A review of total energy-momenta in GR with a positive cosmological constant*

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Abstract

A review is given of the various approaches to and expressions for total energy-
momentum and mass in the presence of a positive cosmological constant in Einstein’s
field equations, together with a discussion of the key conceptual questions, main
ideas and techniques behind them.

1 Introduction

As a consequence of the equivalence principle, following the lesson of the Galileo–Eötvös
experiments, gravity is known to have a universal nature and the spacetime metric plays a
double role: both as the field variable for gravity and as the geometric background for the
dynamics of the matter fields at the same time. A direct consequence of the lack of any
non-dynamical geometric background is that any local expression for the gravitational
energy-momentum is necessarily pseudotensorial or, in the tetrad formulation of the
theory, Lorentz gauge dependent (see e.g. [1, 2, 3]). Although all these local quantities
can be recovered from (various forms of) a single geometric object on the linear frame
bundle (or of its subbundles) (see e.g. [4, 5, 6, 7]), the gravitational energy-momentum
density in the spacetime is not well defined – it cannot be localized to points. The
gravitational energy-momentum is non-local in nature, and can be well defined only if
it is associated to extended domains in spacetime. Thus, it must be a measure of total
energy-momentum, i.e. when the domain is an infinitely large part of the spacetime (for

* Dedicated to our friend, Jörg Frauendiener, on the occasion of his 60th birthday.

Ach, wie schön muss sich’s ergehen,
Dort in Newmans Himmelsraum.
Und Lichtkugeln auf den Höhen –
O wie lab’n sie meinen Traum !

– from Ein theoretisch–physikalischer Schiller–Traum by N. V. Mitskievitch
example, the whole spacetime itself), or quasi-local, when the domain is only a compact subset of the spacetime.

From pragmatic points of view the significance of these quantities is given by the positivity of the corresponding total (or quasi-local) energy/mass, or at least their boundedness from below. In fact, some of these quantities have already been proven to be useful tools in geometric analysis (e.g. of asymptotic structure of spacetime or black-hole uniqueness), in stability investigations of general relativistic gravitating systems, in numerical calculations (e.g. to control errors), and so on.

The usual form of the total energy-momentum of a localized gravitating system is a 2-surface integral of some local ‘superpotential’ $U$, which also depends (linearly) on some vector field $K^a$ or (quadratically) on some spinor field $\lambda^A$ representing ‘asymptotic translations’ in the spacetime:

$$P_a K^a := \oint_S U(K) dS, \quad \text{or} \quad P_a \sigma_{AB}^a \lambda^A \bar{\lambda}^{B'} := \oint_S U(\lambda, \bar{\lambda}) dS. \quad (1.1)$$

Here $K^a$ and $\lambda^A$ are the components of the ‘asymptotic translation’ $K^a$ and its spinor constituent $\lambda^A$, respectively, in some basis of the space of (candidate) asymptotic translations$^1$ and $\sigma_{AB}^a$, $a = 0, 1, A, B = 0, 1$, are the standard $SL(2, \mathbb{C})$ Pauli matrices. Thus, the energy-momentum 4-vector is an element of the dual space of the space of asymptotic translations. Therefore, to have a well defined total energy-momentum expression, we should specify: (1) the domain of integration $S$ (i.e. the choice of what to consider as the physical system); (2) the ‘superpotential’ $U$; and (3) the ‘generator’ ($K^a$ or $\lambda^A$) of the quantity in question (i.e. the definition of the asymptotic translations). Different choices for these yield different expressions with different properties.

Total mass (rather than energy-momentum) can be associated not only with localized sources, but with closed universes with non-negative $\Lambda$ as well. In this case the domain of integration is a Cauchy surface (rather than a closed 2-surface).

The aim of the present paper is to review the total energy-momentum/mass constructions in the presence of a strictly positive cosmological constant $\Lambda$. However, to put them in perspective and to see the roots of certain key ideas behind the actual constructions, and also to motivate how to choose the domain, the superpotential and the generator field in items (1)-(3) above, we briefly recall the analogous constructions in the $\Lambda = 0$ and $\Lambda < 0$ cases where these ideas appeared first. This part of the review is far from being complete. For a review of the quasi-local constructions, see e.g. [8].

1.1 On total energy-momenta with $\Lambda = 0$

1.1.1 The domain of integration

If the cosmological constant is zero, then the spacetime describing the gravitational ‘field’ of a localized source is expected to be ‘asymptotically flat’ in some well-defined sense, and hence its global structure at large distances is expected to be similar to that of the Minkowski spacetime. (For the global properties of the latter, see e.g. [9].) In fact, formally the conserved Arnowitt–Deser–Misner (ADM) energy-momentum [10] is based on a 2-surface integral on the boundary at infinity of an (appropriately defined) asymptotically flat spacelike hypersurface that extends to spatial infinity. In Minkowski spacetime the $t = \text{const}$ hyperplanes, denoted by $\Sigma_t$, provide a foliation of the spacetime.

$^1$As we shall see, this can be a problematic notion with $\Lambda \neq 0$. 

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by such global Cauchy surfaces, while their (common) boundary at infinity is the \( r \to \infty \) limit of the 2-spheres \( S_{t,r} := \{ t = \text{const} , \ r = \text{const} \} \) (see Fig. 1.i.). Clearly, this boundary is \textit{spatially separated} from the world tube of the source.

Figure 1: Foliations of the Minkowski spacetime: i. The hypersurfaces \( \Sigma_t \) form a global foliation. Each of the leaves is a global Cauchy surface with \( \mathbb{R}^3 \) topology, and all these extend to spatial infinity \( \mathcal{I}^0 \). These hypersurfaces are both intrinsically and extrinsically asymptotically flat. The \( t = \text{const} \) hyperplanes in the Cartesian coordinates are like this. ii. The hypersurfaces \( \Sigma_\tau \), which are only \textit{partial} Cauchy surfaces, extend to the future null infinity \( \mathcal{I}^+ \) and foliate both the spacetime and \( \mathcal{I}^+ \). These hypersurfaces are intrinsically asymptotically hyperboloidal and the extrinsic curvature is asymptotically proportional to the intrinsic metric. For fixed \( T > 0 \), in the Cartesian coordinates, the hypersurfaces \( \tau := t - \sqrt{T^2 + r^2} = \text{const} \) have this character. iii. However, for fixed \( \tau \) and variable \( T \), the hypersurfaces \( \Sigma_T \), given by \( T^2 := (t - \tau)^2 - r^2 = \text{const} \), foliate only the chronological future of the point \((\tau,0,0,0)\) but do not provide a foliation of (even an open subset of) \( \mathcal{I}^+ \).

