Directional differentiability, coexhausters, codifferentials and polyhedral DC functions.

Majid E. Abbasov

St. Petersburg State University, SPbSU, 7/9 Universitetskaya nab., St. Petersburg, 199034 Russia.
Institute for Problems in Mechanical Engineering of the RAS 61, Bolshoj pr. V.O., St. Petersburg, 199178
m.abbasov@spbu.ru, abbasov.majid@gmail.com

November 29, 2021

Abstract

Codifferentials and coexhausters are used to describe nonhomogeneous approximations of a nonsmooth function. Despite the fact that coexhausters are modern generalizations of codifferentials, the theories of these two concepts continue to develop simultaneously. Moreover, codifferentials and coexhausters are strongly connected with DC functions. In this paper we trace analogies between all these objects, and prove the equivalence of the boundedness and optimality conditions described in terms of these notions. This allows one to extend the results derived in terms of one object to the problems stated via the other one. Another contribution of this paper is the study of connection between nonhomogeneous approximations and directional derivatives and formulate optimality conditions in terms of nonhomogeneous approximations.

Introduction

Among the variety of approaches of nonsmooth analysis the method of quasidifferential stands out due to its constructiveness. One important advantage of this approach is that all the tools and methods can be built and used not only theoretically but also in practical problems. The approach goes back to the early 80-th when Demyanov, Rubinov and Polyakova proposed and studied the notion of quasidifferentials. Quasidifferentials are pairs of convex compact sets that enable one to represent the directional derivative of a function at a point in a form of sum of maximum and minimum of a linear functions. Quasidifferentials enjoy full calculus, that grants the calculation of quasidifferentials for a rich variety of functions. Such functions are also called quasidifferentiable.

Polyakova and Demyanov derived optimality conditions in terms of these objects and also showed how to find the directions of steepest descent and ascent when these conditions are not satisfied. This paved a way for constructing new
optimization algorithms. An interesting example of the application of quasidifferential calculus can be found in [6], where the authors use this tool to solve a complex optimization problem appearing in the area of Chebyshev approximation.

In some cases, however, these algorithms experience convergence problems [7]. This happens due to the fact that quasidifferentiable set-valued mapping is not continuous in Hausdorff metric. Similar results have been reported with exhausters [8–10] which can be viewed as a generalization of quasidifferentials [17].

To overcome this drawback Demyanov and Rubinov in the mid 90-th introduced the notion of codifferentials [18]. Codifferential is a pair of convex compact sets that provides the representation of the approximation of the studied function in a neighborhood of a given point in the form of sum of minimum and maximum of affine functions.

Coexhausters arose as a generalization of codifferentials [19]. A class of coexhausterable functions is wider than the class of codifferentiable functions. Coexhausters are families of convex compact sets which are used to represent the approximation of a considered function in a neighborhood of a point as a sum of MaxMin or MinMax of affine functions. The formulas of calculus for codifferentials and coexhausters have been derived as well as optimality conditions in terms of these tools [20–22].

The usage of continuously codifferentiable and coexhausterable functions guaranteed stability and convergence of numerical algorithms, but positive homogeneity property was lost in this path. In this paper we address this issue by study the optimality conditions in terms of inhomogeneous approximations.

It must be noted that codifferentials and coexhausters have strong connection with DC functions [23–27]. Therefore the problem of studying the connection between all these notions is of high interest. It can enable us to extend results derived in terms of one object to the problems stated via another one.

The paper is organized as follows. In Section 1 we establish connection between directional derivatives and nonhomogeneous approximations of a function. Then we give definitions of codifferentials and coexhausters and connect these notions with the class of difference of polyhedral convex functions. In Section 2 we present Polyakova’s (see [28]) boundedness condition in terms of codifferentials. We prove that this condition is equivalent to the condition of boundedness stated in [29] in terms of coexhausters. In Section 3 we describe Demyanov’s optimality conditions in terms of coexhausters and Polyakova’s optimality conditions in terms of codifferentials. We demonstrate that these conditions are equivalent. All the presented results are also considered from DC functions point of view.

