Towards an exact solution of FRW type of spacetime
with a generalised Chaplygin gas - an alternative approach

D Panigrahi
Sree Chaitanya College, Habra 743268, India
E-mail: dibyendupanigrahi@yahoo.co.in

Abstract. The evolution of a universe modelled by a generalised Chaplygin gas with an
equation of state $p = -\frac{\alpha}{\rho}$ is studied for a Robertson Walker type of spacetime. The resulting
field equations are highly non linear in the scale factor which is a key equation of this work.
Using this equation previous authors have explained dust dominated and accelerating universe
at the two extreme cases. But to our knowledge we have not as yet come across any explicit
solution of scale factor as function of time. To avoid such incompleteness, we have taken the
first order approximation of the key equation and subsequently have found out the time explicit
exact solutions of scale factor. This solution approaches $\Lambda$CDM model for large scale factor
and the desirable feature of acceleration flip occurs in this case. We have also made a detailed
analysis of flip time both analytically and graphically. Further the whole situation is studied in
the backdrop of well known Raychaudhury equation and a comparison made with the previous
results.

1. Introduction

Three discoveries in the last century have radically changed our understanding of the universe -
as opposed to the idea of Einstein’s static universe Hubble and Slipher(1927) showed that it is
expanding. Secondly CMBR as also primordial nucleosynthesis analysis in the sixties point to an
initial hot dense state of the universe, which has been expanding for the last 13.5 Gyr. Finally,
if we put faith in Einstein’s theory and FRW type of model then as standardised candles type Ia
supernova suggest [1] that the universe is undergoing accelerated expansion with baryonic matter
contributing only five percent of the total energy budget. Later data from CMBR probes [2]
also point to the same finding. This has naturally led a vast chunk of cosmology community
to embark on a quest to attempt to explain the cause of the apparent acceleration. The vexed
question in this field is the possible identification of the processes likely to be responsible for
triggering the late inflation. Researchers are plainly divided into two broad groups - either
modification of the original Einstein’s theory or introduction of any exotic type of fluid like a
cosmological constant or a quintessential type of scalar field. But the popular explanation with
the help of a cosmological constant is beset with serious theoretical problems because absence
of acceleration at redshifts $z \geq 1$ implies that the required value of the cosmological constant
is approximately 120 orders of magnitude smaller than its natural value in terms of Planck
scale [3]. As for the alternative quintessential field [4] we do not in fact have a theory that would
explain, not to mention predict, the existence of a scalar field fitting the bill without violating the realistic energy conditions. Moreover we can not generate this type of a scalar field from any basic principles of physics. Other alternatives include k-essence [5], tachyon [6], phantom [7] and quintom [8]. So there has been a resurgence of interests among relativists, field theorists, astrophysicists and people doing astroparticle physics both at theoretical and experimental levels to address the problems emanating from the recent extra galactic observations without involving any mysterious form of scalar field by hand but looking for alternative approaches based on sound physical principles. Alternatives include, among others, higher curvature theory, axionic field and also Brans- Dicke field. Some people attempted to look into the problem from a purely geometric point of view - an approach more in line with Einstein’s spirits. For example, Wanas [9] introduced torsion while Neupane [10, 11] modifies the spacetime with a warped factor in 5D spacetime in a brane like cosmology and finally addition of extra spatial dimensions in physics as an offshoot of prediction from the string theory [12, 13, 14, 15, 16]. While this torsion inspired inflation has certain desirable features the problem with Wanas’ model is that the geometry is no longer Riemannian. Further a good number of people [17, 18, 19, 20] have done away with the concept of homogeneity itself and have argued that accelerating model and consequent introduction of exotic matter field have to be invoked only in FRW type of cosmology.

