Completeness of Wronskian Bethe equations for rational $\mathfrak{gl}_{m|n}$ spin chains

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Abstract: We consider rational integrable supersymmetric $\mathfrak{gl}_{m|n}$ spin chains in the defining representation and prove the isomorphism between a commutative algebra of conserved charges (the Bethe algebra) and a polynomial ring (the Wronskian algebra) defined by functional relations between Baxter Q-functions that we call Wronskian Bethe equations. These equations, in contrast to standard nested Bethe equations, admit only physical solutions for any value of inhomogeneities and furthermore we prove that the algebraic number of solutions to these equations is equal to the dimension of the spin chain Hilbert space (modulo relevant symmetries).

Both twisted and twist-less periodic boundary conditions are considered, the isomorphism statement uses, as a sufficient condition, that the spin chain inhomogeneities $\theta_\ell$, $\ell = 1, \ldots, L$ satisfy $\theta_\ell + \hbar \neq \theta_{\ell'}$ for $\ell < \ell'$. Counting of solutions is done in two independent ways: by computing a character of the Wronskian algebra and by explicitly solving the Bethe equations in certain scaling regimes supplemented with a proof that the algebraic number of solutions is the same for any value of $\theta_\ell$. In particular, we consider the regime $\theta_{\ell+1}/\theta_\ell \gg 1$ for the twist-less chain where we succeed to provide explicit solutions and their systematic labelling with standard Young tableaux.
Notations

Typical values of indices

- \(a, b\) 1 to \(m\)
- \(i, j\) 1 to \(\hat{n}\) (hat is omitted sometimes)
- \(\alpha, \beta\) from the set \(\{1, \ldots, m, \hat{1}, \ldots, \hat{n}\}\)
- \(\ell\) 1 to \(L\)

Parameters

- \(z_\alpha\) twist eigenvalues, \(z_a \equiv x_a, z_1 \equiv y_1\)
- \(\theta_\ell\) inhomogeneities (as variables)
- \(\hat{\theta}_\ell\) inhomogeneities (fixed number)
- \(\chi_\ell\) elementary symmetric polynomials
- \(\hat{\chi}_\ell = \chi_\ell(\theta_1, \ldots, \hat{\theta}_L)\)
- \(\lambda_1, \ldots, \lambda_m, \nu_1, \ldots, \nu_n\)
- \(\lambda'_1, \lambda'_2, \ldots\)
- \(\lambda'_1, \lambda'_2, \ldots\)
- \(\bar{\lambda}_1, \ldots, \bar{\lambda}_m, \bar{\nu}_1, \ldots, \bar{\nu}_n\)

Lie algebra

- \(\mathfrak{gl}_{m|n}\) symmetry of the system (broken to Cartan in the twisted case)
- \(E_{\alpha\beta}\) abstract generators and defining representation
- \(\mathcal{E}_{\alpha\beta}\) global spin chain action
- \(\Lambda^+ = (\lambda_1, \lambda_2, \ldots)\) Young diagram \(\equiv\) integer partition (typically of \(L\))
- \(\lambda'_1, \lambda'_2, \ldots\) transposed partition, \(h_{\Lambda^+} := \lambda'_1\)
- \(\Lambda = [\lambda_1, \ldots, \lambda_m, \nu_1, \ldots, \nu_n]\) fundamental weight (eigenvalues of \(\mathcal{E}_{\alpha\alpha}\))
- \((\lambda_1, \ldots, \lambda_m, \nu_1, \ldots, \nu_n)\) shifted weight (describes \(\Lambda^+\) with marked point)

Spin chain

- \(V\) Hilbert space of the spin chain \(\simeq (\mathbb{C}^m)^{\otimes L}\)
- \(V_\Lambda\) subspace of \(V\) spanned by states of weight \(\Lambda\)
- \(V^+_\Lambda\) subspace of \(V\) spanned by highest weight states of irreps \(\Lambda^+\)
- \(U_\Lambda\) either \(V_\Lambda\) or \(V^+_\Lambda\)
- \(d_\Lambda\) dimension of \(U_\Lambda\)

Bethe and Wronskian algebras

- \(\hat{c}_k^{(d)}, \hat{c}_\ell\) operators acting on spin chain, coefficients in Baxter Q-operators, \(e.g.\)
  \[Q_k = u^{\lambda_k}(1 + \frac{2}{a} \hat{c}_k^{(d)} + \ldots)\]
- \(c_k^{(d)}, c_\ell\) abstract variables and/or eigenvalues of \(\hat{c}_k^{(d)}, \hat{c}_\ell\)
- \(\mathcal{B}_\Lambda\) Bethe algebra restricted to \(U_\Lambda\) (generated by \(\hat{c}_k\), \(\hat{c}_\ell\), a \(\mathbb{C}[\chi]\)-module
- \(\mathcal{B}_\Lambda(\hat{\theta})\) specialised Bethe algebra (for spin chain representation at point \(\hat{\theta}\))
- \(\mathcal{B}_\Lambda(\hat{\chi})\) specialised Bethe algebra (for symmetrised representation at point \(\hat{\chi}\))
- \(\mathcal{W}_\Lambda\) Wronskian algebra (generated by \(c_\ell\) subject to Wronskian Bethe equations)
- \(\mathcal{W}_\Lambda(\hat{\chi})\) specialised Wronskian algebra

Functional relations conventions

- \(u\) spectral parameter
- \(\hbar\) Unit of discrete shift in \(e.g.\) Baxter equation, typically \(\hbar = \pm i, \pm 1, \pm 2\)
- \(f^{[n]}\) \(f^{[n]} \equiv f(u + \frac{1}{n}n), f^\pm \equiv f^{[\pm 1]}\)
- \(f \propto g\) \(f\) and \(g\), as functions of \(u\), are equal up to a normalisation
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1 Introduction

Rational integrable spin chains are one of the first quantum integrable systems that were discovered and studied. In fact, their simplest SU(2) representative was introduced and solved, by means of coordinate Bethe Ansatz, in the seminal paper of Hans Bethe [1].

In this article we consider periodic integrable spin chains of length $L$ constructed using the $\mathfrak{gl}_{m|n}$-invariant rational R-matrix, and with spin chain nodes being in fundamental (defining) representation of $\mathfrak{gl}_{m|n}$. The parameters defining the model is the twist matrix $G$ and inhomogeneities $\theta_1, \ldots, \theta_L$. We cover the cases when $G$ is either equal to the identity (twist-less case) or is diagonalisable with distinct eigenvalues $x_1, \ldots, x_m, y_1, \ldots, y_n$ (generic twisted case).

Spectrum of the commuting charges that form the so-called Bethe algebra $B$ can be encoded into rational symmetric combinations of the Bethe roots $u_k^{(\alpha)}$. Equations defining the values of $u_k^{(\alpha)}$ shall be called Bethe equations, and their most known presentation is given by nested Bethe Ansatz equations (NBAE) which is the following relation between fractions [2–5]

$$\prod_{\ell=1}^{L} \frac{u_k^{(\alpha)} - \theta_\ell - \frac{c_{\alpha\lambda_1} + c_{\alpha\lambda_2}}{2} \hbar \delta_{\alpha_1}}{u_k^{(\alpha)} - \theta_\ell + \frac{c_{\alpha\lambda_1} + c_{\alpha\lambda_2}}{2} \hbar \delta_{\alpha_1}} = \frac{z_{\alpha+1}}{z_{\alpha}} \prod_{1 \leq \beta \leq m+n-1 \atop 1 \leq l \leq M_\beta \atop (\beta,l) \neq (\alpha,k)} \frac{u_k^{(\alpha)} - u_l^{(\beta)} + \frac{\hbar^2 c_{\alpha\beta}}{2}}{u_k^{(\alpha)} - u_l^{(\beta)} - \frac{\hbar^2 c_{\alpha\beta}}{2}}. \quad (1.1)$$

Here $\alpha \in \{1, 2, \ldots, m+n-1\}$ and all $k \in \{1, \ldots, M_\alpha\}$, we denote $z_{\alpha} = x_\alpha$ for $1 \leq \alpha \leq m$, and $z_{\alpha} = y_{\alpha-m}$ for $m+1 \leq \alpha \leq m+n$, and $\hbar$ is a non-zero complex number (typical choices are $i, 1, 2$). Finally $c_{\alpha\beta}$ is the Cartan matrix of the $\mathfrak{sl}_{m|n}$ subalgebra of $\mathfrak{gl}_{m|n}$. It is equal e.g. to $\begin{pmatrix} -2 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ for $\mathfrak{sl}(2|2)$, the expression for other ranks should be obvious from this example. This expression of the Cartan matrix is written in the so-called distinguished grading of $\mathfrak{gl}_{m|n}$ but other gradings are also possible [6, 7], the corresponding equations are obtained via duality transformations, and we briefly mention them in Section 5.4.
Obvious questions arising are whether each solution of the Bethe equations describes some physical state, which we call the \textit{faithfulness} property, and whether all physical states can be described in this way, which we call the \textit{completeness} property. In particular, one often asks whether the number of solutions to the Bethe equations is equal to the dimension of the Hilbert space, probably after some obvious symmetries are factored out. In the literature, these properties are typically covered by the name of completeness, however the precise meaning of the word varies.

Quite surprisingly, despite the fundamental nature of these questions, they were properly resolved only in 2009 for the \textit{gl}_2 case and in 2013 for the \textit{gl}_m case by Mukhin, Tarasov and Varchenko [8, 9]. Completeness and faithfulness were also recently proven for \textit{gl}_{4|1} by Huang, Lu, and Mukhin [10, 11]. Proving completeness and faithfulness for an arbitrary rank \textit{gl}_{m|n} case is the subject of the current paper. For formal proofs, we build on ideas of [9] and add several new insights, even for the bosonic \textit{gl}_m subcase, to achieve the result. Besides formal proofs, we also give a recipe to explicitly label solutions.

Counting of solutions was first time addressed already in [1] using the so-called string hypothesis, and later on this approach was extended to \textit{gl}_m [12], \textit{gl}_{2|1} [13] and \textit{gl}_{2|2} [14] cases. Further study of combinatorics implied by string hypothesis for \textit{gl}_m spin chains led to formulation of the Kerov-Kirillov-Reshetikhin bijection [15, 16] between rigged configurations of Bethe strings and (in case of spin chains in the defining representation) standard Young tableaux. Although counting assuming string hypothesis leads to correct numbers, the hypothesis is strictly speaking wrong as one can show by a more detailed analysis and explicit counter-examples, see e.g. [17, 18]. Hence this approach, after all, does not accomplish its original thought application – proving the completeness of rational Bethe equations. Instead, ideas of [15, 16] became extremely fruitful and were further generalised in various applications of algebraic combinatorics, in particular in the context of “combinatorial” integrability that can be viewed as the crystal limit \( q \to 0 \) of \( q \)-deformed (XXZ-type) spin chains, see e.g. [19, 20] and references therein.

Apart from combinatorial challenge, analysing solutions of (1.1) has clear technical complications. First, solutions with coinciding Bethe roots generically do not correspond to physical states and then they should be discarded. However, there are cases when such solutions should be kept [21–23]. Second, the so-called exceptional solutions with \( u = u' \pm h \) and/or \( u = \theta \pm \frac{b}{2} \) (case \( a \) in [21]) render relation (1.1) singular. Some of the exceptional solutions are physical and some of them or not, and, for instance, their behaviour upon change of twist or inhomogeneities can decide for their physicality. Tracing this behaviour becomes a burden, especially at higher rank. To our knowledge, only homogeneous \textit{gl}_2 case was properly understood [24]. At higher ranks, non-physicality can be also hidden in non-physical exceptional solutions of dual Bethe equations even if the original Bethe equations appear as being free from any singularities [25].

It is then not surprising that one should look for a different set of equations instead of (1.1) to prove completeness [26], and it is indeed the case for the proof in [9] where a very elegant Wronskian condition was used. Define the finite-difference Wronskian between any
number of $k$ functions as

$$W(F_1, \ldots, F_k) \equiv \det_{1 \leq i,j \leq k} F_i(u + \hbar(\frac{k+1}{2} - j)), \quad (1.2)$$

introduce $m$ monic polynomials $q_a = u^{M_a}$, $a = 1, \ldots, m$ of degree $M_a$. Then eigenstates of the Bethe algebra of the $\mathfrak{gl}_m$ spin chain are in one-to-one correspondence with solutions of

$$\frac{W(x_1^{u/h} q_1^{1/2}, \ldots, x_m^{u/h} q_m^{1/2})}{W(x_1^{u/h}, \ldots, x_m^{u/h})} = \prod_{\ell=1}^L (u - \theta_\ell) \quad (1.3)$$

which should be considered as equations on coefficients $c^{(k)}_{a|\Theta}$. The statement as formulated holds for the case when all $x_a$ are pairwise distinct. The twist-less case will be considered in Section 2.5.

We shall call (1.3) Wronskian Bethe equations (WBE). They are equivalent to NBAE (with coinciding Bethe roots solutions being discarded) for generic values of $\theta_\ell$ but, in contrast to (1.1), smoothly work at any values of inhomogeneities, and this includes an important physical case of homogeneous spin chain with all $\theta_\ell = 0$. Relation to Bethe roots of (1.1) is given by

$$W(x_1^{u/h} q_1^{1/2}, \ldots, x_m^{u/h} q_m^{1/2}) = W(x_1^{u/h}, \ldots, x_m^{u/h}) = \prod_{k=1}^a (u - u_k^{(a)}), \quad M_a = \sum_{b=1}^a M_{b|\Theta}.$$  

WBE are natural in the logic of the analytic Bethe Ansatz [27, 28] (though in early works in this formalism their significance was not recognised), while NBAE are often associated with the (nested) coordinate or algebraic Bethe Ansatz [29, 30] (though they are derived via analytic Bethe Ansatz as well [31, 32]). The details of the completeness and faithfulness questions depend on the chosen approach.

The nested coordinate/algebraic Bethe Ansatz is indeed an ansatz to build an eigenfunction and Bethe equations appear as the necessary consistency conditions for the ansatz to succeed. Faithfulness is then the question of sufficiency of these conditions. In physics literature it is often considered as granted, and then only the question of completeness remains which reduces to asking how many solutions to the Bethe equations are there. If there are enough of solutions we shall construct enough of eigenfunctions. The generic position faithfulness indeed follows rather straightforwardly from the ansatz itself, but a full systematic proof covering all exceptional situations would be much harder to achieve. To our knowledge, such a proof using a direct algebraic Bethe Ansatz framework was done only very recently for the $\mathfrak{gl}_2$ homogeneous case [33]. In the context of separation of variables, a new type of ansatz to build eigenstates in higher-rank systems emerged [34–36]. Its faithfulness for $\mathfrak{gl}_m$ chains is proven in the case of non-degenerate twist and assuming $\theta_\ell - \theta_\ell' \neq h\mathbb{Z}$.

The analytic Bethe Ansatz is actually not an ansatz to build wave functions. It often

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\footnote{We use notations suited for our generalisation to supersymmetric case. They are different from those in [9].}
departs from considering the Bethe algebra – a set of commuting operators which satisfy various functional relations as functions of the spectral parameter $u$ that were extensively studied [37–41]. WBE is one of (or a consequence of) these relations with $Q_{a|\emptyset} := x_a^{u/\hbar} q_{a|\emptyset}$ being the renown Q-operators, the first example of such an operator is due to Baxter [42]. In the analytic Bethe Ansatz approach, faithfulness is non-trivial to demonstrate even in generic position, and it is an important part of our paper to prove it. On the other hand, one is certain that each of Q-operators eigenvalues satisfy WBE. Hence the question of completeness becomes equivalent to the question of whether the Bethe algebra generated by Q-operators contains the full set of commuting charges, i.e. whether the eigenvalues of Q-operators are sufficient to fully parameterise the Hilbert space. We shall resolve this question positively by explicitly counting the number of solutions of WBE. Hence we still face the question of counting as in the algebraic/coordinate Bethe Ansatz scenario, however the statement that is proven as a consequence of counting is a bit different.

In physical applications, inhomogeneities are often set to $\theta_\ell = 0$. However, keeping inhomogeneities as parameters that we are going to vary is decisive for the approach discussed in this paper. To start with, they are regulators that put our system to a generic position. One can even explore regimes where Bethe equations can be solved explicitly which provides a very explicit way to count solutions. In the twisted case, such a regime is $\left| \frac{\theta_{\ell+1} - \theta_\ell}{\hbar} \right| \gg 1$ when we label solutions using binomial expansions, and in the twist-less case such a regime is $\frac{\theta_{\ell+1}}{\theta_{\ell}} \gg 1$ when we can label solutions using standard Young tableaux (SYT), the result first time stated in [43].

Furthermore, we show that the Bethe algebra can be generated by only $L$ generators. There are also $L$ inhomogeneities which allows us to prove generic position faithfulness statements using a rigorous version of a “number of variables equals number of parameters” argument.

Finally, it can be demonstrated that all properties can be made polynomial in $\chi_\ell$ – elementary symmetric polynomials in inhomogeneities. Algebraically this shall be formalised by proving that certain properly designed objects are free $\mathbb{C}[\chi_1, \ldots, \chi_L]$-modules. This implies that general position completeness and faithfulness statements can be specialised to any numerical value of $\chi_\ell$.

The paper is organised as follows.

In Section 2, we recall all the necessary known results about Yangian, Bethe algebra, and Q-operators. The fact that Q-operators belong to the Bethe algebra on the level of representation is proven in Appendix C. We conclude the section with formulation of the supersymmetric twisted and twist-less version of WBE (1.3).

Sections 3 and 4 provide proofs of completeness and faithfulness.

Section 3 proves completeness. We introduce the concept of Wronskian algebra (a polynomial ring subjected to WBE as a constraint), prove that it is a free $\mathbb{C}[\chi]$-module, and explicitly count its rank using Hilbert series (a.k.a. character, index, partition function).

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2The twist values $x_a, y_i$ are always kept fixed however.
confirming that it coincides with the dimension of the (corresponding) Hilbert space. By a
standard argument, the rank of the Wronskian algebra is the number of solutions to WBE
counted with multiplicities. The proof of freeness essentially uses the so-called properness
property of WBE which is proven in Appendix D.1 and Appendix D.2.

Section 4 proves faithfulness. More accurately, faithfulness means that the Bethe algebra
is a faithful representation of the Wronskian algebra. Actually, because a representation
is a surjective map it is also an isomorphism. The proof is first done for generic values of
inhomogeneities (over the polynomial ring \( \mathbb{C}[\chi] \)) and then is specialised to numerical values
of \( \chi_\ell \). Important results allowing one to specialise at any numerical value are covered in
Appendix B which builds substantially on the approach of [9].

Sections 5 and 6 aim to make the obtained results more practical.

Section 5 discusses various way to parameterise the Bethe algebra in the twist-less
case. In particular, we demonstrate an isomorphism of the restricted Bethe algebras \( B_\Lambda \)
for \( gl_{m|n} \) spin chains with different \( m, n \). The isomorphism class depends only on the Young
diagram \( \Lambda^+ \), and \( B_\Lambda \) is also isomorphic to the Q-system on this diagram. We also explain
how this formalism is mapped to NBAE.

Section 6 considers regimes \( \frac{\theta_{\ell+1} - \theta_\ell}{\hbar} \gg 1, \frac{\theta_{\ell+1}}{\theta_\ell} \gg 1 \) and shows how to explicitly find sol-
lutions of WBE in these regimes. Some technical questions are postponed to Appendix D.3.

In Section 7, we summarise the results and then discuss their immediate applications.
This includes an algorithm to solve Bethe equations (including at \( \chi_\ell = 0 \)) with solutions
being labelled with SYT, and applications to the Gaudin model and to the separation of
variables program. We conclude the section with a review of a relation between the Bethe
algebra and a quantum cohomology ring.

The paper uses substantially results and terminology from algebraic geometry and
commutative algebra while the target audience includes researchers with no appropriate
background. To alleviate the issue, we illustrate the discussion with numerous examples,
including a comprehensive case study in Appendix B.5, and supplement the paper with
Appendix A containing mostly a textbook material applied to the concrete problem that we
consider. Section 3.2 also summarises textbook knowledge about multiplicity of solutions
but we decided to keep it in the main text given its importance for the paper.

2 Definitions and basic properties

As often happens in mathematical physics, it will be useful to recast a physical question into
a problem in representation theory. A spin chain should be considered as a representation of
the \( gl_{m|n} \) Yangian. Commuting Hamiltonians belong to its certain commutative subalgebra
known as the Bethe algebra \( B \), and the completeness question is closely related to explicit
realisation of this algebra by Baxter polynomials subjected to constraints.

This section collects definitions of the above-mentioned objects.

2.1 \( gl_{m|n} \) Lie superalgebra, shifted and fundamental weights

Let us recall some essential facts about the \( gl_{m|n} \) Lie superalgebra [44]. Assign parity \( \bar{a} = 0 \)
for any “bosonic” index \( a \in \{1, \ldots, m\} \) and parity \( \bar{i} = 1 \) for any “fermionic” index \( i \in \{1, \ldots, n\} \).
The irreps appearing in the tensor powers of the defining algebra such that the inner corner of the fat hook attains the Young diagram boundary, \( \lambda \), with tuples of shifted weights \( \hat{\lambda} = \lambda_1, \ldots, \lambda_m, \hat{\nu} = \nu_1, \ldots, \nu_r \). We choose the marked point to be \( m' = m - r, n' = n - r \) for \( r \in \mathbb{Z}_{\geq 0} \), the role of \( r \) is to reduce diagonally the rank of the \( \mathfrak{gl}_m \) algebra such that the inner corner of the fat hook attains the Young diagram boundary, see Figure 1. The explicit relation between the shifted weights and the shape of \( \Lambda^+ \) is

\[
\hat{\lambda}_a = \lambda_{a'} - a - n + m, \quad a = 1, 2, \ldots, m - r,
\hat{\nu}_i = \nu_{i'} - i - m + n, \quad a = 1, 2, \ldots, n - r,
\]

where \( (\hat{\lambda}_1, \hat{\lambda}_2, \ldots) \) is the integer partition forming shape \( \Lambda^+ \), and \( (\nu'_1, \nu'_2, \ldots) \) is the integer partition of the transposed diagram. \( r = \min_k (k|\Delta_m-k + k - n \geq 0) \).

Most of the results do not depend on the choice of the marked point, this is demonstrated in Section 5. We made the choice of the diagonal reduction only for easier connections with the results already known in the literature.

\[\text{Usage of terminology "Hilbert space" is customary for quantum systems, however we do not discuss any scalar products in this work, except in Section 7.2.}
\]

by the Weyl vector and with the additional shift by \(-1\) of \( \hat{\nu} \) to get a symmetric description.
Figure 1: One-to-one correspondence between shifted weights $(9, 6, 4, 0|3, 2)$ and a Young diagram with a marked point (red dot). Other points on the boundary (black dots) and hence other sets of the shifted weights can be chosen. The outlined option is diagonally shifted from the corner of the $m|n$-hook that is linked to the $\mathfrak{gl}_{m|n}$ representation theory [44]: the Young diagrams that fit into the hook exactly describe finite-dimensional irreps. The Young diagrams that touch the hook corner correspond to the so-called long (typical) irreps, and the diagrams that do not touch the corner correspond to the so-called short (atypical) irreps.

For spin chains with twisted boundary conditions, and for generic diagonal twist, only the Cartan subalgebra of $\mathfrak{gl}_{m|n}$ is the symmetry of the system, and states are then described using the fundamental weight. We define it as the tuple $\Lambda = [\lambda_1, \ldots, \lambda_m|\nu_1, \ldots, \nu_n]$, where $E_{aa} v = \lambda_a v$, $E_{ii} v = \nu_i v$.

Dictated by the symmetry of the problem, we introduce restrictions of the Hilbert space to the weight subspaces

$$V \supset V_\Lambda \supset V_\Lambda^+.$$  \hfill (2.4)

$V_\Lambda$ is defined as the space of all vectors with fundamental weight $\Lambda$. $V_\Lambda^+$ is defined as the space of the highest-weight vectors for all irreps with Young diagram $\Lambda^+$ inside $V$. For concreteness we choose the standard order $1 < 2 < \ldots < m < 1 < \ldots < n$ in which case the fundamental weight $\Lambda$ of the highest-weight vectors of the irrep $\Lambda^+$ is given by the rule

$$\lambda_a = \lambda_a, \quad \nu_i = \max(0, \lambda_i' - m).$$  \hfill (2.5)

As is explained on page 67, any other choice of the total order would lead to an isomorphic description and to the same conclusions albeit explicit realisation of $V_\Lambda^+$ and certain related objects will be modified.

We shall use notation $U_\Lambda$ instead of $V_\Lambda^+$ or $V_\Lambda$ when discussion equally applies to both subspaces $V_\Lambda^+$ and $V_\Lambda$. 

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2.2 Yangian

The Yangian $\mathcal{Y}(\mathfrak{gl}_{m|n})$ is a quasi-triangular Hopf algebra. We summarise below its properties which will be relevant for us, see e.g. [47] for a more detailed discussion.

Let $\alpha, \beta \in \{1, \ldots, m\} \cup \{\hat{1}, \ldots, \hat{n}\}$ and $k = 1, 2, \ldots$. The Yangian’s generators $t_{\alpha\beta}^{(k)}$ are collected, via formal series in $\hbar/ut_{\alpha\beta}(u) = \delta_{\alpha\beta}1 + \hbar u t_{\alpha\beta}(u) + \ldots$ into “monodromies” $t_{\alpha\beta}(u)$ whose parity is equal to $\bar{\alpha} + \bar{\beta}$.

Quasi-triangularity is an RTT-type relation that reads in component form as

$$[t_{\alpha\beta}(u), t_{\gamma\delta}(v)] = \hbar \left( \frac{(-1)^{\bar{\alpha} + \bar{\beta} + \bar{\gamma}}}{u - v} (t_{\gamma\delta}(u)t_{\alpha\beta}(v) - t_{\alpha\beta}(v)t_{\gamma\delta}(u)) \right).$$

From Hopf algebra structures, we will only need the co-product $\Delta(t_{\alpha\beta}(u)) = \sum \gamma t_{\alpha\gamma} \otimes t_{\gamma\beta}$.

To realise the Yangian representation on the spin chain, consider first the evaluation homomorphism $\text{ev}_{\theta}: t_{\alpha\beta}(u) \mapsto \delta_{\alpha\beta}1 + \hbar u E_{\alpha\beta} + \ldots$.

We shall call (2.9) the spin chain representation of Yangian.

This representation contains global action of $\mathfrak{gl}_{m|n}$ on the spin chain, as defined in Section 2.1, in the first non-trivial coefficient of the $\hbar/u$ expansion:

$$\text{ev}_{\theta}(t_{\alpha\beta}(u)) = \delta_{\alpha\beta}1 + (-1)^{\bar{\alpha}} \frac{\hbar}{u} E_{\alpha\beta} + \ldots$$

For $T_{\alpha\beta} \equiv Q_\theta(u)t_{\alpha\beta}(u)$, where $Q_\theta(u) = \prod_{\ell=1}^L (u - \theta_\ell)$, $\text{ev}_{\theta}(T_{\alpha\beta}) \equiv Q_\theta \text{ev}_{\theta}(t_{\alpha\beta})$ are polynomials in $u$ of degree at most $L$. Note that $\text{ev}_{\theta}(T_{\alpha\beta})$ are also polynomials in $\theta_{\ell}$. Construction (2.9) corresponds to the graphics commonly used to define the monodromy matrix of a spin chain from Lax operators: $\text{ev}_{\theta}(T_{\alpha\beta}) = \frac{1}{u_{q_1} u_{q_2} \cdots u_{q_L}}$, and $\theta_{\ell}, \ell = 1, \ldots, L$ are commonly known as the spin chain inhomogeneities.

We will use $T_{\alpha\beta} = Q_\theta t_{\alpha\beta}$ to denote Yangian generators as well as their images $\text{ev}_{\theta}(T_{\alpha\beta})$. The context shall make it clear which meaning is being used. Note that $\text{ev}_{\theta}$ is not a faithful map and hence not all algebra-level results subdue the representation-level properties.

For discussion of this paper, it will be often important to consider $\theta_{\ell}$ as unevaluated
commuting variables. When it is the case, the image of the map \( ev_\theta \) are endomorphisms with polynomial coefficients:\(^5\)

\[
ev_\theta : Y(\mathfrak{gl}_{m|n}) \longrightarrow (\text{End}(\mathbb{C}^{m|n}))^\otimes L \otimes \mathbb{C}[\theta],
\]

where \( \mathbb{C}[\theta] \equiv \mathbb{C}[\theta_1, \theta_2, \ldots, \theta_L] \) is the polynomial ring in variables \( \theta_i \). Such operators naturally act on \( V := V \otimes \mathbb{C}[\theta] \). Such a description also appears in the context of a Hecke algebra, see page 65.

If we are interested in inhomogeneities having particular numerical values, in which case we typically denote them by \( \theta_i \), then we get representation of Yangian in a more standard sense

\[
ev_\theta : Y(\mathfrak{gl}_{m|n}) \longrightarrow (\text{End}(\mathbb{C}^{m|n}))^\otimes L.
\]

If we need to emphasise that (2.12) but not (2.11) is being used, we shall refer to (2.12) as *spin chain representation at point* \( \theta \equiv (\theta_1, \ldots, \theta_L) \).

### 2.3 Bethe algebra

The below-defined Bethe algebra \( \mathcal{B} \) is a commutative subalgebra of \( Y(\mathfrak{gl}_{m|n}) \) which depends on a constant \( \text{GL}(m|n) \) group matrix \( G \) dubbed twist. We restrict ourselves to the case when \( G \) is diagonalisable and furthermore choose a reference frame that diagonalises \( G \), so \( \mathcal{B} \) actually depends only on the eigenvalues \(^6\) \( x_1, \ldots, x_m, y_1, \ldots, y_n \) of \( G \). In practice, we will consider only two opposing cases: of generic twist, when all \( x_i, y_i \) are distinct, and of no-twist when \( G = 1 \). Considering intermediate cases is possible but combinatorially bulky \(^7\).

The Bethe algebra \( \mathcal{B} \) is defined as the algebra that is polynomially generated by the transfer matrices \( T_\mu \) in covariant representations of \( \mathfrak{gl}_{m|n} \) labelled with integer partitions or equivalently Young diagrams \( \mu \). By “polynomially generated” we mean that elements of \( \mathcal{B} \) are finite degree polynomials in \( \hat{d}_k \) – coefficients of the (a priori formal) \( h/u \) expansion

\[
\mathcal{T}_\mu = \chi_\mu(G)u^{\Lambda_\mu} \cdot (1 + \hat{d}_1 \frac{h}{u} + \ldots),
\]

where \( \chi_\mu(G) \) is the character of \( G \) in representation \( \mu \).

Transfer matrices \( T_\mu \) can be constructed using fusion from \( T_{\alpha\beta} \) \(^{31, 48, 49}\) and hence are defined on the level of Yangian as well as its representation \( ev_\theta \). When we descend to the representation level, \( T_\mu(u) \) is a degree-\( L|\mu| \) polynomial in \( u \), so the \( h/u \) expansion truncates.

Let \((1^a)\) denote the Young diagram consisting of one column of height \( a \), and \((s)\) – the Young diagram consisting of one row of width \( s \). To avoid discussing fusion in detail, we note that the \( T_\mu \), and hence the Bethe algebra, can be polynomially generated from \( T_{(1^a)} \).

---

\(^5\)Here we allow freedom of speech and consider \( ev_\theta(Y(\mathfrak{gl}_{m|n})) \) in the sense of \( ev_\theta(T_{\alpha\beta}) \).

\(^6\)Matrices of the \( \text{GL}(m|n) \) group have entries belonging to a Grassmann algebra, hence their eigenvalues are in principle not complex numbers. Our discussion will assume that twists are complex numbers nevertheless. One can then check that the results still hold for any twists of type \( x_a = A_a + n_a, y_i = B_i + n_i \) where \( A, B \in \mathbb{C} \) and \( n \) - even nilpotent elements of the Grassmann algebra, assuming that \( A_a, B_i \) are pairwise distinct.

\(^7\)Analytic structure of Q-functions for partially degenerate twists, which is an essential ingredient for the completeness statements, was explored in detail in [41].
or from \( T(s) \), \( a, s = 1, 2, \ldots \) using the determinant Cherednik-Bazhanov-Reshetikhin (CBR) formula [50–52] – the Yangian version of Jacobi-Trudi identities for characters \( \chi_{\mu}(G) \), while \( T_{(1^s)} \) and \( T_{(s)} \) are compactly defined through monodromies \( T_{\alpha\beta} \) as [53]

\[
\text{Ber} \left[ \mathbb{I} - DT(u)GD \right] = \sum_{a=0}^{\infty} (-1)^a D^a \; T_{(1^a)}(u) \; D^a, \quad (2.13a)
\]

\[
\frac{1}{\text{Ber} \left[ \mathbb{I} - DT(u)GD \right]} = \sum_{s=0}^{\infty} D^s \; T_{(s)}(u) \; D^s, \quad (2.13b)
\]

where \( D \equiv e^{-\frac{1}{\hbar} \partial_u} \).

The l.h.s. of (2.13) is defined as follows. For \( M = \mathbb{I} - DTG \), introduce the notation \( M = \begin{pmatrix} [M]_A & [M]_B \\ [M]_C & [M]_D \end{pmatrix} \) and \( M^{-1} = \begin{pmatrix} [M^{-1}]_A & [M^{-1}]_B \\ [M^{-1}]_C & [M^{-1}]_D \end{pmatrix} \), where \( [M]_A \) is the \( m \times m \) block of \( M \), \( [M^{-1}]_D \) is the \( n \times n \) block of \( M^{-1} \) etc. Then the Berezinian is defined as \( \text{Ber}(M) = \det [M]_A \det [M^{-1}]_D \), where the determinants of blocks “A” (resp. “D”) are defined through a column-ordered (resp. line-ordered) expansion e.g. \( \det [M]_A = \epsilon_{a_1 \ldots a_m} ([M]_{a_1 \ldots a_m}) \). Although \( M \) is a matrix with non-commutative entries \( M_{\alpha\beta} \), the entries satisfy the (supersymmetric version of) Manin relations \( M_{\alpha\beta, M_{\gamma\delta}} = (-1)^{\bar{\alpha}\bar{\beta} + \bar{\gamma}\bar{\delta}} M_{\alpha\beta, M_{\gamma\delta}} \) which ensure that the above-defined Berezinian can only change sign if columns/rows of \( M \) are permuted. From earlier works, we mention that Berezinians in the context of \( Y(\mathfrak{g}l_{m|n}) \) were introduced in [54], and generalise similar constructions in the \( Y(\mathfrak{g}l_n) \) case [55–57], see also [47].

The physical Hamiltonian of the system is an element of the Bethe algebra and it is usually chosen to be \( \mathbb{H} = \partial_u \log T_{(1)}(u) \). The algebraic equivalent of the statement that the Bethe algebra contains all commuting charges is the statement that it is a maximal commutative subalgebra that contains \( \mathbb{H} \). The question about maximality can be asked on the level of the Yangian algebra or of its spin chain representation. On the level of the algebra, in the bosonic \( Y(\mathfrak{g}l_n) \) case, polynomial combinations of \( T_{(1^s)} \) indeed generate, for non-degenerate twist, a maximal commutative subalgebra of \( Y(\mathfrak{g}l_n) \) [47, 55] but it seems an equivalent statement was not proved for the supersymmetric case. To our knowledge, a comprehensive study of the Bethe algebra on the Yangian algebra level is still lacking.

