Global existence for 3D non-stationary Stokes flows with Coulomb’s type friction boundary conditions

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ABSTRACT
In this paper, we study non-stationary viscous incompressible fluid flows with non-linear boundary slip conditions given by a subdifferential property of friction type. More precisely we assume that the tangential velocity vanishes as long as the shear stress remains below a threshold $F$, that may depend on the time and the position variables but also on the stress tensor, allowing to consider Coulomb’s type friction laws. An existence and uniqueness theorem is obtained first when the threshold $F$ is a data and sharp estimates are derived for the velocity and pressure fields as well as for the stress tensor. Then an existence result is proved for the non-local Coulomb’s friction case using a successive approximation technique with respect to the shear stress threshold.

1. Introduction

In the study of fluid flow, it is usually assumed that the fluid sticks to the boundary of the flow domain, leading to the so-called no-slip boundary condition. Such a behaviour has been mathematically justified by considering the microscopic asperities of the boundary (see [1–5]). Unfortunately experiments show that more complex boundary conditions may occur, especially in the case of non-wetting, hydrophobic or chemically patterned surfaces [6–10], leading to linear slip conditions of Navier type [11–13] or non-linear slip conditions of friction type when the tangential fluid velocity does not vanish only when a threshold is reached [14]. This last class of boundary conditions has been considered first for Bingham fluids in [15]. Then they have been introduced for incompressible Newtonian fluid flows by Fujita during his lectures at Collège de France in 1993 [16] and subsequently studied by Fujita who proved existence and uniqueness for the stationary Stokes problem and by Saito who established some regularity properties for the solutions [17–19]. See also [20] for shape optimization issues.

More precisely they consider boundary conditions given by a subdifferential property, i.e.

$$
\begin{align*}
{v} &= 0 \quad \text{on } \Gamma \setminus \Gamma_0 \text{ (no-slip condition on } \Gamma \setminus \Gamma_0), \\
{v}_n &= 0, \quad -\sigma_t \in F\partial(|v_t|) \quad \text{on } \Gamma_0 \text{ (slip condition on } \Gamma_0)
\end{align*}
$$

where the boundary of the flow domain is splitted into $\Gamma = \Gamma_0 \cup (\Gamma \setminus \Gamma_0)$, $F$ is a given positive function on $\Gamma_0$, $\partial(|\cdot|)$ is the subdifferential of the function $|\cdot|$, $v_n, v_t$ and $\sigma_t$ are the normal component of the velocity, the tangential component of the velocity and the shear stress, respectively. By using
the definition of the subdifferential of a convex function [21], we may rewrite the boundary condition on $\Gamma_0$ as

$$v_n = 0, \quad |\sigma_t| \leq F \quad \text{on } \Gamma_0$$

and

$$|\sigma_t| < F \implies v_t = 0,$$

$$|\sigma_t| = F \implies \exists \lambda \geq 0 \text{ s.t. } v_t = -\lambda \sigma_t \quad \text{on } \Gamma_0$$

which can be interpreted as a Tresca’s friction condition on $\Gamma_0$ [22].

For the unsteady Stokes problem, existence has been established by Fujita [23] when the density of body forces is equal to zero using the non-linear semigroup theory and for regularity properties the reader is referred to Saito and Fujita [24]. See also [25,26] for global existence for time-dependent flows with Tresca’s friction boundary conditions.

The purpose of this paper is to extend these results to unsteady problems with non-vanishing external forces and to more general friction boundary conditions, like Coulomb’s friction boundary conditions, where the threshold $F$ may depend on stress tensor $\sigma$. Indeed, for solids in contact with a sliding planar surface $\Gamma_0$, Coulomb established experimentally [27] that

$$v_n = 0, \quad |\sigma_t| \leq k|\sigma_n| \quad \text{on } \Gamma_0$$

and

$$|\sigma_t| < k|\sigma_n| \implies v_t = s,$$

$$|\sigma_t| = k|\sigma_n| \implies \exists \lambda \geq 0 \text{ s.t. } v_t = s - \lambda \sigma_t \quad \text{on } \Gamma_0$$

where $\sigma_n$ is the normal component of the stress vector, $s$ is the sliding velocity of the surface and $k > 0$ is a friction coefficient. Hence we get

$$v_n = 0, \quad -\sigma_t \in F \partial \left(|v_t - s|\right) \quad \text{on } \Gamma_0$$

with $F = k|\sigma_n|$.

In order to be able to take into account also possible anisotropic friction, we will consider in this paper shear stress thresholds of the form $F = F(x', t, \sigma)$. More precisely we consider a non-stationary Stokes flow described by the system

$$\frac{\partial v}{\partial t} - \text{div}(2\mu D(v)) + \nabla p = f \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

$$\text{div}(v) = 0 \quad \text{in } \Omega \times (0, T), \quad (1.2)$$

with the initial condition

$$v(0) = v^0 \quad \text{in } \Omega. \quad (1.3)$$

Here $[0, T]$ is a given non-trivial time interval, $v$ and $p$ Denote, respectively, the velocity and the pressure of the fluid, $\mu \in \mathbb{R}_+^*$ is its viscosity, $f$ is the density of body forces and $D(v)$ is the strain rate tensor defined as

$$D(v) = (d_{ij}(v))_{1 \leq i,j \leq 3}, \quad d_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right), \quad 1 \leq i,j \leq 3.$$
Motivated by lubrication or extrusion/injection problems, we assume that the fluid domain $\Omega$ is given by

$$\Omega = \{(x',x_3) \in \mathbb{R}^2 \times \mathbb{R} : x' \in \omega, \ 0 < x_3 < h(x')\},$$

where $\omega$ is a non-empty open-bounded subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary, and $h$ is a Lipschitz continuous function which is bounded from above and from below by some positive real numbers. We decompose the boundary of $\Omega$ as $\partial \Omega = \Gamma_0 \cup \Gamma_L \cup \Gamma_1$, with $\Gamma_0 = \{(x',x_3) \in \overline{\Omega} : x_3 = 0\}$, $\Gamma_1 = \{(x',x_3) \in \overline{\Omega} : x_3 = h(x')\}$ and $\Gamma_L$ the lateral part of the boundary. Let us denote by $n = (n_1,n_2,n_3)$ the unit outward normal vector to $\partial \Omega$, and by $u \cdot w$ (resp. $|u|$) the Euclidean inner product (resp. the Euclidean norm) of vectors $u$ and $w$. We define the normal and the tangential velocities on $\partial \Omega$ by

$$\nu_n = v \cdot n = \sum_{i=1}^{3} \nu_i n_i, \quad \nu_t = (\nu_{ti})_{1 \leq i \leq 3} \quad \text{with} \quad \nu_{ti} = v_i - \nu_n n_i \quad 1 \leq i \leq 3$$

and the normal and the tangential components of the stress tensor $\sigma = -pI + 2\mu D(v)$ by

$$\sigma_n = \sum_{i,j=1}^{3} \sigma_{ij} n_i n_j, \quad \sigma_t = (\sigma_{ti})_{1 \leq i \leq 3} \quad \text{with} \quad \sigma_{ti} = \sum_{j=1}^{3} \sigma_{ij} n_j - \sigma_{in} n_i \quad 1 \leq i \leq 3.$$

We introduce a function $s : \Gamma_0 \rightarrow \mathbb{R}^2$ and a function $g : \partial \Omega \rightarrow \mathbb{R}^3$ such that

$$\int_{\Gamma_L} g_n \, d\gamma = 0, \quad g = 0 \text{ on } \Gamma_1, \quad g_n = 0 \text{ and } g_t = g - g_n n = (s,0) \text{ on } \Gamma_0.$$

We assume that the upper part of the fluid domain is fixed while the lower part is moving with a shear velocity given by $s \zeta(t)$, where $\zeta : [0,T] \rightarrow \mathbb{R}$ is such that $\zeta(0) = 1$. Then the fluid velocity satisfies the following non-homogeneous boundary condition on $\Gamma_1 \cup \Gamma_L$

$$v = g\zeta \quad \text{on} \quad (\Gamma_1 \cup \Gamma_L) \times (0,T). \quad (1.4)$$

We assume furthermore that the flow satisfies a generalized dry friction law on $\Gamma_0$, i.e.

$$\nu_n = 0 \quad \text{on} \quad \Gamma_0 \times (0,T), \quad (1.5)$$

$$|\sigma_t| \leq F(x',t,\sigma) \quad \text{on} \quad \Gamma_0 \times (0,T) \quad (1.6)$$

and

$$|\sigma_t| < F(x',t,\sigma) \quad \implies \nu_t = (s\zeta,0), \quad \text{on} \quad \Gamma_0 \times (0,T)$$

$$|\sigma_t| = F(x',t,\sigma) \quad \implies \exists \lambda \geq 0 \text{ s.t. } \nu_t = (s\zeta,0) - \lambda \sigma_t \quad \text{on} \quad \Gamma_0 \times (0,T) \quad (1.7)$$

where $F$ is a given non-negative mapping on $\Gamma_0 \times (0,T) \times \mathbb{R}^{3 \times 3}$.

2. Mathematical formulation of the problem

In order to get a variational formulation of the problem we introduce the following functional spaces

$$\mathcal{V}_0 = \left\{ \varphi \in (H^1(\Omega))^3 : \varphi = 0 \text{ on } \Gamma_1 \cup \Gamma_L, \varphi_n = 0 \text{ on } \Gamma_0 \right\},$$
\[ V = \{ \varphi \in \mathcal{V}_0 : \text{div}(\varphi) = 0 \text{ in } \Omega \}, \]

endowed with the norm of \( (H^1(\Omega))^3 \), and

\[ L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_{\Omega} q \, dx = 0 \right\} \]

endowed with the norm of \( L^2(\Omega) \). We assume that

\[ f \in L^2(0, T; (L^2(\Omega))^3), \quad (2.1) \]

\[ \mu \in \mathbb{R}^*_+, \quad \zeta \in C^\infty([0, T], \mathbb{R}) \text{ such that } \zeta(0) = 1, \quad (2.2) \]

with \( T > 0 \), and we define

\[ a(u, v) = \int_{\Omega} 2\mu D(u) : D(v) \, dx \]

\[ = \int_{\Omega} 2\mu \sum_{i,j=1}^3 d_{ij}(u)d_{ij}(v) \, dx \quad \forall (u, v) \in (H^1(\Omega))^3 \times (H^1(\Omega))^3. \]

From Korn’s inequality [28] we infer that there exists \( \alpha > 0 \) such that

\[ \alpha \| u \|^2_{H^1(\Omega)} \leq \int_{\Omega} 2\mu D(u) : D(u) \, dx \leq 2\mu \| u \|^2_{L^2(\Omega)} \quad \forall u \in \mathcal{V}_0. \quad (2.3) \]

Moreover, in order to deal with homogeneous boundary conditions on \( \Gamma_1 \cup \Gamma_L \), we assume that there exists an extension of \( g \) to \( \Omega \), denoted by \( G_0 \), such that

\[ G_0 \in (H^2(\Omega))^3, \quad \text{div}(G_0) = 0 \text{ in } \Omega, \quad G_0 = g \text{ on } \partial \Omega, \quad (2.4) \]

and we let

\[ \tilde{v} = v - G_0 \zeta. \]

By multiplying (1.1) by a test-function \( \varphi \chi \), with \( \varphi \in \mathcal{V}_0 \) and \( \chi \in \mathcal{D}(0, T) \), a formal integration by part leads to

\[ -\int_0^T \int_{\Omega} \text{div}(2\mu D(v)) \cdot \varphi \chi \, dx \, dt + \int_0^T \int_{\Omega} \nabla p \cdot \varphi \chi \, dx \, dt \]

\[ = -\int_0^T \int_{\Omega} \text{div}(\sigma) \cdot \varphi \chi \, dx \, dt \]

\[ = \int_0^T \int_{\Omega} 2\mu \sum_{i,j=1}^3 d_{ij}(v) \frac{\partial \varphi_i}{\partial x_j} \, dx \, dt - \int_0^T \int_{\Omega} p \text{div}(\varphi) \chi \, dx \, dt \]

\[ - \int_0^T \int_{\partial \Omega} \sum_{i,j=1}^3 \sigma_{ij} \phi_i n_j \chi \, dy \, dt. \]
Then we infer from (1.5)–(1.7) that
\[ \sigma_t \cdot (v - G_0 \zeta) + \mathcal{F}(x', t, \sigma)|v - G_0 \zeta| = 0 \quad \text{on } \Gamma_0 \times (0, T) \]
and recalling that \( \varphi \in \mathcal{V}_0 \) we get
\[
\int_0^T \int_{\Omega} \sum_{i,j=1}^3 \sigma_{ij} \varphi_i \eta_j \, dx \, dt = \int_0^T \int_{\Gamma_*} \sigma_t \cdot \varphi \chi \, dx' \, dt
\]
\[
\geq - \int_0^T \int_{\Gamma_*} \mathcal{F}(x', t, \sigma)|\tilde{v} + \varphi \chi| \, dx' \, dt
\]
\[
+ \int_0^T \int_{\Gamma_*} \mathcal{F}(x', t, \sigma)|\tilde{v}| \, dx' \, dt.
\]
Since we expect \( \tilde{v} \) to take its values in \( L^2(0, T; \mathbf{V}) \) these integrals make sense if \( \mathcal{F}(x', t, \sigma) \in L^2(0, T; L^2(\Gamma_0)) \).

Under this assumption we consider the following problem

**Problem (P)** Find
\[ \tilde{v} \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; (L^2(\Omega))^3), \quad p \in H^{-1}(0, T; L^2(\Omega)) \]
such that, for all \( \varphi \in \mathcal{V}_0 \) and for all \( \chi \in \mathcal{D}(0, T) \), we have
\[
\left\langle \frac{d}{dt} (\tilde{v}, \varphi), \chi \right\rangle_{D'(0, T), D(0, T)} - \left\langle (p, \text{div}(\varphi)), \chi \right\rangle_{D'(0, T), D(0, T)}
\]
\[
+ \int_0^T a(\tilde{v}, \varphi \chi) \, dt + \Psi_\mathcal{F}(\tilde{v} + \varphi \chi) - \Psi_\mathcal{F}(\tilde{v}) \geq \left( f, \varphi \right)_{D'(0, T), D(0, T)}
\]
\[
- \int_0^T a(G_0 \zeta, \varphi \chi) \, dt - \left( \left( G_0 \frac{\partial \zeta}{\partial t}, \varphi \right)_{D'(0, T), D(0, T)}
\]
\]
where \( \Psi_\mathcal{F} \) is given by
\[
\Psi_\mathcal{F}(u) = \int_0^T \int_{\Gamma_*} \mathcal{F}(x', t, \sigma)|u(x', t)| \, dx' \, dt \quad \forall u \in L^2(0, T; (L^2(\Gamma_0))^3)
\]
together with the initial condition
\[ \tilde{v}(0, \cdot) = \tilde{v}^0 = v^0 - G_0 \zeta(0) = v^0 - G_0, \]
where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( (L^2(\Omega))^3 \). Let us emphasize that we identify \( \tilde{v} + \varphi \chi \) and \( \tilde{v} \) with their trace on \( \Gamma_* \) in the definition of \( \Psi_\mathcal{F}(\tilde{v} + \varphi \chi) \) and \( \Psi_\mathcal{F}(\tilde{v}) \).

