The shape of invasion percolation clusters in random and correlated media

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Abstract. The shape of two-dimensional invasion percolation clusters are studied numerically for both non-trapping (NTIP) and trapping (TIP) invasion percolation processes. Two different anisotropy quantifiers, the anisotropy parameter and the asphericity, are used for probing the degree of anisotropy of clusters. We observe that, in spite of the difference in scaling properties of NTIP and TIP, there is no difference in the values of anisotropy quantifiers of these processes. Furthermore, we find that, in completely random media, the invasion percolation clusters are on average slightly less isotropic than standard percolation clusters. Introducing isotropic long-range correlations into the media reduces the isotropy of the invasion percolation clusters. The effect is more pronounced for the case of persisting long-range correlations. The implication of boundary conditions on the shape of clusters is another subject of interest. Compared to the case of free boundary conditions, IP clusters of conventional rectangular geometry turn out to be more isotropic. Moreover, we see that in conventional rectangular geometry the NTIP clusters are more isotropic than the TIP clusters.

Keywords: other numerical approaches, percolation problems (theory), fractal growth (theory)

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1. Introduction

Invasion percolation (IP) [1–3] is a dynamical percolation process, primarily developed to describe the evolution of the interface between two immiscible fluids in a random porous medium. In this process, the advance of the interface is modeled as a result of a series of discrete single jumps of the invader (displacing fluid) into previously defender (displacing fluid) occupied sites through the least resistant path. The defender can be treated as an incompressible fluid. This means that, once a portion of it gets surrounded, a trap forms and the invader cannot penetrate it further. This variant of invasion percolation is called invasion percolation with trapping (TIP). On the other hand, in non-trapping invasion percolation (NTIP), which applies for compressible fluids, the invading fluid can potentially enter any region occupied by the defender. IP has also been used for modeling corrosion and intrusion [4], simulating the melt infiltration process [5] and studying random behavior of market prices [6]. In addition to these applications, there are some purely scientific interests in the subject. After all, IP is one of the simplest parameter-free models which exhibits self-organized criticality [7, 8].

Like standard percolation [9], invasion percolation generates self-similar fractal clusters. But, unlike standard percolation, the growth process described above produces only a single connected cluster. So far, much of the efforts have been devoted to investigation of the critical exponents [10–12] and scaling properties of this cluster [13]. The statistics of invaded sites and the distribution of sizes of trapped clusters in TIP have been studied too [2, 3, 14]. The shape of IP clusters has remained an open question.

The shape of random fractals is an important physical property that has been studied for several models including lattice animals and percolation clusters [15–17], Ising clusters [18], random walk [19], Eden clusters [20], bond trees [21] and aggregates with tunable fractal dimension [22]. All these studies show that anisotropy is an intrinsic property of fractal aggregates. Generally speaking, the shape of a $D$-dimensional cluster is determined by $R_1^2 \geq R_2^2 \geq \cdots \geq R_D^2$, where $R_i^2$'s are the eigenvalues (the principal radii of
The shape of invasion percolation clusters in random and correlated media

Gyration) of the cluster radius of gyration tensor

\[ \mathbf{G} = \sum_{i=1}^{N} (\vec{x}_i^2 \mathbf{1} - \vec{x}_i \vec{x}_i^T). \]  

(1)

In the above definition, \( \vec{x}_i \) is the distance of invaded site \( i \) from the center of mass and \( N \) is the size of the cluster. If all the \( R_i^2 \) are equal, the cluster is spherically symmetric. Otherwise, it is anisotropic and we can probe the degree of its anisotropy by defining a proper cluster anisotropy quantifier based on the variations in the \( R_i^2 \) [17], which have the following asymptotic form:

\[ \langle R_i^2 \rangle = r_i N^{2\nu} (1 + a_i N^{-\theta} + b_i N^{-1} + \cdots) \]  

(2)

where \( \nu \) is the leading scaling exponent and is equal to the inverse of \( D \), the fractal dimension of clusters. The leading analytic correction-to-scaling term is proportional to \( N^{-1} \) and \( N^{-\theta} \) represents the leading non-analytic correction-to-scaling term. The coefficients \( r_i, a_i \) and \( b_i \) are all independent of \( N \) [15].

