Frobenius algebras and ambidextrous adjunctions

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Abstract

In this paper we explain the relationship between Frobenius objects in monoidal categories and adjunctions in 2-categories. Specifically, we show that every Frobenius object in a monoidal category $M$ arises from an ambijunction (simultaneous left and right adjoints) in some 2-category $D$ into which $M$ fully and faithfully embeds. Since a 2D topological quantum field theory is equivalent to a commutative Frobenius algebra, this result also shows that every 2D TQFT is obtained from an ambijunction in some 2-category. Our theorem is proved by extending the theory of adjoint monads to the context of an arbitrary 2-category and utilizing the free completion under Eilenberg-Moore objects. We then categorify this theorem by replacing the monoidal category $M$ with a semistrict monoidal 2-category $\mathcal{M}$, and replacing the 2-category $D$ into which it embeds by a semistrict 3-category. To state this more powerful result, we must first define the notion of a ‘Frobenius pseudomonoid’, which categorifies that of a Frobenius object. We then define the notion of a ‘pseudo ambijunction’, categorifying that of an ambijunction. In each case, the idea is that all the usual axioms now hold only up to coherent isomorphism. Finally, we show that every Frobenius pseudomonoid in a semistrict monoidal 2-category arises from a pseudo ambijunction in some semistrict 3-category.

1 Introduction

In this paper we aim to illuminate the relationship between Frobenius objects in monoidal categories and adjunctions in 2-categories. One of the results we prove is that:

Every Frobenius object in any monoidal category $M$ arises from simultaneous left and right adjoints in some 2-category into which $M$ fully and faithfully embeds.

To indicate the two-handedness of these simultaneous left and right adjoints we refer to them as ambidextrous adjunctions following Baez [3]. We sometimes refer to an ambidextrous adjunction as an ambijunction for short.

Intuitively, the relationship between Frobenius objects and adjunctions can best be understood geometrically. This geometry arises naturally from the language of 2-categorical string diagrams [14,34]. In string diagram notation, objects $A$ and $B$ of the 2-category $\mathcal{D}$ are depicted as 2-dimensional regions which we sometimes shade to differentiate between different objects:

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The morphisms of $\mathcal{D}$ are depicted as one dimensional edges. Thus, if $L: A \to B$ and $R: B \to A$ are morphisms in $\mathcal{D}$, then they are depicted as follows:

```
  A \quad B
  \downarrow \quad \downarrow
L \quad R
```

and their composite $RL: A \to A$ as:

```
  A \quad B
  \downarrow \quad \downarrow
L \quad R
\circ \quad \circ
A \quad B
  \downarrow \quad \downarrow
L \quad R
```

As a convenient convention, the identity morphisms of objects in $\mathcal{D}$ are not drawn. This convention allows for the identification:

```
A = 1_A A
```

of string diagrams.

The 2-morphisms of $\mathcal{D}$ are drawn as 0-dimensional vertices or as small discs if we want to label them. Hence, the unit $i: 1 \Rightarrow RL$ and counit $e: LR \Rightarrow 1$ of an adjunction $A \xrightarrow{\overset{L}{\leftarrow}} B$ are depicted as:

```
A
\downarrow
L  R
```

```
A
\downarrow
L  R
```

However, using the convention for the identity morphisms mentioned above and omitting the labels we can simplify these string diagrams as follows:

```
A
\downarrow
L  R
```

```
A
\downarrow
L  R
```

We can also express the axioms for an adjunction, often referred to as the triangle identities or zig-zag identities, by the following equations of string diagrams:

```
L
\quad L = \quad L
```

```
R
\quad R = \quad R
```

Early work on homological algebra [11,27] and monad theory [10,21] showed that an adjunction $A \xrightarrow{\overset{L}{\leftarrow}} B$ endows the composite morphism $RL$ with a monoid structure in the monoidal category.
Hom\((A, A)\). This monoid in Hom\((A, A)\) can be vividly seen using the language of 2-categorical string diagrams. The multiplication on \(RL\) is defined using the unit for the adjunction as seen below:

\[
\begin{array}{cccc}
  & L & R & L & R \\
A & A & & & \\
  L & R & & & \\
  & & & & \\
  & & & & \\
L & R & R & L & R \\
& & & & \\
  & A & A & & \\
  & L & R & & \\
\end{array}
\]

and the unit for multiplication is

\[
\begin{array}{cccc}
  & & & \\
  & A & & \\
  & L & R & \\
\end{array}
\]

the unit of the adjunction. The associativity axiom:

\[
\begin{array}{cccc}
  & L & R & L & R \\
A & A & & & \\
  L & R & & & \\
  & & & & \\
  & & & & \\
L & R & R & L & R \\
& & & & \\
  & A & A & & \\
  & L & R & & \\
\end{array}
\]

follows from the interchange law in the 2-category \(D\), and the unit laws:

\[
\begin{array}{cccc}
  & L & R & L & R \\
A & A & & & \\
  L & R & & & \\
  & & & & \\
  & & & & \\
L & R & R & L & R \\
& & & & \\
  & A & A & & \\
  & L & R & & \\
\end{array}
\]

follow from the triangle identities in the definition of an adjunction.

Starting with an adjunction \(\begin{array}{ccc}
  & A & B \\
  \leftarrow & L & \rightarrow \end{array}\) where \(L\) is the right adjoint produces the color inverted versions of the diagrams above. The unit \(j: 1_B \Rightarrow LR\) and counit \(k: RL \Rightarrow 1_A\) would appear as:

\[
\begin{array}{cccc}
  & & & L & R \\
  & & & A & L \\
  & R & L & & \\
\end{array}
\]

In this case \(RL\) becomes a comonoid in Hom\((A, A)\) whose comultiplication is given by the diagram:

\[
\begin{array}{cccc}
  & L & R & L & R \\
A & A & & & \\
  L & R & & & \\
  & & & & \\
  & & & & \\
L & R & R & L & R \\
& & & & \\
  & A & A & & \\
  & L & R & & \\
\end{array}
\]

and whose counit is:

\[
\begin{array}{cccc}
  & L & R & L & R \\
A & A & & & \\
  L & R & & & \\
  & & & & \\
  & & & & \\
L & R & R & L & R \\
& & & & \\
  & A & & & \\
  & L & R & & \\
\end{array}
\]
the counit of the right adjunction. By similar diagrams as those above, the coassociativity and counit axioms follow from the axioms of a 2-category and the axioms of an adjunction.

When the morphisms $L$ is both left and right adjoint to $R$ the object $RL$ of $\text{Hom}(A, A)$ is both a monoid and a comonoid. These structures satisfy compatibility conditions, known as the Frobenius identities, making $RL$ into a Frobenius object. Indeed, the Frobenius identities:

\begin{align*}
L_R & \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \si...
1  INTRODUCTION

generating a given monad. This can be attributed to the lack of an object in \( K \) to play the role of the Eilenberg-Moore category of algebras (or the lack of a Kleisli object, but we will focus on Eilenberg-Moore objects in this paper). When such an object does exist it is called an Eilenberg-Moore object for the monad \( T \). The existence of Eilenberg-Moore objects in a 2-category \( K \) is a completeness property of the 2-category in question. In particular, \( K \) has Eilenberg-Moore objects if it is finitely complete as a 2-category [31,33].

Recall that every bicategory is biequivalent to a strict 2-category, and hence every monoidal category is biequivalent to a strict monoidal category. Let \( M \) be a monoidal category and denote as \( \Sigma(M) \) the suspension of a strictification of \( M \). Then since the 2-category \( \Sigma(M) \) has only one object, say \( \bullet \), a Frobenius object in \( M \) is just a Frobenius monad on the object \( \bullet \) in the 2-category \( \Sigma(M) \). It is tempting to use Eilenberg and Moore’s theorem on adjoint monads to conclude that this Frobenius monad arose from an ambijunction, but their construction used the fact the 2-category \( K \) was \textbf{Cat}. Since \textbf{Cat} is finitely complete as a 2-category, this allows the construction of Eilenberg-Moore objects which are a crucial ingredient in Eilenberg and Moore’s result. Considering Frobenius monads in \( \Sigma(M) \), the strictification of the suspension of the monoidal category \( M \), it is apparent that the required Eilenberg-Moore object is unlikely to exist: the 2-category \( \Sigma(M) \) has only one object! Fortunately, there is a categorical construction that enlarges a 2-category into one that has Eilenberg-Moore objects. This is known as the Eilenberg-Moore completion and it will be discussed in greater detail later. The important aspect to bear in mind is that this construction produces a 2-category together with an ambijunction generating our Frobenius object.

Frobenius objects have found tremendous use in topology, particularly in the area of topological quantum field theory. A well known result going back to Dijkgraaf [9] states that 2-dimensional topological quantum field theories are equivalent to commutative Frobenius algebras, see also [1,22]. Our result then indicates that:

\textbf{Every 2D topological quantum field theory arises from an ambijunction in some 2-category.}

More recently, higher-dimensional analogs of Frobenius algebras have begun to appear in higher-dimensional topology. For example, instances of categorified Frobenius structure have appeared in 3D topological quantum field theory [36], Khovanov homology — the homology theory for tangle cobordisms generalizing the Jones polynomial [20], and the theory of thick tangles [25].

In all of the cases mentioned above, the higher-dimensional Frobenius structures can be understood as instances of a single unifying notion — a ‘Frobenius pseudomonoid’. A Frobenius pseudomonoid is a categorized Frobenius algebra — a monoidal category satisfying the axioms of a Frobenius algebra up to coherent isomorphism. Being inherently categorical, our approach to solving the problem of constructing adjunctions from Frobenius objects suggests a quite natural procedure for not only defining a Frobenius pseudomonoid\(^2\), but more importantly, for showing that:

\textbf{Every Frobenius pseudomonoid in a semistrict monoidal 2-category arises from a pseudo ambijunction in a semistrict 3-category.}

The categorified theorem as stated above takes place in the context of a semistrict 3-category, also referred to as a \textbf{Gray}-category. We take this as a sufficient context for the generalization since every tricategory or weak 3-category is triequivalent to a \textbf{Gray}-category [12]. A \textbf{Gray}-category can be defined quite simply using enriched category theory [19]. Specifically, a \textbf{Gray}-category is a category enriched in \textbf{Gray}. Although a more explicit definition of a \textbf{Gray}-category can be given, see for instance Marmolejo [28], we will not be needing it for this paper. Adjunctions as well as monads generalize to this context and are called pseudoadjunctions and pseudomonads, respectively. They consist of the usual data, where the axioms now hold up to coherent isomorphism. In the context of an arbitrary \textbf{Gray}-category we extend the notion of mateship under

\(^2\)Note that our definition of Frobenius pseudomonoid nearly coincides with the notion given by Street [35]; the slight difference is that certain isomorphisms in our definition are made explicit.
adjunction to the notion of mateship under pseudoadjunction. Eilenberg-Moore objects and the
Eilenberg-Moore completion also make sense in this context, so we are able to categorify Eilen-
berg and Moore’s theorem on adjoint monads, as well as our theorem relating Frobenius objects
to ambijunctions, to the context of an arbitrary Gray-category.

We remark that Street has demonstrated that the condition for a monoidal category to be a
Frobenius pseudomonoid is identical to the condition of $*$-autonomy [35]. These $*$-autonomous
monoidal categories are known to have an interesting relationship with quantum groups and quan-
tum groupoids [8]. Combined with our result relating Frobenius pseudomonoids to pseudo am-
bijunctions, the relationship with $*$-autonomous categories may have implications to quantum
groups, as well as the field of linear logic where $*$-autonomous categories are used extensively.

2 Adjoint monads and Frobenius objects

2.1 Preliminaries

In this section we review the concepts of adjunctions and monads in an arbitrary 2-category along
with some of the general theory needed later on. A good reference for much of the material
presented in this section is [17].

Definition 1. An adjunction $i,e : F \dashv U : A \to B$ in a 2-category $K$ consists of

• morphisms $U : A \to B$ and $F : B \to A$, and

• 2-morphisms $i : 1 \Rightarrow UF$ and $e : FU \Rightarrow 1$,

such that

\[
\begin{array}{ccc}
U & \xrightarrow{i} & UFU \\
\downarrow \quad & & \downarrow e \\
U & \xrightarrow{=} & U
\end{array}
\quad \quad \text{and} \quad \quad
\begin{array}{ccc}
F & \xrightarrow{e} & FUF \\
\downarrow \quad & & \downarrow i \\
F & \xrightarrow{=} & F
\end{array}
\]

commute.

Proposition 2. If $i,e : F \dashv U : A \to B$ and $i',e' : F' \dashv U' : B \to C$ are adjunctions in the
2-category $K$, then $FF' \dashv U'U$ with unit and counit:

\[
\begin{array}{c}
i \quad := \quad 1 \xrightarrow{i'} U'F' \xrightarrow{U'iF'} U'UFF'
\end{array}
\]

\[
\begin{array}{c}
e \quad := \quad FF'U'U \xrightarrow{Fe'} FU \xrightarrow{=} 1
\end{array}
\]

Proof. We must verify that the triangle identities are satisfied. The proof is as follows:
where the inner squares commute by the interchange law of the 2-category $K$.

We recall the notion of mateship under adjunction.

**Definition 3.** Let $i, e: F \dashv U: A \to B$ and $i', e': F' \dashv U': A' \to B'$ in the 2-category $K$. It was shown by Kelly and Street [17] that if $a: A \to A'$ and $b: B \to B'$, then there is a bijection between 2-morphisms $\xi: bU \Rightarrow U'a$ and 2-morphisms $\zeta: bF \Rightarrow aF$, where $\zeta$ is given in terms of $\xi$ by the composite:

$$\zeta = F'b \xrightarrow{F'\xi} F'bUF \xrightarrow{F'eF} F'U'aF \xrightarrow{e'aF} aF$$

and $\xi$ is given in terms of $\zeta$ by the composite:

$$\xi = bU \xrightarrow{i'bU} U'F'bU \xrightarrow{U'aF} U'aFU \xrightarrow{U'ae} U'a.$$

Under these circumstances we say that $\xi$ and $\zeta$ are mates under adjunction and we sometimes write $\xi \dashv \zeta$.

The naturality of this bijection can be expressed as an isomorphism of certain double categories, see Proposition 2.2 [17]. In both cases, the objects of the double categories are those of $K$. The horizontal arrows are the morphisms of $K$ with the usual composition and the vertical arrows are the adjunctions in $K$ with the composition given in Proposition 2. In the first double category, a square with sides $a: A \to A'$, $b: B \to B'$, $i, e: F \dashv U: A \to B$, and $i', e': F' \dashv U': A' \to B'$ is a 2-cell $\xi: bU \Rightarrow U'a$. In the second double category a square with the same sides is a 2-cell $\zeta: bF \Rightarrow aF$. The isomorphism between these two double categories makes precise the idea that the association of mateship under adjunction respects composites and identities both of adjunctions and of morphisms in $K$.

**Definition 4.** A monad $T = (T, \mu, \eta)$ in a 2-category $K$ on the object $B$ of $K$ consists of an endomorphism $T: B \to B$ together with 2-morphisms:

- multiplication for the monad: $\mu: T^2 \Rightarrow T$, and
- unit for the monad: $\eta: 1 \Rightarrow T$,

such that

$$T \xrightarrow{T\eta} TT \xleftarrow{\eta T} T \xrightarrow{\mu} T$$

and

$$T^3 \xrightarrow{\mu T \eta} T^2 \xleftarrow{T \mu} T^2 \xrightarrow{T \mu} T$$

commute.

