SYMMETRIC INVERSE TOPOLOGICAL SEMIGROUPS
OF FINITE RANK $\leq n$

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Abstract. We establish topological properties of the symmetric inverse topological semigroup of finite transformations $\mathcal{I}^n_\lambda$ of the rank $\leq n$. We show that the topological inverse semigroup $\mathcal{I}^n_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups. Also we prove that a topological semigroup $S$ with countably compact square $S \times S$ does not contain the semigroup $\mathcal{I}^n_\lambda$ for infinite cardinal $\lambda$ and show that the Bohr compactification of an infinite topological symmetric inverse semigroup of finite transformations $\mathcal{I}^n_\lambda$ of the rank $\leq n$ is the trivial semigroup.

In this paper all topological spaces will be assumed to be Hausdorff. We shall follow the terminology of [1, 2, 5, 11]. If $A$ is a subset of a topological space $X$, then we denote the closure of the set $A$ in $X$ by $\text{cl}_X(A)$. By $\omega$ we denote the first infinite cardinal.

A semigroup $S$ is called an inverse semigroup if every $a$ in $S$ possesses an unique inverse, i.e. if there exists an unique element $a^{-1}$ in $S$ such that

$$aa^{-1}a = a \quad \text{and} \quad a^{-1}aa^{-1} = a^{-1}.$$ 

A map which associates to any element of an inverse semigroup its inverse is called the inversion.

A topological (inverse) semigroup is a topological space together with a continuous semigroup operation (and an inversion, respectively). Obviously, the inversion defined on a topological inverse semigroup is a homeomorphism. If $S$ is a semigroup (an inverse semigroup) and $\tau$ is a topology on $S$ such that $(S, \tau)$ is a topological (inverse) semigroup, then we shall call $\tau$ a semigroup (inverse) topology on $S$.

If $S$ is a semigroup, then by $E(S)$ we denote the band (the subset of all idempotents) of $S$. On the set of idempotents $E(S)$ there exists a natural partial order: $e \leq f$ if and only if $ef = fe = e$.

Let $X$ be a set of cardinality $\lambda \geq 1$. Without loss of generality we can identify the set $X$ with the cardinal $\lambda$. A function $\alpha$ mapping a subset $Y$ of $X$ into $X$ is called a partial transformation of $X$. In this case the set $Y$ is called the domain of $\alpha$ and is denoted by $\text{dom}\alpha$. Also, the set $\{x \in X \mid y\alpha = x \text{ for some } y \in Y\}$ is called the range of $\alpha$ and is denoted by $\text{ran}\alpha$. The cardinality of $\text{ran}\alpha$ is called the rank of $\alpha$ and denoted by $\text{rank}\alpha$. For convenience we denote by $\emptyset$ the empty transformation, that is a partial mapping with $\text{dom}\emptyset = \text{ran}\emptyset = \emptyset$.

Let $\mathcal{I}(X)$ denote the set of all partial one-to-one transformations of $X$ together with the following semigroup operation:

$$x(\alpha \beta) = (x\alpha)\beta \quad \text{if} \quad x \in \text{dom}(\alpha \beta) = \{y \in \text{dom}\alpha \mid y\alpha \in \text{dom}\beta\}, \quad \text{for} \quad \alpha, \beta \in \mathcal{I}(X).$$

The semigroup $\mathcal{I}(X)$ is called the symmetric inverse semigroup over the set $X$ (see [2]). The symmetric inverse semigroup was introduced by V. V. Wagner [15] and it plays a major role in the theory of semigroups.

Put

$$\mathcal{I}^\infty_\lambda = \{\alpha \in \mathcal{I}(X) \mid \text{rank}\alpha \text{ is finite}\}, \quad \text{and} \quad \mathcal{I}^n_\lambda = \{\alpha \in \mathcal{I}(X) \mid \text{rank}\alpha \leq n\},$$

