The time interpretation of expected utility theory

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Decision theory is the model of individual human behavior employed by neoclassical economics. Built on this model of individual behavior are models of aggregate behavior that feed into models of macroeconomics and inform economic policy. Neoclassical economics has been fiercely criticized for failing to make meaningful predictions of individual and aggregate behavior, and as a consequence has been accused of misguiding economic policy. We identify as the Achilles heel of the formalism its least constrained component, namely the concept of utility. This concept was introduced as an additional degree of freedom in the 18th century when it was noticed that previous models of decision-making failed in many realistic situations. At the time, only pre-18th century mathematics was available, and a fundamental solution of the problems was impossible. We re-visit the basic problem and resolve it using modern techniques, developed in the late 19th and throughout the 20th century. From this perspective utility functions do not appear as (irrational) psychological re-weightings of monetary amounts but as non-linear transformations that define ergodic observables on non-ergodic growth processes. As a consequence we are able to interpret different utility functions as encoding different non-ergodic dynamics and remove the element of human irrationality from the explanation of basic economic behavior. Special cases were treated in [1]. Here we develop the theory for general utility functions.

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The first three sections are concerned with putting this work in context, and a brief summary of relevant aspects expected utility theory. The novel technical part starts in Section IV.

I. POSITIONING

The present document is concerned with decision theory, part of the foundation of formal economics. It is therefore worth our while to spell out where in this vast context we feel our contribution is located. It addresses the most formal part of economics, something that is often called neoclassical economics. Broadly speaking this is the part of economics that builds simple quantitative models of economic processes, analyzes these models mathematically, and interprets their behavior by giving real-world meaning to model variables.

This approach to thinking about economic issues became particularly dominant in the second half of the 20th century. Soon after the rise of its popularity it began to be fiercely criticized. We take these criticisms very seriously and interpret them as an indication that something is fundamentally wrong in the way we conceptualize economic problems in the neoclassical approach.

It is certainly true that some of the predictions of neoclassical economics clash with observations. Paradoxes, that is, apparent internal inconsistencies stubbornly remain in the field (examples are the St. Petersburg paradox or the Equity Premium Puzzle). This situation may elicit different responses, for example

1. we can think of it as a normal part of science in progress. Of course there are unresolved problems – finding their solutions is the job of the economic researcher.

2. we may conclude that the pen-and-paper approach using models simple enough for analytical solution makes the representation of people too simplistic. Instead of analyzing such models, it has been argued, we should turn to numerical work and build in-silico worlds of agents with more complex, more realistic behaviour.

3. we may reject the entire scientific approach, whether analytic or numerical. Proponents of this position argue that economic questions are fundamentally moral, not scientific, and that a scientific approach is bound to miss the most important aspects of the problem.

We consider all three responses valid but not mutually exclusive. Every discipline has open problems, and it would be foolish to dismiss an approach only because it has not resolved every problem it encountered. Turning to computer simulations is part of every scientific discipline – when simple models fail and more complex models are not analytically tractable, of course we should use computers. Nor can we dismiss the argument that building a good society entails more than building an economically wealthy society, and that mathematical models only elucidate the consequences of a set of axioms but cannot prove the validity, let alone the moral validity, of the axioms themselves.
The treatment we present here is most informative with respect to perspective 1. We agree with the neoclassical approach in the following sense: we believe that simple mathematical models can yield meaningful insights. We ask precisely how the failures of neoclassical economics may be interpreted as a flaw in the formalism that can be corrected. Such a flaw indeed exists, buried deep in the foundations of formal economics: often expectation values are taken where time averages would be appropriate. In this sense, formal economics has missed perhaps the most important property of decisions: they are made in time and affect the future. They are not made in the context of co-existing possibilities across which resources may be shared. We find reflections of this missing element, for instance in the criticism of “short termism” that is often levelled against neo-classical economics. Indeed, an approach that disregards time in this precise way will result in a formalism that is overly focused on the short term. For example, such a formalism will not provide an understanding of the fundamental benefits of cooperation.

We are led by this analysis to a correction of the formalism capable of resolving a number of very persistent problems. The work in the present paper is part of implementing the correction. It also helps clarify the relationship between existing work in neo-classical economics and our own work. Overall, we propose to re-visit and re-develop the entire formalism from a more nuanced basis that gives the concept of time the central importance it must have if the formalism is to be of use to humans and collections of humans whose existence is inescapably subject to time.

