NONLOCAL SYMMETRIES, CONSERVATION LAWS, AND RECURSION OPERATORS OF THE VERONESE WEB EQUATION

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Abstract. We study the Veronese web equation $u_y u_{tx} + \lambda u_x u_{ty} - (\lambda + 1)u_t u_{xy} = 0$ and using its isospectral Lax pair construct two infinite series of nonlocal conservation laws. In the infinite differential coverings associated to these series, we describe the Lie algebras of the corresponding nonlocal symmetries. Finally, we construct a recursion operator and explore its action on nonlocal shadows. The operator provides a new shadow which serves as a master-symmetry.

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Introduction

This work finalizes our research of Lax-integrable (i.e., admitting Lax pairs with non-vanishing spectral parameter) linearly degenerate (in the sense of [11]) 3D equations, see [1, 2, 3, 4, 12, 13]. We deal here with the Veronese web equation (VWE)

\[ u_y u_{tx} + \lambda u_x u_{ty} - (\lambda + 1) u_t u_{xy} = 0, \tag{1} \]

which is a generic case of the so-called ABC-equation, \( A + B + C = 0 \), introduced in [20] (see also [10, 18]). Here \( \lambda \neq 0 \) is a real parameter. This equation determines three-dimensional Veronese webs that appear in the study of three-dimensional bi-Hamiltonian systems, see [8] and references therein. In [10] a one-to-one correspondence between three-dimensional Veronese webs and Lorentzian Einstein–Weyl structures of hyper-CR type was found. The latter are parametrized by the solutions of the hyper-CR equation

\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \tag{2} \]

which is a symmetry reduction of Plebański’s second heavenly equation, see [9]. In [23] it was shown that equations (1) and (2) are related by a Backlund transformation. This transformation produces, inter alia, non-local conservation laws for equation (2) from local conservation laws of equation (1), see [19].

Equation (1) admits a Lax pair with a non-vanishing spectral parameter [20], see also [7], and we use the scheme applied before in similar situations: we expand this Lax pair in formal series with respect to the spectral parameter and “cut” the result into two infinite-dimensional coverings called the positive and negative ones \( \tau^+ \) and \( \tau^- \), resp., see Section 1. Then, in Section 2 we give a full description of the Lie algebras formed by nonlocal symmetries in these coverings. The arising algebras are infinite-dimensional and possess quite an interesting structures, to our opinion. In Section 3 we construct two mutually inverse recursion operators and study their action on shadows of nonlocal symmetries. An interesting feature of the VWE which distinguishes it from other linearly degenerate equations is that the recursion operators generate a new symmetry in the Whitney product of \( \tau^+ \) and \( \tau^- \). This is a master-symmetry.

In the subsequent exposition, we use definitions and constructions from the geometric theory of PDEs [6] and its nonlocal version [10]. A concise exposition of the necessary material can be found in the preliminary parts of [12] or [4]. Everywhere below we omit the proofs that are accomplished by straightforward computations.

1. The Veronese web equation and its coverings

Rewrite Equation (1) in the form

\[ u_{ty} = \frac{(\lambda + 1)u_t u_{xy} - u_y u_{tx}}{\lambda u_x} \tag{3} \]

and choose the functions

\[ x, \ y, \ t, \ u_x, \ u_y, \ u_{x^k y^l}, \ u_{x^k y^{l+1}}, \ k \geq 0, \ l > 0, \]

for internal coordinates on the infinite prolongation \( \mathcal{E} \) of VWE, where

\[ u_{x^i y^j t^k} = \frac{\partial^{i+j+k} u}{\partial x^i \partial y^j \partial t^k}. \]

Then the total derivatives on \( \mathcal{E} \) acquire the form

\[
\begin{align*}
D_x &= \frac{\partial}{\partial x} + \sum_k u_{x^k+1} \frac{\partial}{\partial u_{x^k}} + \sum_{k,l} \left( u_{x^{k+1} y^l} \frac{\partial}{\partial u_{x^{k+1} y^l}} + u_{x^k+1 l} \frac{\partial}{\partial u_{x^k y^{l+1}}} \right), \\
D_y &= \frac{\partial}{\partial y} + \sum_k u_{x^k y} \frac{\partial}{\partial u_{x^k}} + \sum_{k,l} \left( u_{x^k y^{l+1}} \frac{\partial}{\partial u_{x^k y^{l+1}}} + D^k_x D^{-1}_t (R) \frac{\partial}{\partial u_{x^k y^l}} \right), \\
D_t &= \frac{\partial}{\partial t} + \sum_k u_{x^k t} \frac{\partial}{\partial u_{x^k}} + \sum_{k,l} \left( D^k_x D^{-1}_t (R) \frac{\partial}{\partial u_{x^k y^l}} + u_{x^k y^{l+1}} \frac{\partial}{\partial u_{x^k y^{l+1}}} \right),
\end{align*}
\]

where \( R \) is the right-hand side of (3).
The Lax pair for the VWE is
\[ w_t = \frac{\mu(\lambda + 1)}{\lambda(\mu + 1)} u_x u_y, \quad w_y = \frac{\mu u_y}{\lambda u_x}, \]  
(5)
where \( \mu \in \mathbb{R} \) is the spectral parameter. Expanding \( w = \sum_i \mu^i w_i \), one obtains
\[ w_{i,t} = \frac{(\lambda + 1)}{\lambda} \frac{u_t}{u_x} w_{i-1,x} - w_{i-1,t}, \quad w_{i,y} = \frac{1}{\lambda} \frac{u_y}{u_x} w_{i-1,x}, \]
(6)
where \( i \in \mathbb{Z} \). Setting \( w_i = 0 \) for \( i < 0 \), we obtain the positive covering; if we set \( w_i = 0 \) for \( i > 0 \) the negative covering arises.

