Correspondence, Canonicity, and Model Theory for Monotonic Modal Logics

Abstract. We investigate the role of coalgebraic predicate logic, a logic for neighborhood frames first proposed by Chang, in the study of monotonic modal logics. We prove analogues of the Goldblatt-Thomason Theorem and Fine’s Canonicity Theorem for classes of monotonic neighborhood frames closed under elementary equivalence in coalgebraic predicate logic. The elementary equivalence here can be relativized to the classes of monotonic, quasi-filter, augmented quasi-filter, filter, or augmented filter neighborhood frames, respectively. The original, Kripke-semantic versions of the theorems follow as a special case concerning the classes of augmented filter neighborhood frames.

Keywords: modal logic, canonicity, Fine’s theorem, Goldblatt-Thomason theorem, neighborhood frames

1. Introduction

Monotonic modal logics generalize normal modal logics by dropping the K axiom $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ and instead requiring only that $\vdash \phi \rightarrow \psi$ imply $\vdash \Box \phi \rightarrow \Box \psi$. There are a number of reasons for relaxing the axioms of normal modal logics and considering monotonic modal logics. For instance, monotonic modal logics are considered more appropriate to describe the ability of agents or systems to make certain propositions true in the context of games and open systems [21, 22, 1]. The standard semantics for monotonic modal logics is provided by monotonic neighborhood frames (see, e.g., [13]).

Just as the first-order language with a relation symbol is a useful correspondence language for Kripke frames, it is natural to consider what would be a useful correspondence language for monotonic neighborhood frames. Litak et al. [16] studied coalgebraic predicate logic (CPL) as a logic that plays that role and proved a characterization theorem in the style of van Benthem and Rosen [23]. In this article, we continue that path for monotonic neighborhood frames and prove variants of the Goldblatt-Thomason theorem [12] and the Fine canonicity theorem [6] in the setting of coalgebraic predicate logic.
We will deal with a relativized notion of CPL-elementarity, relativized to subclasses of the class of monotonic neighborhood frames. There are several important subclasses to consider: the class of filter neighborhood frames, providing a more general semantics [11, 10] for normal modal logics than relational semantics; the class of quasi-filter neighborhood frames, providing a semantics for regular modal logics; the class of augmented quasi-filter neighborhood frames, providing a less general semantics for regular modal logics; and the class of augmented filter neighborhood frames, which are Kripke frames in disguise [4, 18].

| Subclass                | Closed under ...                                                                 |
|-------------------------|----------------------------------------------------------------------------------|
| monotonic               | supersets                                                                        |
| quasi-filter            | superset, intersections of nonempty finite families of neighborhoods             |
| augmented quasi-filter  | superset, intersections of nonempty families of neighborhoods                    |
| filter                  | superset, intersections of finite families of neighborhoods                       |
| augmented filter        | superset, intersections of families of neighborhoods                             |

Table 1. Classes of monotonic neighborhood frames and their definitions

The analogue of the Goldblatt-Thomason theorem in this article is that a class of monotonic neighborhood frames closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 1 is modally definable if and only if it is closed under disjoint unions, bounded morphic images, and generated subframes, and it reflects ultrafilter extensions; and the analogue of Fine’s theorem we will prove states that a sufficient condition for the canonicity of a monotonic modal logic is that it is complete with respect to the class of monotonic neighborhood frames it defines and that that class is closed under CPL-elementarity relative to any of the classes of neighborhood frames in Table 1.

The relevance of coalgebraic predicate logic in this article is that many monotonic modal logics define classes of monotonic neighborhood frames that are CPL-elementary. For instance, the monotonic modal logics axiomatized by formulas of the form

\[(\text{purely propositional positive formula}) \rightarrow \langle \text{positive formula} \rangle \quad (1)\]

are determined by CPL-elementary classes of monotonic neighborhood frames (see Remark 2.5). In addition, relative to the class of augmented quasi-filter frames, all monotonic modal logics axiomatized by Sahlqvist formulas
are CPL-elementarily determined (see Example 2.6). Further discussion regarding the relevance of this language in the context of Fine’s theorem is in Remark 4.5.

Since augmented filter frames are Kripke frames in disguise (see also Example 2.6), our result regarding classes elementary relative to the class of augmented filter frames generalizes the original, Kripke-semantic Goldblatt-Thomason theorem. Also, our Goldblatt-Thomason theorem concerns elementary classes like the original theorem, whereas some existing Goldblatt-Thomason theorems such as [14] or [15] deal with classes closed under ultrafilter extensions.

The article is organized as follows. In § 2, we recall standard concepts in the semantics of monotonic modal logic and introduce the language for neighborhood frames. In § 3, we give an overview of the model theory of neighborhood frames for this language. We also define a two-sorted first-order language (Definition 3.12) and a translation of coalgebraic predicate logic into it (Proposition 3.14), which are used later to explain the existence of ℵ₀-saturated models of languages of coalgebraic predicate logic (Proposition 3.17). In § 4, we prove the main lemmas of this article. In § 5, we give the applications of the main lemmas, which are analogues of the Goldblatt-Thomason Theorem and Fine’s Canonicity Theorem.

The presentation of the results in this article does not presuppose the reader’s prior knowledge of coalgebras or coalgebraic predicate logic.

2. Preliminaries

2.1. Languages and structures

In this subsection, we recall standard definitions in neighborhood semantics of modal logic and the language coalgebraic predicate logic introduced in [3] and [16] to describe neighborhood frames.

We define languages of coalgebraic predicate logic relative to sets of nonlogical symbols here; this is so that we can use expansions of the smallest language in proofs in § 4.

Definition 2.1.

(i) Let σ be the set of atomic formulas of some language of first-order logic. The language of coalgebraic predicate logic L based on σ is the least set of formulas containing σ and closed under Boolean combinations, existential quantification, and formation of formulas of the form $x \Box[y; \phi]$, where $\phi \in L$, and $x$ and $y$ are variables. To save space, we
sometimes write \(x \Box y \phi\) or even \(x \Box \Box y : \phi\). For a language \(L_0\) of first-order logic, the *language of coalgebraic predicate logic based on \(L_0\)* is defined to be the language of coalgebraic predicate logic based on the set of atomic formulas of \(L_0\). We write \(L_=\) for the language of coalgebraic predicate logic based on the empty language, i.e., the language with just the equality symbol.

(ii) Let \(L_0\) be a language of first-order logic and \(L\) the language of coalgebraic predicate logic based on \(L_0\). An \(L\)-structure \(F = (F,N_F)\) is an \(L_0\)-structure \(F\) with an additional datum \(N^F : F \to \mathcal{P}(\mathcal{P}(F))\), where \(\mathcal{P}\) is the powerset operation. The map \(N^F\) is called the *neighborhood function* of \(F\). A set \(U \in N^F(w)\) is called a *neighborhood* of \(w\). If \(L_0\) is the empty first-order language, the \(L\)-structures are exactly the *neighborhood frames*.

(iii) A neighborhood frame \(F\) is *monotonic* if for every \(w \in F\) the family \(N^F(w)\) is closed under supersets. \(F\) is a *quasi-filter* neighborhood frame if for every \(w \in F\) the family \(N^F(w)\) is closed under intersections of nonempty finite families of neighborhoods. \(F\) is a *filter* neighborhood frame if it is a quasi-filter frame and for every \(w \in F\) the family \(N^F(w)\) is nonempty. \(F\) is an *augmented quasi-filter* neighborhood frame if for every \(w \in F\) the family \(N^F(w)\) is either empty or a *principal upset* in the Boolean algebra \(\mathcal{P}(F)\), i.e., there exists \(U_0 \subseteq F\) such that \(U \in N^F(w) \iff U_0 \subseteq U\). Finally, \(F\) is an *augmented filter* neighborhood frame if for every \(w \in F\) the family \(N^F(w)\) is a principal upset.

