Renormalon Ambiguities in NRQCD Operator Matrix Elements

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Abstract

We analyze the renormalon ambiguities that appear in factorization formulas in QCD. Our analysis contains a simple argument that the ambiguities in the short-distance coefficients and operator matrix elements are artifacts of dimensional-regularization factorization schemes and are absent in cutoff schemes. We also present a method for computing the renormalon ambiguities in operator matrix elements and apply it to a computation of the ambiguities in the matrix elements that appear in the NRQCD factorization formulas for the annihilation decays of S-wave quarkonia. Our results, combined with those of Braaten and Chen for the short-distance coefficients [1], provide an explicit demonstration that the ambiguities cancel in the physical decay rates. In addition, we analyze the renormalon ambiguities in the Gremm-Kapustin relation and in various definitions of the heavy-quark mass.
I. INTRODUCTION

In Quantum Chromodynamics (QCD), it is often useful to describe physical processes involving more than one distance scale by making use of a factorization formalism. In such a formalism, a physical observable is written as a sum of products of short-distance coefficients with long-distance operator matrix elements. The short-distance coefficients may be calculated as a perturbative series in the strong coupling constant $\alpha_s$, evaluated at the short-distance scale. The operator matrix elements contain all of the sensitivity of the physical observable to low-momentum (infrared) processes. Because of this infrared (IR) sensitivity, the operator matrix elements are generally not amenable to a perturbative calculation and are usually determined by comparison of physical quantities with experimental values or through nonperturbative methods, such as lattice QCD. Some well-known examples of this approach are the light-cone expansion in deep-inelastic scattering, the factorization expressions for hard hadron-hadron cross sections, Heavy-Quark Effective Theory (HQET), which is useful in the study of heavy-light mesons, and Nonrelativistic Quantum Chromodynamics (NRQCD), which is useful in the study of heavy quarkonium.

The conventional wisdom is that the perturbation expansions for the short-distance coefficients in QCD factorization formulas are, at best, asymptotic series [2]. One can see why this might be the case by examining the behavior of the perturbation series in the context of a simple model, which gives the leading large-$N_f$ behavior of the theory. This model has seen extensive applications in recent years in such topics as the Operator Product Expansions (OPE) for $e^+e^-$ annihilation and $\tau$ decay [3–6], Heavy Quark Effective Theory (HQET) [7–10], and the NRQCD factorization formalism [1]. In this model, which we call the “bubble-chain model,” one generates the “perturbation series” by inserting into the gluon propagator in a one-loop diagram all numbers of order-$\alpha_s$ fermion-loop vacuum-polarization corrections. The resulting perturbation series gives the order-$\alpha_s$ fermion-loop contribution to the running of the coupling constant. In this expression, one then replaces the fermion-loop contribution to the one-loop QCD beta function $\beta_0$ with the full one-loop QCD beta function. As we will describe later, in the bubble-chain model, the perturbation series diverges, with terms growing as the factorial of the order.

One can attempt to resum the series by carrying out a Borel transformation. In the Borel plane, the factorial growth of the series corresponds to singularities, known as renormalons. If the renormalon singularities lie on the positive real axis, then they make the inverse Borel transform ill-defined. These ambiguities in the inverse Borel transform are potentially important because they have the same nominal size as corrections in the QCD factorization formulas that go as powers of the ratio of scales.

In a well-defined theory, physical observables, in contrast with short-distance coefficients, are unambiguous. Of course, if one were to attempt to compute a physical observable entirely in perturbation theory, without making use of a factorization formalism, then one might encounter ambiguities, owing to the failure of the perturbation series to converge. However, such ambiguities are not intrinsic to the observable. They are artifacts of the method of computation and would be absent if one were to compute the observable in terms of nonperturbative expressions, such as lattice path integrals.

When one computes a physical observable by making use of a factorization scheme that is defined in perturbation theory, such as one that is based on dimensional regularization,
then the short-distance coefficients and operator matrix elements can contain ambiguities that arise in consequence of the defining perturbation series. It is these ambiguities that arise from the factorization scheme that are the focus of this paper.

If one were to attempt to compute an operator matrix element entirely in perturbation theory, then additional ambiguities, which have nothing to do with the factorization scheme, might arise. Such ambiguities are not intrinsic to the matrix element, since they would not be present if one defined the matrix element nonperturbatively and evaluated it in terms of nonperturbative expressions. As we shall describe later in this paper, one can isolate the ambiguities that arise from the factorization scheme by computing the difference between an unambiguous definition of the operator matrix element (for example, a lattice definition) and an ambiguous definition (for example, one based on dimensional regularization). In the remainder of this paper, when we refer to ambiguities in operator matrix elements, we mean only those ambiguities that arise from the factorization scheme.

Because physical observables are unambiguous, the renormalon ambiguities in the short-distance coefficients must be cancelled by corresponding ambiguities in the operator matrix elements, to the level of accuracy of the factorization procedure. Factorization procedures that are based entirely on an underlying effective field theory, such as those in the light-cone expansion, OPE, HQET, and NRQCD, are, in principal, valid to all orders in an expansion in the inverse of the large scale in the process. In contrast, factorization procedures for hadron-hadron-induced hard-scattering processes, for example, the Drell-Yan process, are known to fail for contributions that are subleading in the inverse of the large scale \([11]\). In the discussions in the remainder of this paper, we have in mind the factorization procedures that are based entirely on underlying effective field theories. However, our results may also apply to other factorization procedures at the level of leading orders in the inverse of the large scale.

If one determines operator matrix elements by comparing factorized expressions for physical observables with results from experiment, then the cancellation of renormalon ambiguities holds by construction, for those observables. On the other hand, if one determines operator matrix elements through a calculation in the underlying effective field theory, as we do in this paper, then the cancellation of renormalon ambiguities is a nontrivial confirmation of the accuracy of the effective field theory in reproducing the low-momentum behavior of the full theory and of the consistency of the definitions of the short-distance coefficients and operator matrix elements.

In this paper, we investigate the relations between the renormalon ambiguities in the short-distance coefficients and the renormalon ambiguities in the operator matrix elements. In our analysis, it is clear that the short-distance coefficients and the operator matrix elements display no ambiguities in a factorization scheme in which ultraviolet (UV) divergences in the operator matrix elements are controlled with a regulator, such as lattice regularization, that has an explicit cutoff in the loop momenta \([12,13]\). In contrast, in a dimensional-regularization scheme, both the short-distance coefficients and the operator matrix elements contain renormalon ambiguities, but they cancel in physical observables \([10]\). We give a prescription for calculating the renormalon ambiguities in the dimensionally-regulated operator matrix elements, and we apply this method to the computation of the renormalon ambiguities in the NRQCD matrix elements that appear in the decays of S-wave quarkonia in the leading and first subleading orders in the expansion in the heavy-quark velocity. Compar-
with a calculation of the ambiguities in the short-distance coefficients for the decays of S-wave quarkonia [1], we find that the renormalon ambiguities cancel in the physical decay rates, as expected.

The remainder of this paper is organized as follows. In Sec. II, we review the bubble-chain model and Borel transformation and discuss, in general, the origins of renormalons in short-distance coefficients. Here we present a concise argument to show that renormalon ambiguities in the short-distance coefficients arise from the low-momentum regions of loop integrals. Sec. III contains a discussion of the interplay between the factorization scheme and the renormalon ambiguities. In this section we clarify the fact that renormalon ambiguities in the short-distance coefficients and operator matrix elements are artifacts of dimensional-regularization factorization schemes and are absent in cutoff schemes. In Sec. IV, we describe our method for computing the renormalon ambiguities in the operator matrix elements. We use this method, in Sec. V, to calculate the renormalon ambiguities in the NRQCD matrix elements for decays of S-wave quarkonia. In Sec. VI, we examine the renormalon content of the various quantities that appear in the Gremm-Kapustin [14] relation. We also discuss the ambiguities in various definitions of the heavy-quark mass. Finally, we summarize our results and discuss their implications in Sec VII.

II. RENORMALONS IN THE BUBBLE-CHAIN MODEL

As we have mentioned, the bubble-chain model, gives the exact large-$N_f$ behavior of QCD. In it, one generates the “perturbation series” by inserting into the gluon propagator in a one-loop diagram all numbers of order-$\alpha_s$ fermion-loop vacuum-polarization corrections. That is, one replaces the factor $\alpha_s(\mu^2)$ in the one-loop diagram with

$$\alpha_s(\mu^2) \sum_{n=0}^{\infty} \left[ -\beta_0 \alpha_s(\mu^2) \ln(l^2 e^C/\mu^2) \right]^n$$

$$= \frac{\alpha_s(\mu^2)}{1 + \beta_0 \alpha_s(\mu^2) \ln(l^2 e^C/\mu^2)}$$

$$\equiv \alpha_s^{(1)}(l^2),$$

where $l$ is the Euclidean momentum of the gluon in the one-loop expression, $C$ is a renormalization-scheme-dependent constant ($C = -5/3$ in the $\overline{MS}$ scheme), $\beta_0$ is the one-loop contribution to the QCD beta function, and we identify the expression (1) as the one-loop running coupling constant $\alpha_s^{(1)}(l^2)$. At this stage, $\beta_0$ contains only the fermion-loop contribution. However, in the bubble-chain model, one includes the effects of gluons on the running of the coupling by promoting $\beta_0$ to the complete QCD expression $\beta_0 = (33/2 - N_f)/(6\pi)$.

Now, suppose that a part of the original one-loop integral has the IR behavior

$$\int_0^{\Lambda} dl l^m.$$  

Here we have introduced a cutoff $\Lambda$ on the magnitude of $l$, in order to focus on the low-momentum region. We assume that the integral is IR finite ($m \geq -1$), since any infrared divergences are absorbed into the operator matrix elements in the factorization procedure.
If one inserts the expression (1a) into the one-loop integral (2) and integrates term by term, then it is easy to see, using
\[
\int_0^\mu dl \ln^n(l^2/\mu^2) = 2^n \mu^{n+1} (-1)^n n! / (m + 1)^{n+1},
\]
that terms in the resulting series grow as \(n!\). Hence, the series is only asymptotic.

One way to associate a well-defined function with an asymptotic series is through the Borel transform. If a function \(f(\alpha_s)\) has a power-series expansion
\[
f(\alpha_s) = \sum_{n=0}^{\infty} a_n \alpha_s^n,
\]
then its Borel transform is defined by
\[
\tilde{f}(t) = a_0 \delta(t) + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} a_n t^{n-1}.
\]

In this paper, we indicate the Borel transform of a quantity \(f\) either as \(\tilde{f}\) or \(B[f]\). Clearly, the series (5) for the Borel transform has better convergence properties than the original power series (4). The function \(f(\alpha_s)\) that generates the series (4) can be recovered from the Borel transform \(\tilde{f}(t)\) through the inverse Borel transform
\[
f(\alpha_s) = \int_0^\infty dt \ e^{-t/\alpha_s} \tilde{f}(t),
\]
provided that the integral (6) is sufficiently convergent and that the Borel transform (5) is well-defined along the positive real axis.

