The importance of precession in modelling the direction of the final spin from a black-hole merger

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Abstract. The prediction of the spin of the black hole resulting from the merger of a generic black-hole binary system is of great importance to study the cosmological evolution of supermassive black holes. Several attempts have been recently made to model the spin via simple expressions exploiting the results of numerical-relativity simulations. Here, I first review the derivation of a formula, proposed in Ref. [1], which accurately predicts the final spin magnitude and direction when applied to binaries with separations of hundred or thousands of gravitational radii. This makes my formula particularly suitable for cosmological merger-trees and N-body simulations, which provide the spins and angular momentum of the two black holes when their separation is of thousands of gravitational radii. More importantly, I investigate the physical reason behind the good agreement between my formula and numerical relativity simulations, and nail it down to the fact that my formula takes into account the post-Newtonian precession of the spins and angular momentum in a consistent manner.

The dynamics of black-hole (BH) binaries is a very complex problem which has been solved only very recently through time-expensive numerical-relativity (NR) calculations. However, it has been shown that the dimensionless spin of the remnant from a BH binary merger, $a_{\text{fin}} = S_{\text{fin}}/M_{\text{fin}}^2$, can be described with simple prescriptions based on point particles [3, 4, 5], on fits to the NR data [6, 7, 8, 9, 10], or on a combination of the two approaches [11]. These formulas are useful because they provide information over the entire 7-dimensional space of parameters for BH binaries in quasi-circular orbits, namely: the mass ratio $q = M_2/M_1$ and the six components of the initial dimensionless spin vectors $a_{1,2} = S_{1,2}/M_{1,2}^2$. Such parameter space could in principle be investigated entirely via NR calculations; in practice, however, the simulations are very expensive and restricted to $q = 0.1–1$. Also, these formulas have applications in astrophysics, where they could provide information on the properties of massive-star binary systems; in cosmology, where supermassive BHs (SMBHs) are believed to assemble through accretion and mergers; in gravitational-wave astronomy, where the a priori knowledge of the final spin can help the detection.

While the different expressions for the spin norm, $|a_{\text{fin}}|$, are in good agreement with the results of NR simulations, the predictions for the final spin direction, $\hat{a}_{\text{fin}} = a_{\text{fin}}/|a_{\text{fin}}|$, do not agree well with one another and are all essentially imprecise when the binaries are widely separated. This is because all expressions are built from and model the typical NR binaries and hence the dynamics of the last few orbits before the merger. Because it does not account systematically for the precession of the orbital angular momentum $L$, the prediction for $\hat{a}_{\text{fin}}$ depends on the separation of the binary and is therefore of little use for applications, such as cosmological merger-trees or N-body simulations, that provide the spins of the two BHs at separations of thousands of gravitational radii. Although one could use the PN equations to
evolve a widely-separated binary to a separation of few gravitational radii and then apply the formulas, this makes the formulas difficult to use and implement. In Ref. [1] we followed instead a different approach and showed that it is possible to derive a formula for the final spin that takes into account the precession of the spins and that is, therefore, applicable to binaries with arbitrary separations.

The derivation of the formula, for more details about which we refer the reader to Ref. [1], is based on the following assumptions:

(i) When the spins are parallel to $\mathbf{L}$ (either aligned or antialigned), the formula must reduce to the fit of the NR results presented in Refs. [1, 11], namely

$$a_{\text{fin}} = \hat{a} + \hat{a} \nu (s_4 \hat{a} + s_5 \nu + t_0) + \nu (2 \sqrt{3} + t_2 \nu + t_3 \nu^2),$$

where $\nu \equiv M_1 M_2 / (M_1 + M_2)^2$ is the symmetric mass ratio, $\hat{a} \equiv (a_1 + a_2 q^2) / (1 + q^2)$, and

$$s_4 = -0.1229 \pm 0.0075, \quad s_5 = 0.4537 \pm 0.1463,$$
$$t_0 = -2.8904 \pm 0.0359, \quad t_2 = -3.5171 \pm 0.1208, \quad t_3 = 2.5763 \pm 0.4833.$$  

(ii) The mass $M_{\text{rad}}$ radiated to gravitational waves can be neglected i.e., $M_{\text{fin}} = M \equiv M_1 + M_2$. The reason why assumption (i) is reasonable here is that $M_{\text{rad}}$ is largest for aligned binaries but these are also the ones fitted by expression (1). In this way, the mass losses are automatically accounted for by the values of the coefficients $t_0, t_2, t_3, s_4$, and $s_5$.

