SAUSAGES AND BUTCHER PAPER

DANNY CALEGARI

Abstract. For each \( d > 1 \) the \textit{shift locus of degree} \( d \), denoted \( S_d \), is the space of normalized degree \( d \) polynomials in one complex variable for which every critical point is in the attracting basin of infinity under iteration. It is a complex analytic manifold of complex dimension \( d - 1 \).

We are able to give an explicit description of \( S_d \) as a complex of spaces over a contractible \( \tilde{A}_{d-2} \) building, and to describe the pieces in two quite different ways:

1. (combinatorial): in terms of dynamical extended laminations; or
2. (algebraic): in terms of certain explicit ‘discriminant-like’ affine algebraic varieties.

From this structure one may deduce numerous facts, including that \( S_d \) has the homotopy type of a CW complex of real dimension \( d - 1 \); and that \( S_3 \) and \( S_4 \) are \( K(\pi,1) \)s.

The method of proof is rather interesting in its own right. In fact, along the way we discover a new class of complex surfaces (they are complements of certain singular curves in \( \mathbb{C}^2 \)) which are homotopic to locally CAT(0) complexes; in particular they are \( K(\pi,1) \)s.

Contents

1. Introduction \hfill 1
2. The shift locus \hfill 4
3. Elaminations \hfill 6
4. Butcher Paper \hfill 10
5. Formal shift space \hfill 12
6. Degree 2 \hfill 15
7. Degree 3 \hfill 15
8. Degree 4 and above \hfill 21
9. Sausages \hfill 27
10. Fundamental Groups \hfill 37
11. Acknowledgments \hfill 39
References \hfill 39

1. Introduction

For each \( d > 1 \) the \textit{shift locus of degree} \( d \), denoted \( S_d \), is the space of normalized (i.e. monic with roots summing to zero) degree \( d \) polynomials in one complex variable for which every critical point is in the attracting basin of infinity under iteration. A polynomial in \( S_d \) is called a \textit{shift polynomial}. These are the polynomials...
whose dynamics are the easiest to understand; perhaps in compensation, their parameter spaces appear to be extremely complicated. Much is known about the geometry and topology of $S_d$ and much is still mysterious.

The main point of this paper is to describe a canonical decomposition of $S_d$ (and some equivalent spaces) into pieces, giving $S_d$ the explicit structure of a ‘complex of spaces’ over a rather nice space (a contractible $\tilde{A}_{d-2}$ building) and to give two, quite different, descriptions of the pieces.

One description is combinatorial, in terms of certain iterated fiber bundles resp. their orbifolded quotients that we call *monkey prisms* resp. *monkey turnovers*. In this description, the fibers and their monodromy are encoded quite explicitly in objects called *dynamical laminations*; the word ‘lamination’ here is shorthand for ‘extended lamination’ — a lamination with ‘extra’ structure. Elaminations are related to the sorts of laminations used elsewhere in holomorphic dynamics (see e.g. [28]) but are in some ways quite different. Their definitions and basic properties are given in §3 and they are a key tool throughout the remainder of the paper.

The other description is algebraic, in terms of certain complex affine varieties, which arise as moduli spaces of maps between infinite nodal genus 0 surfaces called *sausages*. The relationship between sausages and shift polynomials is of an essentially topological nature, so that although both objects and their moduli spaces carry natural complex analytic structures, the maps between them do not respect this structure. This seems to be unavoidable: the shift locus is a highly transcendental object, whereas the moduli spaces we construct are algebraic.

One interesting consequence of this relationship between these two ways of seeing the shift locus is that information which is obscure on one side can become transparent on the other. Here is an example. In degree 3, the Shift Locus can be described (up to homotopy) as a space obtained from the 3-sphere by drilling out a trefoil knot, and gluing in a bundle over $S^1$ whose fiber is a disk minus a Cantor set. This Cantor set can be thought of as an infinite nested intersection $K = \cap E_n$ of subsets of the disk, where each $E_n$ is itself a finite union of disks. The monodromy permutes each $E_n$, and it is a fact (Theorem 7.9) that the orbits of this permutation are cycles whose lengths are powers of 2. The only proof of this that I know is to interpret the permutation action of the monodromy as an action on the roots of a certain polynomial obtained by iterated quadratic extensions.

The value of mathematical machinery is that it can prove theorems whose statement does not mention the machinery. As a consequence of our structure theorems we are able to deduce some facts about the topology of the shift locus, especially in low degrees. In particular:

**Homotopy Theorem.** $S_d$ has the homotopy type of a $(d - 1)$-complex (i.e. a complex of half the real dimension of $S_d$ as a manifold). For $d = 3$ or 4 it is a $K(\pi, 1)$. For $d = 3$ it is homotopic to a $\text{CAT}(0)$ 2-complex.

This is an amalgamation of Theorems 7.3, 8.7 and 8.12. In fact, it is plausible that $S_d$ is a $K(\pi, 1)$ in every degree.

In degree 3 we are able to give an extremely explicit description. $S_3$ is homeomorphic to a product $X_3 \times \mathbb{R}$ where $X_3$ is a 3-manifold obtained from $S^3$ by drilling out a right-handed trefoil, and gluing in a bundle $D_\infty \to N_\infty \to S^1$ where each fiber $D_\infty(t)$ over $t \in S^1$ is a disk minus a Cantor set. In fact, we are able to give a completely explicit description of $D_\infty$ and its monodromy in terms of an object...
called the \textit{Tautological Elamination}. There is one Tautological Elamination $\Lambda_T(t)$ for each $t$. These elaminations vary continuously in the so-called collision topology (defined in §3.2), and the $D_\infty(t)$ are obtained by an operation called \textit{pinching}. Finally, the monodromy is completely described by the formula $\mathcal{T}_t \Lambda_T(s) = \Lambda_T(s + t)$ where $\mathcal{T}_t$ is an explicit flow on the space of elaminations.

In words: the monodromy on $D_\infty$ is the composition of infinitely many fractional Dehn twists in a disjoint collection of circles, associated to the elamination $\Lambda_T$ in a concrete manner. The combinatorics of $\Lambda_T$ is rather complicated and beautiful; Theorem 7.9 and §9.3 describe some of its properties.

One intermediate result that we believe is interesting in its own right, is the discovery of a new class of affine complex surfaces which are $K(\pi,1)$s:

\textbf{Regular Value Theorem.} \textit{Let $Y_n$ be the space of degree 3 polynomials $z^3 + pz + q$ for which $n$ specific complex values (e.g. the $n$th roots of unity) are regular values. Then $Y_n$ is homotopic to a locally CAT(0) complex, and consequently is a $K(\pi,1)$.}

Even the case $n = 2$ is new, so far as we know.

1.1. \textbf{Apology.} ‘Butcher’ in the title of this paper and throughout is a rather inelegant pun on the name Böttcher which, Curt McMullen informs me, translates to \textit{cooper} in English (i.e. a maker of casks). However etymologically misguided, I have decided to keep ‘butcher’ for the sake of the sausages.

1.2. \textbf{Other Work.} I would like to compare and connect the constructions and techniques in this paper to prior and ongoing work of other mathematicians. First and foremost I would like to emphasize the resemblance of elements of the theory of dynamical elaminations to the DeMarco–Pilgrim theory of \textit{pictographs} as explained in [22] (to the degree that I understand them). In fact, DeMarco, sometimes in collaboration with Pilgrim or McMullen, has developed a sophisticated and intricate picture of the shift locus over many years and papers; e.g. [21, 20]. The fact that $S_d$ has the homotopy type of a $(d - 1)$-complex follows from DeMarco’s thesis [19], where it is proved that $S_d$ is a Stein manifold. I wish I better understood the relationship between her work and the point of view we develop here.

Recently, Blokh et. al. [2] have developed a theory of laminations to parameterize the pinching of components of (higher degree) Mandelbrot sets. I believe there is a family resemblance of their laminations to the tautological elamination we introduce in §7.1 and its variants and completions in higher degree, but the precise relationship is unclear.

The significance of configuration-space techniques (e.g. braiding of roots, attractors, etc.) to complex dynamics has been apparent at least since the work of McMullen [25] and Goldberg–Keen [23]. This is a vast story that I only touch on briefly in §10.2.

Branner–Hubbard [8], in a tour de force, found a detailed description of much of the parameter space of degree 3 polynomials. In particular, they showed that $S_3$ (away from a piece with easily understood topology) has the structure of a bundle over a circle (up to homotopy) whose fiber has free fundamental group. This is perfectly parallel to our Theorem 7.4. However, in their theory (which is more concretely tied to polynomials) the monodromy is completely opaque, and the culmination of their description (in §11.4) is only meant to indicate how formidable an explicit computation would be. Whereas in our theory, we have a completely
explicit (d)description of the fiber (it is the disk obtained by pinching the tautological elimination) and the monodromy (rotation by $F_t$).

2. The shift locus

Fix an integer $d > 1$, and let $f(z) = \sum b_j z^{d-j}$ be a complex polynomial of degree $d$, so that $b_0 \neq 0$. A change of variables $z \to \alpha z + \beta$ with $\alpha \in \mathbb{C}^*$ conjugates $f$ to a polynomial

$$f(z) = \sum \frac{b_j}{\alpha} (\alpha z + \beta)^{d-j} - \frac{\beta}{\alpha} = \alpha^{d-1} b_0 z^d + \alpha^{d-2} (d \beta b_0 + b_1) z^{d-1} + \cdots$$

Setting $\alpha = b_0^{1/(d-1)}$ and $\beta = -b_1/db_0$ we can put $f$ in normal form

$$f(z) = z^d + a_2 z^{d-2} + a_3 z^{d-3} + \cdots + a_d$$

There is non-uniqueness in the choice of $\alpha$; different choices differ by multiplication by a $(d-1)$st root of unity $\zeta$, which multiplies the coefficient $a_j$ by $\zeta^{d-j-1}$.

**Definition 2.1** (Shift locus). The shift locus of degree $d \geq 2$, denoted $S_d$, is the space of normalized degree $d$ polynomials $f$ for which every critical point of $f$ is in the attracting basin of infinity.

The critical points of $f$ are the roots of $f'$. To say a point $c$ is in the attracting basin of infinity means that the iterates $c, f(c), f^2(c), \cdots$ converge to infinity.

Note that the property of being in the shift locus is expressed in purely dynamical terms. Thus we could define $S_d$ to be the space of conjugacy classes of polynomials with a certain dynamical property. The relationship between that definition and the one we adopt comes down to an ambiguity of $\mathbb{Z}/(d-1)\mathbb{Z}$ in the representation of a conjugacy class by a normalized polynomial.

The coefficients of a normalized degree $d$ polynomial embed $S_d$ as a subset of $\mathbb{C}^{d-1}$. It is clear that $S_d$ is open, since for any polynomial $f$ the punctured disk $E(R) := \{ z : |z| > R \}$ is in the attracting basin of infinity for sufficiently big $R$ (depending continuously on $f$), and $f$ is in $S_d$ if and only if there is some integer $n$ so that $f^n(c) \in E(R)$ for all critical points $c$.

Recall the following definition:

**Definition 2.2** (Julia Set). The Julia set $J_f$ of a polynomial $f$ is the closure of the set of repelling periodic orbits of $f$.

The complement of $J_f$ in the Riemann sphere is the Fatou set $\Omega_f$; it is the maximal (necessarily open) set on which $f$ and all its iterates together form a normal family. Actually, it is perhaps more natural to take this to be the definition of the Fatou set, and to define the Julia set to be its complement. The Julia set and the Fatou set are both totally invariant (i.e. $f(J_f) = J_f = f^{-1}(J_f)$) and similarly for $\Omega_f$). The Julia set is always nonempty and perfect. See e.g. Milnor [26], § 4.

**Proposition 2.3.** A polynomial $f$ is in the shift locus if and only if the Julia set $J_f$ is a Cantor set on which the action of $f$ is uniformly expanding.

**Proof.** If $J_f$ is a Cantor set, its complement is connected and is therefore equal to the attracting basin of infinity. If $f$ is uniformly expanding on $J_f$ then $|f'|$ is bounded below on $J_f$ by a positive constant, so $J_f$ can’t contain any critical points and $f$ is in the shift locus.
Conversely, suppose \( f \) is in the shift locus. Since \( \infty \) is an attracting fixed point, there is a connected neighborhood \( U \) of \( \infty \) with \( f(U) \subset U \). Because \( f \) is a polynomial, \( \infty \) is its own unique preimage under \( f \); it follows by induction that for each \( n \), the set \( V_n := f^{-n}(U) \) is both forward-invariant and connected (because each component contains \( \infty \)). Because \( f \) is in the shift locus, there is an \( n \) so that all the critical points are contained in \( V_n \). Let \( K \) be the complement of \( V_n \), so that \( K \) is a finite union of disks.

Because all the critical points are in \( V_n \), each point in \( K \) has exactly \( d \) distinct preimages; these vary continuously as a function of \( K \), and since each component \( D \) of \( K \) is simply-connected, \( f^{-1}|D \) has \( d \) well-defined continuous branches with disjoint image. By the Schwarz Lemma the branches of \( f^{-1} \) are uniformly contracting in the hyperbolic metric on each component of \( K \); thus the diameters of the components of \( f^{-n}(K) \) converge (at a geometric rate) to zero, so that \( \Lambda := \cap_n f^{-n}(K) \) is totally disconnected and \( f \) is uniformly expanding on \( \Lambda \).

Evidently the complement of \( \Lambda \) is the basin of infinity, so \( \mathcal{J}_f = \Lambda \). Since \( \mathcal{J}_f \) is always perfect, it is a Cantor set, and \( f \) is uniformly expanding on \( \mathcal{J}_f \), as claimed.

Example 2.4 (Mandelbrot set). A quadratic polynomial \( z \to z^2 + c \) has 0 as its unique critical point. The set of \( c \in \mathbb{C} \) for which 0 is not in the basin of infinity of \( z \to z^2 + c \) is called the Mandelbrot set \( M \); see Figure 1. Thus \( M \) is the complement of \( S_2 \) in \( \mathbb{C} \). The connectivity of the Mandelbrot set (proved by Douady and Hubbard [18]) is equivalent to the fact that \( S_2 \) is homeomorphic to an (open) annulus.

Example 2.5 (Discriminant complement). Let \( f(z) \) be any degree \( d \) polynomial with distinct roots (i.e. for which 0 is not a critical value). Then \( g(z) := \lambda f(z) \) is (conjugate to a polynomial) in the shift locus for \( |\lambda| \gg 1 \). To see this, let \( U \) be any neighborhood of infinity for which \( f(U) \) does not contain 0. Then for sufficiently large \( |\lambda| \) we have \( g(U) \subset U \) (so that \( U \) is contained in the attracting basin of infinity for \( g \)). Furthermore, \( g \) and \( f \) have the same critical points, so for sufficiently large \( |\lambda| \) we have \( g(c) \in U \) for every critical point \( c \) of \( g \).
We can think of this as showing that near infinity, $S_d$ is ‘nearly equal’ to the complement of the discriminant locus $\mathbb{C}^{d-1} - \Delta$. We shall elaborate on this remark in the sequel.

**Example 2.6 (Cantor set $J_f$).** In degree two, $J_f$ is a Cantor set precisely when $f$ is in the shift locus, but for degree bigger than two it is possible for $J_f$ to be a Cantor set for $f$ not in the shift locus.

For example, consider the polynomial $f(z) := \alpha z(z - 1)^2$ with $\alpha$ real and positive. The fixed points are 0 and $\beta_{\pm} := 1 \pm \sqrt{1 - (\alpha - 1)/\alpha}$ and the critical points are $1/3$ and 1. Since $f(1) = 0$, the polynomial $f$ is never in the shift locus. If $f(1/3) > \beta^+$ then $f^{-1}([0, \beta^+])$ is real and properly contained in $[0, \beta^+]$, and $J_f = \cap_n f^{-n}([0, \beta^+])$ is a totally real Cantor set. This happens for $\alpha > 9$.

In the limiting case $\alpha = 9$, the Julia set $J_f$ is the real interval $[0, 4/3]$.

Suppose $f$ is in the shift locus, so that $J_f$ is a Cantor set, equal to the complement of the basin of infinity. Then $f$ has $d$ distinct fixed points, all in $J_f$.

Because the dynamics of $f$ on $J_f$ is expanding, it is structurally stable there. So if $f_t$ is a family of polynomials in the shift locus with Julia sets $J_{f_t}$, there are open sets $U(t)$ containing $J_{f_t}$, and maps $\varphi_t : U(0) \to U(t)$ conjugating $f_t|U(t)$ to $f_0|U(0)$. In particular, we obtain a *monodromy representation* $\rho$ from the fundamental group $\pi_1(S_d)$ to the *mapping class group* of $\mathbb{C} -$ Cantor set. This is an example of a so-called *big mapping class group*; see e.g. [30] for background and an introduction to the theory of such groups.

The dynamics of any $f$ on $J_f$ is conjugate to the action of the shift on the space of one-sided sequences in a $d$ letter alphabet; this justifies the name. One way to see this is to take a compact $K$ containing $J_f$ in its interior for which $f|K$ has $d$ inverse branches $f_1, \cdots, f_d$, and the $f_j(K)$ are disjoint subsets of the interior of $K$. Then $J_f$ is in bijection with the set of right infinite words in the $\{f_j\}$.

