Yang-Mills Cosmologies
and Collapsing Gravitational Sphalerons

G.W. Gibbons
Alan R. Steif

Department of Applied Mathematics and Theoretical Physics
Cambridge University
Silver St.
Cambridge, CB3 9EW
United Kingdom
gwg1@amtp.cam.ac.uk
ars1001@amtp.cam.ac.uk

ABSTRACT: Cosmological solutions with a homogeneous Yang-Mills field which oscillates and passes between topologically distinct vacua are discussed. These solutions are used to model the collapsing Bartnik-McKinnon gravitational sphaleron and the associated anomalous production of fermions. The Dirac equation is analyzed in these backgrounds. It is shown explicitly that a fermion energy level crosses from the negative to positive energy spectrum as the gauge field evolves between the topologically distinct vacua. The cosmological solutions are also generalized to include an axion field.
1. Introduction

A few years ago the Einstein-Yang-Mills field equations were shown to admit spherically symmetric particlelike solutions in which the Yang-Mills configuration is bound by gravity in unstable equilibrium [1]. These solutions correspond to gravitational analogs of the standard model sphaleron solution [2]. In other words, they are saddlepoint solutions that lie on an energy barrier in field configuration space separating vacua with different topological charge. Just as the ordinary sphaleron mediates transitions between these vacua and leads to anomalous fermion production so should its gravitational analog. Consider a gauge field which vanishes as $t \to -\infty$ and is given by a pure gauge with unit Chern-Simons number as $t \to \infty$. According to the anomaly equation, the change in fermion number is given by the change in Chern-Simons number: $\Delta n = \int F \wedge F d^4x$. Ordinarily, even in a time-dependent background particles and anti-particles are created in pairs. However, as discussed in [3], this is not necessarily the case for fermions. For slowly varying fields, the anomalous production of fermions has a simple description in terms of the spectral flow of the Dirac operator. Consider the spectrum in the background of the sequence of instantaneous static field configurations with the initial and final gauge field configurations being pure gauge but with their Chern-Simons numbers differing by $n$. The initial and final spectra should coincide with the spectrum in the absence of a gauge field. However, the anomalous production is realized by $n$ energy levels crossing from negative to positive values of the energy. Thus, assuming one has begun in the vacuum with the negative energy Dirac sea filled, afterwards, there would be $n$ positive energy levels occupied corresponding to the creation of $n$ particles. Since the sphaleron configuration is the midpoint in the transition, one would expect that there would be a zero energy normalizable fermion mode in the sphaleron background. Indeed, this was shown to be the case for the Bartnik-McKinnon solution [4].

For the sphaleron of the standard model, one may view the time dependent process as being initiated by a high-energy collision of particles and ending with the fields going off to infinity. For the gravitational sphaleron, in addition to this scenario, there is the possibility of the configuration undergoing gravitational collapse and forming a horizon and singularity. Thus, the anomalously produced fermions could either enter the horizon or go off to infinity. [One should note that the third possibility of the fermions remaining bound outside the black hole is excluded by the no-hair theorem.] To calculate the fraction of fermions that go off to infinity, one should study the evolution of the fermion zero-mode. Thus, one should solve the time-dependent Dirac equation in the background of
the collapsing Bartnik-McKinnon solution. This is of course a difficult problem for which little progress can be made analytically.

In this paper, we study cosmological solutions to the Einstein-Yang-Mills equations with closed spatial slices with the motivation that they might serve as an approximation for the interior of the collapsing sphaleron. Since $SU(2)$ and the spatial sections are both three-spheres, there is a natural ansatz for a homogeneous $SU(2)$ gauge field involving a single function of time, $f$. The energy momentum tensor of the gauge field corresponds to pure radiation, and the spacetime to the Tolman universe. In contrast with abelian electromagnetism where the only way to obtain a pure radiation stress tensor is by some averaging procedure, with a non-abelian gauge field, one can obtain a pure radiation stress tensor directly from a classical solution. As we will discuss, these cosmological solutions share features of the sphaleron. The time evolution of $f$ corresponds to motion in a double well with minima at the topologically distinct vacua. There is an unstable gauge field configuration at the top of the energy barrier which corresponds to the reduction of the four-dimensional meron solution to three-dimensions. There is a fermion zero mode in the background of this configuration with the same internal structure as the zero mode in the Bartnik-McKinnon solution. Moreover, we show explicitly that as the field evolves between the topologically distinct vacua there is a single fermion level crossing corresponding to the anomalous production of one particle.

