DOMINATED PESIN THEORY:
CONVEX SUM OF HYPERBOLIC MEASURES

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ABSTRACT. In the uniformly hyperbolic setting it is well known that the measure supported on periodic orbits is dense in the convex space of all invariant measure. In this paper we consider the reverse question, in the non-uniformly hyperbolic setting: assuming that some ergodic measure converges to a convex sum of hyperbolic ergodic measures, what can we deduce on the initial measures?

To every hyperbolic measure $\mu$ whose stable/unstable Oseledets splitting is dominated we associate canonically a unique class $H(\mu)$ of periodic orbits for the homoclinic relation, called its intersection class. In a dominated setting, we prove that a convex sum of finitely many ergodic hyperbolic measures of the same index is accumulated by ergodic measures if, and only if, they share the same intersection class. This result also holds if the measures fail to be ergodic but are supported on hyperbolic sets.

We provide examples which indicate the importance of the domination assumption.

1. INTRODUCTION

1.1. Quick presentation of the results. The space $M(f)$ of invariant measures by a homeomorphism $f$ of a compact metric space is a compact metric space (for the the weak* topology) and is convex. The ergodic measures are the extremal points of this convex set and any invariant measure can be written as a convex sum of ergodic measures which is unique (up to 0-measure) and called its disintegration in ergodic measures or ergodic decomposition.

Nevertheless, a typical picture of hyperbolic dynamics it that the ergodic measure associated to periodic orbits may be dense in $M(f)$. More precisely, if you consider a shift or a subshift of finite type, there are periodic orbits following an arbitrary itinerary. Hence, given any ergodic measures $\mu_1, \ldots, \mu_k$, there are periodic orbits which follow a given proportion of time a typical point of the measure $\mu_1$ in an orbit segment long enough for approaching $\mu_1$ and then follow $\mu_2$ and so on, so that the measure obtained at the period is arbitrarily close to a given convex combination of the $\mu_i$. Now, if $f$ is a diffeomorphism on a manifold and if $\Lambda$ is an invariant (uniformly) hyperbolic basic set, the existence of a Markov partitions allows us to transfer this property to the set $M(\Lambda)$ of invariant measures supported on $\Lambda$.

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For understanding the non-hyperbolic dynamical systems, one often starts the study by considering the hyperbolic parts contained in $f$. For instance, if $p$ and $q$ are hyperbolic periodic points of the same index, one says that they are homoclinically related if the stable and unstable manifold of $p$ cut transversely the unstable and stable (respectively) manifolds of $q$. A famous result by Smale shows that, if $p$ and $q$ are homoclinically related, then they are contained in a hyperbolic basic set. Thus, any convex combination of the invariant measures associated to $p$ and $q$ is accumulated by periodic orbits.

In this paper, we are interested in some kind of converse to this property. Given a diffeomorphism $f$ of a compact manifold $M$ and given two ergodic measures $\mu_1$ and $\mu_2$: what can we say about the dynamical behavior of $f$ if we know that a convex combination $\alpha \mu_1 + (1 - \alpha) \mu_2$ is the limit of measures associated to periodic orbits?

In Section 8 we give examples showing that this approximation of a convex combination does not imply anything about homoclinic relations, even in the $\mathbb{R}$-analytic setting, if the dynamic fails to be dominated.

Our main result states that, if the stable/unstable splitting is dominated, then the approximation of convex combinations by ergodic measures is equivalent to transverse homoclinic intersection. The idea consists in controlling the size of the Pesin’s invariant manifold at the time the orbits are following generic points of one or the other measure, and to check that this size is enough for getting transverse intersections. Usually, Pesin theory does not hold in the $C^1$ setting (see [24, 7]). However, [1] shows that Pesin theory holds for $C^1$-diffeomorphism if one assume that the stable/unstable splitting is dominated, which is precisely our setting. This explains why our statements hold for $C^1$-diffeomorphisms. Let us emphasize that our results are not perturbations results: we are not creating periodic orbits, measures, or homoclinic intersections by using some perturbation lemma, but prove that they already exist.

Let us now present precisely this result. Let $f : M \to M$ be a diffeomorphism and $\Lambda$ be an invariant compact set. Assume that $f$ admits a dominated splitting $T\Lambda M = E \oplus F$ over $\Lambda$: the bundles $E$ and $F$ are $df$-invariant and there is a Riemann metric over $M$ so that $df$ expands the vectors in $E_x$ strictly less than $F_x$. We call $\dim E$ the $s$-index of the splitting. Let $\mu$ be an ergodic measure supported on $\Lambda$. On says that $\mu$ is hyperbolic if its Lyapunov exponents do not vanish. We call the number of negative Lyapunov exponents the $s$-index of $\mu$. The stable/unstable splitting of $\mu$ is the Oseledets splitting $E^s \oplus E^u$ defined over the generic points of $\mu$, where $E^s$ (respectively $E^u$) is the sum of the Lyapunov spaces corresponding to the negative (respectively positive) Lyapunov exponents. If $\mu$ supported on $\Lambda$ is hyperbolic and if the $s$-index of $\mu$ is equal to the $s$-index of the dominated splitting $E \oplus F$, then $E^s_x = E_x$ and $E^u_x = F_x$ for generic points $x$ for $\mu$. One then says that the stable/unstable splitting of $\mu$ is dominated.

If $\mu$ is an ergodic hyperbolic measure whose stable/unstable splitting is dominated, then [11 Proposition 1.4] (see Proposition 1.4) shows that every generic point $x$ of $\mu$ is the limit of periodic orbits approaching $\mu$ in the weak* topology, and approaching the support of $\mu$ in the Hausdorff topology; these periodic orbits are hyperbolic of the same $s$-index as $\mu$ and the size of their invariant manifolds is uniform; as a consequence they are all (but finitely many) homoclinically related.

We consider the homoclinic relation on the set of hyperbolic periodic orbit. It is a equivalence relation and we call intersection classes the equivalence classes for the
homoclinic relation. Corollary proves that the periodic orbits in Proposition all belong to the same intersection class \( H(\mu) \) which only depends on the measure \( \mu \), and is called the intersection class of \( \mu \).

We are now ready for stating our first main result.

**Theorem 1.** Let \( f \) be a \( C^1 \) diffeomorphism and \( \Lambda \) be a compact invariant set with a dominated splitting \( T\Lambda M = E \oplus F \). Let \( \tilde{\Lambda} \) be the maximal invariant set in a neighborhood of \( \Lambda \) so that the dominated splitting \( E \oplus F \) extends on \( \tilde{\Lambda} \).

Let \( \mu_1, \ldots, \mu_k \in M(\tilde{\Lambda}) \) be \( k \) ergodic hyperbolic measures of \( s \)-index equal to \( \dim E \), where \( k > 1 \).

There exists \( s_1, \ldots, s_k \in (0, 1) \) such that \( \sum_{j=1}^k s_j = 1 \) and \( s_1\mu_1 + \cdots + s_k\mu_k \) is accumulated by ergodic measures \( \nu_n \in M(\tilde{\Lambda}) \) if, and only if, the intersection classes \( H(\mu_1), \ldots, H(\mu_k) \) coincide.

If the intersection classes \( H(\mu_j), j \in \{1, \ldots, k\} \) coincide then every measure in the convex hull \( \text{ch}(\mu_1, \ldots, \mu_k) \) is accumulated by measures supported on periodic orbits belonging to \( H(\mu_1) \). Therefore we get the following straightforward corollary:

**Corollary 1.1.** With the hypotheses of Theorem the two following properties are equivalent:

- there exists \( s_1, \ldots, s_k \in (0, 1) \) such that \( \sum_{j=1}^k s_j = 1 \) and \( s_1\mu_1 + \cdots + s_k\mu_k \) is accumulated by ergodic measures \( \nu_n \in M(\tilde{\Lambda}) \);
- for all \( t_1, \ldots, t_k \in [0, 1] \) such that \( \sum_{j=1}^k t_j = 1 \) the measure \( t_1\mu_1 + \cdots + t_k\mu_k \) is accumulated by measures \( \gamma_n \in M(\tilde{\Lambda}) \) supported on hyperbolic periodic orbits of \( s \)-index \( \dim E \) and belonging to \( H(\mu_1) \).

A natural question is how Theorem 1 can be generalized in the case where the measures \( \mu_j \) are not assumed to be ergodic (but are still “hyperbolic”). We give here a first answer in that direction, when the measure are supported on some hyperbolic set:

**Theorem 2.** Let \( f \) be a \( C^1 \) diffeomorphism of \( M \). Let \( \Lambda \subset M \) be a compact \( f \)-invariant set with dominated splitting \( T\Lambda M = E \oplus F \). Let \( \tilde{\Lambda} \) be the maximal invariant set in a neighborhood of \( \Lambda \) so that the dominated splitting \( E \oplus F \) extends on \( \tilde{\Lambda} \). Let \( k > 1 \) and \( \Lambda_1, \ldots, \Lambda_k \subset \Lambda \) be \( k \) pairwise disjoint uniformly hyperbolic transitive\(^2\) invariant compact sets of \( s \)-index \( \dim E \) and denote \( V_i = M(\Lambda_i), i = 1, \ldots, k \).

Then \( \Lambda_1, \ldots, \Lambda_k \) are pairwise homoclinically related if, and only if, there exists a measure \( \mu \in \text{ch}(V_1 \cup \cdots \cup V_k) \setminus (V_1 \cup \cdots \cup V_k) \) which is accumulated by ergodic measures \( \nu_n \in M(\tilde{\Lambda}) \).

Exactly as for Theorem we can show that if one measure of the interior of the convex hull \( \text{ch}(V_1 \cup \cdots \cup V_k) \setminus (V_1 \cup \cdots \cup V_k) \) is accumulated by ergodic measures in \( M(\tilde{\Lambda}) \) then the whole convex hull \( \text{ch}(V_1 \cup \cdots \cup V_k) \) is contained in the closure of periodic orbits homoclinically related with \( \Lambda_1 \).

\(^1\)With this terminology, the classical homoclinic classes are the closures of these intersection classes. See also Remark for similar, but \textit{a priori} unrelated, concepts.

\(^2\)As in many recent works, an invariant compact set is called transitive if it is the closure of a positive orbit. This notion is equivalent to the notion of topological ergodicity.

\(^3\)We leave it as an exercise to see that this set could empty if the sets \( \Lambda_i \) are not pairwise disjoint.
A first step for proving Theorems 1 and 2 consists in proving the (non-uniform) hyperbolicity of the measures $\nu_n$ approaching the non-ergodic measure $\mu$. We will see that this non-uniform hyperbolicity of the measures $\nu_n$ has some uniform strength.

**Proposition 1.2.** Let $f$ be a $C^1$ diffeomorphism of $M$. Let $\Lambda \subset M$ be a compact $f$-invariant set with dominated splitting $T_\Lambda M = E \oplus F$ of $s$-index $\dim E$. Let $\tilde{\Lambda}$ be the maximal invariant set in a neighborhood of $\Lambda$ so that the dominated splitting $E \oplus F$ extends on $\tilde{\Lambda}$.

1. Let $\mu_1, \ldots, \mu_k \in \mathcal{M}(\Lambda)$ be $k$ ergodic hyperbolic measures of $s$-index equal to $\dim E$. Then, there is $\varkappa > 0$ and $\lambda > 0$ with the following property: Let $\nu \in \mathcal{M}(\tilde{\Lambda})$ so that $\nu$ belongs to the $\varkappa$-neighborhood $\mathrm{ch}_\varkappa(\{\mu_1, \ldots, \mu_k\})$ of the simplex $\{s_1 \mu_1 + \cdots + s_k \mu_k : \sum_{j=1}^k s_j = 1\}$. Then $\nu$ is hyperbolic of $s$-index $\dim E$ and the maximal Lyapunov exponent $\lambda^+_E(\nu)$ of $\nu$ in $E$ and the minimal Lyapunov exponent $\lambda^-_F(\nu)$ of $\nu$ in $F$ satisfy

$$\lambda^+_E(\nu) < -\lambda < 0 < \lambda < \lambda^-_F(\nu).$$

2. Let $\Gamma \subset \Lambda$ be a compact invariant hyperbolic set of $s$-index $\dim E$ and denote by $\mathcal{M}(\Gamma)$ the set of measures supported in $\Gamma$.

Then, there are $\varkappa > 0$ and $\lambda > 0$ with the following property: Let $\nu \in \mathcal{M}(\tilde{\Lambda})$ so that $\nu$ belongs to the $\varkappa$-neighborhood of $\mathcal{M}(\Gamma)$ then,

$$\lambda^+_E(\nu) < -\lambda < 0 < \lambda < \lambda^-_F(\nu),$$

where $\lambda^+_E(\nu), \lambda^-_F(\nu)$ are the maximal and minimal Lyapunov exponents of $\nu$ in $E$ and $F$, respectively.

The proof of the above proposition will be sketched in Section 3.1 and completed at the end of Section 3.

As said above, in Section 3 we will give examples of smooth diffeomorphisms having hyperbolic saddle fixed points $p_1, p_2$ such that a convex combination of their Dirac measures is approached by hyperbolic periodic orbits, even with exponents far away from 0, and whose homoclinic and intersection classes are disjoint (see Theorem 1). This illustrates the importance of the domination assumption of our results.

1.2. **Motivation and historical setting.** The question about ergodic measures associated to periodic orbits being dense in the space $\mathcal{M}(f)$ of $f$-invariant measures has been studied also in a more abstract setting, that is, without a priori assuming that there is a differentiable hyperbolic dynamics present. Perhaps, among the first attacking this general question was Sigmund [26, 27] who showed that provided the dynamical system satisfies a so-called periodic orbit specification property then the ergodic measures and, in particular, the periodic orbit measures are dense in $\mathcal{M}(f)$. Sigmund’s theorem applies to basic sets of axiom A diffeomorphisms [26, 27]. Roughly speaking, this property says that given an arbitrary number of arbitrarily long orbit segments, one can find a periodic orbit which stays $\varepsilon$-close to each of those segments and between one segment and the next one needs a fixed number of iterations which only depends on $\varepsilon$ (see [27] for details). In the context of a basic set of an axiom A diffeomorphism, the existence of a Markov partition guarantees a symbolic 1-1 description of essentially all orbits. The shadowing property enables to
arbitrarily connect given (“specified”) orbit pieces of arbitrary length given by certain symbolic sequences. Concatenating these symbolic sequences infinitely often, leads to periodic shadowing orbits. As the Markov partition can be chosen with arbitrarily small diameter, this procedure enables the “specification” and production of periodic orbits with an arbitrary precision. Sigmund’s specification property has been verified also in a number of topological dynamical systems such as, for example, topologically mixing subshifts of finite type [12] and continuous topologically mixing maps on the interval [5, 9]. An earlier related result has been obtained by Parthasarathy [21].

The shadowing lemma holds more generally for transitive (uniformly) hyperbolic sets with some caution: if Λ is a topologically transitive hyperbolic set, it admits a compact neighborhood in which the maximal invariant set ˜Λ is a topologically transitive hyperbolic set, and the pseudo orbits in Λ are shadowed by orbits in ˜Λ: as a consequence, the convex hull of all invariant measure supported in Λ is contained in the closure of measures supported on periodic orbits contained in Λ.

For a general diffeomorphism, the periodic orbit specification property beyond a uniformly hyperbolic context is often difficult to verify or fails to hold true. There exist various extensions of Sigmund’s result studying weaker versions. Perhaps the most interesting line of extension in the spirit of our paper are the so-called approximate product property [22] and g-almost product property [23] initiated by Pfister and Sullivan. Their variations of the specification property, roughly speaking, “allows to make some number of errors” in the resulting shadowing orbits up to some number of iterations which decays sufficiently fast when the length of the specified orbits growths to infinity. Under any of their conditions they show that in the space of invariant measures ergodic measures are still dense.

Much closer to our approach in the present paper is perhaps Hirayama [16] who studied a general $C^{1+α}$ topologically mixing diffeomorphism preserving a ergodic hyperbolic probability measure. He shows that then there exists a measurable set Γ of full measure $μ$ such that the set of all measures supported on periodic hyperbolic orbits is dense in the (convex) space of invariant measures supported by Γ, meaning that Γ has full measure for any measure in this space.

All these studies tried to established density of ergodic (periodic) measures. However, to the best of our knowledge, little is known about dynamical systems which do not have such a property. Clearly, one has to disregard cases where the system dynamically splits into “basic pieces” such as, for example, attractor-repeller pairs or a similar family of unrelated disjoint compact invariant sets. More precisely, Conley theory [10] divides the dynamics of any homeomorphism on a compact metric space into chain recurrent classes. A non-ergodic measure $μ$ cannot be approached by ergodic measures if it is not supported on a unique chain recurrent class, and in particular if its disintegration in ergodic measures gives positive weight in measures supported in distinct chain recurrent classes.