While the above mentioned alternatives to explain away the observed acceleration of the current phase have both positive and negative aspects the one that caught the attention of large number of workers is the introduction of a Chaplygin type of gas as new matter field to simulate a sort of dark energy. The form of the matter field is later generalised through the addition of an arbitrary constant as exponent over the mass density [21, 22] and is generally referred to as generalised chaplygin gas(GCG) [23]. In the present work we have revisited the dynamics of the FRW model taking GCG as matter field and have tried to discuss some as yet unexplored region and have got some interesting results. We have organised the paper as follows: In section 2 we have talked about the filed equations and equation of state only. In section 3 the mathematical formulation is given and we have ended up with a hypergeometric solution and also an effective equation of state as

\[ \rho = W(t)p \]

So depending on initial conditions our model mimics both \( \Lambda CDM \) and quiescence models and the evolution is also shown graphically. In section 4 we discuss the interesting feature in our analysis where we have taken the first order approximation of the field equation as key equation and subsequently found out the exact solutions. We are not aware of attempts of similar kind in the past literature. We have also made a detailed analysis of flip time both analytically and graphically in this section. In section 5 these conclusions are checked in the framework of well known Raychaudhury equation. The paper ends with a discussion in section 6.

2. Field Equations
We consider a spherically symmetric flat homogeneous spacetime given by

\[ ds^2 = dt^2 - a^2(t) \left\{ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\} \]

(1)

where the scale factor, \( a(t) \) depends on time only. A comoving coordinate system is taken such that \( u^0 = 1, u^i = 0 \) (\( i = 1, 2, 3 \)) and \( g^{\mu\nu}u_\mu u_\nu = 1 \) where \( u_i \) is the 4-velocity. The energy momentum tensor for a dust distribution in the above defined coordinates is given by

\[ T^\mu_\nu = (\rho + p)\delta^\mu_\nu - p\delta^\mu_\nu \]

(2)

where \( \rho(t) \) is the matter density and \( p(t) \) the isotropic pressure. The independent field equations for the metric (1) and the energy momentum tensor (2) are given by

\[ \frac{3}{a^2} \frac{d^2a}{dt^2} = \rho \]

(3)
\[
\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = -p
\]

(4)

From the conservation law we get

\[
\nabla \nu T_{\mu \nu} = 0
\]

(5)

which, in turn, yields for the line element (1)

\[
\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0
\]

(6)

At this stage we consider a generalised Chaplygin type of gas (GCG) obeying an equation of state \[23\]

\[
p = -\frac{A}{\rho^\alpha}
\]

(7)

where \(A\) is a positive universal constant related to \(A_s = \frac{A}{\rho_0}\) \& \(A_s > 0\) \[24\] and the quantity \(\rho_0\) represents the present density of GCG. \(\alpha\) is also a positive constant belonging to the range \(0 < \alpha \leq 1\). This range ensures that the sound velocity \(c_s^2 = \alpha \frac{A}{\rho^{1+\alpha}}\) does not exceed the velocity of light.

With the help of equations (6) \& (7) a little mathematics shows that the expression for density comes out to be

\[
\rho(a) = a^{-3} \left[ 3(1 + \alpha) \int Aa^{3(1+\alpha)-1} \, da + c \right]^{\frac{1}{1+\alpha}}
\]

(8)

where \(c\) is an integration constant. The above equation (8) yields a first integral as

\[
\rho = \left[ A + \frac{c}{a^{3(1+\alpha)}} \right]^{\frac{1}{1+\alpha}}
\]

(9)

Plugging in the expression of \(\rho\) from equations (3) and (9) we finally get

\[
3\frac{\dot{a}^2}{a^2} = \left[ A + \frac{c}{a^{3(1+\alpha)}} \right]^{\frac{1}{1+\alpha}}
\]

(10)

3. Cosmological dynamics

It is very difficult to get the exact temporal behaviour of the scale factor, \(a(t)\) from the equation (10) in a closed form because integration yields elliptical solution only and we get hypergeometric series. However, the equation (10) does give significant information under extremal conditions as briefly discussed below.

