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A note on Hilbert-Kunz multiplicity

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1 Introduction

This is a joint work with Prof. Kei-ichi Watanabe in Nihon University; see [WY].

Throughout this talk, let $(A, m, k)$ be a Noetherian local ring of characteristic $p > 0$. Put $d := \dim A \geq 1$. Let $\widehat{A}$ denote the $m$-adic completion of $A$, and let $\text{Ass}(A)$ (resp. $\text{Min}(A)$) denote the associated prime ideals (resp. minimal prime ideals) of $A$. Moreover, unless specified, let $I$ denote an $m$-primary ideal of $A$ and $M$ a finite $A$-module.

First, we recall the notion of Hilbert-Kunz multiplicity which was defined by Kunz [Ku1]; see also Monsky [Mo], Huneke [Hu].

Definition 1.1 The Hilbert-Kunz multiplicity $e_{HK}(I, M)$ of $M$ with respect to $I$ is defined as follows:

$$e_{HK}(I, M) := \lim_{q \to \infty} \frac{\lambda_A(M/I^qM)}{q^d},$$

where $q = p^e$ and $I^q = (a^q \mid a \in I)A$. For simplicity, we put $e_{HK}(I) := e_{HK}(I, A)$ and $e_{HK}(A) := e_{HK}(m)$.

The following question is fundamental but still open.

Question 1.2 Is $e_{HK}(I)$ always a rational number?

• Known Results.

(1.3.1) Let $e(I)$ be the multiplicity of $A$ with respect to $I$. Then we have the following inequalities:

$$\frac{e(I)}{d!} \leq e_{HK}(I) \leq e(I).$$

(1.3.2) $e_{HK}(I) \geq e_{HK}(A) \geq 1$.

(1.3.3) Put $\text{Assh}(A) = \{P \in \text{Spec}(A) \mid \dim A/P = d\}$. Then

$$e_{HK}(I, M) = \sum_{P \in \text{Assh}(A)} e_{HK}(I, A/P) \cdot l_{AP}(M_P).$$

For example, if $A$ is a local domain and $B$ is a torsion free $A$-module of rank $r$, then $e_{HK}(I, B) = r \cdot e_{HK}(A)$. 
(1.3.4) (Kunz [Ku2]) For any prime ideal \( P \in \text{Spec}(A) \) such that height \( P + \dim A/P = \dim A \), we have 
\[
e_{HK}(A_P) \leq e_{HK}(A).
\]

(1.3.5) If \( A \) is a regular local ring, then 
\[e_{HK}(I) = \lambda(A/I).
\]

(1.3.6) If \( I \) is a parameter ideal, then 
\[e_{HK}(I) = e(I).
\]

(1.3.7) We recall the notion of tight closure. An element \( x \in A \) is said to be in the tight closure \( I^* \) of \( I \) if there exists an element \( c \in A^0 \) such that for all large \( q = p^e \), \( cx^q \in I^{[q]} \), where \( A^0 := A \setminus \bigcup\{P \mid P \in \text{Min}(A)\} \).

Let \( I, J \) be \( \mathfrak{m} \)-primary ideals such that \( I \subseteq J \). Then if \( I^* = J^* \), then \( e_{HK}(I) = e_{HK}(J) \). Furthermore, if, in addition, \( \hat{A} \) is equidimensional and reduced, then the converse is also true.

(1.3.8) ([WY] or [BCP]) Let \( (A, \mathfrak{m}) \subseteq (B, \mathfrak{n}) \) be a module-finite extension of local domains. Then 
\[
e_{HK}(I, A) = \frac{[B/\mathfrak{n} : A/\mathfrak{m}]}{[Q(B) : Q(A)]} \cdot e_{HK}(IB, B),
\]
where \( Q(A) \) denotes the fraction field of \( A \).

Question 1.4 If \( \text{pd}_A A/I < \infty \), then does the same formula as that in (1.3.5) hold?

• Background and Questions.