The Borel transform of the series (1a) is
\[
\tilde{\alpha}_s^{(1)}(t) = \frac{(\mu^2/eC)^u}{\Lambda^{2u}},
\]
where \(u = \beta_0 t\). If we insert the Borel transform (5) into the one-loop integral (2), we obtain
\[
\left(\mu^2/eC\right)^u \int_0^\Lambda dl \ln^{2u} = \left(\mu^2/eC\right)^u \Lambda^{m+1-2u}.
\]

where \(\Lambda\) is an ultraviolet cutoff, and, in the spirit of dimensional regularization, we have defined the integral for \(\text{Re } u > (m + 1)/2\) by analytic continuation. The integral has a pole in the complex \(u\) plane at \(u = (m + 1)/2\). This singularity appears even though the original one-loop integral is perfectly well-behaved. A pole in \(u\), such as this one, that is associated with the running of \(\alpha_s\) is called a renormalon.

In order to understand the origins of this renormalon singularity more fully, let us re-examine the integral (8). Applying an IR cutoff \(\lambda\), we have

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1If the power series (4) is only asymptotic, then there is not a unique function \(f(\alpha_s)\) that generates it. The inverse Borel transform picks out a function with particular analyticity properties (15).
\begin{equation}
\left(\frac{\mu^2}{e^C}\right)^u \int_\Lambda^\Lambda \frac{dl^{m-2u}}{l^{m-2u}} = \left(\frac{\mu^2}{e^C}\right)^u \frac{\Lambda^{m+1-2u} - \lambda^{m+1-2u}}{m+1-2u}.
\end{equation}

If \( \Lambda \) and \( \lambda \) are finite positive numbers, then the integral (9) is well defined, even at the point \( u = (m + 1)/2 \). Furthermore, the inverse Borel transform is convergent, provided that

\begin{equation}
\alpha_s(\mu^2) \beta_0 \ln[\mu^2/(\lambda^2 e^C)] < 1,
\end{equation}

which is identical to the condition that the integration over \( l \) never passes through the Landau pole in (11). Hence, we conclude that the renormalon singularity in the \( u \) plane arises from the region of integration near \( l = 0 \), which we have excluded by introducing the IR cutoff \( \lambda \). The renormalon singularity is a signal that, because of the growth of the running coupling, the perturbation series breaks down near zero loop momentum. A renormalon that arises from the region near zero loop momentum is called an IR renormalon.

The renormalon in (8) is on the positive real axis. Hence, the inverse Borel transform is not unique. One can obtain various values, depending on the prescription used to deform the integration contour near the pole. For example, one can deform the integration contour in (11) into the complex plane so that it runs above the pole or below the pole, or one can take a linear combination of these two contours, such as the principal value. These prescriptions all differ by amounts that are proportional to the residue of the integrand of (6) at the pole. If the pole in \( \tilde{f}(t) \) is at the point \( t^* = u^*/\beta_0 \) and has residue \( R^* \), the ambiguity in \( f(\alpha_s) \) has the form

\begin{equation}
\Delta f(\alpha_s) = K \left(2\pi R^*\right) e^{-t^*/\alpha_s(\mu^2)},
\end{equation}

where \( K \) is a constant of order unity. In \( \alpha_s(\mu^2) \), the scale \( \mu \) is typically chosen to be the largest scale in the physical process. For large \( \mu \)

\begin{equation}
\alpha_s(\mu) \approx \frac{1}{\beta_0 \ln(\mu^2/\Lambda^2_{\text{QCD}})}.
\end{equation}

Inserting this expression into (11), we see that the renormalon ambiguity is given approximately by

\begin{equation}
\Delta f \approx K(2\pi R^*) \left(\frac{\Lambda^2_{\text{QCD}}}{\mu^2}\right)^{u^*}.
\end{equation}

Thus, the ambiguity is suppressed as a power of \( \mu^2 \). However, such an ambiguity can be of practical importance, since it is of the same nominal size as power corrections in the factorization formulas.

We have seen that IR renormalons can arise from the region of integration near zero loop momentum. In an analogous fashion, UV renormalons can arise from the region of integration near infinite loop momentum. The loop integration extends to infinity in unregulated, UV-finite integrals and in UV-divergent integrals in dimensional regularization. In a renormalizable theory, the loop integration in a short-distance coefficient is, at worst, logarithmically divergent in the UV region when \( u = 0 \). (The integrand that we have displayed in the expression (8) is an approximate form that is valid only in the IR region.) In the case of a logarithmic UV divergence, the pole at \( u = 0 \) is identical to the dimensional-regularization
pole in $\epsilon = (4 - D)/2$, where $D$ is the number of space-time dimensions, and it is removed in the standard minimal-subtraction renormalization procedure. The remaining pieces of the integrand, which have a convergent power count in the large-momentum region when $u = 0$, may contribute additional renormalons. However, by virtue of their convergent power count, these pieces yield renormalons only on the negative real $u$ axis. Renormalons on the negative real $u$ axis do not introduce ambiguities into the inverse transform in Eq. (1).

Another potential source of renormalons in the short-distance coefficients is the UV renormalization procedure that is applied to full QCD. If the renormalization counterterms involve loop integrations down to zero momentum, then the counterterms can contain renormalons. In this paper, we assume that a minimal-subtraction procedure ($\overline{MS}$) or modified minimal-subtraction procedure ($MS$) is used, so that the renormalization counterterms are poles in $\epsilon$ or constants and contain no IR renormalons.

III. FACTORIZATION SCHEME DEPENDENCE OF RENORMALONS

In a factorization formalism, the domain of integration of a loop integral in full QCD is partitioned between the operator matrix elements and the short-distance coefficients. This partitioning allocates the low-momentum part of the loop integral, including the IR divergences, to the operator matrix elements and allocates the remaining high-momentum part of the loop integral to the short-distance coefficients, which contain no IR divergences.

The partitioning arises in the matching of the effective theory to full QCD. For example, in the on-shell matching procedure, on-shell quark and gluon amplitudes in the full theory are equated to the same amplitudes in the effective theory. Each amplitude in the effective theory can be written as a sum of products of short-distance coefficients with matrix elements in the on-shell quark and gluon states. Low-momentum parts of the amplitudes in the full theory, including all of the IR divergences, correspond to matrix elements in the effective theory. These matrix-element contributions can be factored from the full amplitudes. Then, the remaining IR-finite parts of the full amplitudes correspond to the coefficient functions. In the perturbative implementation of the matching procedure, which is familiar, for example, from treatments of the QCD-improved parton model, the matrix-element contributions are subtracted from the full-QCD amplitudes order-by-order in $\alpha_s$. Then, the remainders from this subtraction procedure are identified with perturbative contributions to the short-distance coefficients in the effective theory.

The amount of the full-QCD amplitude that resides in the matrix elements and, hence, in the short-distance coefficients, is controlled by a UV regulator that is imposed on the operator matrix elements. The UV cutoff plays the rôle of a factorization scale, and the

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2Throughout this paper, we speak of the short-distance coefficients as being dimensionally regulated, lattice regulated, etc. What we mean is that these are the short-distance coefficients that correspond, in the factorization formalism, to operator matrix elements that are dimensionally regulated, lattice-regulated, etc. Of course, one can apply an IR regulator to the short-distance coefficients as an intermediate step in calculations. However, dependences on such IR regulators cancel in the matching conditions, and the IR behavior of the short-distance coefficients is
choice of UV regulator is known as the “factorization scheme.” The consistency of the underlying effective field theory guarantees that physical quantities are independent of the factorization scheme, provided that one works to sufficient accuracy in the effective field theory.

One class of UV regulator consists of those regulators in which the cutoff is manifest and alters the form of the integrand at large momentum to make it more convergent. We call such regulators “cutoff” regulators. Pauli-Villars regulators, lattice regulators, and explicit bounds on the magnitude of the Euclidean momentum are all examples of cutoff regulators. For such regulators, power UV divergences manifest themselves as explicit power dependences of the matrix elements on the UV cutoff. For our purposes, the crucial property of cutoff regulators is that they do not alter the behavior of the integral below the cutoff.\(^3\)

In contrast, in dimensional regularization in \(D\) dimensions, the cutoff enters only implicitly through an overall factor that goes to unity when \(2\epsilon = 4 - D\) vanishes. After one discards poles in \(\epsilon\), one finds that logarithmic divergences are cut off by the dimensional-regularization scale.\(^4\) However, terms with a power-divergent power count that are homogeneous in the integration momentum vanish.\(^5\) Because of this property, dimensional regularization potentially alters the behavior of the integral below the cutoff.

In a cutoff factorization scheme, the loop-integration for the IR-finite parts of the short-distance coefficients does not extend to zero momentum. The part of the integration below the cutoff resides in the operator matrix elements, to the extent that the effective theory accurately reproduces the low-momentum behavior of the full theory. Hence, in a cutoff factorization scheme, the short-distance coefficients are free of a given IR renormalon ambiguity, provided that the effective action retains enough powers of the small scale in the calculation to reproduce the low-momentum behavior that is characteristic of such a renormalon. The absence of renormalons in a cutoff factorization scheme has been discussed in a number of publications. Some of the earliest discussions are in Refs. \([12,13]\).

In a dimensional-regularization factorization scheme, it is conventional to integrate the IR-finite parts of the short-distance coefficients down to zero momentum. Consequently, the short-distance coefficients contain renormalon ambiguities. The IR-finite parts of the

\(^3\) Lattice regulators and Pauli-Villars regulators do alter the behavior of the integral below the cutoff, but only through terms that scale as inverse powers of the cutoff. In an effective field theory, one compensates for this effect, restoring the behavior of the theory below the cutoff to the required level of accuracy, by including in the effective action corresponding terms that scale as inverse powers of the cutoff.

\(^4\) When we speak of dimensionally-regularized matrix elements in this paper, we assume that such poles, and possibly some associated constants, have been discarded. That is, by dimensionally-regulated matrix elements, we really mean \(\overline{MS}\) or \(MS\) matrix elements.

