ESSENTIAL DIMENSION OF PROJECTIVE ORTHOGONAL AND SYMPLECTIC GROUPS OF SMALL DEGREE

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In this paper, we study the essential dimension of classes of central simple algebras with involutions of index less or equal to 4. Using structural theorems for simple algebras with involutions, we obtain the essential dimension of projective and symplectic groups of small degree.

Key Words: Central simple algebra; Essential dimension; Involution; Projective orthogonal group; Projective symplectic group.

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1. INTRODUCTION

The essential dimension of an algebraic structure is a numerical invariant which measures the minimum number of parameters required to define the structure. Here, an algebraic structure means a functor from the category of field extensions over a fixed field to the category of sets. There are numerous such examples including $G$-torsors which assigns to a field extension the set of isomorphism classes of $G$-torsors, where $G$ is an algebraic group over a field.

An effective way to compute the essential dimension of $G$-torsors is to use a bijection between isomorphism classes of $G$-toror and isomorphism classes of the related algebras (or structures). For example, $S_n$-torsors is bijective to the set of classes of étale algebras of rank $n$, $O_n$-torsors is bijective to the set of classes of quadratic forms of rank $n$, $PGL_n$-torsors is bijective to the set of classes of central simple algebras of degree $n$, etc. Once we know some information on the algebras or structures, one can estimate the essential dimension based on the bijection. For instance, the essential dimension of $PGL_2$-torsors is less or equal to 2 as any quaternion algebra can be defined by 2 parameters. For details on étale algebras, quadratic forms, and central simple algebras, see [5], [4], [2].

In the present paper, we compute, in a uniform way, upper bounds for the essential dimensions of $PGSp_{2n}$, $PGO_{2n}$, and $PGO^{+}_{2n}$ for odd $n$ (Corollary 3.2), and we determine the essential dimension of $PGSp_{4}$, $PGO_{4}$, and $PGO^{+}_{4}$ (Theorem 3.7). To do this, we use the bijections between torsors under these groups and algebras...
with involution (see Section 2). Together with this, we construct classifying varieties corresponding to these algebras with involution to estimate essential dimension.

A part of our main results concerning $\text{PGSp}_{2n}$ for odd $n$ and $n = 2$ was known before: Corollary 3.2 (i) was obtained in [9] and Theorem 3.7 (iii) can be recovered from the exceptional isomorphism $\text{PGSp}_4 \simeq \text{O}_7^+$ as $\text{ed}(\text{O}_7^+) = 4$ by [11, Theorem 10.3]. However, we would like to emphasize that our approach provides a systematic construction of explicit classifying varieties for these groups including $\text{PGO}_{2n}$ and $\text{PGO}_{2n}^+$ for odd $n$ and $n = 2$.

The rest of the paper is structured as follows. In Section 2, we recall basic definitions and facts concerning simple algebras with involutions including exceptional isomorphisms and essential dimension. In Section 3, we prove the main results: Corollary 3.2 and Theorem 3.7.

2. PRELIMINARIES

In this section we recall some basic results on simple algebras with involution and essential dimension, which will be used throughout this paper.

2.1. Simple Algebras with Involutions

Let $F$ be a field, $A$ a central simple $F$-algebra, and $(\sigma, f)$ a quadratic pair on $A$ (see [8, 5.B]). A morphism of algebras with quadratic pair $\phi : (A, \sigma, f) \rightarrow (A', \sigma', f')$ is an $F$-algebra morphism $\phi : A \rightarrow A'$ such that $\sigma' \circ \phi = \phi \circ \sigma$ and $f \circ \phi = f'$. For any field extension $K/F$, we write $(A, \sigma, f)_K$ for $(A \otimes_F K, \sigma \otimes \text{Id}_K, f_K)$, where $f_K : \text{Sym}(A_K, \sigma_K) \rightarrow K$.

For $n \geq 2$, let $D_n$ denote the category of central simple $F$-algebras of degree $2n$ with quadratic pair, where the morphisms are $F$-algebra isomorphisms which preserve the quadratic pairs and let $A_1^\sigma$ denote the category of quaternion algebras over an étale quadratic extension of $F$, where the morphisms are $F$-algebra isomorphisms. Then, there is an equivalence of groupoids

$$D_2 \equiv A_1^\sigma,$$  \hspace{1cm} (1)

see [8, Theorem 15.7].

