On a class of critical $N$-Laplacian problems

Tsz Chung Ho and Kanishka Perera
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne, FL 32901, USA
tho2011@my.fit.edu & kperera@fit.edu

Abstract

We establish some existence results for a class of critical $N$-Laplacian problems in a bounded domain in $\mathbb{R}^N$. In the absence of a suitable direct sum decomposition of the underlying Sobolev space to which the classical linking theorem can be applied, we use an abstract linking theorem based on the $\mathbb{Z}_2$-cohomological index to obtain a nontrivial critical point.

1 Introduction

In this paper we establish some existence results for the class of critical $N$-Laplacian problems

\[
\begin{aligned}
-\Delta_N u &= h(u) e^\alpha |u|^{N'} \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

(1.1)

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $\alpha > 0$, $N' = N/(N-1)$ is the Hölder conjugate of $N$, and $h$ is a continuous function such that

\[
\lim_{|t| \to \infty} h(t) = 0
\]

(1.2)

and

\[
0 < \beta := \liminf_{|t| \to \infty} th(t) < \infty.
\]

(1.3)

$^*\text{MSC2010:}$ Primary 35J92, Secondary 35B33, 35B38

$^{Key \ Words \ and \ Phrases:}$ critical $N$-Laplacian problems, existence, critical points, linking, $\mathbb{Z}_2$-cohomological index
This problem is motivated by the Trudinger-Moser inequality

$$\sup_{u \in W^{1,N}_0(\Omega), \|u\| \leq 1} \int_{\Omega} e^{\alpha_N |u|^N} \, dx < \infty,$$

(1.4)

where $W^{1,N}_0(\Omega)$ is the usual Sobolev space with the norm

$$\|u\| = \left( \int_{\Omega} |\nabla u|^N \, dx \right)^{1/N},$$

and

$$\alpha_N = N \omega_{N-1}^{1/(N-1)},$$

and

$$\omega_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)}$$

is the area of the unit sphere in $\mathbb{R}^N$ (see Trudinger [14] and Moser [10]). Problem (1.1) is critical with respect to this inequality and hence lacks compactness. Indeed, the associated variational functional satisfies the Palais-Smale compactness condition only at energy levels below a certain threshold (see Proposition 2.1 in the next section).

In dimension $N = 2$, problem (1.1) is semilinear and has been extensively studied in the literature (see, e.g., [2, 3, 4, 6]). In dimensions $N \geq 3$, this problem is quasilinear and has been studied mainly when

$$G(t) := \int_0^t h(s) e^{\alpha |s|^N} \, ds \leq \lambda |t|^N$$

for some $\lambda \in (0, \lambda_1)$ (see, e.g., [1, 5, 9]). Here

$$\lambda_1 = \inf_{u \in W^{1,N}_0(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^N \, dx}{\int_{\Omega} |u|^N \, dx}$$

(1.6)

is the first eigenvalue of the eigenvalue problem

$$\begin{cases}
-\Delta_N u = \lambda |u|^{N-2} u & \text{in } \Omega \\
\quad u = 0 & \text{on } \partial\Omega.
\end{cases}$$

(1.7)

The case $h(t) = \lambda |t|^{N-2} t$ with $\lambda > 0$, for which $\beta = \infty$, was recently studied in Yang and Perera [13]. The remaining case, where $N \geq 3$, $\lambda \geq \lambda_1$, and $\beta < \infty$, does not seem to have been studied in the literature. This case is covered in our results here, which are for large $\beta < \infty$ and allow $N \geq 3$ and $\lambda \geq \lambda_1$ in (1.5).

Let $d$ be the radius of the largest open ball contained in $\Omega$. Our first result is the following theorem.
Theorem 1.1. Assume that $\alpha > 0$, $h$ satisfies (1.2) and (1.3), and $G$ satisfies

\[
G(t) \geq -\frac{1}{N} \sigma_0 |t|^N \quad \text{for } t \geq 0, \tag{1.8}
\]

\[
G(t) \leq \frac{1}{N} (\lambda_1 - \sigma_1) |t|^N \quad \text{for } |t| \leq \delta \tag{1.9}
\]

for some $\sigma_0 \geq 0$ and $\sigma_1, \delta > 0$. If

\[
\beta > \frac{1}{N\alpha^{N-1}} \left( \frac{N}{d} \right)^N e^{\sigma_0/(N-1) \kappa}, \tag{1.10}
\]

where $\kappa = \frac{1}{N!} \left( \frac{N}{d} \right)^N$, then problem (1.1) has a nontrivial solution.

In particular, we have the following corollary for $\sigma_0 = 0$.

Corollary 1.2. Assume that $\alpha > 0$, $h$ satisfies (1.2) and (1.3), and $G$ satisfies

\[
G(t) \geq 0 \quad \text{for } t \geq 0, \tag{1.12}
\]

\[
G(t) \leq \frac{1}{N} (\lambda_1 - \sigma_1) |t|^N \quad \text{for } |t| \leq \delta
\]

for some $\sigma_1, \delta > 0$. If

\[
\beta > \frac{1}{N\alpha^{N-1}} \left( \frac{N}{d} \right)^N, \tag{1.13}
\]

then problem (1.1) has a nontrivial solution.

Corollary 1.2 should be compared with Theorem 1 of do Ó [9], where this result is proved under the stronger assumption $h(t) \geq 0$ for $t \geq 0$.

To state our second result, let $(\lambda_k)$ be the sequence of eigenvalues of problem (1.7) based on the $\mathbb{Z}_2$-cohomological index that was introduced in Perera [11] (see Proposition 2.3 in the next section). We have the following theorem.

Theorem 1.3. Assume that $\alpha > 0$, $h$ satisfies (1.2) and (1.3), and $G$ satisfies

\[
G(t) \geq \frac{1}{N} (\lambda_{k-1} + \sigma_0) |t|^N \quad \forall t, \tag{1.11}
\]

\[
G(t) \leq \frac{1}{N} (\lambda_k - \sigma_1) |t|^N \quad \text{for } |t| \leq \delta \tag{1.12}
\]
for some $k \geq 2$ and $\sigma_0, \sigma_1, \delta > 0$. Then there exists a constant $c > 0$ depending on $\Omega$, $\alpha$, and $k$, but not on $\sigma_0$, $\sigma_1$, or $\delta$, such that if

$$\beta > \frac{1}{\alpha^{N-1}} \left( \frac{N}{d} \right)^N e^{c/\sigma_0^{N-1}},$$

then problem $\text{(1.1)}$ has a nontrivial solution.

