Allan Kaufman’s contributions to plasma wave theory

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Abstract. A brief review is presented of the contributions of Allan Kaufman to the theory of plasma waves. These contributions have been rich and various, characterized by high quality and a sense of what makes a problem both interesting and important. We include a brief summary of work prior to the mid-1980’s, but the primary emphasis will be on more recent work concerning the use of phase space methods in the theory of linear plasma waves and mode conversion. One goal of the paper is to place Allan’s contributions in the wider context of semi-classical methods. We will emphasize the underlying intuitions rather than providing a detailed mathematical exposition which can be found in the literature cited.

1. Introduction

The authors have had the great privilege of working with Allan Kaufman for nearly twenty years and can attest to the fact that he combines deep physical intuition about plasmas, with a strong mix of geometrical and analytical skills. Both of the authors were attracted to work with Allan because of his track record of working on important physics problems, and that more elusive quality we usually call ‘style’. In this paper we take the opportunity afforded by the KaufmanFest 07 conference proceedings to step back and provide a look at Allan’s contributions in the area of plasma wave theory. Our goal is not to give a detailed technical discussion of his achievements in this area, which can be found in the papers cited, but to instead provide an overview of his many years of productive work. This will show how keen Allan’s instincts have been for finding interesting problems, and to highlight the elegance of his approach to physics. Allan’s work on the closely related areas of action principles, geometrical perturbation theory, and gyrokinetics, are described elsewhere in these proceedings.

Because this summary is intended to describe Allan’s contributions in plasma wave theory, rather than providing a detailed review of this topic, we apologize in advance for researchers whose papers are not cited here.

Allan has made seminal contributions in the areas of wave chaos, wave kinetic equations, ponderomotive effects, quasilinear diffusion and mode conversion in nonuniform plasmas. For many of these topics, Allan was one of the first in the plasma community to contribute. For others, Allan’s contributions are often considered to be among the clearest or most insightful. Further, he was often responsible for keeping plasma physics ‘out front’ among the other fields of physics, developing ideas and tools that have since been adopted by others, or where other fields have yet to catch up.
Although we will attempt to provide a reasonably full historical survey of Allan’s most significant contributions in plasma wave theory, we assert the authors’ prerogative to focus on areas that we are most familiar with. Hence, we will break the historical survey into two ‘epochs’: before, and after, the mid-1980’s. Prior to the mid-80’s, Allan’s work was largely concerned with nonlinear effects, as was much of the plasma community. The authors can recall as graduate students learning the general opinion that linear theory was basically ‘done’ and all the ‘real’ problems were nonlinear. Hence, it was a bit of shock to find out this was not the case, and that there were still many problems in linear wave theory that demanded attention. Among these were problems in vector WKB theory, such as covariant formulations, wave-kinetic equations for incoherent fields, and how to deal with regions where WKB methods break down such as caustics, mode conversion and gyroresonance. By the end of the 1980’s, Allan’s work in linear wave theory took on the distinctive flavor of modern ‘semi-classical’ analysis, with its emphasis on ray phase-space methods.

Given how much of Allan’s work in plasma wave theory has centered on the use of WKB methods, it is interesting to note that Allan received his PhD from the University of Chicago under the direction of Gregor Wentzel, the ‘W’ in WKB. Hence, Allan has a ‘Wentzel number’ of one. Allan’s recollections of his work with Wentzel are described in his memoirs, also a part of this volume.

The outline of our paper is as follows: In Section 2 we provide a quick review of Allan’s contributions to plasma wave theory prior to 1985. This break is somewhat arbitrary, but it marks a point where Allan’s approach to the theory of waves began to incorporate and develop new ideas from semi-classical analysis. Hence, the work after this time becomes more informed by the notion of ray phase space. His research interests prior to this time are discussed in his memoirs, so we will pass over them quickly. One pattern that emerges from this survey is how intellectually ambitious Allan has been, and how open he is to new ideas and methods. As a result, Allan has often been among the first in plasma theory to exploit new mathematical ideas, or concepts and tools from other fields.

Much of Allan’s work since 1985 has concerned WKB theory, and extensions of that theory that are required in various circumstances, so in Section 3 we provide an exceedingly brief summary of WKB methods and ray tracing. Particular emphasis is placed on new ideas that have been introduced since the 1960’s.

In Section 4 we describe Allan’s contributions, in collaboration with Steve McDonald, to the theory of ‘quantum chaos’ (although Allan’s work was actually done in the context of classical wave equations) and a closely related derivation of the wave-kinetic equation for electromagnetic waves. The wave-kinetic equation makes the important assumption that there are no non-adiabatic (mode conversion or resonance crossing) effects, though it did allow for the presence of weak sources and dissipation. The vector character of the problem introduces significant complications that do not occur in the scalar wave problem, and the approach taken by Kaufman and McDonald introduces some key ideas that will reappear in much of Allan’s subsequent work.

In Section 5 we describe what Lazar Friedland has called Allan’s ‘conversion to mode conversion’. In this section we stick to a configuration space viewpoint of conversion and draw heavily upon some notes provided to us by Professor Friedland concerning his work with Allan.

In Section 6, we highlight a key insight of Allan’s work: that the geometrical picture of mode conversion greatly simplifies when it is viewed in ray phase space, especially in spatial dimensions higher than one. Friedland and Kaufman showed that the mode conversion transmission coefficient involves the Poisson bracket of the two uncoupled dispersion functions. The appearance of the Poisson bracket was an important clue that Allan pursued in later publications. This led to a beautiful synthesis of ideas: 1] Friedland’s congruent reduction method and a local linearization of the dispersion matrix can be used to develop a local $2 \times 2$ PDE governing the two resonant modes. The approach is completely general and can
be applied to any $N$-component wave equations, whether of integral or differential, or mixed integral/differential form. Given the reduction to the local $2 \times 2$ form, one can then use various transformations to find a complete solution. These transformations are generalizations of Fourier transformations (metaplectic transformations) which arise in the theory of linear canonical transformations. Metaplectic transformations entered plasma theory in the work of Littlejohn concerning wave packet formulations. The use of phase space ideas provides a very natural geometric picture of what is going on and points toward generalizations. The exchange of ideas among the Berkeley plasma theory group, led by Allan and Robert Littlejohn has been very fruitful over the years. For example, Littlejohn's work on wave packet propagation, the role of the Heisenberg-Weyl group (the group of phase space shifts) and the notion of metaplectic transformations, has had a very profound influence on our approach to mode conversion.

The idea that phase space is the natural place to view conversion has informed all of Allan’s subsequent work in the area, and forms the basis for our collaborative efforts with him. In Section 7, we describe some of these further efforts to develop the phase space theory of conversion by summarizing a series of papers which Allan’s group of collaborators has produced, as well as other researchers in this area. These include 1) the incorporation of kinetic effects and Budden-type resonant absorption, 2) the development by our group, joined by André Jaun, of a practical algorithm for incorporating mode conversion into ray-tracing codes in realistic geometries and plasma models, and 3) a new approach to the theory of operator symbols that provides interesting perspectives on phase space representations of wave problems.

We end with a summary of active lines of research currently being pursued by our group and others.

2. Contributions to plasma wave theory prior to 1985
An INSPEC search (which only goes back to 1969) using Allan’s name returned 107 hits. In this section we will give a very brief summary of the most significant papers prior to about 1985. The papers after 1985 will considered in later sections.

2.1. Manley-Rowe relations
Bruce Cohen, Ken Watson and Allan collaborated on a PRL highlighting the importance of the Manley-Rowe relations in beat heating of plasmas [1]. The following scenario was considered: two laser beams with a difference frequency that is nearly equal to the plasma frequency can resonantly interact to produce longitudinal plasma oscillations. These can then induce transitions to other transverse modes. It is the nonlinear damping of the longitudinal mode that heats the plasma. The importance of action conservation, as summarized by the Manley-Rowe relations, was emphasized. Allan returned to the subject of the Manley-Rowe relations in a later paper with Alain Brizard [2], where non-eikonal wave fields are treated. In the later paper, the Manley-Rowe relations are shown to follow from a Noether symmetry for an appropriate action principle. The use of action principles and symmetries to derive conservation laws is a recurring theme is Allan’s work.

2.2. The introduction of action-angle variables for tokamak linear susceptibility and quasilinear diffusion
In a 1972 paper appearing in Physics of Fluids, Allan introduced an action-angle formalism for the study of the plasma susceptibility and quasilinear diffusion in tokamak geometry [3]. In a toroidal plasma with axial symmetry, the three adiabatically invariant actions of a particle are the magnetic moment, the canonical angular momentum, and the toroidal flux enclosed by the drift surface. Resonant interactions between particles and the normal modes of collective oscillations produce mode growth or decay and random changes in the actions. This random walk is represented by a diffusion equation in action space. Both the diffusion tensor and the growth
rate depend upon a coupling coefficient which represents the work done by a normal-mode field eigenfunction on the current density of an unperturbed particle orbit. In this paper, Allan was unable to include the reaction of the background field, due to resonant diffusion, and was always concerned about this. In later work with Huanchun Ye [4] (see also Huanchun’s PhD thesis) these self-consistent effects were included. By that time the use of variational principles, guiding center and oscillation center Lagrangian Lie-transforms and Noether methods had become part of Allan’s toolkit.

2.3. Hamiltonian chaos and stochastic heating
In 1975 Gary Smith and Allan wrote a PRL on the chaotic acceleration of particles in a coherent wave field [5]. Surface-of-section plots were used to show that varying the wave amplitude leads to a transition from adiabatic to chaotic particle motion and to the onset of heating. As an application of these concepts, the authors showed the possibility of heating the tail of the ion distribution in a magnetized plasma by an intermediate-frequency acoustic wave. This paper, along with those of Charles Karney of Princeton, helped to introduce the concept of deterministic chaos into plasma theory.

2.4. Development of the Hamiltonian Lie transform
The development of the Hamiltonian Lie transform was carried out with John Cary (1977) and led to a 1981 paper applying this new theoretical tool to the study of ponderomotive effects [6]. This method involves the application of the Lie-transform perturbation technique to the Hamiltonian formulation of the Vlasov equation. A canonical change of coordinates is carried out to new variables in which the high-frequency has been removed. In this system the distribution function evolves according to a ponderomotive Hamiltonian, which is the kinetic generalization of the ponderomotive potential. It was shown that the ponderomotive Hamiltonian can easily be determined from the well-known linear susceptibility (a result now known as the $K-\chi$ Theorem).

