On quasi bi-slant Lorentzian submersions from LP-Sasakian manifolds

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Abstract

At this work, quasi bi-slant Lorentzian submersions from LP-Sasakian manifolds onto Riemannian manifolds have been studied. Further, the geometry of leaves of the distributions, integrability conditions and totally geodesic conditions have also been discussed. Finally, we construct some examples of this setting.

Keywords: LP-Sasakian manifolds, slant submersions, Lorentzian submersions, quasi bi-slant Lorentzian submersions.

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1. Introduction

Differential geometry is one of the most popular branch of mathematics and physics from ancient days. There are several topics in differential geometry which have very important applications in both, mathematics and physics [2, 14, 20]. Immersions and submersions are some of them. The properties of Riemannian submersions become an interesting subject in complex geometry as well as in contact geometry.

The theory of Riemannian submersions was first established by O'Neill [24] and Gray [8]. In 1976, Watson [32] introduced almost Hermitian submersions within almost Hermitian manifolds. In 1985, Chinea [5] generalized the idea of almost Hermitian submersion to different sub-classes of the almost contact manifolds. There are so many important and interesting results about Riemannian and almost Hermitian submersion which are studied at [4, 6, 30]. Recently, slant submersions, semi-invariant submersions as well as semi-slant submersions from almost Hermitian manifolds on Riemannian manifolds have been studied in [21, 27, 28], respectively. Several types of Riemannian submersions between Riemannian manifolds endowed with various constructures were investigated by several geometers ([1, 3, 12, 13, 19, 26, 29]). In 2016, Sahin et al. [31] proved decomposition theorems for hemi-slant Riemannian submersions from Hermitian manifolds on Riemannian manifolds.

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Magid [16] and Falcitelli et al. [7], introduced the theory of Lorentzian submersions. Matsumoto [17] started the idea of $\text{LP}$-Sasakian manifolds, while in 1992, related subject is investigated by Mihai and Rosca [18]. Recently, Gunduzalp [9] and Gunduzalp and Sahin [10] studied paracontact and Lorentzian almost paracontact structures. Kumar et al. [15] defined and studied conformal semi-slant submersions from $\text{LP}$-Sasakian manifolds onto Riemannian manifolds. Very recently, Prasad et al. [23] introduced the concept of quasi bi-slant submersions from Kaehler manifold on the Riemannian manifold.

In this research we undertake our work as follows. In Section 2, we present several main informations relating to quasi bi-slant Lorentzian submersion. At Section 3, certain interesting outcomes on quasi bi-slant Lorentzian submersions from an $\text{LP}$-Sasakian manifold onto the Riemannian manifold are obtained and studied the geometry of leaves of distributions that are included at this submersion. In the same section, certain conditions are obtained of similar submersions to become totally geodesic. Finally, some non-trivial examples for such submersions have constructed.

2. Preliminaries

The $n$-dimension smooth manifold $M$ admitting $\varphi$ the $(1, 1)$-tensor field, $\zeta$: the structural vector field, $\eta$: the 1-form and $g$: the Lorentzian metric named the Lorentzian para Sasakian (in brief, $\text{LP}$-Sasakian) manifold [11, 25] satisfies:

\begin{align}
\varphi^2 &= I + \eta \otimes \zeta, \quad \varphi \circ \zeta = 0, \quad \eta \circ \varphi = 0, \\
\eta(\zeta) &= -1, \quad g(\cdot, \zeta) = \eta(\cdot), \\
g(\varphi \cdot, \varphi \cdot) &= g + \eta \otimes \eta, \quad g(\varphi \cdot, \cdot) = g(\cdot, \varphi \cdot), \\
\nabla \zeta &= \varphi, \\
(\nabla_X \varphi)Y &= \eta(Y)X + g(X, Y)\zeta + 2\eta(X)\eta(Y)\zeta,
\end{align}

choosing $X, Y$ at $M$, where $\nabla$ denotes Levi-Civita connection respecting to Lorentzian metric $g$.

In the $\text{LP}$-Sasakian manifold, clearly

$$\text{rank}(\varphi) = n - 1.$$

Now, in case

$$\Phi(X, Y) = \Phi(Y, X)$$

for all $X, Y$ on $M$, then $\Phi$ is called symmetric $(0, 2)$ tensor field, where $\Phi(X, Y) = g(X, \varphi Y)$.

**Lemma 2.1.** Suppose $W$ is a subspace of dimension $\geq 1$ in the Lorentz vector space. Then the following are equivalent:

1. $W$ is timelike, hence is itself a Lorentz vector space;
2. $W$ includes two linearly independent null vectors;
3. $W$ contains a timelike vector.

**Lemma 2.2.** Suppose $W$ is a subspace of Lorentz vector space $V$ and Suppose $g$ is the metric (scalar product) of $V$, therefore the possible cases for $W$ are:

1. $g|_W$ is positive definite, then $W$ is the inner product space;
2. $g|_W$ is non-degenerate of index 1, therefore $W$ is timelike;
3. $g|_W$ is degenerate, therefore $W$ is lightlike.

**Lemma 2.3.** Let $Z$ be the subspace spanned by the timelike vector in Lorentz vector space $V$, therefore the subspace $Z^\perp$ is spacelike and $V$ is a direct sum of $Z$ and $Z^\perp$. 
This argument shows, more generally, that the subspace \( W \) is timelike if and only if \( W^\perp \) is spacelike. Since \((W^\perp)^\perp = W\).

\( W \) is lightlike if and only if \( W^\perp \) is lightlike.

**Lemma 2.4.** For the subspace \( W \) of the Lorentz vector space, the coming statements are equivalent:

1. \( W \) is lightlike, that is, degenerate;
2. \( W \) includes the null vector but not timelike vector;
3. \( W \cap A = \mathcal{L} - 0 \), where \( \mathcal{L} \) is the one dimensional subspace and \( A \) is the null cone of \( V \), which means

\[ \mathcal{L} = W \cap W^\perp. \]

Note that we denote \((M, \varphi, \xi, \eta, g_M) : \) the almost contact metric manifold, \((\mathbf{N}, g_N) : \) the Riemannian manifold and \( \ker h_s : \) the vertical distribution of \( h \) in \( M \). To use later, we recall the following definitions.

**Definition 2.5** ([22]). The Riemannian submersion \( h : (M, \varphi, \xi, \eta, g_M) \to (\mathbf{N}, g_N) \) is named an invariant Riemannian submersion in case

\[ \varphi(\ker h_s) = \ker h_s. \]

**Definition 2.6** ([19]). Suppose \( h : (M, \varphi, \xi, \eta, g_M) \to (\mathbf{N}, g_N) \) is a Riemannian submersion such that (in brief, s.t.) \( \varphi(\ker h_s) \subseteq (\ker h_s)^\perp \). Therefore, \( h \) is called the anti-invariant Riemannian submersion.

