One invariant measure and different Poisson brackets for two nonholonomic systems

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Abstract
We discuss the nonholonomic Chaplygin and the Borisov-Mamaev-Fedorov systems, for which symplectic forms are different deformations of the square root from the corresponding invariant volume form. In both cases second Poisson bivectors are determined by $L$-tensors with non-zero torsion on the configurational space, in contrast with the well known Eisenhart-Benenti and Turiel constructions.

1 Introduction
At the beginning of the 20th century S.A. Chaplygin showed that a two degree of freedom nonholonomic system possessing invariant measure can be reduced to Hamiltonian after a suitable change of time [10, 11]. Using this process of the Chaplygin hamiltonization we can get the usual Hamilton-Jacobi equation, variables of separation, the Abel-Jacobi equations, quadratures, etc [1, 4, 6, 8, 9, 13, 17].

In bi-hamiltonian geometry separability is invariant geometric property of the distribution defined by mutually commuting independent functions $H_1, \ldots, H_n$ [20, 22, 23]. In fact there is neither Hamilton-Jacobi equation, no time which describes only some partial parametrization of geometric objects. So, in this paper we want to show how these standard bi-Hamiltonian geometric methods may be directly applied to nonholonomic systems without any change of time. The second aim is to discuss a deformation of the Turiel construction [25], which appears only in the nonholonomic case and gives rise to some interesting modifications of the standard conformal Killing tensors that lie at the heart of classical Eisenhart-Benzenti theory of separability [2, 3, 14].

As an example, we will consider a rolling of dynamically asymmetric and balanced ball over an absolutely rough fixed sphere with radius $a$. At $a \to \infty$ one gets a Chaplygin problem on a non-homogeneous sphere rolling over a horizontal plane without slipping [10]. Thus, we are able to compare bi-Hamiltonian methods with the chaplygin hamiltonization.

Let $\omega = (\omega_1, \omega_2, \omega_3)$ be an angular velocity vector of the rolling ball. Its mass, inertia tensor and radius will be denoted by $m$, $I = \text{diag}(I_1, I_2, I_3)$ and $b$, respectively. According to [5], the angular momentum $M = (M_1, M_2, M_3)$ of the ball with respect to the contact point with the sphere is equal to

$$M = (I + dE)\omega - d(\gamma, \omega)\gamma,$$

(1.1)

Here $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the unit normal vector to the fixed sphere at the contact point, $E$ is the unit matrix and $(,)$ means the standard scalar product in $\mathbb{R}^3$. All these vectors are expressed in the so-called body frame, which is firmly attached to the ball, and its axes coincide with the principal inertia axes of the ball.

For further use we rewrite the relation (1.1) in the following equivalent form

$$\omega = A_M M = \left( A + d g(\gamma) A (\gamma \otimes \gamma) A \right) M,$$

(1.2)
where
\[ A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = (I + dE)^{-1}, \]
and
\[ g(\gamma) = \frac{1}{1 - d(\gamma, A\gamma)}. \]  
(1.3)

According to [10, 5], there is no slip nonholonomic constraint associated with the zero velocity in the point of contact. It allows us to reduce equation of motion to the following form
\[ \dot{M} = M \times \omega, \quad \dot{\gamma} = \kappa \gamma \times \omega. \]  
(1.4)

where \( \times \) means vector product in \( \mathbb{R}^3 \) and \( \kappa = a/(a + b) \).

At any \( \kappa \) there are three integrals of motion
\[ H_1 = (M, \omega), \quad H_2 = (M, M), \quad C_1 = (\gamma, \gamma), \]  
(1.5)

and invariant measure
\[ \mu = \sqrt{g(\gamma)} \, d\gamma dM. \]  
(1.6)

If \( \kappa = \pm 1 \) there is one more integral of motion
\[ C_2 = (\gamma, BM), \quad B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix} = \text{tr}A^{-1} + (\kappa - 1)A^{-1}. \]  
(1.7)

At \( d = 0 \) we have \( g(\gamma) = 0 \) and \( \omega = AM. \) In this case \( \mu \) is a standard volume form with constant density and, therefore, equations (1.4) describe some Hamiltonian flow. Namely, at \( \kappa = 1 \) equations (1.4) can be identified with the Euler-Poisson equations describing the rotation of a rigid body around a fixed point, whereas at \( \kappa = -1 \) equations (1.4) describe the so-called Contensou model of Fleuriais gyroscope [12].

At \( d \neq 0 \) equations (1.4) describe the rolling of a dynamically nonsymmetric sphere over a horizontal plane or a fixed sphere without slipping. The case \( \kappa = 1 \) is the so-called Chaplygin system [10] and case \( \kappa = -1 \) will be referred as the Borisov-Mamaev-Fedorov system. A detailed description of these nonholonomic systems may be found in [5, 6, 7, 8, 9, 13].

In [7, 8, 9] authors change time variable in the equations of motion (1.4) first and only then study the Poisson structure of the resulting equations. In the Hamiltonian mechanics transformation of time can drastically change almost all the invariant geometric properties of the initial system, such as the Lagrangian foliation, compatible Poisson structures, Lax and \( r \)-matrices, bi-Hamiltonian construction of the variables of separation, etc [18, 19]. For the nonholonomic system it can change even the initial Hamilton function [1, 13, 17].

Our main aim is to get a family of Poisson brackets associated with the invariant measure (1.6) in framework of the bi-Hamiltonian geometry, i.e. without any change of time. It means that we for a while forget about the equations of motion (1.4) and try to solve the following geometric equations
\[ PdC_{1,2} = 0, \quad (PdH_1, dH_2) = \{H_1, H_2\} = 0, \quad [P, P] = 0, \]  
(1.8)

where [\( ., . \)] is the Schouten bracket, with respect to the Poisson bivector \( P \).

So, in our approach the Hamiltonization process is equivalent to a search of the Poisson structure satisfying to equations (1.8), i.e.

Hamiltonization \( \iff \) Poisson bracket

because using this bracket we can always get new Hamiltonian system
\[ \frac{d}{dt'} z_k = \{H_1, z_k\}, \]
with new time \( t' \) and the same integrals of motion in involution.
Remark 1 In our case at $d = 0$ we have Hamiltonian system with canonical Poisson bracket. So, Hamiltonization is equivalent to existence of the proper deformations of this canonical Poisson bracket. Obstacles to such deformations are well known, see, for instance, the geometric quantization theory.

The principal disadvantage is that equations (1.8) have infinitely many solutions [20, 21, 22]. So, in order to get any particular solution we have to set aside an invariance and to narrow the search space.

1.1 Spherical coordinates

In order to narrow the search space we will use the notion of natural Poisson bivectors on the Riemannian manifolds [22]. In this case we have to reduce our initial phase space to the cotangent bundle of the unit two-dimensional Poisson sphere.