2.5. Beat Hamiltonians and ponderomotive potentials
With Shayne Johnston and George Johnston, Allan recognized that the ponderomotive potential is the oscillation-center Hamiltonian [7] and a novel approach to the theory of nonlinear mode coupling in hot magnetized plasma was developed. The formulation retains the conceptual simplicity of the familiar ponderomotive-scalar-potential method, but removes the approximations. The essence of the approach is a canonical transformation of the single-particle Hamiltonian, designed to eliminate those interaction terms which are linear in the fields. The new entity (the ‘oscillation centre’) then has no first-order jittering motion, and generalized ponderomotive forces appear as nonlinear terms in the transformed Hamiltonian. This viewpoint is applied to derive a compact symmetric formula for the general three-wave coupling coefficient in hot uniform magnetized plasma, and to extend the conventional ponderomotive-scalar-potential method to a case with momentum dependence in the ponderomotive hamiltonian.

2.6. The dissipative bracket
In a 1984 paper [8], Allan showed that the concept of a Hamiltonian system (i.e., the combination of a Hamiltonian function and a Poisson-bracket structure) could be generalized to include a wide class of dissipative processes. This was an idea that was simultaneously and independently developed by Morrison [9] and Grmela [10] applied to different areas. Evolution of any observable is generated jointly by a Hamiltonian, with an entropy-conserving Poisson bracket and an entropy, with an energy-conserving dissipative bracket. This approach yields many of the standard kinetic equations such as those representing particle collisions, three-wave interactions, and wave-packet resonances.
2.7. Recognition of the fundamental importance of the action principle

Through a series of papers with several authors [11, 12, 13, 14], Allan became a firm believer in the fundamental importance of action principles. Whitham’s averaged-Lagrangian method was a strong influence [15]. These ideas were developed further in Bruce Boghosian’s thesis (1984-6) [16]. Action principles play a central role in these papers and provide a very elegant method for developing covariant formulations of plasma wave problems, including relativistic effects. Noether symmetries play an important role in the study of conservation laws. In addition, through use of the Lagrangian Lie transform, the action principle becomes a powerful tool for perturbation calculations.

During these years, Allan recognizes that his greatest influence was that of Robert Littlejohn, who introduced him to symplectic geometry, noncanonical brackets, Lagrangian variational principles, and the phase-space Lagrangian Lie-transform perturbation method. As we’ll discuss in later sections, Robert continues to have a great influence on Allan’s work.

This completes our summary of the most significant of Allan’s pre-1985 papers in plasma wave theory. We now move on to the years when Allan became deeply involved in the application of WKB methods in plasma wave theory, and various extensions of WKB theory. Prior to discussing Allan’s contributions, we start with a brief summary of the underlying ideas.

3. WKB theory: a very short synopsis

In these informal notes we will use the terms ‘WKB theory’, ‘ray tracing’, and ‘short-wavelength asymptotics’ interchangeably. The more general term ‘semi-classical analysis’ we will use to include extensions of WKB theory that are required in the vicinity of caustics or mode conversion, where the WKB approximation is invalid.

3.1. Preliminaries: uniform and static media

We begin by discussing the uniform case, which can be dealt with using Fourier methods. Suppose we start with a scalar wave equation of the form:

\[ D(i\partial_t, -i\nabla)\psi(x; t) = 0, \]  

where \( \nabla \) is the gradient operator and \( x = (x_1, \ldots, x_n) \) \((n, the number of spatial dimensions, will be 1, 2 or 3 depending upon the problem of interest). Note that this PDE might be of infinite order (i.e. a pseudodifferential equation). We assume that the wave operator is self-adjoint, implying that (1) can be derived using the action principle:

\[ \mathcal{I} = \int dt \, d^n x \, \psi^*(x; t) D(i\partial_t, -i\nabla)\psi(x; t). \]  

The wave equation (1) is recovered via the stationarity condition \( \delta \mathcal{I} / \delta \psi^* = 0 \). The use of action principles in WKB problems is a central theme in much of Allan’s work as we’ll see.

The wave equation is supplemented by some initial/boundary data which we will leave unspecified here, but note that satisfaction of the initial/boundary conditions requires a superposition of the plane wave solutions. Inserting the ansatz \( \psi(x, t) = \exp(ik \cdot x - i\omega t) \) with \( k = (k_1, \ldots, k_n) \) into the wave equation (1), we arrive at the condition \( D(\omega, k) = 0 \), which defines the dispersion manifold. Because the operator \( D(i\partial_t, -i\nabla) \) is self-adjoint, the function \( D(\omega, k) \) will be a real function of real \( \omega \) and \( k \). We concern ourselves only with real roots of \( D(\omega, k) = 0 \) here, but note that complex roots are often of interest in tunneling problems.

The condition \( D(\omega, k) = 0 \) implies that, in the \( n + 1 \)-dimensional space \((\omega, k)\), only those points which lie on the \( n \)-dimensional surface defined by \( D(\omega, k) = 0 \) can be associated with freely propagating waves. The \( n \)-dimensional dispersion manifold is thus implicitly defined. We can, if we choose, use local coordinates on this surface and write an explicit relation between \( \omega \)
and $k = (k_1, \ldots, k_n)$. For example, if we choose to use $(k_1, \ldots, k_n)$ as the local coordinates on the dispersion surface, we then solve $D(\omega, k) = 0$ for $\omega$ which locally defines $\omega = \Omega(k)$, which is an example of a dispersion relation. The implicit condition $D(\omega, k) = 0$ can have multiple branches (even an infinite number in some cases), while a dispersion relation should be understood to refer to a particular branch (using a particular choice of local coordinates).

In uniform media, the phase level sets, $\theta(x, t) = k \cdot x - \omega t$, change with time in a simple way. By considering how level sets of the phase move with time, it is possible to identify a characteristic ‘velocity’ associated with their evolution. The convention is to ask how points on level sets move in a direction normal to the phase fronts themselves. The phase velocity will be denoted $v_p$ and is defined by

$$v_p \equiv \frac{\omega}{|k|} \hat{k}.$$  

The phase velocity will play an important role in our discussion of mode conversion because resonant interaction between two wave types requires phase matching, hence they must have the same phase velocities.

Now consider a nearly monochromatic wavetrain, or a wave packet, with a well defined carrier oscillation, but still in a uniform and static background. This type of solution can be constructed by superposing plane wave solutions that lie close to one another on the same branch of $D(\omega, k) = 0$. Denote the carrier as $(\omega_0, k_0)$ (noting that $D(\omega_0, k_0) = 0$), and assume a new ansatz of the form

$$\psi(x, t) = A(x, t)e^{i k_0 \cdot x - \omega_0 t}.$$  

Inserting this into the wave equation (1) we find that the amplitude $A(x, t)$ must satisfy

$$D(\omega_0 + i \partial_t, k_0 - i \nabla)A(x, t) = 0.$$  

This is still completely general, we have simply carried out a change of the dependent variable from $\psi \to A$. If we now insist that $A(x, t)$ is a slowly varying function of its arguments, then we can expand the wave operator in powers of the derivatives to find at leading order:

$$D(\omega_0 + i \partial_t, k_0 - i \nabla)A(x, t) \approx \left[ \frac{\partial D}{\partial \omega} \partial_t - \frac{\partial D}{\partial k} \cdot \nabla \right] A(x, t) \approx 0,$$

where we have used $D(\omega_0, k_0) = 0$ and the derivatives of $D(\omega, k)$ are evaluated at $(\omega_0, k_0)$. Note that $\partial D/\partial k = (\partial D/\partial k_1, \ldots, \partial D/\partial k_n)$. Provided $\partial D/\partial \omega \neq 0$, we can recast the evolution equation for the envelope into a standard advection equation:

$$[\partial_t + v_g \cdot \nabla] A(x, t) = 0, \quad v_g \equiv -\left( \frac{\partial D}{\partial \omega} \right)^{-1} \frac{\partial D}{\partial k};$$

where the group velocity, denoted $v_g$, has appeared. Note that the group velocity is associated with the evolution of the amplitude (equivalently the envelope) of the wave, hence it determines how energy and action are transported by the wave. Note, also, the narrow-banded assumption is critical. For dispersive systems, broad banded solutions do not have well-defined signal pathways.

The generalization of the above discussion to $N$-component wave equations proceeds as follows: The wave equation is now of the form

$$D(i \partial_t, -i \nabla) \cdot \Psi(x, t) = 0,$$

or, in component form:

$$\sum_{n=1}^N D_{mn}(i \partial_t, -i \nabla)\psi_m(x, t) = 0.$$
We still assume self-adjointness. This system has plane-wave solutions of the form: 
\[ \hat{e}(\omega, k) \exp(ik \cdot x - i\omega t) \]. Inserting this ansatz, the wave equation becomes:
\[ D(\omega, k) \cdot \hat{e}(\omega, k) = 0, \] (10)
where \( D(\omega, k) \) is a self-adjoint \( N \times N \) matrix for all real \( \omega \) and \( k \). Therefore, plane-wave solutions of the wave equation can only exist for those values of \( (\omega, k) \) for which \( D(\omega, k) \) has a null eigenvalue (\( \hat{e}(\omega, k) \) is the associated eigenvector). We can test for a null eigenvalue by taking the determinant, leading to the definition of the dispersion manifold for multicomponent wave equations:
\[ D(\omega, k) \equiv \det(D(\omega, k)) = 0, \] (11)
a result that is both familiar and deceptive because by moving so quickly we have slipped some things under the rug. Let’s proceed a bit more slowly.

First, the dispersion matrix \( D(\omega, k) \) is well-defined at each \( (\omega, k) \) and is self-adjoint everywhere, with well-defined eigenvalues and eigenvectors. Therefore, at each \( (\omega, k) \) we can decompose the dispersion matrix onto an orthonormal basis composed of its \( N \) eigenvectors (i.e. solutions of \( D \cdot \hat{e}_\alpha = \lambda_\alpha \hat{e}_\alpha \)):
\[ D(\omega, k) = \sum_{\alpha=1}^{N} \lambda_\alpha(\omega, k)\hat{e}_\alpha(\omega, k)\hat{e}_\alpha^\dagger(\omega, k). \] (12)
Generically, the eigenvalues of a hermitian matrix will be non-degenerate implying the \( N \) eigenvectors are automatically orthogonal. If two (or more) of the eigenvalues of \( D(\omega, k) \) are degenerate, however, things get more interesting. This is because if \( \hat{e}_1 \) and \( \hat{e}_2 \) have the same eigenvalue, then so does any linear combination of them. Therefore, it would appear that we have complete freedom in choosing a basis in the degenerate subspace. Mathematically speaking, this is certainly true for any fixed hermitian matrix. However, we must always keep in mind that we are dealing with a family of hermitian matrices, namely one for each point \( (\omega, k) \). If the matrix is degenerate at the point \( (\omega, k) \), then it generically will \textit{not} be degenerate at a neighboring point \( (\omega + \Delta \omega, k + \Delta k) \). At this neighboring point there is therefore no freedom to choose the eigenvectors for decomposition. (More precisely, the only freedom we have is the choice of phase for \( \hat{e}(\omega, k) \).) Hence, at a point of degeneracy it would seem reasonable that one should choose eigenvectors that smoothly connect to their non-degenerate neighbors. (This is the essential idea behind degenerate perturbation theory in quantum mechanics.) In one-dimensional parameter spaces this approach leads to a well-defined choice of basis vector at the point of degeneracy. But in higher dimensions it does not because the result often depends sensitively upon the direction one uses to approach the degeneracy. Hence, there is no choice of polarization eigenvector at the point of degeneracy that will smoothly connect to all of the surrounding non-degenerate neighbors.

