The Cauchy problem for operator-Boussinesq equations
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Abstract
In this paper, the existence and uniqueness of solution of the Cauchy prob-
lem for abstract Boussinesq equation is obtained. By applying this result, the
Cauchy problem for systems of Boussinesq equations of finite or infinite orders
are studied.
Key Word: Boussinesq equations, Semigroups of operators, Hyperbolic-
operator equations; cosine operator functions, Operator-valu ed multipliers
AMS: 35Lxx, 35Mxx, 47Lxx, 47Axx
1. Introduction
The subject of this paper is to study the local existence and unique ness of
solution of the Cauchy problem for the following Boussinesq-operator equation
\[ u_{tt} - Lu_{tt} + Au = f(u), \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]\n(1.1)
\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \]\n(1.2)
where \( A \) is a linear operator in a Banach space \( E \), \( u(x, t) \) is the \( E \)-valued un-
known function, \( f(u) \) is the given nonlinear function, \( \varphi(x) \) and \( \psi(x) \) are the
given initial value functions, subscript \( t \) indicates the partial derivative with
respect to \( t \), \( n \) is the dimension of space variable \( x \) and \( L \) is an elliptic operator
in \( \mathbb{R}^n \) with constant coefficients. Since the Banach space \( E \) is arbitrary and \( A \) is
a possible linear operator, by choosing \( E \) and \( A \) we can obtain numerous classes
of generalized Boussinesq type equations which occur in a wide variety of phys-
ical systems, such as in the propagation of longitudinal deformation waves in
an elastic rod, hydro-dynamical process in plasma, in materials science which
describe spinodal decomposition and in the absence of mechanical stresses (see
\[ 1 - 4 \]). For example, if we choose \( E = \mathbb{C}, \mathbb{L} = \Delta \) and \( A = -\Delta \) we obtain the
scalar Cauchy problem for generalized Boussinesq type equation
\[ u_{tt} - \Delta u_{tt} - \Delta u = f(u), \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]\n(1.3)
\[ u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x). \]\n(1.4)
The equation (1.3) arises in different situations (see \[ 1, 2 \]). For example, for
\( n = 1 \) it describes a limit of a one-dimensional nonlinear lattice \[ 3 \], shallow-
water waves \[ 4, 5 \] and the propagation of longitudinal deformation waves in an
elastic rod \[ 6 \]. Rosenau \[ 7 \] derived the equations governing dynamics of one, two
and three-dimensional lattices. One of those equations is (1.3). In [8], [9] the existence of the global classical solutions and the blow-up of the solutions of the initial boundary value problem and Cauchy problem (1.3) − (1.4) are obtained. Here, by inspiring [8] and [9], the Cauchy problem for Boussinesq operator equation is considered. Note that, differential operator equations were studied e.g. in [10-42, 53]. Cauchy problem for abstract hyperbolic equations were treated e.g. in [11-20] and for abstract Boussinesq equations studied in [35, 36]. In this paper, we obtain the local existence and uniqueness of small-amplitude solution of the Cauchy problem for abstract Boussinesq equations with general elliptic principal part. The strategy is to express the abstract Boussinesq equation as an integral equation with operator coefficient. To treat the nonlinearity as a small perturbation of the linear part of the equation, the contraction mapping theorem is used. Also, a priori estimates on $E-$valued $L^p$ norms of solutions of the linearized version are utilized. The key step is the derivation of the uniform estimate of the solutions of the linearized Boussinesq-operator equation. Modern analysis methods, particularly abstract harmonic analysis, operator theory, interpolation of Banach Spaces, embedding theorems in abstract Sobolev-Lions spaces are the main tools implemented to carry out the analysis.

In order to state our results precisely, we introduce some notations and some function spaces.

**Definitions and Background**

Let $E$ be a Banach space. $L^p (\Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$
\|f\|_{L^p} = \left( \int_\Omega \|f(x)\|_E^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,
$$

$$
\|f\|_{L^\infty} = \text{ess sup}_{x \in \Omega} \|f(x)\|_E.
$$

The Banach space $E$ is called an UMD-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} \, dy$$

is bounded in $L^p (R, E), p \in (1, \infty)$ (see e.g. [43]). UMD spaces include e.g. $L^p$, $l_p$ spaces and Lorentz spaces $L_{pq}$ for $p, q \in (1, \infty)$.

Let

$$S_\psi = \{ \lambda \in \mathbb{C}, \ | \arg \lambda | \leq \omega, \ 0 \leq \omega < \pi \},
$$

$$S_{\omega, \kappa} = \{ \lambda \in S_\omega, \ | \lambda | > \kappa > 0 \}.
$$

A closed linear operator $A$ is said to be positive in a Banach space $E$ if $D(A)$ is dense on $E$ and $\| (A + \lambda I)^{-1} \|_{B(E)} \leq M (1 + |\lambda|)^{-1}$ for any $\lambda \in S_\omega,
$0 \leq \omega < \pi$, where $I$ is the identity operator in $E$, $B(E)$ is the space of bounded linear operators in $E$; $D(A)$ denote domain of the operator $A$. It is known [44, §1.15.1] that there exist fractional powers $A^\theta$ of a positive operator $A$. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$
\|u\|_{E(A^\theta)} = \left(\|u\|^p + \|A^\theta u\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty, \ 0 < \theta < \infty.
$$

A closed linear operator $A$ in a Banach space $E$ belong to $\sigma(C_0, \omega, E)$ (see [11], § 11.2) if $D(A)$ is dense on $E$, the resolvent $(A - \lambda^2 I)^{-1}$ exists for $\text{Re} \lambda > \omega$ and

$$
\left\| (A - \lambda^2 I)^{-1} \right\|_{B(E)} \leq C_0 |\text{Re} \lambda - \omega|^{-1}.
$$

Let $E_1$ and $E_2$ be two Banach spaces. $(E_1, E_2)_{\theta, p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ denotes the interpolation spaces obtained from $\{E_1, E_2\}$ by $K$-method [44, §1.3.2].

