STRONG CONVERGENCE RATE OF RUNGE–KUTTA METHODS AND SIMPLIFIED STEP-N EULER SCHEMES FOR SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS

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Abstract. This paper focuses on the strong convergence rate of both Runge–Kutta methods and simplified step-N Euler schemes for stochastic differential equations driven by multi-dimensional fractional Brownian motions with $H \in \left(\frac{1}{2}, 1\right)$. Based on the continuous dependence of both stage values and numerical schemes on driving noises, order conditions of Runge–Kutta methods are proposed for the optimal strong convergence rate $2H - \frac{1}{2}$. This provides an alternative way to analyze the convergence rate of explicit schemes by adding ‘stage values’ such that the schemes are comparable with Runge–Kutta methods. Taking advantage of this technique, the optimal strong convergence rate of simplified step-N Euler scheme is obtained, which gives an answer to a conjecture in [3] when $H \in \left(\frac{1}{2}, 1\right)$. Numerical experiments verify the theoretical convergence rate.

1. Introduction

In this paper, we investigate the strong convergence rate of numerical schemes for the following stochastic differential equation (SDE)

\begin{equation}
    dY_t = V(Y_t) dX_t = \sum_{l=1}^{d} V_l(Y_t) dX^l_t, \quad t \in (0, T],
\end{equation}

\begin{equation}
    Y_0 = y \in \mathbb{R}^m,
\end{equation}

where $X_t = (X^1_t, \cdots, X^d_t) \in \mathbb{R}^d$ with $X^1_t = t$ and $X^2_t, \cdots, X^d_t$ being independent fractional Brownian motions with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. The well-posedness is interpreted through Young’s integral or fractional calculus (see [5, 14, 18] and references therein) pathwisely.

The fractional Brownian motion (fBm) $\{B_t^H\}_{t \in [0,T]}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a centered Gaussian process with continuous sample paths. Its covariance satisfies

\begin{equation}
    \mathbb{E}\left[B^H_s B^H_t\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad \forall \ s, t \in [0, T],
\end{equation}

where $H \in (0, 1)$ is the Hurst parameter. The fBm is a semi-martingale and Markovian process only when $H = \frac{1}{2}$, that is, standard Brownian motion. Otherwise, the process exhibits long-range or short-range dependence when $H > \frac{1}{2}$ or $H < \frac{1}{2}$.

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respectively. It fits better than Markovian ones in models of economics, fluctuations of solids, hydrology and so on, which motivates numerous researches (see e.g. [15]). However, it brings more obstacles in both the simulation of noises and analysis of optimal strong convergence rate.

On the one hand, nontrivial covariance causes difficulties in simulating iterated integrals of fBms from Taylor expansion in multi-dimensional case. For the standard Brownian case, iterated integrals can be simulated by specific independent and identically distributed Gaussian random variables. For the case $H \neq \frac{1}{2}$, other techniques need to be explored. An implementable choice is substituting the $N$-level iterated integral of $X$ by $\frac{1}{N!}(\Delta X_k)^N$ directly, $N \geq 2$. The corresponding numerical schemes are called simplified step-$N$ Euler schemes (see [1, 3, 4, 7]). Another way is taking advantage of internal stages values to design Runge–Kutta methods. These methods are derivative free and can be particularly chosen as structure-preserving methods or stability preserving methods (see [6, 8, 16] and references therein).

On the other hand, without properties of martingale, approaches to analyze the convergence rate for schemes in fractional setting are different from those via the fundamental convergence theorem in standard Brownian case. In [3], authors analyze the modified Milstein scheme, which is called simplified step-2 Euler scheme in this paper. They smooth noises by piecewise-linear approximations, i.e., the Wong–Zakai approximations, and obtain the pathwise convergence rate $(H - \frac{1}{2})^{-\frac{1}{3}}$ in Hölder norm for any $H \in (\frac{1}{3}, 1)$. They conjecture that the optimal rate in supremum norm is $2H - \frac{1}{2}$ based on the strong convergence rate of the Lévy area of $X$. In [9], by estimates for the Lévy area type processes, authors prove the optimal strong convergence rate of Crank-Nicolson scheme is $2H - \frac{1}{2}$ in general. Namely, denote by $Y^n$ the numerical solution with time step $h = \frac{T}{n}$, then

\[
\sup_{t \in [0,T]} \|Y_t - Y^n_t\|_{L^p(\Omega)} \leq C h^{2H - \frac{1}{2}}, \quad p \geq 1.
\]

(1.3)

It inspires us a potential approach to gain the optimal global strong convergence rate for the schemes under study without utilizing the Wong–Zakai approximations.

Our main idea is regarding general Runge–Kutta methods as implicit ones determined through the internal stage values. We show that the constructed continuous versions of internal stage values and numerical solutions are continuously dependent on the driving noise $X$. This robustness coincides with the property of the exact solution. Combining the estimates of internal stage values and that of iterated integrals of $X$ (see [9, 11]), we obtain order conditions of the optimal strong convergence rate for Runge–Kutta methods. Namely, if the coefficients of a Runge–Kutta method satisfy (4.5), then

\[
\sup_{t \in [0,T]} \|Y_t - Y^n_t\|_{L^p(\Omega)} \leq C h^{2H - \frac{1}{2}}, \quad p \geq 1.
\]

(1.4)

Notice that condition (4.5) can be satisfied by Crank-Nicolson scheme and we give the $L^p(\Omega)$-estimate for the error in supremum norm.

Furthermore, we compare simplified step-$N$ schemes with Runge-Kutta methods satisfying condition (4.5). By means of adding internal stage values to explicit simplified step-$N$ Euler schemes, we express these schemes in an implicit way. This approach leads us to the optimal strong convergence rate $2H - \frac{1}{2}$ by avoiding the estimation of the Wong–Zakai approximations. Our result gives an answer to the
conjecture in [3] for $H \in (\frac{1}{2}, 1)$. Numerical experiments are represented to verify this optimal rate. For the case $H \in (\frac{1}{2}, \frac{1}{3})$ where equation (1.1) is understood in the rough path framework (see e.g. [2, 5, 14]), the optimal strong convergence rate of simplified step-2 Euler scheme is still an open problem. We refer to related works [1, 13] for more details.

The paper is organized as follows. In Section 2, we recall some definitions and results about fractional calculus and fBms. We prove the solvability of implicit Runge–Kutta methods and the continuous dependence of continuous versions of methods under study with respect to driving noises in Hölder semi-norm in Section 3. The order conditions for Runge–Kutta methods are derived for the strong convergence rate $2H - \frac{1}{2}$ in Section 4. Comparing the simplified step-2 Euler schemes with a Runge–Kutta method satisfying (4.5), we get the same strong convergence rate, which coincides with the conjecture given by [3]. Numerical experiments are performed in Section 5.

2. Preliminaries

In this section, we introduce some notations, definitions and results about fractional calculus and fBms. They are essential for us to prove the properties and strong convergence rates of numerical schemes in subsequent sections. We use $C$ as a generic constant which could be different from line to line.

2.1. Fractional calculus. Denote by $C([0, T]; \mathbb{R}^d)$ the space of continuous functions from $[0, T]$ to $\mathbb{R}^d$. For any $f \in C([0, T]; \mathbb{R}^d)$, $0 \leq s < t \leq T$ and $0 < \beta \leq 1$, the $\beta$-Hölder semi-norm of $f$ is defined by

$$
\|f\|_{s, t, \beta} = \sup \left\{ \frac{|f_v - f_u|}{|v - u|^{\beta}}, s \leq u < v \leq t \right\},
$$

where $|\cdot|$ is the Euclid norm in $\mathbb{R}^d$. Especially, we use $\|f\|_\beta := \|f\|_{0, T, \beta}$ for short.

The Hölder semi-norm can be expressed in an integral form by the Besov-Hölder embedding, which is a corollary from Garsia-Rademich-Rumsey inequality.

Lemma 2.1. ([5, Corollary A.2] Let $q > 1$, $\alpha \in (\frac{1}{q}, 1)$ and $f \in C([0, T]; \mathbb{R}^d)$. Then there exists a constant $C = C(\alpha, q)$ such that for all $0 \leq s < t \leq T$,

$$
\|f\|_{s, t, \alpha - \frac{1}{q}} \leq C \int_s^t \int_s^t \frac{|f_u - f_v|^q}{|u - v|^{1 + q\alpha}} du dv.
$$

Let $f \in C([s, t]; \mathbb{R})$ be $\beta$-Hölder continuous on $[s, t] \subseteq [0, T]$ with $1/2 < \beta < 1$ and $g : [s, t] \to \mathbb{R}$ be a step function defined by $g_t = g_0 \mathbf{1}_{(0)} + \sum_{k=0}^{n-1} g_k \mathbf{1}_{(t_k, t_{k+1})}$ with $s = t_0 < t_1 < \cdots < t_n = t$. The integral of $g$ with respect to $f$ can be defined piecewisely:

$$
\int_s^t g_t df_t := \sum_{k=0}^{n-1} g_k (f_{t_{k+1}} - f_{t_k}).
$$

For any $1/2 < \alpha < 1$, according to fractional calculus (see e.g. [18, Section 2]), it has the characterization:

$$
\int_s^t g_t df_t = (-1)^\alpha \int_s^t D_s^\alpha g_t D_t^{1-\alpha} F_t, dr.
$$
Here \((-1)^α = e^{-iα}\), \(F_r := f_r - f_t\), \(D^α_x g_r\) and \(D^{1-α}_x F_r\) are fractional Weyl derivatives of the order \(α\) and \(1 − α\) respectively:

\[
(D^α_x g)_{r} := \frac{1}{Γ(1 − α)} \left( \frac{g_r}{(r − s)^α} + α \int_{s}^{r} \frac{g_r - g_u}{(r - u)^α+1} \, du \right),
\]

\[
(D^{1-α}_x F)_{r} := \frac{(-1)^{1-α}}{Γ(α)} \left( \frac{F_r}{(t − r)^1-α} + (1 − α) \int_{r}^{t} \frac{F_r - F_u}{(u − r)^2-α} \, du \right).
\]

2.2. A priori estimate for the solution and iterated integrals. In the sequel, we denote by \(C^N_0(\mathbb{R}^m; \mathbb{R}^M)\) the space of bounded and \(N\)-times continuously differentiable functions \(V : \mathbb{R}^m \rightarrow \mathbb{R}^M\) with bounded derivatives.