The geometry of $S_d$ is very complicated. For $d = 2$ the space $S_2$ is the complement in $\mathbb{C}$ of the Mandelbrot set; showing that $S_2$ is conformal to a punctured disk is equivalent to showing that the Mandelbrot set is connected. The main goal of this paper is to develop tools to describe the topology of $S_d$ for higher $d$.

### 3. Elaminations

In this section we introduce the concept of an *elamination*. Laminations, as introduced by Thurston, are a key tool in low-dimensional geometry, topology and dynamics; see e.g. [27], Chapter 8.5. The reader already familiar with laminations can think of the term ‘elamination’ as an abbreviation for ‘extended lamination’, or ‘enhanced lamination’ — an ordinary lamination with some extra structure.

Elaminations are an essential combinatorial tool that will be used throughout the sequel, especially beginning with §4 so throughout this section we just spell out the basic theory, deferring the connection to dynamics until the sequel. There are some points of contact between elaminations — and in particular the ‘collision topology’ on the space $\mathcal{EL}$ — to the theory partially developed by Thurston in [29]; but there are many points of difference, and it seems pointless to try to force the two theories into a common framework.

Elaminations (and laminations for that matter) have several more-or-less equivalent identities, and it is useful to be able to move back and forth between them.
By abuse of notation, we will often use the same symbol or term to refer to the underlying abstract object or any of its equivalent manifestations.

We fix the following notation here and throughout the rest of the paper: let $\mathbb{D}$ denote the closed unit disk in the complex plane $\mathbb{C}$, and let $\mathbb{E} := \mathbb{C} - \mathbb{D}$ denote its open exterior.

**Definition 3.1** (Circle Lamination). A leaf is a finite subset of the unit circle of cardinality at least 2. A leaf is simple if it consists of 2 points; a leaf of multiplicity $n$ consists of $n + 1$ points.

A circle lamination is a set of leaves, no two of which have 2 element subsets that are linked. A circle lamination is simple if all its leaves are simple.

Most authors require laminations to be closed in the space of finite subsets of $S^1$ (in the Hausdorff topology), but we explicitly do not require this.

**Definition 3.2** (Geodesic Lamination). A simple geodesic leaf is a complete geodesic in $\mathbb{D}$ with its hyperbolic metric. A geodesic leaf of multiplicity $n > 1$ is an ideal $(n + 1)$-gon.

A geodesic lamination is a set of geodesic leaves no two of which cross in $\mathbb{D}$. A geodesic lamination is simple if all its leaves are simple.

Every ideal $(n + 1)$-gon in $\mathbb{D}$ determines an unordered set of $n + 1$ endpoints in $S^1$ and conversely. Two $(n + 1)$-gons in $\mathbb{D}$ cross if and only if two pairs of their endpoints link in $S^1$. Thus there is a natural correspondence between circle laminations and geodesic laminations.

**Definition 3.3** (Elamination). For each $z \in \mathbb{E}$ we let $\ell(z)$ denote the straight line segment from $z/|z|$ to $z$. We call $\ell(z)$ a radial segment. The height of the segment $\ell(z)$ is $\log(|z|)$.

An extended leaf of height $h > 0$ is the union of a geodesic leaf in $\mathbb{D}$ (the vein) with radial segments in $\mathbb{E}$ (the tips) all of height $h$, attached at the endpoints of the vein. An extended leaf is simple if the vein is simple.

An extended lamination, or elamination for short, is a set of extended leaves with the following properties:

1. lamination: distinct leaves have distinct veins, and the set of all veins of all leaves forms a geodesic lamination (called the vein of the elamination);
2. properness: there are only finitely many extended leaves with height $\geq \epsilon$ for any $\epsilon > 0$ (thus every elamination has only countably many leaves); and
3. saturation: to be defined below.

Let us now explain the meaning of saturation. Let $\Lambda$ be an elamination, and let $\ell$ be a leaf with height $h$. Let $pq$ be an oriented edge of $\ell$, and let $L$ be the finite set of leaves of $\Lambda$ on the positive side of $pq$ with height $\geq h$. Let $L_p$ (resp. $L_q$) denote the subset of $L$ of leaves with an endpoint with the same argument as $p$ (resp. $q$). Since leaves of $\Lambda$ do not cross, and distinct leaves have distinct veins, the leaves $L_p$ are ordered by how they separate each other from $pq$: thus if $L_p$ is nonempty there is a closest $\ell_p \in L_p$ to $pq$ (and similarly for $L_q$). A leaf $\ell_p$ (resp. $\ell_q$) if it exists, is called an elder sibling for $\ell$ at $p$ (resp. at $q$).

Saturation means the following two conditions hold for every $\ell$:

1. an elder sibling of $\ell$ has height $h'$ strictly bigger than $h$; and
2. if $L_p$ is nonempty so is $L_q$ and vice versa; and furthermore $\ell_p = \ell_q$. 
We say that a leaf $\ell$ is saturated by an elder sibling. Another way to say this is that if the vein of $\ell$ shares one endpoint with the vein of a taller leaf $\ell'$, and there are no other $\ell''$ (also taller than $\ell$) in the way, then the vein of $\ell$ actually shares two endpoints with $\ell'$.

3.1. Pinching. Let $\Lambda$ be an elamination. We define an operation called *pinching* that associates to $\Lambda$ a Riemann surface $\Omega$ obtained from $E$ by suitable cut and paste along the tips of $\Lambda$.

**Construction 3.4 (Pinching).** Let $\Lambda$ be an elamination. For each leaf $\lambda$ with multiplicity $n$ and with tips $\sigma_0, \sigma_1, \ldots, \sigma_n$ enumerated in cyclic order in $S^1$, cut open $E$ along the $\sigma_j$ and glue the right side of each $\sigma_j$ to the left side of $\sigma_{j-1}$ (indices taken mod $n+1$) by a Euclidean isometry. The resulting Riemann surface $\Omega$ is said to be obtained from $\Lambda$ by pinching. We also write $\Omega = E \mod \Lambda$.

**Lemma 3.5 (Planar).** $\Omega$ obtained from an elamination $\Lambda$ by pinching is planar.

**Proof.** This is equivalent to the fact that the leaves do not cross. □

By construction, the function $\log |\cdot| : E \to (0, \infty)$ is preserved under pinching, and therefore descends to a well-defined proper function on $\Omega$ that we refer to as the *height function* or sometimes as the *Green’s function*, and denote $h$. Furthermore, $d\arg$ is a well-defined 1-form on $\Omega$, so the level sets of the height function are finite unions of metric graphs. We sometimes denote $d\arg$ by $d\theta$. In fact, the combination $dh + id\theta$ is just the image of $d\log(z)$ on $E$, which makes sense because this 1-form is preserved by cut-and-paste. By abuse of notation therefore we sometimes write $dh + id\theta = d\log(z)$. This 1-form has a zero of multiplicity $m$ for each leaf of multiplicity $m$.

**Definition 3.6 (Monkey pants).** A *monkey pants* is a (closed) disk with at least two (open) subdisks removed. If $P$ is a monkey pants, a function $\pi : P \to [t_1,t_2]$ is *monkey Morse* if it is a submersion away from finitely many points in the interior which are all saddles or monkey saddles, and if $\pi^{-1}(t_2)$ is equal to a distinguished boundary component $\partial^+ P$ (the *waist*) and $\pi^{-1}(t_1)$ is equal to the other components $\partial^- P$ (the *cuffs*).

Let $\Omega$ be the Riemann surface associated to an elamination. If $0 < t_1 < t_2$ are numbers not equal to the height of any leaf, then $\Omega([t_1,t_2]) := h^{-1}[t_1,t_2] \subset \Omega$ is a monkey pants, and $h$ restricted to $\Omega([t_1,t_2])$ is monkey Morse. There is one saddle point for each simple leaf with height in $[t_1,t_2]$, and one monkey saddle with multiplicity equal to the multiplicity of a non-simple leaf.

Suppose $\Lambda$ is a finite elamination, which pinches $E$ to $\Omega$. Then $\Omega$ is a plane minus $n+1$ disks, where $n$ is the number of leaves of $\Lambda$ counted with multiplicity. If $t$ is the least height of leaves of $\Lambda$, then $\Omega((0,t))$ is a disjoint union of $n+1$ annuli whose inner ‘boundary components’ (where $h \to 0$) can be compactified by $n+1$ circles. We refer to this collection of circles as $S^1 \mod \Lambda$. Thus: just as $E$ is compactified (away from $\infty$) by $S^1$, the surface $E \mod \Lambda$ is compactified (away from $\infty$) by $S^1 \mod \Lambda$.

3.2. Push over and amalgamation. Denote the set of elaminations by $\mathcal{EL}$. We would like to define a natural topology on $\mathcal{EL}$. In a nutshell, a family of elaminations...
Λ_t in \( \mathcal{EL} \) varies continuously if and only if the Riemann surfaces \( \Omega_t = \mathbb{E} \mod \Lambda_t \) do.

Because of properness, an elamination \( \Lambda \) has only finitely many leaves of height bigger than any positive \( \epsilon \). When these leaves have disjoint veins, it is obvious what it means to say that they vary continuously in a family: it just means that the heights and arguments vary continuously.

When two leaves of different heights collide, the shorter leaf becomes saturated by the taller (which becomes at that moment its elder sibling); if we continue the motion in the obvious way, the shorter leaf becomes unsaturated as it moves away from the taller leaf, and the net result is that the shorter leaf has been pushed over the taller one. The meaning of this is illustrated in Figure 2.

![Figure 2](image2)

**Figure 2.** Pushing a shorter leaf over a taller one; at the intermediate step the shorter leaf is saturated by the taller one.

When two leaves of the same height collide, saturation dictates that they must become amalgamated into a common leaf; see Figure 3.

![Figure 3](image3)

**Figure 3.** When two simple leaves of the same height collide, they amalgamate to form a leaf of multiplicity 2.

We now define a topology on \( \mathcal{EL} \) called the **collision topology**.

**Definition 3.7** (Collision Topology). A family of elaminations \( \Lambda_t \) varies continuously in \( \mathcal{EL} \) in the collision topology if every finite subset of leaves varies continuously when they are disjoint, and varies by push over or amalgamation when they collide.

The whole point of the collision topology is that it is compatible with pinching.

**Lemma 3.8** (Continuous quotient). If \( \Lambda_t \) varies continuously in \( \mathcal{EL} \) then \( \Omega_t \) vary continuously as Riemann surfaces.

**Proof.** The only thing to check is that push over and amalgamation are continuous under pinching; but this is essentially by definition. \( \square \)
4.1. Böttcher Coordinates. Let \( f(z) := z^d + a_2 z^{d-2} + \cdots + a_d \) be a degree \( d \) polynomial in normal form. Lucjan Böttcher, a Polish mathematician who worked in Lvov in the beginning of the 20th century, showed [3] that \( f \) is conjugate to \( z \to z^d \) in a neighborhood of infinity:

**Proposition 4.1** (Böttcher Coordinates). Let \( f(z) := z^d + a_2 z^{d-2} + \cdots + a_d \) be a degree \( d \) polynomial in normal form. Then \( f \) is holomorphically conjugate to \( z \to z^d \) on some neighborhood of infinity.

For a proof see e.g. Milnor [26], Thm. 9.1.

4.2. Holomorphic 1-form. Let’s let \( \phi \) be the holomorphic conjugacy promised by Proposition 4.1 normalized so that \( \phi f \phi^{-1}(z) = z^d \) near infinity. The map \( \phi \) is only defined in a neighborhood of infinity, but we can extend it inductively over larger and larger domains by using the functional equation. Recall that \( \mathcal{E} \) denotes the exterior of the closed unit disk in \( \mathbb{C} \); i.e. \( \mathcal{E} \) is the basin of infinity of \( z \to z^d \).

The function \( \log z \) is not single-valued on \( \mathcal{E} \), but its differential \( dz/z \) is. The map \( z \to z^d \) pulls back \( dz/z \) to \( d \cdot dz/z \) (we use the notation \( d \cdot \) to indicate multiplication by the degree \( d \) to distinguish it from the exterior derivative of forms). If we define \( \alpha := \phi^* dz/z \) in a neighborhood of infinity, we can extend \( \alpha \) uniquely to all of the Fatou set \( \Omega_f \) by iteratively solving \( f^* \alpha = d \cdot \alpha \). Thus \( \alpha \) is a holomorphic 1-form on \( \Omega_f \) with zeroes at the critical points of \( f \) and their preimages.

4.3. Horizontal/Vertical foliations. The real and imaginary parts of \( \alpha \) and \( dz/z \) give rise to foliations on \( \Omega_f \) and on \( \mathcal{E} \) related by \( \phi \) near infinity. We call these the horizontal and the vertical foliations respectively.

On \( \mathcal{E} \) these foliations are nonsingular; the horizontal leaves are the circles \( |z| = \infty \) and the vertical leaves are the rays \( \arg(z) = \) constant. The corresponding foliations on \( \Omega_f \) have saddle singularities at simple critical points and their preimages, and monkey saddle singularities at critical points (and their preimages) of multiplicity bigger than one (as roots of \( f' \)). Evidently \( \phi \) may be extended by analytic continuation along every nonsingular vertical leaf, and along every singular leaf from infinity until the first singularity. These singularities are critical points and their preimages; this is a proper subset of \( \Omega_f \).

4.4. Construction of the dynamical elimination. Let \( L_f \subset \Omega_f \) be the complement of this (maximal) domain of definition of \( \phi \), and \( L \subset \mathcal{E} \) the complement of \( \phi(\Omega_f - L_f) \). These subsets are both closed and backwards invariant. The complements \( \Omega_f - L_f \) and \( \mathcal{E} - L \) are open, simply connected, and dense. The set \( L \) consists of a countable collection of radial segments; in the generic case there are exactly two such segments \( \ell(q^\pm) \) for each critical or pre-critical point \( p \). One may think of \( q^\pm \) as the ‘image’ of \( p \) under \( \phi \). If \( c \) is a simple critical point with image \( v = f(c) \) then \( \phi(v) \) will have \( d \) preimages under \( z \to z^d \), whereas \( v \) will only have \( d - 1 \) preimages under \( f \); the two of the preimages of \( \phi(v) \) that correspond to \( c \) are \( q^\pm \).

**Example 4.2.** If \( f \) has real coefficients, \( \phi \) preserves the real axis. Thus the vertical leaves with \( \arg(\phi(z)) \in \pi d^{-n} \mathbb{Z} \) consist of the \( z \) with \( f'(z) \) real. The polynomial \( f(z) := z^3 + 3z + 3^{-1/2} \) has critical points at \( \pm i \) with initial forward orbit

\[
\pm i \to 3^{-1/2} \pm 2i \to -23 \cdot 3^{-3/2} \approx -4.42635
\]
Figure 4 shows some vertical leaves in $\Omega_f$ and in $E$ in the preimage of the negative real axis. $L_f$ and $L$ are in red. The set $L_f \cup J_f$ is a dendrite.

Note that $\arg(\phi(f^2(i))) = \pi$ and $\arg(\phi(f(i))) = \pi/3$. The absolute value $|\phi(i)|$ is well-defined, and equal to approximately 1.18, but $\arg(\phi(i))$ is multi-valued, and takes values $7\pi/9$ and $\pi/9$.

One may repair this multi-valuedness of $\phi$ by doing cut-and-paste on $E$: cut open $E$ along the segments $L$ and reglue edges in pairs, so that each copy of $\ell(q^+)$ is glued to a copy of $\ell(q^-)$ in the unique manner which is orientation-reversing and compatible with the dynamics $z \to z^d$. The result is a new Riemann surface $\Omega$ on which the map $z \to z^d$ on $E - L$ extends uniquely to a holomorphic degree $d$ map $F : \Omega \to \Omega$ and for which $\phi : \Omega_f - L_f \to E - L$ extends to a holomorphic isomorphism $\phi : \Omega_f \to \Omega$ conjugating $f$ to $F$.

Another way to say this is that $L$ is the set of tips of a simple elamination $\Lambda$, with one leaf for each pair $\ell(q^\pm)$. And $\Omega$ is precisely the Riemann surface obtained from $\Lambda$ by pinching, together with the 1-form $dz/z$ whose real and imaginary parts are the (derivatives of) height and argument respectively.

When one talks about constructing a Riemann surface by gluing Euclidean polygons, one sometimes says the Riemann surface is built ‘from paper’ (see e.g. [14]). As a mnemonic therefore, and by abuse of homonymy, we say that $\Omega$ is built from butcher paper.

In case some critical points are not simple, there might be three (or more) segments in $L$ associated to some (pre)-critical points, and some segment $\ell(q^\pm)$ associated to a critical point $c$ might be a subsegment of some precritical $\ell(r^\pm)$ associated to another critical point. Exactly as in the simple case, these sets form the tips of the leaves of an elamination $\Lambda$ (no longer simple) and $\Omega = E \mod \Lambda$.

**Definition 4.3** (Dynamical Elamination). The elamination $\Lambda$ obtained from $f$ as above is called the *dynamical elamination* associated to $f$.

If we need to stress the dependence of $\Lambda$ on $f$ we denote it $\Lambda(f)$.