In Section 2, we review Robertson-Walker spacetimes with a homogeneous Yang-Mills field. In Section 3, these solutions are extended to include an axion, or antisymmetric tensor field. In Section 4, we begin our study of fermions in this homogeneous time-dependent background. Since the spatial slices are three spheres ($S^3$), spinor harmonics on $S^3$ are reviewed. The coupling to the gauge field is then included. We find a fermion zero mode in the background of the $f = 1/2$ saddlepoint solution. It is shown explicitly that as the gauge field evolves between vacua, a negative energy level arises from the Dirac sea, crosses zero energy at the saddlepoint configuration, and then enters the positive energy spectrum.

2. Yang-Mills Cosmologies

In this section, we consider cosmological solutions with a homogeneous Yang-Mills field as energy source. Since we wish to match the interior cosmological solution onto an exterior solution initially at rest, we take closed spatial sections because only for $k = 1$
is there a point at which the solution is momentarily static. Assume a Robertson-Walker ansatz
\[ ds^2 = -dt^2 + a^2(t)dΩ_3^2 \]
\[ = a^2(\eta)(-d\eta^2 + dΩ_3^2) \] (2.1)
with \( S^3 \) spatial sections where \( dΩ_3^2 \) is the round metric on \( S^3 \) and where \( \eta \) conformal time with \( \frac{d\eta}{dt} = a^{-1} \). The metric (2.1) is conformal to the Einstein static universe. In addition to the spacetime metric, consider an \( SU(2) \) Yang-Mills field \( A_μ^a \) where \( a \) is a Lie algebra index with associated field strength \( F \equiv dA - ie/2[A, A] \) where \( A \equiv A_μdx^μ \) is the Lie-algebra valued one-form and \( e \) the gauge coupling constant. Under a gauge transformation, \( U(x) \), the fields transform as \( A \rightarrow U^{-1}AU + i/e U^{-1}dU \) and \( F \rightarrow U^{-1}FU \). As we now discuss, since the group manifold of \( SU(2) \) is \( S^3 \) and coincides with the spatial sections, there is a natural ansatz for the gauge field which shares the symmetries of \( dΩ_3^2 \), the spatial metric.

Before discussing the gauge field ansatz, we review invariant forms on \( S^3 \). The natural imbedding of \( S^3 \) into the group manifold of \( SU(2) \) is given by
\[ g = \left( \begin{array}{ccc} x^4 + ix^3 & x^2 + ix^1 & x^4 + ix^1 \\ -x^2 + ix^1 & x^4 - ix^3 & \end{array} \right) = x^4 + ix^i \tau^i, \quad \det g = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1 \] (2.2)
where \( \tau^i \) are the Pauli spin matrices. \( SO(4) \) transformations are induced by its two fold cover \( SU(2)_L \times SU(2)_R \) corresponding to multiplication of \( g \) on the left and right by \( SU(2) \). Left invariant one-forms are given by (or more precisely, given by the pull back to \( S^3 \) of)
\[ e_i^L = -i\text{Tr} \tau_i g^{-1} dg = 2(x^4dx^i - x^i dx^4 + \epsilon_{ijk}x^j dx^k), \quad i = 1, 2, 3 \] (2.3)
and obey \( de_i^L = \frac{1}{2} \epsilon_{ijk}e_j^L \wedge e_k^L \). Under right multiplication the forms \( e_i^L \) transform in the adjoint. The right invariant one-forms are \( e_i^R = i\text{Tr} \tau^i dgg^{-1} \) and obey \( de_i^R = \frac{1}{2} \epsilon_{ijk}e_j^R \wedge e_k^R \). \( e_i^L \) and \( e_i^R \) can be written as \( e_i^L = \epsilon_{ijk}(M_{jk} + *M_{jk}) \) and \( e_i^R = \epsilon_{ijk}(M_{jk} - *M_{jk}) \), the self-dual and anti-self-dual parts of \( M_{ij} = x^i dx^j - x^j dx^i \) where \( *M_{ij} \equiv \frac{1}{2} \epsilon_{ijkl}M_{kl} \). The left and right invariant vector fields are dual to (2.3) and given by
\[ E_i^L = 1/2(x^4 \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^4} + \epsilon_{ijk}x^j \frac{\partial}{\partial x^k}) \]
\[ E_i^R = -1/2(x^4 \frac{\partial}{\partial x^i} - x^i \frac{\partial}{\partial x^4} - \epsilon_{ijk}x^j \frac{\partial}{\partial x^k}), \] (2.4)
\( E_3^L \) and \( E_3^R \) generate the transformation \((z^1, z^2) \rightarrow (e^{i\theta/2}z^1, e^{i\theta/2}z^2)\) and \((z^1, z^2) \rightarrow (e^{-i\theta/2}z^1, e^{i\theta/2}z^2)\) respectively where \( z^1 \equiv x^4 + ix^3 \) and \( z^2 \equiv x^1 + ix^2 \). \( L_i = -iE_i^L \) and \( R_i = -iE_i^R \) obey the \( SU(2) \) commutation relations \([L_i, L_j] = i\epsilon_{ijk}L_k, [R_i, R_j] = i\epsilon_{ijk}R_k \),
and $[L_i, R_j] = 0$. The round metric is the pull back to $S^3$ of the flat four dimensional metric
\[ ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2 \]
and in terms of the matrix $g$ (2.2) takes the form
\[ ds^2 = -\frac{1}{2} \text{Tr} (g^{-1} dg)^2 = \frac{1}{4} e^L_i \otimes e^L_i = \frac{1}{4} e^R_i \otimes e^R_i. \] (2.6)

