Semi-simple enlargement of the $\mathfrak{bms}_3$ algebra from a $\mathfrak{so}(2,2) \oplus \mathfrak{so}(2,1)$ Chern-Simons theory

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Abstract

In this work we present a BMS-like ansatz for a Chern-Simons theory based on the semi-simple enlargement of the Poincaré symmetry, also known as AdS-Lorentz algebra. We start by showing that this ansatz is general enough to contain all the relevant stationary solutions of this theory and provides with suitable boundary conditions for the corresponding gauge connection. We find an explicit realization of the asymptotic symmetry at null infinity, which defines a semi-simple enlargement of the $\mathfrak{bms}_3$ algebra and turns out to be isomorphic to three copies of the Virasoro algebra. The flat limit of the theory is discussed at the level of the action, field equations, solutions and asymptotic symmetry.
1 Introduction

Different Chern-Simons (CS) (super)gravity models based on extensions and deformations of the Poincaré and AdS algebras have been recently introduced in the literature [1–10]. In general, apart from the vielbein and spin connection, these theories include new gauge fields which appear as a direct consequence of the enlargement of the symmetry. Although such models represent interesting gravity theories that extend General Relativity (GR), not much is known yet about their solutions and their physical interpretations. In three-dimensions, some progress has been recently carried out, for the so-called Maxwell algebra [11–13] and its semisimple version [14], also known as AdS-Lorentz algebra. The general solution in the stationary sector was reported in [15] and, as a direct consequence of a symmetry enhancement, it depends on three arbitrary functions for which no gauge fixing was discussed. The study of non-stationary solutions with null boundary in a CS gravity theory with Maxwell symmetry, was first reported in [16]. The general solution was found by means of a suitable ansatz for the gauge connection consisting in the standard Bondi-Metzner-Sachs-van der Burg (BMS) gauge for the space-time metric [17] plus a suitable choice for the extra field content.

Asymptotic symmetries in gravitational theories, on the other hand, have attracted great attention in the last decades due to their relation to different aspects of quantum gravity [18–25]. Along these lines, three-dimensional gravity is of particular interest. Despite the absence of local degrees of freedom, CS gravities in three dimensions present a rich boundary dynamics and provide toy models that could realize the bulk/boundary duality beyond the AdS/CFT scenario [26]. In fact, in the case of three-dimensional asymptotically AdS Einstein gravity, it
is possible to define an infinite number of conserved charges at spatial infinity, which span a central extension of the two-dimensional conformal algebra [28]. In the asymptotically flat case, an infinite dimensional asymptotic symmetry algebra can be found at null infinity, given by the $\text{bms}_3$ algebra [29]. Different generalizations of these results for supersymmetric and higher spin extensions of three-dimensional Einstein gravity, have been found in the last years [30–38]. In [16], the authors studied the surface charge algebra of a CS gravity theory with Maxwell symmetry. In this case, asymptotic symmetry is given by an enlargement of the $\text{bms}_3$ algebra found in three-dimensional GR.

In this paper, we extend these results for the case of the semi-simple version of the Poincaré algebra [14], also known as AdS-Lorentz algebra. We construct a BMS-like gauge for the field content of the theory and find that the asymptotic symmetry of conserved charges at null infinity, which turns out to be given by a semi-simple enlargement of the $\text{bms}_3$ algebra. This novel infinite-dimensional symmetry was recently introduced as an expansion of the Virasoro algebra in [39]. These considerations also indicate how to gauge fix the arbitrary functions of the stationary solution found in [15], in such a way that it is contained in the solution space defined by the boundary conditions proposed here. By consistency, we also show that a flat limit of the asymptotic symmetry recovers the previous results found in [16].

This paper is organized as follows. First, in Section 2, we review the main properties of the CS theory for the AdS-Lorentz algebra. In Section 3.1, we propose a special gauge fixing for the stationary solution in such a way that it can be recovered as particular case of the BMS solution that we construct in Section 3.2. In Section 4, we find the asymptotic algebra associated to the AdS-Lorentz CS gravity and show that, under a change of basis, it can be written as a sum of three Virasoro algebras. Some aspects about the flat limit are also discussed. Finally, in Section 5 we conclude with some final comments.

2 Three-dimensional AdS-Lorentz gravity

2.1 The AdS-Lorentz algebra

A semi-simple enlargement of the Poincaré symmetry is described by the AdS-Lorentz algebra [40,41]. Such symmetry can be obtained by introducing an additional set of generators $Z_a$, which render the translations in the Poincaré algebra non-abelian. The commutation relations of this algebra read

\[
\begin{align*}
[J_a, J_b] &= \epsilon_{abc} J^c, \\
[P_a, P_b] &= \epsilon_{abc} Z^c, \\
[J_a, Z_b] &= \epsilon_{abc} Z^c, \\
[Z_a, Z_b] &= \frac{1}{\ell^2} \epsilon_{abc} Z^c, \\
[J_a, P_b] &= \epsilon_{abc} P^c, \\
[Z_a, P_b] &= \frac{1}{\ell^2} \epsilon_{abc} P^c,
\end{align*}
\]

(2.1)

where $a = 0, 1, 2$ is a Lorentz index. This algebra can be shown to be equivalent to the direct sum of three copies of the Lorentz algebra. Indeed, three copies of the form

\[
\begin{align*}
[J^\pm_a, J^\pm_b] &= \epsilon_{abc} J^{\pm c}, \\
[\hat{J}_a, \hat{J}_b] &= \epsilon_{abc} \hat{J}^c,
\end{align*}
\]

(2.2)
reproduce (2.1) considering the redefinitions

\[
\begin{align*}
Z_a &= \frac{1}{\ell^2} (J_a^+ + J_a^-), \\
P_a &= \frac{1}{\ell} (J_a^+ - J_a^-), \\
J_a &= \hat{J}_a + J_a^+ + J_a^-.
\end{align*}
\]

On the other hand, the algebra (2.1) can be written as the direct sum \( \mathfrak{s}\mathfrak{o}(2, 2) \oplus \mathfrak{s}\mathfrak{o}(2, 1) \), reason why it is usually referred as AdS-Lorentz algebra. In fact, one could define

\[
\begin{align*}
\tilde{J}_a &= \ell^2 Z_a, \\
\tilde{P}_a &= P_a, \\
\tilde{Z}_a &= J_a - \ell^2 Z_a,
\end{align*}
\]

so that a direct sum of the Lorentz and AdS algebras appears.

