1 R^{[2]m} R^m \chi^{[2]}(\Omega^{[2]}) + \frac{3d_0}{m} R^{[2]m} R^{[2]} \chi^{[2]}(\Omega^{[2]}) H_{d[4]} . \]

Using the on-shell relations \( h_m h_m R^{[m]} = 0 \) and \( h_m R^{[m]} = 0 \), one can show that the following identity holds:
\[ [-2R^{[2]m} R^m \chi^{[2]}(\Omega^{[2]}) + R^{[m]} R^{[2]} \chi^{[2]}(\Omega^{[2]})] H_{d[4]} = 0 . \]

Thus, we have to put
\[ a_1 = \frac{3a_0}{2m} . \] (4.47)

At the same time, for the \( \chi^{[2]} \)-transformations we obtain
\[ \delta_0 L_0 + \delta_0 L_1 = \left[ 4a_2 K^{[2]m} R^m \chi^{[2]} + \frac{2d_0}{m} K^{[m]} R[k]_m \chi^{[2]} \right] H_{d[4]} . \]

Again, using the on-shell relations \( h_m K^{[m]} = 0 \) and \( h_m R^{[m]} = 0 \), one can show that
\[ [K^{[2]m} R^m + K^{[m]} R[k]_m] H_{d[4]} = 0 . \]

Therefore, we set
\[ a_2 = \frac{a_0}{2m} = \frac{a_1}{3} . \] (4.48)

In particular, the last relation fixes the ambiguity in the free Lagrangian giving us
\[ a_1 = - \frac{9}{4(d - 3)m^2} \Rightarrow a_0 = - \frac{3}{2(d - 3)m} . \] (4.49)

**Going from St"uckelberg to ASV.** We have already mentioned that the St"uckelberg formulation is related to the ASV one through the partial gauge fixing, see (3.24) and below. It is instructive to see how this procedure works in the interacting case. First of all, using the fact that for any non-zero \( \lambda \) we have \( \delta f^a \sim z^a \), we can choose the gauge \( f^a = 0 \). Then, the corresponding torsion equation
\[ \tilde{K}^a = h_m \Omega^ma - m \Phi^a + e^m \Omega^m = 0 \] (4.50)
gives us
\[ \Phi^a = \frac{1}{m} (h_m + e_m) \Omega^ma . \] (4.51)

As a consequence, the second torsion equation \( R^a = 0 \) does not carry any new information, leaving us with two non-trivial curvatures only
\[ \tilde{R}^{[3]} = D \Omega^{[3]} + \omega^a m \Omega^{md} + \lambda^2 (h^a + e^a) \Omega^{[2]} , \]
\[ \tilde{R}^{[2]} = D \Omega^{[2]} + \omega^a m \Omega^ma - (h_m + e_m) \Omega^{md} . \]

Here we have made the rescaling \( \Omega^{[2]} \to m \Omega^{[2]} \) and \( K^{[2]} \to m K^{[2]} \) in accordance with the fact that \( \Omega^{[2]} \) now plays the role of a physical field. After such a rescaling, the deformed Riemann tensor has the form
\[ \tilde{R}^{[2]} = R^{[2]} + m a_0 [\Omega^{[2]m} \Omega^a - \Omega^{[2]m} \Omega^a] = 2 \lambda^2 \Omega^{[2]m} \Omega^m , \]
while the interacting Lagrangian can be written as follows:
\[ L = a_1 [\tilde{R}^{[2]} m \tilde{R}^{[2]}_m + \lambda^2 \tilde{K}^{[2]} \tilde{K}^{[2]}] H_{d[4]} + \frac{1}{4 \lambda^2} \tilde{R}^{[2]} \tilde{R}^{[2]} H_{d[4]} , \]
which is to be compared with the genuine ASV action (4.70).

---

7 The deformed curvatures introduced above are associated with corresponding \( \delta, \) see section 2.
**Flat limit.** According to the general analysis of flat limit of higher-spin cubic actions in AdS$_d$ done in [15], one can always rescale the fields and dimensionful coupling constants in such a way that the flat limit of the AdS$_d$ action retains only the quadratic kinetic terms and the cubic highest derivative terms. Of course, the AdS$_d$-covariant derivatives are replaced by the flat partial ones. In our case, the cubic vertex that will survive in the flat limit is the four-derivative one given by (4.22), (4.25) that are Abelian. Note that there is always a freedom in adding total derivative terms and making field-redefinitions, cf (4.25) and (4.26).

### 4.3. ASV formulation and Fradkin–Vasiliev approach

In this section, we would like to test gravitational interactions for the simplest case of spin-2, 3, 4. 4.3. ASV formulation and Fradkin–Vasiliev approach

As recalled in section 3, the two fields \( \Omega^{[2]}_a, \Omega^{[3]}_a \) correspond to the one-forms needed to describe an irreducible and unitary [2, 1]-type gauge field in AdS$_d$ within the ASV formulation.

Quadratic corrections to curvatures are made by replacing the background tetrad \( h^a \) and Lorentz spin-connection \( \sigma^{a,b} \) with \( h^a + \epsilon^a \) and \( \sigma^{a,b} + \omega^{a,b} \), respectively. Quadratic contributions to the torsion and Riemann curvature are determined from the most general 1-type fields read, (3.16)–(3.17),

\[ T^a = D e^a + e_b \omega^{ba} - g \Omega^{[2]} \Omega [2], \]  
\[ R^{ab} = D \omega^{ab} + \sigma^{a,c} \sigma^{b,c} - \Lambda \delta e^a \delta e^b - 2 \Lambda g \Omega^{ac} \Omega^b - g \Omega^{[2]} \Omega [2], \]  
\[ K^{[2]} = D \Omega^{[2]} - e_b \Omega^{[2]} + \omega^{a,b} \Omega [2], \]  
\[ R^{[3]} = D \Omega^{[3]} + \Lambda \delta e^a \Omega^{[2]} + \omega^{a,b} \Omega [2], \]

The Yang–Mills-like gauge transformations are

\[ \delta e^a = d \xi^a + \xi_b \omega^{ab} - e_b \xi^{ab} + g \eta^{[2]} \Omega [2] - \xi \Omega^{[2]} \eta [2], \]  
\[ \delta \omega^{ab} = d \xi^{ab} - \xi^a \omega^c + \omega^{ac} \xi^b + \Lambda (\xi^a e^b - \xi^b e^a) + 2 \Lambda g \eta^{ac} \Omega^b - \Lambda g \eta^{ac} \Omega [2] + \eta^{[2]} \eta [2], \]  
\[ \delta \Omega^{[2]} = d \eta^{[2]} - \xi_b \Omega^{[2]} - e_b \eta^{[2]} + \omega^{a,b} \eta^{[2]}, \]  
\[ \delta \Omega^{[3]} = d \eta^{[3]} - \Lambda \xi^a \Omega [2] + \Lambda e^a \eta [2] - \xi_b \Omega [2] + \omega^{a,b} \eta [2], \]

and accordingly, for the curvatures:

\[ \delta T^a = \xi_b R^{ab} - T_b \xi^{ab} + g \eta^{[2]} K [2] - \xi \Omega^{[2]} \eta [2], \]  
\[ \delta R^{ab} = -\xi^a R^{bc} + R^{ac} \xi^b + \Lambda (\xi^a T^b - T^a \xi^b) + 2 \Lambda g \eta^{ac} K^b - 2 \Lambda g \eta^{ac} \Omega^b + \eta^{[2]} \eta [2], \]  
\[ \delta K^{[2]} = \xi_b R^{[2]} - T_b \eta^{[2]} - \xi^a \Omega^b - \eta^b \Omega^a + R^{a,b} \eta [2], \]  
\[ \delta R^{[3]} = -\Lambda \xi^a K^{[2]} + \Lambda T^a \eta^{[2]} - \xi^a R^{[2]} + R^{[2]} \eta^{[2]}, \]

