On the facet pivot simplex method for linear programming II: a linear iteration bound

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Abstract

The Hirsch Conjecture stated that any $d$-dimensional polytope with $n$ facets has a diameter at most equal to $n - d$. This conjecture was disproved by Santos (A counterexample to the Hirsch Conjecture, Annals of Mathematics, 172(1) 383-412, 2012). The implication of Santos’ work is that all vertex pivot algorithms cannot solve the linear programming problem in the worst case in $n - d$ vertex pivot iterations.

In the first part of this series of papers, we proposed a facet pivot method. In this paper, we show that the proposed facet pivot method can solve the canonical linear programming problem in the worst case in at most $n - d$ facet pivot iterations. This work was inspired by Smale’s Problem 9 (Mathematical problems for the next century, In Arnold, V. I.; Atiyah, M.; Lax, P.; Mazur, B. Mathematics: frontiers and perspectives, American Mathematical Society, 271-294, 1999).

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1 Introduction

The linear programming (LP) problem has been extensively studied since Dantzig invented the simplex method in the 1940s [5]. Among all feasible solutions in the corresponding polytope, the vertex simplex method checks only a small set of feasible solutions, the vertices of the polytope, to find an optimal solution. As a matter of fact, the vertex simplex method does not check all vertices, it starts from an initial vertex, and searches only for a better vertex in every iteration until an optimal solution is found. This brilliant strategy is very successful and the method solves LP problem efficiently in practice. Although the vertex set is much smaller than the set of feasible solutions, when the problem size increases, the number of vertices increases exponentially fast.

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Although vertex pivot method normally checks a small subset of all vertices for an optimal solution, people were curious how many iterations are needed in the worst case to find an optimal solution given the size of the problem. The best scenario is equivalent to determine the so-called diameter of the polytope, which has been defined in numerous books and papers. For any two vertices $\mathbf{x}^*$ and $\mathbf{y}^*$ on a convex polytope $P$, they can be connected by a path composed of a series of edges. The diameter of $P$ is the integer that is the smallest number of edges between any two vertices $\mathbf{x}^*$ and $\mathbf{y}^*$ on $P$, which defines the shortest path between $\mathbf{x}^*$ and $\mathbf{y}^*$. Therefore, if one finds the diameter of the $d$-dimensional convex polytopes $P$ which have $n$ facets, then the best vertex simplex algorithm will need at least the number of iterations that is equal to the diameter of the convex polytopes to find an optimal solution in the worst case. The famous Hirsch conjecture (see [6]) states that for $n > d \geq 2$, diameter of $P$ is less than $n - d$, which would be excellent. Unfortunately, after about 50 years of many researchers’ effort, this conjecture was disproved by Santos [19]. An even worse scenario about Dantzig’s most negative rule was described by Klee and Minty [15]. They found that the iteration number used to find the optimal solution in the worst case for Dantzig’s simplex method increases exponentially as a function of the problem size $d$. Although there are a number of variants of Dantzig’s pivot rules, analysis similar to Klee and Minty’s idea showed that almost all popular simplex algorithms using different pivot rules will need a number of iterations given as some exponential function of the problem size $[1, 8, 9, 10, 18]$.

These pessimistic results about the vertex simplex method made people think about different ways to solve the LP problem. In 1979, Khachiyan [14] proved that the ellipsoidal method needs a number of iterations bounded by a polynomial function of the size of the rational data inputs to solve the LP problem. However, people quickly realized that the ellipsoidal method is not computationally efficient [2]. Karmarkar in 1984 announced his interior point method (IPM) for LP problem [13] that converges in a number of iterations bounded by a polynomial function of the problem size. The bounds established for IPMs also depend on either the size of the rational data inputs or the prescribed accuracy requirement. The interior-point method since then became very popular and it still is as of today [25, 27]. However, Smale [21] pointed out that both the ellipsoidal method and the interior-point method are not “strongly polynomial” and he asked “Is there a polynomial time algorithm over the real numbers which decides the feasibility of the linear system of inequalities $A\mathbf{x} \geq \mathbf{b}$?” as his Problem 9.

One idea to determine the best possible iteration bound of vertex type simplex method is to obtain a tight upper bound of the diameter of the convex polytope. Currently, all upper bounds are sub-exponential. The first bound in terms of $n$ and $d$ was $n^{2+\log_2 d}$ given by Kalai [11], which was improved by Todd [23] to $(n - d)^{\log_2 d}$. The best upper bound so far is by Sukegawa [22] which is $(n - d)^{\log_2 O(d/\log d)}$. There is still a big gap between these bounds and the Hirsch conjecture (although Hirsch conjecture is disproved, experts believe that a polynomial bound should exist [20]). A different type of bounds considers not only the problems size $n$ and $d$ but also the condition of the data set. For example, Bonifas et al. [3] derived an upper bound $O(d^{3.5}\Delta^2 \ln(d\Delta))$ where $\Delta$ is the largest absolute value among all $(n - 1) \times (n - 1)$ sub-determinants.
of $A$. This author revealed a different upper bound $O(d^3 \Delta \ln(d))$ if $\det(A^*) \geq 0.5$ or $O\left(\frac{d^3 \Delta}{\det(A^*)} \ln\left(\frac{d}{\det(A^*)}\right)\right)$ if $\det(A^*) < 0.5$, where $\det(A^*)$ is the smallest absolute value among all $n \times n$ sub-determinants of $A$.

Currently there is no vertex pivot simplex algorithm that is proven to be able to achieve any of the aforementioned bounds, even though these bounds are not polynomial, for general LP problems. Therefore, researchers are still looking for new pivot algorithms. For example, Vitor and Easton [24] and this author [28] proposed double pivot algorithms that update two variables at a time. An iteration bound is established for the algorithm in [28], which is better than the similar bounds established in [12, 30] (for special LP problems such as Markov decision problem and LP with uni-modular data set, these bounds are “strongly polynomial” in terms of $n$ and $d$). Very recently, Liu et al. published their brilliant facet pivot algorithm [16]. In the first part of this series of papers [29], this author proposed an improved facet pivot algorithm suitable for computer code implementation and demonstrated its superiority by performing extensive numerical tests.

In this paper, we show that one of the facet pivot simplex algorithms described in [29] (the first part of this series of papers) finds an optimal solution in at most $n - d$ iterations. Surprisingly, this upper bound is exactly the same as Hirsch conjecture predicted. However, Hirsch conjecture is applied to the vertex pivot method, which requires the iterate moves from one vertex to the next vertex along an edge of the polytope and all iterates are basic feasible solutions, while the iterate in the facet pivot algorithm jumps among the basic solutions and iterates are not feasible until an optimal solution is found. Therefore, the path of the iteration in the facet pivot simplex algorithm is unrelated to diameter of the polytope and all estimated upper bounds of the diameter of the polytope are not applicable to the facet pivot simplex algorithms.

The remainder of the paper is organized as follows. Section 2 briefly presents the facet pivot algorithm for the linear programming problem. Section 3 proves that the facet pivot algorithm terminates in at most $n - d$ iterations. Section 4 concludes the paper with some remarks.

