MAXIMAL EQUIVARIANT COMPACTIFICATIONS

MICHAEL MEGRELIŠVILI

Dedicated to the centennial of Yu. M. Smirnov’s birth

Abstract. Let $G$ be a locally compact group. Then for every $G$-space $X$ the maximal $G$-proximity $\beta_G$ can be characterized by the maximal topological proximity $\beta$ as follows:

$$A \beta_G B \iff \exists V \in N_{e} \quad VA \beta V B.$$ 

Here, $\beta_G: X \to \beta_G X$ is the maximal $G$-compactification of $X$ (which is an embedding for locally compact $G$ by a result of J. de Vries), $V$ is a neighborhood of $e$ and $A \beta_G B$ means that the closures of $A$ and $B$ do not meet in $\beta_G X$.

Note that the local compactness of $G$ is essential. This theorem comes as a corollary of a general result about maximal $U$-uniform $G$-compactifications for a useful wide class of uniform structures $U$ on $G$-spaces for not necessarily locally compact groups $G$. It helps, in particular, to derive the following result. Let $(U_1, d)$ be the Urysohn sphere and $G = \text{Iso}(U_1, d)$ is its isometry group with the pointwise topology. Then for every pair of subsets $A, B$ in $U_1$, we have

$$A \beta_G B \iff \exists V \in N_{e} \quad d(VA, VB) > 0.$$ 

More generally, the same is true for any $\aleph_0$-categorical metric $G$-structure $(M, d)$, where $G := \text{Aut}(M)$ is its automorphism group.

Contents

1. Introduction 1
2. Proximities and equivariant Smirnov’s Theorem 4
3. Uniform $G$-spaces 6
4. $G$-compactifications and proximities 12
5. Additional notes about maximal $G$-compactifications 16
6. Some open questions 17
References 17

1. Introduction

A topological transformation group ($G$-space) is a continuous action of a topological group $G$ on a topological space $X$. Compactifiability of Tychonoff topological spaces means the existence of topological embeddings into compact Hausdorff spaces. For the compactifiability of $G$-spaces we require, in addition, the continuous extendability of the original action. Compactifiable $G$-spaces are known also as $G$-Tychonoff spaces.

Compactifications of $G$-spaces is a quite an active research field. We do not intend here to give a comprehensive bibliography but try to refer the interested readers to some
publications, where \( G \)-compactifications play a major role. See, for example, R. Brook [8], J. de Vries [50, 51, 52, 53, 54, 55], Yu.M. Smirnov [41, 42, 43, 44], Antonyan–Smirnov [4], Smirnov–Stoyanov [45], L. Stoyanov [24, 25, 26, 30, 31, 32], Dikranjan-Prodanov-Stoyanov [10], Megrelishvili–Sanchis [14], V. Pestov [37, 38], J. van Mill [35], A. Sokolovskaya [46], Google–Megrelishvili [15], Kozlov–Chatyrko [22], N. Antonyan [1], Karasev–Kozlov [17], Ibarlucia–Megrelishvili [16] (and many additional references in these publications).

Compactifications of a Tychonoff space \( X \) can be described in several ways:

- Banach subalgebras of \( C^b(X) \) (Gelfand-Kolmogoroff 1-1 correspondence);
- Completion of totally bounded uniformities on \( X \) (Samuel compactifications);
- Proximities on \( X \) (Smirnov compactifications).

It is well known (see for example [8, 4, 52, 53, 25]) that the first two correspondences admit dynamical generalizations in the category of \( G \)-spaces. Instead of continuous bounded functions, we should use special subalgebras of generalized right uniformly continuous functions (in other terminology, \( \pi \)-uniform functions) and instead of precompact uniformities, we need now precompact equiuniformities (Definition 3.2).

For every Tychonoff \( G \)-space \( X \), the algebra \( \text{RUC}_G(X) \) of all right uniformly continuous bounded functions on \( X \) induces the corresponding Gelfand (maximal ideal) space \( \beta_GX \subset \text{RUC}_G(X)^* \) and the maximal \( G \)-compactification

\[
\beta_G : X \to \beta_GX.
\]

For locally compact groups \( G \), all Tychonoff \( G \)-spaces admit proper compactifications, as was established by de Vries [53]. So, in this case, the map \( \beta_G \) is a topological embedding. However, in general it is not true. Resolving a question of de Vries [50], we proved in [27] that there exist noncompactifiable \( G \)-spaces (even for Polish group actions on Polish spaces).

Moreover, answering an old problem due to Smirnov, an extreme example was found by V. Pestov [38] by constructing a countable metrizable group \( G \) and a countable metrizable non-trivial \( G \)-space \( X \) for which every equivariant compactification is a singleton.

One of the most general (and widely open) attractive problems is

**Problem 1.1.** Clarify the structure of maximal \( G \)-compactifications \( \beta_GX \) of remarkably naturally defined \( G \)-spaces \( X \).

First of all, note that \( \beta_GX \) (for nondiscrete \( G \)) usually is essentially “smaller” than \( \beta X \). For instance, let \( G \) be a metrizable topological group which is not precompact. Then the canonical action \( G \times \beta G \to \beta G \) is continuous iff \( G \) is discrete (Proposition 5.1).

**Problem 1.2.** [16 Question 1.3(b)] Study the greatest \( G \)-compactification \( \beta_G : X \to \beta_GX \) of (natural) Polish \( G \)-spaces \( X \). In particular: when is \( \beta_GX \) metrizable?

Recall that the Čech–Stone compactification \( \beta X \) of any metrizable non-compact space \( X \) cannot be metrizable. In contrast, for several naturally defined “massive actions”, \( \beta_GX \) might be metrizable; sometimes even having a nice transparent geometric presentation. Perhaps the first example of this kind was a beautiful result of L. Stoyanov [27, 38]. He established that the greatest \( U(H) \)-compactification of the unit sphere \( S_H \) in every infinite dimensional Hilbert space \( H \) is the weakly compact unit ball, where \( G = U(H) \) is the unitary group of \( H \) in its standard strong operator topology.

One of the important sufficient conditions when a \( G \)-space \( X \) is \( G \)-compactifiable is the existence of a \( G \)-invariant metric on \( X \). This was proved first by Ludescher–de Vries [23].
Another possibility to establish that such \((X, d)\) is \(G\)-Tychonoff is to observe that in this case \textit{Gromov compactification} \(\gamma: (X, d) \to \gamma(X)\) is a \(G\)-compactification which is a \(d\)-\textit{uniform} topological embedding; see the explanation in [31] using the RUC property of the distance functions \(x \mapsto d(x, \cdot)\) (one may assume that \(d\) is bounded).

Pestov raised several questions in [38] about a possible coincidence between the maximal \(G\)-compactification and the Gromov compactification for some natural geometrically defined isometric actions (Urysohn sphere and Gurarij sphere, among others). These problems were studied recently in [16] (with a positive answer in the case of the Urysohn sphere and a negative answer for the Gurarij sphere).

Remark 1.3. We collect here some old and new nontrivial concrete examples when \(\beta_G X\) is metrizable, usually admitting also a geometric realization.

1. (L. Stoyanov [17,38,10]) Let \(X := S_H\) be the unit sphere of the infinite-dimensional separable Hilbert space \(H\) with the unitary group \(G := U(H)\). Then \(\beta_G X\) is the weak compact unit ball \((B_H, w)\) of \(H\).

2. [16] \textit{Urysohn sphere} \((U_1, d)\) with its isometry group \(G = \text{Iso}(U_1)\). Then \(\beta_G U_1\) can be identified with its Gromov compactification. Moreover, \(\beta_G U_1\) can be identified with the compact space \(K(U_1)\) of all Katetov functions on \(X\).

3. [16, Theorem 4.11] The maximal \(G\)-compactification of the unit sphere \(S_\Theta\) in the Gurarij Banach space \(\Theta\) (where \(G\) is the linear isometry group) is metrizable and does not coincide with its Gromov compactification. \(\beta_G(S_\Theta)\) can be identified with the compact space \(K_1^1(\Theta)\) of all normalized Katetov convex functions on \(\Theta\). These results are strongly related to some properties of the Gurarij space studied by I. Ben Yaacov [5] and Ben Yaacov–Henson [7].

4. (Proved in [16, Theorem 4.14] thanks to an observation of Ben Yaacov) Let \(B_p\) be the unit ball of the classical Banach space \(V_p := L_p[0, 1]\) for \(1 \leq p < \infty, p \notin 2\mathbb{N}\). Then for the linear isometry group \(\text{Iso}_1(V_p)\), the maximal \(G\)-compactification \(\beta_G B_p\) is the Gromov compactification of the metric space \(B_p\).

