SIMULTANEOUS EMBEDDINGS OF FINITE DIMENSIONAL DIVISION ALGEBRAS

LOUIS ROWEN AND DAVID J SALTMAN

A celebrated theorem of P.M. Cohn \[C\] says that for any two division rings (not necessarily finite dimensional) over a field $F$, their amalgamated product over $F$ is a domain which can be embedded in a division ring. Note that even with the two initial division rings begin finite dimensional over their centers, the resulting division ring is never finite dimensional over its center. Perhaps this led Lance Small to ask the following question. We say $D/F$ is a division algebra when $D$ is a division ring finite dimensional over its center $F$. Assume $F_1$ and $F_2$ are fields with the same characteristic. Small asked whether any two division algebras $D_1/F_1$ and $D_2/F_2$ can be embedded in some third division algebra $E/F$.

We start with a surprisingly straightforward counterexample in the next section, but then show that a positive solution exists for division algebras finitely generated over a common subfield which is either algebraically closed or the prime subfield (Theorem 2.9).

1. A COUNTEREXAMPLE

Suppose, first of all, that $D_1/F_1$ is a f.d. division algebra embedded in the division algebra $E/F$, so $F_1$ and $F$ share the same prime subfield $P$. There is a tower of subalgebras $F \subseteq F_1 F \subseteq D_1 F \subseteq E$, where $F_1 F$ must be an amalgamation of $F_1$ and $F$ (meaning that it is the field of fractions of an image of the tensor product $F_1 \otimes_P F$).

Getting more specific, suppose $p_1 \neq p_2$ are primes. Let $G$ be the infinite cyclic profinite $p_2$-group, the inverse limit of all $\mathbb{Z}/p_2^n\mathbb{Z}$. Take $F_1/Q$ Galois with group $G$. (For example, $F_1$ could be contained in the infinite extension of $Q$ obtained by adjoining all $p_2^n$ roots of 1.) Note that $G$ has no finite subgroups. The field extension $F_1 F/F$ has Galois group a subgroup of $G$, and must be finite dimensional (being inside $E/F$), and so must be trivial. That is, $F_1 F = F$, implying $F_1 \subseteq F$.

Next, take $D_1/Q$ of degree $p_1$ and let $D_1 = D_1' \otimes_Q F_1$, a division algebra since $F_1/Q$ is a pro-$p_2$ extension, and let $D_2/Q$ be any division algebra split by $F_1$. For example, there is a cyclic degree $p_2$ extension $L/Q$ such that $L \subseteq F_1$. By class field theory there is a degree $p_2$ division algebra with maximal subfield $L$.

Proposition 1.1. There is no division algebra $E/F$ containing both $D_1$ and $D_2$.

Proof. If $E/F$ contained both $D_1$ and $D_2$, then $F_1 \subseteq F$ is central, by the above paragraph, so $E$ contains $D_2 F_1$ which is a division algebra but also a homomorphic image of the split algebra $D_2 \otimes_Q F_1$ and thus is commutative, a contradiction.

The rationale for this example is that the centers are incompatible in some sense.
2. Positive results

Remark 2.1. Since every division algebra is a tensor product of division algebras of prime power degree, it is natural to ask that if $D_1/F_1$ and $D_2/F_2$ are division algebras over respective degrees $p^{k_1}$ and $p^{k_2}$, then can $D_1/F_1$ and $D_2/F_2$ be embedded into a single division algebra $E/F$ of degree $p^t$, and is there a bound for $t$ in terms of $t_1$ and $t_2$? What would be the best bound?

We approach the problem via [Sa]. First let us fix some notation. We write $\text{index}(D)$ for the (Schur) index of $D$. Fixing $r > 1$, let $UD(F,n)/Z(F,n)$ denote the generic division division algebra of degree $n$ over $F$ in $r$ indeterminates. We write $Z$ for $Z(F,n)$. When $L/F$ is a cyclic Galois extension of dimension $n$ and $a \in F$, $\Delta(L/F,a)/F$ denotes the $F$-central cyclic algebra having maximal subfield $L$, together with some element $z$ inducing the automorphism generating $\text{Gal}(L/F)$, satisfying $z^n = a$. We begin with some lemmas.

