DECREASING EQUISINGULAR APPROXIMATIONS WITH ANALYTIC SINGULARITIES

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ABSTRACT. In this note, for the multiplier ideal sheaves with weights \( \log \sum_i |z_i|^{a_i} \), we present the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities.

1. INTRODUCTION

The multiplier ideal sheaf related to a plurisubharmonic function plays an important role in complex geometry and algebraic geometry, which was widely discussed (see e.g. [21, 15, 18, 6, 7, 2, 8, 13, 19, 20, 3]). We recall the definition as follows.

Let \( \varphi \) be a plurisubharmonic function (see [5, 16, 17]) on a complex manifold. It is known that the multiplier ideal sheaf \( I(\varphi) \) was defined as the sheaf of germs of holomorphic functions \( f \) such that \( |f|^2 e^{-2\varphi} \) is locally integrable (see [3]).

Demailly established the decreasing equisingular approximations \( \varphi_m \) of weight \( \varphi \), which are smooth outside analytic subvarieties (see [3]), where "equisingular" means \( I(\varphi) = I(\varphi_m) \) holds for any \( m \). Then it was asked: can one choose the decreasing equisingular approximations with analytic singularities (see [3])?

In [10], by constructing an example and using the sharp lower bound of log canonical threshold [9] to check the example, a negative answer to the above question was presented. Furthermore, it is natural to ask

**Question 1.1.** For a (given) class of weights, can one give a characterization of the weights which have decreasing equisingular approximations with analytic singularities?

Let \((z_1, \cdots, z_n)\) be the coordinates on \( \mathbb{C}^n \). In this article, we answer Question 1.1 for the class of weights \( \{ \log \sum_{i=1}^m |z_i|^{a_i} : m \leq n \text{ and } a_i > 0 \text{ for any } 1 \leq i \leq m \} \).

**Theorem 1.1.** The weight \( \varphi = \log \sum_{i=1}^m |z_i|^{a_i} \) has decreasing equisingular approximations with analytic singularities near \( o \) if and only if one of the following statements holds

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2.1. \( \varphi \) has analytic singularity near \( o \), i.e., there exists \( c \in \mathbb{R}^+ \) such that \( \frac{a_i}{c} \in \mathbb{Q}^+ \)
for any \( i \in \{1, \cdots, m\} \):

(2) the equation \( \sum_{i=1}^{m} \frac{a_i}{c} = 1 \) has no positive integer solutions.

Considering the complex lines through \( o \), one can obtain the equivalence between
"\( \varphi \) has analytic singularity near \( o \)" and "there exists \( c \in \mathbb{R}^+ \) such that \( \frac{a_i}{c} \in \mathbb{Q}^+ \)
for any \( i \in \{1, \cdots, m\} \)" in Theorem 1.1.

**Remark 1.1.** Note that if \( c \notin \mathbb{Q}^+ \), then statement (2) in Theorem 1.1 holds. Then
statement (1) in Theorem 1.1 can be replaced by
"(1') \( a_i \in \mathbb{Q}^+ \) for any \( i \in \{1, \cdots, m\} \)."

2. Preparations

In the present section, we recall and present some results which will be used to
prove Theorem 1.1

### 2.1. The multiplier ideal with weight \( \log \max_{1 \leq i \leq m} |z_i|^{a_i} \)

Let \( \varphi = \log \max_{1 \leq i \leq m} |z_i|^{a_i} \), where \( m \leq n \) and \( a_i > 0 \) for any \( 1 \leq i \leq m \). Let
\( V_{r_0} = \{ \max(|z_{m+1}|, \cdots, |z_n|) < \frac{1}{2} \log r_0 \} \).

**Lemma 2.1.** Let \( f = \sum_{a} b_{a}z^{a} \) (Taylor expansion) be a holomorphic function on
a neighborhood \( U \ni o \), such that \( |f|^2 e^{-2\varphi} \) is integrable on \( U \). Then for any \( \alpha \),

\[
\int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |f|^2 e^{-2\varphi} \geq \int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |b_{a}z^{a}|^2 e^{-2\varphi}
\]

holds for any \( r > 0 \) and \( r_0 > 0 \) small enough such that \( \{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0} \subset U \),
which implies that \( |a_{\alpha}z^{\alpha}|^2 e^{-2\varphi} \) is integrable near \( o \) for any \( \alpha \).

It follows from Lemma 2.1 that the following lemma holds.

