Fractons from vector gauge theory

Leo Radzihovsky and Michael Hermele
Department of Physics and Center for Theory of Quantum Matter
University of Colorado, Boulder, CO 80309
(Dated: February 25, 2019)

Motivated by the prediction of fractonic topological defects in a quantum crystal, we utilize a reformulated elasticity-duality to derive a novel U(1) vector gauge theory description of fractons. The fracton order and restricted mobility simply emerge as a result of gauge invariance. At low energies this vector gauge theory reduces to the previously studied fractonic symmetric tensor gauge theory. We construct the corresponding fractonic lattice model and discuss its various generalizations.

Introduction. Motivated by continued interest in topological quantum matter and by a search of fault-tolerant quantum memory, recent studies have led to fascinating developments in an exotic class of quantum spin-liquid models[1–7]. These are characterized by many nontrivial properties, the most unusual of which is the sub-extensive ground state degeneracy on a torus and the existence of quasi-particles, dubbed “fractons”, that exhibit restricted mobility. Namely, there are quasi-particles confined to zero-, one- and/or two-dimensional subspaces of the full three-dimensional space of the model. While such fracton phases were originally discovered in fully gapped phases of commuting projector lattice spin Hamiltonians, it was more recently pointed out[8], that fractonic charges are also realized in gapless phases of U(1) symmetric tensor gauge theories[9].

In a parallel development, it was observed by one of us (L.R.)[10] that such restricted quasi-particle mobility is strongly reminiscent of the immobile disclinations and glide-only dislocations in an ordinary two-dimensional (2D) crystal, described by a symmetric strain tensor field. Indeed, utilizing a generalization of boson-vortex duality[11,12], this conjecture was explicitly demonstrated[13], showing that a 2+1D quantum crystal is dual to a symmetric tensor gauge theory, with disclinations and dislocations mapping onto fractonic charges and their dipolar bound states, and with stress tensor $\sigma_{ik}$ and momentum vector $\pi_k$ fields respectively corresponding to the electric tensor $E_{ik}$ and magnetic vector $B_k$ fields.

Motivation and results. An important source of insight into fracton physics has been to relate apparently exotic fracton states to more familiar quantum phases of matter. Indeed, the fracton-elasticity duality is an example of such a relationship. Related progress has also been made for certain gapped fracton phases, via a construction of these phases in terms of coupled layers of ordinary 2D topologically ordered states[14,15]. So far there is a relative paucity of relationships between gapless fracton phases and better understood phases or theories. Most U(1) tensor gauge theories are not dual to elasticity, and even for those that are, developing alternative viewpoints is highly desirable.

Remarkably, the fracton-elasticity duality itself contains the seed of another such point of view. There is a sense in which elasticity, formulated in terms of a symmetrized strain tensor $u_{ik} = 1/2(\partial_i u_k + \partial_k u_i)$ is a system of two “spin” flavors of XY models, joined together via “spin-space” coupling, as in systems with spin-orbit interaction. However, in the absence of spin-space coupling, such a system dualizes to two independent flavors of U(1) vector gauge theory, and lacks fractonic charges. It is thus natural to ask whether fractonic tensor gauge theories can be formulated in terms of coupled vector gauge theories and if so, what minimal ingredients are required for such coupling. Some progress has been made along these lines from a different point of view, starting with a vector U(1) gauge theory and gauging certain global symmetries to obtain a fractonic tensor gauge theory[16]. We make contact with this result below in greater detail.

In this Letter, we first utilize a reformulated fracton-elasticity duality to derive a 2+1D U(1) vector gauge theory that hosts fractonic charges and is equivalent at low energy to the rank-2 symmetric tensor U(1) gauge theory with scalar charge, dubbed the “scalar charge theory”. We then discuss a lattice version of the same theory, which allows the scalar charge theory to be understood starting from two decoupled vector U(1) gauge theories, and condensing certain charged loops. Finally, we discuss generalizations of our lattice construction that provide constructions of a new class of fractonic tensor gauge theories as coupled vector gauge theories.