The following lemma shows the well-posedness of (1.1), which means that the solution is continuously dependent on the driving noises in Hölder semi-norm, where almost all sample paths of \(X\) are \(β\)-Hölder continuous for any \(β \in (0, H)\). In the next section, we will show that the numerical schemes we consider could inherit a similar property.

**Lemma 2.2.** (see e.g. [5] Theorem 10.14) If \(V \in C^1_0(\mathbb{R}^m; \mathbb{R}^{m \times d})\) and \(1/2 < β < H\), then there exists a unique solution of (1.1) satisfying almost surely that

\[
\|Y\|_β \leq C(V, β, T) \max \left\{ \|X\|_β, \|X\|_β^{1/β} \right\},
\]

\[
\|Y\|_∞ \leq |y| + C(V, β, T) \max \left\{ \|X\|_β, \|X\|_β^{1/β} \right\},
\]

where \(\|Y\|_∞ := \{ |Y_u|, \ 0 \leq u \leq T \}\). Moreover, for some \(C_0 > 0\) and \(0 \leq s < t \leq T\) such that \(\|X\|_β|t − s|^{β} \leq C_0\), the estimate can be improved to

\[
\|Y\|_{s,t,β} \leq C(V, β, T, C_0)\|X\|_β.
\]

To get the strong convergence rate of numerical schemes, we recall some results from [9] [11]. For a numerical scheme, we apply the uniform partition of the interval \([0, T]\) with step size \(h = \frac{T}{n}\), \(n \in \mathbb{N}_+\) and denote \(t_k = kh\), \(k = 0, \ldots, n\).

**Lemma 2.3.** (see [9][11]) Let \(X^i_0 = t\) and \(X^i_1, \ldots, X^i_d\) be independent fBms with \(H > 1/2\). Then for any \(n \in \mathbb{N}_+\), it holds for any \(0 \leq t_i < t_j \leq T\) and \(p ≥ 1\) that

\[
\left\| \sum_{k=i}^{j-1} \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dX^1_u dX^2_s - \int_{t_k}^{t_{k+1}} \int_{s}^{t} dX^3_u dX^4_s \right] \right\|_{L^p(Ω)} \leq C |t_j - t_i|^{1/2} h^{2H - 1/2},
\]

\[
\left\| \sum_{k=i}^{j-1} \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} dX^1_u dX^2_s - \int_{t_k}^{t_{k+1}} \int_{s}^{t} dX^3_u dX^4_s \right] \right\|_{L^p(Ω)} \leq C |t_j - t_i|^{1/2} h^{H + 1/2},
\]

where \(C = C(p)\) above is independent of \(n\). Moreover, for any \(l_1, \ldots, l_N \in \{1, \ldots, d\}\), it holds that

\[
\left\| \sum_{k=i}^{j-1} \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s_{l_1}} \cdots \int_{t_k}^{s_{l_N}} dX^{l_{N'}}_{u_{N'}} \cdot \cdot \cdot dX^{l_1}_{u_{l_1}} dX^{l_2}_{u_{l_2}} dX^{l_3}_{u_{l_3}} \right] \right\|_{L^p(Ω)} \leq C |t_j - t_i|^{1/2} h^r,
\]
where \( r = N''H + N' - N'' - 1 \) when \( N'' = 2 \{ i : i \neq 1 \} \) is even, \( r = N''H + N' - N'' - H \) when \( N'' \) is odd, and \( C = C(p) \) is independent of \( n \).

Lemma [2.3] shows estimates for the Lévy area type processes and multiple integrals of \( X \). In particular, if \( N' = N'' = 2 \), then \( r = 2H - 1 < 2H - \frac{1}{2} \). This implies that the convergence rate of the 2nd-level iterated integrals of \( X \) in the form of (2.1) is higher than that in the form of (2.3).

**Lemma 2.4.** ([11, Proposition 8]) Let \( f \) be \( \beta \)-Hölder continuous stochastic process in \( L^{2p}(\Omega) \) with \( \frac{1}{2} < \beta < H \) and \( p \geq 1 \), i.e.,

\[
\sup \left\{ \frac{\| f_u - f_v \|_{L^{2p}(\Omega)}}{(v-u)^\beta}, \ 0 \leq u < v \leq T \right\} < \infty.
\]

If a sequence of stochastic processes \( \{ g_n \}_{n \in \mathbb{N}_+} \) satisfies \( g_n(t_i) = \sum_{k=0}^{i-1} \xi_{n,k} \) and

\[
\| g_n(t_j) - g_n(t_i) \|_{L^{2p}(\Omega)} \leq C|t_j - t_i|^\frac{1}{2}, \quad \forall \ 0 \leq t_i < t_j \leq T,
\]

then

\[
\left\| \sum_{k=i}^{j-1} f_{t_k} \xi_{n,k} \right\|_{L^p(\Omega)} \leq C|t_j - t_i|^\frac{1}{p}, \quad \forall \ 0 \leq t_i < t_j \leq T.
\]

Constants \( C = C(p, f) \) above are all independent of \( n \).

3. Solvability and dependence on driving noises

3.1. Runge–Kutta methods. For \( n \in \mathbb{N}_+ \), denote the time step \( h = \frac{T}{n} \) and \( t_k = kh, k = 0, \ldots, n \). We consider an \( s \)-stage Runge–Kutta method of (1.1):

\[
Y^n_{t_{k+1},i} = Y^n_{t_k} + \sum_{j=1}^{s} a_{ij} V(Y^n_{t_{k+1,j}}) \Delta X_k,
\]

\[
Y^n_{t_{k+1}} = Y^n_{t_k} + \sum_{i=1}^{s} b_i V(Y^n_{t_{k+1,i}}) \Delta X_k,
\]

with \( i, j = 1, \ldots, s, k = 0, \ldots, n-1, \Delta X_k = X_{t_{k+1}} - X_{t_k} \in \mathbb{R}^d \) and \( Y^n_0 = y \in \mathbb{R}^m \). Here \( Y^n_{t_{k+1},i}, i = 1, \ldots, s \), are called stage values of this Runge–Kutta method.

If the method is an implicit one, such as the midpoint scheme, the solvability of (3.1)–(3.2) should be taken into consideration. For SDEs driven by Brownian motions, the classical technique is to truncate each increment of Brownian motions to make increments become bounded, and give the solvability of implicit methods and convergence rates in mean square sense. However, this truncation technique is not suitable for fBms since their increments have nontrivial covariance. Based on Brouwer’s theorem, Proposition [3.1] ensures the solvability of implicit Runge–Kutta methods in almost surely sense.

**Proposition 3.1.** If \( V \in C^0_b(\mathbb{R}^m; \mathbb{R}^{m \times d}) \), then for arbitrary time step \( h > 0 \), initial value \( y \) and coefficients \( \{ a_{ij}, b_i : i, j = 1, \ldots, s \} \), the \( s \)-stage Runge–Kutta method (3.1)–(3.2) has at least one solution for almost every \( \omega \).

*Proof.* Fix \( h > 0 \) and \( Y^n_{t_k} \in \mathbb{R}^{2m} \).
Let \( Z_1, \ldots, Z_s \in \mathbb{R}^{2m} \) and \( Z = (Z_1^\top, \ldots, Z_s^\top)^\top \in \mathbb{R}^{2ms} \). We define a map \( \phi: \mathbb{R}^{2ms} \to \mathbb{R}^{2ms} \) with
\[
\phi(Z) = (\phi(Z)_1^\top, \ldots, \phi(Z)_s^\top)^\top,
\]
\[
\phi(Z)_i = Z_i - Y^n_i - \sum_{j=1}^s a_{ij} V(Z_j) \Delta X_k(\omega), \quad i = 1, \ldots, s.
\]
It then suffices to prove that \( \phi(Z) = 0 \) has at least one solution, which implies the solvability of (3.1) and thus the solvability of the Runge–Kutta method. Let
\[
c = \max\{|a_{ij}| : i, j = 1, \ldots, s\}, \quad \nu = \sup_{y \in \mathbb{R}^m} |V(y)|
\]
and
\[
R = \sqrt{s} Y^n_t + s \sqrt{\text{scv}} |\Delta X_k(\omega)| + 1,
\]
we have that for any \( |Z| = R \),
\[
Z^\top \phi(Z) = \sum_{i=1}^s Z_i^\top \left[ Z_i - Y^n_i - \sum_{j=1}^s a_{ij} V(Z_j) \Delta X_k(\omega) \right]
\geq |Z| \left( |Z| - \sqrt{s} Y^n_t - s \sqrt{\text{scv}} |\Delta X_k(\omega)| \right) > 0.
\]
We aim to show that \( \phi(Z) = 0 \) has a solution in the ball \( B_R = \{Z : |Z| \leq R\} \). Assume by contradiction that \( \phi(Z) \neq 0 \) for any \( |Z| \leq R \). We define a continuous map \( \psi \) by
\[
\psi(Z) = -\frac{R \phi(Z)}{|\phi(Z)|}.
\]
Since \( \psi: B_R \to B_R \), \( \psi \) has at least one fixed point \( Z^* \) such that \( Z^* = \psi(Z^*) \) and \( |Z^*| = R \). This leads to a contradiction since
\[
|Z^*|^2 = \psi(Z^*)^\top Z^* = -\frac{R \phi(Z^*)^\top Z^*}{|\phi(Z^*)|} < 0.
\]
Therefore, \( \phi \) has at least one solution.

