A CONJECTURAL CONNECTION BETWEEN $R^*(C^g_n)$ AND $R^*(\mathcal{M}^{rt}_{g,n})$

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INTRODUCTION

Let $\mathcal{M}^{rt}_{g,n}$ be the moduli space of stable $n$-pointed curves of genus $g > 1$ with rational tails. This space is a partial compactification of the space $\mathcal{M}_{g,n}$, classifying smooth curves of genus $g$ together with $n$ ordered distinct points. It sits inside the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. We also consider the space $C^g_n$ classifying smooth curves of genus $g$ with not necessarily distinct $n$ ordered points. There is a natural proper map from $\mathcal{M}^{rt}_{g,n}$ to $C^g_n$ which contracts all rational components. Tautological classes on these spaces are natural algebraic cycles reflecting the geometry of curves. Their definition and analysis on $\mathcal{M}_g$ and $\overline{\mathcal{M}}_g$ was started by Mumford in the seminal paper [4]. These were later generalized to pointed spaces by Faber and Pandharipande [2]. In this short note we study the connection between tautological classes on $\mathcal{M}^{rt}_{g,n}$ and $C^g_n$. We show that there is a natural filtration on the tautological ring of $\mathcal{M}^{rt}_{g,n}$ consisting of $g - 2 + n$ steps. A conjectural dictionary between tautological relations on $\mathcal{M}^{rt}_{g,n}$ and $C^g_n$ is presented. Our conjecture predicts that the space of relations in $R^*(\mathcal{M}^{rt}_{g,n})$ is generated by relations in $R^*(C^g_n)$ together with a class of relations obtained from the geometry of blow-ups. This conjecture is equivalent to the the independence of certain tautological classes in Chow. We prove the analogue version of our conjecture for the Gorenstein quotients of tautological rings.

Conventions 0.1. We consider algebraic cycles modulo rational equivalence. All Chow groups are taken with $\mathbb{Q}$-coefficients.

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1. Tautological algebras

1.1. The moduli spaces. Let $\pi : C_g \to M_g$ be the universal smooth curve of genus $g > 1$. For an integer $n > 0$ the $n$-fold fiber product of $C_g$ over $M_g$ is denoted by $C_g^n$. It parameterizes smooth curves of genus $g$ together with $n$ ordered points. We also consider the space $M_{g,n}^r$ which classifies stable $n$-pointed curves of genus $g$. This moduli space classifies nodal curves of arithmetic genus $g$ consisting of a unique component of genus $g$. This implies that all other components are rational. The marking points belong to the smooth locus of the curve. Every marking and a nodal point is called special. By the stability condition we require that every rational component has at least 3 special points. As a result the corresponding moduli point has finitely many automorphisms and the associated stack is of Deligne-Mumford type. Recall that the boundary $M_{g,n}^r \setminus M_{g,n}$ is a divisor with normal crossings. It is a union of irreducible divisor classes $D_I$ parameterized by subsets $I$ of the set $\{1, \ldots, n\}$ having at least 3 elements. The generic point of the divisor $D_I$ corresponds to a nodal curve with 2 components. One of its components is of genus $g$ and the other component is rational. The set $I$ refers to the markings on the rational component.

1.2. Tautological classes. Tautological classes are natural algebraic cycles on the moduli space. Their original definition on the spaces $M_g$ and its compactification $\overline{M}_g$ is given by Mumford in [4]. The definition of tautological algebras of moduli spaces of pointed curves is due to Faber and Pandharipande [2]. We will recall the definition below.

**Definition 1.1.** Let $g, n$ be integers so that $2g - 2 + n > 0$. The system of tautological rings is defined to be the set of smallest $\mathbb{Q}$-subalgebras $R^*(\overline{M}_{g,n})$ of the Chow rings $A^*(\overline{M}_{g,n})$ which is closed under push-forward via all maps forgetting points and all gluing maps.

**Definition 1.2.** The tautological ring $R^*(M_{g,n}^r)$ of $M_{g,n}^r$ is the image of $R^*(\overline{M}_{g,n})$ via the restriction map.