Let $\mathbb{N}$ denote the set of natural numbers. A set $\Phi \subset B(E_1, E_2)$ is called R-bounded (see e.g. [10]) if there is a positive constant $C$ such that for all $T_1, T_2, ..., T_m \in \Phi$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$
\int_\Omega \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_\Omega \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,
$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $\Omega$. The smallest $C$ for which the above estimate holds is called a $R$-bound of the collection $\Phi$ and denoted by $R(\Phi)$.

Let $h$ be same parameter with $h \in Q \subset \mathbb{C}$. A set $\Phi_h \subset B(E_1, E_2)$ is called uniform $R$-bounded if there is a constant $C$ independent on $h$ such that

$$
\int_\Omega \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_\Omega \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy.
$$

for all $T_1(h), T_2(h), ..., T_m(h) \in \Phi_h$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$. It is implies that $\sup_{h \in Q} R(\Phi_h) \leq C$.

The positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $L_A = \{\xi (A + \xi I)^{-1} : \xi \in S^2\}$, $0 \leq \omega < \pi$ is $R$-bounded.

Let

$$
\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \ D^{\alpha} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \ |\alpha| = \sum_{k=1}^n \alpha_k.
$$

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embedded into $E$. Let $\Omega$ be a domain in $R^n$ and $m$ be a positive integer.
$W^{m,p}(\Omega; E_0, E)$ denotes the space of all functions $u \in L^p(\Omega; E_0)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k} \in L^p(\Omega; E)$, $1 \leq p \leq \infty$ with the norm

$$\|u\|_{W^{m,p}(\Omega; E_0, E)} = \|u\|_{L^p(\Omega; E_0)} + \sum_{k=1}^n \left\|\frac{\partial^m u}{\partial x_k}\right\|_{L^p(\Omega; E)} < \infty.$$ 

For $E_0 = E$ the space $W^{m,p}(\Omega; E_0, E)$ denotes by $W^{m,p}(\Omega; E)$.

Let $L^{s,p}(R^n; E)$, $-\infty < s < \infty$ denotes the $E$-valued Liouville-Sobolev space of order $s$ which is defined as:

$$L^{s,p} = L^{s,p}(R^n; E) = (I - \Delta)^{-\frac{s}{2}} L^p(R^n; E)$$

with the norm

$$\|u\|_{L^{s,p}} = \left\|(I - \Delta)^{-\frac{s}{2}} u\right\|_{L^p(R^n; E)}.$$ 

It clear that $L^{0,p}(R^n; E) = L^p(R^n; E)$. It is known that if $E$ is a UMD space, then $L^{m,p}(R^n; E) = W^{m,p}(R^n; E)$ for positive integer $m$ (see e.g. [45, § 15]).

$L^{s,p}(R^n; E_0, E)$ denote the Liouville-Lions type space i.e.,

$$L^{s,p}(R^n; E_0, E) = \{u \in L^{s,p}(R^n; E) \cap L^q(R^n; E_0) \mid \|u\|_{L^{s,p}(R^n; E)} + \|u\|_{L^{s,p}(R^n; E_0)} < \infty\},$$

Let $S(R^n; E)$ denote $E$-valued Schwartz class, i.e., the space of $E$-valued rapidly decreasing smooth functions on $R^n$, equipped with its usual topology generated by seminorms. Let $S'(R^n; E)$ denote the space of all continuous linear operators $L : S(R^n; E) \to E$, equipped with the bounded convergence topology. Recall $S(R^n; E)$ is norm dense in $L^p(R^n; E)$ when $1 \leq p < \infty$.

Let $L^q_*(E)$ denote the space of all $E$-valued function space such that

$$\|u\|_{L^q_*(E)} = \left(\int_0^\infty \|u(t)\|_{L^q(E)}^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty, 1 \leq q < \infty, \quad \|u\|_{L^{q}_*(E)} = \sup_{0 < t < \infty} \|u(t)\|_{E^*}.$$ 

Let $s = (s_1, s_2, ..., s_n)$ and $s_k > 0$. Let $F$ denote the Fourier transform. Fourier-analytic representation of $E$-valued Besov space on $R^n$ are defined as:

$$B_{p,q}^s(R^n; E) = \left\{u \in S'(R^n; E) \mid \|u\|_{B_{p,q}^s(R^n; E)} = \left\|F^{-1} \sum_{k=1}^n \left|\xi_k\right|^{s_k} e^{-t|\xi|^2} F u\right\|_{L^q_*(L^p(R^n; E))} < \infty\right\},$$

$$p \in (1, \infty), q \in [1, \infty], \quad s_k > s_k.$$ 

It should be note that, the norm of Besov space does not depends on $s_k$. See ([44, § 2.3] for the scalar case, i.e., $E = \mathbb{C}$).
Let $B_{p,q}^s (\mathbb{R}^n; E_0, E)$ denote the space $L^p (\mathbb{R}^n; E_0) \cap B_{p,q}^s (\mathbb{R}^n; E)$ with the norm
\[ \|u\|_{B_{p,q}^s (\mathbb{R}^n; E_0, E)} = \|u\|_{L^p (\mathbb{R}^n; E_0)} + \|u\|_{B_{p,q}^s (\mathbb{R}^n; E)} < \infty. \]

The embedding theorems in vector valued spaces play a key role in the theory of DOEs. For estimating lower order derivatives we use following embedding theorem that is obtained from [32, Theorem 1]:

**Theorem A.** Suppose the following conditions are satisfied:

1. $E$ is a UMD space and $A$ is an $R$-positive operator in $E$;
2. $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a $n$-tuples of nonnegative integer number and $s$ is a positive number such that
   \[ \kappa = \frac{n (\frac{1}{p} - \frac{1}{q}) \mu}{2} \leq 1, 0 \leq \mu \leq 1 - \kappa, 1 < p \leq q < \infty, 0 < h \leq h_0, \] where $h_0$ is a fixed positive number;

Then the embedding $D^\alpha L^{s,p} (\mathbb{R}^n; E (A), E) \subset L^q (\mathbb{R}^n; E \left( A^{1-\kappa-\mu} \right))$ is continuous and for $u \in L^{s,p} (\mathbb{R}^n; E (A), E)$ the following uniform estimate holds
\[ \|D^\alpha u\|_{L^q (\mathbb{R}^n; E (A^{1-\kappa-\mu})]} \leq h^\mu \|u\|_{L^{s,p} (\mathbb{R}^n; E (A), E)} + h^{-(1-\mu)} \|u\|_{L^p (\mathbb{R}^n; E)}. \]

In a similar way as [31, Theorem A0] and by reasoning as [46, Theorem 3.7] we obtain:

**Proposition A.** Let $1 < p \leq q \leq \infty$ and $E$ be UMD space. Suppose $\Psi_h \in C^n (\mathbb{R}^n \setminus \{0\}; B (E))$ and there is a positive constant $K$ such that
\[ \sup_{h \in Q} R \left( \left\{ |\xi|^{\beta} + n \left( \frac{1}{p} - \frac{1}{q} \right) \right\} D^\beta \Psi_h (\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \beta_k \in \{0, 1\} \right) \leq K. \]

Then $\Psi_h$ is a uniformly bounded collection of Fourier multiplier from $L^p (\mathbb{R}^n; E)$ to $L^q (\mathbb{R}^n; E)$.

