Revisiting Modified Greedy Algorithm for Monotone Submodular Maximization with a Knapsack Constraint

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Abstract
Monotone submodular maximization with a knapsack constraint is NP-hard. Various approximation algorithms have been devised to address this optimization problem. In this paper, we revisit the widely known modified greedy algorithm. First, we show that this algorithm can achieve an approximation factor of $0.405$, which significantly improves the known factor of $0.357$ given by Wolsey [31] or $(1-1/e)/2 \approx 0.316$ given by Khuller et al. [15]. More importantly, our analysis uncovers a gap in Khuller et al.’s proof for the extensively mentioned approximation factor of $(1-1/\sqrt{e}) \approx 0.393$ in the literature to clarify a long time of misunderstanding on this issue. Second, we enhance the modified greedy algorithm to derive a data-dependent upper bound on the optimum. We empirically demonstrate the tightness of our upper bound with a real-world application. The bound enables us to obtain a data-dependent ratio typically much higher than $0.405$ between the solution value of the modified greedy algorithm and the optimum. It can also be used to significantly improve the efficiency of algorithms such as branch and bound.

1 Introduction
A set function $f: 2^V \rightarrow \mathbb{R}$ is submodular [26] if for all $S, T \subseteq V$, it holds that $f(S) + f(T) \geq f(S \cup T) + f(S \cap T)$. Alternatively, defining $f(v \mid S) := f(S \cup \{v\}) - f(S)$ as the marginal gain of adding an element $v \in V$ to a set $S \subseteq V$, an equivalent definition of a submodular set function $f$ is that $f(v \mid S) \geq f(v \mid T)$ for all $S \subseteq T$ and $v \in V \setminus T$. The latter form of definition describes the concept of diminishing return in economics. The function $f$ is monotone nondecreasing if and only if $f(S) \leq f(T)$ for all $S \subseteq T$ (or equivalently $f(v \mid S) \geq 0$).

Many well known combinatorial optimization problems are essentially submodular maximization, such as maximum facility location [1, 6], Max-Cut and Max-DiCut [10, 11]. In addition, a growing number of problems in real-world applications of artificial intelligence and machine learning are also shown to be submodular maximization. These problems include data subset selection [17, 30], feature selection [16, 33], viral marketing [14], sensor placement [18, 19], document summarization [23, 24] and image segmentation [7, 13], etc.

In this paper, we study monotone submodular maximization with a knapsack constraint, which is defined as follows:

$$\max_{S \subseteq V} f(S) \text{ s.t. } \sum_{v \in S} c(v) \leq b, \quad (1)$$

where $f$ is a monotone nondecreasing submodular set function\textsuperscript{1} and $c(v)$ represents the cost of element $v$. Without loss of generality, we may assume that the cost of each element does not exceed $b$, since elements

\textsuperscript{1}We assume that function $f$ is normalized, i.e., $f(\emptyset) = 0$, and is given via a value oracle.
with cost greater than \( b \) do not belong to any feasible solution. This optimization problem has already found great utility in the aforementioned applications.

Since this optimization problem is NP-hard in general, various approximation algorithms have been proposed. For a special cardinality constraint where the costs of all elements are identical, i.e., \( c(v) = 1 \) for every element \( v \in V \), Nemhauser et al. [26] proposed a simple hill-climbing greedy algorithm that can provide \((1 - 1/e)\)-approximation (we say that an algorithm provides \( \alpha \)-approximation, where \( \alpha \leq 1 \), if it always obtains a solution of value at least \( \alpha \) times the value of an optimal solution). However, for the general knapsack constraint, the approximation factor of such a greedy algorithm is unbounded. Wolsey [31] found that slightly modifying the original greedy algorithm can provide an approximation ratio of \((1 - 1/e^3) \approx 0.357\), where \( \beta \) is the unique root of the equation \( e^z = 2 - x \). Khuller et al. [15] studied the budgeted maximum coverage problem (a special case of monotone submodular maximization with a knapsack constraint where the function value is always an integer), and derived two approximation factors, i.e., \((1 - 1/e)/2 \approx 0.316\) and \((1 - 1/\sqrt{e}) \approx 0.393\), for the modified greedy algorithm developed by Wolsey [31]. We note that the factor of \((1 - 1/\sqrt{e})\) is extensively mentioned in the literature, but unfortunately their proof was flawed as pointed out by Zhang et al. [34]. It becomes an open question whether the modified greedy algorithm can achieve an approximation ratio at least \((1 - 1/\sqrt{e})\).

Khuller et al. [15] also developed a partial enumeration greedy algorithm that improves the approximation factor to \((1 - 1/e)\), which was later shown to be applicable to the general problem (1) [28]. However, this algorithm requires \( O(n^5) \) (where \( n = |V| \) is the total number of elements in the ground set \( V \)) function value computations, which is not scalable. We focus on the scalable modified greedy algorithm of \( O(n^2) \) [15] and conduct a comprehensive analysis on its worst-case approximation guarantee. Based on the monotonicity and submodularity, we derive several relations governing the solution value and the optimum. Leveraging these relations, we establish an approximation ratio of 0.405, which significantly improves the factor of \((1 - 1/e^3) \approx 0.357\) given by Wolsey [31] or \((1 - 1/e)/2 \approx 0.316\) given by Khuller et al. [15]. More importantly, our analysis uncovers a critical gap in the proof for the factor of \((1 - 1/\sqrt{e}) \approx 0.393\) given by Khuller et al. [15] to clarify a long time of misunderstanding in the literature.

In addition, we enhance the modified greedy algorithm to derive a data-dependent upper bound on the optimum. We empirically demonstrate the tightness of our bound with a real-world application of viral marketing in social networks. The bound enables us to obtain a data-dependent ratio typically much higher than 0.405 between the solution value of the modified greedy algorithm and the optimum. It can also be used to significantly improve the efficiency of algorithms such as branch and bound as shown by our experimental evaluations.

2 Modified Greedy Algorithm and Approximation Guarantees

For the unit cost version of the optimization problem defined in (1), a simple greedy algorithm that chooses the element with the largest marginal gain in each iteration can achieve an approximation factor of \((1 - 1/e)\) [26]. Inspired by this elegant algorithm, for the general cost version, it is natural to apply a similar greedy algorithm according to cost-effectiveness. That is, we pick in each iteration the element that maximizes the ratio \( \frac{f(v|S_g)}{c(v)} \) based on the selected element set \( S_g \). Unfortunately, this simple greedy algorithm has an unbounded approximation factor. Consider, for example, two elements \( u \) and \( v \) with \( f(\{u\}) = 1 \), \( f(\{v\}) = 2\varepsilon \), \( c(u) = 1 \) and \( c(v) = \varepsilon \), where \( \varepsilon \) is a small positive number. When \( b = 1 \), the optimal solution is \( \{u\} \) while the greedy heuristic picks \( \{v\} \). The approximation factor for this instance is \( 2\varepsilon \), and is therefore unbounded.

