Representability and Boxicity of Simplicial Complexes

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Abstract
Let $X$ be a simplicial complex on vertex set $V$. We say that $X$ is $d$-representable if it is isomorphic to the nerve of a family of convex sets in $\mathbb{R}^d$. We define the $d$-boxicity of $X$ as the minimal $k$ such that $X$ can be written as the intersection of $k$ $d$-representable simplicial complexes. This generalizes the notion of boxicity of a graph, defined by Roberts. A missing face of $X$ is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$. We prove that the $d$-boxicity of a simplicial complex on $n$ vertices without missing faces of dimension larger than $d$ is at most $\left\lfloor \frac{n}{d} \right\rfloor / (d + 1)$. The bound is sharp: the $d$-boxicity of a simplicial complex whose set of missing faces form a Steiner $(d, d + 1, n)$-system is exactly $\frac{n}{d} / (d + 1)$. One of the main ingredients in the proof is the following bound on the representability of a complex: let $V_1, \ldots, V_k$ be subsets of $V$ such that $V_i \notin X$ for all $1 \leq i \leq k$, and for any missing face $\tau$ of $X$ there is some $1 \leq i \leq k$ satisfying $|\tau \setminus V_i| \leq 1$. Then, $X$ can be written as an intersection $X = \bigcap_{i=1}^k V_i$, where, for all $1 \leq i \leq k$, $X_i$ is a $(|V_i| - 1)$-representable complex. In particular, $X$ is $\left(\sum_{i=1}^k (|V_i| - 1)\right)$-representable.

Keywords Representability · Boxicity · Leray number · Convex set · Simplicial complex · Steiner system

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1 Introduction

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of sets. The intersection graph of $\mathcal{F}$ is the graph on vertex set $[n]$, whose edges are the pairs $\{i, j\}$ for $1 \leq i < j \leq n$ such that $F_i \cap F_j \neq \emptyset$. A graph $G = (V, E)$ is called an interval graph if it is isomorphic to the intersection graph of a family of compact intervals in the real line.

Let $G$ be a graph. The boxicity of $G$, denoted by $\text{box}(G)$, is the minimal integer $k$ such that $G$ can be written as the intersection of $k$ interval graphs. Equivalently, $\text{box}(G)$ is the minimal $k$ such that $G$ is isomorphic to the intersection graph of a family of axis-parallel boxes in $\mathbb{R}^k$. The notion of boxicity was introduced by Roberts in [9]. The following result was first proven by Roberts in [9], and later rediscovered by Witsenhausen in [14]:

**Theorem 1.1** (Roberts [9], Witsenhausen [14, Thm. 1]) Let $G$ be a graph with $n$ vertices. Then

$$\text{box}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$  

Moreover, $\text{box}(G) = n/2$ if and only if $G$ is the complete $(n/2)$-partite graph with sides of size 2.

Let $\mathcal{F} = \{F_1, \ldots, F_n\}$ be a family of sets. The nerve of $\mathcal{F}$ is the simplicial complex

$$N(\mathcal{F}) = \left\{ \sigma \subset [n] : \bigcap_{i \in \sigma} F_i \neq \emptyset \right\}.$$  

Let $X$ be a simplicial complex. We say that $X$ is $d$-representable if it is isomorphic to the nerve of a family $\mathcal{C}$ of compact convex sets in $\mathbb{R}^d$. We call the family $\mathcal{C}$ a representation of $X$ in $\mathbb{R}^d$. The representability of $X$, denoted by $\text{rep}(X)$, is the minimal $d$ such that $X$ is $d$-representable.

**Definition 1.2** Let $X$ be a simplicial complex and $d \geq 1$. The $d$-boxicity of $X$, denoted by $\text{box}_d(X)$, is the minimal $k$ such that $X$ can be written as the intersection of $k$ $d$-representable simplicial complexes.

Let $G = (V, E)$ be a graph. The clique complex of $G$, denoted by $X(G)$, is the simplicial complex on vertex set $V$ whose simplices are the cliques in $G$, that is, the sets $U \subset V$ satisfying $\{u, w\} \in E$ for all $u, w \in U$ such that $u \neq w$.

Let $\mathcal{B} = \{B_1, \ldots, B_n\}$ be a family of axis-parallel boxes in $\mathbb{R}^k$. It is well known that any $t$ boxes $B_{i_1}, \ldots, B_{i_t}$ have a point in common if and only if $B_{i_j} \cap B_{i_r} \neq \emptyset$ for every $1 \leq j < r \leq t$. Therefore, the nerve $N(\mathcal{B})$ is exactly the clique complex of the intersection graph of $\mathcal{B}$. So, for any graph $G$, we have $\text{box}(G) = \text{box}_1(X(G))$. Thus, we can see the parameters $\text{box}_d(X)$ as higher dimensional generalizations of the boxicity of a graph.

Let $X$ be a simplicial complex on vertex set $V$. A missing face of $X$ is a set $\tau \subset V$ such that $\tau \notin X$ but $\sigma \in X$ for any $\sigma \subsetneq \tau$. Let $h(X)$ be the maximal dimension of a missing face of $X$. Note that a complex $X$ satisfies $h(X) = 0$ if and only if it is...
a simplex, and it satisfies \( h(X) = 1 \) if and only if it is the clique complex of some graph \( G \) (the missing faces of \( X(G) \) are the edges of the complement graph of \( G \)).

A family \( \mathcal{F} \) of subsets of size \( k \) of a set \( V \) of size \( n \) is called a Steiner \((t, k, n)\)-system if any subset of \( V \) of size \( t \) is contained in exactly one set of \( \mathcal{F} \). If any subset of \( V \) of size \( t \) is contained in at most one set of \( \mathcal{F} \), then \( \mathcal{F} \) is called a partial Steiner \((t, k, n)\)-system. A Steiner \((2, 3, n)\)-system is also called a Steiner triple system.

In [14, Thm. 2], Witsenhausen extended Theorem 1.1, proving that any simplicial complex \( X \) with \( n \) vertices whose missing faces are all of dimension exactly \( d \) has \( d \)-boxicity at most \( \left\lfloor \frac{n^d}{d+1} \right\rfloor \). On the other hand, he showed in [14, Thm. 3] that a complex \( X \) whose missing faces form a Steiner triple system (in particular, \( h(X) = 2 \)) has \( 2 \)-boxicity at least \( \frac{n^2}{3} \). Here, we extend Witsenhausen’s lower bound to all values of \( d \), and prove an improved upper bound, matching the lower bound.