2 The facet pivot simplex method for the canonical LP problem

In this paper, we consider the linear programming problem:

$$\begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad A x \geq b, \\
& \quad u \geq x \geq \ell,
\end{align*}$$

where $A \in \mathbb{R}^{m \times d}$, $b \in \mathbb{R}^{m \times 1}$, $u \in \mathbb{R}^{d \times 1}$, $\ell \in \mathbb{R}^{d \times 1}$, and $c \in \mathbb{R}^{d \times 1}$ are given, and $x \in \mathbb{R}^{d \times 1}$ is a vector of the decision variables. Let $n = m + 2d$. It has been shown in [16, 29] that Problem (1) can be represented as the following canonical form of the LP
problem:

\[
\begin{align*}
\min & \quad c^T x, \\
\text{subject to} & \quad Ax \geq b,
\end{align*}
\]

where \( A \in \mathbb{R}^{n \times d} \), \( b \in \mathbb{R}^{n \times 1} \), and \( c \in \mathbb{R}^{d \times 1} \) are given, and \( x \in \mathbb{R}^{d \times 1} \) is a vector of the decision variables. In addition, \( c \) can be expressed as

\[
c^T = y_c^0 A_{B^0}, \quad y_c^0 \geq 0,
\]

where \( A_{B^0} \in \mathbb{R}^{d \times d} \) is a known full rank sub-matrix of \( A \) (\( A_{B^0} \) is a diagonal matrix whose diagonal elements are \( \pm 1 \)) and \( y_c^0 \geq 0 \) is a known coefficient vector (\( y_c^0_i = \pm c_i \)).

In the remainder of the paper, we consider the canonical form (2) and show that the number of iterations for the facet pivot simplex algorithm proposed in [29] to find an optimal solution is bounded by a linear function of \( n - d \).

Let \( a_i \) be the \( i \)-th row of \( A \) and \( b_i \) be the \( i \)-th element of \( b \). The \( i \)-th half space (constraint) of the polytope, together with other half spaces, defines the \( i \)-th facet\(^1\) represented by \( (a_i, b_i) \). Let \( B^k = \{i_1, \ldots, i_d\} \) be the index set of rows of \( A \) such that \( a_{i_1}, \ldots, a_{i_d} \) form \( d \) independent rows of \( A \) at the \( k \)-th iteration. We call \( B^k \) the base of \( A \) at the \( k \)-th iteration. Let \( N^k = \{1, \ldots, n\} \setminus B^k \) be the index set of the non-base of \( A \) at the \( k \)-th iteration. We further denote by \( A_{B^k} \) the sub-matrix of \( A \) whose rows' indices are in \( B^k \), and by \( A_{N^k} \) the sub-matrix of \( A \) whose rows' indices are in \( N^k \). Similarly, we denote by \( b_{B^k} \) the sub-vector of \( b \) whose rows' indices are in \( B^k \) and by \( b_{N^k} \) the sub-vector of \( b \) whose rows' indices are in \( N^k \). Then, we denote by \( x^k \) the basic solution of (2) at the \( k \)-th iteration, which is the solution of the following linear systems equations:

\[
A_{B^k} x^k = b_{B^k}.
\]

A solution \( x^k \) of (2) is feasible if

\[
Ax^k \geq b.
\]

A basic feasible solution is both a basic solution and a feasible solution. The following theorem provides the conditions such that a basic feasible solution of (2) is an optimal solution of (2).

**Theorem 2.1 ([16, 29])** Let \( x^k \) be a basic feasible solution of (2) which satisfies (4) and (5). If

\[
c^T = y_c^k A_{B^k}, \quad y_c^k \geq 0,
\]

then, \( x^k \) is an optimal solution of (2).

In view of (3), condition (6) is satisfied at the beginning of the iteration. The main idea of the facet pivot simplex algorithm is to update base \( B^k \) such that \( y_c^k \geq 0 \) is

\(^1\)We will abuse the notation by saying that \( a_i \) is the \( i \)-th facet of the polytope when there is no confusion introduced.
maintained while continuously improving the feasibility of the constraints. When a basic solution is feasible, then an optimal solution is found. Let
\[ \sigma_i(x^k) = a_i x^k - b_i, \quad i = 1, \ldots, n, \tag{7} \]
be the measure of the \( i \)-th constraint infeasibility at the \( k \)-th iteration. If \( a_p x^k \geq b_p \) for all \( a_p \) with \( p \in N^k \), i.e., \( x^k \) is feasible, then, an optimal solution is found. The iteration stops here. Otherwise, we may select the entering facet using the so-called the least index rule from the constraint, i.e.,
\[ p = \min_i \{ i \mid \sigma_i < 0, \quad i \in N^k \}, \tag{8} \]
where \( p \in N^k \) is the index of the entering row and \( a_p \) is a row of \( A_{N^k} \) and a row of \( A_{B^{k+1}} \), i.e., \( a_p \) is not in the base of \( B^k \) but in the base of \( B^{k+1} \). Since \( A_{B^k} \) is full rank (this is true for \( B^0 \) and for the general case \( B^k \) as discussed in Theorem 2.3), the entering facet \( a_p \) can be expressed as
\[ a_p = \sum_{j \in B^k} y_{pj}^k a_j = y_p^T A_{B^k}. \tag{9} \]

The following theorem provides the existence conditions for the LP problem (2).

**Theorem 2.2 ([16, 29])** Let \( B^k \) be the base of (2) at \( k \)-th iteration, denote by \( x^k \) the basic (but infeasible) solution of (2), i.e., \( A_{B^k} x^k = b_{B^k} \). If the following conditions
(a) \( a_p x^k < b_p \) for the entering row \( p \in N^k \), and
(b) \( y_{pj}^k \leq 0 \) for all \( j \in B^k \) in (9)
hold, then, there is no feasible solution for Problem (2).

If the conditions of Theorem 2.2 are satisfied, the problem is infeasible, the iteration stops here. Otherwise, there exist at least a \( j \in B^k \) such that \( y_{pj}^k > 0 \). Because \( A_{B^k} \) is full rank, we may express \( c \) as:
\[ c^T = \sum_{j \in B^k} y_{cj}^k a_j = y_c^T A_{B^k}, \quad y_c^k \geq 0, \tag{10} \]
\( y_c^k \geq 0 \) is true for \( B^0 \) and for any \( k \geq 0 \) if we select the leaving row (facet) \( q \) using the least index rule:
\[ \frac{y_{cq}^k}{y_{pq}^k} = \min_j \left\{ \min \left\{ \frac{y_{cj}^k}{y_{pq}^k} \middle| y_{pj}^k > 0, \quad j \in B^k \right\} \right\}, \tag{11} \]
where condition \( y_{pq}^k > 0 \) holds because the problem has at least a feasible solution, and if there are several indices that achieve the first minimum in (11), the least index is selected as \( q \). In view of (9), we have
\[ a_q = \frac{1}{y_{pq}^k} a_p + \sum_{j \in B^k \setminus \{q\}} \left( -\frac{y_{pj}^k}{y_{pq}^k} \right) a_j = y_q^{k+1T} A_{B^{k+1}}. \tag{12} \]
It is worthwhile to emphasize that the leaving facet $a_q$ at the $k$-th iteration is in the base $B^k$ but not in the base $B^{k+1}$. Using the pivot rule (11), we can show that $A_{B^k}$ is full rank for all $k$ as long as $A_{B^0}$ is full rank (which is guaranteed when we convert (1) to (2)).

**Theorem 2.3 ([29])** Assume that the rows of $A_{B^k}$ are independent, then the rows of $A_{B^{k+1}}$ are also independent if pivot rule (11) is used.