Lemma 2.2. There is a field $K(t) \supset Z$ and a degree $n$ cyclic extension $L/K(t)$ such that $L/F$ is rational, and $UD(F,n) \otimes_Z K(t) = \Delta(L/K(t),t)$.

Proof. Write $Z = F(X \oplus Y)^{S_n}$ as usual (see [Sa] p. 322). Let $C_n \subset S_n$ be generated by the $n$ cycle $(1,2,\ldots,n)$. Over $C_n$, $Y \cong M \oplus Z$ where $M$ is a free $C_n$ lattice. We can set $L = F(X \oplus M)$, $K = L^{C_n}$, and take $t$ to be the generator of $Z$.

Lemma 2.3. Suppose $F$ is a field and $D/F$ is an division algebra. Set

$$A = D \otimes_F UD(F,n)^b = (D \otimes_F Z) \otimes_Z UD(F,n)^b.$$ 

Then $\text{index}(A)$ is the degree of $D$ times $n/(n,b) = \text{index}(UD(F,n)^b)$.

Proof. Since $UD(F,n)$ has index equal exponent, the index of any power is equal to the exponent. In fact, if $b = b'(n,b)$ and $n = n'(n,b)$ then $(b',n') = 1$. Thus $UD(F,n)^b = (UD(F,n)^{(n,b)})^b$ and the index and exponent of $UD(F,n)^b$ is the same as that of $(UD(F,n)^{(n,b)})$. That is, we may assume $b/n$.

By Lemma 2.2 there is a field $K(t) \supset Z$ and a degree $n$ cyclic extension $L/K(t)$ such that $D \otimes_F L$ is a division algebra and $UD(F,n) \otimes_Z K(t) = \Delta(L/K(t),t)$. Of course, by Galois theory, $\Delta(L/K(t),t)^b$ is equal in the Brauer group to $\Delta(L/K(t),t)$ where $L/L'$ has degree $b$. Finally, $(D \otimes_K K(t))^b \otimes_{K(t)} \Delta(L'/K(t),t)$ is a division algebra via twisted polynomial rings.

We are in the game of embedding division algebras into bigger division algebras. The key method is the following.

Theorem 2.4. Suppose $D/K$ is a division algebra of degree $a$ and $K/F$ has degree $b$. Assume $E/F$ is a division algebra of degree $N = nab$. Then $D$ is isomorphic to a subalgebra of $E$ over $F$ if and only if $(E \otimes_F K) \otimes_K D^a$ has (Schur) index dividing $n$. Furthermore, if this index divides $n$ then it is equal to $n$.

Proof. Suppose $D \subset E$. In particular, $K \subset E$ and so $E \otimes_F K$ has index $N/b$ and we set $E'/K$ to be the associated division algebra which is the centralizer of $K$ in $E$. Then $D \subset E'$ and we take $D'$ to be its centralizer, implying $E' = D \otimes_K D'$. Since $D'$ has degree $n$, we have proven one direction.

Conversely, suppose $\text{index}((E \otimes_F K) \otimes_K D^a)$ divides $n$. Then $\text{index}(E \otimes_F K)$ divides $na$, implying $\text{index}(E \otimes_F K) = na$ since $\text{index}(E \otimes_F K) \geq \frac{N}{b} = na$. Thus,
for all \(i\) product over \(\bar{\mathbb{F}}\) be the stabilizer of \(gH\) of \(H\). Let \(K/F\) be a central simple algebra and \(n\) an integer dividing the degree of \(A/K\). Then \(A\) and only if \(A \otimes_K K'\) has index dividing \(n\).

Next we set \(\mathcal{W}_n(A)\) to be the Weil transfer to \(F\) of \(\mathcal{W}_n(A)\), so for \(F' \supset F\), \(\mathcal{W}_n(A)\) has an \(F'\) point if and only if \(\mathcal{W}_n(A)\) has an \(\mathcal{W}_F F'\) point (and in fact there is a natural correspondence). Let \(\mathcal{W}_n(A)\) denote the field of fractions of \(\mathcal{W}_n(A)\). Then \(\text{index}(A \otimes_F K \mathcal{W}_n(A))\) divides \(n\).