**Theorem 1.3** Let \( X \) be a simplicial complex with \( n \) vertices, satisfying \( h(X) \leq d \). Then

\[
\text{box}_d(X) \leq \left\lfloor \frac{1}{d+1} \binom{n}{d} \right\rfloor.
\]

Moreover, if \( h(X) = d \), then \( \text{box}_d(X) = \left(\frac{n}{d}\right)/(d+1) \) if and only if the missing faces of \( X \) form a Steiner \((d, d+1, n)\)-system.

Let \( \mathbb{F} \) be a field. For \( k \geq 0 \), let \( \tilde{H}_k(X) \) be the \( k \)-th reduced homology group of \( X \) with coefficients in \( \mathbb{F} \). We say that \( X \) is \( d \)-Leray if for any induced subcomplex \( Y \) of \( X \), \( \tilde{H}_k(Y) = 0 \) for all \( k \geq d \). The Leray number of \( X \), denoted by \( L(X) \), is the minimal \( d \) such that \( X \) is \( d \)-Leray. It is a well-known fact that for any complex \( X \),

\[
L(X) \leq \text{rep}(X).
\]

That is, any \( d \)-representable complex is \( d \)-Leray (see e.g. [4,13]). To prove the equality case in Theorem 1.3 we will need the following result.

**Theorem 1.4** Let \( X \) be a complex whose set of missing faces is a partial Steiner \((d, d+1, n)\)-system \( \mathcal{M} \). Then, \( X \) cannot be written as the intersection of less than \( |\mathcal{M}| \) \( d \)-Leray complexes. On the other hand, the \( d \)-boxicity of \( X \) is at most \( |\mathcal{M}| \). As a consequence,

\[
\text{box}_d(X) = |\mathcal{M}|.
\]

As it was proven by Rödl [10], for any \( d \geq 1 \), there are partial Steiner \((d, d+1, n)\)-systems of size \((1-o(1))\binom{n}{d}/(d+1)\). Therefore, the bound in Theorem 1.3 is asymptotically tight. Moreover, by a well-known result of Keevash [7], there exist Steiner \((d, d+1, n)\)-systems for infinitely many values of \( n \). Thus, the equality case in Theorem 1.3 is achieved for infinitely many values of \( n \).

The upper bound in Theorem 1.3 follows as a consequence of the next result:
Theorem 1.5  Let $X$ be a simplicial complex on vertex set $V$. Let $V_1, \ldots, V_k$ be subsets of $V$ satisfying $V_i \not\in X$ for all $i \in [k]$, such that for any missing face $\tau$ of $X$ there exists some $i \in [k]$ satisfying $|\tau \setminus V_i| \leq 1$. Then $X$ can be written as an intersection

$$X = \bigcap_{i=1}^{k} X_i,$$

where, for all $i \in [k]$, $X_i$ is a $(|V_i| - 1)$-representable complex. In particular, $X$ is $(\sum_{i=1}^{k} (|V_i| - 1))$-representable.

The paper is organized as follows. In Sect. 2 we present the necessary background on simplicial complexes that we will later need. In Sect. 3 we prove some simple results about the missing faces and the representability of intersections of complexes. Section 4 contains the proof of Theorem 1.4. In Sect. 5 we prove Theorem 1.5. In Sect. 6 we prove our main result, Theorem 1.3. In Sect. 7 we present some related open problems.

2 Preliminaries

For any set $U$, the complete complex on vertex set $U$ is the complex

$$2^U = \{\sigma : \sigma \subset U\}.$$

For $0 \leq k \leq |U| - 1$, the complete $k$-dimensional skeleton on vertex set $U$ is the complex

$$\{\sigma \subset U : |\sigma| \leq k + 1\}.$$

Let $X$ be a simplicial complex on vertex set $V$. For $U \subset V$, the subcomplex of $X$ induced by $U$ is the complex

$$X[U] = \{\sigma \in X : \sigma \subset U\}.$$

Let $\mathcal{M}$ be the set of missing faces of $X$. Let

$$\Gamma(X) = \left\{\mathcal{N} \subset \mathcal{M} : \bigcup_{\tau \in \mathcal{N}} \tau \neq V\right\}.$$

Note that $\Gamma(X)$ is a simplicial complex on vertex set $\mathcal{M}$. The homology groups of $X$ and $\Gamma(X)$ are related as follows:

Theorem 2.1  (Björner, Butler, Matveev [2, Thm. 2])  Let $V$ be a finite set and $X$ be a simplicial complex on vertex set $V$. If $X$ is not the complete complex on $V$, then for all $k \geq 0$,

$$\tilde{H}_k(X) \cong \tilde{H}_{|V| - k - 3}(\Gamma(X)).$$
Proposition 3.1 Let \( X \) be a \( d \)-Leray complex. Then \( h(X) \leq d \).

Remark 2.2 In its original formulation, [2, Thm. 2] relates the homology groups of \( X \) to the cohomology groups of \( \Gamma(X) \): \( H_k(X) \cong H_{|V| \geq k-3}(\Gamma(X)) \). However, since we work here with homology with coefficients in a field, we have an isomorphism \( H_{|V| \geq k-3}(\Gamma(X)) \cong \tilde{H}_{|V| \geq k-3}(\Gamma(X)) \).

Finally, we will need the following simple property (see e.g. [4]):

Lemma 2.3 Let \( X \) be a \( d \)-Leray complex. Then \( h(X) \leq d \).