However, our goal is to describe the Bethe algebra represented on the spin chain in which case it can be understood much better. In particular, the quantum Berezinian defined

\[
q\text{Ber} \equiv D^{2(m-n)} \text{Ber} \left[ DT(u)GD \right]
\]

and which is known to generate the center of the Yangian [54, 58] can be expressed, at least on the level of representation, as a ratio of transfer matrices \( q\text{Ber} \propto \left( T_{(m+1)}(u) \right)^{[m-n]} \) [59] and hence, by Hamilton-Cayley, belongs to the Bethe algebra.

It will be one of our results that, under mild assumptions on \( \theta \)’s, that the Bethe algebra is a maximal commutative subalgebra of \( \text{End}(V) \) – the algebra of all linear transformations
of the spin chain Hilbert space. Hence the Bethe algebra contains all possible conserved charges of the system.

2.4 Q-operators

Despite there are infinitely many $T_{(sa)}$ and $T_{(s)}$ in the expansions (2.13), we need finitely many functions of $u$ to generate the Bethe algebra. This can be seen for instance by analysing the CBR formula. On the level of representation, probably the most economic way to demonstrate this fact is to express transfer matrices through Baxter Q-operators that were explicitly constructed as operators acting on the spin chain in [60–62]. The Q-operators are not elements of the Yangian, but they do belong to the Bethe algebra in the representation $ev_q$, in particular they are matrices whose coefficients are polynomials in $\theta_k$, see Appendix C.

The Q-operators generate the Bethe algebra as follows $^8$ [37, 63, 64]

$$T_{(sa)} = u^{a_s L} \chi_{a_s}(G)(1 + d_1 \frac{h}{u} + \ldots)$$

$$\propto \frac{1}{Q^{-s[a]}} \prod_{k=1}^{a} \prod_{l=1}^{s} Q_{\emptyset \emptyset}^{[a+s+2-2k-2]} (\operatorname{Ber} G)^{u/h} \times \left\{ \begin{array}{ll}
Q_{b_1 \ldots b_m}^{[m-n-s]} Q_{b_{a+1} \ldots b_m}^{[-s]} & s \geq a - m + n \\
Q_{b_1 \ldots b_m}^{[m-n-a]} Q_{b_{a+1} \ldots b_m}^{[a]} & a \geq s + m - n
\end{array} \right\},$$

(2.15)

where $\epsilon$ denotes the Levi-Cevita antisymmetric tensor, summation over repeated indices is performed, and the $\propto$ symbol involves a proportionality factor which is identified by imposing that the coefficient of the highest degree of $T_{(sa)}$ (as a polynomial in $u$) is the character $\chi_{(sa)}(G)$.

In total, there are $2^{m+n}$ Q-operators. They are labelled as $Q_{A|I}$, where $A$ is a multi-index from $\{1, \ldots, m\}$ and $I$ is a multi-index from $\{1, \ldots, n\}$. $Q_{A|I}$ are anti-symmetric w.r.t. permutations in $A$ and $I$, and polynomial up to an exponential prefactor (as in 1.3):

$$Q_{A|I} \propto \prod_{a \in A} \frac{u/a}{u/h} Q_{A|I},$$

(2.16)

where the proportionality factor $^9$ in “$\propto$” is fixed by the condition that each $q_{A|I}$ is a monic polynomial in the variable $u$.

They Q-operators satisfy the following QQ-relations

$$Q_{A_{ab}|I} Q_{A|I} = W(Q_{A_{aA}|I}, Q_{A_{bB}|I}),$$

(2.17a)

$$Q_{A_{aA}|I} Q_{A|I} = W(Q_{A_{aA}|I_i}, Q_{A|I}),$$

(2.17b)

$$Q_{A_{ij}|I} Q_{A|I} = W(Q_{A_{ji}|I}, Q_{A_{Ij}}),$$

(2.17c)

$^8$Equation (2.15) expresses $T$ for the so-called rectangular representations, where the Young diagram $(sa) \equiv (s, s, s, \ldots, s)$ is of rectangular shape; representations $(1^s)$, $(s)$ are special subcases. Generalisation to the arbitrary representations is known [63].

$^9$One can set the proportionality factor to be equal to one when $|A| + |I| \leq 1$, which fixes this factor for other values of $A$ and $I$ due to the relations (2.17). It is explicitly spelled out in e.g. [41].
Furthermore, \( Q_{\emptyset|\emptyset} = 1 \) and all \( Q \)-operators can be written explicitly in terms of \( Q_{a|i} \), \( Q_{a|i} \), and \( Q_{a|i} \) [41]:

\[
Q_{(a|a+c)} \propto (Q^{[c]}_{[1]})^a Q^{[c]-1}_{[0]} \cdots Q^{[1-c]}_{[1]} Q^{[c]}_{[1]} \cdots Q^{[1-c]}_{[0]} ,
\]

(2.18)

For compactness, we used exterior forms \( Q_{[a|b]} = \frac{1}{2\pi i} \sum Q_{a1...a|1...i} \) \( \psi_0^0 \cdots \psi_0^i \cdots \psi_i^b \) with \( \psi_0^0, \psi_1^1 \) being auxiliary Grassmann variables.

We shall need the expression for \( Q_{\emptyset|\emptyset} = Q_{1...m|1...n} \) which explicitly is the following determinant

\[
Q_{\emptyset|\emptyset}(u) = (-1)^{n(m-n)} \begin{vmatrix}
Q^{[m-n]}_{1|1} & Q^{[m-n]-1}_{1|2} & \cdots & Q^{[-(m-n)+1]}_{1|\emptyset} \\
Q^{[m-n]}_{1|n} & Q^{[m-n]-1}_{1|\emptyset} & \cdots & Q^{[-(m-n)+1]}_{1|\emptyset} \\
\vdots & \vdots & \ddots & \vdots \\
Q^{[m-n]}_{m|1} & Q^{[m-n]-1}_{m|\emptyset} & \cdots & Q^{[-(m-n)+1]}_{m|\emptyset}
\end{vmatrix},
\]

(2.19)

where, without loss of generality, it was assumed that \( m \geq n \).

One should also note that the coefficients \( Q_{\emptyset|\emptyset} \), \( Q_{\emptyset|i} \), and \( Q_{a|i} \) are related by (2.17b):

\[
Q_{a|i}^{+} - Q_{a|i}^{-} = Q_{a|i}^{+} Q_{a|i}^{-}.
\]

(2.20)

Finally, it can be derived from (2.15), (2.13) and CBR formulae that \( Q_{\emptyset|\emptyset} \) satisfies

\[
Q_{\emptyset|\emptyset}^{-} = q \text{Ber} Q_{\emptyset|\emptyset}^{+}.
\]

(2.21)

and hence \( Q_{\emptyset|\emptyset} \) belongs to the center of the Yangian.

2.5 Quantisation condition (Wronskian Bethe equations)

The essential property for the description of the Bethe algebra is the explicit analytic structure of \( Q \)-operators (a.k.a rational analytic Bethe Ansatz) which is known from results of [60, 61, 65], and can be also derived using logic of Appendix C.

The definition of \( Q \)-operators depends on a gauge choice. Below we write expressions in one particular gauge which is suitable for our goals.

For the central element \( Q_{\emptyset|\emptyset} \) it is possible to directly compute its explicit value which

\[
Q_{\emptyset|\emptyset}(u) \propto (\text{Ber} G)^{u/\theta} Q_{\emptyset}(u).
\]

(2.22)

This property is an important aspect of the Bethe algebra and it is essentially equivalent to the set of Bethe equations as it will become clear below.

We set \( \text{Ber} G = 1 \) for convenience. It only affects the overall normalisation of transfer matrices and hence is inessential.

---

10 Several other discrepancies in conventions can be present across the literature. First, the shift in the spectral parameter can be present. Second, the role of the \( Q \)-operators and their Hodge duals \( Q^{A|J} = \cdots e^{A' e'' J} Q_{J,J} \) can be swapped. Third, a permutation of indices \( 1 \cdots m |1 \cdots n \) can be used.
To write expressions for the other Q-operators, we need to restrict the representation space to a certain subspace. The generic twist and twist-less cases should be treated separately.

**Twisted case**

The Cartan generators $\mathcal{E}_{\alpha\alpha}$ of the global $\mathfrak{gl}_{m|n}$ action (2.10) commute with $\mathcal{B}$ (and belong to it). To define analytic properties of the Q-operators in a useful manner, we need to restrict to an eigenspace of $\mathcal{E}_{\alpha\alpha}$ which is the weight space $V_\Lambda$ defined after (2.4). The Bethe algebra restricted to this subspace shall be denoted as

$$B_\Lambda := B|_{V_\Lambda}. \quad (2.23)$$

Upon restriction to $V_\Lambda$, the polynomial operators $q_{A|J}$ of (2.16) read

$$q_{A|J} = u^{M_{A|J}} + \sum_{k=1}^{M_{A|J}} \hat{c}^{(k)}_{A|J} u^{M_{A|J} - k}. \quad (2.24)$$

These are monic polynomials of degree $M_{A|J}$ with operator-valued coefficients $\hat{c}^{(k)}_{A|J}$. The diagonalisation of $\hat{c}^{(k)}_{A|J}$ is the subject of Bethe Ansatz. The degree $M_{A|J}$ has fixed value on each $V_\Lambda$, which can for instance be identified by explicit computations following Appendix C (see [61]), and they turn out to be equal to the (“magnon” numbers), i.e., they have the following expression in terms of the fundamental weight $\Lambda = [\lambda_1, \ldots, \lambda_m|\nu_1, \ldots, \nu_n]$ that defines $V_\Lambda$:

$$M_{A|J} = \sum_{a=1}^{m} \lambda_a + \sum_{j=1}^{n} \nu_j. \quad (2.25)$$

For generic twist, (2.20) is a non-degenerate system of linear equations in the coefficients $\hat{c}^{(k)}_{a|\emptyset}$ that fixes $\hat{c}^{(k)}_{a|\emptyset}$ and thus $Q_{a|\emptyset}$ uniquely. Hence all the Q-operators are generated by the single-index Q-functions $Q_{a|\emptyset}$, $Q_{\emptyset|j}$. There are precisely $L$ coefficients $\hat{c}^{(k)}_{a|\emptyset}$ and $\hat{c}^{(k)}_{\emptyset|j}$ as one can quickly conclude from (2.25) and the invariant value of the total charge

$$\sum_{a=1}^{m} \lambda_a + \sum_{j=1}^{n} \nu_j = L. \quad (2.25)$$

We can use $C_\Lambda$ – the set of all coefficients $\hat{c}^{(k)}_{a|\emptyset}$ and $\hat{c}^{(k)}_{\emptyset|j}$ – to polynomially generate $Q_{\emptyset|\emptyset}$ using (2.19). This operation is a supersymmetric generalisation of the Wronskian determinant in (1.3) and shall be denoted as $\text{SW}(C_\Lambda)(u)$,

$$\text{SW}(C_\Lambda)(u) = \prod_{\ell=1}^{L} (u - \theta_{\ell}). \quad (2.26)$$

Note that we chose the normalisation of $\text{SW}$ such that the leading $u^L$ term is monic.

We call (2.26) the quantisation condition or the Wronskian Bethe equations (WBE). Its important feature is that it provides exactly $L$ equations on $L$ variables (elements of the set $C_\Lambda$) and that it contains $L$ free parameters $\theta_{\ell}$. We shall denote this system of $L$
equations as

\[ \text{SW}_\ell(c) = \chi_\ell, \quad \ell = 1, 2, \ldots, L, \]  

(2.27)

where \( \chi_\ell \) are elementary symmetric polynomials of \( \theta_1, \theta_2, \ldots, \theta_L \). Dependence on inhomogeneities only through their symmetric combinations \( \chi_\ell \) will be very important in our studies. Quite often, we will consider \( \chi_\ell \) as independent variables instead of inhomogeneities.

We shall consider the quantisation condition as an equation both on the level of operators denoted uniformly as \( \hat{c}_\ell \), \( \ell = 1, \ldots, L \) and on the level of abstract variables denoted as \( c_\ell \). We shall show eventually that any \( c_\ell \) solving (2.26) provides eigenvalues for \( \hat{c}_\ell \)'s.

Example:

Consider a \( GL(3) \) spin chain of length \( L = 3 \), and the weight subspace \( V_\Lambda \) with \( \Lambda = [2, 1, 0] \). The Q-system is parameterised by

\[
Q_1 \propto x_1^{u/\hbar} \times (u^2 + \hat{c}_1(1) u + \hat{c}_1(0)), \\
Q_2 \propto x_2^{u/\hbar} \times (u + \hat{c}_2(0)), \\
Q_3 \propto x_3^{u/\hbar}.
\]

(2.28)

\[ \text{SW} \propto W(Q_1, Q_2, Q_3) = \det_{1 \leq a, b \leq 3} Q_a(u + \hbar(2 - b)). \]

Set for simplicity \( x_3 = 3, x_2 = 2, x_1 = 1 \), then (2.27) becomes explicitly

\[
\begin{aligned}
(\hat{c}_1(0) \hat{c}_2(0) - \hbar(\hat{c}_1(0) + \frac{5}{2} \hat{c}_1(1) \hat{c}_2(0)) + \frac{\hbar^2}{2}(9\hat{c}_1(1) + 7\hat{c}_2(0)) - \frac{15}{2}\hbar^3) &= -\theta_1 \theta_2 \theta_3 \\
+u(\hat{c}_1(0) + \hat{c}_1(1) \hat{c}_2(0) - \hbar(-\frac{7}{2} \hat{c}_1(1) - 5\hat{c}_2(0)) + \frac{25}{2}\hbar^2) &= +u(\theta_1 \theta_2 + \theta_1 \theta_3 + \theta_2 \theta_3) \\
+u^2(\hat{c}_1(1) + \hat{c}_2(0) - 6\hbar) &= +u^2(-\theta_1 - \theta_2 - \theta_3) \\
+u^3 &= +u^3
\end{aligned}
\]

which yields us three equations satisfied by \( c_1(0), c_1(1), c_2(0) \), both on the level of operators and their eigenvalues.

Counting the number of solutions is easy in this example: one can derive a cubic equation on \( c_2(0) \) with the cubic term that never vanishes and furthermore we observe that \( c_1(1), c_1(0) \) follow uniquely if we fix the value \( c_2(0) \). So there are always three solutions which is the dimension of \( V_\Lambda \). For generic values of \( \theta_\ell \), all solutions are distinct.

Example:

Consider a \( gl(1|1) \) spin chain of length \( L = 3 \), and \( V_\Lambda \) with \( \Lambda = [\lambda|\nu] = [2|1] \). We
parameterise the Q-system by
\[
Q_{1|\varnothing} \propto x^{+u/\hbar} \times (u^2 + c_{1|\varnothing}^{(1)} u + c_{1|\varnothing}^{(0)}),
\]
\[
Q_{\varnothing|1} \propto y^{-u/\hbar} \times (u + c_{\varnothing|1}^{(0)}).
\]
(2.30)

In this case \(Q_\theta = SW \propto Q_{1|1}\), and one needs to compute \(Q_{1|1}\) from the finite-difference equation \(Q_{1|1}(u + \hbar/2) - Q_{1|1}(u - \hbar/2) = Q_{1|\varnothing}^{1} Q_{\varnothing|1}\) which supplies the equations on \(c_{1|\varnothing}^{(1)}, c_{1|\varnothing}^{(0)}\), and \(c_{\varnothing|1}^{(0)}\). For \(x = 3, y = 1\) they are
\[
u^3 - u^2(\chi_1 - 3\hbar) + u(\chi_2 - 2\chi_1\hbar + \frac{3}{4}\hbar^2) - (\chi_3 - \chi_2\hbar + \frac{1}{4}\chi_1\hbar^2 - \frac{1}{4}\hbar^3)
\]
\[
= (u^2 + c_{1|\varnothing}^{(1)} u + c_{1|\varnothing}^{(0)})(u + c_{\varnothing|1}^{(0)}),
\]
(2.31)

where \(\chi_1 = \theta_1 + \theta_2 + \theta_3, \chi_2 = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3, \chi_3 = \theta_1\theta_2\theta_3\).

Counting solutions for this example is even simpler, and it is a good demonstration of when a supersymmetric system can be advantageous for finding the spectrum of the Bethe algebra. The l.h.s. of (2.31) is a degree-three polynomial with all coefficients known through parameters of the theory. It has three roots, and one of these roots should be \(u = -c_{\varnothing|1}^{(0)}\) which fixes \(c_{\varnothing|1}^{(0)}\). Values for other \(\chi\)'s follow. Hence there are three solutions again which is indeed the dimension of the weight subspace.

**Twistless case**

For the twistless case, the symmetry of the system is enhanced as all the generators \(E_{\alpha\beta}\) commute with the Bethe algebra. Now, the Cartan subalgebra of \(g(\mathfrak{m}|\mathfrak{n})\) does not belong to the Bethe algebra, and so the latter acting on the spin chain is definitely not maximal commutative. However, again, if we restrict ourselves to the weight subspace \(V_\Lambda^+\), maximal commutativity on this subspace will follow from completeness.

\[
B_\Lambda := B|_{V_\Lambda^+}.
\]
(2.32)

We will typically drop the superscript + and denote the restricted Bethe algebra as \(B_\Lambda\).

The Q-operators were constructed in [60, 61, 65] for the case of generic twist. Taking the twist-less limit is quite a tricky procedure [66] which was analysed substantially in [41]. The result of this analysis is that the below-presented properties that define the twistless Q-system remain true at the level of operators.

**Long representations** Consider first a situation when \(\Lambda\) is a long representation of \(g(\mathfrak{m}|\mathfrak{n})\). The Young diagram of such a representation touches the corner of a fat hook, consider for instance the situation with \(m \to m - r, n \to n - r\) in Figure 1.

In this case, the Q-operators \(Q_{A|J}\) are (normalised to be) monic polynomials in \(u\) of
degree $M_{A|J}$,\footnote{For comparison with other literature, it might be needed to re-label Q-functions using the maps $a \to m + 1 - a, i \to n + 1 - i$. One checks the notation by asking for which $a, i$ $Q_{a|i}$ is a polynomial of the smallest degree. In our conventions, it is $Q_{m|n}$.}
\begin{equation}
M_{A|J} = \sum_{a \in A} \tilde{\lambda}_a + \sum_{j \in J} \tilde{\nu}_j - \frac{|A|(|A| - 1)}{2} - \frac{|J|(|J| - 1)}{2} + |A||J|,
\end{equation}
where $\hat{\lambda}_a, \hat{\nu}_i$ are the shifted weights defined as in Figure 1.

We will need mostly
\begin{equation}
M_{a|\emptyset} = \hat{\lambda}_a, \quad M_{\emptyset|i} = \hat{\nu}_i, \quad M_{a|i} = \hat{\lambda}_a + \hat{\nu}_i + 1. \tag{2.34}
\end{equation}

A few modifications have to be made to obtain an equivalent of (2.26). First, we notice that (2.20) fixes $Q_{a|i}$ only up to an additive constant and hence $c_{a|i}^{(0)}$ are new independent parameters used in the computation of $Q_{\emptyset|\emptyset}$. Second, the computation of $Q_{\emptyset|\emptyset}(u)$ and of other physically relevant quantities such as transfer matrices is invariant under the transformations
\begin{equation}
Q_{\emptyset|i} \to Q_{\emptyset|i} + \alpha Q_{\emptyset|j}, \quad Q_{a|\emptyset} \to Q_{a|\emptyset} + \beta Q_{b|\emptyset} \tag{2.35}
\end{equation}
We impose inequalities $i \geq j$ and $a \geq b$ to preserve the polynomial degrees (2.34). We fix these symmetry transformations by putting to zero all the coefficients $c_{\emptyset|i}^{(\nu_i)}$ for $j \leq i$ and $c_{a|i}^{(\nu_i)}$ for $b \leq a$.

A straightforward counting shows that the set consisting of all non-zero $c_{a|\emptyset}^{(k)}, c_{\emptyset|i}^{(k)}$ combined together with $c_{a|i}^{(0)}$ gives us exactly $L$ variables. Denote this set by $C \Lambda$. $Q_{\emptyset|\emptyset} = Q_\emptyset$ is unambiguously and polynomially reconstructed from $C \Lambda$ according to (2.19) supplemented with (2.20), we denote the corresponding operation again as $SW(C \Lambda)(u)$ though explicit polynomial realisation of SW is different now.

In this modified setting, (2.26) holds.

Example:

Consider a $\mathfrak{gl}(3)$ spin chain of length $L = 3$, and consider states in the representation $\Lambda^+ = \mathfrak{gl}(3)$. By the recipe of Figure 1, $\hat{\lambda}_1 = 4, \hat{\lambda}_2 = 2, \hat{\lambda}_3 = 0$. Then one generates the Bethe algebra by three Q-functions
\begin{align}
Q_1 &= u^4 + c_1^{(3)} u^3 + c_1^{(1)} u, \\
Q_2 &= u^2 + c_1^{(1)} u, \\
Q_3 &= 1. \tag{2.36a}
\end{align}
We fixed $c_1^{(2)} = c_1^{(0)} = c_2^{(0)} = 0$ using symmetries of the system. The Wronskian condition (2.26) which is $\det_{1 \leq a, b \leq 3} Q_a(u + (2 - b)h) \propto Q_\emptyset$ provides three equations to be
satisfied by $c_1^{(3)}, c_1^{(1)}, c_2^{(1)}$:

$$c_1^{(1)} - h^2 (c_2^{(1)} - c_1^{(3)}) = 8\lambda_3, \quad 3c_1^{(3)}c_2^{(1)} - 2h^2 = 8\lambda_2, \quad 3c_1^{(3)} + 6c_2^{(1)} = -8\lambda_1, \quad (2.37)$$

it has two solutions.

Example:

Consider a $\mathfrak{gl}(1|1)$ spin chain of length $L = 3$, again in the representation $\Lambda^+ = \mathbb{P}$. By the recipe of Figure 1, $\lambda = 1, \nu = 2$. Then we use (2.33) to deduce the degree of $Q$-functions and get

$$Q_{1|0} = u + \hat{c}_{1|0}, \quad Q_{0|1} = u + \hat{c}_{0|1}. \quad (2.38a)$$

Equation $Q_{1|1}^+ - Q_{1|1}^- = Q_{1|0}Q_{0|1}$ provides $Q_{1|1}$ up to an additive constant $c_{1|1}$,

$$Q_{1|1} \propto u^3 + \frac{3}{2}(\hat{c}_{1|0} + \hat{c}_{0|1})u^2 + (3\hat{c}_{1|0}\hat{c}_{0|1} - \frac{1}{4}h^2)u + \hat{c}_{1|1}, \quad (2.39)$$

which together with $c_{1|0}$ and $c_{0|1}$ yields three variables that generate the Bethe algebra. The Wronskian condition (2.26) is $Q_{1|1} \propto Q_\theta$, it implies the equations on $c$’s:

$$c_{1|0} + c_{0|1} = -\frac{2}{3}\lambda_1, \quad c_{1|0}c_{0|1} = \frac{1}{3}\lambda_2 + \frac{1}{12}h^2, \quad c_{1|1} = -\lambda_3 \quad (2.40)$$

that have two solutions.

**Short representations** The Young diagram of a short representation does not touch the internal corner of the $m|n$ fat hook. Define $r$ according to Figure 1. Introduce sets $\mathbf{A} = \{m, m - 1, \ldots, m - r + 1\}$ and $\mathbf{J} = \{n - 1, \ldots, n - r + 1\}$, and label all $Q$-functions as $Q_{AA_0|JJ_0}$, where $A_0$ is a multi-index from $\mathbf{A}$ and $J_0$ is a multi-index from $\mathbf{J}$. Then the properties of the $Q$-functions can be described as follows [41]: If $|A_0| = |J_0|$ then $Q_{AA_0|JJ_0} = Q_{A|J}$ and, if $|A_0| \neq |J_0|$ then $Q_{AA_0|JJ_0}$ are not uniquely defined in the twist-less limit but also such $Q$-operators appear in the physically-relevant quantities, such as transfer matrices and $Q_{\mathbf{2}|\mathbf{2}}$, in the combinations that vanish in the twist-less limit.

The described property allows us to restrict the $\mathfrak{gl}(m|n)$ Q-system to the $\mathfrak{gl}(m - r, n - r)$ $Q$-system defined as $Q_{A|J}^{\text{rest}} = Q_{AA_0|JJ_0}$ which has the property $Q_{\mathbf{2}|\mathbf{2}}^{\text{rest}} = Q_{A|J} = Q_{\mathbf{2}|\mathbf{2}} = 1$ and $Q_{\mathbf{2}|\mathbf{2}}^{\text{rest}} = Q_{\mathbf{2}|\mathbf{2}}^{\text{rest}}$. This subsystem is sufficient to generate the Bethe algebra. Since an originally short representation becomes long from the point of view of $\mathfrak{gl}(m - r, n - r)$ subalgebra and since the polynomial degrees are correctly captured by (2.33), we can use the same logic as for the long representations and formulate the supersymmetric Wronskian condition (2.26) using $C_\Lambda$ of the $\mathfrak{gl}(m - r, n - r)$ Q-system.

Example:

The representation $\mathbb{P}$ can be considered as a short one of the $\mathfrak{gl}(2|2)$ algebra. Then
A = \{2\}, J = \{2\}, and so all the physical information is contained in the functions \(Q_{12|12} = Q_{1|1}, Q_{12|2} = Q_{1|\emptyset}, Q_{2|12} = Q_{\emptyset|1}, Q_{2|2} = Q_{\emptyset|\emptyset}\). The Wronskian is given by
\[
SW = Q_{1|1}Q_{2|2} - Q_{1|2}Q_{2|1}.
\] (2.41)

While \(Q_{1|2}\) and \(Q_{2|1}\) are not uniquely defined in the twistless limit, any prescription would imply that at least either \(Q_{1|2}\) or \(Q_{2|1}\) vanish and so their product vanishes as well. Given that \(Q_{2|2} = Q_{\emptyset|\emptyset} = 1\), \(SW = Q_{\emptyset}\) implies \(Q_{1|1} = Q_{\emptyset}\) which fully parallels the above-described \(\mathfrak{gl}(1|1)\) example.

3 Completeness

So far we introduced \(B_\Lambda\) – the restriction of the Bethe algebra to the weight subspace \(U_\Lambda\) (which is \(V_\Lambda\) or \(V_\Lambda^\top\)). It is generated by the restriction of the Q-operators who in turn are (twisted) polynomials of the spectral parameter. We also selected precisely \(L\) coefficients of these polynomials assembled into the set \(C_\Lambda\) and explained how they are used to generate the whole \(B_\Lambda\).

From now on, the elements of \(C_\Lambda\) are labelled in a uniform manner as \(c_\ell, \ell = 1, \ldots, L\). It will be important to articulate what notation \(c_\ell\) means exactly. If it is an explicit matrix acting on \(U_\Lambda\) then we denote it as \(\hat{c}_\ell\). In contrast, we agree to denote by \(c_\ell\) without hat abstract commuting variables that have, by definition, only one property: they satisfy Wronskian Bethe equations (2.27).

A one-to-one correspondence between \(c_\ell\) solving WBE and eigenvalues of \(\hat{c}_\ell\) will be established later, in Section 4. In the current section, we show that WBE have precisely the right number of solutions \(d_\Lambda = \dim_C U_\Lambda\). This property is usually referred to as completeness, why this naming is justified was discussed in the introduction.

3.1 Analytic description

First, we develop some intuition about analytic description of the Wronskian Bethe equations \(SW_\ell(c) = \chi_\ell\). Think about them as a polynomial map
\[
SW : \mathbb{C}^L \rightarrow \mathbb{C}^L,
\]
\[
(c_1, \ldots, c_L) \mapsto (\chi_1 = SW_1(c), \ldots, \chi_L = SW_L(c)).
\] (3.1)

Denote by \(C \simeq \mathbb{C}^L\) the domain of definition of the map and by \(\mathcal{X} = SW(C)\) its image.

Surjectivity. The SW map is in fact surjective, that is \(\mathcal{X} \simeq \mathbb{C}^L\) which means that the Wronskian relations (2.27) have at least one solution for any complex value \(\tilde{\chi}_\ell\) of \(\chi_\ell\). Indeed, matrix coefficients of \(\hat{c}_\ell\) are polynomials in \(\theta_\ell\), e.g. by construction of Baxter Q-operators, and so they are defined for any numerical value \(\tilde{\theta}_\ell\). Furthermore \(\hat{c}_\ell\) commute and so they have at least one common eigenvector \(u(\tilde{\theta})\). Eigenvalues of \(\hat{c}_\ell\) on this vector satisfy (2.27) and so they provide a solution to \(SW_\ell(c) = \tilde{\chi}_\ell\) for \(\tilde{\chi}_\ell = \chi_\ell(\tilde{\theta})\).
Critical and regular points. Denote by $C_{\text{crit}}$ the set of all the critical (degeneration) points $c_\ell$ where the differential of $SW$ is not invertible. Its image $X_{\text{crit}} \equiv SW(C_{\text{crit}})$ shall be called the set of critical values. Using e.g. Sard’s theorem one states that $X_{\text{crit}}$ is of measure zero in $X$. The complement to $C_{\text{crit}}$, resp. $X_{\text{crit}}$, shall be called domain of regular points (solutions), resp. values (parameters). Restricted to the regular points, the map $SW$ is locally a diffeomorphism, i.e. for each point $c_\ell \notin C_{\text{crit}}$ there is a neighbourhood of $SW(c_\ell)$ where $SW$ can be smoothly inverted. This implies that all solutions to the Bethe equations are distinct in a neighbourhood around a regular value $\chi_\ell$. This also shows that in such a neighbourhood the fibers of $SW$ are all finite and of the same cardinality ($SW$ is polynomial and so it cannot have infinite discrete fibers).

Properness. All solutions $c_\ell$ are bounded at any finite value of $\theta_\ell$’s or, in more abstract terms, the inverse image of a compact set is compact. SW is then said to be proper. This very important technical point is proved in two independent ways: using the fact that $Q$-operators have bounded spectrum, as is explained in the remark on page 32; and by a direct analysis of the equations themselves, in Appendix D.1 for the twisted case and Appendix D.2 for the twist-less case.

Path-connectivity $X_{\text{crit}}$ can be easily described as $\det \frac{\partial^2 SW}{\partial c_\ell^2} = 0$ which is just a polynomial equation on $c_\ell$ that, obviously, defines a domain of (complex) co-dimension 1.

This implies that any two solutions $c_\ell$ and $c'_\ell$ can be connected by a smooth path $\gamma$ that avoids the singular domain $C_{\text{crit}}$. We can always choose $\gamma$ such that its image $SW(\gamma)$ also passes only through regular values of $\chi_\ell$. Note that one or both points $c_\ell$ and $c'_\ell$ can actually belong to $C_{\text{crit}}$. So any singular solution can be obtained as a limit of regular solutions. Sporadic solutions, defined as solutions that exist only for some subspace of values $\chi_\ell$ cannot exist neither by the same argument.

We in particular conclude that for any choice of $\chi_\ell$, the number of solutions of Bethe equations is less or equal to $d_\Lambda$, where $d_\Lambda$ is defined as number of solutions at regular values of $\chi_\ell$ (this number does not depend on $\chi_\ell \notin X_{\text{crit}}$ since the regular domain of $X$ is path-connected).

Finiteness By definition, a map is called finite if it is proper and its fibers at all points are finite. So SW is an example of such a map, this property will be used later.

As we have established, all solutions to the Bethe equations (2.27) are distinct for $\chi_\ell$ being in the regular domain of $X$. Some solutions coincide if $\chi_\ell \in X_{\text{crit}}$, and so the number of distinct solutions is smaller. It is typical to count solutions with multiplicities in such a case. When we deal with equations in several variables, the notion of multiplicity requires an appropriate formalism to be introduced which is our next goal.

---

12For a simple counter-example, consider a system of equations $x(x - 1) = 0, \theta x = 0$. For all $\theta \neq 0$ there is only one solution $x = 0$. However, for $\theta = 0$ there is one extra sporadic solution $x = 1$.

13The concept of finite morphism is usually defined in a more general set-up using a rather abstract algebraic formalism. Here we are working with analytic varieties when the general “algebraic” definition is equivalent to the “topological” definition that we are using, see [67, 68].
3.2 How to count solutions with multiplicity

Starting from now, we will gradually introduce an algebraic formalism to analyse the Wronskian Bethe equations. We will be using standard terminology from commutative algebra which is briefly summarised in Appendix A.1.

Let us introduce a polynomial ring \( \mathcal{W}_\Lambda \) that shall be called the Wronskian algebra and which is defined as follows. Consider \( \mathbb{C}[\chi][c] \), the algebra of polynomials in variables \( \chi_1, \chi_2, \ldots, \chi_L, c_1, \ldots, c_L \) and \( \mathcal{I}_\Lambda = \langle \chi_1 - \text{SW}_1(c), \chi_2 - \text{SW}_2(c), \ldots, \chi_L - \text{SW}_L(c) \rangle \) — the ideal generated by the equations (2.27). Then

\[
\mathcal{W}_\Lambda := \mathbb{C}[\chi][c]/\mathcal{I}_\Lambda .
\]

(3.2)

Over \( \mathbb{C} \), \( \mathcal{W}_\Lambda \) is obviously isomorphic to \( \mathbb{C}[c] \). However, additionally, it is also naturally a \( \mathbb{C}[\chi] \)-module. Namely, one defines action of \( \chi_\ell \) on \( \mathcal{W}_\Lambda \cong \mathbb{C}[c] \) as follows: we multiply elements of \( \mathcal{W}_\Lambda \) by \( \chi_\ell \) and then replace \( \chi_\ell \) with \( \text{SW}_\ell(c) \).

To link the \( \mathbb{C}[\chi] \)-module structure with the Wronskian map from the previous section we note that \( \mathbb{C}[\chi] \) is the coordinate ring of \( \mathcal{X} \) and \( \mathbb{C}[\chi] \) is the coordinate ring of \( \mathcal{X} \). The map

\[
\text{SW}^* : \mathbb{C}[\chi] \to \mathbb{C}[c], \quad \chi_\ell \mapsto \text{SW}_\ell(c)
\]

used in the definition of \( \mathbb{C}[\chi] \)-action on \( \mathcal{W}_\Lambda \) is a pullback of (3.1).

Number of solutions to the Wronskian Bethe equations appears as follows in the algebraic context. We specialise the Wronskian algebra to the complex point \( \bar{\chi} \) where we would like to count the solutions. Specialisation is defined as

\[
\mathcal{W}_\Lambda(\bar{\chi}) := \mathcal{W}_\Lambda/\langle \chi_\ell - \bar{\chi}_\ell \rangle \cong \mathbb{C}[c]/(\text{SW}_\ell(c) - \bar{\chi}_\ell) .
\]

(3.4)

Then, it is a standard result that the number of solutions of a polynomial set of equations \( P_i(x_1, \ldots, x_n) = 0, \ i = 1, \ldots, m \) is equal to the dimension of the quotient ring \( \mathcal{R} = \mathbb{C}[x]/(P_1, \ldots, P_m) \) (as a vector space over \( \mathbb{C} \)). Moreover, in the case when solutions degenerate, the dimension of the quotient ring is used as a definition \(^{14}\) of the algebraic number of solutions (i.e. counted with multiplicity). In our case, the quotient ring in question is \( \mathcal{R} = \mathcal{W}_\Lambda(\bar{\chi}) \), and so the algebraic number of solutions of the Wronskian Bethe equation at point \( \bar{\chi} \) is equal to \( \text{dim}_\mathbb{C} \mathcal{W}_\Lambda(\bar{\chi}) \). Since at all points \( \bar{\chi} \) the number of solutions is finite \( \text{dim}_\mathbb{C} \mathcal{W}_\Lambda(\bar{\chi}) < +\infty \).