A comonad is defined by reversing the directions of the 2-cells:

**Definition 5.** A comonad $G = (G, \delta, \varepsilon)$ in a 2-category $K$ on the object $B$ of $K$ consists of an endomorphism $G: B \to B$ together with 2-morphisms:

- comultiplication for the comonad: $\delta: G \Rightarrow G^2$, and
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2.1 Preliminaries

- counit for the comonad: \( \varepsilon : G \Rightarrow 1 \),

such that

\[
\begin{array}{ccc}
G & \xrightarrow{Gz} & G \\
\downarrow \delta & & \downarrow G \\
G & \xrightarrow{\varepsilon} & G
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
G^3 & \xrightarrow{\delta G} & G^2 \\
G & \xleftarrow{G \delta} & G \\
\downarrow \delta & & \downarrow \delta \\
G & \xrightarrow{\varepsilon} & G
\end{array}
\]

commute.

A complete treatment of monads in this generality is presented in [24, 31]. It is clear that if \( i,e : F \dashv U : A \to B \) is an adjunction in \( K \), then \( (UF,UeF,i) \) is a monad on \( B \). We now recall a result due to Eilenberg-Moore [10], proven in the context \( K = \text{Cat} \), that easily generalizes to arbitrary \( K \).

**Proposition 6.** Let \( \mathbb{T} = (T, \mu, \eta) \) be a monad on an object \( B \) in a 2-category \( K \) such that the endomorphism \( T : B \to B \) has a specified right adjoint \( G \) with counit \( \sigma : TG \Rightarrow 1 \) and unit \( \iota : 1 \Rightarrow GT \). Then \( G = (G, \varepsilon, \delta) \) is a comonad where \( \varepsilon \) and \( \delta \) are the mates under adjunction of \( \eta \) and \( \mu \) with the explicit formulas being:

\[
\varepsilon = \sigma \eta G \\
\delta = G^2 \sigma G^2 \mu G ; TG ; \iota G
\]

and \( G \) is said to be a comonad right adjoint to the monad \( \mathbb{T} \), denoted \( \mathbb{T} \dashv G \).

**Proof.** This statement immediately follows from the composition preserving property of the bijection of mates under adjunction. Since \( T \dashv G, \mu \dashv \delta, \eta \dashv \varepsilon \), and mateship under adjunction preserves the various composites, then because \( (T, \mu, \eta) \) satisfies the monad axioms, \( (G, \delta, \varepsilon) \) will satisfy the comonad axioms.

**Definition 7.** A monad \( \mathbb{T} \) in the 2-category \( K \) is called a Frobenius monad if it is equipped with a morphism \( \varepsilon : T \Rightarrow 1 \) such that \( \varepsilon ; \mu \) is the counit for an adjunction \( T \dashv T \).

The notion of a Frobenius monad (or Frobenius standard construction as it was originally called) was first defined by Lawvere [26]. In Street [35] several definitions of Frobenius monad are given and proven equivalent. If one regards the monoidal category \( \text{Vect} \) as a one object 2-category \( \Sigma(\text{Vect}) \), then a Frobenius monad in \( \Sigma(\text{Vect}) \) is just the usual notion of a Frobenius algebra.

**Definition 8.** An action of the monad \( \mathbb{T} \) on a morphism \( s : A \to B \) in the 2-category \( K \) is a 2-morphism \( \nu : Ts \Rightarrow s \) such that

\[
\begin{array}{ccc}
s \xrightarrow{\mu s} Ts \\
\downarrow \iota & & \downarrow \iota \\
s & & s
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
T^2 s & \xrightarrow{\mu s} & Ts \\
\downarrow \nu & & \downarrow \nu \\
Ts & \xrightarrow{s} & s
\end{array}
\]

commute. A morphism \( s \) together with an action is called a \( \mathbb{T} \)-algebra (with domain \( A \)).

For any morphism \( s : A \to B \) in \( K \), \( Ts \) with action \( \mu s : T^2 s \Rightarrow Ts \) is a \( \mathbb{T} \)-algebra. For reasons that will soon become apparent we call the \( \mathbb{T} \)-algebra \( (Ts, \mu s) \) a free \( \mathbb{T} \)-algebra. The traditional notion of \( \mathbb{T} \)-algebra corresponds to the notion presented above when \( K = \text{Cat} \) and \( A \) is the one object category. In this case we identify the map \( s : 1 \to B \) with its image.

**Definition 9.** Let \( \mathbb{T} \) be a monad in \( K \). For each \( A \) in \( K \) define the category \( \mathbb{T}\text{-Alg}_A \) whose objects are \( \mathbb{T} \)-algebras, and whose morphisms between \( \mathbb{T} \)-algebras \( (s, \nu) \) and \( (s', \nu') \) are those 2-morphisms
Given a morphism $K: A' \to A$ in $\mathcal{K}$, one can define a change of base functor $K: \mathcal{T}\text{-Alg}_A \to \mathcal{T}\text{-Alg}_{A'}$. If $h: (s, \nu) \to (s', \nu')$ is in $\mathcal{T}\text{-Alg}_A$, then its image under $K$ is $hK: (sK, \nu K) \to (s'K, \nu'K)$. If $k: K \Rightarrow K'$ in $\mathcal{K}$ then we get a natural transformation $\hat{k}: \hat{K} \Rightarrow \hat{K}'$ such that $\hat{k}(s, \nu) = sk$. In fact, this shows that the construction of $\mathcal{T}$-algebras defines a 2-functor $\mathcal{T}\text{-Alg}: \mathcal{K}^{\text{op}} \to \mathcal{C}$.

As with the case when $\mathcal{K} = \mathcal{C}$, we have a forgetful functor:

$$
U^-_A: \mathcal{T}\text{-Alg}_A \to \mathcal{K}(A, B)
$$

such that $s: A \to B$ to the 2-morphism $\eta s$. The counit $\epsilon^-_A: F^-_A U^-_A \Rightarrow 1$ is the natural transformation that assigns to each $\mathcal{T}$-algebra $(s, \nu)$ the morphism of $\mathcal{T}$-algebras given by $\nu: Ts \Rightarrow s$. This adjunction exists for every $A$ in $\mathcal{K}$. In fact, we have the following:

**Proposition 10.** The collection of adjunctions

$$
i^-_A, e^-_A: F^-_A \Rightarrow U^-_A: \mathcal{T}\text{-Alg}_A \to \mathcal{K}(A, B)
$$

defined for each $A$ in $\mathcal{K}$ defines an adjunction

$$
i^-, e^+: F^+ \Rightarrow U^+: \mathcal{T}\text{-Alg} \to \mathcal{K}(-, B)
$$

in the 2-category $[\mathcal{K}^{\text{op}}, \mathcal{C}]$ consisting of 2-functors $\mathcal{K}^{\text{op}} \to \mathcal{C}$, 2-natural transformations between them, and modifications.

**Proof.** We have already shown above that $\mathcal{T}\text{-Alg}$ is a 2-functor. It is also clear that the collection of natural transformations $U^-_A$ define a 2-natural transformation $\mathcal{T}\text{-Alg} \Rightarrow \mathcal{K}(-, B)$. To see that the collection of $F^-_A$ define a 2-natural transformation $F^+: \mathcal{K}(-, B) \Rightarrow \mathcal{T}\text{-Alg}$ we must verify the 1-naturality:

$$
\begin{array}{ccc}
\mathcal{K}(A, B) & \xrightarrow{F^-_A} & \mathcal{T}\text{-Alg} \\
\mathcal{K}(K, B) \downarrow & & \downarrow \hat{K} \\
\mathcal{K}(A', B) & \xrightarrow{F^-_{A'}} & \mathcal{T}\text{-Alg} \\
\end{array}
$$

which commutes since both functors map $h: s \Rightarrow s'$ to $ThK: TsK \Rightarrow Ts'K$, and we must verify the 2-naturality:

$$
\begin{array}{ccc}
\mathcal{K}(A, B) & \xrightarrow{F^-_A} & \mathcal{T}\text{-Alg}_A \\
\mathcal{K}(K, B) & \xrightarrow{F^-_{A'}} & \mathcal{T}\text{-Alg}_{A'} \\
\mathcal{K}(K', B) & \xrightarrow{F^-_{A'}} & \mathcal{T}\text{-Alg}_{A'} \\
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{K}(A, B) & \xrightarrow{F^-_A} & \mathcal{T}\text{-Alg}_A \\
\mathcal{K}(A', B) & \xrightarrow{F^-_{A'}} & \mathcal{T}\text{-Alg}_{A'} \\
\mathcal{K}(K', B) & \xrightarrow{F^-_{A'}} & \mathcal{T}\text{-Alg}_{A'} \\
\end{array}
$$
This equality holds since both natural transformations assign to the morphism $s: A \to B$ the morphism of $\mathcal{T}$-algebras $Tsk: TsK \to TsK'$.

Next we verify that the collection of $i^*_T$ define a modification $i^*: 1 \Rightarrow U^T F^T$. Consider the diagrams below:

$$
\begin{array}{ccc}
K(A, B) & \xrightarrow{F_A} & K(A', B) \\
\downarrow \alpha^*_A & & \downarrow \alpha^*_A \\
\eta^*_A F^T_A & & \eta^*_A F^T_A
\end{array}
\quad
\begin{array}{ccc}
K(A, B) & \xrightarrow{F_A} & K(A', B) \\
\downarrow \beta^*_A & & \downarrow \beta^*_A \\
\eta^*_A F^T_A & & \eta^*_A F^T_A
\end{array}
\quad
\begin{array}{ccc}
K(A, B) & \xrightarrow{F_A} & K(A', B) \\
\downarrow \gamma^*_A & & \downarrow \gamma^*_A \\
\eta^*_A F^T_A & & \eta^*_A F^T_A
\end{array}
$$

Both of these natural transformations assign to the morphism $s: A \to B$ the morphism $\eta s K$ of $\mathcal{T}$-algebras. Hence, $i^*$ is a modification. Similarly one can check that the collection of $e^*_{A\beta}$ define a modification $e^*: F^T U^T \Rightarrow 1$. Since the coherence axioms for an adjunction are verified pointwise we have shown that $i^*, e^*: F^T \Rightarrow U^T: \mathcal{T}-\text{Alg} \to \mathcal{K}(-, B)$ is an adjunction in $[\mathcal{K}^\text{op}, \mathbf{Cat}]$.

**Definition 11.** We say that an Eilenberg-Moore object exists for a monad $\mathcal{T}$ if the 2-functor $\mathcal{T}-\text{Alg}: \mathcal{K}^\text{op} \to \mathbf{Cat}$ is representable. An Eilenberg-Moore object for the monad $\mathcal{T}$ is then just a choice of representation for the 2-functor $\mathcal{T}-\text{Alg}$, that is an object $B^\mathcal{T}$ of $\mathcal{K}$ together with a specified 2-natural isomorphism from $\mathcal{T}-\text{Alg}$ to the 2-functor $\mathcal{K}(-, B^\mathcal{T})$.

If an Eilenberg-Moore object exists for a monad $\mathcal{T}$ then by the enriched Yoneda lemma, or 2-categorical Yoneda lemma as it is sometimes referred to in this context, the adjunction of Proposition 10 arises from an adjunction $\iota^*, e^*: F^T \Rightarrow U^T: B \to B^T$ in $\mathcal{K}$.

Given a comonad $\mathcal{G}$ in $\mathcal{K}$ we can define the category $\mathcal{G}-\text{CoAlg}_A$ of $\mathcal{G}$-coalgebras $(s, \bar{\nu})$ and maps of $\mathcal{G}$-coalgebras by reversing the directions of the 2-cells in the definition of $\mathcal{T}-\text{Alg}_A$ and substituting the appropriate data for $\mathcal{G}$. If $K: A' \to A$ then we can define a change of base functor $\bar{K}: \mathcal{G}-\text{CoAlg}_A \to \mathcal{G}-\text{CoAlg}_{A'}$ sending $h: (s, \bar{\nu}) \to (s, \bar{\nu})$ to $hK: (sK, \bar{\nu}K) \to (sK, \bar{\nu}K)$. Further, if $h: K \to K'$ in $\mathcal{K}$ we can define the natural transformation $\bar{K}$ which sends $(s, \bar{\nu})$ to $sk: (sK, \bar{\nu}K) \to (sK', \bar{\nu}K')$. Hence we have a 2-functor $\mathcal{G}-\text{CoAlg}: \mathcal{K}^\text{op} \to \mathbf{Cat}$. As before, there is a forgetful 2-natural transformation $U^\mathcal{G}: \mathcal{G}-\text{CoAlg} \to \mathcal{K}(-, B)$. However, in this case, $U^\mathcal{G}$ has a right adjoint $F^\mathcal{G}$. An Eilenberg-Moore object for a comonad is just a choice of representation for the 2-functor $\mathcal{G}-\text{CoAlg}$. If an Eilenberg-Moore object for $\mathcal{G}$ does exist then, again by the 2-categorical Yoneda lemma, the adjunction $\iota^*, e^*: F^\mathcal{G} \Rightarrow U^\mathcal{G}: \mathcal{G}-\text{CoAlg} \to \mathcal{K}(-, B)$ arises from an adjunction $\iota^*, e^*: F^\mathcal{G} \Rightarrow U^\mathcal{G}: B \to B^\mathcal{G}$ in $\mathcal{K}$.

### 2.2 Adjoint monads

Given $\mathcal{T} \dashv \mathcal{G}$ in $\mathbf{Cat}$, Eilenberg and Moore [10] showed that mateship under adjunction of action and coaction defines an isomorphism $B^\mathcal{T} \cong B^\mathcal{G}$ of categories between the Eilenberg-Moore category of $\mathcal{T}$-algebras for the monad $\mathcal{T}$ and the Eilenberg-Moore category of $\mathcal{G}$-coalgebras for the comonad $\mathcal{G}$. In this section we continue the program for the formal theory of monads begun by Street [31]. In particular, we extend the classical theory of adjoint monads developed by Eilenberg and Moore to the context of an arbitrary 2-category.

**Lemma 12.** Let $\mathcal{T}$ be a monad on $B \in \mathcal{K}$. If $\iota, \sigma: T \dashv \mathcal{G}$ and $\mathcal{G}$ is equipped with the comonad structure $\mathcal{G}$ from Proposition 10 then the category $\mathcal{T}-\text{Alg}_A$ is isomorphic to the category $\mathcal{G}-\text{CoAlg}_A$ and this isomorphism commutes with the forgetful functors to $K(A, B)$.

**Proof.** Define a functor $\mathcal{M}_A: \mathcal{T}-\text{Alg}_A \to \mathcal{G}-\text{CoAlg}_A$ by sending $h: (s, \bar{\nu}) \to (s', \bar{\nu}')$ to $h: (s, \bar{\nu}) \to (s', \bar{\nu}')$ where $\nu \dashv \bar{\nu}$ and $\nu' \dashv \bar{\nu}'$. The explicit formulas for $\nu$ and $\nu'$ are $\nu \dashv \nu'$ and $\nu' \dashv \nu'$. Note that $h: s \to s'$ corresponds via mateship under adjunction to itself, and $T h$ corresponds via mateship to $G h$. Hence $(s, \bar{\nu})$ is a $\mathcal{G}$-coalgebra since $(s, \nu)$ is a $\mathcal{T}$-algebra and the association of mates preserves composites by the remarks following Definition 10. Since $\mathcal{M}_A$ is the identity on morphisms it is clear that composites and identities are preserved.
We define the inverse functor $\overline{M}_A: G_{\text{CoAlg}} \to T_{\text{Alg}}$ again using mateship under adjunction. The $G$-coalgebra $(s, \nu)$ is sent to the $T$-algebra $(s, \nu')$ where $\nu' \dashv \nu$. The explicit formula for $\nu'$ is $\sigma s.T\nu$. On morphisms $\overline{M}_A$ is the identity. The fact that $\overline{M}_A$ and $\overline{M}_A$ are inverses follows from the triangle identities for the adjunction $\iota, \sigma: T \dashv G$. Clearly, $U_G A \overline{M}_A = U_T A$ since $\overline{M}_A$ maps $s: A \to B$ to itself and is the identity on morphisms.

