PARABOLIC GENERATING PAIRS OF GENUS-ONE 2-BRIDGE KNOT GROUPS

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ABSTRACT. We show that any parabolic generating pair of a genus-one hyperbolic 2-bridge knot group is equivalent to the upper or lower meridian pair. As an application, we obtain a complete classification of the epimorphisms from 2-bridge knot groups to genus-one hyperbolic 2-bridge knot groups.

1. INTRODUCTION

In [1, Theorem 4.3], Adams proved that the fundamental group of a finite volume hyperbolic manifold is generated by two parabolic transformations if and only if it is homeomorphic to the complement of a 2-bridge link $K(r)$ which is not a torus link. This refines the result of Boileau and Zimmermann [8, Corollary 3.3] that a link in $S^3$ is a 2-bridge link if and only if its link group is generated by two meridians. Adams also proved that any parabolic generating pair of a hyperbolic 2-bridge link consists of meridians. This means that any such pair is represented by an arc properly embedded in the exterior $E(K(r))$, together with a pair of meridional loops on $\partial E(K(r))$ passing through the endpoints of the arc. The meridian pair represented by the upper (resp., lower) tunnel forms a parabolic generating pair, and is called the upper meridian pair (resp., the lower meridian pair). He also proved that each hyperbolic 2-bridge link group admits only finitely many distinct parabolic generating pairs up to conjugacy [1, Corollary 4.1] and moreover that, for the figure-eight knot group, the upper and lower meridian pairs are the only parabolic generating pairs up to conjugacy [1, Corollary 4.6]).

These results were generalized to all 2-bridge links by Agol [3]. In fact, he classified all two parabolic generator Kleinian groups and their parabolic generating pairs. To this end, he proved that for any properly embedded arc in $E(K(r))$ which is not properly homotopic to the upper tunnel nor the lower tunnel, the subgroup of the link group of $K(r)$ generated by the meridian pair represented by the arc is a free group, by using the checkerboard surfaces and Klein-Maskit combination theorem.

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The purpose of this paper, however, is to give an alternative proof of this result for genus-one hyperbolic 2-bridge knots by using small cancellation theory and a geometric observation suggested by Michel Boileau to us [5]. Recall that a 2-bridge knot has genus one if and only if it is equivalent to $K(r)$ with

$$r = [2m, \pm 2n] := \frac{1}{2m \pm \frac{1}{2n}},$$

where $m$ and $n$ are positive integers.

We now describe our strategy. It is well-known that any parabolic generating pair of a 2-bridge link group $G(K(r))$ determines a strong inversion, $h$, of the link $K(r)$, i.e., an orientation-preserving involution of $S^3$ preserving $K(r)$ setwise such that the fixed point set $\text{Fix}(h)$ is a circle intersecting each component of $K(r)$ in two points. The key observation, which Boileau brought to us, is that the parabolic generating pair is represented by an arc component of $\text{Fix}(h) \cap E(K(r))$ (see Proposition 2.1).

Every 2-bridge link admits a diagram which has a $\mathbb{Z}/2\mathbb{Z}$-symmetry as in Figure 1. Let $h_1$ (resp., $h_2$) be the $\pi$-rotations about the horizontal (resp., vertical) axis in the projection plane. If the slope $r = q/p$ satisfies the condition $q^2 \not\equiv 1 \pmod{p}$, then any strong inversion $h$ of $K(r)$ is strongly equivalent to one of the two standard inversions $h_1$ and $h_2$, namely $h$ is conjugate to $h_1$ or $h_2$ by a homeomorphism of $(S^3, K(r))$ which is pairwise isotopic to the identity (cf. [19, Proposition 3.6] and the proof of Corollary 2.2). We may assume that $\text{Fix}(h_1)$ contains the upper tunnel, $\tau_1$, and $\text{Fix}(h_2)$ contains the lower tunnel, $\tau_2$. (See [2] [4] [11] for interesting related results.) Now suppose further that $K(r)$ is a knot, and let $\tau'_1$ be the component of $\text{Fix}(h_1) \cap E(K(r))$ different from $\tau_1$. We call the meridian pairs represented by $\tau'_1$ and $\tau'_2$, respectively, the long upper meridian pair and the long lower meridian pair (see Figure 1). The main ingredient of this paper is a combinatorial proof of the following theorem based on small cancellation theory.

**Theorem 1.1.** Let $r = [2m, \pm 2n]$, where $m$ and $n$ are positive integers, and let $(x_\ell, y_\ell)$ be the long upper meridian pair or the long lower meridian pair for $K(r)$. Then the subgroup of $G(K(r))$ generated by $x_\ell$ and $y_\ell$ is a free group.

Since there is a homeomorphism from $(S^3, K(r))$ with $r = [2m, \pm 2n]$ to $(S^3, K(r'))$ with $r' = [2n, \pm 2m]$, which maps the long lower meridian pair of $K(r)$ to the long upper meridian pair of $K(r')$, it is enough to prove the theorem only for the long upper meridian pair. Thus throughout the remainder of this paper, $(x_\ell, y_\ell)$ denotes the long upper meridian pair for $K(r)$ as illustrated in Figure 1.

In the special case when $r = [2m, -2m]$, we have yet another equivalence class of strong inversions, which is represented by the strong inversion, $h_3$, illustrated by Figure 2 (see also Figure 4 and [19, Proposition 3.6]).
The long upper meridian pair \( \{x_\ell, y_\ell\} \) of \( K(r) \) with \( r = [4, 6] \). To be precise, \( x_\ell \) (resp., \( y_\ell \)) is represented by the left (resp., right) lasso together with an almost vertical line joining the end point of the lasso with the base point of \( E(K(r)) \). Note that the upper tunnel is the short subarc, with both endpoints in \( K(r) \), of the horizontal central line in the projection plane.

Additional symmetry of \( K(r) \) for \( r = [2m, -2m] \) (\( m = 2 \)). \( \text{Isom}^+(S^3 - K(r)) \cong \langle g, h_1 | g^4, h_1^2, (gh_1)^2 \rangle \), where \( g = (\pi/2\text{-rotation about } \eta) \circ (\pi\text{-rotation about } \xi) \), \( h_1 = \pi\text{-rotation around } \gamma \), and \( h_3 = gh_1 = \pi\text{-rotation around } \beta \).

**Theorem 1.2.** Let \( r = [2m, -2m] \), where \( m \) is an integer \( \geq 2 \), and let \( h_3 \) be the strong inversion of \((S^3, K(r))\) as in the above. Then, for each of the arc components of \( \text{Fix}(h_3) \cap E(K(r)) \), the subgroup of \( G(K(r)) \) generated by the meridian pair represented by the arc is a proper subgroup of \( G(K(r)) \).

In fact, it is not difficult to extend Theorem 1.2 to all hyperbolic 2-bridge knots \( K(q/p) \) with \( q^2 \equiv 1 \) (mod \( p \)). We also believe that Theorem 1.1 can be extended to all hyperbolic 2-bridge links by a similar method, but the arguments would become much more complicated.
These two theorems enable us to recover a special case of Agol’s result [3].

**Theorem 1.3.** Let \( K(r) \) be a genus-one hyperbolic 2-bridge knot, namely \( r = [2m, 2n] \) with \( m, n \in \mathbb{N} \) or \( r = [2m, -2n] \) with \( m, n \in \mathbb{N} \) and \( (m, n) \neq (1, 1) \). Then the upper and lower meridian pairs are the only parabolic generating pairs of the knot group of \( K(r) \) up to equivalence.

For the precise definition of the equivalence relation in the above corollary, see the first paragraph of Section 2. Together with the result of [6, Corollary 1.3] on epimorphisms from 2-bridge knot groups and the characterization by [14, Main Theorem 2.4] of upper meridian pair preserving epimorphisms between 2-bridge link groups, the above corollary implies the following theorem.

**Theorem 1.4.** Let \( K(r) \) be a genus-one hyperbolic 2-bridge knot and \( K(\tilde{r}) \) be a 2-bridge knot. Then there is an epimorphism from \( G(K(\tilde{r})) \) onto \( G(K(r)) \) if and only if \( \tilde{r} \) or \( \tilde{r} + 1 \) belongs to (i) the \( \hat{\Gamma}_r \)-orbit of \( r \) or \( \infty \) or (ii) the \( \hat{\Gamma}_r' \)-orbit of \( r' \) or \( \infty \). Here (a) \( \hat{\Gamma}_r \) (resp., \( \hat{\Gamma}_r' \)) is the subgroup of the automorphism group of the Farey tessellation generated by the reflections in the Farey edges with endpoints \( \infty \) or \( r \) (resp., \( r' \)), and (b) \( r' = q'/p' \), where \( qq' \equiv 1 \pmod{p} \); \( p \) and \( q \) are relatively prime integers such that \( r = q/p \).