The on-mass-shell conditions for [2, 1]-type fields read, (3.16)–(3.17),

\[ K^{[2]} = h_b h_b C^{[2]}, \quad R^{[3]} = h_b h_b C^{[3]}, \]
while the spin-2 sector gives the constraints
\[
T^{a}_{0} = 0, \quad R^{a[b]}_0 = h_{b} h_{a} \mathcal{W}^{a_{1} a_{2}} b_{b}, \tag{4.69}
\]
where the linearized quantities are indicated by calligraphic symbols. The so(d − 1, 1) tensors \(C^{a [2]}, b [2], C^{a [3]}, b [2], \mathcal{W}^{a [2]}, b [2]\) are irreducible tensors of symmetry types \([2, 2], [3, 2]\) and \([2, 2]\), respectively.

We take the following ansatz for the action
\[
S[e^{a}, \omega^{ab}, \Omega^{[2]}, \Omega^{[3]}] = \frac{1}{2} \int (R^{a} a \wedge R^{b} b + a_{1} K^{a} a \wedge K^{b} b + a_{2} R^{a} a \wedge R^{b} b \wedge H_{a a b}) \wedge H_{a a b}, \tag{4.70}
\]
where it is understood that the quartic terms are neglected at this order in perturbation. The variation of the above action can be evaluated using (4.64)–(4.67), keeping only terms bilinear in the fields and linear in the gauge parameter. In other words, after taking the gauge variation inside the action, the curvatures are replaced by their linearized expressions that are then constrained according to (4.68) and (4.69).

Denoting
\[
A = \int H_{d[0]} C_{a b, c d} \eta^{a_{1}} d \mathcal{W}^{a_{1} a_{2}}, \tag{4.71}
\]
\[
B = \int H_{d[0]} C_{a b, c d} \eta^{a_{2}} d \mathcal{W}^{a_{2} a_{3}}, \tag{4.72}
\]
\[
C = \int H_{d[0]} \mathcal{W}_{a_{2} a_{3} a_{4}} \xi^{a_{4}} \mathcal{W}^{a_{4} a_{2}}, \tag{4.73}
\]
the Fradkin–Vasiliev consistency condition gives the following constraint on the free parameters entering the action \(S[e^{a}, \omega^{ab}, \Omega^{[2]}, \Omega^{[3]}]\):
\[
[2 \Lambda g - a_{1}] A + \left[ g - \frac{a_{2}}{2} \right] B + \left[ \frac{a_{1}}{2} - \frac{a_{2} \Lambda}{2} \right] C = 0. \tag{4.74}
\]
This admits the solution
\[
a_{1} = 2 g \Lambda, \quad a_{2} = 2 g. \tag{4.75}
\]
We see that the ratio \(a_{1} / a_{2}\) is completely fixed by the consistency of action (4.70).

Within the manifestly AdS-covariant ASV formulation, the computations are basically the same, but one has to take into account a smaller number of terms as some of them join together into single AdS-covariant objects.

One important remark is that switching on gravitational interactions dictates the relative coefficient \(a_{1} / a_{2}\) in a way that manifestly AdS-covariant ASV action acquires the most simple form
\[
S = \frac{1}{2} \int R^{a_{1} a_{2}} a \wedge R^{a_{1} a_{2}} b \wedge H_{a a b}. \tag{4.76}
\]
Unfortunately, it is impossible to take a meaningful flat limit because of discontinuity in the number of physical degrees of freedom. Note that the two fields of ASV formulation correspond to fields similar to Lorentz spin-connection rather than tetrad-like fields, which can be excluded if \(\Lambda \neq 0\) as explained in section 3. Even at the linearized level, the action for \(\Omega^{a} a_{1}\) and \(\Omega^{a} a_{2}\) reduces to a nonunitary theory because no additional gauge symmetry reappears.
5. Gravitational interactions of the type-[2, 1] field in flat space: metric-like Stückelberg formulation and BV-BRST approach

In section 4.2, we have seen that, in the flat limit, no non-Abelian interaction could survive in the Stückelberg action. In this section, we discuss in the metric-like formalism, the flat limit of the interacting Lagrangian obtained in this paper by addressing the related problem of determining all the possible interactions that the hook field can have with gravity in flat space. We show that indeed it is not possible to build non-Abelian interactions in flat space, thereby strengthening the results of section 4.2.

The couplings will involve the following three types of gauge fields: a [2, 1]-type field $T_{\mu\nu,\rho} = -T_{\nu\mu,\rho}$ (we use the antisymmetric convention in this section), the graviton and the Stückelberg companion of the hook field which is of the same symmetry type as the graviton, i.e. in metric-like formalism it is a rank-2 symmetric gauge field. There are not many cubic terms that can consistently couple these fields in a flat background and we will show that none of them is compatible with the existence of a non-Abelian gauge algebra at the first nontrivial order, in sharp contrast to what happens for totally symmetric gauge fields. It means that it is definitely the cosmological constant that is responsible for the non-Abelian nature of the interactions we have presented in this paper.

In flat space, the problem of the self-interactions for arbitrary type-[p, q] gauge fields was thoroughly studied in [50–52] via the cohomological reformulation of the Nöther procedure for constructing consistent interactions [59]. For any [p, q]-type gauge fields in flat space, all the relevant cohomological groups have been computed in [50–52] to which we refer for more details.

Without entering too much into the details of the antifield formulation for [p, q]-type gauge fields in flat space [51, 52], we will give here a list of the various possible cubic couplings between a [2, 1]-type gauge field and a set of two different gravitons, the physical graviton and the Stückelberg companion of the [2, 1]-type gauge field. We will show that there is no way to build non-Abelian cubic vertices among the three species of fields considered here if there is at least one hook field occurring in the vertex. In the case of the cubic coupling between colored gravitons, it is a result of [78] that there is no nontrivial non-Abelian interactions mixing colored gravitons. Therefore, in the case of interactions between the Stückelberg field and the physical graviton, there is no possibility for non-Abelian interactions.

Sector of the mixed-symmetry gauge field. The spectrum of fields and antifields in the sector of the [2, 1]-type gauge field is given by

- the fields $T_{\alpha\beta,\gamma}$ with ghost number ($gh$) zero and antifield number ($antigh$) zero;
- the ghosts $S_{\alpha\beta} = S_{(\alpha\beta)}$ and $A_{\alpha\beta} = A_{(\alpha\beta)}$ with $gh = 1$ and $antigh = 0$;
- the ghosts of ghosts $B_\alpha$ with $gh = 2$ and $antigh = 0$, which appear because of the reducibility relations;
- the antifields $T^*_{\alpha\beta,\gamma}$, with ghost number minus one $gh = -1$ and $antigh = 1$;
- the antifields $S^*_{\alpha\beta}$ and $A^*_{\alpha\beta}$ with $gh = -2$ and $antigh = 2$;
- the antifields $B^*_{\alpha}$ with $gh = -3$ and $antigh = 3$.

