Well-Posedness for Semi-Relativistic Hartree
Equations of Critical Type

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Abstract

We prove local and global well-posedness for semi-relativistic, nonlinear Schrödinger equations $i\partial_t u = \sqrt{-\Delta + m^2} u + F(u)$ with initial data in $H^s(\mathbb{R}^3)$, $s \geq 1/2$. Here $F(u)$ is a critical Hartree nonlinearity that corresponds to Coulomb or Yukawa type self-interactions. For focusing $F(u)$, which arise in the quantum theory of boson stars, we derive a sufficient condition for global-in-time existence in terms of a solitary wave ground state. Our proof of well-posedness does not rely on Strichartz type estimates, and it enables us to add external potentials of a general class.

1 Introduction

In this paper we study the Cauchy problem for nonlinear Schrödinger equations with kinetic energy part originating from special relativity. That is, we consider the initial value problem for

$$i\partial_t u = \sqrt{-\Delta + m^2} u + F(u), \quad (t, x) \in \mathbb{R}^{1+3}, \tag{1.1}$$

where $u(t, x)$ is complex-valued, $m \geq 0$ denotes a given mass parameter, and $F(u)$ is some nonlinearity. Here the operator $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{\xi^2 + m^2}$ in Fourier space.

Such “semi-relativistic” equations have (though not Lorentz covariant in general) interesting applications in the quantum theory for large systems of self-interacting, relativistic bosons. Equation (1.1) arises, for instance, as an effective description of boson stars, see, e.g., [ES05, LYS7], where $F(u)$ is a focusing Hartree nonlinearity given by

$$F(u) = \left(\frac{\lambda}{|x|}\ast |u|^2\right)u, \tag{1.2}$$

with some constant $\lambda < 0$ and $\ast$ as convolution. Motivated by this physical example with focusing self-interaction of Coulomb type, we address the Cauchy problem for equation (1.1) and a class of Hartree nonlinearities including (1.2). In fact, we shall prove well-posedness for initial data $u(0, x) = u_0(x)$ in $H^s = H^s(\mathbb{R}^3)$, $s \geq 1/2$; see Theorems 1-3 below.

Let us briefly point out a decisive feature of the example cited in (1.2) above. Apart from its physical relevance, the nonlinearity given by (1.2) leads to an $L^2$-critical equation as
indicated by the fact that the coupling constant $\lambda$ has to be dimensionless. In consequence of this, $L^2$-smallness of the initial datum enters as a sufficient condition for global-in-time solutions. More precisely, we derive for $u_0 \in H^s$, $s \geq 1/2$, the following criterion implying global well-posedness
\[
\int_{\mathbb{R}^3} |u_0(x)|^2 \, dx < \int_{\mathbb{R}^3} |Q(x)|^2 \, dx.
\] (1.3)
This condition holds irrespectively of the parameter $m \geq 0$ in (1.1); see Theorem 2 below. Here $Q \in H^{1/2}$ is a positive solution (ground state) for the nonlinear equation
\[
\sqrt{-\Delta} Q + \left( \frac{\lambda}{|x|} * |Q|^2 \right) Q = -Q,
\] (1.4)
which gives rise to solitary wave solutions, $u(t, x) = e^{it}Q(x)$, for (1.1) with $m = 0$. In fact, it can be shown that criterion (1.3) guaranteeing global-in-time solutions in the focusing case is optimal in the sense that there exist solutions, $u(t)$, with $\|u_0\|^2_2 > \|Q\|^2_2$, which blow up within finite time; see [Len05] for a proof.

Furthermore, criterion (1.3) can be linked with established results as follows. First, it is reminiscent to a well-known condition derived in [W83] for global well-posedness of nonrelativistic Schrödinger equations with focusing, local nonlinearity (see also [NO92] for Hartree nonlinearities). Second, criterion (1.3) is in accordance with a sufficient stability condition proved in [LY87] for the related time-independent problem (i.e., a static boson star); see [FL04] for a more details concerning known results on Hartree equations.

We now give an outline of our methods. The proof of well-posedness presented below does not rely on Strichartz (i.e., space-time) estimates for the propagator, $e^{-it\sqrt{-\Delta + m^2}}$, but it employs sharp estimates (e.g., Kato’s inequality (3.6) below) to derive local Lipschitz continuity of $L^2$-critical nonlinearities of Hartree type. Local well-posedness then follows by standard methods for abstract evolution equations. Furthermore, global well-posedness is derived by means of a-priori estimates and conservation of charge and energy whose proof requires a regularization method.

This paper is organized as follows.

• In Section 2 we introduce a class of critical Hartree nonlinearities including (1.2). First, we state Theorems 1 and 2 that establish local and global well-posedness in energy space $H^{1/2}$ for this class of nonlinearities. In Theorem 3 we extend these results to $H^s$, for every $s \geq 1/2$. Finally, external potentials are included, i.e., we consider
\[
i\partial_t u = \left( \sqrt{-\Delta + m^2} + V \right) u + F(u),
\] (1.5)
where $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is given. In Theorem 4 we state local and global well-posedness for (1.5) with initial datum $u(0, x) = u_0(x)$ in the appropriate energy space. Assumption imposed below on $V$ is considerably weak and implies that $\sqrt{-\Delta + m^2} + V$ defines a self-adjoint operator via its form sum.

• The main results (i.e., Theorems 1–4) are proved in Section 3.

• Appendix A contains useful facts about fractional derivatives, a discussion of ground states, and some details of the proofs.

2
Notation

Throughout this text, the symbol $*$ stands for convolution on $\mathbb{R}^3$, i.e.,

$$(f * g)(x) := \int_{\mathbb{R}^3} f(x - y)g(y) \, dy,$$

and $L^p(\mathbb{R}^3)$, with norm $\| \cdot \|_p$ and $1 \leq p \leq \infty$, denotes the usual Lebesgue $L^p$-space of complex-valued functions on $\mathbb{R}^3$. Moreover, $L^2(\mathbb{R}^3)$ is associated with the scalar product defined by

$$\langle u, v \rangle := \int_{\mathbb{R}^3} u(x)v(x) \, dx.$$

For $s \in \mathbb{R}$ and $1 \leq p \leq \infty$, we introduce fractional Sobolev spaces (see, e.g., [BL76]) with their corresponding norms according to

$$H^{s,p}(\mathbb{R}^3) := \{ u \in \mathcal{S}'(\mathbb{R}^3) : \| u \|_{H^{s,p}} := \| \mathcal{F}^{-1}[(1 + \xi^2)^{s/2}\mathcal{F}u]\|_p < \infty \},$$

where $\mathcal{F}$ denotes the Fourier transform in $\mathcal{S}'(\mathbb{R}^3)$ (space of tempered distributions). In our analysis, the Sobolev spaces $H^s(\mathbb{R}^3) := H^{s,2}(\mathbb{R}^3)$, with norms $\| \cdot \|_{H^s}$, will play an important role.

In addition to the common $L^p$-spaces, we also make use of local $L^p$-space, $L^p_{\text{loc}}(\mathbb{R}^3)$, with $1 \leq p \leq \infty$, and weak (or Lorentz) spaces, $L^p_{w}(\mathbb{R}^3)$, with $1 < p < \infty$ and corresponding norms given by

$$\| u \|_{p,w} := \sup_{\Omega} |\Omega|^{-1/p'} \int_{\Omega} |u(x)| \, dx,$$

where $1/p + 1/p' = 1$ and $\Omega$ denotes an arbitrary measurable set with Lebesgue measure $|\Omega| < \infty$; see, e.g., [LL01] for this definition of $L^p_{w}$-norms. Note that $L^p(\mathbb{R}^3) \subsetneq L^p_{w}(\mathbb{R}^3)$, for $1 < p < \infty$.