However, the undecomposability property of the chain recurrent classes is a very weak property and it is easy to build examples where non-ergodic measures are not approached by ergodic one. Another natural candidate for being an “elementary piece” of a diffeomorphisms are the homoclinic classes of the hyperbolic periodic orbit, which are the closure of their homoclinic intersection. For $C^1$-generic
diffeomorphisms, the homoclinic classes are the chain recurrence classes containing periodic points, leading to the impression that, in that setting the basic pieces of the dynamics are well defined. The homoclinic classes are the closure of an increasing sequence of hyperbolic basic sets, leading to a good understanding of at least a part of the dynamics contained in it. However, the ergodic theory inside homoclinic classes is in general not understood. [6, Conjecture 2] proposes that for $C^1$-generic diffeomorphisms the ergodic measures supported in a homoclinic class are approached by periodic orbits contained in the class. In the opposite direction, in [13, 14] there can be found an example of a diffeomorphism having a non trivial homoclinic class with a hyperbolic periodic orbit which is isolated from the other ergodic measure supported on the class. We note that this example fits into the hypothesis of Theorem 1. But, besides the existence of examples, it remains a priori unclear what general mechanism causes that the closure of the set of invariant ergodic probability measures splits into distinct components. According to our main results, we can now give a refined statement towards the above conjecture: under the assumptions of $C^1$ domination, for an individual diffeomorphism any ergodic hyperbolic measure can be approached only by periodic orbits contained in its intersection class.

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2. Preliminaries

2.1. Hyperbolicity and dominated splitting. By Oseledets’ multiplicative ergodic theorem, given $\mu \in \mathcal{M}$, for $\mu$-almost every $x$ there are a positive integer $s(x) \leq \dim M$, a $df$-invariant splitting

$$T_x M = \bigoplus_{i=1}^{s(x)} E^i_x,$$

and numbers $\chi_1(x) < \ldots < \chi_{s(x)}(x)$, called the Lyapunov exponents of $x$, such that for all $i = 1, \ldots, s(x)$ and $v \in E^i_x \setminus \{0\}$ we have

$$\chi_i(x) = \lim_{|n| \to \infty} \frac{1}{n} \log \|df^n_x(v)\|.$$

We call such points $x$ Lyapunov regular with respect to $f$ (see for example [20] for details on Lyapunov regularity). Moreover, $\chi_i(\cdot)$ are $\mu$-measurable functions and we denote by

$$\chi_i(\mu) \overset{\text{def}}{=} \int \chi_i(x) d\mu(x).$$

the Lyapunov exponents of the measure $\mu$ (observe that we allow $\mu$ to be non-ergodic). For a Lyapunov regular point $x$ let us denote by $E^s_x$ (by $E^u_x$) the span of all subspaces of $T_x M$ that correspond to a negative Lyapunov exponent (a positive Lyapunov exponent). The stable index or simply $s$-index (unstable index or $u$-index) of a Lyapunov regular point $x$ is the dimension of $E^s_x$ (of $E^u_x$).

We say that $\mu$ is hyperbolic if for $\mu$-almost every $x$ there is $1 \leq \ell = \ell(x) < s(x)$ such that

$$\chi_\ell(x) < 0 < \chi_{\ell+1}(x)$$

(there are negative and positive but no zero Lyapunov exponents). If $\mu$ is ergodic then $s(\cdot)$, $\chi_i(\cdot)$, as well as the dimensions of $E^s(\cdot)$ and $E^u(\cdot)$ are constant almost
everywhere. Correspondingly to what is defined above, the stable index (unstable index) of a hyperbolic measure is the stable index (unstable index) of almost every Lyapunov regular point.

A \( f \)-invariant set \( \Gamma \subset M \) is hyperbolic if there exists a \( df \)-invariant splitting \( E^s \oplus E^u = T_\Gamma M \) of the tangent bundle and positive constants \( C \) and \( \lambda \) such that for every \( x \in \Gamma \) for every \( n \geq 0 \) we have

\[
\|df^N_x(v)\| \leq C e^{n\lambda} \|v\| \quad \text{for all } v \in E^s_x,
\]

\[
\|df^{-n}_x(w)\| \leq C e^{n\lambda} \|w\| \quad \text{for all } w \in E^u_x.
\]

Note that for a compact \( f \)-invariant hyperbolic set, up to a smooth change of metric, we can assume \( C = 1 \).

A set \( \Gamma \subset M \) is locally maximal if there exists an open neighborhood \( U \) of \( \Gamma \) such that \( \Gamma = \bigcap_{k \in \mathbb{Z}} f^k(U) \). A set \( \Gamma \subset M \) is transitive if it is the closure of a positive orbit. Recall that a set is basic (with respect to \( f \)) if it is compact, invariant, transitive, locally maximal, and hyperbolic.

Given a \( f \)-invariant set \( \Gamma \), a \( df \)-invariant splitting \( T_\Gamma M = E \oplus F \) is dominated if there exists \( N \geq 1 \) such that for every point \( x \in \Gamma \), every unitary vectors \( v \in E^s_x \) and \( w \in F^u_x \) we have

\[
\|df^N_x(v)\| \leq \frac{1}{2} \|df^N_x(w)\|
\]

and if \( \dim E^s_x \) (and hence \( \dim F^u_x \)) does not depend on \( x \in \Gamma \). It will be denoted by \( E \oplus_{<} F \).

Note that a dominated splitting \( E \oplus_{<} F \) is always continuous and extends to the closure \( \Gamma \) of \( \Gamma \). Moreover, for a sufficiently small neighborhood \( V \) of \( \Gamma \), considering the maximal invariant set \( \Gamma \) in \( V \), there is a unique dominated splitting on \( \Gamma \) which extends \( E \oplus_{<} F \) (see [8] for more details). We call such \( \Gamma \) a dominated extension of \( (\Gamma, E \oplus_{<} F) \).

If \( \Gamma \subset \Lambda \) is a hyperbolic set then its associated splitting \( E^s \oplus E^u = T_\Lambda M \) is dominated.

Given a dominated splitting \( E \oplus_{<} F \), it may happen that the bundles \( E \) and \( F \) can be further decomposed into subbundles satisfying a domination condition and there always exists a (unique) finest dominated sub-splitting in the sense that it is not further decomposable. For \( \mu \in M \) an ergodic measure, the bundles of any dominated splitting on the support of \( \mu \) can be written, for \( \mu \)-generic points, as sums of the Oseledets spaces of groups of increasing Lyapunov exponents. We will, however, in this paper disregard such finer splittings. We only consider a dominated splitting into two bundles which separates positive and negative exponents alone. These subbundles are denoted \( E^s \) and \( E^u \), respectively, and if this splitting is dominated we say that the stable/unstable splitting of \( \mu \) is dominated.

We note that, if \( E \oplus_{<} F \) is a dominated splitting defined over \( \text{supp } \mu \) which has the same \( s \)-index such as \( \mu \) then the stable/unstable splitting \( E^s \oplus E^u \) over \( \text{supp } \mu \) is dominated.

Consider a dominated splitting \( E \oplus_{<} F = T_\Lambda M \). Observe that the function sequences \( (\psi_n)_n \) given by

\[
(1) \quad \psi_n(x) = \log \|df^n_{x,E^s}\|
\]
is subadditive, that is, for every \(n, m \geq 1\) we have \(\psi_{n+m} \leq \psi_n + \psi_m \circ f^n\). Observe also that the sequence \((\phi_n)_n\) given by
\[
\phi_n(x) = -\log \| df^n / f_n(x) \|^{-1} = \log \| df^{-n} / f_{-n}(x) \|
\]
is also subadditive. Hence, given \(\mu \in \mathcal{M}\), by Kingmann’s ergodic theorem we have
\[
\lambda^+_E(\mu) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \int \log \| df^n / f_E \| \, d\mu = \inf_{n \geq 1} \frac{1}{n} \int \log \| df^n / f_E \| \, d\mu
\]
\[
\lambda^-_F(\mu) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{1}{n} \int \int \log \| df^n / f_F \|^{-1} \, d\mu = \sup_{n \geq 1} \frac{1}{n} \int \int \log \| df^n / f_F \|^{-1} \, d\mu
\]
and we call these numbers the maximal Lyapunov exponent of \(\mu\) in \(E\) and the minimal Lyapunov exponent of \(\mu\) in \(F\), respectively. If \(\mu\) was ergodic, then for \(\mu\)-almost every \(x\) we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \| df^n / f_E \| = \lambda^+_E(\mu), \quad \lim_{n \to \infty} \frac{1}{n} \log \| df^n / f_F \|^{-1} = \lambda^-_F(\mu).
\]
We will study these exponents in more detail in Section 3.3.

2.2. Invariant manifolds in the \(C^1\) dominated setting. The usual Pesin theory requires a \(C^{1+\alpha}\) regularity of the dynamics (see \textsuperscript{24, 7} which show that Pesin theory usually does not hold in the \(C^1\)-setting). We use here the \(C^1\)-dominated Pesin theory developed in \textsuperscript{11}, which holds only when the stable/unstable bundle is dominated. From this theory, we derive in particular the size of local stable/unstable manifolds.

Let \(f\) be a \(C^1\) diffeomorphism. The (Pesin) stable set at a point \(x \in M\) is defined by
\[
W^s(x) \overset{\text{def}}{=} \{ y \in M : \lim_{n \to \infty} \frac{1}{n} \log d(f^n(y), f^n(x)) < 0 \}.
\]
The (Pesin) unstable set at \(x\) is defined to be the stable set at \(x\) with respect to \(f^{-1}\) and denoted by \(W^u(x)\).

In this section we continue to assume that \(\Lambda\) is compact \(f\)-invariant and that its tangent bundle carries a dominated splitting \(T\Lambda M = E \oplus F\). We fix cone fields \(\mathcal{C}^E, \mathcal{C}^F\) around the line bundles \(E, F\) which are strictly invariant. More precisely, for every \(x \in \Lambda\) the open cone \(\mathcal{C}^E_x \subset T_x M\) contains \(E_x\), is transverse to \(F_x\), and the image of its closure under \(df_x\) is contained in \(E_{f(x)}\). Analogously we define \(\mathcal{C}^F\) being invariant with respect to \(df^{-1}_x\). Such cone fields can be extended to cone fields on a small open neighborhood of \(\Lambda\), keeping all the above given properties.

Given \(x \in \Lambda\) and \(\delta > 0\), we say that a set \(D\) is a \(C^1\) stable disk of radius \(\delta\) centered at \(x\) if there is a \(C^1\) map \(\varphi\) from a ball of radius \(\delta\) centered in \(0\) in \(E_x\) to \(F_x\) such that \(D\) is the graph of the map \(v \mapsto \exp_x(v + \varphi(v))\) and so that the tangent space of \(D\) at each point is tangent to the cone field \(\mathcal{C}^E\). Analogously, we define a \(C^1\) unstable disk.

**Definition 2.1 (\(\delta_0\) and \(\delta_1\)).** By the Plaque family theorem (see \textsuperscript{17} Theorem 5.5) or \textsuperscript{11} there exist some \(\delta_0\) and a continuous family of \(C^1\) stable (unstable) disks centered at points \(x \in \Lambda\) of radius \(\delta_0\) which is locally invariant. Moreover, by choosing \(\delta_0\) small enough, one can ensure that these disks are tangent to the cone field \(\mathcal{C}^E\) (\(\mathcal{C}^F\)). By shrinking if necessary the size of the plaques one may assume that every stable plaque is transverse to any unstable plaque at any point of intersection.
For the following we will fix such families and denote them by $\{D^E_x\}_{x \in \Lambda}$ and $\{D^F_x\}_{x \in \Lambda}$.

Given $N \geq 1$ and $\varepsilon > 0$, let $\delta_1 = \delta_1(N, \varepsilon) \in (0, \delta_0)$ be so small so that for any $x \in \Lambda$, for every $y \in D^E_x$ satisfying $d(y, x) < \delta_1$ for every $v \in T_y D^E_x$ and $w \in T_y D^E_x$ we have

$$
||df^N_y(v)|| \leq e^{N\varepsilon} ||df^N_y|| ||v||, \quad ||df^N_y(w)|| \leq e^{N\varepsilon} (||df^N_y\|^{-1}) ||w||.
$$

In the whole paper, when we consider a set $\Lambda$ (or $\tilde{\Lambda}$) with a dominated splitting $E \oplus_F F$, we always endow it implicitly with a continuous family of locally invariant plaques. We say that a point $x \in \Lambda$ has a (local) stable (resp. unstable) manifold of size $\delta$ if there is a disc $D$ of radius $\delta$ centered at $x$ in the plaque $D^E_x$ (resp. $D^F_x$) and contained in the stable set of $x$.

Just by transversality of the stable and unstable plaques families $\{D^E_x\}_{x \in \Lambda}$ and $\{D^F_x\}_{x \in \Lambda}$ one gets the following lemma which will be often used in this paper.

**Lemma 2.2.** For every $\delta > 0$ there exists $\eta = \eta(\delta) > 0$ such that for every pair of points $x, y \in \Lambda$ which satisfy $d(x, y) \leq \eta$ and which both have $C^1$ stable and unstable disks of radius at least $\delta$ these disks intersect transversally and cyclically, that is, the stable disk of $x$ intersects the unstable disk of $y$ and vice versa.

### 2.3. The space of measures and convex hulls.

Consider the space of all Borel probability measures on $\Lambda$. It is well known that equipped with the weak* topology it is a compact metrizable topological space [29, Chapter 6.1]. If $\{\varphi_n\}_{n \geq 1}$ is a dense subset of the space $C^0(\Lambda)$ of continuous functions on $\Lambda$ then

$$
D(\mu, \nu) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{2^n ||\varphi_n||_{\infty}} \left| \int \varphi_n \, d\mu - \int \varphi_n \, d\nu \right|, \quad ||\varphi_n||_{\infty} = \sup_{x \in \Lambda} |\varphi_n(x)|
$$

provides a metric on this space giving the weak* topology. We will use the fact that in the weak* topology $\mu_n \to \mu$ if, and only if, for every $\varphi \in C^0(\Lambda)$ we have $\int \varphi \, d\mu_n \to \int \varphi \, d\mu$.

Recall that the subspace $M(\Lambda) \subset M(f)$ of all $f$-invariant Borel probability measures on $\Lambda$ is a non-empty Choquet simplex (see [29, Chapter 6.2]). In particular, it is convex and compact. The extreme points of $M$ are the ergodic measures. Any point in a Choquet simplex is represented by a unique probability measure on the extreme points – this point of view is often taken to show the ergodic decomposition of nonergodic measures.

Recall that the **convex hull** $\text{ch}(A)$ of a subset $A$ of the locally convex space $M(\Lambda)$ is the smallest convex set containing $A$. Moreover, we have

$$
\text{ch}(A) = \bigcup_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \lambda_j \mu_j : \lambda_j \in [0, 1], \sum_{j=1}^{n} \lambda_j = 1, \mu_j \in A \right\}
$$

(see [28, Theorem 5.2(i)-(ii)]). Given $\varkappa > 0$ and $V_1, V_2 \subset M(\Lambda)$, we denote

$$
\text{ch}_{\varkappa}(V_1 \cup V_2) \overset{\text{def}}{=} \left\{ \mu \in M(\Lambda) : D(\mu, \nu) < \varkappa \text{ for some } \nu \in \text{ch}(V_1 \cup V_2) \right\}.
$$

### 3. Hyperbolic measures in convex hulls.

In this section we would first like to understand how hyperbolicity of ergodic measures passes on to hyperbolicity of measures on their convex hull as well as to ergodic measures which approach this convex hull in the weak* topology. In particular, we will prove Proposition 12.
The idea of the proof is very close to the one of the upper semi-continuity of the largest Lyapunov exponent of ergodic measures. We explain the idea of the proof with some detail in Section 3.1. Then in Section 3.2 we will derive some abstract results about subadditive sequences of continuous functions. Finally, we will apply a “uniformization step” inspired by [1] to deal with the limits (4) defining the maximal/minimal Lyapunov exponents. At the end of Section 3.3 we will arrive at a version of [1, Lemma 8.4] which is adapted to our needs.

3.1. Idea of the proof. By definition of the weak* topology, the average of a continuous function varies (uniformly) continuously with the measure. Proposition 1.2 would be very easy if the Lyapunov exponents were given by averages of continuous functions.

The Lyapunov exponents are not given by integrating a continuous function. However, the largest Lyapunov exponent of an ergodic measure $\mu$ can be approached from above by the averages of a decreasing sequence of continuous functions, namely $\frac{1}{n_i} \log \| df^{n_i} \|$ where $n_i$ divides $n_{i+1}$. This shows the upper semi-continuity of the largest Lyapunov exponent of ergodic measures, and the lower semi-continuity of its smallest Lyapunov exponent. In the general setting (without the domination hypothesis) this semi-continuity is in the bad direction for proving the hyperbolicity of the ergodic measures in the neighborhood of an hyperbolic measure.