3 Intersection of Simplicial Complexes

In this section we prove some basic results about the missing faces and the representability of intersections of complexes:

Proposition 3.1 Let \( X_1, \ldots, X_k \) be simplicial complexes on vertex set \( V \), and \( X = \bigcap_{i=1}^k X_i \). For each \( i \in [k] \), let \( M_i \) be the set of missing faces of \( X_i \), and let \( M \) be the set of missing faces of \( X \). Then, \( M \) is the set of inclusion minimal elements of \( \bigcup_{i=1}^k M_i \). As a consequence, we obtain

\[
h(X) \leq \max_{i \in [k]} h(X_i).
\]

Proof Let \( \tau \in M \). Since \( \tau \not\in X \), then there exists some \( j \in [k] \) such that \( \tau \not\in X_j \). Let \( \sigma \subseteq \tau \). Since \( \tau \) is a missing face of \( X \), we have \( \sigma \in X = \bigcap_{i=1}^k X_i \). In particular, \( \sigma \in X_j \). Hence, \( \tau \) is a missing face of \( X_j \). That is, \( \tau \in M_j \subset \bigcup_{i=1}^k M_i \). Moreover, \( \tau \) does not contain any other face of \( \bigcup_{i=1}^k M_i \). Otherwise, there exists some \( r \in [k] \) and \( \sigma \in M_r \) such that \( \sigma \not\subset \tau \). Since \( \sigma \not\in X_r \), then \( \sigma \not\in X \). But this is a contradiction to \( \tau \) being a missing face of \( X \).

Now, let \( \tau \) be an inclusion minimal element of \( \bigcup_{i=1}^k M_i \). Then \( \tau \in M_j \) for some \( j \in [k] \). In particular, \( \tau \not\in X_j \), and therefore \( \tau \not\in X \). Now, let \( \sigma \subset \tau \). Assume for contradiction that \( \sigma \not\in X \). Then, there exists some \( r \in [k] \) such that \( \sigma \not\in X_r \). So, there exists some \( \eta \in M_r \) such that \( \eta \subset \sigma \subset \tau \). This is a contradiction to \( \tau \) being inclusion minimal in \( \bigcup_{i=1}^k M_i \). So, \( \sigma \in X \). Therefore, \( \tau \) is a missing face of \( X \). Since \( M \subset \bigcup_{i=1}^k M_i \), we obtain (3.1). \( \square \)

Lemma 3.2 Let \( X_1, \ldots, X_k \) be simplicial complexes on vertex set \( V \). If \( X_i \) is \( d_i \)-representable for each \( i \in [k] \), then \( \bigcap_{i=1}^k X_i \) is \( (\sum_{i=1}^k d_i) \)-representable.

Proof For \( i \in [k] \), let \( \{C^i_v\}_{v \in V} \) be a representation of \( X_i \) in \( \mathbb{R}^{d_i} \). For \( v \in V \), let

\[
C_v = C^1_v \times C^2_v \times \cdots \times C^k_v.
\]

We will show that \( C = \{C_v\}_{v \in V} \) is a representation of \( \bigcap_{i=1}^k X_i \) in \( \mathbb{R}^{d_1+\cdots+d_k} \).

Note that the sets \( C_v \) are convex, and for any \( \sigma \subset V \),

\[
\bigcap_{v \in \sigma} C_v = \left( \bigcap_{v \in \sigma} C^1_v \right) \times \cdots \times \left( \bigcap_{v \in \sigma} C^k_v \right).
\]

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Let $\sigma \subset V$. If $\sigma \in \bigcap_{i=1}^{k} X_i$, then $\sigma \in X_i$ for all $i \in [k]$. Hence, $\bigcap_{v \in \sigma} C_v \neq \emptyset$ for all $i \in [k]$. So, by (3.2), $\bigcap_{v \in \sigma} C_v \neq \emptyset$. If $\sigma \notin \bigcap_{i=1}^{k} X_i$, then there exists some $i \in [k]$ such that $\sigma \notin X_i$. Therefore, $\bigcap_{v \in \sigma} C_v = \emptyset$. Thus, by (3.2), $\bigcap_{v \in \sigma} C_v = \emptyset$. Hence, $C$ is a representation of $\bigcap_{i=1}^{k} X_i$ in $\mathbb{R}^{d_1+\cdots+d_k}$.

4 Lower Bounds on $d$-Boxicity

In this section we prove Theorem 1.4. For the proof we will need the following simple lemma, which is a generalization of [14, Lem. 3]:

**Lemma 4.1** Let $A$, $B$ be two finite sets, such that $|A| = |B| = d + 1$ and $|A \cap B| < d$. Let $V = A \cup B$. Let $X$ be a simplicial complex on vertex set $V$ that has $A$ and $B$ as missing faces, and such that for any other missing face $\tau$ of $X$, $\tau \cup A = V$ and $\tau \cup B = V$. Then, there exists some $k \geq d$ such that $\tilde{H}_k(X) \neq 0$.

**Proof** Let $\mathcal{M}$ be the set of missing faces of $X$. Let $\Gamma(X)$ be the simplicial complex

$$\Gamma(X) = \left\{ \mathcal{N} \subset \mathcal{M} : \bigcup_{\tau \in \mathcal{N}} \tau \neq V \right\}.$$ 

By assumption, $A \cup B = V$, and for any missing face $\tau \in \mathcal{M} \setminus \{A, B\}$, $A \cup \tau = V$ and $B \cup \tau = V$. Therefore, both $A$ and $B$ are isolated vertices of the complex $\Gamma(X)$. In particular, $\Gamma(X)$ is disconnected. That is,

$$\tilde{H}_0(\Gamma(X)) \neq 0.$$ 

By Theorem 2.1, we have

$$\tilde{H}_{|V|-3}(X) = \tilde{H}_0(\Gamma(X)) \neq 0.$$ 

Since $|A \cap B| < d$, we have

$$|V| - 3 = |A| + |B| - |A \cap B| - 3 \geq 2(d + 1) - (d - 1) - 3 = d.$$ 

**Proof of Theorem 1.4** Assume we can write $X$ as

$$X = \bigcap_{i=1}^{s} X_i,$$

where, for all $i \in [s]$, $X_i$ is a $d$-Leray complex. For each $i \in [s]$, let $\mathcal{M}_i$ be the set of missing faces of $X_i$. By Proposition 3.1, $\mathcal{M}$ is the set of inclusion minimal elements in $\bigcup_{i=1}^{s} \mathcal{M}_i$. Since all the elements of $\mathcal{M}$ are of size $d + 1$, and all the elements of $\mathcal{M}_i$ are of size at most $d$.
We will show that $Y$ satisfies the conditions of Lemma 4.1: Note that $\tau_1$ and $\tau_2$ are missing faces of $Y$, the vertex set of $Y$ is $\tau_1 \cup \tau_2$, and $|\tau_1| = |\tau_2| = d + 1$. Moreover, since $\mathcal{M}$ is a partial Steiner $(d, d + 1, n)$-system, we have $|\tau_1 \cap \tau_2| < d$. It is left to show that any other missing face $\tau$ of $Y$ (if such a missing face exists) satisfies $\tau \cup \tau_1 = \tau_1 \cup \tau_2$ and $\tau \cup \tau_2 = \tau_1 \cup \tau_2$.

