AN EQUIVARIANT PULLBACK STRUCTURE OF TRIMMABLE GRAPH C*-ALGEBRAS

FRANCESCA ARICI, FRANCESCO D’ANDREA, PIOTR M. HAJAC, AND MARIUSZ TOBOLSKI

ABSTRACT. We prove that the graph C*-algebra $C^*(E)$ of a trimmable graph $E$ is $U(1)$-equivariantly isomorphic to a pullback C*-algebra of a subgraph C*-algebra $C^*(E'')$ and the C*-algebra of functions on a circle tensored with another subgraph C*-algebra $C^*(E')$. This allows us to unravel the structure and K-theory of the fixed-point subalgebra $C^*(E)^{U(1)}$ through the (typically simpler) C*-algebras $C^*(E'), C^*(E'')$ and $C^*(E'')^{U(1)}$. As examples of trimmable graphs, we consider one-loop extensions of the standard graphs encoding respectively the Cuntz algebra $O_2$ and the Toeplitz algebra $T$. Then we analyze equivariant pullback structures of trimmable graphs yielding the C*-algebras of the Vaksman–Soibelman quantum sphere $S^{2n+1}_q$ and the quantum lens space $L^3_q(l; 1, l)$, respectively.

CONTENTS

1. Introduction and preliminaries 2
1.1. Introduction 3
1.2. Preliminaries 4
2. Trimmmable graph C*-algebras 6
2.1. A $K_1$-generator for trimmable graphs without sinks 7
2.2. An equivariant pullback structure 7
2.3. Mayer–Vietoris exact sequences in K-theory 10
3. Examples 10
3.1. A one-loop extension of the Cuntz algebra $O_2$ 10
3.2. A one-loop extension of the Toeplitz algebra $T$ 12
3.3. The Vaksman–Soibelman quantum spheres and projective spaces 13
3.4. Quantum lens spaces and quantum teardrops 17
Acknowledgements 21
References 21

2010 Mathematics Subject Classification. 46L80, 46L85.
Graph C*-algebras are remarkable examples of “operator algebras that one can see” [35]. In particular, they proved to be extremely useful in determining the K-theory of non-commutative deformations of interesting topological spaces. They come equipped with a natural circle action (called the gauge action), and their gauge-invariant subalgebras describe some fundamental examples of noncommutative topology [9, 28, 29].

The goal of this paper is to present $U(1)$-C*-algebras from a large class of trimmable graph C*-algebras as $U(1)$-equivariant pullbacks, so as to determine a pullback structure of their fixed-point subalgebras. The pullback structure yields a Mayer–Vietoris six-term exact sequence in K-theory allowing us to express the even-K-group of the fixed-point subalgebra of a trimmable graph C*-algebra in terms of the K-theory of simpler C*-algebras. More precisely, we have:

**Theorem.** Let $E$ be a $v$-trimmable graph. Denote by $E'$ the subgraph of $E$ obtained by removing the unique outgoing edge (loop) of $v$, and by $E''$ the subgraph of $E'$ obtained by removing the vertex $v$ and all edges ending in $v$. Then there exist $U(1)$-equivariant *-homomorphisms making the following diagram

\[
\begin{array}{c}
C^*(E) \\
\downarrow \quad \downarrow \\
C^*(E') \otimes C(S^1) \\
\downarrow \quad \downarrow \\
C^*(E'') \otimes C(S^1) \\
\end{array}
\]

a pullback diagram of $U(1)$-C*-algebras. Here $C^*(E)$ and $C^*(E'')$ are considered as $U(1)$-C*-algebras with respect to the gauge action, whereas the tensor product algebras are viewed as $U(1)$-C*-algebras with respect to the standard $U(1)$-action on $C(S^1)$.

**Corollary.** The following Mayer–Vietoris six-term sequence in K-theory is exact.

\[
\begin{array}{c}
K_0(C^*(E)^{U(1)}) \quad K_0(C^*(E'')^{U(1)}) \oplus K_0(C^*(E')) \quad K_0(C^*(E'')) \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
K_1(C^*(E'')) \quad K_1(C^*(E')) \quad 0.
\end{array}
\]

The K-theory of graph C*-algebras is very well developed by now [15, 21, 36, 35, 10], so it is hard to make a substantial contribution in this area. However, our approach to the matter does bring its own benefits:

- The K-theory of fixed-point subalgebras of higher-rank graph C*-algebras [25] can be quite difficult to study. It was shown in [22] how equivariant pullback structures of graph C*-algebras provided by the above theorem can be used to reduce the K-theory computations for higher-rank graphs to far more doable K-theory calculations for usual graphs.

---

1. Introduction and preliminaries
• Although the K-groups of the gauge-invariant subalgebra of a graph C*-algebra are, in principle, computable for any graph [20], the above corollary makes this computation more effective by shifting it to simpler graphs.

• When the gauge-invariant subalgebra of a graph C*-algebra is only Morita equivalent to a graph C*-algebra, it is tricky to describe it explicitly, so having it as a pullback C*-algebra (see Corollary 2.5) remedies the problem whenever the gauge-invariant subalgebra of the trimmed graph C*-algebra is known.

• The Milnor connecting homomorphism \( \partial_{10} \) in the above corollary unravels generators of the even K-group the gauge-invariant subalgebra of a graph C*-algebra, so it might useful even when the gauge-invariant algebra is again a graph C*-algebra.

A one-loop extension of any finite graph \( E'' \) automatically becomes a trimmable graph \( E \), and \( E' \) is simply a one-sink extension of \( E'' \) [36]. Thus we can “grow” trimmable graphs from any graph, and use the above corollary to determine the even K-groups. In one case we take as \( E'' \) the standard graph encoding the Cuntz algebra \( \mathcal{O}_2 \), and in another case we take as \( E'' \) the standard graph encoding the Toeplitz algebra \( \mathcal{T} \). Then, in both cases, the gauge-invariant subalgebra \( C^*(E)^{U(1)} \) has infinite-rank \( K_0 \)-group.

On the other hand, we use our approach to study trimmable graphs naturally occurring in noncommutative topology, such as graphs giving the C*-algebras of the Vaksman–Soibelman quantum sphere \( S^2_{q}^{2n+1} \) and the quantum lens space \( L^3_q(l, 1, l) \), respectively. The gauge-invariant subalgebra of the former C*-algebra defines the Vaksman–Soibelman quantum complex projective space \( \mathbb{C}P^n_q \), and the gauge-invariant subalgebra of the latter C*-algebra defines the quantum teardrop \( WP^1_q(1, l) \). Although much is already known (generators included) about both the K-theory of \( C(\mathbb{C}P^n_q) \) and \( C(WP^1_q(1, l)) \), revealing pullback structures of these C*-algebras allows us to view projections whose classes generate the even K-groups as Milnor idempotents, thus rendering the noncommutative vector bundles they define “locally trivial” and given by a clutching of trivial noncommutative vector bundles.

1.1. Introduction. Pushout diagrams in topology provide a systematic way of “gluing” of topological spaces, in particular in the category of compact Hausdorff spaces. The latter is dualized by the Gelfand transform to the category of commutative unital C*-algebras with pushout diagrams of spaces translated into pullback diagrams of algebras.

Our motivating example came from the following \( U(1) \)-equivariant pushout diagram concerning spheres and the \( 2n \)-ball:

\[
\begin{array}{ccc}
S^{2n-1} & \\
\uparrow & \uparrow \\
S^{2n-1} \times S^1 & \\
\downarrow & \downarrow \\
\downarrow & \\
B^{2n} \times S^1 & S^{2n+1} \\
\end{array}
\]
The equivariance of the above diagram allows it to descend to the quotient spaces:

\[
\begin{array}{c}
\mathbb{C}P^n \\
\mathbb{C}P^{n-1} \\
S^{2n-1} \\
\end{array}
\]

Thus we obtain a diagram that manifests the CW-complex structure of complex projective spaces \( \mathbb{C}P^n \).

For \( n = 1 \), a \( q \)-deformed version of the diagram (1.1) was considered in [26] and proved to be a \( U(1) \)-equivariant pullback diagram of \( \mathbb{C}^* \)-algebras. This pullback structure provided therein an alternative way for computing an index pairing for noncommutative line bundles over the standard Podleś quantum sphere [34, 23]. For \( n = 2 \), an analogous result was called for in [22] in the context of the multipullback noncommutative deformation of complex projective spaces. This led to obtaining the general result for an arbitrary \( n \) in this paper: a \( U(1) \)-equivariant pullback structure of the \( \mathbb{C}^* \)-algebra of the Vaksman–Soibelman quantum sphere \( S^{2n+1}_q \) is a prototype of the main theorem herein.

Our main theorem is based on a general concept of a trimmable graph [24]: a finite graph \( E \) is \( \tilde{v} \)-trimmable iff it consists of a subgraph \( E'' \) emitting at least one edge to an external vertex \( \tilde{v} \) whose only outgoing edge \( \tilde{e} \) is a loop and such that all edges other than \( \tilde{e} \) that end in \( \tilde{v} \) begin in a vertex emitting an edge that ends not in \( \tilde{v} \). A trimmable graph \( E \) can be trimmed to its subgraph \( E'' \). There is an intermediate subgraph \( E' \) of \( E \) that is a one-sink extension of \( E'' \). Much as one defines a one-sink extension, we can define a one-loop extension, so that a one-loop extension of any finite graph is automatically a trimmable graph.