Namely, we can avoid the solution of the first equations $PdC_{1,2} = 0$ in (1.8) using the slightly modified Euler variables

$$
\gamma_1 = \sin \phi \sin \theta, \quad M_1 = \frac{1}{b_1} \left( \frac{\sin \phi}{\sin \theta} (\cos \theta p_\phi + p_\psi) - \cos \phi p_\theta \right),
$$

$$
\gamma_2 = \cos \phi \sin \theta, \quad M_2 = \frac{1}{b_2} \left( \frac{\cos \phi}{\sin \theta} (\cos \theta p_\phi + p_\psi) + \sin \phi p_\theta \right),
$$

$$
\gamma_3 = \cos \theta, \quad M_3 = -\frac{p_\phi}{b_3},
$$

such as

$$
C_1 = (\gamma, \gamma) = 1, \quad C_2 = (\gamma, BM) = p_\psi.
$$

In the generic case the remaining equations in (1.8) have infinitely many solutions [20]. In order to find at list one particular solution we impose the following restriction

$$
C_2 = (\gamma, BM) = p_\psi = 0.
$$

In this case we have natural integrals of motion $H_{1,2}$ (1.5) and, therefore, we can solve our equations using the notion of natural Poisson bivectors [22].

At $\kappa = 1$ coordinates $(\phi, \theta)$ in (1.9) are usual spherical coordinates on the unit sphere $S^2$ at

$$
\kappa = 1, \quad b_1 = b_2 = b_3 = \text{tr} A^{-1} = 1.
$$

At $\kappa = -1$ we replace parameters $a_i$ and $J_i = a_i^{-1}$ on $b_i$

$$
\kappa = -1, \quad a_1 = \frac{2}{b_2 + b_3}, \quad a_2 = \frac{2}{b_1 + b_3}, \quad a_3 = \frac{2}{b_1 + b_2},
$$

in order to get more foreseeable formulas.

After the change of variables (1.9), at $\kappa = \pm 1$ we obtain two different dynamical systems on the common phase space $\mathcal{M}$ which is topologically equivalent to cotangent bundle $T^*S^2$ to the sphere. These systems have a common invariant volume form (1.6) and two different Poisson structures, see next Section.

In order to show the difference between the Chaplygin and Borisov-Mamaev-Fedorov sys-
tems we present one equation of motion

\[
\begin{align*}
\kappa = 1, \quad & \dot{\phi} = \frac{(a_1 - a_2) \sin 2\phi}{2} \left( \cos \theta - \frac{g(a_3 - a_1 \sin^2 \phi - a_2 \cos^2 \phi) \sin 2\phi}{2} \right) p_\phi \\
& - \left( \frac{g(a_1 - a_2)^2 \sin^2 2\phi \sin^2 \theta}{4} + a_1 \cos^2 \phi + a_2 \sin^2 \phi \right) p_\theta \\
\kappa = -1, \quad & \dot{\phi} = \frac{1}{b_1 b_2 (b_1 + b_3) (b_2 + b_3)} \left[ \left( \frac{b_3 (b_1 - b_2) \sin 2\phi \cos \theta}{\sin \theta} \right) \sin \theta \\
& - \frac{g(b_1 - b_2) (b_3 (b_2^2 \cos^2 \phi + b_2^2 \sin^2 \phi) - b_2^2 b_3^2 \sin 2\phi \sin 2\theta)}{b_3 (b_1 + b_2) (b_1 + b_3) (b_2 + b_3)} \right] p_\phi \\
& + \left( 2(b_3 (b_1 \sin^2 \phi + b_2 \cos^2 \phi) + b_1 b_2) - \frac{g(b_3 (b_1 - b_2)^2}{(b_1 + b_3) (b_2 + b_3) \sin^2 2\phi \sin^2 \theta} \right) p_\theta \right].
\end{align*}
\]

and one integral of motion

\[
\begin{align*}
\kappa = 1, \quad & H_2 = \frac{1}{\sin^2 \theta} p_\phi^2 + p_\theta^2, \\
\kappa = -1, \quad & H_2 = \left( \frac{b_3^2 \cos^2 \phi + b_2^2 \sin^2 \phi \cos^2 \theta}{b_1^2 b_2^2 \sin^2 \theta} + \frac{1}{b_3^2} \right) p_\phi^2 + \left( \frac{b_3^2 - b_2^2}{b_1^2 b_2^2 \sin \theta} \right) p_\phi p_\theta \\
& + \frac{b_2^2 \sin^2 \phi + b_2^2 \cos^2 \phi}{b_1^2 b_2^2} p_\theta^2.
\end{align*}
\]

Of course, any calculations for the Borisov-Mamaev-Fedorov systems require more efforts and large-scale resources in comparison to the same calculations for the Chaplygin system.

2 Invariant measure and Poisson brackets

Let \( \mathcal{M} \) be a smooth symplectic manifold endowed with a symplectic form \( \Omega \) which in the Darboux coordinates

\[
z = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)
\]

reads as

\[
\Omega = dp_1 \wedge dq_1 + \ldots dp_n \wedge dq_n. \tag{2.1}
\]

The volume form \( \Omega^2 \) on \( \mathcal{M} \) is invariant under all hamiltonian diffeomorphisms by the Liouville theorem.

If we have another invariant volume form \( \mu \) on the same manifold \( \mathcal{M} \), we can get another symplectic form \( \Omega_\mu \) taking a formal square root on \( \mu \), because

\[
\mu = \Omega^2_\mu. \tag{2.2}
\]

However, in our case invariant volume form \( \mu = \sqrt{g} \lambda \tag{1.6} \) is invariant with respect to the non-
hamiltonian flow \( \tag{1.3} \) and, therefore, we have to deform its formal square root \( \tag{2.2} \). We will describe these deformations using Poisson bivectors, instead of the corresponding symplectic forms.

We rewrite the Poisson bivector \( P \) associated with the canonical symplectic form \( \Omega \tag{2.1} \) in the following tensor form

\[
P = \begin{pmatrix}
0 & L_{ij} \\
-L_{ij} & \sum_{k=1}^{n} \left( \frac{\partial L_{k1}}{\partial q_j} - \frac{\partial L_{kj}}{\partial q_i} \right) p_k
\end{pmatrix} = \begin{pmatrix}
0 & \text{Id} \\
-\text{Id} & 0
\end{pmatrix}, \tag{2.3}
\]

\[
4
\]
where \( L \) is an identity \((1,1)\) tensor field on a configurational space.

We use such unusual notation because any torsionless \((1,1)\) tensor field \( L'(q_1, \ldots, q_n) \) on a configurational space \( Q \) with coordinates \( q_1, \ldots, q_n \) determines another Poisson bivector

\[
P' = \begin{pmatrix}
0 & L'_{ij} \\
-L'_{ij} & \sum_{k=1}^{n} \left( \frac{\partial L'_{ki}}{\partial q_j} - \frac{\partial L'_{kj}}{\partial q_i} \right) p_k
\end{pmatrix}
\]  

on \( \mathcal{M} \), according to \([25]\). The corresponding Poisson brackets read as

\[
\{ q_i, q_j \}' = 0, \quad \{ q_i, p_j \}' = L'_{ij}, \quad \{ p_i, p_j \}' = \sum_{k=1}^{n} \left( \frac{\partial L'_{ki}}{\partial q_j} - \frac{\partial L'_{kj}}{\partial q_i} \right) p_k.
\]

The vanishing of \( L' \) torsion entails that \( P' \) is a Poisson bivector compatible with \( P \), i.e.

\[
[P, P] = [P', P] = [P', P] = 0.
\]

The torsion of the \((1,1)\) tensor field \( A \) equals to zero, if for any vector fields \( X, Y \)

\[
T_A(X, Y) \equiv L A X Y - A (L A X Y + L A Y X - A L X Y) = 0, \quad \forall X, Y.
\]

Here \( L_X \) means the Lie derivative along \( X \).