In general, there is an example such that \( e_{HK}(I) = e(I) \); for instance, let \( q \) be a minimal reduction of \( m \). If \( q^* = m \), then we have \( e_{HK}(m) = e_{HK}(q) = e(q) = e(m) \). However, we have no example such that \( \frac{e(I)}{d!} = e_{HK}(I) \). On the other hand, if \( A = k[[X_1, \ldots, X_d]]^{(r)} \), then 
\[e_{HK}(A) = \frac{1}{r} \left( \frac{d + r - 1}{r - 1} \right) \text{ and } e(A) = r^{d-1}.
\]

Thus if we tend \( r \) to \( \infty \), then the limit \( \frac{e_{HK}(A)}{e(A)} \) tends to \( \frac{1}{d!} \). So we consider the following question.

Question 1.5 Is there a constant number \( \alpha > 0 \) depending on \( d = \dim A \) alone such that 
\[e_{HK}(I) \geq \frac{e(I)}{d!} + \alpha?\]

On the other hand, in [WY], we proved the following theorem.

Theorem 1.6 [WY, Theorem (1.5)] If \( A \) is an unmixed (i.e. \( \text{Ass}(\hat{A}) = \text{Assh}(\hat{A}) \)) local ring with \( e_{HK}(A) = 1 \), then it is regular.
In the above theorem, we cannot remove the assumption that $A$ is “unmixed”. For instance, if $e(A) = 1$, then $e_{HK}(A) = 1$. We now consider the case of Cohen-Macaulay local rings. Then the following question is a natural extension of the above theorem.

**Question 1.7** If $A$ is a Cohen-Macaulay local ring with $e_{HK}(A) < 2$, then is it F-regular?

The following conjecture is related to the above questions.

**Conjecture 1.8** Let $A$ be a quasi-unmixed (i.e. $\text{Min}(\hat{A}) = \text{Assh}(\hat{A})$) local ring. Then $e_{HK}(I) \geq \lambda(A/I^{*})$ for any $m$-primary ideal $I$.

Further, if $A$ is a Cohen-Macaulay local ring then $e_{HK}(I) \geq \lambda(A/I)$ for any $m$-primary ideal $I$.

## 2 A positive answer to Question 1

Throughout this section, let $A$ be a Noetherian local ring with $\dim A = 2$ and suppose that $k = A/m$ is infinite. The following theorem is a main result in this section.

**Theorem 2.1** (cf. [WY, Section 5]) Suppose $\dim A = 2$. Then for any $m$-primary ideal $I$, we have

$$e_{HK}(I) \geq \frac{e(I) + 1}{2} > \frac{e(I)}{2}.$$

First, we consider the case of Cohen-Macaulay local rings. Now suppose that $A$ is Cohen-Macaulay. Let $I$ be an $m$-primary ideal and $J$ its minimal reduction, that is, $J = (a, b)$ is a parameter ideal of $A$ and $I^{n+1} = JI^{n}$ for some $n \geq 1$.

**Lemma 2.2** Suppose that $A$ is Cohen-Macaulay, $1 \leq s < 2$ and $q = p^{s}$. We define $I^{x} = I^{\lfloor x \rfloor}$ for any positive real number $x$. Then we have

1. $\lambda_{A}(A/I^{(s-1)q}) = \frac{e(I)}{2}((s-1)q^{2} + o(q^{2}))$, where $f(q) = o(q^{2})$ means $\lim_{q \to \infty} \frac{f(q)}{q^{2}} = 0$.

2. $\lambda_{A}\left(\frac{I^{sq} + J^{[q]}}{J^{[q]}}\right) = \frac{e(I)}{2}(2-s)^{2}q^{2} + o(q^{2})$.

**Proof.** Put $n = [(s-1)q]$ and $\epsilon = (s-1)q - n$.

1. $\lambda_{A}(A/I^{(s-1)q}) = \lambda_{A}(A/I^{n}) = \frac{e(I)}{2}n^{2} + f(n)$, where $\lim_{n \to \infty} \frac{f(n)}{n^{2}} = 0$.