We may observe that, for any solution of problem (P), the stress tensor \( \sigma = -p \text{Id} + 2 \mu D(\tilde{v} + G_0 \zeta) \) belongs to \( H^{-1/2}(\partial \Omega) \). Thus we cannot consider directly the Coulomb's friction case described by \( \mathcal{F}(\cdot, \cdot, \sigma) = k|\sigma_n| \) since \( \sigma_n \) is not necessarily well defined on \( \Gamma_0 \) and \( |\sigma_n| \) does not belong to \( L^2(0, T; L^2(\Gamma_0)) \). This kind of difficulty appears also in the study of frictional contact problems in solid mechanics and it has been encompassed by replacing \( \sigma_n \) by some regularization \( \sigma_n^* \). This idea introduced by Duvaut [22,29] has led to the so-called non-local Coulomb's friction law. The regularization procedure \( \sigma_n \mapsto \sigma_n^* \) is built using a linear continuous operator from \( H^{-1/2}(\partial \Omega) \) to \( L^2(\Gamma_0) \) which fits the mechanical meaning of \( \sigma_n \) that is defined as the ratio of a force by a surface.
Namely, for the static case in solid mechanics, it is easily proved by duality arguments that $\sigma_n$ belongs to $H^{-1/2}(\partial\Omega)$ and $\sigma_n^*$ is obtained by convolution of $\sigma_n$ with a smooth non-negative function [30]. See also [31] in the framework of non-Newtonian fluids.

In our case, for the unsteady flow problem, we have to deal with two additional difficulties. Indeed, in order to define the normal trace of $\sigma$ on $\partial\Omega$, we have to establish some regularity properties for $\div(\sigma)$. But, if we choose $\varphi \in (\mathcal{D}(\Omega))^3$ and $\chi \in \mathcal{D}(0, T)$ in (2.5) we get

$$\left\langle \frac{d}{dt} \left( \bar{v}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} + \int_0^T a(\bar{v} + G_0 \zeta, \varphi \chi) \, dt - \left\langle (p, \div(\varphi)), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} \geq \left\langle (f, \varphi), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} - \left\langle \left( G_0 \frac{\partial \zeta}{\partial t}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)}$$

i.e.

$$\left\langle \frac{d}{dt} \left( \bar{v}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} + \int_0^T \int_{\Omega} \sum_{i,j=1}^3 \sigma_{ij} \frac{\partial \psi_i}{\partial x_j} \varphi \chi \, dx \, dt \geq \int_0^T \int_{\Omega} f \varphi \chi \, dx \, dt - \int_0^T \int_{\Omega} G_0 \frac{\partial \zeta}{\partial t} \varphi \chi \, dx \, dt.$$

Hence the regularity of $\div(\sigma)$ is ‘governed’ by the regularity of $\frac{\partial \zeta}{\partial t}$. Moreover, since $\sigma \in H^{-1}(0, T; (L^2(\Omega))^3)$, we may only expect $\sigma_n$ to belong to $H^{-1}(0, T; H^{-1/2}(\partial\Omega))$. It follows that we will need to regularize $\sigma_n$ not only with respect to the space variable as in [22,29,30] but also with respect to the time variable. Of course, in the case of an evolutionary problem, it will be a nonsense to propose a convolution of $\sigma_n$ with respect to the time variable on $[0, T]$ and the most natural regularization seems to replace $\sigma_n$ by $\sigma_n^*$ and then to regularize $\sigma_n^*$ by a kind of truncated convolution on each time interval $[0, t], t \in [0, T]$, leading to a space-time non-local friction law described by

$$\mathcal{F}(\chi', t, \sigma) = k \int_0^t S(t - s) |\sigma_n^*(\chi', s)| \, ds$$

for almost every $\chi' \in \Gamma_0$ and for all $t \in [0, T]$, where $S$ is a non-negative smooth real function.

Let us emphasize that $S$ can also be interpreted as the kernel of some history-dependent shear stress threshold. Such kind of friction laws have been recently developed in the framework of solid mechanics (see [32] and the references therein for instance).

The outline of our paper is as follows. In the next section we consider the Tresca’s case when the mapping $\mathcal{F}$ is a given non-negative function of $(\chi', t)$ and does not depend on $\sigma$. We will establish an existence and uniqueness result for problem $(P)$ as well as some estimates of the solution, using a Yosida’s approximation of the slip condition (Theorem 1). Then, under some compatibility assumptions on the initial velocity $v_0$, we prove additional regularity properties and sharp estimates for $\frac{\partial \bar{v}}{\partial t}, p$ and $\sigma$ (Theorem 2 and Proposition 1). Then, in Section 4 we consider the generalized Coulomb’s friction case described by

$$\mathcal{F}(\chi', t, \sigma) = \mathcal{F}^0(\chi', t) + \mathcal{F}^\sigma(\chi', t) \int_0^t S(t - s) |\sigma_n^*(\chi', s)| \, ds$$

for almost every $\chi' \in \Gamma_0$ and for all $t \in [0, T]$. If $\mathcal{F}^\sigma \equiv 0$ we recover the Tresca’s friction case and when $\mathcal{F}^0 \equiv 0$ and $\mathcal{F}^\sigma \equiv k$, with $k > 0$, we obtain the space-time non-local friction law introduced previously.
For this generalized Coulomb’s friction law we prove an existence result by applying a successive approximation technique with respect to the shear stress threshold.

3. The Tresca’s friction case

Let us assume from now on that \( F \) does not depend on its third argument, i.e.

\[
F(x', t, \sigma) = \ell(x', t) \text{ on } \Gamma_0 \times (0, T),
\]

with

\[
\ell \in L^2(0, T; L^2(\Gamma_0; \mathbb{R}^+)) \tag{3.1}
\]

Let \( H \) be the closure in \( L^2(\Omega) \) of \( \{ \phi \in (D(\Omega))^3; \ div(\phi) = 0 \} \).

**Theorem 1:** Let assumptions (2.1)–(2.2)–(2.4)–(3.1) hold. Then, for all \( \tilde{v}^0 \in H \), problem \( P \) admits an unique solution. Furthermore, \( \frac{\partial \tilde{v}}{\partial t} \in L^2(0, T; V) \).

**Proof:** For the sake of notational simplicity, let us denote \( X \) instead of \( X^3 \) for any functional space \( X \).

For any \( \varepsilon > 0 \) we consider the following approximate problem \( (P_\varepsilon) \)

**Problem \( P_\varepsilon \)**

Find

\[
\tilde{v}_\varepsilon \in L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega)), \ p_\varepsilon \in H^{-1}(0, T; L^2_0(\Omega))
\]

such that, for all \( \varphi \in \mathcal{V}_0 \) and for all \( \chi \in D(0, T) \), we have

\[
\left\{ \begin{array}{l}
\left\langle \frac{d}{dt}(\tilde{v}_\varepsilon, \varphi), \chi \right\rangle_{D'(0,T),D(0,T)} - \left\langle (p_\varepsilon, \text{div}(\varphi)), \chi \right\rangle_{D'(0,T),D(0,T)} \\
+ \int_0^T a(\tilde{v}_\varepsilon, \varphi \chi) \, dt + \int_0^T \int_{\Gamma_0} \ell \frac{\tilde{v}_\varepsilon \chi}{\sqrt{\varepsilon^2 + |\tilde{v}_\varepsilon|^2}} \chi \, dx \, dt = \left\langle (f, \varphi), \chi \right\rangle_{D'(0,T),D(0,T)} \\
- \int_0^T a(G_0 \varphi, \varphi \chi) \, dt - \left\langle \left( G_0 \frac{\partial \varphi}{\partial t}, \varphi \right), \chi \right\rangle_{D'(0,T),D(0,T)}
\end{array} \right. \tag{3.2}
\]

with the initial condition

\[
\tilde{v}_\varepsilon(0, \cdot) = \tilde{v}^0. \tag{3.3}
\]

As a first step we solve \( P_\varepsilon \) using Galerkin’s method. Recalling that \( H = \{ \phi \in L^2(\Omega); \ div(\phi) = 0 \text{ in } \Omega; \ \phi_n = 0 \text{ on } \partial \Omega \} \) (see [33] for instance) we infer that there exists a Hilbertian basis \( (w_i)_{i \geq 1} \) of \( H \) which is orthonormal for the inner product of \( L^2(\Omega) \), and such that \( (w_i)_{i \geq 1} \) is also a Hilbertian basis of \( V \) which is orthogonal for the inner product of \( H^1(\Omega) \). Then, for all \( m \geq 1 \), we look for a function \( \tilde{v}_{\varepsilon m} \) given by

\[
\tilde{v}_{\varepsilon m}(t, x) = \sum_{j=1}^m g_{\varepsilon j}(t) w_j(x), \quad \forall t \in (0, T), \ \forall x \in \Omega,
\]
such that, for all $k \in \{1, \ldots, m\}$, we have

$$
\left( \frac{\partial \tilde{v}_{em}}{\partial t}, \tilde{v}_{em} \right) + a(\tilde{v}_{em}, w_k) + \int_{\Omega} \ell \left( \tilde{v}_{em} \cdot w_k \right) - a(G_0 \xi, w_k) - \left( G_0 \frac{\partial \xi}{\partial t}, w_k \right) \quad \text{a.e. in } (0, T)
$$

(3.4)

with the initial condition

$$
\tilde{v}_{em}(0, \cdot) = \tilde{v}_m^0
$$

(3.5)

where $\tilde{v}_m^0$ is the orthogonal projection of $\tilde{v}_m^0$ in $(L^2(\Omega))^3$ on Span$\{w_1, \ldots, w_m\}$.

Using Caratheodory's theorem (see [34]), we obtain that (3.4)–(3.5) admits a unique maximal solution $\tilde{v}_{em} \in W^{1,2}(0, \tau_m; V)$, with $\tau_m \in (0, T]$. As usual we may establish some a priori estimates independent of $m$ and $\varepsilon$ which allow us to extend this solution to the whole interval $[0, T]$. 

**Lemma 1:** Assume that (2.1), (2.2), (2.4), (3.1) hold and that $\tilde{v}_m^0 \in H$. Then, for all $m \geq 1$, the problem (3.4)–(3.5) admits a unique solution $\tilde{v}_{em} \in W^{1,2}(0, T; V)$ which satisfies the following estimates:

$$
\|\tilde{v}_{em}\|_{L^\infty(0,T;L^2(\Omega))} \leq C_1
$$

(3.6)

$$
\|\tilde{v}_{em}\|_{L^2(0,T;H^1(\Omega))} \leq C_1
$$

(3.7)

where $C_1$ is a positive constant independent of $m$ and $\varepsilon$.

**Proof:** We multiply Equation (3.4) by $g_{ek}(t)$ and we add from $k = 1$ to $m$. Observing that $\ell$ is a non-negative mapping we obtain

$$
\left( \frac{\partial \tilde{v}_{em}}{\partial t}, \tilde{v}_{em} \right) + \int_{\Omega} 2\mu D(\tilde{v}_{em}) : D(\tilde{v}_{em}) \, dx
$$

$$
\leq (f, \tilde{v}_{em}) - \int_{\Omega} 2\mu D(G_0 \xi) : D(\tilde{v}_{em}) \, dx - \left( G_0 \frac{\partial \xi}{\partial t}, \tilde{v}_{em} \right) \quad \text{a.e. in } (0, \tau_m).
$$

We integrate from 0 to $s$, with $0 < s < \tau_m$. Then, using (2.3) and Young's inequalities, we obtain

$$
\frac{1}{2} \|\tilde{v}_{em}(s)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \int_0^s \|\tilde{v}_{em}\|_{H^1(\Omega)}^2 \, dt \leq \frac{1}{2} \|\tilde{v}_{em}(0)\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^s \|f\|_{L^1(\Omega)}^2 \, dt
$$

$$
+ \frac{2\mu^2}{\alpha} \|G_0\|_{H^1(\Omega)}^2 \int_0^s \|\xi\|^2 \, dt + \frac{1}{2} \|G_0\|_{L^2(\Omega)}^2 \int_0^s \left| \frac{\partial \xi}{\partial t} \right|^2 \, dt + \int_0^s \|\tilde{v}_{em}\|_{L^2(\Omega)}^2 \, dt.
$$

Recalling that $\tilde{v}_{em}(0)$ is defined as the orthogonal projection of $\tilde{v}_m^0$ in $L^2(\Omega)$ on Span$\{w_1, \ldots, w_m\}$, we have

$$
\frac{1}{2} \|\tilde{v}_{em}(s)\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \int_0^s \|\tilde{v}_{em}\|_{H^1(\Omega)}^2 \, dt \leq C'_1 + \int_0^s \|\tilde{v}_{em}\|_{L^2(\Omega)}^2 \, dt,
$$

(3.8)

where $C'_1$ is given by

$$
C'_1 = \frac{1}{2} \|\tilde{v}_m^0\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_0^T \|f\|_{L^2(\Omega)}^2 \, dt + \frac{2\mu^2}{\alpha} \|G_0\|_{H^1(\Omega)}^2 \int_0^T \|\xi\|^2 \, dt
$$

$$
+ \frac{1}{2} \|G_0\|_{L^2(\Omega)}^2 \int_0^T \left| \frac{\partial \xi}{\partial t} \right|^2 \, dt.
With Grönwall’s lemma, we get
\[
\|\tilde{\nu}_{em}(s)\|_{L^2(\Omega)}^2 \leq 2C'_1 \exp\left(2s\right) \leq 2C'_1 \exp\left(2T\right) \quad \forall s \in [0, \tau_m).
\] (3.9)

By definition of the maximal solution, we may conclude that \(\tau_m = T\) and (3.6) follows from (3.9). By inserting (3.9) in (3.8) with \(s = T\), we obtain (3.7). \(\square\)

As a corollary, we obtain also a uniform estimate of \(\frac{\partial \tilde{\nu}_{em}}{\partial t}\) in \(L^2(0, \tau; V')\).