An alternative procedure to obtain the AdS-Lorentz symmetry is through the semigroup expansion method [42]. As was shown in [43], the AdS-Lorentz algebra can be seen as an S-expansion of the AdS algebra. As a consequence, the non-vanishing components of an invariant tensor for the AdS-Lorentz algebra can be obtained using Theorem VII.2 of [42]. Indeed, the relevant invariant tensors of rank 2 are given by

\[
\begin{align*}
\langle J_a J_b \rangle &= \mu_0 \eta_{ab}, \\
\langle P_a P_b \rangle &= \frac{\mu_2}{\ell^2} \eta_{ab}, \\
\langle J_a P_b \rangle &= \frac{\mu_1}{\ell} \eta_{ab}, \\
\langle J_a Z_b \rangle &= \frac{\mu_2}{\ell^2} \eta_{ab}, \\
\langle P_a Z_b \rangle &= \frac{\mu_1}{\ell^3} \eta_{ab}, \\
\langle Z_a Z_b \rangle &= \frac{\mu_2}{\ell^4} \eta_{ab}, \\
\langle Z_a P_b \rangle &= \frac{\mu_1}{\ell^3} \eta_{ab},
\end{align*}
\]

where \( \mu_0, \mu_1 \) and \( \mu_2 \) can be redefined as

\[
\begin{align*}
\mu_0 &\rightarrow \alpha_0, \\
\mu_1 &\rightarrow \alpha_1 \ell, \\
\mu_2 &\rightarrow \alpha_2 \ell^2,
\end{align*}
\]

such that the flat limit \( \ell \rightarrow \infty \) is well defined and leads to the invariant tensor of the Maxwell group. Here, \( \alpha_0, \alpha_1 \) and \( \alpha_2 \) are real arbitrary constants. Note that the flat limit in (2.1) reproduces the commutation relations of the Maxwell algebra. This limit has also been discussed in the context of supergravity [44–45] and higher-spin theory [46]. As an ending remark, it is worth it to mention that supersymmetric extensions of the AdS-Lorentz algebra have been useful in order to restore supersymmetry invariance of supergravity with boundary [47–49].

### 2.2 Chern-Simons gravity

Now we consider a three-dimensional CS gravity theory invariant under the algebra (2.1). The starting point is the CS action

\[
I [A] = \frac{\kappa}{4\pi} \int_M \langle \text{Ad} A + \frac{2}{3} A^3 \rangle,
\]

(2.7)
where $A$ is the gauge connection one-form and $\langle \cdots \rangle$ denotes the invariant trace. Here, $\kappa = \frac{1}{\ell^2}$ is related to the gravitational constant $G$. In particular, the AdS-Lorentz gauge connection one-form reads

$$A = \omega^a J_a + e^a P_a + \sigma^a Z_a, \quad (2.8)$$

where $\omega^a$ is the spin connection one-form, $e^a$ is the vielbein and $\sigma^a$ is the gauge field associated with the non-abelian $Z_a$ generator. In terms of the gauge field components, the CS gravity action invariant under the AdS-Lorentz symmetry reads

$$I_{AdS-L} = -\kappa \frac{1}{4\pi} \int_{\mathcal{M}} \left[ \alpha_0 \left( \omega^a d\omega_a + \frac{1}{3} \epsilon^{abc} \omega_a \omega_b \omega_c \right) + \alpha_1 \left( 2 R_a e^a + 2 \frac{2}{\ell^2} F^a e^a + \frac{1}{3 \ell^2} \epsilon^{abc} e_a e_b e_c \right) + \alpha_2 \left( T^a e_a + \frac{1}{\ell^2} \epsilon^{abc} e_a \sigma_b e_c + 2 R^a \sigma_a + \frac{2}{\ell^2} F^a \sigma_a \right) - d \left( \alpha_1 (\omega^a + \sigma^a) e_a + \alpha_2 \omega^a \sigma_a \right) \right], \quad (2.9)$$

where we have considered the AdS-Lorentz gauge field one-form (2.8), the invariant tensor (2.5) and the algebra (2.1) in the CS general expression (2.7). Here, $R^a = d\omega^a + \frac{1}{2} \epsilon^{abc} \omega_b \omega_c$ corresponds to the Lorentz curvature two-form, $T^a = D\omega e^a$ is the torsion two-form and $F^a = D\omega \sigma^a + \frac{1}{2\ell^2} \epsilon^{abc} \sigma_b \sigma_c$ is the curvature two-form related to the gauge field $\sigma^a$. One can see that the usual gravitational CS term appears with the coupling constant $\alpha_0$, while the Einstein-Hilbert term is related to the coupling constant $\alpha_1$. Then, a natural choice is $\alpha_1 = 1$. Of particular interest is the presence of the extra field $\sigma^a$, which explicitly appears with the coupling constants $\alpha_1$ and $\alpha_2$. Such gauge field contributes to the dynamics of the other fields. Indeed, the equations of motion are given by

$$\delta \omega^a : \quad 0 = \alpha_0 R_a + \left( T_a + \frac{1}{\ell^2} \epsilon^{abc} \sigma^b e^c \right) + \alpha_2 \left( F_a + \frac{1}{2} \epsilon^{abc} e^b e^c \right),$$

$$\delta e^a : \quad 0 = \left( R_a + \frac{1}{\ell^2} F_a + \frac{1}{2\ell^2} \epsilon^{abc} e^b e^c \right) + \alpha_2 \left( T_a + \frac{1}{\ell^2} \epsilon^{abc} \sigma^b e^c \right), \quad (2.10)$$

$$\delta \sigma^a : \quad 0 = \frac{1}{\ell^2} \left( T_a + \frac{1}{\ell^2} \epsilon^{abc} \sigma^b e^c \right) + \alpha_2 \left( R_a + \frac{1}{\ell^2} F_a + \frac{1}{2\ell^2} \epsilon^{abc} e^b e^c \right).$$