Moreover, if we consider the full subgroupoid $1A_1^\sigma$ of $A_1^\sigma$ whose objects are $F$-algebras of the form $Q \times Q'$, where $Q$ and $Q'$ are quaternion algebras over $F$, and the full subgroupoid $1D_2$ of $D_2$ whose objects are central simple algebras over $F$ with quadratic pair of trivial discriminant, then the equivalence in (1) specialize to the following equivalence of subgroupoids:

$$1D_2 \equiv 1A_1^\sigma,$$  \hspace{1cm} (2)

see [8, Corollary 15.12].

For $n \geq 1$, we denote by $C_n$ be the category of central simple $F$-algebras of degree $2n$ with symplectic involution, where the morphisms are $F$-algebra isomorphisms which preserve the involutions.
Example 2.1. Let $\mathcal{F} : \text{Sets} \to \text{Sets}$ be a functor from the category $\text{Sets}$ of sets, and let $p$ be a prime. We denote by $\text{ed}(\mathcal{F})$ and $\text{ed}_p(\mathcal{F})$ the essential dimension and essential $p$-dimension of $\mathcal{F}$, respectively. We refer to [5, Def. 1.2] and [10, Sec. 1] for their definitions. Let $G$ be an algebraic group over $F$. The essential dimension $\text{ed}(G)$ (respectively, essential $p$-dimension $\text{ed}_p(G)$) of $G$ is defined to be $\text{ed}(H^1(\mathcal{F}(E)/G))$ (respectively, $\text{ed}_p(H^1(\mathcal{F}(E)/G))$), where $H^1(E,G)$ is the nonabelian cohomology set with respect to the finitely generated faithfully flat topology (equivalently, the set of isomorphism classes of $G$-torsors) over a field extension $E$ of $F$.

A morphism $\mathcal{F} \to \mathcal{T}$ from $\text{Fields}/F$ to $\text{Sets}$ is called $p$-surjective if for any $E \in \text{Fields}/F$ and any $x \in \mathcal{F}(E)$, there is a finite field extension $L/E$ of degree prime to $p$ such that $x_L \in \text{Im}(\mathcal{F}(L) \to \mathcal{T}(L))$. A morphism of functors $\mathcal{F} \to \mathcal{T}$ from $\text{Fields}/F$ to $\text{Sets}$ is called surjective if for any $E \in \text{Fields}/F$, $\mathcal{F}(E) \to \mathcal{T}(E)$ is surjective. Obviously, any surjective morphism is $p$-surjective for any prime $p$. Such a surjective morphism gives an upper bound for the essential ($p$)-dimension of $\mathcal{F}$ and a lower bound for the essential (p)-dimension of $\mathcal{F}$,

$$\text{ed}(\mathcal{F}) \leq \text{ed}(\mathcal{T}) \quad \text{and} \quad \text{ed}_p(\mathcal{F}) \leq \text{ed}_p(\mathcal{T});$$

see [5, Lemma 1.9] and [10, Proposition 1.3]. In particular, if such a functor $\mathcal{F}$ is represented by a scheme over $F$, then it is called a classifying scheme for $\mathcal{F}$.

2.2. Essential Dimension

Let $\mathcal{F} : \text{Fields}/F \to \text{Sets}$ be a functor from the category $\text{Fields}/F$ of field extensions over $F$ to the category $\text{Sets}$ of sets, and let $p$ be a prime. We denote by $\text{ed}(\mathcal{F})$ and $\text{ed}_p(\mathcal{F})$ the essential dimension and essential $p$-dimension of $\mathcal{F}$, respectively. We refer to [5, Def. 1.2] and [10, Sec. 1] for their definitions. Let $G$ be an algebraic group over $F$. The essential dimension $\text{ed}(G)$ (respectively, essential $p$-dimension $\text{ed}_p(G)$) of $G$ is defined to be $\text{ed}(H^1(\mathcal{F}(E)/G))$ (respectively, $\text{ed}_p(H^1(\mathcal{F}(E)/G))$), where $H^1(E,G)$ is the nonabelian cohomology set with respect to the finitely generated faithfully flat topology (equivalently, the set of isomorphism classes of $G$-torsors) over a field extension $E$ of $F$.