The continuum Hamiltonian density we obtain is given by

$$\mathcal{H} = \frac{1}{2} C |\hat{E}_k|^2 + \frac{1}{2} (\nabla \times \hat{A}_k)^2 + \frac{1}{2} K |\hat{e}|^2 + \frac{1}{2} (\nabla \times \hat{\alpha} + \hat{A}_e)^2 - \hat{A}_k \cdot J_k - \hat{e} \cdot j . \tag{1}$$

which involves three U(1) vector gauge fields with electric fields $\hat{E}_k$ and $\hat{e}$ and corresponding vector potentials $\hat{A}_k$ and $\hat{\alpha}$, with flavors $k = x, y$. We denote the corresponding charge densities by $p_k$ and $\rho$, and currents by $J_k$ and $j$, with $p_k$ and $J_k$ referred to as the dipole charge and dipole current, respectively. The gauge field variables
satisfy the canonical commutation relations ($\hbar = 1$),
\begin{align}
[\hat{A}_{ik}, \hat{E}_{jl}] &= -i\delta_{ij} \delta_{kl} \delta^2(\mathbf{x} - \mathbf{x}'), \\
[\hat{u}_i, \hat{e}_j] &= -i\delta_{ij} \delta^2(\mathbf{x} - \mathbf{x}').
\end{align}
Moreover, $\hat{A}_a = \epsilon_{ik} \hat{A}_{ai}$, and we use a short-hand notation where the curl of a 2D vector field implicitly its scalar $z$-component, i.e., \( \nabla \times \mathbf{a} = \hat{z} \cdot \nabla \times \mathbf{a} \). The Hamiltonian is supplemented by the Gauss’s laws,
\begin{align}
\nabla \cdot \hat{E}_k &= p_k - \hat{e}_k, \\
\nabla \cdot \hat{e} &= \rho.
\end{align}
Crucially, the components of the electric field $\hat{e}_k$ appear as additional dipole charge in the Gauss’s law (4).

The fractonic nature of the $\rho$ charges can be seen by observing that moving such a charge requires creating or destroying field lines of the $\hat{e}$ electric field, but since these field lines themselves carry gauge charge, a single “piece” of field line cannot be locally created or destroyed. The immobility of these charges is also manifest in that gauge invariance requires the current $\mathbf{j}$ to vanish identically, as we elaborate below. We next turn to the derivation of this fractonic coupled vector gauge theory and its connection to the previously studied tensor scalar charge theory.

**Derivation.** To this end, we begin with a Hamiltonian density
\begin{equation}
\hat{\mathcal{H}} = \frac{1}{2} \pi_i^2 + \frac{1}{2} C_{ijkl} \hat{u}_{ij} \hat{u}_{kl}
\end{equation}
of a 2+1D quantum crystal formulated in terms of the phonon field $\hat{u}_k$ and its canonically conjugate momentum $\hat{\pi}_k$. For simplicity we will take the elastic tensor $C_{ijkl}$ to be $C_{ijkl} = C \delta_{ik} \delta_{jl}$. The generalization to an arbitrary $C_{ijkl}$ is straightforward.

As demonstrated in Ref[13] the Hamiltonian (6) is dual to a symmetric tensor gauge theory and thus exhibits fractonic charges (disclinations) and subdimensional dipoles (dislocations). To get to an equivalent flavored vector gauge theory description, we reformulate above elastic Hamiltonian Eq. (6) in terms of “minimally” coupled quantum XY models, introducing the orientational bond-angle field, $\theta$ and its canonically conjugate angular momentum density $\hat{L}$. The Hamiltonian density is given by
\begin{equation}
\hat{\mathcal{H}} = \frac{1}{2} \pi_k^2 + \frac{1}{2} C(\partial_i \hat{u}_k - \hat{\theta}_{ik})^2 + \frac{1}{2} \hat{L}^2 + \frac{1}{2} K(\nabla \hat{\theta})^2.
\end{equation}
The low-energy equivalence of this Hamiltonian can be straightforwardly seen by integrating out the orientational bond angle field $\hat{\theta}$, that, like an effective gauge field, Higgses out the antisymmetric part of the unsymmetrized strain $\partial_i \hat{u}_k$, at low energies (below Higgs mass set by $C$) reducing it to a symmetrized strain, $\hat{u}_{ik}$.