**Lemma 2:** Assume that (2.1), (2.2), (2.4), (3.1) hold and that \(\tilde{\nu}^0 \in H\). Then there exists a positive real number \(C_2\), independent of \(m\) and \(\epsilon\), such that
\[
\left\|\frac{\partial \tilde{\nu}_{em}}{\partial t}\right\|_{L^2(0,T;V')} \leq C_2.
\] (3.10)

**Proof:** Let \(\varphi \in V\). For all \(m \geq 1\), we define \(\varphi_m\) as the orthogonal projection with respect to the inner product of \(H^1(\Omega)\) of \(\varphi\) on Span\(\{w_1, \ldots, w_m\}\). With (3.4) we get
\[
\left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi_m\right) = -\int_{\Omega} 2\mu D(\tilde{\nu}_{em}) \cdot D(\varphi_m) \, dx - \int_{\Gamma_0} \frac{\ell - \tilde{\nu}_{em} \cdot \varphi_m}{\sqrt{\varepsilon^2 + |\tilde{\nu}_{em}|^2}} \, dx' + (f, \varphi_m) - \int_{\Omega} 2\mu D(G_0\xi) \cdot D(\varphi_m) \, dx - \left(\frac{\partial \xi}{\partial t}, \varphi_m\right) \quad \text{a.e. in } (0, T). \quad (3.11)
\]

We estimate all the terms in the right-hand side of the previous equality, we obtain
\[
\left|\left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi_m\right)\right| \leq 2\mu \|\tilde{\nu}_{em}\|_{H^1(\Omega)} \|\varphi_m\|_{H^1(\Omega)} + \|\ell\|_{L^2(\Gamma_0)} \|\varphi_m\|_{L^2(\Gamma_0)} + \|f\|_{L^2(\Omega)} \|\varphi_m\|_{L^2(\Omega)} + 2\mu |\xi| \|G_0\|_{H^1(\Omega)} \|\varphi_m\|_{H^1(\Omega)} + \left|\frac{\partial \xi}{\partial t}\right| \|G_0\|_{L^2(\Omega)} \|\varphi_m\|_{L^2(\Omega)} \quad \text{a.e. in } (0, T).
\]

As \((w_i)_{i \geq 1}\) is an orthogonal family of \(L^2(\Omega)\) and \(\varphi_m\) is the orthogonal projection with respect to the inner product of \(H^1(\Omega)\) of \(\varphi\) on Span\(\{w_1, \ldots, w_m\}\), we have \(\|\varphi_m\|_{H^1(\Omega)} \leq \|\varphi\|_{H^1(\Omega)}\) and
\[
\left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi_m\right) = \left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi_k\right) \quad \forall k \geq m.
\]

Moreover \((w_i)_{i \geq 1}\) is a Hilbertian basis of \(V\), so the sequence \((\varphi_k)_{k \geq 1}\) converges strongly to \(\varphi\) in \(H^1(\Omega)\) and we get
\[
\left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi_m\right) = \left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi\right).
\]

Then, we obtain
\[
\left|\left(\frac{\partial \tilde{\nu}_{em}}{\partial t}, \varphi\right)\right| \leq 2\mu \|\tilde{\nu}_{em}\|_{H^1(\Omega)} \|\varphi\|_{H^1(\Omega)} + c(y_0) \|\ell\|_{L^2(\Gamma_0)} \|\varphi\|_{H^1(\Omega)} + \left|\frac{\partial \xi}{\partial t}\right| \|G_0\|_{L^2(\Omega)} \|\varphi\|_{H^1(\Omega)} \quad \text{a.e. in } (0, T),
\]
where \( c(\gamma_0) \) is the norm of the trace operator \( \gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma_0) \). Hence
\[
\left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{V'} \leq 2\mu \| \tilde{v}_{em} \|_{H^1(\Omega)} + c(\gamma_0) \| \ell \|_{L^2(\Gamma_0)} + \| f \|_{L^2(\Omega)} + 2\mu |\xi| \| G_0 \|_{H^1(\Omega)} + \left| \frac{\partial \xi}{\partial t} \right| \| G_0 \|_{L^2(\Omega)} \quad \text{a.e. in } (0, T)
\]
and the conclusion follows from the estimates of Lemma 1.

Now we can pass to the limit as \( m \) tends to \(+\infty\). Indeed there exists a subsequence of \((\tilde{v}_{em})_{m \geq 1}\), still denoted \((\tilde{v}_{em})_{m \geq 1}\), such that
\[
\tilde{v}_{em} \rightharpoonup \tilde{v}_e \quad \text{weakly star in } L^\infty(0, T; L^2(\Omega)) \quad \text{and weakly in } L^2(0, T; V)
\]
and
\[
\frac{\partial \tilde{v}_{em}}{\partial t} \rightharpoonup \frac{\partial \tilde{v}_e}{\partial t} \quad \text{weakly in } L^2(0, T; V').
\]
Using Aubin’s lemma we infer that, possibly extracting another subsequence,
\[
\tilde{v}_{em} \rightarrow \tilde{v}_e \quad \text{strongly in } L^2(0, T; H^s(\Omega))
\]
with \( \frac{1}{2} < s < 1 \) and thus
\[
\tilde{v}_{em} \rightarrow \tilde{v}_e \quad \text{strongly in } L^2(0, T; L^2(\Gamma_0)) \quad \text{and a.e. on } \Gamma_0 \times (0, T). \tag{3.12}
\]

Let \( \chi \in L^2(0, T) \) and \( \varphi \in V \). For all \( m \geq 1 \) we define again \( \varphi_m \) as the orthogonal projection of \( \varphi \) with respect to the inner product of \( H^1(\Omega) \) on \( \text{Span}\{w_1, \ldots, w_m\} \). We multiply (3.11) by \( \chi \), we integrate on \( (0, T) \) and we pass to the limit as \( m \) tends to \(+\infty\). We get
\[
\int_0^T \left\langle \frac{\partial \tilde{v}_e}{\partial t}, \varphi \right\rangle_{V', V} \chi \, dt + \int_0^T \int_\Omega 2\mu D(\tilde{v}_e) : D(\varphi \chi) \, dx \, dt \\
+ \int_0^T \int_{\Gamma_0} \ell \frac{\tilde{v}_e \cdot \varphi}{\sqrt{\varepsilon^2 + |\tilde{v}_e|^2}} \chi \, dx' \, dt = \int_0^T (f, \varphi \chi) \, dt \\
- \int_0^T \int_\Omega 2\mu D(G_0 \xi) : D(\varphi \chi) \, dx \, dt - \int_0^T \left( G_0 \frac{\partial \xi}{\partial t}, \varphi \chi \right) \, dt. \tag{3.13}
\]
Furthermore, using Simon’s lemma and possibly extracting another subsequence, we have
\[
\tilde{v}_{em} \rightarrow \tilde{v}_e \quad \text{strongly in } C^0([0, T]; \mathcal{H}),
\]
for any Banach space \( \mathcal{H} \) such that \( L^2(\Omega) \subset \mathcal{H} \subset V' \) with continuous injections and compact embedding of \( L^2(\Omega) \) into \( \mathcal{H} \). Recalling that
\[
\tilde{v}_{em}(0) = \tilde{v}_m^0 \rightarrow \tilde{v}^0 \quad \text{strongly in } L^2(\Omega)
\]
we infer that \( \tilde{v}_e(0) = \tilde{v}^0 \).

Finally, using De Rham’s theorem, we obtain that there exists \( p_e \) such that \((\tilde{v}_e, p_e)\) is a solution of problem \((P_e)\). Indeed, possibly modifying \( \tilde{v}_e \) on a negligible subset of \([0, T]\), we have \( \tilde{v}_e \in \)
\( C^0([0, T]; H) \). Hence for all \( t \in [0, T] \), we may define \( F_\varepsilon \in C^0([0, T]; \mathbf{H}^{-1}(\Omega)) \) by

\[
\{F_\varepsilon(t), \varphi\}_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = -(\overline{v}_\varepsilon(t) - \overline{v}_0, \varphi) - \int_0^t \int_\Omega 2\mu D(\overline{v}_\varepsilon) + G_0 \zeta : D(\varphi) \, dx \, dt
+ \int_0^t (f, \varphi) - \int_0^t \left( G_0 \frac{\partial \zeta}{\partial t}, \varphi \right) \, dt \quad \forall \varphi \in \mathbf{H}_0^1(\Omega), \; \forall t \in [0, T].
\]

With (3.13) we obtain that

\[
\{F_\varepsilon(t), \varphi\}_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = 0
\]

for all \( \varphi \in \mathbf{H}_0^1(\Omega) \) such that \( \text{div}(\varphi) = 0 \) and we infer that there exists \( \pi_\varepsilon \in C^0([0, T]; L_0^2(\Omega)) \) such that

\[
\{F_\varepsilon(t), \varphi\}_{\mathbf{H}^{-1}(\Omega), \mathbf{H}_0^1(\Omega)} = \{\nabla \pi_\varepsilon(t), \varphi\}_{\mathbf{D}'(\Omega), \mathbf{D}(\Omega)} \quad \forall \varphi \in \mathbf{D}(\Omega), \; \forall t \in [0, T].
\]  \tag{3.14}

Let us denote now by \( p_\varepsilon \) the time derivative of \( \pi_\varepsilon \) in the distribution sense. From (3.14) we obtain

\[
\frac{d}{dt} (\overline{v}_\varepsilon, \varphi) + a(\overline{v}_\varepsilon + G_0 \zeta, \varphi) + \{\nabla p_\varepsilon, \varphi\}_{\mathbf{D}'(\Omega), \mathbf{D}(\Omega)} = (f, \varphi) - \left( G_0 \frac{\partial \zeta}{\partial t}, \varphi \right) \, dt \quad \forall \varphi \in \mathbf{D}(\Omega). \tag{3.15}
\]

in \( \mathbf{D}'(0, T) \).

**Lemma 3:** We have \( p_\varepsilon \in H^{-1}(0, T; L_0^2(\Omega)) \) and there exists a constant \( C_3 \) independent of \( \varepsilon \) such that

\[
\|p_\varepsilon\|_{H^{-1}(0, T; L_0^2(\Omega))} \leq C_3.
\]

**Proof:** With (3.15) we get

\[
\{(p_\varepsilon, \text{div}(\varphi)), \chi\}_{\mathbf{D}'(0, T), \mathbf{D}(0, T)} = - \int_0^T (\overline{v}_\varepsilon, \varphi) \, \chi' \, dt + \int_0^T a(\overline{v}_\varepsilon + G_0 \zeta, \varphi) \, \chi \, dt
- \int_0^T (f, \varphi) \, \chi \, dt + \int_0^T \left( G_0 \frac{\partial \zeta}{\partial t}, \varphi \right) \, \chi \, dt \quad \forall \varphi \in \mathbf{D}(\Omega), \; \forall \chi \in \mathbf{D}(0, T).
\]

By density the same equality is still valid for all \( \varphi \in \mathbf{H}_0^1(\Omega) \). Now let \( \tilde{w} \in L_0^2(\Omega) \) and

\[
w = \tilde{w} - \frac{1}{|\Omega|} \int_\Omega \tilde{w} \, dx.
\]

By construction we have \( w \in L_0^2(\Omega) \) and \( \|w\|_{L_0^2(\Omega)} \leq \|\tilde{w}\|_{L_0^2(\Omega)} \). Moreover there exists a linear continuous operator \( P : L_0^2(\Omega) \to \mathbf{H}_0^1(\Omega) \) such that

\[
P(w) = \varphi \in \mathbf{H}_0^1(\Omega), \quad \text{div}(\varphi) = w \quad \forall w \in L_0^2(\Omega).
\]
(see [35]). It follows that

\[
\|\langle p_e, w \rangle, \chi \rangle_{D'(0,T), D(0,T)} - \|\langle p_e, \text{div}(\varphi), \varphi \rangle \|_{D'(0,T), D(0,T)}
\]

\leq \|\varphi_{\varepsilon}\|_{L^2(0,T;L^2(\Omega))} \|P(w)\|_{L^2(0,T;L^2(\Omega))} + 2\mu \|\varphi_{\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \sqrt{T} \|G_0\|_{H^1(\Omega)} \|\xi\|_{C^0([0,T])} \|P(w)\|_{L^2(0,T;H^1(\Omega))}
\]

\[
+ \left( \|f\|_{L^2(0,T;L^2(\Omega))} + \sqrt{T} \|G_0\|_{L^2(\Omega)} \left\| \frac{\partial \xi}{\partial t} \right\|_{C^0([0,T])} \right) \|P(w)\|_{L^2(0,T;L^2(\Omega))} + \sqrt{T} \|G_0\|_{L^2(\Omega)} \left\| \frac{\partial \xi}{\partial t} \right\|_{C^0([0,T])} \right) \|P(w)\|_{L^2(0,T;L^2(\Omega))}
\]

\forall \chi \in D(0,T).

Since the estimates obtained in Lemma 1 are independent of \(m\) and \(\varepsilon\), we infer that the sequence \((\varphi_{\varepsilon})_{\varepsilon > 0}\) is bounded in \(L^2(0, T; V) \cap L^\infty(0, T; H^2(\Omega))\). Finally, using the continuity of the operator \(P\) and the density of \(D(0, T) \otimes L^2(\Omega)\) into \(H^1_0(0, T; L^2(\Omega))\), we may conclude. \(\Box\)

It follows that \(\sigma_{\varepsilon} = -p_e \text{Id} + 2\mu D(\varphi_{\varepsilon} + G_0 \xi)\) belongs to \(H^{-1}(0, T; (L^2(\Omega))^{3 \times 3})\) and is bounded in \(H^{-1}(0, T; (L^2(\Omega))^{3 \times 3})\) uniformly with respect to \(\varepsilon\). Moreover, we have

\[
\left\langle \frac{d}{dt} (\varphi_{\varepsilon}, \varphi), \chi \right\rangle_{D'(0,T), D(0,T)} + \int_0^T a(\varphi_{\varepsilon} + G_0 \xi, \varphi, \chi) \, dt
\]

\[-\left\langle (p_e, \text{div}(\varphi), \varphi) \right\rangle_{D'(0,T), D(0,T)} = \left\langle (f, \varphi, \chi) \right\rangle_{D'(0,T), D(0,T)}
\]

\[-\left\langle (G_0 \frac{\partial \xi}{\partial t}, \varphi, \chi) \right\rangle_{D'(0,T), D(0,T)} \quad \forall \varphi \in D(\Omega), \forall \chi \in D(0,T)
\]

i.e.

\[
\int_0^T (\varphi_{\varepsilon}, \varphi) \, dt - \int_0^T (\text{div}(\varphi_{\varepsilon}), \varphi) \, dt
\]

\[= \int_0^T (f, \varphi) \, dt - \int_0^T (G_0 \frac{\partial \xi}{\partial t}, \varphi) \, dt \quad \forall \varphi \in D(\Omega), \forall \chi \in D(0,T)
\]

(3.16)

and we infer that \(\text{div}(\sigma_{\varepsilon})\) belongs to \(H^{-1}(0, T; L^2(\Omega))\) and is bounded in \(H^{-1}(0, T; L^2(\Omega))\) uniformly with respect to \(\varepsilon\).