In nonuniform media, this nasty behavior near a degeneracy is precisely what leads to the breakdown of the WKB approximation. This is because, as we discuss in the following section, in vector WKB the polarization is assumed to be transported adiabatically following a ray. This polarization vector is usually taken to be a local null-eigenvector of the dispersion matrix, following the intuitively reasonable argument that what is true globally for the uniform case should be true locally in the nonuniform one. However, in the vicinity of a degeneracy, where two or more of the eigenvalues of the dispersion matrix approach zero, the null eigenvectors are badly behaved for precisely the same reason as in the uniform case. Therefore, they change \textit{rapidly} following a ray and the adiabatic assumption breaks down. We will discuss at some length methods that have been developed to overcome this difficulty while retaining the power of WKB methods away from the degeneracy.
3.2. Non-uniform and time-varying media

All of the material in this section has been covered in great detail in earlier papers [19, 18, 17]. We include a brief heuristic introduction to the ideas for completeness.

Suppose the background plasma is no longer uniform in space and that it varies in time as well. This significantly complicates the problem. However, if we can treat these new effects as a perturbation, then we can use plane waves as local solutions with slowly varying parameters. Intuitively, what we would like to do is argue that the dispersion function $D(\omega, k)$ now becomes local, $D(t, \omega, x, k)$. The correspondence $k \rightarrow -i\nabla$ still holds. The new complication is that $\nabla$ does not commute with $x$ so we must be very careful about ordering issues. The Weyl symbol calculus provides a systematic approach for dealing with these matters, and we’ll discuss it momentarily.

Let’s write the scalar wave equation (1) in the following form:

$$D(t, i\epsilon \partial_t, x, -i\epsilon \nabla)\psi(x,t) = 0. \quad (13)$$

The formal expansion parameter $\epsilon$ has been introduced to clarify the asymptotic balances about to be invoked. The Weyl calculus ensures that this wave operator is symmetrized (for example the product $xk$ corresponds to the operator $(\hat{x}\hat{k} + \hat{k}\hat{x})/2$). We still assume the wave operator is self-adjoint, therefore there is an action principle:

$$I = \int dt d^n x \psi^*(x,t)D(t, i\epsilon \partial_t, x, -i\epsilon \nabla)\psi(x,t) = 0. \quad (14)$$

The wave equation (13) arises from the stationarity condition $\delta I/\delta \psi^* = 0$.

Guided by the earlier calculation for the uniform medium, we insert the ansatz $\psi(x,t) = A(x,t)\exp(i\theta(x,t)/\epsilon)$ and find:

$$D(t, -\theta_t + i\epsilon \partial_t, x, \nabla \theta - i\epsilon \nabla)A(x,t) = 0. \quad (15)$$

At this stage no approximations have been made, we have only changed the dependent variable. Expand in powers of $\epsilon$ to find:

$$[D(t, -\theta_t, x, \nabla \theta) + \epsilon i 2 \left(\frac{\partial D}{\partial \omega} \partial_t + \partial_t \frac{\partial D}{\partial \omega} - \epsilon i 2 \left(\frac{\partial D}{\partial k} \cdot \nabla + \nabla \cdot \frac{\partial D}{\partial k}\right)\right) A(x,t)] = O(\epsilon^2). \quad (16)$$

We now insist that these expressions vanish order-by-order in $\epsilon$.

At $O(\epsilon^0)$ we have $D(t, -\theta_t, x, \nabla \theta) = 0$, which is a nonlinear PDE (the Hamilton-Jacobi equation) for the unknown phase function $\theta(x,t)$. Defining the local frequency $\omega \equiv -\theta_t$ and wavevector $k = \nabla \theta$ we see that we can interpret the Hamilton-Jacobi equation as a requirement that the local phase function must be chosen so the local dispersion relation $D(t, \omega, x, k) = 0$ is satisfied.

At $O(\epsilon)$ we get a linear PDE for the envelope $A(x,t)$. It should be emphasized, however, that the coefficients of this PDE involve the derivatives of the dispersion function at the points $\omega = -\theta_t$ and $k = \nabla \theta$, and hence it appears to require that the solution of the Hamilton-Jacobi equation is already known. In fact, the situation isn’t quite so bad. This is because both the Hamilton-Jacobi equation and the amplitude transport equation can be solved using ray tracing methods (following directly in Hamilton’s footsteps). Once the rays are known, the equation for the envelope can be recast as an ODE describing how the amplitude changes along any given ray.

It must be emphasized that in spatial dimensions higher than one it is not sufficient to follow only single rays. One must follow an entire ray family in order to transport phase and amplitude. We will discuss this important point further when we discuss modern developments.
because there is a very natural geometrical interpretation of this notion that leads to the concept of a Lagrange manifold.

The first order equation (16) can be recast as a conservation law for wave action by brute force. However, Allan showed that it is possible to derive this in a more elegant fashion, and to gain important insight into the origin of the conservation law by returning to the action principle (14). Insert the eikonal ansatz directly into the action principle and keep only the leading order terms. This leads to the modified action principle:

$$I' = \int dt \, d^nx \, D(t, -\theta_t, x, \nabla \theta) |A|^2(x, t) = 0. \tag{17}$$

Stationarity under the variation $\delta I / \delta A^* = 0$ gives the Hamilton-Jacobi equation. The action principle is invariant under a global shift by a constant phase $\theta \rightarrow \theta + \theta_0$. Therefore, by Noether’s theorem, there is an associated conserved quantity. This conserved quantity is the wave action, and the associate wave action conservation law, which is usually derived by direct calculation, is revealed to be due to an underlying symmetry of the theory.

Returning for the moment to the rays: the rays equations are written in terms of a ray parameter $\sigma$ (for a derivation, see the papers cited at the beginning of this section):

$$\dot{x} = \frac{dx}{d\sigma} = -\frac{\partial D}{\partial k}, \quad \dot{k} = \frac{dk}{d\sigma} = \frac{\partial D}{\partial x}. \tag{18}$$

Recall that $x$ and $k$ are vectors. The ray parameter and the physical time are related via $dt/d\sigma = \partial D / \partial \omega$. The phase increment $\Delta \theta$ following a ray that starts at the point $x_0$ with wavevector $k_0$ is given by first following the evolution generated by Hamilton’s equations to find $(x(\sigma; x_0, k_0), k(\sigma; x_0, k_0))$. This information is now used to compute $\dot{x}$ along the ray and an integration performed to find $\Delta \theta = \int d\sigma \, k \cdot \dot{x}$. Because Hamilton’s equations satisfy uniqueness (at least they do for the smooth types of dispersion functions of most interest in physics), rays will not cross in phase space. However, they can cross in $x$-space. This leads to singular behavior in the amplitude transport equation (caustics). Therefore, in general, the Hamilton-Jacobi equation does not have global solutions, only local ones. The treatment of caustics is discussed in [20].

WKB theory for vector wave equations can be brutally summarized by saying that if we now start with a wave equation of the form:

$$D(t, i\epsilon \partial_t, x, -i \epsilon \nabla) \cdot \Psi(x; t) = 0; \tag{19}$$

we insert an ansatz of the form:

$$\Psi(x; t) = e^{i\theta(x; t)/\epsilon} A(x; t) \hat{e}(x; t). \tag{20}$$

At leading order in $\epsilon$ this requires:

$$D(t, \omega = -\theta_t, x, k = \nabla \theta) \cdot \hat{e}(x; t) = 0. \tag{21}$$

This requires that $\theta(x; t)$ is such that $D(t, \omega = -\theta_t, x, k = \nabla \theta)$ has a null eigenvalue, with $\hat{e}(x; t)$ the associated null eigenvector. The existence of a null eigenvalue can be tested for using the determinant, hence the Hamilton-Jacobi equation for multicomponent wave problems is:

$$D(t, \omega = -\theta_t, x, k = \nabla \theta) \equiv \det(D(t, \omega = -\theta_t, x, k = \nabla \theta)) = 0. \tag{22}$$

As in the scalar case, this equation in general has only local solutions, so something must be done to connect the local solutions together to find the global solution. We note also that the
condition (21) determines the direction of $\hat{e}(x, t)$, but not its phase. A very elegant approach to this topic is one of Allan’s many contributions that we discuss in later sections.

Before ending this very rudimentary discussion, we note that the ideas presented here are also readily applicable in the quantum setting and what we have described is essentially the approach taken in adiabatic perturbation theory. These methods underly what has come to be called EBK quantization, but it must be supplemented by methods for dealing with the localized breakdown of the WKB approximation. These breakdowns occur in two different ways:

(i) Near caustics the family of rays used to construct the solutions have bad projections from phase space back onto configuration space, hence the phase assignment at $x$ is not well-defined and the amplitude can become be singular. A variety of authors have contributed to the theory here, and we point the interested reader to [20] as a point of entry to the literature.

(ii) Near degeneracies, the null eigenvectors of $D(t, \omega, x, k)$ will not be well-behaved and will change significantly with slight changes of position in phase space (see the earlier discussion for the uniform medium vector theory). In the quantum mechanics literature, this type of breakdown is often called a ‘non-adiabatic’ effect. The literature there stretches back to the 1930’s in the work of Landau, Zener, Stückelberg and Teller. Reviews of this older literature have appeared recently [21, 22] and Littlejohn has recently discussed Teller’s contributions to this topic [23]. In the plasma literature this phenomena is known as mode conversion. We emphasize that the underlying mathematics of these problems are essentially the same, even though the physical interpretations are very different. We will discuss mode conversion much more in a later section because it has been a major focus of Allan’s work in the last twenty years.

Up to this point we have been pursuing a very heuristic approach to the topic in order to emphasize that the ideas are very intuitive and flow directly from experience with the theory of uniform media. The time has come to discuss some of the more modern ideas that have been introduced. Allan’s work in plasma wave theory for the last twenty years has been informed by these newer ideas, which put the intuitive approach onto a much firmer mathematical foundation.