II. EPISTEMOLOGY

We begin with some remarks on rationality. Economics is the only science that frequently states that it assumes rational behavior. The comparison with physics is illuminating.

1. A strong though rarely articulated assumption in physics is that observed behavior can be explained, in the sense that it follows rules, laws, or tendencies that – once identified – enable us to predict and comprehend the behavior of a given system. This assumption is a fundamental belief. It is assumed that the world, or rather very little isolated bits of the world, can be understood. Without this assumption it would not be sensible to try to understand the behavior of physical systems.

2. In physics we proceed by specifying a model of the observed behavior, that is, a mathematical analog, our guess of the rules governing the physical system. For instance, we might say that electrons are point particles with mass $9.1 \times 10^{-31}$ kg that repel one another with $1/r^2$ Coulomb force.

b Similarly, in economics we proceed by specifying our model of human behavior. For instance, we might say that humans choose the action that maximizes the expectation value of their monetary wealth.

3. We now confront our model with observations. No observation will be exactly as predicted by the model. No two electrons will be observed to repel each other with $1/r^2$ Coulomb force. There are too many other electrons around, and protons and gravity and countless perturbations. Nonetheless, the model is useful because it makes more or less sensible predictions of large groups of electrons. The behavior of a billion billion billion electrons over here and a billion billion billion electrons over there may be well described as the behavior of many electrons repelling each other with $1/r^2$ Coulomb force. But we may find a realm where the electrons behave irrationally. For instance, a lump of $9.1 \times 10^{-31}$ kg of matter should be able to absorb any amount of energy. But as it turns out, electrons bound to a nucleus only accept certain fixed amounts of energy. This presents a dilemma to the physicist. He now has a choice between i) declaring electrons as behaving irrationally, i.e. giving up the search for an explanation, and ii) declaring his model as deficient in the regime of interest and search for another model. Often a pretty good mathematical description of the irrational behavior is easily found but is perceived as a mathematical trick, just a description with no inherent meaning. Some years or centuries later an intuition evolves in a new context, and the previously purely formal model (the mathematical trick) now appears as a natural part of a bigger picture.

b Similarly, observations of human behavior will not be exactly as predicted. There are too many idiosyncratic and circumstantial factors involved. No single person will be observed to maximize the expectation value of his wealth consistently. Nonetheless, an overall tendency may be predictable – a majority of people may prefer a 50/50 chance of receiving $2 or losing $1 over no change in their wealth. But we may find a realm where
people behave consistently irrationally. Perhaps few people will prefer a 50/50 chance of winning $20,000 or losing $10,000 over no change in their wealth. Again, the scientist has a choice between i) giving up the fundamental belief that made him a scientist in the first place and declaring humans to be irrational, and ii) declaring his model deficient in the new regime and look for a better model. In the example we mentioned, a new model was quickly found in the early 18th century. While human behavior is not well described as maximizing the expectation value of wealth, it is quite well described as maximizing the expectation value of changes in the logarithm of wealth. Where the logarithm comes from is unclear – the psychological label “risk aversion” is attached to it but that’s just a label. In essence, this is a mathematical trick that seems to work well, just as Planck’s trick of quantizing energy worked well. Following the development of quantum mechanics, Planck’s trick doesn’t seem so strange any more. An intuition has arisen around it. The story of this paper is the story of the equivalent development in decision theory. Following the formulation of the concept of ergodicity, the logarithm – the mathematical trick that saved decision theory – does not seem so strange any more. We identify the use of the logarithm as a different model of rationality: it is rational to maximize average wealth growth over time under the null model of multiplicative growth; it is not rational to maximize the mathematical expectation of wealth. The two models give similar predictions for small monetary amounts, but entirely different predictions when the amounts involved approach the expected growth; it is not rational to maximize the average growth rate of wealth, were those decisions to be repeated indefinitely. In other words we work with a form of homo economicus.

For a decision maker facing a choice between different courses of action, the workflow of expected utility theory is as follows

1. Imagine everything that could happen under the different actions:
   
   Associate with any action \( A, B, C \ldots \) a set of possible future events \( \Omega_A, \Omega_B, \Omega_C \ldots \). 
   
2. Estimate how likely the different consequences of each action are and how they would affect your wealth:
   
   For set \( \Omega_A \), associate a probability \( p(\omega_A) \) and a change in wealth \( \Delta w_{\omega_A} \) with each elementary event \( \omega_A \in \Omega_A \), and similarly for all other sets.

3. Specify how much these outcomes would affect your happiness:
   
   Define a utility function, \( u(w) \), that only depends on wealth and describes the decision maker’s risk preferences.