Let us consider now \(\varphi \in \mathcal{V}_0\) in (3.16). Recalling that \(\sigma_{\varepsilon}\) and \(\text{div}(\sigma_{\varepsilon})\) belong to \(H^{-1}(0, T; (L^2(\Omega))^{3 \times 3})\) and \(H^{-1}(0, T; L^2(\Omega))\), respectively, we may apply Green’s formula (see [35]) and we get

\[
\int_0^T (\varphi_{\varepsilon}, \varphi') \, dt + \int_0^T a(\varphi_{\varepsilon} + G_0 \xi, \varphi) \, dt - \int_0^T (p_e, \text{div}(\varphi)) \, dt
\]

\[-\int_0^T \sum_{i,j=1}^3 \sigma_{ijx} n_j \varphi_i \, dx \, dt
\]

\[= \int_0^T (f, \varphi) \, dt - \int_0^T (G_0 \frac{\partial \xi}{\partial t}, \varphi) \, dt \quad \forall \chi \in D(0,T).
\]

But for any \(\varphi \in \mathcal{V}_0\) we have

\[
\int_{\partial \Omega} \varphi \cdot n \, d\gamma = 0
\]
and there exists $\tilde{\varphi} \in H^1(\Omega)$ such that

$$\tilde{\varphi} = \varphi \text{ on } \partial \Omega, \quad \text{div}(\tilde{\varphi}) = 0 \text{ in } \Omega$$

since $\Omega$ is connected (see [35]). Then we get

$$- \int_0^T (\tilde{\nu}_e, \tilde{\varphi})' \, dt + \int_0^T a(\tilde{\nu}_e + G_0 \xi, \tilde{\varphi}) \, dt - \int_0^T \int_{\Gamma_0} \sum_{i,j=1}^3 \sigma_{ij} n_j \tilde{\varphi}_i \chi \, dx' \, dt$$

$$= \int_0^T (f, \tilde{\varphi}) \, dt - \int_0^T \left(G_0 \frac{\partial \xi}{\partial t}, \tilde{\varphi}\right) \, dt \quad \forall \chi \in D(0, T).$$

By comparing with (3.13), we obtain

$$- \int_0^T \int_{\Gamma_0} \sum_{i,j=1}^3 \sigma_{ij} n_j \tilde{\varphi}_i \chi \, dx' \, dt = \int_0^T \int_{\Gamma_0} \ell \frac{\tilde{\nu}_e \cdot \varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \chi \, dx' \, dt \quad \forall \chi \in D(0, T).$$

Owing that $\tilde{\varphi} = \varphi$ on $\Gamma_0$ we infer that

$$- \int_0^T \int_{\Gamma_0} \sum_{i,j=1}^3 \sigma_{ij} n_j \varphi_i \chi \, dx' \, dt = \int_0^T \int_{\Gamma_0} \ell \frac{\tilde{\nu}_e \cdot \varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \chi \, dx' \, dt \quad \forall \chi \in D(0, T)$$

and finally $(\tilde{\nu}_e, p_\varepsilon)$ is a solution of problem $(P_\varepsilon)$.

Now, observing that

$$\int_0^T \int_{\Gamma_0} \ell \left(\sqrt{\varepsilon^2 + |\tilde{\nu}_e + \varphi|^2} - \sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}\right) \, dx' \, dt \geq \int_0^T \int_{\Gamma_0} \ell \frac{\tilde{\nu}_e \cdot \varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \chi \, dx' \, dt$$

we get from (3.2) the following variational inequality

$$\left\langle \frac{d}{dt} (\tilde{\nu}_e, \varphi), \chi \right\rangle_{D'(0,T),D(0,T)} - \left\langle (p_\varepsilon, \text{div} (\varphi)), \chi \right\rangle_{D'(0,T),D(0,T)} + \int_0^T a(\tilde{\nu}_e, \varphi \chi) \, dt$$

$$+ \int_0^T \int_{\Gamma_0} \ell \sqrt{\varepsilon^2 + |\tilde{\nu}_e + \varphi|^2} \, dx' \, dt - \int_0^T \int_{\Gamma_0} \ell \sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2} \, dx' \, dt$$

$$\geq \left\langle (f, \varphi), \chi \right\rangle_{D'(0,T),D(0,T)} - \int_0^T a(G_0 \xi, \varphi \chi) \, dt$$

$$- \left\langle \left(G_0 \frac{\partial \xi}{\partial t}, \varphi\right), \chi \right\rangle_{D'(0,T),D(0,T)} \quad \forall \varphi \in \mathcal{V}_0, \forall \chi \in D(0, T).$$

(3.17)

Let us pass now to the limit as $\varepsilon$ tends to zero. Since the estimates obtained in Lemmas 1–3 are independent of $m$ and $\varepsilon$, we infer that the sequences $(\tilde{\nu}_e)_{\varepsilon > 0}$, $\left(\frac{\partial \tilde{\nu}_e}{\partial t}\right)_{\varepsilon > 0}$ and $(p_\varepsilon)_{\varepsilon > 0}$ are bounded in $L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))$, $L^2(0, T; V')$ and $H^{-1}(0, T; L^2_0(\Omega))$, respectively. Hence we have the following convergence results

$$\tilde{\nu}_e \rightharpoonup \tilde{v} \text{ weakly star in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; V),$$
\[ \frac{\partial \tilde{v}_e}{\partial t} \to \frac{\partial \tilde{v}}{\partial t} \text{ weakly in } L^2(0, T; V'), \]

\[ \tilde{p}_e \to \tilde{p} \text{ weakly in } H^{-1}(0, T; L^2_0(\Omega)). \]

Moreover, possibly extracting another subsequence, we have

\[ \tilde{v}_e \to \tilde{v} \text{ strongly in } L^2(0, T; L^2(\Gamma_0)), \]

and

\[ \tilde{v}_e \to \tilde{v} \text{ strongly in } C^0([0, T]; H), \]

for any Banach space \( H \) such that \( L^2(\Omega) \subset H \subset V' \) with continuous injections and compact embedding of \( L^2(\Omega) \) into \( H \). Then we may use the same arguments as previously to pass to the limit as \( \varepsilon \) tends to zero in problem (3.17). Indeed, for the boundary term, we may apply the following property

\[
\left| \int_0^T \int_{\Gamma_0} \ell \sqrt{\varepsilon^2 + |\tilde{v}_e + \varphi \chi|^2} \, dx' \, dt - \int_0^T \int_{\Gamma_0} \ell |\tilde{v} + \varphi \chi| \, dx' \, dt \right| \\
\leq \| \ell \|_{L^2(0,T;L^2(\Gamma_0))} \left( \| \tilde{v}_e - \tilde{v} \|_{L^2(0,T;L^2(\Gamma_0))} + \varepsilon \sqrt{T \text{ meas}(\Gamma_0)} \right)
\]

for all \( \varphi \in V_0 \) and for all \( \chi \in D(0, T) \), which allows us to conclude that \((\tilde{v}, p)\) is a solution of problem (P).

There remains now to prove uniqueness. Let \((\tilde{v}_1, p_1)\) and \((\tilde{v}_2, p_2)\) be two solutions of problem (P). Since \( \frac{\partial \tilde{v}_i}{\partial t} \in L^2(0, T; V') \), \( i = 1, 2 \), we may rewrite (2.5) as

\[
\int_0^T \left( \frac{\partial \tilde{v}_i}{\partial t}, \varphi \chi \right)_{V', V} \, dt + \int_0^T a(\tilde{v}_i + G_0 \xi, \varphi \chi) \, dt \\
+ \int_0^T \int_{\Gamma_0} \ell \left( |\tilde{v}_i + \varphi \chi| - |\tilde{v}_i| \right) \, dx' \, dt \geq \int_0^T (f, \varphi \chi) \, dt - \int_0^T \left( G_0 \frac{\partial \xi}{\partial t}, \varphi \chi \right) \, dt
\]

for any \( \varphi \in V \) and \( \chi \in D(0, T) \). By density of \( D(0, T) \otimes V \) into \( L^2(0, T; V) \) we may replace \( \varphi \chi \) by \( (\tilde{v}_j - \tilde{v}_i)I_{[i,j]} \) with \( i, j \in \{1, 2\} \), \( i \neq j \) and \( s \in [0, T] \). By adding the two variational inequalities and using (2.3) we get

\[
\frac{1}{2} \int_0^s \frac{d}{dt} \|\tilde{v}_1 - \tilde{v}_2\|_{L^2(\Omega)}^2 \, dt + \alpha \int_0^s \|\tilde{v}_1 - \tilde{v}_2\|_{H^1(\Omega)}^2 \, dt \leq 0.
\]

As \( \tilde{v}_1(0) = \tilde{v}_2(0) = \tilde{v}^0 \), we obtain

\[
|\tilde{v}_1(s) - \tilde{v}_2(s)|_{L^2(\Omega)}^2 \leq 0 \quad \forall s \in [0, T].
\]

Then, with (2.5) we have

\[
\left( (p_1 - p_2, \text{div}(\varphi)), \chi \right)_{D'(0,T),D(0,T)} = 0 \quad \forall \varphi \in H^1_0(\Omega), \forall \chi \in D(0, T).
\]
Now let $\tilde{w} \in L^2(\Omega)$ and

$$w = \tilde{w} - \frac{1}{|\Omega|} \int_{\Omega} \tilde{w} \, dx \in L^2_0(\Omega).$$

There exists $\varphi = P(w) \in H^1_0(\Omega)$ such that $\text{div}(\varphi) = w$ (see [35]) and thus

$$\langle (p_1 - p_2, \tilde{w}), \chi \rangle_{D'(0,T),D(0,T)} = \langle (p_1 - p_2, w), \chi \rangle_{D'(0,T),D(0,T)} = 0 \quad \forall \chi \in D(0,T).$$

By density of $D(0,T) \otimes L^2(\Omega)$ into $H^1_0(0,T; L^2(\Omega))$ we get

$$\langle p_1 - p_2, \eta \rangle_{H^{-1}(0,T; L^2(\Omega)), H^1_0(0,T; L^2(\Omega))} = 0 \quad \forall \eta \in H^1_0(0,T; L^2(\Omega))$$

and thus $p_1 = p_2$. □

Let us assume now that the following compatibility condition for the initial velocity is satisfied

$$v^0 \in H^2(\Omega), \quad \text{div}(v^0) = 0 \text{ in } \Omega, \quad v^0 = g \text{ on } \partial\Omega \quad (3.18)$$

and

$$\frac{\partial v^0}{\partial x_3} = 0 \text{ on } \Gamma_0. \quad (3.19)$$

Then we may choose $G_0 = v^0$ and the initial condition (2.6) becomes

$$\tilde{v}(0,\cdot) = v^0 - G_0 = \tilde{v}^0 = 0 \in H.$$ 

Let us assume also that

$$f \in W^{1,2}(0,T; L^2(\Omega)), \quad \ell \in W^{1,2}(0,T; L^2(\Gamma_0; \mathbb{R}^+)). \quad (3.20)$$

Then we can prove further regularity properties for the unique solution of problem (P).

**Theorem 2:** Let assumptions (2.2)–(3.18)–(3.19)–(3.20) hold. Then the unique solution $(\tilde{v}, p)$ of problem (P) satisfies the following regularity properties

$$\frac{\partial \tilde{v}}{\partial t} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; \mathcal{V}), \quad p \in L^\infty(0,T; L^2_0(\Omega))$$

and

$$\frac{\partial^2 \tilde{v}}{\partial t^2} \in L^2(0,T; \left[H^1_{0,\text{div}}(\Omega)\right]^*)$$

with $H^1_{0,\text{div}}(\Omega) = \{ \varphi \in H^1(\Omega) : \varphi = 0 \text{ on } \partial\Omega \text{ and } \text{div}(\varphi) = 0 \text{ in } \Omega \}$.

**Proof:** Let us adopt the same notations as in the previous proof. Recalling that the trace operator maps $H^1(\Omega)$ into $L^4(\partial\Omega)$ (see [36] for instance), we infer from (3.20) that, for all $\varepsilon > 0$ and for all $m \geq 1$, we have $\tilde{v}_{\varepsilon m} \in W^{2,2}(0,T; \text{Span}\{w_1, \ldots, w_m\})$ and (3.4) holds for all $t \in [0,T]$. We may
differentiate all the terms of (3.4) with respect to the time variable and we obtain

\[ \begin{align*}
\left( \frac{\partial^2 \widetilde{v}_{em}}{\partial t^2}, w_k \right) + a \left( \frac{\partial \widetilde{v}_{em}}{\partial t}, w_k \right) + \int_{\Gamma_0} \frac{\partial \ell}{\partial t} \frac{\widetilde{v}_{em} \cdot w_k}{\sqrt{\varepsilon^2 + |\widetilde{v}_{em}|^2}} \, dx' \\
+ \int_{\Gamma_0} \ell \left( \frac{\partial \widetilde{v}_{em}}{\partial t}, w_k \right) - \left( \frac{\partial \widetilde{v}_{em}}{\partial t}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) \frac{\left( \varepsilon^2 + |\widetilde{v}_{em}|^2 \right)^{3/2}}{\left( \varepsilon^2 + |\widetilde{v}_{em}|^2 \right)^{3/2}} \, dx' \\
= \left( \frac{\partial f}{\partial t}, w_k \right) - a \left( \nu^0 \frac{\partial \zeta}{\partial t}, w_k \right) - \left( \nu^0 \frac{\partial^2 \zeta}{\partial t^2}, w_k \right) \quad \text{a.e. in } (0, T),
\end{align*} \]  

for all \( k \in \{1, \ldots, m\} \). Now we multiply (3.21) by \( g_k'(t) \) and we add from \( k = 1 \) to \( m \). We obtain

\[ \int_{\Gamma_0} \ell \left( \frac{|\partial \widetilde{v}_{em}|^2}{\sqrt{\varepsilon^2 + |\widetilde{v}_{em}|^2}} - \frac{\left( \varepsilon^2 + |\widetilde{v}_{em}|^2 \right)^{3/2}}{\left( \varepsilon^2 + |\widetilde{v}_{em}|^2 \right)^{3/2}} \right) \, dx' \geq \int_{\Gamma_0} \ell \varepsilon^2 \frac{|\partial \widetilde{v}_{em}|^2}{\left( \varepsilon^2 + |\widetilde{v}_{em}|^2 \right)^{3/2}} \, dx' \geq 0 \]

and with (2.3) we get

\[ \begin{align*}
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{\alpha}{4} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 & \leq - \int_{\Gamma_0} \frac{\partial \ell}{\partial t} \frac{\widetilde{v}_{em} \cdot \frac{\partial \widetilde{v}_{em}}{\partial t}}{\sqrt{\varepsilon^2 + |\widetilde{v}_{em}|^2}} \, dx' \\
+ \left( \frac{\partial f}{\partial t}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) - a \left( \nu^0 \frac{\partial \zeta}{\partial t}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) - \left( \nu^0 \frac{\partial^2 \zeta}{\partial t^2}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) \quad \text{a.e. in } (0, T).
\end{align*} \]  

