$D_4$-symmetric Maps with Hidden Euclidean Symmetry

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ABSTRACT

Bifurcation problems in which periodic boundary conditions (PBC) or Neumann boundary conditions (NBC) are imposed often involve partial differential equations that have Euclidean symmetry. In this case posing the bifurcation problem with either PBC or NBC on a finite domain can lead to a symmetric bifurcation problem for which the manifest symmetries of the domain do not completely characterize the constraints due to symmetry on the bifurcation equations. Additional constraints due to the Euclidean symmetry of the equations can have a crucial influence on the structure of the bifurcation equations. An example is the bifurcation of standing waves on the surface of fluid layer. The Euclidean symmetry of an infinite fluid layer constrains the bifurcation of surface waves in a finite container with square cross section because the waves satisfying PBC or NBC can be shown to lie in certain finite-dimensional fixed point subspaces of the infinite-dimensional problem. These constraints are studied by analyzing the finite-dimensional vector fields obtained on these subspaces by restricting the bifurcation equations for the infinite layer. Particular emphasis is given to determining which bifurcations might reveal observable effects of the rotational symmetry of the infinite layer. It turns out that a necessary condition for this possibility to arise is that the subspace for PBC must carry a reducible representation of the normalizer subgroup acting on the subspace. This condition can be met in different ways in both codimension-one and codimension-two bifurcations.

keywords: bifurcation, symmetry, boundary conditions, hidden symmetry

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I. Introduction

For nonlinear maps $\mathcal{F}$ on an infinite-dimensional linear space $E_{n,2}(\mathbb{R}^2)$ that are equivariant with respect to an action of the Euclidean group $\mathcal{E}(2)$ on $E_{n,2}(\mathbb{R}^2)$, we study the finite-dimensional maps (or vector fields),

$$\mathcal{F}|_{\text{Fix}(\Sigma)},$$

that arise when $\mathcal{F}$ is restricted to the fixed point subspaces $\text{Fix}(\Sigma)$ of certain physically interesting isotropy subgroups $\Sigma$. These finite-dimensional maps are equivariant with respect to a subgroup of $\mathcal{E}(2)$, but in general are also constrained by the original $\mathcal{E}(2)$ symmetry of $\mathcal{F}$. The restricted map (1) is thus described as having hidden Euclidean symmetry.

Bifurcation problems arising in fluids and other continuum systems motivate this study; especially recent experimental work on the bifurcation of surface waves\cite{1,2}. In experiments reported in 1989, Simonelli and Gollub observed the appearance of standing waves on the surface of a fluid layer subjected to a periodic vertical oscillation. This transition is a parametric instability corresponding to a period doubling bifurcation for the stroboscopic map\cite{1}. The experiments were performed on a fluid layer in a square container, but surprisingly the observed standing wave patterns sometimes failed to break symmetry as predicted from the bifurcation theory based on the square symmetry of the container. Guided by work on related bifurcation problems\cite{3-6}, Crawford et al. pointed out that under idealized conditions translation symmetries could constrain the solutions to the fluid equations describing the waves even though these symmetries did not respect the boundary conditions of the problem\cite{7}. Recent work has emphasized that the rotational symmetry of the plane also imposes constraints on the solution in an entirely analogous fashion\cite{2}.

The idealized theoretical model describes the motion of an incompressible, irrotational
and inviscid fluid whose state is specified by two fields \( \phi(x,y,z,t) \) and \( \eta(x,y,t) \); the velocity potential \( \phi \) determines the velocity of the fluid \( \mathbf{u} = \nabla \phi \) and \( \eta \) describes the height of the free surface \( z = \eta(x,y,t) \) at the horizontal position \( (x,y) \). Each of these fields is assumed to satisfy a Neumann boundary condition (NBC), e.g. \( \hat{n} \cdot \nabla \eta(x,y,t) = 0 \), at the sidewalls of the container. In addition, incompressibility of the fluid implies that \( \phi \) satisfies Laplace’s equation, \( \nabla^2 \phi = 0 \), which can be reduced to the two-dimensional Helmholtz equation, \( \nabla^2_\perp \psi(\mathbf{r}) + \kappa^2 \psi(\mathbf{r}) = 0 \) by separating variables \( \phi(x,y,z) = \psi(x,y) \cosh \kappa(h+z) \). Here \( \nabla^2_\perp \) is the two-dimensional Laplacian and \( h \) denotes the depth of the layer. A more detailed discussion of this model is given elsewhere; for our purposes it is sufficient to note that the linear wave frequencies depend on the geometry of the container only through the eigenvalue \( \kappa^2 \) which determines the linear degeneracy of the critical eigenvalue near \(-1\) for a given period doubling instability.

By virtue of the NBC, solutions to the equations in the physical cross section \( \Omega \) of the container can be smoothly extended by reflection across the sidewalls to functions on a larger square domain \( \tilde{\Omega} \). Let \( \Omega \) correspond to the domain \( 0 \leq x,y \leq \pi \), then \( \tilde{\Omega} \) corresponds to \( -\pi \leq x,y \leq \pi \), and the extended functions, defined by

\[
\eta(-x,y) \equiv \eta(x,y) \quad \text{for} \quad 0 \leq x \leq \pi \\
\eta(x,-y) \equiv \eta(x,y) \quad \text{for} \quad 0 \leq y \leq \pi,
\]

still satisfy the fluid equations and moreover obey periodic boundary conditions on \( \tilde{\Omega} \). Hence, from among all the solutions to the fluid equations on \( \tilde{\Omega} \) with periodic boundary conditions (PBC), the physically relevant solutions can be picked out by requiring that \( \eta \) and \( \phi \) are even under reflection across \( x = 0 \) and \( y = 0 \). In a similar fashion, a solution on \( \tilde{\Omega} \) with PBC can always be extended to functions on the entire plane \( \mathbb{R}^2 \) by periodic replication. These functions then satisfy the fluid equations on all of \( \mathbb{R}^2 \). Conversely, if we start with
the solutions to the fluid equations on $\mathbb{R}^2$, then the initial periodic solutions on $\tilde{\Omega}$ can be picked out simply by requiring $\eta$ (and $\phi$) to satisfy

$$\eta(x, y) = \eta(x + 2\pi, y) = \eta(x, y + 2\pi).$$

(4)

The second extension to an infinite fluid layer on $\mathbb{R}^2$ leads to a bifurcation problem with $\mathcal{E}(2)$ symmetry. The solutions on $\Omega$ to the bifurcation problem of direct physical interest are a small subset of the solutions on $\mathbb{R}^2$ but clearly must arise in a manner consistent with the Euclidean symmetry of the extended system. A normal form for the period doubling bifurcations generating the surface waves in $\Omega$ cannot be expected to correctly describe these bifurcations unless it incorporates any constraints due to the “hidden” translation and rotation symmetry of the infinite fluid layer. Such normal forms can be constructed by beginning with Euclidean symmetric maps appropriate to the infinite layer and then restricting these maps to the fixed point subspaces associated with the isotropy subgroups determined by PBC and NBC in (1) and (2), respectively. Some work in this direction has been done to understand the consequences of the hidden translation symmetry for the normal form of the stroboscopic map.\textsuperscript{8} Subsequent experiments by Gollub and Lane have detected the effects of hidden translation symmetries by breaking these symmetries in a controlled fashion. Their results agree with the predictions of the normal form analysis.\textsuperscript{8} The analysis of how and when the hidden Euclidean symmetry affects the normal forms for these bifurcations is extended in this paper; in particular we investigate whether there are cases in which one could hope to distinguish between the effects of the hidden translations and the hidden rotations. It turns out that the specific waves studied in the experiments of Gollub and Lane are not affected by the hidden rotation symmetries, but there are codimension-one and codimension-two bifurcations where it may be possible to detect the presence of the hidden rotational symmetry. This is one of the interesting experimental issues in this topic.
The Euclidean group $E(2) : \mathbb{R}^2 \to \mathbb{R}^2$ is generated by reflection $\gamma_2 \cdot (x, y) \to (y, x)$, translations $T_{(a, b)} \cdot (x, y) \to (x + a, y + b)$, and rotations $R(\phi)$

$$R(\phi) \cdot (x, y) \to (x', y')$$

where

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$  \hspace{1cm} (6)

The transformation $\gamma \in E(2)$ acts on the fluid state $(\phi, \eta)$ in the usual way

$$((\gamma \cdot \phi)(\vec{r}, z), (\gamma \cdot \eta)(\vec{r}, z)) = (\phi(\gamma^{-1} \cdot \vec{r}, z), \eta(\gamma^{-1} \cdot \vec{r}, z)).$$  \hspace{1cm} (7)

In terms of this action, the solutions in (4) satisfying PBC are those fixed by the subgroup $B_P = \{ T_{(\pm 2\pi, 0)}, T_{(0, \pm 2\pi)} \}$. Similarly the restriction to solutions satisfying NBC on $\Omega$ selects those that are fixed by the larger subgroup $B_N = \{ \tilde{\gamma}_1, \gamma_3, T_{(\pm 2\pi, 0)}, T_{(0, \pm 2\pi)} \}$ where

$$\tilde{\gamma}_1 \cdot (x, y) \equiv \gamma_2 R(3\pi/2) \cdot (x, y) = (-x, y)$$

$$\gamma_3 \cdot (x, y) \equiv \gamma_2 \cdot \tilde{\gamma}_1 \cdot \gamma_2 \cdot (x, y) = (x, -y).$$  \hspace{1cm} (8)

The stroboscopic map for the infinite fluid layer commutes with the action of $E(2)$ in (7). In addition, this map has a fixed point $(\phi, \eta) = (0, 0)$ corresponding to a featureless surface, i.e. the fixed point is $E(2)$ invariant. The linearization of the stroboscopic map at $(0, 0)$ thus commutes with the action of $E(2)$ and the eigenspaces of the linearization transform under representations of $E(2)$ determined by (7).

Let $\Lambda_\vec{k}$ denote an eigenvector of the linearized map corresponding to the critical eigenvalue near $-1$; for the ideal fluid

$$\Lambda_\vec{r}(\vec{r}) = \begin{pmatrix} \eta(\kappa) \\ \phi(\kappa) \end{pmatrix} e^{i\vec{k} \cdot \vec{r}},$$  \hspace{1cm} (10)

\footnote{The subgroup is specified by listing its generators.}
and $\gamma \in E(2)$ acts on $\Lambda_{\vec{k}}$ by
\[(\gamma \cdot \Lambda_{\vec{k}})(\vec{r}) = \Lambda_{\vec{k}}(\gamma^{-1} \cdot \vec{r}).\] (11)

The fluid fields are real-valued requiring the eigenvector to satisfy $(\Lambda_{\vec{k}})^* = \Lambda_{-\vec{k}}$. The Euclidean invariance of the fixed point implies that the eigenvalues of the linearization depend only on the magnitude $|\vec{k}| = \kappa$ of the wavevector; the specific value of $\kappa$ associated with the critical eigenvalue varies with the frequency of the external vibration. For fixed $\vec{k}$, there are two linearly independent eigenvectors corresponding to eigenvalues that in general are distinct. The infinite-dimensional eigenspace $E_{\kappa^2}(\mathbb{R}^2)$, associated with the critical eigenvalue of the linearized stroboscopic map, contains $\Lambda_{\vec{k}}$ and its images under $E(2)$.

In the actual experiments, the dissipation in the fluid causes a decay of transient behavior in the wave motion and simplifies the time-asymptotic dynamics. When we consider the bifurcation problem on the finite domains $\tilde{\Omega}$ (with PBC) and $\Omega$ (with NBC) then the critical eigenspaces $E_{\kappa^2}(\tilde{\Omega})$ and $E_{\kappa^2}(\Omega)$, respectively, are finite-dimensional and we assume the dissipation permits a reduced description of the bifurcation on a finite-dimensional center manifold. Restricting the stroboscopic map to a center manifold gives a finite-dimensional map which can be locally represented as a map on the critical eigenspace. Let $\tilde{F}$ and $f$ denote these finite-dimensional maps for the domains $\tilde{\Omega}$ and $\Omega$ respectively
\[
\tilde{F} : E_{\kappa^2}(\tilde{\Omega}) \rightarrow E_{\kappa^2}(\tilde{\Omega}) \quad \text{(12)}
\]
\[
f : E_{\kappa^2}(\Omega) \rightarrow E_{\kappa^2}(\Omega). \quad \text{(13)}
\]

The stroboscopic map for the problem posed on $\tilde{\Omega}$ is obtained by simply applying the map for the infinite layer to fluid states that are spatially periodic, i.e. those states which are invariant under the subgroup $B_P$. Since the infinite layer map has $E(2)$ symmetry, the finite-dimensional map $\tilde{F}$ on the center manifold for PBC will be constrained by this original $E(2)$ symmetry. These constraints can enter in two ways. First, $E_{\kappa^2}(\tilde{\Omega})$ is invariant
under a subgroup of $\mathcal{E}(2)$ and $\tilde{F}$ will be equivariant with respect to this subgroup. In addition, the form of $\tilde{F}$ may be further constrained by symmetries in $\mathcal{E}(2)$ that do not leave $E_{\kappa^2}(\tilde{\Omega})$ invariant. These latter constraints arise simply from the fact that $\tilde{F}$ is constructed from maps that are $\mathcal{E}(2)$-equivariant. The nature of these latter constraints can be studied by considering the form of $\mathcal{E}(2)$-equivariant maps $F$ on the infinite-dimensional eigenspace $E_{\kappa^2}(\mathbb{R}^2)$ which commute with the representation of $\mathcal{E}(2)$ carried by the eigenspace.

The connections between the three classes of maps $F$, $\tilde{F}$ and $f$ follow from the isotropy subgroups $B_P$ and $B_N$ that characterize the solutions with PBC and NBC respectively. The eigenspaces are nested $E_{\kappa^2}(\mathbb{R}^2) \supset E_{\kappa^2}(\tilde{\Omega}) \supset E_{\kappa^2}(\Omega)$, and for $\tilde{\Omega}$ and $\Omega$ they are fixed point subspaces: $E_{\kappa^2}(\tilde{\Omega}) = \text{Fix}(B_P)$ and $E_{\kappa^2}(\Omega) = \text{Fix}(B_N)$. The requirement that $\tilde{F}$ and $f$ should be consistent with the Euclidean symmetry of the infinite layer is formalized by demanding that they be realizable as the restrictions of an $\mathcal{E}(2)$-equivariant map $F$ on $E_{\kappa^2}(\mathbb{R}^2)$ to appropriate subspaces of $E_{\kappa^2}(\mathbb{R}^2)$:

$$\tilde{F} = F|_{\text{Fix}(B_P)} \quad (14)$$
$$f = F|_{\text{Fix}(B_N)} = \tilde{F}|_{\text{Fix}(B_N)}. \quad (15)$$

These restricted maps are not $\mathcal{E}(2)$-equivariant rather they commute with certain subgroups of $\mathcal{E}(2)$. For an isotropy subgroup $\Sigma \subset \mathcal{E}(2)$, the normalizer $N(\Sigma)$ is the maximal subgroup of $\mathcal{E}(2)$ that leaves $\text{Fix}(\Sigma)$ invariant, e.g. for $B_P \subset \mathcal{E}(2)$

$$N(B_P) = \{ \gamma \in \mathcal{E}(2) | \gamma : \text{Fix}(B_P) \to \text{Fix}(B_P) \}. \quad (16)$$

Each of the restricted maps is equivariant with respect to the normalizer on the appropriate fixed point subspace: $\tilde{F}$ commutes with $N(B_P)$ and $f$ commutes with $N(B_N)$. This hierarchy of domains, eigenspaces and symmetries is summarized in Table 1. Note that it is the map $f$ which models the bifurcations in the physical experiment.
Table 1. Hierarchy of Bifurcation Problems

| Domain | Boundary Conditions | Eigenspace | Map | Map Symmetry |
|--------|---------------------|------------|-----|--------------|
| $\mathbb{R}^2$ | none | $E_{\kappa^2}(\mathbb{R}^2)$ | $F$ | $\mathcal{E}(2)$ |
| $\tilde{\Omega}$ | PBC | $E_{\kappa^2}(\tilde{\Omega}) = \text{Fix}(B_P)$ | $\tilde{F} = F|_{\text{Fix}(B_P)}$ | $N(B_P)$ |
| $\Omega$ | NBC | $E_{\kappa^2}(\Omega) = \text{Fix}(B_N)$ | $f = F|_{\text{Fix}(B_N)}$ | $N(B_N)$ |

Notes:
PBC are periodic boundary conditions and NBC are Neumann boundary conditions.