Working with a path-integral formulation, the corresponding Lagrangian density is given by
\begin{equation}
\mathcal{L} = \pi_k \partial_t u_k + L \partial_t \theta - \sigma_k \cdot \nabla u_k + \sigma_\theta \theta - \tau \cdot \nabla \theta + \frac{1}{2} C^{-1} \sigma_k^2 - \frac{1}{2} \pi_k^2 - \frac{1}{2} L^2 + \frac{1}{2} K^{-1} \tau^2,
\end{equation}
where $\sigma_k \equiv \epsilon_{ik} \sigma_{ik}$ and we decoupled the elastic and orientational terms via Hubbard-Stratanovich vector fields, stress $\sigma_k$ and torque $\tau$.

For a complete description, in addition to the single-valued (smooth) Goldstone mode degrees of freedom, $\theta^e$ and $u_k^e$, we must also include topological defects. A disclination defect is defined by a nonsingle-valued bond angle with winding $\oint d\theta^s = 2\pi s/n$ around the disclination position, or equivalently in a differential form,
\begin{equation}
\nabla \times \nabla \theta^s = \frac{2\pi s}{n} \delta^2(\mathbf{r}) \equiv \rho(\mathbf{r}).
\end{equation}
The integer disclination charge $s$ corresponds to an integer-multiple of $2\pi/n$ missing (added) wedge of atoms for $s > 0$ ($s < 0$) in a $C_n$ symmetric crystal, with most common case of a hexagonal lattice, $n = 6$.

A dislocation is a point vector defect, around which the displacement $u_k$ is not single-valued, with winding $\oint du_k = b_k$, or equivalently in a differential form,
\begin{equation}
\nabla \times \nabla u_k^e = b_k \delta^2(\mathbf{r}) \equiv b_k(\mathbf{r}).
\end{equation}
An elementary dislocation is a dipole of $\pm 2\pi/n$ disclinations and is characterized by a 2D Burgers vector charges, $b_k$, that takes values in the lattice. An edge dislocation corresponds to a ray of missing or extra lattice sites, with a Burgers vector lying in the 2d plane of the crystal. A nontrivial configuration of dislocations, $b(\mathbf{r})$ can also contribute to a disclination density, given by $\pi_k(\mathbf{r}) = \hat{z} \cdot \nabla \times b(\mathbf{r})$, with a single disclination corresponding to an end point of a ray of dislocations.

Expressing the phonon and bond-angle fields in terms of corresponding singular (s) and elastic (e) parts
\begin{equation}
u_k = u_k^e + u_k^s, \quad \theta = \theta^e + \theta^s,
\end{equation}
and integrating over the elastic parts, gives constraints
\begin{align}
\partial_t \pi_k - \nabla \cdot \sigma_k &= 0, \\
\partial_t L - \nabla \cdot \tau &= \sigma_a,
\end{align}
that encode momentum and angular momentum conservations, respectively, with
\begin{equation}
\mathcal{L} = \pi_k \partial_t u_k^e + L \partial_t \theta^s - \sigma_k \cdot \nabla u_k^e + \sigma_\theta \theta^s - \tau \cdot \nabla \theta^s + \frac{1}{2} C^{-1} \sigma_k^2 - \frac{1}{2} \pi_k^2 - \frac{1}{2} L^2 + \frac{1}{2} K^{-1} \tau^2.
\end{equation}
To solve the first, linear momentum conservation equation, we introduce dual magnetic and electric fields,
\[ \pi_k = \epsilon_{kj} B_j, \quad \sigma_{ik} = -\epsilon_{ij\ell} E_{j\ell}, \]
in terms of which the constraint takes the form of a \( k \)-flavored Faraday equations,
\[ \partial_t B_k + \nabla \times E_k = 0. \]
As in standard electrodynamics and in boson-vortex duality [10] can be solved by \( k \)-flavored vector \( A_k \) and scalar \( A_{0k} \) gauge potentials,
\[ B_k = \nabla \times A_k, \quad E_k = -\partial_t A_k - \nabla A_{0k}. \]
We emphasize that, in contrast to the symmetric tensor approach [13, 17], here, the \( k \) vector gauge field \( A_k \) has components \( A_{ik} \) that form an unsymmetrized tensor field.

