Holographic Entanglement Entropy in 2D Holographic Superconductor via $AdS_3/CFT_2$

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The aim of the present letter is to find the holographic entanglement entropy (HEE) in 2D holographic superconductors (HSC). Indeed, it is possible to compute the exact form of this entropy due to a advantage of approximate solutions inside normal and superconducting phases with backreactions. By making the UV and IR limits applied to the integrals, an approximate expression is obtained. In case the software cannot calculate minimal surface integrals analytically it offers the possibility to proceed with a numerical evaluation of the corresponding terms. We’ll understand how the area formula incorporates the structure of the domain wall approximation. We conclude that the wider belt angle corresponds to a larger surface holographic surface. We see that HEE changes linearly with belt angle.

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Introduction Our contemporary physical questions are appearing a bit harder. On the one hand the Anti-de Sitter space/Conformal Field Theory (AdS/CFT) conjecture gives an abstract and still largely conjectural approach which applies in very general situations [1]. It stated: weakly coupled gravitational models at AdS bulk are dual to a strongly coupled CFT on boundary. This means that the strongly coupled quantum systems may correspond precisely to black holes. Gauge/gravity dualities is an frequent application, particularly seen in those systems with strongly coupling, like type II superconductors [2-3]. The AdS/CFT movement seems particularly adept in its innovative approach to reality. Its areas of research interest include holographic superconductors, Quark-Gluon plasma, and superconductor/superfluid in condensed matter physics, particularly using qualitative approaches [4-7]. AdS/CFT has been used recently to produce a realistic model for entanglement quantum systems [2-3] (with conformal field theory descriptions) with some success [10-24], as a geometric approach. In order to address these issues, we consider two possible partitions $A$ (set $A$) and $B$ (complementary set) of a single quantum system upon which an Hilbert space $\mathcal{H}_A \times \mathcal{H}_B$ may be based. We consider the Von-Neumann entropy $S_X = -Tr_X (\rho \log \rho)$ the best of the best for statistical description, where $Tr$ is the quantum trace of quantum operator $\rho$ over quantum basis $X$. If we compute $S_A$ and $S_{A'}$, this is extremely useful to see $S_A = S_{A'}$. A further consequence, however, is that Von-Neumann entropies are now more likely to identify it with a region, the boundary of $\partial A$ [22]. Here the HEE is precisely defined by the , but these geometries appear arbitrary. More recently, studies on the role of analytical methods in computation of the EE have been initiated. [24-25] It must be specially an outstanding note in the role of this type of entropy to be specially computed to lower dimensional quantum systems as its $AdS_3/CFT_2$ picture. Using a specially designed gravitational dual, we use EE to explain our 2D dynamical phase transitions. We’ll investigate the reduced HEE of a strip geometry (belt) in three dimensional AdS background. It can be calculated analytically in terms of the cutoff of length. The HEE and total length (angle) is approximated by a function for which the minimal surface integral is analytically solvable. After the normal form for the zero temperature has been derived, the extent to which the criticality regime $T \sim T_c$ may be solved analytically is covered. In case the hand cannot calculate surface integrals analytically it offers the possibility to proceed with a numerical evaluation of the corresponding integrals. The variation in aggregation of the HEE illustrates a magnify like ability to adapt to different phase transitions. At this point in the system the superconducting phase is prefreed to the normal phase, thereby presenting the minimal surface area.

Model for 2D HSC The following action combines the accuracy of AdS bulk modeling with the 2D quantum system on boundary [22-31]:

$$S = \int d^3x \sqrt{-g} \left[ \frac{1}{2\kappa^2} (R + \frac{2}{L^2}) - \frac{1}{4} F_{ab} F_{ab} - |\nabla \phi - iA\phi|^2 - m^2 |\phi|^2 \right].$$  \hspace{1cm} (1)

Here, $\kappa$ defines the three dimensional gravitational constant $\kappa^2 = 8\pi G_3$, the Newton constant $G_3$, $L$ is the AdS radius, $m^2 = m_\phi^2 \in (-1, \infty)$ mass of scalar field, and
\[ g = |g_{\mu\nu}|. \] For more accurate information we may also choose to fix a metric to AdS bulk over a given range of coordinates:

\[ ds^2 = -f(r)e^{-\beta(r)}dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_4^2. \] (2)

