Metrology on effective Rabi frequency in Raman transition

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The interacting model describing the Raman transition is important in the study of atom-photon interactions. For convenience of analytical analysis, the kinetic term is often omitted. We first study the approximation by the fidelity approach, and identify the region of validity in parameter space of the mass and momentum. As the momentum of the radiation field decreases or the atomic mass increases, the approximation becomes valid. We further find that the inclusion of the kinetic term will enhance the precision measurement of the effective frequency. We aim in this article. From the perspective of QFI, we try to understand the influence of the kinetic energy term in the Hamiltonian under different parameters. Among the above research, we notice that the kinetic energy term is omitted in some cases (e.g., Raman-Nath regime). However, for many cases it can’t be directly omitted (Bragg regime). We try to figure out the mechanism in a general light-atom interaction Hamiltonian.

Here we use the Hamiltonian which considers the inhomogeneous field, and this means that the spatial motion of the atomic center of mass plays an important role in the interacting Hamiltonian between the field and the atom. In this article, we deal with this kind of Hamiltonians with or without the kinetic energy terms to figure out whether this term is crucial for the metrology of the effective frequency in Raman transition. We find the omission of the kinetic term will decrease the QFI, which means we can’t measure the effective frequency precisely. What’s more, we further work out the corresponding CFIs to see how the specific measurements match the theoretical maximum. Our analytical results will help determine when the omission is valid in different regimes of parameters and the comparison of QFI and CFI will make us know if some specific measurements can attain the best sensitivity under certain parameters.

This article is divided into four parts. The first part is the following section, where we compare the final states with or without kinetic energy term directly through the fidelity. The above two cases become similar when the ratio between the momentum of the radiation field and the atomic mass goes down, which means the kinetic energy becomes less important. Then, in the next section, we give our main results about the influence on QFI when the kinetic energy term is eliminated. We compare the two cases with or without the kinetic energy term via changing the parameters and we find that the above ratio is also crucial for the Fisher information. The third part mainly talks about the possibility of attaining QFI in measurement (CFI) without the kinetic energy term. The last part discusses QFI and CFI when the kinetic energy term is present.

Actually, we can see that the ratio between the momentum of the radiation field and the atomic mass is critical.
throughout the article. Furthermore, the less the ratio is, the more the Fisher information (both cases and both QFI and CFI) will be. Decreasing the ratio is necessary for the enhancement of the metrology of effective Rabi frequency. What’s more, we find that if the initial state is Guassian type in the position space, the increasement of the variance of the Guassian state will enhance the Fisher information while taking the kinetic energy into account.

II. DIRECT COMPARISON OF FINAL STATES WITH RESPECT TO KINETIC ENERGY TERM

In this section, we will briefly discuss the effect of kinetic energy term in the Hamiltonian on the fidelity. We will utilize the system of the effective two-level dynamics in the Raman transition. In the frame rotating at the laser frequency, our Hamiltonian will be written as

\[
\hat{H}_1 = \frac{\hbar^2}{2m} \Delta \hat{b}^\dagger \hat{b} + \frac{\hbar \Omega}{2} \left( |b\rangle \langle a| e^{i k_0 \hat{z}} + |a\rangle \langle b| e^{-i k_0 \hat{z}} \right),
\]

(1)

where \( \delta \) refers to two-photon detuning and is typically set as \( \frac{\hbar k_0}{2m} \) to nearly satisfy the two-photon resonance condition \([29]\). The momentum is in \( \hat{z} \) direction. \( \Omega \) is two-photon Rabi frequency \( \frac{\hbar \Omega}{\Delta} \), where the single-photon detuning \( \Delta \) should be much larger than \( \delta \), as depicted by resonance. The intermediate level has been omitted, and the energy of level \( |a\rangle \) has been set to zero. The parameters are set as \( \hbar \sim 10^{-34} \text{ J-s}, \Omega \sim 50 \text{ s}^{-1} \), \( m = 1.44 \times 10^{-25} \text{ kg}, k_0 \sim 2 \times 10^7 \text{ m}^{-1} \), then \( \delta \sim 1.39 \times 10^5 \text{ s}^{-1} \). Those parameters are all in SI \([30, 31]\), and we will always use these parameters below if we don’t specifically mention it. In fact, we will see below these initial parameters aren’t good enough for the metrology of the effective Rabi frequency, however, we will work out the better region of specific parameters to enhance the measurement precision.