The paper is organized as follows. To make it self-contained, we start by recalling some results on graph \( \mathbb{C}^* \)-algebras in the preliminaries. Then we proceed to Section 2 where we prove the main result and study general K-theoretical benefits of decomposing a trimmable graph \( \mathbb{C}^* \)-algebra into simpler building blocks. The last section is devoted to four examples of two different types. The first two of them are in the spirit of abstract graph algebras (one-loop extensions of the Cuntz algebra \( O_2 \) and the Toeplitz algebra \( T \)), whereas the remaining two are in the spirit of noncommutative topology (\( q \)-deformations of spheres, balls, complex projective spaces, lens spaces, teardrops).

1.2. Preliminaries.

In this section, we recall some general results from the theory of graph \( \mathbb{C}^* \)-algebras. Our main references are [1, 4, 35]. We adopt the conventions of [4], i.e., the roles of source and range map are exchanged with respect to [35].

Let \( E \) be a directed graph, \( E_0 \) the set of vertices, \( E_1 \) the set of edges, \( s : E_1 \to E_0 \) and \( r : E_1 \to E_0 \) the source and range map respectively. A directed graph is called row-finite if \( s^{-1}(v) \) is a finite set for every \( v \in E_0 \). It is called finite if both sets \( E_0 \) and \( E_1 \) are finite. A sink is a vertex \( v \) with no outgoing edges, that is \( s^{-1}(v) = \{ e \in E_1 : s(e) = v \} = \emptyset \).
By a path $e$ of length $|e| = k \geq 1$ we mean a directed path, i.e. a sequence of edges $e := e_1 \ldots e_k$, with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, k - 1$. We view vertices as paths of length zero. We denote the set of all paths in $E$ by $\text{Path}(E)$. We extend the maps $r$ and $s$ to $\text{Path}(E)$ by setting $s(e) := s(e_1)$ and $r(e) := r(e_k)$ for all $e$ of length $k \geq 1$, and $s(v) := v = r(v)$ for all paths $v$ of length zero.

**Definition 1.1** (Graph C*-algebra). The graph C*-algebra $C^*(E)$ of a row-finite graph $E$ is the universal C*-algebra generated by mutually orthogonal projections $\{P_v : v \in E_0\}$ and partial isometries $\{S_e : e \in E_1\}$ satisfying the Cuntz–Krieger relations:

\begin{align*}
(\text{CK1}) & \quad S_e^*S_e = P_{r(e)} \quad \text{for all } e \in E_1, \text{ and} \\
(\text{CK2}) & \quad \sum_{e \in E_1 : s(e) = v} S_eS_e^* = P_v \quad \text{for all } v \in E_0 \text{ that are not sinks.}
\end{align*}

The datum $\{S, P\}$ is called a Cuntz–Krieger $E$-family.

On the notational side, if $E'$ is a subgraph of $E$ and $\{S, P\}$ is the Cuntz–Krieger $E$-family, we will slightly abuse notation and denote the Cuntz–Krieger $E'$-family by $\{S, P\}$ as well.

Any graph C*-algebra $C^*(E)$ can be endowed with a natural circle action

$$\alpha : U(1) \rightarrow \text{Aut}(C^*(E)),$$

called the gauge action. Using the universality of $C^*(E)$, it is defined by being set on the generators:

$$\alpha_\lambda(P_v) = P_v, \quad \alpha_\lambda(S_e) = \lambda S_e, \quad \text{where } \lambda \in U(1), \ v \in E_0, \ e \in E_1.$$

The fixed-point subalgebra under the gauge action is an AF-subalgebra of the form (e.g., see [35, Corollary 3.3]):

$$C^*(E)^{U(1)} = \text{sp} \left\{ S_xS_y^* : x, y \in \text{Path}(E), \ r(x) = r(y), \ |x| = |y| \right\}.$$  

Here for a path $x := x_1x_2 \ldots x_n$ we set $S_x := S_{x_1}S_{x_2} \ldots S_{x_n}$, and for a path $v$ of length 0 we put $S_v := P_v$.

The gauge action is a central ingredient in the gauge-invariant uniqueness theorem proved by an Huef and Raeburn [1, Theorem 2.3] in the context of Cuntz–Krieger algebras [16], and then generalized to graph C*-algebras of row-finite graphs by Bates, Pask, Raeburn and Szymański [4, Theorem 2.1]. This theorem, together with the universality of graph C*-algebras with respect to the Cuntz–Krieger relations, is an essential tool in proving that a given C*-algebra is isomorphic to a graph C*-algebra. We give here a slight reformulation of the result, more suitable for the purposes of this work.

**Theorem 1.2** (Gauge-invariant uniqueness theorem [1, 4, 35, Theorem 2.2]). Let $E$ be a row-finite graph with the Cuntz–Krieger family $\{S, P\}$, let $A$ be a C*-algebra with a continuous action of $U(1)$ and $\rho : C^*(E) \rightarrow A$ a $U(1)$-equivariant *-homomorphism. If $\rho(P_v) \neq 0$ for all $v \in E_0$, then $\rho$ is injective.

To understand gauge-invariant ideals of graph C*-algebras, we need to introduce two kinds of subsets of the set of vertices. Recall that given two vertices $v, w \in E_0$, whenever $w$ is reachable from $v$, that is whenever there is a path from $v$ to $w$, we write $v \geq w$, and we write $v \geq w$ if $v > w$ or $v = w$. A subset $H$ of $E_0$ is called hereditary iff $v \geq w$
and \( v \in H \) imply \( w \in H \). A hereditary subset \( H \) is saturated iff every vertex which feeds into \( H \) and only into \( H \) is again in \( H \). We denote by \( \overline{H} \) the saturation of a hereditary subset \( H \), that is the smallest saturated subset containing \( H \).

It follows from [4, Lemma 4.3] that, for any hereditary subset \( H \), the (algebraic) ideal generated by \( \{ P_v : v \in H \} \) is of the form

\[
I_E(H) = \text{span} \left\{ S_x S_y^* : x, y \in \text{Path}(E), \ r(x) = r(y) \in \overline{H} \right\}.
\]

Equation (1.4) will play an essential role in the proof of Theorem 2.4.

By [4, Theorem 4.1 (a)], given a row-finite graph \( E \), the gauge-invariant ideals in the graph algebra \( C^*(E) \) are in one-to-one correspondence with saturated hereditary subsets of \( E_0 \). By [4, Theorem 4.1 (b)], quotients by (closed) ideals generated by saturated hereditary subsets can be realised also as graph \( C^* \)-algebras by constructing a quotient graph. Given a saturated hereditary subset \( H \) of \( E_0 \), the quotient graph \( E/H \) is the graph obtained by removing from \( E \) all the vertices in \( H \) and all the edges whose range is in \( H \), i.e. \( (E/H)_0 := E_0 \setminus H \) and \( (E/H)_1 := \{ e \in E_1 : r(e) \notin H \} \). As a consequence, we have a \( U(1) \)-equivariant isomorphism

\[
C^*(E)/I_E(H) \cong C^*(E/H),
\]

where \( \overline{I_E(H)} \) is the norm closure of \( I_E(H) \).

\[\text{2. Trimmable graph } C^* \text{-algebras}\]

The following notion of a trimmable graph was introduced in [24, Definition 2.1] in the context of Leavitt path algebras.

**Definition 2.1 ([24]).** Let \( E \) be a finite graph with a distinguished vertex \( \bar{v} \) emitting a loop \( \bar{e} \). A graph \( E \) is called \( \bar{v} \)-trimmable iff the pair \( (E, \bar{v}) \) satisfies the following conditions

\[
\begin{align*}
&(\text{T1}) \quad s^{-1}(\bar{v}) = \{\bar{e}\}, \quad r^{-1}(\bar{v}) \setminus \{\bar{e}\} \neq \emptyset, \\
&(\text{T2}) \quad \forall \ v \in s(r^{-1}(\bar{v}) \setminus \{\bar{e}\}) : \ s^{-1}(v) \setminus r^{-1}(\bar{v}) \neq \emptyset.
\end{align*}
\]

We call \( C^*(E) \) a \( \bar{v} \)-trimmable graph \( C^* \)-algebra iff \( E \) is \( \bar{v} \)-trimmable.

Note that conditions (T1) and (T2) imply that \( \{\bar{v}\} \) is a saturated hereditary subset of \( E_0 \). Furthermore, if (T2) is not satisfied, i.e. for some \( v \in s(r^{-1}(\bar{v}) \setminus \{\bar{e}\}) \) the set difference \( s^{-1}(v) \setminus r^{-1}(\bar{v}) \) is empty, then the quotient map \( C^*(E) \to C^*(E/\{\bar{v}\}) \) would not be well defined as it would map all elements \( S_y \), where \( y \in r^{-1}(\bar{v}) \), to zero, thus violating the Cuntz-Krieger relations for \( C^*(E/\{\bar{v}\}) \).

There is an ample supply of trimmable graphs because, given any finite graph \( E'' \), we can create a trimmable graph \( E \) by taking a one-loop extension of \( E'' \). We define one-loop extensions in the spirit of one-sink extensions defined in [36, Definition 1.1].