Although our primary interest is the type of maps $\tilde{F}$ and $f$ that can arise by restriction from $F$, we first define $E_{\kappa^2}(\mathbb{R}^2)$ and consider the $\mathcal{E}(2)$-symmetric map $F$. Our focus is on the eigenspaces $E_{\kappa^2}(\mathbb{R}^2)$ characteristic of codimension-one bifurcations for the infinite fluid layer.

A. Representations of $\mathcal{E}(2)$ on $E_{\kappa^2}(\mathbb{R}^2)$

The eigenvector $\Lambda_{\tilde{k}}$ satisfies PBC on $\tilde{\Omega}$ if $\tilde{k}$ has integer components: $(k_x, k_y) \in \mathbb{Z}^2$. When such eigenfunctions are extended by periodic replication to $\mathbb{R}^2$ we get functions that are not square-integrable. Following Melbourne, we define an infinite-dimensional eigenspace $E_{\kappa^2}(\mathbb{R}^2)$ so that it contains the eigenspace $E_{\kappa^2}(\tilde{\Omega})$ and transforms under the representation determined by (11):

\[
(T_{(a,b)} \cdot \Lambda_{\tilde{k}})(\vec{r}) = e^{-i(ak_x + bk_y)}\Lambda_{\tilde{k}}(\vec{r}),
\]

\[
(R(\phi) \cdot \Lambda_{\tilde{k}})(\vec{r}) = \Lambda_{\tilde{k}'}(\vec{r}),
\]

where $\tilde{k}' = R(\phi) \cdot \tilde{k}$ from (5),

\[
(\gamma_2 \cdot \Lambda_{\tilde{k}})(\vec{r}) = \Lambda_{\tilde{k}'}(\vec{r}),
\]
where \( \vec{k}' = (k_y, k_x) \).

Since Euclidean transformations do not change the length of \( |\vec{k}| \), sums of the form

\[
\Lambda(\vec{r}) = \sum_{\vec{k} \in A(\kappa)} a(\vec{k}) \Lambda_{\vec{k}}(\vec{r})
\]  (20)

must be elements of \( E_{\kappa^2}(\mathbb{R}^2) \). In this sum \( A(\kappa) \) denotes the circle of \( \vec{k} \) vectors with length \( \kappa \), and we assume \( a(\vec{k}) = 0 \) for all but a finite set of points in \( A(\kappa) \); we also require \( a(\vec{k})^* = a(-\vec{k}) \) so that \( \Lambda(\vec{r}) \) is real. These finite sums form a linear vector space \( X(\kappa) \) which can be given the norm

\[
\| \Lambda(\vec{r}) \| = \sum_{\vec{k} \in A(\kappa)} |a(\vec{k})|.
\]  (21)

We define the critical eigenspace \( E_{\kappa^2}(\mathbb{R}^2) \) as the closure of \( X(\kappa) \) with respect to this norm \( E_{\kappa^2}(\mathbb{R}^2) \equiv \overline{X(\kappa)} \). Since eigenvalues for different values of \( \kappa \) are typically unequal, this choice for \( E_{\kappa^2}(\tilde{\Omega}) \) is satisfactory for codimension-one bifurcations. When two (or more) parameters are varied, then mode interactions can occur for which the sum in (20) should include eigenvectors \( \Lambda_{\vec{k}} \) with unequal values of \( |\vec{k}| \). The action of \( \mathcal{E}(2) \) in (17) - (19) defines a representation on \( E_{\kappa^2}(\mathbb{R}^2) \) in the obvious way. Melbourne has shown that this representation is absolutely irreducible.

**Theorem I.1** (Melbourne) Let \( L : E_{\kappa^2}(\mathbb{R}^2) \rightarrow E_{\kappa^2}(\mathbb{R}^2) \) be a linear map which commutes with the representation of \( \mathcal{E}(2) \) in (17) - (19), then \( L \) must be a scalar multiple of the identity.

**Proof.**

Let

\[
L \Lambda_{\vec{k}}(\vec{r}) = \sum_{\vec{k}' \in A(\kappa)} c_{\vec{k}'}(\vec{k}') \Lambda_{\vec{k}'}(\vec{r})
\]  (22)
describe the action of $L$ on $\Lambda_\vec{k}(\vec{r})$. When applied to $\Lambda_\vec{k}(\vec{r})$ the assumption that $L$ commutes with an arbitrary translation $T_\vec{p}$ requires $e^{-i\vec{k} \cdot \vec{p}} c_{\vec{k}}(\vec{k}') = e^{-i\vec{k}' \cdot \vec{p}} c_{\vec{k}}(\vec{k}')$ for arbitrary $\vec{p}$. Thus $c_{\vec{k}}(\vec{k}')$ must vanish unless $\vec{k} = \vec{k}'$: $c_{\vec{k}}(\vec{k}') \equiv \delta_{\vec{k},\vec{k}'} C(\vec{k})$. When $L$ is applied to a general element of $E_{\kappa^2}(\mathbb{R}^2)$, this simplification implies

$$L \Lambda(\vec{r}) = \sum_{\vec{k}' \in A(\kappa)} a(\vec{k}') C(\vec{k}') \Lambda_{\vec{k}'}$$

where $\Lambda(\vec{r})$ is given by (20). Acting alone, the remaining generators $\mathcal{R}(\phi)$ and $\gamma_2$ generate the subgroup $\text{O}(2)$, and the assumption that $\gamma L \Lambda(\vec{r}) = L \gamma \Lambda(\vec{r})$ for $\gamma \in \text{O}(2)$ implies $C(\vec{k}')$ must be an $\text{O}(2)$ invariant function $C(\gamma \cdot \vec{k}') = C(\vec{k}')$. The $\text{O}(2)$ invariance of $C$ means that it depends only on the magnitude of $\vec{k}'$ and hence is a $\kappa$-dependent constant $C(\vec{k}') = \sigma(\kappa)$. Thus our linear operator is simply $L = \sigma(\kappa) I$. □

B. $\mathcal{E}(2)$-equivariant maps on $E_{\kappa^2}(\mathbb{R}^2)$

A second result due to Melbourne that characterizes the nonlinear $\mathcal{E}(2)$-equivariant maps on $E_{\kappa^2}(\mathbb{R}^2)$. Let $\mathcal{F}: E_{\kappa^2}(\mathbb{R}^2) \rightarrow E_{\kappa^2}(\mathbb{R}^2)$ be a Euclidean-symmetric homogeneous polynomial map of degree $p$. That is we assume

$$\mathcal{F}(\alpha \Lambda) = \alpha^p \mathcal{F}(\Lambda) \quad \text{for any } \alpha \in \mathbb{C}$$

and

$$\gamma \cdot \mathcal{F}(\Lambda) = \mathcal{F}(\gamma \cdot \Lambda) \quad \text{for any } \gamma \in \mathcal{E}(2).$$

If we represent $\Lambda$ as in (20), then the homogeneity of $\mathcal{F}$ (24) implies that it must have the form

$$\mathcal{F}(\Lambda) = \sum_{\vec{k} \in A(\kappa)} \Lambda_{\vec{k}}(\vec{r}) \left[ \sum_{\vec{k}_1' \in A(\kappa)} \sum_{\vec{k}_2' \in A(\kappa)} \ldots \sum_{\vec{k}_p' \in A(\kappa)} a(\vec{k}_1') a(\vec{k}_2') \ldots a(\vec{k}_p') P(\vec{k}, \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p') \right].$$

(26)
Theorem I.2 (Melbourne) $\mathcal{F}$ will have Euclidean symmetry (25) if $P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p)$ satisfies
\[ P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p) = 0 \text{ if } \vec{k} \neq \vec{k}_1 + \vec{k}_2 + \ldots + \vec{k}_p; \] (27)
and
\[ P(\gamma \cdot \vec{k}, \gamma \cdot \vec{k}_1, \gamma \cdot \vec{k}_2, \ldots, \gamma \cdot \vec{k}_p) = P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p) \text{ for all } \gamma \in O(2) \subset E(2). \] (28)

In addition, if $\mathcal{F}$ is real-valued then $P$ must be real-valued.

Proof.

The Euclidean group $E(2) = O(2) \circ T(2)$ is the semi-direct product of $O(2)$ and the group of translations $T(2)$, and the conditions (28) and (27) correspond to these two components.

1. The effect of an arbitrary translation $T_{\vec{v}}$ on $\Lambda$ in (20) is to replace $a(\vec{k})$ by $a(\vec{k}) e^{-i\vec{k} \cdot \vec{v}}$. Thus $T_{\vec{v}} \cdot \mathcal{F}(\Lambda) = \mathcal{F}(T_{\vec{v}} \cdot \Lambda)$ requires
\[ [e^{-i\vec{v} \cdot (\vec{k}_1 + \vec{k}_2 + \ldots + \vec{k}_p - \vec{k})} - 1] P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p) = 0 \] (29)
for arbitrary $\vec{v}$; hence if $\vec{k} \neq \vec{k}_1 + \vec{k}_2 + \ldots + \vec{k}_p$, then $P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p) = 0$.

2. For $\gamma \in O(2)$, we evaluate $\gamma \cdot \mathcal{F}$ using the invariance of the inner product $\vec{k} \cdot \vec{r} = (\gamma \vec{k}) \cdot (\gamma \vec{r})$
\[ \gamma \cdot \mathcal{F}(\Lambda) = \sum_{\vec{k} \in A(\kappa)} \Lambda_{\vec{k}}(\vec{r}) \left[ \sum_{\vec{k}'_1 \in A(\kappa)} \ldots \sum_{\vec{k}'_p \in A(\kappa)} a(\vec{k}'_1) \ldots a(\vec{k}'_p) P(\gamma^{-1} \cdot \vec{k}, \vec{k}'_1, \ldots, \vec{k}'_p) \right], \] (30)
and then from
\[ (\gamma \cdot \Lambda)(\vec{r}) = \sum_{\vec{k} \in A(\kappa)} a(\gamma^{-1} \cdot \vec{k}) \Lambda_{\vec{k}}(\vec{r}), \] (31)
we obtain
\[
\mathcal{F}(\gamma \cdot \Lambda) = \sum_{\vec{k} \in A(\kappa)} \Lambda_{\vec{k}}(\vec{r}) \left[ \sum_{\vec{k}_1' \in A(\kappa)} \ldots \sum_{\vec{k}_p' \in A(\kappa)} a(\vec{k}_1') \ldots a(\vec{k}_p') P(\vec{k}, \gamma \cdot \vec{k}_1', \ldots, \gamma \cdot \vec{k}_p') \right].
\]

Thus \(\gamma \cdot \mathcal{F}(\Lambda) = \mathcal{F}(\gamma \cdot \Lambda)\) requires that \(P\) is an \(O(2)\)-invariant function:
\[
P(\gamma \cdot \vec{k}, \gamma \cdot \vec{k}_1, \gamma \cdot \vec{k}_2, \ldots, \gamma \cdot \vec{k}_p) = P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p).
\]

3. Note that for the reflection \(\vec{r} \rightarrow -\vec{r}\), this gives
\[
P(-\vec{k}, -\vec{k}_1, -\vec{k}_2, \ldots, -\vec{k}_p) = P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p);
\]
so given a real-valued map \(\mathcal{F}^* = \mathcal{F}\) for which \(P\) must satisfy \(P(-\vec{k}, -\vec{k}_1, -\vec{k}_2, \ldots, -\vec{k}_p)^* = P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p)\) the reflection symmetry (33) implies \(P\) is also real-valued.

\[\square\]

II. Periodic boundary conditions and hidden rotation symmetry

If \(\mathcal{F}\) is the Euclidean symmetric stroboscopic map on \(E_{\kappa^2}(\mathbb{R}^2)\), then the restriction in (14) to \(\text{Fix}(B_P)\), yields the stroboscopic map \(\tilde{\mathcal{F}}\) describing the bifurcation of solutions satisfying PBC on \(\tilde{\Omega}\). Since the intersection of the circle \(A(\kappa)\) with the integer lattice \(\mathbb{Z}^2\),
\[
\tilde{A}(\kappa) = A(\kappa) \cap \mathbb{Z}^2,
\]
contains a finite number of points, the eigenspace \(E_{\kappa^2}(\tilde{\Omega})\) will always be finite-dimensional. Although by construction the restricted map is equivariant with respect to \(N(B_P)\), the representation of \(N(B_P)\) is not faithful because of the kernel \(B_P\), and it is the quotient \(N(B_P)/B_P\) that describes the nontrivial action of \(N(B_P)\). One can show quite generally that \(N(B_P)/B_P\) is the semi-direct product \(\tilde{D}_4 + T^2\) of \(\tilde{D}_4\) and \(T^2\) where \(T^2\) denotes the translations \(\mathcal{T}(\theta_1, \theta_2)\) with \((\theta_1, \theta_2)\) taken to be \(2\pi\)-periodic variables and \(\tilde{D}_4 = \{\tilde{\gamma}_1, \gamma_2\}\) is
generated by diagonal reflection and reflection in \( x \). Depending on \( \kappa^2 \), the representation of \( \tilde{D}_4 \hat{+} T^2 \) obtained on \( \text{Fix}(B_P) \) may be reducible or irreducible.\(^3\) Thus \( \tilde{F} \) is always a \( \tilde{D}_4 \hat{+} T^2 \)-symmetric map although the representation involved varies.

It is important to understand whether the hidden Euclidean symmetry of \( \mathcal{F} \) imposes constraints on the form of \( \tilde{F} \) beyond the obvious constraint of \( \tilde{D}_4 \hat{+} T^2 \)-equivariance. Recall that when a (compact) Lie group \( \Gamma \) acts on a linear vector space \( V \), the set of all (smooth) \( \Gamma \)-symmetric vector fields on \( V \) has the structure of a module over the ring of \( \Gamma \)-invariant functions on \( V \). Let \( R(\Gamma) \) and \( \tilde{M}(\Gamma) \) denote this ring and module, respectively.\(^4\)

For \( \Gamma = \tilde{D}_4 \hat{+} T^2 \) acting on \( V = E_{\kappa^2}(\tilde{\Omega}) \), the set of all \( \tilde{D}_4 \hat{+} T^2 \)-equivariant maps is the module \( \tilde{M}(\tilde{D}_4 \hat{+} T^2) \), and the restriction in (14) always yields an element of \( \tilde{M}(\tilde{D}_4 \hat{+} T^2) \). The issue of hidden constraints poses the inverse problem: if \( \tilde{F} \in \tilde{M}(\tilde{D}_4 \hat{+} T^2) \) is taken to be an arbitrary \( \tilde{D}_4 \hat{+} T^2 \)-equivariant map on \( E_{\kappa^2}(\tilde{\Omega}) \) can one always obtain such an \( \tilde{F} \) by the restriction in (14)? The answer depends on whether or not the representation of \( \tilde{D}_4 \hat{+} T^2 \) carried by \( E_{\kappa^2}(\tilde{\Omega}) \) is irreducible.