On the other hand, if the localized system emits radiation and the total energy-momentum is expected to reflect the dynamical aspects of the system, e.g. to give account of the energy carried away by the radiation, then that should be associated with a hypersurface extending to the \textit{null infinity} of the spacetime. In fact, the Trautman–Bondi–Sachs (TBS) energy-momentum \[ \text{[11, 12, 13]} \] is an integral on the boundary at infinity of an outgoing \textit{null hypersurface} that defines a retarded time instant. One may also think of this 2-surface as the boundary at infinity of an \textit{asymptotically hyperboloidal} spacelike hypersurface whose asymptote is just the null hypersurface. For example, in Minkowski spacetime, the family of spacelike hypersurfaces \( \tau := t - \sqrt{T^2 + r^2} = \text{const} \) for some constant \( T > 0 \), denoted by \( \Sigma_\tau \), provides a foliation of the whole spacetime (see Fig. 1.ii.). \( \Sigma_\tau \) intersects future null infinity at the retarded time instant \( u = \tau \). The induced metric \( h_{ab} \) on them is of constant curvature with the scalar curvature \( R = -6/T^2 \), and the extrinsic curvature is \( \chi_{ab} = h_{ab}/T \). Hence, these are \textit{hyperboloidal}, but only \textit{partial} Cauchy surfaces\footnote{It might be interesting to note that the line element of the flat metric with the global foliations \( \Sigma_t \) and \( \Sigma_\tau \) takes the Friedman–Robertson–Walker (FRW) form with \( k = 0 \) and \( k = -1 \), respectively. On the other hand, for fixed \( \tau \) and variable \( T \) the hypersurfaces \( \Sigma_T := \{ T = \text{const} \} \) foliate only the chronological}. The \( r \to \infty \) limit of the 2-spheres \( S_{r,r} := \{ \tau = \text{const} , \ r = \text{const} \} \) is \textit{null separated} from the source in the centre \( r = 0 \) at \( t = \tau \). This energy-momentum
depends on the retarded time instant \( u \) that the hypersurface defines, and, in particular, the mass, defined to be the Lorentzian length of the TBS energy-momentum 4-vector, is a monotonically decreasing function of \( u \) ("Bondi’s mass-loss"). To derive this, a foliation of the null infinity and the asymptotic solutions of the field equations are needed.

In the conformal approach of Penrose \[14, 15\] the future null infinity, as a smooth boundary \( \mathcal{I}^+ \), is attached to the conformal spacetime. In this framework the asymptotic flatness is defined by the compactifiability of the spacetime by such a boundary. Since \( \Lambda = 0 \), \( \mathcal{I}^+ \) is a null hypersurface in the conformal spacetime, its null geodesic generators are shear-free (and, in a certain conformal gauge, divergence-free, too); and it is assumed to have \( S^2 \times \mathbb{R} \) topology. The advantage of this approach to asymptotic flatness is that the techniques of local differential geometry can be used to study the asymptotic properties of the fields and spacetime (see also \[17, 18, 15, 19\]).

Nevertheless, the existence of null infinity as a smooth boundary of the conformal spacetime restricts the fall-off rate of the physical metric: it should tend to the flat metric as \( 1/r \). For a treatment of asymptotically flat spacetimes with slower (e.g. logarithmic) fall-off metrics, see, e.g. \[22\].

1.1.2 The asymptotic translations

In Minkowski spacetime the four independent translational Killing fields are defined in a geometric/algebraic way: they are constant vector fields; and, also, they form a commutative ideal in the Poincaré algebra of the Killing fields. At spatial infinity their restriction to the boundary 2-surface at infinity yields constant vector fields there. Since the isometries of the spacetime must take \( \mathcal{I}^+ \) to itself, the vector fields that they determine on \( \mathcal{I}^+ \) must be tangent to \( \mathcal{I}^+ \). In particular, the vector fields that the translations yield on \( \mathcal{I}^+ \) are all proportional to the tangent of the null geodesic generators of \( \mathcal{I}^+ \). However, in contrast to the spatial infinity case, the factor of proportionality is not constant, but rather is a linear combination of the first four ordinary spherical harmonics.

In fact, these vector fields, the so-called Bondi–Metzner–Sachs (BMS) translational vector fields, can be characterized completely in terms of the conformal structure of \( \mathcal{I}^+ \). This yields the notion of asymptotic translations even in general spacetimes admitting \( \mathcal{I}^+ \) \[13, 17, 18, 15\]; and these coincide with the (equivalence classes of the) asymptotically constant solutions of the asymptotic Killing equation \[20\]. The BMS translational vector fields can be characterized by their Weyl or 2-component spinor constituents, too: they are proportional to the constituent spinor of the tangent of the null geodesic generators of \( \mathcal{I}^+ \), and the factor of proportionality is a linear combination of spin-weight 1/2 spherical harmonics. These can be recovered as the solutions of various linear partial differential equations on the cut. (For a list of these, see the appendix of \[23\].)
1.1.3 The superpotential

In the literature several different forms of the integrand in (1.1) are known both for the ADM [10] and TBS [11, 12, 13] energy-momenta: they can be given (1) by certain expansion coefficients in the asymptotic expansion of the components of the (spatial or spacetime) metric as a series of 1/r in an asymptotic Cartesian or retarded null coordinate system; or (2) by the traditional superpotential of some classical (e.g. Einstein’s) energy-momentum pseudotensor (or, in the tetrad formalism of the theory, by an SO(1, 3) gauge dependent energy-momentum complex) in some appropriate coordinate system (or Lorentz frame). Also, (3) one can write the spacetime metric as the sum of the flat spacetime metric $\bar{g}_{ab}$ and some ‘correction field’ $\gamma_{ab}$, and then rewrite the exact Einstein equations by the linearized Einstein tensor, built from $\gamma_{ab}$ on the background $\bar{g}_{ab}$, and an effective energy-momentum tensor. The $U$ in (1.1) is the contraction of the superpotential for this effective energy-momentum tensor and a translational Killing field of the background metric. (4) The 2-surface integral (1.1) can also be sought in the form of the boundary term in the Hamiltonian of the theory, in which the lapse and the shift are the timelike and spacelike part of $K^a$, respectively. (5) These total energy-momenta can also be written as a Komar or linkage integral, based on the vector field $K^a$ (see [24]); or, (6) using the field equations, they can be re-expressed by certain parts of the curvature tensor.