4. Aggregate the possible changes of happiness for any given event:
   
   Compute the expected changes in utility associated with each available action, \( \langle \Delta u_A \rangle = \sum_{\Omega_A} p(\omega_A)u(w + \Delta w(\omega_A)) - u(w) \), and similarly for actions \( B, C \ldots \).

### III. EXPECTED UTILITY THEORY

Expected utility theory is the bedrock of neoclassical economics. It provides the discipline’s answer to the fundamental decision problem of how to choose between different sets of uncertain outcomes. The generality of the framework is all-encompassing. Everything in the past is certain, whereas everything in the future comes with a degree of uncertainty. Any decision is about choosing one uncertain future over alternative uncertain futures, wherefore expected utility theory is behind the answer of neoclassical economics to any problem involving human decision making.

To keep the discussion manageable, we restrict it to financial decisions, i.e. we will not consider the utility of an apple or of a poem but only utility differences between different dollar amounts. We restrict ourselves to situations where any non-financial attendant circumstances of the decision can be disregarded. In other words we work with a form of homo economicus.

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5. Pick the action that makes you happiest:
   The option with the highest expected utility change is the decision maker’s best choice.

Each step of this process has been criticized, but we assume that all steps are possible. This does not reflect a personal opinion that they are unproblematic in reality but is a methodological choice. By overlooking some undeniable but possibly solvable difficulties we are able to inspect and question aspects at a deeper level of the formalism. Thus we assume that all possible future events, associated probabilities and changes in wealth are known, that a suitable utility function is available, and that the mathematical expectation of utility changes is the mathematical object whose ordering reflects preferences among actions. For simplicity we also make the common assumption that the time between taking an action and experiencing the corresponding change in wealth is independent of the action taken.

Having accepted the basic premises of expected utility theory we acknowledge a remaining criticism. Expected utility theory may not be useful in practice. Of course usefulness can only be assessed if we know what we want to achieve. One aim of decision theory may be to genuinely help real people make decisions. On this score expected utility theory is limited. It is designed to ensure consistency in an individual’s choices, but judged against criteria other than the risk preferences of the individual the theory may produce consistently bad choices. For example, decision theory is not designed to find the decisions that lead to the fastest growth in wealth; the decisions it recommends are those that maximize the mathematical expectation of a model of the decision maker’s happiness. For a gambling addict, for instance, these decisions may lead to bankruptcy. Expected utility theory will recognize the individual as addicted to gambling, and conclude that he will be happiest behaving recklessly. It is a laissez-faire approach to decision theory. Such an approach is not illegitimate, however its limitations must be borne in mind. For instance, when designing policy it is no use to recognize that a financial institution that takes larger risks than are good for systemic stability is happiest when doing so. For any given decision maker it requires a utility function that can only be estimated by querying the decision maker, possibly about simpler choices that we believe he can assess more easily. Preferences of the decision maker are thus an input to the formalism. The output of the formalism is also a preference, namely the action that makes the decision maker happiest. In other words, the output is of the same type as the input, which makes the framework circular. It may help the decision maker by telling him which action is most consistent with other actions he has taken or knows he would take in other situations.

We will interpret the basic findings of expected utility theory in a different light. We will remove the circularity, for better or worse, and using our model of rational behavior show that rationality according to our axioms under a reasonable model of wealth dynamics is equivalent to expected utility theory with commonly used utility functions. Some researchers consider this an irrelevant contribution because in that case we might just continue using expected utility theory. We disagree and consider our contribution an important step forward because it motivates new questions and provides answers that are not circular.

The range of questions we can answer in this way is surprising to us. Examples are: how does an investor choose the leverage of an investment? How can we resolve the St. Petersburg paradox? How do people choose to cooperate? Why do insurance contracts exist? How can we make sense of the recent changes in observed economic inequality? Do economic systems change from one phase to another under different tax regimes?

We have variously referred to our approach as “dynamical” or “time-based” or as recognizing disequilibrium or non-ergodicity. The best term to refer to our perspective may be “ergodicity economics” – in every problem we have treated we have asked whether the expectation values of key variables were meaningful, in particular how they were related to time averages.

IV. TECHNICAL

We repeat our two axioms.

1. Human behavior can be understood. It follows a rationale and is in that sense rational.
2. We explore the following model of this rationale. Humans make decisions so that the growth rate of their wealth would be maximized over time were those decisions repeated indefinitely.