Interestingly, a small modification to the greedy algorithm, referred to as MGreedy (Algorithm 1), achieves a constant approximation factor [15, 31]. Specifically, in addition to \( S_g \) obtained from the greedy heuristic (Lines 1–5), the algorithm also finds an element \( v^* \) that maximizes \( f(\{v\}) \) (Line 6), and then chooses the better one between \( S_g \) and \( \{v^*\} \) (Lines 7 and 8). Wolsey [31] showed that MGreedy achieves an approximation factor of 0.357. Later, Khuller et al. [15] gave two approximation factors of \((1 - 1/e)/2\) and \((1 - 1/\sqrt{e})\) that can be achieved by MGreedy for the budgeted maximum coverage problem, but
Algorithm 1: MGreedy

1. initialize $S_g \leftarrow \emptyset$, $V' \leftarrow V$;
2. while $V' \neq \emptyset$ do
3.   find $u \leftarrow \arg \max_{x \in V'} \left\{ f(x|S_g) \right\}$;
4.   if $c(S) + c(u) \leq b$ then $S_g \leftarrow S_g \cup \{u\}$;
5.   update the search space $V' \leftarrow V' \setminus \{u\}$;
6. $v^* \leftarrow \arg \max_{x \in V} f(v)$;
7. $S_m \leftarrow \arg \max_{S \in \{v^*, S_g\}} f(S)$;
8. return $S_m$;

Unfortunately, their proof for the factor of $(1 - 1/\sqrt{e})$ was flawed as pointed out by Zhang et al. [34]. In this paper, we establish an improved approximation factor of 0.405 for MGreedy.

Theorem 1. Let $\alpha^\perp$ be the root of $(1 - \alpha^\perp) \ln(1 - \alpha^\perp) + (2 - 1/e)(1 - 2\alpha^\perp) = 0$ satisfying $\alpha^\perp > 0.405$. The MGreedy algorithm achieves an approximation factor of $\alpha^\perp$.

To our knowledge, this is the first work giving a constant factor achieved by MGreedy that is even larger than $(1 - 1/\sqrt{e}) \approx 0.393$, which not only significantly improves the known factor of $(1 - 1/e^2) \approx 0.357$ given by Wolsey [31] or $(1 - 1/e)/2 \approx 0.316$ given by Khuller et al. [15] but also clarifies a long time of misunderstanding regarding the factor of $(1 - 1/\sqrt{e})$ in the literature.

3 Proof of Theorem 1

The key idea of our proof is that we derive several relations governing the solution value and the optimum by carefully characterizing the properties of MGreedy, and utilize these relations to construct an optimization problem whose optimum is a lower bound on the approximation factor of MGreedy. Then, it remains to show that the optimum of our newly constructed optimization problem is no less than 0.405. Our analysis procedure can be used as a general approach for analyzing the approximation guarantees of algorithms. In the following, we start the proof with a useful lemma.

Lemma 1. For any monotone nondecreasing submodular (and non-negative) set function $f$, denote $S^* \subseteq S_g$ as the intermediate element set constructed by the greedy heuristic after a certain number of iterations, and let $A^*$ be the element set abandoned so far due to budget violation, i.e., $A^* := (V \setminus V') \setminus S^*$, where $V$ is the ground element set and $V'$ is the remaining element set defined in Algorithm 1. Given any element set $T$, if $T \cap A^* = \emptyset$, we have

$$f(S^*) \geq (1 - e^{-c(S^*)/c(T)}) \cdot f(T). \quad (2)$$

Proof. The lemma directly holds when $f(S^*) \geq f(T)$. In the following, we consider $f(S^*) \leq f(T)$. Let $u_i$ be the $i$-th element added to $S^*$ by the greedy heuristic and $S_i := \{u_1, u_2, \ldots, u_i\}$ for any $0 \leq i \leq s = |S^*|$. According to the greedy rule, for any $i \leq s - 1$ and any $v \in T \setminus S_i$, we have $\frac{f(u_{i+1}|S_i)}{c(u_{i+1})} \geq \frac{f(v|S_i)}{c(v)}$ since $T \cap A^* = \emptyset$. Thus, by the monotonicity and submodularity of $f$, we have

$$f(S_i) + c(T) \cdot \frac{f(u_{i+1}|S_i)}{c(u_{i+1})} \geq f(S_i) + \sum_{v \in T \setminus S_i} f(v|S_i) \geq f(T).$$

Rearranging it yields

$$f(T) - f(S_{i+1}) \leq \left(1 - \frac{c(u_{i+1})}{c(T)}\right) \cdot \left(f(T) - f(S_i)\right) \leq e^{-c(u_{i+1})/c(T)} \cdot (f(T) - f(S_i)).$$

In the appendix, we provide an explanation of the problem in Khuller et al.’s proof and show a correct proof for the factor of $(1 - 1/\sqrt{e})$.  

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where the second inequality is because $1 - x \leq e^{-x}$ for any $x \geq 0$ and $f(T) \geq f(S^*) \geq f(S_i)$. Recursively,

$$f(T) - f(S^*) \leq e^{-\sum_{i=0}^{n-1} \frac{c(S_{i+1})}{c(S_i)}} \cdot f(T).$$

As a result, we have

$$f(S^*) \geq (1 - e^{-\sum_{i=0}^{n-1} \frac{c(S_{i+1})}{c(S_i)}}) \cdot f(T) = (1 - e^{-c(S^*)/c(T)}) \cdot f(T).$$

This completes the proof.

Lemma 1 gives a lower bound on the function value of the intermediate greedy solution. Based on Lemma 1, we derive a lower bound on the worst-case approximation of MGREEDY. In particular, we define an optimization problem to characterize the approximation factor of MGREEDY.