One might object to calling the language in Definition 2.1(i) the language of “coalgebraic predicate logic” since this is essentially what Chang introduced in [3] whereas the language in Litak et al. [16] is applicable to general coalgebras. We use the name “coalgebraic predicate logic” in this article because Chang’s language does not have a name, and technically speaking we do not use Chang’s syntax, which imposes a more strict rule regarding variables bound by modal operators.

**Example 2.2.** For a topological space \(X = (X,\tau)\), we associate a neighborhood frame \(X^* = (X,N)\) defined by

\[U \in N(w) \iff w \in U^\circ,\]

where \(^\circ\) denotes topological interior. We call a monotonic neighborhood frame of the form \(X^*\) a *topological neighborhood frame*. Recall the satisfaction
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predicate $\models_{\text{top}}$ for topological semantics and the satisfaction predicate $\models_{\text{nbhd}}$ for neighborhood semantics (see, e.g., [4, 27] for more details):

$$M, w \models_{\text{top}} \Box \phi \iff w \in \{ w' \mid M, w' \models_{\text{top}} \phi \}^\circ$$

and

$$M', w \models_{\text{nbhd}} \Box \phi \iff \{ w' \mid M, w' \models_{\text{nbhd}} \phi \} \in N(w),$$

where $M$ is a topological model, $M'$ is a neighborhood model, and $N$ is the neighborhood function of the neighborhood frame of $M'$. It is then easy to see that for every $w \in X$, every modal formula $\phi$, every topological model $M$ based on $X$, and every neighborhood model $N$ based on $X^*$, if the valuations of $M$ and $N$ are the same, then

$$M, w \models_{\text{top}} \phi \iff N, w \models_{\text{nbhd}} \phi.$$
for a monotonic neighborhood frame $F$ and $w \in F$. By the monotonicity of $F$, the usual minimum valuation argument (see, e.g., [2]) applies: the B axiom is valid at $w$ if and only if its consequent is true under the minimum valuation that makes its antecedent true, which is the valuation that sends $p$ to the set $\{w\}$. The latter condition is expressible by a formula in $L_\geq$:

$$w \Box_y (\neg y \Box z \neq w).$$

**Remark 2.5.** It can be shown likewise that modal formulas of the form (1) have frame correspondents relative to the class of monotonic neighborhood frames. A formula of the form (1) is what is called a KW formula in [13] and axiomatizes a monotonic modal logic complete with respect to the class of monotonic neighborhood frames that it defines. Hence, the monotonic modal logics axiomatized by such formulas are determined by CPL-elementary classes (see Definition 4.1) of monotonic neighborhood frames.

**Example 2.6.** Consider the 4 axiom $\Box p \to \Box \Box p$. We show that this modal formula has a local frame correspondent relative to the class of augmented quasi-filter neighborhood frames in the same language $L_\geq$ as above. Consider the validity of the 4 axiom for an augmented quasi-filter neighborhood frame $F$ and $w \in F$. If $w \in F$ is impossible, i.e., $N^F(w) = \emptyset$, then the 4 axiom is valid at $w$. Note that by monotonicity $w$ is impossible if and only if $F \notin N^F(w)$, i.e., $F \models \neg w \Box y y = y$. Otherwise, we can again use the minimum valuation argument. Here, the minimum interpretation of $p$ that makes the antecedent true is $R[w]$ because $F$ is an augmented quasi-filter neighborhood frame, where $R \subseteq F \times F$ is the binary relation defined by

$$xRy \iff \{z \in F \mid z \neq y\} \notin N^F(x) \iff F \models \neg x \Box z z \neq y.$$  

(2)

To summarize, the 4 axiom has the local frame correspondent

$$\neg w \Box[y: y = y] \lor (w \Box[y: y = y] \land w \Box[y_1: y_1 \Box[y_2: \neg y_2 \Box[z: z \neq w]]]).$$

In fact, since the accessibility relation $R$ and the set of impossible worlds are definable in $L_\geq$ as we have seen above, the first-order frame correspondence language in [20] translates into $L_\geq$, and thus all Sahlqvist formulas have frame correspondents in $L_\geq$ relative to the class of augmented quasi-filter neighborhood frames.

The displayed formula (2) can be used to define the class of augmented quasi-filter neighborhood frames by coalgebraic predicate logic as well. Write $R[w]$ for the set of $y \in F$ satisfying (2) for an arbitrary monotonic neighborhood frame $F$ and $x \in F$. We see that a monotonic neighborhood frame
$F$ is augmented quasi-filter if and only if either $N^F(w)$ is impossible, or $R[w] \subseteq N^F(w)$ for every $w \in F$, i.e., $F$ satisfies the $L_\square$-sentence

$$\forall x[(x \square y) = y \rightarrow x \square y \neg(x \square z z \neq y)].$$

Indeed, we have seen the “only if” direction in the last paragraph; to see the “if” direction, observe that $R[w] = \bigcap N^F(w)$.

Example 2.7. Recall that for a topological space $X$ the specialization preorder of $X$ is the preorder $\leq$ on $X$ defined by

$$x \leq y \iff x \in \overline{\{y\}},$$

where $\overline{\{\}}$ denotes topological closure. A space $X$ is $T_0$ if and only if $\leq$ is a partial order, and $X$ is $T_1$ if and only if $\leq$ is a discrete partial order. Note that the specialization preorder of a topological space $X$ is “definable” in coalgebraic predicate logic in the sense that $x \leq y \iff X^* \models \neg x \square z z \neq y$. (3)

Hence, the images under $*$ of the classes of $T_0$ and $T_1$ spaces are CPL-elementary relative to the class of topological neighborhood frames: $X$ is $T_0$ if and only if $X^* \models \forall z \forall w(z \leq w \land w \leq z \rightarrow w = z)$, and $X$ is $T_1$ if and only if $X^* \models \forall z \forall w(z \leq w \rightarrow w = z)$, where $x \leq y$ abbreviates the formula of coalgebraic predicate logic on the right-hand side of the displayed formula (3).

Definition 2.8. Let $F$ and $F'$ be neighborhood frames. A function $f : F \rightarrow F'$ is a bounded morphism if for each $w \in F$:

$$f^{-1}(U') \in N^F(w) \implies U' \in N^{F'}(f(w)) \quad \text{("forth")}$$

and

$$U' \in N^{F'}(f(w)) \implies f^{-1}(U') \in N^F(w). \quad \text{("back")}$$

Lemma 2.9 ([5]). Let $F$ and $F'$ be monotonic neighborhood frames and $f : F \rightarrow F'$ be a function that satisfies the “forth” condition. Suppose in addition that for all $U' \in N^{F'}(f(w))$ there exists $U \in N^F(w)$ such that $f(U) \subseteq U'$. Then $f$ is a bounded morphism.

Proof. By assumption, if $U' \in N^{F'}(w)$, then there exists $U$ such that $f^{-1}(U') \supseteq U \in N^G(w)$; by monotonicity, we have $f^{-1}(U') \in N^G(w)$. □

Note that bounded morphisms between monotonic neighborhood frames clearly satisfy the assumption of this lemma.
2.2. Algebraic concepts

In this subsection, we recall some standard definitions from the algebraic treatment of modal logic; for the standard notions that we do not define here, see [28].