**Theorem 13 (The adjoint monad theorem).** Let $T$ be a monad in $K$ with $T \dashv G$ and denote the induced comonad of Proposition 6 as $G_{\text{CoAlg}}$. Then there is a 2-natural isomorphism $M: T_{\text{Alg}} \to G_{\text{CoAlg}}$ making the following diagram commute. Furthermore, if one exists, an Eilenberg-Moore object $B_T$ for the monad $T$ serves as an Eilenberg-Moore object $B_G$ for the comonad $G$. So that the above diagram arises via the 2-categorical Yoneda lemma from the commutative diagram:

$$
\begin{array}{ccc}
T_{\text{Alg}} & \xrightarrow{M} & G_{\text{CoAlg}} \\
\downarrow U_T & & \downarrow U_G \\
\mathcal{K}(\cdot, B) & \xrightarrow{\eta} & \mathcal{K}(\cdot, B)
\end{array}
$$

in $K$.

**Proof.** We show that the collection of natural isomorphisms $M_A$ defined in Lemma 12 define a 2-natural isomorphism $M: T_{\text{Alg}} \to G_{\text{CoAlg}}$. The 1-naturality of $M$ follows from the fact that if $K: A' \to A$, then

$$
\overline{K}.M_A(s, \nu) = (sK, (g\nu.\iota s)K) = (sK, g\nu K.\iota sK) = M_A.\overline{K}(s, \nu).
$$

The 2-naturality of $M$ follows from the fact that $M_A$ is the identity on morphisms. Hence, $M$ is a 2-natural transformation. From Lemma 12 it is clear that $M$ commutes with the forgetful functors since this is verified pointwise.

If $B_T$ is an Eilenberg-Moore object for the monad $T$, then we have a choice of 2-natural isomorphism $\mathcal{K}(\cdot, B_T) \cong T_{\text{Alg}}$. Composing this 2-natural isomorphism with the 2-natural isomorphism $M$ equips $B_T$ with the structure of an Eilenberg-Moore object for the comonad $G$. Since the 2-natural isomorphism $M$ commutes with the forgetful 2-natural isomorphisms $U_T$ and $U_G$, it is clear that their images under the 2-categorical Yoneda lemma will make the required diagram commute.

This theorem shows that if the monad $T$ has an adjoint comonad $G$, and if the Eilenberg-Moore objects exists, then the ‘forgetful morphism’ $U^*: B^T \to B$ has not only a left adjoint $F^T$, but also a right adjoint $\overline{M}F_G^T$. We can also extend the classical converse of this theorem to show that if a morphism has both a left and right adjoint, then the induced monad and comonad are adjoint.

**Theorem 14.** Let $B \xrightarrow{L_1} C$ and $B \xrightarrow{L_2} C$ (or $L_1 \dashv R \dashv L_2$) be specified adjunctions in the 2-category $K$. Also, let $T_1$ be the monad on $B$ induced by the composite $RL_1$, and $T_2$ be the comonad on $B$ induced by the composite $RL_2$. Then $T_1 \dashv T_2$ via a specified adjunction determined from the data defining the adjunctions $L_1 \dashv R \dashv L_2$. 




Proof. Let $i_1, e_1 : L_1 \dashv R_1 : C \to B$ and $i_2, e_2 : R_2 \dashv L_2 : B \to C$, then it follows from the composition of adjunctions that $RL_1 \dashv RL_2$ with $i = Ri_2L_1, i_2 : 1 \Rightarrow RL_2RL_1$ and $\sigma = e_2Re_1L_2 : RL_1RL_2 \Rightarrow 1$. The triangle identities follow from the triangle identities for the pairs $(i_1, e_1)$ and $(i_2, e_2)$. It remains to be shown that $\mu = L_1eiR$ is mates under adjunction with $\delta = Ri_2L_2$ and $\eta = i_1$ is mates with $\varepsilon = e_2$.

The mate to $e_2$ is given by the composite:

$$1 \xrightarrow{i_1} RL_1 \xrightarrow{Re_1L_2} RL_2RL_1 \xrightarrow{e_2RL_1} RL_1$$

but, by one of the triangle identities, this is just the map $i_1$. The mate to $i_1$ is given by the composite:

$$RL_2 \xrightarrow{i_1RL_2} RL_1RL_2 \xrightarrow{Re_1L_2} RL_2 \xrightarrow{e_2} 1$$

and by the other triangle identity is equal to $e_2$. Hence $\eta \vdash \varepsilon$. In a similar manner it can be shown that $\mu \dashv \delta$ using multiple applications of the triangle identities.

As previously discussed, Street uses the classical version of this theorem to show that a Frobenius monad in $\text{Cat}$ is always induced by ambijunction in $\text{Cat}$. Furthermore, he shows that the converse of this theorem is also true — corresponding to a Frobenius monad in $\text{Cat}$ there always exists an ambijunction generating it. We will now extend this result to include the case of Frobenius monads in an arbitrary 2-category. The most significant impediment to this extension is the lack of Eilenberg-Moore objects in 2-categories that are not finitely complete. This leads us to the free completion of a 2-category under Eilenberg-Moore objects to be discussed in the next section. Note that this free completion was not needed in Street’s construction since $\text{Cat}$ already has Eilenberg-Moore objects.

### 2.3 Eilenberg-Moore completions

As we saw in the introduction, one of the aims of this paper is show that every Frobenius object in any monoidal category arises from an ambijunction in some 2-category. To prove this, one is tempted to apply Theorem 13. However, when regarding a Frobenius object in a monoidal category as a Frobenius monad on the suspension of the monoidal category caution must be exercised. The 2-category $Σ(M)$ has only one object. Thus, it is unlikely that the Eilenberg-Moore objects, supposed to exist in Theorem 13 actually exists in $Σ(M)$.

Street [24] has shown that an Eilenberg-Moore object can be considered as a certain kind of weighted limit. He has also shown that the weight is finite in the sense of [16]. In The Formal Theory of Monads. II. [24], Lack and Street use this result to show that one can define $EM(K)$: the free completion under Eilenberg-Moore objects of the 2-category $K$. Since the free completion under a class of colimits is more accessible than completions under the corresponding limits, Lack and Street first construct $Kl(K) —$ the free completion under Kleisli objects. They then take $EM(K)$ to be $Kl(K^{op})^{op}$. Since a Kleisli object is a colimit, to construct $Kl(K)$ one must complete $K$ embedded in $[K^{op}, \text{Cat}]$, by Yoneda, under the class of $Φ$-colimits, where $Φ$ consists of the weights for Kleisli objects. This amounts to taking the closure of the representables under $Φ$-colimits [19]. By the theory of such completions, we obtain a 2-functor $Z : K \to EM(K)$ with the property that for any 2-category $L$ with Eilenberg-Moore objects, composition with $Z$ induces an equivalence of categories between the 2-functor category $[K, L]$ and the full subcategory of the 2-functor category $[EM(K), L]$ consisting of those 2-functors which preserve Eilenberg-Moore objects [24]. Furthermore, the theory of completions under a class of colimits also tells us that $Z$ will be fully faithful.

The Eilenberg-Moore completion can also be given a concrete description. The object of $EM(K)$ are the monads in $K$ and the morphisms are the usual morphisms of monads. Hence, a morphism from $T = (T : B \to B, µ, η)$ to $T' = (T' : C \to C, µ', η')$ in $EM(K)$ is a morphism...
F: B → C and a 2-morphisms φ: T'F ⇒ FT of K satisfying two equations:

\[
\begin{align*}
T'T'F & \xrightarrow{T'\phi} T'FT \xrightarrow{\phi} FT \xrightarrow{T} TTT \\
TF & \xrightarrow{\phi} FT \\
T'T'F & \xrightarrow{T'\phi} T'FT \xrightarrow{\phi} FT \xrightarrow{T} TTT \\
\end{align*}
\]

A crucial observation made by Lack and Street is that the 2-morphisms in EM(K) are not the 2-morphisms of the 2-category of monads. Rather, a 2-morphism from (F, φ) to (F', ψ) in EM(K) consists of a 2-morphism f: F → F'T satisfying

\[
\begin{align*}
T'F & \xrightarrow{\phi} FT \xrightarrow{fT} F'TT \\
F & \xrightarrow{\psi} F'TT \xrightarrow{\phi} FT \xrightarrow{T} T'T \\
\end{align*}
\]

2.4 EM-Completions in Vect

The Eilenberg-Moore completion may seem rather substantial, so in order to gain some insight into this procedure we briefly discuss the implications of this completion for Vect. We will then construct the adjunction that generates a given monad in Σ(Vect). The objects of EM(Σ(Vect)) will be the monads in Σ(Vect). In this case, a monad in Σ(Vect) is an algebra in the traditional sense of linear algebra — a vector space equipped with an associative, unital multiplication. For the duration of this example ‘algebra’ is to be interpreted in this sense; not in the sense of an algebra for a monad. A morphism in EM(Σ(Vect)) from an algebra A1 to an algebra A2 amounts to a vector space V together with a linear map V ⊗ A2 → A1 ⊗ V such that

\[
\begin{align*}
V \otimes A2 & \xrightarrow{\delta \otimes A2} A1 \otimes V \otimes A2 \xrightarrow{A1 \otimes \phi} A1 \otimes A1 \otimes V \\
V \otimes m2 & \xrightarrow{\phi} A1 \otimes V \\
V \otimes A2 & \xrightarrow{V \otimes \iota2} V \otimes A2 \xrightarrow{\phi} A1 \otimes V \\
\end{align*}
\]

commute, where (m1, \iota1) and (m2, \iota2) are the multiplication and unit for the algebras A1 and A2 respectively. This might be described as a left-free bimodule: a vector space V with a right A2 action on A1 ⊗ V given by A1 ⊗ V ⊗ A2 → A1 ⊗ A1 ⊗ V. This action makes A1 ⊗ V into a (A1, A2)-bimodule. The composite of morphisms (V, φ): A1 → A2 and (V', φ'): A2 → A3 is given by (V ⊗ V', φ ⊗ V' ⊗ φ'): A1 → A3 — the left-free bimodule A1 ⊗ V ⊗ V'.

A 2-morphism in EM(Σ(Vect)) from (V, φ) ⇒ (V', ψ) is a linear map ρ: V → A1 ⊗ V' making

\[
\begin{align*}
V \otimes A2 & \xrightarrow{\phi} A1 \otimes V \xrightarrow{A1 \otimes \rho} A1 \otimes A1 \otimes V' \\
\rho \otimes A2 & \xrightarrow{\phi} A1 \otimes V \xrightarrow{A1 \otimes \rho} A1 \otimes A1 \otimes V' \\
A1 \otimes V' \otimes A2 & \xrightarrow{A1 \otimes \psi} A1 \otimes A1 \otimes V' \xrightarrow{m1 \otimes V'} A1 \otimes V' \\
\end{align*}
\]

commute. This amounts to saying that a 2-morphism is just a bimodule homomorphism of left-free bimodules. To summarize:

Every Frobenius algebra in Vect will be shown to arise from an ambijunction in the 2-category EM(Σ(Vect)) consisting of: algebras, left-free bimodules, and bimodule homomorphisms.
Recall that the Eilenberg-Moore completion was obtained from the Kleisli completion as $K(\mathcal{K}^{op})^{op}$. Hence, a similar description of the Kleisli completion of $\Sigma(\textbf{Vect})$ can be given in terms of right-free bimodules.

Ambijunctions in the Eilenberg-Moore completion of $\Sigma(\textbf{Vect})$ correspond to the notion of a Frobenius extension familiar to algebraists, see for example [15]. For an algebra $A$ over the field $k$ we have the inclusion map $\iota: k \rightarrow A$. The category of $A$-modules corresponds to the category of algebras for the monad $A$ in $\Sigma(\textbf{Vect})$. The restriction functor $\text{Res}: A\mod A \rightarrow k\mod k$ has left and right adjoint functors: the induction functor $\text{Ind}(M) = A \otimes_k M$ and coinduction $\text{CoInd}(M) = \text{Hom}_k(A, M)$. When $A$ is a Frobenius algebra in $\textbf{Vect}$ these functors are isomorphic defining an ambijunction generating $A$.

### 2.5 Frobenius monads and ambijunctions

**Lemma 15.** Let $T = (T, \mu, \eta, \varepsilon)$ be a Frobenius monad on $\mathcal{K}$ with $\iota, \varepsilon, \mu: T \dashv T$. For notational convenience, denote the induced comonad of Proposition 6 on $T$ as $G$. Then the 2-natural isomorphism $\mathcal{M}$ of Theorem 13 satisfies the commuting diagram

$$
\begin{array}{ccc}
\mathcal{T}\text{-Alg} & \xrightarrow{\mathcal{M}} & G\text{-CoAlg} \\
\downarrow F^T & & \downarrow F^G \\
\mathcal{K}( -, B) & & \mathcal{K}( -, B)
\end{array}
$$

**Proof.** By Theorem 13 all we must show is that $\mathcal{M}F^T = F^G$. This equality can be verified pointwise. Let $s: A \rightarrow B$ so that

$$
\mathcal{M}_A F^T_A(s) = (Ts, T\mu s, \iota s)
$$

$$
F^G_A(s) = (Ts, \delta s) = (Ts, T^2(\varepsilon, \mu)s, T^2 \mu Ts, T^2s, T^2s, T^2s).
$$

The required equality follows by the commutativity of the following diagram:

$$
\begin{array}{cccc}
T^3 s & \xrightarrow{T\mu s} & T^2 s & \xrightarrow{T\varepsilon} & T^2 s \\
\downarrow T^3 s & & \downarrow T^2 s & & \downarrow T^2 s \\
T^3 s & \xrightarrow{T^2 \mu s} & T^4 s & \xrightarrow{T^2 \varepsilon} & T^4 s \\
\downarrow T^3 s & & \downarrow T^4 s & & \downarrow T^4 s \\
T^3 s & \xrightarrow{T^2 \mu s} & T^4 s & \xrightarrow{T^2 \varepsilon} & T^4 s
\end{array}
$$

Stated with the fact that if $h: s \Rightarrow s'$, then $F^T(h) = Th = F^G(h)$ and the fact that $\mathcal{M}$ is the identity on morphisms.$\square$

**Theorem 16.** Given a Frobenius monad $(\mathbb{T}, \mu, \eta, \varepsilon)$ on an object $B$ in $\mathcal{K}$, then in $\text{EM}(\mathcal{K})$ the left adjoint $F^\mathbb{T}: B \rightarrow B^\mathbb{T}$ to the forgetful functor $U^\mathbb{T}: B^\mathbb{T} \rightarrow B$ is also right adjoint to $U^\mathbb{T}$ with counit $\varepsilon$. Hence, the Frobenius monad $\mathbb{T}$ is generated by an ambidextrous adjunction in $\text{EM}(\mathcal{K})$.

**Proof.** Identify $\mathbb{T}$ with its fully faithful image via the 2-functor $Z: \mathcal{K} \rightarrow \text{EM}(\mathcal{K})$; then an Eilenberg-Moore object $B^\mathbb{T}$ for the monad $\mathbb{T}$ exists in $\text{EM}(\mathcal{K})$. Let $G$ denote the induced comonad.

---

3This description of the Eilenberg-Moore completion and Kleisli completion was explained to the author by Steve Lack.
structure on $T$ given in Proposition 6. Then by the adjoint monad theorem, the object $B^T$ serves as an Eilenberg-Moore object $B^G$ for the comonad $G$ and we have the commutative diagram:

$$
\begin{array}{ccc}
B^T & \xrightarrow{M} & B^G \\
\downarrow{U^T} & & \downarrow{U^G} \\
B & & B
\end{array}
$$

By the remarks following Definition 11 we have $i^T, e^T : F^T \dashv U^T : B \to B^T$ and $i^G, e^G : F^G \dashv U^G : B \to B^G$.