Let us estimate now the right side of (3.22). We obtain

\[ \left| \int_{\Gamma_0} \frac{\partial \ell}{\partial t} \frac{\widetilde{v}_{em} \cdot \frac{\partial \widetilde{v}_{em}}{\partial t}}{\sqrt{\varepsilon^2 + |\widetilde{v}_{em}|^2}} \, dx' \right| \leq c(\gamma_0) \left\| \frac{\partial \ell}{\partial t} \right\|_{L^2(\Gamma_0)} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)} \leq \frac{\alpha}{4} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 + \frac{c(\gamma_0)^2}{\alpha} \left\| \frac{\partial \ell}{\partial t} \right\|_{L^2(\Gamma_0)}^2 \]

where we recall that \( c(\gamma_0) \) is the norm of the trace operator from \( H^1(\Omega) \) to \( L^2(\Gamma_0) \),

\[ \left( \frac{\partial f}{\partial t}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) \leq \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\Omega)} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)} \leq \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2, \]

\[ \left| a \left( \nu^0 \frac{\partial \zeta}{\partial t}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) \right| \leq 2 \mu \left\| \frac{\partial \zeta}{\partial t} \right\|_{H^1(\Omega)} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)} \]

\[ \leq \frac{\alpha}{4} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 + \frac{4 \mu^2}{\alpha} \left\| \frac{\partial \zeta}{\partial t} \right\|_{H^1(\Omega)}^2 \left\| \nu^0 \right\|_{H^1(\Omega)}^2, \]

\[ \left| \left( \nu^0 \frac{\partial^2 \zeta}{\partial t^2}, \frac{\partial \widetilde{v}_{em}}{\partial t} \right) \right| \leq \frac{\partial^2 \zeta}{\partial t^2} \left\| \nu^0 \right\|_{L^2(\Omega)} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)} \]

\[ \leq \frac{\partial^2 \zeta}{\partial t^2} \left\| \nu^0 \right\|_{L^2(\Omega)}^2 + \frac{1}{4} \left\| \frac{\partial \widetilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2. \]
Gathering all these estimates, we infer

\[
\frac{1}{2} \frac{d}{dt} \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2 + \alpha \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 \leq A_1 + \frac{1}{2} \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2 \quad \text{a.e. in } (0, T)
\]

with

\[
A_1 = \frac{c(\gamma_0)^2}{\alpha} \left\| \frac{\partial \ell}{\partial t} \right\|_{L^2(\Gamma_0)}^2 + \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{4 \mu^2}{\alpha} \left\| \frac{\partial \xi}{\partial t} \right\|_{H^1(\Omega)}^2 + \left\| \frac{\partial^2 \xi}{\partial t^2} \right\|_{L^2(\Omega)}^2 \| v^0 \|_{H^1(\Omega)}^2.
\]

Let \( s \in [0, T] \). By integration we have

\[
\left\| \frac{\partial \tilde{v}_{em}}{\partial t}(s) \right\|_{L^2(\Omega)}^2 + \alpha \int_0^s \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 \, dt \\
\leq \left\| \frac{\partial \tilde{v}_{em}}{\partial t}(0) \right\|_{L^2(\Omega)}^2 + 2 \int_0^s A_1 \, dt + \int_0^s \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{L^2(\Omega)}^2 \, dt.
\]

Moreover, taking \( t = 0 \) in (3.4) we obtain

\[
\left( \frac{\partial \tilde{v}_{em}}{\partial t}(0), w_k \right) = -a(\tilde{v}_{em}(0) + v^0 \xi(0), w_k) - \int_{\Gamma_0} \ell(0) \frac{\tilde{v}_{em}(0) \cdot w_k}{\sqrt{\varepsilon^2 + |\tilde{v}_{em}(0)|^2}} \, dx' \\
+ (f(0), w_k) - \left( v^0 \frac{\partial \xi}{\partial t}(0), w_k \right) \quad \forall k \in \{1, \ldots, m\}.
\]

Reminding that \( \xi(0) = 1 \) and \( G_0 = v^0 \), we get \( \tilde{v}_{em}(0) = \tilde{v}_{m}^0 = 0 \) for all \( m \geq 1 \) and

\[
\int_{\Gamma_0} \ell(0) \frac{\tilde{v}_{em}(0) \cdot w_k}{\sqrt{\varepsilon^2 + |\tilde{v}_{em}(0)|^2}} \, dx' = 0.
\]

We multiply the previous equality by \( \phi_{ek}(0) \) and we add from \( k = 1 \) to \( m \). With Green’s formula and (3.18)–(3.19), we get

\[
\left\| \frac{\partial \tilde{v}_{em}}{\partial t}(0) \right\|_{L^2(\Omega)}^2 = \int_\Omega 2 \mu \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (d_{ij}(v^0)) \frac{\partial \tilde{v}_{em}}{\partial t}(0) \, dx \\
+ (f(0), \frac{\partial \tilde{v}_{em}}{\partial t}(0)) - \left( v^0 \frac{\partial \xi}{\partial t}(0), \frac{\partial \tilde{v}_{em}}{\partial t}(0) \right).
\]

It follows that

\[
\left\| \frac{\partial \tilde{v}_{em}}{\partial t}(0) \right\|_{L^2(\Omega)} \leq A_0 = 2 \sqrt{3} \mu \| v^0 \|_{H^1(\Omega)} + \| f(0) \|_{L^2(\Omega)} + \left\| \frac{\partial \xi}{\partial t}(0) \right\| \| v^0 \|_{L^2(\Omega)}.
\]
Finally, with (3.23) we get

\[
\left\| \frac{\partial \tilde{v}_{em}}{\partial t}(s) \right\|_{L^2(\Omega)}^2 + \alpha \int_0^s \left\| \frac{\partial \tilde{v}_{em}}{\partial t} \right\|_{H^1(\Omega)}^2 \, dt \\
\leq \left( A_0^2 + \frac{2c(\gamma_0)^2}{\alpha} \right) \left\| \frac{\partial \ell}{\partial t} \right\|_{L^2(\Omega)}^2 + \frac{2}{\alpha} \left\| \frac{\partial f}{\partial t} \right\|_{L^2(\Omega)}^2 \\
+ \frac{8\mu^2}{\alpha} \left\| \frac{\partial \tilde{v}}{\partial t} \right\|_{C([0,T])}^2 \left\| v^0 \right\|_{H^1(\Omega)}^2 + 2T \left\| \frac{\partial \xi}{\partial t} \right\|_{C([0,T])}^2 \left\| v^0 \right\|_{L^2(\Omega)}^2 \exp(s) \quad \forall s \in [0, T].
\]

Hence \( \frac{\partial \tilde{v}_{em}}{\partial t} \) is bounded in \( L^\infty(0, T; L^2(\Omega)) \) and in \( L^2(0, T; H^1(\Omega)) \) uniformly with respect to \( m \) and \( \varepsilon \).

It follows that we can pass to the limit as \( m \) tends to +\( \infty \). We obtain

\[
\frac{\partial \tilde{v}_{em}}{\partial t} \rightarrow \frac{\partial \varepsilon}{\partial t} \quad \text{weakly star in} \quad L^\infty(0, T; L^2(\Omega)) \quad \text{and weakly in} \quad L^2(0, T; V). \tag{3.24}
\]

Hence \( \tilde{v}_{\varepsilon} \in W^{1,\infty}(0, T; L^2(\Omega)) \cap W^{1,2}(0, T; V) \) and with a straightforward adaptation of Lemma 3 we get

\[
\left| \langle \tilde{v}_{\varepsilon}, w \rangle_{D'(0, T), D(0, T)} \right| \leq \left\| \frac{\partial \tilde{v}_{\varepsilon}}{\partial t} \right\|_{L^\infty(0, T; L^2(\Omega))} \left\| P(w) \right\|_{L^1(0, T; L^2(\Omega))} \\
+ 2\mu \left( \left\| \tilde{v}_{\varepsilon} \right\|_{L^1(0, T; H^1(\Omega))} + \left\| v^0 \right\|_{H^1(\Omega)} \right) \left\| \xi \right\|_{C^0([0, T])} \left\| P(w) \right\|_{L^1(0, T; H^1(\Omega))} \\
+ \left( \left\| f \right\|_{L^1(0, T; L^2(\Omega))} + \left\| v^0 \right\|_{L^2(\Omega)} \right) \left\| \frac{\partial \xi}{\partial t} \right\|_{C^0([0, T])} \left\| P(w) \right\|_{L^1(0, T; L^2(\Omega))} \\
\quad \forall \chi \in D(0, T).
\]

Hence \( p_{\varepsilon} \in L^\infty(0, T; L^2_0(\Omega)) \) and the sequence \( \left( \frac{\partial \tilde{v}_{em}}{\partial t}, p_{\varepsilon} \right)_{\varepsilon>0} \) is uniformly bounded in \( L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V) \times L^\infty(0, T; L^2_0(\Omega)) \). Moreover, for all \( \varphi \in V \) and \( m \geq 1 \) we may define \( \varphi_m \) as the orthogonal projection of \( \varphi \) with respect to the inner product of \( H^1(\Omega) \) on \( \text{Span}\{w_1, \ldots, w_m\} \). We multiply (3.21) by \( \chi \) and we integrate on \( (0, T) \). We obtain

\[
\begin{align*}
&\int_0^T \left( \frac{\partial \tilde{v}_{em}}{\partial t}, \varphi_m \right) \chi' \, dt + \int_0^T a \left( \frac{\partial \tilde{v}_{em}}{\partial t}, \varphi_m \right) \chi \, dt \\
&+ \int_0^T \int_{\Gamma_0} \frac{\partial \ell}{\partial t} \frac{\tilde{v}_{em} \cdot \varphi_m}{\sqrt{\varepsilon^2 + |\tilde{v}_{em}|^2}} \chi \, dx' \, dt \\
&+ \int_0^T \int_{\Gamma_0} \ell \left( \frac{\frac{\partial \tilde{v}_{em}}{\partial t} \cdot \varphi_m}{\sqrt{\varepsilon^2 + |\tilde{v}_{em}|^2}} - \frac{\left( \tilde{v}_{em} \cdot \frac{\partial \tilde{v}_{em}}{\partial t} \right)}{(\varepsilon^2 + |\tilde{v}_{em}|^2)^{3/2}} \right) \chi \, dx' \, dt \\
&= \int_0^T \left( \frac{\partial f}{\partial t}, \varphi_m \right) \chi \, dt - \int_0^T a \left( v^0 \frac{\partial \xi}{\partial t}, \varphi_m \right) \chi \, dt \\
&- \int_0^T \left( v^0 \frac{\partial^2 \xi}{\partial t^2}, \varphi_m \right) \chi \, dt \quad \text{for all} \ m \geq 1. \tag{3.25}
\end{align*}
\]
Since the trace operator is a linear continuous mapping from $H^1(\Omega)$ into $L^4(\partial\Omega)$ we infer from (3.24) that

\[
\frac{\partial \tilde{\nu}_m}{\partial t} = \frac{\partial \tilde{\nu}_e}{\partial t} \quad \text{weakly in } L^2(0, T; \mathbf{L}^4(\Gamma_0))
\]

under the convention that we identify $\frac{\partial \tilde{\nu}_m}{\partial t}$ (resp. $\frac{\partial \tilde{\nu}_e}{\partial t}$) with its trace on $\Gamma_0$. Moreover, with (3.12) we have

\[
\ell \left( \frac{\varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} - \frac{\varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \right) = \ell \left( \frac{\varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} - \frac{\varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \right) \quad \text{strongly in } L^2(0, T; \mathbf{L}^{4/3}(\Gamma_0)).
\]

Hence we can pass to the limit in all the terms of (3.25) and we get

\[
- \int_0^T \left( \frac{\partial \tilde{\nu}_e}{\partial t}, \varphi \right) \chi \, dt + \int_0^T a \left( \frac{\partial \tilde{\nu}_e}{\partial t}, \varphi \right) \chi \, dt + \int_0^T \int_{\Omega_0} \frac{\partial \ell}{\partial t} \frac{\tilde{\nu}_e - \varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \chi \, dx \, dt
\]

\[
+ \int_0^T \int_{\Gamma_0} \ell \left( \frac{\partial \tilde{\nu}_e}{\partial t} \cdot \varphi - \frac{\tilde{\nu}_e \cdot \varphi}{\sqrt{\varepsilon^2 + |\tilde{\nu}_e|^2}} \right) \chi \, dx \, dt
\]

\[
= \int_0^T \left( \frac{\partial f}{\partial t}, \varphi \right) \chi \, dt - \int_0^T a \left( \nu^0 \frac{\partial \xi}{\partial t}, \varphi \right) \chi \, dt - \int_0^T \left( \nu^0 \frac{\partial^2 \xi}{\partial t^2}, \varphi \right) \chi \, dt
\]

for all $\varphi \in V$ and for all $\chi \in D(0, T)$. Let us choose now $\varphi \in H^1_{\text{div}}(\Omega)$. We obtain

\[
\left\langle \frac{d}{dt} \left( \frac{\partial \tilde{\nu}_e}{\partial t}, \varphi \right), \chi \right\rangle_{D(0, T), D(0, T)} + \int_0^T a \left( \frac{\partial \tilde{\nu}_e}{\partial t}, \varphi \right) \chi \, dt
\]

\[
= \int_0^T \left( \frac{\partial f}{\partial t}, \varphi \right) \chi \, dt - \int_0^T a \left( \nu^0 \frac{\partial \xi}{\partial t}, \varphi \right) \chi \, dt
\]

\[
- \int_0^T \left( \nu^0 \frac{\partial^2 \xi}{\partial t^2}, \varphi \right) \chi \, dt \quad \forall \chi \in D(0, T)
\]

and we infer from the previous estimates that $\frac{\partial^2 \tilde{\nu}_e}{\partial t^2}$ belongs to $L^2(0, T; (H^1_{\text{div}}(\Omega))^{'})$ and is uniformly bounded with respect to $\varepsilon$ in $L^2(0, T; (H^1_{\text{div}}(\Omega))^{'})$.

Finally, using the techniques of Theorem 1, we may conclude that the unique solution $(\tilde{\nu}, p)$ of problem (P) satisfies

\[
\frac{\partial \tilde{\nu}}{\partial t} \in L^\infty(0, T; \mathbf{L}^2(\Omega)) \cap L^2(0, T; V), \quad p \in L^\infty(0, T; \mathbf{L}^2(\Omega))
\]

and

\[
\frac{\partial^2 \tilde{\nu}}{\partial t^2} \in L^2(0, T; (H^1_{\text{div}}(\Omega))^{'})
\]

As a corollary we obtain
Proposition 1: Let us assume that (2.2)–(3.18)–(3.19)–(3.20) hold. Then the stress tensor $\sigma = -pI + 2\mu D(\vec{\nu} + v_0\xi)$ belongs to $L^\infty(0, T; (L^2(\Omega))^3 \times 3)$ and $\text{div}(\sigma)$ belongs to $L^\infty(0, T; L^2(\Omega))$.