First, we recall basic definitions regarding the algebraic treatment of monotonic modal logic.

**Definition 2.10.** A monotonic Boolean algebra expansion (BAM for short) \( A = (A, \square^A) \) is a Boolean algebra \( A \) with an additional datum \( \square^A : A \to A \), a function that is monotonic, i.e., for all \( a, b \in A \) we have \( a \leq b \implies \square^A(a) \leq \square^A(b) \).

**Lemma 2.11.** Let \( F \) be a monotonic neighborhood frame. The function \( \square^F : \mathcal{P}(F) \to \mathcal{P}(F) \) defined by

\[
X \mapsto \{ w \in F \mid X \in N^F(w) \}
\]

is monotonic.

**Definition 2.12 ([5]).** The complex algebra \( F^+ \) of a monotonic neighborhood frame \( F \) is the BAM \( (\mathcal{P}(F), \square^F) \), where \( \mathcal{P}(F) \) is the Boolean algebra of the powerset of \( F \).

**Proposition 2.13.** Let \( F \) and \( F' \) be monotonic neighborhood frames. A function \( f : F \to F' \) is a bounded morphism if and only if \( f^+ : F^+ \to F^+ \) defined by \( f^+(X) = f^{-1}(X) \) is a homomorphism.

Since this article concerns canonicity, we need to recall definitions regarding canonical extensions.

**Definition 2.14.** Let \( B \) be a Boolean algebra. The canonical extension \( B^\sigma \) of \( B \) is the Boolean algebra of the powerset of the set \( \text{Uf}(B) \) of ultrafilters in \( B \). An element of \( B^\sigma \) of the form \( [a] := \{ u \in \text{Uf}(B) \mid a \in u \} \) for a fixed \( a \in B \) is called clopen. Meets and joins of clopen elements of \( B^\sigma \) are closed and open, respectively. The sets of closed and open elements of \( B^\sigma \) are denoted \( K(B^\sigma) \) and \( O(B^\sigma) \), respectively.

**Proposition 2.15.** For a Boolean algebra \( B \), the map \( [-] : B \to B^\sigma \) is an embedding.

**Proof.** See, e.g., [28].

**Definition 2.16 (see, e.g., [28]).**
(i) Let $A = (A, \Box)$ be a BAM. The canonical extension $A^\sigma = (A^\sigma, \Box^\sigma)$ of $A$ is the canonical extension of the Boolean algebra $A$ expanded by the function $\Box^\sigma$, where
\[
\Box^\sigma(u) = \bigvee_{u \supseteq K(A^\sigma)} \bigwedge_{x \subseteq a \in A} \Box(a).
\]

(ii) A set $\Delta$ of modal formulas is canonical if for every BAM $A \models \Delta$ we have $A^\sigma \models \Delta$.

**Proposition 2.17.** For a BAM $A = (A, \Box)$, the function $\Box^\sigma$ is monotonic, and thus the canonical extension $A^\sigma = (A^\sigma, \Box^\sigma)$ is again a BAM.

**Proof.** See, e.g., [28].

**Remark 2.18.** Canonical extensions can be defined for larger classes of algebras. We stick to BAMs in this article since they admit the most natural definition for $\Box^\sigma$, among other reasons.

**Definition 2.19 ([13]).**

(i) Let $A$ be a BAM. The ultrafilter frame of $A$ is a neighborhood frame $(Uf(A), N^\sigma)$ with $N^\sigma$ defined by
\[
U \in N^\sigma(u) \iff \exists K \subseteq U \forall a \in A([a] \supseteq K \Rightarrow \Box(a) \in u), \tag{4}
\]
where $u \in Uf(A)$, and $K$ ranges over closed elements of $A^\sigma = \mathcal{P}(UfA)$. We denote the ultrafilter frame of $A$ by $Uf(A)$.

(ii) Let $F$ be a monotonic neighborhood frame. The ultrafilter extension $Uf^\sigma F$ of $F$ is $Uf(F^+)$. 

**Proposition 2.20.** Let $A$ be a BAM.

(i) $Uf(A)$ is monotonic.

(ii) $(Uf(A))^+ = A^\sigma$.

Finally, we define a few notions necessary to state our Goldblatt-Thomason theorem.

**Definition 2.21 ([13]).** For a disjoint family $((F_i, N^i) \mid i \in I)$ of monotonic neighborhood frames, the disjoint union of the family is $(F, N)$, where $F = \bigsqcup_i F_i$ and $N$ is a neighborhood function defined by $U \in N(w) \iff U \cap F_i \in N^i(w)$. A monotonic neighborhood frame $F$ is a bounded morphic image of another $F'$ if there is a surjective bounded morphism $F' \to F$. A monotonic neighborhood frame $F$ is a generated subframe of another $F'$ if $F \subseteq F'$, and the inclusion map $F \hookrightarrow F'$ is a bounded morphism.
3. Model theory of neighborhood frames

In this section, we recall as well as develop results in the model theory of neighborhood frames and coalgebraic predicate logic.

3.1. Standard concepts in first-order model theory

Here, we define concepts that have counterparts in classical first-order model theory.

Definition 3.1. Let $L$ be a language of coalgebraic predicate logic, $F$ an $L$-structure, and $A \subseteq F$. A subset $X \subseteq F$ is $A$-definable in $F$ if there is an $L$-formula $\phi(x; \bar{y})$ and a tuple $\bar{a}$ of elements of $A$ (notation: $\bar{a} \in A$) such that $X = \phi(F; \bar{a})$. A subset $X$ is definable in $F$ if it is $F$-definable in $F$.

Definition 3.2.

(i) A set of $L$-sentences is called an $L$-theory.

(ii) Let $L$ be a language of coalgebraic predicate logic and $F$ be an $L$-structure. The full $L$-theory $\text{Th}_L(F)$ of $F$ is the set of $L$-sentences $\phi$ such that $F \models \phi$.

(iii) Two $L$-structures $F, F'$ are $L$-elementarily equivalent, or $F \equiv_L F'$, if $\text{Th}_L(F) = \text{Th}_L(F')$.

For the rest of this section, we fix a language $L$ of coalgebraic predicate logic and a monotonic $L$-structure $F$. We also let $T = \text{Th}_L(F)$.

Definition 3.3. Let $A \subseteq F$. We write $\text{Def}(F/A)$ for the Boolean algebra of $A$-definable subset in $F$, its operations being the set-theoretic ones. We also think of $\text{Def}(F/A)$ as a BAM whose monotone operation $\Box$ is defined by

$$\Box(\phi(F)) = (\Box \phi)(F)$$

for an $L(A)$-formula $\phi(x)$, where $(\Box \phi)(x)$ is the $L$-formula $x \Box_y \phi(y)$. (It is easy to see that $\Box : \text{Def}(F/A) \to \text{Def}(F/A)$ is well defined here. This is true of similar definitions that appear in later parts of the article.)

It is easy to see that $\text{Def}(F/A)$ is a subalgebra of $F^+$ as a BAM.

Proposition 3.4. Assume $F' \models T$. Then $\text{Def}(F/\emptyset)$ and $\text{Def}(F'/\emptyset)$ are isomorphic as BAMs.

Types play an important rôle in the proof of the original theorem of Fine as well as in this article.
Definition 3.5.

