Tangle addition and the knots-quivers correspondence

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Abstract

We prove that the generating functions for the one row/column colored HOMFLY-PT invariants of arborescent links are specializations of the generating functions of the motivic Donaldson-Thomas invariants of appropriate quivers that we naturally associate with these links. Our approach extends the previously established tangles-quivers correspondence for rational tangles to algebraic tangles by developing gluing formulas for HOMFLY-PT skein generating functions under Conway’s tangle addition. As a consequence, we prove the conjectural links-quivers correspondence of Kucharski–Reineke–Stošić–Sułkowski for all arborescent links.

1 Introduction

The knots-quivers correspondence of Kucharski–Reineke–Stošić–Sułkowski [10, 11] proposes a relation between the colored HOMFLY-PT polynomials of knots and the motivic Donaldson-Thomas invariants of symmetric quivers. The main prediction is that the generating function of the (anti-) symmetrically colored reduced HOMFLY-PT polynomials $P_j(K)$ of any framed oriented knot $K$ can be expressed as

$$\sum_{j \geq 0} P_j(K)x^j = \sum_{d=(d_1,\ldots,d_m) \in \mathbb{N}^m} (-1)^Rq^Sd_1q^Aq^Qd_1\cdots d_m \left[\frac{d_1 + \cdots + d_m}{d_1,\ldots,d_m}\right]_{d_1+\cdots+d_m}$$

for some $m \geq 1$, $R, S, A \in \mathbb{Z}^m$ and a symmetric integer matrix $Q$ of size $m \times m$, where square brackets indicate quantum multinomial coefficients and $\mathbb{N} := \mathbb{Z}_{\geq 0}$. Such expressions compactly encode the growth behavior of the colored HOMFLY-PT invariants (and also colored Jones polynomials), which is of central interest in quantum topology.

Assuming the matrix $Q$ has non-negative entries, it can be interpreted as the adjacency matrix of a quiver. As explained in [10], the right-hand side of (1) then appears as a specialisation of the motivic DT-series of the quiver, i.e. the $\mathbb{Z} \times \mathbb{N}^m$-graded Hilbert–Poincaré series of the corresponding cohomological Hall algebra [9, 5]. Moreover, the specialisation data is conjectured to be determined by the Poincaré polynomial of the reduced triply-graded Khovanov–Rozansky homology [8] of $K$.

For geometric interpretations of the knots-quivers correspondence and relations with colored HOMFLY-PT homology [27] we refer to [6, 22, 21]. Possibly related connections between enumerative geometry and quantum knot invariants have been studied in [14, 26, 4, 15, 18, 19].

1 This can always be achieved at the expense of a framing change on $K$. 

The knots-quivers conjecture has been verified for all knots with at most six crossings and the infinite families of \((2, 2n + 1)\) torus knots and twist knots in [10]. These verifications proceeded by ad hoc constructions of generating function data as required by (1) for each such knot.

In [23] we verified the knots-quivers conjecture and its natural extension to a links-quivers conjecture for all 2-bridge links in a systematic way. Motivated by the idea of \textit{skein theory with variable color}, we introduced a way of encoding the generating functions of HOMFLY-PT invariants of 4-ended tangles in a “quiver form” analogous to (1). We then showed how to explicitly construct such quiver forms for all rational tangles by induction on the crossing number, building on previous work in [26, 25]. Finally, we verified that they provide generating function data as required by (1) upon closing the rational tangle into a 2-bridge link.

Furthermore, while expressions of the form (1) are certainly not unique, not even for a fixed size \(m \in \mathbb{N}\), the inductive description from [23] assigns to each rational tangle, and thus also to its closure, a distinguished quiver form expression.

The purpose of this paper is to extend the relationship between 4-ended tangles and quivers beyond rational tangles and prove the links-quivers conjecture for a larger class of links.

**Theorem 1.1.** There exists a family \(QT_4\) of 4-ended framed oriented tangles with the following properties:

- \(QT_4\) contains the trivial 2-strand tangle.
- \(QT_4\) is closed under diffeomorphisms of \((B^3, \partial B^3, \{4 \text{ pts}\})\).
- \(QT_4\) is closed under Conway’s tangle addition [3], the binary operation of gluing two 4-ended tangles at pairs of boundary points as follows:

\[
\begin{array}{c}
\tau_1 \ \tau_2 \\
\tau_1 \tau_2
\end{array}
\]

- The appropriate analogue of the knots-quivers correspondence (2) holds for any link obtained by closing off a tangle in \(QT_4\).

The orbit of the trivial tangle under under diffeomorphisms of the 3-ball that preserve the tangle boundary set-wise is called the set of rational tangles. Its closure under tangle addition is called the set of algebraic tangles, which were introduced by Conway in the context of link enumeration [3]. Links obtained as closures of algebraic tangles are called arborescent, or algebraic in the sense of Conway [2].

**Corollary 1.2.** The links-quivers correspondence holds for all arborescent links and, in particular, all Montesinos and pretzel links.

In particular, this provides an algorithm to compute one row/column colored HOMFLY-PT invariants for these classes of links. For alternative approaches see e.g. [16, 17, 1].

\footnote{This is to distinguish them from a different notion of algebraic links, which arise as links of plane curve singularities.}
To summarise, our results in this paper show that the tangles-quivers correspondence is robust under skein-theoretic operations, at least as far as 4-ended tangles and homogeneous one row/column colorings are concerned. This is surprising, given that geometric and physical models for the large color limit of HOMFLY-PT invariants usually work in the closed setting of knots and links. However, in the 4-ended tangle case see [25] for elements of a geometric model for categorified colored HOMFLY-PT invariants and [12, 13] for a detailed study in the uncolored, categorified $\mathfrak{sl}_2$ case.

Extensions to tangles with more endpoints and different colors, and connections to categorified invariants are interesting subjects for future work.

Structure of this paper

The family $QT_4$ is constructed in Section 2 and the gluing result is proved in Section 3. In Section 3.7 we show that gluing is at worst bilinear in the size of the input quiver data. Section 1 contains examples for pretzel tangles that suggest that the minimal quiver data of a sum tangle is typically smaller than the worst case estimate.

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2 HOMFLY-PT partition function

Let $L$ be a link with $c$ components. Consider the reduced $\otimes^j$-colored HOMFLY-PT invariant $P_j(L)$ for $j \geq 0$ and form the generating function $P(L) = \sum_{j \geq 0} P_j(L)x^j$.

Definition 2.1. The HOMFLY-PT partition function $P(L)$ of a link with $c$ components is said to be in quiver form, if it is presented as:

$$P(L) = \sum_{d \in \mathbb{N}^m} (-q)^S a^d q^d Q \left[\frac{|d|}{d} \right] (q^2)^{1-c|d|} x^{|d|}.$$ \hspace{1cm} (2)

for some $m \geq 1$, $S, A \in \mathbb{Z}^m$ and $Q$ a symmetric $m \times m$ integer matrix. Here $|d|$ denotes the sum of the entries of the vector $d$, $(q^2)^k = \prod_{i=1}^k (1 - q^{2i})$ is a $q$-Pochhammer symbol, and

$$\left[\frac{d_1 + \cdots + d_m}{d_1, \ldots, d_m} \right] := \frac{(q^2)^{d_1 + \cdots + d_m}}{(q^2)^{d_1} \cdots (q^2)^{d_m}}$$

is a quantum multinomial coefficient.

If $L$ is a knot and $P(L)$ is in quiver form, then the coefficients of $x^j$ are manifestly polynomial. Note, that (2) is slightly more rigid than (1) since we require $R = S$. If $L$ is a link with $c \geq 2$ components, then the coefficients exhibit the expected standard denominator $(q^2)^{c-1}$ instead.

As explained in [23, Section 4.3], the generating function for the symmetrically colored HOMFLY-PT invariants can be obtained from $P(L)$ by a simple change of variables.

2.1 4-ended tangles and their HOMFLY-PT partition functions

Let $T_4$ denote the set of framed, oriented tangles with exactly 4 boundary points, which we represent by tangle diagrams as shown below, with boundary points labeled by intercardinal directions. Let $\bullet T_4$ be the subset of diagrams for which the SW boundary point is oriented inward. Within $\bullet T_4$ we distinguish three different configurations of boundary orientations, which we call boundary types:

$$\begin{array}{cccc}
\text{NW} & \text{NE} & \text{UP} : & \text{OP} : & \text{RI} : \\
\text{SW} & \text{SE} & \uparrow & \downarrow & \downarrow \uparrow.
\end{array}$$

We distinguish the types UP and RI even though they are related by a rotation.

