Finite Dimensional Schwinger Basis, Deformed Symmetries, Wigner Function, and an Algebraic Approach to Quantum Phase

T. Hakioğlu

Physics Department, Bilkent University, 06533 Ankara, Turkey

Abstract

Schwinger’s finite (D) dimensional periodic Hilbert Space representations are studied on the toroidal lattice with specific emphasis on the deformed oscillator subalgebras and the generalized representations of the Wigner function. These subalgebras are shown to be admissible endowed with the non-negative norm of Hilbert space vectors. Hence, they provide the desired canonical basis for the algebraic formulation of the quantum phase problem. Certain equivalence classes in the space of labels are identified within each subalgebra, and connections with area preserving canonical transformations are studied. The generalised representations of the Wigner function are examined in the finite dimensional cyclic Schwinger basis. These representations are shown to conform to all fundamental conditions of the generalized phase space Wigner distribution. As a specific application of the Schwinger basis, the number-phase unitary operator pair is studied and, based on the admissibility of the underlying q-oscillator subalgebra, an algebraic approach to the unitary quantum phase operator is established. Connections with the Susskind-Glogower-Carruthers-Nieto phase operator formalism as well as the standard action-angle Wigner function formalisms are examined in the infinite period limit. The concept of continuously shifted Fock basis is introduced to facilitate the Fock space representations of the Wigner function.
I. INTRODUCTION

Recently, finite dimensional quantum group symmetries find increasing physical applications in condensed matter systems. The Landau problem is known to have the $\omega_\infty$ symmetry in the algebra satisfied by magnetic translation operators. An $sl_q(2)$ realisation of the same problem has also been recently studied. The finite dimensional representations of these algebras are parameterised by a discrete set of labels on a two-dimensional toroidal lattice $\mathbb{Z}_D \times \mathbb{Z}_D$. The action of the group elements on the Hilbert space vectors is cyclic, the periodicity of which is determined by the dimension of the corresponding algebra. In the Landau problem the periodicity is directly connected with the degeneracy of the Landau levels in the ground state. In a more general framework, similar algebraic structures have been examined a long time ago by Schwinger in the unitary cyclic representations of the Weyl-Heisenberg (WH) algebra. Recently Floratos has examined the WH algebra parameterised by the labeling vectors on the toroidal lattice in terms of $D^2 - 1$ unitary traceless generators as a convenient representation of $su(D)$. More generally, the elements of the discrete and finite dimensional WH algebra are generators of the area preserving diffeomorphisms on $\mathbb{Z}_D \times \mathbb{Z}_D$ which are known to respect the Fairlie-Fletcher-Zachos sine algebra. The infinite dimensional extension of this is the group of infinitesimal area-preserving diffeomorphisms which has been examined by Arnold in the theory of phase space formulation of classical Hamiltonian flow. With the connection to area preserving diffeomorphisms, the finite dimensional WH algebra defines, in the quantum domain, the set of linear canonical transformations on the discrete canonical phase space pair, the generalised coordinate and the momentum. This has been observed as an emerging presymplectic structure preserving the discrete phase space area of which the connection with the classical symplectic structure is established in the continuous limit as the dimension of the algebra is extended to infinity. A more general frame for unitary cyclic reps of finite dimensional algebras, of which a special case is the WH algebra, is Schwinger’s finite, special-unitary-canonical basis. Schwinger’s approach proves to be a generalised realisation for the group of discrete area preserving
transformations on the 2-torus. This basis has been used indirectly in various applications to physics, particularly to condensed matter and field theory related problems such as the discretised versions of the Chern-Simons theory, the dynamics of Bloch electrons in two dimensions interacting with a constant uniform magnetic field, the quantum Hall effect etc. Most of these applications refer to the discrete WH algebra although the results can be equally valid using the more general Schwinger basis which will be discussed first in Sec.II.

In this work we will follow a different route than the standard applications above and demonstrate that the Schwinger basis also provides an algebraic approach to the canonical phase space formulation of the celebrated quantum phase problem. As the first step in this route, the subalgebraic realisations of Schwinger’s unitary operator basis will be constructed in Sec.III.A,B with a particular emphasis on the realisations in terms of the q-oscillator. It will be shown that in finite dimensions, these deformed oscillator realisations naturally lead to an admissible (i.e. non-negative) cyclic spectrum by the natural emergence of a positive Casimir operator. The net effect of the positive Casimir operator is to shift the spectrum to the admissible ranges, viz., a strong condition on the non-negative norm of the vectors in the Hilbert space. The crucial role played by the admissible cyclic representations in the canonical formulation of the quantum phase problem will be examined. In order to complete the picture, we also briefly discuss the well-known $u_q(sl(2))$ subalgebraic realisations of the Schwinger operator basis.

The equivalence classes and their connection to canonical transformations on the discrete lattice will be discussed in Sec.III.C. Sec.IV is devoted to the application of the Schwinger basis in the Wigner-Kirkwood construction of the Wigner function. It will be shown that this construction complies with all fundamental properties of the Wigner function. In Sec.V we explore the applications of Schwinger’s formalism in the unitary finite dimensional number-phase operator basis. In this context, we elaborate more on the q-oscillator subalgebraic realisations of Sec.III.B. In the first three subsections A,B,C of Sec.VI we examine the infinite dimensional limit of the number-phase basis, the q-oscillator subalgebra and the Wigner
function respectively. There, it will be shown that as the dimension of the unitary number-phase operator algebra is extended to infinity, the conventional phase operator formalism of Susskind-Glogower- Carruthers-Nieto is recovered. We consider this as the first step to establish the desired unification of the quantum phase problem with the canonical action-angle quantum phase space formalism. The admissible q-oscillator subalgebra is also investigated in the $D \to \infty$ limit and shown to have a linear equidistant spectrum accompanied by a typical spectral singularity at $D = \infty$. This singular behaviour is examined using Fujikawa’s index theorem.

The representations of the Wigner function in the phase and number eigenbases are investigated in the finite and infinite Hilbert space dimensions. Within this unification scheme the concept of continuously shifted finite dimensional Fock basis is introduced in Sec.V.D. It is suggested that this concept facilitates the formulation of the Wigner function in the Fock eigenbasis. In the following, we start our discussion with a short study of Schwinger’s cyclic unitary operator basis.

## II. FINITE DIMENSIONAL SCHWINGER OPERATOR BASIS

In this formulation, one considers a unitary cyclic operator $\hat{U}$ acting on a finite dimensional Hilbert space $\mathcal{H}_D$ spanned by a set of orthonormal basis vectors $\{|u\rangle_k\}_{k=0,...,(D-1)}$ with the cyclic property $\hat{U}^D = \mathbb{1}$ as

$$\hat{U} |u\rangle_k = |u\rangle_{k+1}, \quad |u\rangle_{k+D} = |u\rangle_k, \quad k \langle u | u' \rangle = \delta_{k,k'}.$$  \hspace{1cm} (1)

In the $\{|u\rangle_k\}$ basis, $\hat{U}$ is represented by

$$\hat{U} = \sum_{k=0}^{D-1} |u\rangle_{k+1} \langle u|.$$  \hspace{1cm} (2)

The action of $\hat{U}$ corresponds to a rotation in $\mathcal{H}_D$. The axis of rotation is along the direction in $\mathcal{H}_D$ given by the vector $|v\rangle_\ell$ of which the direction remains invariant under the action of $\hat{U}$ as
\[ \hat{U}|v\rangle_\ell = e^{i\gamma_0 \ell} |v\rangle_\ell , \quad |v\rangle_\ell = \frac{1}{\sqrt{D}} \sum_{k=0}^{D-1} v_k^{(\ell)} |u\rangle_k , \quad 0 \leq \ell \leq D - 1 \]  

where \( v_k^{(\ell)} = e^{-i\gamma_0 k \ell} \) and \( \gamma_0 = 2\pi/D \). On the other hand it was shown by Schwinger that the new set \{ \(|v\rangle_\ell \) \}_{\ell=0,...,(D-1)} also forms an orthonormal set of vectors i.e. \( \langle \ell' | v \rangle_\ell = \delta_{\ell,\ell'} \), for which one can define a second unitary operator \( \hat{V} \) such that \( \hat{V}^D = I \) and

\[
\hat{V}|v\rangle_\ell = |v\rangle_{\ell+1} , \quad |v\rangle_{\ell+D} = |v\rangle_\ell
\]

where \( k \in \mathbb{Z} \) and \( 0 \leq k \leq (D-1) \). The basis vectors \{ \(|u\rangle_k \) \}_{k=0,...,(D-1)} and \{ \(|v\rangle_\ell \) \}_{\ell=0,...,(D-1)} define two equivalent and **conjugate** representations in the sense that the representation in the \{ \(|u\rangle_k \) \}_{k=0,...,(D-1)} basis in Eq. (3) is complemented by

\[
|u\rangle_k = \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} u_\ell^{(k)} |v\rangle_\ell \quad \text{where} \quad u_\ell^{(k)} = e^{i\gamma_0 k \ell} = v_k^{(\ell)^*}.
\]

The corresponding operators \( \hat{U} \) and \( \hat{V} \) satisfy

\[
\hat{U}^{m_1} \hat{V}^{m_2} = e^{i\gamma_0 m_1 m_2} \hat{V}^{m_2} \hat{U}^{m_1} \quad \text{and} \quad \hat{U}^{m_1+D} = \hat{U}^{m_1} \quad \text{and} \quad \hat{V}^{m_2+D} = \hat{V}^{m_2}.
\]

An operator \( \Psi \), of which the projection in the \(|u\rangle_k \) representation is \( \Psi(u_k) \), is given in the \(|v\rangle_\ell \) representation as \( \tilde{\Psi}(v_\ell) \). These two conjugate representations are then connected by

\[
\Psi(u_k) = \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} k \langle u | v \rangle_\ell \tilde{\Psi}(v_\ell) \quad \text{where} \quad k \langle u | v \rangle_\ell = e^{-i\gamma_0 \ell k}.
\]

In analogy with the elements of the discrete Wigner-Kirkwood basis, we now define the operator

\[
\hat{S}_{\bar{m}} \equiv e^{-i\gamma_0 m_1 m_2/2} \hat{U}^{m_1} \hat{V}^{m_2} = e^{i\gamma_0 m_1 m_2/2} \hat{V}^{m_2} \hat{U}^{m_1},
\]

where \( \bar{m} = (m_1, m_2) \). We now represent the transformation in Eq’s (3) and (4) between \{ \(|v\rangle_\ell \) \}_{0 \leq \ell \leq (D-1)} and \{ \(|u\rangle_k \) \}_{0 \leq k \leq (D-1)} bases using the unitary Fourier operator \( \hat{F} \) defined as

\[
\{ |u\rangle_k \} \equiv \hat{F} \{ |v\rangle_\ell \} \quad \text{and} \quad \{ |u\rangle_k \} \equiv \hat{F}^{-1} \{ |v\rangle_\ell \}, \text{where} \quad \hat{F}^{-1} = \hat{F}^{-1}.
\]