We may choose a temperature for CFT from our AdS black hole:

\[ T = \frac{f'(r_+)}{4\pi}. \] (3)

We can adapt any conventional information by substituting static symmetry to adapt our metric to best showcase the normal state of our system in the absence of scalar field \( \phi \) for bulk:

\[ A_4 = A(r)dt, \quad \phi \equiv \phi(r). \] (4)

We can also use static symmetry to adapt our metric to best showcase the normal state of our system in the absence of scalar field \( \phi \) for bulk:

\[ f(r) = k + \frac{r^2}{L^2} - \kappa^2\mu^2 \log r, \]
\[ A(r) = \rho + \mu \log r. \] (5,6)

where \( k = -\frac{r^2}{L^2} + \kappa^2\mu^2 \log r_+, \mu, \rho \) correspond to the chemical potential and charge density in the dual field theory and \( \kappa^2 = 8\pi G_3 \). Here, \( r_+ \) is the radius \( r \) of the event horizon \( f(r_+) = 0 \) for a AdS black hole. The fields \( A_\mu, \phi \) will satisfy regularity if they satisfy these auxiliary boundary conditions:

\[ A(r_+) = 0, \quad \phi'(r_+) = \frac{m^2}{f'(r_+)} \phi(r_+), \] (7)

and the metric ansatz satisfies:

\[ f'(r_+) = \frac{2r_+}{L^2} - 2\kappa^2 r_+ \left[ m^2 \phi(r_+)^2 + \frac{1}{2}e^{\beta(r_+)} A'(r_+) \frac{\partial \beta(r_+)}{f'(r_+)} \right] \] (8)
\[ \beta'(r_+) = -4\kappa^2 r_+ \left[ \frac{A'(r_+)^2}{f'(r_+)^2}e^{\beta(r_+)} + \phi'(r_+) \right] \] (9)

The AdS asymptotic expansions for the fields, \( 30 \), require the values of:

\[ \beta \rightarrow 0, \quad f(r) \sim \frac{r^2}{L^2}, \quad A(r) \sim \mu \log r, \]
\[ \phi(r) \sim \frac{<O_->}{r^{\Delta_-}} + \frac{<O_+>}{r^{\Delta_+}}, \text{ as } r \rightarrow \infty. \] (10)

The use of \( \Delta_\pm \) can denote conformal dimensiones \( \Delta_\pm = 1 \pm \sqrt{1 + m^2} \). Let \( <O_\pm> \) denote the standard vacuum expectation values (VEV) of dual operators \( O_\pm \) in CFT. A change of variable \( z = \frac{r}{r_+} \) has been applied, which essentially simplifies the forms of the equations of motion:

\[ \phi'' + \frac{\phi'}{z} \left[ 1 + \frac{z f'}{f} - \frac{z \beta'}{2} \right] + \frac{r_+^2 \phi}{z^4} \left[ \frac{A'^2 e^{\beta}}{f^2} - m^2 \right] = 0, \] (11)
\[ A'' + \frac{A'}{z} \left[ 1 - \frac{z \beta'}{2} \right] - 2\kappa^2 \frac{A \phi'}{z f} = 0, \] (12)
\[ \beta' - 4\kappa^2 z^2 \left[ A^2 \phi^2 e^{\beta} \right] + \frac{z \beta'^2}{f} = 0, \] (13)
\[ f' = \frac{2r_+^2}{L^2 z^3} - \kappa^2 z e^\beta A^2 - \frac{2\kappa^2 m r_+^2 \phi^2}{z^3} \] (14)
\[ - \frac{2\kappa^2 r_+^2}{z^3} \left[ A^2 \phi^2 e^{\beta} \right] + \frac{f \phi'^2}{r_+^4} = 0, \]

Solving the above equation with \( \phi \neq 0, T < T_c \) is the principal purpose of the superconductivity program \( 29 \).