In view of (12), we have the following lemma:

**Lemma 2.1** Let $p$ be the index of the entering facet and $q$ be the index of the leaving facet at $k$-th iteration, then,$a_q = y_{qp}^{k+1} A_{B^{k+1}}$. In addition, let $j \in B^k \setminus \{q\}$, then, the following equality and inequality hold

$$y_{qp}^{k+1} = \frac{1}{y_{pq}^k} > 0, \quad y_{qj}^{k+1} = -\frac{y_{pj}^k}{y_{pq}^k}.$$  

(13)

The selection of the leaving facet guarantees that $y_{c}^{k+1} \geq 0$ (condition in (6)) holds because $c^T$ can be represented by the vectors in the new base

$$B^{k+1} = \{p\} \cup B^k \setminus \{q\}$$  

(14)

as follows:

$$c^T = y_{cq}^k a_q + \sum_{j \in B^k \setminus \{q\}} y_{cj}^k a_j$$

$$= \frac{y_{cq}^k}{y_{pq}^k} a_p - \sum_{j \in B^k \setminus \{q\}} \frac{y_{pj}^k y_{cq}^k}{y_{pq}^k} a_j + \sum_{j \in B^k \setminus \{q\}} y_{cj}^k a_j$$

$$= \frac{y_{cq}^k}{y_{pq}^k} a_p + \sum_{j \in B^k \setminus \{q\}} \left( y_{cj}^k - \frac{y_{pj}^k y_{cq}^k}{y_{pq}^k} \right) a_j$$

(15)

$$:= y_{c}^{k+1} A_{B^{k+1}}.$$  

(16)

Since $q \in B^k$ and $y_{cq}^k \geq 0$ is maintained before the $k$-th iteration, this means $\frac{y_{cq}^k}{y_{pq}^k} \geq 0$ because $y_{pq}^k > 0$ according to (11). Also, it must have $\left( y_{cj}^k - \frac{y_{pj}^k y_{cq}^k}{y_{pq}^k} \right) \geq 0$ for all $j \in B^k$ because of (11). This proves that $y_{c}^{k+1} \geq 0$.

**Theorem 2.4 ([16, 29])** Let $B^k$ be the base of (2) in the $k$-th iteration. Denote by $x^k$ the basic solution of (2) corresponding to $B^k$, i.e., $A_{B^k} x^k = b_{B^k}$, and by $x^{k+1}$ the basic solution of (2) corresponding to $B^{k+1}$, i.e., $A_{B^{k+1}} x^{k+1} = b_{B^{k+1}}$. Assume that

(a) the entering facet is $p \in N^k$, i.e., $a_p x^k < b_p$,
(b) the leaving facet $q \in B^k$ and $y_{pq}^k > 0$, and

c) the leaving facet $q$ is determined by (11),

then,

(i) $c$ can be expressed as (15) with $y_{c}^{k+1} \geq 0$.

(ii) The following relation holds

$$c^T x^{k+1} - c^T x^k = \frac{y_{pq}^k}{y_{pq}^k} (b_p - a_p x^k) \geq 0. \quad (17)$$

(iii) If $y_{pq}^k > 0$ and $y_{pj}^k \leq 0$ for all $j \in B^k \setminus \{q\}$, then, $a_q x \geq b_q$ is a redundant constraint.

Summarizing the discussion in this section, the facet pivot simplex algorithm can be stated as follows:

**Algorithm 2.1**

1: Data: Matrices $A$, and vectors $b$ and $c$.
2: Given initial $B^0$ and $y_c^0 \geq 0$, compute the initial basic solution $x^0$.
3: Compute the constraints violation determinants $\sigma_i$ using (7).
4: while exist at least a $\sigma_i^k < 0$ in $i \in N^k$ do
5: Select the entering facet (row) $a_p$ using the least/lowest index rule (8).
6: Given $a_p$, compute $y_p^k$ (i.e., $y_{pr}^k$) by solving linear systems of equations (9).
7: if there is no feasible solution (Theorem 2.2) then
8: Exit the loop and report “there is no feasible solution”.
9: end if
10: Select leaving facet (row) $a_q$ by using (11).
11: Update base using (14).
12: Express $c$ in $B^{k+1}$ using (15), i.e., $y_{cp}^{k+1} = \frac{y^k_c}{y_{pq}^k}$ and $y_{cj}^{k+1} = (y_{cj}^k - y_{pj}^k \frac{y^k_c}{y_{pq}^k})$.
13: if leaving facet (row) $a_q$ is redundant (Theorems 2.4) then
14: Remove the $q$-th constraint from the constraint set.
15: end if
16: Compute the updated solution $x^{k+1}$ by solving $A_{B^{k+1}} x = b_{B^{k+1}}$
17: Compute the constraints violation determinants $\sigma_i^k$ using (7).
18: $k \leftarrow k + 1$.
19: end while

**Theorem 2.5** ([16, 29]) There is no cycling for Algorithm 2.1, i.e., there is no repeated bases in all iterations for Algorithm 2.1.
3 Polynomial iteration bound for the facet pivot simplex method

Our strategy is to show that once a facet leaves the base $B^k$ at the $k$-th iteration, it will never come back to the bases in the future iterations. First, we state a fact that is based on the definition of the basic solution:

**Lemma 3.1** Assume that a facet $a_d$ is in the bases of $B^k$ and $B^{k+\ell}$ for any integers $k$ and $\ell \geq 1$, and $x^k$ and $x^{k+\ell}$ are the basic solutions of (2) at the $k$-th and $(k+\ell)$-th iterations. Then, we have

$$a_d x^k = a_d x^{k+\ell} = b_d. \quad (18)$$

The next lemma indicates that the leaving facet (constraint) at the $k$-th iteration $a_q$ ($q \in B^k \cap N^{k+1}$) is not only a feasible constraint at the $k$-th iteration but also a feasible constraint at the $(k+1)$-th iteration, i.e., $a_q x^k = b_q$ and $a_q x^{k+1} > b_q$, where $x^k$ and $x^{k+1}$ are the basic solutions at the $k$-th and the $(k+1)$-th iterations, i.e., $A_{B^k} x^k = b_{B^k}$, and $A_{B^{k+1}} x^{k+1} = b_{B^{k+1}}$.

**Lemma 3.2** ([29]) Let $a_q \in B^k$ be the leaving facet at the $k$-th iteration. Then, we have

$$a_q x^{k+1} > a_q x^k = b_q. \quad (19)$$

**Remark 3.1** This lemma indicates that the leaving facet $a_q$ at the $k$-th iteration will not be the entering facet at the $(k+1)$-th iteration.

We divide the rest discussion into two scenarios.

3.1 Case 1: The entering facet will be the leaving facet in the next iteration

Let $p$ and $s$ be indices of the entering facets at the $k$-th and the $(k+1)$-th iterations respectively, let $q$ and $p$ be the indices of the leaving facets at the $k$-th and the $(k+1)$-th iterations respectively. The following lemma follows directly from the assumption that the entering facet will be the leaving facet in the next iteration.