Recall that there is a proper map $M_{g,n}^r \to C_g^n$ which contracts all rational components of the curve.

**Definition 1.3.** The tautological ring $R^*(C_g^n)$ of $C_g^n$ is defined as the image of $R^*(M_{g,n}^r)$ via the push-forward map $A^*(M_{g,n}^r) \to A^*(C_g^n)$.

There is a more explicit presentation of the tautological ring of $C_g^n$ and $M_{g,n}^r$. Let $\pi : C_g \to M_g$ be the universal curve of genus $g$ as
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before. Denote by $\omega$ its relative dualizing sheaf. Its class in the Picard group of $C^n_g$ is denoted by $K$. This defines the class $K_i$ for every $1 \leq i \leq n$ on $C^n_g$. We also denote by $d_{i,j}$ the class of the diagonal for $1 \leq i < j \leq n$. The kappa class $\kappa_i$ is the push-forward class $\pi_\ast(K^{i+1})$. According to the vanishing results proven by Looijenga [3] the class $\kappa_i$ is zero when $i > g - 2$. We also have that tautological groups of $C^n_g$ vanish in degrees higher than $g - 2 + n$. The tautological group of $C^n_g$ and $M_{g,n}^{rt}$ in top degree $g - 2 + n$ are one dimensional. There are several other proofs of these facts. See [1] for a survey. We denote the pull-back of these classes to $M_{g,n}^{rt}$ along the contraction map by the same letters. The proof of the following fact is left to the reader:

**Lemma 1.4.**

(1) The tautological ring of $C^n_g$ is the $\mathbb{Q}$-subalgebra of its Chow ring $A^\ast(C^n_g)$ generated by kappa classes $\kappa_i$ for $1 \leq i \leq g - 2$ together with the classes $K_i$ and $d_{i,j}$ for $1 \leq i < j \leq n$.

(2) The tautological ring of $M_{g,n}^{rt}$ is the $\mathbb{Q}$-subalgebra of $A^\ast(M_{g,n}^{rt})$ generated by $\kappa_i$ for $1 \leq i \leq g - 2$, the divisor classes $K_i$, $d_{i,j}$ and $D_I$ for $1 \leq i < j \leq n$ and subsets $I$ of the set $\{1, \ldots, n\}$ having at least 3 elements.

**Remark 1.5.** The Hodge bundle $E$ on the moduli space also defines tautological classes. Recall that the bundle $E$ on $M_g$ is the locally free sheaf of rank $g$ whose fiber over a moduli point $[C] \in M_g$ is the space of holomorphic differentials on $C$. Chern classes of the Hodge bundle are also natural cycles. The Grothendieck-Riemann-Roch computation by Mumford [4] shows that lambda classes are expressed in terms of kappa classes and belong to the tautological ring.

For a fixed curve $C$ the connection between tautological classes on the Fulton-MacPherson compactification of $C^n$ and the space of curves with rational tails are explained in our previous works [8, 9, 10]. Based on the same picture we can see the connection between tautological classes on $M_{g,n}^{rt}$ and $C^n_g$. The idea is to view $M_{g,n}^{rt}$ as the relative Fulton-MacPherson compactification of the space $C^n_g$ over $M_g$ for $g \geq 2$. In genus one we need to consider the base $M_{1,1}$ instead. To get a uniform description we assume that $g > 1$ below. From this picture one can think of the divisor classes $D_I$ for $|I| \geq 3$ as exceptional divisors which appear in the process of blow-ups. Notice that the pull-back homomorphism identifies the Chow ring of $C^n_g$ with a subalgebra of $A^\ast(M_{g,n}^{rt})$. We use this identification and view $R^\ast(C^n_g)$ as a subalgebra of $R^\ast(M_{g,n}^{rt})$. The following statement follows from the discussion above:
Proposition 1.6. The tautological ring \( R^*(\mathcal{M}_{g,n}^r) \) of \( \mathcal{M}_{g,n}^r \) is an algebra over \( R^*(\mathcal{C}_g^n) \) generated by the divisor classes \( D_I \) for subsets \( I \) of \( \{1, \ldots, n\} \) with at least 3 elements.

