Examples of naked singularity formation
in higher-dimensional Einstein-vacuum spacetimes

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Abstract

The vacuum Einstein equations in 5+1 dimensions are shown to admit solutions describing
naked singularity formation in gravitational collapse from nonsingular asymptotically locally flat
initial data. We present a class of specific examples with spherical extra dimensions. Thanks to
the Kaluza-Klein dimensional reduction, these spacetimes are constructed by lifting continuously
self-similar solutions of the 4-dimensional Einstein-scalar field system with a negative exponential
potential. Their existence provides a new test-bed for weak cosmic censorship in higher-dimensional
gravity. In addition, we point out that a similar attempt of embedding Christodoulou’s well-
known solutions for massless scalar fields fails to capture formation of naked singularities in 4+1
dimensions, due to a diverging Kretschmann scalar in the initial data.

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I. INTRODUCTION

In the process of gravitational collapse, singularities can emerge under a broad range of circumstances. They are either covered by trapped surfaces (black holes) or being visible to far-away observers (naked singularities). Regarding these two outcomes, the weak cosmic censorship conjecture \cite{1, 2} claims that generically all singularities should be hidden inside black holes. Despite huge effort, the validity of this conjecture still remains one of the most important open problems in classical general relativity (see recent reviews \cite{2–6}). Nevertheless, from a different perspective, significant progress has been made on the mathematical studies of formation of trapped surfaces in vacuum spacetimes without symmetry assumption, notably in the recent monumental work by Christodoulou \cite{7} and related development \cite{8–15}.

In parallel to advancement in 4 dimensions, string theory and brane-world models \cite{16} have greatly promoted the study of general relativity in higher dimensions. By entering dimensions higher than 4, novel topologies and horizon geometries can arise, generating a large variety of black objects \cite{17, 18}. Particularly on the dynamical aspect of the vacuum Einstein equations, Choptuik’s type of critical phenomenon \cite{19, 20} was discovered in collapse of 5-dimensional gravitational waves via numerical simulation \cite{21}. The spacetime has the symmetry of deformed 3-spheres (Bianchi IX) so that Birkhoff’s theorem can be evaded. At the threshold of black hole formation, the critical evolution is expected to give rise to a naked singularity. Also in 5 dimensions, the Gregory-Laflamme instability \cite{22} of black strings has been shown numerically to result in naked singularities through a process of self-similar cascading \cite{23}. Clearly, higher-dimensional spacetimes allow more possibilities that cosmic censorship could be challenged.

In searching of naked singularities, a powerful way to construct solutions in higher dimensions is by dimensional reduction \cite{24, 25}. Originally proposed by Kaluza and Klein, the procedure requires that all coordinate dependence of extra dimensions be dropped. The spacetime is assumed to be the product of a 4-dimensional Lorentzian manifold and a homogeneous space representing extra dimensions. With this approach, the dimensional reduction on the vacuum Einstein equations leads to 4-dimensional Einstein gravity coupled with a scalar field with or without a negative exponential potential, depending on the geometry of the extra dimensions. Under spherical symmetry, the Einstein-scalar field system without a
potential (Klein-Gordon) has been thoroughly investigated by Christodoulou. His analysis of \([26]\) established the first example of naked singularity formation from regular asymptotically flat initial data with a suitable matter field (see also \([27]\)). This result has been recently extended to include the negative exponential potential (Liouville type) by two of us \([28]\), and new examples that bear similar strength to Christodoulou’s original one were constructed.