Using these definitions inside the second constraint, [13] reduces the latter to
\[ \partial_t (L - A_0) - \nabla \cdot (\tau - \hat{z} \times A_0) = 0, \]
This continuity equation is then solved by introducing another set of vector \( a \) and scalar \( a_0 \) gauge fields, giving
\[ L = \nabla \times a + A_a, \]
\[ \tau_k = \epsilon_{kj} (\partial_t a_j + \partial_j a_0 - A_{0j}). \]

Integrating over the zeroth components of the gauge field (scalar potentials, \( A_{0k}, a_0 \)), exactly enforces the associated Gauss’s laws, [15] and allows us to read off the Hamiltonian, [1] and the commutation relations, [2].

The unusual Gauss’s law (4) couples three vector gauge theories, with \( k \)-th component of the \( e \) field acting as an additional source of charge in the Gauss’s law for \( E_k \). We also note that taking the divergence on the second index \( k \) of the Gauss’s law for \( E_k \), [4] and using the second law for \( e \) to eliminate \( \nabla \cdot e \) from the resulting right hand side gives,
\[ \partial_t \partial_k E_{ik} = \tilde{\rho}. \]
This thereby recovers the generalized Gauss’s law of scalar-charge tensor gauge theory, with \( \tilde{\rho} \equiv -\rho + \nabla \cdot p \) the total charge contribution, [13] that encodes the additional dipole conservation responsible for immobility of fracto-tons.

Using these gauge fields to eliminate \( \sigma, \pi, \tau \) and \( L \), gives an effective Lagrangian density
\[ \tilde{\mathcal{L}} = \mathcal{L}_M + A_k \cdot J_k - A_{0k}(p_k - e_k) + a \cdot j - a_0 \rho, \]
where \( \mathcal{L}_M \) is the Maxwell part and the charges contributions are obtained by integrating by parts and using the definitions of disclination and dislocations topological defects, Eqs. (10). In above, the dipole charge \( p_k \) is given by the dislocation density, \( p_k = \epsilon_{ik} b_i = (\hat{z} \times b)_k \), fracton charge by the disclination density, \( \rho \), and the corresponding currents are given by
\[ J_k = \epsilon_{ik} \hat{z} \times (\partial_t \nabla u_i - \nabla \partial_i u_t), \]
\[ j = \hat{z} \times (\partial_t \nabla \theta - \nabla \partial_i \theta), \]
that, in the 2+1D notation compactly organize into 3-vectors, \( J_{ik} = (J_k, p_k - e_k), j \equiv (j, \rho) \).

The Maxwell part of the dual Lagrangian is given by,
\[ \mathcal{L}_M = \frac{1}{2} C^{-1}(\partial_t A_k + \nabla A_{0k})^2 - \frac{1}{2}(\nabla \times A_k)^2 \]
\[ + \frac{1}{2} K^{-1}(\partial_t a_k + \partial_k a_0 - A_{0k})^2 - \frac{1}{2}(\nabla \times a + A_a)^2. \]
Putting these together and using Hubbard-Stratonovich transformations to introduce independent electric fields, canonically conjugate to the corresponding vector potentials, we obtain,
\[ \tilde{\mathcal{L}} = -E_k \cdot (\partial_t A_k + \nabla A_{0k}) - e \cdot (\partial_t a + \nabla a_0) \]
\[ - \frac{1}{2} K|e|^2 - \frac{1}{2}(\nabla \times a + A_a)^2 + A_k \cdot J_k - A_{0k}(p_k - e_k) + a \cdot j - a_0 \rho. \]

Integrating over the zeroth components of the gauge field (scalar potentials, \( A_{0k}, a_0 \)), exactly enforces the associated Gauss’s laws, [15] and allows us to read off the Hamiltonian, [1] and the commutation relations, [2].