Note that the antifield number is sometimes also called the ‘antighost number’. The BRST differential for the free theory takes the simple form

$$s = \delta + \gamma.$$  \hfill (5.1)
A grading is associated with each of these differentials: $γ$ increases by one unit the ‘pure ghost number’ denoted $puregh$ while the Koszul–Tate differential $δ$ increases the antighost number $antigh$ by one unit. The ghost number $gh$ is defined by

$$gh = puregh - antigh.$$  \hspace{1cm} (5.2)

The action of the differentials $γ$ and $δ$ on all the fields of the formalism is displayed in table 1 that indicates also the pureghost number, antighost number, ghost number and Grassmannian parity of the various fields.

| $Z$    | $γ(Z)$ | $δ(Z)$ | $puregh(Z)$ | $antigh(Z)$ | $gh(Z)$ | Parity |
|--------|--------|--------|-------------|-------------|---------|--------|
| $T_{αβγ}$ | $γT_{αβγ}$ | 0      | 0           | 0           | 0       | 0      |
| $S_{αβ}$   | $δS_{αβ}$  | 6$δθ_αB_{β|}$ | 1          | 0           | 1       | 1      |
| $A_{αβ}$   | $2δθ_αB_{β|}$ | 0      | 1           | 0           | 1       | 1      |
| $B_α$      | 0       | 0      | 2           | 0           | 2       | 0      |
| $T^∗_{αβγ}$ | $−16δθ_α^{[4]}K_{γ[3],[2]}$ | 0      | 1           | 0           | 1       | 1      |
| $S^{∗αβ}$ | $−2δθ_α^{[2]}T^∗_{γαβ,γ}$ | 0      | 2           | 0           | 2       | 0      |
| $A^{∗αβ}$ | $−6δθ_α^{[2]}T^∗_{γαβ,γ}$ | 0      | 2           | 0           | 2       | 0      |
| $B^{∗α}$ | $6δθ_αS^{∗αβ} + 2δθ_αA^{∗αβ}$ | 0      | 3           | −3         | 1       |        |

Following the results of [50, 52], in order to perturbatively deform the solution $W^{(0)}$ of the master equation for the free theory $(W^{(0)}W^{(0)})_{A,B} = 0$ (where $(·,·)_{A,B}$ denotes the antibracket) into $W = W^{(0)} + gW^{(1)} + \cdots$ with $W^{(1)} = \int d^3x \ (a_0 + a_1 + a_2 + a_3)$ where $gh(a_1) = 0$ and $antigh(a_1) = i$, one has to solve the cocycle equation $\alpha + dc = 0$ where $a = a_0 + a_1 + a_2 + a_3$ and $c = c_2 + c_1 + c_0$. The descent of equations coming from the decomposition of $\alpha + dc = 0$ with respect to the antifield number is

$$\delta a_1 + γ a_0 + dc_0 = 0,$$  \hspace{1cm} (5.6)

$$\delta a_2 + γ a_1 + dc_1 = 0,$$  \hspace{1cm} (5.7)

$$\delta a_3 + γ a_2 + dc_2 = 0,$$  \hspace{1cm} (5.8)

$$γ a_3 = 0.$$  \hspace{1cm} (5.9)

In order to solve this system, one starts with $a_3$ that must belong to the cohomological group $H(γ)$ and plug it into the equation $\delta a_3 + γ a_2 + dc_2 = 0$ that must be solved for $a_2$. If that is possible, one has thereby ‘lifted’ or ‘integrated’ $a_3$ to $a_2$. In case such an integration is not obstructed, one plugs $a_2$ into the next equation $\delta a_2 + γ a_1 + dc_1 = 0$ and tries to solve it for...
a₁. If a₂ can be lifted to an a₁, one finally tries to solve δa₁ + γa₂ + dc₀ = 0 for a₀ which is the vertex appearing in the deformed Lagrangian. The deformations of the gauge algebra appear in a₂ while a₁ contains the deformations of the gauge transformations. The element a₃ contains information about the deformation of the reducibility transformations.

Important ingredients for the construction of the various elements {ai}, i = 0, 1, 2, 3, are the following cohomological groups:

(i) H(γ), the cohomology of γ, is isomorphic to the algebra \{f([Kαaαβ], [Φ⁺], [Cμ, δhμCν]), [Φ⁺]}\} of functions of the generators, where H[αaαβ] = δhμAαβ, [Φ⁺] denotes collectively all the antifields and their derivatives and similarly [Kαaαβ] denotes the curvature tensor and all its derivatives.

(ii) The cohomology groups H[3](δ|d) vanish in the antifield number q strictly greater than three: H[3](δ|d) ≅ 0 for q > 3.

(iii) A complete set of representatives of H[2](δ|d) is given by the antifields Bμν conjugate to the ghost of ghosts Bμ, i.e. \(δa_2 + δb_2 = 0 \Rightarrow a_2 = λ_μ - B^{μν}dx^0 ∧ dx^1 ∧ \cdots ∧ dx^{d-1} + δb_2 = 0 \Rightarrow a_2 = λ_μ - C^{μν}dx^0 ∧ dx^1 ∧ \cdots ∧ dx^{d-1} + δb_3 + db_4\) where the λ_μ’s are constants.

(iv) The cohomological group H[1](δ|d) vanishes if one considers cochains a that have no explicit x-dependence (as it is necessary for constructing Poincaré-invariant Lagrangians).

**Sector of the graviton.** In the sector of the graviton fields, the relevant cohomological analysis was performed in [78]. On top of the spin-2 gauge field hμν, the BRST-BV spectrum includes ghost Cμ associated with the linearized diffeomorphisms together with the antifields hμν and Cμν. We have summarized the action of the various relevant differentials on these fields in table 2.

In this sector, the relevant cohomological groups are

(i) H(γ), the cohomology of γ, that is isomorphic to the algebra \{f([Kαaαβ], [Φ⁺], [Cμ, δhμCν]), [Φ⁺], δhμCν)]\} of functions of the generators;

(ii) H[2](δ|d) ≅ 0 for q > 2;

(iii) H[3](δ|d) given by the antifields Cμν conjugate to the ghosts Cμ, i.e. \(δa_2 + δb_2 = 0 \Rightarrow a_2 = λ_μ - C^{μν}dx^0 ∧ dx^1 ∧ \cdots ∧ dx^{d-1} + δb_2 = 0 \Rightarrow a_2 = λ_μ - C^{μν}dx^0 ∧ dx^1 ∧ \cdots ∧ dx^{d-1} + δb_3 + db_4\) where the λ_μ’s are constants.

**Couplings [1, 1]–[1, 1]–[1, 1].** As we said above, there are no non-Abelian vertices mixing nontrivially several kinds of gravitons, therefore the only couplings we can introduce are the Born–Infeld coupling with six derivatives (that does not appear in the analysis in AdSp), and the following four-derivative coupling in d ≥ 5:

\[ a₀ \sim \delta \phi^{[5]}_d hʒ^a K^{[2]}_d K^{[2]}_d \]  

(5.10)
in four derivatives and it contributes via (4.25) to making the AdS$_d$ non-Abelian vertex in the St"uckelberg formulation.