The (non-trivial) irreducible representations of \( \tilde{D}_4 \hat{+} T^2 \) have dimension four or eight and may be classified by two mode numbers \( \vec{k} = (l, n) \) with \( l \geq n \geq 0 \) giving the components of a specific wavevector in \( \tilde{A}(\kappa) \).\(^11\) There are two sets of distinct four-dimensional irreducible representations:

1. For \( l = n > 0 \), \( \tilde{A}(\kappa) = \{ \pm \vec{k}_1, \pm \vec{k}_2 \} \) where \( \vec{k}_1 = (l, l) \) and \( \vec{k}_2 = (l, -l) \). Then \( \kappa^2 = 2l^2 \) and

\[
E_{\kappa^2}(\tilde{\Omega}) = \{ z_1 \Lambda_{k_1} (\vec{r}) + z_2 \Lambda_{k_2} (\vec{r}) + cc \mid (z_1, z_2) \in \mathbb{C}^2 \}, \quad (35)
\]

carries the representation of \( \tilde{D}_4 \hat{+} T^2 \) generated by

\[
\tilde{\gamma}_1 \cdot (z_1, z_2) = (z_2, \overline{z}_1) \quad (36) \\
\tilde{\gamma}_2 \cdot (z_1, z_2) = (z_1, \overline{z}_2) \quad (37)
\]

13
\[ T(\theta_1, \theta_2) \cdot (z_1, z_2) = (e^{-i(\theta_1 + \theta_2)} z_1, e^{-i(\theta_1 - \theta_2)} z_2). \] (38)

2. For \( l > n = 0, \tilde{A}(\kappa) = \{ \pm \vec{k}_1, \pm \vec{k}_2 \} \) where \( \vec{k}_1 = (l, 0) \) and \( \vec{k}_2 = (0, l) \). Then \( \kappa^2 = l^2 \) and

\[ E_{\kappa^2}(\tilde{\Omega}) = \{ \tilde{\Omega} = (z_1 \Lambda_{\vec{k}_1}^{-}(\vec{r}) + z_2 \Lambda_{\vec{k}_2}^{-}(\vec{r}) + cc)| (z_1, z_2) \in \mathbb{C}^2 \}, \] (39)

carries the representation of \( \tilde{D}_4 \hat{\otimes} T^2 \) generated by

\[ \tilde{\gamma}_1 \cdot (z_1, z_2) = (\bar{z}_1, \bar{z}_2) \] (40)
\[ \gamma_2 \cdot (z_1, z_2) = (z_2, z_1) \] (41)
\[ T(\theta_1, \theta_2) \cdot (z_1, z_2) = (e^{-i\theta_1} z_1, e^{-i\theta_2} z_2). \] (42)

Note that \( T(\pi/l, \pi/l) \) is in the kernel of (35) but not of (39), hence the two sets of representations must be inequivalent.

The eight-dimensional representations correspond to \( l > n > 0 \) with \( \tilde{A}(\kappa) = \{ \pm \vec{k}_1, \pm \vec{k}_2, \pm \vec{k}_3, \pm \vec{k}_4 \} \) where \( \vec{k}_1 = (l, n), \vec{k}_2 = (l, -n), \vec{k}_3 = (n, l), \) and \( \vec{k}_4 = (n, -l) \). Now \( \kappa^2 = l^2 + n^2 \) and

\[ E_{\kappa^2}(\tilde{\Omega}) = \{ (z_1 \Lambda_{\vec{k}_1}^{-}(\vec{r}) + z_2 \Lambda_{\vec{k}_2}^{-}(\vec{r}) + z_3 \Lambda_{\vec{k}_3}^{-}(\vec{r}) + z_4 \Lambda_{\vec{k}_4}^{-}(\vec{r}) + cc)| (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \} \] (43)

transforms under the following representation of \( \tilde{D}_4 \hat{\otimes} T^2 \):

\[ \tilde{\gamma}_1 \cdot (z_1, z_2, z_3, z_4) = (\bar{z}_2, \bar{z}_1, \bar{z}_4, \bar{z}_3) \] (44)
\[ \gamma_2 \cdot (z_1, z_2, z_3, z_4) = (z_2, \bar{z}_4, z_1, \bar{z}_2) \] (45)
\[ T(\theta_1, \theta_2) \cdot (z_1, z_2, z_3, z_4) = (e^{-i(\theta_1 + \theta_2)} z_1, e^{-i(\theta_1 - \theta_2)} z_2, e^{-i(n\theta_1 + l\theta_2)} z_3, e^{-i(n\theta_1 - l\theta_2)} z_4). \] (46)

We shall assume that \( (l, n) \) are relatively prime although this is not required for irreducibility. Our analysis will depend only on the invariant functions \( R(\tilde{D}_4 \hat{\otimes} T^2) \) and equivariant vector fields \( \tilde{M}(\tilde{D}_4 \hat{\otimes} T^2) \) for a given representation. For two irreducible representations \( (l, n) \) relatively prime and \( (l', m) = mn \), the sets \( R(\tilde{D}_4 \hat{\otimes} T^2) \) and \( \tilde{M}(\tilde{D}_4 \hat{\otimes} T^2) \) are the same. Our main result is the following theorem.
Theorem II.1 Assume \( E_{\kappa^2} (\tilde{\Omega}) \) carries an irreducible representation of \( \tilde{D}_4 + T^2 \), and \( \tilde{F} : E_{\kappa^2} (\tilde{\Omega}) \rightarrow E_{\kappa^2} (\tilde{\Omega}) \) is a \( \tilde{D}_4 + T^2 \)-equivariant map, then there is a \( \mathcal{E}(2) \)-equivariant map \( \mathcal{F} : E_{\kappa^2} (\mathbb{R}^2) \rightarrow E_{\kappa^2} (\mathbb{R}^2) \) such that

\[
\tilde{F} = \mathcal{F}|_{E_{\kappa^2}(\tilde{\Omega})}. \tag{47}
\]

Proof.

The theorem follows from lemma II.1 and lemma II.7 below which treat the four-dimensional and eight-dimensional cases respectively. \( \square \)

The content of this theorem is not new to experts in the subject but it seems that a proof has never been published.

A. Four-dimensional irreducible representations of \( \tilde{D}_4 + T^2 \)

For the four-dimensional representations, Gomes has given a general representation of \( \tilde{D}_4 + T^2 \)-equivariant maps \( \tilde{F} \) on \( E_{\kappa^2} (\tilde{\Omega}) \):

\[
\tilde{F}(z_1, z_2) = \tilde{p} (\tilde{N}, \tilde{\rho}) \left( \begin{array}{c} z_1 \\ z_2 \end{array} \right) + \tilde{q} (\tilde{N}, \tilde{\rho}) \left( \begin{array}{c} z_1 |z_2|^2 \\ z_2 |z_1|^2 \end{array} \right) + \tilde{r} (\tilde{N}, \tilde{\rho}) \left( \begin{array}{c} z_1 |z_1|^2 \\ z_2 |z_2|^2 \end{array} \right) \tag{48}
\]

where \( \tilde{p} \), \( \tilde{q} \), and \( \tilde{r} \) are arbitrary real-valued functions of the basic \( \tilde{D}_4 + T^2 \) invariants \( \tilde{N} = |z_1|^2 + |z_2|^2 \) and \( \tilde{\rho} = |z_1|^2 |z_2|^2 \). By simply checking that the three basic equivariants in (18) and the basic invariants \( \tilde{N} \) and \( \tilde{\rho} \) can be extended to Euclidean equivariants and invariants on \( E_{\kappa^2}(\mathbb{R}^2) \) one can prove the existence of an \( \mathcal{E}(2) \)-equivariant \( \mathcal{F} \) on \( E_{\kappa^2}(\mathbb{R}^2) \) which restricts to the general \( \tilde{F} \) as in (14). However, when we consider the eight-dimensional irreducible representation and higher-dimensional reducible representations associated with mode interactions, it becomes increasingly laborious to enumerate such a Hilbert basis for the \( \tilde{D}_4 + T^2 \) invariants and generators for the \( \tilde{D}_4 + T^2 \) equivariants. Thus in this simpler case we illustrate an alternative approach which requires only a study of the \( T^2 \) invariants of a given representation of \( \tilde{D}_4 + T^2 \).
Lemma II.1  Assume \( E_{\kappa^2}(\tilde{\Omega}) \) carries a four-dimensional irreducible representation of \( \tilde{D}_4 \cdot T^2 \) as in (33) or (39), and \( \tilde{F} : E_{\kappa^2}(\tilde{\Omega}) \to E_{\kappa^2}(\tilde{\Omega}) \) is a \( \tilde{D}_4 \cdot T^2 \)-equivariant map, then there is a \( \mathcal{E}(2) \)-equivariant map \( F : E_{\kappa^2}(\mathbb{R}^2) \to E_{\kappa^2}(\mathbb{R}^2) \) such that \( \tilde{F} = F|_{E_{\kappa^2}(\tilde{\Omega})} \).

Proof.

1. The reflections \( \tilde{\gamma}_1 \) and \( \gamma_2 \) imply \( \tilde{F} \) may be written in terms of a single function \( F_1(z_1, z_2) \). Translation symmetry of \( \tilde{F} \) then implies that \( F_1 \) is simply related to the \( T^2 \) invariants. Let

\[
\tilde{F}(z_1, z_2) = \begin{pmatrix} F_1(z_1, z_2) \\ F_2(z_1, z_2) \end{pmatrix}; \tag{49}
\]

for representations (33) associated with equal wavenumbers \( \tilde{\gamma}_1 \cdot \tilde{F}(z_1, z_2) = \tilde{F}(\tilde{\gamma}_1 \cdot (z_1, z_2)) \) requires \( F_2(z_1, z_2) = F_1(z_2^*, z_1^*)^* \) and \( \gamma_2 \cdot \tilde{F}(z_1, z_2) = \tilde{F}(\gamma_2 \cdot (z_1, z_2)) \) implies

\[
\begin{align*}
F_1(z_1, z_2) &= F_1(z_1, z_2^*) \\
F_1(z_1, z_2) &= F_1(z_1^*, z_2^*). \tag{50}
\end{align*}
\]

Translation symmetry, \( \mathcal{T}(\theta_1, \theta_2) \cdot \tilde{F}(z_1, z_2) = \tilde{F}(\mathcal{T}(\theta_1, \theta_2) \cdot (z_1, z_2)) \), requires

\[
e^{-i\theta_1(z_1)} F_1(z_1, z_2) = F_1(e^{-i\theta_1(z_2)}, e^{-i\theta_2(z_1)}); \tag{51}
\]

which states that \( z_1^* F_1(z_1, z_2) \) is a translation-invariant function. Thus for this representation all equivariant maps have the form

\[
\tilde{F}(z_1, z_2) = \begin{pmatrix} F_1(z_1, z_2) \\ F_1(z_2, z_1) \end{pmatrix} \tag{52}
\]

where \( F_1 \) satisfies (49) and \( (z_1^* F_1) \) is \( T^2 \)-invariant.

2. For representations (39) associated with zero wavenumbers, the same considerations show that \( \tilde{F} \) must have the form

\[
\tilde{F}(z_1, z_2) = \begin{pmatrix} F_1(z_1, z_2) \\ F_1(z_2, z_1) \end{pmatrix} \tag{53}
\]
where $F_1$ satisfies $F_1(z_1, z_2) = F_1(z_1, z_2^*) = F_1(z_1^*, z_2^*)$ and $(z_1^* F_1)$ is $T^2$-invariant.

3. For both types of irreducible representations, the quadratic invariants $\sigma_1 = |z_1|^2$ and $\sigma_2 = |z_2|^2$ provide a Hilbert basis for $T^2$-invariant functions. Thus, in both cases, the general monomial $F_1$ defining a homogeneous symmetric map $\tilde{F}(z_1, z_2)$ of degree $p$ is

$$F_1(z_1, z_2) = z_1^{\sigma_1^m} \sigma_2^n$$

with $p = 2(m + n) + 1$. (The form of the resulting map $\tilde{F}$ depends on the representation considered.)

4. Now consider the form of the $\tilde{D}_4 + T^2$-equivariant map obtained when we restrict an $E(2)$-equivariant map $\mathcal{F}(\Lambda)$ on $E_{\kappa^2}(\mathbb{R}^2)$ to the four-dimensional eigenspace $E_{\kappa^2}(\tilde{\Omega})$ defined in (35) or (39). In either case the restriction is obtained in same manner: $\Lambda \in E_{\kappa^2}(\mathbb{R}^2)$ belongs to $E_{\kappa^2}(\tilde{\Omega})$ if $a(\vec{k})$ in (55) has the form

$$a(\vec{k}) = z_1 \delta_{\vec{k}, \vec{k}^1} + z_2 \delta_{\vec{k}, \vec{k}^2} + z_1^* \delta_{\vec{k}^1, \vec{k}} + z_2^* \delta_{\vec{k}^2, \vec{k}}$$

where the definition of $\vec{k}^1$ and $\vec{k}^2$ depends on the representation. Inserting $a(\vec{k})$ from (55) into the right hand side of (26) gives

$$\mathcal{F}|_{E_{\kappa^2}(\mathbb{R}^2)} = \sum_{\vec{k}' \in \tilde{\Omega}(\kappa)} a'(\vec{k}') \Lambda_{\vec{k}'}(\vec{r})$$

where

$$a'(\vec{k}') = \sum_{\vec{k}'_1 \in \tilde{\Omega}(\kappa)} \ldots \sum_{\vec{k}'_p \in \tilde{\Omega}(\kappa)} a(\vec{k}'_1) a(\vec{k}'_2) \ldots a(\vec{k}'_p) P(\vec{k'}, \vec{k}^1_1, \vec{k}^2_2, \ldots, \vec{k}^p_p)$$
with $\tilde{A}(\kappa) = \{ \pm \vec{k}_1, \pm \vec{k}_2 \}$. Since $a'(\vec{k})$, defined in this way, must also have the form \( (\kappa) \), we can read off the function $F_1(z_1, z_2)$ for the resulting $\tilde{D}_4 + T^2$-equivariant map by setting $\vec{k}' = \vec{k}_1$ in (57):

$$F_1(z_1, z_2) = \sum_{\vec{k}_1' \in \tilde{A}(\kappa)} \ldots \sum_{\vec{k}_p' \in \tilde{A}(\kappa)} a(\vec{k}_1') a(\vec{k}_2') \ldots a(\vec{k}_p') P(\vec{k}_1', \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p'). \quad (58)$$

5. Comparing (54) and (58) shows that to prove that a given $\tilde{D}_4 + T^2$-equivariant map can be obtained by restriction from a $E(2)$-symmetric map on $E_{\kappa^2}(\mathbb{R}^2)$, it is sufficient to check the existence of $P(\vec{k}_1', \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p')$ satisfying both (27) and (28) and yielding

$$z_1 \sigma^m_1 \sigma^n_2 = \sum_{\vec{k}_1' \in \tilde{A}(\kappa)} \ldots \sum_{\vec{k}_p' \in \tilde{A}(\kappa)} a(\vec{k}_1') \ldots a(\vec{k}_p') P(\vec{k}_1', \vec{k}_1', \ldots, \vec{k}_p') \quad (59)$$

for arbitrary integers $m \geq 0$ and $n \geq 0$. The $O(2)$ invariance of $P$ (28) can be manifestly assured by constructing it from the inner products of its arguments. The condition for $T^2$-equivariance (27) can be checked once $P$ satisfies (59).