1 Directional differentiability, codifferentials, coexhausters and polyhedral DC functions

Let a function $f: \mathbb{R}^n \to \mathbb{R}$ be given. The function $f$ is called directionally differentiable at a point $x \in \mathbb{R}^n$ if for every $\Delta \in \mathbb{R}^n$ there exists the final limit

$$f'(x, \Delta) = \lim_{\alpha \downarrow 0} \frac{f(x + \alpha \Delta) - f(x)}{\alpha}.$$
The value $f'(x, \Delta)$ is called the directional derivative of the function $f$ at the point $x \in \mathbb{R}^n$ in the direction $\Delta \in \mathbb{R}^n$. Directional derivative allows us to formulate necessary conditions for a minimum and maximum (see [1]).

**Theorem 1.** Let a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be directionally differentiable at a point $x_\ast \in \mathbb{R}^n$. For the point $x_\ast$ to be a minimizer of the function $f$ on $\mathbb{R}^n$ it is necessary that

$$f'(x_\ast, \Delta) \geq 0 \quad \forall \Delta \in \mathbb{R}^n.$$  \hspace{1cm} (1)

**Theorem 2.** Let a function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be directionally differentiable at a point $x_\ast \in \mathbb{R}^n$. For the point $x_\ast$ to be a maximizer of the function $f$ on $\mathbb{R}^n$ it is necessary that

$$f'(x_\ast, \Delta) \leq 0 \quad \forall \Delta \in \mathbb{R}^n.$$  \hspace{1cm} (2)

A point $x_\ast$ satisfying condition (1), is called an inf-stationary point of the function $f$. A point $x_\ast$ satisfying condition (2), is called a sup-stationary point of the function $f$.

Let the following expansion holds

$$f(x + \Delta) = f(x) + h_x(\Delta) + o_x(\Delta) \quad \forall \Delta \in \mathbb{R}^n,$$  \hspace{1cm} (3)

for a continuous directionally differentiable function $f$, where

$$\lim_{\alpha \downarrow 0} \frac{o_x(\alpha \Delta)}{\alpha} = 0 \quad \forall \Delta \in \mathbb{R}^n.$$

Due to the continuity of $f$ we have $h_x(0_n) = 0$ at any point $x$. Therefore $h_x(\Delta)$ is directionally differentiable at the origin and

$$f'(x, \Delta) = h_x'(0_n, \Delta).$$

If $h_x(\Delta)$ is positively homogenous as a function of $\Delta$ we have

$$h_x'(0_n, \Delta) = h_x(\Delta) \quad \forall \Delta \in \mathbb{R}^n,$$

i.e. in this case we can state results similar to Theorems [1] and [2] by replacing $f'(x, \Delta)$ with $h_x(\Delta)$.

If $h_x(\Delta)$ is not positively homogenous then

$$h_x'(0_n, \Delta) \neq h_x(\Delta),$$

but inequality $h_x(\Delta) \leq 0$ implies $h_x'(0_n, \Delta) \leq 0$ while inequality $h_x(\Delta) \geq 0$ implies $h_x'(0_n, \Delta) \geq 0$. Therefore in this case the condition

$$h_x(\Delta) \geq 0 \quad \forall \Delta \in \mathbb{R}^n.$$  

is sufficient for $x_\ast$ to be an $inf$-stationary point, while the condition

$$h_{x_\ast}(-\Delta) \leq 0 \quad \forall \Delta \in \mathbb{R}^n.$$

is sufficient for $x^\ast$ to be a $sup$-stationary point.
Having a specific form of the approximation \( h_x(\Delta) \), we can describe conditions of minimum in terms of objects that define the form and use these conditions to construct optimization algorithms. This is the case in smooth case where \( h_x(\Delta) \) can be presented as the inner product of the gradient and the direction \( h_x(\Delta) = \langle \nabla f(x), \Delta \rangle \). For a nonsmooth function a linear approximation is not applicable and one have to work with more complicated forms.

The function \( f : X \to \mathbb{R} \) is called codifferentiable at a point \( x \), if there exist convex compact sets \( \overline{\partial} f(x) \subset \mathbb{R}^{n+1} \) and \( \partial f(x) \subset \mathbb{R}^{n+1} \) such that

\[
    h_x(\Delta) = \max_{[a,v] \in \partial f(x)} [a + \langle v, \Delta \rangle] + \min_{[b,w] \in \overline{\partial} f(x)} [b + \langle w, \Delta \rangle],
\]

where

The pair \( Df(x) = [\partial f(x), \overline{\partial} f(x)] \) is called a codifferential of the function \( f \) at the point \( x \). Recall that a codifferential is a pair of sets in the space \( \mathbb{R}^{n+1} \).