We construct the continuous version (3.3) and (3.4) for the Runge–Kutta method, taking advantages of the stage values \( Y^n_{t_k,i} \). Indeed, the continuous version of \( Y^n_{t_k,i} \) comes after that of \( Y^n_{t_k} \). For \( t \in (t_k, t_{k+1}] \), \( [t] := t_{k+1} \). In particular, \( t = t_k \) if and only if \( t = [t] \) for some \( k = 0, \ldots, n \). The continuous version reads
\[
Y^n_{t,k} := Y^n_{(t_k, t_{k+1})} + \sum_{j=1}^s \int_{(t_k, t_{k+1})} a_{ij} V(Y^n_{[s], j}) dX_s, \quad i = 1, \ldots, s, \tag{3.3}
\]
\[
Y^n_t := y + \sum_{i=1}^s \int_0^t b_i V(Y^n_{[s], i}) dX_s, \tag{3.4}
\]
where \( s \vee t \) denotes the maximum of \( s \) and \( t \).

To estimate the Hölder semi-norm of \( Y^n \) and \( Y^n_{i,i} \), we first introduce the discrete Hölder semi-norm for \( f \in C([0,T], \mathbb{R}^d) \):
\[
\|f\|_{s,t,\beta,n} := \sup \left\{ \frac{|f_u - f_v|}{|u - v|^{\beta}}, \quad s \leq u < v \leq t, \quad u = [u]^n, \quad v = [v]^n \right\},
\]
\[
\|f\|_{\beta,n} := \sup \left\{ \frac{|f_u - f_v|}{|u - v|^{\beta}}, \quad 0 \leq u < v \leq T, \quad u = [u]^n, \quad v = [v]^n \right\}.
\]

**Lemma 3.2.** If \( V \in C^0_b(\mathbb{R}^m; \mathbb{R}^{m \times d}) \), then for any \( n \in \mathbb{N}_+ \) and \( 1/2 < \beta < H \), \( \|Y^n\|_{\beta,n} \) and \( \|Y^n_{i,i}\|_{\beta,n} \), \( i = 1, \ldots, s \), are all finite almost surely. More precisely,
\[
\|Y^n\|_{\beta,n} \leq C(d, m, n, c, \nu, s) \|X\|_\beta < \infty, \quad a.s.,
\]
\[
\|Y^n_{i,i}\|_{\beta,n} \leq C(d, m, n, c, \nu, s) \|X\|_\beta < \infty, \quad a.s.,
\]
where \( C \) is a constant depending on \( d, m, n, c, \nu, s \) and \( \beta \).
with $c = \max\{|a_{ij}|, |b_i| : i, j = 1, \cdots, s\}$ and $\nu = \sup_{y \in \mathbb{R}^m} |V(y)|$.

Inspired by [10], we give the following two lemmas as a priori estimates for the continuous version [3.3]-[3.4]. Based on them, Proposition 3.5 shows that the stage values $Y^n_i$ is continuously dependent with respect to the driving noises in Hölder semi-norm and so is $Y^n$.

**Lemma 3.3.** Let $\alpha$, $\beta$ and $\beta'$ satisfy $\beta' > \alpha > 1 - \beta$. Then for any $s, t \in [0, T]$ such that $s < t$ and $s = \lfloor s \rfloor^n$, there exists a constant $C = C(\alpha, \beta, \beta', T)$ such that

$$
\int_s^t (r-s)^{\alpha + \beta - 1} \int_s^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr \leq C(t-s)^{\beta + \beta'}.
$$

**Proof.** Suppose $T = 1$ without loss of generality. By the definition of $\lfloor r \rfloor^n$, we have

$$
\int_s^t (r-s)^{\alpha + \beta - 1} \int_s^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr
$$

$$
= \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} \int_{[s]^{n}}^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr
$$

$$
= \frac{1}{\lfloor s \rfloor^n + 1/n} \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} \left( \int_{[s]^{n}}^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \right) dr
$$

$$
=: I_1 + I_2.
$$

For the first term, since $r-u > \frac{1}{n}$ and $[r] - [u] < r-u + \frac{1}{n}$, we have

$$
I_1 = \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} \int_{[s]^{n}}^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr
$$

$$
\leq \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} \int_{[s]^{n}}^r \frac{2^{\beta'}(r-u)^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr
$$

$$
\leq C \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} (r-s)^{\beta' - \alpha} dr
$$

$$
\leq C \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} (t-s)^{\beta' - \alpha} dr
$$

$$
\leq C (t-s)^{\beta + \beta'}.
$$

For the second term,

$$
I_2 = \int_{[s]^{n+1/n}}^t (r-s)^{\alpha + \beta - 1} \int_{[r]^{n+1/n}}^r \frac{([r] - [u])^{\beta'}}{(r-u)^{\alpha + 1}} du \, dr
$$

$$
\leq C (t-s)^{\alpha + \beta - 1} \left( \frac{2}{n} \right)^{\beta'} \int_{[s]^{n+1/n}}^t (r-[r]+\frac{1}{n})^{-\alpha} dr
$$

$$
\leq C (t-s)^{\alpha + \beta - 1} \left( \frac{2}{n} \right)^{\beta'} (\frac{t-s}{n})^{\frac{1}{n}-\alpha+1}
$$

$$
\leq C (t-s)^{\beta + \beta'}.
$$

$\square$
Lemma 3.4. Let $\beta$ and $\beta'$ satisfy $\beta + \beta' > 1$. Let $g \in C^1_b([1,\infty); \mathbb{R})$, $x \in C([s,t]; \mathbb{R})$ and $z \in C([s,t]; \mathbb{R}^n)$. If $\|x\|_\beta$ and $\|z\|_{\beta',n}$ are all finite for any $n \in \mathbb{N}_+$, then for any $s = [s]^n$ and $t = [t]^n$,

$$
\int_s^t g(z_{[r]^n})dx_r \leq C(g, \beta, \beta', T)(1 + \|z\|_{s,t,\beta',n}(t-s)^{\beta'})\|x\|_\beta(t-s)^\beta.
$$

Proof. Considering the equivalence of norms in $\mathbb{R}^m$, we suppose $m = 1$ here for simplicity without loss of generality. Let $\alpha$ satisfy $\alpha < \beta'$ and $\beta + \alpha > 1$. According to the characterization of the integral in Section 2

$$
\int_s^t g(z_{[r]^n})dx_r = \int_s^t D^\alpha_{s+}g(z_{[r]^n})D^{1-\alpha}_{t-}(x_r - x_t)dr.
$$

Combining the fractional Weyl derivatives, we have, for $s < r < t$,

$$
|D^\alpha_{s+}g(z_{[r]^n})| \leq C\left(\left|\frac{g(z_{[r]^n})}{(r-s)^\alpha}\right| + \int_s^r \left|\frac{g(z_{[r]^n}) - g(z_{[u]^n})}{(r-u)^{\alpha+1}}\right|du\right)
$$

$$
\leq C\left(\frac{1}{(r-s)^\alpha} + \int_s^r \frac{|z_{[r]^n} - z_{[u]^n}|}{(r-u)^{\alpha+1}}du\right)
$$

$$
\leq C\left(\frac{1}{(r-s)^\alpha} + \|z\|_{s,t,\beta',n}\int_s^r \frac{([r]^n - [u]^n)^{\beta'}}{(r-u)^{\alpha+1}}du\right),
$$

and

$$
|D^{1-\alpha}_{t-}(x_r - x_t)| \leq C\left(\left|\frac{x_r - x_t}{(t-r)^{1-\alpha}}\right| + \int_r^t \left|\frac{x_t - x_u}{(u-r)^2}\right|du\right)
$$

$$
\leq C\|x\|_\beta(t-r)^{\alpha+\beta-1}.
$$

Using Lemma 3.3 we obtain

$$
\int_s^t g(z_{[r]^n})dx_r
$$

$$
\leq \int_s^t |D^\alpha_{s+}g(z_{[r]^n})D^{1-\alpha}_{t-}(x_r - x_t)|dr
$$

$$
\leq C\int_s^t \left(\frac{1}{(r-s)^\alpha} + \|z\|_{s,t,\beta',n}\int_s^r \frac{([r]^n - [u]^n)^{\beta'}}{(r-u)^{\alpha+1}}du\right)\|x\|_\beta(t-r)^{\alpha+\beta-1}dr
$$

$$
\leq C(1 + \|z\|_{s,t,\beta',n}(t-s)^{\beta'})\|x\|_\beta(t-s)^\beta.
$$

□
Proposition 3.5. If $V \in C^1_b(\mathbb{R}^m; \mathbb{R}^{m \times d})$ and $1/2 < \beta < H$, then for any $n \in \mathbb{N}_+$,

$$
\sum_{i=1}^{s} \| Y^n_{i} \|_{\beta} \leq C(c, s, V, \beta, T) \max \left\{ \| X \|_{\beta}, \| X \|_{\beta}^{1/\beta} \right\},
$$

$$
\sum_{i=1}^{s} \| Y^n_{i} \|_{\infty} \leq s|y| + C(c, s, V, \beta, T) \max \left\{ \| X \|_{\beta}, \| X \|_{\beta}^{1/\beta} \right\},
$$

$$
\| Y^n \|_{\beta} \leq C(c, s, V, \beta, T) \max \left\{ \| X \|_{\beta}, \| X \|_{\beta}^{1/\beta + 1} \right\},
$$

$$
\| Y^n \|_{\infty} \leq |y| + C(c, s, V, \beta, T) \max \left\{ \| X \|_{\beta}, \| X \|_{\beta}^{1/\beta + 1} \right\},
$$

where $c = \max\{ |a_{ij}|, |b_i| : i, j = 1, \ldots, s \}$.