Let us note that, unlike the Maxwell CS theory, there is no limit or gauge field-free configuration allowing to recover GR, without cosmological constant, from the AdS-Lorentz CS gravity theory. Nevertheless, as was shown in [50, 52], the AdS-Lorentz symmetry belongs to a family of algebras denoted as $\mathfrak{c}_m$ [53] which, under certain conditions, conduce to the Pure Lovelock (PL) gravity theory. In particular, the AdS-Lorentz algebra, which corresponds to $\mathfrak{c}_4$, is a good candidate to recover the PL theory, that is the usual AdS CS theory in three dimensions [50]. Indeed, considering only the Euler type CS term (proportional to $\alpha_1$) and setting $\sigma_a = 0$, the equation of motion becomes

$$R_a + \frac{1}{2\ell^2} \epsilon^{abc} e^b e^c = 0,$$

$$T_a = 0. \quad (2.11)$$
Naturally, when the gauge field $\sigma_a$ is turned on, the field equations are modified. Since the $\alpha_0$ and $\alpha_2$ are still arbitrary, the equations (2.10) can be written as
\begin{align*}
R_a &= 0, \\
T_a + \frac{1}{\ell^2} \epsilon_{abc} \sigma^b e^c &= 0, \\
F_a + \frac{1}{2} \epsilon_{abc} e^b e^c &= 0.
\end{align*}
\hfill (2.12)

As we shall see, the new field affects not only the asymptotic sector of the spacetime but also the asymptotic charges.

\section{Solutions}

In this section we study field equations solutions in three-dimensional gravity with local AdS-Lorentz symmetry. To this purpose we shall deal with two different representations of the Minkowski metric. First, we focus on the stationary solutions introduced in \cite{15}, where the Minkowski metric is taken to be on its diagonal form $\bar{\eta} = \text{diag}(-1, 1, 1)$. To avoid confusion, gauge fields whose indices are raised and lowered with this metric, will be denoted with bars. Second, we approach the solution in the BMS gauge, where the Minkowski metric is written in the light-cone representation,
\begin{equation}
\eta_{ab} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\end{equation}
\hfill (3.1)

\subsection{Stationary solutions in ADM form}

In this subsection we consider the BTZ-type solution for the equations of motion (2.12). We also calculate the conserved charges of the theory, which, as we will see, are affected by the presence of the non-abelian gauge field $\sigma^a$.

We will consider the solution of the equations of motion (2.12) as was first presented in \cite{15}. The ADM form of the metric is
\begin{equation}
\text{ds}^2 = -N^2 \text{dt}^2 + \frac{dr^2}{N^2} + r^2 (d\varphi + N_\varphi \text{dt})^2,
\end{equation}
\hfill (3.2)

where
\begin{equation}
N^2 = -M + \frac{J^2}{4r^2} + \frac{r^2}{\ell^2}, \quad N_\varphi = -\frac{J}{2r^2}.
\end{equation}
\hfill (3.3)

Here $M$ and $J$ are integration constants. The vielbein can be chosen as
\begin{equation}
\begin{aligned}
\bar{e}^0 &= N \text{dt}, \\
\bar{e}^1 &= N^{-1} \text{dr}, \\
\bar{e}^2 &= r (d\varphi + N_\varphi \text{dt}).
\end{aligned}
\end{equation}
\hfill (3.4)
With this choice, the solution found in [15] for the spin-connection one-form and the non-abelian gauge field \( \bar{\sigma} \) reads

\[
\bar{\omega}^0 = a\xi dt + \rho dr + \xi d\phi,
\bar{\omega}^1 = aH dt + \chi dr + H d\phi,
\bar{\omega}^2 = aG dt + F dr + G d\phi,
\]

(3.5)

\[
\bar{\sigma}^0 = -\ell^2 \left[ a\xi dt + \rho dr + (\xi - N) d\phi \right],
\bar{\sigma}^1 = -\ell^2 \left[ aH dt + \left( \chi + \frac{N\phi}{N} \right) dr + H d\phi \right],
\bar{\sigma}^2 = -\ell^2 \left[ \left( -\frac{r}{\ell^2} + aG \right) dt + F dr + (G - rN\phi) d\phi \right],
\]

(3.6)

where it was defined

\[
\xi = \sqrt{G^2 + H^2 + b},
\rho = \frac{H' + \xi F}{G},
\chi = \frac{GG' + HH'}{G\xi} + \frac{HF}{G}.
\]

(3.7)

Note that the solution depends on three arbitrary functions \( F(r) \), \( G(r) \) and \( H(r) \) and two additional constants \( a \) and \( b \).

We choose to fix the arbitrary functions as

\[
G = rN\phi, \quad F = \frac{rN\phi}{N^2} + \frac{N'}{N}, \quad H = \frac{b + G^2 - N^2}{2N}
\]

(3.8)

since for these particular values, the stationary solution can be recover as as a particular point of the non-stationary solution (see next subsection).

Let us now focus on the calculation of the Noether charges. As it is very well-known, in CS gravity these are given by [54, 55]

\[
Q[\xi] = \frac{\kappa}{4\pi} \int_{\partial\Sigma} \langle \alpha_\xi A \rangle,
\]

(3.9)

where \( \xi^i \) are the asymptotic Killing vectors, and the charges are evaluated at the asymptotic infinity and at a constant time slice \( \partial\Sigma \). Considering the gauge field (2.8), it becomes

\[
Q[\xi] = \frac{\kappa}{4\pi} \lim_{r \to \infty} \int_{0}^{2\pi} d\phi \xi^i \left[ \alpha_0 \bar{\omega}^a_{\phi} \bar{\sigma}_{ai} + \bar{\omega}^a_{\phi} \bar{e}_{ai} + \bar{e}^a_{\phi} \bar{\omega}_{ai} + \frac{1}{\ell^2} (\bar{e}^a_{\phi} \bar{\sigma}_{ai} + \bar{\sigma}^a_{\phi} \bar{e}_{ai}) \right.
\]

\[
+ \alpha_2 \left( \bar{\omega}^a_{\phi} \bar{\sigma}_{ai} + \bar{e}^a_{\phi} \bar{e}_{ai} + \bar{\sigma}^a_{\phi} \bar{\omega}_{ai} + \frac{1}{\ell^2} \bar{\sigma}^a_{\phi} \bar{e}_{ai} \right).
\]