### 3.3. Semi-classical extensions: Geometrical ideas come to the fore

The field of semi-classical analysis has developed significantly since the 1960’s and the literature is wide and varied. It is very much alive as a field of research in mathematics and theoretical physics. Some of the ideas reach back to the 1930’s in the work of Moyal, Weyl and the later work of Wigner. But, some of the ideas are newer, especially the geometric flavor of the modern approach. In fact, semi-classical analysis is an area where the mathematical fields of geometry, algebra and analysis come together. What follows is an admittedly skewed picture of this field, colored by the authors’ particular viewpoint. No attempt has been made to carry out a full survey, which would have resulted in a very different paper, and we apologize to those whose work is not mentioned. Here we simply summarize some of the key ideas that will be of use describing Allan’s work in the area.

#### 3.3.1. Weyl symbols of operators

In his book on group theoretical methods in quantum mechanics [24], Weyl described a method for associating operators on Hilbert space with functions on classical phase space:

$$\hat{A} \leftrightarrow a(q, p).$$

Recall that a linear operator is defined abstractly by its action on vectors that live in a Hilbert space. It can be represented by a matrix if we choose a particular basis in the space. The symbol is simply a different way of representing the operator, but now on the vector space of functions on classical phase space.
The mapping \( \hat{A} \mapsto a(q, p) \) can be given explicitly. For example, using the \( q \)-representation of the operator we can express the symbol as:

\[
a(q, p) = \int d^n s \ e^{-i p \cdot s} |q + \frac{s}{2}|^n |q - \frac{s}{2}|
\]

In this formula, and in what follows, we ignore normalizing factors of \( 2\pi \) in order to focus on the conceptual relations and use the notation \( q = (q_1, \ldots, q_n) \) and \( p = (p_1, \ldots, p_n) \). Depending upon the circumstance, we might also use an extended phase space to include time and frequency.

To go backward from the symbol to the operator, we first take the Fourier transform of the symbol:

\[
\hat{a}(\sigma, \tau) = \int d^n q d^n p \ e^{i(\sigma q + \tau p)} a(q, p).
\]

Now use the transform of the symbol as a set of expansion coefficients over the set of phase space shifts generated by the operators \( \hat{q} \) and \( \hat{p} \). In essence, we do an inverse Fourier transform but, instead of using \( q \) and \( p \), use the operators \( \hat{q} \) and \( \hat{p} \):

\[
\hat{A} = \int d^n \sigma d^n \tau e^{-i(\sigma \hat{q} + \tau \hat{p})} \hat{a}(\sigma, \tau).
\]

Technicalities aside, the mapping \( \hat{A} \mapsto a(q, p) \) is essentially one-to-one and invertible. This is extremely useful, because it allows us to gain insight into the action of the operator \( \hat{A} \) by considering its ‘classical’ counterpart \( a(q, p) \). The mapping is also ‘topological’ in the sense that it preserves the notion of closeness: operators that are ‘close’ will map to symbols that are ‘close’. Hence, we can develop approximations to operators by using series expansions in the related space of symbols.

Because this mapping preserves neighborhood relationships, it must also reflect the fact that operators generally don’t commute. Hence, the symbol of the operator \( \hat{A} \hat{B} \) cannot be equal to the symbol of \( \hat{B} \hat{A} \). It is possible, however, to derive by direct calculation using the definition (24) how the symbol of the product \( \hat{A} \hat{B} \) can be written in terms of the symbols of \( \hat{A} \) and \( \hat{B} \), denoted \( a(q, p) \) and \( b(q, p) \) respectively, a result that goes back to Moyal. See, for example, Steve McDonald’s Physics Report on the subject [25]. The result is:

\[
\hat{A} \hat{B} \mapsto a(q, p) \ast b(q, p),
\]

where the ‘\( \ast \)’ product can be written as a Moyal series:

\[
a(q, p) \ast b(q, p) \equiv a(q, p) \ \exp \left[ \frac{i}{2} \left( \partial_q \cdot \partial_p - \partial_q \cdot \partial_p \right) \right] b(q, p).
\]

This result is completely general and relies only upon the formal definition of the mapping from operators to symbols. As we will see in Section 4.2, it was used in a very elegant way by McDonald and Kaufman to derive the wave kinetic equation for incoherent electromagnetic fields including weak sources and sinks [26]. To our knowledge, this was the first application of Weyl’s ideas in plasma wave theory.

We note a very important example of an operator symbol. If we have a wave function \( |\Psi\rangle \) in Hilbert space and construct its projector \( |\Psi\rangle \langle \Psi| \), then using the definition (24) a short calculation shows that the symbol of this operator is the Wigner function. This function has properties that make it useful for the study of waves in phase space. This will be covered in a moment.

Before moving on, we mention that our colleague Nahum Zobin has developed a theory of symbols based upon group theoretical ideas which shows that many results in the symbol calculus have a group theoretic origin. The product rule for symbols, for example, is simply a convolution theorem, but for a non-commutative group: the Heisenberg-Weyl group. (The more familiar convolution theorem from Fourier analysis is related to the translation group on configuration space, which is commutative.) The interested reader is referred to Steve Richardson’s PhD. thesis [27].
3.3.2. Pseudodifferential operators  The Weyl symbol calculus provides a further useful tool for the study of plasma waves: it allows us to convert integral operators to pseudodifferential form. In this form, the application of WKB methods becomes far more transparent. Suppose we start with a vector wave equation written in integral form:

$$\sum_m \int dt' d^nx' D_{nm}(t,t',x,x') \psi_m(x',t') = 0. \quad (29)$$

Is not immediately apparent how to apply WKB methods to this system. However, by first computing the symbol of the kernel using the Weyl prescription (here we have to use the extended phase space to include \((q,p) = (t,x,\omega,k)\)), it is possible to construct a pseudodifferential operator whose action on the Hilbert space of wave functions \(|\Psi\rangle\) is essentially the same as that of the original integral form.

A simple example of this idea can be given. The shift operator, whose action on a wavefunction \(\psi(x)\) is defined via \(\hat{T}_a \psi(x)\) \(\equiv \psi(x-a)\) can also be expressed in pseudodifferential form:

$$\psi(x-a) = e^{-a\partial_x} \psi(x), \quad (30)$$

where the right hand side (the Taylor series) is written in pseudodifferential form.

Using the Weyl calculus, the integral form of the wave equation can be recast into the pseudodifferential form:

$$\sum_m D_{nm}(t,i\partial_t,x,-i\nabla) \psi_m(x,t) = 0, \quad (31)$$

as assumed in earlier sections. This can be understood as simply a different way of characterizing the same operator. See [28] for details. To our knowledge, Allan Kaufman and Robert Littlejohn and their students were the first to bring these ideas into plasma physics.

We mentioned earlier the important fact that the symbol-operator mapping preserves notions of closeness. If it did not it would be of limited use. This suggests the following question: suppose we Taylor expand the symbol of \(\hat{A}\) about a fixed, but arbitrary, point in phase space:

$$a(q,p) \approx a(q_0,p_0) + (q - q_0) \cdot a_{q0} + (p - p_0) \cdot a_{p0} + \ldots \quad (32)$$

Now truncate this expansion. This is a new symbol and it has a related operator, \(\hat{A}'\), that is close to the original operator in some sense:

$$\hat{A}' = a(q_0,p_0) \hat{1}d + (\hat{q} - q_0) \cdot a_{q0} + (\hat{p} - p_0) \cdot a_{p0}, \quad (33)$$

where \(\hat{1}d\) is the identity operator in the relevant Hilbert space. The operators \(\hat{q}\) and \(\hat{p}\) are multicomponent position and momentum operators. For example, in the \(q\)-representation \(\hat{q}\) becomes multiplication by \((t,x)\) and \(\hat{p}\) becomes \((i\partial_t,-i\nabla)\). We now ask the question: In what sense is \(\hat{A}'\) close to \(\hat{A}\)? In ordinary function approximation, truncation of Taylor series leads to another function that is ‘close’ to the original one only within some region. What is the related region where \(\hat{A}\) and \(\hat{A}'\) agree? The region of agreement lies in the Hilbert space where the two operators act. The two operators ‘agree’ in their action on wave functions \(|\psi\rangle\) that lie in that region. These functions are local plane waves, or more precisely, they are wave packets. For example, in the \(q\)-representation they are wave functions of the form:

$$\langle q | \psi \rangle = \tilde{\psi}(q-q_0)e^{iq\cdot(q-q_0)} \quad (34)$$

where the envelope \(\tilde{\psi}\) is a smooth function of its argument. This notion can be developed much further. By an appropriate choice of envelope function it is possible to construct complete sets
of states. Using the Weyl calculus, the action of any operator can be then decomposed into its much simpler local action (in Hilbert space) on each of these wavepackets. This makes wave evolution look much more like the evolution of a classical gas. We refer the reader to Littlejohn’s 1986 review of his wavepacket formalism [28].

Thus, symbols provide a powerful means for constructing approximations to general operators, but the approximate form of the operator must be understood to act on wave functions that are localized in a particular region of phase space. In the mathematics literature, this is called microlocal analysis because it is local both in $q$ and $p$.

3.3.3. Eikonal waves and Lagrange manifolds  The previous discussion concerning symbols and pseudodifferential operators was very general. Now we consider a very special class of solutions of the wave equation: those which are of eikonal form. This introduces several important new ideas. It has been known since the time of Hamilton that in order to solve multidimensional wave equations using ray methods one must use a family of rays. The initial conditions for this family are determined by the initial/boundary conditions of the wave equation, and these conditions must be consistent with the search for a WKB solution. (For example, if we drive an antenna at the boundary of a plasma with a broadband space and time dependence then we should not expect to find the solution in the interior of the plasma to be of the narrowbanded eikonal form). The rays that evolve from WKB-type initial conditions (using Hamilton’s equations), fill out a smooth surface that has a dimension that is always equal to the dimension of the space $(t,x)$, and hence one-half the dimension of the full phase space $(t,x,\omega,k)$. This is because eikonality, which assumes there is a well-defined phase function $\theta(x,t)$, implies that at each $(x,t)$ there are a well-defined frequency and wave-vector: $(\omega(x,t) = -\partial \theta / \partial t, k(x,t) = \nabla \theta)$. This is valid so long as $\theta(x,t)$ is well-defined. What happens when this is no longer true? At caustics the ray evolution in phase space is still well-defined, but a problem has developed in the projection of the ray family back down to $x$-space. It is reasonable to ask, since the ray family is perfectly well behaved in phase space, whether it might be possible to choose a different projection (say to $k$-space) to find the way around the problem. This turns out to be possible.