We note that in contrast to the scalar charge theory, \( E_{ik} \) is not a symmetric tensor, but effectively becomes symmetric at low energy as we demonstrate below.

We note that the above Lagrangian is invariant under a deformed gauge transformation,
\[ A_k \rightarrow A_k + \nabla \chi_k, \]
\[ A_{0k} \rightarrow A_{0k} - \partial_t \chi_k, \]
\[ a_k \rightarrow a_k + \partial_k \phi - \chi_k, \]
\[ a_0 \rightarrow a_0 - \partial_t \phi, \]
with [29] ensuring that \( \nabla \times a + A_a \) is gauge invariant.

We observe that under this \( \chi_k \) gauge transformation the current source terms in (21) shifts by
\[ -\chi_k j_k + \partial_t \chi_k \tilde{\rho}_k + \nabla \chi_k \cdot J_k, \]
where \( \tilde{p}_k = p_k - e_k \) is the effective dipole density, that is a combination of microscopic dipoles (dislocations) and electric field generated by pairs of fracton charges (disclinations). Requiring gauge invariance, then leads to the dipole continuity equation

\[
\partial_t \tilde{p}_k + \nabla \cdot \mathbf{J}_k = -j_k, \tag{32}
\]

violated by a nonzero fracton current \( j \). This is quite analogous to a violation of vacancies/interstitials conservation by motion of dipoles (i.e., by a nonzero dislocations current \( \mathbf{J}_k \)). Thus, in the absence of gapped dipoles and charges, we find that for on-shell processes,

\[
j = 0, \tag{33}
\]

e.g., isolated disclinations are immobile fractonic charges.

We conclude by demonstrating that in fact this dual coupled vector \( U(1) \) gauge theory, at low energies is indeed equivalent to the symmetric tensor gauge theory. To this end, we observe that the enlarged gauge redundancy allows us to completely eliminate \( a_k \) from the Lagrangian, by choosing \( \chi_k = a_k \). Returning to the Lagrangian, \( \chi \) is the term \( \frac{1}{2} (\nabla \times a + A_a)^2 \) reduces to \( \frac{1}{2} A_k^2 \), thereby gapping out the antisymmetric component \( A_{ik} = \epsilon_{ij} A_{jk} \).

Thus, at energies well below this gap, \( \epsilon_{ik} A_{ik} \approx 0 \) and the active degrees of freedom reduce to a symmetric component of \( A_{ik} \). Furthermore, for \( a = 0 \) (eliminated by the gauge choice), the electric field term reduces to \( \frac{1}{2} K^{-1} (\partial t a + \nabla a_0 - A_{0k})^2 \rightarrow \frac{1}{2} K^{-1} (\nabla a_0 - A_{0k})^2 \), enforcing \( A_{0k} = \partial_t a_0 \). Thus at low energies, this reduces the Lagrangian exactly to that of the symmetric tensor gauge theory.

**Lattice model and charged loop condensation.** We now consider a lattice version of the Hamiltonian Eq. (1); an analogous construction in the 3D case appeared in \( \text{(18)} \).

We use the resulting model to obtain a physical picture of the Hilbert space of the scalar charge theory upon taking the limit \( K \rightarrow \infty \). We take a different point of view, considering the phases of the Hamiltonian Eq. (37). When \( U_c \gg K_c \), we put the \( \hat{e} \) gauge field into its confining phase, with \( \hat{e} \approx 0 \), and electric field loops costing an energy proportional to their length. In this limit \( \hat{e} \) can be integrated out and we wind up with two decoupled non-compact \( U(1) \) vector gauge theories, with two gapless photon modes.

Starting from this two-photon phase, we increase \( K_c \), thus increasing the fluctuations of the \( \hat{e} \) loops. After \( K_c \) is raised above a critical value, the \( \hat{e} \) loops proliferate and condense. This is analogous to “p-string condensation” in the coupled layer construction of the X-cube fracton model, because each \( \hat{e} \) loop consists of charges of the \( E_k \) gauge fields. We can access the condensed phase by taking \( K_c \) large and expanding the cosine in the last term; the resulting Gaussian model is simply a lattice regularization of the continuum Eq. (1), and is identical at low energy to the scalar charge tensor gauge theory. This can also be shown directly on the lattice; details will be presented elsewhere.