At the end of the introduction, we would like to point out that if we remove the condition that generating pairs consist of parabolic elements, then it is proved by Heusener and Porti [13] that every hyperbolic knot admits infinitely many generating pairs up to Nielsen equivalence. Moreover, the same conclusion for torus knots, especially non-hyperbolic 2-bridge knots, had been proved by Zieschang [23].

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2. **Strong inversions associated with parabolic generating pairs**

Let \( K \) be a link in \( S^3 \) and \( E(K) \) the exterior of \( K \), namely the complement of an open regular neighborhood of \( K \). An essential simple loop in \( \partial E(K) \) is called a **meridian** if it bounds a disk on the (closed) regular neighborhood of \( K \). An element of the link group \( G(K) = \pi_1(E(K)) \) which is freely homotopic to a meridian is also called a **meridian**. By a **meridian pair** of \( K \), we mean an unordered pair \( \{x, y\} \) of meridians of \( G(K) \). Two meridian pairs \( \{x, y\} \) and \( \{x', y'\} \) are said to be **equivalent** if \( \{x', y'\} \) is equal to \( \{x^{\varepsilon_1}, y^{\varepsilon_2}\} \) for some \( \varepsilon_1, \varepsilon_2 \in \{\pm 1\} \) up to simultaneous conjugacy.

Note that there is a bijective correspondence between the set of meridian pairs up to equivalence and the set of arcs properly embedded in \( E(K) \) up to proper homotopy. Here a proper arc \( \gamma \) in \( E(K) \) corresponds to a meridian pair \( \{x, y\} \) which is obtained as follows. Pick an interior point, \( q \), in \( \gamma \), and divide \( \gamma \) into two subarcs \( \gamma_1 \) and \( \gamma_2 \) such that \( \gamma_1 \cap \gamma_2 = \{q\} \). For \( i = 1, 2 \), let \( C_i \) be a meridian passing through
the end point of $\gamma_i$ in $\partial E(K)$. Let $\delta$ be an arc in $E(K)$ joining $q$ with the base point of $E(K)$. Then the pair of based loops $\{\delta \cup \gamma_i \cup C_i\}_{i=1,2}$ gives the meridian pair $\{x,y\}$ corresponding to the arc $\gamma$.

If $K$ is hyperbolic, then a meridian pair $\{x,y\}$ is identified with a pair of parabolic transformations of the hyperbolic 3-space. We are interested only in the case where $\{x,y\}$ generate a non-elementary Kleinian group, namely the case where the parabolic fixed points of $x$ and $y$ are not identical. Then the geodesic joining the parabolic fixed points of $x$ and $y$ descends to a proper geodesic in the hyperbolic manifold $S^3 - K$ and hence determines a proper arc in $E(K)$, where we identify $E(K)$ with the complement of an open cusp neighborhoods. By the correspondence between the fundamental group and the covering transformation group, we see that this arc corresponds to the meridian pair $\{x,y\}$.

Boileau [5] informed us of the following fact, which had been observed by Adams [1].

**Proposition 2.1.** Let $K(r)$ be a hyperbolic 2-bridge link, and let $\{x,y\}$ be a parabolic generating pair of the link group $G(K(r))$. Then there is a strong inversion $h$ of $K(r)$ such that $(h_*x,h_*y) = (x^{-1},y^{-1})$ and that $\{x,y\}$ is a meridian pair corresponding to an arc component of $\text{Fix}(h) \cap E(K(r))$. Here $h_*$ denotes the automorphism of $G(K(r))$ induced by $h$.

**Proof.** Let $K(r)$ be a hyperbolic 2-bridge link, and let $\{x,y\}$ be a parabolic generating pair. Then, by assumption, $x$ and $y$ are identified with parabolic transformations. Since $\{x,y\}$ generates non-elementary group $G(K(r))$, $x$ and $y$ have distinct parabolic fixed points. Let $\eta$ be the order 2 elliptic transformation whose axis is the geodesic, $\tilde{c}$, joining the two parabolic fixed points. Then $\eta x \eta = x^{-1}$ and $\eta y \eta = y^{-1}$ (cf. [21, Section 5.4]) and therefore $\eta$ descends to an orientation-preserving involution, $h$, of $S^3 - K(r)$, such that the restriction of $h$ to $\partial E(K(r))$ is a hyper-elliptic involution. Thus $h$ extends to a strong inversion of $K(r)$, which we continue to denote by the same symbol $h$. Now recall the result [1, Theorem 4.3] that $\{x,y\}$ is a meridian pair. Thus by the observation made before this proposition, we obtain the desired result. \qed

**Corollary 2.2.** For a hyperbolic 2-bridge knot $K(r)$ with $r = q/p$, the following hold.

1. If $q^2 \not\equiv 1 \pmod{p}$, then any parabolic generating pair of $G(K(r))$ is equivalent to either the upper, lower, long upper or long lower meridian pair.
2. If $q^2 \equiv 1 \pmod{p}$, then for any parabolic generating pair of $G(K(r))$, one of the following holds.
   (i) It is equivalent to either the upper, lower, long upper or long lower meridian pair.
   (ii) There is an automorphism of $G(K(r))$ which carries it to a parabolic generating pair represented by one of the two arc components of $\text{Fix}(h_3) \cap E(K(r))$, where $h_3$ is the strong inversion, as illustrated in Figure 2.
Proof. Suppose that \( q^2 \not\equiv 1 \pmod{p} \). Then the orientation-preserving isometry group \( \text{Isom}^+(S^3 - K(r)) \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) (see [9, 20]). By Tollefson’s theorem [22] or the orbifold theorem [7, 12], this implies that \( h \) is strongly equivalent to one of the two strong inversions \( h_1 \) and \( h_2 \) introduced in the introduction. Hence, we obtain the conclusion by Proposition 2.1.

Suppose that \( q^2 \equiv 1 \pmod{p} \). Then \( \text{Isom}^+(S^3 - K(r)) \) is isomorphic to the dihedral group \( D_8 \) of order 8 (see [9, 20]). To be precise, \( \text{Isom}^+(S^3 - K(r)) \cong \langle g, h_1 \mid g^4, h_1^2, (gh_1)^2 \rangle \), where \( g \) and \( h_1 \) are as illustrated in Figure 2. By using this fact and Tollefson’s theorem [22] or the orbifold theorem [7, 12], we see that any strong inversion \( h \) of \( K(r) \) is strongly equivalent to \( g^i h_1 \) with \( 0 \leq i \leq 3 \). Since \( g^0 h_1 = h_1 \), \( g h_1 = h_3 \), \( g^2 h_1 = g h_1 g^{-1} (= h_2) \) and \( g^3 h_1 = g h_3 g^{-1} \), we obtain the conclusion by Proposition 2.1. \( \square \)

3. Wirtinger generators and long upper/lower meridian pairs

Throughout this paper, we use the following notation: For an element \( x \) in a group, we denote \( x^{-1} \) by \( \bar{x} \). For two elements \( x \) and \( y \) of a group and for a positive integer \( k \), we define \( \langle xy \rangle^k \) to be the alternative product of \( x \) and \( y \) of length \( k \). Namely:

\[
\langle xy \rangle^k = \begin{cases} 
(xy)^\frac{k}{2} & \text{if } k \text{ is even;} \\
(xy)^\frac{k-1}{2} x & \text{if } k \text{ is odd.}
\end{cases}
\]

We also define \( \langle xy \rangle^{-k} \) to be \( (\langle xy \rangle^k)^{-1} \).

Consider the genus-one 2-bridge knot \( K(r) \) with \( r = [2m, \pm 2n] \), where \( m \) and \( n \) are positive integers. Let \( c_i \) \((-m \leq i \leq m + 1)\) and \( d_j \) \((0 \leq j \leq 2n)\) be the Wirtinger generators of the knot group \( G(K(r)) \) as illustrated in Figure 3(a) and (b) according to whether \( r = [2m, 2n] \) and \([2m, -2n]\), respectively. Here, we follow the convention of [10]. Namely, we assume that the base point of \( E(K(r)) \) lies far above the projection plane, and the symbol, say \( c_i \), denotes the element of the knot group represented by an oriented short arc passing under the arc of the knot diagram with label \( c_i \) in a left-right direction, together with a pair of straight almost vertical arcs joining the endpoints of the short arc with the base point of \( E(K(r)) \).