The function \( f \) is continuous, therefore from (4) (for \( \Delta = 0 \)) it follows that

\[
    \max_{[a,v] \in \partial f(x)} a + \min_{[b,w] \in \overline{\partial} f(x)} b = 0.
\]

Since a codifferential function is not uniquely defined at a point, without loss of generality we can rewrite equality (4) as

\[
    \max_{[a,v] \in \partial f(x)} a = \min_{[b,w] \in \overline{\partial} f(x)} b = 0.
\]

A function \( f \) is called continuously codifferentiable at a point \( x \) if it is codifferentiable in some neighborhood of the point \( x \) and there exists a codifferential mapping \( Df(x) = [\partial f(x), \overline{\partial} f(x)] \) which is continuous in the Hausdorff metric at the point \( x \).

Polyhedral codifferential is of high importance for many applications and therefore we concentrate on this case in the rest of the paper, i.e.

\[
    \partial f(x) = \text{co} \{ [a_i, v_i] \mid i \in I \}, \quad \overline{\partial} f(x) = \text{co} \{ [b_j, w_j] \mid i \in J \},
\]

where \( I \) and \( J \) are finite index sets.

Expression (4) implies

\[
    h_x(\Delta) = \max_{[a,v] \in \partial f(x)} \min_{[b,w] \in \overline{\partial} f(x)} [a + b + \langle v + w, \Delta \rangle] = \max_{C \in \overline{E}(x)} \min_{[b,w] \in C} [b + \langle w, \Delta \rangle], \tag{7}
\]

where

\[
    \overline{E}(x) = \{ C \subset \mathbb{R}^{n+1} \mid C = [a, v] + \overline{\partial} f(x), \ [a, v] \in \partial f(x) \}.
\]

Similarly we can get the representation

\[
    h_x(\Delta) = \min_{[b,w] \in \overline{\partial} f(x)} \max_{[a,v] \in \partial f(x)} [a + b + \langle v + w, \Delta \rangle] = \min_{C \in \overline{E}(x)} \max_{[a,v] \in C} [a + \langle v, \Delta \rangle], \tag{8}
\]

where

\[
    \overline{E}(x) = \{ C \subset \mathbb{R}^{n+1} \mid C = [b, w] + \overline{\partial} f(x), \ [b, w] \in \overline{\partial} f(x) \}.
\]
The functions

\[
\max_{C \in \mathcal{E}(x)} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle] \quad \text{and} \quad \min_{C \in \mathcal{E}(x)} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle]
\]

represent approximations of the increment of the function \( f \) in a neighbourhood of \( x \). The usage of continuously codifferentiable functions introduced above allows one to guarantee stability and convergence of numerical algorithms.

The notion of codifferential was introduced in [13] where necessary optimality conditions were stated. Via expansions (7) and (8) we obtain the following generalization of the codifferential notion.

Let a function \( f \) be continuous at a point \( x \in X \). We say that at the point \( x \) the function \( f \) has an upper coexhauster if the following expansion holds:

\[
\min_{C \in \mathcal{E}(x)} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle], \quad (9)
\]

where \( \mathcal{E}(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \). The set \( \mathcal{E}(x) \) is called an upper coexhauster of \( f \) at the point \( x \).

We say that at the point \( x \) the function \( f \) has a lower coexhauster if the following expansion holds:

\[
\max_{C \in \mathcal{E}(x)} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle], \quad (10)
\]

where \( \mathcal{E}(x) \) is a family of convex compact sets in \( \mathbb{R}^{n+1} \). The set \( \mathcal{E}(x) \) is called a lower coexhauster of the function \( f \) at the point \( x \).

The function \( f \) is continuous, therefore from (9) and (10) we have

\[
\min_{C \in \mathcal{E}(x)} \max_{[a, v] \in C} a = \max_{C \in \mathcal{E}(x)} \min_{[b, w] \in C} b = 0. \tag{11}
\]

The notion of coexhauster was introduced in [8, 9]. Similar to the case of codifferentiable functions, we can consider continuous upper and lower coexhauster mappings.

It is important to notice that DC functions are codifferentiable and have upper and lower coexhausters. This means that DC functions can be studied via the rich theory of codifferentials and coexhausters. Let us illustrate this.