Definition 2.2. For any tangle $\tau \in \bullet T_4$ and for $j \in \mathbb{N}$, we consider the $\otimes^j$-colored HOMFLY-PT invariant $(\tau)_j$, an element of the free $\mathbb{Q}[a^{\pm 1}](q)$-module generated by the following basic webs for in the HOMFLY-PT skein theory (see [22, Section 2.3] for a concise summary and [17, 2, 24] for background):

$$\begin{align*}
\text{UP}[j,k] &= \begin{array}{c}
\uparrow \uparrow \downarrow \downarrow \\
\uparrow \downarrow \uparrow \downarrow \\
\downarrow \uparrow \downarrow \uparrow \\
\downarrow \downarrow \uparrow \uparrow \\
\end{array}, & \text{OP}[j,k] &= \begin{array}{c}
\uparrow \downarrow \uparrow \downarrow \\
\uparrow \downarrow \downarrow \uparrow \\
\downarrow \uparrow \uparrow \downarrow \\
\downarrow \downarrow \uparrow \uparrow \\
\end{array}, & \text{RI}[j,k] &= \begin{array}{c}
\uparrow \downarrow \downarrow \uparrow \\
\uparrow \downarrow \uparrow \downarrow \\
\downarrow \uparrow \downarrow \uparrow \\
\downarrow \uparrow \uparrow \downarrow \\
\end{array}, & 0 \leq k \leq j.
\end{align*}$$

We will write $P(\tau) = \sum_{j \geq 0} (\tau)_j$ for the generating function of these invariants and call it the HOMFLY-PT partition function.
In [23] Theorem 3.4 we proved that for a rational tangle \( \tau \in \bullet T_4 \) of boundary type \( X \), the partition function \( P(\tau) \) can be written in the form:

\[
P(\tau) = \sum_{d \in \mathbb{N}^{m+n}} (-q)^{S \cdot d} A \cdot d_q \cdot d^{Q \cdot d} \left[ \begin{array}{c} |d_a| \\ d_a \\ \end{array} \right] \left[ \begin{array}{c} |d_i| \\ d_i \\ \end{array} \right] X[|d|, |d_a|]
\]  

(3)

where \( S, A \in \mathbb{Z}^{m+n} \) and \( Q \in \mathbb{Z}^{(m+n) \times (m+n)} \) depend on the tangle. The entries of the subvectors \( d_a \in \mathbb{N}^m \) and \( d_i \in \mathbb{N}^n \) of \( d \) are called the active and inactive summation indices.

Expressions such as (3) are suitable for studying the HOMFLY-PT partition functions of rational tangles. However, for more general 4-ended tangles, in particular those with closed components, we need a more flexible notion.

**Definition 2.3.** If \( \tau \in \bullet T_4 \) of boundary type \( X \) with \( c \) closed components, then we say that \( P(\tau) \) is in quiver form if:

\[
P(\tau) = \sum_{d \in \mathbb{N}^{m+n}} (-q)^{S \cdot d} A \cdot d_q \cdot d^{Q \cdot d} \left[ \begin{array}{c} |d_a| \\ d_a \\ \end{array} \right] \left[ \begin{array}{c} |d_i| \\ d_i \\ \end{array} \right] (q^2)^{|d|} X[|d|, |d_a|]
\]  

(4)

We say that \( P(\tau) \) is in active quiver form, and write \( \tau \in \mathcal{F}_a(X) \), if:

\[
P(\tau) = \sum_{d \in \mathbb{N}^{m+n}} (-q)^{S \cdot d} A \cdot d_q \cdot d^{Q \cdot d} \left[ \begin{array}{c} |d_a| \\ d_a \\ \end{array} \right] \left[ \begin{array}{c} |d_i| \\ d_i \\ \end{array} \right] \frac{(q^2)^{|d|}}{d} X[|d|, |d_a|]
\]  

(5)

We say that \( P(\tau) \) is in inactive quiver form, and write \( \tau \in \mathcal{F}_i(X) \), if:

\[
P(\tau) = \sum_{d \in \mathbb{N}^{m+n}} (-q)^{S \cdot d} A \cdot d_q \cdot d^{Q \cdot d} \left[ \begin{array}{c} |d_a| \\ d_a \\ \end{array} \right] \left[ \begin{array}{c} |d_i| \\ d_i \\ \end{array} \right] \frac{(q^2)^{|d|}}{d} X[|d|, |d_a|]
\]  

(6)

In all cases we ask that \( m, n \in \mathbb{N} \) with \( m + n \geq 1 \), \( S, A \in \mathbb{Z}^{m+n} \) and \( Q \in \mathbb{Z}^{(m+n) \times (m+n)} \). Such data will be recorded as a triple, for example in the case of (3) by

\[
X, \left( \frac{4 - c}{c} \right)^{S \cdot \Delta}, \frac{Q_+}{Q_-}
\]

which indicates the boundary type, the presence of additional \( q \)-Pochhammer symbols, as well as the vectors \( S, A \in \mathbb{Z}^{m+n} \) and the matrix \( Q \in \mathbb{Z}^{(m+n) \times (m+n)} \) in block decomposition according to active (+) and inactive (−) summation indices.

**Definition 2.4.** Consider the following operations on \( T_4 \):

\[
T \left( \begin{array}{c} \vdots \\ 1 \\ \vdots \\ \end{array} \right) := \begin{array}{c} \vdots \\ \vdots \\ \end{array}, \quad R \left( \begin{array}{c} \vdots \\ 1 \\ \vdots \\ \end{array} \right) := \begin{array}{c} \vdots \\ 1 \\ \end{array}
\]

Note that these operations preserve the orientation on the SW boundary point, and thus restrict to endomorphisms of \( \bullet T_4 \). We will also write \( T^{-1} \) and \( R^{-1} \) for the inverse operations, which are given by gluing on the respective inverse crossings.
Definition 2.5. We will consider six refined types of tangles \( \tau \in \mathcal{T}_4 \), which encode boundary types and connectivity between boundary points:

\[
\begin{align*}
UP_{\text{par}} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
OP_{ud} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
RI_{\text{par}} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
UP_{cr} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
OP_{lr} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
RI_{cr} : & \quad \begin{array}{|c|}
\hline
\vdots \\
\hline
\end{array} \\
\end{align*}
\]

For example, an \( UP_{\text{par}} \) tangle has one strand directed from the SW to the NW boundary point and the other strand directed from the SE to the NE boundary point, and possibly additional closed components.

2.2 The family of tangles \( QT_4 \)

In this section we define a family of 4-ended tangles \( QT_4 \subset T_4 \) that satisfy a tangles-quivers conjecture that implies the links-quivers conjecture for their closures. As before, we will focus on tangles with the inwards orientation on the SW boundary point.

Definition 2.6. Let \( \bullet QT_4 \subset \bullet T_4 \) denote the family of 4-ended tangles \( \tau \) that satisfy the following conditions, depending on their type:

1. \( \tau \in \mathcal{F}_a(UP) \) if \( \tau \) is of type \( UP_{\text{par}} \),
2. \( \tau \in \mathcal{F}_a(OP) \) if \( \tau \) is of type \( OP_{ud} \),
3. \( \tau \in \mathcal{F}_i(OP) \) if \( \tau \) is of type \( OP_{lr} \),
4. \( \tau \in \mathcal{F}_i(RI) \) if \( \tau \) is of type \( RI_{\text{par}} \),
5. \( C\tau \in QT_4 \) for any \( C \in \{T, T^{-1}, R, R^{-1}\} \) if \( \tau \) is of type \( UP_{cr} \) or \( RI_{cr} \).

Let \( QT_4 \subset T_4 \) be the family of those 4-ended tangles \( \tau \), for which there exists a rotation \( r \) of the plane that takes \( \tau \) to \( r(\tau) \in QT_4 \).

We have the following symmetries.

Lemma 2.7. If \( \tau \in QT_4 \), then so is

1. the mirror image \( -\tau \in QT_4 \),
2. the reflection across the SW-NE-diagonal \( r_d(\tau) \in QT_4 \),
3. the \( \pi \)-rotation around the vertical axis \( r_v(\tau) \in QT_4 \) for type \( UP \),
4. the \( \pi \)-rotation around the horizontal axis \( r_h(\tau) \in QT_4 \) for type \( RI \).