Moreover, for some $C_0 > 0$ and $0 \leq s < t \leq T$ such that $\| X \|_{\beta}(t - s)^\beta \leq C_0$, the estimate can be improved to

$$
\sum_{i=1}^{s} \| Y^n_{i} \|_{s, t, \beta} \leq C(c, s, V, \beta, T, C_0) \| X \|_{\beta},
$$

$$
\sum_{i=1}^{s} \| Y^n_{i} \|_{s, t, \beta} \leq C(c, s, V, \beta, T, C_0) \| X \|_{\beta}.
$$

Proof. We first take $s = \lfloor s \rfloor$ and $t = \lfloor t \rfloor$, then Lemma 3.4 yields

$$
\left| Y^n_{i} - Y^n_{s, i} \right| \leq \sum_{j=1}^{s} \int_{0}^{\lfloor (t-h) \rfloor} b_j V(Y^n_{\lfloor r \rfloor, j})dX_r + \int_{\lfloor (t-h) \rfloor}^{\lfloor t \rfloor} a_{ij} V(Y^n_{\lfloor r \rfloor, j})dX_r - \int_{0}^{\lfloor (s-h) \rfloor} b_j V(Y^n_{\lfloor r \rfloor, j})dX_r - \int_{\lfloor (s-h) \rfloor}^{\lfloor s \rfloor} a_{ij} V(Y^n_{\lfloor r \rfloor, j})dX_r
$$

$$
\leq \sum_{j=1}^{s} \left| \int_{\lfloor s \rfloor}^{\lfloor (t-h) \rfloor} b_j V(Y^n_{\lfloor r \rfloor, j})dX_r \right| + \left| \int_{\lfloor (t-h) \rfloor}^{\lfloor t \rfloor} a_{ij} V(Y^n_{\lfloor r \rfloor, j})dX_r \right|
$$

$$
+ \left| \int_{\lfloor s \rfloor}^{\lfloor (s-h) \rfloor} b_j V(Y^n_{\lfloor r \rfloor, j})dX_r \right| + \left| \int_{\lfloor (s-h) \rfloor}^{\lfloor s \rfloor} a_{ij} V(Y^n_{\lfloor r \rfloor, j})dX_r \right|
$$

$$
\leq C(c, V, \beta, T)(1 + \sum_{j=1}^{s} \| Y^n_{j} \|_{s, t, \beta, n}(t - s)^\beta) \| X \|_{\beta}(t - s)^\beta.
$$

Summing up above inequalities for all $i = 1, \ldots, s$ and dividing both sides by $(t - s)^\beta$, we have

$$
\sum_{i=1}^{s} \| Y^n_{i} \|_{s, t, \beta, n} \leq C(c, s, V, \beta, T) \vee 1(1 + \sum_{i=1}^{s} \| Y^n_{i} \|_{s, t, \beta, n}(t - s)^\beta) \| X \|_{\beta}
$$

$$
= C_1(1 + \sum_{i=1}^{s} \| Y^n_{i} \|_{s, t, \beta, n}(t - s)^\beta) \| X \|_{\beta}.
$$

If $n \geq 2T(2C_1 \| X \|_{\beta})^{1/\beta}$, then there exist $N_0 \in \mathbb{N}_+$ and $N_1 = \frac{N_0 T}{n}$ such that

$$
(2C_1 \| X \|_{\beta})^{−1/\beta} \leq 2N_1 \leq 2(2C_1 \| X \|_{\beta})^{−1/\beta}.
$$
When $t - s = N_1$, considering the choice for $N_1$, we get
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta,n} \leq 2C_1\|X\|_{\beta}. \]

When $t - s > N_1$,
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta,n} \leq C\left(\left\lceil \frac{t - s}{N_1} \right\rceil + 1\right) \sup_{r = \lfloor r \rfloor \leq t_{n-1}} \sum_{i=1}^{s} \|Y_{i}^{n}\|_{r,r+N_1,\beta,n} \frac{N_1^\beta}{(t - s)^{\beta}}, \]
where $[t]$ means the largest integer which is not larger than $t$. So we obtain
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta,n} \leq C\left(\frac{T}{N_1} + 1\right)^{1-\beta} C_1\|X\|_{\beta} \]
\[ \leq C\left(\|X\|_{\beta}^{1/\beta} + 1\right)\|X\|_{\beta} \]
\[ \leq C \max \left\{ \|X\|_{\beta}, \|X\|_{\beta}^{1/\beta} \right\}, \]
where the second inequality is from the choice of $N_1$. If $n < 2T(2C_1\|X\|_{\beta})^{1/\beta}$, the definition of $Y_{n,i}^{n}$ leads to
\[ \sum_{i=1}^{s} |Y_{n,i}^{n} - Y_{s,i}^{n}| \leq C(t - s)\left(\frac{T}{n}\right)^{-1+\beta} \|X\|_{\beta}, \]
and then
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta,n} \leq C\|X\|_{\beta}^{1/\beta}. \]

Now we can take any $0 \leq s < t \leq T$ and suppose $t \in (t_k, t_{k+1}]$, then
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta,n} \leq C\|X\|_{\beta} + \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t_{n},\beta,n} \leq C \max \left\{ \|X\|_{\beta}, \|X\|_{\beta}^{1/\beta} \right\}. \]

Therefore,
\[ \|Y_{n,i}^{n}\|_{\beta} \leq C(c, s, V, \beta, T) \max \left\{ \|X\|_{\beta}, \|X\|_{\beta}^{1/\beta} \right\}, \]
\[ \|Y_{n,i}^{n}\|_{\infty} \leq |\beta| + C(c, s, V, \beta, T) \max \left\{ \|X\|_{\beta}, \|X\|_{\beta}^{1/\beta} \right\}. \]

Moreover, if $C_0 \in (0, 1/C_1)$, then for any $0 \leq s < t \leq T$ such that $\|X\|_{\beta} |t - s|^{\beta} \leq C_0$, it can be improved to
\[ \sum_{i=1}^{s} \|Y_{n,i}^{n}\|_{s,t,\beta} \leq C(C_1, \beta, T, C_0)\|X\|_{\beta}.
On the other hand, since
\[
|Y^n_s - Y^n_t| \leq \sum_{i=1}^n \left| \int_s^t b_i V(Y^n_{\lfloor r \rfloor}, j) dX_r \right|
\]
\[
\leq C(1 + \sum_{j=1}^n \|Y^n_{\lfloor t \rfloor}, j(t - s)^\beta) \|X\|_\beta(t - s)^\beta
\]
\[
\leq C(c, s, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|_\beta^{1/\beta+1} \right\} (t - s)^\beta,
\]
we obtain that
\[
\|Y^n\|_\beta \leq C(c, s, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|_\beta^{1/\beta+1} \right\},
\]
\[
\|Y^n\|_\infty \leq |y| + C(c, s, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|_\beta^{1/\beta+1} \right\},
\]
and for any \( 0 \leq s < t \leq T \) such that \( \|X\|_\beta |t - s|^{\beta} \leq C_0 \in (0, 1/C_1) \),
\[
\|Y^n\|_{s,t,\beta} \leq C(c, s, V, \beta, T, C_0) \|X\|_\beta.
\]

3.2. Simplified step-N Euler schemes. Fix an integer \( N \geq 2 \). For \( n \in \mathbb{N}_+ \), denote the time step \( h = \frac{T}{n} \) and \( t_k = kh, \ k = 0, \cdots, n \). The simplified step-N Euler scheme of (1.1) is:

\[
Y^n_{t_{k+1}} = Y^n_{t_k} + \sum_{w=1}^N \sum_{l_1, \cdots, l_N = 1}^d \mathcal{Y}_{l_{1}} \cdots \mathcal{Y}_{l_{N}} I(Y^n_{t_k}) \frac{\Delta X^n_{l_{1}} \cdots \Delta X^n_{l_{N}}}{N!},
\]

where every \( \mathcal{Y}_i \) is identified with the first order differential operator \( \sum_q V^q(y) \frac{\partial}{\partial y^q} \).

Note that \( Y^n_{t_{k+1}} \) may stand for numerical solutions of different schemes in different subsections if there is no confusion. We consider its corresponding continuous version as before. For \( t \in (t_k, t_{k+1}) \), \( [t]_n := t_k \) and for \( t = 0, \ [t]_n := 0 \). Then the continuous version of (3.5) is

\[
Y^n_t = y + \int_0^t \sum_{w, l_1, \cdots, l_N} \mathcal{Y}_{l_{1}} \cdots \mathcal{Y}_{l_{N}} I(Y^n_{[s]_n}) \frac{\left( X^n_{[s]_n} - X^n_{[t]_n} \right) \cdots \left( X^n_{[s]_n} - X^n_{[t]_n} \right)}{N!} \partial X^n_{l_{1}}.
\]

Similar to the analysis for Runge–Kutta methods, the continuous dependence of \( Y^n \) on the driving noises in H"older semi-norm is obtained.

Lemma 3.6. If \( V \in C_b^{N-1}(\mathbb{R}^m; \mathbb{R}^{m \times d}) \), then for any \( n \in \mathbb{N}_+ \) and \( 1/2 < \beta < H \), \( \|Y^n\|_{\beta, n} \) are all finite almost surely.