\(^5\) We use the term “homogenous” to denote expressions that are a pure power or a pure power times logarithms.
short-distance coefficients correspond to the power-UV-divergent parts of the operator matrix elements. Thus, because of the constraints imposed by the matching conditions, the integration of the IR-finite parts of the short-distance coefficients down to zero momentum demands, for consistency, that the UV-power-divergent parts of the matrix elements be set to zero. As we will explain in Sec. IVB, there is a prescription for regulating the operator matrix elements dimensionally in which such power UV divergences are set to zero. This prescription implies that the dimensionally-regulated matrix elements do not completely reproduce the behavior of the full theory below the factorization scale.

Matrix elements are completely determined, in principle, in terms of short-distance coefficients and physical observables. Therefore, the absence of ambiguities in the cutoff short-distance coefficients implies that the cutoff matrix elements are also ambiguity-free. This is consistent with the fact that lattice-regulated matrix elements have an unambiguous definition in terms of path integrals in the effective theory. In a cutoff scheme, the low-momentum content of the theory resides completely in the operator matrix elements.

In a dimensional-regularization scheme, the short-distance coefficients contain ambiguities. Therefore, the dimensionally-regulated operator matrix elements must contain ambiguities that cancel those in the short-distance coefficients. The appearance of renormalon ambiguities in the short-distance coefficients and operator matrix elements in the dimensional-regularization scheme has no physical significance: It is a factorization-scheme artifact. We conclude that the factorial growth in the perturbation series for the short-distances coefficients is cancelled by the factorial growth in the perturbation series that relate cutoff-regulated matrix elements to dimensionally-regulated matrix elements, provided that one works to the same order in $\alpha_s$ in both quantities, and provided that one works to sufficient accuracy in the underlying effective field theory [10].

Let us explain more fully what we mean by sufficient accuracy in the underlying effective field theory. Generally, an operator matrix element scales, in its leading behavior, as a power of the small scale in the two-scale problem. For example, in the NRQCD factorization formalism for heavy-quarkonium decay and production [16], matrix elements scale as powers of $p = mv$, where $p$ and $v$ are the typical heavy-quark-antiquark relative momentum and velocity, and $m$ is the heavy-quark mass. To achieve the cancellation of a renormalon at a given value of $u$, it is crucial to retain matrix elements of sufficiently high order in the small scale to reproduce the IR power behavior that gives rise to that renormalon. Furthermore, if one is computing a matrix-element renormalon ambiguity in the context of an effective field theory, it is necessary to retain terms in the effective action of sufficiently high accuracy in the small scale to reproduce the corresponding IR power behavior. The action of an effective theory may depend on both the cutoff and the large scale. For example, in HQET and NRQCD, the action depends on $m$, as well as the cutoff. Hence, the small scale may appear in the ratio of the small scale to the large scale or in the ratio of the small scale to the cutoff. In such cases, it is necessary to retain terms of sufficient accuracy in both the expansion in inverse powers of the large scale and the expansion in inverse powers of the cutoff.
A. Short-distance expression for the ambiguity

To identify the renormalon ambiguities in a dimensionally regulated matrix element, we make use of the fact that corresponding cutoff-regulated matrix element is free of ambiguities. Then, instead of computing the dimensionally-regulated matrix element itself, we compute only the effects of a change of regularization scheme from cutoff to dimensional.

The effects of a change of regularization are contained in finite renormalizations of the operators in the underlying effective theory. In general, in order to work out these finite renormalizations, it is necessary to identify all of the renormalization counterterms in the effective theory and to compute their finite coefficients. However, at the one-loop level, this amounts merely to computing the difference between the dimensionally-regulated and cutoff amplitudes. (In both the dimensionally-regulated and cutoff amplitudes, we use the dimensionally-regulated expression for the vacuum-polarization insertions in the gluon propagator.)

Because the renormalization parts of an operator matrix element are short-distance quantities, we can analyze them perturbatively. Furthermore, the renormalization parts are independent of the external states. Therefore, we are free to make a choice of the external states that is convenient for a perturbative analysis, namely, states consisting of on-shell elementary quanta (quarks, gluons, etc.).

We note that, for purposes of computing the ambiguities in the dimensionally-regulated matrix elements, any reference cutoff regulator will do. At the one-loop level, a particularly convenient choice of cutoff regulator is a simple cutoff on the magnitude of the three-momentum of the gluon in the loop. While such a cutoff is not, in general, consistent with gauge invariance, it is compatible with the QED-like nature of the gauge invariance at one loop. With this choice, we compute

$$\langle K \rangle_{\text{dim}} - \langle K \rangle_{\text{cutoff}} = \int_0^\lambda d^3l I_K - \int d\lambda \int_0^\lambda d^3l I_K = \int d\lambda \int_0^\text{dim} d^3l I_K,$$

where $K$ is an operator, $I_K$ is the integrand corresponding to its matrix element between on-shell states in one-loop perturbation theory, and “dim” and “cutoff” denote dimensional and cutoff ultraviolet regularization, respectively. We write “dim” as the upper limit of integration to indicate dimensional regularization of UV divergences. $\lambda$ is a cutoff on the magnitude of the 3-momentum $l$ and corresponds roughly to the IR cutoff $\lambda$ discussed in Sec. II. We assume that $\lambda$ is sufficiently large that the condition (10) is satisfied, so that the integration on the right side of Eq. (14) does not pass through the Landau pole.

The expression (14), being a renormalization-scheme dependence, is IR finite, as is apparent from the form on the right side of the equation. However, it may happen that

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6In fact, the dimensionally-regulated matrix elements have only a perturbative definition, either in terms of physical observables or in terms of matrix elements in some other scheme, such as lattice regularization, that does have a nonperturbative definition.
the separate terms in the left and middle expressions in Eq. (14) require an IR regulator. Then, one must, for consistency with the complete IR-finite expression, impose the same IR regulator on both terms.

The expression on the right side of Eq. (14) is the basis for our calculation of the renormalon ambiguities. One could equally well use the middle expression in Eq. (14), taking advantage of the fact, explained in Sec. IV B, that the dimensionally-regulated integral vanishes for IR-finite terms in the integrand, so that one need compute only the cutoff integral. However, we wish to emphasize the insensitivity of the renormalon ambiguities to the low-energy physics by working with the right side of Eq. (14), which contains no low-momentum contribution. With this computational procedure, the dimensionally-regulated matrix elements contain only UV renormalons.

### B. Dimensional-regularization prescription

Let us now explain more precisely what we mean by dimensional regularization of operator matrix elements. It is well-known that dimensionally-regulated expressions can sometimes be ambiguous when the integrand involves a limiting procedure. For example, the result for a dimensionally-regulated integral can be changed by carrying out formal manipulations involving power series in the integration variable. Consequently, it is necessary to specify how the integrand is to be arranged before the dimensional regularization is imposed.

In computing the short-distance coefficients in the dimensional-regularization scheme, it is conventional to integrate IR-finite terms down to zero loop momentum. As we have discussed in Sec. IV B, such an approach corresponds to setting to zero the UV power divergences in the operator matrix elements. Therefore, for consistency with conventional calculations of the short-distance coefficients, we wish to follow a procedure for the dimensional regularization of the operator matrix elements in which UV power divergences are set to zero. Such a procedure is usually employed in regulating, for example, the matrix elements for parton distributions [17].

One applies this dimensional regularization procedure to QCD corrections to the matrix elements of operators between on-shell quark and gluon states. Such expressions appear, for example, in Eq. (14), and also in the matching conditions between full QCD and the effective field theory that fix the short-distance coefficients. One first decomposes the QCD corrections to the operator matrix element into a linear combination of the tree-level operators in the effective theory. (In the case of NRQCD, this amounts to Taylor expanding the integrand with respect to the external momenta and taking appropriate linear combinations to form quantities of definite orbital angular momentum, spin, and color.) All of the dependence on the external momenta now resides in the tree-level operators. The coefficients of the tree-level operators contain the integration over the loop momentum \( l \) and are now independent

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7It is interesting to note that the cutoff integral in the middle expression in Eq. (14) contains IR renormalons, whereas the right side of Eq. (14) contains UV renormalons. Thus, we see that UV renormalons can be replaced by IR renormalons through the manipulation of quantities that vanish in dimensional regularization.
of the external momenta. In some effective theories, such as NRQCD, the coefficients of the bare operators may not be homogeneous in $l$. In that case, one further expands the integrand in a power series in $l$ divided by the large scale ($m$ in the case of NRQCD). Then, each term in the series is homogeneous in $l$. Now, any terms that are IR finite (that is, UV power divergent) require no IR regulator. Hence, such terms are scale invariant, and their integrals over all $l$ vanish in dimensional regularization. We note that this formal procedure is equivalent to absorbing the UV-power divergences into a renormalization of the operator matrix elements.

In light of these general considerations, we see that the application of Eq. (14) requires some care. To obtain a result that is consistent with conventional calculations of the short-distance coefficients in the dimensional-regularization scheme, it is essential to use a procedure for computing the dimensionally-regulated matrix elements, such as the one that we have specified, in which the UV power divergences vanish.

V. NRQCD FACTORIZATION FORMALISM

In order to establish our notation and conventions, we now give a brief review of the NRQCD factorization formalism for heavy-quarkonium decays.

In the nonrelativistic limit, in which the heavy-quark (or antiquark) velocity $v$ is much less than unity, full QCD is described accurately by the effective theory NRQCD. In particular, NRQCD is convenient for describing heavy-quarkonium bound states. In such systems, there are three important energy scales: the heavy-quark mass $\sim m$, the heavy-quark 3-momentum $\sim mv$, and the heavy-quark energy $\sim mv^2$. The NRQCD effective theory is constructed from full QCD by integrating out the effects from energies of order $m$ or larger. Since NRQCD describes only the effects at energy scales less than $m$, there is no heavy-quark-antiquark pair creation in the effective theory. That implies, to first approximation, that the heavy-quark and heavy-antiquark fields are decoupled. Hence, it is convenient to express the effective action in terms of Pauli two-component spinors—one for the heavy-quark field and one for the heavy-antiquark field.