![Figure 1. Raman transition in \( \Lambda \) system](image)

In order to simplify the calculations under the evolution of the above Hamiltonian, we need to use a unitary transformation \([28]\)

\[
\hat{W} = |b\rangle \langle e^{i k_0 \hat{z}} + |a\rangle \langle e^{-i k_0 \hat{z}} \rangle
\]

(2)

to get a su(2)-type effective Hamiltonian

\[
\hat{H}_{e2} = \hat{W}^\dagger \hat{H}_1 \hat{W} = \frac{\hbar^2}{2m} \frac{k_0^2}{8m} |b\rangle \langle b| - \frac{\hbar k_0}{2m} \hat{p} \langle a| - |b\rangle \langle a| - \frac{\hbar \Omega}{2} \hat{p} + \frac{k_0^2}{4m} \sigma_z + \frac{\hbar \Omega}{2} \sigma_x.
\]

(3)

Then the evolution operator can be written as

\[
\hat{U}_1 = e^{-i \hat{H}_1 t} = \hat{W} e^{-i \hat{H}_{e2} t} \hat{W}^\dagger.
\]

(4)

The advantage of the above effective Hamiltonian is that it has eliminated the \( \hat{z} \) spatial component, which doesn’t commute with \( \hat{z} \)-direction momentum \( \hat{p} \). Then we try to omit the kinetic energy term, which suits for relatively large coupling strength and atomic mass or small evolution time. The Hamiltonian can be written as

\[
\hat{H}_2 = -\hbar \delta |b\rangle \langle b| + \frac{\hbar \Omega}{2} (|b\rangle \langle a| e^{i k_0 \hat{z}} + |a\rangle \langle b| e^{-i k_0 \hat{z}}),
\]

(5)

to get an effective su(2)-type Hamiltonian

\[
\hat{H}_{e2} = \frac{\hbar^2 k_0^2}{2m} |b\rangle \langle b| + \frac{\hbar \Omega}{2} (|b\rangle \langle a| + |a\rangle \langle b|) - \frac{k_0^2}{4m} \sigma_z + \frac{\hbar \Omega}{2} \sigma_x.
\]

(6)

This effective Hamiltonian doesn’t even consist of momentum operator, which means we can obtain a simpler expression. Then we have the evolution operator

\[
\hat{U}_2 = e^{-i \hat{H}_2 t} = \hat{W} e^{-i \hat{H}_{e2} t} \hat{W}^\dagger.
\]

(7)

Before we calculate the quantum Fisher information, we might as well compare the two final states. Here, we assume the initial state is \( |\psi_i\rangle = |a\rangle |\psi_0\rangle \), where the motional state in position space is

\[
|z| \psi_0\rangle = \exp(-z^2/2\sigma^2)(\pi \sigma^2)^{1/4},
\]

(8)

where \( \sigma \sim 4 \times 10^{-5} \text{ m} \) \([29]\). To show the exact influence of the kinetic energy term on the evolved final state, we calculate the fidelity

\[
F = |\langle \psi_1 | \psi_2 \rangle|^2 = |\langle \psi_i U_1^\dagger U_2 | \psi_i \rangle|^2 = |\langle \psi_0 | (a | e^{-i \hat{H}_1 t} e^{-i \frac{\hbar \Omega}{2} \hat{p} \sigma_z} | a\rangle | \psi_0 \rangle|^2|
\]

(9)
and give the contour plot in the following figure. We can see the general trend, which is that the fidelity become larger as $m$ increases or $k_0$ goes down, leading to the valid omission of kinetic energy in the Hamiltonian. We will see this trend satisfies our calculation of QFI below.