This argument holds in restriction to any $df$-invariant continuous subbundle, as are the bundles $E$, $F$ in a dominated splitting $E \oplus F$. For proving the hyperbolicity of $s$-index $\text{dim } E$-measures $\nu_n$ supported on $\Lambda$, we need to show that the largest exponent in $E$ is negative and that the smallest Lyapunov exponent in $F$ is positive. This time, the semi-continuity is in the good direction!

To simplify the exposition, we present the proof of item 1. of Proposition 1.2 for two measures only. Given two hyperbolic ergodic measures $\mu_1$, $\mu_2$ supported on $\Lambda$, we will fix an iterate $N$ large enough so that the averages of $\frac{1}{N} \log \| df^N \|$ and $\frac{1}{N} \log \| df^{-N} \|$ approach simultaneously the weakest Lyapunov exponents of $\mu_1$ and $\mu_2$ on $E$ and $F$, respectively, and in particular are larger (in absolute value) than some constant $\lambda$. Thus these estimates continue to hold for any (non-ergodic) measure $s\mu_1 + (1-s)\mu_2$, $s \in [0, 1]$ and, by continuity of the integral of continuous function, also in a neighborhood of this segment. Now, this will give bounds of the weakest Lyapunov exponents of ergodic measures $\nu$ supported on $\Lambda$ and contained in $\text{ch}_\nu(\{\mu_1, \mu_2\})$. See Corollaries 3.3 and 3.4.

In the case we consider a compact invariant hyperbolic set $\Gamma$ in $\Lambda$, that is, to prove item 2. of Proposition 1.2 we need to find a large iterate $N$ so that $\frac{1}{N} \log \| df^N \|$ and $\frac{1}{N} \log \| df^{-N} \|$ are negative on $\Gamma$. For that we use uniform hyperbolicity. Now it remains to observe that the integral of the (continuous on $\Lambda$) functions $\frac{1}{N} \log \| df^N \|$ and $\frac{1}{N} \log \| df^{-N} \|$ are still negative in a sufficiently small neighborhood of $\text{M}(\Gamma)$ in $\text{M}(\Lambda)$. See Lemma 3.5 and Corollary 3.6.

3.2. On convergence of subadditive sequences. In the course of this subsection only, let $f$ be a general homeomorphism of a compact metric space $M$. Let $(\psi_n)_{n=1}^\infty$ be a subadditive sequence (with respect to $f$) of continuous functions $\psi_n : M \to \mathbb{R}$, in the sense that, for every $n, m \geq 1$ one has

$$\psi_{n+m} \leq \psi_n + \psi_m \circ f^m.$$
We always assume here that the sequence \((\psi_n/n)_n\) is uniformly bounded from below.

Recall that, given \(\mu \in \mathcal{M}(f)\), by Kingman’s subadditive ergodic theorem, \(\psi_n/n\) converges \(\mu\)-almost everywhere to a measurable function \(\psi\). Moreover, \(\psi^+ = \max\{0, \psi\}\) is \(\mu\)-integrable and

\[
\psi(\mu) \overset{\text{def}}{=} \lim_{n \to \infty} \int \frac{1}{n} \psi_n \, d\mu = \inf_{n \geq 1} \int \frac{1}{n} \psi_n \, d\mu
\]

(the limit could be \(-\infty\)). In particular, for every \(\mu \in \mathcal{M}\) and every \(n \geq 1\) we have

\[
\psi(\mu) \leq \int \frac{1}{n} \psi_n \, d\mu.
\]

If \(\mu\) is ergodic then for \(\mu\)-almost every \(x\) we have \(\psi(\mu) = \lim_{n \to \infty} \psi_n(x)/n\). Moreover, as we assume that \(\psi_n/n\) are uniformly bounded from below then, in fact, \(\psi\) is integrable and

\[
\psi(\mu) = \int \psi \, d\mu.
\]

**Lemma 3.1.** Let \(\mu_1, \mu_2 \in \mathcal{M}(f)\) be two ergodic \(f\)-invariant probability measures. Then for every \(\gamma > 0\) there are a positive integer \(n_0 = n_0(\gamma, \mu_1, \mu_2)\) and a positive number \(\varkappa = \varkappa(\gamma, \mu_1, \mu_2, n_0)\) such that for every \(s \in [0, 1]\) and for every \(\nu \in \mathcal{M}(f)\) at distance \(D(\nu, s\mu_1 + (1-s)\mu_2) < \varkappa\), we have

\[
\lambda - \frac{\gamma}{2} < \int \frac{1}{n_0} \psi_{n_0} \, d\nu < \lambda + \frac{\gamma}{2},
\]

where \(\lambda = s\psi(\mu_1) + (1-s)\psi(\mu_2)\).

**Proof.** Given \(\gamma > 0\), for \(i = 1, 2\) there exist \(n_i = n_i(\gamma, \mu_i) \geq 1\) such that for every \(N \geq n_i\) we have

\[
\psi(\mu_i) \leq \int \frac{1}{N} \psi_N \, d\mu_i < \psi(\mu_i) + \frac{\gamma}{6}.
\]

Let \(n_0 = \max\{n_1, n_2\}\). Now choose \(\varkappa = \varkappa(\gamma, \mu_1, \mu_2, n_0) > 0\) sufficiently small such that for every \(s \in [0, 1]\) and every \(\nu \in \mathcal{M}(f)\) satisfying \(D(\nu, s\mu_1 + (1-s)\mu_2) < \varkappa\) we have

\[
\left| \int \frac{1}{n_0} \psi_{n_0} \, d\nu - \int \frac{1}{n_0} \psi_{n_0} (s\mu_1 + (1-s)\mu_2) \right| < \frac{\gamma}{6}.
\]

This implies the claimed property. \(\square\)

Finally, the following result turns the asymptotic growth of the subadditive function sequence into an estimate for the convergence of a certain Birkhoff average. If \(\psi_n\) would be the functions \([1]\) or the functions \([2]\) and if we would be only interested in estimating the maximal in \(E\) and minimal in \(F\) Lyapunov exponents \([3]\), respectively, then Lemma \([3.1]\) above would be completely sufficient as exponents are constant along orbits and the problem with ergodicity of some iterate \(n_0\) of \(f\) does not appear (see the proof of Lemma \([3.2]\) below). However, for estimating the size of stable/unstable manifolds using the approach in Section \([4]\) we will need to control Birkhoff averages (compare the hypotheses \([8]\) in Proposition \([4.3]\)) and hence require the following lemma.
Lemma 3.2. Let $\nu$ be an $f$-ergodic measure, $n_0$ a positive integer, $\lambda$ a number, and $\gamma$ a positive number such that

$$\int \frac{1}{n_0} \psi_{n_0} \, d\nu < \lambda + \frac{\gamma}{2}. $$

Then with

$$N_0 \overset{\text{def}}{=} \gamma^{-1} 4 n_0 \sup \{\psi_1\}$$

for every $N \geq N_0$ for $\nu$-almost every $x$ we have

$$\lim_{k \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \psi_N(f^{\ell N}(x)) = \int \frac{1}{N} \psi_N \, d\nu < \lambda + \gamma.$$ 

Proof. We follow the method of proof of [1 Lemma 8.4]. We have to study the Birkhoff sums relative to the dynamics of $f^N$ for $N \geq N_0$. Let us first look at $f^N$. It may happen that $\nu$ is not ergodic for $f^N$ and hence decomposes as $\nu = \frac{1}{m} (\nu_1 + \ldots + \nu_m)$ where $m = m(\nu) \geq 2$ divides $n_0$ and each $\nu_i$ is an $f^{n_0}$-ergodic probability measure such that $\nu_{i+1} = f_\gamma \nu_i$ for each $i \pmod{m}$. In this case let $A_1 \cup \ldots \cup A_m$ be a measurable partition of $\Lambda$ such that $f(A_i) = A_{i+1}$ for each $i \pmod{m}$ and $\nu_i(A_i) = 1$. Since $\nu$ is the average of $\psi_{n_0}/n_0$ over the $f^{n_0}$-ergodic components $\nu_i$, by hypothesis there exists an index $i_0 = i_0(\nu) \in \{1, \ldots, m\}$ such that

$$\int \frac{1}{n_0} \psi_{n_0} \, d\nu_{i_0} < \lambda + \frac{\gamma}{2}. $$

Applying Kingman to $(f, \nu)$, for $\nu$-almost every $x$ we obtain

$$\lim_{n \to \infty} \frac{1}{n} \psi_n(x) \leq \int \frac{1}{n_0} \psi_{n_0} \, d\nu_{i_0} < \lambda + \frac{\gamma}{2}. $$

On the other hand, relative to $(f^{n_0}, \nu_{i_0})$ for $\nu$-almost every $x$ in particular there is $j = j(x) \in \{0, \ldots, n_0 - 1\}$ such that $f^j(x) \in A_{i_0}$ and the Birkhoff sum of $\psi_{n_0}/n_0$ at $f^j(x)$ converges to some number smaller than $\lambda + \gamma/2$. Given $N \geq N_0$, we decompose the $f$-orbit of $x$ of length $N$ into segments of length $n_0$ as

$$(x, f(x), \ldots, f^{j-1}(x)), $$

$$(f^j(x), \ldots, f^{j+n_0-1}(x)), \ldots, (f^{j+(r-2)n_0}(x), \ldots, f^{j+(r-1)n_0-1}(x)) $$

$$(f^{j+(r-1)n_0}(x), \ldots, f^{N-1}(x)) $$

with $j + rn_0 \geq N$, where all points $f^j(x), f^{n_0}(f^j(x)), \ldots, f^{(r-1)n_0}(f^j(x))$ belong to $A_{i_0}$. Using subadditivity, we obtain

$$\psi_N(x) \leq \sum_{\ell=0}^{j-1} \psi_1(f^\ell(x)) + \sum_{\ell=0}^{r-1} \psi_{n_0}(f^{j+\ell n_0}(x)) + \sum_{\ell=j+(r-1)n_0}^{N-1} \psi_1(f^\ell(x)). $$

We continue decomposing the orbit segments

$$(f^N(x), \ldots, f^{2N-1}(x)), \ldots, (f^{(k-1)N}(x), \ldots, f^{kN-1}(x)) $$

the same way and obtain

$$\sum_{\ell=0}^{k-1} \psi_N(f^{\ell N}(x)) \leq 2kn_0C + \sum_{\ell=0}^{k-1} \psi_{n_0}(f^{j+\ell n_0}(x)).$$
From $f^j(x) \in A_{i_0}$ we conclude
\[
\lim_{k \to \infty} \frac{1}{kN_0} \sum_{\ell=0}^{k-1} \psi_{\iota_0}(f^{\ell\iota_0}(f^j(x))) = \int \frac{1}{\iota_0} \psi_{\iota_0} \, d\nu_{\iota_0} < \lambda + \frac{\gamma}{2}.
\]
Hence, we obtain for $\nu$-almost every $x$
\[
\lim_{r \to \infty} \frac{1}{rN} \sum_{\ell=0}^{r-1} \psi_N(f^{\ell N}(x)) \leq \frac{2N_0C}{N} + \int \frac{1}{\iota_0} \psi_{\iota_0} \, d\nu_{\iota_0} < \lambda + \gamma.
\]
This proves the lemma. \qed

3.3. Uniformed contraction and expansion. We now return to consider a $C^1$ diffeomorphism $f$ in the locally maximal compact invariant set $\Lambda \subset M$. Let $\hat{\Lambda}$ be the maximal invariant set in a neighborhood of $\Lambda$ with a dominated extension.

We study the maximal Lyapunov exponent in $E$ and the minimal Lyapunov exponent in $F$ in a given dominated splitting $E \oplus_F F = T_{\hat{\Lambda}}M$ defined in (3).

Crucial in our approach that the considered functions $\psi_n$ defined in (1) (and the functions in (2)) are continuous in $\hat{\Lambda}$.

Corollary 3.3. Let $\mu_1$ and $\mu_2$ be hyperbolic ergodic measures supported in $\Lambda$ and so that their stable/unstable splitting is the dominated splitting $E \oplus_F F$.

Then for every $\gamma > 0$ there are a positive integer $N_0 = N_0(\gamma, \mu_1, \mu_2)$ and a positive number $\zeta = \zeta(\gamma, \mu_1, \mu_2)$ such that for every $N \geq N_0$ for every $s \in [0, 1]$ and $f$-ergodic $\nu \in \mathcal{M}(\hat{\Lambda})$ satisfying $D(\nu, s\mu_1 + (1-s)\mu_2) < \zeta$, for $\nu$-almost every $x$ we have
\[
\lim_{k \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \|df^N_{/E_x}f_{\iota_0}(x)\| = \int \frac{1}{N} \log \|df^N_{/E_x}\| \, d\nu < \chi^+ + \gamma,
\]
where $\chi^+ \defeq s\lambda^+_E(\mu_1) + (1-s)\lambda^+_E(\mu_2)$, and for $\nu$-almost every $x$ we have
\[
\chi^- - \gamma < \lim_{k \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \|\left(\frac{df^N_{/F_x}}{f_{\iota_0}(x)}\right)^{-1}\|^{-1} = \int \frac{1}{N} \log \|\left(\frac{df^N_{/F_x}}{f_{\iota_0}(x)}\right)^{-1}\|^{-1} \, d\nu,
\]
where $\chi^- = s\lambda^-_F(\mu_1) + (1-s)\lambda^-_F(\mu_2)$.

Proof. Observe that the sequence $\psi_n = \log \|df^n_{/E}\|$ is a subadditive sequence of continuous functions on $\hat{\Lambda}$ so that $\psi_n/n$ for all $n$ is uniformly bounded from below as follows:
\[
\frac{\psi_n}{n} > \log \min \|df\|^{-1}.
\]
Now apply Lemmas 3.1 and 3.2 to obtain the first result.

To show the second result follows analogously. Just observe that the sequence of continuous functions $\psi_n = \log \|\left(\frac{df^N_{/F_x}}{f_{\iota_0}(x)}\right)^{-1}\|$ is subadditive with $\psi_n/n$ bounded from below by $- \log \max \|df\|$. \qed

Corollary 3.4. Let $\mu_1$ and $\mu_2$ be two hyperbolic ergodic measures supported in $\Lambda$ and so that their stable/unstable splitting is the dominated splitting $E \oplus_F F$.

Then there exists a positive number $\zeta$ such that every ergodic $\mu \in \text{ch}_s(\{\mu_1, \mu_2\})$ in $\mathcal{M}(\hat{\Lambda})$ is hyperbolic with the same $s$-indices such as $\mu_1, \mu_2$.  

Proof. Let us choose \( \gamma \) small enough such that
\[
\gamma < \frac{1}{2} \min \{ |\lambda_E(\mu_i)|, \lambda_F(\mu_i): i = 1, 2 \}.
\]
Then let \( \nu = \nu(\gamma, \mu_1, \mu_2) > 0 \) be as in Corollary 3.3 and check that the bound in this corollary is negative and bounds the maximal Lyapunov exponent in \( E \) from above and positive and bounds the minimal Lyapunov exponent in \( F \) from below, respectively. \( \square \)

The convex hull of more general (compact) sets may be more difficult to analyze. However, we have the following result for the convex hull of the simplices of two hyperbolic sets provided that there is an ambient dominated splitting.

**Lemma 3.5.** Let \( \Lambda \subset M \) be a compact \( f \)-invariant set with dominated splitting \( T_{\Lambda}M = E \oplus F \). Let \( \Gamma \subset \Lambda \) be a compact invariant uniformly hyperbolic set of \( s \)-index \( \dim E \) and denote \( V = \mathcal{M}(\Gamma) \).