To see how this definition comes about in practice, consider the regular representation of the (finite-dimensional) quotient ring \( \mathcal{R} \) which is a map from elements of the ring to the ring endomorphisms defined by the ring multiplication. We can describe this map in terms of explicit matrices. Let \( b_1, \ldots, b_r \) be some basis elements of \( \mathcal{R} \). Then, for any \( X \in \mathcal{R} \), one has \( X b_i = \sum_{j=1}^r \tilde{X}_{ij} b_j \), where \( \tilde{X}_{ij} \in \mathbb{C} \). The regular representation maps \( X \) to the matrix \( \tilde{X} \)

\(^{14}\) Again, the general definition of multiplicity in the full formalism of algebraic geometry is much more intricate but in our case it is equivalent to the one we use.
whose components are $X_{ij}$.

**Example:**

Consider $\mathcal{R} = \mathbb{C}[x]/\langle x^2 - ax + b \rangle$. Elements $x$ and 1 span $\mathcal{R}$, choose them as basis elements. Then one has $x \cdot x = x^2 = ax - b$, $x \cdot 1 = x$ and so

$$\tilde{x} = \begin{pmatrix} 0 & 1 \\ -b & a \end{pmatrix}.$$  \hfill (3.5)

It is easy to prove that the image of the regular representation is isomorphic to the algebra $\mathcal{R}$. This allows us to understand properties of a polynomial ring in a more familiar setting of a matrix algebra that we denote as $\tilde{\mathcal{R}}$.

By the isomorphism, $P_i(\tilde{x}_1, \ldots, \tilde{x}_n) = 0$ for $i = 1, 2, \ldots, m$. So all joint eigenvalues of $\tilde{x}_1, \ldots, \tilde{x}_n$ are solutions of the set of equations. And each solution should be one of the joint eigenvalues (to see this, take $\sum_j (\tilde{x}_k)_j b_j = x_k b_i$ and evaluate $x_k$ and $b_i$, who are polynomials in $x_k$, to numerical values corresponding to the solution of interest).

Hence, when $\tilde{x}_\ell$ are diagonalisable then it is clear that the number of solutions is equal to the size of the matrix which is the same as the dimension of the quotient ring. Moreover, all solutions should be distinct (otherwise, isomorphism between $\tilde{\mathcal{R}}$ and $\mathcal{R}$ won’t hold).

When $\tilde{x}_\ell$ are not diagonalisable, intuitively this corresponds to existence of Jordan blocks and to degeneration of the solutions, multiplicity of degeneration would be the size of a Jordan block, example is (3.5) for $a = b = 0$.

Commuting matrices are however not simultaneously jordanisable and the mentioned intuitive picture should be slightly updated. Any commuting set of matrices, $\tilde{\mathcal{R}}$ in our case, allows for the Dunford-Jordan-Chevalley decomposition, namely there is a basis where all the matrices take the form $D + N$, where $D$ is diagonal and $N$ is upper-triangular with zeros on diagonal. Moreover, all elements of $N$ form a subalgebra in $\tilde{\mathcal{R}}$ known as the nil-radical $\text{Nil}(\tilde{\mathcal{R}})$ which is the ideal of all nilpotent elements of the ring. The quotient algebra $\text{diag}(\tilde{\mathcal{R}}) \equiv \tilde{\mathcal{R}}/\text{Nil}(\tilde{\mathcal{R}})$ is isomorphic to the algebra of matrices from $D$.

Resorting to the regular representation was of course optional, the concepts of the nil-radical $\text{Nil}(\mathcal{R})$ and the quotient $\text{diag}(\mathcal{R})$ exist for any (commutative) ring. In summary, one has a short exact sequence

$$0 \longrightarrow \text{Nil}(\mathcal{R}) \longrightarrow \mathcal{R} \longrightarrow \text{diag}(\mathcal{R}) \longrightarrow 0.$$  \hfill (3.6)

$\dim_\mathbb{C} \text{diag}(\mathcal{R})$ is precisely the number of distinct solutions to the polynomial equations. $\dim_\mathbb{C} \text{Nil}(\mathcal{R})$ counts the amount of degeneration in solutions, and

$$\dim_\mathbb{C} \mathcal{R} = \dim_\mathbb{C} \text{diag}(\mathcal{R}) + \dim_\mathbb{C} \text{Nil}(\mathcal{R})$$  \hfill (3.7)

is the total number of solutions counted with multiplicity.

We would like to emphasise that $\dim_\mathbb{C} \mathcal{R}$ is both the dimension of the quotient ring and the dimension of its regular representation (size of matrices). Eigenspaces of $\tilde{\mathcal{R}}$ are all
of dimension one and $\hat{R}$ is a maximal commutative subalgebra of $\text{End}(R)$. This remark will become important in our study of the Bethe algebra.

### 3.3 Wronskian algebra is a free $\mathbb{C}[\chi]$-module

The following very powerful result can be proven about the Wronskian algebra:

**Proposition 3.1.** $W_\Lambda$ is a free $\mathbb{C}[\chi]$-module.

**Proof.** $SW^*$ (3.3) is a ring morphism from $\mathbb{C}[\chi]$ to $\mathbb{C}[c]$ making $W_\Lambda$ a $\mathbb{C}[\chi]$-algebra and therefore a $\mathbb{C}[\chi]$-module. On the other hand, we can view $SW$ (3.1) as an algebraic morphism from the variety $\mathcal{C} \simeq \mathbb{A}^L$ (for the current discussion, the affine space $\mathbb{A}^L$) to $X \simeq \mathbb{A}^L$. We know that all the fibres of $SW$ are finite sets and are therefore of dimension 0. Moreover, $\mathbb{A}^L$, as an algebraic variety, is regular and (therefore) Cohen-Macaulay. Then, by a general result (sometimes called “miracle flatness theorem”) $SW^*$ is a flat ring morphism and so $W_\Lambda$ is a flat $\mathbb{C}[\chi]$-module, see for example [69, 70]. Since $\mathbb{C}[\chi]$ is Noetherian and $W_\Lambda$ is finitely-generated as a $\mathbb{C}[\chi]$-module (because $SW$ is finite) it is actually projective [71]. Finally, by the Quillen–Suslin theorem [72] it is free.

The above proof looks very short, however it is based on several abstract results from algebraic geometry. In appendix A we provide an elementary study of the Wronskian algebra which helps in understanding the logic behind the above proof.

By definition of a free module, $W_\Lambda$ has a basis – a collection of elements $b_1, \ldots, b_r$ such that any other element $a \in W_\Lambda$ is represented in a unique way as a linear combination

$$a = k_1 b_1 + \ldots + k_r b_r,$$

where $k_i \in \mathbb{C}[\chi]$. It is easy to prove that $b_1, \ldots, b_r$ remains a basis after specialisation (3.4) which leads to the immediate corollary of Proposition 3.1:

**Corollary 3.2.** The algebraic number of solutions of the Wronskian Bethe equations is the same for any value of $\bar{\chi}_\ell$.

We denote this number by $d_\Lambda$.

### 3.4 Number of solutions via Hilbert series

Let us find the value of $d_\Lambda$ explicitly. With the help of Proposition 3.1, counting is reduced to a simple dimensional analysis as we shall describe now.

If one chooses a rule by which we assign degree 0 to the identity element and some positive integer degrees to other elements of a ring $R$ then we can define the ring filtration

$$\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_k \subset \mathcal{F}_{k+1} \subset \ldots,$$
where $\mathcal{F}_k$ is the vector subspace of $\mathcal{R}$ (over $\mathbb{C}$) spanned by all elements of degree not exceeding $k$. Grading assignment should be compatible with the ring structure meaning that for any $k, k'$ and any ring element $r_k$ of degree $k$ one has $r_k \mathcal{F}_{k'} \subset \mathcal{F}_{k+k'}$.

A useful characterisation of a ring is given by its Hilbert series defined as

$$
\text{ch}_R(t) = \sum_{k=0}^{\infty} \dim_{\mathbb{C}} (\mathcal{F}_k/\mathcal{F}_{k-1}) t^k.
$$

Since the Wronskian ring $\mathcal{W}_\Lambda$ is $\mathbb{C}$-isomorphic to $\mathbb{C}[c_1, \ldots, c_L]$, computing its Hilbert series is particularly simple. It is just given by

$$
\text{ch}_{\mathcal{W}_\Lambda}(t) = \prod_{a=1}^{L} \frac{1}{1 - t^\deg c_a}.
$$

Recall that $c_\ell$ is a selected subset of $c^{(k)}_{\Lambda \cup J}$. Define $\deg c^{(k)}_{\Lambda \cup J} = k$, cf. (2.24). Then we have

**Lemma 3.3.** The Hilbert series of the Wronskian algebra $\mathcal{W}_\Lambda$ of the twisted system for $\Lambda = [\lambda_1, \ldots, \lambda_m; \nu_1, \ldots, \nu_n]$ is

$$
\text{ch}_\Lambda(t) := \text{ch}_{\mathcal{W}_\Lambda}(t) = \prod_{a=1}^{m} \prod_{k=1}^{\lambda_a} \frac{1}{1 - t^k} \prod_{i=1}^{n} \prod_{k=1}^{\nu_i} \frac{1}{1 - t^k}.
$$

The Hilbert series of the Wronskian algebra $\mathcal{W}_\Lambda$ of the $\mathfrak{gl}_{m|n}$ twist-less system with $\Lambda = \Lambda^+$ being a Young diagram depends on the Young diagram $\Lambda^+$ alone and is given by

$$
\text{ch}_{\Lambda^+}(t) := \text{ch}_{\mathcal{W}_\Lambda}(t) = \prod_{(a,s) \in \Lambda^+} \frac{1}{1 - h_{a,s}},
$$

where the product runs over all boxes of $\Lambda^+$ and $h_{a,s}$ is the hook length at box $(a,s)$.

**Proof.** The result in the twisted case is immediately obvious from (2.25) and the fact that $c_1, \ldots, c_L$ are precisely all $c^{(k)}_{\Lambda \cup J}$ and $c^{(k)}_{\emptyset \cup i}$. For the twist-less case, one needs to perform a little analysis on precisely what $c^{(k)}_{\Lambda \cup J}$ generate the Wronskian algebra. One can do it by filling the boxes of $\Lambda^+$ with degrees of the variables $c_\ell$ in a special way, this procedure outlined in Figure 2 clearly establishes a bijection with the lengths of the corresponding hooks.

To make the above-introduced filtration compatible with the $\mathbb{C}[\chi]$-action on $\mathcal{W}_\Lambda$, one needs to define $\deg \chi_\ell = \ell$. This is a simple reflection of the fact that the Wronskian relations $\chi_\ell = \text{SW}_\ell(c)$ come from equating coefficients between polynomials in $u$.

Because $\mathcal{W}_\Lambda$ is a free $\mathbb{C}[\chi]$-module, $\dim \mathcal{W}_\Lambda(\bar{\chi}) = \frac{\text{ch}_R(t)}{\text{ch}_{\mathbb{C}[\chi]}(t)|_{t=1}}$ for any $\bar{\chi}_\ell$. We then compute $\text{ch}_{\mathbb{C}[\chi]}(t) = \prod_{\ell=1}^{L} \frac{1}{1 - t^\ell}$ and easily conclude
Figure 2: Degrees of $c_n$ that generate the twist-less Wronskian algebra. On the left: The blue numbers show degrees of $c_n^{(k)}_a$, where $a = 1, 2, \ldots$ is the corresponding row of the Young diagram. The red numbers show degrees of $c_n^{(k)}_i$, where $i = 1, \ldots$ is the corresponding column. Crosses mean the terms $c_n^{(k)}_a$ and $c_n^{(k)}_i$ that are excluded by symmetry reasons, see the discussion after (2.34). For instance, the term $c_n^{(9)}_1$ (the constant term of $Q_n^1$) is cancelled in the linear combination $Q_n^1 + \alpha Q_n^4$. On the right: the same blue/red numbers compressed to the right/bottom after the crosses are removed. Note that this arrangement corresponds to the hook lengths. The green rectangular area has exactly as many boxes as number of physically-relevant functions $Q_n^a$. The constant terms of these functions, $c_n^{(k)}_a$, with $k = \deg Q_n^a = \lambda_a + \lambda_i + 1$, are independent variables used in generation of the Wronskian algebra. Their degrees, shown in green, match the hook lengths as well.

Theorem 3.4. The number of solutions of the Wronskian Bethe equations counted with algebraic multiplicity is equal to

$$\dim W_\Lambda(\bar{\chi}) = \dim V_\Lambda = \frac{L!}{\prod_{a=1}^m \lambda_a! \prod_{i=1}^n \nu_i!} \quad \text{for the twisted case},$$  

(3.12a)

$$\dim W_\Lambda^+(\bar{\chi}) = \dim V_\Lambda^+ = \frac{L!}{\prod_{(a,s)\in \Lambda} h_{a,s}} \quad \text{for the twist-less case}.$$  

(3.12b)

In summary, the number of solutions to the Wronskian Bethe equations is the correct one and hence the equations are complete for any numerical choice of the inhomogeneities.

4 Faithfulness

Until now, we have concentrated on the Wronskian algebra $W_\Lambda$ to study the properties of the Bethe equations (2.27). In particular we have shown that they have the expected number of solutions counted with multiplicity. Now we will establish a bijective correspondence between these solutions and eigenvalues (in degenerate cases, trigonal blocks) of the Bethe algebra. Mathematically, this correspondence is formulated as an isomorphism between the Wronskian algebra and the Bethe algebra. The map $\varphi : c_\ell \mapsto \hat{c}_\ell$ from the Wronskian algebra to the Bethe algebra is obviously surjective, and it is its injectivity (faithfulness) that we need to prove.

In this section, we shall first establish the isomorphism over the polynomial ring $\mathbb{C}[\chi]$ (which in practice means “in general position”) and then prove that the isomorphism also
holds for any numerical value $\bar{\chi}_\ell$ of $\chi_\ell$. A sufficient condition for the isomorphism to hold is that inhomogeneities that solve $\bar{\chi}_\ell = \chi_\ell(\bar{\theta}_1, \ldots, \bar{\theta}_L)$ satisfy $\theta_\ell + h \neq \theta_\ell'$ for $\ell < \ell'$.

### 4.1 Isomorphism between Wronskian and Bethe algebra

Recall that $B_\Lambda$ – the Bethe algebra restricted to $U_\Lambda$ – can be viewed as an algebra of operators generated by $c_\ell$. As we learned from the previous section, it is beneficial to first keep inhomogeneities as indeterminates, and that it is what we are going to do with the Bethe algebra as well. Then $B_\Lambda$ is naturally a subalgebra of $\text{End}(U_\Lambda) \otimes \mathbb{C}[\theta]$, however one should be careful because we define $B_\Lambda$ as the algebra generated from $c_\ell$ by considering any polynomials in $c_\ell$ with coefficients from $\mathbb{C}$, and not from $\mathbb{C}[\theta]$. This is a non-trivial remark because of the following

**Lemma 4.1.** If $p \in \mathbb{C}[\theta]$ and $p \times 1 \in B_\Lambda$ then $p$ is a symmetric polynomial in inhomogeneities.

**Proof.** Recall that the spin chain vector space is a tensor product of $L$ copies of $\mathbb{C}^m$. Introduce $r_\ell(\theta) = (\theta_\ell - \theta_{\ell+1})P_{\ell,\ell+1} + h \, \mathbf{1}$, where $P_{\ell,\ell+1}$ is the graded permutation of two copies of $\mathbb{C}^m$ at the $\ell$’th and the $(\ell + 1)$’th position of $(\mathbb{C}^m)^\otimes L$. This is an intertwining operator that satisfies

$$r_\ell \, ev(\theta_1, \ldots, \theta_\ell, \theta_{\ell+1}, \ldots, \theta_L) \, (T_{ij}) = ev(\theta_1, \ldots, \theta_{\ell+1}, \theta_\ell, \ldots, \theta_L) \, (T_{ij}) \, r_\ell$$

which is nothing but the Yang-Baxter equation in the physical channel graphically represented as

![Yang-Baxter equation](image)

If we also introduce $\Pi_\ell$ – permutation of inhomogeneities $\theta_\ell$ and $\theta_{\ell+1}$ in $\mathbb{C}[\theta]$ then $\Pi_\ell r_\ell$ commutes with the Yangian action and hence with the Bethe algebra. Therefore, if there is any equation $P(\hat{c}_1, \ldots, \hat{c}_L) = p(\theta_1, \ldots, \theta_L) \times \mathbf{1}$ that holds so will hold $P(\hat{c}_1, \ldots, \hat{c}_L) = p(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(L)}) \times \mathbf{1}$ for any $\sigma \in S_L$. Since inhomogeneities are independent, this is only consistent if $p$ is a symmetric polynomial.

To be prudent, we notice that derivation of $P(\hat{c}_1, \ldots, \hat{c}_L) = p(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(L)}) \times \mathbf{1}$ emerges from the following argument. $0 = \Pi_\ell r_\ell(P(\hat{c}_1, \ldots, \hat{c}_L) - p(\theta_1, \ldots, \theta_\ell, \theta_{\ell+1}, \ldots, \theta_L) \times \mathbf{1})\Pi_\ell r_\ell = (P(\hat{c}_1, \ldots, \hat{c}_L) - p(\theta_1, \ldots, \theta_{\ell+1}, \theta_\ell, \ldots, \theta_L) \times \mathbf{1})(r_\ell \Pi_\ell)^2 = A \times (\Pi_\ell r_\ell)^2$. $(\Pi_\ell r_\ell)^2 = (h^2 - (\theta_\ell - \theta_{\ell+1})^2) \times \mathbf{1}$. Then, because components of the matrix $A$ are polynomials in $\theta_\ell$, $0 = A \times (h^2 - (\theta_\ell - \theta_{\ell+1})^2)$ is only possible if $A = 0$.

We see that for instance $\theta_\ell \times \mathbf{1}$ does not belong to the Bethe algebra, except for $L = 1$. But any symmetric polynomials in inhomogeneities (times the identity operator) are elements of $B_\Lambda$ because of (2.27). So $B_\Lambda$ is naturally a $\mathbb{C}[\chi]$-algebra.

Recall now also the definition of the Wronskian algebra $W_\Lambda$ (3.2) which is a polynomial algebra generated by $c_\ell$ and which is also a $\mathbb{C}[\chi]$-algebra.

There is a potential difference between $W_\Lambda$ and $B_\Lambda$. The generators of the Wronskian algebra, by definition, satisfy only (2.27). The generators of the Bethe algebra a certain explicit operators and they could in principle satisfy some other additional constraints. However, we can show that they do not.
Theorem 4.2. The map $\varphi$ defined as

$$\varphi : \mathcal{R} \longrightarrow \mathcal{B}_\Lambda, \quad \varphi : c_\ell \mapsto \hat{c}_\ell$$

(4.2)

is an isomorphism of $\mathbb{C}[\chi]$-algebras.

In other words, $\hat{c}_\ell$ not only satisfy (2.27) but any polynomial relation between $\hat{c}_\ell$ with coefficients in $\mathbb{C}[\chi]$ should follow from (2.27).

The proof below makes precise the following argument: as there are as many variables $c_\ell$ as the parameters $\chi_\ell$ in (2.27), there cannot exist an extra relation between the variables because it would imply a relation between the parameters which are known to be independent.

Proof. We need to show that the exhibited map $\varphi$ is a well-defined (consistent) morphism and that it is surjective and injective. It is well-defined because Q-operators form a commutative algebra and they satisfy (2.27) from the very derivation of this relation. It is surjective because $\hat{c}_\ell$ generate $\mathcal{B}_\Lambda$.

The non-trivial part is the injectivity (faithfulness). To prove this we take an element $P \in \mathcal{W}_\Lambda$ (so just a polynomial in the variables $c_\ell$ and $\chi_\ell$ modulo relations in the ideal) and show that $\varphi(P) = 0$ implies $P = 0$.

Note that $P$ can be viewed as a polynomial in $c_\ell$ with constant coefficients as all occurrences of $\chi_\ell$ can be replaced by $SW_\ell^{-1}(c)$. Since $\varphi(P(c)) = P(\hat{c})$, $P$ has to vanish every time when $c_\ell$ are eigenvalues of $\hat{c}_\ell$ on a joint eigenvector. Then it suffices to construct enough of such eigenvalues to conclude that $P = 0$.

To this end consider $\chi_\ell \notin \mathcal{X}_{\text{crit}}$. There exists a neighbourhood $\mathcal{O}_{\chi_\ell}$ where all the solutions of (2.27) are distinct and can be parameterised by $d_\Lambda$ diffeomorphisms $SW_\ell^{-1}$ from $\mathcal{O}_{\chi_\ell}$ to $d_\Lambda$ non-intersecting open sets $U_i$ in $\mathcal{C}$.

We know that for all points of $\mathcal{O}_{\chi_\ell}$ the Bethe algebra has at least one common eigenvector and that the corresponding eigenvalues of $\hat{c}_\ell$ provide a solution of (2.27). By choosing in some way exactly one eigenvector at each point of $\mathcal{O}_{\chi_\ell}$, we create a disjoint partition of $\mathcal{O}_{\chi_\ell}$ into $d_\Lambda$ sets $\mathcal{O}_i$ corresponding to points of $\mathcal{O}_{\chi_\ell}$ where the common eigenvector gives the $i$-th solution. The closure (in $\mathcal{O}_{\chi_\ell}$) of one of the $\mathcal{O}_i$’s, say $\mathcal{O}_1$, contains an $L$-dimensional ball $\mathcal{O}$. This is proved as follows \cite{16}. Consider $\lambda$ the Lebesgue measure on $\mathcal{O}_{\chi_\ell}$ normalised to 1. Then either $\lambda(\mathcal{O}_1) = 1$ and therefore $\mathcal{O}_1 = \mathcal{O}_{\chi_\ell}$ since its complementary in $\mathcal{O}_{\chi_\ell}$ is an open set of measure zero or $\lambda(\mathcal{O}_1) < 1$ in which case by restricting to its complementary in $\mathcal{O}_{\chi_\ell}$ we are brought back to the same problem but with $d_\Lambda - 1$ sets. We conclude by induction.

Then $P$ vanishes on $SW_1^{-1}(\mathcal{O}_1)$ and since the zeros of a polynomial form a closed set and $SW_1^{-1}$ is a diffeomorphism on $\mathcal{O}_{\chi_\ell}$ it will also vanish on $SW_1^{-1}(\hat{0}_1)$ which contains an $L$-dimensional ball. $P$ being a polynomial in $L$ variables thus implies $P = 0$. \hfill \Box

\cite{16}This can be also proven by arguing that the common eigenvector can be chosen continuously in which case taking closure is unnecessary as well.
One may ask whether there are some additional polynomial relations between $\hat{c}_\ell$ with coefficients being non-symmetric polynomials of inhomogeneities. This is impossible either which is a slightly updated version of Lemma 4.1, see Appendix A.4.

4.2 What can happen upon specialisation

Now we shall consider what happens with isomorphism when inhomogeneities $\theta_\ell$ get concrete numerical values. We call this procedure specialisation at point $\bar{\theta}$.

Specialisation of the Bethe algebra $B_\Lambda(\bar{\theta})$ is replacing $\theta_\ell$ with $\bar{\theta}_\ell$ in all matrix entries of the operators $\hat{c}_\ell$. On the other hand, specialisation of the Wronskian algebra is

$$W_\Lambda(\bar{\theta}) \equiv W_\Lambda(\bar{\chi}_\ell = \chi_\ell(\bar{\theta})) \simeq W_\Lambda/(\chi - \bar{\chi}).$$

(4.3)

Its image under the map $\varphi$ is $B_\Lambda/(\chi(\theta) - \bar{\chi}) \times 1$ which explicitly means the following: replace $\chi_\ell$ with its numerical value each time it multiplies some matrix belonging to $B_\Lambda$. This operation is less restrictive than specialisation of the Bethe algebra and hence one can state that the morphism

$$\varphi_{\bar{\theta}} : W_\Lambda(\bar{\theta}) \longrightarrow B_\Lambda(\bar{\theta})$$

(4.4)

is surjective but may have a non-zero kernel. We denote by $\Theta_{\text{not}}$ the set of $\bar{\theta}$ when $\varphi_{\bar{\theta}}$ is not an isomorphism.

Example:

Consider a Wronskian algebra $W$ realised by relations $c_1 + c_2 = \chi_1$ and $c_1c_2 = \chi_2$. It is a free $\mathbb{C}[\chi_1, \chi_2]$-module, for the basis one can choose $1, c_1$, and the ring multiplication rule follows from $c_1^2 - \chi_1 c_1 + \chi_2 = 0$. Then $\hat{c}_1 = \begin{pmatrix} 0 & 1 \\ -\chi_2 & \chi_1 \end{pmatrix}$.

Consider two “Bethe algebras” $B^{\text{good}}$ and $B^{\text{bad}}$, with, respectively,

$$\hat{c}_1^{\text{good}} = \begin{pmatrix} \theta_1 & 1 \\ 0 & \theta_2 \end{pmatrix}, \quad \text{and} \quad \hat{c}_1^{\text{bad}} = \begin{pmatrix} \bar{\theta}_1 & 0 \\ 0 & \bar{\theta}_2 \end{pmatrix}.$$ 

(4.5)

They are both $\mathbb{C}[\chi_1, \chi_2]$-isomorphic, as algebras, to $W$. Note however that they realise non-isomorphic representations over $\mathbb{C}[\chi_1, \chi_2]$, i.e. there is no intertwiner matrix mapping $c_1^{\text{good}}$ to $c_1^{\text{bad}}$ whose coefficients are polynomial in $\chi_1, \chi_2$.

If we specialise at any point where $\bar{\theta}_1 \neq \bar{\theta}_2$, the corresponding $\varphi_{\bar{\theta}}$ would be an algebra isomorphism both for $B^{\text{good}}(\bar{\theta})$ and $B^{\text{bad}}(\bar{\theta})$, also there is obviously an intertwiner over $\mathbb{C}$ between “good” and “bad” representations making them isomorphic.

Now, let us specialise to a point $\bar{\theta}_1 = \bar{\theta}_2$. The specialised Wronskian algebra becomes a two-dimensional algebra over $\mathbb{C}$ generated by $1, c_1$ and relation $(c_1 - \bar{\theta}_1)^2 = 0$. It is isomorphic to the algebra generated by $\hat{c}_1 = \begin{pmatrix} 0 & 1 \\ -\bar{\theta}_1^2 & 2\bar{\theta}_1 \end{pmatrix}$ which cannot be diagonalised, cf. (3.5).

The Bethe algebra $B^{\text{good}}(\bar{\theta})$ is also two-dimensional and isomorphic to $W(\bar{\theta})$, \ldots
whereas $B^{\text{bad}}(\bar{\theta})$ is one-dimensional, and $c_1 - \bar{\theta}_1$ is in the kernel of $\varphi_{\bar{\theta}}$.

The difference between “good” and “bad” cases is in the presence of the nilpotent piece

$$
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
$$

in $\hat{c}_1^{\text{good}}$ which becomes an element of $B(\bar{\theta})$ each time $\bar{\theta}_1 = \bar{\theta}_2$. This well illustrates what happens in the general situation. As $B_\Lambda(\bar{\theta})$ is a commutative algebra, we can define the short exact sequence (3.6) for it. Then we can state the following

**Theorem 4.3.** In the map between the two sequences

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{Nil}(W_\Lambda(\bar{\theta})) & \longrightarrow & W_\Lambda(\bar{\theta}) & \longrightarrow & \text{diag}(W_\Lambda(\bar{\theta})) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Nil}(B_\Lambda(\bar{\theta})) & \longrightarrow & B_\Lambda(\bar{\theta}) & \longrightarrow & \text{diag}(B_\Lambda(\bar{\theta})) & \longrightarrow & 0 \\
\end{array}
$$

(4.6)

$\varphi^{\text{diag}}_{\bar{\theta}}$ is an isomorphism for any $\bar{\theta}$.

The isomorphism $\varphi^{\text{diag}}_{\bar{\theta}}$ literally means that each distinct solution of the Wronskian Bethe equations is in one-to-one correspondence with eigenspaces of the Bethe algebra (there are no non-physical solutions).

In the regular case when $\chi(\bar{\theta}) \notin X_{\text{crit}}$, Nil($W_\Lambda(\bar{\theta})$) = 0 and so the lemma implies that the Wronsksian and the Bethe algebras are isomorphic. Hence $\chi(\theta_{\text{not}}) \subset X_{\text{crit}}$.

In the degenerate case $\chi(\bar{\theta}) \in X_{\text{crit}}$, the non-isomorphism between the Wronskian and Bethe algebras, if present, can be only due to $\varphi^{\text{nil}}_{\bar{\theta}}$ having non-zero kernel. Roughly speaking, one can only lose information about Jordan block structure. The Bethe algebra could in principle have eigenspaces of dimension higher than 1 while this never happens with $\hat{W}_\Lambda$, see Section 3.2.

**Proof.** First note that $\varphi^{\text{nil}}_{\bar{\theta}}$ and hence $\varphi^{\text{diag}}_{\bar{\theta}}$ are well-defined because the nil-radical is an ideal and the image of a nilpotent element is nilpotent.

Take a regular value $\bar{\theta}$ ($\chi(\bar{\theta}) \notin X_{\text{crit}}$) and consider a linear combination $X$ of $c_\ell$’s taking pairwise distinct values at the $d$ solutions $\bar{\theta}$ of (2.27) at $\bar{\theta}$ (this is possible since all the solutions are distinct). Then $(X_i^1)_{0 \leq i < d - 1}$ is a basis of $W_\Lambda(\bar{\theta})$ and by continuity it will remain a basis upon specialisation to any $\theta$ in some open neighbourhood $O_{\bar{\theta}}$. Other choices of local bases are possible but for convenience we will work with this one.

Suppose $\varphi_{\bar{\theta}}$ is not an isomorphism for all $\theta \in O_{\bar{\theta}}$. Then $(X_i^1)_{0 \leq i < d - 1}$ are not linearly dependent for all $\theta \in O_{\bar{\theta}}$. Construct columns from the $d^2$ components of the matrices $X^i$ and combine the columns into a $d^2 \times d$ matrix. Linear dependence implies that all of the $d \times d$ minors of this matrix vanish on $O_{\bar{\theta}}$. Since these minors are polynomials in $\theta_\ell$, this means that they are zero as polynomials. This in turn provides a non-trivial relation $\sum_{i=1}^{d-1} r_i(\theta_\ell) \hat{X}^i = 0$ with $r_i \in \mathbb{C}[\theta]$.

---

\(^{17}\)We do not assume $d = d_\Lambda$ to make this proof independent of the counting result of Section 3.4.
Now we would like to be able to take \( r_i \in \mathbb{C}[\chi] \). To this end we use the braiding property (4.1) which implies \( \sum_{i=1}^{d-1} r_i(\theta_{\sigma(i)}) \hat{X}^i = 0 \) for any \( \sigma \in S_d \). Thus we can replace the \( r_i \)'s by their symmetric part. To ensure that it is non-zero for at least one of them we can multiply the relation we started with by \( \prod_{\sigma \in S_L \setminus \text{Id}} r_k(\theta_{\sigma(i)}) \) for some non-zero \( r_k \) and then take the symmetric part.

In the end we obtain a non-zero polynomial \( P \) with coefficients in \( \mathbb{C}[\chi] \) of degree smaller or equal to \( d - 1 \) such that \( P(\hat{X}) = 0 \). But by the isomorphism (4.2) this implies that \( P(X) \in \mathcal{I}_\Lambda \). Specialising \( \mathcal{W}_\Lambda \) at a point of \( \mathcal{O}_{\tilde{\theta}} \) where one of the coefficients of \( P \) does not vanish we obtain a contradiction with the fact that \( (X^i)_{0 \leq i \leq d-1} \) must be a basis at that point.

Therefore \( \varphi_{\tilde{\theta}} \) is an isomorphism for at least one regular value \( \tilde{\theta}' \in \mathcal{O}_{\tilde{\theta}} \). By path-connectivity this immediately propagates to all regular values. Indeed, \( \varphi_{\tilde{\theta}} \) can cease to be an isomorphism only if the dimension of the Bethe algebra drops but since the spectrum of \( Q \)-operators (the set of roots of their characteristic polynomials) is continuous in \( \theta \) this can only happen when two solutions cross, that is, at singular points.

At singular points, by continuity of the spectrum, solutions of the Wronskian Bethe equations are still in bijection with the spectrum of \( Q \)-operators. The only information that can be lost is the multiplicity of the solutions. Then considering \( \varphi_{\tilde{\theta}} \) up to nilpotent parts restores isomorphism.

Note that the set of \( \tilde{\theta} \) where \( \varphi_{\tilde{\theta}}(X^i) \) for \( 0 \leq i \leq d-1 \) do not form a basis is a priori not related to the set of \( \tilde{\theta} \) for which \( \chi(\tilde{\theta}) \in \mathcal{X}_{\text{crit}} \). Hence we can typically expect that \( \chi(\Theta_{\text{not}}) \) is of measure zero inside \( \mathcal{X}_{\text{crit}} \).

**Remark.** The continuity of the spectrum of \( Q \)-operators combined with the above theorem provides an immediate proof that \( \mathcal{S} W \) is proper as was previously announced. There is no circular argument as we did not use properness in the proof above.

### 4.3 Specialisation of the isomorphism

Although the set \( \Theta_{\text{not}} \) where the Wronskian and the Bethe algebras are not isomorphic is constrained to be, most likely, in a measure zero subset of critical values of \( \tilde{\theta} \), we still do not have means to locate \( \Theta_{\text{not}} \). This is unsatisfactory because we cannot guarantee to be outside \( \Theta_{\text{not}} \) for physically interesting cases, for instance when all inhomogeneities coincide. In this section we will provide an explicit constraint on \( \Theta_{\text{not}} \). Since the required formalism is quite heavy we will only present the logic behind it and the final results. The technical details are postponed to Appendix B.

The main conceptual step is the following. Although the Wronskian and the Bethe algebras were shown to be isomorphic in Theorem 4.2, there is an important qualitative difference between them. Namely, the Bethe algebra is represented by matrices and so it naturally acts on a vector space (the spin chain Hilbert space), whereas the Wronskian algebra is abstractly defined by generators and relations and does not admit such a representation. The only natural space on which \( \mathcal{W}_\Lambda \) could possibly act is itself (this is the
so-called regular representation). We would like to build an isomorphism between this representation of $\mathcal{W}_\Lambda$ and the physical representation of $B_\Lambda$.

To build such an isomorphism, a standard procedure is to try to find a cyclic vector. By definition, for a given algebra $\mathcal{A}$ that acts on some vector space $V$, a vector $\omega$ is said to be cyclic if the action of $\mathcal{A}$ on $\omega$ spans $V$. Then $V$ is said to be a cyclic $\mathcal{A}$-module. Equivalently, $\omega$ is cyclic if the map $\psi_\omega: \mathcal{A} \rightarrow V$, $A \mapsto A \cdot \omega$ is surjective. If moreover it is injective, it is an isomorphism identifying $V$ with the regular representation of $\mathcal{A}$.