**Definition 2.7** ([11]). The Riemannian submersion \( h : (M, \varphi, \xi, \eta, g_M) \to (\mathbf{N}, g_N) \) is called the semi-invariant Riemannian submersion in case there is the distribution \( \mathcal{D}_1 \subseteq \ker h_s \), s.t.,

\[ \ker h_s = \mathcal{D}_1 \oplus \mathcal{D}_2 \ominus < \xi >, \quad \text{and} \quad \varphi(\mathcal{D}_1) = \mathcal{D}_1, \varphi(\mathcal{D}_2) \subseteq (\ker h_s)^\perp, \]

where \( \mathcal{D}_2 \) is orthogonal complementary distribution to \( \mathcal{D}_1 \) at \( \ker h_s \).

Suppose the complementary orthogonal subbundle to \( \varphi(\ker h_s) \) in \( (\ker h_s)^\perp \) is denoted by \( \mu \). Therefore we get

\[ (\ker h_s)^\perp = \varphi(\mathcal{D}_2) \oplus \mu. \]

Clearly, \( \mu \) is the invariant subbundle of \( (\ker h_s)^\perp \) respecting to the almost contact constructor \( \varphi \).

**Definition 2.8** ([9]). The Riemannian submersion \( h : (M, \varphi, \xi, \eta, g_M) \to (\mathbf{N}, g_N) \) is called a slant submersion, in case for all \( X \neq 0 \in (\ker h_s)_p \), \( p \in M \), the angle \( \theta(X) \) within \( \varphi X \) and the space \( (\ker h_s)_p \) is constant. The angle \( \theta \) is called the slant angle of the submersion and in case \( \theta \in (0, \frac{\pi}{2}) \), therefore \( h \) is named the proper slant submersion.

**Definition 2.9** ([22]). The Riemannian map \( h : (M, \varphi, \xi, \eta, g_M) \to (\mathbf{N}, g_N) \) named the semi-slant Riemannian map in case there are three orthogonal complementary distributions \( \mathcal{D}_1, \mathcal{D}_2 \) and \( < \xi > \) in \( \ker h_s \), s.t.,

\[ \ker h_s = \mathcal{D}_1 \oplus \mathcal{D}_2 \ominus < \xi >, \quad \varphi(\mathcal{D}_1) = \mathcal{D}_1, \]

and the angle \( \theta = \theta(X) \) (called a semi-slant angle) between \( \varphi X \) as well as the space \( (\mathcal{D}_2)_p \) is constant of \( X \neq 0 \in (\mathcal{D}_2)_p \) for \( p \in M \), where \( \mathcal{D}_1 \oplus \mathcal{D}_2 \ominus < \xi > \) is an orthogonal decomposition for \( \ker h_s \).

**Definition 2.10** ([31]). Suppose \((M, g_M, J)\) is the almost Hermitian manifold and \((\mathbf{N}, g_N)\) is the Riemannian manifold. The Riemannian submersion \( h : (M, g_M, J) \to (\mathbf{N}, g_N) \) named the hemi-slant submersion in case

\[ \ker h_s = \mathcal{D}_0 \oplus \mathcal{D}^\perp. \]

The distribution \( \mathcal{D}_0 \) is slant with an angle \( \theta \) (named a hemi-slant angle) and \( \mathcal{D}^\perp \) is anti-invariant.
Definition 2.11 ([9]). Suppose \((M, g_M)\) be a Lorentzian manifold and \((B, g_B)\) a Riemannian manifold. A Lorentzian submersion is a map \(h : (M, g_M) \rightarrow (B, g_B)\) which is onto and satisfies the following three conditions.

\((A_1)\) \(h_*\) is onto for all \(p \in M\).
\((A_2)\) The fibers \(h^{-1}(b)\) are semi-Riemannian (Lorentzian) submanifolds of \(M\) for each \(b \in B\).
\((A_3)\) \(h_*\) preserves scalar products of horizontal vectors.

Now, the concept of a quasi bi-slant Lorentzian submersion from LP-Sasakian manifolds onto Riemannian manifolds is introduced:

Definition 2.12. Suppose \((M, \varphi, \zeta, \eta, g_M)\) is the LP-Sasakian manifold as well as \((N, g_N)\) is the Riemannian manifold. The Lorentzian submersion

\[h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N)\]

named the quasi bi-slant Lorentzian submersion in case there are four mutually orthogonal distributions \(D, D_1, D_2\) and \(<\zeta, \cdot>\), s.t.,

\[(i)\) \(\ker h_* = D \oplus_{\text{orth}} D_1 \oplus_{\text{orth}} D_2 \oplus_{\text{orth}} < \zeta >;\)
\[(ii)\) \(\varphi(D) = D\), which means \(D\) is invariant;
\[(iii)\) \(\varphi(D_1) \perp D_2\) and \(\varphi(D_2) \perp D_1;\)
\[(iv)\) for any \(X \neq 0 \in (D_1)_p, p \in M\), the angle \(\theta_1\) within \(\varphi X\) and \((D_1)_p\) is constant and independent of the choice of point \(p\) and \(X\) in \((D_1)_p;\)
\[(v)\) for all \(Z \neq 0 \in (D_2)_q, q \in M\), the angle \(\theta_2\) within \(\varphi Z\) and \((D_2)_q\) is constant and independent of the choice of point \(q\) and \(Z\) in \((D_2)_q\).

The angles \(\theta_1\) and \(\theta_2\) named slant angles of \(h\), where \(D, D_1\) and \(D_2\) are spacelike subspaces and \(\ker h_*\) is Lorentzian subspace.

Thus it is noted that:

\[(a)\) In case \(\dim D \neq 0\) and \(\dim D_1 = \dim D_2 = 0\), therefore \(h\) is invariant submersion.
\[(b)\) In case \(\dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 = 0\), therefore \(h\) is proper semi-slant submersion.
\[(c)\) In case \(\dim D = 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 = 0\), therefore \(h\) is slant submersion with slant angle \(\theta_1\).
\[(d)\) In case \(\dim D = \dim D_1 = 0\) and \(\dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}\), therefore \(h\) is slant submersion with slant angle \(\theta_2\).
\[(e)\) In case \(\dim D_1 \neq 0, \dim D = 0, \theta_1 = \frac{\pi}{2}\) and \(\dim D_2 = 0\), therefore \(h\) is the anti-invariant submersion.
\[(f)\) In case \(\dim D_1 \neq 0, \dim D \neq 0, \theta_1 = \frac{\pi}{2}\) and \(\dim D_2 = 0\), therefore \(h\) is semi-invariant submersion.
\[(g)\) In case \(\dim D_1 \neq 0, \dim D = 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 \neq 0, \theta_2 = \frac{\pi}{2}\), therefore \(h\) is the hemi-slant submersion.
\[(h)\) In case \(\dim D_1 \neq 0, \dim D = 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}\), therefore \(h\) is the bi-slant submersion.
\[(i)\) In case \(\dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 \neq 0, \theta_2 = \frac{\pi}{2}\), therefore \(h\) can be called a quasi-hemi-slant submersion.
\[(j)\) In case \(\dim D \neq 0, \dim D_1 \neq 0, 0 < \theta_1 < \frac{\pi}{2}\) and \(\dim D_2 \neq 0, 0 < \theta_2 < \frac{\pi}{2}\), therefore \(h\) is proper quasi bi-slant submersion.