Let $\tau \neq \tau_1, \tau_2$ be a missing face of $Y$. That is, $\tau$ is a missing face of $X_i$ that is contained in $\tau_1 \cup \tau_2$. Let $k = |\tau_1 \cap \tau_2|$, $t = |\tau_1 \cap \tau_2 \cap \tau|$, $t_1 = |\tau \setminus \tau_2|$, and $t_2 = |\tau \setminus \tau_1|$. Since $\tau \in \mathcal{M}_i \subset \mathcal{M}$, we obtain, by the maximality of $|\tau_1 \cap \tau_2|$, 

$$t_1 + t = |\tau \cap \tau_1| \leq k \quad \text{and} \quad t_2 + t = |\tau \cap \tau_2| \leq k.$$ 

We obtain 

$$d + 1 = |\tau| = t_1 + t_2 + t \leq 2k - t.$$ 

That is, 

$$t \leq 2k - d - 1.$$ 

Hence,

$$|\tau \setminus (\tau_1 \cap \tau_2)| = t_1 + t_2 = d + 1 - t \geq d + 1 - 2k + d + 1 = 2(d + 1 - k).$$

So, $\tau \setminus (\tau_1 \cap \tau_2)$ is a subset of size $t_1 + t_2 \geq 2(d + 1 - k)$ of the set $(\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)$. But $|(\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)| = 2(d + 1 - k)$. Therefore, $\tau \setminus (\tau_1 \cap \tau_2) = (\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)$. Hence, we have 

$$\tau \cup \tau_1 = (\tau \setminus (\tau_1 \cap \tau_2)) \cup \tau_1 = ((\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)) \cup \tau_1 = \tau_1 \cup \tau_2,$$
and similarly
\[ \tau \cup \tau_2 = (\tau \setminus (\tau_1 \cap \tau_2)) \cup \tau_2 = ((\tau_1 \cup \tau_2) \setminus (\tau_1 \cap \tau_2)) \cup \tau_2 = \tau_1 \cup \tau_2. \]

So, by Lemma 4.1, \( \tilde{H}_r(Y) \neq 0 \) for some \( r \geq d \). But this is a contradiction to the fact that \( X_i \) is \( d \)-Leray.

Since any \( d \)-representable complex is \( d \)-Leray, we obtain
\[ \text{box}_d(X) \geq |M|. \]

On the other hand, it is easy to show that \( \text{box}_d(X) \leq |M| \): Let \( V \) be the vertex set of \( X \). For each \( \tau \in M \), let \( X_\tau \) be the simplicial complex on vertex set \( V \) whose only missing face is \( \tau \). It is easy to check that the complex \( X_\tau \) is \( d \)-representable (for example, we may assign to each vertex in \( \tau \) one of the facets of a simplex \( P \) in \( \mathbb{R}^d \), and assign to all of the vertices in \( V \setminus \tau \) the simplex \( P \) itself). Since \( X = \bigcap_{\tau \in M} X_\tau \), we obtain \( \text{box}_d(X) \leq |M| \). \( \square \)

5 Upper Bounds on Representability

In this section we prove Theorem 1.5. We will need the following simple lemma:

**Lemma 5.1** Let \( P \subset \mathbb{R}^d \) be a convex polytope. Let \( F_1, \ldots, F_m \) be faces of \( P \), and let \( p_1, \ldots, p_k \) be points in \( P \) such that \( p_i \notin F_j \) for all \( i \in [k] \) and \( j \in [m] \). Then, there exists a convex polytope \( P' \subset P \) such that \( P' \cap F_j = \emptyset \) for all \( j \in [m] \) and \( p_i \in P' \) for all \( i \in [k] \).

**Proof** Let \( P' = \text{conv} \{p_1, \ldots, p_k\} \). Let \( j \in [m] \) and let \( H \) be a hyperplane supporting \( F_j \). That is, \( H \cap P = F_j \), and \( P \) is contained in one of the closed half-spaces \( H^+ \) defined by \( H \). Now, since the points \( p_1, \ldots, p_k \) belong to \( P \setminus F_j \), they must all lie in the interior of \( H^+ \). Therefore, their convex hull \( P' \) is also contained in the interior of \( H^+ \). Since \( F_j \) lies on the boundary \( H \) of \( H^+ \), we have \( P' \cap F_j = \emptyset \), as wanted. \( \square \)

**Theorem 5.2** Let \( X \) be a simplicial complex on vertex set \( V \). Let \( U \subset V \) be such that \( U \notin X \) and for any missing face \( \tau \) of \( X \), \( |\tau \setminus U| \leq 1 \). Then, \( X \) is \( (|U| - 1) \)-representable.

**Proof** Let \( d = |U| - 1 \). Let \( P \) be a simplex in \( \mathbb{R}^d \). Assign to each vertex \( u \in U \) a facet \( F_u \) of \( P \). For \( \sigma \subset U \), let
\[ F_\sigma = \bigcap_{u \in \sigma} F_u \]
(where we understand that \( F_\emptyset = P \)). Note that, unless \( \sigma = U \), \( F_\sigma \) is a non-empty face of the simplex \( P \). For \( \sigma \subset U \), let \( p_\sigma \) be a point in the relative interior of \( F_\sigma \). Then, for any \( \eta \subset U \) and \( \sigma \subset U \), \( p_\sigma \in F_\eta \) if and only if \( \eta \subset \sigma \). Now we build a representation \( \{F'_v\}_{v \in \mathcal{V}} \) of \( X \) in \( \mathbb{R}^d \), as follows.

We divide into two cases:
1. Let $u \in U$. Let $\eta \subset U$ and $\sigma \subset U$ be such that $u \in \sigma \cap \eta$, $\eta \notin X$, and $\sigma \in X$. Note that $F_\eta$ is a face of $F_u$ and $p_\sigma \in F_u$. Also, since $X$ is a simplicial complex, we must have $\eta \not\subset \sigma$, and therefore $p_\sigma \notin F_\eta$. Hence, by Lemma 5.1, there exists a convex polytope $F'_\eta \subset F_u$ such that $F'_\eta \cap F_\eta = \emptyset$ for all $\eta \subset U$ such that $u \in \eta$ and $\eta \notin X$, and $p_\sigma \in F'_\eta$ for all $\sigma \subset U$ such that $u \in \sigma$ and $\sigma \in X$.

