Dynamics of interacting bosons in a double-well potential

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Abstract – We present effects of an external driving on the tunneling dynamics of interacting bosons confined in a double-well potential. At large values of a periodic driving potential, the dynamics become chaotic, with a distinct difference between tuning of the amplitude or the phase of the driving term with regard to the route leading to chaos. For example, we find that a controlled increase in the amplitude of the driving term with a fixed phase leads to quasiperiodic route to chaos. While tuning of the frequency with a fixed (large) amplitude leads to a crisis-induced intermittency route to a chaotic dynamics, where the system intermittently visits different attractors in the phase space. Surprisingly, a chirp frequency superposed on a harmonic signal suppresses the chaotic motion, thereby yielding an orderly dynamics, pretty similar to a (damped) Rabi oscillation.

Periodically driven systems, for example ultracold atoms in a periodically shaken optical lattice have received unprecedented attention following the realization of exotic behaviour of such systems. A paradigmatic example of these systems is the kicked-top or kicked-rotor model where a particle moving on a ring is subjected to periodic kicks \cite{1}. The display of the transition from integrability to chaotic behaviour in certain limits \cite{2,3} and dynamical Anderson localization \cite{2,4}, dynamical stabilization both in classical and quantum mechanics, coherent control of the phase transition from superfluid to Mott insulator \cite{5}, and parameter-controlled adiabatic transformation of a static Bose-Einstein condensate (BEC) into a dynamical Floquet condensate \cite{6} are few prominent features of these systems. Hence they facilitate investigation of a particularly rich research area for the classical chaos and quantum chaos communities \cite{7}.

A substantial volume of literature exists in the field of driven BECs \cite{8,9}. Particularly, in the context of an optical lattice, a periodic driving can control the tunneling and interaction parameters, thereby providing the possibilities of manipulating the many-body states and quantum phases of the condensates \cite{10–12}. Such periodically driven systems have been experimentally realized to induce many-body localization \cite{13,14} and other exotic physical phenomena.

The double-well trap is especially a paradigm model to study the fundamental tunneling properties in atoms. The system of bosons in a double-well trap with driven interactions is identical to a two-site Bose Hubbard model (BHM) with time-varying interaction potential \cite{3,11}. At zero or low values of the strength of the interaction potential, the atoms execute Rabi oscillations. However, the scenario changes to a far from being Rabi-like, that is, a chaotic dynamics as the interaction strength is enhanced. Depending on the system and forcing strength there can be several routes to chaos, namely, quasiperiodicity \cite{15,16}, period doubling \cite{17}, intermittency \cite{18}, crisis-induced intermittency \cite{19}, and Ruelle-Takens-Newhouse \cite{20}.

Motivated by such exciting possibilities, in this paper, we consider a system of bosons in a double-well trap interacting via a harmonically driven interaction potential. We uncover distinct routes to chaotic dynamics using this setup. Furthermore, we show that an additional chirped-frequency drive leads to a systematic control of the chaoticity of the system. A chirp is essentially a sinusoidal signal whose phase changes instantaneously at each time step and its correlation properties resemble an impulse function. In fact, adding chirp into the signal is an efficient tool in echolocation systems, such as radars and sonars etc. On the experimental front, the tunability of the rate of collisions in ultracold Rb atoms induced by
near-resonant frequency chirp light, or the formation of these Rb molecules via photo-association technique have gathered momentum [21].

In this paper, we use the harmonic and the chirp disturbances as agents that induce or suppress chaotic fluctuations in the dynamics. We present different routes to chaos arising in the presence of different natures of the time-varying interaction potential. In general we notice two kinds of route to chaos depending on the nature of the forcing: 1) quasiperiodic route to chaos in the situation where, frequency of the harmonic modulation (ω) is kept fixed at a particular value and the interparticle interaction strength is varied, and 2) the crisis-induced intermittency route to chaos for the system where the interaction strength is kept fixed, while ω is varied within a certain range.