The important tool for using this construction is the following result ([Sa]) about index reduction, for which we need to introduce more notation. Let \(K/F\) be finite separable with Galois closure \(\bar{K}/F\). Let \(G\) be the Galois group of \(\bar{K}/F\) and \(H \subset G\) the subgroup corresponding to \(K/K\). If \(r\) is the degree of \(A/K\), then we can define an “action” of the \(G\) module \(\mathbb{R} = (\mathbb{Z}/r\mathbb{Z})[G/H]\) as follows. Let \(\hat{A} = A \otimes_K \bar{K}\), so \(\bar{K}\) has a natural semilinear action on \(\hat{A}\) and for any \(g \in G\) we can define the \(g\) twist \(g\hat{A}\). Of course, for \(g \in G\), \(g\hat{A}^{-1}\) has a natural semilinear action on \(g\hat{A}\).

For \(\alpha \in \bar{R}\), define \(H_\alpha = \{g \in G | g\alpha = \alpha\}\). Define \(K(\alpha) = \hat{K}^{H_\alpha}\). Write

\[
\alpha = \sum n_{gH} gH;
\]
then the \(n_{gH}\) are constant on \(H_\alpha\)-orbits. Fix a coset \(gH\) and set \(e = n_{gH}\). Let \(L \subset H_\alpha\) be the stabilizer of \(gH\). Let \(\mathcal{O} = \{gH\}\) be the orbit of \(H_\alpha\) containing \(gH\) so \(e = n_{gH}\) for all \(i\). Then \(L\) acts naturally on \(g(\hat{A})\) and \(H_\alpha\) acts on \(B_{gH}\) which is the tensor product over \(\hat{K}\) of \(g_i(\hat{A})\), one for each \(g_iH\) in \(\mathcal{O}\).

Now we let \(gH\) vary, one for each \(H_\alpha\) orbit. Tensor over \(\hat{K}\) all the \(B_{gH}\) defined above and call the resulting \(\hat{K}\) algebra \(B\). Note that \(\hat{K}\) is the center of \(B\). Define \(A^\alpha\) to be the \(H_\alpha\) invariant subring of \(B\). Then \(A^\alpha\) has center \(K(\alpha)\).

Finally, for \(\alpha = \sum n_{gH} gH\) as above, define

\[
|\alpha| = \prod_{gH} \frac{n}{(n, n_{gH})}.
\]

**Theorem 2.5.** ([Sa] p. 332). Notation as above, suppose \(B/F\) is any central simple algebra (over \(F\)). Then the index of \(B \otimes_F F_n(A)\) is the gcd of all the integers

\[
\text{index}(B \otimes_F A^\alpha)[K(\alpha) : F]| |\alpha|,
\]

taken over all \(\alpha \in \bar{R}\).

We actually need a double version of the above result. Let us assume that \(K/F\) and \(K'/F\) are finite separable with Galois closures \(\bar{K}/F\) and \(\bar{K}'/F\) and corresponding groups \(G \supset H\) and \(G' \supset H'\). For convenience we may assume that \(\bar{K}/F\) and \(\bar{K}'/F\) are linearly disjoint. Let \(A/K\) and \(A'/K'\) be central simple algebras and let \(F_{n,n'}(A, A')\) denote the join of the fields \(F_n(A)\) and \(F_{n'}(A')\) over \(F\). If \(A'/K'\) has degree \(r'\) set \(\bar{R}' = (\mathbb{Z}/r'\mathbb{Z})[G'//H']\) as above. If \(\alpha \in R\) and \(\beta \in R'\) set \(K(\alpha, \beta) = K(\alpha) \otimes_F K'(\beta)\).
Finally write \( \beta = \sum_{gH'} m_{gH'} gH' \in R' \) and set
\[
|\beta| = \prod_{gH'} \frac{n'}{(n', m_{gH'})}.
\]

**Theorem 2.6.** Suppose \( B/F \) is a central simple algebra and set
\[ B(\alpha, \beta) = B \otimes_F K(\alpha, \beta). \]
Then the (Schur) index \( i := \text{index}(B \otimes_F F_{n,v}(A, A')) \) is the gcd of the integers
\[
\text{index} \left( (B(\alpha, \beta) \otimes_{K(\alpha, \beta)} A^\alpha \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (A^\beta \otimes_{K(\beta)} K(\alpha, \beta)) \right) [K(\alpha, \beta) : F][\alpha] | \beta |,
\]
ranging over all \( \alpha \in R \) and \( \beta \in R' \).