Theorem 1.3 should be compared with Theorem 1.4 of de Figueiredo et al. \[3, 4\], where this result is proved in the case $N = 2$ under the additional assumption that $0 < 2G(t) \leq th(t) e^{at^2}$ for all $t \in \mathbb{R} \setminus \{0\}$. However, the linking argument used in \[3, 4\] is based on a splitting of $H^1_0(\Omega)$ that involves the eigenspaces of the Laplacian, and this argument does not extend to the case $N \geq 3$ where the $N$-Laplacian is a nonlinear operator and therefore has no linear eigenspaces. We will prove Theorem 1.3 using an abstract critical point theorem based on the $\mathbb{Z}_2$-cohomological index that was proved in Yang and Perera \[15\] (see Section 2.4).

In the proofs of Theorems 1.1 and 1.3, the inner radius $d$ of $\Omega$ comes into play when verifying that certain minimax levels are below the compactness threshold given in Proposition 2.1.

2 Preliminaries

2.1 A compactness result

Weak solutions of problem $\text{(1.1)}$ coincide with critical points of the $C^1$-functional

$$E(u) = \frac{1}{N} \int_\Omega |\nabla u|^N dx - \int_\Omega G(u) dx, \quad u \in W^{1,N}_0(\Omega).$$

We recall that a (PS)$_c$ sequence of $E$ is a sequence $(u_j) \subset W^{1,N}_0(\Omega)$ such that $E(u_j) \to c$ and $E'(u_j) \to 0$. Proofs of Theorem 1.1 and Theorem 1.3 will be based on the following compactness result.

**Proposition 2.1.** Assume that $\alpha > 0$ and $h$ satisfies $\text{(1.2)}$ and $\text{(1.3)}$. Then for all $c \neq 0$ satisfying

$$c < \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1},$$

every (PS)$_c$ sequence of $E$ has a subsequence that converges weakly to a nontrivial solution of problem $\text{(1.1)}$.

**Proof.** Let $(u_j) \subset W^{1,N}_0(\Omega)$ be a (PS)$_c$ sequence of $E$. Then

$$E(u_j) = \frac{1}{N} \|u_j\|^N - \int_\Omega G(u_j) dx = c + o(1) \quad (2.1)$$
and
\[ E'(u_j) u_j = \|u_j\|^{N} - \int_{\Omega} u_j h(u_j) e^{\alpha |u_j|^N'} dx = o(\|u_j\|). \quad (2.2) \]

First we show that \((u_j)\) is bounded in \(W^{1,N}_0(\Omega)\). Multiplying \((2.1)\) by \(2\) and subtracting \((2.2)\) gives
\[ \|u_j\|^N + \int_{\Omega} \left( u_j h(u_j) e^{\alpha |u_j|^N'} - 2NG(u_j) \right) dx = 2Nc + o(\|u_j\| + 1), \]
so it suffices to show that \(th(t) e^{\alpha |t|^{N'}} - 2NG(t)\) is bounded from below. Let \(0 < \varepsilon < \beta/(2N + 1)\). By \((1.2)\) and \((1.3)\), for some constant \(C_\varepsilon > 0,\)
\[ |G(t)| \leq \varepsilon e^{\alpha |u_j|^N'} + C_\varepsilon \quad (2.3) \]
and
\[ th(t) e^{\alpha |t|^{N'}} \geq (\beta - \varepsilon) e^{\alpha |t|^{N'}} - C_\varepsilon \quad (2.4) \]
for all \(t\). So
\[ th(t) e^{\alpha |t|^{N'}} - 2NG(t) \geq [\beta - (2N + 1) \varepsilon] e^{\alpha |t|^{N'}} - (2N + 1) C_\varepsilon, \]
which is bounded from below.

Since \((u_j)\) is bounded in \(W^{1,N}_0(\Omega)\), a renamed subsequence converges to some \(u\) weakly in \(W^{1,N}_0(\Omega)\), strongly in \(L^p(\Omega)\) for all \(p \in [1, \infty)\), and a.e. in \(\Omega\). We have
\[ E'(u_j) v = \int_{\Omega} |\nabla u_j|^{N-2} \nabla u_j \cdot \nabla v dx - \int_{\Omega} v h(u_j) e^{\alpha |u_j|^N'} dx \to 0 \quad (2.5) \]
for all \(v \in W^{1,N}_0(\Omega)\). By \((1.2)\), given any \(\varepsilon > 0\), there exists a constant \(C_\varepsilon > 0\) such that
\[ |h(t) e^{\alpha |t|^{N'}}| \leq \varepsilon e^{\alpha |t|^{N'}} + C_\varepsilon \quad \forall t. \quad (2.6) \]
By \((2.2)\),
\[ \sup_j \int_{\Omega} u_j h(u_j) e^{\alpha |u_j|^N'} dx < \infty, \]
which together with \((2.4)\) gives
\[ \sup_j \int_{\Omega} e^{\alpha |u_j|^N'} dx < \infty. \quad (2.7) \]

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For \( v \in C_0^\infty(\Omega) \), it follows from (2.6) and (2.7) that the sequence \( (v h(u_j) e^{\alpha |u_j|^N'}) \) is uniformly integrable and hence

\[
\int_\Omega v h(u_j) e^{\alpha |u_j|^N'} dx \to \int_\Omega v h(u) e^{\alpha |u|^N'} dx
\]

by Vitali’s convergence theorem, so it follows from (2.5) that

\[
\int_\Omega |\nabla u|^{N-2} \nabla u \cdot \nabla v dx - \int_\Omega v h(u) e^{\alpha |u|^N'} dx = 0.
\]

Then this holds for all \( v \in W_0^{1,N}(\Omega) \) by density, so the weak limit \( u \) is a solution of problem \((1.1)\).

Suppose that \( u = 0 \). Then

\[
\int_\Omega G(u_j) dx \to 0
\]

since (2.3) and (2.7) imply that the sequence \( (G(u_j)) \) is uniformly integrable, so (2.1) gives

\[
\|u_j\| \to (Nc)^{1/N}.
\]

Let \( Nc < \nu < (\alpha_N/\alpha)^{N-1} \). Then \( \|u_j\| \leq \nu^{1/N} \) for all \( j \geq j_0 \) for some \( j_0 \). Let \( q = \alpha_N/\alpha \nu^{1/(N-1)} > 1 \). By the Hölder inequality,

\[
\left( \int_\Omega u_j h(u_j) e^{\alpha |u_j|^N'} dx \right)^{1/p} \leq \left( \int_\Omega |u_j h(u_j)|^p dx \right)^{1/p} \left( \int_\Omega e^{q|u_j|^N'} dx \right)^{1/q},
\]

where \( 1/p + 1/q = 1 \). The first integral on the right-hand side converges to zero since \( h \) is bounded and \( u_j \to 0 \) in \( L^p(\Omega) \), and the second integral is bounded by (1.4) since

\[
q\alpha |u_j|^N = \alpha_N |\tilde{u}_j|^N,
\]

where \( \tilde{u}_j = u_j/\nu^{1/N} \) satisfies \( \|\tilde{u}_j\| \leq 1 \) for \( j \geq j_0 \), so

\[
\int_\Omega u_j h(u_j) e^{\alpha |u_j|^N'} dx \to 0.
\]

Then \( u_j \to 0 \) by (2.2) and hence \( c = 0 \) by (2.8), contrary to assumption. So \( u \) is a nontrivial solution.