(3.10)
where we have set $\alpha_1 = 1$. Using this formula, we compute the conserved charges of the solution (3.4-3.6), associated with asymptotic invariance under time translations $\xi = \partial_t$ (mass $m$) and rotations $\xi = \partial_\phi$ (angular momentum $j$). These read

$$
m \equiv Q[\partial_t] = \frac{k}{2}[\alpha_0 ab + M + \alpha_2 (\ell^2 ab - J)],$$

$$
j \equiv Q[\partial_\phi] = \frac{k}{2}[\alpha_0 b - J + \alpha_2 \ell^2 (b + M)].$$

(3.11)

As in the Maxwell case [16] the non-abelian gauge field contributes to the mass and angular momentum of the solution and therefore modifies the asymptotic sector.

### 3.2 Solutions in the BMS gauge

As was previously discussed, there is a base where the AdS-Lorentz symmetry can be written as the direct sum $\mathfrak{so}(2,2) \oplus \mathfrak{so}(2,1)$. Thus, a trick can be used to define suitable boundary conditions for the theory: we can go to the direct product base, where the torsionless fields $\tilde{e}^a$ and $\tilde{\omega}^a$ can be set to obey the standard Brown-Henneaux AdS asymptotic behaviour [28] and $\tilde{\sigma}^a$ can be considered simply as a flat Lorentz connection. Subsequently we can go back to the original AdS-Lorentz base. This is the strategy that we will adopt in the following analysis.

For later convenience, the asymptotically flat geometries in the direct product base will be described in three-dimensional BMS gauge [56,57], where the manifold is parameterized by the local coordinates $x^\mu = (u, r, \phi)$. The metric takes the form

$$
ds^2 = \left(\mathcal{M} - \frac{r^2}{\ell^2}\right) du^2 - 2dudr + N d\phi du + r^2 d\phi^2,
$$

(3.12)

were $u$ is the retarded time coordinate and the boundary is located at $r = \text{const}$. Considering the off-diagonal Minkowski metric, this can be written in terms of the vielbein as

$$
ds^2 = 2\tilde{e}^0 \tilde{e}^1 + (\tilde{e}^2)^2.
$$

(3.13)

In particular $\{\tilde{\omega}^a, \tilde{e}^a\}$ can be set to obey the standard AdS asymptotic behaviour,

$$
\begin{align*}
\tilde{\omega}^0 &= \frac{1}{2}\mathcal{M} d\phi + \frac{1}{2\ell^2}N du - \frac{r^2}{2\ell^2} d\phi,
\tilde{\omega}^1 &= d\phi,
\tilde{\omega}^2 &= \frac{r}{\ell^2} du,
\end{align*}
$$

(3.14)

$$
\begin{align*}
\tilde{e}^0 &= -dr + \frac{1}{2}\mathcal{M} du + \frac{1}{2}N d\phi - \frac{r^2}{2\ell^2} du,
\tilde{e}^1 &= du,
\tilde{e}^2 &= r d\phi,
\end{align*}
$$

while the $\tilde{\sigma}^a$ field can be written in the form

$$
\begin{align*}
\tilde{\sigma}^0 &= \frac{1}{2}\mathcal{L} d\phi, \\
\tilde{\sigma}^1 &= d\phi, \\
\tilde{\sigma}^2 &= 0.
\end{align*}
$$

(3.15)
Moreover, since the solutions of the Einstein equations with negative cosmological constant are also solutions of CS gravity invariant under the AdS-Lorentz group (in the considered base), we can write the well-known results for the metric in asymptotically AdS three-dimensional gravity,

$$\dot{M}(u, \phi) = \frac{1}{\ell^2} N'(u, \phi), \quad \dot{N}(u, \phi) = M'(u, \phi),$$

(3.16)

along with the extra condition

$$\mathcal{L} = \mathcal{L}(\phi).$$

(3.17)

In eq. (3.16) the prime and dot denote the derivatives with respect to the coordinates $\phi$ and $u$, respectively.

Since we are interested in the base where the Maxwell symmetry is found as a flat limit of the AdS-Lorentz one, we need the solution in the BMS gauge for our original gauge fields $(e^a, \omega^a, \sigma^a)$. It is straightforward to show by means of (2.4) that these fields are related to the previous ones ($\tilde{e}^a, \tilde{\omega}^a, \tilde{\sigma}^a$) as follows

$$e^a = \tilde{e}^a, \quad \omega^a = \tilde{\sigma}^a, \quad \sigma^a = \ell^2 (\tilde{\omega}^a - \tilde{\sigma}^a).$$

(3.18)

Thus, we have that the field equations (2.12) coming from the AdS-Lorentz CS gravity action are satisfied by the following components of the gauge fields,

$$\omega^0 = \frac{1}{2} \left( M - \frac{\alpha}{\ell^2} \right) d\phi,$$

$$\omega^1 = d\phi,$$

$$\omega^2 = 0,$$

$$e^0 = -dr + \frac{1}{2} \left( M - \frac{r^2}{\ell^2} \right) du + \frac{1}{2} N d\phi,$$

$$e^1 = du,$$

$$e^2 = rd\phi,$$

$$\sigma^0 = \frac{1}{2} N du + \frac{1}{2} (\alpha - r^2) d\phi,$$

$$\sigma^1 = 0,$$

$$\sigma^2 = r du,$$

(3.19)

where we have defined $\alpha = \ell^2 (M - L)$. In addition,

$$\dot{M}(u, \phi) = \frac{\dot{\alpha}}{\ell^2}(u, \phi), \quad \dot{N}(u, \phi) = M'(u, \phi), \quad N'(u, \phi) = \dot{\alpha}(u, \phi).$$

(3.20)

