Demonstration of topologically path-independent anyonic braiding in a nine-qubit planar code: supplementary material

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Published 28 February 2019

This document provides supplementary information to "Demonstration of topologically path-independent anyonic braiding in a nine-qubit planar code," https://doi.org/10.1364/optica.6.000264.

1. QUANTUM CIRCUIT FOR REALIZING THE GROUND STATE AND ANYONIC BRAIDING

The planar code ground state and anyonic braiding in Fig. S1(b) can be mapped to a quantum circuit following the methods described by Han and co-workers [1]. The quantum circuit can be decomposed into the following four subroutines: (1) ground state preparation, (2) anyon creation, (3) anyon braiding, and (4) anyon annihilation. Our photonic experimental setup follows the same basic procedure as the quantum circuit. The only difference is that the ground state can be prepared by manipulating an entangled state generated by spontaneous parametric down-conversion (SPDC) (see Section 2), rather than applying a series of controlled-Z gates in the quantum circuit. The anyon creation, anyon braiding and anyon annihilation operations are implemented by a series of single-qubit gates, which can be easily realized by a combination of quarter-wave plates (QWP) and half-wave plates (HWP). The m anyon is operated through the three loops described in Fig. 1(b). Figure S1(a) shows one of the three loops (loop-3 (3-4-5-7-8-9)), which can be realized by the operation $X_2 X_8 X_5 X_3 X_4$ as shown in Fig. S1(b). For further details, we refer the reader to Ref. [1].

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Fig. S1. The detailed quantum circuit for realization of the ground state and the associated anyon braiding.
2. EXPERIMENTAL DETAILS

A. Preparation of the ground state

In the experiment, the ground state is created in 5 steps as follows:

**Step 1.** Create 3 pairs of entangled-states $|H_3H_5⟩ + |V_3V_5⟩$, $|H_5H_6⟩ + |V_5V_6⟩$, $|H_7H_8⟩ + |V_7V_8⟩$ and two qubits $|H_4⟩$, $|V_4⟩$ using 3 sandwiched-BBOs and a single-BBO (and some half wave plates) respectively.

**Step 2.** Apply the $H$ gate on photons 3 and 4, and let photons 3 and 4 interfere on the PBS$_S$, and post-select the events where there is exactly one photon exiting each output. This creates the GHZ-state $|H_4H_5H_6⟩ + |V_4V_5V_6⟩$. Using a similar method, we could also create the GHZ-state $|H_2H_4H_8⟩ + |V_2V_4V_8⟩$.

**Step 3.** Perform rotations on two GHZ-states to obtain $|+1+3+4⟩ + |−1−3−4⟩$ and $|+5+6+8⟩ + |−5−6−8⟩$. Then we let photon 4 and 5 interfere on the PBS$_S$ and post-select the state

$$|φ⟩ = |H_1H_5H_4H_6H_7H_8H_9⟩ + |V_1V_5V_4V_6V_7V_8V_9⟩ + |H_1V_5V_4V_6V_7V_8V_9⟩ + |V_1H_5V_4V_6V_7V_8V_9⟩ + |H_1V_5V_4V_6V_7V_8V_9⟩ + |V_1H_5V_4V_6V_7V_8V_9⟩.$$

(S1)

**Step 4.** Let photons 7 and 8 interfere on the PBS$_S$ to create the state:

$$|φ⟩ = |H_1H_3H_5H_6H_7H_8H_9⟩ + |V_1V_3V_5V_6V_7V_8V_9⟩ + |H_1V_3V_5V_6V_7V_8V_9⟩ + |V_1H_3V_5V_6V_7V_8V_9⟩ + |H_1V_3V_5V_6V_7V_8V_9⟩ + |V_1H_3V_5V_6V_7V_8V_9⟩,$$

(S2)