In this paper, we will consider the uplift of these 4-dimensional naked singularity solutions and decide whether or not their higher-dimensional descendants can represent formation of naked singularity in vacuum gravitational collapse. In particular, we will point out certain complications (loss of regularity) that such inheritance can be broken. Our objective is to show the following main result:

*In 5+1 dimensional spacetimes with the spatial symmetry of \(S^2 \times S^2\) orthogonal to a radial direction, for the vacuum Einstein equations \(\hat{R}_{\mu \nu} = 0\), there exist nonsingular asymptotically locally flat initial data that lead to naked singularity formation.*

The paper is organized as follows. In Sec. II, we present our higher-dimensional metric and review properties of the solutions to be embedded. For the negative exponential potential, we examine the 6-dimensional solutions in Sec. III from the aspects of apparent horizons, the Kretschmann scalar, asymptotic behavior, and homothetic symmetry. Sec. IV comments on a similar treatment of Christodoulou’s solutions in 5 dimensions. Concluding remarks are made in Sec. V. We use the same notations of \([28]\) throughout the paper. For the conceptual framework involved in this work, one may refer to \([29]\) (Chs. 1 & 2), which contains a quick survey of general relativity from the point of view of the Cauchy problem.

### II. THE LIFTED SOLUTIONS

We start with the following warped product metric (hatted) in \(4 + n\) dimensions with a parameter \(\kappa\):

\[
\begin{align*}
\hat{d}s^2 &= \exp \left( 2 \sqrt{\frac{n}{n+2} \phi} \right) d s^2 + \exp \left( - \frac{4}{\sqrt{n(n+2)} \phi} \right) d s_{n}^2, \\
&= d\hat{s}^2 = -g(x)\hat{g}(x) du^2 - 2g(x) du dr + r^2 d\Omega^2, \\
\phi(u, r) &= \hat{h}(x) - \kappa \ln(-u), \quad \kappa > 0, \quad x = -\frac{r}{u} \geq 0, \quad u < 0.
\end{align*}
\]
Here the 4-dimensional metric $ds^2$ (unhatted) is written in retarded Bondi coordinates, and $d\Omega^2$ denotes the metric of a unit 2-sphere and $ds_n^2$ a compactifying space $K$ having a constant Ricci scalar $R_n = -4V_0$ with $V_0 = 0$ or $-1$. For simplicity, we assume $K = S^n$ or $T^n$. By virtue of consistent Kaluza-Klein dimensional reduction on $ds_n^2$ [24, 25, 30–32], one can verify that the vacuum Einstein equations for (1) correspond to the equations of motion from an effective action on the metric $ds^2$ and the scalar field $\phi$ (dilaton):

$$L = \sqrt{-g}\left(\frac{R}{4} - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right), \quad V = V_0 \exp\left(\frac{2\phi}{\kappa}\right),$$

(4)

$$\kappa = \begin{cases} > 0, & \text{if } V_0 = 0 \quad (R_n = 0, K = T^n), \\ \sqrt{\frac{n}{n+2}} < 1, & \text{if } V_0 = -1 \quad (R_n = 4, K = S^n) \end{cases}.$$  

(5)

Hence the effect of the extra dimensions is tantamount to actuating a scalar field in 4-dimensional Einstein gravity, of which the potential $V(\phi)$ depends on the choice of $K$. Introducing new unknown variables ($' = d/dx$)

$$y(s) = \frac{\bar{g}}{g}, \quad \frac{\zeta(s)}{2(1 - \zeta(s))} = \frac{x}{\bar{g}}, \quad \gamma(s) = x\bar{h}', \quad x = e^s \geq 0,$$

(6)

we can reduce the Einstein equations to a 3-dimensional autonomous system of first-order ordinary differential equations ($' = d/ds = x d/dx$) [28]:

$$\dot{y} = y(\gamma + \kappa)(3\gamma\zeta - 2\gamma + \kappa\zeta) \frac{1 - \zeta}{1 - \zeta},$$

(7)

$$\dot{\zeta} = -\left[2(\gamma + \kappa)^2 + 1 - \kappa^2\right] \zeta^2 + [((\gamma + \kappa)^2 + 1 - \kappa^2)] \zeta,$$

(8)

$$\dot{\gamma} = \gamma^3 + (\kappa^{-1} + 2\kappa) \gamma^2 + \left(1 - \frac{\kappa^2\zeta}{1 - 2\zeta}\right) \gamma - \frac{1 - \zeta}{\kappa(1 - 2\zeta)} \left(1 - \frac{1}{y}\right).$$

(9)