Then

\[
|u\rangle_k \overset{\hat{F}}{\rightarrow} |v\rangle_k \overset{\hat{F}}{\rightarrow} |u\rangle_{-k} \overset{\hat{F}}{\rightarrow} |v\rangle_{-k} \overset{\hat{F}}{\rightarrow} |u\rangle_k ,
\]

\[
|u\rangle_k \overset{\hat{F}^{-1}}{\rightarrow} |v\rangle_{-k} \overset{\hat{F}^{-1}}{\rightarrow} |u\rangle_{-k} \overset{\hat{F}^{-1}}{\rightarrow} |v\rangle_k \overset{\hat{F}^{-1}}{\rightarrow} |u\rangle_k .
\]
The Eq’s (9) produce a Fourier automorphism at the operator level as

\[ \hat{U} \xrightarrow{\hat{F}} \hat{V} \xrightarrow{\hat{F}^{-1}} \hat{U}^{-1} \xrightarrow{\hat{F}} \hat{V}^{-1} \xrightarrow{\hat{F}} \hat{U} , \]  

Next, we define a transformation \( R_{\pi/2} \) in the space of the lattice vector \( \vec{m} \) such that \( R_{\pi/2} : (m_1, m_2) \rightarrow (-m_2, m_1) \). It is possible to show that

\[ \hat{F} \hat{S} \hat{m} \hat{F}^{-1} = \hat{S}_{R_{\pi/2} \vec{m}} , \quad \hat{F}^4 = I , \quad \text{and} \quad R_{\pi/2}^4 = I . \]  

Eq’s (11) imply that Eq. (8) is invariant under simultaneous operation \( \hat{F} \) and \( R_{\pi/2}^{-1} \). \( \hat{S}_m \) has the properties

\[ \hat{S}_m^\dagger = \hat{S}_{-\vec{m}} \]  

\[ Tr\{ \hat{S}_\vec{m} \} = D \delta_{\vec{m}, \vec{0}} \]  

\[ \hat{S}_\vec{m} \hat{S}_{\vec{m}'} = e^{i \gamma_{\vec{m} \times \vec{m}'}/2} \hat{S}_{\vec{m} + \vec{m}'} \]  

\[ (\hat{S}_\vec{m} \hat{S}_{\vec{m}''}) \hat{S}_{\vec{m}'''} = \hat{S}_\vec{m} (\hat{S}_{\vec{m}'} \hat{S}_{\vec{m}''}) \]  

\[ \hat{S}_{\vec{0}} = I \]  

\[ \hat{S}_\vec{m} \hat{S}_{-\vec{m}} = I \]  

where \( \vec{m} \times \vec{m}' = (m_1 m_2' - m_2 m_1') \). Using Eq’s (8) and (12) it is possible to see that

\[ (\hat{S}_\vec{m})^D = \hat{S}_{D\vec{m}} = \hat{S}_{-D\vec{m}} = (-1)^{Dm_1 m_2} I \]  

where \( \hat{S}_{D\vec{m}} \) commutes with all elements \( \hat{S}_{\vec{m}'} \) for all \( \vec{m} \) and \( \vec{m}' \). With the associativity condition in Eq’s (12) satisfied, the unitary Schwinger operator basis \( \hat{S}_\vec{m} \) defines a discrete projective representation of the Heisenberg algebra parameterised by the discrete phase space vector \( \vec{m} \) in \( \mathbb{Z}_D \times \mathbb{Z}_D \). Excluding \( \vec{m} = \vec{0} \) and if \( D \) is a prime number, the elements of the basis \( \hat{U}^{m_1} \hat{V}^{m_2} \) form a complete set of \( D^2 - 1 \) unitary traceless matrices providing an irreducible representation for \( su(D) \). If \( D \) is not a prime, then the prime decomposition of \( D \) as \( D = D_1 D_2 \ldots D_l \ldots \), as shown in Ref.s [4,13], permits the study of a physical system with a number of quantum degrees of freedom with each degree of freedom expressed in terms of an independent Schwinger basis with the cyclic property determined by the particular prime
factor $D_j$. In what follows, we will assume that $D$ is a prime number representing a single degree of freedom. Exceptional cases will be independently mentioned as needed.

The eigenspace of $\hat{S}_{\vec{m}}$ is spanned by the eigenvectors $|\vec{m}, r\rangle_{\{0 \leq r \leq (D - 1)\}}$ with eigenvalues $\lambda_r(\vec{m})$. Using Eq.'s (11) and (12) we expand the eigenvectors $|\vec{m}, r\rangle$ where $0 \leq r \leq (D - 1)$ in, for instance, the $|u\rangle_k$ basis with coefficients $e^{(r)}_k(\vec{m}) \equiv k\langle u|\vec{m}, r\rangle$. From this definition and Eq. (11) it is clear that the coefficients are periodic, i.e., $e^{(r)}_k(\vec{m}) = e^{(r)}_{k+D}(\vec{m})$. The coefficients and eigenvalues are then determined by the recursion

$$\lambda_r(\vec{m}) e^{(r)}_k(\vec{m}) = e^{-i\beta_k(m_1, m_2)} e^{(r)}_{k-m_1}(\vec{m}) , \quad \text{where } \beta_k(m_1, m_2) = \gamma_0 m_2 (2k - m_1)/2 . \quad (14)$$

which yields

$$\lambda_r(\vec{m}) = e^{i\pi m_1 m_2} e^{-i\frac{2\pi r}{D}} , \quad e^{(r)}_k(\vec{m}) = \left\{ \prod_{n=0}^{M-1} \lambda_r(\vec{m}) e^{-i\beta_{k-nm_1}(m_1, m_2)} \right\} e^{(r)}_{k+1}(\vec{m}) \quad (15)$$

where $M = [\text{mod}(D) + 1]/m_1$ and $M \in \mathbb{Z}$. In deriving Eq. (14) from (11) we used the periodicity property $e^{(r)}_k(\vec{m}) = e^{(r)}_{k+D}(\vec{m})$. It should be noted that the diagonal representations $|\vec{m}, r\rangle$ of $\hat{S}_{\vec{m}}$ in the $|u\rangle_k$ and $|v\rangle_k$ bases are equivalent and consistent with Eq.'s (11) and (12) only for the case when $D$ is a prime number. We will come back to Eq. (14) when we examine the q-oscillator subalgebraic realisations of the Schwinger basis in Sec. III.B. We now turn to the subalgebraic structure of the Schwinger basis.

**III. THE DEFORMED SUBALGEBRAIC STRUCTURE**

It is well-known that the $\hat{S}_{\vec{m}}$ basis has an explicit deformed algebraic structure. Defining the operators $\hat{D}_{\vec{m}} = D/2\pi \hat{S}_{\vec{m}}$, the commutator

$$[\hat{D}_{\vec{m}}, \hat{D}_{\vec{n}}] = i \frac{2}{\gamma_0} \sin(\frac{\gamma_0}{2} \vec{m} \times \vec{n}) \hat{D}_{\vec{m}+\vec{n}} , \quad (16)$$

describes the Fairlie-Fletcher-Zachos sine algebra. The generators of the algebra $\hat{J}_{\vec{m}}$ can be represented by the Weyl matrices.
with \( \hat{J}_m = \omega^{m_1m_2/2} g^{m_3} h^{m_2} \) satisfying \( h g = \omega g h, \ g^D = h^D = \mathbb{I}, \) with \( \omega^D = 1 \) and \( \omega = e^{i\gamma_0}. \)

With these at hand, it is possible to verify that \([\hat{J}_m, \hat{J}_n] = \frac{i}{2} 2/\gamma_0 \sin(\gamma_0/2 \vec{m} \times \vec{n}) \hat{J}_{\vec{m}+\vec{n}}.\)

The deformed \( u_q(sl(2)) \) subalgebraic realisations of the sine algebra have been under extensive investigation recently, based on the magnetic translation operator basis. In the following we will present a brief account of this symmetry in the more general Schwinger basis.

### A. The \( u_q(sl(2)) \) subalgebraic realisation

We define the operators \( \hat{A} \) and \( \hat{A}^\dagger \) as

\[
\hat{A} \equiv d \hat{S}_{\vec{m}} + d' \hat{S}_{\vec{m}'}, \quad \hat{A}^\dagger \equiv d^* \hat{S}_{-\vec{m}} + d'^* \hat{S}_{-\vec{m}'}
\]

where \( d \) and \( d' \) satisfy

\[
d d' = d^* d' = -(p^{1/2} - p^{-1/2})^{-2}, \quad p = e^{-i\gamma_0 \vec{m} \times \vec{m}'}.
\]

We find that

\[
\hat{A} \hat{S}_{\vec{m}-\vec{m}'} = p \hat{S}_{\vec{m}-\vec{m}'} \hat{A}, \quad \hat{A} \hat{S}_{\vec{m}-\vec{m}'}^\dagger = p^{-1} \hat{S}_{\vec{m}-\vec{m}'}^\dagger \hat{A}
\]

\[
\hat{S}_{\vec{m}-\vec{m}'} = s_p p^{j_3}, \quad \text{where} \quad \hat{A} \hat{J}_3 \equiv (\hat{J}_3 + 1) \hat{A}.
\]

and \( s_p = e^{-i\pi \vec{m} \times \vec{m}'} = p^{D/2}, \) such that Eq. (13) holds. It is also possible to realise in Eq’s (20) that \( \hat{S}_{\vec{m}} \hat{S}_{-\vec{m}'} = \tilde{s}_p p^{j_3} \) such that \( \tilde{s}_p = e^{-i\pi \gamma_0 (D-1) \vec{m} \times \vec{m}'} = p^{(D-1)/2}. \)

For both cases, a direct calculation yields

\[
[\hat{A}, \hat{A}^\dagger] = -\frac{p^{j_3+D/2} - p^{-j_3-D/2}}{p^{1/2} - p^{-1/2}} \equiv -[\hat{J}_3 + \frac{D}{2}],
\]
which together with Eq’s (20) implies an \( u_{p/2}(sl(2)) \) symmetry defined by the elements \( \hat{A}, \hat{A}^\dagger, \hat{J}_3 \). The Casimir operator for this subalgebra is given by

\[
C_p = \hat{A}^\dagger \hat{A} + \left[ \frac{1}{2}(\hat{J}_3 + \frac{D}{2} - \frac{1}{2}) \right]^2 = \hat{A}\hat{A}^\dagger + \left[ \frac{1}{2}(\hat{J}_3 + \frac{D}{2} + \frac{1}{2}) \right]^2
\]

(22)

where \([x]\) is formally given in Eq. (21). The Hilbert space is spanned by the vectors \(|j, j_3\rangle\) where \(\hat{J}_3 |j, j_3\rangle = j_3 |j, j_3\rangle\) with \(-j \leq j_3 \leq j\). If the lowest weight representations exist such that \(\hat{A} |j, -j\rangle \equiv 0\), then \(j\) is determined by the value of the Casimir operator as

\[
\hat{C}_p |j, -j\rangle = \left[ \frac{1}{2}(\vec{D}/2 - 1/2 - j) \right]^2 |j, -j\rangle.
\]

The lowest weight representations are obtained by successive operations of \(\hat{A}^\dagger\) on the state \(|j, -j\rangle\). These representations are \(D\) dimensional for the particular case \(j = (D - 1)/2\) such that \(\hat{A}^\dagger |j, -j + (D - 1)\rangle = \hat{A}^\dagger |j, j\rangle = 0\), where the highest and lowest weight reps coincide. In this case the representations are cyclic with period \(D\). For this case, the Casimir operator vanishes. We close this section by referring to the extensive applications of the \(u_q(sl(2))\) symmetry, for instance in Ref’s. [9–12] and move on to another subalgebraic realisation of the Schwinger basis.