**Definition 2.2.** Let \( E'' \) be a finite graph. A finite graph \( E \) is called a one-loop extension of \( E'' \) iff the following conditions are satisfied:

1. \( E'' \) is a subgraph of \( E \),
2. \( E_0 \setminus E''_0 = \{\bar{v}\} \) (there is only one vertex outside of \( E'' \)).
(3) \(s^{-1}(\bar{v}) = \{\bar{e}\}\) and \(r(\bar{e}) = \bar{v}\) (the only edge outgoing from \(\bar{v}\) is a loop),
(4) \(r^{-1}(\bar{v}) \setminus \{\bar{e}\} \neq \emptyset\) (there is at least edge connecting \(E''\) with \(\bar{v}\)),
(5) if \(v\) is a sink in \(E''\), then it remains a sink in \(E\) (equivalent to the condition (T2)).

Note that, for any trimmable graph \(E\), there is an intermediate graph \(E'\) that is a subgraph of \(E\) and a one-sink extension of \(E''\).

2.1. A \(K_1\)-generator for trimmable graphs without sinks. Given a \(\bar{v}\)-trimmable graph \(E\), the Cuntz–Krieger relations imply that the partial isometry associated to the loop \(\bar{e}\) based at \(\bar{v}\) is a normal operator. This fact can be used to construct a distinguished class in \(K_1(C^*(E))\).

**Proposition 2.3.** Let \(E\) be a \(\bar{v}\)-trimmable graph without sinks. Then \(K_1(C^*(E))\) contains a copy of \(\mathbb{Z}\) generated by the class of the unitary

\[U = S_{\bar{e}} + (1 - S_{\bar{e}}S_{\bar{e}}').\]

**Proof.** Let \(v_i, i = 1, \ldots, n - 1\), be the vertices of \(E\) different from \(\bar{v}\) (recall that \(E\) is finite).

By the trimmability conditions (T1) and (T2) in Definition 2.1, the incidence matrix for the graph \(E\) is an \(n \times n\) matrix of the form

\[A_E = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
A(1, 0) & & & & & \\
\vdots & & & & & \\
A(n - 1, 0) & & & & & \\
\end{pmatrix}.
\]

Here \(\bar{v}\) vertex is listed first, \(A_{E''}\) is the incidence matrix of the quotient graph \(E'' := E/\{\bar{v}\}\) and \(A(i, 0) := \#\{e \in E^1 : r(e) = \bar{v}, s(e) = v_i\}\). The actual values of the \(A(i, 0)\) do not matter for our proof. The above implies that the first column of the matrix \(1 - A_E^t\) contains only zeros. This means that the vector \((1, 0, \ldots, 0)\) generates a copy of \(\mathbb{Z}\) inside \(\ker(1 - A_E^t)\). Since the graph \(E\) has no sinks, the \(C^*\)-algebra \(C^*(E)\) is isomorphic to the Cuntz–Krieger algebra of the edge matrix \(A_E\) of the graph, so \(\ker(1 - A_E^t) \cong K_1(C^*(E))\) by [15, Proposition 3.1].

Next, let us consider the partial isometry \(S_{\bar{e}}\) associated to the loop \(\bar{e}\) based at the distinguished vertex \(\bar{v}\). Since \(\bar{v}\) does not emit any other edge besides the loop \(\bar{e}\), we deduce from the Cuntz–Krieger relations that \(S_{\bar{e}}\) is a normal element. Then one readily checks that the element \(U := S_{\bar{e}} + (1 - S_{\bar{e}}S_{\bar{e}}')\) is unitary. The argument outlined in [37, Section 2] allows us to conclude that \(U\) generates a copy of \(\mathbb{Z}\) inside \(K_1(C^*(E))\) corresponding to the vector \((1, 0, \ldots, 0)\). \[\blacksquare\]

2.2. An equivariant pullback structure. In this section, we prove that every \(\bar{v}\)-trimmable graph \(C^*\)-algebra \(C^*(E)\) is \(U(1)\)-equivariantly isomorphic to the pullback \(C^*\)-algebra of \(C^*(E') \otimes C(S^1)\) and \(C^*(E'')\) over \(C^*(E'') \otimes C(S^1)\), where \(E'\) is the subgraph of \(E\) obtained by removing the loop \(\bar{e}\) and \(E'' := E/\{\bar{v}\}\). For this statement, we need to introduce \(U(1)\)-equivariant \(*\)-homomorphisms which give the aforementioned pullback structure.
We begin with the map that dualizes the gauge action $\alpha$, namely the gauge coaction $\delta$:

$$\delta : C^*(E) \longrightarrow C^*(E) \otimes C(S^1),$$

given on generators by

$$\delta(S_e) = S_e \otimes u, \quad \delta(P_v) = P_v \otimes 1, \quad \text{for all } e \in E_1 \text{ and } v \in E_0.$$ 

Here we denote by $u$ the standard unitary generator of $C(S^1)$ and by $\{S, P\}$ the Cuntz–Krieger $E$-family. The gauge coaction is $U(1)$-equivariant with respect to the gauge action on $C^*(E)$ and the action on $C^*(E) \otimes C(S^1)$ given by the standard action on the rightmost tensorand.

The singleton set $\{\bar{v}\}$ is a saturated hereditary subset of both $E_0$ and $E'_0$ (note that $E/\{\bar{v}\} = E'/\{\bar{v}\}$) and, by the one-to-one correspondence of gauge-invariant ideals and saturated hereditary subsets, the quotient maps

$$\pi_1 : C^*(E) \longrightarrow C^*(E''), \quad \pi_2 : C^*(E') \longrightarrow C^*(E''),$$

are $U(1)$-equivariant with respect to the gauge actions on $C^*(E)$, $C^*(E')$ and $C^*(E'')$. In the forthcoming theorem, we will consider a $^\ast$-homomorphism

$$\pi_2 \otimes \text{id} : C^*(E') \otimes C(S^1) \longrightarrow C^*(E'') \otimes C(S^1)$$

viewed as a $U(1)$-equivariant map with respect to the standard $U(1)$-action on $C(S^1)$.

Finally, using the condition (T1), one readily verifies that the assignment

$$f(P_v) = P_v \otimes 1, \quad v \in E_0, \quad f(S_e) = \begin{cases} P_{\bar{e}} \otimes u & \text{if } e = \bar{e}, \\ S_e \otimes u & \text{if } e \in E'_1, \end{cases}$$

defines a map

$$f : C^*(E) \longrightarrow C^*(E') \otimes C(S^1),$$

as it preserves the relations (CK1)-(CK2) of the graph $^\ast$-algebra $C^*(E)$. It is equally straightforward to check that $f$ is equivariant with respect to the gauge action on $C^*(E)$ and the action on $C^*(E') \otimes C(S^1)$ given by the standard action on the rightmost tensorand.

**Theorem 2.4.** Let $E$ be a $\bar{v}$-trimmable graph, $E'$ the subgraph of $E$ obtained by removing the unique outgoing loop $\bar{e}$, and $E''$ the subgraph of $E'$ obtained by removing the vertex $\bar{v}$ and all edges ending in $\bar{v}$. Then the following diagram of the above-defined $U(1)$-equivariant $^\ast$-homomorphisms

$$\begin{array}{ccc}
C^*(E) & \xrightarrow{f} & C^*(E') \otimes C(S^1) \\
\pi_1 & & \pi_2 \otimes \text{id} \\
\delta & & \\
C^*(E'') & \xleftarrow{\pi_2} & C^*(E'') \otimes C(S^1)
\end{array}$$

(2.1)

is a pullback diagram of $U(1)$-$^\ast$-algebras.
Proof. The above diagram is clearly commutative, so there is a \(*\)-homomorphism \(F\) mapping \(C^*(E)\) into the pullback \(C^*\)-algebra. Injectivity of \(F\) follows from injectivity of \(f\), which is a consequence of the gauge-invariant uniqueness theorem (Theorem 1.2). Furthermore, since \(\pi_1\) and \(\pi_2 \otimes \text{id}\) are surjective, using a \(C^*\)-algebraic incarnation [33, Proposition 3.1] of a well-known characterization of when a commutative diagram is a pullback diagram (e.g. see [24, Lemma 4.1]), to prove the surjectivity of \(F\), it only remains to check whether

\[
\ker(\pi_2 \otimes \text{id}) \subseteq f(\ker(\pi_1)). \tag{2.2}
\]

Again, we observe that \(\{\bar{v}\}\) is a saturated hereditary subset of both \(E_0\) and \(E'_0\). Therefore, it generates gauge-invariant ideals \(I_E(\bar{v})\) and \(I_{E'}(\bar{v})\) in \(C^*(E)\) and \(C^*(E')\) respectively. It follows from (1.4) and (1.5) that

\[
\ker(\pi_1) = I_E(\bar{v}) \quad \text{and} \quad \ker(\pi_2) = I_{E'}(\bar{v}),
\]

so every element in \(I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}]\) is a linear combination of elements of the form

\[
\left(\sum_{i=1}^{n} k_i S_{x_i} S_{y_i}^*\right) \otimes u^m, \quad k_i \in \mathbb{C}, \quad x_i, y_i \in \text{Path}(E'), \quad r(x_i) = r(y_i) = \bar{v}, \quad m \in \mathbb{Z}.
\]

