\textbf{\textit{\gamma-}\textit{FUNCTIONS OF REPRESENTATIONS AND LIFTING}}

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\textbf{Abstract.} Let $F$ be a local non-archimedian field and let $\psi : F \to \mathbb{C}^\ast$ be a non-trivial additive character of $F$. To this data one associates a meromorphic function $\gamma_\psi : \pi \mapsto \gamma_\psi(\pi)$ on the set of irreducible representations of the group $\text{GL}(n,F)$ in the following way. Consider the invariant distribution $\Phi_\psi := \psi(\text{tr}(g))|\text{det}(g)|^n|dg|$ on $\text{GL}(n,F)$, where $|dg|$ denotes a Haar distribution on $\text{GL}(n,F)$. Although the support of $\Phi_\psi$ is not compact, it is well-known that for generic irreducible representation $\pi$ of $\text{GL}(n,F)$ the action of $\Phi_\psi$ in $\pi$ is well-defined and thus it defines a number $\gamma_\psi(\pi)$. These gamma-functions and the associated L-functions were studied by J. Tate for $n = 1$ and by R. Godement and H. Jacquet for arbitrary $n$.

Let now $G$ be the group of points of an arbitrary quasi-split reductive algebraic group over $F$ and let $G^{\vee}$ be the Langlands dual group. Let also $\rho : G^{\vee} \to \text{GL}(n,\mathbb{C})$ be a finite-dimensional representation of $G^{\vee}$. The local Langlands conjectures predict the existence of a natural map $l_\rho : \text{Irr}(G) \to \text{Irr}(\text{GL}(n,F))$. Assuming that a certain technical condition on $\rho$ (guaranteeing that the image of $l_\rho$ does not lie in the singular set of $\gamma_\psi$) is satisfied, one can consider the meromorphic function $\gamma_{\psi,\rho}$ on $\text{Irr}(G)$, setting $\gamma_{\psi,\rho}(\pi) = \gamma_\psi(l_\rho(\pi))$.

The main purpose of this paper is to propose a general framework for an explicit construction of the functions $\gamma_{\psi,\rho}$. Namely, we propose a conjectural scheme for constructing an invariant distribution $\Phi_{\psi,\rho}$ on $G$, whose action in every $\pi \in \text{Irr}(G)$ is given by multiplication by $\gamma_{\psi,\rho}$. Surprisingly, this turns out to be connected with certain geometric analogs of M.Kashiwara’s crystals (cf. [3]).

We work out in detail several examples. As a byproduct we obtain a conjectural formula for the lifting $l_\rho(\theta)$ where $\theta$ is a character of an arbitrary maximal torus in $\text{GL}(n,F)$. In some of these examples we also give a definition of the corresponding local L-function and state a conjectural $\rho$-analogue of the Poisson summation formula for Fourier transform. This conjecture implies that the corresponding global L-function has meromorphic continuation and satisfies a functional equation.

\section{Introduction and statement of the results}

\subsection{The Langlands program and the lifting}

Let $F$ be either local or finite field, $\overline{F}$ – its separable closure. We will denote algebraic varieties over $F$ and maps between them by boldface letters. The corresponding ordinary letters will denote the associated sets of $F$-points (and the induced maps between them).

Let $G$ be a connected reductive algebraic group over $F$, $G = G(F)$ its group of $F$-points, $\text{Irr}(G)$ the set of isomorphism classes of complex irreducible representations of $G$. For simplicity we shall assume in the introduction that $G$ is split, although
most of our results generalize easily to quasi-split groups. To $G$ one associates a connected complex algebraic group $G^\vee$, called the Langlands dual group.

**Example.** If $G = \text{GL}(n)$ then $G^\vee = \text{GL}(n, \mathbb{C})$.

The local Langlands conjectures can now be summarized as follows. To the field $F$ one associates a pro-algebraic group $\mathfrak{g}_F$ over $\mathbb{C}$, which is very closely related with the Galois group $\text{Gal}(\overline{F}/F)$ (in the case when $F$ is a local field it is called the Weil-Deligne group).

The Langlands conjecture says that

1) There exists a finite-to-one map from $\text{Irr}(G)$ to the set of all $G^\vee$-orbits on $\text{Hom}(\mathfrak{g}_F, G^\vee)$.

2) If $G = \text{GL}(n, F)$ then the above map is a bijection.

**Remark.** This conjecture is known for finite field (due to Lusztig). It is known for local fields when $G$ is a torus (local class field theory) and for for $\text{GL}(n)$ (cf. [20], [12] and [13]).

Let now $\rho : G^\vee \to \text{GL}(n, \mathbb{C})$ be an algebraic representation of $G^\vee$. Then the above conjecture implies that there exists a map $l_\rho : \text{Irr}(G) \to \text{Irr}(\text{GL}(n, F))$.

Our main task is to try to propose a conjecture for an explicit description of the map $l_\rho$ and to formulate certain $\rho$-analogue of the Poisson summation formula.

In the rest of this paper, until Section 9 we assume that $F$ is a local field. Moreover, we assume that the characteristic of $F$ is good with respect to $G$ and that the Lie algebra $\mathfrak{g}$ of $G$ possesses a non-degenerate invariant form. We denote by $\mathcal{H}(G)$ the Hecke algebra of $G$. By definition when $F$ is non-archimedian (resp. archimedian) this is the algebra of locally constant (resp. $C^\infty$) compactly supported distributions on $G$. We shall choose a Haar measure $|dg|$ and thus identify $\mathcal{H}(G)$ with the space $C^\infty_c(G)$ of locally constant (resp. $C^\infty$ when $F$ is archimedian) compactly supported functions on $G$.

### 1.2. $\gamma$-functions.

We will not try to construct explicitly a representation $l_\rho(\pi), \pi \in \text{Irr}(G)$ but give an indirect description of the lifting $l_\rho$.

For this purpose we define a complex-valued function $\gamma$ on $\text{Irr}(\text{GL}(n, F))$.

Choose (once and for all) a non-trivial additive character $\psi : F \to \mathbb{C}^\ast$. Let $|dg|$ denote the unique Haar measure on $G = \text{GL}(n, F)$ such that the Fourier transform $\phi \mapsto \mathcal{F}(\phi)$ defined by the formula

$$
\mathcal{F}(\phi)(y) = \int_G \phi(x)\psi(\text{tr}(xy))|\text{det}(x)|^n|dx|
$$

is unitary.

Consider the distribution $\Phi_\psi(g) := \psi(\text{tr}(g))|\text{det}(g)|^n|dg|$ on $\text{GL}(n, F)$. Clearly, the distribution $\Phi = \Phi_\psi$ is invariant under the adjoint action. Let now $(\pi, V)$ be an
irreducible representation of $G$. Consider the integral

$$\pi(\Phi) = \int_G \pi(g)\Phi(g)$$

(1.2)

One can show (cf. \[11\]) that the integral \eqref{eq:1.2} is convergent (although not absolutely convergent) for generic $\pi \in \text{Irr}(G)$. Therefore, for generic $\pi$ the operator $\pi(\Phi)$ is well-defined and it is equal to multiplication by a scalar $\gamma(\pi)$. One can consider $\gamma$ as a meromorphic function on $\text{Irr}(G)$ (cf. Section \[3,4\] for the definition of this notion).

Let now $G$ and $\rho$ be as above and assume that we know the lifting map $l_\rho$. Then we define a function $\gamma_{\psi,\rho} = \gamma_\rho$ on the set $\text{Irr}(G)$ by $\gamma_\rho = l_\rho^*(\gamma)$. In other words any irreducible representation $\pi$ of $G$ we define

$$\gamma_\rho(\pi) = \gamma(l_\rho(\pi))$$

(1.3)

The function $\gamma_\rho$ is a well-defined meromorphic function on $\text{Irr}(G)$ provided that $l_\rho(\pi)$ does not lie in the singular set of $\gamma$ for generic $\pi \in \text{Irr}(G)$. In order to guarantee this we shall always (except for Section \[3\]) assume that $\rho$ satisfies the following condition: there exists a cocharacter $\sigma : \mathbb{G}_m \to Z(G^\vee)$ of the center $Z(G^\vee)$ of the group $G^\vee$ such that $\rho \circ \sigma = \text{Id}$. Note that $\sigma$ can be regarded as a character of $G$.

It is easy to show (assuming that $l_\rho$ satisfies certain natural properties) that there exists an ad-invariant distribution $\Phi_\rho = \Phi_{\rho,G}$ on $G$ such that

$$\gamma_\rho(\pi) = \pi(\Phi_\rho) \cdot \text{Id}$$

(1.4)

We see from \eqref{eq:1.4} that a lifting $l_\rho$ of a representation $\rho$ of the dual group $G^\vee$ determines an invariant distribution $\Phi_\rho$ on $G$. Since the map $l_\rho$ collapses the $L$-packets we assume that the function $\Phi_{\rho,G}$ is stable. Our main purpose is to give a conjectural description of the function $\Phi_{\rho,G}$. In general we can only propose a framework for such a description. Sometimes we can make our suggestion precise and in some simplest cases when the lifting is known we can check that our definition is correct.

1.3. The Fourier transform $\mathcal{F}_\rho$. Assume now that we know the lifting $l_\rho$ and therefore can construct the distribution $\Phi_\rho$ on $G = G(F)$. In Section \[3\] we define certain Fourier-type transform operator $\mathcal{F}_\rho$ acting in the space of functions on $G$. Namely, for $\phi \in \mathcal{H}(G)$ we define

$$\mathcal{F}_\rho(\phi) = |\sigma|^{-l-1}(\Phi_\rho * \phi')$$

(1.5)

where $\phi'(g) = \phi(g^{-1})$ and $l$ is the semi-simple rank of $G$.

In the case when $G = \text{GL}(n)$ and $\rho$ is the standard representation of $G^\vee \simeq \text{GL}(n)$ the operator $\mathcal{F}_\rho$ coincides with the usual Fourier transform in the space of functions on the space $\mathbf{M}_n(F)$ of $n \times n$-matrices over $F$.
We conjecture that \( \mathcal{F}_\rho \) extends to a unitary operator acting in the space \( L^2_\rho(G) = L^2(G, |\sigma|^{1+1}|dg|) \) where \(|dg|\) is a Haar measure on \( G \) and that in the case when \( \rho \) is non-singular the space \( S_\rho(G) \) is stable under \( L^2_\rho(G) \).

Note that in Section 8 we give a definition of \( \Phi_\rho \) (and thus of \( \mathcal{F}_\rho \)) in certain cases where \( l_\rho \) is not known.

1.4. Local \( L \)-functions. Until now we have discussed only the local lifting. But the only known formulation of the local lifting conjecture is to interpret it as a local part of the global lifting conjecture. There are two approaches to a proof of the global lifting conjecture. The first approach is based on the Trace formula and the second one on the study of \( L \)-functions. So it is natural to try to give a direct definition of the \( L \)-function \( L(l_\rho(\pi), s) \) for \( \pi \in \text{Irr}(G) \) in terms of \( \rho \) and \( \pi \). Assume that we know the distribution \( \Phi_\rho \). In such a case we present a conjecture for a direct definition of the \( L \)-function \( L(l_\rho(\pi), s) \). To any representation \( \rho \) of \( G^\vee \) we associate a certain subspace \( S_\rho \) of the space of functions on \( G \) such that \( S_\rho \) contains the space \( S(G) \) of smooth compactly supported functions on \( G \). For any representation \( \pi \in \text{Irr}(G) \) we define the function \( L(l_\rho(\pi), s) \) as the common denominator of rational operator-valued functions \( \pi_s(\phi) \) where \( \phi \in S_\rho \) and \( \pi_s(f) := \int_{g \in G} \phi(g)|\sigma(g)|^s|\pi(g)|dg \). In the case when \( G = \text{GL}(n) \) and \( \rho \) is the standard representation of \( G^\vee \simeq \text{GL}(n) \) the space \( S_\rho \) coincides with the space of Schwartz-Bruhat functions on the space \( M_n \) of \( n \times n \)-matrices and our definition of the \( L \)-function coincides with the definition from [11].

1.5. Conjectural applications to automorphic \( L \)-functions. Let now \( K \) denote a global field and let also \( A_K \) be the corresponding adele ring. For a place \( p \) of \( K \) we denote by \( K_p \) the corresponding local completion of \( K \). In [11] R. Godement and H. Jacquet defined the \( L \)-function \( L(\pi, s) \) for every automorphic representation \( \pi \) of \( \text{GL}(n, A_K) \). This \( L \)-function is defined as the product of the corresponding local \( L \)-functions over all places of \( K \). It is shown in [11] that \( L(\pi, s) \) is meromorphic and has a functional equation. The main tools in the construction and the proof are the space \( \mathcal{S}(M_n(F)) \) of Schwartz-Bruhat functions on \( n \times n \)-matrices, the Fourier transform acting in this space and the Poisson summation formula.

Assume now that we are given a group \( G \) and a representation \( \rho \) of \( G^\vee \). R. Langlands (cf. [18]) conjectured that one could define an \( L \)-function \( L_\rho(\pi, s) \) attached to every automorphic representation \( \pi \) of \( G(A_K) \) as a product of local factors. We conjecture that these local factors are equal to ones defined in Section 1.4. In Section 5 we define a global version \( \mathcal{S}_\rho(G(A_K)) \) and define a global \( \rho \)-analogue \( \mathcal{F}_\rho \) of the Fourier transform. By the definition the space \( \mathcal{S}_\rho(G(A_K)) \) is the span of functions \( \rho \) which are products \( \phi = \bigotimes \phi_p \) of local factors. We conjecture the existence of an \( \mathcal{F}_\rho \)-invariant distribution \( \varepsilon_\rho \) on \( \mathcal{S}_\rho(G(A_K)) \) such that
1) \[\varepsilon_\rho(\phi) = \sum_{g \in G(K)} \phi(g) \quad (1.6)\]

if some local factor \(\phi_p\) has compact support on \(G_p\).

2) For any \(\phi \in S_\rho(G(\mathbb{A}_K))\)

\[\varepsilon_\rho(\phi) = \varepsilon_\rho(F_\rho(\phi)) \quad (1.7)\]

Existence of such \(\varepsilon\) can be thought of as a \(\rho\)-analogue of the Poisson summation formula. In the case when \(G = \text{GL}(n)\) and \(\rho\) is the standard representation of \(G^\vee \simeq \text{GL}(n)\) our conjecture specializes to the usual Poisson summation formula. The validity of our \(\rho\)-analogue of the Poisson summation formula would imply that the automorphic \(L\)-function \(L_\rho(\pi, s)\) has meromorphic continuation with only a finite number of poles and satisfies a functional equation.

1.6. Application to lifting. In Section 5 we give a construction of the distribution \(\Phi_\rho\) in the case when \(G = \text{GL}(n) \times \text{GL}(m)\) for arbitrary \(n\) and \(m\) and when \(\rho\) is equal to the tensor product of the standard representations of \(\text{GL}(n, \mathbb{C})\) and \(\text{GL}(m, \mathbb{C})\). As a byproduct we get a conjectural formula for \(l_\rho(\theta)\) where \(\theta\) is a character of an arbitrary maximal torus in \(\text{GL}(n, F)\).

1.7. Idea of the construction of \(\Phi_\rho\). In Section 4 we check that in the case when \(G\) is a split torus \(T\) the lifting and the distribution \(\Phi_{\rho, T}\) on \(T\) are well defined for any representation \(\rho_T\) of \(T^\vee\), satisfying the condition from Section 1.2. Moreover, one can construct an algebraic variety \(Y_{\rho, T}\), a map \(p_T : Y_{\rho, T} \to T\), a function \(f_T\) on \(Y_{\rho, T}\) and a top-form \(\omega_T \in \Gamma(\Omega^{\text{top}}, Y_{\rho, T})\) such that the distribution \(\Phi_{\rho, T}\) is equal to the push-forward \((p_T)_!(\psi(f_T)|_{\omega_T})\) where \(|_{\omega_T}|\) is the measure on \(Y_{\rho, T}\) associated with the form \(\omega_T\) (cf. [22]). We say that \(\Phi_{\rho, T} = (Y_{\rho, T}, f_T, \omega_T)\) is an algebraic-geometric distribution representing the distribution \(\Phi_{\rho, T}\) and say that the distribution \(\Phi_{\rho, T}\) is a materialization of an algebraic-geometric distribution \(\Phi_{\rho, T}\).