(2.6) is the bi-invariant metric with isometry group $SU(2)_L \times SU(2)_R$ corresponding to left and right translations.

There is a natural ansatz for the gauge field on $S^3$ which takes the form
\[ A = i e f(\eta) g^{-1} dg = -\frac{1}{e} f(\eta) e^L_i \tau_i. \] (2.7)

Since (2.7) is invariant under left translation and changes by a global gauge transformation under right translation, the gauge field is homogeneous sharing the symmetry group of the $S^3$ spatial slices. $f$ and $\eta$ are dimensionless while $e^2$ has units of $[M]^{-1}[L]^{-1}$. The field strength associated with (2.7) is
\[ F = F^i \tau_i, \quad F^i = -\frac{f}{e} d\eta \wedge e^L_i + \frac{1}{2e} f(f-1) \epsilon^i_{jk} e^L_j \wedge e^L_k. \] (2.8)

From this expression or directly from (2.7), one observes that $f = 0$ and $1$ correspond to vacuum, or pure gauge field, configurations. For a general (non-vacuum) configuration, the Chern-Simons number is given by the integral $N_{CS} = \int_{S^3} \omega_3$ of the three-form
\[ \omega_3 = \frac{e^2}{8\pi^2} \text{Tr}(A \wedge dA - \frac{2ie}{3} A \wedge A \wedge A) \] (2.9)

satisfying $d\omega_3 = \frac{e^2}{8\pi^2} \text{Tr} F \wedge F$. Substituting (2.7) in (2.9) and using the fact that (2.6) implies that $\frac{1}{8} \int e_1 \wedge e_2 \wedge e_3 = 2\pi^2$, the area of the unit 3-sphere, one eventually finds $N_{CS} = 3 f^2 (1 - 2/3 f)$. The $f = 1$ configuration has unit topological, or Chern-Simons, number corresponding to the fact that the map $g$ has unit winding number.

The gauge field should satisfy the Yang-Mills field equation $*D * F = D_\mu F^{\mu \nu} = 0$ where $D \equiv d - ie[A,]$ is the gauge covariant derivative and $*$ is the Hodge dual. Substituting in the ansatz (2.7), one obtains an equation of motion for $f$
\[ \frac{\partial^2 f}{\partial \eta^2} = -\frac{\partial}{\partial f} V, \quad V \equiv 2(f^2 - f^2). \] (2.10)
The field equation is independent of the scale factor $a(\eta)$ since the Yang-Mills field equation is conformally invariant. (2.10) describes a particle moving in a double well potential $V$ with minima at the pure gauge configurations $f = 0$ and 1. These are the only pure gauge configurations within this ansatz. There is a conserved (dimensionless) energy type quantity $E = \frac{1}{2} \dot{f}^2 + V(f)$ for the equation of motion (2.10). From the form of the field strength above, one can identify the electric and magnetic field strength with the kinetic and potential energy respectively. In addition to the pure gauge configurations, the $f = 1/2$ configuration at the top of the energy barrier with $E = E_0 \equiv 1/8$ is also a static solution which is unstable and has non-zero (purely magnetic) field strength [7]. This solution in fact corresponds to the reduction to three dimensions of the four dimensional Euclidean Yang-Mills meron solution [5]. Using the conserved energy, $E$, one can integrate to obtain $f(\eta)$ in terms of elliptic functions. For $E = E_0$, there are two solutions starting and finishing at the top of the energy barrier that take the simple form

$$f(\eta) = \frac{1}{2} \pm \frac{\sqrt{2} \exp \sqrt{2} \eta}{1 + \exp 2\sqrt{2} \eta}.$$  

