SPHERICAL FUNCTIONS FOR SMALL \( K \)-TYPES

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Abstract. For a connected semisimple real Lie group \( G \) of non-compact type, Wallach introduced a class of \( K \)-types called small. We classify all small \( K \)-types for all simple Lie groups and prove except just one case that each elementary spherical function for each small \( K \)-type \((\pi, V)\) can be expressed as a product of hyperbolic cosines and a Heckman-Opdam hypergeometric function. As an application, the inversion formula for the spherical transform on \( G \times_K V \) is obtained from Opdam’s theory on hypergeometric Fourier transforms.

1. Introduction

Let \( G \) be a connected real semisimple Lie group with finite center. Let \( G = KAN \) be an Iwasawa decomposition of \( G \). \( K \)-bi-invariant \( C^\infty \) functions on \( G \) are called spherical functions and are important in the analysis of functions on the Riemannian symmetric space \( G/K \). In particular, a key role is played by those spherical functions that are elementary. Here a spherical function \( \phi \) is called elementary if it is non-zero and satisfies the functional equation

\[
\int_K \phi(xky)dk = \phi(y)\phi(x) \quad \text{for any } x, y \in G
\]

(\( dk \) is the normalized Haar measure on \( K \)), or equivalently, if it takes 1 at \( 1_G \in G \) and is a joint eigenfunction of the algebra \( D \) of the invariant differential operators on \( G/K \) (cf. [HC1, Hel2]). It is well known that the spherical transform (also called the Harish-Chandra transform) defined by elementary spherical functions essentially gives the irreducible decomposition of \( L^2(G/K) \).

Now suppose \((\pi, V)\) is any irreducible unitary representation of \( K \) (a \( K \)-type for short). When we consider the analysis of sections of the vector bundle \( G \times_K V \) in a parallel way to the case of \( G/K \) (which corresponds to the trivial \( K \)-type), there naturally appears a notion of elementary spherical functions for \((\pi, V)\). Unfortunately the general theory for such functions, which has been developed by [G, War, Ti, GV] and others, has some inevitable complexity. But it can be considerably reduced when the algebra \( D^\pi \) of the invariant differential operators on \( G \times_K V \) is commutative (cf. [Ca]). We know from [De, Theorem 3] that \( D^\pi \) is commutative if and only if \( V \) decomposes multiplicity-freely as

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an $M$-module ($M$ is the centralizer of $A$ in $K$). In what follows we assume that $(\pi, V)$ satisfies this condition.

**Definition 1.1.** An $\text{End}_C V$-valued $C^\infty$ function $\phi$ on $G$ is called $\pi$-spherical if

\begin{equation}
\phi(k_1gk_2) = \pi(k_2^{-1})\phi(g)\pi(k_1^{-1}) \quad \text{for any } g \in G \text{ and } k_1, k_2 \in K.
\end{equation}

The space of $\pi$-spherical functions is denoted by $C^\infty(G, \pi, \pi)$. A $\pi$-spherical function $\phi$ is called elementary when it is non-zero and satisfies

\begin{equation}
\int_K \phi(xky) \text{Tr} \pi(k)dk = \frac{1}{\dim V} \phi(x)\phi(y) \quad \text{for any } x, y \in G.
\end{equation}

As we see in §3.2 in detail, $D^\pi$ naturally acts on $C^\infty(G, \pi, \pi)$.

**Theorem 1.2 ([Ca, Theorem 3.8]).** A given $\phi \in C^\infty(G, \pi, \pi)$ is elementary if and only if it takes $\text{id}_V$ at $1_G$ and is a joint eigenfunction of $D^\pi$.

The theorem is certainly central when we investigate analytical properties of elementary $\pi$-spherical functions. However, to get even a little of explicit results, two more things seem necessary, namely, the structure of $D^\pi$ and a version of Chevalley restriction theorem for $C^\infty(G, \pi, \pi)$. As shown below, both of them are available if $(\pi, V)$ is small in the sense of Wallach.

**Definition 1.3 ([Wall, §11.3]).** A $K$-type $(\pi, V)$ is called small if $V$ is irreducible as an $M$-module.

In this paper we restrict ourselves to the study of elementary $\pi$-spherical functions for small $K$-types. So hereafter let $(\pi, V)$ be a small $K$-type. First, we have a lot of the same results as in the case of the trivial $K$-type. For example, there is a generalization of the Harish-Chandra isomorphism by Wallach (Theorem 3.1):

\[ \gamma^\pi : D^\pi \cong S(a_C)^W. \]

Here $S(a_C)$ is the symmetric algebra for the complexification of the Lie algebra $a$ of $A$. (As usual, when a capital English letter denotes a Lie group, the corresponding German small letter denotes its Lie algebra.) $W$ is the Weyl group defined, as usual, by $W = \tilde{M}/M$ with $\tilde{M}$ being the normalizer of $A$ in $K$. $S(a_C)^W$ is the subalgebra of $S(a_C)$ consisting of the $W$-invariants. Furthermore we have

**Theorem 1.4.** For any $\lambda \in a_C^*$ (the dual linear space of $a_C$), there exists a unique $\phi^\pi_\lambda \in C^\infty(G, \pi, \pi)$ such that

\begin{equation}
\phi^\pi_\lambda(1_G) = \text{id}_V \quad \text{and} \quad D \phi^\pi_\lambda = \gamma^\pi(D)(\lambda) \phi^\pi_\lambda \quad \text{for any } D \in D^\pi.
\end{equation}

Moreover, $\phi^\pi_\lambda$ is real analytic on $G$. Thus the elementary $\pi$-spherical functions are parameterized by $\lambda \in W \setminus a_C^*$. 

The proof of the theorem is given in §3.3, together with two integral formulas for $\phi_\lambda^\pi$.

Now, since one easily sees from (1.1) the restriction $\phi|_A$ of any $\phi \in C^\infty(G, \pi, \pi)$ to $A$ takes values in $\text{End}_M V$ and since the $C$-algebra $\text{End}_M V$ is isomorphic to $C$ by Schur’s lemma, $\phi|_A$ is identified with a $C$-valued $C^\infty$ function on $A$. Let $\Upsilon^\pi(\phi) \in C^\infty(a)$ be the composition of $\phi|_A$ with $\exp : a \to A$.

**Theorem 1.5** (the Chevalley restriction theorem). The restriction map $\Upsilon^\pi$ is a linear bijection between $C^\infty(G, \pi, \pi)$ and $C^\infty(a)^W$.

The proof is given in §3.1. Through this bijection, (1.3) is translated into a condition on $\Upsilon^\pi(\phi^\pi_\lambda)$ to be a joint eigenfunction of a commuting family of differential operators on $a$. The family consists of the $\pi$-radial parts of $D \in D^\pi$. The $\pi$-radial part of the Casimir operator, which has a prominent role in the family, is expressed in a uniform way by use of a parameter $\kappa^\pi$, whose precise definition is given in §3.4, is a $W$-invariant function on the set $\Sigma = \Sigma(g, a)$ of restricted roots. (In general, a Weyl group invariant function on a root system is called a multiplicity function.)

Of course, we could proceed further with our study analogously to the case of the trivial $K$-type, which would include calculation of the $c$-function for each individual $(\pi, V)$ by rank-one reduction. But we take an alternative route, attaching importance to the fact that in almost all cases the system of differential equations for $\Upsilon^\pi(\phi^\pi_\lambda)$ coincides with a hypergeometric system of Heckman and Opdam [HO] up to twist by a nowhere-vanishing function. In general, their hypergeometric system is defined for any triple of a root system $\Sigma'$ in $a^*$, a multiplicity function $k$ on $\Sigma'$, and $\lambda \in a^*_C$. ($\Sigma'$ spans $a^*$ by definition. Throughout the paper a root system is assumed crystallographic.) If $k$ satisfies a certain regularity condition, their system has a unique Weyl group invariant analytic solution $F(\Sigma', k, \lambda)$ such that $F(\Sigma', k, \lambda; 0) = 1$. The solution $F(\Sigma', k, \lambda)$ is called a hypergeometric function associated to the root system $\Sigma'$, which is thought of as a deformation of the elementary spherical function for the trivial $K$-type by an arbitrary complex root multiplicity $k$. When $\dim a^* = 1$, $F(\Sigma', k, \lambda)$ reduces to a Jacobi function, which is studied by Koornwinder and Flensted-Jensen prior to [HO] (cf. [Koo]). A Jacobi function is essentially a Gauss hypergeometric function. We review the definition and fundamental properties of $F(\Sigma', k, \lambda)$ in more detail in §4.

To state our main result, we fix some notation. Let $\Sigma^+$ be the positive system corresponding to $N$. For any $\alpha \in \Sigma$ let $g_\alpha$ be the restricted root space and put $m_\alpha = \dim g_\alpha$. Then $m : \Sigma \ni \alpha \mapsto m_\alpha \in \mathbb{Z}_{>0}$ is a multiplicity function on $\Sigma$. We put

$$\delta_{G/K} = \prod_{\alpha \in \Sigma^+} \left| \frac{\sinh \alpha}{||\alpha||} \right|^{m_\alpha}.$$
Here we consider $a$ and $a^\ast$ as inner product spaces by the Killing form $B(\cdot, \cdot)$ of $g$. Likewise, for any root system $\Sigma'$ in $a^\ast$ and multiplicity function $k$ on $\Sigma'$ we put

$$\delta(\Sigma', k) = \prod_{\alpha \in \Sigma'^+} \left| \frac{\sinh(\alpha/2)}{|\alpha/2|} \right|^{2k_\alpha}$$

where $\Sigma'^+$ is some positive system of $\Sigma'$. The main result of this paper is the following:

**Theorem 1.6.** Suppose $\pi, V$ is a small $K$-type of a non-compact real simple Lie group $G$ with finite center. If $G$ is a simply-connected split Lie group $\tilde{G}_2$ of type $G_2$, we further suppose $\pi$ is not the small $K$-type $\pi_2$ specified in Theorem 2.2. Then there exist a root system $\Sigma^\pi$ in $a^\ast$ and a multiplicity function $k^\pi$ on $\Sigma^\pi$ such that

$$\mathcal{Y}^\pi(\phi_\lambda^\pi) = \delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma^\pi, k^\pi)^{\frac{1}{2}} F(\Sigma^\pi, k^\pi, \lambda) \quad \text{for any } \lambda \in a^\ast_\mathbb{C}.$$

The proof of the theorem is divided into two large steps. As the first step, we derive in §3 a simple condition on $\Sigma^\pi$ and $k^\pi$ for the validity of (1.6) under the assumption that $\Sigma^\pi \subset \Sigma \cup 2\Sigma$ (Proposition 5.9). This condition consists of only a few equations between $k^\pi$, $m$ and $\kappa^\pi$. Thus $\kappa^\pi$ encodes all the information on $(\pi, V)$ needed for our purpose. As the second step, we determine the values of $\kappa^\pi$ for each small $K$-type $(\pi, V)$ of each non-compact simple real Lie group $G$ in order to find a pair of $\Sigma^\pi$ and $k^\pi$ using the condition in the first step. The existence of such a pair is actually confirmed in each case except $\pi_2$ of $\tilde{G}_2$. In this process all small $K$-types are classified for each $G$. This generalizes a result of Lee which classifies small $K$-types for each split real simple Lie group (cf. [Le1, Le2]). The case-by-case analysis in this step is carried out in §6. We also prove in §6.9 that in the case of $\pi_2$ of $\tilde{G}_2$, (1.6) never holds for any choice of $\Sigma^\pi$ and $k^\pi$.

As a detailed explanation of Theorem 1.6, the concrete information obtained in the second step is summarized in §2. That is, the classification of small $K$-types and one or two possible choices of $\Sigma^\pi$ and $k^\pi$ for each small $K$-type (except $\pi_2$ of $\tilde{G}_2$). Now, for each our choice of $\Sigma^\pi$ and $k^\pi$, the factor $\delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma^\pi, k^\pi)^{\frac{1}{2}}$ in (1.6) extends to a nowhere-vanishing real analytic function on $a$. Indeed, this can be written as a product of hyperbolic cosines (Proposition 5.4), whose concrete form in each case is also presented in §2.

If $G$ is of Hermitian type, a small $K$-type is nothing but a one-dimensional unitary representation of $K$ (§2.6, Theorem 6.11). Hence in this case Theorem 1.6 restates results of [He2, Chapter 5] and [Sh1]. If $G$ has real rank one, then the hypergeometric function in (1.6) is a Jacobi function. Hence in the rank one case, one can see some known results are essentially equivalent to Theorem 1.6. For example, a result of [FJ] for the one-dimensional $K$-types of the universal cover of $SU(p, 1)$, that of [CP] for the lowest-dimensional non-trivial small $K$-types of $\text{Spin}(2p, 1)$ ($p \geq 2$) and that for the small $K$-types of $\text{Sp}(p, 1)$ obtained by [T3, Sh2] and [DP].

According to Oshima [Os], a commuting family of $W$-invariant differential operators on $a$ is necessarily equal to a system of hypergeometric differential operators up to a
gauge transform under the conditions: (1) $W$ is of a classical type; (2) the symbols of the operators span $S(\mathfrak{a}_C)^W$ (complete integrability); and (3) the operators have regular singularities at every infinity. In view of this, our result is not so surprising since the family of $\pi$-radial parts of $D \in D^\pi$ satisfies (2) and (3). Also, the exceptional case in Theorem 1.6 suggests the possibility that there might be a new class of completely integrable systems associated with the Weyl group of type $G_2$.

Now, Formula (1.6) enables us to reduce a large part of analytic theory for elementary $\pi$-spherical functions to the one for Heckman-Opdam hypergeometric functions. For example, Harish-Chandra’s $c$-function for $G \times K V$ equals a scalar multiple of the $c$-function for Heckman-Opdam hypergeometric functions (Theorem 7.2), and the $\pi$-spherical transform (the spherical transform for $G \times K V$) is a composition of a multiplication operator and a hypergeometric Fourier transform introduced by [Op3] (Theorem 7.4). Using them we can obtain the explicit formula of Harish-Chandra’s $c$-function and the inversion formula of the $\pi$-spherical transform. These applications are discussed in §7.

2. Detailed description of the main result

In this section we list all small $K$-types for each non-compact real simple Lie group $G$ up to equivalence. (Actually the classification is given for each real simple Lie algebra of non-compact type since a small $K$-type for $G$ is always lifted to that for any finite cover of $G$.) In addition, for each small $K$-type $\pi$ other than the one exception stated in §1, we present one or two combinations of $\Sigma^\pi$ and $k^\pi$ for which (1.6) is valid. Though there may be some or infinitely many other choices of such $\Sigma^\pi$ and $k^\pi$ (cf. §6.2), we do not pursue all the possibilities. The results of this section will be proved in §6.

2.1. The trivial $K$-type. First of all, the trivial $K$-type $(\pi, V)$ is small for any $G$. In this case, (1.6) holds with $\Sigma^\pi = 2\Sigma$ and $k^\pi : \Sigma^\pi \ni 2\alpha \mapsto \frac{m_\alpha}{2}$, whence $\delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma^\pi, k^\pi)^{\frac{1}{2}} = 1$ and $Y^\pi(\phi^\sigma) = F(\Sigma^\pi, k^\pi, \lambda)$. In the rest of this section we basically treat only non-trivial small $K$-types.

2.2. Simple Lie groups having no non-trivial small $K$-type. There is no non-trivial small $K$-type in each of the following cases:

- $G$ is a complex simple Lie group;
- $\mathfrak{g} \simeq \mathfrak{sl}(p, \mathbb{H})$ ($p \geq 2$);
- $\mathfrak{g} \simeq \mathfrak{sp}(p, q)$ ($p \geq q \geq 2$);
- $\mathfrak{g} \simeq \mathfrak{so}(2r + 1, 1)$ ($r \geq 1$);
- $\mathfrak{g} \simeq \mathfrak{e}_6(-26)$ (E IV);
- $\mathfrak{g} \simeq \mathfrak{f}_4(-20)$ (F II).

2.3. The case $\mathfrak{g} = \mathfrak{sp}(p, 1)$ ($p \geq 1$). Suppose $G = \text{Sp}(p, 1)$ ($p \geq 1$) and $K = \text{Sp}(p) \times \text{Sp}(1)$. Then $G$ is simply-connected. Let $\text{pr}_1$ and $\text{pr}_2$ be the projections of $K$ to $\text{Sp}(p)$ and $\text{Sp}(1)$ respectively. For the irreducible representation $(\pi_n, \mathbb{C}^n)$ of $\text{Sp}(1) \simeq \text{SU}(2)$ of
dimension $n = 1, 2, \ldots$, the $K$-type $\pi_n \circ pr_2$ is small. If $p = 1$ then $\pi_n \circ pr_2$ is also a small $K$-type. There are no other small $K$-types. Let $\Sigma = \{\pm \alpha, \pm 2\alpha\}$ if $p \geq 2$ and $\Sigma = \{\pm 2\alpha\}$ if $p = 1$. Let $\pi = \pi_n \circ pr_2$ if $p \geq 2$ and $\pi = \pi_n \circ pr_1$ or $\pi_n \circ pr_2$ if $p = 1$. Then putting $\Sigma^\pi = \{\pm 2\alpha, \pm 4\alpha\}$, $k^\pi_{2\alpha} = 2p - 1 \pm n$ and $k^\pi_{4\alpha} = \frac{1}{2} \mp n$, we have (1.6) and $\delta^{-\frac{1}{2}}_{G/K} \delta(\Sigma^\pi, k^\pi) \frac{1}{2} = (\cosh \alpha)^{-1} \mp n$.

2.4. The case $g = \mathfrak{so}(2r, 1)$ ($r \geq 2$). Suppose $G = \text{Spin}(2r, 1)$ ($r \geq 2$) and $K = \text{Spin}(2r)$. Then $G$ is simply-connected. For $s = 0, 1, 2, \ldots$, the irreducible representation $\pi_s^\pm$ of $K = \text{Spin}(2r)$ with highest weight $(s/2, \ldots, s/2, \pm s/2)$ in the standard notation is small. There are no other small $K$-types. Let $\Sigma = \{\pm \alpha\}$. Let $\pi = \pi_s^\pm$. Then putting $\Sigma^\pi = \Sigma \cup 2\Sigma = \{\pm \alpha, \pm 2\alpha\}$, $k^\pi_{\alpha} = -s$ and $k^\pi_{2\alpha} = r + s - \frac{3}{2}$, we have (1.6) and $\delta^{-\frac{1}{2}}_{G/K} \delta(\Sigma^\pi, k^\pi) \frac{1}{2} = (\cosh \frac{\sigma}{2})^s$.

2.5. The case $g = \mathfrak{so}(p, q)$ ($p > q \geq 3$). Suppose $G$ is the double cover of $\text{Spin}(p, q)$ ($p > q \geq 3$). Thus $K = \text{Spin}(p) \times \text{Spin}(q)$ and $G$ is simply-connected. Let $pr_1$ and $pr_2$ be the projections of $K$ to $\text{Spin}(p)$ and $\text{Spin}(q)$ respectively. We may assume $\Sigma = \{\pm e_i | 1 \leq i \leq q\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq q\}$ for some orthogonal basis $\{e_i | 1 \leq i \leq q\}$ of $\mathfrak{a}^*$ with $||e_i|| = \cdots = ||e_q||$.

(i) Let $\sigma$ denote the spin representation of $\text{Spin}(q)$ if $q$ is odd, and either of two half-spin representations of $\text{Spin}(q)$ if $q$ is even. Then $\pi = \sigma \circ pr_2$ is a small $K$-type. For this $\pi$, we can choose $\Sigma^\pi = \{\pm 2e_i | 1 \leq i \leq q\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq q\}$ and $k^\pi$ with $k^\pi_{\pm 2e_i} = \frac{1}{2}$. Let $\Sigma = \{\pm e_i \pm e_j | 1 \leq i < j \leq q\}$ and $\delta^{-\frac{1}{2}}_{G/K} \delta(\Sigma^\pi, k^\pi) \frac{1}{2} = \prod_{1 \leq i < j \leq q} (\cosh \frac{e_i - e_j}{2} \cosh \frac{e_i + e_j}{2})^{-\frac{1}{2}}$.

(ii) In addition, if $p$ is even and $q$ is odd, then $\pi = \sigma \circ pr_1$ with either half-spin representation $\sigma$ of $\text{Spin}(p)$ is a small $K$-type. For this $\pi$, we can choose $\Sigma^\pi = \{\pm e_i \pm 2e_i | 1 \leq i \leq q\} \cup \{\pm e_i \pm e_j | 1 \leq i < j \leq q\}$ and $k^\pi$ with $k^\pi_{\pm e_i \pm e_j} = -p - q$, $k^\pi_{\pm 2e_i} = \frac{1}{2}$. Let $\Sigma = \{\pm e_i \pm e_j | 1 \leq i < j \leq q\}$ and $\delta^{-\frac{1}{2}}_{G/K} \delta(\Sigma^\pi, k^\pi) \frac{1}{2} = \prod_{1 \leq i < j \leq q} (\cosh \frac{e_i}{2})^{-p+q} \prod_{1 \leq i < j \leq q} (\cosh \frac{e_i + e_j}{2})^{-\frac{1}{2}}$.

There are no other non-trivial small $K$-types.

2.6. The Hermitian type. Suppose $G$ is a non-compact simple Lie group of Hermitian type. Let

$$\Sigma_{\text{long}} = \{\alpha \in \Sigma | \alpha \text{ with the longest length}\} \quad \text{and} \quad \Sigma_{\text{middle}} = \Sigma \setminus (\frac{1}{2} \Sigma_{\text{long}} \cup \Sigma_{\text{long}}).$$

A $K$-type $(\pi, V)$ is small if and only if $\dim V = 1$. Hence the set of small $K$-types is naturally identified with a rank one lattice in $\sqrt{-1}Q^*$, where $\sqrt{-1}$ denotes the center of $\mathfrak{k}$. Let $G_{\text{alg}}$ be the analytic subgroup for $g$ in the connected, simply-connected, complex Lie group with Lie algebra $\mathfrak{g}$. Let $\pi_0 \in \sqrt{-1}Q^*$ be a generator of the rank one lattice in the case when $G = G_{\text{alg}}$. Then any $K$-type $\pi$ is identified with $\nu \pi_0 \in \sqrt{-1}Q^*$ for some $\nu \in Q$.

For this $\pi$, we can choose $\Sigma^\pi = \Sigma_{\text{long}} \cup 2\Sigma_{\text{middle}} \cup 2\Sigma_{\text{long}}$ and $k^\pi$ with

$$\begin{cases} k^\pi_{\alpha} = \frac{1}{2} m_\alpha \pm \nu, & k^\pi_{2\alpha} = \frac{1}{2} \mp \nu \quad \text{for} \ \alpha \in \Sigma_{\text{long}}, \\ k^\pi_{2\alpha} = \frac{1}{2} m_\alpha \quad \text{for} \ \alpha \in \Sigma_{\text{middle}} \end{cases}$$
so that (1.6) holds and \( \delta_{G/K}^{-\frac{1}{2}} \delta(K^\pi, k^\pi)^\frac{1}{2} = \prod_{\alpha \in \Sigma_{\text{long}} \cap \Sigma^+} (\cosh \frac{\alpha}{2})^{\frac{1}{2}} \).