Additionally if $V_0 \neq 0$, the function $\bar{h}$ is subject to

$$(\gamma + \kappa)^2 \frac{1 - 2\zeta}{1 - \zeta} = 1 + \kappa^2 - \frac{1 - 2V_0x^2e^{2\bar{h}/\kappa}}{y}.$$ 

(10)

It is straightforward to verify that (7–10) together imply the equation $\gamma = x\bar{h}'$ in (6). If $V_0 = 0$ as in the case of massless scalar fields ($K = T^n$), the equation (10) without $\bar{h}$ then serves as a constraint to the system (7–9), generating a 2-dimensional submanifold in the phase space.

With a combined analytical and numerical approach, the dynamical system (7–9) and the associated 4-dimensional metric (2) have been systematically studied in [28], which extends the earlier results by Christodoulou [26] and Brady [27] on the special case of massless scalar
fields. It is established in 4 dimensions that solutions describing naked singularity formation (see Fig. 1 and 2) are represented by the integral curves of (7-9) that run continuously from the initial starting point

\[ O : \ y = 1, \ \ \zeta = 0, \ \ \gamma = 0, \ \ s = -\infty, \] (11)

to the ending points (forming a line)

\[ P_0(y_0) : \ y = y_0 \begin{cases} = \frac{1}{1+\kappa^2} & \text{if } V_0 = 0, \\ > \frac{1}{1+\kappa^2} & \text{if } V_0 = -1, \end{cases} \ \ \zeta = 1, \ \ \gamma = -\kappa, \ \ s = +\infty. \] (12)

The condition (11) is imposed by regularity at the symmetry center \( r = 0, \ u < 0 \) \((x = 0)\). Requiring the solution curves to terminate at the line \( P_0(y) \) ensures that the future light cone \( u = 0 \) of the curvature singularity can escape from the center. Lack of an apparent horizon in the spacetime is guaranteed by

\[ g^{\mu\nu} r_{,\mu} r_{,\nu} = y(s) \neq 0, \ \ -\infty \leq s \leq +\infty. \] (13)

Fulfilling all these conditions, a 2-parameter family of naked singularity solutions can exist provided

\[ 0 < \kappa^2 < 1/3 \ \ \text{for } V_0 = 0, \] (14)

or

\[ 0 < \kappa^2 < 1 \ \ \text{for } V_0 = -1. \] (15)

Note that among these two possibilities, \( \kappa \) is a free parameter in the former but fixed by the potential (4) in the latter (cf. (5)). Furthermore, for each solution in the family, there lies a midpoint \( s = s_\ast \) such that the integral curve reaches a singular point of the ODEs:

\[ P_s(y_s) : \ y = y_s \begin{cases} = \frac{1}{1+\kappa^2} & \text{if } V_0 = 0, \\ > \frac{1}{1+\kappa^2} & \text{if } V_0 = -1, \end{cases} \ \ \zeta = \frac{1}{2}, \ \ \gamma = \frac{1 - y_s}{y_s \kappa^2}, \ \ s = s_\ast. \] (16)

It bears the significance that the past light cone of the central singularity is given by \(-r/u = x_\ast = \exp(s_\ast)\). By identifying the asymptotic behavior of the solutions as \( s \to +\infty \), one can deduce that in the limit \( u \to 0 \), a curvature singularity forms at the center \( \{26, 28\} \) with the Ricci scalar obeying

\[ R \to \frac{1}{r^2} \left[ 2(y_0 - 1) \left( \frac{1}{\kappa^2} - 2 \right) - 4y_0 \kappa^2 \right], \ \ \text{as } u \to 0 \ (x \to +\infty). \] (17)
Meanwhile, the scalar field diverges logarithmically at \( r = 0 \):

\[
\phi \rightarrow \begin{cases} 
-\kappa \ln r & \text{if } V_0 = 0, \\
-\kappa \ln r + \frac{\kappa}{2} \ln \left( \frac{1 - (1 + \kappa^2) y_0}{2V_0} \right) & \text{if } V_0 = -1,
\end{cases}
\] as \( u \to 0 \).