**Proof.** The basic idea here is to apply Theorem 2.5 twice, noting that \([K(\alpha, \beta) : F] = [K(\alpha) : F][K'(\beta) : F] \). Put \( B' = B \otimes_F F_n(A) \). Then, by Theorem 2.5 using the fact that \([F_n K'(\beta) : F_n(\alpha)] = [K'(\beta) : F], i \) is the gcd of all integers
\[
\text{index} \left( (B' \otimes_{F_n(A)} F_n K'(\beta)) \otimes_{F_n K'(\beta)} (A^\beta \otimes_{K'(\beta)} F_n K'(\beta)) \right) [K'(\beta) : F][\beta],
\]
where \( F_n K'(\beta) \) is the join of \( F_n(A) \) and \( K'(\beta) \) over \( F \). Note that \( F_n K'(\beta) \) is the function field of \( W_n(A \otimes_F K(\beta)) \) which is the \( K(\alpha, \beta)/K'(\beta) \) transfer of \( V_n(A \otimes_F K'(\beta)) \). Now by Theorem 2.5 again, using \( B' \) instead of \( B \), each
\[
\text{index} \left( (B' \otimes_{F_n(A)} F_n K'(\beta)) \otimes_{F_n K'(\beta)} (A^\beta \otimes_{K'(\beta)} F_n K'(\beta)) \right)
\]
is the gcd of
\[
\text{index} \left( (B \otimes_F A^\alpha) \otimes_{K(\alpha)} (A^\beta \otimes_{K(\beta)} F_n K'(\beta)) \right) [K(\alpha, \beta) : K'(\beta)][\alpha]
\]
and the result follows.

Now suppose we already know that \( D_1/K_1 \) and \( D_2/K_2 \) are division algebras of degrees \( d_i \) and \( K_i/F \) is separable of degree \( e_i \). Set \( m_i = d_i e_i \) and let \( N \) be any multiple of the lcm of \( m_1^2 \) and \( m_2^2 \). Recall that \( F'/F \) is called a regular field extension if \( F' \) is finitely generated as a field over \( F \). \( F' \) is a finite separable extension of a purely transcendental extension of \( F \), and \( F \) is algebraically closed in \( F' \).

**Theorem 2.7.** There is a regular field extension \( F' \supset F \) and a division algebra \( E'/F' \) of degree \( N \) such that \( D_i \subset E' \) is compatible with \( F \subset F' \) for \( i = 1, 2 \).