**Proof.** First, in a similar way as in [31, Theorem A0] we show that $\Psi_h$ is a uniformly bounded collection of Fourier multiplier from $L^p (\mathbb{R}^n; E)$ to $L^q (\mathbb{R}^n; E)$. Moreover, by Theorem A1 we get that, for $s \geq n \left( \frac{1}{p} - \frac{1}{q} \right)$ the embedding $L^{s,p} (\mathbb{R}^n; E) \subset L^q (\mathbb{R}^n; E)$ is continuous. From these two fact we obtain the conclusion.

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_\alpha$.

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priory estimates for solution of the linearized problem (1.1)-(1.2). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1)-(1.2). In Section 4, the existence, uniqueness and a priory estimates for solution of Cauchy problem for finite and infinite system of Boussinesq equation is derived.

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context. When we want to specify the dependence of such a constant on a parameter, say $h$, we write $C_h$.

2. Estimates for linearized equation

In this section, we make the necessary estimates for solutions of initial value problems for the linearized abstract Boussinesq equation

$$u_{tt} - Lu_{tt} + Au = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (0, T), \quad (2.1)$$

$$u(x, 0) = \varphi(x), \quad u_t(x, 0) = \psi(x), \quad (2.2)$$

where

$$Lu = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad a_{ij} \in \mathbb{C}.$$

**Condition 2.0.** Assume $L$ is an elliptic operator, i.e., there are positive constants $M_1$ and $M_2$ such that

$$M_1 |\xi|^2 \leq L(\xi) \leq M_2 |\xi|^2$$

for $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, where

$$|\xi|^2 = \sum_{k=1}^{n} \xi_k^2, \quad L(\xi) = \sum_{i,j=1}^{2} a_{ij} \xi_i \xi_j.$$

Let

$$X_p = L^p(R^n; E), \quad Y_s,p = L^{s,p}(R^n; E), \quad Y_s,p^\infty = L^{s,p}(R^n; E) \cap L^\infty(R^n; E).$$

**Condition 2.1.** Assume:

1. $E$ is an UMD space and linear operator $A$ belongs to $\sigma(C_0, \omega; E)$;
2. $\varphi, \psi \in Y_{s,p}^\infty$ and $g(\cdot, t) \in Y_{s,p}^\infty$ for $t \in (0, T)$ and $s > \frac{n}{p}$ for $1 < p < \infty$.

First we need the following lemmas

**Lemma 2.1.** Suppose the Conditions 2.0, 2.1 hold. Then problem (2.1) – (2.2) has a generalized solution.

**Proof.** By using of Fourier transform we get from (2.1) – (2.2):

$$\hat{u}_{tt} (\xi, t) + A_\xi \hat{u} (\xi, t) = [1 + L(\xi)]^{-1} \hat{g} (\xi, t), \quad (2.3)$$

$$\hat{u} (\xi, 0) = \hat{\varphi} (\xi), \quad \hat{u}_t (\xi, 0) = \hat{\psi} (\xi), \quad \xi \in \mathbb{R}^n, \quad t \in (0, T),$$

where $\hat{u}(\xi, t)$ is a Fourier transform of $u(x, t)$ with respect to $x$, where

$$A_\xi = [1 + L(\xi)]^{-1} A, \quad \xi \in \mathbb{R}^n.$$ 

By virtue of [11, §11.2, 11.4] (or [12-20]) we obtain that $A_\xi$ is a generator of a strongly continuous cosine operator function and problem (2.3) has a unique solution for all $\xi \in \mathbb{R}^n$, moreover, the solution can be written as

$$\hat{u} (\xi, t) = C(t, \xi, A) \hat{\varphi} (\xi) + S(t, \xi, A) \hat{\psi} (\xi) +$$

$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t (t - s)^{n-1} S(t, \xi, A) \hat{g} (\xi, s) \, ds, \quad t \in (0, T).$$

By virtue of [11, §11.2, 11.4] (or [12-20]) we obtain that $A_\xi$ is a generator of a strongly continuous cosine operator function and problem (2.3) has a unique solution for all $\xi \in \mathbb{R}^n$, moreover, the solution can be written as

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$$+ \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^t (t - s)^{n-1} S(t, \xi, A) \hat{g} (\xi, s) \, ds, \quad t \in (0, T).$$
\[ \int_{0}^{t} S(t - \tau, \xi, A) \left[ 1 + L(\xi) \right]^{-1} \hat{g}(\xi, \tau) \, d\tau, \quad t \in (0, T), \]

where \( C(t, \xi, A) \) is a cosine and \( S(t, \xi, A) \) is a sine operator-functions (see e.g. \([11]\)) generated by parameter dependent operator \( A_\xi \). From (2.4) we get that, the solution of the problem (2.1) – (2.2) can be expressed as

\[ u(x, t) = S_1(t, A) \varphi(x) + S_2(t, A) \psi(x) + \]

\[ + (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix\xi} S(t - \tau, \xi, A) \left[ 1 + L(\xi) \right]^{-1} \hat{g}(\xi, \tau) \, d\tau d\xi, \quad t \in (0, T), \] (2.5)

where \( S_1(t, A) \) and \( S_2(t, A) \) are linear operators in \( E \) defined by

\[ S_1(t, A) \varphi = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix\xi} \hat{C}(t, \xi, A) \hat{\varphi}(\xi) \, d\xi, \] (2.6)

\[ S_2(t, A) \psi = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{ix\xi} S(t, \xi, A) \hat{\psi}(\xi) \, d\xi. \]

**Lemma 2.2.** Suppose the Conditions 2.0, 2.1 hold. Then the solution of the problem (2.1) – (2.2) satisfies the following estimate

\[ \left( \|u\|_{X_\infty} + \|u_t\|_{X_\infty} \right) \leq C \left( \|\varphi\|_{Y_{r,p}} + \|\varphi\|_{X_1} \right) \] (2.7)

\[ + \|\psi\|_{Y_{r,p}} + \|\psi\|_{X_1} + \int_{0}^{t} \left( \|g(\cdot, \tau)\|_{Y_{r,p}} + \|g(\cdot, \tau)\|_{X_1} \right) \, d\tau \]

uniformly in \( t \in [0, T] \).