2.2 \( \mathbb{Z}_2 \)-cohomological index

The \( \mathbb{Z}_2 \)-cohomological index of Fadell and Rabinowitz \[8\] is defined as follows. Let \( W \) be a Banach space and let \( \mathcal{A} \) denote the class of symmetric subsets of \( W \setminus \{0\} \). For \( A \in \mathcal{A} \), let \( \overline{A} = A/\mathbb{Z}_2 \) be the quotient space of \( A \) with each \( u \) and \( -u \) identified, let \( f : \overline{A} \to \mathbb{R}P^\infty \) be
the classifying map of $\mathcal{A}$, and let $f^* : H^*(\mathbb{RP}^\infty) \to H^*(\mathcal{A})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of $A$ is defined by

$$i(A) = \begin{cases} \sup \{ m \geq 1 : f^*(\omega^{m-1}) \neq 0 \}, & A \neq \emptyset \\ 0, & A = \emptyset, \end{cases}$$

where $\omega \in H^1(\mathbb{RP}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[\omega]$. For example, the classifying map of the unit sphere $S^{m-1}$ in $\mathbb{R}^m$, $m \geq 1$ is the inclusion $\mathbb{RP}^{m-1} \subset \mathbb{RP}^\infty$, which induces isomorphisms on $H^q$ for $q \leq m-1$, so $i(S^{m-1}) = m$.

The following proposition summarizes the basic properties of the cohomological index (see Fadell and Rabinowitz [8]).

**Proposition 2.2.** The index $i : A \to \mathbb{N} \cup \{0, \infty\}$ has the following properties:

(i) **Definiteness:** $i(A) = 0$ if and only if $A = \emptyset$.

(ii) **Monotonicity:** If there is an odd continuous map from $A$ to $B$ (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism.

(iii) **Dimension:** $i(A) \leq \dim W$.

(iv) **Continuity:** If $A$ is closed, then there is a closed neighborhood $N \in A$ of $A$ such that $i(N) = i(A)$. When $A$ is compact, $N$ may be chosen to be a $\delta$-neighborhood $N_\delta(A) = \{ u \in W : \text{dist}(u, A) \leq \delta \}$.

(v) **Subadditivity:** If $A$ and $B$ are closed, then $i(A \cup B) \leq i(A) + i(B)$.

(vi) **Stability:** If $SA$ is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, then $i(SA) = i(A) + 1$.

(vii) **Piercing property:** If $A$, $A_0$ and $A_1$ are closed, and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous map such that $\varphi(-u, t) = -\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$ and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$.

(viii) **Neighborhood of zero:** If $U$ is a bounded closed symmetric neighborhood of $0$, then $i(\partial U) = \dim W$.

### 2.3 Eigenvalues

Eigenvalues of problem (1.7) coincide with critical values of the functional

$$\Psi(u) = \frac{1}{\int_{\Omega} |u|^N \, dx}, \quad u \in S = \left\{ u \in W_0^{1,N}(\Omega) : \int_{\Omega} |\nabla u|^N \, dx = 1 \right\}.$$  

We have the following proposition (see Perera [11] and Perera et al. [12, Proposition 3.52 and Proposition 3.53]).
Proposition 2.3. Let $F$ denote the class of symmetric subsets of $S$ and set

$$
\lambda_k := \inf_{M \in F} \sup_{u \in M} \Psi(u), \quad k \in \mathbb{N}.
$$

Then $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty$ is a sequence of eigenvalues of problem (1.7).

Moreover, if $\lambda_{k-1} < \lambda_k$, then

$$
i(\Psi^{\lambda_{k-1}}) = i(S \setminus \Psi_{\lambda_k}) = k - 1,
$$

where $\Psi^a = \{u \in S : \Psi(u) \leq a\}$ and $\Psi_a = \{u \in S : \Psi(u) \geq a\}$ for $a \in \mathbb{R}$.

We will also need the following result of Degiovanni and Lancelotti ([7, Theorem 2.3]).

Proposition 2.4. If $\lambda_{k-1} < \lambda_k$, then $\Psi^{\lambda_{k-1}}$ contains a compact symmetric set $C$ of index $k - 1$ that is bounded in $C^1(S)$.

2.4 An abstract critical point theorem

We will use the following abstract critical point theorem proved in Yang and Perera [15, Theorem 2.2] to prove Theorem 1.3. This result generalizes the linking theorem of Rabinowitz [13].

Theorem 2.5. Let $E$ be a $C^1$-functional defined on a Banach space $W$ and let $A_0$ and $B_0$ be disjoint nonempty closed symmetric subsets of the unit sphere $S = \{u \in W : \|u\| = 1\}$ such that

$$
i(A_0) = i(S \setminus B_0) < \infty. \quad (2.9)
$$

Assume that there exist $R > \rho > 0$ and $\omega \in S \setminus A_0$ such that

$$
\sup E(A) \leq \inf E(B), \quad \sup E(X) < \infty,
$$

where

$$
A = \{sv : v \in A_0, 0 \leq s \leq R\} \cup \{R\pi((1 - t)v + t\omega) : v \in A_0, 0 \leq t \leq 1\},
$$

$$
B = \{\rho u : u \in B_0\},
$$

$$
X = \{sv + tw : v \in A_0, s, t \geq 0, \|sv + tw\| \leq R\},
$$

and $\pi : W \setminus \{0\} \to S, u \mapsto u/\|u\|$ is the radial projection onto $S$. Let

$$
\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = \text{id}_A\},
$$

and set

$$
c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).
$$

Then $\inf E(B) \leq c \leq \sup E(X)$, and $E$ has a (PS)$_c$ sequence.
2.5 Moser sequence

For \( j \geq 2 \), let

\[
\omega_j(x) = \begin{cases} 
\frac{(\log j)^{(N-1)/N}}{N}, & |x| \leq d/j \\
\frac{\log (d/|x|)}{(\log j)^{1/N}}, & d/j < |x| < d \\
0, & |x| \geq d 
\end{cases}
\]

(2.10)

(see Moser [10]).