Thus we get

$$\lambda_{A}(A/I^{(s-1)q}) = \frac{e(I)}{2}((s-1)q - \epsilon)^{2} + o(q^{2}) = \frac{e(I)}{2}(s-1)^{2}q^{2} + o(q^{2}).$$

2. $\lambda_{A}\left(\frac{I^{sq} + J^{[q]}}{J^{[q]}}\right) \leq \lambda_{A}\left(\frac{J^{sq} + J^{[q]}}{J^{[q]}}\right) + \lambda_{A}\left(\frac{I^{sq}}{J^{[q]}}\right)$.
First, we estimate the second term. Since \( e(I) = e(J) \), we have

\[
\lambda_A(I^{sq}/J^{sq}) = \lambda_A(A/J^{sq}) - \lambda_A(A/I^{sq}) = o(q^2).
\]

Next, we estimate the first term.

\[
\lambda_A \left( \frac{J^{sq} + J'[q]}{J'[q]} \right) \leq \sum_{l=n}^{2q} \left\{ (x, y) \in \mathbb{Z}^2 \mid 0 \leq x, y \leq q-1, x + y = l \right\} \times \lambda_A(A/J) + o(q^2)
\]

\[
= \frac{1}{2} (2q - sq) e(I) + o(q^2).
\]

Q.E.D.

Lemma 2.3 Suppose that \( A \) is Cohen-Macaulay. Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( A \) and \( J \) a minimal reduction of \( I \). If \( I/J \) is generated by \( r \) elements (i.e. \( r \geq \mu_A(I) - 2 \)), then we have

\[
\lambda_A(I^{[q]}/J^{[q]}) \leq \frac{r}{2(r+1)} e(I) \cdot q^2 + o(q^2).
\]

Moreover, if \( J^* \subseteq I \) and \( I/J^* \) is generated by \( r \) elements, the same result holds.

Proof. Let \( s \) be any real number such that \( 1 \leq s < 2 \). Then

\[
\lambda_A \left( \frac{I^{[q]}}{J^{[q]}} \right) \leq \lambda_A \left( \frac{I^{[q]} + Isq}{J^{[q]} + Isq} \right) + \lambda_A \left( \frac{J^{[q]} + Isq}{J^{[q]}} \right) =: (E1) + (E2).
\]

Since we can write as \( I = Au_1 + \cdots + Au_r + J \), we get

\[
(E1) \leq \sum_{i=1}^{r} \lambda_A \left( \frac{u_i^q A + J^{[q]} + Isq}{J^{[q]} + Isq} \right) = \sum_{i=1}^{r} \lambda_A \left( \frac{A}{(J^{[q]} + Isq)} : u_i^q \right)
\]

\[
\leq r \cdot \lambda_A \left( \frac{A}{I^{(s-1)q}} \right) = r \cdot \frac{e(I)}{2} (s-1)^2 q^2 + o(q^2) \quad \text{by (2.2)}.
\]

On the other hand, by (2.2) again, \( (E2) = \frac{e(I)}{2} (2-s)^2 q^2 + o(q^2) \). Thus

\[
\lambda_A \left( \frac{J^{[q]}}{J^{[q]}} \right) \leq \frac{e(I)}{2} q^2 \left\{ (r+1)s^2 - 2(r+2)s + (r+4) \right\} + o(q^2).
\]

Put \( s = \frac{r+2}{r+1} \), and we get the required inequality.

Further, the last statement follows from the fact \( \lambda_A(A/J^{[q]}) = \lambda_A(A/(J^*)^{[q]}) + o(q^2) \).

Q.E.D.

Next proposition easily follows from the above lemma.

Proposition 2.4 Suppose that \( A \) is Cohen-Macaulay. Let \( I \) be an \( \mathfrak{m} \)-primary ideal of \( A \) and \( J \) a minimal reduction of \( I \). If \( I/J \) is generated by \( r \) elements then we have

\[
e_{HK}(I) \geq \frac{r + 2}{2(r + 1)} \cdot e(I).
\]

Moreover, if \( J^* \subseteq I \) and \( I/J^* \) is generated by \( r \) elements (i.e. \( r \geq \mu_A(I/J^*) = \lambda_A(I/J^* + Im) \)), the same result holds.
We now give a proof of Theorem (2.1). First, we suppose that $A$ is Cohen-Macaulay and let $J$ be a minimal reduction of $m$. Since

$$e(I) - 1 = \lambda_A(m/J) = \lambda_A(I/J) + \lambda_A(m/I) \geq \lambda_A(I/J + Im) + \lambda_A(m/I),$$

we have $e(I) - 1 \geq e(I) - 1 - \lambda_A(m/I) \geq \mu_A(I/J)$. By virtue of Proposition (2.4), we get

$$e_{HK}(I) \geq \frac{r + 2}{2(r + 1)} \cdot e(I) \geq \frac{e(I) + 1}{2e(I)} \cdot e(I) = \frac{e(I) + 1}{2},$$

where $r = e(I) - 1 - \lambda_A(m/I)$.

