Robust quantum state recovery from amplitude damping within a mixed states framework

Saeideh Shahrokh Esfahani, Zeyang Liao and M Suhail Zubairy

Institute for Quantum Science and Engineering (IQSE) and Department of Physics and Astronomy, Texas A&M University, College Station, TX 77843-4242, USA

E-mail: zubairy@physics.tamu.edu

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Abstract

Due to interaction with the environment, a quantum state is subjected to decoherence which becomes one of the major problems in many quantum systems. Amplitude damping is one of the most important decoherence processes. Here, we show that general two-qubit mixed states undergoing amplitude damping can be almost completely restored using a reversal procedure. This reversal procedure through CNOT and Hadamard gates could also protect the entanglement of two-qubit mixed states from general amplitude damping. We also propose a robust recovery scheme to protect the quantum states when the decay parameters or the input quantum states are not completely known.

Keywords: recovery scheme, mixed states, fidelity, extended scheme, quantum measurement, Monte Carlo, concurrence

1. Introduction

Due to the inevitable interaction with the environment, a quantum system can entangle with the environment and subsequently become decoherent. This decoherence is a fundamental obstacle to the successful transfer of quantum information and to practical quantum computation. A number of effective approaches have been proposed to suppress the decoherence effect. One way for protecting a quantum state from decoherence is based on the existence of decoherence-free subspaces (DFSs) which require special symmetry properties of the interaction Hamiltonian. In quantum computation, this procedure, i.e. the utilization of DFSs to protect quantum states, is called ‘error-avoiding code’ [1]. ‘Quantum error correction code’ (QECC) is another way to suppress the decoherence effect. In QECC, the logical quantum bit (qubit) is encoded in a larger Hilbert space of several physical qubits and the correction process is performed by constructing proper measurements and correction operations [2, 3]. Other methods such as the quantum Zeno effect [4, 5] and dynamical decoupling [6, 7] are also widely used to mitigate decoherence and to protect the quantum state. Amplitude damping is a consequence of the coupling of a system with a reservoir and therefore it is a fundamental source of noise in many systems [8]. For example, an atom placed in a vacuum can undergo spontaneous emission. Similarly, a field state inside a leaky cavity can interact with vacuum modes outside the cavity and consequently lose its coherence. In recent years, several strategies have been proposed to protect quantum states from amplitude damping. Three widely used strategies to protect the quantum state from amplitude damping are: (1) weak measurement reversal [9–12], (2) uncollapsing a quantum state towards the ground state and the reversal measurement [13], and (3) utilization of quantum gates to restore a qubit state in a weak measurement [14, 15]. Quantum state recovery based on quantum gates can be implemented in a shorter time compared with weak measurement reversal. It is shown that a one-qubit state in a weak measurement can be completely recovered by applying Hadamard and CNOT gates on the system qubit and an auxiliary qubit [14]. This method is generalized to recover an arbitrary two-qubit pure state undergoing amplitude damping [15].
In this paper, we show that this method can be generalized to protect arbitrary two-qubit mixed states which can have many more free parameters as compared to a pure superposition state. We also consider the case when the input density matrix and the damping parameter are not completely known and propose a robust recovery scheme. We test our recovery schemes by generating arbitrary mixed states via Monte Carlo simulations and obtain the optimal parameters to recover an unknown quantum state. Here, we restrict our analysis to zero temperature, as we are only interested in reversing the noise due to quantum fluctuation.

The paper is organized as follows: in section 2, we calculate the damped matrix of an arbitrary two-qubit mixed state. In section 3, we demonstrate how we use the proposed recovery scheme to reverse the damping effect and recover the quantum state. In section 4, an extended scheme [15] is applied to amplify the protection proposed in section 3.2. In section 5 robust recovery under uncertainty of the input states and damping parameters is studied. Finally we summarize the results.