5. [16, Theorem 4.4] For every \(\aleph_0\)-categorical metric structure \((M, d)\), the maximal \(G\)-compactification of \((M, d)\), with \(G := \text{Aut}(M)\), can be identified with the space \(S_1(M)\) of all 1-types over \(M\) (and, in particular, is metrizable).

6. (see Examples 5.4 below) Let \(X = (\mathbb{Q}, \leq)\) be the rationals with the usual order but equipped with the discrete topology. Consider any dense subgroup \(G\) of the automorphism group \(\text{Aut}(\mathbb{Q}, \leq)\) with the pointwise topology (for instance, Thompson’s group \(F\)). In this case \(\beta_G X\) is a metrizable linearly ordered compact \(G\)-space, the actions and \(\beta_G: X \to \beta_G\) are order preserving, where \(\beta_G X\) is an inverse limit of finite linearly ordered spaces \(\mathbb{Q}/St_F\), where \(F \subset \mathbb{Q}\) is finite, \(St_F\) is the stabilizer subgroup and \(\mathbb{Q}/St_F\) is the orbit space.

Whenever \(\alpha: X \to Y\) is a compactification, one of the natural questions is which subsets \(A, B\) of \(X\) are “far” with respect to \(\alpha\). This means that the closures of their images do not meet in \(Y\). This is the most basic idea of classical \textit{proximity spaces}. See Section 2 for a description of the role of proximities and Smirnov’s Theorem. This theorem shows that for every \(G\)-compactification \(\alpha: X \to Y\) and \(\alpha\)-far subsets \(A, B\) of \(X\), there exists a sufficiently small neighborhood \(V \in \mathcal{N}_\alpha\) of the identity in \(G\) such that \(VA, VB\) are also \(\alpha\)-far.

A natural question arises about the converse direction: when does this condition guarantee that we have a \(G\)-compactification? We show that this holds for proximities induced by a certain rich class of uniform structures on \(G\)-spaces (see Theorem 4.11). This leads to one of the main results of this paper which is to describe maximal equivariant compactification...

\(^{1}\)K. Kozlov proved in [21] that \(\beta_G U_1\) is homeomorphic to the Hilbert cube.
of locally compact group actions (Theorem 4.8). The local compactness of \( G \) is necessary. Indeed, there exist a Polish \( G \)-compactifiable \( G \)-space \( X \) with a Polish acting group \( G \) and \( G \)-invariant closed \( G \)-subsets \( A, B \) in \( X \) such that \( A\beta_G B \) (see Example 4.10 and Remark 4.9).

A more special general problem is

**Problem 1.4.** For which metric \( G \)-spaces \( (X, d) \) is the following condition satisfied for every subsets \( A, B \) in \( X \)

\[
A \beta_G B \iff \exists V \in N_e \quad d(VA, VB) > 0.
\]

Using a result from [16], we positively answer Problem 1.4 for an important class of metric \( G \)-spaces. Namely, for \( 8_0 \)-categorical metric structures \( (M, d) \), where \( G := \text{Aut} (M, d) \) is its automorphism group. In particular, this is true for the Urysohn sphere \( U_1 \) (Theorem 4.6).

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2. **Proximities and equivariant Smirnov’s Theorem**

**Proximities and proximity spaces.** In 1908, F. Riesz first formulated a set of axioms to describe the notion of closeness of pair of sets. The most useful version of proximity was introduced and studied by V.A. Efremovich [11]. We follow the setting of [36].

**Definition 2.1.** Let \( X \) be a nonempty set and \( \delta \) be a relation in the set of all its subsets. We write \( A\delta B \) if \( A \) and \( B \) are \( \delta \)-related and \( A\delta B \) if not. The relation \( \delta \) will be called a **proximity** on \( X \) provided that the following conditions are satisfied:

- \((P1)\) \( A \cap B \neq \emptyset \) implies \( A\delta B \).
- \((P2)\) \( A\delta B \) implies \( B\delta A \);
- \((P3)\) \( A\delta B \) implies \( A \neq \emptyset \);
- \((P4)\) \( A\delta (B \cup C) \) iff \( A\delta B \) or \( A\delta C \);
- \((P5)\) If \( A\delta B \) then there exist \( C \subset X \) such that \( A\delta C \) and \( (X \setminus C)\delta B \).

A pair \( (X, \delta) \) is called a **proximity space**. Two sets \( A, B \subset X \) are **near** (or **proximal**) in \( (X, \delta) \) if \( A\delta B \) and far (or, remote) if \( A\delta B \). We say a subset \( A \subset X \) is strongly contained in \( B \subset X \) with respect to \( \delta \) (or, \( B \) is a \( \delta \)-neighborhood of \( A \)) if \( A\delta (X \setminus B) \) and write: \( A \Subset B \). In Definition 2.1 one can replace \((P5)\) by the following axiom:

- \((P5')\) If \( A\delta B \) then there exist subsets \( A_1 \) and \( B_1 \) of \( X \) such that \( A \Subset A_1 \), \( B \Subset B_1 \) and \( A_1 \cap B_1 = \emptyset \).

Every proximity space \( (X, \delta) \) induces a topology \( \tau := \text{top}(\delta) \) on \( X \) by the closure operator:

\[ cl_\delta (A) := \{ x \in X : x\delta A \}. \]

The topology \( \text{top}(\delta) \) is Hausdorff iff the following condition satisfied:

- \((P6)\) If \( x, y \in X \) and \( x\delta y \) then \( x = y \).

Every (separated) proximity space \( (X, \delta) \) is completely regular (resp., Tychonoff) with respect to the topology \( \text{top}(\delta) \). A proximity \( \delta \) of \( X \) is called **continuous** (or, more precisely, a \( \tau \)-continuous proximity) if \( \text{top}(\delta) \subset \tau \). In the case of \( \text{top}(\delta) = \tau \), we say that \( \delta \) is a compatible proximity on the topological space \( (X, \tau) \).

Like compactifications, the family of all proximities on \( X \) admits a natural partial order. A proximity \( \delta_1 \) **dominates** \( \delta_2 \) (and write \( \delta_2 \leq \delta_1 \)) iff for every \( A\delta_1 B \) we have \( A\delta_2 B \).
Example 2.2.

(1) Let $Y$ be a compact Hausdorff space. Then there exists a unique compatible proximity on the space $Y$ defined by

$$A \delta B \iff \text{cl}(A) \cap \text{cl}(B) \neq \emptyset.$$ 

(2) Let $X$ be a Tychonoff space. The relation $\delta_B$ defined by

$$A \beta B \iff \exists f \in C(X) \text{ such that } f(A) = 0 \text{ and } f(B) = 1$$

is a proximity which corresponds to the greatest compatible uniformity on $X$. The proximity $\beta$ comes from the Čech-Stone compactification $\beta: X \to \beta X$.

(3) A Hausdorff topological space $X$ is normal iff the relation

$$A \alpha B \iff \text{cl}(A) \cap \text{cl}(B) \neq \emptyset$$

defines a proximity relation on the set $X$. Then $A \delta_n B \iff A \beta B$.

Smirnov’s Theorems. Let $\alpha: X \to Y$ be a compactification. Denote by $\delta_{\alpha}$ the corresponding initial proximity on $X$ defined via the canonical proximity $\delta_Y$ of $Y$. More precisely, for subsets $A, B$ of $X$ we define $A \delta_{\alpha} B$ if $\alpha(A) \delta_{\gamma} \alpha(B)$, i.e., if $\text{cl}(\alpha(A)) \cap \text{cl}(\alpha(B)) = \emptyset$.

Conversely every continuous proximity $\delta$ on a topological space induces a totally bounded uniformity $\mathcal{U}_{\delta}$. Now the completion gives Smirnov’s compactification $c_{\delta}: (X, \delta) \to c_{\delta}X$. It is equivalent to the Samuel compactification with respect to the uniformity $\mathcal{U}_{\delta}$. This leads to a description of compactifications in terms of proximities (see, for example, [10, 36, 12]).

Fact 2.3. (Smirnov’s classical theorem) Let $X$ be a topological space. Assigning to any compactification $\alpha: X \to Y$ the proximity $\delta_{\alpha}$ on $X$ gives rise to a natural one-to-one order preserving correspondence between all compactifications of $X$ and all continuous proximities on the space $X$.

In the case of $G$-spaces, it was initiated by Smirnov himself, extending in [4] his old classical purely topological results from Tychonoff spaces to the case of group actions.

For proximities of $G$-compactifications we simply say $G$-proximity.

Fact 2.4. (Smirnov’s theorem for group actions) In Smirnov’s bijection (Fact 2.3), $G$-proximities are exactly proximities $\delta$ which satisfy the following two conditions:

1. $(G$-invariant) $gA \delta gB$ for every $A \delta B$ and $g \in G$;
2. (compatible with the action) if $A \delta B$ then there exists $U \in \mathcal{U}_{\delta}$ such that $UA \cap UB = \emptyset$.