The NRQCD effective Lagrangian can be used to reproduce the full theory to any desired accuracy in $v$. We can write it, up to terms of relative accuracy $v^3$, as

$$L_{\text{NRQCD}} = L_0 + \delta L,$$

where

$$L_0 = \psi^\dagger \left(iD_0 + \frac{D^2}{2m}\right) \psi + \chi^\dagger \left(iD_0 - \frac{D^2}{2m}\right) \chi,$$

is the leading-order NRQCD effective Lagrangian, and

$$\delta L = \frac{c_1}{8m^3}[\psi^\dagger (D^2)^2 \psi - \chi^\dagger (D^2)^2 \chi] + \frac{c_2}{8m^2}[\psi^\dagger (D \cdot gE - gE \cdot D) \psi + \chi^\dagger (D \cdot gE - gE \cdot D) \chi] + \frac{c_3}{8m^2}[\psi^\dagger (iD \times gE - gE \times iD) \cdot \sigma \psi + \chi^\dagger (iD \times gE - gE \times iD) \cdot \sigma \chi] + \frac{c_4}{2m}[\psi^\dagger (gB \cdot \sigma) \psi - \chi^\dagger (gB \cdot \sigma) \chi].$$
contains the relative order-$v^3$ corrections. Here, $D^\mu = \partial^\mu + igA^\mu$, $\psi$ is the two-component Pauli field that destroys a heavy quark, $\chi$ is the two-component Pauli field that creates a heavy antiquark, and $c_1$, $c_2$, $c_3$, and $c_4$ are dimensionless short-distance coefficients. They are determined by matching amplitudes in the effective theory with those in full QCD. At tree level, $c_1 = c_2 = c_3 = c_4 = 1$. There is a spin symmetry in the lowest order Lagrangian \cite{10}, since it is independent of the spin of the heavy quark and heavy antiquark. This symmetry is violated by the terms in the higher-order corrections \cite{11} that contain $\sigma$.

In heavy-quarkonium annihilation decays, several distance scales are involved. The annihilation of the heavy quark and antiquark occurs at the short-distance scale $1/m$, whereas the dynamics of quark-antiquark binding involves, principally, the long-distance scales $1/mv$ and $1/mv^2$. The NRQCD factorization formalism \cite{10} separates the physical effects that occur at the scale $1/m$ from those at the longer-distance scales. Since effects from energies of order $m$ are integrated out in obtaining the effective theory, the details of the annihilation are not described in NRQCD. However, the amplitude for a heavy quark and antiquark first to annihilate and then to be recreated is taken into account through effective local four-fermion interactions. The coefficient of a four-fermion operator contains the information about the effects at the scale $m$; the matrix element of the four-fermion operator in a quarkonium state contains the information about the long-distance effects.

The imaginary parts of the four-fermion interactions are related, through the optical theorem, to the total decay rate. Hence, the heavy-quarkonium annihilation decay rate can be written in the factored form \cite{10}

$$\Gamma(H) = \frac{1}{2M_H} \sum_{mn} C_{mn} \langle H | O_{mn} | H \rangle ,$$

(18)

where $C_{mn}$ is a short-distance coefficient that is proportional to the imaginary part of the coefficient, in the effective action, of the four-fermion operator, and $M_H$ is the mass of the state $H$. The matrix elements $\langle H | O_{mn} | H \rangle$ are expectation values of local 4-fermion operators in the quarkonium state $H$. These local operators have the general structure

$$O_{mn} = \psi^\dagger K_m \chi \chi^\dagger K_n \psi ,$$

(19)

where $K_n$ and $K_m$ are direct products of a color matrix (1 or $T^a$), a spin matrix (1 or $\sigma^i$), and a polynomial in the gauge-covariant derivative $D$ and in the field strengths $E$ and $B$.

If $K_m$ and $K_n$ are color-singlet operators, one can use the vacuum-saturation approximation \cite{10} to simplify the matrix elements. This approximation is obtained by inserting a complete set of intermediate states between the quark-antiquark bilinears and retaining only the vacuum hadronic state. The result is

$$\langle H | O_{mn} | H \rangle \approx \langle H | \psi^\dagger K_m \chi | 0 \rangle \langle 0 | \chi^\dagger K_n \psi | H \rangle .$$

(20)

Hence, a matrix element can be factored into a product of the matrix elements $\langle H | \psi^\dagger K_m \chi | 0 \rangle$ and $\langle 0 | \chi^\dagger K_n \psi | H \rangle$. In the case of hadronic decays, this approximation is valid up to corrections of relative order $v^4$ \cite{10}. For electromagnetic decays, the insertion of a vacuum projection operator is exact, since there are no hadrons in the final state.

Generally, the NRQCD matrix elements are nonperturbative in nature. However, it is important to realize that they are fully determined, in principal, by the NRQCD effective theory. In fact, they can be computed by making use of the lattice formulation of NRQCD \cite{20}. 

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VI. COMPUTATION OF THE $U = 1/2$ RENORMALON IN NRQCD MATRIX ELEMENTS

In this section, we apply the method described in Sec. [IV] to compute the $u = 1/2$ renormalons, which yield the ambiguities that are leading in $\Lambda_{\text{QCD}}/\mu$, for the hadron-to-vacuum matrix elements $\langle 0|\chi^\dagger \sigma \psi |J/\psi\rangle$, $\langle 0|\chi^\dagger \psi |\eta_c\rangle$, $\langle 0|\chi^\dagger D^2 \psi |J/\psi\rangle/m^2$, $\langle 0|\chi^\dagger D^2 \psi |\eta_c\rangle/m^2$. These are the matrix elements that arise in leading and next-to-leading order in $v^2$ in the decays of the S-wave charmonium states. Following Ref. [1], we employ a shorthand notation for these matrix elements:

$$\langle K \rangle_\psi \equiv \epsilon \cdot \langle 0|\chi^\dagger \sigma \psi |J/\psi(\epsilon)\rangle,$$  
$$\langle K \rangle_\eta \equiv \langle 0|\chi^\dagger \psi |\eta_c\rangle,$$  
$$\langle K_{D^2} \rangle_\psi \equiv (1/m^2) \epsilon \cdot \langle 0|\chi^\dagger D^2 \psi |J/\psi(\epsilon)\rangle,$$  
$$\langle K_{D^2} \rangle_\eta \equiv (1/m^2) \langle 0|\chi^\dagger D^2 \psi |\eta_c\rangle.$$  

As we have discussed in Section [IV], the renormalon ambiguity of the NRQCD matrix elements [Eq. (14)] is insensitive to the low-energy behavior of the theory. It is, therefore, independent of the external states. Thus, we can evaluate the operator matrix elements in free $c\bar{c}$ states, using perturbative NRQCD (pNRQCD). That is, we use pNRQCD to evaluate the matrix elements

$$\langle K \rangle_V \equiv \epsilon \cdot \langle 0|\chi^\dagger \sigma \psi |c\bar{c}(\epsilon)\rangle,$$  
$$\langle K \rangle_P \equiv \langle 0|\chi^\dagger \psi |c\bar{c}\rangle,$$  
$$\langle K_{D^2} \rangle_V \equiv (1/m^2) \epsilon \cdot \langle 0|\chi^\dagger D^2 \psi |c\bar{c}(\epsilon)\rangle,$$  
$$\langle K_{D^2} \rangle_P \equiv (1/m^2) \langle 0|\chi^\dagger D^2 \psi |c\bar{c}\rangle.$$  

In each state, we take the quark and antiquark to be at rest and on their mass shells. The latter choice makes the calculation manifestly gauge invariant.

A. Feynman rules

In the pNRQCD calculation, we take for the unperturbed Lagrangian the noninteracting part of leading-order-in-$v$ NRQCD Lagrangian (16). The interactions in Eq. (16), as well as the higher-order-in-$v$ corrections in Eq. (17) are treated as perturbations. The Feynman rules may be derived by standard methods. The rules for the quark and antiquark propagators and the “Abelian” parts of the quark-gluon interaction vertices are shown in Table [I]. By “Abelian” we mean the terms that contain no color-matrix commutators. For our computation, which is at the one-loop level, the non-Abelian terms do not contribute.

We also need the rules for the gluon propagators with all numbers of fermion-loop vacuum polarization insertions. Each renormalized fermion-loop vacuum-polarization yields a factor

$$i\beta_0 \alpha_s \left[ \ln \left( \frac{-l^2 - i\epsilon}{\mu^2} \right) + C \right] (-l^2 g^{\mu\nu} + l^\mu l^\nu),$$  

where $l$ is the gluon momentum. (As we have already mentioned, $\beta_0$ contains only the fermion-loop contribution at this stage, but eventually we promote $\beta_0$ to the complete QCD
The sum over all numbers of such vacuum-polarization insertions is a geometric series, which is easily computed for the gluon propagator in a given gauge. In a covariant gauge, the propagator with all numbers of vacuum-polarization insertions is

\[ iD_{\text{cov}}^{\mu\nu} = \sum_{n=0}^{\infty} \beta_0^n \alpha_s^n \left[ -\ln \left( \frac{-l^2 - i\epsilon}{\mu^2} \right) - C \right]^n \frac{i(-g^{\mu\nu} + l^\mu l^\nu/l^2)}{l^2 + i\epsilon} - \xi \frac{i l^\mu l^\nu/l^2}{l^2 + i\epsilon} , \]

where we have suppressed color indices. Similarly, in the Coulomb gauge, the Coulomb-gluon propagator is

\[ iD_C = \sum_{n=0}^{\infty} \beta_0^n \alpha_s^n \left[ -\ln \left( \frac{-l^2 - i\epsilon}{\mu^2} \right) - C \right]^n i \frac{l^2}{l^2} , \]

and the transverse-gluon propagator is

\[ iD_T^{ij} = \sum_{n=0}^{\infty} \beta_0^n \alpha_s^n \left[ -\ln \left( \frac{-l^2 - i\epsilon}{\mu^2} \right) - C \right]^n \frac{i(\delta_{ij} - l_i l_j/l^2)}{l^2 + i\epsilon} . \]

In each instance, the \( \alpha_0^s \) term is the free-gluon propagator. Here, and throughout this paper, we use Greek letters for Minkowski-space indices and Roman letters for 3-space indices. We do not distinguish between upper and lower 3-space indices; both correspond to the upper Minkowski-space index in covariant quantities and the lower Minkowski-space in contravariant quantities.

It is now straightforward to obtain the Borel transforms of \( \alpha_s \) times the propagators in Eqs. (24). They are

\[ iB[\alpha_s D_{\text{cov}}^{\mu\nu}] = \left( \frac{\mu^2}{eC} \right)^u \frac{i(-g^{\mu\nu} + l^\mu l^\nu/l^2)}{(l^2 - i\epsilon)^{1+u}} \]

\[ - \xi \frac{i l^\mu l^\nu/l^2}{(l^2 - i\epsilon)^{1+u}} , \]

\[ iB[\alpha_s D_C] = \left( \frac{\mu^2}{eC} \right)^u \frac{i}{l^2 (l^2 - i\epsilon)^u} , \]

\[ iB[\alpha_s D_T^{ij}] = - \left( \frac{\mu^2}{eC} \right)^u \frac{i(\delta_{ij} - l_i l_j/l^2)}{(l^2 - i\epsilon)^{1+u}} . \]

**B. The computation**

Now we apply the method of Sec. [14] to calculate the \( u = 1/2 \) ultraviolet renormalons associated with the matrix elements in Eq. (22). For each perturbative correction to these matrix elements, we compute the right side of Eq. (14), using the convention of Sec. [15.3] for the form of the integrand in dimensional regularization. We consider only the ambiguities that are proportional to the lowest-order matrix elements \( \langle \mathcal{K} \rangle_V \) and \( \langle \mathcal{K} \rangle_P \). To obtain the
mixing into these matrix elements, we set the external momentum equal to zero in perturbative corrections. We find it most convenient to carry out the calculation in the Coulomb gauge.