2. Amplitude damping of two-qubit mixed state

Amplitude damping is an important type of decoherence. Single-qubit amplitude damping can be mathematically described by the following mappings:

\[
\begin{align*}
|0\rangle|0\rangle_E &\rightarrow |0\rangle|0\rangle_E, \\
|1\rangle|0\rangle_E &\rightarrow \sqrt{1-p}|1\rangle|0\rangle_E + \sqrt{p}|0\rangle|1\rangle_E.
\end{align*}
\]

where \( p \in [0, 1] \) is the possibility of decaying of the excited state, and \( S \) (E) denotes the system (environment). For example, within the Weisskopf–Wigner approximation [8], the decaying probability of an atom interacting with a vacuum field is given by \( \sqrt{1-p} = e^{-\Gamma t} \) with \( \Gamma \) being the spontaneous decay rate of the atom. Similarly, the damping of a field in a cavity is given by \( \sqrt{1-p} = e^{-\kappa t} \) with \( \kappa \) being the leakage rate of the cavity. For a general single-qubit mixed state \( \rho \), amplitude damping can also be written as

\[
\rho \rightarrow \varepsilon_{AD}(\rho) = A_0 \rho A_0^+ + A_1 \rho A_1^+
\]

where the amplitude damping operations are given by

\[
A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}.
\]

An arbitrary two-qubit mixed state can be written as

\[
\rho_i = \begin{pmatrix} a & e & f & g \\ e^* & b & h & i \\ f^* & h^* & c & j \\ g^* & i^* & j^* & d \end{pmatrix}.
\]

The amplitude damping of an arbitrary two-qubit mixed state can be calculated by the following procedures. First an arbitrary two-qubit mixed state can also be written as

\[
\rho = \sum_{i,j,m=0}^{1|} \alpha_{ijmn} |ij\rangle \langle mn|.
\]

Each element \( |ij\rangle \langle mn| \) can be written as two-qubit direct products \( |i\rangle \langle m| \otimes |j\rangle \langle n| \). Next we apply amplitude damping operations on each qubit which yield

\[
|ij\rangle \langle mn| \rightarrow [A_0 |i\rangle \langle m| A_0^+ + A_1 |j\rangle \langle n| A_1^+] \otimes [A_0 |j\rangle \langle n| A_0^+ + A_1 |i\rangle \langle m| A_1^+].
\]

After applying amplitude damping operations on each element, we obtain a two-qubit amplitude-damped state given by

\[
\rho_q = \begin{pmatrix} a + bp + cp + p^2d & e_jq + pj_jq \langle f_jq + ip_jq \langle g_jq \langle \end{pmatrix} + \begin{pmatrix} bq + pdq \quad hq \quad iq_jq \\
\end{pmatrix} + \begin{pmatrix} g_jq \quad i^*q_jq \quad f_jq_jq \quad dq_jq \end{pmatrix}.
\]

where \( q = 1 - p \).

3. Two-qubit mixed states recovery

In this section, we propose a method to recover the damped quantum mixed states in equation (6) to the initial quantum mixed states in equation (4). This method has recently been introduced for two-qubit pure state [15]. In this model, we use the circuit diagram outlined in figure 1. Two auxiliary qubits, both in the \( |0\rangle \) state initially, are added. First, we apply a Hadamard gate with angle \( \theta \) for each ancilla qubit.

\[
H_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}
\]

The ancilla qubits \( A_1 \) and \( A_2 \) after passing through the Hadamard gate will change to:

\[
\rho_{A_1} = \rho_{A_2} = \begin{pmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{pmatrix}.
\]

The state of the whole system, after combining ancilla qubits to the damped system in equation (6), can be written as:

\[
\rho_{Ad} = \rho_{A_1} \otimes \rho_{A_2} \otimes \rho_{AD}.
\]

Afterwards, we apply two CNOT gates onto each pair of the system and ancilla qubits:

The final state is then given by

\[
\rho_f = UC_1 \otimes UC_2 \cdot UC_1 \otimes UC_2 \otimes UC_1^+,
\]

where the operations \( UC_1 \) and \( UC_2 \) are given by \( UC \) table 1. Finally, we make measurements on the two ancilla qubits. If

Figure 1. A schematic view of the recovery process proposed in [15], generalized herein for the mixed states setting.
the ancilla qubits are both in $|0\rangle$ state, the recovery process is successful. Otherwise, the recovery process fails. Since the ancilla qubits are measured to be in $|0\rangle$ state, the state of the whole system becomes

$$
\rho_f = (P_{A1} \otimes P_{A2}) \rho_1 (P_{A2}^\dagger \otimes P_{A1}^\dagger)
$$

(11)

where the projection operators $P_{A1} = |0\rangle \langle 0| \otimes \frac{I}{2}$ and $P_{A2} = I \otimes |0\rangle \langle 0| \otimes \frac{I}{2}$ being a two-by-two unit matrix. The reduced system density matrix is $\rho_f = T_{A1,A2}(\rho_1)$ where $T_{A1,A2}$ denotes the partial trace over the ancilla qubits.

