PoARX Modelling for Multivariate Count Time Series

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Abstract

This paper introduces multivariate Poisson autoregressive models with exogenous covariates (PoARX) for modelling multivariate time series of counts. We obtain conditions for the PoARX process to be stationary and ergodic before proposing a computationally efficient procedure for estimation of parameters by the method of inference functions (IFM) and obtaining asymptotic normality of these estimators. Lastly, we demonstrate an application to count data for the number of people entering and exiting a building, and show how the different aspects of the model combine to produce a strong predictive model. We conclude by suggesting some further areas of application and by listing directions for future work.

1 Introduction

The abundance of data brought about by the digital revolution has increased the availability of time series of counts. Such data appear in many areas, including statistics, econometrics, and the social and physical sciences. For independent count data, generalised linear models (McCullagh and Nelder, 1989) are widely used. The most popular distribution is the Poisson distribution, which has attractive properties and is in some respects the count analogue of the Gaussian distribution. One restrictive property of the Poisson distribution however is that the mean and the variance are equal – this is rarely observed in applications. Naturally, many alternatives have been proposed, see Cameron and Trivedi (2013) for a comprehensive review. In particular, the most common departures from the Poisson distribution are models based on the negative binomial distribution, hurdle models, zero-inflated models, Poisson-Normal mixture models, and finite mixtures models. Fokianos (2012) considers integer-valued autoregressive models for count time series and discusses estimation for both the Poisson model and the negative binomial model. Whilst the negative binomial model can account for over-dispersion present in the data, we have yet to mention a fix for under-dispersed data. McShane et al. (2008) developed a count model based on the Weibull distribution that can handle both under-dispersed and over-dispersed data. Building on this idea, Kharrat et al. (2018) extended this approach to create a rich and flexible family of renewal count distributions, which greatly extends the toolbox of distributions available for modelling count data.

While for independent data the focus is on the provision of suitable distributions, in time series modelling the dependence presents additional challenges. Models developed for modelling the dynamics of (continuous) time series often provide adequate results for count data. The classic examples are ARMA models (Box and Jenkins 1970) and their multivariate extensions, which can be dealt efficiently with state space methods (Durbin and Koopman 2012). A fruitful approach, employed in ARCH and GARCH models (Engle 1982; Bollerslev 1986), uses a separate equation to model directly the dependence of the variance on the past. In

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order to improve the predictive accuracy, the aforementioned models have been augmented with additional exogenous covariates. ARMAX models (Hannan and Deistler 1988; Likothanassis and Demiris 1998) allowed covariates to be added to processes following an ARMA model, while GARCH-X (Engle 2002) added the same feature to GARCH models. Shephard and Sheppard (2010) introduced HEAVY models to improve prediction in high-frequency data, while Hansen et al. (2012) developed the Realised GARCH model, a class of GARCH-X models for returns with an integrated model for realized measures of volatility. There have been many efforts to extend the continuous GARCH model to the multivariate case, summarised by Bauwens et al. (2006). These fall into three categories: direct generalisations of the univariate GARCH model (VEC, BEKK and factor models), linear combinations of univariate GARCH models (generalised orthogonal models and latent factor models), and nonlinear combinations of univariate GARCH models (DCC, GDC and copula-GARCH models).

The above models do not make specific provision for the non-negativity and integer-valued nature of count data. One approach has been to use the generalised linear model (GLM) methodology for time series data with an appropriate distribution, see Kedem and Fokianos (2002) for more details. Another approach is to use a thinning operator to imitate ARMA models. These models are called integer autoregressive moving average (INARMA) models and details can be found in Wei (2008). Furthermore, an integer-valued analogue of the GARCH model was proposed by Ferland et al. (2006), called INGARCH, which uses Poisson deviates rather than normal innovations. Fokianos et al. (2009) also used the GARCH model for inspiration, as they aspired to create a Poisson model for integer-valued time series containing an autoregressive feedback mechanism similar to the volatility in GARCH models. They called this model the Poisson autoregressive model and later the properties were extended to negative binomial autoregressive models by Christou and Fokianos (2014). Agosto et al. (2016) proposed a class of dynamic Poisson models allowing for additional (exogenous) covariates to strengthen the predictions. This was referred to as the Poisson autoregressive model with exogenous covariates (PARX).

All models for count data mentioned so far are univariate. Whilst the Poisson distribution has been widely used for univariate count models, multivariate generalisations have been relatively sparse so far. Nouye et al. (2017) provide a summary of multivariate (Poisson) distributions for count data, with methods including multivariate extensions of a parametric (Poisson) distribution and copula modelling using univariate (Poisson) marginal distributions. For example, Liu (2012) formulates a bivariate Poisson integer-values GARCH (BINGARCH) model using the parametric bivariate Poisson distribution and argues that, given a suitable multivariate Poisson distribution, his framework is capable of dealing with the multivariate case. For predicting the scores of football matches, Koopman and Lit (2015) have applied a parametric bivariate Poisson model. McHale and Scarf (2011) have used Frank’s copula with Poisson and negative binomial marginal distributions, and Boshnakov et al. (2017) have used Frank’s copula with Weibull count distributions as marginal distributions.

Our interest in this article lies in the modelling of multivariate count data. We use a copula approach to extend the (univariate) PARX model of Agosto et al. (2016) to multivariate count time series. This approach is flexible and tractable. Use of covariates in the Poisson model offers clear potential for better modelling and by including the time series covariates we allow over-dispersed data to be considered by our model. Implementation in R (R Core Team 2017) is available in the developmental package PoARX (Halliday and Boshnakov 2018).

This paper is organised as follows. Section 2 introduces the multivariate PoARX model and gives stationarity and ergodicity conditions. In Section 3 we discuss estimation of parameters by the method of inference functions (IMF) and obtain asymptotic results for the resulting estimators. Next, we consider prediction in Section 4 looking at the generating functions for future horizons. Then we demonstrate an application of the PoARX model in Section 5 by analysing a bivariate time series of count data from Ihler et al. (2006). The time series represent the number of people entering and exiting a building on the University of California, Irvine
UCI) campus. Exogenous covariates, such as the occurrence of a meeting or conference are included in the model to aid predictive accuracy. We summarise our findings in Section 6 and outline suggestions for future work.

2 The multivariate PoARX model

In this section we present the new class of models, introducing the necessary background material about the univariate PoARX model and copulas, before focusing on the two-dimensional case and generalising to higher dimensions. For the purpose of this article we focus on using Frank’s copula to capture dependence between time series, but any suitable copula could be used.

2.1 The univariate PoARX model

First, a note on terminology – Agosto et al. (2016) use the abbreviation PARX for this model but we prefer PoARX since it seems to suggest more clearly “Poisson” and avoids confusion with other meanings of “P” in similar abbreviations. For example, PAR is often used to mean periodic autoregression.

Let \( \{Y_t; t = 1, 2, \ldots\} \) denote an observed time series of counts, so that \( Y_t \in \{0, 1, 2, \ldots\} \) for all \( t = 1, 2, \ldots \). Further, let \( x_{t-1} \in \mathbb{R}^r \) denote a vector of additional covariates considered for inclusion in the model. We say that \( \{Y_t\} \) is a univariate PoARX(p,q) process and write \( \{Y_t\} \sim \text{PoARX}_1(p,q) \), if its dynamics can be written as follows:

\[
Y_t | F_{t-1} \sim \text{Poisson}(\lambda_t),
\]

\[
\lambda_t = \omega + \sum_{l=1}^{p} \alpha_l Y_{t-l} + \sum_{l=1}^{q} \beta_l \lambda_{t-l} + \eta \cdot x_{t-1},
\]

where \( \sigma \)-field of past knowledge, \( \sigma\{Y_1, \ldots, Y_{t-1}, \lambda_1, \ldots, \lambda_{t-1}, x_1, \ldots, x_{t-1}\} \), Poisson(\( \lambda \)) denotes a Poisson distribution with intensity parameter \( \lambda, \omega \geq 0 \) is an intercept term, \( \{\alpha_1, \ldots, \alpha_p\} \) and \( \{\beta_1, \ldots, \beta_q\} \) are non-negative autoregressive coefficients, and \( \eta \) is a vector of non-negative coefficients for the exogenous covariates. Thus, the model for the intensity, \( \lambda_t \), uses the past \( p \) values of the process, the past \( q \) values of the intensity and the covariates.

In order to ensure that the process is stationary and ergodic with polynomial moments of a given order, we place two further restrictions on the model (Agosto et al., 2016). Firstly, the autoregressive coefficients must obey the following condition,

\[
\max_{\{p,q\}} \sum_{i=1}^{p+q} (\alpha_i + \beta_i) < 1.
\]

Additionally, we require that each component of the exogenous covariates, denoted \( x_t(k) \) to avoid confusion later, follows a Markov structure, that is,

\[
x_t(k) = g(x_{t-1}(k), \ldots, x_{t-m}(k); \epsilon_t), \quad k = 1, \ldots, r,
\]

for some \( m > 0 \) and some function \( g(\mathbf{x}, \epsilon) \) with vector \( \mathbf{x} \) independent of the observed \( Y_t \) and unobserved \( \lambda_t \), and with \( \epsilon_t \) an i.i.d. error term.

