Nonlocal Dynamics of \( p \)-Adic Strings

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Abstract
We consider the construction of Lagrangians that might be suitable for describing the entire \( p \)-adic sector of an adelic open scalar string. These Lagrangians are constructed using the Lagrangian for \( p \)-adic strings with an arbitrary prime number \( p \). They contain space-time nonlocality because of the d’Alembertian in argument of the Riemann zeta function. We present a brief review and some new results.

1 Introduction

Nonlocal field theory with an infinite number of derivatives has recently attracted much attention. It is mainly based on ordinary and also on \( p \)-adic string theory, which emerged in 1987 \cite{1}. Various kinds of \( p \)-adic strings have been considered, but the most interesting are strings whose worldsheet is \( p \)-adic while all other properties are described by real and complex numbers. Four-point scattering amplitudes of open scalar ordinary and \( p \)-adic strings are connected at the tree level by their product, which is a constant. Ordinary and \( p \)-adic strings are treated on an equal footing in this product (see, e.g. \cite{2,3} for a review). Some other \( p \)-adic structures have also been investigated and \( p \)-adic mathematical physics was established (see \cite{4} for a recent review).

Unlike for ordinary strings, there is an effective nonlocal field theory for the open scalar \( p \)-adic strings with a Lagrangian \cite{5,6} describing four-point scattering amplitudes and all higher ones at the tree-level. It is worth noting that this Lagrangian does not contain \( p \)-adic numbers explicitly, but only the
prime number \( p \) which can be regarded as either a real or a \( p \)-adic parameter. Because this Lagrangian is simple and exact at the tree level, it has been essentially used in the last decade and many aspects of \( p \)-adic string dynamics have been considered, compared with the dynamics of ordinary strings, and applied to nonlocal cosmology (see, e.g. [7, 8, 9, 10, 11] and the references therein).

This paper contains a review and some new results related to constructing a Lagrangian with the Riemann zeta function for the entire \( p \)-adic sector of an open scalar string. Requiring of the Riemann zeta function in the Lagrangian is motivated by the fact that it appears in the product over \( p \) of all four-point \( p \)-adic scalar string amplitudes. In constructing possible Lagrangians, we start from the Lagrangian for a single \( p \)-adic open scalar string. An interesting approach to a field theory and cosmology based on the Riemann zeta function was proposed in [12].

2 Setup: \( p \)-Adic Open Scalar Strings

The \( p \)-adic string theory started analogously to ordinary string theory with scattering amplitudes. Let \( v \in V = \{\infty, 2, 3, \ldots, p, \ldots\} \). The crossing symmetric Veneziano amplitude for scattering of two open scalar strings is defined by the Gel’fand-Graev-Tate beta function

\[
A_v(a, b) = g_v^2 \int_{\mathbb{Q}_v} |x|^a \left| x \right|^{b-1} \mathrm{d}x, \quad (1)
\]

where \( \mathbb{Q}_p \) is the \( p \)-adic number field and \( a = -\alpha(s) = -\frac{s}{2} - 1 \), \( b = -\alpha(t) \) and \( c = -\alpha(u) \) are complex-valued kinematic variables with the condition \( a + b + c = 1 \). We note that the variable \( x \) in the integrands is related to the string worldsheet: the worldsheets of ordinary and \( p \)-adic strings are respectively treated by real and \( p \)-adic numbers (see, e.g. [2, 3] and [13] for the basic properties of \( p \)-adic numbers and their functions). Hence \( p \)-adic strings differ from ordinary strings only by the \( p \)-adic treatment only of the worldsheet. Integrating in (1), one obtains

\[
A_\infty(a, b) = g_\infty^2 \frac{\zeta(1-a)}{\zeta(a)} \frac{\zeta(1-b)}{\zeta(b)} \frac{\zeta(1-c)}{\zeta(c)}, \quad (2)
\]

\[
A_p(a, b) = g_p^2 \frac{1 - p^{a-1}}{1 - p^{-a}} \frac{1 - p^{b-1}}{1 - p^{-b}} \frac{1 - p^{c-1}}{1 - p^{-c}}, \quad (3)
\]
where $\zeta$ is the Riemann zeta function. Expression (2) is for the ordinary case and (3) is for the $p$-adic case.