Define O’Neill’s tensors \(\mathcal{J}\) and \(\mathcal{A}\) as

\[
\mathcal{A}_E L = \mathcal{H} \nabla_{\mathcal{J}E} V L + \nabla \mathcal{J} E \mathcal{H} L, \tag{2.6}
\]

\[
\mathcal{J}_E L = \mathcal{H} \nabla_{\mathcal{J}E} V L + \nabla \mathcal{J} E \mathcal{H} L \tag{2.7}
\]
for all vector fields $E, L$ at $M$, where $\nabla$ defines Levi-Civita connection of $g_M$. Clearly, $T_E$ and $A_E$ are skew-symmetric operators at the tangent bundle of $M$ reversing vertical and horizontal distributions. Using equations (2.6) and (2.7), results in

$$\nabla_X Y = T_X Y + \nabla_Y X,$$

$$\nabla_X V = T_X V + H_X (V),$$

$$\nabla_Y X = A_Y X + \nabla_X Y,$$

$$\nabla_Y W = T_Y W + A_Y W,$$

for all $X, Y \in \Gamma(\ker h_*)$ and $V, W \in \Gamma(\ker h_*)^\perp$, where $H_X (V) = A_Y X$, in case $V$ is basic. It can be easily observed that $T$ works at the fibers as the second fundamental form, where $A$ works on horizontal distribution and measures obstruction to the integrability of the same distribution.

Clearly, for $q \in M, U \in \mathcal{V}_q$ and $Z \in \mathcal{K}_q$

$$A_{U\mathbf{L}}, \ T_Z : T_q M \rightarrow T_q M$$

are skew-symmetric, such that

$$g_M (A_{U\mathbf{L}} E, L) = -g_M (E, A_{U\mathbf{L}} L) \quad \text{and} \quad g_M (T_Z E, L) = -g_M (E, T_Z L)$$

for each $E, L \in T_q M$. Since $T_Z$ is skew-symmetric, therefore it is observed that $h$ has totally geodesic fibres if and only if $T \equiv 0$.

**Definition 2.13.** Let $M$ and $M'$ be two smooth manifolds. Let $\nabla$ and $\nabla'$ be connections on $M$ and $M'$, respectively. A smooth map $h : M \rightarrow M'$ is called connection preserving map if $h_* (\nabla_X Y) = \nabla'_{h_* X} (h_* Y)$

for all vector fields $X, Y$ on $M$.

A smooth map $h : M \rightarrow M'$ is called geodesic preserving map if for each geodesic $\sigma$ in $M$, $h \circ \sigma$ is geodesic in $M'$.

It is known that if a map is connection preserving then it is also the geodesic preserving. Geodesic preserving map is also called totally geodesic map.

We also know if $M$ and $M'$ be two smooth manifolds and $h$ be a diffeomorphism from $M$ onto $M'$, then for a connection $\nabla'$ on $M'$ there exist unique connection $\nabla$ on $M$ such that $h$ is connection preserving map.

Suppose $(M, \varphi, \xi, \eta, g_M)$ is an LP-Sasakian manifold, $(N, g_N)$ is the Riemannian manifold and $h : M \rightarrow N$ is a smooth map. Therefore the second fundamental form of $h$ is

$$(\nabla h_*)(U, V) = \nabla^h_{h_* U} h_* V - h_* (\nabla_U V), \quad \text{for } U, V \in \Gamma(T_p M),$$

where $\nabla$ denotes Levi-Civita connection of the metrics $g_M$ and $g_N$ and $\nabla^h$ is the pullback connection.

The differentiable map $h : M \rightarrow N$ is totally geodesic in case

$$(\nabla h_*)(U, V) = 0, \quad \text{for all } U, V \in \Gamma(TM).$$

Now the following lemma can be proved as in [3].

**Lemma 2.14.** Suppose $h$ is the Lorentzian submersion from the LP-Sasakian manifold $(M, \varphi, \xi, \eta, g_M)$ on Riemannian manifold $(N, g_N)$, therefore we get

(i) $(\nabla h_*)(V, W) = 0$;

(ii) $(\nabla h_*)(X, Z) = -h_* (T_X Z) = -h_* (\nabla_X Z)$;

(iii) $(\nabla h_*)(V, X) = -h_* (\nabla_V X) = -h_* (A_{\nabla X})$, where $V, W$ are horizontal vector fields and $X, Z$ are vertical vector fields.
3. Quasi Bi-Slant Lorentzian submersions

Throughout this section, we take \((M, \varphi, \zeta, \eta, g_{\mathcal{M}})\) to be a LP-Sasakian manifold and \((N, g_{\mathcal{N}})\) to be a Riemannian manifold.

Suppose \(h : (M, \varphi, \zeta, \eta, g_{\mathcal{M}}) \rightarrow (N, g_{\mathcal{N}})\) is the quasi bi-slant Lorentzian submersion. Therefore, we get

\[ TM = \ker h_* \oplus_{\text{orth}} (\ker h_*)^\perp. \]

Here, for all vector field \(Z \in \Gamma(\ker h_*),\) we choose

\[ Z = PZ + QZ + RZ - \eta(Z)\zeta, \tag{3.1} \]

where \(P, Q\) and \(R\) indicates to the projection morphisms of \(\ker h_*\) on \(D, D_1\) and \(D_2\), in the same order.

Choosing \(Z \in \Gamma(\ker h_*),\) we set

\[ \varphi Z = \psi Z + \omega Z, \tag{3.2} \]

where \(\psi Z \in \Gamma(\ker h_*)\) and \(\omega Z \in \Gamma(\omega D_1 \oplus \omega D_2).\) From (3.1) and (3.2), we get

\[ \varphi Z = \psi(PZ) + \omega(PZ) + \psi(QZ) + \omega(QZ) + \psi(RZ) + \omega(RZ). \]

Since \(\varphi D = D,\) therefore \(\omega PZ = 0.\) Hence we obtain

\[ \varphi Z = \psi(PZ) + \omega QZ + \psi RZ + \omega RZ. \]

Thus we have

\[ \varphi(\ker h_*) = D \oplus (\psi D_1 \oplus \psi D_2) \oplus (\omega D_1 \oplus \omega D_2), \]

where \(\oplus\) defines orthogonal direct sum.