**Lemma 2.2.** The following two statements are equivalent
(1) \( \mathcal{I}(t_1 \varphi)_o = \mathcal{I}(t_2 \varphi)_o \);
(2) \( \mathcal{I}(t_1 \varphi)_o \cap \{(x^{\alpha}, o)\}_{\alpha \in \mathbb{N}^n} = \mathcal{I}(t_2 \varphi)_o \cap \{(x^{\alpha}, o)\}_{\alpha \in \mathbb{N}^n} \).

There is a standard calculation as follows

\[
\int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |z_1^{\alpha_1} \cdots z_n^{\alpha_n}|^2 =
\int_{\{ \{z_1 < r^{\frac{1}{a_1}} \} \cap \cdots \cap \{z_m < r^{\frac{1}{a_m}} \} \cap V_{r_0}} |z_1^{\alpha_1} \cdots z_n^{\alpha_n}|^2
\]

\[
= \int_{\{ \{z_1 < r^{\frac{1}{a_1}} \} \cap \cdots \cap \{z_m < r^{\frac{1}{a_m}} \} \cap V_{r_0}} |z_1^{\alpha_1} \cdots z_m^{\alpha_m}|^2
\times \int_{V_{r_0}} |z_{m+1}^{\alpha_{m+1}} \cdots z_n^{\alpha_n}|^2
\]

\[
= \pi^m \prod_{i=1}^{m} \frac{z_i^{\alpha_i + 1}}{a_i} \prod_{i=1}^{m} (\alpha_i + 1) \int_{V_{r_0}} |z_{m+1}^{\alpha_{m+1}} \cdots z_n^{\alpha_n}|^2,
\]

which implies the following lemma.

**Lemma 2.3.** \( (z_1^{\alpha_1} \cdots z_n^{\alpha_n}, o) \in \mathcal{I}(\varphi)_o \) if and only if \( \sum_{i=1}^{m} \frac{\alpha_{i+1}}{a_i} > 1 \).
Proposition 2.1. Assume that the equation $\sum_{i=1}^{m} \frac{x_i}{a_i} = 1$ has no integer solutions. Then there exists $\varepsilon > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$

$$I((1-\varepsilon)\varphi)_o = I(\varphi)_o$$

(2.2)

holds.

Proof. Note that there for any $a_i > 0$ for any $1 \leq i \leq m$, there exist finite $\alpha \in \mathbb{N}^m$, such that $\sum_{i=1}^{m} \frac{a_{i-1}}{a_i} \leq 1$. As the equation $\sum_{i=1}^{m} \frac{a_{i-1}}{a_i} = 1$ has no integer solutions, then there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, the equality

$$\{(x_i)_{1 \leq i \leq m} : \sum_{i=1}^{m} \frac{x_i}{a_i} > 1 - \varepsilon \& x_i \in \mathbb{N}^+\} = \{(x_i)_{1 \leq i \leq m} : \sum_{i=1}^{m} \frac{x_i}{a_i} > 1 \& x_i \in \mathbb{N}^+\}$$

(2.3)

holds. Combining with (2.2) one can obtain that

$$I((1-\varepsilon)\varphi)_o \cap \{(z^n, o)\}_{a \in \mathbb{N}^n} = I(\varphi)_o \cap \{(z^n, o)\}_{a \in \mathbb{N}^n}$$

Then it follows from Lemma (2.2) that the following remark holds. □

2.2. Maximal equisingular weights and minimal integrations. One can define the maximal equisingular weight as follows

**Definition 2.1.** A weight $\varphi_{max}$ was called maximal equisingular near $z_0$ if for any plurisubharmonic function $\varphi \geq \varphi_{max} + O(1)$ near $z_0$ satisfying $I(\varphi)_z = I(\varphi_{max})_z$, the inequality $\varphi = \varphi_{max} + O(1)$ holds near $z_0$.

It follows from Definition 2.1 that for any plurisubharmonic function $\varphi$ near $z_0$,

$$I(\varphi_{max})_z = I(\max\{\varphi, \varphi_{max} + C\})_z$$

(2.4)

holds for any $C > 0$.

**Remark 2.2.** Let $\varphi$ be a maximal equisingular weight near $o$. Then the following two statements are equivalent

(1) $\varphi$ has decreasing equisingular approximations with analytic singularities near $o$.

(2) $\varphi$ has analytic singularities near $o$.

Let $F$ be a holomorphic function on pseudoconvex domain $D \subset \mathbb{C}^n$ (see [5]) containing the origin $o \in \mathbb{C}^n$.