We suppose that an individual’s wealth evolves over time according to a stochastic process. This is a departure from classical decision theory, where wealth is supposed to be described by a random variable without dynamic. To turn a gamble into a stochastic process and enable the techniques we have developed, a dynamic must be assumed, that is, a mode of repetition of the gamble, see 1.

The individual is required to choose one from a set of alternative stochastic processes, say \( x(t) \) and \( x^*(t) \). We suppose that this is done by considering how the decision maker would fare in their long-time limits.

At each decision time, \( t_0 \), our individual acts to maximise subsequent changes in his wealth by selecting \( x(t) \) so that if he waits long enough his wealth will be greater under the chosen process than under the alternative process with certainty. Mathematically speaking, there exists a sufficiently large \( t \) such that the probability of the chosen \( x(t) \) being greater than \( x^*(t) \) is arbitrarily close
to one,
\[ \forall \varepsilon, x^*(t) \quad \exists \Delta t \quad \text{s.t.} \quad \mathbb{P}(\Delta x > \Delta x^*) > 1 - \varepsilon, \quad (1) \]
where \( 0 < \varepsilon < 1 \) and
\[ \Delta x \equiv x(t_0 + \Delta t) - x(t_0), \quad (2) \]
with \( \Delta x^* \) similarly defined.

The criterion is necessarily probabilistic since the quantities \( \Delta x \) and \( \Delta x^* \) are random variables and it might be possible for the latter to exceed the former for any finite \( \Delta t \). Only in the limit \( \Delta t \to \infty \) does the randomness vanish from the system.

Conceptually this criterion is tantamount to maximising \( \lim_{\Delta t \to \infty} \{\Delta x\} \) or, equivalently, \( \lim_{\Delta t \to \infty} \{\Delta x/\Delta t\} \). However, neither limit is guaranteed to exist. For example, consider a choice between two geometric Brownian motions,
\[ dx = x(\mu dt + \sigma dW), \quad (3) \]
\[ dx^* = x^*(\mu^* dt + \sigma^* dW), \quad (4) \]
with \( \mu > \sigma^2/2 \) and \( \mu^* > \sigma^2/2 \). The quantities \( \Delta x/\Delta t \) and \( \Delta x^*/\Delta t \) both diverge in the limit \( \Delta t \to \infty \) and a criterion requiring the larger to be selected fails to yield a decision.

To overcome this problem we introduce a montonically increasing function of wealth, which we call suggestively \( u(x) \). We define:
\[ \Delta u \equiv u(x(t_0 + \Delta t)) - u(x(t_0)); \quad (5) \]
\[ \Delta u^* \equiv u(x^*(t_0 + \Delta t)) - u(x^*(t_0)). \quad (6) \]

The monotonicity of \( u(x) \) means that the events \( \Delta x > \Delta x^* \) and \( \Delta u > \Delta u^* \) are the same. Taking \( \Delta t > 0 \) allows this event to be expressed as \( \Delta u/\Delta t > \Delta u^*/\Delta t \), whence the decision criterion in (Eq. 1) becomes

\[ \forall \varepsilon, x^*(t) \quad \exists \Delta t \quad \text{s.t.} \quad \mathbb{P}(\Delta u/\Delta t > \Delta u^*/\Delta t) > 1 - \varepsilon. \quad (7) \]

Our decision criterion has been recast such that it focuses on the rate of change
\[ r \equiv \frac{\Delta u}{\Delta t}, \quad (8) \]
As before, it is conceptually similar to maximising
\[ \bar{r} \equiv \lim_{\Delta t \to \infty} \left\{ \frac{\Delta u}{\Delta t} \right\} = \lim_{\Delta t \to \infty} \{r\}. \quad (9) \]
If \( x(t) \) satisfies certain conditions, to be discussed below, then the function \( u(x) \) can be chosen such that this limit exists. We shall see that \( \bar{r} \) is then the time-average growth rate mentioned in Section I. For the moment we leave our criterion in the probabilistic form of (Eq. 7) but to continue the discussion we assume that the limit (Eq. 9) exists.