**Lemma 2.** It holds that $f(S_u) \geq \alpha^* \cdot f(OPT)$, where $\alpha^*$ is the minimum of the following optimization problem with respect to $\alpha, x_1, x_2, x_3, x_4, x_5, x_6$.

$$\min \alpha \quad \text{s.t.} \quad \alpha \geq x_1, \quad \alpha \geq x_1 + (1 - e^{(x_4 + x_6 - 1)/x_4})x_3, \quad \alpha \geq x_2, \quad x_1 \geq (1 - 1/e)(1 - 2\alpha) + (x_4 + x_5 + x_6 - 1)x_2/x_5, \quad x_1 \geq 1 - e^{-x_4}, \quad x_1 + x_2 + x_3 \geq 1, \quad x_1 + x_2/x_5 \geq 1, \quad x_4 + x_5 \geq 1, \quad \alpha, x_1, x_2, x_3, x_4, x_5, x_6 \in [0, 1].$$

**Proof.** Let $u$ (resp. $v$) be the first (resp. second) element in OPT considered by the greedy heuristic but not added to the element set $S_u$ due to budget violation.\(^3\) Let $S_u$ be the element set constructed until $u$ is considered by the greedy heuristic, and let OPT\(^*\) := OPT \setminus $S_u \cup \{u\}$. To simplify the notations, define $\alpha_u := f(S_u)/f(OPT), f_S := f(S_u)/f(OPT), f_u := f(u|S_u)/f(OPT), f_v := f(v|\emptyset)/f(OPT), f_f := f(OPT|S_u)/f(OPT), c_S := c(S_u)/b, c_u := c,u, c_v := c(v), and c_f := c(OPT)/b$. In what follows, we show that $\alpha = \alpha_u, x_1 = f_S, x_2 = f_u, x_3 = f_f, x_4 = c_S, x_5 = c_u, and x_6 = c_v$ are always feasible to the optimization problem defined in the lemma, which indicates that $f(S_u) \geq \alpha^* \cdot f(OPT)$.

By the algorithm definition, $\alpha_u \geq f_S$ and $\alpha_u \geq f_u$ (i.e., Constraints (4) and (6)). In addition, for any element set $T \subseteq V \setminus S_u$, let $f(T) := f(T | S_u)$. It is easy to verify that $f(\cdot)$ is also a nondecreasing monotone submodular set function. Let $S_v$ be the element set constructed until $v$ is considered by the greedy heuristic, and let $S_v' := S_v \setminus S_u$. Then, according to Lemma 1, we have $f(S_v') \geq (1 - e^{-c(S_v')/c(OPT)})f(OPT)$. Meanwhile, $c(S_u) + c(S_v') + c(v) > b$ and $c(OPT') \leq b - c(u) < c(S_u)$, which indicates that $c(S_v')/c(OPT') \geq (1 - c_S - c_v)/c_S$. Thus, $\alpha_u \geq f_S + f(S_v')/f(OPT) \geq f_S + (1 - e^{c(S_u) + c(u)}f(OPT') f\text{' (i.e., Constraint (5))}.

Let OPT\(^*\) := OPT \setminus \{u, v\}. Given any $x$, let $f(S(x)) := f(S_i) + f(u_{i+1} | S_i) \cdot (x - c(S_i))/c(u_{i+1})$ with index $i$ satisfying $c(S_i) \leq x < c(S_{i+1})$. Using a similar argument of Lemma 1, it is easy to get that $f(S(x^o)) \geq (1 - 1/e)f(OPT^o)$, where $x^o := c(OPT^o)$. Meanwhile, $f(S_u) - f(S(x^o)) \geq (c(S_u) - x^o) \cdot f(u | S_u)/c(u)$ due to submodularity and the greedy rule. In addition, $f(OPT^o) + f(v | \emptyset) + f(v | \emptyset) \geq f(OPT)$, which indicates that $f(OPT^o) \geq (1 - 2\alpha_u)f(OPT)$. We also note that $c(OPT^o) + c(u) + c(v) = c(OPT) \leq b$, which indicates that $c(S_u) - x^o \geq c(S_u) + c(u) + c(v) - b$. Finally, we have $f_S \geq (1 - 1/e)(1 - 2\alpha_u) + (c_S + c_u + c_v - 1)f_u/c_u$ (i.e., Constraint (7)).

\(^3\)If $u$ does not exist, we consider $u$ as a dummy element such that $c(u) = 0$ and $f(u | S)/c(u) = 0$ given any $S$, and so as for $v$.\)
Again, by setting $T = \text{OPT}$ in Lemma 1, we directly have $f_S \geq 1 - e^{-c_S}$ (i.e., Constraint (8)). Due to monotonicity and submodularity, we also know that $f_S + f_u + f' \geq 1$ and $f_S + f_u/c_u \geq 1$ (i.e., Constraints (9) and (10)). Meanwhile, $c_S + c_u > 1$ due to budget violation (i.e., Constraint (11)). Finally, by definition, the values of $\alpha_m$, $f_S$, $f_u$, $f'$, $c_S$, $c_u$, and $c_\perp$ are all in the range of $[0, 1]$ (i.e., Constraint (12)).

As can be seen, all constraints are satisfied, and hence the lemma immediately concludes.

**Lemma 3.** We have $\alpha^* \geq \alpha^+$, where $\alpha^+$ is defined in Theorem 1.

**Proof.** We first consider the case $x_4 + x_6 - 1 \geq 0$. We obtain from (7) and (8) that

$$x_1 \geq (1 - 1/e)(1 - 2\alpha) + x_2, \quad (13)$$

and

$$-\ln(1 - x_1) \geq x_4. \quad (14)$$

Then, $x_5 \times (10) + (13) + (1 - x_1) \times ((11) + (14))$ gives

$$x_1 - (1 - x_1) \ln(1 - x_1) \geq 1 - x_1 + (1 - 1/e)(1 - 2\alpha).$$

Rearranging yields

$$(1 - x_1)(\ln(1 - x_1) + 2) - 1/e - 2(1 - 1/e)\alpha \leq 0. \quad (15)$$

When $\alpha \geq 1 - e^{-3}$, we directly have $\alpha \geq \alpha^+$. When $\alpha \leq 1 - e^{-3}$, we know that $x_1 \leq 1 - e^{-3}$ by (4). Then, $(1 - x_1)(\ln(1 - x_1)) + 2)$ decreases along with $x_1$, which indicates that $(1 - \alpha)(\ln(1 - \alpha) + 2) - 1/e - 2(1 - 1/e)\alpha \leq 0$. Note that the left hand side is equivalent to $(1 - \alpha)\ln(1 - \alpha) + (2 - 1/e)(1 - 2\alpha)$, which strictly decreases along with $\alpha$ when $\alpha \leq 1 - e^{-3}$. This implies that $\alpha \geq \alpha^+$.