6. For both four-dimensional representations, we have

$$\begin{pmatrix}
    \vec{k}_1 \cdot \vec{k}_1 & \vec{k}_1 \cdot \vec{k}_2 \\
    \vec{k}_2 \cdot \vec{k}_1 & \vec{k}_2 \cdot \vec{k}_2
\end{pmatrix} = \begin{pmatrix}
    \kappa^2 & 0 \\
    0 & \kappa^2
\end{pmatrix} \quad (60)$$

so that up to an overall sign there are only two inner products that occur for vectors in $\tilde{A}(\kappa)$. Define $I_1(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, \kappa^2}$ and $I_2(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, 0}$ to describe these two products; using $I_1$ and $I_2$, we can easily construct $P$ to satisfy (59). A few examples illustrate the point. For $m = n = 0$, set $P(\vec{k}_1', \vec{k}_1') = I_1(\vec{k}_1', \vec{k}_1')$ which yields

$$z_1 = \sum_{\vec{k}_1' \in \tilde{A}(\kappa)} a(\vec{k}_1') P(\vec{k}_1', \vec{k}_1'), \quad (61)$$
and clearly satisfies the conditions (27) and (28) required by Euclidean symmetry. We add a factor of $\sigma_1$ by choosing

$$P(k_1, k_1', k_2, k_3') = I_1(k_1, k_1') I_1(k_1, k_2') I_1(k_2', -k_3')$$  \quad (62)$$

which gives

$$z_1 \sigma_1^2 = \sum_{k_1' \in \tilde{A}(\kappa)} \sum_{k_2' \in \tilde{A}(\kappa)} \sum_{k_3' \in \tilde{A}(\kappa)} a(k_1') a(k_2') a(k_3') P(k_1, k_1', k_2', k_3').$$  \quad (63)$$

Similarly choosing

$$P(k_1, k_1', k_2, k_3') = I_1(k_1, k_1') I_2(k_1, k_2') I_1(k_2', -k_3')/2$$  \quad (64)$$

gives

$$z_1 \sigma_2^2 = \sum_{k_1' \in \tilde{A}(\kappa)} \sum_{k_2' \in \tilde{A}(\kappa)} \sum_{k_3' \in \tilde{A}(\kappa)} a(k_1') a(k_2') a(k_3') P(k_1, k_1', k_2', k_3').$$  \quad (65)$$

For fixed $\vec{k}_1$ there are two solutions to $\vec{k}_1 \perp \vec{k}_2'$ so we need the factor of $1/2$ in (64). The requirement of translation symmetry (27) is obviously satisfied for the functions in (62) and (64). By appending additional factors of $I_1$ and $I_2$ with appropriately specified arguments one can arrange for an arbitrary number of factors of $\sigma_1$ and $\sigma_2$, respectively, without losing the translation symmetry (27). \Box

\section{Eight-dimensional irreducible representations of $\tilde{D}_4 + T^2$}

The corresponding result for the eight-dimensional representations is proved similarly. The analysis of a $\tilde{D}_4 + T^2$ equivariant can be reduced to the study of the $T^2$ invariants for the representation. The argument is longer however because the ring of $T^2$ invariants is more complicated.
For the representation (44) - (46) with \( l > n > 0 \) relatively prime, let \( z \equiv (z_1, z_2, z_3, z_4) \) and consider an arbitrary \( \tilde{D}_4 + T^2 \) equivariant map

\[
\tilde{F}(z) = \begin{pmatrix}
F_1(z) \\
F_2(z) \\
F_3(z) \\
F_4(z)
\end{pmatrix}.
\] (66)

The reflection symmetry, \( \tilde{\gamma}_1 \cdot \tilde{F}(z) = \tilde{F}(\tilde{\gamma}_1 \cdot (z)) \), requires \( F_2(z) = F_1(\tilde{\gamma}_1 \cdot z)^* \) and \( F_4(z) = F_3(\tilde{\gamma}_1 \cdot z)^* \), and \( \gamma_2 \cdot \tilde{F}(z) = \tilde{F}(\gamma_2 \cdot z) \) implies \( F_3(z) = F_1(\gamma_2 \cdot z) \) and

\[
F_1(z) = F_1(z^*). \tag{67}
\]

The translation symmetry, \( T_{(\theta_1, \theta_2)} \cdot \tilde{F}(z) = \tilde{F}(T_{(\theta_1, \theta_2)} \cdot (z)) \), requires \( e^{-i(l\theta_1 + n\theta_2)} F_1(z) = F_1(T_{(\theta_1, \theta_2)} \cdot z) \) which states that \( z_1^* F_1(z) \) is a translation-invariant function. Thus for this representation all equivariant maps have the form

\[
\tilde{F}(z) = \begin{pmatrix}
F_1(z) \\
F_1(\gamma_3 \cdot z) \\
F_1(\gamma_2 \cdot z) \\
F_1(\tilde{\gamma}_1 \gamma_2 \cdot z)
\end{pmatrix} \tag{68}
\]

where \( F_1 \) satisfies (67) and \( (z_1^* F_1) \) is \( T^2 \)-invariant. As in the four-dimensional case we determine a Hilbert basis which generates the \( T^2 \)-invariant functions and then analyze the equivariants (68) in light of this basis.

1. Generators for the \( T^2 \)-invariants

It is sufficient to enumerate the generators for the monomial invariants, a problem that has been previously considered by Gomes.\textsuperscript{12,13} We obviously require the elementary quadratic invariants

\[
\sigma_i = |z_i|^2 \quad i = 1, 2, 3, 4; \tag{69}
\]

for the analysis of the more complicated invariants it is convenient to define

\[
\omega_i^{\nu_i} \equiv \begin{cases} z_i^{\nu_i} & \text{if } \nu_i \geq 0 \\ z_i^{-|\nu_i|} & \text{if } \nu_i < 0 \end{cases} \tag{70}
\]
Lemma II.2 (Gomes 12, 13) A $T^2$-invariant monomial $M(z) = z_1^{\mu_1} z_2^{\mu_2} z_2^{\mu_2} z_3^{\mu_3} z_4^{\mu_4}$ can be written in the form

\[ M(z) = \sigma_1^{\nu_1} \sigma_2^{\nu_2} \sigma_3^{\nu_3} \sigma_4^{\nu_4} m(z) \]  

(71)

where $\nu_i = \min (\mu_i, \mu_i')$ and $m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4}$. The exponents $(\nu_1, \nu_2, \nu_3, \nu_4)$ are given by

\[ \nu_1 = -(bl + an)/2 \quad \nu_2 = (bl - an)/2 \]  

(72)

\[ \nu_3 = (al + bn)/2 \quad \nu_4 = (al - bn)/2 \]  

(73)

where the integers $(a, b)$ are arbitrary except that $(a, b)$ must be even for $(l + n)$ odd, and for $(l + n)$ even, $(a, b)$ must have the same parity but may be either even or odd. No more than one exponent $\nu_i$ can be zero.

Proof.

1. The reduction to (71) only requires extracting that all factors of the form $z_i z_i^*$. The $T^2$-invariance of $M(z)$ implies $m(z)$ must be $T^2$-invariant, this in turn requires $(\nu_1, \nu_2, \nu_3, \nu_4)$ satisfy

\[ l(\nu_1 + \nu_2) + n(\nu_3 + \nu_4) = 0 \]  

(74)

\[ n(\nu_1 - \nu_2) + l(\nu_3 - \nu_4) = 0. \]  

(75)

2. If any two exponents are set to zero, then (74) and (75) force the remaining two to vanish. Hence no more than one exponent can be zero. For example, if $\nu_3 = \nu_4 = 0$, then $\nu_1 + \nu_2 = 0$, and $\nu_1 = \nu_2$ hence $\nu_1 = \nu_2 = 0$.

3. Since $l$ and $n$ are assumed relatively prime, the first condition (74) requires that $(\nu_1 + \nu_2)$ contain a factor of $n$ and $(\nu_3 + \nu_4)$ contain a factor of $l$. 


\[(\nu_1 + \nu_2) = -an \text{ and } (\nu_3 + \nu_4) = al; \text{ similarly the second condition } (73) \text{ requires } (\nu_1 - \nu_2) = -bl \text{ and } (\nu_3 - \nu_4) = bn.\]

4. Solving for \((\nu_1, \nu_2, \nu_3, \nu_4)\) in terms of the integers \((a, b)\) yields (72) - (73).

Obviously the four numerators on the right hand side in (72) - (73) must be even integers, this constrains the allowed parities for \((a, b)\). For \((l + n)\) odd, \(l\) and \(n\) must have opposite parity so both \(a\) and \(b\) must be even; for \((l + n)\) even, \(l\) and \(n\) must be odd and it is only necessary for \((a, b)\) to have the same parity.

\[\square\]

Thus the Hilbert basis contains the elementary quadratic invariants, and also additional higher order “mixed” invariants which generate the remaining \(m(z)\) factor in (71). In determining generators for the mixed invariants, we exploit the fact that \(T^2\) is a normal subgroup of \(\tilde{D}_4\tilde{+}T^2\), that is, if \(T(\theta_1, \theta_2) \in T^2\) then \(\gamma T(\theta_1, \theta_2) \gamma^{-1} \in T^2\) for any \(\gamma \in \tilde{D}_4\tilde{+}T^2\). This is obviously true if \(\gamma\) is a translation and we can easily check that it holds for \(\tilde{\gamma}_1\) and \(\gamma_2\), the generators of \(\tilde{D}_4\): \(\tilde{\gamma}_1 T(\theta_1, \theta_2)\tilde{\gamma}_1^{-1} = T(-\theta_1, \theta_2)\) and \(\gamma_2 T(\theta_1, \theta_2) \gamma_2^{-1} = T(\theta_2, \theta_1)\). The normality of \(T^2\) has the following useful consequence.

**Lemma II.3** If \(M(z)\) is a \(T^2\)-invariant function and \(\gamma \in \tilde{D}_4\) then \(M(\gamma \cdot z)\) is also a \(T^2\)-invariant function. In particular, since \(\tilde{\gamma}_1 \gamma_2 \tilde{\gamma}_1 \gamma_2 \cdot z = z^*\), it follows that \(M(z^*)\) is \(T^2\)-invariant.

**Proof.**

We wish to show that \(M(\gamma T(\theta_1, \theta_2) \cdot z) = M(\gamma \cdot z)\) for arbitrary \((\theta_1, \theta_2)\). Since \(T^2\) is a normal subgroup, for any \(\gamma \in \tilde{D}_4\) and any \(T(\theta_1, \theta_2)\) we must have \(\gamma T(\theta_1, \theta_2) \gamma^{-1} = \)
Table 2. $\tilde{D}_4$ acting on $E_{\kappa^2}(\Omega)$ and the $\tilde{D}_4$-orbit of $m(z) = z_1^*\alpha z_3^\delta z_4^\sigma$

| $\gamma \in \tilde{D}_4$ | $\gamma \cdot z$         | $m(\gamma \cdot z)$ | $(\nu_1, \nu_2, \nu_3, \nu_4)$ |
|------------------------|--------------------------|----------------------|---------------------------------|
| $I$                    | $(z_1, z_2, z_3, z_4)$   | $z_1^*\alpha z_3^\delta z_4^\sigma$ | $(-\alpha, 0, \delta, \sigma)$ |
| $\tilde{\gamma}_1$    | $(z_2^*, z_1^*, z_4^*, z_3^*)$ | $z_2^\alpha z_3^\delta z_4^\sigma$ | $(0, \alpha, -\delta, -\delta)$ |
| $\gamma_2$            | $(z_3, z_4^*, z_1^*, z_2^*)$ | $z_1^\delta z_2^* z_3^\alpha$ | $(-\sigma, -\alpha, 0)$ |
| $\tilde{\gamma}_1\gamma_2$ | $(z_4^*, z_3^*, z_2^*, z_1^*)$ | $z_1^\sigma z_2^\delta z_4^\alpha$ | $(0, -\delta, 0, \alpha)$ |
| $\gamma_3$            | $(z_2^*, z_1^*, z_4^*, z_3^*)$ | $z_2^\alpha z_3^\sigma z_4^\delta$ | $(0, -\alpha, \sigma, \delta)$ |
| $\tilde{\gamma}_1\gamma_3$ | $(z_1^*, z_2^*, z_3^*, z_4^*)$ | $z_1^\alpha z_3^\delta z_4^\sigma$ | $(\alpha, 0, -\delta, -\sigma)$ |
| $\gamma_2\tilde{\gamma}_1$ | $(z_4^*, z_3^*, z_2^*, z_1^*)$ | $z_1^\sigma z_2^* z_4^\alpha$ | $(-\sigma, -\delta, 0, \alpha)$ |
| $\tilde{\gamma}_1\gamma_2\tilde{\gamma}_1$ | $(z_3^*, z_4^*, z_1^*, z_2^*)$ | $z_1^\delta z_2^\sigma z_3^\alpha$ | $(-\delta, \sigma, \alpha, 0)$ |

Notes:
Here $z = (z_1, z_2, z_3, z_4)$ denotes a point in $E_{\kappa^2}(\Omega)$, $I$ stands for the identity element, and $\gamma_3 \equiv \gamma_2\tilde{\gamma}_1\gamma_2$. The exponents $(\alpha, \delta, \sigma)$ are positive integers.

$T_{(\theta', \theta_2)}$ for some $(\theta'_1, \theta'_2)$. Thus we have by direct calculation

$$M(\gamma T_{(\theta_1, \theta_2)} \cdot z) = M(\gamma T_{(\theta_1, \theta_2)} \gamma^{-1} \gamma \cdot z) = M(T_{(\theta'_1, \theta'_2)} \gamma \cdot z) = M(\gamma \cdot z)$$

(76)

where the last equation follows from the assumed $T^2$-invariance of $M(z)$. $\square$

For example in Table 2 we show the $\tilde{D}_4$ orbit of the monomial $m(z) = z_1^*\alpha z_3^\delta z_4^\sigma$. If the exponents $\alpha, \delta, \sigma$ are chosen so that $m(z)$ is $T^2$-invariant, then by Lemma II.3 each monomial on the orbit of $m(z)$ is also $T^2$-invariant.
Table 3. $\tilde{D}_4$ acting on $E_{n^2}(\Omega)$ and the $\tilde{D}_4$-orbits of monomials $m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4}$

| $\gamma \in \tilde{D}_4$ | $m_1(\gamma \cdot z)$ | $m_2(\gamma \cdot z)$ | $m_3(\gamma \cdot z)$ | $m_4(\gamma \cdot z)$ |
|--------------------------|------------------------|------------------------|------------------------|------------------------|
| $I$                      | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ | $z_1^\alpha z_2^\beta z_3^\sigma z_4^\delta$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ |
| $\tilde{\gamma}_1$      | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ |
| $\gamma_2$               | $z_1^\delta z_2^\sigma z_3^\alpha z_4^\beta$ | $z_1^\delta z_2^\sigma z_3^\alpha z_4^\beta$ | $z_1^\delta z_2^\sigma z_3^\alpha z_4^\beta$ | $z_1^\delta z_2^\sigma z_3^\alpha z_4^\beta$ |
| $\tilde{\gamma}_1 \gamma_2$ | $z_1^\sigma z_2^\delta z_3^\beta z_4^\alpha$ | $z_1^\sigma z_2^\delta z_3^\beta z_4^\alpha$ | $z_1^\sigma z_2^\delta z_3^\beta z_4^\alpha$ | $z_1^\sigma z_2^\delta z_3^\beta z_4^\alpha$ |
| $\gamma_3$               | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ | $z_1^\beta z_2^\alpha z_3^\sigma z_4^\delta$ |
| $\tilde{\gamma}_1 \gamma_3$ | $z_1^\delta z_2^\sigma z_3^\beta z_4^\alpha$ | $z_1^\delta z_2^\sigma z_3^\beta z_4^\alpha$ | $z_1^\delta z_2^\sigma z_3^\beta z_4^\alpha$ | $z_1^\delta z_2^\sigma z_3^\beta z_4^\alpha$ |
| $\gamma_2 \tilde{\gamma}_1$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ | $z_1^\alpha z_2^\beta z_3^\delta z_4^\sigma$ |
| $\tilde{\gamma}_1 \gamma_2 \tilde{\gamma}_1$ | $z_1^\sigma z_2^\delta z_3^\alpha z_4^\beta$ | $z_1^\sigma z_2^\delta z_3^\alpha z_4^\beta$ | $z_1^\sigma z_2^\delta z_3^\alpha z_4^\beta$ | $z_1^\sigma z_2^\delta z_3^\alpha z_4^\beta$ |

Notes:
Here $z = (z_1, z_2, z_3, z_4)$ denotes a point in $E_{n^2}(\Omega)$, $I$ stands for the identity element, and $\gamma_3 \equiv \gamma_2 \gamma_1 \gamma_2$. The exponents $(\alpha, \beta, \delta, \sigma)$ are positive integers. There are $2^4 = 16$ distinct ways to choose the signs of the exponents $(\nu_1, \nu_2, \nu_3, \nu_4)$; each possibility occurs twice in the table. The boldfaced entries comprise one set of the 16 possibilities.
In light of Lemma II.3, we seek a sub-basis of $T^2$ invariants $\{\sigma_i\}_{i=1}^J$ with the property that collectively the $D_4$ orbits of the sub-basis provide a Hilbert basis for the $T^2$-invariants.\footnote{A different strategy for calculating translation invariants has been described by Comes as part of an investigation of hidden symmetry in bifurcation problems with rectangular symmetry.\cite{comes_1}}