Local approximation for many DC function can be presented as the difference of polyhedral convex functions, i.e. in the form

\[
h_x(\Delta) = \max_{i \in I} [a_i + \langle v, \Delta \rangle] - \min_{j \in J} [-b_j - \langle w, \Delta \rangle], \quad (12)
\]

where \( I \) and \( J \) are finite index sets. We can rewrite (12) in the form

\[
h_x(\Delta) = \max_{i \in I} [a_i + \langle v, \Delta \rangle] + \min_{j \in J} [-b_j - \langle w, \Delta \rangle]
\]

\[
= \max_{[a, v] \in \mathcal{E}h} [a + \langle v, \Delta \rangle] + \min_{[b, w] \in \mathcal{E}h} [b + \langle w, \Delta \rangle]
\]

\[
= \max_{C \in \mathcal{E}} \min_{[b, w] \in C} [b + \langle w, \Delta \rangle]
\]

\[
= \min_{C \in \mathcal{E}} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle], \tag{13}
\]

5
where \( \delta h = \text{co} \{ [a_i, v_i] \mid i \in I \} \) and \( \delta h = \text{co} \{ [-b_j, -w_j] \mid i \in I \} \),

\[
E = \{ \text{co} \{ [a_i, v_i, j \in J] \mid i \in I \}, \\
E = \{ \text{co} \{ [a_i, b_j, v_i, j \in J] \mid i \in I \mid j \in J \}.
\]

For the sake of shortness we will use notation \( h(\Delta) \) instead of \( h_x(\Delta) \) in what follows.

### 2 Boundedness conditions

Local approximation \( h \) is often used for construction of optimization algorithms. Therefore, in the minimization problems, it is essential that the approximation is bounded from below. Polyakova derived this condition in terms of codifferential in \[28\].

**Theorem 3** (Polyakova). *For the function*

\[
h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] + \min_{j \in J} [b_j + \langle w_j, \Delta \rangle],
\]

*where* \( I \) *and* \( J \) *are finite index sets, to be bounded from below it is necessary and sufficient that for any* \( j \in J \) *the condition*

\[-w_j \in \text{co} \{ v_i \mid i \in I \}\]

*holds.*

The same condition was obtained in \[29\] in terms of coexhausters.

**Theorem 4** (Abbasov). *For the function*

\[
h(\Delta) = \min_{C \in E} \max_{a, v \in C} [a + \langle v, \Delta \rangle]
\]

*to be bounded from below it is necessary and sufficient that the condition*

\[C \bigcap L \neq \emptyset \quad \forall C \in \overline{E},\]

*is satisfied, where* \( L = \{(a, 0_n) \mid a \in \mathbb{R} \}.\)

Based on Theorems 3 and 4 we can state and prove general result which connects polyhedral DC-functions, codifferentials and coexhausters.

**Theorem 5.** *For the function*

\[
h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] - \min_{j \in J} [b_j + \langle w_j, \Delta \rangle],
\]

*where* \( I \) *and* \( J \) *are finite index sets, to be bounded from below it is necessary and sufficient that one of the following equivalent conditions hold*

\[w_j \in \text{co} \{ v_i \mid i \in I \}, \quad \forall j \in J\]

*(14) or*

\[C \bigcap L \neq \emptyset \quad \forall C \in \overline{E}\]

*(15) where* \( L = \{(a, 0_n) \mid a \in \mathbb{R} \} \) *and*

\[E = \{ C \mid C = \text{co} \{ a_i - b_j, v_i - w_j \}, i \in I \}, j \in J \}.
\]
Proof. Since the function \( h \) can be rewritten in the form
\[
h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] + \min_{j \in J} [-b_j - \langle w_j, \Delta \rangle],
\]
boundedness of \( h \) yields immediately from Theorems 3 and 4.

Prove that conditions (14) and (15) are equivalent. Let condition (15) hold. Then we have
\[
0_n \in \text{co}\{v_i - w_j \mid i \in I\} \quad \forall j \in J.
\]
This implies that for an arbitrary \( j \in J \) there exists \( \lambda_i, i \in I \) such that
\[
\sum_{i \in I} \lambda_i = 1,
\]
\[
\lambda_i \geq 0 \quad \forall i \in I,
\]
for which holds the condition
\[
\sum_{i \in I} \lambda_i(v_i - w_j) = 0.
\]
Therefore for we have
\[
\sum_{i \in I} \lambda_i v_i = w_j,
\]
what implies (14).

To prove that (15) follows from (14) we can run the same proof backwards.

Similar theorem can be stated for the upper boundedness conditions.