**Step 5.** By using a Mach-Zehnder interferometer (MZI), we encode the spatial mode of photon 1 as qubit-2, to obtain the state

$$|φ⟩ = |H_1U_2H_3H_5H_6H_7H_8H_9⟩ + |V_1U_2V_3V_5V_6V_7V_8V_9⟩ + |H_1U_2V_3V_5V_6V_7V_8V_9⟩ + |V_1U_2H_3H_5H_6H_7H_8H_9⟩ + |H_1D_2V_3V_5V_6V_7V_8V_9⟩ + |V_1U_2H_3H_5H_6H_7H_8H_9⟩ + |H_1D_2V_3V_5V_6V_7V_8V_9⟩ + |V_1D_2H_3H_5H_6H_7H_8H_9⟩ + |H_1U_2V_3V_5V_6V_7V_8V_9⟩ + |V_1D_2H_3H_5H_6H_7H_8H_9⟩.$$

(S3)

where $U$ and $D$ are the upper and lower path of the MZI. Encoding $H$ and $D$ as qubit 0, and encode $V$ and $U$ as qubit 1, we obtain the ground state

$$|φ⟩ = |000000000⟩ + |111000000⟩ + |110111000⟩ + |001111100⟩ + |110110111⟩ + |001110111⟩ + |000011111⟩ + |111001111⟩.$$

(S4)

B. Preparation of the excited state

By performing different operations on the ground state, we can create an excited state or a superposition of the state with and without the $e$ anyons. The excited state is created by performing the $Z_3$ operator on photon 3 and the $X_4$ operator on photon 4, which can be realized using a HWP of $0°$ and $45°$, respectively.

The superposition in equation (6) of the main text is created by performing the $\sqrt{Z_3}$ operator on photon 3 and the $X_4$ operator on photon 4, which can be realized using QWP of $0°$ and HWP of $45°$, respectively.

C. Measurement of photonic qubit

In our experiment, two kinds of photonic qubit, polarization qubit and spatial qubit, are used. The measurement methods for these two kinds of photonic qubit are as follows.

**Polarization qubit measurement:** The experimental device for measuring polarization qubits consists of HWP, PBS and single-photon detectors, as shown in Fig. S3(a). The Pauli-X measurement is implemented when the angle of HWP is $22.5°$, and the Pauli-Z measurement is implemented when the angle of HWP is $0°$.

**Spatial qubit measurement:** The experimental device for measuring spatial qubits consists is shown in Fig. S3(b). The Pauli-X measurement is implemented via placing the Double-BS and setting the angle of HWP to $22.5°$. The Pauli-Z measurement is implemented via removing the Double-BS and setting the angle of HWP to $0°$.
3. OBSERVABLE FOR READING OUT THE ANYONIC PHASE

In this section discuss the observable
\[
P(\theta) = |+\theta \rangle_{123} \langle +\theta|_{123}
\] (S5)
which is used to read out the anyonic phase \(\phi\) in the main text. In general we desire an observable \(\mathcal{O}(\theta)\) that takes the form
\[
\langle \mathcal{O}(\theta) \rangle = f(\theta - \phi)
\] (S6)
where the expectation value is taken with respect to the state
\[
|\chi\rangle = \frac{1}{\sqrt{2}} \left( |\phi\rangle + e^{i\phi} |e_1, e_2\rangle \right) = \frac{1}{\sqrt{2}} \left( 1 + e^{i\phi} Z_3 \right) |\phi\rangle.
\] (S7)
Such a quantity is convenient since the phase can be read out easily by a displacement of the function \(f\).