**Deceleration Parameter:** At the early stage of the cosmological evolution when the scale factor \(a(t)\) is relatively small, the second term of right hand sight of the last equation (10) dominates which has been already discussed in the literature \[23\]. So we will be very brief on this point. From the expression of the deceleration parameter, \(q\) we get

\[
q = -\frac{1}{H^2 a} \frac{d}{dt} (H^{-1}) - 1 = -\frac{1}{2} + \frac{3}{2} \frac{p}{\rho}
\]

(11)

where \(H\) is the Hubble constant. With the help of the Equation of State (EoS) given by (7) we find
\[ q = \frac{1}{2} - \frac{3A}{2} \frac{1}{\rho^{a+1}} \]  

which via equation (9) gives

\[ q = \frac{1}{2} - \frac{3A}{2} \left[ A + \frac{c}{a^{3(1+\alpha)}} \right]^{-1} \]  

Again at flip time, i.e. when \( q = 0 \) the scale factor becomes

\[ a = \left( \frac{c}{2A} \right)^{\frac{1}{3(1+\alpha)}} \]  

Again, in terms of redshift \( (1 + z = \frac{a}{a_0}) \) where \( a_0 \) is the scale factor of present time, we may write the equation (12) in the following form

\[ q = \frac{1}{2} - \frac{3A}{2} \left[ A + \frac{c(1 + z)^{3(1+\alpha)}}{a_0^{3(1+\alpha)}} \right]^{-1} \]  

and the redshift at flip \( (z_f) \) is given by

\[ z_f = \left( \frac{2A}{c} \right)^{\frac{1}{3(1+\alpha)}} a_0 - 1 \]  

\[ \begin{align*} 
(a) \text{ The variations of } q \text{ and } \rho \\
(b) \text{ The variations of } q \text{ and } z 
\end{align*} \]

\[ \text{Figure 1. Both the figures show that late flip occur for higher values of } \alpha. \text{ Taking } A = 3, c = 1. \]

As the universe expands \( \rho \) decreases with time such that the second term in the equation(12) increases pointing to the occurrence of a flip when the density attains a critical value given by

\[ \rho = \rho_{flip} = (3A)^{-\frac{1}{1+\alpha}}. \]

This flip density \( \rho_{flip} \) depends on the exponent \( \alpha \) such that at the larger value of \( \alpha \) the flip density decreases, i.e., flip occurs at a lower density, i.e., it occurs at a later time. We know from Lima’s work [25] that the value of \( \alpha \) is restricted to \( 0.9 < \alpha < 1 \) to make the acceleration a recent phenomenon. This finding is particularly relevant to our case in the sense that higher values of \( \alpha \) signify a lower \( \rho_{flip} \), i.e., more recent accelerating phase. The above analysis is in conformity with the nature of \( q \sim \rho \) curve in figure-1(a).

Again from equation (16) we have seen that the redshift at flip \( (z_f) \) depends on \( A, c \) and \( \alpha \) (see figure-1(b)). A little analysis shows for \( z_f > 0 \), the value of \( A \) will be \( A > \frac{c}{2} \) which is independent on \( \alpha \).

Now we discuss the extremal cases to understand the evolution of the universe.
CASE A : At the early stage when the scale factor, \(a(t)\) is very small the equation (13) reduces to

\[ q = \frac{1}{2} \]  

representing a dust dominated universe.

CASE B : Using the equations (7) & (9) straightforward calculations yield an effective EoS at the late stage of evolution as

\[ p = \rho \left[ -A \rho^{(1+\alpha)} \right] = \left[ -1 + \frac{1}{A} \frac{c}{a^{3(1+\alpha)}} \right] \rho = W(t) \rho \]  

where

\[ W(t) = -1 + \frac{1}{A} \frac{c}{a^{3(1+\alpha)}} \]  

which is a function of time only. This is clearly at variance with the earlier works [23] where the effective EoS shows no time dependence. We also find that at the late stage of evolution as \(a(t) \to \infty\), \(W(t) \to -1\) so we asymptotically get \(p = -\rho\) from this Chaplygin type of gas, which corresponds to an empty universe with cosmological constant such that the equation (11) implies that the deceleration parameter, \(q\) reduces to \(-1\). Interestingly \(W(t)\) always remains greater than \(-1\), thus avoiding the undesirable feature of big rip.