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for \( n = 1, 2, 3, \ldots \). Obviously, \( \mathcal{I}_\infty \) and \( \mathcal{I}_n^\alpha (n = 1, 2, 3, \ldots) \) are inverse semigroups, \( \mathcal{I}_\infty \) is an ideal of \( \mathcal{I}(X) \), and \( \mathcal{I}_n^\alpha \) is an ideal of \( \mathcal{I}_\infty \), for each \( n = 1, 2, 3, \ldots \). Further, we shall call the semigroup \( \mathcal{I}_\infty^\alpha \) the symmetric inverse semigroup of finite transformations and \( \mathcal{I}^\alpha_n \) the symmetric inverse semigroup of finite transformations of the rank \( \leq n \). The elements of semigroups \( \mathcal{I}_\infty^\alpha \) and \( \mathcal{I}_n^\alpha \) are called finite one-to-one transformations (partial bijections) of the set \( X \). By

\[
\begin{pmatrix}
  x_1 & x_2 & \cdots & x_n \\
  y_1 & y_2 & \cdots & y_n
\end{pmatrix}
\]

we denote a partial one-to-one transformation which maps \( x_1 \) onto \( y_1 \), \( x_2 \) onto \( y_2 \), \ldots, and \( x_n \) onto \( y_n \), and by 0 the empty transformation. Obviously, in such case we have \( x_i \neq x_j \) and \( y_i \neq y_j \) for \( i \neq j \) (\( i, j = 1, 2, 3, \ldots, n \)).

Let \( \lambda \) be a non-empty cardinal. On the set \( B_\lambda = \lambda \times \lambda \cup \{0\} \), where \( 0 \notin \lambda \times \lambda \), we define the semigroup operation “\( . \)” as follows

\[
(a, b) \cdot (c, d) = \begin{cases} 
(a, d), & \text{if } b = c, \\
0, & \text{if } b \neq c,
\end{cases}
\]

and \( (a, b) \cdot 0 = 0 \cdot (a, b) = 0 \cdot 0 = 0 \) for \( a, b, c, d \in \lambda \). The semigroup \( B_\lambda \) is called the semigroup of \( \lambda \times \lambda \)-matrix units (see [2]). Obviously, for any cardinal \( \lambda > 0 \), the semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) is isomorphic to \( \mathcal{I}_\lambda^1 \).

**Definition 1** ([8, 13]). Let \( \mathcal{G} \) be a class of topological semigroups. A topological semigroup \( S \in \mathcal{G} \) is called \( H \)-closed in the class \( \mathcal{G} \) if \( S \) is a closed subsemigroup of any topological semigroup \( T \in \mathcal{G} \) which contains \( S \) as a subsemigroup. If \( \mathcal{G} \) coincides with the class of all topological semigroups, then the semigroup \( S \) is called \( H \)-closed.

We remark that in [13] \( H \)-closed semigroups are called maximal.

**Definition 2.** [9, 14] Let \( \mathcal{G} \) be a class of topological semigroups. A topological semigroup \( S \in \mathcal{G} \) is called absolutely \( H \)-closed in the class \( \mathcal{G} \) if any continuous homomorphic image of \( S \) into \( T \in \mathcal{G} \) is \( H \)-closed in \( \mathcal{G} \). If \( \mathcal{G} \) coincides with the class of all topological semigroups, then the semigroup \( S \) is called absolutely \( H \)-closed.

**Definition 3.** [9, 14] Let \( \mathcal{G} \) be a class of topological semigroups. A semigroup \( S \) is called algebraically \( h \)-closed in \( \mathcal{G} \) if \( S \) with the discrete topology \( \mathcal{G} \) is absolutely \( H \)-closed in \( \mathcal{G} \) and \( (S, \mathcal{G}) \in \mathcal{G} \). If \( \mathcal{G} \) coincides with the class of all topological semigroups, then the semigroup \( S \) is called algebraically \( h \)-closed.

Absolutely \( H \)-closed semigroups and algebraically \( h \)-closed semigroups were introduced by Stepp in [14]. There they were called absolutely maximal and algebraic maximal, respectively.

Gutik and Pavlyk established in [10] topological properties of infinite topological semigroups of \( \lambda \times \lambda \)-matrix units \( B_\lambda \). They showed that an infinite topological semigroup of \( \lambda \times \lambda \)-matrix units \( B_\lambda \) does not embed into a compact topological semigroup, every non-zero element of \( B_\lambda \) is an isolated point of \( B_\lambda \), and \( B_\lambda \) is algebraically \( h \)-closed in the class of topological inverse semigroups.