### B. The spectrum shifted admissible q-oscillator realisation

Let’s now consider the \(\hat{A}\) and \(\hat{A}^\dagger\) operators in Eq. (18) where \(d\) and \(d'\) are constant to be redetermined for the q-oscillator realisation. Using Eq’s (12) we construct the q-commutator

\[
\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger \hat{A} = (|d|^2 + |d'|^2) (1 - q)
+ d d'^* (e^{-i\gamma_0 \vec{m} \times \vec{m}' - q}) \hat{S}_{-\vec{m}'} \hat{S}_{\vec{m}} + d' d^* (e^{i\gamma_0 \vec{m} \times \vec{m}' - q}) \hat{S}_{-\vec{m}} \hat{S}_{\vec{m}'}
\]

(23)

where \(\vec{m} \times \vec{m}' \neq (modD)\). Here, \(|q| = 1\) and is otherwise arbitrary at this level. Eq’s (23) can be written as

\[
\hat{A}\hat{A}^\dagger - q\hat{A}^\dagger \hat{A} = (|d|^2 + |d'|^2) (1 - q) + \hat{Q}, \quad \text{for} \quad q = e^{\pm i\gamma_0 \vec{m} \times \vec{m}'}
\]

(24)

where

\[
\hat{Q} = \begin{cases} 
  d d'^* (q^{-1} - q) \hat{S}_{-\vec{m}'} \hat{S}_{\vec{m}} , & \text{if} \quad q = e^{i\gamma_0 \vec{m} \times \vec{m}'} \\
  d' d^* (q^{-1} - q) \hat{S}_{-\vec{m}} \hat{S}_{\vec{m}'} , & \text{if} \quad q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}.
\end{cases}
\]

(25)
It can be shown that

\[ \hat{A} \hat{Q} = q^{-1} \hat{Q} \hat{A} , \quad q = e^{\pm i \gamma_0 \vec{m} \times \vec{m}'} \] (26)

which implies that a generalised number operator \( \hat{N} \) can be defined in such a way that

\[ \hat{A} \hat{N} \equiv (\hat{N} + 1) \hat{A} \quad \text{and} \quad \hat{Q} = c_q q^{-\hat{N}} , \]

where \( c_q \) is a proportionality constant whose value depends on the choice of \( d \) and \( d' \). Eq. (26) implies that \( \hat{A} \hat{D}, \hat{A}^\dagger \hat{D} \) commute with all elements of the algebra. Since the cases \( q \) and \( q^{-1} \) give rise to identical results as far as the algebra is concerned, we only examine the case \( q = e^{-i \gamma_0 \vec{m} \times \vec{m}'} \). In order to determine \( c_q \) we first make the choice

\[ dd'^* = \frac{1}{q^{-1} - q} = \frac{1}{2i \sin(\gamma_0 \vec{m} \times \vec{m}')} \quad \text{hence} \quad |d||d'| = \frac{1}{2|\sin(\gamma_0 \vec{m} \times \vec{m}')|} . \] (27)

The constants \( d, d' \) are also undetermined up to a constant overall phase factor. Choosing their magnitudes symmetrically we can determine the real positive shift constant \( C \) as

\[ C = |d|^2 + |d'|^2 = \frac{1}{|\sin(\gamma_0 \vec{m} \times \vec{m}')|} . \] (28)

The first one in Eq's (25) leads to the same result in (28). From Eq's (25) we have \( \hat{Q}^D = c_q^D q^{-D \hat{N}} \). Then, making use of \( q^D \equiv 1 \) and Eq's (12) and (13), we find that

\[ c_q = e^{i \gamma_0 (D-1) \vec{m} \times \vec{m}'}/2 = q^{-(D-1)/2} . \]

It can be seen that the net effect of the pure phase \( c_q \) is to shift the spectrum of \( \hat{N} \) by an overall constant \((D-1)/2\). Hence, \( \hat{Q} = q^{-\hat{N}-(D-1)/2} \).

With the generalised number operator as defined below Eq. (26), we have

\[ \hat{A} \hat{A}^\dagger - q \hat{A}^\dagger \hat{A} = C (1 - q) + q^{-\hat{N}-(D-1)/2} \]

\[ \hat{A} \hat{N} = (\hat{N} + 1) \hat{A} , \quad \hat{A}^\dagger \hat{N} = (\hat{N} - 1) \hat{A}^\dagger . \] (29)

Eq's (29) describe the q-oscillator algebra with its spectrum shifted by the positive constant \( C \) as

\[ \hat{A}^\dagger \hat{A} = C + [\hat{N}] , \quad \text{where} \quad [\hat{N}] = \frac{q^{\hat{N}+(D-1)/2} - q^{-\hat{N}-(D-1)/2}}{q - q^{-1}} \] (30)

where \( 0 \leq ||\hat{A}^\dagger \hat{A}|| \) as required, and, \( C \) is identified with the central invariant, which plays a crucial role in the existence of the admissible cyclic reps of the q-oscillator algebra endowed
with a positive spectrum. In Eq’s (29), the existence of the lowest (highest) weight vectors such that $\hat{A} |n_0\rangle = \hat{A}^\dagger |n_0 + D - 1\rangle = 0$ crucially depends on the specific values of $D$ and $\vec{m} \times \vec{m}'$. The condition for the existence of such $|n_0\rangle$ is given by $C = -[n_0]$. For $C$ as given by (28), it can be checked in Eq. (30) that this condition is violated for $D$ being an odd number. If $D$ is an even number, such reps are permitted for $\vec{m} \times \vec{m}' = D/(\text{mod}D)$, however in that case they are not irreducible. For $D$ being a prime other than two, the situation is the same as when $D$ is odd. We now examine how the q-oscillator algebra generators $\hat{A}, \hat{A}^\dagger$ and $\hat{N}$ act in the eigenspace of $\hat{S}_{\vec{m}}$ operators. We first observe that if $|\vec{m} - \vec{m}', r\rangle$ is an eigenstate of $\hat{S}_{\vec{m} - \vec{m}'}$ with eigenvalue $\lambda_r(\vec{m} - \vec{m}')$ for $0 \leq r \leq D - 1$,

\begin{equation}
\hat{S}_{\vec{m} - \vec{m}'} |\vec{m} - \vec{m}', r\rangle \equiv \lambda_r(\vec{m}, \vec{m}') |\vec{m} - \vec{m}', r\rangle
\end{equation}

where the second and third equations can be deduced from Eq’s (12). In the second and third equations, $g_r(\vec{m}, \vec{m}')$ and $f_r(\vec{m}, \vec{m}')$ are pure phase factors to be determined. Using Eq’s (18) we compare the action of $\hat{A}$ in the q-oscillator eigenbasis $|n\rangle$ and in the eigenbasis $|\vec{m}, r\rangle$ as

\begin{equation}
\hat{A} |\vec{m} - \vec{m}', r\rangle = (dg + d' f) |\vec{m} - \vec{m}', r - \vec{m} \times \vec{m}'\rangle
\end{equation}

where it is directly implied that a unit shift in $n$ corresponds to a shift of $r$ in units of $\vec{m} \times \vec{m}'$. Since $\vec{m} \times \vec{m}' \neq (\text{mod}D)$ by construction, the set of integers $n \vec{m} \times \vec{m}'$ for $0 \leq n \leq (D - 1)$ is the same as $n$ itself. Then, all eigenvectors in the q-oscillator and the Schwinger bases are connected on a one-to-one basis with successive operations of $\hat{A}$ and $\hat{A}^\dagger$. Since the eigenbasis $\{|\vec{m}, r\rangle\}_{0 \leq r \leq (D - 1)}$ is normalised, Eq’s (32) imply that

\begin{equation}
|\langle d \ g + \ d' \ f \rangle|^2 = C + [n].
\end{equation}

We then apply Eq’s (27) and (28) to obtain

\begin{equation}
\frac{|g|^2 + |f|^2 - i(g f^* - g^* f)}{2 |\sin(\gamma_0 \vec{m} \times \vec{m}')|} = \frac{|1 + \sin[\gamma_0 (n + (D - 1)/2) \vec{m} \times \vec{m}']|}{|\sin[\gamma_0 \vec{m} \times \vec{m}']|}.
\end{equation}
Since $|g| = |f| = 1$, Eq. (34) yields

$$g_r(\vec{m}, \vec{m}') = f_r^*(\vec{m}, \vec{m}') = e^{i\gamma_0 (n+(D-1)/2)\vec{m}\times\vec{m}'} ,$$

where it can be considered that $r = n \vec{m} \times \vec{m}'$. Comparing $\lambda_r(\vec{m})$ in Eq’s (15) with the first equation in (31) we find that $\lambda_r(\vec{m}, \vec{m}') = e^{i\gamma_0 (n-D/2)\vec{m}\times\vec{m}'}$. Eq’s (31–35) indicate that the admissible cyclic representations of the q-oscillator realisation for a fixed value of the deformation parameter $q \neq 1$ have one-to-one correspondence with the diagonal representations of the Schwinger basis for a fixed but arbitrary non-collinear vectors $\vec{m}, \vec{m}'$.

The admissible q-oscillator subalgebraic structure of the Schwinger basis has not been taking too much attention. The $su_q(2)$ realisations of two shifted and mutually commuting q-oscillators in the Schwinger boson representation has been studied by Fujikawa recently. It will be demonstrated in Sec.V that this particular realisation plays a crucial role in the canonical formulation of the quantum phase problem.

C. Equivalence Classes and Canonical Transformations on the Lattice

In both the q-oscillator and the $u_{p/2}(sl(2))$ cases examined here, there are sets of equivalence classes $E_{\vec{m}\times\vec{m}'}$ incorporating those sets of subalgebras parameterised by different lattice vector pairs $\vec{m}$ and $\vec{m}'$ such that the deformation parameter remains invariant under unitary transformations within each such class.