Moreover, Suppose \(V \in \Gamma(D_1)\) and \(W \in \Gamma(D_2),\) therefore \(g_{\mathcal{M}}(V, W) = 0.\) Now from the Definition 2.12 (iii), we have \(g_{\mathcal{M}}(\varphi V, W) = g_{\mathcal{M}}(V, \varphi W) = 0.\) Now, we consider

\[ g_{\mathcal{M}}(\psi V, W) = g_{\mathcal{M}}(\varphi V - \omega V, W) = g_{\mathcal{M}}(\varphi V, W) = 0. \]

In Similar way, we have \(g_{\mathcal{M}}(V, \psi W) = 0.\) Suppose \(Z \in \Gamma(D)\) and \(Y \in \Gamma(D_1).\) Therefore we get

\[ g_{\mathcal{M}}(\psi Y, Z) = g_{\mathcal{M}}(\varphi Y - \omega Y, Z) = g_{\mathcal{M}}(\varphi Y, Z) = -g_{\mathcal{M}}(Y, \varphi Z) = 0, \]

as \(D\) is invariant, which means \(\varphi Z \in \Gamma(D).\) Similarly, for \(Z \in \Gamma(D)\) and \(X \in \Gamma(D_2),\) we obtain \(g_{\mathcal{M}}(\psi X, Z) = 0.\) From above equations, we have

\[ g_{\mathcal{M}}(\psi Z, \psi W) = 0, \quad \text{and} \quad g_{\mathcal{M}}(\omega Z, \omega W) = 0 \]

for any \(Z \in \Gamma(D_1)\) and \(W \in \Gamma(D_2).\) So, we can write \(\psi D_1 \cap \psi D_2 = \{0\}, \omega D_1 \cap \omega D_2 = \{0\}.\) If \(\theta_2 = \frac{\omega}{\mu},\) then \(\psi R = 0\) and \(D_2\) is anti-invariant, which means \(\varphi(D_2) \subseteq (\ker h_*)^\perp.\) Here we present \(D_2\) as \(D^\perp.\) In addition, we have

\[ \varphi(\ker h_*) = D \oplus \psi D_1 \oplus \omega D_1 \oplus \varphi D^\perp, \]

where \(\oplus\) defines orthogonal direct sum. Since \(\omega D_1 \subseteq (\ker h_*)^\perp,\) \(\omega D_2 \subseteq (\ker h_*)^\perp,\) so it is obtained that

\[ (\ker h_*)^\perp = \omega D_1 \oplus \omega D_2 \oplus \mu, \]

where \(\mu\) is orthogonal complement of \((\omega D_1 \oplus \omega D_2)\) at \((\ker h_*)^\perp.\) Also for all \(V \in \Gamma(\ker h_*)^\perp,\) we set

\[ \varphi V = CV + BV, \tag{3.3} \]

where \(CV \in \Gamma(\mu)\) and \(BV \in \Gamma(\ker h_*)\).
Span(ζ) = ⟨ζ⟩ determines timelike vector field distribution. In case the spacelike vector field X is orthogonal to ζ, therefore \( g(\varphi X, \varphi X) = g(X, X) > 0 \), thus \( \varphi X \) is spacelike and hence \( \psi X \) is also spacelike. Wirtinger angle \( \theta \) is written as

\[
\cos \theta = \frac{g(\varphi X, \psi X)}{|\varphi X| |\psi X|}.
\]

Since \( g|_{\ker h_*} \) is non-degenerate metric of index 1 at all points of \( M \), therefore \( (\ker h_*)_\perp \) is timelike subspace of \( T_xM \) at any point of \( M \), and so \( (\ker h_*)_\perp \) is spacelike subspace of \( T_xM \) at all points \( x \in M \).

**Lemma 3.1.** Let \( h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N) \) be the quasi bi-slant Lorentzian submersion. Therefore we got

\[
\psi^2 V + B \omega V = V + \eta(V)\zeta, \quad \omega \psi V + C \omega V = 0, \quad \omega BW + C^2 W = W, \quad \psi BW + BCW = 0,
\]

for all \( V \in \Gamma(\ker h_*) \) and \( W \in \Gamma(\ker h_*)_\perp \).

**Proof.** By making use of the equations (2.1), (3.2), and (3.3), Lemma 3.1 follows.

**Lemma 3.2.** Let \( h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N) \) be the quasi bi-slant Lorentzian submersion. Therefore, we got

(i) \( \psi^2 V = (\cos^2 \theta_1)V \),

(ii) \( g_M(\psi V, \psi W) = \cos^2 \theta_1 g_M(V, W) \),

(iii) \( g_M(\omega V, \omega W) = \sin^2 \theta_1 g_M(V, W) \),

for all \( V, W \in \Gamma(D_1) \).

**Proof.**

(i) Let \( h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N) \) be the quasi bi-slant Lorentzian submersion with the quasi bi-slant angle \( \theta_1 \). Therefore, for \( V(\neq 0) \in \Gamma(D_1) \), we have

\[
\cos \theta_1 = \frac{|\psi V|}{|\varphi V|}.
\]

and

\[
\cos \theta_1 = \frac{g_M(V, \psi V)}{|V| |\psi V|}.
\]

By making use of (2.1), (2.3), and (3.2), we have

\[
\cos \theta_1 = \frac{g_M(\psi V, \psi V)}{|\varphi V| |\psi V|},
\]

\[
\cos \theta_1 = \frac{g_M(V, \psi^2 V)}{|\varphi V| |\psi V|}.
\]

From the equations (3.4) and (3.5), we get \( \psi^2 V = (\cos^2 \theta_1)V \), for \( V \in \Gamma(D_1) \).

(ii) For all \( V, W \in \Gamma(D_1) \), by the use of equations (2.1), (2.3), (3.2), and Lemma 3.2 (i), we have

\[
g_M(\psi V, \psi W) = g_M(\varphi V - \omega V, \psi W) = g_M(V, \psi^2 W) = \cos^2 \theta_1 g_M(V, W).
\]

(iii) By using the equations (2.3), (3.2), and Lemma 3.2 (i) and (ii), Lemma 3.2 (iii) follows.

Similarly, the coming Lemma is obtained.

**Lemma 3.3.** Suppose \( h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N) \) is the quasi bi-slant Lorentzian submersion. Therefore, we have

(i) \( \psi^2 Z = (\cos^2 \theta_2)Z \);
Proof. By the use of equations (3.6)-(3.9) and (3.14)-(3.17), Lemma 3.5 follows. for all \( Z, U \in \Gamma(D_2) \).

Lemma 3.4. Suppose \( h : (M, \varphi, \xi, \eta, g_M) \rightarrow (N, g_N) \) is the quasi bi-slant Lorentzian submersion. Therefore, we get

\begin{align*}
\nabla_X \psi Y + T_X \omega Y - \psi \nabla_X Y - B T_X Y &= g_M(X, Y) \xi + \eta(Y) X + 2 \eta(X) \eta(Y) \xi, \\
T_X \psi Y + \mathcal{H} \nabla_X \omega Y &= \omega \nabla_X Y + C T_X Y, \\
\nabla_U BV + A_U CV - g_M(CU, V) \xi &= \psi A_U V + B \mathcal{H} \nabla_U V, \\
\nabla_U BV + \mathcal{H} \nabla_U CV &= \omega A_U V + C \mathcal{H} \nabla_U V, \\
\nabla_X BU + T_X CU &= \psi T_X U + B \mathcal{H} \nabla_X U, \\
\nabla_X BU + \mathcal{H} \nabla_X CU &= \omega T_X U + C \mathcal{H} \nabla_X U, \\
\n\nabla_U \psi X + A_V \omega X &= B A_U V + \psi \nabla_U V, \\
\nA_V \psi X + \mathcal{H} \nabla_U \omega X - \eta(X) V &= C A_V X + \omega \nabla_U X,
\end{align*}

for all \( X, Y \in \Gamma(\ker h_+) \) and \( U, V \in \Gamma(\ker h_+) \). 