**Lemma 3.3** Assume that the entering facet $a_p$ at the $k$-th iteration is the leaving facet at the $(k+1)$-th iteration, $a_q$ is the leaving facet at the $k$-th iteration, and $a_s$ is the entering facet at the $(k+1)$-th iteration. Then, the sets of bases $B^k$, $B^{k+1}$, and $B^{k+2}$ satisfy the following relations.

$$B^{k+1} = B^k \cup \{p\} \setminus \{q\}, \quad B^{k+1} \setminus \{p\} = B^k \setminus \{q\}, \quad (20)$$
and

\[ B^{k+2} = B^{k+1} \cup \{s\} \setminus \{p\} = B^k \cup \{s\} \setminus \{q\}. \] (21)

Equations of (20) and (21) imply that \( a_p \) is the entering facet at the \( k \)-th iteration and is the leave facet at the \((k + 1)\)-th iteration. Since \( a_s \) can be expressed as

\[ a_s = \sum_{j \in B^{k+1} \setminus \{p\}} y_{sj}^{k+1} a_j + y_{sp}^{k+1} a_p = \sum_{j \in B^{k+1}} y_{sj}^{k+1} a_j, \] (22)

we have

\[ a_p = \frac{1}{y_{sp}} a_s - \sum_{j \in B^{k+1} \setminus \{p\}} \frac{y_{sj}^{k+1}}{y_{sp}} a_j = \sum_{j \in B^{k+2}} y_{jp}^{k+2} a_j. \] (23)

Since \( a_q \) is not in the bases \( B^{k+1} \) and \( B^{k+2} \), using (23) in the following derivation, we have

\[ a_q = \sum_{j \in B^{k+2}} y_{qj}^{k+2} a_j \] (24)

\[ = \sum_{j \in B^{k+1}} y_{qj}^{k+1} a_j \] (25)

\[ = \sum_{j \in B^{k+1} \setminus \{p\}} y_{qj}^{k+1} a_j + y_{qp}^{k+1} a_p \]

\[ = \sum_{j \in B^{k+1} \setminus \{p\}} \left[ y_{qj}^{k+1} - y_{qp}^{k+1} \left( \frac{y_{sj}^{k+1}}{y_{sp}^{k+1}} \right) \right] a_j + y_{qp}^{k+1} a_s \] (26)

Comparing (24) and (26), we have the following lemma.

**Lemma 3.4** Assume that \( a_p \) and \( a_q \) are the entering and the leaving facets at the \( k \)-th iteration respectively, and \( a_s \) and \( a_p \) are the entering and leaving facets at the \((k + 1)\)-th iteration respectively. Let \( y_{qj}^{k+1} \) for \( j \in B^{k+1} \) and \( y_{qj}^{k+2} \) for \( j \in B^{k+2} \) be defined as in (25) and (24) respectively, and let \( y_{sj} \) for \( j \in B^{k+1} \) be defined as in (22). Then, the following relations hold.

\[ y_{qs}^{k+2} = \frac{y_{qp}^{k+1}}{y_{sp}^{k+1}}, \quad y_{qj}^{k+2} = y_{qj}^{k+1} - y_{qp}^{k+1} \left( \frac{y_{sj}^{k+1}}{y_{sp}^{k+1}} \right), \quad \text{for } j \in B^{k+1} \setminus \{p\}. \] (27)

In addition, \( y_{qs}^{k+2} > 0 \).
Proof: We need only to show that \( y_{qp}^{k+1} > 0 \). Since \( p \) is the index of the entering facet and \( q \) is the index of the leaving facet at the \( k \)-th iteration, it follows from Lemma 2.1 that \( y_{qp}^{k+1} = \frac{1}{y_{pq}} > 0 \). Since \( s \) is the index of the entering facet and \( p \) is the index of the leaving facet at the \((k+1)\)-th iteration, it follows from (11) that \( y_{sp}^{k+1} > 0 \). Therefore, we have \( y_{qs}^{k+2} > 0 \). □

Next, we extend Lemma 3.2 one step further for Case 1.

Lemma 3.5 Let \( a_q \) be the leaving facet at the \( k \)-th iteration. Let \( x^{k+2} \) be the basic solution of (2) at the \((k+2)\)-th iteration, i.e., \( A_{B^{k+2}}x^{k+2} = b_{B^{k+2}} \). Then, we have

\[
a_q x^{k+2} > b_q.
\] (28)

Therefore, \( a_q \) is not the entering facet at the \((k+2)\)-th iteration.

Proof: According to Remark 3.1, \( a_q \) is not the entering facet at the \((k+1)\)-th iteration, therefore, \( a_q \) can be expressed by the rows of \( B^{k+2} \). Substituting (22) into (24) yields

\[
a_q = \sum_{j \in B^{k+2} \setminus \{s\}} y_{qj}^{k+2} a_j + y_{qs}^{k+2} a_s.
\]

Multiplying \( x^{k+2} \) to the right of the items in the parentheses of (29) and using Lemma 3.1 with the fact that \( B^{k+1} \setminus \{p\} = B^{k+2} \setminus \{s\} \) yield

\[
\sum_{j \in B^{k+1} \setminus \{p\}} y_{sj}^{k+1} a_j x^{k+2} + y_{sp}^{k+1} a_p x^{k+2}
= \sum_{j \in B^{k+1} \setminus \{p\}} y_{sj}^{k+1} a_j x^{k+2} + y_{sp}^{k+1} a_p x^{k+2} + y_{sp}^{k+1} b_p - y_{sp}^{k+1} b_p
= \sum_{j \in B^{k+1} \setminus \{p\}} y_{sj}^{k+1} a_j x^{k+2} + y_{sp}^{k+1} b_p + y_{sp}^{k+1} (a_p x^{k+2} - b_p)
= \sum_{j \in B^{k+1}} y_{sj}^{k+1} a_j x^{k+1} + y_{sp}^{k+1} (a_p x^{k+2} - b_p)
= a_s x^{k+1} + y_{sp}^{k+1} (a_p x^{k+2} - b_p),
\] (30)

we used (22) in the derivation of the last equation of (30). Multiplying \( x^{k+2} \) to the right of the both sides of (29), and using (30), Lemma 3.1, (24), the first equation of (27) yield

\[
a_q x^{k+2} = \sum_{j \in B^{k+2} \setminus \{s\}} y_{qj}^{k+2} a_j x^{k+2} + y_{qs}^{k+2} \left( \sum_{j \in B^{k+1} \setminus \{p\}} y_{sj}^{k+1} a_j + y_{sp}^{k+1} a_p \right) x^{k+2}
\]
from Lemma 2.1, we have
\[ y^k(\ldots) = \text{at}(k+1) \]
and
\[ a \]
In view of Lemma 3.2, it follows that
\[ \text{Corollary 3.1} \]
For any integer
\[ a \]
Let
\[ B \]
Now, we extend Lemma 3.5 to
\[ a \]
This corollary will be used in the derivation of the follow lemma.
\[ \text{Lemma 3.6} \]
For any integer
\[ a \]
Let
\[ a \]
Then, we have
\[ a \]
(31)
In view of Lemma 3.2, it follows that
\[ a \]
Since
\[ a \]
Moreover, from Lemma 2.1, we have
\[ y^k + 1 = \frac{1}{y_{pq}} > 0. \]
Therefore, the last expression of (31) is greater than
\[ a \]
This proves
\[ a \]
Now, we extend Lemma 3.5 to
\[ a \]
For any integer
\[ a \]
This corollary will be used in the derivation of the follow lemma.
\[ \text{Corollary 3.1} \]
For any integer
\[ a \]
Let
\[ a \]
Then, the following relations hold.
\[ B^{k+\ell} = B^{k+\ell-1} \cup \{p\} \setminus \{t\}, \quad B^{k+\ell} \setminus \{p\} = B^{k+\ell-1} \setminus \{t\}, \]
and
\[ B^{k+\ell+1} = B^{k+\ell} \cup \{s\} \setminus \{p\} = B^{k+\ell-1} \cup \{s\} \setminus \{t\}, \quad B^{k+\ell+1} \setminus \{s\} = B^{k+\ell} \setminus \{p\}. \]
Proof: By using induction, we assume (a) \( y_{qp}^{k+\ell} > 0 \) holds (which is true for \( \ell = 1 \) according to Lemma 2.1 because \( a_p \) is the entering facet at the \((k+\ell-1)\)-th iteration), (b) \( a_q x^{k+\ell} > b_q \), and (c) \( a_p \not\in B^{k+\ell+1} \) (which is true for \( \ell = 1 \) due to Lemma 3.2 because \( a_p \) is not the entering facet at the \((k+1)\)-th iteration) for \( \ell \geq 1 \). Since \( a_p \) is the entering facet at the \((k+\ell)\)-th iteration, \( s \not\in B^{k+\ell} \), \( a_s \) can be expressed by the facets in \( B^{k+\ell} \) as

\[
a_s = \sum_{j \in B^{k+\ell} \setminus \{p\}} y_{sj}^{k+\ell} a_j + y_{sp}^{k+\ell} a_p = \sum_{j \in B^{k+\ell}} y_{sj}^{k+\ell} a_j.
\]  