Since $U^G F^G$ generates the comonad $G$ and $\varepsilon$ is the counit for the comonad $G$, it is clear that $e^G = \varepsilon$ above. All that remains to be shown is that $\mathcal{M} F^T = F^G$. This follows by the 2-categorical Yoneda lemma applied to Lemma 16. Hence, the Frobenius monad $T = U^T F^T$ is generated by an ambijunction $F^T \dashv U^T \dashv F^T$ in $\mathbf{EM}(K)$.

**Theorem 17.** Let $i, e, j, k : F \dashv U \dashv F : B \to C$ be an ambidextrous adjunction in the 2-category $K$. Then the monad $(UF, UiF, e)$ generated by the adjunction is a Frobenius monad with $\varepsilon = k$.

**Proof.** All we must show is that $UF \dashv UF$ with counit $k.UiF$. Define the unit of the adjunction to be $UjFi$. The zig-zag identities follow from the zig-zag identities for $(i, e)$ and $(j, k)$.

**Corollary 18.** If $B \xrightarrow{F} C$ is a specified ambijunction in the 2-category $K$, then $UF$ is a Frobenius object in the strict monoidal category $K(B, B)$.

**Proof.** By Theorem 16 $UF$ defines a Frobenius monad on the object $B$ in $K$. As explained above, this is simply a Frobenius object in the monoidal category $K(B, B)$.

**Corollary 19.** A Frobenius object in a monoidal category $M$ yields an ambijunction in $\mathbf{EM}(\Sigma(M))$, where $\Sigma(M)$ is the 2-category obtained by the strictification of the suspension of $M$.

**Proof.** Recall that a monad on an object $B$ in a 2-category $K$ can be thought of as a monoid object in the monoidal category $K(B, B)$. Similarly, a comonad on $B$ is just a comonoid object in $K(B, B)$. Regarding $M$ as a one object 2-category $\Sigma(M)$, a Frobenius object in $M$ is simply a Frobenius monad in $\Sigma(M)$. Applying Theorem 16 completes the proof.

**Corollary 20.** Every Frobenius algebra in the category $\mathbf{Vect}$ arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.

**Proof.** This follows immediately from Corollary 16 and the discussion in subsection 2.4.

**Corollary 21.** Every 2D topological quantum field theory, in the sense of Atiyah [2], arises from an ambidextrous adjunction in the 2-category whose objects are algebras, morphisms are bimodules of algebras, and whose 2-morphisms are bimodule homomorphisms.

**Proof.** Since a 2D topological quantum field theory is equivalent to a commutative Frobenius algebra [1, 22], the proof follows from Corollary 20.
3 CATEGORIFICATION

In this section we extend the theory of the previous section to the context of Gray-categories. Gray is the symmetric monoidal closed category whose underlying category is 2-Cat; the category whose objects are 2-categories, and whose morphisms are 2-functors. Gray differs from 2-Cat in that Gray has a more interesting monoidal structure than the usual cartesian monoidal structure on 2-Cat. A Gray-category, also known as a semistrict 3-category, is defined using enriched category theory [19] as a category enriched in Gray. The unusual tensor product in Gray, or ‘Gray tensor product’, has the effect of equipping a Gray-category K with a cubical functor $M : K(B,C) \times K(A,B) \to K(A,C)$ for all objects $A,B,C$ in $K$. This means that if $f : F \Rightarrow F'$ in $K(A,B)$, and $g : G \Rightarrow G'$ in $K(B,C)$, then, rather than commuting on the nose, we have an invertible 3-cell $M_{g,f}$ — denoted $g_f$ following Marmolejo [28] — in the following square:

$\begin{array}{ccc}
GF & \xrightarrow{g_F} & G'F \\
Gf & \downarrow & \downarrow g_f \\
GF' & \xrightarrow{g_{F'}} & G'F'.
\end{array}$

We take this notion to be a sufficiently general extension since every tricategory or weak 3-category is triequivalent to a Gray-category [12].

The proof of the adjoint monad theorem relied heavily on the notion of mates under adjunction and the fact that this relationship respected composites of morphisms and adjunctions. In order to categorify this theorem we will first have to categorify the notion of mates under adjunction to the notion of mates under pseudoadjunction. In this case, rather than a bijection between certain morphisms, we will have an equivalence of Hom categories. The naturality of this equivalence will also be discussed.

In Section 3.2 we define the notion of a pseudomonad in a Gray-category and review some of the basic theory. Using the notion of mateship under pseudoadjunction it is shown that if a pseudomonad has a specified pseudoadjoint $G$, then $G$ is a pseudocomonad. All of the theorems from the previous section are then extended into this context and the notion of a Frobenius pseudomonad and Frobenius pseudomonoid are given. The main result that every Frobenius pseudomonoid arises from a pseudo ambijunction is then proven as a corollary of the categorified version of the Eilenberg-Moore adjoint monad theorem in Section 3.3.

3.1 Pseudoadjunctions

We begin with the definition of a pseudoadjunction given by Verity in [37] where they were called locally-adjoint biadjoint pairs. For more details see also the discussion by Lack where the ‘free living’ or ‘walking’ pseudoadjunction is defined [23].

Definition 22. A pseudoadjunction $I, E, i, e : F \dashv_p U : A \to B$ in a Gray-category $K$ consists of:

- morphisms $U : A \to B$ and $F : B \to A$,
- 2-morphisms $i : 1 \Rightarrow UF$ and $e : FU \Rightarrow 1$, and
- coherence 3-isomorphisms

$\begin{array}{ccc}
& U & \\
& \downarrow i & \downarrow u e \\
U & \Rightarrow & U \\
& 1 & \downarrow 1 \\
& F & \Rightarrow F
\end{array}$

and

$\begin{array}{ccc}
& F & \\
& \downarrow e F \\
F U F & \Rightarrow & F \\
& 1 & \downarrow 1 \\
& F & \Rightarrow F
\end{array}$
such that the following two diagrams are both identities:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
FU & \xrightarrow{F^*U} & FUU \\
1 & \xrightarrow{F^*U} & 1 \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
UF & \xrightarrow{UFU} & UFU \\
1 & \xrightarrow{UFU} & 1 \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

We will sometimes denote a pseudoadjunction as \( F \dashv_p U \) and say that the morphism \( U \) is the right pseudoadjoint of \( F \). Likewise, \( F \) is said to be the left pseudoadjoint of \( U \).

**Proposition 23.** If \( I, E, i, e : F \dashv_p U : A \to B \) and \( I', E', i', e' : F' \dashv_p U' : B \to C \), then \( FF' \dashv_p U'U \) with

\[
\begin{align*}
\bar{i} & := 1 \xrightarrow{i''} U'F'U'F' \xrightarrow{U'EF'} U'UFU'F' \\
\bar{e} & := FF'U'U \xrightarrow{F\bar{e}'U'} FU \xrightarrow{e} 1 \\
\end{align*}
\]

and

\[
\begin{align*}
\bar{I} & := U'UFU'U \\
\bar{E} & := FF'U'F'F' \xrightarrow{F\bar{e}'F'} FUFU'F' \\
\end{align*}
\]

**Proof.** The proof is given in [13] although it is a routine verification and can be checked directly.

**Proposition 24.** Let

- \( I, E, i, e : F \dashv_p U : A \to B \), and
- \( I', E', i', e' : F' \dashv_p U' : A' \to B' \)

in the Gray-category \( K \). If \( a : A \to A' \) and \( b : B \to B' \), then there is an equivalence of categories \( K(bU, U'a) \simeq K(F'b, aF) \) given by:

\[
\begin{align*}
\Theta : K(bU, U'a) & \to K(F'b, aF) \\
\xi & \mapsto \zeta = F'b \xrightarrow{F'b_i} F'bUF \xrightarrow{F'\xi F} F'U'aF \xrightarrow{e'aF} aF \\
\omega : \xi_1 \Rightarrow \xi_2 & \mapsto F'b \xrightarrow{F'b_i} F'bUF \xrightarrow{F'\omega F} F'U'aF \xrightarrow{e'aF} aF \\
\end{align*}
\]

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and

\[ \Phi : K(F'b, aF) \to K(bU, U'a) \]
\[ \zeta \mapsto \xi = bU \xrightarrow{i \zeta} U' F'b U \xrightarrow{U' \zeta} U'aFU \xrightarrow{U'a e} U'a \]
\[ \varrho : \zeta_1 \Rightarrow \zeta_2 \mapsto bU \xrightarrow{i \zeta_1} U' F'b U \xrightarrow{U' \gamma} U'aFU \xrightarrow{U'a e} U'a. \]

**Proof.** It is clear that \( \Theta \) is a functor from its definition above. That is, \( \Theta \) preserves composites of 3-morphisms along 2-morphisms in \( K \). Let \( \xi \) be an object of \( K(bU, U'a) \) so that \( \Phi \Theta(\xi) \) is given by the composite

\[ bU \xrightarrow{i \zeta} U' F'b U \xrightarrow{U' \zeta} U' F'U'aFU \xrightarrow{U'e \zeta} U' aFU \xrightarrow{U'a e} U'a. \]

Define an isomorphism \( \gamma_\xi : \xi \Rightarrow \Phi \Theta(\xi) \) by the diagram

which is invertible because \( I' \), \( E \) and the structural maps in the Gray-category are invertible. It is straightforward to check the naturality of this isomorphism. Let \( \omega : \xi_1 \Rightarrow \xi_2 \); then \( \gamma_{\zeta_1} \circ \omega = \Phi \Theta(\omega) \circ \gamma_{\zeta_2} \) by the invertibility of \( I' \), \( E \) and the axioms of a Gray-category. The isomorphism \( \gamma_{\zeta} : \zeta \Rightarrow \Theta \Phi(\zeta) \), for \( \zeta \) in \( K(F'b, aF) \), is given by:

By similar arguments as above this isomorphism is natural. \( \square \)

Using this equivalence of categories we extend the notion of mateship under adjunction to the notion of mateship under pseudoadjunction. We now express the naturality conditions this equivalence satisfies:

**Proposition 25.** Consider the collection of pseudoadjunctions and morphisms:
• $I, E, i, e: F \rightarrow_{p} U: A \rightarrow B,$
• $I', E', i', e': F' \rightarrow_{p} U': A' \rightarrow B',$
• $I'', E'', i'', e'': F'' \rightarrow_{p} U'': A'' \rightarrow B'',$ and
• $a: A \rightarrow A', a': A' \rightarrow A'', b: B \rightarrow B', b': B' \rightarrow B'',$
in the Gray-category $K$. Let

\[
\Theta: K(bU, U'a) \rightarrow K(F'b, aF)
\]
\[
\Phi: K(F'b, aF) \rightarrow K(bU, U' a)
\]
\[
\Theta': K(b'U', U''a) \rightarrow K(F''b', a'F')
\]
\[
\Phi': K(F''b', a'F') \rightarrow K(b'U', U''a')
\]
\[
\Theta: K(b'bU, U''a) \rightarrow K(F''b'b, a'aF)
\]
\[
\Phi: K(F''b'b, a'aF) \rightarrow K(b'bU, U''a)
\]

be the functors from Proposition 3.1 defining the relevant equivalences of categories. Then there exists a natural isomorphism $W$ between the following pasting composites of functors:

\[
a'\Theta(-), \Theta'(-) b : K(bU, U'a) \times K(b'U', U''a') \rightarrow K(F''b'b, a'aF)
\]
\[
\Theta(-, b') : K(bU, U'a) \times K(b'U', U''a') \rightarrow K(F''b'b, a'aF),
\]

and a natural isomorphism $Y$ between the pasting composites:

\[
\Phi'(-) a, \Phi(-) : K(F'b, aF) \times K(F''b', a'F') \rightarrow K(b'bU, U''a')
\]
\[
\Phi(a', - , - b) : K(F'b, aF) \times K(F''b', a'F') \rightarrow K(b'bU, U''a'').
\]

**Proof.** Let $\xi \in K(bU, U'a)$ and $\xi' \in K(b'U', U''a')$, then $W(\xi \times \xi')$ is given by the following pasting composite of invertible 3-morphisms:

If $\omega: \xi_1 \Rightarrow \xi_2$ and $\omega': \xi'_1 \Rightarrow \xi'_2$ then the naturality of $W$ follows from the axioms of the cubical functor defining the Gray tensor product. Given $\zeta \in K(F'b, aF)$ and $\zeta' \in K(F''b', a'F')$ the
natural isomorphism $Y$ can be defined similarly:

and by similar arguments, $Y$ can be shown to be natural.

We will find it necessary later in this paper to refer to the natural isomorphisms $W$ and $Y$ within the context of some specified choices of composable pseudoadjunctions and morphisms $a, a', b, b'$. We will refer to these isomorphisms generically as $W$ and $Y$, even though we will consider many different choices of pseudoadjunctions and morphisms. This is possible since these natural isomorphisms exist for every possible choice of this data. Furthermore, the specific pseudoadjoints and morphisms should be clear from the context. When no confusion is likely to arise we will also denote $\Theta', \Phi', \Psi'$, and $\Phi$ simply as $\Theta$ and $\Phi$, respectively. Note in particular that when $a = 1_A, b = 1_B, a' = 1_A, b' = 1_B$, then we have $\Theta(\xi) \cdot \Theta(\xi') \cong \Theta(\bar{\xi}, \bar{\xi}')$, and similarly for $\Phi$.

**Proposition 26.** Consider the collection of pseudoadjunctions and morphisms:

- $I, E, i, e: F \dashv_p U: A \to B$,
- $I'_1, E'_1, i'_1, e'_1: F' \dashv_p U': A' \to B'$,
- $I_1, E_1, i_1, e_1: F_1 \dashv_p U_1: B \to C$,
- $I'_1, E'_1, i'_1, e'_1: F'_1 \dashv_p U'_1: B' \to C'$, and
- $a: A \to A', b: B \to B', c: C \to C'$

in the Gray-category $\mathcal{K}$. Let

$$
\begin{align*}
\Theta: K(bU, U'a) &\to K(F'b, aF) & \Phi: K(F'b, aF) &\to K(bU, U'a) \\
\Theta_1: K(cU_1, U'_1 b) &\to K(F'_1 c, bF_1) & \Phi_1: K(F'_1 c, bF_1) &\to K(cU_1, U'_1 b) \\
\bar{\Theta}: K(cU_1 U, U'_1 U'a) &\to K(F'_1 F'_1 c, aFF_1) & \bar{\Phi}: K(F'_1 F'_1 c, aFF_1) &\to K(cU_1 U, U'_1 U'a)
\end{align*}
$$

be the functors from Proposition 27 defining the relevant equivalences of categories. Here $\bar{\Theta}$ and $\bar{\Phi}$ are the equivalence corresponding to the composite pseudoadjunction defined in Proposition 26.