![Contour plot of the fidelity between the final states of two cases.](image)

**III. THE EFFECT OF KINETIC ENERGY TERM ON QFI**

Now let us turn back to the calculation of the QFI. For the $su(2)$-type Hamiltonian, a general solution for the QFI has been discussed [16]. We can define a Hermitian operator which is independent of the initial state

$$H_1 = i(\partial \hat{U} \hat{U}^\dagger_1)\hat{U}_1.$$  

(11)

If the initial state is pure state, the QFI can be expressed as [17]

$$F = 4 \langle \psi_{in} | \Delta^2 H_1 | \psi_{in} \rangle.$$  

(12)

In fact, momentum representation will be helpful to our computation as you can see in the Appendix. We still use the initial state $|\psi_{in}\rangle$. After the evolution of the Hamiltonian, we finally get the integral formalism of QFI

$$F = \frac{\sigma}{\hbar \sqrt{\pi}} \int_{-\infty}^{\infty} dp \, e^{-\frac{p^2}{2\sigma^2}} \left[ \left( \frac{\Omega^2 (\sin \omega t - \omega t)}{\omega^3} - \frac{\sin \omega t}{\omega} \right)^2 + \left( \frac{k \omega (\sin \omega t - \omega t)}{m \omega^3} \right)^2 + \left( \frac{(k \omega) (1 - \cos \omega t)}{m \omega^2} \right)^2 \right],$$  

(13)

where

$$\omega = \sqrt{\left( \frac{k \omega}{m} \right)^2 + \Omega^2}.$$  

(14)

The whole procedure is similar to the above case. We still use the initial state $|\psi_{in}\rangle = |a\rangle|\psi_0\rangle$, so the calculation for the QFI still uses the expression (12). When the kinetic energy term is absent, the calculation will be much easier and we can get a specific expression for the QFI as is written in the Appendix. After the evolution of the Hamiltonian, the QFI at $t$ will be:

$$F = \frac{1}{8m^3 \omega^6} \left[ 2\hbar^2 k_0^2 \sin^2 \omega t + 8m^3 t^2 \Omega^6 + \hbar^2 k_0^2 m^2 (3 + 4 \omega t \sin \omega t + \cos 2 \omega t) \right].$$  

(16)

where

$$\omega' = \frac{1}{2} \sqrt{\frac{\hbar^2 k_0^4}{m^2} + 4 \Omega^2}.$$  

(17)

We compare it with the QFI with kinetic energy term in the following figures, and we will focus on the situations where the parameters are set such that the former (QFI without kinetic energy) increases from nearly zero to almost approach the latter (QFI with kinetic energy). Other cases out of the range will be trivial as will be seen from Figure 3.

![Figure 3. (a) QFI with (solid line) or without (dashed lines) kinetic energy using $m = 10^{-21} \times 6 \times 10^{-22} \times 2 \times 10^{-22} \text{ kg}$ while remaining the others the default. (b) QFI with (solid line) or without (dashed lines) kinetic energy setting $k_0 = 10^5, 3 \times 10^6, 5 \times 10^7 \text{ m}^{-1}$ while remaining the others the default.](image)
As we stated before, in the settings of original parameters, the QFI with the kinetic energy is much larger than that without the kinetic energy term, which means the direct omission of the kinetic energy in the Hamiltonian is unreasonable for measuring the effective Rabi frequency. From the figures, we can read that when \( m < 10^{-22} \) or \( k_0 > 5 \times 10^5 \) the latter almost vanishes compared to the former. Only when \( m > 10^{-21} \) or \( k_0 < 10^5 \) can the latter approach the former, which satisfies the fact that the ratio between the momentum of the field and the atomic mass is crucial. However, unlike the case without the kinetic energy term, in Figure 4 we show how the Gaussian state’s variance \( \sigma \) influence the former (the variance doesn’t influence the latter from its expression):

![Figure 4. QFI with respect to \( \sigma \), here we set \( t = 1s \) and remain the others unchanged.](image)

### IV. PRACTICAL MEASUREMENT OF EFFECTIVE RABI FREQUENCY WITHOUT KINETIC ENERGY

In fact, we don’t know if the QFI will be saturated in practical measurement. Although the QFI gives the theoretical best sensitivity, we still need to choose a particular measurement scheme to attain it. The CFI gives the sensitivity of the measurement of a particular observable and the information of the parameter (in our case \( \Omega \)) is hidden in the probability distribution of the outcome of observable. So we will find a real observable and check the parameter sensitivity which the measurement of given observable provides. For simplicity, we will firstly check out the case without kinetic energy term in this section.

