Spin and New Physical Field

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Abstract

It is shown that the interactions between the fermion and the gravitational fields are due to the torsion field. The torsion field is considered to be a potential one, like the electromagnetic and gravitational fields. The field equations are obtained, which describe the interactions of the torsion field with the conventional physical fields. The general covariant Lagrangian of the gravitational field, based on the torsion field, is derived. Experiment is proposed to test the theory.

Introduction

It is commonly assumed that the Einstein metric potential of the gravitational field is a part of all equations of theoretical physics and thus the gravitational field affects all the physical processes. This statement (currently known as the principle of universality of gravitational interactions) has a fundamental significance for understanding the structure of the physical notions and laws, which forms a framework for all the physical processes. The relativistic wave equation for the electron was suggested by Dirac in 1928. The question naturally arises how to reconcile his equation with General Relativity. This problem was investigated in the works of Fock and Ivanenko [1] and Weyl [2]. The main result of this consideration is that Einstein's gravitational potential does not enter into the Dirac equation. Thus, in the microphysical realm the principle of universality of gravitational interactions met with the difficulties of principal character that are not overcome till now for the following reasons. To save the principle of universality of gravitational interactions it was suggested that a true gravitational potential is a field (vierbein or tetrad) that defines the general covariant form of the Dirac equation. However, it
is quite evident that the validity of this assumption is not indisputable from the logical point of view. Indeed, we have no reasons to doubt that Einstein’s gravitational potential is a primary notion and hence tetrads may be (and should be) incorporated into the framework of Einstein’s gravity theory among the other possible origins of the gravitational field with proper energy–momentum tensor. It is quite clear, but so far as we know such formulation of the problem was not considered in literature. The main reason for this is that the principle of universality of gravitational interactions is related to another very interesting problem, namely the existence of a new physical field due to spin.

With the discovery of the spin (spin of the electron) it was suggested that spin is the source of a new field (known in our days as a torsion field). The notion of torsion is causally related to asymmetric connection which was mentioned by Eddington who pointed out on some applications of this idea in microphysics. Torsion as antisymmetric part of asymmetric affine connection was introduced by Cartan. He hinted that the torsion might be connected with the intrinsic angular momentum of matter. Later the different aspects of the new concept were developed in different directions in the enormous number of publications (see, e.g., the original papers of Eddington and Cartan [3], [4] and [5],[6] for extensive review and references on various aspects of the theory with torsion field). But despite all this efforts the physical meaning of the new field and its possible link with spin is still an open problem.

Thus, spin and gravity are related to the two actual problems. One of them deals with the description of the spinor fields in framework of the metrical theory of gravity and the other with the existence of a new physical field tightly connected with the quantum–mechanical notion of spin. In this paper we show that both problems are interrelated and on this ground we propose their simple solution. The essence of our approach is based on the assumption that a spinning matter interacts with the gravitational field indirectly and the torsion field is an interface between the gravitational and fermion fields. Of course, this means that the principle of universality of gravitational interactions does not hold valid anymore and that is why an experiment is proposed to decide between the metrical and tetrad theories of gravity.
1 The Dirac equation in general covariant form

Let $\mathbb{C}^4$ be a linear space of columns of four complex numbers $\psi_1, \psi_2, \psi_3, \psi_4$. Linear transformations in this space can be presented by the complex matrices $(4 \times 4)$. The set of all invertible $(4 \times 4)$ complex matrices forms a group denoted by $GL(4, \mathbb{C})$. Dirac’s $\gamma^\mu$ matrices belong to $GL(4, \mathbb{C})$ and obey anticommutation relations

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu},$$

where $\eta^{\mu\nu}$ are structure constants of the Poincaré group, $\eta^{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. From $\gamma^\mu$ one can construct sixteen linear independent matrices that form a basis of the Lie algebra of $GL(4, \mathbb{C})$. This basis is especially important since the matrices $S_{\mu\nu} = \frac{1}{2}\gamma^\mu \gamma^\nu$ form the basis of the Lie algebra of the Lorentz group (subgroup of $GL(4, \mathbb{C})$). Thus, a spinor is an element of the space $\mathbb{C}^4$. For better understanding it should be noted that in the space $\mathbb{C}^3$ there are no matrices like $\gamma^\mu$.