Set

\[
(1) \quad a := c_1 \quad \text{and} \quad b := c_0^{-1}
\]

Then \( \{a, b\} \) is a generating pair of \( G(K(r)) \) which is identical with that of \( G(K(r)) \) in [14, Section 3]. To see this, let \( B_1 \) be a small regular neighborhood of the upper tunnel, and \( B_2 \) be the closure of the complement of \( B_1 \) in \( S^3 \). Then \((B_1, B_1 \cap K(r))\) and \((B_2, B_2 \cap K(r))\) are identified with the rational tangles of slopes \( \infty \) and \( r \), respectively. Thus, from the description of the generator system in [14, Section 3], we see that \( \{a, b\} \) is the generating pair given in it. Hence

\[
G(K(r)) \cong \langle a, b \mid u_r \rangle,
\]
where $u_r$ is the cyclically reduced word in $a$ and $b$ in Lemma 3.1. Namely,
\begin{equation}
(2) 
    u_r = a\hat{u}_r b\hat{u}_r^{-1} \quad \text{with} \quad \hat{u}_r = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}},
\end{equation}
where $r = q/p$ with $(p, q) = (4mn \pm 1, 2n)$ and $\epsilon_i = (-1)^{\lfloor iq/p \rfloor}$. Here $\lfloor t \rfloor$ is the greatest integer not exceeding $t$.

Let $f := h_1 h_2$ be the involution of $(S^3, K(r))$ induced by the $\pi$-rotation about the axis which intersects the projection plane, in Figure 3 orthogonally at the central point. We also use the same symbol $f$ to denote the automorphism of $G(K(r))$ induced by the involution $f$. Recall that $(x_\ell, y_\ell)$ denotes the long upper meridian pair of $G(K(r))$ as illustrated in Figure 1. Then we can easily observe the following lemma.

**Lemma 3.1.** We have $(f(a), f(b)) = (b^{-1}, a^{-1})$, $(f(x_\ell), f(y_\ell)) = (y_\ell, x_\ell)$ and $f(c_i) = c_{1-i} \quad (-m \leq i \leq m + 1)$, $f(d_j) = d_{2n+1-j} \quad (1 \leq j \leq 2n)$.

The following lemma can be easily verified by a standard calculation on Wirtinger presentation, where the second formula is obtained from the first formula by using Lemma 3.1.
Lemma 3.2. For $1 \leq i \leq m + 1$, the following hold.

$$c_i = \begin{cases} 
(ab)^i(ab)^{i-1} & \text{if } i \text{ is even;} \\
(ab)^{i-1}(ab)^i & \text{if } i \text{ is odd.}
\end{cases}$$

For $1 \leq i \leq m$, the following hold.

$$c_{-i} = \begin{cases} 
(ba)^i(ba)^{i+1} & \text{if } i \text{ is even;} \\
(ba)^{i+1}(ba)^i & \text{if } i \text{ is odd.}
\end{cases}$$

As shown in Figure 3, we have

$$\begin{align*}
(d_0, d_1) &= \begin{cases} 
(c_m, c_{-m}) & \text{if } r = [2m, 2n]; \\
(c_{-m}, c_m) & \text{if } r = [2m, -2n].
\end{cases}
\end{align*}$$

Lemma 3.3. The member $y_\ell$ of the long upper meridian pair $(x_\ell, y_\ell)$ is given by the following formula.

$$y_\ell = \begin{cases} 
(d_1 d_0)^n b(d_1 d_0)^{-n} & \text{if } m \text{ is even}, \\
(d_1 d_0)^n b(d_1 d_0)^{-n} & \text{if } m \text{ is odd.}
\end{cases}$$

The other member $x_\ell$ is obtained from $y_\ell$ by replacing $(a, b)$ with $(b^{-1}, a^{-1})$.

Proof. We prove the formula only in the case where $r = [2m, \pm 2n]$ with $m$ even. (The other case is proved similarly.) Observe from Figures 1 and 3 that $y_\ell = w\bar{b}w$, where

$$w = \begin{cases} 
(d_n d_{n-1}) \cdots (d_2 d_1) & \text{if } n \geq 2 \text{ is even;} \\
(d_n d_{n-1}) \cdots (d_3 d_2) d_1 & \text{if } n \geq 1 \text{ is odd.}
\end{cases}$$

On the other hand, we can observe from Figure 3 that $\bar{d}_2 d_2 = d_2 \bar{d}_2 = d_1 d_0$. Hence we obtain the formula. The last assertion is a consequence of Lemma 3.1.

By using Lemmas 3.2, 3.3 and the identities (1), (3), we can express $d_0, d_1, x_\ell$ and $y_\ell$ as words in $\{a, b\}$.

Convention 3.4. In the remainder of this paper, we use the symbols $d_0, d_1, x_\ell$ and $y_\ell$ to denote the reduced words in $\{a, b\}$ obtained as in the above.

For the definition of reduced words, see Section 4 below.

4. S-sequences of Long Upper/Lower Meridian Pairs

We first recall basic terminology in combinatorial group theory and the definition of $S$-sequences and cyclic $S$-sequences introduced in [14, Section 4]. Let $X$ be a set. By a word in $X$, we mean a finite sequence $x^{\varepsilon_1} x^{\varepsilon_2} \cdots x^{\varepsilon_i}$ where $x_i \in X$ and $\varepsilon_i = \pm 1$. We call $x_1^{\varepsilon_1}$ and $x_i^{\varepsilon_i}$ the initial letter and terminal letter of the word, respectively. For two words $u, v$ in $X$, by $u \equiv v$ we denote the visual equality of $u$ and $v$, meaning
that if \( u = x_1^{e_1} \cdots x_t^{e_t} \) and \( v = y_1^{\delta_1} \cdots y_m^{\delta_m} \) \((x_i, y_j \in X; e_i, \delta_j = \pm 1)\), then \( t = m \) and \( x_i = y_i \) and \( e_i = \delta_i \) for each \( i = 1, \ldots, t \). The length of a word \( v \) is denoted by \(|v|\). A word \( v \) in \( X \) is said to be reduced if \( v \) does not contain \( xx^{-1} \) or \( x^{-1}x \) for any \( x \in X \). A cyclic word is defined to be the set of all cyclic permutations of a cyclically reduced word. By \((v)\) we denote the cyclic word associated with a cyclically reduced word \( v \). Also by \((u) \equiv (v)\) we mean the visual equality of two cyclic words \((u)\) and \((v)\). In fact, \((u) \equiv (v)\) if and only if \( v \) is visually a cyclic shift of \( u \).

**Definition 4.1.** (1) Let \( v \) be a reduced word in \( \{a, b\} \). Decompose \( v \) into

\[
v \equiv v_1v_2 \cdots v_t,\]

where, for each \( i = 1, \ldots, t - 1 \), all letters in \( v_i \) have positive (resp., negative) exponents, and all letters in \( v_{i+1} \) have negative (resp., positive) exponents. Then the sequence of positive integers \( S(v) := (|v_1|, |v_2|, \ldots, |v_t|) \) is called the \( S \)-sequence of \( v \).

(2) Let \((v)\) be a cyclic word in \( \{a, b\} \). Decompose \((v)\) into

\[
(v) \equiv (v_1v_2 \cdots v_t),
\]

where all letters in \( v_i \) have positive (resp., negative) exponents, and all letters in \( v_{i+1} \) have negative (resp., positive) exponents (taking subindices modulo \( t \)). Then the cyclic sequence of positive integers \( CS(v) := ((|v_1|, |v_2|, \ldots, |v_t|)) \) is called the cyclic \( S \)-sequence of \((v)\). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

(3) A reduced word \( v \) in \( \{a, b\} \) is said to be alternating if \( a^{\pm 1} \) and \( b^{\pm 1} \) appear in \( v \) alternately, i.e., neither \( a^{\pm 2} \) nor \( b^{\pm 2} \) appears in \( v \). A cyclic word \((v)\) is said to be alternating if all cyclic permutations of \( v \) are alternating. In the latter case, we also say that \( v \) is cyclically alternating.

**Notation 4.2.** For \( x, y \in \{a^{\pm 1}, b^{\pm 1}\} \) and for a word \( w \) in \( a, b \), we write \( w \equiv x \cdots y \) if \( w \) is an alternating word which begins with \( x \) and ends with \( y \). It should be noted that an alternating word is determined by the initial letter and the \( S \)-sequence.