(18)

In future sections, we will determine whether this basic picture of 4-dimensional naked singularity spacetimes as illustrated in Fig. 1 may carry over to higher dimensions.

![Spacetime diagram of the naked singularity formation near the symmetry center.](image)

**FIG. 1.** A spacetime diagram of the naked singularity formation near the symmetry center. The self-similar spacetime is obtained by solving the autonomous system (7-9) from \( x = 0 \) to \( x = +\infty \) (the coordinate line \( x = -r/u \) sweeping from \( r = 0 \) to \( u = 0 \)). The line \( x = x^* \) \( (x = +\infty) \) marks the past (future) light cones of the central curvature singularity at \( r = 0 \), \( u = 0 \). There is no apparent horizon. The curvature invariants (the Ricci and Kretschmann scalars) are well-behaved before \( u = 0 \) at the center and beyond.

**III. NAKED SINGULARITY FORMATION IN 6 DIMENSIONS**

Returning to the metric (11), we first consider the uplift of the 4-dimensional naked singularity solutions with \( V_0 = -1 \). We will show that the resulting spacetimes inherit major properties of the lower-dimensional ones, and thus can represent formation of naked singularities as well (cf. Fig. 1). To make our calculation concrete, we focus on the simplest case.
FIG. 2. Numerical integration of the autonomous system \((7-9)\) with \(V_0 = -1\) and \(\kappa = 1/\sqrt{2}\). Integral curves from the initial point \(O\) \((x = 0)\) first reach the singular curve \(P_s(y)\) \((x = x_*, \zeta = 1/2)\), and then they bifurcate and continue to move towards the line \(P_0(y)\) \((x = +\infty)\). The plot shows such bifurcation drawn from one sampling point on \(P_s(y)\).

with \(\mathcal{K} = S^2, \kappa = 1/\sqrt{2}\) and take

\[
ds^2_{n=2} = \frac{1}{2} d\bar{\Omega}^2
\]

with \(d\bar{\Omega}^2\) the metric of a unit 2-sphere. An extra factor \(1/2\) appears above so that the Ricci scalar of \(\mathcal{K}\) follows \(R_{n=2} = 4\) (cf. (5)). Under the assumptions, the metric (1) acquires the spatial symmetry of \(S^2 \times S^2\).

A. Apparent horizon

The radial null geodesics of the metrics (1) and (2) are both determined by \(u = \text{const}\) for outgoing null rays and

\[
\frac{dr}{du} = -\frac{\bar{g}}{2},
\]

for incoming null rays (for construction of double-null coordinates, see [26], Sec. 4). They correspond respectively to the future-directed null vector fields \(l\) and \(n\):

\[
l = \sqrt{\frac{g}{\bar{g}}} \partial_r, \quad n = \frac{2}{\sqrt{g\bar{g}}} \partial_u - \frac{g}{\bar{g}} \partial_r, \quad ds^2(l, n) = -2,
\]
which are normal to a 4-dimensional spacelike surface \( \Sigma = S^2 \times S^2 \) parameterized by \( r, u = \text{const} \). We introduce the null second fundamental forms \( \chi \) and \( \chi \) of the surface \( \Sigma \), which are defined by

\[
\chi(X, Y) := d \hat{s}^2(\nabla_X n, Y), \quad \chi(X, Y) := d \hat{s}^2(\nabla_X l, Y),
\]

for two arbitrary vectors \( X \) and \( Y \) tangent to \( \Sigma \). Their traces measure the expansions of the two radial null geodesic congruences along \( l \) and \( n \). In this setting, an apparent horizon (marginally outer trapped surface [33–37]) consists of a collection of \( \Sigma \)'s such that

\[
\text{tr} \chi \bigg|_{\Sigma} = 0, \quad \text{tr} \chi \bigg|_{\Sigma} \leq 0.
\]

Thus it boils down to calculating these two quantities. The result reads

\[
\text{tr} \chi = \hat{g}^{\alpha\beta} \chi_{\alpha\beta} = \frac{2}{r} \sqrt{\frac{\hat{g}}{g}}, \quad \text{tr} \chi = \hat{g}^{\alpha\beta} \chi_{\alpha\beta} = -\frac{2}{r} \sqrt{\frac{\hat{g}}{g}},
\]

for general input of \( g = g(u, r), \hat{g} = \hat{g}(u, r), \) and \( \phi = \phi(u, r) \) to the metric (1) with (19).

