REGULAR, UNIT-REGULAR, AND IDEMPOTENT ELEMENTS OF SEMIGROUPS OF TRANSFORMATIONS THAT PRESERVE A PARTITION

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Abstract. Let $X$ be a set and $T_X$ be the full transformation semigroup on $X$. For a partition $P$ of $X$, we consider semigroups $T(X, P) = \{f \in T_X | (\forall X_i \in P)(\exists X_j \in P) X_i f \subseteq X_j\}$, $\Sigma(X, P) = \{f \in T(X, P) | (\forall X_i \in P) Xf \cap X_i \neq \emptyset\}$, and $\Gamma(X, P) = \{f \in T_X | (\forall X_i \in P)(\exists X_j \in P) X_i f = X_j\}$. We characterize unit-regular elements of both $T(X, P)$ and $\Sigma(X, P)$ for finite $X$. We discuss set inclusion between $\Gamma(X, P)$ and certain semigroups of transformations preserving $P$. We characterize and count regular elements and idempotents of $\Gamma(X, P)$. For finite $X$, we prove that every regular element of $\Gamma(X, P)$ is unit-regular and also calculate the size of $\Gamma(X, P)$.

1. Introduction

We assume that the reader is familiar with the basic terminology and facts of combinatorics and semigroup theory. Throughout the paper, let $X$ denote a set containing at least three elements, let $P = \{X_i | i \in I\}$ denote a partition of $X$, and let $T_X$ denote the full transformation semigroup on $X$. For a subset $A \subseteq X$, we denote by $Af$ the image of $A$ under $f \in T_X$. We say that a map $f \in T_X$ preserves a partition $P$ of $X$ if for every $X_i \in P$, there exists $X_j \in P$ such that $X_i f \subseteq X_j$.

In 1994, Pei \cite{Pei1994} introduced the subsemigroup

$T(X, P) = \{f \in T_X | (\forall X_i \in P)(\exists X_j \in P) X_i f \subseteq X_j\}$

of $T_X$. If $P$ is a trivial partition of $X$, then it is obvious that $T(X, P) = T_X$. Pei proved in \cite{Pei1994} Theorem 2.8 that $T(X, P)$ is the semigroup of all continuous selfmaps on $X$ endowed with the topology having $P$ as a basis. Since then the semigroup $T(X, P)$ and its subsemigroups, for instance, $\Sigma(X, P) = \{f \in T(X, P) | (\forall X_i \in P) Xf \cap X_i \neq \emptyset\}$ and the group of units $S(X, P)$ of $T(X, P)$ have received great attention from many semigroup theorists (see, e.g., \cite{Dolinka2010, Dolinka2011, Dolinka2013, Dolinka2015, Dolinka2021, Dolinka2022, Dolinka2023, Dolinka2024, Dolinka2025, Dolinka2027, Dolinka2029, Dolinka2030}).

There have also been several studies focused on the regular elements and also the idempotents of semigroups of transformations that preserve a partition. It is worth mentioning that the semigroup $T_X$ is regular (cf. \cite[p.33]{Dolinka2010}). Pei \cite{Pei1994} characterized the regular elements of $T(X, P)$, and then concluded that the semigroup $T(X, P)$ is regular if and only if $P$ is a trivial partition of $X$. Purisang and Rakbud \cite{Purisang2014} characterized the regularity of $\Sigma(X, P)$. Dolinka et al. characterized as well as enumerated the idempotents of $T(X, P)$ for a finite set $X$ in \cite{Dolinka2010} and \cite{Dolinka2011} for the

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uniform and non-uniform cases, respectively. The authors [27] characterized as well as enumerated the idempotents of \( \Sigma(X, P) \) for an arbitrary set \( X \) and a finite set \( X \), respectively. The cardinal of semigroups of transformations that preserve a partition have also been determined (see, e.g., [14, 27, 28]).

For a partition \( P \) of an arbitrary set \( X \), let
\[
\Gamma(X, P) = \{ f \in T_X | (\forall X_i \in P)(\exists X_j \in P) X_i f = X_j \}.
\]
It is easy to see that \( \Gamma(X, P) \) is a subsemigroup of \( T(X, P) \). If \( X \) is a finite set and \( P \) is a uniform partition of \( X \), a number of interesting properties of \( \Gamma(X, P) \) have been obtained. For example, Pei proved in [22, Theorem 4.1] that \( \Gamma(X, P) \) is exactly the semigroup of all closed selfmaps on \( X \) endowed with the topology having \( P \) as a basis. Pei next proved in [22, Theorem 4.1] that the semigroup \( \Gamma(X, P) \) is regular. Moreover, Pei [22, Theorem 4.2] obtained an upper bound for the rank of \( \Gamma(X, P) \). Finally, Araújo and Schneider [3, Theorem 1.1] computed the rank of \( \Gamma(X, P) \).

The rest of the paper is organized as follows. In the next section, we introduce notations, definitions, and results which are used within the paper. In Section 3, we give a characterization of the unit-regular elements of the semigroups \( T(X, P) \) and \( \Sigma(X, P) \) for a finite set \( X \). In Section 4, we discuss the set inclusion between \( \Gamma(X, P) \) and certain existing semigroups of transformations that preserve the partition \( P \). In Section 5, we give a characterization of the regular elements as well as the idempotents of \( \Gamma(X, P) \). For a finite set \( X \), we further prove that every regular element of \( \Gamma(X, P) \) is unit-regular. In Section 6, we count the number of elements, idempotents, and regular elements of \( \Gamma(X, P) \) for a finite set \( X \).

2. PRELIMINARIES AND NOTATIONS

The purpose of this section is to introduce notations, definitions, and results that we will use within the paper. We refer the reader to the standard books [5, 19] for additional information from combinatorics and semigroup theory, respectively.

Unless otherwise specified, we will use capital letter to denote nonempty subset, calligraphic letter to denote collection of subsets, and small letter to denote set-element, map, or positive integer. The letter \( I \) will be reserved for an arbitrary indexing set. We denote by \( \mathbb{N} \) the set of all positive integers. For \( m \in \mathbb{N} \), the symbol \( I_m \) denote the subset \( \{1, \ldots, m\} \). The size or cardinality of a set \( A \) is the number of elements in \( A \), and it is denoted by \(|A|\). A finite set of size \( n \in \mathbb{N} \) is called an \( n \)-element set or \( n \)-set. We write \( A \setminus B \) to denote the set of all elements \( x \in A \) such that \( x \notin B \). We denote by \( \binom{n}{r} \) the number of \( r \)-subsets of an \( n \)-set.