Gutik, Lawson and Repovš in [7] introduced the conception of semigroups with a tight ideal series and there they investigated their closure in semitopological semigroups, partially inverse semigroups with continuous inversion. Also they derived related results about the nonexistence of (partial) compactifications of topological semigroups with a tight ideal series. As a corollary they show that the symmetric inverse semigroup of finite transformations \( \mathcal{I}_n^\alpha \) of the rank \( \leq n \) is algebraically closed in the class of inverse (semi)topological semigroups with continuous inversion. Since semigroups with a tight ideal series are not preserved by homomorphisms ([7, Lemma 19]), naturally arises the following question: is the symmetric inverse semigroup of
finite transformations $I^k_\lambda$ of the rank $\leq n$ is algebraically $h$-closed in the class of topological inverse semigroups?

In this paper we shall show that for every infinite cardinal $\lambda$ the finite symmetric inverse semigroup $I^k_\lambda$ of the rank $\leq n$ has topological properties similar to the infinite semigroup of matrix units $B_\lambda$ as a topological semigroup. We show that the topological inverse semigroup $I^k_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups. Also we prove that a topological semigroup $S$ with countably compact square $S \times S$ does not contain the semigroup $I^k_\lambda$ for infinite cardinal $\lambda$ and show that the Bohr compactification of an infinite topological symmetric inverse semigroup of finite transformations $I^k_\lambda$ of the rank $\leq n$ is the trivial semigroup.

The main results of this paper were announced in \[12\].

**Theorem 4.** For any positive integer $n$ the semigroup $I^k_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups.

**Proof.** In the case $\lambda < \omega$ the assertion of the theorem is obvious. Suppose now that $\lambda \geq \omega$. We shall prove the assertion of the theorem by induction.

Theorem 14 from \[10\] implies that the semigroup $I^1_\lambda$ is algebraically $h$-closed in the class of all topological inverse semigroups. We suppose that the assertion of the theorem holds for $n = 1, 2, \ldots, k - 1$ and we shall prove that it is true for $n = k$.

Suppose to the contrary, that there exist a topological inverse semigroup $S$ and continuous homomorphisms $h$ from the semigroup $I^k_\lambda$ with the discrete topology into $S$ such that $(I^k_\lambda)h$ is a non-closed subsemigroup of $S$. Since a homomorphic image of an inverse semigroup is an inverse semigroup, Proposition II.2 of \[4\] implies that $cl_S((I^k_\lambda)h)$ is a topological inverse semigroup. Therefore, without loss of generality we can assume that $(I^k_\lambda)h$ is a dense inverse subsemigroup of $S$.

Let $x \in S \setminus (I^k_\lambda)h$ and $W(x)$ be an open neighbourhood of the point $x$. Since the semigroup $I^{k-1}_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups, without loss of generality we can assume that $W(x) \cap (I^{k-1}_\lambda)h = \emptyset$.

Suppose that $x$ is an idempotent of $S$. Then there exists an open neighbourhood $V(x) \subseteq W(x)$ such that $V(x) \cdot V(x) \subseteq W(x)$. Then since the neighbourhood $V(x)$ contains infinitely many points from $(I^k_\lambda)h \setminus (I^{k-1}_\lambda)h$ we have that $(V(x) \cdot V(x)) \cap (I^{k-1}_\lambda)h \neq \emptyset$. A contradiction to the assumption $W(x) \cap (I^{k-1}_\lambda)h = \emptyset$. Therefore we have $x \cdot x \neq x$.

Since $I^{k-1}_\lambda$ is an inverse subsemigroup of $I^k_\lambda$ Proposition II.2 \[4\] implies that $x^{-1} \notin S \setminus (I^k_\lambda)h$. Since $S$ is a topological inverse semigroup and the semigroup $I^{k-1}_\lambda$ is algebraically $h$-closed in the class of topological inverse semigroups, there exist open neighbourhoods $V(x)$ and $V(x^{-1})$ of the points $x$ and $x^{-1}$, respectively, such that

$$V(x) \cdot V(x^{-1}) \cdot V(x) \subseteq W(x), \quad V(x) \cap (I^{k-1}_\lambda)h = \emptyset, \quad V(x^{-1}) \cap (I^{k-1}_\lambda)h = \emptyset,$$

and

$$V(x) \subseteq W(x).$$

We observe that the set $V(x) \cap (I^k_\lambda)h$ is infinite, otherwise we have that $x \notin cl_S((I^k_\lambda)h)$.