Let’s assume a transformation $R^q_{\vec{m},\vec{m}';\vec{m}^*,\vec{m}''}$ whose effective action is to map the pair $\vec{m}, \vec{m}'$ into a new one $\vec{m}^*, \vec{m}''$ in $\mathbb{Z}_D \times \mathbb{Z}_D$ as

$$R^p_{\vec{m},\vec{m}';\vec{m}^*,\vec{m}''} f(\vec{m}, \vec{m}') = f(\vec{m}^*, \vec{m}'')\ (36)$$

such that $\vec{m} \times \vec{m}' = \vec{m}^* \times \vec{m}''$, hence $\vec{m}, \vec{m}'; \vec{m}^*, \vec{m}'' \in E_{\vec{m}\times\vec{m}'}$. Here $f$ represents an arbitrary function. If $R^p_{\vec{m},\vec{m}';\vec{m}^*,\vec{m}''}$ is represented in Eq. (36) by the $2 \times 2$ integer matrix $R$, then the $R$ matrix satisfies

$$R^t \ P \ R = \ P , \quad \text{where} \quad P = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(37)
with $\det R = +1$. Here $R^t$ corresponds to the ordinary transpose of $R$. Eq. (37) implies that both $\vec{m}$ and $\vec{m}'$ are to be transformed by the same transformation

$$R f(\vec{m}, \vec{m}') = f(R \vec{m}, R \vec{m}') = f(\vec{m}^*, \vec{m}'^*),$$

and besides the unimodularity of $R$ there is no further restriction. Hence, $R$ is an element of $sl(2, \mathbb{Z}_D)$. The product $\vec{m} \times \vec{m}'$ corresponds to the exact cocycle in the Schwinger operator basis which is proportional to the discrete phase space area spanned by the vectors $\vec{m}, \vec{m}'$. Hence, $R$ plays the role of a class of area-preserving canonical transformations. As a result of the projective character of the Schwinger basis, any unitary transformation acting on the basis elements preserves the phase space area hence the symplectic structure as described by the matrix $P$ in Eq. (37). The action of $R$ on the lattice is then equivalent to the reflection of such unitary transformations in the operator space.

At this point, we find it necessary to mention briefly that there are implications of the equivalence classes in the construction of the generalised coproduct $\Delta(\cdot \otimes \cdot)$ for two deformed subalgebras parameterised by different lattice vectors. Let’s denote by $\hat{X}_1, \hat{X}_1^\dagger, \hat{H}_1$ and $\hat{X}_2, \hat{X}_2^\dagger, \hat{H}_2$ as the generators in two such algebras from the same equivalence class. It is possible to write for their tensor product algebra a generalised coproduct as

$$\Delta(\hat{X}^\otimes) = \hat{X}_1 \otimes p^{\hat{H}_2/2} + p^{-\hat{H}_1/2} \otimes \hat{X}_2,$$

$$\Delta(\hat{X}^{\dagger \otimes}) = \hat{X}_1^\dagger \otimes p^{\hat{H}_2/2} + p^{-\hat{H}_1/2} \otimes \hat{X}_2^\dagger,$$

$$\Delta(\hat{H}^\otimes) = \hat{H}_1 \otimes \mathbb{I} + \mathbb{I} \otimes \hat{H}_2,$$

where $\Delta(\hat{X}^\otimes), \Delta(\hat{X}^{\dagger \otimes}), \Delta(\hat{H}^\otimes)$ respect the same deformed algebra.

Keeping their labels on the lattice explicit, we now consider all operators $\hat{A}_{\vec{n}, \vec{n}'}, \hat{A}_{\vec{n}, \vec{n}'}^\dagger$ and $\hat{S}_{\vec{n}-\vec{n}'}$ on the translated lattice by $\vec{r}$ such that $\vec{n} = \vec{m} + \vec{r}$ and $\vec{n}' = \vec{m}' + \vec{r}$. The algebra on this translated lattice space is given by

$$\hat{A}_{\vec{m}+\vec{r}, \vec{m}'+\vec{r}} \hat{S}_{\vec{m}-\vec{m}'} = p' \hat{S}_{\vec{m}-\vec{m}'} \hat{A}_{\vec{m}+\vec{r}, \vec{m}'+\vec{r}}, \quad \hat{A}_{\vec{m}+\vec{r}, \vec{m}'+\vec{r}}^\dagger \hat{S}_{\vec{m}-\vec{m}'} = p'^{-1} \hat{S}_{\vec{m}-\vec{m}'} \hat{A}_{\vec{m}+\vec{r}, \vec{m}'+\vec{r}}^\dagger$$

$$\hat{S}_{\vec{m}-\vec{m}'} = s_{p'} p'^{1/3}, \quad p' = p e^{i \gamma_0 \delta \alpha}, \quad \delta \alpha = \vec{r} \times (\vec{m} - \vec{m}')$$

Here, $p$ is given in Eq. (13). The Eq’s (40) define the elements of $u_{p'^{1/2}}(sl(2))$ with a different deformation parameter $p'$. It is clear that translations on the lattice are not in the class of
area-preserving transformations defined above, and they cannot be realised by any unitary transformation on the Schwinger basis. Such transformations act as a bridge between the two projective representations characterised by two different cocycles. In our formalism here, this effectively corresponds to transforming the elements of those subalgebras belonging to one equivalence class $E_{\vec{m} \times \vec{m}'}$ into those of the other one $E_{\vec{n} \times \vec{n}'}$. In the example of Eq. (41), these two subalgebras are $u_{p'/2}(sl(2))$ and $u_{p/2}(sl(2))$ with deformations $p$ and $p' = p e^{i\gamma_0 \delta \alpha}$ respectively.

We now shift our attention to a more general structure of linear canonical transformations implicitly generated by $R$ on the lattice. The similarity transformation induced by the Fourier operator $\hat{F}$ in Eqs. (9–11) has been shown in Sec. II to effectively generate the simplest example of canonical transformations, i.e. a $\pi/2$ rotation on $\mathbb{Z}_D \times \mathbb{Z}_D$. Let’s now seek general canonical transformations on the lattice generated by an operator $\hat{G}$ such that

$$\hat{G} \hat{S}_{\vec{m}} \hat{G}^{-1} = \hat{S}_{R \vec{m}} = \hat{S}_{\vec{m}'}$$

where $R = \begin{pmatrix} s_1 & t_1 \\ s_2 & t_2 \end{pmatrix} \in sl(2, \mathbb{Z}_D)$ (41)

with $\vec{s} \times \vec{t} = \det R = 1$, where $\vec{s} = (s_1, s_2)$ and $\vec{t} = (t_1, t_2)$ are two vectors on $\mathbb{Z}_D \times \mathbb{Z}_D$. Such a transformation $\hat{G}$ can be given more explicitly in the $\hat{U}, \hat{V}$ basis by

$$\hat{G} \hat{U} \hat{G}^{-1} = \hat{S}_{\vec{s}} \quad \hat{G} \hat{V} \hat{G}^{-1} = \hat{S}_{\vec{t}}$$

(42)

Using Eq’s (42) and the results in Sec. II, the action of the $\hat{G}$ operator on the basis vectors $\{|u\rangle_k\}_{0 \leq k \leq (D-1)}$ and $\{|v\rangle_k\}_{0 \leq k \leq (D-1)}$ can be found to be

$$\hat{G} |v\rangle_k = |\vec{s}, k\rangle \quad \hat{G} |u\rangle_k = |\vec{t}, -k\rangle$$

(43)

where, similarly to the first one of Eq’s (31), $|\vec{s}, k\rangle$ and $|\vec{t}, -k\rangle$ are the eigenvectors of $\hat{S}_{\vec{s}}$ and $\hat{S}_{\vec{t}}$ with eigenvalue indices $k$ and $-k$ respectively. Hence $\hat{G}$ converts the vectors in the eigenbasis of $\hat{U}$ and $\hat{V}$ into those in $\hat{S}_{\vec{s}}$ and $\hat{S}_{\vec{t}}$ respectively. The similarity transformation in Eq’s (9–11) is a special case of the transformation in (41) and (42) for $\vec{s} = (0, 1)$ and $\vec{t} = (-1, 0)$. 

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IV. APPLICATIONS TO THE WIGNER-KIRKWOOD BASIS AND THE GENERALISED WIGNER FUNCTION

We now consider the discrete Wigner-Kirkwood operator basis $\hat{\Delta}(\vec{V})$ acting on the quantum phase space spanned by the vectors $\vec{V} = (V_1, V_2)$. Phase space representations in the Wigner-Kirkwood and the Schwinger bases are connected by the dual form

$$\hat{\Delta}(\vec{V}) = \frac{1}{D^2} \sum_{\vec{m}} e^{-i\gamma_0 (\vec{m} \times \vec{V})} \hat{S}_{\vec{m}}, \quad \hat{S}_{\vec{m}} = \int d\vec{V} e^{i\gamma_0 (\vec{m} \times \vec{V})} \hat{\Delta}(\vec{V})$$

(44)

where $\vec{m} \times \vec{V} = m_1 V_2 - m_2 V_1$ and the range of the integral over $\vec{V}$ is the entire 2-torus. Similar constructions in the discrete formalism have also been made, for instance in Ref’s. [8,13].

The Wigner function $W_\psi(\vec{V})$ is defined as the projection of $\Delta(\vec{V})$ in a physical state $|\psi\rangle$ as

$$W_\psi(\vec{V}) = \langle \psi | \hat{\Delta}(\vec{V}) | \psi \rangle.$$  

(45)

Any operator $\hat{F}(\hat{U}, \hat{V})$ with $\|\hat{F}\| < \infty$ can then be associated with a classical function $f(\vec{V})$ as

$$\frac{1}{D} \langle \psi | \hat{F} | \psi \rangle = \int d\vec{V} f(\vec{V}) W_\psi(\vec{V}), \quad f(\vec{V}) = Tr\{ \hat{F} \hat{\Delta}^\dagger(\vec{V}) \}; \quad d\vec{V} = dV_1 dV_2$$

(46)

Hereupon, the particular normalization we will use is based on $\int_0^{2\pi} dx e^{ixn} = 2\pi \delta_{n,0}$ and in the continuous limit $\lim_{D \to \infty} \sum_{n=0}^{D-1} e^{ixn} = 2\pi \delta(x)$.

Let’s now consider the action of $\hat{F}$ in Sec.II. The action of the Fourier operator $\hat{F}$ on the Wigner-Kirkwood basis can be found using Eq. (44) to be $\hat{F} \hat{\Delta}(\vec{V}) \hat{F}^{-1} = \hat{\Delta}(R_{\pi/2}^{-1} : \vec{V})$, where $R_{\pi/2}^{-1} : \vec{V} = (V_2, -V_1)$ with $R_{\pi/2}$ as given in Eq. (11). This is one of the simplest non-trivial canonical transformations corresponding to the rotation of the vector $\vec{V}$ by $\pi/2$ on the quantum phase space. As an extension of the finite transformations generated by the operator $\hat{F}$, one can find in Eq. (11) explicit unitary transformations generated by $\hat{G}$ of which the reflections on the quantum phase space are linear canonical ones on the quantum phase space observables.