2.7. The case \( \Sigma \) is of type \( F_4 \). Let \( G \) be a simply-connected real simple Lie group with \( \Sigma \) of type \( F_4 \). We exclude the complex simple Lie group of type \( F_4 \), which is covered in §2.2. There are the following four possibilities:

| \( \mathfrak{g} \) | \( \mathfrak{f}_4(4) \) (FI) | \( \mathfrak{e}_6(2) \) (E II) | \( \mathfrak{e}_{7(-5)} \) (E VI) | \( \mathfrak{e}_{8(-24)} \) (E IX) |
|---|---|---|---|---|
| \( \mathfrak{t} \) | \( \mathfrak{sp}(3) \oplus \mathfrak{su}(2) \) | \( \mathfrak{su}(6) \oplus \mathfrak{su}(2) \) | \( \mathfrak{so}(12) \oplus \mathfrak{su}(2) \) | \( \mathfrak{e}_7 \oplus \mathfrak{su}(2) \) |

Thus \( K \) is the direct product of a simple compact group \( K_1 \) and \( K_2 := \text{SU}(2) \). Let \( \mathfrak{pr}_2 \) denote the projection \( K \to \text{SU}(2) \). Let \( (\sigma, \mathcal{C}^2) \) be the irreducible representation of \( \text{SU}(2) \) of dimension 2. Then \( \pi := \sigma \circ \mathfrak{pr}_2 \) is the only non-trivial small \( K \)-type. Let \( \Sigma_{\text{short}} \cup \Sigma_{\text{long}} \) be the division of \( \Sigma \) according to the root lengths. Putting \( \Sigma^\pi = 2\Sigma_{\text{short}} \cup \Sigma_{\text{long}}, k^\pi_\alpha = \frac{1}{2}m_\alpha \) for \( \alpha \in \Sigma_{\text{short}} \) and \( k^\pi_\alpha = \frac{1}{2} \) for \( \alpha \in \Sigma_{\text{long}} \), we have (1.6) and \( \delta_{G/K}^{-\frac{1}{2}} \delta(K^\pi, k^\pi)^\frac{1}{2} = \prod_{\alpha \in \Sigma_{\text{long}} \cap \Sigma^+} (\cosh \frac{\alpha}{2})^{-\frac{1}{2}} \).

2.8. Split Lie groups with simply-laced \( \Sigma \). Let \( G \) be a split real simple Lie group. We assume its restricted root system \( \Sigma \) is simply-laced with rank \( \geq 2 \). We also assume \( G \) is simply-connected. (For example, if \( \mathfrak{g} = \mathfrak{sl}(p, \mathbb{R}) \) then \( G \) is the double cover of \( \text{SL}(p, \mathbb{R}) \).

| \( \mathfrak{g} \) | \( \mathfrak{sl}(p, \mathbb{R}) \) \((p \geq 3)\) | \( \mathfrak{so}(p, p) \) \((p \geq 3)\) | \( \mathfrak{e}_6(6) \) (E I) | \( \mathfrak{e}_{7(7)} \) (E V) | \( \mathfrak{e}_{8(8)} \) (E VIII) |
|---|---|---|---|---|---|
| \( \Sigma \) | \( A_{p-1} \) | \( D_p \) | \( E_6 \) | \( E_7 \) | \( E_8 \) |
| \( K \) | \( \text{Spin}(p) \) | \( \text{Spin}(p) \times \text{Spin}(p) \) | \( \text{Sp}(4) \) | \( \text{SU}(8) \) | \( \text{Spin}(16) \) |

(\( \mathfrak{sl}(4, \mathbb{R}) \cong \mathfrak{so}(3, 3) \))

**Theorem 2.1** ([Le2] Theorem 1). Let \( \sigma \) be the spin representation of \( \text{Spin}(p) \) \((p \geq 3)\) if \( p \) is odd and either of two half-spin representations of \( \text{Spin}(p) \) if \( p \) is even. Then \( \sigma \) is a small \( K \)-type when \( \mathfrak{g} = \mathfrak{sl}(p, \mathbb{R}) \) and \( \sigma \circ \mathfrak{pr} \) is a small \( K \)-type when \( \mathfrak{g} = \mathfrak{so}(p, p) \) and \( \mathfrak{pr} \) is either of two projections of \( K \) to \( \text{Spin}(p) \). If \( \mathfrak{g} = \mathfrak{e}_6(6) \), then the 8-dimensional standard (vector) representation of \( \text{Sp}(4) \) is a small \( K \)-type. If \( \mathfrak{g} = \mathfrak{e}_{7(7)} \), then the 8-dimensional standard representation of \( \text{SU}(8) \) and its contragredient are small \( K \)-types. If \( \mathfrak{g} = \mathfrak{e}_{8(8)} \), then the pullback of the 16-dimensional standard representation of \( \text{SO}(16) \) to \( \text{Spin}(16) \) is a small \( K \)-type. There are no other non-trivial small \( K \)-types.

Let \( \pi \) be one of the small \( K \)-types stated above. Then putting \( \Sigma^\pi = \Sigma \) and \( k^\pi_\alpha = \frac{1}{2} \) \((\alpha \in \Sigma)\), we have (1.6) and \( \delta_{G/K}^{-\frac{1}{2}} \delta(K^\pi, k^\pi)^\frac{1}{2} = \prod_{\alpha \in \Sigma^+} (\cosh \frac{\alpha}{2})^{-\frac{1}{2}} \).

2.9. The split Lie group of type \( G_2 \). Let \( G \) be the simply-connected split real simple Lie group \( \tilde{G}_2 \) of type \( G_2 \). Then \( K = K_1 \times K_2 \) with \( K_1 \simeq K_2 \simeq \text{SU}(2) \). Let \( \mathfrak{t} \) be a Cartan subalgebra of \( \mathfrak{k} \). The root system \( \Delta_\mathfrak{k} \) for \((\mathfrak{t}_\mathbb{C}, \mathfrak{c}_\mathbb{C})\) is written as \( \{ \pm \alpha_1 \} \cup \{ \pm \alpha_2 \} \). We assume \( \{ \pm \alpha_1 \} \) is the root system of \( ((\mathfrak{t}, \mathfrak{c})(\mathfrak{t} \cap \mathfrak{t}_\mathbb{C}) \) \((i = 1, 2) \) and \( ||\alpha_1|| \leq ||\alpha_2|| \) with respect to the norm induced from \( B(\cdot, \cdot) \). Let \( \mathfrak{pr}_i \) be the projection of \( K \) to \( K_i \) \((i = 1, 2) \).
Theorem 2.2 ([Le2, Theorem 1]). Let \( \sigma \) denote the irreducible representation of \( SU(2) \) of dimension 2. Then both \( \pi_1 := \sigma \circ \text{pr}_1 \) and \( \pi_2 := \sigma \circ \text{pr}_2 \) are small \( K \)-types. There are no other non-trivial small \( K \)-types.

If \( \pi = \pi_1 \), then putting \( \Sigma^\pi = \Sigma \) and \( k^\pi_\alpha = \frac{1}{2} (\alpha \in \Sigma^\pi) \), we have \( (1.6) \) and \( \delta^{-\frac{1}{2}} \hat{\delta}(\Sigma^\pi, k^\pi)^{\frac{1}{2}} = \prod_{\alpha \in \Sigma^+}(\cosh \frac{\alpha}{2})^{-\frac{1}{2}} \). In contrast, if \( \pi = \pi_2 \), then \( (1.6) \) never holds for any choice of \( \Sigma^\pi \) and \( k^\pi \).

3. Spherical functions

Suppose \( (\pi, V) \) is a small \( K \)-type of a connected real semisimple Lie group \( G = KAN \) with finite center. In this section we study general properties of (elementary) \( \pi \)-spherical functions.

3.1. The Chevalley restriction theorem. Let \( \theta \) be the Cartan involution corresponding to \( K \). The differential of \( \theta \) for \( \mathfrak{g} \) is denoted by the same symbol. Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{s} \) be the Cartan decomposition by \( \theta \).

Let us prove Theorem 1.5. Suppose first \( \phi \in C^\infty(G, \pi, \pi) \). Then \( T^\pi(\phi) \) is \( W \)-invariant since for \( w = \tilde{w}M \in W \) (\( \tilde{w} \in \check{M} \)) and \( H \in \mathfrak{a} \) we have

\[
T^\pi(\phi)(wH) = \phi(\tilde{w}(\exp H)\tilde{w}^{-1}) = \pi(\tilde{w})\phi(\exp H)\pi(\tilde{w}^{-1}) = \phi(\exp H)\pi(\tilde{w})\pi(\tilde{w}^{-1}) = T^\pi(\phi)(H).
\]

Here we note \( \phi(\exp H) \) is a scalar operator. This shows \( T^\pi \) maps \( C^\infty(G, \pi, \pi) \) into \( C^\infty(\mathfrak{a})^W \). The injectivity of \( T^\pi \) is clear from \( G = KAK \). In order to show the surjectivity, take any \( f \in C^\infty(\mathfrak{a})^W \). Then by the ordinary Chevalley restriction theorem ([Diss]), \( f \) extends to some \( \tilde{f} \in C^\infty(\mathfrak{s})^K \). Now \( \phi(k \exp X) := \tilde{f}(X)\pi(k^{-1}) (k \in K, \ X \in \mathfrak{s}) \) is a \( \pi \)-spherical function satisfying \( T^\pi(\phi) = f \). The theorem is thus proved.

3.2. The algebra of invariant differential operators. Let \( G \times_K V \) be the homogeneous vector bundle on \( G/K \) associated to \( (\pi, V) \). The space \( C^\infty(G \times_K V) \) of \( C^\infty \) sections of \( G \times_K V \) is identified with

\[
\{ f : G \rightarrow C^\infty V \mid f(xk) = \pi(k^{-1})f(x) \ \text{for} \ x \in G \ \text{and} \ k \in K \},
\]
on which \( g \in G \) acts by \( \ell(g) : f \mapsto f(g^{-1}. \cdot) \). Accordingly, \( \text{Hom}_K(V, C^\infty(G \times_K V)) \) is identified with \( C^\infty(G, \pi, \pi) \) by

\[
\text{Hom}_K(V, C^\infty(G \times_K V)) \simeq (C^\infty(G \times_K V) \otimes V^*)^K \\
\simeq \{ \phi : G \rightarrow C^\infty V \otimes V^* \mid \phi(k_1gk_2) = \pi(k_2^{-1}) \otimes \pi^*(k_1) \phi(g) \} \\
\simeq C^\infty(G, \pi, \pi).
\]

Here \( (\pi^*, V^*) \) is the contragredient representation of \( (\pi, V) \). Let \( U(\mathfrak{g}_C) \) be the complexified universal enveloping algebra of \( \mathfrak{g} \) and \( U(\mathfrak{g}_C)^K \) its subalgebra consisting of the \( \text{Ad}(K) \)-invariant elements. As a left-invariant differential operator on \( G \), each element of \( U(\mathfrak{g}_C)^K \).
acts on $C^\infty(G \times_K V)$. This defines a homomorphism from $U(\mathfrak{g}_C)^K$ to the algebra $D^\pi$ of $\ell(G)$-invariant differential operators on $G \times_K V$. It is known that the homomorphism is onto and its kernel equals $U(\mathfrak{g}_C)^K \cap U(\mathfrak{g}_C)\ker \pi^*$ (cf. [De]). Here $\ker \pi^*$ denotes the kernel of $\pi^*: U(\mathfrak{k}_C) \to \text{End}_C V^*$. Thus we have $D^\pi \simeq U(\mathfrak{g}_C)^K / U(\mathfrak{g}_C)^K \cap U(\mathfrak{g}_C)\ker \pi^*$. Note that $U(\mathfrak{g}_C)^K$ and $D^\pi$ act on $C^\infty(G, \pi, \pi)$ naturally.

Now $U(\mathfrak{g}_C) \simeq U(n_C) \otimes U(\mathfrak{a}_C) \otimes U(\mathfrak{k}_C)$ by the Poincaré-Birkhoff-Witt theorem. Let us consider the following linear map:

$$\gamma^\pi: U(\mathfrak{g}_C) \simeq 1 \otimes U(\mathfrak{a}_C) \otimes U(\mathfrak{k}_C) \oplus n_C U(\mathfrak{g}_C) \xrightarrow{\text{projection}} 1 \otimes U(\mathfrak{a}_C) \otimes U(\mathfrak{k}_C) \simeq S(\mathfrak{a}_C) \otimes U(\mathfrak{k}_C) \xrightarrow{(f(\lambda) \to f(\lambda + \rho)) \otimes \pi^*} S(\mathfrak{a}_C) \otimes \text{End}_C V^*,$$

where $\rho := \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \in \mathfrak{a}^*$. Note that $\gamma^\pi(D) \in S(\mathfrak{a}_C) \otimes \text{End}_M V^*$ for any $D \in U(\mathfrak{g}_C)^M$ and that $S(\mathfrak{a}_C) \otimes \text{End}_M V^* \simeq S(\mathfrak{a}_C) \otimes \mathbb{C} \simeq S(\mathfrak{a}_C)$ since $(\pi^*, V^*)$ is also a small $K$-type. Hence we consider $\gamma^\pi(D) \in S(\mathfrak{a}_C)$ when $D \in U(\mathfrak{g}_C)^M$. The following generalization of Harish-Chandra’s celebrated exact sequence is given in [Wal] §11.3.3:

**Theorem 3.1.** The restriction of $\gamma^\pi$ to $U(\mathfrak{g}_C)^K$ is an algebra homomorphism into $S(\mathfrak{a}_C)^W$ and the sequence

$$0 \to U(\mathfrak{g}_C)^K \cap U(\mathfrak{g}_C)\ker \pi^* \to U(\mathfrak{g}_C)^K \xrightarrow{\gamma^\pi} S(\mathfrak{a}_C)^W \to 0$$

is exact.

This in particular induces an algebra isomorphism $D^\pi \simeq S(\mathfrak{a}_C)^W$ (also denoted by $\gamma^\pi$).

3.3. **Joint eigenfunctions and integral formulas.** Suppose $\lambda \in \mathfrak{a}^*_C$ and put

$$\mathcal{A}(G \times_K V, \lambda) = \{ f \in C^\infty(G \times_K V) \mid Df = \gamma^\pi(D)(\lambda) f \text{ for any } D \in D^\pi \}.$$  

This is a $\ell(G)$-submodule of the space $\mathcal{A}(G \times_K V)$ of the real analytic sections of $G \times_K V$. In fact, letting $\Omega_\mathfrak{g}$ and $\Omega_\mathfrak{k}$ respectively the Casimir elements of $U(\mathfrak{g}_C)$ and $U(\mathfrak{k}_C)$ relative to the Killing form $B(\cdot, \cdot)$, we see $\mathcal{A}(G \times_K V, \lambda)$ is annihilated by the elliptic operator $\Omega_\mathfrak{g} - 2\Omega_\mathfrak{k} - \gamma^\pi(\Omega_\mathfrak{g} - 2\Omega_\mathfrak{k})(\lambda)$ on $G$. Any $\pi$-spherical function $\phi^\pi_\lambda$ satisfying (1.3) is an $\text{End}_C V$-valued real analytic function since it is identified with an element of $\text{Hom}_K(V, \mathcal{A}(G \times_K V, \lambda))$.

We shall prove Theorem [1.4] with the following integral formula:

$$\phi^\pi_\lambda(g) = \int_K e^{(\lambda + \rho)(A(kg))} \pi(u(kg)^{-1}k) dk \tag{3.1}$$

Here for $g \in G$ we define $A(g) \in \mathfrak{a}$ and $u(g) \in K$ to be the unique elements such that $g \in N \exp A(g)u(g)$. It is easy to see $\phi^\pi_\lambda$ defined by (3.1) is a $\pi$-spherical function. We
claim this $\phi_\lambda^\pi$ satisfies (1.3). Indeed, it is clear that $\phi_\lambda^\pi(1_G) = \text{id}_V$. Also, one easily sees $\psi_\lambda^\pi(g) := e^{(\lambda+\rho)(A(g))} \pi^*(u(g))$ is an $\text{End}_G V^*$-valued $C^\infty$ function on $G$ satisfying

\[
\psi_\lambda^\pi(gk_2) = \psi_\lambda^\pi(g) \pi^*(k_2) \quad \text{for } (g, k_2) \in G \times K,
\]

\[
\psi_\lambda^\pi(namg) = e^{(\lambda+\rho)(\log g)(m)} \pi^*(g) \psi_\lambda^\pi(g) \quad \text{for } (n, a, m, g) \in N \times A \times M \times G,
\]

\[
D \psi_\lambda^\pi = \gamma^\pi(D)(\lambda) \psi_\lambda^\pi \quad \text{for } D \in U(\mathfrak{g}_C)^K.
\]

The claim follows since $\phi_\lambda^\pi(g) = \int_K t^\pi \psi_\lambda^\pi(kg) \pi(k) dk$.

To complete the proof, we must show the uniqueness of $\phi_\lambda^\pi$. To do so, suppose $\phi$ is a $\pi$-spherical functions satisfying (1.3). Define a linear map $\Phi : U(\mathfrak{g}_C) \ni D \mapsto D(\phi_\lambda^\pi - \phi)(1_G) \in \text{End}_G V$. Then we have

\[
\Phi(XD) = -\Phi(D)(\pi(X)) \quad \text{and} \quad \Phi(DX) = -\pi(X)\Phi(D) \quad \text{for any } X \in \mathfrak{t}_C \text{ and } D \in U(\mathfrak{g}_C).
\]

This implies $\Phi([X, D]) = [\pi(X), \Phi(D)]$ and hence $\Phi(\text{Ad}(k)D) = \pi(k)\Phi(D)\pi(k^{-1})$ for $k \in K$. Since $\pi(\mathfrak{u}(\mathfrak{t}_C)) = \text{End}_G V$ by Burnside’s theorem, (3.2) also implies $\Phi(U(\mathfrak{g}_C))$ is a two-sided ideal of $\text{End}_G V$. Hence $\Phi(U(\mathfrak{g}_C))$ is either $\text{End}_G V$ or $\{0\}$. In order to show the former case never happens, assume $\Phi(D) = \text{id}_V$ for some $D$. Then with $\tilde{D} := \int \text{Ad}(k)D \, dk \in U(\mathfrak{g}_C)^K$ we have $\Phi(\tilde{D}) = \int \pi(k)\Phi(D)\pi(k^{-1}) \, dk = \int_K \text{id}_V \, dk = \text{id}_V$. On the other hand, $\Phi(D) = \tilde{D}(\phi_\lambda^\pi - \phi)(1_G) = \gamma^\pi(\tilde{D})(\lambda) \phi_\lambda^\pi(1_G) - \gamma^\pi(\tilde{D})(\lambda) \phi(1_G) = \gamma^\pi(\tilde{D})(\lambda) \text{id}_V - \gamma^\pi(\tilde{D})(\lambda) \text{id}_V = 0$, contradicting the last calculation. Thus $\Phi(U(\mathfrak{g}_C)) = \{0\}$ and $D(\phi_\lambda^\pi - \phi)(1_G) = 0$ for any $D \in U(\mathfrak{g}_C)$. Since $\phi_\lambda^\pi - \phi$ is real analytic, we have $\phi_\lambda^\pi - \phi = 0$, the desired uniqueness.

**Proposition 3.2.** For any $\lambda \in \mathfrak{a}_C^*$ one has

\[
(3.3) \quad \phi_\lambda^\pi(\theta g) = \phi_{-\lambda}^\pi(g) \quad (g \in G),
\]

\[
(3.4) \quad T^\pi(\phi_\lambda^\pi)(-H) = T^\pi(\phi_{-\lambda}^\pi)(H) \quad (H \in \mathfrak{a}).
\]

**Proof.** Note $\phi_\lambda^\pi(\theta g) \in C^\infty(G, \pi, \pi)$. Take $\tilde{w}_0 \in \tilde{M}$ so that $w_0 = \tilde{w}_0 M$ is the longest element of $W$. Then it easily follows from the definition of $\gamma^\pi$ that $\gamma^\pi(\theta D)(\lambda) = \gamma^\pi(\theta \text{Ad}(\tilde{w}_0)D)(\lambda) = \gamma^\pi(D)(-w_0 \lambda) = \gamma^\pi(D)(-\lambda)$ for any $D \in U(\mathfrak{g}_C)^K$. Thus Theorem 1.4 implies (3.3) and hence (3.4). $\square$

**Proposition 3.3.** For any $\lambda \in \mathfrak{a}_C^*$ one has

\[
(3.5) \quad t^\lambda \phi_\lambda^\pi(g^{-1}) = \phi_{-\lambda}^\pi(g) \quad (g \in G),
\]

\[
(3.6) \quad T^\pi(\phi_\lambda^\pi) = T^\pi(\phi_{-\lambda}^\pi).
\]

**Proof.** We use $\psi_\lambda^\pi(g)$ defined above. For each $g \in G$ put $\Psi_g(x) := t^\lambda \psi_{-\lambda}^\pi(xg) \psi_\lambda^\pi(x)$. Since

\[
\Psi_g(namx) = e^{2\rho(\log a)} \Psi_g(x) \quad \text{for } (n, a, m, x) \in N \times A \times M \times G,
\]

use $\psi_\lambda^\pi(g)$ defined above. For each $g \in G$ put $\Psi_g(x) := t^\lambda \psi_{-\lambda}^\pi(xg) \psi_\lambda^\pi(x)$. Since

\[
\Psi_g(namx) = e^{2\rho(\log a)} \Psi_g(x) \quad \text{for } (n, a, m, x) \in N \times A \times M \times G,
\]
we have \( \int_K \Psi_\varphi(k) dk = \int_K \Psi_\varphi(kg^{-1}) dk \) by [Hel2] Ch. I, Lemma 5.19. Hence by (3.1)

\[
\phi^\pi_\chi(g) = \int_K t^\pi_\chi(kg) \psi_\chi(k) dk = \int_K t^\pi_\chi(k) \psi_\chi(kg^{-1}) dk \\
= t^\chi(\int_K t^\pi_\chi(kg^{-1}) \psi_\chi(k) dk) = t^\chi(g^{-1}).
\]

Thus we get (3.5).

Now for any \( a \in A \) it follows from (3.3) and (3.5) that

\[
\phi^\pi_\chi(a) = \phi^\pi_\chi(a^{-1}) = t^\chi(a),
\]

which proves (3.6). \( \square \)

Remark 3.4. The uniqueness of \( \phi^\pi_\chi \) is also obtained by general theory (cf. [Ca] Theorem 3.8], [GK] Theorem 1.45]). By use of (3.5) we can rewrite (3.1) as

\[
(3.7) \quad \phi^\pi_\chi(g) = \int_K e^{(\lambda - \rho)(H(k))} \pi(k \kappa (gk)^{-1}) dk.
\]

Here given \( x \in G \), define \( \kappa(x) \in K \) and \( H(x) \in \mathfrak{a} \) by \( x \in \kappa(x) e^{H(x)} N \). Formula (3.1) (or (3.7)) is a special case of the integral representations of elementary spherical functions (or more generally Eisenstein integrals) given by Harish-Chandra (cf. [War] §6.2.2, §9.1.5], [Ca] (42], [Kn1] (14.20)).