Since $S$ is a topological inverse semigroup, the set $V(x^{-1}) \cap (I^k_\lambda)h$ is infinite too. Let $V = (V(x) \cap (I^k_\lambda)h)^h$ and $V^* = (V(x^{-1}) \cap (I^k_\lambda)h)^h$. Then the sets $V$ and $V^*$ are infinite, and we have $V \cap I^{k-1}_\lambda = \emptyset$ and $V^* \cap I^{k-1}_\lambda = \emptyset$. Therefore $V \cdot V^* \cdot V \cap I^{k-1}_\lambda \neq \emptyset$ and hence $((V)h \cdot (V^*)h \cdot (V)h) \cap (I^{k-1}_\lambda)h \neq \emptyset$. But

$$((V)h \cdot (V^*)h \cdot (V)h) \subseteq V(x) \cdot V(x^{-1}) \cdot V(x) \subseteq W(x),$$

a contradiction to the assumption $W(x) \cap (I^{k-1}_\lambda)h = \emptyset$. The obtained contradiction implies the assertion of the theorem. $\square$

Theorem \[4\] implies
Corollary 5. Let $n$ be any positive integer and let $\tau$ be any inverse semigroup topology on $S^n$. Then $(S^n, \tau)$ is an absolutely $H$-closed topological inverse semigroup in the class of topological inverse semigroups.

The following theorem generalizes Theorem 10 from [10].

**Theorem 6.** A topological semigroup $S$ with countably compact square $S \times S$ does not contain an infinite countable semigroup of matrix units.

**Proof.** Suppose to the contrary: there exists a topological semigroup $S$ with countably compact square $S \times S$ such that $S$ contains an infinite countable semigroup of $\omega \times \omega$-matrix units $B_\omega$. We numerate elements of a set $X$ of cardinality $\omega$ by non-negative integers, i. e., $X = \{0, \alpha_1, \alpha_2, \ldots\}$. Then we consider the sequence $\{(\alpha_0, \alpha_n), (\alpha_n, \alpha_0)\}_{n=1}^\infty$ in $B_\omega \times B_\omega \subset S \times S$. The countable compactness of $S \times S$ guarantees that this sequence has an accumulation point $(a, b) \in S \times S$. Since $(\alpha_0, \alpha_n) \cdot (\alpha_n, \alpha_0) = (\alpha_0, \alpha_0)$, the continuity of the semigroup operation on $S$ guarantees that $ab = (\alpha_0, \alpha_0)$. By Lemma 4 [10], every non-zero element of the semigroup of $\omega \times \omega$-matrix units $B_\omega$ endowed with the topology induced from $S$ is an isolated point in $B_\omega$. So, there exists a neighbourhood $O((\alpha_0, \alpha_0)) \subseteq S$ of the point $(\alpha_0, \alpha_0) \in B_\omega$ containing no other points of the semigroup $B_\omega$. Since $ab = (\alpha_0, \alpha_0)$, the points $a, b$ have neighborhoods $O(a), O(b) \subset S$ such that $O(a) \cdot O(b) \subset O((\alpha_0, \alpha_0))$. Since $a$ is an accumulation point of the sequence $(\alpha_0, \alpha_n)$, there exists a positive integer $n$ such that $(\alpha_0, \alpha_n) \in O(a)$. Similarly there exists a positive integer $m > n$ such that $(\alpha_m, \alpha_0) \in O(b)$. Then $(\alpha_0, \alpha_n) \cdot (\alpha_m, \alpha_0) = 0 \in O(a) \cdot O(b) \cap B_\omega = (\alpha_0, \alpha_0)$, which is a contradiction. \square

Since every infinite semigroup of $\lambda \times \lambda$-matrix units $B_\lambda$ contains the semigroup $B_\omega$, Theorem 6 implies

**Theorem 7.** A topological semigroup $S$ with countably compact square $S \times S$ does not contain an infinite semigroup of matrix units.

Theorem 6 implies

**Corollary 8 ([10, Theorem 10]).** An infinite semigroup of matrix units does not embed into a compact topological semigroup.

A semigroup homomorphism $h : S \to T$ is called annihilating if $(s)h = (t)h$ for all $s, t \in S$.

A semigroup $S$ is called congruence-free if it has only two congruences: identical and universal [2]. Obviously, a semigroup $S$ is congruence-free if and only if every homomorphism $h$ of $S$ into an arbitrary semigroup $T$ is an isomorphism “into” or is an annihilating homomorphism.