Notice that we don’t need to involve the divisor classes \( D_I \) for subsets \( I \) with 2 elements. This is because the pull-back of the divisor \( d_{i,j} \) is the sum \( \sum_{i,j \in I} D_I \). We will call a divisor \( D_I \) for a subset \( I \) with \( |I| \geq 3 \) an exceptional divisor.

1.3. Intersection pairings. The one dimensionality of the tautological groups of \( \mathcal{C}_g^n \) and \( \mathcal{M}_{g,n}^r \) in degree \( g - 2 + n \) enables us to define a pairing on tautological classes. It is possible to relate intersection matrices of the pairings for \( \mathcal{M}_{g,n}^r \) and \( \mathcal{C}_g^n \). This connects the Gorenstein quotient of tautological algebras and leads to a conjectural dictionary between tautological relations on these spaces. Here we give a brief description of the method and refer to [10] for the details. However we warn the reader that some of definitions are slightly different.

We introduce a collection of monomials which additively generate tautological groups of \( \mathcal{M}_{g,n}^r \). These elements are called standard monomials. To define them we need to associate a directed graph \( G \) to any non-zero monomial

\[
\begin{equation}
\label{eq:7}
v := a(v) \cdot D(v)
\end{equation}
\]

which is a product of a class \( a(v) \) in \( R^*(\mathcal{C}_g^n) \) with a product \( D(v) := \prod_{r=1}^m D_{I_r}^v \) of exceptional divisors \( D_{I_r} \) for \( 1 \leq r \leq m \).

Definition 1.7. The directed graph \( G \) associated to the monomial \( v \) in (7) consists of a collection of vertices and edges. Vertices of the graph \( G \) correspond to the subsets \( I \) when \( D_I \) is a factor of \( D \). There is an edge from a vertex \( I \) to a vertex \( J \) when \( J \) is a proper subset of \( I \) and it is maximal with this property. Every minimal vertex is called a root of \( G \).

Notice that the graph \( G \) only depends on the monomial \( D \). It is straightforward to see that there is no loop in the resulting graph. The graph \( G \) can be disconnected in general. The degree of a vertex \( I \) is the number of outgoing edges. To every non-zero monomial \( v \in R^*(\mathcal{M}_{g,n}^r) \) as in (7) we associate a subset \( S \) of the set \( \{1, \ldots, n\} \) as follows: Consider the associated graph \( G \). Denote by \( J_1, \ldots, J_s \) the set of roots of \( G \). Let \( \alpha_r \in J_r \) be the smallest element. We define the set \( S \) as follows:

\[
\begin{equation}
\label{eq:8}
S := \{\alpha_1, \ldots, \alpha_s\} \cup (\cap_{i=1}^m I_i).
\end{equation}
\]
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**Definition 1.8.** Let $v$ be a non-zero monomial as in (7) and $\mathcal{G}$ be its associated graph. We say that $v$ is standard if $a(v) \in R^*(C^n_g)$ and for every vertex $I_r$ of $\mathcal{G}$ the following inequality holds:

$$i_r \leq \min(|I_r| - 2, |I_r| - |\cup_{J \subseteq I_r} I_s| + \deg(I_r) - 2).$$

**Proposition 1.9.** Tautological groups of $\mathcal{M}^{rt}_{g,n}$ are additively generated by standard monomials.

**Proof.** The statement follows from a collection of relations. These relations can be used to write elements in terms of standard monomials. They can be divided into 3 classes:

- Let $I$ be a subset of the set $\{1, \ldots, n\}$ with at least 3 elements. For any $i, j \in I$ and $k \in \{1, \ldots, n\} \setminus I$ we have the following relations:

  $$(d_{i,j} + K_j) \cdot D_I = 0, \quad (d_{i,k} - d_{j,k}) \cdot D_I = 0.$$

- For each subset $I$ of the set $\{1, \ldots, n\}$ with at least 3 elements we have the following relation:

  $$\prod_{i \neq j \in I} (d_{i,j} - \sum_{I \subseteq J} D_J) = 0,$$

  where $i \in I$ is arbitrary.