**Proof.** Set \( n_i = N/m_i \), noting that \( n_i \) is a multiple of \( m_i \). Set
\[
E = UD(F, N)
\]
and we will extend \( Z \) so that the \( D_i \) embed in the base extension of \( E \). To achieve this we set \( E' = F_{n_1 n_2}(A_1, A_2) \) where
\[
A_i = (D_i^\circ \otimes_{K_i} K_i Z) \otimes_{K_i Z} (E \otimes_Z K_i Z).
\]
Note that \((E \otimes_Z K_i Z)\) is just the generic division algebra over \( K_i \). Also note that \( E' = E \otimes_Z F' \) has both \( D_i \) embedded because we have suitably reduced the index of both \( A_i \). The problem is to show that \( E' \) is a division algebra, i.e., that \( \text{index}(E') = N \), and for this we apply Theorem 2.6 In applying this theorem, note that the degree of \( A_1 \) is \( N d_1 \), and so is a multiple of \( n_1 \). We make a similar comment about the degree of \( A_2 \).
To apply Theorem 2.6, we need to get a handle on
\[ \text{index}((E \otimes Z K(\alpha, \beta))^{1+\alpha+\beta} \otimes (D_1^{-\alpha} \otimes K(\alpha, \beta)) \otimes (D_2^{-\beta} \otimes K'(\beta) ZK(\alpha, \beta))), \]
where the unsubscripted tensors are over \( ZK(\alpha, \beta) \). Write \( \alpha = \sum n_g gH \) and set \( a = \sum n_g gH \) and similarly for \( \beta \) and \( b \). Note that \( E \) is not moved by either Galois group so \( (E \otimes Z K(\alpha, \beta))^{1+\alpha+\beta} \) is \( E^{1+\alpha+\beta} \otimes Z K(\alpha, \beta)Z \) which has index \( N/(N, 1 + a + b) \) which we define to be \( N_{a,b} \). By Lemma 2.3 the above index is
\[ \text{index}((D_1^{-\alpha} \otimes K(\alpha, \beta)) \otimes K(\alpha, \beta)) \otimes K'(\beta) K(\alpha, \beta)) N_{a,b} \]
and so we want to show that
\[ \text{index}((D_1^{-\alpha} \otimes K(\alpha, \beta)) \otimes K(\alpha, \beta)) \otimes K'(\beta) K(\alpha, \beta)) N_{a,b} \]
We show the needed divisibility prime by prime. So assume, for \( p \) prime that \( p^a \)
divides \( N \) exactly, in the sense that \( \frac{N}{p} \) is prime to \( p \). Likewise, assume that \( p^b \)
divides \( n_i \) exactly. Since \( N = n_1 d_1 e_i \) and \( n_i \) is a multiple of \( d_i e_i \) we have \( 2t_i \geq s \) and also \( t_1 + t_2 \geq s \). If \( 1 + a + b \) is prime to \( p \) we are done. Thus we assume \( p \) divides
\[ 1 + \sum_{gH} n_g gH + \sum_{gH'} m_g gH' \]
and this implies that at least one summand in
\[ \sum_{gH} n_g gH + \sum_{gH'} m_g gH' \]
is prime to \( p \). If any term in \( (\sum_{gH} n_g gH + \sum_{gH'} m_g gH') \) is divisible by \( p^s \), thus, if two terms in \( (\sum_{gH} n_g gH + \sum_{gH'} m_g gH') \) are prime to \( p \), then \( p^{2t_1} \) or \( p^{t_1 + t_2} \) or \( p^{2t_2} \) divide \( |\alpha||\beta| \) and again we are done.

Thus we assume that \( p \) is prime to exactly one summand in \( (\sum_{gH} n_g gH + \sum_{gH'} m_g gH') \). Replacing \( \alpha \) by \( g^{-1} \alpha \), we assume that only \( n_H \) is prime to \( p \). It follows that \( H_\alpha \) fixes the trivial coset \( H \) and so \( H_\alpha \leq H \), implying \( K(\alpha) \supseteq K \). Set \( \alpha' = \alpha - n_H H \); thus,
\[ |\alpha'| = \prod_{gH \neq H} \frac{n}{(n, n_g H)}. \]
We know that \( K(\alpha) = [K(\alpha) : K][K : F] \). Write \( s = s_1 + s_2 + s_3 \) where \( p^{s_1} \)
is the exact power of \( p \) dividing \( n \), \( p^{s_2} \) is the exact power dividing \( d_1 \), and \( p^{s_3} \) is the exact power dividing \( e_1 \). Note that \( p^{s_1} \)
divides \( \frac{n}{(n, n_H)} \), and of course \( p^{s_3} \) divides \( [K : F] \). Thus it suffices to show that \( p^{s_2} \) divides
\[ \text{index}(D^{s_2})[K(\alpha, \beta) : K]|\alpha'||\beta|, \]
where
\[ D^{s_2} = (D_1^{-\alpha} \otimes K(\alpha, \beta)) \otimes K(\alpha, \beta)) \otimes K'(\beta) K(\alpha, \beta)). \]
We will prove in fact that \( (D_1^{-\alpha} \otimes K(\alpha, \beta)) \otimes K(\alpha, \beta)) \otimes K'(\beta) K(\alpha, \beta)) \)
is divisible by \( d_1 \). Note that
\[ (D_1^{-\alpha} \otimes K(\alpha, \beta)) = (D_1 \otimes K(\alpha, \beta)) \otimes K(\alpha, \beta)) \otimes K'(\beta) K(\alpha, \beta)). \]
We need to estimate some indices. Of course \( D_1 \) has index \( d_1 \), and so over \( \bar{K} \bar{K}', \)
g\( D_1)^{n_H} \) has index dividing \( \frac{d_1}{(d_1, n_H)} \). Then \( D_1^{\alpha'} \otimes K(\alpha, \beta)) \) has index dividing
\[ \prod_{gH \neq H} \frac{d_1}{(d_1, n_g H)}. \]
Similarly \( D_2^{\beta'} \otimes K'(\beta) K(\alpha, \beta) \) has index dividing \( \prod_{gH'} \frac{d_2}{(d_2, m_H')}. \)
We need a trivial lemma.
Lemma 2.8. Suppose $a$ divides $b$. Then $a/(a,d)$ divides $b/(b,d)$.