1.1. The positive covering. The defining equations for the positive covering \( \tau^+ : \mathcal{E}^+ \rightarrow \mathcal{E} \) are
\[ q_{i,t} = \frac{(\lambda + 1)}{\lambda} \frac{u_t}{u_x} q_{i-1,x} - q_{i-1,t}, \quad q_{i,y} = \frac{u_y q_{i-1,x}}{\lambda u_x}, \]
(7)
where \( i \geq 1 \) and we formally set \( q_0 = x \). Nonlocal variables in the covering \( \tau^+ : \mathcal{E}^+ \rightarrow \mathcal{E} \) are \( q_i^k \), \( i \geq 1, k \geq 0 \) (in particular, \( q_i^0 = q_i \)), and the total derivatives on \( \mathcal{E}^+ \) are
\[ \tilde{D}_x = D_x + \sum_{i,k} q_i^k \frac{\partial}{\partial q_{i,x}^k}, \quad \tilde{D}_y = D_y + \sum_{i,k} \tilde{D}_x^k(q_{i,y}) \frac{\partial}{\partial q_{i,x}^k}, \quad \tilde{D}_t = D_t + \sum_{i,k} \tilde{D}_x^k(q_{i,t}) \frac{\partial}{\partial q_{i,x}^k}, \]
(8)
where \( q_{i,y} \) and \( q_{i,t} \) are given by Equations (7).

Consider the tower of coverings
\[ \mathcal{E}^+ \rightarrow \ldots \rightarrow \mathcal{E}_{i+1}^+ \rightarrow \mathcal{E}_{i}^+ \rightarrow \ldots \rightarrow \mathcal{E}_0^+ = \mathcal{E}, \]
where nonlocal variables in \( \mathcal{E}_{i}^+ \) are \( q_{\alpha,x}^k, 1 \leq \alpha \leq i, k \geq 0 \).

**Proposition 1.** The 2-forms
\[ \omega_{i+1}^k = \left( \tilde{D}_x^k(q_{i+1,y}) dy + \tilde{D}_x^k(q_{i+1,t}) dt \right) \land dx \]
are linearly independent 2-component conservation laws on \( \mathcal{E}_{i}^+ \).

When proving this statement, as well as Proposition 2, we use the following fact (see [14]): Let \( \mathcal{E} \) be a differentially connected equation (i.e., an equation such that the only functions on \( \mathcal{E} \) annihilated by all total derivatives are constants) and let \( \tau_\Omega : \mathcal{E}_\Omega \rightarrow \mathcal{E} \) be an Abelian covering associated with a system of conservation laws \( \Omega = \{ \omega_1, \ldots, \omega_s \} \). Then these conservation laws are linearly independent if and only if the covering equation is differentially connected as well.

**Proof.** The proof uses a double induction: on \( i \) and on \( k \) for each \( i \). Obviously, VWE is differentially connected; denote by \( \mathcal{K}_s \) the space of functions that are annihilated by \( \tilde{D}_x, \tilde{D}_y, \tilde{D}_t \) and have jet order \( \leq s \). Let \( f \in \mathcal{K}_s \).

**Step 1** \((i = 1, k = 0)\). We have
\[ \tilde{D}_y = D_y + \frac{1}{\lambda} \frac{u_y}{u_x} \frac{\partial}{\partial q_1}, \]
in this case, from where it follows that \( s = 0 \), i.e., \( f = f(x,y,t,u,q_1) \). Hence, \( f \) must be invariant w.r.t.
\[ \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + \frac{1}{\lambda} \frac{u_y}{u_x} \frac{\partial}{\partial q_1}. \]
But \( f \) is independent of \( u_x \); consequently, it does not depend on \( q_1 \).

**Step 2** \((i = 1, k > 0)\). One has
\[ \tilde{D}_y = D_y + \frac{1}{\lambda} \sum_{a=0}^{k} \tilde{D}_x^a \left( \frac{u_y}{u_x} \right) \frac{\partial}{\partial q_{1,x}^a}. \]
The maximal jet order of the nonlocal summand is \( k + 1 \) and the variables of this order are \( u_{x,k+1} \) and \( u_{x^k,y} \). Using expression (4) for \( \tilde{D}_y \), we see that
\[ Z_1 = \left[ \frac{\partial}{\partial u_{x^{k+1}}}, \tilde{D}_y \right] = \frac{\lambda + 1}{\lambda} \frac{u_t u_{xy}}{u_x} - \frac{u_y u_{xt}}{u_x} \frac{\partial}{\partial u_{x^k t}} - \frac{1}{\lambda} \frac{u_y}{u_x} \frac{\partial}{\partial q_{1,x}^k}. \]
On the other hand, since the coefficients of $\tilde{D}_y$ do not depend on $u_{y^l}$, $l > 1$, the function $f \in \mathcal{K}_s$ cannot depend on $u_{y^l}$, $l > 0$. Consequently, it must be invariant w.r.t.

$$Z_2 = \left[ \frac{\partial}{\partial u_y}, Z_1 \right] = \frac{\lambda + 1}{\lambda} \frac{u_x}{u_x} \frac{\partial}{\partial u_{x^l}} - \frac{1}{\lambda} \frac{1}{u_x^2} \frac{\partial}{\partial q_{1,x}^k}.$$ 

But the field $\partial/\partial q_{1,x}^k$ is a linear combination of $Z_1$ and $Z_2$; hence, $f$ does not depend on $q_{1,x}^k$.