![Fig. 1: Lattice geometry of the model. The underlying lattice, x-lattice and y-lattice are shown in black, red and blue, respectively.](image)

We consider the following lattice Hamiltonian analogous to Eq. (1):

\[
H = \frac{U_E}{2} \left[ \sum_{\ell \in L_x} E_{x\ell}^2 + \sum_{\ell \in L_y} E_{y\ell}^2 \right] + \frac{U_e}{2} \sum_{\ell \in L} e_{\ell}^2 \\
+ \frac{K_E}{2} \left[ \sum_{\square_x} (\nabla \times A_x)^2 + \sum_{\square_y} (\nabla \times A_y)^2 \right] \tag{37}
\]

Here, \( L, L_x \) and \( L_y \) are the sets of links in the underlying lattice, x-lattice and y-lattice, respectively. Similarly \( \square \) and \( \square_x \) denote plaquettes of the underlying lattice and the \( k \)-lattice, respectively. The expression \( (\nabla \times a) \square \) is the lattice line integral of \( a \) taken counterclockwise around the perimeter of the plaquette \( \square \). To understand the last term, note that a single \( x \)-lattice link \( \ell_x \) and single \( y \)-lattice link \( \ell_y \) passes through the center of each plaquette \( \square \). \( A_{xy} \) and \( A_{yx} \) reside on \( \ell_x \) and on \( \ell_y \), respectively.

In the 3D case, Ref. [16] showed that one obtains the Hilbert space of the scalar charge theory upon taking the limit \( K \rightarrow \infty \). We take a different point of view, considering the phases of the Hamiltonian Eq. (37). When \( U_c \gg K_c \), we put the \( \hat{e} \) gauge field into its confining phase, with \( \hat{e} \approx 0 \), and electric field loops costing an energy proportional to their length. In this limit \( \hat{e} \) can be integrated out and we wind up with two decoupled non-compact \( U(1) \) vector gauge theories, with two gapless photon modes.
Generalizations. We briefly state some generalizations of the above lattice construction, a detailed analysis of which will be presented elsewhere. We go to the 3D cubic lattice, introducing an underlying cubic lattice and three cubic \((k = x, y, z)\), whose vertices are centered on \(k\)-directed links of the underlying lattice. We introduce a compact \((U(1))\) gauge field on each lattice with electric fields \(e\) and \(E_k\), writing \(E_{ik} \equiv \langle E_k \rangle\). We consider Gauss’s laws of the form

\[
\nabla \cdot e = 0 \tag{38} \
\sum_{i=x,y,z} M_{ik} \Delta_i E_{ik} = e_k, \tag{39} 
\]

where \(\Delta_i\) is the lattice derivative operator and \(M\) is a \(3 \times 3\) matrix with \(m\) in the diagonal entries and \(n\) in the off-diagonal entries, with \(m\) and \(n\) positive, relatively prime integers. \(m = n = 1\) corresponds to an isotropic Gauss’s law for the \(E_k\) electric fields, with \(e_k\) carrying \(E_k\) electric charge, while \(m \neq n\) makes the Gauss’s laws anisotropic. Starting from a phase where \(e\) loops are confined and the three \(E_k\) gauge fields are deconfined, condensing \(e\) loops produces the deconfined phase of the \((m, n)\) scalar charge tensor gauge theories introduced in [19]. These theories have a symmetric tensor electric field \(E_{ij}^s\) obeying the Gauss’s law \(\sum_{i,j} M_{ij} \Delta_i \Delta_j E_{ij}^s = 0\).

This construction can be further generalized by adding another vector gauge field on the underlying lattice, with electric field \(\mathbf{E}\), and supplementing the above Gauss’s laws with

\[
\nabla \cdot \mathbf{E} = \sum_i E_{ii}. \tag{40} 
\]

Construction a Hamiltonian for these gauge fields and taking the coefficients of all the Maxwell terms large, so that all cosines can be expanded, one obtains the traceless version of the \((m, n)\) scalar charge theory, which obeys the additional constraint \(\sum_i E_{ii} = 0\).