3.4. The multiplicity function \( \kappa^\pi \). For any \( \lambda \in \mathfrak{a}^* \) let \( H_\lambda \in \mathfrak{a} \) be the unique element such that \( \lambda = B(H_\lambda, \cdot) \). For each \( \alpha \in \Sigma \) we choose an orthonormal basis \( \{ X_\alpha^{(1)}, \ldots, X_\alpha^{(m_\alpha)} \} \) of \( \mathfrak{g}_\alpha \) with respect to the inner product \( -\frac{||\alpha||^2}{2} B(\cdot, \cdot) \). Note that \( [X_\alpha^{(i)}, \theta X_\alpha^{(j)}] = -\alpha^\vee \) for \( i = 1, \ldots, m_\alpha \) \( (\alpha^\vee := \frac{2H_\alpha}{||\alpha||^2}) \).

Lemma 3.5. The element \( \sum_{i=1}^{m_\alpha} (X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 \in U(\mathfrak{g}_\alpha) \) does not depend on the choice of \( \{ X_\alpha^{(i)} \} \) and is \( \mathrm{Ad}(M) \)-invariant. Moreover for any \( \tilde{\varphi} \in \tilde{M} \) we have \( \mathrm{Ad}(\tilde{\varphi}) \sum_{i=1}^{m_\alpha} (X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 = \sum_{i=1}^{m_\alpha} (X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 \) with \( w := \tilde{\varphi} M \in W \).

Proof. The element \( \sum_{i=1}^{m_\alpha} (X_\alpha^{(i)})^{\otimes 2} \in \mathfrak{g}_\alpha^{\otimes 2} \) is independent of the choice of \( \{ X_\alpha^{(i)} \} \) and is \( M \)-invariant since \( M \) acts isometrically on \( \mathfrak{g}_\alpha \). Now the first assertion follows since \( \mathfrak{g}_\alpha^{\otimes 2} \ni X \otimes Y \mapsto (X + \theta X)(Y + \theta Y) \in U(\mathfrak{g}_\alpha) \) is \( M \)-linear. The second assertion is also immediate since \( \{ \mathrm{Ad}(\tilde{\varphi}) X_\alpha^{(i)} \} \) is an orthonormal basis of \( \mathfrak{g}_{w\alpha} \). \( \square \)

Since \( \pi \) is small, it follows from Schur’s lemma that \( \frac{1}{m_\alpha} \sum_{i=1}^{m_\alpha} \pi(X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2 \) is a scalar operator. We denote this value by \( \kappa^\pi_\alpha \), namely

\[
\kappa^\pi_\alpha \mathrm{id}_V = \frac{1}{m_\alpha} \sum_{i=1}^{m_\alpha} \pi(X_\alpha^{(i)} + \theta X_\alpha^{(i)})^2.
\]

By the second statement of Lemma 3.5, \( \Sigma \ni \alpha \mapsto \kappa^\pi_\alpha \) is a multiplicity function. The next lemma is useful to calculate \( \kappa^\pi_\alpha \)'s in various examples:
Lemma 3.6. Suppose $\alpha \in \Sigma$. For any $X_\alpha \in \mathfrak{g}_\alpha$ such that $-\frac{||\alpha||^2}{2} B(X_\alpha, \theta X_\alpha) = 1$ it holds that

$$\kappa_\alpha^\pi = \frac{1}{\dim V} \text{Tr}(\pi(X_\alpha + \theta X_\alpha)^2).$$

Proof. By the following theorem, $\text{Tr}(\pi(X_\alpha + \theta X_\alpha)^2)$ takes a constant value for any $X_\alpha$ in the unit sphere of $\mathfrak{g}_\alpha$. The rest of the proof is easy. \hfill \Box

Theorem 3.7 ([Kos Theorem 2.1.7]). For any $\alpha \in \Sigma$ with $m_\alpha > 1$, $M_0$ (the identity component of $M$) acts transitively on the unit sphere of $\mathfrak{g}_\alpha$.

Note $M_0$ acts trivially on $\mathfrak{g}_\alpha$ for $\alpha \in \Sigma$ with $m_\alpha = 1$.

Proposition 3.8. For any $\alpha \in \Sigma$, $\kappa_\alpha^\pi \leq 0$. Furthermore, $\kappa_\alpha^\pi = 0$ if and only if $\pi(X_\alpha + \theta X_\alpha) = 0$ for any $X_\alpha \in \mathfrak{g}_\alpha$.

Proof. Suppose $\alpha \in \Sigma$ and take $X_\alpha \in \mathfrak{g}_\alpha$ so that $-B(X_\alpha, \theta X_\alpha) = \frac{2}{||\alpha||^2}$. Then $\pi(X_\alpha + \theta X_\alpha)$ is diagonalizable since it is skew-Hermitian with respect to the inner product of $V$. With a basis $\{v_1, \ldots, v_n\}$ of $V$ and real numbers $\lambda_1, \ldots, \lambda_n$, write

$$\pi(X_\alpha + \theta X_\alpha)v_j = \sqrt{-1}\lambda_j v_j \quad (j = 1, \ldots, n).$$

From Lemma 3.6 we have

$$\kappa_\alpha^\pi = -\frac{1}{\dim V} \sum_{j=1}^n \lambda_j^2.$$ 

Hence $\kappa_\alpha^\pi \leq 0$ and the equality holds if and only if $\pi(X_\alpha + \theta X_\alpha) = 0$. This, together with Theorem 3.7 proves the proposition. \hfill \Box

3.5. The radial part of the Casimir operator. Let $\Omega_m$ and $\Omega_a$ be the Casimir elements of $U(\mathfrak{m}_\mathbb{C})$ and $U(\mathfrak{a}_\mathbb{C})$ relative to $B(\cdot, \cdot)$, respectively. Note $\pi(\Omega_m)$ is a scalar operator. We denote its value by $\varpi^\pi$.

Theorem 3.9. For any $\phi \in C^\infty(G, \pi, \pi)$ it holds that

$$\mathcal{T}^\pi((\Omega_\mathfrak{g} - \varpi^\pi) \phi) = \left( \Omega_\alpha + \sum_{\alpha \in \Sigma^+} m_\alpha \left( \coth \alpha H_\alpha - \frac{\kappa_\alpha^\pi ||\alpha||^2}{4 \cosh^2 \frac{\alpha}{2}} \right) \right) \mathcal{T}^\pi(\phi)$$
on $\mathfrak{a}_{\text{reg}} := \{ H \in \mathfrak{a} | \alpha(H) \neq 0 \text{ for any } \alpha \in \Sigma \}$.

Proof. Suppose $H \in \mathfrak{a}_{\text{reg}}$. It follows from [HC2 Lemma 22] and [War Proposition 9.1.2.11] (see also [Sh1 Proposition 2.3]) that the equality

$$\Omega_\mathfrak{g} = \Omega_\alpha + \Omega_m + \sum_{\alpha \in \Sigma^+} \left( m_\alpha \coth \alpha H_\alpha + \frac{||\alpha||^2}{4} \sum_{i=1}^{m_\alpha} \left( \frac{1}{\sinh^2 \alpha H_\alpha} (X^{(i)}_\alpha + \theta X^{(i)}_\alpha)^2 ight. 

- \frac{2 \coth \alpha H_\alpha}{\sinh \alpha H_\alpha} \left( \text{Ad}(e^{-H}) (X^{(i)}_\alpha + \theta X^{(i)}_\alpha)) (X^{(i)}_\alpha + \theta X^{(i)}_\alpha) 

\left. + \frac{1}{\sinh^2 \alpha H_\alpha} \left( \text{Ad}(e^{-H}) (X^{(i)}_\alpha + \theta X^{(i)}_\alpha) \right)^2 \right) \right)$$

on $\mathfrak{a}_{\text{reg}}$.
Hence we calculate
\[ \sum_{i=1}^{m_\alpha} ((\text{Ad}(e^{-H})(X^{(i)}_\alpha + \theta X^{(i)}_\alpha))(X^{(i)}_\alpha + \theta X^{(i)}_\alpha) \phi)(e^H) = \sum_{i=1}^{m_\alpha} \pi(X^{(i)}_\alpha + \theta X^{(i)}_\alpha) \phi(e^H) \pi(X^{(i)}_\alpha + \theta X^{(i)}_\alpha) \]
\[ = \phi(e^H) \sum_{i=1}^{m_\alpha} \pi(X^{(i)}_\alpha + \theta X^{(i)}_\alpha)^2 \]
\[ = m_\alpha \kappa_\alpha^\pi \mathcal{Y}^\pi(\phi)(H) \]
and in the same way
\[ \sum_{i=1}^{m_\alpha} ((X^{(i)}_\alpha + \theta X^{(i)}_\alpha)^2 \phi)(e^H) = \sum_{i=1}^{m_\alpha} ((\text{Ad}(e^{-H})(X^{(i)}_\alpha + \theta X^{(i)}_\alpha))^2 \phi)(e^H) = m_\alpha \kappa_\alpha^\pi \mathcal{Y}^\pi(\phi)(H). \]
Hence we calculate
\[ \mathcal{Y}^\pi((\Omega_0 - \varpi^\pi) \phi)(H) - \Omega_\alpha \mathcal{Y}^\pi(\phi)(H) \]
\[ = ((\Omega_0 - \Omega_\alpha - \Omega_\mu) \phi)(e^H) \]
\[ = \sum_{\alpha \in \Sigma^+} m_\alpha \left( \coth \alpha(H) H_\alpha \right) \]
\[ + \frac{\kappa_\alpha^\pi ||\alpha||^2}{4} \left( \frac{1}{\sinh^2 \alpha(H)} - 2 \cosh \alpha(H) \right) \mathcal{Y}^\pi(\phi)(H) \]
\[ = \sum_{\alpha \in \Sigma^+} m_\alpha \left( \coth \alpha(H) H_\alpha - \frac{\kappa_\alpha^\pi ||\alpha||^2}{4 \cosh^2 \alpha(H)} \right) \mathcal{Y}^\pi(\phi)(H). \]

3.6. Radial parts of general invariant differential operators. Let \( \mathcal{B} \) be the unital algebra of functions on \( a_{\text{reg}} \) generated by \( (1 \pm e^\alpha)^{-1} \) \( (\alpha \in \Sigma^+) \). The Weyl group \( W \) acts on \( \mathcal{B} \) naturally. The algebra of differential operators on \( a_{\text{reg}} \) with coefficients in \( \mathcal{B} \) is identified with \( \mathcal{B} \otimes S(a_C) \) as a linear space. Let \( a_- := \{ H \in a \mid \alpha(H) < 0 \text{ for any } \alpha \in \Sigma^+ \} \) and \( N \Sigma^+ := \{ \sum_{\alpha \in \Sigma^+} n_\alpha \alpha \mid n_\alpha \in \mathbb{N} \}. \) Then each element \( f \in \mathcal{B} \) is expanded in a power series \( \sum_{\mu \in N \Sigma^+} a_\mu e^\mu \) which absolutely converges on \( a_- \). We define \( \mathcal{M} \) to be the maximal ideal of \( \mathcal{B} \) consisting of the power series without constant term.

Proposition 3.10. For any \( D \in U(g_C)^K \) there exists a unique differential operator \( \Delta^\pi(D) \in \mathcal{B} \otimes S(a_C) \) such that for any \( \phi \in C^\infty(G, \pi, \pi) \)
\[ \mathcal{Y}^\pi(D \phi) = \Delta^\pi(D) \mathcal{Y}^\pi(\phi) \]
on \( a_{\text{reg}} \). Moreover, \( \Delta^\pi(D) \) is \( W \)-invariant and is of the form
\[ \Delta^\pi(D) = \gamma^\pi(D)(-\rho) + \sum_{j=1}^{k} f_j E_j \quad \text{with } f_1, \ldots, f_k \in \mathcal{M} \text{ and } E_1, \ldots, E_k \in S(a_C). \]
Remark 3.11. We call $\Delta^\pi(D)$ in the theorem the $\pi$-radial part of $D$. It follows from Theorem 3.9 that

\begin{equation}
\Delta^\pi(\Omega_\phi - \mathfrak{w}^\pi) = \Omega_\alpha + \sum_{\alpha \in \Sigma^+} m_\alpha \left( \coth \alpha \text{H}_\alpha - \frac{\kappa^\pi||\alpha||^2}{4 \cosh^2 \frac{\alpha}{2}} \right).
\end{equation}

**Proof of Proposition 3.10.** First, we claim that for an arbitrary $D \in U(\mathfrak{g}_\mathbb{C})$ (not necessarily $K$-invariant) there exist $f_j \in \mathcal{M}$, $E_j \in S(\mathfrak{a}_\mathbb{C})$ and $U_j, U_j' \in U(\mathfrak{t}_\mathbb{C})$ ($j = 1, \ldots, k$) such that

\begin{equation}
D = D' + \sum_{j=1}^k f_j(H) (\text{Ad}(e^{-H})(U_j)) E_j U_j'
\end{equation}

for any $H \in \mathfrak{a}_{\text{reg}}$, where $D' \in U(\mathfrak{a}_\mathbb{C}) \otimes U(\mathfrak{t}_\mathbb{C})$ is a unique element such that $D - D' \in \mathfrak{n}_\mathbb{C} U(\mathfrak{g}_\mathbb{C})$. Indeed, since for each $\alpha \in \Sigma^+$ and $i = 1, \ldots, m_\alpha$ one has

\[ X^{(i)}_a = \frac{e^{\alpha(H)}}{1 - e^{2\alpha(H)}} \text{Ad}(e^{-H})(X^{(i)}_a + \theta X^{(i)}_a) - \frac{e^{2\alpha(H)}}{1 - e^{2\alpha(H)}} (X^{(i)}_a + \theta X^{(i)}_a), \]

the claim can be easily shown by induction on the degree of $D$.

Next, for any $H \in \mathfrak{a}_{\text{reg}}$ the linear map

\[ \eta_H : S(\mathfrak{a}_\mathbb{C}) \otimes (U(\mathfrak{t}_\mathbb{C}) \otimes U(\mathfrak{m}_\mathbb{C})) \ni E' \otimes U \otimes U' \mapsto (\text{Ad}(e^{-H})(U)) E' U' \in U(\mathfrak{g}_\mathbb{C}) \]

is a well-defined bijection. Note $\eta_H$ is an $M$-homomorphism. Now suppose $D \in U(\mathfrak{g}_\mathbb{C})^K$. Since $D$ and $D'$ are $M$-invariant, we have

\[ \sum_{j=1}^k f_j(H) E'_j \otimes U_j \otimes U'_j = \eta_H^{-1}(D - D') \in S(\mathfrak{a}_\mathbb{C}) \otimes (U(\mathfrak{t}_\mathbb{C}) \otimes U(\mathfrak{m}_\mathbb{C})) U(\mathfrak{t}_\mathbb{C}))^M. \]

For each $j = 1, \ldots, k$, take $U_{j,\nu}$ and $U'_{j,\nu} \in U(\mathfrak{t}_\mathbb{C})$ ($\nu = 1, \ldots, k_j$) so that

\[ \int_M \text{Ad}(m)(U_j) \otimes \text{Ad}(m)(U'_j) \, dm = \sum_{\nu=1}^{k_j} U_{j,\nu} \otimes U'_{j,\nu} \in (U(\mathfrak{t}_\mathbb{C}) \otimes U(\mathfrak{t}_\mathbb{C}))^M, \]

where $dm$ is the normalized Haar measure on $M$. Then we can rewrite (3.11) in the following way:

\[ D = D' + \sum_{j=1}^k \eta_H \left( f_j(H) E'_j \otimes \sum_{\nu=1}^{k_j} U_{j,\nu} \otimes U'_{j,\nu} \right) = D' + \sum_{j=1}^k f_j(H) \sum_{\nu=1}^{k_j} (\text{Ad}(e^{-H})(U_{j,\nu})) E'_j U'_{j,\nu}. \]

Now suppose $\phi$ is any $\pi$-spherical function. Since $U^t \phi = \mathbb{1}_\phi \pi^s(U)$ for any $U \in U(\mathfrak{t}_\mathbb{C})$, we have $D'^t \phi = \gamma^\pi(D)(-\rho)^t \phi$ and hence $D' \phi = \gamma^\pi(D)(-\rho) \phi$, where $\gamma^\pi(D)(-\rho) \in S(\mathfrak{a}_\mathbb{C})$ acts on $\mathbb{1}_\phi$ and $\phi$ as a differential operator. On the other hand, we have

\[ \sum_{\nu=1}^{k_j} (\text{Ad}(e^{-H})(U_{j,\nu})) E'_j U'_{j,\nu} \mathbb{1}_\phi(e^H) = \sum_{\nu=1}^{k_j} \pi^s(U_{j,\nu}) (E'_j \phi(e^H)) \pi^s(U_{j,\nu}) = \pi^s \left( \sum_{\nu=1}^{k_j} U_{j,\nu} U'_{j,\nu} \right) E'_j \phi(e^H). \]
Here the second equality holds since \( i\phi(e^H) \) and \( E_j^{(i)} \phi(e^H) \) are scalar operators. Note that \( \pi^*\left( \sum_{\nu=1}^{k_j} U_{j,\nu} U_{j,\nu}' \right) \in \text{End}_M V^* \) is also a scalar operator. Let \( E_j \in S(a_C) \) be the product of this scalar value and \( E_j' \) \((j = 1, \ldots, k)\). Then the operator \( \Delta^a(D) \) defined by (3.9) satisfies (3.8).

Finally, since for each \( w \in W \) any compactly supported \( C^\infty \) function on \( \{ e^H \mid H \in wa_- \} \) is (uniquely) extended to a \( \pi \)-spherical function on \( G \) by Theorem 1.5 we get the uniqueness of \( \Delta^a(D) \). The \( W \)-invariance of \( \Delta^a(D) \) easily follows from this. \( \square \)

**Remark 3.12.** Since the action of \( D \in U(g_C)^K \) on \( \pi \)-spherical functions factors through \( U(g_C)^K \to D^\pi \): \( \{ \Delta^a(D) \mid D \in U(g_C)^K \} \) is a commutative subalgebra of \( (\mathcal{R} \otimes S(a_C))^W \). We denote this subalgebra by \( \Delta^a(D^\pi) \).

Theorems 1.4, 1.5 and Proposition 3.10 imply

**Corollary 3.13.** Suppose \( \lambda \in a^*_C \). The subspace of \( C^\infty(a)^W \) consisting of those \( f \) satisfying

\[
\Delta^a(D) f = \gamma^a(D)(\lambda) f \quad \text{for any } D \in U(a_C)^K
\]

equals \( C Y^a(\phi^a_\lambda) \) and is a subspace of \( \mathcal{A}(a) \) (the space of real analytic functions on \( a \)).

4. HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS

In this section \( a \) denotes any finite-dimensional linear space with inner product \( B(\cdot, \cdot) \). Let \( (\cdot, \cdot) \) be the symmetric bilinear form on \( a_C^* \) induced from \( B(\cdot, \cdot) \). Let \( \Sigma' \) be a (possibly non-reduced) crystallographic root system in \( a^* \). Its Weyl group is denoted by \( W' \). In a series of papers starting from [HO], Heckman and Opdam define and study the hypergeometric function \( F(\Sigma', k, \lambda) \in \mathcal{A}(a) \) associated to \( \Sigma' \). Here \( k \) is a \( \mathbb{C} \)-valued multiplicity function on \( \Sigma' \) with some regularity condition and \( \lambda \in a^*_C \). The hypergeometric function \( F(\Sigma', k, \lambda) \) is a natural generalization of \( Y^a(\phi^a_\lambda) \) with the trivial \( K \)-type \( \pi \), allowing the root multiplicities \( m \) to be generic complex numbers \( k \). In this section we review the definition and some fundamental properties of \( F(\Sigma', k, \lambda) \). We refer the reader to [Op4, Hec2] for details.

4.1. Hypergeometric differential operators. Let \( K(\Sigma') \) be the space of multiplicity functions on \( \Sigma' \). This is a linear space with dimension equal to the number of the \( W' \)-orbits in \( \Sigma' \). Regarding \( a \) as an Abelian Lie algebra equipped with an inner product, we define \( O_\alpha \) and \( H_\alpha \) \((\alpha \in \Sigma')\) as in \( \S 3 \). Fix a positive system \( \Sigma'^+ \subset \Sigma' \). Let \( \mathcal{R}' \) denote the unital algebra generated by \((1 - e^{\alpha})^{-1} \) \((\alpha \in \Sigma'^+)\). Let \( \mathcal{M}' \) be the maximal ideal of \( \mathcal{R}' \) consisting of those \( f \) that are expanded in the form \( f = \sum_{\mu \in \mathbb{N}\Sigma'^+ \setminus \{0\}} a_{\mu} e^{\mu} \) on \( a_- := \{ H \in a \mid \alpha(H) < 0 \mbox{ for any } \alpha \in \Sigma'^+ \} \). As in \( \S 3.6 \), \( \mathcal{R}' \otimes S(a_C) \) denotes the algebra of differential operators with coefficients in \( \mathcal{R}' \).
Suppose \( k \in \mathcal{K}(\Sigma') \). Put

\[
L(\Sigma', k) = \Omega_\alpha + \sum_{\alpha \in \Sigma^+} k_\alpha \coth \frac{\alpha}{2} H_\alpha,
\]

which belongs to \((\mathcal{R}' \otimes S(\mathfrak{a}_C))^W\).

**Remark 4.1.** In the setting of \( \mathfrak{S} \), let \( \pi \) be the trivial \( K \)-type. Then the radial part of the Casimir operator given by \( \text{(3.10)} \) equals \( L(2\Sigma, k) \) with \( k_{2\alpha} = m_\alpha \) (\( \forall \alpha \in \Sigma \)).

Note \( D \mapsto \delta(\Sigma', k)^\frac{1}{2} \circ D \circ \delta(\Sigma', k)^{-\frac{1}{2}} \) with \( \delta(\Sigma', k) \) in \( \text{(1.5)} \) defines an algebra automorphism of \((\mathcal{R}' \otimes S(\mathfrak{a}_C))^W\).

**Proposition 4.2** ([HO Proposition 2.2], [Hec2 Theorem 2.1.1]). Putting \( \rho(k) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} k_\alpha \alpha \), we have

\[
\delta((\Sigma', k)^\frac{1}{2} \circ (L(\Sigma', k) + (\rho(k), \rho(k)))) \circ \delta(\Sigma', k)^{-\frac{1}{2}} = \Omega_\alpha + \sum_{\alpha \in \Sigma^+} \frac{k_\alpha(1 - k_\alpha - 2k_{2\alpha})||\alpha||^2}{4 \sinh^2 \frac{\alpha}{2}}.
\]

Here \( k_{2\alpha} = 0 \) if \( 2\alpha \notin \Sigma^+ \).

Since \( \mathcal{M}' \otimes S(\mathfrak{a}_C) \) is an ideal of \( \mathcal{R}' \otimes S(\mathfrak{a}_C) \), the linear map

\[
\gamma_\rho(k) : \mathcal{R}' \otimes S(\mathfrak{a}_C) = S(\mathfrak{a}_C) \oplus \mathcal{M}' \otimes S(\mathfrak{a}_C) \xrightarrow{\text{projection}} S(\mathfrak{a}_C) \xrightarrow{f(\lambda) \mapsto f(\lambda + \rho(k))} S(\mathfrak{a}_C)
\]

is an algebra homomorphism.