(2.11)

The coupling of the Yang-Mills field to gravity is governed by the energy-momentum tensor

$$T_{\mu\nu} = 3 \sum_{i=1}^{3} (F_{\mu\alpha}^i F^{i\alpha} - \frac{1}{4} g_{\mu\nu} F_{i\alpha\beta}^i F^{i\alpha\beta}).$$

(2.12)

Thus, each term in (2.12) behaves locally like the stress tensor of a Faraday flux tube with longitudinal tension and perpendicular pressure. For our ansatz, the three flux tubes are mutually orthogonal and have equal strengths. Thus, the energy-momentum tensor is of the form of pure radiation $P = \rho/3$ where the energy density is given by $\rho = \rho_0/a^4$, $\rho_0 = 12 \frac{P}{e}$. The solution to Einstein’s equation for pure radiation is a Tolman universe with scale factor in conformal time given by $a(\eta) = (32\pi E)^{1/2} l_0 \sin \eta$. Here, $l_0 = G^{1/2} e^{-1}$ defines the only length scale in the system. By contrast, $m_0 = G^{-1/2} e^{-1}$ sets the scale of the mass of the Bartnik-McKinnon solutions.

The scale factor $a(\eta)$ has the familiar $a \sim t^{1/2}$ form for small time for a radiation dominated universe. These solutions describe an expanding universe of radiation in which the Yang-Mills field constituting the radiation oscillates in the potential well given in (2.10). We note that even though $f = 1/2$ is a static solution to the Yang-Mills equation with the field strength, $F_{\mu\nu}^i$, independent of time, the scale factor is time dependent. The lifetime
of the universe is $\Delta \eta = \pi$, or $\Delta t \sim \sqrt{E_0}$. The number of oscillations in the potential well in the lifetime of the universe can be obtained by integrating $f$. For large $E$, one has

$$\Delta \eta = \int \frac{df}{\sqrt{2(E - V(f))}} \propto E^{-1/4}, \quad (2.13)$$

the integral being taken between the turning points. Thus, the number of oscillations in the lifetime of the universe grows as $N \sim E^{1/4}$ for large $E$. As $E$ tends to $E_0$ from above, the period of an oscillation diverges as $\Delta \eta \sim (E - E_0)^{-1/2}$.

As stated earlier, one motivation for studying these solutions is that they might serve as a model for the interior of the collapsing Bartnik-McKinnon sphaleron just as a simple model of the gravitational collapse of a star is provided by a portion of a $k = 1$ pressure free Friedmann universe. There are, however, certain difficulties with modelling the interior by a homogeneous Yang-Mills field. First, unlike the case of a pressure free perfect fluid, even if the stress tensor were initially homogeneous and isotropic, there is no reason to suppose that it will remain so. Second, the Bartnik-McKinnon solution more closely approximates a shell of Yang-Mills gauge field rather than a ball, and numerical studies suggest that the shell becomes thinner as the configuration collapses. Third, since the Yang-Mills field possesses non-zero pressure, there will be a pressure jump, and therefore, unlike pressureless dust one cannot simply attach the homogeneous interior directly onto the exterior vacuum. This last problem might be resolved provided one could smooth out the boundary between the homogeneous solution and the exterior vacuum.

3. Axion Field

In this section, we consider the addition of an axion field, $H_{\mu\nu\rho}$. The natural ansatz for this field which shares the symmetries of $S^3$ is given by

$$H = h(\eta)e^3 \quad (3.1)$$

where $e^3$ is the volume form on $S^3$ with metric $d\Omega^2$. The equations of motion for $H$ are $d*H = 0$ and $dH = 0$. (3.1) automatically satisfies the first, while the second implies that $h$ is a constant. The energy-momentum tensor for $H$ is

$$T_{\mu\nu} = H_{\mu\lambda\rho}H^{\lambda\rho}_{\nu} - \frac{1}{6}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}. \quad (3.2)$$