**Theorem 6.** For the function
\[
h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] - \max_{j \in J} [b_j + \langle w_j, \Delta \rangle],
\]
where \( I \) and \( J \) are finite index sets, to be upper bounded it is necessary and sufficient that one of the following equivalent conditions hold
\[
v_i \in \text{co}\{w_j \mid j \in J\}, \quad \forall i \in I \quad \text{(16)}
\]
or
\[
C \cap L \neq \emptyset \quad \forall C \in \mathcal{E} \quad \text{(17)}
\]
where \( L = \{(a,0_n) \mid a \in \mathbb{R}\} \) and
\[
\mathcal{E} = \{C \mid C = \text{co}\{[a_i - b_j, v_i - w_j], j \in J\}, i \in I\}.
\]

Now let us demonstrate how these results works.

**Example 2.1.** Consider the function
\[
h(\Delta) = \max\{2\Delta - 4, 0, -2\Delta - 4\} - \max\{\Delta - 1, 0, -\Delta - 1\}.
\]

Fig. 1 shows that \( h \) is bounded from below.
Since $h$ can be rewritten as
\[ h(\Delta) = \max\{2\Delta - 4, 0, -2\Delta - 4\} + \min\{-\Delta + 1, 0, \Delta + 1\}, \]
the function has a codifferential of the form (see Fig. 2 a)
\[ \partial h = \co\left\{\left(-\frac{4}{2}, 0, \frac{-4}{-2}\right)\right\}, \quad \partial^h = \co\left\{\left(-\frac{1}{1}, 0, \frac{1}{1}\right)\right\}, \]
and an upper coexhauster $\mathcal{E} = \{C_1, C_2, C_3\}$, where
\[ C_1 = \co\left\{\left(-\frac{3}{1}, \frac{-3}{1}\right)\right\}, \quad C_2 = \co\left\{\left(-\frac{4}{2}, 0, \frac{-4}{-2}\right)\right\}, \]
\[ C_3 = \co\left\{\left(-\frac{3}{3}, \frac{1}{1}, \frac{-3}{-1}\right)\right\} \]
(see Fig. 2 b).

Sets $\partial h$ and $\partial^h$ are polyhedrons. We have $v_1 = 2, v_2 = 0, v_3 = -2, w_1 = -1, w_2 = 0, w_3 = 1$. Therefore $\co\{v_i | i = 1, 3\} = [-2, 2]$ and $w_j \in [-2, 2]$ for all $j = 1, 3$. It is obvious (see Fig. 2 b) that $C_i \cap L \neq \emptyset$ for all $i = 1, 3$. This means that both conditions (14) and (15) are satisfied here.

### 3 Optimality conditions

Now let us proceed to the minimality conditions in terms of coexhausters and codifferentials.
Theorem 7 (Demaynov [15]). For the inequality
\[ h(\Delta) = \min_{C \in \mathcal{E}} \max_{[a, v] \in C} [a + \langle v, \Delta \rangle] \geq 0 \]
to be valid for all \( \Delta \in \mathbb{R}^n \) it is necessary and sufficient that the condition
\[ C \cap L_+ \neq \emptyset \quad \forall C \in \mathcal{E}, \]
where \( L_+^0 = \{(a, 0_n) \mid a \geq 0\} \), holds.

Theorem 8 (Polyakova [28]). For the inequality
\[ h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] + \min_{j \in J} [b_j + \langle w_j, \Delta \rangle] \geq 0 \]
to be valid for all \( \Delta \in \mathbb{R}^n \) it is necessary and sufficient that the condition
\[ \text{co}\{(a_i, v_i) \mid i \in I\} \cap \text{co}\{(-b_j, -w_j), (0, -w_j)\} \neq \emptyset \quad \forall j \in J \]
holds.

Theorems 7 and 8 can be used to show the connection between polyhedral DC-functions, codifferentials and coexhausters.

Theorem 9. Let the function
\[ h(\Delta) = \max_{i \in I} [a_i + \langle v_i, \Delta \rangle] - \max_{j \in J} [b_j + \langle w_j, \Delta \rangle] \]
be given, where \( I \) and \( J \) are finite index sets. Then the following statements are equivalent

1. The inequality \( h(\Delta) \geq 0 \) holds for all \( \Delta \in \mathbb{R}^n \).

2. The condition
\[ \text{co}\{(a_i, v_i) \mid i \in I\} \cap \text{co}\{(-b_j, -w_j), (0, -w_j)\} \neq \emptyset \quad (18) \]
holds for all \( j \in J \).