Remark 2 In the framework of the Eisenhart-Benenti theory tensor field \( L' \) yields special conformal Killing tensor called the Benenti \( L \)-tensor, Killing-Stäckel space, Stäckel web etc \([2, 3]\). Of course, we can try to transfer the corresponding geometric machinery to nonholonomic theory.

One possible generalization of the Turiel construction \([2, 3]\) has been proposed in \([22]\). Here we consider some other generalizations related with the nonholonomic integrable systems.

In our case \( n = 2 \) and the Darboux coordinates on \( \mathcal{M} = T^*S^2 \) are standard spherical coordinates \((1,9)\), thus,

\[
q_1 = \phi, \quad q_2 = \theta, \quad p_1 = p_\phi, \quad p_2 = p_\theta.
\]  

At \( d = 0 \) and \( \kappa = \pm 1 \) we have the constant invariant measure and, therefore, integrals of motion \( H_{1,2} \) are in involution

\[
\{ H_1, H_2 \} = 0, \quad d = 0, \quad \kappa = \pm 1,
\]

with respect to the Poisson brackets associated with the canonical bivector \( P \) \((2.3)\).

2.1 Case \( \kappa = 1 \)

At \( d \neq 0 \) and \( \kappa = 1 \) substituting another torsionless tensor field

\[
L_g = \frac{1}{\sqrt{g}} L = \frac{1}{\sqrt{g}} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix},
\]  

into the definitions \((2.4)\), one gets the desired solution of the equations \((1.8)\)

\[
P_g = \frac{1}{\sqrt{g}} \begin{pmatrix}
0 & 0 & 1 & 0 \\
* & 0 & 0 & 1 \\
* & * & 0 & \frac{1}{2} \left( \frac{\partial \ln g}{\partial \phi} p_\phi - \frac{\partial \ln g}{\partial \theta} p_\theta \right) \\
* & * & * & 0
\end{pmatrix}
\]  

\[\text{(2.7)}\]
Remark 3  This deformation has a similar form with the well-known relation between the modular vector fields
\[ X_{g\mu} = X_{\mu} - X_{ln g} \]
associated with the volume forms \( \mu \) and \( \nu = g\mu \) [15, 26].

Remark 4  At \( d = 0 \) function \( g \) (1.3) equals to unit and, therefore, at this limit one gets standard canonical Poisson bivector \( P = \lim_{d \to 0} P_g \).

In terms of initial variables \((\gamma, M)\) this Poisson bivector \( P_g \) (2.7) has been obtained in [6].

Proposition 1  [6] Integrals of motion \( H_1, H_2 \) (1.5) are in involution with respect to the Poisson bracket associated with the Poisson bivector \( P_g \) (2.7)
\[ \{H_1, H_2\}_g = 0, \quad d > 0, \quad \kappa = 1. \]
The corresponding volume form
\[ \nu = P_g^{-2} = -2g dq dp \]
is invariant with respect to a Hamiltonian flow associated with new time \( t_g \) defined by
\[ \frac{dt_g}{dt} z_k = \{H_1, z_k\}_g, \quad k = 1, \ldots, 4. \]

We can easily relate new and old time variables
\[ dt_g \simeq \sqrt{g} dt \quad (2.8) \]
because at \( \kappa = 1 \) initial equations of motion are equal to
\[ \frac{d}{dt} z_k = \sqrt{g} \left\{ H_1, z_k \right\}_g. \quad (2.9) \]

Transformation of time (2.8) has been proposed by Chaplygin in [10]. Namely this process is to be referred to as the Chaplygin Hamiltonization, see [6, 8, 9, 13, 17].

Remark 5  One of the global invariants in Poisson geometry is a modular class. It is an obstruction to the existence of a measure in \( M \) which is invariant under all hamiltonian flows [15, 16, 26]. So, in fact it is a geometric obstruction to the Hamiltonization process.

For the manifold \( M \) endowed with a Poisson bivector \( P \), its modular class is an element of the first Poisson cohomology group. In Section 3 we discuss some elements of the second Poisson cohomology group and the corresponding Poisson bivectors \( P' \) compatible with \( P \), which allows us to get variables of separation without Hamiltonization.

2.2  Case \( \kappa = -1 \)

It is easy to see, that at \( \kappa = -1 \) the integrals of motion \( H_{1,2} \) (1.5) do not commute with respect to the Poisson brackets associated with bivector \( P_g \) (2.7)
\[ \{H_1, H_2\}_g \neq 0, \quad d > 0, \quad \kappa = -1. \]

So, we have to propose another deformation of the canonical Poisson structure applicable to the Borisov-Mamaev-Fedorov system.

Let us try to solve our geometric equations
\[ (PdqH_1, dqH_2) \equiv \{H_1, H_2\} = 0, \quad [P, P] = 0, \quad (2.10) \]
by "brute force" method, using similar to (2.7) anzats
\[ P = \begin{pmatrix}
0 & 0 & f(\phi, \theta) & 0 \\
* & 0 & 0 & h(\phi, \theta) \\
* & * & 0 & u(\phi, \theta)p_\phi + v(\phi, \theta)p_\theta \\
* & * & * & 0
\end{pmatrix}. \]

As a result we have the following
Proposition 2 At $\kappa = -1$ the integrals of motion $H_{1,2}$ (1.5) are in involution with respect to the Poisson brackets associated with the Poisson bivector

$$P_\eta = \begin{pmatrix} 0 & L_{\eta ij} \\ -L_{\eta ij} & \sum_{k=1}^{n} (1 + \eta) \frac{\partial L_{\eta ki}}{\partial q_j} - \frac{1}{(1 + \eta)} \frac{\partial L_{\eta kj}}{\partial q_i} p_k \end{pmatrix},$$

(2.11)

where

$$L_\eta = \frac{1}{\sqrt{g}} \begin{pmatrix} 1 & 0 \\ 0 & 1 + \eta \end{pmatrix}$$

and

$$\eta = \frac{2 \sin^2 \theta (b_3^2 - (b_1 + b_2)b_3 + b_1b_2)}{b_3^2(d+1)(b_1 + b_2) - 2}. \quad (2.12)$$

Here $L_\eta$ is the (1,1) tensor field with non-zero torsion, in contrast with the tensor field from the Turiel construction (2.4).

The proof is straightforward.

Tensor field $L_\eta$ may be considered as an additional deformation $L_g$ (2.6) by function $\eta$ depending only on variable $\theta$, parameters $d$ and $b_k$, such as

$$\lim_{d \to 0} \eta = 0 \quad \Rightarrow \quad \lim_{d \to 0} P_\eta = P.$$  

Moreover, $\eta = 0$ for the axially symmetric ball at $b_3 = b_1$ or $b_3 = b_2$.