Proof: The first part of the result is an immediate consequence of Theorem 2. Now let us choose $\varphi \in (\mathcal{D}(\Omega))^3$ and $\chi \in \mathcal{D}(0, T)$. With (2.5) we have

$$\int_0^T \int_\Omega \frac{\partial \vec{\nu}}{\partial t} \varphi \, dx \, dt + \int_0^T \int_\Omega \sum_{ij=1}^3 \sigma_{ij} \frac{\partial \varphi_i}{\partial x_j} \chi \, dx \, dt = \int_0^T \int_\Omega f \varphi \, dx \, dt - \int_0^T \int_\Omega \nu^0 \frac{\partial \xi}{\partial t} \varphi \, dx \, dt$$

and thus

$$\left| \int_0^T \int_\Omega \sum_{ij=1}^3 \sigma_{ij} \frac{\partial \varphi_i}{\partial x_j} \chi \, dx \, dt \right| \leq \left( \left\| \frac{\partial \vec{\nu}}{\partial t} \right\|_{L^\infty(0, T; L^2(\Omega))} + \|f\|_{L^\infty(0, T; L^2(\Omega))} \right)$$

$$+ \left\| \frac{\partial \xi}{\partial t} \right\|_{C^0([0, T]; L^2(\Omega))} \|\nu^0\|_{L^1(0, T; L^2(\Omega))}. $$

Hence $\text{div}(\sigma) \in (L^1(0, T; L^2(\Omega)))' = L^\infty(0, T; L^2(\Omega))$. \qed

4. The generalized Coulomb’s friction case

Let us assume now that $\mathcal{F}$ can be decomposed as

$$\mathcal{F}(x', t, \sigma) = \mathcal{F}^0(x', t) + \mathcal{F}^\sigma(x', t) \int_0^t S(t - s) |\mathcal{R}(\sigma^3(\cdot, s))(x')| \, ds \quad (4.1)$$

for almost every $x' \in \Gamma_0$ and for all $t \in [0, T]$, where $\mathcal{R}$ is a regularization operator which will be described below, $\sigma^3$ is the vector $(\sigma_{ij})_{1 \leq i \leq 3}$ and

$$\mathcal{F}^0 \in W^{1,2}(0, T; L^2(\Gamma_0; \mathbb{R}^+)), \quad \mathcal{F}^\sigma \in W^{1,p}(0, T; L^2(\Gamma_0; \mathbb{R}^+)) \text{ with } p > 2, \quad (4.2)$$

and

$$S \in C^1(\mathbb{R}^+; \mathbb{R}^+). \quad (4.3)$$

Then, possibly modifying $\mathcal{F}^0$ and $\mathcal{F}^\sigma$ on a negligible subset of $[0, T]$, we have

$$\mathcal{F}^0 \in C^0([0, T]; L^2(\Gamma_0; \mathbb{R}^+)), \mathcal{F}^\sigma \in C^0([0, T]; L^2(\Gamma_0; \mathbb{R}^+)).$$
Similarly, recalling that \( f \in W^{1,2}(0, T; L^2(\Omega)) \), possibly modifying \( f \) on a negligible subset of \([0, T]\), we have also \( f \in C^0([0, T]; L^2(\Omega)) \).

We will prove an existence result for problem \((P)\) using a successive approximation technique. Assuming that problem \((P)\) admits a solution \((\tilde{v}, \tilde{p})\) on some interval \([0, \tau]\), with \( \tau \in [0, T) \), we consider the corresponding friction threshold \( \mathcal{F}(\cdot, \cdot, \tilde{\sigma}) \) and we extend it to \( \ell_0 \in W^{1,2}(0, T; L^2(\Gamma_0, \mathbb{R}^+)) \) such that \( \mathcal{F}(\cdot, \cdot, \tilde{\sigma}) = \ell_0|_{[0, \tau]} \) (see (4.7)–(4.8)). Then starting from this given \( \ell_0 \) we construct a sequence \((\tilde{v}_k, p_k)_{k \geq 0}\) such that \((\tilde{v}_k, p_k)\) is solution to problem \((P_k)\) below for all \( k \geq 0 \), with an iteration procedure given by (4.9), and we prove that there exists \( \tau' > \tau \) such that the restriction of \((\tilde{v}_k, p_k)\) to \([0, \tau']\) converges to a solution \((\tilde{v}_*, p_*)\) of problem \((P)\) which extends \((\tilde{v}, \tilde{p})\) to \([0, \tau']\). Finally, observing that \( \tau' \) is independent of \( \tau \) (see (4.10)) we will reach \([0, T]\) by a finite induction argument.

Indeed, for any given Tresca’s friction threshold \( \ell_k \in W^{1,2}(0, T; L^2(\Gamma_0; \mathbb{R}^+)) \), we consider

**Problem \((P_k)\)** Find

\[
\tilde{v}_k \in L^2(0, T; \mathbf{V}) \cap L^\infty(0, T; L^2(\Omega)), \quad p_k \in H^{-1}(0, T; L^2(\Omega))
\]

such that, for all \( \varphi \in \mathcal{V}_0 \) and for all \( \chi \in \mathcal{D}(0, T) \), we have

\[
\begin{align*}
\left\langle \frac{d}{dt}(\tilde{v}_k, \varphi), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} - \left\langle \langle p_k, \text{div}(\varphi) \rangle, \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} \\
+ \int_0^T a(\tilde{v}_k, \varphi \chi) \, dt + \int_0^T \int_{\Gamma_0} \ell_k [\tilde{v}_k + \varphi \chi - |\tilde{v}_k|] \, d\mathbf{x}' \, dt \\
\geq \left\langle \langle f, \varphi \rangle, \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)} - \int_0^T a(\varphi^0, \varphi \chi) \, dt - \left\langle \left( \nu^0 \frac{\partial \xi}{\partial t}, \varphi \right), \chi \right\rangle_{\mathcal{D}'(0, T), \mathcal{D}(0, T)}
\end{align*}
\]  

(4.4)

and

\[
\tilde{v}_k(0, \cdot) = \tilde{v}^0 = 0.
\]  

(4.5)

With the results of Section 3 we know that problem \((P_k)\) admits an unique solution and \( \sigma = -p_k I + 2\mu D(\tilde{v}_k + \nu_0 \xi) \), belongs to \( L^\infty(0, T; \{L^2(\Omega) \}^{3 \times 3}) \) with \( \text{div}(\sigma) \in L^\infty(0, T; L^2(\Omega)) \) for all \( i \in \{1, 2, 3\} \) denote by \( \sigma^i_k \) the vector \((\sigma^i_{ijk})_{1 \leq j, k \leq 3}\) and we introduce the following functional space

\[
E(\Omega) = \{ \tilde{\sigma} \in L^2(\Omega); \text{div}(\tilde{\sigma}) \in L^2(\Omega) \}
\]

endowed with the norm \( \| \cdot \|_{E(\Omega)} \) given by

\[
\| \tilde{\sigma} \|_{E(\Omega)} = \left( \| \tilde{\sigma} \|_{L^2(\Omega)}^2 + \| \text{div}(\tilde{\sigma}) \|_{L^2(\Omega)}^2 \right)^{1/2} \quad \forall \tilde{\sigma} \in E(\Omega).
\]

With the previous results we know that \( \sigma^i_k \in L^\infty(0, T; E(\Omega)) \) for all \( i \in \{1, 2, 3\} \) and for all \( k \geq 0 \). Let us recall that there exists a trace operator \( \gamma_n : E(\Omega) \to H^{-1/2}(\partial \Omega) \) such that, for all \( \tilde{\sigma} \in E(\Omega) \), the following Green’s formula holds

\[
\int_{\Omega} \text{div}(\tilde{\sigma}) \psi \, dx + \int_{\Omega} \tilde{\sigma} \cdot \nabla \psi \, dx = \left\langle \gamma_n(\tilde{\sigma}), \psi \right\rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} \quad \forall \psi \in H^1(\Omega)
\]
and $γ_n(\tilde{σ})$ is called the normal component of $\tilde{σ}$ on $\partialΩ$ (see [35] for instance). Then, following [30] we introduce a regularization operator $R : E(Ω) → C^0(Γ_0)$ given by

$$R(\tilde{σ})(x') = \{γ_n(\tilde{σ}), f_{x'}\}_{H^{-1/2}(\partialΩ), H^{1/2}(Ω)}$$

$$= \int_Ω \text{div}(\tilde{σ})f_{x'}\, dx + \int_Ω \tilde{σ} \cdot \nabla f_{x'}\, dx \quad ∀x' \in Γ_0, ∀\tilde{σ} \in E(Ω)$$

(4.6)

where $f$ is a function belonging to $C^∞(Ω)$ and $f_{x'} : Ω → R$ is defined by $f_{x'}(x) = f(x', x' - x)$ for all $x \in Ω$ and for all $x' \in Γ_0$. Since $γ_n(\tilde{σ}) = \tilde{σ} \cdot n$ on $\partialΩ$ for all $\tilde{σ} \in (D(Ω))^3$ and $n = (0, 0, -1)$ on $Γ_0$, we obtain that

$$σ_n = \sum_{i,j=1}^3 σ_{ij}n_in_j = -γ_n(σ^3) \quad \text{on } Γ_0$$

for any $σ \in (D(Ω))^3 \times 3$. Then we let

$$ℓ_{k+1}(x', t) = F(x', t, σ_k) = F^0(x', t) + F^σ(x', t) \int_0^t S(t - s)|R(σ_k^3(s, s))(x')| \, ds$$

for almost every $x' ∈ Γ_0$ and for all $t ∈ [0, T]$, thus $ℓ_{k+1} ∈ W^{1,2}(0, T; L^2(Γ_0; R^+))$. Starting from a given $ℓ_0 ∈ W^{1,2}(0, T; L^2(Γ_0; R^+))$, we construct a sequence $(\tilde{v}_k, p_k, σ_k)_{k>0}$ such that

$$\tilde{v}_k ∈ W^{1,∞}(0, T; L^2(Ω)) \cap W^{1,2}(0, T; V),$$

$$p_k ∈ L^∞(0, T; L^2(Ω))$$

and we expect that $(\tilde{v}_k, p_k)_{k>0}$ converges towards a solution of problem (P).

More precisely, let us assume that problem (P) admits a solution $(\tilde{v}, \tilde{p})$ on some time interval $[0, τ]$, with $0 ≤ τ < T$, and let us construct an extension $(\tilde{v}_*, p_*)$ of $(\tilde{v}, \tilde{p})$ to some interval $[0, τ']$ such that $(\tilde{v}_*, p_*)$ is a solution of problem (P) on $[0, τ']$ and $τ' - τ$ is independent of $τ$.

Let $\tilde{σ} = -\tilde{p}d + 2μD(\tilde{v} + v^0 r)$ and assume that $\tilde{σ} ∈ L^∞(0, τ; (L^2(Ω))^3 \times 3)$, $\text{div}(\tilde{σ}) ∈ L^∞(0, τ; L^2(Ω))$. We define $ℓ_0$ by

$$ℓ_0(x', t) = F^0(x', t) + F^σ(x', t) \int_0^t S(t - s)|R(σ^3(s, s))(x')| \, ds$$

(4.7)

for almost every $x' ∈ Γ_0$ and for all $t ∈ [0, τ]$. Then $(\tilde{v}, \tilde{p})$ is also the unique solution of problem (P) with Tresca’s friction threshold $F(\cdot, \cdot, \tilde{σ}) = ℓ_0$ and we extend now $ℓ_0$ to $[τ, T]$ by

$$ℓ_0(x', t) = F^0(x', t) + F^σ(x', τ) \int_0^τ S(τ - s)|R(σ^3(s, s))(x')| \, ds$$

(4.8)

for almost every $x' ∈ Γ_0$ and for all $t ∈ [τ, T]$. Let us observe that in this formula $\tilde{σ}$ is taken from the solution $(\tilde{v}, \tilde{p})$ on $[0, τ]$.

Let $(\tilde{v}_0, p_0)$ be the unique solution of problem $(P_0)$ on $[0, T]$. Then $(\tilde{v}_0, p_0)$ satisfies also problem (P) with $F(\cdot, \cdot, \tilde{σ}) = ℓ_0(0, τ]$, on $[0, τ]$ and, by uniqueness of the solution (see Theorem 1), we infer that $\tilde{v}_0 = \tilde{v}$ and $p_0 = \tilde{p}$ on $(0, τ)$. 
Next we choose this extended Tresca’s friction threshold $\ell_0 \in W^{1,2}(0, T; L^2(\Gamma_0; \mathbb{R}^+))$ as the starting point of the iteration procedure described previously i.e. we let $(\ell_k)_{k \geq 0}$ be given by

$$\ell_{k+1}(x', t) = \mathcal{F}(x', t, \sigma_k) = \mathcal{F}^0(x', t) + \mathcal{F}^\sigma(x', t) \int_0^t S(t-s) |\mathcal{R}(\sigma_k^2(\cdot, s))(x')| \, ds \quad (4.9)$$

for almost every $x' \in \Gamma_0$ and for all $t \in [0, T]$, with $\sigma_k = -p_k \text{Id} + 2\mu D(\tilde{v}_k + v^0 \xi)$ for all $k \geq 0$. Let $(\tilde{v}_1, p_1)$ be the unique solution of problem $(P_1)$ on $[0, T]$. We observe that $\sigma_0 = \tilde{\sigma}$ on $(0, \tau)$, hence $\ell_1 = \ell_0$ on $(0, \tau)$ and we obtain $\tilde{v}_1 = \tilde{v}_0 = \tilde{v}$ and $p_1 = p_0 = \tilde{p}$ on $(0, \tau)$. By an immediate induction we get

$$\ell_k = \ell_0, \quad \tilde{v}_k = \tilde{v}, \quad p_k = \tilde{p} \quad \text{on} \ (0, \tau), \ \text{for all} \ k \geq 0.$$

Now let $\tau' \in (\tau, T]$. Using the results of Theorems 1, 2 and Proposition 1 we obtain that there exists a positive real number $C_{\text{data}} = C_{\text{data}}(\mu, \xi, f, v^0)$, depending only on the data $\mu, \xi, f$ and $v^0$, such that

$$\|\sigma_k^2\|_{L^\infty(0, \tau'; E(\Omega))} \leq \sqrt{2} C_{\text{data}} \left( 1 + \|\ell_k\|_{W^{1,2}(0, \tau'; L^2(\Gamma_0))} \right) \quad \forall k \geq 0.$$ 

Moreover, let $\tilde{\ell}_k = \ell_k - \mathcal{F}^0$ for all $k \geq 0$. We have:

**Proposition 2:** Under the previous assumptions, there exist a constant $C_{\tau}$, depending only on $\tau$, and a constant $C'_{\text{data}} = C'_{\text{data}}(\mu, \xi, f, v^0)$, depending only on the data $\mu, \xi, f$ and $v^0$, such that

$$\|\tilde{\ell}_{k+1}\|_{W^{1,2}(0, \tau'; L^2(\Gamma_0))} \leq C_{\tau} + C'_{\text{data}} \left( \sqrt{\tau' - \tau} + (\tau' - \tau)^{\frac{p_0^2}{2p}} \right) \|\tilde{\ell}_k\|_{W^{1,2}(0, \tau'; L^2(\Gamma_0))}$$

for all $k \geq 0$.