Using this formula, Corollary 3.1, and expressing \( a_q \) by the facets in \( B^{k+\ell+1} \) as

\[
a_q = \sum_{j \in B^{k+\ell+1} \setminus \{s\}} y_{qj}^{k+\ell+1} a_j + y_{qs}^{k+\ell+1} a_s
\]  

\[
= \sum_{j \in B^{k+\ell+1} \setminus \{s\}} y_{qj}^{k+\ell+1} a_j + y_{qs}^{k+\ell+1} \left( \sum_{j \in B^{k+\ell} \setminus \{p\}} y_{sj}^{k+\ell} a_j + y_{sp}^{k+\ell} a_p \right)
\]  

\[
= \sum_{j \in B^{k+\ell+1} \setminus \{p\}} \left( y_{qj}^{k+\ell+1} + y_{qs}^{k+\ell+1} y_{sj}^{k+\ell} \right) a_j + y_{qs}^{k+\ell+1} y_{sp}^{k+\ell} a_p = \sum_{j \in B^{k+\ell}} y_{qj}^{k+\ell} a_j.
\]  

This shows

\[
y_{qp}^{k+\ell} = y_{qs}^{k+\ell+1} y_{sp}^{k+\ell}, \quad y_{qj}^{k+\ell} = y_{qs}^{k+\ell+1} y_{sj}^{k+\ell}, \quad j \in B^{k+\ell} \setminus \{p\}.
\]  

Therefore, we have

\[
y_{qs}^{k+\ell+1} = \frac{y_{qp}^{k+\ell}}{y_{sp}^{k+\ell}}.
\]  

By assumption, we have \( y_{qp}^{k+\ell} > 0 \). Since \( s \) is the index of the entering facet and \( p \) is the index of the leaving facet at \((k+\ell)\)-th iteration, \( y_{sp}^{k+\ell} > 0 \) according to (11). This shows \( y_{qs}^{k+\ell+1} > 0 \). Multiplying \( x^{k+\ell+1} \) to the right of the items in the parentheses of (36) and using (39) yield

\[
\sum_{j \in B^{k+\ell+1} \setminus \{p\}} y_{sj}^{k+\ell} a_j x^{k+\ell+1} + y_{sp}^{k+\ell} a_p x^{k+\ell+1}
\]  

\[
= \sum_{j \in B^{k+\ell+1} \setminus \{s\}} y_{sj}^{k+\ell} a_j x^{k+\ell+1} + y_{sp}^{k+\ell} a_p x^{k+\ell+1} + y_{sp}^{k+\ell} b_p - y_{sp}^{k+\ell} b_p
\]  

\[
= \sum_{j \in B^{k+\ell+1} \setminus \{p\}} y_{sj}^{k+\ell} b_j + y_{sp}^{k+\ell} b_p + y_{sp}^{k+\ell} \left( a_p x^{k+\ell+1} - b_p \right)
\]  

\[
= \sum_{j \in B^{k+\ell+1}} y_{sj}^{k+\ell} a_j x^{k+\ell} + y_{sp}^{k+\ell} \left( a_p x^{k+\ell+1} - b_p \right)
\]  

\[
= a_s x^{k+\ell} + y_{sp}^{k+\ell} \left( a_p x^{k+\ell+1} - b_p \right).
\]  

Multiplying \( x^{k+\ell+1} \) to the right of the both sides of (36) and using (40) and (39) yield

\[
a_q x^{k+\ell+1} = \sum_{j \in B^{k+\ell+1} \setminus \{s\}} y_{qj}^{k+\ell+1} a_j x^{k+\ell+1} + y_{qs}^{k+\ell+1} \left( \sum_{j \in B^{k+\ell} \setminus \{p\}} y_{sj}^{k+\ell} a_j + y_{sp}^{k+\ell} a_p \right) x^{k+\ell+1}
\]  

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\[ \begin{align*}
\sum_{j \in B^{k+\ell+1}\{s\}} y_{qj} y_{k+\ell+1}^j a_j x^k y_{k+\ell+1}^j + y_{qs}^k y_{k+\ell+1}^s a_s x^k y_{k+\ell+1}^s + y_{qs}^k y_{k+\ell+1}^p a_p x^k y_{k+\ell+1}^p - b_p \end{align*} \]

According to the assumption, the inequalities \( y_{qj}^k y_{k+\ell+1}^j a_j x^k y_{k+\ell+1}^j > 0 \) and \( a_q x_{k+\ell+1}^q > b_q \) hold. Since \( a_p \) is the entering facet at the \((k+\ell)\)-th iteration, it follows from Lemma 3.2 that \( a_p x_{k+\ell+1}^p - b_p > 0 \). This proves \( a_q x_{k+\ell+1}^q > b_q \), i.e., \( a_q \) is not the entering facet of \( B^{k+\ell+1} \), therefore, \( a_q \not\in B^{k+\ell+2} \).

3.2 Case 2: Entering facet will not be the leaving facet in the next iteration

Let \( p \) and \( s \) be the indices of the entering facets at the \( k \)-th iteration and the \((k+1)\)-th iteration, and \( q \) and \( t \) be the indices of the leaving facets at the \( k \)-th iteration and the \((k+1)\)-th iteration, respectively. Denote \( a_p \) and \( a_q \) be the entering facet and the leaving facet at the \( k \)-th iteration, and \( a_s \) and \( a_t \) be the entering facet and leaving facet at the \((k+1)\)-th iteration, respectively. The following lemma follows directly from the assumption that the entering facet will not be the leaving facet in the next iteration.

Lemma 3.7 Let \( a_p \) and \( a_q \) be the entering facet and the leaving facet at the \( k \)-th iteration, \( a_s \) and \( a_t \) be the entering facet and the leaving facet at the \((k+1)\)-th iteration, respectively. The bases of \( B^k \), \( B^{k+1} \), and \( B^{k+2} \) satisfy the following relations.

\[ B^{k+1} = B^k \cup \{p\} \setminus \{q\}, \quad B^{k+1} \setminus \{p\} = B^k \setminus \{q\}, \quad (42) \]

and

\[ B^{k+2} = B^{k+1} \cup \{s\} \setminus \{t\} = B^k \cup \{p, s\} \setminus \{q, t\}. \quad (43) \]

Conditions of (42) and (43) imply that \( a_p \) is the entering facet at the \( k \)-th iteration but is not the leave facet at the \((k+1)\)-th iteration. The key idea in the proof of this case is to use an extension of Farkas lemma by Ky Fan [7, Theorem 4], which is named as the nonhomogeneous Farkas Lemma [4, Theorem 1.3].