**Lemma 4.4.** The assignment $\Phi : f \to \Lambda(f)$ is a continuous function from $\mathcal{S}_d$ to $\mathcal{E}\mathcal{L}$ that we call the butcher map.
Proof. The Fatou sets $\Omega_f$ together with their vertical/horizontal foliations vary continuously as a function of $f$. Since $\Lambda(f)$ can be recovered from $\Omega_f$ under the identification of $E \mod \Lambda(f)$ with $\Omega_f$, and since we defined the topology on $EL$ so that the inverse of pinching is continuous, the lemma follows.

5. Formal shift space

In this section we shall characterize the dynamical elaminations $\Lambda(f)$ that arise from shift polynomials by the construction in §4.4 and describe an inverse map. The existence of this inverse is the Realization Theorem 5.4, due essentially to DeMarco–McMullen, although we express things in rather different language.

In this section we use logarithmic coordinates and fix the notation $\log(z) = r + i\theta$ for $z \in E$, so that $r \in \mathbb{R}^+$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and we denote the radial segment associated to $z$ by $\ell(r, \theta)$. In $(r, \theta)$ coordinates, the map $z \to z^d$ acts as multiplication by $d$. We call $r$ the height and $\theta$ the angle of the segment $\ell(r, \theta)$.

5.1. Dynamical Elaminations. The geometry and combinatorics of $L$ is best expressed in the language of elaminations. Let’s fix the degree $d$ in what follows.

Definition 5.1 (Critical data). A (degree $d$) critical leaf is an extended leaf whose tips have angles that are equal mod $2\pi d^{-1}$.

If $C_1, \ldots, C_e$ is a finite set of degree $d$ critical leaves, we say the critical multiplicity of $C_j$ is equal to its ordinary multiplicity, minus 1 for every $C_k$ with greater height which shares a pair of ideal points with $C_j$.

A (degree $d$) critical set is a finite elamination consisting of degree $d$ critical leaves $C_1, \ldots, C_e$ whose critical multiplicities sum to $d - 1$.

The map $z \to z^d$ acts on radial segments by $\ell(r, \theta) \to \ell(dr, d\theta)$. This induces a (partially) defined action on extended leaves, that might reduce multiplicity if distinct tips have angles that differ by a multiple of $2\pi d^{-1}$. If $\lambda$ is a leaf for which all tips have angles that differ by a multiple of $2\pi d^{-1}$, the image of $\lambda$ under $z \to z^d$ is undefined. For instance, $z \to z^d$ is undefined on any critical leaf. If $P$ is a leaf, we denote its image under $z \to z^d$ by $P^d$.

Definition 5.2 (Dynamical Elamination). A dynamical elamination $L$ is an elamination containing a finite subset of leaves $C$ which is a degree $d$ critical set, and such that $z \to z^d$ maps $L - C$ to $L$ in a $d$ to $1$ manner. We say $L$ is generated by $C$.

Figure 5 indicates a simple dynamical elamination of degree 3.

Proposition 5.3 (Dynamical elamination). Let $C$ be a degree $d$ critical set. Then there is a unique dynamical elamination $L$ generated by $C$.

Proof. Recall that the notation $S^1 \mod C$ denotes the result of pinching the unit circle along $C$. From the definition of a critical set, $S^1 \mod C$ is the union of $d$ disjoint circles, each canonically isomorphic to $\mathbb{R}/\frac{1}{d}\mathbb{Z}$ (with respect to the angle coordinates it inherits from $S^1$). Thus the map $z \to z^d$ maps each of these circles isomorphically to the unit circle. An extended leaf in $S^1 \mod C$ canonically pulls back to an extended leaf on the unit circle by taking the preimage of the tips to be the tips of the preimage. We may therefore inductively construct $L$ as the union of $L_n$ where $L_0 = C$ and $L_j$ is obtained from $L_{j-1}$ by taking the preimages of $L_j$ in $S^1 \mod C$ and pulling back to an elamation on $S^1$. Uniqueness is clear. □
We refer to the preimages of the critical leaves as precritical leaves, and we say that the depth of a precritical leaf \( P \) is the number of iterates of the dynamical map which take it to some \( C_i \).

5.2. **Realization.** Let \( L \) be a degree \( d \) dynamical elimination generated by \( C \), and let \( \Omega \) be the Riemann surface obtained from \( L \) by pinching. The map \( z \to z^d \) induces a degree \( d \) proper holomorphic map \( F \) from \( \Omega \) to itself with \( d - 1 \) critical points counted with multiplicity, which are the endpoints of the tips of the \( C \).

The Realization Theorem says that the action of \( F \) on \( \Omega \) is holomorphically conjugate to the action of some (unique) shift polynomial \( f \) on its Fatou set.

**Theorem 5.4** (Realization). Let \( C \) be a degree \( d \) critical set with dynamical elimination \( L \) and associated Riemann surface \( F : \Omega \to \Omega \). Then there is a unique conjugacy class of degree \( d \) polynomial \( f \) in the shift locus for which \( f|\Omega_f \) is holomorphically conjugate to \( F|\Omega \).

Essentially the same theorem is proved by DeMarco–McMullen [21], Thm. 7.1 although in different language, and with quite a different proof. One distinctive feature of our proof of Theorem 5.4 is that it finds the desired embedding of \( \Omega \) in \( \mathbb{C}P^1 \) by a rapidly convergent algorithm; we expect this might be useful e.g. for computer implementation.

**Proof.** The Riemann surface \( \Omega \) has one isolated puncture (corresponding to \( \infty \)) and a Cantor set \( J \) of ends (the ‘image’ of the unit circle under iterated cut-and-paste along \( \partial^+ L \)). The map \( F \) extends holomorphically over the isolated puncture; we claim that it also extends (uniquely, holomorphically) over \( J \). The resulting
extension will be a degree $d$ holomorphic self-map from a sphere to itself, which is conjugate to a polynomial.

We now explain how to extend the dynamics of $F$ over $J$ holomorphically. Let $X$ be the subset of $\Omega$ consisting of points with height $\leq t$ where $t$ is less than the height of any critical leaf, and let $Y$ be the closure of $X - F^{-1}(X)$. Then $Y$ is a (typically disconnected) compact planar surface with outer boundary $\partial^+ Y := \partial X$, and inner boundary $\partial^- Y := \partial Y - \partial^+ Y$. The map $F : \partial^- Y \to \partial^+ Y$ is a $d$-fold covering map for which every component maps homeomorphically to its image; thus we may define $F_1, \ldots, F_d : \partial^+ Y \to \partial^- Y$ to be branches of $F^{-1}$ with disjoint images whose union is $\partial^- Y$.

Suppose that $\partial^+ Y = \partial X$ has $e$ components. Let $D$ denote the disjoint union of $e$ copies of the unit disk $\mathbb{D}$. We would like to find a holomorphic embedding $\psi : X \to D$, so that $J := D - \psi(X)$ is a Cantor set, and so that $F$ (or, really, its conjugate by $\psi$) extends holomorphically over $J$.

Let $T$ denote the Teichmüller space of holomorphic embeddings $\psi : Y \to D$ taking components of $\partial^+ Y$ to components of $\partial D$, and normalized to take fixed values on three marked points on each component. We define a skinning map $\sigma : T \to T$ as follows. Given $\psi$, we cut out $D - \psi(Y)$ and sew in $d$ copies of $D$ by gluing their boundaries to $\psi(\partial^- Y)$ along the identifications

\[
\partial D \xrightarrow{\psi^{-1}} \partial^+ Y \xrightarrow{F_j} \partial^- Y \xrightarrow{\psi} \psi(\partial^- Y)
\]

We then uniformize the resulting surface $D'$ to obtain a holomorphic identification $D' \to D$, and the restriction of this uniformization to $Y$ (which we identify with its image in $D'$ under $\psi$) is $\sigma(\psi)$. The skinning map is holomorphic, and therefore distance non-increasing in the Teichmüller metric. In fact it is evidently strictly distance decreasing; furthermore, orbits are easily seen to be bounded. Thus $\sigma$ is uniformly strictly distance decreasing, and there is a (unique) fixed point (actually convergence to the fixed point is easy to see directly by considering moduli of accumulating annuli around points of $J$).

By construction, this fixed point gives the desired embedding of $X$ and extension of $F$. □

We denote by $\mathcal{DL}_d$ the space of degree $d$ dynamical elaminations, thought of as a subspace of $\mathcal{EL}$. Theorem 5.4 produces a continuous inverse to the butcher map $\Phi : S_d \to \mathcal{EL}$ called the realization map $\Psi : \mathcal{DL}_d \to S_d$; in particular, the spaces $S_d$ and $\mathcal{DL}_d$ are homeomorphic.

The location of the tips of the critical leaves define local holomorphic coordinates on $\mathcal{DL}_d$ giving it the structure of a complex manifold. With respect to these coordinates, $\Phi$ and $\Psi$ are holomorphic; thus $\mathcal{DL}_d$ and $S_d$ are isomorphic as complex manifolds.

5.3. Squeezing. There is a free proper $\mathbb{R}$ action on $\mathcal{DL}_d$ which simultaneously multiplies the heights of the critical leaves by some fixed positive real number $e^t$. We call this transformation squeezing, and refer to the $\mathbb{R}$ action as the squeezing flow.

Since the squeezing flow is (evidently) proper, it gives $\mathcal{DL}_d$ the structure of a global product:

**Corollary 5.5.** Each $\mathcal{DL}_d$ is homeomorphic to a product $\mathcal{DL}_d = X_d \times \mathbb{R}$ where $X_d$ is a real manifold of dimension $2d - 3$. 
For concreteness, we may think of \(X_d\) as the subspace of \(\mathcal{DL}_d\) where the largest critical height is equal to 1.

5.4. Rotation. If \(P\) is a leaf in \(L\), we let \(e^{i2\pi t}P\) denote the result of rotating \(P\) anticlockwise through \(t\), mod leaves of greater height. This makes sense unless \(P\) collides with a leaf of the same height. If \(P\) and \(Q\) are leaves of different height, the operations of rotating \(P\) and rotating \(Q\) commute.

If \(L\) is a dynamical elamination of degree \(d\) with distinct critical leaves, let \(L_j\) be the critical leaf \(C_j\) and its preimages. Suppose no two critical leaves have heights whose ratio is a power of \(d\); we say \(L\) has generic heights. Then for a vector \(s := s_1, \cdots, s_{d-1}\) of real numbers we can simultaneously rotate all the leaves of each \(L_j\) of height \(h\) through angle \(hs_j\), mod leaves of greater height; since leaves of the same height are all rotated through the same angle, they never collide and this operation is well-defined. Denote the result by \(F_sL := \bigcup_j e^{i2\pi hs_j}L_j\).

Lemma 5.6 (Torus orbits). If \(L\) is a degree \(d\) dynamical elamination with generic heights \(h(C)\), then \(F_sL \in \mathcal{DL}_d\). Furthermore the orbit map \(\mathbb{R}^{d-1} \to \mathbb{D}L_d\) factors through a torus \(T_L := \mathbb{R}^{d-1}/\Gamma_L\) where \(\Gamma_L\) is contained in \(d^{-n}h(C)^{-1}\mathbb{Z}^{d-1}\) for some \(n\).

Proof. By induction, for each precritical leaf \(P\) we have \((e^{i\theta}P)^d = e^{i\theta d}P^d\) mod leaves of greater height. Thus \(F_sL\) is a degree \(d\) dynamical elamination.

For each critical leaf \(C_j\) the angles of \(C_j\) vary continuously in a component of \(S^1\) mod leaves of greater height. Since the angles of these leaves of greater height all differ by multiples of \(d^{-n}\) for some fixed \(n\), the length of this component is a multiple of \(\mathbb{R}/d^{-n}\mathbb{Z}\). The lemma follows. \(\square\)

6. Degree 2

Our goal in the sequel is to investigate the topology and combinatorics of \(S_d\). As a warm-up, and in order to introduce the main ideas in a relatively clean context, we describe in the next few sections the special cases of degrees 2, 3 and 4. After developing the theory of the past few sections, the case of degree 2 is almost a triviality.

Theorem 6.1 (Douady–Hubbard [18]). The space \(S_2\) is holomorphically equivalent to a punctured disk.

Proof. A degree 2 dynamical elamination \(L\) is generated by a single (necessarily simple) critical leaf \(C\). The tips of \(C\) are of the form \(\ell(z)\) and \(\ell(-z)\) for some \(z \in \mathbb{E}\). Since every other leaf of \(L\) has smaller height than \(C\), the number \(z^2\) is a continuous function of \(\mathcal{DL}_2\), and conversely we can recover \(C\) and therefore \(L\) from \(z^2\). Hence \(\mathcal{DL}_2\) is holomorphically isomorphic to the quotient of \(\mathbb{E}\) by \(\pm 1\). \(\square\)

Corollary 6.2. The Mandelbrot Set \(M\) (i.e. the complement of \(S_2\) in \(\mathbb{C}\)) is connected.

7. Degree 3

7.1. The Tautological Elamination. Throughout this section we refer to the angles of a leaf \(P\) of an elamination as the arguments of the tips divided by \(2\pi\); thus angles take values in the circle \(S^1 = \mathbb{R}/\mathbb{Z}\).
For some small $\epsilon > 0$ and angles $t, s \in S^1$ let $L(t,s)$ be the degree 3 dynamical elamination with simple critical leaves $C_1, C_2$ where $C_1$ has height 1 and angles $\{t, t + 1/3\}$, and $C_2$ has height $1 - \epsilon$ and angles $\{s, s + 1/3\}$. Note that this forces $s \in (t + 1/3, t + 2/3)$.

If we fix $t$ and vary $s$ in $(t + 1/3, t + 2/3)$, then whenever $3^n s$ is equal to $t$ or $t + 1/3$, the leaf $C_2$ collides with a leaf $P$ of $L(t,s)$ which is a depth $n$ preimage of $C_1$. We define an elamination $\Lambda_T(t)$ whose leaves are the union of the leaves $P^3$ over all $P$ in all $L(t,s)$ of this kind.

**Example 7.1.** Let $t = 0$ and $s = 5/9$. Thus $C_1$ has angles $\{0, 1/3\}$ and $C_2$ has angles $\{5/9, 8/9\}$. There is a unique leaf $P$ with angles $\{s = 5/9, s'\}$ which collides with $C_2$ for which $P^3 = C_1$ and neither $P$ nor $P^3$ crosses $C_1$ or $C_2$ (actually, because $P$ is saturated by $C_2$, it has angles $\{s = 5/9, s', 8/9\}$ but we ignore this point, since the tips with angles $5/9$ and $8/9$ become equal in $P^3$ and it is the leaf $P^3$ that is in $\Lambda_T(0)$). The leaf $P^3$ has angles $\{3s = 2/3, 3s'\}$; since $9s' = 1/3 \mod \mathbb{Z}$, for $P^3$ not to cross $C_1$ or $C_2$ we must have $3s' = 7/9$. Thus, in order for $P$ not to cross $C_1$ or $C_2$ we must have $s' = 16/27$. See Figure 6.

![Figure 6. P and P^3 (in blue) have angles {5/9, 16/27, 8/9} and {2/3, 7/9}.](image)

The leaf $P^3$ with height $1/3$ and angles $\{2/3, 7/9\}$ is therefore a leaf of $\Lambda_T(0)$.

**Definition 7.2** (Tautological Elamination). Fix $t \in S^1$. The **tautological elamination** $\Lambda_T(t)$ is the union of $P^3$ over all leaves $P \in L(t,s)$ in the preimage of $C_1$ over all values of $s$ at which $C_2 \in L(t,s)$ collides with $P$.

If $P \in L(t,s)$ is a depth $n$ preimage of $C_1$ that collides with $C_2$, we refer to its image $P^3 \in \Lambda_T(t)$ as a depth $(n - 1)$ leaf of $\Lambda_T(t)$.

**Proposition 7.3.** For all $t$, $\Lambda_T(t)$ is an elamination. Furthermore, $\Lambda_T(t+s) = e^{i2\pi hs} \Lambda_T(t)$ for any $t, s$.

**Proof.** As we vary $C_2$ fixing its height, the preimages of $C_1$ are occasionally pushed over preimages of $C_2$ of greater height. But a depth 1 preimage $P$ of $C_1$ has height $1/3$, which is greater than the height of any preimage of $C_2$, so $P$ is only pushed over $C_2$ itself. Since the angles of $C_2$ differ by $1/3$, pushing $P$ over $C_2$ does not change its image $P^3$. So we can simply add $P^3$ to $\Lambda_T(t)$. 
Now imagine shrinking the height of $C_2$ to $1/3(1-\epsilon)$ and then varying its angles again. The depth 1 preimages of $C_1$ pinch the unit circle into smaller circles, and $C_2$ is confined to a single component. Since $C_1$ now has height $< 1/3$, the depth 2 preimages $Q$ of $C_1$ in this component have bigger height than any preimage of $C_2$, so they stay fixed until they collide with $C_2$, and we can simply add the $Q^3$ to $\Lambda_T(t)$. In other words: the depth 2 leaves of $\Lambda_T(t)$ are the cubes of the depth 2 preimages of $C_1$ in the component of $S^3$ pinched along the depth 1 preimages of $C_1$ containing $C_2$. It follows that these leaves are disjoint, and do not cross depth 1 leaves.