However, (7) there is a ‘universal superpotential’, built from a Dirac spinor or a pair of 2-component spinors, by means of which the classical (e.g. the pseudotensorial) superpotentials could be recovered (see the Introduction); and both the ADM and TBS energy-momenta can be re-expressed. This is the Nester–Witten 2-form [25, 26], given in terms of any two 2-component spinors [28] by

$$u(\lambda, \bar{\mu})_{ab} := \frac{i}{2} \{ \bar{\mu}_A \nabla_{BB}' \lambda_A - \bar{\mu}_{B'} \nabla_{AA}' \lambda_B \}. \quad (1.2)$$

Although this is a complex-valued 2-form, its integral on any closed, orientable 2-surface $S$ defines a Hermitian bilinear form on the space of the spinor fields on $S$.

To ensure the existence of the integrals in (1.1), and, in the traditional formulation, also the independence of the ADM energy-momentum of the coordinate system/background Minkowski metric, non-trivial fall-off conditions should be imposed both on the matter fields and the geometric data ($h_{ab}, \chi_{ab}$) on the hypersurface. The latter should tend asymptotically to the trivial flat or hyperboloidal data on the hypersurfaces $\Sigma_{\tau}$ and $\Sigma$, respectively.

The key properties of the ADM and TBS energy-momenta are that they are future pointing and timelike vectors (a property we will call ‘positivity’), provided the energymomentum tensor of the matter fields satisfies the dominant energy condition on the spacelike hypersurface $\Sigma$ whose boundary at infinity is $S$; and the vanishing of these energy-momenta is equivalent to the flatness of the domain of dependence $D(\Sigma)$ of $\Sigma$ and the vanishing of the matter fields (‘rigidity’) [27, 28, 25, 28, 29]. These results together, known as the positive energy or positive mass theorem, hold true even in the presence of black holes [30]. Probably the simplest proof of this theorem is based on the use of spinors, the superpotential (1.2) and the Witten-type gauge condition for the spinor fields on the hypersurface $\Sigma$ [29]. The rigorous mathematical proof of the existence and uniqueness of the solution of the Witten equation on asymptotically flat spacelike hypersurfaces could be based on the techniques of [31]. The analogous results on asymptotically hyperboloidal hypersurfaces are given in [32].
The spacetimes describing the history of closed universes are defined to be those globally hyperbolic spacetimes that admit compact Cauchy surfaces. Clearly, these do not represent the gravitational field of localized sources, and total energy-momentum of the form (1.1) cannot be associated with them. Nevertheless, their total mass can be introduced using the (common) 3-surface integral form of the ADM and TBS energy-momenta. Although its non-negativity is trivial by construction (given the assumption that the dominant energy condition holds), still it has a non-trivial rigidity property: this total mass is zero precisely when the spacetime is locally flat, the matter fields are vanishing and the Cauchy surface is a 3-torus [33, 34].

1.2 On total energy-momentum with $\Lambda < 0$

1.2.1 The domain of integration

If $\Lambda < 0$, then the gravitational field of a localized source is expected to be represented by a spacetime of asymptotically constant negative curvature, i.e. by some ‘asymptotically anti-de Sitter’ spacetime. The anti-de Sitter spacetime itself [9] is the universal covering space of the vacuum spacetime with constant negative curvature $R = 4\Lambda < 0$. Its conformal boundary $\mathcal{I}$ is a timelike hypersurface in its conformal completion with topology $S^2 \times \mathbb{R}$. In the standard global coordinates $(t, r, \theta, \phi)$ [9] the $t = \text{const}$ spacelike hypersurfaces, denoted by $\Sigma_t$, are partial Cauchy surfaces and intersect $\mathcal{I}$ in 2-spheres. They provide a foliation of $\mathcal{I}$ as well [4] (see Fig. 2.i.). The induced metric on them is of constant negative curvature with scalar curvature $R = 2\Lambda$ and vanishing extrinsic curvature, i.e. these hypersurfaces are intrinsically hyperboloidal and extrinsically flat.

The traditional definition of asymptotically anti-de Sitter spacetimes is based on the decomposition of the spacetime metric into the sum of the anti-de Sitter metric $\tilde{g}_{ab}$ and some ‘correction term’ $\gamma_{ab}$ [35]; but the latter tensor field should be required to satisfy appropriate fall-off conditions. However, adapting the key ideas of the conformal approach to infinity to the present case, the asymptotically anti-de Sitter spacetimes could be defined in a fully geometric way as in [36], by the conformal compactifiability of the spacetime. Since $\Lambda < 0$ the conformal boundary $\mathcal{I}$ is timelike, the trace-free part of its extrinsic curvature is vanishing and the trace can be made to be vanishing in some conformal gauge. However, its intrinsic metric at this point is not restricted further, and could still be completely general. Thus, to restrict the asymptotic properties of the spacetime to be similar to that of the anti-de Sitter, the conformal boundary $\mathcal{I}$ is usually required to be topologically $S^2 \times \mathbb{R}$ and intrinsically conformally flat [36].

The latter condition is called the ‘reflective boundary condition’ [37], and is equivalent to the vanishing of the magnetic part of the conformally rescaled Weyl curvature [38], determined by the spacelike normal of $\mathcal{I}$, on $\mathcal{I}$. Nevertheless, this boundary condition excludes any gravitational energy-flux through the conformal boundary, and, as Friedrich stresses (see [39], and especially [40]), this boundary condition is one choice among many. Other ‘natural’ boundary conditions, when incoming and/or outgoing energy flux through $\mathcal{I}$ is allowed, are also possible [40].

The domain $\mathcal{S}$ of integration in (1.1) is a closed spacelike 2-surface, which may also be called a ‘cut’, in $\mathcal{I}$. In the exact anti-de Sitter spacetime such a cut is, for example,

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4The anti-de Sitter line element can be rewritten in FRW form with $k = -1$, but the leaves of this latter foliation would foliate only globally hyperbolic proper subsets of the spacetime, say $D(\Sigma_t)$ for given $t$ (see Fig. 2.ii.).
the intersection $\Sigma_t \cap \mathcal{I}$. Since, however, $\mathcal{I}$ is timelike, there is no way to distinguish the ADM and TBS type energy-momenta: the specific properties of the total energy-momentum associated with the cuts of $\mathcal{I}$ depend crucially on the boundary conditions on $\mathcal{I}$.

1.2.2 The asymptotic translations

The Lie algebra of the Killing fields is the anti-de Sitter Lie algebra $so(2,3)$. Since, however, $so(2,3)$ is semi-simple, there is no way to single out ‘translations’ in a canonical algebraic way; nor, since this spacetime does not admit any non-trivial constant vector field, is there a way to single out ‘translations’ in a canonical geometric manner either. Nevertheless, the coordinate $t$ is timelike and the components of the metric in the global coordinates $(t, r, \theta, \phi)$ are independent of $t$, so the Killing field $(\partial/\partial t)^a$ is often interpreted as the time translational Killing field (though, for example, it is not a geodesic vector field). The Killing fields of the spacetime extend to conformal Killing fields on infinity, and, in particular, $(\partial/\partial t)^a$ extends to a timelike one on $\mathcal{I}$.