Next, we consider the case $x_4 + x_6 - 1 \leq 0$. We prove $\alpha \geq \alpha^+$ by contradiction. Assume on the contrary that $\alpha < \alpha^+$. Then, $x_1 < \alpha^+$ and $x_2 < \alpha^+$. We can get from (5) and (9) that

$$x_4 + x_6 - 1 \geq \ln(1 - \frac{\alpha - x_1}{x_3}) \cdot x_4 \geq \ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) \cdot x_4.$$

Combining it with (7) and (11) gives

$$x_1 \geq (1 - \frac{1}{e})(1 - 2\alpha) + x_2(1 + \frac{x_4 + x_6 - 1}{1 - x_4}) > (1 - \frac{1}{e})(1 - 2\alpha^+) + x_2(1 + \ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) \cdot \frac{x_4}{1 - x_1}). \quad (16)$$

Note that as $\ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) \leq 0$, the left hand side of the above inequality decreases along with $x_4$ when $x_4 \in [0, 1)$. By (4) and (14), we have $x_4 \leq -\ln(1 - \alpha^+)$. Combining with (16) gives

$$x_1 > (1 - \frac{1}{e})(1 - 2\alpha^+) + x_2(1 + \ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) \cdot \frac{-\ln(1 - \alpha^+)}{1 + \ln(1 - \alpha^+)}). \quad (17)$$

Meanwhile, $x_5 \times (10) + (1 - x_1) \times ((11) + (14))$ gives

$$x_2 \geq (1 - x_1)(1 + \ln(1 - x_1)), \quad (18)$$

where the right hand side decreases along with $x_1$ when $x_1 \in [0, \alpha^+]$. Together with (4), we have

$$x_2 \geq (1 - \alpha^+)(1 + \ln(1 - \alpha^+)).$$

Let $g(x_2) := x_2(1 + \ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) \cdot \frac{-\ln(1 - \alpha^+)}{1 + \ln(1 - \alpha^+)})$ subject to $x_2 \in [x_2^0, \alpha^+]$, where $x_2^0 := (1 - \alpha^+)(1 + \ln(1 - \alpha^+))$. Taking the derivative of $g(x_2)$ with respect to $x_2$ gives

$$g'(x_2) = 1 + \ln\left(\frac{1 - \alpha^+ - x_2}{1 - x_1 + x_2} \right) + \frac{x_2(x_1 - \alpha^+)}{(1 - x_1 + x_2)(1 - \alpha^+ - x_2)} \cdot \frac{-\ln(1 - \alpha^+)}{1 + \ln(1 - \alpha^+)}. \quad (19)$$

Observe that $g'(x_2)$ decreases along with $x_2$. Thus, $g(x_2) \geq \min\{g(x_2^0), g(\alpha^+)\}$. Define

$$\tilde{g}(x_1) := g(\alpha^+) - g(x_2^0).$$

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Taking the derivative of $\tilde{g}(x_1)$ with respect to $x_1$ gives
\[
\tilde{g}'(x_1) = \left( \frac{\alpha}{1-x_1 - \alpha} - \frac{x_1}{1-x_1 - x_2} \right) \cdot -\frac{\ln(1-\alpha)}{1+\ln(1-\alpha)} = \frac{(\alpha-x_1)(1-x_1)}{(1-x_1 - \alpha)(1-x_1 - x_2)} \cdot -\frac{\ln(1-\alpha)}{1+\ln(1-\alpha)} > 0.
\]

Meanwhile, by (18), we can get that $\alpha > (1-x_1)(1+\ln(1-x_1))$, which indicates that $x_1 > 0.32$. Hence, $\tilde{g}(x_1) \geq \tilde{g}(0.32)$. One can verify that
\[
\tilde{g}(0.32) = \alpha (1 + \ln(\frac{\alpha}{0.68-\alpha})) \cdot -\frac{\ln(1-\alpha)}{1+\ln(1-\alpha)} - x_1 \frac{1+\ln(1-\alpha)}{1-x_1-x_2} > 0.
\]
This implies that $g(x_2) \geq g(x_1)$. Therefore, (17) can be further relaxed to
\[
x_1 > (1 - 1/e)(1 - 2\alpha) + x_2 \frac{1+\ln(1-\alpha)}{1-x_1-x_2}. \tag{19}
\]

Furthermore, define $\hat{g}(x_1) := x_2 \frac{1+\ln(1-\alpha)}{1-x_1-x_2} - x_1$ subject to $x_1 \in [0, \alpha]$. Taking the derivative of $\hat{g}(x_1)$ with respect to $x_1$ gives
\[
\hat{g}'(x_1) = \frac{x_2}{1-x_1-x_2} \cdot -\frac{\ln(1-\alpha)}{1+\ln(1-\alpha)} - 1 \leq \frac{x_2}{1-\alpha} \cdot -\frac{\ln(1-\alpha)}{1+\ln(1-\alpha)} - 1 = 0,
\]
which indicates that $\hat{g}(x_1) \geq \hat{g}(\alpha) = x_2 - \alpha$. Putting it together with (19) gives
\[
0 > (1 - 1/e)(1 - 2\alpha) + x_2 \alpha - \alpha = (1 - \alpha) \ln(1 - \alpha) + (2 - 1/e)(1 - 2\alpha) = 0.
\]
This shows a contradiction and completes the proof. \(\Box\)

Proof of Theorem 1. Combining Lemmas 2 and 3 immediately concludes Theorem 1. \(\Box\)

4 Data-Dependent Upper Bound

The constant approximation factor 0.405 established above gives a lower bound on the worst-case solution quality over all problem instances. In this section, we enhance the modified greedy algorithm to derive a data-dependent upper bound on the optimum. The upper bound allows us to obtain a potentially tighter data-dependent ratio between the solution value of modified greedy and the optimum for individual problem instances.

Specifically, given a set $S$, let $v_1, v_2, \ldots$ be the sequence of elements in $V \setminus S$ in the descending order of $f(v \mid S)/c(v)$. Let $r$ be the lowest index such that the total cost of the elements $\{v_1, v_2, \ldots, v_r\}$ is larger than $b$, i.e., $c^* := \sum_{i=1}^{r-1} c(v_i) \leq b$ and $c^* + c(v_r) > b$. We define $\Delta(b \mid S)$ as
\[
\Delta(b \mid S) := \sum_{i=1}^{r-1} f(v_i \mid S) + f(v_r \mid S) \cdot \frac{b - c^*}{c(v_r)}, \tag{20}
\]
which is an upper bound on the largest marginal gain on top of $S$ subject to the budget $b$. Specifically, let $w_i = f(v_i \mid S)$ and $c_i = c(v_i)$, then $\Delta(b \mid S)$ is the optimum of a linear program $\max \sum_i w_i x_i$ subject to $\sum_i c_i x_i \leq b$ and $0 \leq x_i \leq 1$ for any $i$. On the other hand, the largest marginal gain $\max_{c(T) \leq b} \sum_{v \in T} f(v \mid S)$ is the optimum of the corresponding integer linear program. Thus, $\Delta(b \mid S)$ is an upper bound on $\max_{c(T) \leq b} \sum_{v \in T} f(v \mid S)$.

