ON THE EXTENDED AND ALLAN SPECTRA AND TOPOLOGICAL RADII

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Abstract. In this paper we prove that the extended spectrum \( \Sigma(x) \), defined by W. Żelazko, of an element \( x \) of a pseudo-complete locally convex unital complex algebra \( A \) is a subset of the spectrum \( \sigma_A(x) \), defined by G.R. Allan. Furthermore, we prove that they coincide when \( \Sigma(x) \) is closed. We also establish some order relations between several topological radii of \( x \), among which are the topological spectral radius \( R_t(x) \) and the topological radius of boundedness \( \beta_t(x) \).

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1. INTRODUCTION

A complex algebra \( A \) with a topology \( \tau \) is a locally convex algebra if it is a Hausdorff locally convex space and its multiplication \((x, y) \to xy\) is jointly continuous. The topology of \( A \) can be given by the family of all continuous seminorms on \( A \).

Throughout this paper \( A = (A, \tau) \) will be a locally convex complex algebra with unit \( e \), \( A' \) its topological dual and \( \{\|\cdot\|_\alpha : \alpha \in \Lambda\} \) the family of all continuous seminorms on \( A \).

An element \( x \in A \) is called bounded if for some non-zero complex number \( \lambda \), the set \( \{(\lambda x)^n : n = 1, 2, \ldots\} \) is a bounded set of \( A \). The set of all bounded elements of \( A \) is denoted by \( A_0 \).

For \( x \in A \) define the radius of boundedness \( \beta(x) \) of \( x \) by

\[
\beta(x) = \inf \left\{ \lambda > 0 : \left\{ \left( \frac{x}{\lambda} \right)^n : n \geq 1 \right\} \text{ is bounded} \right\}
\]

adopting the usual convention that \( \inf \emptyset = \infty \). Henceforth we shall use this convention without further mention.
Notice that \( \lambda_0 > 0 \) and \( \left\{ \left( \frac{x}{\lambda_0} \right)^n : n \geq 1 \right\} \) bounded imply \( \| (\frac{x}{\lambda})^n \|_{\alpha} \to 0 \) for all \( |\lambda| > \lambda_0 \) and \( \alpha \in \Lambda \). Using this fact it is easy to see that \( \beta(x) = \beta_0(x) \), where

\[
\beta_0(x) = \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \left( \frac{x}{\lambda} \right)^n = 0 \right\}.
\]

In [1], by \( B_1 \) it is denoted the collection of all subsets \( B \) of \( A \) such that:

(i) \( B \) is absolutely convex and \( B^2 \subset B \),
(ii) \( B \) is bounded and closed.

For any \( B \in B_1 \), let \( A(B) \) be the subalgebra of \( A \) generated by \( B \). From (i) we get

\[
A(B) = \{ \lambda x : \lambda \in \mathbb{C}, x \in B \}.
\]

The formula

\[
\| x \|_B = \inf \{ \lambda > 0 : x \in \lambda B \}
\]
defines a norm in \( A(B) \), which makes it a normed algebra. It will always be assumed that \( A(B) \) carries the topology induced by this norm. Since \( B \) is bounded in \( (A, \tau) \), the norm topology on \( A(B) \) is finer than its topology as a subspace of \( (A, \tau) \).

The algebra \( A \) is called \textit{pseudo-complete} if each of the normed algebras \( A(B) \), for \( B \in B_1 \), is a Banach algebra. It is proved in [1, Proposition 2.6] that if \( A \) is sequentially complete, then \( A \) is pseudo-complete.

In [1], it is also introduced by G. R. Allan the \textit{spectrum} \( \sigma_A(x) \) of \( x \in A \) as the subset of the Riemann sphere \( \mathbb{C}_\infty = \mathbb{C} \cup \{ \infty \} \) defined as follows:

(a) for \( \lambda \neq \infty \), \( \lambda \in \sigma_A(x) \) if and only if \( \lambda e - x \) has no inverse belonging to \( A_0 \),

(b) \( \infty \in \sigma_A(x) \) if and only if \( x \notin A_0 \).

In [1, Corollary 3.9] it is proved that \( \sigma_A(x) \neq \emptyset \) for all \( x \). We shall call \( \sigma_A(x) \) the \textit{Allan spectrum}.