**Proof.** Let \( N \in \mathbb{N} \) and

\[ \Pi_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \leq N \}, \quad \Pi'_N = \{ \xi : \xi \in \mathbb{R}^n, \ |\xi| \geq N \}. \]

It is clear to see that

\[ \|u(\cdot, t)\|_{L^{\infty}(\mathbb{R}^n; E)} = \| F^{-1} \hat{u}(\xi, t)\|_{L^{\infty}(\mathbb{R}^n; E)} \leq \]

\[ \| F^{-1} C(t, \xi, A) \hat{\varphi}(\xi)\|_{X_{\infty}} + \| F^{-1} S(t, \xi, A) \hat{\psi}(\xi)\|_{X_{\infty}} \leq \]

\[ \| F^{-1} C(t, \xi, A) \hat{\varphi}(\xi)\|_{L^{\infty}(\Pi_N; E)} + \| F^{-1} S(t, \xi, A) \hat{\psi}(\xi)\|_{L^{\infty}(\Pi_N; E)} \] (2.8)

\[ + \| F^{-1} C(t, \xi, A) \hat{\varphi}(\xi)\|_{L^{\infty}(\Pi'_N; E)} + \| F^{-1} S(t, \xi, A) \hat{\psi}(\xi)\|_{L^{\infty}(\Pi'_N; E)}; \]
\[
\|F^{-1}C(t, \xi, A) \hat{\varphi}(\xi)\|_{L^\infty(W_N;E)} + \|F^{-1}S(t, \xi, A) \hat{\psi}(\xi)\|_{L^\infty(W_N;E)} = \\
= \|F^{-1} (1 + L(\xi))^{-\hat{s}} C(t, \xi, A) (1 + L(\xi))^{\hat{s}} \hat{\varphi}(\xi)\|_{L^\infty(W_N;E)} + \\
\|F^{-1} (1 + L(\xi))^{-\hat{s}} S(t, \xi, A) (1 + L(\xi))^{\hat{s}} \hat{\psi}(\xi)\|_{L^\infty(W_N;E)}. \\
\]

Using the Hölder inequality we have
\[
\|F^{-1}C(t, \xi, A) \hat{\varphi}(\xi)\|_{L^\infty(W_N;E)} + \|F^{-1}S(t, \xi, A) \hat{\psi}(\xi)\|_{L^\infty(W_N;E)} \leq \tag{2.10}
\]
\[
C \left[\|\varphi\|_{X_1} + \|\psi\|_{X_1}\right].
\]
By using the resolvent properties of operator \(A\), representation of \(C(t, \xi, A)\), \(S(t, \xi, A)\) and the Condition 2.0 we get
\[
\|\xi|^{\alpha+\hat{s}} \|D^\alpha \left[(1 + L(\xi))^{-\hat{s}} C(t, \xi, A)\right]\|_{B(E)} \leq C_1,
\]
\[
\|\xi|^{\alpha+\hat{s}} \|D^\alpha \left[(1 + L(\xi))^{-\hat{s}} S(t, \xi, A)\right]\|_{B(E)} \leq C_2, \tag{2.11}
\]
for \(s > \frac{p}{q}\) and all \(\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)\), \(\alpha_k \in \{0, 1\}\), \(\xi \in \mathbb{R}^n\), \(\xi \neq 0\), \(t \in [0, T]\). By Proposition A1 from (2.9) and Condition 2.0 we get that, the operator-valued functions \((1 + L(\xi))^{-\hat{s}} C(t, \xi, A)\), \((1 + L(\xi))^{-\hat{s}} S(t, \xi, A)\) are \(L^p(\mathbb{R}^n; E) \rightarrow L^\infty(\mathbb{R}^n; E)\) Fourier multipliers uniformly in \(t \in [0, T]\). Then by Minkowski's inequality for integrals, the semigroups estimates (see e.g. [11-12]) and (2.9) we obtain
\[
\|F^{-1}C(t, \xi, A) \hat{\varphi}(\xi)\|_{L^\infty(W_N;E)} + \|F^{-1}S(t, \xi, A) \hat{\psi}(\xi)\|_{L^\infty(W_N;E)} \leq \tag{2.12}
\]
\[
C \left[\|\varphi\|_{Y_{s,p}} + \|\psi\|_{Y_{s,p}}\right].
\]
By reasoning as the above we get
\[
\left\| F^{-1} \int_0^t S(t - \tau, \xi, A) [1 + L(\xi)]^{-1} \hat{g}(\xi, \tau) d\tau \right\|_{X_{\infty}} \leq \tag{2.13}
\]
\[
C \int_0^t \left(\|g(., \tau)\|_{Y_{s}} + \|g(., \tau)\|_{X_1}\right) d\tau.
\]
By differentiating, in view of (2.6) we obtain from (2.5) the estimate of type (2.10), (2.12), (2.13) for \(u_t\).