(iii) The norms $|S_1|, |S_2|, |\tilde{\ell}|$ do not depend on the separation of the binary $r$, with $\tilde{\ell}$ being

$$\tilde{\ell}(r) \equiv \tilde{S}_{\text{fin}} - [S_1(r) + S_2(r)] = \tilde{L}(r) - J_{\text{rad}}(r),$$

where $S_1(r), S_2(r)$ and $L(r)$ are the spins and the orbital angular momentum at the separation $r$ and $J_{\text{rad}}(r)$ is the angular momentum radiated from $r$ to the merger. While the constancy of $|S_1|$ and $|S_2|$ is a very good assumption for BHs, the constancy of $|\tilde{\ell}|$ is heuristic and based on the idea that the merger takes place at an “effective” innermost stable circular orbit (ISCO), so that $|\tilde{\ell}|$ can be interpreted as the residual orbital angular momentum contributing to $S_{\text{fin}}$.

(iv) The final spin $S_{\text{fin}}$ is parallel to the initial total angular momentum $J(r_{\text{in}}) = S_1(r_{\text{in}}) + S_2(r_{\text{in}}) + L(r_{\text{in}})$.

(v) The angle between $\mathbf{L}$ and $S \equiv S_1 + S_2$ and the angle between $S_1$ and $S_1$ are constant during the inspiral, although $\mathbf{L}$ and $S$ precess around $J$.

Assumptions (iv) and (v) are motivated by the PN approximation. It has been in fact shown by Ref. [13] that within the adiabatic approximation the secular angular-momentum losses via gravitational radiation are along $J$. This is because as $L$ rotates around $J$, the emission orthogonal to $J$ averages out. Therefore, $S$ and $L$ essentially precess around the direction $\hat{J}$, which remains nearly constant (cf. the detailed discussion in Ref. [13]), and the angle between $\mathbf{L}$ and $S$ and that between the two spins remain constant as well.

(vi) When the initial spin vectors are equal and opposite and the masses are equal, the spin of the final BH is the same as for nonspinning binaries. Besides being physically reasonable – if the spins are equal and opposite, their contributions are expected to cancel out – this assumption is confirmed by NR simulations and by the leading-order PN spin-spin and spin-orbit couplings.

1 The number of free parameters of the fit is actually four, since $t_0, t_2, t_3, s_4$ and $s_5$ must satisfy the constraint

$$a_{\text{fin}} = \frac{\sqrt{3}}{2} + t_2 \frac{16}{64} + t_3 \frac{64}{1229} = 0.68646 \pm 0.00004,$$

which follows from the results obtained by Ref. [12] for equal-mass non-spinning BHs.
These assumptions are sufficient to derive an expression for both the magnitude and the direction of the final spin [1]. In particular, the final spin norm is given by

\[ |\mathbf{a}_{\text{fin}}| = \frac{1}{(1+q)^2} \left[ |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 q^4 + 2 |\mathbf{a}_2| |\mathbf{a}_1| q^2 \cos \alpha + 2 (|\mathbf{a}_1| \cos \beta + |\mathbf{a}_2| q^2 \cos \gamma) |\ell| q + |\ell|^2 q^2 \right]^{1/2}, \tag{4} \]

where

\[ |\ell| = 2\sqrt{3} + t_2 \nu + t_3 \nu^2 + \frac{s_4}{(1+q^2)^2} (|\mathbf{a}_1|^2 + |\mathbf{a}_2|^2 q^4 + 2 |\mathbf{a}_1| |\mathbf{a}_2| q^2 \cos \alpha) + \left( \frac{s_3 \nu + t_0 + 2}{1+q^2} \right) (|\mathbf{a}_1| \cos \beta + |\mathbf{a}_2| q^2 \cos \gamma), \]

and the angles \( \alpha, \beta, \gamma \) are defined by

\[ \cos \alpha \equiv \hat{\mathbf{a}}_1 (r_{\text{in}}) \cdot \hat{\mathbf{a}}_2 (r_{\text{in}}), \quad \cos \beta \equiv \hat{\mathbf{a}}_1 (r_{\text{in}}) \cdot \hat{\mathbf{L}} (r_{\text{in}}), \quad \cos \gamma \equiv \hat{\mathbf{a}}_2 (r_{\text{in}}) \cdot \hat{\mathbf{L}} (r_{\text{in}}). \tag{5} \]