In the remainder of this section, we describe the \( S \)-sequences of the words \( d_0, d_1, x_t \) and \( y_t \) in Convention [3.4]

**Lemma 4.3.** Suppose \( r = [2m, 2n] \). Then \( d_0 \equiv a \) or \( \bar{a} \cdots a \) according to whether \( m = 1 \) or \( m \geq 2 \), and \( d_1 \equiv b \cdots b \). Moreover, their \( S \)-sequences are given as follows.

(1) If \( m \) is even, then \( S(d_0) = (m, m - 1) \) and \( S(d_1) = (m, m + 1) \).

(2) If \( m \) is odd, then \( S(d_0) = (1) \) or \( (m - 1, m) \) according to whether \( m = 1 \) or \( m \geq 3 \), and \( S(d_1) = (m + 1, m) \).
Proof. By the formula (3) and Lemma 3.2 we have the following identity in the free group $F(a,b)$ with free basis \{a, b\}:

$$d_0 = c_m = \begin{cases} (\bar{a}\bar{b})^m(ab)^{m-1} & \text{if } m \text{ is even;} \\ (\bar{a}\bar{b})^{m-1}(ab)^m & \text{if } m \text{ is odd.} \end{cases}$$

Since the last two words on the right hand side in the above identity are alternating and therefore reduced, we obtain the assertion for the reduced word $d_0$. Similarly, by the formula (3) and Lemma 3.2 again, we have the following identity in $F(a,b)$:

$$d_1 = c_{-m} = \begin{cases} (ba)^m(\bar{b}\bar{a})^{m+1} & \text{if } m \text{ is even;} \\ (ba)^{m+1}(\bar{b}\bar{a})^m & \text{if } m \text{ is odd.} \end{cases}$$

Hence we obtain the assertion for $d_1$. \[\square\]

**Lemma 4.4.** Suppose $r = [2m, -2n]$. Then $d_0 \equiv b \cdots \bar{b}$ and $d_1 \equiv a$ or $\bar{a} \cdots a$ according to whether $m = 1$ or $m \geq 2$. Moreover, their $S$-sequences are given as follows.

1. If $m$ is even, then $S(d_0) = (m, m+1)$ and $S(d_1) = (m, m-1)$.
2. If $m$ is odd, then $S(d_0) = (m+1, m)$ and $S(d_1) = (1)$ or $(m-1, m)$ according to whether $m = 1$ or $m \geq 3$.

Proof. This is proved by using the formula (3) and Lemma 3.2 as in Lemma 4.3. \[\square\]

**Proposition 4.5.** Suppose $r = [2m, 2n]$, and set $\varepsilon = (-1)^n$. Then the following hold.

1. $x_\ell \equiv w_x a \bar{w}_x$ and $y_\ell \equiv w_y \bar{b} \bar{w}_y$, where $w_x$ and $w_y$ are the alternating words such that $w_x \equiv \bar{a} \cdots \bar{b}$, $w_y \equiv b \cdots a$ and

$$S(w_x) = S(w_y) = \begin{cases} (m, (n-1)(2m), m) & \text{if } n \geq 2; \\ (m, m) & \text{if } n = 1. \end{cases}$$

(2) $x_\ell \equiv \bar{a} \cdots a$, $y_\ell \equiv b \cdots \bar{b}$, and

$$S(x_\ell^{\varepsilon}) = S(y_\ell^{\varepsilon}) = \begin{cases} (m, (n-1)(2m), m, m+1, (n-1)(2m), m) & \text{if } n \geq 2; \\ (m, m, m+1, m) & \text{if } n = 1. \end{cases}$$

(3) For any non-zero integer $k$, the words $w_x a^k \bar{w}_x$ and $w_y \bar{b}^k \bar{w}_y$ are reduced, and we have the identities $x_\ell^k = w_x a^k \bar{w}_x$ and $y_\ell^k = w_y \bar{b}^k \bar{w}_y$ in $F(a,b)$. Moreover, each of $w_x a^k \bar{w}_x$ and $w_y \bar{b}^k \bar{w}_y$ is alternating if and only if $|k| = 1$.

In the above proposition, the symbol “$k(2m)$” represents $k$ successive $2m$’s.

Proof. Note that $x_\ell = f(y_\ell)$ is obtained from $y_\ell$ by replacing $a^{\pm 1}$ with $b^{\mp 1}$ by Lemma 3.1. Moreover, the assertion (3) is an immediate consequence of the assertions (1) and (2). Thus we prove (1) and (2) for $y_\ell$.\[\square\]
Suppose first that \( m \) is even. Then \( y_t = \langle d_1 \bar{d}_0 \rangle^n \bar{b} \langle d_1 \bar{d}_0 \rangle^{-n} \) in \( F(a, b) \) by Lemma 3.3. By Lemma 4.3 we see \( d_1 \bar{d}_0 \equiv (b \cdots b)(\bar{a} \cdots a) \equiv b \cdots a \) and \( S(d_1 \bar{d}_0) = (m, 2m, m) \).

If \( n \) is even, then these identities imply that \( \langle d_1 \bar{d}_0 \rangle^n \equiv b \cdots a \) and \( S(\langle d_1 \bar{d}_0 \rangle^n) = (m, (n-1)(2m), m) \). (This follows from the following fact: Since the initial letter and the terminal letter of \( d_1 \bar{d}_0 \) are \( b \) and \( a \), respectively, the terminal component, \( m \), of the \( S \)-sequence of the \( i \)-th factor \( d_1 \bar{d}_0 \) is amalgamated with the initial component, \( m \), of the \( S \)-sequence of the \( i + 1 \)-th factor \( d_1 \bar{d}_0 \) to form a component \( 2m \) of \( S(\langle d_1 \bar{d}_0 \rangle^n) \) for each \( i \) with \( 1 \leq i \leq n/2 \).) Set \( w_y \) to be the alternating word \( \langle d_1 \bar{d}_0 \rangle^n \) in \( \{a, b\} \).

Then \( y_t = w_y \bar{b} \bar{w}_y \) in \( F(a, b) \) and \( w_y \bar{b} \bar{w}_y \equiv (b \cdots a) \bar{b} (a \cdots \bar{b}) \equiv b \cdots \bar{b} \) is alternating. Hence \( y_t \equiv w_y \bar{b} \bar{w}_y \). Moreover, we can observe that \( S(w_y) \) and \( S(y_t) \) are of the desired form. These arguments also work even when \( n = 1 \), if we discard the entry \( (n-1)(2m) = 0(2m) \).

Suppose next that \( m \) is odd. Then \( y_t = \langle d_1 \bar{d}_0 \rangle^n b \langle d_1 \bar{d}_0 \rangle^{-n} \) in \( F(a, b) \) by Lemma 3.3. By similar arguments, we obtain the desired result by setting \( w_y \) to be the alternating word \( \langle d_1 \bar{d}_0 \rangle^n \) or the alternating word obtained from \( \langle d_1 \bar{d}_0 \rangle^n \) by removing the last letter according to whether \( n \) is even or odd. \( \square \)

**Proposition 4.6.** Suppose \( r = [2m, 2n] \), and set \( \varepsilon = (-1)^n \). Then the following hold.

1. \( x_t \equiv w_x a \bar{w}_x \) and \( y_t \equiv w_y \bar{b} \bar{w}_y \), where \( w_x \) and \( w_y \) are the alternating words such that \( w_x \equiv b \cdots b^\varepsilon \), \( w_y \equiv \bar{a} \cdots \bar{a}^\varepsilon \), except when \( (m, n) = (1, 1) \), and such that

\[
S(w_x) = S(w_y) = \begin{cases} 
(m, (n-1)(2m), m-1) & \text{if } m \geq 2 \text{ and } n \geq 2; \\
(m, m-1) & \text{if } m \geq 2 \text{ and } n = 1; \\
(1, (n-1)(2)) & \text{if } m = 1 \text{ and } n \geq 2; \\
(1) & \text{if } m = 1 \text{ and } n = 1.
\end{cases}
\]

In the exceptional case when \( (m, n) = (1, 1) \), we have \( w_x \equiv b \) and \( w_y \equiv \bar{a} \).