The \textit{Allan spectral radius} \( r_A(x) \) of \( x \) is defined by

\[
r_A(x) = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \},
\]

where \( |\infty| = \infty \).

On the other hand, W. Želazko defined in [4] the concept of \textit{extended spectrum} of \( x \in A \) in the way that we now recall.

As usual

\[
\sigma(x) = \{ \lambda \in \mathbb{C} : \lambda e - x \notin G(A) \},
\]

where \( G(A) \) is the set of all invertible elements of \( A \). The resolvent

\[
\lambda \to R(\lambda, x) = (\lambda e - x)^{-1}
\]
is then defined on \( \mathbb{C} \setminus \sigma(x) \), but it is not always a continuous map. Put

\[
\sigma_d(x) = \{ \lambda_0 \in \mathbb{C} \setminus \sigma(x) : R(\lambda, x) \text{ is discontinuous at } \lambda = \lambda_0 \},
\]
and
\[ \sigma_\infty(x) = \begin{cases} \emptyset & \text{if } \lambda \to R(1, \lambda x) \text{ is continuous at } \lambda = 0, \\ \infty & \text{otherwise.} \end{cases} \]

Then the extended spectrum of \( x \) is the set
\[ \Sigma(x) = \sigma(x) \cup \sigma_d(x) \cup \sigma_\infty(x). \]

It is proved in [4, Theorem 15.2] that if \( A \) is complete, then \( \Sigma(x) \) is a non empty set of \( \mathbb{C}_\infty \) for every \( x \), and the extended spectral radius \( R(x) \) is defined by
\[ R(x) = \sup \{ |\lambda| : \lambda \in \Sigma(x) \}. \]

We shall not assume that \( A \) is complete. Nevertheless, from now on we assume that \( \Sigma(x) \) is a non empty set of \( \mathbb{C}_\infty \) for every \( x \in A \).

2. COMPARISON OF \( \Sigma(x) \) AND \( \sigma_A(x) \)

**Theorem 2.1.** If \( A \) is pseudo-complete, then \( \Sigma(x) \subset \sigma_A(x) \) for any \( x \in A \).

**Proof.** Let \( \lambda \notin \sigma_A(x) \) with \( \lambda \neq \infty \), then \( \lambda \notin \sigma(x) \) and \( R(\lambda, x) \) is bounded. Hence \( R(\lambda, x) \in A(B) \) for some \( B \in B_1 \) ([1, Proposition 2.4]).

For any \( \mu \in \mathbb{C} \), we have that \((\mu e - x) = (\lambda e - x) + (\mu - \lambda) e\). Let \( 0 < \gamma < \| R(\lambda, x) \|_B^{-1} \), then for \( |\mu - \lambda| < \gamma \), the formula
\[ S_n(\mu) = R(\lambda, x) - (\mu - \lambda) R(\lambda, x)^2 + (\mu - \lambda)^2 R(\lambda, x)^3 - \ldots + (-1)^n (\mu - \lambda)^n R(\lambda, x)^{n+1}, \]
defines a Cauchy sequence in the Banach algebra \( A(B) \). Therefore, it converges in \( A(B) \) to \( R(\mu, x) \).

Given \( \varepsilon > 0 \), there exists \( 0 < \delta < \gamma \) such that
\[ \| S_n(\mu) - R(\lambda, x) \|_B < |\mu - \lambda| \| R(\lambda, x) \|_B^2 \frac{1}{1 - \gamma \| R(\lambda, x) \|_B} < \varepsilon \]
for all \( n \) if \( |\lambda - \mu| < \delta \), which implies that \( \| R(\mu, x) - R(\lambda, x) \| \leq \varepsilon \) if \( |\lambda - \mu| < \delta \).

Hence \( R(\mu, x) \to R(\lambda, x) \) as \( \mu \to \lambda \), in \( A(B) \) and also in \((A, \tau)\), therefore \( \lambda \notin \sigma_d(x) \).

Thus, \( \lambda \notin \Sigma(x) \).