The symbol $\Delta = \sum_{i=1}^3 \partial^2_{x_i}$ stands for the usual Laplacian on $\mathbb{R}^3$, and $\sqrt{-\Delta + m^2}$ is defined via its symbol $\sqrt{x^2 + m^2}$ in Fourier space. Besides the operator $\sqrt{-\Delta + m^2}$, we also employ Riesz and Bessel potentials of order $s \in \mathbb{R}$, which we denote by $(-\Delta)^{s/2}$ and $(1 - \Delta)^{s/2}$, respectively; see also Appendix A.

Except for theorems and lemmas, we often use the abbreviations $L^p = L^p(\mathbb{R}^3)$, $L^p_w = L^p_{w}(\mathbb{R}^3)$, and $H^s = H^s(\mathbb{R}^3)$. In what follows, $a \lesssim b$ always denotes an inequality $a \leq cb$, where $c$ is an appropriate positive constant that can depend on fixed parameters.

2 Main Results

We consider the following initial value problem

$$\begin{cases}
i \partial_t u = \sqrt{-\Delta + m^2} u + \left(\frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2\right) u, \\
u(0, x) = u_0(x), \quad u : [0, T) \times \mathbb{R}^3 \to \mathbb{C},
\end{cases}$$

(2.1)

where $m \geq 0$, $\lambda \in \mathbb{R}$, and $\mu \geq 0$ are given parameters. Note that $|\lambda|$ could be absorbed in the normalization of $u(t, x)$, but we shall keep $\lambda$ explicit in the following; see also [ES05] for this convention.
Our particular choice of the Hartree type nonlinearities in (2.1) is motivated by the fact that (2.1) can be rewritten as the following system of equations

\[
\begin{aligned}
  i\partial_t u &= \sqrt{-\Delta + m^2} u + \Psi u, \\
  (\mu^2 - \Delta)\Psi &= 4\pi\lambda|u|^2, \\
  u(0, x) &= u_0(x),
\end{aligned}
\]

(2.2)

where \(\Psi = \Psi(t, x)\) is real-valued and \(\Psi(t, x) \to 0\) as \(|x| \to \infty\). This reformulation stems from the observation that \(e^{-\mu|x|/4\pi|x|}\) is the Green’s function of \((\mu^2 - \Delta)\) in \(\mathbb{R}^3\); see Appendix A.1. System (2.2) now reveals the physical intuition behind (2.1), i.e., the function \(u(t, x)\) corresponds to a “positive energy wave” with instantaneous self-interaction that is either of Coulomb or Yukawa type depending on whether \(\mu = 0\) or \(\mu > 0\), respectively. To prove well-posedness we shall, however, use formulation (2.1) instead, and we refer to facts from potential theory only when estimating the nonlinearity.

2.1 Local Well-Posedness

Let us begin with well-posedness in energy space, i.e., we assume that \(u_0 \in H^{1/2}\) holds in (2.1). The following Theorem 1 establishes local well-posedness in the strong sense, i.e., we have existence and uniqueness of solutions, their continuous dependence on initial data, and the blow-up alternative. The precise statements is as follows.

Theorem 1. Let \(m \geq 0, \lambda \in \mathbb{R}, \) and \(\mu \geq 0\). Then initial value problem (2.1) is locally well-posed in \(H^{1/2}(\mathbb{R}^3)\). This means that, for every \(u_0 \in H^{1/2}(\mathbb{R}^3)\), there exist a unique solution

\[
u \in C^0([0, T); H^{1/2}(\mathbb{R}^3)) \cap C^1([0, T); H^{-1/2}(\mathbb{R}^3))
\]

and it depends continuously on \(u_0\). Here \(T \in (0, \infty]\) is the maximal time of existence, where we have that either \(T = \infty\) or \(T < \infty\) and \(\lim_{t \to T} \|u(t)\|_{H^{1/2}} = \infty\) holds.

Remark. Continuous dependence means that the map \(u_0 \mapsto u \in C^0(I; H^{1/2})\) is continuous for every compact interval \(I \subset [0, T)\).

2.2 Global Well-Posedness

The local-in-time solutions derived in Theorem 1 extend to all times, by virtue of Theorem 2 below, provided that either \(\lambda \geq 0\) holds (corresponding to a repulsive nonlinearity) or \(\lambda < 0\) and the initial datum is sufficiently small in \(L^2\).

Theorem 2. The solution of (2.1) derived in Theorem 1 is global in time, i.e., we have that \(T = \infty\) holds, provided that one of the following conditions is met.

i) \(\lambda \geq 0\).

ii) \(\lambda < 0\) and \(\|u_0\|_2^2 < \|Q\|_2^2\), where \(Q \in H^{1/2}(\mathbb{R}^3)\) is a strictly positive solution (ground state) of

\[
\sqrt{-\Delta} Q + \left(\frac{\lambda}{|x|} + |Q|^2\right) Q = -Q.
\]

Moreover, we have the estimate \(\|Q\|_2^2 > \frac{4}{\pi|\lambda|}\).
Remarks. 1) Notice that condition ii) implies global well-posedness for (2.1) irrespectively of $m \geq 0$.

2) Due to the scaling behavior of (2.3), the function $Q_a(x) = a^{3/2}Q(ax)$, with $a > 0$, yields another ground state with $\|Q_a\|_2 = \|Q\|_2$ that satisfies

$$\sqrt{-\Delta} Q_a + \left(\frac{\lambda}{|x|} + |Q_a|^2\right)Q_a = -aQ_a.$$  \hspace{1cm} (2.4)

We refer to Appendix A.2 for a discussion of $Q \in H^{1/2}$.

3) Condition ii) resembles a well-known criterion derived in [We83] for global-in-time existence for $L^2$-critical nonlinear (nonrelativistic) Schrödinger equations.

4) It is shown in [Len05] that criterion (1.3) for having global-in-time solutions in the focusing case is optimal in the sense that there exist solutions, $u(t)$, with $\|u_0\|_2^2 > \|Q\|_2^2$, which blow up within finite time.

2.3 Higher Regularity

We now turn to well-posedness of (2.1) in $H^s$, for $s \geq 1/2$, which is settled by the following result.

**Theorem 3.** For every $s \geq 1/2$, the conclusions of Theorems 1 and 2 hold, where $H^{1/2}(\mathbb{R}^3)$ and $H^{-1/2}(\mathbb{R}^3)$ in Theorem 1 are replaced by $H^s(\mathbb{R}^3)$ and $H^{s-1}(\mathbb{R}^3)$, respectively.

**Remark.** For $s = 1$, this result is needed in [ES05] for a rigorous derivation of (2.1) with Coulomb type selfinteraction (i.e., $\mu = 0$) from many-body quantum mechanics.

2.4 External Potentials

Now we consider the following extension of (2.1) that arises by adding an external potential:

$$\begin{cases} i\partial_t u = \left(\sqrt{-\Delta} + m^2 + V\right)u + \left(\frac{\lambda e^{-\mu|x|}}{|x|} + |u|^2\right)u, \\ u(0, x) = u_0(x), \quad u : [0, T) \times \mathbb{R}^3 \rightarrow \mathbb{C}, \end{cases}$$  \hspace{1cm} (2.5)

where $m \geq 0$, $\lambda \in \mathbb{R}$, $\mu \geq 0$ are given parameters, and $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ denotes a preassigned function that meets the following condition.

**Assumption 1.** Suppose that $V = V_+ + V_-$ holds, where $V_+$ and $V_-$ are real-valued, measurable functions with the following properties.

i) $V_+ \in L^1_{\text{loc}}(\mathbb{R}^3)$ and $V_+ \geq 0$.

ii) $V_-$ is $\sqrt{-\Delta}$-form bounded with relative bound less than 1, i.e., there exist constants $0 \leq a < 1$ and $0 \leq b < \infty$, such that

$$|\langle u, V_- u \rangle| \leq a \langle u, \sqrt{-\Delta} u \rangle + b \langle u, u \rangle$$

holds for all $u \in H^{1/2}(\mathbb{R}^3)$. 