2.2 Copulas

Copulas provide a well-defined approach to model multivariate data, with the dependence structure considered separately from the univariate margins (Joe, 2005). A copula, \( C \), is a multivariate distribution function with all univariate margins having the \( U(0,1) \) distribution (Joe, 1997).
More specifically, let $U_i \sim U(0, 1)$ for $i = 1, \ldots, K$, be uniformly distributed random variables, not necessarily independent. Their joint distribution function is the copula

$$C(u_1, \ldots, u_K) = \Pr(U_1 \leq u_1, \ldots, U_K \leq u_K), \quad 0 \leq u_1, \ldots, u_K \leq 1.$$  

In particular, the copula $C$ is a function mapping the $K$-dimensional unit cube, $[0, 1]^K$, onto the interval $[0, 1]$. Note that the distribution corresponding to the copula is also called a copula.

The dependence structure for the random variables $U_1, \ldots, U_K$ is contained in $C$, parametrised by a dependence parameter $\rho$, which can be a vector. Copula theory has developed from a theorem by Sklar (1959), which states that any multivariate distribution can be represented as a function of its marginals.

**Theorem 1** (Sklar’s Theorem). Let $F$ be a joint distribution function with marginals $F_1, \ldots, F_K$. Then there exists a copula $C: [0, 1]^K \to [0, 1]$ such that

$$F(y_1, \ldots, y_K) = C(F_1(y_1), \ldots, F_K(y_K)), \quad y_1, \ldots, y_K \in \mathbb{R}. $$

Copulas allow for flexible joint modelling of multivariate data whilst retaining control over the dependence structure between the variables. Whilst the copula must act upon uniform random variables, it is straightforward to apply the probability integral transform (Angus, 1994) to the respective variables before fitting the copula to find the dependence parameter.

An important class of copulas are called Archimedean copulas. They are developed using Laplace transforms and mixtures of powers of univariate densities to create multivariate distributions. They have many nice properties and can be constructed easily (Nelsen, 2006) from a generator function $\varphi(t)$ and its pseudo-inverse, $\varphi^{-1}(t)$, defined as follows.

**Definition 1** (Pseudo-Inverse). Let $\varphi$ be a continuous, strictly decreasing function from $I = [0, 1]$ to $[0, \infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of $\varphi$ is:

$$\varphi^{-1}(t) = \begin{cases} \varphi^{-1}(t) & 0 \leq t \leq \varphi(0), \\ 0 & \varphi(0) \leq t \leq \infty. \end{cases}$$

The pseudo-inverse, $\varphi^{-1}$, is continuous and non-increasing on $[0, \infty]$ and strictly decreasing on $[0, \varphi(0)]$. If $\varphi(0) = \infty$, then $\varphi^{-1}(t) = \varphi^{-1}(t)$.

An Archimedean copula in $K$ dimensions is constructed by the following equation, given a generator function $\varphi(\cdot)$ (Joe, 1997).

$$C(u_1, \ldots, u_K) = \varphi^{-1}\left(\sum_{i=1}^{K} \varphi(u_i)\right).$$  

(4)

To ensure that this satisfies the conditions for a copula, see the conditions placed on $\varphi(\cdot)$ and $\varphi^{-1}(\cdot)$ in McNeil and Neslehová (2009).

Frank’s copula (Nelsen, 2006) is one example of an Archimedean copula where the dependence parameter can take any value except zero in the two-dimensional case ($\rho \in \mathbb{R} \setminus \{0\}$). This is an advantage of Frank’s copula over many other common Archimedean copulas, as we can account for both positive and negative dependence. The generator function is

$$\varphi_\rho(t) = -\log\left(\frac{\exp(-\rho t) - 1}{\exp(-\rho) - 1}\right),$$  

(5)

and its pseudo-inverse can be written explicitly as

$$\varphi_\rho^{-1}(t) = \varphi_\rho^{-1}(t) = -\frac{1}{\rho} \log (1 + \exp(-t)(\exp(-\rho) - 1)).$$  

(6)
By substituting these functions into Equation (4) we obtain Frank’s copula. Since \( \varphi_\rho(0) = \infty \), Equation (4) is true for all \( t \geq 0 \). We use the subscript \( \rho \) to distinguish Frank’s copula from the general case.

In higher dimensions, the dependence parameter is limited to values in \((0, \infty)\), but in any case the limit as \( \rho \to 0 \) corresponds to independence. Indeed, from the easily verifiable limits \( \lim_{\rho \to 0} \varphi_\rho(t) = -\log(t) \) and \( \lim_{\rho \to 0} \varphi_\rho^{-1}(t) = \exp(-t) \), it follows that

\[
\lim_{\rho \to 0} C_\rho(u_1, \ldots, u_K) = \exp \left( - \sum_{i=1}^{K} - \log(u_i) \right) \\
= \exp \left( \log \left( \prod_{i=1}^{K} u_i \right) \right) \\
= \prod_{i=1}^{K} u_i,
\]

which is the joint cumulative density function of independent \( U(0,1) \) random variables.

To conclude the discussion of copulas, we give the probability mass function (pmf) for \( K \)-dimensional discrete distributions (Nelsen 2006). In the discrete case the copula is no longer unique due to the presence of stepwise marginal distribution functions (Joe 2014). Despite this issue, copula models are still valid constructions for discrete distributions (Genest and Nešlehová 2007). The pmf is given as

\[
\Pr(Y_1 = y_1, \ldots, Y_K = y_K) = \sum_{l_1=0}^{y_1} \cdots \sum_{l_K=0}^{y_K} (-1)^{l_1+\cdots+l_K} \Pr(Y_1 \leq l_1, \ldots, Y_K \leq l_K) \\
= \sum_{l_1=0}^{1} \cdots \sum_{l_K=0}^{1} (-1)^{l_1+\cdots+l_K} C(F_1(y_1-l_1), \ldots, F_K(y_K-l_K)),
\]

where \( C \) is any copula from Sklar’s theorem.

### 2.3 The bivariate PoARX model

We start with the two-dimensional case since it is of interest on its own and the notation is somewhat simpler. Let \( \{Y_t = (Y_t^1, Y_t^2), \ t = 1, 2, \ldots\} \) be a bivariate time series of counts with associated exogenous covariates \( \{x_{t-1}^j = (x_{t-1}^j(1), x_{t-1}^j(2))^\top, j = 1, 2\} \). Then the collection of exogenous covariates associated with \( Y_t \) is the matrix

\[
x_{t-1} = (x_{t-1}^1, x_{t-1}^2)^\top = \begin{bmatrix} x_{t-1}^1(1) & x_{t-1}^1(2) \\ x_{t-1}^2(1) & x_{t-1}^2(2) \end{bmatrix}.
\]

We say that \( \{Y_t\} \) is a bivariate PoARX\((p,q)\) process and write \( \{Y_t\} \sim \text{PoARX}_2(p,q) \), if each of the component time series is a univariate PoARX process (see Equation (1)) and the joint conditional distribution is a copula Poisson.

More formally, let \( D(\lambda, \lambda^2; \rho) \) be a bivariate distribution based on Frank’s copula with dependency parameter \( \rho \) and marginals \( \text{Poisson}(\lambda) \) and \( \text{Poisson}(\lambda^2) \). Let also \( \{Y_t^1\} \) and \( \{Y_t^2\} \) be univariate PoARX processes with intensities \( \lambda_j^1 \), for \( j = 1, 2 \). Letting \( \lambda_t = (\lambda_t^1, \lambda_t^2) \), denote by \( \mathcal{F}_{t-1} \) the \( \sigma \)-field generated by all past observations and exogenous covariates:

\[
\mathcal{F}_{t-1} = \sigma\{Y_{1-}, \ldots, Y_{t-1}, \lambda_{1-}, \lambda_{2-}, x_{1-}, \ldots, x_{t-1}\}.
\]

The process \( \{Y_t = (Y_t^1, Y_t^2), \ t = 1, 2, \ldots\} \) is a PoARX\(_2(p,q)\) process if the conditional distribution of \( Y_t \) is

\[
Y_t \mid \mathcal{F}_{t-1} \sim D(\lambda_t; \lambda_t^2; \rho),
\]
where $\lambda_1^j, \lambda_2^j$ are the intensities of $\{Y_1^j\}$ and $\{Y_2^j\}$, respectively, with dynamics specified by the equations:

$$ Y_i^j \mid F_{t-1} \sim \text{Poisson}(\lambda_i^j), \quad j = 1, 2; $$

$$ \lambda_i^j = \omega^j + \sum_{l=1}^{p} \alpha_i^j Y_{i-1}^j + \sum_{l=1}^{q} \beta_i^j \lambda_{i-l}^j + \eta^j \cdot x_{i-l}^j, \quad j = 1, 2; $$

where $\alpha_i^j, \beta_i^j \geq 0$ denote coefficients for the past values of the observations and intensities respectively, $\eta^j$ denotes the vector of (non-negative) coefficients for the exogenous covariates, and $\omega^j \geq 0$ denotes an (optional) intercept term.

From the above specifications it follows that the (bivariate) conditional distribution function of $Y_t$ is

$$ F(y; \lambda, \rho) = C_p(F_1(y^1; \lambda^1), F_2(y^2; \lambda^2)), $$

where $C_p$ is Frank's copula function and $F_1$ and $F_2$ are the distribution functions of the Poisson marginals, i.e.

$$ F_j(x; \mu) = \sum_{k=0}^{x} e^{-\mu} \frac{\mu^k}{k!}, \quad j = 1, 2. $$

### 2.4 The multivariate PoARX model

The extension to the multivariate case is straightforward. Let $\{Y_t = (Y_1^t, \ldots, Y_K^t), \ t = 1, 2, \ldots \}$ be a multivariate time series and let $\{x_{i-1}^j = (x_{i-1}^{1j}(1), \ldots, x_{i-1}^{Kj}(r))\}$ be the matrix of exogenous covariates associated with $Y_t$. We say that $\{Y_t\}$ is a PoARX process and write $\{Y_t\} \sim \text{PoARX}_K(p, q)$, if each of the component time series is a univariate PoARX process and the joint conditional distribution is a copula Poisson. Let the intensities of PoARX processes be $\{\lambda_i^j; \ t = 1, 2, \ldots, j = 1, \ldots, K\}$ and be denoted using $\lambda_i = (\lambda_1^i, \ldots, \lambda_K^i)$.