In the case of $B_\Lambda$ acting on $U_\Lambda \otimes \mathbb{C}[\theta]$, it turns out that $\psi_\omega$ is injective for any nonzero vector $\omega$ as shown in Lemma B.7. Unfortunately, the image of $\psi_\omega$ is not $U_\Lambda \otimes \mathbb{C}[\theta]$ except probably for $L = 1$. This can already be seen by the fact that $B_\Lambda$ as an algebra involves only symmetric polynomials $\chi_\ell$ whereas the matrix coefficients of $\hat{c}_\ell$ are from $\mathbb{C}[\theta]$. Nevertheless, for a specific and unique (up to normalisation) choice of $\omega$ one can explicitly describe the image of $\psi_\omega$ as a certain subspace $\mathcal{U}_S^S \subset U_\Lambda \otimes \mathbb{C}[\theta]$ invariant under an action of the symmetric group $S_L$ commuting with the Yangian (Lemma B.8). Thus the regular representation of $\mathcal{W}_\Lambda$ can be identified with the representation $\mathcal{U}_S^S$ of $B_\Lambda$.

The above remarks are the tools to prove a powerful and explicit constraint on $\Theta_{\text{not}}$. Since it is a central result to us, we first recall the definitions of all the objects.

$\mathcal{W}_\Lambda(\bar{\chi})$ is the specialised Wronskian algebra at point $\bar{\chi} \equiv (\bar{\chi_1}, \ldots, \bar{\chi_L}) \in \mathcal{X} \simeq \mathbb{C}^L$. It is defined as $\mathcal{W}_\Lambda(\bar{\chi}) := \mathbb{C}[c_1, \ldots, c_L]/\mathcal{I}$, where $\mathcal{I} := (\mathcal{S}W_1 - \bar{\chi}_1, \ldots, \mathcal{S}W_L - \bar{\chi}_L)$ is an ideal in $\mathbb{C}[c_1, \ldots, c_L]$. For the definition of $\mathcal{S}W_\ell$ see (2.27). Also denote $\hat{\mathcal{I}} := \varphi(\mathcal{I})$, where $\varphi$ is the map (4.2), and $\hat{\mathcal{J}} := (\bar{\chi}_1 - \bar{\chi}_1, \ldots, \bar{\chi}_L - \bar{\chi}_L)$ considered as a (maximal) ideal of $\mathbb{C}[\chi]$.

$B_\Lambda(\bar{\theta})$ is the Bethe subalgebra of the Yangian in the spin chain representation at point $\bar{\theta} = (\bar{\theta_1}, \ldots, \bar{\theta_L})$ restricted to the weight subspace $U_\Lambda$. It is generated by operators $\hat{c}_1, \ldots, \hat{c}_L$. Matrix entries of these operators are polynomials in inhomogeneities $\theta_\ell$ that are being set to values $\bar{\theta}_\ell$.

$$\chi_\ell(\bar{\theta}) := \sum_{1 \leq i_1 < \ldots < i_\ell \leq L} \theta_{i_1} \ldots \theta_{i_\ell} \text{ are the elementary symmetric polynomials of degree } \ell, \ell = 1, \ldots, L.$$

Let us first state an immediate consequence of Theorem 4.2 and the discussion above.

**Lemma 4.4.** i) $\varphi$ induces an isomorphism of $\mathbb{C}$-algebras $\mathcal{W}_\Lambda(\bar{\chi}) \simeq B_\Lambda/\hat{\mathcal{I}}$.

ii) $\psi_\omega$ induces an isomorphism of representations $\mathcal{W}_\Lambda(\bar{\chi}) \simeq \mathcal{U}_S^S/\hat{\mathcal{J}} \cdot \mathcal{U}_S^S$.

**Proof.** One easily checks that indeed $\psi_\omega(\hat{\mathcal{I}}) = \hat{\mathcal{I}} \cdot \omega = \hat{\mathcal{J}} \cdot B_\Lambda \cdot \omega = \hat{\mathcal{J}} \cdot \mathcal{U}_S^S$. \qed

This does not seem to be a very helpful statement since it is not clear how one should interpret the abstract quotients $B_\Lambda/\hat{\mathcal{I}}$ and $\mathcal{U}_S^S/\hat{\mathcal{J}} \cdot \mathcal{U}_S^S$. However, Theorem B.9 implies the following

**Theorem 4.5.** If $\bar{\chi}_\ell = \chi_\ell(\bar{\theta}_1, \ldots, \bar{\theta}_L)$ and $\bar{\theta}_\ell + h \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$ then

$$\text{ev}_\bar{\theta} : \mathcal{U}_S^S/\hat{\mathcal{J}} \cdot \mathcal{U}_S^S \to U_\Lambda, \quad [v] \mapsto v(\bar{\theta}_1, \ldots, \bar{\theta}_L)$$

(4.7)

is an isomorphism of the Bethe algebra representations. \quad \square
Here $ev_{\bar{\theta}}$ denotes the map\(^{18}\) induced by $Ev_{\bar{\theta}}$, the evaluation of vectors of $U^S_\Lambda \subset U_\Lambda \otimes \mathbb{C}[\theta]$ at $\bar{\theta}$. As $Ev_{\bar{\theta}}(J \cdot U^S_\Lambda) = 0$, (4.7) is well-defined. Since the Bethe algebra is represented by $B_\Lambda/\hat{I} \simeq W_\Lambda(\bar{\chi})$ in $\text{End}(U^S_\Lambda/J \cdot U^S_\Lambda)$ and by $B_\Lambda(\bar{\theta})$ in $\text{End}(U_\Lambda)$ we thus obtain

**Theorem 4.6.** If $\bar{\chi}_\ell = \chi(\bar{\theta}_1, \ldots, \bar{\theta}_L)$ and $\bar{\theta}_\ell + h \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$ then

$$\varphi_{\bar{\theta}} : W_\Lambda(\bar{\chi}) \longrightarrow B_\Lambda(\bar{\theta}), \quad c_\ell \longmapsto \hat{c}_\ell \quad (4.8)$$

is an algebra isomorphism over $\mathbb{C}$.

This result improves Theorem 4.3 by giving an explicit condition under which not only the diagonal but also the nilpotent parts of the Wronskian and Bethe algebras are isomorphic. Note that values of $\bar{\chi}_\ell$ are not restricted in any way, there is only a restriction which solution of $\bar{\chi}_\ell = \chi(\bar{\theta}_1, \ldots, \bar{\theta}_L)$ can be taken.

Assuming the condition $\bar{\theta}_\ell + h \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$ is satisfied, the construction above has several immediate consequences. Composing $ev_{\bar{\theta}} \circ \psi_\omega$ or, equivalently, acting with $W_\Lambda(\bar{\chi})$ on $\omega(\bar{\theta})$ via $\varphi_{\bar{\theta}}$ we obtain

**Corollary 4.7.** The spin chain representation of the Bethe algebra restricted to $U_\Lambda$ is isomorphic to the regular representation of $W_\Lambda(\bar{\chi})$.

This result is important for separation of variables, as is discussed in Section 7.4.

**Corollary 4.8.** $B_\Lambda(\bar{\theta})$ is a maximal commutative subalgebra of $\text{End}(U_\Lambda)$.

**Proof.** $\tilde{W}_\Lambda(\bar{\chi})$ – the image of generators $W_\Lambda(\bar{\chi})$ under the regular representation – is a maximal commutative subalgebra of $\text{End}_\mathbb{C}(W_\Lambda(\bar{\chi}))$ (this is true for any regular representation, see Section 3.2).

In physical terms, this means that the Bethe algebra contains all commuting charges of the system.

**Corollary 4.9.** If $B_\Lambda(\bar{\theta})$ is diagonalisable then its spectrum is simple.

**Proof.** A diagonalisable commutative algebra of matrices whose spectrum is not simple is not maximal commutative.

In particular, the Bethe algebra has simple spectrum every time it is invariant under a Hermitian conjugation which is often the case in physical applications. Simplicity of the spectrum allows us to introduce a new way of classifying solutions by continuously deforming inhomogeneities, see Section 7.2.

**Example:**

Let us anticipate on the example detailed in Appendix B.5 corresponding to a $gl_2$ non-twisted spin chain of length $L = 3$. The Bethe algebra on the two-dimensional highest-weight subspace $V^{[\varnothing]}$ is generated (as a $\mathbb{C}[\chi]$-module) by the identity and the

\(^{18}\) not to confuse with $ev_\theta$ in (2.9)
non-trivial operator
\[ c = \begin{pmatrix} 2\chi_1 - \sqrt{3}(\theta_1 - \theta_3) & \chi_1 - 3\theta_2 - \sqrt{3}h \\ \chi_1 - 3\theta_2 + \sqrt{3}h & 2\chi_1 + \sqrt{3}(\theta_1 - \theta_3) \end{pmatrix} \] (4.9)

As long as \( c \) is not proportional to the identity, the evaluated Bethe algebra will be of dimension 2 and therefore isomorphic to the Wronskian algebra. We see that it can only happen if \( \hbar = 0 \), that is in the Gaudin limit, and moreover \( \bar{\theta}_1 = \bar{\theta}_2 = \bar{\theta}_3 \). Note that if \( \hbar \neq 0 \) or if \( \hbar = 0 \) and \( \bar{\theta}_i \) are not all equal, we still have isomorphism even though the hypothesis of Theorem 4.6 is not always satisfied.

### 4.4 Construction of Bethe vectors

Let us finally comment how to use the established isomorphism to construct in bijective way eigenstates of the Bethe algebra from solutions of the Bethe equations. This gives us the practical meaning of the words “completeness” and “faithfulness”.

We can always take an element of the Wronskian algebra \( X \) such that \( (X^k)_{1 \leq k \leq d_{\Lambda}} \) forms a basis. In other words the polynomial of smallest degree such that \( P(X) = 0 \) in \( W_\Lambda(\bar{\chi}) \) is of degree \( d_{\Lambda} \). Its roots \( x_i, i = 1, \ldots, d_{\Lambda} \) is a way to encode the solutions of Bethe equations (2.26).

By the established isomorphism, \( P \) is both the characteristic and minimal polynomial of the matrix \( \hat{X} \in B_{\Lambda}(\bar{\theta}) \) which means that any smaller-degree polynomial of \( \hat{X} \) yields a non-zero matrix. Let \( x_i \) be an eigenvalue of \( \hat{X} \). Then \( \frac{\det(\lambda - \hat{X})}{\lambda - x_i} \) is a polynomial in \( \lambda \). Take a cyclic vector \( \omega \) and define
\[ v_{x_i} = \frac{\det(\lambda - \hat{X})}{\lambda - x_i} \mid_{\lambda = \hat{X}} \omega. \] (4.10)

Since \( \det(\lambda - \hat{X})_{\lambda = \hat{X}} = 0 \), one has \( (\hat{X} - x_i)v_{x_i} = 0 \) and so \( v_{x_i} \) is an eigenvector of \( \hat{X} \) with the eigenvalue \( x_i \).

We emphasise that as \( B_{\Lambda}(\bar{\theta}) \) is isomorphic to the regular representation of the Wronskian algebra all eigenspaces are one-dimensional and so all eigenvectors with eigenvalue \( x_i \) are collinear with \( v_{x_i} \), which guarantees the bijection.

In case of degeneration of solutions, different Jordan blocks of \( \hat{X} \) must have distinct eigenvalues. If \( x_i \) is a root of multiplicity \( n \) then \( v_{x_i}^{(n)} = \frac{\det(\lambda - \hat{X})}{(\lambda - x_i)^m} \mid_{\lambda = \hat{X}} \omega, m \leq n \), provide a Jordan basis for \( \hat{X} \). Since \( \hat{X} \) generates \( B_{\Lambda}(\bar{\theta}) \) this basis will also trigonalise the Bethe algebra.

---

19The existence of such an \( X \) is obvious at non-degenerate points \( \bar{\theta} \) and otherwise follows from the analysis in Appendix A.2. We introduced \( X \) for clarity, but the discussed construction of the Bethe algebra eigenstates can be also formulated in a way that does not rely on the existence of \( X \).

20Corollary 4.7 implies that under the usual assumption on \( \bar{\theta} \) such a vector always exists. By continuity, this implies that any generic vector will also be cyclic.
5 Various parameterisations of the Bethe algebra

Although we proved that the number of solutions of the Wronskian Bethe equations is the correct one, we did not develop any intuition about how these solutions are organised. We address this issue in the next two sections by proposing techniques to systematically label solutions.

5.1 Restriction and extension of Q-systems

We introduced a restricted Q-system on page 20 to cover the case of short representations: for special sets $\mathbf{A}, \mathbf{J}$, $Q^\text{rest}_{\mathbf{A} | \mathbf{J}} := Q_{\mathbf{A} | \mathbf{J}}$. Now, we remark that the Q-functions $Q^\text{rest}_{\mathbf{A} | \mathbf{j}} = Q_{\mathbf{A} | \mathbf{J}, \mathbf{j}}$ satisfy the QQ-relations (2.17) and $Q^\text{rest}_{\mathbf{j} | \emptyset} = Q_{\mathbf{j} | \emptyset}$ for any sets $\mathbf{A}, \mathbf{J}$.

If $Q_{\mathbf{A} | \mathbf{J}} = 1$ then one can interpret the restricted Q-system as a Q-system of a smaller $\mathfrak{gl}(m' | n')$ algebra, where $m' = m - |\mathbf{A}|$, $n' = n - |\mathbf{J}|$. The condition $Q_{\mathbf{A} | \mathbf{J}} = 1$ is of course non-trivial to demand. By counting degrees of polynomials according to (2.33) we see that this is possible if $(m, n)$ is outside of the Young diagram or if $(m, n)$ is situated on the boundary of the Young diagram such that $(m', n')$ is also on the boundary. We note that the degrees of the restricted Q-functions come out to be given by (2.33) for $\mathfrak{gl}(m' | n')$ Q-system.

Let us now construct an opposite to the restriction procedure. For simplicity consider an “elementary” move. Define extension from a $\mathfrak{gl}(m | n - 1)$ Q-system to the $\mathfrak{gl}_{m | n}$ Q-system as follows:

$$Q^\text{ext}_{\mathbf{A} | \mathbf{j}, n} = Q_{\mathbf{A} | \mathbf{j}, n} , \quad Q^\text{ext}_{\mathbf{j} | \emptyset, m} = Q^\text{ext}_{\mathbf{j} | \emptyset, m} = 1 ,$$

supplemented with the requirement that the Q-functions that do not contain $\mathbf{n}$ are fixed by consistency of the QQ-relations. An example of this extension is depicted in Figure 3 using a Hasse diagram [40].

**Lemma 5.1.** The extension (5.1) is always possible and moreover it defines all the Q-functions $Q^\text{ext}_{\mathbf{a} | \mathbf{j}, n}$ uniquely up to symmetries.

**Proof.** Let us find $Q^\text{ext}_{\mathbf{a} | \emptyset, n}, Q^\text{ext}_{\mathbf{a} | \mathbf{j}, n}, Q^\text{ext}_{\mathbf{a} | \mathbf{j}, n}$ by the prescribed construction. Since $Q_{\mathbf{j} | \emptyset} = 1$, one computes $Q^\text{ext}_{\mathbf{a} | \emptyset} = (Q^\text{ext}_{\mathbf{a} | \mathbf{n}})^+ - Q^\text{ext}_{\mathbf{a} | \emptyset}$ using the known value of $Q^\text{ext}_{\mathbf{a} | \mathbf{n}} = Q_{\mathbf{a} | \emptyset}$. To find $Q^\text{ext}_{\mathbf{j} | \emptyset}$ one solves $(Q^\text{ext}_{\mathbf{j} | \emptyset})^+ - (Q^\text{ext}_{\mathbf{j} | \emptyset})^- = Q^\text{ext}_{\mathbf{j} | \emptyset}$. The polynomial solution is fixed up to an additive constant, but we remind that $Q^\text{ext}_{\mathbf{j} | \emptyset} \to Q^\text{ext}_{\mathbf{j} | \emptyset} + \alpha Q^\text{ext}_{\mathbf{j} | \emptyset}$ is a symmetry of the twist-less Q-systems. Finally, we use $(Q^\text{ext}_{\mathbf{a} | \mathbf{j}})^- (Q^\text{ext}_{\mathbf{a} | \mathbf{n}}) - (Q^\text{ext}_{\mathbf{a} | \emptyset})^- (Q^\text{ext}_{\mathbf{a} | \emptyset})^- = Q^\text{ext}_{\mathbf{a} | \emptyset}$ which is a consequence of (2.17) to uniquely fix $Q^\text{ext}_{\mathbf{a} | \emptyset}$ from the already identified quantities. One can now check that the above-constructed Q’s satisfy $(Q^\text{ext}_{\mathbf{a} | \mathbf{j}})^+ - (Q^\text{ext}_{\mathbf{a} | \mathbf{j}})^- = Q^\text{ext}_{\mathbf{a} | \mathbf{n}} Q^\text{ext}_{\mathbf{a} | \emptyset}$ and so they properly generate the whole Q-system. \( \square \)

A caveat of the extension procedure is that $Q^\text{ext}_{\mathbf{a} | \emptyset} = 0$ if $Q^\text{ext}_{\mathbf{a} | \mathbf{n}}$ is a constant. This does not happen however if both points $(\mathbf{m}, \mathbf{n} - 1)$ and $(\mathbf{m}, \mathbf{n})$ belong to the boundary of the Young diagram.

The analogous definitions and statements can be made also for the extension from the $\mathfrak{gl}(m - 1 | n)$ to the $\mathfrak{gl}_{m | n}$ Q-system.
Figure 3: Hasse diagram for extension of a \( \mathfrak{gl}(2) \) Q-system (blue square) to the \( \mathfrak{gl}(2|1) \) Q-system (blue and red squares). One considers nodes of the blue square as known, supplements this data with \( Q_{\emptyset}^{\text{ext}} = 1 \) condition, and finds the rest by QQ-relations. Note that if \( Q_2 = 1 \) then \( Q_{2|\emptyset}^{\text{ext}} = 0 \). In this case, \( Q_{\emptyset} \) contains all physical information and we can restrict both \( \mathfrak{gl}(2) \) and \( \mathfrak{gl}(2|1) \) systems to the \( \mathfrak{gl}(1) \) system (green line) that consists of \( Q_{\emptyset}^{\text{rest}} = Q_{\emptyset} = Q_{\emptyset|\emptyset}^{\text{ext}} \) and \( Q_{\emptyset}^{\text{rest}} = Q_2 = Q_{2|1}^{\text{ext}} = 1 \).

5.2 Isomorphism of twist-less \( B_{\Lambda} \) across \( \mathfrak{gl}_{m|n} \) algebras of various ranks

An important conclusion from the made observations is that if \( \mathfrak{gl}(m'|n') \subset \mathfrak{gl}(m|n) \) and both points \((m', n')\) and \((m, n)\) belong to the boundary of the Young diagram then restriction and extension procedures of the Q-system are inverse of one another and hence both Q-systems contain precisely the same physical information.

By performing a sequence of restrictions and extensions, we conclude that all the \( \mathfrak{gl}(m'|n') \) Q-systems with \((m', n')\) being on the boundary of \( \Lambda^+ \) (black and red dots in Figure 1) are bijectively related. Moreover, this is done by utilising only polynomial operations. Hence we conclude the following.

**Lemma 5.2.** In the twist-less case, all the Bethe algebras \( B_{\Lambda} \) generated by the \( c_{\ell} \) of \( \mathfrak{gl}(m|n) \) Q-systems, where \((m, n)\) is any point on the boundary of the Young diagram \( \Lambda^+ \), are isomorphic as \( \mathbb{C}[\chi] \)-algebras.

We note that arguments leading to this conclusion do not require isomorphism of the Wronskian and Bethe algebra.

**Example:**
The \( \mathfrak{gl}(3) \) and \( \mathfrak{gl}(1|1) \) Q-systems for representation \( \mathbb{P} \) from the previous examples are related by the prescribed procedure:

\[
\begin{align*}
\mathfrak{gl}(3|0) & \overset{\text{rest}}{\longrightarrow} \mathfrak{gl}(2|0) \overset{\text{ext}}{\longrightarrow} \mathfrak{gl}(2|1) \overset{\text{rest}}{\longrightarrow} \mathfrak{gl}(1|1), 
\end{align*}
\]
the extension bit of which is outlined in Figure 3. Hence $Q^{gl(3)}_1$, $Q^{gl(3)}_2$ (and $Q^{gl(3)}_3 = 1$) should contain the same physical information as $Q^{gl(1)}_{1\emptyset}$, $Q^{gl(1)}_{\emptyset1}$ (and $Q^{gl(1)}_{11} = Q_\emptyset$) and expressible through one another.

By performing the restriction and extension transformation as described above one finds

$$Q^{gl(3)}_2 \propto \Psi(Q^{gl(1)}_{\emptyset1}),$$

$$Q^{gl(3)}_1 \propto \Psi \left( \frac{W(Q^{gl(1)}_{1\emptyset}, \Psi(Q^{gl(1)}_{\emptyset1}))}{Q^{gl(1)}_{1\emptyset}} \right),$$

where $\Psi(A) = B$ if $B^+ - B^- = A$ and the ratio in (5.3b) is a polynomial (Bethe equations ensure that the euclidean remainder of the numerator and the denominator is the zero polynomial). $\Psi$ is defined up to addition of a constant, there are three such constants in (5.3) which corresponds to three symmetries in the $gl(3)$ system restricted to $v \mathbb{P}$. We fixed the symmetry in (2.36) by setting certain $c_i^{(k)}$ to zero, and we should set the constants of integration to the value that reproduces this choice.

Explicitly in terms of $c_i$’s, the transformation (5.3) becomes

$$c^{(3)}_1 = 4c_{1\emptyset},$$

$$c^{(1)}_1 = (2c_{\emptyset1} - 4c_{1\emptyset})h^2 - 8c_{11},$$

$$c^{(1)}_2 = 2c_{\emptyset1}.$$

By reversing (5.2), we find that

$$Q^{gl(1)}_{\emptyset1} \propto Q^{gl(3)}_{23},$$

$$Q^{gl(1)}_{1\emptyset} \propto W(Q^{gl(3)}_1, Q^{gl(3)}_2, Q^{gl(3)}_3, u),$$

$$Q^{gl(1)}_{11} \propto Q^{gl(3)}_{123}.$$

that becomes in terms of $c$’s

$$c_{\emptyset1} = \frac{c^{(1)}_2}{2},$$

$$c_{1\emptyset} = \frac{c^{(3)}_1}{4},$$

$$c_{11} = \frac{(c^{(1)}_2 - c^{(3)}_1)h^2 - c^{(1)}_1}{8}.$$

Transformation (5.6) is the inverse of (5.4). The fact that inverse of a nonlinear polynomial transformation is still polynomial is of course non-trivial, and it holds because of the relations satisfied by $c$’s.

By Lemma 5.2, we can use the Q-system which corresponds to $gl(m = h_{A+}, n = 0)$, where $h_{A+}$ is the height of Young diagram, and hence is purely bosonic. Hence we can
use in principle results from [9] for bosonic Bethe algebras to formulate and prove the completeness and faithfulness statements for supersymmetric spin chains 21, but we choose to not rely on this relation to [9], highlight instead novel important features of the system, and to produce results in a way that does not use the nested Bethe Ansatz.

5.3 Q-system on Young diagrams

One of the interesting features is that the Bethe algebra can be also generated from the so-called Q-system on a Young diagram which was introduced in [73], extensively used in application to the AdS/CFT spectral problem [74, 75] and more recently was used in several other studies, see e.g. [76, 77].

The Q-system on a Young diagram is a collection of monic polynomials $Q_{a,s}$ defined as

$$Q_{a,s} \propto Q_{a+1,a+2,...,m|s+1,s+2,...,n},$$  \hspace{1cm} (5.7)

where the Q-function on the right-hand side is a member of the $\mathfrak{gl}(m|n)$ Q-system and $(m,n)$ belongs to the boundary of Young diagram. It is clear from the extension-restriction procedure that $Q_{a,s}$ does not depend on $m,n$. In particular, $Q_{0,0} = Q_\theta$.

$Q_{a,s}$ are naturally assigned to the nodes of the $\mathbb{Z}_2$ lattice bounded by the Young diagram shape, $Q_{a,s} = 1$ on the boundary of the diagram (black/red dots in Figure 1). The QQ-relations between $Q_{a,s}$ are

$$Q_{a+1,s+1}Q_{a,s} \propto W(Q_{a+1,s}, Q_{a,s+1}),$$  \hspace{1cm} (5.8)

this follows from (2.17b).

Example:

For the Young diagram $\mathfrak{P}$, $Q_{a,s}$ that are not equal to 1 are $Q_{1,0} = Q_{\mathfrak{gl}(1|1)} = Q_{2}^{(2)} = Q_{23}^{(3)}$, $Q_{0,1} = Q_{1}^{(1)}$, and $Q_{0,0} = Q_\theta$. There is only one non-trivial relation $Q_{0,1}Q_{1,0} = Q_{0,0}^{-} - Q_{0,0}^{+}$, so there is no significant difference between this simple Q-system and the above-discussed example of a $\mathfrak{gl}(1|1)$ Q-system. Systems for larger Young diagrams are more interesting of course.

The Q-system on a Young diagram $\Lambda^+$ is polynomially generated from the Bethe algebra $\mathcal{B}_\Lambda$ as it follows directly from (5.7). The converse is also true [73]:

Lemma 5.3. Consider the Q-system on a Young Diagram $\Lambda^+$, and choose any $m,n$ that lie on the boundary of $\Lambda^+$. Then, up to symmetry, one can uniquely construct a solution of the $\mathfrak{gl}(m|n)$ QQ-relations such that (5.7) holds and that the coefficients of the $Q_{A,J}$ are polynomial functions of the coefficients of the $Q_{a,s}$.

21It is also possible to establish a bijection between bosonic and supersymmetric Q-systems in the presence of twist. To this end one first extends the original Q-system to a larger one with a partially degenerate twist where an analog of the Young diagram boundary and hence a possibility to move along it emerges.
So the Q-system on a Young diagram is yet another description of the Bethe algebra restricted to $V^+_\Lambda$. We shall benefit from it for counting purposes.

We will now prove Lemma 5.3 in a bit different way compared to [73]. Later we are going to rely on the techniques introduced in this proof.

**Proof.** It suffices to find all $Q^{gl(m)}_{a|\emptyset}$ for the $gl(m|0)$ Q-system, where $m = h_\Lambda$ is the height of the Young diagram. Then we can use extensions and restrictions to get all other $gl(m'|n')$ Q-systems. To this end, let us extend $gl(m|0)$ to the $gl(m|n)$ Q-system, where $n = \lambda_1$ is the width of Young diagram. Upon extension $Q^{gl(m)}_{a|\emptyset} = Q^{gl(m|n)}_{a|\emptyset}$. If $(m,n)$ lies outside of the Young diagram boundary, the extended Q-system contains non-physical Q-functions making the extension procedure non-unique, but this will not induce an ambiguity in fixing $Q^{gl(m)}_{a|\emptyset}$.

In the following, superscript $gl(m|n)$ will be omitted.

One has $Q_{\emptyset|\emptyset} = Q_{m,0} = 1$ which is very suitable for applying the so-called bosonisation trick [64]. The bosonisation trick is the observation that the Q-functions $B_{\Sigma}: = Q_{\emptyset|\emptyset}$, where $\emptyset$ meansthecomplementarysetto $J$, satisfytheQQ-relationsofthebosonic $gl(n+m)$ system:

$$B_{\Sigma_1,\beta}B_{\Sigma_2} \propto W(B_{\Sigma_1}, B_{\Sigma_2}), \quad (5.9)$$

where $\Sigma$ is a multi-index and $\alpha, \beta$ are indices from the set $\{1, \ldots, m, \hat{1}, \ldots, \hat{n}\}$. This immediately follows from (2.17) and the definition of $B$.

Since $B_{\emptyset} = Q_{\emptyset|\emptyset} = 1$, the relations (5.9) are solved by $B_{\Sigma} = W(B_{\alpha_1}, B_{\alpha_2}, \ldots, B_{\alpha_k})$, where $\Sigma = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and so (5.7) becomes

$$Q_{a,s} \propto W(B_{a+1}, \ldots, B_m, B_{\hat{1}}, \ldots, B_{\hat{s}}). \quad (5.10)$$

We are going to solve these equations for $B_{\alpha}$, this is done in a unique way modulo admissible symmetry transformations. Note that our main interest is $B_{a} = Q^{gl(m)}_{a|\emptyset}$.

From (2.34), $\deg B_a = \deg Q^{gl(m)}_{a|\emptyset} = \lambda_a + m - a$. Furthermore, remark the following property of the Wronskian determinant: If $W(P_1, \ldots, P_k) \propto 1$ for some polynomials $P_1, \ldots, P_k$ then, modulo permutations, $\deg P_r = r - 1$ for $r = 1, \ldots, k$. And so, by examining (5.10) for $(a,s)$ being on the boundary of Young diagram, where $Q_{a,s} = 1$, we conclude that all $B_a$ for $\alpha = 1, \ldots, n + m$ should have distinct degrees from 0 till $m + n - 1$. The degrees $d_i \equiv \deg B_i$’s satisfy $d_1 \leq d_2 \leq \ldots$. Their assignment rule is explained in Figure 4.

We can actually set $B_i = u^{d_i}$ because any subleading orders in polynomials $B_i$ do not affect physically relevant Q-functions (i.e. those that survive restrictions that make short representations long, see page 20, in particular $Q_{a,s}$). Also, using (2.35), we can restrict
Figure 4: Rectangular lattice of size $m \times n$ represents the $gl_{m|n}$ Q-system used in the proof of Lemma 5.3. We associate the collection of all $Q_{A|J}$ with $|A| = m - a, |J| = n - s$ to the node $(a,s)$ of the lattice. When $(a,s)$ is on the Young diagram $\Lambda^+$, $Q_{a,s}$ is the smallest degree polynomial among $Q_{A|J}$ situated at the down-right corner or, using bosonisation, by $B_i = Q_{\emptyset|i}$ and $B_a = Q_{a|\emptyset}$ situated in the down-left corner. Degrees of polynomials $B_a, B_i$ are given by the Manhattan distance from $(m,1)$ to the appropriate points on the boundary of the Young diagram. For instance, $6_i$, the fifth number with subscript $i$, means that $\deg B_{\hat{5}} = 6$. Correspondingly $8_a$ means $\deg B_3 = 8$.

Finally, we recursively fix coefficients $c_{a,s}$ following the serpentine path in Figure 4. The point $(a, s)$ on the path is used to fix $c_{a,s}$. Any other $c_{a',s'}$ present in the equation (5.10) are fixed from the previous recursion steps. $c_{a,s}$ appears only linearly in the r.h.s. of (5.10) and is multiplied by the non-zero prefactor $W(u^{d+1}, B_{a+2}, \ldots, B_m, u^{d_1}, \ldots, u^{d_s}) = g Q_{a+1,s+1}$, where $g$ is a non-zero constant, and so solution is unique. To confirm that it exists, recall that equation (5.8) has a solution by assumptions of the lemma. But an arbitrary solution to this equation forms a one-parametric space $Q_{a,s} = Q_{a,s}^{(0)} + \tilde{c}_{a,s} Q_{a+1,s+1}$ parameterised by $\tilde{c}_{a,s}$. We can set $Q_{a,s}^{(0)} = W(B_{a+1} - c_{a,s} u^{d+1}, B_{a+2}, \ldots, B_3)$ and then $\tilde{c}_{a,s} = g c_{a,s}$.

We remark that $B_a$ can be fixed uniquely up to symmetries from (5.10) by considering only $a = 0, \ldots, n - 1$ and $s = 0$, but the above proof asserts that the solution is polynomial if only if the Q-system on Young diagram has a polynomial solution. What is typically redundant is the number of equations (5.8) needed to ensure polynomiality of the Q-system. A conjecture about what is the minimal set of equations needed was given in [73]. For $gl(2)$ spin chains, it was proven [33] that taking four equations with $a = 0, 1, s = 0, 1$ exactly suffices if $\theta_l = 0$.

5.4 Relation to Nested Bethe equations and quantum eigenvalues

This topic was discussed numerously in the literature, and the lore started probably after [37]. It is worth emphasising [78] where the connection between Q-systems and Bethe equations via the choice of different Kac-Dynkin paths was elucidated. In this section

If we assign grading to $c_{a,s}$ as is done in Section 3.4 then $\deg c_{a,s}$ is equal to $h_{a+1,s+1}$ – the hook length, cf. Figure 2.
we summarise the known results and complement them with discussion that focuses on completeness questions.

For a $\mathfrak{g}(m|n)$ spin chain, choose any permutation of the sequence $12\ldots m1\ldots n$, it shall be called a choice of the nesting path. Define by $\leftarrow k$ the sequence of the first $k$ letters from the nesting path. For instance, if we chose $21\hat{2}1\hat{1}$ for the $\mathfrak{g}(2|2)$ case, then $\leftarrow 1 = 2$, $\leftarrow 2 = \hat{1}$, $\leftarrow 3 = 2\hat{1}\hat{2}$, $\leftarrow 4 = 2\hat{1}\hat{2}$.

Nested Bethe equations are the equations on zeros of $Q_{\leftarrow k}$, $k = 1, \ldots, m+n-1$, where we mean e.g. $Q_{\leftarrow 3} = Q_{2\hat{1}\hat{2}}$. Note also that, independently of the path choice, $Q_{\leftarrow (m+n)} = Q_\theta$ – this is the fixed $Q$-function which plays the role of the source term.

The equations are derived as follows. Let $a, b$ denote some indices from the set $\{1, \ldots, m\}$, and $i, j$-some indices from the set $\{\hat{1}, \ldots, \hat{n}\}$. Then (2.17) imply

$$Q_{\leftarrow (k+1)} Q_{\leftarrow (k-1)} \propto W(Q_{\leftarrow k}, Q_{\leftarrow (k-1)b}), \quad \text{for } \leftarrow (k+1) = \leftarrow (k-1)ab, \quad (5.12a)$$

$$Q_{\leftarrow (k+1)} Q_{\leftarrow (k-1)} \propto W(Q_{\leftarrow (k+1)}, Q_{\leftarrow (k-1)}), \quad \text{for } \leftarrow (k+1) = \leftarrow (k-1)aj, \quad (5.12b)$$

$$Q_{\leftarrow (k+1)} Q_{\leftarrow (k-1)} \propto W(Q_{\leftarrow k}, Q_{\leftarrow (k-1)}ij), \quad \text{for } \leftarrow (k+1) = \leftarrow (k-1)ij. \quad (5.12c)$$

Take (5.12a) (or (5.12c)), make a shift $u \to u + \frac{\hbar}{2}$ and evaluate it at a zero of $Q_{\leftarrow k}$. Do the same for $u \to u - \frac{\hbar}{2}$ and divide the two evaluated equations by one another. One gets “bosonic” nested Bethe equations

$$\frac{Q_{\leftarrow (k+1)}^+ Q_{\leftarrow (k-1)}^+}{Q_{\leftarrow (k+1)}^- Q_{\leftarrow (k-1)}^-} = \frac{Q_{\leftarrow k}^+}{Q_{\leftarrow k}^-} \quad \text{at zeros of } Q_{\leftarrow k}. \quad (5.13a)$$

By evaluating (5.12b) at zeros of $Q_{\leftarrow k}$, one gets $W(Q_{\leftarrow k}, Q_{\leftarrow (k-1)j}) = 0$ which can be written in the “fermionic” nested Bethe equations form

$$\frac{Q_{\leftarrow (k+1)}^+ Q_{\leftarrow (k-1)}^-}{Q_{\leftarrow (k+1)}^- Q_{\leftarrow (k-1)}^+} = 1 \quad \text{at zeros of } Q_{\leftarrow k}. \quad (5.13b)$$

For $\leftarrow k = \leftarrow (k-1)\alpha_k$, we remind that $Q$-functions are actually (twisted) polynomials and denote them as $Q_{\leftarrow k} \propto \prod_{\beta \in \leftarrow k} z_{\beta}^{(-1)^{\alpha_k}u/\hbar} M_k \prod_{l=1}^{M_k} (u - u_l^{(k)})$, equations (1.13) read

$$\frac{z_{\alpha_{k+1}}}{z_{\alpha_k}} \prod_{(k', l') \neq (k, l)} \frac{u_l^{(k)} - u_{l'}^{(k')}}{u_l^{(k)} - u_{l'}^{(k')}} + \frac{h}{2} c_{k, k'} = 1 \quad (5.13)$$

where the Cartan matrix $c_{k, k'}$ has $c_{k, k} = -(-1)^{\alpha_k} - (-1)^{\alpha_{k+1}}$, $c_{k, k+1} = c_{k+1, k} = (-1)^{\alpha_{k+1}}$ and all other coefficients equal to zero. In the particular case $\alpha_k = k$, if we rename the label $(k)$ as $(\alpha)$ where $\alpha := m + n - k$ and we remember that $Q_{\leftarrow (m+n)} = Q_\theta$, then the equation (5.13) becomes exactly $^2$ (1.1).