If one considers $\psi_1, \psi_2, \psi_3, \psi_4$ as a set of complex scalar fields on the space–time manifold then a spinor field emerges on the manifold which is a basis of irreducible representation of the group $GL(4, \mathbb{C})$. It is not difficult to understand that $GL(4, \mathbb{C})$ is a group of internal symmetry since its transformations touch only functions of the spinor field and do not affect the coordinates. In other words, spin symmetry is an internal symmetry.

Now we consider general covariant formulation of the Dirac equation in the Minkowski space–time. We shall follow the fundamental physical principle that physical laws, like geometrical relationships, depend neither on the choice of the coordinate system nor on the choice of the basis in the studied vector spaces. With respect to an arbitrary curvilinear system of coordinates Minkowski space–time is characterized by the metric

$$ds^2 = g_{ij}dx^idx^j$$

of the Lorentz signature, which satisfies the equation $R_{ijkl} = 0$. At given $g_{ij}$, the generators of the group of space-time symmetry can be presented as a set of linear independent solutions of general covariant system of equations (Killing’s equations)

$$P^i \partial_i g_{jk} + g_{ik} \partial_j P^i + g_{ji} \partial_k P^i = 0$$

for a vector field $P^i$. In the case of the Minkowski metric we have ten the linear independent solutions of the Killing equations, which are denoted $P^i_\mu$. 
and $M_{i}^{\mu} = -M_{i}^{\mu}$, and hence the Greek indices enumerate a vector fields and take the values 0, 1, 2, 3, like coordinate Latin indices.

It is well-known that the generators of the Poincaré group

$$P_{\mu} = P_{\mu}^{i} \frac{\partial}{\partial x^{i}}, \quad M_{\mu \nu} = M_{\mu \nu}^{i} \frac{\partial}{\partial x^{i}}$$

satisfy the following commutation relations

$$[P_{\mu}, P_{\nu}] = 0, \quad (1)$$

$$[P_{\mu}, M_{\nu \lambda}] = \eta_{\mu \nu} P_{\lambda} - \eta_{\mu \lambda} P_{\nu}. \quad (2)$$

It is evident that all these relations are general covariant and that the operators $P_{\mu} = P_{\mu}^{i} \frac{\partial}{\partial x^{i}}$ transform a scalar field into the scalar one.

Now we shall show that the general covariant Dirac equation has the form

$$i \gamma^{\mu} P_{\mu} \psi = \frac{mc}{\hbar} \psi, \quad (3)$$

where $\psi$ is a column of four complex scalar fields in question and $P_{\mu}$ are the generators of space–time translations. Dirac’s equation is covariant with respect to the general coordinate transformations. It is also clear that the equation (3) is equivalent to the equation

$$i \tilde{\gamma}^{\mu} P_{\mu} \psi = \frac{mc}{\hbar} \psi,$$

if $\tilde{\gamma}^{\mu} = S \gamma^{\mu} S^{-1}$, where $S \in GL(4, \mathbb{C})$ (the equation (3) is covariant with respect to the transformations of the group $GL(4, \mathbb{C})$). Since Dirac’s equation (3) is general covariant we can use any system of coordinates. In the preferred system of coordinates the equation (3) has a customary form.