Arnold introduced the concept of a Lagrange manifold to capture this essential idea: a Lagrange manifold is a surface in phase space that has a dimension that is one-half that of the full phase space and at each point on the surface the tangent vectors all have zero symplectic product. This implies that the surface locally looks like a configuration space. The notion of a Lagrange manifold is now a central idea in the geometric theory of Hamiltonian mechanics and semiclassical analysis. Simply put, Lagrange manifolds are the only surfaces in phase space that could be associated with ray families of some wave equation. This is a new idea that is needed in multidimensional problem but is a triviality in one spatial dimension. In one spatial dimension, with a two-dimensional phase space, a ray is a lagrange manifold. It is also fills the dispersion surface. In spatial dimensions greater than one, however, this is no longer true. In $2n$-dimensional phase space, a ray is still one-dimensional, but a Lagrange manifold is $n$-dimensional and a dispersion manifold $(2n-1)$-dimensional.

As mentioned earlier, WKB theory will break down near caustics. However, the Lagrange manifold associated with the ray family is perfectly well behaved. Even though it might develop folds or other caustic singularities under projection, it will always remain a Lagrange manifold because the surface is generated by Hamilton’s equations. This allows for a very systematic and elegant approach to classifying the types of caustic singularities that can develop and local normal forms for their analysis [20].

With these ideas in place, we now return to Allan’s post-1985 work in plasma wave theory. Following Lazar Friedland’s suggestion, we break this into ‘pre’- and ‘post’-conversion.
4. Prior to ‘conversion’

4.1. Wave chaos in stadia

In a 1979 PRL Allan and Steve McDonald described solutions of the two-dimensional Helmholtz equation for a stadium-shaped cavity [29]. The underlying classical system is chaotic, and the question arose of how the related quantum system behaved. In this context, the notion of ’quantum chaos’ concerns the nature of wave functions and eigenvalues when the short-wave-limit Hamiltonian has stochastic trajectories. In the 1979 PRL, it was shown that the probability distribution of eigenvalue separations, $P(\Delta E)$, is a Wigner distribution (characteristic of a random Hamiltonian), in contrast to the clustering found for a separable equation. The eigenfunctions exhibited a random pattern for the nodal curves, with isotropic distribution of local wave vectors. This was among the first papers in what became the very active area of quantum chaos. It was followed almost a decade later by a much longer paper giving more details of the calculations [30]. The local spatial correlation function $\langle \psi_n(x + \frac{1}{2}s)\psi_n(x - \frac{1}{2}s) \rangle$ and the probability distribution $P_n(\psi)$ of wave amplitude for normal modes $\psi_n$ were computed and compared with predictions based on semiclassical arguments applied to the nonintegrable classical Hamiltonian.

4.2. Allan’s work with Steve McDonald on the wave-kinetic equation

As mentioned earlier, to our knowledge Allan Kaufman and Robert Littlejohn were the first to use ideas from the Weyl calculus in plasma wave theory. Allan and Steve McDonald used these concepts in a very elegant paper [31] to show how to derive the wave-kinetic equation for incoherent electromagnetic waves. Because we have already described the Weyl product law for operators, we can describe this important paper very succinctly. Suppose we start with a wave equation for the electric field, written in abstract form as:

$$\sum_{m=1}^{N} \hat{D}_{nm}|E_m\rangle = |j_n\rangle; \quad n = 1, 2, 3,$$

(35)

where sources have been included on the right hand side. We use the Dirac notation because the logic of the approach is clearer. For concreteness, we can construct the $x$-representation of the wave equation by projecting onto $\langle x, t |$ and inserting a complete set of states:

$$\sum_{m=1}^{3} \int dt'dn x' \langle x, t | \hat{D}_{nm} x', t' \rangle |E_m \rangle = \langle x, t | j_n \rangle, \quad n = 1, 2, 3;$$

(36)

or

$$\sum_{m=1}^{3} \int dt'dn x' \hat{D}_{nm}(t, t', x, x') E_m(x'; t') = j_n(x; t), \quad n = 1, 2, 3.$$

(37)

The wave operator has an anti-hermitian part due to dissipation, as required by the Kramers-Kronig relations. The dissipation is treated perturbatively. Returning to the abstract notation, the adjoint form of the wave equation is:

$$\sum_{m=1}^{N} \langle E_m | \hat{D}_{mn}^\dagger \rangle = \langle j_n |; \quad n = 1, 2, 3.$$

(38)

Separating the wave operator into hermitian and anti-hermitian pieces (denoted $D'$ and $D''$ respectively) and introducing a formal small parameter to keep track of orderings, we have (keeping in mind that $|E\rangle$ and $|j\rangle$ are 3-component objects):

$$[\hat{D}' + \epsilon \hat{D}''] \cdot |E\rangle = \epsilon |j\rangle.$$

(39)
We can take the outer product with the adjoint equation to get, after a little manipulation:

\[
\hat{D}' + \epsilon \hat{D}'' \cdot |E\rangle \langle E| = \epsilon |j\rangle \langle j| \cdot \left( \hat{D}' + \epsilon \hat{D}'' \right)^{-1}.
\] (40)

This is an equation for the spectral tensor of the electric field $|E\rangle$, written in a representation-free manner.

Kaufman and McDonald now use the Weyl calculus to compute the symbol of this equation. They then invoke the Moyal formula (28) for the product of symbols. This expresses the equation in terms of the spectral tensors of $E(x; t)$ and the source $j(x; t)$, now given as distributions (Wigner tensors) on phase space. Up to this point, the calculation is completely general, though formal. They now proceed to use the small parameter $\epsilon$ to set up an asymptotic expansion. All quantities are expanded in powers of $\epsilon$ and like powers collected. Weak sources and weak dissipation means the conservative part of the free evolution dominates at leading order. One still must deal with the fact that the Moyal expansion is infinite order in derivatives. If the spectral tensors are smooth functions on phase space, then one can introduce an ordering in the Moyal series which assumes each order of derivative introduces a power of $\epsilon$. Strictly speaking, this is not true. Wigner functions are typically distributions, not smooth functions. However, for a scalar problem if the wave field is incoherent, rather than eikonal, the Wigner function can be smooth along the dispersion manifold, even if it is still sharply confined to the dispersion surface itself. This is because the fine structure off the dispersion manifold that appears in the Wigner function for coherent waves is assumed to phase mix away when incoherent fields are used. Therefore, the Moyal series is assumed to have a natural ordering in $\epsilon$ at each order of derivative.

The symbol of $\hat{D}'$ is the dispersion matrix $D'(t, \omega, x, k)$. It is self-adjoint at each point in the phase space and assumed to be a smooth function of its arguments. Therefore, at each point in phase space it has a set of real eigenvalues that are well defined. Because of the assumption that the sources are weak, at leading order it can be assumed that waves must lie on a dispersion manifold where one of the eigenvalues is zero. Kaufman and McDonald assume that there are no degeneracies (hence mode conversion is not allowed in this treatment). With these assumptions, it is possible to simplify the vector problem to a scalar one by projecting onto the null subspace associated with the mode in question. Then, assuming that the distribution function fills the dispersion manifold and is smooth along it, they show that the Moyal series has as its next order term a transport equation. They identify this as the wave-kinetic equation for the wave action.

4.3. Phase-space action principles

In 1987 Allan wrote a paper describing a phase-space Lagrangian action principle and the generalized $K - \chi$ Theorem [32]. In this paper the covariant coupled equations equations for plasma dynamics and the Maxwell field were expressed using a phase-space action principle. Carrying out a gauge-invariant Lagrangian Lie transform, the linear interaction is transformed to the bilinear beat Hamiltonian. This leads directly to the generalized linear susceptibility.

4.4. Covariant vector WKB

Many of the research ideas that Allan and co-workers had pursued in the previous decade came together in a very beautiful and exceedingly concise paper with Huanchun Ye and Yukkei Hui [33]. In this letter a Lorentz-covariant action principle for vector waves is introduced. Using the methods of the Weyl symbol calculus, this action principle is then rewritten in terms of quantities defined on ray phase space (e.g., the Wigner tensor and the dispersion tensor). Insertion of the eikonal ansatz is then shown to lead to the hamilton ray equations which define the Lagrange manifold of associated rays. At next order the wave action conservation law is recovered and shown to be due to a Noether symmetry. In addition, a Lorentz-covariant
transport law for the polarization phase is derived. The polarization phase transport has an interesting mathematical structure, and these results were followed up by Littlejohn and Flynn who showed that, although the phase transport equations themselves were not gauge invariant, the resulting quantization conditions were [34].

In the work described so far resonances were excluded from consideration. Removing this limitation leads us to the next topic: mode conversion.

5. The conversion to mode conversion

When two or more of the eigenvalues of the dispersion matrix are nearly zero the associated null eigenvectors will depend sensitively upon their position in the ray phase space, for reasons described in previous sections. The WKB approximation breaks down because of this and a new type of asymptotic approximation must be developed near the degeneracy. It is assumed that WKB is valid away from the region of degeneracy, hence the problem reduces to 1) finding the appropriate local description of the wave dynamics in the vicinity of the conversion, and 2) fitting this local solution onto the incoming and outgoing WKB solutions. The picture is that an incoming ray ‘splits’ into two outgoing rays and some type of connection formula must be applied ray-by-ray. In the immediate vicinity of the conversion, however, the local wave fields will not be well-described by the WKB solutions, so this simple picture should be viewed as heuristic only.

There is, of course, a huge literature on this topic because it occurs throughout physics. It can be found to play a role in fields as diverse as RF heating of fusion plasmas [35], ionospheric physics [36], ocean waves [37, 38, 39], magnetohelioseismology [40], atomic, molecular and optical physics [41], and neutrino physics [42]. While there is a large physics literature on conversion in one dimension (see [43, 44, 45, 46, 47, 48] and references therein), and WKB-type methods were applied to multicomponent wave equations as early as Rayleigh, there has been comparatively little attention given to multidimensional conversion in multicomponent wave equations. Exceptions to this in the physics literature are the papers of Bernstein and Friedland [49], Friedland [50], Friedland and Kaufman [51], Littlejohn and co-workers [52], Tracy and Kaufman [53, 54], and Krasniak and Tracy [55, 56], and Nassiri-Mofakham [57]. In the mathematics literature, of particular note is the work of Braam and Duistermaat [58, 59], Colin de Verdière [60, 61], and Emmrich and Weinstein [62, 63].

In the following sections we will recount Allan’s ‘conversion to conversion’, as recalled by Lazar Friedland who played a key role in exciting Allan’s interest in the topic, followed by a discussion of some of the applications and further developments that followed Allan’s conversion.