HEE proposal: Following to the proposal \( 3, 3 \), suppose a field theory in \( (d-1)D \) has a gravitational dual embedded in \( AdS_{d+1} \) bulk. The holographic algorithm is then used to compute the entanglement entropy of a region of space \( \tilde{A} \) and its complement from the \( AdS_{d+1} \) geometry of bulk:

\[ S_{\tilde{A}} \equiv S_{\text{HEE}} = \frac{Area(\gamma_{\tilde{A}})}{4G_{d+1}}. \] (15)

We first compute the the minimal \( (d-1)D \) mini-super surface \( \gamma_{\tilde{A}} \). It had been proposed to extend \( \gamma_{\tilde{A}} \) to bulk, but with criteria to keep surfaces with same boundary \( \partial \gamma_{\tilde{A}} \) and \( \partial A \). The equal boundary is the leading technique working to compute HEE via AdS/CFT .

Several options for the parametrization of the \( \tilde{A} \) are available as well as different choices for the \( \partial \gamma_{\tilde{A}} \) in the bulk. We discuss the possibility of computing HEE of 2D systems using parametric representation in one degree of freedom \( \tilde{A} := \{ t = t_0, -\theta_0 \leq \theta \leq \theta_0, r = r(\theta) \} \). Minimization can be used on Lagrangian which has a simple Beltrami form:

\[ L \equiv \sqrt{r'^2 + \frac{r'(\theta)^2}{f(r)}} \] (16)

Using the Beltrami identity, since \( \partial_\theta L = 0 \), computations were taken by using a constant quantity \( \mathcal{L} - r'\partial_r \mathcal{L} = C \) designed for \( \mathcal{L} \).

In this case, we define half of the total length (angle)\( \theta_0 \) and HEE in the more conventional forms, using the shorter list enumerated here:

\[ \theta_0 = \int_0^{\theta_0} d\theta = \int_0^{\theta_0} \frac{C dr}{r \sqrt{f(r)(r^2 - C^2)}} \] (17)
\[ S_{\text{HEE}} = \frac{1}{2G_3} \int_0^{\theta_0} r dr \sqrt{f(r)(r^2 - C^2)} \] (18)

The aim this letter is to evaluate the sensitivity of \( 17, 18 \) in the bulk of acute regimes of temperature \( T = 0, T \gg T_c, T \lesssim T_c \).
Sharp domain wall approximation: Studies are currently underway to develop and evaluate (17,18) using numerical algorithms. Our aim is to evaluate the (17,18) as an analytic tool. The (17,18) are determined from the domain wall approximation analysis (19). Suppose we have a sharp phase transition between two patches of the AdS space time. We will try to locate it at \( r = r_{DW} \). Here \( r_{DW} \) defines the position of the domain wall in the AdS radial direction. Indeed, in the massless limit, \( m^2 \), there exists an intermediate radius \( -\infty < r_m < 0 \) such that \( \phi'(r_m) = 0 \). The ideal candidate for \( r_{DW} \) should be \( r_{DW} = r_m \). We always assume that \( r_{DW} < 0 \). So we assume that the following form is a perfectly good tool for analytical evaluation:

\[
C \equiv \frac{r^2}{\sqrt{r^2 + r^2(\theta)^2}} = \begin{cases} \frac{L}{r}, & r > r_{DW} \\ \frac{L_{IR}}{r}, & r < r_{DW} \end{cases}. \tag{19}
\]

In the previous equation we took into account two different AdS radii, \( L \) and \( L_{IR} \) in each region. With the previous considerations, (17,18) are easily integrated,

\[
\int_0^{\theta_0} d\theta = \theta_0 = \theta_{IR} + \theta_{UV}, \tag{20}
\]

\[
\theta_{IR} = \int_{r_{UV}}^{r_{DW}} \frac{L_{IR} dr}{r \sqrt{\sqrt{2} - L_{IR}^2}}, \tag{21}
\]

\[
\theta_{UV} = \int_{r_{UV}}^{r_{DW}} \frac{L dr}{r \sqrt{\sqrt{2} - L^2}}. \tag{22}
\]

here \( r_{*} \) denotes the “turning” point of the minimal surface \( \gamma_{\tilde{A}} \). It is defined by \( r'(\theta)|_{r=r_{*}} = 0 \). So we can choose to engage turning point with \( r_{*} \) or \( C \). We replaced the integrating out to \( r = \infty \) by integrating out to large positive radius \( r_{UV} \). Indeed, we assume that \( r_{UV} \) stands out for UV cutoff (28).