Recall that the minimal integration related to ideal $I \subset \mathcal{O}_o$ was defined in [11] as follows

$$C_{F,I}(D) := \inf \{ \int_D |\tilde{F}|^2((\tilde{F} - F)_o) \in I \& \tilde{F} \in \mathcal{O}(D) \}.$$

Let $\varphi$ be a plurisubharmonic function on $D$. In [11], the following concavity of $G(-\log r) := C_{F,I(\varphi)}(\{\varphi < \frac{1}{r} \log r\})$ was presented.

**Proposition 2.1.** [11] Assume that $G(0) < +\infty$. Then $G(-\log r)$ is concave with respect to $r \in (0,1]$, which implies that

(1) the inequality $G(-\log r) \geq rG(0)$ holds for any $r \in (0,1]$;

(2) the equality $G(-\log r) = rG(0)$ holds for some $r \in (0,1)$ if and only if the equality holds for any $r \in (0,1]$.

It follows from the dominated convergence theorem that $G(0) < +\infty$ implies

$$\lim_{r \to 0+} G(-\log r) = 0.$$
Lemma 2.4. Let \( \varphi_1 \leq \varphi_2 < 0 \) be plurisubharmonic functions on \( D \). Assume that \( C_{F,I(\varphi_2)_o}(D) < +\infty \), and \( (F,o) \not\in I(\varphi_1)_o \). Then the following three statements are equivalent:

1. \( \varphi_1 = \varphi_2 \) on \( D \);
2. the quality \( \{ \varphi_1 < \log r \} \subseteq \{ \varphi_2 < \log r \} \) holds for any \( r \in (0,1] \); 
3. the equality \( C_{F,I(\varphi_1)_o}(\{ \varphi_1 < \log r \}) = C_{F,I(\varphi_2)_o}(\{ \varphi_2 < \log r \}) \) holds for any \( r \in (0,1] \).

Proof. It suffices to prove that (3) \( \Rightarrow \) (2). We prove it by contradiction: if not, then \( \{ \varphi_1 < \log r \} \subseteq \{ \varphi_2 < \log r \} \) holds for some \( r \). It follows from \( C_{F,I(\varphi_2)_o}(D) < +\infty \), and Lemma 2.2 in [11] there exists (unique) \( \tilde{F} \in O(\{ \varphi_2 < \log r \}) \) such that

\[
\int_{\{ \varphi_1 < \log r \}} |\tilde{F}|^2 = C_{F,I(\varphi_1)_o}(\{ \varphi_1 < \log r \}) \quad \text{and} \quad (\tilde{F} - F, o) \not\in I(\varphi_1)_o.
\]

Then we have

\[
C_{F,I(\varphi_1)_o}(\{ \varphi_1 < \log r \}) = \int_{\{ \varphi_1 < \log r \}} |\tilde{F}|^2 > \int_{\{ \varphi_2 < \log r \}} |\tilde{F}|^2 \geq C_{F,I(\varphi_2)_o}(\{ \varphi_2 < \log r \}),
\]

which contradicts statement (3) in the present Lemma. \( \square \)

By Proposition 2.1 and Lemma 2.4, one can obtain the following remark.

Remark 2.3. If equality \( G(-\log r) = rG(0) \) holds for some \( r \in (0,1) \), then for any plurisubharmonic function \( \varphi_2 \geq \varphi \) on \( D \) satisfying \( I(\varphi_2)_{z_0} = I(\varphi)_{z_0} \), the equality \( \varphi_2 = \varphi \) holds on \( D \).

Proof. Proposition 2.1 shows that if equality \( G(-\log r) = rG(0) \) holds for some \( r \in (0,1) \), then \( G(-\log r) = rG(0) \) holds for any \( r \in (0,1) \). By Lemma 2.4 \( (\varphi \sim \varphi_1) \) and \( \varphi_2 \geq \varphi \), it follows that \( \varphi_2 = \varphi \) holds on \( D \). \( \square \)

Lemma 2.5. Let \( \varphi_1 \) and \( \varphi_2 \) be two plurisubharmonic functions near \( z_0 \), such that \( \varphi_1 \leq \varphi_2 + O(1) \). If \( I(\varphi_1)_{z_0} = I(\varphi_2)_{z_0} \), then

\[
I(\varphi_1)_{z_0} = I(\varphi_2)_{z_0} = I(\max\{\varphi_1,\varphi_2 + C\})_{z_0}
\]

holds for any \( C \in \mathbb{R} \).