Proof. Mirroring acts by inverting the variable \( q \), which preserves the existence of active resp. inactive quiver forms as required. The reflection interchanges \( OP[j, k] \) with \( OP[j, j - k] \) and \( UP[j, k] \) with \( RI[j, j - k] \), and thus also active with inactive variables. The \( \pi \)-rotations simply preserve the generating functions, because the corresponding basis webs are invariant by virtue of the “square switch” web relation. Thus all symmetries considered here preserve the conditions in Definition 2.6. \( \square \)
The choice of the rotation $r$ in the Definition 2.6 of $QT_4$ is immaterial.

**Lemma 2.8.** Let $r$ be a rotation in the plane that takes $\tau \in \mathcal{T}_4$ to $r(\tau) \in \mathcal{T}_4$. Then we have

$$\tau \in \mathcal{QT}_4 \iff r(\tau) \in \mathcal{QT}_4.$$  

*Proof.* The only relevant rotations are the $\pi/2$-rotation interchanging $UP[j,k]$ and $RI[j,j-k]$ and swapping active and inactive summation indices, and the $\pi$-rotation which preserves generating functions of type $OP$. Using Lemma 2.7 it is straightforward to check that both rotations preserve the conditions in Definition 2.6. $\square$

The next lemma was a key result in [23].

**Lemma 2.9.** We have the following:

1. if $\tau \in F_a(UP)$, then $T^{\pm 2}\tau \in F_a(UP)$, $RT\tau \in F_i(OP)$ and $R^{\pm 1}\tau \in F_a(UP)$,
2. if $\tau \in F_a(OP)$, then $T^{\pm 2}\tau \in F_a(OP)$, $RT\tau \in F_i(RI)$ and $R^{\pm 1}\tau \in F_a(OP)$,
3. if $\tau \in F_i(OP)$, then $R^{\pm 2}\tau \in F_i(OP)$, $TR\tau \in F_a(UP)$ and $T^{\pm 1}\tau \in F_i(OP)$,
4. if $\tau \in F_i(RI)$, then $R^{\pm 2}\tau \in F_i(RI)$, $TR\tau \in F_a(OP)$ and $T^{\pm 1}\tau \in F_i(RI)$.

*Proof.* For positive crossing operations, these properties are proven in Lemma 4.5, 4.6 and 4.7 of [23]. The proof for negative crossing operations is analogous. $\square$

**Lemma 2.10.** If $\tau$ is of type $UP_{cr}$ or $RI_{cr}$, then we have:

$$\tau \in \mathcal{QT}_4 \iff T\tau \in \mathcal{QT}_4 \iff T^{-1}\tau \in \mathcal{QT}_4 \iff R\tau \in \mathcal{QT}_4 \iff R^{-1}\tau \in \mathcal{QT}_4.$$  

In particular, the four conditions in item 5 of Definition 2.6 are equivalent.

*Proof.* For the sake of concreteness, suppose $\tau$ is of type $UP_{cr}$, and so $T^{\pm 1}\tau$ is of type $UP_{par}$ and $R^{\pm 1}\tau$ is of type $OP_{tr}$. Then Lemma 2.9 provides the following equivalences and implications:

$$T\tau \in F_a(UP) \iff T^{\pm 2}\tau \in F_a(UP) \iff R^{\pm 2}\tau \in F_i(OP) \iff R^{-1}\tau \in F_i(OP) \iff TR\tau \in F_a(UP).$$

Any of these conditions is equivalent to all the others, and thus also to $\tau \in \mathcal{QT}_4$. $\square$

**Proposition 2.11.** We have $T^{\pm 1}\mathcal{QT}_4 \subset \mathcal{QT}_4$ and $R^{\pm 1}\mathcal{QT}_4 \subset \mathcal{QT}_4$.

*Proof.* This follows from Lemma 2.9 and Lemma 2.10. $\square$

**Lemma 2.12.** $QT_4$ is closed under attaching arbitrary crossings between pairs of neighbouring boundary points.
Proof. Let $\tau \in \mathrm{QT}_4$. Suppose the crossing is to be attached between two boundary points which are not both incoming. Instead of attaching the crossing, we first rotate the free incoming boundary point into the SW position and obtain a tangle $r(\tau) \in \mathrm{•QT}_4$. Then we attach the crossing, which results in a tangle $\tau' \in \mathrm{•QT}_4$ by Proposition 2.11. After rotating back, the desired tangle is $r^{-1}(\tau') \in \mathrm{QT}_4$.

If the crossing is to be attached between the two incoming boundary points, we again rotate $\tau$ until we get a UP-tangle $r(\tau) \in \mathrm{•QT}_4$. Then by Lemma 2.7 attaching the crossing on the bottom produces a tangle in $\mathrm{•QT}_4$ if and only if attaching the crossing at the top does, but the latter is again covered by Proposition 2.11 and we proceed as before.

**Theorem 2.13.** $\mathrm{QT}_4$ is closed under diffeomorphisms of the 3-ball, which fix the boundary set-wise.

**Proof.** $\mathrm{QT}_4$ is invariant under isotopies relative to the boundary by skein theory. Further, the corresponding mapping class group is generated by twists on the boundary, which are covered by Lemma 2.12 and mirroring, which is covered by Lemma 2.7.

**Theorem 2.14.** $\mathrm{QT}_4$ contains all rational tangles.

**Proof.** The trivial tangle is in $\mathrm{QT}_4$ and any rational tangle is built from it by successively attaching crossings between pairs of neighbouring boundary points.

In Section 3 we will prove that $\mathrm{QT}_4$ contains all algebraic 4-ended tangles.

### 2.3 From tangle quivers to link quivers

If a framed, oriented link $L$ is obtained from a tangle $\tau \in \mathrm{T}_4$ by connecting the northern boundary points to the southern boundary points (resp. eastern to western) by arcs in the plane, then we write $L = \mathrm{Cl}_{NS}(\tau)$ (resp. $L = \mathrm{Cl}_{EW}(\tau)$) and say that $L$ is a closure of $\tau$.

**Lemma 2.15.** Let $L$ be a framed, oriented link, obtained as the closure of $\tau \in \mathrm{QT}_4$, then $P(L)$ can be written in quiver form (2).

**Proof.** Up to isotopy, we may assume $L = \mathrm{Cl}_{NS}(\tau)$ for a tangle $\tau \in \mathrm{•QT}_4$ of type UP with $c$ closed components. This may require a change of framing, which, however, does not affect the existence of a quiver form for $P(L)$, see [23, Section 4.3].

If $\tau$ is of type $\mathrm{UP}_{\text{par}}$, then $L$ has $c + 2$ components. Applying the closure rule from [23, Section 2.3.3] to $P(\tau)$ directly produces an expression for $P(L)$ in quiver form.

If $\tau$ is of type $\mathrm{UP}_{\text{cr}}$, then $L$ has only $c + 1$ components and more care is needed. In this case $T^{-1}\tau$ is of type $\mathrm{UP}_{\text{par}}$ and in $\mathrm{•QT}_4$, and thus $P(T^{-1}\tau)$ may be assumed to be in active quiver form. Now the closure-after-top-crossing rule from [23, Lemma 4.8] implies that $P(L)$ can be written in quiver form. 

\[ \]
3 Adding 4-ended tangles

We will now consider the binary addition operation on 4-ended tangles, which is given by gluing along pairs of adjacent boundary points, provided the orientations are compatible there.

\[ + : \left( \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \right) \rightarrow \begin{array}{c} \tau_1 \\ \tau_2 \end{array} \]

Our goal for this section is to prove the following theorem.

**Theorem 3.1.** Let \( \tau_1, \tau_2 \in QT_4 \) with orientations such that \( \tau_1 + \tau_2 \) is defined. Then \( \tau_1 + \tau_2 \in QT_4 \).

For the purpose of deciding whether a glued tangle \( \tau_1 + \tau_2 \) belongs to \( QT_4 \) we may assume \( \tau_1 + \tau_2 \in \bullet T_4 \). (Otherwise we would move it in such a position using a diffeomorphism of the 3-ball and Theorem 2.13.) With the orientation of the SW boundary point fixed as incoming, there exist only five orientation configurations:

\[
\begin{align*}
&RI \quad RI, \quad OP \quad OP, \quad OP \quad OP, \quad UP \quad OP, \quad UP \quad UP
\end{align*}
\]

We will only consider the first and the second configuration. The fourth configuration is a rotated version of the second, and the fifth is related to the third by a Reidemeister 2 move.

\[
R \left( \begin{array}{c} OP \\ OP \end{array} \right) = \begin{array}{c} OP \\ OP \end{array} \in \begin{array}{c} OP \\ OP \end{array}
\]

Finally, the third configuration can be reduced to the second at the expense of twisting the boundary points.

\[
R \left( \begin{array}{c} OP \\ OP \end{array} \right) = \begin{array}{c} OP \\ OP \end{array} \in \begin{array}{c} OP \\ OP \end{array}
\]

Theorem 3.1 thus follows from the following.