If \( \infty \notin \sigma_A(x) \), then \( x \) is bounded and there exists \( r > 0 \) such that the idempotent set \( \{ (\frac{x}{r})^n : n \geq 1 \} \) is bounded. The closed absolutely convex hull \( B \) of \( \{ (\frac{x}{r})^n : n \geq 1 \} \) belongs to \( B_1 \). Consider the Banach algebra \( A(B) \). Since \( \| \frac{x}{r} \|_B < 1 \) for every \( |\beta| > r \), we obtain
\[ R\left( 1, \frac{x}{\beta} \right) = e + \frac{x}{\beta} + \left( \frac{x}{\beta} \right)^2 + \ldots \]
in the Banach algebra \( A(B) \).
Since
\[ \left\| R \left( 1, \frac{x}{\beta} \right) - e \right\|_B \to 0 \]
as \(|\beta| \to \infty\), we have that \( R \left( 1, tx \right) \to e \) as \( t \to 0 \), in \( A(B) \) and hence in \((A, \tau)\) as well. Thus \( R \left( 1, tx \right) \) is continuous in \( t = 0 \) and \( \infty \notin \Sigma(x) \). 

**Lemma 2.2.** Suppose \( A \) is pseudo-complete and let \( x \in A \) be such that the extended spectral radius \( R(x) < \infty \). Then for each \( f \in A' \) the function \( F(\lambda) = f \left( R \left( 1, \lambda x \right) \right) \) is holomorphic in the open disc \( D(0, \delta) \), with \( \delta = \frac{1}{R(x)} \), where \( D(0, \delta) = \mathbb{C} \) when \( R(x) = 0 \). Furthermore,

\[ F^{(n)}(\lambda) = n! f \left( R \left( 1, \lambda x \right)^{n+1} x^n \right) \quad (2.1) \]

for every \( \lambda \in D(0, \delta) \) and \( n = 0, 1, 2, \ldots \). In particular,

\[ F^{(n)}(0) = n! f(x^n) \]

for all \( n \geq 0 \).

**Proof.** We have that \( \lambda \notin \Sigma(x) \) whenever \(|\lambda| > R(x)\). This implies that the function

\[ \lambda \to R \left( 1, \lambda x \right) \]
is continuous in the open disc \( D = D(0, \delta) \). By definition \( F^{(0)}(0) = f(e) \) and \( F(\lambda) = f \left( R \left( 1, \lambda x \right) \right) \) is holomorphic in \( D \) since

\[ F'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{f \left( R \left( 1, \lambda x \right) \right) - f \left( R \left( 1, \lambda_0 x \right) \right)}{\lambda - \lambda_0} = \]

\[ = \lim_{\lambda \to \lambda_0} f \left( \frac{R \left( 1, \lambda x \right) R \left( 1, \lambda_0 x \right) (\lambda - \lambda_0) x}{\lambda - \lambda_0} \right) = \]

\[ = f \left( R \left( 1, \lambda_0 x \right)^2 x \right) \]

for every \( \lambda_0 \in D \).

It is easy to obtain (2.1) by induction. \( \square \)

**Theorem 2.3.** If \( A \) is pseudo-complete, then for any \( x \in A \) we have that \( \Sigma(x) = \sigma_A(x) \) if \( \Sigma(x) \) is closed in \( \mathbb{C}_\infty \).

**Proof.** Let \( x \in A \) and assume that \( \Sigma(x) \) is closed, then by Theorem 2.1 we only have to prove that \( \lambda_0 \notin \Sigma(x) \) implies \( \lambda_0 \notin \sigma_A(x) \).

Let \( \lambda_0 \notin \Sigma(x) \), with \( \lambda_0 \neq \infty \), then \( \lambda_0 e - x \in G(A) \). We shall show that \( (\lambda_0 e - x)^{-1} \) is bounded. Since \( \Sigma(x) \) is closed, then there exists an open disc \( D(\lambda_0) \) around \( \lambda_0 \) such that \( \lambda e - x \in G(A) \) if \( \lambda \in D(\lambda_0) \) and \( R(\lambda, x) \) is continuous at \( \lambda = \lambda_0 \). Using the identity

\[ (\lambda e - x)^{-1} - (\lambda_0 e - x)^{-1} = (\lambda_0 - \lambda) (\lambda e - x)^{-1} (\lambda_0 e - x)^{-1} , \]
we obtain
\[ \lim_{\lambda \to \lambda_0} \frac{R(\lambda, x) - R(\lambda_0, x)}{\lambda - \lambda_0} = -R(\lambda_0, x)^2. \]
Then for any \( f \in A' \) we get
\[ \lim_{\lambda \to \lambda_0} \frac{f(R(\lambda, x)) - f(R(\lambda_0, x))}{\lambda - \lambda_0} = -f(R(\lambda_0, x)^2), \]
which implies that \( R(\lambda, x) \) is weakly holomorphic in \( \lambda = \lambda_0 \). By [1, Theorem 3.8 (i)] we obtain that \((\lambda_0 e - x)^{-1}\) is bounded in \( A \). Therefore, \( \lambda_0 \notin \sigma_A(x) \).