Then there exists a natural isomorphism $V$ between the following pasting composites of functors:

$$
\Theta(-): F_1 \cup \Theta_1(-) : K(bU, U'a) \times K(cU_1, U'_1 b) \to K(F'_1 F'_1 c, aFF_1)
$$
$$
\bar{\Theta}(U'_1 - . - U) : K(bU, U'a) \times K(cU_1, U'_1 b) \to K(F'_1 F'_1 c, aFF_1),
$$

and a natural isomorphism $X$ between the pasting composites:

$$
U'_1 \Phi(-) . \Phi_1(-) U : K(F'b, aF) \times K(F'_1 c, bF_1) \to K(cU_1 U, U'_1 Ua)
$$
$$
\bar{\Phi}(-) F_1 F'(-) : K(F'b, aF) \times K(F'_1 c, bF_1) \to K(cU_1 U, U'_1 Ua).$$
Proof. Let $\xi \in K(bU, U'a)$ and $\xi_1 \in K(cU_1, U_1'b)$, then $V(\xi \times \xi_1)$ is given by the following pasting composite of invertible 3-morphisms:

Since this 3-isomorphism is composed entirely of Gray-naturality isomorphisms, it is clear that $V$ is a natural isomorphisms. Given $\zeta \in K(Fb, aF)$ and $\zeta_1 \in K(F_1c, bF_1)$ the natural isomorphism $X$ can be defined similarly:

As with the isomorphisms $W$ and $Y$ in Proposition 26 we will find it necessary to generically refer to the natural isomorphisms $V$ and $X$ even though we may consider many different choices of pseudoadjunctions and composable morphisms $a, b, c$. Again, this is allowed because these natural isomorphisms exist for every choice of this data.

Before moving on to the theory of pseudomonads we first collect a result about the functors $\Theta$ and $\Phi$.

Proposition 27. Let $\Theta$ and $\Phi$ be as in Proposition 24 with $I, E, i, e: F \Rightarrow_p U = I', E', i', e': F' \Rightarrow_p U'$ and $a = 1_A$ and $b = 1_B$. Then in the category $K(U, U)$ the object $\Phi(1_F)$ is isomorphic to the object $1_U$, and in the category $K(F, F)$ the object $\Theta(1_U)$ is isomorphic to the object $1_F$.

Proof. The isomorphisms are $I^{-1}$ and $E$ respectively.

3.2 Pseudomonads

Here we present the theory of pseudomonads. For more details see [23, 28, 29].

Definition 28. A pseudomonad $T = (T, \mu, \eta, \lambda, \rho, \alpha)$ on an object $B$ of the Gray-category $K$ consists of an endomorphism $T: B \to B$ together with:

- multiplication for the pseudomonad: $\mu: T^2 \Rightarrow T$,
- unit for the pseudomonad: $\eta: 1 \Rightarrow T$, and
such that the following two equations are satisfied:

\[
\begin{align*}
T^4 & \xrightarrow{T^2 \mu} T^3 & T^3 & \xleftarrow{T^2 \mu} T^2 \\
\mu T^2 & \xrightarrow{T \alpha} T^2 & \mu T^2 & \xrightarrow{T \alpha} T^2
\end{align*}
\]

This definition was given by F. Marmolejo in [28] and can be understood as a pseudomonoid (in the sense of [7]) in \(K(B, B)\). An elegant treatment of pseudomonads is presented in [23] where the ‘free living’ or ‘walking’ pseudomonad is defined. A pseudocomonad \(G\) induces a pseudomonad \(\bar{G}\) by reversing the directions of the 2-cells in the definition of a pseudomonad. A pseudoadjunction

**Proposition 29 (Lack [23]).** A pseudoadjunction \(F \dashv_p U : B \to C\) in the Gray-category \(K\) induces a pseudomonad \((UF, i, UeF, IF, UE, Ue^{-1})\) on the object \(B\) in \(K\).

**Proposition 30.** Let \(T = (T, \mu, \eta, \lambda, \rho, \alpha)\) be a pseudomonad on an object \(B\) in a Gray-category \(K\) such that the endomorphism \(T : B \to B\) has a specified right pseudoadjoint \(G\) with counit \(\sigma : TG \to 1\), unit \(\iota : 1 \to GT\), and coherences \(\Upsilon : \sigma T. T \iota \to 1\) and \(\Sigma : 1 \to G \sigma . G\). Then mateship under pseudoadjunction, together with the natural isomorphisms in Propositions 28 and 26, define a pseudocomonad \(\bar{G} = (G, \varepsilon, \delta, \bar{\lambda}, \bar{\rho}, \bar{\alpha})\) on \(G\) with explicit formulas:

\[
\begin{align*}
\varepsilon & := \Phi(\eta) = \sigma. \eta G \\
\delta & := \Phi(\mu) = G^2 \sigma. G^2 \mu. G. G T. G \times G
\end{align*}
\]
Under these circumstances \( G \) is said to be a pseudocomonad right pseudoadjoint to the pseudomonad \( T \), denoted \( T \dashv_p G \).

**Proof.** Mateship under pseudoadjunction preserves composites along morphisms and pseudoadjoints up to natural isomorphism by Propositions 25 and 26. Therefore because \( T = (T, \mu, \eta, \lambda, \rho, \alpha) \) is a pseudomonad, \( G = (G, \varepsilon, \delta, \bar{\lambda}, \bar{\rho}, \bar{\alpha}) \) defines a pseudocomonad on \( B \).

**Definition 31.** A pseudomonad \( T \) in the Gray-category \( K \) is called a Frobenius pseudomonad if it is equipped with a map \( \varepsilon : T \to 1 \) such that \( \varepsilon.\mu \) is the counit for a specified pseudoadjunction \( T \dashv_p T \).

We use this notion of Frobenius pseudomonad to define a Frobenius pseudomonoid in a Gray-monoid or semistrict monoidal 2-category. A Gray-monoid is just a one object Gray-category.

In particular, if \( K \) is a Gray-category and \( B \) is an object of \( K \), then \( K(B, B) \) is a Gray-monoid. A Frobenius pseudomonad on \( B \) is then just a Frobenius pseudomonoid in the Gray-monoid \( K(B, B) \). This definition of Frobenius pseudomonoid takes the minimalists approach, a pseudomonoid equipped with the specified pseudoadjoint structure that enables one to construct a pseudocomonoid structure. For a more explicit description of this definition see Street’s work [35].

One may prefer the definition of a Frobenius pseudomonoid to be symmetrical: a pseudomonoid, and a pseudocomonoid subject to compatibility conditions. In the sequel to this paper we explain the relationship between these two perspectives which turn out to be equivalent in a precise sense [25].

We now describe the generalization of algebras for a monad and construct the 2-category of pseudoalgebras based at \( A \) for a pseudomonad \( T \). Pseudoalgebras for a 2-monad were first explicitly defined by Street [32] and were well known to the Australian category theory community at that time [18]. For a treatment using the powerful machinery of Blackwell-Kelly-Power [5], see [4]. The treatment we give here follows Marmolejo [28].

**Definition 32.** Let \( T \) be a pseudomonad in the Gray-category \( K \) and let \( A \) be an object of \( K \). We define a pseudoalgebra based at \( A \) for the pseudomonad \( T \) to consist of:

- a morphism \( s : A \to B \),
- 2-morphisms \( \nu : Ts \Rightarrow s \), and
- 3-isomorphisms

\[
\begin{array}{ccc}
  s & \xrightarrow{\eta s} & Ts \\
  \downarrow \nu & & \downarrow \\
  s & \xrightarrow{T \nu} & Ts \\
\end{array}
\]

and

\[
\begin{array}{ccc}
  s & \xleftarrow{\chi} & Ts \\
  \downarrow \nu & & \downarrow \\
  s & \xleftarrow{T \chi} & Ts \\
\end{array}
\]

such that the following two equations are satisfied:

\[
\begin{array}{ccc}
  T^3 s & \xrightarrow{T^2 \nu} & T^2 s \\
  \downarrow \mu T^2 & & \downarrow T \mu T \\
  T^2 s & \xrightarrow{T \mu} & Ts \\
\end{array} = \begin{array}{ccc}
  T^3 s & \xrightarrow{T^2 \nu} & T^2 s \\
  \downarrow \mu T^2 & & \downarrow T \mu T \\
  T^2 s & \xrightarrow{T \mu} & Ts \\
\end{array}
\]

\[
\begin{array}{ccc}
  T^2 s & \xrightarrow{\mu \nu} & Ts \\
  \downarrow \chi & & \downarrow \\
  T^2 s & \xrightarrow{T \chi} & Ts \\
\end{array} = \begin{array}{ccc}
  T^2 s & \xrightarrow{\mu \nu} & Ts \\
  \downarrow \chi & & \downarrow \\
  T^2 s & \xrightarrow{T \chi} & Ts \\
\end{array}
\]

\[
\begin{array}{ccc}
  T^2 s & \xrightarrow{\mu} & s \\
  \downarrow \nu & & \downarrow \\
  T^2 s & \xrightarrow{T \nu} & Ts \\
\end{array} = \begin{array}{ccc}
  T^2 s & \xrightarrow{\mu} & s \\
  \downarrow \nu & & \downarrow \\
  T^2 s & \xrightarrow{T \nu} & Ts \\
\end{array}
\]

\[
\begin{array}{ccc}
  T s & \xrightarrow{\chi} & Ts \\
  \downarrow \nu & & \downarrow \\
  T s & \xrightarrow{T \chi} & Ts \\
\end{array}
\]

\[
\begin{array}{ccc}
  s & \xrightarrow{\eta s} & T s \\
  \downarrow \nu & & \downarrow \\
  s & \xrightarrow{T \nu} & Ts \\
\end{array}
\]
satisfying the following two equations:

\[
\begin{align*}
Ts & \xrightarrow{T\eta s} T^2 s \\
\mu_s & \quad \mu'_{s'}
\end{align*}
\]

\[
\begin{align*}
Ts & \xrightarrow{T\nu s} T^2 s \\
\nu & \quad \nu'
\end{align*}
\]

It is clear that for any morphism \( r: A \rightarrow B \) in \( K, Tr \) with action \( \mu r: T^2 r \Rightarrow Tr \) and coherence \( \lambda r: \mu r.\eta Tr \Rightarrow Tr \) and \( \alpha r: \mu r.\mu t \Rightarrow \mu r.\mu Tr \) is a pseudoalgebra based at \( A \). We call the pseudoalgebra \( Tr \) a free pseudoalgebra.

**Definition 33.** Let \( T\text{-Alg}_A \) be the 2-category whose objects are pseudoalgebras based at \( A \) for the pseudomonad \( T \). A morphism \( (h, \varrho): (s, \nu, \psi, \chi) \rightarrow (s', \nu', \psi', \chi') \) in \( T\text{-Alg}_A \) consists of a 2-morphism \( h: s \Rightarrow s' \) in \( K \) (a morphism in \( K(A, B) \)), together with an invertible 3-morphism

\[
\begin{align*}
Ts' \xrightarrow{\mu'_{s'}} & \xrightarrow{T\mu s'} Ts' \\
\nu' & \quad \nu
\end{align*}
\]

satisfying the following two equations:

\[
\begin{align*}
Ts & \xrightarrow{T\eta s} T^2 s \\
\mu_s & \quad \mu'_{s'}
\end{align*}
\]

\[
\begin{align*}
Ts & \xrightarrow{T\nu s} T^2 s \\
\nu & \quad \nu'
\end{align*}
\]

A 2-morphism \( \xi: (h, \varrho) \Rightarrow (h', \varrho'): (\psi, \chi) \rightarrow (\psi', \chi') \) in \( T\text{-Alg}_A \) is a 3-morphisms \( \xi: h \Rightarrow h' \) such that the following condition is satisfied:

\[
\begin{align*}
Ts & \xrightarrow{T\xi s} T^2 s \\
\nu & \quad \nu'
\end{align*}
\]
Marmolejo has shown that given a morphism $K: A' \to A$ in $\mathbb{K}$, one can define a change of base 2-functor $\hat{k}: \mathbb{T}_{\text{-}\text{Alg}} \to \mathbb{T}_{\text{-}\text{Alg}}$. If $\xi: (h, \varrho) \Rightarrow (h', \varrho')$: $(s, \nu, \psi, \chi) \to (s', \nu', \psi', \chi')$ is in $\mathbb{T}_{\text{-}\text{Alg}}$, then its image under $\hat{k}$ is $\xi K: (hK, \varrho K) \Rightarrow (h'K, \varrho'K)$: $(sK, \nu K, \psi K, \chi K) \to (s'K, \nu'K, \psi'K, \chi'K)$. If $k: K \Rightarrow K'$ in $\mathbb{K}$ then we get a pseudo natural transformation $\hat{k}: K \Rightarrow K'$ such that $\hat{k}((s, \nu, \psi, \chi)) = (sk, \nu_k^{-1})$ and $\hat{k}(h, \varrho) = h_k$. If $\kappa: k \Rightarrow k': K \Rightarrow K'$, then $\kappa(s, \nu, \psi, \chi) = sk\kappa$ defines a modification $\hat{\kappa}: \hat{k} \Rightarrow \hat{k}'$ in $\mathbb{K}$. In fact, this shows that the construction of $\mathbb{T}$-pseudoalgebras defines a $\mathbb{Gray}$-functor $\mathbb{T}_{\text{-}\text{Alg}}: \mathbb{K}^{\text{op}} \to \mathbb{Gray}$.

For every object $A$ in $\mathbb{K}$ there is a forgetful 2-functor $U^A_{\mathbb{T}}: \mathbb{T}_{\text{-}\text{Alg}} \to \mathbb{K}(A, B)$

$$U^A_{\mathbb{T}}: \mathbb{T}_{\text{-}\text{Alg}} \to \mathbb{K}(A, B)$$

$$(s, \nu, \psi, \chi) \mapsto s \quad (h, \varrho) \mapsto h \quad \xi: h \Rightarrow h' \mapsto \xi.$$

This assignment extends to a $\mathbb{Gray}$-natural transformation $U^T: \mathbb{T}_{\text{-}\text{Alg}} \to \mathbb{K}(\cdot, B)$. In Proposition 3.24 we will define a left pseudoadjoint $F^A$ to the 2-functor $U^A_{\mathbb{T}}$, see also [28]. In Theorem 3.25 we will show that this left pseudoadjoint $F^A$ extends to $\mathbb{Gray}$-natural transformation $F^T: \mathbb{K}(\cdot, B) \to \mathbb{T}_{\text{-}\text{Alg}}$ left pseudoadjoint to $U^T$ in the $\mathbb{Gray}$-category $[\mathbb{K}^{\text{op}}, \mathbb{Gray}]$ described below.

Recall that $\mathbb{Gray}$ is the symmetric monoidal closed category whose closed structure is given by the internal hom in $\mathbb{Gray}$. Hence, for $\mathbb{Gray}$-functors $F, G: \mathbb{K} \to \mathbb{L}$ the internal hom $\mathbb{Gray}(F, G)$ in $\mathbb{Gray}$ is the 2-category consisting of 2-functors, pseudo natural transformations, and modifications. It is a standard result from enriched category theory that $\mathbb{Gray}$-categories, $\mathbb{Gray}$-functors, and $\mathbb{Gray}$-natural transformations form a 2-category written $\mathbb{Gray}_{\text{-}\text{Cat}}$ [6, 19]. Furthermore, since $\mathbb{Gray}$ is a complete symmetric monoidal closed category, if $\mathbb{K}$ is small, then the category of $\mathbb{Gray}$-functors and $\mathbb{Gray}$-natural transformations can be provided with the structure of a $\mathbb{Gray}$-category, written $[\mathbb{K}, \mathbb{L}]$.