Everything is now set up to make the link to expected utility theory. Perhaps (Eq. 9) is the same as the rate of change of the expectation value of \( \Delta u \)
\[ \langle \Delta u \rangle \frac{\Delta t}{\Delta t} = \langle r \rangle. \quad (10) \]
We could then make the identification of \( u(x) \) being the utility function, noting that our criterion is equivalent to maximizing the rate of change in expected utility. We note \( \Delta u \) and hence \( r \) are random variables but \( \langle r \rangle \) is not. Taking the long-time limit is one way of removing randomness from the problem, and taking the expectation value is another. The expectation value is simply another limit: it’s an average over \( N \) realizations of the random number \( \Delta u \), in the limit \( N \to \infty \). The effect of removing randomness is that the process \( x(t) \) is collapsed into the scalar \( \Delta u \), and consistent transitive decisions are possible by ranking the relevant scalars. In general, maximising \( \langle r \rangle \) does not yield the same decisions as the criterion espoused in (Eq. 7). This is only the case for a particular function \( u(x) \) whose shape depends on the process \( x(t) \). Our aim is to find these pairs of processes and functions. When using such \( u(x) \) as the utility function, expected utility theory will be consistent with optimisation over time. It is then possible to interpret observed behavior that is found to be consistent with expected utility theory using the utility function \( u(x) \) in purely dynamical terms: such behavior will lead to the fastest possible wealth growth over time.

We ask what sort of dynamic \( u(x) \) must follow so that \( \bar{r} = \langle r \rangle \) or, put another way, so that \( r \) is an ergodic observable, in the sense that its time and ensemble averages are the same [3] p. 32].

We start by expressing the change in utility, \( \Delta u \), as a sum over \( M \) equal time intervals,
\[ \Delta u \equiv u(t_0 + \Delta t) - u(t_0) \quad (11) \]
\[ = \sum_{m=1}^{M} [u(t_0 + m\delta t) - u(t_0 + (m - 1)\delta t)] \quad (12) \]
\[ = \sum_{m=1}^{M} \delta u_m(t), \quad (13) \]
where \( \delta t \equiv \Delta t/M \) and \( \delta u_m(t) \equiv u(t_0 + m\delta t) - u(t_0 + (m - 1)\delta t) \). From (Eq. 9) we have
\[ \bar{r} = \lim_{\Delta t \to \infty} \left\{ \frac{1}{\Delta t} \sum_{m=1}^{M} \delta u_m \right\}. \quad (14) \]
\[ = \lim_{M \to \infty} \left\{ \frac{1}{M} \sum_{m=1}^{M} \delta u_m \right\}, \quad (15) \]
keeping \( \delta t \) fixed. From (Eq. 10) we obtain
\[ \langle r \rangle = \lim_{N \to \infty} \left\{ \frac{1}{N} \sum_{n=1}^{N} \frac{\Delta u_n}{\Delta t} \right\}. \quad (16) \]
where each $\Delta u_m$ is drawn independently from the distribution of $\Delta u$.

We now compare the two expressions (Eq. 15) and (Eq. 16). Clearly the value of $\bar{r}$ in (Eq. 15) cannot depend on the way in which the diverging time period is partitioned, so the length of interval $\delta t$ must be arbitrary and can be set to the value of $\Delta t$ in (Eq. 16), for consistency we then call $\delta u_m(t) = \Delta u_m(t)$. Expressions (Eq. 15) and (Eq. 16) are equivalent if the successive additive increments, $\Delta u_m(t)$, are distributed identically to the $\Delta u_n$ in (Eq. 16), which requires only that they are stationary and independent.

Thus we have a condition on $u(t)$ which suffices to make $\bar{r} = \langle r \rangle$, namely that it be a stochastic process whose additive increments are stationary and independent. This means that $u(t)$ is, in general, a Lévy process. Without loss of realism we shall restrict our attention to processes with continuous paths. According to a theorem stated in [9] p. 2 and proved in [10] Chapter 12 this means that $u(t)$ must be a Brownian motion with drift,

$$\text{du} = a_u dt + b_u dW, \quad (17)$$

where $dW$ is the infinitesimal increment of the Wiener process.

By arguing backwards we can address concerns regarding the existence of $\bar{r}$. If $u$ follows the dynamics specified by (Eq. 17), then it is straightforward to show that the limit $\bar{r}$ always exists and takes the value $a$. Consequently the decision criterion (Eq. 7) is equivalent to the optimisation of $\bar{r}$, the time-average growth rate. The process $x(t)$ may be chosen such that (Eq. 17) does not apply for any choice of $x(u)$. In this case we cannot interpret expected utility theory dynamically, and such processes are likely to be pathological.

This gives our central result:

**Theorem 1.** For any invertible utility function $u(x)$ a class of corresponding wealth processes $dx$ can be obtained such that the (linear) rate of change in the expectation value of net changes in utility is the time-average growth rate of wealth.