Note that in the limit $\ell \to \infty$, the solution (3.19) reduces to the Maxwell one presented in [16]. Indeed, from (3.20) we obtain in this limit the following conditions: $\mathcal{M} = \mathcal{M}(\phi)$, $\mathcal{N} = u \mathcal{M}' + J(\phi)$ and $\alpha = \frac{\ell^2}{2} (\mathcal{M}'' + u \mathcal{J}' + \mathcal{Z}(\phi)).$

The solution (3.19) matches the stationary one studied in the previous subsection, when the functions $\mathcal{M}$, $\mathcal{N}$ and $\alpha$ are constants, namely

$$\mathcal{M}(u, \phi) = M, \quad \mathcal{N}(u, \phi) = -J, \quad \alpha(u, \phi) = \ell^2 (b + M).$$

(3.21)
along with the appropriate gauge fixing (3.8) for the arbitrary functions \((F,G,H)\). Furthermore, recalling that we have defined \(\alpha = \ell^2(M - L)\), the last expression in (3.21) implies that the two solutions coincides when \(L(\phi) = -b\).

The first and second expressions of (3.21) are straightforward while the third one can be obtained by proceeding in a similar way as in \([16]\). In the mentioned work, it was defined a matrix \(K^a_b\), such that \(e^a_\mu = K^a_b(x) \bar{e}^b_\mu\) and \(\sigma^a_\mu = K^a_b \bar{\sigma}^b_\mu\). It can be shown that the same relations are valid in this case, with the same matrix defined as in \([16]\)

\[
K^a_b = e^a_\mu \bar{e}^\mu_b = \begin{pmatrix}
-N^{-1} & \frac{1}{2N} \left( N^2 \ell^2 - N^2 \right) & \frac{1}{2N} \left( N^2 \ell^2 - N^2 \right) & rN\phi \\
-N^{-1} & 0 & -N^{-1} & rN\phi \\
rN\phi N^{-1} & rN\phi N^{-1} & 1 & 0
\end{pmatrix}, \tag{3.22}
\]

but with \(N\) and \(N\phi\) given by eq. (3.3). At the particular point \(\mathcal{M}(u, \phi) = M\) and \(\mathcal{N}(u, \phi) = -J\), where the solution in the BMS gauge reduces to the stationary one, the remaining arbitrary function \(\alpha(u, \phi)\), or equivalently \(L(\phi)\), also become constant, that is

\[
\mathcal{M}(u, \phi) = M, \quad \mathcal{N}(\phi) = -J, \quad \mathcal{L}(\phi) = -b. \tag{3.23}
\]

Then, it is quite clear that a solution in the BMS gauge (3.19) contains the stationary solution (3.4-3.6) in the CS gravity theory invariant under AdS-Lorentz symmetry, in the particular point where (3.8) and (3.23) are satisfied.

### 4 Asymptotic Symmetry and flat behaviour

In order to compute the asymptotic algebra associated to the AdS-Lorentz CS gravity, we have to consider suitable fall-off conditions for the gauge fields at infinity. Using (3.19), it is direct to evaluate the gauge connection \(A\) in the BMS gauge,

\[
A = \frac{1}{2} \mathcal{N}(u, \phi) du + \frac{1}{2} \alpha(u, \phi) d\phi - \frac{r^2}{2} d\phi Z_0 + rduZ_2 \\
+ \left(-dr + \frac{1}{2} \mathcal{M}(u, \phi) du + \frac{1}{2} \mathcal{N}(u, \phi) d\phi - \frac{r^2}{2\ell^2} du\right) P_0 + duP_1 + r\phi P_2, \tag{4.1}
\]

Furthermore, the radial dependence of the connection (4.1), can be gauged away by an appropriate gauge transformation,

\[
A = h^{-1}dh + h^{-1}ah, \tag{4.2}
\]

where \(h = e^{-rF_h}\). Then, using the Baker-Campbell-Hausdorff formula and the identity \(h^{-1}dh = -dr P_0\), we obtain the following asymptotic field

\[
a = \frac{1}{2} \mathcal{N}(u, \phi) du + \frac{1}{2} \alpha(u, \phi) d\phi Z_0 + \left(\frac{1}{2} \mathcal{M}(u, \phi) du + \frac{1}{2} \mathcal{N}(u, \phi) d\phi\right) P_0 \\
+ duP_1 + \left(\frac{1}{2} \mathcal{M}(u, \phi) - \frac{1}{2\ell^2} \alpha(u, \phi)\right) d\phi J_0 + d\phi J_1. \tag{4.3}
\]
The next step consists in finding gauge transformations, $\delta A = d\lambda + [A, \lambda]$, that preserve our boundary conditions (4.1). To this purpose, let us consider the following gauge parameter

$$\Lambda = h^{-1} \lambda h, \quad \lambda = \chi^a (u, \phi) J_a + \varepsilon^a (u, \phi) P_a + \gamma^a (u, \phi) Z_a. \quad (4.4)$$

In particular, gauge transformations of the connection $A$ with gauge parameter $\Lambda$ imply $r$-independent gauge transformations of $a$ with gauge parameter $\lambda$. The variation of the asymptotic field (4.3) reads

$$\delta \lambda a = \frac{1}{2} \left( \delta \lambda \mathcal{N} (u, \phi) du + \delta \lambda \alpha (u, \phi) d\phi \right) Z_0 + \frac{1}{2} \left( \delta \lambda \mathcal{M} (u, \phi) du + \delta \lambda \mathcal{N} (u, \phi) d\phi \right) P_0 + \frac{1}{2} \left( \delta \lambda \mathcal{M} (u, \phi) d\phi - \frac{1}{\ell^2} \delta \lambda \alpha (u, \phi) d\phi \right) J_0, \quad (4.5)$$

and has to be equal to a gauge transformation of the form

$$\delta \lambda a = D\lambda = d\lambda + [a, \lambda]. \quad (4.6)$$

Considering the angular component of the boundary field $a$ and replacing (4.4) and (4.3) in (4.6), we find