Then there is a positive number \( \chi \) such that for all sufficiently small \( \gamma > 0 \) there are a positive integer \( n_0 = n_0(\Gamma) \) and a positive number \( \kappa = \kappa(\gamma, \Gamma) > 0 \) such that for every \( \nu \in \text{ch}_\kappa(V) \cap \mathcal{M}(\Lambda) \) we have
\[
\int \frac{1}{n_0} \log \| df^n_E \| \, d\nu < -\chi + \frac{\gamma}{2} \quad \text{and} \quad \chi - \frac{\gamma}{2} < \int \frac{1}{n_0} \log \| (df^n_F)^{-1} \|^{-1} \, d\nu.
\]
Moreover, if \( \nu \) is \( f \)-ergodic then for \( \nu \)-almost every \( x \)
\[
\lim_{n \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \| df^N_{f^\ell(x)} \| < -\chi + \gamma
\]
and
\[
\lim_{n \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \| (df^N_{f^\ell(x)})^{-1} \|^{-1} > \chi - \gamma.
\]

**Proof.** For \( n \geq 1 \) define \( \psi_n(x) \) by \( \| df^n_E \| \). Since \( \Gamma \) is compact \( f \)-invariant and hyperbolic, there is \( n_0 = n_0(\Gamma) \geq 1 \) such that
\[
\chi \overset{\text{def}}{=} -\max_x \frac{1}{n_0} \psi_{n_0}(x) < 0.
\]

By definition of \( \chi \), hence for every \( \mu \in V \) (not necessarily ergodic), by Kingman’s subadditive ergodic theorem, we have
\[
\lim_{n \to \infty} \int \frac{1}{n} \psi_n \, d\mu = \inf_{n \geq 1} \int \frac{1}{n} \psi_n \, d\mu \leq \int \frac{1}{n_0} \psi_{n_0} \, d\mu \leq -\chi.
\]

Let \( \gamma \in (0, \chi/6) \). Now choose \( \nu = \nu(\gamma, \Gamma) > 0 \) such that for every \( \mu \in V \) and \( \nu \in \mathcal{M}(\Lambda) \) satisfying \( D(\nu, \mu) < \nu \) we have
\[
\left| \int \frac{1}{n_0} \psi_{n_0} \, d\nu - \int \frac{1}{n_0} \psi_{n_0} \, d\mu \right| < \frac{\gamma}{4}.
\]
This implies the first claimed property. The statement of the first limit is a consequence of Lemma 3.2

The remaining properties are analogous. \( \square \)
Corollary 3.6. Let $\Lambda \subset M$ be a compact $f$-invariant set with dominated splitting $T_\Lambda M = E \oplus F$. Let $\Lambda_1, \ldots, \Lambda_k \subset \Lambda$ be $k$ compact invariant uniformly hyperbolic sets of $s$-index $\dim E$ and denote $V_i = M(\Lambda_i)$, $i = 1, \ldots, k$.

Then there exists a positive number $\varkappa$ such that every ergodic $\mu \in \operatorname{ch}_\mu(V_1 \cup \ldots \cup V_k)$ in $M(\Lambda)$ is hyperbolic of $s$-index $\dim E$.

Proof. It suffices to apply Lemma 3.2 and Lemma 3.5 to $\Gamma = \Lambda_1 \cup \ldots \cup \Lambda_k$. \qed

Proof of Proposition 1.2. Item 1. in the case of $k = 2$ measures follows from Corollaries 3.3 and 3.4. The case for general $k$ is similar. Item 2. follows from Lemma 3.5 and Corollary 3.6. \qed

3.4. General convex sums of hyperbolic measures. We can show a result similar to Lemma 3.1 under a more general hypothesis considering measures approximating any kind of “convex combination” of ergodic measures, that is, any kind of ergodic decomposition. It is not required for the present paper but interesting in its own right.

Lemma 3.7. Suppose that $\mu \in M(f)$ is an $f$-invariant probability measure such that its ergodic decomposition $\mu = \int \mu_\theta \, d\tau(\theta)$ satisfies $\psi(\mu_\theta) < 0$ for $\tau$-almost every $\theta$.

Then for every sufficiently small $\gamma > 0$ there is a positive integer $n_0 = n_0(\mu, \gamma)$ and a positive number $\varkappa = \varkappa(\mu, \gamma, n_0)$ such that for every $f$-ergodic $\nu$ satisfying $D(\nu, \mu) < \varkappa$ we have

$$
\psi(\mu) - \frac{\gamma}{2} < \int \frac{1}{n_0} \psi_{n_0}(x) \, d\nu < \psi(\mu) + \frac{\gamma}{2} < 0.
$$

Proof. Observe that $\mu \mapsto \psi(\mu)$ defined above is upper semi-continuous and hence measurable.

Given $\mu \in M(f)$ satisfying the hypothesis of the lemma, with ergodic decomposition $\mu = \int \mu_\theta \, d\tau(\theta)$ we have

$$
\psi(\mu) = \int \psi \, d\mu = \int \left( \int \psi \, d\mu_\theta \right) \, d\tau(\theta) = \int \psi(\mu_\theta) \, d\tau(\theta),
$$

where $\mu_\theta \in M_{\operatorname{erg}}(f)$ are ergodic measures. By assumption, the integrand in the latter integral is negative for $\tau$-almost every $\theta$. Hence, $\psi(\mu)$ is negative, say less than $-\gamma$ for some $\gamma \in (0, C)$, where

$$
C \stackrel{\text{def}}{=} \max \psi_1.
$$

Given $\varepsilon > 0$, define

$$
\Sigma_\varepsilon \stackrel{\text{def}}{=} \{ \mu \in M_{\operatorname{erg}}(f): \psi(\mu) \leq -2\varepsilon \}.
$$

By the above, since $\psi(\mu_\theta) < 0$ for $\tau$-almost every $\theta$, $\int_{\Sigma_\varepsilon} \psi(\mu_\theta) \, d\tau(\theta) \lesssim \psi(\mu)$ and $\tau(\Sigma_\varepsilon) \to 1$ as $\varepsilon \to 0$. Choose $\varepsilon > 0$ sufficiently small such that $\tau(\Sigma_\varepsilon) > 1 - \gamma/(32C)$ and

$$
\int_{\Sigma_\varepsilon} \psi(\mu_\theta) \, d\tau(\theta) \leq \psi(\mu) + \frac{\gamma}{8}.
$$

Given $n \geq 1$, define

$$
\Sigma_{\varepsilon, n} \stackrel{\text{def}}{=} \{ \mu \in M_{\operatorname{erg}}(f): \psi(\mu) \leq \int \frac{1}{N} \psi_N \, d\mu < \psi(\mu) + \frac{\gamma}{32} \text{ for every } N \geq n \}.
$$
By this definition and using the simple fact $\tau(\Sigma_{\varepsilon,n}) \leq 1$, for $n \geq 1$ we have
\[
\int_{\Sigma_{\varepsilon,n}} \psi(\mu_\theta) \, d\tau(\theta) \leq \int_{\Sigma_{\varepsilon,n}} \left( \int \frac{1}{n} \psi_n \, d\mu_\theta \right) \, d\tau(\theta) < \int_{\Sigma_{\varepsilon,n}} \psi(\mu_\theta) \, d\tau(\theta) + \frac{\gamma}{32}
\]
and, using again that $\psi(\mu_\theta)$ are nonpositive and that $\Sigma_{\varepsilon,n}$ is an increasing family in $n$, when $n \to \infty$ we have
\[
\int_{\Sigma_{\varepsilon,n}} \psi(\mu_\theta) \, d\tau(\theta) \searrow \int_{\Sigma_\varepsilon} \psi(\mu_\theta) \, d\tau(\theta) \quad \text{and} \quad \tau(\Sigma_{\varepsilon,n}) \nearrow \tau(\Sigma_\varepsilon).
\]
For the complementary set $\Sigma^c_{\varepsilon,n}$ of $\Sigma_{\varepsilon,n}$, by subadditivity of $\psi_n$ we can estimate
\[
\int_{\Sigma^c_{\varepsilon,n}} \psi(\mu_\theta) \, d\tau(\theta) \leq \int_{\Sigma^c_{\varepsilon,n}} \left( \int \frac{1}{n} \psi_n \, d\mu_\theta \right) \, d\tau(\theta) \leq \tau(\Sigma^c_{\varepsilon,n}) C \searrow (1 - \tau(\Sigma_\varepsilon)) C
\]
as $n \to \infty$. Now choose $n_0 \geq 1$ sufficiently large such that for $n \geq n_0$ we have
\[
\int \frac{1}{n_0} \psi_{n_0} \, d\mu = \int \left( \int \frac{1}{n_0} \psi_{n_0} \, d\mu_\theta \right) \, d\tau(\theta)
= \int_{\Sigma_\varepsilon} \left( \int \frac{1}{n_0} \psi_{n_0} \, d\mu_\theta \right) \, d\tau(\theta) + \int_{\Sigma_{\varepsilon,n}} \left( \int \frac{1}{n_0} \psi_{n_0} \, d\mu_\theta \right) \, d\tau(\theta)
\leq \int \psi(\mu_\theta) \, d\tau(\theta) + \frac{\gamma}{16} + (1 - \tau(\Sigma_\varepsilon)) C + \frac{\gamma}{32}
< \psi(\mu) + \frac{\gamma}{8} + \frac{\gamma}{8}.
\]
Moreover, as a consequence of Kingman [\ref{kingman}], we also have
\[
\psi(\mu) \leq \int \frac{1}{n_0} \psi_{n_0} \, d\mu.
\]
Finally choose $\varkappa = \varkappa(\gamma, \mu, n_0) > 0$ sufficiently small such that for every $\nu$ satisfying $D(\nu, \mu) < \varkappa$ we have
\[
\left| \int \frac{1}{n_0} \psi_{n_0} \, d\nu - \int \frac{1}{n_0} \psi_{n_0} \, d\mu \right| < \frac{\gamma}{4}.
\]
This implies the claimed property. \hfill \square

4. Stable and Unstable Manifolds

Let $f$ be a $C^1$ diffeomorphism, $\Lambda$ a compact invariant set with a dominated splitting $E \oplus F$. The function defined below is the main ingredient in \cite{[1]} for controlling the size of the invariant manifolds.

**Definition 4.1.** Let $N$ be a positive integer. Let $\lambda > 0$. For $x \in \Lambda$ we define
\[
C^{E,N}_{-\lambda}(x) \overset{\text{def}}{=} \sup_{k \geq 0} \left( e^{-kN(-\lambda)} \prod_{\ell=0}^{k-1} \| df^N_{f^\ell(x)} \| \right)
\]
whenever this makes sense, with the convention $\prod_{\ell=0}^{k-1} \| df^N_{f^\ell(x)} \| = 1$ for $k = 0$. In the same way, we define
\[
C^{F,N}_{-\lambda}(x) \overset{\text{def}}{=} \sup_{k \geq 0} \left( e^{-kN(-\lambda)} \prod_{\ell=0}^{k-1} \| df^{-N}_{f^{-\ell}(x)} \| \right).
\]
As a consequence of Corollary 3.3 we get the following.
Lemma 4.2. \(\text{Let } \mu_1 \text{ and } \mu_2 \text{ be hyperbolic ergodic measures supported in } \Lambda \text{ and so that their stable/unstable splitting is the dominated splitting } E \oplus \mathcal{F}. \text{ Let } \chi = \min \{|\lambda^+_E(\mu_1)|, \lambda^-_E(\mu_1), i = 1, 2\}. \text{ For } \gamma \in (0, \chi) \text{ let } N_0 = N_0(\gamma, \mu_1, \mu_2) \text{ and } \chi = \chi(\gamma, \mu_1, \mu_2) > 0 \text{ be as provided by Corollary 3.3.}

Then, for any } N \geq N_0, \text{ for any ergodic measure } \mu \in \text{ch}_{\chi}(\{\mu_1, \mu_2\}) \text{ for } \mu\text{-almost every point } x \text{ the numbers } C^-_{-\chi+\gamma}(x) \text{ and } C^+_{-\chi+\gamma}(x) \text{ are well defined (that is } < +\infty). \]

Proof. \(\text{It suffices to observe that, according to Corollary 3.3, the products in (6) and (7) both decay exponentially in } k \text{ with a rate smaller than } -\chi + \gamma < 0. \)

The following result determines in particular the minimal size of the stable and unstable disks for a generic point of an ergodic hyperbolic measure. For the proof of our main results, we will need the explicit determination of this size which we extract from the proof of [11, Proposition 8.9 in Section 8].

Proposition 4.3 ([11 Proposition 8.9]). \(\text{Let } f \text{ be a } C^1 \text{ diffeomorphism, } \Lambda \text{ a compact invariant set with a dominated splitting } E \oplus \mathcal{F}, \text{ and } \mu \text{ an ergodic invariant hyperbolic probability measure supported in } \Lambda \text{ and so that } E \oplus \mathcal{F} \text{ is its stable/unstable splitting.}

\text{Suppose that } \lambda > 0 \text{ is a lower bound for the absolute values of the maximal and minimal Lyapunov exponents of } \mu \text{ in } E \text{ and } \mathcal{F}, \text{ respectively}

\[0 < \lambda \leq \min \{|\lambda^+_E(\mu)|, \lambda^-_E(\mu)|\} \]

\text{Let}

\[1 < C_f \overset{\text{def}}{=} \max_{x \in \Lambda} \left\{ \log \|df_x\|, \log \|(df_x)^{-1}\| \right\} \]

\text{Fix } \epsilon \in (0, \lambda/3). \text{ Let } N \geq 1 \text{ be sufficiently large such that for } \mu\text{-almost every } x \text{ we have}

\[\lim_{k \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \|df^{N}_{f^{\ell}(x)}\| < -\lambda + \epsilon. \tag{8}\]

\text{Let } \delta \in (0, C_f^{-1} \delta_1) \text{, where } \delta_1 = \delta_1(N, \epsilon) \text{ was defined in Definition 2.1.}

\text{Then for } \mu\text{-almost every point } x, \text{ there exist injectively immersed } C^1\text{-manifolds } W^G_{\text{loc}}(y) \text{ of dimension } \text{dim } G \text{ tangent to } G_y, \text{ } G = E, \mathcal{F}, \text{ which are stable manifold and unstable manifold, respectively: for } \mu\text{-almost every } x, \text{ there exists a local stable manifold } W^E_{\text{loc}}(x) \subset W^E(x) \text{ and a local unstable manifold } W^F_{\text{loc}}(x) \subset W^F(x), \text{ where}

\[W^E(x) = \{ y: d(f^n(y), f^n(x)) e^{n\lambda} \to 0 \text{ if } n \to \infty \}, \]

\[W^F(x) = \{ y: d(f^{-n}(y), f^{-n}(x)) e^{n\lambda} \to 0 \text{ if } n \to \infty \}. \]

\text{Moreover, the local stable manifold contains a } C^1 \text{ stable disk and the local unstable manifold contains a } C^1 \text{ unstable disk centered at } x, \text{ of radius}

\[L^*_x(\epsilon) \overset{\text{def}}{=} \delta(C^*_{-\lambda+\epsilon}(x))^{-1} \]

\text{and tangent to } \mathcal{F}^*, \text{ for } * = E, F \text{ respectively, and}

\[W^E(x) = \bigcup_{n \geq 0} f^{-n}(W^E_{\text{loc}}(f^n(x))), \quad W^F(x) = \bigcup_{n \geq 0} f^{-n}(W^F_{\text{loc}}(f^n(x))). \]

We will use the following slightly strengthened version of the statement of [11 Proposition 1.4] which is contained in its proof in [11]. This is an ersatz to Katok’s horseshoe construction (see [18 Supplement S.5]) in the } C^1 \text{ dominated setting.
Proposition 4.4. Let $f$ be a $C^1$ diffeomorphism, $\Lambda$ a compact invariant set with a dominated splitting $E \oplus F$, and $\mu$ an ergodic invariant hyperbolic probability measure supported in $\Lambda$ and of s-index $\dim E$.

Then for every $\sigma \in (0, 1)$ there exists a set $\Gamma_\sigma$ and a number $\delta = \delta(\sigma, \mu) > 0$ such that $\mu(\Gamma_\sigma) > 1 - \sigma$ and for every point in $\Gamma_\sigma$ there is a sequence $(p_n)_n$ of periodic hyperbolic points of s-index $\dim E$ such that:

- $p_n$ converges to $x$ as $n \to \infty$;
- the orbit of $p_n$ converges to the support of $\mu$ in the Hausdorff topology;
- the invariant measures supported on the orbit of $p_n$ converge to $\mu$ in the weak$^*$ topology;
- all $p_n$ have stable and unstable local manifolds of size at least $\delta$.

As a consequence, all but finitely many of the points $p_n$ are pairwise homoclinically related.

Proof. In the proof of [11, Proposition 1.4] considers the set $X = X(c)$ of points for which

$$C^{E,N}_{-\chi+\gamma}(x) < c \quad \text{and} \quad C^{F,N}_{-\chi+\gamma}(x) < c,$$

where $\chi - \gamma$ is the number $\rho$ and $N$ is $\nu$ in [11].

For $c$ large, the measure $\mu(X)$ is positive and Lemma [12] implies that $\mu(X)$ tends to 1 as $c \to +\infty$; we denote by $\Gamma_\sigma$ the set $X(c)$ so that $\mu(X(c)) > 1 - \sigma$. Then, [11] shows that every point $x \in X(c)$ is accumulated by hyperbolic periodic orbits $p_n$ of s-index $\dim E$ so that the orbit of $p_n$ shadows the orbit of $x$ until the period of $p_n$; this proves that the orbits tend to the support of $\mu$ in the Hausdorff topology, and the associated measure tend to $\mu$.

Furthermore, the size of the local invariant manifolds of $p_n$ is larger than some constant which depends only on $X$.

5. Intersection classes

Definition 5.1 (Intersection class). Let $f$ be a diffeomorphism of a compact manifold $M$. Let $\text{Per}_{\text{hyp}}$ be the set of periodic hyperbolic points in $M$. On $\text{Per}_{\text{hyp}}$ we consider the relation $\sim_H$ of being homoclinically related, that is, two periodic hyperbolic points $p \sim_H q$ if, and only if, the invariant manifolds of their orbits meet cyclically and transversely. This defines an equivalence relation on the set $\text{Per}_{\text{hyp}}$. The equivalence classes for $\sim_H$ are called the intersection classes.