**Theorem 3.2.** Suppose that \( \tau_1, \tau_2 \in \bullet QT_4 \) are either both of type RI, or \( \tau_1 \) is of type OP and \( \tau_2 \) of type UP. Then we have \( \tau_1 + \tau_2 \in \bullet QT_4 \).

The proof of this theorem will occupy the remainder of this section. For given \( \tau_1, \tau_2 \in \bullet QT_4 \), we know that \( P(\tau_1) \) and \( P(\tau_2) \) (or the partition functions for tangles that differ by the twisting of two boundary points) admit very special expressions, depending on the type and connectivity of the tangle. Our task will be to show that the same holds for the glued tangle \( \tau_1 + \tau_2 \). The first step is to compute how the skein basis elements glue.

3.1 Gluing of basis webs

This section details how the horizontal gluings of basic webs from Definition 2.2 expand into linear combinations of basic webs. The proofs of the lemmas here are straightforward diagrammatic computations using the local relation in the HOMFLY-PT skein theory (see e.g. [23, Fig. 1]) and therefore omitted.

In the following we will encounter certain linear expressions in the variables \( k, l, \) and \( j \), which we indicate by the generic expression \( \# \) if their precise form is immaterial. Similarly we write \( \#\# \) for quadratic expressions in these variables.
Lemma 3.3. The result of gluing the skein basis elements $RI[j, k]$ and $RI[j, l]$ can be expressed in terms of $RI$ basis elements as follows:

$$RI[j, k] + RI[j, l] = \sum_{0 \leq t} M_{[k', l']}^{[k, l]} [t, k' - l - t] RI[j, k + l - j + t]$$

where we write $k' = j - k$ and $l' = j - l$, and $M$ is of the form $q^\#$. Note that the non-zero summands occur only for $0 \leq t \leq \min(k', l')$.

Lemma 3.4. The result of gluing the skein basis elements $OP[j, k]$ and $UP[j, l]$ can be expressed in terms of $UP$ basis elements as follows:

$$OP[j, k] + UP[j, l] = \frac{\alpha^2 q^\#}{(q^2)_t} \sum_{0 \leq t} M_{[k', l']}^{[k, l]} [t, k' - l - t] OP[j, k + l + t - j]$$

where we again write $k' = j - k$ and $l' = j - l$ and $M$ is of the form $q^\#$. Note that the non-zero summands occur only for $\max(0, k' - l') \leq t \leq \min(l, k')$.

In an intermediate step, we will also need a lemma for gluings of type $OP + OP$.

Lemma 3.5. The result of gluing the skein basis elements $OP[j, k]$ and $OP[j, l]$ can be expressed in terms of $OP$ basis elements as follows:

$$OP[j, k] + OP[j, l] = \sum_{0 \leq t} M_{[k', l']}^{[k, l]} [t, k' - l - t] \frac{(\alpha^2 q^\#)}{(q^2)_t} OP[j, k + l + t - j]$$

$$= \sum_{0 \leq t} M_{[k', l']}^{[k, l]} [t, k' - l - t] \frac{(q^2)^{k' + l' - t}}{(q^2)^t} \frac{(\alpha^2 q^\#)}{q^{t}} OP[j, k + l + t - j]$$

where we again write $k' = j - k$ and $l' = j - l$, and $M, M'$ are of the form $q^\#$. Note that the non-zero summands occur only for $\max(0, j - k - l) \leq t \leq \min(k', l')$.

3.2 Some quantum algebra

This section introduces notation and lemmas that are useful in manipulating expressions of HOMFLY-PT partition functions.

Here we will use capital letters such as $A$ to denote tuples of integers (typically non-negative) $A = (A_1, \ldots, A_n)$, whose elements are indicated by subscripts (sometimes more than one). We use corresponding lower-case letters $a := \#A$ to denote their cardinalities and the notation $|A| = \sum_{a \in A} A$. If $A$ and $B$ are such that $\#A = \#B$, then $A \leq B$ means $A_i \leq B_i$ for $1 \leq i \leq \#A$ and $0 \leq A$ means that $0 \leq A_i$ for $1 \leq i \leq \#A$.

We use two basic lemmas: the first one is used for splitting $|A|$ into two smaller pieces, and another one is used for resummations where we have two sets of summation indices $A$ and $B$ such that $|A| = |B|$.

Lemma 3.6. Let $A = (A_1, \ldots, A_n) \in \mathbb{N}^n$, and $0 \leq t \leq |A|$. Then we have

$$[\frac{|A|}{A}] = \sum_{At \in \mathbb{N}^n} q^t [\frac{|A|}{At}] [\frac{|A| - t}{At^c}]$$

where $At^c = (At_1^c, \ldots, At_n^c) := (A_1 - At_1, \ldots, A_n - At_n)$. 

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The non-zero contributions to the above sum are for tuples $At := (At_1, \ldots, At_a)$ such that $|At| = t$ and $At^c \in \mathbb{N}^a$, since otherwise the multinomial coefficients would be zero. The notation $At$ and $At^c$ is intended to indicate that the entries of these tuple refine the entries of $A$ into components that do or do not contribute to $t$, respectively. Lemma 3.6 is a special case of the following.

**Lemma 3.7 (Lemma 4.6 [10]).** Let $A = (A_1, \ldots, A_a) \in \mathbb{N}^a$, $B = (B_1, \ldots, B_b) \in \mathbb{N}^b$ with $|A| = |B|$. Then we have:

$$\left[\frac{|A|}{B}\right] = \sum_{AB \in \mathbb{N}^{ab} \atop |AB_{\alpha, \beta}| = B_{\beta}} q^{|A| - |AB|},$$

(7)

The non-zero contributions to the above sum are for tuples $AB \in \mathbb{N}^{ab}$ with entries $AB_{\alpha, \beta}$, with $1 \leq \alpha \leq a$ and $1 \leq \beta \leq b$, such that $|AB_{\alpha, \beta}| = A_{\alpha}$ and $|AB_{*, \beta}| = B_{\beta}$. Also note that $|AB| = |A| = |B|$. Here the notation $AB$ is intended to indicate a refinement of the entries of the tuple $A$ according to the contribution of the entries of $B$.

As a corollary, we get the following.

**Lemma 3.8.** Let $A \in \mathbb{N}^a$ and $B \in \mathbb{N}^b$ with $|A| = |B|$. Then

$$\left[\frac{|A|}{A}\right] \left[\frac{|B|}{B}\right] = \sum_{AB \in \mathbb{N}^{ab} \atop |AB_{\alpha, \beta}| = A_{\alpha}} q^{|A| - |AB|}.$$

Next we need the $q$-Pochhammer symbol $(x^2; q^2)_k = \prod_{i=0}^{k-1} (1 - x^2 q^{2i})$, which generalises $(q^2; q^2)_k = (q^2)_k$.

**Lemma 3.9 ([10] Lemma 4.5]).** For any $d_1, \ldots, d_k \geq 0$, we have:

$$\frac{(x^2; q^2)_{d_1 + \ldots + d_k}}{(q^2)_{d_1} \cdots (q^2)_{d_k}} = \sum_{\alpha_1 + \beta_1 = d_1 \atop \alpha_k + \beta_k = d_k} \frac{(-x^2 q^{-1})^{\alpha_1 + \ldots + \alpha_k} q^{\alpha_1^2 + \ldots + \alpha_k^2 + 2 \sum_{i=1}^{k-1} \alpha_{i+1}(d_i + \ldots + d_k)}}{(q^2)_{\alpha_1} \cdots (q^2)_{\alpha_k} (q^2)_{\beta_1} \cdots (q^2)_{\beta_k}}.$$

Having recalled these basic lemmas, we now assemble them into more specialised tools for rewriting generating functions.