As a consequence, optimizing expected changes in such utility functions is equivalent to optimizing the time-average growth, in the sense of Section IV, under the corresponding wealth process.

The origin of optimizing expected utility can be understood as follows: in the 18th century, the distinction between ergodic and non-ergodic processes was unknown, and all stochastic processes were treated by computing expectation values. Since the expectation value of the wealth process is an irrelevant mathematical object to an individual whose wealth is modelled by a non-ergodic process the available methods failed. The formalism was saved by introducing a non-linear mapping of wealth, namely the utility function. The (failed) expectation value criterion was interpreted as theoretically optimal, and the non-linear utility functions were interpreted as a psychologically motivated pattern of human behavior. Conceptually, this is wrong.

Optimization of time-average growth recognizes the non-ergodicity of the situation and computes the appropriate object from the outset—a procedure whose building blocks were developed beginning in the late 19th century. It does not assume anything about human psychology and indeed predicts that the same behavior will be observed in any growth-optimizing entities that need not be human.

### A. Examples

Equation (18), creates pairs of utility functions $u(x)$ and dynamics $dx$. In discrete time, two such pairs were investigated in [1], namely cases 1. and 2. below.
The trivial linear utility function corresponds to additive wealth dynamics (Brownian motion),

\[ u(x) = x \quad \leftrightarrow \quad dx = a_x dt + b_x dW. \tag{19} \]

2. Logarithmic utility

Introduced by Bernoulli in 1738 [11], the logarithmic utility function is in wide use and corresponds to multiplicative wealth dynamics (geometric Brownian motion),

\[ u(x) = \ln(x) \quad \leftrightarrow \quad dx = x \left[ \left( a_u + \frac{1}{2} b_u^2 \right) dt + b_u dW \right]. \tag{20} \]

In practice the most useful case will be multiplicative wealth dynamics. But to demonstrate the generality of the procedure, we carry it out for a different special case that is historically important.

3. Square-root (Cramer) utility

The first utility function ever to be suggested was the square-root function \( u(x) = x^{1/2} \), by Cramer in a 1728 letter to Daniel Bernoulli, partially reproduced in [11]. This function is invertible, namely \( x(u) = u^2 \), so that (Eq. 18) applies. We note that the square root, in a specific sense, sits between the linear function and the logarithm: \( \lim_{x \to \infty} x^{1/2} = 0 \) and \( \lim_{x \to 0} \frac{\ln(x)}{x^{1/2}} = 0 \). Since linear utility produces additive dynamics and logarithmic utility produces multiplicative dynamics, we expect square-root utility to produce something in between or some mix. Substituting for \( x(u) \) in (Eq. 18) and carrying out the differentiations we find

\[ u(x) = x^{1/2} \quad \leftrightarrow \quad dx = \left( 2a_u x^{1/2} + b_u^2 \right) dt + 2b_u x^{1/2} dW. \tag{21} \]

The drift term contains a multiplicative element (by which we mean an element with \( x \)-dependence) and an additive element. We see that the square-root utility function that lies between the logarithm and the linear function indeed represents a dynamic that is partly additive and partly multiplicative.

(Eq. 21) is reminiscent of the Cox-Ingersoll-Ross model [12] in financial mathematics, especially if \( a_u < 0 \). Similar dynamics, i.e., with a noise amplitude that is proportional to \( \sqrt{x} \), are also studied in the context of absorbing-state phase transitions in statistical physics [13] [14]. That a 300-year-old letter is related to recent work in statistical mechanics is not surprising; the problems that motivated the development of decision theory, and indeed of probability theory itself are far-from equilibrium processes. Methods to study such processes were only developed in the 20th century and constitute much of the work currently carried out in statistical mechanics.

VI. UTILITY FUNCTION FROM A DYNAMIC

We now ask under what circumstances the procedure in (Eq. 18) can be inverted. When can a utility function be found for a given dynamic? In other words, what conditions does the dynamic \( dx \) have to satisfy so that optimization over time can be represented by optimization of expected net changes in utility \( u(x) \)?

We ask whether a given dynamic can be mapped into a utility whose increments are described by Brownian motion, (Eq. 17).