Remark 2.5. The compatibility condition (2) can be replaced by the following (formally stronger) assumption:

$(2^{\text{str}})$ if $A \delta B$ then there exists $V \in \mathcal{U}_{\delta}$ such that $VA \delta VB$.

In order to see that $(2)$ implies $(2^{\text{str}})$, apply the axiom $(P5')$ (from Definition 2.4). Then for $A \delta B$ there exist subsets $A_1$ and $B_1$ of $X$ such that $A \subseteq A_1$, $B \subseteq B_1$ and $A_1 \delta B_1$. By $(2)$ there exists $V \in \mathcal{U}_{\delta}$ such that $VA \subseteq A_1$, $VB \subseteq B_2$. Therefore, $VA \delta VB$.

The same can be derived also by results of [15] Section 5.2, where a natural generalization of Smirnov’s theorem (Fact 2.3) for semigroup actions was obtained.

Note that the $G$-invarianstness of $\delta_{\alpha}$ guarantees that the $G$-action on $X$ can be extended to a $G$-action on the compactification $Y$ such that all $g$-translations are continuous. That is, we have a continuous action $G_{\text{discr}} \times Y \to Y$, where $G_{\text{discr}}$ is the group $G$ with the discrete topology (however, see Fact 3.15 and Theorem 3.18 below).
Remark 2.6. Let $X$ be a locally compact Hausdorff space. Then the following relation
\[ A \delta \alpha B \iff \text{cl}(A) \cap \text{cl}(B) = \emptyset \]
deﬁnes a compatible proximity on $X$ which suits the (one-point) Alexandrov compactiﬁcation. If $X$ is a $G$-space then this is a $G$-proximity. This explains Fact 3.15.2 below.

Remark 2.7. Let $G/H$ be a coset $G$-space with respect to the left action $\pi: G \times G/H \to G/H$ and a closed subgroup $H$. The relation $\delta_R$ deﬁned by
\[ A \delta R B \iff U \in \mathcal{U}_R(G) : U \cap B \neq \emptyset \]
is a compatible proximity on $G/H$. In fact, it is a $G$-proximity (see Fact 3.15.1 below).

**Uniform spaces and the corresponding proximity.** Recall the following standard lemma about the basis of a uniform structure (deﬁned by the entourages – reﬂexive binary relations) in the sense of A. Weil.

**Lemma 2.8.** (see, for example, [39 Prop. 0.8]) An abstract set $\mathcal{B}$ of entourages on $X$ is a basis of some uniformity $\mathcal{U}$ iﬀ the following conditions are satisﬁed:

1. $\forall \varepsilon \in \mathcal{B} \quad \Delta_X \subset \varepsilon$;
2. $\forall \varepsilon \in \mathcal{B} \quad \exists \delta \in \mathcal{B} \quad \delta \subset \varepsilon^{-1}$;
3. $\forall \varepsilon, \delta \in \mathcal{B} \quad \exists \gamma \in \mathcal{B} \quad \gamma \subset \varepsilon \cap \delta$ (\(\mathcal{B}\) is a ﬁlterbase);
4. $\forall \varepsilon \in \mathcal{B} \quad \exists \delta \in \mathcal{B} \quad \delta \circ \delta \subset \varepsilon$.

The corresponding induced uniformity is just the ﬁlter generated by $\mathcal{B}$. Each uniformity $\mathcal{U}$ on $X$ deﬁnes a topology $\text{top}(\mathcal{U})$ on $X$ as follows: a subset $A \subset X$ is open iﬀ for each $a \in A$ there exists $\varepsilon \in \mathcal{U}$ such that $\varepsilon(a) \subset A$, where $\varepsilon(x) := \{y \in X : (x, y) \in \varepsilon\}$.

$\text{top}(\mathcal{U})$ is Hausdorff iﬀ $\cap \{\varepsilon : \varepsilon \in \mathcal{B}\} = \Delta$. If otherwise not stated, we always consider only Hausdorff completely regular (i.e., Tychonoff) topological spaces, Hausdorff uniformities and proper compactiﬁcations $\alpha: X \to Y$ (i.e., $\alpha$ is an embedding).

**Definition 2.9.** Let $\mathcal{U}$ be a uniformity on $X$. Then the relation $\delta_\mathcal{U}$ deﬁned by
\[ A \delta \mathcal{U} B \iff \varepsilon \cap (A \times B) \neq \emptyset \quad \forall \varepsilon \in \mathcal{U} \]
is a proximity on $X$ which is called the proximity induced by the uniformity $\mathcal{U}$.

Always, $\text{top}(\mathcal{U}) = \text{top}(\delta_\mathcal{U})$. Conversely, every proximity $\delta$ on a topological space deﬁnes canonically a totally bounded compatible uniformity $\mathcal{U}_\delta$.

We say that a proximity $\nu$ on $X$ is $\mathcal{U}$-uniform if $\nu \preceq \delta_\mathcal{U}$.

3. **Uniform $G$-spaces**

**Definition 3.1.** Let $\pi: G \times X \to X$ be a group action. A uniformity $\mathcal{U}$ on $X$ is:

1. **equicontinuous** if (the set of all translations is equicontinuous)
\[ \forall x_0 \in X \quad \forall \varepsilon \in \mathcal{U} \quad \exists \delta \in N_{x_0} \quad (gx_0, gx) \in \varepsilon \quad \forall x \in \delta \quad \forall g \in G; \]
2. **uniformly equicontinuous** if (the set of all translations is uniformly equicontinuous)
\[ \forall \varepsilon \in \mathcal{U} \quad \exists \delta \in \mathcal{U} \quad (gx, gy) \in \varepsilon \quad \forall (x, y) \in \delta \quad \forall g \in G; \]
3. if the conditions (1) or (2) are true for a subset $P \subseteq G$ then we say that $P$ acts equicontinuously or uniformly equicontinuously, respectively.

If $d$ is a $G$-invariant metric on $X$, then the corresponding uniform structure $\mathcal{U}(d)$ is a very natural case of a uniformly equicontinuous uniformity.

**Definition 3.2.** Let $\pi: G \times X \to X$ be an action of a topological group $G$ on a set $X$ and $\mathcal{U}$ is a uniform structure on $X$. 

Proposition 3.4. Let \( \mathcal{U} \) be an equiuniformity on a \( G \)-space \( X \). Then \( \delta_{\mathcal{U}} \) is a \( G \)-proximity (hence, the corresponding Smirnov compactification and also the Samuel compactification \( s(X, \mathcal{U}) \to sX \) are proper \( G \)-compactifications).

Proof. Let \( A \delta_{\mathcal{U}} B \). By Definition 2.9, there exists an entourage \( \varepsilon \in \mathcal{U} \), such that \( (A \times B) \cap \varepsilon = \emptyset \). Fix \( g_0 \in G \). Then we claim that there exist \( \varepsilon' \in \mathcal{U} \) and a neighborhood \( V \) of \( g_0 \) in \( G \) such that \( V^{-1}A \) and \( V^{-1}B \) are \( \varepsilon' \)-far (this means that \( V^{-1}A \not\delta_{\mathcal{U}} V^{-1}B \)).

Since \( \mathcal{U} \) is an equiuniformity, it follows that for \( \varepsilon \in \mathcal{U} \) we can choose \( \varepsilon' \in \mathcal{U} \) and \( V \in N_{g_0} \) such that

\[
(3.1) \quad (x, y) \in \varepsilon' \implies (g_1x, g_2y) \in \varepsilon, \quad \forall g_1, g_2 \in V.
\]

Now we claim that \( (V^{-1}A \times V^{-1}B) \cap \varepsilon' = \emptyset \). Assuming the contrary, we get

\[
\varepsilon' \cap (V^{-1}A \times V^{-1}B) \neq \emptyset \implies \exists (x, y) \in (V^{-1}A \times V^{-1}B) : (x, y) \in \varepsilon'.
\]
Therefore by definition of $V^{-1}A$ and $V^{-1}B$, we conclude
\[ \exists g', g'' \in V : (g'x, g''y) \in A \times B. \]
On the other hand by Formula (3.1) for $g', g'' \in V$, we have
\[ (x, y) \in \varepsilon' \implies (g'x, g''y) \in \varepsilon. \]
This means
\[ \forall \varepsilon \in U : (g'x, g''y) \in (A \times B) \cap \varepsilon. \]
Hence $(A \times B) \cap \varepsilon \neq \emptyset$, a contradiction. \qed

**Fact 3.5.** (Completion theorem [30]) Let $(X, \mathcal{U}) \in \text{Unif}^G$. Then the action $G \times X \to X$ continuously can be extended to the action on the completion $\hat{G} \times \hat{X} \to \hat{X}$, where $(\hat{X}, \hat{\mathcal{U}}) \in \text{Unif}^G$ and $\hat{G}$ is the Raikov completion of $G$.