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We mention that Assumption 1 implies that \( \sqrt{-\Delta + m^2} + V \) leads to a self-adjoint operator on \( L^2 \) via its form sum. Furthermore, the energy space given by

\[
X := \left\{ u \in H^{1/2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) |u(x)|^2 \, dx < \infty \right\}
\]

is complete with norm \( \| \cdot \|_X \), and its dual space is denoted by \( X^* \). We refer to Section 3.4 for more details on \( \sqrt{-\Delta + m^2} + V \) and \( X \).

After this preparing discussion, the extension of Theorems 1 and 2 for the initial value problem (2.5) can be now stated as follows.

**Theorem 4.** Let \( m \geq 0, \lambda \in \mathbb{R}, \mu \geq 0, \) and suppose that \( V \) satisfies Assumption 1. Then (2.5) is locally well-posed in the following sense. For every \( u_0 \in X \), there exists a unique solution

\[
u \in C^0([0, T); X) \cap C^1([0, T); X^*)
\]

and it depends continuously on \( u_0 \). Here \( T \in (0, \infty] \) is the maximal time of existence such that either \( T = \infty \) or \( T < \infty \) and \( \lim_{t \uparrow T} \| u(t) \|_X = \infty \) holds. Moreover, we have that \( T = \infty \) holds, if one of the following conditions is satisfied.

\( i) \lambda \geq 0. \)

\( ii) \lambda < 0 \) and \( \| u_0 \|_2^2 < (1 - a) \| Q \|_2^2 \), where \( Q \) is the ground state mentioned in Theorem 2 and \( 0 \leq a < 1 \) denotes the relative bound introduced in Assumption 1.

**Remarks.** 1) To meet Assumption 1 for \( V_+ \), we can choose, for example, \( V_+(x) = |x|^{2\beta} \), with \( \beta \geq 0 \); or even super-polynomial growth such as \( V_+(x) = e^{x^2} \). Note that Assumption 1 for \( V_- \) is satisfied (by virtue of Sobolev inequalities), if

\[
|V_-(x)| \leq \frac{c}{|x|^{1-\epsilon}} + d
\]

holds for some \( 0 < \epsilon \leq 1 \) and constants \( 0 \leq c, d < \infty \). In fact, we can even admit \( \epsilon = 0 \) provided that \( c < 2/\pi \) holds, as can be seen from inequality 3.3 below.

2) Since we avoid Strichartz estimates in our well-posedness proof below, we only need that \( V_+ \) belongs to \( L^1_{\text{loc}} \). In contrast to this, compare, for instance, the conditions on \( V \) in \([YZ04]\) for deriving Strichartz type estimates for \( e^{-it(-\Delta + V)} \) in order to prove local well-posedness for (nonrelativistic) nonlinear Schrödinger equations with external potentials.

## 3 Proof of the Main Results

In this section we prove Theorems 1-4. Although Theorem 4 generalizes Theorems 1 and 2, we postpone the proof of Theorem 4 to the final part of this section.

### 3.1 Proof of Theorem 1 (Local Well-Posedness)

Let \( u_0 \in H^{1/2} \) be fixed. In view of (2.1) we put

\[
A := \sqrt{-\Delta + m^2} \quad \text{and} \quad F(u) := \left( \frac{\lambda e^{-\mu|x|}}{|x|} * |u|^2 \right) u,
\]

3.1
and we consider the integral equation

\[ u(t) = e^{-itA}u_0 - i \int_0^t e^{-i(t-\tau)A}F(u(\tau)) \, d\tau. \]  

(3.2)

Here \( u(t) \) is supposed to belong to the Banach space

\[ Y_T := C^0([0, T); H^{1/2}(\mathbb{R}^3)), \]  

(3.3)

with some \( T > 0 \) and corresponding norm \( \|u\|_{Y_T} := \sup_{t \in [0, T]} \|u(t)\|_{H^{1/2}} \). The proof of Theorem \( \text{(ii)} \) is now organized in two steps as follows.

**Step 1: Estimating the Nonlinearity**

We show that the nonlinearity \( F(u) \) is locally Lipschitz continuous from \( H^{1/2} \) into itself. This is main point of our argument for local well-posedness and it reads as follows.

**Lemma 1.** For \( \mu \geq 0 \), the map \( J(u) := (e^{-\mu|x|/|x|} + |u|^2)u \) is locally Lipschitz continuous from \( H^{1/2}(\mathbb{R}^3) \) into itself with

\[ \|J(u) - J(v)\|_{H^{1/2}} \lesssim (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}})\|u - v\|_{H^{1/2}}, \]

for all \( u, v \in H^{1/2}(\mathbb{R}^3) \).

**Proof of Lemma 1.** We prove the claim for \( \mu = 0 \) and \( \mu > 0 \) in a common way, so let \( \mu \geq 0 \) be fixed. For \( s \in \mathbb{R} \), it is convenient to introduce

\[ D^s := (\mu^2 - \Delta)^{s/2}. \]

Note that due to the equivalence

\[ \|u\|_2 + \|D^{1/2}u\|_2 \lesssim \|u\|_{H^{1/2}} \lesssim \|u\|_2 + \|D^{1/2}u\|_2, \]

it is sufficient to estimate the quantities

\[ I := \|J(u) - J(v)\|_2 \quad \text{and} \quad II := \|D^{1/2}[J(u) - J(v)]\|_2, \]

where \( I \) is needed only if \( \mu = 0 \). Using now the identity

\[ J(u) - J(v) = \frac{1}{2} \left[ (e^{-\mu|x|/|x|} * (|u|^2 - |v|^2))(u + v) + (e^{-\mu|x|/|x|} * (|u|^2 + |v|^2))(u - v) \right] \]

together with Hölder’s inequality (which we tacitly apply from now on), we find that

\[ I \lesssim \|e^{-\mu|x|/|x|} * (|u|^2 - |v|^2))(u + v)\|_2 + \|e^{-\mu|x|/|x|} * (|u|^2 + |v|^2))(u - v)\|_2 \]

\[ \lesssim \|e^{-\mu|x|/|x|} * (|u|^2 - |v|^2)\|_{L^6} \|u + v\|_3 + \|e^{-\mu|x|/|x|} * (|u|^2 + |v|^2)\|_{L^\infty} \|u - v\|_2. \]  

(3.4)
Observing that $e^{-\mu|x|} \in L^3_u$ holds, the first term of right-hand side of (3.4) can be bounded by means of the weak Young inequality (see, e.g., [LL01]) as follows
\[
\| \frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 - |v|^2)\|_6 \lesssim \| \frac{e^{-\mu|x|}}{|x|} \|_{3, w} \|u\|^2 - |v|^2\|_{6/5} \lesssim \|u + v\|_3 \|u - v\|_2. \tag{3.5}
\]
The second term occurring in (3.4) can be estimated by noting that
\[
\| \frac{e^{-\mu|x|}}{|x|} \ast |u|^2 \|_\infty \lesssim \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x - y|} dx \lesssim \|(-\Delta)^{1/4}u\|_2^2,
\tag{3.6}
\]
which follows from the operator inequality $|x - y|^{-1} \leq (\pi/2)(-\Delta x - y)^{1/2}$ (see, e.g., [Kat80], Section V.5.4) and translational invariance, i.e., we use that $\Delta x - y = \Delta x$ holds for all $y \in \mathbb{R}^3$.
Combining now (3.5) and (3.6) we find that
\[
I \lesssim \|u + v\|_3^2 \|u - v\|_2 + (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}}) \|u - v\|_2
\lesssim (\|u\|^2_{H^{1/2}} + \|v\|^2_{H^{1/2}}) \|u - v\|_{H^{1/2}},
\]
where we make use of the Sobolev inequality $\|u\|_3 \lesssim \|u\|_{H^{1/2}}$ in $\mathbb{R}^3$.