If, however, the ‘asymptotically anti-de Sitter spacetimes’ are defined simply by the existence of a (timelike) conformal boundary $\mathcal{I}$ (possibly with $S^2 \times \mathbb{R}$ topology) but without any further condition on its intrinsic conformal geometry, then in general $\mathcal{I}$ does not admit any conformal isometry. Hence, the spacetime cannot admit any ‘asymptotic Killing vector’ either. On the other hand, the requirement of the intrinsic conformal
flatness of $\mathcal{I}$, i.e. the reflective boundary condition, guarantees the maximal number of conformal Killing fields, and their Lie algebra is isomorphic to the anti-de Sitter algebra $so(2,3)$ \cite{36}.

1.2.3 The superpotential

As in the $\Lambda = 0$ case (in subsection 1.1.3), several different superpotentials can (and, in fact, have been) used in (1.1) to associate energy-momentum with cuts of the timelike conformal boundary $\mathcal{I}$. For example, that could be based on the use of the effective energy-momentum tensor built from $\gamma_{ab}$ mentioned in (3) of subsection 1.1.3 \cite{35}; or on another explicitly given superpotential \cite{41}, obtained from a Hamiltonian. Also, the precise fall-off conditions for $\gamma_{ab}$ are determined which make the resulting expressions unambiguously defined. (Without this, the expression of \cite{35} would suffer from coordinate ambiguities.) The integrand used in \cite{36} is the electric part of the rescaled Weyl curvature, contracted with a conformal Killing vector of $\mathcal{I}$; and this was shown in \cite{42} to coincide with the ‘renormalized’ quasi-local mass of Penrose \cite{15} (after subtracting the cosmological constant term), calculated on a cut of $\mathcal{I}$. The latter two are manifestly coordinate-free, and all these expressions, when they are well-defined, give the same result. The Nester–Witten 2-form (1.2) in its ‘renormalized’ form is used in \cite{30}, where a Witten-type proof of the positivity of energy, and also of rigidity, are also given. In all these investigations the intrinsic conformal flatness of $\mathcal{I}$ was assumed.

1.3 The need for total energy-momenta with $\Lambda > 0$

The simplest explanation of the observed accelerating expansion of the Universe \cite{43, 44} is the presence of a strictly positive cosmological constant in Einstein’s field equations. Thus, the future history of our Universe is asymptotically de Sitter. Also, the positivity of $\Lambda$ is the basis of the conformal cyclic cosmological (CCC) model of Penrose \cite{45}. In the analysis of the asymptotic structure of these spacetimes, the total energy-momenta could still provide useful tools. In fact, the need (and three potential expressions) for the total TBS type energy was already raised by Penrose in \cite{46}. Since then several papers devoted to the question of total energy-momentum in asymptotically de Sitter spacetimes have appeared. The aim of the present paper is to give a review of these results.

In section 2 we review the properties of general asymptotically de Sitter spacetimes; and discuss the important special spacetimes that are asymptotic to de Sitter, with special emphasis on their foliations, symmetries and the fields on the background de Sitter spacetime. Then, in section 3 we discuss ADM type energy-momentum expressions, while sections 4 and 5 are devoted to the TBS type expressions and a suggestion for the total mass in closed universes, respectively. The signature of the spacetime metric is chosen to be $(+, -, -, -)$, and Einstein’s equations are written in the form $R_{ab} - \frac{1}{2}Rg_{ab} = -\kappa T_{ab} - \Lambda g_{ab}$ with $\kappa = 8\pi G$. 
### 2 The structure of asymptotically de Sitter spacetimes

#### 2.1 The de Sitter spacetime

#### 2.1.1 Foliations

The de Sitter spacetime is the constant positive curvature solution of the vacuum Einstein equations with scalar curvature $R = 4\Lambda > 0$. (For a summary of its key geometric properties, see [47], and for a detailed discussion of its global structure, especially its Penrose diagram, see [9].) Its line element in the global coordinates $(t, r, \theta, \phi)$ is

$$ds^2 = dt^2 - \alpha^2 \cosh^2(t/\alpha) \left( dr^2 + \sin^2 r \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right),$$

where $\alpha^2 := 3/\Lambda$, and the range of the coordinates are $t \in \mathbb{R}$, $r \in [0, \pi]$ and $(\theta, \phi) \in S^2$. This metric has the FRW form with $k = 1$, scale function $S(t) = \alpha \cosh(t/\alpha)$ and foliation by global Cauchy surfaces $\Sigma_t := \{ t = \text{const} \}$ with $S^3$ topology (see Fig. 3.i.). Then, in the coordinates $(\tau, r, \theta, \phi)$ with $\tau := 2 \arctan(\exp(t/\alpha)) - \pi/2$, the future/past conformal boundary, $\mathcal{I}^\pm$, is given by $\tau = \pm \pi/2$ and

$$ds^2 = S^2 \left( d\tau^2 - dr^2 - \sin^2 r \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right).$$

![Figure 3: Foliations of the de Sitter spacetime: i. The hypersurfaces $\Sigma_t$ form a global foliation. The leaves $\Sigma_t$ are global Cauchy surfaces with $S^3$ topology. The $t = \text{const}$ hypersurfaces in the global coordinates of the de Sitter spacetime are like this, which are metric spheres. ii. The hypersurfaces $\Sigma_{\hat{t}}$ foliate the ‘steady state’ part of (or ‘Poincaré patch’ in) the (half) de Sitter spacetime. Their topology is $\mathbb{R}^3$, they are intrinsically asymptotically flat and their extrinsic curvature is asymptotic to a nonzero value. Their one-point compactification yields a ‘spatial infinity’ $i^0$, which is a point of the future conformal boundary $\mathcal{I}^+$. These hypersurfaces do not foliate $\mathcal{I}^+$. (See also Figure 17 of [9].)](image)

The (half) de Sitter spacetime, often called the ‘steady state universe’ or ‘Poincaré patch’, has another foliation with intrinsically flat spacelike hypersurfaces as well [9] (see Fig. 3ii.). These are the $\Sigma_{\hat{t}} := \{ \hat{t} = \text{const} \}$ hypersurfaces, where
\[
\hat{t} := \alpha \ln \left( \sinh \left( \frac{t}{\alpha} \right) + \cosh \left( \frac{t}{\alpha} \right) \cos r \right), \quad \hat{x} := \frac{\alpha}{\tanh \left( \frac{t}{\alpha} \right) + \cos r} \sin r \sin \theta \cos \phi, \\
\hat{y} := \frac{\alpha}{\tanh \left( \frac{t}{\alpha} \right) + \cos r} \sin r \sin \theta \sin \phi, \quad \hat{z} := \frac{\alpha}{\tanh \left( \frac{t}{\alpha} \right) + \cos r} \sin r \cos \theta. \tag{2.1}
\]