**Lemma 3.10.** Given any function of the form

$$P = \sum_{\begin{array}{c} AB \in \mathbb{N}^{ab} \\ B^c \in \mathbb{N}^b \\ t \leq |A|, t \leq |B| \end{array}} \left[\frac{|A|}{AB}\right] \left[\frac{|B|}{B^c}\right] f(A, B, t)$$

we can instead sum over $ABt \in \mathbb{N}^{ab}$, $At^c \in \mathbb{N}^a$, and $Bt^c \in \mathbb{N}^b$, with $|ABt| = t$, $|At^c| = |A| - t$, and $|Bt^c| = |B| - t$ and write

$$P = \sum_{\begin{array}{c} ABt \in \mathbb{N}^{ab} \\ At^c \in \mathbb{N}^a \\ Bt^c \in \mathbb{N}^b \end{array}} \left[\frac{t}{ABt}\right] \left[\frac{|A| - t}{At^c}\right] \left[\frac{|B| - t}{Bt^c}\right] f'(ABt, At^c, Bt^c)$$

where $f'$ arises from substitutions and a monomial factor from $f$. 

Proof. We first apply Lemma 3.6 and get

$$[\frac{|A|}{A}] = \sum_{AT \in \mathbb{N}^a} q^\# [\frac{f}{AT} \frac{|A| - t}{At}]$$

Since we also have \(|B| = b \geq t = |A|\), we can apply Lemma 3.7 to the tuples \((|B| - t, At_1, \ldots, At_a)\) and \(B\). Thus we get

$$[\frac{|B|}{B}] = \sum_{ABT \in \mathbb{N}^b} q^\# \prod_{\alpha=1}^{a} [\frac{AT}{ABt_{\alpha,s}}] \cdot [\frac{|B| - t}{B}]$$

with constraints described in Lemma 3.7. In combination, these expressions achieve the desired rewriting of \(P\) as far as multinomial coefficients are concerned. To rewrite \(f(A, B, t)\), note that every entry of \(A\) (resp. \(B\)) can be expressed as the sum of an entry of \(At^c\) (resp \(Bt^c\)) and \(b\) (resp. \(a\)) entries from the tuple \(ABt\). Moreover, \(t = |ABt|\). Thus \(f(A, B, t)\) can be re-expressed as a function of the entries of \(ABt\), \(At^c\), and \(Bt^c\). The function \(f'(A', B', T')\) results from this re-expression, also taking into account the monomial scaling \(\#\).

\[\square\]

**Lemma 3.11.** Given any function of the form

$$P = \sum_{A' \in \mathbb{N}^a, B' \in \mathbb{N}^b, C \in \mathbb{N}^c, D \in \mathbb{N}^d} \frac{[|C|]}{|C|} \frac{[|D|]}{|D|} f(A', B', C, D)$$

we can instead sum over \(A'C \in \mathbb{N}^{ac}, B'D \in \mathbb{N}^{bd}\), and \(S \in \mathbb{N}^{cd}\) and write

$$P = \sum_{A'C \in \mathbb{N}^{ac}, B'D \in \mathbb{N}^{bd}, S \in \mathbb{N}^{cd}} \frac{[|C|]}{|C|} \prod_{\alpha=1}^{a} [\frac{A'_{\alpha,s}}{A'C_{\alpha,s}}] \cdot \prod_{\beta=1}^{b} [\frac{B'_\beta}{B'D_{\beta,s}}] f'(A'C, B'D, S)$$

where \(f'\) arises from substitutions and a monomial factor from \(f\).

**Proof.** Since \(|C| = s + |A'|\) with \(s \geq 0\), we can apply Lemma 3.7 to the tuples \((s, A'_1, \ldots, A'_a)\) and \(C\):

$$[\frac{|C|}{C}] = \sum_{S' \in \mathbb{N}^c, A'C \in \mathbb{N}^{ac}} q^\# [\frac{s}{S'}] \prod_{\alpha=1}^{a} [\frac{A'_{\alpha,s}}{A'C_{\alpha,s}}]$$

The non-zero contributions to this sum have \(|S'| = s\), and so we also have \(|D| = |S'| + |B'|\). Now we apply Lemma 3.7 to the tuples \((S'_1, \ldots, S'_c, B'_1, \ldots, B'_b)\) and \(D\)

$$[\frac{|D|}{D}] = \sum_{S'D \in \mathbb{N}^{cd}, B'D \in \mathbb{N}^{bd}} q^\# \prod_{\gamma=1}^{c} [\frac{S'_\gamma}{S'D_{\gamma,s}}] \cdot \prod_{\beta=1}^{b} [\frac{B'_{\beta,s}}{B'D_{\beta,s}}]$$

By combining these two equations, setting \(S := S'D\), and re-expressing \(f\) in these new summation indices, we obtain the claimed expression for \(P\).

\[\square\]
The two previous lemmas combine in the following useful way.

**Corollary 3.12.** Given any function of the form

\[ P = \sum_{A \in \mathbb{N}^a, B \in \mathbb{N}^b, C \in \mathbb{N}^c, D \in \mathbb{N}^d} [A]^\tau [B]^\tau [C]^\tau [D]^\tau f(A, B, C, D, t) \]

we can instead sum over \( S \in \mathbb{N}^{cd} \), \( ABt \in \mathbb{N}^{ab} \), \( A'C \in \mathbb{N}^{ac} \), and \( B'D \in \mathbb{N}^{bd} \), and write

\[ P = \sum_{S \in \mathbb{N}^{cd}, ABt \in \mathbb{N}^{ab}, A'C \in \mathbb{N}^{ac}, B'D \in \mathbb{N}^{bd}} [S]^t [A']^\tau [B']^\tau f'(S, ABt, A'C, B'D) \]

where \(|ABt| = t\), \(|S| = |C| - |A| + t = |D| - |B| + t\), \(|A'C| = |A| - t\), \(|B'D| = |B| - t\), and \(f'\) arises from substitutions and a monomial factor from \(f\).

**Proof.** Apply Lemma 3.10 and then Lemma 3.11 with \( A' = At'\) and \( B' = Bt'\). \(\square\)

### 3.3 RI-RI gluing

Let \( \tau_1 \) and \( \tau_2 \) be tangles of type \( RI \). Suppose for now that both tangles have generating functions in quiver form. From now on we use symbols \( A_1 \) and \( A_2 \) instead of \( d_a \) for vectors of active summation indices, and \( I_1 \) and \( I_2 \) instead of \( d_i \) for vectors of inactive indices. Then (8) for \( \tau_1 \) and \( \tau_2 \) take the form:

\[
\begin{align*}
P(\tau_1) & = \sum_{A_1 \in \mathbb{N}^{a_1}, I_1 \in \mathbb{N}^{i_1}} M_1 [A_1]^{I_1} (q^2)^{-c_1} RI |A_1| + |I_1|, |A_1| \\
P(\tau_2) & = \sum_{A_2 \in \mathbb{N}^{a_2}, I_2 \in \mathbb{N}^{i_2}} M_2 [A_2]^{I_2} (q^2)^{-c_2} RI |A_2| + |I_2|, |A_2| 
\end{align*}
\]

where \( M_1 \) and \( M_2 \) stand for a monomials of the form \((-q)^a a^# q^#\), with exponents depending linearly, resp. quadratically on the summation indices in the tuples \( A_i \) and \( I_i \), and \( c_i \) is the number of closed components in \( \tau_i \).

Then we use Lemma 3.3.2 to compute the following expression for \( P(\tau_1 + \tau_2)\):

\[
\sum_{A_1 \in \mathbb{N}^{a_1}, A_2 \in \mathbb{N}^{a_2}, I_1 \in \mathbb{N}^{i_1}, I_2 \in \mathbb{N}^{i_2}} M [A_1]^{I_1} [A_2]^{I_2} (q^2)^{-c_{12}} RI |A_1| - |I_1| + t \]

where we write \( c := c_1 + c_2 \) and \( M \) denotes a monomial in \( a \) and \( q \) with exponents linear, resp. quadratic in the summation indices. Note that non-zero summands only occur for \( 0 \leq |I_1| - t \leq |A_2| \) and \( 0 \leq |I_2| - t \leq |A_1| \). Thus we can apply...
Corollary 3.12 with $A = I_1$, $B = I_2$, $C = A_2$, and $D = A_1$. After contracting the quantum multinomial coefficients we obtain

$$P(\tau_1 + \tau_2) = \sum_{S \in \mathbb{N}^{n+1} \cup \mathbb{N}^{n+1}} M' \left[ \frac{S}{|S|} \right] \left[ \sum_{I_1 I_2 t, I_1 t \in \mathbb{N}^{n+1}} \left[ \sum_{I_1 t_2 A_2 I_2 t_2 A_1} \left[ \sum_{I_1 I_2} \right] \left( q^2 \right)^{-c} \right] RI[j, |S|] \right]$$  \hspace{1cm} (9)

Here we have $|I_1 I_2 t| = t$, $|I_1 t_2 A_2| = |I_1| - t$, $|I_2 t_2 A_1| = |I_2| - t$, $|S| = |A_1| - |I_2| + t$, and $j = |I_1 I_2 t| + |I_1 t_2 A_2| + |I_2 t_2 A_1| + |S|$. Note that (9) is almost in quiver form, except for the extra factor $\left[ \frac{|I_1| + |I_2|}{|I_1||I_2|} \right]$.