Let \( X \) be a nonempty set, and let \( m, k \in \mathbb{N} \) such that \( m \geq k \). A partition of \( X \) is a collection of nonempty disjoint subsets, called blocks, whose union is \( X \). A partition is called trivial if it has only singleton blocks or a single block. A partition \( P \) is called uniform if all the blocks of \( P \) have the same size; otherwise, \( P \) is called non-uniform. A partition is said to be \( m \)-partition if it has exactly \( m \) blocks. If an \( m \)-partition has exactly \( k \) distinct sizes of blocks, we say that the partition is an \((m, k)\)-partition. A \( k \)-subpartition of an \( m \)-partition \( P \) is a subcollection of \( P \) containing \( k \) blocks. The number of partitions of an \( m \)-element set into \( k \) blocks is denoted by \( S(m, k) \), and is called the Stirling number of the second kind (cf.
A selfmap on a set $A$ is a map from $A$ to itself. The composition of maps will be denoted by juxtaposition. Let $f, g \in T_X$ and $x \in X$. We will write $xf$ to denote the image of $x$ under $f$, and compose maps from left to right: $x(fg) = (xf)g$. The domain, codomain, and range of a map $\alpha$ will be denoted by $\text{dom}(\alpha)$, $\text{codom}(\alpha)$, and $\text{ran}(\alpha)$, respectively. The rank of a map $\alpha$, denoted by $\text{rank}(\alpha)$, is the cardinality of $\text{ran}(\alpha)$. The pre-image of $B \subseteq X$ under $f$ is denoted by $Bf^{-1} = \{x \in X \mid xf \in B\}$. If $A, B \subseteq X$ such that $Af \subseteq B$, then there is a map $h : A \rightarrow B$ defined by $xh = xf$ for all $x \in A$ and, in this case, we say that the map $h$ is induced by $f$.

Let $S$ be a semigroup with identity. An element $a \in S$ is said to be regular if there exists $b \in S$ such that $a = aba$; otherwise, $a$ is called irregular. If each element of $S$ is regular, then we say that the semigroup $S$ is regular. The set of all regular elements of $S$ will be denoted by $\text{Reg}(S)$. If $A \subseteq S$, then we write $\text{Reg}(A)$ for the set $A \cap \text{Reg}(S)$. An element $a$ of $S$ is said to be unit-regular if there exists a unit $u \in S$ such that $a = auu$. Note that every unit-regular element of $S$ is contained in $\text{Reg}(S)$. If each element of $S$ is unit-regular, then we say that the semigroup $S$ is unit-regular. An element $a \in S$ is called an idempotent if $a^2 = a$. The set of all idempotents of $S$ is denoted by $E(S)$. Note that $E(S) \subseteq \text{Reg}(S)$. If $A \subseteq S$, then we write $E(A)$ for the set $A \cap E(S)$. It is worth mentioning that $f \in T_X$ is an idempotent if and only if $f$ acts as the identity map on its image set (cf. [8] p.6).

3. Unit-regular elements of $T(X, \mathcal{P})$

In this section, we give a characterization of the unit-regular elements of the semigroup $T(X, \mathcal{P})$ for a finite set $X$. Using this, we also characterize the unit-regular elements of the semigroup $\Sigma(X, \mathcal{P})$ for a finite set $X$. Recall that an element $a$ of a semigroup with identity is unit-regular if there exists a unit $u$ such that $a = auu$. The definition of unit-regular elements of a semigroup with identity is a natural one. The notion of unit-regular elements was first appeared in the context of rings [12]. There have been a number of interesting works on the unit-regular elements and unit-regular semigroups (see, e.g., [4 6 7 9 16 18 31]).

It is worth mentioning that the semigroup $T_X$ is unit-regular if and only if $X$ is a finite set (cf. [9] Proposition 5)). Thus, if $\mathcal{P}$ is a trivial partition of a finite set $X$ then $T(X, \mathcal{P})$ is unit-regular. Pei [23] gave a characterization of the regular elements of $T(X, \mathcal{P})$. However, there can be a regular element in $T(X, \mathcal{P})$ for a nontrivial partition $\mathcal{P}$ of $X$ which is not unit-regular as shown in the following example.

**Example 3.1.** Let $\mathcal{P} = \{\{1\}, \{2, 3\}\}$ be a partition of $X = \{1, 2, 3\}$. Define a selfmap $f : X \rightarrow X$ by

$$f = \left(\begin{array}{ccc}1 & 2 & 3 \\2 & 1 & 1\end{array}\right).$$

It is clear that $f \in T(X, \mathcal{P})$. Since $f = fff$, it follows that $f$ is a regular element of $T(X, \mathcal{P})$. We can also observe that there are only two selfmaps in $T(X, \mathcal{P})$, namely $f$ and $g$ such that $f = fff$ and $f = fgf$, where $g : X \rightarrow X$ is
defined by

\[ g = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}. \]

Note that both selfmaps \( f \) and \( g \) on \( X \) are not bijective. Hence \( f \) is not a unit-

regular element of \( T(X, \mathcal{P}) \).

Let us now recall a definition from [26].

**Definition 3.2.** (cf. [26]) Let \( \mathcal{P} = \{X_i | i \in I\} \) be a partition of a set \( X \), and let \( f \in T(X, \mathcal{P}) \). Then the *character* of \( f \), denoted by \( \chi(f) \), is a selfmap \( \chi(f) : I \to I \) defined by

\[ i \chi(f) = j \text{ whenever } X_i f \subseteq X_j. \]

If \( X \) is a finite set, the selfmap \( \chi(f) \) has also been discussed and denoted by \( f^\top \) (see [1, 10, 11]).

The next remark is simple.

**Remark 3.3.** Let \( \mathcal{P} = \{X_i | i \in I\} \) be a partition of a set \( X \), and let \( f \in \Gamma(X, \mathcal{P}) \). Then

\[ i \chi(f) = j \text{ if and only if } X_i f = X_j. \]

The following theorem characterizes the unit-regular elements of the semigroup \( T(X, \mathcal{P}) \) for a finite set \( X \).

**Theorem 3.4.** Let \( \mathcal{P} = \{X_i | i \in I_m\} \) be an \( m \)-partition of a finite set \( X \), and let \( f \in T(X, \mathcal{P}) \). Then \( f \) is unit-regular if and only if for each \( i \in I_m \chi(f) \) there exists \( j \in I_m \) such that \( |X_i| = |X_j| \) and \( X_i \cap X f = X_j f \).

**Proof.** Suppose first that \( f \in T(X, \mathcal{P}) \) is unit-regular. Then there exists \( g \in S(X, \mathcal{P}) \) such that \( f = fgf \). Let \( i \in I_m \chi(f) \). Since \( g \in S(X, \mathcal{P}) \), there exists \( j \in I_m \) such that \( X_i g \subseteq X_j \) and so \( |X_i| = |X_j| \) by [27] Lemma 3.6 (ii). We now verify that \( X_i \cap X f = X_j f \).