This coordinate system covers only the ‘steady state’ part of the de Sitter spacetime, for which \( \sinh (t/\alpha) + \cosh (t/\alpha) \cos r > 0 \), and the line element takes the form

\[
ds^2 = d\hat{t}^2 - \exp(2\hat{t}/\alpha) \left( d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2 \right).
\]

Thus, this line element has the FRW form with \( k = 0 \) and scale function \( S(\hat{t}) = \exp(\hat{t}/\alpha) \). Hence, the extrinsic curvature of the hypersurfaces \( \Sigma_{\hat{t}} \) is a pure trace, and the mean curvature is the positive constant \( \chi = 3/\alpha \). All these hypersurfaces reach \( \mathcal{I}^+ \) at the point \( r = \pi \), anti-podal to the origin, where they are, in fact, tangent to \( \mathcal{I}^+ \). Thus, the anti-podal point provides a one-point-compactification of these hypersurfaces, and this can be interpreted as the ‘spatial infinity’ of the flat 3-spaces. Clearly, this foliation is analogous to the foliation of the Minkowski spacetime by the \( t = \text{const} \) hyperplanes. Nevertheless, the hypersurfaces \( \Sigma_{\hat{t}} \) are extrinsically hyperboloidal; and none of these hypersurfaces is a global Cauchy surface, because no future inextendible non-spacelike curve terminating at the anti-podal point intersects any of these \( \Sigma_{\hat{t}} \). These are Cauchy surfaces only for the steady state part of the de Sitter spacetime.

In contrast to this foliation, a neighbourhood of \( \mathcal{I}^+ - \{(\pi, 0, 0)\} \) can be foliated by spacelike hypersurfaces which foliate \( \mathcal{I}^+ - \{(\pi, 0, 0)\} \), too (see e.g. [32]). Such a foliation can be based on the coordinate system \((u, w, \theta, \phi)\), where, in terms of the global coordinates above, \( u := \alpha(\tau - r) \) and \( w := (\pi/2 - \tau)/\alpha \). Then \( \mathcal{I}^+ \) is just the \( w = 0 \) hypersurface, and the \( u = 0 \) ‘origin cut’ of \( \mathcal{I}^+ \) is the \( r = \pi/2 \) maximal 2-sphere. Then, for
a fixed positive $W$ and $\tau := u - Ww$, let us define $\Sigma_\tau := \{ \tau = \text{const} \}$. These are spacelike hypersurfaces, $-\alpha \pi / 2 \leq \tau \leq \alpha \pi / 2$, and $\Sigma_\tau \cap \cal I^+$ is the cut $r = \pi / 2 - \tau / \alpha$. (Although these hypersurfaces are not smooth at $r = 0$ [in the original coordinates], these conical singularities can be smoothed out.) Hence, $\{ \Sigma_\tau \}$ provides, in fact, a foliation of the conformal boundary of $\cal I$ (see Fig. 4.ii.). One can show [32] that in the de Sitter spacetime their intrinsic geometry is asymptotically hyperboloidal and their extrinsic curvature is asymptotically proportional to their intrinsic metric.

The analog of the three special foliations shown by Figures 3.i., 3.ii. and 4.iii. are used in asymptotically de Sitter spacetimes in a total mass construction in closed universes (section 5), and in the definition of ADM type (section 3) and of TBS type energy-momenta (section 4), respectively. The foliation shown by Fig. 4.iv. (like the ones shown by Fig. 4.ii. in the Minkowski and by Fig. 4.ii. in the anti-de Sitter cases) does not seem to provide an appropriate framework in which the mass-loss property of a TBS type energy-momentum could potentially be proven.

### 2.1.2 The Killing and conformal Killing fields

The de Sitter spacetime is of maximal symmetry: the Lie algebra of its Killing and conformal Killing vectors is $so(1, 4)$, the so-called de Sitter Lie algebra, and $so(2, 4)$, respectively. Since both are semi-simple, generators of ‘translations’ cannot be defined in a natural geometric or algebraic way. On the other hand, there is an analytical characterization of a basis of proper conformal Killing vectors: these are gradients (see e.g. [37]) and any conformal Killing vector is a sum of a gradient conformal Killing vector and a Killing vector.

Clearly, the six independent Killing vectors that are tangent to the hypersurfaces $\Sigma_t$ are spacelike, and the remaining four are timelike on some open subset, and spacelike on the interior of its complement. However, since $\cal I^+$ is spacelike and the isometries map $\cal I^+$ to itself, every spacetime Killing field must be spacelike or zero on $\cal I^+$. For example, the Killing field

$$K^0_{e4} = -\cos r(\nabla_e t) + \alpha \sinh \left( \frac{t}{\alpha} \right) \cosh \left( \frac{t}{\alpha} \right) \sin r(\nabla_e r),$$

given in the global coordinates, is timelike precisely on the domain where $\cos^2 r > \tanh^2 (t/\alpha)$ holds. In the $t \to \infty$ limit the closure of this domain reduces to the two points $(0, 0, 0)$ and $(\pi, 0, 0)$ of the future conformal boundary, where $K^0_{e4}$ vanishes. The remaining three Killing vectors behave in a similar way. Hence, no Killing field is timelike on any open neighbourhood of some asymptotic end of the hypersurfaces $\Sigma_t$ of the intrinsically flat foliation of the steady state part of the de Sitter spacetime. In particular, the Killing field

$$K^a := \left( \frac{\partial}{\partial t} \right)^a + \hat{x} \left( \frac{\partial}{\partial x} \right)^a + \hat{y} \left( \frac{\partial}{\partial y} \right)^a + \hat{z} \left( \frac{\partial}{\partial z} \right)^a, \quad (2.2)$$

given in the coordinates (2.1) on the steady state part of the de Sitter spacetime, is timelike precisely for $\exp(2t/\alpha)(\hat{x}^2 + \hat{y}^2 + \hat{z}^2) < 1$. A simple (although less explicit)

---

5The hypersurfaces $\Sigma_T := \{ T = \text{const} \}$, given in the coordinates $(T, R, \theta, \phi)$ defined by $\cosh(T/\alpha) := \cosh(t/\alpha) \cos r$, $\sinh(T/\alpha) \cosh R := \sinh(t/\alpha)$ (see e.g. [48]), foliate only the disjoint union of the past domain of dependence of the $r \in [0, \pi/2]$ and the $r \in (\pi/2, \pi]$ pieces of $\cal I^+$. Moreover, all of these hypersurfaces intersect $\cal I^+$ in the 2-sphere $r = \pi/2$, rather than giving a foliation of $\cal I^+$ (see Fig. 4.iii.).
demonstration of the non-existence of a \textit{globally timelike} Killing field in a neighbourhood of \( \mathcal{I}^+ \) is given in \[47\].