We remark that Equality $e_{HK}(I) = (e(I) + 1)/2$ implies $I = m$.

Next, we consider about general local rings. Since $e_{HK}(I) = e_{HK}(I\hat{A})$ and $e(I) = e(I\hat{A})$, we may assume that $A$ is complete. Moreover, since

$$e_{HK}(I) = \sum_{P \in \text{Assh}(A)} e_{HK}(I, A/P) \cdot \lambda_A(P),$$

we may assume that $A$ is a complete local domain. Let $B$ be the integral closure of $A$ in its fraction field. Then $B$ is a complete normal local domain and a finite $A$-module; thus it is a two-dimensional Cohen-Macaulay local ring. Let $n$ be an unique maximal ideal of $B$ and put $t = [B/n : A/m]$. Then we have

$$e_{HK}(I) = t \cdot e_{HK}(IB, B), \quad e(I) = t \cdot e_{HK}(IB, B).$$

Thus by the argument in the Cohen-Macaulay case, we get

$$e_{HK}(I) = t \cdot e_{HK}(IB, B) \geq t \cdot \frac{e_{HK}(IB, B) + 1}{2} \geq \frac{e_{HK}(I) + 1}{2}.$$

**Corollary 2.5** If $A$ is a non-Cohen-Macaulay, unmixed local ring (with $\dim A = 2$), then

$$e_{HK}(I, A) > \frac{e(I) + 1}{2}$$

for any $m$-primary ideal $I$ of $A$.

**Proof.** By the above proof, we may assume that $A$ is a complete local domain. With the same notation as in the proof of Theorem, $B$ is a torsion free $A$-module. If $\mu_A(B) = 1$, then $B \cong A$; this contradicts the assumption that $A$ is not Cohen-Macaulay. Thus $\lambda_A(B/mB) = \mu_A(B) \geq 2$.

When $t := [B/n : A/m] = 1$, since $\lambda_B(B/mB) = \lambda_A(B/mB) \geq 2$, we have $IB \subseteq mB \subset n$. Hence

$$e_{HK}(I) = e_{HK}(IB, B) > \frac{e(IB) + 1}{2} = \frac{e(I) + 1}{2}.$$

On the other hand, when $t \geq 2$, we have

$$e_{HK}(I) \geq \frac{e(I) + t}{2} > \frac{e(I) + 1}{2} \quad \text{Q.E.D.}$$
Corollary 2.6 Let $A$ be a local ring with $\dim A = 2$. Then

(1) When $e(A) = 1$, we have $e_{HK}(A) = 1$.

(2) When $e(A) \geq 2$, we have $e_{HK}(A) \geq \frac{3}{2}$.

3 Local rings with small Hilbert-Kunz multiplicity

In this section, we consider Question (1.7) in case of local rings with $\dim A = 2$. In order to state the main theorem, we recall the notion of $\mathcal{F}$-regular rings. A local ring $A$ is said to be $\mathcal{F}$-regular (resp. $\mathcal{F}$-rational) if $I^* = I$ for every ideal (resp. parameter ideal) $I$ of $A$. We are now ready to state the main theorem, which is a slight generalization of Theorem (5.4) in [WY].

Theorem 3.1 (cf. [WY, Theorem (5.4)]) Let $A$ be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$. Then

(1) $1 < e_{HK}(A) < 2$ if and only if $\hat{A}$ is an $\mathcal{F}$-rational double point, that is, $\hat{A} \cong k[[X, Y, Z]]/(f)$, where $f$ is given by the list below (3.2).