2. Let $v \in V \setminus U$. Let $\eta \subset U$ and $\sigma \subset U$ be such that $\eta \cup \{v\} \notin X$ and $\sigma \cup \{v\} \in X$.

Since $X$ is a simplicial complex, we must have $\eta \not\subset \sigma$; hence, $p_\sigma \notin F_\eta$. Therefore, by Lemma 5.1, there exists a convex polytope $F'_v \subset P$ such that $F'_v \cap F_\eta = \emptyset$ for all $\eta \subset U$ such that $\eta \cup \{v\} \notin X$ and $p_\sigma \in F'_v$ for all $\sigma \subset U$ such that $\sigma \cup \{v\} \in X$.

We will show that the family $\{F'_v\}_{v \in V}$ is a representation of $X$.

First, let $\sigma \in X$. Let $\sigma_1 = \sigma \cap U$. Since $\sigma_1 \in X$ and $U \notin X$, we have $\sigma_1 \subset U$. So, for any $u \in \sigma_1$, we have

$$p_{\sigma_1} \in F'_u.$$ 

Moreover, for any $v \in \sigma \setminus \sigma_1$, since $\sigma_1 \cup \{v\} \subset \sigma \in X$, we have

$$p_{\sigma_1} \in F'_v.$$ 

Hence,

$$p_{\sigma_1} \in \bigcap_{v \in \sigma} F'_v.$$ 

In particular, $\bigcap_{v \in \sigma} F'_v \neq \emptyset$.

Now, let $\sigma \subset V$ be such that $\sigma \notin X$. Then, there exists some missing face $\tau$ of $X$ such that $\tau \subset \sigma$. By assumption, we have $|\tau \setminus U| \leq 1$. We divide into two cases:

1. Assume $\tau \subset U$. Then, on the one hand, we have

$$\bigcap_{u \in \tau} F'_u \subset \bigcap_{u \in \tau} F_u = F_\tau.$$ 

On the other hand, for all $u \in \tau$, by the definition of $F'_u$, we have $F'_u \cap F_\tau = \emptyset$. Hence,

$$\bigcap_{u \in \tau} F'_u = \emptyset.$$ 

2. Assume that $|\tau \setminus U| = 1$. Let $w$ be the unique vertex in $\tau \setminus U$. Then

$$\bigcap_{u \in \tau \setminus \{w\}} F'_u \subset \bigcap_{u \in \tau \setminus \{w\}} F_u = F_{\tau \setminus \{w\}}.$$
But, since \((\tau \setminus \{w\}) \cup \{w\} = \tau \notin X\), we obtain, by the definition of \(F'_w\), \(F'_w \cap F_{\tau \setminus \{w\}} = \emptyset\). Hence,

\[
\bigcap_{v \in \tau} F'_v = F'_w \cap \bigcap_{u \in \tau \setminus \{w\}} F'_u \subset F'_w \cap F_{\tau \setminus \{w\}} = \emptyset.
\]

In both cases we obtain \(\bigcap_{v \in \tau} F'_v = \emptyset\), and therefore

\[
\bigcap_{v \in \sigma} F'_v \subset \bigcap_{v \in \tau} F'_v = \emptyset.
\]

So, \(\{F'_v\}_{v \in \sigma}\) is a representation of \(X\) in \(\mathbb{R}^d = \mathbb{R}^{|U| - 1}\), as wanted. \(\square\)

The proof of Theorem 5.2 is based on ideas developed by Wegner in his thesis [12] (as presented in [5,11]). Indeed, we can think of Theorem 5.2 as an extension of the following result of Wegner [12]:

**Theorem 5.3** Let \(X\) be a simplicial complex with \(n\) vertices. Then \(X\) is \((n - 1)\)-representable. Moreover, if \(X\) is not the complete \((n - 2)\)-dimensional skeleton, then it is \((n - 2)\)-representable.

**Proof** If \(X\) is the complete complex, then it is trivially \(0\)-representable. Otherwise, let \(U = V\). Since \(V \notin X\) and \(|\tau \setminus V| = 0 \leq 1\) for any missing face \(\tau\) of \(X\), then by Theorem 5.2, \(X\) is \((n - 1)\)-representable. If \(X\) is not the complete \((n - 2)\)-dimensional skeleton, then there exists some \(U \subset V\) of size \(n - 1\) such that \(U \notin X\). Since \(|V \setminus U| \leq 1\), then \(|\tau \setminus U| \leq 1\) for any missing face \(\tau\) of \(X\). Hence, by Theorem 5.2, \(X\) is \((n - 2)\)-representable. \(\square\)

**Proof of Theorem 1.5** For \(i \in [k]\), let \(\mathcal{M}_i\) be the set consisting of all the missing faces \(\tau\) of \(X\) such that \(|\tau \setminus V_i| \leq 1\). Let

\[
X_i = \{\sigma \subset V : \tau \notin \sigma \text{ for all } \tau \in \mathcal{M}_i\}.
\]

Note that \(X = \bigcap_{i=1}^k X_i\). Indeed, if \(\sigma \in X\), then \(\sigma\) does not contain any missing face of \(X\); in particular, for all \(i \in [k]\), \(\sigma\) does not contain any \(\tau \in \mathcal{M}_i\). Therefore, \(\sigma \in \bigcap_{i=1}^k X_i\). On the other hand, if \(\sigma \notin X\), then \(\tau \subset \sigma\) for some missing face \(\tau\) of \(X\). By the assumption of the theorem, there exists some \(i \in k\) such that \(\tau \in \mathcal{M}_i\). So, \(\sigma \notin X_i\) and therefore \(\sigma \notin \bigcap_{i=1}^k X_i\).