Substituting in (3.1), one finds that $T_{\mu\nu}$ is of the form of a perfect fluid with $P = \rho = h^2/a^6$. 
Let us now find the solution for the scale factor, \( a \), with the combined Yang-Mills and axion fields as energy source. The Friedmann equation is

\[
\frac{\dot{a}}{a^2} + \frac{1}{a^2} = \frac{8\pi G}{3} \rho \tag{3.3}
\]

or in conformal time,

\[
\left( \frac{da}{d\eta} \right)^2 + a^2 = \frac{8\pi G}{3} \rho a^4 \tag{3.4}
\]

where the energy density, \( \rho \), is the sum of the Yang-Mills and axion contributions \( \rho = \rho_{YM} + \rho_{AXION} = 12E/e^2a^{-4} + h^2a^{-6} \). (3.4) can be integrated exactly to yield the scale factor in closed form

\[
a(\eta) = ((\alpha^2 + \beta)^{1/2} \sin 2(\eta - \eta_0) + \alpha)^{1/2},
\]

\[
\sin 2\eta_0 = \frac{\alpha}{\sqrt{\alpha^2 + \beta}}, \quad \alpha \equiv \frac{16\pi G E}{e^2}, \quad \beta \equiv \frac{8\pi G h^2}{3}. \tag{3.5}
\]

The solution still has an expansion followed by a contraction. Near the singularity, however, the contribution of \( H \) to the energy density dominates with the scale factor vanishing as \( a \sim t^{1/3} \) rather than \( a \sim t^{1/2} \) as in the Tolman universe.

4. Fermions

We now consider fermions propagating in the Einstein-Yang-Mills backgrounds discussed above [9]. The Dirac equation for a massless Dirac fermion coupled to gravity and a Yang-Mills field is given by

\[
i\gamma^\mu (\nabla_\mu - ieA_\mu T^a)\Psi = 0 \tag{4.1}
\]

where \( \nabla_\mu \) is the covariant derivative and \( T^a \) are generators for the gauge group. The massless Dirac equation is conformally invariant. That is, given a solution (\( \Psi, g_{\mu\nu}, A_\mu \)) in \( d \) spacetime dimensions, there is another solution (\( \Omega^{d-2} \Psi, \Omega^{-2}g_{\mu\nu}, A_\mu \)). Thus, if \( \Psi \) is a solution in the spacetime (2.1), \( \tilde{\Psi} = a^{3/2}\Psi \) is a solution to the Dirac equation on \( S^3 \times R \), the Einstein static universe (ESU). The Dirac equation (4.1) on ESU takes the form

\[
(i\gamma^0 \frac{\partial}{\partial \eta} + i\partial_3 + eA^a T^a)\tilde{\Psi} = 0 \tag{4.2}
\]
where $D_3$ is the Dirac operator on $S^3$. Consider the chiral representation for the gamma matrices

$$
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}
$$

(4.3)

where $\sigma^i$ are Pauli spin matrices.

In the absence of the gauge field, solving (4.2) reduces to finding spinor harmonics on $S^3$. Before considering spinors, we review scalar harmonics on $S^3$. The scalar spherical harmonics satisfying $\nabla_3^2 f_l = \lambda_l f_l$ are given by

$$
f_l = \alpha_{i_1 \ldots i_l} n^{i_1} \cdots n^{i_l}, \quad n^i = x^i/r, \quad \alpha_{i_1 \ldots i_l} = 0 \quad \text{(traceless)}
$$

(4.4)

$$
\lambda_l = -l(l+2)
$$

where $\nabla_3^2$ is the Laplacian on $S^3$. This is easily checked using the form of the four-dimensional flat space Laplacian

$$
\nabla_4^2 = \frac{\partial^2}{\partial r^2} + \frac{3}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \nabla_3^2
$$

(4.5)

and using $\nabla_3^2 (r^l f_l) = 0$ from (4.3). [In $n$-dimensions, the harmonics are also given by (4.3), but with eigenvalues $\lambda_l = -l(l+n-1)$.] Since $SU(2)_L \times SU(2)_R$ is the two-fold cover of $SO(4)$, $\alpha_{i_1 \ldots i_l}$ can be written in terms of spinor components as $\alpha_{A_1 \hat{A}_1 \ldots A_l \hat{A}_l}$ where undotted and dotted components correspond to left and right factors of $SU(2)$ respectively. The fact that $\alpha_{i_1 \ldots i_l}$ is totally symmetric and traceless implies that $\alpha_{A_1 \hat{A}_1 \ldots A_l \hat{A}_l}$ is totally symmetric in dotted and undotted indices separately. Since the $\frac{k}{2}$ representation of $SU(2)$ is given by rank $k$ totally symmetric $SU(2)$ tensors, the harmonics transform in the $(\frac{l}{2}, \frac{l}{2})$ representation of $SU(2)_L \times SU(2)_R$. [Note that the harmonics for a squashed sphere form a representation of some subgroup of $SU(2)_L \times SU(2)_R$.] From (2.4), one can show

$$
E_i^L E_i^L = E_i^R E_i^R = \nabla_3^2
$$

(4.6)

If one diagonalizes $L_3$, and $R_3$ with eigenvalues $m$ and $n$, then the harmonics can be expressed as monomials in $z_1, z_2, \bar{z}_1, \bar{z}_2$ of degree $l$. The eigenvalues $m$ and $n$ of these monomials are determined from those of $z_1, z_2, \bar{z}_1, \bar{z}_2$. They are $m = \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$ and $n = -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$.