Now we have found enough to provide some valuable insights into the connection between the space–time and internal transformations. Consider again the generators of the internal Lorentz group $S_{\mu \nu} = \frac{1}{4} (\gamma_{\mu} \gamma_{\nu} - \gamma_{\nu} \gamma_{\mu})$ and pay attention to the commutation relations

$$[\gamma_{\mu}, S_{\nu \lambda}] = \eta_{\mu \nu} \gamma_{\lambda} - \eta_{\mu \lambda} \gamma_{\nu}. \quad (4)$$

Comparing (2) and (4) it is not difficult to verify that the operators

$$L_{\mu \nu} = M_{\mu \nu} + S_{\mu \nu}$$

commute with the Dirac operator $D = i \gamma^{\mu} P_{\mu}$ and satisfy the commutation relations of the Poincaré group. Thus, in the Minkowski space–time there is a relation between the internal symmetry group and the space–time symmetry group. The consequence is that Dirac’s equation (3) is invariant with respect to the transformations of the Poincaré group.
2 Generalization of the Dirac theory

Our goal here is to consider natural extension of the Dirac theory assuming that now $P^i_\mu$ are simply components of the four linear independent vector fields. In this case there is no strict relation between the internal and space–time symmetry but instead a new possibility is opened. We keep the internal symmetry on the same ground as in the starting situation but attempt to expand the space–time symmetry to the diffeomorphism group. It may be possible if we can establish a system of equations for the fields $\psi$ and $P^i_\mu$ that is invariant with respect to the transformations of the diffeomorphism group.

Idea of physical interpretation of the new field $P^i_\mu$ can be derived from the principle of universality of gravitational interactions or from the other arguments which will be considered in the following section. If we claim that the principle of universality of gravitational interactions is held valid, then we should demand that a quadruplet of the linear independent vector fields is a potential of the gravitational field. Furthermore, we suggest that a new gravitational potential enters into the Dirac equation in the form considered above and into the equations of other fields in the form of the metric tensor associated with a new gravitational potential

$$\tilde{g}_{ij} = \eta_{\mu\nu}P^\mu_i P^\nu_j,$$

where $\eta_{\mu\nu}$ are the structure constants of the Poincaré group and $P^\mu_i$ is also a quadruple of covector fields, which is inverse to $P^i_\mu$

$$P^\mu_i P^\mu_j = \delta^i_j, \quad P^\mu_i P^\nu_i = \delta^\mu_\nu. \quad (5)$$

Taking into account this postulate (which is alternative to the original Einstein’s idea) we need to derive the field equations for the new gravitational potential. There are two distinct approaches to this problem. One of them follows the line that equations for the $P^i_\mu$ are defined completely by an associative metric $\tilde{g}_{ij}$, i.e. in the Einstein-Hilbert action we simply replace $g_{ij}$ by $\tilde{g}_{ij}$. However, this postulate indicates that a theory of the new field $P^i_\mu$ should be invariant with respect to the local transformations of the form

$$\tilde{P}^i_\mu = L^i_k P^k_\mu,$$

where $L^i_k$ is a tensor field of the type (1,1), such that

$$L^i_k L^j_l \tilde{g}_{ij} = \tilde{g}_{kl}.$$
This condition is fulfilled in some modification of the Dirac equation (see [1],[2], [7],[8]). The second approach is based on the other Lagrangian and characterized by the absence of local symmetry group defined above. In this case Dirac’s equation (3) needs no modification (See for example [9],[10] and further references therein).

For the clarity we should emphasize that there are three successful alternative theories of gravity. The first theory, originated by Einstein, has a metric tensor as the basic structure. The problem (which is not solved till now) is how to incorporate Dirac’s theory into the framework of this description of the gravity physics. In the second and third approaches the last problem is solved by the method, which is not faultless from a logical point of view. Our goal here is to formulate a theory of interactions of the gravitational field with the spinor field in the framework of the Einstein metrical theory of gravity.

Thus, in the present work we state that a quadruplet of linear independent vector fields is a new physical field, which should be incorporated into the framework of the Einstein gravity theory as an origin of the gravitational field with a proper energy–momentum tensor. Below we shall consider this possibility in more details. Of course, this idea leads to the violation of the principle of universality of gravitational interactions but not to a modification of the theory of the gravitational field (known as General Relativity).