Remark 6 At present we don’t know any physical meaning of the function $\eta(\theta)$ and the geometric explanation of the deformation (2.11). It will be interesting to understand the relations between $L_\eta$ and the theory of Killing tensors with non-zero torsion.

The Poisson bivector $P_\eta$ (2.11) may be rewritten as follows

$$P_\eta = \frac{1}{\sqrt{g}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ * & 0 & 0 & (1 + \eta) \\ * & * & 0 & -\frac{1}{2} \left( (1 + \eta) \frac{\partial \ln g}{\partial \theta} p_\phi - \frac{\partial \ln g}{\partial \phi} p_\theta \right) \\ * & * & * & 0 \end{pmatrix}.$$

The corresponding volume form

$$\nu_\eta = P_\eta^{-2} = -\frac{2g}{(1 + \eta)} dqdp$$

is a more complicated deformation of the invariant volume form (1.6) introduced in [27]

$$\mu = \sqrt{g} dqdp.$$  

This new volume form is invariant with respect to a Hamiltonian evolution associated with new time $t_\eta$ defined by

$$\frac{d}{dt_\eta} z_k = \{H_1, z_k\}_\eta, \quad k = 1, \ldots, 4.$$  

Relation between the initial and new time variables is also more complicated then in (2.8), because at $\kappa = -1$ initial equations of motion (1.4) read as

$$\frac{d}{dt} z_k = \frac{\sqrt{g}}{2} (b_1 + b_2 + b_3 + w_1) \{H_1, z_k\}_\eta - \sqrt{g} (1 + w_2) \{H_2, z_k\}_\eta. \quad (2.13)$$
Here
\[ w_1 = \frac{\eta}{1 + \eta} \frac{(b_1 + b_2) \left( b_3 (b_1 \cos^2 \phi + b_2 \sin^2 \phi) - b_1 b_2 \right)}{(b_1 - b_3)(b_2 - b_3)} \] (2.14)
\[ w_2 = \frac{\eta}{1 + \eta} \frac{b_3 (b_1 \cos^2 \phi + b_2 \sin^2 \phi) - b_1 b_2}{(b_1 - b_3)(b_2 - b_3)}. \]

So, without the intermediate time transformation we did not get conformally Hamiltonian system from [8, 9], because we consider, in fact, two different systems with geometric point of view, see [4, 13, 17]. The modern theory of conformally Hamiltonian systems may be found in [16].

### 3 Second Poisson brackets

In this Section we want to get another solution \( P' \) of the equations (1.8, 2.10), which is compatible with the first solution \( P \) obtained earlier, i.e.
\[ [P, P'] = 0. \]

Compatible bivectors \( P' \) are the 2-cocycles in the Poisson cohomology defined by \( P \) on the Poisson manifold \( M \), whereas the Lie derivatives of \( P \) along vector field \( X \)
\[ P' = \mathcal{L}_X P \]
are 2-coboundaries. So, in order to get the desired solution of (1.8, 2.10) we will use the Lie derivatives along the vector fields \( X \) with linear in momenta entries.

In bi-Hamiltonian geometry equations of motion usually have the following form
\[ \frac{d}{dt} z_k = s_1 \{H_1, z_k \} + s_2 \{H_2, z_k \}' , \quad (3.1) \]
where \( \{., .\}' \) is the second Poisson bracket associated with \( P' \) and \( s_{1,2} \) are some functions on dynamical variables [23, 22].

If \( s_1 = 0 \) and \( s_2 = \text{const} \) we have a bi-Hamiltonian dynamical system. If \( s_1 = 0 \) and \( s_2 \) is arbitrary, one gets the so-called quasi bi-Hamiltonian system. At \( s_{1,2} \neq 0 \) we have bi-integrable dynamical system [20, 22]. So, the equations of motion (2.15) for the Borisov-Mamaev-Fedorov system have the standard bi-Hamiltonian form.

#### 3.1 Case \( d = 0 \) and \( \kappa = 1 \)

At \( d = 1 \) we have the hamiltonian flow (1.4) associated with the canonical Poisson bivector \( P' \). It is easy to prove that the integrals of motion \( H_{1,2} \) are in bi-involution
\[ \{H_1, H_2 \} = \{H_1, H_2 \}' = 0 , \quad d = 0 , \]
with respect to canonical Poisson brackets associated with bivectors \( P \) and \( P' \) determined by the following (1,1) torsionless tensor field [22]:
\[ L = \begin{pmatrix}
  a_1 \cos^2 \phi + a_2 \sin^2 \phi & \frac{(a_1 - a_2) \sin 2\phi \cos \theta}{2} \\
  \frac{(a_1 - a_2) \sin 2\phi}{2} \cos \theta \sin \theta & a_3 \sin^2 \theta + (a_1 \sin^2 \phi + a_2 \cos^2 \phi) \cos^2 \theta
\end{pmatrix} . \] (3.2)

The Turiel bivector \( P' \) may be rewritten as the Lie derivative \( P' = \mathcal{L}_Y P \) of the canonical bivector \( P \) along the vector field \( Y = \sum Y^j \partial_j \) with the following entries
\[ Y^{1,2} = 0 , \quad \begin{pmatrix} Y^3 \\ Y^4 \end{pmatrix} = - L'^\top \begin{pmatrix} p_\phi \\ p_\theta \end{pmatrix} . \] (3.3)
Here \( L'^T \) stands for the transpose of the matrix \( L' \).

**Remark 7** The corresponding volume form \( \lambda' = P'^{-2} \) is invariant with respect to the new time defined by

\[
\frac{d}{dt'} z_k = \{ H_1, z_k \}', \quad k = 1, \ldots, 4.
\]

It is neither bi-Hamiltonian nor quasi bi-Hamiltonian system \cite{22} and, therefore, this reparametrization of time looks like the Hamiltonization for the Borisov-Fedorov system.

The eigenvalues \( u, v \) of the recursion operator \( N = P'P^{-1} \) are the roots of the following polynomial

\[
B(\lambda) = (\lambda - u)(\lambda - v) = \lambda^2 - \text{tr} (L'L^{-1}) \lambda + \frac{\det L'}{\det L} = 0. \tag{3.4}
\]

Of course, in this case coordinates \( u, v \) are the standard elliptic coordinates on the sphere defined by

\[
\frac{(\lambda - u)(\lambda - v)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)} = \frac{\gamma_1^2}{\lambda - a_1} + \frac{\gamma_2^2}{\lambda - a_2} + \frac{\gamma_3^2}{\lambda - a_3}. \tag{3.5}
\]

### 3.2 Case \( d = 0 \) and \( \kappa = -1 \)

At \( d = 0 \) and \( \kappa = -1 \) the integrals of motion \( H_{1,2} \) \cite{13} are in bi-involution with respect to canonical Poisson bracket and the second bracket associated with bivector \( \hat{P}' \) \cite{23} defined by the following \((1,1)\) tensor field

\[
\hat{L}' = \begin{pmatrix}
  c_1 \cos^2 \phi + c_2 \sin^2 \phi & \frac{(c_1 - c_2) \sin 2\phi \cos \theta}{2} \\
  \frac{(c_1 - c_2) \sin 2\phi}{2} \cos \theta \sin \theta & c_3 \sin^2 \theta + (c_1 \sin^2 \phi + c_2 \cos^2 \phi) \cos^2 \theta
\end{pmatrix}, \tag{3.6}
\]

where \( c_i = a_i/b_i \). The eigenvalues of the recursion operator \( \hat{N} = \hat{P}'P^{-1} \) coincide with standard elliptic coordinates on the sphere

\[
\gamma_i = \sqrt{\frac{(u - c_i)(v - c_i)}{(c_j - c_i)(c_k - c_i)}}, \quad i \neq j \neq k, \quad c_i = \frac{a_i}{b_i}. \tag{3.7}
\]

The conjugated momenta \( p_u, p_v \) are defined by standard relations

\[
M_i = \frac{1}{b_i} 2\varepsilon_{ijk} \gamma_j \gamma_k (c_j - c_k) \left( (c_i - u)p_u - (c_i - v)p_v \right), \tag{3.8}
\]

where \( \varepsilon_{ijm} \) is a completely antisymmetric tensor.