Equations of motion for the SU(2) generators. – We consider a system of interacting bosons in a double-well trap. In a two-mode approximation, the Bose-Hubbard Hamiltonian for a system of N interacting bosons occupying the weakly coupled lowest-energy states (thereby enforcing the energies involved in system dynamics to be negligible corresponding to the excitation energies [22]) can be written as [3]

\[ \hat{H} = \frac{\epsilon}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{\gamma}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) - \frac{c(t)}{2}(\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2), \]

(1)

where \( \hat{a}_1^\dagger \) (\( \hat{a}_2^\dagger \)) are the creation (annihilation) operators of bosons in the first (second) wells, \( \epsilon \) is the coupling between the modes (i.e., the tunneling parameter and \( \epsilon < 0 \) here), \( \gamma \) is the energy difference between the quantum states and \( c(t) \) is the driving term that disturbs the system which is assumed to be time-dependent in our case (elaborated below). For \( c(t) = 0 \), we recover the familiar BHM. The time-dependent interaction term has been studied earlier and periodic modulations for some or each of the terms in eq. (1) have received some attention as well. In contrast to the above, we consider a chirp modulation in the driving term, in addition to a periodic component which we denote by, \( c(t) = c_0 \cos(\omega t + \beta t^2) \), where \( c_0 \) is the amplitude of the disturbance, \( \omega \) and \( \beta \) denote harmonic and chirp modulations, respectively. Here, both the energy scales, namely \( \gamma \) and \( c_0 \) are measured in units of \( \epsilon \).

Clearly, the Hamiltonian in eq. (1) commutes with the total number of particles \( N(= \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2) \) and hence \( N \) is conserved. Further the Hamiltonian can be written in a symmetric fashion (upto a constant term depending on \( N \)) of the form

\[ \hat{H} = \frac{\epsilon}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1) + \frac{\gamma}{2}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) - \frac{c(t)}{4}(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2)^2. \]

(2)

To solve eq. (2), it is convenient to introduce the SU(2) generators, namely the components of the angular momentum, \( \hat{L}_x \), \( \hat{L}_y \) and \( \hat{L}_z \) as, \( \hat{L}_x = \frac{\epsilon}{2}(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1); \hat{L}_y = \frac{\gamma}{2}(\hat{a}_1^\dagger \hat{a}_2 - \hat{a}_2^\dagger \hat{a}_1); \hat{L}_z = \epsilon\gamma(\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) \) where the angular-momentum operators, \( \hat{L}_i \) obey the usual commutation relations, \( \{\hat{L}_i, \hat{L}_j\} = i\hbar \hat{L}_k \) (\( \hbar = 1 \)). With the above assumptions the Hamiltonian assumes the form

\[ \hat{H} = \epsilon \hat{L}_x + \gamma \hat{L}_z - c(t)\hat{L}_z^2, \]

(3)

The z-component, that is \( \hat{L}_z \) denotes the difference in population density between the two wells. The dynamics of the system is obtained by computing the equation of motion (EOM), \( \frac{d}{dt} \hat{L}_i = \frac{1}{\hbar}[\hat{L}_i, \hat{H}] \).

In order to solve the EOMs it is necessary to take the expectation values of the operators, \( \hat{L}_i \). However, even that has a difficulty as the expectation values of the first-order operators, namely \( \langle \hat{L}_i \rangle \) depend on the second-order correlations, such as, \( \langle \hat{L}_i \hat{L}_j \rangle \) [23], which in turn would depend on higher-order correlations, such as \( \langle \hat{L}_i \hat{L}_j \hat{L}_k \rangle \) and so on. This will eventually render BBGKY hierarchy of EOMs for the expectation values. To have a tractable closed set of equations, the hierarchy must be terminated somewhere.

The lowest-order truncation retains only the first-order correlations, \( \langle \hat{L}_i \rangle \). This enables defining a Bloch vector \( \vec{J} = (J_x, J_y, J_z) \) with the components, \( J_i \equiv \langle \hat{L}_i \rangle /N/2 \). The EOMs hence read as

\[ \frac{d}{dt} J_z = -\gamma J_y + 2\alpha(J_y J_z), \]
\[ \frac{d}{dt} J_y = \gamma J_z - \epsilon J_x - 2\alpha(J_x J_z), \]
\[ \frac{d}{dt} J_x = \epsilon J_y, \]