Now it is time to consider a geometrical meaning of the new field $P^i_\mu$. As it is well known the Einstein gravitational potential has a simple geometrical interpretation as an element of length in the Riemann space–time. In 1917 Levi–Civita proposed how to introduce a parallel transport as an internal notion on the Riemann manifold and opened a way for generalizations.

Consider a vector field $V^i(x)$. Equation of local parallel transport from a point $x^i$ to a point $x^i + dx^i$ has in general the form

$$dV^i(x) = -L^i_jk(x)V^k(x)dx^j,$$

where functions $L^i_jk(x)$ are components of a new geometrical object on the manifold, called a linear or affine connection $L$. Under a parallel transport along the infinitesimal close curve the change of the vector is equal to the quantity

$$\Delta V^k = -B^k_{ijl}V^l dx^i \delta x^j,$$
where
\[ B^k_{ijl} = \partial_i L^k_{jl} - \partial_j L^k_{il} + L^k_{im} L^m_{jl} - L^k_{jm} L^m_{il} \]  
(7)
is a tensor field of the type (1,3), called the Riemann tensor of the affine connection \( \bar{L}^i_{jk} \).

From (6) it follows that under a coordinate mapping
\[ \bar{x}^i = \bar{x}^i(x), \quad x^i = x^i(\bar{x}), \]
the transformation law for a \( L^i_{jk} \) has the form
\[ \bar{L}^i_{jk} = \frac{\partial \bar{x}^i}{\partial x^l} \left( L^l_{mn} \frac{\partial x^m}{\partial \bar{x}^j} \frac{\partial x^n}{\partial \bar{x}^k} + \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^k} \right). \]
(8)

We shall say that a geometrical quantity is irreducible if it is possible to find linear combinations of its components which themselves constitute a new geometrical quantity. It is very important that under the coordinate mappings a linear connection is a reducible quantity. It is easy to see from the expansion
\[ L^i_{jk} = \frac{1}{2} (L^i_{jk} + L^i_{kj}) + \frac{1}{2} (L^i_{jk} - L^i_{kj}). \]
From (8) it follows that a symmetrical part of the affine connection
\[ \{i_{jk}\} = \frac{1}{2} (\Gamma^i_{jk} + \Gamma^i_{kj}), \]
is again the affine connection and the antisymmetrical part,
\[ H^i_{jk} = \frac{1}{2} (\Gamma^i_{jk} - \Gamma^i_{kj}), \]
(9)
transforms as a tensor field of the type (1,2). This tensor field is called the torsion tensor. Thus, we have two independent irreducible fields. It is very important that with respect to the first field (symmetric affine connection) a metric can be considered as potential of this new field in accordance with the relation
\[ \{i_{jk}\} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_j g_{lk} - \partial_l g_{jk}). \]
(10)
Expression on the right hand side (10) was first found by Christoffel. For comparison, the formula
\[ F_{ij} = \partial_i A_j - \partial_j A_i \]
provides the link between a vector potential of the electromagnetic field and a tensor of this field. If we take a tensor field (for example \( g_{ij} \)) and form
its derivatives ($\partial_i g_{jk}$) then these derivatives are neither the components of a tensor of any geometrical object. But if one can form new geometrical object from these derivatives then this process can be called natural derivative. For example, symmetrical affine connection can be considered as a natural derivative with respect to the metric tensor and a bivector of the electromagnetic field as a natural derivative of the covector field. With this transparent presentation we can put forward the idea that torsion field (being new physical field) should be (like the electromagnetic and the gravitational field) a potential field. Now the clear problem is to show that the torsion tensor is a natural derivative with respect to the unknown field (potential of the torsion field). We state that a quadruplet of linear independent vector fields forms the potential of the torsion field because

$$H^i_{jk} = P^i_\mu (\partial_j P^\mu_k - \partial_k P^\mu_j).$$

(11)

So, a torsion tensor is actually a natural derivative of the $P^i_\mu$.