(2) $e_{HK}(A) = 2$ if and only if $A$ satisfies either one of the following conditions:

(a) $A$ is not $\mathcal{F}$-regular with $e(A) = 2$.

(b) $\hat{A} \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

Corollary 3.2 Let $A$ be an unmixed local ring with $\dim A = 2$. If $e_{HK}(A) < 2$, then $\hat{A}$ is isomorphic to the completion of the ring $k[X, Y]^G$ where $G$ is a finite subgroup of $SL_2(k)$. In particular, $A$ is a module-finite subring of $k[[X, Y]]$ and $e_{HK}(A) = 2 - \frac{1}{|G|}$.

In fact, $|G|$ is given by the following table.

| type  | $f$               | $|G|$ |
|-------|-------------------|------|
| $(A_n)$ | $f = xy + z^{n+1}$ | $n + 1$ | $n \geq 1$ |
| $(D_n)$ | $f = x^2 + yz^2 + y^{n-1}$ | $4(n-2)$ | $n \geq 4$, $p \geq 3$ |
| $(E_6)$ | $f = x^2 + y^3 + z^4$ | $24$ | $p \geq 3$ |
| $(E_7)$ | $f = x^2 + y^3 + yz^3$ | $48$ | $p \geq 5$ |
| $(E_8)$ | $f = x^2 + y^3 + z^5$ | $120$ | $p \geq 7$ |

From now on, let $A$ be an unmixed local ring with $\dim A = 2$. In order to prove the above theorem, we give several lemmas.

Lemma 3.3 If $1 < e_{HK}(A) < 2$, then $\hat{A}$ is an integral domain with $e(\hat{A}) = 2$ and $\hat{A}_P$ is regular for any prime ideal $P \neq \mathfrak{m}\hat{A}$.
Proof. We may assume that $A$ is complete. First, we observe that $e(A) = 2$. Actually, it follows from Theorem (2.1).

Next, we show that $A$ is a local domain with isolated singularity. For any prime ideal $P \neq m$, we have $e_{HK}(A_P) \leq e_{HK}(A) < 2$. Since $e_{HK}(A_P)$ must be a positive integer, we have $e_{HK}(A_P) = 1$. Hence $A_P$ is regular.

On the other hand, $\# \text{Ass}(A) = \# \text{Assh}(A) = 1$. Actually, if $\# \text{Assh}(A) \geq 2$,

$$2 > e_{HK}(A) = \sum_{P \in \text{Assh}(A)} e_{HK}(A_P) \cdot \lambda_{A_P}(A) \geq \# \text{Assh}(A) \geq 2$$

gives a contradiction. Hence $\# \text{Ass}(A) = 1$. Therefore $A$ is a local domain. Q.E.D.

Corollary 3.4 Let $A$ be a Cohen-Macaulay local ring with $e(A) = 2$ and suppose that $\hat{A}$ is reduced. Then

1. If $A$ is F-regular, then $e_{HK}(A) < 2$.
2. If $A$ is not F-regular, then $e_{HK}(A) = 2$.

Proof. Let $q$ be a minimal reduction of $m$. Since $A$ is Cohen-Macaulay, we have $\lambda_A(A/q) = e(A) = 2$; thus $q^* = q$ or $q^* = m$, because $q \subseteq q^* \subseteq m$.

When $q^* = q$, since $A$ is Gorenstein, $A$ must be F-regular. Moreover, since $m \neq q^*$ and $\hat{A}$ is reduced, we get

$$e_{HK}(A) := e_{HK}(m) < e_{HK}(q^*) = e_{HK}(q) = e(q) = 2.$$  

On the other hand, when $q^* = m$, $A$ is not F-regular and $e_{HK}(A) = e_{HK}(q) = 2$. Q.E.D.

We now give an outline of the proof of Theorem (3.1). Let $A$ be an unmixed local ring with $\dim A = 2$ and suppose $k = \overline{k}$.

Step 1. When $A$ is a complete Cohen-Macaulay local ring with $e_{HK}(A) < 2$, it is an F-rational double point.

Proof. In fact, by Lemma (3.3), $A$ is a complete local domain with $e(A) = 2$. Thus Corollary (3.4) implies that $A$ is F-regular. Then $A$ is given by the list in Corollary (3.2).