The angle \( \theta_{\text{fin}} \) between the final spin and the initial orbital angular momentum \( \mathbf{L}(r_{\text{in}}) \) is instead

\[ \cos \theta_{\text{fin}} = \hat{\mathbf{L}} (r_{\text{in}}) \cdot \hat{\mathbf{J}} (r_{\text{in}}). \tag{6} \]

We will now test our expressions (4) and (5) for \( |\mathbf{a}_{\text{fin}}| \) and our expression (6) for \( \theta_{\text{fin}} \) against the NR simulations for generic binaries (i.e., with spins not parallel to \( \mathbf{L} \)) published so far. Also, we will compare our predictions (AEI) with those of similar formulas suggested by Refs. [4] (BKL), [11] (AEI old) and [8] (FAU). The comparison consists of two steps. First, we compare the different predictions using as input the initial data of the NR simulations, in which the binaries have small separations \( r_{\text{in}} \lesssim 10 M \). Second, using binaries at large separations \( r_{\text{in}} \lesssim 2 \times 10^4 M \), for which the dynamics starts being dominated by gravitational-wave emission and thus of direct relevance for cosmological investigations. More precisely, we evolve the NR initial configurations back in time up to a separation of \( 2 \times 10^4 M \) using the PN equations in the quasi-circular limit\(^2\), calculating the predictions of the different formulas at each step and considering the maximum error for each formula. (The results that we present, however, do not change significantly if we integrate only up to a separation of \( \sim 200 M \).)

The upper left panel of Fig. 1 shows the predictions of the various formulas for \( |\mathbf{a}_{\text{fin}}| \), when applied to the small-separation configurations corresponding to the initial data of the NR simulations (see caption for details). In particular, it reports the error \( ||\mathbf{a}_{\text{fin},\text{NR}}| - |\mathbf{a}_{\text{fin},*}|| \), where “*” stands either for “AEI” (which, as mentioned in Ref. [1], gives the same predictions for \( |\mathbf{a}_{\text{fin}}| \) as “AEI old”), “FAU” or “BKL”. The lower left panel shows instead the maximum error when the configurations are evolved back in time up to \( r_{\text{in}} = 2 \times 10^4 M \). Although the AEI expression is slightly better, all the formulas give accurate predictions for the final spin norm, both for small and large separations. Note that the larger errors for the indices 1–10, which correspond to the simulations of Ref. [14] with small mass ratios \( (q = 0.13–0.17) \), are most likely due to the larger truncation errors affecting those simulations (see Ref. [14], sec. IIIA).

The situation is very different when considering the final spin direction. In particular, the right panels of Fig. 1 report the inclination-angle error, \( ||\theta_{\text{fin},*} - \arccos [\hat{\mathbf{L}} (r_{\text{in}}) \cdot \hat{\mathbf{a}}_{\text{fin},\text{NR}}]| | \), for the data in the left panels, but for those of Ref. [17], for which the final spin direction was not published. Clearly, when considering small-separation binaries (upper right panel), our new formula performs slightly better than the “BKL” and “AEI old” formulas, but it is not better

\(^2\) In particular, following Ref. [2], we use the precession equations for the spins and the angular momentum at 2 PN order and the rate of change of the frequency at 3.5 PN order (with spin terms included up to 2 PN order).
than the “FAU” formula. Indeed, the latter is exact by construction for the indices 34–66. This is because for such data the final spin direction has not been published and it has been here reconstructed using the FAU formula applied to the configurations of Table II in Ref. [8]. However, when considering large-separation binaries (lower right panel), our new formula clearly performs much better than all the other ones. For instance, our error for $\theta_{\text{fin}}$ is below 7 degrees, for any separation, while it can be as large as 70 degrees with the older formulas. (The “steps” in the lower right panel reflect the different sequences of Table I of Ref. [14].)