In terms of these Darboux-Nijenhuis variables \( u, v \) and \( p_u, p_v \), our Poisson bivectors look like

\[
\begin{pmatrix}
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
 -1 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
  0 & 0 & u & 0 \\
 0 & 0 & 0 & v \\
 -u & 0 & 0 & 0 \\
 0 & -v & 0 & 0
\end{pmatrix},
\]

whereas in terms of initial physical variables, first bivector \( P \) reads as

\[
P = \frac{1}{b_1 b_2 b_3} \begin{pmatrix}
  0 & 0 & 0 & 0 & b_1 b_3 \gamma_3 & -b_1 b_2 \gamma_2 \\
  * & 0 & 0 & -b_2 b_3 \gamma_3 & 0 & b_2 b_1 \gamma_1 \\
  * & * & 0 & b_3 b_2 \gamma_2 & -b_3 b_1 \gamma_1 & 0 \\
  * & * & * & 0 & b_3^2 M_3 & -b_3^2 M_2 \\
  * & * & * & * & 0 & b_1^2 M_1
\end{pmatrix}, \tag{3.9}
\]
It is evident that any functions \( f_1(u) \) and \( f_2(v) \) are variables of separation as well. So, the following trivial point transformation

\[
u \to f_1(u), \quad v \to f_2(v)
\]

preserves the separability property of distribution defined by the functions \( H_{1,2} \). Namely, according to [20], these integrals are in involution with respect to the Poisson brackets associated with the following bivectors

\[
P' = \begin{pmatrix}
0 & 0 & f_1(u) & 0 \\
0 & 0 & 0 & f_2(v) \\
-f_1(u) & 0 & 0 & 0 \\
0 & -f_2(v) & 0 & 0
\end{pmatrix}.
\]

Of course, in terms of initial physical variables these bivectors have more complicated form. For instance, tensor field

\[
L'' = \frac{1}{\zeta} \left[ \hat{L}' + 2 \begin{pmatrix}
\rho & 0 \\
0 & \rho
\end{pmatrix} \right]
\]

where

\[
\zeta = \cos^2 \theta + \frac{b_3(b_2 + b_1) \cos \theta}{(b_1b_2 + b_3(1 + 2 \sin^2 \phi + b_2 \cos^2 \phi)) \sin^2 \theta}, \quad \rho = \frac{\cos^2 \theta}{b_1b_2 + b_3(1 + 2 \sin^2 \phi + b_2 \cos^2 \phi)} - \frac{\cos^2 \theta + \cos^2 \phi \sin^2 \theta}{b_1(b_2 + b_2)} \frac{\sin^2 \phi \cos^2 \theta + 1}{b_2(1 + b_3)},
\]

yields bivector (3.11) associated with new variables of separation (3.10) defined by

\[
\begin{align*}
f_1(u) &= -\frac{2(ub_1(b_2 + b_3) - 2)ub_2(b_1 + b_3)}{ub_1b_2(b_1 + b_3)(b_2 + b_3)(ub_3(b_1 + b_2) - 2)}, \\
f_2(v) &= -\frac{2(vb_1(b_2 + b_3) - 2)vb_2(b_1 + b_3)}{vb_1b_2(b_1 + b_3)(b_2 + b_3)(vb_3(b_1 + b_2) - 2)}.
\end{align*}
\]

It is natural that the canonical transformations (3.10) preserve the first bivector and change the second bivector \( \hat{P}' \) simultaneously with coefficients \( s_{1,2} \) in the equations of motion (3.11). Of course, some geometric properties of these equations are invariant with respect to such transformations.

**Proposition 3** At \( d = 0 \) and \( \kappa = -1 \) there does not exist nontrivial linear in momenta Poisson bivector \( P'' \), which is compatible with the canonical ones, such that \( s_1 = 0 \) in (3.1).

So, at \( \kappa = -1 \) the dynamical system (1.4) is only bi-integrable, whereas at \( \kappa = 1 \) it is bi-Hamiltonian.

By adding equations (3.1) with \( s_1 = 0 \) and compatibility condition \([P, P''] = 0\) to the initial equations (1.8, 2.10) one gets an overdetermined system of algebro-differential equations. If the entries of \( P'' \) are linear nonhomogeneous polynomials in momenta, then the system has only trivial solution \( P'' = 0 \).

**Remark 8** According to [5], at \( d = 0 \) dynamical systems with \( \kappa = \pm 1 \) are related to each other by the Poisson map \( M \to BM \) and the trivial change of time

\[
t \to -t.
\]

In Proposition 3 we proved that even such seemingly harmless transformation leads to a loss of very important geometric property. Namely, after this change of time the new system becomes non bi-Hamiltonian with respect to initial integrals of motion.
3.3 Chaplygin system, $\kappa = 1$

According to [24], let us introduce the vector field $X = \sum X^j \partial_j$ with the following entries

$$X^i = 0, \quad X^{i+3} = \left[ \gamma \times A_g(\gamma \times M) \right]_i, \quad i = 1, 2, 3.$$  \hspace{1cm} (3.13)

where $A_g$ is the following $3 \times 3$ matrix

$$A_g = A + dg(\gamma) A \left( \gamma \otimes \gamma \right) A.$$  

entering into the angular velocity definition (1.2).

**Proposition 4** [24] The Lie derivative of $P_g$ (2.7) along the vector field $X$ (3.13) is the desired second solution of the equations (1.3) compatible with the first solution

$$P'_g = \mathcal{L}_X P_g,$$  \hspace{1cm} (3.14)

so that the integrals of motion $H_{1,2}$ (1.5) are in bi-involution

$$\{H_1, H_2\}_g = [H_1, H_2]_g = 0,$$  \hspace{1cm} (3.15)

with respect to a pair of the corresponding compatible Poisson brackets.