Then by using (2.10), (2.12), (2.13) we get the estimate (2.7).
Lemma 2.3. Assume the Conditions 2.0, 2.1 are hold. Then the solution of the problem (2.1)−(2.2) satisfies the following uniform estimate

\[ (\|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}}) \leq C \left( \|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|g(\cdot,\tau)\|_{Y^{s,p}} d\tau \right). \] (2.14)

Proof. From (2.4) we have the following estimate

\[ \left( \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u} \right\|_{X^p} + \left\| F^{-1} \left( 1 + |\xi|^2 \right)^{\frac{s}{2}} \hat{u}_t \right\|_{X^p} \right) \leq \]

\[ C \left\{ \left\| F^{-1} (1 + |\xi|)^{\frac{s}{2}} C(t, \xi, A) \hat{\varphi} \right\|_{X^p} + \left\| F^{-1} (1 + |\xi|)^{\frac{s}{2}} S(t, \xi, A) \hat{\psi} \right\|_{X^p} \right\} + \]

\[ \int_0^t \left( (1 + |\xi|)^{\frac{s}{2}} S(t - \tau, \xi, A) [1 + L(\xi)]^{-1} g(\cdot, \tau) \right) \| \right\|_{X^p} d\tau \right) \] (2.15)

By construction of operator-valued functions \( C(t, \xi, A), \) \( S(t, \xi, A) \) and in view of Proposition A1 and Condition 2.0 we get that \( C(t, \xi, A) \) and \( S(t, \xi, A) \) are \( L^p(R^n;E) \) Fourier multipliers uniformly in \( t \in [0, T] \). So, the estimate (2.15) by using the Minkowski's inequality for integrals implies (2.14).

From Lemmas 2.1-2.3 we obtain

**Theorem 2.1.** Let the Condition 2.1 hold. Then problem (2.1)−(2.2) has a unique solution \( u \in C^{(2)}([0, T];Y^{s,p}_1) \) and the following estimates holds

\[ \|u\|_{X^\infty} + \|u_t\|_{X^\infty} \leq C \left( \|\varphi\|_{Y^{s,p}} + \|\varphi\|_{X^1} \right) \] (2.16)

\[ + \|\psi\|_{Y^{s,p}} + \|\psi\|_{X^1} + \int_0^t \left( \|g(\cdot, \tau)\|_{Y^{s,p}} + \|g(\cdot, \tau)\|_{X^1} \right) d\tau, \]

\[ \|u\|_{Y^{s,p}} + \|u_t\|_{Y^{s,p}} \leq C \left( \|\varphi\|_{Y^{s,p}} + \|\psi\|_{Y^{s,p}} + \int_0^t \|g(\cdot, \tau)\|_{Y^{s,p}} d\tau \right) \] (2.17)

uniformly in \( t \in [0, T] \).

Proof. From Lemma 2.1 we obtain that, problem (2.1)−(2.2) has a unique generalized solution. From the representation of solution (2.5) and Lemmas 2.2, 2.3 we get that there is a solution \( u \in C^{(2)}([0, T];Y^{s}_1) \) and estimates (2.16), (2.17) hold.

3. Initial value problem for nonlinear equation
In this section, we will show the local existence and uniqueness of solution for the Cauchy problem (1.1), (1.2).

For the study of the nonlinear problem (1.1) − (1.2) we need the following lemmas

**Lemma 3.1** (Abstract Nirenberg’s inequality). Let \( E \) be an UMD space. Assume that \( u \in L_p (\Omega; E), \ D^m u \in L_q (\Omega; E), \ p, q \in (1, \infty) \). Then for \( i \) with \( 0 \leq i \leq m, \ m > \frac{n}{q} \) we have

\[
\| D^i u \|_p \leq C \| u \|_p^{1-\mu} \sum_{k=1}^{n} \| D_k^m u \|_{q}^{\mu}, \tag{3.1}
\]

where

\[
\frac{1}{r} = \frac{i}{m} + \mu \left( \frac{1}{q} - \frac{m}{n} \right) + (1-\mu) \frac{1}{p}, \quad \frac{i}{m} \leq \mu \leq 1.
\]

**Proof.** By virtue of interpolation of Banach spaces [44, §1.3.2], in order to prove (3.1) for any given \( i \), one has only to prove it for the extreme values \( \mu = \frac{i}{m} \) and \( \mu = 1 \). For the case of \( \mu = 1 \), i.e., \( \frac{1}{r} = \frac{i}{m} + \frac{1}{q} - \frac{m}{n} \) the estimate (3.1) is obtained from Theorem A1. The case \( \mu = \frac{i}{m} \) is derived by reasoning as in [47, §2] and in replacing absolute value of complex-valued function \( u \) by the \( E \)-norm of \( E \)-valued function.

Note that, for \( E = \mathbb{C} \) the lemma considered by L. Nirenberg [47].

Using the chain rule of the composite function, from Lemma 3.1 we can prove the following result

**Lemma 3.2.** Let \( E \) be an UMD space. Assume that \( u \in W^{m,p} (\Omega; E) \cap L^{\infty} (\Omega; E) \), and \( f (u) \) possesses continuous derivatives up to order \( m \geq 1 \). Then \( f (u) - f (0) \in W^{m,p} (\Omega; E) \) and

\[
\| f (u) - f (0) \|_p \leq \left\| f^{(1)} (u) \right\|_\infty \| u \|_p,
\]

\[
\| D^k f (u) \|_p \leq C_0 \sum_{j=1}^{k} \left\| f^{(j)} (u) \right\|_\infty \| u \|_\infty^{j-1} \| D^k u \|_p, \quad 1 \leq k \leq m, \tag{3.2}
\]

where \( C_0 \geq 1 \) is a constant.

For \( E = \mathbb{C} \) the lemma coincide with the corresponding inequality in [48].