Theorem 1 from [3] implies that the semigroup $B_\lambda$ is congruence-free for every cardinal $\lambda \geq 2$ and hence Theorem 6 implies

**Theorem 9.** Every continuous homomorphism from an infinite topological semigroup of matrix units into a topological semigroup $S$ with countably compact square $S \times S$ is annihilating.

Theorem 9 implies

**Corollary 10 ([10, Theorem 12]).** Every continuous homomorphism from an infinite topological semigroup of matrix units into a compact topological semigroup is annihilating.

**Theorem 11.** Let $\lambda \geq \omega$ and $n$ be a positive integer. Then every continuous homomorphism of the topological semigroup $S_\lambda^n$ into a topological semigroup $S$ with countably compact square $S \times S$ is annihilating.

**Proof.** We shall prove the assertion of the theorem by induction. By Theorem 9 every continuous homomorphism of the topological semigroup $S_\lambda^n$ into a topological semigroup $S$ with countably compact square $S \times S$ is annihilating. We suppose that the assertion of the theorem holds for $n = 1, 2, \ldots, k - 1$ and we shall prove that it is true for $n = k$. \square
Obviously it is sufficiently to show that the statement of the theorem holds for the discrete semigroup \( J_k \). Let \( h: J_k \to S \) be arbitrary homomorphism from \( J_k \) with the discrete topology into a topological semigroup \( S \) with countably compact square \( S \times S \). Then by Theorem 9, the restriction \( h_{| J_1^i} : J_1^i \to S \) of homomorphism \( h \) onto the subsemigroup \( J_1^i \) of \( J_k \) is an annihilating homomorphisms. Let \((J_1^1) = h_{| J_1^1} = (J_1^1)h = e\), where \( e \in E(S) \). We fix any \( \alpha \in J_k \) with \( \text{ran}(\alpha) = i \geq 2 \). Let \( \alpha = \left( \begin{array}{ccc} x_1 & x_2 & \cdots & x_i \\ y_1 & y_2 & \cdots & y_i \end{array} \right) \) (where \( x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_i \in X \) for some set \( X \) of cardinality \( \lambda \)). We fix \( y_1 \in X \) and define subsemigroup \( T_{y_1} \) of \( J_k \) as follows:

\[
T_{y_1} = \left\{ \beta \in J_k \mid \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) \cdot \beta = \beta \cdot \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) = \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) \right\}.
\]

Then the semigroup \( T_{y_1} \) is isomorphic to the semigroup \( J_k^{-1} \), the element \( \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) \) is zero of \( T_{y_1} \) and hence by induction assumption we have \( \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) h = (\beta)h \) for all \( \beta \in T_{y_1} \).

Since \( \left( \begin{array}{c} y_1 \\ y_1 \end{array} \right) \in J_1^1 \), we have that \( (\beta)h = (0)h \) for all \( \beta \in T_{y_1} \). But \( \alpha = \alpha \gamma \), where \( \gamma = \left( \begin{array}{ccc} y_1 & y_2 & \cdots \\ y_1 & y_2 & \cdots \\ \vdots & \vdots & \ddots \end{array} \right) \in T_{y_1} \), and hence we have

\[
(\alpha)h = (\alpha \gamma)h = (\alpha)h \cdot (\gamma)h = (\alpha)h \cdot (0)h = (\alpha \cdot 0)h = (0)h = e.
\]

This completes the proof of the theorem. \( \square \)

Theorem 11 implies

**Theorem 12.** Let \( \lambda \geq \omega \) and \( n \) be a positive integer. Then every continuous homomorphism of the topological semigroup \( J_\lambda^n \) into a compact topological semigroup is annihilating.

Recall [3] that a Bohr compactification of a topological semigroup \( S \) is a pair \((\beta, B(S))\) such that \( B(S) \) is a compact semigroup, \( \beta: S \to B(S) \) is a continuous homomorphism, and if \( g: S \to T \) is a continuous homomorphism of \( S \) into a compact semigroup \( T \), then there exists a unique continuous homomorphism \( f: B(S) \to T \) such that the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\beta} & B(S) \\
g \downarrow & & \downarrow f \\
T & & 
\end{array}
\]

commutes.

Theorem 12 and Theorem 2.44 [1, Vol. I] imply

**Theorem 13.** If \( \lambda \geq \omega \) and \( n \) is a positive integer, then the Bohr compactification of the topological semigroup \( J_\lambda^n \) is a trivial semigroup.

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