Having constructed the scalar harmonics, we now consider spinor harmonics on $S^3$. Like the scalar harmonics, spinor harmonics are most simply expressed in terms of quantities in the four-dimensional imbedding space. The four-dimensional Euclidean flat space Dirac operator in spherical polar coordinates is

$$
\bar{\phi} = \gamma^r \left( \frac{\partial}{\partial r} + \frac{3}{2} \frac{1}{r} \right) + \frac{1}{r} \bar{D}_3
$$

(4.7)
The spinor harmonics satisfying \( D_3 \Psi_l = \lambda_l \gamma^r \Psi_l \) are given by

\[
\Psi_l = T^\alpha_{i_1\ldots i_l} n^{i_1} \cdots n^{i_l}, \quad \gamma^i_{\alpha\beta} T^\beta_{i_1\ldots i_l} = 0
\]

\[\lambda_l = -(l + \frac{3}{2})\]  

where \( T^\alpha_{i_1\ldots i_l} \) is constant with \( \alpha \) a Dirac spinor index and \( i \) a vector index. This is easily checked using (4.7) and that (4.8) implies \( \hat{\theta} (\gamma^r \Psi_l) = 0 \). \( T^\alpha_{i_1\ldots i_l} \) can be decomposed into tensors \( T_{AA_1\hat{A}_1\ldots A_l\hat{A}_l} \) with \( l + 1 \) undotted and \( l \) dotted indices and tensors \( T_{\hat{A}_1\ldots A_l\hat{A}_1\ldots\hat{A}_l} \) with \( l \) undotted and \( l + 1 \) dotted indices. (4.8) implies that they are totally symmetric tensors. Therefore, the spinor harmonics fall into the \((l,0), (l,1), \ldots, (l,l)\) representations of \( SU(2)_L \times SU(2)_R \) with \( l = 0, 1, \ldots \). Since one has \( \{D_3, \gamma^r\} = 0 \), the harmonics with positive eigenvalues can be obtained by applying \( \gamma^r \) to (4.8).

The spinor harmonics can also be expressed in terms of an orthonormal frame \( e^\mu_i \) on \( S^3 \). The Dirac operator is then given by

\[
i D_3 = i \gamma^\mu D_3 \mu, \quad D_3 \mu = \partial_\mu + \frac{1}{2} \omega^{ij}_\mu \Sigma^{ij}, \quad \Sigma^{ij} = -[\gamma^i, \gamma^j]/4
\]

where \( \Sigma^{ij} \) are the Lorentz generators in the spinor representation with the Dirac matrices (4.3) satisfying \( \{\gamma^i, \gamma^j\} = -2\delta^{ij} \) and \( \omega^{ij}_\mu \), the spin connection. As usual Latin and Greek indices refer to tangent and curved space and are related by \( e^\mu_i \). If we let the left-invariant vector fields (2.4) \( e^\mu_i = 2E_i^L \mu \) define the orthonormal frame, then, from (2.3), the dual one-forms \( e^i_\mu = \frac{1}{2} e^{L_i} \mu \) satisfy \( de_i = \epsilon_{ijk} e_j \wedge e_k \). We can now read off the spin connection \( \omega_{ij} = \epsilon_{ijk} e^k \). Substituting the spin connection and the Dirac matrices in (4.9), one eventually finds

\[
-i D_3 = \begin{pmatrix} 0 & \hat{H}_0 \\ -\hat{H}_0 & 0 \end{pmatrix}, \quad \hat{H}_0 \equiv 4S \cdot L + 3/2, \quad S \cdot L \equiv S^i L^i
\]

with \( S^i = \sigma^i/2 \) and \( L^i = -i E^L_i \). \( \hat{H}_0 \) commutes with \( R_i \) and \( J_i = L_i + S_i \). The spinor harmonics fall into the \((l,0), (l,1), \ldots, (l,l)\) representations of \( SU(2)_J \times SU(2)_R \) and can be labelled by \((l,n,j_\pm \equiv \frac{l}{2} \pm \frac{1}{2}, j_3)\), the eigenvalues of \( R^2, R_3, J^2 \), and \( J_3 \). They take the form of products of 2-component spinors and scalar harmonics. Since \( S \cdot L = (J^2 - L^2 - S^2)/2 = (j(j + 1) - \frac{l}{2}(\frac{l}{2} + 1) - \frac{3}{4})/2 \), we find that the spectrum of \( \hat{H}_0 \) is