**Proposition 4.3** ([Hec1 Theorem 3.11], [Hec2 Theorem 1.3.12]). The restriction of \( \gamma_\rho(k) \) to the subalgebra

\[
(\mathcal{R}' \otimes S(\mathfrak{a}_C))^{W', L(\Sigma', k)} := \{ D \in (\mathcal{R}' \otimes S(\mathfrak{a}_C))^W \mid [L(\Sigma', k), D] = 0 \}
\]

is an algebra isomorphism onto \( S(\mathfrak{a}_C)^{W'} \). In particular \( (\mathcal{R}' \otimes S(\mathfrak{a}_C))^{W', L(\Sigma', k)} \) is a commuting family of differential operators.

One easily sees \( \mathcal{R}' \), \( L(\Sigma', k) \), \( (\mathcal{R}' \otimes S(\mathfrak{a}_C))^{W', L(\Sigma', k)} \) and the restriction of \( \gamma_\rho(k) \) to \( (\mathcal{R}' \otimes S(\mathfrak{a}_C))^{W', L(\Sigma', k)} \) do not depend on the choice of \( \Sigma^+ \). Now suppose \( \lambda \in \mathfrak{a}_C^\ast \) and let us consider the following three conditions for \( f \in \mathcal{A}(\mathfrak{a}) \):

(HG1) \( f \in \mathcal{A}(\mathfrak{a})^W \);

(HG2) \( Df = \gamma_\rho(k)(D)(\lambda)f \) for any \( D \in (\mathcal{R}' \otimes S(\mathfrak{a}_C))^{W', L(\Sigma', k)} \);

(HG3) \( f(0) = 1 \).

As we see below these conditions characterize \( F(\Sigma', k, \lambda) \).
4.2. Definition of the hypergeometric functions. Put

\begin{equation}
\hat{c}(\Sigma', k, \lambda) = \prod_{\alpha \in \Sigma'_+} \frac{\Gamma(\lambda(\alpha') + \frac{1}{2}k_{\frac{1}{2}\alpha})}{\Gamma(\lambda(\alpha') + \frac{1}{2}k_{\frac{1}{2}\alpha} + k_{\alpha})},
\end{equation}

\begin{equation}
c(\Sigma', k, \lambda) = \frac{\hat{c}(\Sigma', k, \lambda)}{\hat{c}(\Sigma', k, \rho(k))},
\end{equation}

where \(k_{\frac{1}{2}\alpha} = 0\) if \(\frac{1}{2}\alpha \notin \Sigma'^+\).

For any \(\lambda \in \mathbb{C}_+\) with the property

\[2(\lambda, \beta) + (\beta, \beta) \neq 0\quad \text{for any } \beta \in \mathbb{N}\Sigma'^+ \setminus \{0\},\]

there is a unique formal series in the form

\[\Phi(\Sigma', k, \lambda) = e^{\lambda + \rho(k)} + \sum_{\mu \in \mathbb{N}\Sigma'^+ \setminus \{0\}} a_{\mu} e^{\lambda + \rho(k) + \mu} \quad (a_{\mu} \in \mathbb{C})\]

such that

\[L(\Sigma', k) \Phi(\Sigma', k, \lambda) = \gamma_{\rho(k)}(L(\Sigma', k))(\lambda) \Phi(\Sigma', k, \lambda).\]

The series actually converges absolutely on \(a_{\cdot}\). Thus for a generic \(\lambda\)

\begin{equation}
\hat{F}(\Sigma', k, \lambda) = \sum_{w \in W} \hat{c}(\Sigma', k, -w\lambda)\Phi(\Sigma', k, w\lambda).
\end{equation}

is a well-defined real analytic function on \(a_{\cdot}\). We have immediately from (4.6) that \(\hat{F}(\Sigma', k, w\lambda) = \hat{F}(\Sigma', k, \lambda)\) for \(w \in W'\) and that for \(H \in a_{\cdot}\) and a generic \(\lambda\) satisfying \((\text{Re } \lambda, \alpha) < 0\) \((\forall \alpha \in \Sigma')\)

\begin{equation}
\lim_{t \to \infty} e^{t(-\lambda - \rho(k))}(H) \hat{F}(\Sigma', k, \lambda; tH) = \hat{c}(\Sigma', k, -\lambda).
\end{equation}

**Theorem 4.4** ([Op1, Theorem 2.8], [Hec2, §4]). There exists a \(W\)-invariant open neighborhood \(U\) of \(0 \in a_{\cdot}\) such that \(\hat{F}(\Sigma', k, \lambda; H)\) extends to a holomorphic function on \(K(\Sigma') \times a^*_C \times (a + \sqrt{-1}U)\). Moreover \(f := \hat{F}(\Sigma', k, \lambda; H)\) satisfies (HG1) and (HG2) for each \((k, \lambda) \in K(\Sigma') \times a^*_C\).

**Theorem 4.5** ([Op2, Theorem 6.1]). \(\hat{F}(\Sigma', k, \lambda; 0) = \hat{c}(\Sigma', k, \rho(k))\) for any \((k, \lambda) \in K(\Sigma')\).

In particular \(\hat{c}(\Sigma', k, \rho(k))\) is an entire holomorphic function on \(K(\Sigma')\).

**Definition 4.6.** Put

\[K_{\text{reg}}(\Sigma') = \{k \in K(\Sigma') \mid \hat{c}(\Sigma', k, \rho(k)) \neq 0\}.\]

For each \((k, \lambda) \in K_{\text{reg}}(\Sigma') \times a^*_C\) we define

\[F(\Sigma', k, \lambda) = \hat{c}(\Sigma', k, \rho(k))^{-1} \hat{F}(\Sigma', k, \lambda).\]

This is a real analytic function on \(a_{\cdot}\) satisfying (HG1) and (HG3).
4.3. Regularity conditions on $k$. Suppose $k \in K(\Sigma')$. For $H \in a$ a so-called Cherednik operator is defined by

$$T_k(H) = \partial(H) + \sum_{\alpha \in \Sigma^+} \frac{k_\alpha(H)}{1-e^{-\alpha}}(1-r_\alpha) - \rho(k)(H)$$

where $\partial(H)$ is the $H$-directional derivative and $r_\alpha$ is the orthogonal reflection in $\alpha = 0$. The operator $T_k(H)$ acts on various function spaces including $\mathcal{A}(a)$ and the algebra $\mathcal{A}_0$ of formal power series at $0 \in a$. Note $\mathcal{A}(a) \subset \mathcal{A}_0$. By [Ch1], $T_k(H)$'s $(H \in a)$ are commutative and $T_k(\cdot)$ extends to an algebra homomorphism $S(a_C) \to \text{End}_C \mathcal{A}_0$ (see also [Ch2, Theorem 2.1]). Suppose $\lambda \in a_C^\ast$. In view of [Op3, Theorem 2.12], $f \in A$ satisfies (HG1)–(HG3) if and only if $f$ satisfies the following:

(HG1’) $f \in \mathcal{A}_0^{W'}$, (HG2’) $T_k(D)f = D(\lambda)f$ for any $D \in S(a_C)^{W'}$. (HG3’) $f(0) = 1$.

These conditions can be applied to any $f \in \mathcal{A}_0$. Define a bilinear form $\langle \cdot, \cdot \rangle_k : S(a_C)^{\times} \times \mathcal{A}_0 \to \mathbb{C}$ by $\langle D, f \rangle_k = (T_k(D)f)(0)$ and put

$$l-\text{Rad}_k = \{D \in S(a_C) \mid \langle D, f \rangle_k = 0 \quad (\forall f \in \mathcal{A}_0)\};$$

$$r-\text{Rad}_k = \{f \in \mathcal{A}_0 \mid \langle D, f \rangle_k = 0 \quad (\forall D \in S(a_C))\}.$$  

The Dunkl operator for $H \in a$ with multiplicity parameter $k$ ([Du1]) is defined by

$$\hat{T}_k(H) = \partial(H) + \sum_{\alpha \in \Sigma^+} \frac{k_\alpha(H)}{\alpha}(1-r_\alpha).$$

The space $\mathcal{P}(a)$ of polynomial functions on $a$ is identified with $S(a_C)$ via $B(\cdot, \cdot)$, so that $\hat{T}_k(H)$’s act on $S(a_C)$. This action also extends to an algebra homomorphism $\hat{T}_k(\cdot) : S(a_C) \to \text{End}_C S(a_C)$. Define a bilinear form $(\cdot, \cdot)_k$ on $S(a_C)^{\times} \times S(a_C)$ by $(D_1, D_2)_k = (\hat{T}_k(D_1)D_2)(0)$. By [Du2, Theorem 3.5], $(\cdot, \cdot)_k$ is symmetric.

**Theorem 4.7.** The following conditions on $k \in K(\Sigma')$ are all equivalent:

1. $k \in K_{\text{reg}}(\Sigma')$;
2. $(\cdot, \cdot)_k$ is non-degenerate;
3. $l-\text{Rad}_k = \{0\}$;
4. $r-\text{Rad}_k = \{0\}$;
5. for any $\lambda \in a_C^*$ it holds that any $f \in \mathcal{A}_0 \setminus \{0\}$ satisfying (HG1') and (HG2') takes a non-zero value at $0$;
6. for any $\lambda \in a_C^*$ there exists some $f \in \mathcal{A}(a)$ satisfying (HG1), (HG3);
7. there exists a Zariski dense subset $Z \subset a_C^*$ such that for any $\lambda \in Z$ there exists some $f \in \mathcal{A}_0$ satisfying (HG1'), (HG3)
Corollary 4.8. For any \( k \in K_{\text{reg}}(\Sigma') \) and \( \lambda \in a_+^\times \), \( F(\Sigma', k, \lambda) \) is the unique real analytic function on a satisfying \([HG1] [HG2] [HG3] \) (or equivalently \([HG1'] [HG2'] [HG3'] \)). In particular the definition of \( F(\Sigma', k, \lambda) \) is independent of the choice of \( \Sigma'^+ \).

**Proof.** The uniqueness follows from \([5]\) \( \square \)

**Corollary 4.9.** \( F(\Sigma', k, \lambda; -H) = F(\Sigma', k, -\lambda; H). \)

**Proof.** Let \( T_k^- (\xi) \ (\xi \in a) \) stand for the Cherednik operator defined by using \( -\Sigma' \) as a positive system. For \( f \in \mathcal{A}(a) \) define \( \sigma f \in \mathcal{A}(a) \) by \( (\sigma f)(H) = f(-H) \). Observe that \( T_k^- (\xi) \sigma f = \sigma T_k(-\xi) f \) for any \( \xi \in a \). Hence for any \( f \) satisfying \([HG2'] \) we have

\[
T_k^- (D)\sigma f = \sigma T_k(D(-)) f = D(-\lambda)\sigma f \quad \text{for any } D \in S(a^*_C)^{W'}. 
\]

Thus the desired equality follows from Corollary 4.8 \( \square \)

**Corollary 4.10.** Suppose \( k \in K_{\text{reg}}(\Sigma') \). Then for \( H \in a_+ := -a_- \) and a generic \( \lambda \) satisfying \((\text{Re} \lambda, \alpha) > 0 \ (\alpha \in \Sigma') \) we have

\[
\lim_{t \to \infty} e^{t(-\lambda+\rho(k))(H)} F(\Sigma', k, \lambda; tH) = c(\Sigma', k, \lambda). 
\]

**Proof.** Immediate from \([4.7]\) and the previous corollary. \( \square \)

**Remark 4.11.** The prototype of the theorem is a similar result for Opdam’s generalized Bessel functions obtained by Dunkl, de Jeu and Opdam \([DJO, \S4]\). They also give for each type of irreducible \( \Sigma' \) a very explicit description of the singular set of those \( k \) which do not satisfy \([2]\). By the theorem this singular set equals \( K(\Sigma') \setminus K_{\text{reg}}(\Sigma') \).

The rest of this subsection is devoted to the proof of Theorem 4.7. For \( d = 0, 1, 2, \ldots \) let \( S_d(a_C) \) be the subspace of \( S(a_C) \approx \mathcal{P}(a) \) consisting of homogeneous polynomials of degree \( d \). Since \( \tilde{T}_k(H) \) is a homogeneous operator of degree \(-1\) for any \( H \in a \setminus \{0\} \), \( S_d(a_C) \perp S_e(a_C) \) with respect to \( (\cdot, \cdot)_k \) if \( d \neq e \). Now any \( f \in \mathcal{A}_0 \) decomposes into the sum \( f = \sum_{d \geq 0} f_d \) of its homogeneous parts \( f_d \in S_d(a_C) \). We define \( \text{ord } f = \min\{d \mid f_d \neq 0\} \in \mathbb{N} \cup \{\infty\} \) and put \( \mathcal{A}_{0, > d} = \{f \in \mathcal{A}_0 \mid \text{ord } f > d\} \). Then we have

\[
\mathcal{A}_0 / \mathcal{A}_{0, > d} \simeq S_{\leq d}(a_C) := \bigoplus_{e \leq d} S_e(a_C).
\]

The next lemma is easily observed:

**Lemma 4.12.** Suppose \( f \in \mathcal{A}_0 \setminus \{0\} \) with \( d = \text{ord } f \). Then for any \( H \in a \), \( \text{ord}(T_k(H)f) \geq d - 1 \) and

\[
T_k(H)f \equiv \tilde{T}_k(H)f_d \pmod{\mathcal{A}_{0, > d-1}}. 
\]

For \( d = 0, 1, 2, \ldots \) let \( (\cdot, \cdot)_k^d \) be the restriction of \( (\cdot, \cdot)_k \) to \( S_{\leq d}(a_C) \times S_{\leq d}(a_C) \). Since \( \dim S_{\leq d}(a_C) < \infty \), the left and right radicals of \( (\cdot, \cdot)_k^d \) have the same dimension. Let us consider an auxiliary condition:

\[(3') \ (\cdot, \cdot)_k^d \text{ is non-degenerate for each } d = 0, 1, 2, \ldots \]
Proof of (2)⇒(3)⇒(4). Suppose (2). For any $D \in S(a_C) \setminus \{0\}$ with $\deg D = d$, we can take $f \in S_d(a_C)$ so that $(D,f)_k \neq 0$. Then from the lemma above we have $(D,f)_k = (T_k(D)f)(0) = (T_k(D)f)(0) = (D,f)_k \neq 0$, proving (3).

Next, one easily sees (3)⇒(3′)⇒(4′) since $S_{\leq d}(a_C) \perp \mathcal{A}_0$ with respect to $\langle \cdot, \cdot \rangle_k$ by the lemma.

To deduce (2) from (3′) take an arbitrary $f \in S_d(a_C) \setminus \{0\}$ with $d = 0,1,2,\ldots$. Then (3′) assures the existence of $D \in S_{\leq d}(a_C)$ such that $(D,f)_k \neq 0$. Since $\text{ord} f = d$, the lemma again implies $(D,f)_k = (D,f)_k \neq 0$.

The implication (1)⇒(6)⇒(7) is obvious.

Lemma 4.13. For any $k_0 \in K(\Sigma')$, $\hat{F}(\Sigma',k_0;\lambda,H)$ is a non-trivial function in $(\lambda,H) \in a_C^* \times a$.

Proof. Because there is a non-empty open subset $U_{k_0} \subset a_C^*$ such that (4.7) holds for any $(\lambda,H) \in U_{k_0} \times a$ and $k = k_0$.

Hence by Theorem 4.5 we have (5)⇒(1).

Proof of (7)⇒(3). Suppose $\text{l-Rad}_k \supseteq D \neq 0$. Since $\text{l-Rad}_k$ is an ideal of $S(a_C)$, $D' := \prod_{w \in W'} w(D)$ is a non-zero $W'$-invariant element in $\text{l-Rad}_k$. Now suppose (7) is true. Then we can find $\lambda \in Z$ such that $D'(\lambda) \neq 0$. But for a function $f \in \mathcal{A}_0$ of [HG1] (HG’), we have $(D',f)_k = (T_k(D')f)(0) = D'(\lambda)f(0) = D'(\lambda) \neq 0$. This contradicts that $D' \in \text{l-Rad}_k$.

The proof is complete if we show (4)⇒(5). To do so, we needs the graded Hecke algebra $H = H(\Sigma'^+,k)$ by [Lu]. The algebra $H$ is isomorphic to $C \mathcal{W}' \otimes S(a_C)$ as a $C$-linear space. Here the group algebra $C \mathcal{W}'$ of $W'$ and $S(a_C)$ are identified with subalgebras of $H$ by $w \mapsto w \otimes 1$ and $D \mapsto 1 \otimes D$. These two subalgebras relate to each other as follows:

$w \cdot D = w \otimes D$ for any $w \in W'$ and $D \in S(a_C)$;

$H \cdot r_\alpha = r_\alpha \cdot r_\alpha(H) - (k_\alpha + 2k_\alpha)\alpha(H)$ for any simple root $\alpha \in \Sigma'^+$ and any $H \in a_C$.

The center of $H$ equals $S(a_C)^{W'}$ ([Lu] Theorem 6.5]).

Thanks to [Ch2] Theorem 2.4, $H \ni w \otimes D \mapsto w \cdot T_k(D) \in \text{End}_C \mathcal{A}_0$ defines an algebra homomorphism (also denoted by $T_k(\cdot)$). Let $Cv_0$ be a one-dimensional right trivial $W'$-module. Then $Cv_0 \otimes_{C \mathcal{W}'} H$ is a right $H$-module. Identifying $S(a_C)$ with $Cv_0 \otimes_{C \mathcal{W}'} H$ by $D \mapsto v_0 \otimes D$, we consider $\langle \cdot, \cdot \rangle_k$ is a bilinear form on $Cv_0 \otimes_{C \mathcal{W}'} H \times \mathcal{A}_0$. Observe that $\langle h, \cdot \rangle_k = \langle \cdot, h \rangle_k$ for any $h \in H$. For $d = 0,1,2,\ldots$, $v_0 \otimes S_{\leq d}(a_C)$ is a $W'$-submodule of $Cv_0 \otimes_{C \mathcal{W}'} H$ and

$v_0 \otimes S_{\leq d}(a_C) / v_0 \otimes S_{\leq d-1}(a_C) \simeq S_d(a_C)$ with natural right $W'$-module structure.

From this one easily sees
Lemma 4.14. The subspace of $\mathbb{C}v_0 \otimes_{G/K} H$ consisting of the right $W'$-invariants is $v_0 \otimes S(aC)^W$.

Proof of (4) Let $\lambda \in a_C^+$ and suppose $f \in \mathcal{A}_0$ satisfies \((\text{HG}1')\) and $f(0) = 0$. Take an arbitrary $D \in S(aC)$ . Then by the last lemma there exists $D \in S(aC)^W$ such that $v_0 \otimes D = \frac{1}{\#W'} \sum_{w \in W'} v_0 \otimes D \cdot w$. Hence we have
\[
\langle D, f \rangle_k = \langle v_0 \otimes D, f \rangle_k = \frac{1}{\#W'} \sum_{w \in W'} \langle v_0 \otimes D, w \cdot f \rangle_k = \frac{1}{\#W'} \sum_{w \in W'} \langle v_0 \otimes D \cdot w, f \rangle_k
\]
\[
= \langle v_0 \otimes \tilde{D}, f \rangle_k = (T_k(\tilde{D})f)(0) = \tilde{D}(\lambda)f(0) = 0.
\]
This shows $f \in r\text{-Rad}_k$. \qed

5. Matching conditions

We return to the setting of §4. Thus $(\pi, V)$ is a small $K$-type of a connected non-compact real semisimple Lie group $G$ with finite center. The purpose of this section is to get an easy and concrete condition on a root system $\Sigma^\pi$ and a multiplicity function $k^\pi$ for the validity of (1.6).

5.1. Coincidence of differential operators. Let $\Sigma'$ be a root system in $a^*$ ($a = \text{Lie} A$) and $k$ a multiplicity function on $\Sigma'$. In addition, we suppose $\Sigma' \subset \Sigma \cup 2\Sigma$ and the Weyl group $W'$ for $\Sigma'$ equals $W$. Let $\Sigma'^+ := \Sigma' \cap (\Sigma^+ \cup 2\Sigma^+)$.

Proposition 5.1. With $\delta_{G/K}$ in (3.14) it holds that
\[
\delta_{G/K}^2 \circ (\Delta^\pi(\Omega_g - \varpi^\pi)) + ||\rho||^2 \circ \delta_{G/K}^{-2}
\]
\[
= \Omega_\alpha + \sum_{\alpha \in \Sigma^+} m_\alpha \coth \alpha H_\alpha \left(\frac{-\kappa_\alpha}{\sinh^2 \frac{\alpha}{2}} + \frac{2 - m_\alpha - 2m_{2\alpha} + 4\kappa_\alpha}{\sinh^2 \alpha} \right).
\]

Proof. Letting $\Sigma' = 2\Sigma$ and $k_{2\alpha} = \frac{1}{2}m_\alpha$ ($\alpha \in \Sigma$), we have
\[
L(\Sigma', k) = \Omega_\alpha + \sum_{\alpha \in \Sigma^+} m_\alpha \coth \alpha H_\alpha, \quad \delta(\Sigma', k) = \delta_{G/K} \quad \text{and} \quad \rho(k) = \rho.
\]
Hence it follows from (3.10) and the Proposition 4.2 that
\[
\delta_{G/K}^2 \circ (\Delta^\pi(\Omega_g - \varpi^\pi)) + ||\rho||^2 \circ \delta_{G/K}^{-2} = \Omega_\alpha + \sum_{\alpha \in \Sigma^+} m_\alpha ||\alpha||^2 \left(\frac{-\kappa_\alpha}{\sinh^2 \frac{\alpha}{2}} + \frac{2 - m_\alpha - 2m_{2\alpha} + 4\kappa_\alpha}{\sinh^2 \alpha} \right).
\]
Using the equality
\[
\frac{1}{\cosh^2 \frac{\alpha}{2}} = \frac{1}{\sinh^2 \frac{\alpha}{2}} - \frac{4}{\sinh^2 \alpha}
\]
we get the proposition. \qed
Let us consider general $\Sigma'$ and $k$ again. The algebra homomorphism $\gamma_{\rho(k)} : \mathcal{R}' \otimes S(\mathfrak{a}_C) \to S(\mathfrak{a}_C)$ defined by (4.3) can be regarded as a part of the algebra homomorphism

$$\gamma_{\rho(k)} : \mathcal{R} \otimes S(\mathfrak{a}_C) \to S(\mathfrak{a}_C) \oplus \mathcal{H} \otimes S(\mathfrak{a}_C) \xrightarrow{\text{projection}} S(\mathfrak{a}_C) \xrightarrow{f(\lambda) \mapsto f(\lambda + \rho(k))} S(\mathfrak{a}_C).$$

**Lemma 5.2.** The subalgebra

$$(\mathcal{R} \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k) := \{ D \in (\mathcal{R} \otimes S(\mathfrak{a}_C))^\text{W} \mid [L(\Sigma', k), D] = 0 \}$$

coincides with $(\mathcal{R}' \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k)$.

**Proof.** By the same argument as in [Hec2] §1.2, we can prove the restriction of $\gamma_{\rho(k)}$ to $(\mathcal{R} \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k)$ is injective homomorphism into $S(\mathfrak{a}_C)^\text{W}$. Hence the lemma follows from Proposition 4.3 (Recall $W = W'$ by assumption.)