First we identify the $u = 1/2$ ultraviolet renormalons associated with the matrix elements $\langle K_{D^2} \rangle_V$ and $\langle K_{D^2} \rangle_P$. Consider the vertex corrections to these matrix elements that arise from a Coulomb-gluon exchange with two $A_0$ vertices. These corrections contain pieces proportional to the lower-order matrix elements $\langle K \rangle_V$ and $\langle K \rangle_P$, respectively, which are obtained by setting the external momentum equal to zero. Since the $A_0$ vertices respect the NRQCD spin symmetry, the vector and pseudoscalar renormalization constants of proportionality are equal. They are given by

$$
\delta \tilde{Z}_{V,D^2} = \delta \tilde{Z}_{P,D^2} = 4\pi i C_F \left( \frac{\mu^2}{e^C} \right)^u \int_{\lambda}^{\text{dim}} \frac{d^4 l}{(2\pi)^4} \frac{1}{l_0 - l^2/(2m) + i\epsilon} \frac{l^2}{m^2} \frac{1}{-l_0 - l^2/(2m) + i\epsilon} \times \frac{1}{l^2 (-l^2 - i\epsilon)^u},
$$

where $\int_{\lambda}^{\text{dim}} d^4 l$ is shorthand for $\int_{l_0}^{\infty} \int_{\lambda}^{\text{dim}} d^3 l$, and $C_F = 4/3$. To evaluate this integral, we first integrate over $l_0$, using the contour-integral method. It is easy to see that the cut contribution has a power count such that it cannot contribute a pole at $u = 1/2$. The contribution from the quark (or antiquark) pole yields

$$
\delta \tilde{Z}_{V,D^2} = \delta \tilde{Z}_{P,D^2} \sim -\frac{16\pi}{3m} \left( \frac{\mu^2}{e^C} \right)^u \int_{\lambda}^{\text{dim}} \frac{d^3 l}{(2\pi)^3} \frac{1}{l^2/(4m^2)} \left[ 1 - l^2/(4m^2) \right]^{-1}.
$$

Now we expand the factor $[1 - l^2/(4m^2)]^{-1}$ in a power series in $l^2/m^2$. Only the first term in the series has the correct power count to yield a pole at $u = 1/2$. Integrating over $l$, we find the $u = 1/2$ renormalon:

$$
\delta \tilde{Z}_{V,D^2} = \delta \tilde{Z}_{P,D^2} \sim -\frac{8}{3\pi m} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u},
$$

where we have used the fact that the contribution from the upper limit of the integration vanishes, since the integral is UV convergent in dimensional regularization. We note that diagrams involving transverse gluons, higher-order NRQCD vertices or the gauge fields in the $D^2$ operator all contribute with the wrong power count to yield a pole at $u = 1/2$. (In fact, contributions involving transverse gluons and the part of an $\nabla \cdot A_i$ vertex that is proportional to $l$ vanish in the Coulomb gauge.) From (13), we obtain the $u = 1/2$ renormalon ambiguities in the NRQCD matrix elements:

$$
\Delta \langle K_{D^2} \rangle_\psi = -K \frac{8}{3\beta_0} \frac{\Lambda_{\text{QCD}}}{e^{C/2m}} \langle K \rangle_\psi,
$$

$$
\Delta \langle K_{D^2} \rangle_\eta = -K \frac{8}{3\beta_0} \frac{\Lambda_{\text{QCD}}}{e^{C/2m}} \langle K \rangle_\eta.
$$

Next we identify the $u = 1/2$ ultraviolet renormalons in the matrix elements $\langle K \rangle_V$ and $\langle K \rangle_P$ associated with the mixing of these matrix elements into themselves.
First consider the heavy-quark and heavy-antiquark self-energy diagrams. It can be seen, by counting powers of momentum, that the diagrams involving a transverse gluon or higher-order NRQCD vertices do not yield a pole at $u = 1/2$. The diagrams involving a Coulomb gluon contribute to the $u = 1/2$ renormalon through the renormalization constant of the quark (antiquark) wavefunction. For a heavy quark with external energy $E$ and momentum $p$, the Borel transform of the self-energy diagram is given by

$$\tilde{\Sigma}(E, p) = 4\pi i C_F \int \frac{d^4 l}{(2\pi)^4} \frac{1}{E + l_0 - (p - l)^2/2m + i\epsilon} \frac{1}{l^2(-l^2 - i\epsilon)^u}. \quad (30)$$

The Borel transform of the renormalization constant of the wave function is

$$\delta \tilde{Z}_2 = \left. \frac{\partial \tilde{\Sigma}(E, p)}{\partial E} \right|_{E=p=0} = -4\pi i C_F \int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l_0 - l^2/2m + i\epsilon)^2} \frac{1}{l^2(-l^2 - i\epsilon)^u}. \quad (31)$$

The antiquark self-energy differs from the quark self-energy only in the sign of the $l_0$ term in the quark (antiquark) propagator. Hence, these contributions are equal under a change of integration variables $l_0 \rightarrow -l_0$. It is useful to symmetrize the integrand under this change of variables. Then, the sum of the contributions of the quark and antiquark wavefunction renormalizations yields the renormalization coefficient

$$\delta \tilde{Z}_{\text{wf}} = -4\pi i C_F \left( \frac{\mu^2}{e^C} \right)^u \int_{\lambda} \frac{d^4 l}{(2\pi)^4} \frac{1}{(l_0 - l^2/2m + i\epsilon)^2} \frac{1}{l^2(-l^2 - i\epsilon)^u}. \quad (32)$$

If we carry out the $l_0$ contribution by contour integration, the individual cut contributions yield poles at $u = 1/2$. However, it is easy to see that the sum of the cut contributions from the quark and antiquark terms does not have a pole at $u = 1/2$. Computing the double-pole contribution and integrating over $l$, we obtain

$$\delta \tilde{Z}_{\text{wf}} \sim \frac{2}{3\pi m} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u}. \quad (33)$$

Next we consider the vertex correction involving a Coulomb gluon with $A_0$ vertices. From the right side of Eq. (14), we find that this vertex correction yields the renormalization coefficients

$$\delta \tilde{Z}_{V,a} = \delta \tilde{Z}_{P,a} = -4\pi i C_F \left( \frac{\mu^2}{e^C} \right)^u \int_{\lambda} \frac{d^3 l}{(2\pi)^3} \frac{1}{l_0 - l^2/(2m) + i\epsilon} \frac{1}{l^4(-l^2 - i\epsilon)^u}. \quad (34)$$

If we carry out the $l_0$ integration by the contour method, we see that that cut contribution does not contain a pole at $u = 1/2$. However, the contribution of the quark (or antiquark) pole is

$$\delta \tilde{Z}_{V,a} = \delta \tilde{Z}_{P,a} \sim \frac{16\pi m}{3} \left( \frac{\mu^2}{e^C} \right)^u \int_{\lambda} \frac{d^3 l}{(2\pi)^3} \frac{1}{l^4} \frac{1}{l^2 - l^4/(4m^2)\epsilon}^u. \quad (35)$$
Expanding the last denominator in a power series in $l^2/m^2$, we find that the leading term contains a pole at $u = -1/2$, but not at $u = 1/2$. The second term in the power-series expansion of the last denominator in Eq. (35) does yield a pole at $u = 1/2$:

$$\delta \tilde{Z}_{V,a} = \delta \tilde{Z}_{P,a} \sim \frac{4u}{3\pi m} \left( \frac{\mu^2}{eC} \right)^u \int_\lambda^{\text{dim}} \frac{d^3l}{(2\pi)^3} \frac{1}{l^{2+2u}}.$$  (36)

Carrying out the integration over $l$, we obtain

$$\delta \tilde{Z}_{V,a} = \delta \tilde{Z}_{P,a} \sim -\frac{1}{3\pi m} \left( \frac{\mu^2}{eC} \right)^u \frac{1}{1 - 2u}.  \ (37)$$

There is also a vertex correction involving a Coulomb gluon with $A_0$ vertices and a $D^4/(8m^3)$ correction in either the quark or antiquark propagator. The resulting renormalization coefficients are

$$\delta \tilde{Z}_{V,b} = \delta \tilde{Z}_{P,b} = 4\pi iC_F \left( \frac{\mu^2}{eC} \right)^u \int_\lambda^{\text{dim}} \frac{d^4l}{(2\pi)^4} \frac{l^4}{4m^2} \frac{1}{(l_0 - l^2/(2m) + i\epsilon)^2} \frac{1}{-l_0 - l^2/(2m) + i\epsilon} \times \frac{1}{l^2(-l^2 - i\epsilon)^u}, \ (38)$$

where we have doubled the contribution with a $D^4/(8m^3)$ insertion in the quark propagator in order to account for the equal contribution with a $D^4/(8m^3)$ insertion in the antiquark propagator. Integrating over $l_0$ by closing the contour in the upper half-plane, we obtain

$$\delta \tilde{Z}_{V,b} = \delta \tilde{Z}_{P,b} \sim \frac{4\pi}{3m} \left( \frac{\mu^2}{eC} \right)^u \int_\lambda^{\text{dim}} \frac{d^3l}{(2\pi)^3} \frac{1}{l^{2+2u}}, \ (39)$$

where, as usual, we have dropped the cut contribution, which does not yield a pole at $u = 1/2$. Carrying out the integration, we identify the $u = 1/2$ renormalon:

$$\delta \tilde{Z}_{V,b} = \delta \tilde{Z}_{P,b} \sim -\frac{2}{3\pi m} \left( \frac{\mu^2}{eC} \right)^u \frac{1}{1 - 2u}. \ (40)$$