**Step 3** ($i > 1, k = 0$). From Equations (8) and (7) we see that

$$\tilde{D}_y = D_y + \frac{1}{\lambda} \sum_{\beta=1}^i \sum_{\alpha=0}^{k_\beta} \tilde{D}_x^\alpha \left( \frac{u_y q_{\beta-1,x}^1}{u_x} \right) \frac{\partial}{\partial q_{\beta,x}^3} = D_y + \frac{1}{\lambda} \sum_{\beta=1}^i \sum_{\alpha=0}^{k_\beta} \left( \frac{u_y q_{\beta-1,x}^1}{u_x} + \alpha \tilde{D}_x \left( \frac{u_y}{u_x} \right) q_{\beta-1,x}^1 + \cdots \right) \frac{\partial}{\partial q_{\beta,x}^3}.$$

This means that the inequalities

$$k_1 > k_2 > \cdots > k_i \tag{9}$$

hold. Consequently, the maximaljet order of the coefficients appears at the term

$$\frac{1}{\lambda} \tilde{D}_x^{k_1} \left( \frac{u_y}{u_x} q_{1,x}^1 \right) \frac{\partial}{\partial q_{1,x}^{k_1}},$$

which means that repeating the reasoning of Step 2 one can prove that $f \in \mathcal{K}_s$ is independent of the variables $q_{1,y}^{k_1}, \ldots, q_{1,x}^{k_{i-1}}$, i.e., $k_1 = k_2$, which is impossible by (9).

**Step 4** ($i > 1, k > 0$). The proof at this step repeats literally the one accomplished at Step 2 for $i = 1$.

The result is proved. \qed

### 1.2. The negative covering

The negative covering $\tau^- : \mathcal{E}^- \rightarrow \mathcal{E}$ is defined by the system

$$r_{i,t} = \frac{(\lambda + 1) u_{r_{i-1,y}} - r_{i-1,t}}{u_y}, \quad r_{i,x} = \frac{\lambda u_x r_{i-1,y}}{u_y} \tag{10}$$

where $i \geq 1$ and $r_0 = y$. Nonlocal variables in $\mathcal{E}^-$ are $r_{i,y}^k, i \geq 1, k \geq 0$ (in particular, $r_{i,x}^0 = r_i$), while the total derivatives take the form

$$\tilde{D}_x = D_x + \sum_{i,k} \tilde{D}_y^k (r_{i,x}) \frac{\partial}{\partial r_{i,y}^k}, \quad \tilde{D}_y = D_y + \sum_{i,k} r_{i,y}^{k+1} \frac{\partial}{\partial r_{i,y}^k}, \quad \tilde{D}_t = D_t + \sum_{i,k} \tilde{D}_x^k (r_{i,x}) \frac{\partial}{\partial r_{i,y}^k}, \tag{11}$$

where $r_{i,x}, r_{i,y}$ are given by Equations (10).

Similar to the positive case, we consider the tower

$$\mathcal{E}^- \longrightarrow \ldots \longrightarrow \mathcal{E}^-_{i-1} \longrightarrow \mathcal{E}^-_i \longrightarrow \ldots \longrightarrow \mathcal{E}^-_0 = \mathcal{E},$$

where nonlocal variables in $\mathcal{E}^-_i$ are $r_{i,y}^k, 1 \leq \alpha \leq i, k \geq 0$.

#### Proposition 2

The 2-forms

$$\theta_{i+1}^k = \left( \tilde{D}_y^k (r_{i+1,x}) \, dx + \tilde{D}_y^k (r_{i+1,t}) \, dt \right) \wedge dy$$

are linearly independent 2-component conservation laws on $\mathcal{E}^-_i$.

**Proof.** The proof of this statement is similar to the one of Proposition 1 but we must work with the field $\tilde{D}_x$ instead of $\tilde{D}_y$. \qed
2. Lie Algebras of Nonlocal Symmetries

Local symmetries of $\mathcal{E}$ are solutions to the linearization
\[ \ell_{\mathcal{E}}(\varphi) = 0 \]  
(12)
of Equation (11), where
\[ \ell_{\mathcal{E}} = (u_yD_xD_t + u_xtD_y) + \lambda(u_xD_tD_y + u_yD_x) - (\lambda + 1)(u_tD_xD_y + u_xyD_t) \]  
(13)
they form a Lie algebra w.r.t. the Jacobi bracket $\{\cdot, \cdot\}$ denoted by $\text{sym}(\mathcal{E})$. The corresponding vector field on $\mathcal{E}$ is the evolution derivation
\[ \mathbf{E}_\varphi = \sum D_\sigma(\varphi) \frac{\partial}{\partial u_\sigma}, \]
where summation is done over all the internal coordinates $u_\sigma$ on $\mathcal{E}$.

Direct computations lead to the following

**Proposition 3.** The algebra $\text{sym}(\mathcal{E})$ is spanned by the elements
\[ \varphi_1(T) = Tu_t, \quad \varphi_2(X) = Xu_x, \quad \varphi_3(Y) = Y u_y, \quad \varphi_4(U) = U, \]
where $T = T(t)$, $X = X(x)$, $Y = Y(y)$, $U = U(u)$ are arbitrary smooth functions. The non-zero commutators are
\[ \{\varphi_1(T), \varphi_1(T)\} = \varphi_1([\tilde{T}, T]), \quad \{\varphi_2(X), \varphi_2(\tilde{X})\} = \varphi_2([\tilde{X}, X]), \]
\[ \{\varphi_3(Y), \varphi_3(\tilde{Y})\} = \varphi_3([\tilde{Y}, Y]), \quad \{\varphi_4(U), \varphi_4(\tilde{U})\} = \varphi_4([\tilde{U}, U]), \]
where $[Z, \tilde{Z}]$ denotes $Z \partial \tilde{Z}/\partial z - \tilde{Z} \partial Z/\partial z$ for any functions $Z$ and $\tilde{Z}$ in $z$.