\[
\hat{H}_0 |l,n,j_\pm,j_3> = \mu^+_l |l,n,j_\pm,j_3>, \quad j_\pm = \frac{l}{2} \pm \frac{1}{2}
\]

\[
\mu^+_l = l + \frac{3}{2}, \quad l = 0, 1, \ldots
\]

\[
\mu^-_l = -l - \frac{1}{2}, \quad l = 1, 2, \ldots
\]
We thus have recovered the spectrum (4.8). The spectrum of $\hat{H}_0$ is symmetric about zero. The lowest lying states in the positive energy spectrum are $|0, 0, \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}> = |\pm > (S_3 |\pm > = \frac{1}{2} |\pm >)$ with eigenvalue $\frac{3}{2}$ corresponding to the two components of the constant spinor (with respect to the left invariant frame). The highest energy states in the negative energy spectrum are $|\frac{1}{2}, \frac{1}{2}, 0, 0, 0> = (\bar{z}_1 |+ > - z_2 |>)/\sqrt{2}$ and $|\frac{1}{2}, -\frac{1}{2}, 0, 0, 0> = (\bar{z}_2 |+ > - z_1 |>)/\sqrt{2}$ with eigenvalue $-\frac{3}{2}$.

We now consider the Dirac equation (4.2) with a non-zero gauge field background. Let the fermions transform in the fundamental representation of the gauge group so that $T^a = \frac{\tau^a}{2}$ with $\tau^a$, the Pauli spin matrices. From (2.7), we have $e^\mu_i A^a_\mu = -2f/e\delta^a_i$ implying $\sigma^i e^\mu_i A^a_\mu T^a = -4f/eS \cdot T$. The full time-dependent Dirac equation (4.2) now becomes

$$i\frac{\partial \tilde{\psi}_{\pm}}{\partial \eta} = \mp \hat{H} \tilde{\psi}_{\pm}, \quad \hat{H} = \hat{H}_0 + 4f(\eta)S \cdot T$$

with $\hat{H}_0$ given in (4.10) and where we have decomposed $\tilde{\Psi} = \left( \begin{array}{c} \tilde{\psi}_+ \\ \tilde{\psi}_- \end{array} \right)$ into $\tilde{\psi}_{\pm}$, left and right chiral components each carrying two-component Lorentz and internal indices. For a vanishing gauge field ($f = 0$), $\hat{H}$ reduces to $\hat{H}_0$. The $f = 1$ configuration is pure gauge with Chern-Simons number unity, and thus, the spectrum of $\hat{H}$ should be identical to $\hat{H}_0$. Indeed, we can redefine $\vec{L}' \equiv \vec{L} + \vec{T}$, and the spectrum in the absence of the gauge field is recovered. Since $\hat{H}_0$ and the interaction $\hat{H}_I \equiv 4fS \cdot T$ do not commute, one cannot in general solve (4.12). However, it is possible to do so within the $s$-wave sector where $L_i = R_i = 0$. In this sector, there are the singlet (or hedgehog) $\chi_1$ and triplet $\chi_2$ states satisfying $(S + T)^2 \chi_1 = 0$ and $(S + T)^2 \chi_2 = 2\chi_2$. The time-dependent single and triplet wave functions are found from (4.10) and (4.12) and by performing the conformal transformation back to the Robertson-Walker spacetime [9]. One finds

$$\psi_{1,2}^{\pm} = a^{-3/2} \chi_{1,2} \exp \pm i \int_0^\eta E_1(\eta) d\eta, \quad E_1(\eta) = \frac{3}{2} - 3f(\eta)$$

Let us now consider the adiabatic approximation where the Hamiltonian, $\hat{H}$, is treated instantaneously. The energies of the single and triplet states are given by $E_1(\eta)$ and $E_2(\eta)$. Now as $f$ evolves between the vacua $f = 0$ and $f = 1$, we see that the singlet (hedgehog) state passes from the lowest lying state in the positive energy spectrum, $E_1 = \frac{3}{2}$, to the highest lying state in the negative energy spectrum, $E_1 = -\frac{3}{2}$. At the midpoint of the
transition corresponding to the unstable \( f = 1/2 \) solution, the hedgehog \( \chi_1 \) is a zero energy mode \([10]\). As pointed out earlier, \( f = 1/2 \) is the analog of the sphaleron, and like the Bartnik-McKinnon sphaleron, it has a fermion zero mode. Since \( E_2 \) remains positive for \( 0 < f < 1 \), there is no level crossing in the triplet state. In fact, since the interaction \( 2S \cdot T \) is bounded between \(-3/2\) and \(1/2\), among all sectors the only level crossing occurs for the singlet state. This is consistent with the anomaly equation according to which if the Chern-Simons number of the gauge field changes by unity, there will be one level crossing in each chiral sector. For Dirac fermions, the level crossing cancels between chiral sectors.