(i) The Stone space $S_1(T)$ of 1-types over $\emptyset$ for $T$ is the ultrafilter frame $\text{Uf}(\text{Def}(F/\emptyset))$ of $\text{Def}(F/\emptyset)$. (Note that if $F'$ is such that $\text{Th}_L(F') = T$, then $\text{Uf}(\text{Def}(F'/\emptyset)) = \text{Uf}(\text{Def}(F/\emptyset))$ and that $S_1(T)$ is, therefore, defined uniquely regardless of the choice of $F \models T$.) We consider $S_1(T)$ as a topological space whose open subsets are exactly the open elements of $(\text{Uf}(\text{Def}(F/\emptyset)))^+ = (\text{Def}(F/\emptyset))^\sigma$. An element $p \in S_1(T)$ is called a 1-type over $\emptyset$.

(ii) Likewise, we let $S^F_1(A) = \text{Uf}(\text{Def}(F/A))$. An element $p \in S^F_1(A)$ is called a 1-type over $A$.

(iii) A set $\Sigma(x)$ of $L(A)$-formulas with one variable, say, $x$, is called a partial 1-type over $A$. We write $\Sigma(F)$ for the set $\{w \in F \mid \forall \phi \in \Sigma F \models \phi(w)\}$.

Convention 3.6. We identify a 1-type $p$ over $A$ with the partial 1-type $\{\phi(x; \bar{a}) \mid \phi(F; \bar{a}) \in p, \bar{a} \in A\}$ over $A$. In fact, this is closer to how types are usually defined in classical model theory and is what types are in [16]. Likewise, we write $[\phi]$ for the clopen set $[X]$ in a Stone space of 1-types if $\phi$ defines $X$.

Given a partial type $\Sigma(x)$, the intersection $\bigcap_{\phi \in \Sigma} [\phi]$ is a closed set in the Stone space of 1-types.

Definition 3.7.

(i) A partial 1-type $\Sigma(x)$ over $A$ is deductively closed if $[\phi] \supseteq \bigcap_{\psi \in \Sigma} [\psi] \implies \phi \in \Sigma$.

(ii) For a deductively closed partial 1-type $\Sigma(x)$, we write $E_\Sigma$ for the closed set $\{p \mid p \supseteq \Sigma\} = \bigcap_{\phi \in \Sigma} [\phi]$.

Proposition 3.8. Let $w \in F$ and $A \subseteq F$. The family $\text{tp}^F(w/A)$ of $A$-definable subsets of $F$ containing $w$ is an ultrafilter in $\text{Def}(F/A)$ and thus a 1-type over $A$.

Definition 3.9.

(i) Let $A \subseteq F$. An element $w \in F$ realizes $p \in S^F_1(A)$, or $w \models p$, if $\text{tp}^F(w/A) = p$. The 1-type $p$ is realized in $F$ if there is $w \in F$ with $w \models p$.

(ii) The $L$-structure $F$ is $\aleph_0$-saturated if for every finite $A \subseteq F$, every $p \in S^F_1(A)$ is realized in $F$. 

3.2. Model theory specific to neighborhood frames

In this section, we study the model theory of neighborhood frames while we relate it to the classical model theory.

Definition 3.10. Let $L$ be a language of coalgebraic predicate logic based on $L_0$ and $F$ an $L$-structure. The essential part $F^e$ of $F$ is the $L$-structure whose reduct to $L_0$ is the same as that of $F$ and whose neighborhood function $N^e$ is defined by

$$U \in N^e(w) \iff U \text{ is definable in } F \text{ and } U \in N^F(w)$$

for $w \in F^e$.

Proposition 3.11 ([3]). Let $L$ be a language of coalgebraic predicate logic and $F, G$ be $L$-structures. Suppose $F^e \equiv G^e$.

(i) $F \equiv_L G$.

(ii) If $F$ is $\aleph_0$-saturated, so is $G$. $\blacksquare$

We define a class of languages of first-order logic, one for each language of coalgebraic predicate logic.

Definition 3.12 ([14],[24, Definition 9]). Let $L$ be an arbitrary language of coalgebraic predicate logic and $L_0$ the language of first-order logic on which $L$ is based. We define the language $L^2$ to be the two-sorted first-order language whose sorts are the state sort and neighborhood sort and whose atomic formulas are those in $L_0$, recast as formulas in which constants and variables belong to the state sort, together with $xNU$ and $x \in U$, where $x$ and $U$ are variables for the state sort and the neighborhood sort, respectively. (In general, we will use lowercase variables for the state sort and uppercase variables for the neighborhood sort.)

Definition 3.13. Let $L$ be a language of coalgebraic predicate logic and $F$ an $L$-structure. Given a family $\mathcal{S} \subseteq \mathcal{P}(F)$ that contains all definable subsets of $F$, we write $(F, \mathcal{S})$ for the $L^2$-structure $G$. The domain of the state sort of $G$ is that of $F$, and the domain of the neighborhood sort of $G$ is $\mathcal{S}$. The $L^2$-structure $G$ interprets all nonlogical symbols of $L^2$ but $N$ and $\in$ in the same way as $F$. Finally, we have $(w, U) \in N^G \iff U \in N^F(w)$ and $(w, U) \in G \iff w \in U$. A family $\mathcal{S}$ is large for $F$ if $U \in \mathcal{S}$ whenever there is $w \in F$ with $U \in N^F(w)$. 
Proposition 3.14 ([3, 16]). Let $L$ be a language of coalgebraic predicate logic. Let $(-)^2 : L \to L^2$ be the translation that commutes with Boolean combinations and satisfies
\[
(\exists x \phi)^2 = \exists x(\phi^2)
\]
\[
(x \Box_y \phi)^2 = \exists U [\forall y(y \in U \leftrightarrow \phi^2(y)) \land xNU].
\]
Let $S \subseteq \mathcal{P}(F)$ be a family that contains all definable subsets of $F$. Then for every $L$-formula $\phi$ and $\bar{a} \in F$ we have
\[
F \models \phi(\bar{a}) \iff (F, S) \models \phi^2(\bar{a}).
\]

Remark 3.15. Note that the same two-sorted language $L^2$ is considered in [14] even though their transformation of neighborhood frames into $L^2$-structures there is different from ours. While in [14] a neighborhood frame $F$ is always associated with the structure $M$ for $L^2$, whose neighborhood sort consists of those subsets of $F$ that are neighborhoods of some state of $F$, we do not impose such a restriction here. In addition, there is a third language for neighborhood frames used before as a model correspondence language [24, Definition 12] for neighborhood and topological semantics of modal logic and for the study of model theory of topological spaces [7] in general. This is also a fragment of the two-sorted language introduced above and, in fact, contains the image of the embedding of coalgebraic predicate logic into the two-sorted language [29].

Lemma 3.16. Let $L$ be a language of coalgebraic predicate logic and $F$ an $L$-structure. Let $G$ be an $L^2$-structure that is an elementary extension of $(F, \mathcal{P}(F))$. There exists an $L$-structure $G'$ whose domain is that of the state sort of $G$ and a family $S \subseteq \mathcal{P}(G')$ that satisfies the following:

(i) $S$ contains all definable subsets in $G'$.

(ii) $S$ is large for $G'$.

(iii) $G \equiv (G', S)$.