**Corollary 3.6.** Let $G_1 \subseteq G$ be a dense subgroup of $G$. Then for every Tychonoff $G$-space $X$ the maximal equivariant compactifications $\beta_G X$ and $\beta_{G_1} X$ are the same.

Every compact $G$-space (with its unique uniform structure) is equiuniform.

**Corollary 3.7.** For totally bounded uniformities we have the coincidence $\text{EUnif}^G = \text{Unif}^G$.

The following lemma with full proofs can be found only in my dissertation [26].

**Lemma 3.8.** [25, 26, 28] Let $X$ be a $G$-space with a topologically compatible uniformity $\mathcal{U}$. Assume that $\mathcal{U}$ is quasiuniform. Then there exists a topologically compatible uniformity $\mathcal{U}^G \subseteq \mathcal{U}$ on $X$ such that $\mathcal{U}^G$ is bounded. Furthermore,

1. if $(X, \mathcal{U}) \in \text{Unif}^G$ then $(X, \mathcal{U}^G) \in \text{EUnif}^G$;
2. if $(X, \mu) \in \text{EUnif}^G$ and $\mu \subseteq \mathcal{U}$ then $\mu \subseteq \mathcal{U}^G$.
3. if $\mathcal{U}$ is totally bounded and $(X, \mathcal{U}) \in \text{Unif}^G$, then $\mathcal{U} = \mathcal{U}^G$ (is an equiuniformity);
4. if $\mathcal{U}$ and $G$ are metrizable, then $\mathcal{U}^G$ is also metrizable.

**Proof.** For every $U \in N_\varepsilon$ and $\varepsilon \in \mathcal{U}$, consider
\[ (3.2) \quad [U, \varepsilon] := \{(x, y) \in X \times X : \exists u_1, u_2 \in U \quad (u_1 x, u_2 y) \in \varepsilon\}. \]
Then $\Delta_X \subseteq \varepsilon \subseteq [U, \varepsilon]$ and $[U_1, \varepsilon_1] \subseteq [U_2, \varepsilon_2]$ for every $U_1 \subseteq U_2, \varepsilon_1 \subseteq \varepsilon_2$.
It follows that $[U_1 \cap U_2, \varepsilon_1 \cap \varepsilon_2] \subseteq [U_1, \varepsilon_1] \cap [U_2, \varepsilon_2]$. It is also easy to see that if $U = U^{-1}$ and $\varepsilon^{-1} = \varepsilon$ are symmetric, then also $[U, \varepsilon^{-1}] = [U, \varepsilon]$ is symmetric.

The system $\alpha := \{[U, \varepsilon] \mid U \subseteq N_\varepsilon, \varepsilon \in \mathcal{U}\}$ is a filter base on the set $X \times X$.

Define by $\mathcal{U}^G$ the corresponding filter generated by $\alpha$. We show that $\mathcal{U}^G$ is a uniformity on the set $X$. The conditions (1),(2),(3) of Lemma 2.8 are satisfied. We have to show only condition (4) for the members of the base $\alpha$.

Let $[U, \varepsilon] \in \alpha$. We have to show that there exists $[V, \delta] \in \alpha$ such that
\[ [V, \delta] \circ [V, \delta] \subseteq [U, \varepsilon]. \]
Choose $\varepsilon_1 \in \mathcal{U}$ such that $\varepsilon_1^2 \subseteq \varepsilon$. By the quasiuniformity condition for $\varepsilon_1$ there exist $\delta \in \mathcal{U}$ and $V \in N_\varepsilon$ such that
\[ (x, y) \in \delta, v \in V \implies (vx, vy) \in \varepsilon_1. \]
Without loss of generality (by properties of topological groups), we can assume in addition that
\[ V = V^{-1}, \quad V^2 \subseteq U. \]
We check now that $[V, \delta]^2 \subseteq [U, \varepsilon]$. Let $(x, y), (y, z) \in [V, \delta]$. Then there exist $v_1, v_2, v_3, v_4 \in V$ such that
\[ (v_1 x, v_2 y), (v_3 x, v_4 y) \in \delta. \]
Then by (3.3) (taking into account that $V = V^{-1}$), we get
\[(v_2^{-1}v_1x, y) \in \varepsilon_1, \quad (y, v_3^{-1}v_4z) \in \varepsilon_1.\]

Therefore,
\[(v_2^{-1}v_1x, v_3^{-1}v_4z) \in \varepsilon_1 \circ \varepsilon_1 \subseteq \varepsilon.\]

Since $v_2^{-1}v_1$ and $v_3^{-1}v_4$ both are in $V^{-1}V \subseteq U$, we conclude that $(x, z) \in [U, \varepsilon]$.

It is easy to see the other axioms. So $\mathcal{U}^G$ is a uniformity.

**The uniformity $\mathcal{U}^G$ is topologically compatible with $X$.** That is, $\text{top}(\mathcal{U}^G) = \text{top}(\mathcal{U})$. Clearly, $\varepsilon \subseteq [U, \varepsilon]$ for all $U \in N_\varepsilon, \varepsilon \in \mathcal{U}$. Hence, $\mathcal{U}^G \subseteq \mathcal{U}$. As to the inverse direction $\mathcal{U}^G \supseteq \mathcal{U}$, one may show that for every $\varepsilon \in \mathcal{U}$ and $x_0 \in X$ there exist $[V, \delta] \in \alpha$ such that $[V, \delta](x_0) \subseteq \varepsilon(x_0)$. Indeed, using the continuity of the action, one may choose $\delta, \gamma \in \mathcal{U}$ and $V \in N_\varepsilon$ such that:

a) $g \gamma(x_0) \subseteq \varepsilon(x_0)$ \quad $\forall g \in V$

b) $(gx_0, x_0) \in \delta$ \quad $\forall g \in V$

c) $\delta^2 \subseteq \gamma, V = V^{-1}, \gamma = \gamma^{-1}$

Now, if $y \in [V, \delta](x_0)$ then $(v_1y, v_2x_0) \in \delta$ for some $v_1, v_2 \in V$. We obtain that $(v_1y, x_0) \in \delta \circ \delta \subseteq \gamma$. Hence, $v_1y \in \gamma(x_0)$. So, $y \in v_1^{-1} \gamma(x_0) \subseteq \varepsilon(x_0)$.

$\mathcal{U}^G$ is bounded. Indeed, $(x, ux) \in [U, \varepsilon]$ for every $x \in X$ and every $u \in U$ (because if we choose $u_1 := u, u_2 := e \in U$ then $(u_1x, u_2ux) = (ux, ux) \in \Delta_X \subseteq \varepsilon$).

$\mathcal{U}^G$ is saturated if $\mathcal{U}$ is saturated. Let $g_0 \in G$. Then $g_0^{-1}Ug_0 \in N_\varepsilon$ for every $U \in N_\varepsilon$ and $g_0^{-1}\varepsilon \in \mathcal{U}$ for every $\varepsilon \in \mathcal{U}$ because $\mathcal{U}$ is saturated. Now observe that $g_0[U, \varepsilon] = [g_0^{-1}Ug_0, g_0^{-1}\varepsilon]$.

The assertions (1), (2) and (4) easily follow now from the construction. In order to check (3) consider the completion $(X, \mathcal{U}) \mapsto (\hat{X}, \hat{\mathcal{U}})$, which, in fact is a compactification because $\mathcal{U}$ is totally bounded. According to the completion theorem [30] we have $(\hat{X}, \hat{\mathcal{U}}) \in \text{Unif}^G$ and the action $G \times \hat{X} \to \hat{X}$ is continuous. Now, by Remark 3.3.1, we obtain that $\mathcal{U}$ is an equiuniformity. Hence, by (2) we conclude that $\mathcal{U} = \mathcal{U}^G$. \[\square\]

The equivalence of (1) and (2) in the following result is an old result which goes back at least to R. Brook [8] and J. de Vries [30].

**Theorem 3.9.** [26] Let $X$ be a Tychonoff $G$-space. The following are equivalent:

1. $X$ is $G$-Tychonoff.
2. $(X, \mathcal{U}) \in \text{Unif}^G$ for some compatible uniform structure $\mathcal{U}$ on $X$.
3. There exists a compatible uniform structure $\mathcal{U}$ on $X$ which is quasibounded.