Let now \(G\) be an arbitrary reductive group. We conjecture that the \(\text{Ad}\)-invariant distribution \(\Phi_\rho\) is stable and therefore comes from a distribution \(\tilde{\Phi}_\rho\) on the set \(G/\text{Ad}(F)\) of \(F\)-rational points of the geometric quotient \(G/\text{Ad} = \text{Spec}(\mathcal{O}(G))^G\) of \(G\) by the adjoint action. Since \(G/\text{Ad} = T/W\) where \(T \subset G\) is a maximal split torus and \(W\) is the Weyl group of \(G\) we can consider \(\tilde{\Phi}_\rho\) as a distribution on the set \((T/W)(F)\). We conjecture that the distribution \(\Phi_{\rho, T}\) is a materialization of an algebraic-geometric distribution \(\Phi_\rho\) which is a “descent” of \(\Phi_{\rho_T, T}\) where \(\rho_T\) is the restriction of \(\rho\) on \(T^\vee\). More precisely we conjecture that there exists an action of the Weyl group \(W\) on \(\Phi_{\rho, T}\) such that the distribution \(\Phi_{\rho, T}\) on the set \(T/W(F)\) is the materialization of the algebraic-geometric distribution \(\Phi_\rho\) on \(T/W\) which is a ”descent” of \(\Phi_{\rho_T, T}\) (we shall explain the “descent” procedure carefully in Section 7).

In Section 8 we give explicit formulas for this action in a number of cases.
This paper is organized as follows. In Section 2 we formulate the Langlands lifting conjecture and discuss several examples. In Section 3 we define $\gamma$-functions and formulate their conjectural properties. In Section 4 we give an explicit construction of the lifting for the case when our reductive group is a split torus $T$ and we write an explicit formula for the corresponding distribution $\Phi_{\rho,T}$. Section 5 and Section 6 are devoted respectively to the definition of the Schwartz space $S_{\rho}$ and an analogue of the Poisson summation formula for the operator $F_{\rho}$. In Section 7 we develop the notion of an algebraic-geometric distribution. Using this notion we reduce (conjecturally) the problem of constructing the distribution $\Phi_{\rho}$ to a purely algebraic question.

Section 8 is devoted to the discussion of this question in the following case. Let $m,n$ be two positive integers. Define the group

$$G(m,n) = \{(A,B) \in \text{GL}(m) \times \text{GL}(n) \mid \det(A) = \det(B)\}.$$  \hspace{1cm} (1.8)

The dual group $G(m,n)^\vee$ is the quotient of $\text{GL}(m,\mathbb{C}) \times \text{GL}(n,\mathbb{C})$ by the subgroup consisting of all pairs of matrices $(t\text{Id}_m, t^{-1}\text{Id}_n)$ for $t \in \mathbb{C}^\ast$. Hence the tensor product $\rho_m \otimes \rho_n$ of the standard representations of $\text{GL}(m,\mathbb{C})$ and $\text{GL}(n,\mathbb{C})$ descends to a representation $\rho$ of $G(m,n)^\vee$. The corresponding $\gamma$-function is essentially constructed in [14].

In Section 8 we give a construction of $\Phi_{\rho}$ in this case. Using [3] one can check that our definition gives rise to (almost) the same $\gamma$-function as the one defined in [14]. As an application we give conjectural formulas for the lifting problem discussed in Section 1.6.

In Section 9 we explain how to construct $\Phi_{\rho}$ in the case when the field $F$ is finite. More precisely, in this case we construct a perverse sheaf $\Phi_{\rho}$ on $G$ such that $\Phi_{\rho}$ is obtained from it by taking traces of the Frobenius morphism in the fibers.

Finally, the Appendix (by V. Vologodsky) contains a proof of some result about non-archimedian oscillating integrals used throughout the paper.

1.9. **Notations.** Everywhere, except for section Section 9, $F$ denotes a local field. If $F$ is non-archimedian then we denote by $O_F \subset F$ its ring of integers and by $q$ the number of elements in the residue field of $F$.

If $X$ is either a totally disconnected topological space or a $C^\infty$-manifold we denote by $C(X)$ the space of $\mathbb{C}$-valued continuous functions on $X$. We denote by $C_c^\infty(X)$ the space of all compactly supported $\mathbb{C}$-valued functions on $X$ which are:

1) locally constant if $X$ is totally disconnected;

2) $C^\infty$ if $X$ is a $C^\infty$-manifold.

We shall denote algebraic varieties over $F$ by boldface letters (e.g. $G,X,...$). The corresponding ordinary letters ($G, X,...$) will denote the corresponding sets of $F$-points.

Let $X$ be a smooth algebraic variety over $F$ and $\omega \in \Gamma(X, \Omega^{\text{top}}(X))$, where $\Omega^{\text{top}}$ denotes the sheaf of differential forms of degree $\dim(X)$ on $X$. According to [25] to $\omega$ one can associate a distribution $|\omega|$. 

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If $G$ is a topological group (resp. a Lie group) then we denote by $\text{Irr}(G)$ the set of isomorphism classes of irreducible algebraic (cf. $[3]$) representations of $G$ (resp. the set of equivalence classes of continuous irreducible Banach representations of $G$ with respect to infinitesimal equivalence – cf. $[26]$).

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2. **Lifting**

2.1. **Unramified lifting.** Let $F$ be a non-archimedian local field and $G$ a quasi-split group over $\mathcal{O}_F$, which can be split over an unramified extension of $F$ of degree $k$ (we assume that $k$ is minimal with this property). Then the cyclic group $\mathbb{Z}/k\mathbb{Z}$ acts naturally on $G^\vee$. We denote the action of the generator of $\mathbb{Z}/k\mathbb{Z}$ by $g \mapsto \kappa(g)$.

The group $G^\vee$ acts on itself by $g : x \mapsto gx\kappa(g)^{-1}$. Let $S_G$ denote the set of closed orbits on $G^\vee$ with respect to the above action.

Let $\text{Irr}_{un}(G)$ denote the set of isomorphism classes of irreducible unramified representations of $G$, i.e. the set of irreducible representations of $G$ which have a non-zero $G(\mathcal{O}_F)$-invariant vector. Then one has the natural identification $\text{Irr}_{un}(G) \simeq S_G$.

Let $\rho : G^\vee \rtimes \mathbb{Z}/k\mathbb{Z} \to \text{GL}(n)$ be a representation. The map $x \mapsto (x, \kappa)$ gives rise to a well-defined map from $S_G$ to the set of semi-simple conjugacy classes in $G^\vee \rtimes \mathbb{Z}/k\mathbb{Z}$. If we compose this map with $\rho$ we get a map from $S_G$ to $S_{\text{GL}(n)}$.

Consider now the composite map

$$\text{Irr}_{un}(G) \to S_G \to S_{\text{GL}(n)} \to \text{Irr}_{un}(\text{GL}(n, F)).$$

We will denote this map by $l_{\rho, un}$ and call it *the unramified lifting*.

2.2. **The global lifting.** Let $K$ be a global field, $\mathbb{A}_K$ – its ring of adeles. We denote by $\mathcal{P}(K)$ the set of places of $K$. For every $p \in \mathcal{P}(K)$ we let $K_p$ be the corresponding local completion of $K$.

Let now $G$ be a quasi-split reductive group over $K$, which splits over a finite separable extension of $K$ with Galois group $\Gamma$. As is well-known any irreducible representation $\pi$ of the group $G(\mathbb{A}_K)$ can be uniquely written as a restricted tensor product $\otimes_{p \in \mathcal{P}(K)} \pi_p$ where $\pi_p \in \text{Irr}(G(K_p))$ and $\pi_p \in \text{Irr}_{un}(G(K_p))$ for almost all $p$.

We denote by $\text{Aut}(G, K)$ the set of irreducible automorphic representations of $G(\mathbb{A}_K)$.

The group $\Gamma$ acts naturally on $G^\vee$. Hence we can consider the semidirect product $G^\vee \rtimes \Gamma$. For almost all $p \in \mathcal{P}(K)$ the group $G$ can be split over an unramified extension of $K_p$ of degree $k_p$ and in this case the group $G^\vee \rtimes \mathbb{Z}/k_p\mathbb{Z}$ is naturally embedded in $G^\vee \rtimes \Gamma$.

Hence every representation $\rho : G^\vee \rtimes \Gamma \to \text{GL}(n)$ defines a representation of $G^\vee \rtimes \mathbb{Z}/k_p\mathbb{Z}$ for almost all $p$. 

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Definition 2.3. Let $\pi = \otimes \pi_p$ be an automorphic representation of $G(\mathbb{A}_F)$. The global lifting $l_\rho(\pi)$ is an automorphic representation $l_\rho(\pi) = \otimes l_\rho(\pi)_p$ of the group $GL(n, \mathbb{A}_F)$ such that for every place $p \in \mathcal{P}(F)$ such that $\pi_p$ is unramified one has $l_\rho(\pi)_p = l_{\rho, un}(\pi_p)$ where $l_{\rho, un}$ is as in Section 2.4.

Remark. It follows from the strong multiplicity one theorem for $GL(n)$ (cf. [22]) that if $l_\rho(\pi)$ exists then it is unique.

Conjecture 2.4. 1. For every $G$, $\pi$ and $\rho$ as above the lifting $l_\rho(\pi)$ exists.
2. Let $G$ be a connected quasi-split reductive group over a local field $F$, which splits over a normal separable extension $L/F$ such that $Gal(L/F) = \Gamma$. Then for every representation $\rho : G^\vee \rtimes \Gamma \to GL(n, \mathbb{C})$ there exists a map $l^F_\rho : Irr(G) \to Irr(GL(n, F))$ such that for any global field field $K$, a place $p \in \mathcal{P}(K)$ such that $F = K_p$, a quasi-split group $G'$ over $K$ as in Section 2.2 such that $G'_p \simeq G$ and every $\pi \in Aut(G', K), \pi = \otimes \pi_p$, the $p$-th component of the automorphic representation $l_\rho(\pi)$ is equal to $l^F_\rho(\pi_p)$.

We say that $\rho$ is liftable if Conjecture 2.4 holds for $\rho$.

Remark. It is easy to see that the local lifting $l_\rho$ is uniquely determined by the conditions of Conjecture 2.4.

2.5. Example. One of the examples that we would like to consider in this paper is the following. Let $E/F$ be a separable extension of $F$ of degree $n$. Let $T_E = Res_{E/F} \mathbb{G}_m$ where $Res_{E/F}$ denotes the functor of restriction of scalars. Let $\Gamma_F$ denote the absolute Galois group of $F$. $\Gamma_F$ acts naturally on the set of all embeddings of $E$ into $\overline{F}$, which is a finite set with $n$ elements. Hence we get a homomorphism $\alpha_E : \Gamma_F \to S_n$ which is defined uniquely up to $S_n$-conjugacy. Let $\Gamma = \mathfrak{Z}(\alpha_E)$. By the definition $\Gamma$ is a subgroup of $S_n$ which is also a quotient of $\Gamma_F$. Moreover, it is clear that the torus $T_E$ splits over the Galois extension $L/F$ where $Gal(L/F) = \Gamma$.

The dual torus $T^\vee$ is isomorphic to $(\mathbb{C}^*)^n$ and the group $\Gamma$ acts on it by the restriction of the standard action of $S_n$ on $(\mathbb{C}^*)^n$ to $\Gamma$. Hence the standard embeddings of $(\mathbb{C}^*)^n$ and $S_n$ to $GL(n, \mathbb{C})$ give rise to a homomorphism $\rho_T : T^\vee \rtimes \Gamma \to GL(n, \mathbb{C})$. One of the purposes of this paper is to give a conjectural construction of $l_\rho$ in this case.

3. Gamma-functions and Bernstein’s center

3.1. Bernstein’s center. Let $G$ be a reductive algebraic group over $F$, $G = G(F)$. We denote by $\mathcal{M}(G)$ the category of smooth representations of $G$.

Recall that the Bernstein center $\mathcal{Z}(G)$ of $G$ is the algebra of endomorphisms of the identity functor on the category $\mathcal{M}(G)$. It was shown by Bernstein that $\mathcal{Z}(G)$ is isomorphic to the direct product of algebras of the form $\mathcal{O}(\Omega)$ where $\Omega$ is a finite-dimensional irreducible complex algebraic variety. We shall denote the set of all $\Omega$ as above by $\mathcal{C}$. By $\text{Spec}(\mathcal{Z}(G))$ we shall mean the disjoint union of all $\Omega \in \mathcal{C}$. It is shown in [2] that the natural map $\text{Irr}(G) \to \text{Spec}(\mathcal{Z}(G))$ sending every irreducible...
representation to its infinitesimal character is finite-to-one and moreover is generically one-to-one.

The following result (also due to J. Bernstein) gives a different interpretation of $\mathcal{Z}(G)$. Let $\Phi$ be a distribution on $G$. We say that $\Phi$ is essentially compact if for every function $\phi \in \mathcal{H}(G)$ one has $\Phi \ast \phi \in \mathcal{H}(G)$. Given two essentially compact distributions $\Phi_1, \Phi_2$ one defines their convolution $\Phi_1 \ast \Phi_2$ by setting

$$\Phi_1 \ast \Phi_2(\phi) = \Phi_1(\Phi_2 \ast \phi)$$

(3.1)

It is easy to see that $\Phi_1 \ast \Phi_2$ is again an essentially compact distribution.

**Lemma 3.2.** The algebra $\mathcal{Z}(G)$ is naturally isomorphic to the algebra of invariant essentially compact distributions.

3.3. **Example.** Let $G = \text{GL}(n)$, $G = \text{GL}(n, F)$. Fix any $a \in F^*$ and a non-trivial additive character $\psi : F \to \mathbb{C}^*$. Let $G_a = \{ g \in G | \det(g) = a \}$. Then a choice of a Haar measure on $\text{SL}(n, F)$ defines a measure on $G_a$ for every $a \in F^*$ (since $G_a$ is a principal homogeneous space over $\text{SL}(n, F)$). We denote this measure by $d_a g$. Define a distribution $\Phi_a$ on $G$ supported on $G_a$ by

$$\Phi_a(\phi) = \int_{G_a} \phi(g)\psi(\text{tr}(g))d_a g$$

(3.2)

for any $\phi \in C_c^\infty(G)$.

One can verify that $\Phi_a$ is essentially compact.

3.4. **Rational functions on $\text{Irr}(G)$.** Let $\mathcal{K}(G)$ denote the “field of fractions” of $\mathcal{Z}(G)$, i.e. the associative algebra $\prod \mathcal{K}(\Omega)$, where $\Omega$ runs over connected components of $\text{Spec}(\mathcal{Z}(G))$ as above and $\mathcal{K}(\Omega)$ is the field of fractions of $\mathcal{O}(\Omega)$. Since generically $\text{Spec}(\mathcal{Z}(G))$ and $\text{Irr}(G)$ are identified we shall refer to elements of $\mathcal{K}(G)$ as rational functions on $\text{Irr}(G)$.

3.5. **$\sigma$-regular central elements.** For an associative algebra $\mathcal{A}$ we let $\mathcal{M}(G, \mathcal{A})$ denote the category of $G$-modules endowed with an action of $\mathcal{A}$ by endomorphisms of the $G$-module structure (we call it the category of $(G, \mathcal{A})$-modules).

Let $\mathcal{A} = \mathbb{C}[t, t^{-1}], \mathcal{B} = \mathbb{C}((t))$. Assume that we are given an $F$-rational character $\sigma : G \to \mathbb{G}_m$. Then we can define an action of $G$ on $\mathcal{A}$ and $\mathcal{B}$ by requiring that

$$g \cdot t = |\sigma(g)|t$$

(3.3)

for every $g \in G$.

Define the functor $F_\mathcal{A} : \mathcal{M}(G, \mathcal{A}) \to \mathcal{M}(G, \mathcal{B})$ such that $F_\mathcal{A} : V \mapsto V \otimes \mathcal{B}$. Set

$$\mathcal{Z}_\sigma(G) = \text{End} F_\mathcal{A} (\text{the End is taken in the category of functors from } \mathcal{M}(G, \mathcal{A}) \text{ to } \mathcal{M}(G, \mathcal{B})).$$

We would like now to identify $\mathcal{Z}_\sigma(G)$ with a subspace of $\mathcal{K}(G)$. The assignment $\pi \mapsto \pi \otimes |\sigma|^{\log_a(z)}$ gives rise to an action of $\mathbb{C}^*$ on $\text{Irr}(G)$ (here $z \in \mathbb{C}^*$). It is easy to
see that this action descends to a well-defined algebraic action of \( \mathbb{C}^* \) on \( \text{Spec}(\mathcal{Z}(G)) \). For every \( f \in \mathcal{K}(G) \) we can write
\[
f(z \cdot x) = \sum f_n(x) z^n
\]
for some \( f_n \in \mathcal{K}(G) \).

**Lemma 3.6.** \( \mathcal{Z}_\sigma(G) \) is naturally isomorphic to the space of all \( f \in \mathcal{K}(G) \) such that \( f_n \in \mathcal{Z}(G) \) for every \( n \in \mathbb{Z} \).

Lemma 3.6 explains the term “\( \sigma \)-regular”.

### 3.7. \( \sigma \)-compact distributions

Let now \( \Phi \) be a distribution on \( G \) and let \( G_n = \sigma^{-1}(\pi^n \mathcal{O}^*) \). \( G_n \) is an open subset of \( G \). For every \( n \in \mathbb{Z} \) we define a new distribution \( \Phi_n \) on \( G \) by \( \Phi_n = \chi_n \Phi \), where \( \chi_n \) is the characteristic function of \( G_n \).