Observe that $\sum_{v \in \text{OPT}\setminus S} f(v \mid S)$ is no more than the latter. Therefore, we have
\[
f(S) + \Delta(b \mid S) \geq f(\text{OPT}\cup S) \geq f(\text{OPT}). \tag{21}
\]

To incorporate into MGREEDY, we choose the smallest upper bound $\Lambda$ over all the intermediate sets constructed by the greedy heuristic, i.e.,
\[
\Lambda := \min_i \{f(S_i) + \Delta(b \mid S_i)\}, \tag{22}
\]

This shows a contradiction and completes the proof. \(\Box\)
We carry out experiments on two applications to demonstrate the effectiveness of our upper bound. All the topics in data mining in recent years. In this application, we consider influence maximization on a social network. Viral marketing in social networks [14] is one of the most important topics in data mining in recent years. We show in the appendix.) Next, we conduct experiments to show that the data-dependent ratio is usually smaller than $f(S_m)/0.357$.

**Theorem 2.** Let $\alpha'$ be the root of $(1 - \alpha') \cdot (\ln(1 - \alpha') + 2) - 1 = 0$ satisfying $\alpha' > 0.357$. We have $f(OPT) \leq \Lambda \leq f(S_m)/\alpha' \leq f(OPT)/\alpha'$.

**Proof.** The first and third inequalities are straightforward. In the following, we prove $\Lambda < f(S_m)/\alpha'$.

The inequality is trivial if $\Lambda = f(S_m)$. Suppose $\Lambda > f(S_m)$. Let $S_k = \{u_1, u_2, \ldots, u_k\}$ be the element set constructed by the greedy heuristic when the first element $u_{k+1}$ from $V'$ is considered but not added to $S_k$ due to budget violation. For any $i = 0, 1, \ldots, k$ and any element $v \in V'$, by the greedy rule, it holds that $\frac{f(u_{i+1} | S_i)}{c(u_{i+1})} \geq \frac{f(v | S_i)}{c(v)}$. Thus,

$$f(S_i) + b \cdot \frac{f(u_{k+1} | S_k)}{c(u_{k+1})} \geq f(S_i) + \Delta(b \mid S_i) \geq \Lambda. \tag{23}$$

Using an analogous argument to the proof of Lemma 1, we can get that $f(S_k) \geq (1 - e^{-c(S_k)/b}) \cdot \Lambda$. This implies that

$$c(S_k)/b \leq -\ln(1 - f(S_k)/\Lambda) \leq -\ln(1 - f(S_m)/\Lambda).$$

In addition, we can directly obtain from (23) that $f(S_k) + b \cdot \frac{f(u_{k+1} | S_k)}{c(u_{k+1})} \geq \Lambda$. This implies that

$$c(u_{k+1})/b \leq f(u_{k+1} \mid S_k)/(\Lambda - f(S_k)) \leq f(S_m)/(\Lambda - f(S_m)).$$

By the algorithm definition, we know that $c(S_k) + c(u_{k+1}) > b$. Putting it together gives

$$f(S_m)/(\Lambda - f(S_m)) - \ln(1 - f(S_m)/\Lambda) \geq 1.$$ 

Define $g(x) := x/(1 - x) - \ln(1 - x) - 1$ subject to $x \in [0, 1)$. One can see that $g(x)$ increases along with $x$. Thus, the minimum $x$ satisfying $g(x^*) \geq 0$ is achieved at $g(x^*) = 0$ such that $(1 - x^*) \cdot (\ln(1 - x^*) + 2) - 1 = 0$. Therefore, $f(S_m)/\Lambda \geq x^* = \alpha'$. This completes the proof.

Figure 1 shows that the data-dependent ratio of $f(S_m)$ to $\Lambda$ is guaranteed to be larger than 0.357 for any problem instance, which is again tighter than the factor of $(1 - 1/e)/2 \approx 0.316$ given by Khuller et al. [15] and matches that given by Wolsey [31]. (For the unit cost version, the factor can be improved to $(1 - 1/e)$ as shown in the appendix.) Next, we conduct experiments to show that the data-dependent ratio is usually much larger than 0.357 or 0.405 in practice, which demonstrates the tightness of our upper bound $\Lambda$. Figure 1 depicts the relationship among $f(S_m)$, $\Lambda$, and $f(OPT)$.

## 5 Experiments

We carry out experiments on two applications to demonstrate the effectiveness of our upper bound. All the experiments are conducted on a Windows machine with an Intel Core 2.6GHz i7-7700 CPU and 32GB RAM.

**Viral marketing in social networks.** Viral marketing in social networks [14] is one of the most important topics in data mining in recent years. In this application, we consider influence maximization on a social network.
network $G = (V, E)$ with a set $V$ of vertices (representing users) and a set $E$ of edges (representing connections among users). The goal is to seed some users with incentives (e.g., discount, free samples, or monetary payment) to boost the revenue by leveraging the word-of-mouth effects on other users. We adopt the widely-used influence diffusion model called the independent cascade model [14]. Each edge $(u, v)$ is associated with a propagation probability $p_{u,v}$. Initially, the seed vertices $S$ are active, while all the other vertices are inactive. When a vertex $u$ first becomes active, it has a single chance to activate each inactive neighbor $v$ with success probability $p_{u,v}$. This process repeats until no more activation is possible. The influence spread $f(S)$ of the seed set $S$ is the expected number of active vertices produced by the above process. Kempe et al. [14] show that $f(S)$ is nondecreasing monotone submodular. We consider budgeted influence maximization that aims to find a vertex set $S$ maximizing $f(S)$ with the total cost $c(S)$ capped by a budget $b$, where each vertex $v$ is associated with a distinct cost $c(v)$ and $c(S) = \sum_{v \in S} c(v)$.