Remark 5.2. For every hyperbolic periodic point, its homoclinic class is the closure of its intersection class.

- A given homoclinic class $C$ of a hyperbolic periodic point $p$ may contain several distinct intersection classes. For instance, $C$ may contain points of $s$-indices different from the $s$-index of $p$. More surprising, $C$ may contain hyperbolic periodic points $q$ of the same $s$-index as $p$ but not homoclinically related with $p$: [13] [13] (see also [19]) provide such examples where the homoclinic class of $q$ is strictly contained in $C$.

5Note that we do not consider the closure of the set of the transverse intersection points. To avoid confusion with other common usage of terms we avoid to call such an equivalence class a homoclinic class, although this would be an appropriate name.
Let Λ be a compact invariant set with a dominated splitting \( E_r \oplus F \). We consider a small compact neighborhood \( U \) of \( Λ \) so that this dominated splitting extends on the maximal invariant set \( \hat{Λ} \) in \( U \). Let us call \( \hat{Λ} \) a dominated extension of \( Λ \).

For any ergodic hyperbolic measure \( μ \) of index \( \dim E \) supported in \( Λ \), Proposition 4.4 asserts that for \( μ \)-almost every \( x \) there exists a positive number \( δ = δ(x) \) and a sequence \( (p_n)_{n \geq 1} \) of periodic hyperbolic points \( p_n \in \hat{Λ} \) of index \( \dim E \) such that for every \( n \) the stable and unstable local manifolds of \( p_n \) each have diameter at least \( δ \) and that \( \lim_{n \to \infty} p_n = x \).

**Definition 5.3.** We denote by \( ℐ \) the set of points \( x \in Λ \) so that there exists \( δ \) and a sequence \( (p_n)_{n \geq 1} \) of periodic hyperbolic points \( p_n \in \hat{Λ} \) of index \( \dim E \) such that for every \( n \) the stable and unstable local manifolds of \( p_n \) each have diameter at least \( δ \) and that \( \lim_{n \to \infty} p_n = x \).

**Remark 5.4.** Observe that \( ℐ \) depends on the choice of the dominated extension of \( Λ \). We will write \( ℐ(Λ) \) if we would like to emphasize this dependence.

Note that \( ℐ \) is invariant by \( f \).

**Lemma and Definition 5.5.** For every \( x \in ℐ \), there is a unique intersection class, that we denote by \( H_x \), with the following property:

Consider \( δ > 0 \) and a sequence \( (p_n)_{n \geq 1} \) of periodic hyperbolic points \( p_n \in \hat{Λ} \) of index \( \dim E \) such that for every \( n \) the stable and unstable local manifolds of \( p_n \) each have diameter at least \( δ \) and that \( \lim_{n \to \infty} p_n = x \). As the size of the invariant manifolds of the \( p_n \) is larger than \( δ \) and as these invariant manifolds are tangent the the cone fields given by the dominated splitting, one gets that all but finitely many of the \( p_n \) are pairwise homoclinically related, hence belong to the same intersection class. Let us denote it by \( H_p \).

Let \( δ' > 0 \) and \( \hat{q} = (q_n)_{n \geq 1} \) be a sequence of periodic hyperbolic points \( q_n \in \hat{Λ} \) of index \( \dim E \) such that for every \( n \) the stable and unstable local manifolds of \( q_n \) each have diameter at least \( δ' \) and that \( \lim_{n \to \infty} q_n = x \).

Then, as the invariant manifolds of both \( p_n \) and \( q_n \) are larger than \( \min(δ, δ') \), for \( n \) large enough, \( p_n \) and \( q_n \) are homoclinically related, implying \( H_{\hat{q}} = H_p \). \( \square \)

**Remark 5.6.** If \( x \in Λ \) is a hyperbolic periodic point of index \( \dim E \) then \( H_x \) is the intersection class of \( x \) (that is, the equivalence class of \( x \) for \( \sim_H \)).

**Remark 5.7.** The class \( H_x \) does not depend on the dominated extension of \( Λ \), once \( x \) belongs to \( ℐ \). More precisely, let \( Λ_1 \) and \( Λ_2 \) be two dominated extensions of \( Λ \) and assume that \( x \in ℐ(Λ_1) \cap ℐ(Λ_2) \). Then the class \( H_x \) given by Lemma 5.5 is the same when applying it to \( Λ_1 \) or to \( Λ_2 \).

Note that for any ergodic invariant hyperbolic measure \( μ \) supported in \( Λ \) and any \( σ > 0 \), the set \( Γ_σ \) provided by Proposition 4.4 belongs to \( ℐ(Λ) \) for any dominated extension \( \hat{Λ} \) of \( Λ \). For that note that the periodic points \( p_n \) provided in this proposition belong to \( \hat{Λ} \) for \( n \) large enough. As a consequence, for \( x \in Γ_σ \) we can refer to \( H_x \) without indicating the choice of the dominated extension.
Definition 5.8 (Having same intersection class). On the set \( \mathcal{R} \) we define an equivalence relation \( \approx \) saying that

\[ x \approx y \quad \text{if, and only if,} \quad H_x = H_y. \]

This indeed defines an equivalence relation. We denote by \( [x]_\approx \) the equivalence class containing \( x \).

Observe that each such equivalence class \( [x]_\approx \) is an \( f \)-invariant subset of \( \mathcal{R} \).

Corollary 5.9. Let \( f \) be a \( C^1 \) diffeomorphism, \( \Lambda \) a compact invariant set with a dominated splitting \( E \oplus F \) of \( s \)-index \( \dim E \), and \( \mu \) an ergodic invariant hyperbolic probability measure supported in \( \Lambda \) and of \( s \)-index \( \dim E \) (in other words, \( \mu \) satisfies the hypotheses of Proposition 4.4).

Then there exists an intersection class, that we denote by \( H(\mu) \), such that \( H(\mu) = H_x \) for \( \mu \)-almost all \( x \). In other words there is \( x \in \mathcal{R} \) so that

\[ \mu([x]_\approx) = 1. \]

Proof. As each equivalence class \( [x]_\approx \) is \( f \)-invariant and \( \mu \) is assumed to be ergodic, it is enough to prove that there is \( x \in \mathcal{R} \) with \( \mu([x]_\approx) = 1 \).

Let \( \sigma \in (0,1) \) so that the set \( \Gamma_\sigma \) announced by Proposition 4.4 has positive measure.

Claim 5.10. There is \( \eta > 0 \) so that if \( x, y \in \Gamma_\sigma \) satisfy \( d(x, y) < \eta \) then \( y \in [x]_\approx \).

Proof. Fix a small compact neighborhood \( U \) so that the dominated splitting \( E \oplus F \) extends on the maximal invariant set \( \hat{\Lambda} \) in \( U \).

By definition of \( \Gamma_\sigma \), there are \( \delta > 0 \) so that for every \( x, y \in \Gamma_\sigma \) there are sequences of hyperbolic periodic orbits \( (p_n)_n \) and \( (q_n)_n \) contained in \( \hat{\Lambda} \), converging to \( x \) and \( y \) respectively, and whose invariant manifolds all have a size larger than \( \delta \).

In particular for \( n \) large enough the distance between \( p_n \) and \( q_n \) is much smaller than \( \delta \) and hence \( p_n \) and \( q_n \) are homoclinically related. This implies that the intersection classes \( H_x \) and \( H_y \) coincide. This proves the claim. \( \square \)

The above claim implies that the set \( \Gamma_\sigma \) is contained in finitely many equivalence classes of \( \approx \). Therefore, at least one class \( [x]_\approx \) has positive weight of \( \mu \). This proves the corollary. \( \square \)

A straightforward consequence is the following.

Corollary 5.11. For \( \mu \) satisfying the hypotheses of Proposition 4.4 the closure \( \overline{H(\mu)} \) of the intersection class \( H(\mu) \) contains the support of \( \mu \).

Remark 5.12. Finally let us point out that there exist other concepts in the literature which are related to the above defined intersection class (besides the already mentioned concept of a homoclinic class, see Remark 5.2). They characterize, however, in general quite distinct objects and are used for distinct purposes.

One related concept has been studied for example in [25] (see also [8]). For completeness we will provide its definition. In the context of \( f \) being a \( C^{1+\alpha} \) diffeomorphism, Pesin theory guarantees that for every ergodic hyperbolic measure \( \mu \) there is a Borel set \( \hat{\mathcal{R}} \) of full measure such that for every \( x \in \mathcal{R} \) the stable and
unstable sets $W^s(x)$ and $W^u(x)$ are immersed manifolds (see [4] for details). Let $p$ be a periodic hyperbolic point and denote by $O(p)$ its orbit. Define

$$H^s_{\text{erg}}(p) \overset{\text{def}}{=} \{ x \in \hat{R} : W^s(x) \text{ transversally intersects } W^u(O(p)) \},$$

$$H^u_{\text{erg}}(p) \overset{\text{def}}{=} \{ x \in \hat{R} : W^u(x) \text{ transversally intersects } W^s(O(p)) \},$$

$$H_{\text{erg}}(p) = H^s_{\text{erg}}(p) \cap H^u_{\text{erg}}(p).$$

In [25], the set $H_{\text{erg}}(p)$ is called the ergodic homoclinic class of $p$. It has been introduced to study ergodicity of smooth measures and it is shown in [25] that if $f$ preserves a smooth measure $m$ and if both sets $H^s_{\text{erg}}(p)$ and $H^u_{\text{erg}}(p)$ both have positive measure, then $H_{\text{erg}}(p) = H^s_{\text{erg}}(p) = H^u_{\text{erg}}(p)$ is an ergodic component of $m$.

In our notation, we have

$$H_p = H_{\text{erg}}(p) \cap \text{Per}_{\text{hyp}}$$

and this set is contained in the homoclinic class of $p$.

6. Proof of Theorem 1

Let $f$ be a $C^1$ diffeomorphism, $\Lambda$ be a compact invariant set with a dominated splitting $TM|_{\Lambda} = E \oplus_c F$. We fix a maximal invariant set $\hat{\Lambda}$ in a neighborhood of $\Lambda$ so that the dominated splitting $E \oplus_c F$ extends on $\hat{\Lambda}$.

6.1. Part “if” of Theorem 1: assuming that the intersection classes coincide. Assume that $\mu_1, \ldots, \mu_k$ are ergodic hyperbolic measures of $s$-index equal to $\dim E$ such that $H(\mu_1) = \ldots = H(\mu_k)$. Applying Proposition 4.4 to each of these measures $\mu_j$, there is a sequence of periodic hyperbolic points $(p_{j,n})$ such that the invariant measures $\nu_{j,n}$ supported on the orbit of $p_{j,n}$ converge to $\mu_j$ in the weak* topology. Moreover, by Remark 5.7 and Corollary 5.9 eventually all points $p_{j,n}$ belong to $H(\mu_j)$.

Let $s_1, \ldots, s_k \in (0, 1)$ such that $\sum_{j=1}^k s_j = 1$. As the intersection classes $H(\mu_j)$ all coincide, the periodic points $p_{j,n}$ are all pairwise homoclinically related. Hence, for every $n \geq 1$ there is a basic set $\Delta_n$ which contains all $p_{j,n}$, $j = 1, \ldots, k$. Then there is a sequence $(p_{n,m})_m$ of periodic points $p_{n,m}$ in $\Delta_n$ whose associated invariant measures $\eta_{n,m}$ accumulate at the measure $s_1\mu_{1,n} + \cdots + s_k\mu_{k,n}$. By diagonalization, there is a sequence $\eta_{n,m}$ tending to $s_1\mu_1 + \cdots + s_k\mu_k$, concluding.

6.2. Part “only if” of Theorem 1: assuming existence of one accumulated convex combination. We consider $k$ ergodic measures $\mu_1, \ldots, \mu_k$ supported in $\Lambda$ which are hyperbolic with $s$-index equal to $\dim E$. Furthermore, we assume that there exists $s_1, \ldots, s_k \in (0, 1)$ with $\sum_{j=1}^k s_j = 1$ such that the measure $s_1\mu_1 + \cdots + s_k\mu_k$ is accumulated by ergodic measures $\nu_n \in \mathcal{M}(\hat{\Lambda})$ where $\hat{\Lambda}$ is some dominated extension of $\Lambda$.

Corollary 5.9 associates intersections classes $H(\mu_1), \ldots, H(\mu_k)$ to the measures $\mu_1, \ldots, \mu_k$, respectively. The aim of this section is to prove Theorem 1 that is, to show that

$$H(\mu_1) = H(\mu_j), \quad \forall j \in \{1, \ldots, k\}.$$  

According to Corollary 3.4, for $n$ large enough the ergodic measures $\nu_n$ are hyperbolic of $s$-index $\dim E$. According to Proposition 4.4 each measure $\nu_n$ is accumulated by a sequence $(\nu_{n,m})_m$ of measures supported on periodic orbits contained in $\hat{\Lambda}$ and of $s$-index $\dim E$. Thus, there is $(m_n)_n$ so that the measures $\nu_{n,m_n}$ are
supported in $\tilde{\Lambda}$, tend to $s_1\mu_1 + \cdots + s_k\mu_k$ in the weak*-topology. Thus, one can assume that the measures $\nu_n$ are supported on the orbits of hyperbolic periodic points $q_n$ of $s$-index $\dim E$.

Thus for proving Theorem 1 it remains to prove the following:

**Theorem 3.** Let $f$ be a $C^1$ diffeomorphism, $\Lambda$ be a compact invariant set with a dominated splitting $TM|_{\Lambda} = E \oplus F$. We fix a set $\tilde{\Lambda}$ which is maximal invariant in a neighborhood of $\Lambda$ so that the dominated splitting $E \oplus F$ extends on $\tilde{\Lambda}$. We consider $k > 1$ and $k$ ergodic measures $\mu_1, \ldots, \mu_k$ supported on $\Lambda$ which are hyperbolic with $s$-index equal to $\dim E$. Finally we assume that there exists $s_1, \ldots, s_k \in (0, 1)$, with $\sum_1^k s_j = 1$, such that the measure $s_1\mu_1 + \cdots + s_k\mu_k$ is accumulated by measures $\nu_n \in \mathcal{M}(\tilde{\Lambda})$ supported on the orbits of periodic points $q_n$.

Then the intersection classes $H(\mu_j)$, $j \in \{1, \ldots, k\}$ all coincide and $q_n \in H(\mu_1)$ for any $n$ large enough.

### 6.3. Idea of the proof of Theorem 3

The proof consists in showing that the invariant manifold of the orbit of $q_n$, for $n$ large enough, intersects transversally the invariant manifolds of some periodic point in $H(\mu_i)$, $i = 1, \ldots, k$. As the orbit of $q_n$ approaches the measure $s_1\mu_1 + \cdots + s_k\mu_k$, it passes a proportion of time close to $s_i$ following generic points of $\mu_i$ for every $i$.

Proposition 4.4 provides sets $\Gamma_\sigma^i$ with $\mu_i(\Gamma_\sigma^i) > 1 - \sigma$ so that points in $\Gamma_\sigma^i$ are approached by periodic points in $H(\mu_i)$ whose invariant manifolds have a size larger than a given $\delta > 0$. By Lemma 2.2 there is some $\eta > 0$ so that any other hyperbolic periodic point of index $\dim E$, contained in $\Lambda$, at distance less than $\eta$ from a point in $\Gamma_\sigma^i$, and having invariant manifold of size at least $\delta$ is homoclinically related with the points in $H(\mu_i)$ and hence belongs to $H(\mu_i)$.

On the other hand, the orbit of $q_n$ admits times for which its invariant manifolds have a size larger than $\delta$. It remains to prove that the orbit of $q_n$ admits times so that simultaneously:

1. the stable (resp. unstable) manifold is large,
2. the distance to $\Gamma_\sigma^i$ is very small.

For proving these simultaneous properties, the strategy could be to show that both properties happen with a sufficiently large proportion of time for ensuring an intersection. The proportion of times close to $\Gamma_\sigma^i$ is almost $s_1$ (for $n$ large and $\sigma$ small). The Pliss lemma will ensure the existence of a positive proportion of iterates on the orbit of $q_n$ at which the stable manifold is large. However, this proportion does not tend to 1 even if our requirement for the stable (resp. unstable) manifold being large tends to 0: if $s_1$ is small, this strategy is not enough for concluding.

The idea for solving this problem consists in applying the Pliss lemma to the sequence of times where the orbit of $q_n$ is far from the generic points of $\mu_i$ with a good (i.e. uniform) hyperbolic behavior, $i \geq 2$. This times *far from the good times for $\mu_i$, $i \geq 2$, have a proportion close to $s_1$ and are mostly close to the good points for $\mu_i$.* Pliss provides hyperbolic times for this sequence, where we forget the times close to good points for $\mu_i$, $i \geq 2$. This allows to get such hyperbolic times close to good points for $\mu_1$ (in particular close to $\Gamma_\sigma^1$).