**Proof.** (1) $\Rightarrow$ (2) Let $X$ be $G$-Tychonoff. Consider a proper $G$-compactification $\alpha : X \mapsto Y$. Since the natural uniformity on $Y$ is an equiuniformity, then it induces on $X$ a (precompact) equiuniformity $\mathcal{U}$. So, $(X, \mathcal{U}) \in \text{EUnif}^G$ with respect to some compatible uniformity $\mathcal{U}$. Now recall that $\text{EUnif}^G \subseteq \text{Unif}^G$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let $\xi$ be a quasibounded uniformity on $X$. Then one may easily find a stronger quasibounded uniformity $\mathcal{U}$ which, in addition, is $G$-saturated. Indeed, the system $\Sigma := \{gx : g \in G, \varepsilon \in \mathcal{U}\}$, where $gx := \{(gx, gy) \in X \times X : (x, y) \in \varepsilon\}$ is a subbase of a filter of subsets in $X \times X$. Denote by $\mathcal{U}$ the corresponding filter generated by $\Sigma$. Then $\mathcal{U}$ is a saturated uniformity on $X$, $\xi \subseteq \mathcal{U}$ and $\text{top}(\xi) = \text{top}(\mathcal{U})$. If $\Sigma$ is quasibounded or bounded, it is straightforward to show (use that the conjugations are continuous in any topological group), then $\mathcal{U}$ respectively is quasibounded or bounded.
Hence, we obtain \((X, \mathcal{U}) \in \text{Unif}^G\). Then \((X, \xi) \in \text{EUnif}^G\) according to Lemma 3.11. By Proposition 3.14, \(\delta_{\mathcal{U}}\) is a \(G\)-proximity (hence, the corresponding Smirnov’s compactification is a \(G\)-compactification).

\textbf{Corollary 3.10.} (Ludescher–de Vries [23]) Every continuous uniformly equicontinuous action of a topological group \(G\) on \((X, \mathcal{U})\) is \(G\)-Tychonoff. In particular, it is true if \(X\) admits a \(G\)-invariant metric.

\textbf{Lemma 3.11.} Let \(\pi : G \times X \to X\) be a continuous action and \(\Sigma := \{d_i\}_{i \in I}\) be a bounded system of pseudometrics on \(X\) such that the induced uniform structure \(\mathcal{U}\) on \(X\) is topologically compatible. Assume that \(M\) is a family of nonempty subsets in \(G\) such that:

\[\forall A \in M \ \forall d_i \in \Sigma \ \forall x, y \in X \ \ d_{A, i}(x, y) := \sup_{g \in A} d_i(gx, gy) < \infty.\]

Define by \(\Sigma_M\) the system of pseudometrics \(\{d_{A, i} : A \in M, i \in I\}\) on \(X\). Let \(\xi = \xi(\Sigma, M)\) be the corresponding uniform structure on \(X\) generated by the system \(\Sigma_M\).

1. If every \(A \in M\) acts equicontinuously on \((X, \mathcal{U})\), then \(\text{top}(\xi) = \text{top}(\mathcal{U})\).
2. If for every \(A \in M\) there exist \(V \in \mathcal{N}_e\) and \(B \in M\) such that \(AV \subseteq B\), then the action is \(\xi\)-quasibounded.
3. If there exists \(A \in M\) such that \(e \in A\), then \(\mathcal{U} \subseteq \xi\).

\textbf{Proof.} (1) and (3) are straightforward.

(2) Observe that if \(A \subseteq B\) then \(d_{A, i}(x, y) \leq d_{B, i}(x, y)\). Therefore, if \(VA \subseteq B\) then

\[d_{A, i}(vx, vy) = \sup_{g \in A} d_i(gvx, gvy) \leq \sup_{g \in AV} d(tx, ty) \leq \sup_{g \in B} d(tx, ty) = d_{B, i}(x, y).\]

For every triple \(A \in M, i \in I, \varepsilon > 0\) (such triples control the natural uniform subbase of \(\xi\)), we have

\[d_{B, i}(x, y) < \varepsilon \Rightarrow d_{A, i}(vx, vy) < \varepsilon \ \forall v \in V.\]

\textbf{Theorem 3.12.} (de Vries [54, 55] and also [26]) Let \(G \times X \to X\) be a continuous action. Suppose that a neighborhood \(U\) of \(e\) acts equicontinuously on \(X\) with respect to some compatible uniformity \(\mathcal{U}\). Then \(X\) is \(G\)-Tychonoff.

\textbf{Proof.} By Theorem 3.11, it is enough to show that there exists a compatible finer uniformity \(\xi \supseteq \mathcal{U}\) on the topological space \(X\) which is quasibounded.

By our assumption there exists a neighborhood \(U\) of \(e\) in \(G\) and a compatible uniformity \(\xi\) on the topological space \(X\) such that \(U\) acts equicontinuously on \((X, \mathcal{U})\). Choose a sequence \(U_n \in \mathcal{N}_e\) such that \(U_n^{-1} = U_n, U_{n+1}^2 \subseteq U_n \subseteq U\) for every \(n \in \mathbb{N}\). Now define inductively the sequence \(M := \{V_n\}_{n \in \mathbb{N}}\) of subsets in \(G\) where

\[V_n := U_1 U_2 \cdots U_n.\]

Choose also a family of pseudometrics \(\Sigma := \{d_i\}_{i \in I}\) on \(X\) such that \(\Sigma\) generates the uniformity \(\mathcal{U}\). One may assume that \(d_i \leq 1\) for every \(i\). Now we define the uniformity \(\xi\) as in Lemma 3.11 generated by the system of pseudometrics \(\Sigma_M\).

For every Tychonoff space \(X\) there exists the greatest compatible uniformity on \(X\). We denote it by \(\mathcal{U}_{\text{max}}\).

\textbf{Theorem 3.13.} [26] Let \(G\) be a locally compact group. Then for every \(G\)-space \(X\) we have \((X, \mathcal{U}_{\text{max}}) \in \text{Unif}^G\).\hfill \(\Box\)

\textbf{Proof.} According to the proof of Theorem 3.11 there exists a compatible finer uniformity \(\xi \supseteq \mathcal{U}_{\text{max}}\) on the topological space \(X\) which is quasibounded. Then by the maximality of \(\mathcal{U}_{\text{max}}\) we have \(\xi = \mathcal{U}_{\text{max}}\). Therefore \(\mathcal{U}_{\text{max}}\) is quasibounded. In fact, again by the maximality property we obtain that \(\mathcal{U}_{\text{max}}\) is also saturated. Hence, \((X, \mathcal{U}_{\text{max}}) \in \text{Unif}^G\). \hfill \(\Box\)
Combining Theorems 3.9 and 3.13, one directly gets the following well-known important result of de Vries:

**Fact 3.14.** (J. de Vries [53]) Let $G$ be a locally compact group. Then every Tychonoff $G$-space is $G$-compactifiable.

**Fact 3.15.** Here we list several sufficient conditions of $G$-compactifiability. Some of these results were already mentioned above.

1. [50] Every coset $G$-space $G/H$.
2. [50] Every locally compact $G$-space $X$.
3. [53] Every $G$-space $X$, where $G$ is locally compact.
4. [28, 29] Let $G$ and $X$ both be topological groups and $\alpha: G \times X \to X$ is a continuous action by group automorphisms. Then $X$ is $G$-Tychonoff (see Remark 3.3.4 and Corollary 4.2).
5. [28] For every metric $G$-space $(X, d)$, where $G$ is a Baire space and every $g$-translation is $d$-uniform we have $(X, \mathcal{U}(d)) \in \text{Unif}^G$ (and $X$ is $G$-compactifiable).
6. [28] If $G$ is Baire then every metrizable $G_{\text{discr}}$-compactification of a $G$-space $X$ is a $G$-compactification.
7. [49] Every $G$-space $X$, where the action is algebraically transitive, $X$ is Baire and $G$ is $\aleph_0$-bounded. More generally, every $d$-open action.
8. [23] Every metric space $X$ with a $G$-invariant metric. More generally, every continuous uniformly equicontinuous action of a topological group $G$ on $(X, \mathcal{U})$.
9. (53, 55 and also 26) Let $G \times X \to X$ be a continuous action. Suppose that a neighborhood $U$ of $e$ acts equicontinuously on $X$ with respect to some compatible uniformity $\mathcal{U}$. Then $X$ is $G$-Tychonoff.
10. (Theorem 3.18 below) Every ordered $G_{\text{discr}}$-compactification of a $G$-space $X$ is a $G$-compactification.

### 3.1. Linearly ordered $G$-compactifications

In this section, a linearly ordered topological space (LOTS) $X$ will mean that $X$ is a topological space which topology is just the interval topology for some linear order on $X$. We show that every linearly ordered $G_{\text{discr}}$-compactification of a $G$-space $X$ with the interval topology is necessarily a $G$-compactification.

Recall the following result of V. Fedorchuk which gives an analog of Smirnov’s theorem for linearly ordered compactifications.