Analogously to the previous section, let $\mathcal{D}(\lambda_1^1, \ldots, \lambda_K^K; \rho)$ be a multivariate distribution based on Frank’s copula with marginal distributions $\text{Poisson}(\lambda_1^1), \ldots, \text{Poisson}(\lambda_K^K)$ and dependency parameter $\rho$. Let also

$$ C_p(u_1, \ldots, u_K) = \varphi_{p}^{-1}\left(\sum_{k=1}^{K} \varphi_{p}(u_k)\right), $$

(8)

where $\varphi_{p}$ and $\varphi_{p}^{-1}$ are the generator function and its pseudo-inverse of the Frank’s copula from Equations (3) – (6). Before stating the entire behaviour of the multivariate model, the distribution function corresponding to $\mathcal{D}(\lambda_1^1, \ldots, \lambda_K^K; \rho)$ is

$$ F(y; \lambda, \rho) = C_p(F_1(y^1; \lambda^1), \ldots, F_K(y^K; \lambda^K)). $$

(9)

The conditional distribution of $Y_t$ is a Frank’s copula distribution

$$ Y_t \mid F_{t-1} \sim \mathcal{D}(\lambda_1^1, \ldots, \lambda_K^K; \rho), $$

(10a)

where $F_{t-1}$ denotes the $\sigma$-field defined by all previous observations and exogenous covariates, $\sigma\{Y_{t-p}, \ldots, Y_{t-1}, \lambda_{t-q}, \ldots, \lambda_{t-l}, x_{t}, \ldots, x_{t-1}\}$, where each term contains information on all components of the time series. As before, the dynamics of the components of $Y_t$ are specified by the equations:

$$ Y_i^j \mid F_{t-1} \sim \text{Poisson}(\lambda_i^j), \quad j = 1, \ldots, K; $$

(10b)

$$ \lambda_i^j = \omega^j + \sum_{l=1}^{p} \alpha_i^j Y_{i-1}^j + \sum_{l=1}^{q} \beta_i^j \lambda_{i-l}^j + \eta^j \cdot x_{i-l}^j, \quad j = 1, \ldots, K; $$

(10c)

where $\alpha_i^j, \beta_i^j \geq 0$ denote coefficients for the past values of the observations and intensities respectively, $\eta^j$ denotes the vector of (non-negative) coefficients for the exogenous covariates, and $\omega^j \geq 0$ denotes an (optional) intercept term. For each univariate process, the two conditions in Equations (2) and (3) must hold.
2.5 Properties of multivariate PoARX

Here we prove stationarity and ergodicity of PoARX models using the properties of univariate PoARX processes, developed in [Agosto et al. 2016], and \( \tau \)-weak dependence. \( \tau \)-weak dependence is a stability concept developed by Doukhan and Wintenberger [2008] for Markov chains that implies stationarity and ergodicity. To aid the establishment of asymptotic properties later, it is advantageous to express each PoARX process in terms of a sequence of independent Poisson realisations. Specifically, introduce \( \{ N^j_t(\cdot), t = 1, 2, \ldots \} \) for \( j = 1, 2, \ldots, K \) and let each set be a sequence of independent Poisson processes of unit intensity, such that \( Y^j_t \) is equal to \( N^j_t(\lambda^j_t) \), the number of events in the time interval \([0, \lambda^j_t]\). Then the model can be rewritten as

\[
Y^j_t = N^j_t(\lambda^j_t), \quad j = 1, 2, \ldots, K,
\]

\[
\lambda^j_t = \omega^j + \sum_{l=1}^{p} \alpha^j_{l} Y^j_{t-l} + \sum_{l=1}^{s} \beta^j_{l} \lambda^j_{t-l} + \eta^j, \quad x^j_{t-1},
\]

assuming all terms used to initialise, \( \{ Y_0, Y_{-1}, \ldots, Y_{-p}, \lambda_0, \lambda_{-1}, \ldots, \lambda_{-q} \} \) are known and fixed, noting that each \( \{ Y_t \} \) and \( \{ \lambda_t \} \) is a \( K \)-dimensional vector. Now, we impose a simpler Markov structure to help state and prove the results,

\[
x^j_t(k) = g^j_t \left( x^j_{t-1}(k); \epsilon^j_t \right), \quad j = 1, \ldots, K, \quad k = 1, \ldots, r.
\]

However, the statements hold for the more general structure found in Equation (3). We also make three assumptions similar to those found in [Agosto et al. 2016] for the univariate model.

**Assumption 1 (Markov)** The innovations \( \epsilon^j_t \) and Poisson processes \( N^j_t(\cdot) \) are i.i.d. for all \( j = 1, 2, \ldots, K \).

**Assumption 2 (Exogenous Stability)**

\[
E \left[ \left| \left| g^j( x^j; \epsilon^j_t ) - g^j( \bar{x}^j; \epsilon^j_t ) \right| \right|^s \right] \leq \kappa \left| x^j - \bar{x}^j \right|^s
\]

for some \( \kappa < 1 \) and \( E \left[ \left| g^j(0; \epsilon^j_t) \right| \right]^s < \infty \) for all \( j = 1, 2, \ldots, K \), for some \( s \geq 1 \).

**Assumption 3 (PoARX Stability)** \( \sum_{i=1}^{\max(p, q)} (\alpha^j_i + \beta^j_i) < 1 \), for each \( j = 1, 2, \ldots, K \).

In the formulae below the operator vec has its usual meaning. For a matrix \( A \), vec\( (A) \) is a (column) vector obtained by stacking the columns of \( A \) on top of each other. As a shorthand, vec\( (A_1, \ldots, A_m) \) is equivalent to the more verbose vec\( (\text{vec}(A_1), \ldots, \text{vec}(A_m)) \).

**Theorem 2.** Under Assumptions 1 – 3 and the Markov assumption in Equation (12), there exists a weakly dependent stationary and ergodic solution, \( X^*_t = \text{vec}( (Y^*_t, \lambda^*_t, x^*_{t-1}) ) \), to Equations (10). The solution is such that \( E( \left| X^*_t \right|^s ) < \infty \), where \( s \geq 1 \) is found in Assumption 2, \( Y^*_t = (Y^1_t, \ldots, Y^K_t) \) and \( \lambda^*_t = (\lambda^*_1, \ldots, \lambda^*_K) \) are \( K \)-vectors, and \( x^*_{t-1} = (x^1_{t-1}, \ldots, x^K_{t-1}) \) is a \( K \times r \) matrix.

**Proof.** See [A].

A consequence of Theorem 2 is that it allows PoARX models to use the (weak) law of large numbers (LLN) for stationary and ergodic processes. To ensure the correct analysis of asymptotic behaviour, we need to be able to use the LLN for any initialisation, rather than a set of fixed initial values. Lemma [11] extends the LLN to hold for this case. The proof is no different to the univariate case in [Agosto et al. 2016], where the reader is directed to [Kristensen and Rahbek 2015].
Lemma 1. Let $X_t = \text{vec} \left( (Y_t, \lambda_t, x_{t-1})^T \right)$ be a process satisfying $X_t = F(X_{t-1}; \xi_t)$ with $\xi_t$ i.i.d, $E|F(x; \xi) - F(x'; \xi)|^r \leq M |x - x'|^r$, and $E|F(0; \xi)|^r < \infty$. For any function $h(x)$ satisfying:

(i). $||h(x)||^{1+\delta} \leq M (1 + ||x||^r)$ for some $M, \delta > 0$,

(ii). for some $c > 0$ there exists $L_c > 0$ such that $||h(x) - h(x')|| \leq L_c ||x - x'||$ for $||x - x'|| < c$, it holds that

$$\frac{1}{T} \sum_{t=1}^{T} h(X_t) \xrightarrow{P} E(h(X_t^T)),$$

as $T \to \infty$.

Proof. See Kristensen and Rahbek (2015), or apply the main result from Lindner and Szimayer (2005).

3 Estimation

Here we describe how the PoARX model can be estimated. We also provide asymptotic results for the estimated parameters.

We consider the model specified by Equations (10), where we denote the unknown parameters by $\vartheta$. Then with $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_p)^\top$, $\beta^j = (\beta^j_1, \ldots, \beta^j_q)^\top$, and $\nu^j = (\nu^j_1, \ldots, \nu^j_r)^\top$,

$$\vartheta = \left( \omega^j, (\alpha^j)^\top, (\beta^j)^\top, (\nu^j)^\top \right)^\top, \quad \omega^j = \left( \omega_1^j, \ldots, \omega^K^j, (\alpha^K)^\top, (\beta^K)^\top, (\nu^K)^\top, \rho \right)^\top,$$

where $\theta^j \in \Theta^j \subset [0, \infty)^{1+p+q+r}$.