Next we consider a vertex correction involving a Coulomb gluon with one $A_0$ vertex and one $D \cdot E$ vertex. It yields the renormalization coefficients

$$\delta \tilde{Z}_{V,c} = \delta \tilde{Z}_{P,c} = 4\pi iC_F \left( \frac{\mu^2}{eC} \right)^u \int_\lambda^{\text{dim}} \frac{d^4l}{(2\pi)^4} \frac{l^2}{4m^2} \frac{1}{l_0 - l^2/(2m) + i\epsilon} \frac{1}{-l_0 - l^2/(2m) + i\epsilon} \times \frac{1}{l^2(-l^2 - i\epsilon)^u}, \ (41)$$

where we have taken into account the two diagrams obtained by interchanging the $A_0$ vertex and the $D \cdot E$ vertex. This expression differs from (26) by an overall factor $1/4$. Thus from (28), we obtain the contribution to the $u = 1/2$ renormalon:

$$\delta \tilde{Z}_{V,c} = \delta \tilde{Z}_{P,c} \sim \frac{2}{3\pi m} \left( \frac{\mu^2}{eC} \right)^u \frac{1}{1 - 2u}. \ (42)
Finally, we consider the vertex correction involving a transverse gluon with two $\sigma \cdot B$ vertices. These vertices violate the spin symmetry, and, so, the vertex correction depends on the spin structure of the original matrix element. The resulting renormalization coefficients are

$$\frac{\delta \tilde{Z}_{V,d}}{\delta \tilde{Z}_{P,d}} \left\{ \Sigma = \sum \right\} = -4\pi i C_F \left( \frac{\mu^2}{e^C} \right)^u \int_\lambda^{\text{dim}} \frac{d^4 l}{(2\pi)^4} \frac{1}{2m} \frac{(\sigma \times l)_i}{l_0 - l^2/(2m) + i\epsilon} \frac{1}{-l_0 - l^2/(2m) + i\epsilon} \times \frac{(\sigma \times l)_j}{2m} \frac{\delta_{ij} - l_0 l_j/l^2}{(-l^2 - i\epsilon)^{1+u}}, \tag{43}$$

where

$$\Sigma = \set{\sigma, 1}. \tag{44}$$

Integrating over $l_0$ by closing the contour in the upper half-plane and simplifying the integrand by averaging over the angles of $l$, we obtain

$$\frac{\delta \tilde{Z}_{V,d}}{\delta \tilde{Z}_{P,d}} \left\{ \Sigma \right\} \sim \frac{8\pi}{9m} \sigma_i \Sigma \sigma_i \left( \frac{\mu^2}{e^C} \right)^u \int_\lambda^{\text{dim}} \frac{d^3 l}{(2\pi)^3} \frac{1}{l^{2+2u}}, \tag{45}$$

where we have discarded the cut contribution, which does not contain a pole at $u = 1/2$. Carrying out the $l$ integration, we can identify the $u = \frac{1}{2}$ renormalon:

$$\frac{\delta \tilde{Z}_{V,d}}{\delta \tilde{Z}_{P,d}} \left\{ \right\} \sim \frac{1}{\pi m} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u} \left\{ \begin{array}{ll} +4/9 & \\ -4/3 & \end{array} \right\}, \tag{46}$$

It may be seen, from power-counting arguments, that the diagrams we have discussed so far are the only ones that contribute a pole at $u = 1/2$. Adding the contributions in Eqs. (33), (37), (40), (42), and (46) we obtain

$$\frac{\delta \tilde{Z}_V}{\delta \tilde{Z}_P} \left\{ \right\} \sim \frac{1}{\pi m} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u} \left\{ \begin{array}{ll} 7/9 & \\ -1 & \end{array} \right\}. \tag{47}$$

Therefore, from Eq. (13), we see that the ambiguities in the operators $\mathcal{K}_\psi$ and $\mathcal{K}_\eta$ are

$$\Delta \langle \mathcal{K}_\psi \rangle = -K \frac{7}{9\beta_0} \frac{\Lambda_{QCD}}{e^{C/2m}} \langle \mathcal{K}_\psi \rangle, \tag{48a}$$

$$\Delta \langle \mathcal{K}_\eta \rangle = K \frac{1}{\beta_0} \frac{\Lambda_{QCD}}{e^{C/2m}} \langle \mathcal{K}_\eta \rangle. \tag{48b}$$

The results of our calculations of the $u = 1/2$ renormalon in the operator matrix elements for S-wave charmonium decay are summarized in Table II.

We note that in NRQCD, in contrast with HQET and with the light-cone expansion, the renormalon ambiguities scale differently from the matrix elements themselves. For example, the $u = 1/2$ renormalons computed in this section have ambiguities that scale as $\Lambda_{QCD}/m$, relative to the matrix elements of lowest order in $\nu$, but the matrix elements themselves scale relatively as $\nu^0$ or $\nu^2$. The differences in scaling behavior between the renormalons and
the matrix elements arise because the dynamics of heavy-quarkonium system depends on the scales $m v$ and $m v^2$, rather than the scale $\Lambda_{\text{QCD}}$.

In Ref. [1], Braaten and Chen studied the large-order asymptotic behavior of the perturbation series for short-distance coefficients in the NRQCD factorization formulas for the decays $J/\psi \rightarrow e^+e^-$ and $\eta_c \rightarrow \gamma\gamma$. They calculated the Borel transforms of the short-distance coefficients of the leading (in $v^2$) and first subleading matrix elements. By requiring that the renormalon ambiguities in the short-distance coefficients be cancelled by the renormalon ambiguities in the NRQCD matrix elements, they deduced the $u = 1/2$ renormalon ambiguities in the matrix elements. Comparing the results obtained in Ref. [1] with the results of our direct calculation of the matrix-element ambiguities in Eqs. (29) and (48), we find agreement. Thus, the general principle of the cancellation of renormalon ambiguities between the short-distance coefficients and the operator matrix elements is supported by this specific example in context of the NRQCD factorization formalism.

VII. RENORMALONS IN THE GREMM-KAPUSTIN RELATION AND THE QUARK MASS

In this section, we discuss the renormalon content of the matrix elements that appear in the Gremm-Kapustin relation [14] and the renormalon content of various definitions of the heavy-quark mass.

The Gremm-Kapustin relation is an equality between NRQCD matrix elements that relates the heavy-quark kinetic-energy operator to the energy of the quarkonium meson. It is correct at leading order in $v$. One form of the Gremm-Kapustin relation follows immediately from the leading-order equations of motion of the NRQCD Lagrangian. In the case of the $J/\psi$ state, for example, we have

\begin{equation}
\langle 0 | D^2 \sigma \psi | J/\psi \rangle = -m \langle 0 | i \partial_0 (\chi^\dagger \sigma \psi) | J/\psi \rangle.
\end{equation}

There are analogous relations for states with different spin and orbital-angular-momentum quantum numbers.

The $u = 1/2$ renormalon in the matrix element on the left side of Eq. (13) is proportional to the matrix element of the lowest order operator $\chi^\dagger \sigma \psi$. The constant of proportionality is given by $m^2$ times Eq. (28). The right side of Eq. (13) is proportional to the NRQCD energy of the quark-antiquark state $E_{\psi}$:

\begin{equation}
\langle 0 | i \partial_0 (\chi^\dagger \sigma \psi) | J/\psi \rangle = E_{\psi} \langle 0 | \chi^\dagger \sigma \psi | J/\psi \rangle.
\end{equation}

The mixing of $i \partial_0 (\chi^\dagger \sigma \psi)$ into the lower-order operator $\chi^\dagger \sigma \psi$ is obtained by evaluating the QCD corrections to the matrix element $\langle 0 | i \partial_0 (\chi^\dagger \sigma \psi) | c \bar{c} \rangle$, with the quark and antiquark

\footnote{The matrix elements of Braaten and Chen differ from the ones that we consider here by a factor $\sqrt{2M_H}$, where $M_H$ is the quarkonium mass. However, $M_H$ is ambiguity free, so this change in normalization has no effect on the sizes of the ambiguities relative to the lowest-order-in-$v$ matrix elements.}
taken on shell and at zero external momentum. Beyond tree level, the mass-shell energy at zero momentum is shifted from zero to

\[ E_0 = \Sigma(E, p)|_{E=p=0}, \tag{51} \]

where \( \Sigma(E, p) \) is the quark (or antiquark) self-energy (see Eq. (30)). This shift in the mass-shell position yields a mixing of \( i\partial_0(\chi^\dagger \sigma)\psi \) into \( \chi^\dagger \sigma \psi \), with a constant of proportionality \( 2E_0 \). (The factor of two comes from the sum of the equal quark and antiquark energy shifts.) Using Eq. (14), we can compute the renormalization coefficient for the mixing of \( 2E_0 \) into unity in passing from the cutoff regulator to dimensional regularization. The contribution to the Borel transform of that coefficient that comes from the graph containing a Coulomb gluon with two \( A_0 \) vertices is given by

\[ \delta \tilde{Z}_{V,\partial_0} = \delta \tilde{Z}_{E_\psi} = 2\delta \tilde{Z}_{E_0} = 4\pi i C_F \int_\lambda^{\text{dim}} \frac{d^4l}{(2\pi)^4} \left[ \frac{1}{l_0 - l^2/2m + i\epsilon} + \frac{1}{-l_0 - l^2/2m + i\epsilon} \right] \frac{1}{l^2 (-l^2 - i\epsilon)^u}, \tag{52} \]

where we have taken into account both the quark and antiquark contributions, which has the effect of symmetrizing the integrand under \( l_0 \to -l_0 \). We have made explicit the fact that, to leading order in \( v \), \( \delta \tilde{Z}_{V,\partial_0} = \delta \tilde{Z}_{E_\psi} \), since, in Eq. (50), any regulator dependence in \( \langle 0|\chi^\dagger \sigma \psi|J/\psi \rangle \) is suppressed by the factor \( E_\psi \), which is of order \( v^2 \). The contribution in Eq. (52) is equal to \(-m\) times the contribution in Eq. (26) and yields a pole at \( u = 1/2 \):

\[ \delta \tilde{Z}_{V,\partial_0} = \delta \tilde{Z}_{E_\psi} = 2\delta \tilde{Z}_{E_0} \sim -\frac{8}{3\pi} \left( \frac{\mu^2}{eC} \right)^u \frac{1}{1 - 2u}. \tag{53} \]

Hence, the left and right sides of Eq. (19) have equal \( u = 1/2 \)-renormalon content.