Lemma 3.7. Let \( \alpha, \beta \) and \( \beta' \) satisfy \( \beta' > 1 - \beta \). Then for any \( s, t \in [0, T] \) such that \( s < t \) and \( s = [s]_n \), there exists a constant \( C = C(\alpha, \beta, \beta', T) \) such that

\[
\int_s^t (t - r)^{\alpha+1-\beta} \int_s^r \frac{(r - u)^{\beta'}}{(r - u)^{\alpha+1}} dudr \leq K(t - s)^{\beta+\beta'}.
\]

Proof. Similar to the proof of Lemma 3.3.
Lemma 3.8. Let $\beta$ and $\beta'$ satisfy $\beta + \beta' > 1$. If $g \in C^1_c(\mathbb{R}^m; \mathbb{R})$, $x^l \in C([s,t]; \mathbb{R}^m)$, $l \in \{1, \ldots, d\}$, and $z \in C([s,t]; \mathbb{R}^m)$. If $\|x^l\|_\beta$, $\|z\|_{\beta', n}$ are all finite for any $l \in \{1, \ldots, d\}$, $n \in \mathbb{N}_+$, then for any $w \geq 2$, $s = [s]^n$ and $t = [t]^n$,

$$
\left| \int_s^t g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) dx_r \right| \leq (g(\beta, \beta', T)(1 + \|z\|_{s,t,\beta', n}(t-s)^{\beta'})) \|x^l\|_\beta \cdots \|x^l\|_\beta(t-s)^{\beta w},
$$

where $l_1, \ldots, l_w \in \{1, \ldots, d\}$.

Proof. Let $\Phi(r) = g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n})$. Taking $\alpha$ such that $\alpha < \beta'$ and $\beta + \alpha > 1$, we first estimate the left fractional Weyl derivative of $\Phi$:

$$
|D^\alpha_s \Phi_r| \leq \frac{\Phi_r - \Phi_u}{(r-s)^\alpha} + \alpha \int_s^r \frac{\Phi_r - \Phi_u}{(r-u)^\alpha+1} du.
$$

For the first term,

$$
\left| \frac{\Phi_r - \Phi_u}{(r-s)^\alpha} \right| \leq C \|x^l\|_\beta \cdots \|x^l\|_\beta \left( \frac{r}{n} \right)^{\beta(w-1)} (r-s)^{-\alpha}.
$$

For the second term, we decompose $\Phi_r - \Phi_u$ into

$$
\begin{align*}
&g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) - g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) \\
&+ g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) - g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) \\
&\quad + \cdots \\
&+ g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) - g(z_{[r]^n})(x^l_{[r]^n} - x^l_{[r]^n}) \cdots (x^l_{[r]^n} - x^l_{[r]^n}) \\
&= : I_1 + I_2 + \cdots + I_w.
\end{align*}
$$

We analyze each of them by

$$
\begin{align*}
|I_1| \leq C \|x^l\|_\beta \cdots \|x^l\|_\beta \left( \frac{T}{n} \right)^{\beta(w-2)} \left[ (\lceil r \rceil - \lfloor u \rfloor)^\beta + (\lfloor r \rfloor - \lfloor u \rfloor)^\beta \right], \\
|I_2| \leq C \|x^l\|_\beta \cdots \|x^l\|_\beta \left( \frac{T}{n} \right)^{\beta(w-2)} \left[ (\lceil r \rceil - \lfloor u \rfloor)^\beta + (\lfloor r \rfloor - \lfloor u \rfloor)^\beta \right], \\
&\quad \cdots \\
|I_w| \leq C \|x^l\|_\beta \cdots \|x^l\|_\beta \left( \frac{T}{n} \right)^{\beta(w-1)} \|z\|_{s,t,\beta'} (\lfloor r \rfloor - \lfloor u \rfloor)^{\beta'}. 
\end{align*}
$$

Combining Lemma 3.7 and arguments in Lemma 3.4 we conclude the proof. \qed

Proposition 3.9. If $V \in C^0_c(\mathbb{R}^m; \mathbb{R}^{m \times d})$ and $1/2 < \beta < H$, then for any $n \in \mathbb{N}_+$,

$$
\|Y^n\|_\beta \leq C(N,V,\beta,T) \max \left\{ \|X\|_\beta, \|X\|_{\beta}^{N-1+1/\beta} \right\},
$$

$$
\|Y^n\|_\infty \leq |y| + C(N,V,\beta,T) \max \left\{ \|X\|_\beta, \|X\|_{\beta}^{N-1+1/\beta} \right\}.
$$

Moreover, for some $C_0 > 0$ and $0 \leq s < t \leq T$ such that $\|X\|_\beta |t-s|^{\beta} \leq C_0$, the estimate can be improved to

$$
\|Y^n\|_{s,t,\beta} \leq C(N,V,\beta,T,C_0) \|X\|_\beta.
$$
Proof. Take \( s = [s]n \) and \( t = [t]n \). Lemma 3.8 yields
\[
|Y^n_t - Y^n_s| \leq C(N, V, \beta, T) \sum_{w=1}^{N} \|X^w\|_\beta (t-s)^{\beta w} \left[ 1 + \|Y^n\|_{s,t,\beta,n}(t-s)^{\beta} \right]
\]

\[
= C_1 \sum_{w=1}^{N} \|X^w\|_\beta (t-s)^{\beta w} \left[ 1 + \|Y^n\|_{s,t,\beta,n}(t-s)^{\beta} \right],
\]

since \( \|X^1\|_\beta \cdots \|X^k\|_\beta \leq \|X\|_\beta \). Dividing both sides by \((t-s)^\beta\), we have
\[
\|Y^n\|_{s,t,\beta,n} \leq C_1 \sum_{w=1}^{N} \|X^w\|_\beta (t-s)^{\beta (w-1)} \left[ 1 + \|Y^n\|_{s,t,\beta,n}(t-s)^{\beta} \right].
\]

If \( n \geq 2T(2NC_1\|X\|_\beta)^{1/\beta} \), then there exist \( N_0 \in \mathbb{N}_+ \) and \( N_1 = \frac{N_0 T}{n} \) such that
\[
(2NC_1\|X\|_\beta)^{-1/\beta} \leq 2N_1 \leq 2(2NC_1\|X\|_\beta)^{-1/\beta}.
\]

When \( t-s = N_1 \), considering the choice for \( N_1 \), we get
\[
C_1 \|X\|_\beta (t-s)^{\beta w} \leq \frac{1}{2N}, \quad \forall \ w = 1, \ldots, N.
\]

So
\[
\|Y^n\|_{s,t,\beta,n} \leq 2C_1 \sum_{w=1}^{N} \|X^w\|_\beta (t-s)^{\beta (w-1)}.
\]

When \( t-s > N_1 \),
\[
\|Y^n\|_{s,t,\beta,n} \leq C \left( \frac{t-s}{N_1} + 1 \right) \sup_{r = \lfloor r \rfloor \leq t_{n-1}} \|Y^n\|_{r,r+N_1,\beta,n}(t-s)^{\beta}.
\]

So we have that if \( n \geq 2T(2NC_1\|X\|_\beta)^{1/\beta} \),
\[
\|Y^n\|_{s,t,\beta,n} \leq C \max \left\{ \|X\|_\beta, \|X\|_\beta^{N-1+1/\beta} \right\}.
\]

If \( n < 2T(2NC_1\|X\|_\beta)^{1/\beta} \), the definition of \( Y^n \) leads to
\[
\|Y^n\|_{s,t,\beta,n} \leq C\|X\|_\beta^{1/\beta}.
\]

Now we can take any \( 0 \leq s < t \leq T \),
\[
\|Y^n\|_{s,t,\beta} \leq C\|X\|_\beta^N + \|Y^n\|_{s,t\|X\|_\beta^N} \leq C \max \left\{ \|X\|_\beta, \|X\|_\beta^{N-1+1/\beta} \right\}.
\]

Therefore,
\[
\|Y^n\|_\beta \leq C(c, s, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|_\beta^{N-1+1/\beta} \right\},
\]
\[
\|Y^n\|_\infty \leq |s| + C(c, s, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|_\beta^{N-1+1/\beta} \right\}.
\]

Moreover, if \( C_0 \in (0, (C_1N)^{-1}) \), then for any \( 0 \leq s < t \leq T \) such that \( \|X\|_\beta|t-s|^\beta \leq C_0 \), it can be improved to
\[
\|Y^n\|_{s,t,\beta} \leq C(N, V, \beta, T, C_0)\|X\|_\beta.
\]

\[\square\]
Remark 3.10. Note that Fernique’s lemma implies that \( \| X \|_{L^p(\Omega)}^N < \infty \) for any \( p \geq 1 \) and \( N \in \mathbb{N}_+ \) (see e.g. [10, Remark 3.2]).

