FOUR OBSERVATIONS ON \( n \)-TRIVIALITY AND BRUNNIAN LINKS

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Abstract. Brunnian links have been known for a long time in knot theory, whereas the idea of \( n \)-triviality is a recent innovation. We illustrate the relationship between the two concepts with four short theorems.

In 1892, Brunn introduced some nontrivial links with the property that deleting any single component produces a trivial link. Such links are now called Brunnian links. (See Rolfsen [7]). Ohyama [5] introduced the idea of a link which can be independantly undone in \( n \) different ways. Here “undo” means to change some set of crossings to make the link trivial. “Independant” means that once you change the crossings in any one of the \( n \) sets, the link remains trivial no matter what you do to the other \( n-1 \) sets of crossings. Philosophically, the ideas are similar because, after all, once you delete one component of a Brunnian link the result is trivial no matter what you do to the other components. We shall prove four theorems that make the relationship between Brunnian links and \( n \)-triviality more precise.

We shall show (Theorem 1) that an \( n \)-component Brunnian link is \((n-1)\)-trivial; (Theorem 2) that an \( n \)-component Brunnian link with a homotopically trivial component is \( n \)-trivial; (Theorem 3) that an \((n-k)\)-component link constructed from an \( n \)-component Brunnian link by twisting along \( k \) components is \((n-1)\)-trivial; and (Theorem 4) that a knot is \((n-1)\)-trivial if and only if it is “locally \( n \)-Brunnian equivalent” to the unknot. At the end of the paper we sketch a proof of Theorem G, which generalizes Theorems 1–3.

The property of \( n \)-triviality is closely related to Vassiliev invariants. It is not hard to show that if a link (or braid, string link, knotted graph, etc) is \( n \)-trivial, then its Vassiliev invariants of order \(< n \) vanish. Also, it follows from the work of a number of different authors that a knot is \( n \)-trivial if and only if its Vassiliev invariants of order \(< n \) vanish. (See for example [8] or Habiro [2].)

A link will be a tame, oriented link in oriented \( S^3 \). We shall assume that the components of a link are ordered, although the choice of ordering is essentially irrelevant. Equivalence of links is up to the ambient isotopy, and we work with regular diagrams in the usual fashion. A link of \( n \) components is Brunnian if every \((n-1)\)-component

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sublink is trivial. If $L$ is an $n$-component link and $T \subset \{1, 2, \ldots, n\}$, then we denote by $L_T$ the sublink of $L$ obtained by deleting the components with indices in $T$. Thus an $n$-component link $L$ is Brunnian if and only if $L_T$ is trivial for all nonempty $T \subset \{1, 2, \ldots, n\}$.

Suppose a link $L$ has a diagram with $n$ disjoint sets of crossings $S_1, S_2, \ldots, S_n$. (The $n$ here is not necessarily related to the number of components.) If $T \subset \{1, 2, \ldots, n\}$, then we denote by $L(T)$ the link obtained from $L$ by changing all the crossings in $\bigcup_{i \in T} S_i$. The link is said to be $n$-trivial if it has such a diagram with $L(T)$ trivial for all $T \neq \emptyset$. Note that $n$-trivial implies $(n-1)$-trivial for $n > 0$. (In some of the literature, $n$-trivial is defined to be what we call here $(n+1)$-trivial.) Figure 1 shows two 2-trivial links, the Borromean Rings and the Whitehead Link. One possible way to choose the sets $S_i$ is indicated with letters “A” and “B”.

If $L$ is a link, then we denote the mirror image of $L$ by $\hat{L}$. If $T \subset \{1, 2, \ldots, n\}$, then we set $\overline{T} = \{1, 2, \ldots, n\} - T$.