$^2$Note that due to the relabelling $\alpha := m + n - k$, the Cartan matrices $c_{\alpha, \beta}$ and $c_{k, k'}$ in equations (5.13) and (1.1) differ by a re-ordering of their rows and columns.
In particular we see that the choice of the Cartan matrix, or equivalently of the Kac-Dynkin diagram, see e.g. [79], is implied by the choice of the nesting path [78]. Namely, the Kac-Dynkin diagram should be a chain of \( m + n - 1 \) nodes where the \( k \)'th node is fermionic (crossed) if the \( k \)'th and the \( (k+1) \)'th letters have different grading and is bosonic (blanc) otherwise.

For twist-less systems, two comments are due. First, a twist-less Q-system is invariant under symmetry transformations (2.35) and this ambiguity can propagate to Bethe equations if the nesting path is generic. To avoid this happening, we restrict ourselves only to those paths for which \( a \) is to the left from \( b \) if \( a > b \), and \( i \) is to the left of \( j \) if \( i > j \). Such a choice ensures that if \( Q_{A|J} = Q_{+k} \) for some \( k \) then \( Q_{A|J} \) is the polynomial of the smallest degree among all \( Q_{A'|J'} \) with \( |A'| = |A|, |J'| = |J| \). Hence the distinguished subclass of the nesting paths is naturally realised by paths across Young diagrams, with \( Q_{a,s} = Q_{+k} \), \( a = m - |A|, s = n - |J| \), and so we can reformulate (5.13) using Q-functions from the Young diagram Q-system. Second, for short representations, a part of the nesting path lies outside of the Young diagram. We should define \( Q_{a,s} = Q_{+k} = 1 \) if \( (a,s) \) is on the path but outside of the Young diagram to get correct interpretation in terms of the nested Bethe equations.

**Example:**

The Q-system on the depicted Young diagram leads, by the choice of the Kac-Dynkin path, to Bethe equations of \( \mathfrak{gl}_4 \) spin chain of length \( L = 18 \). The momentum carrying Bethe roots are zeros of \( Q_{1,0} \). The Bethe roots on the nested levels are those of \( Q_{2,0} \) and \( Q_{3,0} \).

Bethe equations are written as

\[
\prod_{i=1}^{L} \frac{u_i^{(k)} - \theta_i + \frac{c_{12} + c_{11}}{2} \delta_{k,1}}{u_i^{(k)} - \theta_i - \frac{c_{12} + c_{11}}{2} \delta_{k,1}} = (-1)^{c_{kk'}} \prod_{i=1}^{3} \prod_{j'=1}^{M_{i'}} \frac{u_j^{(k')_i} - u_j^{(k')_i}}{u_j^{(k')_{i'}} - u_j^{(k')_{i'}}} - \frac{1}{2} c_{kk'}, \quad k = 1, 2, 3, \quad i = 1, \ldots, M_i(5.15)
\]

\[
c_{kk'} = \begin{pmatrix}
-2 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & -2 \\
\end{pmatrix},
\]

\[
Q_{1,0} = \prod_{i=1}^{M_1} (u - u_i^{(1)}), \quad M_1 = 10,
\]

\[
Q_{2,0} = \prod_{i=1}^{M_2} (u - u_i^{(2)}), \quad M_2 = 4,
\]

\[
Q_{3,0} = \prod_{i=1}^{M_3} (u - u_i^{(3)}), \quad M_3 = 2.
\]

It is possible to perform the so called duality transformation [6, 39, 80–82] – to pass from

\[
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & 0 \\
\end{pmatrix},
\]

\[
Q_{0,1} = \prod_{i=1}^{M_1} (u - u_i^{(1)}), \quad M_1 = 14,
\]

\[
Q_{1,1} = \prod_{i=1}^{M_2} (u - u_i^{(2)}), \quad M_2 = 7,
\]

\[
Q_{1,2} = \prod_{i=1}^{M_3} (u - u_i^{(3)}), \quad M_3 = 4.
\]
the nested Bethe equations for one nesting path to the equations for another path, typically when change of the path is an elementary permutation. It can be done by lifting (5.13) to the QQ-relations and then by descending to another path.

Concerning completeness, we start by commenting on relation between (5.13) and (5.12). The Bethe equations (5.13) involve ratios which can become of type 0/0 for certain class of solutions known as exceptional solutions, see e.g. a in [21]. We should provide some regularisation prescription to treat them properly. Furthermore, if $Q_{k}$ contains a double zero, i.e. coinciding Bethe roots, we are losing information when passing from (5.12) to (5.13). Indeed, we should consider also a derivative of (5.12) at a double zero which provides an extra constraint in addition to the nested Bethe equations. Let us note that double zeros indeed can exist as we can always collide roots by fine-tuning values of inhomogeneities or twist [22]. Moreover, the fine-tuned points coincide sometimes with the physically-relevant case of $\theta_{\ell} = 0$ [21, 23]²⁴.

Based on the above comments, (5.12) look as more appropriate equations than (5.13). The Wronskian relations (5.12) along a nesting path are closely related to description in terms of the quantum eigenvalues [6, 31, 37, 83]. For clarity, we introduce them in the example of $\mathfrak{gl}_{2|2}$ spin chain in a concrete grading choice - $2\tilde{2}11$. Consider a generating functional

$$
\frac{1}{1 - D\Lambda_1 D} (1 - D\Lambda_2 D) \frac{1}{1 - D\Lambda_2 D} \tag{5.16}
$$

that can be viewed as a way to factorise $\text{Ber} [\mathbb{I} - DT(u)G D]$ which we used in (2.13), and so $\Lambda_\alpha$ are called quantum eigenvalues (of the monodromy matrix). $\Lambda_\alpha$ commute between themselves and are expressed in terms of $Q$-functions as

$$
\Lambda_2 = \frac{Q_{2|2}^{[2]} \Lambda_2}{Q_{2|2}^{[2]}}, \quad \Lambda_2 = \frac{Q_{2|2}^{[2]} \Lambda_2}{Q_{2|2}^{[2]}}, \quad \Lambda_1 = \frac{Q_{2|2}^{[3]} \Lambda_1}{Q_{2|2}^{[3]}}, \quad \Lambda_1 = \frac{Q_{2|2}^{[3]} \Lambda_1}{Q_{2|2}^{[3]}}. \tag{5.17}
$$

The general rule is

$$
\Lambda_{\alpha} = \frac{Q_{\alpha \leftarrow k}^{[\pm 2-|A|+|J|]} Q_{\alpha \leftarrow (k-1)}}{Q_{\alpha \leftarrow k}^{[-|A|+|J|]} Q_{\alpha \leftarrow (k-1)}}, \tag{5.18}
$$

where $\alpha$ is on the $k$’th position of the nesting path, and $\leftarrow k = A|J$, $\leftarrow (k-1) = A'|J'$, the upper sign corresponds to $\alpha = 0$ and the lower sign - to $\alpha = 1$. While $\Lambda_\alpha$ depend on the choice of the nesting path, the generating functional (5.16) does not which follows from the QQ-relations (2.17b)²⁵.

Given that (5.16) generates transfer matrices, $\Lambda_2 - \Lambda_3$ should not have poles at zeros of $Q_2$, $\Lambda_1 + \Lambda_2$ should not have poles at zeros of $Q_{2|2}^+$ etc, these conditions is another way

²⁴ the observed cases are however for $\mathfrak{gl}_{k}$ chains in a higher spin representation
²⁵ In [10], the same concepts and statements are expressed more formally. There, population is the same as Q-system reviewed on page 14 and onwards, and reproduction procedure is the same as the above-mentioned duality transformations.
to generate Bethe equations.

Both QQ-relations along the path and no-poles condition for combinations of quantum eigenvalues (which are also path-dependent) are less constraining than the Wronskian condition (2.26). What is happening, polynomiality should be insured for all choices of paths, that is for all Q-functions. This requirement is achieved by (2.26) or equivalent to it formulations. We remark that even if a solution of (5.12) looks normal (i.e. it has no coinciding Bethe roots or roots separated by $\frac{1}{2}\hbar$) it might be still not physical because of the problems with polynomiality happening when we try to change the path.\footnote{This was observed by one of the authors and C. Marboe [25] while computing the AdS/CFT spectrum for [74, 75] and. Curiously, attempts to mitigate this issue lead to the formulation of the Q-system on Young diagram [73].}

6 Labelling solutions

In this section we solve Wronskian Bethe equations explicitly in a special regime, when all $\theta_{\ell}$ are far away from one another. Then one can continuously deform $\theta_{\ell}$ to any desired values thus obtaining a way to label solutions. A practical application of this labelling approach is demonstrated in Section 7.2. The labelling approach provides an alternative physics-style proof to the statement that the number of solutions is the right one.

6.1 Twisted case, labelling with multinomial expansion

Consider the regime when $|\theta_{\ell} - \theta_{\ell'}| \sim \Lambda$, and $\Lambda$ is large. This limit was used many times in the literature including for counting purposes, see e.g. [9, 84].

By rescaling

$$u \to u/\Lambda, \quad \theta_{\ell} \to \theta_{\ell}/\Lambda, \quad \hbar \to \hbar/\Lambda,$$

(6.1)

which is a symmetry of the Wronskian equations, we can consider the $\hbar \to 0$ limit with all $\theta_{\ell}$ being finite and distinct instead of $\Lambda \to \infty$. When $\hbar \to 0$ we can neglect shifts of the spectral parameter in the polynomial piece of the Baxter function $Q = z^{-u/\hbar} q(u)$, $q(u + \hbar) \simeq q(u) + \mathcal{O}(\hbar)$, this will be confirmed below. So all QQ-relations simply become of structure $QQ = QQ$ and equation (2.26) simplifies to

$$\prod_{a=1}^{n} q_{a}[\varnothing](u) \prod_{i=1}^{m} q_{\varnothing}[\varnothing](u) = \prod_{\ell=1}^{L} (u - \theta_{\ell}).$$

(6.2)

The number of solutions of the last equation which is an equation on $c_{a}[\varnothing], c_{\varnothing}[\varnothing]$ is easily counted to be

$$\frac{L!}{\prod_{a=1}^{n} \lambda_{a}! \prod_{i=1}^{m} \nu_{i}!} = \dim V_{\Lambda}.$$

(6.3)

The value $\hbar = 0$ is quite special, therefore let us ensure that the conclusion about number of solutions holds also for $\hbar \neq 0$.\footnotemark
Lemma 6.1. For distinct $\theta_{\ell}$, the number of solutions of (2.26) in some neighbourhood of $h = 0$ is given by (6.3) and the solutions are in one-to-one correspondence, by analytic continuation in $h$, with solutions at $h = 0$.

Proof. Let us treat $h$ as a parameter and consider $SW$ as a smooth map from $C \times C$ to $X$. For our discussion, it will be safe to use $\theta_{\ell}$ instead of $\chi_{\ell}$ to (locally) parameterise $X$. Take a solution $\bar{c}$ of (6.2) corresponding to a point $\bar{\theta}$ with pairwise distinct coordinates. It is easy to check that the differential of $SW$ with respect to the first $L$ coordinates (the $c_{\ell}$ variables) at the point $\bar{c}, h = 0$ is an invertible $L \times L$ matrix. Indeed it just reduces to the differential of the smooth map from $C^{L}$ to $C^{L}$ defined by (6.2) which is clearly non-degenerate if $\bar{\theta}$ are distinct. Therefore we can apply the analytic implicit function theorem to conclude that all the solutions at $h = 0$ can be analytically extended to solutions in a neighbourhood of $h = 0$.

It remains to show that all solutions for some $h \neq 0$ can be obtained by extending from $h = 0$. This is equivalent to establishing a version of properness, namely that all sequences of solutions $\bar{c}^{(n)}$ corresponding to $\bar{\theta}$ and $h^{(n)} \to 0$ as $n \to \infty$ remain bounded. Assuming the contrary and rewriting (2.26) in terms of the roots $u_{i}$ of the Q-functions, we can extract a subsequence of solutions $\bar{u}^{(n)}$ such that every root either converges to a finite limit or diverges to infinity. But then one of the equations of (2.26) will contain a monomial with all the diverging roots and only those. This can be seen by the fact that at $h = 0$, (2.26) reduces to (6.2). This monomial will therefore grow faster than any other possible monomial as $n \to \infty$. Since $h^{(n)} \to 0$ and $\bar{\theta}$ is finite we arrive at a contradiction.

6.2 Twist-less case, labelling with standard Young tableaux

Using the $h \to 0$ limit is not sufficient in the absence of twist as there is no longer fast-oscillating term $z_{a}^{u/h}$ in Q-functions and so the QQ relations won’t simplify to the structure $QQ = QQ$ \footnote{They will reduce to those of the Gaudin model, see Section 7.3}. Hence we use a stronger regime when

$$\theta_{L} \gg \theta_{L-1} \gg \ldots \gg \theta_{1} \gg 1.$$ \hspace{1cm} (6.4)

The technical analysis of this limit is given in Appendix D, and here we describe its combinatorial outcome. Similarly to the twisted case, each root of each Q-function $Q_{a,s}$ “sticks to” a specific inhomogeneity, in the sense that it is proportional to this inhomogeneity, the statement holds at the leading order of (6.4). However, in contrast to the twisted case, the coefficients of proportionality are not equal to one and an arbitrary distribution of the roots between the inhomogeneities is not allowed. Namely, when $\theta_{L}$ is large compared to the other inhomogeneities, the Q-functions must exhibit behaviour \footnote{\$\sim\$ designates an equality at the leading order of the corresponding expansion, for this case – the large-$\theta_{L}$ expansion. Equality is verified by comparing coefficients of polynomials in $u$.}$

$$Q_{a,s}(u) \sim (u - N_{a,s}^{(L)} \theta_{L}) \tilde{Q}_{a,s}(u)$$ \hspace{1cm} (6.5)
for certain \((a, s)\) and \(\tilde{Q}_{a, s}(u) \sim \tilde{Q}_{a, s}(u)\) for the other \((a, s)\), such that \(\tilde{Q}_{a, s}\) form a Q-system on a Young diagram which is obtained from \(\Lambda^+\) by removing one box. \(N_{a, s}^{(L)}\) are numerical coefficients fixed below. It is clear that (6.5) will hold precisely for those \((a, s)\) for which \(\deg Q_{a, s} = \deg \tilde{Q}_{a, s} + 1\), and these are the points satisfying \(a \leq \bar{a}, s \leq \bar{s}\) if the box \((\bar{a}, \bar{s})\) is being removed.

After removing one box, we end up with a Q-system for a spin chain of length \(L - 1\). Now we repeat the argument with \(\theta_{L-1} \to \infty\) and so on and recursively fully disentangle the Young diagram. We associate a number \(T_{a, s}\) to each box of the Young diagram which is equal to the length of the spin chain at which the box \((a, s)\) decouples. These numbers range from 1 to \(L\) and increase across each row and each column, i.e. they form a standard Young tableau (SYT) \(T\) of shape \(\Lambda^+\). An equivalent statement on the level of NBAE was made in [43].

For a given SYT \(T\), the solution \(Q_{a, s}\) at the leading order of (6.4) is then given by

\[
Q_{a, s} \sim Q_{a, s}^{\text{lead}} = \prod_{\bar{a} > a, \bar{s} > s} \left( u - \frac{N_{a, s}^{(T, i)}}{T_{a, s}} \theta_{T_{a, s}} \right),
\]

see Figure 5.

Let us now fix coefficients \(N_{a, s}^{(L)}\). To this end recall that \(Q_{a, s}\) are defined as monic polynomials and restore the normalisation in (5.8)

\[
Q_{a+1, s}Q_{a, s+1} = \frac{Q_{a, s}^+Q_{a+1, s+1}^+ - Q_{a, s}^+Q_{a+1, s+1}^-}{h(\deg Q_{a, s} - \deg Q_{a+1, s+1})}.
\]

The key identity we will need is the relation of the polynomial degrees of \(Q_{a, s}\) to the hook length \(h_{a, s}\) of the Young diagram box \((a, s)\)

\[
\deg Q_{a, s} - \deg Q_{a+1, s+1} = h_{a+1, s+1}.
\]

Note that \(h_{a, s}\) will change when we remove certain boxes from the Young diagram. Denote therefore by \(h_{a, s}^{(\ell)}\) the hook length in the diagram with \(\ell\) boxes which appears in the recursive procedure. Also, denote by \(Q_{a, s}^{(\ell)}\) the Q-functions on this diagram.

Let us remove the box \((\bar{a}, \bar{s})\) in the recursive procedure. This means that there are
currently ℓ = T_a,s boxes in the diagram, and one has Q_{a,s}^{(ℓ)} \sim (u - N_{a,s}^{(ℓ)} θ_ℓ)Q_{a,s}^{(ℓ-1)} for all pairs (a, s) with a ≤ ă, s ≤ ă and Q_{a,s}^{(ℓ)} \sim Q_{a,s}^{(ℓ-1)} otherwise. Consider the regime u ≪ θ_ℓ for which (u + κ ℏ N_{a,s} θ_ℓ) ≃ -N_{a,s} θ_ℓ for any finite κ and so (6.7) simplify providing consistency relations between N_{a,s}^{(ℓ)}:

\[ N_{a+1,s}^{(ℓ)}N_{a,s+1}^{(ℓ)} = N_{a,s}^{(ℓ)}N_{a+1,s+1}^{(ℓ)}, \quad a < ă, s < ă, \]
\[ N_{a,s+1}^{(ℓ)} = N_{a,s}^{(ℓ)}\frac{h_{a+1,s+1}^{(ℓ)}}{h_{a+1,s+1}^{(ℓ)}}, \quad a = ă, s < ă, \]
\[ N_{a+1,s}^{(ℓ)} = N_{a,s}^{(ℓ)}\frac{h_{a+1,s+1}^{(ℓ)}}{h_{a+1,s+1}^{(ℓ)}}, \quad a < ă, s = ă. \]  

(6.9)

There is no constraint at point a = ă, s = ă, but instead we have to set N_{0,0}^{(ℓ)} = 1 since Q_{0,0}^{(L)} = Q_{θ}, and so Q_{0,0}^{(ℓ)} = (u - θ_ℓ)Q_{0,0}^{(ℓ-1)}. This normalisation allows to find all N_{a,s}^{(ℓ)} with no ambiguities:

\[ N_{a,s}^{(ℓ)} = \prod_{a'=1}^{a} h_{a',s}^{(ℓ)} - 1 \prod_{s'=1}^{s} h_{a,s'}^{(ℓ)} - 1, \quad a ≤ ă, s ≤ ă. \]  

(6.10)

We emphasise that this solution depends on the choice of T through the condition T_a,s = ℓ. In fact, there are no two distinct Q_{a,s}^{lead} that have Bethe roots proportional to the same inhomogeneities.

We hence confirmed that Q_{a,s}^{lead} is explicitly and bijectively fixed by standard Young tableaux which, we remind, is the dimension over C of V_A^+ on which the Bethe algebra is restricted.

Since inhomogeneities are not bounded in the regime (6.4), one still needs to perform work to show that solutions in this regime are bijectively linked to solutions at finite values of inhomogeneities. This is done in Appendix D.3. In summary, we have the following result

**Lemma 6.2.** For Λ_ℓ := \frac{θ_ℓ}{θ_ℓ}, ℓ = 1, ..., L - 1, being large enough but finite, solutions of the Q-system on a Young diagram Λ^+ and hence of the Wronskian Bethe equations (2.26) at point θ are bijectively labelled with standard Young tableaux, where a solution associated with the tableau T approaches

\[ Q_{a,s} \sim \prod_{ă > a, s > ă} \left( u - \bar{θ}_{T_{a,s}} \prod_{a'=1}^{a} h_{a',s}^{(T_{a,s})} - 1 \prod_{s'=1}^{s} h_{a,s'}^{(T_{a,s})} - 1 \right) \]  

when Λ_ℓ approaches infinity.

\[ \square \]

7 Summary and applications

7.1 Completeness, faithfulness, and maximality of the Bethe algebra

In this paper we proved completeness of Wronskian Bethe equations (WBE) and faithfulness of the map from the Wronskian to the Bethe algebra, for the case of both twisted and
twist-less supersymmetric spin chains.

Completeness on the level of equations is the statement that the algebraic number of solutions of the WBE is the ‘right one’, i.e. it is equal to the dimension of the weight space \( U_\Lambda \) (as a vector space over \( \mathbb{C} \)). We proved the statement for arbitrary numerical values of \( \chi_\ell \) – elementary symmetric polynomials in inhomogeneities \( \theta_\ell \). The paper actually contains two independent proofs. The first one is based on character computation presented in Section 3.4 which is valid because the Wronskian algebra \( \mathcal{W}_\Lambda \) is a free \( \mathbb{C}[\chi] \)-module, by Lemma 3.1. The second is based on the explicit solution counting in the limits \( \left| \frac{\theta_{\ell+1}-\theta_\ell}{\hbar} \right| \gg 1 \) (twisted case) and \( \frac{\theta_{\ell+1}}{\theta_\ell} \gg 1 \) (twist-less case). The fact that this counting is valid for finite (but probably large) values of inhomogeneities is summarised in Lemma 6.1 and Lemma 6.2; the fact that the algebraic number of solutions remains the same for any values of inhomogeneities is a consequence of freeness but also we show this using more elementary arguments in Lemma A.2.

Faithfulness and hence bijectivity of the map established in Theorem 4.2 allows one to transfer algebraic properties of the Wronskian algebra \( \mathcal{W}_\Lambda \) to the Bethe algebra \( B_\Lambda \). The Bethe algebra over \( \mathbb{C}[\chi] \) and restricted to the weight subspace \( U_\Lambda \) is a polynomial ring defined by Wronskian Bethe equations. Furthermore, for the twist-less case, \( B_\Lambda \) depends on the Young diagram alone and does not depend on the rank of \( \mathfrak{gl}_m \). Its formulation in terms of a Q-system on a Young diagram directly follows from the results of [41, 73] although this fact was not explained there and we filled in the gap in Section 5. Using the bosonisation trick, we also found a novel very explicit way (5.10) to parameterise functions \( Q_{a,s} \) using Wronskian determinants which is the main technical tool for analysing the \( \frac{\theta_{\ell+1}}{\theta_\ell} \gg 1 \) regime.

The faithfulness property holds also for specialisation of \( \chi_\ell \) to any numerical value \( \bar{\chi}_\ell = \chi_\ell(\bar{\theta}_1, \ldots, \bar{\theta}_L) \), whereas inhomogeneities should probably satisfy the constraint \( \bar{\theta}_\ell + \hbar \neq \bar{\theta}_{\ell'} \) for \( \ell < \ell' \). For almost any values of \( \bar{\chi}_\ell \), this follows already from Theorem 4.3 which uses very general properties of the WBE. However, to get really arbitrary values of \( \bar{\chi}_\ell \), a more refined analysis is performed in Appendix B which relies on properties of the Yangian and its representations. This analysis builds on ideas of [9] generalising them to supersymmetric case, with notable exception of Lemma B.7 that establishes cyclicity of the Bethe algebra representation on the level of \( \mathbb{C}[\chi] \)-modules.

Completeness and faithfulness combined insure that \( B_\Lambda \) is a maximal commutative subalgebra of \( \text{End}(U_\Lambda) \) which should be viewed as the completeness property on the Bethe algebra level. Each distinct solution of WBE bijectively corresponds to a joint eigenstate of commuting charges. The word “distinct” means that even in the case when solutions degenerate, the eigenspace corresponding to the coinciding solutions is still one-dimensional. This does not contradict maximality of the Bethe algebra as the latter becomes non-diagonalisable in the degeneration case with the size of the corresponding trigonal block equal to the degree of degeneration.

\[29\] If solutions of WBE are non-degenerate this constraint is not needed. For \( \chi \in \mathcal{X}_{\text{crit}} \), it might be needed but we did not analyse precisely when, so we keep it as a sufficient requirement. Analysing when it is necessary would probably require exploration of a Yangian representation theory beyond techniques developed in the paper.
The results of this paper are likely to be generalisable for spin chains in arbitrary highest-weight representation of $\mathfrak{gl}_m$. Given our preliminary studies, an analog of the quantisation condition (2.26) will not be sufficient but generalisation of Q-system on Young diagram techniques for the twist-less system should work. Q-systems on Young diagrams can be also defined for non-compact spin chains [74] and they suggest an explicit isomorphism map between restricted Bethe algebras $\mathcal{B}_\Lambda$ for non-compact spin chains and compact spin chains. The isomorphism class of $\mathcal{B}_\Lambda$ should depend only on the extended Young diagram introduced in [74, 87]. Performing the suggested program should prove completeness of quantum spectral curve for $\mathcal{N}=4$ SYM [88, 89] which is confirmed so far by an extensive analysis of QSC solutions in [74, 75].

### 7.2 Simplicity of the spectrum and controlled numerical solution

Choose normalisation $\hbar = i$ and consider the twist-less case and real values of inhomogeneities $\theta_\ell$. The Bethe algebra is invariant under Hermitian conjugation in this case and hence is diagonalisable. On the other hand, diagonalisation is impossible if there are coinciding solutions which immediately implies that restriction of the Bethe algebra to a weight subspace $V^+_\Lambda$ has simple spectrum, cf. Corollary 4.9.

The considered scenario contains both homogeneous spin chain with $\theta_\ell = 0$ and the spin chain decoupling limit $\theta_{\ell+1}/\theta_\ell \gg 1$ where WBE can be solved explicitly and labelled with standard Young tableaux. We can continuously connect homogeneous spin chain and the spin chain in the decoupling limit while keeping $\theta_\ell$ real and in this way unambiguously label solutions of homogeneous Bethe equations by SYT, this makes precise and proves a conjecture made in [43].

We have realised this idea numerically. Parameterise inhomogeneities as $\bar{\theta}_\ell = \Lambda^{\ell-1} - 1$, $\ell = 1, \ldots, L$. For a chosen tableau $\mathcal{T}$, start with the solution (6.11) in the decoupling regime $\Lambda \gg 1$ and then incrementally decrease $\Lambda$ until it reaches the point $\Lambda = 1$. While changing $\Lambda$ we require that (6.7) are always satisfied. The numerical realisation turned out to be very stable for any choice of $\mathcal{T}$ that we tried. For $L \lesssim 20$, we are able to produce, for a given $\mathcal{T}$, a numerical solution with an 80-digit precision in less than three minutes, the speed is obviously much faster for shorter chains. Further substantial optimisation of the code should be possible.

The details of implementation and the code will be published in a separate work. Here we give one illustration. For the example of Q-system on Young diagram on page 43, choose

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$^{30}$A unified approach for twisted and non-twisted Bethe subalgebras of $Y(\mathfrak{gl}_n)$ has been put forward in [85]. Using these ideas and representation-theoretic arguments, powerful completeness-type results have been proven for the $\mathfrak{gl}_2$ case in [86].
a standard Young tableau

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 8 & 9 & 10 & 14 & 18 \\
4 & 6 & 7 & 13 & 16 & 17 \\
5 & 11 \\
12 & 15
\end{array}
\]

Then we get the following numerical solution

Roots of \(Q_{1,0}\) (blue), \(Q_{2,0}\) (green), \(Q_{3,0}\) (red) are shown

Roots of \(Q_{0,1}\) (blue), \(Q_{1,1}\) (green), \(Q_{1,2}\) (red) are shown

To our knowledge, the proposed approach is the first example when we have a systematic and proven to be unambiguous way to control all solutions of the Bethe equations for systems of size where direct brute-forcing (e.g. by numerical diagonalisation of Hamiltonian matrix) is unlikely to be practical. For instance, there are 2148120 different standard Young tableaux of the same shape (and hence distinct solutions of the Bethe equations with the same magnon numbers) as in the above example. Overall, the length \(L = 18\) \(\mathfrak{gl}_4\) chain has Hilbert space of dimension \(\sim 6.8 \times 10^{10}\) comprising 81662152 irreps of \(\mathfrak{gl}_4\), there is one solution of WBE per irrep.

To compare with other approaches, all solutions of \(\mathfrak{gl}_2\) chain for \(L = 14\) were reported in [90]. Yet, at this length the Hilbert space is of dimension 16384 and Hamiltonian being a sparse matrix [91] can be diagonalised numerically. In [92] solutions with large magnon numbers were studied quite systematically but only particular classes of solutions were controlled. For spin chains of rather large length, the low-energy excitations around the antiferromagnetic vacuum are also of numerical interest, see e.g. [93]. It would be interesting to explore whether we can apply the proposed techniques in this regime and improve the existing tools which rely on the string hypothesis.

7.3 Gaudin model

The \(\mathfrak{gl}_n\) Gaudin model [29, 94] can be obtained as the \(\hbar \to 0\) limit of the non-twisted \(\mathfrak{gl}_n\) spin chain. Formally, this just amounts to replacing the discrete Wronskian by an actual Wronskian. On the representation theory side the spin chain will no longer be a representation of the Yangian \(Y(\mathfrak{gl}_n)\) but of the current algebra \(\mathfrak{gl}_n[u]\) (e.g. in the terminology of
Completeness of the Bethe Ansatz for the $\mathfrak{gl}_n$ Gaudin model has been proven in [95] under the assumption that all inhomogeneities are pairwise distinct. The philosophy of the proof and the end result is similar to [9]. As far as we know, for the supersymmetric $\mathfrak{gl}_{m|n}$ Gaudin model, completeness is proven for generic values of $\theta_\ell$ [96] in the twist-less case, and for any pair-wise distinct $\theta_\ell$ for the twisted case [97].

In our construction, the Bethe equations and $Q$-operators admit a well-defined $\hbar \to 0$ limit and an analogue of Theorem 4.2 can be proven along the same lines. Similarly, all the results relying solely on the analytic properties of the map $SW$ will also be true in the Gaudin case. In particular, the algebraic number of solutions will not depend on $\tilde{\chi}$ and will be equal to $d_\Lambda$. Specialisation of the isomorphism will also hold generically using the same arguments as in Theorem 4.3.

To prove further constraints on specialisation as in Theorem 4.6, one way would be to investigate the representation theory of $\mathfrak{gl}_{m|n}[u]$. But instead of doing that, we can re-formulate statements for the Bethe algebra $B_\Lambda$ of the $\mathfrak{gl}_{m|n}$ system as statements for the non-supersymmetric $\mathfrak{gl}_{\Lambda^+}$ system. This is based on the results of Section 5 for isomorphisms of Bethe algebras and the discussion on page 67 for the existence of a cyclic vector. They apply in the $\hbar \to 0$ limit as well. Using [95], one then confirms that the specialisation of the isomorphism also holds in the case of the supersymmetric Gaudin model for pairwise distinct inhomogeneities which is a naive $\hbar \to 0$ limit of Theorem 4.6 and related statements.

### 7.4 Separation of variables

To construct a basis that factorises wavefunctions of eigenstates of the Bethe algebra, Maillet and Niccoli proposed [98] to repeatedly act with transfer matrices on a reference state. One can reach factorisation also by choosing other Bethe algebra members that depend on the spectral parameter $u$. This idea was fruitfully used recently alongside with other related tools in application to rational spin chains [36, 59, 84, 99–102]. Currently, an SoV basis was constructed for $\mathfrak{gl}_m$ spin chains in arbitrary finite-dimensional representation [101] as

$$\langle x \rangle = \langle 0 | \prod_{\ell=1}^L \prod_{k=1}^{m-1} \det_{1 \leq i,j \leq k} Q_\ell(x_{kj}^\ell)$$

(7.1)

and for $\mathfrak{gl}_{m|n}$ spin chains in the defining representation [59] as

$$\langle x \rangle = \langle 0 | \prod_{\ell=1}^L \prod_{(1)}(\theta_\ell)^{d_\ell}$$

(7.2)

Here $x_{kj}^\ell = \theta_\ell + h m_{kj}^\ell$, where $m_{kj}^\ell$ are integers forming Gelfand-Tsetling patterns and defining what is $\langle x \rangle$, and $d_\ell$ are integers from $0 \leq d_\ell \leq m + n - 1$ also defining what is $\langle x \rangle$. There are exactly as many choices for $m_{kj}^\ell$ and $d_\ell$ as the dimension of the corresponding Hilbert space.
An important technical challenge of this approach is to prove that the construction as above indeed produces a basis of the Hilbert space. It was resolved in the mentioned works, however only spin chains with generic twist were considered and certain restrictions on admissible values of inhomogeneities apply.

We can now give an alternative insight on resolving this challenge. By Theorem 4.5, the representation of the Bethe algebra is isomorphic to the regular representation of the Wronskian algebra. In particular there is always a cyclic vector. One can choose the cyclic vector as a reference state \( \langle 0 | \varphi (b) \rangle \) form a basis as long as \( b \) form a basis in the Wronskian algebra. The Wronskian algebra is a polynomial algebra with Plücker-type relations. Then the basis question reduces to the questions very similar to those of projective geometry. This naturally links to the last topic we would like to review.

### 7.5 Geometric representation theory and Bethe/Gauge correspondence

So far we have concentrated on the isomorphism between the restricted Bethe algebra \( B_\Lambda \) and the polynomial ring \( \mathcal{W}_\Lambda = \mathbb{C}[\chi][c]/\mathcal{I}_\Lambda \). It turns out that for \( \mathfrak{gl}_n \) spin chains the Bethe algebra can be realised in a third, purely geometric way \([103–106]\). Consider the manifold \( \mathcal{F}_\Lambda \) of all the flags associated to the partition \( \Lambda^+ = (\Lambda_1, \Lambda_2, \ldots) \) that is chains of vector spaces

\[
\{0\} = V_0 \subsetneq V_1 \subsetneq \ldots \subsetneq V_{m-1} \subsetneq V_m = \mathbb{C}^L
\]

such that \( \dim V_a/V_{a-1} = \Lambda_i \) for all \( 1 \leq a \leq m \). \( \mathcal{F}_\Lambda \) admits a natural action of \( \text{GL}(L) \).