Note that the influence diffusion is a random process. We use the advanced sampling technique in [27] to estimate the influence spread in which 200 random Monte-Carlo subgraphs are generated. We experiment with four real datasets from [20, 21] with millions of vertices, namely, Pokec (1.6M vertices and 30.6M edges), Orkut (3.1M vertices and 117.2M edges), LiveJournal (4.8M vertices and 69.0M edges), and Twitter (41.7M vertices and 1.5G edges). As in [14], we set $p_{u,v}$ of each edge $(u, v)$ to the reciprocal of $v$’s in-degree, and set $c(v)$ proportional to $v$’s out-degree to emulate that popular users require more incentives to participate.

Due to massive data sizes, we cannot compute the true optima. To better visualize the tightness of different bounds on the optimum, we measure the ratios of the solution values obtained by MGreedy to the upper bounds, e.g., $f(S_n)/\Lambda$, which represent the approximation guarantees achieved by MGreedy. We note that Leskovec et al. [22] developed an upper bound of $f(S_k) + \Delta(b \mid S_k)$ in our notations on the optimum. For comparisons, we evaluate both the ratios obtained for our upper bound $\Lambda$ and the upper bound developed by Leskovec et al. [22]. Figure 2 shows the results. Note that a larger ratio represents a tighter upper bound. We observe that the ratio calculated by our upper bound is usually better than 0.9, which is much larger than both the constant factor of 0.405 and the ratio calculated by Leskovec et al.’s bound. This demonstrates that our upper bound $\Lambda$ is quite close to the optimum for the tested cases.

**Branch-and-bound algorithm for budgeted maximum coverage.** Tight bounds are valuable to
advancing algorithmic efficiency. Consider the information retrieval problem where one is given a bipartite graph constructed between a set \( V \) of objects (e.g., documents, images etc.) and a bag of words \( W \). There is an edge \( c_{v,w} \) if the object \( v \in V \) contains the word \( w \in W \). A natural choice of the function \( f \) has the form \( |\Gamma(X)| \), where \( \Gamma(X) \) is the neighborhood function that maps a subset of objects \( X \subseteq V \) to the set of words \( \Gamma(X) \subseteq W \) presented in the objects. Meanwhile, assigning an object \( v \in X \) will incur a cost \( c(v) \). Intuitively, one may want to maximize the diversity (i.e., the number of words) by selecting a set of objects subject to a cost budget \( b \). This problem can be seen as budgeted maximum coverage. As a proof-of-concept, we use synthetic data that define \( |V| = 100 \) and \( |W| = 100 \) and randomly generate an edge between \( v \in V \) and \( w \in W \) with probability \( p = 0.02 \). We report the average results of 10 instances.

We compare the branch-and-bound algorithm using our upper bound \( \Lambda \) (called “Our method”) against the data-correcting algorithm (called DCA) \([12]\) which is a branch-and-bound algorithm for maximizing a submodular function. In particular, in each branch of a search lattice \([A, B]\), the branch-and-bound algorithm needs to find an upper bound on the value of any candidate solution \( S \) satisfying \( A \subseteq S \subseteq B \) and \( c(S) \leq b \). To achieve this goal, we first compute an upper bound \( \Lambda' \) on the optimum of \( \max_{T \subseteq B \setminus A} \{f(T \mid A) : c(T) \leq b - c(A)\} \) as \( f(T \mid A) \) is also a monotone submodular function. Then, \( f(A) + \Lambda' \) is an upper bound on the optimum of branch \([A, B]\). On the other hand, DCA uses \( f(A) + \Delta(b - c(A) \mid A) \) as the upper bound, which is always (much) looser than ours. DCA considers homogeneous costs only and thus we set \( c(v) = 1 \) for each object \( v \). We manually terminate the algorithm if it cannot finish within 2 hours. Table 1 shows the running time of DCA and our algorithm when the cost budget \( b \) increases from 1 to 10. As can be seen, our algorithm can find the optimal solution within 2 seconds for all the cases tested whereas DCA runs 1–4 orders of magnitude slower than our algorithm when \( b \leq 5 \) and even fails to find the solution within 2 hours when \( b \geq 6 \).

### 6 Related Work

Nemhauser et al. \([26]\) studied monotone submodular maximization with a cardinality constraint, and proposed a greedy heuristic that achieves an approximation factor of \((1 - 1/e)\). For this problem, Nemhauser and Wolsey \([25]\) showed that no polynomial algorithm achieves an approximation factor that exceeds \((1 - 1/e)\). Feige \([9]\) further showed that even maximum coverage (which is a special submodular function) cannot be approximated in polynomial time within a ratio of \((1 - 1/e + \varepsilon)\) for any given \( \varepsilon > 0 \), unless \( P = NP \). Leveraging the notion of curvature, Conforti and Cornuéjols \([5]\) obtained an improved upper bound \((1 - e^{-\varepsilon})/\kappa_f\), where \( \kappa_f := 1 - \min_{v \in V} f(v\setminus\{v\})/f(v) \in [0, 1] \) measures how much \( f \) deviates from modularity. By utilizing multilinear extension \([4]\), Sviridenko et al. \([29]\) proposed a continuous greedy algorithm that can further improve the approximation ratio to \((1 - \kappa_f/e - \varepsilon)\) at the cost of increasing time complexity from \( O(kn) \) to \( \tilde{O}(n^4) \), where \( k \) is the maximum cardinality of elements in the optimization domain. Different from these studies, we focus on the more general problem of monotone submodular maximization with a knapsack constraint, for which the greedy heuristic does not have any bounded approximation guarantee.

Wolsey \([31]\) proposed a modified greedy algorithm of \( O(n^2) \), referred to as MGREEDY, gave a constant approximation factor of 0.357 for the problem of monotone submodular maximization with a knapsack constraint. Khuller et al. \([15]\) showed that MGREEDY can achieve an approximation guarantee of \((1 - 1/\sqrt{e})\) for the budgeted maximum coverage problem. This factor, after being extensively mentioned in the literature, is suddenly pointed out by Zhang et al. \([34]\) to be problematic due to the flawed proof. Such a long time of misunderstanding on the factor of \((1 - 1/\sqrt{e})\) becomes a critical issue needed to be solved urgently. In
this paper, we show that the MGREEDY algorithm can achieve an improved constant approximation ratio of 0.405 through a careful analysis, which answers the open question—the worst-case approximation guarantee of MGREEDY is better than $(1 - 1/\sqrt{e})$. In addition, we also enhance the MGREEDY algorithm to derive a data-dependent upper bound on the optimum, which slightly increases the time complexity of MGREEDY by a multiplicative factor of $\log n$. We theoretically show that the ratio of the solution value obtained by MGREEDY to our upper bound is always larger than 0.357, which is again tighter than the approximation factor given by Khuller et al. [15] and matches that given by Wolsey [31]. We note that Leskovec et al. [22] developed an upper bound of $f(S^*_g) + \Delta(b \mid S^*_g)$ in our notations, which is always looser than ours. Unlike our upper bound with worst-case guarantees, the relationship between $f(S^*_g) + \Delta(b \mid S^*_g)$ and $f(S_m)$ is unclear. As has been demonstrated in the experiments, our upper bound is significantly tighter than that developed by Leskovec et al. [22].