It remains to estimate $II$. To do so, we appeal to the generalized (or fractional) Leibniz rule (see Appendix A.1) leading to
\[
II \lesssim \|D^{1/2}\left(\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 - |v|^2)\right)(u + v)\|_2
\+
\|D^{1/2}\left(\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 + |v|^2)\right)(u - v)\|_2
\lesssim \|D^{1/2}\left(\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 - |v|^2)\right)\|_6 \|u + v\|_3
\+
\|\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 - |v|^2)\|_\infty \|D^{1/2}(u + v)\|_6
\+
\|D^{1/2}\left(\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 + |v|^2)\right)\|_6 \|u - v\|_3
\+
\|\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 + |v|^2)\|_\infty \|D^{1/2}(u - v)\|_6. \tag{3.7}
\]

By referring to Appendix A.1 we notice that $\frac{e^{-\mu|x|}}{|x|} \ast f$ can be expressed as $D^{-2}f = (\mu^2 - \Delta)^{-1}f$ in $\mathbb{R}^3$ (here $f \in \mathcal{S}(\mathbb{R}^3)$ is initially assumed, but our arguments follow by density). Thus, the first term of the right-hand side of (3.7) is found to be
\[
\|D^{1/2}\left(\frac{e^{-\mu|x|}}{|x|} \ast (\|u\|^2 - |v|^2)\right)\|_6 \lesssim \|D^{1/2-2}\left(\|u\|^2 - |v|^2\right)\|_6
\lesssim \|D^{-3/2}\left(\|u\|^2 - |v|^2\right)\|_6
\lesssim \|G_{3/2}^\mu \ast (\|u\|^2 - |v|^2)\|_6
\lesssim \|G_{3/2}^\mu \|_{2, w} \|u\|^2 - |v|^2\|_{3/2}
\lesssim \|u + v\|_3 \|u - v\|_3. \tag{3.8}
\]
where we use weak Young’s inequality together with the fact that $D^{-3/2}f$ corresponds to $G^\mu_{3/2} \ast f$ with some $G^\mu_{3/2} \in L^2_w(\mathbb{R}^3)$; see \ref{A.1}. The $\| \cdot \|_\infty$-part of the second term occurring in (3.7) can be estimated by using the Cauchy-Schwarz inequality and (3.6) once again:

$$\| \frac{e^{-\mu|x|}}{|x|} \ast (|u|^2 - |v|^2) \|_\infty \leq \| \frac{1}{|x|} \ast (|u|^2 - |v|^2) \|_\infty$$

$$\lesssim \sup_{y \in \mathbb{R}^3} \left| \int_{\mathbb{R}^3} \frac{|u(x)|^2 - |v(x)|^2}{|x-y|} \, dx \right|$$

$$\lesssim \sup_{y \in \mathbb{R}^3} \left| (u(x) + v(x)), \frac{1}{|x-y|}(u(x) - v(x)) \right|$$

$$\lesssim \| (-\Delta)^{1/4}(u + v) \|_2 \| (-\Delta)^{1/4}(u - v) \|_2$$

$$\lesssim \| u \|_{H^{1/2}} + \| v \|_{H^{1/2}} \| u - v \|_{H^{1/2}}$$

(3.9)

The remaining terms in (3.7) deserve no further comment, since they can be estimated in a similar fashion to all estimates derived so far. Thus, we conclude that

$$\| J(u) - J(v) \|_{H^{1/2}} \lesssim I + II \lesssim (\| u \|_{H^{1/2}}^2 + \| v \|_{H^{1/2}}^2) \| u - v \|_{H^{1/2}}$$

and the proof of Lemma 1 is now complete. \qed

**Remarks.** 1) The proof of Lemma 1 relies on (3.6) in a crucial way. Employing just the Sobolev embedding $H^{1/2} \subset L^2 \cap L^3$ (in $\mathbb{R}^3$) together with the (non weak) Young inequality is not sufficient to conclude that $\| \frac{e^{-\mu|x|}}{|x|} \ast |u|^2 \|_\infty < \infty$ whenever $u \in H^{1/2}$.

2) The proof of Lemma 1 fails for “super-critical” Hartree nonlinearities $J(u) = (|x|^{-\alpha} \ast |u|^2)u$, where $1 < \alpha < 3$. Thus, the choice $\alpha = 1$ represents a borderline case when deriving local Lipschitz continuity in energy space $H^{1/2}$.

**Step 2: Conclusion**

Returning to the proof of Theorem 1 we note that $A$ defined in (3.1) gives rise to a self-adjoint operator $L^2$ with domain $H^1$. Moreover, its extension to $H^{1/2}$, which we denote by $A : H^{1/2} \to H^{-1/2}$, generates a $C^0$-group of isometries, $\{e^{-itA} \}_{t \in \mathbb{R}}$, acting on $H^{1/2}$. Local well-posedness in the sense of Theorem 1 now follows by standard methods for evolution equations with locally Lipschitz nonlinearities. That is, existence and uniqueness of a solution $u \in Y_T$ for the integral equation (3.2) is deduced by a fixed point argument, for $T > 0$ sufficiently small. The equivalence of the integral formulation (3.2) and the initial value problem (2.1), with $u_0 \in H^{1/2}$, as well as the blow-up alternative can also be deduced by standard arguments; see, e.g., \cite{Paz83, CH98} for general theory on semilinear evolution equations. Finally, note that $u \in C^1([0,T); H^{-1})$ follows by equation (2.1) itself. The proof of Theorem 1 is now accomplished.

**3.2 Proof of Theorem 2** (Global Well-Posedness)

The first step taken in the proof of Theorem 2 settles conservation of energy and charge that are given by

$$E[u] := \frac{1}{2} \int_{\mathbb{R}^3} u(x) \sqrt{-\Delta + m^2} u(x) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{\lambda e^{-\mu|x|}}{|x|} \ast |u|^2 \right)(x) |u(x)|^2 \, dx$$

(3.10)
\[ N[u] := \int_{\mathbb{R}^3} |u(x)|^2 \, dx, \]  
(3.11)
respectively. After deriving the corresponding conservation laws (where proving energy conservation requires a regularization), we discuss how to obtain a-priori bounds on the energy norm of the solution.

Step 1: Conservation Laws

**Lemma 2.** The local-in-time solutions of Theorem 1 obey conservation of energy and charge, i.e.,
\[ E[u(t)] = E[u_0] \quad \text{and} \quad N[u(t)] = N[u_0], \]
for all \( t \in [0, T) \).

**Proof of Lemma 2.** Let \( u \) be a local-in-time solution derived in Theorem 1 and let \( T \) be its maximal time of existence. Since \( u(t) \in H^{1/2} \) holds, we can multiply (2.1) by \( \text{i} \overline{u}(t) \) and integrate over \( \mathbb{R}^3 \). Taking then real parts yields
\[ \frac{d}{dt} N[u(t)] = 0 \quad \text{for} \quad t \in [0, T), \]  
(3.12)
which shows conservation of charge.

At a formal level, conservation of energy follows by multiplying (2.1) with \( \hat{u}(t) \in H^{-1/2} \) and integrating over space, but the paring of two elements of \( H^{-1/2} \) is not well-defined. Thus, we have to introduce a regularization procedure as follows; see also, e.g., [Caz03, GV00] for other regularization methods for nonlinear (nonrelativistic) Schrödinger equations. Let us define the family of operators
\[ M_\varepsilon := (\varepsilon A + 1)^{-1}, \quad \text{for} \quad \varepsilon > 0, \]  
(3.13)
where the operator \( A = \sqrt{-\Delta + m^2} \geq 0 \) is taken from (3.1). Consider the sequences of embedded spaces
\[ \ldots \hookrightarrow H^{3/2} \hookrightarrow H^{1/2} \hookrightarrow H^{-1/2} \hookrightarrow H^{-3/2} \ldots \]  
(3.14)
It is easy to see (by using functional calculus) that the following properties hold.

a) For \( \varepsilon > 0 \) and \( s \in \mathbb{R} \), we have that \( M_\varepsilon \) is a bounded map from \( H^s \) into \( H^{s+1} \).

b) \( \| M_\varepsilon u \|_{H^s} \leq \| u \|_{H^s} \) whenever \( u \in H^s \) and \( s \in \mathbb{R} \).

c) For \( u \in H^s \) and \( s \in \mathbb{R} \), we have that \( M_\varepsilon u \to u \) strongly in \( H^s \) as \( \varepsilon \downarrow 0 \).