4. STRONG CONVERGENCE RATE

4.1. Runge–Kutta methods. The order conditions on coefficients of Runge–Kutta methods are derived in this section to ensure the strong convergence rate is \( 2H - \frac{1}{2} \). For simplicity, we omit the range of indices in summations if it is not confusing.

\[
Y_t - Y^n_t = \left[ \int_0^t V(Y_s) dX_s - \int_0^t V(Y^n_s) dX_s \right]
+ \left[ \int_0^t V(Y^n_s) dX_s - \int_0^t \sum_i b_i V(Y^n_{[s],i}) dX_s \right]
= : L_t + R_t.
\]

For the first term, the Taylor expansion yields

\[
L_t = \int_0^t V(Y_s) dX_s - \int_0^t V(Y^n_s) dX_s
= \sum_{i=1}^d \int_0^t \int_0^1 \nabla V_i(\theta Y_s + (1 - \theta) Y^n_s)(Y_s - Y^n_s) d\theta dX_s.
\]

For the second term, fix any \( t = [t^n] \), we have

\[
R_t = \int_0^t V(Y^n_s) dX_s - \int_0^t \sum_i b_i V(Y^n_{[s],i}) dX_s
= \sum_{k=0}^{t^n / T - 1} \int_{t_k}^{t_k + 1} \left[ V(Y^n_s) - \sum_i b_i V(Y^n_{t_{k+1},i}) \right] dX_s.
\]

For any \( i = 1, \cdots, s \), denote by \( Y_{t^n,i}^q \) and \( Y_{t,n,i}^{q,i} \) the \( q \)-th component of \( Y_t^n \) and \( Y_{t^n,i}^n \), respectively, \( q = 1, \cdots, m \). We apply the Taylor expansion to \( V(Y^n_{t_{k+1},i}) \) at \( Y^n_{t_k} \) or \( Y^n_{t_{k+1},i} \), then

\[
V(Y^n_{t_{k+1},i}) = V(Y^n_{t_k}) + \int_0^1 \sum_q \partial_q V(\theta Y^{n}_{t_{k+1},i} + (1 - \theta) Y^n_{t_k}) \left( Y^{q,i}_{t_{k+1},i} - Y^{n,q}_{t_{k+1},i} \right) d\theta
= V(Y^n_{t_{k+1}}) + \int_0^1 \sum_q \partial_q V(\theta Y^{n}_{t_{k+1},i} + (1 - \theta) Y^n_{t_{k+1}}) \left( Y^{q,i}_{t_{k+1},i} - Y^{n,q}_{t_{k+1}} \right) d\theta,
\]
where $\partial_q$ denotes the partial differential operator with respect to the $q$-th variable. Assume $\eta \in [0, 1]$, then for any $s \in (t_k, t_{k+1}]$

$$V(Y^n_s) - \sum_i b_i V(Y^n_{t_k, i})$$

$$= \eta \left[ V(Y^n_s) - \sum_i b_i V(Y^n_{t_k}) \right]$$

$$- \eta \sum_{i,q} b_i \int_0^1 \partial_q V(\theta Y^n_{t_k, i}) + (1 - \theta) Y^n_{t_k, q} - Y^n_{t_k, i})d\theta$$

$$+ (1 - \eta) \left[ V(Y^n_s) - \sum_i b_i V(Y^n_{t_k}) \right]$$

$$- (1 - \eta) \sum_{i,q} b_i \int_0^1 \partial_q V(\theta Y^n_{t_k, i}) + (1 - \theta) Y^n_{t_k, q} - Y^n_{t_k, i})d\theta$$

$$= : \eta R^1_s + \eta R^2_s + (1 - \eta) R^3_s + (1 - \eta) R^4_s.$$

Since (3.1) implies

$$R^1_s = V(Y^n_s) - \sum_i b_i V(Y^n_{t_k})$$

$$= \left[ V(Y^n_{t_k}) + \int_0^s \sum_q \partial_q V(\theta Y^n_{t_k} + (1 - \theta) Y^n_{t_k} - Y^n_{t_k})d\theta \right] - \sum_i b_i V(Y^n_{t_k})$$

$$= \left[ V(Y^n_{t_k}) + \sum_{q,i} \partial_q V(Y^n_{t_k}) \int_{t_k}^s b_i V(Y^n_{t_k, i})dX_u + E^1_s \right] - \sum_i b_i V(Y^n_{t_k})$$

we propose the first condition that $\sum_{i=1}^{n} b_i = 1$. Then

(4.1)

$$\int_{t_k}^{t_{k+1}} R^1_s dX_s = \int_{t_k}^{t_{k+1}} \sum_{q,i} \partial_q V(Y^n_{t_k}) \int_{t_k}^s b_i V(Y^n_{t_k+1, i})dX_u dX_s + \int_{t_k}^{t_{k+1}} E^1_s dX_s$$

with $E^1_s$ denoting the remainder term of $R^1_s$, so as $E^2_s, E^3_s, E^4_s$ in the following analysis. Similarly,

(4.2)

$$\int_{t_k}^{t_{k+1}} R^3_s dX_s = - \int_{t_k}^{t_{k+1}} \sum_{q,i} \partial_q V(Y^n_{t_k}) \int_{t_k}^{t_{k+1}} b_i V(Y^n_{t_k+1, i})dX_u dX_s + \int_{t_k}^{t_{k+1}} E^3_s dX_s.$$

For $R^2_s$ and $R^4_s$, it follows from (3.1) and (3.2) that

$$Y^n_{t_{k+1}, i} - Y^n_{t_k} = \sum_j a_{ij} V(Y^n_{t_{k+1, j}}) \Delta X_k,$$

$$Y^n_{t_{k+1}, i} - Y^n_{t_k} = - \sum_j b_j V(Y^n_{t_{k+1, j}}) \Delta X_k + \sum_j a_{ij} V(Y^n_{t_{k+1, j}}) \Delta X_k.$$
Then,
\[
\int_{t_k}^{t_{k+1}} R^2_s \, dX_s = - \int_{t_k}^{t_{k+1}} \sum_{i,q,j} b_i \partial_q V(Y^n_{t_{k+1},i}) \left[ \int_{t_k}^{t_{k+1}} a_{ij} V^q(Y^n_{t_{k+1},j}) \, dX_u \right] \, dX_s + \int_{t_k}^{t_{k+1}} E^2_s \, dX_s,
\]
\[
\int_{t_k}^{t_{k+1}} R^4_s \, dX_s = - \int_{t_k}^{t_{k+1}} \sum_{i,q,j} b_i \partial_q V(Y^n_{t_{k+1},i}) \left[ \int_{t_k}^{t_{k+1}} (a_{ij} - b_j) V^q(Y^n_{t_{k+1},j}) \, dX_u \right] \, dX_s + \int_{t_k}^{t_{k+1}} E^4_s \, dX_s.
\]

Taking the Taylor expansion to both \(V(Y^n_{t_{k+1},i})\) and \(V(Y^n_{t_{k+1},j})\) at \(Y^n_{t_k}\) in above two expressions and choosing \(\eta = \frac{\delta}{2}\), we propose another condition \(\sum_{j=1}^n b_j(\sum_{j=1}^n b_j - 2a_{ij}) = 0\) such that terms contain 2nd-level iterated integrals of \(X\) in the form of the Lévy area type processes \(2.3\) vanish. Therefore, the leading term of \(R_t\) only appears in \((4.1)\) and \((4.2)\), which contains 2nd-level iterated integrals of \(X\) in the form of \(2.3\), \(2.2\). Using the Taylor expansion again to \(V(Y^n_{t_k})\) and \(V(Y^n_{t_{k+1}})\) at \(Y^n_{t_{k+1},i}\) in \((4.1)\) and \((4.2)\), we obtain the leading term of \(R_t\) is

\[
(4.3) \quad R^\text{lead}_t = \frac{1}{2} \sum_{k=0}^{nT/2-1} \sum_{i,q,l,l'} b_i \left[ \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \partial_q V_l(Y^n_{t_{k+1},i}) V_{l'}(Y^n_{t_{k+1},j}) \, dX_u \, dX_s \right]
\]
\[
(4.4) \quad - \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \partial_q V_l(Y^n_{t_{k+1},i}) V_{l'}(Y^n_{t_{k+1},j}) \, dX_u \, dX_s.
\]

The remainder \(R_t - R^\text{lead}_t\) contains 3rd-level iterated integrals of \(X\) in each interval \((t_k, t_{k+1})\) in \(2.3\).

**Theorem 4.1.** Suppose \(V \in C^3_1(\mathbb{R}^m; \mathbb{R}^m \times d)\) and \(H > 1/2\). Denote \(c_i = \sum_{j=1}^n a_{ij}\). If it holds that

\[
(4.5) \quad \sum_{i=1}^n b_i = 1 \quad \text{and} \quad \sum_{i=1}^n b_i c_i = 1/2,
\]

then the strong convergence rate of Runge–Kutta method for \((1.1)\) is \(2H - \frac{1}{2}\). More precisely, there exists a constant \(C\) independent of \(n\) such that

\[
(4.6) \quad \left\| \sup_{t \in [0,T]} |Y_t - Y^n_t| \right\|_{L^p(\Omega)} \leq Ch^{2H - \frac{1}{2}}, \quad p \geq 1,
\]

where \(h = \frac{T}{n}\) and \(Y^n_t\) is defined by \((3.3)\).