**Definition 3.8.** We say that \( \Phi \) is \( \sigma \)-compact if the following two conditions are satisfied:
1. \( \Phi_n \) is essentially compact for every \( n \in \mathbb{Z} \).
2. For every \( \phi \in \mathcal{H}(G) \) there exists a rational function \( F_\phi : \mathbb{C} \to \mathcal{H}(G) \) such that for every \( g \in G \) one has
\[
\sum_{n \in \mathbb{Z}} z^n (\Phi_n * \phi)(g) = F_\phi(z)(g)
\]

The following lemma is straightforward from the definitions and Lemma 3.2.

**Lemma 3.9.** The space of all \( \sigma \)-compact distributions is naturally isomorphic to \( \mathcal{Z}_\sigma(G) \).

### 3.10. An example

Let \( G = \text{GL}(n, F) \). Choose a non-trivial additive character \( \psi : F \to \mathbb{C}^* \) and set \( \Phi_\psi(g) = \psi(\text{tr}(g)) |\det(g)|^n |dg| \), where \( |dg| \) is a Haar measure on \( G \). Let also \( \sigma : \text{GL}(n) \to \mathbb{G}_m \) be given by \( \sigma(g) = \det(g) \).

**Proposition 3.11.** \( \Phi_\psi \) is \( \sigma \)-compact.

**Proof.** Let \( d_\omega g = |\det|^{n} |dg| \) be the “additive” measure on \( G \) and let also \( \mathcal{S}(M_n) \) denote the space of Schwartz-Bruhat functions on \( n \times n \)-matrices \( M_n \) over \( F \). We shall regard \( \mathcal{S}(M_n) \) as a subspace of \( C(G) \). Consider the Fourier transform \( \mathcal{F} : \mathcal{S}(M_n) \to \mathcal{S}(M_n) \) given by
\[
\mathcal{F}(\phi)(y) = \int_G \phi(x) \psi(\text{tr}(xy)) d_\omega x
\]
It is easy to see that \( \mathcal{F}(\phi) = (\Phi_\psi * \phi) \zeta^{-n} \), where \( \zeta \phi(x) = \phi(x^{-1}) \).

Let us now show that conditions 1 and 2 above hold. Without loss of generality we may assume that \( \text{supp} \phi \subset G_0 \). Then \( \zeta^{-n} \Phi_n * \phi = \mathcal{F}(\phi) \chi_{-n} \). Since \( \mathcal{F}(\phi) \in \mathcal{S}(M_n) \) it follows that \( \mathcal{F}(\phi) \chi_{-n} \in \mathcal{H}(G) \). The verification of the second conditions is left to the reader. \( \square \)
We shall denote by \( \gamma \) the rational function on \( \text{Irr}(\text{GL}(n, F)) \) which corresponds to \( \Phi \).

### 3.12. Lifting and the distributions \( \Phi_{\psi, \rho} \)

Let now \( \rho : G^\vee \to \text{GL}(n, \mathbb{C}) \) be a finite-dimensional representation of the Langlands dual group \( G^\vee \). The local lifting conjecture predicts the existence of the natural map \( l_\rho : \text{Irr}(G) \to \text{Irr}(\text{GL}(n, F)) \). Assuming that we know \( l_\rho \) we can try to define a rational function \( \gamma_{\psi, \rho} \) by setting \( \gamma_{\psi, \rho}(\pi) = \gamma_{\psi}(l_\rho(\pi)) \). However, it might happen that the image of \( l_\rho \) lies entirely inside the singular set of \( l_\rho \). To guarantee that this does not happen we introduce the following notion.

**Definition 3.13.** We say that \( \rho \) is admissible if the following conditions hold.

1. \( \text{Ker}(\rho) \) is connected.
2. There exists a character \( \sigma : G \to \mathbb{G}_m \) defined over \( F \) such that \( \rho \circ \sigma = \text{Id} \) \( (3.7) \) where we regard \( \sigma \) as a cocharacter of the center \( \mathbb{Z}(G^\vee) \) of \( G^\vee \) and \( I_n \in \text{GL}(n, \mathbb{C}) \) denotes the identity matrix.

It is easy to see that if we assume that \( \rho \) is admissible and liftable then \( \gamma_{\psi, \rho} \) satisfies the conditions of Lemma 3.6. Thus we can construct the corresponding \( \sigma \)-compact distribution \( \Phi_{\psi, \rho, G} \). Our main task will be to try to construct the distribution \( \Phi_{\psi, \rho, G} \) explicitly (we will sometimes drop the indices \( \psi \) and \( G \), when it does not lead to a confusion).

### 3.14. An example.

One of the examples that we would like to study in this paper is the following. Fix a sequence \( n_1, ..., n_k \) of natural numbers. Let \( G = G(n_1, ..., n_k) \) be the subgroup of \( \text{GL}(n_1) \times ... \times \text{GL}(n_k) \) consisting of \( k \)-tuples of non-degenerate matrices \( (A_1, ..., A_k) \) such that \( \det A_i = \det A_j \) for all \( 1 \leq i, j \leq k \). The dual group \( G^\vee \) is the quotient of \( \text{GL}(n_1) \times ... \times \text{GL}(n_k) \) by the subgroup consisting of the \( k \)-tuples \( (t_1 \text{Id}_{n_1}, t_2 \text{Id}_{n_2}, ..., t_k \text{Id}_{n_k}) \) with \( t_1 ... t_k = 1 \). Hence the tensor product representation \( \rho_{n_1} \otimes ... \otimes \rho_{n_k} \) of \( \text{GL}(n_1) \times ... \times \text{GL}(n_k) \) factors through \( G^\vee \) and we define \( \rho = \rho_{n_1} \otimes ... \otimes \rho_{n_k} \). We shall refer to \( \rho \) as standard representation of \( G^\vee \). It is easy to see that \( \rho \) is an embedding and that \( \rho \circ \sigma = \text{Id} \) where \( \sigma = \text{det} \circ \rho \).

*Example.* Suppose that \( k = 1, n_1 = n \). In this case \( G = \text{GL}(n, F) \) and \( \Phi_{\psi, \rho} \) is the same as in Section 3.10.

### 3.15. Compatibility with induction.

Let \( P \subset G \) be a parabolic subgroup of \( G \) defined over \( F \), \( M \) – the corresponding Levi subgroup. Let \( \delta_P \) be the determinant of the action of \( M \) on the Lie algebra of the unipotent radical \( U_P \) of \( P \).

For every \( \phi \in \mathcal{H}(G) \) we can construct a new function \( r_P(\phi) \) on \( G/U_P \)

\[
r_P(\phi)(g) = \int_{U_P} \phi(gu)|du| \quad (3.8)
\]
Note that $M$ acts on $G/U_P$ on the right.

Let now $\rho$ be a representation of $G^\vee$ as above. Since $M^\vee$ is naturally a subgroup of $G^\vee$ we can regard $\rho$ as a representation of $M^\vee$. We conjecture that the following property holds for the distributions $\Phi_{\psi,\rho,G}$ and $\Phi_{\psi,\rho,M}$:

**Conjecture 3.16.**

$$r_P(\Phi_{\psi,\rho,G} \ast \phi) = r_P(\phi) \ast (\Phi_{\psi,\rho,M}|_dP)^{-1/2}).$$  \hspace{1cm} (3.9)

3.17. Assume that $\rho = \rho_1 \oplus \rho_2$ is a direct sum of two admissible representations, such that the corresponding character $\sigma$ is the same for both representations. The following result can be easily deduced from the definitions.

**Lemma 3.18.** One has

$$\Phi_{\psi,\rho} = \Phi_{\psi,\rho_1} \ast \Phi_{\psi,\rho_2}. \hspace{1cm} (3.10)$$

3.19. $\gamma$-functions determine the lifting. Let $G = \text{GL}(n-1) \times \text{GL}(n)$ and the character $\sigma$ be given by $\sigma(g_{n-1}, g_n) = \text{det}(g_{n-1})$. In [14] Ch.7 a rational function $\gamma_{n-1,n}$ was defined on the subset of non-degenerate representations of the group $G$. One can show that $\gamma_{n-1,n}$ extends to a $\sigma$-regular central element for $G$. Hence there exists a $\sigma$-regular distribution $\Phi_{n-1,n}$ on the group $G(n-1,F) \times \text{GL}(n,F)$ such that $\gamma_{n-1,n} = \Phi_{n-1,n}$. In [14] the analogous function $\gamma_{m,n}$ on $\text{Irr}(\text{GL}(m,F) \times \text{GL}(n,F))$ was defined. It is also easy to see that it comes from a $\sigma$-regular distribution $\Phi_{m,n}$ on the group $G(n,F)$.

As was shown in [14] for any non-degenerate representation $\pi$ of $\text{GL}(n,F)$ the rational function $\gamma_{\pi} = \gamma_{n-1,n}(\pi, \ast)$ on $\text{Irr}(\text{GL}(n-1,F))$ determines uniquely the representation $\pi$. Moreover [14] contains an explicit recipe for the construction of the representation $\pi$ in terms of the function $\gamma_{\pi}$. It is clear from this recipe that we have to know the function $\gamma_{\pi}$ only up to multiplication by a constant.

3.20. Let now $G$ be arbitrary and $\rho : G^\vee \to \text{GL}(n, \mathbb{C})$ be an admissible representation. For any positive integer $m$ we denote by $\rho_m : G^\vee \times \text{GL}(m, \mathbb{C}) \to \text{GL}(mn, \mathbb{C})$ the representation given by $\rho_m(g^\vee, r) = \rho(g^\vee) \otimes r$. Assume that both $\rho$ and $\rho_m$ are liftable. For any $\pi \in \text{Irr}(G)$ we define rational functions $\gamma'_\pi, \gamma''_\pi$ on $\text{Irr}(\text{GL}(m,F))$ by $\gamma'_\pi(\sigma) = \gamma_{\rho_m}(\pi, \sigma), \gamma''_\pi(\sigma) = \gamma_{n,m}(l_\rho(\pi), \sigma)$

**Proposition 3.21.** For any $\pi \in \text{Irr}(G)$ the two rational functions $\gamma'_\pi$ and $\gamma''_\pi$ on $\text{Irr}(\text{GL}(m,F))$ coincide.

This Proposition follows from the main result of [12].

**Corollary 3.22.** For any $\rho : G^\vee \to \text{GL}(n, \mathbb{C})$ such that the representation $\rho_{n-1}$ is liftable, the lifting $l_\rho$ is determined by the knowledge of the function $\gamma_{\rho_{n-1}}(\pi, \ast)$ on $\text{Irr}(\text{GL}(n-1,F))$. Moreover it is sufficient to know the function $\gamma_{\rho_{n-1}}(\pi, \ast)$ only up to a multiplication by a constant.
4. The case of split tori

In this section we assume that $F$ is non-archimedian.

4.1. Lifting for split tori. Almost the only case when $l_\rho$ is known for every representation $\rho$ is the case when $G$ is equal to a split torus $T$. In this section we show how to carry out our program in this easy case.

We start with an explicit description of the lifting $l_\rho$.

We can write $\rho$ in the form

$$\rho = \bigoplus_{i=1}^{n} \lambda_i \tag{4.1}$$

where $\lambda_i$ are characters of $T^\vee$. By the definition of $T^\vee$ we can regard $\lambda_i$ as cocharacters $\lambda_i : \mathbb{G}_m \to T$ of $T$. Let $\Lambda = X_*(T)$ be the group of cocharacters of $T$. We define $\text{supp}(\rho) \subset \Lambda$ as the union of $\lambda_i$, $1 \leq i \leq n$.

Let $\theta$ be a character of $T$ and $\chi_i = \theta \circ \lambda_i$ be the corresponding characters of $F^*$. Then for every $i = 1, \ldots, n$ there exists $z_i \in \mathbb{R}_{>0}$ such that $|\chi_i(t)| = z_i^{\nu_\rho(t)}$ for every $t \in F^*$.

Let us assume that the ordering $\lambda_1, \ldots, \lambda_n$ is chosen in such a way that

$$z_1 \leq z_2 \leq \ldots \leq z_n \tag{4.2}$$

Let $B_n$ be the subgroup of upper triangular matrices in $GL(n, F)$. We can define the character $\chi : B_n \to \mathbb{C}^*$ by setting

$$\chi(b) = \prod_{i=1}^{n} \chi_i(b_{ii}) \tag{4.3}$$

where $b_{ii}$ are the diagonal entries of $b$.

Consider the unitarily induced representation $l_{B_n}^{GL(n, F)} \chi$. It is well-known (cf. [7]) that this representation has a unique irreducible quotient. We define this quotient to be $l_\rho(\theta)$. As follows from the theory of Eisenstein series this definition satisfies the conditions of Conjecture 2.4.

4.2. $\gamma$-functions for split tori. Let $T$ be as above,

$$\rho = \bigoplus_{i=1}^{n} \lambda_i \tag{4.4}$$

an admissible representation of $T^\vee$. We will give now an explicit description of the distribution $\Phi_{\rho, T}$. Let $T^\vee = \mathbb{G}_m^n$ be the corresponding torus in $GL(n)$. Thus we get a homomorphism $p_{\rho}^\vee : T^\vee \to T^\vee$. Let $T_\rho$ be the split torus over $F$ dual to $T^\vee$ (thus $T_\rho = (F^*)^n$). We then get an $F$-rational map $p_\rho : T_\rho \to T$.

Consider the top degree differential form $\omega_\rho := dt_1 \ldots dt_n$ on $T_\rho$. and the function $f_\rho : T_\rho \to \mathbb{A}^1$ where $f_\rho(t_1, \ldots, t_n) = t_1 + \ldots + t_n$. 13
For every $r \in \mathbb{R}_{\geq 0}$ we denote by $T_\rho(r)$ the set $\{t \in T_\rho| |f_\rho(t)| \leq r\}$.

**Proposition 4.3.**  
1. For every $r \in \mathbb{R}_{\geq 0}$ and for every open compact subset $C \subset T$ the integral
   $$\Phi_\rho(C) = \int \psi(f_\rho(t))|\omega_\rho|$$
   (4.5)
   is absolutely convergent.
2. For every $C$ as above the limit
   $$\lim_{r \to \infty} \Phi_\rho(C)$$
   (4.6)
   exists. We denote this limit by $\Phi_\rho(C)$. We also denote by $\Phi_\rho$ the corresponding distribution on $T$.
3. For every character $\theta$ of $T$ there exists $s_0(\theta) \in \mathbb{R}$ such that for every $s \in \mathbb{C}$ such that $\Re(s) > s_0(\theta)$ the convolution $\Phi_\rho * \theta|\sigma|^s$ is absolutely convergent and
   $$\Phi_\rho * \theta|\sigma|^s = \gamma_{\rho,T}\theta|\sigma|^s$$
   (4.7)
4. $\Phi_\rho$ is $\sigma$-regular.

Our next goal is to describe explicitly the space $S_\rho$ in the case of a torus.

**4.4. The Fourier transform $F_\rho$.** We assume now that the representation $\rho$ is faithful. This assumption will be kept until the end of Section 8.

Consider the map $F_\rho : C^\infty_c(T) \to C^\infty(T)$ given by
   $$\phi \mapsto |\sigma|^{-1}\Phi_\rho * \phi$$
   (4.8)
where $\phi(x) = \phi(x^{-1})$.

Let also $L^2_\rho(T) = L^2(T,|\sigma|d^*t)$. The following lemma is straightforward.

**Lemma 4.5.** $F_\rho$ extends to a unitary automorphism of $L^2_\rho(T)$.

We now set
   $$S_\rho(T) = C^\infty_c(T) + F_\rho(C^\infty_c(T)) \subset L^2_\rho(T)$$
   (4.9)
(the reader should compare this definition with the definition of a Schwartz space $S(X)$ in [3]).

**4.6. The function $C_\rho$.** Let us give an example of a function in $S_\rho(T)$. Let $T_0 \subset T$ be the maximal compact subgroup of $T$. Then we can identify the quotient $T/T_0$ with the lattice $\Lambda$ of cocharacters of $T$.

Let
   $$\rho = \bigoplus_{i=1}^n \lambda_i$$
   (4.10)
be an admissible representation of $T^\vee$. We denote the set of all $\lambda_i$ above by $\text{supp}(\rho)$. 
Let us define a $T_0$-invariant function $S_\rho$ on $T$ (i.e. a function on $T/T_0 = \Lambda$ by setting

$$C_\rho(\mu) = \# \{(a_1, ..., a_n) \in \mathbb{Z}_n^+ \mid a_1\lambda_1 + ... + a_n\lambda_n = \mu\}$$

(4.11)

**Lemma 4.7.** $C_\rho \in S_\rho(T)$.

### 4.8. The semigroup $\overline{T}_\rho$.

We now want to exhibit certain locality properties of the space $S_\rho(T)$. For this we have to introduce first an additional notation. Let $\Lambda_\rho$ be the sub-semigroup of $\Lambda$ generated by $\lambda \in supp(\rho)$, $\mathcal{O}_\rho$ be the group algebra of $\Lambda_\rho$ over $F$ and $\overline{T}_\rho := \text{Spec} \mathcal{O}_\rho$. It is easy to see that $\overline{T}_\rho$ is an $F$-semi-group which contains $T$ as an open dense subgroup.