Now, the NRQCD energy is related to the physical mass of the charmonium state \( M_\psi \) as

\[ M_\psi = E_\psi - 2E_0 + 2m_{\text{pole}}, \tag{54} \]

where \( m_{\text{pole}} \) is the heavy-quark pole mass. In a theory without confinement, such as Quantum Electrodynamics, both \( m_{\text{pole}} \) and \( E_0 \) have nonperturbative definitions in terms of the poles in the heavy-quark propagators in the full theory and the effective theory, respectively. In a confining theory, such definitions are problematic; only the linear combination \( m_{\text{pole}} - E_0 \) has a nonperturbative definition in terms of quarkonium matrix elements. Therefore, we absorb \( E_0 \) into redefinition of \( m_{\text{pole}} \):

\[ m'_{\text{pole}} = m_{\text{pole}} - E_0, \tag{55} \]

which implies that

\[ 2m'_{\text{pole}} = M_\psi - E_\psi. \tag{56} \]

Using this redefinition, we can rewrite the Gremm-Kapustin relation (19) in the more conventional form

\[ \langle 0|\chi^\dagger D^2 \sigma \psi|J/\psi \rangle = -m(M_\psi - 2m'_{\text{pole}})\langle 0|\chi^\dagger \sigma \psi|J/\psi \rangle. \tag{57} \]
In dimensional regularization, $E_0$ vanishes and $m'_\text{pole}$ is equal to $m_{\text{pole}}$. In the literature, $m_{\text{pole}}$ and $m'_\text{pole}$ are used interchangeably. However, it is important to bear in mind the differences between $m_{\text{pole}}$ and $m'_\text{pole}$. In a confining theory, $m_{\text{pole}}$ is defined only in perturbation theory, whereas $m'_\text{pole}$ is defined, through Eq. (50) and Eq. (54), in terms of operator matrix elements. In a cutoff scheme, $m'_\text{pole}$ exhibits a dependence, through $E_\psi$ on the UV cutoff, diverging as its first power. (In a dimensional-regularization scheme, this power dependence of $m'_\text{pole}$ on the cutoff disappears.) $m'_\text{pole}$ also depends on the effective theory in which it is defined. In contrast, $m_{\text{pole}}$ is independent of the cutoff and the effective theory. This is true, by definition, in a nonconfining theory, where $m_{\text{pole}}$ is the physical mass, and it is enforced in perturbation theory in a confining theory.

In Eq. (54), UV renormalons are absent in $E_0$ and $E_\psi$ in cutoff regularization and cancel, by Eq. (53), in dimensional regularization. Since $E_0$ vanishes in dimensional regularization, the IR renormalons in $E_0$ must be equal in magnitude and opposite in sign to the UV renormalons $E_\psi$, being an operator matrix element, has no IR renormalons. Therefore, we conclude that $m_{\text{pole}}$ has IR renormalon ambiguities, and that near $u = 1/2$

$$m'_\text{pole} \sim \frac{4}{3\pi} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u}.$$ (58)

Of course, we expect $m_{\text{pole}}$ to contain IR renormalon ambiguities, since the counterterm for on-shell mass renormalization involves a loop integration down to zero momentum. It is important to recognize that $m_{\text{pole}}$ is neither a short-distance coefficient nor an operator matrix element. Rather, $m_{\text{pole}}$ is a quantity that is defined perturbatively in terms of the pole in the heavy-quark propagator. Consequently, $m_{\text{pole}}$ is factorization-scheme independent, and there are renormalon ambiguities in $m_{\text{pole}}$, even if one employs a cutoff factorization scheme. Of course, one could absorb renormalon ambiguities in $m_{\text{pole}}$ into cutoff matrix elements of operators in NRQCD. However, the residual renormalon-free short-distance coefficient would not be equal to the position of the pole in the heavy-quark propagator.

We remark that, in a dimensional-regularization factorization scheme, $m_{\text{pole}} = m'_\text{pole}$, and, so,

$$m'_\text{pole} \sim \frac{4}{3\pi} \left( \frac{\mu^2}{e^C} \right)^u \frac{1}{1 - 2u}.$$ (59)

As with $m_{\text{pole}}$, $m_{\text{MS}}$ is neither a short-distance coefficient nor an operator matrix element. $m_{\text{MS}}$ is related to the bare mass $m_0$ of full QCD as

$$m_{\text{MS}} = m_0 + \delta m_{\text{MS}}.$$ (60)

These remarks also are relevant in HQET. $\tilde{\Lambda}$, the heavy-light meson energy in HQET, is analogous to $E_\psi$ in NRQCD. In HQET $m'_\text{pole} = M_{\text{meson}} - \tilde{\Lambda}$, where $M_{\text{meson}}$ is the physical meson mass.

This is an example of the replacement of IR renormalons with UV renormalons through the manipulation of a quantity that vanishes in dimensional regularization.
$m_0$ is a parameter of full QCD and is independent of any ambiguities associated with factorization; $\delta m_{\overline{MS}}$ consists of poles in $\epsilon$ and associated constants, which contain no ambiguities. Hence, in contrast with $m_{\text{pole}}$, $m_{\overline{MS}}$ contains no ambiguities.

Now we wish to demonstrate that our results for the ambiguities in $E_\psi$, $E_0$ and $m_{\text{pole}}$ are consistent with the fact that $m_{\overline{MS}}$ has no ambiguities. First we re-express $m_{\text{pole}}$ in terms of the heavy-quark $\overline{MS}$ mass $m_{\overline{MS}}$:

$$\Delta m = m_{\text{pole}} - m_{\overline{MS}} = \Sigma^{\text{QCD}}(p)|_{p \cdot \gamma = m_{\text{pole}}} - \delta m_{\overline{MS}}.$$  \hspace{1cm} (61)

where $\Sigma^{\text{QCD}}(p)$ is the heavy-quark self-energy correction in full QCD and $\delta m_{\overline{MS}}$ is the usual $\overline{MS}$ mass counterterm $[\delta m_{\overline{MS}} = (3C_F \alpha_s m/4\pi)(1/\epsilon - \gamma_E + \ln 4\pi)$ in order $\alpha_s]$. Thus, we can rewrite Eq. (54) as

$$E_\psi = M_\psi - 2m_{\overline{MS}} + (2E_0 - 2\Delta m).$$  \hspace{1cm} (62)

Since the expressions for the full QCD heavy-quark energy shift $\Sigma^{\text{QCD}}(p)|_{p \cdot \gamma = m_{\text{pole}}}$ and the NRQCD energy shift $E_0$ [Eq. (51)] have, by construction, identical behavior at small loop momentum, their difference $\Delta m - E_0$ has no support at small loop momentum. Therefore, $\Delta m - E_0$ contains no IR renormalon ambiguities. $E_\psi$, being an operator matrix element, has no IR renormalon ambiguities. Furthermore, $\Delta m$ is free of UV renormalon ambiguities because the counterterm $\delta m_{\overline{MS}}$ removes the poles at $u = 0$, which correspond to the UV renormalons for a logarithmically divergent quantity such as $\Sigma^{\text{QCD}}$. We also see, from Eq. (52), that the quantity $E_\psi - 2E_0$ is free of UV renormalon ambiguities. Thus, in Eq. (62), $m_{\overline{MS}}$ must be free of both UV and IR renormalon ambiguities.

We note that the absence of renormalon ambiguities in $m_{\overline{MS}}$ implies that Eq. (58) is in agreement with the standard result [7] for the ambiguity of $m_{\text{pole}}$ relative to $m_{\overline{MS}}$.

VIII. SUMMARY AND DISCUSSION

Factorization formalisms for QCD are of great computational utility because they allow one to separate short-distance, perturbative physics from long-distance, nonperturbative effects. However, the perturbation series for the short-distance coefficients in QCD are, at best, asymptotic. Therefore, perturbative calculations of QCD processes are ultimately limited in precision.

One reason for the nonconvergence of the perturbation series is that, even in the case of “IR safe” quantities, loop integrations typically extend down to zero momentum. In that region, the running coupling becomes large and perturbation theory fails. In the bubble-chain model (described in Sec. [4]), the coefficients in the perturbation series grow as the factorial of the order in $\alpha_s$, leading to renormalon singularities in the Borel transform and to ambiguities in the Borel summation of the series.

By employing a cutoff factorization scheme, in which loop momenta never become small, one can eliminate renormalon ambiguities from the perturbation series for the short-distance coefficients. We wish to emphasize that the renormalon ambiguities are emblematic of other nonperturbative effects that could arise in the low-momentum, long-distance region and which are not, as yet, well-understood. A cutoff factorization scheme excludes this nonperturbative region from the coefficient functions. In a dimensional-regularization factorization
scheme, one integrates all IR-finite terms in the integrand down to zero momentum. Hence, dimensionally-regulated short-distance coefficients contain renormalon ambiguities.

Physical observables are free of ambiguities, in the sense described in Section I. This implies that ambiguities in the factorized expression for a physical observable must cancel between the short-distance coefficients and the operator matrix elements (provided that the factorization procedure and the underlying effective field theory are sufficiently accurate as an expansion in inverse powers of the large scale in the process). That is, the ambiguities are merely artifacts of the factorization procedure.

The presence of ambiguities in the dimensionally-regulated short-distance coefficients implies that at least some of the dimensionally-regulated matrix elements contain ambiguities. Since operator matrix elements are determined completely, in principal, in terms of short-distance coefficients and physical observables, the absence of ambiguities in the cutoff short-distance coefficients implies that the cutoff matrix elements are free of ambiguities. Lattice regularization is one example of a cutoff regulator and, indeed, the QCD operator matrix elements are defined unambiguously in lattice regularization.

The cancellation of ambiguities in a physical observable holds, by construction, if one infers the matrix elements through a comparison of theoretical expressions with observables. If one determines the matrix elements through a calculation in an underlying effective field theory, as we have done in this paper, then the cancellation of ambiguities requires that the effective theory reproduce the low-momentum behavior of the original theory.

We stress that, in computing QCD corrections to the dimensionally-regulated matrix elements, it is essential to adopt a convention that is consistent with the standard computations of the short-distance coefficients, in which IR-finite expressions are integrated down to zero loop momentum. This convention, which is described in detail in Sec. IV B, involves writing the loop corrections as linear combinations of operators in the underlying effective theory and then expanding the loop integrals in powers of the loop momentum divided by the large scale in the factorization formalism. The net effect is to remove from the matrix elements the power UV divergences that correspond to the IR-finite terms in the short-distance coefficients.

In this paper, we have presented a method for computing the renormalon ambiguities in dimensionally-regulated matrix elements. The method exploits the fact that cutoff matrix elements are unambiguous, which implies that one can compute the ambiguities in dimensionally-regulated matrix elements by computing the differences between the cutoff and dimensionally-regulated matrix elements. These differences are short-distance quantities and, hence, can be computed in perturbation theory.

We have used our method to compute the $a = 1/2$ renormalon ambiguities in the matrix elements that appear in the NRQCD factorization expressions for the annihilation decays of S-wave heavy quarkonium at the leading and first subleading orders in $v$. On comparison with the results of Braaten and Chen [1] for the renormalon ambiguities in the corresponding short-distance coefficients, we find that the ambiguities cancel in the physical decay rates, as expected. The Gremm-Kapustin relation between operator matrix elements is also consistent with the ambiguities that we find. Our result for the ambiguity in $m_{\text{pole}}$ is consistent with the ambiguity in the expression for $m_{\text{pole}}$ in terms of $m_{\overline{\text{MS}}}$ given in Ref. [6].