Let

\[
X = L^p (R^n; E), \quad Y = W^{2,p} (R^n; E (A), E), \quad E_0 = (X, Y)_{\frac{2}{p}, p}.
\]

**Remark 3.1.** By using J.Lions-I. Petree result (see e.g. [49] or [44, §1.8]) we obtain that the map \( u \to u (t_0), \ t_0 \in [0, T] \) is continuous from \( W^{2,p} (0, T; X, Y) \) onto \( E_0 \) and there is a constant \( C_1 \) such that

\[
\| u (t_0) \|_{E_0} \leq C_1 \| u \|_{W^{2,p} (0, T; X, Y)}, \quad 1 \leq p \leq \infty.
\]
Here, we define the space $Y(T) = C([0, T]; Y^2_{Y\infty})$ equipped with the norm defined by

$$
\|u\|_{Y(T)} = \max_{t \in [0, T]} \|u\|_{Y^2} + \max_{t \in [0, T]} \|u_t\|_{X_{\infty}}, \ u \in Y(T).
$$

It is easy to see that $Y(T)$ is a Banach space. For $\varphi, \psi \in Y^2$, let

$$
M = \|\varphi\|_{Y^2} + \|\varphi\|_{X_{\infty}} + \|\psi\|_{Y^2} + \|\psi\|_{X_{\infty}}.
$$

**Definition 3.1.** For any $T > 0$ if $u, \psi \in Y^2_{Y\infty}$ and $u \in C([0, T]; Y^2_{Y\infty})$ satisfies the equation (1.1) – (1.2) then $u(x, t)$ is called the continuous solution or the strong solution of the problem (1.1) – (1.2). If $T < \infty$, then $u(x, t)$ is called the local strong solution of the problem (1.1) – (1.2). If $T = \infty$, then $u(x, t)$ is called the global strong solution of the problem (1.1) – (1.2).

**Condition 3.1.** Assume:

1. the operator $A$ generates continuous cosine operator function in UMD space $E$;
2. $\varphi, \psi \in Y^2_{Y\infty}$ and $1 < p < \infty$ for $\frac{2}{p} < 2$;
3. the function $u \rightarrow f(u): R^n \times [0, T] \times E_0 \rightarrow E$ is a measurable in $(x, t) \in R^n \times [0, T]$ for $u \in E_0$; $f(x, t, .,.)$ is continuous in $u \in E_0$ for $x \in R^n$, $t \in [0, T]$ and $f(u) \in C^{(3)}(E_0; E)$.

Main aim of this section is to prove the following result:

**Theorem 3.1.** Let the Condition 3.1 hold. Then problem (1.1) – (2.2) has a unique local strange solution $u \in C^{(2)}([0, T_0]; Y^2_{Y\infty})$, where $T_0$ is a maximal time interval that is appropriately small relative to $M$. Moreover, if

$$
\sup_{t \in [0, T_0]} (\|u\|_{Y^2} + \|u_t\|_{X_{\infty}} + \|u_t\|_{Y^2} + \|u_{tt}\|_{X_{\infty}}) < \infty \quad (3.3)
$$

then $T_0 = \infty$.

**Proof.** First, we are going to prove the existence and the uniqueness of the local continuous solution of the problem (1.1) – (1.2) by contraction mapping principle. Consider a map $G$ on $Y(T)$ such that $G(u)$ is the solution of the Cauchy problem

$$
G_{tt}(u) - L G_{tt}(u) + A G(u) = f(G(u)), \ x \in R^n, \ t \in (0, T), \quad (3.4)
$$

$$
G(u)(x, 0) = \varphi(x), \ G_t(u)(x, 0) = \psi(x).
$$

From Lemma 3.2 we know that $f(u) \in L^p(0, T; Y^2_{Y\infty})$ for any $T > 0$. Thus, by Theorem 2.1, problem (3.4) has a unique solution which can be written as

$$
G(u)(t, x) = S_1(t, A) \varphi(x) + S_2(t, A) \psi(x) +
$$

$$
+ \int_0^t F^{-1} S(t - \tau, \xi, A) [1 + L(\xi)]^{-1} f(u)(\xi, \tau) d\tau, \ t \in (0, T). \quad (3.5)
$$
From Lemma 3.2 it is easy to see that the map \( G \) is well defined for \( f \in C^{(2)}(X_0; E). \) We put
\[
Q(M; T) = \{ u \mid u \in Y(T), \|u\|_{Y(T)} \leq M + 1 \}.
\]

First, by reasoning as in [9] let us prove that the map \( G \) has a unique fixed point in \( Q(M; T) \). For this aim, it is sufficient to show that the operator \( G \) maps \( Q(M; T) \) into \( Q(M; T) \) and \( G: Q(M; T) \to Q(M; T) \) is strictly contractive if \( T \) is appropriately small relative to \( M \).

Consider the function \( \bar{f}(\xi): [0, \infty) \to [0, \infty) \) defined by
\[
\bar{f}(\xi) = \max_{|x| \leq \xi} \left\{ \left\| f^{(1)}(x) \right\|_E, \left\| f^{(2)}(x) \right\|_E \right\}, \quad \xi \geq 0.
\]

It is clear to see that the function \( \bar{f}(\xi) \) is continuous and nondecreasing on \([0, \infty)\). From Lemma 3.2 we have
\[
\|f(u)\|_{Y^{2,p}} \leq \left\| f^{(1)}(u) \right\|_{X^{\infty}} \|u\|_{X^p} + \left\| f^{(1)}(u) \right\|_{X^{\infty}} \|Du\|_{X^p} + C_0 \left[ \left\| f^{(1)}(u) \right\|_{X^{\infty}} \|u\|_{X^p} + \left\| f^{(2)}(u) \right\|_{X^{\infty}} \|u\|_{X^p} \left\| D^2u \right\|_{X^p} \right] \leq 2C_0\bar{f}(M + 1)(M + 1)\|u\|_{Y^{2,p}}.
\]

By using Theorem 2.1 we obtain from (3.5):
\[
\|G(u)\|_{X^{\infty}} \leq \|\varphi\|_{X^{\infty}} + \|\psi\|_{X^{\infty}} + \int_0^t \|f(u(\tau))\|_{X^{\infty}} \quad (3.7)
\]
\[
\|G(u)\|_{Y^{2,p}} \leq \|\varphi\|_{Y^{2,p}} + \|\psi\|_{Y^{2,p}} + \int_0^t \|f(u(\tau))\|_{Y^{2,p}} d\tau. \quad (3.8)
\]

Thus, from (3.6) – (3.8) and Lemma 3.2 we get
\[
\|G(u)\|_{Y(T)} \leq M + T(M + 1) \left[ 1 + 2C_0(M + 1)\bar{f}(M + 1) \right].
\]

If \( T \) satisfies
\[
T \leq \left\{ (M + 1) \left[ 1 + 2C_0(M + 1)\bar{f}(M + 1) \right] \right\}^{-1}. \quad (3.9)
\]