Since \( i \in I_m \chi(f) \), there exists \( k \in I_m \) such that \( X_k f \subseteq X_i \). Recall that \( X_i g \subseteq X_j \) and \( f = fgf \). We then obtain

\[ X_k f = (X_k f) g f \subseteq X_i (g f) = (X_i g) f \subseteq X_j f. \]

It concludes that \( X_j f \subseteq X_i \) and subsequently \( X_j f \subseteq X_i X f \). On the other hand, let \( y \in X_i \cap X f \). Then there exists \( x \in X \) such that \( x f = y \). Since \( y \in X_i \) and \( X_i g \subseteq X_j \), we have \( yg \in X_j \). Since \( f = fgf \), we then obtain

\[ y = x f = (x f) g f = (yg) f \in X_j f. \]

Thus \( X_i \cap X f \subseteq X_j f \) and consequently \( X_i \cap X f = X_j f \). Since \( i \in I_m \chi(f) \) is an arbitrary element, the necessary part of the proof is complete.

Conversely, assume that the condition holds and we need to find a bijection \( g \in S(X, \mathcal{P}) \) such that \( f = fgf \).

If \( i \in I_m \chi(f) \), by hypothesis, there exists \( j \in I_m \) such that \( |X_i| = |X_j| \) and \( X_i \cap X f = X_j f \). For each \( x' \in X_j f \), we arbitrarily fix an element \( x'' \) in \( \{x \in X_j | x f = x'\} \). Note that \( x' \in X_i, x'' \in X_j, \) and \( x'' f = x' \). Since \( |X_i| = |X_j| \), we can choose a bijective map \( g_i : X_i \to X_j \) such that \( x' g_i = x'' \).

If \( i \in I_m \setminus I_m \chi(f) \), then we consider the following two possibilities separately.
Case \((X_i = \text{codom}(g_j))\) for some \(j \in I_m \chi^{(f)}\): Then we choose a bijective map \(h_i\) from \(X_i\) onto a block \(X_k\), where \(k \in I_m \chi^{(f)}\) and \(X_k\) is not the image of some previously defined bijective map.

Case \((X_i \notin \text{codom}(g_j))\) for all \(j \in I_m \chi^{(f)}\): Then we choose the identity map on the block \(X_i\).

Using these bijective maps on blocks of \(\mathcal{P}\), we now define a selfmap \(g: X \to X\) by setting

\[
xg = \begin{cases} 
x_i g_i, & \text{if } x \in X_i \text{ where } i \in I_m \chi^{(f)}; 
x_i h_i, & \text{if } x \in X_i \text{ where } i \in I_m \setminus I_m \chi^{(f)} \text{ and } X_i = \text{codom}(g_j) \text{ for some } j \in I_m \chi^{(f)}; 
x, & \text{otherwise.}
\end{cases}
\]

Observe that \(g \in \text{S}(X, \mathcal{P})\). We finally show that \(f = f g f\). Let \(x \in X\). Then \(x f \in X_i\) for some \(i \in I_m \chi^{(f)}\). Set \(x f = y\) and \(y g_i = z\). Then \(z f = y\) by definition of bijection \(g_i\). We therefore obtain

\[x (f g f) = y (g f) = z f = y = x f.\]

Since \(x \in X\) is an arbitrary element, we have \(f = f g f\). Hence \(f\) is a unit-regular of the semigroup \(T(X, \mathcal{P})\). This completes the proof. \(\square\)

If \(X\) is a finite set, by using Theorem 3.4 and Corollary 3.5 of [27], we prove the following proposition which characterizes the unit-regular elements of the semigroup \(\Sigma(X, \mathcal{P})\).

**Proposition 3.5.** Let \(\mathcal{P} = \{X_i | i \in I_m\}\) be an \(m\)-partition of a finite set \(X\), and let \(f \in \Sigma(X, \mathcal{P})\). Then \(f\) is unit-regular if and only if \(|X_i| = |X_j|\) whenever \(i \chi^{(f)} = j\).

**Proof.** Suppose first that \(f \in \Sigma(X, \mathcal{P})\) is unit-regular. Since \(\Sigma(X, \mathcal{P}) \subseteq T(X, \mathcal{P})\), by Theorem 3.4 for each \(j \in I_m \chi^{(f)}\) there exists \(i \in I_m\) such that \(|X_i| = |X_j|\) and \(X f \cap X_j = X_i f\). It simply concludes that \(i \chi^{(f)} = j\). Since \(\mathcal{P}\) is an \(m\)-partition, by [27, Corollary 3.5], the selfmap \(\chi^{(f)}\) is bijective on \(I_m\) and hence \(i \chi^{(f)} = j\).

Conversely, suppose that \(|X_i| = |X_j|\) whenever \(i \chi^{(f)} = j\). By definition of \(\chi^{(f)}\), we have \(X_i f \subseteq X_j\). Since \(\mathcal{P}\) in an \(m\)-partition of finite set \(X\) and \(f \in \Sigma(X, \mathcal{P})\), by [27, Corollary 3.5], the selfmap \(\chi^{(f)}\) is bijective on \(I_m\). It follows that \(X_i f = X_j \cap X f\). Hence \(f\) is unit-regular by Theorem 3.4. \(\square\)

4. Set Inclusion

In this section, we discuss the set inclusion between \(\Gamma(X, \mathcal{P})\) and certain known semigroups of transformations that preserve the partition \(\mathcal{P}\). If \(X\) is a finite set, we also observe that the intersection of the semigroups \(\Gamma(X, \mathcal{P})\) and \(\Sigma(X, \mathcal{P})\) is exactly the group of units \(S(X, \mathcal{P})\) of the semigroup \(T(X, \mathcal{P})\). We begin with the following interesting theorem, whose proof is similar to the proof of Theorem 4.1 in [22].

**Theorem 4.1.** Let \(\mathcal{P}\) be a partition of an arbitrary set \(X\). Then \(\Gamma(X, \mathcal{P})\) is the semigroup of all closed selfmaps on \(X\) endowed with topology having \(\mathcal{P}\) as a basis.

Note that \(\Gamma(X, \mathcal{P}) \subseteq T(X, \mathcal{P}) \subseteq \mathcal{T}_X\). Moreover, we have the following obvious remark.
Remark 4.2. Let \( P \) be a trivial partition of a set \( X \).

(i) If \( P \) has singleton blocks, then \( \Gamma(X,P) = T_X \).

(ii) If \( P \) has a single block, then \( \Gamma(X,P) \) is the semigroup of all surjective transformations on \( X \).

(iii) If \( X \) is an \( n \)-element set and \( P \) has a single block, then \( \Gamma(X,P) \) is the symmetric group on \( X \).

The next proposition provides a necessary condition for a map in \( T(X,P) \) to be in \( \Gamma(X,P) \).

Proposition 4.3. Let \( P = \{X_i| i \in I\} \) be a partition of a set \( X \), and let \( f \in T(X,P) \). If \( f \in \Gamma(X,P) \), then there exists \( X_i \in P \) such that \( |X_i| = |X_if| \).