Two main numerical techniques are commonly used for probing the shape of random clusters. In the first method, proposed by Family et al [15], an asymmetry measure, \( A_N = R_D^2 / R_1^2 \), called the anisotropy parameter of an \( N \)-site cluster is evaluated. The quantity \( A_N \), when properly averaged over all clusters with the same size, is denoted by \( \langle A_N \rangle \) and is an estimate of the anisotropy parameter of \( N \)-site clusters in the ensemble. The case \( \langle A_N \rangle = 1 \) corresponds to spherical symmetry. For anisotropic objects, \( \langle A_N \rangle \) is less than unity (the term ‘anisotropy parameter’ may be misleading; the shape of the cluster is more isotropic for larger values of \( A_N \)). The asymptotic behavior of \( \langle A_\infty \rangle \) is obtained by taking the limit \( N \to \infty \). Using this method for two dimensions, Family et al, observed for the first time that percolation clusters are not isotropic and estimated \( \langle A_\infty \rangle \approx 0.4 \) as the asymptotic value for the anisotropy of infinitely large percolation clusters.

The method introduced by Family et al has this advantage that, besides the shape of clusters, it provides an unbiased way of evaluating the non-analytical correction-to-scaling exponent [9,15]. Nevertheless, it is difficult to treat analytically. A more tractable approach has been suggested by Aronovitz et al [23] and Rudnick et al [24] based on the definition of the asphericity \( \Delta_D \) as

\[ \Delta_D = \frac{D}{D-1} \frac{\text{Tr} \mathbf{Q}^2}{(\text{Tr} \mathbf{G})^2} \]  

(3)

where \( \mathbf{Q} = \mathbf{G} - \overline{R^2} \mathbf{I} \) and \( \overline{R^2} = D^{-1} \text{Tr} \mathbf{G} \). Written in terms of \( R_i^2 \) in two dimensions, this becomes

\[ \Delta_2 = \frac{(R_1^2 - R_2^2)^2}{(R_1^2 + R_2^2)^2}. \]  

(4)

For an isotropic cluster this quantity is equal to zero. For an ensemble of clusters the asphericity \( \overline{\Delta}_2 \) is defined to be

\[ \overline{\Delta}_2 = \frac{\langle (R_1^2 - R_2^2)^2 \rangle}{\langle (R_1^2 + R_2^2)^2 \rangle} \]  

(5)

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in which $\langle \cdots \rangle$ denotes an ensemble average of the quantity. Note that this quantity is different from $\langle \Delta_2 \rangle$, the ensemble average of $\Delta_2$. Using this method, Quandt et al [18] obtained the value $\Delta_2 = 0.325 \pm 0.006$ for the asymptotic asphericity of two-dimensional percolating clusters, showing again that percolation clusters are not isotropic.

In this paper we study the shape of IP clusters by evaluating both the asphericity and the anisotropy parameter. The plan of the work is as follow. After describing the simulation method in section 2, we present the results of our extensive numerical simulations of the NTIP and TIP processes for completely random media in section 3. The effect of boundary conditions are examined in section 4. Section 5 contains our estimations of the shape of IP clusters when isotropic long-range correlations are introduced into the medium. The paper is concluded with section 6.

2. Method

Let us consider a sufficiently large (effectively infinite) square lattice with linear size $L$, and assign to each of lattice sites a random resistance $r$ drawn from an arbitrary distribution $D(r)$. Starting from the center of the lattice as a single-site invaded cluster, we follow the growth of the IP cluster by making a series of single jumps per time step to the least resistance neighbor of the cluster. Obviously, the list of the next-nearest neighbors increases rapidly with time. For the TIP process, we should also consider the possibility of the formation of traps and discard all the trapped sites from the list of cluster neighbors. In this work, the trapping rule has been implemented by using the Hoshen–Kopelman algorithm [25]. The search for traps is time-consuming and makes TIP simulations much slower than NTIP simulations.

For each cluster of an arbitrary size $N$, we evaluate $R_1^2$ and $R_2^2$, the principal radii of gyration of the cluster, via diagonalization of the cluster radius of gyration tensor $G$. The shape of the cluster is then characterized by evaluating its asphericity or anisotropy parameter, as described previously. Following the growth of the IP cluster in time, we may calculate these values for clusters of any desired size. To achieve highly accurate results, we estimate the mean values by sampling the growth of an IP cluster in a large number of media. The condition of effectively infinite medium requires that none of the IP clusters of a given size $N$ touches any boundary of the medium. More precisely, the linear size of the lattice, $L$, should be large enough such that all the possible configurations, including the most anisotropic ones, can potentially appear within the lattice boundaries. Otherwise, our sampling will be biased in favor of more isotropic clusters.