(4)

where the coupling \( \alpha = c(t)\frac{\hbar}{\gamma} = c_0\frac{\hbar}{\gamma}\cos(\omega t + \beta t^2) \equiv \kappa\cos(\omega t + \beta t^2) \). Here \( \kappa = c_0\frac{\hbar}{\gamma} \) is the modified interaction strength. Equations (4) denote the dynamics of the single-particle Bloch vector \( \vec{J} \) whose length \( |\vec{J}| \) is conserved and can be taken as unity. To solve eqs. (4) requires an initial condition, for which we can assume that at \( t = 0 \) the bosons populate both the wells equally, that is, half of them in each well, resulting in \( J_z = 0 \). Thus considering one of the possible initial condition as \( \vec{J} = (0,1,0) \). Hence the tunneling dynamics can be obtained by solving eqs. (4).

To re-validate that the first-order truncation procedure outlined above is indeed sufficient, we have computed the exact dynamics by solving the time-dependent Schrödinger equation with the Hamiltonian in eq. (1) in the Fock basis for a fixed (few) number of bosons. The exercise yields convincing evidence for a strikingly similar tunneling characteristics, including the emergence of chaotic dynamics that are emphasized and carefully analyzed in the next section. Since computing the exact dynamics for larger number of particles is computationally expensive, it is sensible to resort to the solution of the truncated EOMs for our purpose.
Results. - In order to have an intuitive idea about the dynamical behaviour of the system in a double-well potential, we systematically investigate the time evolution of population imbalance between the wells, \( J_\tau(t) \). To get a deeper understanding of the trajectory of the system and onset of chaotic behaviour therein, the power spectral density (PSD) is analyzed which is defined as,

\[
PSD = \frac{1}{2\pi N} |J_z(N, f, \tau)|^2,
\]

where \( J_z(N, f, \tau) \) is the discrete Fourier transform of the population imbalance, \( J_\tau(t) \) evaluated at \( t = k\tau \) (\( k = 0, 1, \ldots, N \) and \( N \) is the length of the discrete time series). Further the corresponding phase space projections of \( J_z(t + \tau) \) and \( J_\tau(t) \) are plotted to distinguish different kinds of dynamics as the function of the driving term, \( \alpha \), that emerge in a double-well potential.

Static interaction, \( \alpha = \kappa \). First we consider the case of static interaction without the presence of any time variation in \( \alpha \). In the absence of interaction, \( \kappa = 0 \), the usual Rabi oscillation is observed (fig. 1(a)) where the particles periodically move back and forth among the two wells [8].

As the interaction strength, \( \kappa \), is gradually increased one observes damped Rabi oscillations, which is evident from fig. 1(b)–(d) where the amplitudes of \( J_z(t) \) diminish indicating that only a fraction of the atoms move from one well to another [8].

In fig. 2 we show the PSD of the temporal evolution of the population dynamics. We find that without any interaction, the fundamental frequency of the system appears at \( f_1 \approx 1.6 \), which is benchmarked as the frequency of Rabi oscillations.

With the inclusion of the interaction term, the fundamental frequency gets shifted to larger values due to the change in the period of oscillations of the system (damped Rabi oscillations). For example, in fig. 2(b) \( f_1 \) gets shifted to 2.26 for \( \kappa = 1 \). In addition, we observe that for non-zero values of \( \kappa \), PSD exhibits two peaks instead of one. However, a closer look reveals that the second frequency present in the plot at 6.82 is simply a harmonic (or an overtone) of the fundamental frequency of the system, that is, \( f_2 \approx 6.82 \approx 3f_1 \). Similarly in fig. 2(c) and (d), respectively for \( \kappa = 4 \) and \( \kappa = 7 \), two peaks are observed at \( f_1 = 4 \) and \( f_2 = 12 \) and \( f_1 = 5.18 \) and \( f_2 = 15.53 \). Again they satisfy \( f_2 \approx 3f_1 \). Thus it is clear that upon increasing (only) the strength of the interaction potential (and no time variation yet), the dynamics of bosons in a double well involve only one frequency as it should be. This is also apparent from the dynamics presented in fig. 1.