Proof. If \( f \in \Gamma(X,P) \), then \( X_i f \in P \) for all \( X_i \in P \). It follows that \( |X_i f| \leq |X_i| \) for each \( X_i \in P \). Let \( X_k \in P \) such that \( |X_k| \leq |X_j| \) for all \( X_j \in P \). Since \( X_k \) is a block of the smallest size, we then have \( |X_k| = |X_k f| \). This completes the proof. \( \square \)

Lemma 4.4. If \( P \) is a partition of a set \( X \), then \( S(X,P) \subseteq \Gamma(X,P) \).

Proof. Let \( f \in S(X,P) \). Then, by \([27\text{ Lemma } 3.6(i)]\), we have \( X_i f \in P \) for all \( X_i \in P \). It simply concludes that \( f \in \Gamma(X,P) \). Hence the result follows. \( \square \)

There is now a natural curiosity that whether or not every map of \( \Gamma(X,P) \) belongs to \( S(X,P) \). We give a partial answer affirmatively to the question in the following proposition.

Proposition 4.5. Let \( P = \{X_i| i \in I_m\} \) be an \( m \)-partition of a finite set \( X \), and let \( f \in \Gamma(X,P) \). If \( \chi(f) \) is a bijective map, then \( f \in S(X,P) \).

Proof. By hypothesis, it suffices to show that \( f \) is surjective. Let \( y \in X \). Then there exists \( j \in I_m \) such that \( y \in X_j \). Since the map \( \chi(f) \) is bijective, there exists \( i \in I_m \) such that \( i y f = j \). Then, by Remark 3.3 it follows that \( X_i f = X_j \) and so \( y \in X_i f \). Hence \( f \) is surjective. This completes the proof. \( \square \)

If \( X \) is an arbitrary set, the following example shows that the above Proposition 4.5 need not be true.

Example 4.6. Let \( X = \mathbb{Z} \). Consider the partition \( P = \{X_1, X_2\} \) of \( X \), where \( X_1 \) and \( X_2 \) denote the set of nonnegative integers and the set of negative integers, respectively. Define a selfmap \( f: X \to X \) by \( xf = \lfloor x/2 \rfloor \), where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).

Clearly \( f \in T(X,P) \). Note that \( 0 f = 1 f = 0 \). It follows that \( f \) is not injective and so \( f \notin S(X,P) \). Observe that the map \( \chi(f) \) is bijective and so \( f \in \Sigma(X,P) \) by \([27\text{ Corollary } 3.5]\). One can also verify in a routine manner that \( f \in \Gamma(X,P) \).

The following corollary is an immediate consequence of Proposition 4.5 and \([27\text{ Lemma } 3.6(ii)]\).

Corollary 4.7. Let \( P = \{X_i| i \in I_m\} \) be an \( m \)-partition of a finite set \( X \), and let \( f \in \Gamma(X,P) \) such that the map \( \chi(f) \) is bijective. If \( X_i f = X_j \), then \( |X_i| = |X_j| \).

If \( X \) is a finite set, we now prove that the intersection of the semigroups \( \Gamma(X,P) \) and \( \Sigma(X,P) \) is exactly the group of units of the semigroup \( T(X,P) \) in the following proposition.
Proposition 4.8. Let $P = \{X_i | i \in I_m\}$ be an $m$-partition of a finite set $X$. Then
\[ \Gamma(X, P) \cap \Sigma(X, P) = S(X, P). \]

Proof. Let $f \in S(X, P)$. Then we have $f \in \Gamma(X, P)$ by Lemma 4.4. Moreover, since $f$ is bijective, it follows that $Xf = X$. Therefore, $Xf \cap X_i \neq \emptyset$ for all $X_i \in P$. Hence $f \in \Sigma(X, P)$ and consequently $S(X, P) \subseteq \Gamma(X, P) \cap \Sigma(X, P)$.

On the other hand, let $f \in \Gamma(X, P) \cap \Sigma(X, P)$. Since $f \in \Sigma(X, P)$, the selfmap $\chi(f)$ on $I_m$ is bijective by [27, Corollary 3.5]. Moreover, since $f \in \Gamma(X, P)$, we have $f \in S(X, P)$ by Proposition 4.5. Hence $\Gamma(X, P) \cap \Sigma(X, P) \subseteq S(X, P)$. This completes the proof. □

If $X$ is an arbitrary set, the above Proposition 4.8 is, in general, not true as one can quickly see from Example 4.6.

5. Regular elements and Idempotents of $\Gamma(X, P)$

In this section, we give a characterization of the regular elements as well as the idempotents of the semigroup $\Gamma(X, P)$ in the respective subsections. Let us first recall a definition and lemma from [27].

Definition 5.1. [27, Definition 5.1] Let $P$ be a partition of a set $X$. A block map is a map whose domain and codomain are the blocks of $P$.

Lemma 5.2. [27, Lemma 5.2] Let $P = \{X_i | i \in I\}$ be a partition of a set $X$, and let $f \in T_X$. Then $f \in T(X, P)$ if and only if there exists a unique indexed family $B(f, I)$ of block maps induced by $f$, where
\[ B(f, I) = \{f_i | f_i \text{ is induced by } f \text{ and } \text{dom}(f_i) = X_i \text{ for each } i \in I\}. \]

The next remark is simple.

Remark 5.3. If $f \in \Gamma(X, P)$, then each block map of the family $B(f, I)$ is surjective.

5.1. Regular elements. In this subsection, we first give a characterization of the regular elements of the semigroup $\Gamma(X, P)$. We next observe that if $P$ is a partition of a finite set $X$ containing at most two blocks, then $\Gamma(X, P)$ is a regular semigroup. For a finite set $X$, we also prove that every regular element of the semigroup $\Gamma(X, P)$ is unit-regular.

If $P$ is a uniform partition of a finite set $X$, we know that the semigroup $\Gamma(X, P)$ is regular (cf. [22, Theorem 4.1]). However, for a non-uniform partition $P$ of $X$, the semigroup $\Gamma(X, P)$ is, in general, not regular as the following example shows.

Example 5.4. Let $P = \{\{1\}, \{2\}, \{3, 4\}\}$ be a non-uniform partition of $X = \{1, 2, 3, 4\}$. Define a selfmap $f: X \to X$ by
\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 2 \end{pmatrix}. \]

It is clear that $f \in \Gamma(X, P)$. We now show that $f$ is irregular. Assume, to the contrary, that there exists $g \in \Gamma(X, P)$ such that $f = fgf$. Since $3f = 2$ and $f = fgf$, we then obtain
\[ 3(fgf) = 2 \implies (2g)f = 2. \]
It concludes that $2g = 3$ or $2g = 4$. But, in either case, the image of $\{2\} \in P$ under $g$ does not belong to $P$. This contradicts our assumption that $g \in \Gamma(X, P)$. Hence $f$ is an irregular element of $\Gamma(X, P)$.

The following theorem characterizes the regular elements of the semigroup $\Gamma(X, P)$ in terms of certain block maps.