**Proposition 5.3.** Suppose for a choice of $\Sigma'$ and $k$ the equality

$$(5.2) \quad \delta(\Sigma', k) \frac{1}{2} o (L(\Sigma', k) + (\rho(k), \rho(k))) o \delta(\Sigma', k) - \frac{1}{2} = \delta_{G/K} o (\Delta^\pi(\Omega_\theta - \omega^\pi) + ||\rho||^2) o \delta_{G/K}$$

holds, namely, the operators in (4.2) and (5.1) coincide. Then we have

$$(\mathcal{R}' \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k) = \delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K} o \Delta^\pi(D^\pi) o \delta_{G/K}^\pi \delta(\Sigma', k)^{\frac{1}{2}}.$$

Moreover, for any $D \in U(\mathfrak{g}_C)^\text{K}$ it holds that

$$\gamma_{\rho(k)}(\delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K} o \Delta^\pi(D) o \delta_{G/K}^\pi \delta(\Sigma', k)^{\frac{1}{2}}) = \gamma^\pi(D).$$

**Proof.** Define an algebra homomorphism $\tau : U(\mathfrak{g}_C)^\text{K} \to (\mathcal{R} \otimes S(\mathfrak{a}_C))^\text{W}$ by

$$\tau(D) = \delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K} o \Delta^\pi(D) o \delta_{G/K}^\pi \delta(\Sigma', k)^{\frac{1}{2}}.$$

Suppose $D \in U(\mathfrak{g}_C)^\text{K}$. Then Proposition 3.10 implies $\gamma^\pi(D) = \gamma_\rho o \Delta^\pi(D)$, while one easily sees $\gamma_{\rho(k)}(\delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K} o E o \delta_{G/K}^\pi \delta(\Sigma', k)^{\frac{1}{2}}) = \gamma_\rho(E)$ for any $E \in \mathcal{R} \otimes S(\mathfrak{a}_C)$. Thus we have $\gamma_{\rho(k)} o \tau(D) = \gamma^\pi(D)$, the second assertion of the proposition. Now it holds that

$$[L(\Sigma', k), \tau(D)] = \tau([\Omega_\theta - \omega^\pi + ||\rho||^2 - (\rho(k), \rho(k)), D]) = 0.$$ 

This shows $\tau(U(\mathfrak{g}_C)^\text{K}) \subset (\mathcal{R} \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k) = (\mathcal{R}' \otimes S(\mathfrak{a}_C))^\text{W,L}(\Sigma', k)$. But these two subalgebras actually coincide by Theorem 3.1 Proposition 4.3 and the second assertion.

Since the functions $\sinh^{-\frac{1}{2}} \frac{\alpha}{2}$ ($\alpha \in \Sigma \cup 2\Sigma$) are linearly independent, (5.2) holds if and only if

$$(5.3) \quad -m_\alpha \kappa_\alpha^\pi + \frac{1}{2} m_\alpha^2 (1 - \frac{1}{2} m_\alpha - m_\alpha + 2 \kappa_\alpha^\pi) = k_\alpha (1 - k_\alpha - 2k_{2\alpha}) \quad \text{for any } \alpha \in \Sigma \cup 2\Sigma.$$

Here we suppose $m_\alpha = \kappa_\alpha^\pi = 0$ for $\alpha \notin \Sigma$ and $k_\alpha = 0$ for $\alpha \notin \Sigma'$. 
Proposition 5.4. The function \( \delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma', k)^\frac{1}{2} \in \mathcal{A}(a_{\text{reg}}) \) extends to a real analytic function on some open set \( U \) containing \( a_{\text{reg}} \cup \{0\} \) if and only if

\[
\frac{m_\alpha + m_{2\alpha}}{2} = k_\alpha + k_{2\alpha} + k_{4\alpha} \quad \text{for any } \alpha \in \Sigma \setminus 2\Sigma.
\]

If this is the case then

\[
\delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma', k)^\frac{1}{2} = \prod_{\alpha \in \Sigma' \setminus 2\Sigma} \left( \cosh \frac{\alpha}{2} \right)^{-k_\alpha} (\cosh \alpha)^{k_{4\alpha} - \frac{m_{2\alpha}}{2}}.
\]

In particular, \( \delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma', k)^\frac{1}{2} \) extends to a nowhere-vanishing real analytic function on \( a \) taking \( 1 \) at \( 0 \in a \).

Proof. The first statement is immediate since for each \( \alpha \in \Sigma \cup 2\Sigma \) we have the expansion

\[
\sinh(\alpha/2) = \frac{\alpha}{||\alpha||} \left( 1 + \frac{(\alpha/2)^2}{3!} + \frac{(\alpha/2)^4}{5!} + \cdots \right).
\]

Next, \( (5.5) \) holds since for each \( \alpha \in \Sigma' \setminus 2\Sigma^+ \)

\[
\left| \sinh \alpha \right| \left| \frac{m_\alpha}{2} \right| \sinh(2\alpha) \left| \frac{m_{2\alpha}}{2} \right| \left| \frac{\sinh(\alpha/2)}{2\alpha} \right| \left| \frac{\sinh(2\alpha)}{2\alpha} \right| \left| \frac{\alpha}{||\alpha||} \right| ^{k_\alpha} \left| \frac{\alpha}{||\alpha||} \right| ^{k_{2\alpha}} \left| \frac{\alpha}{||\alpha||} \right| ^{k_{4\alpha}}
= \left( \cosh \frac{\alpha}{2} \right)^{-k_\alpha} (\cosh \alpha)^{k_{4\alpha} - \frac{m_{2\alpha}}{2}}.
\]

Theorem 5.5. Suppose \( (5.3) \) and \( (5.4) \) hold for a choice of \( \Sigma' \) and \( k \). Then \( k \in \mathcal{K}_{\text{reg}}(\Sigma') \) and it holds that

\[
\mathcal{T}^\pi(\phi_\lambda^\pi) = \delta_{G/K}^{-\frac{1}{2}} \delta(\Sigma', k)^\frac{1}{2} F(\Sigma', k, \lambda) \quad \text{for any } \lambda \in a_\ast^{'\text{reg}}.
\]

Proof. Suppose \( \lambda \in a_\ast^{'\text{reg}} \). By Proposition 5.4, \( \delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K}^{\frac{1}{2}} \mathcal{T}^\pi(\phi_\lambda^\pi) \) extends to a real analytic function on \( a \) taking \( 1 \) at \( 0 \in a \). This is clearly \( W \)-invariant. Also, it follows from Corollary 3.13 and Proposition 5.3 that this function satisfies \( (\text{HG2}) \) in §4.1. Thus \( \delta(\Sigma', k)^{-\frac{1}{2}} \delta_{G/K}^{\frac{1}{2}} \mathcal{T}^\pi(\phi_\lambda^\pi) \) satisfies \( (\text{HG1}), (\text{HG3}) \) for any \( \lambda \in a_\ast^{\text{reg}} \). Hence \( k \in \mathcal{K}_{\text{reg}}(\Sigma') \) by the implication \( (6) \Rightarrow (1) \) in Theorem 4.4. Finally, \( (5.6) \) follows from Corollary 4.8.

In the notation of Theorem 1.6 our result is summarized as follows. Let \( \Sigma^\pi \) be a root system in \( a^\ast \) and \( k^\pi \) a multiplicity function on \( \Sigma^\pi \). Then \( (1.6) \) holds if the following conditions are satisfied:

(MC1) \( \Sigma^\pi \subset \Sigma \cup 2\Sigma \);
(MC2) \( (5.3) \) holds with \( (\Sigma', k) = (\Sigma^\pi, k^\pi) \);
(MC3) \( (5.4) \) holds with \( (\Sigma', k) = (\Sigma^\pi, k^\pi) \).
Note that if \([\text{MC3}]\) is true, then each root in \(\Sigma\) is proportional to some root in \(\Sigma^\pi\). Hence the Weyl group of \(\Sigma^\pi\) equals \(W\) under \([\text{MC1}]\) and \([\text{MC3}]\). It is not so hard to observe that under \([\text{MC1}]\) \([\text{MC2}]\) holds only if both \([\text{MC2}]\) and \([\text{MC3}]\) are true. (We do not use this fact in the paper.) Conditions \([\text{MC1}]\) \([\text{MC3}]\) will be further simplified after we look into the structure of \((\pi, V)\) more precisely.

5.2. The associated split semisimple subgroup. Let \(\mathfrak{b}\) be a Cartan subalgebra of \(\mathfrak{m} = \text{Lie} \, M\). Then \(\mathfrak{b}_\mathbb{C} + \mathfrak{a}_\mathbb{C}\) is a Cartan subalgebra of \(\mathfrak{g}_\mathbb{C}\). A root \(\mu\) for \((\mathfrak{g}_\mathbb{C}, \mathfrak{b}_\mathbb{C} + \mathfrak{a}_\mathbb{C})\) is called real when \(\mu|_{\mathfrak{b}_\mathbb{C}} = 0\). We denote the set of real roots by \(\Sigma_{\text{real}}\), which is naturally identified with a subset of \(\Sigma\). A restricted root \(\alpha \in \Sigma\) belongs to \(\Sigma_{\text{real}}\) if and only if \(\mathfrak{m}_\alpha\) is odd (cf. [Kn2, Chapter X, Exercises F]). Now

\[
\mathfrak{g}^\mathfrak{b} = \mathfrak{a} + \mathfrak{b} + \sum_{\alpha \in \Sigma_{\text{real}}} \mathfrak{g}^\mathfrak{b}_\alpha
\]

is a reductive subalgebra of \(\mathfrak{g}\). Its semisimple part is

\[
\mathfrak{g}_{\text{split}} := [\mathfrak{g}^\mathfrak{b}, \mathfrak{g}^\mathfrak{b}] = \mathfrak{a}_{\text{split}} + \sum_{\alpha \in \Sigma_{\text{real}}} \mathfrak{g}^\mathfrak{b}_\alpha = \sum_{\alpha \in \Sigma_{\text{real}}} \mathfrak{R} \mathfrak{H}_\alpha,
\]

which is a split semisimple Lie algebra with Cartan subalgebra \(\mathfrak{a}_{\text{split}}\) ([Kn2, Chapter VII, §5]). The restricted root system of \(\mathfrak{g}_{\text{split}}\) is identified with \(\Sigma_{\text{real}}\). Let \(G_{\text{split}}\) be the analytic subgroup for \(\mathfrak{g}_{\text{split}}\) (the associated split semisimple subgroup). Let \(M_0\) denote the identity component of \(M\). If we put \(K_{\text{split}} = K \cap G_{\text{split}}\), \(M_{\text{split}} = M \cap K_{\text{split}}\), then \(M_{\text{split}}\) is the centralizer of \(\mathfrak{a}_{\text{split}}\) in \(K_{\text{split}}\). Furthermore \(M_{\text{split}}\) normalizes \(M_0\), and \(M = M_0 M_{\text{split}}\) (cf. [Kn2, Theorem 7.52]).

**Proposition 5.6.** The restriction of \((\pi, V)\) to \(M_0\) is isomorphic to the direct sum of some copies of an irreducible representation of \(M_0\): \(V|_{M_0} \cong U^{\otimes r}\). Let \(\mu\) be an extremal \(\mathfrak{b}_\mathbb{C}\)-weight of \(U\) and put

\[
(5.8) \quad V_\mu = \{v \in V \mid \pi(H)v = \mu(H)v \ (\forall H \in \mathfrak{b})\}.
\]

Then \(\pi(K_{\text{split}}) V_\mu \subset V_\mu\) and \(V_\mu\) is irreducible as an \(M_{\text{split}}\)-module. That is, \((\pi|_{K_{\text{split}}}, V_\mu)\) is a small \(K_{\text{split}}\)-type of \(G_{\text{split}}\) with dimension \(r\).

**Proof.** Let \(\mu\) be the highest weight of \(V|_{M_0}\) with respect to a lexicographical order of \(\mathfrak{b}^*\) and define \(V_\mu\) by \((5.8)\). Since \(K_{\text{split}} \subset G\), we have \(\pi(K_{\text{split}}) V_\mu \subset V_\mu\). Let \(E \subset V_\mu\) be an irreducible \(M_{\text{split}}\)-submodule. Since \(M_{\text{split}}\) normalizes \(M_0\), \(\pi(U(\mathfrak{m}_\mathbb{C})) E\) is \(M\)-stable and hence is equal to \(V\). By the highest weight theory, \(E = V_\mu \cap \pi(U(\mathfrak{m}_\mathbb{C})) E = V_\mu\). Thus \(V_\mu\) is an irreducible \(M_{\text{split}}\)-module. By the highest weight theory again, we see if \(U\) is an irreducible \(M_0\)-module with highest weight \(\mu\) then \(V|_{M_0} \cong U^{\otimes r}\) with \(r = \dim V_\mu\). \(\square\)

**Corollary 5.7.** For any \(\alpha \in \Sigma\) with even \(m_\alpha\) we have \(\kappa^\pi_\alpha = 0\).
Proof. Let $\Delta_m$ be the root system for $(m_C, b_C)$. Suppose $\alpha \in \Sigma$ has an even root multiplicity. Then $\alpha \notin \Sigma_{\text{real}}$ and all $b_C$-weights of $m_C$-module $(g_\alpha)_C$ are not zero. Hence by the representation theory of complex reductive Lie algebra, any $b_C$-weight of $(g_\alpha)_C$ (or equivalently, that of $\{X_\alpha + \theta X_\alpha \mid X_\alpha \in (g_\alpha)_C\}$) is outside the root lattice $\mathbb{Z}\Delta_m$. Let $\mu$ be as in Proposition 5.6. Then by the proposition, $\mu - \lambda$ belongs to $\mathbb{Z}\Delta_m$ for any $b_C$-weight $\lambda$ of $V$. But this means the difference of $\mu$ and any $b_C$-weight of the $M$-submodule $\{\pi(X_\alpha + \theta X_\alpha)v \mid X_\alpha \in g_\alpha, v \in V\} \subset V$ is also inside $\mathbb{Z}\Delta_m$. It is possible only when $\{\pi(X_\alpha + \theta X_\alpha)v \mid X_\alpha \in g_\alpha, v \in V\} = \{0\}$. Hence Proposition 3.8 implies $\kappa_\alpha^\pi = 0$. \[\square\]

Corollary 5.8. Let $V_\mu$ be as in Proposition 5.6 and put $(\pi_\mu, V_\mu) = (\pi|_{K_{\text{split}}}, V_\mu)$. For any $\alpha \in \Sigma$ with $m_\alpha = 1$, we have $\alpha \in \Sigma_{\text{real}}$ and $\kappa_\alpha^\pi = \kappa_\alpha^\mu$.

Proof. Suppose $\alpha \in \Sigma$ with $m_\alpha = 1$ is given. One has $\alpha \in \Sigma_{\text{real}}$ and $g_\alpha \subset g_{\text{split}}$. Take $X_\alpha \in g_\alpha$ so that $-\frac{|\alpha|^2}{2} B(X_\alpha, \theta X_\alpha) = 1$. Then $\kappa_\alpha^\mu \text{id}_V = \pi(X_\alpha + \theta X_\alpha)^2$ by definition. Note the normalization condition for $X_\alpha$ is rewritten as $[X_\alpha, [X_\alpha, \theta X_\alpha]] = 2X_\alpha$, which is common to both $g$ and $g_{\text{split}}$. Hence $\kappa_\alpha^\pi \text{id}_{V_\mu} = \pi_\mu(X_\alpha + \theta X_\alpha)^2 = \pi(X_\alpha + \theta X_\alpha)^2|_{V_\mu} = \kappa_\alpha^\pi \text{id}_{V_\mu}$.

\[\square\]

5.3. Simplifying matching conditions.

Proposition 5.9. Let $R$ be a complete system of representatives for the $W$-orbits of $\Sigma \setminus 2\Sigma$. Let $\Sigma'$ be a root system in $a^*$ satisfying (MC1). Then a multiplicity function $k^\pi$ on $\Sigma'$ satisfies (MC2) and (MC3) if and only if the following are valid:

(1) for any $\alpha \in R$ with $m_{2\alpha} = 0$

$$
\begin{align*}
 k_\alpha^\pi &= m_\alpha - 1 \pm \sqrt{(m_\alpha - 1)^2 - 4m_\alpha \kappa_\alpha^\pi}, \\
 k_{2\alpha}^\pi &= \frac{1 \mp \sqrt{(m_{2\alpha} - 1)^2 - 4m_{2\alpha} \kappa_{2\alpha}^\pi}}{2}.
\end{align*}
$$

(2) for any $\alpha \in R$ with $m_{2\alpha} > 0$

$$
\begin{align*}
 k_\alpha^\pi &= 0, \\
 k_{2\alpha}^\pi &= \frac{m_\alpha + m_{2\alpha} - 1 \pm \sqrt{(m_{2\alpha} - 1)^2 - 4m_{2\alpha} \kappa_{2\alpha}^\pi}}{2}, \\
 k_{4\alpha}^\pi &= \frac{1 \mp \sqrt{(m_{4\alpha} - 1)^2 - 4m_{4\alpha} \kappa_{4\alpha}^\pi}}{2},
\end{align*}
$$

or

$$
\kappa_{2\alpha}^\pi = \frac{1}{4} m_{2\alpha} - \frac{1}{2} \quad \text{and} \quad \begin{align*}
 k_\alpha^\pi &= m_\alpha + m_{2\alpha} - 1, \\
 k_{2\alpha}^\pi &= 1 - \frac{m_\alpha + m_{2\alpha}}{2}, \\
 k_{4\alpha}^\pi &= 0.
\end{align*}
$$
5.9. 

Proof. Suppose \( \alpha \in R \) and \( m_{2\alpha} > 0 \). Then \( m_\alpha \) is even (cf. [Hec11] Chapter X, Exercises F]) and \( \kappa_\alpha^\pi = 0 \) by Corollary 5.7. Thus those parts of (5.3) and (5.4) that relate to \( \alpha, 2\alpha \) and \( 4\alpha \) are reduced to:

\[
\begin{align*}
0 &= k_\alpha^\pi (1 - k_\alpha^\pi - 2k_{2\alpha}^\pi), \\
-2m_{2\alpha}k_{2\alpha}^\pi + \frac{1}{4}m_\alpha (1 - \frac{1}{2}m_\alpha - m_{2\alpha}) &= k_{2\alpha}^\pi (1 - k_{2\alpha}^\pi - 2k_{4\alpha}^\pi), \\
\frac{1}{2}m_{2\alpha} (1 - \frac{1}{2}m_{2\alpha} + 2k_{2\alpha}^\pi) &= k_{2\alpha}^\pi (1 - k_{2\alpha}^\pi), \\
\frac{1}{2} (m_\alpha + m_{2\alpha}) &= k_\alpha^\pi + k_{2\alpha}^\pi + k_{4\alpha}^\pi.
\end{align*}
\]

(5.9)

In addition, since \( \Sigma^\pi \) is a root system, either \( k_\alpha^\pi \) or \( k_{2\alpha}^\pi \) is zero. Hence by an elementary argument (5.9) is still reduced to the condition in (2). The condition in (1) for \( \alpha \in R \) with \( m_{2\alpha} = 0 \) is obtained in a similar way. \( \square \)

6. Case-by-case analysis

In this section, all the results in §2 will be proved through case-by-case analysis. We start with some preparation. Let \( G \) be a non-compact real simple Lie group with finite center. Note that \( G \) is connected by definition.

Lemma 6.1. Suppose \( \mathfrak{t}_1 \) is an ideal of \( \mathfrak{k} \) such that \( \mathfrak{t}_1 \subset \mathfrak{m} \). Then \( \mathfrak{t}_1 = \{0\} \).

Proof. By assumption, one has for any \( k \in K \)

\[
[\mathfrak{t}_1, \text{Ad}(k)a] = \text{Ad}(k)[\text{Ad}(k^{-1})\mathfrak{t}_1, a] = \text{Ad}(k)[\mathfrak{t}_1, a] \subset \text{Ad}(k)[\mathfrak{m}, a] = \{0\}.
\]

But since \( \mathfrak{s} = \bigcup_{k \in K} \text{Ad}(k)a \), \( \mathfrak{t}_1 \) is an ideal of \( \mathfrak{g} \), which must be \( \{0\} \) since \( \mathfrak{t}_1 \subset \mathfrak{k} \neq \mathfrak{g} \). \( \square \)

Corollary 6.2. The Lie algebra \( \mathfrak{k} \) is generated by \( \{X_\alpha + \theta X_\alpha | X_\alpha \in \mathfrak{g}_\alpha (\alpha \in \Sigma)\} \).

Proof. Note \( \mathfrak{t} = \mathfrak{m} \oplus \sum_{\alpha \in \Sigma} \{X_\alpha + \theta X_\alpha | X_\alpha \in \mathfrak{g}_\alpha \} \) and \( \{X_\alpha + \theta X_\alpha | X_\alpha \in \mathfrak{g}_\alpha (\alpha \in \Sigma)\} \) is \( M \)-stable for each \( \alpha \in \Sigma \). It follows that the Lie subalgebra \( \mathfrak{t}_0 \) generated by \( \{X_\alpha + \theta X_\alpha | X_\alpha \in \mathfrak{g}_\alpha (\alpha \in \Sigma)\} \) is an ideal of \( \mathfrak{k} \). Hence the orthogonal complement \( \mathfrak{t}_1 \) of \( \mathfrak{t}_0 \) in \( \mathfrak{k} \) with respect to \( B(\cdot, \cdot) \) is an ideal satisfying the assumption of Lemma 6.1. Thus we get \( \mathfrak{t}_1 = \{0\} \) and \( \mathfrak{t}_0 = \mathfrak{k} \). \( \square \)

6.1. The trivial \( K \)-type.

Proposition 6.3. A small \( K \)-type \( (\pi, V) \) is trivial if and only if

\[
(6.1) \quad \kappa_\alpha^\pi = 0 \quad \text{for any } \alpha \in \Sigma.
\]

Proof. Note \( \text{Ker}_\mathfrak{k} \pi := \{X \in \mathfrak{k} | \pi(X) = 0\} \) is an ideal of \( \mathfrak{k} \) (and in particular, it is a subalgebra). If we assume (6.1), then \( \text{Ker}_\mathfrak{k} \pi = \mathfrak{t} \) by Proposition 3.8 and Corollary 6.2, showing \( \pi \) is trivial. The converse is clear from Proposition 3.8. \( \square \)

The result on the trivial \( K \)-type stated in §2 readily follows from (6.1) and Proposition 5.9.
6.2. Complex simple Lie groups. Let $G$ be a complex simple Lie group and $(\pi, V)$ a small $K$-type of $G$. Then for any $\alpha \in \Sigma$ we have $m_\alpha = 2$, and hence $\kappa_\pi^2 = 0$ by Corollary 5.7. Thus by Proposition 6.3 $(\pi, V)$ is the trivial $K$-type. Moreover the right-hand side of (5.1) equals $\Omega_\alpha$. From [Hel2, Chapter IV, §5, No.2] we have

$$
\Upsilon_{\pi}(\phi_\pi^\lambda) = \prod_{\alpha \in \Sigma^+} \frac{\rho(H_\alpha)}{\lambda(H_\alpha)} \sum_{w \in W} \text{sgn}(w)e^{w\lambda} \sum_{w \in W} \text{sgn}(w)e^{w\rho}.
$$

Put $\Sigma^\pi = c\Sigma$ and $k^\pi \equiv 1$ with any $c > 0$. Then one easily has for any $\lambda \in \mathfrak{a}_C^*$

$$
F(\Sigma^\pi, k^\pi, \lambda) = \prod_{\alpha \in \Sigma^+} \frac{\rho(H_\alpha)}{\lambda(H_\alpha)} \sum_{w \in W} \text{sgn}(w)e^{w\lambda} \sum_{w \in W} \text{sgn}(w)e^{w\rho},
$$

$$
\tilde{\delta}^{\frac{1}{2}}_{G/K} \tilde{\delta}(\Sigma^\pi, k^\pi)^{\frac{1}{2}} = \prod_{\alpha \in \Sigma^+} \frac{\rho(H_\alpha)}{\lambda(H_\alpha)} \sum_{w \in W} \text{sgn}(w)e^{w\rho}.
$$

Hence (1.6) holds for infinitely many combinations of $\Sigma^\pi$ and $k^\pi$.