In enumerating this sub-basis we have the elementary invariant $\sigma_1 = |z_1|^2$, whose $D_4$ orbit consists of the four quadratic invariants \((39)\), and a variable set of mixed invariants whose number and definition depends on the representation \((l, n)\). Three of these can be given explicitly:

$$
\sigma_5(z) = \begin{cases} 
    z_1^{snl} z_3^{(l^2+n^2)/2} z_4^{(l^2-n^2)/2} & \text{if } (l + n) \text{ even} \\
    z_1^{2nl} z_3^{(l^2+n^2)} z_4^{(l^2-n^2)} & \text{if } (l + n) \text{ odd} 
\end{cases} \quad (77)
$$

$$
\sigma_6(z) = \begin{cases} 
    (z_1^*)^{(l+2n)/2} z_2^{(l-n)/2} z_3^{(l+n)/2} z_4^{(l-n)/2} & \text{if } (l + n) \text{ even} \\
    (z_1^*)^{(l-n)/2} z_2^{(l+n)/2} z_3^{(l-n)} z_4^{(l-n)} & \text{if } (l + n) \text{ odd} 
\end{cases} \quad (78)
$$

$$
\sigma_7(z) = (z_1^* z_2^*)^n (z_3 z_4)^l. \quad (79)
$$

In addition the sub-basis generally involves invariants that are most simply described in a somewhat implicit fashion. Given a positive integer $\beta$, define the integers \((a_\beta, b_\beta)\) and \((a'_\beta, b'_\beta)\) as follows. Let $a_\beta$ denote the smallest positive integer (for fixed $\beta$) such that $b_\beta \equiv (2\beta + na_\beta)/l$ is an integer with the same even/odd parity as $a_\beta$. Similarly let $b'_\beta$ denote the smallest positive integer (for fixed $\beta$) such that $a'_\beta \equiv (2\beta + lb'_\beta)/n$ is an integer with the same even/odd parity as $b'_\beta$. Using \((a_\beta, b_\beta)\) and \((a'_\beta, b'_\beta)\) we define the two $T^2$-invariants:

$$
m_1(z, a_\beta, b_\beta) \equiv (z_1^*)^{(b_\beta l+a_\beta n)/2} z_2^{(b_\beta l-a_\beta n)/2} z_3^{(a_\beta l+b_\beta n)/2} z_4^{(a_\beta l-b_\beta n)/2}. \quad (80)
$$

$$
m_2(z, a'_\beta, b'_\beta) \equiv (z_1^*)^{(b'_\beta l+a'_\beta n)/2} (z_2^*)^{(a'_\beta l+b'_\beta n)/2} z_3^{(a'_\beta l+b'_\beta n)/2} z_4^{(a'_\beta l-b'_\beta n)/2}. \quad (81)
$$
Lemma II.4 For the eight-dimensional representation \((44) - (46)\) with \(l > n > 0\) relatively prime, the invariants \(\{\sigma_i\}_{i=1}^7\), \(\{m_1(z, a_\beta, b_\beta)\}_{\beta=1}^N\), and \(\{m_2(z, a'_\beta, b'_\beta)\}_{\beta=1}^{n-1}\) where

\[
N = \begin{cases} 
(l - n - 2)/2 & \text{if } (l + n) \text{ even} \\
 l - n - 1 & \text{if } (l + n) \text{ odd.} 
\end{cases}
\] (82)

form a sub-basis for the \(T^2\)-invariants. If \(N = 0\) the \(m_1\) invariants are absent and if \(n = 1\) the \(m_2\) invariants are absent.

Proof.

This result follows from Lemma II.2 and Lemmas II.5, II.6 below. \(\square\)

As an illustration of Lemma II.4 let \((l, n) = (4, 3)\), then we have \(N = 0\) in (82) so there are no invariants of \(m_1\) type. There are two invariants of \(m_2\) type corresponding to \(\beta = 1, 2\) with \((a'_1, b'_1) = (6, 4)\) and \((a'_2, b'_2) = (4, 2)\) respectively. The sub-basis consists of the quadratic invariants \(\sigma_1, \ldots, \sigma_4\) and

\[
\begin{align*}
\sigma_5 &= z_1^{24} z_3^{25} z_4^7 \\
\sigma_6 &= (z_1^*)^7 z_2^7 z_3 z_4 \\
\sigma_7 &= (z_1^* z_2^*)^3 (z_3 z_4)^4 \\
\sigma_8 &= m_2(z, 6, 4) = (z_1^*)^{17} z_2^{18} z_3^6 z_4^6 \\
\sigma_9 &= m_2(z, 4, 2) = (z_1^*)^{10} (z_2^*)^2 z_3^{11} z_4^5.
\end{align*}
\] (83)

When \(\tilde{D}_4\) is applied to this sub-basis, we obtain a Hilbert basis with 36 distinct elements.

We now give two results that characterize a set of generators for the mixed invariants.

**Lemma II.5** 1. All the \(T^2\)-invariant monomials \(m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4}\) with exactly one zero exponent \(\nu_i = 0\) are generated by applying \(\tilde{D}_4\) to the invariant \(\sigma_5(z)\).
2. All the $T^2$-invariant monomials $m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4}$ with $\nu_i \neq 0$ for $i = 1, 2, 3, 4$ are generated by applying $\tilde{D}_4$ to the following two classes of monomial invariants:

(a) Class $m_1$:

$$m_1(z, a, b) = z_1^{\alpha} z_2^{\beta} z_3^{\delta} z_4^{\sigma}$$  \hspace{1cm} (84)

with

$$\alpha = (bl + an)/2 \quad \beta = (bl - an)/2$$  \hspace{1cm} (85)
$$\delta = (al + bn)/2 \quad \sigma = (al - bn)/2$$  \hspace{1cm} (86)

where the positive integers $(a, b)$ satisfy

$$a \geq b > \frac{n}{l} a > 0.$$  \hspace{1cm} (87)

(b) Class $m_2$:

$$m_2(z, a, b) = z_1^{\alpha} z_2^{\beta} z_3^{\delta} z_4^{\sigma}$$  \hspace{1cm} (88)

with

$$\alpha = (bl + an)/2 \quad \beta = (an - bl)/2$$  \hspace{1cm} (89)
$$\delta = (al + bn)/2 \quad \sigma = (al - bn)/2$$  \hspace{1cm} (90)

where the non-negative integers $(a, b)$ satisfy

$$\frac{n}{l} a > b \geq 0.$$  \hspace{1cm} (91)

Proof.

1. First consider monomials $m(z)$ with exactly one zero exponent. From Table 2 it is clear that the $\tilde{D}_4$ orbit of every such monomial must contain a point with $\nu_2 = 0$ (in fact two such points). Thus we can restrict attention to
invariants \( m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4} \) with \( \nu_2 = 0 \) and then apply \( \tilde{D}_4 \) to generate the remaining possibilities. In addition we can always arrange that \( \nu_1 < 0 \) by applying \( \gamma_2 \tilde{\gamma}_1 \) if necessary. From (72) - (73), setting \( \nu_2 = 0 \) requires \( bl = an \) which in turn requires that \( b \) contain a factor of \( n \) and \( a \) a factor of \( l: a = dl \) and \( b = dn \). Hence the remaining exponents become \( \nu_1 = -dnl \), \( \nu_3 = d(l^2 + n^2)/2 \), and \( \nu_4 = d(l^2 - n^2)/2 \); \( d < 0 \) is required to obtain \( \nu_1 < 0 \).

Thus all of these invariants have the form
\[
\left[ (z_1^*)^{nl} z_3^{(l^2+n^2)/2} z_4^{(l^2-n^2)/2} \right]^d;
\] (92)

if \((l + n)\) is even then \(d = 1\) gives the lowest order result and if \((l + n)\) is odd then we must take \(d = 2\). This yields the definition of \( \sigma_5 \) in (77), and proves the first part of the lemma.

2. Now consider monomials \( m(z) = \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4} \) with all exponents non-zero.

There are \(2^4 = 16\) ways to choose the signs of \((\nu_1, \nu_2, \nu_3, \nu_4)\), however we need only consider the four possibilities: \( m_1(z) = z_1^{*\alpha} z_2^{*\beta} z_3^{*\delta} z_4^{*\sigma}, \) \( m_2(z) = z_1^{*\alpha} z_2^{*\beta} z_3 z_4^{*\sigma}, \) \( m_3(z) = z_1^{*\alpha} z_2^{*\beta} z_3^{*\delta} z_4^{*\sigma}, \) and \( m_4(z) = z_1^{*\alpha} z_2^{*\beta} z_3^{*\delta} z_4^{*\sigma}; \) the remaining twelve possibilities can be generated by applying \( \tilde{D}_4 \), cf. Table 3.

3. Of these four possibilities, only the first two can satisfy the requirement of \( T^2 \)-invariance. For \( m_3 \), the relations (72) - (73) become \( \alpha = (bl + an)/2, \beta = -(bl - an)/2, \delta = -(al + bn)/2, \) and \( \sigma = (al - bn)/2. \) Therefore \( \alpha + \beta = an \) and \( \delta + \sigma = -bl \) imply \( a > 0 \) and \( b < 0 \), so that \( \alpha > 0 \) and \( \delta > 0 \) require \( |a| > |b| \) and \( |a| < |b| \), respectively. This is impossible.

Similarly for \( m_4, T^2 \)-invariance requires \( \alpha = (bl + an)/2, \beta = -(bl - an)/2, \delta = -(al + bn)/2, \) and \( \sigma = -(al - bn)/2; \) now \( \alpha + \beta = an \) and \( \delta + \sigma = -al \) require \( a > 0 \) and \( a < 0 \) respectively. Thus the monomials of type \( m_3 \) and
$m_4$ cannot be $T^2$-invariant.

4. For the $m_1$ monomials, translation invariance requires (85) - (86). In this case $\alpha + \beta = bl$ and $\delta + \sigma = al$ imply $a > 0$ and $b > 0$, respectively. In conjunction with $\beta > 0$ and $\sigma > 0$, this yields

$$\frac{al}{n} > b > \frac{an}{l} > 0; \quad (93)$$

finally applying $\tilde{\gamma}_1 \gamma_2 \tilde{\gamma}_1$ interchanges the roles of $a$ and $b$ so we can replace (93) with (87).

5. For the $m_2$ monomials, translation invariance requires (89) - (90). Now $\alpha + \beta = an$ implies $a > 0$, and $\beta > 0$ requires $an/l > b$. Although the sign of $b$ is not determined, by applying $\gamma_3$ we can arrange for $b$ to be non-negative. These requirements are summarized in (91). This proves the second part of the lemma. \(\Box\)

We now describe how to generate the remaining two classes $m_1$ and $m_2$ from a finite number of invariants. The answer depends on whether $(l + n)$ is odd or even, and involves the invariants defined in (80) and (81).

**Lemma II.6**  
1. For $(l + n)$ even:

   (a) The $T^2$-invariant monomials of the form $m_1$ in (84) can be generated by $\sigma_5$, $\sigma_6$ and $m_1(z, a_\beta, b_\beta)$ for $\beta = 1, 2, \ldots, (l - n - 2)/2$.

   (b) The $T^2$-invariant monomials of the form $m_2$ in (88) can be generated by $\sigma_5$, $\sigma_7$, and $m_2(z, a'_\beta, b'_\beta)$ for $\beta = 1, 2, \ldots, n - 1$.

2. For $(l + n)$ odd:
(a) The $T^2$-invariant monomials of the form $m_1$ in (84) can be generated by $\sigma_5$, $\sigma_6$, and $m_1(z, a_\beta, b_\beta)$ for $\beta = 1, 2, \ldots, (l - n) - 1$.

(b) The $T^2$-invariant monomials of the form $m_2$ in (88) can be generated by $\sigma_5$, $\sigma_7$, and $m_2(z, a'_{\beta}, b'_{\beta})$ for $\beta = 1, 2, \ldots, n - 1$.

Proof.

We analyze the classes $m_1$ and $m_2$ separately.

1. Invariants of $m_1$ form.

(a) Assume $(l + n)$ is odd; in this case $(a, b)$ must be even. Given $m_1(z, a, b)$, if $\beta = (bl - an)/2 \geq l - n$ then we can extract a factor of $\sigma_6$: $m_1(z, a, b) = \sigma_6 m_1(z, a - 2, b - 2)$, and reduce $\beta$ by $(l - n)$. Thus we need only consider invariants $m_1(z, a, b)$ such that $\beta < l - n$. All of these can be written as $m_1(z, a, b) = \sigma_5^k m_1(z, a_\beta, b_\beta)$ for some integer $k \geq 0$. Hence we require $\sigma_5$, $\sigma_6$, and the invariants $m_1(z, a_\beta, b_\beta)$ for $\beta = 1, 2, \ldots, (l - n) - 1$.

(b) Assume $(l + n)$ is even, then $(l, n)$ are each odd and $(a, b)$ may be either even or odd. Given $m_1(z, a, b)$, if $\beta = (bl - an)/2 \geq (l - n)/2$ then either $a = b = 1$ or $a \geq b > 1$. In the former case $m_1(z, 1, 1) = \sigma_6$; in the latter case we can extract a factor of $\sigma_6$: $m_1(z, a, b) = \sigma_6 m_1(z, a - 1, b - 1)$, and reduce $\beta$ by $(l - n)/2$. Thus we need only consider invariants $m_1(z, a, b)$ such that $\beta < (l - n)/2$. All of these can be written as $m_1(z, a, b) = \sigma_5^k m_1(z, a_\beta, b_\beta)$ for some integer $k \geq 0$. Hence we require $\sigma_5$, $\sigma_6$, and the invariants $m_1(z, a_\beta, b_\beta)$ for $\beta = 1, 2, \ldots, (l - n - 2)/2$.

2. Invariants of $m_2$ form. Given $m_2(z, a, b)$, if $\beta = (an - bl)/2 \geq n$ then we can extract a factor of $\sigma_7$: $m_2(z, a, b) = \sigma_7 m_2(z, a - 2, b)$, and reduce $\beta$
by $n$. Thus we need only consider invariants $m_2(z, a, b)$ such that $\beta < n$. All of these can be written as $m_2(z, a, b) = \sigma_5^k m_2(z, a'_\beta, b'_\beta)$ for some integer $k \geq 0$. Hence we require $\sigma_5$, $\sigma_7$, and the invariants $m_2(z, a'_\beta, b'_\beta)$ for $\beta = 1, 2, \ldots, (n - 1)$.

\[\square\]

This completes the proof of Lemma II.4 enumerating a sub-basis for the $T^2$-invariants.

2. Extension from $E_{\kappa^2}(\tilde{\Omega})$ to $E_{\kappa^2}(\mathbb{R}^2)$

We return to the main issue: whether an arbitrary $\tilde{D}_4 + T^2$ symmetric map can be obtained by restriction from an $\mathcal{E}(2)$-equivariant map.

Lemma II.7 Assume $E_{\kappa^2}(\tilde{\Omega})$ carries an eight-dimensional irreducible representation of $\tilde{D}_4 + T^2$ as in (14) - (16). Let $\tilde{F} : E_{\kappa^2}(\tilde{\Omega}) \to E_{\kappa^2}(\tilde{\Omega})$ denote a $\tilde{D}_4 + T^2$-equivariant map, then there is a $\mathcal{E}(2)$-equivariant map on $E_{\kappa^2}(\mathbb{R}^2)$, $\mathcal{F} : E_{\kappa^2}(\mathbb{R}^2) \to E_{\kappa^2}(\mathbb{R}^2)$, such that

\[\tilde{F} = \mathcal{F}|_{E_{\kappa^2}(\tilde{\Omega})}.\] (94)

Proof.

1. The argument follows the proof of lemma II.1. Recall from (68) that $\tilde{F}$ has the form

\[\tilde{F}(z) = \begin{pmatrix} F_1(z) \\ F_1(\gamma_3 \cdot z) \\ F_1(\gamma_2 \cdot z) \\ F_1(\tilde{\gamma}_1 \gamma_2 \cdot z) \end{pmatrix}\] (95)

where $F_1$ satisfies $F_1(z) = F_1(z^*)^*$ and $z_1^* F_1(z)$ is $T^2$ invariant. Without loss of generality we assume $z_1^* F_1(z)$ is a $T^2$-invariant monomial, then

\[z_1^* F_1(z) = \sigma_1^{\nu_1} \sigma_2^{\nu_2} \sigma_3^{\nu_3} \sigma_4^{\nu_4} \omega_1^{\nu_5} \omega_2^{\nu_6} \omega_3^{\nu_7} \omega_4^{\nu_8}\] (96)
from (71). Any $T^2$-invariant monomial can appear on the right hand side provided it contains a factor of $z_1^*$.  