Harmonic modulation. Now we include the harmonically driven term in the interparticle interaction to see the effect on the dynamics of the system (that is, \( \omega \neq 0 \) and \( \beta = 0 \), such that \( \alpha = \kappa \cos(\omega t) \)). Let us further divide this scenario into two cases and explore the dynamics: first by keeping \( \omega \) constant (say at \( \omega = 1 \)) and by varying \( \kappa \),...
and in the other, keeping $\kappa$ fixed at a certain value and varying $\omega$. We show that the two scenarios correspond to two different routes to chaos, namely the following.

i) Fixed phase, varying amplitude: In the first case we keep a fixed $\omega = 1$ and vary $\kappa$. With a gradual increase of the interaction strength, $\kappa$, it is observed that for $\kappa = 0.3$, the time evolved population imbalance, $J_z(t)$, shows the appearance of a dual periodicity (fig. 3(b)), that is, both slow and fast oscillations are present, a hallmark signature of periodically driven systems. One of the periodicities corresponds to the natural dynamics with the other denoting the driving frequency. The slow oscillations become more frequent with increase in values for $\kappa$ (see fig. 3(c) and (d)). However the observed dual periodicity in the behaviour of $J_z(t)$ gradually disappears at large values of $\kappa$. For very high values of $\kappa$ ($\kappa = 7$), it is observed that the behaviour of $J_z(t)$ becomes completely aperiodic in nature (fig. 3(f)). There is an onset of chaotic dynamics in the system. Hence $\kappa$ is fixed at this value, namely $\kappa = 7$ for subsequent discussion.

To probe deeper into the above scenario, we compute PSD and systematically analyze the presence of different frequencies therein. In fig. 4 we show PSD for the case when the frequency is fixed ($\omega = 1$) and $\kappa$ is varied in the range that is from $\kappa = 0$ to $\kappa = 7$. Without interaction, PSD peaks at the fundamental frequency $f_1 = 1.6$ (figs. 2(a) and 4(a)). For small and finite values of the interaction strength (for example, $\kappa = 0.3$), two peaks in PSD appears apart from $f_1$. One at $f \approx 1.6$ and another at $f \approx 0.05$ (marked in fig. 4(b)). These two frequencies are simply the harmonics of $f_1$. Upon further increase of $\kappa$, that is to $\kappa = 0.8$ results into the appearance of the peaks in PSD at frequencies $f_2 = 1.15$ and $f_1 = 1.6$. Since $f_1$ and $f_2$ are independent, this indicates the involvement of two frequencies of oscillations in the time evolution of $J_z(t)$, which is a signature of the quasiperiodic nature of the system. With further increase in the interaction strength, we find that more frequencies near $f_1$ and $f_2$ start getting populated. Finally, for very large interaction strength, that is, $\kappa = 7$, PSD shows an exponential decay [20] with fully populated pattern over the entire frequency range (fig. 4(f)). This is the distinguishing feature of the onset of chaos. Thus for $\kappa = 7$, the observed aperiodic time evolution of $J_z(t)$ displays chaotic behaviour for the tunneling dynamics (see fig. 3(f)). Hence we obtain a series of transitions, namely, from periodic to quasiperiodic and finally to chaotic dynamics as the strength of the harmonic interaction is enhanced. Thus fixing $\omega$ and varying $\kappa$ results in a quasiperiodic route to chaos.

Next we plot the phase space projection defined by $J_z(t + \tau)$ vs. $J_z(t)$ to establish the observed chaotic behaviour in the system. Without any interaction ($\kappa = 0$), the phase space projection shows an usual elliptical trajectory (Rabi) depicting periodic behaviour of the system (fig. 5(a)). While for the quasiperiodic regime (small, but finite $\kappa$), the elliptical trajectory is observed with distinct width (see fig. 5(b), (c)) denoting a quasiperiodic behaviour. Finally, for $\kappa = 7$, the phase space trajectory traces out a path around a multiple set of attractors that depict the onset of a chaotic behaviour (fig. 5(d)). Thus the phase plots corroborate the information obtained from the spectral densities.
ii) Fixed amplitude, varying phase: In the second case, we fix $\kappa = 7$ and subsequently vary $\omega$ between 0 and 1. In fig. 6 we show the temporal evolution of the population imbalance, $J_z(t)$ in this case. For very small values of $\omega$ (say $\omega = 0.002$), we find that initially $J_z(t)$ exhibits damped Rabi oscillations with a single frequency (fig. 6(a)). On further increase in $\omega$, say for $\omega = 0.008$ onwards, we observe the appearance of multiple intermediate states in $J_z(t)$. The system fluctuates about the intermediate states for some time before making transition to another states. The switching between the states happens to be intermittently (fig. 6(b)). This feature becomes more distinct with further increase in $\omega$ and occurs for smaller time spans (see fig. 6(c)). Finally at $\omega = 1$, the dynamics of $J_z(t)$ become completely aperiodic (see fig. 6(d)) as we have noted earlier. However the route to chaos appears to be different in this case. In what follows we will systematically analyze the appearance of chaos in the system by analyzing PSD for this case.