In spherical coordinates this bivector looks like a deformation of the Turiel construction (2.4)

$$P'_g = \left( \begin{array}{c c c} 0 & L'_{gij} \\ -L'_{gij} & \sum_{k=1}^n \left( x_{ki} \frac{\partial L'_{gki}}{\partial q_j} - y_{kj} \frac{\partial L'_{gkj}}{\partial q_i} \right) p_k \end{array} \right).$$  \hspace{1cm} (3.16)

Similar to the tensor field $L_g$ (2.12), this $(1,1)$ tensor field

$$L'_g = \sqrt{g} 

L' - \frac{d \sqrt{g} \sin^2 \theta}{1 - da_3} \begin{pmatrix} a_1 a_2 - a_3 (a_1 \cos^2 \phi + a_2 \sin^2 \phi) & 0 \\
0 & -a_1^2 + a_3 \sin^2 \phi + a_2 \cos^2 \phi \end{pmatrix},$$

has a non-zero torsion too.

Functions $x_{ki}, y_{kj}$ depending only on the coordinates $\phi, \theta$ can be easily restored from the relation (3.14), which in spherical coordinates reads as

$$P'_g = \mathcal{L}_Z P_g, \quad Z^{1,2} = 0, \quad \begin{pmatrix} Z_1 \\
Z_3 \\
Z_4 \end{pmatrix} = -\sqrt{g} L'_g \begin{pmatrix} p_0 \\
p_\phi \\
p_\theta \end{pmatrix}.$$  \hspace{1cm} (3.17)

Here $L'_g \top$ stands for the transpose of the matrix $L'_g$.

Using this second Poisson structure for the Chaplygin system we can rewrite the equations of motion (1.3) in the following form

$$\frac{d}{dt} z_k = \left( \frac{1 - da_3}{2} \sqrt{g} \right) \begin{pmatrix} \frac{s_1}{a_3} (H_1, z_k)'_g + (H_2, z_k)'_g \end{pmatrix},$$

where

$$s_1 = -1 - da_3 + a_3^2 \sin^2 \theta - (a_3 (a_1 + a_2) - a_1 a_2) \cos^2 \theta - d(a_1 a_2 - a_1 - a_2)a_3^2 \frac{da_1 a_2 a_3 - a_1 a_2 \cos^2 \theta - a_3 (a_1 \cos \phi + a_2 \sin \phi) \sin^2 \theta}{da_1 a_2 a_3 - a_1 a_2 \cos^2 \theta - a_3 (a_1 \cos \phi + a_2 \sin \phi) \sin^2 \theta}.$$  

The eigenvalues $u_g, v_g$ of the recursion operator $N_g = P'_g P_g^{-1}$ are defined by the relation

$$\frac{(\lambda - u_g)(\lambda - v_g)}{(\lambda - a_1)(\lambda - a_2)(\lambda - a_3)} = g(\gamma) \left( \frac{\gamma^2 (1 - da_1)}{\lambda - a_1} + \frac{\gamma^2 (1 - da_2)}{\lambda - a_2} + \frac{\gamma^2 (1 - da_3)}{\lambda - a_3} \right),$$  \hspace{1cm} (3.18)
which bears a resemblance to the usual definition \(3.5\) of the elliptic coordinates on the sphere. Namely these variables have been obtained by Chaplygin who used hamiltonization \(10\).

As above, using the trivial point transformations \(3.10\) we can change this second Poisson bracket and the equations of motion \(3.1\) associated with this bracket. For instance, let us consider another \((1,1)\) tensor field

\[
L''_g = \frac{1}{\zeta \sqrt{g}} \left[ L' + \frac{1}{1 - da_3} \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} \right]
\]

where

\[
\zeta = da_1a_2a_3 + a_1a_2 \cos^2 \theta + a_3(a_1 \cos^2 \phi + a_2 \sin^2 \phi) \sin^2 \theta,
\]

\[
\rho_1 = \left( da_3 - \sin^2 \phi \cos^2 \theta - \cos^2 \phi \right)a_1 + \left( da_3 - \cos^2 \phi \sin^2 \theta - 1 \right)a_2 - a_3 \sin^2 \theta,
\]

\[
\rho_2 = \rho_1 + d \sin^2 \theta(a_1 - a_3)(a_2 - a_3).
\]

Substituting this tensor field into the Lie derivative \(3.17\), one gets a new Poisson bivector \(P''_g\) with the following properties.

**Proposition 5** At \(\kappa = 1\) the initial equations of motion \(\{1.4\}\) have the following form

\[
\frac{d}{dt} z_k = \sqrt{g} \left( \{H_1, z_k\}_g = \frac{1 - da_3}{2} \{H_2, z_k\}_g \right),
\]

where \(\{\ldots\}_g\) is the Poisson bracket associated with bivector \(P''_g\).

So, equations of motion for the nonholonomic Chaplygin system are conformally Hamiltonian equations with respect to both the first bracket \(\{\ldots\}_g\) with first integral of motion \(H_1 \ (2.9)\) and the second bracket \(\{\ldots\}_g''\) with second integral of motion \(H_2 \ (3.20)\).

### 3.4 Borisov-Mamaev-Fedorov system, \(\kappa = -1\)

As usual, at \(\kappa = -1\) there exist many linear in momenta solutions of the equations \(2.10\), which are related to each other by point canonical transformations \(\lambda_i \rightarrow f_i(\lambda_i) \ (3.10)\), where \(\lambda_i\) are the Darboux-Nijenhuis coordinates, i.e. the eigenvalues of the recursion operator.

All these solutions have the form

\[
P'_\eta = \begin{pmatrix} 0 & L'_{\eta ij} \\ -L'_{\eta ij} & \sum_{k=1}^n \left( x_{ki} \frac{\partial L'_{\eta ki}}{\partial q_j} - y_{kj} \frac{\partial L'_{\eta kj}}{\partial q_i} \right) p_k \end{pmatrix}.
\]

Let us consider only one solution associated with a relatively simple tensor field

\[
L'_\eta = \frac{1}{\sqrt{g}} \left[ L' + \begin{pmatrix} \alpha & 0 \\ 0 & (1 + \eta)\alpha + \frac{2 \eta (b_1 + b_2 + b_3)}{(b_1 + b_2)(b_1 + b_3)(b_2 + b_3)} \end{pmatrix} \right],
\]

depending on the arbitrary number \(\alpha\). As above, this tensor field has a non zero torsion at generic \(\alpha\).

Functions \(x_{ki}, y_{kj}\) depending only on the coordinates \(\phi, \theta\) can be easily restored from the other definition of the same bivector

\[
P'_\eta = L_Z P_\eta,
\]

where the entries of the vector field \(Z = \sum Z^j \partial_j\) are equal to

\[
Z^{1,2} = 0, \quad \begin{pmatrix} Z^3 \\ Z^4 \end{pmatrix} = -\sqrt{g} L''_\eta \begin{pmatrix} 1 & 0 \\ 0 & (1 + \eta)^{-1} \end{pmatrix} \begin{pmatrix} p_\phi \\ p_\theta \end{pmatrix}.
\]

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Remark 9 It is easy to see, that at \( \kappa = \pm 1 \) entries of the Liouville vector field \( Z \) can be rewritten in the common form

\[
Z = - \begin{pmatrix} 0 & 0 \\ L_\sigma \top L_\sigma^{-1} & 0 \end{pmatrix} z, \quad \sigma = g, \eta,
\]

where \( Z = (Z^1, Z^2, Z^3, Z^4) \) is the vector of the entries vector field, whereas \( z = (q_1, q_2, p_1, p_2) \) is the vector of Darboux coordinates. Geometric origin of this new construction is unclear of yet.