The probability mass function of the copula PoARX model, derived from the cumulative mass function as rectangle probabilities (compare to Equation (7)), is

$$\text{Pr}(y_1^j = y_1^j, \ldots, y^K_j = y^K_j) = \prod_{l_1=0}^{1} \cdots \prod_{l_K=0}^{1} (-1)^{l_1+\cdots+l_K} C_\rho \left( F_1(y_1^j - l_1; \lambda_1^j), \ldots, F_K(y^K_j - l_K; \lambda^K_j) \right),$$

with $C_\rho(\cdot)$ representing Frank’s copula and

$$F_j(x; \mu) = \sum_{k=0}^{x} e^{-\mu \frac{k}{k!}}, \quad j = 1, \ldots, K.$$

The conditional log-likelihood for $\vartheta$ given the multivariate observations $y_1, \ldots, y_n$ with initial values $y_0$ and $\lambda_0$ (denoted by the $\sigma$-field $\mathcal{F}_0$) is given by the following.

$$l(\vartheta) = \sum_{t=1}^{n} \log \left( \text{Pr}( (y_1^j, \ldots, y^K_j)^\top | \mathcal{F}_{t-1}; \vartheta) \right)$$

$$= \sum_{t=1}^{n} l_t(\vartheta).$$

The maximum likelihood estimator (MLE) is

$$\hat{\vartheta} = \text{arg max}_{\vartheta \in \Theta} l(\vartheta).$$

However, with the large dimension of $\Theta$ it is computationally more feasible to use a two-stage procedure known as the method of inference functions (IFM), developed by Joe (2005). The idea
of IFM is to estimate the marginal parameters separately from the dependence parameter, hence reducing the dimension of the unknown parameters in each maximisation process. To perform this we need the marginal log-likelihoods. When we consider the observations \( y_1, \ldots, y_K \) for each \( j = 1, \ldots, K \) separately, the marginal log-likelihood for \( \theta^j \) can be written as

\[
l_j(\theta^j) = \sum_{t=1}^{n} \log \left( \Pr(y_{jt} \mid F_{t-1}; \theta^j) \right) = -\lambda^j_t + y^j_t \log(\lambda^j_t) - \log(y^j_t!),
\]

with \( \lambda^j_t \) calculated using Equation (10c).

The IFM method is more explicitly stated as follows,

\begin{itemize}
  \item[(a)] the log-likelihoods \( l_j(\cdot) \) of the \( K \) univariate marginals are independently maximised to produce estimates \( \hat{\theta}^1, \ldots, \hat{\theta}^K \);
  \item[(b)] the function \( \ell(\hat{\theta}^1, \ldots, \hat{\theta}^K, \rho) \) is maximised over \( \rho \) to obtain \( \hat{\rho} \).
\end{itemize}

Before we state the main result of this section we make a reference to the large sample properties of univariate PoARX obtained by Agosto et al. (2016). In order to analyse these properties, conditions were imposed on the parameters and the exogenous covariates.

**Assumption 4** The space of possible parameters for each marginal distribution \( j, \Theta_j \), is compact for all \( j = 1, \ldots, K \). This means that for all \( \theta^j = (\omega^j, \alpha^j, \beta^j, \eta^j) \in \Theta^j \), \( \beta^j \leq \beta^j_{\text{min}} \), for each \( i = 1, \ldots, q \), and \( \omega^j \geq \omega^j_{\text{min}} \) for some constants \( \omega^j_{\text{min}} > 0 \) and \( \beta^j_{\text{min}} > 0 \) with \( \sum_{i=1}^{q} \beta^j_{\text{min}} < 1 \).

**Assumption 5** The polynomials \( A^j(z) := \sum_{i=1}^{p} \alpha^j_{0,i} z^i \) and \( B^j(z) := 1 - \sum_{i=1}^{q} \beta^j_{0,i} z^i \) have no common roots; and for any \( \alpha \neq 0 \) and \( g \neq 0 \), \( \sum_{i=1}^{p} a_i Y_{i,t}^{\alpha} + \sum_{i=1}^{q} g_i x_{i,t}^{g} \) has a non-degenerate distribution. This should be true for each \( j = 1, \ldots, K \).

Using Assumptions 1 – 5 we can obtain consistency of the maximum likelihood estimators of the parameters for the \( j \)th univariate PoARX component based on Equation (13). Equivalently, we can state that the IFM estimator (from part (a) of the IFM procedure) of the multivariate PoARX model is consistent. Furthermore, if \( \theta^j \in \text{int} \Theta^j \), then

\[
\sqrt{n}(\hat{\theta}^j - \theta^j) \overset{d}{\to} N \left( 0, H^{-1}(\theta^j) \right), \quad H(\theta^j) := -E \left( \frac{\partial^2 l_j(\theta^j)}{\partial \theta^j \partial \theta^j} \right),
\]

where \( l_j(\theta^j) \) denotes the marginal likelihood function evaluated at the stationary solution. The proof is equivalent to the proof of Theorem 2 in Agosto et al. (2016).

Lastly, from the theory of inference functions (Godambe 1991; Joe 2005), we can deduce an asymptotic result for the IFM estimate of \( \rho \),

\[
\sqrt{n}(\hat{\rho} - \rho_0) \overset{d}{\to} N \left( 0, H^{-1}(\rho_0) \right), \quad H(\rho) := -E \left( \frac{\partial^2 l_j}{\partial \rho \partial \rho} \right),
\]

We can now state our result about the asymptotic behaviour of the IMF estimator of \( \vartheta \), the full vector of parameters.

**Theorem 3.** Suppose that Assumptions 1 – 5 hold with \( s \geq 2 \) and the true value of \( \vartheta \) is denoted by \( \vartheta_0 \). Then \( \vartheta \) is consistent and if \( \vartheta \in \text{int} \Theta \),

\[
\sqrt{n}(\hat{\vartheta} - \vartheta_0) \overset{d}{\to} N \left( 0, V \right),
\]

where details of asymptotic covariance matrix \( V \) can be found in the proof.

**Proof.** See [3]
4 Forecasting

Forecasting with PoARX models is to some extent similar to the forecasting of GARCH-X processes [Hansen et al., 2012]. Predictions for the intensities can be obtained recursively using Equation (10c) and the property $E(Y_t^j | F_{t-1}) = \lambda_t^j$. This procedure also gives point predictions for the process. However, there is substantial difference when predictive distributions are required.

One-step ahead forecasts at time $t$ of the intensities $\lambda_{t+1}^j, \ldots, \lambda_{t+h-1}^j$, given information $F_t$, parameters $\theta^j$, and covariates $x_t$ are:

$$
\lambda_{t+1}^j = \omega^j + \sum_{i=1}^p \alpha_i^j y_{t+1-i} + \sum_{i=1}^3 \beta_i^j \lambda_{t+1-i}^j + \eta^j \cdot x_t^i, \quad j = 1, \ldots, K. \quad (15)
$$

By the specifications of the model, the one-step ahead marginal predictive distributions are Poisson with predicted intensities computed above, i.e. for each $j = 1, \ldots, K$,

$$
P(Y_{t+1}^j = y | F_t) = \frac{\lambda^y \exp(-\lambda)}{y!},
$$

where $\lambda = \lambda_{t+1}^j$. The joint predictive distribution is obtained by substituting the predicted intensities in Equation (10).

For multi-step-ahead forecasts, the procedure is not so straightforward. Firstly, the computation of the $h$-step-ahead forecast at time $t$ assumes that the exogenous covariates $x_t, \ldots, x_{t+h-1}$ are known. In practice, these will often need to be replaced by their own forecasts or projections. This is not a problem when the covariates are leading indicators, see the example in Section 5. With a slight abuse of notation we use $\lambda_{t+h+1}^j$ to represent the “intensity for horizon $h$ conditional on $F_t$ and $x_t, \ldots, x_{t+h-1}$”. We let this knowledge be denoted by the $\sigma$-field $\mathcal{G}_t$.

Agosto et al. (2016) assume that the predictive distribution for any horizon $h$ follows a Poisson distribution, $Y_{t+h+1}^j \sim \text{Poisson}(\lambda_{t+h+1}^j)$, and use it to obtain prediction intervals. However, we show below that the predictive distributions for $h \geq 2$ are not necessarily Poisson. Rather than compute the probabilities directly, we use an approach similar to Boshnakov (2009) who derived predictive distributions (for a different class of models) using conditional characteristic functions. Since the Poisson distribution is discrete, it is more convenient to use probability generating functions.

The probability generating functions can be calculated as follows, starting with $h = 2$. For a time series $Y_t$ following a PoARX process with intensity $\lambda_t$, we can write $\lambda_{t+2} | \mathcal{G}_t = c_{t+2} + \alpha_1 y_{t+1}$, where $c_{t+2}$ is measurable w.r.t. $\mathcal{G}_t$. In the derivation below we will need the following result:

$$
E(\exp((-1+z)\alpha_1 y_{t+1}) | \mathcal{G}_t) = \sum_{k=0}^{\infty} \frac{\lambda_{t+1}^j}{k!} \exp(-\lambda_{t+1}) \exp((-1+z)\alpha_1 k)
= \exp(-\lambda_{t+1}) \sum_{k=0}^{\infty} \frac{(\lambda_{t+1} e^{(-1+z)\alpha_1})^k}{k!}
= \exp(-\lambda_{t+1}) \exp(\lambda_{t+1} e^{(-1+z)\alpha_1})
= \exp\left(\lambda_{t+1}(-1 + e^{-(1+z)\alpha_1})\right). \quad (16)
$$

The 2-step ahead forecast has the following generating function ($P_2(z)$ depends also on $t$ but
we omit that to keep the notation transparent):

\[ P_2(z) = E(z^{Y_{t+2}} | G_t) \]
\[ = E(E(z^{Y_{t+2}} | G_{t+1}) | G_t) \]
\[ = E(\exp((-1 + z)\lambda_{t+2}) | G_t) \]
\[ = \exp((-1 + z)c_{t+2})E(\exp((-1 + z)\alpha_1Y_{t+1}) | G_t) \]
\[ = \begin{cases} 
\exp((-1 + z)c_{t+2}) & \text{if } \alpha_1 = 0, \\
\exp((-1 + z)c_{t+2})\exp(\lambda_{t+1}(-1 + \exp((-1 + z)\alpha_1))) & \text{if } \alpha_1 \neq 0.
\end{cases} \]

We can see that if \( \alpha_1 \neq 0 \), then \( P_2(z) \) is not Poisson, by the uniqueness property of generating functions. The joint distribution can be obtained by computing analogously the joint probability generating functions.