**Proof.** Notice that condition \((4.5)\) ensures the expression of \(R^\text{lead}_t\) in the form of \((4.3)\). Then Lemma \(2.3\) and Lemma \(2.4\) combined with Proposition \(3.5\) lead to

\[
(4.7) \quad \left\| R^\text{lead}_t - R^\text{lead}_s \right\|_{L^p(\Omega)} \leq C(t-s)^{H/2} h^{2H - \frac{1}{2}}, \quad p \geq 1, \quad t = \lceil t \rceil, \quad s = \lceil s \rceil.
\]

Similarly, based on \((2.3)\), we have

\[
\left\| (R_t - R^\text{lead}_t) - (R_s - R^\text{lead}_s) \right\|_{L^p(\Omega)} \leq C(t-s)^{H/2} h^{2H}, \quad p \geq 1, \quad t = \lceil t \rceil, \quad s = \lceil s \rceil.
\]
Next, for the estimate of $L_t$, recall that
\[
L_t = \sum_{i=1}^{t} \int_{0}^{t} \nabla V_i(\theta Y_s + (1 - \theta) Y^n_s)(Y_s - Y^n_s) d\theta dX^i_s.
\]
We introduce two linear equations defined through $S^i_s$. Let matrices $\Lambda^n$ and $\Gamma^n$ satisfy the linear equations:
\[
\Lambda^n I = I + \sum_{i=1}^{t} S^i_s \Lambda^n dX^i_s,
\]
\[
\Gamma^n I = I - \sum_{i=1}^{t} \Gamma^n S^i_s dX^i_s,
\]
where $I \in \mathbb{R}^{m \times m}$ denotes the identity matrix. Using the chain rule, we know that $\Lambda^n \Gamma^n = I$. Applying Proposition 3.3, Remark 3.10 and [10, Lemma 3.1 (ii)], we have
\[
(4.8)
\]
\[
\max \left\{ \|\Lambda^n\|_{L^p(\Omega)}, \|\Lambda^n\|_{L^p(\Omega)}, \|\Gamma^n\|_{L^p(\Omega)}, \|\Gamma^n\|_{L^p(\Omega)} \right\} \leq C, \quad p \geq 1.
\]
It can be verified that $Y_t - Y^n_t = \Lambda^n \int_0^t \Gamma^n dR_s$, so the Hölder inequality impies that
\[
\|Y - Y^n\|_{L^p(\Omega)} \leq \left\| \Lambda^n \right\|_{L^p(\Omega)} \left\| \int_0^t \Gamma^n dR_s \right\|_{L^p(\Omega)} \leq C, \quad p \geq 1.
\]
Let $f^n_t = n^{2H - \frac{1}{2}} \int_0^t \Gamma^n dR_s$. It suffices to prove that $\|f^n\|_{L^q(\Omega)} \leq C$, for any $q \geq 1$. If there exists some $k \in \{1, \cdots, n\}$ such that $0 \leq s < t_k < t \leq T$, we decompose $\int_0^t \Gamma^n dR_u$ by
\[
\int_s^t \Gamma^n dR_u = \int_s^{[s]} \Gamma^n dR_u + \int_{[s]}^{[t]} \Gamma^n dR_u + \int_{[t]}^t \Gamma^n dR_u
\]
\[
= \int_s^{[s]} \Gamma^n dR_u + \int_{[s]}^{[t]} \Gamma^n dR_u + \int_{[t]}^t \Gamma^n dR_u + \int_{[s]}^{[t]} \Gamma^n dR_u.
\]
For the first term, combining the definitions of $Y^n_t$ and $R_t$, we have
\[
\int_{[s]}^{[x]} \Gamma^n u dR_u = \int_{[s]}^{[x]} \Gamma^n u \left[ V(Y^n_u) - \sum_i b_i V(Y^n_{[u]}^{n,i}) \right] dX_u.
\]
By the Taylor expansion and the property of Young's integral (see e.g. [13]), for any $\frac{1}{2} < \beta < H$,
\[
\left| \int_{[s]}^{[x]} \Gamma^n u dR_u \right| \leq C(\beta, V, T) \|X\|_{\beta}^2 (\|\Gamma^n\|_{\infty} + \|\Gamma^n\|_{\beta}) |[s]^{n} - s|^\beta h^{\beta}
\]
\[
\leq C(\beta, V, T) \|X\|_{\beta}^2 (\|\Gamma^n\|_{\infty} + \|\Gamma^n\|_{\beta}) |t - s|^\beta - 2(\beta - \frac{1}{2}) n^{-(2H - \frac{1}{2})}.
\]
For the second term, according to (4.7)- (4.8) and Lemma [2.1] we have
\[
\left\| n^{2H-\frac{1}{2}} \int_{[s]}^{[t]} \Gamma_{[u]}^{n} dR_{u} \right\|_{L^{q}(\Omega)} \leq C|t-s|^\frac{1}{2}.
\]
For the third term, combining the definitions of \( \Gamma_{n}^{t} \) and \( R_{t} \), we know that it contains the 3rd-level iterated integrals of \( X \), then
\[
\left\| n^{2H} \int_{[s]}^{[t]} d\Gamma_{[u]}^{n} dR_{u} \right\|_{L^{q}(\Omega)} \leq C|t-s|^\frac{1}{2}.
\]
For the fourth term, using similar arguments as the first one, we have
\[
\left| \int_{[t]}^{k} \Gamma_{u}^{n} dR_{u} \right| \leq C(\beta, V, T) \|X\|_{\beta}^{2}(\|\Gamma^{n}\|_{\infty} + \|\Gamma^{n}\|_{\beta})|t-s|^\frac{1}{2}(2H-\beta)n^{-2(H-\frac{1}{2})}.
\]
If \( t_{k} \leq s < t \leq k+1 \), it holds that
\[
\left| \int_{s}^{t} \Gamma_{u}^{n} dR_{u} \right| \leq C(\beta, V, T) \|X\|_{\beta}^{2}(\|\Gamma^{n}\|_{\infty} + \|\Gamma^{n}\|_{\beta})|t-s|^\frac{1}{2}(2H-\beta)n^{-2(H-\frac{1}{2})}.
\]
Therefore, for any \( 0 \leq s < t \leq T \) and \( q \geq 1 \), we obtain
\[
\|f^{n}_{t} - f^{n}_{s}\|_{L^{q}(\Omega)} \leq C(|t-s|^\frac{1}{2} + |t-s|^\frac{1}{2})(2H-\beta)n^{-2(H-\frac{1}{2})}.
\]
If \( q > 4 \), we take \( \beta \) such that \( \max\{\frac{1}{q}, \frac{1}{2q} \} < \beta < H \) and take \( \alpha = \frac{1}{2} - \frac{1}{q} \) such that \( \alpha \in (\frac{1}{q}, \frac{1}{2}) \), then Lemma [2.1] yields that
\[
\mathbb{E}\left[\|f^{n}\|_{\infty}^{q}\right] \leq T^{\alpha q - 1}\mathbb{E}\left[\|f_{t}^{n}\|_{\alpha - \frac{1}{q}}^{q}\right]
\]
\[
\leq C \int_{0}^{T} \int_{0}^{T} \mathbb{E}\left[\|f^{n}_{t} - f^{n}_{s}\|^{q}\right]dtds
\]
\[
\leq C \int_{0}^{T} \int_{0}^{T} \frac{|t-s|^\frac{1}{2} + |t-s|^\frac{1}{2}(2H-\beta)n^{-2(H-\frac{1}{2})}}{dtds}
\]
\[
\leq C.
\]

\[\square\]

Remark 4.2. If the noise is one-dimensional or the diffusion term satisfies the following commutative condition
\[
\sum_{q} \partial_{q} V_{l} V_{l}'^{q} = \sum_{q} \partial_{q} V_{l} V_{l}'^{q}, \quad 1 < l < l' \leq d,
\]
then Fubini’s theorem shows
\[
R_{t}^{\text{lead}} = \frac{1}{2} \sum_{k=0}^{\lfloor t/T \rfloor - 1} \sum_{i,q} \sum_{l \neq 1} b_{i} \left[ \partial_{q} V_{i}(Y_{t_{k+1,i}}^{n})V_{l}^{q}(Y_{t_{k+1,i}}^{n}) - \partial_{q} V_{i}(Y_{t_{k+1,i}}^{n})V_{l}^{q}(Y_{t_{k+1,i}}^{n}) \right]
\]
\[
\left[ \int_{t_{k}}^{t_{k+1}} \int_{s}^{t_{k+1}} \partial_{q} X_{u}^{l} dudX_{u}^{l} - \int_{t_{k}}^{t_{k+1}} \int_{s}^{t_{k}} \partial_{q} X_{u}^{l} dudX_{u}^{l} \right].
\]
As a result, the strong convergence rate in (4.6) is \( H \frac{1}{2} + \frac{1}{2} \) from (2.2).

Remark 4.3. As \( H \) goes to \( \frac{1}{2} \), the rate tends to \( \frac{1}{2} \) in Theorem [4.1] and to 1 in Remark [4.2] respectively. This is consistent with classical results for SDEs driven by standard Brownian motions in Stratonovich sense.
Remark 4.4. If \( d = 1 \) or both drift and diffusion terms satisfy the commutative condition
\[
\sum_q \partial_q V_l V_{l'}^q = \sum_q \partial_q V_{l'} V_l^q, \quad 1 \leq l < l' \leq d,
\]
then \( R_t^{lead} = 0 \) and the strong convergence rate is \( 2H \) from (4.3). In other words, we recover the conditions for order 2 for deterministic ODEs if we take \( H = 1 \) formally when \( X_t = t \).

Remark 4.5. For more general case, if the Hurst parameter of \( X^i \) is \( H_i, \ i = 1, \ldots, d, \) satisfying \( H_1 \geq \cdots \geq H_d > \frac{1}{2} \), Lemma 4.3 and Theorem 4.1 with some revisions imply that the strong convergence rate is \( H_{d-1} + H_d - \frac{1}{2} \).

4.2. Simplified step-\( N \) Euler schemes. For simplicity, we take \( N = 2 \) in the following. Indeed, our approach gives the same strong convergence rate \( 2H - \frac{1}{2} \) of simplified step-\( N \) Euler schemes for \( N \geq 2 \).