- Let $1 \leq r_1 < \cdots < r_{k+1} \leq n$ be a sequence of numbers. To every such sequence associate a polynomial as follows:

  $$P(t) := \prod_{i=2}^{r_1} (t + d_{1,i}) \cdot \prod_{j=1}^{k} (t + d_{1,r_j+1}).$$

  Consider the subsets $I_0 = \{1, \ldots, r_{k+1}\}$ and $I_i = \{r_i + 1, \ldots, r_{i+1}\}$ of the set $\{1, \ldots, n\}$. We have the following relation:

  $$P(- \sum_{I_0 \subseteq I} D_I) \cdot \prod_{i=1}^{k} D_{I_i} = 0.$$

  Similar relations found in previous works by studying intersection rings of blow-ups. It was shown that these vanishings all follow from the well-known formula for $\psi$-classes in genus zero together with a trivial vanishing for non intersecting boundary divisors.

$\square$
1.4. The filtration of the tautological ring. There is a natural way to define a decreasing filtration on the tautological ring of $\mathcal{M}_{g,n}^{rt}$. This filtration consists of $g - 2 + n$ steps. We prove a collection of vanishings in terms of this filtration. These vanishings are our key tool in relating tautological relations on $\mathcal{C}^*_g$ and $\mathcal{M}_{g,n}^{rt}$. The definition of a similar filtration was formulated after a question by Looijenga in connection with our work [8] in genus one.

Definition 1.10. Let $v$ be a standard monomial as given in (7) and $J_1, \ldots, J_s$ be roots of the associated graph. The integer $p(v)$ is defined as follows:

$$p(v) := \deg a(v) + \sum_{r=1}^{s} |J_r| - s.$$  

The subspace $\mathcal{F}^p R^k(\mathcal{M}_{g,n}^{rt}) \subset R^k(\mathcal{M}_{g,n}^{rt})$ is defined to be the $\mathbb{Q}$-vector space generated by standard monomials $v$ of degree $k$ satisfying $p(v) \geq p$.

It follows from the definition that the group $\mathcal{F}^{g-1+n} R^*(\mathcal{M}_{g,n}^{rt})$ vanishes. We define a preorder on the collection of subsets of the set $\{1, \ldots, n\}$ as follows: Let $I, J$ be two such subsets. We say that $I < J$ when $|J| < |I|$ or when $|I| = |J|$ and $I \neq J$. This induces a preorder on the set of exceptional divisors. We say that $D_I < D_J$ when $I < J$. This induces a preorder on the set of standard monomials using Lexicographic order. We say that $w \ll v$ if every factor of $D(w)$ is less than every factor of $D(v)$. We also make the convention that $w \ll v$ holds when $D(w) = 1$.

Proposition 1.11. Let $v \in R^d(\mathcal{M}_{g,n}^{rt})$ and $w \in \mathcal{F}^p R^*(\mathcal{M}_{g,n}^{rt})$ be such that $v \ll w$. If $p + d > g - 2 + n$ then the intersection product $v \cdot w$ is zero.

Proof. We may assume that $v \cdot w := a \cdot D$ is a standard monomial. Denote by $J_1, \ldots, J_s$ the collection of roots of its associated graph. Consider the factorization of the $D$ part of $v$:

$$D(w) := \prod_{r=1}^{m} D_{I_r}^r.$$  

The condition in 1.8 gives a bound for the power $i_r$ of $D_{I_r}$ for each $1 \leq r \leq m$. Combining these conditions together with the assumption $p + d > g - 2 + n$ gives that

$$\deg(a) > g - 2 + n - \sum_{i=1}^{s} |J_i| + s + m.$$
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We can write the product $a \cdot D$ as $b \cdot D$ for some $b$ which is the pull-back of a class from $R^*(C^k_g)$, where

$$k = n - \sum_{i=1}^{s} |J_i| + s.$$ 

This means that the degree of $b$ is bigger than $g - 2 + k + m$, which is at least $g - 2 + k$. The element $b$ vanishes according to the vanishing \([3]\) proven by Looijenga.

\[ \square \]

Remark 1.12. In Proposition 8.10. in [8] a similar statement was proved for the restriction of tautological classes to the fiber $C[n]$ associated to a fixed curve $C$. The argument utilised the geometry of the Fulton-MacPherson space $C[n]$.