3. The shape of IP clusters in random media

First we consider the shape of IP clusters in completely random media, i.e. when $D(r)$ is chosen to be a uniform distribution. We have followed the growth of IP clusters in 50 000 different samples and calculate $R_1^2$ and $R_2^2$ for selected values of cluster size in the range $32 < N \leq 32 768$. The values of $N$ s have chosen such that for each block of factor of two in size (e.g. $32 < N \leq 64$, $64 < N \leq 128$, $128 < N \leq 256$) there are 10 equally spaced $N$ s in the logarithmic scale. For each cluster size $N$, the anisotropy parameter $\langle A_N \rangle$ has been calculated by averaging the ratio $R_1^2/R_2^2$ over different samples. Then, the results have been lumped together at the block centers. This procedure not
The shape of invasion percolation clusters in random and correlated media

Figure 1. Variation of $\langle A_N \rangle$, the anisotropy parameter of equilibrium percolation clusters (diamonds) and NTIP clusters (circles), in the range $32 \leq N \leq 32768$. For each factor of 2 in size, the results have been lumped together. The size of error bars is smaller than the icons used. The absolute value of error in each $\langle A_N \rangle$ is less than 0.001 for percolation clusters. In the NTIP process, this quantity is less than 0.0005, since the number of samples at each block has been much larger than the percolation case.

only helps to eliminate correction-to-scaling for small clusters [18], but it produces new data points which are usually less correlated than the original data [26]. The same method has been applied for computing $\langle (R_1^2 - R_2^2)^2 \rangle$ and $\langle (R_1^2 + R_2^2)^2 \rangle$ to obtain the asphericity parameter $\Delta^2$ at the center of each block. The behavior of anisotropy quantifiers of NTIP clusters are depicted in figures 1 and 2. For comparison, the anisotropy quantifiers of equilibrium percolation clusters are included too. These clusters have been generated using the Alexandrowicz method [27] which was later modified by Grassberger [28]. In this method, one starts with a single-site cluster at the lattice. One of its nearest neighbors (perimeter sites) is chosen randomly. This site is occupied with a probability $p_c = 0.592746$, the percolation threshold of a square lattice. The process continues until the number of perimeter sites becomes zero. Only at this point is the radius of the gyration tensor computed. We have generated 400000 equilibrium percolation clusters of size $32 < N \leq 32768$ and compute the ensemble averages $\langle A_N \rangle$, $\langle (R_1^2 - R_2^2)^2 \rangle$ and $\langle (R_1^2 + R_2^2)^2 \rangle$ within each block.

We observe that, when $N > 2^{10} = 1024$, the variation in all curves becomes very small, such that for $N > 2^{12} = 4096$, all the curves are effectively flat. This means...
Figure 2. Variation of $\Delta_2$, the asphericity parameter of equilibrium percolation clusters (diamonds) and NTIP clusters (circles), in the range $32 \leq N \leq 32768$. The size of error bars is smaller than the icons used. The absolute error in asphericity is not constant and decreases as $N$ increases. It is less than 0.001 in the NTIP process and slightly greater than 0.001 for equilibrium percolation clusters, when $N$ is large ($N > 4096$).

The effect of correction-to-scaling for both NTIP and percolation clusters is negligible and the anisotropy quantifiers have saturated. At this limit, the anisotropy parameters of NTIP and percolation clusters fluctuates around $0.337 \pm 0.001$ and $0.389 \pm 0.002$, respectively. On the other hand, the asymptotic value of the asphericity of NTIP clusters is $\overline{\Delta}_2 = 0.401 \pm 0.002$, while for percolation clusters we find $\overline{\Delta}_2 = 0.322 \pm 0.002$. These observations demonstrate that NTIP clusters are less isotropic than standard percolation clusters. This is an interesting result, because NTIP and standard percolation clusters have the same self-similarity dimension ($D = 0.18959 \pm 0.0001$) and hence belong to the same universality class [2, 12].