Clearly, one of the five \textit{conformal} Killing fields, namely \( K_0^e = \cosh(t/\alpha)(\nabla_e t) \), is everywhere timelike, but the remaining four are timelike only on some open subset. For example,

\[
K_4^e = \sinh\left(\frac{t}{\alpha}\right) \cos r (\nabla_e t) - \alpha \cosh\left(\frac{t}{\alpha}\right) \sin r (\nabla_e r)
\]

is timelike precisely when \( \cosh^2(t/\alpha) \cos^2 r > 1 \). In the \( t \to \infty \) limit this reduces to the disjoint union of the \( r \in [0, \pi/2) \) and \( r \in (\pi/2, \pi] \) hemispheres of \( \mathcal{I}^+ \). This vector field is future pointing on the \( r \in (0, \pi/2) \), but past pointing on the \( r \in (\pi/2, \pi] \) hemisphere.

The remaining three conformal Killing fields behave in a similar way. Rewriting the contravariant form of them in the \((\tau, r, \theta, \phi)\) coordinates we find that all these are orthogonal to \( \mathcal{I}^+ \) in the conformal spacetime:

\[
K_0^a = \frac{1}{\alpha} \left( \frac{\partial}{\partial \tau} \right)^a, \quad K_4^a \approx \cos r K_0^a, \quad K_1^a \approx \sin r \sin \theta \cos \phi K_0^a, \quad K_2^a \approx \sin r \sin \theta \sin \phi K_0^a, \quad K_3^a \approx \sin r \cos \theta K_0^a; \tag{2.3}
\]

where \( \approx \) means ‘equal at the points of \( \mathcal{I}^+ \)’. Therefore, the structure of the conformal Killing vectors at the conformal boundary is similar to that of the BMS translations at future null infinity in asymptotically flat spacetimes, though now the number of the conformal Killing vectors is five. Another (twistor theoretical) demonstration of this statement is given in \[47\], where all the 15 conformal Killing vectors are constructed from the independent solutions of the twistor equation, too. Using Dirac spinors, some of these results are also discussed in \[49\]. Some of the ADM type energy-momenta are based on the use of the asymptotic Killing vectors, while others depend on the use of the asymptotic conformal Killing vectors.

### 2.2 Asymptotically de Sitter spacetimes

#### 2.2.1 Fields on de Sitter background

In Yang–Mills theory on the de Sitter background the meaning of the vanishing of the magnetic field strength on \( \mathcal{I}^+ \), i.e. the meaning of the analog of the condition of the intrinsic conformal flatness of \( \mathcal{I}^+ \), is investigated in \[21\]. It is shown that this condition removes half the degrees of freedom of the Yang–Mills field; and all the ten de Sitter fluxes, built from the energy-momentum tensor and the ten conformal Killing fields of \( \mathcal{I}^+ \), vanish. A systematic investigation of the Maxwell fields on and of the linear gravitational perturbation of the (steady state part of the) de Sitter spacetime are given in \[50\]. The covariant phase space method is used to obtain a Hamiltonian for them, in which the ‘time translation’ is a Killing field which is future pointing timelike only in the intersection of the chronological past of the point \((0,0,0)\) of \( \mathcal{I}^+ \) and the steady state universe, but spacelike on the interior of the rest of the steady state universe. The resulting energy flux through \( \mathcal{I}^+ \) can be \textit{negative and arbitrarily large}. The requirement of the vanishing of the magnetic field strength, and of the magnetic part of the rescaled Weyl curvature on \( \mathcal{I}^+ \), removes half the degrees of freedom of the Maxwell and linear gravitational perturbations, respectively, and yields vanishing energy flux through \( \mathcal{I}^+ \) \[50\]. Here the ‘energy’ is defined by using the Killing field \((2.2)\). For a single, isolated source
located at the origin, and in the absence of incoming gravitational waves through the null boundary of the steady state part of the de Sitter spacetime, which is the physical situation of principal interest in the body of work here referred to, this energy is proven to be positive. Under the same assumptions, and following a tour de force of calculation, in [51] a generalisation of the Einstein quadrupole formula is derived and discussed. The summary of these results is given in [52, 53]. For the calculation of the Penrose mass at a cut of $\mathcal{I}^+$ in the linearized theory, see [47] and subsection 4.1 below.

2.2.2 The general definition of asymptotically de Sitter spacetimes

The geometric definition of asymptotically de Sitter spacetimes is based on the idea of conformal compactifiability of the physical spacetime by attaching a boundary hypersurface to it [15]. Explicitly, it is assumed that there is a manifold $\tilde{M}$ with non-empty boundary $\partial\tilde{M}$, a Lorentzian metric $\tilde{g}_{ab}$ on $\tilde{M}$ and a smooth function $\Omega : \tilde{M} \to [0, \infty)$ such that (1) $\tilde{M} - \partial\tilde{M}$ is diffeomorphic to (and hence identified with) $M$; (2) $\tilde{g}_{ab} = \Omega^2 g_{ab}$ on $M$; (3) the boundary is just $\partial\tilde{M} = \{\Omega = 0\}$ and $\nabla_a \Omega$ is nowhere vanishing on $\partial\tilde{M}$; (4) $g_{ab}$ solves Einstein’s equations, $R_{ab} - \frac{1}{2} R g_{ab} = -\kappa T_{ab} - \Lambda g_{ab}$, on $M$, where $\Lambda > 0$; (5) $\tilde{T}^a_{\ b} := \Omega^{-3} T^a_{\ b}$ can be extended to $\tilde{M}$ as a smooth field.

A systematic investigation of the asymptotic structure of spacetimes satisfying these conditions is given in [32, 21]. In particular, the asymptotic form of spacetime metric, the Newman–Penrose spin coefficients, the various pieces of the curvature and the energy-momentum tensor, the geometry of asymptotically hyperboloidal hypersurfaces, etc; and also the asymptotic field equations for them are determined. The conformal boundary $\partial M$ is necessarily spacelike, so it consists of the future and past boundary, $\mathcal{I}^+$ and $\mathcal{I}^-$, respectively. (In what follows, we concentrate only on $\mathcal{I}^+$. In a cosmological setting $\mathcal{I}^-$ will usually not be present, being instead replaced by an initial singularity.) The trace-free part of its extrinsic curvature is vanishing and, in an appropriate conformal gauge, its trace can be made zero; but its intrinsic conformal geometry is still completely unrestricted. Therefore, such a general asymptotically de Sitter spacetime cannot admit any of the usual ten de Sitter Killing fields even asymptotically, otherwise they would extend to conformal Killing fields on $\mathcal{I}^+$. Thus, as the term is usually used, asymptotically de Sitter spacetimes are not the spacetimes that are asymptotic to de Sitter spacetime. The latter form a very special subset of the former.