To highlight the role played by the precession of the spins in the prediction of the final spin direction, in the left panel of Fig. 2 we focus on the binary “SP3” of Ref. [15] (index 41 in Fig. 1). In this configuration the spins precess strongly and the final spin flips relative to $\mathbf{S}(r_{\text{in}})$. In particular, $S_1$ and $S_2$ are initially parallel, orthogonal to $\mathbf{L}$ and have: $a_1 = a_2 \approx 0.5$, $q = 1$. The filled hexagon in Fig. 2 is the numerical result as obtained with an initial separation $r_{\text{in}} \approx 6.6 M$, while different lines show the angle $\theta_{\text{fin}}$ obtained with the various formulas as a function of the binary separation. More specifically, using the initial conditions of the NR simulation we have evolved the “SP3” binary back in time using the PN equations up to a separation of $2 \times 10^4 M$, applying the various formulas at each step to compute $\theta_{\text{fin},*} = \arccos[\hat{L}(r_{\text{in}}) \cdot \hat{a}_{\text{fin},*}]$. The line “PN+NR” reports instead the angle $\arccos[\hat{L}(r_{\text{in}}) \cdot \hat{a}_{\text{fin},NR}]$. In the right panel, we consider the configuration “Q13TH00” of Ref. [14] (index 1 in Fig. 1). This configuration features a smaller, non-spinning BH and a larger, spinning BH with spin $a \approx 0.81$ lying on the equatorial plane and

![Figure 1](image-url)

**Figure 1.** *Left:* The upper panel shows the error $||a_{\text{fin},NR} - a_{\text{fin},*}||$ (“**” being either “AEI”, “FAU” or “BKL”) of the various formulas for the final spin norm, when applied to the small-separation configurations corresponding to the initial data of the simulations of Refs. [14] (indices 1-40), [15] (indices 41-42), [16] (index 43), [8] (indices 44-76) and [17] (indices 77-84). The lower panel shows instead the maximum error $||a_{\text{fin},NR} - a_{\text{fin},*}||$ when the configurations are evolved back in time up to large separations of $r = 2 \times 10^4 M$. Although the new AEI expression is slightly better, all formulas give accurate predictions for $|a_{\text{fin}}|$, both for small and large separations. Note that the larger errors for the indices 1–10 are due to the larger truncation errors affecting those simulations (see text for details). *Right:* The same as in the left panel but for the inclination angle error $|\theta_{\text{fin},*} - \arccos[\hat{L}(r_{\text{in}}) \cdot \hat{a}_{\text{fin},NR}]|$ and without the data of Ref. [17], for which the final spin direction has not been published. The new AEI expression is accurate both for small and large separations, while the other ones become imprecise for large separations.
Figure 2. Left: Predictions of the different formulas for $\theta_{\text{fin}}$ for the binary “SP3” of Ref. [15] as a function of the initial separation $r_{\text{in}}$. Shown with the “NR+PN” curve is the angle $\arccos \left[ \hat{L}(r_{\text{in}}) \cdot \hat{a}_{\text{fin,NR}} \right]$, where $\hat{L}(r_{\text{in}})$ is obtained by integrating the PN equations starting from the initial conditions of the NR simulation, while the filled hexagon shows $\theta_{\text{fin}}$ as computed by the NR simulation. Right: The same as in the left panel, but for the configuration “Q13TH00” of Ref. [14]. Clearly, only our formula reproduces the “NR+PN” results to within 2-3 degrees for any $r_{\text{in}}$, while the other formulas have errors that grow very large for separations $r_{\text{in}} \gtrsim 200M$.

orthogonal to the separation ($r_{\text{in}} \approx 6.4M$). The mass ratio is $q \approx 0.13$, which makes this binary interesting as it allows to test the behavior of the various formulas in a region of parameter space which so far has been sampled only sparsely by NR simulations, which usually deal with $q \approx 1$. (This is because the cost of a NR simulation grows as $1/q^2$, a factor $1/q$ coming from the slow inspiral of a small mass ratio binary, and another factor $1/q$ appearing because if the mass ratio is small, one needs much resolution to resolve the two very different scales of the problem.)