3. The condition
\[ C \cap L_+ \neq \emptyset \quad (19) \]
holds for all \( C \in \mathcal{E} \), where \( L_+^0 = \{(a, 0_n) \mid a \geq 0\} \) and
\[ \mathcal{E} = \{C \mid C = \text{co}\{[a_i - b_j, v_i - w_j], i \in I\}, j \in J\}. \]

Proof. We only need to prove the equivalence of conditions (18) and (19), since the rest parts of the proof follows immediately from Theorems 7 and 8.

First of all note that according to (6) we have \( a_i \leq 0 \) and \( b_i \leq 0 \) for any \( i \in I \) and \( j \in J \).

Let condition (18) be valid. Choose an arbitrary \( j \in J \). Then there exists \( \lambda_i, i \in I \) such that
\[ \begin{cases} \sum_{i \in I} \lambda_i = 1, \\ \lambda_{ij} \geq 0 \quad \forall i \in I, \end{cases} \]
for which we have
\[
\begin{align*}
\sum_{i \in I} \lambda_i a_i & \geq b_j, \\
\sum_{i \in I} \lambda_i v_i & = w_j,
\end{align*}
\]
whence
\[
\begin{align*}
\sum_{i \in I} \lambda_i (a_i - b_j) & \geq 0, \\
\sum_{i \in I} \lambda_i (v_i - w_j) & = 0,
\end{align*}
\]
This immediately brings us to (19).

Since all the above steps of the proof can be reversed, we conclude that (19) implies (18).

Remark 3.1. Condition (18) can be rewritten in terms of codifferentials as
\[
\mathcal{d}h \bigcap \text{co}\{-b_j, -w_j\}, \text{co}\{(a_i, v_i)| i \in I\} \neq \emptyset \forall j \in J,
\]
where \( \mathcal{d}h = \text{co}\{(b_j, w_j) | j \in J\} \).

Similar result can be stated for maximum conditions.

Theorem 10. Let the function
\[
h(\Delta) = \max_{i \in I}[a_i + (v_i, \Delta)] - \max_{j \in J}[b_j + (w_j, \Delta)]
\]
be given, where \( I \) and \( J \) are finite index sets. Then the following statements are equivalent
1. The inequality \( h(\Delta) \leq 0 \) holds for all \( \Delta \in \mathbb{R}^n \).
2. The condition
\[
\text{co}\{(b_j, w_j) | j \in J\} \bigcap \text{co}\{(a_i, v_i), (0, v_i)\} \neq \emptyset
\]
holds for all \( i \in I \).
3. The condition
\[
\mathcal{C} \bigcap L^- \neq \emptyset
\]
holds for all \( \mathcal{C} \in \mathcal{E} \), where \( L^- = \{(a, 0_n) | a \leq 0\} \) and
\[
\mathcal{E} = \{C | C = \text{co}\{[a_i - b_j, v_i - w_j]_j \in J, i \in I\}\).
\]

Remark 3.2. Condition (21) can be rewritten in terms of codifferentials as
\[
\mathcal{d}h \bigcap \text{co}\{-a_i, -v_i\}, \text{co}\{(0, -v_i)| i \in I\} \neq \emptyset \forall i \in I,
\]
where \( \mathcal{d}h = \text{co}\{(a_i, w_i) | i \in I\} \).
Example 3.1. Let $f: \mathbb{R} \to \mathbb{R}$ be a function of the form

$$f(x) = \max\{-x^2 + 2x, -x^2 - 2x, 0\} - \max\{x - 1, -x - 1, 0\}.$$

Fig. 3 shows that the point $x_1 = 0$ is the local minimum of the function while the point $x_2 = 1$ is a local maximum.