**Proposition 4.9.** Let $\phi$ be a function on $T$. Then $\phi \in S_\rho(T)$ if and only if $\phi$ satisfies the following conditions:

1. The closure of $supp(\phi)$ in $\overline{T}_\rho$ is compact.
2. For every $x \in \overline{T}_\rho$ there exists a neighbourhood $U_x$ of $x$ in $\overline{T}_\rho$ and a function $\phi' \in S_\rho(T)$ such that

$$\phi|_{U_x} = \phi'|_{U_x}$$

(4.12)

Proposition 4.9 says that one can determine whether a function $\phi$ lies in $S_\rho(T)$ looking at its local behaviour around points of $\overline{T}_\rho$. We will discuss the notion of a local space of functions in Section 5 in more detail.

For every $x \in \overline{T}_\rho$ we denote by $Z_x$ the stabilizer of $x$ in $T \times T$. If $x \in \overline{T}_\rho$ we define an $Z_x$-module $S_{\rho,x}$ as the quotient $S_\rho(G)/S_{\rho,x}(T)$ where $S_{\rho,x}(T)$ the space of all functions from $S_\rho(G)$, which vanish in some neighbourhood of $x$.

**Lemma 4.10.** There exists unique $Z_x$-invariant functional $\varepsilon_x : S_{\rho,x} \to \mathbb{C}$ such that $\varepsilon_x(C_\rho) = 1$.

Thus given $\phi \in S_\rho(T)$ we can produce a function $\phi^\varepsilon$ on $\overline{T}_\rho$ setting $\phi^\varepsilon(x) = \varepsilon_x(\phi)$. However, the function $\phi^\varepsilon$ is not locally constant in general.

### 4.11. Mellin transform.

We now going to give yet another (equivalent) definition of $S_\rho(T)$ using the *Mellin transform* on $T$. Let $X(T) = \text{Hom}(T, \mathbb{C}^*)$. Then $X(T)$ has the natural structure of an algebraic variety, whose irreducible components are parametrized by $\text{Hom}(T_0, \mathbb{C}^*) = \text{Hom}(T_0, S^1)$. Choose a Haar measure $d^*t$ on $T$.

Let $\Lambda$ be the lattice of cocharacters of $T$. As before one can identify $\Lambda$ with $T/T_0$ where $T_0 \subset T$ is the maximal compact subgroup of $T$. We denote by $X^\text{un}(T) \subset X(T)$ the subset of unitary characters of $T$.

We denote by $X_0(T)$ the component of $X(T)$, consisting of characters whose restriction to $T_0$ is trivial. One has the natural identification $X_0(T) \simeq T^\vee$ (where $T^\vee$ denotes the dual torus to $T$ over $\mathbb{C}$). For any $\lambda \in supp(\rho)$ and choose a lift $\lambda'$ of $\lambda$ to
Then we denote by $p_\lambda$ the regular function on $X(T)$ defined by

$$p_\lambda(\chi) = \begin{cases} 
\chi(\lambda') & \text{if } \chi \circ \lambda|_{\sigma^*} = 1 \\
0 & \text{otherwise.}
\end{cases}$$

Clearly, this definition does not depend on the choice of $\lambda'$.

For $\phi \in C_c^\infty(T)$ we define the Mellin transform $M(\phi)$ of $\phi$ as a function on $X(T)$ given by

$$M(\phi)(\chi) = \int_T \phi(t) \chi(t) d^* t. \quad (4.13)$$

The following lemma is standard.

**Lemma 4.12.** $M$ defines an isomorphism between $C_c^\infty(T)$ and $O(X(T))$ which extends to an isomorphism between $L^2(T)$ and $L^2(X^\text{un}(T))$.

Let $\rho$ be an admissible representation of $T^\vee$ and $F_{\psi, \rho}$ denote the operator defined above. The following result is proved by a straightforward calculation.

**Theorem 4.13.** A function $\phi \in L^2_\rho(T)$ lies in $S_\rho(T)$ if and only if $\prod_{1 \leq i \leq n} (p_{\lambda_i} - 1) M(\phi)$ is a regular function on $X(T)$.

Let us now discuss the connection of the space $S_\rho(T)$ with the corresponding local $L$-functions.

**Lemma 4.14.**

1. For every $\phi \in S_\rho(T)$ and every character $\chi$ of $T$ the integral

$$Z(\phi, \chi, s) = \int_T \phi(t) \chi(g) |\sigma(t)|^s d^* t \quad (4.14)$$

is absolutely convergent for $\Re(s) >> 0$.

2. $Z(\phi, \chi, s)$ has a meromorphic continuation to the whole of $\mathbb{C}$ and defines a rational function of $q^s$.

3. The space of all $Z(\phi, \chi, s)$ as above is a finitely generated non-zero fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$. We shall denote this ideal by $J_\chi$. Let also $L_\rho(s, \chi)$ to be the unique generator of $J_\chi$ of the form $P(q^s)^{-1}$, where $P$ is a polynomial such that $P(0) = 1$.

4. $L_\rho(\chi, s) = \prod_{1 \leq i \leq n} L(\chi \circ \lambda_i, s)$ where $L(\chi \circ \lambda_i, s)$ denotes the corresponding Tate’s $L$-function (cf. [24]).

5. **Local spaces of functions**

We now want to generalize some constructions of the preceding Section to the case of arbitrary reductive group $G$. Let us remind that we assume now that the representation $\rho$ is admissible and faithful.
In this section we assume that we have constructed the corresponding distribution $\Phi_{\psi, \rho}$ which satisfies the assumptions of Section 3.

5.1. **Saturations.** Let $\overline{X}$ be a topological set and $X \subset \overline{X}$ an open dense subset. Given a space $L$ of functions $\overline{X}$ we say that $L$ is local if there exists a sheaf $\mathcal{L}$ on $\overline{X}$ and an embedding $\mathcal{L} \in i_*(C(X))$ such that $L = \Gamma_c(\overline{X}, \mathcal{L})$ where $C(X)$ is the sheaf of continuous functions on $X$.

It is easy to see that when the space $X$ is totally disconnected then locality of $L$ is equivalent to the following two conditions:

1) $C_c(X) \subset L \subset C(\overline{X})$

2) $L$ is closed under multiplication by elements of $C(\overline{X})$.

Given a subspace $V \subset C(\overline{X})$ we denote by $P^V$ the presheaf on $\overline{X}$ such that $\Gamma(U, P^V)$ is the subspace of $\phi \in C(U \cap X)$ consisting of all $\phi \in C(U \cap X)$ such that there exists $v \in V$ for which $v|_{U \cap X} = \phi$. Let $\mathcal{L}_V$ be the associated sheaf. It is clear that we have the natural embedding $L^V \hookrightarrow i_*(C(X))$. We set $V = \Gamma_c(\overline{X}, \mathcal{L}_V)$ and call $V$ the saturation of $V$ with respect to $X$.

5.2. **$C^\infty$-version.** Let now $\overline{X}$ be a closed subset of a $C^\infty$-manifold $Y$. Let $d(\cdot, \cdot)$ be a metric on $\overline{X}$ coming from a Riemannian metric on $Y$.

Let $i: X \hookrightarrow \overline{X}$ be an open dense embedding of a $C^\infty$-manifold $X$ into $\overline{X}$. Let also $\mathcal{L} \subset i_* C^\infty(X)$ be a subsheaf. We denote by $\Gamma^a_s(\overline{X}, \mathcal{L})$ the space of $C^\infty$-functions $\phi$ on $X$ such that for any $x \in \overline{X}$ and any $N > 0$ there exists an open neighbourhood $U$ of $x$ and a section $l \in \Gamma(U, \mathcal{L})$ such that for any $y \in U$ one has $\phi(y) - l(y) \leq C d(x, y)^N$ where $C$ is a constant (i.e. $C$ does not depend on $y$).

**Example.** Let $X = \overline{X} = (0, 1)$ and let $\mathcal{L}$ be the sheaf of polynomial functions on $X$. Then $\Gamma^a_s(\overline{X}, \mathcal{L}) = C^\infty_c(0, 1)$.

Given a subset $V$ of $C^\infty(X)$ we can define its saturation $\overline{V}$ as $\Gamma^a_s(\overline{X}, \mathcal{L}_V)$ where $\mathcal{L}_V$ is defined as in Section 5.1.

5.3. **The Fourier transform $\mathcal{F}_\rho$.** In this subsection we would to define certain "twisted" analog of the Fourier transform in the space of functions of $G$ attached to every $\rho$ as above. As before we denote by $L^2_\rho(G)$ the space of $L^2$-functions on $G$ with respect to the measure $|\sigma|^{l+1}|dg|$ where $l$ is the semi-simple rank of $G$ and $|dg|$ is a Haar measure on $G$.

For every $\phi \in C^\infty_c(G)$ we define a new function $\mathcal{F}_\rho(\phi)$

$$\mathcal{F}_\rho(\phi) = |\sigma|^{-l}(\Phi_{\psi, \rho} * \phi)$$

(5.1)

where $\phi(g) = \phi(g^{-1})$. In what follows we will assume the validity of the following conjecture.
Conjecture 5.4. 1. $F_\rho$ extends to a unitary operator on the space $L^2_\rho(G)$ and $S_\rho$ is $F_\rho$-invariant.

Example. Let $G = \text{GL}(n, F)$ and $\rho$ be the standard representation of $G^\vee = \text{GL}(n)$. In this case $L^2_\rho(G)$ is the same as $L^2(M_n)$ with respect to the additive measure on $M_n$ and $F_\rho$ coincides with the Fourier transform on $M_n$ (where we identify $M_n$ with the dual vector space by means of the form $\langle A, B \rangle = \text{tr}(AB)$).

5.5. The semigroup $\overline{G}_\rho$. We would like to embed $G$ as an open subset in a larger affine variety $\overline{G}_\rho$ in a $G \times G$-equivariant way.

The variety $\overline{G}_\rho$ is an algebraic semigroup, containing $G$ as an open dense subgroup. It is well-known (cf. [23]) that in order to define such a semigroup one needs to exhibit a subcategory of the category of finite-dimensional $G$-modules, closed under subquotients, extensions and tensor products.

We will define now the category of $\rho$-positive representations of $G$ which will satisfy the above properties. Let $\lambda : T^\vee \to \mathbb{G}_m$ be a character, which has non-zero multiplicity in $\rho$. We can regard $\lambda$ as a cocharacter of $T$. We say that a $G$-module $V$ is $\rho$-positive if for every $\lambda$ as above the representation $\rho \circ \lambda$ of $\mathbb{G}_m$ is isomorphic to a direct sum of characters of the form $t \mapsto t^i$ for $i \geq 0$. It is clear that the category of $\rho$-positive representations satisfies the properties discussed above and thus defines a semi-group $\overline{G}_\rho$.

5.6. The space $S_\rho(G)$. Let

$$V_\rho = C_c^\infty(G) + F_\rho(C_c^\infty(G)) \subset L^2_\rho(G)$$

(5.2)

Unlike in the case of the torus, one can show that the space $V_\rho$ is almost never local (this is not so for example when $G = \text{GL}(2)$ and $\rho$ is the standard representation of $G^\vee \simeq \text{GL}(2, \mathbb{C})$). We define $S_\rho(G)$ to be the saturation of $V_\rho$ (the definition makes sense both for archimedean and non-archimedean $F$).

5.7. The function $C_\rho$. Let us now exhibit certain explicit element in $S_\rho(T)$. In what follows we choose a square root $q^{1/2}$ of $q$.

5.7.1. The Satake transform. Let $K = G(\mathcal{O})$ and let $\mathcal{H}_K = C^\infty_c(K\backslash G/K)$ be the corresponding Hecke algebra. Recall that there exists the natural isomorphism $S : \mathcal{H}_K \simeq \mathcal{O}(T^\vee)^W \simeq \mathcal{O}(G^\vee)^G$, where $\mathcal{O}(T^\vee)^W$ is the algebra of $W$-invariant regular functions on $T^\vee$. This isomorphism is characterized as follows. One can identify $T^\vee$ with the group of unramified characters of the torus $T$. For every $\lambda \in T^\vee$ let $i(\lambda)$ denotes the corresponding representation of $G$ obtained by normalized induction of the character $\lambda$ from $B$ to $G$. Let $v_\lambda \in i(\lambda)$ be the (unique up to a constant) non-zero $K$-invariant vector in $i(\lambda)$. Then for every $\phi \in C^\infty_c(K\backslash G/K)$ one has

$$(\phi|dg|) \cdot v_\lambda = S(\phi)(\lambda)v_\lambda$$

(5.3)
5.7.2. The function $C_\rho$. Let $\rho$ be as above. Let us also denote by $2\delta_\rho^G$ the sum of positive coroots of $G$. Set
\[
f_{i,\rho}(\lambda) = \text{tr}(\lambda \cdot 2\delta_\rho^G(q^{-1/2}), \text{Sym}^i \rho)
\] for every $i \in \mathbb{Z}_+$.

We define now
\[
C_\rho = \sum_{i=0}^{\infty} S^{-1}(f_{i,\rho})
\] (5.5)

It is easy to see that the non-degeneracy assumptions on $\rho$ imply that $C_\rho$ is a well-defined locally constant function on $G$, which however does not have compact support.

**Lemma 5.8.** $C_\rho \in S_\rho(G)$ and $\mathcal{F}_\rho(C_\rho) = C_\rho$.

The lemma follows easily from Conjecture 3.16 from Section 3.

**Conjecture 5.9.** There exists unique up to a constant $\mathbb{Z}_x$-invariant functional $\varepsilon_x : S_{\rho,x} \rightarrow \mathbb{C}$.

**Example.** Let $G = \text{GL}(n)$ and take $\rho$ to be the standard representation of $G^\vee \cong \text{GL}(n)$. Then $G_\rho = M_n$ – the semi-group of $n \times n$-matrices. Also one has $S_{\rho}(G) = S(M_n)$ – the space of Schwartz-Bruhat functions on $M_n$. In this case $S_{\rho,x}$ is one-dimensional for every $x$ and the functional $\varepsilon_x$ is equal to the evaluation of a locally constant function at $x$.

5.10. Local $L$-functions. Here we give a definition of the local $L$-factor for every admissible non-singular representation $\rho$.

Let $\pi$ be an irreducible representation of $G$. We denote by $M(\pi) \subset C^\infty(G)$ the space of all matrix coefficients of $\pi$.

**Conjecture 5.11.**
1. For every $\phi \in S_{\rho}(G)$ and every $m \in M(\pi)$ the integral
\[
Z(\phi, m, s) = \int_G \phi(g)m(g)|\sigma(g)|^s|dg|
\] (5.6)
is absolutely convergent for $\Re(s) >> 0$.

2. $Z(\phi, m, s)$ has a meromorphic continuation to the whole of $\mathbb{C}$ and defines a rational function of $q^s$.

3. The space of all $Z(\phi, m, s)$ as above is a finitely generated non-zero fractional ideal of the ring $\mathbb{C}[q^s, q^{-s}]$. We shall denote this ideal by $J_{\pi}$.

We now define $L_{\rho}(s, \pi)$ to be the unique generator of $J_{\pi}$ of the form $P(q^{-s})^{-1}$, where $P$ is a polynomial such that $P(0) = 1$. 
6. The Poisson summation formula

Let as before $K$ be a global field and $\mathcal{P}(K)$ be the set of places of $K$. We also denote by $\mathbb{A}_K$ the adele ring of $K$.

Let $G$ be a split reductive algebraic group over $K$ and $\rho : G^\vee \to \text{GL}(n, \mathbb{C})$ be as before. For every place $p \in \mathcal{P}(K)$ we can consider the Schwartz space $\mathcal{S}_\rho = \mathcal{S}_\rho(G(F_p))$ with the distinguished function $\mathcal{C}_\rho$ in it. We now define the space $\mathcal{S}_\rho(G(\mathbb{A}_K))$ as the restricted tensor product of the spaces $\mathcal{S}_\rho$ with respect to the functions $\mathcal{C}_\rho$.

Choose now a non-trivial character $\psi : \mathbb{A}_K \to \mathbb{C}^*$ such that $\psi|_K$ is trivial. Then $\psi$ defines an additive character $\psi_p$ of $K_p$ for every $p$. The Fourier transform

$$F_{\rho,\psi} = \prod_{p \in \mathcal{P}(K)} F_{\rho,p}$$  \hspace{1cm} (6.1)

acts on the space $\mathcal{S}_\rho(G(\mathbb{A}_K))$.