$$\delta \lambda a_\phi = \left( \gamma^0 - \frac{\alpha}{2} \chi^2 - \frac{N}{2} \varepsilon^2 - \frac{M}{2} \gamma^2 \right) Z_0 + \left( \gamma^1 + \gamma^2 \right) Z_1 + \left( \gamma^2 + \frac{\alpha}{2} \chi^1 + \frac{N}{2} \varepsilon^1 + \frac{M}{2} \gamma^1 - \gamma^0 \right) Z_2 + \left[ \varepsilon^0 - \frac{N}{2} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \frac{M}{2} \varepsilon^2 \right] P_0 + \left( \varepsilon^1 + \varepsilon^2 \right) P_1 + \left[ \varepsilon^2 - \frac{N}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{M}{2} \varepsilon^1 - \varepsilon^0 \right] P_2 + \left[ \chi^0 + \frac{1}{2} \left( \frac{\alpha}{\ell^2} - \mathcal{M} \right) \chi^2 \right] J_0 + \left[ \chi^2 + \frac{1}{2} \left( \mathcal{M} - \frac{\alpha}{\ell^2} \right) \chi^1 - \chi^0 \right] J_2. \quad (4.7)$$

From (4.5) and (4.7), one immediately sees that the arbitrary functions appearing in the asymptotic field must satisfy the following relations

$$\delta \lambda \mathcal{M} + \frac{\delta \lambda \alpha}{\ell^2} = 2\chi^0 - \mathcal{M} \chi^2 + \frac{\alpha}{\ell^2} \chi^2, \quad \delta \lambda \mathcal{N} = 2\varepsilon^0 - \mathcal{N} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \mathcal{M} \varepsilon^2, \quad (4.8)$$

while the gauge parameters satisfy

$$\gamma^1 + \gamma^2 = 0, \quad \gamma^2 + \frac{\alpha}{2} \chi^1 + \frac{N}{2} \varepsilon^1 + \frac{M}{2} \gamma^1 - \gamma^0 = 0, \quad \varepsilon^1 + \varepsilon^2 = 0, \quad \varepsilon^1 + \frac{N}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{M}{2} \varepsilon^1 - \varepsilon^0 = 0, \quad (4.9)$$

$$\chi^1 + \chi^2 = 0, \quad \chi^2 + \frac{M}{2} \chi^1 - \frac{\alpha}{2\ell^2} \chi^1 - \chi^0 = 0.$$
On the other hand, the variation of the $u$-component of the gauge field $a$ is given by

$$
\delta \lambda a_u = \left[ \dot{\gamma}^0 - \frac{N}{2} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \frac{M}{2} \varepsilon^2 \right] Z_0 + \left( \dot{\gamma}^1 + \varepsilon^2 \right) Z_1 \\
+ \left[ \dot{\gamma}^2 + \frac{N}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{M}{2} \varepsilon^1 - \varepsilon^0 \right] Z_2 \\
+ \left[ \dot{\varepsilon}^0 - \frac{M}{2} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \frac{N}{2 \ell^2} \varepsilon^2 \right] P_0 + \left( \dot{\varepsilon}^1 + \chi^2 + \frac{\gamma^2}{\ell^2} \right) P_1 \\
+ \left[ \dot{\varepsilon}^2 + \frac{M}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{N}{2 \ell^2} \varepsilon^1 - \chi^0 - \frac{\gamma^0}{\ell^2} \right] P_2 + \dot{\chi}^a J_a.
$$

Comparing (4.10) with the $u$-component of (4.5), we obtain

$$
\begin{align*}
\delta \lambda \mathcal{M} &= 2\dot{\varepsilon}^0 - \mathcal{M} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \frac{N}{\ell^2} \varepsilon^2, \\
\delta \lambda \mathcal{N} &= 2\dot{\varepsilon}^0 - \mathcal{N} \left( \frac{\gamma^2}{\ell^2} + \chi^2 \right) - \mathcal{M} \varepsilon^2,
\end{align*}
$$

(4.11)

together with the condition $\dot{\chi}^a = 0$. Furthermore, (4.10) implies that the other parameters satisfy the following equations

$$
\begin{align*}
\dot{\gamma}^1 + \varepsilon^2 &= 0, \\
\dot{\gamma}^2 + \frac{N}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{M}{2} \varepsilon^1 - \varepsilon^0 &= 0, \\
\dot{\varepsilon}^1 + \chi^2 + \frac{\gamma^2}{\ell^2} &= 0, \\
\dot{\varepsilon}^2 + \frac{M}{2} \left( \frac{\gamma^1}{\ell^2} + \chi^1 \right) + \frac{N}{2 \ell^2} \varepsilon^1 - \chi^0 - \frac{\gamma^0}{\ell^2} &= 0.
\end{align*}
$$

(4.12)

Equations (4.9) and (4.12) can be solved for all the gauge parameters in terms of $\chi^1 = Y(\phi)$, $\varepsilon^1 = T(\phi)$ and $\gamma^1 = R(\phi)$, where $Y$, $T$ and $R$ are arbitrary functions of $\phi$. This leads to the following solution of (4.8) and (4.11) for the transformation laws of $\mathcal{M}$, $\mathcal{N}$ and $\alpha$:

$$
\begin{align*}
\delta \mathcal{M} &= \mathcal{M}' Y + 2\mathcal{M}' Y' - 2\mathcal{Y}'' + \frac{2}{\ell^2} \left( \mathcal{M} R' + \mathcal{N}' T' - R'' + \frac{R M'}{2} + \frac{T N'}{2} \right), \\
\delta \mathcal{N} &= \mathcal{M}' T + 2\mathcal{M}' T' - 2\mathcal{T}'' + \mathcal{N}' Y + 2\mathcal{N}' Y' + \frac{1}{\ell^2} (2\mathcal{N}' R' + R N'), \\
\delta \alpha &= \mathcal{M}' R + 2\mathcal{M}' R' - 2\mathcal{R}'' + \mathcal{N}' T + 2\mathcal{N}' T' + \alpha' Y + 2\alpha Y'.
\end{align*}
$$

(4.13)