Lemma 3.8 (Farkas-Fan Lemma) Denote \( A^T = [a_1^T, a_2^T, \ldots, a_n^T] \) with \( a_i \in \mathbb{R}^{1 \times d} \) \( (i = 1, 2, \ldots, n) \), \( A \in \mathbb{R}^{n \times d} \), \( b \in \mathbb{R}^n \), and \( d \in \mathbb{R}^d \) be the constant matrix and vectors. Consider the linear system

\[ Ax \geq b. \quad (44) \]
The inequality

\[ x^T d \geq b_0 \]  (45)

is satisfied by every vector \( x \) satisfying system (44) if and only if there exists a nonnegative vector \( y \in \mathbb{R}^n \) and \( y \neq 0 \) such that

\[ d = A^T y, \quad \text{and} \quad b_0 \leq y^T b. \]  (46)

Since \( a_s \) is the entering facet at the \((k + 1)\)-th iteration, and \( \{p, t\} \in B^{k+1} \), using Lemma 3.7, we can write

\[ a_s = \sum_{j \in B^{k+1}} y_{sj}^{k+1} a_j = \sum_{j \in B^{k+1} \setminus \{p, t\}} y_{sj}^{k+1} a_j + y_{sp}^{k+1} a_p + y_{st}^{k+1} a_t = \sum_{j \in B^k \setminus \{q, t\}} y_{sj}^{k+1} a_j + y_{sp}^{k+1} a_p + y_{st}^{k+1} a_t. \]  (47)

This gives

\[ a_t = \frac{1}{y_{st}^{k+1}} a_s - \frac{y_{sp}^{k+1}}{y_{st}^{k+1}} a_p - \sum_{j \in B^k \setminus \{q, t\}} \frac{y_{sj}^{k+1}}{y_{st}^{k+1}} a_j. \]  (48)

Since \( a_q \) can be expressed either in base \( B^{k+1} \) or in base \( B^{k+2} \), we can write

\[ a_q = \sum_{j \in B^{k+1}} y_{qj}^{k+1} a_j = \sum_{j \in B^{k+1} \setminus \{p\}} y_{qj}^{k+1} a_j + y_{qp}^{k+1} a_p = \sum_{j \in B^k \setminus \{q, t\}} y_{qj}^{k+2} a_j = \sum_{j \in B^{k+1} \setminus \{t\}} y_{qj}^{k+2} a_j + y_{qs}^{k+2} a_s = \sum_{j \in B^k \setminus \{q, t\}} y_{qj}^{k+2} a_j + y_{qp}^{k+2} a_p + y_{qs}^{k+2} a_s. \]  (49)

Parallel to Lemma 3.4 in Case 1, for Case 2, we have

**Lemma 3.9** Assume \( a_p \) and \( a_s \) are the entering facets at the \( k \)-th iteration and the \((k + 1)\)-th iteration, and \( a_q \) and \( a_t \) are the leaving facets at the \( k \)-th iteration and the \((k + 1)\)-th iteration, respectively. Let \( y_{qj}^{k+1} \) for \( j \in B^{k+1} \) and \( y_{qj}^{k+2} \) for \( j \in B^{k+2} \) be defined as in (49), respectively; let \( y_{sj}^{k+1} \) for \( j \in B^{k+1} \) be defined as in (47). Then, the following relations hold.

\[ y_{qj}^{k+2} = \frac{y_{qj}^{k+1}}{y_{st}^{k+1}}, \quad y_{pq}^{k+2} = y_{pq}^{k+1} \left( \frac{y_{sj}^{k+1}}{y_{sp}^{k+1}} \right), \quad \text{for} \quad j \in B^{k+1} \setminus \{t\}. \]  (50)

In addition, we have \( y_{qj}^{k+1} \geq 0 \), therefore, \( y_{pq}^{k+1} \geq 0 \) and \( y_{qs}^{k+2} \geq 0 \).
Proof: Since \( a_q \) can be represented in \( B^{k+2} \) and in \( B^{k+1} \), using (48) and (49), we have

\[
a_q = \sum_{j \in B^{k}\{q,t\}} y_{qj}^{k+2} a_j + y_{qp}^{k+2} a_p + y_{qs}^{k+2} a_s
\]

\[
= \sum_{j \in B^{k}\{q,t\}} y_{qj}^{k+1} a_j + y_{qp}^{k+1} a_p + y_{qt}^{k+1} a_t
\]

\[
= \sum_{j \in B^{k}\{q,t\}} \left[ y_{qj}^{k+1} - y_{qj} y_{qt}^{k+1} \right] a_j + \left[ y_{qp}^{k+1} - y_{qt} y_{qt}^{k+1} \right] a_p + y_{qt}^{k+1} a_s.
\]

Comparing (51) with (52) yields

\[
y_{qs}^{k+2} = \frac{y_{qt}^{k+1}}{y_{st}^{k+1}}, \quad y_{qj}^{k+2} = y_{qj}^{k+1} - \left( \frac{y_{qt}^{k+1}}{y_{st}^{k+1}} \right) y_{sj}^{k+1}, \quad \text{for} \quad j \in B^{k+1} \setminus \{t\}.
\]

Since \( s \) is the index of entering facet and \( t \) is the index of leaving facet at the \((k+1)\)-th iteration, from (11), it follows \( y_{st}^{k+1} > 0 \). To show \( y_{qt}^{k+1} \geq 0 \), we use the Farkas-Fan Lemma by setting \( A = A_{B^{k+1}}, \ b = b_{B^{k+1}}, \ b_0 = b_q, \ d^T = a_q, \ x = x^{k+1}, \) and \( y = y_q^{k+1} \) in the Farkas-Fan Lemma. Since \( x^{k+1} \) is the basic solution of (2) at \( B^{k+1} \), it follows that \( A_{B^{k+1}} x^{k+1} = b_{B^{k+1}} \), i.e., the inequality (44) in Farkas-Fan Lemma holds. In view of Lemma 3.2, it follows that \( a_q x^{k+1} > b_q \) or equivalently \( d^T x > b_0 \), i.e., the inequality (45) in Farkas-Fan Lemma holds. Farkas-Fan Lemma claims that conditions of (46) \( A^T y = d \iff A_{B^{k+1}}^T y_q^{k+1} = a_q^T \) and \( y \geq 0 \iff y_q^{k+1} \geq 0 \) hold, which is equation (12) with \( y_q^{k+1} \geq 0 \). Therefore, it must have \( y_{qt}^{k+1} \geq 0 \). In view of (53), we conclude that \( y_{qs}^{k+2} \geq 0 \). Since \( y \geq 0 \), the inequality in (46) can be written as

\[
b_0 \leq y^T b \leq y^T A_{B^{k+1}} x^{k+1} \leq a_q x^{k+1} = d^T x^{k+1},
\]

which again is (45).

The next lemma extends the result of Lemma 3.2 one step further for Case 2.