2. \( x_t \equiv b \cdots b \), \( y_t \equiv \bar{a} \cdots a \), and

(i) If \( m \geq 2 \), then

\[
S(x_t^\varepsilon) = S(y_t^\varepsilon) = \begin{cases} 
(m, (n-1)(2m), m, m-1, (n-1)(2m), m) & \text{if } n \geq 2; \\
(m, m, m-1, m) & \text{if } n = 1.
\end{cases}
\]
(ii) If $m = 1$, then

$$S(x_\ell^x) = S(y_\ell^x) = \begin{cases} (1, (n-1)\langle 2\rangle, 3, (n-2)\langle 2\rangle, 1) & \text{if } n \geq 3; \\ (1, 2, 3, 1) & \text{if } n = 2; \\ (1, 2) & \text{if } n = 1. \end{cases}$$

(3) For any non-zero integer $k$, the words $w_x a^k \bar{w}_x$ and $w_y b^k \bar{w}_y$ are reduced, and we have the identities $x_\ell^x = w_x a^k \bar{w}_x$ and $y_\ell^k = w_y b^k \bar{w}_y$ in $F(a, b)$. Moreover, each of $w_x a^k \bar{w}_x$ and $w_y b^k \bar{w}_y$ is alternating if and only if $|k| = 1$.

Proof. As in the proof of Proposition 4.3 we have only to prove the assertions (1) and (2) for $y_\ell$.

Suppose first that $m$ is even. Then $y_\ell = \langle d_1 \bar{d}_0 \rangle^n b \langle d_1 \bar{d}_0 \rangle^{-n}$ in $F(a, b)$ by Lemma 4.3. By Lemma 4.3 we see $d_1 \bar{d}_0 = (\bar{a} \cdots a)(b \cdots \bar{b}) = \bar{a} \cdots \bar{b}$ and $S(d_1 \bar{d}_0) = (m, 2m, m)$.

If $n$ is even, then these identities imply that $\langle d_1 \bar{d}_0 \rangle^n \equiv \bar{a} \cdots \bar{b}$ and $S(\langle d_1 \bar{d}_0 \rangle^n) = (m, (n-1)(2m), m)$. Set $w_y$ to be the alternating word in $\langle a, b \rangle$ obtained from the alternating word $\langle d_1 \bar{d}_0 \rangle^n$ by deleting the last letter $\bar{b}$. Then $w_y \equiv \bar{a} \cdots \bar{a}$, $S(w_y) = (m, (n-1)(2m), m-1)$ and $y_\ell = \langle d_1 \bar{d}_0 \rangle^n b \langle d_1 \bar{d}_0 \rangle^{-n} = (a \cdots b)\bar{b}(b \cdots \bar{a})$ reduces to the alternating word $w_y b \bar{w}_y \equiv (a \cdots \bar{a})\bar{b}(a \cdots a) \equiv a \cdots a$. Hence $y_\ell \equiv w_y b \bar{w}_y$. Moreover, we can observe that $S(w_y)$ and $S(y_\ell)$ are of the desired form.

If $n$ is odd and $\geq 3$, then we have $\langle d_1 \bar{d}_0 \rangle^n \equiv \langle d_1 \bar{d}_0 \rangle^{n-1}d_1 \equiv (\bar{a} \cdots \bar{b})(a \cdots a) \equiv \bar{a} \cdots a$ and $S(\langle d_1 \bar{d}_0 \rangle^n) = (m, (n-1)(2m), m-1)$. Set $w_y$ to be the reduced alternating word $\langle d_1 \bar{d}_0 \rangle^n$. Then $y_\ell = w_y b \bar{w}_y$ and $w_y b \bar{w}_y \equiv (a \cdots a)\bar{b}(a \cdots a) \equiv a \cdots a$ is an alternating word. Hence $y_\ell \equiv w_y b \bar{w}_y$. Moreover, we can observe that $S(w_y)$ and $S(y_\ell)$ are of the desired form. These arguments also work even when $n = 1$, if we discard the entry $(n-1)(2m) = 0(2m)$.

Suppose next that $m$ is odd. Then $y_\ell = \langle d_1 \bar{d}_0 \rangle^n b \langle d_1 \bar{d}_0 \rangle^{-n}$ in $F(a, b)$ by Lemma 4.3. Since $d_1 = \langle \bar{a}b \rangle^m \langle ab \rangle^{m-1}$ and $d_0 = \langle ba \rangle^{m+1} \langle \bar{b}a \rangle^m$ by Lemma 4.3 we see

$$\langle d_1 \bar{d}_0 \rangle^n \equiv \begin{cases} \bar{a} \cdots \bar{b} & \text{if } m \geq 3 \text{ and } n \text{ is even;} \\ \bar{a} \cdots a & \text{if } m \geq 3 \text{ and } n \text{ is odd;} \\ \bar{a} \cdots \bar{a} & \text{if } m = 1 \text{ and } n \text{ is even;} \\ \bar{a} \cdots \bar{a} & \text{if } m = 1 \text{ and } n \geq 3 \text{ is odd;} \\ \bar{a} & \text{if } m = 1 \text{ and } n = 1. \end{cases}$$

Set $w_y$ to be the alternating word $\langle d_1 \bar{d}_0 \rangle^n$ or the alternating word obtained from $\langle d_1 \bar{d}_0 \rangle^n$ by deleting the last letter $\bar{b}$, according to whether $\langle d_1 \bar{d}_0 \rangle^n$ ends with $a^{\pm 1}$ or $\bar{b}$. Then we can see as in the previous cases that the desired results hold. \qed
5. Proof of Theorem \[14\]

In this section, we prove Theorem \[14\] by using the small cancellation theory. For standard terminologies in the small cancellation theory, we refer the readers to [17], Chapter V and [14] Sections 5 and 6.

Recall the presentation \(G(K(r)) \cong \langle a, b \mid u_r \rangle\), where \(u_r\) is the cyclically reduced word in \(a\) and \(b\) given by the formula (2). Then the symbol \(S\) (resp., \(CS(r)\)) denotes the \(S\)-sequence \(S(u_r)\) of \(u_r\) (resp., cyclic \(S\)-sequence \(CS(u_r)\)) of \((u_r)\). In [14], we have proved that the sequence \(S(r)\) has a canonical decomposition \((S_1, S_2, S_1, S_2)\) and established various properties of the decomposition. We summarize the key facts which are used in the proof of Theorem \[14\].

In the remainder of this section, by a piece, we mean a piece relative to the symmetrized set of relators generated by \(u_r\) (see [14] Definition 5).

**Proposition 5.1.** The canonical decomposition \((S_1, S_2, S_1, S_2)\) of the sequence \(S(r)\) satisfies the following conditions.

1. Each \(S_i\) is symmetric and occurs only twice in the cyclic sequence \(CS(r)\).
2. If \(v\) is a subword of the cyclic word \((u_r)\) which is a product of 3 pieces but is not a product of \(t\) pieces with \(t \leq 3\), then \(v\) contains a subword, \(v'\), such that \(S(v') = (S_1, S_2, \ell)\) or \(S(v') = (\ell, S_2, S_1)\), for some \(\ell \in \mathbb{Z}_+\).
3. Suppose \(r = [2m, 2n]\). Then \(CS(r) = \langle S_1, S_2, S_1, S_2 \rangle\) with \(S_1 = (2m + 1)\) and \(S_2 = (2n - 1)(2m)\). Moreover, if \(v\) is a subword of \((u_r^{\pm 1})\) such that \(S(v) = (1, \ell)\) or \((\ell, 1)\) for some \(\ell\) with \(1 \leq \ell \leq m\), then \(v\) is a piece.
4. Suppose \(r = [2m, -2n] = [2m - 1, 1, 2n - 1]\). Then \(CS(r) = \langle S_1, S_2, S_1, S_2 \rangle\) with \(S_1 = ((2n - 1)(2m))\) and \(S_2 = (2m - 1)\). Moreover, if \(v\) is a subword of \((u_r^{\pm 1})\) such that \(S(v)\) is of one of the following form, then \(v\) is a piece: \(\ell\) with \(1 \leq \ell \leq 2m\), \((1, \ell)\) with \(1 \leq \ell \leq m\), \((\ell, 1)\) with \(1 \leq \ell \leq m\), \((k(2m), 1)\) with \(0 \leq k \leq 2m - 2\), \((1, k(2m))\) with \(0 \leq k \leq 2n - 3\), or \((1, k(2m))\) with \(0 \leq k \leq 2n - 2\).

**Proof.** (1) This is a part of [14] Proposition 4.5.

(2) This follows from the proof of [15] Lemma 3.3. In the lemma, pieces of the symmetrized set of relators generated by a power \(u_r^k\) with \(k \geq 2\) of the relator \(u_r\) is treated. However, the same argument also works when \(k = 1\). (See also [16] the proof of Corollary 3.25.)

(3), (4) The first assertions are nothing other than [16] Lemma 3.16(1),(3)]. The second assertions follow from the characterization of pieces described in [14] Lemma 5.3(2-c)] and [14] Lemma 5.2].