![Graph of the function $h$ in Example 2.](image)

Let us check conditions for a minimum at $x_1$ via Theorem 9 and conditions for a maximum at $x_2$ via Theorem 10. We start with an expansion of $f$ in the neighborhood of $x$.

$$f(x + \Delta) = f(x) + \max\{-x^2 + 2x - f(x) + (-2x + 2)\Delta, -x^2 - 2x - f(x) + (-2x - 2)\Delta, -f(x)\} - \max\{x - 1 + \Delta, -x - 1 - \Delta, 0\} + o(\Delta),$$

where $\lim_{\Delta \to 0} \frac{o(\Delta)}{\Delta} = 0$. Hence

$$\partial h_x = \text{co} \left\{ \left( -x^2 + 2x - f(x) \right), \left( -x^2 - 2x - f(x) \right), \left( -f(x) \right) \right\},$$

$$\bar{\partial} h_x = \text{co} \left\{ \left( -x + 1 \right), \left( x + 1 \right), \left( 0 \right) \right\}.$$

A codifferential at the point $x_1$ has the form (see Fig. 4 a)

$$\partial h_{x_1} = \text{co} \left\{ \left( 0, \frac{1}{2}, -\frac{1}{2} \right), \left( 0, 0 \right) \right\}, \quad \bar{\partial} h_{x_1} = \text{co} \left\{ \left( -1, 1, 0 \right) \right\}.$$

For an upper coexhauster we have $\bar{E}(x_1) = \{ C_1, C_2, C_3 \}$, where

$$C_1 = \text{co} \left\{ \left( 1, 1 \right), \left( -1, 3 \right), \left( -1, 1 \right) \right\}, \quad C_2 = \text{co} \left\{ \left( 1, 3 \right), \left( -1, 1 \right), \left( 1, 1 \right) \right\},$$

$$C_3 = \text{co} \left\{ \left( 0, 2 \right), \left( -0, 2 \right), \left( 0, 0 \right) \right\}$$

(see Fig. 4 b).
Conditions (19) and (20) hold which means that $x_1$ is an inf-stationary point.

Now proceed to point $x_2$. A codifferential at this point has the form

$$d h_{x_2} = \text{co} \begin{Bmatrix} (0, 0), (-4, 0), (-1, 0) \end{Bmatrix}, \quad dh_{x_2} = \text{co} \begin{Bmatrix} (0, -1), (2, 1), (0, 0) \end{Bmatrix},$$

(see Fig. 4 a), whence for a lower coexhauster we have $E(x_2) = \{C_4, C_5, C_6\}$, where

$$C_4 = \text{co} \begin{Bmatrix} (-1, 0), (2, 0), (0, 0) \end{Bmatrix}, \quad C_5 = \text{co} \begin{Bmatrix} (-4, -5), (-1, 2), (-4, 3) \end{Bmatrix},$$

$$C_6 = \text{co} \begin{Bmatrix} (-1, -1), (1, 1), (1, 0) \end{Bmatrix}$$

(see Fig. 5 b).

Despite the fact that $x_2$ is a sup-stationary point we see that neither condition (22) nor condition (23) are fulfilled here.

4 Conclusion

We identified the connection between directional derivative and nonhomogeneous approximations. Based on these connections, we reformulated optimality conditions in terms of such approximations.
Theorems that unite boundedness and optimality conditions in terms of codifferentials, coexhausters and difference of polyhedral convex functions were derived. It must be noted that in the case of difference of polyhedral convex function $f$, expansion (3) does not contain $o_2(\Delta)$ since this summand equals to zero and therefore all the conditions described in Section 3 are necessary and sufficient conditions of global optimality. At the same time, if we deal with a function which is not the difference of polyhedral convex functions but can be approximated in that form, results of Section 3 are only sufficient conditions of stationarity. This was demonstrated at point $x_2$ in Example 3.1.

The intention of this paper is to widen the facilities of researchers in solving nondifferentiable optimization problems and to make closer specialist working in different branches of nonsmooth analysis.

Acknowledgements

Results in Section 3 were obtained in the Institute for Problems in Mechanical Engineering of the Russian Academy of Sciences with the support of Russian Science Foundation (RSF), project No. 20-71-10032.