Since curvature and torsion are tightly connected then it is natural to suppose that a geometrical Lagrangian for these fields has the form $L = B$, where $B$ is a scalar that can be constructed from the Riemann tensor (7). Indeed, from (7), (9), (10), (11) it follows that

$$B^l_{ijk} = R^l_{ijk} + \nabla_i H^l_{jk} - \nabla_j H^l_{ik} + H^l_{im} H^m_{jk} - H^l_{jm} H^m_{ik},$$

(12)

where

$$R^l_{ijk} = \partial_i \{^l_{jk}\} - \partial_j \{^l_{ik}\} + \{^l_{im}\} \{^m_{jk}\} - \{^l_{jm}\} \{^m_{ik}\}$$

(13)

is the well–known curvature tensor and $\nabla_i$ stands for the covariant derivative with respect to the connection (10),

$$\nabla_i H^l_{jk} = \partial_i H^l_{jk} + \{^l_{im}\} H^m_{jk} - \{^m_{ij}\} H^l_{mk} - \{^m_{ik}\} H^l_{jm}.$$

By contraction we derive from (12) a more simple tensor

$$B_{jk} = B^i_{ijk} = R_{jk} + \nabla_i H^i_{jk} - \nabla_j H^i_{ik} + H^i_{im} H^m_{jk} - H^i_{jm} H^m_{ik},$$

(14)

where $R_{jk}$ is the Ricci tensor. From (14) it follows that a required scalar is

$$B = g^{jk} B_{jk} = R + g^{jk} H^l_{jm} H^m_{jl} - \nabla_j H^j,$$

where $R$ is scalar curvature (the Einstein-Hilbert Lagrangian of the gravitational field), and $H^j = g^{jk} H^i_{jk}$. We see that the Lagrangian of the torsion field itself is given by the formula

$$L_t = \frac{1}{2} g^{jk} H^l_{jm} H^m_{kl}.$$

(15)
As a byproduct of our consideration we shall derive now a new general covariant form of the Lagrangian of the gravitational field. Motivation for our study is well–known ( The Einstein-Hilbert Lagrangian $R$ contains a second order derivatives of $g_{ij}$ and this leads to the known difficulties [11]).

Consider a binary tensor field

$$B_{jk}^i = P_\mu ^i \nabla_j P^\mu _k = \Gamma _{jk}^i - \{ ^i _{jk} \}, \quad (16)$$

where

$$\Gamma _{jk}^i = P_\mu ^i \partial _j P^\mu _k \quad (17)$$

is the canonical connection for which the Riemann tensor is equal to zero identically. Setting

$$\Gamma _{jk}^i = \{ ^i _{jk} \} + \Gamma _{jk}^i - \{ ^i _{jk} \} = \{ ^i _{jk} \} + B _{jk}^i$$

and following closely the line defined by (12), (13), (14), we derive the relation

$$0 = R ^{jk} + \nabla _i B _{jk}^i - \nabla _j B _{ik}^i + B _{im} ^m B _{jk}^m - B _{jm} ^m B _{ik}^m.$$ 

From the last formula it follows that

$$R + \nabla _i (g ^{jk} B _{jk}^i - g ^{ik} B _{lk}^l) = g ^{jk} (B _{jm} ^m B _{ik}^i - B _{im} ^m B _{jk}^m).$$

Thus, the Einstein-Hilbert Lagrangian is equivalent to the Lagrangian

$$L_q = \frac{1}{2} g ^{jk} (B _{jm} ^m B _{ik}^i - B _{im} ^m B _{jk}^m).$$

This Lagrangian is based on the torsion field and may be more convenient in the quantum theory of the gravitational field.