Proof. Using equations (2.1), (2.2), (2.5), (2.8)-(2.11), we can easily get the equations (3.6)-(3.13). \( \square \)

Now, we define

\begin{align*}
(\nabla_X \psi) W &= \nabla_X \psi W - \psi \nabla_X W, \\
(\nabla_X \omega) W &= \mathcal{H} \nabla_X \omega W - \omega \nabla_X W, \\
(\nabla_X C) Y &= \mathcal{H} \nabla_X CY - C \mathcal{H} \nabla_X Y, \\
(\nabla_X B) Y &= \nabla_X BY - B \mathcal{H} \nabla_X Y,
\end{align*}

for all \( V, W \in \Gamma(\ker h_+) \) and \( X, Y \in \Gamma(\ker h_+) \).

Lemma 3.5. Let \( h : (M, \varphi, \xi, \eta, g_M) \rightarrow (N, g_N) \) be the quasi bi-slant Lorentzian submersion. Therefore, we get

\begin{align*}
(\nabla_X \varphi) W &= B T_V W - T_V \omega W + g_M(V, W) \xi + 2 \eta(V) \eta(W) \xi + \eta(W) V, \\
(\nabla_X \omega) W &= C T_V W - T_V \psi W, \\
(\nabla_X C) Y &= \omega A_X Y - A_X BY, \\
(\nabla_X B) Y &= \psi A_X Y - A_X CY + g_M(X, Y) \xi,
\end{align*}

for all \( V, W \in \Gamma(\ker h_+) \) and \( X, Y \in \Gamma(\ker h_+) \). 

Proof. By the use of equations (3.6)-(3.9) and (3.14)-(3.17), Lemma 3.5 follows. \( \square \)

Now, in case tensors \( \varphi \) and \( \omega \) are parallel respecting to \( \nabla \) at \( M \), therefore

\[ B T_V W = T_V \omega W - g_M(V, W) \xi - 2 \eta(V) \eta(W) \xi - \eta(W) V, \]

and

\[ C T_V W = T_V \psi W \]

for all \( V, W \in \Gamma(TM) \).

Theorem 3.6. Let \( h : (M, \varphi, \xi, \eta, g_M) \rightarrow (N, g_N) \) be the proper quasi bi-slant Lorentzian submersion. Therefore, the invariant distribution \( D \) is integrable if and only if

\[ g_M(T_X \varphi Y - T_Y \varphi X, \omega Q Z + \omega R Z) = -g_M(\nabla_X \varphi Y - \nabla_Y \varphi X, \psi Q Z + \psi R Z) \]

for all \( X, Y \in \Gamma(D) \) and \( Z \in \Gamma(D_1 \oplus D_2 \oplus \langle \xi \rangle) \).
Proof. For $X, Y \in \Gamma(D)$, and $Z \in \Gamma(D_1 \oplus D_2 \oplus < \zeta >)$, by the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.2), we have

$$g_M([X, Y], Z) = g_M(\nabla_X \varphi Y, \varphi Z) - g_M(\nabla_Y \varphi X, \varphi Z) - \eta(Z)\eta(\nabla_X Y) + \eta(Z)\eta(\nabla_Y X),$$

$$= g_M(\nabla_X \varphi Y, \varphi Z) - g_M(\nabla_Y \varphi X, \varphi Z),$$

$$= g_M(\xi X \varphi Y - \xi Y \varphi X, \omega RZ + \omega QZ) + g_M(- \nabla_Y \varphi X + \nabla_X \varphi Y, \psi QZ + \psi RZ),$$

this proof is completed. \hfill \Box

Theorem 3.7. Let $h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N)$ is the proper quasi bi-slant Lorentzian submersion. Then the slant distribution $D_1$ is integrable if and only if

$$g_M(\xi W \omega \psi Z - \xi Z \omega \psi W, U) = g_M(\xi Z \omega W - \xi W \omega Z, \varphi PU + \psi RU) + g_M(\xi(\nabla_Z \omega W - \nabla_W \omega Z, \omega RU)$$

for all $Z, W \in \Gamma(D_1)$ as well as $U \in \Gamma(D \oplus D_2 \oplus < \zeta >)$.

Proof. For any $Z, W \in \Gamma(D_1)$ and $U \in \Gamma(D \oplus D_2 \oplus < \zeta >)$, we have

$$g_M([Z, W], U) = g_M(\nabla_Z W, U) - g_M(\nabla_W Z, U).$$

By the use of equations (2.1)-(2.5), (2.8), (2.9), (3.1), and (3.2) and Lemma 3.2, it is obtained that

$$g_M([Z, W], U) = g_M(\varphi \nabla_Z W, \varphi U) - g_M(\varphi \nabla_W Z, \varphi U),$$

$$= g_M(\nabla_Z \varphi W, \varphi U) - g_M(\nabla_W \varphi Z, \varphi U),$$

$$= g_M(\nabla_Z \psi W, \varphi U) + g_M(\nabla_Z \omega W, \varphi U) - g_M(\nabla_W \psi Z, \varphi U) - g_M(\nabla_W \omega W, \varphi U),$$

$$= \cos^2 \theta_1 g_M(\nabla_Z W, U) - \cos^2 \theta_1 g_M(\nabla_W Z, U) + g_M(\xi Z \omega \psi W - \xi W \omega \psi Z, U)$$

$$+ g_M(\xi(\nabla_Z \omega W + \xi Z \omega W, \varphi PU + \psi RU + \omega RU)$$

$$- g_M(\xi(\nabla_W \omega Z + \xi W \omega Z, \varphi PU + \psi RU + \omega RU).$$

Now, we have

$$\sin^2 \theta_1 g_M([Z, W], U) = g_M(\xi Z \omega W - \xi W \omega Z, \varphi PU + \psi RU) + g_M(\xi(\nabla_Z \omega W - \xi W \omega Z, \omega RU)$$

$$+ g_M(\xi(\nabla_Z \omega W - \xi W \omega Z, \varphi PU + \psi RU + \omega RU),$$

This proof is completed. \hfill \Box

Similarly, the coming theorem is presented.

Theorem 3.8. Let $h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N)$ is the proper quasi bi-slant Lorentzian submersion. Therefore the slant distribution $D_2$ is integrable if and only if

$$g_M(\xi Y \omega \varphi X - \xi X \omega \varphi Y, Z) = g_M(\xi(\nabla_X \omega Y - \xi(\nabla_Y \omega X, \omega QZ) + g_M(\xi X \omega Y - \xi Y \omega X, \varphi PZ + \psi QZ)$$

for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D \oplus D_1 \oplus < \zeta >)$.

Proposition 3.9. Suppose $h : (M, \varphi, \zeta, \eta, g_M) \rightarrow (N, g_N)$ is the proper quasi bi-slant Lorentzian submersion. Therefore the vertical distribution $(\ker h_\ast)$ does not determines the totally geodesic foliation at $M$.