In fact, $\mathcal{I}^+$ admits ten linearly independent conformal Killing vectors precisely when its intrinsic geometry is conformally flat; which is equivalent to the vanishing of the magnetic part of the rescaled Weyl tensor on $\mathcal{I}^+$. This condition appears to be unreasonably strong: investigations of linear gravitational perturbations of the de Sitter background reveal that this condition would remove half the gravitational degrees of freedom [21, 50] (see subsection 2.2.1 above) and in the full (nonlinear) theory this is confirmed. The existence of a large number of spacetimes with positive $\Lambda$ admitting a general future conformal boundary has been demonstrated by Friedrich [51]. He shows that the free data for the Einstein equations consist of a pair of symmetric tensors $(h_{ab}, \mathcal{E}_{ab})$ with $h_{ab}$ the Riemannian metric of $\mathcal{I}^+$ and $\mathcal{E}_{ab}$ the electric part of the rescaled Weyl tensor at $\mathcal{I}^+$, subject to

$$h^{ab} \mathcal{E}_{ab} = 0, \quad D^a \mathcal{E}_{ab} = 0$$

and with the freedom $(h_{ab}, \mathcal{E}_{ab}) \mapsto (\Theta^2 h_{ab}, \Theta^{-1} \mathcal{E}_{ab})$ for positive functions $\Theta$ on $\mathcal{I}^+$. (Here $D_a$ is the covariant derivative on $\mathcal{I}^+$ defined by $h_{ab}$.) These results, together with those
in [39, 40], show the essential difference between the role of the boundary conditions in the \( \Lambda < 0 \) and \( \Lambda > 0 \) cases: while in the \( \Lambda < 0 \) case some boundary condition must be specified on \( \mathcal{I} \) to have a well posed initial-boundary value problem for the field equations, in the \( \Lambda > 0 \) case similar boundary conditions on \( \mathcal{I}^+ \) restrict the freely specifiable part of the Cauchy data.

A detailed discussion of the various global and asymptotic properties of some special asymptotically de Sitter solutions (viz. the Schwarzschild–de Sitter, Kerr–de Sitter, Vaidya–de Sitter and Friedman–Robertson–Walker) is also given in [21]. In particular, the existence and properties of the horizons, the global topology of the conformal boundary, the group of the globally defined isometries and so on are clarified. Motivated by these special cases, one may wish to impose further conditions, for example the geodesic completeness of the conformal boundary.

3 ADM type total energy-momenta

The definition of the ADM type energy-momenta are based on spacelike hypersurfaces that extend to ‘spatial infinity’; i.e. which are analogous to the foliation of the steady state part of the de Sitter spacetime by intrinsically flat hypersurfaces, given by (2.1). Thus, strictly speaking, such an ADM type energy-momentum is associated with a point \( p \) of the conformal boundary \( \mathcal{I}^+ \) (still conveniently thought of as ‘spatial infinity’), and the construction itself is based on a spacelike hypersurface \( \Sigma \) that is a Cauchy surface for the past domain of dependence of \( \mathcal{I}^+ - \{ p \} \) in the spacetime. These hypersurfaces are tangent to \( \mathcal{I}^+ \) at \( p \), and hence \( \Sigma \) is asymptotically intrinsically flat and extrinsically asymptotically hyperboloidal. If the intrinsic conformal geometry of \( \mathcal{I}^+ \) is homogeneous, e.g. when it is conformally flat, then the energy-momentum is independent of \( p \), but in general it may depend on it. It is obvious how to generalize these ideas to allow finitely many ‘spatial infinities’.

3.1 The Abbott–Deser energy-momentum

The Abbott–Deser energy-momentum [35] is based on the decomposition \( g_{ab} = \bar{g}_{ab} + \gamma_{ab} \) of the spacetime metric into the sum of the de Sitter metric \( \bar{g}_{ab} \) and some tensor field \( \gamma_{ab} \). Then Einstein’s equations can be written in the form \( L G_{ab} = \Lambda \gamma_{ab} - \kappa t_{ab} \), where \( L G_{ab} \) denotes the linearized Einstein tensor on the de Sitter background, built from \( \gamma_{ab} \), and \( t_{ab} \) is the sum of the matter energy-momentum tensor and the correction to the linearized Einstein tensor, being quadratic and higher order in \( \gamma_{ab} \). By the contracted Bianchi identity \( \bar{\nabla}_a t^a_{\ b} = 0 \) holds, where \( \bar{\nabla}_a \) is the covariant derivative in the de Sitter geometry and index raising and lowering are defined by the background metric. Thus, \( t_{ab} \) plays the role of an effective energy-momentum tensor. Therefore, for any Killing vector \( \bar{K}^a \) of the de Sitter background the contraction \( \bar{K}_a t^{ab} \) is \( \bar{\nabla}_b \)-divergence-free, and hence the integral

\[
Q[\bar{K}] := \int_{\Sigma} \bar{K}_a t^{ab} \bar{\xi}_b d\Sigma
\]  

is independent of the hypersurface provided it is finite and free of the ambiguity in the decomposition \( g_{ab} = \bar{g}_{ab} + \gamma_{ab} \), which seems unlikely in general. Here \( \bar{\xi}_a \) is the \( \bar{g}_{ab} \)-unit normal to, and \( d\Sigma \) is the induced volume element on \( \Sigma \). In addition to the implicit assumption that \( Q[\bar{K}] \) is well defined, it is also assumed that the conformal boundary
$\mathcal{I}^+$ is intrinsically conformally flat; otherwise the Killing fields $\bar{K}^a$ could not extend to conformal Killing vectors of $\mathcal{I}^+$ and they, as solutions of the asymptotic Killing equations in a neighbourhood of $\mathcal{I}^+$, would be ambiguously defined. By the conservation $\nabla_b(\bar{K}^a t^{ab}) = 0$ the integral $Q[\bar{K}]$ can be rewritten as a 2-surface integral of an appropriate superpotential on the ‘spatial infinity’ of $\Sigma$ (see also [55, 56, 57]).