How are the $A_N$'s distributed? To answer this question we have calculated $P(A)$, the normalized distribution of $A$ for a specified cluster size, say $N = 10000$. To this end, we divided the entire range of $[0, 1]$ into 50 bins with equal width $\delta = 0.02$ and counted the number of clusters with the anisotropy parameter in the range $[A - \delta, A]$. It is seen from figure 3 that the distribution is asymmetric and quite broad with a peak approximately located at $A \simeq 0.2$, which means the most probable configurations are those for which $R_1/R_2 \approx \sqrt{0.2} = 0.45$. Our calculation also shows that the fluctuation in $A$ (not shown) is approximately equal to 0.19. Furthermore, we observed that the shape
Figure 3. $P(A)$, the normalized distribution of $A$ for NTIP clusters of size $N = 10000$.

The shape of invasion percolation clusters in random and correlated media

of $P(A)$ (and consequently the fluctuation) is almost independent of cluster size $N$, if $N$ is not too small.

We have also evaluated the asphericity and the anisotropy parameter of TIP clusters for cluster sizes in the range $100 < N < 1000$. The results are presented in figure 4. As is seen from the figure, there is no difference in the shape of TIP and NTIP clusters although the self-similarity dimension of these processes differs from each other ($D = 1.825 \pm 0.005$ for TIP in square lattices [12]). Both $\langle R_1^2 \rangle$ and $\langle R_2^2 \rangle$ have the same leading exponents $2\nu$ (equation (2)) and hence the anisotropy does not involve it. The equivalence of the anisotropy quantifiers, therefore, indicates that, in addition to the value of $a_1/a_2$, the ratio of correction-to-scaling terms is equal in these processes.

It is worth mentioning that the anisotropy quantifiers are independent of the orientation of the principal axes of the cluster, which might be arbitrarily oriented. In fact, the underlying ensembles of clusters are isotropic themselves [17]. However, isotropy of an ensemble only implies that a given cluster conformation will appear with equal probability in arbitrary orientations [16]. The observed anisotropy in the shape of clusters is a result of spontaneous fluctuations in shape about the expected isotropic shape. We may relate it to the nature of the dynamics of invasion percolation. As shown by Furuberg et al [3], the advance of the interface occurs by invading local areas in *bursts*; once a new site is invaded, the interface tends to stay in that vicinity. Quantitatively, they found that the most probable growth after a time $t$ occurs at a distance $d_t \sim t^{1/z}$, where $z$ is the dynamic exponent. Naturally, this local growth might amplify any small fluctuations in the ratio of $R_1^2/R_2^2$. 
The shape of invasion percolation clusters in random and correlated media

Figure 4. Comparison between the asphericity (upper curve) and the anisotropy parameter (lower curve) of TIP (circles) and NTIP (dots) clusters.

4. The shape of IP clusters in conventional geometry

In the more conventional simulations of invasion percolation processes, the host lattice is assumed to be a $L \times 2L$ rectangular lattice, and instead of the center, the invasion process starts from one of the smaller lattice edges. The outlet or sink is located on the opposite edge and the other two lattice edges are assumed to be impermeable. The growth process stops at breakthrough, when the invader reaches the outlet. In this situation, the IP cluster connects the inlet and outlet through a single, continuous path. The properties of this sample spanning cluster (SSC) within the central $L \times L$ part of the lattice, i.e. far from inlet and outlet [3], have been the subject of intense research.

To estimate the asymptotic value of the anisotropy parameter of the central part of the SSC, we generated 10,000 samples for each of lattice sizes, $L = 64, 128, 256$ and 2000 samples of size $L = 512$. The mean anisotropy parameter $A_L$ is then computed for each $L$. In this geometry, the mass of SSC varies in different realizations even when $L$ is fixed. For example, in ordinary TIP the mass of the central part of SSC is $N = (5.4 \pm 1.1) \times 10^4$ for $L = 512$. Nevertheless, since $N$ is very large itself, this variation does not affect the value of $A_\infty$ via correction-to-scaling terms. In fact, our simulations show that $A_L$ does not depend on $L$, if $L$ is sufficiently large. The obtained value of $A_\infty$ is 0.64 for the NTIP process and 0.57 for the TIP process. Compared to the previous case, the shape of the SSC in both NTIP and TIP has turned out to be more isotropic. This is because, in this case, the growth process continues even after the IP cluster touches the boundaries of
the central $L \times L$ frames. The difference between the shape of $A_\infty$ in this geometry is a consequence of the trapping rule which limits the growth of the SSC in the TIP process.