Proof. Note that $F$ satisfies extensionality:
\[
(F, \mathcal{P}(F)) \models \forall U \forall V [\forall x(x \in U \leftrightarrow x \in V) \rightarrow U = V].
\]
By $L^2$-elementarity, so does $G$. Let $G'$, $S^G$ be the domains of the state sort and the neighborhood sort of $G$, respectively. Let $i : S^G \to \mathcal{P}(G')$ be defined by
\[
i(U) = \{ w \in G' \mid G \models w \in U \}.\]
By the extensionality of $G$, $i$ is injective. Let $\mathcal{S}$ be the range of $i$. Define the neighborhood function $N^{G'}$ by

$$i(U) \in N^{G'}(w) \iff G \models wNU.$$  

Let $\phi(x; \bar{y})$ be an $L$-formula and $X := \phi(G', \bar{a})$ be a definable set in $G'$, where $\bar{a} \in G'$. Note that the $L^2$-structure $(F, \mathcal{P}(F))$ satisfies comprehension:

$$(F, \mathcal{P}(F)) \models \forall \bar{y} \exists U \forall x(\phi^{\sharp}(x; \bar{y}) \leftrightarrow x \in U).$$

So does $G$. Let $U$ witness the satisfaction by $G$ of the existential formula $\exists U \forall x(\phi^{\sharp}(x; \bar{a}) \leftrightarrow x \in U)$. It can easily be seen that $i(U) = \phi(G', \bar{a})$.

It is easy to see that $\mathcal{S}$ is large for $G'$ and that $G \cong (G', \mathcal{S})$. $
$

**Proposition 3.17.** Let $L$ be a language of coalgebraic predicate logic and $F$ an $L$-structure. There exists an $L$-structure $G$ such that $G \equiv_L F$ and that $G$ is $\aleph_0$-saturated.1  

**Proof.** Consider the $L^2$-structure $(F, \mathcal{P}(F))$, and take an elementary extension $G_0$ of $(F, \mathcal{P}(F))$ that is $\aleph_0$-saturated. By Lemma 3.16(iii), take an $L$-structure $G$ and $\mathcal{S} \subseteq \mathcal{P}(G)$ with $G_0 \cong (G, \mathcal{S})$. Suppose that $A \subseteq G$ is finite. Let $p \in S^G(A)$ be arbitrary. Let $\Sigma^2$ be the partial type $\{\phi^x \mid \phi \in p\}$ over $A$ in $L^2$. Since $p$ is a proper filter in $\text{Def}(F/A)$, the type $\Sigma^2$ is consistent by Proposition 3.14. Thus, by the $\aleph_0$-saturation of $G_0$, we can take $w \in G_0$ realizing $\Sigma^2$. By Proposition 3.14, we have $w \models p$. $
$
We now introduce the notion of quasi-ultraproducts as we will use it to give a proof of Fine’s theorem at the end of this article.

**Definition 3.18 ([3, 16]).** Let $L$ be a language of coalgebraic predicate logic based on $L_0$ and $(F_i)_{i \in I}$ be a family of monotonic $L$-structures. Suppose that $D$ is an ultrafilter over $I$. Let $\prod_D F_i$ be the ultraproduct of $(F_i)_i$ as $L_0$-structures modulo $D$. A subset $A \subseteq \prod_D F_i$ is induced by a family $(A_i)_{i \in J}$ if $J \in D$, $A_i \subseteq F_i$ for $i \in J$, and

$$a \in A \iff a(i) \in A_i \text{ for all } i \in J.$$

1Our use of both coalgebraic predicate logic and first-order logic makes phrases such as “elementarily equivalent” and “$\aleph_0$-saturation” potentially ambiguous because we have two different classes of definitions, one from the previous subsection and the other standard in classical model theory. Note, however, that (expansions of) neighborhood frames are never structures of any language of first-order logic and that first-order structures are never $L'$-structures for any language $L'$ of coalgebraic predicate logic. Hence, for example, if $L'$ is a language of coalgebraic predicate logic, and $F$ is an $L'$-structure, then whenever we say that $F$ is $\aleph_0$-saturated, we mean what we stated in Definition 3.9(ii), with $L$ in the definition being $L'$.
A quasi-ultraproduct of $(F_i)_i$ modulo $D$ is a monotonic $L$-structure that is the $L_0$-structure $\prod_D F_i$ equipped with a neighborhood function $N$ that satisfies

$$A \in N(w) \iff A_i \in N^i(w(i)) \text{ for all } i \in J,$$

whenever $w \in \prod_D F_i$, and $A$ is induced by $(A_i)_{i \in J}$. A class $\mathcal{K}$ of monotonic neighborhood frames admits quasi-ultraproducts if whenever $(F_i)_i$ is a family of neighborhood frames from $\mathcal{K}$, a quasi-ultraproduct of $(F_i)_i$ exists in $\mathcal{K}$.

**Proposition 3.19** ([16, 3]).

1. Each class of the classes in Table 1 admits quasi-ultraproducts.
2. Let $(F_i)_{i \in I}$ be a family of monotonic $L$-structures for a language $L$ of coalgebraic predicate logic. If $F_i$ satisfies a theory $T$ for all $i \in I$, so does a quasi-ultraproduct of $(F_i)_i$.

**Proof.**

1. By Remark 4.2, it suffices to prove this for the class of monotonic neighborhood frames, the class of quasi-filter frames. This could be done by using the machinery introduced in Litak et al. [16], but it is easy to prove it directly in the following way.

Let $\mathcal{K}_0$ be either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames. Let $(F_i)_i$ be a family of neighborhood frames in $\mathcal{K}_0$. Let $N^i$ be the neighborhood function of $F_i$. Define the neighborhood function $N$ on $\prod_D F_i$ as follows: a subset $U \subseteq \prod_D F_i$ is in $N(w)$ if and only if there is a set $A \subseteq U$ induced by $(A_i)_{i \in J}$ with $A_i \in N^i(w(i))$ for all $i \in J$. It is easy to see that this indeed defines a quasi-ultraproduct and that if each $F_i$ is in $\mathcal{K}_0$ then so is the quasi-ultraproduct.

2. The usual argument by induction works; see Litak et al. [16].

**4. Proof of the main lemmas**

In this section we prove the main lemmas of this article. Recall that $L_\equiv$ is the language of coalgebraic predicate logic based on the empty language of first-order logic.

**Definition 4.1.** Let $\mathcal{K}_0$ be a class of monotonic neighborhood frames. A class $\mathcal{K}$ of monotonic neighborhood frames is **CPL-elementary relative to $\mathcal{K}_0$** if there is an $L_\equiv$-theory $T$ with

$$\mathcal{K} = \{ F \in \mathcal{K}_0 \mid F \models T \}.$$

Two monotonic neighborhood frames $F$ and $F'$ are CPL-elementarily equivalent relative to $K_0$ if $F, F' \in K_0$ and $\text{Th}_{L_\omega}(F) = \text{Th}_{L_\omega}(F')$.

**Remark 4.2.** The class of filter frames is CPL-elementary relative to the class of quasi-filter frames (see Definition 4.1), and the class of augmented filter frames is CPL-elementary relative to the class of augmented quasi-filter frames; indeed, they are both defined by the same $L_\omega$-sentence $\forall x \Box y y = y$. Furthermore, by the second paragraph of Example 2.6, the class of augmented quasi-filter frames is CPL-elementary relative to the class of monotonic frames. Therefore, the main lemma in this section concerns the classes of monotonic and quasi-filter neighborhood frames, respectively, which suffice for the purpose of the main results (Theorems 5.1 and 5.4), which deal with any of the classes in Table 1.

**Lemma 4.3.** Let $F$ be a monotonic neighborhood frame, and let $G$ and $G'$ be $(L_2)^2$- and $L_\omega$-structures, respectively, obtained by elementarily extending $F$ as in Lemma 3.16.