Remarkably, the same expressions also hold for the metric \( d \hat{s}^2 (\Sigma = S^2) \) itself, as well as when \( K = T^1 (\Sigma = S^2 \times T^1, \text{cf. (47)}) \). This is due to certain key cancelations enabled by the Kaluza-Klein reduction. Hence identically to the situation in 4 dimensions (cf. (13)), absence of the apparent horizon is signaled by \( \hat{g}/g = y \neq 0 \) in the lifted spacetime. For similar discussions on such invariance of trapping within the Kaluza-Klein dimensional reduction, one can see [38, 39].

**B. Kretschmann scalar**

Much like its 4-dimensional counterpart, the lifted spacetime can maintain nonsingular for \( u < 0 \), both at the center and beyond. A scalar-valued curvature singularity first occurs at \( r = 0, u = 0 \). To see this, one should examine the Kretschmann scalar \( \hat{K} = \hat{R}^{\mu\nu\rho\sigma} \hat{R}_{\mu\nu\rho\sigma} \) particularly at three key locations in the spacetime, i.e., \( x = 0, +\infty \) and \( x_+ \). In terms of \( y, \zeta \) and \( \gamma \), the full expression of \( \hat{K} \) is given by

\[
\hat{K} = 12 \left[ -6(2\zeta - 1)^2 \gamma^4 - 2\sqrt{2}(9\zeta - 4)(2\zeta - 1)\gamma^3 - \zeta(25\zeta - 12)\gamma^2 \right.
\]

\[
-4\sqrt{2}\zeta^2 \gamma - 9\zeta^2 + 16\zeta - 8 \right] y^2 + \left[ -8(2\zeta - 1)(\zeta - 1)\gamma^2 \right.
\]

\[
-4\sqrt{2}\zeta(\zeta - 1)\gamma + 16(\zeta - 1)^2 \right] y - 8(\zeta - 1)^2 \}
\]

\[
r^2(\zeta - 1) \left[ 2\gamma(\gamma + \sqrt{2})(2\zeta - 1)y - (\zeta - 2)y + 2(\zeta - 1) \right],
\]

\( (25) \)
where we have used the equations (7-9) to eliminate all the derivatives of \( y, \zeta \) and \( \gamma \).

Near the initial point \( O \) with \( x = 0 \), the system (7-9) admits the following Taylor series solution [28]:

\[
\begin{align*}
y &= 1 - c_T x^2 - 2c_T x^3 + O(x^4), \\
\zeta &= 2x - 4x^2 + (2c_T + 8 - \kappa^2) x^3 + O(x^4), \\
\gamma &= \kappa x + \left( \frac{c_T}{\kappa} + \kappa \right) x^2 + \left[ \frac{3c_T}{\kappa} + (c_T + 1)\kappa - \frac{\kappa^3}{2} \right] x^3 + O(x^4),
\end{align*}
\]

with a parameter \( c_T < \kappa^2/3 \) (\( c_T = \kappa^2/3 \) for \( V_0 = 0 \)). Plugging it into (25), we obtain

\[
\hat{K} = \frac{18(1 - 4c_T)^2}{(1 - 6c_T)u^2} (1 + 2x + O(x^2)), \quad c_T < \frac{\kappa^2}{3} = \frac{1}{6},
\]

which stays finite and continuous at \( r = 0 \) for finite \( u < 0 \). Similarly, near an ending point \( P_0(y_0) \) with \( x = +\infty \), the solution satisfies [28]

\[
y \to y_0 > \frac{1}{1 + \kappa^2} = \frac{2}{3}, \quad (\zeta - 1)e^{(1 - \kappa^2)s} \to c_2 < 0,
\]

\[
(\gamma + \kappa)e^{(1 - \kappa^2)s} \to c_2 \left( \frac{1 - y_0}{y_0\kappa^3} + \kappa \right), \quad \text{as} \ s \to +\infty,
\]

where \( c_2 \) is a constant. These asymptotic limits result in

\[
\hat{K} \to \frac{12y_0^2}{(3y_0 - 2)r^2}, \quad \text{as} \ s \to +\infty \ (u \to -). \tag{29}
\]