**Lemma 3.10** Let \( a_q \in B^k \) be the leaving facet at the \( k \)-th iteration. Then, we have

\[
a_q x^{k+2} \geq b_q.
\]
Proof: Let $p$ and $s$ be the indices of the entering facets at the $k$-th and the $(k+1)$-th iterations, and $q$ and $t$ be the indices of the leaving facets at the $k$-th and the $(k+1)$-th iterations, respectively. Denote by $x^{k+1}$ and $x^{k+2}$ the basic solution of (2) at the $(k + 1)$-th and the $(k + 2)$-th iterations, respectively. Since $p \in B^{k+1} \cap B^{k+2}$, it follows that $a_p x^{k+2} = a_p x^{k+1} = b_p$. Using (47) and Lemma 3.7, $a_q$ can be expressed by the base $B^{k+2}$ as follows:

$$a_q = \sum_{j \in B^{k+2}} y_{qj}^{k+2} a_j = \sum_{j \in B^{k} \setminus \{p,s\}} y_{qj}^{k+2} a_j + \sum_{j \in B^{k} \setminus \{q,t\}} y_{qj}^{k+2} a_j + y_{qp}^{k+2} a_p + y_{qs}^{k+2} a_s$$

Multiplying $x^{k+2}$ to the right on both sides of (57) yields

$$a_q x^{k+2} = \sum_{j \in B^{k} \setminus \{q,t\}} y_{qj}^{k+2} a_j x^{k+2} + \sum_{j \in B^{k} \setminus \{q,t\}} y_{qj}^{k+2} a_j x^{k+2}$$

In view of Lemma 3.7, $j \in B^{k} \setminus \{q,t\}$ means $j \in B^{k+2} \setminus \{p,s\}$, and vice versa, therefore, two summations in (58) can be expressed as

$$\sum_{j \in B^{k} \setminus \{q,t\}} y_{qj}^{k+2} a_j x^{k+2} = \sum_{j \in B^{k} \setminus \{q,t\}} y_{qj}^{k+2} b_j + \sum_{j \in B^{k} \setminus \{q,t\}} y_{sj}^{k+1} a_j x^{k+2} + \sum_{j \in B^{k} \setminus \{q,t\}} y_{sj}^{k+1} b_j \quad (59)$$

Since $a_p \in B^{k+2}$, it follows that

$$y_{qp}^{k+2} a_p x^{k+2} = y_{qp}^{k+2} b_p, \quad y_{sp}^{k+1} a_p x^{k+2} = y_{sp}^{k+1} b_p \quad (60)$$

Since $s$ is the index of the entering facet at the $(k + 1)$-th iteration and $t$ is the index of the leaving facet at the $(k+1)$-th iteration, in view of (11), it must have $y_{st}^{k+1} > 0$. Since $t \in B^{k+1}$ and $t \notin B^{k+2}$, according to Lemma 3.2, it must have $a_t x^{k+2} > b_t$. Summarizing the discussions below equation (58), we can rewrite the items inside the bracket of (58) as

$$\sum_{j \in B^{k} \setminus \{q,t\}} y_{sj}^{k+1} a_j x^{k+2} + y_{sp}^{k+1} a_p x^{k+2} + y_{st}^{k+1} a_t x^{k+2}$$

$$= \sum_{j \in B^{k} \setminus \{q,t\}} y_{sj}^{k+1} b_j + y_{sp}^{k+1} b_p + y_{st}^{k+1} b_t + y_{st}^{k+1} (a_t x^{k+2} - b_t)$$
Lemma 3.11

Let \( \beta \) any iteration of \( k \) and leaving facets at the \( k \) and the indices of the entering and leaving facets at the \( k \) and \( k+1 \). The last inequality follows from the facts that \( a_i x^{k+1} > b_q \) (using Lemma 3.2, since \( a_q \) is the leaving facet at the \( k \)-th iteration), \( y_{qs}^{k+2} \geq 0 \) (Lemma 3.9), \( y_{st}^{k+1} > 0 \) (Formulas (11)), and \( a_i x^{k+2} - b_t \) (using Lemma 3.2, since \( a_t \) is the leaving facet at the \( (k+1) \)-th iteration).

Using induction, we can prove the following lemma:

**Lemma 3.11** Let \( \alpha_i \) and \( \beta_i \) represent the indices of the entering and leaving facets at any iteration of \( i = k, k+1, \ldots, k+\ell \). For example, \( \alpha_1 = p \) and \( \beta_1 = q \) represent the indices of the entering and leaving facets at the \( k \)-th iteration, \( \alpha_2 = s \) and \( \beta_2 = t \) represent the indices of the entering and leaving facets at the \( (k+1) \)-th iteration, \( \alpha_{\ell-2} = e \) and \( \beta_{\ell-2} = f \) represent the indices of the entering and leaving facets at the \( (k+\ell-2) \)-th iteration, and \( \alpha_{\ell-1} = u \) and \( \beta_{\ell-1} = v \) represent the indices of the entering and leaving facets at the \( (k+\ell-1) \)-th iteration. Then,

\[
a_i x^{k+i} \geq b_q, \text{ for } i = 1, \ldots, k+\ell, \text{ where } \ell \geq 1. \tag{63}
\]

**Proof:** First, we assume by induction that \( a_i x^{k+\ell-1} \geq b_q \) hold. Note that \( a_q \) can be expressed as

\[
a_q^T = \sum_{j \in \mathbb{B}^k \setminus \{a_q \}} y_j^{k+\ell-1} a_j = A_q^{T} B_{k+\ell-1} q^{k+\ell-1}. \tag{64}
\]

Denote \( y = y_q^{k+\ell-1}, A = A_{p+\ell-1}, b = b_{B_{k+\ell-1}}, b_0 = b_q, d^T = a_q, \) and \( x = x^{k+\ell-1} \). Since \( x^{k+\ell-1} \) be the basic solution of (2) at \( (k+\ell-1) \)-th iteration, it follows that \( A_{B_{k+\ell-1}} x^{k+\ell-1} = b_{B_{k+\ell-1}} \) which can be written as

\[
A x = b. \tag{65}
\]
From the inductive assumption \( a_q x^{k+\ell-1} \geq b_q \), we have
\[ d^T x \geq b_0. \] (66)

From (65) and (66), using Farkas-Fan lemma, we conclude that (64) holds with \( y = y_q^{k+\ell-1} \geq 0 \). This shows that
\[ y_{qv}^{k+\ell-1} \geq 0 \] (67)
holds. By induction, we will show that \( a_q x^{k+\ell} \geq b_q \). Following exactly the same argument used in the proof of Lemmas 3.9 and 3.10, it is straightforward to prove the claim. We provide the details here for completeness. First, we have following relations.

\[ B^{k+\ell-1} = B^{k+\ell-2} \cup \{e\} \setminus \{f\}, \quad B^{k+\ell-1} \setminus \{e\} = B^{k+\ell-2} \setminus \{f\}, \] (68)

and
\[ B^{k+\ell} = B^{k+\ell-1} \cup \{u\} \setminus \{v\} = B^{k+\ell-2} \cup \{e, u\} \setminus \{f, v\}. \] (69)

Since \( a_u \) is the entering facet at the \((k+\ell-1)\)-th iteration, and \( \{e, v\} \in B^{k+\ell-1} \), \( a_u \) can be expressed by the facets in \( B^{k+\ell-1} \) as
\[ a_u = \sum_{j \in B^{k+\ell-1}} y_{qj}^{k+\ell-1} a_j = \sum_{j \in B^{k+\ell-1}\setminus\{e, v\}} y_{qj}^{k+\ell-1} a_j + y_{ue}^{k+\ell-1} a_e + y_{uv}^{k+\ell-1} a_v. \] (70)