Allan’s conversion by Lazar Friedland

This section is adapted directly from notes provided by Lazar Friedland in the Fall of 2007. Lazar writes:

“In 1979-1982 I joined Ira Bernstein at Yale to work on further development of his general geometric optics (multi-dimensional WKB) formalism in space/time varying plasmas [49]. Formally, the geometric optics theory comprised a perturbation expansion with respect to the small parameter characterizing the slow variation of the background plasma. The perturbation scheme used a formal decomposition of the local (Hermitian) dielectric tensor in the wave propagation problem in terms of its eigenvectors and eigenvalues. A particular geometric optics mode was associated with vanishing of one of the eigenvalues (the nondegenerate case) and both the amplitude and the eikonal phase of the wave could be calculated by solving a set of ordinary differential equations along the characteristics (geometric optics rays) in the problem. A different version of the theory dealt with a degenerate plasma case, when two eigenvalues of the dielectric
tensor vanished simultaneously in some plasma region. In this case, the wave problem could be again reduced to a set of ODEs along the rays. However, in contrast to a \textit{single} amplitude equation in the nondegenerate case, the degeneracy yielded \textit{two coupled} amplitude equations, corresponding to the wave components along the two zero-eigenvectors.

In addition to these multidimensional theories, there existed an independent line of research of the so called \textit{mode conversion} problem, where a wave propagating in a nonuniform plasma could pass a resonance and excite another wave, so that far away from the localized resonant region, the two waves propagate independently. Two approaches to this problem were adopted in the early 80s, both limited to one-dimensional problems only. The first approach by Stix and Swanson [64] was based on a generalization of the tunneling problem. They associated a \textit{single}, higher order ODE for the wave amplitude to mode conversion events and studied its asymptotic solutions via the Laplace transform. The second approach, suggested by Cairns and Lashmore-Davies [65], related the mode conversion problem to local dispersion relations of the form \(|k - k_a(x)][k - k_b(x)] - \eta^2 = 0\). This equation allowed to guess a system of coupled differential equations for the amplitudes of the two modes, i.e.

\[
\begin{align*}
    dA_a/dx - ik_a(x)A_a &= i\eta A_b, \\
    dA_b/dx - ik_b(x)A_b &= i\eta^* A_a,
\end{align*}
\]

to describe the mode conversion from, say mode \(a\) to mode \(b\) at the resonance point \(x_0\), such that \(k_a(x_0) = k_b(x_0)\). Both approaches seem to be heuristic and unrelated and did not suggest a mechanism of taking into account higher dimensionality and such geometric optics effect as the ray divergence. Nevertheless, the approach of Cairns and Lashmore-Davies appeared to be more physical, because it could be also associated with the dispersion matrix

\[
D = \begin{pmatrix}
    k - k_a & \eta \\
    \eta & k - k_a
\end{pmatrix},
\] (41)

taking into account the multi-component structure of the coupled mode propagation problem. Furthermore, having \textit{two} coupled amplitude equations seemed to agree with my knowledge from the geometric optics theory of having a system of coupled amplitude equations in degenerate plasmas \((k_a(x_0) = k_b(x_0)\) is the point of degeneracy in the case of \(\epsilon = 0\)). Thus, I have concluded at the time that the next goal of the multidimensional WKB theory should be the development of the self-consistent extension of the perturbation approach to mode conversion in space/time varying media. Motivated by this goal, I developed a “renormalized” geometric optics theory of mode conversion in 1985 (see Ref. [50]). The theory was still one-dimensional, and the dielectric tensor was again decomposed in terms of its eigenvectors and eigenvalues. However, the formalism yielded a reduction of the embedded coupled mode system in plasma regions where locally, as one follows a mode along a ray in a nondegenerate plasma, an additional eigenvalue of the dielectric tensor becomes small, i.e., one enters a region of \textit{near degeneracy}. We have also understood that the use of the reduced \(3 \times 3\) dielectric tensor in mode conversion problems was too restrictive and started employing an unreduced \(N \times N\) dispersion matrix \(D\) in studying propagation of multicomponent waves with possible mode conversion events. One year later, we have also suggested another approach to mode conversion [66], which, instead of the decomposition of the large rank dispersion matrix, used a procedure similar to Gauss elimination and related to congruence transformations (see below). The starting point of this approach was a
general, unreduced vector-amplitude equation put in the following (conservative) form (the case of $1D$ in $x$)

$$iD \cdot A = -\partial D/\partial k \cdot dA/dx - \frac{1}{2} \frac{d(\partial D/\partial k)}{dx} \cdot A.$$  \hspace{1cm} (42)

We then showed that one can reduce the order of this system by eliminating the amplitude components one by one, so that the form of the equation for the remaining components at each reduction step remains unchanged and only $D$ changes and its rank decreases. By the end of this reduction procedure, one arrived at either a single equation of the aforementioned form for the last remaining amplitude component in the nondegenerate plasma regions (the final reduced $D$ being a scalar) or remained with a system of two coupled amplitude equations in nearly degenerate regions, where the reduced $D$ was a $2 \times 2$ matrix.

I have presented these theories at the APS meeting in 1985 and, for the first time, met Allan Kaufman, who came to my poster to discuss the new developments in the geometric optics theory. At that time, Allan and his student McDonald were working on the Weyl representation for electromagnetic waves [26] and was interested in our new eikonal-type theories. Shortly after this encounter, in 1986, he invited me to spend my first (after the tenure at the Hebrew University) sabbatical at LBL and join forces in studying the mode conversion paradigm. This sabbatical turned out to be most successful and resulted in three important extensions of the theory. Firstly, we have developed a self-consistent multidimensional mode conversion theory [67], yielding the generic form of coupled mode equations embedded in multicomponent wave propagation problems:

$$V_a \cdot \partial A_a/\partial x - i(x \cdot R^a)A_a = i\eta A_b,$$

$$V_b \cdot \partial A_b/\partial x - i(x \cdot R^b)A_b = i\eta^* A_a,$$  \hspace{1cm} (43)

where $x = (t, \vec{r})$ was defined around the resonance, $V_{a,b}$ and $R^{a,b}$ were the 4-group velocities and space/time dispersion coefficients associated with the two modes at the resonance, while $\eta$ was the off-diagonal element of the associated $2 \times 2$ dispersion matrix. We have also found the asymptotic solution of this system predicting a simple formula for the transmission coefficient of the incident wave flux through the mode conversion region, $T = \exp(-2\pi |\eta|^2 / |B|)$, with $B$ being the Poisson bracket $B = R^a \cdot V_b - R^b \cdot V_a$. Secondly, we have put the above-mentioned order reduction procedure [66] on a firm mathematical bases in 4D space/time varying plasmas showing its relation to congruence transformations [51]. In the same work we have also proved the generic nature of mode conversion events in multicomponent multidimensional wave propagation problems. Finally, in the same year, seeking mathematical simplicity and compactness, Allan developed a phase-space theory of multidimensional mode conversion [68].

On my return to Jerusalem after the sabbatical, I started working on the effect of the nonlinear frequency/wave vector shifts in mode conversion and three wave interactions, leading to a new autoresonant wave interaction paradigm [69]. At the same time, Allan and his colleagues (R. Littlejohn, E. Tracy, A. Brizard, J. Morehead, G. Flynn, and A. Jaun), continued working on further developing the linear mode conversion theory and its applications. Some of their important results are the wave-kinetic formulation of incoherent mode conversion [70], the normal forms for linear mode conversion [43], the oceanographic mode conversion [37], the introduction of the ray helicity concept [88, 71], and applications to tokamak plasmas [17].
In summary, the above mentioned developments during a relatively short period of time (1985-1987), have changed and shaped the research interests of Allan Kaufman, myself and other people for the next two decades. I feel very fortunate of being a part of these developments.”

This ends Lazar’s summary of how he contributed to Allan’s conversion to mode conversion.

6. Phase space methods in mode conversion

Allan’s subsequent work on mode conversion came to rely heavily upon ray phase space ideas. He was led to this formulation upon noting that the transmission coefficient involved the Poisson bracket of the two dispersion functions for the resonant waves. This suggested that symplectic geometry was playing a role that needed to be understood. In addition, he drew heavily upon the (then) recent work of Littlejohn on wavepacket dynamics and the metaplectic transformation.

The phase space viewpoint was first presented in a paper with Friedland [68] which we will discuss in some detail momentarily because the ideas introduced have informed all of the authors’ subsequent work with Allan in this area. But, first, let’s consider why mode conversion is naturally viewed in ray phase space. A more complete discussion of Allan’s thinking on these matters can be found in [19]. This paper is not well known, but shows how far along Allan’s thoughts were at that time.

Consider a wave packet moving in more than one spatial dimension in a nonuniform plasma. The envelope is transported by the local group velocity, while the phase evolves according to the local phase velocity. The directions associated with these local velocities are, in general, completely unrelated. Mode conversion is a resonance between two modes that local have the same phase velocity, but the group velocities are not required to have any particular relation to one another. Furthermore, in general the coupling coefficient will depend upon the location in phase space, which leads to the possibility of ray helicity and conical intersections of dispersion manifolds, rather than ‘avoided crossings’. These conical intersections are generic in higher dimension and cannot occur in one dimensional problems. This will also be discussed in the following sections. We first discuss briefly Littlejohn’s wavepacket formalism and the metaplectic transformation.

6.1. Littlejohn’s wave packet theory and the metaplectic group

Here we discuss related work by Allan’s former student Robert Littlejohn because the ideas will become of central importance in Allan’s later work on mode conversion. This was independent work of great beauty and is based upon Weyl’s insight that all operators, including evolution operators associated with wave equations, can be decomposed into phase space shifts using irreducible representations of the Heisenberg-Weyl group (this is an example of what is now called non-commutative harmonic analysis). Each phase space shift has a simple action and if the wave function is decomposed on a wave packet basis, the full wave evolution now looks like that of a gas of particles [72].

In addition to shifts on phase space, Littlejohn considers linear canonical transformations. These arise in the study of wave packet evolution because when the hamiltonian flow is linearized about a reference orbit, the nearby orbits evolve in time via a family of linear canonical transformations. Linear canonical transformations are of the form (x and k should be thought of as n-dimensional column vectors, and A, B, C and D are n × n real matrices):

\[
\begin{pmatrix}
q \\
p
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
x \\
k
\end{pmatrix}
\] (44)

Requiring this to be a canonical transformation implies the the matrix is symplectic. The symplectic matrices form a group. The Heisenberg-Weyl group and the linear symplectic group
are related to one another in a manner analogous to translations and rotations in \( n \)-dimensional configuration spaces.

Consider now wavefunctions on configuration space \( \psi(x) \). Translations in \( x \) are easy to understand. Translations in \( k \) are multiplications by \( \exp(ik \cdot x) \). Therefore, the Heisenberg-Weyl group is represented by general combinations of shifts in \( x \), multiplication by a plane wave, and multiplication by an overall constant phase. How is the group of linear canonical transformations to be represented in this space? By analogy with rotations, there is a set of unitary operators acting on the Hilbert space of wavefunctions \( \psi \) which satisfy the same product law as the symplectic matrices. This set of operators forms a group, and the transformations are called the metaplectic transformations. They should be thought of as generalizations of Fourier transforms. They are extremely useful because they allow us to take any pair of operators that are linear in \( \hat{x} \) and \( \hat{k} \) that do not commute and convert them into a ‘position’ and ‘momentum’ pair. In the appropriate representation, this can then be used to greatly simplify the local wave equation, as will be discussed in the following sections.