**Definition 3.16.** [13] Let $\leq$ be a linear order on $X$. A proximity $\delta$ on $X$ is said to be an ordered proximity (with respect to $\leq$) if $\delta$ induces the interval topology $\tau_{\leq}$ on $X$ and the following two properties are satisfied:

1. (op1) for every $x < y$ we have $(-\infty, x] \delta [y, +\infty)$;
2. (op2) for every $A \delta B$ there exists a finite number $O_i$, $i \in \{1, 2, \cdots, n\}$ of open $\leq_X$-convex subsets $^2$ such that $A \subset \bigcup_{i=1}^n O_i \subset X \setminus B$.

**Fact 3.17.** (V. Fedorchuk [13]) Let $\alpha: X \to Y$ be a compactification of a LOTS $X$ and $\delta$ be the corresponding proximity on $X$. The following conditions are equivalent:

1. There exists a linear order $\leq_Y$ on $Y$ such that $Y$ is LOTS.
2. The proximity $\delta$ of $\alpha$ is an ordered proximity with respect to the linear order $\leq_X$ on $X$ inherited from $\leq_Y$.

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$^2$as usual, $C$ is said to be convex if $a, b \in C$ implies that the interval $(a, b)$ is a subset of $C$
Note that if $\alpha: X \to Y$ is a compactification, where $(Y, \tau)$ is compact with respect to some linear order $\leq_Y$ on $Y$ (i.e., $\tau = \tau_{\leq_Y}$), then the subspace topology on $X$, in general, is stronger than the interval topology of the inherited order $\leq_X$ on $X$. The coincidence $\tau_{\leq_X} = \tau|_X$ we have iff the proximity $\delta$ of $\alpha$ is an ordered proximity.

**Theorem 3.18.** Let $X$ be a linearly ordered space with the interval topology of a linear order $\leq$. Let $G \times X \to X$ be a continuous action which preserves the order $\leq$. Assume that $\delta$ is an ordered proximity of a linearly ordered compactification $\alpha: (X, \leq) \to (Y, \leq_Y)$ such that $\alpha$ is a $G$-compactification (i.e., $\delta$ is $G$-invariant). Then $\alpha$ is a $G$-compactification.

**Proof.** Since $\delta$ is already $G$-invariant, it is enough to show (by Smirnov’s theorem, Fact 2.4) that the proximity is compatible with the action. That is, if $A \delta B$ then there exists $U \in \mathcal{N}_e$ such that $UA \cap B = \emptyset$. In condition (b) of Fact 3.17, we may assume that the open convex subsets $O_i$ are disjoint. Let $A_i := A \cap O_i$ for every $i \in \{1, 2, \cdots, n\}$. Then $A_i = A \setminus \bigcup_{i \neq i_0} O_i$ is closed in $X$. Since we have finitely many $i$, it is enough to prove the following (this will cover also condition (a) in the definition of ordered proximity).

**Claim.** Let $A$ be a closed subset of $X$ and $O$ is an open $\leq$-convex subset of $X$ which contains $A$. Then there exists $U \in \mathcal{N}_e$ such that $UA \subset O$.

**Proof:** We may assume that $O \neq \emptyset, O \neq X$. Consider other cases for convex open subsets.

(a) $O = (-\infty, b)$. Two subcases:

(a1) If $[b, \infty)$ is open. By the continuity of the action, there exists $U \in \mathcal{N}_e$ such that $Ub \subset [b, \infty)$. For every $b \leq x$ and every $u \in U$ we have $b \leq ub \leq ux$ (action preserves the order). Hence, $U[b, \infty) \subset [b, \infty)$.

(a2) If $[b, \infty)$ is not open. That is, $b$ is not an internal point of $[b, \infty)$. Then every open neighborhood of $b$ meets $(-\infty, b)$. Choose a neighborhood $W$ of $b$ which is disjoint with the closed set $A$. Since $X$ carries the interval topology $\tau_{\leq}$, one may assume that $W = (c, d)$. Then $a < x$ for every $a \in A$ and every $x \in (c, d)$. Choose $x_0 \in (c, d) \cap (-\infty, b)$. There exists $U \in \mathcal{N}_e$ such that $U^{-1} = U$ and $UX_0 \subset (c, d)$. Then $a \leq c < ux_0 < ux$ for every $x \in [b, \infty)$ and $u \in U$. So, $UA = U^{-1}A$ and $[b, \infty)$ are disjoint.

(b) $O = (b, \infty)$. This case is completely similar to (a).

(c) $O = (-b_1, b_2)$. Combine (a), (b) taking the intersection of two neighborhoods of $e$.

(d) $O = (-\infty, b]$. Then, since $O$ is open, $b$ is an internal point of $(-\infty, b]$. There exists $U \in \mathcal{N}_e$ such that $Ub \subset (-\infty, b]$. Then $UA \subset U(-\infty, b] \subset (-\infty, b]$.

(e) $O = [b, \infty)$. Similar to (d).

(f) $O = [b_1, b_2]$. Combine (d) and (e). $\square$

### 4. $G$-Compactifications and Proximities

**Theorem 4.1.** Let $(X, \mathcal{U}) \in \text{Unif}^G$, where $G$ is an arbitrary topological group. Then the following rule

\[ A \nu B \iff \forall V \in \mathcal{N}_e \quad VA \cap \mathcal{U} \cap VB \]

defines a $G$-proximity on $X$ such that $\delta_{\mathcal{U}|G} = \nu$ and it is the greatest $\mathcal{U}$-uniform $G$-compactification of $X$ (that is, $\rho \leq \nu$ for any $G$-proximity $\rho \leq \delta_{\mathcal{U}}$).
Proof. We have to show that $\delta_{U\mathcal{C}} = \nu$. That is, $\nu$ coincides with the canonical proximity $\delta_{U\mathcal{C}}$ of the uniformity $U^{G}$ (where $U^{G}$ is defined in Lemma 3.8).

Let $A$ and $B$ be $U^{G}$-far subsets in $X$. That is, $A \delta_{U\mathcal{C}} B$. Lemma 3.8 guarantees that $(X, U^{G}) \in E\text{Unif}^{G}$. Since $U^{G}$ is an equiuniformity, its proximity $\delta_{U\mathcal{C}}$ is a $G$-proximity by Proposition 3.4. Hence,

\[ \exists V \in N_{\epsilon} \quad VA \overline{\delta_{U\mathcal{C}}} VA. \]

Since $U^{G} \subseteq U$, we have $\delta_{U\mathcal{C}} \leq \delta_{U}$. Therefore, $VA \delta_{U} VB$. By definition of $\nu$ this means that $A \mathcal{C} B$.

In the converse direction, we assume now that $A \mathcal{C} B$. That is, $VA \mathcal{C} VB$ for some $V \in N_{\epsilon}$. There exists $\epsilon \in U$ such that

\[ \epsilon \cap (VA \times VB) = \emptyset. \]

This implies that

\[ [V, \epsilon] \cap (A \times B) = \emptyset, \]

where, as in Lemma 3.8, $[V, \epsilon] = \{(x, y) \in X \times X : \exists v_{1}, v_{2} \in U \quad (v_{1}x, v_{2}y) \in \epsilon\}$. By definition of the uniformity $U^{G}$, this means that $A \delta_{U\mathcal{C}} B$. So, we can conclude that $\nu = \delta_{U\mathcal{C}}$ and $\nu$ is a $G$-proximity (because, $\delta_{U\mathcal{C}}$ is).

Since, $U^{G} \subseteq U$, we have $\nu = \delta_{U\mathcal{C}} \leq \delta_{U}$. This shows that the $G$-compactification $\nu$ of $X$ is $\mathcal{U}$-uniform.

Finally we show the maximality property of $\nu$. Let $\rho$ be any $\mathcal{U}$-uniform $G$-compactification and $A \mathcal{C} B$. Then by Smirnov’s theorem [24] (and Remark [25]), there exists $V \in N_{\epsilon}$ such that $VA \mathcal{C} VB$. Since $\rho$ is $\mathcal{U}$-uniform, we have $\rho \leq \delta_{U}$. Therefore, $VA \mathcal{C} VB$ holds. So, by Equation 4.1, we can conclude that $VA \mathcal{C} VB$. This means that $\rho \leq \delta_{U\mathcal{C}} = \nu$. □

Corollary 4.2. Let $G$ and $X$ both be topological groups and $\alpha : G \times X \to X$ is a continuous action by group automorphisms. Then the following condition

\[ A \mathcal{C} B \Leftrightarrow \exists V \in N_{\epsilon}(X) \quad \exists U \in N_{\epsilon}(G) \quad V \{g(A)\}_{g \in U} \cap V \{g(B)\}_{g \in U} = \emptyset \]

defines a $G$-proximity which is the greatest $\mathcal{R}(X)$-uniform $G$-compactification, where $\mathcal{R}(X)$ is the right uniformity of $X$.