Proof. Suppose we have $X \in \Gamma(\ker h_\ast)$ and $Z \in \Gamma(\ker h_\ast)^\perp$, by the use of (2.4) we get

$$g_M(\nabla_X \zeta, Z) = g_M(\varphi X, Z),$$

as $g_M(\varphi X, Z) \neq 0$, so $g_M(\nabla_X \zeta, Z) \neq 0$ for some $X$ and $Z$. Hence, $(\ker h_\ast)$ is not defining a totally geodesic foliation at $M$. \hfill \Box
Theorem 3.10. Suppose \( h : (M, \varphi, \xi, \eta, g_M) \to (N, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore the distribution \((\ker h_\ast) - < \xi, >\) determines the totally geodesic foliation at \( M \) if and only if

\[
g_M(\mathcal{T}_U PV + \cos^2 \theta_1 \mathcal{T}_U QV + \cos^2 \theta_2 \mathcal{T}_U RV, X) = -g_M(\mathcal{H}_\xi \mathcal{T}_U \omega \psi PV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi QV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi RV, X) g_M(\mathcal{T}_U \omega V, BX) - g_M(\mathcal{H}_\xi \mathcal{T}_U \omega V, CX)
\]

for any \( U, V \in \Gamma(\ker h_\ast) - < \xi, > \) and \( X \in \Gamma(\ker h_\ast)^\perp \).

Proof. For all \( U, V \in \Gamma(\ker h_\ast) - < \xi, > \) and \( X \in \Gamma(\ker h_\ast)^\perp \), by the use of equations (2.2), (2.3), and (3.1), we have

\[
g_M(\nabla_U V, X) = g_M(\nabla_U \varphi PV, \varphi X) + g_M(\nabla_U \varphi QV, \varphi X) + g_M(\nabla_U \varphi RV, \varphi X).
\]

Using equations (2.3), (2.10), (2.11), (3.1), (3.2), Lemma 3.2, and Lemma 3.3, we have

\[
g_M(\nabla_U V, X) = g_M(\mathcal{T}_U PV, X) + \cos^2 \theta_1 g_M(\mathcal{T}_U QV, X) + \cos^2 \theta_2 g_M(\mathcal{T}_U RV, X)
+ g_M(\mathcal{H}_\xi \mathcal{T}_U \omega \psi PV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi QV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi RV, X) + g_M(\nabla_U (\omega PV + \omega QV + \omega RV), \varphi X).
\]

Since \( \omega PV + \omega QV + \omega RV = \omega V \) and \( \omega PV = 0 \), thus we have

\[
g_M(\nabla_U V, X) = g_M(\mathcal{T}_U PV + \cos^2 \theta_1 \mathcal{T}_U QV + \cos^2 \theta_2 \mathcal{T}_U RV, X)
+ g_M(\mathcal{H}_\xi \mathcal{T}_U \omega \psi PV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi QV + \mathcal{H}_\xi \mathcal{T}_U \omega \psi RV, X)
+ g_M(\mathcal{T}_U \omega V, BX) + g_M(\mathcal{H}_\xi \mathcal{T}_U \omega V, CX),
\]

this proof is completed. \( \square \)

Theorem 3.11. Suppose \( h : (M, \varphi, \xi, \eta, g_M) \to (N, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore, the horizontal distribution \((\ker h_\ast)^\perp\) does not demonstrates the totally geodesic foliation at \( M \).

Proof. Suppose \( Z, V \in \Gamma(\ker h_\ast)^\perp \), and by the use of equation (2.4), we got

\[
g_M(\nabla_Z V, \xi) = -g_M(V, \nabla_Z \xi) = -g_M(V, \varphi Z),
\]

as \( g_M(V, \varphi Z) \neq 0 \), therefore \( g_M(\nabla_Z V, \xi) \neq 0 \) for some \( V \) and \( Z \). Hence, \((\ker h_\ast)^\perp\) does not demonstrates a totally geodesic foliation at \( M \). \( \square \)

Proposition 3.12. Suppose \( h : (M, \varphi, \xi, \eta, g_M) \to (N, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution \( D \) does not demonstrates the totally geodesic foliation on \( M \).

Proof. For all \( U, V \in \Gamma(D) \), using equation (2.4), we got

\[
g_M(\nabla_U V, \xi) = -g_M(V, \varphi U),
\]

since \( g_M(V, \varphi U) \neq 0 \), so \( g_M(\nabla_U V, \xi) \neq 0 \) for some \( U \) and \( V \). Hence \( D \) is not defining the totally geodesic foliation on \( M \). \( \square \)

Theorem 3.13. Suppose \( h : (M, \varphi, \xi, \eta, g_M) \to (N, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution \( D \oplus < \xi, > \) demonstrates the totally geodesic foliation if and only if

\[
g_M(\mathcal{T}_X \varphi PY, \omega RZ + \omega QZ) = -g_M(\mathcal{V} \nabla_X \varphi PY, \psi QZ + \psi RZ),
\]

and

\[
g_M(\mathcal{T}_X \varphi PY, CV) = -g_M(\mathcal{V} \nabla_X \varphi PY, BV),
\]

for all \( X, Y \in \Gamma(D \oplus < \xi, >) \), \( Z = QZ + RZ \in \Gamma(D_1 \oplus D_2) \) and \( V \in \Gamma(\ker h_\ast)^\perp \).
Proof. For all \(X, Y \in \Gamma(\mathbb{D} < \zeta >), Z = QZ + RZ \in \Gamma(\mathbb{D}_1 \oplus \mathbb{D}_2)\) and \(V \in \Gamma(\ker h_\alpha)^{\bot}\), the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.2), gives
\[
g_M(\nabla_X Y, Z) = g_M(\nabla_X \varphi Y, \varphi Z) = g_M(\nabla_X \varphi PY, \varphi QZ + \varphi RZ) \\
= g_M(\mathcal{J}_X \varphi PY, \omega RZ + \omega QZ) + g_M(\nabla \nabla_X \varphi PY, \psi QZ + \psi RZ).
\]
Now, again the use of equations (2.1)-(2.5), (2.8), (3.1), and (3.3), leads to
\[
g_M(\nabla_X Y, V) = g_M(\nabla_X \varphi Y, \varphi V) = g_M(\nabla_X \varphi PY, BV + CV) = g_M(\nabla \nabla_X \varphi PY, BV) + g_M(\mathcal{J}_X \varphi PY, CV),
\]
this proof is completed. \(\square\)

Proposition 3.14. Suppose \(h : (\mathbb{M}, \varphi, \zeta, \eta, g_M) \rightarrow (\mathbb{N}, g_N)\) is the proper quasi bi-slant Lorentzian submersion. Therefore the distribution \(D_1\) does not defines a totally geodesic foliation at \(M\), where \(i = 1, 2\).