The objects of $[\mathbb{K}, \mathbb{L}]$ are the $\mathbb{Gray}$-functors $F, G: \mathbb{K} \to \mathbb{L}$, and the morphisms are the $\mathbb{Gray}$-natural transformations between them. The 2-category $\mathbb{Gray}_{\text{-}\text{Nat}}(F, G)$ of $\mathbb{Gray}$-natural transformations is given by the following equalizer:

$$\mathbb{Gray}_{\text{-}\text{Nat}}(F, G) \xrightarrow{\Pi_{A \in \mathbb{K}} \mathbb{L}(FA, GA)} \Pi_{A', A'' \in \mathbb{K}} \mathbb{K}(A', A'') \xrightarrow{\Pi_{A \in \mathbb{K}} \mathbb{L}(FA', GA')}$$

where $u$ and $v$ are the morphisms corresponding via adjunction and symmetry to the morphisms$^4$:

$$(\Pi_A \mathbb{L}(FA, GA)) \otimes \mathbb{K}(A', A'') \xrightarrow{\Pi_A \mathbb{L}(FA, GA) \otimes \Pi_A \mathbb{L}(FA, GA) \otimes \mathbb{K}(A', A'') \otimes \mathbb{K}(A', A'')} (\Pi_A \mathbb{L}(FA, GA))$$

$$(\Pi_A \mathbb{L}(FA', GA')) \otimes \mathbb{L}(FA', GA') \xrightarrow{(\Pi_A \mathbb{L}(FA, GA)) \otimes (s \nu, \psi, \chi) \Rightarrow (s \nu, \psi, \chi)} (\Pi_A \mathbb{L}(FA, GA))$$

$$(\Pi_A \mathbb{L}(FA', GA')) \otimes \mathbb{L}(FA', GA') \otimes \mathbb{L}(FA', GA') \otimes \mathbb{L}(FA', GA')$$

$$(\Pi_A \mathbb{L}(FA', GA')) \otimes \mathbb{L}(FA', GA') \otimes \mathbb{L}(FA', GA') \otimes \mathbb{L}(FA', GA')$$

$$\mathbb{L}(FA', GA') \otimes \mathbb{L}(FA', GA')$$

We will refer to the morphisms and 2-morphisms of the 2-category $\mathbb{Gray}_{\text{-}\text{Nat}}(F, G)$ as $\mathbb{Gray}$-modifications and $\mathbb{Gray}$-perturbations respectively. This terminology should not be interpreted to mean some sort of ‘$\mathbb{Gray}$ enriched modification’ or ‘$\mathbb{Gray}$ enriched perturbation’ since there is no such notion as a $\mathbb{V}$-modification or $\mathbb{V}$-perturbation for arbitrary enriching category $\mathbb{V}$.

Let $\alpha, \beta: F \Rightarrow G: \mathbb{K} \to \mathbb{Gray}$ be $\mathbb{Gray}$-natural transformations with $\mathbb{K}$ a small $\mathbb{Gray}$-category. A $\mathbb{Gray}$-modification $\theta: \alpha \Rightarrow \beta$ assigns to each object $A$ of $\mathbb{K}$ a pseudo natural transformation

$^4$Here we are using the notation for $\mathbb{V}$-functors and $\mathbb{V}$-natural transformations given in [6].
Let \( \varphi: \alpha \Rightarrow \beta: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray} \) be a pseudomonad in the \text{Gray}-category \( \mathcal{K} \). Then the following equality holds:

\[
\begin{align*}
\Omega: & \xrightarrow{\varphi} \alpha \Rightarrow \beta: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray} \\
\alpha & \xrightarrow{\varphi} \beta \Rightarrow \gamma: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray} \\
\beta & \xrightarrow{\varphi} \gamma \Rightarrow \delta: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray} \\
\delta & \xrightarrow{\varphi} \delta \Rightarrow \epsilon: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray}
\end{align*}
\]

If \( \Omega: \varphi: \alpha \Rightarrow \beta: F \rightarrow G: \mathcal{K} \rightarrow \text{Gray} \) are \text{Gray}-modifications, a \text{Gray}-perturbation assigns to each object \( A \in \mathcal{K} \) a modification \( \Omega_A: \varphi_A \Rightarrow \theta_A \) such that if \( \kappa: k \Rightarrow k': K \rightarrow K': A' \rightarrow A'' \) in \( \mathcal{K} \), then the following equality holds:

\[
\begin{align*}
\varphi_A & \xrightarrow{\Omega} \varphi' \Rightarrow \theta': F' \rightarrow G': \mathcal{K} \rightarrow \text{Gray} \\
\theta_A & \xrightarrow{\Omega} \theta' \Rightarrow \theta''': F'' \rightarrow G'' : \mathcal{K} \rightarrow \text{Gray}
\end{align*}
\]

**Proposition 34.** Let \( \mathcal{T} \) be a pseudomonad in the \text{Gray}-category \( \mathcal{K} \). Then the forgetful 2-functor \( U^A_\mathcal{T}: \mathcal{T}-\text{Alg}_A \rightarrow \mathcal{K}(A,B) \) has a left pseudoadjoint \( F^A_\mathcal{T}: \mathcal{K}(A,B) \rightarrow \mathcal{T}-\text{Alg}_A \) in the \text{Gray}-category \text{Gray} given by sending each object \( r \) of \( \mathcal{K}(A,B) \) to the corresponding free pseudoalgebra \( (Tr, \mu_r, \lambda_r, \alpha_r) \), each morphism \( h: r \rightarrow r' \) to \( (Th, \mu_{h}^{-1}) \), and each 2-morphism \( \xi: h \Rightarrow h' \) to \( T\xi: Th \Rightarrow Th' \).

**Proof.** First we show that \( F^A_\mathcal{T} \) is a 2-functor. It is clear that if \( r: A \rightarrow B \) in \( \mathcal{K} \), then \( (Tr, \mu_r, \lambda_r, \alpha_r) \) is a pseudoalgebra based at \( A \). If \( h: r \Rightarrow r': A \rightarrow B \) then \( Th \) is a morphism of pseudoalgebras with \( \mu_{h}^{-1} \) playing the role of the invertible 2-morphism \( \varrho \). Indeed, the axioms:

\[
\begin{align*}
T^2r & \xrightarrow{\eta_{Tr}} TT^2r \\
T^2r & \xrightarrow{\mu_{Tr}} TTh \\
T^2r & \xrightarrow{\lambda_r} TT^2r \\
T^2r & \xrightarrow{1} TTr \\
T^2r & \xrightarrow{Th} Tr \\
T^2r & \xrightarrow{\mu_{r'}} TTr' \\
T^2r & \xrightarrow{\lambda_{r'}} TT^2r' \\
T^2r & \xrightarrow{1} TTr' \\
T^2r & \xrightarrow{Th} Tr'
\end{align*}
\]
follow directly from the axioms of a Gray-category. If $\xi : h \Rightarrow h'$ is a 2-morphism in $\mathcal{K}(A, B)$ then $T\xi$ is a 2-morphism of pseudoalgebras since the equation

$\mu_r \circ (\mu_s \circ \lambda_s \circ \alpha_s) = (\mu_s \circ \lambda_s \circ \alpha_s) \circ \mu_r$ is satisfied, again, because of the axioms of a Gray-category. It is straightforward to verify that all composites and identities are preserved.

Since $\mathcal{K}$ is a Gray-category, $\mathcal{K}(A, B)$ is an object of Gray, i.e., a 2-category. It is clear that if $\mathcal{T}$ is a pseudomonad on $B$ in $\mathcal{K}$, then $\mathcal{K}(A, \mathcal{T})$ is a pseudomonad on the 2-category $\mathcal{K}(A, B)$ in the Gray-category Gray. We now show that $F_A^* \mathcal{T}$ is the left pseudoadjoint of $U_A^* \mathcal{T}$. Define the pseudo natural transformation $i^*_A: 1 \Rightarrow U_A^* F_A^* \mathcal{T}_A$ to be the unit $i^*_A(s) = \eta s$ and $i^*_A(h) = h \eta^{-1}$. We define $e^*_A: F_A^* U_A^* \Rightarrow 1$ on any pseudoalgebra $(s, \nu, \psi, \chi)$ to be the morphism of pseudoalgebras given by setting $h = \nu$ and $\varrho = \chi$. The relevant axioms for a morphism of pseudoalgebras follow using the axioms of the pseudoalgebra $(s, \nu, \psi, \chi)$ and the pseudomonad $\mathcal{T}$. To establish the pseudo naturality of $e^*_A$ let $(h, \varrho): (s, \nu, \psi, \xi) \Rightarrow (s', \nu', \psi', \xi')$ be a map of pseudoalgebras. Then the pseudo naturality 2-cell filling the square:

$\begin{array}{ccc}
(Ts, \mu s, \lambda s, \alpha s) & \xrightarrow{(Th, \mu_r^{-1})} & (Ts, \mu s', \lambda s', \alpha s') \\
(\nu, \chi) \downarrow & & \downarrow (\nu', \chi') \\
(s, \nu, \psi, \chi) & \xrightarrow{(h, \varrho)} & (s', \nu', \psi', \chi')
\end{array}$

is given by the 3-isomorphism $\varrho$.

We now describe the coherence modifications for this pseudoadjunction in Gray. Define $I^*_A: 1_{U_A^*} \Rightarrow U_A^* F_A^* \mathcal{T}_A$ on the pseudoalgebra $(s, \nu, \psi, \chi)$ to be $\psi^{-1}$. It is easy to check that this map defines a modification between pseudo natural transformations $1_{U_A^*}$ and $U_A^* F_A^* \mathcal{T}_A$. We define $E_A^*: c_A^* F_A^* \Rightarrow 1_{E_A^*}$ on the map $s$ to be the 2-homomorphism of pseudomonads $\varrho^{-1}s$. The fact that this map is a modification follows from the pseudo naturality of $\mathcal{K}(A, \varrho)$.

To establish the coherence of $I_A^*$ and $E_A^*$ consider the diagram below:

The commutativity of this diagram can be deduced from the second coherence condition in the
The definition of a pseudoalgebra. The other coherence law for a pseudoadjunction,

\[
\begin{array}{ccc}
T^2 s & 1 & \eta_{\alpha} \\
\downarrow & & \\
\eta_{\alpha} & & s
\end{array}
\]

\[
\begin{array}{ccc}
T s & \mu_{\alpha} & \eta_{\alpha} \\
\downarrow & & \\
\eta_{\alpha} & & s
\end{array}
\]

can be deduced from the axioms of a pseudomonad, see Proposition 8.1 of [28].

**Theorem 35.** The collection of pseudoadjunctions:

\[
\begin{array}{ccc}
I^T, E^T, \iota^T, e^T : F^T A \rightarrow U^T A \rightarrow T-Alg \rightarrow K(A, B)
\end{array}
\]

defined for each \( A \) in Proposition 34 extend to a pseudoadjunction

\[
\begin{array}{ccc}
I^T, E^T, \iota^T, e^T : F^T \rightarrow U^T \rightarrow T-Alg \rightarrow K(-, B)
\end{array}
\]

in the \( \text{Gray} \)-category \([K^{op}, \text{Gray}]\). In particular, \( F^T \) is a \( \text{Gray} \)-natural transformation, \( \iota^T, e^T \) are \( \text{Gray} \)-modifications, and \( I^T, E^T \) are \( \text{Gray} \)-perturbations.

**Proof.** Let \( \kappa : k \Rightarrow k' : K \Rightarrow K ; A' \rightarrow A \) in the \( \text{Gray} \)-category \( K \). To check that \( F^T \) is a \( \text{Gray} \)-natural transformation we must check that:

\[
\begin{array}{ccc}
K(A, B) & \xrightarrow{F^T_A} & T-Alg \\
\downarrow \kappa & & \downarrow \hat{K} \\
K(A', B) & \xrightarrow{F^T_{A'}} & T-Alg
\end{array}
\]

commutes. Let \( \xi : h \Rightarrow h' : s \Rightarrow s' \) in \( K(A, B) \). Then

\[
\hat{K} F^T_A (\xi) = T \xi (K (ThK, \mu_{\alpha}^{-1} K) \Rightarrow (Th'K, \mu_{\alpha}^{-1} K) : (TsK, \mu_{s} K, \lambda sK, \alpha sK) \Rightarrow (Ts's'K, \mu_{s'} K, \lambda s'K, \alpha s'K))
\]

\[
= T \xi (K (ThK, \mu_{\alpha}^{-1} K) \Rightarrow (Th'K, \mu_{\alpha}^{-1} K) : (TsK, \mu_{s} K, \lambda sK, \alpha sK) \Rightarrow (Ts's'K, \mu_{s'} K, \lambda s'K, \alpha s'K))
\]

\[
= F^T_A K (\xi).
\]

Next we show that the collection of \( e^T_A \) define a \( \text{Gray} \)-modification \( e^T : F^T U^T \Rightarrow 1_{T-Alg} \). To see that \( e^T \) is a \( \text{Gray} \)-modification we must verify the following equality:

\[
\begin{array}{ccc}
F^T_A U^T_A \hat{K} & \xrightarrow{e^T_A, \hat{K}} & F^T_A U^T_A \hat{K}' \\
\downarrow k & & \downarrow k \hat{e}^{-1}_A \\
\hat{K} & \xrightarrow{k} & \hat{K}'
\end{array}
\]

\[
\begin{array}{ccc}
\hat{K} F^T_A U^T_A & \xrightarrow{\hat{K} e^T_A} & \hat{K}' F^T_A U^T_A \\
\downarrow k & & \downarrow k \\
\hat{K} & \xrightarrow{k} & \hat{K}'
\end{array}
\]

Note that

\[
F^T_A U^T_A \hat{K}(s, \nu, \psi, \chi) = (TsK, \mu sK, \lambda sK, \alpha sK) = \hat{K} F^T_A U^T_A (s, \nu, \psi, \chi)
\]

\[
F^T_A U^T_A \hat{K}'(s, \nu, \psi, \chi) = (TsK', \mu sK', \lambda sK', \alpha sK') = \hat{K}' F^T_A U^T_A (s, \nu, \psi, \chi).
\]

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Furthermore, \( F_A^T U_A^T k(s, \nu, \psi, \chi) = (Tsk, \mu_{sk}^{-1}) = (Tsk, \mu_{sk}^{-1}k) = k F_A^T U_A^T(s, \nu, \psi, \chi) \), and \( e_A^T K(s, \nu, \psi, \chi) = (\nu K, \chi K) = K e_A^T(s, \nu, \psi, \chi) \). Hence, the desired equality is satisfied by the Gray axioms asserting the following equality:

To show that the collection of pseudo natural transformations \( i_A^T \) define a Gray-modification \( i^T : 1_{K(-, B)} \Rightarrow U^T F^T \) we must verify the following equality:

\[
\begin{align*}
\mathcal{K}(K, B) & \xrightarrow{\kappa(k, B)} \mathcal{K}(K', B) \\
\mathcal{K}(K, B) & \xrightarrow{\bar{\kappa}_{A'}(\kappa(k, B), i_A^T) \bar{\kappa}_{K(B)}(K, B) \bar{\kappa}_{K(K', B)}(K, B) \bar{\kappa}_{K(K', B)}(K, B)} \mathcal{K}(K', B) \\
\end{align*}
\]

Note that

\[
U_A^T F_A^T \mathcal{K}(K, B)(s) = TsK = \mathcal{K}(K, B) U_A^T F_A^T(s)
\]

and

\[
U_A^T F_A^T \mathcal{K}'(K, B)(s) = TsK' = \mathcal{K}'(K', B) U_A^T F_A^T(s).
\]

Furthermore,

\[
i_A^T \mathcal{K}(K, B)(s) = (\eta sK) = \mathcal{K}(K, B) i_A^T(s)
\]

and

\[
U_A^T F_A^T \mathcal{K}(k, B)(s) = (Ts k) = \mathcal{K}(k, B) U_A^T F_A^T(s).
\]

Hence, the desired equality follows from the Gray axioms asserting the equality

\[
\begin{align*}
\mathcal{K}(K, B) & \xrightarrow{\kappa(k, B)} \mathcal{K}(K', B) \\
\mathcal{K}(K, B) & \xrightarrow{\bar{\kappa}_{A'}(\kappa(k, B), i_A^T) \bar{\kappa}_{K(B)}(K, B) \bar{\kappa}_{K(K', B)}(K, B) \bar{\kappa}_{K(K', B)}(K, B)} \mathcal{K}(K', B) \\
\end{align*}
\]
In order to show that the collection of $I_A^T$ define a Gray-perturbation we must establish the following equality:

This clearly follows from the Gray axioms. To establish that the collection of $E_A^T$ define a Gray-perturbation we must verify the following equality of pseudoalgebra 2-homomorphisms:

Again this equality above is follows directly from the Gray axioms. The fact that this data defines a pseudoadjunction in the Gray-category $[K^{op}, \text{Gray}]$ follows from Proposition since the coherence conditions are verified pointwise.