We will suppose that \( r_{*} < r_{DW} \). It means that the minimal surface drift onto the IR region. We can rewrite HEE as we like:

\[
S_{HEE} = \frac{1}{2L_{\gamma}} \left[ S_{IR} + S_{UV} \right],
\]

\[
S_{IR} = \int_{r_{*}}^{r_{DW}} \frac{dr}{\sqrt{r_{IR}^2 - L_{IR}^2}}, \tag{24}
\]

\[
S_{UV} = \int_{r_{UV}}^{r_{DW}} \frac{dr}{\sqrt{r_{UV}^2 - L^2}}. \tag{25}
\]

In both cases IR, UV, geometry of AdS has imposed tight constraints on the metric:

\[
f(r) \to f_{IR} = 1, \text{ as } r \to -\infty. \tag{26}
\]

\[
f(r) \to f_{UV} = \frac{r^2}{L^2}, \text{ as } r \to \infty, \tag{27}
\]

Perhaps we’ll compute the \( \theta_{IR}, \theta_{UV} \) for another purpose:

\[
\theta_{IR} = i \log \left[ \frac{r_{*}}{r_{DW} \sqrt{L_{IR}^2 - r_{*}^2}} - L_{IR} \right], \tag{28}
\]

\[
\theta_{UV} = \frac{\sqrt{r_{UV}^2 - L^2} r_{UV} - L^2}{r_{DW}}. \tag{29}
\]

The entanglement entropy can be computed as the following:

\[
S_{IR} = \sqrt{r_{DW}^2 - L_{IR}^2} - \sqrt{r_{*}^2 - L_{IR}^2}, \tag{30}
\]

\[
S_{UV} = \frac{i L^2}{L_{UV}} \log \left[ \frac{r_{UV} \sqrt{L_{UV}^2 - r_{UV}^2} - L_{UV}^2}{L_{UV}} \right]. \tag{31}
\]

UV limit: we first consider \( r_{*} > r_{DW} \). This showed that \( \gamma_{\tilde{A}} \) were already embedding deeply into the AdS\(_3\) with boundary \( r \to \infty \). To get a rough approximation of how much entropy in the \( \gamma_{\tilde{A}} \) would be in UV limit, we simplify the problem by putting \( \theta_{IR} = S_{IR} = 0 \) and identifying \( r_{DW} = r_{*} \). Eqs. (24) and (25) we obtain:

\[
\theta_0 = -\frac{\pi}{2} - \sqrt{\sqrt{2} - L^2} - \frac{r_{UV}}{2r_{UV}}, \text{ as } r_{*} \to \infty, \tag{32}
\]

\[
\lim_{r_{*} \to \infty} S_{IR}^{\text{finite}} = r_{DW}, \tag{33}
\]

\[
S_{UV} = \frac{L^2}{L_{UV}} \log \left( \frac{i L_{UV} + \sqrt{L_{UV}^2 - r_{UV}^2} - L_{UV}^2}{r_{UV}} \right). \tag{34}
\]

Perhaps not surprisingly, \( S_{HEE/IR} = \frac{r_{DW}}{2L_{\gamma}} \) is exactly the result we would have computed if we were purely in the IR theory. It’s interesting to note that IR limit can’t follow the similar UV form. However according to note above the BH area term were calculated in IR limit is \( S_{HEE} = S_{BH} = \frac{r_{DW}}{2L_{\gamma}} + \frac{L^2}{2L_{\gamma}}, \) so there is a difference for this regime. Note there were major differences in the BH term in both regimes.

HEE close to the \( T < T_c \) in the absence of scalar field \( \phi(z) = 0 \). The normal phase can even be achieved from a list of functions:

\[
\phi_0 = \beta_0 = 0, \quad A_0 = -\mu_c \log z, \tag{35}
\]

\[
f_0 = \frac{r_{*}^2}{L^2} (z^2 - 1) + \kappa^2 \mu_c^2 \log z. \tag{36}
\]
The EE between $\tilde{A}$ and its complement is given by:

$$ s_{\tilde{A}} = 4G_3S_{\text{HEE}} = 2r_+^2 \int_{2\pi r_+}^{r_+} \frac{r dr}{\sqrt{f(r)(r^2 - r_+^2)}} $$

(39)

The technique of this computation was a rewrite of $s_{\tilde{A}}$ to better coordinate $z$:

$$ s_{\tilde{A}} = 2r_+ r_* \int_{z_{UV}}^{z_{+}} \frac{dz}{z^3 \sqrt{f(z)(z^2 - z_+^2)}} $$