It follows from (11) that \( y_{uv}^{k+\ell-1} > 0 \), and from (70), we have
\[ a_v = \frac{1}{y_{uv}^{k+\ell-1}} a_v - \frac{y_{ue}^{k+\ell-1}}{y_{uv}^{k+\ell-1}} a_e - \sum_{j \in B^{k+\ell-1}\setminus\{e, v\}} \frac{y_{qj}^{k+\ell-1}}{y_{uv}^{k+\ell-1}} a_j. \] (71)

By the induction assumption, \( a_q x^{k+\ell-1} \geq b_q \), it follows that \( a_q \) is not the entering facet at the \((k+\ell-1)\)-th iteration, i.e., \( a_q \notin B^{k+\ell} \). Using (68), (69), and (71), we can express \( a_q \) by the facets in \( B^{k+\ell-1} \) or the facets in \( B^{k+\ell} \) as
\[ a_q = \sum_{j \in B^{k+\ell}} y_{qj}^{k+\ell} a_j = \sum_{j \in B^{k+\ell-2}\setminus\{f, v\}} y_{qj}^{k+\ell} a_j + y_{qe}^{k+\ell} a_e + y_{qu}^{k+\ell} a_u \] (72)
\[ = \sum_{j \in B^{k+\ell-1}} y_{qj}^{k+\ell-1} a_j = \sum_{j \in B^{k+\ell-2}\setminus\{f, v\}} y_{qj}^{k+\ell-1} a_j + y_{qe}^{k+\ell-1} a_e + y_{qu}^{k+\ell-1} a_u \]
\[ + y_{qv}^{k+\ell-1} \left[ \frac{1}{y_{uv}^{k+\ell-1}} a_v - \frac{y_{ue}^{k+\ell-1}}{y_{uv}^{k+\ell-1}} a_e - \sum_{j \in B^{k+\ell-1}\setminus\{e, v\}} \frac{y_{qj}^{k+\ell-1}}{y_{uv}^{k+\ell-1}} a_j \right] \]
Comparing (72) and (73), we have
\[ y_{k+\ell} = \frac{y_{k+\ell-1}}{y_{uv}^{k+\ell-1}}, \quad y_{k+\ell} = y_{k+\ell-1} - y_{ju}^{k+\ell-1}y_{uv}^{k+\ell-1}, \quad j \in B^{k+\ell} \setminus \{u\}. \]

In view of (67), we have \( y_{pu}^{k+\ell-1} \geq 0 \). Since \( u \) is the index of the entering facet and \( v \) is the index of the leaving facet at \((k + \ell - 1)\)-th iteration, \( y_{uv}^{k+\ell-1} > 0 \) according to (11). This shows \( y_{pu}^{k+\ell} \geq 0 \). Using (73) and (70), we can express \( a_q \) as
\[
a_q = \sum_{j \in B^{k+\ell-2}} y_{k+\ell}^{j}a_j + y_{k+\ell}^{qe}a_e + y_{k+\ell}^{qu} \left[ \sum_{j \in B^{k+\ell-1}} y_{uj}^{k+\ell-1}a_j + y_{ue}^{k+\ell-1}a_e + y_{uv}^{k+\ell-1}a_v \right].
\]

Multiplying \( x^{k+\ell} \) to the right of the items in the bracket of (75), and using Lemma 3.1, \( B^{k+\ell} \setminus \{u\} = B^{k+\ell-1} \setminus \{v\}, v \in B^{k+\ell-1} \setminus \{v\} \), and formula (70), we have
\[
\sum_{j \in B^{k+\ell-1} \setminus \{v\}} y_{uj}^{k+\ell-1}a_j x^{k+\ell} + y_{uv}^{k+\ell-1}a_v x^{k+\ell}
= \sum_{j \in B^{k+\ell-1} \setminus \{v\}} y_{uj}^{k+\ell-1}a_j x^{k+\ell} + y_{uv}^{k+\ell-1}a_v x^{k+\ell} - y_{uv}^{k+\ell-1}b_v
= \sum_{j \in B^{k+\ell-1} \setminus \{v\}} y_{uj}^{k+\ell-1}b_j + y_{uv}^{k+\ell-1}b_v + y_{uv}^{k+\ell-1}(a_v x^{k+\ell} - b_v)
= \sum_{j \in B^{k+\ell-1} \setminus \{v\}} y_{uj}^{k+\ell-1}a_j x^{k+\ell-1} + y_{uv}^{k+\ell-1}(a_v x^{k+\ell} - b_v)
= a_v x^{k+\ell-1} + y_{uv}^{k+\ell-1}(a_v x^{k+\ell} - b_v).
\]

Multiplying \( x^{k+\ell} \) to the right of the both sides of (75), and using (76), \( B^{k+\ell-2} \setminus \{f, v\} = B^{k+\ell} \setminus \{e, u\}, e \in B^{k+\ell-1} \setminus B^{k+\ell} \) (\( a_e x^{k+\ell-1} = b_e = a_e x^{k+\ell} \)), (72), and (74), we have
\[
a_q x^{k+\ell} = \sum_{j \in B^{k+\ell-2}} y_{k+\ell}^{j}a_j x^{k+\ell} + y_{k+\ell}^{qe}a_e x^{k+\ell}
+ y_{qu}^{k+\ell} \left[ \sum_{j \in B^{k+\ell-1} \setminus \{e, v\}} y_{uj}^{k+\ell-1}a_j + y_{ue}^{k+\ell-1}a_e + y_{uv}^{k+\ell-1}a_v \right] x^{k+\ell}
= \sum_{j \in B^{k+\ell-2} \setminus \{f, v\}} y_{k+\ell}^{j}b_j + y_{qe}^{k+\ell}b_e + y_{qu}^{k+\ell} \left[ a_e x^{k+\ell-1} + y_{uv}^{k+\ell-1}(a_v x^{k+\ell} - b_v) \right].
\]
According to the induction assumption, \( a_q x^{k+\ell} \geq b_q \) holds. From (67), the inequalities \( y_{qv}^{k+\ell} \geq 0 \) holds. Since \( a_v \) is the leaving facet at \((k+\ell-1)\)-th iteration, it follows from Lemma 3.2 that \( a_v x^{k+\ell} - b_v > 0 \). Therefore, the last expression of (77) is greater than or equal to \( b_q \). This proves \( a_q x^{k+\ell} \geq b_q \).

3.3 The general case

Summarizing the discussions of the two cases, we obtain the main result:

Theorem 3.1 Let \( a_q \in B^k \) be the leaving facet at the \( k \)-th iteration. Then, for any integer \( \ell \) satisfying \( \ell > 0 \), we have

\[
a_q x^{k+\ell} \geq b_q.
\] (78)

This theorem indicates that for any facet, if it leaves the base, it will never come back to the base because it will be feasible after it leaves the base of (2). Since the initial base has \( d \) facets and the system has \( n \) facets, there are at most \( n-d \) facet pivots, i.e., it needs at most \( n-d \) pivots for the facet pivot method to find a feasible solution, which is an optimal solution because the algorithm maintains \( y_c^k \geq 0 \), and according to Theorem 2.1 this feasible solution must be optimal. We present this conclusion as the following theorem:

Theorem 3.2 For linear programming problem (2). It needs at most \( n-d \) facet pivot iterations for Algorithm 2.1 to find the optimal solution.

4 Conclusion

In this paper, we have shown that the proposed facet algorithm converges fast even in the worst case. The total number of iterations in the worst case is bounded by a linear function of \( n-d \), where \( n \) is the number of constraints and \( d \) is the dimension of the related polytope.
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