The properties of a generalised phase-space Wigner function have been enlisted by Hillery et al.[21] under several fundamental conditions. Most of these conditions can be checked by
employing the appropriate canonical transformations $\hat{G}$ and the corresponding $R$. In the following we will check these conditions for Eq. (44) using the properties of the Schwinger basis.

i) The Wigner function is real: $W_{\psi}(\vec{V}) = W_{\psi}^*(\vec{V})$

Using the first equation in (12) it can easily be proven that $\Delta(\vec{V})$ is a self-adjoint operator. Hence, $W_{\psi}(\vec{V})$ is real.

ii) Integration over one phase space variable $V_i$ yields the marginal probability distribution of the physical state in the eigenbasis of the other variable $V_j$: $\int dV_i W_{\psi}(\vec{V}) = |\langle V_j |\psi \rangle|^2$

To prove this property using Eq. (44), perform the integral over $V_i$ to obtain $D\delta_{m_j,0}$. Then express $\hat{S}_{\vec{m} \cdot \delta_{m_j,0}}$ in the $\hat{U}, \hat{V}$ basis where only the $m_j$'th power of $\hat{U}$ or $\hat{V}$ appears. Write $\hat{U}$ or $\hat{V}$ raised to the power $m_j$ in terms of its eigenbasis using Eq. (1–3) or (4,5). Following the summation, perform the summation over the eigenvector index $k$ to obtain the proof. Note that this condition is true for any canonically transformed $\vec{V} = (V_1, V_2)$ such that $V_i \rightarrow (R : \vec{V})_i, V_j \rightarrow (R : \vec{V})_j$, which can be easily done using $\hat{G}$ and $R$ in Eq. (41).

iii) $W_{\psi}(\vec{V})$ should be covariant under Galilean translations on the lattice.

Since the phase space spanned by the vectors $\vec{m}$ is discrete, the translations are generated by the integer powers of $\hat{U}$ and $\hat{V}$ operators as $\hat{U}^{n_1} |u\rangle_k = |u\rangle_{k+n_1}$ and $\hat{V}^{n_2} |v\rangle_k = |v\rangle_{k+n_2}$. In the Galilean translated physical state $|\psi'\rangle = (\hat{U}^{-n_1} \hat{V}^{-n_2}) |\psi\rangle$, the Wigner function is given by

$$W_{\psi'}(\vec{V}) = \langle \psi | (\hat{U}^{-n_1} \hat{V}^{-n_2}) \Delta(\vec{V}) (\hat{U}^{n_1} \hat{V}^{n_2}) |\psi\rangle$$

(47)

where the upper and lower cases correspond to the translations performed independently in either the $|u\rangle_k$ or the $|v\rangle_k$ basis. Using the properties of the $\hat{U}$ and $\hat{V}$ operators as well as Eq's (12) it can be shown that

$$W_{\psi'}(\vec{V}) = W_{\psi}(\vec{V}')$$

where $\vec{V}' = (V_1 + n_1, V_2)$. (48)

Hence, Eq. (44) is covariant under Galilean translations on the lattice.

iv) $W_{\psi}(\vec{V})$ should be covariant under space and/or time inversions.
To prove this, we assume that the time inversion is defined by \((m_1, m_2) \xrightarrow{T^*} (m_1, -m_2)\) and the space inversion is given by \((m_1, m_2) \xrightarrow{P} (-m_1, -m_2)\). The time inversion is a \(\det T^* = -1\) type improper canonical transformation. Following a similar derivation in the time inverted, i.e. \(|\psi'\rangle = T^* |\psi\rangle\), or space inverted, i.e. \(|\psi'\rangle = \hat{P} |\psi\rangle\), physical state \(|\psi'\rangle\), it is possible to see that

\[
W_{T^*}\psi(\vec{V}) = W_{\psi}(\vec{V}''), \quad \vec{V}'' = (V_1, -V_2)
\]

\[
W_{\hat{P}}\psi(\vec{V}) = W_{\psi}(\vec{V}''), \quad \vec{V}' = (-V_1, -V_2)
\]

(49)

In particular we notice that the transformation corresponding to space inversion is identical to the successive operations of the Fourier operator in Eq. (11) twice, viz., \(\hat{P} = \hat{F}^2\).

v) If \(W_{\psi}(\vec{V})\) and \(W_{\psi'}(\vec{V})\) are two Wigner functions corresponding to the physical states \(|\psi\rangle\) and \(|\psi'\rangle\) respectively, then

\[
\int d\vec{V} W_{\psi}(\vec{V}) W_{\psi'}(\vec{V}) = \frac{1}{D} |\langle \psi | \psi' \rangle|^2.
\]

(50)

We present the proof starting from

\[
\int d\vec{V} W_{\psi}(\vec{V}) W_{\psi'}(\vec{V}) = \frac{1}{D^2} \sum_{\vec{m}, \vec{m}'} \int d\vec{V} e^{-i\gamma_0 (\vec{m} + \vec{m}') \cdot \vec{V}} \langle \psi | \hat{S}_{\vec{m}} | \psi \rangle \langle \psi' | \hat{S}_{\vec{m}'} | \psi' \rangle
\]

(51)

We then express \(|\psi\rangle\) and \(|\psi'\rangle\), for instance in the \(\{|u\rangle_k\}_{0 \leq k \leq (D-1)}\) basis as

\[
|\psi\rangle = \sum_k \psi_k |u\rangle_k, \quad |\psi'\rangle = \sum_k \psi'_k |u\rangle_k.
\]

(52)

The \(\vec{V}\) integral yields \(D^2 \delta_{\vec{m}, -\vec{m}'}\). Then, using \(\hat{S}_{\vec{m}} |u\rangle_k = e^{-i\gamma_0 / 2(2k+1)} |u\rangle_{k+m_2}\) we obtain the right hand side of Eq. (50).

vi) If \(\hat{Y}\) and \(\hat{Z}\) are two dynamical operators of \(\hat{U}\) and \(\hat{V}\), then

\[
\frac{1}{D} \text{Tr} \{\hat{Y} \hat{Z}\} = \int d\vec{V} y(\vec{V}) z(\vec{V})
\]

(53)

where \(y(\vec{V})\) and \(z(\vec{V})\) are classical functions on the phase space corresponding to \(\hat{Y}\) and \(\hat{Z}\).

The proof of this condition can be done using Eq. (16) and \(\text{Tr} \{\hat{S}_{\vec{m}}\} = D \delta_{\vec{m}, \vec{0}}\).

We thus suggest that the realisations of the generalised Wigner-Kirkwood basis in terms of the elements of the Schwinger basis as expressed in Eq. (14) satisfies all fundamental conditions to represent the Wigner function in a more generalised form.
The connection between the unitary transformations in the Schwinger basis and canonical area preserving ones on the quantum phase space have been intensively studied recently. We refer to Ref. [8] for a detailed analysis of this connection. The Wigner function on $\mathbb{Z}_D \times \mathbb{Z}_D$ has been examined by Wooters and applications to action-angle case and the problems therein have been recently studied in detail by Bizzaro and Vaccaro.

The discrete Wigner function we examined in this section is based on the particular normalization adopted in Eq’s (44), (i.e. the $1/D^2$ factor in the first equation). Using a different normalization, it is also possible to examine the case where one of the two (or both) continuous phase space variables $\vec{V} = (V_1, V_2)$ is (are) replaced by the discrete ones. The former is more convenient in the case where the canonical variables correspond to the action-angle pair, whereas the latter should be used when the discrete phase space variables are considered on equal footing (i.e. canonical linear discrete coordinate and momentum). It should be noted that in Sec. V and VI we will use the normalization adopted for the action-angle variables and replace the $1/D^2$ factor in Eq’s (44) by $1/(2\pi D)$ in order to obtain the conventional action-angle Wigner function in the continuous limit.

V. APPLICATIONS TO THE UNITARY NUMBER-PHASE BASIS AND CONNECTION WITH THE QUANTUM PHASE PROBLEM

It is known that a finite dimensional admissible cyclic algebra

$$\hat{a} |n\rangle = f(n)^{1/2} |n - 1\rangle, \quad n \neq 0$$
$$\hat{a}^\dagger |n\rangle = f(n + 1)^{1/2} |n + 1\rangle, \quad n \neq (D - 1)$$
$$\hat{a} |0\rangle = f(0)^{1/2} \beta |D - 1\rangle, \quad |\beta| = 1; |D\rangle \equiv |0\rangle$$
$$\hat{a}^\dagger |D - 1\rangle = f(D)^{1/2} \beta^* |0\rangle,$$

$$0 \leq f(n) , \quad n \in \mathbb{Z} \ (modD)$$

provides a well-defined algebraic basis for the quantum phase operator. Here $\hat{a}$ and $\hat{a}^\dagger$ are spectrum lowering and raising operators and $f(n)$ is a generalised spectrum with the
cyclic property that $f(n + D) = f(n)$. The admissibility condition is enforced by the last equation in (54).

The unitary phase operator $\hat{\mathcal{E}}_\phi$ is given in the generalised cyclic number basis by

$$\hat{\mathcal{E}}_\phi = \sum_{n=0}^{D-1} |n - 1\rangle \langle n|, \quad |n + D\rangle \equiv |n\rangle \quad \text{for all} \quad n$$

(55)

where the discrete phase eigenvalues and eigenstates are

$$\hat{\mathcal{E}}_\phi |\phi\rangle_\ell = e^{i\gamma_0 \ell} |\phi\rangle_\ell \quad |\phi\rangle_\ell = \frac{1}{\sqrt{D}} \sum_{n=0}^{D-1} e^{i\gamma_0 n\ell} |n\rangle, \quad |\phi\rangle_{\ell + D} \equiv |\phi\rangle_{\ell}$$

(56)

with $0 \leq \ell \leq D - 1$. The phase eigenbasis is orthonormal and resolves the identity as

$$\ell \langle \phi | \phi\rangle_\ell = \delta_{\ell, \ell} \quad \text{and} \quad I = \sum_{\ell=0}^{D-1} |\phi\rangle_\ell \langle \phi | \ell \right.$$  

(57)

We now define the unitary operator $\hat{\mathcal{E}}_N = e^{-i\gamma_0 \hat{N}}$ with $\hat{N}$ describing the number operator such that $\hat{N} |n\rangle = n |n\rangle$. Then, $\hat{\mathcal{E}}_N = e^{-i\gamma_0 \hat{N}}$ has

$$\hat{\mathcal{E}}_N |\phi\rangle_\ell = |\phi\rangle_{\ell - 1}, \quad \hat{\mathcal{E}}_N |n\rangle = e^{-i\gamma_0 n} |n\rangle \quad \text{where} \quad |n\rangle = \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} e^{-i\gamma_0 n \ell} |\phi\rangle_\ell$$

(58)

The properties of the unitary phase and number operators $\hat{\mathcal{E}}_\phi$ and $\hat{\mathcal{E}}_N$ have been recently studied from this algebraic point of view. Here, in addition to these properties, they also establish a particular application of Schwinger’s operator basis. Among the four equivalent choices in Eq’s (10), we examine the particular case

$$\left(\hat{U} \atop \hat{\mathcal{V}}\right) \Rightarrow \left(\hat{\mathcal{E}}_N \atop \hat{\mathcal{E}}_\phi\right).$$