Recall that the simplified step-2 Euler scheme is
\[
Y^n_{t_{k+1}} = Y^n_{t_k} + \sum_l V(Y^n_{[s]_l}) \Delta X^n_k + \frac{1}{2} \sum_{l,l',q} \partial_q V_l (Y^n_{[s]_l}) V_{l'}^q (Y^n_{[s]_l}) \Delta X^n_k \Delta X^n_{l'}. \tag{4.9}
\]
The corresponding continuous version is
\[
Y^n = y + \int_0^t V(Y^n_{[s]_l}) dX_s + \frac{1}{2} \int_0^t \sum_{l,l',q} \partial_q V_l (Y^n_{[s]_l}) V_{l'}^q (Y^n_{[s]_l}) \left( X^n_{l'} - X^n_{l'} \right) dX^n_{l'}. \tag{4.10}
\]

To gain a sharp convergence order of the simplified step-2 Euler scheme, we compare it with the following 2-stage Runge–Kutta method (the Heun’s method)
\[
Z^n_{k+1,1} = Z^n_k, \tag{4.11}
\]
\[
Z^n_{k+1,2} = Z^n_k + V(Z^n_{k+1,1}) \Delta X_k, \tag{4.12}
\]
\[
Z^n_{t_{k+1}} = Z^n_k + \frac{1}{2} V(Z^n_{k+1,1}) \Delta X_k + \frac{1}{2} V(Z^n_{k+1,2}) \Delta X_k, \tag{4.13}
\]
which satisfies condition (4.5). We introduce two similar stage values for \( Y^n_{t_{k+1}} \):
\[
Y^n_{t_{k+1,1}} = Y^n_k, \quad Y^n_{t_{k+1,2}} = Y^n_k + V(Y^n_{t_{k+1,1}}) \Delta X_k.
\]
Notice that
\[
V_l (Y^n_{t_{k+1,2}}) = V_l (Y^n_k) + \left[ \int_0^1 \sum_{q,l'} \partial_q V_l (\theta Y^n_{t_{k+1,2}}) (1 - \theta) Y^n_{t_k} d\theta \right] V_{l'}^q (Y^n_k) \Delta X^n_{l'}
\]
and
\[
\partial_q V_l (\theta Y^n_{t_{k+1,2}}) (1 - \theta) Y^n_{t_k} = \partial_q V_l (Y^n_k) + \sum_{q,l',l''} \left[ \int_0^1 \partial_q V_l (\theta' \theta Y^n_{t_{k+1,2}}) (1 - \theta) Y^n_{t_k} + (1 - \theta') Y^n_{t_k} d\theta' \right] \left( \theta V_{l'}^q (Y^n_k) \Delta X^n_{l'} \right)
\]
\[
= \partial_q V_l (Y^n_k) + G^n_{q,l,k} (\theta).
\]
Therefore,
\[
V(Y^n_{t_{k+1},2}) = V(Y^n_{t_k}) + \sum_{q,l'} \left[ \partial_q V(Y^n_{t_k}) + \int_0^1 G^n_{q,l,k}(\theta) d\theta \right] V^\beta(Y^n_{t_k}) \Delta X^l_k
\]
\[
= V(Y^n_{t_k}) + \sum_{q,l'} \partial_q V(Y^n_{t_k}) V^\beta(Y^n_{t_k}) \Delta X^l_k + \sum_{q,l'} \left[ \int_0^1 G^n_{q,l,k}(\theta) d\theta \right] V^\beta(Y^n_{t_k}) \Delta X^l_k
\]
\[
= V(Y^n_{t_k}) + \sum_{q,l'} \partial_q V(Y^n_{t_k}) V^\beta(Y^n_{t_k}) \Delta X^l_k - G^n_{t_{k+1}}.
\]

Scheme 4.9 and its continuous version 4.10 can be transformed as

\[(4.14)\]
\[
Y^n_{t_{k+1}} = Y^n_{t_k} + \frac{1}{2} V(Y^n_{t_{k+1},1}) \Delta X_k + \frac{1}{2} V(Y^n_{t_{k+1},2}) \Delta X_k + \frac{1}{2} \sum_l G^n_{t_{k+1}} \Delta X^l_k,
\]
\[(4.15)\]
\[
Y^n_t = y + \frac{1}{2} \int_0^t V(Y^n_{[s],1}) dX^l_s + \frac{1}{2} \int_0^t V(Y^n_{[s],2}) dX^l_s + \frac{1}{2} \sum_l \int_0^t G^n_{t,s} dX^l_s,
\]
where \(\frac{1}{2} \sum_l \int_0^t G^n_{t,s} dX^l_s\) contains 3rd-level iterated integrals of \(X\). We define the continuous versions for the stage values \(Y^n_{t_{k+1},1}\) and \(Y^n_{t_{k+1},2}\):

\[
Y^n_{t,1} = Y^n_{(t-h)\vee 0},
\]
\[
Y^n_{t,2} = Y^n_{(t-h)\vee 0} + \int_{(t-h)\vee 0}^t V(Y^n_{[s],1}) dX_s.
\]

The continuous dependence of \(Y^n_{t,1}\) and \(Y^n_{t,2}\) on the driving noises follows from Proposition 3.9.

**Proposition 4.6.** If \(V \in C^N_b(\mathbb{R}^m; \mathbb{R}^m \times d)\) and \(1/2 < \beta < H\), then for any \(n \in \mathbb{N}_+\),
\[
\|Y^n_{t,2}\|_\beta \leq C(N, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|^{N-1+1/\beta}_\beta \right\},
\]
\[
\|Y^n_{t,2}\|_\infty \leq |y| + C(N, V, \beta, T) \max \left\{ \|X\|_\beta, \|X\|^{N-1+1/\beta}_\beta \right\}.
\]

Moreover, for some \(C_0 > 0\) and \(0 \leq s < t \leq T\) such that \(\|X\|_\beta |t - s|^\beta \leq C_0\), the estimate can be improved to
\[
\|Y^n_{s,t,\beta}\| \leq C(N, V, \beta, T, C_0) \|X\|_\beta.
\]

Based on 4.14-4.15 and arguments in subsection 4.1, we obtain that the simplified step-2 Euler scheme has the same leading term as the one of scheme
where the modified Milstein scheme has higher order if \( H \). Compared with the optimal convergence rate \( \gamma \) of the modified Euler scheme \([10]\),

\( \gamma = 2H - \frac{1}{2} \) when \( H \in (\frac{1}{2}, 1) \), \( \gamma = 1 \) when \( H = \frac{3}{4} \), \( \gamma = 1 \) when \( H \in (\frac{3}{4}, 1) \), the modified Milstein scheme has higher order if \( H \in (\frac{3}{4}, 1) \).

**Theorem 4.7.** If \( N \geq 2 \), \( V \in C^1_b(\mathbb{R}^m; \mathbb{R}^m \times d) \) and \( H > 1/2 \), then the simplified step-2 Euler scheme as in Theorem 4.1.

\[
Y^{n+1}_t = Y^n_{t_k} + h \sum_{i=1}^{n} \int_{t_{k-1}}^{t_k} \partial_q V_i(Y^n_{t_{k-1}}, s) dW_s,
\]

\( t \in [t_k, t_{k+1}) \).

Remark 4.8. Our result indicates that the modified Milstein scheme is superior to the classical Euler method \([17]\), whose convergence rate is \( 2H - 1 \), \( H \in (\frac{1}{2}, 1) \). Compared with the optimal convergence rate \( \gamma \) of the modified Milstein scheme \([10]\),

\( \gamma = 2H - \frac{1}{2} \) when \( H \in (\frac{1}{2}, \frac{3}{4}) \), \( \gamma = 1 \) when \( H = \frac{3}{4} \), \( \gamma = 1 \) when \( H \in (\frac{3}{4}, 1) \), the modified Milstein scheme has higher order if \( H \in (\frac{3}{4}, 1) \).
Figure 1. Maximum mean-square error (MMSE) vs. stepsize

5. Numerical experiments

In this section, we give an example to verify our main theorems. Consider

\[ dY_t = 3 \sin(Y_t) dt + 3 \cos(Y_t) dX^2_t + 3 \sin(Y_t) dX^3_t, \quad t \in (0, 1], \]

\[ Y_0 = 5, \]

where \( X^2 \) and \( X^3 \) are independent fBms with Hurst parameter \( H > \frac{1}{2} \). We compare the following three numerical schemes: simplified step-2 Euler scheme and two Runge–Kutta methods with coefficients expressed in the Butcher tableaus below

\[
\begin{array}{c|ccc}
  & 1/2 & 1/2 \\
 0 &  & \\
 1/2 & 1/2 & \\
 1/2 & 0 & 1/2 \\
 1 & 0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
  & 1/6 & 2/6 & 2/6 & 1/6 \\
 0 & \\
 1/2 & 1/2 & \\
 1/2 & 0 & 1/2 & \\
 1 & 0 & 0 & 1 \\
\end{array}
\]
In other words, the first method is the implicit midpoint scheme and the second one is a 4-stage Runge–Kutta method satisfying conditions for order 4 in deterministic case. Both of them satisfy condition (4.5). Theorems 4.1 and 4.7 indicate that their maximum mean-square convergence rate is \(2H - \frac{1}{2}\), i.e.,

\[
\left\| \max_{1 \leq k \leq n} |Y_{t_k} - Y_{t_k}^n| \right\|_{L^2(\Omega)} \leq C h^{2H - \frac{1}{2}},
\]

which is consistent with numerical results in Figure 1.2. For each scheme, we use the numerical solution with time step \(h = 2^{-13}\) as the approximated ‘exact solution’ for comparison. The number of sample paths is 1000.

Remark 5.1. As mentioned in the introduction, the rate \(2H - \frac{1}{2}\) is optimal since only increments of fBms are used in the methods under study. This fact is illustrated in Figure 1 that the 4-stage Runge-Kutta method shows the same order as other ones. Therefore, Runge–Kutta methods with stage \(s = 1, 2\) and the step-2 Euler scheme are enough for this rate. It is still an open problem to construct numerical schemes with orders higher than \(2H - \frac{1}{2}\), in which case efficient simulation of iterated integrals of multi-dimensional fBms should also be taken into consideration to make schemes implementable.

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