2.1. The algebra $\text{sym}_{\tau^+}(\mathcal{E})$. To find the Lie algebra $\text{sym}_{\tau^+}(\mathcal{E})$ of nonlocal symmetries in the positive covering $\tau^+$, one needs to solve the following system:
\[ \tilde{\ell}_{\mathcal{E}}(\varphi) = 0, \]
\[ \tilde{D}_t(\varphi^i) = \frac{\lambda + 1}{\lambda} \left( \frac{u_x \tilde{D}_t(\varphi) - u_t \tilde{D}_x(\varphi)}{u_x^2} q_{i-1,x} + \frac{u_t}{u_x} \tilde{D}_x(\varphi^{i-1}) \right) - \tilde{D}_t(\varphi^{i-1}), \]
\[ \tilde{D}_y(\varphi^i) = \frac{1}{\lambda} \left( \frac{u_x \tilde{D}_y(\varphi) - u_y \tilde{D}_x(\varphi)}{u_x^2} q_{i-1,x} + \frac{u_y}{u_x} \tilde{D}_x(\varphi^{i-1}) \right), \]
(14)
where $\tilde{\ell}_{\mathcal{E}}$ denotes the natural lift of the operator (13) from $\mathcal{E}$ to $\mathcal{E}^+$. Solutions of (14) are denoted by
\[ \Phi = [\varphi, \varphi^1, \ldots, \varphi^i, \ldots], \]
and to any such a $\Phi$ there corresponds the vector field
\[ S_{\Phi} = \mathbf{E}_\varphi + \sum_{i,k} \tilde{D}_x^k(\varphi^i) \frac{\partial}{\partial q_{i,x}} \]
on $\mathcal{E}^+$. Solutions of the first equation in (14) are called nonlocal $\tau^+$-shadows. In particular, local symmetries can be considered as shadows in any covering. If $\varphi$ is a shadow and there exists a nonlocal symmetry $\Phi = [\varphi, \varphi^1, \ldots]$ then we say that this shadow lifts to the covering. Nonlocal symmetries $\Phi$ with $\varphi = 0$ (i.e., with trivial shadows) are called invisible. Given $\Phi$ and $\bar{\Phi}$, one can define their bracket by
\[ \{\Phi, \bar{\Phi}\} = S_{\Phi}(\bar{\Phi}) - S_{\bar{\Phi}}(\Phi), \]
where the action is component-wise.

Consider the vector field
\[ X = \sum_{i=0}^{\infty} (i + 1)q_{i+1} \frac{\partial}{\partial q_i}, \]
(15)
(recall that $q_0 = x$) and set
\[ P_0(X) = X, \quad P_j(X) = \frac{1}{j}X(P_{j-1}(X)), \quad j \geq 1. \]
(16)

**Proposition 4.** All the local symmetries of the VWE can be lifted to the positive covering.
Proof. Denote by $\Phi_\alpha = [\varphi_\alpha, \varphi_\alpha^1, \ldots, \varphi_\alpha^4]$, $\alpha = 1, \ldots, 4$, the lifts to be constructed and set
\[
\varphi_1^\prime(T) = T q_{1,t}, \quad \varphi_2^\prime(X) = X q_{1,x} - P_1(X), \quad \varphi_3^\prime(Y) = Y q_{1,y}, \quad \varphi_4^\prime(U) = 0.
\]
A direct computation shows that the functions $\Phi_1 = \Phi_1(T)$, $\Phi_2 = \Phi_2(X)$, $\Phi_3 = \Phi_3(Y)$, and $\Phi_4 = \Phi_4(U)$ are the desired lifts.

We now construct two series of $\tau^+$-nonlocal symmetries.

The first one, denoted by $\Psi_k^+ = [\psi_k^+, \psi_k^{1,-}, \ldots, \psi_k^{-,+}, \ldots]$, $k \geq 2$, arises as follows. The symmetries $\Psi_2^+$, $\Psi_3^+$, and $\Psi_4^+$ are introduced “by hand”:
\[
\begin{align*}
\psi_2^+ &= (2q_2 - q_1(q_{1,x} - 1)) u_x, \\
\psi_2^{1,i} &= -(i + 2)q_{i+2} - (i + 1)q_{i+1} + 2q_2 q_{i,x} + q_1 (q_{i+1,x} - (q_{1,x} - 1)q_{i,x}) \\
&\quad + \frac{1}{\lambda} ((i + 1)q_{i+1} + iq_i - q_1q_{i,x}); \\
\psi_3^+ &= (3q_3 - 2q_2 q_{1,x} - q_1(q_{2,x} - q_{1,x}^2 + 1)) u_x, \\
\psi_3^{1,i} &= -(i + 3)q_{i+3} + (i + 1)q_{i+1} + 3q_3 q_{i,x} + 2q_2 (q_{i+1,x} - q_{1,x} q_{i,x}) \\
&\quad + q_1 (q_{i+2,x} - q_{1,x} q_{i+1,x} - (q_{2,x} - q_{1,x}^2 + 1)q_{i,x}) \\
&\quad + \frac{1}{\lambda} ((i + 2)q_{i+2} - iq_i - 2q_2 q_{i,x} - q_1(q_{i+1,x} - q_{1,x} q_{i,x})); \\
\psi_4^+ &= (4q_4 - 3q_3 q_{1,x} - 2q_2 (q_2 - q_{2,x}^2)) - q_1 (q_3 - 2q_1 q_{2,x} + q_{1,x}^2 - 1)) u_x, \\
\psi_4^{1,i} &= -(i + 4)q_{i+4} - (i + 1)q_{i+1} + 4q_4 q_{i,x} + 3q_3 (q_{i+1,x} - q_{1,x} q_{i,x}) \\
&\quad + 2q_2 (q_{i+2,x} - q_{1,x} q_{i+1,x} - (q_{2,x} - q_{1,x}^2)q_{i,x}) \\
&\quad + q_1 (q_{i+3,x} - q_{1,x} q_{i+2,x} - (q_{2,x} - q_{1,x}^2) q_{i+1,x} - (q_{3,x} - 2q_1 q_{2,x} + q_{1,x}^2 - 1)q_{i,x}) \\
&\quad + \frac{1}{\lambda} ((i + 3)q_{i+3} + iq_i - 3q_3 q_{i,x} - 2q_2 (q_{i+1,x} - q_{1,x} q_{i,x}) \\
&\quad - q_1 (q_{i+2,x} - q_{1,x} q_{i+1,x} - (q_{2,x} - q_{1,x}^2) q_{i,x}));
\end{align*}
\]
For $k > 4$, we set
\[
\Psi_k^+ = \frac{1}{k - 4} \left( \{ \Psi_{k-2}^+, \Psi_2^+ \} - (k - 3) \Psi_{k-1}^+ + (-1)^k \Psi_3^+ \right)
\]
\[
+ \frac{1}{\lambda} \left( (k - 4) \Psi_{k-1}^+ + (k - 3) \Psi_{k-2}^+ + (-1)^k \Psi_2^+ \right).
\]