(59)

Using this, and following (8), we construct the operators $\hat{S}_{\vec{m}}$ in the number-phase basis as

$$\hat{S}_{\vec{m}} \equiv e^{-i1/2 \gamma_0 m_1 m_2} \hat{\mathcal{E}}_{N}^{m_1} \hat{\mathcal{E}}_\phi^{m_2}, \quad \text{where} \quad \hat{\mathcal{E}}_{N}^{m_1} \hat{\mathcal{E}}_\phi^{m_2} = e^{i\gamma_0 m_1 m_2} \hat{\mathcal{E}}_\phi^{m_2} \hat{\mathcal{E}}_{N}^{m_1}.$$

(60)

All properties of the cyclic Schwinger unitary operator basis studied in Sec. II and III are satisfied in the unitary number-phase basis. In addition to these properties, a strong limitation exists on the admissibility of the representations in $\mathcal{H}_D$ to make the mapping in (59) an acceptable one.
The q-oscillator algebra in Sec.III.A defined by the elements $\hat{A}, \hat{A}^\dagger$, and $\hat{N}$ for a fixed $\vec{m}$ and $\vec{m}'$ with 
$q = e^{\pm i\gamma_0 \vec{m} \times \vec{m}'}$ and $\gamma_0 = 2\pi/D$ is an admissible cyclic algebra which provides a natural realisation of Eq’s (54) with $\hat{a} \rightarrow \hat{A}$, $\hat{a}^\dagger \rightarrow \hat{A}^\dagger$, and $\hat{N}$. In this case, the admissible algebra in Eq’s (54) is given by the shifted q-oscillator algebra in Eq’s (29) where 

$$f(n) \rightarrow [n] + C = \frac{q^{n+(D-1)/2} - q^{-n-(D-1)/2}}{q - q^{-1}} + C,$$

$$C = \frac{1}{|\sin(\gamma_0 \vec{m} \times \vec{m}')|} \neq 0. \quad (61)$$

Now, let’s consider a real cyclic operator $F(\hat{N})$ with $0 \leq \|F(\hat{N})\|$ such that $F(\hat{N}) = F(\hat{N} + D)$ of which the eigenvalues in the number basis $\{|n\rangle\}_{0 \leq n \leq (D-1)}$ are given by $f(n)$. We consider the expansion of $F(\hat{N})$ as

$$F(\hat{N}) = \frac{1}{D} \sum_{k=0}^{D-1} \tilde{f}_k \cdot q^{-\hat{N}k}, \quad q = e^{-i\gamma_0 \vec{m} \times \vec{m}'} . \quad (62)$$

The sets of integers $\{k \vec{m} \times \vec{m}'; \vec{m} \times \vec{m}' \neq (modD)\}_{0 \leq k \leq (D-1)}$ and $\{k\}_{0 \leq k \leq (D-1)}$ are equivalent for any $\vec{m}, \vec{m}'$. Thus, Eq’s (62) is nothing but the operator Fourier expansion of $F(\hat{N})$. Using Eq. (25), and the fact that $\vec{m}$ and $\vec{m}'$ are not to be collinear, the operator $q^{-\hat{N}}$ can be realised as the third element $c_q^{-1} \hat{S}_{-\vec{m}'} \hat{S}_{\vec{m}}$ of the q-oscillator subalgebra. Hence, Eq. (62) can be equivalently written as

$$F(\hat{N}) = \frac{1}{D} \sum_{k=0}^{D-1} \tilde{f}_k \cdot \hat{S}_{k(\vec{m} - \vec{m}')} , \quad \tilde{f}_k = Tr\{ \hat{S}_{k(\vec{m} - \vec{m}')}^\dagger F(\hat{N}) \} = \sum_{n=0}^{D-1} e^{i\gamma_0 kn} f(n) \quad (63)$$

where we redefined $\tilde{f}_k$ as $\tilde{f}_k \rightarrow \tilde{f}_k c_q^{-1} q^{1/2} = \tilde{f}_k q^{D/2}$. Since the vectors $\vec{m}, \vec{m}'$ are fixed but indetermined, Eq. (63) is the expansion of $F(\hat{N})$ in an arbitrary but fixed q-oscillator subalgebra based on a fixed $\vec{m}$ and $\vec{m}'$ of the Schwinger basis with the deformation parameter $q = e^{-i\gamma_0 \vec{m} \times \vec{m}'}$.

As a specific application of Sec.IV, and making use of the correspondence in (59), we construct the Schwinger realisation of the discrete Wigner-Kirkwood operator basis in the number-phase space as

$$\hat{\Delta}(J, \theta) = \frac{1}{2\pi D} \sum_{\vec{m}} e^{i(\gamma_0 m_1 J - m_2 \theta)} e^{-i1/2 \gamma_0 m_1 m_2} \hat{E}^{m_1}_{\vec{N}} \hat{E}^{m_2}_{\phi} , \quad (64)$$
where we used the particular $1/(2\pi D)$ normalization to examine the action-angle Wigner function and $J, \theta$ are introduced as the generalised action-angle variables as a physical realisation of the phase space vector $\vec{V} \rightarrow (\theta/\gamma_0, J)$ in Eq’s (44). The change in the normalization factor from Eq. (44) to Eq. (64) is then simply the Jacobian of the transformation $d\vec{V} \rightarrow dI d\theta$. The Wigner-Kirkwood basis $\hat{\Delta}(J, \theta)$ has the cyclic property that $\hat{\Delta}(J, \theta) = \hat{\Delta}(J(mod D), \theta(mod 2\pi))$. Let’s now insert the identity operator in (57) on both sides of the basis operators in (64). Using Eq’s (56) and (58) repeatedly $m_2$ and $m_1$ times, Eq. (64) becomes

$$\hat{\Delta}(J, \theta) = \frac{1}{2\pi D} \sum_{\vec{m}} \sum_{\ell=0}^{D-1} e^{i(\gamma_0 m_1 J-m_2 \theta)} e^{i\gamma_0 \ell m_2} e^{i\gamma_0 m_1 m_2 / 2} |\phi\rangle_{\ell+\ell m_1} \langle \phi|.$$

(65)

The action-angle Wigner function in any particular finite dimensional Hilbert space state $|\psi\rangle$ is then given as in (44) by

$$W_\psi(J, \theta) = \langle \psi | \Delta(J, \theta) | \psi \rangle,$$

(66)

with all required conditions for the generalised Wigner function satisfied. In Sec.VI.C we will examine the continuous limit of Eq. (66) as $D \rightarrow \infty$.

VI. THE LIMIT TO CONTINUUM

The large $D$ limit of the sine algebra has been extensively studied initially, for instance in Ref.s [5,6], and later by many other workers. We will not present these results here. We will also consider the $D \rightarrow \infty$ limit with the condition that $D$ remains a prime number.

A. The Number-Phase Basis

In the limit $D \rightarrow \infty$ the spectra of $\hat{U}$ and $\hat{V}$ become arbitrarily dense and approach a continuously uniform distribution on the unit circle. Hence, for both unitary operators, the strong convergence is clearly guaranteed from those with discrete spectra to those with
In particular, the continuous limits of $\hat{E}_\phi$ and $\hat{E}_N$ will be identified as

$$
\begin{align*}
\lim_{D \to \infty} \hat{E}_N^{m_1} & \to \hat{E}_N^\gamma \equiv e^{-i\gamma N}, \quad \text{where} \quad \gamma \equiv \lim_{D \to \infty} \frac{2\pi m_1}{D} \in \mathbb{R} \\
\lim_{D \to \infty} \hat{E}_\phi^{m_2} & \equiv \hat{E}_\phi^{m_2} \quad 0 \leq m_2 < \infty, \quad m_2 \in \mathbb{Z}.
\end{align*}
$$

(67)

where $\hat{E}_N$ and $\hat{E}_\phi$ are now corresponding unitary operators with continuous spectra. On the other hand, in the limit to continuity we must restrict the physical states that $\hat{E}_\phi$ and $\hat{E}_N$ act upon to those everywhere differentiable and continuous functions in the infinite dimensional Hilbert space. For all such acceptable states $|\psi\rangle$, the condition for weak convergence

$$
\lim_{D \to \infty} \| (\hat{E}_N^{m_1} - \hat{E}_N^{\gamma}) |\psi\rangle \|^2 < \epsilon \quad \text{where} \quad \epsilon < 0^+ \text{ (arbitrarily small)},
$$

(68)

and similarly for $\hat{E}_\phi$, must be respected. In particular, it was shown in Ref. [25] that the eigenstates of $\hat{E}_N$ and $\hat{E}_\phi$ are good examples of such $|\psi\rangle$ and the convergence in (68) in the limit $D \to \infty$ is known to exist. Considering the $D \to \infty$ limit of Eq. (56) and (58), the eigenstates of $\hat{E}_N$ and $\hat{E}_\phi$ are

$$
|n\rangle = \int \frac{d\phi}{\sqrt{2\pi}} e^{-i\phi n} |\phi\rangle, \quad |\phi\rangle = \frac{1}{\sqrt{2\pi}} \lim_{D \to \infty} D^{-1/2} \sum_{n=0}^{D-1} e^{i\phi n} |n\rangle,
$$

(69)

where we have defined

$$
\lim_{D \to \infty} |n\rangle = |n\rangle, \quad 0 \leq n < \infty \\
\lim_{D \to \infty} \frac{1}{\sqrt{D}} |\phi\rangle = |\phi\rangle, \quad \phi = \lim_{D \to \infty} \frac{2\pi \ell}{D} \in \mathbb{R}, \quad \text{and} \quad 0 \leq \phi < 2\pi,
$$

(70)

with the proper normalizations $\langle \phi' | \phi \rangle = \delta(\phi - \phi')$ and $\langle n' | n \rangle = \delta_{n',n}$. Remember that the periodic boundary conditions are still valid in the limit (i.e. $|\phi\rangle \equiv |\phi + 2\pi\rangle$ and $|n\rangle = \lim_{D \to \infty} |n + D\rangle$). For a generally acceptable state $|\psi\rangle = \sum_{\ell=0}^{D-1} \psi_\ell |\phi\rangle_\ell$ with $\| |\psi\rangle \| = 1$, a similar weak convergence condition as in (68) stated for the phase operator requires

$$
\lim_{D \to \infty} \| (\hat{E}_\phi - \hat{E}_\phi) |\psi\rangle \|^2 = \lim_{D \to \infty} \sum_{\ell=0}^{D-1} |\psi_\ell (e^{i\gamma_0 \ell} - e^{i\phi}) |^2 < \epsilon.
$$

(71)

Since $|\psi_\ell| \leq 1$, and the convergence

$$
\lim_{D \to \infty} \| (\hat{E}_\phi - \hat{E}_\phi) |\psi\rangle \|^2 = \lim_{D \to \infty} \sup \left\{ | (e^{i\gamma_0 \ell} - e^{i\phi}) |^2 : 0 \leq \ell < (D-1) \right\} < \epsilon
$$

(72)
is guaranteed because of Eq’s (69) and (70), the only condition for the existence for such acceptable states is that in the limit $D \to \infty$, the wavefunction $\psi_\ell$ is sufficiently well behaved and everywhere differentiable. Once the weak convergence condition in Eq. (68) is satisfied for an acceptable state $|\psi\rangle$ expressed in one basis (i.e. in $|n\rangle$ or $|\phi\rangle$), the weak convergence in the other basis is guaranteed by the Eq’s (69).