The asymptotic structure of the AdS-Lorentz CS gravity is contained in the transformation laws of the functions $\mathcal{M}$, $\mathcal{N}$ and $\alpha$. Indeed, the charge algebra of the AdS-Lorentz theory can be computed following the Regge-Teitelboim approach [58]. In particular, the variation of the charge generators in a three-dimensional CS theory is given by [19]

$$
\delta Q [\lambda] = \frac{k}{2\pi} \int d\phi \left\langle \lambda \delta a_\phi \right\rangle,
$$

(4.14)

where the non-vanishing components of the invariant tensor for the AdS-Lorentz algebra are given by (2.15). Then, using (4.13), we find modulo boundary terms:

$$
\begin{align*}
\delta Q [Y, T, R] &= -\frac{k}{4\pi} \int d\phi \left[ Y \left( \alpha_0 \delta \mathcal{M} + \left( \alpha_2 - \frac{\alpha_0}{\ell^2} \right) \delta \alpha + \alpha_1 \delta \mathcal{N} \right) \\
&+ T \left( \alpha_2 \delta \mathcal{N} + \alpha_1 \delta \mathcal{M} \right) + R \left( \alpha_2 \delta \mathcal{M} + \frac{\alpha_1 \delta \mathcal{N}}{\ell^2} \right) \right].
\end{align*}
$$

(4.15)
Since this expression is linear in the variation of the functions, it can be directly integrated as
\[
Q[Y, T, R] = -\frac{k}{4\pi} \int d\phi \left[ Y \left( \alpha_0 M + \left( \alpha_2 - \frac{\alpha_0}{\ell^2} \right) \alpha + \alpha_1 N \right) + f \left( \alpha_2 N + \alpha_1 M \right) + h \left( \alpha_2 M + \frac{\alpha_1 N}{\ell^2} \right) \right].
\]
(4.16)

4.1 Charge algebra

Following [58], the Poisson algebra of the conserved charges can be evaluated by looking at their variations under gauge transformations, i.e.,
\[
\delta_{\Lambda_2} Q[\Lambda_1] = \{ Q[\Lambda_1] , Q[\Lambda_2] \}.
\]
(4.17)

From the expression (4.16), we see that it is possible to define independent charges for each parameter \( Y, T \) and \( R \) as
\[
j[Y] = \frac{k}{4\pi} \int d\phi Y \left( \alpha_0 M + \left( \alpha_2 - \frac{\alpha_0}{\ell^2} \right) \alpha + \alpha_1 \right),
\]
\[
p[T] = \frac{k}{4\pi} \int d\phi T \left( \alpha_2 N + \alpha_1 M \right),
\]
\[
z[R] = \frac{k}{4\pi} \int d\phi R \left( \alpha_2 M + \frac{\alpha_1 N}{\ell^2} \right).
\]
(4.18)

The Poisson brackets of these charges can be found using (4.17) and (4.13), leading to
\[
\{ j[Y_1], j[Y_2] \} = j \left[ [Y_1, Y_2] \right] - \frac{k \mu_0}{2\pi} \int d\phi Y_1 Y_2''',
\]
\[
\{ j[Y], p[T] \} = p \left[ [Y, T] \right] - \frac{k \mu}{2\pi} \int d\phi YT''',
\]
\[
\{ j[Y], z[R] \} = z \left[ [Y, R] \right] - \frac{k \nu}{2\pi} \int d\phi Y R''',
\]
(4.19)
\[
\{ p[T_1], p[T_2] \} = z \left[ [T_1, T_2] \right] - \frac{k \nu}{2\pi} \int d\phi T_1 T_2''',
\]
\[
\{ p[T], z[R] \} = \frac{p}{\ell^2} \left[ [T, R] \right] - \frac{k \nu}{2\pi \ell^2} \int d\phi TR''',
\]
\[
\{ z[R_1], z[R_2] \} = \frac{z}{\ell^2} \left[ [R_1, R_2] \right] - \frac{k \nu}{2\pi \ell^2} \int d\phi R_1 R_2''',
\]
where here \([x, y] = xy' - yx'\) denotes the Lie bracket. By expanding the functions \( Y(\phi), T(\phi) \) and \( R(\phi) \) in Fourier modes and defining
\[
J_m = j[e^{im\phi}] , \ P_m = p[e^{im\phi}] , \ Z_m = z[e^{im\phi}],
\]

13
the algebra (4.19) takes the following form:

\[ i \{ \mathcal{J}_m, \mathcal{J}_n \} = (m - n) \mathcal{J}_{m+n} + \frac{c_1}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{J}_m, \mathcal{P}_n \} = (m - n) \mathcal{P}_{m+n} + \frac{c_2}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{P}_m, \mathcal{P}_n \} = (m - n) \mathcal{Z}_{m+n} + \frac{c_3}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{J}_m, \mathcal{Z}_n \} = (m - n) \mathcal{Z}_{m+n} + \frac{c_3}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{P}_m, \mathcal{Z}_n \} = \frac{1}{\ell^2} (m - n) \mathcal{P}_{m+n} + \frac{c_2}{12 \ell^2} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{Z}_m, \mathcal{Z}_n \} = \frac{1}{\ell^2} (m - n) \mathcal{Z}_{m+n} + \frac{c_3}{12 \ell^2} m^3 \delta_{m+n,0}, \]

(4.20)

where the central charges \( c_1, c_2 \) and \( c_3 \) are related to the invariant tensor constants \( \mu_0, \mu_1 \) and \( \mu_2 \) defined in (2.5) as \( c_i = 12 k \mu_i - 1 \). This structure corresponds to an infinite-dimensional enhancement of the AdS-Lorentz algebra. In the same way as the AdS-Lorentz algebra defines a semi-simple enlargement of the Poincaré symmetry, the algebra (4.20) defines a semi-simple enhancement of the AdS-Lorentz algebra. In the same way as the AdS-Lorentz algebra defines the asymptotic symmetry of the AdS-Lorentz CS gravity theory in three space-time dimensions. One can see that the AdS-Lorentz algebra is a finite subalgebra spanned by the generators \( \{ \mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_{-1}, \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_{-1}, \mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_{-1} \} \). It is worth noting that the algebra (4.20) is isomorphic to the direct product of three-copies of the Virasoro algebra. In face, by considering the following redefinitions of the generators