Hence, the scalar \( \hat{K} \) diverges at \( r = 0 \), but remains bounded elsewhere along \( u = 0 \) (the Cauchy horizon). Related to this, the same fall-off behavior also holds for other fixed \( u < 0 \):

\[
\hat{K} = \frac{12y_0^2}{(3y_0 - 2)r^2} + o(r^{-2}) \to 0, \quad \text{as} \ r \to +\infty. \tag{30}
\]

Now move on to \( x = x_* \). To capture the asymptotic behavior of the solutions, we introduce a new independent variable \( t \) to the system (7-9) such that

\[
\frac{ds}{dt} = -2(1 - 2\zeta). \tag{31}
\]

In terms of \( t \), the singular point \( P_*(y_*) \) turns into a critical point of the transformed equations [28]. Therefore, we can perform standard linearization with

\[
y = y_* + x_1(t), \quad \zeta = \frac{1}{2} + x_2(t), \quad \gamma = \frac{1 - y_*}{y_*\kappa^3} + x_3(t), \quad y_* > \frac{1}{1 + \kappa^2} = \frac{2}{3}. \tag{32}
\]

The eigenvalues of the linearized system are

\[
\lambda_1 = \kappa^2, \quad \lambda_2 = 1 - \kappa^2, \quad \lambda_3 = 0. \tag{33}
\]

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The zero eigenvalue reflects the fact that $P_s(y)$ forms a curve. Given that $\kappa = 1/\sqrt{2}$, the first two eigenvalues coincide:

$$\lambda_1 = \lambda_2 = \frac{1}{2},$$  \hspace{1cm} (34)

Accordingly, the solution in a neighbourhood of $s = s_*$ $(t = -\infty)$ are of the form (cf. \cite{26}, Sec. 2)

$$y = y_* - \frac{4(5y_* - 4)(3y_* - 4)a_1}{y_*} e^{t/2} + \cdots, \hspace{1cm} (35)$$

$$\zeta = \frac{1}{2} + a_1 e^{t/2} + \cdots, \hspace{1cm} (36)$$

$$\gamma = \frac{2\sqrt{2}(1 - y_*)}{y_*} + \left[ a_2 - \frac{4\sqrt{2}(3y_* - 4)a_1}{y_*} \right] e^{t/2} + \cdots, \hspace{1cm} (37)$$

where $a_{1,2}$ are two parameters and higher-order terms in $x_{1,2,3}$ are omitted. Consequently, we obtain

$$\hat{K} \rightarrow \frac{12y_*^2}{(3y_* - 2)r^2}, \hspace{1cm} \text{as} \hspace{1cm} s \rightarrow s_* \ (t \rightarrow -\infty), \hspace{1cm} (38)$$

which resembles the limit (29). Further numerical tests by us agree with (38).

For other fixed values of $x > 0$, the equation (25) implies that

$$\hat{K} \propto \frac{1}{r^2} \rightarrow 0, \hspace{1cm} \text{as} \hspace{1cm} r \rightarrow +\infty, \hspace{1cm} (39)$$

where the boundedness of the proportionality coefficient can be verified by numerical calculation for individual solutions.