As pointed earlier the key equation (10) is not amenable to explicit solution as function of time. So cosmological variables like sale factor, flip time etc. can not be explicitly obtained. To avoid such type of incompleteness or to get the flip time etc, an alternative approach has been attempted in the next section.

4. An alternative approach :

As we are considering a late evolution of our model the second term of right hand sight(RHS) in the equation (10) is almost negligible compared to the first term. We know that the Chaplygin gas equation of state explains only from dust dominated to present accelerating universe. So the scale factor should be large enough and it may not be inappropriate to consider only the first order approximation of the binomial expansion of RHS of the equation (10). Here we find an exact solution of the first order approximation of the equation (10). Authors of this work are not aware of attempts of similar kind in any earlier work. So this is clearly a new result.

Now from equation (10) we get as first order approximation, the equation at the late stage of evolution becomes (neglecting higher order terms)

\[ 3 \frac{a^2}{a^2} = A^{1+\alpha} + \frac{c}{(1+\alpha)A^{1+\alpha}} a^{-3(1+\alpha)} \]  

Solving the equation (20) we get an explicit solution of the scale factor as follows

\[ a(t) = a_0 \sinh^n \omega t \]  

where, \(a_0 = \left( \frac{B}{A^{1+\alpha}} \right)^{\frac{1}{1+\alpha}}\); \(n = \frac{2}{3(1+\alpha)}\) and \(\omega = \sqrt{3} \frac{c}{2(1+\alpha)A^{1+\alpha}}\). In figure-2 we have seen the evolution of the scale factor \(a\) with time \(t\) for different values of \(\alpha\). Now using equation (3), (4) and (21) we can write the expression of \(p\) and \(\rho\) as follows.

\[ \rho = 3n^2 \omega^2 \coth^2 \omega t \]
Figure 2. The variation of $a$ and $t$ for different values of $\alpha$ with $A = 3.0$ & $c = 1$

and

$$p = n\omega^2 \left\{ (2 - 3n) \coth^2 \omega t - 2 \right\}$$

The effective equation of state is given by

$$W = \frac{p}{\rho} = \frac{2 - 3n}{n^2} - \frac{2}{3} \tanh^2 \omega t$$

From equation (24) it is seen that for $n = \frac{2}{3}$, $W$ always negative and it approaches from 0 to $-1$ which gives rise to a Phantom type of cosmology. But it is seen that $\alpha$ vanishes for $n = \frac{2}{3}$. We are dealing with GCG model where $0 < \alpha < 1$ should be maintained. So we may not consider this value of $n$. It is interesting to note that the identical result follows from equation (19).

Figure 3. The variation of $W$ with $t$ are shown in this figure. Taking $\omega = 1$. This figure clearly shows that $w$ always greater than $-1$.

From equation (21) we get the deceleration parameter
The equation (25) gives that the exponent $n$ determines the evolution of $q$. A little analysis of equation (25) shows that (i) if $n > 1$ we get only acceleration, no flip occurs in this condition. But for $n > 1$ gives $-\frac{1}{3} > \alpha$, which is physically unrealistic, since we know that $\alpha > 0$. (ii) Again, if $0 < n > \frac{2}{3}$ it gives early deceleration and late acceleration and in this condition $\alpha > 0$, so the desirable feature of flip occurs which agrees with the observational analysis for positive values of $\alpha$.