**Proposition 2.6.** We have

\[
\int_{\Omega} \omega_j^m \ dx = \frac{m! \omega_{N-1}^{1-m/N} d^N}{N^{m+1} (\log j)^{m/N}} \left[ 1 - \frac{1}{j^{N}} \sum_{l=1}^{m} \frac{(N \log j)^{m-l}}{(m-l)!} \right], \quad m = 1, \ldots, N
\]

(2.11)

and

\[
\int_{\Omega} |\nabla \omega_j|^m \ dx = \begin{cases} 
\frac{\omega_{N-1}^{1-m/N} d^{N-m}}{(N - m) (\log j)^{m/N}} \left( 1 - \frac{1}{j^{N-m}} \right), & m = 1, \ldots, N - 1 \\
1, & m = N.
\end{cases}
\]

(2.12)

**Proof.** We have

\[
\int_{\Omega} \omega_j^m \ dx = \frac{\omega_{N-1}^{1-m/N} d^N}{(\log j)^{m/N}} \left[ I_m + \frac{(\log j)^m}{N j^N} \right],
\]

where

\[
I_m = \int_{1/j}^{1} (-\log s)^{m} s^{N-1} \ ds.
\]

We have

\[
I_1 = \frac{1}{N^2} \left[ 1 - \frac{1}{j^{N}} (N \log j + 1) \right],
\]

and integrating by parts gives the recurrence relation

\[
I_m = \frac{m}{N} I_{m-1} - \frac{(\log j)^m}{N j^N}, \quad m \geq 2.
\]

So

\[
I_m = \frac{m!}{N^{m+1}} \left[ 1 - \frac{1}{j^{N}} \sum_{l=0}^{m} \frac{(N \log j)^{m-l}}{(m-l)!} \right],
\]

and (2.11) follows. The integral in (2.12) is easily evaluated. \( \square \)
2.6 A limit calculation

We will need the following limit in the proof of Theorem 1.1.

**Proposition 2.7.** We have

\[
\lim_{n \to \infty} \int_0^1 ne^{-n(t-t^{'N'})} dt = N.
\]

**Proof.** Let \( f_n(t) = ne^{-n(t-t^{'N'})} \) and set \( t_0 = (N'-1)/(N'-1) \). For \( t \neq t_0 \),

\[
f_n(t) = g_n(t) - \frac{d}{dt} \left( \frac{e^{-n(t-t^{'N'})}}{1 - N't^{N'-1}} \right),
\]

where

\[
g_n(t) = \frac{N'(N'-1)t^{N'-2}e^{-n(t-t^{'N'})}}{(1 - N't^{N'-1})^2}.
\]

Fix \( \delta \) so small that \( 0 < \delta < t_0 < 1 - \delta < 1 \) and write

\[
\int_0^1 f_n(t) dt = \int_0^\delta f_n(t) dt + \int_\delta^{1-\delta} f_n(t) dt + \int_{1-\delta}^1 f_n(t) dt.
\]

By (2.13),

\[
\int_0^\delta f_n(t) dt = \int_0^\delta g_n(t) dt - \frac{e^{-n(\delta-\delta^{'N'})}}{1 - N'\delta^{N'-1}} + 1.
\]

For all \( t \in (0, \delta) \), \( g_n(t) \to 0 \) as \( n \to \infty \) and \( |g_n(t)| \leq N'(N'-1)t^{N'-2}/(1 - N'\delta^{N'-1})^2 \), so \( \int_0^\delta g_n(t) dt \to 0 \) by the dominated convergence theorem. So \( \int_0^1 f_n(t) dt \to 1 \) by (2.13). A similar calculation shows that \( \int_{1-\delta}^1 f_n(t) dt \to N - 1 \). On the other hand, it is easily seen that \( \int_\delta^{1-\delta} f_n(t) dt \to 0 \). So \( \int_0^1 f_n(t) dt \to N \) by (2.14).

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1 by showing that the functional \( E \) has the mountain pass geometry with the mountain pass level \( c \in (0, (1/N)(\alpha_N/\alpha)^{N-1}) \) and applying Proposition 2.1.

**Lemma 3.1.** There exists a \( \rho > 0 \) such that

\[
\inf_{\|u\|=\rho} E(u) > 0.
\]
Proof. Since (1.2) implies that \( h \) is bounded, there exists a constant \( C_\delta > 0 \) such that
\[
|G(t)| \leq C_\delta |t|^{N+1} e^{\alpha |t|^N'} \quad \text{for } |t| > \delta,
\]
which together with (1.9) gives
\[
\int_{\Omega} G(u) \, dx \leq \frac{1}{N} (\lambda_1 - \sigma_1) \int_{\Omega} |u|^N \, dx + C_\delta \int_{\Omega} |u|^{N+1} e^{\alpha |u|^N'} \, dx. \tag{3.1}
\]
By (1.6),
\[
\int_{\Omega} |u|^N \, dx \leq \frac{\rho^N}{\lambda_1}, \tag{3.2}
\]
where \( \rho = ||u||. \) By the Hölder inequality,
\[
\int_{\Omega} |u|^{N+1} e^{\alpha |u|^N'} \, dx \leq \left( \int_{\Omega} |u|^{2(N+1)} \, dx \right)^{1/2} \left( \int_{\Omega} e^{2\alpha |u|^N'} \, dx \right)^{1/2}. \tag{3.3}
\]
The first integral on the right-hand side is bounded by \( C \rho^{2(N+1)} \) for some constant \( C > 0 \) by the Sobolev embedding theorem. Since \( 2\alpha |u|^N' = 2\alpha \rho^N' |\tilde{u}|^N' \), where \( \tilde{u} = u/\rho \) satisfies \( ||\tilde{u}|| = 1 \), the second integral is bounded when \( \rho^N' \leq \alpha_N/2\alpha \) by (1.4). So combining (3.1)–(3.3) gives
\[
\int_{\Omega} G(u) \, dx \leq \frac{1}{N} \left( 1 - \frac{\sigma_1}{\lambda_1} \right) \rho^N + O(\rho^{N+1}) \quad \text{as } \rho \to 0.
\]
Then
\[
E(u) \geq \frac{1}{N} \frac{\sigma_1}{\lambda_1} \rho^N + O(\rho^{N+1}),
\]
and the desired conclusion follows from this for sufficiently small \( \rho > 0 \).

We may assume without loss of generality that \( B_d(0) \subset \Omega \). Let \( (\omega_j) \) be the sequence of functions defined in (2.10).

Lemma 3.2. We have
\begin{enumerate}[(i)]
\item \( E(t\omega_j) \to -\infty \) as \( t \to \infty \) for all \( j \geq 2 \),
\item \( \exists j_0 \geq 2 \) such that
\[ \sup_{t \geq 0} E(t\omega_{j_0}) < \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1}. \]
\end{enumerate}
Proof. (i) Fix \(0 < \varepsilon < \beta\). By (1.3), \(\exists M_\varepsilon > 0\) such that
\[
\text{th}(t)e^{\alpha |t|^N} > (\beta - \varepsilon)e^{\alpha |t|^N} \quad \text{for } |t| > M_\varepsilon.
\] (3.4)
Since \(e^{\alpha |t|^N} > \alpha^{2N-2} t^{2N}/(2N-2)!\) for all \(t\), then there exists a constant \(C_\varepsilon > 0\) such that
\[
\text{th}(t)e^{\alpha |t|^N} \geq \frac{1}{(2N-2)!} (\beta - \varepsilon) \alpha^{2N-2} t^{2N} - C_\varepsilon |t|
\] (3.5)
and
\[
G(t) \geq \frac{2N - 1}{(2N)!} (\beta - \varepsilon) \alpha^{2N-2} t^{2N} - C_\varepsilon |t|
\] (3.6)
for all \(t\). Since \(\|\omega_j\| = 1\) and \(\omega_j \geq 0\), then
\[
E(t\omega_j) \leq \frac{t^N}{N} - \frac{2N - 1}{(2N)!} (\beta - \varepsilon) \alpha^{2N-2} t^{2N} \int_\Omega \omega_j^{2N} dx + C_\varepsilon t \int_\Omega \omega_j dx,
\]
and the conclusion follows.