Remark 3.13. The gluing rule can be made very explicit in triple notation. For example, if the input data (8) is given by

$$P(\tau_1) := \left[ RI, \left( -c_1 \frac{S_1^2 + A_1^2}{Q_1}, Q_1 \right) \right], \quad P(\tau_2) := \left[ RI, \left( -c_2 \frac{S_2^2 + A_2^2}{Q_2}, Q_2 \right) \right]$$

then $P(\tau_1 + \tau_2)$ in the form of (9) (with an extra q-binomial coefficient) can be expressed via the data

$$\left[ RI, \left( -\frac{S_1^2 + S_2^2}{Q_1 Q_2}, Q_1, Q_2 \right), \left( Q_1^2, Q_1^2, Q_2^2, Q_1^2, Q_2^2 \right) \right] + C$$

where the matrix $C$ contains certain off-diagonal correction terms. For example, if $P(\tau_1)$ and $P(\tau_2)$ have (inactive) quiver forms with 2 active and 2 inactive variables each, then

$$C = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$

3.4 OP-UP gluing

Let $\tau_1$ and $\tau_2$ be tangles of type OP and UP respectively. Suppose for now that both tangles have generating functions in quiver form:

$$P(\tau_1) = \sum_{A_1 \in \mathbb{N}^{n_1}} M_1 \left[ \frac{A_1}{A_1}, I_1 \right] \left( q^2 \right)^{-c_1} \left[ A_1 \right] \left[ |I_1| + |A_1| \right] |A_1| \left| OP \left[ |A_1| + |I_1|, |A_1| \right] \right|$$  \hspace{1cm} (10)

$$P(\tau_2) = \sum_{A_2 \in \mathbb{N}^{n_2}} M_2 \left[ \frac{A_2}{A_2}, I_2 \right] \left( q^2 \right)^{-c_2} \left[ A_2 \right] \left[ |I_2| + |A_2| \right] |A_2| \left| UP \left[ |A_2| + |I_2|, |A_2| \right] \right|$$

Then we use Lemma 3.14 to compute the following expression for $P(\tau_1 + \tau_2)$:
Note that the present quantum multinomial coefficients imply that the non-zero contributions to the sum satisfy $0 \leq |I_1| - t \leq |I_2|$ and $0 \leq |A_2| - t \leq |A_1|$, and so we can apply Corollary 3.12 with $A = A_2$, $B = I_1$, $C = A_1$, and $D = I_2$. After contracting the quantum multinomial coefficients we obtain the following expression for $P(\tau_1 + \tau_2)$.

$$
\begin{align*}
\sum_{S \in N^{n_1} \cup \tau \in N^{n_2}} M'[A_2t^eA_1] [A_2t^eA_1+(I_2^eS)](a^2q^#)_{t^e} U_P[j, A_2t^eA_1] & \\
\sum_{\tau \in N^{n_1} \cup \tau \in N^{n_2}} M[A_1] [A_2] [I_1] [I_2] (a^2q^#)_{t^e} U_P[j, A_1] & \\
\sum_{\tau \in N^{n_1} \cup \tau \in N^{n_2}} M[A_1] [A_2] [I_1] [I_2] (a^2q^#)_{t^e} U_P[j, A_2]
\end{align*}
$$

Here we have $|A_2t^eA_1| = |A_2| - t$, $|A_2t^eA_1| = |I_2| - t$, $|I_2t^eA_1| = |I_1| - t$, $|S| = |I_2| - |I_1| + t$, and $j = |A_2t^eA_1| + |A_2t^eA_1| + |I_2t^eA_1| + |S|$. Note that (11) is almost in quiver form, except for the extra factor $(a^2q^#)_{t^e}$ and possible an extra factor of $(q^2)^j$, depending on the number of components of $\tau_1 + \tau_2$.

### 3.5 OP-OP gluing

Let $\tau_1$ and $\tau_2$ be tangles of type OP. Suppose for now that both tangles have generating functions in quiver form:

$$
P(\tau_1) = \sum_{A_1 \in N^{n_1}} M_1[A_1] [I_1] (q^2)_{I_1} OP[j, A_1] + |I_1|, |A_1|)
$$

$$
P(\tau_2) = \sum_{A_2 \in N^{n_2}} M_2[A_2] [I_2] (q^2)_{I_2} OP[j, A_2] + |I_2|, |A_2|)
$$

Then we use Lemma 3.3 to compute the following expression for $P(\tau_1 + \tau_2)$:

$$
\sum_{\tau \in N^{n_1} \cup \tau \in N^{n_2}} M[A_1] [A_2] [I_1] [I_2] (a^2q^#)_{t^e} OP[j, A_1] + |A_2| + t - j
$$

Note that the present quantum multinomial coefficients imply that the non-zero contributions to the sum satisfy $0 \leq |I_1| - t \leq |A_2|$ and $0 \leq |I_2| - t \leq |A_1|$, and so we can apply Corollary 3.12 with $A = I_2$, $B = I_1$, $C = A_1$, and $D = A_2$. After contracting the quantum multinomial coefficients we obtain the following expression for $P(\tau_1 + \tau_2)$.
Theorem 3.2 have HOMFLY-PT partition functions in quiver form, then the partition function for $\tau_1 + \tau_2$ is very close to being in quiver form. We will now refine this observation to assemble a proof of Theorem 3.2. We consider the cases $RI-RI$ and $OP-UP$ separately.

### Proof of Theorem 3.2 for RI-RI gluing

Suppose that $\tau_1, \tau_2 \in \bullet QT_4$ are both of type $RI$. Let $c_i$ denote the number of closed components of $\tau_i$ and observe that $\tau_1 + \tau_2$ has $c_1 + c_2$ closed components. Now there are four connectivity configurations to consider.

1. Both $\tau_1$ and $\tau_2$ are of type $RI_{par}$. Then $\tau_1 + \tau_2$ will also be of type $RI_{par}$. By assumption, both $P(\tau_1)$ and $P(\tau_2)$ may be assumed to be in inactive quiver form, i.e. the expressions in (8) may be assumed to have an additional factor $(q^2)_{|I_i|}$ each. The partition function for the glued tangle, rewritten as in (9), is manifestly in quiver form, except that it carries an extra factor of $(q^2)_{|I_i| + |I_j|}$.

Since the length of this $q$-Pochhammer symbol $|I_i| + |I_j|$ is greater than the sum $|I_1| + |I_2| - t$ of the new inactive variables, it can be shortened (at the expense of splitting the summation indices from the set $I_1I_2t$ via Lemma 2.10) to bring $P(\tau_1 + \tau_2)$ into inactive quiver form, as required for a tangle of type $RI_{par}$. Thus $\tau_1 + \tau_2 \in \bullet QT_4$.

2. If $\tau_1$ is of type $RI_{par}$ but $\tau_2$ is of type $RI_{cr}$, then $\tau_1 + \tau_2$ will also be of type $RI_{cr}$. Lemma 2.10 shows that $\tau_2 \in \bullet QT_4 \iff R\tau_2 \in \bullet QT_4$, but $R\tau_2$ is of type $RI_{par}$, so we can use the first case to deduce $R(\tau_1 + \tau_2) = \tau_1 + \tau_2 \in \bullet QT_4$. Now, again by Lemma 2.10 this is equivalent to $\tau_1 + \tau_2 \in \bullet QT_4$.