Conjecture 1.13. The space of relations in $R^*(\mathcal{M}^r_{g,n})$ is generated by relations in $R^*(C^n_g)$ together with the vanishing of:

- All relations of the form $x \cdot D_I$, where $I \subseteq \{1, \ldots, n\}$ is a subset with at least 3 elements and $x$ is of the form $K_i + d_{i,j}$ or $d_{i,k} - d_{j,k}$ for some $i, j \in I$ and $k \in \{1, \ldots, n\} \setminus I$.
- All products $D_I \cdot D_J$ for non-intersecting divisors $D_I$ and $D_J$.

Remark 1.14. As a consequence let $D$ be a standard monomial which is a product of exceptional divisors. Let $S$ be the subset of the set $\{1, \ldots, n\}$ defined as in (8). This conjecture in particular implies that the map

$$A^*(C^n_g) \rightarrow A^*(\mathcal{M}^r_{g,n}) \quad \alpha \rightarrow \alpha \cdot D$$

defines an injection when $\alpha$ is a tautological class. Conjecture 1.13 also implies that there is no non-trivial relation among standard monomials with different $D$ parts.

1.5. Gorenstein quotients.

Definition 1.15. The Gorenstein quotient $G^*(\mathcal{M}^r_{g,n})$, (resp. $G^*(C^n_g)$) is defined as the quotient of the tautological ring $R^*(\mathcal{M}^r_{g,n})$, (resp. $R^*(C^n_g)$) modulo elements which define the zero map via the intersection pairings. The filtration defined in 1.10 induces a filtration on $G^*(\mathcal{M}^r_{g,n})$, (resp. $G^*(C^n_g)$) in a natural way.

We have identified $R^*(C^n_g)$ with a subalgebra of $R^*(\mathcal{M}^r_{g,n})$. Under this identification we can view $G^*(C^n_g)$ as a subalgebra of $G^*(\mathcal{M}^r_{g,n})$. This follows from the following lemma:
Lemma 1.16. The natural inclusion of $R^*(C^n_g)$ into $R^*(\mathcal{M}^{rt}_{g,n})$ induces an injection from $G^*(C^n_g)$ to $G^*(\mathcal{M}^{rt}_{g,n})$

Proof. Let $x \in R^*(C^n_g)$ be an element of degree $k$ which pairs to zero with all elements of degree $g - 2 + n - k$ in $R^*(C^n_g)$. We need to show that $x$ pairs to zero with all elements $y$ of degree $g - 2 + n - k$ in $R^*(\mathcal{M}^{rt}_{g,n})$. We may assume that $y := a(y) \cdot D(y)$ is standard. The product $x \cdot y$ vanishes if $D(y) = 1$ by our assumption. We prove the same statement when $D(y) \neq 1$. It is straightforward to see that $x \cdot y$ can be written as a product $v \cdot w$ for some $v$ and $w$ satisfying the condition given in Proposition 1.11. Therefore the product $x \cdot y$ vanishes. This means that there is a well-defined non-zero map from $G^*(C^n_g)$ to $G^*(\mathcal{M}^{rt}_{g,n})$. The statement follows since Gorenstein algebras don't have non-trivial quotients. □

Notice that intersection pairings on $R^*(C^n_g)$ induce pairings on its Gorenstein quotient. That gives an involution $*$ on the algebra $G^*(C^n_g)$.