A morphism $\mathcal{F} \to \mathcal{T}$ from $\text{Fields}/F$ to $\text{Sets}$ is called $p$-surjective if for any $E \in \text{Fields}/F$ and any $x \in \mathcal{F}(E)$, there is a finite field extension $L/E$ of degree prime to $p$ such that $x_L \in \text{Im}(\mathcal{F}(L) \to \mathcal{T}(L))$. A morphism of functors $\mathcal{F} \to \mathcal{T}$ from $\text{Fields}/F$ to $\text{Sets}$ is called surjective if for any $E \in \text{Fields}/F$, $\mathcal{F}(E) \to \mathcal{T}(E)$ is surjective. Obviously, any surjective morphism is $p$-surjective for any prime $p$. Such a surjective morphism gives an upper bound for the essential ($p$)-dimension of $\mathcal{F}$ and a lower bound for the essential (p)-dimension of $\mathcal{F}$,

$$\text{ed}(\mathcal{F}) \leq \text{ed}(\mathcal{T}) \quad \text{and} \quad \text{ed}_p(\mathcal{F}) \leq \text{ed}_p(\mathcal{T});$$

see [5, Lemma 1.9] and [10, Proposition 1.3]. In particular, if such a functor $\mathcal{F}$ is represented by a scheme over $F$, then it is called a classifying scheme for $\mathcal{F}$.

Example 2.1. Let $(M_2(F), \gamma) \in C_1$, where $\gamma$ is the canonical involution on $M_2(F)$. As $(M_2(K), \gamma_K) \simeq (M_2(F), \gamma) \otimes K$ for any field extension $K/F$, we have $\text{ed}((M_2(F), \gamma)) = 0$.

Assume that $\text{char}(F) \neq 2$. The exact sequence

$$1 \to \mu_2 \to \text{Sp}_2 \to \text{PGSp}_2 \to 1$$

induces the connecting morphism $\hat{\partial} : H^1(\mathcal{F}(E)/G_2) \to \text{Br}_2(\mathcal{F}(E)/G_2)$ which sends a pair $(Q, \gamma)$ of a quaternion algebra with canonical involution to the Brauer class $[Q]$. As this morphism is nontrivial, by [5, Corollary 3.6] we have $\text{ed}((\text{PGSp}_2)) \geq 2$ (or by Lemma 3.5). Consider the morphism $G_2^0 \to C_2$ defined by $(x,y) \mapsto ((x,y), \gamma)$, where $(x,y)$ is a quaternion algebra and $\gamma$ is the canonical involution. As this morphism is surjective, by (5) we have $\text{ed}(C_2) \leq 2$, thus $\text{ed}((\text{PGSp}_2)) = 2$. This can be recovered from the exceptional isomorphism $\text{PGSp}_2 \simeq O^+_3$. 
3. ESSENTIAL DIMENSION OF PROJECTIVE ORTHOGONAL AND SYMPLECTIC GROUPS ASSOCIATED TO SIMPLE ALGEBRAS OF INDEX \( \leq 4 \)

The aim of this section is to provide upper bounds for the essential dimension of projective orthogonal and symplectic groups of degree \( 2n \), where \( n \) is odd (Corollary 3.2), and to compute the essential dimension of projective orthogonal and symplectic groups of degree 4 (Theorem 3.7).

First, we compute upper bounds for the essential dimension of certain classes of simple algebras with involutions of index less or equal to 2 in the following proposition. For any field extension \( K/F \) and any integer \( n \geq 3 \), we write \( \text{QH}^+_n(K) \) (respectively, \( \text{QH}^-_n(K) \)) for the set of isomorphism classes of objects of the form \((M_n(Q), \sigma_h)\),

where \( Q \) is a quaternion algebra over \( K \) together with its canonical involution \( \gamma \), and where \( \sigma_h \) is the adjoint involution with respect to a nondegenerate hermitian form (respectively, skew-hermitian form) \( h \) on \( Q^n \), with respect to \( \gamma \). Similarly, if \( n \) is odd, then we write \( \text{QH}^+_n(K) \) for the set of isomorphism classes of objects of the form \((M_n(Q), \sigma_h)\), where \( \sigma_h \) is the adjoint involution with respect to a nondegenerate skew-hermitian form \( h \) on \( Q^n \), with respect to \( \gamma \) such that \( \text{disc}(\sigma_h) = 1 \).