In addition to the modified greedy algorithm, Khuller et al. [15] also gave a partial enumeration greedy heuristic that can achieve $(1 - 1/e)$-approximation, which was later shown to be also applicable to the general submodular functions by Sviridenko [28]. Recently, Yoshida [32] proposed a continuous greedy algorithm achieving a curvature-based approximation guarantee of $(1 - \kappa_f/e - \varepsilon)$. However, the time complexities of the partial enumeration greedy algorithm and the continuous greedy algorithm are as high as $O(n^3)$ and $O(n^5)$, respectively. These algorithms [28, 32] are hard to apply in practice. Some recent work [2, 8] proposed algorithms with $(1 - 1/e - \varepsilon)$-approximation. These algorithms are again impractical due to the high dependency on $\varepsilon$, i.e., $(\log n/\varepsilon)^{O(1/\varepsilon)}n^2$ [2] and $(1/\varepsilon)^{O(1/\varepsilon)}n \log^2 n$ [8], which are of theoretical interests only.

7 Conclusion

In this paper, we show that MGREEDY can achieve an approximation factor of 0.405 for monotone submodular maximization with a knapsack constraint. This factor not only significantly improves the known factor of 0.357 or $(1 - 1/e)/2 \approx 0.316$ but also uncovers a critical gap on the misunderstood factor of $(1 - 1/\sqrt{e}) \approx 0.393$ in the literature. We also derive a data-dependent upper bound on the optimum that is guaranteed to be smaller than a multiplicative factor of $\frac{1}{0.357}$ to the solution value obtained by MGREEDY. Empirical comparisons for the application of viral marketing in social networks show that our bound is quite close to the optimum. It remains an open question to study whether the approximation factor of 0.405 is completely tight.

A Analysis of $(1 - 1/\sqrt{e})$ Approximation Guarantee

Khuller et al. [15] claimed that the modified greedy algorithm, referred to as MGREEDY, achieves an approximation guarantee of $(1 - 1/\sqrt{e})$, but unfortunately their proof was flawed as pointed out by Zhang et al. [34]. We provide here a brief explanation of the problem in the proof of [15, Theorem 3]. When showing that $f(S_g) \geq (1 - 1/\sqrt{e}) \cdot f(\text{OPT})$ when $c(S_g) \geq b/2$, where $S_g$ is the set obtained by the greedy heuristic and OPT is the optimal solution, the proof relies on $c(S_t) \geq b/2$, where $S_t \subseteq S_g$ is an intermediate set constructed by the greedy heuristic when the first element from OPT is selected for consideration but not added to $S_g$ due to budget violation. However, there is a gap here, since $c(S_g) \geq b/2$ does not imply $c(S_t) \geq b/2$. Interested readers are referred to the detailed analysis by Zhang et al. [34].

In this section, we provide a correct proof for the factor of $(1 - 1/\sqrt{e})$. We again utilize our general approach for analyzing approximation guarantees of algorithms by solving an optimizing problem that characterizes the relations between the solution value and the optimum.

Theorem 3. The MGREEDY algorithm achieves an approximation factor of $(1 - 1/\sqrt{e})$.

To prove Theorem 3, we start with the following useful lemma.

Lemma 4. Given any element set $T$, the greedy heuristic returns $S_g$ subject to a budget constraint $b$ satisfying

$$f(S_g) \geq (1 - c(T)/b) \cdot f(T).$$
Proof. The lemma is trivial when \( T \subseteq S_g \). Suppose \( T \setminus S_g \neq \emptyset \). Let \( S_t = \{u_1, u_2, \ldots, u_t\} \) be the element set constructed by the greedy heuristic when the first element from \( T \) is considered but not added to \( S_t \) due to budget violation. Due to submodularity and the greedy rule, we have

\[
\frac{f(u_1 | S_0)}{c(u_1)} \geq \frac{f(u_2 | S_1)}{c(u_2)} \geq \cdots \geq \frac{f(u_t | S_{t-1})}{c(u_t)} \geq \max_{v \in T'} \frac{f(v | S_t)}{c(v)} \geq \frac{f(T' | S_t)}{c(T')},
\]

where \( T' = T \setminus S_t \). Observe that \( f(T) \leq f(S_t) + f(T' | S_t) \). Meanwhile,

\[
f(S_t) = \sum_{i=1}^t f(u_i | S_{t-1}) \geq \sum_{i=1}^t \left( c(u_i) \cdot \frac{f(T' | S_t)}{c(T')} \right) = c(S_t) \cdot \frac{f(T' | S_t)}{c(T')}.
\]

Note that by the algorithm definition, \( c(S_t) + c(T') > b \). Therefore,

\[
f(T) \leq \left( 1 + \frac{c(T')}{b - c(T')} \right) \cdot f(S_t) \leq \frac{b}{b - c(T')} \cdot f(S_t).
\]

Rearranging it concludes the lemma. \( \square \)

Based on Lemma 4, we derive a lower bound on the worst-case approximation of MGreedy.