Inductively, shrink the height of $C_2$ to $3^{-n}(1-\epsilon)$. It is confined to a component of $S^1$ pinched along the depth $\leq n$ preimages of $C_1$, and as it moves around this component, it collides with some depth $(n+1)$ preimages $R$ of $C_1$ and we add $R^3$ to $\Lambda_T(t)$. It follows (as before) that these leaves are disjoint and do not cross leaves of depth $\leq n$. This proves that $\Lambda_T(t)$ is an elamination.

To see how $\Lambda_T(t)$ varies with $t$, shrink $C_2$ down to the height of a depth $n$ preimage $P$ it has just collided with. Then rotate $C_1$ and simultaneously rotate $C_2$ at speed $3^{-n}$ (modulo leaves of greater height) so that it continues to collide with $P$.

Figure 7 depicts subsets of the tautological elaminations up to depth six associated to $\theta_1 = 1/12$ in units where the unit circle has length 1.

**Figure 7.** Tautological elaminations $\Lambda_T(1/12)$ to depths 1, 2, 3, 4 and 6

### 7.2. Topology of $S_3$.

Let $\Omega(t) = E \mod \Lambda_T(t)$, and let $D_\infty(t)$ be the subsurface of $\Omega(t)$ of height $\leq 3(1-\epsilon)$. Then $D_\infty(t)$ is a disk minus a Cantor set, and as $t$ varies, the $D_\infty(t)$ vary by ‘rotating’ the level sets of height $h$ through angle $ht/3$. By Proposition 7.3 this family of motions for $t \in [0, 1]$ induces a mapping class $\varphi$ of $D(0)$ to itself. The mapping torus $N_\infty$ of $\varphi$ is the total space of a fiber bundle over $S^1$ whose fiber over $t$ is $D_\infty(t)$.

Figure 8 shows a tautological elamination $\Lambda_T(5/6)$ and the disk $D_\infty$ obtained by pinching it (to depth 7). These pictures were generated by the program **shifty** which pinches elaminations recursively one leaf at a time, instead of simultaneously pinching all leaves of fixed depth. Thus the picture of $D_\infty$ is only a combinatorial approximation, and is not conformally accurate.

**Theorem 7.4 (Topology of $S_3$).** The space $S_3$ is homeomorphic to a product $X_3 \times \mathbb{R}$ where $X_3$ is the 3-manifold obtained from the 3-sphere $S^3$ by drilling out a neighborhood of a right-handed trefoil and inserting the mapping torus $N_\infty$, so that the longitude intersects the circle $\partial D_\infty(t)$ at angle $t$.

**Proof.** This follows more or less directly from the definitions. Let’s examine the subspace $Y_3$ of $X_3$ for which $h(C_1) = 1$ and $h(C_2) \leq (1-\epsilon)$. If we fix $\theta_1$ and
the height \( h := h(C_2) \) then we obtain a (1-dimensional) subspace \( \Gamma(\theta_1, h) \) of \( Y_3 \). Evidently \( \Gamma(\theta_1, h) \) is obtained from the circle of possible \( \theta_2 \) values \([\theta_1 + 1/3, \theta_1 + 2/3]\)/endpoints by suitable cut and paste. By multiplying angles by 3 we can identify this space of \( \theta_2 \) values with the unit circle \( S^1 \); so \( \Gamma(\theta_1, h) \) is obtained from \( S^1 \) by cut and paste. We claim it is precisely equal to the result of cut and paste along the leaves of \( \Lambda_T(\theta_1) \) of height \( > h \).

To see this, think about a component \( \gamma \) of \( \Gamma(\theta_1, h) \); its preimage \( \tilde{\gamma} \) in \( S^1 \) is a union of segments. The discontinuities of \( \theta_2 \) in \( \Gamma(\theta_1, h) \) occur precisely when \( C_2 \) is pushed over a precritical leaf of \( C_1 \) of height \( > h \); thus the boundary of each component of \( S^1 - \tilde{\gamma} \) is a precritical leaf \( P \) of \( C_1 \) so that \( C_2 \) collides with \( P \) in some dynamical elamination \( L(\theta_1, s) \). But then by definition \( P^3 \) is a leaf of \( \Lambda_T(\theta_1) \), and all leaves of \( \Lambda_T(\theta_1) \) arise this way. This proves the claim, and shows that \( Y_3 \) is homeomorphic to \( N_\infty \).

It remains to show that \( X_3 - Y_3 \) is homeomorphic to the complement of the right handed trefoil. For each \( h \in (1/3, 1) \) the slice of \( X_3 \) for which \( h(C_2) = h \) is just a torus \( T \), with coordinates \( \theta_1 \in S^1 \) and \( \theta_2 \in [\theta_1 + 1/3, \theta_1 + 2/3] \)/endpoints. When \( h = 1 \) we can no longer distinguish \( C_1 \) and \( C_2 \), so this torus is quotiented out by the involution switching \( \theta_1 \) and \( \theta_2 \) coordinates; the quotient is a circle bundle over an interval with orbifold endpoints of orders 2 and 3 — see Figure 9. Thus \( X_3 - Y_3 \) is a circle bundle over a disk with two orbifold points, one of order 2 and one of order 3; this is the standard Seifert fibered structure on \( S^3 - \text{trefoil}. \)

7.3. Geometry and topology of \( X_3 \). Let \( \Lambda_T(\theta_1, n) \) denote the finite elamination consisting of the leaves of \( \Lambda_T(\theta_1) \) of depth \( \leq n \) (i.e. they correspond in the construction of the tautological elamination to depth \( n \) preimages of \( C_1 \)).

Let \( \Omega_n(\theta_1) \) be the Riemann surface obtained by pinching \( \Lambda_T(\theta_1, n) \) and let \( D_n(\theta_1) \) be the subsurface of height \( 3(1 - \epsilon) \). Then each \( D_{n+1}(\theta_1) \) is obtained by pinching \( D_n(\theta_1) \) along the depth \( (n + 1) \) leaves, and we can think of \( D_\infty(\theta_1) \) as the limit. Likewise we can define mapping tori \( N_n \) which are \( D_n(\theta_1) \) bundles over the \( \theta_1 \) circle \( S^1 \).
Figure 9. Quotient of the torus $T$ by the involution switching $\theta_1$ and $\theta_2$ is a circle bundle over an interval with orbifold endpoints of orders 2 and 3.

Let $M_n$ denote the result of inserting $N_n$ into the right-handed trefoil complement in $S^3$. Then $M_n$ is a link complement, $S^3 - K_n$ where $K_0$ is the trefoil itself and each $K_{n+1}$ is obtained from $K_n$ by (a rather simple) satellite of its components. The limit $K_\infty = S^3 - X_3$ is a Cantor set bundle over $S^1$; one sometimes calls such objects Solenoids.

We now state and prove two theorems, which describe $S_3$ in geometric resp. topological terms. The geometric statement is that $S_3$ is homotopic to a locally CAT(0) 2-complex. This means a 2-dimensional CW complex (in the usual sense) with a path metric of non-positive curvature; see e.g. [9] for an introduction to the theory of CAT(0) spaces.

The most important corollary of this structure for us is that a locally CAT(0) complex is a $K(\pi,1)$; the proof is a generalization of the usual proof of the Cartan–Hadamard theorem for complete Riemannian manifolds of nonpositive curvature (which are themselves examples of locally CAT(0) spaces). Thus (for example) $\pi_1(S_3)$ is torsion free, and has vanishing homology with any coefficients in dimension greater than 2.

**Theorem 7.5 (CAT(0) 2-complex).** $S_3$ is a $K(\pi,1)$ with the homotopy type of a locally CAT(0) 2-complex.

**Proof.** Up to homotopy, we can take $M_0$ to be the spine of the trefoil complement; this is the mapping torus of a theta graph by an order three isometry that permutes the edges by a cyclic symmetry. It can be thickened slightly to $M_0$ by gluing on a metric product (flat) torus times interval. Each $M_n$ has boundary a union of totally geodesic flat tori, and each $M_{n+1}$ is obtained by gluing a flat annulus whose boundary components are parallel geodesics in $\partial M_n$ (circlewise, the endpoints of a leaf of the tautological elamination of depth $(n + 1)$) and then gluing a flat torus times interval on each resulting boundary component to thicken. The union is homeomorphic to $X_3$.

Simply gluing the spines at each stage without thickening gives a homotopic complex which is evidently CAT(0).

**Corollary 7.6.** $\pi_1(S_3)$ is torsion-free, and homology with any coefficients vanishes in dimension greater than 2.
The topological statement is that $X_3$ is homeomorphic to a Solenoid complement of a particularly simple kind: one obtained as an infinite increasing union of iterated cables.

**Theorem 7.7 (Link complement).** The degree 3 shift locus $S_3$ is homeomorphic to $X_3 \times \mathbb{R}$ where $X_3$ is $S^3$ minus a Solenoid $K_\infty$ obtained as a limit of a sequence of links $K_n$ where

1. $K_0$ is the right-handed trefoil; and
2. Each component $\alpha$ of $K_n$ gives rise to new components $\alpha_0 \cup \alpha_c$ of $K_{n+1}$, where $\alpha_0$ is the core of a neighborhood of $\alpha$ (i.e. we can think of it just as $\alpha$ itself) and $\alpha_c$ is a finite collection of $(p_\alpha, q_\alpha)$ cables of $\alpha_0$, for suitable $p_\alpha, q_\alpha$.

**Proof.** The only thing to prove is the second bullet point. Let $\alpha$ be a component of $K_n$. The boundary of a tubular neighborhood of $\alpha$ is the mapping torus of a finite collection of boundary circles of $D_n(0)$ which are permuted by the monodromy $\varphi$. Let $m$ be the least power of $\varphi$ that takes one such boundary component $\gamma \subset \partial^- D_n(0)$ to itself. Then $\varphi^m$ acts on $\gamma$ by rotation through $2\pi p_\alpha/q_\alpha$.

The depth $(n+1)$ leaves of $\Lambda_T$ on the component $\gamma$ form a finite elamination permuted by $\varphi^m$. Think of this as determining a finite geodesic lamination of $D$. The complementary components are in bijection with the components $\gamma_j$ of $\partial^- D_{n+1}(0)$ obtained by pinching $\gamma$, and we must understand how $\varphi^m$ acts on them. A finite order rotation of $D$ has a unique fixed point — the center. So there is a unique component $\gamma_0$ invariant under $\varphi^m$, and all the other components are freely permuted with period $q_\alpha$. Evidently under taking mapping tori $\gamma_0$ is associated to the core $\alpha_0$ and the other $\gamma_j$ are associated to components $\alpha_c$ which are all $(p_\alpha, q_\alpha)$ cables of $\alpha_0$. \qed

**Corollary 7.8 (Homology of $S_3$).** $H_1$ and $H_2$ of $S_3$ (and of $\pi_1(S_3)$) is free abelian on countably infinitely many generators. $H_0 = \mathbb{Z}$ and $H_n = 0$ for all $n > 2$.

In fact, it is possible to get more precise information about the denominators $q_\alpha$, and in fact we are able to show:

**Theorem 7.9 (Powers of 2).** The orbit lengths under $\varphi$ of the cuffs of $D_n$ (and hence all denominators $q_\alpha$ in Theorem 7.7) are powers of 2.

In fact, the proof of Theorem 7.9 goes via arithmetic, and will be given in §9 technically, the proof is a consequence of Theorem 9.20 and Example 9.10. We do not actually know a direct combinatorial proof of this theorem in terms of the combinatorics of the tautological elamination, and believe it would be worthwhile to try to find one. We explore the combinatorics of the tautological elamination further in §9.5.

The tautological elamination has exactly $3^{n-1}$ leaves of depth $n$ and therefore $(3^n - 1)/2$ leaves of depth $\leq n$. It follows that $D_n$ is a disk with $(3^n + 1)/2$ holes. However, the monodromy $\varphi$ permutes these nontrivially, and $K_n$ has one component for each orbit.

The links $K_n$ have 1, 2, 5, 11 components for $n = 0, 1, 2, 3$, though the degrees with which these components wrap around the cores of their parents are quite complicated. Thickened neighborhoods of $K_n$ for $n = 0, 1, 2$ are depicted in Figure 10.
8. Degree 4 and above

8.1. Weyl chamber. As in the case of degree 3, we set $S_4 = X_4 \times \mathbb{R}$ where $X_4$ is the quotient of $S_4$ by the orbits of the squeezing flow.

Order the critical heights with multiplicity so that $h_1 \geq h_2 \geq h_3$ and define a map $\rho : X_4 \to \mathbb{R}^3$ with coordinates $t_j := -\log_4 h_j$. If we identify $X_4$ with the subspace for which $h_1 = 1$ then $t_1 = 0$ and the image of $\rho$ is the subset of $(t_2, t_3) \in \mathbb{R}^2$ with $0 \leq t_2 \leq t_3$. Another normalization is to set $\sum t_j = 0$ in which case the image of $\rho$ may be identified with the Weyl chamber $W$ associated to the root system $A_2$.

Within this chamber we have a further stratification. Define $t_{ij} := t_i - t_j$ and refer to the level sets $t_{ij} = n \in \mathbb{Z}$ as walls. The walls define a cell decomposition $\tau$ of $W$ into right angled triangles with dual cell decomposition $\tau'$.

We shall describe a natural partition of $X_4$ into manifolds with corners $X_4(v)$, for vertices $v$ of $\tau$, where $X_4(v)$ is defined to be the preimage under $\rho$ of the cell of $\tau'$ dual to $v$. These submanifolds are typically disconnected, and the way their components are glued up in $X_4$ will give $X_4$ the structure of a complex of spaces over a contractible $\tilde{A}_2$ building.

8.2. Two partitions. Let’s suppose critical leaves are simple, and we label them $C_j$ compatibly with the ordering on heights.

There are two combinatorially distinct ways for $C_1$ to sit in the circle: the angles of the segments are either antipodal, or they are distance $1/4$ apart (remember we are working in units where the circle has total length 1). When $h(C_1)$ is strictly larger than the other $h(C_j)$ the leaf $C_1$ is the unique leaf of greatest height. Thus the difference of the angles is locally constant; it follows that the subset of $X_4$ where $h(C_2) < 1$ is disconnected. In fact, it is easy to see it has exactly two components according to the placement of $C_1$.

Where $C_1$ is an antipodal leaf, it pinches the unit circle into two circles of length $1/2$, each bisected by one of $C_2$ and $C_3$. The restriction of the dynamical elimination in each of each of these length $1/2$ circles is symmetric under the antipodal map.

When $C_1$ is not antipodal, it pinches the unit circle into circles of length $1/4$ and $3/4$, with $C_2$ and $C_3$ both contained in the longer circle. The leaf $C_2$ pinches this circle into circles of length $1/2$ and $1/4$, and $C_3$ divides the length $1/2$ circle antipodally.
8.3. Monkey prisms, monkey turnovers. Let’s fix a generic \((t_2, t_3)\) in the interior of \(W\), so that none of \(t_2, t_3, t_3 - t_2\) are integers. Denote the fiber of \(\rho\) over \((t_2, t_3)\) by \(T(t_2, t_3)\). These fibers are disjoint union of 3-tori, orbits of the \(\mathbb{R}^3\) action \(\mathcal{F}_s\) on \(\mathcal{DL}_{3}\) described in Lemma 5.6. These tori piece together to form a product throughout each open triangle of \(\tau\). We let \(\theta_j\) (taking values in \(\mathbb{R}^3\) mod a suitable lattice) denote angle coordinates on one of these tori.

As we pass through a wall where some \(t_{ij} \in \mathbb{N}\), circle factors in these tori pinch as follows. The angle coordinates \(\theta\) and the log height coordinates \(t\) determine a dynamical elamination. When \(t_{ij} = n\) the circle parameterized by \(\theta_i\) is pinched along the precritical leaves of \(C_j\) of depth \(n\). As we move around in the fiber, the dynamical elamination varies by a rotation, so the way in which the \(\theta_i\) circle pinches depends only on which component we are in, and the value of the local coordinates \(\theta_j\) with \(j < i\). In other words, the structure locally is that of a certain kind of iterated fiber bundle called a monkey bundle.

Recall from Definition 8.1 the terms monkey pants and monkey Morse functions.

**Definition 8.1 (Monkey bundle).** A monkey bundle of order \(n\) consists of the following data:

1. A finite sequence of fiber bundles \(\Omega_2 \to E_2 \to S^1\) and \(\Omega_j \to E_j \to E_{j-1}\) for \(3 \leq j \leq n\) where each \(\Omega_j\) is a monkey pants;
2. A map \(\pi_j : E_j \to [0, 1]\) whose restriction to each \(\Omega_j\) fiber is monkey Morse; and such that
3. If \(E := E_n\) is the total space, and \(\pi : E \to [0, 1]^{n-1}\) denotes the map whose factors restrict to \(\pi_j\) on each \(E_j\), then for each \(j\) the image of the critical points in the \(\Omega_j\) fibers is a collection of affine hyperplanes.