For \( h > 2 \) the above calculation can be extended by repeatedly using the property of the iterated conditional expectation. It can also be expressed recursively as follows:

\[ P_h(z) = E(z^{Y_{t+h}} | G_t) \]
\[ = E(E(z^{Y_{t+h}} | G_{t+h-1}) | G_t) \]
\[ = E(P_h(z) | G_t). \]

Clearly, for \( h \geq 2 \) the forecast distribution is not necessarily Poisson. Nevertheless, we have that

**Lemma 2.** \( E(Y_{t+h} | G_t) = E(\lambda_{t+h} | G_t) = : \lambda_{t+h} | t \)

**Proof.** For \( h = 1 \), the claim follows from the specification of the model. For \( h > 1 \) we can use Equation (10c) and iterated conditional expectations to find that

\[ E(Y_{t+h} | G_t) = E(E(Y_{t+h} | G_{t+h-1}) | G_t) \]
\[ = E(\lambda_{t+h} | G_t). \]

Therefore, we can generate \( h \)-step ahead forecast of the intensity with the following equation,

\[ \lambda_{t+h} | t = \omega + \sum_{l=1}^{p} \alpha_l Y_{t+h-l} | t + \sum_{l=1}^{q} \beta_l \lambda_{t+h-l} | t + \eta \cdot x_{t+h-1}. \]  

(17)

where

\[ Y_{t+h} | t = \begin{cases} 
\lambda_{t+k} & \text{if } k > 0, \\
Y_{t+k} & \text{if } k \leq 0.
\end{cases} \]

Prediction intervals can be obtained by computing the probabilities from the probability generating functions discussed above. Since these are probably feasible only for small horizons, simulation would be a more practical alternative. To obtain a prediction interval for \( Y_{t+h}^* \), simulate a trajectory of the PoARX time series until time \( t+h \), resulting in one simulated value \( Y_{t+h}^* \). Repeating this process \( B \) times allows access to the quantiles from which we can obtain a prediction interval for the time series. Simulating a joint predictive region is an area for further work and not discussed here.

5 Applications

We illustrate the use of PoARX models with a data set from Ihler et al. (2006), who used it in their work on event detection. The computations were done with R (R Core Team, 2017) using the implementation of the PoARX models in package PoARX (Halliday and Boshnakov, 2018).
5.1 Data

The data contains counts of the estimated number of people that entered and exited a building over thirty-minute intervals of a UCI campus building. Counts were recorded by an optical sensor at the front door starting from the end of 23/07/2005 until the end of 05/11/2005. The data has periodic tendencies but is also influenced by events within the building causing an influx of traffic. Originally, the data was used to build a novel event detection framework under a Bayesian scheme. The counts of people going into ($N^I(t)$) and out of ($N^O(t)$) the building were both assumed to follow Poisson distributions and were used in a model to detect the occurrence of an event. Three weeks worth of the data in question is shown in Figure 1. In total, there are 5040 observations, which corresponds to 15 weeks of data.

Figure 1: Three weeks of counts for people entering and exiting a UCI campus building.

In this application, we will estimate the number of people entering and exiting the building using the Poisson distribution in the spirit of [Ihler et al., 2006]. The basis of model predictions will be the lagged values of the observations and mean value, as well as some exogenous covariates. These covariates are all indicator variables, representing the following. The first is a “weekday” indicator, that takes value 1 when the day is Monday – Friday. This corresponds to an uplift for working days. The second indicator is a “daytime” indicator, taking value 1 when the time is between 07:30 and 19:30, representing an uplift in the traffic during working hours.
The third indicator is associated with the presence of an event occurring. For the flow count into the building, the variable takes the value 1 when an event will occur in the next hour. For the flow out of the building, the variable takes the value 1 in the hour after an event finished. These represent the arrival and departure of people coming to the building for the event. We will investigate whether the use of Frank’s copula, hence the capturing of any positive or negative dependence, improves the prediction of the number of people entering and exiting the building.

5.2 Estimation and in-sample model evaluation
We fit four types of models to the data in an attempt to find the best predictive model. We first fit a model with no covariates - it uses only the time series aspects to predict upcoming counts. Model 1 uses this approach and treats the two counts independently, whereas model 2 fits the joint distribution of the flows using Frank’s copula. We then add covariates to the models, seeking to improve the predictive accuracy of the two models. As mentioned, there are three covariates available for each time series. Model 3 uses the covariates along with the assumption of independence, whilst Model 4 uses Frank’s copula with the covariates.

To assess the quality of our models, we used 5-fold cross validation (Stone, 1974) on a training set to produce a cross-validated log score (Bickel, 2007). This was also the performance metric used to select the lagged values of the observations and means. Since we are modelling time series, we cannot leave out a fold that occurs in the middle of the data (thus disrupting the time series). Hence we choose overlapping folds, aggregating the log scores of predictions for each observation. Using the first 4000 observations of the building data as a training set, we use 2000 observations in each fold of the cross-validation. The observations not used to estimate the model are used for evaluation. The log score is calculated as follows. Let \( r = (r_1, \ldots, r_n) \) be a vector of probabilities for \( i = 1, \ldots, n \) observed events. Then the log score is

\[
L(r) = \sum_{i=1}^{n} \log(r_i).
\]

For analysis, the lagged values chosen differed slightly for each time series. For the number of people entering the building (\( N^I(t) \)), we chose to use 4 lagged values for the observations (lags 1, 2, 48, 336) and 1 lagged value for the means (lag 1). Lagged values from the previous 2 observations represent the flow of people within the last hour, whilst the lag of 48 corresponds to the same time point on the previous day, and 336 to the same time point on the same day in the previous week. For the number of people exiting the building (\( N^O(t) \)) we used the same 4 lagged values for the observations (lags 1, 2, 48, 336) but included an extra lag for the mean values (lags 1, 48). These were chosen based on the cross-validated log scores. In Table I we present the values of the coefficients of the fitted models, where lags are sorted in increasing size (in other words \( \alpha_3 \) corresponds to the observations lag 48). The standard errors of parameters in Models 1 and 3 are of the order \( 10^{-4} \), and in Models 2 and 4 are of the order \( 10^{-5} \) or \( 10^{-6} \). This means that \( \beta^O_2 \) is not statistically significant in every model except Model 2, but when a new model is fitted without this variable we find that the strength of the predictions decreases. For this reason, we choose to keep the 48th lagged mean in our models.

In Table II we present the cross-validated log score, AIC (Akaike, 1974), and BIC (Schwarz, 1978) of the four models. Looking firstly at the information criteria, they both suggest that the best model is Model 4, which includes covariates and dependence. Further, it seems that adding the covariates to the model improved the strength of both the model fitted with an independence assumption (Model 2 vs. Model 1) and the model using Frank’s copula (Model 4 vs. Model 3). It also appears that the models using Frank’s copula (Models 2 and 4) are better fits to the data than the independent case (Models 1 and 3, respectively).

However, we are interested in predictive accuracy, so we look mainly at the log scores. Firstly we notice that Model 2 appears to be the best model, while Model 1 is second. It seems as though the addition of the covariates weakens the fit of the model, despite the parameters of the
Table 1: Fitted models

| Coefficient \ Model | 1     | 2     | 3     | 4     |
|---------------------|-------|-------|-------|-------|
| \( \omega^I \)     | 0.079 | 0.079 | 0.019 | 0.019 |
| \( \alpha_1^I \)    | 0.390 | 0.390 | 0.396 | 0.396 |
| \( \alpha_2^I \)    | 0.137 | 0.137 | 0.113 | 0.113 |
| \( \alpha_3^I \)    | 0.054 | 0.054 | 0.048 | 0.048 |
| \( \alpha_4^I \)    | 0.275 | 0.275 | 0.256 | 0.256 |
| \( \beta_1^I \)     | 0.142 | 0.142 | 0.140 | 0.140 |
| \( \eta_1^I \)      | -     | -     | 0.102 | 0.102 |
| \( \eta_2^I \)      | -     | -     | 0.229 | 0.229 |
| \( \eta_3^I \)      | -     | -     | 5.684 | 5.684 |
| \( \omega^O \)      | 0.129 | 0.129 | 0.035 | 0.035 |
| \( \alpha_1^O \)    | 0.347 | 0.347 | 0.342 | 0.342 |
| \( \alpha_2^O \)    | 0.163 | 0.163 | 0.153 | 0.152 |
| \( \alpha_3^O \)    | 0.049 | 0.049 | 0.045 | 0.045 |
| \( \alpha_4^O \)    | 0.264 | 0.264 | 0.255 | 0.255 |
| \( \beta_1^O \)     | 0.161 | 0.161 | 0.136 | 0.136 |
| \( \beta_2^O \)     | 2.05e-04 | 2.05e-04 | 9.24e-10 | 9.24e-10 |
| \( \eta_1^O \)      | -     | -     | 0.153 | 0.153 |
| \( \eta_2^O \)      | -     | -     | 0.299 | 0.299 |
| \( \eta_3^O \)      | -     | -     | 2.500 | 2.500 |
| \( \rho \)          | -     | 2.545 | -     | 2.642 |

relevant models being significantly greater than zero, statistically speaking. Furthermore, using this metric, we deduce that the use of Frank’s copula improves the predictions compared to those using the independence assumption. The smallest score and therefore the worst performance is found in the results from Model 3. This model contains covariates along with the independence assumption. However, since the two counts share common covariates, the assumption of independence is violated and we would speculate that this is the reason for the extreme score.