By choosing $\theta = \tan^{-1}(1/\sqrt{2})$, the final state after the recovery process for the system can be calculated to be

$$
\rho_f = \frac{1}{1 + (1 - a + d)p + p^2d} \left[ \begin{array}{cccc}
a + bp + cp + p^2d & e + pj & f + ip & g \\
e^* + j^*p & b + pd & h & i \\
f^* + i^*p & h^* & c + pd & j \\
g^* & i^* & j^* & d
\end{array} \right]
$$

(12)

From equation (12) we can obtain $\rho_f = (\rho_i + \rho_{err})/N$ where $\rho_i$ is the initial state, $N = 1 + (1 - a + d)p + p^2d$ is the normalization factor, and $\rho_{err}$ is the recovering error matrix which is given by

$$
\rho_{err} = \left[ \begin{array}{cccc}
bp + cp + p^2d & jp & ip & 0 \\
j^*p & dp & 0 & 0 \\
i^*p & 0 & pd & 0 \\
0 & 0 & 0 & 0
\end{array} \right].
$$

(13)

Thus, the system is not completely recovered but is restored to the initial input density matrix plus an error term. When $p = 0$, $\rho_f = \rho_i$ which is expected. Here, we have neglected the amplitude damping of the system qubits and ancilla qubit during the recovery process. This is a good approximation as the recovery process is based on quantum gates which can be implemented much faster than the decoherence time of the system. For example, a CNOT gate can be implemented by the cavity-QED system proposed by [14] where the interaction time is around 20 $\mu$s while the atom and field relaxation time are 30 ms and 1 ms, respectively [16]. In the following subsections, we quantitatively analyze how the quantum state is restored using fidelity and quantum concurrence.

### 3.1. Fidelity

One way to measure how a quantum state is recovered is by calculating the fidelity between the recovered and the initial states. The fidelity function between two quantum mixed states is defined by [17]

$$
F(\rho_i, \rho_f) = |Tr(\sqrt{\sqrt{\rho_i} \rho_f \sqrt{\rho_i}})|^2,
$$

(14)

where $\rho_i$ and $\rho_f$ are the initial and final states, respectively. In this paper, the fidelity between the damped state and the initial state is $F_d = F(\rho_i, \rho_f)$, and the fidelity between the recovered state and the initial state is $F_r = F(\rho_i, \rho_r)$.

In figures 2(a) and (b), we depict the recovering fidelities as a function of the decaying probability $p$ for two mixed states given by:

$$
\rho_f = \left[ \begin{array}{cccc}
0.4 & 0 & 0 & 0.25 \\
0 & 0.1 & 0 & 0 \\
0 & 0 & 0.3 & 0 \\
0.25 & 0 & 0 & 0.2
\end{array} \right],
$$

$$
\rho_f = \left[ \begin{array}{cccc}
0.6 & 0 & 0 & 0.25 \\
0 & 0.12 & 0 & 0 \\
0 & 0 & 0.11 & 0 \\
0.25 & 0 & 0 & 0.17
\end{array} \right].
$$

(15)

From the figures, we see that the fidelities of the recovered states are higher than those of the damped states which indicates that our recovery scheme also works for two-qubit mixed states.

To justify whether our method works for general two-qubit mixed states, we also perform numerical calculation of the average fidelity of the damped states and the recovered states over a large ensemble. To do so, we randomly generate a large ensemble of two-qubit mixed states using the method shown in [18–20] in which they would obey the required properties of a valid density matrix from a certain probability distribution. For each decaying probability $p$, $\theta$ is chosen to be $\tan^{-1}(1/\sqrt{1 - p})$ and the average fidelity of the damped and recovered states is shown in figure 3 where we can see...
that our recovery scheme can effectively restore the general two-qubit mixed state.

3.2. Entanglement protection from amplitude damping

In this subsection, we study whether the quantum entanglement of the two-qubit mixed state can be protected by our scheme or not. The quantum entanglement of a two-qubit mixed state can be calibrated by quantum concurrence which is defined as [21]

\[ C(\rho) \equiv \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \]

where \( \lambda_1, \ldots, \lambda_4 \) are the eigenvalues in decreasing order of the Hermitian matrix \( R(\rho) = \sqrt{\rho} \rho \sqrt{\rho} \) with \( \rho = (\sigma_x \otimes \sigma_x) \rho^\# (\sigma_x \otimes \sigma_x) \).