C. Asymptotic local flatness

With $\kappa = 1/\sqrt{2}$, the limits in (28) imply that

$$\frac{gg}{x} \rightarrow \frac{4c_2^2}{y_0}, \hspace{1cm} \frac{g}{\sqrt{x}} \rightarrow - \frac{2c_2}{y_0}, \hspace{1cm} xe^{\sqrt{2}\phi} \rightarrow \frac{\sqrt{3y_0 - 2}}{-2u}, \hspace{1cm} \text{as} \hspace{1cm} x \rightarrow +\infty. \hspace{1cm} (40)$$

Introducing new coordinates

$$\bar{r}^2 = r, \hspace{1cm} \bar{t} - \bar{r} = \bar{u} = 2c_2\sqrt{-u} < 0, \hspace{1cm} (41)$$
thus for finite \( u < 0 \) with \( r \) (equivalently \( x \)) being large, we have
\[
d s^2 = e^{\sqrt{2}\phi} \left( -g(x)\hat{g}(x) \, du^2 - 2g(x) \, du \, dr + r^2 \, d\Omega^2 \right) + \frac{e^{-\sqrt{2}\phi}}{2} \, d\hat{\Omega}^2 \tag{42}
\]
\[
\approx -\frac{2\sqrt{3}y_0 - 2}{y_0} \left( \frac{e^2 du^2 - 2c_2 dud\tilde{r}}{-u} \right) + \frac{\sqrt{3}y_0 - 2}{2} \tilde{r}^2 d\Omega^2 + \frac{\tilde{r}^2}{\sqrt{3}y_0 - 2} d\hat{\Omega}^2 \tag{43}
\]
\[
= -\frac{2\sqrt{3}y_0 - 2}{y_0} \left( d\tilde{u}^2 + 2d\tilde{u}d\tilde{r} \right) + \tilde{r}^2 \left( \frac{\sqrt{3}y_0 - 2}{2} d\Omega^2 + \frac{1}{\sqrt{3}y_0 - 2} d\hat{\Omega}^2 \right) \tag{44}
\]
\[
= \frac{2\sqrt{3}y_0 - 2}{y_0} (-d\tilde{t}^2 + d\tilde{r}^2) + \tilde{r}^2 \left( \frac{\sqrt{3}y_0 - 2}{2} d\Omega^2 + \frac{1}{\sqrt{3}y_0 - 2} d\hat{\Omega}^2 \right). \tag{45}
\]
Then note that by stereographic projection, \( \tilde{r}^2 d\Omega^2 = (d\tilde{x}^2 + d\tilde{y}^2)/[1 + (\tilde{x}^2 + \tilde{y}^2)/4\tilde{r}^2] \rightarrow d\tilde{x}^2 + d\tilde{y}^2 \) for local \( \tilde{x} \) and \( \tilde{y} \) as \( \tilde{r} \rightarrow +\infty \). Combining these approximations with the decaying Kretschmann scalar in (30), we comment that the lifted spacetime is asymptotically locally flat with non-trivial topology \( S^2 \times S^2 \) at spatial infinity.

D. Homothetic Killing vector

With the assumption of (19) and \( \kappa = 1/\sqrt{2} \), one can check that the lifted metric \( ds^2 \) \( (\hat{g}_{\mu\nu}) \) possesses the same homothetic Killing vector \( \xi \) as \( ds^2 \ (g_{\mu\nu}) \) \[26, 28\]:
\[
\xi = u \partial_u + r \partial_r, \quad L_\xi g_{\mu\nu} = 2g_{\mu\nu}, \quad L_\xi \hat{g}_{\mu\nu} = \hat{g}_{\mu\nu}. \tag{46}
\]
Therefore, the 6-dimensional naked singularity spacetimes with spherical extra dimensions are continuously self-similar.

IV. COMMENT ON 5 DIMENSIONS

From the previous discussion, one would anticipate that lifting the solutions with \( V_0 = 0 \) for massless scalar fields might also generate new naked singularity solutions in higher dimensions (see [40] for discussions on lifting solutions with \( \kappa = 0 \)). However, we will show that this turns out to be false. The reason lies in that, unlike the 6-dimensional example, the Kretschmann scalar blows up at \( x = x_* \), i.e., along the past light cone of the central singularity. Hence in such a spacetime, a curvature singularity can pre-exist before the central singularity emerges later at \( u = 0 \).