Every $\phi \in \mathcal{S}_\rho(G(\mathbb{A}_K))$ gives rise to a function on $G(\mathbb{A}_K)$.

**Conjecture 6.1.** a) There exists a $G(K)$-invariant functional $\varepsilon : \mathcal{S}_\rho(G(\mathbb{A}_K)) \to \mathbb{C}$ such that

1) $\varepsilon(\phi) = \varepsilon(F_{\rho}(\phi))$ for any $\phi \in \mathcal{S}_\rho(G(\mathbb{A}_K))$.

2) Let $\phi = \prod \phi_p \in \mathcal{S}_\rho(G(\mathbb{A}_K))$. Assume that there exists a place $p_0 \in \mathcal{P}(K)$ such that $\phi_{p_0} \in \mathcal{H}(G)$. Then

$$\varepsilon(\phi) = \sum_{g \in G(K)} \phi(g)$$  \hspace{1cm} (6.2)

b) The functional $\varepsilon$ is supported on $\overline{G}_p(K)$. In other words, assume that we are given $\phi \in \mathcal{S}_\rho(G(\mathbb{A}_K))$ such that for every $g \in \overline{G}_p(K)$ there exists a neighbourhood $U_g$ of $g$ such that $\phi(x) = 0$ for every $x \in U_g \cap G(K)$. Then $\varepsilon(\phi) = 0$.

When $G = \text{GL}(n)$ and $\rho$ is the standard representation, the statement of Conjecture 6.1 is the standard Poisson summation formula. In this case $\phi$ is a smooth function on the space $\text{M}_n(\mathbb{A}_K)$ of adelic $n \times n$ matrices and one has

$$\varepsilon(\phi) = \sum_{g \in \text{M}_n(K)} \phi(g)$$  \hspace{1cm} (6.3)

We do not know any explicit formula for $\varepsilon$ in general.

Let now $\pi$ be an irreducible automorphic representation of $G(\mathbb{A}_K)$, $\pi_p$ – its component at $p \in \mathcal{P}(S)$. One can consider the global $L$-function

$$L_{\rho}(\pi, s) = \prod_{p \in \mathcal{P}(S)} L_{\rho}(\pi_p, s)$$  \hspace{1cm} (6.4)

Using the arguments of [11] is easy to see that the validity of the above conjectures implies the meromorphic continuation and functional equation of the corresponding automorphic $L$-functions.
7. "Algebraic" integrals over local fields

In the next two sections we assume that the local field \( F \) has characteristic 0.
Let \( X \) be an algebraic variety over \( F \) of dimension \( d \), \( X = X(F) \). Recall that (cf. [25]) any \( \omega \in \Gamma(X, \Omega^{\text{top}}) \) defined over \( F \) defines a measure \(|\omega|\) on \( X \).

7.1. Locally integrable functions. We say that a function \( \phi : X \to \mathbb{C} \) is locally integrable if for every point \( x \in X \) and any top-form \( \omega \in \Gamma(U, \Omega^{d}) \) defined in a compact neighbourhood \( U \subset X \) of \( x \) the integral

\[
\int_{U} |\phi(y)||\omega(y)| (7.1)
\]

is convergent.

Theorem 7.2. (V. Vologodsky) Let \( f : X \to A^{1} \) be a proper morphism and let \( \omega \in \Gamma(X, \Omega^{d}) \). Suppose that both are defined over \( F \). Then there exist an open compact subgroup \( K \) of \( O_{F}^{*} \) and a \( K \)-invariant function \( \alpha(x) \) such that distribution \( f_{!}(|\omega|) \) on \( F \) is equal to \( \alpha(x)dx \) for \( |x| > 0 \).

Theorem 7.2 is proved in the Appendix.

Corollary 7.3. Let \( \psi : F \to \mathbb{C}^{*} \) be a non-trivial character. Then the integral

\[
\int_{X} |\psi(f)||\omega| := \int_{F} f_{!}(|\omega|) \cdot \psi := \lim_{N \to \infty} \int_{|x| < N} f_{!}(|\omega|) \psi \tag{7.2}
\]

is convergent.

7.4. Algebraic-geometric distributions. Let as before \( X \) be an algebraic variety over \( F \).

Definition 7.5. By an algebraic-geometric distribution on \( X \) we mean a quadruple \( \Phi = (Y, f, p, \omega) \) where

1) \( Y \) is a smooth algebraic variety over \( F \)
2) \( p : Y \to X, f : Y \to A^{1} \) are morphisms defined over \( F \)
3) \( \omega \in \Gamma(Y, \Omega^{\text{top}}_{Y}) \)

By an isomorphism between two algebraic-geometric distributions \( \Phi = (Y, f, p, \omega) \) and \( \Phi' = (Y', f', p', \omega') \) we shall mean a birational isomorphism between \( Y \) and \( Y' \) preserving all the structure.

Given an algebraic-geometric distribution \( \Phi \) we can try to define a usual distribution \( \Phi = \Phi_{\psi} \) on \( X \) by writing

\[
\Phi = p_{!}(\psi(f)|\omega|) = \lim_{N \to \infty} p_{!}(\psi(f)|\omega|)\big|_{\{y \in Y \mid |f(y)| \leq N\}} \tag{7.3}
\]

Corollary 7.3 guarantees that in a number of cases (7.3) makes sense. For example one can show that the following result holds.
Proposition 7.6. Let an algebraic-geometric distribution $\Phi$ as above be given. Assume that there exists an open dominant embedding $Y \hookrightarrow \tilde{Y}$ such that $\tilde{Y}$ is smooth and such that $p, f$ and $\omega$ extend to $\tilde{Y}$. Assume furthermore that the morphism $p \times f : \tilde{Y} \to X \times \mathbb{A}^1$ is proper. Then $\Phi$ given by (7.3) is a well-defined distribution on $X$.

We say that an algebraic-geometric distribution $\Phi$ is finite if it satisfies the conditions of Proposition 7.6. Also, we call $\Phi$ weakly finite if it is finite over the generic point of $X$. We also say that $\Phi$ is analytically finite if it is weakly finite and for any finite extension $E/F$ the corresponding distribution $\Phi$ defined a priori on a dense open subset of $X(E)$ is locally $L^1$ on the whole of $X(E)$. In this case we call $\Phi$ the materialization of $\Phi$.

Lemma 7.7. Let $X = T$ be a split torus over $F$ and and let $\rho$ be an admissible representation of $T^\vee$. Let $\Phi_{\rho,T} = (T, f_\rho, p_\rho, \omega_\rho)$ be given by the formulas of Section 4.2. Then $\Phi_{\rho,T}$ is analytically finite and the distribution $\Phi_{\rho,T}$ is the materialization of $\Phi_{\rho,T}$.

7.8. Reduction of algebraic-geometric distributions. Let $\Phi = (Y, f, p, \omega)$ be an algebraic-geometric distribution and let $V$ be a vector-group defined over $F$. We say that $V$ acts on $\Phi$ if we are given a free action of $V$ on $Y$ such that

1) $p$ and $\omega$ are $V$-invariant.

2) Let $q : Y \to Z = Y/V$ be the quotient map. Then for any $z \in Z$ the restriction of $f$ to $q^{-1}(z)$ is an affine function.

Let $V^*$ denote the dual vector space to $V$. We denote by $s : Z \to V^*$ the map sending every $z \in Z$ to the linear part of $f|_{q^{-1}(z)}$. Set

$$\overline{Y} = \{ z \in Z | s(z) = 0 \}. \quad (7.4)$$

We say that an action of $V$ on $\Phi$ is non-degenerate if for any $\overline{y} \in \overline{Y}$ the differential $ds : T_{\overline{y}}\overline{Y} \to V^*$ is onto. In this case the variety $\overline{Y}$ is smooth and for any $\overline{y} \in \overline{Y}$ we get the natural isomorphism $T_{\overline{y}}Z/T_{\overline{y}}Y \simeq V^*$. Hence for any $\overline{y} \in \overline{Y}$ and any $y \in q^{-1}(\overline{y})$ we have the natural isomorphisms

$$\Lambda^{\text{top}}(T_{\overline{y}}Y) \simeq \Lambda^{\text{top}}(T_{\overline{y}}Z) \otimes \Lambda^{\text{top}}(V) \simeq \Lambda^{\text{top}}(T_{\overline{y}}Y) \otimes \Lambda^{\text{top}}(T_{\overline{y}}Z/T_{\overline{y}}Y) \otimes \Lambda^{\text{top}}(V) \simeq \Lambda^{\text{top}}(T_{\overline{y}}Y) \otimes \Lambda^{\text{top}}(V^*) \otimes \Lambda^{\text{top}}(V) \simeq \Lambda^{\text{top}}(T_{\overline{y}}Y).$$

Therefore, for any $\overline{y} \in \overline{Y}$, $y \in q^{-1}(\overline{y})$, the restriction of $\omega$ to $T_{\overline{y}}Y$ defines an element $\overline{\omega} \in \Lambda(T_{\overline{y}}Y)^*$. Since $\omega$ is $V$-invariant, it follows that $\overline{\omega}$ does not depend on the choice of $y \in q^{-1}(\overline{y})$. It is easy to see that there exists a top-degree form $\overline{\omega}$ on $\overline{Y}$ such that for any $\overline{y} \in \overline{Y}$ one has $(\overline{\omega})_{\overline{y}} = \overline{\omega}_{\overline{y}}$.

By the definition, the restriction of $f$ to $q^{-1}(Y)$ is $V$-invariant. So, $f|_{q^{-1}(Y)} = \overline{f} \circ q$ for some function $\overline{f}$ on $\overline{Y}$. Also, since $p$ is $V$-invariant we have $p|_{q^{-1}(Y)} = \overline{p} \circ q$ for some morphism $p : \overline{Y} \to X$. 22
We call the algebraic-geometric distribution $\Phi = (Y, p, f, \omega)$ the reduction of $\Phi$.

Let us see the effect of this reduction procedure on materializations. Assume that $\Phi$ is analytically finite and denote by $\Phi$ its materialization. For simplicity we shall assume that $F$ is non-archimedian. Let $V_n$ be a sequence of open compact subgroups of $V$, such that

$$V = \bigcup_{n=1}^{\infty} V_n$$  (7.5)

Let also $Y_n$ be an increasing covering of $Y$ by $V_n$-invariant open compact subsets. Consider the sequence $\Phi_n, n = 1, 2, \ldots$ of distributions on $X$ where

$$\Phi_n(\phi) = \int_{Y_n} \psi(f)p^*(\phi)|\omega|.$$  (7.6)

Lemma 7.9. The sequence $\Phi_n$ weakly converges to $\Phi$.

When $\dim V = 1$ this is proved in [10](§1). The general case is treated similarly.

7.10. $W$-equivariant distributions. Let now $G$ be a reductive algebraic group over $F$ and let $T$ be its Cartan group. The Weyl group $W$ acts naturally on $T$ and we have the natural $F$-rational morphism $s : G \to T/W$. Let $G_r$ denote the set of regular elements in $G$. Then the restriction of $s$ to $G_r$ is a smooth map. In the rest of this section $s$ will always denote the restriction of $s$ to $G_r$.

Let now $\Phi_T = (Y_T, f_T, p_T, \omega_T)$ be an algebraic-geometric distribution on $T$. We assume that $p_T$ is a dominant map. By a $W$-equivariant structure on $\Phi$ we shall mean a birational action of $W$ on $Y$ such that

1) $f_T$ is invariant under this action.
2) $p_T$ intertwines the action of $W$ on $Y_T$ and on $T$.
3) One has $w^*(\omega_T) = (-1)^l(w)\omega_T$, where $l : W \to \mathbb{Z}$ is the length function.

Since $W$ is a finite group we can always find an open dense subset of $Y_T$ on which the action $W$ is biregular. We can also assume that this subset is chosen in such a way that it is contained in the set of regular semi-simple elements. Note that birational modifications of $Y_T$ (such as passing to an open subset) do not change the distribution $\Phi$ given by (7.3). We also assume in the sequel that $p_T$ is a smooth map.

Assume now that we are given a $W$-equivariant algebraic-geometric distribution $\Phi_T$ as above. Then we may construct an algebraic-geometric distribution $\Phi = (Y_G, f_G, p_G, \omega_G)$ over $G_r$ in the following way. We may assume that $W$ acts biregularly on $Y_T$. Then we define

$$Y_G = G_r \times_{T/W} Y_T/W$$  (7.7)

We let $p_G$ be the projection on the first multiple. Condition 1 above implies that $f_T$ descends to a function on $Y_T/W$ and hence defines a function $f_G$ on $Y_G$. 

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The differential form $\omega_G$ on $Y_G$ is defined as follows. Let $y \in Y_G$ and let $g = p_G(y)$. Choose a Borel subgroup $B$ of $G$ containing $g$ and let $U$ be its unipotent radical. Then we have the canonical isomorphism $B/U \simeq T$ and we set $t = g \mod U \in T$. Let $y = (g, y'')$ where $y'' \in Y_T/W$. Since $g$ is a regular and semi-simple there exists unique $y' \in P_T^{-1}(t)$ such that $y' \mod W = y''$.

Let $O_g$ denote the orbit of $g$ under the adjoint action. Then we have the short exact sequence

$$0 \to T_y Y_T \to T_y Y_G \to T_g O_g \to 0 \quad (7.8)$$

where $T_y Y_G, T_y Y_T, T_g O_g$ are the corresponding tangent spaces.

The choice of $B$ induces a symplectic form $\alpha_B$ on $T_g O_g$. The tensor product $\alpha_{B}^{\dim O_g/2} \otimes \omega_{T'|T_y Y_T}$ defines an element in $\Lambda^{\dim Y_G} (T_y Y_G)^{\ast}$. It is easy to see that condition 3 above implies that this element does not depend on the choice of $B$ and we define it to be $\omega_G|T_y Y_G$.

**Conjecture 7.11.** 1. Let $G$ be a split reductive group over $F$ and let $\rho : G^{\vee} \to GL(n, \mathbb{C})$ be an admissible representation of $G^{\vee}$. Let $\Phi_{\rho, T}$ be the algebraic-geometric distribution as in Lemma [7.7]. Then there exists a $W$-equivariant structure on $\Phi_{\rho, T}$ such that the corresponding distribution $\Phi_{\rho, G}$ is analytically finite and the underlying distribution on $G$ is equal to $\Phi_{\rho, G}$ introduced in Section 3.

Let $P \subset G$ be a parabolic subgroup and let $L$ be the corresponding Levi factor. The representation $\rho$ defines canonically a representation $\rho_L$ of $L^{\vee}$, with the same restriction to $T^{\vee}$. Note that the Weyl group $W_L$ is naturally a subgroup in $W$.

2. The restriction of the $W$-action on $\Phi_{\rho, T}$ to $W_L$ is equal to the corresponding $W_L$-action on $\Phi_{\rho, L}$. 

**Remark.** We shall see some examples of the above $W$-equivariant structure in some examples. Surprisingly, the corresponding $W$-action on $T_\rho$ turns out to be rather complicated and it is related to the theory of geometric crystals (cf. [3]).

8. **The case of $G(m,n)$ and its applications**

Let $G$, $T$ and $\rho$ be as above. In Section 4 we have constructed a finite algebraic-geometric distribution $\Phi_{\rho, T} = (T_\rho, f_\rho, p_\rho, \omega_\rho)$ on $T$. It is explained in Section 7 that in order to get from it a finite algebraic-geometric distribution $\Phi_{\rho, G}$ one has to define a birational action of the Weyl group $W$ on $T_\rho$ satisfying certain conditions. We do not know how to construct this action in general. In this section we give an explicit construction of this action for the group $G = G(n,m)$ (cf. Section 3.14).

8.1. **The case $m = 2$.** We now want to define the action of $W$ on $T_\rho$ in the case when $m = 2$. The Weyl group $W$ of $G = G(2,n)$ is isomorphic to $\mathbb{Z}_2 \times S_n$. In this
subsection we are going to give an explicit formula for the action of the first factor by using part 2 of Conjecture 7.11.

Let \( T \) be as before a maximal torus of \( G \). It can be identified with the variety of all collections \((a_1, a_2, b_1, \ldots, b_n)\), where \( a_i, b_j \in \mathbb{G}_m \) and \( a_1a_2 = b_1b_2 \).

The torus \( T_\rho \) consists of all matrices \( t = (t_{ij}) \) where \( i = 1, 2 \) and \( j = 1, \ldots, n \). The map \( p_\rho \) is given by

\[
p_\rho((t_{ij})) = (t_{11}t_{12}\ldots t_{1n}, t_{21}t_{22}\ldots t_{2n}, t_{11}t_{21}, t_{12}t_{22}, \ldots, t_{1n}t_{2n}) \quad (8.1)
\]

The function \( f_\rho \) sends the matrix \((t_{ij})\) to \( \sum_{i,j} t_{ij} \) and the differential form \( \omega_\rho \) is up to a sign equal to \( \prod dt_{ij} \).