**Proof of Theorem \[14\].** Suppose on the contrary that the subgroup of \(G(K(r))\) with \(r = [2m, \pm 2n]\) generated by \(x_\ell\) and \(y_\ell\) is not a free group. Then there is a non-trivial relation consisting of \(x_\ell\) and \(y_\ell\). We may assume after conjugacy that \(x_\ell^{k_1}y_\ell^{l_1} \cdots x_\ell^{k_1}y_\ell^{l_1} = 1\) in \(G(K(r))\), where each \(k_i\) and \(l_i\) are non-zero integers.
By Propositions 4.5(3) and 4.6(3), the word \( x_\ell^{k_1} y_{\ell_1} \cdots x_\ell^{k_\ell} y_{\ell_\ell} \) is represented by the following cyclically reduced word \( w \) in \( \{a, b\} \):

\[
(4) \quad w := w_x a^{k_1} \bar{w}_x w_y \bar{l}_1 \bar{w}_y \cdots w_x a^{k_\ell} \bar{w}_x w_y \bar{l}_\ell \bar{w}_y.
\]

Since \( w = 1 \) in \( G(K(r)) \), there is a reduced van Kampen diagram \( (M, \phi) \) over \( G(K(r)) = \langle a, b | u_r \rangle \) such that \( (\phi(\partial M)) \equiv (w) \). Namely, \( M \) is a map, i.e., a finite 2-dimensional cell complex embedded in \( \mathbb{R}^2 \), and \( \phi \) is a function assigning to each oriented edge \( e \) of \( M \), as a label, a reduced word \( \phi(e) \) in \( \{a, b\} \) such that the following conditions are satisfied.

(i) If \( e \) is an oriented edge of \( M \) and \( e^{-1} \) is the oppositely oriented edge, then \( \phi(e^{-1}) \equiv \phi(e)^{-1} \).

(ii) For any boundary cycle \( \delta \) of any face of \( M \), \( \phi(\delta) \) is a cyclically reduced word such that \( (\phi(\delta)) \equiv (u_+^k) \). (If \( \alpha = e_1, \ldots, e_n \) is a path in \( M \), we define \( \phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_n) \).)

By [14, Corollary 6.2], \( M \) is a \([4,4]\)-map (cf. [14, Definition 7]). Then by the Curvature Formula of Lyndon and Schupp (see [17, Corollary V.3.4]), we have

\[
\sum_{v \in \partial M} (3 - d_M(v)) \geq 4,
\]

where \( d_M(v) \) is the degree of a vertex \( v \in \partial M \) in \( M \). This inequality yields the following Claim 1 (cf. [14, Claim in the proof of Theorem 6.3]).

**Claim 1.** In \( \partial M \), there exist at least four more vertices of degree 2 than vertices of degree at least 4.

**Claim 2.** Any two of degree 2 vertices cannot lie consecutively on \( \partial M \).

**Proof of Claim 2.** Suppose the contrary. Then \((\phi(\partial M)) \equiv (w)\) contains a subword \( z \) such that \( z \) is a subword of \((u_+^k)\) which is a product of three pieces but is not a product of \( t \) pieces with \( t \leq 2 \) (see [14, Convention 1]). Note that \( z \) is a subword of the cyclic word \((u_+^k)\) and hence it is alternating. On the other hand, we see from Propositions 4.5(3) and 4.6(3) that the words \( w_x a^{k_1} \bar{w}_x w_y b^{-1} \bar{w}_y \) or \( w_y b^{-1} \bar{w}_y w_x a^{k_\ell} \bar{w}_x \) with \( k, l \neq 0 \) are alternating if and only if \( |k| = |l| = 1 \). Hence \( z \) is a subword of the cyclically alternating cyclic word \((w') := (x_\ell^{e_1,x} y_{\ell_1}^{e_1,y} \cdots x_\ell^{e_\ell,x} y_{\ell_\ell}^{e_\ell,y})\), where \( \varepsilon_{i,x} = k_i/|k_i| \) and \( \varepsilon_{i,y} = l_i/|l_i| \). We will show that this cannot be possible in each case.

**Case 1:** \( r = [2m, 2n] \). By Proposition 4.5(2), \( x_\ell \equiv \bar{a} \cdots a \) and \( y_\ell \equiv \bar{b} \cdots b \). Thus, in the cyclic \( S \)-sequence of \( w' \), the last component, \( m \), of \( S(x_\ell^{e_\ell,x}) \) and the first component, \( m \), of \( S(y_{\ell_\ell}^{e_\ell+1,y}) \) are amalgamated into a component \( 2m \). Similarly, the first component, \( m \), of \( S(x_\ell^{e_1,x}) \) and the last component, \( m \), of \( S(y_{\ell_1}^{e_1+1,y}) \) are amalgamated into a component \( 2m \). Hence, we see by using Proposition 4.5(2) that

\[
CS(w') = ((2n - 1)(2m), (m + 1, m)^{e_1}, \ldots, (2n - 1)(2m), (m + 1, m)^{e_\ell}),
\]
where each \( \varepsilon_i \) is either 1 or \(-1\) and \((m + 1, m)^{-1}\) denotes \((m, m + 1)\). Since \(z\) is a product of three pieces and is not a product of two pieces, we see by Proposition 5.5(2),(3) that \(S(z)\) contains \((2m + 1, (2n - 1)/2m, \ell)\) or \((\ell, (2n - 1)/2m, 2m + 1)\) as a subsequence, for some \(\ell \in \mathbb{Z}_+\). This implies that \(CS(w')\) contains a term bigger than or equal to \(2m + 1\), since \(z\) is a subword of \(w'\). But this is an obvious contradiction to the above formula for \(CS(w')\).

**Case 2.a:** \(r = [2m, -2n]\), where \(m \geq 2\). We see by using Proposition 4.6(2) as in the previous case that

\[
CS(w') = \langle (2n - 1)/2m, (m, m - 1)^{\varepsilon_1}, \ldots, (2n - 1)(2m), (m, m - 1)^{\varepsilon_m} \rangle,
\]

where each \(\varepsilon_i\) is either 1 or \(-1\). Since \(z\) is a product of three pieces and is not a product of two pieces, we see by Proposition 5.5(2),(4) that \(S(z)\) contains \((2n - 1)/2m, 2m - 1, \ell)\) or \((\ell, 2m - 1, (2n - 1)/2m)\) as a subsequence, for some \(\ell \in \mathbb{Z}_+\). This implies that \(CS(w')\) contains a term \(2m - 1\), since \(z\) is a subword of \(w'\). But this is an obvious contradiction to the above formula for \(CS(w')\).

**Case 2.b:** \(r = [2m, -2n]\), where \(m = 1\). By Proposition 4.6(2), \(x_\ell \equiv b \cdots \bar{b}\) and \(y_\ell \equiv \bar{a} \cdots a\). Thus as in the previous case, we see by using Proposition 4.6(2),

\[
CS(w') = \begin{cases} 
\langle (2n - 2)/2, 3^{\varepsilon_1}, \ldots, (2n - 2)/2, 3^{\varepsilon_t} \rangle & \text{if } n \geq 2; \\
\text{every term is 2, 3, or 4} & \text{if } n = 1.
\end{cases}
\]

Since \(z\) is a product of three pieces and is not a product of two pieces, we see by Proposition 5.5(2),(4) that \(S(z)\) contains \((2n - 1)/2, 1, \ell)\) or \((\ell, 1, (2n - 1)/2)\) as a subsequence, for some \(\ell \in \mathbb{Z}_+\). This implies that \(CS(w')\) contains a term 1, since \(z\) is a subword of \(w'\). But this is an obvious contradiction to the above formula for \(CS(w')\).

By Claims 1 and 2, there must be some pair of degree 2 vertices on \(\partial M\) having only degree 3 vertices between them. Decompose \(\partial M\) into paths:

\[
\partial M = p_1q_1 \cdots p_s q_s,
\]

where every vertex lying in the closure of each \(q_i\) has degree 3 and every vertex lying in the interior of each \(p_i\) has degree 2 or degree at least 4. Here some \(q_i\) may be degenerate to a vertex.

Note that \(\phi(p_1q_1 \cdots p_s q_s)\) is not alternating at \(\phi(q_i)\) in the sense that (i) the last letter of \(\phi(p_i)\) and the first letter of \(\phi(p_{i+1})\) are the same letter, \(a^{\pm 1}\) or \(b^{\pm 1}\), and (ii) \(\phi(q_i)\) is equal to \(a^{\pm k}\) or \(b^{\pm k}\) with \(k \geq 0\) accordingly. On the other hand, \(w\) is not alternating precisely at the subwords \(a^{k_j}\) with \(|k_j| \geq 2\) and \(b^{l_j}\) with \(|l_j| \geq 2\) in the expression \((4)\). Hence \(\phi(q_j)\) corresponds to (possibly empty) subword of \(a^{k_j}\) with \(|k_j| \geq 2\) or \(b^{l_j}\) with \(|l_j| \geq 2\).