Let \(i \in [k]\). The set of missing faces of \(X_i\) is exactly \(\mathcal{M}_i\). Moreover, since \(V_i \notin X\), there is some missing face \(\tau\) of \(X\) such that \(\tau \subset V_i\). Since \(|\tau \setminus V_i| = 0 \leq 1\), we have \(\tau \in \mathcal{M}_i\); therefore, \(V_i \notin X_i\). So, by Theorem 5.2, \(X_i\) is \((|V_i| - 1)\)-representable. Finally, by Lemma 3.2, \(X\) is \((\sum_{i=1}^k (|V_i| - 1))\)-representable. \(\square\)

**Remark 5.4** In [6, Thm. 1.2], an upper bound similar to the one in Theorem 1.5 is proven for the Leray number of a simplicial complex. Since \(L(X) \leq \text{rep}(X)\) for any complex \(X\), we can see Theorem 1.5 as a generalization of that result.
6 Boxicity of Complexes Without Large Missing Faces

In this section we prove our main result, Theorem 1.3. First, we will need the following simple results about Steiner systems:

Lemma 6.1 Let \( F \subset 2^V \) be a partial \((d, d + 1, n)\)-Steiner system. Then

\[
|F| \leq \left\lfloor \frac{1}{d + 1} \binom{n}{d} \right\rfloor.
\]  

(6.1)

Moreover, if \(|F| = \frac{\binom{n}{d}}{d + 1}\), then \(F\) is a Steiner \((d, d + 1, n)\)-system.

Proof Since \(F\) is a partial Steiner \((d, d + 1, n)\)-system, then any subset of \(V\) of size \(d\) is contained in at most one element of \(F\). On the other hand, since each \(\sigma \in F\) contains exactly \(d + 1\) subsets of size \(d\), we obtain

\[
(d + 1)|F| \leq \binom{n}{d}.
\]

(6.2)

Therefore (6.1) holds. Now, assume that \(|F| = \frac{\binom{n}{d}}{d + 1}\). Then, equality must hold in (6.2). Thus, each subset of \(V\) of size \(d\) must be contained in exactly one set of \(F\). That is, \(F\) is a Steiner \((d, d + 1, n)\)-system. \(\square\)

Lemma 6.2 Let \(F \subset 2^V\) be a \((d, d + 1, n)\)-Steiner system. Let \(\tau \subset V\) be a set of size at most \(d + 1\) that is not contained in any set of \(F\). Then

\[
|\{\sigma \in F : |\tau \setminus \sigma| = 1\}| \geq d + 1.
\]

Proof Since \(F\) forms a Steiner \((d, d + 1, n)\)-system, then any set of size at most \(d\) is contained in at least one set of \(F\). Therefore, we must have \(|\tau| = d + 1\). Now, let \(\tau_1, \ldots, \tau_{d+1}\) be the subsets of \(\tau\) of size \(d\). Again, since \(F\) is a Steiner system, there exist \(\sigma_1, \ldots, \sigma_{d+1} \in F\) such that \(\tau_i \subset \sigma_i\) for all \(i \in [d + 1]\).

Since \(\tau\) is the only set of size \(d + 1\) containing two or more of the sets \(\tau_1, \ldots, \tau_{d+1}\), but \(\tau \notin F\), we must have \(\sigma_i \neq \sigma_j\) for all \(i \neq j\). Thus,

\[
|\{\sigma \in F : |\tau \setminus \sigma| = 1\}| \geq |\{\sigma_1, \ldots, \sigma_{d+1}\}| = d + 1.
\]

(\(\square\))

The last ingredient needed for the proof of Theorem 1.3 is the following result:

Proposition 6.3 Let \(X\) be a simplicial complex on vertex set \(V\) of size \(n\), satisfying \(h(X) \leq d\). Let \(t\) be the minimum size of a family \(\{\sigma_1, \ldots, \sigma_t\}\) of subsets of size \(d + 1\) of \(V\) satisfying \(\sigma_i \notin X\) for all \(i \in [t]\), such that for any missing face \(\tau\) of \(X\), there exists some \(i \in [t]\) such that \(|\tau \setminus \sigma_i| \leq 1\). Then

\[
t \leq \left\lfloor \frac{1}{d + 1} \binom{n}{d} \right\rfloor.
\]
Moreover, if \( h(X) = d \geq 2 \), then \( t = \binom{n}{d}/(d + 1) \) if and only if the set of missing faces of \( X \) forms a Steiner \((d, d + 1, n)\)-system.

**Proof** Let \( \mathcal{M} \) be the collection of all subsets of \( V \) of size \( d + 1 \) that are not simplices of \( X \). Let \( A \subset \mathcal{M} \) be a maximal (with respect to inclusion) partial Steiner \((d, d + 1, n)\)-system. By Lemma 6.1, we have

\[
|A| \leq \left\lfloor \frac{1}{d + 1} \binom{n}{d} \right\rfloor.
\]

We will show that for any missing face \( \tau \) of \( X \), there exists some \( \sigma \in A \) such that \( |\tau \setminus \sigma| \leq 1 \). Assume for contradiction that there exists some missing face \( \tau \) of \( X \) such that \( |\tau \setminus \sigma| > 1 \) for all \( \sigma \in A \). Let \( \sigma_0 \) be some set in \( \mathcal{M} \) containing \( \tau \). Then \( |\sigma_0 \setminus \sigma| \geq |\tau \setminus \sigma| > 1 \) for all \( \sigma \in A \). Let \( A' = A \cup \{\sigma_0\} \). Let \( \eta \subset V \) be a set of size \( d \). If \( \eta \not\subset \sigma_0 \), then, since \( A \) is a partial Steiner \((d, d + 1, n)\)-system, \( \eta \) is contained in at most one set in \( A' \). If \( \eta \subset \sigma_0 \), then assume for contradiction that \( \eta \subset \sigma \) for some \( \sigma \in A \). Since \( |\sigma_0 \setminus \sigma| > 1 \), we have \( |\sigma_0 \cap \sigma| \leq d - 1 \). But this is a contradiction to the fact that \( \eta \) is a set of size \( d \) contained in \( \sigma_0 \cap \sigma \). So, \( \eta \) is not contained in any set of \( A \).

In both cases, \( \eta \) is contained in at most one set of \( A' \). Therefore, \( A' \subset \mathcal{M} \) is a partial Steiner \((d, d + 1, n)\)-system. But this is a contradiction to the maximality of \( A \).