Proof. For any \(Z, V \in \Gamma(D_1)\), by the use of equation (2.4) we have
\[
g_M(\nabla Z V, \zeta) = -g_M(Z, \varphi V),
\]
since \(g_M(Z, \varphi V) \neq 0\), so \(g_M(\nabla Z V, \zeta) \neq 0\) for some \(V\) and \(Z\). Hence \(D_1\) is not defining the totally geodesic foliation at \(M\), where \(i = 1, 2\). \(\square\)

Theorem 3.15. Suppose \(h : (\mathbb{M}, \varphi, \zeta, \eta, g_M) \rightarrow (\mathbb{N}, g_N)\) is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution \(D_1 \oplus < \zeta >\) demonstrates the totally geodesic foliation if and only if
\[
g_M(\mathcal{J}_Z \omega \psi W, X) = -g_M(\mathcal{J}_Z \omega W, \varphi PX + \psi RX) - g_M(\mathcal{J}_Z \nabla Z \omega W, \omega RX + \eta(W)) g_M(Z, \varphi PX + \psi RX),
\]
and
\[
g_M(\mathcal{J}_Z \nabla \nabla Z \omega \psi W, V) = -g_M(\mathcal{J}_Z \nabla Z \omega W, CV) - g_M(\mathcal{J}_Z \omega W, BV) + \eta(W) g_M(Z, BV),
\]
for all \(Z, W \in \Gamma(D_1 \oplus < \zeta >), X \in \Gamma(\mathbb{D} \oplus \mathbb{D}_2)\) and \(V \in \Gamma(\ker h_\alpha)^{\bot}\).

Proof. For every \(Z, W \in \Gamma(D_1 \oplus < \zeta >), X \in \Gamma(\mathbb{D} \oplus \mathbb{D}_2)\) and \(V \in \Gamma(\ker h_\alpha)^{\bot}\), the use of equations (2.1)-(2.5), (2.9), (3.1), (3.2), and Lemma 3.2 gives
\[
g_M(\nabla Z W, X) = g_M(\nabla Z \varphi W, \varphi X) - \eta(W) g_M(Z, \varphi X) \\
= g_M(\nabla Z \psi W, \varphi X) + g_M(\nabla Z \omega W, \varphi X) - \eta(W) g_M(Z, \varphi PX + \psi RX), \\
= \cos^2 \theta g_M(\nabla Z W, X) + g_M(\mathcal{J} Z \omega \psi W, X) \\
+ g_M(\mathcal{J} Z \omega W, \varphi PX + \psi RX) + g_M(\mathcal{J} \nabla Z \omega W, \omega RX) - \eta(W) g_M(Z, \varphi PX + \psi RX).
\]
Now, we have
\[
\sin^2 \theta g_M(\nabla Z W, X) = g_M(\mathcal{J} Z \omega \psi W, X) + g_M(\mathcal{J} Z \omega W, \varphi PX + \psi RX) \\
+ g_M(\mathcal{J} \nabla Z \omega W, \omega RX) - \eta(W) g_M(Z, \varphi PX + \psi RX).
\]
Next, from equations (2.1)-(2.5), (2.9), (3.2), (3.3), and Lemma 3.2, we have
\[
g_M(\nabla Z W, V) = g_M(\nabla Z \varphi W, \varphi V) - \eta(W) g_M(Z, \varphi V) \\
= g_M(\nabla Z \psi W, \varphi V) + g_M(\nabla Z \omega W, \varphi V) - \eta(W) g_M(Z, \varphi V) \\
= \cos^2 \theta g_M(\nabla Z W, V) + g_M(\mathcal{J} \nabla Z \omega \psi W, V) \\
+ g_M(\mathcal{J} \nabla Z \omega W, CV) + g_M(\mathcal{J} Z \omega W, BV) - \eta(W) g_M(Z, BV).
\]
Now, we have
\[
\sin^2 \theta g_M(\nabla Z W, V) = g_M(\mathcal{J} \nabla Z \omega \psi W, V) + g_M(\mathcal{J} \nabla Z \omega W, CV) + g_M(\mathcal{J} Z \omega W, BV) - \eta(W) g_M(Z, BV),
\]
this proof is completed. \(\square\)
Similarly, we can easily prove the coming theorem.

**Theorem 3.16.** Suppose \( h : (\mathcal{M}, \varphi, \zeta, \eta, g_M) \to (\mathcal{N}, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore, the distribution \( D_2 \oplus \zeta > \) demonstrates the totally geodesic foliation if and only if

\[
g_M(\mathcal{T}_X \omega Y, Z) = g_M(\mathcal{T}_X \omega QY, \varphi PZ + \varphi RZ) + g_M(\mathcal{T}_X \omega QY, \omega RZ) + \eta(Y) g_M(X, \varphi PZ + \varphi RZ),
\]

and

\[
g_M(\mathcal{T}_X \omega \psi Y, V) = -g_M(\mathcal{T}_X \omega Y, CV) - g_M(\mathcal{T}_X \omega Y, BV) + \eta(Y) g_M(X, BV),
\]

for all \( X, Y \in \Gamma(D_2 \oplus \zeta >), Z \in \Gamma(D \oplus D_1) \) and \( V \in \Gamma(\ker h_* \perp) \).

By the use of Proposition 3.9 and Theorem 3.11 one can give the coming theorem.

**Theorem 3.17.** Suppose \( h : (\mathcal{M}, \varphi, \zeta, \eta, g_M) \to (\mathcal{N}, g_N) \) is the proper quasi bi-slant Lorentzian submersion. Therefore, the map \( h \) is not a totally geodesic map.

**Example 3.18.** Consider the differentiable manifold \( \mathbb{R}^{11} \) with coordinates \( (x^1, \ldots, x^5, y^1, \ldots, y^5, z) \) and base field \( \{E_i, E_{5+i}, \zeta\} \) where \( E_i = 2 \frac{\partial}{\partial y_i}, E_{5+i} = 2(\frac{\partial}{\partial x_i} + y^i \frac{\partial}{\partial z}), i = 1, \ldots, 5 \) and contravariant vector field \( \zeta = 2 \frac{\partial}{\partial z} \). Define Lorentzian almost paracontact structure on \( \mathbb{R}^{11} \) as follows:

\[
\varphi(\sum_{i=1}^{5}(X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}) + Z \frac{\partial}{\partial z}) = -\sum_{i=1}^{5} Y_i \frac{\partial}{\partial x_i} - \sum_{i=1}^{5} X_i \frac{\partial}{\partial y_i} + \sum_{i=1}^{5} Y_i y^i \frac{\partial}{\partial z},
\]

\[
\eta = -\frac{1}{2}(dz - \sum_{i=1}^{5} y^i dx^i),
\]

\[
g = -\eta \otimes \eta + \frac{1}{4} \sum_{i=1}^{5} dx^i \otimes dx^i + \sum_{i=1}^{5} dy^i \otimes dy^i,
\]

where \( X_i, Y_i \) and \( Z \) are \( C^\infty \) functions on \( \mathbb{R}^{11} \). Then \( (\mathbb{R}^{11}, \varphi, \zeta, \eta, g) \) is the LP-Sasakian manifold. Suppose \( \mathbb{R}^4 \) is the Riemannian manifold with the Riemannian metric tensor field \( g_{\mathbb{R}^4} \) defined as

\[
g_{\mathbb{R}^4} = \frac{1}{4} \sum_{i=1}^{4} (dv^i \otimes dv^i)
\]

on \( \mathbb{R}^4 \), where \( (v^1, v^2, v^3, v^4) \) is local coordinate system on \( \mathbb{R}^4 \).