6.2. Plasma echoes and the Budden problem

In a series of papers with his graduate student, Huanchun Ye, Allan pursued the use of these new ideas in the context of gyroresonance absorption. As part of Huanchun’s PhD dissertation, Lagrangian Lie transform methods were used to introduce a gyrokinetic description of the plasma and to isolate those regions of the system where gyroresonance was occurring. Following a suggestion of Friedland’s, the crossing of a gyroresonance was treated as a mode conversion between a collective plasma wave (such as a magnetosonic wave) and a ‘gyroballistic’ wave (a solution of the undriven gyrokinetic equation). The entire process can then be viewed in ray phase space. The gyroballistic waves have the interesting property that they propagate slowly in \( x \)-space (they are tied to the drift orbits), but can have their primary propagation in \( k \)-space. This can be understood by considering a family of gyrating ions in a nonuniform magnetic field. A phase pattern imprinted on these local ‘clocks’ will evolve because of the nonuniformity, which leads to the evolution in \( k \)-space. These ideas lead to an appealing explanation of the phenomenon of linear ion-cyclotron echoes, which had been observed by spacecraft [73, 74].

The idea that a Budden-type resonance could be treated in phase space as a two-step conversion process emerged from this work and has been used by our group in a variety of settings ever since.

6.3. Incoherent mode conversion and the use of metaplectic methods

One of the authors (Tracy) recalls first becoming aware of Allan Kaufman’s work while a graduate student at the University of Maryland in the early 1980’s. Word of Allan’s work came through Steve McDonald when he arrived as a post-doc to work with Ed Ott, and through a seminar given by Celso Grebogi. While my thesis work concerned inverse scattering theory and nonlinear waves, I found the ideas in Steve and Celso’s work very interesting and wanted to learn more. Upon graduation, I applied to work with Allan but he already had a post-doc (Shayne Johnston). I decided to visit Berkeley anyway during a visit to the West Coast in the spring of 1984 as it would give me a chance to meet him. As it turned out, Allan was unable to be there when I visited, but I remember the kindness of Wulf Kunkel and Robert Littlejohn, who showed me around campus and arranged for my visit in Allan’s absence. I wound up accepting a position at William and Mary, but kept abreast of what Allan was up to and finally got to meet him at the APS-DPP and Sherwood meetings. In the late 80’s, I was collaborating with Allen Boozer, then at William and Mary, on some problems in geometric optics and decided that I wanted to take the opportunity to make contact again with Allan Kaufman. So, at the APS meeting in 1989 I suggested to Allan that I come for a visit to Berkeley so we could work on problems of
The first problem we worked on concerned multidimensional mode conversion. We chose this because Allan was still pondering the implications of his work with Lazar. Although we have strayed into other areas since then, resonance crossing in one variation or another has been the major theme of our collaboration.

The first task was to compute the converted wave amplitude using Allan and Lazar’s normal form (the Physics Letter they had recently published only treated the transmission coefficient). As Lazar and Allan did in their Physics Letter, we used metaplectic ideas to simplify the coupled PDEs. We verified that the resulting $S$-matrix was unitary. Having the full $S$-matrix available meant that it was possible to treat the situation with incoming disturbances on both channels, hence the outgoing disturbances would be an interference pattern. Because Steve and Allan’s work on wave kinetic equations for incoherent waves was still on our minds it was natural for us to ask whether we could include conversion effects in the wave-kinetic equations. We accomplished this phenomenologically by insisting that away from conversion the two wave types satisfy the action transport equation (the wave kinetic equation), but when one of the rays of type $a$ punctures the dispersion manifold of waves of type $b$ the action has to be split between the two rays [70]. This approach implicitly assumes incoherence because the interference pattern is ignored using a handwaving argument of the type that Marshall Rosebluth used to call a ‘random phrase approximation’. Action conservation is satisfied, however.

In order to put these results on a somewhat firmer footing, we then revisited the solutions of the local $2 \times 2$ wave equation and, without making an assumption of incoherence, asked what the full solutions looked like in detail. Instead of taking the solutions in the $q$- or $p$-representations (which were simple) and using an inverse metaplectic transformation to bring them back to $x$-space (which looked hard in a technical sense, and it was, see [75] for the details), we used the fact that the Wigner function was invariant under metaplectic transformations. This allowed us to compute the Wigner function in whatever representation was easiest. The result showed that, near conversions, the Wigner function has a very complicated interference structure on the ray phase space. Because the Wigner function involves integrals over the whole space-time, parts of the Wigner function are non-causal. However, when an integration is carried out across the dispersion manifold, one recovers causal behavior for the action density [18].

### 6.4. The Berkeley plasma theory group

While Allan and Gene were beginning their collaboration, the second author (Alain Brizard) had arrived at Berkeley and joined the very active ferment of ideas being discussed. Alain’s work in gyrokinetics and action principles is discussed in a separate article, and his work on mode conversion will be covered in a moment.

At this time (in the early 90’s), Allan, Alain and Robert Littlejohn shared a set of offices in Building 4 at LBNL. Robert and Allan’s students were also in Building 4, so there was a very definite buzz of activity. Some of the work Robert was pursuing at this time was directly relevant to what we were working on, so we always took advantage of this situation to ask Robert and his students for comments and suggestions. During these years, it was a great pleasure to get to know some very outstanding young scientists: Steven Creagh, Chris Jarzynski, Greg Flynn, Kevin Mitchell, Huanchun Ye, Dan Cook, Jim Morehead, Stefan Weigert, Matt Cargo and Matthias Reinsch.

Robert’s work with Greg Flynn was of particular interest to us because it concerned the basic theory of mode conversion [43], with applications not only in plasma [76], but also in quantum systems [77]. One particularly impressive result concerned the appearance of conical intersections, rather than avoided crossings, in multidimensional mode conversion. These cannot occur in one dimensional systems and are a truly new feature of the higher-dimensional problem.
They occur because the off-diagonal term of the local $2 \times 2$ wave equation depends upon the point in phase space about which the expansion is taken. Therefore, generically, there will be points in the phase space where the off-diagonal coupling is zero. At such conversions, the entire dispersion matrix vanishes at linear order and the local geometry of the dispersion manifolds in phase space has a cone structure. Exploiting the fact that the dispersion matrix is a $2 \times 2$ hermitian matrix, Robert and Greg used a decomposition in terms of the Pauli matrices:

$$D = \left( \begin{array}{cc} D_{11} & D_{12} \\ D_{12}^* & D_{22} \end{array} \right) = \sum_{\mu} B_\mu \sigma^{\mu},$$

(45)

with $B_0 = (D_{11} + D_{22})/2$, $B_1 = \text{Re}(D_{12})$, $B_2 = \text{Im}(D_{12})$, and $B_3 = (D_{11} - D_{22})/2$. With this reparametrization of the dispersion matrix, the determinant takes the form:

$$\det(D) = B_0^2 - B_1^2 - B_2^2 - B_3^2.$$ 

(46)

The condition $\det(D) = 0$ is now seen to imply that the four-vector $(B_0, B_1, B_2, B_3)$ must lie on the ‘light cone’. The components $B$ are functions on phase space, and the apex of the light cone is the point where all entries of $D$ vanish. (Strictly speaking, it is only a point in four-dimensional phase spaces. In higher dimensions the local geometry will locally be the product of a cone and a plane.) This is important because, near a conical intersection, the outgoing wave field will have a contribution (or deletion, depending upon the channel) that has the form of a gaussian beam rather than being eikonal.

Littlejohn and Flynn then developed some very beautiful arguments concerning the universality of the $S$-matrix connection coefficients, showing that they are almost completely determined by symmetry requirements [34].

Through the 90’s and into the 00’s, we continued to extend this line of effort in various ways, as described in the next section.

A tutorial article in Physics of Plasmas appeared in 2003 that summarized our understanding of ray-based methods in mode conversion up to that point [78].

7. Extensions, elaborations and related work by other groups

7.1. Minority gyroresonance and the computation of the reflected wave in tokamak geometry

Lazar Friedland had pointed out that when a collective wave crosses a minority gyroresonance the mathematical form of the equations wave look very much like that for a mode conversion. This is often overlooked because the particle disturbance, as described by the linearized evolution equation for the perturbed particle distribution, is usually eliminated in favor of the electric field. This leads to the usual singular denominator associated with wave-particle resonance. If one does not eliminate the particle distribution, but instead treats the electric field and particle distribution on an equal mathematical footing, then the resulting equations are similar to the $2 \times 2$ form of a mode conversion. The reflected wave is emitted when the disturbance in the minority-ion distribution has evolved until it satisfies the collective wave dispersion relation once more (this is the same idea that appears in Huanchun Ye’s work with Allan). Strictly speaking, this is only true when the minority ions are cold. When they are warm, then there is an infinite family of minority ‘modes’ that take part in the conversion. Each is resonant at a different spatial position because of Doppler effects and the spatial variation in the local magnetic field strength.

Allan’s student, Dan Cook, pursued these issues and reported on them in a series of papers. The work proceeded along two fronts: a full kinetic treatment of conversion to gyroballistic waves in one-dimension, and conversion to cold gyroballistic waves, but now in two spatial dimensions.

In [79], the conversion to a continuum of gyroballistic waves was considered for a one-dimensional slab model. Tor Flå was visiting the Berkeley group at the time and joined in
this work. This paper gave the first analytic formulas for energy transfer from a magnetosonic wave to the minority Bernstein wave and the absorption profile, for plasma heating via minority gyroresonance. The absorption profile stems from: (1) gyroresonant damping of the Bernstein wave; (2) phase mixing of the kinetic residue. As part of this work, Dan did a direct comparison between the semi-analytic theory and a direct numerical simulation, getting good agreement in the limit of small minority fraction. Computation of the absorption profile using these ideas required developing a means to extract the minority-ion Bernstein wave which arises due to the self-consistent fields associated with the gyroballistic waves. Strictly speaking, when these interactions are included one is no longer dealing with gyroballistic wave, but with Case-van Kampen modes. The method of Case and van Kampen was first developed in a uniform plasma. The extension to non-uniform plasma was due to Bateman and Kruskal, which could be adapted to the present setting.

In [80] the reflected wave in tokamak geometry was constructed using a very simple conceptual model. Phase-space methods were used for the conversion of an incident magnetosonic wave field to a continuum of gyroballistic waves, for their propagation along guiding-center orbits in the poloidal plane of a tokamak, and for their conversion to the two-dimensional magnetosonic reflection field. The wave fronts and amplitude of that field were obtained explicitly. This work did not include a full kinetic treatment.