(ii) Set
\[
H_j(t) = E(t\omega_j) = \frac{t^N}{N} - \int_\Omega G(t\omega_j) dx, \quad t \geq 0.
\]
If the conclusion is false, then it follows from (i) that for all \(j \geq 2\), \(\exists t_j > 0\) such that
\[
H_j(t_j) = \frac{t_j^N}{N} - \int_\Omega G(t_j\omega_j) dx = \sup_{t \geq 0} H_j(t) \geq \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1},
\] (3.7)
\[
H'_j(t_j) = t_j^{N-1} - \int_\Omega \omega_j h(t_j\omega_j) e^{\alpha t_j^{N'} \omega_j^{N'}} dx = 0.
\] (3.8)
Since \(G(t) \geq -C_\varepsilon t\) for all \(t \geq 0\) by (3.6), (3.7) gives
\[
t_j^N \geq t_0^N - N\delta_j t_j,
\] (3.9)
where
\[
t_0 = \left( \frac{\alpha_N}{\alpha} \right)^{(N-1)/N}
\]
and
\[
\delta_j = C_\varepsilon \int_\Omega \omega_j dx \to 0 \quad \text{as } j \to \infty
\] (3.10)
by Proposition \[2.6\]. First we will show that \(t_j \to t_0\).

By (3.9) and the Young’s inequality,

\[
(1 + \nu) t_j^N \geq t_0^N - \frac{N - 1}{\nu^{1/(N-1)}} \delta_j^N \quad \forall \nu > 0,
\]

which together with (3.10) gives

\[
\liminf_{j \to \infty} t_j \geq t_0.
\] (3.11)

Write (3.8) as

\[
t_j^N = \int_{\{t_j \omega_j > M_\varepsilon\}} t_j \omega_j h(t_j \omega_j) e^{\alpha t_j \omega_j^N} dx + \int_{\{t_j \omega_j \leq M_\varepsilon\}} t_j \omega_j h(t_j \omega_j) e^{\alpha t_j \omega_j^N} dx =: I_1 + I_2.
\] (3.12)

Set \(r_j = d e^{-M_\varepsilon (\omega_{N-1} \log j)^{1/N}/j}\). Since \(\liminf t_j > 0\), for all sufficiently large \(j\), \(d/j < r_j < d\) and \(t_j \omega_j(x) > M_\varepsilon\) if and only if \(|x| < r_j\). So (3.4) gives

\[
I_1 \geq (\beta - \varepsilon) \int_{|x| < r_j} e^{\alpha t_j \omega_j^N} dx = (\beta - \varepsilon) \left( \int_{|x| \leq d/j} e^{\alpha t_j \omega_j^N} dx + \int_{d/j < |x| < r_j} e^{\alpha t_j \omega_j^N} dx \right) =: (\beta - \varepsilon) (I_3 + I_4). \] (3.13)

We have

\[
I_3 = \frac{\omega_{N-1}}{N} \left( \frac{d}{j} \right)^N e^{\alpha t_j \log j / \omega_{N-1}^{1/(N-1)}} = \frac{\omega_{N-1}}{N} d^N j^{\alpha (t_j^N - t_0^N)/\omega_{N-1}^{1/(N-1)}}. \] (3.14)

Since \(th(t) e^{\alpha |t|^N} \geq -C_\varepsilon t\) for all \(t \geq 0\) by (3.3),

\[
I_2 \geq -C_\varepsilon t_j \int_{\{t_j \omega_j \leq M_\varepsilon\}} \omega_j dx \geq -\delta_j t_j. \] (3.15)

Combining (3.12)–(3.15) and noting that \(I_4 \geq 0\) gives

\[
t_j^N \geq (\beta - \varepsilon) \frac{\omega_{N-1}}{N} d^N j^{\alpha (t_j^N - t_0^N)/\omega_{N-1}^{1/(N-1)}} - \delta_j t_j.
\]

It follows from this that

\[
\limsup_{j \to \infty} t_j \leq t_0,
\] 13
which together with (3.11) shows that $t_j \to t_0$.

Next we estimate $I_4$. We have

$$I_4 = \int_{\{d/j < |x| < r_j\}} e^{\alpha t_j^N [\log (d/x)]^N / (\omega N \log j)^{1/(N-1)}} \, dx$$

$$= \omega_{N-1} \left( \int_{d/j}^d e^{\alpha t_j^N [\log (d/r)]^N / (\omega N \log j)^{1/(N-1)}} r^{-1} \, dr \right.\right.$$

$$- \left. \int_{r_j}^d e^{\alpha t_j^N [\log (d/r)]^N / (\omega N \log j)^{1/(N-1)}} r^{-1} \, dr \right)$$

$$= \omega_{N-1} d^N \left( \log j \int_0^1 e^{-N t \left[ 1 - (t_j/t_0)^{N'} \right]^{1/(N-1)}} \log j \, dt \right.$$

$$- \left. \int_s^1 e^{-N \left( - \log s \right)^N / (\omega N \log j)^{1/(N-1)}} \, ds \right), \quad (3.16)$$

where $t = \log (d/r) / \log j$, $s = r/d$, and $s_j = r_j/d = e^{-M \log j / \omega N}$, and goes to zero as $j \to \infty$, so the last integral converges to

$$\int_0^1 s^{N-1} \, ds = \frac{1}{N}. \quad (3.16)$$

So combining (3.12)–(3.16) and letting $j \to \infty$ gives

$$t_0^N \geq (\beta - \varepsilon) \omega_{N-1} \frac{d^N}{N} (L_1 + L_2 - 1),$$

where

$$L_1 = \lim_{j \to \infty} \inf e^{-n \left[ 1 - (t_j/t_0)^{N'} \right]},$$

$$L_2 = \lim_{j \to \infty} \int_0^1 ne^{-n \left[ 1 - (t_j/t_0)^{N'} \right]} \, dt,$$

and $n = N \log j \to \infty$. Letting $\varepsilon \to 0$ in this inequality gives