Recall that there is a pair of degree 2 vertices on \(\partial M\) having only degree 3 vertices between them. After a cyclic permutation of indices, we may assume that this occurs
at \(p_1q_1p_2\), namely the last (resp., first) occurring vertex in the interior of \(p_1\) (resp., \(p_2\)) has degree 2. Then a terminal (resp., initial) subword of \(\phi(p_1)\) (resp., \(\phi(p_2)\)) is a subword of the cyclic word \((u^\pm_1)\) which is a product of two pieces but is not a piece in itself (see [14, Convention 1]). Thus we may assume that the following holds. (The other possibility that \(\phi(p_1q_1p_2)\) contains the subword \(\bar{w}_xw_y\bar{b}_j\bar{w}_yw_x\) with \(|l_j| \geq 2\) can be settled by using Lemma [3.11].)

(i) \(\phi(p_1q_1p_2)\) contains the subword \(\bar{w}_yw_xa^{\ell,j,x}\) of \(w\), such that \(|k_j| \geq 2\).

(ii) \(\phi(p_1)\) ends with the alternating subword \(\bar{w}_yw_xa^{\ell,j,x}\).

(iii) \(\phi(q_1) \equiv a^{\ell,j,x}[k_j-2]\).

(iv) \(\phi(p_2)\) begins with the alternating subword \(a^{\ell,j,x}\bar{w}_xw_y\).

From this, we will derive a contradiction in each case.

**Case 1:** \(r = [2m, 2n]\). By using Proposition [4.5](1), we can see

\[
S(\bar{w}_yw_xa^{\ell,j,x}) = \begin{cases} 
(m, (2n - 1)(2m), m, 1) & \text{if } (-1)^n = \varepsilon_{j,x}; \\
(m, (2n - 1)(2m), m + 1) & \text{if } (-1)^n \neq \varepsilon_{j,x}.
\end{cases}
\]

Hence \(S(\phi(p_1))\) ends with \((m + \ell, (2n - 1)(2m), m, 1)\) or \((m + \ell, (2n - 1)(2m), m + 1)\) for some \(\ell \geq 0\) according to whether \((-1)^n = \varepsilon_{j,x}\) or not. Similarly, by using Proposition [4.5] we can see

\[
S(a^{\ell,j,x}\bar{w}_xw_y) = \begin{cases} 
(m + 1, (2n - 1)(2m), m) & \text{if } (-1)^n = \varepsilon_{j,x}; \\
(1, m, (2n - 1)(2m), m) & \text{if } (-1)^n \neq \varepsilon_{j,x}.
\end{cases}
\]

Hence \(\phi(p_2)\) begins with \((m + 1, (2n - 1)(2m), m + \ell)\) or \((1, m, (2n - 1)(2m), m + \ell)\) for some \(\ell \geq 0\) according to whether \((-1)^n = \varepsilon_{j,x}\) or not.

Thus we have shown that the following hold.

(i) If \((-1)^n = \varepsilon_{j,x}\), then \(S(\phi(p_1))\) ends with \(((2n - 1)(2m), m, 1)\).

(ii) If \((-1)^n \neq \varepsilon_{j,x}\), then \(\phi(p_2)\) begins with \((1, m, (2n - 1)(2m))\).

This leads to a contradiction as follows. Suppose that \((-1)^n = \varepsilon_{j,x}\) and so \(S(\phi(p_1))\) ends with \(((2n - 1)(2m), m, 1)\). Recall that \(\phi(p_1)\) ends with a subword, say \(v\), of \((u_r^{\pm 1})\) which is a product of two pieces but is not a piece. Since \(CS(r) = (2m + 1, (2n - 1)(2m), 2m + 1, (2n - 1)(2m))\), we see \(S(v) = (k, 1)\) for some \(k\) with \(1 \leq k \leq m\). But then \(v\) is a piece by Proposition [5.1](3), a contradiction. We also have a similar contradiction in the remaining case when \((-1)^n \neq \varepsilon_{j,x}\) and so \(\phi(p_2)\) begins with \((1, m, (2n - 1)(2m))\).

**Case 2.a:** \(r = [2m, -2n]\), where \(m \geq 2\). As in Case 1, by using Proposition [4.6](1) and the facts that \(\phi(p_1)\) ends with \(\bar{w}_yw_xa^{\ell,j,x}\) and \(\phi(p_2)\) begins with \(a^{\ell,j,x}\bar{w}_xw_y\), we can see that either \(S(\phi(p_1))\) ends with \(((2n - 1)(2m), m - 1, 1)\) or \(S(\phi(p_2))\) begins with \((1, m - 1, (2n - 1)(2m), 2m - 1)\). Noting that \(CS(r) = ((2n - 1)(2m), 2m - 1, (2n - 1)(2m), 2m - 1)\), this implies that there is a subword, \(v\), of \((u_r^{\pm 1})\) which is a product
of two pieces but is not a piece, such that \( S(v) = (k, 1) \) or \( (1, k) \) with \( k \leq m - 1 \). But then \( v \) is a piece by Proposition 5.1(4), a contradiction.

**Case 2.b:** \( r = [2m, -2n] \), where \( m = 1 \) and \( n \geq 2 \). As in the previous case, we can see that either \( S(\phi(p_1)) \) ends with \( ((2n - 2)(2), 1) \) or \( S(\phi(p_2)) \) begins with \( (1, (2n - 2)(2)) \). Noting that \( CS(r) = \langle ((2n - 1)(2), 1, (2n - 1)(2), 1) \rangle \), this implies that there is a subword, \( v \), of \( (u_r^\pm) \) which is a product of two pieces but is not a piece, such that \( S(v) = (k/2, 1) \) with \( 0 \leq k \leq 2n - 2 \), \( (1, k/2, 1) \) with \( 0 \leq k \leq 2n - 3 \), or \( (1, k/2) \) with \( 0 \leq k \leq 2n - 2 \). But, this implies that \( v \) is a piece by Proposition 5.1(4), a contradiction.

**Case 2.c:** \( r = [2m, -2n] \), where \( m = n = 1 \). In this case, by Proposition 4.6(1), \( w_x \equiv b \) and \( w_y \equiv a \). So \( \phi(p_1) \) ends with \( aba^\epsilon_jxa \) and \( \phi(p_2) \) begins with \( a^\epsilon_jxb a \). Suppose that \( \epsilon_j \equiv 1 \). (The other case when \( \epsilon_j = -1 \) can be treated similarly.) Since \( \phi(p_1) \) ends with a subword, \( v_1 \), of \( (u_r^\pm) \) which is a product of two pieces but is not a piece, and since \( \phi(p_2) \) begins with a subword, \( v_2 \), of \( (u_r^\pm) \) which is a product of two pieces but is not a piece, we see by using Proposition 5.1(4) that \( v_1 \equiv ba \) and \( v_2 \equiv ab \). Considering the equality \( v_1 \equiv ba \) together with the facts that \( CS(r) = \langle (2, 1, 2) \rangle \) and every vertex lying in the closure of \( q_1 \) has degree 3, we see that three incoming edges of each vertex lying in the closure of \( q_1 \) must have label \( a, a \) and \( b \), respectively. But then \( bv_2 \equiv bab \) is a subword of \( (u_r^\pm) \), which is a contradiction to \( CS(r) = \langle (2, 1, 2, 1) \rangle \).

The proof of Theorem 1.1 is now completed. \( \square \)

### 6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using the homology of the double branched covering, \( M(K(r)) \), of \( S^3 \) branched over \( K(r) \) and the \( \pi \)-orbifold group \( O(K(r)) \) introduced by Boileau and Zimmermann [8].