6.3. Other simple Lie groups having no non-trivial small $K$-type. We have no non-trivial small $K$-type for those real simple Lie groups $G$ with the following Lie algebras:

| $\mathfrak{g}$ | $\mathfrak{sl}(p, \mathbb{H})$ ($p \geq 2$) | $\mathfrak{so}(2r + 1, 1)$ ($r \geq 1$) | $\mathfrak{e}_6(-26)$ (EIV) |
|---------|-------------------|-------------------|-------------------|
| $\Sigma$ | $A_{p-1}$ | $A_1$ | $A_2$ |
| $m_\alpha$ | 4 | $2r$ | 8 |
| $\mathfrak{t}$ | $\mathfrak{sp}(p)$ | $\mathfrak{so}(2r + 1)$ | $\mathfrak{f}_4$ |

($)\mathfrak{so}(3, 1) \simeq \mathfrak{sl}(2, \mathbb{C}), \mathfrak{sl}(2, \mathbb{H}) \simeq \mathfrak{so}(5, 1)$(

The argument for the first three cases is the same as for the complex case. Suppose $\mathfrak{g} = \mathfrak{f}_{4(-20)}$ and $(\pi, V)$ is a small $K$-type of $G$. Let $\alpha$ is a short restricted root. Then it follows from Corollary 5.7 and Proposition 3.8 that $X_\alpha + \theta X_\alpha \in \text{Ker}_\pi \setminus \{0\}$ for any $X_\alpha \in \mathfrak{g}_\alpha \setminus \{0\}$. Now since $\text{Ker}_\pi$ is an ideal of the simple Lie algebra $\mathfrak{t} \simeq \mathfrak{so}(9)$, one has $\text{Ker}_\pi = \mathfrak{t}$ and hence $(\pi, V)$ is the trivial $K$-type.

6.4. The case $\mathfrak{g} = \mathfrak{sp}(p, q)$. Suppose $G = \mathfrak{sp}(p, q)$ ($p \geq q \geq 1$) and $K = \mathfrak{sp}(p) \times \mathfrak{sp}(q)$. Then $G$ is connected, simply-connected Lie group and $M = M_0$ (cf. [Kn2, Appendix C, §3]). Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the field of quaternions. We use the following
realization:

\[ g = \left\{ \begin{pmatrix} A & B \\ \overline{B} & C \end{pmatrix} \in \mathfrak{gl}(p+q, \mathbb{H}) \mid A \in \mathfrak{gl}(p, \mathbb{H}), C \in \mathfrak{gl}(q, \mathbb{H}), \overline{A} = -A, \overline{C} = -C \right\}, \]

\[ \mathfrak{t} = \left\{ \begin{pmatrix} A & O_{p,q} \\ O_{q,p} & C \end{pmatrix} \mid A \in \mathfrak{gl}(p, \mathbb{H}), C \in \mathfrak{gl}(q, \mathbb{H}), \overline{A} = -A, \overline{C} = -C \right\} \cong \mathfrak{sp}(p) \oplus \mathfrak{sp}(q). \]

\[ \mathfrak{a} = \left\{ H(\mathbf{a}) := \begin{pmatrix} O_{q,q} & O_{q,p-q} & \text{diag}(a_1, \ldots, a_q) \\ O_{p-q,q} & O_{p-q,q} & O_{p-q,q} \\ \text{diag}(a_1, \ldots, a_q) & O_{p-q,q} & O_{q,q} \end{pmatrix} \mid \mathbf{a} = (a_1, \ldots, a_q) \in \mathbb{R}^q \right\}, \]

\[ \mathfrak{m} = \left\{ \begin{pmatrix} \text{diag}(m_1, \ldots, m_q) & O_{q,p-q} & O_{q,q} \\ O_{p-q,q} & Y & O_{p-q,q} \\ O_{q,q} & \text{diag}(m_1, \ldots, m_q) & 0 \end{pmatrix} \mid m_1, \ldots, m_q \in \mathbb{H}, \mathfrak{m}_i + \mathfrak{m}_i = 0 \ (1 \leq i \leq q), \ Y \in \mathfrak{gl}(p-q, \mathbb{H}), \overline{Y} = -Y \right\} \cong \mathfrak{su}(2)^q \oplus \mathfrak{sp}(p-q). \]

Define \( e_i \in \mathfrak{a}^* \) by \( e_i(H(\mathbf{a})) = a_i \ (i = 1, \ldots, q) \). Then \( \Sigma \subset \{ \pm e_i, \pm 2e_i \mid 1 \leq i \leq q \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq q \} \) and the multiplicity of each restricted root is as follows:

| \( g \) | \( \mathfrak{sp}(p, q) \) \( (p \geq q \geq 1) \) |
|---|---|
| \( \mathfrak{m}_{\text{short}} := m_{\pm e_i} \) \( (1 \leq i \leq q) \) | \( q \) |
| \( \mathfrak{m}_{\text{middle}} := m_{\pm e_i \pm e_j} \) \( (1 \leq i < j \leq q) \) | \( 4(p-q) \) |
| \( \mathfrak{m}_{\text{long}} := m_{\pm 2e_i} \) \( (1 \leq i \leq q) \) | \( 4(q \geq 2) \) |
| \( \mathfrak{m}_{\text{short}} := m_{\pm e_i} \) \( (1 \leq i \leq q) \) | \( 3 \) |

Let \( \text{pr}_1 \) and \( \text{pr}_2 \) be the projections of \( K \) to \( \text{Sp}(p) \) and \( \text{Sp}(q) \) respectively.

**Theorem 6.4.** If \( p \geq q \geq 2 \) then \( G = \text{Sp}(p, q) \) has no non-trivial small \( K \)-type. Suppose \( p \geq q = 1 \). Then for the irreducible representation \( (\pi_n, \mathbb{C}^n) \) of \( \text{Sp}(1) \) \( \cong \text{SU}(2) \) of dimension \( n = 1, 2, \ldots \), \( \pi_n \circ \text{pr}_2 \) is a small \( K \)-type of \( G = \text{Sp}(p, 1) \) with \( k_{\text{short}} = 0 \) and \( k_{\text{long}} = -\frac{n^2-1}{3} \). If \( p > q \) then all the small \( K \)-types are constructed in this way. If \( p = q = 1 \) then the other small \( K \)-types are constructed in the same way as above but using \( \text{pr}_1 \) instead of \( \text{pr}_2 \) and \( k_{\text{long}} = k_{\text{long}} \) for any \( n = 1, 2, \ldots \).

**Proof.** Suppose first \( p \geq q \geq 2 \). Then we have a restricted root vector

\[ X_{e_1-e_2} = \begin{pmatrix} 0 & -1 & 0 & -1 & O_{2,q-2} \\ 1 & 0 & O_{p-2,2} & O_{p-2,2} & O_{p-2,2} \end{pmatrix} \in \mathfrak{g}_{e_1-e_2}. \]
Observe that $X_{e_1 - e_2} + \theta X_{e_1 - e_2}$ belongs to neither $\mathfrak{sp}(p)$ nor $\mathfrak{sp}(q)$. Thus there is no proper ideal of $\mathfrak{k}$ that contains $X_{e_1 - e_2} + \theta X_{e_1 - e_2}$. Now, for any small $K$-type $(\pi, V)$ of $G$, $X_{e_1 - e_2} + \theta X_{e_1 - e_2} \in \text{Ker}_\pi \pi$ by Corollary 5.7 and Proposition 3.8. This means $\text{Ker}_\pi \pi = \mathfrak{k}$ and hence $(\pi, V)$ is the trivial $K$-type.

Next suppose $p > q = 1$ Then $\pi_n \circ \text{pr}_2$ is small since $\text{pr}_2(M) = \text{Sp}(1)$. Also, $\kappa_{\text{long}}^{{\pi_n}^{\text{opr}2}} = 0$ by Corollary 5.7 To calculate $\kappa_{\text{long}}^{{\pi_n}^{\text{opr}2}}$ take a root vector 

$$X_{2e_1} = \frac{1}{2} \begin{pmatrix} i & O_{1,p-1} & -i \\ O_{p-1,1} & O_{p-1,p-1} & O_{p-1,1} \\ i & O_{1,p-1} & -i \end{pmatrix} \in \mathfrak{g}_{2e_1},$$

which is normalized as in Lemma 3.6 Under

$$\mathfrak{sp}(1) \ni bi + cj + dk \mapsto \begin{pmatrix} b\sqrt{-1} & c + d\sqrt{-1} \\ -c + d\sqrt{-1} & -b\sqrt{-1} \end{pmatrix} \in \mathfrak{su}(2),$$

does not map to $\kappa_{\text{long}}^{{\pi_n}^{\text{opr}2}}$ maps to $\begin{pmatrix} -\sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}$. Hence $\pi_n \circ \text{pr}_2(X_{2e_1} + \theta X_{2e_1}) \simeq \sqrt{-1} \text{diag}(n-1, n-2, \ldots, n-1)$ and from Lemma 3.6 one has $\kappa_{\text{long}}^{{\pi_n}^{\text{opr}2}} = -\frac{\sqrt{-1}}{3}$. Take any $X_{e_1} \in \mathfrak{g}_{e_1} \setminus \{0\}$. We have $X_{e_1} + \theta X_{e_1} \in \text{Ker}_\pi(\pi \circ \text{pr}_2)$ by Corollary 5.7 and Proposition 3.8. Since $\text{Ker}_\pi(\pi \circ \text{pr}_2) = \text{sp}(p)$ for $n \geq 2$, we see $X_{e_1} + \theta X_{e_1} \in \mathfrak{sp}(p)$ and $\mathfrak{sp}(p)$ is generated by $X_{e_i} + \theta X_{e_i}$ as an ideal of $\mathfrak{k}$. Now, for any small $K$-type $(\pi, V)$ of $G$, $X_{e_1} + \theta X_{e_1} \in \text{Ker}_\pi \pi$ by the same reason as above. This means $\mathfrak{sp}(p) \subseteq \text{Ker}_\pi \pi$ and $\pi$ equals some $\pi_n \circ \text{pr}_2$.

Finally suppose $p = q = 1$. Then $M = \text{SU}(2)$ is diagonally embedded in $K = \text{SU}(2) \times \text{SU}(2)$. Since any $K$-type is given as the exterior tensor product $\pi_m \boxtimes \pi_n$ of two irreducible representations of $\text{SU}(2)$, its restriction to $M$ equals the interior tensor product $\pi_m \otimes \pi_n$. By the representation theory of $\text{SU}(2)$, $\pi_m \otimes \pi_n$ is irreducible if and only if either $\pi_m$ or $\pi_n$ is trivial. Hence $\{\pi_n \circ \text{pr}_i \ | \ i = 1, 2, n = 1, 2, \ldots\}$ is the complete set of small $K$-types. The values of $\kappa_{\pi_n \circ \text{pr}_i}$ are calculated in the same way as in the previous case. □

The result of §2.2 follows from this theorem and what we discussed in §§6.2–6.4. Also, Theorem 6.4 and Proposition 5.9 easily imply the result of §2.3.

6.5. The case $\mathfrak{g} = \mathfrak{so}(p, q)$. Suppose $\mathfrak{g} = \mathfrak{so}(p, q)$ ($p \geq q \geq 1$). (We exclude the cases $\mathfrak{so}(1, 1) \simeq \mathbb{R}$ and $\mathfrak{so}(2, 2) \simeq \mathfrak{sl}(2, \mathbb{R})^{\mathbb{C}}$.) Under the natural inclusion $\mathfrak{so}(p, q) \subset \mathfrak{sp}(p, q)$, $\mathfrak{t}$, $\mathfrak{a}$ and $\mathfrak{m}$ are identified with the intersections of $\mathfrak{so}(p, q)$ and those for $\mathfrak{sp}(p, q)$. In particular, $\mathfrak{t} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$ and

$$\mathfrak{m} = \left\{ \begin{pmatrix} O_{q,q} & O_{q,p-q} & O_{q,q} \\ O_{p-q,q} & Y & O_{p-q,q} \\ O_{q,q} & O_{q,p-q} & O_{q,q} \end{pmatrix} \ | \ Y \in \mathfrak{so}(p-q) \right\} \simeq \mathfrak{so}(p-q).$$

One has $\Sigma \subseteq \{ \pm e_i \ | \ 1 \leq i \leq q \} \cup \{ \pm e_i \pm e_j \ | \ 1 \leq i < j \leq q \}$ and the multiplicity of each restricted root is as follows:
(ii) Suppose a (half-)spin representation $K$ non-trivial denoted by $K$ then we may assume $\text{(Le2, Theorem 1, Lemmas 4.2, 4.3)}$ Theorem 6.5

Taking some finite covering group of $G$ if necessary, we may assume $K = K_1 \times K_2$ with $\mathfrak{t}_1 := \text{Lie } K_1 \simeq \mathfrak{so}(p)$ and $\mathfrak{t}_2 := \text{Lie } K_2 \simeq \mathfrak{so}(q)$. Furthermore, if $\mathfrak{t}_i \simeq \mathfrak{so}(r)$ with $r \geq 3$, then we may assume $K_i \simeq \text{Spin}(r)$ $(i = 1, 2)$. The projections $K \to K_i$ and $\mathfrak{t} \to \mathfrak{t}_i$ are denoted by $\text{pr}_i$ $(i = 1, 2)$.

**Theorem 6.5** (Le2 Theorem 1, Lemmas 4.2, 4.3). (i) Suppose $p = q \geq 3$. Then a non-trivial $K$-type $\pi$ is small if and only if it is equivalent to $\sigma \circ \text{pr}_i$ with $i = 1, 2$ and a (half-)spin representation $\sigma$ of $K_i$. For such $\pi$, $\kappa^\pi_{\text{short}} = 0$ and $\kappa^\pi_{\text{long}} = -\frac{1}{4}$.

(ii) Suppose $p = q + 1$ with odd $q \geq 3$. Then there are three non-trivial small $K$-types: $\pi = \sigma \circ \text{pr}_1$ with either of two half-spin representations $\sigma$ of $K_1 = \text{Spin}(p)$ and $\pi = \sigma \circ \text{pr}_2$ with the spin representation $\sigma$ of $K_2 = \text{Spin}(q)$. One has $\kappa^\pi_{\text{short}} = -1$, $\kappa^\pi_{\text{long}} = -\frac{1}{4}$ in the former case and $\kappa^\pi_{\text{short}} = 0$, $\kappa^\pi_{\text{long}} = -\frac{1}{4}$ in the latter case.

(iii) Suppose $p = q + 1$ with even $q \geq 4$. Then a non-trivial $K$-type $\pi$ is small if and only if it is equivalent to $\sigma \circ \text{pr}_2$ with a half-spin representation $\sigma$ of $K_2 = \text{Spin}(q)$. For such $\pi$, $\kappa^\pi_{\text{short}} = 0$ and $\kappa^\pi_{\text{long}} = -\frac{1}{4}$.

We generalize this result to all cases in the subsequent two theorems.

**Theorem 6.6.** Suppose $p$ is even, $p \geq 4$ and $q = 1$. Then $K = K_1 = \text{Spin}(p)$. Fix a Cartan subalgebra and a system of positive roots of $\mathfrak{t}_C$. For $s = 0, 1, 2, \ldots$, let $\pi_s^\pm$ be the irreducible representation of $K = \text{Spin}(p)$ with highest weight $(s/2, \ldots, s/2, \pm s/2)$ in the standard notation. Then $\pi_s^\pm$ is a small $K$-type with $\kappa^\pi_{\text{short}} = -\frac{s(s+p-2)}{p-1}$. There are no other small $K$-types.

**Remark 6.7.** We already studied $\mathfrak{so}(2r + 1, 1)$ $(r \geq 1)$ in §6.3 and $\mathfrak{so}(4, 1) \simeq \mathfrak{sp}(1, 1)$ in Theorem 6.4. Also, $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{su}(1, 1)$ will be covered in §6.6.

Let $E_{ij}$ denote a matrix whose entry is 1 in the $(i, j)$-position and 0 elsewhere. Let $F_{ij} = E_{ij} - E_{ji}$.

**Proof of Theorem 6.6.** Suppose first $\pi$ is small. Then by Proposition 5.9 there is only one isotypic component in the restriction of $\pi$ to $\mathfrak{m} = \mathfrak{so}(p - 1)$. In view of the branching law for $\mathfrak{so}(p) \downarrow \mathfrak{so}(p - 1)$ (cf. [GW Theorem 8.1.4]), it is possible only when $\pi$ is equivalent to some $\pi_s^\pm$ in the theorem. Conversely, let $\pi = \pi_s^\pm$ with representation space $V$. Then $\pi$ is small since $\pi|_{\mathfrak{m}_0}$ is irreducible by the branching law. To calculate $\kappa^\pi_{\text{short}}$ we take restricted root vectors $X^{(i)} := F_{i,1} - E_{i,p+1} - E_{p+1,i}$ $(2 \leq i \leq p)$ for $e_1 \in \Sigma$. They constitute an orthonormal basis of $\mathfrak{g}_{e_1}$, so that $\kappa^\pi_{\text{short}} \text{id}_V = \frac{1}{p-1} \sum_{i=2}^p \pi(X^{(i)} + \theta X^{(i)})^2$. Now, we assume $H_1, \ldots, H_\frac{p}{2}$ with $H_i := \sqrt{-1}F_{2i,2i-1}$ $(1 \leq i \leq \frac{p}{2})$ constitute a basis of
the Cartan subalgebra of \( \mathfrak{t}_C \) and \( \Delta^+_C := \{ \epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq \frac{p}{2} \} \) is the system of positive roots, where we let \( \{ \epsilon_i \} \) be the dual basis of \( \{ H_i \} \). For \( i = 2, 3, \ldots, \frac{p}{2} \) take vector roots

\[
X_{\epsilon_1 + \epsilon_i} := \frac{1}{2} (F_{2i-1,1} + \sqrt{-1} F_{2i-1,2} + \sqrt{-1} F_{2i,1} - F_{2i,2}) \in (\mathfrak{t}_C)_{\epsilon_1 + \epsilon_i},
\]
\[
X_{\epsilon_1 - \epsilon_i} := \frac{1}{2} (F_{2i-1,1} + \sqrt{-1} F_{2i-1,2} - \sqrt{-1} F_{2i,1} + F_{2i,2}) \in (\mathfrak{t}_C)_{\epsilon_1 - \epsilon_i}.
\]

Then one has for \( i = 2, 3, \ldots, \frac{p}{2} \)

\[
\begin{align*}
[X_{\epsilon_1 + \epsilon_i}, X_{\epsilon_1 - \epsilon_i}] &= [X_{\epsilon_1 + \epsilon_i}, X_{\epsilon_1 - \epsilon_i}] = [X_{\epsilon_1 + \epsilon_i}, X_{\epsilon_1 - \epsilon_i}] = 0, \\
[X_{\epsilon_1 + \epsilon_i}, X_{\epsilon_1 - \epsilon_i}] &= -(H_1 + H_i), \\
X^{(2i-1)} + \theta X^{(2i-1)} &= X_{\epsilon_1 + \epsilon_i} + X_{\epsilon_1 - \epsilon_i} + X_{\epsilon_1 + \epsilon_i} + X_{\epsilon_1 - \epsilon_i}, \\
X^{(2i)} + \theta X^{(2i)} &= -\sqrt{-1} (X_{\epsilon_1 + \epsilon_i} - X_{\epsilon_1 - \epsilon_i} - X_{\epsilon_1 + \epsilon_i} + X_{\epsilon_1 - \epsilon_i}).
\end{align*}
\]

From these we calculate in \( U(\mathfrak{t}_C) \)

\[
\sum_{i=2}^{p} (X^{(i)} + \theta X^{(i)})^2 = (-2\sqrt{-1}H_1)^2 + \sum_{i=2}^{\frac{p}{2}} \left( (X^{(2i-1)} + \theta X^{(2i-1)})^2 + (X^{(2i)} + \theta X^{(2i)})^2 \right)
\]
\[
\equiv -4H_1^2 - 2(p-2)H_1 \mod U \left( \sum_{\alpha \in \Delta^+_C} (\mathfrak{t}_C)_\alpha + U(\mathfrak{t}_C) \sum_{\alpha \in \Delta^+_C} (\mathfrak{t}_C)_\alpha. \right.
\]

Applying this to the highest weight vector of \( V \), we obtain \( (p-1)\kappa^\pi_{\text{short}} = -s(s+p-2) \). \( \square \)

**Theorem 6.8.** (i) Suppose \( p > q = 2 \). Then a \( K \)-type \( \pi \) is small if and only if it is equivalent to \( \tau \circ \text{pr}_2 \) with a one-dimensional representation \( \tau \) of \( K_2 \). For such \( \pi \), \( \kappa^\pi_{\text{short}} = 0 \).

(ii) Suppose \( p \) is even and \( q \) is odd with \( p > q \geq 3 \). Then one has the same results as in the case of Theorem 6.5(ii).

(iii) Suppose \( p > q \geq 3 \) and either \( q \) or \( p-q \) is even. Then a non-trivial \( K \)-type \( \pi \) is small if and only if it is equivalent to \( \sigma \circ \text{pr}_2 \) with a (half-)spin representation \( \sigma \) of \( K_2 = \text{Spin}(q) \). For such \( \pi \), \( \kappa^\pi_{\text{short}} = 0 \) and \( \kappa^\pi_{\text{long}} = -\frac{1}{4} \).

**Proof.** Choosing a Cartan subalgebra \( \mathfrak{b} \subset \mathfrak{m} \) suitably, we may assume

\[
\mathfrak{g}_{\text{split}} = \left\{ \begin{pmatrix} A & O_{q,p-q} & B \\ O_{p-q,q} & O_{p-q,p-q} & O_{p-q,q} \\ iB & O_{p-q,q} & C \end{pmatrix} \in \mathfrak{gl}(p+q, \mathbb{R}) \mid A, C \in \mathfrak{so}(q), \quad B \in \text{Mat}(q, q, \mathbb{R}) \right\} \cong \mathfrak{so}(q, q)
\]
if \( p - q \) is even, and
\[
\mathfrak{g}_{\text{split}} = \begin{cases}
A & O_{p+1,p-q-1} \\
O_{p-q-1,q+1} & O_{p-q-1,p-q-1} \\
B & O_{p-q-1,q} \\
m & C 
\end{cases}
\in \mathfrak{gl}(p + q, \mathbb{R})
\] (1)
\[
\begin{cases}
A \in \mathfrak{so}(q + 1), \\
B \in \mathfrak{Mat}(q + 1, q, \mathbb{R}) \\
C \in \mathfrak{so}(q)
\end{cases}
\]
\[
\simeq \mathfrak{so}(q + 1, q)
\]

if \( p - q \) is odd.