**Figure 1**

**Theorem 1.** An $n$-component Brunnian link is $(n-1)$-trivial.

**Proof:** Take any diagram of an $n$-component Brunnian link $L$. For $1 \leq i \leq n$, let $S_i$ be the set of crossings where the bottom strand is on the $i$th component. (In the Borromean Rings in Figure 1, the crossings labeled “A” and “B” correspond to $S_1$ and $S_2$ chosen in this way.) We need to show that if $T \subset \{1, 2, \ldots, n-1\}$ and $T \neq \emptyset$ then $L(T)$ is trivial. In fact we will see that if $T$ is any proper subset of $\{1, 2, \ldots, n\}$, then $L(T)$ is trivial. Let $T \subset \{1, 2, \ldots, n\}$. In $L(T)$, a strand from component $i \in T$ will always pass over a strand from component $j \in \overline{T}$. Thus $L(T)$ is the disjoint union of an “upper” link and a “lower” link. The lower link is just $L_T$, since no crossings between components with indices in $\overline{T}$ are changed. $L_T$ is a trivial link because $T \neq \emptyset$. The upper link is $\hat{L}_T$, consisting of all components with indices in $T$, with all crossings changed. Since $L_{\overline{T}}$ is trivial (because $\overline{T} \neq \emptyset$), so is $\hat{L}_T$, and therefore so is $L(T)$. □

In particular, an $n$-component Brunnian link $L$ has trivial Vassiliev invariants of order $< n - 1$. This was shown by Kalfagianni and Lin [3] in the special case where $L$ is the plat-closure of a pure braid.
The proof of Theorem 1 suggests that an $n$-component Brunnian link is almost $n$-trivial, the only failure of the sets $S_1, S_2, \ldots S_n$ being when $T = \{1, 2, \ldots n\}$. We cannot hope for an $n$-component Brunnian link to be $n$-trivial in general—this can fail in fact when $n = 2$. Consider a two-component link with each component unknotted and a nonzero linking number between the two components. Such a link is Brunnian, but if it were 2-trivial then it would have vanishing Vassiliev invariants of order $< 2$, and it is well-known that the linking number between two components of a link is a Vassiliev invariant of order 1.

We can, however, add $S_n$ to the argument if we make an additional assumption. We shall say that a component $K$ of a link is homotopically trivial if there is a homotopy in the complement of $L - K$ taking $K$ to a trivial loop. In terms of link diagrams, this means that $L$ may be trivialized by Reidemeister moves and by allowing crossing changes of $K$ with itself. Of the two links in Figure 1, the components of the Whitehead Link are both homotopically trivial, whereas none of the components of the Borromean Rings are.

**Theorem 2.** If $L$ is an $n$-component Brunnian link with a homotopically trivial component, then $L$ is $n$-trivial.

**Proof:** Suppose without loss of generality that the $n$th component $K$ is homotopically trivial. Choose a diagram for $L$ such that there exists a set $R$ of crossings between strands of $K$ such that changing all of the crossings in $R$ trivializes $L$. Let $S_1, S_2, \ldots S_{n-1}$ be as before. Let $S_n$ be the set of all crossings where the bottom strand is on the $n$th component, minus the crossings in $R$. (In the Whitehead link in Figure 1, the crossings labeled “A” and “B” correspond to $S_1$ and $S_2$ chosen in this way.) Let $T \subset \{1, 2, \ldots n\}$ be nonempty. As before, $L(T)$ is now the disjoint union of a lower link and an upper link. The lower link is again $L_T$, and is always trivial (since $T \neq \emptyset$) but it may be empty. In the upper link, all crossings of $L_T$ are changed except those in $R$.

Therefore, the upper link is $\hat{L}_T$ with the crossings in $R$ changed, which is trivial for $T = \{1, 2, \ldots n\}$ by the choice of $R$. For any other $T$, the upper link will be a sublink of this trivial link, and will therefore be trivial. \hfill $\blacksquare$

Let $L$ be an $n$-component Brunnian link, and let $U$ be a proper subset of $\{1, 2, \ldots n\}$. Since $L_U$ is trivial, its components bound disjoint disks. Choose a framing (an integer) for each component of $L_U$. Let $L^U$ be obtained from $L_U$ by twisting along the disk of each component of $L_U$ according to its framing. Note that the components of $L^U$, as with $L_U$, are the components of $L$ whose indices are in $U$.