Proof. Note that \( \varphi_1 \leq \max\{\varphi_1,\varphi_2 + C\} \leq \varphi_2 + O(1) \) holds near \( z_0 \). \( \square \)

Remark 2.3 implies the following sufficient condition of maximal equisingular weight.

Proposition 2.2. Let \( \{D_j\}_{j=1,2,\ldots} \) be a sequence of pseudoconvex subdomains on \( D \) satisfying \( \cap_j D_j = \{o\} \), \( \varphi|_{D_j} < \frac{1}{r} \log r_j \). Let \( F_j \) be holomorphic functions on \( D_j \). If

\[
\frac{1}{r_j' \log r_j'} \int_{\{ \varphi < \log r_j' \}} |F_j|^2 = \frac{C_{F,I(\varphi)_o}(D_j)}{r_j}
\]

(2.6)

holds for any \( j \in \{1,2,\ldots\} \) and some \( r_j' \in (0,r_j) \), then \( \varphi \) is a maximal equisingular weight near \( o \).

Proof. We prove Proposition 2.2 by contradiction: if not, there exists a weight \( \varphi' \) on a neighborhood \( U \ni o \) such that

1. \( \varphi' \geq \varphi \) and \( \lim_{z \to o} (\varphi'(z) - \varphi(z)) = +\infty \);
2. \( I(\varphi')_o = I(\varphi)_o \).
One can find some \( j \) such that \( D_j \subset \subset U \). Then there exists \( N_j \gg 0 \) such that \( \varphi' - N_j < \frac{1}{2} \log r_j \) on \( D_j \).

Consider \( \max \{ \varphi, \varphi' - N_j \} < \frac{1}{2} \log r_j \). It follows from Lemma 2.3 that \( \mathcal{I}(\max \{ \varphi, \varphi' - N_j \})_o = \mathcal{I}(\varphi)_o \). Note that Remark 2.4 (\( \max \{ \varphi, \varphi' - N_j \} = \varphi \)) implies that

\[
\max \{ \varphi, \varphi' - N_j \} = \varphi,
\]

which contradicts \( (1) \lim_{z \to o} (\varphi(z) - \varphi(z)) = +\infty \). Then Proposition 2.2 has been proved.

Equality 2.4 implies the following lemma.

**Lemma 2.6.** Let \( \varphi = \log \max_{1 \leq i \leq m} |z_i|^a_i \) (\( m \leq n \)), where \( a_i > 0 \) for any \( 1 \leq i \leq m \). Let \( \alpha = (a_1, \cdots, a_n) \). Assume that \( \sum_{i=1}^n \frac{a_i + 1}{a_i} \leq 1 \). Then

\[
C_{z^\alpha, \mathcal{I}(\varphi)_o} \{ \{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0} \} = \int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |z^\alpha|^2 = \pi^m \sum_{i=1}^n \frac{a_i + 1}{a_i} \prod_{i=1}^n (a_i + 1) \int_{V_{r_0}} |z_{m+1} \cdots z_n|^2.
\]

(2.7)

**Proof.** Note that for any \( L^2 \) integrable holomorphic function \( z^\alpha + \sum_{\alpha' \neq \alpha} b_{\alpha'} z^{\alpha'} \) on \( \{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0} \), inequality

\[
\int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |z^\alpha + \sum_{\alpha' \neq \alpha} b_{\alpha'} z^{\alpha'}|^2 \geq \int_{\{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0}} |z^\alpha|^2
\]

holds, then one can obtain the present lemma by equality 2.1.

If \( \sum_{i=1}^n \frac{a_i + 1}{a_i} > 1 \), then it follows from Lemma 2.3 that

\[
(z^\alpha, o) \in \mathcal{I}(\varphi)_o,
\]

which implies that

\[
C_{z^\alpha, \mathcal{I}(\varphi)_o} \{ \{ \varphi < \frac{1}{2} \log r \} \cap V_{r_0} \} = 0
\]

holds for any \( r > 0 \) and \( r_0 > 0 \).

It follows from Proposition 2.2 and Lemma 2.6 that

**Remark 2.4.** Assume that \( \sum_{i=1}^m \frac{a_i + 1}{a_i} = 1 \) for some \( \alpha \in \mathbb{N}^m \). Then \( \varphi = \log \max_{1 \leq i \leq m} |z_i|^{a_i} \) is a maximal weight, where \( a_i > 0 \) for any \( 1 \leq i \leq m \).