The actions of the operators in (67) on the infinite dimensional Hilbert space spanned by the vectors in (69) are therefore

$$
\begin{align*}
\hat{\mathcal{E}}^\gamma_N |\phi\rangle &= |\phi - \gamma\rangle , & \hat{\mathcal{E}}^\gamma_N |n\rangle &= e^{-i\gamma n} |n\rangle \\
\hat{\mathcal{E}}^\ell_\phi |n\rangle &= |n - \ell\rangle , & \hat{\mathcal{E}}^\ell_\phi |\phi\rangle &= e^{i\ell \phi} |\phi\rangle .
\end{align*}
$$

(73)

In this continuous limit, Eq. (61) implies that

$$
e^{-i\gamma \hat{N}} \hat{\mathcal{E}}^\ell_\phi = e^{i\ell \phi} \hat{\mathcal{E}}^\ell_\phi e^{-i\gamma \hat{N}} .
$$

(74)

Differentiating (74) with respect to $\gamma$ and considering the limit $\gamma \to 0$ we find that

$$
[\hat{N}, \hat{\mathcal{E}}^\ell_\phi] = -\ell \hat{\mathcal{E}}^\ell_\phi
$$

(75)

which is the Susskind-Glogower-Carruthers-Nieto phase-number commutation relation with $\hat{\mathcal{E}}_\phi$ describing the unitary phase operator with a continuous spectrum as given in (73).

The expansion of Eq. (74) for all orders in $\gamma$ is consistent with the first order term described in Eq. (75). The coefficient of the $O(\gamma^r)$ term reproduces the $r$th order commutation relations between $\hat{N}$ and $\hat{\mathcal{E}}^\ell_\phi$ as $[\hat{N}, [\hat{N}, \ldots, [\hat{N}, \hat{\mathcal{E}}^\ell_\phi]]] \ldots] = (-\ell)^r \hat{\mathcal{E}}^\ell_\phi$. In this respect, Eq. (74) or, more generally, its discrete version in Eq. (3) should be treated as generalized canonical commutation relations.

B. The spectrum shifted q-oscillator

To study the $D \to \infty$ limit of the q-oscillator we first consider, in the numerator of $[n]$ in (61), the equivalence of the sets of integers $\{n \; \overline{m} \times \overline{m}'; \; \overline{m} \times \overline{m}' (mod D)\}_{0 \leq n \leq (D-1)}$ and $\{n\}_{0 \leq n \leq (D-1)}$ for any $\overline{m}, \overline{m}'$. If $\overline{m} \times \overline{m}' \neq 1$, this equivalence amounts to folding the value of
$n\vec{m} \times \vec{m}'$ into the first Brillouin zone $n$ for $0 \leq n \leq (D - 1)$. In the limit, the spectrum is given by

$$f(n) = \lim_{D \to \infty} \frac{1 \pm \sin(\gamma_0 n)}{|\sin(\gamma_0 \vec{m} \times \vec{m}')|}.$$  

(76)

Depending on $\vec{m} \times \vec{m}'$, the sine term in the numerator takes continuous values in the range $[0, 1)$ for $0 \leq n \leq (D - 1)$. Two limiting cases can be identified depending on the basis vectors $\vec{m}, \vec{m}'$ by

$$f(n) = \begin{cases} \lim_{D \to \infty} [1/\gamma_0 \pm n], & \text{if } \vec{m} \times \vec{m}' = 1, \\ \lim_{D \to \infty} [1 \pm \gamma_0 n], & \text{if } \vec{m} \times \vec{m}' = (D - 1)/4 \in \mathbb{Z}. \end{cases}$$  

(77)

The first case is identical to the continuous limit considered by Fujikawa. The spectrum is linear and unbounded, and the admissibility condition implies an unbounded positive shift by $\lim_{D \to \infty} 1/\gamma_0$. This is somewhat an infinitely shifted harmonic oscillator spectrum. Whereas, in the second case in (77), one obtains a continuous, finite, and linear spectrum. The limit $D \to \infty$ has other interesting features. Fujikawa has shown that the vanishing of the index

$$I = \sum_{n=0}^{D-1} \left\{ e^{-f(n)} - e^{-f(n+1)} \right\}$$  

(78)

is a stringent condition for the existence of the unitary phase operator. Using this index condition for the general admissible algebra in (54), it was previously shown that the limit $D \to \infty$ has a singular behaviour in the spectrum at $D = \infty$. This typical transition to a singular behaviour is also visible here if we compare the two indexes in (78) once calculated using Eq.(76) and then (77). The former correctly yields $I = 0$, whereas for the latter $I \neq 0$. Hence, in transition from (76) to (77), the vanishing index condition is violated. This proves that the spectrum as expressed in (77) is not admissible at the limit $D = \infty$. The admissible form of (77) is given by

$$f(n) = \begin{cases} \lim_{D \to \infty} [1/\gamma_0 \pm \sin(\gamma_0 n)], & \text{if } \vec{m} \times \vec{m}' = 1, \\ \lim_{D \to \infty} [1 \pm \sin(\gamma_0 n)], & \text{if } \vec{m} \times \vec{m}' = (D - 1)/4 \in \mathbb{Z} \end{cases}$$  

(79)

so that the vanishing index condition is respected. Thus, we learn that the vanishing index requires the information on the cyclic properties of the algebra to be maintained for all $D$. 


including the transition to infinity. For a more general consideration of the index theorem, we refer to Ref. [28]. Before closing this subsection, we mention as a side remark that the second limiting case in (79) is somewhat similar to tight binding energy spectra in certain condensed matter systems.

C. The Wigner Function in the Phase Eigenbasis

Let’s define in (64) the variables \( \phi = \lim_{D \to \infty} \phi_{\ell} \), \( \phi + \gamma = \lim_{D \to \infty} \phi_{\ell + m_1} \) with \( \phi, \gamma \in \mathbb{R} \) as well as \( |\phi\rangle = \lim_{D \to \infty} \gamma_0^{-1/2} |\phi\rangle_{\ell} \) in accordance with Eq’s (67) and (70). Since \( \phi, \gamma \) are continuous, we can replace the summation over \( m_1 \), in the limit, by an integral over \( \gamma \) such that \( \lim_{D \to \infty} 1/D \sum_{m} \to \lim_{D \to \infty} \sum_{m=0}^{D-1} \int \frac{d\gamma}{2\pi} \). Combining everything, we find for the Wigner function in this limit

\[
W_{\psi}(J, \theta) = \int \frac{d\gamma}{2\pi} e^{iJ\gamma} \langle \psi|\theta - \gamma/2 \rangle \langle \theta + \gamma/2|\psi \rangle
\]

which is the conventional action-angle Wigner function represented in the continuous phase basis. Recently, a similar construction of the continuous Wigner function based on the continuous Weyl-Heisenberg basis was suggested in Ref. [13] as well as in Ref. [29] in very close correspondence with the results obtained here. Eq. (80) can be realised as the action angle analog of Ref. [29]. If one starts in the generalised dual form represented by Eq. (44) with the symmetric normalization, the discrete Weyl-Heisenberg representation of \( \Delta(\tilde{V}) \) is obtained which leads in the continuous limit to Wolf’s Wigner function formulation in Ref. [29]. The continuous Weyl-Heisenberg representation as the standard representation of the Wigner function has also been examined by Schwinger as well as in Ref’s. [8].

D. The continuously Shifted Finite Dimensional Fock Spaces and the Wigner Function in the generalised Fock Representation

Let’s consider the cyclic algebra in (54) with the unitary phase and number operators as defined in Eq’s (59) and (58). We consider the phase operator \( \hat{E}_{\phi}^{\alpha} \) in \( \mathcal{H}_D \) as
\[
\hat{\mathcal{E}}^{-\alpha}_\phi |\phi\rangle_\ell = e^{-i\gamma_0 \ell \alpha} |\phi\rangle_\ell , \quad \text{and} \quad \hat{\mathcal{E}}^{-\alpha}_\phi |n\rangle \equiv |n + \alpha\rangle
\] (81)

where \(\alpha \in \mathbb{R}[0,1)\) and \(|n + \alpha\rangle\) is defined by

\[
|n + \alpha\rangle \equiv \frac{1}{\sqrt{D}} \sum_{\ell=0}^{D-1} e^{-i\gamma_0 (n+\alpha)\ell} |\phi\rangle_\ell .
\] (82)

Since \(\alpha \in \mathbb{R}[0,1)\), the states \(|n + \alpha\rangle\) do not belong to the set of vectors spanning the finite dimensional conventional Fock space \(\mathcal{F}_D\). We now define a continuously shifted finite dimensional Fock space \(\mathcal{F}^{(\alpha)}_D\) where \(|n + \alpha\rangle\) is defined by

\[
\langle n + \alpha|n' + \alpha\rangle = \delta_{n,n'} , \quad \sum_{n=0}^{D-1} |n + \alpha\rangle \langle n + \alpha| = \mathbb{I} .
\] (83)

This implies that for a fixed \(\alpha \in \mathbb{R}\), the shifted Fock space \(\mathcal{F}^{(\alpha)}_D\) is also spanned by a complete orthonormal set of vectors \(|n + \alpha\rangle\) and it can equivalently be used in the generalised Fock representation of a physical state. The overlap between \(\mathcal{F}_D\) and \(\mathcal{F}^{(\alpha)}_D\) clearly respects the condition \(|\langle n|n + \alpha\rangle| \leq 1\) for all \(\alpha \in \mathbb{R}\), and the extreme limits of \(\alpha \to 0\) and \(D \to \infty\) are commutative and well behaved:

\[
|\langle n|n + \alpha\rangle| = \begin{cases} 
\sin \pi \alpha / (\pi \alpha) , & \text{if} \quad D \to \infty , \quad 0 \leq \alpha \leq 1 \\
1 - (1 - 1/D) (\pi \alpha)^2 / 3! , & \text{if} \quad D < \infty , \quad \alpha \to 0 
\end{cases}
\] (84)

Since \(\mathcal{F}_D\) and \(\mathcal{F}^{(\alpha)}_D\) are spanned by cyclic vectors, \(\alpha = 1\) and \(\alpha = 0\) correspond to the identical Fock space representations. The action of the operator \(\hat{\mathcal{E}}^{(\beta)}_\phi\) on the vectors in \(\mathcal{F}^{(\alpha)}_D\) is, therefore, equivalent to a continuous shift of the origin in \(\mathcal{F}^{\alpha}_D\) by \(\beta \in \mathbb{R}\) such that

\[
\hat{\mathcal{E}}^{(\beta)}_\phi : \mathcal{F}^{(\alpha)}_D \to \mathcal{F}^{(\alpha-\beta)}_D .
\] (85)