Step 2. If $A$ is unmixed local ring with $e_{HK}(A) < 2$, then $\hat{A}$ is F-regular.

Proof. We may assume that $A$ is complete. By Lemma (3.3), $A$ is a complete local domain with $e(A) = 2$. Let $B$ the integral closure of $A$ in its fraction field. Then $\lambda_A(B/A) < \infty$ and $B$ is a local domain and is a module-finite extension of $A$. Let $n$ be an unique maximal ideal of $B$. In order to show that $A$ is F-regular it is enough to show $A = B$, for $B$ is Cohen-Macaulay. As $A/m \cong B/n$, we get

$$2 > e_{HK}(A) = e_{HK}(m, B) \geq e_{HK}(n, B) =: e_{HK}(B).$$

According to Step 1, $B$ is F-regular with $e_{HK}(B) = 2 - \frac{1}{|G|}$ and is a module-finite subring of $C = k[[X, Y]]$ such that $|G| = [Q(C) : Q(B)]$. 

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$P$, $Q$, $\text{Alg}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$, $\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{O}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{R}$, $\mathcal{S}$, $\mathcal{T}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$
Now suppose $A \neq B$. Then $H^1_m(A) \cong B/A \neq 0$ and thus $A$ is not Cohen-Macaulay. Further, as $\mu_A(B) \geq 2$, we have $m.B \subseteq n$. Moreover, since both $B$ and $C$ are F-regular rings, we obtain that $I.C \cap B = I$ for any ideal $I$ of $B$. In particular, we have $m.C \subseteq n.C$. Hence we get

$$e_{HK}(A) - e_{HK}(B) = \frac{1}{|G|} \lambda_A(C/m.C) - \frac{1}{|G|} \lambda_A(C/n.C)$$

$$= \frac{1}{|G|} \lambda_A(n.C/m.C) \geq \frac{1}{|G|}.$$

Thus

$$e_{HK}(A) \geq e_{HK}(B) + \frac{1}{|G|} = 2.$$

Thus we conclude that $A = B$ as required. \hfill \Box

Step 3. Let $A$ be a complete Cohen-Macaulay local ring. Then $e_{HK}(A) = 2$ if and only if $A$ is not F-regular with $e(A) = 2$ or $A \cong k[[X^3, X^2Y, XY^2, Y^3]]$.

Proof. If part is easy. But only if part is hard. See [WY, Section5] for details. \hfill \Box

Step 4. Suppose that $A$ is unmixed but not Cohen-Macaulay. Then $e_{HK}(A) = 2$ if and only if $e(A) = 2$.

Proof. If part: If $e(A) = 2$, then $e_{HK}(A) \leq 2$. If $e_{HK}(A) < 2$, then $A$ is Cohen-Macaulay by Step 2. However, this contradicts the assumption. Hence $e_{HK}(A) = 2$.

Only if part follows from Corollary (2.5). \hfill Q.E.D.

In the final of this section, we give the following problem.

**Problem 3.5** Let $A$ be an unmixed local ring with dim $A = 2$. Characterize the ring $A$ which satisfies $e_{HK}(A) = \frac{e(A) + 1}{2}$.

In fact, if $A = k[[X,Y]](e)$ then $e(A) = e$ and $e_{HK}(A) = \frac{e + 1}{2}$. Further, the proof of the above theorem implies that if $e_{HK}(A) = \frac{e(A) + 1}{2}$ and $e(A) \leq 3$ then $A \cong k[[X,Y]]^{\nu(A)}$. Moreover, the following proposition gives a partial answer to this problem.

**Proposition 3.6** If $A$ is an unmixed local ring with $e_{HK}(A) = \frac{e(A) + 1}{2}$, then it is F-rational.

Proof. By Cor (2.5), $A$ is Cohen-Macaulay. Then we show that $A$ has a minimal multiplicity, that is, $\text{emb}(A) = e(A) + \text{dim } A - 1$. Let $q$ be a minimal reduction of $m$. Then since

$$e(A) - 1 = \lambda_A(m/q) \geq \lambda_A(m/q + m^2) = \mu_A(m/q).$$

If $e(A) - 1 > \mu_A(m/q) =: r_0$, then

$$e_{HK}(A) \geq \frac{r_0 + 2}{2(r_0 + 1)} \cdot e(A) > \frac{e(A) + 1}{2};$$
see the proof of Theorem (2.1) for detail. Thus $e(A) - 1 = \mu_A(m/q)$. It follows that $m^2 \subseteq q$; thus $A$ has a minimal multiplicity.