(40)

The vicinity of the critical point $T \lesssim T_c$ may be represented as a place for an equivalent form of integral:

$$ s_{\tilde{A}} = 2r_+ r_* \int_{z_{UV}}^{z_{+}} \frac{dz}{z^3 \sqrt{f_0(z)(z^2 - z_+^2)}} $$

(41)

and

$$ \frac{\theta_0}{2} = r_* \int_{r_+}^{r_*} \frac{dr}{r \sqrt{f(r)(r^2 - r_+^2)}} $$

(42)

$$ = \frac{r_+}{r_*} \int_{z_{UV}}^{z_{+}} \frac{dz}{z \sqrt{f(z)(z^2 - z_+^2)}} $$

At criticality $T \lesssim T_c$,

$$ \frac{\theta_0}{2} = \frac{r_+}{r_{+c}} \int_{z_{UV}}^{z_{+}} \frac{dz}{z \sqrt{f_0(z)(z^2 - z_+^2)}} $$

(43)

where

$$ T_c = \frac{1}{4\pi r_{+c}} \left( 2r_{+c}^2 - \kappa^2 \mu_c^2 \right) $$

(44)

We go on to calculate the critical value of the horizon $r_+$ used by $\{T_c, \mu_c, \kappa^2\}$:

$$ \frac{r_{+c}}{L} = \pi T_c L + \frac{1}{2} \sqrt{4\pi^2 T_c^2 L^2 + 2 \kappa^2 \mu_c^2}. $$

(45)

Parametric estimation of the $\{\mu_c, \kappa^2\}$ using polynomials is needed. We apply series method to estimation of $\frac{1}{\sqrt{f(z)}}$ in $\{\mu_c, \kappa^2\}$ Expansion of the $\frac{1}{\sqrt{f(z)}}$ as follows :

$$ \frac{1}{\sqrt{f_0}} = \sum_{n=0}^{\infty} b_n \frac{(\log z)^n}{(z^2 - 1)^{n+1/2}}, $$

(46)

$$ b_n = \frac{(\kappa \mu_c)^2 n! (-1)^n (1/2) n! \mu_c (2/1) \sqrt{r_+}}{\sqrt{r_+}} $$

(47)

allows us to expand into the series the $\{\mu_c, \kappa^2\}$ We first evaluate a value of integral $I_a^n$ for use in the $\{\mu_c, \kappa^2\}$ :

$$ I_a^n \equiv \int_{z_{UV}}^{z_{+}} \frac{(\log z)^n dz}{z^3 \sqrt{z^2 - z_+^2(z^2 - 1)^{n+1/2}}} $$

for $a = 1, 3$

The aim is to evaluate the integral of $I_a^n$ as a series tool for interval $0 \lesssim z_{UV} < z < z_+ \lesssim 1$:

$$ I_a^n = \sum_{\alpha, \beta, \gamma = 0}^{\infty} \frac{(1/2)_{\alpha + 1/2} \beta \gamma + n}{a! \beta! \gamma! (\gamma + n)} \times \left( 2(\alpha + \beta + n) - a + 3 \right)^{\gamma} \left( \log z_+^{-\alpha} - \log z_{UV}^{-\gamma} \right)^n $$

where $(\alpha + n - 1)!$ is the Pochhammer symbol. We need to carefully evaluate $I_a^n$ with $I_a^n$:

$$ \frac{\theta_0}{2} = r_* \sum_{n=0}^{\infty} b_n I_a^n, $$

(50)

$$ s_{\tilde{A}} = 2r_+ r_* \sum_{n=0}^{\infty} b_n I_a^n. $$

(51)

Indeed, the functions $\theta' \equiv \frac{\theta_0}{2}, s' \equiv \frac{s_+}{r_{+c}}$ have the simple forms in their bi-parametric ($\frac{T_c}{n}, \frac{\mu_c}{\mu}$) list:

$$ s' = \frac{2 T_c}{L} + \sqrt{4 \left( \frac{T_c}{L} \right)^2 + \frac{2 \kappa^2 \mu_c^2}{(\mu_c^2)}} $$

$$ \theta' = \frac{T_c}{L} + \frac{1}{4T_c^2 L^2 + 2 \kappa^2 \mu_c^2} $$

as they may have the parameters $\zeta = \kappa \mu_c = 0.005, L \equiv 1$ and

$$ B_n = \frac{\zeta^n (-1)^n (1/2)_n}{n!} $$

(49)

$$ \times \left( \pi T_c L + \frac{1}{2} \sqrt{4 \pi^2 T_c^2 L^2 + 2 \kappa^2 \mu_c^2} \right)^{n+1/2}. $$