The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - p^{-s}}, \quad s = \sigma + i\tau, \quad \sigma > 1,$$

has an analytic continuation to the entire complex-$s$ plane excluding the point $s = 1$, where it has a simple pole with unit residue. Taking the product of $p$-adic string amplitudes (3) over $p$ and using (4), we obtain (see, e.g. [20])

$$\prod_v A_v(a, b) = \prod_v g_v^2 = \text{const.}$$

The product of $p$-adic amplitudes in (5) diverges [14], but it converges after an appropriate regularization. Requiring that amplitude product (5) be finite implies that the product of coupling constants is finite, i.e. $g_\infty^2 \prod_p g_p^2 = \text{const.}$

There are three interesting possibilities: (i) $g_p^2 = 1$, (ii) $g_p^2 = \frac{p^{2}}{p^{2} - 1}$, which gives $\prod_p g_p^2 = \zeta(2)$, and (iii) $g_p^2 = \frac{|m|_p}{n|_p}$, where $m$ and $n$ are any two nonzero integers, and this gives $g_\infty^2 \prod_p g_p^2 = \frac{|m|_p}{n|_p} \prod_p \frac{|m|_p}{n|_p} = 1$.

It follows from (5) that the ordinary Veneziano amplitude, which is a special function, can be expressed as the product of all inverse $p$-adic counterparts, which are elementary functions. This is a consequence of the Gel’fand-Graev-Tate beta functions and is not a general property of string scattering amplitudes. In the general case, the string amplitude product is a function of kinematic variables.

Another interpretation of expression (5) is related to an adelic string. But an adelic string should have an adelic worldsheet. A scattering amplitude of two open scalar strings with their adelic worldsheets has not yet been obtained. Therefore, the concept of an adelic string with an adelic worldsheet is not well founded and remains questionable. But $p$-adic strings with a $p$-adic worldsheet are well defined, and the string amplitude product for open scalar strings has a useful meaning.

The exact tree-level Lagrangian of the effective scalar field $\varphi$, which describes the open $p$-adic string tachyon, is [5, 6]

$$\mathcal{L}_p = \frac{m^p}{g_p^2} \frac{p^2}{p - 1} \left[-\frac{1}{2} \varphi \frac{2m^p}{p^{p-1}} \varphi + \frac{1}{p + 1} \varphi^{p+1}\right],$$

(6)
where \( p \) is a prime, \( \Box = -\partial_t^2 + \nabla^2 \) is the \( D \)-dimensional d’Alembertian. The corresponding equation of motion for (6) has been investigated by many authors (see, e.g. [9] and the references therein).

We now want to consider construction of Lagrangians that can be used to describe entire \( p \)-adic sector of an open scalar string. In particular, an appropriate such Lagrangian should describe the scattering amplitude, which contains the Riemann zeta function. Consequently, this Lagrangian must contain the Riemann zeta function with the d’Alembertian in its argument. We should therefore seek possible constructions of a Lagrangian that contains the Riemann zeta function and is closely related to \( p \)-adic Lagrangian (6). There are additive and multiplicative approaches; we mainly consider the additive approach below.

## 3 Additive and Multiplicative Approaches

The prime number \( p \) in (6) can be replaced by any natural number \( n \geq 2 \) and the results make sense. We introduce a Lagrangian incorporating all Lagrangians (6) described above but with \( p \) replaced by \( n \in \mathbb{N} \). The corresponding sum of all Lagrangians \( \mathcal{L}_n \) is

\[
L = \sum_{n=1}^{+\infty} C_n \mathcal{L}_n = m^D \sum_{n=1}^{+\infty} \frac{C_n}{g_n^2} \frac{n^2}{n-1} \left[ -\frac{1}{2} \phi n - \frac{n}{2m^2} \phi + \frac{1}{n + 1} \phi^{n+1} \right], \tag{7}
\]

whose concrete form depends on the choice of the coefficients \( C_n \) and coupling constants \( g_n \). We set

\[
\frac{C_n}{g_n^2} \frac{n^2}{n-1} = D_n, \quad n = 1, 2, ...
\]

The following simple cases lead to the Riemann zeta function: \( D_n = 1, D_n = (-1)^{n-1}, D_n = n + 1, D_n = \mu(n), D_n = -\mu(n)(n + 1), \) and \( D_n = (-1)^{n-1}(n + 1) \), where \( \mu(n) \) is the Möbius function.

The case \( D_n = 1 \) was considered in [15, 16] and the case \( D_n = n + 1 \) was investigated in [17].