**Proposition 2.3.** Let \( \varphi = \log \max_{1 \leq i \leq m} |z_i|^{a_i} \), where \( a_i > 0 \) (\( 1 \leq i \leq m \)), and \( m \leq n \). Then the following three statements are equivalent

1. \( \varphi \) is not a maximal equisingular weight near \( o \);
2. there exists \( \varepsilon \in (0, 1) \) such that \( \mathcal{I}(\varphi)_o = \mathcal{I}((1 - \varepsilon) \varphi)_o \);
3. the equation \( \sum_{i=1}^m \frac{a_i}{a_i} = 1 \) has no positive integer solutions.

**Proof.** It follows from Definition 2.4 and Remark 2.4 (respectively) that "(2) \( \Rightarrow (1) \)" and "(3) \( \Rightarrow (2) \)" hold (respectively). Then it suffices to consider "(1) \( \Rightarrow (3) \)".

We prove "(1) \( \Rightarrow (3) \)" by contradiction: if not, then the equation \( \sum_{i=1}^m \frac{a_i}{a_i} = 1 \) has a positive integer solution denoted by \( (a_1 + 1, \cdots, a_m + 1) \). It follows from Remark 2.4 that \( \varphi \) is a maximal equisingular weight near \( o \), which contradicts statement (1). Then we obtain that (1) \( \Rightarrow (3) \) holds.
2.3. Decreasing equisingular approximations with analytic singularities.

Lemma 2.7. Let $\varphi = \log \max_{1 \leq i \leq m} |z_i|^{a_i}$, where $a_i > 0$ ($1 \leq i \leq m$), and $m \leq n$.

If there exists $\varepsilon \in (0, 1)$ such that $\mathcal{I}(\varphi)_o = \mathcal{I}((1 - \varepsilon)\varphi)_o$, then $\mathcal{I}(\varphi)_o$ has decreasing equisingular approximations with analytic singularities near $o$.

Proof. Note that for any $a_i$ there exists rational numbers $a_{i,k} \in ((1 - \varepsilon)a_i, a_i)$ such that $\lim_{k \to +\infty} a_{i,k} = a_i$ holds for any $i \in \{1, \ldots, m\}$, and increasing with respect to $k$. Let $\varphi_k = \log \max_{1 \leq i \leq m} |z_i|^{a_{i,k}}$. Then the sequence $\{\varphi_k\}_k$ with analytic singularities near $o$ are decreasing convergent to $\varphi$ with respect to $k$ on $\{z : \max_{1 \leq i \leq m} |z_i| < 1\}$.

Note that $\varphi \leq \varphi_k \leq (1 - \varepsilon)\varphi$ near $o$ for any $k$, which implies

$\mathcal{I}(\varphi)_o \subseteq \mathcal{I}(\varphi_k)_o \subseteq \mathcal{I}((1 - \varepsilon)\varphi)_o$

holds for any $k$. Then it follows from $\mathcal{I}(\varphi)_o = \mathcal{I}((1 - \varepsilon)\varphi)_o$ that

$\mathcal{I}(\varphi)_o = \mathcal{I}(\varphi_k)_o$

holds for any $k$. Combining with the decreasing property of $\varphi_k$, we obtain the present lemma. □

3. Proof of Theorem

We prove Theorem 1.1 by two steps: sufficiency and necessity respectively.

Step 1. (Sufficiency). It suffices to prove that $(2)$ implies that $\varphi$ has decreasing equisingular approximations with analytic singularities. By Proposition 2.3, it follows that $(2)$ implies that there exists $\varepsilon \in (0, 1)$ such that $\mathcal{I}(\varphi)_o = \mathcal{I}((1 - \varepsilon)\varphi)_o$.

By Lemma 2.7, we obtain that $(2)$ implies that $\varphi$ has decreasing equisingular approximations with analytic singularities. Then the sufficiency has been proved.

Step 2. (Necessity). We prove by contradiction: if not, i.e. neither statement $(1)$ nor statement $(2)$ holds, and $\varphi$ has decreasing equisingular approximations with analytic singularities, then it follows from Proposition 2.3 that $\varphi$ is a maximal weight near $o$. By Remark 2.2, it follows from $\varphi$ has decreasing equisingular approximations with analytic singularities that $\varphi$ has analytic singularity near $o$, which contradicts the assumption that statement $(1)$ does not hold. Then the necessity has been proved.

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