Proof. Observe that $(\{X, \mathcal{R}(X)\}, \alpha) \in \text{Unif}^{G}$. Now apply Theorem 4.1 to $(X, \mathcal{R}(X))$. □

Definition 4.3. Let $(X, d)$ be a metric space and $\pi : G \times X \to X$ is any action with uniform translations. We say that this action is $d$-majored if the greatest $G$-compactification of $X$ is $d$-uniform.

Proposition 4.4. Let $(X, d)$ be a metric space and a $G$-space such that $(X, U(d)) \in \text{Unif}^{G}$ (e.g., $d$ is $G$-invariant). Then the following condition

\[ A \mathcal{C} B \Leftrightarrow \exists V \in N_{\epsilon} \quad d(VA, VB) > 0 \]

defines a $G$-proximity which is the greatest $d$-uniform $G$-compactification of $X$ (and coincides with the proximity of $U(d)^{G}$).

If, in addition, the action is $d$-majored then 4.2 describes the proximity of $\beta_{G}X$.

Proof. Let $U(d)$ be the uniformity of the metric $d$. Its proximity $\delta_{d}$ is defined as follows:

\[ A\delta_{d}B \Leftrightarrow d(A, B) = 0. \]

Now apply Theorem 4.1 to $(X, U(d))$. □
Remark 4.5. Let $X := (\mathbb{U}_1, d)$ be the Urysohn sphere and $G := \text{Iso}(\mathbb{U}_1)$ be the Polish isometry group (pointwise topology). In the joint work [16] with T. Ibarlucia, we prove that for the $G$-space $X$ the maximal $G$-compactification of $X$ is just the Gromov compactification $\gamma(\mathbb{U}_1, d)$ (in particular, $\beta_G(\mathbb{U}_1)$ is metrizable) and $\text{RUC}_G(\mathbb{U}_1)$ is the unital algebra generated by the distance functions. Since $\gamma: (X, d) \to \gamma X$ is a $d$-uniformly continuous topological $G$-embedding, we obtain that the greatest $G$-compactification of $X$ is $d$-uniform. So, the action is $d$-majored.

Theorem 4.6. Let $X := (\mathbb{U}_1, d)$ be the Urysohn sphere and $G := \text{Iso}(\mathbb{U}_1)$. Then for closed subsets $A, B$ in $\mathbb{U}_1$ we have:

$$A \beta_G B \iff \exists V \in \mathcal{N}_e(G) \quad d(VA, VB) > 0.$$  

Proof. Combine Remark 4.5 and Proposition 4.4. □

Remark 4.7. Theorem 4.6 remains true for a large class of all $\aleph_0$-categorical metric $G$-structures $M = (X, d)$ (including Urysohn sphere), where $G := \text{Aut}(M, d)$ is its automorphism group (for definitions, motivation and related tools, we refer to [6] or, [16]). In this case the action is $d$-majored as it follows from [16 Theorem 4.4].

Another condition which guarantees that the action is $d$-majored, is the uniform micro-transitivity of the action in the sense of [16].

**Theorem 4.8.** (Maximal $G$-compactification for locally compact $G$ actions)

Let $G$ be a locally compact group. Then for every $G$-space $X$ the maximal $G$-proximity $\beta_G$ can be characterized by the maximal topological proximity $\beta$ as follows:

$$A \beta_G B \iff \forall V \in \mathcal{N}_e \quad VA \beta VB.$$  

So, if $X$, as a topological space is normal, then we obtain

$$A \beta_G B \iff \forall V \in \mathcal{N}_e \quad Vcl(A) \cap Vcl(B) \neq \emptyset.$$  

Proof. By Theorem 3.13 for the maximal compatible uniformity, we have $(X, \mathcal{U}_{\text{max}}) \in \text{Unif}^G$. In fact, $\mathcal{U}_{\text{max}}$ is the greatest compatible uniformity on $X$. Therefore, its proximality defines just the usual maximal compactification $\beta X$. We use the notation $\beta$ for this proximity. Note that $VA \beta VB$ means that $VA$ and $VB$ cannot be functionally separated. Now, we use Theorem 4.14 in order to complete the proof.

If $X$ is normal then the condition $VA \beta VB$ means that $\text{cl}(VA) \cap \text{cl}(VB) \neq \emptyset$ (see Example 2.2.3). Since $G$ is locally compact, we can suppose that $V$ is compact. Hence $VM$ is closed for every closed subset $M \subset X$. In particularly, we get $Vcl(A) = \text{cl}(VA)$ and $Vcl(B) = \text{cl}(VB)$. □

Remark 4.9. Local compactness of $G$ is essential. Indeed, for every Polish group $G$ which is not locally compact, there exists a second countable $G$-Tychonoff space $X$ and closed disjoint (hence, far in $\beta X$) $G$-invariant subsets $A, B$ such that $A \beta_G B$ (cannot be separated by $\text{RUC}_G$ functions). This follows from the proof of [34 Theorem 4.3].

Since the proof of [34 Theorem 4.3] is quite complicated, we give here a concrete simpler example for the sake of completeness. The idea is similar to [27] (see also [32]).

**Example 4.10.** Let $I = [0, 1]$ be the unit interval and

$$G_1 = H_+ [0, 1] = \{ g \in \text{Homeo} (I) : g(0) = 0, g(1) = 1 \}.$$  

Denote by $\pi_1: G_1 \times I \to I$ the natural action of $G_1$ on $I$. Then $\text{cl}(G_1 O) = [0, 1]$ for every neighborhood of 0.

Let $\{(G_n, I_n, \pi_n) : n \in \mathbb{N} \}$ be a countable system of $\text{ttg}$’s, where each $(G_n, I_n, \pi_n)$ is a copy of $(G_1, I, \pi_1)$. Consider the special equivariant sum $(G, X, \pi)$ of the actions $\pi_n$. So, $G =$
\[ \prod_{k \in \mathbb{N}} G_k \text{ and } X = \bigsqcup_{k \in \mathbb{N}} I_k. \] Clearly, \( G \) is a Polish group and \( X \) is a separable metrizable \( G \)-space. Define two naturally defined subsets \( A, B \) of \( X \), where \( A := \{ i_n(0) : n \in \mathbb{N} \} \) is the set of all left end-points and \( B := \{ i_n(1) : n \in \mathbb{N} \} \) is the set of all right end-points. Then \( X \), being locally compact, clearly is \( G \)-Tychonoff. The subsets \( A, B \) are closed disjoint and \( G \)-invariant subsets in \( X \). So, \( GA \cap GB = \emptyset \). However, they cannot separated by any RUC\(_G\) function. Hence we have \( A \beta_G B \).

**Equivariant normality.** Two subsets \( A, B \) of a \( G \)-space \( X \) are said to be \( \pi \)-disjoint if \( UA \cap UB = \emptyset \) for some \( U \in N_e \). It is equivalent to require that \( VA \cap B = \emptyset \) for some \( V \in N_e \).

We introduced the following definition in order to answer some questions of Yu.M. Smirnov. Among others, to have a generalized *Urysohn Lemma* and a *Tietze extension theorem* for \( G \)-spaces.

**Definition 4.11.** [24] [26] Let \( G \) be a topological group \( G \). A \( G \)-space \( X \) is \( G \)-normal (or, *equinormal*) if for every pair of \( \pi \)-disjoint closed subsets \( A_1, A_2 \) in \( X \) there exists a pair of disjoint neighborhoods \( O_1, O_2 \) in \( X \) such that \( O_1 \) and \( O_2 \) are \( \pi \)-disjoint.

This concept is closely related to \( G \)-proximities as the following result shows.

**Fact 4.12.** The following are equivalent:

1. \( X \) is \( G \)-normal;
2. every pair of \( \pi \)-disjoint closed subsets \( A_1, A_2 \) in \( X \) can be separated by a function \( f \in \text{RUC}_G(X) \);
3. the relation
   \[ A \delta_n B \iff \forall V \in N_e \quad V cl(A) \cap V cl(B) \neq \emptyset \]
   is a proximity on \( X \);
4. the relation from (2) is a \( G \)-proximity on \( X \);
5. the relation from (2) is the maximal \( G \)-proximity \( \beta_G \) on \( X \).

Every \( G \)-normal is \( G \)-Tychonoff. The action of \( G := \mathbb{Q} \) on \( X := \mathbb{R} \) is not \( G \)-normal. By [34], \( G \) is locally compact if and only if every normal \( G \)-space is \( G \)-normal. For some additional properties of \( G \)-normality, we refer to [1] and [32].

**Description of \( \beta_G X \) by filters.**

**Remark 4.13.** For every proximity space \( (X, \delta) \) there exists the *Smirnov’s compactification* \( s_\delta: X \to sX \),

where \( sX \) is the set of all \( \delta \)-ends (maximal centered \( \delta \)-systems) \( \xi \). See [40] for details. Recall that \( \delta \)-system means that every member \( A \in \xi \) is a \( \delta \)-neighborhood of some \( B \in \xi \). That is, \( B \supseteq A \) holds (meaning that \( B \) and \( A^c := X \setminus A \) are \( \delta \)-far).