Then
\[
\|Gu\|_{Y(T)} \leq M + 1.
\]
Therefore, if (3.9) holds, then \( G \) maps \( Q(M; T) \) into \( Q(M; T) \). Now, we are going to prove that the map \( G \) is strictly contractive. Assume \( T > 0 \) and \( u_1, u_2 \in Q(M; T) \) given. We get

\[
G(u_1) - G(u_2) = \int_0^t F^{-1}S(t - \tau, \xi, A) \left[ \hat{f}(u_1)(\xi, \tau) - \hat{f}(u_2)(\xi, \tau) \right] d\tau, \quad t \in (0, T).
\]

By using the mean value theorem, we obtain

\[
\hat{f}(u_1) - \hat{f}(u_2) = \hat{f}^{(1)}(u_2 + \eta_1(u_1 - u_2))(u_1 - u_2),
\]

\[
D_\xi \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] = \hat{f}^{(2)}(u_2 + \eta_2(u_1 - u_2))(u_1 - u_2) D_\xi u_1 + \\
\hat{f}^{(1)}(u_2)(D_\xi u_1 - D_\xi u_2),
\]

\[
D_\xi^2 \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] = \hat{f}^{(3)}(u_2 + \eta_3(u_1 - u_2))(u_1 - u_2)(D_\xi u_1)^2 + \\
\hat{f}^{(2)}(u_2)(D_\xi u_1 - D_\xi u_2)(D_\xi u_1 + D_\xi u_2) + \\
\hat{f}^{(2)}(u_2 + \eta_4(u_1 - u_2))(u_1 - u_2)D_\xi^2 u_1 + \hat{f}^{(1)}(u_2)(D_\xi^2 u_1 - D_\xi^2 u_2),
\]

where \( 0 < \eta_i < 1, \ i = 1, 2, 3, 4 \). Thus using Hölder’s and Nirenberg’s inequality, we have

\[
\left\| \hat{f}(u_1) - \hat{f}(u_2) \right\|_{X_p} \leq \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p}, \quad (3.10)
\]

\[
\left\| \hat{f}(u_1) - \hat{f}(u_2) \right\|_{X_p} \leq \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p}, \quad (3.11)
\]

\[
\left\| D_\xi \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] \right\|_{X_p} \leq (M + 1) \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + D_\xi^2 u_1 \right\|_{Y_{2,p}}^2 + \\
\left\| D_\xi^2 \left[ \hat{f}(u_1) - \hat{f}(u_2) \right] \right\|_{X_p} \leq (M + 1) \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + \tilde{f}(M + 1) \left\| D_\xi u_1 \right\|_{X_p} + \tilde{f}(M + 1) \left\| D_\xi (u_1 - u_2) \right\|_{X_p} \leq \quad (3.12)
\]

\[
C^2 \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + D_\xi^2 u_1 \right\|_{X_p} + 
\]

\[
C^2 \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + D_\xi^2 u_1 \right\|_{X_p} + \quad (3.13)
\]

\[
3C^2 (M + 1)^2 \tilde{f}(M + 1) \left\| u_1 - u_2 \right\|_{X_p} + 2C^2 (M + 1) \tilde{f}(M + 1) \left\| D_\xi^2 (u_1 - u_2) \right\|_{X_p},
\]
where \( C \) is the constant in Lemma 3.1. From (3.10) – (3.11), using Minkowski’s inequality for integrals, Fourier multiplier theorems for operator-valued functions in \( X_p \) spaces and Young’s inequality, we obtain

\[
\|G(u_1) - G(u_2)\|_{Y(T)} \leq \int_0^t \|u_1 - u_2\|_{X_\infty} d\tau + \int_0^t \|u_1 - u_2\|_{Y_{2,p}} d\tau + 
\]

\[
\int_0^t \|f(u_1) - f(u_2)\|_{X_\infty} d\tau + \int_0^t \|f(u_1) - f(u_2)\|_{Y_{2,p}} d\tau \leq T \left[ 1 + C_1 (M + 1)^2 \bar{f}(M + 1) \right] \|u_1 - u_2\|_{Y(T)},
\]

where \( C_1 \) is a constant. If \( T \) satisfies (3.9) and the following inequality

\[
T \leq \frac{1}{2} \left[ 1 + C_1 (M + 1)^2 \bar{f}(M + 1) \right]^{-1},
\]

then

\[
\|G u_1 - G u_2\|_{Y(T)} \leq \frac{1}{2} \|u_1 - u_2\|_{Y(T)}.
\]

That is, \( G \) is a constructive map. By contraction mapping principle we know that \( G(u) \) has a fixed point \( u(x, t) \in Q(M; T) \) that is a solution of the problem (1.1) – (1.2). From (2.5) we get that \( u \) is a solution of the following integral equation

\[
u(t, x) = S_1(t, A) \varphi(x) + S_2(t, A) \psi(x) + \int_0^t F^{-1} S(t - \tau, \xi, A) \left[ 1 + L(\xi) \right]^{-1} \hat{f}(u)(\xi, \tau) \, d\tau, \quad t \in (0, T).
\]

Let us show that this solution is a unique in \( Y(T) \). Let \( u_1, u_2 \in Y(T) \) are two solution of the problem (1.1) – (1.2). Then

\[
\|u - u_2\|_{Y(T)} = \int_0^t \|u - u_2\|_{Y_{2,p}} d\tau.
\]

By definition of the space \( Y(T) \), we can assume that

\[
\|u_1\|_{X_\infty} \leq C_1(T), \quad \|u_1\|_{X_\infty} \leq C_1(T).
\]

Hence, by Lemmas 2.3, Minkowski’s inequality for integrals and Theorem 2.1 we obtain from (3.15)

\[
\|u - u_2\|_{Y_{2,p}} \leq C_2(T) \int_0^t \|u_1 - u_2\|_{Y_{2,p}} d\tau.
\]
From (3.16) and Gronwall’s inequality, we have \( \|u_1 - u_2\|_{Y^{2,p}} = 0 \), i.e. problem (1.1) − (1.2) has a unique solution which belongs to \( Y(T) \). That is, we obtain the first part of the assertion. Now, let \([0, T_0]\) be the maximal time interval of existence for \( u \in Y(T_0) \). It remains only to show that if (3.3) is satisfied, then \( T_0 = \infty \). Assume contrary that, (3.3) holds and \( T_0 < \infty \). For \( T \in [0, T_0) \) we consider the following integral equation

\[
\begin{align*}
\upsilon(x, t) &= S_1(t, A) u(x, T) + S_2(t, A) u_t(x, T) + \\
&\quad \int_0^t F^{-1} S(t - \tau, \xi, A) [1 + L(\xi)]^{-1} \hat{f}(\upsilon)(\xi, \tau) d\tau, \; \; t \in (0, T).
\end{align*}
\]