Proof. For any prime $p$, the power of $p$ dividing $\frac{a}{(a,d)}$ is less than or equal to the power of $p$ dividing $\frac{b}{(b,d)}$.

Let $p^{u_1}$ be the exact power of $p$ dividing $\text{index}(D_1 \otimes_K K(\alpha, \beta))$ and let

$$D^\# = (D_1' \otimes_{K(\alpha)} K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} (D_2' \otimes_{K'(\beta)} K(\alpha, \beta)).$$

Also let $p^{u_2}$ be the exact power of $p$ dividing $\text{index}(D^\#)$. It follows from Lemma 2.8 that $p^{u_2}$ divides $|\alpha'||\beta|$. Also, $D'' = (D_1'' \otimes_K K(\alpha, \beta)) \otimes_{K(\alpha, \beta)} D^\#$. Now let $p^{u_3}$ be the exact power of $p$ dividing

$$\text{index}(D_1'' \otimes_K K(\alpha, \beta)).$$

This is the same as the exact power of $p$ dividing

$$\text{index}(D_1 \otimes_K K(\alpha, \beta)).$$

Set $p^{u_4}$ to be the exact power of $p$ dividing $[K(\alpha, \beta) : K]$. Then $\text{index}(D'')$ is a multiple of $p^{u_1 - u_2}$. Thus $\text{index}(D'')|\alpha'||\beta|$ is a multiple of $p^{u_1}$ and

$$\text{index}(D'')|\alpha'||\beta|[K(\alpha, \beta) : K]$$

is a multiple of $p^{u_1 + u_4}$ which is the exact power of $p$ dividing

$$\text{index}(D_1 \otimes_K K(\alpha, \beta))[K(\alpha, \beta) : K],$$

a multiple of $d_1$. This proves Theorem 2.7.

Finally let $D_i'/K_i'$ be arbitrary such that both $K_i$ are finitely generated over a prime or algebraically closed field $k'$. Let $k$ be the subfield of elements of $K_1$ and $K_2$ algebraic over $k'$. Then $k/k'$ is a finite field extension. In particular, $k$ is perfect. Then we can write $K_i' \supset F_i$ such that $K_i/F_i$ is finite separable and $F_i/k$ is rational. Let $F = q(F_1 \otimes_k F_2)$, $K_i = K_i' \otimes_{F_i} F$ and $D_i = D_i' \otimes_{K_i'} K_i$. We apply Theorem 2.7 to the $D_i$ and conclude:

Theorem 2.9. Suppose $D_i'/K_i'$ are division algebras with the $K_i'$ finitely generated over a common field $k'$ which is either algebraically closed or the prime field. Then the $D_i'$ can be embedded into a common division algebra $E$ finite over its center.

The question remains: What is the lowest possible bound for $\deg E$?

References

[C] Cohn, P.M. The embedding of firs in skewfields, Proc. London Math. Soc. (3) 23 (1971), 193-213.

[FSS] Fein, B., Saltman, D., and Schacher, M. Embedding problems for finite dimensional division algebras, J. Algebra 167 (1994), 588–626.

[Sa] Saltman, D., The Schur Index and Moody’s Theorem, K-Theory 7: 309–332, 1993