The concurrence and the recovered concurrences for the two quantum states used in the previous section are shown in figures 4(a) and (b) respectively. From figures 4(a) and (b) one can see the following features: (1) The concurrence of the recovered state is higher than that of the damped one which indicates that our scheme can protect the quantum entanglement of the two-qubit mixed state from amplitude damping. However, the amount of quantum entanglement does not improve very much. (2) The entanglement vanishes at a special point which is called entanglement sudden death (ESD) [22, 23]. Before the ESD point, the quantum entanglement can be restored by a certain amount. However, beyond the ESD point, the quantum entanglement cannot be improved by the quantum algorithm shown in figure 1 because the recovering scheme shown in figure 1 is essentially a non-unitary local operation.

4. Extended scheme

In the previous section, we show that a quantum state can be recovered with very good fidelity by the scheme shown in figure 1. However, the quantum entanglement cannot be well recovered in that scheme. In this section, we discuss how to improve this scheme. Similar to that of [15], we can significantly improve the fidelity and quantum entanglement by adding a preparation stage before the amplitude damping of a two-qubit mixed state. The extended scheme scenario is depicted in figure 5, which is a mixed-state generalization of the scheme in [15].

The proposed method proceeds as follows: before the initial two-qubit mixed states undergo amplitude damping, we pre-process the system to make it robust against the amplitude damping. To do so, we apply the same quantum circuit as in the recovery part to prepare the initial state. In this stage, the preparation is successful if the ancilla qubits are measured to be |00⟩. After the preparation stage, the system undergoes the damping stage shown in section 2. In the final part, we perform the same recovery procedure as shown in section 3 to recover the quantum state and quantum entanglement.

The quantum state after the preparation stage can be obtained from equation (12), by considering \( p = 0 \) and \( \theta = \theta_1 \) where \( \theta_1 \) is the rotation angle of the Hadamard gate in the preparation step. Then, by denoting \( x \equiv \tan^2 \theta_1 \), the quantum state after the preparation stage is given by

\[
\rho_p =\begin{pmatrix}
\frac{a}{(1 + x)^2} & e\left(\frac{1}{1 + x}\right)^{1/2} & \frac{e^{\ast}}{(1 + x)^2} & f\left(\frac{1}{1 + x}\right)^{1/2} \\
\frac{e^{\ast}}{(1 + x)^2} & \frac{x}{1 + x} & \frac{h}{(1 + x)^2} & \frac{g}{(1 + x)^2} \\
e\left(\frac{1}{1 + x}\right)^{1/2} & \frac{h}{(1 + x)^2} & \frac{e^2}{(1 + x)^2} & \frac{g^2}{(1 + x)^2} \\
f\left(\frac{1}{1 + x}\right)^{1/2} & \frac{g}{(1 + x)^2} & \frac{g^2}{(1 + x)^2} & \frac{e^2}{(1 + x)^2}
\end{pmatrix}.
\]

It is noted that if \( \theta_1 \) is selected such that \( x < 1 \), the system uncollapses toward the ground state as weak measurement [12, 13]. The ground state is less vulnerable to the amplitude damping because it is uncoupled to the environment [13]. In the next stage, the prepared state shown in equation (17) undergoes amplitude damping and the recovery procedure, shown in figure 5. For the recovery stage we determine the rotation angle of the Hadamard gate, \( \theta_2 \), such that \( y = \tan^2 \theta_2 \). Then, as in section 3, we measure the ancilla qubits in |00⟩ states, and finally obtain the recovered density matrix.

We now examine how our extended scheme works compared with the scheme without a preparation stage. In figures 6(a) and (b), we show the quantum entanglement
recovery $C_r$ under different values of $x$. From the figures, we see that $C_r$ in the extended scheme with $x < 1$ can be higher than $C_r$ in the previous scheme without a preparation stage. When $x = 1$, $C_r$ in the extended scheme returns to the previous one. In addition, we notice that the quantum entanglement does not vanish in the extended scheme even beyond the ESD point which never occurs in the previous scheme. The reason why ESD can be inhibited in the extended scheme is that we prepare the initial state in a more robust state by uncollapsing the quantum state toward the ground state before the amplitude damping. The more we uncollapse the system toward the ground state, the larger the ESD point will be. After a certain value, the ESD point can even be eliminated in the whole region. The fidelity of the recovered state can also be significantly improved in the extended scheme (see figures 7(a) and (b)). However, we note that the success probability decreases when $x$ is smaller.