$$q = \frac{1 - n \cosh^2 \omega t}{n \cosh^2 \omega t} \quad (25)$$

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![Figure 4](image-url)  
*Figure 4. The variation of $q$ with $t$ are shown in this figure. Taking $A=3$. This figure clearly shows that late flip occurs at $\alpha = 1.0$.\* Figures-4 shows the variation of $q$ with $t$ for different values of $\alpha$ where flip occurs. It is seen that the flip occurs at different instance for different values of $\alpha$ but this change is not monotonous. We would like to focus on the occurrence of late flip as because all observational evidences suggest that accelerating phase is a recent phenomena. It is interesting to note that the late flip also depends on the value of $A$. Again from the observational evidences [25] we know that the value of $\alpha$ is nearly equal to unity. So we may check the value of $A$ for late flip for $\alpha = 1$. We have chosen three different values of $\alpha$ in figure-4 but it is seen that the late flip occurs at $\alpha = 1$ which is in agreement with the observational analysis. Now the flip time ($t_f$) will be in this case

$$t_f = \frac{1}{\omega} \cosh^{-1} \left( \sqrt{\frac{1}{n}} \right) \quad (26)$$

Using equation (26) we have drawn the figure-5 where the variation of $t_f$ with $\alpha$ for different values of $A$ is shown. It is seen that the variation of $t_f$ with $\alpha$ does not change uniformly. When the value of $\alpha$ is small $t_f$ increases with $\alpha$; after a certain value of $\alpha$, $t_f$ decreases as $\alpha$ increases. That means we get a maximum value of $t_f$ for different value of $A$.

5. Raychaudhuri Equation

It may not be out of place to address and compare the situation discussed in the last section with the help of the well known Raychaudhuri equation [26], which in general holds for any
cosmological solution based on Einstein’s gravitational field equations. With matter field expressed in terms of mass density and pressure Raychaudhury equation reduces to a compact form as

$$\dot{\theta} = -2(\sigma^2 - \omega^2) - \frac{1}{3} \theta^2 - \frac{8\pi G}{2} (\rho + 3p)$$

(27)

in a co moving reference frame. Here $p$ is the isotropic pressure and $\rho$ is the energy density from varied sources. Moreover other quantities are defined with the help of a unit vector $v^\mu$ as under

- The expansion scalar $\theta = v^i; _i$
- $\sigma^2 = \sigma_{ij}\sigma^{ij}$
- The shear tensor $\sigma_{ij} = \frac{1}{2} (v_{ij} + v_{ji}) - \frac{1}{2} (\dot{v}_i v_j + \dot{v}_j v_i) - \frac{1}{3} v^\alpha_{;\alpha} (g_{ij} - v_i v_j)$
- The vorticity tensor $\omega_{ij} = \frac{1}{2} (v_{ij} - v_{ji}) - \frac{1}{2} (\dot{v}_i v_j - \dot{v}_j v_i)$

(28)

We can calculate an expression for effective deceleration parameter as

$$q = \frac{-\dot{H} + H^2}{H^2} = -1 - 3 \frac{\dot{\theta}}{\theta^2}$$

(29)

which allows us to write,

$$\theta^2 q = 6\sigma^2 + 12\pi G (\rho + 3p)$$

(30)

Case-1: With the help of the equations (7), (9) & (30) we finally get,

$$\theta^2 q = 6\sigma^2 + 12\pi G \rho^{-\alpha} \left[ -2A + \frac{c}{a^{3(1+\alpha)}} \right]$$

(31)

In our case as we are dealing with an isotropic rotation free spacetime both the shear scalar and vorticity vanish, i.e., $\sigma = 0$ and $\omega = 0$, the equation (31) now reduces to

$$\theta^2 q = 12\pi G \rho^{-\alpha} \left[ -2A + \frac{c}{a^{3(1+\alpha)}} \right]$$

(32)
It follows from the equation (32) that flip occurs (i.e., at \( q = 0 \)) when
\[
a = \left( \frac{c}{2A} \right) \frac{1}{1 + \alpha}
\] (33)

Now \( q < 0 \), at \( a > \left( \frac{c}{2A} \right) \frac{1}{1 + \alpha} \) i.e., acceleration takes place in this case.