The variants with the Möbius function \( \mu(n) \) are described in [18] and [19]. We recall that its explicit definition is

\[
\mu(n) = \begin{cases} 
0, & n = p^2 m, \\
(-1)^k, & n = p_1 p_2 \cdots p_k, \ p_i \neq p_j, \\
1, & n = 1, \ (k = 0),
\end{cases} \tag{8}
\]
and it is related to the inverse Riemann zeta function by
\[ \frac{1}{\zeta(s)} = \sum_{n=1}^{+\infty} \frac{\mu(n)}{n^s}, \quad s = \sigma + i\tau, \quad \sigma > 1. \] (9)

The corresponding Lagrangian for \( D_n = \mu(n) \) is
\[ L = m^D \left\{ -\frac{1}{2} \phi \left[ \frac{1}{\zeta\left(\frac{2m^2}{2m^2 - 1}\right)} \phi + \int_0^\phi M(\phi) \, d\phi \right] \right\}, \] (10)
where \( M(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^n = \phi - \phi^2 - \phi^3 - \phi^5 + \phi^6 - \phi^7 + \phi^{10} - \phi^{11} - \ldots. \)

For \( D_n = -\mu(n)(n + 1) \) the Lagrangian is
\[ L = m^D \left\{ \frac{1}{2} \phi \left[ \frac{1}{\zeta\left(\frac{2m^2}{2m^2 - 1}\right)} \phi + \frac{1}{\zeta\left(\frac{2m^2}{2m^2 - 1}\right)} \phi^2 F(\phi) \right] \right\}, \] (11)
where \( F(\phi) = \sum_{n=1}^{+\infty} \mu(n) \phi^{n-1} = 1 - \phi - \phi^2 - \phi^4 + \ldots. \)

The case with \( D_n = (-1)^{n-1}(n + 1) \) was recently introduced in [20]. We recall that
\[ \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n^s} = (1 - 2^{1-s}) \zeta(s), \quad s = \sigma + i\tau, \quad \sigma > 0, \] (12)
which has an analytic continuation to the entire complex-\( s \) plane without singularities, i.e. the analytic expression [21] is
\[ (1 - 2^{1-s}) \zeta(s) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (k + 1)^{-s}. \] (13)

At point \( s = 1 \), one has \( \lim_{s \to 1} (1 - 2^{1-s}) \zeta(s) = \sum_{n=1}^{+\infty} (-1)^{n-1} \frac{1}{n} = \log 2. \)

Applying (12) to (7) and using analytic continuation we obtain
\[ L = -m^D \left\{ \frac{1}{2} \phi \left\{ (1 - 2^{2-\frac{s}{2m^2}}) \zeta\left(\frac{2m^2}{2m^2 - 1}\right) + (1 - 2^{1-\frac{s}{2m^2}}) \zeta\left(\frac{2m^2}{2m^2 - 1}\right) \right\} \phi - \frac{\phi^2}{1 + \phi} \right\}. \] (14)
We now consider the case $D_n = (-1)^{n-1}$. The corresponding Lagrangian is

$$L = m^D \left[ -\frac{1}{2} \phi \left( 1 - 2^{1-\frac{n}{2m^2}} \right) \zeta \left( \frac{\Box}{2m^2} \right) \phi + \phi - \frac{1}{2} \log(1 + \phi)^2 \right], \quad (15)$$

The potential is

$$V(\phi) = -L(\Box = 0) = m^D \left[ \frac{1}{4} \phi^2 - \phi + \frac{1}{2} \log(1 + \phi)^2 \right], \quad (16)$$

which has one local maximum $V(0) = 0$ and one local minimum at $\phi = 1$. It is singular at $\phi = -1$, i.e. $V(-1) = -\infty$, and $V(\pm \infty) = +\infty$. The equation of motion is

$$\left( 1 - 2^{1-\frac{n}{2m^2}} \right) \zeta \left( \frac{\Box}{2m^2} \right) \phi = \frac{\phi}{1 + \phi}, \quad (17)$$

which has the two trivial solutions: $\phi = 0$ and $\phi = 1$.

The Riemann zeta function arising in the multiplicative approach is given in the form of product (4). The initial Lagrangian is $p$-adic Lagrangian (6) with $g_p^2 = \frac{p^2}{p^2 - 1}$. The Lagrangian obtained in this approach is similar to (11) above. These two Lagrangians describe the same field theory in the weak field approximation.

### 4 Concluding remarks

In the preceding section, we presented some Lagrangians that can be used to describe the $p$-adic sector of open scalar strings. They contain the Riemann zeta function and are also starting points for interesting examples of the so-call zeta field theory. The corresponding potentials, which are $V(\phi) = -L(\Box = 0)$, and equations of motions are considered in the cited references. All these zeta field theory models contain tachyons.

The most interesting of the above Lagrangians are (14) and (15). Unlike the other Lagrangians, these have no singularity with respect to the d’Alembertian $\Box$, and it is easier to apply the pseudodifferential treatment. This analyticity of the Lagrangian is expected to be useful in its application to nonlocal cosmology, in particular, using linearization procedure (see, e.g., [22] and references therein).
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