5.3 Prediction and out-of-sample model evaluation

As we are interested in the predictive strength of our model, it is a good idea to assess how the model performs predicting observations not in the original sample. Since we only used the first 4000 observations in training, we can use the remaining 1040 observations as a test set. Again using the log score to evaluate the performance, we display the results in Table 3.

From Table 3, we notice that Models 1-3 have similar scores, but Model 4 has a significantly lower log score. This would suggest that the combination of the time series aspects, the covariates and the multivariate modelling produces the most accurate out-of-sample predictions for this kind of data. Focusing on smaller comparisons, we first look at Models 1 and 2. There is a very small increase in performance by removing the independence assumption and using Frank’s copula, but perhaps this is not worth the extra complexity gained from using a copula model.
Table 2: Model training scores from cross-validated fit on 4000 observations

| Model number | Log score | AIC   | BIC   |
|--------------|-----------|-------|-------|
| 1            | -15444    | 30252 | 30334 |
| 2            | -15411    | 29802 | 29891 |
| 3            | -25088    | 29800 | 29920 |
| 4            | -16856    | 29269 | 29395 |

Table 3: Model testing scores based on the 1040 out-of-sample observations

| Model number | Log score |
|--------------|-----------|
| 1            | -4184     |
| 2            | -4182     |
| 3            | -4190     |
| 4            | -4164     |

However between Models 3 and 4, the aforementioned increase in predictive performance is evident, showing that when covariates are considered, the greater accuracy can be obtained using Frank’s copula. Comparing Models 1 and 3 we see that there is a slight decline in predictive performance when the covariates are added. As mentioned earlier, one reason for this could be the violation of the assumption of independence due to the common covariates. However, between Models 2 and 4 the combination of covariates and copula produces the best performance.

6 Conclusion

We introduced the multivariate PoARX model as an extension of the univariate PoARX model. Using previously established properties of the univariate PoARX model and copulas, we showed that our multivariate models inherit similar stability and large sample properties of the univariate case. We also established a law of large numbers.

For estimation of the parameters of multivariate PoARX models, we used the method of inference functions (Joe, 2005), which is computationally more efficient than the maximum likelihood method. We established a central limit theorem for the parameters estimated by IFM.

Our discussion of forecasting, especially predictive distributions for horizons larger than one, seems novel even for the univariate PoARX models. In particular, it is important to point out that the predictive distributions for lags greater than one are not Poisson.

In the example in Section 5 we illustrated the use of bivariate PoARX models for modelling the counts of the number of people entering and exiting a building, using lagged values and covariates. Overall, information criteria and out-of-sample prediction suggested that using both covariates and dependence parameters can provide better models. In this instance, we chose to use k-fold cross-validation coupled with the model assessment tool of the log score. However, this were relatively arbitrary choices, with no clearly defined methodology in place for model assessment in general. Depending on the field of study, some people will use information criteria, some will prefer scoring criteria. We feel that the analysis in Section 5 provides material for further thought and work on model evaluation for count data time series models.

We give here some examples of multivariate count data where multivariate PoARX models could be useful. The univariate PoARX model (or PARX model) has also been used to model the scores of a football match in Angelini and Angelis (2017). They used a univariate PoARX
model for the goals scored by each team in the English Premier League and predicted the score coupling the processes independently. However, it has long been thought that there should be a dependence between teams competing in a match (see Maher (1982) for the seminal paper in this area). Application of our multivariate PoARX model could be used to improve predictions for scores by considering such a dependence. Further applications could consider data modelled by a Poisson autoregressive process, and explore any influence of external factors. Such examples would be the Hyde Park Purse Snatchings and Presidential Vetoes from Brandt and Williams (2000), prices and times of trades made on the New York stock market from Rydberg and Shephard (2001) and the number of transactions per minute for the relevant stock from Fokianos et al. (2009).

There is also plenty of scope for further work. Our class of models uses Frank’s copula to jointly model Poisson marginal distributions. We did not have to use Frank’s copula – if there is a belief that the dependence structure can be captured in a different way, then other copulas can be used. Another direction would be to consider distributions other than Poisson. We are considering the possibility of using the renewal count distributions of Kharrat et al. (2018), mentioned in the introduction, which are implemented in the R package Rcountr (Kharrat and Boshnakov, 2016). Combining these renewal distributions with the ideas found in this paper could lead to a fascinating new family of count time series models. Additionally, exploring a time varying copula structure as seen in Kearney and Patton (2000) may be advantageous in some applications.

7 References

References

A. Agosto, G. Cavaliere, D. Kristensen, and A. Rahbek. Modelling corporate defaults: Poisson autoregression with exogenous covariates (PARX). *Journal of Empirical Finance*, 38:640 – 663, 2016. doi: 10.1016/j.jempfin.2016.02.007.

H. Akaike. A new look at the statistical model identification. *IEEE Transactions on Automatic Control*, 19:716 – 723, 1974. doi: 10.1109/TAC.1974.1100705.

G. Angelini and L. D. Angelis. PARX model for football matches predictions. *Journal of Forecasting*, pages 1 – 13, 2017. doi: 10.1002/for.2471.

J. E. Angus. The probability integral transform and related results. *SIAM Review*, 36(4):652 – 654, 1994.

L. Bauwens, S. Laurent, and J. V. K. Rombouts. Multivariate GARCH models: A survey. *Journal of Applied Econometrics*, 21:79 – 109, 2006. doi: 10.1002/jae.842.

J. E. Bickel. Some comparisons among quadratic, spherical, and logarithmic scoring rules. *Decision Analysis*, 4(2):49 – 65, 2007. doi: 10.1287/deca.1070.0089.

T. Bollerslev. Generalised autoregressive conditional heteroscedasticity. *Journal of Econometrics*, 31:307 – 327, 1986. doi: 10.1016/0304-4076(86)90063-1.

G. N. Boshnakov. Analytic expressions for predictive distributions in mixture autoregressive models. *Statistical & Probability Letters*, 79(15):1704–1709, 2009. doi: 10.1016/j.spl.2009.04.009.

G. N. Boshnakov, T. Kharrat, and I. G. McHale. A bivariate Weibull count model for association football scores. *International Journal of Forecasting*, 33(2):458 – 466, 2017. doi: 10.1016/j.ijforecast.2016.11.006.
G. E. P. Box and G. M. Jenkins. *Time Series Analysis: Forecasting and Control*. Holden–Day, San Francisco, 1970.

P. T. Brandt and J. T. Williams. A linear Poisson autoregressive model: The Poisson AR(p) model. *Political Analysis*, 9(2):164 – 184, 2000. doi: 10.1093/oxfordjournals.pan.a004869.

A. C. Cameron and P. K. Trivedi. *Regression Analysis of Count Data*. Cambridge University Press, Second edition, 2013.

V. Christou and K. Fokianos. Quasi-likelihood inference for negative binomial time series models. *Journal of Time Series Analysis*, 25:55 – 78, 2014. doi: 10.1111/jtsa.12050.

P. Doukhan and O. Wintenberger. Weakly dependent chains with infinite memory. *Stochastic Processes and their Applications*, 118(11):1997 – 2013, 2008. doi: 10.1016/j.spa.2007.12.004.

J. Durbin and S. J. Koopman. *Time Series Analysis by State Space Methods*. Number 38 in Oxford Statistical Science Series. Oxford University Press, Second edition, 2012.

R. F. Engle. Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 50(4):987 – 1008, 1982.

R. F. Engle. Dynamic conditional correlation: A simple class of multivariate generalized autoregressive conditional heteroskedasticity models. *Journal of Business & Economic Statistics*, 20(3):339 – 350, 2002.

R. Ferland, A. Latour, and D. Oraichi. Integer-valued GARCH processes. *Journal of Time Series Analysis*, 27(6):923 – 942, 2006. doi: 10.1111/j.1467-9892.2006.00496.x.

K. Fokianos. Count time series models. In T. S. Rao, S. S. Rao, and C. Rao, editors, *Time Series Analysis: Methods and Applications*, volume 30 of *Handbook of Statistics*, chapter 12, pages 315 – 347. Elsevier, 2012. doi: 10.1016/B978-0-444-53858-1.00012-0. URL http://www.sciencedirect.com/science/article/pii/B9780444538581000120.

K. Fokianos, A. Rahbek, and D. Tjøstheim. Poisson autoregression. *Journal of the American Statistical Association*, 104(488):1430 – 1439, 2009. doi: 10.1198/jasa.2009.tm08270.

V. P. Godambe, editor. *Estimating Functions*. Oxford Statistical Science Series. Oxford University Press, 1991.

J. Halliday and G. N. Boshnakov. *PoARX: Fit PoARX models to multivariate time series*, 2018. R package version 0.3.2 (under development, to be published on CRAN).