To define \( \mathbb{Z}_2 \)-action on \( \Phi_{\rho,T} \) we need to define an involution \( \tau \) on \( T_\rho \) that preserves \( f_\rho \), sends \( \omega_\rho \) to \( -\omega_\rho \) and has the following two properties:

1. For every \( t \in T_\rho \) and every \( j = 1, \ldots, n \) one has \( t_{1j}t_{2j} = \tau(t_{1j})\tau(t_{2j}) \).
2. For every \( t \in T_\rho \) \( t_{11}\ldots t_{1n} = \tau(t_{11})\ldots \tau(t_{1n}) \) and \( t_{21}\ldots t_{2n} = \tau(t_{21})\ldots \tau(t_{2n}) \).

It is easy to see that in the case when \( n > 2 \) there exist many ways to define an involution on \( T_\rho \) which satisfy the compatibility conditions (1) and (2). We will use the conjectural properties of the distribution \( \Phi_{\rho$,G(2,n)} \) to obtain the formula for the involution \( \tau \).

Let \( T_n = \mathbb{G}_m^n \) and let \( N : T_n \to \mathbb{G}_m \) be given by \( N((t_1, \ldots, t_n)) = t_1\ldots t_n \). Define

\[
\mathbb{G}_n = \{(g,t) \in \text{GL}(2) \times T_n | \det(g) = N(t)\} \quad (8.2)
\]

The group \( \mathbb{G}_n \) can be naturally regarded as a Levi subgroup of \( \text{G}(2,n) \). Hence the dual group \( \mathbb{G}_n^\vee \) is a Levi subgroup of \( \text{G}(2,n)^\vee \) and we can restrict the representation \( \rho \) to it. This restriction can be explicitly described as follows. The group \( \mathbb{G}_n^\vee \) is isomorphic to the quotient of \( \text{GL}(2,\mathbb{C}) \times (\mathbb{C}^*)^n \) by the subgroup consisting of elements of the form \((x \cdot \text{Id}, x^{-1}, x^{-1}, \ldots, x^{-1})\). For every \( i = 1, \ldots, n \) we can define a two-dimensional representation \( \rho_i \) of \( \mathbb{G}_n^\vee \), sending \((g, z_1, \ldots, z_n)\) to \( g\bar{z}_i \). Then it is easy to see that

\[
\rho = \bigoplus_{i=1}^n \rho_i. \quad (8.3)
\]

The Cartan group of \( \mathbb{G}_n \) is naturally isomorphic to the Cartan group \( T \) of \( \text{G}(2,n) \) considered in Section 3.1. The Weyl group of \( \mathbb{G}_n \) is equal to the first factor of the Weyl group \( W \) of \( \text{G}(2,n) \).

As follows from (3.10) and (8.3) the distribution \( \Phi_{\rho$,G_n} \) is equal to the convolution

\[
\Phi_{\rho$,G_n}(g, b_1, \ldots, b_n) = (\Phi_{b_1} \ast \ldots \ast \Phi_{b_n})|db_1\ldots db_n| \quad (8.4)
\]

where distributions \( \Phi_b, b \in F^* \) on \( \text{GL}(m, F) \) are defined in Section 3.3. In other words, \( \Phi_{\rho$,G_n} \) can be thought of as a materialization of the algebraic-geometric distribution \( \Phi = (\overline{Y}, \overline{f}, \overline{p}, \overline{\omega}) \), where:

- \( \overline{Y} = \text{GL}(2)^n \)
- \( \overline{f}(g_1, \ldots, g_n) = \text{tr}(g_1) + \ldots + \text{tr}(g_n) \)
• \( \tilde{p}(g_1, \ldots, g_n) = (g_1 \ldots g_n, \det(g_1), \ldots, \det(g_n)) \)
• \( \tilde{\omega} = \det(g_1 \ldots g_n)dg_1 \ldots dg_n \)

where \( dg \) denotes a translation invariant top degree differential form on \( \text{GL}(2) \).

More precisely, for every \( N > 0 \) define an open subset \( \tilde{Y}(N) \) of \( \tilde{Y} \) by

\[
\tilde{Y}(N) = \{(g_1, \ldots, g_n) \in \text{GL}(2)^n | |(g_i)_{\alpha \beta}| \leq N\}
\]  

(8.5)

Let \( \tilde{\mu}_N \) denote the restriction of the distribution \( \psi(\tilde{f})|\tilde{\omega}| \) to \( \tilde{Y}(N) \).

**Lemma 8.2.**

\[
\Phi_{\rho,G_n} = \lim_{N \to \infty} p_n(\tilde{\mu}_N).
\]  

(8.6)

Fix a maximal unipotent subgroup \( U_+ \) in \( \text{GL}(2) \). Then \( U_+ \simeq \mathbb{G}_a \). For every \( g \in \text{GL}(2) \) the function \( u \mapsto \text{tr}(gu) - \text{tr}(g) \) is linear in \( u \). We set \( \text{tr}(gu) - \text{tr}(g) = g_u - u \) (thus \( g \mapsto g_u \) is a function on \( \text{GL}(2) \)).

Let \( V_+ = U_+^{-1} \). Define an action of \( V_+ \) on \( \tilde{Y} \) by setting

\[
(u_1, \ldots, u_{n-1})(g_1, \ldots, g_n) = (g_1u_1^{-1}, u_1g_2u_2^{-1}, \ldots, u_{n-1}g_n)
\]  

(8.7)

This action preserves \( \tilde{p} \) and \( \tilde{\omega} \) and for any \( (g_1, \ldots, g_n) \in \tilde{Y} \) we have

\[
\tilde{f}(ug) - \tilde{f}(g) = \sum_{i=1}^{n-1} u_i((g_{i+1})_1 - (g_i)_1).
\]  

(8.8)

Thus we obtain an action of \( V_+ \) on \( \tilde{\Phi} = (\tilde{Y}, \tilde{f}, \tilde{p}, \tilde{\omega}) \). Let \( \tilde{\Phi}_{U_+} = (\tilde{Y}, \tilde{p}, \tilde{f}, \tilde{\omega}) \) be the \( U_+ \)-reduction of \( \tilde{\Phi} \) (cf. Section 7.3). As follows from Lemma 7.3 the materialization \( \tilde{\Phi}_{U_+} \) is equal to \( \Phi_\rho \).

Let \( B \) be the variety of Borel subgroups of \( \text{GL}(2) \). For any \( g \in \text{GL}(2) \) we denote by \( B^g \subset B \) the subvariety of Borel subgroups containing \( g \).

**Lemma 8.3.** For any regular \( g \in \text{GL}(2) \) the centralizer of \( g \) in \( \text{PGL}(2) \) acts simply transitively on the variety \( B - B^g \).

**Corollary 8.4.** For any two unipotent subgroups \( U_+, U_- \subset \text{GL}(2) \) we have a canonical isomorphism \( \tilde{\Phi}_{U_+} \simeq \tilde{\Phi}_{U_-} \). Therefore, we can write \( \tilde{\Phi} \) instead of \( \tilde{\Phi}_{U_+} \).

Let

\[
\tilde{G}_n = \{(B, g, t) \in B \times G_n | g \in B\}
\]  

(8.9)

We have the natural maps \( r : \tilde{G}_n \to G_n \) (sending \( (B, g, t) \) to \( g \)) and \( m : \tilde{G}_n \to T \) (sending \( (B, g, t) \) to \( g \mod U_B \) where \( U_B \) denotes the unipotent radical of \( B \)). Then \( r \) is a ramified double covering and for any \( (g, t) \in G_n \) one has \( r^{-1}(g, t) = B^g \). We denote by \( \tilde{\Phi} = (\tilde{Y}, \tilde{p}, \tilde{f}, \tilde{\omega}) \) the lift of \( \Phi \) to \( \tilde{G}_n \). By definition \( \tilde{Y} = \tilde{Y} \times \tilde{G}_n \).

Since the generic fiber of \( m \) carries a canonical top form (see Section 7.10) we can define an algebraic-geometric distribution \( m^*(\Phi_{\rho,T}) \) on \( \tilde{G}_n \). We write \( m^*(\Phi_{\rho,T}) = \)
(\(Y'_T, p'_T, f'_T, \omega'_T\)), where \(Y'_T = Y_{\rho,T} \times \tilde{G}_n\). We will construct now an isomorphism \(\mu : \tilde{\Phi} \rightarrow m^*(\Phi_{\rho,T})\).

For any \(B \in B\) such that \(U_+ \not\subset B\) the multiplication map gives rise to birational isomorphisms \(B \times U_+ \rightarrow \text{GL}(2)\) and \(U_+ \times B \rightarrow \text{GL}(2)\). Therefore the subset \(\tilde{Y}_+ := B^{n-1} \times \text{GL}(2) \subset \tilde{Y}\) is a section of the action of \(V_+\) on \(\tilde{Y}\). Fix any element \((g,t;B) \in \tilde{G}_n\) such that \(g\) is regular and semisimple. Since by definition \(g \in B\) we see that \(\tilde{Y}_+ \cap \tilde{p}^{-1}(g,t;B) \subset B^o\). So we can describe the variety \(\tilde{p}^{-1}(g,t;B)\) as the preimage \(s_B^{-1}(0)\) where \(s_B\) is the restriction of the map \(s : \tilde{Y} \rightarrow V^*\) to \(B^n \cap p^{-1}(g,t)\) where \(s\) is as in Section 7.8. In other words we can identify the variety \(\tilde{Y}\) with \(s_B^{-1}(0)\).

Let \(U \subset B\) be the unipotent radical of \(B\) and let \(T_g \subset B\) be the centralizer of \(g\). Then \(B = T_g U\) and \(T_g\) is canonically isomorphic to \(\mathbb{G}_m^2\). Therefore \(T_g^n\) is canonically isomorphic to \(T_{\rho}\).

**Lemma 8.5.** a) The natural projection \(B \rightarrow T_g\) defines an embedding \(\mu : s_B^{-1}(0) \hookrightarrow T_{\rho}\) and the image of \(i\) is equal to \(p_{\rho,T}^{-1}(g,t)\).

In other words we have constructed an isomorphism \(\mu : \tilde{Y} \rightarrow Y'_T\).

b) \(\mu\) defines an isomorphism of algebraic-geometric distributions.

The natural involution \(\theta\) of \(\tilde{G}_n\) over \(G_n\) induces an involution \(\tilde{\theta}\) of \(\tilde{\Phi}\) and we can define an involution \(\theta'\) of \(m^*(\Phi_{\rho,T})\) by \(\theta' = \mu \circ \tilde{\theta} \circ \mu^{-1}\). It is easy to see that the involution \(\theta'\) is \(\text{GL}(2)\)-invariant and therefore it is induced by an involution \(\tau\) on \(\Phi_{\rho,T}\).

Let us give an explicit formula for \(\tau\). For every \(k = 1, \ldots, n\) we define the function \(\Delta_k\) on \(T_{\rho}\) by

\[
\Delta_k(t) = t_{11} \ldots t_{1(k-1)} + t_{11} \ldots t_{1(k-2)} t_{2k} + \ldots + t_{22} \ldots t_{2k}\]  \hspace{1cm} (8.10)

(by definition \(\Delta_1(t) = 1\)).

We also define a rational function \(\eta\) on \(T_{\rho}\) setting

\[
\eta(t) = \frac{t_{11} \ldots t_{1n} - t_{21} \ldots t_{2n}}{\Delta_n(t)}.\]  \hspace{1cm} (8.11)

**Lemma 8.6.** The involution \(\tau\) satisfies

\[
\tau(t)_{21} \tau(t)_{22} \ldots \tau(t)_{2k} = t_{21} \ldots t_{2k} + \Delta_k(t) \eta(t)\]  \hspace{1cm} (8.12)

It is clear that \(\tau\) is uniquely determined by \((8.12)\) and by the conditions (1) and (2) above.

Assume for example that \(n = 2\). Then one can compute the above action explicitly. Namely, in this case we have

\[
T = \{(a_1, b_1, a_2, b_2) \in \mathbb{G}_m^4 \mid a_1 b_1 = a_2 b_2\}\]  \hspace{1cm} (8.13)
Let \( w \in W \) be the involution which interchanges \( a_1 \) and \( b_1 \). Let \( \tau = \tau_w \). Then an explicit calculation shows that

\[
\tau : (t_{11}, t_{12}, t_{21}, t_{22}) \mapsto (t_{21}, t_{12} + t_{21}, t_{22}, t_{11} + t_{22})
\]

(8.14)

Moreover, it is easy to see that in this case \( \tau \) given by (8.14) is the unique birational involution of \( T_\rho \) which satisfies our requirements.

8.7. The case of \( G(n, m) \). Let us now consider the group \( G(n, m) \) for arbitrary \( n \) and \( m \). In this case one can identify \( T_\rho \) with the variety \( (t_{ij}, i = 1, \ldots, m, j = 1, \ldots, n) \) of \( m \times n \) matrices with non-zero entries. We wish to define a birational action of the Weyl group \( W = S_m \times S_n \) on \( T_\rho \). For every \( \alpha = 1, \ldots, m - 1 \) we define a birational involution \( \tau^1_\alpha \) on \( T_\rho \) in the following way:

1) All rows of \( \tau^1_\alpha(t = t_{ij}) \) except for the \( \alpha \) and \( \alpha + 1 \)st are equal to the corresponding rows of \( t \).

2) The \( \alpha \)- and \( (\alpha + 1) \)st rows of \( \tau^1_\alpha(t) \) are obtained from those of \( t \) by means of (8.12).

Similarly, for every \( \beta = 1, \ldots, n - 1 \) we define \( \tau^2_\beta(t) = \tau^1_{\beta}(t') \) where \( t' \) is the transposed matrix to \( t \). The following lemma is straightforward.

Lemma 8.8. The involutions \( \tau^1_\alpha, \tau^2_\beta \) commute with \( p_\rho \), preserve \( f_\rho \) and map \( \omega_\rho \) to \(-\omega_\rho\).

The following theorem is proven in [3] (§ 6.2).

Theorem 8.9. The involutions \( \tau^1_\alpha \) and \( \tau^2_\beta \) define a birational action of the group \( S_m \times S_n \) on \( T_\rho \). In particular, \( \tau^1_\alpha \) commutes with \( \tau^2_\beta \) for any \( \alpha \) and \( \beta \).

Despite the fact the formulas (8.12) are quite explicit we did not manage to prove Theorem 8.9 directly. The proof, given in [3] uses the machinery of geometric crystals.

8.10. \( \gamma \)-functions. Let us still assume that \( G = G(n, m) \). Recall that we have the natural character \( \sigma : G \to \mathbb{G}_m \) (which sends the pair \((A, B)\) of matrices to \( \det(A) = \det(B) \)). Using Theorem 8.9 we can define an algebraic-geometric distribution \( \Phi_\rho,G \) over the generic point of \( G \). We would like to check that it gives the "correct" answer.

Let \( \pi_m, \pi_n \) be generic representations of respectively \( GL(m, F) \) and \( GL(n, F) \). Then in [14] H.Jacquet, I. Piatetski-Shapiro and J. Shalika define a rational function \( \Gamma(\pi_m, \pi_n, s) \) of one complex variable \( s \) (which depends on the choice of a character \( \psi \)) The following result can be deduced from [3].

Theorem 8.11. 1. The materialization \( \Phi_{\rho,G} \) of \( \Phi_{\rho,G} \) defines a \( \sigma \)-compact distribution on the whole of \( G \). We denote by \( \gamma_\rho \) the corresponding rational function on the set \( \text{Irr}(G) \).
2. Assume that \( n \geq m \). Let \( \pi_m, \pi_n \) be as above and let \( \pi \) be an irreducible representation of \( G \) such that \( \text{Hom}_G(\pi, \pi_m \otimes \pi_n) \neq 0 \). Then

\[
\Gamma(\pi_m, \pi_n, s) = \text{sign}(\pi_n) \gamma_{\rho}(\pi| \cdot |^s)
\]

(8.15)

where \( \text{sign}(\pi) \) is the value of the central character of \( \pi_n \) at the matrix \(-\text{Id}_n \in GL(n, F)\).

8.12. Lifting from non-split tori. Let us recall the notations of Section 2.5: let \( E/F \) be a separable extension of \( F \) of degree \( n \). Let \( \alpha_E : \text{Gal}(F/F) \to S_n \) be the corresponding homomorphism. We denote by \( N : T_E \to G_m \) the morphism of algebraic groups coming from the norm map \( N : E^* \to F^* \).

Let \( T_E = \text{Res}_{E/F} G_m,E \) where \( \text{Res}_{E/F} \) denotes the functor of restriction of scalars.