$$\beta \leq \frac{1}{\alpha^{N-1}} \left( \frac{N}{d} \right)^N \frac{1}{L_1 + L_2 - 1}. \quad (3.17)$$
By (3.7), (1.8), and Proposition 2.6,
\[ t_j^N - t_0^N \geq N \int_{\Omega} G(t_j \omega_j) \, dx \geq -\sigma_0 t_j^N \int_{\Omega} \omega_j^N \, dx \geq -\frac{\sigma_0 t_j^N}{\kappa n}, \]
so
\[ \left( \frac{t_j}{t_0} \right)^{N'} \geq \left( 1 + \frac{\sigma_0}{\kappa n} \right)^{-1/(N-1)} \geq 1 - \frac{\sigma_0}{(N-1) \kappa n}. \]
This gives
\[ L_1 \geq e^{-\sigma_0/(N-1) \kappa} \]
and
\[ L_2 \geq \lim_{n \to \infty} \int_{0}^{1} n e^{-n(t-t^{N'}) - \sigma_0 t^{N'}/(N-1) \kappa} \, dt \geq N e^{-\sigma_0/(N-1) \kappa} \]
by Proposition 2.7. So (3.17) gives
\[ \beta \leq \frac{1}{\alpha^{N-1}} \left( \frac{N}{d} \right)^{N} \frac{1}{N e^{-\sigma_0/(N-1) \kappa} - \left( 1 - e^{-\sigma_0/(N-1) \kappa} \right)} \leq \frac{1}{N \alpha^{N-1}} \left( \frac{N}{d} \right)^{N} e^{\sigma_0/(N-1) \kappa}, \]
contradicting (1.10). \hfill \Box

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( j_0 \) be as in Lemma 3.2 (ii). By Lemma 3.2 (i), \( \exists R > \rho \) such that \( E(R \omega_{j_0}) \leq 0 \), where \( \rho \) is as in Lemma 3.1. Let
\[ \Gamma = \left\{ \gamma \in C([0, 1], W^{1,N}_\Omega) : \gamma(0) = 0, \gamma(1) = R \omega_{j_0} \right\} \]
be the class of paths joining the origin to \( R \omega_{j_0} \), and set
\[ c := \inf_{\gamma \in \Gamma} \max_{u \in \gamma([0,1])} E(u). \]
By Lemma 3.1, \( c > 0 \). Since the path \( \gamma_0(t) = tR \omega_{j_0}, t \in [0,1] \) is in \( \Gamma \),
\[ c \leq \max_{u \in \gamma_0([0,1])} E(u) \leq \sup_{t \geq 0} E(t \omega_{j_0}) < \frac{1}{N} \left( \frac{\alpha N}{\alpha} \right)^{N-1}. \]
If there are no (PS)\( c \) sequences of \( E \), then \( E \) satisfies the (PS)\( c \) condition vacuously and hence has a critical point \( u \) at the level \( c \) by the mountain pass theorem. Then \( u \) is a solution of problem (1.1) and \( u \) is nontrivial since \( c > 0 \). So we may assume that \( E \) has a (PS)\( c \) sequence. Then this sequence has a subsequence that converges weakly to a nontrivial solution of problem (1.1) by Proposition 2.1. \hfill \Box

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4 Proof of Theorem 1.3

In this section we prove Theorem 1.3 using Theorem 2.5. We take $A_0$ to be the set $C$ in Proposition 2.4 and $B_0 = \Psi_{\lambda_k}$. Since $i(S \setminus B_0) = k - 1$ by Proposition 2.3, (2.9) holds.

**Lemma 4.1.** There exists a $\rho > 0$ such that $\inf E(B) > 0$, where $B = \{\rho u : u \in B_0\}$.

**Proof.** As in the proof of Lemma 3.1, there exists a constant $C_\delta > 0$ such that

$$|G(t)| \leq C_\delta |t|^{N+1} e^{\alpha |t|^{N'}}$$

for $|t| > \delta$,

which together with (1.12) gives

$$G(t) \leq \frac{1}{N} (\lambda_k - \sigma_1) |t|^N + C_\delta |t|^{N+1} e^{\alpha |t|^{N'}} \forall t. \quad (4.1)$$

For $u \in B_0$ and $\rho > 0$,

$$\int_\Omega |\rho u|^N \leq \frac{\rho^N}{\lambda_k} \quad (4.2)$$

and

$$\int_\Omega |\rho u|^{N+1} e^{\alpha |\rho u|^{N'}} \leq \rho^{N+1} \left( \int_\Omega |u|^{2(N+1)} dx \right)^{1/2} \left( \int_\Omega e^{2\alpha \rho^{N'} |u|^{N'}} dx \right)^{1/2}. \quad (4.3)$$

The first integral on the right-hand side of (4.3) is bounded by the Sobolev embedding theorem, and the second integral is bounded when $\rho^{N'} \leq \alpha N / 2\alpha$ by (1.3). So combining (1.1)–(1.3) gives

$$\int_\Omega G(\rho u) dx \leq \frac{1}{N} \left( 1 - \frac{\sigma_1}{\lambda_k} \right) \rho^N + O(\rho^{N+1}) \quad \text{as } \rho \to 0.$$

Then

$$E(\rho u) \geq \frac{1}{N} \frac{\sigma_1}{\lambda_k} \rho^N + O(\rho^{N+1}),$$

and the desired conclusion follows from this for sufficiently small $\rho$. \hfill \Box

We may assume without loss of generality that $B_d(0) \subset \Omega$. Let $(\omega_j)$ be the sequence of functions defined in (2.10).

**Lemma 4.2.** We have

(i) $E(sv) \leq 0 \ \forall v \in A_0, s \geq 0,$
(ii) for all $j \geq 2$,
\[
\sup \{ E(\pi((1-t)v + t\omega_j)) : v \in A_0, 0 \leq t \leq 1 \} \to -\infty \text{ as } R \to \infty,
\]

(iii) \exists j_0 \geq 2 \text{ such that }
\[
\sup \{ E(sv + t\omega_{j_0}) : v \in A_0, s, t \geq 0 \} < \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1}.
\]

Proof. [i] By (1.11),
\[
E(u) \leq \frac{1}{N} \left[ \int_\Omega |\nabla u|^N dx - (\lambda_{k-1} + \sigma_0) \int_\Omega |u|^N dx \right]. \quad (4.4)
\]
For $v \in A_0$ and $s \geq 0$,
\[
\int_\Omega |sv|^N dx \geq \frac{s^N}{\lambda_{k-1}}
\]
since $A_0 \subset \Psi_{k-1}$, so (4.4) gives
\[
E(sv) \leq - \frac{1}{N} \sigma_0 \frac{s^N}{\lambda_{k-1}} \leq 0.
\]