**Theorem 5.5.** Let $P = \{X_i \mid i \in I\}$ be a partition of an arbitrary set $X$, and let $f \in \Gamma(X, P)$. Then $f$ is regular if and only if for every $j \in I_{\chi(f)}$ there exists $i \in I$ such that the block map $f_i : X_i \to X_j$ of the family $B(f, I)$ is injective.

**Proof.** Suppose first that $f \in \Gamma(X, P)$ is regular. Let $j \in I_{\chi(f)}$. Then there exists $i \in I$ such that $i\chi(f) = j$ and so $X_if = X_j$ by Remark 3.3. We then have a block map $f_i : X_i \to X_j$ in $B(f, I)$. If $f_i$ is injective, then we are done. Otherwise, suppose that $f_i$ is not injective.

Since $f \in \Gamma(X, P)$ is regular, there exists $g \in \Gamma(X, P)$ such that $f = fgf$.

Moreover, since $j \in I_{\chi(f)} \subseteq I$, there exists $l \in I_{\chi(g)}$ such that $j\chi(g) = l$ and so $X_lg = X_l$ by Remark 3.3. We then have a block map $g_j : X_j \to X_l$ in $B(g, I)$.

Recall that $X_if = X_j$, $X_lg = X_l$, and $f = fgf$. We then obtain

$$X_j = X_i(fgf) = (X_if)gf = (X_lg)f = X_lf,$$

and subsequently there is a block map $f_i : X_i \to X_j$ in $B(f, I)$. We now claim that the block map $f_i$ is injective.

Assume, to the contrary, that there exist two distinct elements $x, y \in X_l$ such that $xf = yf$. By Remark 3.3 we know that the block map $g_j : X_j \to X_l$ is surjective. Therefore, for distinct elements $x$ and $y$ of $X_l$, there exist two distinct elements $x', y' \in X_j$ such that $x'g = x$ and $y'g = y$. Write $xf = z \in X_j$. Since $x'$ and $y'$ are distinct elements of $X_j$, the element $z$ may be equal to at most one element of $\{x', y'\}$. We can then verify in a routine manner that $f \neq fgf$ which is a contradiction, and so the necessity follows.

Conversely, suppose that the condition holds and we need to find a map $g \in \Gamma(X, P)$ such that $f = fgf$. By Remark 3.3 we know that every block map of $B(f, I)$ is surjective. Therefore, by hypothesis, for every $j \in I_{\chi(f)}$, there exists $i \in I$ such that the block map $f_i : X_i \to X_j$ is bijective. Denote by $h_j$ the inverse map of the bijective map $f_i : X_i \to X_j$. Note that the inverse map $h_j : X_j \to X_i$ is also bijective. Define a map $g : X \to X$ by

$$xg = \begin{cases} xh_j, & x \in X_j \text{ where } j \in I_{\chi(f)}; \\ x, & \text{otherwise.} \end{cases}$$

It is clear that $g \in \Gamma(X, P)$. One can also verify in a routine manner that $f = fgf$ and so $f$ is a regular element of $\Gamma(X, P)$. This completes the proof.

**Corollary 5.6.** If $P = \{X_i \mid i \in I_m\}$ is a uniform m-partition of a finite set $X$, then the semigroup $\Gamma(X, P)$ is regular.

**Proof.** Let $f \in \Gamma(X, P)$, and let $j \in I_{m\chi(f)}$. Then there exists $i \in I_m$ such that $i\chi(f) = j$ and so $X_if = X_j$ by Remark 3.3. We then have the surjective block map $f_i : X_i \to X_j$ in $B(f, I_m)$. By hypothesis, we know that $|X_i| = |X_j|$. It follows that $f_i$ is injective (cf. [17 Proposition 1.1.3]). Then $f$ is a regular element of $\Gamma(X, P)$.
by Theorem 5.3. Hence, since $f \in \Gamma(X, \mathcal{P})$ is arbitrary, the semigroup $\Gamma(X, \mathcal{P})$ is regular.

If the size of a partition $\mathcal{P}$ of a finite set $X$ is at most two, the following proposition proves that the semigroup $\Gamma(X, \mathcal{P})$ is regular.

**Proposition 5.7.** If $\mathcal{P}$ is a partition of a finite set $X$ such that $|\mathcal{P}| \leq 2$, then the semigroup $\Gamma(X, \mathcal{P})$ is regular.

**Proof.** If $|\mathcal{P}| = 1$, then it is clear that $\Gamma(X, \mathcal{P})$ is the symmetric group on $X$ and so $\Gamma(X, \mathcal{P})$ is regular. Therefore, suppose that $\mathcal{P} = \{X_1, X_2\}$. Without loss of generality, assume that $|X_1| \leq |X_2|$. If $|X_1| = |X_2|$, then $\mathcal{P}$ is a uniform partition and so the semigroup $\Gamma(X, \mathcal{P})$ is regular by Corollary 5.6. Otherwise, we have $|X_1| < |X_2|$.

Let $f \in \Gamma(X, \mathcal{P})$. By Proposition 4.3, we must have $X_1f = X_1$. If $I_2\chi^f = \{1\}$, then we are done by Theorem 5.3. Otherwise, we have $X_2f = X_2$. Then each block map of $B(f, I)$ is injective and so $f$ is regular by Theorem 5.5. Since $f$ is an arbitrary map of $\Gamma(X, \mathcal{P})$, the semigroup $\Gamma(X, \mathcal{P})$ is regular. This completes the proof.

If $X$ is a finite set, the following proposition proves that every regular element of the semigroup $\Gamma(X, \mathcal{P})$ is unit-regular.

**Proposition 5.8.** If $\mathcal{P} = \{X_i | i \in I_m\}$ is an $m$-partition of a finite set $X$, then every regular element of the semigroup $\Gamma(X, \mathcal{P})$ is unit-regular.

**Proof.** Let $f \in \Gamma(X, \mathcal{P})$ be a regular map. Let $j \in m\chi(f)$. Then, by Theorem 5.3, there exists $i \in I_m$ such that the block map $f_i: X_i \to X_j$ of $B(f, I)$ is injective. By Remark 5.3, we know that every block map of the family $B(f, I)$ is surjective. It follows that the block map $f_i \in B(f, I)$ is bijective and so $|X_i| = |X_j|$. Moreover, $X_if = X_if_i = X_j = X_j \cap Xf$. Hence $f$ is unit-regular by Theorem 5.4.

5.2. **Idempotents.** The following theorem characterizes the idempotents of the semigroup $\Gamma(X, \mathcal{P})$ in terms of certain block maps.

**Theorem 5.9.** Let $\mathcal{P} = \{X_i | i \in I\}$ be a partition of an arbitrary set $X$, and let $f \in \Gamma(X, \mathcal{P})$. Then $f$ is an idempotent if and only if for each $i \in I\chi(f)$ the block map $f_i \in B(f, I)$ is the identity map.