Suppose \( p \geq 3 \). Let \( (\tau, V) \) be a non-trivial irreducible representation of \( K_2 \) and consider
the \( K \)-type \( (\pi, V) := (\tau \circ \mathrm{pr}_2, V) \). If \( q = 2 \) then this is always small since \( V \) is one-dimensional. For \( q \geq 3 \), we assert that \((\pi, V)\) is small if and only if \( \tau \) is a (half)-spin representation \( \sigma \) of \( K_2 \) is \( \mathrm{Spin}(q) \) and that \( \kappa_{\text{long}}^\pi = -\frac{1}{2} \). Indeed, if \( \tau = \sigma \), then \((\sigma|_{\mathfrak{pr}_2}|_{K_{\text{split}}}, V)\) is a small \( K \)-type by Theorem 6.5 and \((\sigma|_{\mathfrak{pr}_2}|_{M_{K\text{split}}}, V)\) is irreducible.

Suppose \( (\pi, V) \) is odd. Conversely, if \( (\pi, V) \) is odd, then \( V_{\mu} \) in Proposition 5.6 equals \( V \) since \( \mathfrak{m} \subset \mathfrak{k} \) acts on \( V \) trivially. Hence \((\tau \circ \mathfrak{pr}_2|_{K_{\text{split}}}, V)\) is a small \( K_{\text{split}} \)-type and is non-trivial since \( \mathfrak{pr}_2(K_{\text{split}}) = \mathfrak{pr}_2(K) \). It then follows from Theorem 6.5 that \( \tau \) is a (half)-spin representation. We also have \( \kappa_{\text{long}}^\pi = -\frac{1}{2} \) by Corollary 5.8.

Next, note \( p > q \) in all cases of the theorem and we have a restricted root vector
\[
X_{e_1} := F_{q+1,1} - E_{q+1,p+1} - E_{p+1,q+1} \in \mathfrak{g}_{e_1},
\]
for which \( X_{e_1} + \theta X_{e_1} \in \mathfrak{t}_1 \setminus \{0\} \). It is easy to check there is no proper ideal of \( \mathfrak{t}_1 = \mathfrak{so}(p) \) that contains \( X_{e_1} + \theta X_{e_1} \). (Note \( \mathfrak{so}(p) \) is simple for \( p = 3, 5, 6, \ldots \).) Hence by Proposition 3.8 a small \( K \)-type \( (\pi, V) \) is written as \((\pi, V) = (\tau \circ \mathfrak{pr}_2, V)\) for some irreducible representation \( \tau \) of \( K_2 \) if and only if \( \kappa_{\text{short}}^\pi = \kappa_{\text{long}}^\pi = 0 \). It follows from Corollary 5.7 that if \( p - q \) is even then all small \( K \)-type are of this type. We claim the same thing holds if \( q \) is even. To show this, we may assume \( p \) is odd. If \( (p, q) = (3, 2) \), then \( \mathfrak{so}(3, 2) \simeq \mathfrak{sp}(2, \mathbb{R}) \) and the claim in this case will be shown in the first paragraph of the proof of Theorem 6.11.

Suppose \((p, q)\) is a general combination of an odd \( p \) and an even \( q \) \((p > q \geq 2)\) and let \((\pi, V)\) be any small \( K \)-type. Let \( V_{\mu} \) be as in Proposition 5.6. Since \((\pi|_{K_{\text{split}}}, V_{\mu})\) is a small \( K_{\text{split}} \)-type, it follows from Theorem 6.5 that the claim for \((p, q) = (3, 2)\) that \( \mathfrak{t}_1 \cap \mathfrak{t}_{\text{split}} = \mathfrak{so}(q + 1) \) acts trivially on \( V_{\mu} \). Note that \( F_{2,1} \) is in \( \mathfrak{t}_1 \cap \mathfrak{t}_{\text{split}} \) commutes with \( \mathfrak{m}_\mathbb{C} \) and that \( V = \pi(U(\mathfrak{m}_\mathbb{C}))V_{\mu} \) by Proposition 5.6. Thus \( F_{2,1} \) acts trivially on \( V \). By the simplicity of \( \mathfrak{t}_1 = \mathfrak{so}(p) \), we have \( \mathfrak{t}_1 \subset \mathrm{ker}_\pi \), which proves our claim. Note that \((\pi, V)\) is trivial. We assert that \( \pi|_{K_2} \) is trivial. In fact, if \( \mathfrak{t}_1 \cap \mathfrak{t}_{\text{split}} = \mathfrak{so}(q + 1) \) acts trivially on \( V_{\mu} \) in Proposition 5.6, then the same argument as in the last paragraph implies \( \pi|_{K_2} \) is trivial, a contradiction. Thus the action of \( \mathfrak{t}_1 \cap \mathfrak{t}_{\text{split}} = \mathfrak{so}(q + 1) \) on \( V_{\mu} \) is non-trivial and hence...
that of \( \mathfrak{t}_2 \cap \mathfrak{s}_{\text{split}} = \mathfrak{t}_2 = \mathfrak{so}(q) \) is trivial by Theorem 6.5(ii). (We also see \( \kappa_{\text{long}} = -\frac{1}{4} \) by Corollary 5.8.) Since \( V = \pi(U(\mathfrak{m}_C))V_\mu \) and since \( \mathfrak{t}_2 \) acts trivially on \( V \), proving the assertion. Therefore there exists a non-trivial irreducible representation \( \tau \) of \( \mathfrak{t}_1 = \mathfrak{so}(p) \) such that \( \pi = \tau \circ \text{pr}_1 \). Now, by Proposition 5.6 there is only one isotypic component in the restriction of \( (\tau, V) \) to \( \mathfrak{m} = \mathfrak{so}(p-q) \). In view of the branching laws for the orthogonal Lie algebras (cf. GW Theorems 8.1.3, 8.1.4), it is possible only when \( \tau \) is a half-spin representation. Conversely, let \( (\sigma, V) \) be any of two half-spin representations of \( K_1 = \text{Spin}(p) \). The proof is complete if we can show \( \sigma = \sigma \circ \text{pr}_1 \) is small and \( \kappa_{\text{short}} = -1 \). Let \( \varpi : \text{Spin}(p) \to \text{SO}(p) \) be the canonical projection. Then

\[
\text{pr}_1(M) = \varpi^{-1}\left( \left\{ \left( \begin{array}{cc} \text{diag}(m_1, \ldots, m_q) & O_{q,p-q} \\ O_{p,q} & g \end{array} \right) \right| m_i = \pm 1, \prod_{i=1}^q m_i = 1, g \in \text{SO}(p-q) \right\} \right)
\]

\[
\supset \varpi^{-1}\left( \left\{ \text{diag}(m_1, \ldots, m_p) \right| m_i = \pm 1, \prod_{i=1}^q m_i = \prod_{i=q+1}^p m_i = 1 \right\},
\]

\[
\text{pr}_1(M) \cup \text{pr}_1(M) \cdot \varpi^{-1}(\text{diag}(-1, \ldots, -1))
\]

\[
\supset \varpi^{-1}\left( \left\{ \text{diag}(m_1, \ldots, m_p) \right| m_i = \pm 1, \prod_{i=1}^p m_i = 1 \right\}.
\]

Now \( V \) is irreducible under the action of \( \varpi^{-1}\left( \left\{ \text{diag}(m_1, \ldots, m_p) \right| m_i = \pm 1, \prod_{i=1}^p m_i = 1 \right\} \). (In fact, \( (\sigma, V) \) is a small ‘\( K \)-type’ of the double cover of \( \text{SL}(p, \mathbb{R}) \) by Theorem 2.1.)

But since \( \varpi^{-1}(\text{diag}(-1, \ldots, -1)) \) is contained in the center of \( \text{Spin}(p) \), \( V \) is irreducible as a \( \text{pr}_1(M) \)-module. This proves the smallness of \( \pi = \sigma \circ \text{pr}_1 \). Finally, we can directly check \( X_{e_i} \) in [6.2] is normalized as in Lemma 3.6 and the eigenvalues of \( \pi(X_{e_1} + \theta X_{e_j}) \) are \( \pm \sqrt{-1} \). Thus \( \kappa_{\text{short}} = -1 \).

\[\square\]

All the results stated in 2.5 follow from Theorem 6.8(ii), (iii) and Proposition 5.9.

6.6. The Hermitian type. Let \( G \) be a non-compact real simple Lie group of Hermitian type. There exists a central element \( Z \in \mathfrak{t} \) such that \( J = \text{ad}(Z) \) is a complex structure of \( \mathfrak{s} = \mathfrak{g}^{-\theta} \). Let \( 2e_1, \ldots, 2e_l \) be the longest roots in \( \Sigma^+ \). Then \( \Sigma \subset \{ \pm e_i, \pm 2e_i \mid 1 \leq i \leq l \} \cup \{ \pm e_i \pm e_j \mid 1 \leq i < j \leq l \} \). Put \( \mathfrak{m}_{\text{long}} = \mathfrak{m}_{\pm 2e_i} (1 \leq i \leq l) \), \( \mathfrak{m}_{\text{middle}} = \mathfrak{m}_{\pm e_i \pm e_j} (1 \leq i < j \leq l) \) and \( \mathfrak{m}_{\text{short}} = \mathfrak{m}_{\pm e_i} (1 \leq i \leq l) \). Their values are listed below:

\[
\begin{array}{cccccccc}
\mathfrak{g} & \mathfrak{su}(p, p) & \mathfrak{sp}(p, \mathbb{R}) & \mathfrak{so}^*(4p) & (p \geq 2) & \mathfrak{so}(p, 2) & (p \geq 3) & \epsilon_{7(-25)} & \text{(E VII)} \\
\text{real rank } l & p & p & p & 2 & 3 \\
\mathfrak{m}_{\text{short}} & 0 & 0 & 0 & 0 & 0 \\
\mathfrak{m}_{\text{middle}} & 2 (p \geq 2) & 1 (p \geq 2) & 4 & p - 2 & 8 \\
\mathfrak{m}_{\text{long}} & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\((\mathfrak{su}(1, 1) \simeq \mathfrak{sp}(1, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R}), \mathfrak{su}(2, 2) \simeq \mathfrak{so}(4, 2), \mathfrak{sp}(2, \mathbb{R}) \simeq \mathfrak{so}(3, 2), \mathfrak{so}^*(8) \simeq \mathfrak{so}(6, 2))\)
Take $X_{2e_i} \in \mathfrak{g}_{2e_i}$ so that $-\frac{1}{2}|2e_i|^2 B(X_{2e_i}, \theta X_{2e_i}) = 1$ (1 ≤ $i$ ≤ $l$).

**Lemma 6.9.** By replacing $X_{2e_i}$ with $-X_{2e_i}$ if necessary, we have

\begin{equation}
Z = \frac{1}{2} \sum_{i=1}^{l} (X_{2e_i} + \theta X_{2e_i}) + Y
\end{equation}

for some $Y \in \mathfrak{b}$. Here $\mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{m}$.

**Proof.** It is well known that $\mathfrak{t} := \sum_{i=1}^{l} \mathbb{R}(X_{2e_i} + \theta X_{2e_i}) + \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{t}$. Since $Z \in \mathfrak{t}$, there exist constants $c_1, \ldots, c_l \in \mathbb{R}$ and $Y \in \mathfrak{b}$ such that

\begin{equation}
Z = c_1(X_{2e_1} + \theta X_{2e_1}) + \cdots + c_l(X_{2e_l} + \theta X_{2e_l}) + Y.
\end{equation}

Since

\begin{align*}
[X_{2e_i}, X_{2e_j}] &= [X_{2e_i}, \theta X_{2e_j}] = [H_{2e_i}, X_{2e_j}] = 0 \quad (i \neq j),

[X_{2e_i}, \theta X_{2e_j}] &= -\frac{2}{||2e_i||^2} H_{2e_i}, \quad [H_{2e_i}, X_{2e_j}] = ||2e_i||^2 X_{2e_i},
\end{align*}

we have for $i = 1, \ldots, l$

\begin{equation}
- H_{2e_i} = J^2 H_{2e_i} = \text{ad}(Z)^2 H_{2e_i} = -4c_i^2 H_{2e_i} - c_i ||2e_i||^2 ([Y, X_{2e_i}] - [Y, \theta X_{2e_i}])
\end{equation}

and hence $[Y, X_{2e_i}] = [Y, \theta X_{2e_i}] = 0$ and $c_i = \pm \frac{1}{2}$. \hfill \square

**Corollary 6.10.** One has $\mathfrak{g}_{\pm 2e_i} \subset \mathfrak{g}^M$ for $i = 1, \ldots, l$.

**Proof.** This is clear from (6.3) since $\text{Ad}(m)Z = Z$ for any $m \in M$. \hfill \square

**Theorem 6.11.** Suppose $(\pi, V)$ is a small $K$-type. Then $V$ is one-dimensional and $\kappa_{\text{short}} = \kappa_{\text{middle}} = 0$.

**Proof.** First we assume $\mathfrak{g} = \mathfrak{sp}(p, \mathbb{R})$. Then $\mathfrak{m} = \{0\}$ and $\mathfrak{t} = \sum_{i=1}^{p} \mathbb{R}(X_{2e_i} + \theta X_{2e_i})$ is a Cartan subalgebra of $\mathfrak{t}$. Since $\mathfrak{t} \subset \mathfrak{t}^M$ by Corollary 6.10, $M$ is a finite subgroup of the Cartan subgroup corresponding to $\mathfrak{t}$ and in particular is Abelian. Thus any small $K$-type $\pi$ is one-dimensional. We claim $\kappa_{\text{middle}} = 0$ for such $\pi$. Indeed, $[\mathfrak{t}, \mathfrak{t}]$ equals the orthogonal complement $(\mathbb{R}Z)_{\perp}$ of $\mathbb{R}Z$ in $\mathfrak{t}$ with respect to $B(\cdot, \cdot)$. Since

\begin{equation}
\sum_{\alpha: \text{middle}} \{X_{\alpha} + \theta X_{\alpha} \mid X_{\alpha} \in \mathfrak{g}_{\alpha}\} \perp \sum_{i=1}^{p} \mathbb{R}(X_{2e_i} + \theta X_{2e_i}),
\end{equation}

we have $\kappa_{\text{middle}} = 0$ for such $\pi$.

Next, we assume $\mathfrak{g} = \mathfrak{su}(p, q)$ with $p > q \geq 1$. Then $\mathfrak{g}^* = \mathfrak{so}^*(4p + 2)$ with $p \geq 1$. Then $\mathfrak{g}_{\text{short}} = \mathfrak{g}_{\text{middle}} = \mathfrak{g}_{\text{long}} = \mathfrak{g}_{\text{middle}} = \mathfrak{g}_{\text{middle}}$. Then $\mathfrak{g} = \mathfrak{sp}(p, \mathbb{R})$. Then $\mathfrak{m} = \{0\}$ and $\mathfrak{t} = \sum_{i=1}^{p} \mathbb{R}(X_{2e_i} + \theta X_{2e_i})$ is a Cartan subalgebra of $\mathfrak{t}$. Since $\mathfrak{t} \subset \mathfrak{t}^M$ by Corollary 6.10, $M$ is a finite subgroup of the Cartan subgroup corresponding to $\mathfrak{t}$ and in particular is Abelian. Thus any small $K$-type $\pi$ is one-dimensional. We claim $\kappa_{\text{middle}} = 0$ for such $\pi$. Indeed, $[\mathfrak{t}, \mathfrak{t}]$ equals the orthogonal complement $(\mathbb{R}Z)_{\perp}$ of $\mathbb{R}Z$ in $\mathfrak{t}$ with respect to $B(\cdot, \cdot)$. Since

\begin{equation}
\sum_{\alpha: \text{middle}} \{X_{\alpha} + \theta X_{\alpha} \mid X_{\alpha} \in \mathfrak{g}_{\alpha}\} \perp \sum_{i=1}^{p} \mathbb{R}(X_{2e_i} + \theta X_{2e_i}),
\end{equation}

we have $\kappa_{\text{middle}} = 0$ for such $\pi$.\hfill \square
one sees by Lemma \ref{lem:6.9} that \( \{X_\alpha + \theta X_\alpha \mid X_\alpha \in \mathfrak{g}_\alpha\} \subset [\mathfrak{k}, \mathfrak{k}] \) for any middle \( \alpha \). Since \([\mathfrak{k}, \mathfrak{k}] \subset \text{Ker} \pi\), our claim follows from Proposition \ref{prop:3.8}.

Next, suppose \( \mathfrak{g} \) is a general simple Lie algebra of Hermitian type and \((\pi, V)\) is a small \(K\)-type. We claim \( \kappa^\pi_{\text{short}} = \kappa^\pi_{\text{middle}} = 0 \). This was shown in the previous paragraph for \( \mathfrak{g} = \mathfrak{sp}(p, \mathbb{R}) \) and in Theorem \ref{thm:6.8}(1) for \( \mathfrak{g} = \mathfrak{so}(p, 2) \). For the remaining cases, the claim follows from Corollary \ref{cor:5.7}. Now, by Proposition \ref{prop:3.8}, \( \pi(X_\alpha + \theta X_\alpha) = 0 \) for any restricted root vector \( X_\alpha \) of any \( \alpha \in \Sigma \) with short or middle length. On the other hand, by Corollary \ref{cor:6.10}, each \( \pi(X_{2e_i} + \theta X_{2e_i}) \) is a scalar operator. Hence by Corollary \ref{cor:6.2}, \( \pi(X) \) for any \( X \in \mathfrak{k} \) is a scalar operator. This proves \( V \) is one-dimensional. \( \square \)

Let \( \mathfrak{j} \) and \( \pi_0 \in \sqrt{-1} \mathfrak{j}^* \) be as in \[\ref{eq:2.6}\]. If we identify \( \pi_0 \) with one-dimensional representation of \( \mathfrak{k} \), then it follows from [Hec2, Proposition 5.3.2] that

\[
\pi_0(X_{2e_i} + \theta X_{2e_i}) \in \{\pm \sqrt{-1}\} \quad \text{for } i = 1, \ldots, l.
\]

Hence if (the differentiation of) a small \( K\)-type \( \pi \) is written as \( \pi = \nu \pi_0 \) for some \( \nu \in \mathbb{Q} \), then \( \kappa^\pi_{\text{long}} = -\nu^2 \). This and Proposition \ref{prop:5.9} imply the result stated in \[\ref{eq:2.6}\].

6.7. The case \( \Sigma \) is of type \( F_4 \). Let \( G \) be a simply-connected real simple Lie group with \( \Sigma \) of type \( F_4 \). As in \[\ref{sec:2.7}\] we exclude the complex simple Lie group of type \( F_4 \). Thus \( \mathfrak{g} \) is one of \( \mathfrak{f}_4(4), \mathfrak{e}_6(2), \mathfrak{e}_7(-5) \) and \( \mathfrak{e}_8(-24) \). Among these \( \mathfrak{f}_4(4) \) is of split type. Let \( \Sigma_{\text{short}} \) and \( \Sigma_{\text{long}} \) be as in \[\ref{sec:2.7}\]. From [B] one sees there exists a sequence of embeddings

\[
\mathfrak{f}_4(4) \subset \mathfrak{e}_6(2) \subset \mathfrak{e}_7(-5) \subset \mathfrak{e}_8(-24).
\]

The following table summarizes some necessary data on these Lie algebras:

| \( \mathfrak{g} \) | \( \mathfrak{f}_4(4) \) (FI) | \( \mathfrak{e}_6(2) \) (E II) | \( \mathfrak{e}_7(-5) \) (E VI) | \( \mathfrak{e}_8(-24) \) (EIX) |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| \( \mathfrak{m}_{\text{short}} \) | \( 1 \) | \( 2 \) | \( 4 \) | \( 8 \) |
| \( \mathfrak{m}_{\text{long}} \) | \( 1 \) | \( 1 \) | \( 1 \) | \( 1 \) |
| \( \mathfrak{k} \) | \( \mathfrak{sp}(3) \oplus \mathfrak{su}(2) \) | \( \mathfrak{su}(6) \oplus \mathfrak{su}(2) \) | \( \mathfrak{so}(12) \oplus \mathfrak{su}(2) \) | \( \mathfrak{e}_7 \oplus \mathfrak{su}(2) \) |

Thus in any case \( K \) is the product of two simple compact groups. Let \( K = K_1 \times K_2 \) with \( K_2 = \text{SU}(2) \). Let \( \text{pr}_i : K \to K_i \) be the projection \( (i = 1, 2) \).

**Theorem 6.12** ([Le2 Theorem 1, Lemmas 4.2, 4.3]). Suppose \( \mathfrak{g} = \mathfrak{f}_4(4) \). Let \( (\sigma, \mathbb{C}^2) \) be the irreducible representation of \( \text{SU}(2) \) of dimension 2. Then \( \pi = \sigma \circ \text{pr}_2 \) is the only non-trivial small \( K\)-type. Moreover, \( \kappa^\pi_{\text{short}} = 0 \) and \( \kappa^\pi_{\text{long}} = -\frac{1}{4} \).

We generalize this to

**Theorem 6.13.** The last theorem also holds for \( \mathfrak{g} = \mathfrak{e}_6(2), \mathfrak{e}_7(-5) \) and \( \mathfrak{e}_8(-24) \).