So, for the Borisov-Mamaev-Fedorov system we have a pair of compatible Poisson bivector on \( \mathcal{M} \) and, therefore, this manifold is bi-Hamiltonian.

**Proposition 6** At \( \kappa = -1 \) the integrals of motion \( H_{1,2} \) are in bi-involution

\[
\{ H_1, H_2 \}_\eta = \{ H_1, H_2 \}_\eta' = 0,
\]

with respect to the Poisson brackets associated with bivectors \( P_\eta \) and \( P'_\eta \).

Using the second Poisson brackets \( \{ , \}_\eta' \), we can rewrite the initial equation of motion \( (1.4) \) in the standard form \( (3.11) \) with relatively big coefficients \( s_{1,2} \).

At \( \alpha = 0 \) in \( (3.22) \), the eigenvalues \( u_\eta \) and \( v_\eta \) of the recursion operator \( N_\eta = P'_\eta P'^{-1} \) are defined by the relation

\[
\frac{(\lambda - u_\eta)(\lambda - v_\eta)}{(\lambda - c_1)(\lambda - c_2)(\lambda - c_3 - \delta)} = \frac{1}{\zeta(\gamma)} \left( \begin{array}{c} \gamma_1^2 \\ \frac{\gamma_2^2}{\lambda - c_1} \\ \frac{\beta \gamma_3^2}{\lambda - c_3 - \delta} \end{array} \right),
\]

(3.25)
depending on two constants

\[
\beta = 1 - \frac{2d(b_1 - b_3)(b_2 - b_3)}{b_3(b_3 - 2d)(b_1 + b_2) + 2db_1b_2},
\]

\[
\delta = \frac{4d}{b_3(b_3 - 2d)(b_1 + b_2) + 2db_1b_2} \frac{b_1b_2(b_1 - b_3)(b_2 - b_3)}{b_3(b_1 + b_2)(b_1 + b_3)(b_2 + b_3)},
\]

(3.26)
and one function on \( \theta = \arccos \gamma_3 \)

\[
\zeta(\gamma) = 1 - \frac{2d(b_1 - b_3)(b_2 - b_3)}{b_3(b_3 - 2d)(b_1 + b_2) + 2db_1b_2} \gamma_3^2.
\]

This relation is very close to \( (3.5) \) and \( (3.18) \) and, of course, at \( d = 0 \) coincides with the definition \( (3.7) \) of the elliptic coordinates on the sphere \( \mathcal{S} \) which are variables of separation for the corresponding Hamilton-Jacobi equation.

Now we have to compare this Darboux-Nijenhuis coordinates with the variables of separation obtained in \( [7, 9] \) by hamiltonization process.

**Proposition 7** Variables of separation \( q_{1,2} \) from \( [7, 9] \) are related with the eigenvalues \( u_\eta \) and \( v_\eta \) of the recursion operator \( H_\eta \) by bi-involution similar to \( (3.10) \).

Let us reproduce the definition of variables of separation \( q_{1,2} \) from \( [9] \), see formulae (3.2):

\[
\gamma_i = \sqrt{\frac{\det I}{(J_i - d)J_jJ_k G(q_1, q_2)}} \sqrt{\frac{(q_1 - c_i)(q_2 - c_i)}{(c_j - c_i)(c_k - c_i)}}
\]

(3.27)
where

\[
G(q_1, q_2) = \frac{(b_1 + b_2 - 2d)(b_1 + b_3 - 2d)(b_2 + b_3 - 2d)}{(b_1 + b_2)(b_1 + b_3)(b_2 + b_3)} g.
\]
Substituting this definition into the equation (3.25) one gets the desired point transformation

\[ u_\eta = F(q_1), \quad v_\eta = F(q_2), \]

where

\[ F(q) = \frac{\left( b_3^2 + (b_1 + b_2 - 2d)b_2^2 + b_1b_2b_3 + 2db_1b_2 \right)q - 4d}{(b_1 + b_3)(b_2 + b_3)(d(b_1b_2q - 2) + b_3)}. \]

Associated with the variables \( q_{1,2} \), tensor field \( L'_{\eta} \) in (3.21) is more complicated than tensor field \( L' \) and, for brevity, we omit this expression.

### 4 Separation of variables

In geometry, instead of an additive separation of variables in the partial differential equation called the Hamilton-Jacobi equation, we have some invariant geometric property of the Lagrangian distribution defined by \( n \) independent functions \( H_1, \ldots, H_n \).

Namely, an \( n \)-tuple \( H_1, \ldots, H_n \) of functionally independent functions defines a separable foliation on \( M, \dim M = n \), if there are variables of separation \( (q_1, \ldots, q_n, p_1, \ldots, p_n) \) and \( n \) separated relations of the form

\[ \Phi_i(q_i, p_i, H_1, \ldots, H_n) = 0, \quad i = 1, \ldots, n, \quad \text{with} \quad \det \left[ \frac{\partial \Phi_i}{\partial H_j} \right] \neq 0. \quad (4.28) \]

It simple means, the common level surfaces of \( H_1, \ldots, H_n \) form foliation and every leaf of this foliation may be represented as a direct product of one-dimensional geometric objects defined by separated relations (4.28). Usually we have a direct product of \( n \) algebraic curves, because \( \Phi_i \) are polynomials in \( q_i \) and \( p_i \).

It can be easily shown [20], that condition (4.28) entails the involutivity of \( H_i \) with respect to the compatible Poisson brackets

\[ \{q_i, q_j\}_f = \{p_i, p_j\}_f = 0, \quad \{p_i, q_j\}_f = \delta_{ij} f_j(p_j, q_j), \quad (4.29) \]

depending on the arbitrary functions \( f_1, \ldots, f_n \). In fact, this definition of separability implicitly appeared in the Lagrange proof of the Jacobi theorem, but both Lagrange and Jacobi used only canonical brackets

\[ \{q_i, q_j\} = \{p_i, p_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad (4.30) \]

which belongs to the family (4.29).

In bi-Hamiltonian geometry eigenvalues \( q_1, \ldots, q_n \) of the recursion operator are the desired coordinates of separation or the Darboux-Nijenhuis coordinates. It is a sequence of the fact that the distribution tangent to the foliation defined by \( H_1, \ldots, H_n \) is Lagrangian with respect to the symplectic form \( P^{-1} \) and invariant with respect to recursion operator \( N = P'P^{-1} \). So, if we know these coordinates, then we have to explicitly find the conjugated momenta \( p_1, \ldots, p_n \) and the separated relations (4.28).