In summary, motivated by the fracton-elasticity duality\cite{13, 17}, we utilized its reformulation to derive a fractonic coupled U(1) vector gauge theory representation in terms of \(d + 1\)-coupled gauge fields, where components of one type of electric field act as charges for the remaining \(d\) gauge theories. At low energies this vector description is identical to fractonic scalar charge tensor gauge theory. We used a lattice version of this model to discuss fracton order in terms of proliferation of electric field loops. We also proposed a number of generalizations of this construction, making contact with fractonic tensor gauge theories that are not dual to elasticity.

Acknowledgments. LR thanks Michael Pretko for discussions. The work of LR was supported by the Simons Investigator Award from the Simons Foundation, and by the Soft Materials Research Center under NSF MRSEC Grants DMR-1420736. The work of MH is supported by the U.S. Department of Energy, Office of Science, Basic Energy Sciences (BES) under Award number DESC0014415.

[1] C. Chamon, Phys. Rev. Lett. 94, 040402 (2005), URL https://link.aps.org/doi/10.1103/PhysRevLett.94.040402
[2] S. Bravyi, B. Leemhuis, and B. M. Terhal, Annals of Physics 326, 830 (2011), ISSN 0003-4916, URL http://www.sciencedirect.com/science/article/pii/S0003491610001910
[3] J. Haah, Phys. Rev. A 83, 042330 (2011), URL https://link.aps.org/doi/10.1103/PhysRevA.83.042330
[4] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B 92, 235136 (2015), URL https://link.aps.org/doi/10.1103/PhysRevB.92.235136
[5] S. Vijay, J. Haah, and L. Fu, Phys. Rev. B 94, 235157 (2016), URL https://link.aps.org/doi/10.1103/PhysRevB.94.235157
[6] K. Slagle and Y. B. Kim, Phys. Rev. B 96, 165106 (2017), URL https://link.aps.org/doi/10.1103/PhysRevB.96.165106
[7] R. M. Nandkishore and M. Hermele, Annual Review of Condensed Matter Physics 10, 295 (2019), URL https://doi.org/10.1146/annurev-conmatphys-031218-013604
[8] M. Pretko, Phys. Rev. B 95, 115139 (2017), URL https://link.aps.org/doi/10.1103/PhysRevB.95.115139
[9] A. Rasmussen, Y.-Z. You, and C. Xu, ArXiv e-prints (2016), 1601.08235.
[10] L. Radzihovsky, unpublished (2016).
[11] C. Dasgupta and B. I. Halperin, Phys. Rev. Lett. 47, 1556 (1981), URL https://link.aps.org/doi/10.1103/PhysRevLett.47.1556
[12] M. P. A. Fisher and D. H. Lee, Phys. Rev. B 39, 2756 (1989), URL https://link.aps.org/doi/10.1103/PhysRevB.39.2756
[13] M. Pretko and L. Radzihovsky, Phys. Rev. Lett. 120, 195301 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.120.195301
[14] H. Ma, E. Lake, X. Chen, and M. Hermele, Phys. Rev. B 95, 245126 (2017), URL https://link.aps.org/doi/10.1103/PhysRevB.95.245126
[15] S. Vijay, ArXiv e-prints (2017), 1701.00762.
[16] D. J. Williamson, Z. Bi, and M. Cheng, ArXiv e-prints arXiv:1809.10275 (2018), 1809.10275.
[17] M. Pretko and L. Radzihovsky, Phys. Rev. Lett. 121, 235301 (2018), URL https://link.aps.org/doi/10.1103/PhysRevLett.121.235301
[18] M. C. Marchetti and L. Radzihovsky, Phys. Rev. B 59, 12001 (1999), URL https://link.aps.org/doi/10.1103/PhysRevB.59.12001
[19] D. Bulmash and M. Barkeshli, Phys. Rev. B 97, 235112 (2018), URL https://link.aps.org/doi/10.1103/PhysRevB.97.235112