Therefore, for any missing face \( \tau \) of \( X \) there exists some \( \sigma \in A \) such that \( |\tau \setminus \sigma| \leq 1 \). Hence,

\[
t \leq |A| \leq \left\lfloor \frac{1}{d + 1} \binom{n}{d} \right\rfloor.
\]

Now, assume \( t = \binom{n}{d}/(d + 1) \). Then, we must have \( |A| = t = \binom{n}{d}/(d + 1) \). By Lemma 6.1, \( A \) is a Steiner \((d, d + 1, n)\)-system.

Assume that \( h(X) = d \geq 2 \). We will show that \( A \) is exactly the set of missing faces of \( X \). We may assume that \( n \geq d + 2 \). Otherwise, since \( h(X) = d \), \( X \) must contain a unique missing face of size \( d + 1 \) (that is, \( X \) is a Steiner \((d, d + 1, d + 1)\)-system).

First, we will show that \( A = \mathcal{M} \). Assume for contradiction that there exists some \( \tilde{\tau} \in \mathcal{M} \setminus A \). By Lemma 6.2, there exist \( \sigma_1, \sigma_2 \in A \) such that \( |\tilde{\tau} \setminus \sigma_1| = |\tilde{\tau} \setminus \sigma_2| = 1 \). Since \( |\tilde{\tau}| = d + 1 \), we also have \( |\sigma_1 \setminus \tilde{\tau}| = |\sigma_2 \setminus \tilde{\tau}| = 1 \). Let

\[
A' = A \cup \{\tilde{\tau}\} \setminus \{\sigma_1, \sigma_2\}.
\]

Let \( \tau \) be a missing face of \( X \). We will show that there exists some \( \sigma \in A' \) such that \( |\tau \setminus \sigma| \leq 1 \). We divide the reasoning into the following cases:

1. If \( \tau \) is not contained in any set of \( A \), then, by Lemma 6.2, we have

\[
|\{\sigma \in A' : |\tau \setminus \sigma| = 1\}| \geq |\{\sigma \in A : |\tau \setminus \sigma| = 1\}| - 2 \\
\geq d + 1 - 2 = d - 1 \geq 1.
\]

Therefore, there exists some \( \sigma \in A' \) such that \( |\tau \setminus \sigma| = 1 \).
2. If \( \tau \) is contained in some \( \sigma \in A \setminus \{ \sigma_1, \sigma_2 \} \subset A', \) then \( |\tau \setminus \sigma| = 0 \leq 1. \)
3. If \( \tau \) is contained in \( \sigma_i \) for some \( i \in \{1, 2\} \), then
   \[ |\tau \setminus \tilde{\tau}| = |\sigma_i \setminus \tilde{\tau}| = 1. \]

Since \( |A'| = t - 1 \), this is a contradiction to the minimality of \( t \). Hence, we must have \( A = M \).

Finally, assume for contradiction that there exists some missing face \( \tau \) of \( X \) of size \( |\tau| \leq d \). Let \( \eta \) be a set of size \( d \) containing \( \tau \). Then, since we assumed \( n \geq d + 2 \), we have
\[ |\{ \sigma \subset V : |\sigma| = d + 1, \ \eta \subset \sigma \}| = n - d \geq 2. \]

Note that any \( \sigma \subset V \) such that \( |\sigma| = d + 1 \) and \( \eta \subset \sigma \), is not a simplex of \( X \) (since it contains the missing face \( \tau \)), and therefore belongs to \( M = A \). Hence, \( \eta \) is contained in at least two sets of \( A \), a contradiction to \( A \) being a Steiner \((d, d + 1, n)\)-system. Thus, the set of missing faces of \( X \) is exactly \( A \). \( \square \)

**Proof of Theorem 1.3** Let \( \{V_1, \ldots, V_t\} \) be a family of minimum size of subsets of size \( d + 1 \) of \( V \) such that \( V_i \not\in X \) for all \( i \in [t] \), and such that for any missing face \( \tau \) of \( X \), there exists some \( i \in [t] \) satisfying \( |\tau \setminus V_i| \leq 1 \). By Theorem 1.5, we have \( \text{box}_d(X) \leq t \). So, by Proposition 6.3, we obtain
\[ \text{box}_d(X) \leq t \leq \left\lfloor \frac{1}{d + 1} \binom{n}{d} \right\rfloor. \]

Now, assume that \( h(X) = d \), and the set of missing faces of \( X \) does not form a Steiner \((d, d + 1, n)\)-system. If \( d = 1 \), then it is proven in [14, Thm. 1] that \( \text{box}_1(X) < n/2 \). If \( d \geq 2 \) then, by Proposition 6.3, we have
\[ t < \frac{1}{d + 1} \binom{n}{d}, \]
and therefore
\[ \text{box}_d(X) \leq t < \frac{1}{d + 1} \binom{n}{d}. \]

Finally, assume that the missing faces of \( X \) form a Steiner \((d, d + 1, n)\)-system \( M \). Then, by Theorem 1.4, we have
\[ \text{box}_d(X) = |M| = \frac{1}{d + 1} \binom{n}{d}, \]
as wanted. \( \square \)

**Remark 6.4** In the case \( d = 1 \), the proof of the upper bound in Theorem 1.3 reduces to the proof of Theorem 1.1 presented by Cozzens and Roberts in [3, Cor. 3.7].
7 Concluding Remarks

Let $X$ be a simplicial complex. By Lemma 3.2, we have for any $d \geq 1$,

$$\text{rep}(X) \leq d \cdot \text{box}_d(X).$$

In particular, for $d = 1$, we obtain as a corollary of Theorem 1.1:

**Proposition 7.1** Let $G$ be a graph with $n$ vertices, and let $X(G)$ be its clique complex. Then,

$$\text{rep}(X(G)) \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Moreover, $\text{rep}(X(G)) = n/2$ if and only if $G$ is the complete $(n/2)$-partite graph with all sides of size 2.

The fact that $\text{rep}(X(G)) = n/2$ if $G$ is the complete $(n/2)$-partite graph with sides of size 2 does not follow directly from Theorem 1.1. However, it is easy to check that in this case $X(G)$ is the boundary of the $(n/2)$-dimensional cross-polytope; in particular, it has non-trivial $(n/2 - 1)$-dimensional homology group. Thus, $X(G)$ is not $(n/2 - 1)$-Leray, and therefore is not $(n/2 - 1)$-representable. We conjecture that for $d \geq 1$, the following extension of Proposition 7.1 holds:

**Conjecture 7.2** Let $X$ be a simplicial complex with $n$ vertices satisfying $h(X) \leq d$. Then,

$$\text{rep}(X) \leq \left\lfloor \frac{d \cdot n}{d + 1} \right\rfloor.$$

Moreover, $\text{rep}(X) = d \cdot n/(d + 1)$ if and only if the missing faces of $X$ consist of $n/(d + 1)$ pairwise disjoint sets of size $d + 1$.