Now, the second series $\Xi_k^+(X) = [\xi_k^+(X), \xi_k^{1,-}(X), \ldots, \xi_k^{-,+}(X), \ldots]$, $k \geq 1$, is defined by the relations
\[
\begin{align*}
\xi_1^+(X) &= (X q_{1,x} - X x q_{1}) u_x, \\
\xi_1^{1,i}(X) &= X (q_{1,x} q_{i,x} - q_{i+1,x}) - X_x q_{1} q_{i,x} + P_{i+1}(X) + \frac{1}{\lambda} (X q_{i,x} - P_{i}(X)); \\
\xi_2^+(X) &= (X (q_{2,x} - q_{2,x}^2) + X_x q_{1} q_{1,x} - q_{2} - \frac{1}{2} X_{xx} q_{1,x}^2) u_x, \\
\xi_2^{1,i}(X) &= X (q_{1,x} q_{i+1,x} + (q_{2,x} - q_{2,x}^2) q_{i,x} - q_{i+2,x}) - X_x q_{2} q_{i,x} + q_1 (q_{i+1,x} - q_{1,x} q_{i,x}) \\
&\quad - \frac{1}{2} X_{xx} q_{1,x}^2 q_{i,x} + P_{i+2}(X) + \frac{1}{\lambda} (X (q_{i+1,x} - q_{1,x} q_{i,x}) + X_x q_{1} q_{i,x} - P_{i+1}(X)),
\end{align*}
\]
where the functions $P_i(X)$ are given by relations [10], and
\[
\Xi_k^+(X) = \frac{1}{k - 3} \left( \{ \Xi_{k-2}^+, \Psi_2^+ \} - \Xi_{k-1}^+(X) + \frac{1}{\lambda} (k - 3) (\Xi_{k-1}^+(X) + \Xi_{k-2}^+(X)) \right)
\]
for $k \geq 3$.

Finally, invisible symmetries in $\tau^+$ are
\[
\Phi_k^\text{inv}(X) = [0, \ldots, 0, P_0(X), P_1(X), \ldots],
\]
where \( k \geq 1 \) and \( P_i(X) \) are given by (10), as above.

To describe the Lie algebra structure in \( \text{sym}_+(\mathcal{E}) \), it is convenient to relabel the above introduced symmetries. Namely, we change the generators of \( \text{sym}_+(\mathcal{E}) \) as follows:

\[
\Phi_2(X) \mapsto -\Xi^+_0(X), \quad \Phi^\text{inv}_k(X) \mapsto \Xi^+_{-k}(X), \quad k \geq 1,
\]

and

\[
\Psi^+_k \mapsto (-1)^{k+1}\Psi^+_k, \quad k \geq 2, \quad \Xi^+_k(X) \mapsto (-1)^k\Xi^+_k(X), \quad k \in \mathbb{Z}.
\]

**Proposition 5.** In the new basis, the Lie algebra structure of \( \text{sym}_+(\mathcal{E}) \) is given by the brackets

\[
\{\Psi^+_i, \Psi^+_j\} = (j-i)\left(\Psi^+_i + \frac{1}{\lambda}\Psi^+_i - 1\right) - (j-1)\left(\Psi^+_j + \frac{1}{\lambda}\Psi^+_j - 1\right) + (i-1)\left(\Psi^+_i + \frac{1}{\lambda}\Psi^+_i - 1\right)
\]

for all \( j > i \geq 2 \). One also has

\[
\{\Xi^+_i(X), \Xi^+_j(X)\} = \begin{cases} 
\Xi^+_i(X)\Xi^+_j(X), & i, j \leq 0 \text{ or } i < 0, j > 0, i + j > 0, \\
\Xi^+_i(X)\Xi^+_j(X) + \frac{1}{\lambda}\Xi^+_i j^{-1}(X), & \text{otherwise,}
\end{cases}
\]

\[
i, j \in \mathbb{Z}, \text{ and } \{\Psi^+_i, \Xi^+_j(X)\} = \begin{cases} 
\frac{j\Xi^+_i(X) - \Xi^+_j(X)}{\lambda}, & j \geq 1, \\
\frac{j\Xi^+_i(X) - \Xi^+_j(X)}{\lambda} + \frac{1}{\lambda}\Xi^+_i j^{-1}(x), & j < 1, i + j > 0, \\
\frac{j\Xi^+_i(X) - \Xi^+_j(X)}{\lambda} - \frac{1}{\lambda}\Xi^+_i j^{-1}(X) - \Xi^+_j(X)), & \text{otherwise,}
\end{cases}
\]

\[
i \geq 2, j \in \mathbb{Z}. \text{ All the other commutators vanish.}
\]