Three results are clear from these comparisons. First: all previous formulas provide a reasonable estimate of $\theta_{\text{fin}}$ but only when the binary has a separation $r_{\text{in}} \ll 200M$. Second: Only our formula provides a reasonable estimate at all separations; indeed, the “PN+NR” results are reproduced to within the accuracy of the NR simulation, i.e. 2-3 degrees. All the other formulas, instead, predict the same value for $\theta_{\text{fin}}$ for all $r_{\text{in}}$, because they take as an input the angles $\arccos (\hat{S}_{1,2} \cdot \hat{L})$ and $\arccos (\hat{S}_1 \cdot \hat{S}_2)$, which are constant under the quasi-circular PN evolution [2]. Therefore, the other formulas become rapidly become imprecise when the binary’s separation increases, reaching errors as large as 60 degrees for $r_{\text{in}} \gtrsim 200M$. Third: the error of the old formulas when applied to large separation binaries is larger for small mass ratio systems (e.g. the maximum error is $\sim 13$ degree for the comparable-mass configuration SP3, while it is $\sim 60$ degrees for the small mass-ratio binary Q13TH00). This had to be expected, again on the grounds that the old formulas predict the same $\theta_{\text{fin}}$ for all $r_{\text{in}}$. As a result of this, in fact, the maximum error of the old formulas is roughly given by their prediction for $\theta_{\text{fin}}$ at small separations, because the correct $\theta_{\text{fin}}$ becomes small for large $r_{\text{in}}$ (cf. the “PN+NR” curves in Fig. 2). This prediction can be very large for small $q$ if the angle between $\hat{S}$ and $\hat{L}$ at small separations is large. This can be easily understood by noting that at small separations the old formulas give roughly the same results as our formula: Therefore, for $q \approx 0$ their prediction is
cos(\theta_{in}) \approx \hat{L}(r_{in}) \cdot \hat{S}(r_{in}) \text{ [because for } q \approx 0, \hat{J}(r_{in}) \approx \hat{S}(r_{in})].

We stress that such a good agreement between our formula and the data for the final spin, irrespective of the separation of the binary to which the formula is applied, emerges because we have consistently taken into account the effect of precession through assumptions (iv) and (v), while that effect is not accounted for in the other formulas. For instance, in our earlier formula of Ref. [11] (“AEOl’d”), assumptions (iv) and (v) were replaced by the assumption that the angular momentum emitted during the inspiral is parallel to \(\mathbf{L}(r_{in})\) [cf. assumption (iii) of Ref. [11]]. This is not true unless one neglects spin-orbit precession (as was indeed stressed explicitly in Ref. [11]). A similar assumption was also made in Ref. [4], while Ref. [17] admittedly recognized that their formula might not be valid at arbitrary separations, as it is based on a fit to NR simulations, which, as already mentioned, have initial separations \(r_{in} \sim 10M\).

In conclusion: I have reviewed the assumptions needed to derive a new formula predicting the spin of the BH resulting from the merger of two BHs in quasi-circular orbits and having arbitrary initial masses and spins [1]. This formula includes the effect of the precession of the spins through assumptions (iv) and (v), and can therefore be applied to widely separated binaries, such as those relevant for cosmological applications, for which the other available formulas become imprecise. I stress that requiring that the formulas for the final spin work also when applied to large separation binaries is necessary to ensure that these formulas can be readily used in cosmological applications. Cosmological simulations typically provide the initial spins and angular momentum of SMBH binaries when their separation is still of about 0.1 pc (see Ref. [18] for a recent example), which corresponds to \(2 \times 10^8 M\) if \(M = 10^8 M_\odot\). This separation roughly corresponds to the point at which the evolution of the binary starts being driven by gravitational-wave emission. While it would be in principle possible to use the PN equations and evolve the widely-separated binaries provided by cosmological simulations to small separations, read-off the relevant information and apply the formulas for the final spin, this procedure would be impractical and potentially very time-consuming (cf. again Ref. [18]), which simulates tens of thousands of SMBH binaries in order to get significant statistics).

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