2. On the other hand, when we restrict a $E(2)$-symmetric map on $E_{κ^2}(R^2)$ to the eigenspace $E_{κ^2}(Ω)$ we get a $\tilde{D}_4+T^2$-symmetric map with $F_1$ of the form

$$F_1(z_1, z_2) = \sum_{k_1' ∈ A(κ)} \cdots \sum_{k_p' ∈ A(κ)} a(k_1') a(k_2') \cdots a(k_p') P(k_1', k_1', k_2', \ldots, k_p')$$  \hspace{1cm} (97)

where

$$a(k) = z_1 δ_{k, k_1} + z_2 δ_{k, k_2} + z_3 δ_{k, k_3} + z_4 δ_{k, k_4}$$

$$+ z_1^* δ_{-k, -k_1} + z_2^* δ_{-k, -k_2} + z_3^* δ_{-k, -k_3} + z_4^* δ_{-k, -k_4}$$  \hspace{1cm} (98)

and $\tilde{A}(κ) = \{ ±k_1', ±k_2', ±k_3', ±k_4' \}$ from (34). The function $P(k_1', k_1', k_2', \ldots, k_p')$ satisfies the conditions (27) - (28). Thus the extension of the $\tilde{D}_4+T^2$-symmetric map (95) to a $E(2)$-symmetric map depends on the existence of $P(k_1', k_1', k_2', \ldots, k_p')$ such that

$$\frac{σ_1 μ_1' σ_2 μ_2' σ_3 μ_3' ω_1 μ_1' ω_2 μ_2' ω_3 μ_3' ω_4 μ_4'}{z_1^*} = \sum_{k_1' ∈ A(κ)} \cdots \sum_{k_p' ∈ A(κ)} a(k_1') a(k_2') \cdots a(k_p') P(k_1', k_1', k_2', \ldots, k_p').$$  \hspace{1cm} (99)

We shall check that $P$ can always be found, given any appropriate $T^2$-invariant monomial in (98).

3. The $O(2)$ invariance of $P$ will be assured if it is constructed from the inner products of its arguments. For the eight-dimensional representation, we
have

\[
\begin{pmatrix}
\vec{k}_1 \cdot \vec{k}_1 & \vec{k}_1 \cdot \vec{k}_2 & \vec{k}_1 \cdot \vec{k}_3 & \vec{k}_1 \cdot \vec{k}_4 \\
\vec{k}_2 \cdot \vec{k}_1 & \vec{k}_2 \cdot \vec{k}_2 & \vec{k}_2 \cdot \vec{k}_3 & \vec{k}_2 \cdot \vec{k}_4 \\
\vec{k}_3 \cdot \vec{k}_1 & \vec{k}_3 \cdot \vec{k}_2 & \vec{k}_3 \cdot \vec{k}_3 & \vec{k}_3 \cdot \vec{k}_4 \\
\vec{k}_4 \cdot \vec{k}_1 & \vec{k}_4 \cdot \vec{k}_2 & \vec{k}_4 \cdot \vec{k}_3 & \vec{k}_4 \cdot \vec{k}_4 \\
\end{pmatrix}
= 
\begin{pmatrix}
\kappa^2 & l^2 - n^2 & 2nl & 0 \\
l^2 - n^2 & \kappa^2 & 0 & 2nl \\
2nl & 0 & \kappa^2 & n^2 - l^2 \\
0 & 2nl & n^2 - l^2 & \kappa^2 \\
\end{pmatrix}
\tag{100}
\]

so that for vectors in \( \tilde{A}(\kappa) \) the inner product \( \vec{k}_i \cdot \vec{k}_j \) has only four possible values, up to an overall sign. Define \( I_1(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, \kappa^2} \), \( I_2(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, l^2 - n^2} \), \( I_3(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, 2nl} \), and \( I_4(\vec{k}, \vec{q}) = \delta_{\vec{k} \cdot \vec{q}, 0} \) to describe these four values.

4. The monomial at left in (99) contains a finite number of factors of the coordinates \((z_1, z_2, z_3, z_4)\) and their complex conjugates. If each of these eight coordinates can be individually specified using the functions \( I_i \), then we obviously can construct an \( O(2) \)-invariant \( P \) to yield a given monomial. Note, that in (99) \( \vec{k}_1 \) is always the first argument in \( P \) regardless of which monomial we are given; this allows us to specify factors of \( z_1, z_2, \) and \( z_3 \) using \( I_1, I_2, \) and \( I_3 \), respectively:

\[
\sum_{\vec{k}'_1 \in \tilde{A}(\kappa)} a(\vec{k}'_1) I_i(\vec{k}_1, \vec{k}'_1) = z_i \quad \text{for } i = 1, 2, 3; \tag{101}
\]

similarly using \( I_i(\vec{k}_1, -\vec{k}'_1) \) yields \( z_i^* \). Thus arbitrary monomials in \((z_1, z_2, z_3, z_1^*, z_2^*, z_3^*)\) may be obtained from \( O(2) \)-invariant \( P \).

5. Unfortunately \( z_4 \) and \( z_4^* \) are not uniquely specified by \( I_4 \) in this way, instead we obtain

\[
\sum_{\vec{k}'_1 \in \tilde{A}(\kappa)} a(\vec{k}'_1) I_4(\vec{k}_1, \vec{k}'_1) = z_4 + z_4^* \tag{102}
\]
since $+\vec{k}_4$ and $-\vec{k}_4$ are both perpendicular to $\vec{k}_1$. However our analysis of the $T^2$-invariant monomials shows that $z_4$ and $z_4^*$ appear either together in $\sigma_4 = |z_4|^2$ or separately in one of the higher order mixed invariants. Factors of $\sigma_4$ can be easily specified with $I_1$:

$$\sum_{\vec{k}_1' \in A(\kappa)} \sum_{\vec{k}_2' \in A(\kappa)} a(\vec{k}_1') a(\vec{k}_2') \frac{I_4(\vec{k}_1', \vec{k}_1) I_1(\vec{k}_1', -\vec{k}_2')}{2} = \sigma_4. \quad (103)$$

When $z_4$ or $z_4^*$ appears in through one of the higher order invariants then it will always be in combination with at least two of the other three coordinates ($z_1, z_2, z_3$) or their complex conjugates. Inspection of the matrix of inner product values shows that the combination of $z_4$ or $z_4^*$ with any one of the other three coordinates can be uniquely specified in terms of the $I_i$. For example consider $z_2^* z_4$, since $\vec{k}_1 \cdot \vec{k}_2 = \kappa^2$ and $\vec{k}_2 \cdot \vec{k}_4 = 2nl$ we use $I_2(\vec{k}_1, -\vec{k}_1') I_3(-\vec{k}_1', \vec{k}_2')$:

$$\sum_{\vec{k}_1' \in A(\kappa)} \sum_{\vec{k}_2' \in A(\kappa)} a(\vec{k}_1') a(\vec{k}_2') I_2(\vec{k}_1', -\vec{k}_1') I_3(-\vec{k}_1', \vec{k}_2') = z_2^* z_4. \quad (104)$$

Thus in given any monomial on the left, we can construct an $O(2)$-invariant function $P$ which yields that monomial.

6. It remains to verify that these $O(2)$-invariant functions may be constructed so as to also satisfy the requirement of translation symmetry:

$$P(\vec{k}, \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p') = 0 \text{ unless } \vec{k} = \vec{k}_1' + \vec{k}_2' + \ldots + \vec{k}_p'. \quad (105)$$

It is helpful to formalize the foregoing construction somewhat. Any given monomial in $F_1(z_1, z_2)$:

$$F_1(z_1, z_2) = \frac{\sigma_1^{\nu_1} \sigma_2^{\nu_2} \sigma_3^{\nu_3} \sigma_4^{\nu_4} \omega_1^{\nu_1} \omega_2^{\nu_2} \omega_3^{\nu_3} \omega_4^{\nu_4}}{z_4^{\nu_4}}, \quad (106)$$

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corresponds to a set of distinct wave vectors in \( \tilde{A}(\kappa) \),

\[
F_1(z_1, z_2) \leftrightarrow \{ \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_d \} \subseteq \tilde{A}(\kappa) \quad d \leq p \tag{107}
\]
determined by the distinct amplitudes in (106) i.e. \( z_i \) corresponds to \( \vec{k}_i \) and \( z_i^* \) corresponds to \( -\vec{k}_i \). For example, \( F_1(z_1, z_2) = \sigma_1/z_1^* = z_1 \) corresponds to \( \{ \vec{c}_1 \} = \{ \vec{k}_1 \} \), and \( F_1(z_1, z_2) = \sigma_7/z_1^* = (z_1^*)^{n-1}(z_2^*)^n(z_3z_4)^l \) corresponds to \( \{ \vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{c}_4 \} = \{ -\vec{k}_1, -\vec{k}_2, \vec{k}_3, \vec{k}_4 \} \) if \( n > 1 \) and \( \{ \vec{c}_1, \vec{c}_2, \vec{c}_3 \} = \{ -\vec{k}_2, \vec{k}_3, \vec{k}_4 \} \) if \( n = 1 \). We construct \( P(\vec{k}, \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_d', \ldots, \vec{k}_p') \) by first multiplying factors of \( I_i \) with arguments that specify the inner products amongst the arguments \( \{ \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_d' \} \):

\[
\vec{k}_i \cdot \vec{k}_i' = \vec{k}_i \cdot \vec{c}_i \quad i = 1, \ldots d \tag{108}
\]

\[
\vec{k}_i' \cdot \vec{k}_j' = \vec{c}_i \cdot \vec{c}_j' \quad i, j = 1, \ldots d. \tag{109}
\]

The remaining arguments of \( P \) (if required), \( \{ \vec{k}_d'+1, \ldots, \vec{k}_p' \} \), are specified to equal one of the previous wave vectors \( \{ \vec{k}_1', \ldots, \vec{k}_d' \} \), e.g. \( I_1(\vec{k}_1', \vec{k}_d'+1) \) sets \( \vec{k}_d'+1 = \vec{k}_1' \) and adds a factor of \( a(\vec{k}_1') \). In this way the monomial in (106) is built up. Note that in this construction, \( \{ \vec{k}_d'+1, \ldots, \vec{k}_p' \} \) are completely determined by the arguments \( \{ \vec{k}_1', \ldots, \vec{k}_d' \} \). In particular, when we set \( \{ \vec{k}_1', \ldots, \vec{k}_d' \} = \{ \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_d \} \) this determines a set of wave vectors, \( \{ \vec{k}_1', \ldots, \vec{k}_d', \vec{k}_d'+1, \ldots, \vec{k}_p' \} \), that necessarily satisfy

\[
\vec{k}_1 = \vec{c}_1 + \vec{c}_2 + \ldots + \vec{c}_d + k_{d+1}' \ldots + \vec{k}_p'. \tag{110}
\]

This follows from the assumption that \( z_1^* F_1 \) in (106) is \( T^2 \)-invariant.

7. We now prove that this construction of an \( O(2) \) invariant \( P \) always yields a function that satisfies the requirements of translation symmetry (105) as
well. We begin with a useful reduction: since $|\vec{k}| = \kappa = |\vec{k}_1|$ in (105) we can always rotate the arguments of $P$ so that $\vec{k} = \vec{k}_1$ using the $O(2)$ invariance of $P$:

$$P(\vec{k}, \vec{k}_1, \vec{k}_2, \ldots, \vec{k}_p) = P(\vec{k}_1, \mathcal{R}(\phi_k) \cdot \vec{k}_1, \mathcal{R}(\phi_k) \cdot \vec{k}_2, \ldots, \mathcal{R}(\phi_k) \cdot \vec{k}_p)$$  (111)

where $\phi_k$ is the angle such that $\mathcal{R}(\phi_k) \cdot \vec{k} = \vec{k}_1$. Thus we need only verify (105) for the specific case $\vec{k} = \vec{k}_1$:

$$P(\vec{k}_1, \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p') = 0 \text{ unless } \vec{k}_1 = \vec{k}_1' + \vec{k}_2' + \ldots + \vec{k}_p'.$$  (112)

For most sets of arguments $(\vec{k}_1, \vec{k}_1', \vec{k}_2', \ldots, \vec{k}_p')$, $P = 0$, so we need only examine those sets of vectors for which $P \neq 0$ and verify that $\vec{k}_1 = \vec{k}_1' + \vec{k}_2' + \ldots + \vec{k}_p'$ holds in those cases. By construction $P \neq 0$ when we set $\{\vec{k}_1', \ldots, \vec{k}_d'\} = \{c_1, c_2, \ldots, c_d\}$ and in this case (111) provides the desired relation. In general, however, this is not the only set of wave vectors $\{\vec{k}_1', \ldots, \vec{k}_d'\}$ in $A(\kappa)$ selected by the prescribed inner products in (108) - (109), and it is necessary to verify (112) for all the allowed sets $\{\vec{k}_1', \ldots, \vec{k}_d'\}$ for which $P \neq 0$.

8. The occurrence of additional selected sets of vectors is due to the reflection $\gamma_{k_1} \cdot \vec{k}_1 = \vec{k}_1'$ that fixes the vector $\vec{k}_1'$; in terms of $\gamma_3$ and the angle, $\tan \theta_{ln} = n/l$, we find $\gamma_{k_1} = \mathcal{R}(\theta_{ln}) \gamma_3 \mathcal{R}(-\theta_{ln}) = \mathcal{R}(2\theta_{ln}) \gamma_3$ where $\gamma_3$ is the reflection defined in (9). Setting $\vec{k} = \vec{k}_1'$ in (108), the equation for $\vec{k}_1'$ (for fixed $i$) $\vec{k}_1' \cdot \vec{k}_1' = \vec{k}_1' \cdot \vec{c}_i$ has exactly two solutions in $A(\kappa)$: $\vec{k}_i' = \vec{c}_i$ and $\vec{k}_i' = \gamma_{k_1} \cdot \vec{c}_i$. If $\vec{c}_i \neq \gamma_{k_1} \cdot \vec{c}_i$, then for $\vec{k} = \vec{k}_1'$ there is a second set of vectors selected by the inner products (108) - (109) for which $P \neq 0$:

$$\{\vec{k}_1', \ldots, \vec{k}_d'\} = \{\gamma_{k_1} \cdot \vec{c}_1, \gamma_{k_1} \cdot \vec{c}_2, \ldots, \gamma_{k_1} \cdot \vec{c}_d\}.$$  (113)
For this second set, we obtain

\[ \vec{k}_1 = \gamma_{k_1} \cdot \vec{c}_1 + \gamma_{k_1} \cdot \vec{c}_2 + \ldots + \gamma_{k_1} \cdot \vec{c}_d + \gamma_{k_1} \cdot \vec{k}_{d+1}^{\prime} \ldots + \gamma_{k_1} \cdot \vec{k}_{p}^{\prime} \]  

(114)

by applying \( \gamma_{k_1} \) to (110). This completes the verification of (112), and establishes the translation symmetry of \( P \). ∎

This lemma completes the proof of Theorem II.1.

C. Discussion

In summary, if \( \text{Fix}(B_P) = E_{\kappa^2}(\tilde{\Omega}) \) carries an irreducible representation of the normalizer, then the restriction of Euclidean symmetric maps on \( E_{\kappa^2}(R^2) \) to the subspace \( E_{\kappa^2}(\tilde{\Omega}) \) yields the entire module \( \tilde{M}(\tilde{D}_4 \hat{+} T^2) \). These cases are summarized in Table 4. Thus the Euclidean symmetry of \( \mathcal{F} \) does not imply constraints on \( \tilde{F} = \mathcal{F}|_{\text{Fix}(B_P)} \) beyond the requirement of \( \tilde{D}_4 \hat{+} T^2 \)-equivariance. In these cases, the hidden rotational symmetry has no consequences.