### 3 Field equations for the torsion field

We make small variations in our field quantities $P _\mu ^i$ and calculate the change in the action integral

$$A = \frac{1}{2} \int g ^{jk} H _{jm} ^l H _{kl} ^m \sqrt{g} d^4 x$$

with $g = -Det(g_{ij})$. It is convenient to introduce tensor

$$F _{kj} ^{ij} = g ^{il} H _{lk} ^j - g ^{jl} H _{lk} ^i = H _{kj} ^{ij} - H _{ki} ^{ji}$$
with inverse transformation
\[ H_{jk}^i = \frac{1}{2}(g_{il} F_{mn}^j g_{km} + g_{jl} F_{ki}^m - g_{kl} F_{ji}^m). \]

Since
\[ H_{jk}^i = P_\mu(\partial_j P_k^\mu - \partial_k P_j^\mu) = P_\mu(\nabla_j P_k^\mu - \nabla_k P_j^\mu), \]
we get sequentially
\[ \delta L_t = F_{jk}^l \delta(P_l^\mu \nabla_j P_k^\mu) = F_{jk}^l (\nabla_j P_k^\mu) \delta P_l^\mu + F_{jk}^l P_l^\mu \nabla_j \delta P_l^\mu. \]  \hspace{1cm} (18)

With (5)
\[ \delta P_l^\mu = -P_l^\mu P_k^\mu \delta P_k. \]
By this, the second term in the right hand side of (18) can be presented in the following form
\[ \nabla_j(F_{jk}^l P_l^\mu \delta P_k^\mu) + P_k^\mu(\nabla_j F_{jk}^l + F_{jk}^l P_l^\nu \nabla_j P_k^\nu) \delta P_l^\mu. \]  \hspace{1cm} (19)

From (18) and (19) it follows that the variational principle provides the following equation for the potential of the torsion field
\[ P_k^\mu \nabla_j F_{jk}^l + F_{jk}^l \nabla_j P_k^\mu + F_{jk}^l P_k^\nu \nabla_j P_l^\mu = 0. \]  \hspace{1cm} (20)

It is possible to rewrite this equation in various forms. With (5) and (16) we have
\[ P_l^\nu \nabla_j P_k^\nu = -P_l^\nu \nabla_j P_k^\nu = -B_{jl}^m \nabla_j P_k^\nu = B_{jk}^m P_k^\mu \]
and hence (20) can be written in the following form
\[ \nabla_j F_{jk}^l + B_{jm}^k F_{jk}^m - B_{jl}^m F_{jk}^m = 0. \]  \hspace{1cm} (21)

Let \( \nabla_i \) be a covariant derivative with respect to the canonical connection (17). Since
\[ \nabla_j F_{jk}^l = \nabla_j F_{jk}^l - B_{jl}^i F_{jk}^i - B_{jm}^k F_{jk}^m + B_{jl}^m F_{jk}^m, \]
equations (21) can be presented as follows
\[ (\nabla_j - B_j) F_{jk}^l = 0, \]  \hspace{1cm} (22)
where \( B_i \) is a trace of the binary tensor field (16), \( B_i = B_{ki}^k \).
4 Interactions with known physical fields

In this chapter we shall consider interactions of the torsion field with known fundamental fields, the fermion, the gravitational and the electromagnetic.

Dirac’s Lagrangian has the form

\[ L^D = \frac{i}{2} P^i_\mu \left( \bar{\psi} \gamma^\mu D_i \psi - (D_i \bar{\psi}) \gamma^\mu \psi \right) - m \bar{\psi} \psi, \]  

(23)

where \( P^i_\mu \) is a potential of the torsion field and

\[ D_i \psi = (\partial_i - ie A_i) \psi, \quad D_i \bar{\psi} = (\partial_i + ie A_i) \bar{\psi} \]

as usually. Dirac’s Lagrangian is invariant with respect to the substitutions

\[ \psi \Rightarrow e^{i\lambda} \psi, \quad \bar{\psi} \Rightarrow e^{-i\lambda} \bar{\psi}, \quad A_i \Rightarrow A_i + \partial_i \lambda. \]