We shall use tacitly properties a) – c) in the following analysis.
By means of $\mathcal{M}_\varepsilon$ and noting that $E \in C^1(H^{1/2}; \mathbb{R})$, we can compute in a well-defined way for $t_1, t_2 \in [0, T)$ as follows

$$E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)] = \int_{t_1}^{t_2} \{E'(\mathcal{M}_\varepsilon u), \mathcal{M}_\varepsilon \dot{u}\}_{H^{-1/2}, H^{1/2}} \, dt$$

$$= \int_{t_1}^{t_2} \text{Re} \, \langle AM_\varepsilon u + F(\mathcal{M}_\varepsilon u), -iM_\varepsilon (Au + F(u)) \rangle \, dt$$

$$= \int_{t_1}^{t_2} \text{Im} \left[ \langle AM_\varepsilon u, M_\varepsilon Au \rangle + \langle F(\mathcal{M}_\varepsilon u), M_\varepsilon Au \rangle + \langle AM_\varepsilon u, M_\varepsilon F(u) \rangle + \langle F(\mathcal{M}_\varepsilon u), M_\varepsilon F(u) \rangle \right] \, dt$$

$$=: \int_{t_1}^{t_2} f_\varepsilon(t) \, dt, \quad (3.15)$$

where we write $u = u(t)$ for brevity and recall the definition of $F$ from (3.14). We observe that the first term in $f_\varepsilon(t)$ is the “most singular” part, i.e., if $\varepsilon = 0$ we would have pairing of two $H^{-1/2}$-elements. But for $\varepsilon > 0$ we can use the obvious fact that $\mathcal{M}_\varepsilon A = AM_\varepsilon$ holds and conclude that

$$\text{Im} \langle AM_\varepsilon u, M_\varepsilon Au \rangle = \text{Im} \langle AM_\varepsilon u, AM_\varepsilon u \rangle = 0.$$ Notice that this manipulation is well-defined, because $AM_\varepsilon u$ and $M_\varepsilon Au$ are in $H^{1/2}$ whenever $u \in H^{1/2}$. After some simple calculations, we find $f_\varepsilon(t)$ to be of the form

$$f_\varepsilon(t) = \text{Im} \left[ \langle F(M_\varepsilon u), M_\varepsilon Au \rangle + \langle AM_\varepsilon u, M_\varepsilon F(u) \rangle + \langle F(M_\varepsilon u), M_\varepsilon F(u) \rangle \right]$$

$$= \text{Im} \left[ \langle A^{1/2}F(M_\varepsilon u), A^{1/2}M_\varepsilon u \rangle + \langle A^{1/2}M_\varepsilon u, A^{1/2}M_\varepsilon F(u) \rangle \right.$$

$$\quad \left. + \langle F(M_\varepsilon u), M_\varepsilon F(u) \rangle \right].$$

Since $M_\varepsilon u \to u$ strongly in $H^{1/2}$ as $\varepsilon \downarrow 0$, we can infer, by Lemma 4, that

$$\lim_{\varepsilon \downarrow 0} f_\varepsilon(t) = \text{Im} \left[ \langle A^{1/2}F(u), A^{1/2}u \rangle + \langle A^{1/2}u, A^{1/2}F(u) \rangle + \langle F(u), F(u) \rangle \right]$$

$$= \text{Im} \text{(Real Number)} = 0.$$ To interchange the $\varepsilon$-limit with the $t$-integration in (3.15), we appeal to the dominated convergence theorem. That is, we seek for a uniform bound on $f_\varepsilon(t)$. In fact, by using the Cauchy-Schwarz inequality and Lemma 4 again we find the following estimate

$$|f_\varepsilon(t)| \lesssim |\langle A^{1/2}F(M_\varepsilon u), A^{1/2}M_\varepsilon u \rangle| + |\langle A^{1/2}M_\varepsilon u, A^{1/2}M_\varepsilon F(u) \rangle|$$

$$+ |\langle F(M_\varepsilon u), M_\varepsilon F(u) \rangle|$$

$$\lesssim \|A^{1/2}F(M_\varepsilon u)\|_2 \|A^{1/2}M_\varepsilon u\|_2 + \|A^{1/2}M_\varepsilon u\|_2 \|A^{1/2}M_\varepsilon F(u)\|_2$$

$$+ \|F(M_\varepsilon u)\|_2 \|M_\varepsilon F(u)\|_2$$

$$\lesssim \|u\|_{H^{1/2}}^4 + \|u\|_{H^{1/2}}^6,$$
for all $\varepsilon > 0$. Putting now all together leads to conservation of energy, i.e., we find for all $t_1, t_2 \in [0, T]$ that

$$E[u(t_2)] - E[u(t_1)] = \lim_{\varepsilon \to 0} \left( E[\mathcal{M}_\varepsilon u(t_2)] - E[\mathcal{M}_\varepsilon u(t_1)] \right)$$

$$= \lim_{\varepsilon \to 0} \int_{t_1}^{t_2} f_\varepsilon(t) \, dt = \int_{t_1}^{t_2} \lim_{\varepsilon \to 0} f_\varepsilon(t) \, dt = 0.$$

This completes the proof of Lemma 2. \hfill \Box

**Step 2: A-Priori Bounds**

To fill the last gap towards the global well-posedness result of Theorem 2, we now discuss how to obtain a-priori bounds on the energy norm. By the blow-up alternative of Theorem 1, global-in-time existence follows from an a-priori bound of the form

$$\|u(t)\|_{H^{1/2}} \leq C(u_0). \quad (3.16)$$

First, let us assume that $\lambda \geq 0$ holds. Then, for all $t \in [0, T)$, we find from Lemma 2 and (3.10) that

$$\|(-\Delta)^{1/4} u(t)\|_2 \lesssim E[u(t)] = E[u_0].$$

This implies together with charge conservation derived in Lemma 2 i.e.,

$$\|u(t)\|_2^2 = N[u(t)] = N[u_0] \quad (3.17)$$

an a-priori estimate (3.16). Therefore condition i) in Theorem 2 is sufficient for global existence.

Suppose now a focusing nonlinearity, i.e., $\lambda < 0$ holds, and without loss of generality we assume that $\lambda = -1$ is true (the general case follows by rescaling). Now we can estimate as follows.

$$E[u] = \frac{1}{2} \|(-\Delta + m^2)^{1/4} u\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{e^{-\mu |x|}}{|x|} * |u|^2 \right)(x) |u(x)|^2 \, dx$$

$$\geq \frac{1}{2} \|(-\Delta + m^2)^{1/4} u\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |u|^2 \right)(x) |u(x)|^2 \, dx$$

$$\geq \frac{1}{2} \|(-\Delta)^{1/4} u\|_2^2 - \frac{1}{4K} \|(-\Delta)^{1/4} u\|_2 \|u\|_2^2$$

$$= \left( \frac{1}{2} - \frac{1}{4K} \|u\|_2^2 \right) \|(-\Delta)^{1/4} u\|_2^2, \quad (3.18)$$

where $K > 0$ is the best constant taken from Appendix A.2. Thus, energy conservation leads to an a-priori bound on the $H^{1/2}$-norm of the solution, if

$$\|u_0\|_2^2 < 2K \quad (3.19)$$

holds. In fact, the constant $K$ satisfies

$$K = \frac{\|Q\|_2^2}{2} > \frac{2}{\pi},$$
where \( Q(x) \) is a strictly positive (ground state) solution of
\[
\sqrt{-\Delta} Q - \left( \frac{1}{|x|} * |Q|^2 \right) Q = -Q; \tag{3.20}
\]
see Appendix A.2. Going back to (3.19), we find that
\[
\|u_0\|_2^2 < \|Q\|_2^2 \tag{3.21}
\]
is sufficient for global existence for \( \lambda = -1 \). The assertion of Theorem 2 for all \( \lambda < 0 \) now follows by simple rescaling. The proof of Theorem 2 is now complete.