The definition of CFI is

\[
I = \int P(\lambda|\theta) \left( \frac{\partial \ln P(\lambda|\theta)}{\partial \theta} \right)^2 d\lambda,
\]

where \( P(\lambda|\theta) \) is the probability of obtaining \( \lambda \) when the parameter is \( \theta \) and we measure the observable \( \hat{\Lambda} \). For example, if we carry out the population-difference measurement \( \hat{\Lambda} = \hat{S}_z \), then the CFI will be denoted as

\[
I_1 = \Sigma_{i=a,b}(\partial_0 P_i)^2/P_i.
\]

In the case of continuous variables, such as momentum measurement \( \hat{\Lambda} = \hat{p} \), the analytical expression of the corresponding CFI can be given below

\[
I_2 = \int (\partial_0 P(p))^2/P(p) dp. \tag{20}
\]

To calculate the CFI with the \( \hat{S}_z \), we know

\[
I_1 = \Sigma_{i=a,b}(\partial_0 P_i)^2/P_i = (\partial_0 P_a)^2/\left[ P_a(1 - P_a) \right], \tag{21}
\]

so we can use \( P_a = \langle \psi_a | \psi_a \rangle = |\langle a | \psi_{\text{out}} \rangle|^2 \) to represent the probability in level \( |a\rangle \) of the final state. Then we have

\[
\langle \psi_a \rangle = \langle a | e^{-i \hat{H}_z t} | a \rangle |\psi_0 \rangle = \langle a | W e^{-i \hat{H}_z W^\dagger} | a \rangle |\psi_0 \rangle = \langle a | e^{-i \hat{H}_z t} e^{i \frac{\pi}{2} \hat{J}_z} | a \rangle |\psi_0 \rangle.
\]

So the probability is

\[
P_a = |\langle a | e^{-i \hat{H}_z t} | a \rangle|^2 = \cos^2 c_1 + n_{z1}^2 \sin^2 c_1, \tag{22}
\]

where

\[
c_1 = -\frac{t}{\hbar} \frac{\hbar^4 k_0^4}{16m^2} + \frac{\hbar^2 \Omega^2}{4}, \tag{23}
\]

\[
n_{z1} = \frac{-\hbar^2 k_0^2}{4m} \frac{m \omega'll^2}{\hbar^4 \Omega^2} \frac{1}{\sqrt{1 + \frac{m^2 \omega'll^2}{4} + \frac{k_0^2 \hbar^2}{16}}}. \tag{24}
\]

So we finally get our analytical solution of CFI:

\[
I_1 = \frac{1}{m^2 \omega'll^2 (2m^2 \omega'll^2 + 1) + k_0^2 \hbar^2}. \tag{25}
\]

In the following, we summarize the Fisher information (QFI, CFI for \( \hat{S}_z \)) in Figure 3.

![Figure 5. Fisher Information, in which the parameters are set as \( \hbar \sim 10^{-34} \text{J} \cdot \text{s}, \Omega \sim 50 \text{s}^{-1}, \sigma \sim 4 \times 10^{-5} \text{m}, m \sim 1.44 \times 10^{-25} \text{kg}, k_0 \sim 2 \times 10^7 \text{m}^{-1}. \) Those parameters are all in SI as we presented, and we will always use these parameters below if we don’t specifically mention it.](image)
situations become different when the parameters change, such as $m$ and $k_0$ for different values in Figure 6.

Figure 6. QFI and CFI when the kinetic energy term is absent for different mass $m = 10^{-23}, 10^{-22}, 10^{-21}$ kg while keeping the other parameters the default. We can see that the Fisher information basically become larger as the mass increases.

We obtain when the mass of the atom become sufficiently large, especially larger than $10^{-22}$ kg, the Fisher information become considerable. But atoms with such large mass seem to be difficult to find in reality, so it’s probably a hard task to enhance the metrology precision of $\Omega$ via increasing the atomic mass. However, the parameter $k_0$ also contributes a lot. Change of the parameter of the radiation field could be a better way to improve the Fisher information, as seen from the Figure 7.

Figure 7. QFI and CFI when the kinetic energy term is absent for different wave-vector $k_0 = 10^6, 10^5, 10^4$ m$^{-1}$ while keeping the other parameters the default.

Basically, the ratio $\frac{k_0^2}{m}$ influences the result significantly as seen from the expression. When this value is sufficiently small, the CFI coincides with QFI well and both become substantial. Since the enhancement of the atomic mass is usually unpractical, increasing the wavelength of the radiation field might be helpful to the fre-
frequency measurement. And this value also occupies an important place in the Fisher information with the kinetic energy as we will see. More differences can be obtained using our analytical solutions.