[ii] Fix $0 < \varepsilon < \beta$. As in the proof of Lemma 3.2, \exists M_\varepsilon > 0 such that
\[
th(t) e^{\alpha |t|^N} > (\beta - \varepsilon) e^{\alpha |t|^N} \text{ for } |t| > M_\varepsilon
\]
and there exists a constant $C_\varepsilon > 0$ such that
\[
th(t) e^{\alpha |t|^N} \geq \frac{1}{(2N-2)!} (\beta - \varepsilon) \alpha^{2N-2} t^{2N} - C_\varepsilon |t|
\]
and
\[
G(t) \geq \frac{2N-1}{(2N)!} (\beta - \varepsilon) \alpha^{2N-2} t^{2N} - C_\varepsilon |t|
\]
for all $t$. Let $A_1 = \{ \pi((1-t)v + t\omega_j) : v \in A_0, 0 \leq t \leq 1 \}$. For $u \in A_1$ and $R > 0$, (4.7) gives
\[
E(Ru) \leq \frac{R^N}{N} - \frac{2N-1}{(2N)!} (\beta - \varepsilon) \alpha^{2N-2} R^{2N} \int_\Omega |u|^{2N} dx + C_\varepsilon R \int_\Omega |u| dx.
\]

The set $A_1$ is compact since $A_0$ is compact, so the first integral on the right-hand side is bounded away from zero on $A_1$. Since the second integral is bounded, the desired conclusion follows.
If the conclusion is false, then it follows from (i) and (ii) that for all \( j \geq 2 \), there exist \( v_j \in A_0, s_j \geq 0, t_j > 0 \) such that

\[
E(s_j v_j + t_j \omega_j) = \sup \{ E(sv + t \omega_j) : v \in A_0, s, t \geq 0 \} \geq \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1}.
\]

Set \( u_j = s_j v_j + t_j \omega_j \). Then

\[
E(u_j) = \frac{1}{N} \| u_j \|^N - \int_{\Omega} G(u_j) \, dx \geq \frac{1}{N} \left( \frac{\alpha_N}{\alpha} \right)^{N-1}.
\]  

(4.8)

Moreover, \( \tau u_j \in \{ sv + t \omega_j : v \in A_0, s, t \geq 0 \} \) for all \( \tau \geq 0 \) and \( E(\tau u_j) \) attains its maximum at \( \tau = 1 \), so

\[
\frac{\partial}{\partial \tau} E(\tau u_j) \bigg|_{\tau=1} = E'(u_j) u_j = \| u_j \|^N - \int_{\Omega} u_j h(u_j) e^{\alpha |u_j|^N} \, dx = 0.
\]  

(4.9)

Since \( \| v_j \| = \| \omega_j \| = 1 \) and \( G(t) \geq 0 \) for all \( t \) by (1.11), (4.8) gives

\[
s_j + t_j \geq t_0,
\]

where

\[
t_0 = \left( \frac{\alpha_N}{\alpha} \right)^{(N-1)/N}.
\]

First we show that \( s_j \to 0 \) and \( t_j \to t_0 \) as \( j \to \infty \).

Combining (4.8) with (1.11) gives

\[
\| s_j v_j + t_j \omega_j \|^N \geq (\lambda_{k-1} + \sigma_0) \int_{\Omega} |s_j v_j + t_j \omega_j|^N \, dx + t_0^N.
\]

Set \( \tau_j = s_j/t_j \). Then

\[
\| \tau_j v_j + \omega_j \|^N \geq (\lambda_{k-1} + \sigma_0) \int_{\Omega} |\tau_j v_j + \omega_j|^N \, dx + \left( \frac{t_0}{t_j} \right)^N.
\]  

(4.10)

Since \( (v_j) \) is bounded in \( C^1(\Omega) \), Proposition 2.6 gives

\[
\| \tau_j v_j + \omega_j \|^N \leq \int_{\Omega} (\tau_j |\nabla v_j| + |\nabla \omega_j|)^N \, dx = \tau_j^N \int_{\Omega} |\nabla v_j|^N \, dx + \int_{\Omega} |\nabla \omega_j|^N \, dx
\]

\[
+ \sum_{m=1}^{N-1} \binom{N}{m} \tau_j^{N-m} \int_{\Omega} |\nabla v_j|^{N-m} |\nabla \omega_j|^m \, dx \leq \tau_j^N + c_1 \sum_{m=1}^{N-1} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}
\]

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\[
\int_{\Omega} |\tau_j v_j + \omega_j|^N dx \geq \int_{\Omega} (\tau_j |v_j| - \omega_j)^N dx = \tau_j^N \int_{\Omega} |v_j|^N dx \\
+ \sum_{m=1}^{N} (-1)^m \left( \frac{N}{m} \right) \tau_j^{N-m} \int_{\Omega} |v_j|^{N-m} \omega_j^m dx \geq \frac{\tau_j^N}{\lambda_{k-1}} - c_2 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}
\]
for some constants \( c_1, c_2 > 0 \). So (4.10) gives
\[
\frac{\sigma_0}{\lambda_{k-1}} \tau_j^N + \left( \frac{t_0}{t_j} \right)^N \leq 1 + c_3 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}
\]
(4.11)
for some constant \( c_3 > 0 \), which implies that \( (\tau_j) \) is bounded and
\[
\lim_{j \to \infty} t_j \geq t_0.
\]
(4.12)

Next combining (4.9) with (4.5) and (4.6) gives
\[
\|u_j\|^N = \int_{\{|u_j| > M\}} u_j h(u_j) e^{\alpha |u_j|^N} dx + \int_{\{|u_j| \leq M\}} u_j h(u_j) e^{\alpha |u_j|^N} dx \\
\geq (\beta - \varepsilon) \int_{\{|u_j| > M\}} e^{\alpha |u_j|^N} dx - C_\varepsilon \int_{\{|u_j| \leq M\}} |u_j| dx.
\]
(4.13)

For \(|x| \leq d/j\),
\[
|u_j| \geq t_j \omega_j - s_j |v_j| \geq \frac{t_j}{\omega_{N-1}^{1/N}} \left( (\log j)^{(N-1)/N} - c_4 \tau_j \right)
\]
for some constant \( c_4 > 0 \), and the last expression is greater than \( M_\varepsilon \) for all sufficiently large \( j \) since \( (\tau_j) \) is bounded and \( \lim \inf t_j > 0 \). So
\[
\int_{\{|u_j| > M\}} e^{\alpha |u_j|^N} dx \geq e^{\alpha t_j^{N'} (\log j)^{(N-1)/N} - c_4 \tau_j} \int_{\{|x| \leq d/j\}} dx \\
= \frac{\omega_{N-1} q^N}{N} \int_{\{|x| \leq d/j\}} dx \\
= \frac{\omega_{N-1} q^N}{N} j^{N' \left( (1 - c_4 \tau_j)/(\log j)^{(N-1)/N} \right)^N - t_0^{N'}/\omega_{N-1}^{1/(N-1)}}
\]
for large \( j \). On the other hand,
\[
\int_{\{|u_j| \leq M\}} |u_j| dx \leq \int_{\Omega} (s_j |v_j| + t_j \omega_j) dx \leq c_5 t_j \left[ \tau_j + \frac{1}{(\log j)^{1/N}} \right]
\]

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for some constant $c_5 > 0$ by Proposition 2.6. So \((4.13)\) gives
\[
(\beta - \varepsilon) j^\alpha [t_j^{N'}(1-c_4\tau_j/(\log j)^{(N-1)/N})^{N'-t_0^{N'}/\omega_{N-1}^{1/(N-1)}}] \leq \frac{Nt_j^N(\tau_j + 1)^N}{\omega_{N-1} d^N} + c_6 t_j \left[ \tau_j + \frac{1}{(\log j)^{1/N}} \right]
\]
for some constant $c_6 > 0$. Since $\tau_j$ is bounded, it follows from this that
\[
\limsup_{j \to \infty} t_j \leq t_0,
\]
which together with \((4.12)\) shows that $t_j \to t_0$. Then \((4.11)\) implies that $\tau_j \to 0$, so $s_j = \tau_j t_j - t_0 \to 0$.