**Proof.** Suppose first that $f \in \Gamma(X, \mathcal{P})$ is an idempotent. Let $i \in I\chi(f)$. Since $\Gamma(X, \mathcal{P}) \subseteq T(X, \mathcal{P})$, the block map $f_i \in B(f, I)$ is an idempotent by Proposition 5.4. By Remark 5.3, the block map $f_i \in B(f, I)$ is a surjection. Note that any idempotent map on a set acts as the identity map on its image set (cf. [8, p.6]). Combining these, we conclude that $f_i \in B(f, I)$ is the identity map. Since $i \in I\chi(f)$ is an arbitrary element, this completes the proof of the necessity part.

Conversely, suppose that the condition holds and we need to show that $f$ is an idempotent. It suffices to show that $f$ is the identity map on its image set $Xf$ (cf. [8, p.6]). Let $x \in Xf$. Then there exists $X_j \in \mathcal{P}$ such that $x \in X_j$. Clearly $j \in I\chi(f)$. By hypothesis, the block map $f_j \in B(f, I)$ is the identity map. Hence $xf = xf_j = x$. Since $x \in Xf$ is an arbitrary element, it follows that $f$ is the identity map on the image set $Xf$. This completes the proof.
6. THE CARDINALITY OF $\Gamma(X, \mathcal{P})$, $E(\Gamma(X, \mathcal{P}))$, AND $\text{Reg}(\Gamma(X, \mathcal{P}))$

Throughout this section, $X$ is a finite set and $\mathcal{P}$ is an $(m, k)$-partition of $X$, where $m, k \in \mathbb{N}$ with $m \geq k$. Moreover, $\mathcal{P}$ has $m_i$ blocks of size $n_i$ for each $i \in I_k$ and $n_1 < n_2 < \cdots < n_k$. Thus, $m = m_1 + m_2 + \ldots + m_k$.

The aim of this section is to count the number of elements, idempotents, and regular elements of the semigroup $\Gamma(X, \mathcal{P})$ for a finite set $X$ in the respective subsections. Before we calculate these, we state a remark and prove a simple lemma.

We immediately state the following from Remark 5.3.

Remark 6.1. Let $\mathcal{P} = \{X_i \mid i \in I_m\}$ be a partition of a finite set $X$, and let $f \in \Gamma(X, \mathcal{P})$. If $f_i \in B(f, I_m)$, then $|\text{ran}(f_i)| \leq |\text{dom}(f_i)|$ for all $i \in I_m$.

Lemma 6.2. Let $\mathcal{P}$ be an $(m, k)$-partition of a finite set $X$, and let $r \in I_m$. Let $\mathcal{F}_r$ be the collection of all $r$-subpartitions of $\mathcal{P}$ such that each $r$-subpartition in $\mathcal{F}_r$ contains at least one block of size $n_1$. Then

$$|\mathcal{F}_r| = \sum_{l=1}^{\min\{m_1, r\}} \binom{m_1}{l} \binom{m - l}{r - l},$$

where $m_1$ is the number of blocks in $\mathcal{P}$ of size $n_1$.

Proof. Let $l \in \mathbb{N}$ with $1 \leq l \leq m_1$. Then the number of $r$-subpartitions of $\mathcal{P}$ which contain exactly $l$ blocks of the smallest size is

$$\binom{m_1}{l} \binom{m - l}{r - l}.$$

Since $l$ is arbitrary, by the addition principle, we get

$$|\mathcal{F}_r| = \sum_{l=1}^{\min\{m_1, r\}} \binom{m_1}{l} \binom{m - l}{r - l}$$

and hence the proof is complete. \(\square\)

Notation 6.3. For $r \in I_m$, we denote by $\mu_r$ the size of the collection $\mathcal{F}_r$ obtained in Lemma 6.2. Moreover, we let $\mathcal{F}_r = \{Q_{r1}, \ldots, Q_{r\mu_r}\}$, where each $Q_{rt} \in \mathcal{F}_r$ has $r_t \geq 0$ blocks of size $n_1$.

6.1. The cardinality of $\Gamma(X, \mathcal{P})$. The following theorem counts the number of elements of the semigroup $\Gamma(X, \mathcal{P})$.

Theorem 6.4. Let $\mathcal{P}$ be an $(m, k)$-partition of a finite set $X$. Then

$$|\Gamma(X, \mathcal{P})| = \prod_{i=1}^{k} \left( \sum_{j=1}^{i} m_j (n_j!) S(n_i, n_j) \right)^{m_i}.$$

Proof. From Lemma 6.2 and Remark 6.3 we know that each map $f \in \Gamma(X, \mathcal{P})$ is uniquely determined by the $m$-family $B(f, I_m)$ of surjective block maps. Therefore, it suffices to count the total number of such possible $m$-families $B(f, I_m)$ of surjective block maps. Since $\mathcal{P}$ has $k$ different size blocks, we break up the problem into $k$ subfamilies of surjective block maps according to their domain size.
Let \( i \in I_k \). Since \( P \) has \( m_i \) blocks of size \( n_i \), we begin by counting the number of \( m_i \)-subfamilies of surjective block maps from \( m_i \) distinct blocks of size \( n_i \). By Remark 6.1, we can easily observe that the number of surjective block maps from any fixed \( n_i \)-element block is \( \sum_{j=1}^{i} m_j (n_j)! S(n_i, n_j) \). Therefore, by the multiplication principle, the number of possible \( m_i \)-subfamilies of surjective block maps from \( m_i \) distinct blocks of size \( n_i \) is \( \left( \sum_{j=1}^{i} m_j (n_j)! S(n_i, n_j) \right)^{m_i} \).

Since \( i \in I_k \) is an arbitrary element, the total number of possible \( m \)-families of surjective block maps is

\[
\prod_{i=1}^{k} \left( \sum_{j=1}^{i} m_j (n_j)! S(n_i, n_j) \right)^{m_i}
\]

by the multiplication principle. This completes the proof.

6.2. The cardinality of \( E(\Gamma(X, P)) \). In this subsection, we count the number of idempotents of the semigroup \( \Gamma(X, P) \). We begin by proving the following lemma.

**Lemma 6.5.** Let \( P \) be an \((m, k)\)-partition of a finite set \( X \), and let \( Q \) be an \( r \)-subpartition of \( P \) containing at least one block of size \( n_1 \). Let

\[
A_Q = \{ f \in \Gamma(X, P) : Xf \text{ is the union of all blocks of } Q \}.
\]

Then

\[
|E(A_Q)| = \prod_{i=1}^{k} \left( \sum_{j=1}^{i} r_j (n_j)! S(n_i, n_j) \right)^{m_i-r_i},
\]

where \( r_i \geq 0 \) is the number of blocks in \( Q \) of size \( n_i \) for each \( i \in I_k \).