We draw $s'\left( \frac{T_c}{n}, \frac{\mu_c}{\mu} \right) \text{ vs } \theta_0 \left( \frac{T_c}{n}, \frac{\mu_c}{\mu} \right).$

An example plot of reduced HEE in a system is shown in (41). Numeric analyzes showed a significant smooth relationship between increasing proportions of in the this phase. Seeing an increasing HEE for system, it decided to become a normal conductor than just a superconductor. Increasing temperature to reduce on HEE adds to the system at criticality, thus slowing the superconducting.

We plot isothermal curves of $\{T\}$ for various values of $I_a^n$ in (22). Attending at least one lower temperature regime $T < T_c$ is almost compulsory for superconductivity.

For fixed relative chemical potential $\frac{\mu}{\mu_c}$, we plot (52) as function of $T$ in (49). We observe that at fixed $\frac{\mu}{\mu_c}$, one may increase the $s'(T)$ simply by increasing the $T > T_c$.

Fig (4) shows typical behaviors of $\{T_{\text{cond}}\}$ versus temperature $T$ for fixed $T_c$. These are relatively low temperature, holographic superconductor which can linearly be described for system. It has been suggested that where there is low temperature phase may be able to keep superconductivity with increased reduced entropy $\{T_{\text{cond}}\}$ rises.
FIG. 1: Plot of the surface \( (52) \) versus \( \mu, T \). It shows that \( s' \) is a monotonic-increasing function. It always increasing or remaining constant, and never decreasing. It produces a regular phase of matter for \( T > T_c \).

Regular attendance at these non superconducting phase has proved numerically. Boundary conditions and regular tiny backreactions \( \zeta \) will help keep normal phase for longer. Normal phase increasing the entropy \( (52) \), increases the hardenability of superconductivity.

FIG. 2: Plot of the of the isothermal HEE \( (52) \) for various values of \( T \). For positive values of the chemical potential, \( s' \) is a monotonic-increasing function, always increasing, and never decreasing. The "fixed" chemical potential will give a lower value of \( (52) \) for lower temperatures. At fixed temperature, in an isothermal graph, when \( \frac{\mu}{\mu_c} \) increases, the associated HEE entropy is also a monotonic-increasing function of \( \frac{\mu}{\mu_c} \). Of course, attending at least one lower temperature regime \( T < T_c \) is almost compulsory for superconductivity. It was always realistic to expect that superconducting phase could be in being by lower values of \( \frac{\mu}{\mu_c} \) in isothermal regime.

Figure 3 shows a linearly-dependent of reduced HEE \( (52) \) versus angle \( (53) \).

Figure 4 shows that there are low-impact angle \( (53) \) designed specifically for low temperature and chemical potential. Furthermore, \( \theta' \) is a monotonic-decreasing function of \( \mu, T \).

**HEE in the presence of scalar field**\( \phi(z) \neq 0 \) at \( T \lesssim T_c \): During the critical phase transition, \( \epsilon \equiv< O_\pm > \) is sufficiently tiny to expand functions by the following series forms:

\[
\phi = \sum_{k=1}^{\infty} \epsilon^k \phi_k, \quad A = \sum_{k=0}^{\infty} \epsilon^{2k} A_{2k}, \quad (55)
\]

\[
f = \sum_{k=0}^{\infty} \epsilon^{2k} f_{2k}, \quad \beta = \sum_{k=1}^{\infty} \epsilon^{2k} \beta_{2k}. \quad (56)
\]