It remains to see that the Pliss times for the new sequence are indeed hyperbolic times for $q_n$: we need to add again the times we have disregarded. This times are good times for $\mu_i$, $i \geq 2$, where the hyperbolicity is uniform. This enables us to conclude the argument.
In order to implement this argument we first need to define precisely what means good points for $\mu_i$ and to get a separation between the good points for $\mu_1$ and for $\mu_i$, $i \geq 2$: as the proportions of time close to these sets tends to $s_1$ and $s_2, \ldots, s_k$, respectively, if these sets are far from one another, the proportion of times far from these sets will tend to 0.

In order to simplify the exposition, in the following we will present the proof in the case $k = 2$ only. Thus, from now on

$$s = s_1 \in (0, 1), \text{ and } 1 - s = s_2.$$

The argument for the general case will be a straightforward adaptation, nevertheless we will explain it in Section 6.7.

These choices will be done in Section 6.4 which collects some preliminary considerations. We start by fixing some universal constants, then determine large $\mu_i$-measure compact sets $\Lambda^\sigma_i$ which are disjoint, whose points are uniformly hyperbolic and accumulated by periodic hyperbolic orbits with uniformly bounded stable/unstable manifolds (Lemma 6.1). We determine how close the periodic points have to stick to $\Lambda^\sigma_i$ in order to have homoclinic relation with $H(\mu_i)$. Moreover, we choose sufficiently small neighborhoods of the set $\Lambda^\sigma_i$ to control distortion (Lemma 6.2) and estimate the average visits to these neighborhoods (Lemma 6.3). Then Section 6.5 will use the idea of Pliss times close to $\mu_1$ and far from the good points of $\mu_2$ for concluding.

The arguments will be shown for the subbundle $E$ and the local stable manifolds of the periodic points $q_n$ intersecting the local unstable manifolds of periodic points in $H(\mu_1)$, only. The ones for the subbundle $F$ and the local unstable manifolds are analogous. We will also only show that $q_n \in H(\mu_1)$, that is, we focus only on $\Gamma^1_\sigma$ and $s$; however we will choose our constants simultaneously for $s$ and $1 - s$ (compare (9)) and it will be clear that the very same reasoning then applies also to $\Gamma^2_\sigma$ and $1 - s$.

6.4. Preliminaries.

Universal constants. Let us first choose some universal constants and specify precisely the approximation values. Consider the maximal Lyapunov exponents in $E$ and the minimal Lyapunov exponents in $F$ of $\mu_1$ and $\mu_2$, respectively, and let

$$\chi \overset{\text{def}}{=} \min \{ |\lambda^E_F(\mu_i)|, \lambda^F_E(\mu_i) : i = 1, 2 \}.$$

Let $s \in (0, 1)$ be such that $s\mu_1 + (1 - s)\mu_2$ is accumulated by ergodic measures. Let

$$C_f \overset{\text{def}}{=} \max_{x \in \Lambda} \{ \log \| df_x \|, \log \| (df_x)^{-1} \| \} > 1.$$

Fix any $\gamma \in (0, \chi/4)$ and then choose $\sigma_0 \in (0, 1)$ satisfying

$$\sigma_0 < \frac{1}{3} \min \{ s, 1 - s \}, \quad \sigma_0 < \frac{\gamma}{12 \log C_f - 5 \gamma}, \quad 2\sigma_0(C_f + \chi - \frac{\gamma}{4}) < \frac{\gamma}{4}.$$

Nonuniform hyperbolicity. Let us now determine the approximation of the measure $s\mu_1 + (1 - s)\mu_2$ by an ergodic measure $\nu$ to guarantee that $\nu$ is also hyperbolic. Given $\gamma$, let $N_0 = N_0(\gamma/4, \mu_1, \mu_2) \geq 1$ and $\varkappa = \varkappa(\gamma/4, \mu_1, \mu_2) > 0$ be the numbers
provided by Corollary [3.3] that is, for every \( N \geq N_0 \) and for every \( f \)-ergodic \( \nu \) with 
\[ D(\nu, s\mu + (1-s)\mu_2) < \infty \] for \( \nu \)-almost every \( y \) we have
\[
\lim_{k \to \infty} \frac{1}{kN} \sum_{\ell=0}^{k-1} \log \|df^N_{\ell \cdot E \mu \kappa_N(y)}\| = \int \frac{1}{N} \log \|df^N\| \, d\nu < -\chi + \frac{\gamma}{4}.
\]
In particular
\[
\lambda_+^N(\nu) < -\chi + \frac{\gamma}{4}.
\]
In particular, if \( \nu \) is supported on a periodic orbit of a periodic point \( q \) of period \( \pi(q) \) then this orbit is hyperbolic of \( s \)-index \( \dim E \) and the above holds for every point \( y = f^j(q), \ j \in \{0, \ldots, \pi(q) - 1\} \), on this orbit.

Picking disjoint subsets of the support of \( \mu_i \) with good properties. We now separate the supports of the ergodic measures and choose subsets with good uniform hyperbolic properties.

**Lemma 6.1.** Given \( \gamma > 0 \) and \( \sigma_0 \in (0, 1) \), for every number \( \sigma \in (0, \sigma_0) \) there are disjoint compact sets \( \Lambda_\sigma^i \subset \Lambda, \ i = 1, 2 \), a positive integer \( N_1 = N_1(\sigma, \gamma, \mu_1, \mu_2) \), and a positive number \( \delta_\sigma = \delta_\sigma(\sigma, \mu_1, \mu_2) \) such that for every \( N \geq N_1 \) for every \( x \in \Lambda_\sigma^i \), \( i = 1, 2 \), we have
\[
\frac{1}{N} \log \|df^N_{\ell \cdot E \mu \kappa_N(x)}\| < -\chi + \frac{\gamma}{4}, \quad \chi - \frac{\gamma}{4} < \frac{1}{N} \log \|\left(df^N_{\ell \cdot E \mu \kappa_N(x)}\right)^{-1}\|^{-1}.
\]
Moreover,
- every \( x \in \Lambda_\sigma^i \) is accumulated by periodic hyperbolic points of \( s \)-index \( \dim E \) which have stable/unstable local manifolds of size at least \( \delta_\sigma \),
- \( \mu_i(\Lambda_\sigma^i) \geq 1 - \sigma \),
- \( \Lambda_\sigma^1 \cap \left( \bigcup_{\ell=0}^{N} f^{\ell}(\Lambda_\sigma^2) \right) = \emptyset \), \( \Lambda_\sigma^2 \cap \left( \bigcup_{\ell=0}^{N} f^{\ell}(\Lambda_\sigma^1) \right) = \emptyset \).

**Proof.** We will only show the part for the subbundle \( E \), the one for \( F \) is analogous.

Let \( \sigma \in (0, \sigma_0) \). By hypothesis, \( \varphi_n(x) = \frac{1}{N} \log \|df^N_{\ell \cdot E \mu \kappa_N(x)}\| \) is a sequence of measurable functions which on a measurable set \( A_i \) of full measure \( \mu_i \)-almost every \( x \in A_i \) to a limit function which is bounded from above by \( \lambda_\sigma^N(\mu_i) \). By Egorov’s theorem, there is a measurable subset \( A_i \) satisfying \( \mu_i(A_i) < \sigma/6 \) such that \( \varphi_n \) is converging uniformly on \( A_i \setminus A_i \). Hence there are numbers \( k_i = k_i(\sigma, \mu_i, \gamma) \geq 1 \) such that for every \( x \in A_i \setminus A_i \) for every \( N \geq k_i \) we have
\[
\frac{1}{N} \log \|df^N_{\ell \cdot E \mu \kappa_N(x)}\| < -\chi + \frac{\gamma}{4}.
\]
By the regularity of the Borel probability measures \( \mu_i \) there exist compact sets \( \Omega_\sigma^i \subset A_i \setminus A_i \) such that \( \mu_i(\Omega_\sigma^i) > 1 - \sigma/3 \).

By Proposition [4.4] there exist sets \( \Gamma_\sigma^i \subset \Lambda \) and numbers \( \delta_i = \delta_i(\sigma, \mu_i) \) such that \( \mu_i(\Gamma_\sigma^i) > 1 - \sigma/3 \), that every \( y \in \Gamma_\sigma^i \) is accumulated by periodic hyperbolic points of \( s \)-index \( \dim E \) which have stable and unstable local manifolds of size at least \( \delta_i \). Let
\[
\delta_\sigma \overset{\text{def}}{=} \min\{\delta_1, \delta_2\}.
\]
Now we split the supports of the measures which, in general may be nondisjoint. Since \( \mu_1 \neq \mu_2 \), there exists a continuous function \( \varphi: \Lambda \to \mathbb{R} \) such that \( \int \varphi \, d\mu_1 \neq \int \varphi \, d\mu_2 \). Consider the sets
\[
B_i = B_i(\varphi) \overset{\text{def}}{=} \left\{ x \in \operatorname{supp} \mu_i : \lim_{k \to \infty} \frac{1}{k} \sum_{\ell=0}^{k-1} \varphi(f^\ell(x)) = \int \varphi \, d\mu_i \right\}.
\]
Observe that by ergodicity of \( \mu_i \), each set \( B_i \) is invariant and satisfies \( \mu_i(B_i) = 1 \). By our choice of \( \varphi \), \( B_1 \) and \( B_2 \) are disjoint.

Take
\[
N_1 \overset{\text{def}}{=} \max\{N_0, k_1, k_2\}.
\]

By the regularity of the Borel probability measures \( \mu_i \), there exist compact disjoint sets \( \Lambda_i^* \subset B_i \cap \Omega_{\sigma_i} \cap \Gamma_i \) such that \( \mu_i(\Lambda_i^*) \geq 1 - \sigma_i \), \( i = 1, 2 \).

Finally, observe that by invariance of \( B_i \) for any \( \ell \geq 1 \) we have \( f^\ell(\Lambda_i^*) \subset B_i \). Therefore, \( \Lambda_i^* \cup f(\Lambda_i^*) \cup \ldots \cup f^N(\Lambda_i^*) \) and \( \Lambda_i^*, i \neq j \), are disjoint. \( \square \)

**Stable/unstable manifolds of \( \nu \)-regular points.** For the following let us fix numbers \( \sigma \in (0, \sigma_0) \) and consider the compact disjoint sets \( \Lambda_i^* \), \( i = 1, 2 \), and the numbers \( N_1 = N_1(\sigma, \gamma, \mu_1, \mu_2) \) and \( \delta_\sigma = \delta_\sigma(\sigma, \mu_1, \mu_2) \) provided by Lemma 6.1. Fix \( N \geq N_1 \).

Considering now \( \nu \) which satisfies \( D(\nu, s\mu_1 + (1 - s)\mu_2) < \infty \) and which is supported on a periodic orbit of a periodic point \( q \). By (10) we have \( \lambda^+_{\nu}(\nu) < -\chi + \gamma/4 \).

Let \( \delta_1 = \delta_1(N, \gamma/4) \) as defined in Definition 2.1. Choose now
\[
\delta \in (0, \min\{\delta_\sigma, C_\nu^{-1}\delta_1\}).
\]

Then by Proposition 4.3 for \( \nu \)-almost every \( y \), that is, for every \( y = f^j(q), j \in \{0, \ldots, \pi(q) - 1\} \), the local stable manifold contains a \( C^1 \) stable disk and the local unstable manifold contains a \( C^1 \) unstable disk, each of radius
\[
(12) \quad L_{\gamma/4}^F(y) = \delta \cdot \left( C_{-\chi + \gamma/4}^E(y) \right)^{-1} \quad \text{and} \quad L_{\gamma/4}^E(y) = \delta \cdot \left( C_{-\chi + \gamma/4}^F(y) \right)^{-1},
\]
respectively, where the functions \( C^*_{-\chi + \gamma/4} \), \( * = E, F \), are defined in (6).

Let \( \eta = \eta(\delta) \) be as provided by Lemma 2.2.

**Sticking sufficiently close to subsets to control distortion.** When the orbit of the \( \nu \)-generic point a proportion \( s \) of times stays close to \( \Lambda_1^* \) its contribution to the Birkhoff sum can be approximated by the corresponding values of a point inside \( \Lambda_1^* \): we can control well this distortion if we require that the neighborhoods are sufficiently small. To do so, assume also that \( \eta \) is sufficiently small such that we also have
\[
(13) \quad \max_{x \in \Lambda} \max_{y \in U(x, \eta/2)} \left\{ \log \left( \frac{\|df_{\nu}^N\|}{\|df_{\nu}^F\|} \right), \log \left( \frac{\|df_{\nu}^N\|}{\|df_{\nu}^F\|} \right)^{-1} \right\} < \frac{\gamma}{4}.
\]

We can summarize the obtained distortion control, which we will need below.

**Lemma 6.2.** For every \( \varepsilon \in (0, \eta/2) \) for every \( y \in B(\Lambda_1^*, \varepsilon) \cup B(\Lambda_2^*, \varepsilon) \) we have
\[
\log \|df_{\nu}^N\| < N\left(-\chi + \frac{\gamma}{2}\right), \quad N\left(-\chi + \frac{\gamma}{2}\right) < \log \|df_{\nu}^F\|^{-1}.
\]
Proof. For every $y \in B(\Lambda^1_\nu, \varepsilon)$ there is $x \in \Lambda^1_\nu \subset \Omega^{1}_{\sigma}$ with $d(x, y) < \varepsilon$ and hence we can apply (13) and (11) to obtain

$$\log \|df_{f^{-N}}\| = \log \|df_{f^{-N}}\| + \log \frac{\|df_{f^{-N}}\|}{\|df_{f^{-N}}\|} < N\left( -\chi + \frac{\gamma}{2} \right).$$

The estimate for the subbundle $F$ is analogous. \hfill \Box

Separating well the disjoint subsets $\Lambda^i_\nu$. We would like to control well all iterations of a $\nu$-generic point. In particular, we would like to separated well the iterations in those which are close to $\Lambda^1_\nu$ (a proportion of roughly $s$) and those which are close to $\Lambda^2_\sigma$ (a proportion of roughly $1-s$). To distinguish both, we require that they stick in a given $\eta/2$-neighborhood of the sets $\Lambda^i_\nu$, respectively. Since we have to deal with the iteration $f^N$ and not only $f$, we require that they stick for (at least) $N$ consecutive iterates once they enter the neighborhoods. Moreover, we would like to do so keeping the distortion control.

To do so, choose first $\varepsilon_0 \in (0, \eta/2)$ such that if $z \in B(y, \varepsilon_0) \cap \Lambda$ for some $y \in \Lambda$ then $d(f^\ell(z), f^\ell(y)) < \eta/2$ for every integer $\ell$ with $|\ell| \leq N$. Such $\varepsilon_0$ exists by uniform continuity of $f$ on the compact set $\Lambda$.

By the disjointness properties of the compact sets $\Lambda^i_\nu$ and their iterates, we can choose now $\varepsilon \in (0, \varepsilon_0)$ so small that

$$B(\Lambda^1_\nu, \varepsilon) \cap \bigcup_{\ell=0}^N f^\ell(B(\Lambda^2_\sigma, \varepsilon)) = \varnothing, \quad B(\Lambda^2_\sigma, \varepsilon) \cap \bigcup_{\ell=0}^N f^\ell(B(\Lambda^1_\nu, \varepsilon)) = \varnothing,$$

where we used the notation

$$B(\Lambda^i_\nu, \varepsilon) \overset{\text{def}}{=} \bigcup_{x \in \Lambda^i_\nu} B(x, \varepsilon).$$

Estimate the visiting times of a $\nu$-generic point. Finally, in order to estimate the number of iterates a $\nu$-generic point for the $f$-ergodic measure $\nu$ stays $\varepsilon$-close to $\Lambda^1_\nu$ and $\Lambda^2_\sigma$, respectively, below we will be using the Birkhoff ergodic theorem (in fact, we will have to switch from $f$ to its iterate $f^N$). For that we will estimate the weight which the $(s, 1-s)$-averaged measure $\nu$ gives to the respective neighborhoods.