Note that the previous theorem can also be adapted to the context of a pseudocomonad $G$ on $B$. In this case, one obtains a Gray-functor $G\text{-CoAlg} : K^{op} \to \text{Gray}$. As before there exists a forgetful Gray-natural transformation $U^G : G\text{-CoAlg} \to K(-, B)$. However, in this case, $U^G$ has a right pseudoadjoint $F^G$.

### 3.3 Pseudoadjoint pseudomonads

In Section 2.3 it was explained how thinking of an Eilenberg-Moore object as a weighted limit can be used to construct the free completion of a 2-category under Eilenberg-Moore objects. In this section we will need to generalize the notion of an Eilenberg-Moore object for a monad to an Eilenberg-Moore object for a pseudomonad. It turns out that thinking of an Eilenberg-Moore object as weighted limit will prove useful for this task as well.
Denote the ‘free living monad’ or ‘walking monad’ as $\text{mmd}$, meaning that a monad in a 2-category $\mathcal{K}$ is a 2-functor $\text{mmd} \rightarrow \mathcal{K}$. Street has constructed a 2-functor $J: \text{mmd} \rightarrow \text{Cat}$ with the property that the Eilenberg-Moore object of a monad $T: \text{mmd} \rightarrow \mathcal{K}$ is the $J$-weighted limit of the 2-functor $T^{\circ} [33]$. This idea was used by Lack to construct a Gray-category $\text{psm}$ — the ‘free living pseudomonad’ — such that a pseudomonad $T$ in the Gray-category $\mathcal{K}$ is a Gray-functor $T: \text{psm} \rightarrow \mathcal{K} [23]$. Lack also constructs a Gray-functor $P: \text{psm} \rightarrow \text{Gray}$ with the property that the Eilenberg-Moore object of the pseudomonad $T$ is the $P$-weighted limit of the Gray-functor $T: \text{psm} \rightarrow \mathcal{K}$, denoted $\{P, T\}$.

This limit does not always exist in $\mathcal{K}$, but $\mathcal{K}$ can always be embedded via the Yoneda embedding $Y: \mathcal{K} \rightarrow [\mathcal{K}^{\text{op}}, \text{Gray}]$ where the $P$-weighted limit of $YT$ can be formed. Then the Eilenberg-Moore object of $T$ will exist in $\mathcal{K}$ if and only if $\{P, YT\}$ is representable since the Yoneda embedding must preserve any limits which exist. The Gray-functor $\{P, YT\}$ is just the Gray-functor $T$-$\text{Alg}$ constructed in the previous section. Thus, an Eilenberg-Moore object for the pseudomonad $T$ is just a choice of representation for $T$-$\text{Alg}$.

If $T$-$\text{Alg}$ is representable, then in our previous notation, it will correspond to the Gray-functor $\mathcal{K}(-, B^T)$ where $B^T$ is an Eilenberg-Moore object for the pseudomonad $T$. If an Eilenberg-Moore object for $T$ does exist then the pseudoadjunction $I^T, E^T, i^T, e^T: F^T \multimap_p U^T: T$-$\text{Alg} \rightarrow \mathcal{K}(-, B)$ of Theorem 35 corresponds via the enriched Yoneda lemma to a pseudoadjunction $I, E, i, c: F \multimap_p U: B \rightarrow B^T$ in $\mathcal{K}$.

The limit description of an Eilenberg-Moore object in a Gray-category also facilitates the free completion of an arbitrary Gray-category to one that has Eilenberg-Moore objects. Indeed, because a Gray-category is just a Gray-enriched category, the free-completion is achieved using the theory of enriched category theory [19]. With all of the abstract theory in place, we begin by proving the pointwise version of the categorified adjoint monad theorem. In Theorem 37 we will prove the full result.

**Lemma 36.** If $\Sigma, \Upsilon, \iota, \sigma: T \multimap_p G$, then the 2-category $T$-$\text{Alg}_A$ of pseudoalgebras based at $A$ is 2-equivalent to the 2-category $G$-$\text{CoAlg}_A$ of pseudocoalgebras based at $A$ for the pseudocomonad $G$. Furthermore, this 2-equivalence commutes with the forgetful 2-functors $U^T_A: T$-$\text{Alg}_A \rightarrow \mathcal{K}(A, B)$ and $U^G_A: G$-$\text{CoAlg}_A \rightarrow \mathcal{K}(A, B)$.

**Proof.** This lemma is essentially due to the properties of pseudomates under pseudoadjunction and the fact that this association preserves composites up to natural isomorphism. With $\Theta$ and $\Phi$ as in Proposition 24 define the 2-functor:

$$M_A: T$-$\text{Alg}_A \rightarrow G$-$\text{CoAlg}_A$$

\begin{align*}
(s, \nu, \psi, \chi) \rightarrow (s, \Phi(\nu), \Phi(\eta)s.\Phi(\nu) \xrightarrow{Y} \Phi(\nu.\eta)s \xrightarrow{\Phi(\psi)} \Phi(s) = s, \\
G\Phi(\nu).\Phi(\nu) \xrightarrow{X} \Phi(\nu.\nu) \xrightarrow{\Phi(\chi)} \Phi(\nu.\mu)s \xrightarrow{Y^{-1}} \Phi(\nu)s.\Phi(\nu) \\
(h, g) \rightarrow (h, \Phi(\nu).\Phi(h) \xrightarrow{X} \Phi(\nu.\nu)h \xrightarrow{\Phi(\eta)} \Phi(h.\nu) \xrightarrow{X^{-1}} Gh.\Phi(\nu)) \\
(\xi: (h, g) \Rightarrow (h', g')) \rightarrow (h, X^{-1} \circ \Phi(g) \circ X) \Rightarrow (h', X^{-1} \circ \Phi(\eta) \circ X)
\end{align*}

This data defines a pseudocoalgebra, morphism of pseudoalgebras, and 2-morphism of pseudoalgebras because $\xi: (h, g) \Rightarrow (h', g')$: $(s, \nu, \psi, \chi) \rightarrow (s', \nu', \psi', \chi')$ is a 2-morphism of pseudoalgebras, and mateship under pseudoadjunction preserves all composites up to natural isomorphism.

Since $M_A: h: s \Rightarrow s' \mapsto h: s \Rightarrow s'$ it is clear that $M_A$ preserves 1-morphism identities and to see that $M_A$ preserves composites of 1-morphisms all we must check is its behavior on $g$. For this purpose it will be helpful to have the specific form of $M_A(h, g)$. By plugging in the relevant pseudoadjunctions, one can check that $M_A(h, g) = (h, \Phi(g) \circ G\nu'.(\iota_h))$. Hence, using the definition
of $\Phi(\varrho)$ and the Gray axioms it is easy to verify the following chain of equalities:

$$
M_A(h', g'). M_A(h, g) = (h'. h, G h'. G g. l s \circ G h'. G v'. l h' \circ G g'. l s'. h \circ G v'. l h'. h) \\
= (h'. h, G h'. G g. l s \circ G g'. G t h. l s. h \circ G v'. l h'. h) \\
= M_A(h'. h, g'. t h \circ h', g).
$$

Since $M_A$ maps 2-morphisms to themselves, it is clear that $M_A$ preserves composition of 2-morphisms on the nose as well. Hence, $M_A : T\text{-Alg}_A \to G\text{-CoAlg}_A$ is a 2-functor. Now we define the other 2-functor taking part in the equivalence:

$$
\overline{M}_A : G\text{-CoAlg}_A \to T\text{-Alg}_A
$$

\begin{align*}
(s, \bar{\nu}, \bar{\psi}, \bar{\chi}) & \mapsto (s, \Theta(\bar{\nu}), \Theta(\bar{\nu}). \Theta(s) s \overleftarrow{\Theta(\bar{\psi})} \Theta(s) = s, \\
\Theta(\bar{\nu}). T \Theta(\bar{\nu}) & \overrightarrow{\Theta(\bar{\psi})} \Theta(\bar{\delta}. s) \overleftarrow{\Theta(\bar{\psi})} \Theta(s) = \Theta(\bar{\nu}), \Theta(\bar{\nu}) = h. \Theta(\bar{\nu})
\end{align*}

\begin{align*}
(h, \bar{\varrho}) & \mapsto (h, \Theta(\bar{\varrho}). t h \overrightarrow{\Theta(\bar{\psi})} \Theta(\bar{\varrho}. h) \overleftarrow{\Theta(\bar{\psi})} \Theta(\bar{\varrho}. h) \overrightarrow{\Theta(\bar{\psi})} h) \\
\xi : (h, \bar{\varrho}) \Rightarrow (h', \bar{\varrho}') & \mapsto \xi : (h, V^{-1} \circ \Theta(\bar{\varrho}) \circ V) \Rightarrow (h', V^{-1} \circ \Theta(\bar{\varrho}) \circ V)
\end{align*}

This will define a 2-functor, again by the functoriality of mateship under pseudoadjunction and the axioms of Gray-category. It will be helpful to have the explicit formula for $\overline{M}_A(h, \bar{\varrho})$. By plugging in the relevant pseudoadjunctions one can check that

$$
\overline{M}_A(h, \bar{\varrho}) = (h, \sigma_h. T \varrho \circ \Theta(\bar{\varrho})).
$$

We now show that $M_A$ and $\overline{M}_A$ define a 2-equivalence of 2-categories. Define the 2-natural isomorphism $\Gamma_A : 1_{T\text{-Alg}_A} \Rightarrow \overline{M}_A M_A$ as follows: Denote $\overline{M}_A M_A ((s, \nu, \psi, \chi))$ as $(s, \bar{\nu}, \bar{\psi}, \bar{\chi})$. Define the morphism of pseudoalgebras $\Gamma_{(s, \nu, \psi, \chi)} : (s, \nu, \psi, \chi) \to (s, \bar{\nu}, \bar{\psi}, \bar{\chi})$ by letting $h: s \Rightarrow s$ be the identity, so that $g$ is just a map $\bar{\nu} \Rightarrow \nu$. From the definition of $M_A$ and $\overline{M}_A$ we know that $\nu = \Theta \Phi(\nu)$. Hence we can choose $\varrho$ to be the isomorphism $\tilde{\gamma}_\varrho^{-1} : \Theta(\Phi(\nu)) \Rightarrow \nu$ defined in Proposition 24. The pair $(\nu, \tilde{\gamma}_\varrho^{-1})$ is a morphism of pseudoalgebras by the naturality of the isomorphism $\delta$ of Proposition 24 applied to the 3-morphisms $\psi$ and $\chi$. The explicit form of the isomorphism $\tilde{\gamma}_\varrho^{-1}$ is $\nu. T s \circ \sigma_h^{-1}. T i s$.

To see that $\Gamma_A$ is natural in the one dimensional sense, suppose that $(h, \varrho) : (\psi, \chi) \Rightarrow (\psi', \chi')$ is an arbitrary 1-cell in $T\text{-Alg}_A$. Consider the following diagram:

$$
\begin{array}{ccc}
(s, \nu, \psi, \chi) & \xrightarrow{(h, \varrho)} & (s', \nu', \psi', \chi') \\
\downarrow{(1, \tilde{\gamma}_\varrho^{-1})} & & \downarrow{(1, \tilde{\gamma}_\varrho^{-1})}
\end{array}
$$

Note that since $h: s \Rightarrow s'$ for some $s': A \to B$, the pseudoadjunction determining the mate of $h$ is the identity adjunction so that $h$ is its own mate under pseudoadjunction. Thus, this diagram of pseudoalgebra maps commutes if and only if

$$
h. \gamma_\varrho \circ \bar{\varrho} = \varrho \circ \gamma_\varrho'. T h.
$$

Using the explicit formulas given above we have that

$$
\overline{M}_A M_A (h, \varrho) = (h, \sigma_h^{-1}. T \varrho. T i s \circ \sigma s'. T G \varrho. T i s \circ \sigma s'. T G \varrho'. T i^{-1}_h).
$$
In order to prove the naturality of $\Gamma_A$ we will need the following equalities that all follow directly from the axioms of a Gray-category:
\[
\begin{align*}
    h.\sigma_{\nu}^{-1} \circ \sigma_{h.\nu}^{-1} \cdot TG_{\nu} &= \sigma_{h.\nu}^{-1}
    \\
    \sigma_{h.\nu}^{-1} \circ \sigma_{s'.TG_{\rho}}^{-1} &= \varrho_sTG_{s'} \circ \sigma_{\nu'.Th}^{-1}
    \\
    \sigma_{\nu'.Th} &= \nu'.\sigma_{Th}^{-1} \circ \sigma_{h.\nu}^{-1} \cdot TG_{Th}
    \\
    \sigma_{Th}^{-1} \circ \sigma_{Ts'T_{i_h}^{-1}}^{-1} &= (\sigma.Ti_h)_h^{-1}
    \\
    Th.\Upsilon_{s} \circ (\sigma.Ti_h)_h^{-1} &= Th_s \circ \Upsilon_{s}.Th = \Upsilon_{s'.Th}
\end{align*}
\]

The proof of equation 4 above is as follows:
\[
\begin{align*}
    h.\gamma_{\nu} \circ \tilde{\tilde{\rho}} &= h.\nu.\Upsilon_s \circ h.\sigma_{\nu}^{-1}.Ti_{s} \circ \sigma_{h.\nu}^{-1}.TG_{\nu}.Ti_{s} \circ \sigma_{s'.TG_{\rho}}^{-1}.TG_{s'.Ti_{i_h}^{-1}}^{-1}
    \\
    &= h.\nu.\Upsilon_s \circ \sigma_{h.\nu}^{-1}.Ti_{s} \circ \sigma_{s'.TG_{\rho}}^{-1}.TG_{s'.Ti_{i_h}^{-1}}^{-1}
    \\
    &= h.\nu.\Upsilon_s \circ \sigma_{h.\nu}^{-1}.Ti_{s} \circ \sigma_{\nu'.Th}^{-1}.TG_{s'.Ti_{i_h}^{-1}}^{-1}
    \\
    &= h.\nu.\Upsilon_s \circ \sigma_{h.\nu}^{-1}.Ti_{s} \circ \sigma_{\nu'.(\sigma.Ti_h)_h^{-1}}^{-1}.TG_{s'.Ti_{i_h}^{-1}}^{-1}
    \\
    &= \varrho \circ \nu'.Ti_{s} \circ \sigma_{\nu'.Ti_{i_h}^{-1}}^{-1}.TG_{\nu}.Th (\text{Interchange})
    \\
    &= \varrho \circ \nu'.\Upsilon_{s'.Th} \circ \sigma_{\nu'.Ti_{i_h}^{-1}}^{-1}.TG_{\nu}.Th (\text{Interchange})
    \\
    &= \varrho \circ \gamma_{\nu}.Th
\end{align*}
\]

To see the 2-naturality of $\Gamma_A$ let $\xi: (h, \rho) \Rightarrow (h', \rho')$, then the equality:
\[
\begin{align*}
    (\psi, \chi) &\quad \xymatrix{ (\psi', \chi') & (\tilde{\psi}, \tilde{\chi}) \ar[l]_-{(h, \varrho)} \ar[r]^-{(1, \gamma_{\nu}^{-1})} & (\tilde{\psi}', \tilde{\chi}' \ar[l]_-{(h', \varrho')} } \\
    &\ar@{}[r]|{=} & \\
    &\ar@{}[r]|{=} & \begin{array}{c}
    (\psi, \chi) \ar[r]_-{(1, \gamma_{\nu}^{-1})} & (\psi', \chi') \ar[r]_-{\overline{M_A\overline{M_A}(h, \varrho)}} & (\tilde{\psi}, \tilde{\chi}) \ar[r]_-{\overline{M_A\overline{M_A}(h', \varrho')}} & \end{array}
\end{align*}
\]

follows from the fact that $\overline{M_A\overline{M_A}(\xi)} = \xi$ and the naturality of $\tilde{\tilde{\rho}}$ applied to the 3-morphism $\xi$ in $K$. A 2-natural isomorphism $\overline{\overline{\overline{T}}_A}: \overline{M_A\overline{M_A}} \Rightarrow 1_{\overline{G-CoAlg}}$ can be defined in a similar way.