In the Chaplygin hamiltonization method momenta are defined by complete integrals \( S \) of the Hamilton-Jacobi equation after the corresponding change of time

\[ p_j = \frac{\partial}{\partial q_j} S_j(q_j, \alpha_1, \ldots, \alpha_n). \quad (4.31) \]

On this step one usually gets momenta \( p_i \) which have more complicated Poisson brackets with coordinates \( q_i \) (4.29) instead of standard canonical brackets (4.30).
For instance, let us consider definition of the separated momenta from [9], see formulae (3.14):

\[
M_i = \frac{(J_i - d)^2 \sqrt{(c_j - q_1)(c_j - q_3)} \sqrt{(c_k - q_1)(c_k - q_3)}}{2 \sqrt{G(q_1, q_2)} (u - v)} \times \left( \frac{p_2}{(q_1 - c_j)(q_1 - c_k)} - \frac{p_2}{(q_2 - c_j)(q_2 - c_k)} \right).
\]

(4.32)

It is easy to observe, that at \(d = 0\) variables \(q_{1,2}\) (3.27) coincide with the standard elliptic coordinates \(u, v\) (3.7) on the sphere \(S\), but the corresponding momenta differ from the standard canonical variables \(p\) and \(p\). Moreover, after substituting \(\gamma_i\) (3.8) and \(M_i\) (4.32) into \(C_2 = \sum b_i \gamma_i M_i\) one gets \(C_2 \neq 0\) even at \(d = 0\). So, we suppose that the definition of momenta in [9] contains some misprint and, therefore, we have to define these variables correctly.

### 4.1 Chaplygin system, \(\kappa = 1\)

According to [21, 24], we can use the following recurrence chain

\[
\phi_1 = \{u, H_k\}, \quad \phi_2 = \{u, \phi_1\}, \ldots, \quad \phi_i = \{u, \phi_{i-1}\}, \quad k = 1, 2,
\]

(4.34)

in order to calculate the desired momenta. Namely, in our case this chain breaks down on the third step \(\phi_3 = 0\). It means that \(H_{1,2}\) are the second order polynomials in momenta \(p_u\) and, therefore, we can define this unknown momenta in the following way

\[
p_u = \phi_1 \phi_2
\]

(4.35)

up to the canonical transformations \(p_u \rightarrow p_u + f(u)\). Similar calculation allows us to determine the second momenta \(p_v\).

At \(\kappa = 1\) the results obtained so far can be summarized in the following definition

\[
M_i = \frac{2\epsilon_{ijk}\gamma_i\gamma_k(a_j - a_k)\sqrt{g}}{u - v} \left( (a_i - u)(1 - du)p_u - (a_i - v)(1 - dv)p_v \right).
\]

(4.36)

where

\[
g = \frac{(1 - du)(1 - dv)}{(1 - da_1)(1 - da_2)(1 - da_3)}.
\]

By adding the expressions for \(\gamma_i\)

\[
\gamma_i = \sqrt{\frac{(1 - da_j)(1 - da_k)}{(1 - du)(1 - dv)}}, \quad \sqrt{\frac{(u - a_i)(v - a_i)}{(a_j - a_i)(a_m - a_i)}}, \quad i \neq j \neq k,
\]

(4.37)

obtained from (3.18) to (4.30) one gets the expressions of initial physical variables in terms of canonical separated variables.

By substituting (4.37) and (4.36) into \(H_{1,2}\) (1.5) we can easily prove that variables of separation lie on two copies of the hyperelliptic genus 2 curve defined by the following separated relation

\[
4(1 - dx)(a_1 - x)(a_2 - x)(a_3 - x)y^2 - xH_2 + H_1 = 0, \quad x = u, v, \quad y = p_u, p_v.
\]

(4.38)

It is easy to see, that at \(d = 0\) we obtain the standard elliptic variables on \(T^* S\) and well-known separated relations for the Euler top on the sphere.
4.2 Borisov-Mamaev-Fedorov system, $\kappa = -1$

Let us take the coordinates of separation $q_{1,2}$ \( (3.27) \) and add momenta $p_{1,2}$ to them, in order to obtain a complete set of the canonically conjugated variables \( (4.30) \) with respect to the first Poisson bracket \( (2.11) \).

As above, we can simply calculate these momenta using the recurrence chain $\phi_k \ (4.34)$ associated with the first Poisson brackets \{,\} \( \eta \) \( (2.11) \) at $\kappa = -1$ and rewrite the obtained results \( (4.33) \) in the following form

$$M_i = \frac{2\varepsilon_{ijk}\gamma_j\gamma_k(c_j - c_k)\sqrt{g}}{b_i(q_1 - q_2)} \times \left( (c_i - q_1) \left( 1 - \frac{d(2 - b_1 b_2 q_1)}{b_3} \right) p_1 - (c_i - q_2) \left( 1 - \frac{d(2 - b_1 b_2 q_2)}{b_3} \right) p_2 \right),$$

where $c_i = a_i/b_i$ and

$$g = \frac{\left( 1 - d(b_1 + b_2 + b_3 - 2d) q_1 \right) \left( 1 - d(b_1 + b_2 + b_3 - 2d) q_2 \right)}{(1 - da_1)(1 - da_2)(1 - da_3)} + \frac{8d}{a_1 a_2 a_3} q_1 q_2.$$

These expressions are similar to \( (4.30) \) and at $d = 0$ turn into the standard definitions \( (4.33) \) of the elliptic variables on $T^*\mathbb{S}$, in contrast with expressions \( (4.32) \) from \[9\].

By substituting $\gamma_i \ (3.27)$ and \( (4.39) \) into $H_{1,2} \ (1.5)$ we easily prove that variables of separation for the Borisov-Mamaev-Fedorov system lie on two copies of the hyperelliptic genus 2 curve defined by the following separated relation,

$$4 \left( 1 - \frac{d(2 - b_1 b_2 x)}{b_3} \right)^2 (x - c_1)(x - c_2)(x - c_3)y^2 - \alpha H_2 + \beta H_1 = 0,$$

where $x = q_{1,2}$, $y = p_{1,2}$ and

$$\alpha = db_1 b_2 b_3 x^2 - (b_1 b_2 + b_1 b_3 + b_2 b_3)x + 2$$
$$\beta = \left( d(b_1 b_2 + b_1 b_3 + b_2 b_3) - \frac{(b_1 + b_2)(b_1 + b_3)(b_2 + b_3)}{2} \right) x + b_1 + b_2 + b_3 - 2d.$$

At $d = 0$ this equation coincides with the separated equation for Hamiltonian systems on the sphere separable in elliptic variables, which lie on the elliptic curve instead of hyperelliptic at $d \neq 0$.

5 Conclusion

Using the standard machinery of the bi-Hamiltonian geometry, we reproduce some results from \[5, 7, 9\] obtained in framework of the Chaplygin hamiltonization method. Definitions of the Poisson bivectors for the Borisov-Mamaev-Fedorov system \( (2.11) \), \( (3.21) \) and of the second bivector for the Chaplygin case \( (3.10) \) are completely new. They can be considered as nontrivial deformations of the Turiet and Benenti constructions associated only with nonholonomic dynamical systems.

The explicit form of the separated relations \( (4.40) \) with canonical variables of separation for the Borisov-Mamaev-Fedorov system is also new. Because all the hyperelliptic genus 2 curves are isomorphic to each other, we could use these separated relations \( (4.33) \) and \( (4.40) \) in order to get a mapping between Chaplygin and Borisov-Mamaev-Fedorov systems. We suppose that such mapping may be extended to the case $C_2 \neq 0$, that allows us to get solutions of the equations \( (1.4) \) in generic case.

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