The operator (S5) is a simple quantity that satisfies the above requirements as can be shown below. To evaluate the expectation value it is more convenient to write the projection operator in terms of Pauli operators. Let us first define
\[
\Omega(\theta) = Z \cos \theta + iZX \sin \theta
\] (S8)
which is the Pauli operator that has
\[
|\theta\rangle = \frac{1}{\sqrt{2}} \left( |+\rangle + e^{i\theta} |-\rangle \right) = e^{i\theta/2} \left( \cos \frac{\theta}{2} |0\rangle - i \sin \frac{\theta}{2} |1\rangle \right)
\] (S9)
as its positive eigenvector \(\Omega(\theta)|\theta\rangle = |\theta\rangle\). Then the projection operator (S5) can be written
\[
P(\theta) = \frac{1}{8} \left( 1 + X_1 \right) \left( 1 + X_2 \right) \left( 1 + \Omega_3(\theta) \right)
\]
\[
= \frac{1}{8} \left( 1 + X_1 + X_2 + \Omega_3(\theta) \right)
\]
\[
+ X_1 X_2 + X_1 \Omega_3(\theta) + X_2 \Omega_3(\theta) + X_1 X_2 \Omega_3(\theta)\)
\] (S10)
We may now evaluate the state obtained after each of the terms in (S10) on the state (S7). We obtain
\[
X_1 |\chi\rangle = \frac{1}{\sqrt{2}} \left( X_1 + e^{i\phi} X_1 Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( |m_1\rangle + e^{i\phi} |m_1, e_1, e_2\rangle \right)
\]
\[
X_2 |\chi\rangle = \frac{1}{\sqrt{2}} \left( X_2 + e^{i\phi} X_2 Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( |m_2\rangle + e^{i\phi} |m_2, e_1, e_2\rangle \right)
\]
\[
X_1 X_2 |\chi\rangle = \frac{1}{\sqrt{2}} \left( X_1 X_2 + e^{i\phi} X_1 X_2 Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( |m_1, m_2\rangle + e^{i\phi} |m_1, m_2, e_1, e_2\rangle \right)
\]
\[
X_1 \Omega_3(\theta) |\chi\rangle = \frac{1}{\sqrt{2}} \left( X_1 Z_3 \cos \theta + iX_1 Z_3 X_3 \sin \theta \right) \left( 1 + e^{i\phi} Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( \cos \theta |m_1, e_1, e_2\rangle + e^{i\phi} \cos \theta |m_1\rangle \right)
\]
\[
+ i \sin \theta |m_2, e_1, e_2\rangle - e^{i\phi} \sin \theta |m_2\rangle
\]
\[
X_2 \Omega_3(\theta) |\chi\rangle = \frac{1}{\sqrt{2}} \left( X_2 Z_3 \cos \theta + iX_2 Z_3 X_3 \sin \theta \right) \left( 1 + e^{i\phi} Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( \cos \theta |m_2, e_1, e_2\rangle + e^{i\phi} \cos \theta |m_2\rangle \right)
\]
\[
- i \sin \theta |m_1, e_1, e_2\rangle - e^{i\phi} \sin \theta |m_1\rangle\).
\] (S11)

In the above definition of states involving anyons, we take the convention that the \(m\) anyons are created first, then the \(e\) anyons, e.g. \(|m_1, e_1, e_2\rangle = Z_3 X_3 |\phi\rangle\). All the above terms produce states that are orthogonal to \(|\chi\rangle\), hence these contribute zero to the expectation value. The terms which give a non-zero contribution are the unity term,
\[
\Omega_3(\theta) |\chi\rangle = \frac{1}{\sqrt{2}} \left( Z_3 \cos \theta + iZ_3 X_3 \sin \theta \right) \left( 1 + e^{i\phi} Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( \cos \theta |e_1, e_2\rangle + e^{i\phi} \cos \theta |\phi\rangle \right)
\]
\[
+ i \sin \theta |m_1, m_2, e_1, e_2\rangle - e^{i\phi} \sin \theta |m_1, m_2\rangle\)
\] (S12)
and
\[
X_1 X_2 \Omega_3(\theta) |\chi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( X_1 X_2 Z_3 \cos \theta + iX_3 A_1 \sin \theta \right) \left( 1 + e^{i\phi} Z_3 \right) |\phi\rangle
\]
\[
= \frac{1}{\sqrt{2}} \left( \cos \theta |m_1, m_2, e_1, e_2\rangle + e^{i\phi} \cos \theta |m_1, m_2\rangle \right)
\]
\[
+ i \sin \theta |e_1, e_2\rangle - e^{i\phi} \sin \theta |\phi\rangle\),
\] (S13)
where we have used the fact that \(A_1 |\phi\rangle = X_1X_2X_3 |\phi\rangle = |\phi\rangle\).

The expectation values of these terms are thus
\[
\langle \chi | \Omega_3(\theta) |\chi\rangle = \cos \theta \cos \phi
\]
\[
\langle \chi | X_1 X_2 \Omega_3(\theta) |\chi\rangle = \sin \theta \sin \phi
\] (S14)
Combining the above results together we obtain
\[
\langle \chi | P(\theta) |\chi\rangle = \frac{1}{8} \left( 1 + \cos (\theta - \phi) \right)
\] (S15)
as given in the main text.

REFERENCES

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