To see that this 2-equivalence of 2-categories commutes with the forgetful 2-functors, note that in the above proof we have shown that the 2-equivalence is the identity on the base map $s$ of the pseudoalgebra. Furthermore, in the discussion of naturality we have shown that for any map $(h, \varrho)$ of pseudoalgebras $M_A$ is the identity on $h$ and $M_A$ also acts as the identity on every 3-cell defining a 2-morphism of pseudoalgebras. Thus, by the definition of the forgetful 2-functors $U_A^T$ and $U_A^G$ it is clear that the equivalence $\overline{M_A\overline{M_A}}$ commutes with the forgetful 2-functors. \qed

**Theorem 37 (The categorified adjoint monad theorem).** If $\Sigma, \Upsilon, \iota, \sigma: T \Rightarrow p, G$ in $K$, then the Gray-functors $T_{-Alg}$ and $G_{-Alg}$ are Gray-equivalent in the Gray-category $[K^{op}, \text{Gray}]$. This means that there exists Gray-natural transformations $M: T{-Alg} \rightarrow G{-CoAlg}$, $\overline{M}: G{-CoAlg} \rightarrow T{-Alg}$ and invertible Gray-modifications $\Gamma: 1_{T{-Alg}} \Rightarrow \overline{M}M$, $\overline{T}: MM \Rightarrow 1_{G{-CoAlg}}$. Furthermore, this Gray-equivalence commutes with the forgetful Gray-natural transformations $U^T$ and $U^G$.

**Proof.** Define a Gray-natural transformation $M: T{-Alg} \rightarrow G{-CoAlg}$ which assigns to each object in $K^{op}$ the 2-functor $M_A$ defined in the preceding lemma. To see the naturality of $M$ let

\[
\begin{align*}
    h.\sigma_{\nu}^{-1} \circ \sigma_{h.\nu}^{-1} \cdot TG_{\nu} &= \sigma_{h.\nu}^{-1}
    \\
    \sigma_{h.\nu}^{-1} \circ \sigma_{s'.TG_{\rho}}^{-1} &= \varrho_sTG_{s'} \circ \sigma_{\nu'.Th}^{-1}
    \\
    \sigma_{\nu'.Th} &= \nu'.\sigma_{Th}^{-1} \circ \sigma_{h.\nu}^{-1} \cdot TG_{Th}
    \\
    \sigma_{Th}^{-1} \circ \sigma_{Ts'T_{i_h}^{-1}}^{-1} &= (\sigma.Ti_h)_h^{-1}
    \\
    Th.\Upsilon_{s} \circ (\sigma.Ti_h)_h^{-1} &= Th_s \circ \Upsilon_{s}.Th = \Upsilon_{s'.Th}
\end{align*}
\]
Recall that \( \nu K \) and \( \bar{\gamma} \) (\( \mu s \))

Hence, we have shown that the Gray-natural transformation \( \overline{M} : G\text{-CoAlg} \to T\text{-Alg} \) defines a Gray-natural transformation \( \overline{M} : G\text{-CoAlg} \to T\text{-Alg} \).

Let \( M \), then there exists an invertible \( \gamma(s K, \nu K) = \theta K(\nu K) \) and \( \bar{\gamma}^{-1} K = \bar{\gamma}^{-1} \).

In a similar fashion one can define an invertible Gray-modification \( \overline{M} = M(M M) \Rightarrow 1 \text{-CoAlg} \). Hence, we have shown that the Gray-functors \( T\text{-Alg} \) and \( G\text{-CoAlg} \) are Gray-equivalent in the Gray-category \( [K^{op}, G] \). Lemma \ref{lemma:Gray-equivalence} shows that this Gray-equivalence commutes with the forgetful functors.

**Theorem 38.** Let \( T = (T, \mu, \eta, \lambda, \rho, \alpha, \varepsilon) \) be a Frobenius pseudomonad on the object \( B \) of the Gray-category \( K \) with \( \Sigma, \Upsilon, \iota, \varepsilon, \mu; T \Rightarrow T \). Denote the induced pseudomonad structure on \( K \) by \( G \). Then there exists an invertible Gray-modification \( \Xi : MF^T \Rightarrow F^G \) of the Gray-category \( [K^{op}, G] \). Alternatively, the morphisms \( MF^T \) and \( F^G \) of the Gray-category \( [K^{op}, G] \) are isomorphic.

**Proof.** We begin by defining a 2-natural isomorphism \( \Xi_A : A F^T_A \Rightarrow G : K(A, B) \Rightarrow G\text{-CoAlg}_A \). Recall that \( F^T_A(s) = (T s, \mu s, \lambda s, \alpha s) \) and that \( F^G_A(s) = (T s, \delta s, \rho^{-1} s, \bar{\alpha} s) \) with \( \delta, \rho, \bar{\alpha} \) of the form given in Proposition \ref{prop:Gray-equivalence}. Hence,

\[
M_A F^T_A(s) = (T s, \Phi((\mu s)), \Phi((\lambda s)), \Phi((\alpha s)))
\]

\[
= (T s, T \mu s, \lambda s, \Phi((\lambda s)), \Phi((\alpha s)))
\]

\[
F^G_A(s) = (G s, \delta s, \rho^{-1} s, \bar{\alpha} s) = (T s, T^2(\varepsilon, \mu)sT s, T^2 s, T s, \rho^{-1} s, \bar{\alpha} s).
\]

The double parenthesis are used in order to distinguish which pseudoadjunction is intended. For example, \( \Phi((\mu s)) \) is determined by the pseudoadjunctions with \( F = T, U = T, F' = T^2, U' = G^2 \) and morphisms \( a = b = s \). While \( \Phi((\mu s)) \) is given by the pseudoadjunctions with \( F = U = 1_B, F' = T, U' = T \) and morphisms \( a = b = T s \).

We define the isomorphism of pseudocoalgebras \( (h, \bar{\delta}s) : M_A F^T_A(s) \Rightarrow F^G_A(s) \) by taking \( h = 1_{T s} \).
and $\bar{s}s$ given by the following diagram:

One can verify using the definitions of $\mathcal{M}_A$, $\bar{\psi}$, and $\bar{\chi}$ that this map is indeed a morphism of pseudocoalgebras.

Let $h: s \Rightarrow s'$ in $\mathcal{K}(A, B)$. To establish the 1-naturality of $\Xi_A$ we must show that:

$$\mathcal{M}_A(Ts, \mu_s, \lambda_s, \alpha_s) \xrightarrow{\mathcal{M}_A(Th, \mu_h^{-1})} \mathcal{M}_A(Ts', \mu_{s'}, \lambda_{s'}, \alpha_{s'})$$

commutes where

$$\mathcal{M}_A(Th, \mu_h^{-1}) = \Phi((\mu_h^{-1}) \circ T\mu_{s'}((\mu_h^{-1})) = T\mu_{T_h \cdot sT} \circ T\mu_{s'}((\mu_h^{-1}))$$

This amounts to the equality of the following diagrams:
which are equal by a routine verification using the Gray-category axioms. The 2-naturality of \( \Xi_A \) follows from the fact that both \( F_A^\xi \) and \( F_B^\xi \) map the 2-morphism \( \xi: h \Rightarrow h': s \to s' \) to \( T \xi \), and the fact that the \( 2 \)-functor \( M_A \) is the identity on 2-morphisms of pseudocoalgebras.

The collection of \( \Xi_A \) define a Gray-modification by the commutativity of the following diagrams:

\[
\begin{array}{c}
\xymatrix{ 
\mathcal{M}_A F_A^\xi K(K, B) \ar[r]^{\mathcal{M}_A F_A^\xi K(k, B)} & \mathcal{M}_A F_A^\xi K(K', B) \\
F_A^\xi K(K, B) \ar[r]^{F_A^\xi K(k, B)} & F_A^\xi K(K', B) \\
\Xi_A K(K, B) \ar[d] & \Xi_A K(K', B) \ar[d] & K \Xi_A \ar[d] & K' \Xi_A \\
T \xi_k F_A^\xi K(K, B) \ar[r]_{T \xi_k F_A^\xi K(k, B)} & T \xi_k F_A^\xi K(K', B) \\
\end{array}
\]

which are both equal to

\[
\begin{array}{c}
\xymatrix{ 
(T s K, \mu s K) \ar[r]^{(T s k, \Phi((\mu s)^{-1}))} & (T s K', \mu s K') \\
(T s K, \Theta s K) \ar[d] & (T s K', \Theta s K') \ar[d] & (T s K, \Theta s K) \ar[d] \ar[r]^{(T s k, \Phi(\mu)) s_k} & (T s K', \Theta s K') \\
\end{array}
\]

\[\square\]

**Proposition 39.** Let

\[
\xymatrix{ 
B \ar[r]^{L_1} & C \\
\ar@{=>}[u]^{\varepsilon} & \\
B \ar[r]_{L_2} & C ,
\]

(or \( L_1 \vdash_p R \vdash_p L_2 \)) be pseudoadjunctions in the Gray-category \( K \). Also, let \( T_1 \) be the pseudomonad on \( B \) induced by the composite \( RL_1 \), and \( T_2 \) be the endomorphism on \( B \) induced by the composite \( RL_2 \). Then \( T_1 \vdash_p T_2 \) are pseudoadjoint morphisms, hence \( T_2 \) is with the pseudocomonad structure induced via mateship is a right pseudoadjoint pseudocomonad for the pseudomonad \( T \).

**Proof.** The composites \( RL_1 \) and \( RL_2 \) of pseudoadjoints are pseudoadjoint by Proposition 28. Thus, if we let \( T_2 \) be the pseudomonad on \( B \) determined via mateship from the pseudomonad \( T_1 \) then it is clear that \( T_1 \vdash_p T_2 \).

\[\square\]

**Theorem 40.** If \( I, E, J, K, i, e, j, k: F \vdash_p U \vdash_p F: A \to B \) is a pseudo ambijunction in the Gray-category \( K \), then the induced pseudomonad \( UF \) on \( B \) is Frobenius with \( \varepsilon = k \).

**Proof.** All we must show is that \( UF \vdash_p UF \) with counit \( k UiF \). Define the unit of the pseudo adjunction to be \( U j F i \). Then \( UF \vdash_p UF \) follows by Proposition 28.

We now make use of the fact that every Gray-category \( K \) can be freely completed to a Gray-category \( EM(K) \) where an Eilenberg-Moore object exists for every pseudomonad in \( K \).

**Theorem 41.** Given a Frobenius pseudomonad \( (T, \varepsilon) \) on an object \( B \) in the Gray-category \( K \), then in \( EM(K) \) the left pseudoadjunction \( F^T: B \to B^T \) to the forgetful Gray-functor \( U^T: B^T \to B \) is also right pseudoadjoint to \( U^T \) with counit \( \varepsilon \). Hence, the Frobenius pseudomonad \( T \) is generated by an ambidextrous pseudo adjunction in \( EM(K) \).

**Proof.** In \( EM(K) \) an Eilenberg-Moore object exists for the pseudomonad \( T \). In particular, this means that the Gray-functor \( T-\text{Alg} \) is represented by \( K(-, B^T) \) for some \( B^T \) in \( EM(K) \). Hence, the pseudoadjunction

\[
I^T, E^T, i^T, e^T: F^T \vdash_p U^T: T-\text{Alg} \to K(-, B)
\]
of Theorem 35 arises via the enriched Yoneda lemma from a pseudoadjunction

\[ \mathcal{F}_T^\varepsilon, E_T^\varepsilon, i_T^\varepsilon, e_T^\varepsilon : F_T^\varepsilon \dashv U_T^\varepsilon : B \to B_T^\varepsilon \]

in \( \mathbf{EM}(\mathcal{K}) \). Furthermore, since \( T \) is a Frobenius pseudomonad we can equip the endomorphism \( T \) with the induced pseudocomonad structure of Proposition 30. We denote this pseudocomonad as \( G \). Then the pseudoadjunction:

\[ I_G^G, E_G^G, i_G^G, e_G^G : \mathcal{M}F_G \dashv_p U_G^G : \mathcal{T}-\mathbf{Alg} \to \mathcal{K}(\_ , B) \]

given by the construction of pseudocoalgebras composed with the \( \text{Gray} \)-equivalence \( \mathcal{T}-\mathbf{Alg} \cong G-\mathbf{CoAlg} \) must also arise via the enriched Yoneda lemma from a pseudoadjunction:

\[ I_G^G, E_G^G, i_G^G, e_G^G : U_G^G \mathcal{M} \dashv_p \mathcal{M}F_G^G : B \to B_T^G \]

in \( \mathbf{EM}(\mathcal{K}) \). Since this pseudoadjunction generates the pseudocomonad \( G \), and \( G \) is defined by mateship under the self pseudoadjunction determined by \( \varepsilon \), we have that \( e_G^G = \varepsilon \).

By Theorem 37 we have that \( U_G^G \mathcal{M} = U_T^\varepsilon \). Since \( \mathcal{T}-\mathbf{Alg} \) is representable in \( \mathbf{EM}(\mathcal{K}) \) the isomorphism \( \mathcal{M}F_T^\varepsilon \cong F_G^G \) of Proposition 38 arises via the enriched Yoneda lemma from an isomorphism between the morphisms \( \mathcal{M}F_T^\varepsilon \) and \( F_G^G \) in \( \mathbf{EM}(\mathcal{K}) \). Hence, \( F_T^\varepsilon : B \to B_T^\varepsilon \) is both a left and right pseudoadjoint to \( U_T^\varepsilon \), so that the Frobenius pseudomonad \( T \) is induced from an ambidextrous pseudoadjunction.

**Corollary 42.** A Frobenius pseudomonoid in a semistrict monoidal 2-category \( \mathcal{M} \) (or \( \text{Gray}-\text{monoid} \)) yields a pseudo ambijunction in \( \mathbf{EM}(\Sigma(\mathcal{M})) \), where \( \Sigma(\mathcal{M}) \) is the \( \text{Gray} \)-category obtained from the suspension of \( \mathcal{M} \).

**Proof.** Recall that a Frobenius pseudomonoid in the \( \text{Gray} \)-monoid \( \mathcal{M} \) is just a Frobenius pseudomonad in the \( \text{Gray} \)-category \( \Sigma(\mathcal{M}) \). The result follows by Theorem 41.

**Corollary 43.** If \( \begin{array}{c} F \\ \downarrow \cong \Downarrow \\ U \\ \downarrow \cong \Downarrow \\ C \end{array} \) is a pseudo ambijunction in the \( \text{Gray} \)-category \( \mathcal{K} \), then \( UF \) is a Frobenius pseudomonoid in the semistrict monoidal 2-category \( \mathcal{K}(B, B) \).

**Proof.** By Theorem 40 \( UF \) is a Frobenius pseudomonad on \( B \) in \( \mathcal{K} \). By definition this is a Frobenius pseudomonoid in \( \mathcal{K}(B, B) \).

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