We will show that $A$ is F-rational. Suppose not. Then $q^* \neq q$. Since $m^2 \subseteq q \subseteq q^*$, we have $r_1 := \mu_A(m/q^*) < \mu_A(m/q) = r_0$. Thus by virtue of (2.4), we get

$$e_{HK}(A) \geq \frac{r_1 + 2}{2(r_1 + 1)} \cdot e(A) > \frac{r_0 + 2}{2(r_0 + 1)} \cdot e(A) = \frac{e(A) + 1}{2}.$$ 

This contradicts the assumption. Hence we conclude that $A$ is F-rational. \textbf{Q.E.D.}

4 Extended Rees Rings.

In this section, we consider the following question.

\textbf{Question 4.1} Let $A$ be a local ring and $F = \{F_n\}$ a filtration of $A$. Then does $e_{HK}(A) \leq e_{HK}(G_F(A))$ always hold? Further, when does equality hold?

In order to state our result, we recall the definition of Rees ring, extended Rees ring and the associated graded ring.

Let $A$ be a local ring of $A$ with $d := \dim A \geq 1$. Then $F = \{F_n\}_{n \in \mathbb{Z}}$ is said to be a filtration of $A$ if the following conditions are satisfied:

(a) $F_i$ is an ideal of $A$ such that $F_i \supseteq F_{i+1}$ for each $i$.

(b) $F_i = A$ for each $i \leq 0$ and $m \supseteq F_1$.

(c) $F_i F_j \subseteq F_{i+j}$ for each $i, j$.

For a given filtration $F = \{F_n\}_{n \in \mathbb{Z}}$ of $A$, we define

$$R := R_F(A) := \bigoplus_{n=0}^{\infty} F_n t^n.$$ 

$$S := R'_F(A) := \bigoplus_{n \in \mathbb{Z}} F_n t^n.$$ 

$$G := G_F(A) := \bigoplus_{n=0}^{\infty} F_n / F_{n+1} \cong S / t^{-1} S \cong R / R(1).$$

$R_F(A)$ (resp. $R'_F(A), G_F(A)$) is said to be the Rees (resp. the extended Rees, the associated graded) ring with respect to a filtration $F$ of $A$.

Then our main result in this section is the following theorem.

\textbf{Theorem 4.2} Let $A$ be any local ring with $d := \dim A > 0$ and let $F = \{F_n\}_{n \in \mathbb{Z}}$ be a filtration of $A$. Suppose that $R_F(A)$ is a Noetherian ring with $\dim R_F(A) = d + 1$. Then for any $m$-primary ideal $I$ of $A$ such that $F_1 \subseteq I \subseteq m$, we have

(1) $e_{HK}(I, A) \leq e_{HK}(N, S)$, where $N = (t^{-1}, I, S_+)$. 
(2) If $F_1$ is an $m$-primary ideal, then $e_{HK}(N, S) \leq e_{HK}(G)$.

In particular, if $F_1$ is an $m$-primary ideal, then

$$e_{HK}(A) \leq e_{HK}(S) \leq e_{HK}(G).$$

**Question 4.3** In the above theorem, when does equality hold? How about $e_{HK}(A) \leq e_{HK}(R_F(A))$?

**Example 4.4** Let $A = k[[X, Y]]$ and $I = (X^m, Y^n)$, where $m \geq n \geq 1$. Then

1. $e(R(I)) = n + 1$.
2. $e_{HK}(R(I)) = n + 1 - \frac{n(3m - 1)}{3m^2}$.
3. $e(R'(I)) = n + 2$ (if $n \geq 2$), $= 2$ (otherwise).
4. $e_{HK}(R'(I)) = n + 2 - \frac{n}{m} - \frac{1}{n}$.

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