2.2. The algebra \( \text{sym}_{-+}(\mathcal{E}) \). Computations here go along the same lines as in Subsection 2.1 and we use similar notation below. The defining equations are

\[
\tilde{\xi}_\varphi(\varphi) = 0,
\]

\[
\tilde{D}_t(\varphi^i) = (\lambda + 1)\left(\frac{u_x\tilde{D}_t(\varphi) - u_t\tilde{D}_x(\varphi)}{u^2} r_{i-1,y} + \frac{u_t}{u_y}\tilde{D}_y(\varphi^i - 1)\right) - \tilde{D}_t(\varphi^i - 1),
\]

\[
\tilde{D}_x(\varphi^i) = \lambda\left(\frac{u_y\tilde{D}_x(\varphi) - u_x\tilde{D}_y(\varphi)}{u^2} r_{i-1,y} + \frac{u_x}{u_y}\tilde{D}_y(\varphi^i - 1)\right),
\]

where “tilde” marks operators on \( \mathcal{E}^- \). A solution \( \Phi = [\varphi, \varphi^1, \ldots, \varphi^i, \ldots] \) of (17) corresponds to the vector field

\[
S_\Phi = E_\Phi + \sum_{i,k} \frac{\tilde{D}^k_y(\varphi)}{\partial r_{i,k}}
\]

on \( \mathcal{E}^- \) and the bracket \( \{\Phi, \Phi\} = S_\Phi(\Phi) - S_\Phi(\Phi) \) is defined for such solutions.

To proceed with further constructions, we will introduce the vector field

\[
y = \sum_{i=0}^{\infty} (i + 1)r_{i+1} \frac{\partial}{\partial r_{i}}
\]

and the quantities \( Q_j, j = 0, 1, \ldots \) defined as follows:

\[
Q_0(Y) = Y, \quad Q_j(Y) = \frac{1}{j}y(Q_{j-1}(Y)), \quad j \geq 1,
\]

(recall that \( r_0 = y \)).

**Proposition 6.** All the local symmetries of the WVE can be lifted to the negative covering.

**Proof.** Denote the lifts by \( \Phi_\alpha = [\varphi_\alpha, \varphi^1_\alpha, \ldots, \varphi^i_\alpha, \ldots] \), \( \alpha = 1, \ldots, 4 \), and set

\[
\varphi^1_1(T) = Tr_{i,t}, \quad \varphi^2_1(X) = Xr_{i,x}, \quad \varphi^3_2(Y) = Yr_{i,y} - Q_i(Y), \quad \varphi^4_1(U) = 0.
\]

The rest of proof is a straightforward check. \( \square \)
Let us now construct, similar to the positive case, two series of nonlocal symmetries. The first one \( \Psi^{\text{inv}}_{-} = [\psi^{\text{inv}}_{-}, \psi^{\text{inv}}_{-1}, \ldots, \psi^{\text{inv}}_{-k}, \ldots] \), \( k \geq 2 \), is defined as follows. For \( k = 2, 3, 4 \) we set
\[
\psi_2^{-} = (2r_2 - r_1(r_{1,y} - 1))u_y,
\]
\[
\psi_3^{-} = -(i + 2)r_{i+2} + (i + 1)r_{i+1} + 2r_2r_{i,y} + r_1(r_{i+1,y} - (r_{1,y} - 1)r_{i,y}) + \lambda((i + 1)r_{i+1} + ir_i - r_1r_{i,y}),
\]
\[
\psi_4^{-} = (3r_3 - 2r_2r_{1,y} - r_1(r_{2,y} - r_{1,2,y} + 1))u_y,
\]
where
\[
\psi_i^{-} = -(i + 3)r_{i+3} + (i + 1)r_{i+1} + 3r_3r_{i,y} + 2r_2(r_{i+1,y} - r_{1,y}r_{i,y}) + r_1(r_{i+2,y} - r_1y)r_{i+y} + (r_{2,y} - r_{1,2,y} + 1)r_{i,y}) + \lambda((i + 2)r_{i+2} - ir_i - 2r_2r_{i,y} - r_1(r_{i+1,y} - r_{1,y}r_{i,y})).
\]

For \( k > 4 \) we define
\[
\psi_k^{-} = \frac{1}{k - 4} \left( \{ \psi_{k-2}^{-}, \psi_2^{-} \} - (k - 3)\psi_{k-1}^{-} + (-1)^{k}\psi_3^{-} \right) + \lambda \left( (k - 4)\psi_{k-1}^{-} + (k - 3)\psi_{k-2}^{-} + (-1)^{k}\psi_2^{-} \right).
\]

Introduce the second series \( \Xi^{-}_{-} = [\xi^{-}_{-}(Y), \xi^{-}_{-1}(Y), \ldots, \xi^{-}_{-k}(Y), \ldots] \) now by
\[
\xi_1^{-}(Y) = (Yr_{1,y} - Y_1)r_1u_y,
\]
\[
\xi_2^{-}(Y) = (Yr_{2,y} - r_{1,2,y})u_y,
\]
\[
\xi_3^{-}(Y) = (Yr_{3,y} - r_{2,3,y})u_y,
\]
and for \( k \geq 3 \)
\[
\Xi^{-}_{-} = \frac{1}{k - 2} \left( \{ \Xi^{-}_{k-2}(Y), \psi_2^{-} \} - \Xi^{-}_{k-1}(Y) + \lambda(k - 3)(\Xi^{-}_{k-1}(Y) + \Xi^{-}_{k-2}(Y)) \right).
\]

Invisible symmetries in \( \tau^{-} \) have the form
\[
\Phi^{\text{inv}}_{-} = \left[ 0, \ldots, 0, Q_0(Y), Q_1(Y), \ldots \right]^{k-\text{times}}
\]
where \( k \geq 1 \) and \( Q_j(Y) \) are given by (19).