\[ \mathcal{L}_m = \frac{1}{2} (\ell \mathcal{P}_m + \ell^2 \mathcal{Z}_m), \quad \tilde{\mathcal{L}}_m = \frac{1}{2} (\ell \mathcal{P}_m - \ell^2 \mathcal{Z}_m), \quad \mathcal{L}_{-m} = \mathcal{J}_m - \ell^2 \mathcal{Z}_m, \quad (4.21) \]

three copies of the Virasoro algebra are revealed

\[ i \{ \mathcal{L}_m, \mathcal{L}_n \} = (m - n) \mathcal{L}_{m+n} + \frac{c}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \tilde{\mathcal{L}}_m, \tilde{\mathcal{L}}_n \} = (m - n) \tilde{\mathcal{L}}_{m+n} + \frac{\tilde{c}}{12} m^3 \delta_{m+n,0}, \]

\[ i \{ \mathcal{L}_m, \tilde{\mathcal{L}}_n \} = (m - n) \mathcal{L}_{m+n} + \frac{\tilde{c}}{12} m^3 \delta_{m+n,0}, \]

(4.22)

with the following central charges

\[ c = \frac{1}{2} (\ell c_2 + \ell^2 c_3), \quad \tilde{c} = \frac{1}{2} (\ell c_2 - \ell^2 c_3), \quad \bar{c} = \frac{1}{2} (c_1 - \ell^2 c_3). \]

(4.23)

### 4.2 Flat limit

The vanishing cosmological constant limit \( \ell \to \infty \) can be performed transparently throughout all the steps followed in obtaining the asymptotic symmetry algebra (4.20). In particular, this limit applied to connection (4.11) and their variations (4.7), (4.10) lead to the asymptotic
form of the Maxwell gravity connection introduced in [16]. As was shown previously, this flat behaviour appears at the level of the AdS-Lorentz CS gravity theory only in the \{J_a, P_a, Z_a\} basis. Then, analogously to the \textit{bms}_3 case, one could expect to recover the asymptotic symmetry of the Maxwell CS gravity theory presented in [16,39] as a flat limit of the asymptotic structure of the AdS-Lorentz CS gravity. This is indeed the case.

From (4.20), the flat limit \(\ell \to \infty\) leads to the enlarged and deformed \textit{bms}_3 algebra presented in [16,39]. Note that after such limit, the \textit{bms}_3 algebra is recovered by setting the generators \(Z_n\) and the central charge \(c_3\) to zero. Analogously to the conformal symmetry in the asymptotically AdS case, the flat limit cannot be appropriately applied to the three copies of the Virasoro algebra, but in the basis \{\(J_m, P_m, Z_m\)\}.

The relation between the algebra obtained here and the deformed \textit{bms}_3 can be generalized to other algebras. As was discussed in [39], such infinite-dimensional symmetries belong to a family of infinite-dimensional algebras denoted as \textit{vir}_\(C_k\) which is related to a generalized \textit{bms}_3 algebra (\textit{vir}_\(B_k\)) by an IW contraction. In particular, \(k = 4\) corresponds to our result, while \(k = 3\) reproduces the 2D-conformal from which its flat limit is precisely given by \textit{bms}_3 symmetry. This particular notation is motivated by the fact that they correspond to infinite-dimensional lifts of the \(C_k\) and \(B_k\) symmetries, respectively [53].

5 Comments and possible developments

In this paper we have studied the asymptotic structure of a CS gravity theory invariant under the semi-simple enlargement of the Poincaré algebra. In order to carry out the analysis, a generalization of the three-dimensional BMS gauge familiar from the GR analysis [57] has been considered to include the extra field content present in the AdS-Lorentz connection. We have contrasted our results with the stationary solution already known in ADM coordinantes in ref. [15] and found that it can be recovered as a particular case of our BMS-like extension when a suitable gauge fixing is chosen for the stationary \(\bar{\sigma}\) field in eq. (3.8). Using this generalized ansatz we have defined boundary conditions for the theory and the CS field equations can be solved exactly, which determines the solution space of the theory completely.

The asymptotic symmetry of the theory leads to an infinite dimensional algebra, which defines a semi-simple enlargement of the \textit{bms}_3 symmetry in the same spirit as the AdS-Lorentz algebra is a semi-simple enlargement of the \textit{iso}(2, 1) [14]. This novel infinite dimensional algebra has three central charges and it is isomorphic to three-copies of the Virasoro algebra.

The BMS formulation is known for providing a well-defined flat limit at the level of the asymptotic charges when passing from asymptotically AdS to asymptotically flat GR [56]. This is also the case in our analysis, where the limit \(\ell \to \infty\) leads to the deformed \textit{bms} algebra previously found by the authors as the asymptotic symmetry of a CS theory invariant under the Maxwell algebra. Remarkably, the flat behaviour present in the AdS-Lorentz CS gravity is
inherited to the asymptotic symmetry:

\[
\begin{array}{cccc}
\text{AdS-Lorentz} & \text{asymptotic symmetry} & \rightarrow & \text{Semi-simple} \\
\text{CS gravity} & & & \text{enlargement of} \ bms_3 \\
\downarrow \text{flat limit} & & & \downarrow \text{flat limit} \\
\text{Maxwell} & \text{asymptotic symmetry} & \rightarrow & \text{Enlarged and} \\
\text{CS gravity} & & & \text{deformed} \ bms_3
\end{array}
\]

Such flat limit was already discussed at the level of infinite-dimensional algebras in [39]. However, this is the first report showing that the semi-simple enlargement of the \(bms_3\) algebra, called as generalized Virasoro algebra in [39], is the corresponding asymptotic symmetry of the three-dimensional \(so(2, 2) \oplus so(2, 1)\) CS gravity theory.

The results presented here could be generalized to the supersymmetric extension of the AdS-Lorentz CS gravity. One could expect to obtain a supersymmetric extension of the semi-simple enlargement of the \(bms_3\) algebra. Furthermore, one could argue that the flat behaviour also appear at the supersymmetric level. Indeed, it is known that the AdS-Lorentz CS and the Maxwell CS supergravities are related through a Inönü-Wigner contraction in three spacetime dimensions [8,44,45].

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