Now we show that there exists a constant $c > 0$ depending only on $\Omega$, $\alpha$, and $k$ such that
\[
\beta \leq \frac{1}{\alpha^{N-1}} \left( \frac{N}{d} \right)^N e^{c_0^{N-1}}. \tag{4.15}
\]
The right-hand side of \((4.14)\) goes to $(N/d)^N/\alpha^{N-1}$ as $j \to \infty$. If $\beta \leq (N/d)^N/\alpha^{N-1}$, then we may take any $c > 0$, so suppose $\beta > (N/d)^N/\alpha^{N-1}$. Then for $\varepsilon < \beta - (N/d)^N/\alpha^{N-1}$ and all sufficiently large $j$, \((4.14)\) gives
\[
\frac{t_0}{t_j} \geq 1 - \frac{c_4 \tau_j}{(\log j)^{(N-1)/N}}.
\]
Combining this with \((4.11)\) gives
\[
\frac{\sigma_0}{\lambda_{k-1}} \tau_j^N - \frac{N c_4 \tau_j}{(\log j)^{(N-1)/N}} \leq c_3 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}},
\]
so
\[
\sigma_0 \tau_j^N \leq c_7 \sum_{m=1}^{N} \frac{\tau_j^{N-m}}{(\log j)^{m/N}}
\]
for some constant $c_7 > 0$. Set $\bar{\tau}_j = \tau_j (\log j)^{1/N}$. Then
\[
\sigma_0 \bar{\tau}_j^N \leq c_7 \sum_{m=1}^{N} \bar{\tau}_j^{N-m}. \tag{4.16}
\]
We claim that
\[ \tilde{\tau}_j \leq \frac{c_8}{\sigma_0} \tag{4.17} \]
for some constant \( c_8 > 0 \). Taking \( \sigma_0 \) smaller in (4.11) if necessary, we may assume that \( \sigma_0 \leq 1 \). So if \( \tilde{\tau}_j < 1 \), then (4.17) holds with \( c_8 = 1 \), so suppose \( \tilde{\tau}_j \geq 1 \). Then (4.16) gives
\[
\left( \frac{t_0}{t_j} \right)^N \leq 1 + \frac{c_3}{\log j} \sum_{m=1}^{N} \tilde{\tau}_j^{N-m} \leq 1 + \frac{c_9}{\sigma_0^{N-1} \log j}
\]
for some constant \( c_9 > 0 \), so
\[
\left( \frac{t_0}{t_j} \right)^{N'} \leq \left( 1 + \frac{c_9}{\sigma_0^{N-1} \log j} \right)^{1/(N-1)} \leq 1 + \frac{c_9}{\sigma_0^{N-1} \log j}.
\]
Then
\[
t_j^{N'} \left[ 1 - \frac{c_4 \tilde{\tau}_j}{(\log j)(N-1)/N} \right]^{N'} - t_0^{N'} = t_j^{N'} \left[ \left( 1 - \frac{c_4 \tilde{\tau}_j}{\log j} \right)^{N'} - \left( \frac{t_0}{t_j} \right)^{N'} \right]
\]
\[
\geq t_j^{N'} \left[ \left( 1 - \frac{c_{10}}{\sigma_0 \log j} \right)^{N'} - \left( 1 + \frac{c_9}{\sigma_0^{N-1} \log j} \right) \right] \geq -t_j^{N'} \left( \frac{N' c_{10}}{\sigma_0 \log j} + \frac{c_9}{\sigma_0^{N-1} \log j} \right)
\]
\[
\geq -\frac{c_{11}}{\sigma_0^{N-1} \log j}
\]
for some constants \( c_{10}, c_{11} > 0 \), so
\[
j^{\alpha N'} \left[ (1-c_4 \tilde{\tau}_j/(\log j)(N-1)/N)^{N'} - t_0^{N'} \right]/\omega_j^{N'/(N-1)} \geq e^{-c/\sigma_0^{N-1} \log j} = e^{-c/\sigma_0^{N-1}}
\]
for some constant \( c > 0 \). Combining this with (4.14) and passing to the limit gives
\[
(\beta - \varepsilon) e^{-c/\sigma_0^{N-1}} \leq \frac{1}{\alpha^{N-1}} \left( \frac{N}{d} \right)^N,
\]
and letting \( \varepsilon \to 0 \) gives (4.15). \( \square \)

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Let \( j_0 \geq 2 \) be as in Lemma 4.2 (iii). By Lemma 4.2 (ii), \( \exists R > \rho \) such that
\[
\sup \{ E(R \pi((1-t)v + tw_j_0)) : v \in A_0, 0 \leq t \leq 1 \} \leq 0, \tag{4.18}
\]
where $\rho > 0$ is as in Lemma 4.1. Let

$$A = \{sv : v \in A_0, 0 \leq s \leq R\} \cup \{R\pi((1-t)v + t\omega_j) : v \in A_0, 0 \leq t \leq 1\},$$

$$X = \{sv + t\omega_j : v \in A_0, s, t \geq 0, \|sv + t\omega_j\| \leq R\}.$$ 

Combining Lemma 4.1, (4.18), and Lemma 4.1 gives

$$\sup E(A) \leq 0 < \inf E(B),$$

(4.19)

while Lemma 4.2 (iii) gives

$$\sup E(X) \leq \sup \{E(sv + t\omega_j) : v \in A_0, s, t \geq 0\} < \frac{1}{N} \left(\frac{\alpha N}{\alpha}\right)^{N-1}.$$ 

(4.20)

Let

$$\Gamma = \{\gamma \in C(X, W) : \gamma(X) \text{ is closed and } \gamma|_A = \text{id}_A\},$$

and set

$$c := \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(X)} E(u).$$

By Theorem 2.5, $\inf E(B) \leq c \leq \sup E(X)$, and $E$ has a $(PS)_c$ sequence. By (4.19) and (4.20),

$$0 < c < \frac{1}{N} \left(\frac{\alpha N}{\alpha}\right)^{N-1},$$

so a subsequence of this $(PS)_c$ sequence converges weakly to a nontrivial solution of problem (1.1) by Proposition 2.1.

5 Competing interests declaration

The authors declare no competing interests.

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