Action has the form

\[ A = \int L^D p \, d^4x, \]

where \( p = \text{Det}(P^i_\mu) \). Since

\[ P^i_\mu \partial_j P^j_\mu = \frac{1}{p} \partial_j p, \]

this action leads to the Dirac equations in the presence of external torsion and electromagnetic fields

\[ i P^i_\mu \gamma^\mu (D_i - \frac{1}{2} H_i) \psi = m \psi, \]  

(24)

\[ i P^i_\mu (D_i - \frac{1}{2} H_i) \bar{\psi} \gamma^\mu = -m \bar{\psi}, \]  

(25)

where \( H_i \) is a trace of the strength tensor of the torsion field \( H_i = H^k_{ki} \).

Setting

\[ W^\mu_i = i \left( \bar{\psi} \gamma^\mu D_i \psi - (D_i \bar{\psi}) \gamma^\mu \psi \right), \]

we have \( L^D = P^i_\mu W^\mu_i - m \bar{\psi} \psi \). Hence, from the action

\[ A = \int L^D p \, d^4x + \int L_t \sqrt{g} \, d^4x, \quad g = -\text{Det}(g_{ij}), \]

we derive (in accordance with (21) ) the following equation for the torsion field

\[ \nabla_j F^i_{jk} + B^k_{jm} F^j_{im} - B^m_{jl} F^j_{mk} + W^k_i = 0, \]  

(26)
where
\[ W^k_i = \epsilon P^k_\mu W^\mu_i, \quad \epsilon = p/\sqrt{g}. \]
The equations (24), (25) and (26) explain clearly how the torsion field interacts with the spinor field.

From the equation (26) an interesting relation can be derived. By summing over the indices k and l we get that a trace of \( H_{jk}^i \) satisfies the following equation
\[ \nabla_i H^i = m \bar{\psi} \psi, \tag{27} \]
where \( H^i = g_{ik} H_k \). We conclude that for \( m = 0 \) the interactions of the torsion and spinor fields are characterized by a new conserved quantity. Indeed, this fact simply means that the action is invariant under the mapping
\[ P^\mu_i \rightarrow a P^\mu_i, \quad \psi \rightarrow a^{-\frac{1}{2}} \psi, \]
where \( a \) is dimensionless constant.

Let \( L_M = -\frac{1}{4} F_{ij} F^{ij} \) be a standard Lagrangian of the electromagnetic field. From the action
\[ A = \int L_M \sqrt{g} d^4x + \int L_D p d^4x \]
we derive the equations of the electromagnetic field
\[ \nabla_i F^{ij} + e J^i = 0, \quad J^i = \epsilon P^i_\mu \bar{\psi} \gamma^\mu \psi. \tag{28} \]

We write the joint Lagrangian of the gravitational, torsion and electromagnetic fields
\[ L = \frac{1}{2} R + \frac{1}{2} H^k_i H^l_j g^{ij} - \frac{1}{4} F_{ij} F^{ij} = L_g + L_t + L_M. \tag{29} \]

Varying action
\[ A = \int L \sqrt{g} d^4x \]
with respect to \( g^{ij} \), we get Einstein’s equation
\[ G_{ij} = g_{ij} L_t - H^k_i H^l_j g^{ij} + F_{ik} F_{jl} g^{kl} + g_{ij} L_M \tag{30} \]
with the stress energy–momentum tensor of the torsion field
\[ T_{ij} = g_{ij} L_t - H^k_i H^l_j. \tag{31} \]
Notice that the trace of energy–momentum tensor (31) is not equal to zero.