Again for concreteness of calculation, we focus on the simplest case \( \mathcal{K} = T^1(= S^1) \) in 5 dimensions and take
\[
d s^2_{n=1} = (dx^1)^2 \tag{47}
\]
with $x^1$ the local coordinate on $\mathcal{K}$. Using (10) with $V_0 = 0$, we can remove the unknown $y$ in the equation (9) and obtain a reduced system for $\zeta$ and $\gamma$. It can also be verified that the equations (8-10) together imply (7). To solve this 2-dimensional autonomous system in a neighborhood of the singular point $\mathcal{P}_s(\frac{1}{1+\kappa^2})$, one can apply the same procedure as in (31-37) and obtain (26), Sec. 2; for a comparison of notations, see the Appendix of [28]

$$\zeta = \frac{1}{2} + a_2 e^{(1-\kappa^2)t} + \ldots,$$

$$\gamma = \frac{1}{\kappa} + a_1 e^{\kappa^2 t} + \frac{4a_2 (1 + \kappa^2)}{\kappa^3 (1 - 2\kappa^2)} e^{(1-\kappa^2)t} + \ldots,$$

where the variable $t$ is defined by (31) and $a_{1,2}$ are two parameters. Owing to these asymptotic solutions for $t \to -\infty$ ($s \to s_*$), the scalar $\hat{K}$ (written in terms of $\zeta$ and $\gamma$ without derivatives) diverges according to

$$\lim_{t \to -\infty} e^{(1-2\kappa^2)t} \hat{K} \propto \frac{1}{r^{4-4\kappa/\sqrt{3}}}, \quad 0 < \kappa < \frac{1}{\sqrt{3}}.$$  

(50)

One can also confirm the divergence of $\hat{K}$ at $x = x_*$ via numerical integrations along individual solution curves.

As an aside, we note that the lifted metric $d\hat{s}^2$ with (47) can no longer admit the homothetic Killing vector $\xi = u \partial_u + r \partial_r$ unless $\kappa = 1/\sqrt{3}$, of which the value lies outside the required bound $0 < \kappa^2 < 1/3$ for naked singularities in 4 dimensions.

At first glance, the inconsistency of spacetime regularity (cf. (50)) in the Kaluza-Klein dimensional reduction may appear puzzling. One physical way to resolve it is by changing the conformal frame for $d\hat{s}^2$ [41], which means that one should instead look at the 4-dimensional projection of the full metric $d\hat{s}^2$, i.e., $\exp(2\phi/\sqrt{3}) ds^2$. For this re-scaled metric with the dilaton $\phi$ involved, direct calculation shows that the Kretschmann scalar diverges at $x = x_*$ ($K$ continuous for $ds^2$ except at $r = 0, u = 0$; $K \propto 1/r^4$) in accordance with its 5-dimensional counterpart.

V. CONCLUDING REMARKS

In the context of the Kaluza-Klein dimensional reduction, we have constructed a 2-parameter family of 6-dimensional naked singularity solutions by lifting the 4-dimensional ones in the Einstein-scalar field system with a negative exponential potential. Despite its success, the same construction does not come through in the case of a vanishing potential.
due to a diverging Kretschmann invariant along the incoming null ray $x = x_*$ (hence in the initial data), which raises a caveat for utilizing such lifting/embedding. Regarding the 6-dimensional solutions, it may draw some concern that the spatial infinity has a non-trivial topology $S^2 \times S^2$. However, we should emphasize that the local formation of a singularity near the center does not depend on the spacetime region far from it. Hence it would be interesting to investigate if the central region of the spacetime can be truncated and matched to an appropriate surrounding region that is, for instance, asymptotically Minkowski ($S^4$) or asymptotically Kaluza-Klein (compact extra dimensions). Related to this, we recall that such truncation is necessary for Christodoulou’s solutions \[2\], in order to obtain asymptotically flat initial data. As a final remark, it should be noted that though we have restricted ourselves to 6 and 5 dimensions, our treatment can be generalized to other values of $4 + n \geq 6$ (certain complications may arise). The results with detailed proofs will be reported elsewhere.

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