**Proposition 3.1.** Let \( F \) be a field and \( n \geq 3 \) any integer.

1. \( \text{ed}(\text{QH}^+_n) \leq n + 1 \).
2. \( \text{ed}(\text{QH}^-_n) \leq 3n - 3 \) if \( \text{char}(F) \neq 2 \).
3. \( \text{ed}(\text{QH}^+_n) \leq 3n - 4 \) if \( \text{char}(F) \neq 2 \).

**Proof.** (i) If \( h \) is a hermitian form, then by [7, Proposition (6.2.4)] it can be diagonalized, \( h = \langle t_1, t_2, \ldots, t_n \rangle \) for some \( t_i \in F \). We consider the affine variety

\[
X = \begin{cases} 
\mathbb{G}_m^2 \times \mathbb{A}^{n-1}_F & \text{if } \text{char}(F) \neq 2, \\
\mathbb{G}_m \times \mathbb{A}^n_F & \text{if } \text{char}(F) = 2,
\end{cases}
\]

and define a morphism \( X(K) \rightarrow \text{QH}^+_n(K) \) by

\[
(a, b, t_1, \ldots, t_{n-1}) \mapsto \begin{cases} 
((a, b) \otimes M_n(K), \sigma_{(t_1, \ldots, t_{n-1})}) & \text{if } \text{char}(F) \neq 2, \\
([a, b] \otimes M_n(K), \sigma_{(t_1, \ldots, t_{n-1})}) & \text{if } \text{char}(F) = 2,
\end{cases}
\]

where \( (a, b) \) and \([a, b]\) are quaternion algebras. As a scalar multiplication does not change the adjoint involution, this morphism is surjective. Therefore, by (5), we have \( \text{ed}(\text{QH}^+_n) \leq n + 1 \).

(ii) From now we assume that \( \text{char}(F) \neq 2 \). If \( h \) is a skew-hermitian form, then by [7, Proposition (6.2.4)] it can be diagonalized, \( h = \langle q_1, q_2, \ldots, q_n \rangle \) for some pure quaternions \( q_i \in Q \). Consider the variety \( Y = \mathbb{G}_m^2 \times \mathbb{A}^1 \times \mathbb{A}^{3n-2} \) with coordinates \((a, b, c, t_1, \ldots, t_{3n-6})\) and the conditions

\[
ac^2 + b \neq 0, \quad at_{3k-2}^2 + bt_{3k-1}^2 - abt_{3k}^2 \neq 0 \quad \text{for all } 1 \leq k \leq n - 2.
\]
Define a morphism \( \phi_K : Y(K) \to \text{QH}_n^-(K) \) by
\[
(a, b, c, t_1, \ldots, t_{3n-6}) \mapsto ((a, b) \otimes M_4(K), \sigma_h),
\]
where \( p = i, \ q = ci + j, \ r_k = t_{3k-2}i + t_{3k-1}j + t_{3k}ij \) for \( 1 \leq k \leq n - 2 \), and \( h = \langle p, q, r_1, \ldots, r_{3n-2} \rangle \).

We show that \( Y \) is a classifying variety for \( \text{QH}_n^- \). Suppose that we are given a quaternion \( K \)-algebra \( Q \) and a skew hermitian form \( h = \langle p, q, r_1, \ldots, r_{n-2} \rangle \) for some pure quaternions \( p, q, r_k \). We may assume that \( p, q, r_k \) are not a scalar multiple of one of the quaternions \( p, q, r_k \) since otherwise we can find a classifying variety of dimension smaller than \( Y \) in the same way (i.e., the dimension of the classifying scheme is determined by the dimension of \( Y \)). By reordering of \( p, q, r_k \), we can find a scalar \( c \in K \) such that \( p \) and \( q - cp \) anticommute, thus they are orthogonal. Hence, we have \( Q \simeq \langle a, b \rangle \), where \( p \) and \( q - cp \) are new generators of \( Q \) such that \( p^2 = a \) and \( (q - cp)^2 = b \). For \( 1 \leq k \leq n - 2 \), let
\[
r_k = t_{3k-2}p + t_{3k-1}(q - cp) + t_{3k}p(q - cp)
\]
with \( t_1, t_2, \ldots, t_{3n-6} \in K \). Then \( (M_4(Q), \sigma_h) \simeq (M_4(\langle a, b \rangle), \sigma_h) \) is the image of \( \phi_K \).