5. Robust recovery under uncertainty

In previous sections, we have considered the scenario where we have complete knowledge about the parameters of the apparatus. It means that our model would work in a situation where we know the exact values of the parameters, e.g. we know $p$ and consequently design $\theta$ based on $p$. In this situation, as described above, we can follow the reversal scheme outlined in figures 1 and 5, and use them to reverse the initial mixed states when they undergo amplitude damping.

Another question that has been studied is: What if we aim to design such an apparatus where we face some issues of uncertainty? One of the important issues is uncertainty on $p$. Furthermore, we know that a Hadamard gate angle which works properly for one state may not necessarily be the best one for other states. Below, we depict two scenarios. First, we consider a scenario where we want to design a setup where there is a mismatch in the actual $p$ and the one with which we design the angle. We discuss the effect of this mismatching in section 5.1. Next, in order to overcome the illustrated shortcoming of this mismatch, we propose a robust recovery scheme where we can find an optimal Hadamard gate angle and it can be indeed helpful for battling against the uncertainty on $p$, and also uncertainty around the input state. This approach would be widely applicable, since it requires no initial assumption.

5.1. Uncertainty in $p$

In the previous sections, we assume that the decay parameter $p$ is known which leads to designing $\theta$ such that $\theta = \tan^{-1}(1/\sqrt{q})$. However, in practice, one may not have a complete estimate of $p$, i.e. it is either completely unknown or known up to an interval. Therefore, a legitimate question can be, How can we determine the Hadamard gate angle such that given our uncertainty about $p$, the achieved fidelity would become sufficient? In order to quantify the degrading effect of an unknown $p$, we conduct a numerical simulation study. Suppose that we have a point estimation for the value that $p$ can take, say $\hat{p} = 0.7$. Then, based on this value, we set the Hadamard gate angle using $\theta = \tan^{-1}(1/\sqrt{\hat{q}})$, i.e. $\hat{p} = 61.3^\circ$. Now, we are interested in evaluating the fidelity of this ‘$\theta$-fixed’ recovery scheme across all possible actual values of $p$. Moreover, we would like to see the difference in fidelity with the case with known $p$ and the adaptive selection of $\theta$ as $\theta = \tan^{-1}(1/\sqrt{1 - p})$, i.e. for every $p$ design $\theta$ such that $\theta = \tan^{-1}(1/\sqrt{1 - \hat{p}})$. The simulation results for two mixed density matrices $\rho_1$ and $\rho_2$ are shown in figures 8(a) and (b). We plot the fidelity of the recovered scheme by considering the fixed $\theta$ above (i.e. corresponding to $\hat{p} = 0.7$), along with two other curves, taken from figures 2(a) and (b).

Deducing from figures 8(a) and (b), we can summarize the simulation results by the following two points: (1) For fixed $\theta = 61.3^\circ$, the quantum state is not recovered well on all $p$’s, unless in the range around $p = 0.35$ to 1 for the density matrix corresponding to $\rho_1$ and the range around $p = 0.55$ to 1 for $\rho_2$. (2) Even though the selection of $\theta$ through the tangent formula shows better performance overall, as shown in figures 8(a) and (b), one may find a better $\theta$ for a specific damping probability $p$ than that calculated by $\tan^{-1}(1/\sqrt{1 - \hat{p}})$. It seems that in this situation, where we do not know $p$, choosing a random $p$ to determine the angle $\theta$ is
not a favorable way. Hence, in section 5.2, we define a robust method to solve this problem.

5.2. Unknown $p$ and $\rho$

In the previous subsection, we study how choosing a Hadamard gate angle $\theta$, where we have uncertainty on $p$, would affect the fidelity under different values of $p$. We now want to make our uncertainty broader by assuming uncertainty on both $p$ and the initial quantum state of the system $\rho$. In this scenario, we introduce a robust recovery scheme based on finding an optimal $\theta$ which yields the best average fidelity taken over the distribution.

**Definition 1 (Fidelity robust recovery scheme).** Suppose that the (unknown) density states are governed by a given distribution, i.e., each density matrix also has a probability of occurrence. Then, we define fidelity as a function of $\rho$, $p$ and $\theta$, and denote it by $F(\rho, p, \theta)$. We define the average fidelity over the range of $p$ and $\rho$ as follows:

$$F(\theta) = E_p[E_\rho[F(\rho, p, \theta)]]$$

Then, we define a recovery scheme, fidelity-robust, if its Hadamard gate angle $\theta_{\text{opt}}$ is chosen as follows:

$$\theta_{\text{opt}} = \max \; F(\theta).$$

We call $\theta_{\text{opt}}$ the robust Hadamard gate angle. It should be noted that by averaging we cancel out the roles of unknown $p$ and $\rho$ in the fidelity. This is also called marginalization.