### 3.3 Proof of Theorem 3 (Higher Regularity)

To prove Theorem 3 we need the following generalization of Lemma 1 whose proof is a careful but straightforward generalization of the proof of Lemma 1. We defer the details to Appendix A.3.

**Lemma 3.** For \( \mu \geq 0 \) and \( s \geq 1/2 \), the map \( J(u) := (e^{-\mu |x|} * |u|^2)u \) is locally Lipschitz continuous from \( H^s(\mathbb{R}^3) \) into itself with
\[
\|J(u) - J(v)\|_{H^s} \lesssim (\|u\|_{H^s}^2 + \|v\|_{H^s}^2)\|u - v\|_{H^s}
\]
for all \( u, v \in H^s(\mathbb{R}^3) \). Moreover, we have that
\[
\|J(u)\|_{H^r} \lesssim \|u\|_{H^r}^2 \|u\|_{H^s}
\]
holds for all \( u \in H^s(\mathbb{R}^3) \), where \( r = \max\{s - 1, 1/2\} \).

Local well-posedness of (2.5) in \( H^s \), for \( s > 1/2 \), can be shown now as follows. We note that \( \{e^{-itA}\}_{t \in \mathbb{R}} \), with \( A = \sqrt{-\Delta + m^2} \), is a \( C^0 \)-group of isometries on \( H^s \). Moreover, since the nonlinearity defined in (3.1), is locally Lipschitz continuous from \( H^s \) into itself, local well-posedness in \( H^s \) follows similarly as explained in the proof of Theorem 1 for \( H^{1/2} \). To show global well-posedness in \( H^s \), we prove by induction and Lemma 3 that an a-priori bound on the \( H^{1/2} \)-norm of solution implies uniform bounds on the \( H^s \)-norm on any compact interval \([0, T_s] \subset [0, T]\). This claim follows from (3.2) and the second inequality stated in Lemma 3 by noting that
\[
\|u(t)\|_{H^s} \leq \|e^{-itA}u_0\|_{H^s} + \int_0^t \|e^{-i(t-\tau)A}F(u(\tau))\|_{H^s} \ d\tau \\
\leq \|u_0\|_{H^s} + \int_0^t \|F(u(\tau))\|_{H^s} \ d\tau \\
\lesssim C_1 + C_2 \int_0^t \|u(\tau)\|_{H^s} \ d\tau,
\]
holds, provided that \( \|u(t)\|_{H^r} \lesssim 1 \) for \( r = \max\{s - 1, 1/2\} < s \). Invoking Gronwall’s inequality we conclude that
\[
\|u(t)\|_{H^s} \lesssim e^{C_2 T_s}, \quad \text{for} \quad t \in [0, T_s] \subset [0, T).
\]
Induction now implies that an a-priori bound on \( \|u(t)\|_{H^{1/2}} \) guarantees uniform bounds \( \|u(t)\|_{H^s} \) on any compact interval \( I \subset [0, T) \). Thus, the maximal time of existence of an \( H^s \)-valued solution coincides with the maximal time of existence when viewed as an \( H^{1/2} \)-valued solution. Therefore sufficient conditions for global existence for \( H^{1/2} \)-valued solutions imply global-in-time \( H^s \)-valued solutions. This completes the proof of Theorem 3.
3.4 Proof of Theorem \ref{thm:external-potentials} (External Potentials)

Let \( V = V_+ + V_- \) satisfy Assumption \ref{ass:external-potentials} in Section \ref{sec:assumptions}. We introduce the quadratic form
\[
Q(u, v) := \langle u, \sqrt{-\Delta + m^2} v \rangle + \langle u, V_- v \rangle + \langle u, V_+ v \rangle,
\]
which is well-defined on the set (energy space)
\[
X := \{ u \in L^2(\mathbb{R}^3) : Q(u, u) < \infty \}.
\]
Note that Assumption \ref{ass:external-potentials} also guarantees that \( C_c^{\infty}(\mathbb{R}^3) \subset X \). It easy to show that our assumption on \( V \) implies that the quadratic form \((3.22)\) is bounded from below, i.e., we have
\[
Q(u, u) \geq -M \langle u, u \rangle
\]
holds for all \( u \in X \) and some constant \( M \geq 0 \). By the semi-boundedness of \( Q \), we can assume from now on (and without loss of generality) that
\[
Q(u, u) \geq 0
\]
holds for all \( u \in X \). Since \( Q(\cdot, \cdot) \) is closed (it is a sum of closed forms), the energy space \( X \) equipped with its norm
\[
\| u \|_X := \sqrt{\langle u, u \rangle + Q(u, u)}
\]
is complete, and we have the equivalence
\[
\| u \|_{H^{1/2}} + \| V_+^{1/2} u \|_2 \lesssim \| u \|_X \lesssim \| u \|_{H^{1/2}} + \| V_+^{1/2} u \|_2.
\]
Furthermore, there exists a nonnegative, self-adjoint operator
\[
A : D(A) \subset L^2 \to L^2
\]
with \( X = D(A^{1/2}) \), such that
\[
\langle u, Av \rangle = Q(u, v)
\]
holds for all \( u \in X \) and \( v \in D(A) \); see, e.g., \cite{Kat80}. This operator can be extended to a bounded operator, still denoted by \( A : X \to X^* \), where \( X^* \) is the dual space of \( X \).

To prove now the assertion about local well-posedness in Theorem \ref{thm:external-potentials} we have to generalize Lemma \ref{lem:lip} to the following statement.

\textbf{Lemma 4.} Suppose \( \mu \geq 0 \) and let \( V \) satisfy Assumption \ref{ass:external-potentials}. Then the map \( J(u) := (\frac{e^{-\mu|\cdot|}}{|\cdot|} * |u|^2)u \) is locally Lipschitz continuous from \( X \) into itself with
\[
\| J(u) - J(v) \|_X \lesssim (\| u \|_X^2 + \| v \|_X^2) \| u - v \|_X
\]
for all \( u, v \in X \).

\textbf{Proof of Lemma 4.} By \((3.26)\), it suffices to estimate \( \| J(u) - J(v) \|_{H^{1/2}} \) and \( \| V_+^{1/2} |J(u) - J(v)| \|_2 \) separately. By Lemma \ref{lem:lip} we know that
\[
\| J(u) - J(v) \|_{H^{1/2}} \lesssim (\| u \|_{H^{1/2}}^2 + \| v \|_{H^{1/2}}^2) \| u - v \|_{H^{1/2}}
\]
\[
\lesssim (\| u \|_X^2 + \| v \|_X^2) \| u - v \|_X.
\]
It remains to estimate \( \|V_{+}^{1/2}[J(u) - J(v)]\|_{2} \), which can be achieved by recalling (3.9) and proceeding as follows.

\[
\|V_{+}^{1/2}[J(u) - J(v)]\|_{2} \lesssim \|V_{+}^{1/2}[\left(\frac{e^{-\mu|x|}}{x} \ast (|u|^2 + |v|^2)\right)(u + v)]\|_{2} \\
+ \|V_{+}^{1/2}[\left(\frac{e^{-\mu|x|}}{x} \ast (|u|^2 - |v|^2)\right)(u - v)]\|_{2} \\
\lesssim \left\| \frac{e^{-\mu|x|}}{|x|} \ast (|u|^2 - |v|^2) \right\|_{\infty} \|V_{+}^{1/2}(u + v)\|_{2} \\
+ \left\| \frac{e^{-\mu|x|}}{|x|} \ast (|u|^2 + |v|^2) \right\|_{\infty} \|V_{+}^{1/2}(u - v)\|_{2} \\
\lesssim \|u + v\|_{H^{1/2}}\|u - v\|_{H^{1/2}} \|V_{+}^{1/2}(u + v)\|_{2} \\
+ (\|u\|_{H^{1/2}}^2 + \|v\|_{H^{1/2}}^2) \|V_{+}^{1/2}(u - v)\|_{2} \\
\lesssim (\|u\|_{X}^2 + \|v\|_{X}^2) \|u - v\|_{X}.
\]

This completes the proof of Lemma 4.