In analyzing renormalon ambiguities, it is important to bear in mind that $m_{\text{pole}}$ and $m_{\overline{\text{MS}}}$ are neither short-distance coefficients nor operator matrix elements. $m_{\overline{\text{MS}}}$ is equal to
the sum of the bare mass $m_0$ of full QCD and the $\overline{MS}$ mass counterterm, both of which are free of ambiguities and independent of the factorization scheme. $m_{\text{pole}}$ is a quantity that is defined in perturbation theory by the on-shell renormalization condition for a heavy quark. Because the on-shell renormalization counterterms involve integrations down to zero momentum, $m_{\text{pole}}$ is ambiguous. $m_{\text{pole}}$ is also independent of the factorization scheme.

The accounting of renormalon ambiguities that we have described in this paper can be obscured if one writes operator matrix elements in terms of $m_{\text{pole}}$. For example, the matrix element for the energy of the quarkonium state in NRQCD can be written [Eq. (54)] in terms of the physical quarkonium mass $M_\psi$, $m_{\text{pole}}$, and the NRQCD energy shift $E_0$. Since $E_0$ vanishes in dimensional regularization, one can use this relation to trade ambiguities in the NRQCD quarkonium-energy matrix element for ambiguities in $m_{\text{pole}}$. The analogue of this procedure for HQET is frequently employed in discussions of heavy-light mesons. As a further step, one can remove the ambiguity in $m_{\text{pole}}$ and the corresponding ambiguity in the short-distance coefficients by writing $m_{\text{pole}}$ in terms of $m_{\text{MS}}$.

The renormalization procedure for full QCD can introduce additional IR renormalons into the short-distance coefficients if the renormalization counterterms themselves contain IR renormalons. This occurs whenever the counterterms involve loop integrations down to zero momentum, as is the case in on-shell renormalization. The ambiguities associated with these additional renormalons are not cancelled by ambiguities in the operator matrix elements, but, rather, by ambiguities in the parameters of full QCD. For example, because $m_{\text{pole}}$ is ambiguous, one introduces additional ambiguities into the short-distance coefficients if one expresses them in terms of $m_{\text{pole}}$, rather than, say, $m_{\text{MS}}$. The additional ambiguities in the short-distance coefficients are cancelled by the ambiguities in $m_{\text{pole}}$.

In practice, one determines dimensionally-regulated matrix elements either by comparing perturbative expressions for physical observables with experiment or by computing the perturbative relations between dimensionally-regulated matrix elements and some reference cutoff matrix elements, such as lattice matrix elements. In both methods, the ambiguities in the perturbation series make the determinations of the matrix elements ambiguous. Hence, in order to describe a dimensionally-regulated matrix element, one must give not only its value, but also the order in $\alpha_s$ in which it was determined. At low orders in $\alpha_s$, this last specification may not be important numerically, but at high orders, when the factorial growth of the perturbation series dominates, it is essential.

Finally, let us discuss the relative strengths and weaknesses of the dimensional-regularization and cutoff factorization schemes. Dimensionally-regulated short-distance coefficients can be computed relatively efficiently, but the computation of, for example, lattice-regulated short-distance coefficients, is much more complicated. The dimensionally-regulated short-distance coefficients and matrix elements are ambiguous, whereas the cutoff short-distance coefficients and matrix elements are not. Of course, the presence of renormalon ambiguities in the dimensional-regularization scheme is not a fatal flaw: In physical observables, the factorial growth in the dimensionally-regulated short-distance coefficients cancels against a similar growth in the dimensionally-regulated matrix elements. However, in

\[\footnote{This relation, with $E_0$ set to zero, is often taken as a definition of the pole mass. However, one can neglect $E_0$ only in a dimensional-regularization factorization scheme.} \]
order to achieve this cancellation, one must work to the same order in $\alpha_s$ in the short-distance coefficients and the matrix elements. In addition, it is essential to work to sufficient accuracy in the small scale of the underlying effective field theory \cite{10}. In practice, one cannot achieve a complete cancellation, but one can systematically reduce the renormalon ambiguities by introducing more operator matrix elements and/or including more terms in the effective action. In general, cutoff matrix elements are very sensitive to the cutoff (factorization scale), typically scaling as a power of the cutoff. This sensitivity to the cutoff is cancelled by a similar dependence in the short-distance coefficients, but the cancellation is imperfect at any finite order in perturbation theory. A related difficulty is that the cutoff matrix elements exhibit their nominal sizes only if the cutoff is chosen to be of the order of the small scale in the calculation. In contrast, dimensionally-regulated (\textit{MS}) matrix elements have no power dependence on the cutoff. Nevertheless, at high orders in $\alpha_s$, they must deviate from their nominal sizes because of the factorial growth of the determining perturbation series.

Of course, in the end, the physical observables are independent of the choice of factorization scheme, provided that one works to sufficient accuracy in the small scale of the effective theory and to sufficient and consistent accuracy in $\alpha_s$. However, it is important to bear in mind that the properties of the dimensionally-regulated and cutoff quantities are rather different and that the differences may be significant in practical calculations.

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### TABLE I. Feynman rules for quark and antiquark propagators and interaction vertices in NRQCD

The upper (lower) signs correspond to quarks (antiquarks). We have retained only the “Abelian” terms, i.e., those that contain no color-matrix commutators. Energy and momentum are conserved at the vertices, and we observe the convention in which the antiquark diagrammatic energy and momentum are the negatives of the physical energy and momentum. \( p \) is the incoming (diagrammatic) quark or antiquark momentum, \( p' \) is the outgoing (diagrammatic) quark or antiquark momentum. The gluons have incoming momenta, spatial polarization indices, and color indices \((l_1, i, a), (l_2, j, b), (l_3, k, c), \) and \((l_4, m, d)\), respectively. \( T \) denotes an \( SU(3) \) color matrix in the fundamental representation, normalized to \( \text{tr} t_a t_b = (1/2)\delta_{ab} \). “Perm” denotes all additional permutations obtained by interchanging the momentum, polarization, and color labels of one gluon with those of another gluon.

| Diagrammatic Element | Feynman Rule |
|----------------------|--------------|
| quark propagator     | \( i/\left[ \pm p_0 - p^2/(2m) + i\epsilon \right] \) |
| \( A_0 \) vertex     | \( \mp ig t_a \) |
| \( \nabla \cdot A_i \) vertex | \( ig(p_i + p'_i) t_a/(2m) \) |
| \( A_i \cdot A_j \) seagull vertex | \( -ig^2(\delta_{ij} t_b t_a + \text{perm})/(2m) \) |
| \( \sigma \cdot B \) spatial-gluon vertex | \( ge_{ijk} l_j l_k t_a/(2m) \) |
| \( D \cdot E \) temporal-gluon vertex | \( \pm ig l_i^2 t_a/(8m^2) \) |
| \( D \cdot E \) spatial-gluon vertex | \( \mp ig l_{i10} t_a/(8m^2) \) |
| \( D \times E \cdot \sigma \) temporal-gluon vertex | \( \pm g e_{ijk} p'_i p_j \sigma_k t_a/(4m^2) \) |
| \( D \times E \cdot \sigma \) spatial-gluon vertex | \( \mp g e_{ijk} (p + p') j \sigma_k l_{10} t_a/(8m^2) \) |
| \( D \times E \cdot \sigma \) spatial-temporal seagull vertex | \( \pm g^2 (\epsilon_{ijk} l_j l_k t_b t_a + \text{perm})/(4m^2) \) |
| \( D \times E \cdot \sigma \) spatial-spatial seagull vertex | \( \pm g^2 (\epsilon_{ijk} \sigma_k l_{10} t_b t_a + \text{perm})/(4m^2) \) |
| \( D^4/(8m^3) \) quark-propagator correction | \( i\mathbf{p}^4/(8m^3) \) |
| \( D^4/(8m^3) \) spatial-gluon vertex | \( -ig(\mathbf{p}^2 + \mathbf{p'}^2) (p + p') t_a/(8m^3) \) |
| \( D^4/(8m^3) \) spatial-gluon seagull vertex | \( ig^2 \left\{ \left[ (2p' - l_2) j (2p + l_1) i + (\mathbf{p}^2 + \mathbf{p'}^2) \delta_{ij} \right] t_b t_a + \text{perm} \right\}/(8m^3) \) |
| \( D^4/(8m^3) \) 3-spatial-gluon vertex | \( -ig^3 \left\{ \left[ (2p' - l_3) k \delta_{ij} + (2p + l_1) \delta_{kj} \right] t_c t_b t_a + \text{perm} \right\}/(8m^3) \) |
| \( D^4/(8m^3) \) 4-spatial-gluon vertex | \( ig^4 (\delta_{ij} \xi_{km} t_d t_c t_b t_a + \text{perm})/(8m^3) \) |
| Quantity | Description | Spin Triplet | Spin Singlet |
|----------|-------------|--------------|--------------|
| Correction to $\langle K^2_D \rangle$: | $\delta Z_{V,D^2}/\delta Z_{P,D^2}$ | $A_0$-$A_0$ vertex correction | $+8/3$ | $+8/3$ |
| Corrections to $\langle K \rangle$: | $\delta Z_{wf}$ | Quark wave-function renormalization | $+2/3$ | $+2/3$ |
| | $\delta Z_{V,a}/\delta Z_{P,a}$ | $A_0$-$A_0$ vertex correction | $-1/3$ | $-1/3$ |
| | $\delta Z_{V,b}/\delta Z_{P,b}$ | $A_0$-$A_0$ vertex correction with $D^4$ prop. insertion | $-2/3$ | $-2/3$ |
| | $\delta Z_{V,c}/\delta Z_{P,c}$ | $A_0$-$D \cdot E$ vertex correction | $+2/3$ | $+2/3$ |
| | $\delta Z_{V,d}/\delta Z_{P,d}$ | $\sigma \cdot B$-$\sigma \cdot B$ vertex correction | $+4/9$ | $-4/3$ |
| | $\delta Z_{V}/\delta Z_{P}$ | Total correction to $\langle K \rangle$ | $+7/9$ | $-1$ |

TABLE II. The $u = 1/2$-renormalon contributions in the mixing of the spin-triplet matrix and spin-singlet matrix elements $\langle K^2_D \rangle_V$ and $\langle K^2_D \rangle_P$ into the spin-triplet and spin-singlet matrix elements $\langle K \rangle_V$ and $\langle K \rangle_P$, respectively, and the $u = 1/2$-renormalon contributions in the multiplicative renormalization of the spin-triplet and spin-singlet matrix elements $\langle K \rangle_V$ and $\langle K \rangle_P$. The values displayed are the coefficients of $(1/\pi)(\mu^2/e)^u(1-2u)^{-1}$ times the matrix elements.