Given an element as above, we can show that it belongs to \(f(I_E(\bar{v}))\) using the following equality

\[
\sum_{i=1}^{n} f(S_{x_i} S_{y_i}^{m-\|x_i\|\|y_i\|} S_{y_i}) = \left(\sum_{i=1}^{n} k_i S_{x_i} S_{y_i}^*\right) \otimes u^m.
\]

Hence \(I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}] \subseteq f(I_E(\bar{v}))\). Finally, to prove (2.2), we compute

\[
\ker(\pi_2 \otimes \text{id}) = \ker(\pi_2) \otimes C(S^1) = I_{E'}(\bar{v}) \otimes \mathbb{C}[u, u^{-1}] \subseteq f(I_E(\bar{v})) = f(\ker(\pi_1)).
\]

Here the last equality follows from the fact that the image of a \(C^*\)-algebra under any \(*\)-homomorphism is closed. \(\blacksquare\)

As a corollary, by equivariance of all maps in diagram (2.1) and compactness of \(U(1)\), we obtain a pullback diagram at the level of fixed-point subalgebras:

**Corollary 2.5.** The diagram of \(*\)-homomorphisms

\[
\begin{array}{ccc}
C^*(E)^{U(1)} & \xleftarrow{} & C^*(E''')^{U(1)} \\
\xrightarrow{\bar{\pi}_1} & & \xrightarrow{\bar{\pi}_2} \\
C^*(E') & \xleftarrow{} & C^*(E'')
\end{array}
\]

is a pullback diagram of \(C^*\)-algebras. Here \(\bar{\pi}_1\) and \(\bar{f}\) denote \(*\)-homomorphisms that are restrictions-corestrictions to the fixed-point subalgebras of \(\pi_1\) and \(f\) respectively, and \(\iota\) is the subalgebra inclusion.
2.3. Mayer–Vietoris exact sequences in K-theory. Any pullback diagram of C*-algebras induces a six-term exact sequence in K-theory that goes under the name of Mayer–Vietoris exact sequence (see for example [5, Section 1.3], [6, Section 1.2.3] and [7, Theorem 21.5.1]). In this subsection, we describe the Mayer–Vietoris exact sequence for trimmable graph C*-algebras and their gauge-invariant subalgebras.

Let $E$ be a $\bar{v}$-trimmable graph. The Mayer–Vietoris six-term exact sequence associated to the diagram (2.1) reads

\[
\begin{align*}
K_0(C^*(E)) & \xrightarrow{\partial_{10}} K_0(C^*(E')) \oplus K_0(C^*(E') \otimes C(S^1)) \\
& \xrightarrow{\delta_{10}-(\pi_2 \otimes \text{id})} K_0(C^*(E') \otimes C(S^1)) \\
& \xrightarrow{(\pi_1, f_*)} K_1(C^*(E')) \oplus K_1(C^*(E') \otimes C(S^1)) \\
& \xleftarrow{\delta_{11}} K_1(C^*(E')).
\end{align*}
\]

Here $\partial_{10}$ is a Milnor connecting homomorphism and $\delta_{11}$ is a Bott connecting homomorphism. Furthermore, the pullback diagram (2.3) of fixed-point subalgebras leads to another six-term exact sequence in K-theory:

\[
\begin{align*}
K_0(C^*(E)^{U(1)}) & \xrightarrow{\partial_{10}} K_0(C^*(E'')^{U(1)}) \oplus K_0(C^*(E')) \xrightarrow{\pi_2 \otimes -1} K_0(C^*(E'')) \\
& \xrightarrow{\partial_{01}} K_1(C^*(E'')) \oplus K_1(C^*(E')) \xrightarrow{\pi_1, f_*} K_1(C^*(E''))^{U(1)}.
\end{align*}
\]

Next, since gauge-invariant subalgebras of graph C*-algebras are always AF-algebras, their odd-K-groups vanish, so we obtain a simpler six-term exact sequence:

\[
\begin{align*}
K_0(C^*(E)^{U(1)}) & \xrightarrow{\partial_{10}} K_0(C^*(E'')^{U(1)}) \oplus K_0(C^*(E')) \xrightarrow{\pi_2 \otimes -1} K_0(C^*(E'')) \\
& \xleftarrow{\pi_2 \otimes -1} K_1(C^*(E')) \xrightarrow{\pi_1, f_*} 0.
\end{align*}
\]

3. Examples

3.1. A one-loop extension of the Cuntz algebra $O_2$. Recall that the Cuntz algebra $O_2$ can be viewed as the graph C*-algebra of the graph $\Lambda''$ consisting of two loops starting at the unique vertex $w$. Let us now consider the one-loop extension $\Lambda$ of this graph obtained by adding one outgoing edge from $w$ to $\bar{v}$ (see Figure 1).

\[\text{Figure 1. The graph } \Lambda.\]
Denote by $\Lambda'$ the one-sink extension of $\Lambda''$ obtained by adding one outgoing edge from $w$ to $\bar{v}$. By Theorem 2.4, we have the following $U(1)$-equivariant pullback structure:

\[
\begin{array}{ccc}
\mathcal{O}_2 & \xleftarrow{\delta} & \mathcal{O}_2 \\
C^*(\Lambda) & \xrightarrow{\delta} & C^*(\Lambda') \otimes C(S^1)
\end{array}
\]

The fixed-point subalgebra $\mathcal{O}_2^{U(1)}$ is isomorphic to the CAR algebra $M_{2\infty}(\mathbb{C})$ [12, §1.5]. Hence, by Corollary 2.5, we have another pullback diagram:

\[
\begin{array}{ccc}
\mathcal{O}_2 & \xleftarrow{\delta} & \mathcal{O}_2 \\
C^*(\Lambda) & \xrightarrow{\delta} & C^*(\Lambda')
\end{array}
\]

The diagram (2.6) for this example reads

\[
\begin{array}{ccc}
K_0(C^*(\Lambda)^{U(1)}) & \xrightarrow{\delta} & K_0(M_{2\infty}(\mathbb{C})) \oplus K_0(C^*(\Lambda')) \\
K_1(\mathcal{O}_2) & \xrightarrow{\delta} & K_1(C^*(\Lambda')) \xleftarrow{\delta} 0
\end{array}
\]

Now, to compute $K_0(C^*(\Lambda)^{U(1)})$, we observe that:

1. $K_0(\mathcal{O}_2) = 0 = K_1(\mathcal{O}_2)$ [13], [14, Theorem 3.7, Theorem 3.8],
2. $K_0(M_{2\infty}(\mathbb{C})) \cong \mathbb{Z}[\frac{1}{2}]$, where $\mathbb{Z}[\frac{1}{2}]$ is the group of dyadic rationals (e.g., see [19, Example IV.3.4]),
3. $K_0(C^*(\Lambda')) = \mathbb{Z}[P_{\bar{v}}]$ and $K_1(C^*(\Lambda')) = 0$, where $P_{\bar{v}}$ is the vertex projection of $\bar{v}$.

Here the last statement follows from [36, Lemma 5.2] because $\Lambda'$ is a one-sink extension of $\Lambda''$. Hence, from the diagram (3.3), we conclude:

**Proposition 3.1.** Let $C^*(\Lambda)$ be the graph $C^*$-algebra of the graph given by Figure 1. The gauge-invariant subalgebra $C^*(\Lambda)^{U(1)}$ has the following even-$K$-group:

\[
K_0(C^*(\Lambda)^{U(1)}) \cong K_0(M_{2\infty}(\mathbb{C})) \oplus K_0(C^*(\Lambda')) \cong \mathbb{Z}[-\frac{1}{2}] \oplus \mathbb{Z}[P_{\bar{v}}].
\]

The above result agrees with [32, Proposition 4.1.2], where the K-theory of fixed-point subalgebras of arbitrary Cuntz–Krieger algebras (note that $\Lambda$ is finite and without sinks) was computed using a dual Pimsner–Voiculescu sequence.
3.2. A one-loop extension of the Toeplitz algebra $\mathcal{T}$. Let us now consider an example that goes beyond [32, Proposition 4.1.2] and connects with Section 3.4.2. To this end, we take the standard graph $Q''_2$ encoding the Toeplitz algebra $\mathcal{T}$, that is the graph consisting of two vertices $v^0_0$ and $v^1_0$, one loop $e^0_0$ based at $v^0_0$, and one edge $e^0_01$ from $v^0_0$ to $v^1_0$. Note that $v^1_0$ is a sink, so $\mathcal{T}$ is not a Cuntz–Krieger algebra. Next, we consider the one-loop extension $Q_2$ of this graph obtained by adding one outgoing edge from $v^0_0$ to $v^1_0$ (see Figure 2). Denote by $Q'\mathcal{Q}_2$ the one-sink extension of $Q''_2$ obtained by adding one outgoing edge from $v^0_0$ to $v^1_0$. The C*-algebra of the graph $Q'_2$ is isomorphic to the C*-algebra of the equatorial Podleš quantum sphere $S^2_{q\infty}$ [34, 28].

Figure 2. The graph $Q_2$.