The scenario of approaching to a chaotic trajectory on varying $\omega$ becomes clearer as we analyze PSD. In fig. 7 we show PSD for $\kappa = 7$ and varying the frequency in the range between $\omega = 0.002$ and $\omega = 1$. We find that PSD gets populated in a random fashion from $\omega = 0.002$ onwards (fig. 7(a)) and this random occurrence of a large number of new frequencies becomes more prominent with further increase in $\omega$. Finally for $\omega = 1$ it leads to a chaotic behaviour which is quite evident from the exponential decay of PSD accompanied by the presence of a large number of frequencies (fig. 7(d)). The transition in the dynamics from regular to chaos happens due to the crisis-induced intermittency route to chaos.

Hence both the cases described above advocate chaotic scenario at $\kappa = 7$ and $\omega = 1$, however the routes to this transition to a chaotic regime are clearly different. To summarize, the first case suggests a quasiperiodic route to chaos, while the latter bears fingerprints of an intermittency route to chaos. These are among the central results of our paper, where a controlled variation of different parameters of a driven system renders different pathways leading to chaos.

A more lucid understanding of the above scenario can be offered in the following sense. Ramping up the amplitude ($\kappa$) of the driving term leaving the angular frequency ($\omega$) all the while to have a constant value indicates a systematic way of perturbing the system. The dual, that is, slow and fast oscillations are preserved. Thus the chaos induced in this case is distinct from the one in which the amplitude is kept fixed and $\omega$ is varied. The latter has different ramifications as the frequency associated with the driving term is constantly varied, however the natural frequency of the tunneling from one well to another remains the same. At large driving frequencies (here $\omega = 1$), a chaotic dynamics emerges which is distinct from the one noted in the previous case.

Finally, we study the phase space projection of the population imbalance, $J_z(t)$. For small value of $\omega$ ($\omega = 0.001$),
observed with gradual increase in $\beta$ at this value of the usual elliptical orbit (fig. 8(a)) as the dynamics re-structured the phase space trajectory traces out $\omega$ into torus (quasiperiodic) at $\kappa = 7$. Transition from aperiodic to periodic behaviour is observed with gradual increase in $\beta$.

Fig. 9: (Colour online) $J_z(t)$ for $\omega \neq 0$ and $\beta \neq 0$ with $|\epsilon| = 1$ and $\gamma = 0$. Here we keep $\omega = 1$ and varying $\beta$ gradually for $\kappa = 7$. Transition from aperiodic to periodic trajectory is observed at $\beta = 0$ with $\gamma = 0$ and $\kappa = 7$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$. The orbit is completely chaotic for $\beta = 0.001$ which turns into a periodic one at $\beta = 0.3$.

It is observed that the phase space trajectory traces out the usual elliptical orbit (fig. 8(a)) as the dynamics remain periodic at this value of $\omega$. The regular orbits turns into torus (quasiperiodic) at $\omega = 0.008$ as shown in the fig. 8(b). In fig. 8(c) we find that for $\omega = 0.05$, the system intermittently keeps switching between the four tori (marked as A, B, C, D) which is also termed as crisis-induced intermittency [19]. At large frequency, namely, $\omega = 1$ the orbits turns out to be completely chaotic. Here it appears that the system remains around one of the orbit (torus) for sometime and further makes transitions to others in an intermittent way. This frequent visits to the different intermediate states lead to the chaotic behaviour in the system.