E. J. Hannan and M. Deistler. *The statistical theory of linear systems*, volume 70. SIAM, 1988.

P. R. Hansen, Z. Huang, and H. H. Shek. Realised GARCH: A joint model for returns and realised measures of volatility. *Journal of Applied Econometrics*, 27:877 – 906, 2012. doi: 10.1002/jae.1234.

A. Ihler, J. Hutchins, and P. Smyth. Adaptive event detection with time-varying Poisson processes. In *Proceedings of the 12th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 207 – 216. ACM Press, 2006. doi: 10.1145/1150402.1150428.
D. I. Inouye, E. Yang, G. I. Allen, and P. Ravikumar. A review of multivariate distributions for count data derived from the Poisson distribution. Wiley Interdisciplinary Reviews: Computational Statistics, 9(3), 2017. URL https://arxiv.org/abs/1609.00066.

H. Joe. Multivariate models and dependence concepts. Monographs on Statistics and Applied Probability. Chapman & Hall Ltd, 1997.

H. Joe. Asymptotic efficiency of the two-stage estimation method for copula-based models. Journal of Multivariate Analysis, 94:401 – 419, 2005. doi: 10.1016/j.j multivariate.2004.06.003. URL http://www.sciencedirect.com/science/article/pii/S0047259X04001289.

H. Joe. Dependence Modeling with Copulas. New York: Chapman and Hall/CRC., 2014.

C. Kearney and A. J. Patton. Multivariate GARCH modelling of exchange rate volatility transmission in the European Monetary System. The Financial Review, 41:29 – 48, 2000. doi: 10.1111/j.1540-6288.2000.tb01405.x.

B. Kedem and K. Fokianos. Regression Models for Time Series. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc, 2002.

T. Kharrat and G. N. Boshnakov. Countr: Flexible univariate count models based on renewal processes, 2016. URL https://CRAN.R-project.org/package=Countr R package version 3.2.8.

T. Kharrat, G. N. Boshnakov, I. G. McHale, and R. Baker. Flexible regression models for count data based on renewal processes: The Countr package (under revision). Journal of Statistical Software, 2018.

S. J. Koopman and R. Lit. A dynamic bivariate Poisson model for analysing and forecasting match results in the English Premier League. Journal of the Royal Statistical Society A, 178 (1):167 – 186, 2015. doi: 10.1111/jrssa.12042.

D. Kristensen and A. Rahbek. Quasi-likelihood estimation of multivariate GARCH models: A weak dependence approach. Working Papers, 2015.

S. D. Likothanassis and E. N. Demiris. ARMAX model identification with unknown process order and time-varying parameters. In A. Procházka, J. U. P. W. J. Rayner, and N. G. Kingsbury, editors, Signal Analysis and Prediction, Applied and Numerical Harmonic Analysis. Birkhäuser Inc., 1998.

A. M. Lindner and A. Szimayer. A limit theorem for copulas, 2005. URL http://hdl.handle.net/10419/31052 urn:nbn:de:bvb:19-epub-1802-0.

H. Lui. Some models for time series of counts. PhD thesis, Columbia University, 2012.

M. J. Maher. Modelling association football scores. Statistica Neerlandica, 36(3):109 – 118, 1982. doi: 10.1111/j.1467-9574.1982.tb00782.x.

P. McCullagh and J. A. Nelder. Generalised Linear Models. Number 37 in Monographs on Statistics and Applied Probability. CRC press/Chapman & Hall, Second edition, 1989.

I. G. McHale and P. A. Scarf. Modelling the dependence of goals scored by opposing teams in international soccer matches. Statistical Modelling, 11(3):219 – 236, 2011. doi: 10.1177/1471082X1001100303.

A. J. McNeil and J. Nešlehová. Multivariate Archimedean copulas, d-monotone functions and L1-norm symmetric distributions. The Annals of Statistics, pages 3059 – 3097, 2009. doi: 10.1214/07-AOS556.
B. McShane, M. Adrian, E. T. Bradlow, and P. S. Fader. Count models based on Weibull interarrival times. *Journal of Business & Economic Statistics*, 26(3):369 – 378, 2008. doi: 10.1198/073500107000000278.

M. Meitz and P. Saikkonen. Ergodicity, mixing, and existence of moments of a class of Markov models with applications to GARCH and ACD models. *Econometric Theory*, 24(5):1291 – 1320, 2008.

R. B. Nelsen. *An Introduction to Copulas*. New York: Springer, Second edition, 2006.

R Core Team. *R: A language and environment for statistical computing*. R Foundation for Statistical Computing, Vienna, Austria, 2017. URL https://www.R-project.org/.

T. H. Rydberg and N. Shephard. A modelling framework for the prices and times of trades made on the New York stock exchange. In W. J. Fitzgerald, R. L. Smith, A. T. Walden, and P. C. Young, editors, *Nonlinear and Nonstationary Signal Processing*. Cambridge University Press, 2001. doi: 10.2139/ssrn.164170.

G. Schwarz. Estimating the dimension of a model. *The Annals of Statistics*, 8(2):461 – 464, 1978. doi: 10.1214/aos/1176344136.

N. Shephard and K. Sheppard. Realising the future: Forecasting with high-frequency-based volatility (HEAVY) models. *Journal of Applied Econometrics*, 25:197 – 231, 2010. doi: 10.1002/jae.1234.

A. Sklar. Fonctions de répartition à n dimensions et leurs marges. *Publications de l’Institut de statistique de l’Université de Paris*, 8:229 – 231, 1959.

M. Stone. Cross-validatory choice and assessment of statistical predictions. *Journal of the Royal Statistical Society B*, 36(2):111 – 147, 1974.

C. H. Weiβ. Serial dependence and regression of INARMA models. *Journal of Statistical Planning and Inference*, 138(10):2975 – 2990, 2008. doi: 10.1016/j.jspi.2007.11.009.

### A Proof of Theorem 2

**Proof.** We start with the case $\rho = 0$ (independent time series). As each univariate time series satisfies the assumptions of Theorem 2, we know they are individually stationary and ergodic from [Agosto et al., 2016]. Furthermore, the joint distribution is well defined as the product of each univariate probability. Hence the joint distribution is stationary. Lastly, for sets $A_1, \ldots, A_K \in \mathbb{R}$, we have that

$$P((Y_1^t, \ldots, Y_K^t) \in (A_1, \ldots, A_K) \mid F_{t-l}^1, \ldots, F_{t-l}^K) = P(Y_1^t \in A_1 \mid F_{t-l}^1) \cdots P(Y_K^t \in A_K \mid F_{t-l}^K).$$

Using Theorem 1 from [Agosto et al., 2016], we have that

$$P(Y_j^t \in B \mid F_{t-l}^j) \to P(Y_j^t \in B) \quad \text{as } l \to \infty, \quad \text{for } j = 1, \ldots, K.$$

Hence,

$$P((Y_1^t, \ldots, Y_K^t) \in (A_1, \ldots, A_K) \mid F_{t-l}^1, \ldots, F_{t-l}^K) \to P((Y_1^t, \ldots, Y_K^t) \in (A_1, \ldots, A_K))$$

as $l \to \infty$, for any $A_1, \ldots, A_K \in \mathbb{R}$. 

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This proves that independent PoARX processes are weakly dependent, therefore stationary and ergodic.

Now we move onto the case when $\rho \neq 0$. As before, we know that each time series in a multivariate PoARX model is stationary and ergodic. Using similar arguments to [Meitz and Saikkonen 2008] we show the required joint result. Proving that the joint distribution is stationary is straightforward — when $\rho \neq 0$, the cumulative mass function of the joint model is a simple, well-defined transformation of the univariate time series, as seen for the bivariate case in Equation (18).

$$F(y_{t1}, y_{t2}) = \Pr(Y_t^1 \leq y_{t1}, Y_t^2 \leq y_{t2}) = -\frac{1}{\rho} \log \left(1 + \frac{(\exp(-\rho F_1(y_{t1}^1)) - 1)(\exp(-\rho F_2(y_{t2}^2)) - 1)}{e^{\rho} - 1}\right).$$  \hspace{1cm} (18)$$

To show the ergodicity, we must work harder. We show that the property of $\tau$-weak dependence holds for any number of dimensions using induction.

Start with $K = 2$. Let

$$F_{t-1}^1 = \sigma(Y_{t-1}^1, \lambda_{t-1}^1, x_{t-1,1}, Y_{t-1-1}^1, \lambda_{t-1-1}^1, x_{t-1-1,1}, \ldots)$$

$$F_{t-1}^2 = \sigma(Y_{t-1}^2, \lambda_{t-1}^2, x_{t-1,1}, Y_{t-1-1}^2, \lambda_{t-1-1}^2, x_{t-1-1,1}, \ldots)$$

and consider, for any sets $A, B \in \mathbb{R}$,

$$P((Y_{t}^1, Y_{t}^2) \in (A, B) \mid F_{t-1}^1, F_{t-1}^2)$$

$$= P(Y_{t}^1 \in A \mid Y_{t}^2 \in B, F_{t-1}^1, F_{t-1}^2)P(Y_{t}^2 \in B \mid F_{t-1}^1, F_{t-1}^2)$$

$$= P(Y_{t}^1 \in A \mid Y_{t}^2 \in B, F_{t-1}^1)P(Y_{t}^2 \in B \mid F_{t-1}^2).$$  \hspace{1cm} (19)$$

Using the definition of $\tau$-weak dependence inherited by univariate PoARX processes,

$$P(Y_{t}^2 \in B \mid F_{t-1}^2) \to P(Y_{t}^2 \in B) \text{ as } l \to \infty.$$  

Using Equation (19), $P(Y_{t}^1 \in A \mid Y_{t}^2 \in B, F_{t-1}^2)$ is a simple transformation of $P(Y_{t}^1 \in A \mid F_{t-1}^1)$. As $Y_{t}^1$ is a univariate PoARX process,

$$P(Y_{t}^1 \in A \mid F_{t-1}^1) \to P(Y_{t}^1 \in A) \text{ as } l \to \infty.$$  

By applying the simple transformation for the conditional probability we find that

$$P(Y_{t}^1 \in A \mid Y_{t}^2 \in B, F_{t-1}^1) \to P(Y_{t}^1 \in A \mid Y_{t}^2 \in B) \text{ as } l \to \infty.$$  

Thus, using Equation (19),

$$P((Y_{t}^1, Y_{t}^2) \in (A, B) \mid F_{t-1}^1, F_{t-1}^2) \to P((Y_{t}^1, Y_{t}^2) \in (A, B))$$

as $l \to \infty$, for any $A, B$.