The cube \([0, 1]^{n-1}\) together with the hyperplanes which are the images of fiberwise critical points under \(\pi\) should be thought of as a graphic in the sense of Cerf theory; see e.g. \([15]\). We say that a curve in \([0, 1]^{n-1}\) crosses a hyperplane of the graphic positively if it corresponds to the positive direction in the factor \(\pi_j : E_j \to [0, 1]\) to which the hyperplane is associated.

**Definition 8.2 (Monkey prism; monkey turnover).** Suppose \(E\) is a monkey bundle with projection \(\pi : E \to [0, 1]^{n-1}\). Suppose \(\Delta \subset [0, 1]^{n-1}\) is a convex polyhedron for which there is a vertex \(v \in \Delta\) so that the ray from \(v\) to every other point in \(\Delta\) crosses the graphic in the positive direction. Then we call \(P := \pi^{-1}(\Delta)\) a monkey prism.

Suppose \(\pi : P \to \Delta\) is a monkey prism, and some collection of finite groups act on some boundary strata of \(P\) preserving \(\pi\). Then the quotient space \(Q\) of \(P\) together with the data of its induced projection to \(\Delta\) is called a monkey turnover.

**Lemma 8.3 (Prism is \(K(\pi, 1)\)).** A monkey prism of order \(n\) is a \(K(\pi, 1)\) with the homotopy type of an \(n\)-complex. A monkey turnover of order \(n\) has the homotopy type of an \(n\)-complex.

**Proof.** A monkey pants is homotopic to a graph, and iterated fibrations of \(K(\pi, 1)\)s are \(K(\pi, 1)\)s. Thus a monkey bundle is a \(K(\pi, 1)\) with the homotopy type of an \(n\)-complex.

The universal cover \(\tilde{E}\) of a monkey bundle \(E\) is a (noncompact) manifold with corners, and interior homeomorphic to a product \(\mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \times \mathbb{R}\) where each \(\mathbb{R}^2\) factor has a singular foliation with leaf space an oriented tree.
If $F \subset E$ is a monkey prism associated to a polyhedron $\Delta \subset [0,1]^{n-1}$ then the preimage $\tilde{F} \subset \tilde{E}$ is bounded in each $\mathbb{R}^2$ factor by a collection of lines of the foliation, and is homeomorphic to a disjoint union of $\mathbb{R}^2$s. As we move along a straight ray in $\Delta$ from the distinguished vertex we might cross hyperplanes of the graphic, but by hypothesis we only cross in the positive direction. As we cross a hyperplane, the part of $\tilde{F}$ in some $\mathbb{R}^2$ fibers splits apart, but pieces can never recombine; thus $\tilde{F}$ is homeomorphic to $\mathbb{R}^{2n-1}$ so that $F$ is also a $K(\pi,1)$ with the homotopy type of an $n$-complex.

Since orbifolding is compatible with $\pi$, a monkey turnover also has the homotopy type of an $n$-complex. \hfill \square

From the description of the fibers of $\rho$ and how they pinch as we cross a wall, the following is immediate:

**Lemma 8.4.** Let $\Delta$ be a cell of the dual cellulation $\tau'$. Then $\rho^{-1}(\Delta)$ is a disjoint union of monkey prisms and monkey turnovers with respect to the map $\rho$.

Figure 9 is a simple example of the way a fiber can be quotiented in a monkey turnover.

There does not seem to be any obvious reason why monkey turnovers in generality should be $K(\pi,1)$s. However it will turn out that the turnovers that occur in the partition of $X_4$ are $K(\pi,1)$s. The reason for this is subtle, and only proved in §9.

There is another natural cellulation $\kappa$ of $W$ associated to the subset of walls of the form $t_{i_1} \in \mathbb{N}$; i.e. the walls of the integer lattice in $\mathbb{R}^2$. They decompose $W$ into squares and right-angled triangles. Let $\kappa'$ be the dual cellulation; the cells of $\kappa'$ are triangles, squares and rectangles, and the cells of $\kappa'$ are in bijection with the cells of $\tau'$. Since $\tau$ and $\kappa$ have the same set of vertices, there is a bijection between the top dimensional cells of $\tau'$ and $\kappa'$.

In the sequel it will be convenient to compare the monkey prisms and turnovers associated to $\tau'$ with those associated to $\kappa'$.

**Lemma 8.5** (Equivalent Cells). Let $K$ and $T$ be cells of the cellulations $\kappa'$ and $\tau'$ associated to a vertex $v$. Then the components of $\rho^{-1}(K)$ and of $\rho^{-1}(T)$ are homeomorphic, and are isotopic inside $X_4$.

**Proof.** There is an isotopy of the frontiers of the cells from one to the other which never introduces any new tangency with the graphic. Since fibers are arranged in a product structure away from the graphic, the lemma follows. \hfill \square

The prisms and turnovers associated to cells of $\kappa'$ are naturally homeomorphic to the *moduli spaces* introduced in §9.3.

8.4. $K(\pi,1)$. Decompose $W$ into cells dual to the cellulation by walls; note that typical cells (those dual to interior vertices of $W$) are hexagons. The preimage under $\rho$ of each of these cells is a disjoint union of monkey prisms and monkey turnovers, and the walls in each cell are the graphic. Thus $X_4$ is a *complex of spaces* in the sense of Corson [10]. The associated complex is built from copies of cells of $\tau$ according to the pattern of inclusion of connected components; thus it is an example of an $A_2$ *building*, which comes with an immersion to $W$. See e.g. Brown [10] for an introduction to the theory of buildings.
Theorem 8.6 (Complex of spaces). \(X_4\) is a complex of monkey prisms and monkey turnovers over a contractible \(\tilde{A}_2\) building \(B\).

**Proof.** The direction of pinching is transverse to the walls, so there is a unique path in the building from every point to the origin projecting to a ray in \(W\). \(\square\)

In retrospect, the inductive picture of \(X_3\) we obtained in §7 as an infinite union of knot and link complements, exhibits it as a complex of monkey prisms and monkey turnovers (actually, only one monkey turnover) over a contractible \(\tilde{A}_1\) building (i.e. a tree).

The next theorem is the analog in degree 3 of Theorem 7.5.

**Theorem 8.7** (\(K(\pi, 1)\)). \(S_4\) is a \(K(\pi, 1)\) with the homotopy type of a 3-complex.

We have already seen that the monkey prisms (and consequently also monkey turnovers) in \(X_4\) have the homotopy type of 3-complexes. The same is therefore true of \(X_4\).

\(X_4\) is assembled from monkey prisms and monkey turnovers associated to the vertices of \(B\). The edges and triangles are associated to lower dimensional monkey prisms and turnovers included as facets in the boundary. The monkey prisms and their boundary strata are all \(K(\pi, 1)\)'s by Lemma 8.3 and the inclusions of boundary strata are evidently injective at the level of \(\pi_1\). It remains to show that the same holds for the monkey turnovers.

We defer the proof of this to §9, but for the moment we give some examples to underline how complicated the monkey turnovers can be.

**Example 8.8** \((K(B_4, 1))\). The turnover associated to the vertex \((0, 0)\) homotopy retracts onto the fiber \(\rho^{-1}(0, 0)\). This is the (3 real dimensional) configuration space of degree 4 dynamical elaminations with all critical leaves of height 1. This turns out to be a spine for the configuration space of 4 distinct unordered points in \(\mathbb{C}\); i.e. it is a \(K(B_4, 1)\) (an analogous statement holds in every degree). There are several ways to see this; one elegant method is due to Thurston, and explained in [29]. We shall see a quite different and completely transparent demonstration of this fact in §9.

**Example 8.9** (Star of David). There are two monkey turnovers associated to the vertex \((1, 1)\) of \(\tau\) in \(W\), corresponding to the two combinatorially distinct ways for \(C_1\) to sit in \(S^1\).

When \(C_1\) is antipodal, the leaves \(C_2\) and \(C_3\) sit on either side and do not interact with each other. For each fixed value of \(C_1\) the other two leaves vary as a product \(P \times P\) of pairs of pants. Monodromy around the \(C_1\) circle switches the two factors by an involution.

When \(C_1\) is not antipodal, the leaves \(C_2\) and \(C_3\) may interact, and the topology is significantly more complicated. This component is also a bundle over \(S^1\) whose fiber is a certain 4-manifold \(Y_2\) that we call the Star of David (the explanation for the name will come in §9). It is built from five pieces; two of these pieces are homotopic to trefoil complements (i.e. they are \(K(B_3, 1)\)'s). The other three pieces are homotopic to tori, which attach to the other components along a subspace homotopic to a wedge of two circles; in other words this decomposition does not form an injective complex of \(K(\pi, 1)\)'s. In fact, the fundamental group of \(Y_2\) is obtained from the free product of two \(B_3\)s by adding three commutation relations. The five pieces are illustrated in Figure 11.
Figure 11. One of the two monkey turnovers associated to the vertex (1, 1) is a $Y_2$ bundle over $S^1$, where $Y_2$ is built from five pieces associated to the configurations indicated in the figure. The first two pieces are $K(B_3, 1)$s and the last three are $K(\mathbb{Z}^2, 1)$s. $C_1$ and its preimages with greater height than $C_2, C_3$ are in red.

8.5. Degree $d$. Most of what we have done in this section generalizes to degree $d$ readily. Set $S_d = X_d \times \mathbb{R}$, and order critical heights with multiplicity so that $1 = h_1 \geq h_2 \geq \cdots \geq h_{d-1}$. Define $\rho : X_d \to \mathbb{R}^{d-2}$ with coordinates $t_j := -\log_j h_j$ for $j = 2, \cdots, d-1$. The image of $X_d$ is the Weyl chamber $W$, which is partitioned by walls $t_ij \in \mathbb{Z}$ where $t_ij := t_i - t_j$ into the cells of a cell decomposition $\tau$ with dual decomposition $\tau'$. If we identify $\mathbb{R}^{d-2}$ affinely with the subspace of $\mathbb{R}^{d-1}$ with coordinates summing to 0, then $\tau$ becomes the simplectic honeycomb; see e.g. Coxeter [17]. For example, in degree 5 the cells of $\tau$ are regular tetrahedra and octahedra, and the cells of $\tau'$ are regular rhombic dodecahedra.

Let $\kappa$ be the cellulation defined only by the subset of walls $t_{i1}$ and let $\kappa'$ be the dual cellulation. Then we have:

Lemma 8.10 (Equivalent Cells). Let $K$ and $T$ be cells of the cellulations $\kappa'$ and $\tau'$ associated to a vertex $v$. Then the components of $\rho^{-1}(K)$ and of $\rho^{-1}(T)$ are homeomorphic, and are isotopic inside $X_d$.

Theorem 8.11 (Complex of spaces). $S_d$ is a complex of monkey prisms and monkey turnovers over a contractible $A_{d-2}$-building.

Theorem 8.12 (Homotopy dimension). $S_d$ has the homotopy type of a $(d-1)$-complex (i.e. a complex of half the real dimension of $S_d$ as a manifold).

The proofs are all perfectly analogous to the proofs of Lemma 8.5 Theorem 8.6 and (the relevant part of) Theorem 8.7.

8.6. Tautological Elaminations. It is straightforward to generalize Definition 7.2 to higher degree for the critical leaves of least height. Fix $C_1, C_2, \cdots, C_{d-2}$ at heights $h_1 \geq h_2 \cdots h_{d-2}$, and let $C_{d-1}$ at height $h_{d-2} - \epsilon$ vary. Every time $C_{d-1}$ collides with a leaf $P$ which is a preimage of $C_j$ for $j < d - 1$ we add $P^d$ to the tautological elamination.

It is harder to decide on a definition for the other critical leaves. This is because the elamination associated to $C_j$ depends on the fixed locations of $C_k$ with $k < j$ and an equivalence class of fixed locations of $C_k$ with $k > j$. We explain.

Definition 8.13 (Degree $d$ Tautological Elaminations). Fix a degree $d$ and an index $1 < i \leq d - 1$. Fix locations of leaves $C_j$ for $j \neq i$ where the $C_j$ with $j < i$ have heights $h_1 \geq h_2 \cdots h_{i+1}$, and the $C_j$ with $j > i$ have height 0. We shall define the leaves of the tautological elamination $A_T(C)$ associated to $C := C_1, \cdots, \hat{C}_i, \cdots, C_{d-1}$ of depth $n$. Insert $C_i$ somewhere at height $h_{i+1} - \epsilon$ compatibly with the other...
leaves, and construct the leaves of the dynamical elamination associated to the critical data $C \cup C_i$ which are preimages of $C_j$ up to depth $n$. As we vary $C_i$, the leaves $C_j$ with $j < i$ stay fixed but the $C_j$ with $j > i$ are pushed over $C_i$ and over preimages of higher depth critical leaves. Whenever $C_i$ collides with a preimage $P$ of a higher $C_j$ we add $P^d$ to the tautological elamination.

The $C_j$ with $j > i$ are ‘hidden parameters’; we need them to determine the location of the preimages of greater height, but they do not themselves contribute any leaves to $\Lambda_T$.

As the angles of $C_j$, $j < i$ vary by a vector of parameters $t$ (and $C_j$, $j > i$ are pushed over by this motion) the tautological elaminations vary by the flow $F_i$.

Within each monkey prism the pinching is described by these tautological elaminations. Let’s fix a cell $\tau$ of leaves in a subsequence of the critical data leaves, and construct the leaves of the dynamical elamination associated to the construction of the tautological elamination, set the formal height of $\Lambda$ above. It is possible to define a suitable ‘completion’ of the tautological elamination 

\[ \text{Completed Tautological Elamination}. \]

8.7. **Completed Tautological Elmination.** Fix $C := C_1, \cdots, \hat{C}_i, \cdots, C_{d-1}$ as above. It is possible to define a suitable ‘completion’ of the tautological elamination $\Lambda_T(C)$ as follows.

**Definition 8.14** (Completed Tautological Elmination). Fix $d$ and $C$ as above. In the construction of the tautological elamination, set the formal height of $C_i$ to be equal to 0, and define $L_n$ to be the set of leaves of the form $P^d$ where $P$ is a depth $n$ preimage of $C_i$ that collides with $C_i$ itself.

Although they have height 0, the $L_n$ have a well-defined vein in $\mathbb{D}$. Note that some pairs of leaves of $L_n$ cross each other in $\mathbb{D}$. Nevertheless we can think of $L_n$ as a closed subset of the space of geodesic leaves in $\mathbb{D}$ and take the limit sup $L_\infty := \limsup_{n \to \infty} L_n$ (i.e. there is a leaf in $L_\infty$ for each convergent sequence of leaves in a subsequence of the $L_n$). Then we define the completed tautological elamination associated to $C$ to be $\hat{\Lambda}_T(C) := \Lambda_T(C) \cup L_\infty$.

The leaves of $\hat{\Lambda}_T(C) - \Lambda_T(C)$ are called flat since they have height 0, to distinguish them from the ordinary leaves of $\Lambda_T(C)$.

**Theorem 8.15** (Limit is lamination). *The vein of $\Lambda_T(C)$ is a geodesic lamination* (i.e. leaves of $L_\infty$ do not cross $\Lambda_T(C)$ or each other).

The proof of this will appear in a forthcoming paper.

Pinching along $\Lambda_T(C)$ is the same as pinching along $\Lambda_T(C)$, since the flat leaves all have height zero, so do not actually intrude into $E$. However, it does make sense to pinch the closure $\bar{E} \subset C \cup \infty$ along $\Lambda_T(C)$, exactly as before by cut and paste along the tips of $\Lambda_T(C)$, and then by quotienting the endpoints of the flat leaves to single points. Let’s call the result $\Omega_T(C)$. Because we added limits in the definition of $\Lambda_T(C)$, $\Omega_T(C)$ is Hausdorff. It is a compactification of $\Omega_T(C)$ away from $\infty$, by locally connected spaces (isolated points or monotone quotients of circles).

Notice that this construction is non-vacuous even when $d = 2$; it reproduces Thurston’s quadratic geolamination [28], which is a proposed topological model for...
the boundary of the Mandelbrot set (proposed, since it is famously unknown if the
Mandelbrot set is locally connected).
Thus it seems reasonable to conjecture that the boundary components of \( \Omega_T(C) \)
should parameterize (modulo the question of local connectivity) the boundaries of
the components of the complement of \( \mathbb{S}_d \) in the slice associated to \( C \). Compare
with \([2]\).

9. Sausages

In this section we introduce a completely new way to see the pieces in the building
decomposition of \( X_d \) via algebraic geometry. It will turn out that the monkey prisms
and monkey turnovers in \( X_d \) all become homeomorphic (after taking a product with
an interval) to (rather explicit) complex affine varieties — moduli spaces of certain
objects called \textit{sausage shifts}.

9.1. Sausages: the basic idea. Everyone likes sausages. Now we will see them
made. The basic idea is illustrated in Figure 12.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{sausage_diagram.png}
\caption{making sausages}
\end{figure}

A dynamical elamination is a machine that, by a process of repeatedly pinching
leaves in order of height, extrudes a long, complicated Riemann surface \( \Omega \) (a Fatou
set); by tying this Riemann surface off at periodic values of \( -\log_d h \), we decompose
it into manageable genus zero chunks: sausages.
Thus the Riemann surface \( \Omega \) is tied off into a tree of sausages, and the dynamics
of \( F \) on \( \Omega \) decomposes into polynomial maps between the sausages, whose moduli
spaces are described by (elementary) algebraic geometry.