By Theorem 2.4, we have the following $U(1)$-equivariant pullback structure:

$$
\begin{array}{ccc}
\mathcal{C}^*(Q_2) & \rightarrow & \mathcal{C}(S^2_{q\infty}) \otimes \mathcal{C}(S^1)
\\
\mathcal{T} & \downarrow & \mathcal{T} \otimes \mathcal{C}(S^1)
\\
\mathcal{C}(X) & \leftarrow & \mathcal{C}(S^2_{q\infty})
\end{array}
$$

(3.4)

To compute the fixed-point subalgebra $\mathcal{T}^{U(1)}$, we take advantage of (1.3) and combine it with the fact that $\mathcal{T}$ is the unital universal C*-algebra generated by a single isometry $s$ [11]. Indeed, identifying the isometry $s$ with the sum of partial isometries $S_{e^0_0} + S_{e^0_01}$, one easily computes that $\mathcal{T}^{U(1)} = \text{span} \{s^k(s^*)^k : k \in \mathbb{N}\}$, where $s^0(s^*)^0 = 1$. Hence $\mathcal{T}^{U(1)}$ is a commutative AF-algebra generated by countably many orthogonal projections, so it is isomorphic to the C*-algebra of continuous complex-valued functions on the Cantor set $X$. Thus, by Corollary 2.5, we have another pullback diagram:

$$
\begin{array}{ccc}
\mathcal{C}^*(Q_2)^{U(1)} & \rightarrow & \mathcal{C}(S^2_{q\infty})
\\
\mathcal{C}(X) & \leftarrow & \mathcal{C}(S^2_{q\infty})
\\
\mathcal{T} & \downarrow & \mathcal{T}
\end{array}
$$

(3.5)
The diagram (2.6) for this example reads

\[
\begin{array}{ccc}
K_0(C_*(Q_2)^{U(1)}) & \longrightarrow & K_0(C(X)) \oplus K_0(C(S^2_{q\infty})) \\
\downarrow & & \downarrow \\
K_1(T) & \leftarrow & K_1(C(S^2_{q\infty})) \\
\end{array}
\]

The K-groups of all the algebras involved here except for \( C_*(Q_2)^{U(1)} \) are as follows:

1. \( K_0(T) = \mathbb{Z}[1] \) and \( K_1(T) = 0 \),
2. \( K_0(C(X)) \cong \bigoplus \mathbb{Z} \),
3. \( K_0(C(S^2_{q\infty})) \cong \mathbb{Z}^2 \) and \( K_1(C(S^2_{q\infty})) = 0 \).

Hence there is a short exact sequence

\[
0 \longrightarrow K_0(C_*(Q_2)^{U(1)}) \longrightarrow \bigoplus \mathbb{Z} \longrightarrow \mathbb{Z}[1] \longrightarrow 0,
\]

so \( K_0(C_*(Q_2)^{U(1)}) \) is a countably-generated subgroup of a free abelian group. Thus we have proved:

**Proposition 3.2.** Let \( C_*(Q_2) \) be the graph \( C^*-\)algebra of the graph given by Figure 2. The gauge-invariant subalgebra \( C_*(Q_2)^{U(1)} \) has the following even-K-group:

\[
K_0(C_*(Q_2)^{U(1)}) \cong \bigoplus \mathbb{Z}.
\]

To end with, let us observe that, since any AF-algebra is Morita equivalent to a graph \( C^*-\)algebra by [20, Theorem 1], in principle, one could compute \( K_0(C_*(Q_2)^{U(1)}) \) using [20]. However, such a calculation might still be difficult, and our method reduces the calculation of the even K-group of the gauge-invariant subalgebra of a graph algebra to an analogous calculation for a **simpler** graph algebra. In this particular case, the latter calculation is immediate.

### 3.3. The Vaksman–Soibelman quantum spheres and projective spaces.

In 1991, Vaksman and Soibelman [38] defined a class of odd-dimensional quantum spheres \( S^q_{2n+1} \), where \( n \) is a non-negative integer. Their \( C^*-\)algebras can be viewed as \( q \)-deformations of the \( C^*-\)algebras of continuous functions on odd-dimensional spheres \( S^{2n+1} \), where \( q \in [0,1] \) is a deformation parameter. A decade later, Hong and Szymański [28] showed that in any dimension and for \( q \in [0,1] \) these spheres can be realised as graph \( C^*-\)algebras. In the same paper, they define even-dimensional noncommutative balls \( C(B^q_{2n}) \) using noncommutative double suspension, and in [31] they give their graph \( C^*-\)algebraic presentation.

#### 3.3.1. Spheres.

The \( C^*-\)algebra \( C(S^q_{2n+1}) \) of the \((2n+1)\)-dimensional quantum sphere is isomorphic, for any \( q \in [0,1] \), to the graph \( C^*-\)algebra of the graph \( L_{2n+1} \) [28, Theorem 4.4] (see Figure 3) with

- \( n + 1 \) vertices \( \{v_0, v_1, \ldots, v_n\} \),
- one edge \( e_{i,j} \) from \( v_i \) to \( v_j \) for all \( 0 \leq i < j \leq n \),
- one loop \( e_i \) over each vertex \( v_i \) for all \( 0 \leq i \leq n \).
Throughout this subsection, we denote the Cuntz–Krieger \( L_{2n+1} \)-family by \( \{S, P\} \). To simplify notation, we set \( S_{e_{i,j}} := S_{i,j}, S_{e_j} := S_j \) and \( P_{e_j} := P_j \), where \( 0 \leq i < j \leq n \).

The \( \mathcal{C}^* \)-algebra \( \mathcal{C}(B_{2n}^q) \) of the Hong–Szymański \( 2n \)-dimensional quantum ball [31, 30] can be viewed as the graph \( \mathcal{C}^* \)-algebra of the graph \( \Gamma_{2n} \) obtained from \( L_{2n+1} \) by removing the loop \( e_n \) (see Figure 4).

Since the graph \( L_{2n+1} \) giving the Vaksman–Soibelman \( (2n+1) \)-sphere is \( v_n \)-trimmable, we immediately conclude from Theorem 2.4 that the diagram

\[
\begin{array}{ccc}
C(S_{2n+1}^q) & \xrightarrow{\pi_1^q} & C(S_{2n-1}^q) \\
\downarrow{f^n} & & \downarrow{\delta^n} \\
C(S_{2n-1}^q) & \xrightarrow{\pi_2^q \otimes \text{id}} & C(B_{2n}^q) \otimes C(S^1)
\end{array}
\]

is a pullback diagram of \( U(1) \)-\( \mathcal{C}^* \)-algebras. All the maps in the above diagram are special cases of the maps in the diagram (2.1).

In this example, the six-term exact sequence in K-theory given by the diagram (2.4) reads

\[
\begin{array}{ccc}
K_0(C(S_{2n+1}^q)) & \xrightarrow{(\pi_1^q, f^n_+)} & K_0(C(S_{2n-1}^q)) \oplus K_0(C(B_{2n}^q) \otimes C(S^1)) \\
\oplus K_0(C(S_{2n-1}^q) \oplus C(S^1)) & \xrightarrow{\delta^n} & K_0(C(S_{2n-1}^q) \oplus C(S^1)) \\
\downarrow{\partial_0} \quad \downarrow{\partial_0} & & \downarrow{\partial_1} \\
K_1(C(S_{2n-1}^q) \oplus C(S^1)) & \xleftarrow{\partial_0} & K_1(C(S_{2n-1}^q) \oplus K_1(C(B_{2n}^q) \otimes C(S^1)) \oplus \partial_1 & \xleftarrow{\delta^n} & K_1(C(S_{2n+1}^q)).
\end{array}
\]
This diagram allows us to compute inductively an explicit formula of $K_1$-generators of the quantum odd spheres, which then could be compared with results in [27, Section 4.3] and [10].

3.3.2. Projective spaces. We define the C*-algebra $C(\mathbb{C}P^n_q)$ of the Vaksman–Soibelman quantum complex projective space $\mathbb{C}P^n_q$ [38] as the fixed-point subalgebra of $C(S^{2n+1}_q)$ under the gauge action of $U(1)$. It can be viewed as the graph C*-algebra of the graph $M_n$ consisting of the same vertices $v_0, ..., v_n$ as in the graph $L_{2n+1}$ (vertex projections are gauge invariant), with no loops, and countably many edges between all pairs of vertices [28]. (See Figure 5.)

![Figure 5. The graph $M_n$ of $C(\mathbb{C}P^n_q)$.
](image)

From the equivariance of the diagram (3.8), we conclude:

**Proposition 3.3.** The C*-algebra of the Vaksman–Soibelman quantum complex projective space $C(\mathbb{C}P^n_q)$ has the following pullback structure

\[
\begin{array}{ccc}
C(\mathbb{C}P^n_q) & \xrightarrow{\pi_1^n} & C(\mathbb{C}P^{n-1}_q) \\
\downarrow{\bar{\pi}} & & \downarrow{\bar{\theta}} \\
C(S^{2n-1}_q) & \xleftarrow{\iota^n} & C(B^{2n}_q), \\
\end{array}
\]

where all the maps above are analogous to the ones in the diagram (2.3). Furthermore, we obtain

\[
K_0(C(\mathbb{C}P^n_q)) \cong K_0(C(\mathbb{C}P^{n-1}_q)) \oplus \partial_{10}(K_1(C(S^{2n-1}_q))),
\]

where $\partial_{10}$ is Milnor’s connecting homomorphism.