V. PRACTICAL MEASUREMENT OF EFFECTIVE RABI FREQUENCY WITH KINETIC ENERGY

Now, we will turn to the case with the kinetic energy term to see if the QFI will be saturated. The same as the QFI, the CFI with the kinetic energy term is a little bit cumbersome. To calculate $I_1 = (\partial_3 P_a)^2 / [P_a (1 - P_a)]$, we need to know $|\psi_a⟩ = ⟨a|ψ_{out}⟩$:

$$|\psi_a⟩ = ⟨a|W e^{-i \frac{t}{2} H_{a1}} W^† |a⟩|ψ⟩ = ⟨a|e^{-i \frac{t}{2} \hat{a}} e^{-i \frac{t}{2} \hat{H}_{a1}} e^{i \frac{t}{2} \hat{a}} |a⟩|ψ⟩ = e^{-i \frac{t}{2} \hat{a}} e^{-i \frac{t}{2} (\frac{\Omega^2}{2m} + \frac{k^2 \sigma^2}{4m}) (\cos \sigma^2 c_2 - i n_{2c} \sin \sigma^2 c_2) e^{i \frac{t}{2} \hat{a}} |ψ⟩,$$

(27)

where the operators read

$$c_2 = -\frac{t}{\hbar} \sqrt{\frac{\hbar \Omega^2}{2} + (\frac{\hbar k_0}{2m} \hat{p} - \frac{\hbar^2 k_0^2}{4m})^2},$$

(28)

$$n_{2c} = \frac{\hbar k_0 \hat{p} - \frac{\hbar^2 k_0^2}{4m}}{\sqrt{\left(\frac{\hbar \Omega^2}{2}\right)^2 + (\frac{\hbar k_0}{2m} \hat{p} - \frac{\hbar^2 k_0^2}{4m})^2}}.$$

As a result, we can get the probability:

$$P_a = ⟨ψ | e^{-i \frac{t}{2} \hat{a}} (\cos^2 c_2 + n_{2c}^2 \sin^2 c_2) e^{i \frac{t}{2} \hat{a}} |ψ⟩ = \int dp ⟨ψ | e^{-i \frac{t}{2} \hat{a}} (\cos^2 c_2 + n_{2c}^2 \sin^2 c_2) e^{i \frac{t}{2} \hat{a}} |p⟩ |ψ⟩ = \int dp |⟨ψ | p⟩|^2 (\cos^2 c' + n_{2c}^2 \sin^2 c'),$$

(30)

where

$$c' = -\frac{t}{\hbar} \sqrt{\frac{\hbar \Omega^2}{2} + (\frac{\hbar k_0}{2m} \hat{p})^2},$$

(31)

$$n'_{2c} = \frac{\hbar k_0 \hat{p}}{\sqrt{\left(\frac{\hbar \Omega^2}{2}\right)^2 + (\frac{\hbar k_0}{2m} \hat{p})^2}}.$$

Then we can get the integral formalism of CFI for population difference measurement.

Nevertheless, the CFI in this case is actually very small compared to the QFI, which means the population difference measurement isn’t a good candidate for estimating $Ω$. However, since the QFI, as the maximum of CFI, is large, we can believe there should be some measurement which facilitate the ascent of CFI. If we carry out a measurement which resolves the internal states and the momentum distribution $Λ = (S_x, \hat{p})$ [29, 32], the CFI is

$$I_3 = \sum_{a,b} \int dp \frac{[\partial_3 P_a(p)]^2}{P_b(p)}.$$

(33)

Figure 8 will show more information about the parameter:

![Figure 8](image)

The CFI has become larger, but it’s still small compared with QFI. However, if we change the initial state’s variance, we can finally obtain the Figure 9:

![Figure 9](image)

The variation of Guassian state’s variance won’t change the Fisher information without the kinetic energy term. We conclude with the fact that under some circumstance the kinetic energy term shouldn’t be eliminated directly for the metrology of $Ω$. More cases with much changed parameters can be obtained using our analytical expressions.