(i) If $F$ is monotonic, $X, Y \subseteq G'$ are definable, $X \subseteq Y$, and $X \in N_{G'}(w)$ for $w \in G'$, then $Y \in N_{G'}(w)$.

(ii) If $F$ is an augmented filter frame, then for every $w \in G'$ either $N_{G'}(w)$ is empty or has a minimum element.

**Proof.** For (i), let $L(G')$-formulas $\phi(x; \bar{a})$ and $\psi(x; \bar{b})$ define $X$ and $Y$, respectively. Since $F$ is monotonic, we have

$$(F, \mathcal{P}(F)) \models \forall \bar{y} \forall \bar{z} \forall v [\forall x (\phi(x; \bar{y}) \rightarrow \psi(x; \bar{z}))$$

$$\land v \Box_x \phi(x; \bar{y}) \rightarrow v \Box_x \psi(x; \bar{z})].$$

(5)

Since $(F, \mathcal{P}(F))$ satisfies the $(-)^2$-translation of the right-hand side of the displayed formula (5) by Proposition 3.14, so does $G$. Again by Proposition 3.14,

$$G' \models \forall x (\phi(x; \bar{a}) \rightarrow \psi(x; \bar{b})) \land w \Box_x \phi(x; \bar{a}) \rightarrow w \Box_x \psi(x; \bar{b}).$$

Since $X \in N_{G'}(w)$, we have $\psi(G', \bar{b}) \in N_{G'}(w)$.

For (ii), first observe that the $L^2$-structure $(F, \mathcal{P}(F))$ satisfies the sentence

$$\forall x [\neg \exists U xNU \lor \exists U_0 U(xNU \rightarrow U_0 \subseteq U)],$$

where $\subseteq$ is an abbreviation of the obvious $L^2$-formula. Since $G'$ satisfies the same $L^2$-formula, the claim follows. $\blacksquare$
We are now ready to prove the key lemmas used in the proof of our main result. Our lemmas are analogous to [25, 8.9 Theorem].

**Lemma 4.4.** Let $F$ be a monotonic neighborhood frame. There exists $G \equiv_{L_\infty} F$ such that there is a surjective bounded morphism $f : G \to \text{ue}F$. Moreover, if $\mathcal{K}_0$ is either the class of monotonic neighborhood frames or the class of quasi-filter neighborhood frames, and $F \in \mathcal{K}_0$, then we can take $G \in \mathcal{K}_0$.

The following is the outline of the proof, which comes after this paragraph. We follow the classical proof of [25, 8.9 Theorem] by taking an expansion $L$ of the correspondence language so that every subset of the given frame $F$ will be definable and taking an $\aleph_0$-saturated extension $G$ in that language. However, we need to add more neighborhoods to the neighborhood frame $G$ that is being constructed to make sure that the map from $G$ to the ultrafilter frame of $F$ is a bounded morphism. Much of the proof is dedicated to showing that this construction preserves elementary equivalence in $L$.

**Proof.** Let $L$ be the language of coalgebraic predicate logic based on $\{P_S \mid S \subseteq F\}$, the unary predicates for the subsets of $F$. The neighborhood frame $F$ can be made into an $L$-structure naturally. Let $G_0 \equiv_L F$ be an $\aleph_0$-saturated $L$-structure as obtained by Proposition 3.17. Let $G_1$ be the essential part of $G_0$. Let $G_2$ be the $L$-structure obtained from $G_1$ as follows: for each state $w \in G_1$, add as a neighborhood of $w$ the set $\Sigma(G_1)_w$, where $\Sigma(x)$ is a partial type over a finite set $A \subseteq G_1$ such that $\Sigma(x)$ is deductively closed and that for every $\phi \in \Sigma$ we have $\phi(G_1)_w \subseteq N^{G_1}(w)$. We call such a partial type *good* at $w$. Let $G$ be the $L$-structure identical to $G_2$ except that its neighborhood function $N^G$ is defined by $U \in N^G(w) \iff \exists U_0 \subseteq U \subseteq U_0 \in N^{G_2}(w)$.

Note that a singleton partial type $\Sigma = \{\phi\}$ with $\phi(x) \in L(A)$ is always good at $w \in G_1$ if $\phi(G_1)_w \subseteq N^{G_1}(w)$.

We show that $G \equiv_L F$. By Proposition 3.11, we have $G_1 \equiv_L G_0 \equiv_L F$, so it suffices to see that for every definable $X \subseteq G$ we have $X \in N^G(w) \iff X \in N^{G_1}(w)$. We show $\implies$ (the other direction is easy). By construction, there is either a definable set $Y \subseteq X$ with $Y \in N^{G_1}(w)$ or a partial type $\Sigma(x)$ over a finite set $A$ good at $w$ with $\Sigma(G_1) \subseteq X$. The former is a special case of the latter, so we assume the latter. Let $A'$ be a finite set containing $A$ and the parameters used in the definition of $X$. Let $f' : G_1 \to S^{G_1}_1(A')$ be defined by $f'(w) = \text{tp}^{G_1}(w/A')$. By $\aleph_0$-saturation, $f'$ is a surjection. We show that $f'(\Sigma(G_1)) = E_\Sigma \subseteq S^{G_1}_1(A')$. It is easy to see that $f'(\Sigma(G_1)) \subseteq E_\Sigma$; we show $f'(\Sigma(G_1)) \supseteq E_\Sigma$. Let $p \in E_\Sigma$ be arbitrary. By $\aleph_0$-saturation, take $w \in G_1$ with $f'(w) = p$. Since $p \supseteq \Sigma$, $w \in \Sigma(G_1)$. We have shown that
We show that there is \( f'(\Sigma(G_1)) = E_{\Sigma} \). That \( f'(X) = [X] \) easily follows from the \( \aleph_0 \)-saturation of \( G \) as well. We have \( E_{\Sigma} \subseteq [X] \). By the compactness of \( S^G_1(A') \), we have a finite \( \Sigma_0 \subseteq \Sigma \) for which \( E_{\Sigma_0} \subseteq [X] \). Being the intersection of finitely many clopen sets,

\[
E_{\Sigma_0} = \bigcap_{\phi \in \Sigma_0} [\phi] = [\bigwedge \Sigma_0]
\]

is clopen. Since \( \Sigma \) is good at \( w \), we have \( (\bigwedge \Sigma_0)(G_1) \in N^{G_1}(w) \). We conclude that \( X \in N^{G_1}(w) \) by Lemma 3.16 (i). (See Remark 4.5 for an alternate proof of this fact.)

Since \( F^+ \cong \text{Def}(F/\varnothing) \), we have \( w \in F \cong S_1(T) \), where \( T \) is the full \( L \)-theory of \( F \), which is identical to \( \text{Th}_L(G) \). We show \( f : G \to S_1(T) \) defined by \( f(w) = \text{tp}^G(w/\varnothing) \), which is surjective by \( \aleph_0 \)-saturation, is a bounded morphism. In the rest of the proof, we write \( N^\sigma \) for the neighborhood function of \( S_1(T) \).

**The “forth” condition.** Suppose that \( U \in N^G(w) \). We show that \( f(U) \in N^\sigma(\text{tp}^G(w)) \). By construction, we have either (I) \( U \supseteq \phi(G, \bar{a}) \in N^G(w) \) or (II) \( U \supseteq \Sigma(G) \in N^G(w) \), where \( \phi(x, \bar{y}) \) is an \( L \)-formula, \( \bar{a} \in G \), and \( \Sigma(x) \) is a partial type over a finite set \( A \) good at \( w \). Since (I) is a special case of (II), we will just show (II).