It switches elements in degree $k$ and $g - 2 + n - k$ for every $0 \leq k \leq g - 2 + n$. This involution induces a natural involution on $G^*(\mathcal{M}^{rt}_{g,n})$ as well. It is defined as follows:

Definition 1.17. Let $v \in G^k(\mathcal{M}^{rt}_{g,n})$ be a standard monomial as in Definition 1.8. The dual element $v^*$ is an element in $G^{g-2+n-k}(\mathcal{M}^{rt}_{g,n})$ defined as:

$$v^* := a(v)^* \cdot \prod_{i=1}^{m} D_{I_i}^{j_r},$$

where $j_r := |I_r| - |\bigcup_{I_s \subset I_r} I_s| + \deg(I_r) - 1 - i_r$.

Notice that the defined involution gives a one to one correspondence between standard monomials in degree $k$ and $g - 2 + n - k$. There is a natural way to formulate a connection between $G^*(C^n_g)$ and $G^*(\mathcal{M}^{rt}_{g,n})$ as in Conjecture 1.13.

Theorem 1.18. The analogue of Conjecture 1.13 holds for the Gorenstein quotients of tautological algebras.

Proof. Let $0 \leq k \leq g - 2 + n$ and consider the intersection pairing among tautological classes in degree $k$ and $g - 2 + n - k$. We consider the set of standard monomials of degree $k$ and their duals. According to Proposition 1.11 there is a block structure on the intersection matrix. All blocks below the diagonal vanish. More precisely, let $v_1, v_2$ be standard monomials in $R^t(\mathcal{M}^{rt}_{g,n})$ satisfying the condition $D(v_1) < D(v_2)$. Then the intersection product $v_1 \cdot v_2^*$ can be written
as \( v \cdot w \) for some \( v \) and \( w \) satisfying the condition given in Proposition 1.11. It therefore vanishes. An elementary computation shows that square blocks along the main diagonal are intersection matrices of the pairings for \( C_g^S \) for various subsets \( S \) of the set \( \{1, \ldots, n\} \) up to a constant depending only on the graphs associated to standard monomials: Let \( v_1 \) and \( v_2 \) be two elements with the property \( D(v_1) = D(v_2) \). Denote by \( G \) their associated graph. The intersection number

\[
v_1 \cdot v_2^* \in R^{g-2+n}(\mathcal{M}_{g,n}^{rt}) \cong \mathbb{Q}
\]

and the number

\[
a(v_1) \cdot a(v_2) \in R^{g-2+|S|}(C_g^S) \cong \mathbb{Q}
\]

differ by \((-1)^\epsilon (2g-2)^n-|S|+1\), where

\[
\epsilon = |\bigcup_{r=1}^m I_r| + \sum_{i \in V(G)} \deg(i).
\]

In the identification of the tautological group of \( C_g^n \) in top degree with \( \mathbb{Q} \) we take the generator \( \kappa_{g-2} \cdot \prod_{i=1}^n K_i \). We also identify \( R^*(C_g^n) \) with a subring of \( R^*(\mathcal{M}_{g,n}^{rt}) \) as usual. These matrices are all invertible since we are working with Gorenstein quotients. This shows that there is no more relation among standard monomials. \( \square \)

2. Final remarks

There is no known counterexample to Conjecture 1.18 since we don’t know any example where \( R^*(C_g^n) \) or \( R^*(\mathcal{M}_{g,n}^{rt}) \) is not Gorenstein. However there are counterexamples \([6, 5]\) for larger compactifications. Notice that according to Theorem 1.18 a counterexample to Conjecture 1.13 would immediately show that the tautological ring of \( C_g^n \) is not Gorenstein.

In \([7]\) Pixton introduces a large collection of tautological relations on \( \mathcal{M}_{g,n}^{rt} \). He conjectures that these relations give a complete set of generators among tautological classes. We can restrict Pixton’s relation on \( \mathcal{M}_{g,n}^{rt} \) and \( C_g^n \).

**Question 2.1.** Can one relate Pixton’s relations on \( \mathcal{M}_{g,n}^{rt} \) in terms of his relations on \( C_g^n \) as described in Conjecture 1.13?
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