Lemma 6.3. Given $\varepsilon > 0$ there exists $\sigma > 0$ such that for every $f$-ergodic measure $\nu$ supported on a periodic orbit of a periodic point $q$ of period $\pi(q)$ satisfying $D(\nu, s\mu_1 + (1-s)\mu_2) < \varkappa$, for every $y = f^j(q)$, $j \in \{0, \ldots, \pi(q) - 1\}$ we have

$$\left| \frac{1}{\pi(q)} \operatorname{card} \{ \ell \in \{0, \ldots, \pi(q) - 1\} : f^{\ell N}(y) \in B(\Lambda^1_\nu, \varepsilon) \} - s \right| \leq \sigma,$$

$$\left| \frac{1}{\pi(q)} \operatorname{card} \{ \ell \in \{0, \ldots, \pi(q) - 1\} : f^{\ell N}(y) \in B(\Lambda^2_\sigma, \varepsilon) \} - (1-s) \right| \leq \sigma.$$

Proof. Because of (14) we can choose two continuous functions $\varphi_i$ being equal to 0 outside the set $B(\Lambda^i_\nu, \varepsilon)$ and equal to 1 inside the $B(\Lambda^i_\nu, \varepsilon/2)$, for $i = 1, 2$ respectively. Note that

$$\int \varphi_1 \, d\mu_1 = 1, \quad \int \varphi_1 \, d\mu_2 = 0, \quad \int \varphi_2 \, d\mu_2 = 1, \quad \int \varphi_2 \, d\mu_1 = 0.$$
Suppose that $\varepsilon$ was chosen small enough such that for every $\nu$ satisfying $D(\nu, s\mu_1 + (1-s)\mu_2) < \varepsilon$ we have for all functions

$$\psi \in \{ \varphi_1, \varphi_2, \frac{1}{N} \log \|d\nu^{N}_f\|, \frac{1}{N} \log \|d\nu^{N}_f\|^{-1}\},$$

we have

$$\left| \int \psi \, d\nu - \int \psi \, d(s\mu_1 + (1-s)\mu_2) \right| < \sigma.$$  

So, in particular we obtain

$$\nu(B(\Lambda^1_\sigma, \varepsilon)) - s = \left| \int \varphi_1 \, d\nu - s \right| \leq \sigma, \quad \nu(B(\Lambda^2_\sigma, \varepsilon)) - (1-s) \leq \sigma. \tag{15}$$

To prove the claim, in the case $N = 1$ we simply would apply the Birkhoff ergodic theorem and use the estimates \[\text{(15)}.\] However, we have to consider iterates $f^N$. Clearly, with respect to $f^N$ the orbit of $q$ is also $\pi(q)$-periodic. However (as we had to do in the proof of Lemma 3.2), we have to cope with the case that the orbit (and its measure $\nu$) is not $f^N$-ergodic. In this case, $\nu$ decomposes as $\nu = \frac{1}{n} (\nu_1 + \cdots + \nu_n)$ where $n = n(\nu) \geq 2$ divides $N$ and each $\nu_k$ is an $f^N$-ergodic probability measure such that $\nu_{k+1} = f_\ast \nu_k$ for each $k \pmod{n}$. Without loss of generality, we can assume that the point $p$ is $\nu_1$-generic. We hence conclude

$$\frac{1}{\pi(q)} \text{card}\{ \ell \in \{0, \ldots, \pi(q) - 1\} : f^{\ell N}(q) \in B(\Lambda^1_\sigma, \varepsilon) \} = \frac{N}{n} \nu_1(B(\Lambda^1_\sigma, \varepsilon)) = \nu(B(\Lambda^1_\sigma, \varepsilon))$$

together with the analogous relation for $B(\Lambda^2_\sigma, \varepsilon)$. This proves the claim. \hfill \Box

This finishes the preliminary considerations.

6.5. **Finding good times with large manifolds close to good points of $\mu_1$.**

The key step in the proof of Theorem 3 is now to consider an ergodic measure $\nu$ which sufficiently well approximates $s\mu_1 + (1-s)\mu_2$ (according to the choice of $\varepsilon$ above) and which is supported on a periodic orbit. In this section we will show (Lemma 6.6) that for such choices, there are points on the periodic orbit whose stable local manifolds are large and which simultaneously are close to $\Lambda^1_\sigma$.

Formulas \[\text{(12)}.\] provide explicitly estimates of the (minimal) size of the stable/unstable manifolds for points on such orbits. These functions formally take into account the asymptotic growth of the derivative in the subbundles along the *entire* orbit. On a periodic orbit, however, these numbers are determined by a finite number of iterations only. Observe also that for a periodic orbit an estimate for *almost every* point holds, in fact, for *every* point on this orbit. This is summarized in the following simple fact.

**Lemma 6.4.** For every periodic hyperbolic point $q$ with $s$-index $\dim E$ of period $\pi(q)$ and with maximal Lyapunov exponent in $E$ being less than $-\lambda$ we have

$$C^{-E,N}_{-\lambda}(q) = \max_{0 \leq n \leq \pi(q) - 1} \left\{ e^{n\lambda} \prod_{\ell=0}^{n-1} \left\| d\nu^{N}_f \right\| \right\}.$$  

**Proof.** Denote

$$c_n(q) \overset{\text{def}}{=} e^{\pi(q)n\lambda} \prod_{\ell=0}^{n-1} \left\| d\nu^{N}_f \right\|.$$
Similarly, we obtain continue counting those points on this orbit which are outside \( B \) where

\[
\pi_n = \left| \frac{\log f}{\log \lambda} \right|
\]

Then given a positive integer \( k \) iteration times (the numbers \( 1 < k_1 < \ldots < k_j \) below) on it from where we observe uniform backward contraction. Here we will consider the maximal exponent in the bundle \( E \) and study the backward orbit (relative to \( f^{-1} \)). The Pliss lemma is the following abstract result (see, for example, \cite[Lemma 3.1]{2}).

**Lemma 6.5** (Pliss). Given numbers \( A \geq c_2 > c_1 > 0 \), let \( \theta \overset{\text{def}}{=} (c_2 - c_1)/(A - c_1) \). Then given a positive integer \( n \) and any real numbers \( a_1, \ldots, a_n \) satisfying

\[
\sum_{i=1}^{n} a_i \geq c_2 n \quad \text{and} \quad a_i \leq A \quad \text{for every} \quad 1 \leq i \leq n,
\]

there are \( j > \theta n \) and \( 1 < k_1 < \ldots < k_j \leq n \) so that

\[
\sum_{i=k+1}^{r} a_i \geq c_1(k_r - k) \quad \text{for every} \quad k = 0, \ldots, k_r - 1 \quad \text{and} \quad r = 1, \ldots, j.
\]

We now are ready to announce and prove the key step in the proof of the theorem.

**Lemma 6.6.** For every ergodic measure \( \nu \) which is \( \varepsilon \)-close to \( s\mu_1 + (1-s)\mu_2 \) and supported on a periodic orbit there is a point \( \hat{x} \) on this orbit such that

\[
\begin{align*}
(1) & \quad \hat{x} \in B(\Lambda^1_{s}, \varepsilon) \\
(2) & \quad C^{E,N}_{-\chi,\gamma}(\hat{x}) = \max_{0 \leq k \leq \pi(\hat{x})-1} \left\{ e^{kN(\chi - \gamma)} \prod_{\ell=0}^{k-1} ||df^\ell_{E(\hat{x})}|| \right\} < 1.
\end{align*}
\]

**Proof.** Let \( q \) be a periodic point whose \( f \)-invariant measure \( \nu \) supported on this orbit is \( \varepsilon \)-close to \( s\mu_1 + (1-s)\mu_2 \). By Lemma \ref{lemma:4} the periodic orbit intersects \( B(\Lambda^1_{s}, \varepsilon) \). Hence, exchanging \( q \) for one of its iterates, we can assume that \( q \in B(\Lambda^1_{s}, \varepsilon) \).

In order to invoke the Pliss Lemma \ref{lemma:6}, we now disregard on the \( f^N \)-orbit of \( q \) those iterations which are \( \varepsilon \)-close to \( \Lambda^1_{s} \) and will define a sequence of points \( (x_i)_{i=0}^{L-1} \) for some \( L < \pi(\hat{x}) \) Note that we go backward along the orbit: We start by defining

\[
x_0 = q, \quad x_1 = f^{-N}(q), \quad \ldots, \quad x_r = f^{-rN}(q),
\]

where \( r \geq 0 \) is the largest integer with the property \( f^{-rN}(q) \in B(\Lambda^1_{s}, \varepsilon) \). In particular, \( f^{-(r+1)N}(q) \notin B(\Lambda^1_{s}, \varepsilon) \). Further, \cite{14} implies \( f^{-(r+1)N}(q) \notin B(\Lambda^1_{s}, \varepsilon) \). We continue counting those points on this orbit which are outside \( B(\Lambda^1_{s}, \varepsilon) \) for some \( L < \pi(\hat{x}) \) and will define a sequence of points \( (x_i)_{i=0}^{L-1} \) for some \( L < \pi(\hat{x}) \)

\[
x_{r+1} = f^{-(r+1)N}(q), \quad \ldots, \quad x_k = f^{-kN}(q),
\]
until $k > r$ being the largest integer such that $f^{-kN}(q) \notin B(\Lambda_1^r, \varepsilon) \cup B(\Lambda_2^r, \varepsilon)$. If $m > k$ is the smallest integer such that $f^{-mN}(q) \in B(\Lambda_1^r, \varepsilon)$ then we continue

$$x_{k+1} = f^{-mN}(q), \ldots$$

and repeat defining this sequence of points on this orbit that do not hit the neighborhood of $\Lambda_2^r$ until we have $x_{L-1} = x_0 = q$, where $L$ is the largest integer $L < \pi(q)N$ with this property.

Recall that in Lemma 6.3 we estimated the number of visits of the periodic orbit to the $\varepsilon$-neighborhoods of $\Lambda_1^r$ and $\Lambda_2^r$. Moreover, in Lemma 6.2 we estimated the derivatives in such points. Hence, we can summarize the properties of the defined sequence $(x_i)_{i=0}^{L-1}$ as follows:

- $x_i \notin B(\Lambda_2^r, \varepsilon)$ for every $i = 0, \ldots, L - 1$,
- $x_i \in B(\Lambda_1^r, \varepsilon)$ for at least $(s - \sigma)\pi(x)$ and for at most $(s + \sigma)\pi(x)$ indices $\dim E$ and we have
  $$- \log \|df_{f_{x_i}}^N\| > N\left(\chi - \frac{\gamma}{2}\right),$$
- $x_i \notin B(\Lambda_1^r, \varepsilon)$ for at most $2\sigma\pi(x)$ indices $\dim E$ 
- for every $i = 0, \ldots, L - 1$ we have
  $$- \log \|df_{f_{x_i}}^N\| \leq N \log C_f,$$
- $(s - \sigma)\pi(x) < L < (s + 3\sigma)\pi(x)$.

To apply Lemma 6.5 let consider the following numbers

$$A \overset{\text{def}}{=} N \log C_f,$$
$$c_2 \overset{\text{def}}{=} N\left(\chi - \frac{\gamma}{2}\right) - N\sigma\frac{4(\chi - \frac{\gamma}{2})}{s + 3\sigma} - N\sigma\frac{2\log C_f}{s + 3\sigma},$$
$$c_1 \overset{\text{def}}{=} N\left(\chi - \gamma\right),$$
$$a_i \overset{\text{def}}{=} \begin{cases} - \log \|df_{f_{x_i}}^N\| & \text{if } x_i \in B(\Lambda_1^r, \varepsilon), \\ N \log C_f & \text{if } x_i \notin B(\Lambda_1^r, \varepsilon). \end{cases}$$

With (9) one verifies that $A \geq c_2 > c_1 > 0$. For every $m \geq 1$ we have

$$\sum_{i=1}^{mL} a_i > N\left(\chi - \frac{\gamma}{2}\right) \cdot (s - \sigma)m\pi(x) - N \log C_f \cdot 2\sigma m\pi(x)$$
$$= N\left((s - \sigma)\left(\chi - \frac{\gamma}{2}\right) - 2\sigma \log C_f\right) \cdot m\pi(x) \cdot \frac{L}{L}$$
$$> N\left((s - \sigma)\left(\chi - \frac{\gamma}{2}\right) - 2\sigma \log C_f\right) \cdot \frac{1}{s + 3\sigma} mL$$
$$= c_2 mL.$$

Thus, these numbers satisfy the hypotheses of the Pliss Lemma 6.5. Given

$$\theta \overset{\text{def}}{=} \frac{c_2 - c_1}{A - c_1} \in (0, 1)$$

pick an integer $m > \theta^{-1}$. Thus, applying Lemma 6.5 to the above numbers and $n = mL$, there are numbers $j > \theta mL > L$ and $1 < k_1 < \cdots < k_j \leq mL$ so that for
every \( r = 1, \ldots, j \) and \( k \in \{0, \ldots, k_r - 1\} \)

\[
\sum_{i=k+1}^{k_r} a_i \geq N(\chi - \gamma) \cdot (k_r - k).
\]

We now make the choice of the point \( \hat{x} \). For that let \( k_r \) be the smallest integer with \( k_r > L \). Such number indeed exists because \( j > L \). Observe that, in particular, we have \( x_{k_r} = f^{-\ell N}(x) \) for some \( \ell > \pi(x) \). Then for every \( k = k_r - 1, k_r - 2, \ldots, 0 \) we have

\[
- \log \|df^N_{/E_{x_{k_r}}^r}\| \geq a_{k_r} > N(\chi - \gamma)
\]

\[
- \log \|df^N_{/E_{x_{k_r}}^r}\| - \log \|df^N_{/E_{x_{k_r-1}}^r}\| \geq a_{k_r} + a_{k_r-1} > N(\chi - \gamma) \cdot 2
\]

\[
\ldots
\]

\[
- \sum_{i=0}^{k_r-1} \log \|df^N_{/E_{x_{k_r-i}}^r}\| \geq \sum_{i=1}^{k_r} a_i > N(\chi - \gamma) \cdot k_r.
\]

Let \( \hat{x} = x_{k_r} \). The choice of sequence \( (a_i)_i \) and (16) imply \( \hat{x} \in B(\Lambda^1_\mu, \varepsilon) \), that is, claimed property (1) of the proposition.

In the above arguments, we disregarded the piece of orbit passing close to \( \Lambda^2_\nu \). To estimate \( G_{\bar{x}}^{E, N} \), what remains is to include again into the Birkhoff sum those iterates of the \( f^N \)-orbit which stay in \( B(\Lambda^2_\nu, \varepsilon) \). Observe that by our above choice of \( \varepsilon < \varepsilon_0 \) we have that if some iteration \( x_k = f^{-kN}(\bar{x}) \) is \( \varepsilon_0 \)-close to some point \( y \in \Lambda^2_\nu \subseteq \Omega^2_\nu \) then it stays at least \( N \) further (either backward or forward) iterations \( \varepsilon \)-close to the orbit of \( y \). For each such piece by Lemma 6.2 we obtain

\[
- \log \|df^N_{/E_{f^{-kN}(\bar{x})}}\| > N(\chi - \gamma) \cdot \frac{\varepsilon}{2}.
\]

As we can split such an orbit piece which entirely stays in \( B(\Lambda^2_\nu, \varepsilon) \) into pieces of length \( N \),

\[
\{f^{-kN}(\bar{x}), \ldots, f^{-(k+1)N+1}(\bar{x})\}, \ldots, \{f^{-mN}(x), \ldots, f^{-(m+1)N+1}(x)\}
\]

for some \( m \geq k \) where each piece is \( \varepsilon_0 \)-shadowing some point in \( \Lambda^2_\nu \) which in particular satisfies (11). Hence, with (16)-(17) for every \( k = 0, \ldots, \pi(x) - 1 \) we obtain

\[
- \sum_{\ell=0}^{k-1} \log \|df^N_{/E_{f^{\ell N}(\bar{x})}}\| > N(\chi - \gamma) \cdot k.
\]

In other terms,

\[
\max_{k \in \{0, \ldots, \pi(x) - 1\}} \left\{ e^{kN(\chi - \gamma)} \prod_{\ell=0}^{k-1} \|df^N_{/E_{f^{\ell N}(\bar{x})}}\| \right\} < 1.
\]

This proves claimed property (2) of Lemma 6.6 and finishes the proof. \( \square \)

6.6. **Conclusion of the proof of Theorem 3 for 2 measures.** By Lemma 6.6 and Proposition 4.4 for every measure \( \nu \) which is \( \varepsilon \)-close to \( s\mu_1 + (1 - s)\mu_2 \) and supported on a periodic orbit there is a point \( y_1 \) on this orbit being \( \eta/2 \)-close to \( \Lambda^1_\mu \) and having a stable manifold of size at least \( \delta \). Hence, by Lemma 6.1 there is a periodic hyperbolic point \( p \in H(\mu_1) \) with unstable manifold of size at least \( \delta \) and \( \eta/2 \)-close to \( \Lambda^1_\mu \). Hence, by Lemma 2.2 their manifolds intersect.
Completely analogous, arguing for the subbundle $F$ instead of $E$ and for $f^{-1}$ instead of $f$, the above implies that there is a point $y_2$ on this orbit being $\eta/2$-close to $\Lambda_1^j$, and having an unstable manifold of size at least $\delta$ which intersects the stable manifold of a point in $H(\mu_1)$. This shows that $q \in H(\mu_1)$.