**Proof.** From Lemma 5.2 and Remark 5.3, we know that each idempotent \( f \in E(A_Q) \) is uniquely determined by the \( m \)-family \( B(f, I_m) \) of surjective block maps. Further, from Theorem 5.9, we know that a map \( f \in A_Q \) is idempotent if and only if each block map \( f_i \in B(f, I_m) \) with \( \text{dom}(f_i) \subseteq Xf \) is the identity map. Since \( Q \) is an \( r \)-subpartition of \( P \), it suffices to count the total number of such possible \((m-r)\)-families \( B(f, I_m) \) of surjective block maps from \( (m-r) \) distinct blocks of \( P \setminus Q \). To count it, we break up the problem into \( k \) subfamilies of surjective block maps according to their domain size. Note that \( r = r_1 + \cdots + r_k \), \( m = m_1 + \cdots + m_k \), and \( m-r = (m_1 - r_1) + \cdots + (m_k - r_k) \).

Let \( i \in I_k \). Since \( P \setminus Q \) has \((m_i - r_i)\) blocks of size \( n_i \), we begin by counting the number of possible \((m_i - r_i)\)-subfamilies of surjective block maps from these \((m_i - r_i)\) distinct blocks of size \( n_i \). By Remark 6.1, we can easily observe that the number of surjective block maps from any fixed \( n_i \)-element block of \( P \setminus Q \) is \( \sum_{j=1}^{i} r_j (n_j)! S(n_i, n_j) \).

Recall that \( P \setminus Q \) has \((m_i - r_i)\) blocks of size \( n_i \). By the multiplication principle, the number of possible \((m_i - r_i)\)-subfamilies of surjective block maps from \((m_i - r_i)\) distinct blocks in \( P \setminus Q \) of size \( n_i \) is \( \left( \sum_{j=1}^{i} r_j (n_j)! S(n_i, n_j) \right)^{m_i-r_i} \).

Since \( P \) has \( k \) different size blocks and \( i \in I_k \) is an arbitrary element, by the multiplication principle, we can get the desired number of \((m-r)\)-families of surjective block maps. This completes the proof.
Proof. Let \( r \in I_m \), and let
\[
A_r = \{ f \in \Gamma(X, \mathcal{P}) | \text{ rank } \chi(f) = r \}.
\]
We observe that \( E(A_r) = \sum E(A_{Q_{rt}}) \), where the sum runs over all \( r \)-subpartitions \( Q_{rt} \in \mathcal{F}_r \) and
\[
A_{Q_{rt}} = \{ f \in \Gamma(X, \mathcal{P}) | Xf \text{ is the union of all blocks of } Q_{rt} \}.
\]
Note that \( |\mathcal{F}_r| = \mu_r \). Therefore,
\[
|E(A_r)| = \sum_{t=1}^{\mu_r} E(A_{Q_{rt}}) = \sum_{t=1}^{\mu_r} k \prod_{i=1}^{k} \left( \sum_{j=1}^{i} r_{ij}(n_j!)S(n_i, n_j) \right)^{m_i-r_i} \text{ by Lemma 6.5}
\]
Since \( r \in I_m \) is an arbitrary element, by the addition principle, we obtain
\[
|E(\Gamma(X, \mathcal{P}))| = \sum_{r=1}^{m} \sum_{t=1}^{\mu_r} k \prod_{i=1}^{k} \left( \sum_{j=1}^{i} r_{ij}(n_j!)S(n_i, n_j) \right)^{m_i-r_i}.
\]
This completes the proof. \( \square \)

If \( \mathcal{P} \) is a uniform partition of a finite set \( X \), the following proposition provides a rather simple formula for the size of the set \( E(\Gamma(X, \mathcal{P})) \).

**Proposition 6.7.** Let \( \mathcal{P} \) be a uniform \( m \)-partition of an \( n \)-element set \( X \). Then
\[
|E(\Gamma(X, \mathcal{P}))| = \sum_{r=1}^{m} \binom{m}{r} r^{(m-r)} (q!)^{m-r},
\]
where \( q = \frac{n}{m} \) is the size of a block of \( \mathcal{P} \).

*Proof. Let \( r \in I_m \), and let
\[
A_r = \{ f \in \Gamma(X, \mathcal{P}) | \text{ rank } \chi(f) = r \}.
\]
We observe that \( E(A_r) = \sum E(A_Q) \), where the sum runs over all \( r \)-subpartitions \( Q \) of \( \mathcal{P} \) and
\[
A_Q = \{ f \in \Gamma(X, \mathcal{P}) | Xf \text{ is the union of all blocks of } Q \}.
\]
Note that there are exactly \( \binom{m}{r} \) choices for an \( r \)-subpartition \( Q \) of the \( m \)-partition \( \mathcal{P} \). Therefore, by Lemma 6.5 we obtain \( E(A_r) = \binom{m}{r} r^{(m-r)} (q!)^{m-r} \), where \( q = \frac{n}{m} \).

Since \( r \in I_m \) is an arbitrary element, by the addition principle, we get
\[
|E(\Gamma(X, \mathcal{P}))| = \sum_{r=1}^{m} \binom{m}{r} r^{(m-r)} (q!)^{m-r},
\]
where \( q = \frac{n}{m} \). This completes the proof. \( \square \)
6.3. The cardinality of $\text{Reg}(\Gamma(X, \mathcal{P}))$. In this subsection, we count the number of regular elements of the semigroup $\Gamma(X, \mathcal{P})$.

If $\mathcal{P}$ is a uniform partition of a finite set $X$, we know that the semigroup $\Gamma(X, \mathcal{P})$ is regular (cf. [22, Theorem 4.1]), and so we get the size of the set $\text{Reg}(\Gamma(X, \mathcal{P}))$ by Theorem 6.4. However, note that the semigroup $\Gamma(X, \mathcal{P})$ need not be regular for an arbitrary partition $\mathcal{P}$ of $X$.

For a finite set $X$, let us recall Theorem 5.5 which can be restate immediately as follows.

Remark 6.8. Let $\mathcal{P} = \{X_i \mid i \in I_m\}$ be a partition of a finite set $X$, and let $f \in \Gamma(X, \mathcal{P})$. Then $f$ is regular if and only if for each $j \in I_m\chi(f)$, there exists $i \in I$ such that $|X_i| = |X_j|$ and $X_i f = X_j$.

We now prove the following lemma.

Lemma 6.9. Let $\mathcal{P}$ be an $(m, k)$-partition of a finite set $X$, and let $\mathcal{Q}$ be an $r$-subpartition of $\mathcal{P}$ containing at least one block of size $n_1$. Let

$$A_\mathcal{Q} = \{f \in \Gamma(X, \mathcal{P}) \mid Xf \text{ is the union of all blocks of } \mathcal{Q}\}.$$ 

Then

$$|\text{Reg}(A_\mathcal{Q})| = (r_1!)(m_1, r_1)(n_1)!^{m_1} \prod_{i=2}^{k} \left( \sum_{p=r_i}^{m_i} \binom{m_i}{p}(r_i!)S(p, r_i)(n_i!)^p \left( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \right)^{m_i-p} \right),$$

where $r_i \geq 0$ is the number of blocks in $\mathcal{Q}$ of size $n_i$ for each $i \in I_k$.