VI. CONCLUSION

In summary, we have theoretically investigated the effect of the kinetic energy on the measurement precision of the effective Rabi frequency. Through transforming the original Hamiltonian into a su(2)-type Hamiltonian, we can obtain the analytical expression of the QFI and CFI which indicate the uncertainty of the frequency.
We can list our main results in the follows. Firstly, under certain parameters, the QFI without the kinetic energy term is not favourable for the metrology of the effective Rabi frequency $\Omega$. Considering this case alone, however, some changes for these parameters, e.g., increasing $\frac{k_0^4}{m^2}$, will increase the Fisher information (both QFI and CFI). Secondly, the QFI with kinetic energy is much larger, so we anticipate better CFI. Although we found that the population difference measurement is not adequate for CFI, inclusion of the momentum measurement will help a lot. Some changes on the parameter, e.g., variance of the initial state, will make CFI almost approach QFI. In conclusion, the inappropriate omission of the kinetic energy term will cause a negative effect on measurement of effective Rabi frequency with some parameters. The Fisher information in different regions of parameters can be deduced from our expressions and they can contribute to the precise measurement of the effective Rabi frequency in realistic situations.

ACKNOWLEDGMENTS

Xiaoguang Wang proposed the idea and Xingyu Zhang did the calculations and wrote this article. This work was supported by the National Key Research and Development Program of China (Grants No. 2017YFA0304202 and No. 2017YFA0205700), the NSFC (Grants No. 11875231 and No. 11935012), and the Fundamental Research Funds for the Central Universities through Grant No. 2018FZA3005.

Appendix: Detailed calculations of QFI

We use the crucial result of the article [10]. For the $\text{su}(2)$-type Hamiltonian $H = r(\phi) \cdot J$, we focus on a unitary parametrization $U = e^{-i\hat{H}(t)}$ (here we set $\hbar = 1$ and we can recover it after the whole calculation using dimension analysis). Then we can define a Hermitian operator $\mathcal{H} = i(\partial_\phi U^\dagger)U$ and a velocity vector $v = \frac{d\phi}{dt}$. For an initial pure state $|\psi\rangle$, we have the quantum Fisher information for the final state: $F = 4(\langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2)$, where the

$$\mathcal{H} = \frac{(r \cdot v)(\sin|\rho|t - |\rho|t)}{|\rho|^3} - \frac{\sin|\rho|t}{|\rho|^2} \cdot J + \frac{1 - \cos|\rho|t}{|\rho|^2} (r \times v) \cdot J$$

Case 1: the QFI with the kinetic energy,

$$\mathcal{H}_1 = i(\partial_\rho \hat{U}^\dagger)\hat{U} = i\hat{W}(\partial_\rho e^{i\hat{H}(t)})e^{-i\hat{H}(t)}\hat{W}^\dagger$$

$$= i\hat{W}(\partial_\rho e^{i\hat{r}})e^{-i\hat{r}}\hat{W}^\dagger$$

$$= \hat{W} \mathcal{H}' \hat{W}^\dagger$$

where $\hat{r} = (\Omega, 0, \frac{k_0}{m}\vec{p} - \frac{k_0}{2m})$, $\vec{v} = (1, 0, 0)$.

$$\mathcal{H}' = i(\partial_\Omega e^{i\hat{r}})e^{-i\hat{r}} = \hat{R} \cdot \vec{\sigma}$$

where

$$\hat{R} = \left( \frac{\Omega^2 (\sin(|\rho|t) - |\rho|t)}{2|\rho|^3} - \frac{\sin(|\rho|t)}{2|\rho|^2}, \right.$$

$$\left. \frac{1 - \cos|\rho|t}{2|\rho|^2} \left( \frac{k_0}{m}\vec{p} - \frac{k_0}{2m} \right), \right.$$

$$\left. \Omega (\sin(|\rho|t) - |\rho|t) \left( \frac{k_0}{m}\vec{p} - \frac{k_0}{2m} \right) \right)$$