The further development of these ideas to compute the conversion to the minority-ion Bernstein wave in tokamak geometry required extending the Bateman-Kruskal theory to multidimensions [81]. The generalization to many dimensions and non-trivial geometries required several important new developments: In tokamak geometry particles can be trapped, an effect that is absent in the slab model. Also, the ray propagation dynamics for both the free gyroballistic waves and the collective minority-ion Bernstein wave is far more complicated than in the slab model. In particular, a resonance zone was identified wherein the gyroballistic waves interact strongly and cannot be treated as free. We used the Weyl calculus to construct a local form of the self-consistent gyroballistic equation within the resonance zone. This reduced equation was simplified via a metaplectic transformation. After this simplification, the equation was shown to be of Case-van Kampen type with weak non-uniformities; hence there are no true Case-van Kampen eigenfunctions. Using the Bateman-Kruskal approach, a local Case-van Kampen basis was constructed and the initial-value problem solved.

7.2. Multiple conversion
Because the conversion is local in phase space, this means that multiple conversions can be dealt with in a modular fashion if they are well separated. It is important to realize that this separation is required to be in phase space, not in configuration space. It is possible for two conversions to lie over one another in z-space yet be well separated in phase space because they occur for very different k’s. This is, for example, the case with a Budden-type resonance, as discussed in [82, 83]. In these Budden-type of double-conversions, the signal pathways are all open, hence there is no possibility of interference effects occurring due to rays following multiple pathways.

In some cases, however, multiple conversion can occur in such a way that signal can follow closed circuits in phase space. An example occurs in one spatial dimension when the ray paths are curved and cross (like intersecting parabolas). If the two crossing are sufficiently far apart, they can be treated in modular fashion again and the global input/output relationships can be deduced by repeated application of the single-step conversion formulas [84]. These effects can be particularly dramatic if the two intersection waves are of opposite energy, in which case there is amplification at each crossing and the double-crossing becomes an absolute instability. A simple model with multiple conversions was considered in [85], where it was shown that to get the proper phase matching between the conversions one must go beyond WKB order and
include the leading order effects of the coupling. Once this is done, the interference effects are correctly captured using the modular form. More recently, we have considered the possibility of recirculation that can occur if a tertiary wave is present, in addition to the primary and secondary waves usually associated with a Budden resonance. This tertiary wave might be due, for example, to the presence of energetic particles [86].

Multiple conversion has been an ongoing interest of the group, and we have returned to it several times. For example, when considering the effects of flow on the magnetosonic to ion-hybrid conversion, we found that a virtual cavity can be created and multiple conversions can occur if the flow has sufficient shear [87]. Because of the local Doppler shift caused by the flow, the dispersion relations of waves are modified, and a single Budden resonance may be replaced by a triple resonance. The occurrence of such triplication depends on the layer’s parameters: its location, maximum flow speed, and radial width. We used a standard one-dimensional slab model, with cold-plasma response in the ion-gyrofrequency range, with the example of conversion of a magnetosonic wave to ion-hybrid waves. When triplication occurs, wave energy is converted at three locations instead of one. This possibility must be taken into account in designing conversion scenarios for heating, for current-drive, or for flow-drive.

7.3. Wave emission in nonuniform plasmas
Consider the emission of waves by sources, such as resonant particles. This is usually dealt with by assuming the plasma is uniform near the localized source. From this, one computes the emission spectrum using Fourier methods and propagates this disturbance to the far field then switches to WKB. But, what if the nonuniformity of the medium cannot be neglected near the source? Allan noted that in phase space this is essentially a form of resonance crossing and in a series of papers we used the tools we had developed for analyzing mode conversion in phase space to attack this problem. In [54] we considered the case of emission of a scalar wave by a gyrating particle, while in [55, 56] we considered the case of emission that occurs within a mode conversion region.

7.4. Ray helicity
Robert and Greg’s conical intersection work [77], and their work on the universality of the connection coefficients [34], inspired us to pursue more fundamental issues on conversion. In particular, we considered how to simplify the local $2 \times 2$ wave equation as much as possible by various changes of representation, guided by the proposition that for a ray-based quantity to have a fundamental physical meaning, it must be invariant under the following two groups of transformations, which are used to construct solutions: congruence transformations (which involve linear combinations of components of the multicomponent wave field) and canonical transformations (which act on the ray phase space). In [88] we showed that for conversion between two waves there is a new invariant not previously discussed: the intrinsic helicity of the ray. If the determinant of the dispersion matrix is used to generate rays, then it is possible to show that the ray motion in phase spaces of dimension higher than two will always be a combination of hyperbolic and elliptic motions, and that the combined motion is locally a helix, hence a one-dimensional model is not adequate. This ray helicity can be invariantly defined and enters into the $S$-matrix coefficients (unpublished).

In [89] we applied these ideas to the conversion from a magnetosonic wave to an ion-hybrid wave in tokamak geometry. A cold-plasma model was introduced in this paper which exhibits ray helicity in conversion regions where the density and magnetic field gradients are significantly nonparallel. For illustration, such regions are identified in a model of the poloidal plane of a DT tokamak plasma. In each conversion region, characterized by a six-sector topology, rays in the sector for incident and reflected magnetosonic waves exhibit significant helicity. A detailed
analytic and numerical study of helical rays in this sector was then developed for a “symmetric-wedge” model.

7.5. Applications beyond plasma physics
Darryl Holm suggested that we should consider applications of mode conversion ideas in the theory of equatorial waves and led us to some earlier work on upwelling in the Gulf of Guinea off the coast of West Africa. A resonance had been identified between an equatorially trapped Rossby-Kelvin wave (also known as a Yanai wave) and a coastally trapped Kelvin wave. The coupling of the waves is due to the overlap of their eigenfunctions (normal modes in \( y \), the meridional coordinate). After deriving coupled mode equations from a variational principle, which gives the coupling constant between the two modes at the conversion point, we then used the S-matrix formulas obtaining an analytic expression for the wave-energy conversion coefficient, in terms of the wave frequency and the scale length of the thermocline depth [37]. These ideas were later picked up and used by Tailleux and McWilliams in several papers [38, 39].

Cally and co-workers have used phase space methods in their study of magnetohelioseismology [40] where ray splitting has been found in certain models of the solar atmosphere.

7.6. Algorithms for numerical computation
In addition to studies of more foundational issues, our group has also pursued the development of a practical numerical algorithm [90, 17, 75]. A ray tracing algorithm must allow for the ray splitting that occurs at a mode conversion, and requires solutions of the following subproblems: (1) how to detect conversion regions while following rays; (2) once conversion is detected, how to fit to a generic saddle structure in ray phase space associated with the most common type of conversion (the ‘avoided crossing’ type); (3) given the saddle structure, how to carry out a local projection of the full vector wave equation onto a local two-component normal form that governs the two resonantly interacting waves. This determines both the uncoupled dispersion functions and the coupling constant, which in turn determine the uncoupled WKB solutions; (4) given the normal form of the local two-component wave equation, how to find the particular solution that matches the amplitude, phase, and polarization of the incoming ray, to the amplitude, phase, and polarization of the two outgoing rays: the transmitted and converted rays. This work was carried out in collaboration with André Jaun, who had previously worked on the development of full-wave codes to study conversion in tokamak plasmas, and more recently with Steve Richardson.

7.7. Theoretical foundations and work in progress
The phase space approach that our group has pursued is based upon the Weyl symbol calculus. Our colleague Nahum Zobin has developed a new version of this theory that is based upon group theoretic ideas. This leads to a generalization of the Weyl symbol, and the usual definition (24) is then seen to be a special case. This new formulation of the symbol calculus is based upon the representation theory of the Heisenberg-Weyl group. The theory then leads very naturally to the phase space path integral (a variant of the Feynman path integral, which is on configuration space). The path integral arises when we ask the following question: suppose we wish to solve an evolution equation of the Schrödinger form:

\[
i \hbar \partial_t |\psi\rangle = \hat{H} |\psi\rangle.
\] (47)

Suppose, in addition, we know the symbol of the operator \( \hat{H} \), denoted \( H(x,k) \). The standard way to solve (47) is to exponentiate \( \hat{H} \) to find the evolution operator \( \hat{U}_t = \exp(i\hat{H}t) \). But, this requires a full knowledge of the eigenvalues and eigenvectors of \( \hat{H} \) (so that we can diagonalize it). This information is usually not available. The only other general method for computing the
The evolution operator is to use the symbol calculus. In this approach, we first construct the symbol of $\hat{U}_t$ using our knowledge of the symbol $H(x,k)$. Knowledge of the symbol is tantamount to knowledge of the operator. When this calculation is carried out, the path integral emerges. In practice the path integral cannot be computed because it involves summations over all possible paths, the vast majority of which are wild. However, one usually invokes the limit $\hbar \rightarrow 0$ and considers the leading term in the asymptotics of the path integral, which usually are dominated by the classical (ray, or WKB) paths. See Steve Richardson’s dissertation for details [27].

Suppose we apply these ideas to $2 \times 2$ wave equations. In that case $\hat{H}$ is a $2 \times 2$ operator-valued matrix and $|\psi\rangle$ a two-component spinor. The symbol of $\hat{H}$ is a $2 \times 2$ hermitian matrix. The ‘classical limit’ of the path integral in this circumstance is very subtle when there is a resonance. In fact, the path integral suggests that it is the diagonals of the symbol of $\hat{H}$, that is $H_{11}(x,k)$ and $H_{22}(x,k)$ that are acting as the ray Hamiltonians, not the determinant or the eigenvalues. The choice of representation becomes very important, because in some representations the asymptotic limits will be cleaner than others. This led us to a new definition of the normal form for $2 \times 2$ wave equations: it is a representation where the symbols on the diagonal Poisson-commute with the symbols on the off-diagonals. In this representation, the diagonals can be interpreted as the uncoupled ray Hamiltonians and the off-diagonals are constant along rays generated by the diagonals (because of the condition of Poisson commutativity). This makes it natural to interpret the off-diagonals as coupling constants. We are currently pursuing these ideas, and attempting to construct practical algorithms for generating this normal form representation while following a ray.

Our group is currently working with Allan on several problems: the direct comparison of ray-based and full-wave calculations for multidimensional conversion, the development of phase space representations for full-wave calculations, and the application of modular methods to compute full-wave maps in tokamak geometry. Our extended group of collaborators has been joined by two Ph.D. students, Yanli Xiao and David Johnston, who are working on various aspects of these problems.

8. Summary
It is clear from this selective survey that Allan’s years of work in plasma wave theory have been highly creative and productive. Many new ideas were brought into the field either directly by Allan, or through his close association with Robert Littlejohn. Their students have become productive researchers as well. For the two authors of the current paper, the plasma theory group at Berkeley has always been a stimulating place to visit and we have very much enjoyed being a part of the group.

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