We now consider the effect of the antisymmetric tensor field on the fermions. Consider an interaction of the form

\[
S = ib \int \bar{\Psi} H_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho \Psi \sqrt{g} d^4x
\]

with \( b \) having units \([L]^{1/2}[M]^{-1/2}\). The Dirac equation now becomes

\[
(i\gamma^\mu \nabla_\mu + e\gamma^\mu A_\mu^a T^a + ibH_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho)\Psi = 0.
\]

One can show generally that a co-closed axion field (\( H = \ast d\phi \)) can be absorbed by a local chiral transformation on \( \Psi \) of the form

\[
\Psi \rightarrow \exp (-6ib\gamma^5\phi) \Psi.
\]

For the axion field, (3.1), one finds that \( \phi = \int \frac{b}{a^2} \) and therefore, from (4.13) and (4.16), one finds the time-dependent triplet and single wave functions

\[
\psi_1^+ = a^{-3/2} \chi_1 \exp \pm i \int (E_1(\eta) - \frac{6bh}{a^2})d\eta
\]

\[
\psi_2^+ = a^{-3/2} \chi_2 \exp \pm i \int (E_2(\eta) - \frac{6bh}{a^2})d\eta.
\]

In this paper, we have studied cosmological solutions with a Yang-Mills field as a potential analytic model of the gravitational collapse of a Bartnik-McKinnon sphaleron. This provides a qualitative picture of anomalous fermion production and gives some support to the viewpoint proposed in the previous paper \([4]\). However, it clearly cannot give a detailed quantitative description. This can, presumably, only be achieved by a numerical analysis.

**Acknowledgements**

We would like to thank Peter Aichelburg, Piotr Bizon, and Paulo Moniz for discussions. A.S. wishes to acknowledge the financial support of the SERC.
References

[1] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* **61** (1988) 141.

[2] D. Gal’tsov and M. Volkov, *Phys. Lett.* **B273** (1991) 255; D. Sudarsky and R. Wald, *Phys. Rev. D* **46** (1992) 1453.

[3] R. Wald, *Ann. Phys.* **118** (1979) 490; G. Gibbons, *Phys. Lett. B* **84** (1979) 431; G. Gibbons, *Ann. Phys.* **125** (1980) 98; N. Christ, *Phys. Rev. D* **21** (1980) 1591.

[4] G. Gibbons and A. Steif, *Phys. Lett. B.* **314** (1993) 13.

[5] V. de Alfaro, S. Fubini, and G. Furlan, *Nuovo Cimento* **50** (1979) 4523.

[6] There is a large literature on cosmological solutions with a Yang-Mills field. See for example: J. Tafel, in *Proc. Geometrical and Topological Methods in Gauge Theories*, ed. J. Harnard and S. Shnider (Springer, Berlin, 1980); M. Henneaux, *Journal Math. Phys.* **23** (1982) 830; P. Moniz and J. Mourao, *Class. Quantum Grav.* **8** (1991) 1815; Y. Verbin and A. Davidson, *Phys. Lett. B* **229** (1989) 364; D. Gal’tsov and M. Volkov, *Phys. Lett. B* **256** (1991) 17; O. Bertolami, et. al., *Int. Journal Mod. Phys.* **6** (1991) 4149; M. Bento, O. Bertolami, P. Moniz, J. Mourao, and P. Sa, *Class. Quantum Grav.* **10** (1993) 285; P. Moniz, J. Mourao, and P. Sa, *Class. Quantum Grav.* **10** (1993) 517.

[7] Y. Hosotani, *Phys. Lett. B* **147** (1984) 44.

[8] Z. Zhou, *Helvetica Physica Acta* **65** (1992) 767.

[9] Fermions in Euclidean Yang-Mills wormholes were discussed in S-J. Rey, *Nucl. Phys. B* **336** (1990) 146; O. Bertolami, “Wormhole solutions of Euclidean Yang-Mills with Fermions, 13th International Conference on Group Theoretical Methods, Moscow (1990).

[10] A. Hosoya and W. Ogura, *Phys. Lett. B* **225** (1989) 117.