Returning to the proof of Theorem 4 we simply note that \( \{e^{-itA}\}_{t \in \mathbb{R}} \) is a \( C^0 \)-group of isometries on \( X \), where \( A = \sqrt{-\Delta + m^2} + V \) is defined in the form sense (see above). By Lemma 4 the nonlinearity is locally Lipschitz on \( X \). Thus, local well-posedness now follows in the same way as for Theorem 4.

To establish global well-posedness we have to prove conservation of charge, \( N[u] \), and energy, \( E[u] \), which is for (2.3) given by

\[
E[u] := \frac{1}{2} \int_{\mathbb{R}^3} \bar{u}(x) \sqrt{-\Delta + m^2} u(x) \, dx + \frac{1}{2} \int_{\mathbb{R}^3} V(x) |u(x)|^2 \, dx \\
+ \frac{1}{4} \int_{\mathbb{R}^3} \left( \lambda e^{-\mu|x|} \right) \ast |u|^2(x) |u(x)|^2 \, dx.
\]

As done in Section 3.2 we have to employ a regularization method using the class of operators

\[
\mathcal{M}_\varepsilon := (\varepsilon A + 1)^{-1}, \quad \text{for} \quad \varepsilon > 0,
\]

where we assume without loss of generality that \( \lambda \geq 0 \) holds. The mapping \( \mathcal{M}_\varepsilon \) acts on the sequence of embedded spaces

\[
\ldots X^{+2} \hookrightarrow X^{+1} \hookrightarrow X^{-1} \hookrightarrow X^{-2} \ldots,
\]

with corresponding norms given by \( \|u\|_{X^s} := \|(1 + A)^{s/2}u\|_2 \). Note that \( X = X^{+1} \) (with equivalent norms) and that its dual space obeys \( X^* = X^{-1} \). By using functional calculus, it is easy to show that \( \mathcal{M}_\varepsilon \) exhibits properties that are analog to a) – c) in Section 3.2.

The rest of the argument for proving conservation of energy carries over from Section 3.2 without major modifications. Finally, we mention that deriving a-priori bounds on \( \|u(t)\|_X \) leads to a similar discussion as presented in Section 3.2 while noting that we have to take care that \( V_\varepsilon \) has a relative \((\Delta)^{1/2}\)-form bound, \( 0 \leq \varepsilon < 1 \), introduced in Assumption II.

This completes the proof of Theorem 4.
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A Appendix

A.1 Fractional Calculus

The following result (generalized Leibniz rule) is proved in [GK96] for Riesz and Bessel potentials of order \( s \in \mathbb{R} \), which are denoted by \((-\Delta)^{s/2}\) and \((1 - \Delta)^{s/2}\), respectively. But as a direct consequence of the Milhin multiplier theorem [BL76], the cited result holds for \( D^s := (\mu^2 - \Delta)^{s/2} \), where \( \mu \geq 0 \) is a fixed constant.

Lemma (Generalized Leibniz Rule). Suppose that \( 1 < p < \infty \), \( s \geq 0 \), \( \alpha \geq 0 \), \( \beta \geq 0 \), and \( 1/p_i + 1/q_i = 1/p_i \) with \( i = 1, 2 \), \( 1 < q_i \leq \infty \), \( 1 < p_i \leq \infty \). Then

\[
\|D^s(fg)\|_p \leq c(\|D^{s+\alpha}f\|_{p_1}\|D^{-\alpha}g\|_{q_1} + \|D^{-\beta}f\|_{p_2}\|D^{s+\beta}g\|_{q_2}),
\]

where the constant \( c \) depends on all of the parameters but not on \( f \) and \( g \).

A second fact we use in the proof of our main result is as follows. For \( 0 < \alpha < 3 \) and \( \mu \geq 0 \), the potential operator \( D^{-\alpha} = (\mu^2 - \Delta)^{-\alpha/2} \) corresponds to \( f \mapsto G^\mu_\alpha \ast f \), with \( f \in \mathcal{S}(\mathbb{R}^3) \), and we have that

\[
G^\mu_\alpha \in L^{3/(3-\alpha)}_w(\mathbb{R}^3). \tag{A.1}
\]

To see this, we refer to the inequality and the exact formula

\[
0 \leq G^\mu_\alpha(x) \leq \frac{c_\alpha}{|x|^{3-\alpha}}, \quad \text{for} \quad \mu \geq 0 \quad \text{and} \quad 0 < \alpha < 3, \tag{A.2}
\]

with some constant \( c_\alpha \); these facts can be derived from [Ste70, Section V.3.1]. Now (A.1) follows from \( |x|^{-\sigma} \in L^{3/\sigma}_w(\mathbb{R}^3) \) whenever \( 0 < \sigma < 3 \). Another observation used in Section 2 is the well-known explicit formula

\[
G^\mu_2(x) = \frac{e^{-\mu|x|}}{4\pi|x|}. \tag{A.3}
\]

That is, \((\mu^2 - \Delta)\) in \( \mathbb{R}^3 \) has the Green’s function \( \frac{e^{-\mu|x|}}{4\pi|x|} \) with vanishing boundary conditions.

A.2 Ground States

We consider the functional (see also [LY87])

\[
K[u] := \frac{\|(\Delta)^{1/4}u\|_2^2\|u\|_2^2}{\int_{\mathbb{R}^3} \frac{1}{|x|} |u(x)|^2 |u(x)|^2 \, dx}, \tag{A.4}
\]

which is well-defined for all \( u \in H^{1/2} \) with \( u \neq 0 \). Note that by using (3.6) we can estimate the denominator in \( K[u] \) as follows.

\[
\int_{\mathbb{R}^3} \frac{1}{|x|} |u(x)|^2 |u(x)|^2 \, dx \leq \frac{1}{|x|} \|u\|_\infty^2 \|u\|_2^2 \leq \frac{\pi}{2} \|(\Delta)^{1/4}u\|_2^2 \|u\|_2^2, \tag{A.5}
\]
which leads to the bound
\[
\frac{2}{\pi} \leq K[u] < \infty.
\]  
(A.6)

Indeed, we will see that the estimate from below is a strict inequality. With respect to the related variational problem
\[
K := \inf \left\{ K[u] : u \in H^{1/2}(\mathbb{R}^3), \, u \not\equiv 0 \right\}
\]  
(A.7)

we can state the following result.

**Lemma (Ground States).** There exists a minimizer, \( Q \in H^{1/2}(\mathbb{R}^3) \), for (A.7), and we have the following properties.

i) \( Q(x) \) is a smooth function that can be chosen to be real-valued, strictly positive, and spherically symmetric with respect to the origin. It satisfies
\[
\sqrt{-\Delta} Q - \left( \frac{1}{|x|} * |Q|^2 \right) Q = -Q,
\]  
(A.8)

and it is nonincreasing, i.e., we have that \( Q(x) \geq Q(y) \) whenever \(|x| \leq |y|\).

ii) The infimum satisfies \( K = \|Q\|_2^2 / 2 \) and \( K > 2 / \pi \).