9.2. Definitions.

9.2.1. Tagged Points. Let \( f \) be a holomorphic map between open subsets of \( \mathbb{C} \)
taking \( p \) to \( q \). If \( f'(p) \) is nonzero, \( df \) is a \( \mathbb{C} \)-linear isomorphism from
\( T_p \) to \( T_q \).
Thus after scaling by a suitable positive real number, it induces an isometry of unit
tangent circles. We denote these unit tangent circles by \( U \) and the induced map as
\( Uf : U_p \to U_q \).
If \( p \) is a critical point of multiplicity \( m \), then \( f \) maps infinitesimal round circles
centered at \( p \) to infinitesimal round circles centered at \( q \) by a degree \((m + 1)\) cov-
ering. By abuse of notation we write \( Uf : U_p \to U_q \) for this map. In holomorphic
coordinates for which $f$ is $z \to z^{m+1}$ this map is just multiplication by $(m + 1)$ on $U_0$ (really we are using an implicit identification between the tangent space $T_q$ and its $(m + 1)$st tensor power).

**Definition 9.1** (Tagged Point). A tagged point is a point $p$ together with an element $u_p \in U_p$. The zero tag is the point $0 \in \mathbb{C}$ together with the unit vector $u_0 \in U_0$ tangent to the positive real axis.

If $f$ is a holomorphic map taking a tagged point $p$ to a tagged point $q$ we say it preserves tags if $Uf(u_p) = u_q$. If $f$ is a holomorphic function, a tagged root is a tagged point $p$ with $f(p) = 0$ for which $Uf(u_p)$ is the zero tag.

**9.2.2. Sausages.** Let $T$ be a locally finite rooted tree. Every vertex $v$ but the root has a unique parent — the unique vertex adjacent to $v$ on the unique embedded path in $T$ from $v$ to the root. If $w$ is the parent of $v$ we say $v$ is a child of $w$. Every edge of $T$ is oriented from child to parent.

**Definition 9.2** (Bunch of sausages). Let $T$ be a locally finite rooted tree. A bunch of sausages over $T$ is an infinite nodal genus 0 Riemann surface $S$ made from a copy of $\mathbb{C}P^1$ for each vertex $v$ of $T$ (the sausages, which we denote $\mathbb{C}P^1_v$) and for each $v$ a finite set of marked tagged points $Z_v \subset \mathbb{C}P^1_v - \infty$ and a bijection $\sigma$ from the children of $v$ to the set $Z_v$, so that if $w$ is a child of $v$, the point $\infty$ in the sausage $\mathbb{C}P^1_w$ is attached to the point $\sigma(w) \in Z_v \in \mathbb{C}P^1_v$.

If $T$ is a rooted tree, for each vertex $w$ of $T$ there is a rooted subtree $T_w \subset T$ with root $w$. If $S$ is a bunch of sausages over $T$, then $S_w \subset S$ denotes the bunch of sausages associated to the subtree $T_w$.

A morphism between rooted trees $T, T'$ is a simplicial map $\tau : T \to T'$ taking roots to roots, and directed edges to directed edges. Thus if $w$ is a child of $v$, the image $\tau(w)$ is a child of $\tau(v)$.

**Definition 9.3** (Augmentation). If $T$ is a rooted tree, the augmentation of $T$, denoted $S$, is the rooted tree obtained from $T$ by adding a new root $v'$ and an edge from the root $v$ of $T$ to $v'$. If $S$ is a bunch of sausages over $T$, the augmentation of $S_v$, denoted $S'_v$, is the bunch of sausages over $T'$ obtained by attaching $\mathbb{C}P^1_{v'}$ along $0 = Z_{v'} \subset \infty$ in $Z_v$.

**Definition 9.4** (Polynomial). Let $S$ be a bunch of sausages over a locally finite tree $T$. A degree $d$ polynomial $p$ is a degree $d$ tagged holomorphic map from $S$ to its augmentation $S'$ over a morphism $\tau : T \to T'$. This means that for every vertex $w$ of $T$ there is a polynomial map $p_w : \mathbb{C}P^1_w \to \mathbb{C}P^1_{\tau(w)}$ of degree $d_w$ in normal form taking $Z_w$ to $Z_{\tau(w)}$, and so that

1. if $v$ is the root, the polynomial $p_v$ has degree $d$ and its roots are exactly $Z_v \subset \mathbb{C}P^1_v$, and furthermore as tagged points $Z_v$ are tagged roots of $p_v$;
2. the root polynomial $p_v$ has more than one root; i.e. $p_v$ is not the polynomial $z^d$;
3. for every vertex $w$ with $\tau(w) = u$ the map $p_w : \mathbb{C}P^1_w \to \mathbb{C}P^1_u$ takes $Z_w$ to $Z_u$ as tagged points, and $Z_w$ is the entire preimage $p^{-1}_w(Z_u)$; and
4. if $w$ is the child of $u$ with $\tau(w) = z \in Z_u \subset \mathbb{C}P^1_u$ then the degree $d_w$ of the polynomial $p_w$ is equal to the multiplicity of $z$ as a preimage under $p_u$.

The second bullet point is a kind of nondegeneracy condition: if the root polynomial $p_v$ were $z^d$, then $S$ would already be the augmentation of some other sausage polynomial.
Lemma 9.5. Let $S$ be a bunch of sausages over $T$, and let $p : S \to S'$ be a degree $d$ polynomial over a morphism $\tau : T \to T'$. Then for every vertex $w' \in S'$ the sum of degrees $\sum_{\tau(w) = w'} d_v = d$, and every point in $S'$ has exactly $d$ preimages, counted with multiplicity.

Proof. This is true for the root vertex by bullet (1) from Definition 9.4, and by induction by bullets (2) and (3). \hfill \Box

This lemma justifies the terminology ‘polynomial map’.

Definition 9.6. Let $S$ be a bunch of sausages over $T$, and $p$ a polynomial map of degree $d$. Let $w$ be a vertex of $T$, and let $c \in \mathbb{CP}^1_w - \infty$ be a critical point for $p_w$. We say $c$ is a genuine critical point if one of the following occurs:

1. $c$ is not in $Z_w$; or
2. $c$ is in $Z_w$ but more than one sausage is attached at $c$.

and is false otherwise. In the second case, the multiplicity of $c$ is equal to one less than the number of sausages attached at $c$.

We say $p$ is a degree $d$ shift polynomial and $(S, p)$ is a degree $d$ sausage shift if there are exactly $d - 1$ genuine critical points, counted with multiplicity.

Bullet (2) in the Definition 9.4 is equivalent to saying that the root sausage contains at least one genuine critical point.

If $p$ is a shift polynomial, there is a minimal finite rooted subtree $U \subset T$ containing all the genuine critical points. Thus for $w \in T - U$, every polynomial $p_w$ is degree 1; since it is in normal form it is the identity map $p_w(z) = z$.

Corollary 9.7. Let $S$ be a bunch of sausages over $T$, and let $p$ be a degree $d$ shift polynomial. Then the space $E(T)$ of ends of $T$ is a Cantor set, and the action of $p$ on $E(T)$ is conjugate to the one-sided shift on right-infinite words in a $d$-letter alphabet.

9.2.3. Isomorphism of polynomials. The definition of a sausage polynomial includes data in the form of tags that is essential if we want to construct a map from sausage polynomials to shift polynomials, as we shall do in §9.4. In order for this map to be injective we must quotient out by a (finite) equivalence relation that we now explain.

Let $S$ be a bunch of sausages over a tree $T$, and let $p$ be a degree $d$ polynomial as in Definition 9.4. Let $u$ be a vertex of $T$, let $z \in Z_u \subset \mathbb{CP}^1_u$, and let $w$ be the child of $u$ with $\sigma(w) = z$. If $z$ is a critical point of $p_u$ of multiplicity $m$ then $p_w$ has degree $m + 1$; i.e. the degree of $p_u$ near $z$ agrees with the degree of $p_w$ near infinity. In the sequel we will ‘cut open’ $\mathbb{CP}^1_u$ at $z$ and $\mathbb{CP}^1_w$ at infinity, and sew together the two resulting boundary circles in a dynamically compatible way, lining up the tag at $z$ in $\mathbb{CP}^1_u$ with the positive real axis at infinity in $\mathbb{CP}^1_w$.

The tag at $z$ maps under $p_u$ to the tag at $p_u(z)$; thus given $p_u$ and the choice of tag at $p_u(z)$ we have freedom in the choice of a compatible tag at $z$: different choices differ by multiplication by an $(m + 1)$st root of unity $\zeta$. If we multiply the tag at $z$ by $\zeta$, we must at the same time change the coordinates on $\mathbb{CP}^1_w$, by multiplication by $\zeta$. Changing coordinates on $\mathbb{CP}^1_w$ inductively affects the data associated to $w$ and the subtree $T_w$ and its preimages under $p$ in the obvious way. For example, $p_w(z)$ is replaced by $p_w(\zeta^{-1}z)$, the marked points $Z_w$ are replaced by their preimages $\zeta Z_w$, etc.
We say two sausage shifts are isomorphic if they are related by a finite sequence of modifications of this sort. There are $\prod_{w \in T} \prod_{z \in Z_w} (m(z) + 1)$ polynomials in an isomorphism class, where $m(z)$ is the multiplicity of $z$ as a critical point of $p_w$, and where the product is taken over all $z \in T_w$ for all $w \in T$. Note that for a sausage shift, this product is finite, since all but finitely many $p_w$ have degree 1.

9.3. Moduli spaces. For each fixed combinatorial type of degree $d$ sausage shift, there is an associated moduli space of isomorphism classes with the given combinatorics, parameterized locally by the coefficients of the vertex polynomials $p_w$ of degrees $> 1$. We shall see in Theorem 9.15 that moduli spaces for sausage shifts with generic heights have complex dimension $d - 1$, and in fact they have the natural structure of iterated bundles of complex affine varieties in an obvious way.

This is best explained by examples.

Example 9.8 (Degree 2). The root polynomial $p_r$ is of the form $z^2 - c$ for some nonzero $c$. Since every other polynomial has degree 1 (and is therefore the identity function $z$) $S$ is a rooted dyadic tree, where each parent has two children attached at $\pm \sqrt{c}$. The moduli space of such sausages is evidently $\mathbb{C}^*$. This is homeomorphic (but not holomorphically isomorphic) to $S_2$.

Example 9.9 (Distinct roots). The simplest case in every degree $d$ is that the root polynomial $p_r$ has distinct roots. Then every other polynomial has degree 1 and $S$ is a rooted $d$-adic tree, where each parent has $d$ children attached at the roots of $p_r$. Thus the moduli space is a discriminant complement, and hence a $K(B_d, 1)$.

Example 9.10 (Degree 3). Suppose the root polynomial $p_r$ has two roots, so it is of the form $p_r := (z - c)^2(z + 2c) = z^3 - 3cz^2 + 2c^3$ with $c$ nonzero. The root vertex $v$ has two children $u, w$ where $u$ is attached at the double root $c$ (say). Then $p_w = z$ and $p_u$ has degree 2. Either 0 is a genuine critical point for $p_u$, or $p_u$ is of the form $z^2 + c$ or $z^2 - 2c$. In the latter case $u$ has two children $u', w'$ where $u'$ is attached at 0 and this chain of critical roots $u, u', u''(2), u''(3), \cdots$ continues until $p_u^{(n)} := z^2 + x$ has a genuine critical point (or equivalently, $x \in \mathbb{C} - Z_t$ where $p$ takes the vertex $u^{(n)}$ to $t$). If we ignore tags, the moduli space is a bundle over $\mathbb{C}^*$ (parameterized by the choice of $c$) and whose fiber is $\mathbb{C} - Z_t$.

Notice that the points of $Z_t$ are obtained from $c, -2c$ by repeatedly pulling back under double branch covers of the form $z \rightarrow z^2 + c_j$ where $c_j$ is one of the preimages pulled back so far. The monodromy acts on each of these double branch covers either trivially or by permuting some of the preimages in pairs. It follows that every orbit of the monodromy on $Z_t$ has length a power of 2.

Example 9.11 (Star of David). Suppose that the root polynomial in degree 4 has one simple root and one triple root; i.e. the root polynomial is $p_r := (z - c)^3(z + 3c)$ with $c$ nonzero. The root has two children $u, w$ where $u$ is attached at the triple root $c$ (say). The simplest case is when $c$ and $-3c$ are regular values for $p_u$. Then the moduli space is a bundle over $\mathbb{C}^*$ whose fiber is a copy of $Y_2$, the space of degree 3 polynomials $z^3 + pz + q$ for which two specific distinct complex numbers (in this case $c$ and $-3c$) are regular values. It turns out that this moduli space is homotopic to the monkey turnover described in Example 8.9.

The general structure of moduli spaces should now be starting to become clear. To make a precise statement, we introduce the notion of a Hurwitz Variety.
**Definition 9.12 (Hurwitz Variety).** A degree $d$ Hurwitz variety is an affine complex variety of the following form. Fix a finite set $Q \subset \mathbb{C}$ and a conjugacy class of representation $\sigma$ from $\pi_1(\mathbb{C}P^1 - Q)$ to the symmetric group $S_d$.

The Hurwitz Variety $H(Q, \sigma, d)$ is the space of degree $d$ normalized polynomials of the form $f(z) := z^d + a_2z^{d-2} + \cdots + a_d$ for which $f : \mathbb{C} \to \mathbb{C}$ is a degree $d$ branched cover whose monodromy around $q$ is conjugate to $\sigma(q)$ for all $q \in Q$.

For a permutation $\sigma$ let $|\sigma| = d$—number of orbits. Thus $|\sigma(q)|$ is the multiplicity of $q$ as a critical value of $f$, for each $q \in Q$ and each $f \in H(Q, \sigma, d)$. We establish some basic properties of these varieties:

**Proposition 9.13 (Basic Properties).** Hurwitz varieties $H(Q, \sigma, d)$ satisfy the following basic properties:

1. the dimension of $H(Q, \sigma, d)$ is equal to $d - 1 - \sum_q |\sigma(q)|$;
2. $H(Q, \sigma, d)$ is connected if its dimension is positive;
3. if there is a homeomorphism from $\mathbb{C}P^1$ to $\mathbb{C}P^1$ taking $Q$ to $Q'$ and conjugating $\sigma$ to $\sigma'$ then $H(Q, \sigma, d)$ is homeomorphic to $H(Q', \sigma', d)$.

**Proof.** The first bullet (i.e. dimension count) is elementary.

If we choose a finite subset $P \subset \mathbb{C} - Q$ and extend $\sigma$ to $P$ then we can build a degree $d$ branched cover of $\mathbb{C}P^1$ over $P \cup Q$ with monodromy $\sigma$ at $P \cup Q$. The genus of this branched cover depends only on $\sigma$. Thus the family of covers which are connected and genus 0 form a bundle over the space of pairs $Q \cup P, \sigma$ of a particular combinatorial type, and it is an exercise in finite group theory to show that these fibers are connected when they have positive dimension. Each $H(Q, \sigma, d)$ is a finite branched cover of the associated fiber (the Riemann surface determines the polynomial up to finite ambiguity); this proves the second bullet.

To prove the third bullet, let’s modify our homeomorphism $\varphi : \mathbb{C}P^1 \to \mathbb{C}P^1$ by an isotopy so that it is equal to the identity in a neighborhood of $\infty$, and is $K$-quasiconformal for some $K$. For each $f \in H(Q, \sigma, d)$ we can pull back the Beltrami differential $\mu := \partial\varphi/\partial \varphi$ to $f^*\mu$ and let $\phi : \mathbb{C}P^1 \to \mathbb{C}P^1$ uniquely solve the Beltrami equation for $f^*\mu$, normalized to be tangent to the identity at infinity to second order. Then $\psi(f) := \varphi f \phi^{-1}$ is a normalized polynomial, and by construction it is in $H(Q', \sigma', d)$. Letting $f$ range over $H(Q, \sigma, d)$ defines a homeomorphism $\psi : H(Q, \sigma, d) \to H(Q', \sigma', d)$ as desired. \hfill $\square$

**Example 9.14 (Discriminant Variety).** If we set $Q = \{0\}$ and $\sigma$ the map to the identity element, then $H(\{0\}, \text{id}, d)$ is the space of degree $d$ polynomials in normal form with simple roots. In other words, $H(\{0\}, 0, d)$ is the complement of the discriminant variety, and is a $K(B_d, 1)$.

**Theorem 9.15 (Moduli spaces).** Every moduli space of a degree $d$ sausage shift of a fixed combinatorial type is an algebraic variety over $\mathbb{C}$ which has the structure of an iterated bundle whose base and fibers are all Hurwitz varieties. Furthermore, it has dimension $d - 1$.

**Proof.** Consider a vertex $w$ with parent $u$ and image $v = \tau(w)$. There is a polynomial $p_w : \mathbb{C}P^1 \to \mathbb{C}P^1$ whose degree is equal to the multiplicity of $u$ as a preimage under $p_u$. The points $Z_w$ are the preimages of $Z_v$ under $p_w$, and the number and multiplicity of these points depends on the monodromy of $p_w$ as a branched cover around $Z_v$. Thus for a fixed combinatorial type, the polynomials $p_w$ vary in a
Hurwitz Variety whose data is determined by the polynomials in vertices above \( w \). Changing a tag changes the coordinates on the Hurwitz variety by a (finite) automorphism. Thus the moduli space is an iterated bundle as claimed.