Now consider its cotangent bundle \( T^*\mathcal{F}_\Lambda \) with an action of \( \text{GL}(L) \times \mathbb{C}^* \), where \( \mathbb{C}^* \) acts on the cotangent spaces by multiplication. It is known that the equivariant cohomology ring \( H_{\text{GL}(L) \times \mathbb{C}^*}(T^*\mathcal{F}_\Lambda, \mathbb{C}) \) (with coefficients in \( \mathbb{C} \)) of \( T^*\mathcal{F}_\Lambda \) with this action of \( \text{GL}(L) \times \mathbb{C}^* \) is given by

\[
H_{\text{GL}(L) \times \mathbb{C}^*}^*(T^*\mathcal{F}_\Lambda, \mathbb{C}) = \mathbb{C}[c, \chi, h]/(\prod_{a=1}^m q_a(u) - Q_\theta(u)).
\]

In this identification the \( L + 1 \) parameters \( (\chi_i)_{1 \leq i \leq L} \) and \( h \) correspond to the generators of the maximal torus of \( \text{GL}(L) \times \mathbb{C}^* \). Therefore the ring \( H_{\text{GL}(L) \times \mathbb{C}^*}^*(T^*\mathcal{F}_\Lambda, \mathbb{C}) \) is isomorphic (as a \( \mathbb{C}[\chi, h] \)-module) to \( B_\Lambda \) in the singular twist limit \( x_1 \ll x_2 \ll \ldots \ll x_m \). In general, when \( x_a \) are arbitrary pairwise distinct complex numbers, the twisted Bethe algebra \( B_\Lambda \) can be identified with a quantum deformation of \( H_{\text{GL}(L) \times \mathbb{C}^*}^*(T^*\mathcal{F}_\Lambda, \mathbb{C}) \), the so-called equivariant quantum cohomology ring \( QH_{\text{GL}(L) \times \mathbb{C}^*}^*(T^*\mathcal{F}_\Lambda, \mathbb{C}) \).

This connection is actually a particular case of a more general construction \([103]\) which first appeared in the context of Bethe/Gauge correspondence \([104, 105]\). Starting from any so-called Nakajima quiver variety one can build a Yangian action \(^{33}\) on its equivariant quantum cohomology ring considered as a Hilbert space. Moreover, the action of the

\(^{31}\)It is a general fact that the equivariant cohomology ring of a manifold with an action of a Lie group \( G \) with roots \( (\alpha_i)_{1 \leq i \leq r} \) and Weyl group \( W_G \) is a module over \( \mathbb{C}[\alpha_1, \ldots, \alpha_r]^{W_G} \) (\( W_G \)-invariant polynomials in \( (\alpha_i)_{1 \leq i \leq r} \)).

\(^{32}\)Here \( h \) is considered as a parameter and not a fixed complex number.

\(^{33}\)The Yangian in question is not necessarily \( \text{Y}(\mathfrak{gl}_n) \). The general identification of these geometrically-realised Yangians with known integrable systems and solutions of the Yang-Baxter equation is an open question.
Yangian generators can be expressed as some geometric operations on the classes of the variety. In particular the Baxter Q-operator can be constructed in a purely geometric way [107].

It is still unclear how to properly extend this construction to supersymmetric Yangians [10, 108]. We hope that results of our paper, in particular the isomorphism between the bosonic and supersymmetric case elucidated in Section 5, will be useful for the advancement of the subject.

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A Wronskian algebra - a pedestrian approach

A.1 Some facts from commutative algebra and algebraic geometry

Basic definitions

Study of the polynomial equations can be done in analytic (geometric) or in algebraic way. In the analytic approach, a set of \( m \) polynomial equations in \( n \) variables \( P_i(x_1, \ldots, x_n) = 0 \), \( i = 1, \ldots, m \) is assigned an algebraic variety \( A \) - a set of points \( x \equiv (x_1, \ldots, x_n) \in \mathbb{C}^n \) where equations hold. The algebraic approach attaches to the equations an ideal generated by \( P_\ell \), \( \mathcal{I} = \langle P_\ell \rangle \) which is the set of all possible polynomials in \( n \) variables \( Q \in \mathbb{C}[x_1, \ldots, x_L] \) that can be written in the form \( Q = \sum_\ell q_\ell P_\ell \) for some polynomials \( q_\ell \).

Relation between the two approaches is established by the Hilbert Nullstellensatz: if \( Q \) vanishes on \( A \) then \( Q^r \in \mathcal{I} \) for some integer \( r \). Ideal constructed by all polynomials that vanish on \( A \) is called the radical of \( \mathcal{I} \) and is denoted by \( \sqrt{\mathcal{I}} \).

The algebraic description is more abstract and is less used in physics but it allows one to more accurately formulate some of the properties of the Wronskian algebra. In particular, we can work over fields different to \( \mathbb{C} \), e.g. field of fractions \( \mathbb{C}(\theta) \).
The next concept is to consider functions on $\mathcal{A}$ formalised as the quotient ring
\[ R = \mathbb{C}[x_1, \ldots, x_n]/\mathcal{I}. \] (A.1)

If $\mathcal{I} = \sqrt{\mathcal{I}}$ then $R$ is called the coordinate ring of $\mathcal{A}$. Note that in this paper not all ideals equal to their radicals.

A ring is called to be an integral domain if $ab = 0$ implies $a = 0$ or $b = 0$. The corresponding ideal is then called prime ($ab \in \mathcal{I}$ implies $a \in \mathcal{I}$ or $b \in \mathcal{I}$). Certain complications arise when $R$ is not an integral domain and we note that Wronskian algebras are typically not integral domains when considered as polynomial rings in variables $c$ and $\theta$.

**Polynomial division**

Easiness in study of polynomials in one variable exists mainly due to the unambiguous polynomial division procedure. Recall how it works: let $P$ be a polynomial in $x$ of degree $b$ which is one (of those polynomials) that generates the ideal $\mathcal{I}$ in $\mathbb{C}[x]$. Let $Q$ be any polynomial in $x$. If $Q$ contains a monomial $cx^a$ with $a \geq b$, we represent $Q$ as a combination $Q = cx^{a-b}P + (Q - cx^{a-b}P)$ in which the first term is divisible by $P$ and the second term has no monomial of degree $a$. One performs the same procedure with $P' = Q - cx^{a-b}P$ and continue it recursively until no monomials divisible by $x^b$ remain. So one obtains a representation $Q = qP + r$, where degree of $r$ is strictly smaller than $b$. Both $q$ and $r$ are fixed uniquely. Furthermore, one can guarantee the Bézout's lemma, that is one can find such $\alpha, \beta \in \mathbb{C}[x]$ that $\alpha P_1 + \beta P_2 = \text{GCD}(P_1, P_2)$ for any polynomials $P_1, P_2$, and hence conclude that any ideal in one variable is principal, i.e. it is generated by a single polynomial – GCD of polynomials $P_1, \ldots, P_m$ that generate the ideal.

A practical application in our case would be: if a Bethe algebra is generated by a single operator $\hat{x}$ then this algebra is guaranteed to be isomorphic to a quotient $\mathbb{C}[x]/\mathcal{I}$ where $\mathcal{I}$ is the ideal generated by the minimal polynomial of $\hat{x}$. As we have $L$ generating operators $\hat{c_\ell}$, things are not that simple.

**Gröbner bases**

Many problems in systems with multiple variables arise from difficulties with the polynomial division. First, to even define a division algorithm one needs to introduce a total order on a set of monomials $x^d \equiv \prod_{i=1}^{n} x_i^{d_i}$ that should be an order in which 1 is the smallest monomial and $a < b$ implies $ac < bc$ for any $a, b, c$. A diversity of the monomial orders is available in contrast to the only one option for the single-variable case. We shall use below only lexicographic orders which form a small subset of all possibilities.

After fixing a monomial order and denoting by $P_i$ the generators of the ideal $\mathcal{I}$, one can perform long polynomial division (exclusion of all monomials that are divisible by leading monomials of $P_i$) to represent any polynomial $Q$ as
\[ Q = \sum_i q_i P_i + r. \] (A.2)
Unfortunately, neither the procedure nor its result are unique if \( P_i \) are arbitrary generators, so the division is essentially meaningless.

However, if \( P_i \) form a special set called Gröbner basis then \( r \) is the unique result of the polynomial division \(^{34}\). Hence \( Q \in \mathcal{I} \) iff \( r = 0 \). \( q_i \) are not unique though, but uniqueness of \( r \) suffices for the study of the quotient ring (A.1).

A set of polynomials \( P_i \) forms a Gröbner basis of an ideal \( \mathcal{I} \) if i) they generate \( \mathcal{I} \), ii) the set is closed under computation of S-polynomials, see e.g. [109] for further explanations. If moreover, for all \( i \neq i' \), \( P_i \) does not contain monomials divisible by the leading monomial of \( P_{i'} \) then such a Gröbner basis is called the reduced one and it is unique for the given choice of monomial order. By a Gröbner basis we mean the reduced basis in the following.

**Monomial basis**

Let us fix a Gröbner basis. The set of monomials that can arise in the remainders of polynomial divisions forms a basis in the quotient ring \( \mathcal{R} \) considered as a vector space. This basis shall be called the monomial basis.

We can use the monomial basis to realise the regular representation of an algebra in terms of explicit matrices, see the example on page 24. Such a basis has an important advantage – all computations in it are performed in the original field, and so the coefficients of \( \tilde{x} \) will belong to the same field.

### A.2 \( \mathbb{C}(\chi) \)-module and invariance of solutions multiplicity

Consider the ring of polynomials in \( 2L \) variables \( \mathbb{C}[[\chi]] = \mathbb{C}[\chi_1, \ldots, \chi_L][c_1, \ldots, c_L] \). We define the Wronskian algebra as \( \mathcal{W}_\Lambda = \mathbb{C}[[\chi]]/[\mathcal{I}_\Lambda] \), where \( \mathcal{I}_\Lambda = \langle \mathcal{W}_\ell(c) - \chi_\ell \rangle \) – the ideal generated by Wronskian relations. As we can simply exclude \( \chi_\ell \) using equations \( \chi_\ell = \mathcal{W}_\ell(c) \), \( \mathcal{W}_\Lambda \) is isomorphic to \( \mathbb{C}[c] \) – the ring of polynomials in \( L \) variables. Hence, in particular, \( \mathcal{W}_\Lambda \) is an integral domain and \( \mathcal{I}_\Lambda \) is a prime ideal.

In case of prime ideals, it is quite easy to promote rings to fields. In our case, we shall consider \( \chi_\ell = \mathcal{W}_\ell(c) \) as an equation on \( c_\ell \) in the field of fractions \( \mathbb{C}(\chi) \) and \( \mathcal{W}_\Lambda \) as a ring over \( \mathbb{C}(\chi) \). “Easiness” of promotion lies in the following statement: any polynomial in variables \( c_\ell \) and \( \chi_\ell \) that belongs to \( \mathcal{W}_\Lambda \) considered as an object in a ring over \( \mathbb{C}(\chi) \) would also belong to \( \mathcal{W}_\Lambda \) considered as an object in a ring over \( \mathbb{C}(\chi) \).

When we work over a field of fractions, we can compute a Gröbner basis. Simply, instead of conventional computation in \( \mathbb{C}[c_1, \ldots, c_L]/(\mathcal{W}_\ell(c) - \chi_\ell) \) with numerical \( \tilde{\chi}_\ell \in \mathbb{C} \), we do an equivalent computation in \( \mathbb{C}(\chi)[c_1, \ldots, c_L]/(\mathcal{W}_\ell(c) - \chi_\ell) \) with symbolic \( \chi_\ell \in \mathbb{C}(\chi) \). When Gröbner basis is computed, we can construct the corresponding monomial basis and conclude what is the dimension of \( \mathcal{W}_\Lambda \) (as a vector space over \( \mathbb{C}(\chi) \)) and hence what is the number of solutions of the Wronskian equations. Note that the solutions themselves would typically only exist in an algebraic closure of \( \mathbb{C}(\chi) \). However, computation of the monomial basis can be performed directly in \( \mathbb{C}(\chi) \) and this is the only thing needed.

\(^{34}\)Note however that, in contrast to the one-dimensional case, monomials comprising \( r \) can be still bigger than the leading monomials of \( P_i \), and yet not divisible by the latter. Hence we might be not able to perform a chain of divisions that leads to the Bézout’s lemma. The lemma generically does not hold in multivariable case.
Working over \( \mathbb{C}(\chi) \) is equivalent to considering \( \chi \) in generic position, when no accidental relations happen. When we specialise to a concrete numerical value \( \tilde{\chi} \) of \( \chi \), we are interested whether number of solutions changes. We can formulate (a bit stronger) question from the point of view of the Gröbner basis: does it remain a Gröbner basis upon specialisation?

**Lemma A.1.** Let the Gröbner basis of the ideal \( \mathcal{I}_\Lambda = \langle SW_\ell - \chi \rangle \) in \( \mathbb{C}(\chi)[c] \) w.r.t. some monomial order \( < \) be given by polynomials

\[
s_m = c^m + \sum_{m' < m} p_{mm'}(\chi) c^{m'}, \quad m \in M,
\]

where \( m = (m_1, \ldots, m_L) \), \( c^m \equiv \prod_{\ell=1}^L c^{m_\ell} \), \( M \) is a set of tuples \( m \), and \( p_{mm'} \in \mathbb{C}(\chi) \).

Let \( p_{mm'} \) be finite numbers when evaluated at \( \chi = \tilde{\chi} \in \mathbb{C} \). Then \( \tilde{s}_m = c^m + \sum_{m' < m} p_{mm'}(\tilde{\chi}) c^{m'}, m \in M \), form the Gröbner basis of the ideal \( \mathcal{I}_\Lambda(\tilde{\chi}) = \langle SW_\ell - \tilde{\chi} \rangle \) in \( \mathbb{C}[c] \) for the same monomial order.

In other words, it is safe to specialise a Gröbner basis at those values of \( \chi \) where denominators of \( p_{mm'} \) do not vanish.

**Proof.** To verify the statement first we check that the declared set of \( \tilde{s}_m \) generates \( \mathcal{I}_\Lambda(\tilde{\chi}) \). To this end, use long division in \( \mathbb{C}(\chi)[c_1, \ldots, c_L] \) to write \( SW_\ell - \chi = \sum m q_m(\chi) s_m \). From the algorithm of long division it is clear that \( q_m(\chi) \) are not singular at \( \chi = \tilde{\chi} \) if \( p_{mm'}(\chi) \) are not singular which is the case by the condition of the theorem. Hence \( SW_\ell - \chi = \sum m q_m(\chi) s_m \) can be evaluated and still holds at \( \chi = \tilde{\chi} \). To check that \( \tilde{s}_m \) form a Gröbner basis we need to e.g. compute S-polynomials but this is combinatorially the same exercise as for \( s_m \) since the leading monomials are not affected by specialisation.

Wronskian equations can be obviously specialised at arbitrary point \( \tilde{\chi} \) and so the ring \( \mathcal{W}_\Lambda(\tilde{\chi}) \) is always a well-defined object.

**Theorem A.2.** \( d_\Lambda \equiv \dim_\mathbb{C} \mathcal{W}_\Lambda(\chi) \) does not depend on \( \tilde{\chi} \).

In other words, number of solutions of Wronskian equations counted with multiplicities is always the same, even on the degeneration set \( X_{\text{crit}} \).

**Proof.** We know that the theorem holds for all points \( \tilde{\chi} \notin \text{SW}(D) \) since all solutions of the Wronskian equations are distinct there and so dimension of the quotient ring coincides with number of solutions that we denote as \( d_\Lambda \). We can path-connect any two regular points, number of solutions cannot change along the path, see Section 3.1.

Take \( L \) linearly independent constant vectors \( w_\ell = (w_{\ell 1}, \ldots, w_{\ell L}) \) and define \( x_\ell = \sum_{\ell'} w_{\ell \ell'} c_{\ell'} \). For almost any choice of \( w_\ell \), the Gröbner basis of \( \mathcal{I}_\Lambda \) in \( \mathbb{C}(\chi) \) w.r.t. the
monomial order $x_1 < x_2 < \ldots x_L$ should have the form

$$x_1^{d_A} + a_1^{(d_A-1)}(\chi)x_1^{d_A-1} + \ldots a_1^{(0)}, \quad (A.4a)$$

$$x_2 - \sum_{k=0}^{d_A-1} b_{2k}(\chi)x_1^{k}, \quad \ldots \quad (A.4b)$$

$$x_L - \sum_{k=0}^{d_A-1} b_{Lk}(\chi)x_1^{k}.$$  

Indeed, take a point $\bar{\chi} \notin \mathcal{X}_{\text{crit}}$ for which the conditions of Lemma A.1 hold. At such a point, leading monomials of the Gröbner basis are the same before and after specialisation, and so we can judge about the Gröbner basis from its specialised version. Since $\bar{\chi} \notin \mathcal{X}_{\text{crit}}$, $\check{x}_\ell$ (regular representation of $x_\ell$, written as a matrix in the monomial basis) should have $d_A$ distinct eigenvalues for almost any choice of $\omega_\ell$, and therefore the minimal polynomial equation it satisfies is of degree $d_A$ which is (A.4a). In the chosen lexicographic order this equation should belong to the Gröbner basis. Other variables $x_2, \ldots, x_L$ should satisfy (A.4b) (i.e. they are uniquely fixed if $x_1$ is fixed) otherwise dimension of $\mathcal{W}_A(\bar{\chi})$ would exceed $d_A$.  

Since all solutions of the Wronskian equations are bounded, $a_1^{(\omega)}(\chi)$ cannot have singularities, hence they are simply polynomials in $\chi_\ell$. Coefficients $b_{lk}$ however are rational functions of $\chi_\ell$ that can contain poles. Outside of these poles, the conditions of Lemma A.1 hold and we can perform specialisation asserting that dimension of the specialised polynomial ring is $d_A$.  

It remains to show that for any $\bar{\chi} \in \mathbb{C}^L$, one can choose $\omega_\ell$ such that $b_{lk}$ are not singular at $\bar{\chi}$. To this end, we can actually explicitly express $b_{lk}$ in terms of solutions of the Wronskian system. Let $x_\ell = x_\ell^{(i)}$ be the $i$'th solution. Then polynomials (A.4b) can be rewritten as

$$x_\ell - \sum_{k=0}^{d_A-1} b_{lk}(\chi)x_1^{k} = \frac{\det \begin{vmatrix} x_\ell^{(1)} & x_1^{(1)} & x_1^2 & \ldots \\ x_\ell^{(2)} & x_1^{(2)} & (x_1^{(1)})^2 & \ldots \\ x_\ell^{(3)} & x_1^{(3)} & (x_1^{(2)})^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & x_1^{(1)} & (x_1^{(1)})^2 & \ldots \\ 1 & (x_1^{(2)})^2 & \ldots & \vdots \\ 1 & \ldots & \ldots & \ldots \end{vmatrix}}{\det \begin{vmatrix} x_1^{(1)} & x_1^{(1)} & x_1^{2} & \ldots \\ x_1^{(2)} & x_1^{(2)} & (x_1^{(1)})^2 & \ldots \\ x_1^{(3)} & x_1^{(3)} & (x_1^{(2)})^2 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & x_1^{(1)} & (x_1^{(1)})^2 & \ldots \\ 1 & (x_1^{(2)})^2 & \ldots & \vdots \\ 1 & \ldots & \ldots & \ldots \end{vmatrix}}. \quad (A.5)$$

Indeed, equality of the above polynomials to zero implies $x_\ell = x_\ell^{(i)}$ precisely when $x_1 = x_1^{(i)}$.  

While $x_\ell^{(i)}$ belong to an algebraic closure of $\mathbb{C}(\chi)$, the above ratio of determinants is symmetric under permutations $x_\ell \rightarrow x_\ell^{(i)}$ and hence should be a polynomial in $x_1$ with coefficients in the base field, i.e. $\mathbb{C}(\chi)$. This follows for instance from $\sum_{i=1}^{d_A} f(x_1^{(i)}) = \text{Tr} f(\check{x})$ and basic combinatorial arguments. Of course, one can conclude the same from the fact
that (A.4b) are obtained in the process of computation of the Gröbner basis.

At points $\bar{\chi}$ where all $x_1^{(i)}$ are distinct, denominator of (A.5) is non-zero and hence $b_{\ell k}$ are non-singular. As discussed, for a given regular $\bar{\chi}$, we can adjust $\omega_\ell$ that $x_1$ has non-degenerate solutions.

When $\bar{\chi} \in SW(D)$ all $x_\ell$ degenerate. Then consider a one-parametric smooth path $\chi(t)$ in the space of parameters such that $\chi(t = 0) = \bar{\chi}$ is the degeneration point of interest and $\chi(t \neq 0) \notin \mathcal{X}_{\text{crit}}$. Moreover, one chooses such a path that all $x_\ell^{(i)}$ are distinct along the path for sufficiently small $t$, except for the point $t = 0$ itself.

The value of ratio of determinants in (A.5) is not well-defined at $t = 0$ but it can be computed as the limit $t \to 0$. Since this ratio is a rational function of $\chi_\ell$, the limit, if finite, should produce polynomials (A.4b) specialised at $t = 0$.

To compute the limit, note that all $x_\ell$, for generic enough choice of $\omega_\ell$, satisfy one-variable equations $x_\ell^{d_\Lambda} + a_\ell^{(d_\Lambda)} x_\ell^{d_\Lambda - 1} + \ldots = 0$, where $a_\ell^{(k)}$ are polynomials in $\chi$ and hence are well-defined even at $t = 0$. Define $x_\ell(t)$ as solutions of these equations that coincide with solutions of Wronskian equations for $t \neq 0$; their $t = 0$ value is then defined as continuation $t \to 0$. If $\mu_{i\ell}$ is degree of degeneration of solution $x_\ell^{(i)}$ at $t = 0$ (i.e. $\mu_{i\ell}$ solutions of the one-variable equation on $x_\ell$ coincide at this point) then $x_\ell^{(i)}(t)$ is expanded in the Puiseux series

$$x_\ell^{(i)}(t) = x_\ell^{(i)}(0) + r_{\ell i, 1} t^{1/\mu} + r_{\ell i, 2} t^{2/\mu} + \ldots,$$

(A.6)

where $\mu = \text{LCM}(\mu_{11}, \ldots, \mu_{LL})$.

One should know finitely many terms in the series (A.6) to compute the determinants ratio in (A.5) in the limit $t \to 0$. We can require that for these finitely many terms, for each $k$ and $i$, if at least one $r_{\ell i, k}$ is non-zero then all $r_{\ell i, k}$, $\ell = 1, \ldots, L$, are non-zero. It is sufficient to guarantee that the ratio is finite, while imposing of such a requirement excludes measure zero subspace from acceptable values of $\omega_\ell$. Recall that we already excluded the space of $\omega_\ell$ where degree of a minimal polynomial for $\bar{\chi}$ is less than $d_\Lambda$ which is of measure zero as well. Majority of $\omega_\ell$ are outside of the stated restrictions, and we can choose any valid option to guarantee regularity of $b_{\ell k}$ at $\chi = \chi^{(0)}$ and hence possibility to specialise the Gröbner basis (A.4) at this point thus concluding that $\dim_C \mathcal{W}_\Lambda(\bar{\chi}) = d_\Lambda$.

The proposed proof provides a concrete analogy between Wronskian equations and a polynomial equation in single variable. Indeed, for any point $\bar{\chi} \in \mathcal{X}$, regular or not, we can choose a variable $x_1$ that satisfies (A.4a) and such that there is a neighbourhood $\mathcal{O}_{\bar{\chi}}$ where $b_{\ell k}$ are non-singular which allows one to compute all elements of $\mathcal{W}_\Lambda$ using (A.4b). So a single-variable equation (A.4a) contains all information about $\mathcal{W}_\Lambda$ in the selected neighbourhood.

### A.3 Freeness of $\mathcal{W}_\Lambda$ and trivialisation of a vector bundle

For each $\mathcal{O}_{\bar{\chi}}$, we have a basis generated by powers of $x_1$. Two different bases constructed at $\bar{\chi}$ and $\chi'$ are related by a transition matrix which is regular together with its inverse on the intersection of $\mathcal{O}_{\bar{\chi}}$ and $\mathcal{O}_{\bar{\chi}'}$. Hence we get a structure of a holomorphic bundle with...
fibers being $d_Λ$-dimensional vector spaces over the field $\mathbb{C}$ and with base $X$. The existence of this holomorphic bundle is the same as saying that $\mathcal{W}_Λ$ is a projective $\mathbb{C}[x]$-module. This is the so-called Serre-Swan correspondence [110].

Because the base $X ≃ \mathbb{C}^L$ is contractible, this bundle must be topologically trivial, that is, we can find $d_Λ$ global holomorphic sections forming a basis of the fiber at each point. A much more complicated question, already asked by Serre [110], is whether we can choose these global sections to be polynomials of $\mathcal{W}_Λ$. A positive answer was given by the Quillen-Suslin theorem [72]. This theorem requires that $\mathcal{W}_Λ$ is a finitely generated $\mathbb{C}[x]$-module, i.e. that there exist finitely many elements $\hat{b}_1, \ldots, \hat{b}_{d_Λ}$ such that any element of $\mathcal{W}_Λ$ is their linear combination with coefficients from $\mathbb{C}[x]$. This is easy to see to be the case. Take for instance the finite set of $\hat{d} = L \times d_Λ$ monomials $x^n := x_1^{n_1}, x_2^{n_2}, \ldots, x_L^{n_L}$ with $0 ≤ n_i < d_Λ$, where $x_i$ are the ones from the proof of Theorem A.2. Due to properness, $x_i$ satisfy a degree-$d_Λ$ equations with polynomial coefficients, cf. (A.4a), and hence any higher powers of $x_i$ are expressible as linear combinations of the first $d_Λ$ powers.

Abstractly, the Quillen-Suslin theorem establishes that there are no non-trivial algebraic vector bundles over $\mathbb{C}^L$ or equivalently by the Serre-Swan correspondence, that any finitely generated projective $\mathbb{C}[x]$-module is free, with a basis given by the aforementioned global sections. Applied to $\mathcal{W}_Λ$, this is precisely the statement that it is a free module over $\mathbb{C}[x]$, see (3.8).

A.4 Non-symmetric functions

Most of the time we work with only symmetric combinations $x_\ell$ of inhomogeneities. However, Baxter operators as explicit matrices acting on the spin chain have coefficients from $\mathbb{C}[θ] \equiv \mathbb{C}[θ_1, \ldots, θ_L]$. This prompts us to understand some properties of $\mathbb{C}[θ]$-modules as compared to $\mathbb{C}[x]$-modules. Also, one can consider equations

$$x_\ell(θ_1, \ldots, θ_L) = x_\ell$$ (A.7)

as a toy model for (2.27) with $c_\ell = θ_\ell$ and $SW_\ell(c) = x_\ell(θ)$.

First, we demonstrate how to use the Gröbner basis techniques to conclude that the polynomial ring $\mathbb{C}[θ]$ is a free $\mathbb{C}[x]$-module and count number of solutions to (A.7). To this end denote the elementary symmetric polynomial of degree $\ell$ in $k$ variables $θ_1, \ldots, θ_k$ as $χ_\ell^{(k)}$. Being roots of $\prod_{\ell=1}^k (u - θ_\ell)$, inhomogeneities satisfy the characteristic equations

$$s_k ≡ θ_k^k + \sum_{\ell=1}^k (-1)^n χ_\ell^{(k)} θ_k^{k-\ell} = 0, \quad k = 1, \ldots, L.$$ (A.8)

Now note that the polynomials $χ_\ell^{(k)}$ can be rewritten as polynomials in $χ_\ell(θ) ≡ χ_\ell^{(L)}$, with

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35We are grateful to L. Cassia for pointing out this relation to us
Now let’s make a small formalisation: Treat $\theta_\ell$ and $\chi_\ell$ as independent variables and consider an ideal $I = \langle s_1, \ldots, s_L \rangle$ as an ideal in $\mathbb{C}[\chi][\theta]$. In the quotient ring $\mathbb{C}[\chi][\theta]/I$, excluding $\chi_\ell$ in favour of $\theta_\ell$ is easy. However, we are interested in the opposite – to solve for $\theta_\ell$ in terms of $\chi_\ell$. As this requires an algebraic closure, we won’t do this explicitly but compute a Gröbner basis instead.

**Lemma A.3.** The above-introduced polynomials $s_1, \ldots, s_L$ form the Gröbner basis of the ideal $I = \langle s_1, \ldots, s_L \rangle$ w.r.t. a lexicographic order for which $\theta_1 > \theta_2 > \ldots > \theta_L > \chi_\ell$, for any $\ell$.

**Proof.** First, by definition, $s_k$ generate the ideal $I$. Then, $s_k$ has $\theta_k^k$ as its leading monomial. Indeed, the other monomials are products of $\theta_k^{k'}$ with $k' < k$, powers of $\theta_k$ with $k' > k$, and $\chi_\ell$ which hence are lexicographically smaller than $\theta_k^k$. Finally, as the leading monomials enjoy the property $\text{GCD}(\theta_k^k, \theta_{k'}^{k'}) = 1$, the S-polynomials between $s_k$ and $s_{k'}$ do not produce new relations and so this set of ideal generators is indeed a Gröbner basis.

Conceptually the Gröbner basis tells us how to algorithmically find $\theta_\alpha$ from the values of their symmetric combinations $\chi_\ell$. First one needs to solve (A.9c) for fix $\theta_L$ ($L$ solutions), then one needs to substitute the found value of $\theta_L$ to the equation $s_{L-1} = 0$ and solve it for $\theta_{L-1}$ ($L - 1$ solutions) etc.

Very similarly to the analysis of the Wronskian algebra, we note that the ring $\mathbb{C}[\chi][\theta]/I$ is isomorphic (over $\mathbb{C}$) to $\mathbb{C}[\theta]$, but it is also naturally endowed with the structure of a $\mathbb{C}[\chi]$-module. The computation of the Gröbner basis above immediately implies that this module is free and of rank $L!$. Indeed, the corresponding monomial basis are given by monomials $\theta_2^{n_2}\theta_3^{n_3}\ldots\theta_L^{n_L}$, with $n_\ell < \ell$. Any relations between these monomials is impossible precisely because $s_k$ form a Gröbner basis and leading monomials of $s_k$ do not belong to the monomial basis. Of course we know that $L!$ is an expected number, if to count with multiplicities: equation $\theta^L - \chi_1\theta^{L-1} + \ldots + (-1)^L\chi_L = 0$ has $L$ solutions, and any permutation of solutions is allowed as well.

Finally, let us extend the Theorem 4.2 to the case of non-symmetric polynomials in $\theta_\ell$. Consider first the following example

\[ s_1 = \theta_1 + (-\chi_1 + \sum_{i=2}^{L} \theta_i), \quad (A.9a) \]
\[ s_2 = \theta_2^2 + \theta_2(-\chi_1 + \sum_{i=3}^{L} \theta_i) + (\chi_1 - \sum_{i=3}^{L} \theta_i) \sum_{i=3}^{L} \theta_i - \sum_{3 \leq i < j \leq L} \theta_i \theta_j, \quad (A.9b) \]
\[ \ldots \]
\[ s_L = \theta_L^L + \sum_{\ell=1}^{L-1} (-1)^n \chi_\ell \theta_L^{L-n} + (-1)^L \chi_L, \quad (A.9c) \]
Example:

Let the generators of a Wronskian algebra $\mathcal{W}$ satisfy equations $c_1 + c_2 = \chi_1$, $c_1 c_2 = \chi_2$, and (a hypothetical) Bethe algebra $\mathcal{B}$ is generated by $2 \times 2$ diagonal matrices $\hat{c}_1 = \theta_1 \times I_2$, $\hat{c}_2 = \theta_2 \times I_2$. These Wronskian and Bethe algebras are isomorphic as $\mathbb{C}[\chi]$-modules. Let us now consider the extension of the Wronskian algebra $\mathcal{W}^\theta \simeq \mathcal{W} \otimes \mathbb{C}[\chi] \otimes \mathbb{C}[\theta]$, i.e. consider generators satisfying $c_1 + c_2 = \theta_1 + \theta_2$, $c_1 c_2 = \theta_1 \theta_2$ and treat this algebra as a $\mathbb{C}[\theta]$-module. This is a rank-two $\mathbb{C}[\theta]$-module. In contrast, the Bethe algebra considered as a $\mathbb{C}[\theta]$-module is of rank one.

By Lemma 4.1 we actually know that generators of type $\theta_\ell \times I$ cannot appear in polynomial combinations of $c_\ell$, and so the hypothetical Bethe algebra in the above example cannot exist. The argument is based on the braiding property (4.1).

More generally, we can show that all polynomial relations satisfied $\hat{c}_\ell$, even with non-symmetric coefficients, should follow from the Wronskian algebra in the following sense. We can add non-symmetric polynomials by hand to the Wronskian algebra by considering $\mathcal{W}_\Lambda^\theta \simeq \mathcal{W}_\Lambda \otimes \mathbb{C}[\chi] \otimes \mathbb{C}[\theta]$. Likewise, non-symmetric polynomials (times the identity operator) are not elements of the Bethe algebra by Lemma 4.1, and hence appending them as extra generators is also realised as $\mathcal{B}_\Lambda^\theta \simeq \mathcal{B}_\Lambda \otimes \mathbb{C}[\chi] \otimes \mathbb{C}[\theta]$. Isomorphism between $\mathcal{W}_\Lambda^\theta$ and $\mathcal{B}_\Lambda^\theta$ as $\mathbb{C}[\theta]$-algebras is then obvious from the isomorphism between $\mathcal{B}_\Lambda$ and $\mathcal{W}_\Lambda$ as $\mathbb{C}[\chi]$-algebras.

We also note that $\mathcal{W}_\Lambda^\theta$ and $\mathcal{B}_\Lambda^\theta$ are free as $\mathbb{C}[\theta]$-modules and $\mathbb{C}[\chi]$-modules as follows e.g. from Lemma A.3.

B Cyclicity of representations

The goal of this appendix is to build all the formalism necessary for the proof of Theorems 4.5 and 4.6. There are two reasons why proving isomorphism of the specialised map $\varphi_{\bar{\theta}}$ (4.4) is problematic. First, setting $\theta_\ell$ to numerical values, which is done for the Bethe algebra, is more restrictive than setting their symmetric combinations $\chi_\ell$ to numerical values, which is done for the Wronskian algebra. Second, the specialisation procedure is actually native to the representation of an algebra, not to the algebra alone. Namely, we set to numerical values coefficients of a matrix which is more restrictive than setting to numerical values only the factors that multiply this matrix as a whole.

To overcome these difficulties, we want to “rigidify” the algebra isomorphism (4.2) by also proving isomorphism between certain representations of these algebras. As was already mentioned in Section-4.3, the only natural choice of representation for the Wronskian algebra $\mathcal{W}_\Lambda$ is its regular representation. As for the Bethe algebra, it acts on the $a$ priori unrelated physical space $\text{End}(U_\Lambda) \otimes \mathbb{C}[\theta]$. These two representations are not isomorphic. This is why we need to introduce an alternative Yangian representation dubbed symmetrised representation. Using a cyclic vector argument we prove that its weight (and highest-weight) subspaces $U_\Lambda^S$ are indeed isomorphic to $\mathcal{W}_\Lambda$ as representations of $\mathcal{B}_\Lambda \simeq \mathcal{W}_\Lambda$, which resolves the second difficulty. This symmetrised representation has the virtue to manifestly depend only on symmetric combinations of inhomogeneities. We show that, under
some explicit restriction on $\bar{\theta}$, its specialisation at a point $\bar{\chi}$ is isomorphic to the spin chain representation at a point $\bar{\theta}$ which resolves the first difficulty.

The discussed approach was developed in [9] for $\mathfrak{gl}_n$ spin chains. The below-presented generalisation to the supersymmetric case is conceptually very straightforward. The only difference, apart from the way we present the results, is in the proof of Lemma B.7 which is in line with the ideas of Theorem 4.2.