(3.17)

By virtue of (3.3), for \( T' > T \) we have

\[
\sup_{t \in [0, T)} \left( \|u\|_{Y^{2,p}} + \|u_t\|_{X^\infty} + \|u_{tt}\|_{Y^{2,p}} + \|u_t\|_{X^\infty} \right) < \infty.
\]

By reasoning as a first part of theorem and by contraction mapping principle, there is a \( T^* \in (0, T_0) \) such that for each \( T \in [0, T_0) \), the equation (3.17) has a unique solution \( \upsilon \in Y(T^*) \). The estimates (3.9) and (3.14) imply that \( T^* \) can be selected independently of \( T \in [0, T_0) \). Set \( T = T_0 - \frac{T^*}{2} \) and define

\[
\tilde{u}(x, t) = \begin{cases} 
\upsilon(x, t), & t \in [0, T] \\
\upsilon(x, t - T), & t \in [T, T_0 + \frac{T^*}{2}]
\end{cases}
\]

(3.18)

By construction \( \tilde{u}(x, t) \) is a solution of the problem (1.1)–(1.2) on \([T, T_0 + \frac{T^*}{2}]\) and in view of local uniqueness, \( \tilde{u}(x, t) \) extends \( u \). This is against to the maximality of \([0, T_0)\), i.e. we obtain \( T_0 = \infty \).

4. The Cauchy problem for the system of Boussinesq equation

Consider the Cauchy problem for the following nonlinear system

\[
(u_m)_{tt} - (Lu_m)_{tt} + \sum_{j=1}^N a_{mj} u_j(x, t) = f_m(u), \; \; x \in \mathbb{R}^n, \; \; t \in (0, T),
\]

(4.1)

\[
u_m(x, 0) = \varphi_m(x), \quad \frac{\partial}{\partial t} u_m(x, 0) = \psi_m(x), \; \; m = 1, 2, \ldots, N, \; \; N \in \mathbb{N},
\]

(4.2)

where \( u = (u_1, u_2, \ldots, u_N) \), \( a_{mj} \) are complex numbers, \( \varphi_m(x), \psi_m(x) \) are data functions and

\[
Lu = \sum_{i,j=1}^2 a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j}, \; \; a_{ij} \in \mathbb{C}.
\]
Let
\[ l_q = l_q(N) = \left\{ u = \{u_j\}, \, j = 1, 2, \ldots N, \|u\|_{l_q(N)} = \left( \sum_{j=1}^{N} |u_j|^q \right)^{\frac{1}{q}} < \infty \right\}. \]

(see [44, § 1.18]. Let \( A \) be the operator in \( l_q(N) \) defined by
\[ A = [a_{mj}], \quad a_{mj} = g_m 2^m, \quad m, j = 1, 2, \ldots, N, \quad D(A) = l_q(N) = \]
\[ \left\{ u = \{u_j\}, \, j = 1, 2, \ldots N, \|u\|_{l_q(N)} = \left( \sum_{j=1}^{N} 2^m u_j^q \right)^{\frac{1}{q}} < \infty \right\}. \]

Let
\[ X_{pq} = L^p(R^n; l_q), \quad Y^{s,p,q} = L^s(p, l_q), \quad Y_1^{s,p,q} = L^s(p, l_q) \cap L^1(R^n; l_q), \]
\[ Y_{\infty}^{s,p,q} = L^s(p, l_q) \cap L^\infty(R^n; l_q) \]
and
\[ Y^{2,p,q} = W^{2,p}(R^n; l_q^s, l_q) \]
\[ E_{0q} = B_{p,p}^{2(1 - \frac{1}{p})} \left( R^n; l_q^s, l_q \right). \]

From Theorem 3.1 we obtain the following result

**Theorem 4.1.** Let Conditon 2.0 hold. Assume \( \varphi_m, \psi_m \in Y^{2,p,q} \) and \( 1 < p < \infty \) for \( \frac{p}{2} < 2 \). Suppose the function \( u \to f(u) : R^n \times [0, T] \times E_{0q} \to l_q \)
is a measurable function in \( (x, t) \in R^n \times [0, T] \) for \( u \in E_{0q} \), \( f(x, t, \cdot) \)
and this function is continuous in \( u \in E_{0q} \) for \( x, t \in R^n \times [0, T] \); moreover, \( f(u) \in C^{(3)}(E_{0q}; l_q) \). Then problem (4.1) \( - (4.2) \) has a unique local strange solution \( u \in C^{(2)}([0, T_0); Y_{\infty}^{2,p,q}) \), where \( T_0 \) is a maximal time interval that is appropriately small relative to \( M \). Moreover, if
\[ \sup_{t \in [0, T_0)} \left( \|u\|_{Y^{2,p,q}} + \|u_t\|_{Y^{2,p,q}} + \|u_{tt}\|_{Y^{2,p,q}} \right) < \infty \quad (4.3) \]
then \( T_0 = \infty \).

**Proof.** By virtue of [43], the \( l_q(N) \) is a UMD space. It is easy to see that the operator \( A \) is \( R \)-positive in \( l_q(N) \). Moreover, by interpolation theory of Banach spaces [44, § 1.3], we have
\[ E_{0q} = (W^{2,p}(R^n; l_q^s, l_q), L^p(R^n; l_q))_{\frac{1}{p}, \frac{1}{q}} = B_{p,q}^{2(1 - \frac{1}{p})} \left( R^n; l_q^s(1 - \frac{1}{p}), l_q \right). \]

By using the properties of spaces \( Y^{s,p,q}, Y_{\infty}^{s,p,q}, E_{0q} \) we get that all conditions of Theorem 3.1 are hold, i.e., we obtain the conclusion.
Acknowledgements

The author would like to express a gratitude to Bulent Eryigit for his useful advices in English in preparing of this paper.

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