**Couplings [1,1]–[1,1]–[2,1].** There is a candidate deformation with five derivatives between the hook field $T_{\mu[2],\nu}$ and two gravitons $h^i_{\mu\nu}$, $i = 1, 2$ (in the following we will omit the extra internal index $i$ for simplicity of notation):

$$a_0 = K_{\nu[1]}^\mu K_{\nu[2]}^\mu \partial \mu \partial \nu \epsilon_{\mu[6]} \epsilon^{[6]}$$

(5.11)

where we recall that $K_{\nu[1]}^\mu$ is the curvature tensor for $T_{\mu[2],\nu}$ and $K_{\nu[2]}^\mu$ is the linearized curvature tensor in the spin-2 sector. It is easy to check that this vertex is gauge invariant under linearized transformations, up to a total derivative.

With three derivatives involved, it can be seen that there is only one candidate associated with the following element of the cohomology of $\gamma$: $a_1 = T^*_{\gamma} K_{\mu[2],\nu}^\mu C^\nu$, where $C^\nu$ is the ghost associated with the linearized diffeomorphisms. The element $a_1 \in H(\gamma)$ encodes the information concerning a deformation of the gauge transformations for the hook field; a deformation that does not modify the gauge algebra which therefore remains Abelian. Explicitly, it corresponds to the transformation

$$\delta^{(1)}T_{\mu[1],\nu[1]} = K_{\mu[1],\nu[1]}^\nu \xi^\nu.$$  

(5.12)

To see whether this deformation of the gauge transformations can be integrated to a cubic vertex $a_0$, one has to solve the equation $\delta a_1 + \gamma a_0 = dc_0$ where $\delta$ is the Koszul–Tate differential and $\gamma$ is the differential that implements the gauge transformations. When computing $\delta a_1$, it is possible, up to total derivatives, to make a $\gamma$-exact term $\tilde{a}_0 = \delta^{[4]}_{\mu[4]} \partial_\nu \Phi_{ab}^b K^c_{\mu[2]} T^c_\nu$ appear, but there remains a term that cannot be written as the $\gamma$-exact term up to the total derivative so that in flat spacetime, this vertex is not consistent.

Actually, looking at the classification found in [12], one sees that there is indeed only one vertex, bringing five derivatives, whereas in AdS$_d$, this three-derivative vertex plays an important role in making the non-Abelian interactions we presented above, see (4.28).

**Couplings [1,1]–[2,1]–[2,1].** In terms of the quantities $j^{(a)}_\sigma$, $a = 1, 2, 3$ and $\sigma = 1, 2$, relevant for formula (8.66) of Metsaev in [12] (that formula is reproduced below in 5.13), the coupling [1,1] – [2,1] – [2,1] corresponds to either (1,0) – ($\frac{3}{2}$, $\frac{1}{2}$) – ($\frac{3}{2}$, $\frac{1}{2}$), (1,0) – ($\frac{3}{2}$, $\frac{1}{2}$) – ($\frac{3}{2}$, $\frac{1}{2}$) or (1,0) – ($\frac{3}{2}$, $\frac{1}{2}$) – ($\frac{3}{2}$, $\frac{1}{2}$) since in $d=6$ the hook field receives two Gelfand–Zetlin labels $(s_1, s_2)$ with $|s_2| \leq s_1$, so that the hook field can have $s_2 = 1$ or $s_2 = -1$, depending on it being self-dual or anti-self-dual. In the second case (1,0) – (3/2, 1/2) – (1/2, 3/2), one has $J_1 = \sum_\sigma j^{(a)}_\sigma = 3$, $J_2 = 2$, $\min_\sigma j^{(a)}_1 = \frac{1}{2}$ and $\min_\sigma j^{(a)}_2 = 0$, so that Metsaev’s formula

$$2 \max_{\sigma = 1, 2} (J_\sigma - 2 \min_{a=1,2,3} j^{(a)}_\sigma) \leq k \leq 2 \min_\sigma J_\sigma$$  

(5.13)

gives the solution $k = 4$. The other two cases give no solution. On the other hand, we found the deformation $a_1 = T^*_{\gamma} K_{\mu[2],\nu}^\mu H^{[3]}$ where we recall that $H^{[3]}$ is the element of $H(\gamma)$ that corresponds to the curl of the antisymmetric gauge parameter $A_{[2]}$ for the gauge field $T$ that, in flat space, possesses two independent gauge transformations. This candidate $a_1$, again, does not lead to any non-Abelian algebra since the gauge fields appear through the curvature tensor $K_{\mu[2],\nu}^\mu$. This candidate $a_1$ is integrable and gives a vertex $a_0$ involving four derivatives—in

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9 See the discussion at the beginning of section 8.2 in [12]. Note that the restriction on $k$ in Metsaev’s formula (5.13) was found for the first time in [79]. We are grateful to Metsaev for his explanations and comments.
agreement with the $k = 4$ prediction of formula (5.13)—and that gives a contribution in $\text{AdS}_d$ corresponding to (4.22) and (4.23):

$$a_0 = K_{\mu_1 \mu_2, \nu_1 \nu_2} (\partial^{[\rho} T^{\mu_1 \mu_2]_{\sigma}}_{\rho} \partial^{[\sigma} T^{\nu_1 \nu_2]_{\rho}}_{\rho} - \partial^{[\rho} T^{\mu_1 \mu_2]_{\sigma}}_{\rho} \partial^{[\sigma} T^{\nu_1 \nu_2]_{\rho}}_{\rho}),$$  

(5.14)

$$\delta^{(1)} T_{\mu \nu, \rho} = K_{\mu \nu, \alpha \beta} \partial^{[\rho} A^{\alpha \beta]}.$$  

(5.15)

Using group theory\(^{10}\), it can be seen that this vertex is indeed nontrivial in $d = 6$.

There is yet another vertex, this time with six derivatives: the Born–Infeld vertex simply obtained by contracting the indices of the three linearized curvature tensors. Using group theory again, one can see this time that in $d = 6$ there is no way to contract the three curvature tensors so that the result is nonvanishing. This vertex starts being nontrivial from $d = 7$ on.

Finally, there is the Lorentz minimal coupling between the hook field and the graviton, bringing two derivatives in the Lagrangian. Interestingly enough, this vertex appears through the following only possible nontrivial candidate in $\text{antigh} = 3$:

$$a_3 = B^{\mu} B^{\nu} \partial_{[\mu} C_{\nu]}$$  

(5.16)

that can be integrated via (5.8) to give

$$a_2 = C^{\sigma \mu} \left[ A^{\sigma}_{\mu} + \frac{1}{2} S^{\sigma}_{\mu} \right] \partial_{[\mu} C_{\nu]} + B^{\mu} \partial_{[\mu} h_{\sigma]}. 

(5.17)

When one tries to integrate $a_2$ via (5.7), however, one finds an obstruction to finding $a_1$ so that the Lorentz minimal coupling is inconsistent in flat spacetime. As we have seen, the Lorentz minimal couplings appear in $\text{AdS}_d$ (see section 4.1) and are consistent when added to an appropriate finite tail of higher-derivative vertices, so that the resulting coupling can be called quasi-minimal, like for the gravitational interactions of totally symmetric fields [15].