B.1 Symmetrised Yangian representation

Consider the Yangian spin chain representation at point $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_L)$ defined in Section 2.2. We note that the order of inhomogeneities in $\bar{\theta}$ is often superfluous. Indeed, the operator $r_\ell(\bar{\theta}) = (\bar{\theta}_\ell - \bar{\theta}_{\ell+1}) P_{\ell,\ell+1} + \hbar \mathbb{1}$ satisfies (4.1) evaluated at $\theta = \bar{\theta}$. If $\bar{\theta}_\ell \neq \bar{\theta}_{\ell+1} \pm \hbar$, it is invertible and hence an intertwiner between two representations that differ by permutation of $\bar{\theta}$ and $\bar{\theta}_{\ell+1}$. From here we conclude that the isomorphism class of the representation at point $\bar{\theta}$ is decided only by $\bar{\chi}_\ell$ if there is no $\ell, \ell'$ such that $\bar{\theta}_\ell - \bar{\theta}_{\ell'} = \pm \hbar$.

More generally, the following facts hold for supersymmetric representations of Yangians:

**Proposition B.1.** If $(\bar{\theta}_1, \ldots, \bar{\theta}_L)$ satisfy $\bar{\theta}_\ell + \hbar \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$ then the spin chain representation of $Y(\mathfrak{gl}_m|_n)$ at point $\bar{\theta}$ is cyclic with cyclic vector $\mathbf{e}^+ = \mathbf{e}_1 \otimes \ldots \otimes \mathbf{e}_1$, where $\mathbf{e}_1$ is the highest-weight vector of the defining $\mathfrak{gl}_m|_n$ representation.36

This statement follows from Theorem 5.2 of [111]. $\mathbf{e}^+$ is obviously a highest-weight vector of the Yangian representation, i.e. it satisfies the condition $T_{ij} \mathbf{e}^+ = 0$ for $i < j$. Its weight is given by

$$T_{ii} \mathbf{e}^+ = Q_{\bar{\theta}}(u + \hbar \delta_{i,1}) \mathbf{e}^+. \tag{B.1}$$

**Proposition B.2.** The spin chain representation at point $\bar{\theta}$ is irreducible if there is no such $\ell, \ell'$ that $\bar{\theta}_\ell - \bar{\theta}_{\ell'} = \hbar$.

For the $Y(\mathfrak{gl}_m)$ case, this is a standard result appearing in the study of Kirillov-Reshetikhin modules [112]. For $Y(\mathfrak{gl}_1|_1)$ it was proven in [113], theorem 5. For the $Y(\mathfrak{gl}_m|_n)$ case, it apparently follows from [111], Proposition 5.4. But as this was not stated explicitly we give an alternative argument for irreducibility in a style of statistical lattice models

**Proof.** Let $\mathbb{C}^m|_n$ be the Hilbert space of the $\ell$-th node of the spin chain, consider also the auxiliary space $\mathbb{C}^m|_n$ with basis vectors $\mathbf{e}_\alpha^{\text{aux}}$, $\alpha = 1, 2, \ldots, m+n$. The dual basis vectors of $\mathbf{e}_\alpha^{\text{aux}}$ shall be denoted $\mathbf{e}_\alpha^{\text{aux}}$. Define the R-matrix acting on the tensor product of the mentioned spaces as $R(u - \theta_\ell) = (u - \theta_\ell) \mathbb{1} + \hbar P$, where $P$ is the graded permutation. In the notations of (2.8), $R(u - \theta_\ell) := (u - \theta_\ell) \sum_{\alpha, \beta} e_{\alpha \beta}(t_{\alpha \beta}) \mathbf{e}_{\beta}^{\text{aux}} \otimes \mathbf{e}_\alpha^{\text{aux}}$. The key property we use is that the R-matrix becomes, up to normalisation, the graded permutation if $u = \theta_\ell$.

---

36In the choice of ordering when bosonic indices are considered smaller than fermionic indices.
Introduce $V^\text{aux}$ – the tensor product of $L$ auxiliary spaces spanned by $e_i^\text{aux} = e_i^\text{aux} \otimes \ldots \otimes e_i^\text{aux}$ and define $B = \sum_A e_i^\text{aux} T_{i \alpha L} (\bar{\theta}_L) \times \ldots T_{i \alpha 2} (\bar{\theta}_2) \times T_{i \alpha 1} (\bar{\theta}_1)$. Given the above-mentioned key property, $B$ maps $V$ to $e^+ \otimes V^\text{aux}$ which is easiest to see by a graphical representation of how $B$ acts:

Here each vertical direction corresponds to a node $\mathbb{C}^m_n$ of the spin chain and each horizontal direction corresponds to a tensor factor $\mathbb{C}^m_n$ of $V^\text{aux}$. Intersections are the places where the R-matrices should be applied (considered as maps from South-West to North-East spaces). Red crosses are the places where the corresponding R-matrix becomes the permutation.

The map $B$ is also invertible. Indeed, up to a non-zero factor it reduces to an ordered product of $\frac{L(L-1)}{2}$ R-matrices, e.g. these are three R-matrices marked by the encircled numbers in the image above. Each of these R-matrices is invertible since $\bar{\theta}_\ell - \bar{\theta}_v \neq \hbar$.

Because $B$ is invertible, for any $v \in V$ one can find a vector $v^* \in (V^\text{aux})^*$ such that $(v^*, B)v = e^+$. Then irreducibility follows from Proposition B.1. $\square$

For finite-dimensional irreducible $Y(\mathfrak{gl}_m|n)$ representations, the highest-weight vector exists and unique and the representation is fully determined, up to an isomorphism, by the vector’s weight [114]. Note that the weight of $e^+$ depends only on symmetric combinations of inhomogeneities according to (B.1). Then we can consider the induced representation from $e^+$ which is isomorphic to the spin chain one by the above-mentioned uniqueness but, in contrast to the spin chain realisation is manifestly invariant under permutations of inhomogeneities.

We shall now introduce a different permutation-invariant realisation which does not require irreducibility argument and will be formulated for inhomogeneities being abstract variables.

**Yangian centraliser** Consider the vector space $V \simeq (\mathbb{C}^m_n)^{\otimes L} \otimes \mathbb{C}[\theta_1, \ldots, \theta_L]$ on which the $\mathbb{C}[\theta]$.Yangian representation $ev_\theta$ (2.9) is realised. An interesting question is what is the centraliser of the Yangian action on $V$.

Define operators $S_\ell$ acting on $V$ by

\[
S_\ell = P_{\ell, \ell+1} \Pi_{\ell, \ell+1} - \frac{\hbar}{\theta_\ell - \theta_{\ell+1}} (\Pi_{\ell, \ell+1} - 1), 
\]  

(B.3)
Lemma B.3. For \( \ell = 1, \ldots, L - 1 \) \( S_\ell \) commutes with the \( Y(\mathfrak{gl}_{m|n}) \) action, (i.e. \([S_\ell, T_{\alpha \beta}] = 0\)) and they form a representation of the symmetric group \( S_L \) on \( \mathcal{V} \).

Proof. The commutativity follows from \((\theta_\ell - \theta_{\ell+1})S_\ell = \Pi_{\ell,\ell+1}r_\ell - \hbar \mathbb{1} \) and (4.1). Then, by explicit computation one checks \( S_\ell^2 = \mathbb{1} \) and \((S_\ell S_{\ell+1})^3 = \mathbb{1} \) – the defining relations of \( S_L \).

\[ \square \]

**dAHA** In the limit \( \hbar \to 0 \), \( S_L \) becomes an explicit permutation defined on a graded space that commutes with the action of \( \mathfrak{gl}_{m|n} \). Hence \( \hbar \neq 0 \) should be considered as a generalisation of Schur-Weyl duality to the case of the Yangian algebra. This statement was made mathematically precise for the bosonic \( \mathfrak{gl}_m \) case [115–117]: \((S_\ell)_{1 \leq \ell \leq L - 1}\) together with \((\theta_\ell \times \mathbb{1})_{1 \leq \ell \leq L}\) form a representation of \( \mathcal{H}_L \), the degenerate affine Hecke algebra (dAHA) on \( L \) sites. Moreover, the dAHA and the Yangian form a dual pair – they are maximal mutual centralisers of one another when acting on \( \mathcal{V} \). More formally, one can view \( \mathcal{V} \) as the tensor product of \( S_L \)-modules \( \mathcal{H}_L \otimes S_L \) \((\mathbb{C}^m)^\otimes L\) where \( S_L \) acts on \( \mathcal{H}_L \) as a subalgebra and on \((\mathbb{C}^m)^\otimes L\) by permutation of tensor factors. This point of view is conceptually interesting because to generalise Schur-Weyl duality to supersymmetric Yangians, one does not need to change the defining relations of the dAHA \( \mathcal{H}_L \) but simply to replace the usual action of \( S_L \) on \((\mathbb{C}^m)^\otimes L\) by the graded action on \((\mathbb{C}^m)^\otimes L\) as in (B.3). The full mathematical treatment (for the affine Hecke algebra\(^{37}\)) can be found in [118].

We are only going to use the slightly weaker statement of Lemma B.3 that \( S_\ell \) commute with the Yangian action.

**\( \mathbb{C}[\chi] \)-Yangian module** Define \( \mathcal{V}^S \subset \mathcal{V} \) as the subspace of \( S_\ell \) invariant vectors. As \([S_\ell, T_{AB}] = 0\), the Yangian action is well-defined on \( \mathcal{V}^S \). Multiplication by symmetric polynomials is also well-defined on \( \mathcal{V}^S \) and, moreover, \( \mathbb{C}[\chi] \times \mathbb{1} \) belong to \( ev_\theta(Y(\mathfrak{gl}_{m|n})) \) due to (2.27). Hence we shall call this representation the symmetrised Yangian representation.

**B.2 Symmetrised Bethe modules and their characters**

Since \( S_L \) commutes with the global \( \mathfrak{gl}_{m|n} \) action it is consistent to define \( \mathcal{V}^S = \mathcal{V}^S \cap (V_\Lambda \otimes \mathbb{C}[\theta_1, \ldots, \theta_L]) \) – the weight \( \Lambda \) subspace of \( \mathcal{V}^S \) – and \( \mathcal{V}^{S+} = \mathcal{V}^S \cap (V_\Lambda^+ \otimes \mathbb{C}[\theta_1, \ldots, \theta_L]) \subset \mathcal{V}^S \) – the subspace of \( \mathfrak{gl}_{m|n} \) highest-weight vectors corresponding to the Young diagram \( \Lambda^+ \). These spaces are also naturally \( \mathbb{C}[\chi] \)-modules.

**Characters** For an element \( v \) of \( \mathcal{V}^S \), define its degree as the maximal degree of the monomials in \( \theta_\ell \)'s which occurs in \( v \). Define \( \mathcal{F}_k \mathcal{V}^S \) as the space of all vectors of degree less

\(^{37}\)Recall that the degenerate affine Hecke algebra and the Yangian can be obtained as \( q \simeq 1 + \hbar \) expansions of respectively the affine Hecke algebra \( \mathcal{H}_L(q) \) and \( U_q(\mathfrak{gl}_{m|n}) \).
or equal to $k$. Finally, define the character

$$
\text{ch}(V^S) = \sum_{k=0}^{\infty} (\dim F_k/F_{k-1}) t^k.
$$

(B.4)

Since the $\mathfrak{gl}_{m|n}$ action does not change the degree, we can also define in an analogous way $\text{ch}(V^S_A)$ and $\text{ch}(V^{S+}_A)$.

**Proposition B.4.** $V^S_A$ is a free $\mathbb{C}[\chi]$-module of rank $(\lambda_1 \ldots \lambda_m \ldots \nu_n)$ and its character is given by

$$
\text{ch}(V^S_A) = t^{Y_A} \prod_{a=1}^{m} \prod_{k=1}^{\lambda_a} \frac{1}{1-t^{k}} \prod_{i=1}^{n} \prod_{k=1}^{\nu_i} \frac{1}{1-t^{k}},
$$

(B.5)

where $Y_A := \sum_{i=1}^{n} \nu_i (\nu_i - 1)/2$.

**Proof.** $V^S_A$ is the image of $V_{\Lambda} \otimes \mathbb{C}[\theta]$ by the projector $p := \frac{1}{L!} \sum_{\sigma \in S_L} S_{\sigma}$, where the symmetry group acts with $S_{\sigma}$ on $V_{\Lambda} \otimes \mathbb{C}[\theta]$ as generated from $S_{\ell}$ (B.3). Since the construction is polynomial in $h$ and the $h$-term of $S_{\ell}$ lowers degrees of polynomials the proposition is true iff it is true for $h = 0$. Hence we will consider only the $h = 0$ case. Every element $\sigma \in S_L$ is then represented as $S_{\sigma} = P_{\sigma} \Pi_{\sigma}$, where $P_{\sigma}$ is the graded permutation acting on $V_{\Lambda}$ and $\Pi_{\sigma}$ is the ordinary permutation acting on $\mathbb{C}[\theta]$.

Denote by $e_\alpha$ the standard basis vectors of $\mathbb{C}^{m|n}$ defined by $E_{\beta\gamma} e_\alpha = \delta_{\alpha,\beta} e_\alpha$. The standard spin basis of $V_{\Lambda}$ is indexed by tuples $I = (i_1, \ldots, i_L)$ such that $|\{k : i_k = a\}| = \lambda_a$ and $|\{k : i_k = i\}| = \nu_i$ corresponding to vectors $e_I := \otimes_{i=1}^{L} e_{i_I}$.

Note that it is enough to consider the image of $w \otimes \mathbb{C}[\theta]$ by $p$ with

$$
w = \begin{pmatrix}
e_1 \otimes \ldots \otimes e_1 \otimes \ldots \otimes e_m \otimes \ldots \otimes e_{m+1} \otimes \ldots \otimes e_{m+n} \otimes \ldots \otimes e_{m+n}.
\end{pmatrix}
$$

(B.6)

Indeed, for all $I$ we can always find $\sigma \in S_L$ such that $P_{\sigma} \cdot e_I = e_{\sigma(I)} = \pm w$.

Denote by $S_{\Lambda} := \prod_{a=1}^{m} S_{\lambda_a} \prod_{i=1}^{n} S_{\nu_i}$ the stabiliser of $w$ and by $H = S_L/S_{\Lambda}$ the space of orbits with respect to the right group multiplication. Then the projection by $p$ can be represented as

$$
V^S_A \simeq p(w \otimes \mathbb{C}[\theta]) = \frac{1}{L!} \sum_{[\sigma] \in H} S_{[\sigma]} \cdot (w \otimes R_B \cdot C_F \cdot \mathbb{C}[\theta]),
$$

(B.7)

where $R_B := \sum_{\sigma \in \prod_{a=1}^{m} S_{\lambda_a}} \Pi_{\sigma}$ and $C_F := \sum_{\sigma \in \prod_{i=1}^{n} S_{\nu_i}} (-1)^{|\sigma|} \Pi_{\sigma}$.

To decide about linear independence in $V^S_A$, it is enough to consider one term in the sum $\sum_{[\sigma] \in H} S_{[\sigma]}$, e.g. $[\sigma] = [1]$ since different terms would be proportional to $P_{\sigma} w$ which are linearly independent in $V_{\Lambda}$. Also, $R_B C_F$ commutes with symmetric polynomials and hence we conclude that $V^S_A$ and $R_B C_F \cdot \mathbb{C}[\chi]$ are isomorphic as $\mathbb{C}[\chi]$-modules.
It is easy to describe $R_{B}C_{F} \cdot \mathbb{C}[\theta]$; it is spanned by $Q_{\Lambda} \times \text{Sym}_{\lambda_{1}} \times \ldots \times \text{Sym}_{\nu_{m}}$, where $Q_{\Lambda} := \prod_{i=1}^{n} \prod_{1 \leq k < l \leq \nu_{i}} (\theta_{j_{i}, k} - \theta_{j_{i}, l})$ with $j_{i} := \sum_{a=1}^{m} \lambda_{a} + \sum_{s=1}^{i-1} \nu_{s}$, and Sym$_{k}$ is the space of symmetric polynomials in $k$ variables. $Q_{\Lambda} \times \text{Sym}_{\lambda_{1}} \times \ldots \times \text{Sym}_{\nu_{m}}$ is free under action of $\mathbb{C}[\chi]$, see Appendix A.4, and its character is (B.5). Rank is computed from $\frac{\text{ch}(V_{\Lambda}^{+})}{\text{ch}(\mathbb{C}[\chi])}|_{\ell=1}$. □

**Corollary B.5.** $V_{\Lambda}^{S} = \bigoplus_{\Lambda} V_{\Lambda}^{S}$ is a free $\mathbb{C}[\chi]$-module of rank $(m + n)L$.

**Proposition B.6.** $V_{\Lambda}^{S+}$ is a free $\mathbb{C}[\chi]$-module of rank $\prod_{(a, s) \in \Lambda^{+}} \frac{L!}{h_{a, s}} = \dim V_{\Lambda}^{+}$, where $h_{a, s}$ is the hook length at box $(a, s)$. Its character is given by

$$\text{ch}(V_{\Lambda}^{S+}) = t_{\Lambda}^{\lambda_{1}} \prod_{(a, s) \in \Lambda^{+}} \frac{1}{1 - t^{h_{a, s}}}, \quad \text{(B.8)}$$

where $t_{\Lambda}^{\lambda_{1}} = \sum_{s=1}^{\lambda_{1}} \frac{h_{s}(h_{s} - 1)}{2}$ with $h_{s}$ being the height of the $s$-th column of $\Lambda^{+}$.

Note that $V_{\Lambda}^{S+} \supseteq V_{\Lambda}$ which is consistent with $V_{\Lambda}^{S+} \subset V_{\Lambda}$.

**Proof.** As in the previous proof, it is enough to consider $h = 0$.

The space $V_{\Lambda}^{S+}$ can be constructed from $V \otimes \mathbb{C}[\theta]$ as follows. Take the standard Young tableau $\mathcal{T}$ which is obtained by filling the shape $\Lambda^{+}$ first by filling the boxes defining the weight $\lambda_{1}$, then $\lambda_{2}$, . . . , then $\nu_{m}$. For instance, for $|\lambda_{1}| \nu_{1}, \nu_{2} = [4, 2, 2]$, $\mathcal{T} = \begin{bmatrix} 4 & 2 & 1 \\ 3 & 2 & 1 \\ 6 & 5 & 4 \end{bmatrix}$. Take the normalised 38 Young symmetriser $R_{\mathcal{T}}C_{\mathcal{T}}$, where $R_{\mathcal{T}}$ is the symmetrisation over rows and $C_{\mathcal{T}}$ is the antisymmetrisation over columns. Then $V_{\Lambda}^{S+}$ is the image by the projector $p$ of $S_{R_{\mathcal{T}}C_{\mathcal{T}}}w \otimes \mathbb{C}[\theta]$. Singling out a special vector in $S_{R_{\mathcal{T}}C_{\mathcal{T}}}w$ is possible because $p$ sums over all permutations. Using this feature of $p$ again, and by repeating the same construction as (B.7), one gets

$$V_{\Lambda}^{S+} \simeq p(S_{R_{\mathcal{T}}C_{\mathcal{T}}}w \otimes \mathbb{C}[\theta]) = p(w \otimes \Pi_{C_{\mathcal{T}}} \mathcal{C}[\theta]) = \frac{1}{L!} \sum_{[\sigma] \in H} S_{[\sigma]} \cdot (w \otimes R_{B}C_{F} \Pi_{C_{\mathcal{T}}} \mathcal{C}[\theta]), \quad \text{(B.9)}$$

and so $V_{\Lambda}^{S+}$, as a $\mathbb{C}[\chi]$-module, is isomorphic to $R_{B}C_{F} \Pi_{C_{\mathcal{T}}} \mathcal{C}[\theta]$. We can omit $R_{B}C_{F}$ as it has no kernel when acting on $\Pi_{C_{\mathcal{T}}} \mathcal{C}[\theta]$. Indeed, $\Pi_{C_{\mathcal{T}}} \mathcal{R}_{B}C_{F} \Pi_{C_{\mathcal{T}}} \mathcal{R}_{B}C_{F} = \Pi_{C_{\mathcal{T}}} \mathcal{R}_{B}C_{F}$. But the module $\Pi_{C_{\mathcal{T}}} \mathcal{C}[\theta]$ is the standard application of (reversed) Young symmetriser to a polynomial ring which is well understood, see e.g. [95, 119–121]. It is a free $\mathbb{C}[\chi]$-module with character given by (B.8) 39 Again rank is computed from $\frac{\text{ch}(V_{\Lambda}^{S+})}{\text{ch}(\mathbb{C}[\chi])}|_{\ell=1}$. □

**Young diagram dependence** One may wonder how comes that the character (B.8) and, in fact, the $\mathbb{C}[\chi]$-isomorphism class of $V_{\Lambda}^{S+}$ do not depend on $\mathfrak{g}_{m|n}$ but only on the $\Lambda^{+}$?

38 We normalise all symmetrisations/antisymmetrisations such that they are projectors.

39 This character is an important combinatorial object: $\frac{\text{ch}(V_{\Lambda}^{S+})}{\text{ch}(\mathbb{C}[\chi])}$ is the Kostka-Foulkes polynomial $K_{\mu\nu}(t)$ with $\mu = \Lambda^{+}$ and $\nu = \ell_{1}$, see e.g. [119, 120].
Young diagram $\Lambda^+$. To understand this property, let us extend the underlying symmetry algebra from $\mathfrak{gl}_{m|n}$ to $\mathfrak{gl}_{m'|n'}$, where $m' = \max(m, h_{\Lambda^+})$ and $n' = \max(n, h_{\Lambda^+})$. In addition to $a = 1, \ldots, m$, $i = 1, \ldots, n$, introduce also $\hat{z}$ to label the new indices that appeared due to the extension. In the order $a < i < \hat{z}$, the highest-weight vectors do not involve $v_3$ and hence $\mathcal{V}_{\Lambda^+}^S$ for the extended system is isomorphic to the one of the $\mathfrak{gl}_{m|n}$ system. Now we perform a chain of fermionic duality transformations (odd Weyl reflections) to get any other order in the set $a, i, \hat{z}$ (the order between separately bosonic and fermionic indices will be preserved though). The procedure is described for instance in [87]. It changes the highest-weight vectors, i.e. the actual embedding of $\mathcal{V}_{\Lambda^+}^+$ inside $V$ is modified, but this change is performed by acting with elements of the global $\mathfrak{gl}_{m'|n'} \subset \mathfrak{y}(\mathfrak{gl}_{m'|n'})$ that commutes with the dAHA and in particular with the action of $\mathbb{C}[\chi]$. Since the procedure is invertible it establishes a $\mathbb{C}[\chi]$-isomorphism between two spaces $\mathcal{V}_{\Lambda^+}^S$ that differ by the choice of the order defining the highest-weight vector. The order is bijected to a Manhattan-type path (e.g. the one in Example on page 43), and only those indices that belong to the path participate in the highest-weight vectors. This last observation allows us to choose $m', n'$ to be any pair such that $(m', n')$ lies on the boundary of $\Lambda^+$ (black/red dots of Figure 1) or outside of $\Lambda^+$.

The dependence of $\mathcal{V}_{\Lambda^+}^S$, as a $\mathbb{C}[\chi]$-module, on the Young diagram alone parallels results of Section 5 that show that the isomorphism class of the twist-less $\mathcal{B}_\Lambda$, as a $\mathbb{C}[\chi]$-algebra, only depends on the Young diagram. This is of course not a coincidence because the twist-less $\mathcal{B}_\Lambda$ and $\mathcal{V}_{\Lambda^+}^S$ are isomorphic as $\mathbb{C}[\chi]$-modules which follows from the results of the next subsection.

B.3 Cyclicity of symmetrised Bethe modules

We shall use the notation $\mathcal{U}_\Lambda^S$ to cover both $\mathcal{V}_{\Lambda}^S$ and $\mathcal{V}_{\Lambda^+}^S$ in the discussion below.

We know that $\mathcal{W}_\Lambda$ and $\mathcal{B}_\Lambda$ algebras are isomorphic as $\mathbb{C}[\chi]$-algebras. The goal of this subsection is to show that the regular representation $\mathcal{W}_\Lambda$ of the Wronskian algebra and the symmetrised Bethe module $\mathcal{U}_\Lambda^S$ are isomorphic as $\mathbb{C}[\chi]$-modules. To do so we need a map from $\mathcal{W}_\Lambda$ to $\mathcal{U}_\Lambda^S$ commuting with the action of $\mathcal{W}_\Lambda \equiv \mathcal{B}_\Lambda$. A standard approach is to take a vector $\omega \in U_\Lambda \otimes \mathbb{C}[\theta]$ and to consider the morphism of representations

$$\psi_\omega : \mathcal{W}_\Lambda \longrightarrow U_\Lambda \otimes \mathbb{C}[\theta], \quad c_\ell \mapsto \hat{c}_\ell \omega$$

(B.10)

Of course this map has no reason to be an isomorphism. Nevertheless we can prove the following.

Lemma B.7. For any non-zero vector $\omega \in U_\Lambda \otimes \mathbb{C}[\theta]$, $\psi_\omega$ is injective.

Proof. Take $P \in \mathcal{W}_\Lambda$ such that $\hat{P} \omega = 0$. As before, use Bethe equations to write $P$ as a polynomial in $c_\ell$'s only. Around a regular value $\theta$, we know from our previous results that the Bethe algebra can be fully diagonalised and that the spectrum of the operators $\hat{c}_\ell$ are exactly the solutions of the Bethe equations. Denote by $W(\theta)$ the corresponding $\theta$-dependent change of basis that diagonalises $\hat{c}_\ell$. At least one of the components of $W(\theta) \omega$
has to be non-zero at \( \theta \) and hence in some \( L \)-dimensional ball \( O \) around \( \theta \). Then \( \dot{P}\omega = 0 \)
implies that for one of the solutions \( c_\ell(\theta) \), \( P(c_\ell(\theta)) = 0 \) for \( \theta \in O \). Since \( c_\ell \) is a local
diffeomorphism, \( P \) vanishes on a \( L \)-dimensional ball and thus \( P = 0 \). \( \square \)

We hence see that for all \( \omega \), \( \psi_\omega \) is an isomorphism on its image. Let us take a very
precise \( \omega \) – the vector of the smallest degree \(^{40} \) (as a polynomial in \( \theta_\ell \)) that belongs to \( U^S_{\Lambda} \).

**Lemma B.8.** \( \psi_\omega(W_\Lambda) = U^S_{\Lambda} \). Therefore \( \psi_\omega \) is an isomorphism of \( \mathbb{C}[\chi] \)-modules between
\( W_\Lambda \) and \( U^S_{\Lambda} \).

**Proof.** Injectivity is proven by Lemma B.7. The fact that \( \psi_\omega \) preserves the action of \( \mathbb{C}[\chi] \)
is obvious from the definition. Note also that \( \psi_\omega \) just increases the degree by \( \Upsilon_\Lambda \) (by \( \Upsilon_\Lambda^+ \)
in the non-twisted case) and otherwise preserves the natural filtrations on \( W_\Lambda \) and \( U^S_{\Lambda} \).
Therefore surjectivity of \( \psi_\omega \) follows from the comparison of the corresponding characters
computed in Sections 3.4 and B.2. \( \square \)

### B.4 Specialisations

Let us summarise what has been done so far. On one side, we have the Wronskian algebra
\( W_\Lambda \) acting on itself via the regular representation. On the other side, we have the Bethe
algebra \( B_\Lambda \) acting on the space \( U^S_{\Lambda} \). Moreover the two couples \((W_\Lambda, W_\Lambda)\) and \((B_\Lambda, U^S_{\Lambda})\) are
isomorphic \( \psi_\omega \).

Now consider the ideal \( \mathcal{I} := \langle SW_1 - \bar{\chi}_1, \ldots, SW_L - \bar{\chi}_L \rangle \) of \( W_\Lambda \) and its image \( \hat{\mathcal{I}} \) in
\( B_\Lambda \) under \( \phi \). Automatically, the isomorphisms \((\phi, \psi_\omega)\) will induce isomorphisms between
\((W_\Lambda/\mathcal{I}, W_\Lambda/\mathcal{I})\) and \((B_\Lambda/\hat{\mathcal{I}}, U^S_{\Lambda}/\psi_\omega(\mathcal{I}))\). Moreover, as shown in Proposition 4.4 \( \psi_\omega(\mathcal{I}) = \mathcal{J} \cdot U^S_{\Lambda} \), where \( \mathcal{J} := \langle \chi - \bar{\chi} \rangle \subset \mathbb{C}[\chi] \). Denote \( B_\Lambda(\bar{\chi}) := B_\Lambda/\hat{\mathcal{I}} \) and recall that \( W_\Lambda(\bar{\chi}) := W_\Lambda/\mathcal{I} \).

We also have a third pair in this correspondence, namely \( B_\Lambda(\bar{\theta}) \), the Bethe algebra
evaluated at \( \bar{\theta} \), acting on \( U_\Lambda \). Our final goal is to show that \( \varphi_{\bar{\theta}} : W_\Lambda(\bar{\chi}) \rightarrow B_\Lambda(\bar{\theta}) \). This
is equivalent to showing that \( B_\Lambda(\bar{\theta}) \simeq B_\Lambda(\bar{\chi}) \). Let us emphasise that \( \textit{a priori} \) \( B_\Lambda(\bar{\chi}) \) and
\( B_\Lambda(\bar{\theta}) \) are two different objects.

**Example:**

Consider \( W \) and \( B^{\text{bad}} \) from the example on page 30. \( B^{\text{bad}} \) acts on the space
\( V = \mathbb{C}^2 \otimes \mathbb{C}[\theta_1, \theta_2] \). \( V \) is a \( \mathbb{C}[\chi_1, \chi_2] \)-module of rank four, we can take \( \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \theta_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)
\( \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \theta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) as its basis elements. In this basis
\[
\tilde{c}^{\text{bad}}_1 = \begin{pmatrix}
0 & -\chi_2 & 0 & 0 \\
1 & \chi_1 & 0 & 0 \\
0 & 0 & 0 & -\chi_2 \\
0 & 0 & 1 & \chi_1
\end{pmatrix}.
\tag{B.11}
\]

\(^{40}\)It is unique up to normalisation as follows from (B.5) and (B.8).
Take a vector $\omega = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, where $A, B$ are some polynomials in $\chi_1, \chi_2$. Then $\mathcal{U}^S := \psi_\omega(\mathcal{W})$ is a $\mathbb{C}[\chi_1, \chi_2]$-module of rank two spanned by $\xi_1 := \omega$, and $\xi_2 := A\theta_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B\theta_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathcal{B}^{bad}$ acting on $\mathcal{U}^S$ is $\begin{pmatrix} 0 & -\chi_2 \\ 1 & \chi_1 \end{pmatrix}$ in the basis $\xi_1, \xi_2$.

Specialisation $\mathcal{B}^{bad}(\bar{\chi})$ is two-dimensional for any $\bar{\chi}$ and is clearly $\mathbb{C}$-isomorphic to $\mathcal{W}(\bar{\chi})$, in particular $\mathcal{B}^{bad}(\bar{\chi}) = \begin{pmatrix} 0 & -\bar{\chi}_2 \\ 1 & \bar{\chi}_1 \end{pmatrix}$. Note that the statement is completely independent of the choice of $\omega$. It holds even if $\bar{A}(\bar{\chi}_1, \bar{\chi}_2) = \bar{A}(\bar{\chi}_1, \bar{\chi}_2) = 0$ because $\xi_1 \notin \psi_\omega(I) = (\chi - \bar{\chi})\mathcal{U}^S$, $i = 1, 2$.

We chose $\mathcal{B}^{bad}$ in the example above to explicitly demonstrate that $\mathcal{B}_\Lambda(\bar{\theta})$ and $\mathcal{B}_\Lambda(\bar{\chi})$ can be in principle non-isomorphic.

Instead of showing isomorphism between $\mathcal{B}_\Lambda(\bar{\chi})$ and $\mathcal{B}_\Lambda(\bar{\theta})$ directly, let us show isomorphism between $\mathcal{U}^{S, Y^S}/\mathcal{J} \cdot \mathcal{U}^{S, Y^S}$ and $U_\Lambda$. Since these spaces both carry representations of the Bethe algebra, if one can find an isomorphism commuting with these actions, it would automatically imply $\mathcal{B}_\Lambda(\bar{\chi}) \equiv \mathcal{B}_\Lambda(\bar{\theta})$ as is argued in Section 4.3. The advantage of this strategy is that we can leverage Yangian representation theory to prove such an isomorphism.

To relate the symmetrised Yangian representation at point $\bar{\chi}$ and the spin chain Yangian representation at point $\bar{\theta}$, recall that $\mathcal{V}^S$ is a subspace of $\mathcal{V} := (\mathbb{C}^{\mathbb{Z}_n})^{\otimes L} \otimes \mathbb{C}[\theta_1, \ldots, \theta_L]$ and so we can define a map $\text{Ev}_\bar{\theta} : \mathcal{V}^S \to (\mathbb{C}^{\mathbb{Z}_n})^{\otimes L}$ simply by evaluating all vectors at $\bar{\theta}$. Since $\mathcal{J} \cdot \mathcal{V}^S \subset \text{Ker} \text{Ev}_\bar{\theta}$, this induces a well-defined map

$$\text{ev}_\bar{\theta} : \mathcal{V}^S(\bar{\chi}) \to (\mathbb{C}^{\mathbb{Z}_n})^{\otimes L},$$

where $\mathcal{V}^S(\bar{\chi}) := \mathcal{V}^S/\mathcal{J} \cdot \mathcal{V}^S$. Concretely this just means the following: take a class $[v] \in \mathcal{V}^S(\bar{\chi})$, represent it by some $v \in \mathcal{V}$ and evaluate it at $\bar{\theta}$.

Note that $\mathcal{V}^S(\bar{\chi})$ is the space where the symmetrised Yangian representation at point $\bar{\chi}$ is realised and $(\mathbb{C}^{\mathbb{Z}_n})^{\otimes L}$ is the Hilbert space of the spin chain. We can realise on it the spin chain Yangian representation at point $\bar{\theta}$.

We are ready to formulate the main conceptual result of this appendix which is Proposition 3.5 of [9] generalised to the supersymmetric case.

**Theorem B.9.** Let $\bar{\theta} = (\bar{\theta}_1, \ldots, \bar{\theta}_L)$ be a solution of equations $\chi_\ell(\bar{\theta}) = \bar{\chi}_\ell$ such that $\bar{\theta}_\ell + h \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$. Then $\text{ev}_\bar{\theta}$ is an isomorphism of $\mathcal{Y}(\mathfrak{gl}_{m|n})$ representations.

**Proof.** Since $\text{ev}_\bar{\theta}$ commutes with the Yangian action, it defines a homomorphism from the symmetrised representation to the spin chain representation. Assume $m \geq 1$ and consider the vector $e^+ := e_1^{\otimes L} \in \mathcal{V}^S(\bar{\chi})$. Since $\text{ev}_\bar{\theta} : e^+ \mapsto e^+$, and $e^+$ is a cyclic vector of the spin chain module by Theorem B.1, $\text{ev}_\bar{\theta}$ is surjective. As $\mathcal{V}^S(\bar{\chi})$ and $(\mathbb{C}^{\mathbb{Z}_n})^{\otimes L}$ are of the same dimension by Corollary B.5, $\text{ev}_\bar{\theta}$ is an isomorphism.