For (II), assume that \( U \supseteq \Sigma(G) \in N^G(w) \), where \( \Sigma \) is a partial 1-type over finite \( A \) good at \( w \). Let \( K = r(E_{\Sigma}) \), where \( r : S^G_1(A) \to S_1(T) \) is the closed continuous map dual to the embedding \( \text{Def}(G/\varnothing) \to \text{Def}(G/A) \). Note that \( r(q) = q \cap \text{Def}(G/\varnothing) \) for \( q \in S^G_1(A) \). Being the image of a closed map of a closed set, \( K \) is closed. Recall the equation (4) that defines \( N^\sigma \) to see that it suffices to show (i) that for every \( \chi(x) \in L \) we have \( [\chi] \supseteq K \implies \chi(G) \in N^G(w) \) and (ii) that \( K \subseteq f(U) \). For (i), assume that \( [\chi] \supseteq r(E_{\Sigma}) \), where \( \chi(x) \in L \), and \( [\chi] \) denotes a subset in \( S^G_1(A) \). Take an arbitrary \( q \in E_{\Sigma} \). Then \( r(q) \in r(E_{\Sigma}) \subseteq [\chi] \), so \( \chi \in r(q) \subseteq q \). We have just shown that \( [\chi] \supseteq E_{\Sigma} \), where \( [\chi] \) denotes a subset in \( S^G_1(A) \). By deductive closure \( \chi \in \Sigma \). By construction, \( \chi(G) \in N^G(w) \). For (ii), it suffices to show that arbitrary \( q \in E_{\Sigma} \) can be realized by an element of \( U \). Since \( q \) is a type over a finite set, by \( \aleph_0 \)-saturation, we may take \( v \models q \); this means \( v \models \Sigma \), i.e., \( v \in \Sigma(G) \subseteq U \).

**The “back” condition.** Suppose that \( U' \subseteq S_1(T) \) is in \( N^\sigma(\text{tp}^G(w/\varnothing)) \). We show that there is \( U \subseteq G \) in \( N^G(w) \) such that \( f(U) \subseteq U' \). By the definition of \( N^\sigma \), there is a partial type \( \Sigma(x) \) over \( \varnothing \) good at \( w \) such that
$E_{\Sigma} \subseteq U'$. By construction, $\Sigma(G) \in N^G(w)$. Let $U := \Sigma(G)$. Then for every $v \in U$, the type $tp^{G}(v/\emptyset)$ extends $\Sigma$ and thus is in $E_{\Sigma} \subseteq U'$.

**Closure in relatively CPL-elementary classes.** By construction, $G$ is monotonic.

Suppose that $F$ is a quasi-filter neighborhood frame. Let $w \in G$ and $U, U' \in N^G(w)$ be arbitrary. By construction, there are deductively closed partial types $\Sigma(x), \Sigma'(x)$ over a finite set of parameters both of which are good at $w$ with $\Sigma(G) \subseteq U$ and $\Sigma'(G) \subseteq U'$. The partial type $\Sigma \cup \Sigma'$ is also over a finite set, good at $w$. Moreover, $\Sigma \cup \Sigma'$ is deductively closed since $F$ is a quasi-filter frame. Therefore, we have $(\Sigma \cup \Sigma')(G) = \Sigma(G) \cap \Sigma(G) \subseteq U \cap U'$, so $U \cap U' \in N^G(w)$. We have seen that $G$ is a quasi-filter neighborhood frame.

**Remark 4.5.** In the proof above, we obtain $G$ not only by compactness but also by altering the neighborhoods in an ad-hoc way while maintaining elementary equivalence in $L_\omega^\omega$. There is no reason for us to believe that $G$ has the same theory as $F$ in $L_\omega^2$ or in the languages described in Remark 3.15. This is why we find it difficult to extend our main result to the more expressive languages.

The following is the alternate proof that I announced at the end of the third paragraph of the proof (the concepts that we have not defined have obvious definitions): Suppose $X$ is definable by $\psi(x; A')$ where $\psi \in L$ and $A' \subseteq G$ is a finite set. By $\aleph_0$-saturation of $G_1$, we have $\text{Th}_{L_\omega(A')} \cup \Sigma(x) \models \psi(x, A')$ (otherwise, realize the type $\Sigma(x) \cup \{\neg \psi(x, A')\}$ by some element in $G_1$, which would be in $\Sigma(G_1) \setminus X$.) By compactness, there is finite $\Sigma_0 \subseteq \Sigma$ such that $\Sigma_0(G_1) \subseteq \psi(G_1, A')$. Since $\bigwedge \Sigma_0(x)$ is a single formula of $L$, by deductive closure $\bigwedge \Sigma_0(x) \in \Sigma(x)$. Hence $\bigwedge \Sigma_0(G_1) \in N^G_1(w)$. By Lemma 3.16(i), we have $X = \psi(G_1, A') \in N^{G_1}(w)$ as desired.

5. Applications of the main lemmas

5.1. The Goldblatt-Thomason Theorem

An algebraic argument essentially the same as the classical counterpart can be used to show that a class of monotonic neighborhood frames closed under ultrafilter extensions is modally definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions [15][13, Theorem 7.23]. By applying Lemma 4.4, we obtain the following theorem.
THEOREM 5.1. Let $\mathcal{K}$ be a class of monotonic neighborhood frames that is closed under CPL-elementary equivalence relative to any of the classes in Table 1. $\mathcal{K}$ is modally definable if and only if it is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions.

PROOF. Let $\mathcal{K}$ be a class of monotonic neighborhood frames that is closed under CPL-elementary equivalence relative to a class $\mathcal{K}_0$ in Table 1. We show the “if” case. Suppose that $\mathcal{K}$ is closed under bounded morphic images, generated subframes, and disjoint unions, and it reflects ultrafilter extensions. Apply Lemma 4.4 and Remark 4.2 to conclude that $\mathcal{K}$ is closed under ultrafilter extensions. Note that the hypothesis of [13, Theorem 7.23] is satisfied, and we conclude that $\mathcal{K}$ is modally definable.

EXAMPLE 5.2. As an example, we show that the image $\mathcal{K}$ under $^*$ of the class of discrete topological spaces is modally definable. For a quasi-filter frame $F$, $F$ is a $^*$-image of a discrete topological space if and only if $F \models \forall x \neg x \Box z \neq x$ and $F \models \forall x \Box y y = x$. Hence, $\mathcal{K}$ is CPL-elementary relative to the class of quasi-filter frames, and the Goldblatt-Thomason Theorem is applicable to $\mathcal{K}$. It is easy to check that $\mathcal{K}$ is closed under bounded morphic images, generated subframes, and disjoint unions, so it suffices to show that $\mathcal{K}$ reflects ultrafilter extensions. Assume that for a neighborhood frame $F = (F, N)$ its ultrafilter extension $ueF = (ueF, N^\sigma)$ is in $\mathcal{K}$. We show that $F \in \mathcal{K}$. The class of topological frames is defined by modal formulas $\Box p \land \Box q \rightarrow \Box (p \land q)$, $\Box p \rightarrow p$, and $\Box \Box p \rightarrow \Box p$ [27], so we may assume that $F$ is topological as ultrafilter extensions reflect modally definable classes. Let $w \in F$ be arbitrary, and let $u$ be the principal ultrafilter generated by $w$, so $u \in ueF$. Note that $U \in N^\sigma(u) \iff u \in U$ since $ueF$ is the $^*$-image of a discrete space. Recall the definition of $N^\sigma$ in (4). The singleton $\{u\}$ is in $N^\sigma(u)$, and this has to be witnessed by $K = \emptyset$ or $K = \{u\}$ according to (4) of Definition 2.19.(i).