\textbf{A remark on the non-Abelianization in $\text{AdS}_d$.} We have listed above all the possible couplings between the three types of fields $[[2, 1], [1, 1]]^{(1)}$, $[1, 1]^{(2)}$ and have seen that, in flat spacetime, there is no non-Abelian coupling among them. The remarkable fact is that these couplings, when embedded in $\text{AdS}_d$, become related to each other and contribute to give the non-Abelian interactions we presented here in various forms. Contrary to the totally symmetric case studied in [15], in the flat limit, the St"uckelberg action can give only Abelian interactions. The reason is that the gauge transformations for the St"uckelberg companion of the hook field receives a term, in $\text{AdS}_d$, that vanishes in the flat limit, being proportional to the cosmological constant; see e.g. equations (3.11)–(3.14). Now, in the case of mixed-symmetry fields in $\text{AdS}_d$ in the St"uckelberg formulation, it is no longer true that the linearized gauge transformations can be viewed as an $\text{AdS}$ covariantization of the flat space transformations. We believe that it is responsible for the fact that for mixed-symmetry fields in the St"uckelberg formulation, as opposed to the case of totally symmetric gauge fields, there is the possibility of scaling away the non-Abelian nature of a vertex while at the same time retaining a top vertex.

\textbf{6. Conclusions}

In this paper, we have obtained non-Abelian gravitational interactions for a simple mixed-symmetry gauge field in $\text{AdS}_d$ using various techniques that agree with each other upon partial gauge fixing and trivial field-redefinitions. In the St"uckelberg formulation, the flat limit is smooth also for the cubic action, which strengthens the proposal of [57]. This is not surprising, since the cubic vertices can smoothly be switched on and off by turning the coupling constant,\(^{10}\) See e.g. the Lie program at http://www-math.univ-poitiers.fr/~maavl/LiE/form.html.

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so the fact that the quadratic action allows for a smooth flat limit implies the same for the cubic action.

Starting right away in flat space and addressing the list of all possible cubic couplings, we indeed recuperated the highest-derivative vertices of the AdS\(_d\) action and found that all the flat spacetime vertices give rise to Abelian gauge algebras,—although the gauge transformations may be linear in the gauge fields, which appear then only through the linearized curvature like for the Bel–Robinson or Chaplin–Manton vertices. This means that taking the flat limit of a non-Abelian action for mixed-symmetry fields in AdS\(_d\) trivializes the gauge algebra, in sharp contrast to what happens for totally symmetric gauge fields in AdS\(_d\). This is another instance where mixed-symmetry gauge fields in AdS\(_d\) differ from their totally symmetric cousins.

This is in accordance with the fact that mixed-symmetry gauge fields in AdS\(_d\) have only one genuine differential gauge parameter and not several like in flat space, which makes the non-Abelian coupling problem less constrained. Generically, we expect that with one genuine gauge parameter in the game, one can have non-Abelian interactions, like for totally symmetric fields both in flat space and AdS\(_d\) and for the simple mixed-symmetry field in AdS\(_d\) studied here. However, when one has to deal with more than one gauge symmetry, the problem becomes too restrictive and only Abelian vertices can emerge, like for mixed-symmetry fields in flat space.

It would be interesting to make contact with the appearance of mixed-symmetry fields within string theory through the work [80, 16], where their relevance was exhibited, respectively, via vertex operators in exotic pictures and deconstruction of tensionful string amplitudes around the flat background. Very recently, an interesting connection between string vertex operators and higher-spin theory in the AdS\(_d\) background was made in [81]. It would be very promising to use this setting in order to understand better mixed-symmetry fields in AdS\(_d\) within string theory.

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Appendix. Metric-like formalism for the simplest hook gauge field

In order to make connection with the metric formulation, let us study the Lagrangian describing the simplest hook gauge field around the AdS\(_d\) background, namely the gauge field of Young-symmetry type [2, 1], from a slightly different perspective with respect to what is done in [57]. In particular, we will insist on the role played by gauge-invariant quantities which is more parallel to the frame-like formalism adopted in this paper. We adopt here the manifestly symmetric convention for Young tableaux.

In the base-manifold component notation, the gauge transformations for the dynamical field \(\Phi_{\mu_1\mu_2;\nu}\) and the St"uckelberg field \(x_{\mu_1\mu_2}\) read

\[
\delta\Phi_{\mu_1\mu_2;\nu} = 3 Y^{(2,1)}_{GL(d)} \left[ D_{\mu_1} A_{\mu_2;\nu} + D_{\nu} S_{\mu_1\mu_2} + \lambda g_{\mu_1\mu_2} \xi_{\nu} \right],
\]

\[
\delta x_{\mu_1\mu_2} = 2 D_{(\mu_1} \xi_{\mu_2)} + 2 \lambda S_{\mu_1\mu_2},
\]

(\text{A.1})

(\text{A.2})
where $Y^{(2,1)}_{GL(d)}$ denotes the projector on the Young tableau $(\mu_1, \mu_2, \nu)$ of $GL(d, \mathbb{R})$. Next we construct the so-called curvatures, the basic objects that are invariant under these gauge transformations. They are

$$K_{\mu_1 \mu_2, \nu_1 \nu_2, \rho} = Y^{(2,2)}_{GL(d)} \left[ D_\nu D_\rho \Phi_{\mu_1 \mu_2, \nu_1} - \frac{\lambda}{2} g_{\nu \rho} \Phi_{\mu_1 \mu_2, \nu_1} - \frac{2}{3} \lambda g_{\mu_1 \mu_2} D_\rho X_{\nu_1 \nu_2} \right] \quad (A.3)$$

$$K_{\mu_1 \mu_2, \nu_1 \nu_2} = Y^{(2,2)}_{GL(d)} \left[ D_\nu X_{\mu_1 \mu_2} - \frac{2}{3} \lambda D_\nu \Phi_{\mu_1 \mu_2, \nu_1} \right]. \quad (A.4)$$

From these curvatures, we build the Einstein-like invariant tensors

$$G_{\mu_1 \mu_2, \nu} = 8 K_{\mu_1 \mu_2, \nu \rho} g^{\rho \nu} + 12 Y^{(2,1)}_{GL(d)} \left[ g_{\mu_1 \mu_2} K_{\nu, \rho, \sigma} \right] = \Box \Phi_{\mu_1 \mu_2, \nu} + \cdots, \quad (A.5)$$

$$G_{\mu_1 \mu_2} = 3 \left[ K_{\mu_1 \mu_2, \nu} - \frac{1}{2} g_{\mu_1 \mu_2} K_{\nu, \rho, \sigma} \right] = \Box X_{\mu_1 \mu_2} + \cdots, \quad (A.6)$$

where $\Box$ is the covariant D’Alembertian.

These Einstein-like tensors can be seen to obey the following identities:

$$\nabla^\nu G_{\mu_1 \mu_2, \nu} = - 2 \lambda (d - 3) G_{\mu_1 \mu_2} \equiv 0, \quad (A.7)$$

$$\nabla^\nu G_{\nu \rho, \mu} - \nabla^\nu G_{\nu \rho, \mu} \equiv 0, \quad (A.8)$$

$$g^{\mu_1 \nu_1} G_{\mu_1 \mu_2, \nu} - \frac{2}{\lambda} (d - 3) \nabla^{\mu_1} G_{\mu_1 \nu} \equiv 0. \quad (A.9)$$

It is then natural to propose the following action $S[\Phi, \chi]$ (see also [57]):

$$S[\Phi, \chi] = \frac{1}{2} \int \sqrt{-g} \left[ \Phi_{\mu_1 \mu_2, \nu} G_{\mu_1 \mu_2, \nu} + 3(d - 3) X_{\mu_1 \mu_2} G_{\mu_1 \mu_2} \right] d^d x \quad (A.10)$$

which is invariant under the enhanced $\mathfrak{so}(d - 1, 2)$ gauge transformations (A.1), (A.2) by virtue of the invariance of the Einstein-like tensors and the identities (A.7), (A.8), (A.9) which are nothing but the Nöther identities corresponding to the parameters $S_{\mu_1 \mu_2}$, $A_{\mu, \nu}$ and $\xi_{\mu}$, respectively.