9.3.1. \( K(\pi, 1) \)'s. Hurwitz varieties can apparently be quite complicated, topologically. But at least in low degree we have the following theorem, which is by no means obvious, and which I personally find rather startling:

**Theorem 9.16 (CAT(0) 2-complex).** Every connected Hurwitz variety \( H(Q, \sigma, 3) \) is a \( K(\pi, 1) \) with the homotopy type of a locally CAT(0) 2-complex.

**Proof.** If any point in \( Q \) is a critical value the dimension is 1 or 0 and \( H \) is either homotopic to a graph or to a finite set of points. So the only interesting case is when \( Q \) is a finite set and \( \sigma \) is the constant map to the identity permutation. In other words, if \(|Q| = n\), then \( H(Q, \text{id}, 3) \) is the (two complex dimensional) space \( Y_n \) of degree 3 polynomials \( z^3 + pz + q \) for which the points in \( Q \) are regular values. We show these have the homotopy type of locally CAT(0) 2-complexes (and are therefore \( K(\pi, 1) \)).

First we describe the topology. By the third bullet of Proposition 9.13 we can take \( Q \) to be the set of \( n \)th roots of unity. Then \( Y_n = \mathbb{C}^2 - V \), where \( V \) is the hyperplane in \( \mathbb{C}^2 \) with coordinates \( p, q \) for which \( \prod_j (-4p^2 - 27(q - \zeta_j)^2) = 0 \). By a linear change of coordinates, we can replace this hyperplane by \( \prod_j (x^3 - (y - \zeta_j)^2) = 0 \).

\( V \) intersects the plane \( x = 0 \) in exactly the \( n \)th roots of unity. We foliate the complement of this plane by (real 3-dimensional) open solid tori \( S^1 \times \mathbb{C} \) thought of as a bundle over the circle \( |x| = t \), and let \( V_t \) denote the intersection with \( V \). If we cutoff \( |y| \) at some big \( T \), then we get another solid torus \( |y| = T, |x| \leq t \) and the union is an \( S^3 \). When \( |x| = \epsilon \) is small and positive, \( V_\epsilon \) splits into a union of \( n \) trefoils \( T_j^\epsilon \) (in this \( S^3 \)), each obtained as a narrow cable of the circle \( y = \zeta_j \). The part of \( Y_n \) in the domain \( |x| \leq \epsilon \) is homotopic to a wedge of \( n \) copies of a \( K(B_3, 1) \), one for each trefoil.

When \( 2|x|^{3/2} = |\zeta^j - \zeta^k| \) the trefoils \( T_j^\epsilon \) and \( T_k^\epsilon \) intersect at three points, and when \( |x| \) increases past this value, they become linked. There are no other intersections. The link of a crossing (in \( \mathbb{C}^2 \)) is a Hopf link, and the result of pushing across each such crossing attaches a space to \( Y_n \), homotopic to a 2-torus, attached along a subspace homotopic to a wedge of two circles. In other words, it attaches a 2-cell, whose boundary kills the relator which is the commutator of two meridian circles linking the trefoils at the point of intersection.

For each pair of trefoils \( T_j^\epsilon, T_k^\epsilon \), we may choose Garside generators for \( \pi_1(S^3 - T_j^\epsilon) \) corresponding to these meridian circles (the Garside presentation for \( B_3 \) is of the form \( \langle a, b, c | ab = bc = ca \rangle \)). Thus each pair of trefoils contributes a subgroup of \( \pi_1(Y_n) \) of the form

\[
\langle a, b, c, x, y, z | ab = bc = ca, xy = yz = zx, \langle a, x \rangle = \langle b, y \rangle = \langle c, z \rangle = 1 \rangle
\]

However if we follow this chain of relations around a sequence of three trefoils \( T_j^\epsilon, T_k^\epsilon, T_l^\epsilon \) for which \( j, k, l \) are positively oriented in \( \mathbb{Z} \) mod \( n \) (say), the intersection points of each pair of trefoils is successively displaced by a rotation so that the holonomy of this chain of displacements rotates one third of the way around. Thus for a triple of trefoils with Garside generators \( \langle a, b, c \rangle, \langle n, m, o \rangle \) and \( \langle x, y, z \rangle \), the
commutation relations take the form
\[[a, n], [b, m], [c, o], [n, x], [m, y], [o, z], [x, b], [y, c], [z, a]\]

Here is another way of packaging the same information. Build a graph with vertices at the 3\textsuperscript{rd} roots of unity, and with edges straight line segments between each pair of roots whose ratio is a 3\textsuperscript{rd} root of unity. Then \(\pi_1(Y_n)\) is generated by the edges of this graph, with relations that each triple of edges that form a (n equilateral) triangle are Garside generators for a \(B_3\), and each pair of disjoint edges commutes. Furthermore, \(Y_n\) is homotopic to the presentation 2-complex associated to this presentation. We shall show this 2-complex (or: a closely related and homotopic complex) can be given a CAT(0) structure.

Actually, there is a beautiful trick, that I learned from Jon McCammond, arising from his work with Tom Brady \cite{MC} on the construction of CAT(0) orthoscheme complexes for (certain) braid groups. First replace each Garside presentation \(\langle a, b, c \mid ab = bc = ca \rangle\) by a presentation of the form \(\langle a, b, c, d \mid ab = bc = ca = d \rangle\). A presentation complex can be built from three triangles with edges \(abd^{-1}\) etc. The trick is to make these right angled regular Euclidean triangles — i.e. to set the lengths of \(a, b, c\) to be 1, and the length of \(d\) to be \(\sqrt{2}\). Let \(K\) denote the resulting complex (see Figure 13), and let \(K'\) be the complex built from \(n\) copies of \(K\) (one for each \(B_3\)) and one Euclidean square with edge length 1 for each commutation relation as above. We claim the resulting complex is CAT(0).

![Figure 13. K is obtained from this complex by gluing free edges with the same colors in pairs.](image)

Let’s see why. The complex \(K\) (and \(K'\) for that matter) has one vertex; since these complexes are 2-dimensional and Euclidean, we just need to check that the link of the vertex has no loop of length \(< 2\pi\). The link \(L\) of the vertex of \(K\) is a theta graph, with three edges of length \(\pi\). The intersections with the long edge \(d\) are the vertices of the theta graph, and the intersections with the edges \(a, b, c\) give rise to six points (let’s call these short points), each at distance \(\pi/4\) from some vertex.

The link \(L'\) of \(K'\) is obtained from \(n\) disjoint copies of \(L\) by gluing a 4-cycle with edges of length \(\pi/2\) for each commutation relation. Each such 4-cycle can be thought of as a complete bipartite graph on two sets of two points, and each pair of points is attached to distinct short points in a copy of \(L\). Since short points in \(L\) are all distance \(\pi\) apart, no cycle in the graph associated to two \(B_3\)s and their commutators has length \(< 2\pi\). By the way, this shows that \(\pi_1(Y_2)\) is CAT(0).
There is a simplicial map from $L'$ to the complete graph $K_n$ with edges all of length $\pi/2$ which just collapses each copy of $L$ to a point, and identifies edges between the same pair of copies of $L$. A loop $\gamma$ in $L'$ of length $< 2\pi$ would project to a (possibly immersed) simplicial ’loop’ in $K_n$ of simplicial length at most 3. If the projection has simplicial length 0 then $\gamma$ is contained in a copy of $L$ which we already know has no loops of length $< 2\pi$. Simplicial length 1 is impossible. If the projection of $\gamma$ has simplicial length 2 in $K_n$ then $\gamma$ is contained in a subgraph formed from a pair of copies of $L$ which (as we have just discussed) has no loops of length $< 2\pi$. If the projection of $\gamma$ has simplicial length 3 then it passes through a cycle of three $L$s, and because of the holonomy described above, a length $3\pi/2$ path in $\gamma$ has endpoints on the same copy of $L$ but at different short points. Thus $\gamma$ has length at least $2\pi$ and we are done. \qed

Together with Theorem 9.15 this immediately implies:

**Corollary 9.17.** Every moduli space in degree 4 is a $K(\pi, 1)$.

**Question 9.18.** Is every Hurwitz Variety a $K(\pi, 1)$? Is every Hurwitz Variety homotopic to a CAT(0) complex?

### 9.4. The sausage map.

Let $\hat{S}_d$ be the subspace of $S_d$ for which $\log_d(h_1) \in (-1/2, 1/2)$, where $h_1$ is the greatest critical height, and $\log_d$ denotes log to the base $d$. This space is homeomorphic to $X_d \times (-1/2, 1/2)$, which is to say it is homeomorphic to $\hat{S}_d$ itself.

For $f \in \hat{S}_d$ let $L \subset DL_d$ be the dynamical elamination associated to $f$ by the butcher map, and let $\Omega$ be the Riemann surface obtained by pinching $L$ (so that $\Omega$ is canonically isomorphic to the Fatou set of $f$).

Let $\hat{\Omega}$ be the subspace of $\Omega$ with $\log_d(h) \leq 1/2$ and let $S$ be the quotient space of $\hat{\Omega}$ obtained by collapsing each component with $\log_d(h) \in 1/2 + Z$ to a point (which we call a node).

Each component $V$ of $S$ minus its nodes can be given a (branched) Euclidean structure with horizontal coordinate $\theta$ and vertical coordinate $\nu(h)$, where $\nu : \mathbb{R}^+ - d^{1/2 + Z} \to \mathbb{R}$ is a function that stretches each interval $(d^{n-1/2}, d^{n+1/2})$ to $\mathbb{R}$ by a homeomorphism (depending on $n$) in such a way that the map $z \to z^d$ on $\Omega$ is conformal in the new coordinates.

Let’s explain this in terms of $E$. In logarithmic coordinates $h, \theta$ we can think of $E$ as a half-open Euclidean cylinder which is the product of the unit circle with the positive real numbers. The map $z \to z^d$ becomes multiplication by $d$, which we denote $\times d$. For each integer $n$ let $I_n$ denote the open interval $(d^{n-1/2}, d^{n+1/2})$ and let $A_n$ be the annulus in $E$ where $h \in I_n$, and let $A := \cup_n A_n \subset E$. Thus $E - A$ is a countable set of circles with $\log_d(h) \in 1/2 + Z$. Thus $\times d$ takes $A_n$ to $A_{n+1}$ for each $n$.

Choose (arbitrarily) an orientation-preserving diffeomorphism $\nu_0 : I_0 \to \mathbb{R}$ and for each $n$ define $\nu_n : I_n \to \mathbb{R}$ by $\nu_n(h) := d^n \nu_0(d^{-n} h)$. Thus, by induction, $\nu_{n+1}(dh) = d \nu_n(h)$ for all $n$ and all $h \in I_n$. Then define $\mu : A \to S^1 \times \mathbb{R}$ by $\mu(\theta, h) = (\theta, \nu_n(h))$ for $\theta, h \in A_n$. Thus $\mu$ semi-conjugates $\times d$ on $A$ to $\times d$ on $S^1 \times \mathbb{R}$. If we identify $S^1 \times \mathbb{R}$ conformally with $\mathbb{C}^*$ by exponentiating, then $\mu$ semi-conjugates $\times d$ on $A$ to $z \to z^d$ on $\mathbb{C}^*$. If we keep a separate ‘copy’ $\mathbb{C}_n^* := \mu(A_n)$ for each $n$, then we could say that $\mu$ conjugates $\times d$ on $A$ to the self-map of $\cup_n \mathbb{C}_n^*$ that sends each $\mathbb{C}_n^*$ to $\mathbb{C}_{n+1}^*$ by $z \to z^d$. 
The components of $S$ minus its nodes are obtained from the $A_n$ by cut and paste along segments of $L$, an operation which respects the Euclidean structure both in $h, \theta$ and $\nu(h), \theta$ coordinates.

With respect to this branched Euclidean structure, the closure of each $V$ (i.e. putting the nodes back in) is a compact Riemann surface; in fact, it is isomorphic to $\mathbb{C}P^1$, and it is natural to choose $\infty$ to be the (unique) node of greatest height. Thus $S$ becomes an infinite nodal genus 0 Riemann surface. Furthermore although the quotient map from $\hat{\Omega}$ to $S$ is very far from being holomorphic, the map $z \to z^d$ on $\Omega$ does descends to a holomorphic map $p$ from $S$ to its augmentation giving $S$ the structure of a bunch of sausages, and $p$ the structure of a degree $d$ shift polynomial. Notice that the images of the critical points are precisely the genuine critical points of the sausage polynomial.

Tags are defined at the nodes by identifying the unit tangent bundle at each node with a circle in $\Omega$, and inductively pulling back tags compatibly with the dynamics of $z \to z^d$ so that the tag at the unique node in the root of the augmentation corresponds to the argument $\theta = 0$ (this is well-defined, since $\theta$ takes values in $\mathbb{R}/\mathbb{Z}$ in the subspace of $\Omega$ with $h$ greater than any critical height).

**Theorem 9.19 (Sausage map).** The sausage map is surjective, and is 1–1 on the subspace of $\hat{\Omega}_d$ for which no critical leaf $C_j$ has $\log_d(h_j) \in 1/2 + \mathbb{Z}$. This subspace maps bijectively to the set of isomorphism classes of degree $d$ sausage shifts.

**Proof.** It suffices to define a (continuous) inverse. Here is the construction. Cut open a bunch of sausages along its set of nodes and sew in a copy of the unit tangent circle $U$ at each point. Reparameterize the vertical coordinate on each component by the inverse of $\mu$ (here we must choose the correct branch depending on the combinatorial distance to the root). Each component becomes in this way a bordered Riemann surface. The point $\infty$ in each $\mathbb{C}P^1_w$ gets a canonical tag, namely the vector associated to the positive real axis. Thus we obtain a collection of bordered surfaces, so that each border is a round circle with a tag, and we glue these up respecting arguments and tags. By the definition of isomorphism, the gluing is well-defined on an isomorphism class of sausage shift. The result is a complete planar Riemann surface $\Omega$ with one punctured end, and the sausage polynomial descends to a degree $d$ self-map on $\Omega$ with $(d-1)$ critical points, counted with multiplicity. By the Realization Theorem 5.4 this is the Fatou set of a unique shift polynomial. □

**Theorem 9.20 (Monkey pieces are moduli spaces).** The sausage map induces homeomorphisms from $(-1/2,1/2)$ times the open monkey prisms and monkey turnovers arising in the decomposition in Theorem 8.11 to the moduli spaces of degree $d$ sausage shifts of each fixed combinatorial type.

**Proof.** The factor of $(-1/2,1/2)$ comes from the difference between $\hat{\Omega}_d$ and $X_d$, via orbits of the squeezing flow.

This is a consequence of Theorem 9.19 and Lemma 8.10. Explicitly: the components of the images of the sausage map are (up to this factor of $(-1/2,1/2)$) both the subspaces of $\rho^{-1}(W)$ in the preimage of the cells $\kappa'$, and at the same time they are (by definition) the moduli spaces of generic degree $d$ sausage shifts. □
Together with Corollary \[9.17\] and the discussion in §\[8.4\] this completes the proof of Theorem \[8.7\]. Moreover, together with Example \[9.10\] this completes the proof of Theorem \[7.9\].

9.5. **Sausages and combinatorics of the Tautological Elimination.** We have already seen (Example \[9.10\]) that moduli spaces reveal nontrivial information about the tautological elimination. Let \(\Lambda_T\) denote the (depth 3) tautological elimination for some fixed \(\theta_1\), and let \(\Lambda_{T,n}\) denote the subset of leaves of depth \(\leq n\). We have seen that monodromy permutes the components of \(S^1\) mod \(\Lambda_{T,n}\) in such a way that the orbits have length a power of 2.

We claim that these components all have lengths of the form \(2^m/3^n\) for various \(m\). Fix a sausage polynomial as in Example \[9.10\] where the root vertex \(v\) has \(Z_v = c, -2c\), and where there is a chain of vertices \(u_1, \ldots, u_n\) mapping to vertices \(v = t_1, t_2, \ldots, t_n\) by polynomials \(p_j := z^2 + c_j\) so that 0 is a fake critical point for each \(j < n\) (i.e. \(c_j \in Z_{t_j}\)) and a genuine one for \(j = n\) (\(c_n\) is not in \(Z_{t_n}\)).

The components of \(S^1\) mod \(\Lambda_{T,n}\) associated to sausages of this combinatorial form are in bijection with the points of \(Z_{t_n}\). Each \(w \in Z_{t_n}\) maps by a succession of polynomials of degrees 1 or 2 until it reaches \(c\) or \(-2c\) (which themselves are mapped to 0 by \(p_v\)). The length of a component is multiplied by 1/3 when we pull back a regular value, and is multiplied by 2/3 when we pull back a critical value. This proves the claim.