Suppose $K = \emptyset$. Then (4) implies that $\Box^+ F \emptyset \in u$ among other things (recall that $A$ in (4) is $F^+$ here). However, since $F$ is topological, $\Box^+ F \emptyset = \emptyset$, and it cannot belong to an ultrafilter $u$. Hence, $K = \{u\}$. Again by (4), for all $a \subseteq F$ such that $[a] \supseteq K = \{u\}$, i.e., $a \in u$, we have that $\Box^+ F^+ a \in u$. Let $a = \{w\}$, so $a \in u$. Since the set $u$ is an ultrafilter, we have $a \land \Box^+ F^+ a \neq \emptyset$, that is, $a \cap \{w \in F \mid a \in N(w)\} \neq \emptyset$; this implies $\{w\} \in N(w)$. Since $w$ was arbitrary, we conclude that $F \in \mathcal{K}$. We have shown that $\mathcal{K}$ is modally definable; in fact, it is defined by $p \rightarrow \Box p$ in addition to the definition of the class of topological neighborhood frames.
5.2. Fine’s Canonicity Theorem

By the dual equivalence between monotonic modal logics and varieties of
BAMs [13, Chapter 7], we will state our version of Fine’s Canonicity Theorem
in an algebraic manner. Our presentation of the proof of the theorem is
modeled after that of the classical version of the theorem in [28].

For a class $\mathcal{K}$ of neighborhood frames, we write $\mathcal{K}^+$ for the class \{$F^+ \mid F \in \mathcal{K}$\}.

Lemma 5.3. Let $\mathcal{K}$ be a class CPL-elementary relative to any of the classes
in Table 1. Let $\mathcal{S} \supseteq \mathcal{K}^+$ be the least class of BAMs closed under subalgebras.

1. $\mathcal{S}$ is closed under canonical extensions.

2. $\mathcal{S}$ is closed under ultraproducts.

Proof.

1. Let $A \in \mathcal{S}$. For some $F \in \mathcal{K}$ we have $A \hookrightarrow F^+$. By duality theory [8, 
Theorem 5.4], we have $A^\sigma \hookrightarrow (F^+)^\sigma$. By Lemma 4.4 and Remark 4.2,
there is $G \in \mathcal{K}$ with $(F^+)^\sigma \hookrightarrow G^+$. Thus, we have $A^\sigma \in \mathcal{S}$ by definition.

2. It suffices to do the following: given an ultraproduct $\prod D F_i^+$ where $I$ is an
index set, $D$ is an ultrafilter over $I$, and $(F_i)_i$ is a family of neighborhood
frames in $\mathcal{K}$, we show that the ultraproduct embeds into $(\prod D F_i)^+$, where
$\prod D F_i$ is a quasi-ultraproduct of $(F_i)_i$ modulo $D$. In fact, we show that
$\iota : \prod D F_i^+ \rightarrow (\prod D F_i)^+$ defined by

$$s \in \iota(a) \iff \{i \mid s(i) \in a(i)\} \in D,$$

where $s \in \prod D F_i$ and $a \in \prod D F_i^+$ is a BAM embedding (we do not
write equivalence classes modulo $D$ explicitly; it is easy to see that
$\iota$ is well defined). It can easily be seen that $\iota$ is a Boolean algebra
embedding. We show that $\iota \circ \Box^\text{pu} = \Box^\text{cm} \circ \iota$, where $\Box^\text{pu}$ and $\Box^\text{cm}$ are the
operations of the domain and the target of $\iota$, respectively. Let $N$ be the
neighborhood function of the quasi-ultraproduct. We write $\Box^i$ and $N^i$
for the operation of $F_i^+$. Note that for all $a$ the set $\iota(a)$ is an induced
subset of the quasi-ultraproduct; if we let $\pi_i(A)$ be the projection of an
induced subset $A$ of the quasi-ultraproduct onto the coordinate $i$, then
\{i \mid \pi_i(\iota(a)) = a(i)\} \in D. We now have
\[
s \in (\iota \circ \Box^{\text{pu}})(a) \iff \{i \mid s(i) \in (\Box^{\text{pu}}(a))(i)\} \in D
\iff \{i \mid s(i) \in \Box^i(a(i))\} \in D \quad (*)
\iff \{i \mid s(i) \in \Box^i(\pi_i(\iota(a)))\} \in D
\iff \{i \mid \iota(a) \in N^i(s(i))\} \in D
\iff \iota(a) \in N(s)
\iff s \in (\Box^{\text{cm}} \circ \iota)(a),
\]
where we have the equivalence (*) since
\[
\{i \mid (\Box^{\text{pu}}(a))(i) = \Box^i(a(i))\} \in D.
\]

Theorem 5.4. Let \(\mathcal{K}\) be a class CPL-elementary relative to any of the classes in Table 1. The variety of BAMs generated by \(\mathcal{K}^+\) is canonical, i.e., closed under canonical extensions.

Proof. Recall Remark 4.2. Gehrke and Harding [8] showed that if \(\mathcal{S}\) is a class of BAMs closed under ultraproducts and canonical extensions, then \(\mathcal{S}\) generates a canonical variety. Apply this result for the class \(\mathcal{S}\) in Lemma 5.3 to conclude that the variety generated by \(\mathcal{K}^+\), which is identical to the variety generated by \(\mathcal{S}\), is canonical.

Note that Fine’s original theorem follows as a special case concerning the classes of augmented neighborhood frames.

Example 5.5. Consider the B axiom \(p \rightarrow \Box \neg \Box \neg p\), which we considered in Example 2.4. Recall that it defined a CPL-elementary class \(\mathcal{K}\) relative to the class of monotonic neighborhood frames. By [13, Proposition 6.5], the variety \(\mathcal{V}\) defined by the B axiom is canonical and hence generated by \(\mathcal{K}^+\). By Theorem 5.4, the canonicity of \(\mathcal{V}\) is explained by the CPL-elementarity of \(\mathcal{K}\).

Remark 5.6. By Remark 2.5, Theorem 5.4 can be used to show the canonicity of the monotonic modal logic axiomatized by any formula of the form (1).

6. Open questions

As we mentioned in Remarks 3.15 and 4.5, one could attempt to use a different notion of elementarity in stating and proving the results of this article, but we stuck to coalgebraic predicate logic due to the limitation
of the proof technique we used. A natural question to ask here would be whether there is a more expressive first-order-like logic that admits similar results possibly by a different kind of proof. Another question would be to characterize classes of monotonic neighborhood frames that admit analogues of the Goldblatt-Thomason theorem and Fine’s theorem in the same sense as in the main result of this article. This question leads to another problem of showing results similar to ours for other coalgebras than those discussed in this article.

It was suggested to the author that our version of Fine’s theorem could be proved by using an algebraic result [9], which implies the original, Kripke-semantic version of the theorem. The “proof” proposed contained a gap, and therefore it remains open whether the results in this article follow from the aforementioned algebraic theorem. Even if they can indeed be proved in that manner, we hope that the proof presented here serves our original purpose of investigating the role of coalgebraic predicate logic in the study of monotonic modal logics, especially in the spirit of van Benthem’s program [26] of re-analyzing algebraic arguments occurring in modal logic from a model-theoretic perspective.

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