Although the Einstein-like tensors $G_{\mu_1 \mu_2, \nu}$ and $G_{\mu_1 \mu_2}$ contain both fields $\Phi$ and $\chi$, it can be seen that the Euler–Lagrange equations for the corresponding fields simply are, as expected,

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S[\Phi, \chi]}{\delta \Phi_{\mu_1 \mu_2, \nu}} \equiv G_{\mu_1 \mu_2, \nu}, \quad (A.11)$$

$$0 = \frac{1}{\sqrt{-g}} \frac{\delta S[\Phi, \chi]}{\delta X_{\mu_1 \mu_2}} \equiv 3(d - 3) G_{\mu_1 \mu_2}. \quad (A.12)$$

By taking traces of the latter field equations and inserting the results back in the corresponding equations, one can express the latter as the following zero-Ricci-like equations:

$$K_{\mu_1 \mu_2, \nu \rho} = 0, \quad K_{\mu_1 \mu_2, \nu} = 0. \quad (A.13)$$

As explained in [57], provided the cosmological constant is non-vanishing, the field $\chi$ is a Stückelberg field that can be gauge-fixed to zero inside the action. Then, the remaining field is $\Phi$, invariant under the gauge transformation (A.1) where only the antisymmetric parameter $A_{\mu, \nu}$ is nonzero. The field equation for the gauge-fixed action therefore is equivalent to

$$0 = F_{\mu_1 \mu_2, \nu \rho} \equiv 8 K_{\mu_1 \mu_2, \nu \rho} \big|_{\chi = 0}. \quad (A.14)$$

We may use the residual gauge symmetry (under the gauge transformation (A.1) where only the antisymmetric parameter $A_{\mu, \nu}$ is nonzero) in order to simplify the field equation (A.14).

Introducing the quantity

$$D_{\mu, \nu} = D^\rho \Phi_{\rho, \mu, \nu} + D_{\nu \rho} \Phi_{\rho, \mu} \equiv \partial_{\mu} \Phi_{\rho, \nu} \equiv \partial_{\nu} \Phi_{\rho, \mu} \equiv \partial_{\rho} \Phi_{\mu, \nu}, \quad (A.15)$$
it is easy to see that it transforms like
\[
\delta_\lambda D_{\mu,\nu} = 2 \left( \Box - (d - 2)\lambda^2 \right) A_{\mu,\nu}.
\] (A.16)
The equation \(\delta_\lambda D_{\mu,\nu} = 0\) is the differential equation obtained by Metsaev for a gauge parameter \(A_{\mu,\nu}\) in AdS\(_d\) [82, 83]. Indeed, from the latter work, we know that the differential constraint on the gauge parameter \(|\lambda_k|\) is
\[
\left[ \Box - \lambda^2 (h_k^{|k|} - k + 1) (h_k^{|k|} - k + d) + \lambda^2 \sum_{l=1}^{\nu} h_l^{|l|} \right] |\lambda_k\rangle = 0,
\] (A.17)
where \(k\) indicates, for the Young diagram associated with the gauge field, the maximal number of upper rows which have the same length and \(h_l^{|l|}\) \((l = 1, \ldots, \nu)\) are the lengths of the rows corresponding to the Young diagram associated with the gauge parameter \(|\lambda_k\rangle\). The index \(\nu\) is the integer part of \((d - 2)/2\). In this case, we have \(k = 1\) since the upper block for the gauge field has height one. The Young diagram of the gauge parameter is obtained from the Young diagram of the gauge field by removing one box at the end of the last row of the upper block.

We indeed get an antisymmetric, rank-2 parameter, i.e. \(\Phi_{1\nu} = 1 = h_1^{|1|}, h_1^{|1|} = 0 \forall \nu > 2\) and equation (A.17) indeed reproduces \(\delta_\lambda D_{\mu,\nu} = 0\), cf (A.16).

The field equation on the gauge field \(|\Phi\rangle\) is, from [82, 83],
\[
\left[ \Box - \lambda^2 (h_k - k - 1) (h_k - k - 2 + d) + \lambda^2 \sum_{l=1}^{\nu} h_l \right] |\Phi\rangle = 0,
\] (A.18)
where \(h_l\) \((l = 1, \ldots, \nu)\) denote the lengths of the rows corresponding to the gauge field \(|\Phi\rangle\). For the example at hand, the only nonvanishing entries are \(h_1 = 2 = h_2 = 1\) with \(k = 1\), as we explained before. Therefore, (A.18) gives the equation
\[
[\Box + 3\lambda^2]|\Phi\rangle = 0.
\] (A.19)

This equation can be obtained from (A.14) upon gauge fixing. Indeed, after imposing the gauge-fixing condition \(D_{\mu,\nu} = 0\) we are still allowed to further fix the gauge, provided the gauge parameter satisfies \(\delta_\lambda D_{\mu,\nu} = 0\). With such a gauge parameter, one can set the trace of \(|\Phi\rangle\) to zero, since \(\delta_\lambda |\Phi\rangle = 4D^\mu A_{\mu\nu}\). Then, at that stage, further gauge transformations could be performed, with a gauge parameter still obeying \(\delta_\lambda D_{\mu,\nu} = 0\) and further satisfying \(D^\mu A_{\mu,\nu} = 0\).

Finally, in the gauge where \(\chi_{\mu,\nu}\) is vanishing and where \(\lambda \neq 0\), the divergence of the field equation gives us the following constraint:
\[
0 = D^\nu G_{\nu,\rho}|_{\chi = 0} = (3 - d)\lambda^2 \left( D^\mu \Phi_{\nu,\rho,\mu} + D_{\nu} \Phi'_{\rho,\mu} + g_{\nu,\rho} D^\mu \Phi'_{\mu} \right), \quad \Phi'_\mu = \Phi_{\mu,\nu},
\] (A.20)
which allows us to set the symmetrized divergence \(D^\rho \Phi_{\rho(\mu,\nu)} = -\frac{1}{2} D^\rho \Phi_{\rho,\mu,\nu}\) to zero in the gauge where \(\Phi'\) is zero. Therefore, in the gauge \(D_{\mu,\nu} = 0, \Phi'_{\mu} = 0\), the field is traceless and totally divergence less.

Summarizing, the equations obtained at that point are
\[
D^\rho \Phi_{\rho(\mu,\nu)} = 0, \quad \Phi'_{\mu} = 0, \quad D^\mu A_{\mu,\nu} = 0,
\]
\[
\left[ \Box + 3\lambda^2 \right] |\Phi\rangle = 0, \quad \left[ \Box - (d - 2)\lambda^2 \right] A_{\mu,\nu} = 0,
\]
which, as shown in [82, 83], correctly define the unitary irreducible representation of \(\mathfrak{so}(d - 1, 2)\) associated with the Young diagram (2, 1). We thus showed that the action (A.10), with \(\lambda \neq 0\), correctly describes a massless \((2, 1)\) field propagating in \(\mathfrak{so}(d - 1, 2)\) and corresponding to a unitary irreducible representation of the latter isometry algebra.