**Lemma 5.** It holds that \( f(S_m) \geq \bar{\alpha}^* \cdot f(\text{OPT}) \), where \( \bar{\alpha}^* \) is the minimum of the following optimization problem with respect to \( \alpha, x_1, x_2, x_3 \).

\[
\begin{align*}
\min & \quad \alpha \\
\text{s.t.} & \quad \alpha \geq x_1, \\
& \quad \alpha \geq 1 - x_1 - x_2, \\
& \quad \alpha \geq x_1 + (1 - \frac{x_3}{1 - x_3})x_2, \\
& \quad x_1 \geq 1 - e^{-x_3}, \\
& \quad \alpha, x_1, x_2, x_3 \in [0, 1].
\end{align*}
\]

**Proof.** Let \( u \) be the first element in \( \text{OPT} \) considered by the greedy heuristic but not added to \( S_g \) due to budget violation. Let \( S_u \) be the element set constructed until \( u \) is considered by the greedy heuristic, and let \( \text{OPT}' := \text{OPT} \setminus (S_u \cup \{u\}) \). To simplify the notations, let \( \alpha_m := \frac{f(S_m)}{f(\text{OPT})} \), \( f_S := \frac{f(S_u)}{f(\text{OPT})} \), \( f_u := \frac{f(u | S_u)}{f(\text{OPT})} \), \( f' := \frac{f(\text{OPT}' | S_u)}{f(\text{OPT})} \), \( c_S := \frac{c(S_u)}{b} \), \( c_u := \frac{c(u)}{b} \), and \( c' := \frac{c(\text{OPT})}{b} \). In what follows, we show that \( \alpha = \alpha_m \), \( x_1 = f_S \), \( x_2 = f' \), and \( x_3 = c_S \) are always feasible to the optimization problem defined in the lemma, which indicates that \( f(S_m) \geq \bar{\alpha}^* \cdot f(\text{OPT}) \).

By the algorithm definition, \( \alpha_m \geq f_S \) (i.e., Constraint (25)) and \( \alpha_m \geq f_u \). Due to monotonicity and submodularity, we have \( f_S + f_u + f' \geq 1 \). Thus, \( \alpha_m \geq 1 - f_S - f' \) (i.e., Constraint (26)). In addition, for any element set \( T \subseteq V \setminus S_u \), let \( \bar{f}(T) := f(T | S_u) \). It is easy to verify that \( \bar{f}(\cdot) \) is also a nondecreasing monotone submodular function. Then, by Lemma 4, \( S' := S_g \setminus S_u \) obtained from the greedy heuristic satisfies \( \bar{f}(S') \geq (1 - \frac{c(S_u)}{b - c(S_u)}) \cdot \bar{f}(\text{OPT}) \), which indicates that \( \alpha_m = f_S + \frac{f(S')}{f(\text{OPT})} \geq f_S + (1 - \frac{c}{1 - c}) \cdot f' \). Furthermore, by the algorithm definition, we know that \( c_S + c_u > 1 \) and \( c_u + c' \leq 1 \). Thus, \( c' < c_S \). As a result, \( \alpha_m \geq f_S + (1 - \frac{c}{1 - c}) \cdot f' \) (i.e., Constraint (27)). On the other hand, one can verify that \( f_S \geq 1 - e^{-c_S} \) (i.e., Constraint (28)). Finally, by definition, the values of \( \alpha_m, f_S, f', \) and \( c_S \) are all in the range of \( [0, 1] \) (i.e., Constraint (29)).

As can be seen, all constraints are satisfied, and hence the lemma immediately concludes. \( \square \)

**Lemma 6.** \( \bar{\alpha}^* \geq 1 - 1/\sqrt{e} \).
Theorem 4. For monotone submodular maximization with a cardinality constraint, we have 
\[ u \geq \frac{(2 - 3x_3)\alpha}{1 - x_3 - x_3e^{-x_3}}. \]

Proof. If \( x_3 > 0.5 \), constraints (25) and (28) directly show that \( \alpha \geq 1 - 1/\sqrt{e} \). Next, we consider the case \( 0 \leq x_3 \leq 0.5 \). Then, \((1 - 2x_3) \times (26) + (1 - x_3) \times (27) + x_3 \times (28)\) gives 
\[ (2 - 3x_3)\alpha \geq 1 - x_3 - x_3e^{-x_3}. \]

Define \( g(x) := \frac{1 - x - xe^{-x}}{2 - 3x} \) subject to \( 0 \leq x \leq 0.5 \). Taking the derivative of \( g(x) \) with respect to \( x \) gives 
\[ g'(x) = \frac{1 - (3x^2 - 2x + 2)e^{-x}}{(2 - 3x)^2}. \]
Furthermore, the derivative of \((2 - 3x)^2g'(x)\) with respect to \( x \) is \((3x^2 - 8x + 4)e^{-x}\), which is non-negative when \( 0 \leq x \leq 0.5 \). Thus, \((2 - 3x)^2g'(x)\) achieves its maximum at \( x = 0.5 \), i.e., the maximum is \( 1 - 1.75e^{-0.5} < 0 \). This implies that \( g(x) \) achieves its minimum at \( x = 0.5 \), i.e., the minimum is \( 1 - 1/\sqrt{e} \). Therefore, when \( 0 \leq x_3 \leq 0.5 \), it also holds that \( \alpha \geq 1 - 1/\sqrt{e} \), which concludes the lemma. \( \square \)

Proof of Theorem 3. Combining Lemmas 5 and 6 immediately concludes Theorem 3. \( \square \)

B Upper Bound for Cardinality Constraint

In this section, we show that when the knapsack constraint degenerates to a cardinality constraint, i.e., \( \max_{S \subseteq V} f(S) \) s.t. \(|S| \leq k\), our upper bound \( \Lambda \) is guaranteed to be smaller than \( f(S_k)/(1 - 1/e) \), which matches the tight approximation factor of \((1 - 1/e)\) [25].

Theorem 4. For monotone submodular maximization with a cardinality constraint, we have 
\[ f(OPT) \leq \Lambda \leq f(S_k)/(1 - 1/e) \leq f(OPT)/(1 - 1/e). \]

Proof. By monotonicity, submodularity and the greedy rule, we have 
\[ f(S_i) + k \cdot f(u_{i+1} | S_i) \geq f(S_i) + \Delta(k | S_i) \geq \Lambda, \]
where \( u_i \) is the \( i \)-the element selected by the greedy heuristic and \( S_i := \{u_1, u_2, \ldots, u_i\} \), e.g., \( S_k = S_k \).
Rearranging it yields 
\[ \Lambda - f(S_{i+1}) \leq (1 - 1/k) \cdot (\Lambda - f(S_i)). \]
Recursively, we have 
\[ \Lambda - f(S_k) \leq (1 - 1/k)^k \cdot (\Lambda - f(S_0)) = (1 - 1/k)^k \cdot \Lambda \leq 1/e \cdot \Lambda. \]
Rearranging it completes the proof. \( \square \)

In each iteration of the greedy heuristic, it takes \( O(n) \) time to find \( u \) and the largest \( k \) marginal gains [3], where \( n = |V| \) is the size of ground set. There are \( k \) iterations in the greedy algorithm. Thus, the total complexity of deriving \( \Lambda \) is \( O(kn) \), which remains the same as that of greedy algorithm.

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