**Proof.** The first part of the statement follows from Corollary 2.5. Recall from [38] and [30, 31] that, for all $n$, we have that $K_0(C(S^{2n-1}_q)) = \mathbb{Z}[1]$, $K_1(C(S^{2n-1}_q)) \cong \mathbb{Z}$, and that $K_0(C(B^{2n}_q)) = \mathbb{Z}[1]$ and $K_1(C(B^{2n}_q)) = 0$. Note also that there is a short exact sequence (e.g., see [28])

\[
0 \to I(v_n) \cong K \to C(\mathbb{C}P^n_q) \xrightarrow{\pi_1^n} C(\mathbb{C}P^{n-1}_q) \to 0,
\]
where $\mathcal{K}$ is the C*-algebra of compact operators. Now, the associated six-term exact sequence and the vanishing of $K_1(\mathcal{K})$ imply that $K_0(C(\mathbb{C}P^n_q)) \cong \mathbb{Z}^{n+1}$ and $K_1(C(\mathbb{C}P^n_q)) = 0$, and that the map $\widetilde{\pi}^n_{1,*} : K_0(C(\mathbb{C}P^n_q)) \to K_0(C(\mathbb{C}P^{n-1}_q))$ is surjective.

Let us consider the Mayer–Vietoris six-term exact sequence associated to the pullback diagram (3.9):

\[
\begin{array}{cccc}
K_0(C(\mathbb{C}P^n_q)) & \xrightarrow{\partial_{10}} & K_0(C(\mathbb{C}P^{n-1}_q)) \\
\text{(3.12)} & \downarrow & \downarrow \\
K_1(C(\mathbb{C}P^{n-1}_q)) & \xleftarrow{\pi^n_{1,*}} & K_1(C(\mathbb{C}P^n_q)) & \oplus K_1(C(\mathbb{C}P^{n-1}_q)) & \xrightarrow{\pi^n_{1,*}} & K_0(C(\mathbb{C}P^n_q)).
\end{array}
\]

We are going to prove formula (3.10) by extracting from (3.12) the following split short exact sequence and the vanishing of $\pi^n_{1,*}$:

\[
\begin{array}{cccc}
0 & \rightarrow & K_1(C(\mathbb{C}P^{n-1}_q)) & \xrightarrow{\partial_{10}} & K_0(C(\mathbb{C}P^n_q)) \\
\text{(3.13)} & & & \rightarrow & K_0(C(\mathbb{C}P^n_q)) & \rightarrow & 0.
\end{array}
\]

We already know that $\widetilde{\pi}^n_{1,*}$ is surjective, so to prove the exactness of (3.13), it suffices to show that the kernel of $(\pi^n_{1,*}, \widetilde{f}^n_*)$ is the same as the kernel of $\pi^n_{1,*}$. To this end, since

$$\ker\left(\widetilde{\pi}^n_{1,*}, \widetilde{f}^n_*\right) = \ker\pi^n_{1,*} \cap \ker\widetilde{f}^n_*,$$

we want to show the inclusion $\ker\pi^n_{1,*} \subseteq \ker\widetilde{f}^n_*$. It follows from the pullback diagram (3.9) and the functoriality of K-theory:

$$\ker\pi^n_{1,*} \subseteq \ker(\nu^n_* \circ \pi^n_{1,*}) = \ker(\widetilde{f}^n_* \circ (\pi^n_2)_*) = \ker(\widetilde{f}^n_*).$$

Here the last equality holds because $(\pi^n_2)_*$ is an isomorphism. Finally, the exact sequence (3.13) splits by the freeness of the $\mathbb{Z}$-module $K_0(C(\mathbb{C}P^n_q))$.

3.3.3. Milnor’s clutching construction for generators of $K_0(C(\mathbb{C}P^n_q))$. Recall that there are $(n + 1)$-many projections $P_0, P_1, \ldots, P_n$, in the graph $M_n$ whose graph algebra is $C(\mathbb{C}P^n_q)$. Therefore, since $K_0(C(\mathbb{C}P^n_q)) \cong \mathbb{Z}^{n+1}$ and the $K_0$-group of a graph C*-algebra is generated by its vertex projections (see [10, Proposition 3.8 (1)]), we infer that

\[
(3.14) \quad K_0(C(\mathbb{C}P^n_q)) = \mathbb{Z}[P_0] \oplus \mathbb{Z}[P_1] \oplus \ldots \oplus \mathbb{Z}[P_n].
\]

We will now compute the explicit value of Milnor’s connecting homomorphism $\partial_{10}$ on a generator of $K_1(C(S^{2n-1}_q))$. Let us first recall that, by Proposition 2.3, the generator of $K_1(C(S^{2n-1}_q)) \cong \mathbb{Z}$ is given by the $K_1$-class of the unitary

$$U = S_{n-1} + (1 - P_{n-1}).$$

Next, using the pullback structure of $C(\mathbb{C}P^n_q)$, we follow [17, Section 2.1], that is we find $C, D \in C(B_{q}^{2n})$ such that $\pi_2(C) = U$ and $\pi_2(D) = U^*$:

\[
\begin{align*}
C &= S_{n-1} + S_{n-1,n} + (1 - P_{n-1} - P_n), \\
D &= C^* = S_{n-1}^* + S_{n-1,n}^* + (1 - P_{n-1} - P_n),
\end{align*}
\]

16
and compute the following 2 by 2 matrix with entries in $C(\mathbb{CP}_q^n)$:

$$
(3.15) \quad p_U = \begin{pmatrix}
(1, C(2 - DC)D) & (0, C(2 - DC)(1 - DC)) \\
(0, (1 - DC)D) & (0, (1 - DC)^2)
\end{pmatrix} = \begin{pmatrix}
(1, 1 - P_n) & (0, 0) \\
(0, 0) & (0, 0)
\end{pmatrix}.
$$

By [17, Theorem 2.2], the element $\partial_{10}([U]) = [p_U] - [1]$ is a generator of $K_0(C(\mathbb{CP}_q^n))$. Observe that in the above formula for $p_U$ we can remove all the entries except the top left one without changing its class in $K_0(C(\mathbb{CP}_q^{n-1}))$, namely $[1] - [p_U] = [1] - [p]$, where $p := (1, 1 - P_n) = (1, 1) - (0, P_n)$ is a projection in $C(\mathbb{CP}_q^n)$. Furthermore, since $p$ and $1 - p$ are orthogonal, we have $[p] + [1 - p] = [1]$, so

$$
(3.16) \quad -\partial_{10}([U]) = [1] - [p] = [1 - p] = [(0, P_n)] = [P_n].
$$

Here the rightmost projection $P_n$ is viewed as the vertex projection corresponding to the vertex $v_n$ in the graph of $C(\mathbb{CP}_q^n)$, and the rightmost equality follows from the fact that the isomorphism from $C(\mathbb{CP}_q^n)$ to the pullback $C^*$-algebra of the digaram (3.9) maps $P_n$ to $(0, P_n)$.

Finally, let us observe that the Milnor’s idempotent in the above calculation is $p_U \cong 1 - P_n = P_0 + P_1 + \ldots + P_{n-1}$.

Hence the projective module it defines can be understood as the section module of a noncommutative vector bundle obtained by the Milnor clutching construction.

3.4. Quantum lens spaces and quantum teardrops. Quantum lens spaces, both weighted and unweighted, have been the subject of increasing interest in the last years. Their realisation as graph $C^*$-algebras has been first proven in [29], and then further generalised in [9] under less stringent assumptions. In the rest of the paper, we focus on the three-dimensional quantum lens spaces $L_q^3(l; 1, l)$. In [18], generators for the K-theory and K-homology of multi-dimensional quantum weighted projective spaces were constructed, leading to an extension of the K-theoretic computations for quantum weighted lens spaces in [2] in the Cuntz–Pimsner picture [3].

3.4.1. Lens spaces. Our starting point is the $C^*$-algebra $C(L_q^3(l; 1, l))$ of the quantum lens space $L_q^3(l; 1, l)$. As explained in [9, Example 2.1], $C(L_q^3(l; 1, l))$ can be viewed as the graph $C^*$-algebra of the graph $L_q^3$ (see Figure 6) with

- $l + 1$ vertices $\{v_0^0, v_1^0, \ldots, v_{l-1}^0\}$,
- one edge $e_i^{01}$ from $v_0^0$ to $v_i^1$ for all $0 \leq i \leq l - 1$,
- a loop $e_0^0$ over the vertex $v_0^0$ and one loop $e_i^1$ over each vertex $v_i^1$ for all $0 \leq i \leq l - 1$.

Observe that $C^*(L_q^3) \cong C(S_q^3)$ (e.g., see [28]). Throughout this subsection, we denote the Cuntz–Krieger $L_q^3$-family by $\{S, P\}$. To simplify notation we set $S_{e_0^1} := S_{e_0^0}^1$, $S_{e_j^1} := S_j^1$ and $P_{e_j^1} := P_j^1$, where $0 \leq i \leq 1$ and $0 \leq j \leq l - 1$.