In the case of a given interval, for the (uniformly distributed) unknown $p \in (p_1, p_2)$, we can simplify equation (18), as follows

$$F(\theta) = \frac{1}{p_2 - p_1} \int_{p_1}^{p_2} E_\rho[F(\rho, p, \theta)] dp.$$  

In many situations, it may not be feasible to find a closed-form solution for either (18) or (19). In these cases, one may take numerical approaches for computing expectations, and solving the maximization problem. In the following, we show an example, where we find the robust angle via Monte Carlo simulations.

To examine the performance of the proposed recovery scheme with the robust angle $\theta$, we compare the case with
complete knowledge of $\theta$ (the original scheme) with the $\theta$ obtained from equation (19):

$$T_{\theta} = E_{\theta}[E_{\rho}[F(\rho, p, \theta_{\rho})]].$$  \hspace{1cm} (21)

where $\theta_{\rho} = \tan^{-1}(1/\sqrt{1 - p})$.

We generate random $\rho$ via the Monte Carlo approach with $10^4$ iterations, and $p$ also uniformly varies between 0.1 and 0.9. To find the maximum average fidelity, we grid the range of $\theta$ between 0 and $2\pi$ with steps of $\frac{\pi}{10}$. We summarize the results in table 2.

| \text{CNOT gate I} | \text{CNOT gate II} |
|---------------------|---------------------|
| $UC_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ | $UC_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ |

One can find $\theta_{\text{opt}}$ (among the selected candidate angles) by choosing the maximum fidelity among these numbers. The optimal fidelity, bold in the table, is 0.790 which corresponds to $3\pi/10$. We omit the rest of the numbers after $\pi$, because they periodically repeat. Although 0.790 is smaller than the fidelity 0.838 in the case where we know exactly what $p$ is, it is still larger than the damped fidelity 0.727. One should note that one may achieve better results by a finer grid of $\theta$. Therefore, one may conclude that via the robust recovery scheme, mixed states recovery can be robustly implemented with no knowledge of the underlying $p$ or $\rho$, and we only need to know the distributions governing these two parameters.

6. Summary

In summary, we show several schemes to protect an arbitrary two-qubit mixed state from amplitude damping. The basic scheme without a preparation stage can recover a quantum
state with very high fidelity, but the quantum entanglement of the two-qubit mixed state is not significantly improved. The extended scheme with a preparation stage can recover the two-qubit mixed state with very high fidelity and the quantum entanglement can also be significantly recovered by choosing suitable parameters. Furthermore, the extended scheme can recover a quantum state even beyond the ESD point.

In addition, a recovery scheme is next introduced which takes the system’s uncertainties into account and in turn leads to a robust recovery scheme. We find an optimal angle for recovering a two-qubit mixed state when the quantum state and the decay probability are unknown. This scheme may be very useful for protecting a quantum state from amplitude damping in practice.

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Table 2. Average recovery fidelity for two scenarios. The first row shows the results for the case where $p$ and $\rho$ are assumed unknown. Also the results are repeated periodically until $2\pi$. The second row shows the fidelity for the scenario where the gate angle is chosen with knowledge of $p$ the same as in section 3.1. The third row is the average damped fidelity.

| $F(\theta)$ | 0   | $\pi/10$ | $2\pi/10$ | $3\pi/10$ | $4\pi/10$ | $5\pi/10$ | $6\pi/10$ | $7\pi/10$ | $8\pi/10$ | $9\pi/10$ | $\pi$ |
|------------|-----|---------|----------|----------|----------|----------|----------|----------|----------|----------|------|
| $F_0$      | 0.250 | 0.427  | 0.632    | **0.790**| 0.751    | 0.248    | 0.090    | 0.089    | 0.114    | 0.154    | 0.250|
| $F_r$      | 0.838 | 0.838  | 0.838    | **0.838**| 0.838    | 0.838    | 0.838    | 0.838    | 0.838    | 0.838    | 0.838|
| $F_d$      | 0.727 | 0.727  | 0.727    | **0.727**| 0.727    | 0.727    | 0.727    | 0.727    | 0.727    | 0.727    | 0.727|