We again relabel the generators by
\[
\Phi_3(Y) \mapsto -\Xi^{-}_{-}(Y), \quad \Phi^{\text{inv}}_{-} \mapsto \Xi^{-}_{-}(Y), \quad k \geq 1,
\]
and
\[
\Psi_k^{-} \mapsto (-1)^{k+1}\psi_k^{-}, \quad k \geq 2, \quad \Xi^{-}_{-}(Y) \mapsto (-1)^{k}\Xi^{-}_{-}(Y), \quad k \in \mathbb{Z}.
\]

Then the following statement holds:

**Proposition 7.** The above defined generators enjoy the following relations:
\[
\{ \psi_i^{-}, \psi_j^{-} \} = (j - i)(\psi_{i+j}^{-} + \lambda \psi_{i+j-1}^{-}) - (j - 1)(\psi_{j+1}^{-} + \lambda \psi_j^{-}) + (i - 1)(\psi_{i+1}^{-} + \lambda \psi_i^{-})
\]
for \( j > i \geq 2, \)
\[
\{ \Xi_i^-(Y), \Xi_j^+(Y) \} = \begin{cases} 
\Xi_{i+j}^-([Y, \bar{Y}]), & i \leq 0, j > 0, i + j > 0 \text{ or } j \leq 0, \\
\Xi_{i+j}^+([Y, \bar{Y}]) + \lambda \Xi_{i+j-1}^-([Y, \bar{Y}]), & \text{otherwise}, 
\end{cases}
\]
for all \( i < j \in \mathbb{Z}, \) and
\[
\{ \Psi_i^-, \Xi_j^-(Y) \} = \begin{cases} 
\left\{ \begin{array}{ll}
 j (\Xi_{i+j}^-(Y) - \Xi_{i+j}^+(Y)) + \lambda ((j-1)\Xi_{i+j-1}^+(Y) - \Xi_j^-(Y)) & j \geq 1, \\
 j (\Xi_{i+j}^-(Y) - \Xi_{i+j}^+(Y)) & j < 1, i + j > 0, \\
 j (\Xi_{i+j}^-([Y, \bar{Y}]) + \lambda (\Xi_{i+j-1}^-([Y, \bar{Y}]) - \Xi_j^-(Y))) & \text{otherwise}, 
\end{array} \right.
\end{cases}
\]
for \( j \in \mathbb{Z}, i \geq 2. \)

3. Recursion operators and a master-symmetry

According to the general theory, see [21], recursion operators for symmetries arise as B"acklund auto-transformations of the tangent space \( T\mathcal{E}, \) cf. [16]. In the case of VWE, this BT is
\[
u_x \tilde{D}_t(\zeta) = u_{tx} \zeta - u_x \tilde{D}_t(\eta) + \frac{\lambda + 1}{\lambda} u_t \tilde{D}_x(\eta) - \frac{1}{\lambda} u_{tx} \eta,
\]
\[
u_x \tilde{D}_y(\zeta) = u_{xy} \zeta + \frac{1}{\lambda} u_y \tilde{D}_x(\eta) - \frac{1}{\lambda} u_{xy} \eta.
\]

Proposition 8. Let \( \eta \) be a \( + \)-shadow. Then \( \zeta = \mathcal{R}_+(\eta) \) obtained as a solution of (21) is a shadow as well. Vice versa, if \( \eta \) is a shadow the \( \zeta = \mathcal{R}_-(\eta) \) obtained in the same way is a shadow too.

Proof. To construct a recursion operator for Equation (11) we use the techniques of [25], cf. [20][22][24][17] also. We find a shadow for VWE in the covering [13]. It is of the form \( s = H(w) u_x w_x^{-1} \), where \( H \) is an arbitrary function in \( w \). Since System (15) is invariant with respect to the transformation \( w \mapsto H(w) \), we put, without loss of generality, \( s = u_x w_x^{-1} \). Differentiation of (15) by \( x \) and substitution \( q_x = u_x s^{-1} \) gives another covering
\[
s_1 = \frac{\mu (\lambda + 1)}{\mu + 1} u_x \frac{\lambda - \mu}{\lambda} s_x + \frac{\lambda}{\mu} u_x s + \frac{(\lambda - \mu)}{\mu} u_{xy} s,
\]
\[
s_2 = \frac{\mu u_y}{\mu} s_x + \frac{(\lambda - \mu)}{\mu} u_x s + \frac{(\lambda - \mu)}{\mu} u_{xy} s
\]
for Equation (11). Note that \( s \) is a solution to the linearization (12), (13) of VWE. Now put
\[
s = \sum_{n=-\infty}^{\infty} s_n \mu^n.
\]

Since system (12), (13) is independent of \( \mu \), each \( s_n \) is a solution to this system as well. Substituting (22) to system (12), (13) yields
\[
s_{n+1,t} = \frac{u_{tx}}{u_x} s_{n+1} - s_n, \quad s_n, t + \frac{\lambda + 1}{\lambda} \frac{u_t}{u_x} s_{n,x} - \frac{1}{\lambda} \frac{u_{tx}}{u_x} s_n,
\]
\[
s_{n+1,y} = \frac{u_{xy}}{u_x} s_{n+1} + \frac{1}{\lambda} \frac{u_y}{u_x} s_{n,x} - \frac{1}{\lambda} \frac{u_{xy}}{u_x} s_n.
\]

Relabeling \( s_n = \eta \) and \( s_{n+1} = \zeta \), we obtain the result.

Thus, \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) are mutually inverse recursion operators.

3.1. Action of recursion operators. We now describe the action of the operators \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) in detail. First of all, it immediately follows from (20) that
\[
\mathcal{R}_+(0) = \xi_0^+(X), \quad \mathcal{R}_-(0) = \xi_0^-(Y)
\]
and thus all subsequent actions of \( \mathcal{R}_+ \) and \( \mathcal{R}_- \) are defined modulo addition of \( \xi_0^+(X) \) and \( \xi_0^-(Y) \), respectively.