Let us apply this to the case of \( \beta_G \) for a \( G \)-space \( X \).

- If \( G \) is locally compact then by Corollary 4.13 \( B \beta_G A^c \) if and only if \( VB \) and \( VA^c \) are functionally separated for some \( V \in N_e \).
- Similar results for an arbitrary topological group \( G \) is not true in general. However, it is true if, in addition, the action is \( G \)-normal in the sense of Definition 4.11.
- If \( (M, d) \) is a \( \aleph_0 \)-categorical metric structure, then by Remark 4.7 \( B \beta_G A^c \) if and only if \( d(VA^c, VB) > 0 \) for some \( V \in N_e(\text{Aut } (M)) \).
5. Additional notes about maximal $G$-compactifications

5.1. Coset spaces and the greatest ambit. For every coset $G$-space $X := G/H$, the standard right uniformity $U_r$ (see, for example, [39]) is the largest possible topologically compatible equiuniformity on the $G$-space $G/H$. So, $G/H$ is a $G$-Tychonoff space (de Vries [50]). Moreover, the Samuel compactification of the right uniform space $(G/H, U_r)$ is the greatest (proper) $G$-compactification. In particular, every topological group $G$ is $G$-compactifiable with respect to the standard left action (Book [8]). This $G$-space $\beta G$ is just the greatest ambit of $G$ which is widely used in topological dynamics.

Let us compare this $G$-compactification with the usual topological greatest compactification $G \to \beta G$. The translations in this case are continuous. So, it is a $G_{disc}$-compactification.

**Proposition 5.1.** Let $G$ be a metrizable topological group which is not precompact. Then the canonical action $G \times \beta G \to \beta G$ is continuous if and only if $G$ is discrete.

**Proof.** Let $G$ is not discrete. We have to show that there exists a pair of closed disjoint subsets $A, B$ which are near with respect to right uniformity $U_r$. Since $G$ is not precompact, there exists an infinite uniformly $U_r$-discrete sequence $\{a_n\}_{n=1}^{\infty}$. This means that

$$\exists U_0 \in N_e \text{ } U_0 x_n \cap U_0 x_m = \emptyset \text{ } \forall m \neq n.$$ 

Choose a symmetric neighborhood $V \in N_e$ such that $V^2 \subset U_0$. Since $G$ is metrizable and not discrete, one may choose a sequence $v_n \in V$ such that $\lim v_n = e$ and all members of this sequence are distinct. Define $A := \{a_n\}_{n=1}^{\infty}$, $B := \{v_n a_n\}_{n=1}^{\infty}$. Then $U A \cap B \neq \emptyset$ for all $U \in N_e$. Therefore, $A \beta G B$. On the other hand, $A$ and $B$ are closed disjoint subsets in the normal space $G$. Hence, $A \beta G B$.

Note that if $G$ is a pseudocompact group then $\beta G$ is a topological group naturally containing $G$ (see [9]). So, in this case, $G \times \beta G \to \beta G$ is continuous and $\beta G G = \beta G$.

**Massive actions.**

**Definition 5.2.** Let $\pi : G \times X \to X$ be an action of a topological group $G$ on a uniform space $(X, U)$. We say that the action is $U$-massive if the uniform structure $U^G$ (from Lemma 3.5) is totally bounded.

**Proposition 5.3.** Let $(X, U) \in \text{Unif}^G$. Consider the following conditions:

1. the greatest $U$-uniform $G$-compactification (induced by the proximity $\delta_{U^G}$) of $X$ is metrizable;
2. the action is $U$-massive.

Then always (1) $\Rightarrow$ (2). If, in addition, the uniformity $U$ is metrizable and $G$ is a metrizable topological group, then (2) $\Rightarrow$ (1).

**Proof.** (1) $\Rightarrow$ (2): By Theorem 4.1, the greatest $U$-uniform $G$-compactification of $X$ is the Smirnov compactification of the proximity $\delta_{U^G}$ (which is the same as the Samuel compactification of $U^G$).

Assume the contrary that $U$ is not $G$-massive. Then by Definition 5.2 $U^G$ is not totally bounded. Equivalently, $X$ contains an infinite sequence which is $U^G$-uniformly discrete. This implies that the corresponding Samuel compactification of $U^G$ is not metrizable.

(2) $\Rightarrow$ (1): Since $U$ and $G$ are metrizable, then $U^G$ is also metrizable. By (2), $U^G$ is totally precompact. Then its completion is metrizable. On the other hand, this completion is the greatest $U$-uniform $G$-compactification by Theorem 4.1. \(\square\)
Many naturally defined uniform structures are $G$-massive as it follows from the examples of Remark 1.3 making use of Proposition 5.3.

An extreme (but useful) sufficient condition is the case of the discrete uniform space $(X, \mathcal{U}_{\text{discr}})$. Let us say that the action is strongly $G$-massive if for every finite subset $F \subset X$ the stabilizer subgroup action $St_F \times X \to X$ has finitely many orbits.

**Examples 5.4.** Here we give some examples of strongly $G$-massive actions.

(a) $S_\infty \times \mathbb{N} \to \mathbb{N}$. In this case $\beta G X$ (of the discrete space $X := \mathbb{N}$ with the action of the Polish symmetric group $G := S_\infty$) is the Alexandrov compactification $\mathbb{N} \cup \{\infty\}$.

(b) $X = (\bar{\mathbb{Q}}, \leq)$ the rationals with the usual order but equipped with the discrete topology. Consider the automorphism group $G = \text{Aut}(\bar{\mathbb{Q}}, \leq)$ with the pointwise topology. In this case the action of $G$ on the discrete uniform space $(X, \mathcal{U}_{\text{discr}})$ is $G$-massive. Hence, $\beta G X$ is metrizable by Proposition 5.3. In fact, one may show that $X \to \beta G X$ is a proper $G$-compactification such that $\beta G X$ is a linearly ordered $G$-space. By Corollary 3.3 the same is true for every dense subgroup $G$ of $\text{Aut}(\mathbb{Q}, \leq)$ (for instance, Thompson’s group $F$).

**Sketch:** We use an idea of [33]. Let $F := \{t_1 < t_2 < \cdots < t_m\}$ be a finite chain in $\mathbb{Q}$. Using the ultrahomogeneity of the action $\text{Aut}(\bar{\mathbb{Q}}, \leq) \times (\bar{\mathbb{Q}}, \leq) \to (\bar{\mathbb{Q}}, \leq)$, the corresponding (finite) orbit of the stabilizer subgroup $St_F$ is

$$X_F := \{(-\infty, t_1), (t_1, t_2), (t_2, t_3), \cdots, (t_{m-1}, t_m), (t_m, \infty)\}.$$  

Therefore, the present action is strongly $G$-massive and in particular $\mathcal{U}_{\text{discr}}$-massive ($\mathcal{U}_{\text{discr}}^G$ is totally bounded). The proximity of the uniformity $\mathcal{U}_{\text{discr}}^G$ corresponds to $\beta G X$. On the other hand, the completion of $\mathcal{U}_{\text{discr}}^G$ can be realized as a certain inverse limit $X_\infty = \lim\limits_{\leftarrow} (X_F, I)$ of finite linearly ordered sets $X_F$, where $F \in I$ and the finite orbit space $X_F := X/\text{St}_F$ carries the natural linear order.

(c) A similar result is valid for the circular version of (b). Namely, for the rationals on the circle with its circular order $X = (\mathbb{Q}/\mathbb{Z}, \circ)$, the automorphism group $G = \text{Aut}(\mathbb{Q}/\mathbb{Z}, \circ)$ and its dense subgroups $G$ (for instance, Thompson’s circular group $T$). In this case $\beta G X \setminus X$ is the universal minimal $G$-space $M(G)$.

### 6. Some open questions

The following questions are still open.

**Question 6.1.** (S. Antonyan–M. Megrelishvili) Is it true that $\dim \beta G G = \dim G$ for every locally compact group $G$? What of $G$ is a Lie group?

**Question 6.2.** (Yu.M. Smirnov (see [31] and [3])) Let $G = \mathbb{Q}$ be the topological group of all rationals. Is it true that there exists a Tychonoff $G$-space which is not $G$-Tychonoff?

**Question 6.3.** (H. Furstenberg and T. Scarr (see [31] [38])) Let $G \times X \to X$ be a continuous action with one orbit (that is, algebraically transitive) of a (metrizable) topological group $G$ on a (metrizable) space $X$. Is it true that $X$ is $G$-Tychonoff?

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Department of Mathematics, Bar-Ilan University, 52900 Ramat-Gan, Israel.

Email address: megereli@math.biu.ac.il
URL: http://www.math.biu.ac.il/~megereli