When we turn-on the condensate, \( \phi(z) \neq 0 \), as we expect, analytical expression for HEE is much more harder. Specially at the criticality, \( T \lesssim T_c \), because scalar field \( \phi(z) \) and Maxwell field \( A(z) \) backreacted on the metric functions \( f(z), \beta(z) \). Generally speaking, one cannot solve field equations given in \( (11-15) \) and find the analytic form of \( f(z) \) in a closed form. However, the approximate solutions for \( (11-15) \) will be possible. We start by the following solutions:

\[
\phi(z) = \epsilon \phi_1, A(z) = A_0 + \epsilon^2 A_2, \quad (57)
\]

\[
\beta(z) = \epsilon^2 \beta_2, f(z) = f_0 + \epsilon^2 f_2. \quad (58)
\]

where \( \epsilon \equiv< O_\pm > \). Analytical solutions obtained by
Expansion of the \( \frac{1}{\sqrt{f_0 + \epsilon^2 f_2}} \) as follows:

\[
\frac{1}{\sqrt{f_0 + \epsilon^2 f_2}} = \frac{1}{\sqrt{f_0}} \left( 1 - \frac{1}{2} \frac{\epsilon^2 f_2}{f_0} \right),
\]

we obtain:

\[
\frac{\theta_0}{2} = \frac{r_s}{r_c} \sum_{n=0}^{\infty} b_n \left( I_n^0 - \frac{\epsilon^2}{2} I_1^0 \right),
\]

\[
s_A = 2r_s r_c \sum_{n=0}^{\infty} b_n \left( I_n^0 - \frac{\epsilon^2}{2} I_1^0 \right),
\]

Where

\[
I_n^0 = \int_{zUV}^{z^*} \frac{f_2}{\int_{zUV}^{z^*} \sqrt{f_0 + \epsilon^2 f_2} (z^2 - z^{-2})} dz.
\]

for \( \alpha = 1, 3 \)

We rewrite them in terms of \( \left( \frac{r_s}{r_c}, \frac{\mu}{\mu_c} \right) \) as the following:

\[
s' = \frac{2 r_s}{r_c} + \sqrt{4(\frac{r_s}{r_c})^2 + \frac{2 r_s^2}{r_c^2} (\frac{\mu}{\mu_c})^2}
\]

\[
\cdot \sum_{n=0}^{\infty} B_n \left( I_n^0 - \frac{\epsilon^2}{2} I_1^0 \right)
\]

\[
\theta' = \frac{2 r_s}{r_c} + \sqrt{4(\frac{r_s}{r_c})^2 + \frac{2 r_s^2}{r_c^2} (\frac{\mu}{\mu_c})^2}
\]

\[
\cdot \sum_{n=0}^{\infty} B_n \left( I_n^0 - \frac{\epsilon^2}{2} I_1^0 \right)
\]

Here \( \epsilon_0 \ll 1 \) is a numeric. The second negative term, seem obviously compatible with a superconducting phase in the presence of the scalar field. By decreasing the amount of entropy produced in the supercondor phase, system alters the phase of conductivity.

We plot \( \frac{\theta_0}{2} \) vs. \( \frac{s_A}{a} \). We adjust data as \( \epsilon_0 = 0.05, \kappa \mu_c = 0.005 \). The critical temperature was obtained as \( T_c = 0.2 \). The system evolves from normal phase \( T > T_c \) to the superconductor phase \( T < T_c \) for \( T \approx 0.0179, 0.0173, 0.0165, 0.0152, 0.0132 \). The wider angle \( \theta' \) corresponds to a larger surface holographic surface. We see that HEE \( \theta' \) changes linearly with \( \theta' \).

Summary: The aim of this letter was to investigate the effect of superconductor critical phase transition in 2D models of holographic superconductors on HEE. We investigate aspects of the domain wall approximation and scalar condensate of the transition phases. We proceeded to investigate why the HEE increase with temperature and belt angle in the backreacted and normal AdS3 background. Using the domain wall auxiliary asymptotic boundary conditions, as have been used before we can investigate the evolution of the HEE for this superconductor model. To calculate the HEE in the critical phase.
FIG. 7: Plot of $s'$ (67) as a function of $\theta'$ (68) for $T \lesssim T_c$. We adjust data as $\epsilon_0 = 0.05$. The critical temperature was obtained as $T_c = 0.2$. The system evolves from normal phase $T > T_c$ to the superconductor phase $T \lesssim T_c$ for $T \approx 0.0179, 0.0173, 0.0165, 0.0152, 0.0132$. The wider angle (68) corresponds to a larger surface holographic surface. We see that HEE (67) changes linearly with $\theta'$.

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