For every \( m > 0 \) we set

\[
G_{E,m} = \{(t, g) \in T_E \times GL(m)| \ N(t) = \text{det}(g)\}
\]

(8.16)

As is explained in Section 3.19, in order to define the lifting \( l_E(\theta) \in \text{Irr}(GL(n, F)) \) of a character \( \theta : E^* \to \mathbb{C}^* \), it is enough to define the \( \gamma \)-function \( \gamma(l_E(\theta), \pi_m) \) as a function on \( \text{Irr}(GL(m, F)) \). Moreover, it is enough to know this function only up to a constant.

We are now going to give a conjectural definition of \( \gamma(l_E(\theta), \pi_m) \) by constructing a weakly finite algebraic geometric distribution \( \Phi_{E,m} \) on \( G_{E,m} \) (in fact, we will define it only up to a constant multiple). We conjecture that this distribution is analytically finite and that the corresponding function on \( \text{Irr}(E^*) \times \text{Irr}(GL(m, F)) \) is equal to \( \gamma_{E,m} \).

Choose \( d \in F^* \) such that its image in \( F^*/(F^*)^2 \) is equal to the discriminant of \( E \).

Let us also choose its square root \( \sqrt{d} \) in \( F \).

Let

\[
G'_{E,m} = \{(t_1, \ldots, t_n, g) \in \mathbb{G}_m^n \times GL(m)| t_1 \ldots t_n = \text{det}(g)\}
\]

Using Section 8.7 we define an algebraic-geometric distribution \( \Phi' = (Y', p', f', \omega') \) on \( \mathbb{G}_m^n \times GL(m) \). Moreover, we have the natural \( S_n \)-action on \( Y' \), which is compatible with \( p' \) and leaves \( f' \) (resp. \( \omega' \)) invariant (resp. skew-invariant).

Define a new \( \text{Gal}(F/F) \)-action on \( Y' \) by

\[
g^{\text{new}}(y) = \alpha_E(g^{\text{old}}(y))
\]

(8.18)

This action defines a new \( F \)-rational structure on \( Y' \). We denote by \( Y \) the corresponding \( F \)-variety. It is clear that the map \( p' \) gives rise to an \( F \)-rational morphism \( p : Y \to G_{E,m} \). Also the function \( f' \) and the differential form \( \sqrt{d} \omega' \) give rise to \( F \)-rational function \( f \) and differential form \( \omega \) on \( Y \). Thus we set \( \Phi_{E,m} = (Y, p, f, \omega) \) to be the required algebraic-geometric distribution. We denote by \( \Phi_{E,m} \) the materialization of \( \Phi_{E,m} \).
Clearly, for different choices of \( \sqrt{d} \), the resulting materializations \( \Phi_{E,m} \) will differ only by a multiplication by \( c \in \mathbb{C}^* \). Hence, the above construction suffices in order to determine the lifting uniquely.

Since the local Langlands conjecture is known (see [12], [13] and [20]) we can ask whether our definition of lifting coincides with one which is implied by [12], [13] and [20]. More precisely let \( \chi \) be a character of the group \( T = T(F) \). The local class field theory associates to \( \chi \) a homomorphism \( \theta_\chi : \text{Gal}(\overline{E}/E) \to \mathbb{C}^* \). Let \( \Theta_\chi := \text{Ind}_{\text{Gal}(E/F)}^{\text{Gal}(\overline{E}/E)} \theta_\chi \). Then \( \Theta_\chi \) is an \( n \)-dimensional representation of the group \( \text{Gal}(\overline{F}/F) \).

Since the local Langlands conjecture for \( G = \text{GL}(n) \) is known one associate with \( \Theta_\chi \) an irreducible representation \( \pi_\chi \in \text{Irr}(\text{GL}(n,F)) \). Therefore we can consider the function \( \Gamma(\pi_m, \pi_\chi, s) \) on the set \( \text{Irr} \ \text{GL}(m,F) \). One can ask whether there exists \( c \in \mathbb{C}^* \) such that for any \( \pi_m \in \text{Irr} \ \text{GL}(m,F) \) we have
\[
\gamma(\pi_m, \pi_\chi, 0) = c \chi \otimes \pi_m(\Phi_{E,m}).
\]

9. The case of finite fields

In this section we shall give an explicit conjectural construction of \( \Phi_{\rho,G} \) for any \( G \) and for (almost) any \( \rho \) in the case when the field \( F \) is finite. It turns out that in this case the relevant tool from algebraic geometry which allows to go from the case of a torus to the case of arbitrary group is not the language of algebraic-geometric distributions, but that of \( \ell \)-adic perverse sheaves. This tool allows to avoid constructing an action of \( W \) on \( T_\rho \).

9.1. Notations. In this section we fix a finite field \( F \) with \( q \) elements, a prime number \( l \) different from the characteristic of \( F \), and a non-trivial character \( \psi : F \to \overline{\mathbb{Q}}_l^* \). All representations discussed in the note are over the field \( \overline{\mathbb{Q}}_l \). We denote by \( \mathcal{L}_\psi \) the Artin-Schreier sheaf on \( \mathbb{A}_F^1 \).

Let \( G \) be a reductive algebraic group over \( F \) and let \( T \) be its abstract Cartan group. The torus \( T \) comes equipped with a canonical \( F \)-rational structure. In this section we assume for simplicity that \( G \) is split. Then this \( F \)-rational structure is split too. We will denote by \( \text{Fr} : T \to T \) the corresponding Frobenius morphism. Let \( W \) denote the Weyl group of \( G \). For every \( w \in W \) we set \( \text{Fr}_w = w \circ \text{Fr} \). The morphism \( \text{Fr}_w \) induces a new \( F \)-rational structure on \( T \) and we will denote the corresponding algebraic torus over \( F \) by \( T_w \). It is well-known that there exists an embedding \( T_w \hookrightarrow G \) and that in this way we get a bijection between conjugacy classes of elements in \( W \) and conjugacy classes of \( F \)-rational maximal tori in \( G \).

For a character \( \theta : T_w \to \overline{\mathbb{Q}}_l^* \) we denote by \( R_{\theta,w} \) the corresponding Deligne-Lusztig representation.

For two \( \ell \)-adic complexes \( A \) and \( B \) on \( G \) we can define the convolution complex \( A \ast B \) and by setting
\[
A \ast B = m_t(A \boxtimes B)[\dim G]
\]
(9.1)
where \( m : G \times G \to G \) is the multiplication map.
Let $X$ be an algebraic variety over $F$ and let $\mathcal{F}$ be a complex of $\ell$-adic sheaves on it. By a Weil structure on $\mathcal{F}$ we shall mean an isomorphism $\xi : \text{Fr}^*\mathcal{F} \to \mathcal{F}$ where Fr denotes the geometric Frobenius morphism on $X$. In this case we can define a function $\text{Tr}(\mathcal{F})$ on $X = X(F)$ by setting

$$\text{Tr}(\mathcal{F})(x) = \sum (-1)^i \text{tr}(\xi_x : H^i(\mathcal{F}_x) \to H^i(\mathcal{F}_x))$$ (9.2)

(here $H^i(\mathcal{F}_x)$ denotes the $i$-th cohomology of the fiber of $\mathcal{F}$ at $x$).

Let us choose a square root $q^{1/2}$ of $q$. Then for any Weil sheaf $\mathcal{F}$ on $X$ and any half integer $n$ we can consider the Tate twist $\mathcal{F}(n)$ of $\mathcal{F}$. By the definition one has

$$\text{Tr}(\mathcal{F}(n)) = \text{Tr}(\mathcal{F})q^{-n}$$ (9.3)

9.2. $\gamma$-functions for $\text{GL}(n)$.

9.2.1. The main result. Let $(\pi, V)$ be an irreducible representation of $G = \text{GL}(n, F)$. Choose a non-trivial additive character $\psi : F \to \mathbb{Q}^*_l$ as above and consider the operator

$$\sum_{g \in G} \psi(\text{tr}(g)) \pi(g)(-1)^n q^{-n^2/2} \in \text{End}_G V$$ (9.4)

Since $\pi$ is irreducible, this operator takes the form $\gamma_{G,\psi}(\pi) \cdot \text{Id}_V$ where $\gamma_{G,\psi}(\pi) \in \overline{\mathbb{Q}}_l$ (we will omit the subscripts $G$ and $\psi$ when it does not lead to a confusion). The number $\gamma(\pi) = \gamma_{G,\psi}(\pi)$ is called the gamma-function of the representation $\pi$. The purpose of this subsection is to compute explicitly the gamma-functions of all irreducible representations of $G$.

Let $W \simeq S_n$ denote the Weyl group of $G$.

Fix $w \in W$. For a character $\theta : T_w \to \overline{\mathbb{Q}}_l$ we set

$$\gamma_w(\theta) = (-1)^{n+l(w)} q^{-n^2/2} \sum_{t \in T_w} \psi(\text{tr}(t)) \theta(t) \in \overline{\mathbb{Q}}_l$$ (9.5)

Example. Assume that $w \in S_n$ is a cycle of length $n$. Then $T_w \simeq E^*$ where $E$ is the (unique up to isomorphism) extension of $F$ of degree $n$. In this case $\gamma_w(\theta) = \gamma_E(\theta)$ for any character $\theta$ of $E^*$, where by $\gamma_E(\theta)$ we denote the $\gamma$-function defined as in (9.4) for the group $\text{GL}(1, E) \simeq E^*$.

Theorem 9.3. Assume that an irreducible representation $(\pi, V)$ appears in $R_{\theta, w}$ for some $w$ and $\theta$ as above. Then

$$\gamma(\pi) = (-1)^{l(w)} \gamma_w(\theta)$$ (9.6)

where $R_+$ denotes the set of positive roots of $G$. In particular, $\gamma(\pi) = \gamma(\pi')$ if $\pi$ and $\pi'$ appear in the same virtual representation $R_{\theta, w}$.

The rest of this subsection is occupied with the proof of Theorem 9.3.
9.3.1. **Character sheaves.** Let $G$ be an arbitrary reductive algebraic group over $F$. Let us recall Lusztig’s definition of (some of) the character sheaves. Let $\tilde{G}$ denote the variety of all pairs $(B, g)$, where

- $B$ is a Borel subgroup of $G$
- $g \in B$

One has natural maps $\alpha : \tilde{G} \to T$ and $\pi : \tilde{G} \to G$ defined as follows. First of all, we set $\pi(B, g) = g$. Now, in order to define $\alpha$, let us remind that for any Borel subgroup $B$ of $G$ one has canonical identification $\mu_B : B/U_B \to T$, where $U_B$ denotes the unipotent radical of $B$ (in fact, this is how the abstract Cartan group $T$ is defined). Now we set $\alpha(B, g) = \mu_B(g)$.

Let $L$ be a tame local system on $T$. We define $K_L = \pi_! \alpha^*(L)[\dim G]$. One knows (cf. [21], [19]) that the sheaf $K_L$ is perverse.

Assume now, that for some $w \in W$ there exists an isomorphism $L \simeq Fr_w^*(L)$ and let us fix it. It was observed by G. Lusztig in [21] that fixing such an isomorphism endows $K_L$ canonically with a Weil structure. Lusztig’s definition of this Weil structure was as follows.

Let $j : G_{rs} \to G$ denote the open embedding of the variety of regular semisimple elements in $G$ into $G$.

**Lemma 9.4.**

$$K_L = j_!(K_L|_{G_{rs}})$$ (9.7)

Here $j_!$ denotes the Goresky-MacPherson (intermediate) extension (cf. [1]).

The lemma follows from the fact that the map $\pi$ is small in the sense of Goresky and McPherson.

The lemma shows that it is enough to construct the Weil structure only on the restriction of $K_L$ on $G_{rs}$. The latter now has a particularly simple form. Namely, let $\tilde{G}_{rs}$ denote the preimage of $G_{rs}$ under $\pi$ and let $\pi_{rs}$ denote the restriction of $\pi$ to $\tilde{G}_{rs}$. Then it is easy to see that $\pi_{rs} : \tilde{G}_{rs} \to G_{rs}$ is an unramified Galois covering with Galois group $W$. In particular, $W$ acts on $\tilde{G}_{rs}$ and this action is compatible with the action of $W$ on $T$ in the sense that the restriction of $\alpha$ on $\tilde{G}_{rs}$ is $W$-equivariant.

Now, an isomorphism $L \simeq Fr_w^*(L)$ gives rise to an isomorphism

$$\alpha^*L \simeq (w \circ Fr)^*(\alpha^*L)$$ (9.8)

(here both $w$ and $Fr$ are considered on the variety $\tilde{G}$). Since $\pi_{rs}$ is a Galois covering with Galois group $W$, it follows that one has canonical identification

$$\pi_{rs}!(Fr^*(\alpha^*L)) \simeq \pi_{rs}!(Fr^*(\alpha^*L))$$ (9.9)

Hence from (9.7) and (9.9) we get the identifications

$$Fr^*\pi_{rs!}(\alpha^*L) \simeq \pi_{rs!}(Fr^*(\alpha^*L)) \simeq \pi_{rs!}((w \circ Fr)^*(\alpha^*L)) \simeq \pi_{rs!}(\alpha^*L)$$ (9.10)
which gives us a Weil structure on $\pi_{rs!}(\alpha^*\mathcal{L}) \simeq \mathcal{K}_{L,G}$. Hence we have defined a canonical Weil structure on $\mathcal{K}_{L,G}$.

Let now $\theta : T_w \to \overline{\mathbb{Q}}_l$ be any character. It is well known that one can associate to $\theta$ a one-dimensional local system $\mathcal{L}_\theta$ together with an isomorphism $\mathcal{L}_\theta \simeq \text{Fr}_w^* \mathcal{L}_\theta$. This local system is constructed as follows: consider the sheaf $(\text{Fr}_w)_*(\overline{\mathbb{Q}}_l)$. It is clear that this sheaf has a natural fiberwise action of the group $T_w$. Thus we set $\mathcal{L}_\theta$ to be the direct summand of $(\text{Fr}_w)_*(\overline{\mathbb{Q}}_l)$ on which $T_w$ acts by means of the character $\theta$. The following result is due to G. Lusztig.

**Theorem 9.5.**

$$\text{Tr}(\mathcal{K}_{L,\theta}) = (-1)^{\dim \mathcal{G} \text{ch}(R_{\theta,w})}$$  \hspace{1cm} (9.11)

9.5.1. Now we assume again that $\mathcal{G} = \text{GL}(n)$. Set $\Phi_\psi = \text{tr}^* \mathcal{L}_\psi[n^2](\frac{n^2}{2})$.

In this case it is well-known that for every character $\theta : T_w \to \overline{\mathbb{Q}}_l$ the vector space, spanned by the characters of all irreducible constituents of $R_{\theta,w}$ coincides with the vector space spanned by the functions of the form $\text{Tr}(A)$, where $A$ runs over all direct summands of $\mathcal{K}_{L,\theta}$. Hence Theorem 9.3 follows from the following result.

**Proposition 9.6.**

1. Let $\text{tr}_T : T \to \mathbb{A}^1$ denote the restriction of the trace morphism from $\mathcal{G}$ to $T$. Then for every $\mathcal{L}$ as above one has

$$H_c^i(\text{tr}_T^*(\mathcal{L}_\psi) \otimes \mathcal{L}) = \begin{cases} 0 & \text{if } i \neq n, \\ \text{is one-dimensional if } i = n. \end{cases}$$  \hspace{1cm} (9.12)

We set $H_\mathcal{L} := H_c^n(\text{tr}_T^*(\mathcal{L}_\psi) \otimes \mathcal{L})(\frac{n}{2})$.

2. Let $\theta : T_w \to \overline{\mathbb{Q}}_l$ be a character (for some $w \in W$). Then the natural isomorphism $\text{Fr}_w^* \mathcal{L}_\theta \simeq \mathcal{L}_\theta$ gives rise to a natural endomorphism of $H_{\mathcal{L}_\theta}$ which by abuse of language we will also denote by $\text{Fr}_w$. Then

$$\text{Fr}_w|_{H_{\mathcal{L}_\theta}} = (-1)^{l(w)}\gamma_w(\theta) \cdot \text{Id}$$  \hspace{1cm} (9.13)

3. One has

$$\Phi_\psi \ast \mathcal{K}_{\mathcal{L}} \simeq H_{\mathcal{L}} \otimes \mathcal{K}_{\mathcal{L}}$$  \hspace{1cm} (9.14)

Moreover, if $\mathcal{L}$ is endowed with an isomorphism $\text{Fr}_w^* \mathcal{L} \simeq \mathcal{L}$ then (9.14) is $\text{Fr}_w$-equivariant.

The proof follows easily from standard properties of the Fourier-Deligne transform. The details are left to the reader.