**Proof.** Suppose \( \mathfrak{g} = \mathfrak{f}_4(4) \) and let \( \pi = \sigma \circ \text{pr}_2 \) be as in Theorem \ref{thm:6.12}. Since \( \kappa^\pi_{\text{short}} = 0 \), it follows from Proposition \ref{prop:3.8} that \( \sum_{\alpha \in \Sigma_{\text{short}}} \{X_\alpha + \theta X_\alpha \mid X_\alpha \in \mathfrak{g}_\alpha\} \subset \text{Ker} \pi = \mathfrak{k}_1 \).
Since \( \sum_{\alpha \in \Sigma_{\text{short}}} \{ X_\alpha + \theta X_\alpha \mid X_\alpha \in g_\alpha \} \) is the orthogonal complement of \( \sum_{\alpha \in \Sigma_{\text{long}}} \{ X_\alpha + \theta X_\alpha \mid X_\alpha \in g_\alpha \} \) in \( \mathfrak{t} \) with respect to \( B(\cdot, \cdot) \) and \( \mathfrak{t}_1 \) is that of \( \mathfrak{t}_2 \), one has
\[
\mathfrak{t}_2 \subset \sum_{\alpha \in \Sigma_{\text{long}}} \{ X_\alpha + \theta X_\alpha \mid X_\alpha \in g_\alpha \}.
\]

Now let \( g' \) be one of \( e_6(2), e_7(-5) \) and \( e_8(-24) \). We use with the obvious meanings the notation \( G', K', \mathfrak{t}', g'_\alpha \), and so on. Fix an embedding \( G \to G' \) so that \( \mathfrak{t}' \cap g = \mathfrak{t} \) and \( \alpha' = \alpha \). We claim \( \mathfrak{t}_2 \simeq \mathfrak{su}(2) \) is an ideal of \( \mathfrak{t}' \). Indeed, since \( g_\alpha (= g'_\alpha) \) commutes with \( m' \) for each \( \alpha \in \Sigma_{\text{long}} \),
\[
(6.5) \quad [m', \mathfrak{t}_2] \subset \left[ m', \sum_{\alpha \in \Sigma_{\text{long}}} \{ X_\alpha + \theta X_\alpha \mid X_\alpha \in g_\alpha \} \right] = \{0\}.
\]
This proves our claim since \( m' \) and \( \mathfrak{t} \) generate the Lie algebra \( \mathfrak{t}' \) by Theorem 3.7. Thus \( \mathfrak{t}_2 = \mathfrak{t}_2' \simeq \mathfrak{su}(2) \) and \( K_2 = K_2' \). Since \( \mathfrak{t} = (\mathfrak{t}' \cap \mathfrak{t}) \oplus \mathfrak{t}_2 \) is a decomposition of \( \mathfrak{t} \) into two ideals, we have \( \mathfrak{t}_1 = \mathfrak{t}' \cap \mathfrak{t} \) and \( K_1 \subset K_1' \). Hence \( \pi \) extends to a \( K' \)-type \( \pi' = \sigma \circ pr_2 \). This is small since \( M \subset M' \). Since \( g_\alpha \subset g'_\alpha \) for any \( \alpha \in \Sigma \), we have \( \kappa^\pi = \kappa^\pi \) by Lemma 3.6.

Conversely, let \( \nu \) be any non-trivial small \( K' \)-type. Then it follows from Corollary 5.7 and Proposition 3.8 that \( \sum_{\alpha \in \Sigma_{\text{short}}} \{ X_\alpha + \theta X_\alpha \mid X_\alpha \in g_\alpha \} \subset \text{Ker}_{\mathfrak{t}_1} \nu \cap \mathfrak{t}'_1 \). By the simplicity of \( \mathfrak{t}'_1 \), \( \mathfrak{t}'_1 \subset \text{Ker}_{\mathfrak{t}_1} \nu \) and there exists a non-trivial irreducible representation \( \tau \) of \( K_2' = SU(2) \) such that \( \nu = \tau \circ pr_2 \). Now we claim \( pr_2'(M') = pr_2(M) \). Indeed, since
\[
g'_\text{split} = a + \sum_{\alpha \in \Sigma_{\text{long}}} g_\alpha \subset g,
\]
we have \( G'_\text{split} \subset G \) and \( M'_\text{split} \subset M \). On the other hand, since (6.5) implies \( m' \subset \mathfrak{t}'_1 \), one has \( M'_0 \subset K'_1 \). Hence \( pr_2'(M') = pr_2'(M'_0 M'_\text{split}) = pr_2'(M'_\text{split}) \subset pr_2(M) = pr_2(M) \). Since the opposite inclusion is obvious, we get the claim. Thus \( \nu |_{K} = \tau \circ pr_2 \) is a small \( K \)-type. This is non-trivial since \( \dim \tau > 1 \). By Theorem 6.12 we conclude \( \tau = \sigma \).

These two theorems and Proposition 5.9 imply the result of §2.7.

6.8. Split Lie groups with simply-laced \( \Sigma \). Let \( G \) be one of the simply-connected split real simple Lie groups of type \( A_l \) (\( l \geq 2 \)), \( D_l \) (\( l \geq 3 \)) and \( E_l \) (\( l = 6, 7, 8 \)). Let \( (\pi, V) \) be a small \( K \)-types listed in Theorem 2.1. Then it follows from \[Le2\] Lemma 4.2] that \( \kappa_\pi^{\alpha} = -\frac{1}{4} \) for any \( \alpha \in \Sigma \). (For the type \( D_l \) case, we already know this by Theorem 6.5 \[Le2\] Lemma 4.2, 4.3]) Thus Proposition 5.9 implies the result of §2.8.

6.9. The split Lie group of type \( G_2 \). Let \( G = \hat{G}_2 \), the simply-connected split real simple Lie group of type \( G_2 \). We use the same notation as in §2.9. Let \( \Sigma_{\text{short}} \cup \Sigma_{\text{long}} \) be the division of \( \Sigma \) according to the root lengths. From \[Le2\] Lemmas 4.2, 4.3] the values of \( \kappa_\pi \) and \( \kappa^{\pi_2} \) are as follows:
\[
\begin{array}{c|cc}
\pi & \pi_1 & \pi_2 \\
\hline
\kappa_{\text{short}} & -\frac{1}{4} & -\frac{1}{4} \\
\kappa_{\text{long}} & -\frac{1}{4} & -\frac{1}{4}
\end{array}
\]
Hence if $\pi = \pi_1$ then we have the same result as for the split simply-laced case.

Suppose $\pi = \pi_2$ and let us prove we cannot find any combination of $\Sigma^\pi$ and $k^\pi$ for which (1.6) holds. To do so, assume (1.6) holds for some $(\Sigma^\pi, k^\pi)$. Then $\tilde{\delta}_{G/K}^{\pi/2} \tilde{\delta}(\Sigma^\pi, k^\pi)^{1/2}$ is non-singular at 0 and hence for each $\alpha \in \Sigma$ there exists $\beta \in \Sigma^\pi$ which is proportional to $\alpha$. This implies $\Sigma^\pi$ is of type $G_2$. Now $\tilde{\delta}_{G/K}^{\pi/2} \tilde{\delta}(\Sigma^\pi, k^\pi)^{1/2} F(\Sigma^\pi, k^\pi, \lambda) \in \mathcal{A}(\mathfrak{a}_{\text{reg}})$ is an eigenfunction of both (5.1) and (4.2) with $(\Sigma', k) = (\Sigma^\pi, k^\pi)$. Hence

$$\sum_{\alpha \in \Sigma_{\text{short}} \cap \Sigma^+} \frac{||\alpha||^2}{16} \left( \frac{4}{\sinh^2 \frac{\alpha}{2}} - \frac{32}{\sinh^2 \alpha} \right) + \sum_{\alpha \in \Sigma_{\text{long}} \cap \Sigma^+} \frac{||\alpha||^2}{16 \sinh^2 \frac{\alpha}{2}} \left( \frac{k^\pi_\alpha(1 - k^\pi_\alpha - 2k^\pi_{2\alpha})}{4 \sinh^2 \frac{\alpha}{2}} \right) + C$$

for some constant $C$. Since the members of $\{\sinh^{-2} \frac{\alpha}{2} | \alpha \in \Sigma \cup 2\Sigma_{\text{short}} \cup \Sigma^\pi \} \cup \{1\}$ are linearly independent in $\mathcal{A}(\mathfrak{a}_{\text{reg}})$, we have $k^\pi_\alpha(1 - k^\pi_\alpha - 2k^\pi_{2\alpha}) \neq 0$ for each $\alpha \in \Sigma \cup 2\Sigma_{\text{short}}$. Hence $\Sigma^\pi \subset \Sigma \cup 2\Sigma_{\text{short}}$, a contradiction. The results in §2.9 are thus proved.

7. Spherical Transforms

Let $G$ be a non-compact real simple Lie group with finite center and $(\pi, V)$ a small $K$-type. In this section we apply our main formula (1.6) to the calculation of Harish-Chandra’s $c$-function for $G \times_K V$ and the theory of $\pi$-spherical transform. We note each combination of $\Sigma^\pi$ and $k^\pi$ in §2 is chosen so that

$$\Sigma^\pi \subset \Sigma \cup 2\Sigma.$$  

7.1. Weight functions. The weight functions $\tilde{\delta}_{G/K}$ and $\tilde{\delta}(\Sigma^\pi, k^\pi)$ defined by (1.4) and (1.5) are normalized so that $\tilde{\delta}_{G/K}^{\pi/2} \tilde{\delta}(\Sigma^\pi, k^\pi)^{1/2}$ in (1.6) takes 1 at 0 $\in \mathfrak{a}$. In the literature,

$$\delta_{G/K} = \prod_{\alpha \in \Sigma^+} |2 \sinh \alpha|^{m_\alpha} \quad \text{and} \quad \delta(\Sigma^\pi, k^\pi) = \prod_{\alpha \in \Sigma^\pi \cap \Sigma^+} |2 \sinh \alpha|^{2k^\pi_\alpha}$$

are often used.

Lemma 7.1. Suppose (1.6) and (7.1) are valid for $\Sigma^\pi$ and $k^\pi$. Then

$$\tilde{\delta}_{G/K}^{\pi/2} \tilde{\delta}(\Sigma^\pi, k^\pi)^{1/2} = 2^{e(\Sigma^\pi, k^\pi)} \delta_{G/K}^{\pi/2} \delta(\Sigma^\pi, k^\pi)^{1/2}$$

with

$$e(\Sigma^\pi, k^\pi) = \sum_{\alpha \in \Sigma^\pi \setminus 2\Sigma^+} \left( k^\pi_\alpha - k^\pi_{2\alpha} + \frac{m_{2\alpha}}{2} \right).$$

Proof. By a calculation similar to the one for (5.5) we have

$$\tilde{\delta}_{G/K}^{\pi/2} \tilde{\delta}(\Sigma^\pi, k^\pi)^{1/2} = \prod_{\alpha \in \Sigma^\pi \setminus 2\Sigma^+} \left( 2 \cosh \frac{\alpha}{2} \right)^{k^\pi_\alpha - \frac{m_{2\alpha}}{2}}.$$

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The lemma then follows from this and (5.5).

7.2. Harish-Chandra’s c-function. Let \( a_+ := \{ H \in a \mid \alpha(H) > 0 \text{ for any } \alpha \in \Sigma^+ \} \) and \( a_+^* = \{ \lambda \in a^* \mid \lambda(\alpha^\vee) > 0 \text{ for any } \alpha \in \Sigma^+ \} \). For \( \lambda \in a_+^* + \sqrt{-1}a^* \) put

\[
c^\pi(\lambda) = \int_{\tilde{N}} e^{-(\lambda + \rho)(H(\tilde{n}))} \pi(\kappa(\tilde{n})) d\tilde{n}
\]

where the Haar measure \( d\tilde{n} \) on \( \tilde{N} := \theta N \) is normalized so that

\[
\int_{\tilde{N}} e^{-2\rho(H(\tilde{n}))} d\tilde{n} = 1.
\]

The integral in (7.4) absolutely converges and defines an \( \text{End}_M V \)-valued holomorphic function known as Harish-Chandra’s c-function. This satisfies for any \( H \in a_+ \) and \( \lambda \in a_+^* + \sqrt{-1}a^* \)

\[
\lim_{t \to \infty} e^{t(-\lambda + \rho)(H)} \phi^\pi_\lambda(e^tH) = c^\pi(\lambda).
\]

These things are shown using the integral formula (3.7) in the same way as in the case of the trivial \( K \)-type (cf. \( \text{Hel2} \) Chapter IV, §6, No.6), or are deduced as a special case of the asymptotic behavior of Eisenstein integrals (cf. \( \text{War} \) Theorem 9.1.6.1), \( \text{[Kn1]} \) Theorem 14.7, (14.29)). It is known that \( c^\pi(\lambda) \) extends to a meromorphic function on \( a_+^* \). We regard \( c^\pi(\lambda) \) as a \( C \)-valued function by \( \text{End}_M V \simeq C \).

Theorem 7.2. Suppose (1.6) and (7.1) are valid for \( \Sigma^\pi \) and \( k^\pi \). With \( e(\Sigma^\pi, k^\pi) \) in (7.3) we have

\[
c^\pi(\lambda) = 2e(\Sigma^\pi, k^\pi) c(\Sigma^\pi, k^\pi, \lambda).
\]

(Recall \( c(\Sigma^\pi, k^\pi, \lambda) \) is defined by (4.5).)

Proof. Note that

\[
\lim_{t \to \infty} e^{t\rho(H)} \delta^{-\frac{1}{2}}_{G/K}(tH) = \lim_{t \to \infty} e^{-t\rho(k^\pi)(H)} \delta(\Sigma^\pi, k^\pi; tH)^{\frac{1}{2}} = 1
\]

and that

\[
e^{-\lambda + \rho} \Upsilon^\pi(\phi^\lambda) = 2e(\Sigma^\pi, k^\pi) (e^{\rho} \delta^{-\frac{1}{2}}_{G/K})(e^{-\rho(k^\pi)} \delta(\Sigma^\pi, k^\pi)^{\frac{1}{2}}) e^{-\lambda + \rho(k^\pi)} F(\Sigma^\pi, k^\pi, \lambda).
\]

Hence (7.6) follows from (7.5) and Corollary 4.10. □

7.3. The \( \pi \)-spherical transform. Let \( C^\infty_c(G, \pi, \pi) \) be the subspace of \( C^\infty(G, \pi, \pi) \) consisting of the compactly supported \( \pi \)-spherical functions. For \( \phi_1 \in C^\infty_c(G, \pi, \pi) \) and \( \phi_2 \in C^\infty_c(G, \pi, \pi) \) define the convolution \( \phi_1 \ast \phi_2 \in C^\infty_c(G, \pi, \pi) \) by

\[
(\phi_1 \ast \phi_2)(x) = \int_G \phi_1(g^{-1}x) \phi_2(g) dg,
\]

where \( dg \) is a Haar measure on \( G \). Since \( (\pi, V) \) is a small, \( C^\infty_c(G, \pi, \pi) \) is a commutative algebra by \( \text{[Dc]} \) Theorem 3. For \( \phi \in C^\infty_c(G, \pi, \pi) \) we define its \( \pi \)-spherical transform by

\[
\hat{\phi}(\lambda) = \int_G \phi^*_\lambda(g^{-1}) \phi(g) dg = (\phi^*_\lambda \ast \phi)(1_G),
\]
which is, by (3.1), a holomorphic function on $a_C^*$ taking values in $\text{End}_K V \simeq \mathbb{C}$. For each $\lambda \in a_C^*$

$$C_c^\infty(G, \pi, \pi) \ni \phi \mapsto \hat{\phi}(\lambda) \in \mathbb{C}$$

is an algebra homomorphism since one has $\phi^\vee \ast \phi = \hat{\phi}(\lambda)\phi^\vee$ by Theorem 1.4.

Now we normalize the Haar measure $dH$ on $a$ so that for any compactly supported continuous $K$-bi-invariant function $\psi$ on $G$

$$\int_G \psi(g)dg = \frac{1}{\#W} \int_a \psi(e^H) \delta_{G/K}(H) dH$$

(cf. [Hel2, Ch. I, Theorem 5.8], [GV, Proposition 2.4.6]). Since the trace of the integrand of (7.7) is $K$-bi-invariant, we have using (3.4)

$$\hat{\phi}(\lambda) = \frac{1}{\#W} \int_a T^\pi(\phi^\vee)(-H) T^\pi(\phi)(H) \delta_{G/K}(H) dH$$

$$= \frac{1}{\#W} \int_a T^\pi(\phi^\vee)(H) T^\pi(\phi)(H) \delta_{G/K}(H) dH.$$

Hence by Theorem 1.5 the $\pi$-spherical transform is identified with the integral transform

$$f \mapsto \hat{f}(\lambda) := \frac{1}{\#W} \int_a f(H) T^\pi(\phi^\vee)(H) \delta_{G/K}(H) dH. \quad (7.8)$$

for $f \in C_c^\infty(a)^W$. Here $C_c^\infty(a)^W = \{ f \in C_c^\infty(a) | f \text{ with compact support} \}$.

Let $\Sigma'$ be a root system in $a^*$ and $k$ a multiplicity function on $\Sigma'$. For $f \in C_c^\infty(a)^W$ we define its hypergeometric Fourier transform $\mathcal{F} = \mathcal{F}(\Sigma', k)$ by

$$\mathcal{F}f(\lambda) := \frac{1}{\#W} \int_a f(H) F(\Sigma', k, -\lambda; H) \delta(\Sigma', k; H) dH. \quad (7.9)$$

This makes sense when $\delta(\Sigma', k)$ is locally integrable.

**Remark 7.3.** The hypergeometric Fourier transform is introduced by Opdam [Op3] as the Cherednik transform. If $\Sigma' = 2\Sigma$ and $k_{2\alpha} = m_{\alpha}/2$, then $\mathcal{F}(\Sigma', k)$ equals (7.8) for the trivial $K$-type $\pi$, that is, the Harish-Chandra transform (cf. [GV, Chapter 6], [Hel2, Ch. IV], [War, Chapter 9]). If $\Sigma'$ has real rank one, then $\mathcal{F}(\Sigma', k)$ reduces to the Jacobi transform (cf. [Koo]).

**Theorem 7.4.** Suppose (1.6) and (7.1) are valid for $\Sigma'$ and $k^\pi$. Then $\delta(\Sigma', k^\pi)$ is locally integrable and it holds with $\mathcal{F} = \mathcal{F}(\Sigma', k^\pi)$ and $\varepsilon(\Sigma^\pi, k^\pi)$ in (7.3) that

$$\hat{f} = 2^{\varepsilon(\Sigma^\pi, k^\pi)} \mathcal{F}(f \delta_{G/K}^{1/2} \delta(\Sigma^\pi, k^\pi)^{-1/2}) \quad \text{for any } f \in C^\infty_c(a)^W. \quad (7.10)$$

**Proof.** The local integrability follows from (5.4), while (7.10) is direct from (1.6), (7.2), (7.8) and (7.9). □
7.4. Inversion formulas and Plancherel formulas. We normalize the Haar measure $d\lambda$ on $\sqrt{-1}\mathfrak{a}^*$ so that the Euclidean Fourier transform and its inversion are given by
\[
\hat{f}(\lambda) = \int_{\mathfrak{a}} f(H) e^{-\lambda(H)} dH, \quad f(H) = \int_{\sqrt{-1}\mathfrak{a}^*} \hat{f}(\lambda) e^{\lambda(H)} d\lambda.
\]

On the hypergeometric Fourier transform we have
Theorem 7.5. So suppose (7.1) is not valid. Even in this case, one easily sees that under (7.1) all the statements are immediate from (1.6), (7.2), (7.6), (7.10) and (7.11) unless $\delta(\Sigma^\pi, k^\pi)$ is locally integrable and that Formulas (7.11) hold by replacing $2\pi(\Sigma^\pi, k^\pi)$ with some constant. Hence the corollary follows.

Proof. Under (7.1) all the statements are immediate from (1.6), (7.2), (7.6), (7.10) and Theorem 7.5. So suppose (7.1) is not valid. Even in this case, one easily sees that $\delta(\Sigma^\pi, k^\pi)$ is locally integrable and that Formulas (7.2), (7.6) and (7.10) hold by replacing $2\pi(\Sigma^\pi, k^\pi)$ with some constant. Hence the corollary follows.

As we can see from the list in §2, there exists at least a pair of $\Sigma^\pi$ and $k^\pi$ satisfying (1.6) and (7.11) unless $(\pi, V)$ is one of the following:

1. $(\pi, V)$ for $\mathfrak{g} = \mathfrak{sp}(p,1)$ $(p \geq 1)$;
2. $(\pi^\pm, V)$ in §2.4 for $\mathfrak{g} = \mathfrak{so}(2r,1)$ $(r \geq 2)$;
3. $(\pi, V)$ in §2.5 for $\mathfrak{g} = \mathfrak{so}(p,q)$ $(p > q \geq 3, p : \text{even}, q : \text{odd})$;
(4) \((\pi, V)\) for a Hermitian \(G\):

(5) \((\pi_2, C^2)\) in Theorem 2.2 for \(\tilde{G}_2\).

For (5), the elementary \(\pi\)-spherical functions cannot be expressed by Heckman-Opdam hypergeometric functions. The harmonic analysis in this exceptional case is therefore to be studied separately. But we do not go further with it in this paper.

For (1)–(4), every \(\Sigma_\pi\) chosen in §2 is of type \(BC\). Suppose first that \(G\) has real rank one and let \(\Sigma_\pi = \{\pm \alpha, \pm 2 \alpha\}\). Then the inversion formula and the Plancherel-type formula for the Jacobi transform \(F\) are available under the assumption

\[
k_\alpha, k_{2\alpha} \in \mathbb{R} \quad \text{and} \quad k_\alpha + k_{2\alpha} > -\frac{1}{2},
\]

which is much weaker than that of Theorem 7.5 (cf. [FJ, Appendix 1], [Koo]). From this we can deduce a result on the \(\pi\)-spherical transform for (1), (2), and (4) with \(g = su(p, 1)\) corresponding to Corollary 7.6. In fact, such a result is shown by [DP, Sh2] for (1), by [CP] for (2) with \(s = 1\), and by [FJ] for (4) with \(g = su(p, 1)\). If \(c_\pi(\lambda)\) has zeros in \(a^*_\pi + \sqrt{-1}a^*\) then the inversion and Plancherel-type formulas contain discrete spectra in addition to the same continuous spectrum as in the formulas of Corollary 7.6.

Next, let us consider (4) for \(G\) with higher rank. If the parameter \(\nu\) of the one-dimensional \(K\)-type \(\pi\) given in §2.6 satisfies \(2|\nu| \leq \max\{m_2, 1\}\) for \(\alpha \in \Sigma_{\text{long}}\), then we can apply Theorem 7.5 to deduce Corollary 7.6. The general inversion and Plancherel-type formulas are given by [Hec2, Chapter 5] for \(|\nu| < \frac{1}{2}m_2\) (\(\alpha \in \Sigma_{\text{long}}\)) and by [Sh1] for an arbitrary \(\nu\). Both [Hec2] and [Sh1] employ Rosenberg’s method ([R]) for the classical Harish-Chandra transform. If \(|\nu|\) is sufficiently large, the Plancherel measure contains spectra with lower-dimensional support along with the most continuous spectrum in Corollary 7.6. Possible spectra with lower-dimensional support are obtained by calculating residues of \(c_\pi(\lambda)^{-1}\) in \(a^*_\pi + \sqrt{-1}a^*\) (see [Sh1] for details).

Finally suppose \((\pi, V)\) is (3). In the notation of §2.5 one has from Theorem 7.2 that

\[
c_\pi(\lambda) = C \prod_{i=1}^{q} \frac{\Gamma(\lambda((2e_i)^\nu) + \frac{1}{2})}{\Gamma(\lambda((2e_i)^\nu) + \frac{1}{2}(p - q + 1))} \cdot \prod_{1 \leq i < j \leq q} \frac{\Gamma(\lambda((e_i - e_j)^\nu))\Gamma(\lambda((e_i + e_j)^\nu))}{\Gamma(\lambda((e_i - e_j)^\nu) + \frac{1}{2})\Gamma(\lambda((e_i + e_j)^\nu) + \frac{1}{2})}
\]

with a positive constant \(C\). Thus \(c_\pi(\lambda)\) has no zero in \(a^*_\pi + \sqrt{-1}a^*\). Hence it is very likely that we could prove the same result as Corollary 7.6 for \(\pi\) by Rosenberg’s method. However it is more desirable that a general result on \(F\) for \(\Sigma_\nu\) of type \(BC\) would hold under a weaker assumption than that of Theorem 7.5 and from this we could deduce all the results for (1)–(4) in a uniform way. We will discuss this problem elsewhere.
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