Taking the flat limit \(\lambda \to 0\) in (A.10), we find that the resulting action indeed describes two massless irreducible field giving \(\mathfrak{so}(d - 2)\) degrees of freedom \((2, 1) \oplus (2, 0)\).
References

[1] Fradkin E S and Vasiliev M A 1987 On the gravitational interaction of massless higher spin fields Phys. Lett. B 189 89–95
[2] Fradkin E S and Vasiliev M A 1987 Cubic interaction in extended theories of massless higher spin fields Nucl. Phys. B 291 141
[3] Vasiliev M A 1990 Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions Phys. Lett. B 243 378–82
[4] Vasiliev M A 1992 More on equations of motion for interacting massless fields of all spins in (3+1)-dimensions Phys. Lett. B 285 225–34
[5] Vasiliev M A 2003 Nonlinear equations for symmetric massless higher spin fields in (a)dS(d) Phys. Lett. B 567 139–51 [arXiv:hep-th/0304049]
[6] Fronsdal C 1978 Massless fields with integer spin Phys. Rev. D 18 3624
[7] Fang J and Fronsdal C 1978 Massless fields with half integral spin Phys. Rev. D 18 3630
[8] Fronsdal C 1979 Singletons and massless, integral spin fields on de Sitter space (elementary particles in a curved space) Phys. Rev. D 20 848–56
[9] Fang J and Fronsdal C 1980 Massless, half integer spin fields in de Sitter space Phys. Rev. D 22 1361
[10] Vasiliev M A 2001 Cubic interactions of bosonic higher spin gauge fields in adS(5) Nucl. Phys. B 616 106–62 [arXiv:hep-th/0106200]
[11] Alkalaev K B and Vasiliev M A 2003 N = 1 supersymmetric theory of higher spin gauge fields in adS(5) at the cubic level Nucl. Phys. B 655 57–92 [arXiv:hep-th/0205068]
[12] Metsaev R R 2006 Cubic interaction vertices for massive and massless higher spin fields Nucl. Phys. B 759 147–201 [arXiv:hep-th/0512342]
[13] Buchbinder I L, Fotopoulos A, Petkou A C and Tsulaia M 2006 Constructing the cubic interaction vertex of higher spin gauge fields Phys. Rev. D 74 105018 [arXiv:hep-th/0609082]
[14] Metsaev R R 2007 Cubic interaction vertices for fermionic and bosonic arbitrary spin fields arXiv:0712.3526 [hep-th]
[15] Boulanger N, Leclercq S and Sundell P 2008 On the uniqueness of minimal coupling in higher-spin gauge theory J. High Energy Phys. JHEP08(2008)056 [arXiv:0805.2764]
[16] Sagnotti A and Taronna M 2011 String lessons for higher-spin interactions Nucl. Phys. B 842 299–361 [arXiv:1006.5242]
[17] Manvelyan R, Mkrtchyan K and Ruehl W 2010 General trilinear interaction for arbitrary even higher spin gauge fields Nucl. Phys. B 836 204–21 [arXiv:1003.2877]
[18] Polyakov D 2010 Gravitational couplings of higher spins from string theory Int. J. Mod. Phys. A 25 4623–40 [arXiv:1005.5512]
[19] Vasiliev M 2011 Cubic vertices for symmetric higher-spin gauge fields in (A)dS_a arXiv:1108.5921
[20] Bekaert X, Boulanger N and Sundell P 2010 How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples arXiv:1007.0435
[21] Vasiliev M A 2000 Higher spin symmetries, star-product and relativistic equations in adS space arXiv:hep-th/0002183
[22] Vasiliev M A 2004 Higher spin gauge theories in various dimensions Fortschr. Phys. 52 702–17 [arXiv:hep-th/0401177]
[23] Bekaert X, Cnockaert S, Iazeolla C and Vasiliev M A 2005 Nonlinear higher spin theories on spaces of constant curvature Class. Quantum Grav. 21 2571–93 [arXiv:hep-th/0311254]
[24] Curtright T 1985 Generalized gauge fields Phys. Lett. B 165 304
[25] Aulakh C S, Koh I G and Ouvery S 1986 Higher spin fields with mixed symmetry Phys. Lett. B 173 284
[26] Siegel W and Zwiebach B 1987 Gauge string fields from the light cone Nucl. Phys. B 282 125
[27] Labastida J M F 1989 Massless particles in arbitrary representations of the Lorentz group Nucl. Phys. B 322 185
[28] de Medeiros P and Hull C 2003 Geometric second order field equations for general tensor gauge fields J. High Energy Phys. JHEP05(2003)019 [arXiv:hep-th/0303036]
[29] de Medeiros P 2004 Massive gauge-invariant field theories on spaces of constant curvature Class. Quantum Grav. 21 2571–93 [arXiv:hep-th/0311254]
[30] Alkalaev K B, Shaynkman O V and Vasiliev M A 2004 On the frame-like formulation of mixed-symmetry massless fields in (a)dS(d) Nucl. Phys. B 692 363–93 [arXiv:hep-th/0311164]
[31] Sagnotti A and Tsulaia M 2004 On higher spins and the tensionless limit of string theory Nucl. Phys. B 682 83–116 [arXiv:hep-th/0311257]
[32] Alkalaev K B, Shaynkman O V and Vasiliev M A 2005 Lagrangian formulation for free mixed-symmetry bosonic gauge fields in (a)dS(d) J. High Energy Phys. JHEP08(2005)069 [arXiv:hep-th/0501108]
[33] Alkalaev K B, Shuynkman O V and Vasiliev M A 2006 Frame-like formulation for free mixed-symmetry bosonic massless higher-spin fields in adS(d) arXiv:hep-th/0601225
[34] Bekaert X and Boulanger N 2007 Tensor gauge fields in arbitrary representations of gl(d,r): II. Quadratic actions Commun. Math. Phys. 271 723–73 (arXiv:)
[35] Fotopoulos A and Tsaalia M 2007 Interacting higher spins and the high energy limit of the bosonic string Phys. Rev. D 76 025014 (arXiv:0705.2939)
[36] Buchbinder I L, Krykhtin V A and Takata H 2007 Gauge invariant Lagrangian construction for massive bosonic mixed symmetry higher spin fields Phys. Lett. B 656 253–64 (arXiv:0707.2181)
[37] Restenryak A A 2008 On Lagrangian formulations for mixed-symmetry HS fields on AdS spaces within BFV-BRST approach arXiv:0809.4815
[38] Skvortsov E D 2009 Frame-like actions for massless mixed-symmetry fields in Minkowski space Nucl. Phys. B 808 569–91 (arXiv:0807.0903)
[39] Zinoviev Y M 2009 Toward frame-like gauge invariant formulation for massive mixed symmetry bosonic fields arXiv:0809.3287
[40] Buchbinder I L, Krykhtin V A and Takata H 2007 Interacting higher spins and the high energy limit of the bosonic string Phys. Rev. D 76 025014 (arXiv:0705.2939)
[41] Boulanger N, Iazeolla C and Sundell P 2009 Unfolding mixed-symmetry fields in AdS and the BMV conjecture: I. General formalism J. High Energy Phys. JHEP07(2009)013 (arXiv:0812.3615)
[42] Boulanger N, Iazeolla C and Sundell P 2009 Unfolding mixed-symmetry fields in AdS and the BMV conjecture: II. Oscillator realization J. High Energy Phys. JHEP07(2009)014 (arXiv:0812.4438)
[43] Alkalaev K B, Grigoriev M and Tipunin I Y 2009 Massless Poincaré modules and gauge invariant equations Nucl. Phys. B 823 509–45 (arXiv:0811.3999)
[44] Campoleoni A, Francia D, Mourad J and Sagnotti A 2009 Unconstrained higher spins of mixed symmetry: I. Bose fields Nucl. Phys. B 815 289–367 (arXiv:0810.4350)
[45] Boulanger N, Iazeolla C and Sundell P 2009 Unfolding mixed-symmetry fields in AdS and the BMV conjecture: II. Oscillator realization J. High Energy Phys. JHEP07(2009)014 (arXiv:0812.4438)
[46] Alkalaev K B, Grigoriev M and Tipunin I Y 2009 Massless Poincaré modules and gauge invariant equations Nucl. Phys. B 823 509–45 (arXiv:0811.3999)
[47] Campoleoni A, Francia D, Mourad J and Sagnotti A 2009 Unconstrained higher spins of mixed symmetry: II. Fermi fields arXiv:0904.4447
[48] Zinoviev Y M 2010 Towards frame-like gauge invariant formulation for massive mixed symmetry bosonic fields: II. General Young tableau with two rows Nucl. Phys. B 826 490–510 (arXiv:0907.2140)
[49] Skvortsov E D 2010 Gauge fields in (A)dS within the unfolded approach: algebraic aspects J. High Energy Phys. JHEP01(2010)106 (arXiv:0910.3334)
[50] Alkalaev K B and Grigoriev M 2010 Unified BRST description of AdS gauge fields Nucl. Phys. B 815 289–367 (arXiv:0810.4350)
[51] Boulanger N and Cnockaert S 2004 Consistent deformations of (p,p)-type gauge field theories J. High Energy Phys. JHEP03(2004)031 (arXiv:hep-th/0402180)
[52] Bekaert X, Boulanger N and Cnockaert S 2005 No self-interaction for two-column massless fields J. Math. Phys. 46 012303 (arXiv:04070102)
[53] Alkalaev K 2011 FV-type action for AdS(5) mixed-symmetry fields J. High Energy Phys. JHEP03(2011)031 (arXiv:1011.6109)
[54] Sezgin E and Sundell P 2001 Doubletons and SD higher spin gauge theory J. High Energy Phys. JHEP09(2001)036 (arXiv:hep-th/0105001)
[55] Sezgin E and Sundell P 2001 Towards massless higher spin extension of D = 5, N = 8 gauged supergravity J. High Energy Phys. JHEP09(2001)025 (arXiv:hep-th/0107186)
[56] Zinoviev Y 2011 On electromagnetic interactions for massive mixed-symmetry fields J. High Energy Phys. JHEP03(2011)082 (arXiv:1012.2706)
[57] Brink L, Metsaev R R and Vasiliev M A 2000 How massless are massless fields in adS(d) Nucl. Phys. B 586 183–205 (arXiv:hep-th/0005136)
[58] Barnich G and Henneaux M 1993 Consistent couplings between fields with a gauge freedom and deformations of the master equation Phys. Lett. B 311 123–9 (arXiv:hep-th/9304057)
[59] Berends F A, Burgers G J H and van Dam H 1985 On the theoretical problems in constructing interactions involving higher spin massless particles Nucl. Phys. B 260 295
[60] Henneaux M 1998 Consistent interactions between gauge fields: the cohomological approach Contemp. Math. 219 93 (arXiv:hep-th/9712226)
[61] Vasiliev M A 1988 Equations of motion of interacting massless fields of all spins as a free differential algebra Phys. Lett. B 209 491–7
[62] Lopatin V E and Vasiliev M A 1988 Free massless bosonic fields of arbitrary spin in d-dimensional de Sitter space Mod. Phys. Lett. A 3 257