References

[1] Rockafellar, R. T.: Convex Analysis, Princeton University Press, Princeton, N.J. (1970)
[2] Demyanov, V.F., Polyakova, L.N., Rubinov, A.M.: On one generalization of the concept of subdifferential. In: Abstracts. All-Union Conference on Dynamical Control. Sverdlovsk, 79-84 (1979)
[3] Demyanov, V.F., Rubinov, A.M.: On quasidifferentiable functionals. Soviet Math. Doklady. 21, 13–17 (1980)
[4] Demyanov, V.F., Vasiliev, L.V.: Nondifferentiable Optimization, Springer-Optimization Software, New York (1985)
[5] Demyanov, V.F., Polyakova L.N.: The minimum conditions of a quasidifferentiable function on a quasidifferentiable set (in Russian). Computational Mathematics and Mathematical Physics. 20, 849–856 (1980)
[6] Sukhorukova, N., Ugon, J., Characterisation theorem for best polynomial spline approximation with free knots. Transactions of the American Mathematical Society 369 (9), 6389–6405 (2017)
[7] Demyanov, V.F., Malozemov, V.N.: Introduction to Minimax, J. Wiley, New York (1974)
[8] Demyanov, V.F.: Exhausters of a positively homogeneous function. Optimization. 45, 13–29 (1999)
[9] Demyanov, V.F.: Exhausters and Convexificators – New Tools in Nonsmooth Analysis. In: V. Demyanov and A. Rubinov: (Eds.) Quasidifferentiability and related topics. Dordrecht: Kluwer Academic Publishers, 85–137 (2000)
10. Demyanov, V.F., Rubinov, A.M.: Exhaustive families of approximations revisited. In: From Convexity to Nonconvexity. Nonconvex Optim. Appl. Vol. 55, 43–50. Kluwer Academic, Dordrecht (2001)

11. Demyanov V.F., Roshchina V.A.: Constrained Optimality Conditions in Terms of Proper and Adjoint Exhausters. Appl. Comput. Math., 4, 144–124 (2005)

12. Demyanov, V.F., Roshchina, V.A.: Optimality conditions in terms of upper and lower exhausters. Optimization. 55, 525–540 (2006)

13. Roshchina V.A.: Reducing Exhausters. J. Optim. Theory Appl. 136, 261–273 (2008)

14. Abbasov, M.E., Demyanov, V.F.: Extremum conditions for a nonsmooth function in terms of exhausters and coexhausters. Proceedings of the Steklov Institute of Mathematics. 269, 6–15 (2010)

15. Demyanov, V.F.: Proper exhausters and coexhausters in nonsmooth analysis. Optimization. 61, 1347–1368 (2012)

16. Demyanov, V.F., Abbasov, M.E.: Proper and adjoint exhausters in non-smooth analysis: optimality conditions. J. Global Optim. 56(2), 569–585 (2013)

17. Abbasov, M.E. Comparison Between Quasidifferentials and Exhausters. J Optim Theory Appl 175, 59-75 (2017).

18. Demyanov, V.F., Rubinov, A.M., Constructive Nonsmooth Analysis, Approximation & Optimization, vol. 7. Peter Lang, Frankfurt am Main (1995). iv+416 pp.

19. Abankin, A.E., Unconstrained minimization of H-hyperdifferentiable functions. Comput. Math. Math. Phys. 38(9), 1439–1446 (1998)

20. Abbasov, M. E., Demyanov, V. F. Adjoint Coexhausters in Nonsmooth Analysis and Extremality Conditions, Journal of Optimization Theory and Applications 156, pp. 535–553 (2013)

21. Abbasov, M.E., Second-Order Minimization Method for Nonsmooth Functions Allowing Convex Quadratic Approximations of the Augment. J Optim Theory Appl 171, 666-674 (2016). https://doi.org/10.1007/s10957-015-0796-7

22. Abbasov, M. E. Constrained optimality conditions in terms of proper and adjoint coexhausters (in Russian). Vestnik of St Petersburg University. Applied Mathematics. Computer Science. Control Processes 15(2), pp. 160–172 (2019)

23. Aleksandrov, A. D., Surfaces represented by the differences of convex functions. (Russian) Doklady Akad. Nauk SSSR (N.S.) 72, 613–616 (1950)

24. Hartman P., On functions representable as a difference of convex functions. Pac. J. Math. 9, 707–713 (1959)
[25] Pardalos, P. M., Resende, M. G. C. (eds): Handbook of Applied Optimization. Oxford University Press, Oxford (2002)

[26] Bagirov, A. M., Taheri, S., Ugon, J., Nonsmooth DC programming approach to the minimum sum-of-squares clustering problems. Pattern Recognition. 53, 12–24 (2016)

[27] Bagirov, A. M., Ugon, J., Nonsmooth DC programming approach to clusterwise linear regression: optimality conditions and algorithms. Optimization Methods and Software. 33(1), 194–219 (2018)

[28] Polyakova, L. N., On global unconstrained minimization of the difference of polyhedral functions, Journal of Global Optimization. 50, 179–195 (2011)

[29] Abbasov, M. E., Finding the set of global minimizers of a piecewise affine function, arXiv:2004.06255 (2020)