It also follows from Raychaudhuri equation that our solution is in conformity with early deceleration and late acceleration. This results also agrees with the equation (25) for \( \alpha > 0 \). It is interesting to note that the expression of scale factor at flip expressed by equations (14) and (33) are identical.

**Case 2:** Now we have discussed our alternative approach in the context of Raychaudhuri equation where shear scalar and vorticity vanish because we have consider an isotropic rotation free spacetime. Now using equations (22), (24) and (30) we finally get after straightforward calculation that
\[
\theta^2 q = 72\pi G n \omega^2 \frac{(1 - n \cosh^2 \omega t)}{\sinh^2 \omega t}
\] (34)

We may calculate the flip time from equation (34) as
\[
t_f = \frac{1}{\omega} \cosh^{-1} \left( \frac{1}{\sqrt{n}} \right)
\] (35)
which is identical with the equation (26).

6. Concluding Remarks

We have investigated the late time acceleration with a generalised Chaplygin type of a gas in spherically symmetric homogeneous model. There is a continuing debate on the exact range of the values of the exponent, \( \alpha \) which generalizes the original Chaplygin gas, although most observations point to the value of \( \alpha \) as nearly equal to unity. Our findings are summarised as follows:

The equation (10) is the key equation of this work. But since it is highly nonlinear one can not solve it in an exact form. Previously authors have suggested their prediction under extremal conditions only, i.e., the solution gives initially dust dominated universe and it approaches \( \Lambda \)CDM model for large \( a(t) \), however we can not predict the evolution of this scale factor with time or flip time explicitly. To avoid this incompleteness, an alternate approach is suggested where we have neglected the higher order terms of binomial expansion of RHS of equation (10). The main reason behind it that the scale factor should be large enough at dust dominated universe and it may not be inappropriate if we consider only the first order term of the binomial expansion of RHS of the equation (10) which is shown in (20). We get the solution in exact form shown in the equation (21).

We have first discussed the GCG model in terms of \( q \) at extremal cases. The change in \( q \) with \( \rho \) and \( z \) is studied in this case where we see that the late flip occurs when the value of \( \alpha \) is nearly equal to unity. From the observational findings [25] we know that the value of \( \alpha \) lies between 0.9 to 1.0. We have also shown that for positive \( z_f \) the value of \( A > \frac{c}{2} \) which is independent on \( \alpha \).

We would like to emphasize this work on the alternative approach because here we get the time explicit solution of scale factor after solving the equation (20). We have also calculated the pressure, energy density and the effective equation of state. From equation (27) it is seen that for \( n = \frac{2}{3} \), \( w \) always negative and it approaches from 0 to \(-1\) which gives rise to a Phantom.
type of cosmology. It is interesting to note that the identical result follows from equation (19). Again for $0 < n < \frac{2}{3}$ equation (30) gives $-1 < W < 1$ which shows the desirable feature of flip.

The deceleration factor $q$ was calculated and flip occurs at $\alpha > 0$. The figure-4 shows the variation of $q$ with $t$ where $\alpha$ is treated as a parameter. It is seen that flip occurs at different instances for different values of $\alpha$ but this change is not monotonic. From this graph we see that late flip occurs at $\alpha = 1$. We also calculate the flip time $t_f$ which is shown in equation (26).

The whole exercise is discussed in the context of Raychaudhuri equation. As expected the results are in broad agreement with the previous findings.

The main drawback of the present analysis is that we have not been able so far to constrain the model parameters with the help of observational data as is customary in relevant works in this field. It would also be a nice idea to use redshifts in place of cosmic time in most of the equations particularly in drawing the graphs. That would have been more consistent with the current nomenclature. Both the issues will be addressed in our future work.

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