Proof. From Lemma 5.2 and Remark 5.3 we know that each map $f \in \text{Reg}(A_\mathcal{Q})$ is uniquely determined by the $m$-family $B(f, I_m)$ of surjective block maps. Therefore, it suffices to count the total number of such possible $m$-families $B(f, I_m)$ of surjective block maps. Since $\mathcal{P}$ has $k$ different size blocks, we break up such $m$-families into $k$ number of $m_1$-subfamilies of surjective block maps according to their domain size.

Note that $\mathcal{P}$ has $m_1$ blocks of the smallest size $n_1$. We first count the number of $m_1$-subfamilies of surjective block maps from $m_1$ distinct blocks of size $n_1$. Note that any block in $\mathcal{P}$ of size $n_1$ can be mapped onto any block in $\mathcal{Q}$ of size $n_1$ only. Therefore, all the $m_1$ blocks in $\mathcal{P}$ of size $n_1$ can be mapped onto $r_1$ blocks in $\mathcal{Q}$ of size $n_1$ in $(r_1!)S(m_1, r_1)$ ways. Moreover, in each such way, the number of surjective block maps is $(n_1!)^{m_1}$. Therefore, the number of possible $m_1$-subfamilies of surjective block maps from $m_1$ distinct blocks of size $n_1$ is $(r_1!)S(m_1, r_1)(n_1!)^{m_1}$.

Let $i \in I_k$ be such that $i \geq 2$. Since $\mathcal{P}$ has $m_i$ blocks of size $n_i$, we now count the number of $m_i$-subfamilies of surjective block maps from $m_i$ distinct blocks of size $n_i$. Note that any block in $\mathcal{P}$ of size $n_i$ can be mapped onto any block in $\mathcal{Q}$ of size $n_j$ where $n_j \leq n_i$. By Remark 5.5 note that there are at least $r_i$ blocks in $\mathcal{P}$ of size $n_i$ which will be mapped onto $r_i$ blocks in $\mathcal{Q}$ of size $n_i$, and the remaining at most $(m_i - r_i)$ blocks in $\mathcal{P}$ of size $n_i$ will be mapped onto blocks in $\mathcal{Q}$ of size $n_j$, where $n_j < n_i$.

Let $p$, where $r_i \leq p \leq m_i$, be the number of blocks in $\mathcal{P}$ of size $n_i$ which map onto $r_i$ blocks in $\mathcal{Q}$ of size $n_i$. Clearly, these $p$ blocks can be mapped onto $r_i$ blocks in $(r_i!)S(p, r_i)$ ways. Moreover, in each such way, there are $(n_i!)^p$ surjective
block maps. Note that there are \( \binom{m_i}{p} \) ways to choose \( p \) blocks among \( m_i \) blocks. Therefore, by the multiplication principle, the number of such surjective block maps is \( \binom{m_i}{p}(r_i!)S(p, r_i)(n_i!)^p \).

Further, each of the remaining \( (m_i - p) \) blocks in \( P \) of size \( n_i \) can be mapped onto any block in \( Q \) of size \( n_j \), where \( n_j < n_i \). Note that the number of surjective block maps from a block in \( P \) of size \( n_i \) is \( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \). Therefore, by the multiplication principle, the number of such surjective block maps is \( \left( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \right)^{m_i - p} \). Hence, for that fixed \( p \), the number of \( m_i \)-subfamilies of surjective block maps from \( m_i \) distinct blocks of size \( n_i \) is

\[
\left( \binom{m_i}{p}(r_i!)S(p, r_i)(n_i!)^p \left( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \right)^{m_i - p} \right)
\]

by the multiplication principle. Since \( p \) runs from \( r_i \) to \( m_i \), summing over all admissible \( p \), the total number of possible \( m_i \)-subfamilies of surjective block maps from \( m_i \) distinct blocks of size \( n_i \) is

\[
\sum_{p=r_i}^{m_i} \left( \binom{m_i}{p}(r_i!)S(p, r_i)(n_i!)^p \left( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \right)^{m_i - p} \right).
\]

Since \( i \in I_k \) is an arbitrary element greater than 1, by the multiplication principle, we obtain the total number of possible \( m \)-families of surjective block maps. This completes the proof. \( \square \)

The following theorem counts the number of regular elements of the semigroup \( \Gamma(X, \mathcal{P}) \).

**Theorem 6.10.** Let \( \mathcal{P} \) be an \( (m, k) \)-partition of a finite set \( X \). Then

\[
|\text{Reg}(\Gamma(X, \mathcal{P}))| = \sum_{r=1}^{m} \sum_{t=1}^{\mu_r} (r_t!)S(m_1, r_t)(n_1!)^{m_1} \prod_{i=2}^{k} \left( \sum_{p=r_i}^{m_i} \left( \binom{m_i}{p}(r_i!)S(p, r_i)(n_i!)^p \left( \sum_{j=1}^{i-1} r_j(n_j!)S(n_i, n_j) \right)^{m_i - p} \right) \right),
\]

where \( r_t \geq 0 \) is the number of blocks in \( Q_{rt} \in \mathcal{F}_r \) of size \( n_i \) for each \( i \in I_k \).

**Proof.** Let \( r \in I_m \), and let

\[
A_r = \{ f \in \Gamma(X, \mathcal{P}) \mid \text{rank} \chi^{(f)} = r \}.
\]

We observe that \( \text{Reg}(A_r) = \sum \text{Reg}(A_{Q_{rt}}) \), where the sum runs over all \( r \)-subpartitions \( Q_{rt} \in \mathcal{F}_r \) and

\[
A_{Q_{rt}} = \{ f \in \Gamma(X, \mathcal{P}) \mid Xf \text{ is the union of all blocks of } Q_{rt} \}.
\]
Note that $|\mathcal{F}_r| = \mu_r$. Therefore,
\[
\text{Reg}(A_r) = \sum_{t=1}^{\mu_r} \text{Reg}(A_{Q_{rt}})
\]
\[
= \sum_{t=1}^{\mu_r} \left( (r_{t_1}!) S(m_1, r_{t_1}) (n_1!)^{m_1} \prod_{i=2}^{\mu_r} \left( \sum_{p=r_{t_i}}^{m_i} \left( \frac{m_i!}{p!} (r_{t_i}!) S(p, r_{t_i}) (n_i!)^{p} \left( \sum_{j=1}^{p} r_{t_j} (n_j!) S(n_i, n_j) \right)^{m_i-p} \right) \right) \right)
\]
by Lemma 6.9. Since $r \in J_m$ is an arbitrary element, by the addition principle, we obtain the desired formula of $|\text{Reg}(\Gamma(X, P))|$. This completes the proof. 

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