[63] Skvortsov E D 2008 Mixed-symmetry massless fields in Minkowski space unfolded J. High Energy Phys. JHEP07(2008)004 (arXiv:0801.2268)

[64] Skvortsov E D 2009 Gauge fields in (anti)-de Sitter space and connections of its symmetry algebra J. Phys. A: Math. Theor. 42 385401 (arXiv:0904.2919)

[65] Zinoviev Y M 2009 Note on antisymmetric spin-tensors J. High Energy Phys. JHEP04(2009)035 (arXiv:0903.0262)

[66] Zinoviev Y M 2009 Frame-like gauge invariant formulation for massive high spin particles Nucl. Phys. B 808 185–204 (arXiv:0808.1778)

[67] Fradkin E S and Vasiliev M A 1987 Candidate to the role of higher spin symmetry Ann. Phys. 173 63

[68] Vasiliev M A 2004 Higher spin superalgebras in any dimension and their representations J. High Energy Phys. JHEP12(2004)046 (arXiv:hep-th/0404124)

[69] Vasiliev M A 2004 Higher spin gauge theories in any dimension C. R. Phys. 5 1101–9 (arXiv:hep-th/0409260)

[70] Zinoviev Y M 2001 On massive high spin particles in (a)dS arXiv:hep-th/0108192

[71] Zinoviev Y M 2003 First order formalism for mixed symmetry tensor fields arXiv:hep-th/0304067

[72] Zinoviev Y M 2009 Frame-like gauge invariant formulation for mixed symmetry fermionic fields Nucl. Phys. B 821 21–47 (arXiv:0904.0549)

[73] Biswas T and Siegel W 2002 Radial dimensional reduction: anti-de Sitter theories from flat J. High Energy Phys. JHEP07(2002)005 (arXiv:hep-th/0203115)

[74] Alkalaev K B and Grigoriev M 2011 Unified BRST approach to (partially) massless and massive AdS fields of arbitrary symmetry type arXiv:1105.6111

[75] Grigoriev M and Waldron A 2011 Massive higher spins from BRST and tractor s arXiv:1104.4994

[76] Stelle K S and West P C 1980 Spontaneously broken de Sitter symmetry and the gravitational holonomy group Phys. Rev. D 21 1466

[77] Alkalaev K B 2004 Two-column higher spin massless fields in dS(d) Theor. Math. Phys. 140 1253–63 (arXiv:hep-th/0311212)

[78] Boulanger N, Damour T, Gualtieri L and Henneaux M 2001 Inconsistency of interacting, multigraviton theories Nucl. Phys. B 597 127–71 (arXiv:hep-th/0007220)

[79] Metsaev R R 1993 Cubic interaction vertices of totally symmetric and mixed symmetry massless representations of the Poincaré group in D = 6 space-time Phys. Lett. B 309 39–44

[80] Polyakov D 2010 Interactions of massless higher spin fields from string theory Phys. Rev. D 82 066005 (arXiv:0910.5338)

[81] Polyakov D 2011 A string model for AdS gravity and higher spins arXiv:1106.1558

[82] Metsaev R R 1995 Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time Phys. Lett. B 354 78–84

[83] Metsaev R R 1998 Arbitrary spin massless bosonic fields in d-dimensional anti-de Sitter space arXiv:hep-th/9810231