Clearly, the graph $L_q^3$ is $v_i^1$-trimmable for $0 \leq j \leq l - 1$. Let us choose the vertex $v_{l-1}^1$, and construct a new graph by removing the loop $e_{l-1}^1$. We denote the thus obtained graph by $Q_l$ (see Figure 7). Note that the graph $Q_l$ and its $C^*$-algebra do not depend on our choice of a vertex.
Figure 6. The graph $L^{3}_{l}$.

Figure 7. The graph $Q_{l}$.

By Theorem 2.4, we obtain the following $U(1)$-equivariant pullback structure of the algebra $C(L^{3}_{q}(l; 1, l)) \cong C^{*}(L^{3}_{l})$:

\[
\begin{array}{ccc}
C^{*}(L^{3}_{l}) & \leftarrow & C^{*}(Q_{l}) \otimes C(S^{1}) \\
C^{*}(L^{3}_{l-1}) & \rightarrow & C^{*}(Q_{l}) \otimes C(S^{1})
\end{array}
\]

(3.17)

Here all the maps are analogous to the ones used in Theorem 2.4.

3.4.2. Teardrops. Recall that the C*-algebra $C(WP_{q}^{1}(1, l))$ of the weighted projective space $WP_{q}^{1}(1, l)$ [8] is defined as the $U(1)$-fixed-point subalgebra of $C(L^{3}_{q}(l; 1, l))$. It can be viewed as the graph C*-algebra of the graph $W_{l}$ consisting of the same vertices $v_{0}^{0}$, ..., $v_{l-1}^{1}$ as in the graph $L^{3}_{l}$ (vertex projections are gauge invariant), with no loops, and countably many edges between $v_{0}^{0}$ and all the other vertices [9]. (See Figure 8.)

Figure 8. The graph $W_{l}$ of $C(WP_{q}^{1}(1, l))$. 
As discussed in Section 2.2, from (3.17) we obtain another pullback diagram of fixed-point subalgebras:

\[
\begin{array}{ccc}
  C(WP^1_q(1,l)) & \xleftarrow{\chi_1} & C^*(Q_l) \\
  \text{\xleftarrow{\chi_1}} & & \text{\xrightarrow{\chi_2}} \\
  C(WP^1_q(1,l-1)) & \xrightarrow{\chi_1} & C^*(L^3_{l-1}) \\
  \text{\xrightarrow{\chi_2}} & & \text{\xleftarrow{\chi_1}} \\
  C^*(L^3_{l-1}) & \xrightarrow{\chi_2} & C(S^1) \\
\end{array}
\]  

(3.18)

Here \( \chi_1 \) is the quotient map by the ideal \( \overline{I(v^1_{l-1})} \).

Let us introduce two more maps needed for Lemma 3.4 below. Define \( \chi_2: C^*(Q_l) \to C^*(Q_l/H) \) to be the quotient map given by the ideal generated by the saturated hereditary subset \( H = \{v^1_0, v^1_1, \ldots, v^1_{l-2}\} \), and \( g: C^*(L^3_{l-1}) \to C^*(L^3/H) \) the quotient map given by the ideal generated by \( H \) considered as a subset in \( (L^3_{l-1})_0 \).

**Lemma 3.4.** The following diagram of the above-defined \( U(1) \)-equivariant *-homomorphisms

\[
\begin{array}{ccc}
  C^*(Q_l) & \xleftarrow{\chi_1} & C^*(L^3_{l-1}) \\
  \downarrow{\chi_2} & & \downarrow{\chi_2} \\
  C(S^1) & \xrightarrow{\sigma} & C(S^1) \\
\end{array}
\]  

(3.19)

is a pullback diagram. Here \( \mathcal{T} \) is the Toeplitz algebra and \( \sigma: \mathcal{T} \to C(S^1) \) is the symbol map.

**Proof.** Recall that the Toeplitz algebra \( \mathcal{T} \) is isomorphic to the graph C*-algebra corresponding to the quotient graph \( Q_l/H \) and the isomorphism is given by \( s \mapsto S^0_0 + S^0_{l-1} \) (e.g., see [31]), where \( s \) is the isometry generating \( \mathcal{T} \).

By [33, Proposition 3.1] and the surjectivity of \( g \) and \( \sigma \), it suffices to show that \( \ker \chi_1 \cap \ker \chi_2 = \{0\} \) and that \( \ker \chi_1 \subseteq \ker \sigma \). To prove the first condition, recall that, since \( \ker \chi_1 \) and \( \ker \chi_2 \) are closed ideals in a C*-algebra, we have that \( \ker \chi_1 \cap \ker \chi_2 = \ker \chi_1 \ker \chi_2 \). Next, \( \{v^1_{l-1}\} \) and \( H \) are saturated hereditary subsets of \( (Q_l)_0 \), so

\[
\ker \chi_1 = \overline{I_{Q_l}(v^1_{l-1})} \quad \text{and} \quad \ker \chi_2 = \overline{I_{Q_l}(H)}.
\]

Using (1.4), one can observe that an arbitrary element of \( \ker \chi_1 \ker \chi_2 \) is of the form \( s_\alpha S^\beta S^\gamma S^\delta \), where \( \alpha, \beta \in \text{Path}(Q_l) \) with \( r(\alpha) = r(\beta) = v^1_{l-1} \) and \( \gamma, \delta \in \text{Path}(Q_l) \) with \( r(\gamma) = r(\delta) \in H \). The claim follows from the analysis of all possible paths satisfying the above conditions.

To prove the second condition, notice that \( \ker \sigma = \overline{I_{Q_l/H}(v^1_{l-1})} \). Any element of \( I_{Q_l/H}(v^1_{l-1}) \) is an element of \( I_{Q_l}(v^1_{l-1}) \), and \( \chi_2(S_\alpha) = S_\alpha \) for all \( \alpha \in \text{Path}(Q_l/H) \). Hence

\[
\overline{I_{Q_l/H}(v^1_{l-1})} \subseteq \ker \chi_2.
\]
\[ \chi_2(I_Q(l_1)) \subseteq I_{Q_l/H}(v_{l-1}^1). \] Furthermore, since \( \chi_2 \) is a \(*\)-homomorphism, we can argue as in the proof of Theorem 2.4 to conclude that \( \chi_2(\ker \chi_1) \subseteq \ker \sigma. \)

Remark 3.5. Note that for \( l = 2 \), we get the graph \( Q_2 \) considered in Section 3.2. By Lemma 3.4, the \( \mathbb{C}^* \)-algebra \( \mathbb{C}^*(Q_2) \) has the following \( U(1) \)-equivariant pullback structure:

\[
\begin{array}{ccc}
C^*(Q_l) & \downarrow & T. \\
C(S_q^3) & \downarrow & \\
C(S^1)
\end{array}
\]

The equivariance of the above diagram allows it to descend to the fixed-point subalgebras:

\[
\begin{array}{ccc}
C^*(Q_l)^{U(1)} & \downarrow & T^{U(1)}. \\
C(\mathbb{C}P_q^1) & \downarrow & \\
\mathbb{C}
\end{array}
\]

The Mayer–Vietoris six-term exact sequence in K-theory associated to this diagram gives the K-groups of \( C^*(Q_2)^{U(1)} \) as in Proposition 3.2.

Next, using Lemma 3.4 along with [33, Proposition 2.7] and the analogous reasoning as in the proof of Lemma 3.3, we arrive at:

**Proposition 3.6.** The \( \mathbb{C}^* \)-algebra \( \mathbb{C}(\mathbb{W}P_q^1(1, l)) \) has the following pullback structure

\[
\begin{array}{ccc}
\mathbb{C}(\mathbb{W}P_q^1(1, l)) & \downarrow & T. \\
\mathbb{C}(\mathbb{W}P_q^1(1, l-1)) & \downarrow & \\
\mathbb{C}(S^1)
\end{array}
\]

Furthermore, we obtain

\[
K_0(\mathbb{C}(\mathbb{W}P_q^1(1, l))) \cong K_0(\mathbb{C}(\mathbb{W}P_q^1(1, l-1))) \oplus \partial_{10}(K_1(\mathbb{C}(S^1))),
\]

where \( \partial_{10} \) is Milnor’s connecting homomorphism.
3.4.3. Milnor’s clutching construction for generators of $K_0(C(\mathbb{W}P^1_q(1,l)))$. Recall that there are $(l + 1)$-many projections $P_0^0$, $P_1^0$, ..., $P_{l-1}^1$, in the graph $W_q$ whose graph algebra is $C(\mathbb{W}P^1_q(1,l))$. Therefore, since $K_0(C(\mathbb{W}P^1_q(1,l))) \cong \mathbb{Z}^{l+1}$ and the $K_0$-group of a graph C*-algebra is generated by its vertex projections (see [10, Proposition 3.8 (1)]), we infer that

$$(3.24) \quad K_0(C(\mathbb{W}P^1_q(1,l))) = \mathbb{Z}[P_0^0] \oplus \mathbb{Z}[P_1^1] \oplus \ldots \oplus \mathbb{Z}[P_{l-1}^1].$$

Below we compute the value of Milnor’s connecting homomorphism on the generator $[u] \in K_1(C(S^1))$, where $u$ is the standard generator of $C(S^1)$. This computation closely follows an analogous computation in Section 3.3.3. First, we find $c, d \in T$ such that

$$\sigma(c) = u \quad \text{and} \quad \sigma(d) = u^*,$$

and, using the formula (3.15), we compute the following 2 by 2 matrix with entries in $C(\mathbb{W}P^1_q(1,l))$:

$$p_u = \begin{bmatrix} (1, P_0^0) & (0, 0) \\ (0, 0) & (0, 0) \end{bmatrix}.$$ 

The element $\partial_{10}([u]) = [p_u] - [1]$ is a generator of $K_0(C(\mathbb{W}P^1_q(1,l)))$. We have that $[1] - [p_u] = [1] - [p]$, where $p := (1, P_0^0) = (1, 1) - (0, P_{l-1}^1)$ is a projection in $C(\mathbb{W}P^1_q(1,l))$. Notice that

$$-\partial_{10}([u]) = [1] - [p] = [(0, P_{l-1}^1)] = [P_{l-1}^1],$$

where $P_{l-1}^1$ is viewed as an element of $C(\mathbb{W}P^1_q(1,l))$.