Further, one has
\[
\mathcal{R}_+(\varphi_1(T)) = -\varphi_1(T), \quad \mathcal{R}_-(\varphi_1(T)) = -\varphi_1(T),
\]
\[
\mathcal{R}_+(\varphi_4(U)) = \lambda^{-1} \varphi_4(U), \quad \mathcal{R}_-(\varphi_4(U)) = \lambda \varphi_4(U)
\]
and
\( \mathcal{R}_+(\xi_i^+(X)) = \xi_{i+1}^+(X), \ i \geq 0; \quad \mathcal{R}_+(\xi_i^-(X)) = \xi_{i-1}^-(X), \ i \geq 1; \quad \mathcal{R}_+(\xi_0^-(X)) = 0; \)
\( \mathcal{R}_-(\xi_i^-(X)) = \xi_{i+1}^-(X), \ i \geq 0; \quad \mathcal{R}_-(\xi_i^+(X)) = \xi_{i-1}^+(X), \ i \geq 1; \quad \mathcal{R}_-(\xi_0^+(X)) = 0. \)

Finally,
\[ \mathcal{R}_+(\psi_i^+) = \psi_{i+1}^+, \ i \geq 2; \quad \mathcal{R}_+(\psi_i^-) = \psi_{i-1}^-, \ i \geq 3; \quad \mathcal{R}_+(\psi_i^-) = \psi_1; \]
\( \mathcal{R}_-(\psi_i^-) = \psi_{i+1}^-, \ i \geq 2; \quad \mathcal{R}_-(\psi_i^+) = \psi_{i-1}^+, \ i \geq 3; \quad \mathcal{R}_-(\psi_2^+) = \psi_1. \)  

The new shadow that arises in Equations (23) “lives” in the Whitney product of \( \tau^+ \) and \( \tau^- \) and has the form
\[ \psi_1 = \lambda u_x q_1 + u_y r_1 \]  
and will be studied in Subsection 3.2.

The following diagram illustrates the above described actions:

3.2. Master-symmetry. Let us describe the lift
\[ \Psi_1 = \psi_1 \frac{\partial}{\partial u} + \sum_i \left( \psi_1^{+,i} \frac{\partial}{\partial q_i} + \psi_1^{-,i} \frac{\partial}{\partial r_i} \right) \]
of the shadow \( \psi_1 \) to the Whitney product \( \tau^+ \oplus \tau^- \). To this end we set
\[ \psi_1^{+,i} = i q_i + (i - 1) q_{i-1} + r_1 q_{i,y} - \lambda ((i + 1) q_{i+1} + i q_i - q_1 q_{i,x}), \]
\[ \psi_1^{-,i} = -(i + 1) r_{i+1} - i r_i + r_1 r_{i,y} + \lambda (i r_i + (i - 1) r_{i-1} + q_1 r_{i,x}), \]
for all \( i \geq 1 \), and this is the desired lift. Then the commutators of \( \Psi_1 \) with the already constructed symmetries are as follows:

\[ \{ \Psi_1, \Psi_i^+ \} = \begin{cases} \lambda \Psi_3^+ - (2\lambda - 1) \Psi_2^+ - \frac{1}{\lambda} \Psi_1, & i = 2, \\ \lambda (i - 1) \Psi_i^+ - ((\lambda - 1)i + 1) \Psi_i^+ - i \Psi_i^+ - 1 \Psi_i^+ - \frac{1}{\lambda} \Psi_1, & i > 2, \end{cases} \]

and

\[ \{ \Psi_1, \Psi_i^- \} = \begin{cases} \Psi_3^- + (\lambda - 2) \Psi_2^- - \lambda \Psi_1, & i = 2, \\ (i - 1) \Psi_i^- + ((\lambda - 1)i - \lambda) \Psi_i^- - \lambda (i \Psi_i^- + \Psi_1), & i > 2. \end{cases} \]

Further, we have
\[ \{ \Psi_1, \Xi_i^+ (X) \} = \begin{cases} i (\lambda \Xi_{i+1}^+ (X) - (i (\lambda - 1) + 1) \Xi_i^+(X) - (i - 1) \Xi_{i-1}^+(X)), & i > 0, \\ 0, & i = 0, \\ i (\lambda \Xi_{i+1}^+ (X) - (\lambda - 1) \Xi_i^+ - \Xi_{i-1}^+ (X)), & i \leq -1, \end{cases} \]

and
\[ \{ \Psi_1, \Xi_i^- (Y) \} = \begin{cases} i (\Xi_{i+1}^- (Y) + (i (\lambda - 1) - \lambda) \Xi_i^- (Y) - \lambda (i - 1) \Xi_{i-1}^- (Y)), & i > 0, \\ 0, & i = 0, \\ (i - 1) (\Xi_{i-1}^- (Y) + (\lambda - 1) \Xi_i^- (Y) - \lambda \Xi_{i+1}^- (Y)), & i \leq -1. \end{cases} \]

Note finally, that \( \{ \Psi_1, \Phi_1(T) \} = \{ \Psi_1, \Phi_3(U) \} = 0. \)

Thus we see that \( \Psi_1 \) plays the role of a master-symmetry: taking \( \Psi_2^+, \Psi_2^-, \Xi_{i+1}^+(X), \Xi_{i+1}^-(Y) \) for “seeds” and acting by \( \{ \Psi_1, \cdot \} \), we can obtain the entire hierarchies \( \Psi_i^+, \Psi_i^-, i > 2, \Xi_i^+(X), \Xi_i^-(Y), i > 1 \), and \( \Xi_i^+(X), \Xi_i^-(Y), i < -1. \)

To conclude, let us compare briefly the Lie algebra structures of nonlocal symmetries for all the five linearly degenerate 3D equations studied in [4] and here. All these algebras are infinite-dimensional. For the rdDym equation \( u_{ty} = u_x u_{xy} - u_y u_{xx}, \) the 3D Pavlov equation [2],
the universal hierarchy equation \( u_{yy} = u_t u_{xy} - u_y u_{tx} \) they are graded. The symmetry algebra of the modified Veronese web equation \( u_{yy} = u_t u_{xy} - u_y u_{tx} \) is filtered (almost-graded). The corresponding algebra for the VWE \([1]\) seemingly admits no reasonable grading or filtering and contains a real irremovable parameter \( \lambda \). It will be interesting to study the properties of this algebra in more detail elsewhere.

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