A chirp modulation ($\beta \neq 0$). We now superpose a chirp signal on the harmonic interparticle interaction and investigate what effects it brings on the tunneling dynamics. We continue with $\kappa = 7$ and $\omega = 1$. Now we vary $\beta$ (in units of the square of the energy scale) within a certain range. For very small value of $\beta$ (namely, $\beta = 0.001$), the time evolution of the population imbalance, $J_z(t)$, shows periodic motion that is chaotic in nature (similar to fig. 3(f)). As we gradually increase $\beta$, periodicity in the evolution of $J_z(t)$ is restored partially as seen in fig. 9(b) and (c) for $\beta = 0.009$ and 0.04, respectively. With further increase in $\beta$, for example, $\beta \sim 0.3$, the dynamics of $J_z$ becomes periodic in nature (shows damped Rabi oscillation similar to the case shown in fig. 1(b)–(d)), which is clear from fig. 9(d).

The above feature suggests that the addition of a small amount of chirp signal in the forcing term suppresses the chaotic fluctuations in the tunneling dynamics where the latter was generated by the harmonic term in the driving.

The PSD yields deeper insights into the observed aperiodic-to-periodic transition in the presence of a chirp modulation. For $\beta = 0.001$, PSD shows exponential decay with a large number of frequencies being present, which culminates into a chaotic behaviour (fig. 10(a)). With further increase in $\beta$, say for $\beta = 0.009$, the development of one prominent peak is observed at a frequency $f_1 = 1.98$, along with some fluctuations hinting towards the emergence of an orderly motion. At $\beta = 0.04$, the fluctuations gradually subside, and the prominent frequency shifts to a value $f_1 \sim 1.6$. It may be noted that this is the same frequency observed corresponding to the case of no interaction (similar to fig. 2(a)). Finally, for $\beta = 0.3$, PSD shows a distinct peak precisely at $f_1 = 1.6$ that indicates the dynamics to be orderly in nature that is the atoms execute (damped) Rabi-like oscillations. Thus the chirp modulation brings back the periodic nature to the tunneling dynamics by suppressing the observed chaotic behaviour at large interaction strengths.

A physical feel for the above result can be obtained in the following sense. Since the chirp term involves $\beta t^2$, it oscillates faster than the harmonic term and hence the

Fig. 10: PSD for $\omega \neq 0$ ($\omega = 1$), $\beta \neq 0$ (varying $\beta$), $|\epsilon| = 1$ and $\gamma = 0$ with $\kappa = 7$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$. (a) $\beta = 0.001$, (b) $\beta = 0.009$, (c) $\beta = 0.04$ and (d) $\beta = 0.3$.
system feels as if being subjected to a constant force owing to the fact that the chirp oscillations are significantly faster than the natural frequency of the motion of the particles between the two wells.

To confirm the re-entrant of periodic behaviour, we also study the phase space projection of the population imbalance $J_z(t)$. It is observed that for small value of $\beta$ ($\beta = 0.001$), the phase space orbit appears to be completely chaotic in nature as shown in fig. 11(a). However, the orbit takes the shape of a toroid for $\beta = 0.04$ (fig. 11(c)). Finally, further increase in $\beta$ ($\beta = 0.3$) leads to the emergence of almost periodic orbit rendering to the onset of an orderly motion as shown in fig. 11(d). Similar effects of chirp modulation in frequency on the dynamics of collisions of Rb atoms has been reported in ref. [21].

Conclusions. – We have investigated the tunneling dynamics of a system of bosons in a double-well potential in the presence of a harmonic interaction potential. At large values of the interaction strength, the system makes a transition to a chaotic state. This feature is complemented very well by investigating PSD and phase space projection. In particular, for the case of harmonic interaction potential, different routes to chaos emerge depending on whether the amplitude or the phase of the interaction term is being tuned. In the former case, we observed that the route to chaos is quasiperiodic, while, for the latter case, the route to chaos is through crisis-induced intermittency. Finally we demonstrated that the superposition of a chirp modulation to the harmonic driving term restores periodicity in the tunneling dynamics of bosons in a double-well potential.

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