As Lemma 6.6 holds verbatim for $\mu_2$ instead of $\mu_1$, this also shows that $q \in H(\mu_2)$. This proves that $H(\nu) = H(\mu_1) = H(\mu_2)$.

This finishes the proof of Theorem 3 and hence of Theorem 1.

\[\square\]

6.7. Proof of Theorem 3, general case. The unique place in proof where we truly consider the measures $\mu_i$, $i \neq 1$ is when we pick disjoint subsets with good properties, that is, in Lemma 6.1. Hence, let us state the version of this lemma in the general case:

**Lemma 6.7.** Given $\gamma > 0$ and $\sigma_0 \in (0,1)$, for every number $\sigma \in (0, \sigma_0)$ there are two disjoint compact sets $\Lambda^i_\sigma \subset \Lambda$, $i = 1, 2$, a positive integer $N_1 = N_1(\sigma, \gamma, \mu_1, \ldots, \mu_k)$, and a positive number $\delta_\sigma = \delta_\sigma(\sigma, \mu_1, \ldots, \mu_k)$ such that for every $N \geq N_1$ for every $x \in \Lambda^i_\sigma$, $i = 1, 2$, we have
\[
\frac{1}{N} \log \|df^N_{E_x}\| < -\chi + \frac{\gamma}{4}, \quad \chi - \frac{\gamma}{4} < \frac{1}{N} \log \|df^N_{E_x}\|^{-1} - 1.
\]
Moreover,
- every $x \in \Lambda^i_\sigma$ is accumulated by periodic hyperbolic points of $s$-index $\dim E$ which have stable/unstable local manifolds of size at least $\delta_\sigma$,
- $\mu_1(\Lambda^1_\sigma) \geq 1 - \sigma$, and $\mu_j(\Lambda^2_\sigma) \geq 1 - \sigma$, for $j \geq 2$,
- $\Lambda^1_\sigma \cap \left( \bigcup_{t=0}^N f^t(\Lambda^2_\sigma) \right) = \emptyset$, $\Lambda^2_\sigma \cap \left( \bigcup_{t=0}^N f^t(\Lambda^1_\sigma) \right) = \emptyset$.

Note that the set $\Lambda^2_\sigma$ joins the good part of the generic points of the measures $\mu_j$, $j \geq 2$. The argument of the proof are merely using arguments from measure theory and from ergodic theory. The measure theoretic arguments are identical. For the latter each time we applied the ergodicity of $\mu_2$ now we have to repeat the argument $k - 1$ times, accordingly.

With this modification, the rest of the proof is identical for proving that the periodic points $q_\alpha$, for large $n$, belong to $H(\mu_1)$, and then apply the same argument for the other measures $\mu_j$ in place of $\mu_1$.

7. Proof of Theorem 2

Let $f$ be a $C^1$ diffeomorphism, $\Lambda$ be a compact invariant set with a dominated splitting $TM|_\Lambda = E \oplus F$. We fix a dominated extension $\tilde{\Lambda}$ of $\Lambda$.

First notice that, if $\Lambda_1, \ldots, \Lambda_k$ are hyperbolic transitive sets which are pairwise homoclinically related, then every measure $\mu \in \text{ch}(V_1 \cup \cdots \cup V_k)$ is accumulated by periodic orbits in $\tilde{\Lambda}$, where $V_i = \mathcal{M}(\Lambda_i)$, $i = 1, \ldots, k$. Thus, if we assume the $\Lambda_i$ to be pairwise disjoint, $\text{ch}(V_1 \cup \cdots \cup V_k) \setminus (V_1 \cup \cdots \cup V_k)$ is nonempty so that there is $\mu \in \text{ch}(V_1 \cup \cdots \cup V_k) \setminus (V_1 \cup \cdots \cup V_k)$ which is accumulated by ergodic measures $\nu_n$ in $\mathcal{M}(\tilde{\Lambda})$.

For proving Theorem 2, it remains to prove the converse. Indeed for this converse, the hypothesis that the $\Lambda_i$ are pairwise disjoint is unnecessary and we present the proof without assuming it.
7.1. The case of 2 hyperbolic transitive sets \( \Lambda_1, \Lambda_2 \). We consider compact hyperbolic transitive sets \( \Lambda_1, \Lambda_2 \subset \Lambda \). Let \( V_i = \mathcal{M}(\Lambda_i), \ i = 1, 2 \). We assume that there is a measure \( \mu \) in the convex hull \( \text{ch}(V_1 \cup V_2) \), such that \( \mu \notin V_i \cup V_2 \), and that \( \mu \) is accumulated by ergodic measures \( \nu_n \) supported in \( \Lambda \). The aim of this section is to conclude that \( \Lambda_1 \) and \( \Lambda_2 \) are homoclinically related.

Let us start by two remarks:

- According to Proposition \([1,2]\) for \( n \) large enough the measures \( \nu_n \) are hyperbolic of \( s \)-index \( \dim E \). Hence as explained in Section 6.3 we can assume that the measures \( \nu_n \) are supported on periodic orbits of periodic points \( p_n \).
- If the sets \( \Lambda_1 \) and \( \Lambda_2 \) intersect each other, then they are homoclinically related and we are done. Thus, we can assume that they are disjoint

\[
\Lambda_1 \cap \Lambda_2 = \emptyset.
\]

The proof of Theorem \([2]\) is very similar and, indeed, is much simpler than the proof of Theorem \([1]\). The measure \( \mu \) can be written as \( s\mu_1 + (1 - s)\mu_2, \ s \in (0, 1) \), where \( \mu_i \) are (not necessarily ergodic) measures supported on \( \Lambda_i, \ i = 1, 2 \).

For \( n \) large the orbit of \( p_n \) passes a proportion of time close to \( s \) in an arbitrarily small neighborhood of \( \Lambda_1 \) and a proportion of time close to \( 1 - s \) in an arbitrarily small neighborhood of \( \Lambda_2 \). If some iteration \( f^{n_k}(p_n) \) is close to \( \Lambda_1 \) and has a large stable manifold, as \( \Lambda_1 \) is a hyperbolic set one deduces that the stable manifold of \( f^{n_k}(p_n) \) (hence of \( p_n \)) cuts transversally the unstable manifold of \( \Lambda_1 \) (and, indeed, of the periodic orbits contained in \( \Lambda_1 \)). The same holds true for the unstable manifold and for \( \Lambda_2 \).

Thus, what we need to prove is the existence of large invariant manifolds at the times the orbit of \( p_n \) is close to \( \Lambda_i \). The proof follows almost verbatim the proof of Theorem \([1]\) using \( \Lambda_i \) instead of \( \Lambda_i^s \), with the following changes:

- one chooses the constant \( \chi > 0 \) to be smaller, in absolute value, than any Lyapunov exponents of points in \( \Lambda_1 \cup \Lambda_2 \) and \( \gamma \in (0, \chi/4) \);
- the uniform hyperbolicity of \( \Lambda_1 \cup \Lambda_2 \) ensures the existence of an integer \( N \) so that the inequality \([11]\) holds for any \( x \in \Lambda_1 \cup \Lambda_2 \).

This sketches the proof of the theorem.

\( \square \)

7.2. Adaptation to the case of \( k \) transitive hyperbolic sets \( \Lambda_1, \ldots, \Lambda_k \). We consider \( k > 2 \) and compact hyperbolic transitive sets \( \Lambda_1, \ldots, \Lambda_k \subset \Lambda \). Let \( V_i = \mathcal{M}(\Lambda_i), \ i = 1, \ldots, k \). We assume that there is a measure \( \mu \) in the convex hull \( \text{ch}(V_1 \cup \cdots \cup V_k) \), such that \( \mu \notin V_1 \cup \cdots \cup V_2 \), and that \( \mu \) is accumulated by ergodic measures \( \nu_n \) supported in \( \Lambda \). We want to conclude that the \( \Lambda_i \) are homoclinically related.

The proof is identical to the case \( k = 2 \) if the \( \Lambda_i \) are pairwise disjoint. Let us explain how to conclude when it is not the case.

Assume for instance (up to a re-indexation) that \( \Lambda_{k-1} \cap \Lambda_k \neq \emptyset \).

**Remark 7.1.** Let \( J \) and \( K \) be two transitive hyperbolic compact invariant set of a diffeomorphism \( f \). Assume that \( J \cap K \neq \emptyset \). Then there is a transitive hyperbolic compact invariant set \( L \) so that

\[
J \cup K \subset L
\]

So let denote \( \Lambda_{k-1} \) be a transitive compact set containing \( \Lambda_{k-1} \cap \Lambda_k \) and let denote \( V_{k-1} = \mathcal{M}(\Lambda_{k-1}) \).
Then \( \mu \) belongs to \( \text{ch}(V_1 \cup \cdots \cup V_k) \setminus (V_1 \cup \cdots \cup \tilde{V}_{k-1}) \).

Thus one concludes the proof by an easy induction on \( k \).

8. **Approximation of convex sum of (uniformly) hyperbolic measures, without assuming domination.**

In our main results, we require systematically that the measures \( \nu_n \) are supported on some dominated extension \( \tilde{\Lambda} \) of \( \Lambda \). This domination is necessary for the use of \( C^1 \)-Pesin theory, and a natural question is if we could remove this hypothesis if we assume a higher regularity, \( C^{1+\alpha} \) or \( C^2 \). The aim of this section is to provide smooth examples showing that the domination is a necessary hypothesis even with high regularity, \( C^\infty \) or even analytic.

We provide here two classes of examples.

8.1. **Variations of Bowen’s figure-8.** We start with a first very classical example. We consider an area preserving \( f \) of \( S^1 \times \mathbb{R} \) given by the time-1 map of the Hamiltonian vector given by the Hamiltonian \( H(x, y) = y^2 - \cos(4\pi x) \), where \( S^1 = \mathbb{R}/\mathbb{Z} \). Observe that \( f \) has 4 fixed points at \((i/4, 0), i \in \{1, \ldots, 4\}\): there are two fixed points of center type at \((0, 0)\) and \((1/2, 0)\) and two hyperbolic fixed points of saddle type at \( p_1 = (1/4, 0) \) and \( p_2 = (3/4, 0) \) having the same contraction/expansion eigenvalues. Note that the level set \( \{ (x, y) : H(x, y) = 1 \} = W^s(p_1) \cup W^s(p_2) = W^u(p_1) \cup W^u(p_2) \).

For every \( \varepsilon \in (0, 1) \), the level set \( \{ H = 1 - \varepsilon \} \) are smooth closed simple curves on which \( f \) is conjugate to a (rational or irrational) rotation (with rotation number tending to 0 as \( \varepsilon \to 0 \)). The following is a classical exercise.

**Lemma 8.1.** If \( \mu_n \) is a sequence of \( f \)-invariant probability measures supported on level sets \( \{ H = 1 - \varepsilon_n \} \) with \( \varepsilon_n \to 0 \) as \( n \to \infty \), then \( \mu_n \) converges in the weak* topology to \( \frac{1}{2} \delta_{p_1} + \frac{1}{2} \delta_{p_2} \). The Lyapunov exponents of every ergodic measure \( \mu_n \) are all equal to 0.

The homoclinic and the intersection classes of the points \( p_i \), \( i = 1, 2 \), are trivial and hence disjoint.

Note that this example is not contradicting the results in this paper: the measures \( \mu_n \) are not supported in any dominated extension of the hyperbolic set \( \Lambda = \{ p_1, p_2 \} \).

Let us present now two variations of the above example (compare the middle and the right figure in Figure 1).

1) We can easily modify the map such that the homoclinic class of \( p_1 \) is non-trivial while the one of \( p_2 \) remains trivial (middle figure), keeping all the properties claimed in Lemma 8.1.

2) We can modify the map inside the disk bounded by the heteroclinic connections in such a way that there appears a sequence of hyperbolic periodic saddles whose measures in the weak* topology tend to \( \frac{1}{2} \delta_{p_1} + \frac{1}{2} \delta_{p_2} \) (right figure). In this case that \( f \) is \( C^2 \) the Lyapunov exponents of these saddles tend to 0.

The construction presented below provides an example where the hyperbolic measures approximating a convex combination of two ergodic hyperbolic measures have Lyapunov exponents all uniformly bounded from 0.
8.2. **Blowing up of an Anosov diffeomorphism.** Let \( A \in SL(2, \mathbb{Z}) \) be an hyperbolic linear automorphism with eigenvalues \( 0 < \frac{1}{\lambda} < 1 < \lambda \). Let still denote by \( A : \mathbb{T}^2 \to \mathbb{T}^2 \) the induced linear Anosov diffeomorphism on the torus \( \mathbb{T}^2 \).

The point \( p = (0,0) \in \mathbb{T}^2 \) is a fixed point of \( A \).

Let us denote by \( S \) the (non-orientable) closed surface obtained from \( \mathbb{T}^2 \) by blowing up the point \( p \), and let \( \pi : S \to \mathbb{T}^2 \) be the canonical projection. \( S \) is naturally endowed with an \( \mathbb{R} \)-analytic structure so that the projection \( \pi \) is \( \mathbb{R} \)-analytic. The projection is a diffeomorphism over \( \mathbb{T}^2 \setminus \{(0,0)\} \), and the exceptional fiber \( C = \pi^{-1}(0,0) \) is canonically identified with the circle \( \mathbb{RP}^1 \) (see Figure 2).

A classical result asserts that:

**Lemma 8.2.**

- The diffeomorphism \( f \) admits a (unique) continuous lift \( f_A \) on \( S \), and \( f_A : S \to S \) is an analytic diffeomorphism of \( S \). In particular \( \pi \) induces a \( \mathbb{R} \)-analytic conjugacy between the restriction of \( f_A \) to \( S \setminus C \) and of \( A \) to \( \mathbb{T}^2 \setminus \{p\} \).
- The restriction of \( f_A \) to the exceptional fiber has exactly two fixed points \( p_1 \) and \( p_2 \).
- The points \( p_1 \) and \( p_2 \) are hyperbolic saddle point of \( f_A \), whose eigenvalues are \( 0 < \frac{1}{\lambda^2} < 1 < \lambda \) and \( 0 < \frac{1}{\lambda} < 1 < \lambda^2 \), respectively.
- The stable manifold \( W^s(p_1,f_A) \) is \( C \setminus \{p_2\} \) and the unstable manifold \( W^u(p_2,f_A) \) is \( C \setminus \{p_1\} \). As a consequence, the homoclinic classes and intersection classes of \( p_1 \) and \( p_2 \) are trivial.
- \( W^u(p_1,f_A) \setminus \{p_1\} \) is the lift of \( W^u(p,A) \setminus \{p\} \) and \( W^s(p_2,f_A) \setminus \{p_2\} \) is the lift of \( W^s(p,A) \setminus \{p\} \).

**Theorem 4.** Let \( \mathcal{M}(f_A) \) be the (convex) set of invariant probability measures of \( f_A \). Then a measure \( \mu \in \mathcal{M}(f_A) \) is accumulated in the weak* topology by ergodic measures \( \nu_n \) if, and only if,

- either \( \mu \in \{\delta_{p_1}, \delta_{p_2}\} \), and hence \( \nu_n = \mu \) for \( n \) large enough,
- or \( \mu(p_1) = \mu(p_2) \), in which case \( \mu \) is the limit of a sequence of periodic orbits whose Lyapunov exponents are \( \pm \log \lambda \).
Proof. We use the $\mathbb{R}$-analytic conjugacy between the restriction of $A$ to $\mathbb{T}^2 \setminus \{p\}$ and of $f_A$ to $S \setminus C$. It suffices to observe that if an orbit of $A$ approaches $p$ then its corresponding orbit of $f_A$ passes the same time close to $C$ with approximately half of this time close to $p_1$ and $p_2$, respectively. \hfill \Box

As a straightforward corollary one gets the following.

**Corollary 8.3.** The hyperbolic periodic points $p_1$ and $p_2$ have trivial and disjoint homoclinic and intersection classes.

Nevertheless the measure $\frac{1}{2}\delta_{p_1} + \frac{1}{2}\delta_{p_2}$ is the weak limit of the measures associated to periodic orbits $\gamma_n$ of $f_A$ in $S \setminus C$. In particular the orbits $\gamma_n$ are hyperbolic and their Lyapunov exponents are $\pm \log \lambda$.

**Remark 8.4.** Starting with an Anosov diffeomorphism so that the fixed point $p$ has eigenvalues $\lambda_1 < 1 < \lambda_2$ and blowing up $p$, we get the same result however with measures approximating the convex combination with weights $s$ and $1-s$ satisfying $s/(1-s) = -\log \lambda_1/\log \lambda_2$.

**Remark 8.5.** Multiplying the above dynamics by a Anosov diffeomorphism on $\mathbb{T}^2$ we get an example where both hyperbolic fixed points $p_1$ and $p_2$ have nontrivial but disjoint homoclinic and intersection classes, keeping all the properties claimed above.

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