Let \( h : \mathbb{R}^{11} \to \mathbb{R}^4 \) is the map written as

\[
h(x^1, \ldots, x^5, y^1, \ldots, y^5, z) = (x^2, \sin \theta_1 x^3 + \cos \theta_1 x^4, \sin \theta_2 y^1 - \cos \theta_2 y^2, y^4)
\]

that is quasi bi-slant Lorentzian submersion map which satisfies

\[
\begin{align*}
\bar{X}_1 &= \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}, \\
\bar{X}_2 &= \cos \theta_1 (\frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}) - \sin \theta_1 (\frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z}), \\
\bar{X}_3 &= \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \\
\bar{X}_4 &= \cos \theta_2 \frac{\partial}{\partial y^1} + \sin \theta_2 \frac{\partial}{\partial y^2}, \\
\bar{X}_5 &= \frac{\partial}{\partial y^3}, \\
\bar{X}_6 &= \frac{\partial}{\partial y^4}, \\
\bar{X}_7 &= \zeta = 2 \frac{\partial}{\partial z},
\end{align*}
\]

(\( \ker h_* = (D \oplus D_1 \oplus D_2 \oplus \zeta >) \)).
where

\[ D = < \tilde{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \]
\[ D_1 = \langle \tilde{X}_2 = \cos \theta_1 \left( \frac{\partial}{\partial x^4} + y^3 \frac{\partial}{\partial z} \right) - \sin \theta_1 \left( \frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z} \right), \]
\[ D_2 = < \tilde{X}_1 = \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}, \]
\[ \tilde{X}_4 = \cos \theta_2 \frac{\partial}{\partial y^4} + \sin \theta_2 \frac{\partial}{\partial y^5}, \]
\[ \tilde{X}_6 = \frac{\partial}{\partial y^5} >, \]
\[ \tilde{X}_5 = \frac{\partial}{\partial y^4} >, \]
\[ \langle \zeta > = < \tilde{X}_7 = 2 \frac{\partial}{\partial z} >, \]

and

\[ (\ker h_*)^\perp = < V_1 = \frac{\partial}{\partial x^3} + y^2 \frac{\partial}{\partial z}, V_2 = \sin \theta_1 \left( \frac{\partial}{\partial x^4} + y^3 \frac{\partial}{\partial z} \right) + \cos \theta_1 \left( \frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z} \right), V_3 = \sin \theta_2 \frac{\partial}{\partial y^4} - \cos \theta_2 \frac{\partial}{\partial y^5}, V_4 = \frac{\partial}{\partial y^5} >, \]

with bi-slant angles \( \theta_1 \) and \( \theta_2 \). Also by direct computations, we obtain

\[ h_* V_1 = \frac{\partial}{\partial y^1}, \ h_* V_2 = \frac{\partial}{\partial y^2}, \ h_* V_3 = \frac{\partial}{\partial y^3}, \ h_* V_4 = \frac{\partial}{\partial y^4}. \]

**Example 3.19.** Consider \( R^{11} \) and \( R^4 \) has same structure as in Example 3.18. Suppose \( R^4 \) is the Riemannian manifold with the Riemannian metric tensor field \( g_{R^4} \) defined as

\[ g_{R^4} = \frac{1}{4} \sum_{i=1}^{4} (dv^i \otimes dv^i) \]

on \( R^4 \), where \((v^1, v^2, v^3, v^4)\) is local coordinate system on \( R^4 \). Let \( h : R^{11} \to R^4 \) be the map determined as

\[ h(x^1, \ldots, x^5, y^1, \ldots, y^5, z) = \left( \frac{\sqrt{3}x^1 + x^2}{2}, x^4, y^1, y^3 - y^4 \right) \]

that is quasi bi-slant Lorentzian submersion map which satisfies

\[ \hat{X}_1 = (\frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}) - \sqrt{3} (\frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial z}) , \]
\[ \hat{X}_2 = \frac{\partial}{\partial x^3} + y^3 \frac{\partial}{\partial z}, \]
\[ \hat{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \]
\[ \hat{X}_4 = \frac{\partial}{\partial y^4}, \]
\[ \hat{X}_5 = \frac{\partial}{\partial y^3} + \frac{\partial}{\partial y^5}, \]
\[ \hat{X}_6 = \frac{\partial}{\partial y^5} >, \]
\[ \hat{X}_7 = \zeta = 2 \frac{\partial}{\partial z}, \]
\[ (\ker h_*) = (D \oplus D_1 \oplus D_2 \oplus \langle \zeta > ) , \]

where

\[ D = < \tilde{X}_3 = \frac{\partial}{\partial x^5} + y^5 \frac{\partial}{\partial z}, \]
\[ D_1 = \langle \tilde{X}_2 = \cos \theta_1 \left( \frac{\partial}{\partial x^4} + y^3 \frac{\partial}{\partial z} \right) - \sin \theta_1 \left( \frac{\partial}{\partial x^4} + y^4 \frac{\partial}{\partial z} \right), \]
\[ D_2 = < \tilde{X}_1 = \frac{\partial}{\partial x^1} + y^1 \frac{\partial}{\partial z}, \]
\[ \tilde{X}_4 = \cos \theta_2 \frac{\partial}{\partial y^4} + \sin \theta_2 \frac{\partial}{\partial y^5}, \]
\[ \tilde{X}_6 = \frac{\partial}{\partial y^5} >, \]
\[ \langle \zeta > = < \tilde{X}_7 = 2 \frac{\partial}{\partial z} >, \]}
and

\[
(h_* V_1) = \sqrt{3} \left( \frac{\partial}{\partial x^1} + y \frac{\partial}{\partial z} \right) + \left( \frac{\partial}{\partial x^2} + y^2 \frac{\partial}{\partial z} \right), \quad V_2 = \frac{\partial}{\partial x^1} + y \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial y}, \quad V_4 = \left( \frac{\partial}{\partial y^3} - \frac{\partial}{\partial y^4} \right),
\]

with bi-slant angles \( \theta_1 = \frac{\pi}{6} \) and \( \theta_2 = \frac{\pi}{4} \). Also by direct computations, we obtain

\[
h_* V_1 = 2 \frac{\partial}{\partial y^1}, \quad h_* V_2 = \frac{\partial}{\partial y}, \quad h_* V_3 = \frac{\partial}{\partial y^3}, \quad h_* V_4 = \sqrt{2} \frac{\partial}{\partial y^4}.
\]

It can be easily seen that Theorem 3.11, and Propositions 3.12 and 3.14 are satisfied by the Examples 3.18 and 3.19.

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