This shows $\tau$-weak dependence, hence the bivariate PoARX copula model $(Y_{t}^1, Y_{t}^2)$ is stationary and ergodic.

Assume that this holds for $K = k$. Let $Y_{t}^{1:k} = (Y_{t}^1, \ldots, Y_{t}^k)$. Then the assumption states that $Y_{t}^{1:k}$ is weakly dependent and hence ergodic.

Now we will prove for $K = k + 1$. Let

$$F_{t-1}^j = \sigma(Y_{t-1}^j, \lambda_{t-1}^j, x_{t-1,1}, Y_{t-1-1}^j, \lambda_{t-1-1}^j, x_{t-1-1,1}, \ldots), \quad j = 1, \ldots, k,$$

$$F_{t-1}^{1:j} = \sigma(Y_{t-1}^{1:j}, \lambda_{t-1}^{1:j}, x_{t-1,1}, Y_{t-1-1}^{1:j}, \lambda_{t-1-1}^{1:j}, x_{t-1-1,1}, \ldots), \quad j = 2, \ldots, k,$$
In the calculation of the IFM estimates

\[ \text{Proof.} \]

\text{B Proof of Theorem 3} \]

and for any sets \( A \in \mathbb{R} \), and \( B \in \mathbb{R}^k \), consider the following

\[
P((Y_{t+1}^{k+1} , Y_{t}^{1:k}) \in (A,B) | \mathcal{F}_{t-l}^{k+1} , \mathcal{F}_{t-l}^{1:k}) = P(Y_{t+1}^{k+1} \in A | Y_{t}^{1:k} \in B, \mathcal{F}_{t-l}^{k+1} , \mathcal{F}_{t-l}^{1:k})P(Y_{t}^{1:k} \in B | \mathcal{F}_{t-l}^{1:k})
\]

\[
= P(Y_{t}^{k+1} \in A | Y_{t}^{1:k} \in B, \mathcal{F}_{t-l}^{k+1})P(Y_{t}^{1:k} \in B | \mathcal{F}_{t-l}^{1:k})
\]

Because we know \( Y_{t}^{1:k} \) is weakly dependent from the assumption made, we have that

\[
P(Y_{t}^{k+1} \in A | \mathcal{F}_{t-l}^{1:k}) \rightarrow P(Y_{t}^{1:k} \in A) \text{ as } l \rightarrow \infty.
\]

\[ P(Y_{t}^{k+1} \in A | Y_{t}^{1:k} \in B, \mathcal{F}_{t-l}^{1:k}) \text{ can be thought of as a simple, well-defined transformation of } P(Y_{t}^{k+1} \in A | \mathcal{F}_{t-l}^{1:k}). \text{ As } Y_{t}^{k+1} \text{ is a univariate PoARX process,}
\]

\[
P(Y_{t}^{k+1} \in A | \mathcal{F}_{t-l}^{1:k}) \rightarrow P(Y_{t}^{k+1} \in A) \text{ as } l \rightarrow \infty,
\]

and as a result,

\[
P(Y_{t}^{k+1} \in A | Y_{t}^{1:k} \in B, \mathcal{F}_{t-l}^{1:k}) \rightarrow P(Y_{t}^{k+1} \in A | Y_{t}^{1:k} \in B) \text{ as } l \rightarrow \infty
\]

follows from the transformation. Thus,

\[
P((Y_{t}^{k+1} , Y_{t}^{1:k}) \in (A,B) | \mathcal{F}_{t-l}^{k+1} , \mathcal{F}_{t-l}^{1:k}) \rightarrow P((Y_{t}^{k+1} , Y_{t}^{1:k}) \in (A,B)) \text{ as } l \rightarrow \infty, \text{ for any } A \in \mathbb{R}, B \in \mathbb{R}^k.
\]

This shows that \( Y_{t}^{1:(k+1)} \) is weakly dependent, hence ergodic, so the induction process holds.

We have now proven that the multivariate PoARX model, whether coupled independently or using Frank’s copula, is jointly stationary and ergodic.

\[ \square \]

\section{Proof of Theorem 3}

\begin{proof}

In the calculation of the IFM estimates \( \theta \) we require the separate optimisations of K marginal likelihoods. Each of these marginal likelihoods is a univariate PoARX process, and therefore under Assumptions 1-5 fulfills the requirements of Theorem 2 in Agosto et al. (2016).

Thus, for the parameters in \( \theta^j \) for each \( j = 1, \ldots, K \),

\[
\sqrt{n}(\theta^j - \theta^j_0) \overset{d}{\rightarrow} \mathcal{N} \left( 0, H_j^{-1}(\theta^j_0) \right), \quad H_j(\theta^j) := -\mathbb{E} \left( \frac{\partial^2 l_j(\theta^j)}{\partial \theta \partial \partial(\theta^j)} \right).
\]

First we consider the case of the PoARX models coupled independently, so there is no dependency parameter to estimate. We should assume further here that there exists no condition that allows the observations to become dependent on each other. Since any linear combination of the PoARX models must also follow a normal distribution, we have the following result.

Using \( \theta = \theta_{(\cdot,p)} = (\theta^1, \ldots, \theta^K) \) to denote the set of unknown parameters,

\[
\sqrt{n}(\theta - \theta_0) \overset{d}{\rightarrow} \mathcal{N} \left( 0, V \right).
\]

In this case, \( V \) is a block diagonal matrix, where \( H_j^{-1}(\theta_{j,0}) \) are the non-zero entries.

\[
V = \begin{bmatrix}
H_1^{-1}(\theta_0^1) & 0 & \cdots & 0 \\
0 & H_2^{-1}(\theta_0^2) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_K^{-1}(\theta_0^K)
\end{bmatrix}
\]

\end{proof}
Now, in the case where Frank’s copula is used to jointly model the PoARX models, we require estimation of the $\rho$ using the profile log-likelihood with $\theta = \hat{\theta}$. The regularity conditions for the theory of inference functions (Godambe, 1991) hold for the dependence parameter, so we can use the asymptotic result,

$$\sqrt{n}(\hat{\rho} - \rho_0) \xrightarrow{d} N(0, H^{-1}_\rho(\rho_0)),$$

$$H_\rho(\rho) := -E\left(\frac{\partial^2 l}{\partial \rho^2}\bigg|_{\hat{\theta}^1, \ldots, \hat{\theta}^K, \rho}\right).$$

Collecting all unknown parameters together, the theory of inference functions states that

$$\sqrt{n}(\hat{\vartheta} - \vartheta_0) \xrightarrow{d} N(0, V),$$

for some asymptotic covariance matrix $V$. This matrix $V$ is given by

$$V = (-D^{-1}_g) M_g (-D^{-1}_g)^\top,$$

where $M_g = \text{Cov}(g(Y; \vartheta))$ and $D_g = E\left(\frac{\partial g(Y; \vartheta)}{\partial \vartheta}\right)$ with $g = (\partial l_1/\partial \theta_1, \ldots, \partial l_K/\partial \theta_K, \partial l/\partial \rho)^\top$. Let $J_{jk} = \text{Cov}(g_j, g_k)$ be the covariance matrix between $g_j$ and $g_k$, and $J_{jk} = -E\left(\frac{\partial^2 l}{\partial \theta_j \partial \theta_k}\right)$ for $1 \leq j, k \leq K$. This means that $J_{jj} = H_j(\vartheta^j)$ is the Fisher information matrix for model.

Lastly, we define $I_{mk} = -E\left(\frac{\partial^2 l}{\partial \theta_j \partial \rho}\right)$ for $k = 1, \ldots, K$. With this notation, the matrices can be partitioned as follows,

$$-D_g = \begin{bmatrix} I_{11} & 0 & \ldots & 0 & 0 \\ 0 & I_{22} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & I_{KK} & 0 \\ I_{m1} & I_{m2} & \ldots & I_{mK} & I_{mm} \end{bmatrix}, \quad M_g = \begin{bmatrix} J_{11} & J_{12} & \ldots & J_{1K} & 0 \\ J_{21} & J_{22} & \ldots & J_{2K} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ J_{K1} & J_{K2} & \ldots & J_{KK} & 0 \\ 0 & 0 & \ldots & 0 & J_{mm} \end{bmatrix}.$$

The only non-trivial calculations are $\text{Cov}(g_j, g_d) = 0$ for $j = 1, \ldots, K$. The proof of this can be found in the Appendix of Joe (2005).