**Sketch of Proof.** We present the main ideas for the proof of the preceding lemma. That (A.7) is attained at some real-valued, radial, nonnegative and nonincreasing function \( Q(x) \geq 0 \) can be proved by direct methods of variational calculus and rearrangement inequalities; see also [Wei83] for a similar variational problem for nonrelativistic Schrödinger equations with local nonlinearities. Furthermore, any minimizer, \( Q \in H^{1/2} \), has to satisfy the corresponding Euler-Lagrange equation that reads
\[
\sqrt{-\Delta} Q - \left( \frac{\lambda}{|x|} * |Q|^2 \right) Q = -Q,
\]  
(A.9)

after a suitable rescaling \( Q(x) \mapsto aQ(bx) \) with some \( a, b > 0 \).

Let us make some comments about the properties of \( Q \). Using an bootstrap argument and Lemma \( \# \) for the nonlinearity, it follows that \( Q \) belongs to \( H^s \), for all \( s \geq 1/2 \). Hence it is a smooth function. To see that \( Q(x) \geq 0 \) is strictly positive, i.e., \( Q(x) > 0 \), we rewrite equation (A.9) such that
\[
Q = \left( \sqrt{-\Delta} + 1 \right)^{-1} W,
\]  
(A.10)

where \( W := (|x|^{-1} * |Q|^2)Q \). By functional calculus, we have that
\[
\left( \sqrt{-\Delta} + 1 \right)^{-1} = \int_0^\infty e^{-t} e^{-t\sqrt{-\Delta}} dt.
\]  
(A.11)

Next, we notice by the explicit formula for the kernel (in \( \mathbb{R}^3 \))
\[
e^{-t\sqrt{-\Delta}}(x,y) = \mathcal{F}^{-1}(e^{-t(|\xi|^2/2)})(x-y) = C \cdot \frac{t}{|t|^2 + |x-y|^2},
\]  
with some constant \( C > 0 \); see, e.g., [LL01]. This explicit formula shows that \( e^{-t\sqrt{-\Delta}} \) is positivity improving. This means that if \( f \geq 0 \) with \( f \not\equiv 0 \) then \( e^{-t\sqrt{-\Delta}}f > 0 \) almost
everywhere. Hence \((\sqrt{-\Delta} + 1)^{-1}\) is also positivity improving, by (A.11), and we conclude that \(Q(x) > 0\) holds almost everywhere, thanks to (A.10) and \(W \geq 0\). Moreover, we know that \(Q(x)\) is a nonincreasing, continuous function. Therefore \(Q(x) > 0\) holds in the strong sense, i.e., for every \(x \in \mathbb{R}^3\).

Finally, to see that ii) holds, we consider the variational problem

\[
I_N := \inf \{ E[u] : u \in H^{1/2}(\mathbb{R}^3), \|u\|_2^2 = N \},
\]

(A.12)

where \(N > 0\) is a given parameter and

\[
E[u] = \frac{1}{2} \|(-\Delta)^{1/4} u\|_2^2 - \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |u|^2 \right) |u(x)|^2 \, dx.
\]

Due to the scaling behavior \(E[\alpha^{3/2}u(\alpha \cdot)] = \alpha E[u]\), we have that either \(I_N = 0\) or \(I_N = -\infty\) holds. By noting that

\[
E[u] \geq \left( \frac{1}{2} - \frac{N}{4K} \right) \|(-\Delta)^{1/4} u\|_2^2,
\]

and the fact that equality holds if and only if \(u\) minimizes \(K[u]\), we find that \(I_N = 0\) holds if and only if \(N \leq N_c := 2K\). Moreover, \(I_N = 0\) is attained if and only if \(N = N_c\). Let \(\tilde{Q}\) be such a minimizer with \(\|\tilde{Q}\|_2^2 = N_c\). Thanks to the proof of part i), we can assume without loss of generality that \(\tilde{Q}\) is real-valued, radial, and strictly positive. Calculating the Euler-Lagrange equation for (A.12), with \(N = N_c\), yields

\[
\sqrt{-\Delta} \tilde{Q} - \left( \frac{1}{|x|} * |\tilde{Q}|^2 \right) \tilde{Q} = -\theta \tilde{Q},
\]

for some multiplier \(\theta\), where it is easy to show that \(\theta > 0\) holds. Putting now \(Q(x) = \theta^{-3/2} \tilde{Q}(\theta^{-1}x)\), which conserves the \(L^2\)-norm, leads to a ground state \(Q(x)\) satisfying (A.9). Thus, we have that

\[
K = \|\tilde{Q}\|_2^2 / 2 = \|Q\|_2^2 / 2.
\]

To prove that \(K > 2 / \pi\) holds, let us assume \(K = \pi / 2\). This implies that the first inequality in (A.10) is an equality for \(u(x) = Q(x) > 0\). But this leads to \((|x|^{-1} * |Q|^2)(x) = \text{const.}\), which is impossible.

A.3 Proof of Lemma 3

Proof of Lemma 3. We only show the second inequality derived in Lemma 3 since the first one can be proved in a similar way.

Let \(\mu \geq 0\) and \(s \geq 1/2\). We put \(D^\alpha := (\mu^2 - \Delta)^{\alpha/2}\) for \(\alpha \in \mathbb{R}\). By the generalized Leibniz rule and (3.6), we have that

\[
\|D^s J(u)\|_2 \lesssim \|D^s[(D^{-2} |u|^2)u]\|_2 \\
\lesssim \|D^{s-2} |u|^2\|_{L^1} \|u\|_{q_1} + \|D^{-2} |u|^2\|_{L^\infty} \|D^s u\|_2 \\
\lesssim \|D^{s-2} |u|^2\|_{L^1} \|u\|_{q_1} + \|u\|_{L^2} \|u\|_{H^{s,}},
\]

(A.13)

where \(1 / p_1 + 1 / q_1 = 1 / 2\) with \(1 < p_1, q_1 \leq \infty\). The first term of the right-hand side of (A.13) can be controlled as follows, where we introduce \(r = \max\{s - 1, 1/2\}\).
i) For $1/2 \leq s < 3/2$, we choose $p_1 = 3/s$ and $q_1 = 6/(3 - 2s)$ which leads to
\[
\|D^{s-2}|u|^2\|_{3/s} \|u\|_{6/(3-2s)} \lesssim \|G_{2-s}^{3/(3-2s)} w\| u^2 \|_{3/2} \|u\|_{H^s} \\
\lesssim \|u\|_{H^{1/2}}^2 \|u\|_{H^s} \lesssim \|u\|_{H^r}^2 \|u\|_{H^s},
\]
where we use the weak Young inequality, Sobolev’s inequality $\|u\|_{6/(3-2s)} \lesssim \|u\|_{H^s}$ in $\mathbb{R}^3$, and \((A.1)\) once again.

ii) For $s \geq 3/2$, we put $p_1 = 6$ and $q_1 = 3$ and find
\[
\|D^{s-2}|u|^2\|_{6} \|u\|_{3} \lesssim \|D^{s-3/2}|u|^2\|_{2} \|u\|_{3} \lesssim \|D^{s-3/2}u\|_{6} \|u\|_{3}^2 \\
\lesssim \|D^{s-1}u\|_{2} \|u\|_{3}^2 \lesssim \|u\|_{H^r}^2 \|u\|_{H^s},
\]
while using twice Sobolev’s inequality $\|f\|_{6} \lesssim \|D^{1/2}f\|_{2}$ in $\mathbb{R}^3$.

Putting now all together, we conclude that
\[
\|J(u)\|_{H^s} \lesssim \|J(u)\|_{2} + \|D^s J(u)\|_{2} \\
\lesssim \|u\|_{H^{1/2}}^2 \|u\|_{2} + \|u\|_{H^r}^2 \|u\|_{H^s} \lesssim \|u\|_{H^r}^2 \|u\|_{H^s}.
\]

\[\square\]

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