The dynamic is an arbitrary Itô process

\[ dx = a_x(x) dt + b_x(x) dW, \tag{22} \]

where \( a_x(x) \) and \( b_x(x) \) are arbitrary functions of \( x \). For this dynamic to translate into a Brownian motion for the utility, \( u(x) \) must satisfy the equivalent of (Eq. 18) with the special requirement that the coefficients \( a_u \) and \( b_u \) in (Eq. 17) be constants, namely

\[ du = \left( a_x(x) \frac{\partial u}{\partial x} + \frac{1}{2} b_x^2(x) \frac{\partial^2 u}{\partial x^2} \right) dt + b_x(x) \frac{\partial u}{\partial x} dW. \tag{23} \]

Explicitly, we arrive at two equations for the coefficients

\[ a_u = a_x(x) u' + \frac{1}{2} b_x^2(x) u'' \tag{24} \]

and

\[ b_u = b_x(x) u'. \tag{25} \]

Differentiating (Eq. 25), it follows that

\[ u''(x) = -\frac{b_u b_x'(x)}{b_x^2(x)}. \tag{26} \]

Substituting in (Eq. 21) for \( u' \) and \( u'' \) and solving for \( a_x(x) \) we find the drift term as a function of the noise term,

\[ a_x(x) = \frac{a_u}{b_u} b_x(x) + \frac{1}{2} b_x(x) b_u'(x). \tag{27} \]

In other words, knowledge of only the dynamic is sufficient to determine whether a corresponding utility function exists. We do not need to construct the utility function explicitly to know whether a pair of drift term and noise term is consistent or not.

Having determined for some dynamic that a consistent utility function exists, we can construct it by substituting for \( b_x(x) \) in (Eq. 24). This yields a differential equation for \( u \)

\[ a_u = a_x(x) u' + \frac{b_x^2}{2 u^2} u'' \tag{28} \]

or

\[ 0 = -a_u u'^2 + a_x(x) u'^3 + \frac{b_x^2}{2} u''. \tag{29} \]
Overall, then the triplet noise term, drift term, utility function is interdependent. Given a noise term we can find consistent drift terms, and given a drift term we find a consistency condition (differential equation) for the utility function.

A. Example

Given a dynamic, it is possible to check whether this dynamic can be mapped into a utility function, and the utility function itself can be found. We consider the following example

$$dx = \left(\frac{a_x}{b_x} e^{-x} - \frac{1}{2} e^{-2x}\right) dt + e^{-x} dW. \quad (30)$$

We note that $a_x(x) = \frac{a_x}{b_x} e^{-x} - \frac{1}{2} e^{-2x}$ and $b_x(x) = e^{-x}$. Equation (27) imposes conditions on the drift term $a_x(x)$ in terms of the noise term $b_x(x)$. Substituting in (Eq. 27) reveals that the consistency condition is satisfied by the dynamic in (Eq. 30).

A typical trajectory of (Eq. 30) is shown in Fig. 1.

![FIG. 1: Typical trajectories of the wealth trajectory $x(t)$ described by (Eq. 30), with parameter values $a_u = 1/2$ and $b_u = 1$, and the corresponding Brownian motion $u(t)$. Note that the fluctuations in $x(t)$ become smaller for larger wealth.](image)

Because (Eq. 30) is internally consistent, it is possible to derive the corresponding utility function. Equation (25) is a first-order ordinary differential equation for $u(x)$

$$u'(x) = \frac{b_u}{b_x(x)}, \quad (31)$$

which can be integrated to

$$u(x) = \int_0^x \frac{b_u}{b_x(x)} \, dx + C, \quad (32)$$

with $C$ an arbitrary constant of integration. This constant corresponds to the fact that only changes in utility are meaningful, as was pointed out by von Neumann and Morgenstern [15] – this robust feature is visible whether one thinks in dynamic terms and time averages or in terms of consistent measure-theoretic concepts and expectation values.

Substituting for $b_x(x)$ from (Eq. 30), (Eq. 31) becomes

$$u'(x) = b_u e^x, \quad (33)$$

which is easily integrated to

$$u(x) = b_u e^x + C, \quad (34)$$

plotted in Fig. 2. This exponential utility function is monotonic and therefore invertible, which is reflected in the fact that the consistency condition is satisfied. The utility function is convex. From the perspective of expected-utility theory an individual behaving optimally according to this function would be labelled “risk-seeking.” The dynamical perspective corresponds to a qualitatively different interpretation: Under the dynamic (Eq. 30) the “risk-seeking” individual behaves optimally, in the sense that his wealth will grow faster than that of a risk-averse individual. The dynamic (Eq. 30) has the feature that fluctuations in wealth become smaller as wealth grows. High wealth is therefore sticky – an individual will quickly fluctuate out of low wealth and into higher wealth. It will then tend to stay there.