If \( \infty \notin \Sigma(x) \), then some neighborhood of \( \infty \) does not intersect \( \Sigma(x) \) and we have that \( R(x) < \infty \). Let \( f \in A' \). By Lemma 2.2, the Taylor expansion of \( F(\lambda) = f(R(1, \lambda x)) \) around 0 is
\[ F(\lambda) = f(e) + \lambda f(x) + \frac{2\lambda^2}{2!} f(x^2) + \ldots \]
for \( |\lambda| < \frac{1}{R(x)} \). In particular, \( \lim_{n \to \infty} f(\lambda_0^n x^n) = 0 \) for some \( \lambda_0 > 0 \) and then \( \{f(\lambda_0^n x^n) : n \geq 1\} \) is bounded; therefore \( \{\lambda_0 x^n : n \geq 1\} \) is bounded. Thus \( x \in A_0 \) and \( \infty \notin \sigma_A(x) \).

\[ \square \]

3. COMPARISON BETWEEN TOPOLOGICAL RADII

Let \( x \in A \), we say that \( x \) is topologically invertible if \( \overline{xA} = \overline{Ax} = A \), i.e. for each neighborhood \( V \) of \( e \) there exist \( a_V, a'_V \in A \) such that \( xa_V \in V \) and \( a'_V x \in V \).

The topological spectrum \( \sigma_t(x) \) of \( x \) is the set
\[ \sigma_t(x) = \{ \lambda \in \mathbb{C} : \lambda e - x \text{ is not topological invertible} \}. \]

The topological spectral radius \( R_t(x) \) is defined by
\[ R_t(x) = \sup \{ |\lambda| : \lambda \in \sigma_t(x) \}. \]

Having in mind the definition of \( \beta_0(x) \) we define the topological radius of boundedness \( \beta_t(x) \) of \( x \) by
\[ \beta_t(x) = \inf \left\{ \lambda > 0 : \liminf_n \left\| \left( \frac{x}{\lambda} \right)^n \right\|_\alpha = 0 \text{ for all } \alpha \in \Lambda \right\}. \]

In [2] the first author defined the lower extended spectral radius of \( x \) by
\[ R_*(x) = \sup_{\alpha \in \Lambda} \liminf_n \sqrt[n]{\|x^n\|_\alpha} \]
and in [3] it is proved that if \( A \) is a complete locally convex unital algebra, then for any \( x \in A \) we have \( R_*(x) \leq r_0(x) \), and \( R_*(x) = r_0(x) \) if \( A \) is a unital \( B_0 \)-algebra (metrizable complete locally convex algebra), where
\[ r_0(x) = \inf \{ 0 < r \leq \infty : \text{there exists } (a_n)_0^\infty, a_n \in \mathbb{C}, \text{ such that} \]
\[ \sum a_n \lambda^n \text{ has radius of convergence } r \text{ and} \]
\[ \sum a_n x^n \text{ converges in } A \}
(In [3] this radius is denoted by \( r_0(x) \).)
Here we have the following result.

**Proposition 3.1.** Let \( x \in A \). Then
\[
R_t(x) \leq \beta_t(x) = R_*(x) \leq \beta(x) \leq r_A(x).
\]

*Proof.* The first inequality is obvious if \( \beta_t(x) = \infty \), therefore let \( \beta_t(x) < \infty \). Given \( \lambda > \beta_t(x) \) and \( \alpha \in \Lambda \), there exists a subsequence \( (n_k)_k = (n_k(\alpha))_k \) of the natural sequence \( (n) \) such that \( \lim_{k \to \infty} \left\| \left( \frac{x}{\lambda} \right)^{n_k} \right\|_\alpha = 0 \). Then
\[
\lim_{k \to \infty} \left\| \left( \frac{e}{\lambda} + \frac{x}{\lambda^2} + \ldots + \frac{x^{n_k-1}}{\lambda^{n_k}} \right) (\lambda e - x) - e \right\|_\alpha = 0.
\]
Hence \( \lambda e - x \) is topologically invertible for any such \( \lambda \) and it follows that \( R_t(x) \leq \beta_t(x) \).