Since we can easily get the equation,

$$\langle p|\psi_0\rangle = \int_{-\infty}^{\infty} \langle p|z\rangle \langle z|\psi_0\rangle = \frac{\sqrt{\sigma}}{\sqrt{\pi}} e^{-\frac{p^2}{2\sigma}}$$

and

$$\hat{W}^\dagger|a, \psi_0\rangle = \int_{-\infty}^{\infty} e^{i\frac{k_0}{2\sigma}p} dp \langle p|\psi_0\rangle |a\rangle$$

$$= \int_{-\infty}^{\infty} dp \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{p^2}{2\sigma^2}} e^{-\frac{k_0}{2\sigma^4}p} |a\rangle$$

we can simplify the above as

$$\hat{W}^\dagger|a, \psi_0\rangle = \int_{-\infty}^{\infty} dp \frac{1}{\sqrt{\pi\sigma^2}} e^{-\frac{p^2}{2\sigma^2}} |p + \frac{k_0}{2}\rangle |a\rangle$$

Finally we obtain

$$\langle \mathcal{H}_1 \rangle = -\frac{\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} dp e^{-p^2} \frac{1}{\sigma^2} \langle p + \frac{k_0}{2}\rangle |\hat{R}_z|p + \frac{k_0}{2}\rangle$$

and

$$\langle \mathcal{H}_1^2 \rangle = \frac{\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} dp e^{-p^2 \sigma^2} \langle p + \frac{k_0}{2}\rangle |\hat{R}_z^2|p + \frac{k_0}{2}\rangle$$
Case 2: the QFI without the kinetic energy,

\[ \mathcal{H}_2 = i(\partial_t U) U_2 = i\dot{W}(\partial_t e^{i\mathcal{H}_2}) e^{-it\mathcal{H}_2} \hat{W}^\dagger \]

\[ = i\dot{W}(\partial_\Omega e^{i(\Omega, J)}) e^{-it(\Omega, J)} \hat{W}^\dagger \]

\[ = \hat{W}\mathcal{H}\hat{W}^\dagger \quad (10) \]

where \( J = \vec{\sigma}/2, \vec{r} = (\Omega, 0, -k^2/2m), \vec{v} = \frac{d\vec{r}}{dt} = (1, 0, 0). \)

\[ \mathcal{H} = \Omega(\sin(|\vec{r}|t) - |\vec{r}|t) \frac{\Omega}{2}\sigma_x - \frac{k^2}{4m}\sigma_z \]

\[ - \sin(|\vec{r}|t) \sigma_x - \frac{1}{2} - \cos(|\vec{r}|t) - \frac{k^2}{4m}\sigma_y \]

\[ = \vec{R} \cdot \vec{\sigma} \quad (11) \]

where

\[ \vec{R} = \left( \frac{\Omega^2(\sin(|\vec{r}|t) - |\vec{r}|t)}{2|\vec{r}|^3}, -\sin(|\vec{r}|t) \frac{2|\vec{r}|}{|\vec{r}|^3}, -\frac{k^2(1 - \cos(|\vec{r}|t))}{4m|\vec{r}|^3}, -\frac{\Omega k^2(\sin(|\vec{r}|t) - |\vec{r}|t)}{4m|\vec{r}|^3} \right) \quad (12) \]

Since the Fisher Information \( F = 4\langle (\mathcal{H}_2^2) - (\mathcal{H}_2)^2 \rangle \), the initial state is \( |a, \psi_0\rangle \), where \( \langle \bar{z}|\psi_0\rangle = \frac{1}{\sqrt{2\pi}} \exp(-\frac{\bar{z}^2}{2\pi}) \).

Then due to \( \hat{W}^\dagger |a, \psi_0\rangle = |a\rangle e^{iW}|\psi_0\rangle \), the z-space doesn’t contribute to the final Fisher information.

\[ F = 4\langle (\vec{R} \cdot \vec{\sigma})^2 \rangle - R_z^2 \]

\[ = R_x^2 + R_y^2 \]

\[ = \frac{8m^2}{(k_0^4 + 4m^2\Omega^2)^3} \left[ k_0^8 \left( 1 - \cos \left( \frac{1}{2t} \sqrt{\frac{k_0^4}{m^2} + 4\Omega^2} \right) \right) \right. \]

\[ + k_0^4m^2\Omega^2(4\cos \left( \frac{1}{2t} \sqrt{\frac{k_0^4}{m^2} + 4\Omega^2} \right) + 2t^2\Omega^2 + 3 \]

\[ + 2t\sqrt{\frac{k_0^4}{m^2} + 4\Omega^2} \sin \left( \frac{1}{2t} \sqrt{\frac{k_0^4}{m^2} + 4\Omega^2} \right) + 8m^4t^2\Omega^6 \right] \quad (13) \]

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