VII. WEALTH DISTRIBUTION FROM A DYNAMIC

The dynamical interpretation of expected utility theory makes it particularly simple to compute wealth distributions. A utility function $u(x)$ implies a dynamic $x(t)$, and that dynamic generates a wealth distribution $P_x(x, t)$. We know that $u(t)$ follows a simple Brownian motion, wherefore we know that $u(t)$ is normally distributed according to

$$P_u(u, t) = N (a_u t, b_u^2 t^2). \quad (35)$$

Since we know $P_u(u, t)$, the distribution of $x$ is easily derived. The wealth distribution in a large population, is

$$P_x(x, t) = P_u(u(x), t) \frac{du}{dx}. \quad (36)$$

A. Example of a wealth distribution

The utility function (Eq. 34) corresponds to the example dynamic (Eq. 30). The wealth distribution at any time $t$ can be read off (Eq. 36)

$$P_x(x, t) = \frac{1}{\sqrt{2\pi b_u^2 t^2}} \exp \left(-\frac{(b_u e^x + C - a_u t)^2}{2b_u^2 t^2}\right) b_u e^x, \quad (37)$$
which is shown in Fig. 3. The distribution is sensible given what we know about the dynamic – since fluctuations diminish with increasing wealth many individuals will be found at high wealth (all those that have fluctuated away from low wealth), with a heavy tail towards lower wealth.

![Utility function](image)

**FIG. 2:** Utility function of (Eq. 34), with $b_u = 1$ and $C = 0$. Optimizing the expected change in this utility function also optimizes time-average growth under the corresponding dynamic (Eq. 30). An unusual utility function – like the convex function shown here – reflects unusual dynamics, see text.

![Probability density function](image)

**FIG. 3:** Probability density function of wealth, also known as the wealth distribution, (Eq. 37). This distribution is generated by the wealth dynamic (Eq. 30). The time is fixed to $t = 5$, and we use $a_u = 1/2$, $b_u = 1$, and $C = 0$.

### VIII. UNBOUNDEDNESS OF $u(x)$

The scheme outlined in Section VII is informative for the debate regarding the boundedness of utility functions. A well-established but false belief in the economics literature, due to Karl Menger [16, 17], is that permissible utility functions must be bounded. We have argued previously that boundedness is an unnecessary restriction, and that Menger’s arguments are not valid [11, 13]. Section VII implies that the interpretation of expected utility theory we offer here formally requires unboundedness of utility functions. Bounded functions are not invertible, and Menger’s incorrect result therefore contributed to obscuring the simple natural arguments we present here.

Of course whether $u(x)$ is bounded or not is practically irrelevant because $x$ will always be finite. However, for a clean mathematical formalism an unbounded $u(x)$ is highly desirable.

The problem is easily demonstrated by considering the case of zero noise. Since $u(x)$ always follows a Brownian motion in our treatment, in the zero-noise case it follows

$$du = a_u dt,$$  \hspace{1cm} \text{(38)}

meaning linear growth in time. For $u$ to be bounded, time itself would have to be bounded. Another way to see the problem is inverting $u(x)$ to find $x(u)$. If we require simultaneously linear growth of $u(t)$ in time, and boundedness from above, $\lim_{x \to \infty} u(x) = U_b$, then $x(t)$ has to diverge in the finite time it takes for $u(t)$ to reach $U_b$, namely in $T_b = \frac{U_b}{a_u}$ (assuming for simplicity $u(t = 0) = 0$).

Such features – an end of time or a finite-time singularity of wealth – are inconvenient to carry around in a formalism. Since they have no physical meaning, for simplicity a model without them should be chosen, i.e. unbounded utility functions will be much better. We repeat that Menger’s arguments against unbounded utility functions are invalid and we need not worry about them.

### IX. DISCUSSION

Expected utility theory is an 18th-century patch, applied to a flawed conceptual framework established in the 17th century that made blatantly wrong predictions of human behavior. Because the mathematics of randomness was in its infancy in the 18th century, the conceptual problems were overlooked, and utility theory set economics off in the wrong direction. Without any of the arbitrariness inherent in utility functions it is nowadays possible to give a physical meaning to the non-linear mappings people seem to apply to monetary amounts. These apparent mappings simply encode the non-linearity of wealth dynamics.
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[18] Both Boltzmann and Planck described some of their greatest discoveries as a mere mathematical trick (respectively, taking expectation values under equiprobability of microstates and quantizing blackbody energy).