As in [14] Section 2, we regard \( (S^3, K(r)) \) as the union of two rational tangles \( (B^3, t(\infty)) \) and \( (\bar{B}^3, t(\gamma)) \) of slopes \( \infty \) and \( r \). Here the common boundary \( \partial(B^3, t(\infty)) = \partial(\bar{B}^3, t(\gamma)) \) is identified with the Conway sphere \( (S^2, P) := (\mathbb{R}^2, \mathbb{Z}^2)/H \), where \( H \) is the group of isometries of the Euclidean plane \( \mathbb{R}^2 \) generated by the \( \pi \)-rotations around the points in the lattice \( \mathbb{Z}^2 \). For each rational number \( s \in \mathbb{Q} = \mathbb{Q} \cup \{ \infty \} \), a line of slope \( s \) in \( \mathbb{R}^2 - \mathbb{Z}^2 \) projects to an essential simple loop, denoted by \( \alpha_s \), in \( S := S^2 - P \). Similarly, a line of slope \( s \) in \( \mathbb{R}^2 \) passing through a point \( \mathbb{Z}^2 \) determines an essential simple proper arc, denoted by \( \delta_s \), in \( S := S^2 - P \). The rational number \( s \) is called the slope of \( \alpha_s \) and \( \delta_s \). By the definition of the rational tangles, the loops \( \alpha_\infty \) and \( \alpha_r \) bound disks in \( B^3 - t(\infty) \) and \( B^3 - t(r) \), respectively.

The double branched covering \( M(K(r)) \) of \( (S^3, K(r)) \) is the union of the solid tori \( V_\infty \) and \( V_r \) which are obtained as the double branched coverings of \( (B^3, t(\infty)) \) and \( (B^3, t(r)) \), respectively. Let \( \tilde{\alpha}_0 \) and \( \tilde{\alpha}_\infty \) be lifts in \( \partial V_\infty \) of the simple loops \( \alpha_0 \) and \( \alpha_\infty \), respectively. Then \( \tilde{\alpha}_0 \) and \( \tilde{\alpha}_\infty \) form the meridian and the longitude of \( V_\infty \).
Similarly a lift $\tilde{\alpha}_r$ of $\alpha_r$ in $\partial V_r$ is a meridian of $V_r$. Thus $[\tilde{\alpha}_\infty]$ and $[\tilde{\alpha}_r]$ are the zero elements of $H_1(V_\infty)$ and $H_1(V_r)$, respectively. Since $[\tilde{\alpha}_r] = p[\tilde{\alpha}_0] + q[\tilde{\alpha}_\infty]$ in $H_1(\partial V_\infty)$, where $r = q/p$, we have

$$H_1(M(K(r))) \cong \langle \tilde{\alpha}_0 \mid p[\tilde{\alpha}_0] \rangle \cong \mathbb{Z}/p\mathbb{Z}$$

Recall the $\pi$-orbifold group $O(K(r))$ of the knot $K(r)$, which is defined as the quotient of the knot group $G(K(r))$ by the normal subgroup normally generated by the square of meridians (see [8]). Then $O(K(r))$ is the semidirect product

$$\pi_1(M(K(r))) \rtimes \mathbb{Z}/2\mathbb{Z} \cong H_1(M(K(r))) \rtimes \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$$

and so it is isomorphic to the dihedral group of order $2p$.

We prove Theorem 1.2 by showing that the images in $O(K(r))$ of the groups generated by the meridian pairs in the theorem is a proper subgroup of $O(K(r))$. Let $r$ be the rational number

$$[2m, -2m] = [2m - 1, 2m - 1] = \frac{2m}{4m^2 - 1} = \frac{2m}{(2m + 1)(2m - 1)}.$$

Observe that the involution $h_3$ in Figure 2 is equivalent to the involution in Figure 4(a). Let $\beta_1$ and $\beta_2$ be the components of $\beta - K(r)$ where $\beta = \text{Fix}(h_3)$, as illustrated in Figure 4(a). We can observe that the arcs $\beta_1$ and $\beta_2$ are the proper essential arcs in the Conway sphere $S$ of slopes $s_1 := 1/(2m - 1)$ and $s_2 := 1/(2m + 1)$, respectively. In fact, the involution $h_3$ preserves $S$ and the involution of the Farey tessellation induced by the restriction of $h_3|S$ is the reflection in the geodesic joining $s_1$ and $s_2$ (see Figure 4(b)).

For $i = 1, 2$, let $\{x_i, y_i\}$ be the meridian pair represented by the proper arc $\beta_i$. Then, by the above observation, the subgroup $\langle x_i, y_i \rangle$ of $G(K(r))$ is equal to $\langle x_i, \alpha_{s_i} \rangle$, where $\alpha_{s_i}$ is an element of $G(K(r))$ represented by the simple loop $\alpha_{s_i}$ in $S$ of slope...
Thus the subgroup \( \langle \hat{a}_{s_1} \rangle \) of \( H_1(M(K(r))) \) has order \( 2m + 1 \) and so it is a proper subgroup of \( H_1(M(K(r))) \). Similarly, the subgroup \( \langle \hat{a}_{s_2} \rangle \) of \( H_1(M(K(r))) \) has order \( 2m - 1 \) and so it is a proper subgroup of \( H_1(M(K(r))) \).

On the other hand, the image of \( \langle x_i, y_i \rangle = \langle x_i, \alpha_{s_i} \rangle \) in \( O(K(r)) \) is the semidirect product \( \langle \hat{a}_{s_i} \rangle \rtimes \mathbb{Z}/2\mathbb{Z} \). Hence, it is a proper subgroup of \( O(K(r)) \), and therefore \( \langle x_i, y_i \rangle \) is a proper subgroup of \( G(K(r)) \) for each \( i = 1, 2 \). This completes the proof of Theorem 1.2.

Remark 6.1. (1) We can show that \( \langle x_2, y_2 \rangle \) is a free group by an argument parallel to the proof of Theorem 1.1. However, our method does not work for the subgroup \( \langle x_1, y_1 \rangle \).

(2) Theorem 1.2 can be easily extended to every 2-bridge knot \( K(r) \) with \( r = q/p \) such that \( q^2 \equiv 1 \pmod{p} \). In fact, we can see that, for the additional strong inversion \( h_3 \), the components, \( \beta_1 \) and \( \beta_2 \), of \( \text{Fix}(h_3) \cap E(K(r)) \) are proper essential arcs of slopes \( s_1 = q_1/p_1 \) and \( s_2 = q_2/p_2 \), where \( p = p_1p_2 \) and both \( p_1 \) and \( p_2 \) are greater than 1. Thus we can see that the subgroup of \( O(K(r)) \) generated by the meridian pair represented by \( \beta_i \) is a proper subgroup for \( i = 1, 2 \).

7. Proof of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Immediate from Theorems 1.1 and 1.2 and Corollary 2.2. □

Proof of Theorem 1.4. The if part follows from the if part of [14, Main Theorem 2.4] (which is essentially equivalent to [18, Theorem 1.1]) and the fact that \( G(K(r)) \) is isomorphic to \( G(K(r')) \). So we prove the only if part. Let \( \varphi : G(K(\hat{r})) \to G(K(r)) \) be an epimorphism between 2-bridge knot groups satisfying the assumption of the theorem. By [6, Corollary 1.3], \( \varphi \) maps the upper meridian pair \( \{\hat{a}, \hat{b}\} \) of \( G(K(\hat{r})) \) to peripheral elements of \( G(K(r)) \). Thus \( \{\varphi(\hat{a}), \varphi(\hat{b})\} \) is a parabolic generating pair and hence by Corollary 2.2 it is either (i) the upper or lower meridian pair, (ii) the long upper or long lower meridian pair, or (iii) isomorphic to the upper or lower exceptional pair. However, Theorems 1.1 and 1.2 prohibit the last two possibilities, and hence \( \varphi \) maps the upper meridian pair of \( G(K(\hat{r})) \) to the upper or lower meridian pair of \( G(K(r)) \).

Suppose first that \( \varphi \) maps the upper meridian pair of \( G(K(\hat{r})) \) to the upper meridian pair of \( G(K(r)) \). Then \( \hat{r} \) or \( \hat{r} + 1 \) belongs to the \( \hat{I}_{\varphi} \)-orbit of \( r \) or \( \infty \) by [14, Main Theorem 2.4]. Suppose next that \( \varphi \) maps the upper meridian pair of \( G(K(\hat{r})) \) to the lower meridian pair of \( G(K(r)) \). Note that there is an isomorphism from \( G(K(r)) \) to \( G(K(r')) \) which maps the lower meridian pair of \( G(K(r)) \) to the upper meridian pair of \( G(K(r')) \). Thus the composition of \( \varphi \) and this isomorphism is an epimorphism from \( G(K(\hat{r})) \) to \( G(K(r')) \) which maps the upper meridian pair of
G(K(\tilde{r})) to that of G(K(r')). Hence, by [14, Main Theorem 2.4], \tilde{r} or \tilde{r} + 1 belongs to the \hat{\Gamma}, \omega-orbit of r' or \infty. This completes the proof of Theorem 1.4. □

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