Table 1 shows the number of components of length \(\ell/3^n\) at each depth \(n\) (omitted entries are zeroes).

| \(n\) | 1 | 2 | 2\(^2\) | 2\(^3\) | 2\(^4\) | 2\(^5\) | 2\(^6\) | 2\(^7\) | 2\(^8\) | 2\(^9\) | 2\(^{10}\) | 2\(^{11}\) | 2\(^{12}\) |
|------|---|---|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0    | 1 |   |       |       |       |       |       |       |       |       |       |       |       |
| 1    | 1 | 1 |       |       |       |       |       |       |       |       |       |       |       |
| 2    | 3 | 1 |       |       |       |       |       |       |       |       |       |       |       |
| 3    | 7 | 6 | 0     | 1     |       |       |       |       |       |       |       |       |       |
| 4    | 21| 16| 3     | 0     | 1     |       |       |       |       |       |       |       |       |
| 5    | 57| 51| 13    | 0     | 0     | 1     |       |       |       |       |       |       |       |
| 6    | 171| 149| 39    | 5     | 0     | 0     | 1     |       |       |       |       |       |       |
| 7    | 499| 454| 117   | 23    | 0     | 0     | 0     | 1     |       |       |       |       |       |
| 8    | 1497| 1348| 360   | 66    | 9     | 0     | 0     | 0     | 1     |       |       |       |       |
| 9    | 4449| 4083| 1061  | 207   | 41    | 0     | 0     | 0     | 0     | 1     |       |       |       |
| 10   | 13347| 12191| 3252  | 591   | 126   | 17    | 0     | 0     | 0     | 0     | 1     |       |       |
| 11   | 39927| 36658| 9738  | 1799  | 370   | 81    | 0     | 0     | 0     | 0     | 0     | 1     |       |
| 12   | 119781| 109898| 29292| 5351  | 1125  | 240   | 33    | 0     | 0     | 0     | 0     | 0     | 1     |

Table 1. Number of components of length \(\ell/3^n\) at depth \(n\)

Note that there is a unique component with \(\ell = 2^n\) for each \(n\); this corresponds to the sausages for which \(t_j = u_{j-1}\), \(p_{u_1} = z^2 + c\) and \(p_{u_j} = z^2\) for \(1 < j < n\). The next biggest components have length \(2^\lfloor n/2 \rfloor /3^n\).

The \((n, \ell)\) entry in this table is the number of components of length \(\ell/3^n\) at depth \(n\). If we denote this entry \(N(n, \ell)\) then

\[
\sum \ell \cdot N(n, \ell) = (3^n + 1)/2 \quad \text{and} \quad \sum N(n, \ell) \cdot \ell = 3^n
\]
Example 9.21 (Recurrence). Eric Rains observed the recurrence relation in the first column that
\[ N(2n, 1) = 3 \cdot N(2n - 1, 1) \text{ and } N(2n + 1, 1) = 3 \cdot N(2n, 1) - 2 \cdot N(n, 1) \]
(a similar recurrence holds in higher degree). The proof of this is surprisingly delicate, and will appear in a forthcoming paper [12].

Example 9.22 (Short \( \ell \) sequences). One reason to be interested in the lengths of components of \( S^1 \mod \Lambda_{T,n} \) is that it gives us insight into the geometry of the complement of \( S_3 \). Actually, it is easy enough to describe the picture in arbitrary degree.

For each degree \( d \) the shift complement is \( \mathbb{C}^{d-1} - S_d \). When critical points are simple, order them by height \( h_1 \geq h_2 \geq \cdots \geq h_{d-1} \), and define a butcher’s slice \( B(C_1, \cdots, C_{d-2}) \) to be the subset of \( S_d \) with \( C_1, \cdots, C_{d-2} \) fixed and \( h_{d-1} < h_{d-2} \). There is a tautological elamination \( \Lambda_T(C_1, \cdots, C_{d-2}) \) (see §8.6), and the result of pinching gives a Riemann surface \( \Omega_T \) for which the subset of height \( < h_{d-2} \) is holomorphically equivalent to \( B \).

For the sake of simplicity, let’s suppose \( 1 = h_1 = h_2 = \cdots = h_{d-3} \) so that the leaves of \( \Lambda_T \) of depth \( n \) all have height \( d^{-n} \). A chain of successive components of \( S^1 \mod \Lambda_{T,n} \) with lengths \( \ell_n \cdot d^{-n} \) determines a system of disjoint annuli in the butcher’s slice with moduli \( 1/\ell_n \). So if \( \sum_n 1/\ell_n \) diverges (for instance, if the sequence \( \ell_n \) is bounded), the modulus goes to infinity and the end of \( B \) converges to an isolated point in the complement of the shift locus. Call such an end of \( B \) a small end. All but countably many of the (uncountable) ends of \( B \) are small.

As we exit a small end of \( B \), points in the Julia set collide in the limit to give rise to a non-shift Cantor Julia set (c.f. Example 2.6) also compare with Branner [6]). The local path component of the shift complement containing this limit point has complex dimension \( d-2 \), and is parameterized by the escaping critical points. There are uncountably many of these local path components, parameterized locally by the small ends of \( B \).

Dragging critical points off to the (Cantor) Julia set one by one defines a nested sequence of holomorphic submanifolds of the shift complement, each parameterized by the remaining escaping critical points. When \( C_{j+1} \cdots C_{d-2} \) have been dragged off to \( J_f \), we can define a butcher’s slice by fixing \( C_1, \cdots, C_{j-1} \) and letting \( C_j \) vary; this slice is the subset of height \( < h_{j-1} \) in the Riemann surface \( \Omega_T(C) \) associated to the tautological elamination \( \Lambda_T(C) \) with critical data \( C:=C_1, \cdots, \hat{C}_j, \cdots, C_{d-1} \) for a suitable equivalence class of \( C_{j+1}, \cdots, C_{d-2} \) (see §8.6). Small ends of these butcher’s slices locally parameterize the space of these \( (j-1) \)-dimensional submanifolds.

10. Fundamental Groups

10.1. Braid Groups. Let \( \Delta_d \subset \mathbb{C}^{d-1} \) be the discriminant variety, parameterizing degree \( d \) polynomials in normal form \( z^d + a_2z^{d-2} + \cdots + a_d \) with multiple roots. The group \( \pi_1(\mathbb{C}^{d-1} - \Delta_d) \) acts as permutations of these roots; the permutation representation is a surjective map from \( \pi_1(\mathbb{C}^{d-1} - \Delta_d) \) to the symmetric group \( S_d \).

This map is very far from being injective. A loop in \( \mathbb{C}^{d-1} - \Delta_d \) defines not just a permutation of roots, but a braid: the mapping class represented by the combinatorial manner in which the points move around each other. In other words, there is a monodromy representation \( \text{Mon} : \pi_1(\mathbb{C}^d - \Delta_d) \to B_d \) where \( B_d \) is Artin’s
braid group on $d$ strands. Forgetting the braiding determines a surjection $\text{Art} : B_d \to S_d$.

Thus we obtain a factorization

$$\pi_1(C^{d-1} - \Delta_d) \xrightarrow{\text{Mon}} B_d \xrightarrow{\text{Art}} S_d$$

where the first map is an isomorphism, and the second indicates that $B_d$ is functorially obtained from $S_d$ by the algebraic process of Artinization.

### 10.2. Shift automorphisms

Let $\Sigma_d$ denote the space of right-infinite words on a $d$ letter alphabet; i.e. $\Sigma_d := \{1, \cdots, d\}^\mathbb{N}$. This is a Cantor set in the product topology, and the shift $\sigma$ acts as a $d$ to $1$ expanding map. Let $\hat{S}_d$ denote the group $\hat{S}_d := \text{Aut}(\Sigma_d, \sigma)$; i.e. the group of homeomorphisms of the Cantor set commuting with the shift.

In [1], Blanchard–Devaney–Keen showed that the natural map $\pi_1(S_d) \to \hat{S}_d$ is surjective, in every degree $d$. As before, this is very far from being injective (as we shall shortly see).

Monodromy defines a representation $\text{Mon} : \pi_1(S_d) \to \text{Mod}(\mathbb{C} - \text{Cantor set})$, but this map is certainly not an isomorphism, since $\pi_1(S_d)$ is countable whereas $\text{Mod}(\mathbb{C} - \text{Cantor set})$ has the cardinality of the continuum. Actually, the image can be lifted to $\text{Mod}(\text{Disk} - \text{Cantor set})$, since all shift polynomials (in normal form) are tangent to second order near infinity. Let’s denote the image by $\hat{B}_d$.

Forgetting the braiding defines a surjective homomorphism $A : \text{Mod}(\text{Disk} - \text{Cantor set}) \to \text{Aut}(\text{Cantor set})$, and the image of $\hat{B}_d$ is $\hat{S}_d$. I proved (see [11]) that $\text{Mod}(\text{Disk} - \text{Cantor set})$ is left-orderable, and therefore torsion-free, whereas $\hat{S}_d$ is generated by torsion.

In any case we have a factorization of the Blanchard–Devaney–Keen map as

$$\pi_1(S_d) \xrightarrow{\text{Mon}} \hat{B}_d \xrightarrow{A} \hat{S}_d$$

Neither map seems easy to understand. On the other hand, with Juliette Bavard and Yan Mary He we were able to show:

**Theorem 10.1** (Bavard–Calegari–He). In degree 3 the map $\text{Mon} : \pi_1(S_3) \to \hat{B}_3$ is an isomorphism.

The proof of this theorem shall (hopefully!) appear in a forthcoming paper. The most optimistic conjecture I can make is:

**Conjecture 10.2** (Monodromy Conjecture). The map $\text{Mod} : \pi_1(S_d) \to \hat{B}_d$ is an isomorphism in every degree.

The only real evidence I have in favor of this conjecture is that it is not obviously falsified by the simplest cases I was able to fully analyze.

If $Y = H(Q, \sigma, e)$ is a Hurwitz variety, the preimage of $Q$ under $f \in Y$ is a finite subset of $\mathbb{C}$ whose cardinality is constant as a function of $f$, and therefore we obtain a monodromy map $M : \pi_1(Y) \to B_n$ for suitable $n$ depending on $Y$. If $Y$ is a Hurwitz variety that arises as a fiber of a moduli space, the image of $\pi_1(Y) \to \pi_1(S_d) \to \hat{B}_d$ factors through this $B_n$, so the monodromy conjecture implies that the maps $M$ are injective. In fact, at least in low dimensions, the monodromy conjecture is equivalent to injectivity on these pieces, since both $\pi_1(S_d)$ and $\hat{B}_d$ are built up in understandable ways from these pieces (this is how Theorem 10.1 is proved).
In any case, this is something we can test, since the groups $\pi_1(Y)$ and $B_n$ are rather explicit, especially in low degree.

**Example 10.3** (Star of David). The ‘hard’ pieces in degree 4 are the Star of David and its generalizations as discussed in Theorem 9.16.

Recall the moduli space $Y_2$ from Example 9.11 and the description of its fundamental group in Theorem 9.16. This fundamental group (let’s call it $G$) has a presentation

$$G := \langle a, b, c, x, y, z \mid ab = bc = ca, xy = yz = zx, [a, x] = [b, y] = [c, z] = 1 \rangle$$

The monodromy map to $B_6$ arises by thinking of the generators as the edges of a Star of David in the plane, and taking each generator to the braid that cycles the endpoints of the edge around each other in a narrow ellipse contained in a neighborhood of the edge.

There is an isometric embedding from the CAT(0) complex for $G$ described in Theorem 9.16 to the Brady–McCammond complex for $B_6$, which has been shown to be CAT(0) by Haettel–Kielak–Schwer. If the image is totally geodesic, this would imply that $G \to B_6$ is injective. This seems quite plausible, but we have not checked it.

11. **Acknowledgments**

I would like to thank Laurent Bartholdi, Juliette Bavard, Pierre Deligne, Laura DeMarco, Yan Mary He, Sarah Koch, Jeff Lagarias, Chris Leininger, Jon McCammond, Curt McMullen, Madhav Nori, Kevin Pilgrim, Eric Rains, Alden Walker, Henry Wilton and the anonymous referee for their help. Most of what I know about polynomial dynamics (which is not much) I learned from Sarah and from Curt at various points in time.

I would also like to extend thanks to the students who attended the graduate topics course I taught on this material at the University of Chicago in Winter 2019, and to Sam Kim who solicited some talks and a paper for the celebration of the 25th anniversary of the founding of KIAS, and without whom I might have never been sufficiently motivated to write any of this up.

**References**

[1] P. Blanchard, R. L. Devaney and L. Keen, *The dynamics of complex polynomials and automorphisms of the shift*, Invent. Math. **104** (1991), 545–580

[2] A. Blokh, L. Oversteegen, R. Ptacek and V. Timorin, *Laminational models for some spaces of polynomials of any degree*, Mem. Amer. Math. Soc. **265** (2020), no. 1288

[3] L. Böttcher, *The principal laws of convergence of iterates and their application to analysis* (Russian), Izv. Kazan. Fiz.-Mat. Obsch. 14 (1904), 137–152

[4] M. Boyle, J. Franks and B. Kitchens, *Automorphisms of the One-Sided Shift and Subshifts of Finite Type*, Erg. Thy. Dyn. Sys. **10** (1990), 421–449

[5] T. Brady and J. McCammond, *Braids, Posets and Orthoschemes*, Algebr. Geom. Topol. **10** (2010), no. 4, 2277–2314

[6] B. Branner, *Cubic polynomials: turning around the connectedness locus*, in L. Goldberg and A. Phillips, eds, *Topological Methods in Modern Mathematics*, pp. 391–427. Publish or Perish, 1993.

[7] B. Branner and J. Hubbard, *The iteration of cubic polynomials. I. The global topology of parameter space*, Acta Math. **160** (1988), no. 3-4, 143–206

[8] B. Branner and J. Hubbard, *The iteration of cubic polynomials. II. Patterns and parapatterns*, Acta Math. **169** (1992), no. 3-4, 229–325
[9] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grund. der Math. Wiss., 319 Springer-Verlag, Berlin, 1999
[10] K. Brown, *Buildings*, Springer–Verlag, Berlin, 1988
[11] D. Calegari, *Circular groups, planar groups, and the Euler class*, Proceedings of the Casson Fest, 431–491. Geom. Topol. Monogr. 7, Geom. Topol. Publ., Coventry, 2004
[12] D. Calegari, *Combinatorics of the Tautological Lamination*, preprint, to appear
[13] D. Calegari, *shifty*, computer program; source available on request
[14] A. de Carvalho and T. Hall, *Riemann surfaces out of paper*, Proc. Lond. Math. Soc. (3) 108 (2014), no. 3, 541–574
[15] J. Corf, *La stratification naturelle des espaces de fonctions différentiables réelles et le théorème de la pseudo-isotopie*, Inst. Haut. Études Sci. Publ. Math., 39 (1970) 51–73
[16] J. Corson, *Complexes of groups*, Proc. Lond. Math. Soc. (3) 65 (1) (1992), 199–224
[17] H. Coxeter, *Regular Polytopes*, Third edition. Dover Publications, Inc., New York, 1973.
[18] A. Douady and J. Hubbard, *Itération des polynômes quadratiques complexes*, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 3, 123–126
[19] L. DeMarco, *Dynamics of rational maps: a current on the bifurcation locus*, Math. Res. Lett. 8 (2001), no. 1-2, 57–66
[20] L. DeMarco, *Combinatorics and topology of the shift locus*, Conformal dynamics and hyperbolic geometry, 35–48, Contemp. Math., 573, Amer. Math. Soc., Providence, RI, 2012.
[21] L. DeMarco and C. McMullen, *Trees and the dynamics of polynomials*, Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 3, 337–382
[22] L. DeMarco and K. Pilgrim, *The classification of polynomial basins of infinity*, Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 4, 799—877.
[23] L. Goldberg and L. Keen, *The mapping class group of a generic quadratic rational map and automorphisms of the 2-shift*, Invent. Math. 101 (1990), no. 2, 335–372
[24] T. Haettel, D. Kielak, P. Schwer, *The 6-strand braid group is CAT(0)*, Geom. Dedicata 182 (2016), 263–286
[25] C. McMullen, *Braiding of the attractor and the failure of iterative algorithms*, Invent. Math. 91 (1988), no. 2, 259–272
[26] J. Milnor, *Dynamics in one complex variable*, Third edition. Ann. of Math. Stud., 160. Princeton University Press, Princeton, NJ, 2006.
[27] W. Thurston, *Thurston’s Notes*, available online from MSRI http://library.msri.org/books/gt3m/
[28] W. Thurston, *On the geometry and dynamics of iterated rational maps*, Edited by Dierk Schleicher and Nikita Selinger and with an appendix by Schleicher. Complex dynamics, 3–137, A. K. Peters, Wellesley, MA, 2009.
[29] W. Thurston, H. Baik, Y. Gao, J. Hubbard, T. Lei, K. Lindsey and D. Thurston, *Degree d invariant laminations*, What’s next? the mathematical legacy of William P. Thurston, 259–325, Ann. of Math. Stud., 205, Princeton Univ. Press, Princeton, NJ, 2020.
[30] N. Vlamis, *Big mapping class groups: an overview*, In the tradition of Thurston, 459–496, Springer, Cham, 2020.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, ILLINOIS, 60637
Email address: dannyc@math.uchicago.edu