Therefore, by (5), we have \( \text{ed}(\text{QH}_n^-) \leq 3n - 3 \).

(iii) Assume that \( n = 2m + 1 \) for \( m \geq 1 \). We consider the variety \( Y \) in (ii) with an additional condition
\[
-a(ac^2 + b) \prod_{k=1}^{m-1} at_{3k-2}^2 + bt_{3k-1}^2 - abt_{3k}^2 = \prod_{k=m}^{n-2} at_{3k-2}^2 + bt_{3k-1}^2 - abt_{3k}^2.
\]
We show that this variety with the same morphism \( \phi_K \) in (ii) is a classifying variety for \( \text{QH}_n^- \). Suppose that we are given a quaternion \( K \)-algebra \( Q \) and a skew hermitian form \( h = \langle p, q, r_1, \ldots, r_{n-2} \rangle \) for some pure quaternions \( p, q, r_k \). We do the same procedure as in (ii), so that we have \( (M_4(Q), \sigma_h) \simeq (M_4(\langle a, b \rangle), \sigma_h) \) and (6), where \( p^2 = a \) and \( (q - cp)^2 = b \). As \( \text{disc}(\sigma_h) = 1 \), there is a scalar \( d \in K \) such that
\[
-p^2q^2r_1^2 \cdots r_{m-1}^2 = \left( \frac{d}{r_m^2 \cdots r_{n-2}^2} \right)^2 r_m^2 \cdots r_{n-2}^2.
\]
We set \( f = d/r_m^2 \cdots r_{n-2}^2 \). As a scalar multiplication does not change the adjoint involution, we can modify \( h \) by the scalar \( f \). As \( (a, b) \simeq (f^2a, f^2b) \), \( (M_4(Q), \sigma_h) \) is the image of \( \phi_K \). Therefore, by (5), we have \( \text{ed}(\text{QH}_n^-) \leq 3n - 4 \).

Assume that \( n \) is odd. Then we have
\[
\text{QH}_n^- = C_n, \quad \text{QH}_n^+ = D_n, \quad \text{and} \quad \text{QH}_n^- = 1D_n.
\]
Hence, by [8, Theorem 4.2] and the exceptional isomorphism \( \text{PGO}_6 \simeq \text{PGU}_4 \), we have the following corollary.

**Corollary 3.2.** Assume that \( n \geq 3 \) is odd.

(i) \( \text{ed}(\text{PGSp}_{2n}) \leq n + 1 \).
(ii) \( \mathsf{ed}(\mathsf{PGO}_{2n}) \leq 3n - 3 \) if \( \mathsf{char}(F) \neq 2 \). In particular, \( \mathsf{ed}(\mathsf{PGU}_4) \leq 6 \).

(iii) \( \mathsf{ed}(\mathsf{PGO}_{2n}) \leq 3n - 4 \) if \( \mathsf{char}(F) \neq 2 \).

**Remark 3.3.** In fact, \( \mathsf{ed}(\mathsf{PGSp}_{2n}) = \mathsf{ed}(\mathsf{PGSp}_{2n}) = n + 1 \) for \( n \geq 3 \) odd and \( \mathsf{char}(F) \neq 2 \); the lower bound was obtained by Chernousov and Serre in [6, Theorem 1], and the exact value was obtained by Macdonald in [9, Proposition 5.1].

We shall need the following three lemmas to prove Theorem 3.7.

**Lemma 3.4 ([3, Section 2.6]).** Let \( F \) be a field of characteristic different from 2. Then

\[
\mathsf{ed}_2(\mathsf{PGL}^{\times n}_2) = \mathsf{ed}(\mathsf{PGL}^{\times n}_2) = 2n.
\]

**Proof.** By [5, Lemma 1.11], we have

\[
\mathsf{ed}_2(\mathsf{PGL}^{\times n}_2) \leq \mathsf{ed}(\mathsf{PGL}^{\times n}_2) \leq n \cdot \mathsf{ed}(\mathsf{PGL}_2) = 2n.
\]