Hence, a continuous shift \(\beta\) induced by the operator \(\hat{\mathcal{E}}^{(\beta)}_\phi\) is effectively equivalent to carrying vectors from the Fock space \(\mathcal{F}^{(\alpha)}_D\) into the other one \(\mathcal{F}^{(\alpha-\beta)}_D\), and the limit \(\beta \to 0\) is continuous and analytic. Therefore, Eq. (85) describes an isomorphism between two inequivalent Fock spaces with equal dimensions. The physical implication of the state \(|\alpha\rangle\) is that it corresponds
to the vacuum state in $\mathcal{F}_D^{(\alpha)}$ and, unless $\alpha = 0$, it is not the conventional vacuum $|0\rangle$. The Fock space of the q-oscillator in Eq. (24) is a typical example in which such a vacuum state is observed where we specifically have $\mathcal{F}_D^{(D-1)/2}$. For $D$ being an odd integer, the conventional Fock representations in $\mathcal{F}_D$ are obtained. For $D$ being an even integer, the Fock space of the q-oscillator is $\mathcal{F}_D^{1/2}$ and the vacuum state is $|1/2\rangle$ corresponding to $\alpha = 1/2$. One crucial application of this is to examine the projection of the Wigner-Kirkwood basis onto the shifted Fock space $\mathcal{F}_D^{(\alpha)}$. Let’s now insert the identity operator in (83) on both sides of the unitary number-phase basis operators in (64) yielding

$$\hat{\Delta}(\mathbf{J}, \theta) = \frac{1}{2\pi D} \sum_{\mathbf{m}} e^{i\gamma_0 m_1 J - m_2 \theta} e^{-i\gamma_0 m_1 m_2/2} \left\{ \sum_{n=0}^{D-1} \langle n + \alpha | n + \alpha \rangle \right\} \hat{E}^{m_1}_\mathbf{N} \hat{E}^{m_2}_\psi \left\{ \sum_{n'=0}^{D-1} |n' + \alpha\rangle \langle n' + \alpha| \right\}.$$  

(86)

So far, the continuous shift $\alpha$ was arbitrary. Now, we adopt a particular set of values of $\alpha$ for each $J$ independently in such a way that $2(J - \alpha) \in \mathbb{Z}$. Since Eq.s (83) are valid for all $\alpha \in \mathbb{R}$, this adaptive choice for $\alpha$ does not spoil the properties of the Wigner function studied in Sec. IV. Now, considering the limit $D \to \infty$ and following a similar calculation leading to (80), we obtain the Wigner function

$$W_\psi(J, \theta) = \frac{1}{2\pi} \lim_{D \to \infty} \sum_{m_2=0}^{D-1} e^{-im_2 \theta} \langle \psi | J - m_2/2 \rangle \langle J + m_2/2 | \psi \rangle$$  

(87)

which is expressed in the shifted Fock bases in the limit $D \to \infty$. For the choice of $\alpha$ as $2(J - \alpha) \in \mathbb{Z}$, we have for the basis vectors $\{|J \pm m_2/2; m_2 = odd\} \in \mathcal{F}_D^{(\alpha+1)/2}$ and $\{|J \pm m_2/2; m_2 = even\} \in \mathcal{F}_D^{(\alpha)}$. Note that because of the cyclic property of the vectors in $\mathcal{H}_D$, the shifted Fock space $\mathcal{F}_D^{\alpha+1/2}$ shares the same vectors with $\mathcal{F}_D^{\alpha-1/2}$ for all $D < \infty$ and $\alpha \in \mathbb{R}[0, 1)$. Thus, $\mathcal{F}_D^{\alpha+1/2}$ and $\mathcal{F}_D^{\alpha-1/2}$ are indeed the same shifted Fock space. This discussion implies that if in the summation in (87), the even and odd values of $m_2$ are separated, the Wigner function becomes a sum of two contributions $W^{(even)}$ and $W^{(odd)}$ projected onto $\mathcal{F}_D^{\alpha}$ and $\mathcal{F}_D^{\alpha+1/2}$ for even and odd $m_2$ respectively as

$$W_\psi(J, \theta) = W^{(even)}_\psi(J, \theta) + W^{(odd)}_\psi(J, \theta).$$  

(88)
Since each contribution is based on a different $D$ dimensional shifted Fock basis, they are properly normalised. It is interesting to note that a similar decomposition of the Wigner function in the Fock representation has been recently proposed by Lukš and Peřinová\textsuperscript{30} as well as by Vaccaro\textsuperscript{24} in order to avoid certain superficial anomalies of the Wigner function they use in mixed physical states. Using continuously shifted Fock spaces, the decomposition they propose follows naturally. To elaborate more on the resolution of the anomalous behaviour of the Wigner function using the shifted Fock spaces exceeds our purpose here.

It can be shown that the concept of continuously shifted Fock basis can also be generalised to the continuously shifted discrete Schwinger basis vectors $\{ |u_k \rangle \}$ and $\{ |v_k \rangle \}$. This subtle point certainly deserves much more attention in the generalised formulation of the Wigner function and quantum canonical transformations, which we intend to present in a forthcoming work.

**VII. CONCLUSIONS**

The central theme of this work was to demonstrate that conceptual foundation of the quantum phase lies in the algebraic properties of the canonical transformations on the generalised quantum phase space. In this context:

1) It is shown that the Schwinger operator basis provides subalgebraic realisations of the admissible $q$-oscillators in addition to the known deformed $su(2)$ symmetries labeled by the lattice vectors in $\mathbb{Z}_D \times \mathbb{Z}_D$. The intensively studied magnetic translation operator algebra is a specific physical realisation of Schwinger’s operator algebra. In this context, some interesting physics might be found in the realisation of the shifted $q$-oscillator subalgebra in terms of the magnetic translation operators as applied to the Bloch electron problem. To the author’s knowledge, the nearest approach to this idea has been made by Fujikawa et al. recently [see the second reference in (18)].

2) Certain equivalence classes within each subalgebra, using different lattice labels, are identified in terms of area preserving transformations. A general formulation of such discrete,
linear canonical transformations is presented.

3) The dual form between the Schwinger operator basis and the generalised discrete Wigner-Kirkwood basis is examined and the connection with the general area preserving canonical transformations on $\mathbb{Z}_D \times \mathbb{Z}_D$ is briefly studied.

4) The application of the Schwinger operator basis on the number-phase basis is discussed and shown that it provides an algebraic approach to the formulation of the quantum phase problem. The admissibly shifted q-oscillator realisations of the Schwinger basis are studied from this algebraic point of view. The generalised Wigner-Kirkwood basis is examined in the unitary number-phase basis and the limit to the conventional formulation of the action-angle Wigner function is investigated as the size of the lattice tends to infinity, or reciprocally, as the lattice spacing $2\pi/D$ tends to zero.

5) Finally, much work has to be done on understanding the quantum phase problem within the canonical quantum phase space formalism. This problem is evidently connected with the recent research areas such as classical and quantum integrability, the deformation quantization, theory of nonlinear quantum canonical transformations and the Lie algebraic representations of the Wigner function.

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REFERENCES

1 X. Shen, Int. J. Mod. Phys. A7, 3717 (1992).

2 J. Zak, Phys. Rev. 134, 1602 (1964); E. Brown, Phys. Rev. 133, 1038 (1964); T. Dereli and A. Verći, Phys. Lett. B288, 109 (1992); ibid, J. Phys. A26, 6961 (1993).

3 P.B. Wiegmann and A.V. Zabrodin, Phys. Rev. Lett. 72, 1890 (1994); Guang-Hong Chen, Le-Man Kuang and Mo-Lin Ge, Phys. Rev. B53, 9540 (1996); ibid, Phys. Lett. A213, 231 (1996).

4 J. Schwinger, Proc. Nat. Acad. Sci. 46, 883, 1401 (1960).

5 E.G. Floratos, Phys. Lett. B228, 335 (1989).

6 D. Fairlie, P. Fletcher and C. Zachos, Phys. Lett. B218, 203 (1989); D.B. Fairlie and C. Zachos, Phys. Lett. B224, 101 (1989).

7 V. Arnold, Ann. Inst. Fourier, XVI, no 1, 319 (1966); V. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, Berlin, 1978).

8 R. Aldrovandi and D. Galetti, J. Math. Phys. 31, 2987 (1990).

9 I. I. Kogan, Int. J. Mod. Phys. A9, 3887 (1994).

10 Choon-Lin Ho, J. Phys. A29, L107 (1996).

11 S. Ito, D. Karabali and B. Sakita, Phys. Lett. B296, 143 (1992); B. Sakita, Phys. Lett. B315, 124 (1993).

12 I.I. Kogan, Mod. Phys. Lett. A7, 3717 (1992).

13 D. Galetti and A.F.R. de Toledo Piza, Physica 149 A, 267 (1988).

14 E.P. Wigner, Phys. Rev. 40, 749 (1932).

15 J.G. Kirkwood, Phys. Rev. 44, 31 (1933).
16 R. Balian and C. Itzykson, C.R. Acad. Sc. Paris, 303 773 (1986).

17 H. Weyl, The Theory of Groups in Quantum Mechanics (NY, Dover 1931).

18 K. Fujikawa, L.C. Kwek and C.H. Oh, Mod. Phys. Lett. A 10, 2543 (1995).

19 T. Hakioglu, J. Phys. A 31, 707 (1998); Hong-Chen Fu and Ryu Sasaki, J. Phys. A29, 4049 (1996).

20 K. Fujikawa and Harunobu Kubo, Mod. Phys. Lett. A12, 403 (1997); K. Fujikawa and Harunobu Kubo, Phys. Lett. A239, 21 (1998).

21 M. Hillery, R.F. O’Connell, M.O. Scully and E.P. Wigner, Phys. Rep. 106, 121 (1984).

22 William K. Wooters, Ann. Phys. 176, 1 (1987).

23 João P. Bizzaro, Phys. Rev A49, 3255 (1994).

24 John Vaccaro, Phys. Rev. A52, 3474 (1995).

25 E.C. Lerner, H.W. Huang and G.E. Walters, J. Math. Phys. 11, 1679 (1970)

26 F. Riesz and B.Sz. Nagy, Functional Analysis, (Ungar, Newyork, 1955).

27 L. Susskind and J. Glogower, Physics 1, 49 (1964); P. Carruthers and M.M. Nieto, Phys. Rev. Lett. 14, 387 (1965); ibid, Rev. Mod. Phys. 40, 411 (1965).

28 K. Fujikawa, Phys. Rev. A52, 3299 (1995).

29 K.B. Wolf, Opt. Commun. 132, 343 (1996).

30 A. Lukš and V. Peřinová, Phys. Scr. T48, 94 (1993).