Finally, as we did in Section 3.3.3, let us observe that the Milnor’s idempotent in the above calculation is

$$p_u \cong 1 - P_{l-1}^1 = P_0^0 + P_1^1 + \ldots + P_{l-2}^1.$$

**Acknowledgements.** This work is part of the project *Quantum Dynamics* partially supported by the EU-grant H2020-MSRA-RISE-2015-691246 and by the Polish Government grants 3542/H2020/2016/2 and 328941/PnH/2016. It was initiated while F.D., P.M.H., and M.T. were visiting Penn State University, and they are very grateful to the University for excellent working conditions and financial support. P.M.H. and M.T. are also very thankful to the University of Copenhagen for its financial support and amazing hospitality. Furthermore, F.A., P.M.H., and M.T. are happy to acknowledge a substantial logistic support of the Max Planck Institute for Mathematics in the Science in Leipzig, where much of joint research was carried out. Most importantly, our deepest gratitude goes to the graph C*-algebra gurus Sørn Eilers and Wojciech Szymański for their key technical support. Finally, F.A. would also like to thank Magnus Goffeng and Bram Mesland for valuable conversations about Cuntz–Krieger algebras.

**References**

[1] A. an Huef, I. Raeburn, The ideal structure of Cuntz–Krieger algebras, *Ergodic Theory Dynam. Sys.*, 17 (1997), 611–624.

[2] F. Arici, Gysin Exact Sequences for Quantum Weighted Lens Spaces. In: Wood D., de Gier J., Praeger C., Tao T. (eds) 2016 MATRIX Annals. MATRIX Book Series, vol 1. Springer.
[3] F. Arici, F. D’Andrea, G. Landi, Pimsner algebras and noncommutative circle bundles, In: D. Alpay, F. Cipriani, F. Colombo, D. Guido, I. Sabadini and J.-L. Sauvageot, editors, Noncommutative Analysis, Operator Theory and Applications, Springer (2016).
[4] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C*-algebras of row finite graphs, New York J. Math., 6 (2000), 307–324.
[5] P. F. Baum, P. M. Hajac, R. Matthes, W. Szymański, The K-theory of Heegaard-type quantum 3-spheres, K-Theory, 35 (2005), no. 1-2, 159–186.
[6] P. F. Baum, R. Meyer, The Baum–Connes conjecture, localisation of categories, and quantum groups, in: P. M. Hajac (ed.), Lecture Notes on Noncommutative Geometry and Quantum Groups, EMS Publ. House, to appear.
[7] B. Blackadar. K-theory of operator algebras. 2nd ed., Cambridge University Press, 1998.
[8] T. Brzeziński, S. A. Fairfax, Quantum Teardrops, Commun. Math. Phys., 316 (2012), 151–170.
[9] T. Brzeziński, W. Szymański, The C*-algebras of quantum lens and weighted projective spaces, J. Noncomm. Geom., in press, arXiv:1603.04678.
[10] T. M. Carlsen, S. Eilers, M. Tomforde, Index maps in the K-theory of graph algebras, J. K-Theory, 9 (2012), 385–406.
[11] L. A. Coburn, Weyl’s theorem for nonnormal operators, Michigan Math. J., 13 (1966), 285–288.
[12] J. Cuntz, Simple C*-algebras generated by isometries. Comm. Math. Phys., 57 (1977), no. 2, 173–185.
[13] J. Cuntz, Murray–von Neumann equivalence of projections in infinite simple C*-algebras. Rev. Roumaine Math. Pures Appl., 23 (1978), no. 7, 1011–1014.
[14] J. Cuntz, K-theory for certain C*-algebras. Ann. of Math., 2 113 (1981), no. 1, 181–197.
[15] J. Cuntz, A class of C*-algebras and topological Markov Chains II: reducible chains and the Ext-functor for C*-algebras, Invent. Math., 63 (1981), 25–40.
[16] J. Cuntz, W. Krieger, A class of C*-algebras and topological Markov chains, Invent. Math., 56 (1980), 251–268.
[17] L. Dabrowski, T. Hadfield, P. M. Hajac, R. Matthes, E. Wagner, Index pairings for pullback C*-algebras. Banach Center Publ., 98 (2012), 67–84.
[18] F. D’Andrea, G. Landi, Quantum weighted projective and lens spaces, Commun. Math. Phys., 340 (2015), 325–353, 2015.
[19] K. R. Davidson. C*-algebras by example. Fields Institute Monographs 6 (American Mathematical Society, Providence, RI, 1996).
[20] D. Drinen, Viewing AF-algebras as graph algebras, Proc. Amer. Math. Soc., 128 (2000), no. 7, 1991–2000.
[21] D. Drinen, M. Tomforde. Computing K-theory and Ext for graph C*-algebras. Illinois J. Math., 46 (2002), no. 1, 81–91.
[22] C. Farsi, P. M. Hajac, T. Maszczyk, B. Zieliński, Rank-two Milnor idempotents for the multipullback quantum complex projective plane, 2015. arXiv:1512.08816.
[23] P. M. Hajac. Bundles over quantum sphere and noncommutative index theorem. K-theory, 21 (2000), 141–150.
[24] P. M. Hajac, A. Kaygun, M. Tobolski, A graded pullback structure of Leavitt path algebras of trimmable graphs, 2018. arXiv:1803.10209.
[25] P. M. Hajac, R. Nest, D. A. Pask, A. Sims, B. Zieliński. The K-theory of twisted multipullback quantum odd spheres and complex projective spaces. *J. Noncommut. Geom.*, (2018). DOI: 10.4171/JNCG/292.

[26] P. M. Hajac, E. Wagner, The pullbacks of principal coactions, *Doc. Math.*, 19 (2014), 1025–1060.

[27] E. Hawkins, G. Landi. Fredholm modules for quantum Euclidean spheres. *J. Geom. Phys.*, 49 (2004), 272–293.

[28] J. H. Hong, W. Szymański, Quantum spheres and projective spaces as graph algebras, *Comm. Math. Phys.*, 232 (2002), 157–188.

[29] J.H. Hong, W. Szymański, Quantum lens spaces and graph algebras, *Pac. J. Math.*, 211 (2003), 249–263.

[30] J. H. Hong, W. Szymański, Noncommutative balls and their doubles, *Czech J. Phys.*, 56 (2006), 1173–1178.

[31] J. H. Hong, W. Szymański, Noncommutative balls and mirror quantum spheres, *J. Lond. Math. Soc.*, 77 (2008), 607–626.

[32] D. Pask, I. Raeburn, On the K-theory of Cuntz-Krieger algebras, *Publ. Res. Inst. Math. Sci.*, 32 (1996), no. 3, 415–443.

[33] G. K. Pedersen, Pullback and Pushout Constructions in C*-Algebra Theory, *J. Funct. Anal.*, 167 (1999), 243–344.

[34] P. Podleś, Quantum spheres, *Lett. Math. Phys.* 14 (1987), no. 3, 193–202.

[35] I. Raeburn, *Graph algebras*, CBMS Regional Conference Series in Mathematics 103, American Mathematical Society, Providence, RI, 2005.

[36] I. Raeburn, M. Tomforde, D. Williams, Classification theorems for the C*-algebras of graphs with sinks, *Bull. Austral. Math. Soc.*, 70 (2004), 143–161.

[37] M. Rørdam, Classification of Cuntz–Krieger algebras, *K-theory*, 9 (1995), 31–58.

[38] L. L. Vaksman, Ya. S. Soibelman, Algebra of functions on the quantum group SU(n + 1) and odd-dimensional quantum spheres, *Algebra i Analiz*, 2 (1990), 101–120.
(F. Arici) Max-Planck-Institut für Mathematik in den Naturwissenschaften, Inselstr. 22, 04103 Leipzig, Leipzig, Germany.

E-mail address: francesca.arici@mis.mpg.de

(F. D’Andrea) Università di Napoli “Federico II” and I.N.F.N. Sezione di Napoli, Complesso MSA, Via Cintia, 80126 Napoli, Italy.

E-mail address: francesco.dandrea@unina.it

(P. M. Hajac) Instytut Matematyczny, Polska Akademia Nauk, ul. Śniadeckich 8, Warszawa, 00-656 Poland

E-mail address: pmh@impan.pl

(M. Tobolski) Instytut Matematyczny, Polska Akademia Nauk, ul. Śniadeckich 8, Warszawa, 00-656 Poland

E-mail address: mtobolski@impan.pl