On the other hand, the natural morphism

\[
H^1(-, \mathsf{PGL}^{\times n}_2) \to \mathsf{Dec}_{2n}(-)
\]

is surjective, where \( \mathsf{Dec}_{2n}(K) \) is the set of all decomposable algebras of degree \( 2^n \) over a field extension \( K/F \), hence, by (5) and [3, Section 2.6], we have

\[
2n = \mathsf{ed}_2(\mathsf{Dec}_{2n}) \leq \mathsf{ed}_2(\mathsf{PGL}^{\times n}_2).
\]

**Lemma 3.5.** Let \( F \) be a field of characteristic different from 2. Then

\[
\mathsf{ed}_2(\mathsf{PGO}_2^r), \mathsf{ed}_2(\mathsf{PGSp}_2^r) \geq \begin{cases} 
2 & \text{if } r = 1, \\
4 & \text{if } r = 2, \\
(r - 1)2^{r - 1} & \text{if } r \geq 3.
\end{cases}
\]

**Proof.** Consider the forgetful functors

\[
H^1(-, \mathsf{PGO}_2^r) \to \mathsf{Alg}_{2^r, 2}
\]

and

\[
H^1(-, \mathsf{PGSp}_2^r) \to \mathsf{Alg}_{2^r, 2},
\]

where \( \mathsf{Alg}_{2^r, 2}(K) \) is the set of isomorphism classes of simple algebras of degree \( 2^n \) and exponent dividing 2 over a field extension \( K/F \). These functors are surjective by a theorem of Albert.

It is well known that \( \mathsf{ed}_2(\mathsf{Alg}_{2^2, 2}) = 2 \), \( \mathsf{ed}_2(\mathsf{Alg}_{2^r, 2}) = 4 \). For \( r \geq 3 \), we have \( \mathsf{ed}_2(\mathsf{Alg}_{2^r, 2}) \geq (r - 1)2^{r - 1} \) by [4, Theorem 1.1]. Therefore, by (5), we have the above lower bound for \( \mathsf{ed}_2(\mathsf{PGO}_2^r) \) and \( \mathsf{ed}_2(\mathsf{PGSp}_2^r) \). □
The following Lemma 3.6 (i) was proved by Rowen in [12, Theorem B] (see also [8, Proposition 16.16]), and Lemma 3.6 (ii) was proved by Serhir and Tignol in [13, Proposition]. However, we shall need the explicit forms of involutions on the decomposed quaternions as below.

**Lemma 3.6.** Let \( F \) be a field of characteristic different from 2. Let \((A, \sigma)\) be a central simple \( F \)-algebra of degree 4 with a symplectic involution \( \sigma \).

(i) If \( A \) is a division algebra, then we have

\[
(A, \sigma) \simeq (Q, \sigma|_Q) \otimes (Q', \gamma),
\]

where \( \sigma|_Q \) is an orthogonal involution defined by \( \sigma|_Q(x_0 + x_1 i + x_2 j + x_3 k) = x_0 + x_1 i + x_2 j - x_3 k \) with a quaternion basis \((1, i, j, k)\) for \( Q \) and \( \gamma \) is the canonical involution on a quaternion algebra \( Q' \).

(ii) If \( A \) is not a division algebra, then we have

\[
(A, \sigma) \simeq (M_2(F), \text{ad}_q) \otimes (Q', \gamma),
\]

where \( q \) is a 2-dimensional quadratic form, \( \text{ad}_q \) is the adjoint involution on \( M_2(F) \), and \( \gamma \) is the canonical involution on a quaternion algebra \( Q' \).

**Proof.** (i) By [12, Proposition 5.3], we can choose \( i \in A \setminus F \) such that \( \sigma(i) = i \) and \( [F(i) : F] = 2 \). Let \( \phi \) be the nontrivial automorphism of \( F(i) \) over \( F \). By [12, Proposition 5.4], there is a \( j \in A \setminus F \) such that \( \sigma(j) = j \) and \( ji = \phi(i)j \). Then \( i \) and \( j \) generate a quaternion algebra \( Q \), \( \sigma|_Q(i) = i \), \( \sigma|_Q(j) = j \), and \( \sigma|_Q(k) = -k \) with \( k = ij \). Hence, \( \sigma|_Q \) is an orthogonal involution on \( Q \). By the double centralizer theorem, we have \( A \simeq Q \otimes C_A(Q) \), where \( C_A(Q) \) is the centralizer of \( Q \subset A \) and is isomorphic to quaternion algebra \( Q' \) over \( F \). By [8, Proposition 2.23], the restriction of \( \sigma \) on \( Q' \) is the canonical involution \( \gamma \).