3. Now suppose $\tau_1$ is of type $RI_{cr}$ and $\tau_2$ is of type $RI_{par}$. We write

$$R(\tau_1 + \tau_2) = \tau_1 + R\tau_2 = R\tau_1 + r_h(\tau_2)$$

where $r_h(\tau_2)$ is the result of rotating $\tau_2$ by $\pi$ in the $EW$-axis. By Lemma 2.10 $r_h(\tau_2)$ is again in $\bullet QT_4$ and of type $RI_{par}$, and so is $R\tau_1$. Now we apply the first case to see that $R(\tau_1 + \tau_2) \in \bullet QT_4$ and conclude with a final application of Lemma 2.10 that $\tau_1 + \tau_2 \in \bullet QT_4$.
4. Both $\tau_1$ and $\tau_2$ are of type $RI_{cr}$, then perform a Reidemeister II move in the gluing region to write

$$\tau_1 + \tau_2 = R\tau_1 + R^{-1}r_h(\tau_2).$$

Now both $R\tau_1$ and $R^{-1}r_h(\tau_2)$ are of type $RI_{par}$ and in $\bullet\text{QT}_4$ by assumption, and we again use the first case to conclude $\tau_1 + \tau_2 \in \bullet\text{QT}_4$.

This completes the proof in the case of $RI-RI$ gluing.

\[\square\]

**Proof of Theorem 3.2 for OP-UP gluing.** Suppose that $\tau_1, \tau_2 \in \bullet\text{QT}_4$ with $\tau_1$ of type $OP$ and $\tau_2$ of type $UP$. Furthermore, let $c_i$ denote the number of closed components of $\tau_i$ and set $c = c_1 + c_2$. There are again four connectivity configurations to consider.

1. If $\tau_1$ is of type $OP_{ad}$ and $\tau_2$ of $UP_{par}$, then $\tau_1 + \tau_2$ is of type $UP_{par}$ with $c + 1$ components. By assumption, both $P(\tau_1)$ and $P(\tau_2)$ may be assumed to be in active quiver form, i.e. expressions in (11) may be assumed have an additional factor $(q^2)^{t|A_i|}$ each. The partition function $P(\tau_1 + \tau_2)$ for the glued tangle, rewritten as in (11), is manifestly in quiver form, except that it carries an extra factor of

$$\sum_{l_1, l_2} (a^2 q^\#)_{l_1}(q^2)_{t|A_1|}(q^2)_{l_2}(q^2)_{t|A_2|-t}$$

The last of the $q$-Pochhammer symbols shown on the right-hand side is what is necessary to put $P(\tau_1 + \tau_2)$ in active quiver form after using all other symbols to split summation indices. Thus we have $\tau_1 + \tau_2 \in \bullet\text{QT}_4$.

2. If $\tau_1$ is of type $OP_{tr}$ and $\tau_2$ of $UP_{par}$, then $\tau_1 + \tau_2$ is of type $UP_{par}$ with $c$ components. We proceed as in the first case, with the notable difference that $P(\tau_1)$ may be assumed to be in inactive quiver form. This provides a $q$-Pochhammer factor $(q^2)^{t|A_i|}$ that we use in place of the factor $(q^2)^{t|A_i|}$ to cancel the denominator in the extra factor in (11). Thus we can write $P(\tau_1 + \tau_2)$ in active quiver form and deduce $\tau_1 + \tau_2 \in \bullet\text{QT}_4$.

3. If $\tau_1$ is of type $OP_{ud}$ and $\tau_2$ of $UP_{cr}$, then $\tau_1 + \tau_2$ is of type $UP_{par}$ with $c$ components. We can now perform a Reidemeister II move in the gluing region, to write

$$\tau_1 + \tau_2 = R\tau_1 + L^{-1}\tau_2$$

Here $R\tau_1$ is now of type $UP_{par}$ and $L^{-1}\tau_2$ is $\tau_2$ with a crossing attached on the left, i.e. a rotated version of a tangle of type $OP_{tr}$. Now the $\pi$-rotation $r_v$ reduces the situation to the already established second case.

4. If $\tau_1$ is of type $OP_{tr}$ and $\tau_2$ of $UP_{cr}$, then $\tau_1 + \tau_2$ is of type $UP_{cr}$ with $c$ components. By Lemma 2.10 it suffices to prove that the type $OP_{tr}$ tangle $R(\tau_1 + \tau_2) = \tau_1 + R\tau_2$ is in $\bullet\text{QT}_4$. For this we note that $\tau_1$ and $R\tau_2$ are both of type $OP_{tr}$ and in $\bullet\text{QT}_4$, so $P(\tau_1)$ and $P(R\tau_2)$ may be assumed to be in inactive quiver form, i.e. the expressions in (12) may be assumed have an additional factor $(q^2)^{t|A_i|}$ each. The partition function for the glued tangle, rewritten as in (13), is manifestly in quiver form, except that it carries an extra factor of

$$(a^2 q^\#)_{l_1}+|A_2|-t(a^2 q^\#)_t$$
After using \( (a^2q^\#)_t \) to split some summation indices, the expression for \( P(\tau_1 + R\tau_2) \) is in inactive quiver form, as required for an \( O\!P_{\!R} \) tangle in \( \bullet QT_4 \).

This completes the proof in the case of \( O\!P-UP \) gluing.

3.7 Size estimates for glued quiver data

If \( \tau_1, \tau_2 \in QT_4 \) admit generating functions in quiver form and suppose the sum \( \tau_1 + \tau_2 \) is defined, then it is plausible to expect that the number of summation indices in a quiver form for \( P(\tau_1 + \tau_2) \) is at worst bilinear in the inputs sizes. The purpose of this section is to refine and prove such a statement.

Remark 3.14. Any generating function in active/inactive quiver form can be expanded into quiver form by using the extra \( q \)-Pochhammer symbol to double the number of active/inactive summation indices via Lemma \ref{lem:expansion}. As explained in \cite[Lemma 4.11]{reference} this corresponds to an identity of the following type in triple notation

\[
\begin{bmatrix}
U\!P, \left( \begin{array}{ccc}
S_+ & 1 & A_+ \\
- & S_+ & A_+ \\
\end{array} \right), & \left( \begin{array}{ccc}
Q_+ & Q_+ + L & Q_+ \\
Q_+ & + & Q_+ \\
\end{array} \right) \right] = \\
\begin{bmatrix}
U\!P, \left( \begin{array}{ccc}
1 - cS_+ & A_+ \\
- & S_+ & A_+ \\
\end{array} \right), & \left( \begin{array}{ccc}
Q_+ & Q_+ & Q_+ \\
Q_+ & Q_+ & Q_+ \\
\end{array} \right) \right]
\end{bmatrix}
\]

where 1 indicates matrices with all entries equal to 1, and \( U \) resp. \( L \) denote matrices that have entries 1 strictly above resp. below the diagonal and zeros elsewhere.

Proposition 3.15. Suppose that \( \tau_1, \tau_2 \in QT_4 \) are gluable of types \( RI_{\!par}-RI_{\!par} \), \( O\!P_{\!R}-UP_{\!par} \), or \( O\!P_{\!R}-O\!P_{\!R} \). Use Lemma \ref{lem:expansion} to convert their generating functions in active/inactive quiver form into generating functions of quiver form \( \left( \begin{array}{ccc}
\end{array} \right) \) with \( a_1 \) resp. \( a_2 \) active and \( i_1 \) resp. \( i_2 \) inactive summation indices. Then \( P(\tau_1 + \tau_2) \) can be brought into quiver form with a total of \( (a_1 + i_1)(a_2 + i_2) \) summation indices.

Proof. We will only consider the cases \( RI_{\!par}-RI_{\!par} \) and \( O\!P_{\!R}-UP_{\!cr} \), since \( O\!P_{\!R}-O\!P_{\!R} \) is entirely analogous to the former.

In the case when we add \( \tau_1, \tau_2 \in \bullet QT_4 \) of type \( RI_{\!par} \), we may assume that their generating functions in inactive quiver form \( \left( \begin{array}{ccc}
\end{array} \right) \) have \( a_1 + i_1/2 \) and \( a_2 + i_2/2 \) summation indices respectively (Rewriting into quiver form via Lemma \ref{lem:expansion} doubles the numbers of inactive indices.). When computing \( P(\tau_1 + \tau_2) \) along the steps of the proof of Theorem \ref{thm:main} the expression \( \left( \begin{array}{ccc}
\end{array} \right) \) will have \( (a_1 + i_1/2)(a_2 + i_2/2) \) summation indices. After splitting the summation indices from the set \( I_1I_2I \), we arrive at an inactive quiver form with \( a_1a_2 \) active summation indices and \( a_1i_2/2 + i_1a_2/2 + i_1i_2/2 \) inactive summation indices. By using Lemma \ref{lem:expansion} again, this is converted into an expression in quiver form with \( a_1a_2 \) active summation indices and \( a_1i_2 + i_1a_2 + i_1i_2 \) inactive summation indices. The total number is \( (a_1 + i_1)(a_2 + i_2) \) as claimed.