**Proof:** For all $k \geq 0$ we have

$$\tilde{\ell}_{k+1}(x', t) = \mathcal{F}^\sigma(x', t) \int_0^t S(t-s) |\mathcal{R}(\sigma_k^2(\cdot, s))(x')| \, ds$$

for almost every $x' \in \Gamma_0$ and for all $t \in [0, T]$ and

$$\frac{\partial \tilde{\ell}_{k+1}}{\partial t}(x', t) = \frac{\partial \mathcal{F}^\sigma}{\partial t}(x', t) \int_0^t S(t-s) |\mathcal{R}(\sigma_k^2(\cdot, s))(x')| \, ds$$

$$+ \mathcal{F}^\sigma(x', t) S(0) |\mathcal{R}(\sigma_k^2(\cdot, t))(x')| + \mathcal{F}^\sigma(x', t) \int_0^t S'(t-s) |\mathcal{R}(\sigma_k^2(\cdot, s))(x')| \, ds$$

for almost every $x' \in \Gamma_0$ and for almost every $t \in [0, T]$.

But $\mathcal{F}^\sigma \in W^{1,p}(0, T; L^2(\Gamma_0; \mathbb{R}^+))$, $S \in C^1(\mathbb{R}^+; \mathbb{R}^+)$ and $\mathcal{R} \in \mathcal{L}_c(E(\Omega); C^0(\Gamma_0))$. So there exists three positive real numbers $C_{\mathcal{F}^\sigma}, C_S$ and $C_\mathcal{R}$ such that

$$\|\mathcal{F}^\sigma\|_{L^\infty(0, T; L^2(\Gamma_0))} \leq C_{\mathcal{F}^\sigma}, \quad \left\| \frac{\partial \mathcal{F}^\sigma}{\partial t} \right\|_{L^p(0, T; L^2(\Gamma_0))} \leq C_{\mathcal{F}^\sigma},$$

$$|S(t')| \leq C_S, \quad |S'(t')| \leq C_S \quad \forall t' \in [0, T],$$

$$|\mathcal{R}(\sigma_k^2(\cdot, s))(x')| \leq C_{\mathcal{R}}.$$
and
\[ \| \mathcal{R}(\tilde{\sigma}) \|_{L^\infty(\Gamma_0)} \leq C_{\mathcal{R}} \| \tilde{\sigma} \|_{E(\Omega)} \quad \forall \tilde{\sigma} \in E(\Omega). \]

It follows that
\[ \left\| \hat{\ell}_{k+1} \right\|_{L^2(0,\tau';L^2(\Gamma_0))}^2 = \left\| \int_0^{\tau'} \left\| \mathcal{F}^\sigma(x', t) \int_0^t S(t - s) |\mathcal{R}(\sigma^3_k(\cdot, s))(x')| \, ds \right\|_{L^2(\Gamma_0)}^2 \, dt \right\|_{L^2(\Gamma_0)}^2 + \left\| \frac{\partial \hat{\ell}_{k+1}}{\partial t} \right\|_{L^2(\tau',L^2(\Gamma_0))}^2 \]
\[ \leq \left\| \frac{\partial \hat{\ell}_0}{\partial t} \right\|_{L^2(\tau',L^2(\Gamma_0))}^2 + 3 \int_\tau^{\tau'} \left\| \frac{\partial \mathcal{F}^\sigma}{\partial t}(x', t) \int_0^t S(t - s) |\mathcal{R}(\sigma^3_k(\cdot, s))(x')| \, ds \right\|_{L^2(\Gamma_0)}^2 \, dt \]
\[ + 3 \int_\tau^{\tau'} \left\| \mathcal{F}^\sigma(x', t)S(0) |\mathcal{R}(\sigma^3_k(\cdot, s))(x')| \right\|_{L^2(\Gamma_0)}^2 \, dt \]
\[ + 3 \int_\tau^{\tau'} \left\| \mathcal{F}^\sigma(x', t) \int_0^t S'(t - s) |\mathcal{R}(\sigma^3_k(\cdot, s))(x')| \, ds \right\|_{L^2(\Gamma_0)}^2 \, dt \]

and thus
\[ \left\| \frac{\partial \hat{\ell}_{k+1}}{\partial t} \right\|_{L^2(\tau',L^2(\Gamma_0))}^2 \leq \left\| \frac{\partial \hat{\ell}_0}{\partial t} \right\|_{L^2(\tau',L^2(\Gamma_0))}^2 + 3C^2_{\mathcal{F}}T^2C^2_{\mathcal{R}} \|\sigma^3_k\|_{L^\infty(0,\tau';E(\Omega))} \int_\tau^{\tau'} \left\| \frac{\partial \mathcal{F}^\sigma}{\partial t}(x', t) \right\|_{L^2(\Gamma_0)}^2 \, dt \]
\[ + 3C^2_{\mathcal{F}}C^2_{\mathcal{S}}(\tau' - \tau)(T^2 + 1)C^2_{\mathcal{R}} \|\sigma^3_0\|_{L^\infty(0,\tau';E(\Omega))} \]
\[ \leq \left\| \frac{\partial \hat{\ell}_0}{\partial t} \right\|_{L^2(\tau',L^2(\Gamma_0))}^2 + 3C^2_{\mathcal{F}}C^2_{\mathcal{S}}(T^2 + 1)C^2_{\mathcal{R}} \left( (\tau' - \tau) + (\tau' - \tau) \frac{p-2}{p} \right) \|\sigma^3_k\|_{L^\infty(0,\tau';E(\Omega))}^2. \]

By combining these estimates we get
\[ \left\| \hat{\ell}_{k+1} \right\|_{W^{1,2}(0,\tau';L^2(\Gamma_0))} \leq \left\| \hat{\ell}_0 \right\|_{W^{1,2}(0,\tau';L^2(\Gamma_0))} \]
\[ + C^2_{\mathcal{F}}C^2_{\mathcal{S}}C^2_{\mathcal{R}}2(4T^2 + 3)C^2_{\text{data}} \left( (\tau' - \tau) + (\tau' - \tau) \frac{p-2}{p} \right) \]
\[ \times \left( 1 + \|\mathcal{F}^\sigma\|_{W^{1,2}(0,\tau';L^2(\Gamma_0))} \right)^2 \]
and we may conclude with

\[ C_\tau = \| \tilde{l}_0 \|_{W^{1,2}(0,\tau; L^2(\Gamma_0))} \\
+ C_{\mathcal{F}_0} C_S C_R \sqrt{2(4T^2 + 3)C_{\text{data}}}(\sqrt{T + T^2}) \left( 1 + \| \mathcal{F}^0 \|_{W^{1,2}(0,T; L^2(\Gamma_0))} \right) \]

and

\[ C'_{\text{data}} = C_{\mathcal{F}_0} C_S C_R \sqrt{2(4T^2 + 3)C_{\text{data}}}. \]

Let us fix now \( \tau' > \tau \) such that

\[ \sqrt{\tau' - \tau} + (\tau' - \tau)^{\frac{p-2}{2p}} \leq \frac{1}{2C'_{\text{data}}}. \]

For instance we may choose

\[
\begin{cases}
\tau' = \tau + \frac{1}{(4C'_{\text{data}})^2} & \text{if } \frac{1}{4C'_{\text{data}}} \geq 1, \\
\tau' = \tau + \frac{1}{(4C'_{\text{data}})^{\frac{2p}{p-2}}} & \text{otherwise.}
\end{cases}
\]

By observing that

\[ \| \tilde{\ell}_0 \|_{W^{1,2}(0,\tau'; L^2(\Gamma_0))} \leq C'_\tau \overset{\text{def}}{=} C_\tau + C_{\mathcal{F}_0} C_S C_R \tau \sqrt{T - \tau} \| \tilde{\sigma}^3 \|_{L^\infty(0,\tau; E(\Omega))} \]

we obtain with an immediate induction that

\[ \| \tilde{t}_k \|_{W^{1,2}(0,\tau'; L^2(\Gamma_0))} \leq C'_\tau \sum_{m=0}^k \frac{1}{2^m} \leq 2C'_\tau \quad \forall k \geq 0. \]

It follows that \( \tilde{\nu}_k, \frac{\partial^2 \tilde{\nu}_k}{\partial t^2}, p_k, \sigma_k \) and \( \text{div}(\sigma_k) \) are uniformly bounded in \( W^{1,\infty}(0, \tau'; L^2(\Omega)) \cap W^{1,2}(0, \tau'; V), L^2(0, \tau', \left( H^{1}_{0,\text{div}}(\Omega) \right)'), L^\infty(0, \tau'; L^2(\Omega)), L^\infty(0, \tau'; (L^2(\Omega)^3 \times 3) \text{ and in } L^\infty(0, \tau'; L^2(\Omega)), \) respectively.

Hence, possibly modifying \( \tilde{\nu}_k \) and \( \frac{\partial \tilde{\nu}_k}{\partial t} \) on a negligible subset of \([0, \tau']\), we have \( \tilde{\nu}_k \in C^0([0, \tau']; V) \) and \( \frac{\partial \tilde{\nu}_k}{\partial t} \in C^0([0, \tau']; (H^{1}_{0,\text{div}}(\Omega))'). \) Moreover

\[ \begin{aligned}
\tilde{\nu}_k, \frac{\partial \tilde{\nu}_k}{\partial t} \rightharpoonup \tilde{\nu}_*, \frac{\partial \tilde{\nu}_*}{\partial t} \quad &\text{weakly star in } L^\infty(0, \tau'; L^2(\Omega)) \\
\text{and weakly in } L^2(0, \tau'; V),
\end{aligned} \]

\[ \begin{aligned}
\frac{\partial^2 \tilde{\nu}_k}{\partial t^2} \rightharpoonup \frac{\partial^2 \tilde{\nu}_*}{\partial t^2} \quad &\text{weakly in } L^2(0, \tau'; (H^{1}_{0,\text{div}}(\Omega))') \\
p_k \rightharpoonup p_* \quad &\text{weakly star in } L^\infty(0, \tau'; L^2(\Omega)),
\end{aligned} \]
and
\[
\ell_k \rightharpoonup \ell_0 \quad \text{weakly in } L^2(0, \tau'; L^2(\Gamma_0)).
\]

Using Aubin’s and Simon’s lemmas, and possibly extracting another subsequence, we have also
\[
\tilde{v}_k \to \tilde{v}_0 \quad \text{strongly in } L^2(0, \tau'; L^2(\Omega)),
\]
\[
\frac{\partial \tilde{v}_k}{\partial t} \to \frac{\partial \tilde{v}_0}{\partial t} \quad \text{strongly in } L^2(0, \tau'; L^2(\Omega)),
\]
and
\[
\tilde{v}_k \to \tilde{v}_0 \quad \text{strongly in } C^0([0, \tau']; L^2(\Omega)).
\]

Furthermore, possibly modifying \( \tilde{v}_* \) and \( \frac{\partial \tilde{v}_*}{\partial t} \) on a negligible subset of \([0, \tau')\), we have \( \tilde{v}_0 \in C^0([0, \tau']; V) \)
and \( \frac{\partial \tilde{v}_*}{\partial t} \in C^0([0, \tau'); (H_{0div}(\Omega))^{'}) \). By passing to the limit as \( k \) tends to \(+\infty\) in (4.4)–(4.5) we get
\[
\begin{align*}
\left\langle \frac{d}{dt} \langle \tilde{v}_*, \varphi \rangle, \chi \right\rangle_{D'(0, \tau'), D(0, \tau')} & - \langle (p_*, \text{div}(\varphi)), \chi \rangle_{D'(0, \tau'), D(0, \tau')} \\
& + \int_0^{\tau'} a(\tilde{v}_*, \varphi \chi) \, dt + \int_0^{\tau'} \int_{\Gamma_0} \ell_s (|\tilde{v}_* + \varphi \chi| - |\tilde{v}_*|) \, dx' \, dt \\
& \geq \langle f, \varphi \rangle_{D'(0, \tau'), D(0, \tau')} - \int_0^{\tau'} a(\nu^0 \xi, \varphi \chi) \, dt - \left( \left( \nu^0 \frac{\partial \xi}{\partial t}, \varphi \right) \right)_{D'(0, \tau'), D(0, \tau')}
\end{align*}
\]
for all \( \varphi \in V_0 \) and for all \( \chi \in D(0, \tau') \), with
\[
\tilde{v}_0(0, \cdot) = \tilde{v}_0 = 0.
\]

It follows that \( (\tilde{v}_*, p_*) \) is the unique solution on \([0, \tau')\) of problem (P) with \( \mathcal{F} = \ell_* = \mathcal{F}_0 + \tilde{\ell}_0 \) and
\[
\sigma_* = -p_* Id + 2\mu D(\tilde{v}_* + \nu^0 \xi) \quad \text{satisfies } \sigma_* \in L^\infty(0, \tau'; (L^2(\Omega))^{3 \times 3}).
\]
Moreover, with a straightforward adaptation of Proposition 1, we have also \( \text{div}(\sigma_*) \in L^\infty(0, \tau'; L^2(\Omega)) \).

Let us prove now that

**Lemma 4:** Under the previous assumptions
\[
\ell_*(x', t) = \mathcal{F}(x', t, \sigma_*) = \mathcal{F}_0(x', t) + \mathcal{F}^\sigma(x', t) \int_0^t S(t - s) |R(\sigma_3^3, s))(x')| \, ds
\]
for almost every \( x' \in \Gamma_0 \) and for all \( t \in [0, \tau') \).