In the context of codimension-one bifurcations, where the assumption that \( E_{\kappa^2}(R^2) \) carries an irreducible representation of \( \mathcal{E}(2) \) is appropriate, there are values of \( \kappa^2 \) for which \( \text{Fix}(B_P) \) fails to be \( \tilde{D}_4 \hat{+} T^2 \)-irreducible. As \( \kappa^2 \) increases, the first example of this occurs for \( \kappa^2 = 25 \) where the modes \( \vec{k}_1 = (5, 0) \), \( \vec{k}_2 = (0, 5) \), \( \vec{k}_3 = (4, 3) \), \( \vec{k}_4 = (3, 4) \) are degenerate due to the hidden rotations and \( E_{25}(R^2) = V_1 \oplus V_2 \) decomposes into two \( \tilde{D}_4 \hat{+} T^2 \)-irreducible subspaces with \( V_1 \) spanned by the modes \( (\vec{k}_1, \vec{k}_2) \) and \( V_2 \) spanned by the modes \( (\vec{k}_3, \vec{k}_4) \). This decomposition is discussed at greater length elsewhere. The effect of the hidden rotational symmetry on such a bifurcation will be studied in a separate paper. In the context of codimension-two bifurcations, one can vary parameters to locate mode interactions where the critical eigenspace \( E_{\kappa_1^2}(R^2) \oplus E_{\kappa_2^2}(R^2) \) is characterized by two unequal values of \( \kappa^2 \) and the representation of \( \mathcal{E}(2) \) is reducible. This will necessarily lead to a reducible representation of \( \tilde{D}_4 \hat{+} T^2 \) on \( \text{Fix}(B_P) \) and thus allow at least the possibility that the hidden rotations

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could have observable consequences. Investigation of particular mode interactions is thus another strategy of discovering bifurcations that can reveal the effects of hidden rotations.

Table 4. Normalizer symmetry \( N(B_P)/B_P \) and the module of \( N(B_P) \)-symmetric maps on \( E_{\kappa^2}(\tilde{\Omega}) \) obtained by the restriction \( \mathcal{F}|_{E_{\kappa^2}(\tilde{\Omega})} \)

| \((l, n)\) | \(E_{\kappa^2}(\tilde{\Omega})\) | \(N(B_P)/B_P\) on \(E_{\kappa^2}(\tilde{\Omega})\) | Representation | \(\{\mathcal{F}|_{E_{\kappa^2}(\tilde{\Omega})}\}\) |
|---|---|---|---|---|
| \(l = n > 0\) | \(\mathbb{R}^4\) | \(\tilde{D}_4+T^2\) | irreducible | \(\tilde{M}(\tilde{D}_4+T^2)\) |
| \(l > n = 0\) | \(\mathbb{R}^4\) | \(\tilde{D}_4+T^2\) | irreducible | \(\tilde{M}(\tilde{D}_4+T^2)\) |
| \(l > n > 0\) | \(\mathbb{R}^8\) | \(\tilde{D}_4+T^2\) | irreducible | \(\tilde{M}(\tilde{D}_4+T^2)\) |

**Notes:**
\(\{\mathcal{F}|_{E_{\kappa^2}(\tilde{\Omega})}\}\) represents the set of all maps on \(E_{\kappa^2}(\tilde{\Omega})\) obtained by the restriction \([14]\) from \(E_{\kappa^2}(\mathbb{R}^2)\). When the normalizer \(N(B_P)\) acts irreducibly, the set of \(\tilde{D}_4+T^2\)-symmetric maps obtained by restriction is a module of maximum size: \(\tilde{M}(N(B_P))\).

### III. Neumann boundary conditions and hidden translation symmetry

Our discussion has shown that if the normalizer \(N(B_P)\) acts irreducibly on eigenspace \(E_{\kappa^2}(\tilde{\Omega})\), then the restriction of Euclidean symmetric maps on \(E_{\kappa^2}(\mathbb{R}^2)\) to the subspace \(E_{\kappa^2}(\tilde{\Omega})\) yields precisely the module \(\tilde{M}(\tilde{D}_4+T^2)\). This is the module of symmetric maps associated with the normalizer \(N(B_P)/B_P\) and is the largest collection of maps that one could hope to obtain by this restriction. For these same values of \(\kappa^2\), we now consider the further restriction of \(\tilde{M}(\tilde{D}_4+T^2)\) to the subspace \(\text{Fix}(B_N) = E_{\kappa^2}(\Omega)\) which yields those maps consistent with Neumann boundary conditions on \(\Omega\).
A. Restriction to Fix $(B_N)$

This second restriction is quite analogous to the first, and one might expect to obtain the module $\tilde{M}(N(B_N))$ of $N(B_N)$-symmetric vector fields on $E_{\kappa^2}(\Omega)$. This expectation is borne out for the four-dimensional irreducible representations of $\tilde{D}_4+T^2$ but an exception occurs for the eight-dimensional representation when $(l+n)$ is even.

Table 5. Normalizer symmetry on Fix $(B_N) = E_{\kappa^2}(\Omega)$

| $(l, n)$ | $E_{\kappa^2}(\tilde{\Omega})$ | Fix $(B_N) \subset E_{\kappa^2}(\tilde{\Omega})$ | $N(B_N)/B_N$ on Fix $(B_N)$ |
|----------|---------------------------------|---------------------------------|---------------------------------|
| $l = n > 0$ | $(z_1, z_2) \approx \mathbb{R}^4$ | $(u, u) \approx \mathbb{R}$ | $Z_2(-I)$ |
| $l > n = 0$ | $(z_1, z_2) \approx \mathbb{R}^4$ | $(u_1, u_2) \approx \mathbb{R}^2$ | $\{\gamma_2, T(\pm \pi/l,0)\} \approx D_4$ |
| $l > n > 0$ | $(z_1, z_2, z_3, z_4) \approx \mathbb{R}^8$ | $(u_1, u_1, u_2, u_2) \approx \mathbb{R}^2$ | \begin{aligned} \{\gamma_2, T(\pm \pi/l,0)\} &\approx D_4 \quad (l+n) \text{ odd} \\
Z_2(\gamma_2) \times Z_2(-I) &\approx D_4 \quad (l+n) \text{ even} \end{aligned} |

Notes:
In the third column $u, u_1, \text{ and } u_2$ are real coordinates. In the fourth column $\{\gamma_2, T(\pm \pi/l,0)\}$ denotes the subgroup of $\tilde{D}_4+T^2$ generated by $\gamma_2$ and $T(\pm \pi/l,0)$ whose action on Fix $(B_N)$ is isomorphic to the two-dimensional representation of $D_4$; similarly $\{\gamma_2, T(\pm \pi,0)\}$ is the subgroup generated by $\gamma_2$ and $T(\pm \pi,0)$ which is isomorphic to the same two-dimensional representation of $D_4$.

In this exceptional case the normalizer $N(B_N)$ acts reducibly on $E_{\kappa^2}(\Omega)$, and we only obtain a sub-module of $\tilde{M}(N(B_N))$. The reducibility of $N(B_N)$ is clearly significant: when $E_{\kappa^2}(\tilde{\Omega})$ carries an irreducible representation of $N(B_P)$, then a reducible representation of $N(B_N)$ necessarily implies that some maps in $\tilde{M}(N(B_N))$ cannot be obtained by restriction from $\tilde{M}(\tilde{D}_4+T^2)$.

In all cases considered $E_{\kappa^2}(\Omega)$ is either one or two-dimensional, c.f. Table 5, and the
quotient \( N(B_N)/B_N \) depends on \( (l, n) \). In the one-dimensional case \( l = n \), \( N(B_N)/B_N \) is \( Z_2(-I) \). In the two-dimensional cases when \( l > n = 0 \) or \( l > n > 0 \) with \( (l + n) \) odd, the representation of \( N(B_N) \) is irreducible and \( N(B_N)/B_N \) is isomorphic to the two-dimensional representation of \( D_4 \). This representation is generated on \( (u_1, u_2) \in \mathbb{R}^2 \) by the two reflections

\[
(u_1, u_2) \rightarrow (u_2, u_1) \quad (115)
\]

\[
(u_1, u_2) \rightarrow (u_1, -u_2); \quad (116)
\]

the first generator \((115)\) represents the diagonal reflection \( \gamma_2 \). When \( l > n > 0 \) with \( (l + n) \) even, then \( N(B_N) \) acts reducibly and \( N(B_N)/B_N \) is isomorphic to \( Z_2(\gamma_2) \times Z_2(-I) \) where \( Z_2(\gamma_2) \) is generated by \((115)\).

For the two-dimensional irreducible case the ring \( R(D_4) \) and module \( \tilde{M}(D_4) \) are well known\cite{8,10}. The ring \( R(D_4) = \{N, \Delta\} \) is generated by the invariants, \( N = u_1^2 + u_2^2 \) and \( \Delta = \delta^2 \) where \( \delta = u_2^2 - u_1^2 \), and \( f(u_1, u_2) \in \tilde{M}(D_4) \) can always be expressed in the form

\[
f(u_1, u_2) = P(N, \Delta) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) + Q(N, \Delta) \delta \left( \begin{array}{c} u_1 \\ -u_2 \end{array} \right). \quad (117)
\]

Here \( P(x, y) \) and \( Q(x, y) \) are arbitrary smooth real-valued functions. For the reducible representation of \( N(B_N)/B_N = Z_2(\gamma_2) \times Z_2(-I) \), there is a second quadratic invariant \( \eta = u_1u_2 \), and an arbitrary \( Z_2(\gamma_2) \times Z_2(-I) \)-equivariant has the form

\[
f(u_1, u_2) = P(N, \eta) \left( \begin{array}{c} u_1 \\ u_2 \end{array} \right) + Q(N, \eta) \left( \begin{array}{c} u_2 \\ u_1 \end{array} \right). \quad (118)
\]

**Theorem III.1** Assume \( E_{\kappa^2}(\bar{\Omega}) = \text{Fix}(B_P) \) carries an irreducible representation of \( \tilde{D}_4 + T^2 \), and let \( \tilde{M}(\tilde{D}_4 + T^2) \) denote the module of \( \tilde{D}_4 + T^2 \)-symmetric vector fields on \( E_{\kappa^2}(\bar{\Omega}) \). Let \( \tilde{M}(\tilde{D}_4 + T^2)|_{\text{Fix}(B_N)} \) denote the set of maps obtained by restricting this module to \( \text{Fix}(B_N) \), and let \( \tilde{M}(N(B_N)) \) denote the module of \( N(B_N) \)-symmetric vector fields on \( \text{Fix}(B_N) \). Denote the two-dimensional representation in \((115)\) by \( D_4 \). Then \( \tilde{M}(\tilde{D}_4 + T^2)|_{\text{Fix}(B_N)} \) is a module
of $N(B_N)$-symmetric vector fields. The size of this module depends on the representation of $\tilde{D}_4\dot{+}T^2$ carried by $E_{\kappa^2}(\Omega)$.

1. Four-dimensional representations of $\tilde{D}_4\dot{+}T^2$:

   (a) for $l = n > 0$, $N(B_N) = Z_2(-I)$ and $\tilde{M}(\tilde{D}_4\dot{+}T^2)|_{\text{Fix}(B_N)} = \tilde{M}(N(B_N))$;

   (b) for $l = n = 0$, $N(B_N) = D_4$ and $\tilde{M}(\tilde{D}_4\dot{+}T^2)|_{\text{Fix}(B_N)} = \tilde{M}(N(B_N))$.

2. Eight-dimensional representations of $\tilde{D}_4\dot{+}T^2$:

   (a) for $l = n > 0$ and $(l + n)$ odd, $N(B_N)/B_N = D_4$ and $\tilde{M}(\tilde{D}_4\dot{+}T^2)|_{\text{Fix}(B_N)} = \tilde{M}(N(B_N))$;

   (b) for $l = n > 0$ and $(l + n)$ even, $N(B_N)/B_N = Z_2(\gamma_2) \times Z_2(-I)$ and $\tilde{M}(\tilde{D}_4\dot{+}T^2)|_{\text{Fix}(B_N)}$ is a submodule of $\tilde{M}(N(B_N))$. This submodule is generated over the ring $R(D_4) = \{N, \Delta\}$ by the four $Z_2(\gamma_2) \times Z_2(-I)$ equivariants:

   \[
   E_1(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad E_2(u_1, u_2) = \delta \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix},
   
   E_3(u_1, u_2) = \eta^l \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad E_4(u_1, u_2) = \eta^{l-1} \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}. \tag{119}
   \]

   That is, $\tilde{M}(\tilde{D}_4\dot{+}T^2)|_{\text{Fix}(B_N)}$ is the collection of vector fields of the form

   \[
   f(u_1, u_2) = A_1(N, \Delta) E_1(u_1, u_2) + A_2(N, \Delta) E_2(u_1, u_2)
   + A_3(N, \Delta) E_3(u_1, u_2) + A_4(N, \Delta) E_4(u_1, u_2) \tag{120}
   \]

   where the $A_i(x, y), \ i = 1, 2, 3, 4,$ are arbitrary smooth real-valued functions.

Proof.
1. For each of the four-dimensional representations, Gomes has shown that the vector fields in $\hat{M}(\tilde{D}_4+T^2)$ may be written as

$$\tilde{F}(z_1, z_2) = \tilde{p}(\tilde{N}, \tilde{\rho}) \left( \frac{z_1}{z_2} \right) + \tilde{q}(\tilde{N}, \tilde{\rho}) \left( \frac{z_1|z_2|^2}{z_2|z_1|^2} \right) + \tilde{r}(\tilde{N}, \tilde{\rho}) \left( \frac{z_1|z_1|^2}{z_2|z_2|^2} \right)$$

(121)

where $\tilde{N} = |z_1|^2 + |z_2|^2$ and $\tilde{\rho} = |z_1|^2|z_2|^2$ provide a Hilbert basis for the $\tilde{D}_4+T^2$ invariants and $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ are arbitrary real-valued functions of these basic $\tilde{D}_4+T^2$ invariants.

For the case $l = n > 0$, we set $(z_1, z_2) = (u, u)$ to restrict to the one-dimensional subspace $\text{Fix}(B_N)$, and $\tilde{F}$ restricts to $f(u)$:

$$f(u) = \tilde{F}|_{(z_1, z_2) = (u, u)} = p(u^2) u$$

(122)

with $p(u^2) = \tilde{p}(2u^2, u^4) + \tilde{q}(2u^2, u^4) + \tilde{r}(2u^2, u^4)$. Since any smooth function $p(x)$ can arise in this way, all maps of the form (122) are obtained. This is precisely the module $\hat{M}(Z_2(-I))$, of reflection-symmetric maps in one dimension, or equivalently the module of maps having $N(B_N)$ symmetry since $N(B_N)/B_N = Z_2(-I)$ for this representation, cf. Table 5. For the case $l > n = 0$, we set $(z_1, z_2) = (u_1, u_2)$ to restrict to the two-dimensional subspace $\text{Fix}(B_N)$, and $\tilde{F}$ restricts to $f(u_1, u_2)$:

$$f(u_1, u_2) = \tilde{F}|_{(z_1, z_2) = (u_1, u_2)} = \tilde{p}(N, \rho) \left( \frac{u_1}{u_2} \right) + \tilde{q}(N, \rho) \left( \frac{u_1 u_2^2}{u_2 u_1^2} \right) + \tilde{r}(N, \rho) \left( \frac{u_1^3}{u_2^3} \right)$$

(123)

where $N = u_1^2 + u_2$ and $\rho = (N^2 - \Delta)/4$. Using the relations

$$\begin{pmatrix} u_1 u_2^2 \\ u_2 u_1^2 \end{pmatrix} = \frac{N}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{3\delta}{2} \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$$

(124)

$$\begin{pmatrix} u_1^3 \\ u_2^3 \end{pmatrix} = \frac{N}{2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{\delta}{2} \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}$$

(125)

$f(u_1, u_2)$ can be re-expressed as (117) where $P(N, \Delta) = \tilde{p}(N, \rho) + N[\tilde{q}(N, \rho) + \tilde{r}(N, \rho)]/2$ and $Q(N, \Delta) = [3\tilde{q}(N, \rho) + \tilde{r}(N, \rho)]/2$. Again, any smooth functions $P(x, y)$ and $Q(x, y)$
can arise in this construction so we obtain all maps of the form (117). This is the module of $D_4$-symmetric maps for the two-dimensional irreducible representation of $D_4$; since $N(B_N)/B_N = D_4$ for this case, the maps in (117) give precisely the module associated with the normalizer. This proves the first part of the theorem.