5. The effect of long-range correlations on the shape of IP clusters

In many practical applications, the nature of disorder is not completely random and there are correlations in the properties of the medium [29,30]. To investigate the effect of correlations on the shape of IP clusters, we have considered the case for which the distribution of the resistance of lattice sites obeys the statistics of fractional Brownian motion (FBM) $D_B(x)$ [31,34]. FBM is a stochastic process whose increments are statistically self-similar such that its mean square fluctuation is proportional to an arbitrary power of the spatial displacement $x$:

$$\langle [D_B(x) - D_B(0)]^2 \rangle \sim |x|^{2H} \quad (6)$$

where $H$ is called the Hurst exponent and determines the type of correlations. If $H = 0.5$, the above equation produces the ordinary Brownian motion, which means that in this case there is no correlation between different increments. If $H > 0.5$, then FBM generates positive correlations, i.e. all the points in a neighborhood of a given point obey more or less the same trend. If $H < 0.5$, FBM is anti-persistence, i.e. a trend at a point will not likely be followed in its immediate neighborhood.

The reason that we have chosen the FBM process is twofold. First, FBM generates long-range and, at the same time, isotropic correlations in the field. Therefore, the host lattice retains its isotropy. Second it has been demonstrated that such a process has practical applications in earth sciences and also reservoir engineering, where the permeability field and also the porosity distribution of many real oil reservoirs and aquifers follow FBM statistics [30,32,33].

There are a number of methods which are capable of producing the FBM statistics [32,34]. We have used one of the most popular one, the method of fast Fourier transformation (FFT) filtering which is based on the fact that the power spectrum of FBM is given by

$$S_B(\omega) = \frac{a_0}{(\omega_x^2 + \omega_y^2)^p} \quad (7)$$

where $a_0$ is a numerical constant, $\omega = (\omega_1, \omega_2)$, with $\omega_i$ being the Fourier component in the $i$th direction and $p = H + 1$. In the FFT method, one starts with a white noise $W(x,y)$ defined on the lattice sites. The power spectrum of $W(x,y)$ is constant and independent of frequency. Therefore, filtering $W(x,y)$ with a transfer function $\sqrt{S_B(\omega)}$ generates another noise whose spectral density is proportional to $S_B(\omega)$. The method is straightforward and fast, but it usually produces periodic noises. Therefore, one has to produce a larger lattice and keep only a portion (typically $1/4$ in two-dimensional lattices).

In figure 5 we have reported our estimation of the anisotropy parameters of IP clusters in media obeying the FBM statistics in the range $10 < N < 800$. In this figure, we have compared the value of $A_N$ for three different Hurst exponents, $H = 0.2$ (anti-persistent correlation), $H = 0.8$ (persistent correlation) and $H = 0.5$ (Brownian motion) with the results of completely random media. The data have been obtained from averaging over 30000 samples for each case. Like completely random media, we observed no difference
The shape of invasion percolation clusters in random and correlated media

Figure 5. The anisotropy parameters of IP clusters in media obeying the FBM statistics; $H = 0.2$ (diamonds); $H = 0.5$ (squares) and $H = 0.8$ (triangles). For comparison the data for completely random media (circles) has been included too.

between the shape of NTIP and TIP clusters (not shown). These results indicate that any deviation from complete randomness makes the shape of invasion percolation clusters more anisotropic. Furthermore, we find that IP clusters in the presence of persistent correlations are less isotropic than IP clusters of anti-persistent correlations. Based on what has been explained in the last lines of section 3, these effects can be assigned to the difference between dynamics of invasion percolation in random and correlated media. In fact, we anticipate that the burst-like growth occurs more effectively, may be with different dynamic exponent and amplitude (which depend on the nature of the disorder), resulting in more anisotropy in the shape of clusters. The difference between the shape of clusters for $H = 0.2$ and 0.8 is compatible with this image. The presence of persistent long-range correlations intensifies the burst-like growth and, as a result, IP clusters become more anisotropic in this case.

6. Conclusions

The shape of clusters in IP processes have been probed numerically by evaluating their asphericity and anisotropy parameters. The results indicate that the shape of clusters are the same for both TIP and NTIP processes. This conclusion does not depend on the type of disorder in the host lattice. We found that, similar to other random fractals, generated in a variety of stochastic processes, the invasion percolation clusters are anisotropic.
too. Moreover, we observed that IP clusters are less isotropic than standard percolation clusters. By introducing long-range correlation into the media the clusters became more anisotropic in shape than before. These effects might be explained according to the dynamics of invasion percolation and the burst-like nature of the growth process of IP clusters.

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