**Proof:** Let \( \varphi \in \left( D(\Omega) \right)^3 \) and \( \chi \in D(0, \tau') \). We have
\[
\begin{align*}
\left\langle \frac{d}{dt} \langle \tilde{v}_*, \varphi \rangle, \pm \chi \right\rangle_{D'(0, \tau'), D(0, \tau')} + \int_0^{\tau'} a(\tilde{v}_* + \nu^0 \xi, \pm \varphi \chi) \, dx \, dt \\
- \langle (p_*, \text{div}(\varphi)), \pm \chi \rangle_{D'(0, \tau'), D(0, \tau')} \\
\geq \langle f, \varphi \rangle_{D'(0, \tau'), D(0, \tau')} - \left( \left( \nu^0 \frac{\partial \xi}{\partial t}, \varphi \right) \right)_{D'(0, \tau'), D(0, \tau')}.
\end{align*}
\]
Similarly, for all \( k \geq 0 \) we have also

\[
\left\langle \frac{d}{dt}(\tilde{v}_k, \varphi), \pm \chi \right\rangle_{D'(0,\tau'),D(0,\tau')} + \int_0^{\tau'} a(\tilde{v}_k + v^0 \xi, \pm \varphi \chi) \, dx \, dt \\
- \left\langle (p_k, \text{div}(\varphi)), \pm \chi \right\rangle_{D'(0,\tau'),D(0,\tau')} \\
\geq \left\langle (f, \varphi), \pm \chi \right\rangle_{D'(0,\tau'),D(0,\tau')} - \left\langle \left(v^0 \frac{\partial \xi}{\partial t}, \varphi \right), \pm \chi \right\rangle_{D'(0,\tau'),D(0,\tau')}.
\]

(4.15)

By subtracting (4.14) to (4.15) we obtain

\[
\int_0^{\tau'} \int_{\Omega} \frac{\partial(\tilde{v}_k - \tilde{v}_*)}{\partial t} : \varphi \chi \, dx \, dt + \int_0^{\tau'} \int_{\Omega} \sum_{i=1}^{3} (\sigma_{ijk} - \sigma_{ijk}) \frac{\partial \varphi_i}{\partial x_j} \chi \, dx \, dt = 0 \quad \forall k \geq 0.
\]

Thus

\[
\left\| \text{div}(\sigma_k - \sigma_*) \right\|_{L^2(0,\tau';L^2(\Omega))} = \left\| \frac{\partial(\tilde{v}_k - \tilde{v}_*)}{\partial t} \right\|_{L^2(0,\tau';L^2(\Omega))} \quad \forall k \geq 0.
\]

Since \( \left( \frac{\partial \tilde{v}_k}{\partial t} \right)_{k \geq 0} \) converges strongly in \( L^2(0,\tau';L^2(\Omega)) \) to \( \frac{\partial \tilde{v}_*}{\partial t} \), we infer that \( \left( \text{div}(\sigma_k) \right)_{k \geq 0} \) converges also strongly to \( \text{div}(\sigma_*) \) in \( L^2(0,\tau';L^2(\Omega)) \). Hence

\[
\left\| \frac{\partial \tilde{v}_k}{\partial t} - \frac{\partial \tilde{v}_*}{\partial t} \right\|_{L^2(\Omega)} \to 0 \quad \text{strongly in } L^2(0,\tau'), \\
\left\| \text{div}(\sigma_k) - \text{div}(\sigma_*) \right\|_{L^2(\Omega)} \to 0 \quad \text{strongly in } L^2(0,\tau'),
\]

and we infer that, possibly extracting a subsequence still denoted \((\tilde{v}_k,p_k)_{k \geq 0}\), there exists a negligible subset \( A \) of \((0,\tau')\) such that

\[
\frac{\partial \tilde{v}_k}{\partial t}(t), \text{div}(\sigma_k)(t) \to \frac{\partial \tilde{v}_*}{\partial t}(t), \text{div}(\sigma_*)(t) \quad \text{strongly in } L^2(\Omega), \quad \text{for all } t \in (0, \tau') \setminus A.
\]

(4.16)

On the other hand, \((\tilde{v}_k)_{k \geq 0}\) is bounded in \( W^{1,2}(0,\tau';V) \) and using Helly’s theorem (see [37] for instance) we obtain that, possibly extracting another subsequence, still denoted \((\tilde{v}_k,p_k)_{k \geq 0}\), we have

\[
\tilde{v}_k(t) \to \Lambda(t) \quad \text{weakly in } V, \quad \text{for all } t \in [0,\tau']
\]

(4.17)

with \( \Lambda \in BV(0,\tau';V) \). Then, for all \( \varphi \in V \) and for all \( \chi \in D(0,\tau') \) we have

\[
(\tilde{v}_k(t),\varphi)_{H^1(\Omega)} \chi(t) \to (\Lambda(t),\varphi)_{H^1(\Omega)} \chi(t) \quad \text{for all } t \in [0,\tau']
\]

where \((\cdot,\cdot)_{H^1(\Omega)}\) denotes the inner product of \( H^1(\Omega) \) and

\[
| (\tilde{v}_k(t),\varphi)_{H^1(\Omega)} \chi(t) | \leq ||\chi||_{C^0([0,\tau'])} ||\varphi||_{H^1(\Omega)} ||\tilde{v}_k||_{L^\infty(0,\tau';H^1(\Omega))} \quad \text{for all } t \in [0,\tau']
\]

We may apply Lebesgue’s dominated theorem and we get

\[
\int_0^{\tau'} (\tilde{v}_k(t),\varphi)_{H^1(\Omega)} \chi(t) \, dt \to \int_0^{\tau'} (\Lambda(t),\varphi)_{H^1(\Omega)} \chi(t) \, dt.
\]
It follows that there exists another negligible subset $A'$ of $(0, \tau')$ such that
\[
\Lambda(t) = \widetilde{\nu}_*(t) \text{ in } \mathbf{V}, \text{ for all } t \in (0, \tau') \setminus A'.
\] (4.18)

Recalling that, $\frac{\partial \nu_k}{\partial t}$ belongs to $C^0([0, \tau'], (H^1_0(\Omega), H^1_0(\Omega))) \cap L^\infty(0, \tau', L^2(\Omega))$ for all $k \geq 0$, we infer that $\frac{\partial \nu_k}{\partial t}$ is weakly continuous with values in $L^2(\Omega)$ on $[0, \tau']$ and
\[
\left\| \frac{\partial \nu_k}{\partial t}(t) \right\|_{L^2(\Omega)} \leq \left\| \frac{\partial \nu_k}{\partial t} \right\|_{L^\infty(0, \tau'; L^2(\Omega))} \text{ for all } t \in [0, \tau']
\]
(see Lemma 1.4, p.263 in [33]). Now, let $k \geq 0$. For all $t \in [0, \tau']$ we define $f_k(t) \in H^{-1}(\Omega)$ by
\[
\langle f_k(t), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \left( \frac{\partial \nu_k}{\partial t}(t) + \nu^0 \frac{\partial \xi}{\partial t}(t), \varphi \right) + a(\nu_k(t) + \nu^0 \xi(t), \varphi)
\]
\[
-\langle f(t), \varphi \rangle \quad \forall \varphi \in H^1_0(\Omega).
\]
Then, consider now $\varphi \in H^1_0(\Omega)$. With (4.4), we obtain that
\[
\int_0^{\tau'} \langle f_k(t), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} t(t) \, dt = 0 \quad \forall \chi \in D(0, \tau').
\]
So
\[
\langle f_k(t), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = 0 \quad \text{a.e. in } (0, \tau')
\]
and, using the continuity of the mapping $t \mapsto \langle f_k(t), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$ on $[0, \tau']$, we infer that the previous equality is valid for all $t \in [0, \tau']$. It follows that there exists a mapping $\tilde{p}_k : [0, \tau'] \rightarrow L^2(\Omega)$ such that, for all $t \in [0, \tau']$
\[
\langle f_k(t), \varphi \rangle_{H^{-1}(\Omega), H^1_0(\Omega)} = \langle \nabla \tilde{p}_k(t), \varphi \rangle_{D'(\Omega), D(\Omega)} \quad \forall \varphi \in D(\Omega).
\]
But, for all $t \in [0, \tau']$, we have $\tilde{p}_k(t) \in L^2(\Omega)$ and thus
\[
\langle \nabla \tilde{p}_k(t), \varphi \rangle_{D'(\Omega), D(\Omega)} = -\langle \tilde{p}_k(t), div(\varphi) \rangle_{D'(\Omega), D(\Omega)}
\]
\[
= -\langle \tilde{p}_k(t), div(\varphi) \rangle \quad \forall \varphi \in D(\Omega).
\]
It follows that, for all $t \in [0, \tau']$,
\[
-\langle \tilde{p}_k(t), div(\varphi) \rangle = \left( \frac{\partial \nu_k}{\partial t}(t) + \nu^0 \frac{\partial \xi}{\partial t}(t), \varphi \right) + a(\nu_k(t) + \nu^0 \xi(t), \varphi)
\]
\[
-\langle f(t), \varphi \rangle \quad \forall \varphi \in D(\Omega)
\]
and by density of \( \mathcal{D}(\Omega) \) into \( H^1_0(\Omega) \), the same equality is valid for all \( \varphi \in H^1_0(\Omega) \). With the same arguments as in Theorem 1 and Theorem 2, we obtain also that \( \tilde{p}_k \in L^\infty(0, \tau'; L^2_0(\Omega)) \) and \( p_k = \tilde{p}_k \) in \( L^\infty(0, \tau'; L^2_0(\Omega)) \). Thus possibly modifying \( p_k \) on a negligible subset of \( (0, \tau') \) we have

\[
-(p_k(t), div(\varphi)) = \left( \partial_t \tilde{v}_k(t) + \nu^0 \frac{\partial \zeta}{\partial t}(t), \varphi \right) + a(\tilde{v}_k(t) + \nu^0 \zeta(t), \varphi)
\]

\[
-(f(t), \varphi) \quad \forall \varphi \in H^1_0(\Omega), \quad \forall t \in [0, \tau').
\]

(4.19)

Similarly, possibly modifying \( p_\ast \) on a negligible subset of \( (0, \tau') \), we have

\[
-(p_\ast(t), div(\varphi)) = \left( \partial_t \tilde{v}_\ast(t) + \nu^0 \frac{\partial \zeta}{\partial t}(t), \varphi \right) + a(\tilde{v}_\ast(t) + \nu^0 \zeta(t), \varphi)
\]

\[
-(f(t), \varphi) \quad \forall \varphi \in H^1_0(\Omega), \quad \forall t \in [0, \tau').
\]

(4.20)

Now let \( \tilde{\omega} \in L^2(\Omega) \) and \( w \in L^2_0(\Omega) \) be given by

\[
w = \tilde{\omega} - \frac{1}{|\Omega|} \int_\Omega \tilde{\omega} \, dx.
\]

For all \( k \geq 0 \) and for all \( t \in [0, \tau') \) we have

\[
(p_k(t) - p_\ast(t), \tilde{\omega}) = (p_k(t) - p_\ast(t), w)
\]

\[
= -\left( \frac{\partial \tilde{v}_k}{\partial t} (\cdot, t) - \frac{\partial \tilde{v}_\ast}{\partial t} (\cdot, t), P(w) \right) - a(\tilde{v}_k(t) - \tilde{v}_\ast(t), P(w))
\]

where \( P \) is the linear continuous operator from \( L^2_0(\Omega) \) to \( H^1_0(\Omega) \) such that \( div(P(w)) = w \) for all \( w \in L^2_0(\Omega) \) [35]. With (4.16) and (4.17)–(4.18) we get

\[
\int_\Omega (p_k(t) - p_\ast(t)) \tilde{\omega} \, dx \to 0 \quad \text{for all} \quad \tilde{\omega} \in L^2(\Omega), \quad \text{for all} \quad t \in (0, \tau') \setminus (A \cup A')
\]

which implies that

\[
p_k(t) \rightharpoonup p_\ast(t) \quad \text{weakly in} \quad L^2(\Omega), \quad \text{for all} \quad t \in (0, \tau') \setminus (A \cup A').
\]

(4.21)

Then with (4.16)–(4.18) and (4.21) we may conclude that

\[
\sigma_k(t) \rightharpoonup \sigma_\ast(t) \quad \text{weakly in} \quad E(\Omega), \quad \text{for all} \quad t \in (0, \tau') \setminus (A \cup A').
\]

Using the definition of \( \mathcal{R} \), we obtain that

\[
\mathcal{R}(\sigma^3_\ast) (\cdot, t)(x') = \{ \gamma_n(\sigma^3_\ast(t)), f_n(x') \}_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)}
\]

\[
= \mathcal{R}(\sigma^3_\ast) (\cdot, t)(x') \quad \text{for all} \quad x' \in \Gamma_0, \quad \text{for all} \quad t \in (0, \tau') \setminus (A \cup A').
\]

Moreover, for all \( k \geq 0 \),

\[
|\mathcal{R}(\sigma^3_k(t))(x')| \leq C_{\mathcal{R}} \| \sigma^3_\ast(t) \|_{E(\Omega)} \leq C_{\mathcal{R}} \| \sigma^3_k \|_{L^\infty(0, \tau'; E(\Omega))}
\]

for all \( x' \in \Gamma_0 \), for almost every \( t \in [0, \tau'] \), i.e. there exists a negligible subset \( A_k \) of \( (0, \tau') \) such that

\[
|\mathcal{R}(\sigma^3_k(t))(x')| \leq C_{\mathcal{R}} \| \sigma^3_\ast(t) \|_{E(\Omega)} \leq C_{\mathcal{R}} \| \sigma^3_k \|_{L^\infty(0, \tau'; E(\Omega))}
\]
for all $x' \in \Gamma_0$, for all $t \in (0, \tau') \setminus A_k$.  Let $A'' = A \cup A' \cup (\bigcup_{k \geq 0} A_k)$.  By applying Lebesgue’s dominated theorem, we get

$$|R(\sigma_k^3)(\cdot, t)| \to |R(\sigma_\infty^3)(\cdot, t)|$$

strongly in $L^2(\Gamma_0)$, for all $t \in (0, \tau') \setminus A''$

Thus, observing that $A''$ is also a negligible subset of $(0, \tau')$, we may apply twice Lebesgue’s dominated theorem and we get

$$\int_0^t S(t - s)|R(\sigma_k^3)(\cdot, s)|\, ds \to \int_0^t S(t - s)|R(\sigma_\infty^3)(\cdot, s)|\, ds$$

strongly in $L^2(\Gamma_0)$, for all $t \in [0, \tau']$

and

$$\int_0^* S(\cdot - s)|R(\sigma_k^3)(\cdot, s)|\, ds \to \int_0^* S(\cdot - s)|R(\sigma_\infty^3)(\cdot, s)|$$

strongly in $L^2(0, \tau', L^2(\Gamma_0))$

which allows us to conclude. 

Gathering all the previous results we have obtained that $(\tilde{\nu}_\infty, p_\infty)$ is a solution of problem $(P)$ on $[0, \tau']$ and $(\tilde{\nu}_0, p_0)$ is an extension of $(\nu, p)$ to $[\tau, \tau')$. Indeed, $\ell_k = \ell_0 = F(\cdot, \cdot, \tilde{\sigma})$ on $(0, \tau)$ for all $k \geq 0$, thus $\ell_\infty = F(\cdot, \cdot, \tilde{\sigma})$ on $(0, \tau)$. Moreover $\tau' - \tau$ is independent of $\tau$. Thus, starting from $\tau = 0$, we may conclude with a finite induction argument that

**Theorem 3:** Let assumptions (2.2)–(3.18)–(3.19)–(3.20)–(4.2)–(4.3) hold. Then, the non-local friction problem $(P)$ admits a solution $(\tilde{\nu}, p)$ such that

$$\tilde{\nu} \in W^{1, \infty}(0, T; L^2(\Omega)) \cap W^{1, 2}(0, T; V), \quad p \in L^\infty(0, T; L^2_0(\Omega))$$

and

$$\frac{\partial^2 \tilde{\nu}}{\partial t^2} \in L^2(0, T; \left(H^{1}_0(\Omega)\right)'.

**Remark 4.1:** Let us emphasize that the convergence of this successive approximation technique relies strongly on the regularity properties of the solutions of Tresca’s problem. Hence it seems difficult to extend it to 3D non-stationary Navier–Stokes flows, except under some appropriate smallness condition on data which ensures existence of strong solutions in the Tresca’s case i.e. solutions $(\tilde{\nu}, p)$ such that

$$\tilde{\nu} \in W^{1, \infty}(0, T; L^2(\Omega)) \cap W^{1, 2}(0, T; V), \quad p \in L^2(0, T; L^2_0(\Omega))$$

(see Theorem 4.2 and Proposition 5.1 in [38]). Nevertheless it still does not yield $\sigma \in L^\infty(0, T; E(\Omega))$ and we have to deal with new technical difficulties in defining the space regularization operator. These issues will be addressed in a forthcoming paper.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

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