**Theorem 3.** Let $L$ be an $n$-component Brunnian link, and let $U \subset \{1, 2, \ldots n\}$. Let $L^U$ be obtained by twisting along the components of $L$ (using a fixed but arbitrary framing) whose indices are in $U$, as above. Then $L^U$ is $(n-1)$-trivial.

**Proof:** We may assume without loss of generality that $U = \{1, 2, \ldots k\}$ for some $k < n$. Choose a diagram for $L$ such that components $1, 2, \ldots n - 1$ are all disjoint circles. Arrange the diagram, moreover, so that around the $i$th component, $1 \leq i \leq k$, the diagram looks like the left-hand side of Figure 2. In general there will be an arbitrary number of strands from the $n$th component passing through, not just the three shown. Now we may draw a diagram for $L^U$ by replacing each local picture around the $i$th
component, $1 \leq i \leq k$, by the right-hand side of Figure 2. A single twist is shown, but there is a similar diagram for any integer number of twists. The point is that the twisted strands can always be drawn such that there is a set of crossings, like those marked with an “X” in Figure 2, such that changing those crossings undoes the effect of the twist. For $1 \leq i \leq k$, let $S_i$ be that set of crossings.

For $k < i < n$, let $S_i$ be the set of crossings where the lower strand is on the $i$th component, as before. Because of the way we have chosen the diagram, the top strands of these crossings will all be on the $n$th component. Now observe that for $1 \leq i < n$, changing the crossings in $S_i$ has the same effect as removing the $i$th component from $L$ (for $k < i < n$, “remove” means separate into a disjoint union) and then twisting along any components in $U$ which may be left. But once one component has been removed from $L$ it becomes trivial, and after twisting along or removing any other components, it is still trivial.

\[\text{Figure 2}\]

As an example of Theorem 3, it is not hard to see that twisting once along one component of the Borromean Rings produces the Whitehead Link. Both links are 2-trivial, as noted above.

There have been several notions of $n$th-order equivalence introduced for knots and links in the last few years. There are $n$-equivalence, $n$-similarity, $V_n$-equivalence, and others. All these notions are now known to be equivalent for knots. See Gusarov [1], Habiro [2], Ohyama [6], Ng and Stanford [4], and [8]. We will add yet one more characterization of this same idea in terms of Brunnian string links.

For each positive integer $n$, fix $n$ distinct, ordered points $x_1, x_2, \ldots x_n$ in the two-dimensional disk $I^2$. An $n$-component \textit{string link} is a proper, tame embedding $f_1, f_2, \ldots f_n$ of $n$ disjoint copies of the unit interval $I$ into $I^3 = I^2 \times I$ such that $f_i(0) = (x_i, 0)$ and $f_i(1) = (x_i, 1)$ for all $1 \leq i \leq n$. Equivalence is up to ambient isotopy, fixing the boundary. We work with regular diagrams, as with knots and links. An $n$-component string link $L$ is said to be \textit{Brunnian} if deleting any single component trivializes $L$ (i.e., makes it planar). The string link shown in Figure 3 is Brunnian.
Definition. We say two knots are \textit{locally $n$-Brunnian equivalent} if one can be obtained from the other by a sequence of local replacements of a trivial string link by an $n$-component Brunnian string link. We allow different string links at each each replacement.

Theorem 4. Two knots $K$ and $K'$ are locally $n$-Brunnian equivalent if and only if they are $(n-1)$-equivalent.