(ii) See [13, Proposition]. \( \square \)

Now we apply Lemmas 3.4, 3.5, and 3.6 to prove the main result on projective orthogonal and symplectic groups of degree 4.

**Theorem 3.7.** Let \( F \) be a field of characteristic different from 2.

(i) \( \text{ed}_2(\text{PGO}_4^+) = \text{ed}(\text{PGO}_4^+) = 4 \).

(ii) \( \text{ed}_2(\text{PGO}_4) = \text{ed}(\text{PGO}_4) = 4 \).

(iii) \( \text{ed}_2(\text{PGSp}_4) = \text{ed}(\text{PGSp}_4) = 4 \).

**Proof.** (i) By the exceptional isomorphism (2), we have

\[
\text{PGO}_4^+ = \text{PGL}_2 \times \text{PGL}_2.
\]

The proof follows from Lemma 3.4 with \( n = 2 \).
(ii) By Lemma 3.5, we have $\text{ed}_2(\text{PGO}_4) \geq 4$. For the opposite inequality, we consider the affine variety $X$ defined in $\mathbb{A}^4$ with the coordinates $(a, b, c, e)$ by $e(a^2 - b^2e)(c^2 - e) \neq 0$. Define a morphism $X \to \mathbb{A}^4_1$ by

$$(a, b, c, e) \mapsto (a + bt, c + t),$$

where the one on the right is a quaternion algebra over the étale quadratic extension $F[t]/(t^2 - e)$.

We show that $X$ is a classifying variety for $\mathbb{A}^4_1$. Let $Q = (a + bt, c + dt)$ be a quaternion algebra over an étale quadratic extension $F[t]/(t^2 - e)$. If $d = 0$, we can modify $c$ by $c(1 + t)^2$ as $Q \simeq (a + bt, c(1 + t)^2)$. Hence, we may assume that $d \neq 0$. Similarly, we can assume that $d = 1$, replacing $e$ by $ed^2$. Thus, the morphism $X \to \mathbb{A}^4_1$ is surjective. By (5), we have $\text{ed}(\mathbb{A}^4_1) \leq 4$. Hence, the opposite inequality $\text{ed}(\text{PGO}_4) \leq 4$ comes from the exceptional isomorphism (1) and the canonical bijection (3).

(iii) By Lemma 3.5, we have $\text{ed}_2(\text{PGSp}_4) \geq 4$. For the opposite inequality, we define a morphism $\mathbb{G}_m^4 \to C_2$ by

$$(x, y, z, w) \mapsto ((x, y), \sigma) \otimes ((z, w), \gamma),$$

where $\sigma$ is an involution defined by $\sigma(x_0 + x_1i + x_2j + x_3k) = x_0 + x_1i + x_2j - x_3k$ with a quaternion basis $(1, i, j, k)$ of the quaternion algebra $(x, y)$, and $\gamma$ is the canonical involution on the quaternion algebra $(z, w)$. By Lemma 3.6 (i), this morphism is surjective for any division algebra with symplectic involution $(A, \sigma)$ in $C_2$. If $A$ is not a division algebra, then by using Lemma 3.6 (ii) we can define a classifying variety in the same way. In this case, we have a variety of dimension smaller than 4. Therefore, by (5), we have $\text{ed}(C_2) \leq 4$, and hence the result follows from the canonical bijection (4).

**Remark 3.8.** Assume that $F$ is a field of characteristic 2. By [1, Corollary 2.2], we have $\text{ed}_2(\text{Alg}_{4,2}) = \text{ed}_2(\text{Dec}_4) \geq 3$. As the morphisms (7) and (8) are surjective, we get $\text{ed}_2(\text{PGO}^-_4) \geq 3$ and $\text{ed}_2(\text{PGO}_4) \geq 3$, respectively. On the other hand, the upper bounds in Theorem 3.7 (i) and (ii) still hold, and hence $3 \leq \text{ed}(\text{PGO}^-_4)$, $\text{ed}(\text{PGO}_4) \leq 4$.

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