If \( R_*(x) = \infty \), then \( \beta_t(x) \leq R_*(x) \). Now suppose \( R_*(x) < \mu < \lambda < \infty \). Then given \( \alpha \in \Lambda \) there exists a subsequence \( (n_k)_k = (n_k(\alpha))_k \) of \( (n) \) such that \( n_k \sqrt{\left\| x^{n_k} \right\|_\alpha} < \mu \). Hence \( \liminf n_k \sqrt{\left\| x^{n_k} \right\|_\alpha} > 1 \). On the other hand, \( \lambda > \beta_t(x) \) implies that \( \liminf n_k \sqrt{\left\| x^{n_k} \right\|_\alpha} = 0 \), which contradicts the previous statement. Thus, \( \beta_t(x) = R_*(x) \).

Since \( \beta(x) = \beta_0(x) \) it is clear that \( \beta_t(x) \leq \beta(x) \). Finally, \( \beta(x) \leq r_A(x) \) by [1, Theorem 3.12]. \( \square \)

**Corollary 3.2.** If \( A \) is pseudo-complete, then
\[
R_t(x) \leq R_*(x) = \beta_t(x) \leq \beta(x) = r_A(x) \leq R(x)
\]
for every \( x \in A \).

*Proof.* Let \( x \in A \). We have by [1, Theorem 3.12] that \( \beta(x) = r_A(x) \). Thus we only have to prove that \( \beta(x) \leq R(x) \). This is obvious if \( R(x) = \infty \), so assume that \( R(x) < \infty \), therefore \( \infty \notin \Sigma(x) \). Applying Lemma 2.2 we obtain that the Taylor expansion about 0 of \( F(\lambda) = f(R(1, \lambda x)) \) is
\[
F(\lambda) = f(e) + \lambda f(x) + \frac{2\lambda^2}{2!} f(x^2) + \ldots
\]
for \( f \in A' \) and \( |\lambda| < \frac{1}{R(x)} \).

Then \( \lim_{n \to \infty} f((\lambda x)^n) = 0 \) for any \( 0 < \lambda < \frac{1}{R(x)} \) and \( f \in A' \). In particular, for any such \( \lambda \) the set \( \{(\lambda x)^n : n \geq 1\} \) is weakly bounded and therefore \( \{(\lambda x)^n : n \geq 1\} \) is bounded in \( A \). It follows that \( \lambda \geq \beta(x) \) for every \( \lambda > R(x) \) and then \( \beta(x) \leq R(x) \). \( \square \)

**Proposition 3.3.** If \( A \) is complete, then \( r_A(x) = \beta(x) = R(x) \) for all \( x \in A \).
Proof. It remains to prove that $R(x) \leq \beta(x)$. We can assume that $\beta(x) < \infty$. Let $r > \beta(x)$, then we have that $f \left( (\frac{x}{r})^n \right) \to 0$ for every $f \in A'$, therefore

$$\limsup_{n} r^{\sqrt{n}}|f(x^n)| \leq r$$

for every $f \in A'$. We get from [4, Theorem 15.6] that

$$R(x) = R_2(x) = \sup_{f \in A'} \limsup_{n} r^{\sqrt{n}}|f(x^n)| \leq r.$$ 

Therefore, $R(x) \leq \beta(x)$. 

Remark 3.4. In [2] it is constructed a unital $B_0$-algebra $A$ in which there is an element $x$ such $R_*(x) = 1$ and $R(x) = \infty$. On the other hand, if we consider the non-complete algebra $A = (P(t), \|\cdot\|)$ of all complex polynomials with the norm $\|p(t)\| = \max_{0 \leq t \leq 1} |p(t)|$, then for every $\lambda \neq 0$ we have that $\left\| \left( \frac{t}{\lambda} \right)^n \right\| = \frac{1}{|\lambda|^n}$. Therefore $\beta(t) = 1$, nevertheless $R(t) = \infty$ since $\lambda - t$ does not have an inverse for all $\lambda \in \mathbb{C}$.

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