Thus, the basic equations of the torsion field interacting with the known physical fields are derived. Now it is important to show that these equations are compatible. To this end let us establish the identities for the Lagrangians of the fields in question. For Dirac’s Lagrangian one can derive the identity

$$\partial_j L_D = D_j \bar{\psi} \frac{\delta L_D}{\delta \psi} - \frac{\delta L_D}{\delta \bar{\psi}} D_j \bar{\psi} + \frac{1}{\epsilon} (\nabla_i W^i_j - B^i_{jk} W^k_i + eF_{ji} J^i).$$

(32)

From (32) it follows that the circulation of the energy of the spinning matter is defined by the equation

$$\nabla_i W^i_j - B^i_{jk} W^k_i + eF_{ji} J^i = 0,$$

(33)

when the electromagnetic and torsion fields are present. The energy–momentum tensor $W^i_j$ of the spinning matter is not symmetric (some interesting details of this phenomenon and further references can be found in the review article [5]).

Identity for the Lagrangian of the torsion field may be written as follows

$$\partial_j L_t = \nabla_i S^i_j - B^i_{jk} S^k_j + \nabla^i (H^k_{il} H^l_{jk}),$$

(34)

where

$$S^i_j = \nabla_k F^{ki}_j + B^{i}_{kl} F^{kl}_j - D^l_{kj} F^{ki}_l = P^i_\mu \frac{\delta L_t}{\delta P^\mu_j}.$$ 

It is necessary to illuminate the important points under the derivation of the identity (34). We have

$$\partial_j L_t = F^{ik}_l (\nabla_j P^\mu_k) \nabla_i P^\mu_k + F^{ik}_l P^\mu_k \nabla_j \nabla_i P^\mu_k.$$ 

With Ricci’s identity

$$\nabla_j \nabla_i P^\mu_k = \nabla_i \nabla_j P^\mu_k - R^l_{jk} P^\mu_l$$

we can represent the second term in the right hand side of first relation in the following form

$$\nabla_i (F^{ik}_l P^\mu_k \nabla_j P^\mu_k) - (\nabla_i (F^{ik}_l P^\mu_k)) \nabla_j P^\mu_k - F^{ik}_l R^l_{jk}.$$ 

For the further transformations one needs to use identity

$$F^{ik}_l R^l_{jk} = \nabla_i \nabla_k F^{ik}_j.$$
and relations (5) and (16).

From Einstein’s equation (30) and identity (34) it follows that
\[
\nabla^i G_{ij} = \nabla_i S_i^j - B_{jk}^i S_i^k + F_{jk} \nabla_i F^{ik}.
\]

Since \( \nabla^i G_{ij} = 0 \) identically, the right hand side of the last equation should be equal to zero. From the equations (26), (28) (33) it follows that this is indeed the case. Thus, the theory of dynamical torsion is described by the compatible system of equations.

**Conclusion**

Here we suggest an experiment to test the formulated theory and to make choice between the alternative theories of the gravitational field. It is suggested to measure the gravitational acceleration of electrons and positrons in the Earth gravitational field. The motivation is as follows.

In 1967 Witteborn and Fairbank measured the net vertical component of gravitational force on electrons in vacuum enclosed by a copper tube [12]. This force was shown to be less than 0.09 mg, where \( m \) is the inertial mass of the electron and \( g \) is \( 980 \text{cm/sec}^2 \). They concluded that this result supports the contention that gravity induces an electric field outside a metal surface, of such magnitude and direction that the gravitational force on electrons is cancelled. If this is true, then the positrons will fall in this tube with the acceleration \( a = 2g \). The conclusion from the theory presented here is that electrons and positrons do not interact with the gravitational field directly but only through the torsion field. And the result presented by the measurements may be considered as an estimation for the energy of torsion field generated by electron (and positron). Thus, the measurements of the net vertical component of the force on positrons in vacuum enclosed by a copper tube will have the fundamental significance for understanding of the conceptual basis of contemporary theoretical physics.
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