In the case of \( O\!P_{\!R}-UP_{\!par} \) we start with generating functions in inactive/active quiver form with \( a_1 + i_1/2 \) and \( a_2/2 + i_2 \) summation indices respectively. At the stage of \( \left( \begin{array}{ccc}
\end{array} \right) \) we see a total number of \( (a_1 + i_1/2)(a_2/2 + i_2) \) summation indices, and then use extra \( q \)-Pochhammer symbols of length \( t \) and \( |I_1| \) to obtain an active quiver form with \( a_1a_2/2 \) active and \( i_1a_2 + a_1i_2 + i_1i_2 \) summation indices. A final application of Lemma \ref{lem:expansion} puts \( P(\tau_1 + \tau_2) \) into quiver form with \( a_1a_2 \) active and \( i_1a_2 + a_1i_2 + i_1i_2 \) inactive indices. The total number is again \( (a_1 + i_1)(a_2 + i_2) \).
The minimal numbers of summation indices of generating functions for added tangles are often smaller that suggested by Proposition 3.15. For example, the generating function for a single crossing has 1 active and 1 inactive summation index. However, recall the following result.

**Proposition 3.16 ([23, Proposition 3.5]).** If \( \tau \in \star T_4 \) and \( P(\tau) \) is in quiver form with \( a + i \) summation indices, then \( P(T\tau) \) and \( P(R\tau) \) can also be brought into quiver form with \( (a + i) + i \) and \( a + (a + i) \) summation indices respectively.

In particular, the sizes of the resulting generating functions are smaller than the product of the input sizes. Proposition 3.16 can also be combined with Proposition 3.15 used to deduce size estimates in other gluing situations. We just give one example.

**Corollary 3.17.** Suppose that \( \tau_1, \tau_2 \in QT_4 \) are gluable of types \( RI_{par}, RI_{cr} \) and suppose that \( P(\tau_1) \) and \( P(R^{-1}\tau_2) \) have been brought into quiver form with \( a_1 + i_1 \) and \( a_2 + (i_2 - a_2) \) summation indices respectively by applying Lemma 3.9 to their active quiver form expressions. Then \( P(\tau_2) \) and \( P(\tau_1 + \tau_2) \) can be brought into quiver form with \( a_2 + i_2 \) summation indices and \( a_1a_2 + (a_1i_2 + i_1i_2) \) summation indices respectively.

**Proof.** Proposition 3.15 implies that \( P(\tau_2) \) can be expressed in quiver form of size \( a_2 + i_2 \). For \( P(\tau_1 + \tau_2) \) we first apply Proposition 3.15 to \( \tau_1 \) and \( R^{-1}\tau_2 \) and obtain a quiver form for \( P(\tau_1 + R^{-1}\tau_2) \) of size \( a_1a_2 + (a_1i_2 + i_1i_2 - a_1a_2) \). After applying another operation \( R \), we conclude with \( P(\tau_1 + \tau_2) \) in quiver form with \( a_1a_2 + (a_1i_2 + i_1i_2) \) summation indices.

To get a general size estimate we need the following definition.

**Definition 3.18.** For \( \tau \in \star QT_4 \), the quiver complexity \( qc(\tau) \in \mathbb{N} \) is defined as the minimum of the following numbers

- \( 2a + i \) such that \( P(\tau) \) has an active quiver form with a active and \( i \) inactive variables, if \( \tau \) is of type \( UP_{par} \text{ or } OP_{par} \),

- \( a + 2i \) such that \( P(\tau) \) has an inactive quiver form with a active and \( i \) inactive variables, if \( \tau \) is of type \( OR_{par} \text{ or } RI_{par} \).

In other words, the complexity is the minimal size of a quiver form for \( P(\tau) \), that can be obtained from an active/inactive quiver form. We call such quiver forms tight. For the remaining cases \( UP_{cr} \text{ or } RI_{cr} \) we define \( qc(\tau) \) as the minimum of

- \( 2qc(\tau') \), where \( \tau' \in \{T\tau, T^{-1}\tau, R\tau, R^{-1}\tau\} \).

The next observation follows directly from Proposition 3.16.

**Lemma 3.19.** For any \( \tau \in \star QT_4 \) the generating function \( P(\tau) \) admits a quiver form with at most \( qc(\tau) \) summation indices.

**Lemma 3.20.** For \( \tau \in \star QT_4 \) have \( qc(C\tau) \leq 4qc(\tau) \) for any \( C \in \{T, T^{-1}, R, R^{-1}\} \).
Proof. This follows from the definition if $C\tau$ is of type $UP_{cr}$ or $RI_{cr}$. All other cases follow from [23, Lemmas 4.5, 4.6, 4.7]. We illustrate one of the most interesting cases. Suppose that $C = R^{-1}$ and $\tau$ is of type $UP_{cr}$ with complexity $4a + 2i$ witnessed by an active quiver form of size $a + i$ for $\tau' := T^{-1}\tau$. Then [23, Lemmas 4.6, 4.7] imply that $R^{-1}\tau = R^{-1}T^{-1}T^2\tau'$ has an inactive quiver form with $2(a + 3i) + (a + 2i)$ summation indices, and we get

$$qc(R^{-1}\tau) = qc(R^{-1}T^{-1}T^2\tau') \leq 4a + 7i \leq 16a + 8i = 4qc(\tau).$$

It is likely that the constant can be improved from 4 to 2 using an improved version of [23, Lemmas 4.6] for operations of the form $R^{-1}T$ etc.

**Proposition 3.21.** Suppose that $\tau_1, \tau_2 \in F_T4$ are gluable. Then

$$qc(\tau_1 + \tau_2) \leq 16qc(\tau_1)qc(\tau_2).$$

**Proof.** For the cases from Proposition 3.15 we already know $qc(\tau_1 + \tau_2) = qc(\tau_1)qc(\tau_2)$. In the proof of Theorem 3.22 the remaining five cases are deduced from the those by performing rewrites with at most two crossings, or by absorbing extra $q$-Pochhammer symbols at the expense of at most quadrupling the number of summation indices. In all cases, the complexity bound is observed.

4 Case study on two pretzel tangles

In this section we give examples of quiver form expressions of the generating functions of the $(2,3)$ and $(2,-3)$ pretzel tangles of orientation type $OP-UP$. Both result from adding two rational tangles whose quiver form generating functions require $3 + 1$ and $2 + 1$ summation indices respectively.

An optimistic reading of Section 3.7 suggests an upper bound of $6 + 6$ summation indices for quiver forms and $3 + 6$ summation indices for active quiver forms. In fact, both the $(2,3)$ and the $(2,-3)$ pretzel tangle admit smaller expressions, interestingly of different sizes: $5 + 6$ and $3 + 6$ in quiver form, $5 + 3$ and $3 + 3$ in active quiver form.

4.1 The $(2,3)$-pretzel tangle

The following shows the quiver form generating function data for the skein module element of the $(2,3)$-pretzel tangle. The diagram on the right shows the 11 monomials, 6 active and 5 inactive, in the 1-colored skein evaluation, which gives a lower bound on the size of any quiver describing the generating function. This lower bound

---

3Proposition 3.15 does not cover the $OP_{ud}-UP_{cr}$-case, but the stated upper bound can be deduced for the $(2, -3)$ pretzel tangle. A similar argument for the $(2, 3)$ pretzel tangle produces a worse upper bound, but still a better one than Proposition 3.21.
is achieved by the following data, which was found experimentally.

The display here shows inactive variables after active variables as in [23]. Since the tangle is of type $UP_{par}$, one can also present the generating function data in active quiver form. The 1-colored evaluation gives a lower bound of 3 active and 5 inactive summation indices, which is achieved by the following data.

The two expressions are related by the identity from Remark 3.14.

4.2 The $(2,-3)$-pretzel tangle

The following shows the quiver form generating function data for the skein module element of the $(2, -3)$-pretzel tangle. The diagram on the right shows the 9 monomials, 6 active and 3 inactive, in the 1-colored skein evaluation, which gives a lower bound on the size of any quiver describing the generating function. This lower bound is achieved by the following data, which was also found experimentally.

Just like before, this tangle is of type $UP_{par}$ and admits a generating function in active quiver form. The 1-colored evaluation gives a lower bound of 3 active and 3 inactive generators for such an expression, which is achieved by the following data.

The two expressions are again related by the identity from Remark 3.14.
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