Analogous bounds are known to hold for Leray numbers (see [1, Prop. 5.4]) and for collapsibility numbers (a combinatorial parameter that is bounded from above by the representability of the complex, and bounded from below by its Leray number; see [8, Prop. 3.5]). Conjecture 7.2, if true, would imply both of these results.

The results presented in this paper do not seem suitable for dealing with Conjecture 7.2. One of the simplest examples where our methods fail is the complex $X_{2,7}$, the complex whose set of missing faces forms a Steiner $(2, 3, 7)$-system (usually referred to as the Fano plane). Since any two vertices in $X_{2,7}$ are contained in a missing face, the best bound we can obtain from an application of Theorem 5.2 is $\text{rep}(X_{2,7}) \leq 5$, which is larger than the conjectured bound $\left\lfloor 2 \cdot 7/3 \right\rfloor = 4$. This bound can be proven, however, by the following simple method:

**Lemma 7.3** Let $X$ be a $d$-representable simplicial complex on vertex set $V$. Let $\sigma_1, \sigma_2 \subset V$ be such that $\sigma_1 \cap \sigma_2 \in X$. Then, the complex $X' = X \cup 2^{\sigma_1} \cup 2^{\sigma_2}$ is $(d + 1)$-representable.
Lemma 7.3 gives non-trivial bounds only for complexes with a small number of maximal faces, so it seems unlikely that such a method will be useful in more general cases.

Proof Let $e_1, \ldots, e_{d+1}$ be the standard basis for $\mathbb{R}^{d+1}$. We identify $\mathbb{R}^d$ with the hyperplane $H = \{ x \in \mathbb{R}^{d+1} : x \cdot e_{d+1} = 0 \}$ in $\mathbb{R}^{d+1}$.

Let $P = \{ P_v \}_{v \in V}$ be a representation of $X$ in $\mathbb{R}^d$. Let $x \in \bigcap_{v \in \sigma_1 \cap \sigma_2} P_v \subset H$ (note that $\bigcap_{v \in \sigma_1 \cap \sigma_2} P_v \neq \emptyset$ since $\sigma_1 \cap \sigma_2 \in X$ and $P$ is a representation of $X$). Let $x_1 = x + e_{d+1}$ and $x_2 = x - e_{d+1}$. For $v \in V$, we define

$$ P'_v = \begin{cases} \text{conv} (P_v \cup \{x_1\} \cup \{x_2\}) & \text{if } v \in \sigma_1 \cap \sigma_2, \\ \text{conv} (P_v \cup \{x_1\}) & \text{if } v \in \sigma_1 \setminus \sigma_2, \\ \text{conv} (P_v \cup \{x_2\}) & \text{if } v \in \sigma_2 \setminus \sigma_1, \\ P_v & \text{if } v \notin \sigma_1 \cup \sigma_2. \end{cases} $$

It is left to the reader to check that $\{ P'_v \}_{v \in V}$ is indeed a representation of $X'$. \hfill \Box

Proposition 7.4

$$ \text{rep}(X_{2,7}) \leq 4. $$

Proof We identify the vertex set of $X_{2,7}$ with the set $[7] = \{1, 2, \ldots, 7\}$. Then, the set of missing faces of $X_{2,7}$ is

$$ M = \{ \{1, 2, 3\}, \{1, 4, 5\}, \{1, 6, 7\}, \{2, 4, 7\}, \{3, 4, 6\}, \{2, 5, 6\}, \{3, 5, 7\} \}. $$

It is easy to check that the set of maximal faces of $X_{2,7}$ is the set whose elements are the complements of the sets in $M$:

$$ \{4, 5, 6, 7\}, \{2, 3, 6, 7\}, \{2, 3, 4, 5\}, \{1, 3, 5, 6\}, \{1, 2, 5, 7\}, \{1, 3, 4, 7\}, \{1, 2, 4, 6\}. $$

Let $X_0$ be the complex on vertex set $[7]$ whose set of maximal faces is

$$ \{ \{1, 2, 4\}, \{2, 3, 4, 5\}, \{4, 5, 6, 7\} \}. $$

It can be checked that the following is a representation of $X_0$ in $\mathbb{R}^1$:

$$ P_1 = [0, 1], \quad P_2 = [1, 2], \quad P_3 = [2, 3], \quad P_4 = [0, 5], \quad P_5 = [2, 5], \quad P_6 = P_7 = [4, 5]. $$

Let $X_1 = X_0 \cup 2^{\{1, 2, 5, 7\}} \cup 2^{\{1, 2, 4, 6\}}$. Since $\{1, 2, 5, 7\} \cap \{1, 2, 4, 6\} = \{1, 2\} \in X_0$ then, by Lemma 7.3, $X_1$ is 2-representable. Let $X_2 = X_1 \cup 2^{\{1, 3\}} \cup 2^{\{2, 3, 6, 7\}}$. Since $\{1, 3\} \cap \{2, 3, 6, 7\} = \{3\} \in X_1$ then, by Lemma 7.3, $X_2$ is 3-representable. Finally, let $X_3 = X_2 \cup 2^{\{1, 3, 5, 6\}} \cup 2^{\{1, 3, 4, 7\}}$. Since $\{1, 3, 5, 6\} \cap \{1, 3, 4, 7\} = \{1, 3\} \in X_2$ then, by Lemma 7.3, $X_3$ is 4-representable. But it is easy to check that $X_3$ is in fact the complex $X_{2,7}$. \hfill \Box

Lemma 7.3 gives non-trivial bounds only for complexes with a small number of maximal faces, so it seems unlikely that such a method will be useful in more general cases of the problem.
We conclude with the following problem, whose solution may be a (very modest) step towards Conjecture 7.2:

**Conjecture 7.5** Let \( X_{2,9} \) be the simplicial complex whose missing faces form a Steiner \((2, 3, 9)\)-system (that is, they are the lines of the affine plane of order 3). Then,

\[
\text{rep}(X_{2,9}) \leq 5.
\]

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