Proof: An easy modification of the proof of Theorem 1 shows that if $K'$ is obtained from $K$ by replacing an $n$-component trivial string link with a Brunnian string link, then the two knots are $(n-1)$-similar in Taniyama's sense (see Ohyama [5]). For the converse, observe that the $C_n$ moves and the $*^n$ moves defined by Habiro [2] are both examples of replacing a trivial string link with $(n+1)$ components by a Brunnian string link. □

Remark: Habiro’s results indicate that Theorem 4 is valid for string links, but that for links in general the various notions of $n$th-order equivalence diverge in ways that are not well-understood yet.

We finish with a general theorem of which Theorems 1–3 are special cases. First, we define a \textit{mixed link} to be a string link with some circle components added, as in Figure 4. We allow the cases of only string components (a string link) or only circle components (a link in the usual sense). Then we define \textit{coloring} as a generalization of ordering. (Our coloring has nothing to do with counting representations into finite groups.) An \textit{$n$-color link} is a link together with a color (a number between 1 and $n$) assigned to each component. We require that all $n$ colors be used. A color may have string components, or circle components, or both. An $n$-color link is \textit{Brunnian} if deleting the components of any single color produces a trivial link. It is easy to produce $n$-color Brunnian links which are not Brunnian in the usual sense. For example, take any $n$-component nontrivial Brunnian link $L$ and form $L'$ by replacing any component of $L$ by two parallel unlinked copies of itself. Then $L'$ is not Brunnian in the usual sense. It is, however,
$n$-color Brunnian if the two parallel components are given the same color and each other component is given its own color.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4.png}
\caption{Figure 4}
\end{figure}

Let $L$ be an $n$-color mixed link, and let $k \in \{1, 2, \ldots, n\}$. The color $k$ is said to be \textit{homotopically trivial} if there exists a homotopy of the components of color $k$ to the trivial link in the complement of all the remaining components. In terms of diagrams, this means that $L$ can be trivialized by Reidemeister moves and crossing changes between two strands both of color $k$. For example, if the Borromean Rings in Figure 1 are colored with two colors, then it is easy to see that the color with two components is homotopically trivial, whereas no single component of the link is homotopically trivial.

If $L$ is an $n$-color link and $T \subset \{1, 2, \ldots, n\}$, then we may define $L_U$ to be obtained from $L$ by deleting the components whose color is in $T$. If $U \subset \{1, 2, \ldots, n\}$, and if all the components with color in $U$ are circle components, then we may form $L^U$ as above by choosing a framing for each component with color in $U$ and twisting along each such component according to its chosen framing.

**Theorem G.** Let $L$ be an $n$-color mixed link. Let $U$ be a subset of $\{1, 2, \ldots, n\}$, possibly empty but not equal to $\{1, 2, \ldots, n\}$, such that every component with color in $U$ is a circle component. Let $L^U$ be obtained from $L$ by twisting along the components with colors in $U$ according to some fixed but arbitrary set of framings. Then $L^U$ is $(n-1)$-trivial. Moreover, if one color of $L$ not in $U$ is homotopically trivial, then $L^U$ is $n$-trivial.

The proof of Theorem G is mostly a matter of putting together the proofs of Theorems 1–3. If there is a homotopically trivial color, we may assume that it is $n$. Choose a diagram for $L$ which is planar on the sublink of the first $n-1$ colors. If the $n$th color is homotopically trivial, then choose the diagram so that there exists a set $R$ of crossings between strands of color $n$, such that changing all the crossings in $R$ trivializes $L$. Now draw the diagram for $L^U$ obtained by replacing a disk around each component of color $i$, $1 \leq i \leq k$, with a local twist (or with several twists) as in Figure 2. For $1 \leq i \leq k$, let $S_i$ be the set of crossings in $L^U$ which undoes the effects of the twists around all the components with color $i$. For $k < i < n$, let $S_i$ be the set of crossings whose bottom strand has color $i$. If the color $n$ is homotopically trivial, then let $S_n$ be the set of crossings whose bottom strand has color $n$, minus the crossings in $R$, minus any crossings introduced by the twists along the first $k$ components. Then one only has to check that all the appropriate links $L^U(T)$ are trivial, as in the previous proofs.

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