2. The proof of the second part of the theorem is more complicated, in part because we lack an explicit set of generators for the module $\tilde{M}(\tilde{D}_4 + T^2)$ of $\tilde{D}_4 + T^2$-symmetric vector fields for the eight-dimensional representation. We first note that for $l > n > 0$, the $D_4$ invariants $(N, \Delta)$ and equivariants $E_1(u_1, u_2)$ and $E_2(u_1, u_2)$ always extend to $\tilde{D}_4 + T^2$ invariants and equivariants on $V$. This can be checked by direct construction. The $\tilde{D}_4 + T^2$ invariants $\tilde{N} \equiv \frac{1}{2}(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2)$ and $\tilde{\Delta} \equiv \frac{1}{4}(|z_3|^2 + |z_4|^2 - |z_1|^2 - |z_2|^2)$ restrict to the $D_4$ invariants on $\text{Fix}(B_N)$ when we set $(z_1, z_2, z_3, z_4) = (u_1, u_1, u_2, u_2)$:

$$(N, \Delta) = (\tilde{N}, \tilde{\Delta})|_{\text{Fix}(B_N)}; \quad (126)$$

and the $\tilde{D}_4 + T^2$ equivariants

$$\tilde{E}_1(z) \equiv \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}, \quad \tilde{E}_2(z) \equiv \tilde{N} \left( \begin{array}{c} z_1 \\ z_2 \\ z_3 \\ z_4 \end{array} \right) - 2 \left( \begin{array}{c} |z_1|^2 \\ |z_2|^2 \\ |z_3|^2 \\ |z_4|^2 \end{array} \right) \quad (127)$$

similarly restrict to the generators $E_i(u_1, u_2), \ i = 1, 2,$

$$E_1(u_1, u_2) = \tilde{E}_1|_{\text{Fix}(B_N)} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (128)$$

$$E_2(u_1, u_2) = \tilde{E}_2|_{\text{Fix}(B_N)} = \delta \begin{pmatrix} u_1 \\ -u_2 \end{pmatrix}. \quad (129)$$

Thus all maps of the form

$$f(u_1, u_2) = A_1(N, \Delta) E_1(u_1, u_2) + A_2(N, \Delta) E_2(u_1, u_2) \quad (130)$$

can be obtained regardless of the parity of $(l + n)$; this is precisely the module of all $D_4$ symmetric maps described in (117). When $(l + n)$ is odd, then $N(B_N)/B_N = D_4$.
and the module in (130) is precisely $\tilde{M}(N(B_N))$; hence

$$\tilde{M}(\tilde{D}_4+T^2)|_{\text{Fix}(B_N)} = \tilde{M}(N(B_N)) \quad \text{for} \ (l + n) \ \text{odd.} \quad (131)$$

However, when $(l+n)$ is even, then $N(B_N)/B_N = Z_2(\gamma_2) \times Z_2(-I)$ which is a subgroup of $D_4$; in this case the module in (130) is a submodule of $\tilde{M}(N(B_N))$. It is also a submodule of the set of maps $\tilde{M}(\tilde{D}_4+T^2)|_{\text{Fix}(B_N)}$ obtained by restriction, that is we claim the following inclusions: $\tilde{M}(D_4) \subset \tilde{M}(\tilde{D}_4+T^2)|_{\text{Fix}(B_N)} \subset \tilde{M}(N(B_N))$ for $(l+n)$ even. That $\tilde{M}(\tilde{D}_4+T^2)|_{\text{Fix}(B_N)}$ is strictly larger than $\tilde{M}(D_4)$, when $(l+n)$ is even, follows from the fact that the equivariants $E_3(u_1,u_2)$ and $E_4(u_1,u_2)$ may be obtained by restriction in this case. Recall the general form of $\tilde{D}_4+T^2$ equivariants (18)

$$\tilde{F}(z) = \begin{pmatrix} F_1(z) \\ F_1(\gamma_3 \cdot z) \\ F_1(\gamma_2 \cdot z) \\ F_1(\tilde{\gamma}_1 \gamma_2 \cdot z) \end{pmatrix} \quad (132)$$

and let $z_1^*F_1(z) = \sigma_6(z)$ and $\sigma_1(z)\sigma_6(z)$, using the form of $\sigma_6(z)$ appropriate for even parity, to define two equivariants:

$$\tilde{E}_3(z) \equiv \begin{pmatrix} \sigma_1(z) \sigma_6(z)/z_1^* \\ \sigma_1(\gamma_3 \cdot z) \sigma_6(\gamma_3 \cdot z)/z_3^* \\ \sigma_1(\gamma_2 \cdot z) \sigma_6(\gamma_2 \cdot z)/z_3^* \\ \sigma_1(\tilde{\gamma}_1 \gamma_2 \cdot z) \sigma_6(\tilde{\gamma}_1 \gamma_2 \cdot z)/z_4^* \end{pmatrix}$$

$$= \begin{pmatrix} z_1 (z_1^*)^{(l+n)/2} (z_2^*)^{(l+n)/2} (z_3^*)^{(l-n)/2} (z_4^*)^{(l-n)/2} \\ z_2 (z_1^*)^{(l-n)/2} (z_2^*)^{(l+n)/2} (z_3^*)^{(l-n)/2} (z_4^*)^{(l-n)/2} \\ z_3 (z_1^*)^{(l+n)/2} (z_2^*)^{(l+n)/2} (z_3^*)^{(l-n)/2} (z_4^*)^{(l-n)/2} \\ z_4 (z_1^*)^{(l-n)/2} (z_2^*)^{(l+n)/2} (z_3^*)^{(l-n)/2} (z_4^*)^{(l-n)/2} \end{pmatrix} \quad (133)$$

$$\tilde{E}_4(z) \equiv \begin{pmatrix} \sigma_6(z)/z_1^* \\ \sigma_6(\gamma_3 \cdot z)/z_2^* \\ \sigma_6(\gamma_2 \cdot z)/z_3^* \\ \sigma_6(\tilde{\gamma}_1 \gamma_2 \cdot z)/z_4^* \end{pmatrix}$$
These constructions show that the module in (120) is contained in $\mathbf{M}_B$ field to Fix ($q$ where $\nu$ is sufficient to consider the form of $\tilde{\mathbf{M}} u$. Obviously the solutions (72) - (73) for $(f|_{B\nu} \mid \nu$ whose restriction to Fix ($B\nu$) yields $E_3$ and $E_4$:

$$E_3(u_1, u_2) = \tilde{E}_3|_{\text{Fix}(B\nu)} = \eta^l \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$E_4(u_1, u_2) = \tilde{E}_4|_{\text{Fix}(B\nu)} = \eta^{-1} \begin{pmatrix} u_2 \\ u_1 \end{pmatrix}.$$  \hspace{1cm} (135) \hspace{1cm} (136)

Obviously $E_3$ and $E_4$ belong to $\tilde{M}(\tilde{D}_4 + T^2)|_{\text{Fix}(B\nu)}$, however since these equivariants are not symmetric with respect to $(u_1, u_2) \rightarrow (u_1, -u_2)$ they lack $D_4$ symmetry and are not in $\tilde{M}(D_4)$.

3. These constructions show that the module in (120) is contained in $\tilde{M}(\tilde{D}_4 + T^2)|_{\text{Fix}(B\nu)}$; we now prove that every map in $\tilde{M}(\tilde{D}_4 + T^2)|_{\text{Fix}(B\nu)}$ can be written in the form (120). It is sufficient to consider the form of a $\tilde{D}_4 + T^2$-symmetric vector field (132) when $z_1^* F_1(z)$ is a general $T^2$-invariant monomial (74) $z_1^* F_1(z) = \sigma_1^{\nu_1} \sigma_2^{\nu_2} \sigma_3^{\nu_3} \sigma_4^{\nu_4} \omega_1^{\nu_5} \omega_2^{\nu_6} \omega_3^{\nu_7} \omega_4^{\nu_8}$ where we must have either $\nu_1 \geq 1$ or $\nu_1 \leq -1$ (or both). The restriction of this vector field to Fix ($B\nu$) gives $f(u_1, u_2)$:

$$f(u_1, u_2) = \tilde{F}|_{(z_1, z_2, z_3, z_4) = (a_1, b_1, a_2, b_2)} = \begin{pmatrix} u_1^q u_2^{q'} \\ u_2^q u_1^{q'} \end{pmatrix}.$$  \hspace{1cm} (137)

where $q = 2(\nu_1 + \nu_2) + |\nu_1| + |\nu_2| - 1$ and $q' = 2(\nu_3 + \nu_4) + |\nu_3| + |\nu_4|$. Inspection of the solutions (122) - (123) for $(\nu_1, \nu_2, \nu_3, \nu_4)$ shows that $|\nu_1| + |\nu_2| = \max\{|b|l, |a|n\}$ and $|\nu_3| + |\nu_4| = \max\{|b|n, |a|l\}$; hence when $(a, b)$ are even, then $|\nu_1| + |\nu_2|$ and $|\nu_3| + |\nu_4|$ are even implying that $f(u_1, u_2)$ has $D_4$ symmetry and belongs to $\tilde{M}(D_4)$. Thus we
have only to examine the maps for \((a, b)\) odd, this possibility only arises when \((l + n)\) is even and \((l, n)\) are each odd. Thus we can assume \(|\nu_1| + |\nu_2|\) and \(|\nu_3| + |\nu_4|\) are odd and therefore that \(q\) is even and \(q'\) is odd. Since \(q\) and \(q'\) are unequal, either \(q' > q\) or \(q > q'\). In the first instance, the map in (137) becomes

\[
\begin{pmatrix}
  u_1^q u_2^{q'} \\
  u_2^q u_1^{q'}
\end{pmatrix} = \eta^{q-(l-1)} Z_1(J) \tag{138}
\]

where \(J = q' - q\) is odd and

\[
Z_1(J) = \eta^{l-1} \begin{pmatrix}
  u_2^l \\
  u_1^l
\end{pmatrix} \tag{139}
\]

In the second instance \((q > q')\), the map in (137) becomes

\[
\begin{pmatrix}
  u_1^q u_2^{q'} \\
  u_2^q u_1^{q'}
\end{pmatrix} = \eta^{q'-l} Z_2(J') \tag{140}
\]

where \(J' = q - q'\) is odd and

\[
Z_2(J') = \eta^l \begin{pmatrix}
  u_2^{l'} \\
  u_1^{l'}
\end{pmatrix} \tag{141}
\]

In each case the coefficients \(\eta^{q-(l-1)}\) and \(\eta^{q'-l}\), respectively, are \(D_4\) invariant. Furthermore the equivariants \(Z_1(J)\) and \(Z_2(J')\) can be expressed in terms of \(E_3\) and \(E_4\) with \(D_4\)-invariant coefficients. For low values of \((J, J')\) we have explicitly \(Z_1(1) = E_4(u_1, u_2)\), \(Z_1(3) = N E_4(u_1, u_2) - E_3(u_1, u_2)\), \(Z_2(1) = E_3(u_1, u_2)\), and \(Z_2(3) = N E_3(u_1, u_2) - \eta^2 E_4(u_1, u_2)\); the equivariants for \(J \geq 5\) and \(J' \geq 5\) can be reduced to these low order cases using the identity \(Z(J) = N Z(J - 2) - \eta^2 Z(J - 4)\) which holds for both \(Z_1(J)\) and \(Z_2(J')\). This establishes that each map in \(\tilde{M}(\tilde{D}_4 + T^2)|\text{Fix}(B_N)\) is of the form (120).

4. Finally we check that the module (120) is indeed a submodule of \(\tilde{M}(N(B_N))\) when \((l + n)\) is even. Since \(N(B_N)/B_N = Z_2(\gamma_2) \times Z_2(-I)\) in this case, generators for \(\tilde{M}(N(B_N))\) are easily found: any \(f \in \tilde{M}(N(B_N))\) can be expressed in the form (118)
for arbitrary real-valued functions $P(x, y)$ and $Q(x, y)$. Here $Z_2(\gamma_2) \times Z_2(-I)$ symmetric terms lacking $D_4$ symmetry can occur at any order, however in (120) the lowest order terms lacking $D_4$ symmetry are of degree $(2l - 1)$ in $(u_1, u_2)$:

$$\eta^{l-1} \left( \begin{array}{c} u_2 \\ u_1 \end{array} \right).$$

(142)

This proves the second part of the theorem.

\[ \square \]

This theorem generalizes, and in one case corrects, an earlier analysis of the restriction to $\text{Fix}(B_N)$ for these representations. In reference 8, some terms in the restricted map (120) were overlooked.

B. Discussion

In the above theorem, as summarized in Table 6, two situations arise: either

$$\bar{M}(\bar{D}_4 + T^2)|_{\text{Fix}(B_N)} = \bar{M}(N(B_N))$$

(143)
or $\bar{M}(\bar{D}_4 + T^2)|_{\text{Fix}(B_N)}$ is strictly smaller than $\bar{M}(N(B_N))$. In the latter case, it is clear that the $\bar{D}_4 + T^2$ symmetry of $\bar{F}$ imposes nontrivial constraints on $f = \bar{F}|_{\text{Fix}(B_N)}$ beyond the requirement of $N(B_N)$-equivariance. These bifurcations are thus likely to show effects of the hidden $\bar{D}_4 + T^2$ symmetry. Some of these effects have been discussed elsewhere.\[8\]

However, even if $\bar{M}(N(B_P)|_{\text{Fix}(B_N)}) = \bar{M}(N(B_N))$, the bifurcation of surface waves described by $f$ can reveal the hidden $\bar{D}_4 + T^2$ symmetry. This possibility arises because $N(B_N)$ acting on $E_{\kappa^2}(\Omega)$ can be a larger group than the geometric symmetry group of $\Omega$. Since $\Omega$ is a square domain its geometric symmetry group $\Gamma_\Omega = \{\gamma_1, \gamma_2\}$ is generated by the two reflections $\gamma_1 \cdot (x, y) \to (\pi - x, y)$ and $\gamma_2 \cdot (x, y) \to (y, x)$; this group is always contained
Table 6. Normalizer symmetry $N(B_N)/B_N$ and the module of $N(B_N)$-symmetric maps on $E_{\kappa^2}(\Omega)$ obtained by the restriction $\tilde{M}(\tilde{D}_4 + T^2)|_{E_{\kappa^2}(\Omega)}$

| $(l, n)$ | $E_{\kappa^2}(\Omega)$ | $N(B_N)/B_N$ | Representation | $\tilde{M}(\tilde{D}_4 + T^2)|_{E_{\kappa^2}(\Omega)}$ |
|--------|-----------------|---------------|---------------|----------------|
| $l = n > 0$ | $\mathbb{R}$ | $Z_2(-I)$ | irreducible | $\tilde{M}(Z_2(-I))$ |
| $l > n = 0$ | $\mathbb{R}^2$ | $D_4$ | irreducible | $\tilde{M}(D_4)$ |
| $l > n > 0$ | $\mathbb{R}^2$ | \(\begin{cases} D_4 & (l + n) \text{ odd} \\ Z_2^2 & (l + n) \text{ even} \end{cases}\) | irreducible/reducible | $\subset \tilde{M}(Z_2^2)$ |

Notes:
- Here $Z_2^2$ denotes $Z_2(\gamma_2) \times Z_2(-I)$. When the normalizer $N(B_N)$ acts irreducibly, the module obtained by restriction has maximum size: $\tilde{M}(N(B_N))$; when the normalizer acts reducibly we obtain only a submodule of $\tilde{M}(N(B_N))$.

in the normalizer $\Gamma_\Omega \subseteq N(B_N)$. In the absence of hidden symmetry, one would expect the bifurcation to be described by a map $f \in \tilde{M}(\Gamma_\Omega)$ where $\tilde{M}(\Gamma_\Omega)$ is the module of $\Gamma_\Omega$-symmetric vector fields on $E_{\kappa^2}(\Omega)$. As we have discussed elsewhere, for the four-dimensional representations of $\tilde{D}_4 + T^2$, $\Gamma_\Omega$ is a subgroup of $N(B_N)$, and the module $\tilde{M}(N(B_N))$ is a submodule of $\tilde{M}(\Gamma_\Omega)$. Thus in these cases the effects of hidden $\tilde{D}_4 + T^2$ symmetry are also present. The recent experiments of Gollub and Lane studied the bifurcations for $(l, n) = (2, 0)$ and detected the predicted effects of hidden $\tilde{D}_4 + T^2$ symmetry. A detailed discussion of that work can be found in reference 2.

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