If $m = 0$, one can check that $\mathcal{V}^S$ contains the vector $\prod_{1 \leq \ell < \ell' \leq L} (\bar{\theta}_\ell - \bar{\theta}_{\ell'} + h)e^+$, whose image under $\psi$ is nonzero as long as $\bar{\theta}_\ell + h \neq \bar{\theta}_{\ell'}$ for $\ell < \ell'$. The rest of the proof is the same. \qed
B.5 An explicit case study

Finally, we provide a concrete comprehensive example to illustrate the above-discussed ideas.

Explicit Q-operators Consider the $L = 3$ $\mathfrak{gl}(2)$ spin chain in the absence of twist. By convention, we use the following basis of $(\mathbb{C}^2)^{\otimes 3}$

$$\{(\downarrow\downarrow\downarrow), (\uparrow\downarrow\downarrow), (\downarrow\uparrow\downarrow), (\downarrow\downarrow\uparrow), (\downarrow\uparrow\uparrow), (\uparrow\down\downarrow), (\uparrow\down\uparrow), (\uparrow\up\down), (\uparrow\up\up)\}.$$ (B.13)

In this basis, the periodic (“twist-less”) limit of the $Q$-operators is:

$$Q_0 = 1, \quad Q_1 = \begin{pmatrix} 1 & M_1 \\ M_1 & 1 \end{pmatrix}, \quad Q_{12} = \prod_{i=1}^{3} (u - \theta_i), \quad (B.14)$$

where $M_1$ is the following $3 \times 3$ block matrix

$$M_1 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}_{Ps} + \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}_{Ph} + \frac{1}{6} \begin{pmatrix} 2\theta_2 + 2\theta_3 & h - 2\theta_3 & -h - 2\theta_2 \\ -h - 2\theta_3 & 2\theta_1 + 2\theta_3 & h - 2\theta_1 \\ h - 2\theta_2 & -h - 2\theta_1 & 2\theta_1 + 2\theta_2 \end{pmatrix}_{c_0} \quad (B.15)$$

Notice the projector to symmetric irrep $\Box$ is $\begin{pmatrix} 1 & Ps \\ Ps & 1 \end{pmatrix}$, whereas the projector to hook irreps $2 \times \Box$ is $\begin{pmatrix} 0 & Ph \\ Ph & 0 \end{pmatrix}$.

Also notice that $c_0Ps = 0$, ie $c_0$ only affects the hook irreps.

In addition to these notations, use $\chi_1 = \theta_1 + \theta_2 + \theta_3, \chi_2 = \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3, \chi_3 = \theta_1\theta_2\theta_3$, and get

$$Q_2 = \frac{-6u^4 + 8u^3\chi_1 + u^2(3h^2 - 12\chi_2) + u(-2h^2\chi_1 + 24\chi_3)}{24h} \begin{pmatrix} 1 & Ps \\ Ps & 1 \end{pmatrix} + \left(-\frac{u^3}{2h} + \frac{h\chi_1}{12} - \frac{\chi_3}{h}\right) \begin{pmatrix} 0 & Ph \\ Ph & 0 \end{pmatrix} + \left(\frac{u^2}{4h} - \frac{h}{12}\right) \begin{pmatrix} c_0 & c_0 \\ c_0 & 0 \end{pmatrix}, \quad (B.16)$$
\[ Q_{0,1} = \left( 3\hbar u^2 - 2\hbar \chi_1 u + \hbar \chi_2 + \frac{\hbar^3}{4} \right) \left( \begin{array}{c} 1 \ \ P_1 \\ \ P_1 \end{array} \right) + 3\hbar u \left( \begin{array}{c} 0 \ \ P_0 \\ \ P_0 \end{array} \right) - \frac{\hbar}{2} \left( \begin{array}{c} 0 \ \ c_0 \\ \ c_0 \end{array} \right). \] (B.17)

**Restriction to the hook irreps**  If we restrict to the subspace $2 \times \mathbb{P}$, we obtain $2 \times 2$ matrices written for instance in the basis \[ \left\{ \frac{\sqrt{3} + 3}{6} \left| \uparrow \downarrow \downarrow \right\rangle - \frac{\sqrt{3}}{3} \left| \downarrow \uparrow \downarrow \right\rangle + \frac{\sqrt{3} - 3}{6} \left| \downarrow \downarrow \uparrow \right\rangle, \frac{\sqrt{3} - 3}{6} \left| \uparrow \downarrow \uparrow \right\rangle - \frac{\sqrt{3} + 3}{6} \left| \downarrow \downarrow \downarrow \right\rangle \right\}. \] (B.18)

In this basis, $c_0$ becomes the $2 \times 2$ matrix
\[ c := c_0 = \begin{pmatrix} 2\chi_1 - \sqrt{3}(\theta_1 - \theta_3) & \chi_1 - 3\theta_2 - \sqrt{3}\hbar \\ \chi_1 - 3\theta_2 + \sqrt{3}\hbar & 2\chi_1 + \sqrt{3}(\theta_1 - \theta_3) \end{pmatrix} \] (B.19)
and equations (B.15), (B.16) and (B.17) become respectively
\[ \begin{align*}
Q_{1,1} & = (u - 2\frac{\chi_1}{3})I + \frac{c}{6} = \left( u - \frac{\chi_1}{3} - \frac{\sqrt{3}}{6}(\theta_1 - \theta_3), \frac{\chi_1}{3} - \frac{\theta_2}{2} - \frac{\sqrt{3}}{6}\hbar \right) \\
Q_{2,1} & = \left( -\frac{u^2}{2\hbar} + \frac{\hbar \chi_1}{12} - \frac{\chi_3}{\hbar} \right) I + \left( \frac{u^2}{4\hbar} - \frac{\hbar}{12} \right) c \\
Q_{0,1} & = 3\hbar u - \frac{\hbar}{2}c = 3h \begin{pmatrix} u - \frac{\chi_1}{6} + \frac{\sqrt{3}}{6}(\theta_1 - \theta_3) & -\frac{\chi_1}{6} + \frac{\theta_2}{2} + \frac{\sqrt{3}}{6}\hbar \\ -\frac{\chi_1}{6} + \frac{\theta_2}{2} - \frac{\sqrt{3}}{6}\hbar & u - \frac{\chi_1}{6} - \frac{\sqrt{3}}{6}(\theta_1 - \theta_3) \end{pmatrix}
\end{align*} \] (B.20, B.21, B.22)

**Symmetrised modules**  Let us now compute these operators in the symmetrised Yangian representation. This amounts to finding a $\theta$-dependent change of basis such that all the matrix coefficients $c$, the only non-trivial operator of the Bethe algebra, are symmetric polynomials in $\theta_i$. We will explicitly compute this basis by using the proof of B.8.

For simplicity let us first assume that $\hbar = 0$. The normalised Young symmetriser for $\mathcal{T} = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ is given by
\[ S_T := R_T C_T = \frac{1}{3}(1 + (2 \ 1 \ 3) - (3 \ 2 \ 1) - (2 \ 3 \ 1)). \] (B.23)

We now have to compute $S_T \cdot C[\theta]$ and moreover to find a $\mathbb{C}[\chi]$-basis for it. Start by picking a $\mathbb{C}[\chi]$-basis of $\mathbb{C}[\theta]$. One can for example take Schubert polynomials which in the case of $S_3$ are given by \{1, $\theta_1, \theta_1 + \theta_2, \theta_1^2, \theta_1 \theta_2, \theta_1^2 \theta_2$\}. Since $S_T$ commutes with multiplication by symmetric polynomials we just have to compute its action on Schubert polynomials.

\[ ^{\text{41}} \text{This basis is an orthogonal basis of the 2D subspace, the expression of which is "quite symmetric".} \]

\[ ^{\text{42}} \text{By the way one can check equation (B.5). Indeed } \frac{\text{d}(C[\theta])}{\text{d}(\chi[\chi])} = \frac{(1 - t)(1 - t^2)(1 - t^3)}{(1 - t)^3} = 1 + 2t + 2t^2 + t^3 \text{ which is exactly the character of the Schubert basis.} \]
At this stage we can already check the character formula (B.8). Indeed

$$\text{ch}(\mathcal{V}^{S+}_{(2,1)}) = \frac{t(1-t)(1-t^2)(1-t^3)}{(1-t)^2(1-t^3)} = t(1+t) = t^{\deg \eta_1} + t^{\deg \eta_2}.$$  \hfill (B.25)

To obtain a \(\mathbb{C}[\chi]\)-basis of \(\mathcal{V}^{S+}_{(2,1)}\), it remains to compute \(p(w \otimes \eta_1)\) and \(p(w \otimes \eta_2)\) which can be done straightforwardly.

Now assume \(\hbar \neq 0\). This case is more complicated because now we have to take \(\hbar\) corrections into account. In particular now \(p(w \otimes \eta_1), p(w \otimes \eta_2) \notin \mathcal{V}^{S+}_{(2,1)}\). The reason is that at \(\hbar = 0\) applying the Young symmetriser to the vector or to the polynomial factor of a tensor yields the same result since the action of \(p\) propagates it to the other factor. This is no longer true if \(p\) is deformed by \(\hbar\) because the \(S_L\)-action does not factor into a solely polynomial and a solely vector action anymore.

Nevertheless \(p(w \otimes \eta_1), p(w \otimes \eta_2) \in \mathcal{V}^{S+}_{(2,1)}\) and we have to correct them by some vectors of lower degree such that they belong to \(\mathcal{V}^{S+}_{(2,1)}\). Since \(\deg p(w \otimes \eta_1) = \deg \eta_1 = 1\) it can only be corrected by a vector of degree zero. There is only one such vector in \(\mathcal{V}^S_{(2,1)}\) : the totally symmetric combination \(|\uparrow \downarrow \downarrow\rangle + |\downarrow \uparrow \downarrow\rangle + |\downarrow \downarrow \uparrow\rangle\). Its coefficient can be uniquely fixed by requiring that the corrected vector is highest-weight. By a similar argument we can compute the \(\hbar\) corrections to \(p(w \otimes \eta_2)\). In the end we obtain the following \(\mathbb{C}[\chi]\)-basis of \(\mathcal{V}^{S+}_{(2,1)}\)

\[
\xi_1 := \frac{1}{6}(-2\theta_1 + \theta_2 + \theta_3 - 3\hbar) |\uparrow \downarrow \downarrow\rangle \\
+ \frac{1}{6}(\theta_1 - 2\theta_2 + \theta_3) |\downarrow \uparrow \downarrow\rangle \\
+ \frac{1}{6}(\theta_1 + \theta_2 - 2\theta_3 + 3\hbar) |\downarrow \downarrow \uparrow\rangle \\
\xi_2 := \frac{1}{18}(-3\theta_1 \theta_2 - 3\theta_1 \theta_3 + 6\theta_2 \theta_3 - h(\theta_1 + 4\theta_2 + 4\theta_3)) |\uparrow \downarrow \downarrow\rangle \\
+ \frac{1}{18}(-3\theta_1 \theta_2 + 6\theta_1 \theta_3 + 2\theta_2 \theta_3 - h(4\theta_1 + \theta_2 - 5\theta_3) - 3\hbar^2) |\downarrow \uparrow \downarrow\rangle \\
+ \frac{1}{18}(6\theta_1 \theta_2 - 3\theta_1 \theta_3 - 3\theta_2 \theta_3 - h(5\theta_1 + 5\theta_2 - \theta_3) + 3\hbar^2) |\downarrow \downarrow \uparrow\rangle
\]  \hfill (B.26)

that is, \(\mathcal{V}^{S+}_{(2,1)} = \mathbb{C}[\chi] \xi_1 \oplus \mathbb{C}[\chi] \xi_2\). Changing \(c\) to this basis we finally obtain its symmetrised representative

\[
c^S := \begin{pmatrix}
-\hbar \\
-\chi_2 - \frac{4}{3}h(\chi_1 + h) \\
6
\end{pmatrix}.
\]  \hfill (B.27)

A few remarks are in order. First, note that \(c\) is homogeneous of degree 1 (if we consider \(\deg h = 1\)) whereas \(c^S\) is non homogeneous of degree 2. This has to do with
the fact that $\xi_1$ and $\xi_2$ are $\theta$-dependent and of different degrees. Second, one can check that $c$ and $c^S$ have the same characteristic polynomial and therefore the same spectrum. However there is a crucial difference: if we set $\hbar = 0$ and take $\bar{\chi}_1 = \bar{\chi}_2 = \bar{\chi}_3$ (a potentially “bad” point by Theorem B.9), $c$ becomes proportional to the identity whereas $c^S$ does not, so they cannot be related by a change of basis. In particular the symmetrised Bethe algebra $B_\Lambda(\bar{\chi})$ is still maximal, whereas the evaluated Bethe algebra $B_\Lambda(\bar{\theta})$ is not. Again this has to do with the fact that the basis $(\xi_1, \xi_2)$ is $\theta$-dependent: at this particular point it becomes degenerate as a basis of the physical vector space, but not as a basis of the quotient $V^\mathbb{S} + (2,1)(\bar{\chi})$. Actually, the determinant of the matrix of change of basis between $B_1$ and $(\xi_1, \xi_2)$ is given by
\[ 1 \sqrt{3} (\theta_1 - \theta_2 + \hbar)(\theta_1 - \theta_3 + \hbar)(\theta_2 - \theta_3 + \hbar) \] and it equates to zero precisely at the “bad” points of Theorem B.9. Note however that even at “bad” points (and actually at most of them) $B_\Lambda(\bar{\chi})$ and $B_\Lambda(\bar{\theta})$ are still isomorphic and maximal.

This example shows that a $\mathbb{C}[\chi]$-basis of the symmetrised Yangian representation is quite difficult to compute. As long as the hypothesis of Theorem B.9 is satisfied, the traditional physical frame is perfectly equivalent to the symmetrised one, and can be used without trouble for all practical applications.

### C Q-operators belong to the Bethe algebra

In order to show that the Q functions/operator belong to the Bethe algebra, we will show how to find them from $T$ functions/operators. When $a = 1$, the contraction with the Levi-Civita tensor in the Wronskian expression (2.15) reduces to $(m-1)! \sum b Q_b^{[m-n+s]} Q_b^{[-s]}$, hence we can express the $t$-dependent sum (where $t$ is a free parameter)

\[ S(t) := \sum_b \left( \sum_{s \geq 1} Q_b^{[-m+n-2s]} t^s \right) Q_b \tag{C.1} \]

as an infinite linear combination of the $T_{(s^t)}$: this linear combination looks like $\sum_{s \geq 1} T_{(s^t)}^{[-m+n-s]} t^s$, up to the first terms (when $s < a - m + n$) and up to factors that have no impact on the present argument and would make expressions extremely bulky. These are the factors $Q_b^{|\emptyset|\emptyset}$, Ber $G$, $(m-1)!$ and the proportionality \footnote{This proportionality factor is a supersymmetric version of a Vandermonde determinant of the eigenvalues $z_\alpha$, as can be found from requiring that $q_0$, $q_a$ and $q_i$ are monic.} factor denoted by the symbol $\propto$ in (2.15).

This infinite sum (where $s$ runs from 0 to $+\infty$) converges in the disk $|t| < \min \left| \frac{\text{Ber } G}{z_b} \right|$ and is then analytically continued to $t \in \mathbb{C} \setminus \left\{ \left| \frac{\text{Ber } G}{z_b} \right| 1 \leq b \leq m \right\}$. Indeed if we denote
\[
Q_b = \left( \frac{\text{Ber} \, G}{x_b} \right)^{u/h} \sum_{k=0}^{M_b} c_b^{(k)} \, u^k, \text{ then for } |t| < \min \left| \frac{\text{Ber} \, G}{x_b} \right| \text{ we have:}
\]

\[
\sum_{s \geq 1} Q_b^{[-m+n-2s]} t^s = \left( \frac{\text{Ber} \, G}{x_b} \right)^{-\frac{m+n}{2}} \sum_{j=0}^{\frac{M_b}{2}} (-1)^j \left( \sum_{k=j}^{M_b} \binom{k}{j} (u^{[-m+n]} k-j c_b^{(k)}) \sum_{s \geq 1} s^j \left( \frac{x_b}{\text{Ber} \, G} \right)^s \right)
\]

where the combinatorial factors \( A(j, k) \) are positive integers called “Eulerian numbers”, and where the sum \( \sum_{k=0}^{j-1} A(j, k) \left( \frac{tx_b}{\text{Ber} \, G} \right)^{k+1} \) should be replaced by 1 in the ill-defined case \( j = 0 \).

If the eigenvalues are \( x_b \) are pairwise distinct we deduce that

\[
Q_b \propto \lim_{t \to \frac{\text{Ber} \, G}{x_b}} \left( 1 - \frac{tx_b}{\text{Ber} \, G} \right)^{M_b+1} S(t).
\]

This expression allows one to conclude that, at the level of representations, it belongs to the Bethe algebra. Indeed, the Bethe Algebra forms a linear subspace of the space of operators on the Hilbert space, which is finite dimensional. It is hence topologically closed, hence the sum \( S(t) \) belongs to the Bethe algebra, not only when \( |t| < \min \left| \frac{\text{Ber} \, G}{x_b} \right| \) but even for arbitrary \( t \) by analytic continuation. Then, by taking the limit (C.3) \( Q_b \) belongs to the Bethe Algebra.

Of course, the same argument can be written for \( Q_i \), by focusing on a sum of the form \( \sum_{a \geq 0} T_1^{[-m+n+a]} t^a \), allowing to express all Q operators (with multiple indices).

In [61], explicit computations of such infinite sums and of their limit were performed combinatorially at the level of the representation \( ev \theta \) (for twist with pairwise-distinct eigenvalues, as in the above discussion). This explicit construction of the Q-operators shows that their matrix coefficients are indeed polynomial functions of \( u \) and of the inhomogeneities \( \theta_\ell \), and that they are rational functions of the twist eigenvalues \( z_\alpha \). It also shows that the degree \( M_{\lambda|I} \) of \( q_{\lambda|I} \) is the sum \( M_{\lambda|I} = \sum_{\alpha \in \lambda|I} M_\alpha \) where \( M_\alpha \) is the number of magnons of type \( \alpha \).

More precisely the infinite sum which was computed in [61] is of the form \( \sum_{s \geq 1} T_1^{[s]} t^s \), by contrast with the opposite shifts in the above discussion. Consequently the explicit combinatorial description gives an expression of \( Q_b \) instead of \( Q_b \), and \( Q_b \) was extracted after a few more steps – after \( m-1 \) successive limits – and the whole Q-system is expressed explicitly.
D Details about structural study of Bethe equations

In this section we will adopt analytic point of view of Section 3.1 on inhomogeneities. Namely we will think them as complex numbers that we are going to vary. Then $c_\ell$ – coefficients of (twisted) Baxter polynomials turn out to be algebraic functions of inhomogeneities. The main purpose of this section is to prove properness of Wronskian equations – that all solutions $c_\ell$ are bounded if $\theta_\ell$ (and hence $\chi_\ell$) are bounded. A natural consequence of this analysis will be behaviour of $c_\ell$ when $\theta_\ell$ tend to infinity (in the twist-less case) which we analyse in D.3 to provide the necessary technical results for Section 6.2.

D.1 Properness - twisted case

The feature that ensures that $c_\alpha$ are bounded at finite $\theta_\alpha$ is the fact that $c_\alpha$ are coefficients of polynomials in $u$ and the Wronskian equation (2.26) is an equation on these polynomials.

Assume that there is a point $\bar{\chi} \in X$ by approaching which some of $c_\alpha$ diverge (become unbounded). If $c_\alpha$ is a coefficient of a twisted polynomial $Q(u)$ then divergence of $c_\alpha$ implies divergence of some of the roots of $Q(u)$. Factorise $Q(u)$ in the form $Q = Q \gg Q \ll$, where $Q \gg$ is a polynomial containing all diverging roots, and $Q \ll$ is the function containing the twist prefactor and all finite roots.

Consider first equation $Q_{a|i}^+ - Q_{a|i}^- = Q_{a|i}^+ - (Q_{a|i}^-) + H$, where $H$ is the function that ensures equality. By taking a point $\chi$ sufficiently close to the point $\bar{\chi}$ one can ensure that $H$ is small in the following sense: There exists $R$ such that all roots of $(Q_{a|i}^+) - (Q_{a|i}^-)$ lie inside the circle $|u| = R$, all roots of $Q \gg$ lie outside it, and that absolute value of $H$ is smaller than the absolute value of $Q \gg$ when $|u| = R$. Then by the Rouche’s theorem, number of zeros of $Q_{a|i}^+ - Q_{a|i}^-$ inside and outside the circle is the same as that of $Q_{a|i}^+ - (Q_{a|i}^-)$. Hence existence of large zeros of $Q_{a|i}$ imply existence of large zeros, in the same amount, in the product $Q_{a|\varnothing} Q_{\varnothing|i}$. Distribution of them between $Q_{a|\varnothing}$ and $Q_{\varnothing|i}$ depends on the solution we consider.

Consider now the Wronskian equation (2.26) and recall that super-Wronskian is explicitly the determinant (2.19). Applying the same logic, we write

$$SW(C_\Lambda) = \prod_{a=1}^m Q_{a|\varnothing}^{\gg} \prod_{i=1}^n Q_{\varnothing|i}^{\gg} SW(Q^\ll) + H,$$

and then conclude using the Rouche’s theorem that $Q_\varnothing = SW(Q)$ has large zeros, i.e. the point $\bar{\chi}$ is not the one with finite $\chi_\ell$. For the argument to work, one need to ensure that $SW(Q^\ll)$ is not vanishing but it is straightforward as presence of the twist prefactors $z_i^{u/h}$ ensures that already the leading-$u$ term in $SW(Q^\ll)$ is non-vanishing.
D.2 Properness - twist-less case

To study the twist-less case, we will focus on the bosonised parameterisation of the Q-system on a Young diagram (5.10). In particular, one has

\[ Q_\theta = Q_{0,0} \propto W(B_1, \ldots, B_m). \]  

(D.2)

The last equation contains in principle the full information since the \( \mathfrak{gl}(m|0) \) Q-system, where \( m = h_{\Lambda^+} \) is a possible way to parameterise the Bethe algebra \( B_\Lambda \).

To prove properness, we would like to use an argument similar to that around (D.1), however cancellations in the Wronskian determinant make things more subtle.

Example:

Take \( B_1 = u, B_2 = (u - \Lambda)(u + 1), B_3 = (u - \Lambda)^3 \). Let \( \Lambda \to \infty \) if \( \chi \to \bar{\chi} \).

Formally there are four divergent roots when \( \Lambda \to \infty \). However \[ W(B_1, B_2, B_3) = u^3 + u(3\Lambda - h^2) - \Lambda^2(\Lambda + 3) \] which has three divergent roots.

The issue in the example comes (at least) from the fact that in the decomposition \( Q = Q^{\gg} Q^{\leq}, B_1^\geq \) and \( B_2^\leq \) are polynomials of the same degree (equal to one).

To continue, we do a couple of formalisations.

**Parametric factorisation** Let \( \Lambda \) be a parameter, an we intend to consider \( \Lambda \to \infty \) behaviour of Q-functions. Let \( S \) be a scale function of \( \Lambda \), typically \( \Lambda^\beta \) for some real \( \beta \). We say that \( Q = Q^{\gg} Q^{\leq} \) is the parametric factorisation of the polynomial \( Q \) at scale \( S \) if all roots of the monic polynomial \( Q^{\gg} \) are much larger than \( S \), and all roots of the monic polynomial \( Q^{\leq} \) are compatible or smaller than \( S \). More precisely, for each \( u \) that satisfies \( Q^{\gg} = 0 \) one has \( \lim_{\Lambda \to \infty} S/u = 0 \), and for each \( u \) that satisfies \( Q^{\leq} = 0 \) the \( \Lambda \to \infty \) limit of \( u/S \) is finite.

Then the argument around (D.1) can be formalised in the following lemma.

**Lemma D.1.** Let, for some polynomials \( Q_1, \ldots, Q_a \), \( Q_{12\ldots a} = W(Q_1, \ldots, Q_a) \) and \( Q = Q^{\gg} Q^{\leq} \) is the parametric factorisation at scale \( S \).

If degrees \( \deg Q_1^{\leq}, \deg Q_2^{\leq}, \deg Q_3^{\leq}, \ldots \) are pairwise distinct, and degrees \( \deg Q_1, \deg Q_2, \deg Q_3, \ldots \) are also pairwise distinct then

\[ \deg Q_{12\ldots a}^{\gg} = \sum_{a'=1}^{a} \deg Q_{a'}^{\gg}. \]  

(D.3)

**Proof.** Perform an equivalent of decomposition (D.1) and apply the Rouché’s theorem. An important ingredient is that both \( W(Q_1^{\leq}, \ldots, Q_a^{\leq}) \neq 0 \) and \( W(Q_1, \ldots, Q_a) \neq 0 \) for which sake the restriction on the degrees is imposed.

Now we recall that not all coefficients of \( B_a \) bear physical information as they are subject to the symmetry transformation (2.35). We shall benefit from (2.35) to ensure that a parametric factorisation of \( B_a \) satisfies conditions of the above lemma.
Lemma D.2. Let $B_1, \ldots, B_m$ be monic polynomials with $\deg B_1 < \ldots < \deg B_a$. For any scale $S$, one can find a “rotation”

$$B_a \rightarrow B_a + \sum_{b < a} h_{a,b}B_b,$$  \hspace{1cm} (D.4)

where the $h_{a,b}$’s are complex-valued functions \footnote{More accurately, they are algebraic functions of the inhomogeneities $\theta_\ell$ whose values depend on $\Lambda$.} of $\Lambda$, such that, after the rotation, the degrees of $B_1^\leq S$, $B_2^\leq S$, \ldots, $B_m^\leq S$ are pairwise distinct.

We note that the proof below is constructive and it provides an algorithm to find $h_{a,b}$ explicitly.

Proof. Without loss of generality one can set $S = 1$ in which case we denote the parametric factorisation as $Q = Q^\geq \cdot Q^\leq$. Indeed, we can always perform the rescaling $u \rightarrow u S$.

In the labelling of polynomials $B_a = u^{a} + \ldots + b^{(a)}_k u^k + \ldots + b^{(a)}_b u + b^{(a)}_0$, consider all $b^{(a)}_k$ that have the largest exponent when $\Lambda \rightarrow \infty$ and choose $b^{(a)}_{k_a}$ with the largest $k$ among them. For instance, in $u^3 + \Lambda u^2 + \Lambda^3 u + 2\Lambda^3$, it is $b^{(a)}_1 = \Lambda^3$. Then $\deg B_a^\leq = k$.

If there exist such $a, b$, $b < a$ that $\deg B_a^\leq = \deg B_b^\leq = k$ then perform the transformation $B_a \rightarrow B_a - \frac{b^{(a)}_{k_a}}{b^{(b)}_{k_b}} B_b$. This transformation will affect the parametric factorisation of $B_a$, two things can happen. First, $\deg B_a^\leq$ becomes smaller. Second, all terms with the largest exponent are cancelled out from $B_a$ in which case one gets a new (smaller) largest exponent and new value for $\deg B_a^\leq$ (in principle arbitrarily large, only bounded by the degree of $B_a$).

We repeat recursively the procedure of comparison between all available pairs of $a, b$ and terminate when $\deg B_a^\leq$ become pairwise distinct. The recursion will terminate in finite number of steps and produce a meaningful result for the following reasons: there are finitely many polynomials of finite degree to operate with, the maximal exponents can only decrease in the procedure and they are bounded by zero from below, and $B_a$ cannot vanish entirely as $\deg B_a$ are pairwise distinct and so the leading monomial is never affected by the performed transformations. \qed

Example:

For $S = 1$ and the system in Example on page 77, the rotation is done as follows. First, transformation $B_2 \rightarrow B_2 + \Lambda B_1 = u^2 + u - \Lambda$ drops degree of $B_2^\leq$ to zero. Now both degrees of $B_2^\geq$ and $B_3^\geq$ are zero. We perform transformation $B_3 \rightarrow B_3 - \Lambda^2 B_2$. This drops the maximal exponent in $B_3$ from three to two, and $\deg B_3^\leq$ computed with respect to the new maximal exponent is two. Now all $\deg B_a^\leq$ are pairwise distinct. In summary, we get that the rotated values $B_1 = u, B_2 = u^2 + u - \Lambda, B_3 = u^3 - \Lambda(\Lambda + 3) u^2 + 2\Lambda^2 u$. $B_2$ has two divergent roots, $B_3$ has one divergent root, $W(B_1, B_2, B_3)$ remains unchanged by the performed rotation and it has three divergent roots.
Now we are ready to prove properness as declared on page 22. Assume that there is a finite point $\chi \in X$ by approaching which some coefficients of $B_\alpha$ diverge. We approach this point following some path parameterised by $\Lambda$, and $\Lambda \to \infty$ corresponds to the approach of the point. Choose $S = 1$ and perform the transformation (D.4) to get $deg B_\alpha^\beta$ pairwise distinct. If there are still divergent coefficients after this transformation, we get $deg Q_\beta^\beta > 0$ by Lemma D.1 and (D.2) and hence reach a contradiction. Thus all $B_\alpha$ have finite coefficients. Compute $Q_{\alpha,a,s}$ following (5.10) and use the procedure in the proof of Lemma 5.3 to compute the set $C_\Lambda$ introduced after (2.35). $c_\ell$ appearing in this set are hence non-divergent when we approach $\chi$ which is the properness in the sense of Section 3.

D.3 Labelling solutions with standard Young tableaux – technical details

All solutions approach (6.6)

The key step is to justify the formula (6.5). Consider the situation when $\theta_L = \Lambda$, $\Lambda \to \infty$, and all other $\theta_i$ are finite. In any scaling $S = \Lambda^\beta$ with $0 < \beta < 1$, we rotate $B_\alpha$ to a frame where $deg B_\alpha^\beta$ are pair-wise distinct. By Lemma D.1, there is precisely one $a = a_0$ for which precisely one root of $B_{a_0}$ diverges, and all roots of $B_{a \neq a_0}$ stay finite.

Recall that $deg B_\alpha = \lambda_a + m - a$, and there are also $deg B_i = d_i$. The assignment rules are explained in Figure 4. Since $B_\alpha$ are in the frame with pair-wise distinct $deg B_\alpha^\beta$, if $a_0 \neq m$, it must be $\lambda_{a_0} > \lambda_{a_0+1}$ (equality is impossible). Then, there exists $s_0$ such that $d_{s_0+1} = deg B_{a_0} - 1$ and so the box $(a_0, s_0)$ is a corner box of the Young diagram.

Finally, we consider (5.10) to decide which $Q_{\alpha,a,s}$ have a divergent root. If $a > a_0$ then $Q_{\alpha,a,s}$ does not depend on $B_{a_0}$ and hence has no divergent roots. If $s > s_0$ then all polynomials of degree from 0 to $deg B_{a_0} - 1$ appear in the Wronskian determinant. Then the polynomial structure of $B_{a_0}$ is irrelevant, we can replace it with the leading monomial and so $Q_{\alpha,a,s}$ cannot have divergent roots. Finally if $a \leq a_0$ and $s \leq s_0$ then $B_{a_0}$ is present in the Wronskian and there is no polynomial of degree $deg B_{a_0} - 1$ in the Wronskian, so conditions of the Lemma D.1 are satisfied, so $Q_{\alpha,a,s}$ has precisely one divergent root.

Now we can deduce that roots scale exactly as $\Lambda$. Indeed, for $\beta^\prime \in ]\beta, 1[$, the rotation of Lemma D.2 may change but the $Q$ functions are invariant under this triangular rotation. If the value of $a_0$ changes at scale $\Lambda^{\beta^\prime}$ compared to scale $\Lambda^\beta$, then there would be another corner-box $(a_0^\prime, s_0^\prime) \neq (a_0, s_0)$ such that at scale $\Lambda^{\beta^\prime}$ the $Q$-functions with diverging roots are the nodes with $a \leq a_0^\prime$ and $s \leq s_0^\prime$. This is impossible because all $Q$ functions that diverge at scale $\Lambda^{\beta^\prime}$ also diverge at scale $\Lambda^\beta$. Hence $a_0$ is independent of $\beta \in ]0, 1[$, the number of diverging roots is hence also independent of $\beta$ and the diverging roots scale exactly as $\Lambda$.

Finally, we get (6.5), and also that $\hat{Q}_{\alpha,a,s}$ introduced alongside (6.5) is a $Q$-system on the Young diagram with the box $(a_0, s_0)$ removed.

Unambiguous continuation of the limiting solution (6.6) for each SYT to finite inhomogeneities

To discuss this question, we should not send inhomogeneities one after another to infinity, but do a more smooth realisation of the limit (6.4). Namely, we consider $\alpha_1$, $\alpha_2$, ..., $\alpha_L \in \mathbb{C}^*$ and $\beta_L > \beta_{L-1} > \cdots > \beta_1 > 0$, and we consider $\theta_1 = \alpha_1 \Lambda^{\beta_1}$, $\theta_2 = \alpha_2 \Lambda^{\beta_2}$,
\[ \theta_L = \alpha_L \Lambda^{\beta_L} \text{ when } \Lambda \to +\infty. \] All roots and all coefficients of all Q-polynomials are algebraic functions \(^{45}\) of the parameter \(\Lambda\) and hence have a large-\(\Lambda\) behavior of the form \(\alpha \Lambda^\beta\) which allows applying scaling argumentation from previous sections.

If \(\beta_L\) are spaced apart well, we can recover the same results as if inhomogeneities are sent to infinity one by one. But now, after we know that all solutions of the Q-system approach (6.6), a sharper judgement about possible \(\beta_L\) can be made by observing that the leading order of the large \(\Lambda\) expansion (6.6) is solved by \(^{46, 47}\)

\[ \forall a \leq m, \quad B_a \sim u^{m-a} \prod_{s=1}^{\lambda_a} \left( u - \frac{N_{a-1,s-1}^{(T_{a,s})}}{m-a+s} \theta_{T_{a,s}} \right). \] (D.5)

Let us now parameterise the Q-system by \(v_1, \ldots, v_L\) as follows

\[ \forall a \leq m, \quad B_a = u^{m-a} \prod_{s=1}^{\lambda_a} \left( u - \frac{N_{a-1,s-1}^{(T_{a,s})}}{m-a+s} v_{T_{a,s}} \Lambda^{\beta_{T_{a,s}}} \right). \] (D.6)

Recall that \(\theta_L = \alpha_L \Lambda^{\beta_L}\). Then \(Q_\theta = W(B_1, \ldots, B_m)\) realises a map from \(v_L\) to \(\alpha_L\) which analytically depends on \(1/\Lambda\) for \(\beta_L\) being integers. When \(1/\Lambda = 0\) this map is simply an identity map with obviously non-zero Jacobian. Hence we can apply the analytic implicit function theorem to invert the map. By the theorem, for some neighbourhood of point \(1/\Lambda = 0\), \(v_L\) are smooth functions of \(\alpha_1, \ldots, \alpha_L\) and \(1/\Lambda\) and hence each limiting solution (6.6) can be continued to finite values of inhomogeneities.

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\(^{45}\)For a Q-system on Young diagram, this is an immediate consequence of the QQ-relations. For a Q-system on Hasse diagram, we should restrict symmetry transformations (2.35) to algebraically depend on \(\theta_L\).

\(^{46}\)The computation reduces to the fact that \(W(1, u, \ldots, u^{-1}, u^{k-1}(u^{-\frac{m}{2}} \theta)) \propto (u - \alpha \theta)\).

\(^{47}\)Here \(\lambda_a, a = 1, \ldots, \lambda_m\) form an integer partition whose shape is the Young diagram \(\Lambda^+\).
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