**9.7. The case of arbitrary group.** Let $\mathcal{G}$ be an arbitrary connected reductive algebraic group over $F$. For simplicity we will assume that $\mathcal{G}$ is split. Let $\mathcal{G}^\vee$ denote the Langlands dual group of $\mathcal{G}$ (which is a reductive algebraic group over $\overline{\mathbb{Q}}_l$). Let $\rho : \mathcal{G}^\vee \to \text{Aut}(E)$ be a representation of $\mathcal{G}^\vee$, where $E$ is a vector space over $\overline{\mathbb{Q}}_l$ of dimension $n$. We would like to associate to it a certain function $\pi \mapsto \gamma_\rho(\pi)$ on the set of isomorphism classes of irreducible representations of $G$. This is done as follows.
The group $G^\vee$ comes equipped with a canonical maximal torus $T^\vee$. Let us diagonalize $\rho|_{T^\vee}$. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding characters of $T^\vee$ in $\rho$ (with multiplicities). Let also $M^\vee_\rho$ denote the minimal Levi subgroup containing $\rho(T^\vee)$ and such that the action of $M^\vee_\rho$ is multiplicity free. We denote by $W'$ the group $\text{Norm}_{\text{Aut}(E)}(M^\vee_\rho)/M^\vee_\rho$. We also denote by $W_\rho$ the Weyl group of $\text{Aut}(E)$.

**Lemma 9.8.** $W'$ is naturally a subquotient of $W_\rho$.

**Proof.** Standard. \hfill \Box

Let $T^\vee_\rho \simeq \mathbb{G}_{m,F}^n$ be the Cartan group of $\text{GL}(n)$. Thus we get a natural map $p^\vee_\rho : T^\vee \to T^\vee_\rho$ sending every $t$ to $(\lambda_1(t), \ldots, \lambda_n(t))$. Let now $T_\rho \simeq \mathbb{G}_{m,F}$ denote the dual torus to $T^\vee_\rho$ over $F$ and let $p_\rho : T_\rho \to T$ denote the map, which is dual to $p^\vee_\rho$. Explicitly one has

$$p_\rho(x_1, \ldots, x_n) = \lambda_1(x_1) \cdots \lambda_n(x_n). \quad (9.15)$$

The representation $\rho$ defines the natural homomorphism from $W$ to $W'_\rho$, which by abuse of the language we will also denote by $\rho$. Let now $\pi$ be an irreducible representation of $G$. Assume that $\pi$ appears in some $R_{\theta,w}$ for some $\theta : T_w \to \mathbb{Q}_l^\times$. Let $w'$ be any lift of $\rho(w)$ to $W'_\rho$. Then $p$ induces an $F$-rational map $p_w : T_\rho,w' \to T_w$, hence a homomorphism $p^*_w : T_\rho,w' \to T_w$. Define now

$$\gamma_\rho(\pi) := \gamma(\pi'), \quad (9.16)$$

where $\pi'$ is any irreducible representation of $G_\rho$ which appears in $R_{p^*_w(\theta),w'}$. By Theorem 9.3 one has

$$\gamma_\rho(\pi) = (-1)^{(w')} \gamma_{T_w,p^*_w(\theta)). \quad (9.17)$$

**Lemma 9.9.** The definition of $\gamma_\rho(\pi)$ does not depend on the choice of $w'$.

Let now $\Phi_\rho$ denote the unique central function on $G$ such that for every irreducible representation $(\pi, V)$ of $G$ one has

$$\sum_{g \in G} \Phi_\rho(g)\pi(g) = \gamma_\rho(\pi) \cdot \text{Id}_V \quad (9.18)$$

We would like to compute this function explicitly. We will be able to solve only a slightly weaker problem. Namely, we are going to construct explicitly an $\ell$-adic perverse sheaf $\Phi_{\rho,\psi}$ on $G$ such that

$$\text{Tr}(\Phi_{\rho,\psi}) = \widetilde{\Phi_\rho} \quad (9.19)$$

where $\widetilde{\Phi_\rho}$ is a function which satisfies the following condition:

$$\sum_{g \in G} \tilde{\Phi}_\rho(g)\pi(g) = \gamma_\rho(\pi) \cdot \text{Id} \quad (9.20)$$
for every irreducible ”generic” representation π of G (cf. Theorem 9.11 for the precise statement).

9.9.1. The perverse sheaf Φ_{ρ,ψ}. In what follows we assume that the zero weight does not appear in ρ. Let \( \text{tr}_\rho : T_\rho \to \mathbb{A}^1 \) be given by

\[
\text{tr}_\rho(x_1, ..., x_n) = x_1 + ... + x_n.
\]

Consider the complex \( A_\rho := (p_\rho)_! \text{tr}_\rho^* L_\psi[n](\frac{1}{2}) \) (on \( T \)). It is easy to see that this complex is perverse. We would like to endow this complex with a \( W \)-equivariant structure.

Choose \( w \in W \). We need to define an isomorphism \( \iota_w : w^* (A_\rho) \to A_\rho \). Let (as above) \( w' \) be any lift of \( \rho(w) \) to \( W_\rho \). Then one has

\[
p_\rho(w'(t)) = w(p_\rho(t)).
\]

The sheaf \( \text{tr}_\rho^* L_\psi \) is obviously \( W_\rho \)-equivariant. This, together with (9.22) gives rise to an isomorphism \( \iota'_w : w^* (A_\rho) \to A_\rho \). We now define \( \iota_w := (-1)^{(l(w') - l_w)} \iota'_w \).

**Proposition 9.10.** The isomorphism \( \iota_w \) does not depend on the choice of \( w' \). The assignment \( w \to \iota_w \) defines a \( W \)-equivariant structure on the sheaf \( A_\rho \).

The above \( W \)-equivariant structure on \( A_\rho \) gives rise to the sheaf \( B_\rho = (q_! A_\rho)^W \) on the quotient \( T/W \) where \( q : T \to T/W \) is the canonical map.

Let \( G_r \) be the set of regular elements of \( G \) and let \( i \) be its embedding to \( G \). Let \( s : G_r \to T/W \) be the morphism coming from the identification \( T/W \simeq \mathcal{O}(G)^G \). We define

\[
\Phi_{\rho,ψ} := i_* s^*(B_\rho)[\dim G - \dim T](\frac{\dim G - \dim T}{2})
\]

We also define

\[
\widetilde{\Phi}_\rho = \text{Tr}(\Phi_{\rho,ψ})
\]

By the definition the complex \( \Phi_{\rho,ψ} \) on \( G \) is \( G \)-equivariant with respect to the adjoint action and therefore the function \( \widetilde{\Phi}_\rho \) is central.

9.10.1. The action of \( \widetilde{\Phi}_\rho \) in \( R_{θ,w} \). Recall that a local system \( \mathcal{L} \) on \( T \) is called quasi-regular if for every coroot \( α^\vee : G_m \to T \) the local system \( (α^\vee)^* \mathcal{L} \) is non-trivial. We say that a character \( θ : T_w \to \overline{\mathbb{Q}}_l^\times \) is quasi-regular if the local system \( \mathcal{L}_{θ} \) is quasi-regular.

**Theorem 9.11.** Let \( π \) be an irreducible representation of \( G \), which appears in the Deligne-Lusztig representation \( R_{θ,w} \) for a quasi-regular character \( θ : T_w \to \overline{\mathbb{Q}}_l^\times \). Then

\[
\sum_{g \in G} \Phi_{\rho}(g)π(g) = γ_\rho(π)
\]

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The basic conjecture. In order to proceed we will have to assume that the following conjecture holds.

Choose a maximal unipotent subgroup in $U \subset G$. Denote by $B$ the Borel subgroup of $G$ which normalizes $U$. Let $X = G/U$ and let $r : G \rightarrow X$ denote the natural projection. By the definition one has canonical identification $T \simeq B/U$. Hence $T$ is naturally embedded into $X$.

Conjecture 9.12. The complex $r_! B_p$ vanishes outside of $T$.

One can show that Conjecture 9.12 implies that (9.23) holds for any $\theta$.

10. Appendix (by V. Vologodsky)

10.1. Notations. In this appendix we assume that $F$ is a finite extension of $\mathbb{Q}_p$. We denote by $m$ the maximal ideal of $\mathcal{O}_F$.

Let $C$ be a smooth, quasi-projective curve over $F$, $c_0$ be a $F$-point of $C$. Let $C' = C \setminus c_0$.

Let $f : X \rightarrow C'$ be a smooth, proper morphism of relative dimension $n$ and $\omega$ be a differential form $\omega \in \Omega_{X/C'}^{\top}$ of the top degree.

For an $F$-point $c \in C'(F)$ we denote by $|\omega|$ the measure associated to $\omega$ on the space of $F$-points of the fiber $X_c$ over $c$. Put $V(c) = \int_{X_c(K)} |\omega|$. Choose a coordinate $t$ on a neighborhood of $c_0$ with $t(c_0) = 0$.

Theorem 10.2. There exist integers $n$ and $r$ such that $V(tt') = V(t)$ for all $t \in m^n$, $t' \in (\mathcal{O}_F)^r$.

This theorem implies easily Theorem 7.2.

Proof. First, we can make use of Nagata's theorem to construct a proper, integral scheme $\overline{X}$ over $C$ such that $\overline{X} \times_C C' \simeq X$. By Hironaka's theorem we can blow up $\overline{X}$ to build $\tilde{f} : Y \rightarrow C$ such that the union of the fiber $Y_{c_0}$ over $c_0$ and the closure the zero locus of the differential form $\omega$ on $Y \setminus Y_{c_0}$ is a normal crossing divisor. It follows that for any $F$-point $a$ of $Y_{c_0}$ there exist local coordinates $x_i$ ($i = 0, 1...n$) on a neighborhood $Z_a$ of $a$ satisfying the following conditions:

1) $t(\tilde{f}) = u \prod_i x_i^{n_i}$, for some function $u$ on $Z_a$ with $u(a) \neq 0$ and $n_i \geq 0$

2) $\omega = v \prod_i x_i^{m_i} dx_0 dx_1 dx_2...dx_{n-1}$ on $Z_a \setminus Y_{c_0}$ for some function $v$ on $Z_a$ with $v(a) \neq 0$ and $m_i \in \mathbb{Z}$.

It is clear that for a point $c \in C'(F)$ sufficiently close to $c_0$ the fibers $X_c(F)$ and $Y_c(F)$ have the same measure. Hence we can replace $X$ by $Y \setminus Y_{c_0}$. By the implicit function theorem we can choose a neighborhood $Z'_a$ of $a$ in $p$-adic topology: $a \in Z'_a \subset Z_a$ such that the coordinates $x_i$ give an isomorphism (of sets) $Z'_a \simeq m^l \times m^l \times ... \times m^l$, where $l$ is a positive integer. Moreover, if we choose $l$ sufficiently large the function $(\frac{u(a)}{u})^{m^{-1}}$ is well defined on $Z'_a$, hence, changing $x_i$ for $(\frac{u(a)}{u})^{m^{-1}} x_i$ we can suppose that
on $Z_a'$ one has $t(\tilde{f}) = u(a)\Pi_i x_i^{m_i}$. Without loss of generality we can assume that $|v|$ is constant on $Z_a'$. Since $Y_{q_0}(F)$ is compact it can be covered by finitely many open sets $Z_a'$ satisfying the properties stated above. In fact one can choose them to be disjoint. To prove it we need the following lemma.

**Lemma 10.3.** Let $Q$ be an open, compact subset of the affine space $F^n$. We claim that $Q$ can be covered by finitely many disjoint balls contained in $Q$. (A ball is a subset of the form $B_{r,a} = \{(x_1, ..., x_n) \in K^n, |x_i - a_i| < r\}$).

We apply the lemma to the sets $Z'_q \bigcup_{j<q} Z'_a$ ($q = 1, ..., k$). (By construction each of them is identified with an open compact subset of $F^{n+1}$).

**Proof.** Since $Q$ is compact we can cover it by finitely many balls contained in $Q$. Let $r_0$ be the smallest radius of these balls. Pick an integer $l$ such that $-\log_q r_0 < l$.

Consider the projection $p : Q \rightarrow (F/\mathfrak{m})^n$. The pre-image of each element of $\text{Im} p$ is a ball. These balls constitute the desired covering.

Now the proposition follows from the following simple lemma

**Lemma 10.4.** Let $\omega$ stand for differential form

$$\omega = \prod_i x_i^{m_i} dx_0 dx_1 dx_2 ... d x_{n-1}$$

where $m_i \in \mathbb{Z}$ and

$$V(t) = \int_{\prod_i x_i^{m_i} = t; x_i \in \mathfrak{m}} |\omega|$$

Then $V(t \cdot t') = V(t)$ for any $t \in \mathfrak{m}$, $t' \in (\mathcal{O}_F)^n$.

**References**

[1] A. Belinson, J. Bernstein and P. Deligne, *Faisceaux pervers*, in: Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astrisque, **100** 1982.

[2] J. Bernstein and P. Deligne, *Le centre de Bernstein*, Travaux en Cours, Representations of reductive groups over a local field, 1–32, Hermann, Paris, 1984.

[3] A. Berenstein and D. Kazhdan

[4] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, with a supplement *On the notion of an automorphic representation* by R. P. Langlands. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 189–207, Amer. Math. Soc., Providence, R.I., 1979.

[5] A. Braverman and D. Kazhdan, *On the Schwartz space of the basic affine space*, Selecta Math. (N.S.) **5** (1999), no. 1, 1–28.

[6] J. Bernstein and A. Zelevinsky. *Representations of the group $GL(n,F)$, where $F$ is a local non-Archimedean field* (Russian), Uspehi Mat. Nauk **31** (1976), 5-70.
[7] J. Bernstein and A. Zelevinsky, *Induced representations of reductive $p$-adic groups I*, Ann. Sci. École Norm. Sup. (4) **10** (1977), 441-472.

[8] I. Gelfand, M. Graev and I. Piatetskii-Shapiro, *Representation theory and automorphic functions*, Philadelphia, Pa.-London-Toronto, Ont. 1969

[9] S. Gelfand and D. Kazhdan, *Conjectural algebraic formulas for representations of $\text{GL}(n)$*, in: Sir Michael Atiyah: a great mathematician of the twentieth century. Asian J. Math. **3** (1999), no. 1, 17–48.

[10] I. M. Gelfand and D. Kazhdan, *Representations of the group $\text{GL}(n, K)$ where $K$ is a local field*, in: Lie groups and their representations (Proc. Summer School, Bolyai János Math. Soc., Budapest, 1971), pp. 95–118. Halsted, New York, 1975.

[11] R. Godement and H. Jacquet, *Zeta-functions of simple algebras*, Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972.

[12] M. Harris, *The local Langlands conjecture for $\text{GL}(n)$ over a $p$-adic field, $n < p$*, Invent. Math. **134** (1998), no. 1, 177–210.

[13] M. Harris and R. Taylor,

[14] H. Jacquet, I. Piatetskii-Shapiro, J. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464.

[15] D. Kazhdan, *On lifting*, Lie group representations, II (College Park, Md., 1982/1983), 209–249, Lecture Notes in Math., **1041**, Springer, Berlin-New York, 1984.

[16] D. Kazhdan, *Forms of the principle series for $\text{GL}(n)$*, in: Functional analysis on the eve of the 21st century, Vol. 1 (New Brunswick, NJ, 1993), 153–171, Progr. Math., **131**, Birkhäuser Boston, Boston, MA, 1995

[17] D. Kazhdan, *An algebraic integration*, in: “Mathematics: frontiers and perspectives”, V. Arnold, M. Atiyah, P. Lax and B. Mazur eds., AMS, 2000.

[18] R. P. Langlands, *Problems in the theory of automorphic forms*, Lectures in modern analysis and applications, III, pp. 18–61. Lecture Notes in Math., Vol. 170, Springer, Berlin, 1970.

[19] G. Laumon, *Faisceaux charactères (d’après Lusztig)* (French), Séminaire Bourbaki, Vol. 1988/89. Astérisque **177-178**, (1989), 231–260.

[20] G. Laumon, M. Rapoport and U. Stuhler *D-elliptic sheaves and the Langlands correspondence*, Invent. Math. **113** (1993), no. 2, 217–338.

[21] G. Lusztig, *Character sheaves I*, Adv. in Math. 56 (1985), no. 3, 193–237.

[22] I. Piatetskii-Shapiro, *Multiplicity one theorems*, in: Automorphic forms, representations and $L$-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, pp. 209–212, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[23] E. B. Vinberg, *On reductive algebraic semi-groups*, in: Lie groups and Lie algebras: E. B. Dynkin’s Seminar, 145–182, Amer. Math. Soc. Transl. Ser. 2, **169**, Amer. Math. Soc., Providence, RI, 1995.

[24] J. T. Tate, *Fourier analysis in number fields, and Hecke’s zeta-functions*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), 1967, pp. 305–347.

[25] A. Weil, *Adèles and algebraic groups*, Progress in Mathematics, 23. Birkhäuser, Boston, Mass., 1982

[26] N. Wallach, *Real reductive groups I* Pure and Applied Mathematics, **132**, Academic Press, Inc., Boston, MA, 1988.