According to the traditional prescription [35], the so-called Abbott–Deser (AD) energy is $Q[\bar{K}]$ in which $\bar{K}^a$ is the ‘time translational Killing field’, given explicitly e.g. by (2.2). However, this Killing field is not timelike on any neighbourhood of some asymptotic end of $\Sigma$. (In fact, as we saw in subsection 2.1.2, this holds for any Killing field in a neighbourhood of $\mathcal{I}^+$. ) Nevertheless, this can still be called ‘energy’, but its interpretation is not so obvious. For alternative forms of this energy expression see [21] and (under additional conditions) [57, 48].

Although in certain special cases the positivity of the AD energy can be proven [55, 57, 48], in general it can have any sign [56, 58]. Indeed, in the light of the results of [50] on the energy flux through $\mathcal{I}^+$ of the linear gravitational perturbations preserving the intrinsic conformal flatness of $\mathcal{I}^+$ (see subsection 2.2.1), this notion of energy does not seem to have the rigidity property. To resolve these and above conceptual difficulties, it has already been suggested to use the conformal Killing vector $K_0^a$, given in (2.3), as the generator of the total de Sitter energy [57]. This conformal Killing field is used in the next subsection, too.

### 3.2 A spinorial expression of Kastor and Traschen

Using the original Dirac spinor form of the Nester–Witten form [26, 25], the Witten-type energy positivity proof of [30], originally given for $\Lambda < 0$, has been successfully adapted to the $\Lambda > 0$ case and yielded a positivity argument by Kastor and Traschen in [49] (see also [57]). Here, the energy-momentum tensor of the matter fields is assumed to satisfy the dominant energy condition, and, as the boundary condition for the (modified) Witten equation, the Dirac spinor constituents $\Psi$ of the conformal Killing vectors of the background de Sitter spacetime are used.

A generalization $C^a$ of the conserved current $\bar{K}^a t^{ba}$ of subsection 3.1 built from $t_{ab}$, the ‘perturbation’ $\gamma_{ab}$ and an arbitrary vector field $\xi^a$, is also given in [49]: this $C^a$ is $\nabla_a$-divergence free, and if $\xi^a$ is chosen to be a Killing vector of the de Sitter background, then it reduces to $\bar{K}^a t^{ab}$. Then, it is shown that the flux integral of $C^b$ on $\Sigma$ with the conformal Killing vector $\xi^a$ determined by the Dirac spinor $\Psi$ coincides with the spinorial construction above. Therefore, the present spinorial expression of these authors is a non-trivial generalization of the original construction in [35].

### 3.3 Asymptotically-Schwarzschild/Kerr–de Sitter data: work of Chruściel and collaborators

Chruściel with a range of collaborators has been investigating definitions of mass in a variety of space-times with $\Lambda$ positive, negative or zero for many years. The investigations are typically motivated by adherence to a Hamiltonian approach to space-times, following the monograph [59], analysing Cauchy data to arrive at definitions of mass and momentum. In this subsection we review their work on asymptotically-Schwarzschild–de Sitter data and asymptotically-Kerr–de Sitter data. These are discussed in [60] and [61].
respectively and these in turn rely on families of initial data for the Einstein equations shown to exist in [62] and [63] respectively.

The Schwarzschild–de Sitter metric, sometimes called the Kottler solution, can be defined in space-time dimension $n + 1$ as

$$ds^2 = -V dt^2 + \frac{dr^2}{V} + r^2 \tilde{h}, \quad (3.2)$$

where $\tilde{h}$ is the metric of the standard (round) unit $(n - 1)$-sphere and

$$V = 1 - \frac{2m}{r^{n-2}} - \frac{r^2}{\ell^2}, \quad (3.3)$$

with $m, \ell$ real positive constants. It is straightforward to check that the metric (3.2) satisfies the Einstein equations with cosmological constant $\Lambda = n(n - 1)/(2\ell^2)$.

Typically one wants $\partial/\partial t$ to be a time-like Killing vector at least somewhere, so that $V$ must be positive for some $r$, and it will be, with two positive simple zeroes, provided

$$m^2 \Lambda^{n-2} < c_n := \frac{1}{n^2} \left( \frac{(n-1)(n-2)}{2} \right)^{n-2}. \quad (3.4)$$

On a surface of constant $t$ the metric (3.2) defines the spatial metric

$$g = \frac{dr^2}{V} + r^2 \tilde{h}, \quad (3.5)$$

which is conformally-flat (by spherical symmetry) with constant scalar curvature $R = n(n - 1)/\ell^2$ and necessarily satisfies the Einstein constraint equations with vanishing second fundamental form (so this is time-symmetric data). Given (3.4), the function $V$ has two real distinct positive zeroes in $r$ as noted, but the metric can be extended through both of these (in the style of the Kruskal extension) to obtain a complete, spherically-symmetric, conformally-flat metric with constant scalar curvature. Such metrics are called Delaunay metrics in [62] by analogy with Delaunay surfaces which are complete, rotationally-symmetric, constant mean curvature surfaces in $\mathbb{R}^3$. The Delaunay metrics also arise from singular solutions of the Yamabe problem on $S^n$ (see the discussion in [62]).

A similar but more general class of solutions of the Einstein equations than those in (3.2) was published by Birmingham [64]. The metric looks the same but $\tilde{h}$ is taken to be a Riemannian Einstein metric on a compact $(n - 1)$-manifold with scalar curvature $\tilde{R}$, and $V$ is changed to

$$V = \frac{\tilde{R}}{(n-1)(n-2)} - \frac{2m}{r} - \frac{r^2}{\ell^2}. \quad (3.6)$$

With positive $m$, we need positive $\tilde{R}$ to have $V$ anywhere positive and then constant rescaling and redefinition of $r$ can be used to set $\tilde{R} = (n - 1)(n - 2)$ to recover the previous expression for $V$. Now one can introduce a notion of generalised Delaunay metrics with this $\tilde{h}$ in place of the previous $\tilde{h}$.

In [62] the authors use gluing techniques to produce new time-symmetric data for the Einstein equations which have many exactly Delaunay or generalised Delaunay ends, which is to say that the ends are exactly Delaunay or generalised Delaunay for infinite...
stretches. These ends can then in turn be truncated and glued on to other regions to produce data on connected sums or wormholes. In \[61\], the authors use the Hamiltonian method to assign masses to the simplest case of initial data in this class, a single region with several Delaunay ends. The result is that the Hamiltonian mass of each end is precisely the mass parameter \(m\) in the appropriate metric \((3.5)\) (and \textit{a fortiori} it is positive).

In \[63\], Cortier generalised the work in \[62\] to produce initial data which are deformations of that for Kerr–de Sitter (but only in space-time dimension 4, so the data surface has dimension 3). Again one has an infinite periodic metric but now with a particular nonzero periodic second fundamental form as well. Then in \[61\], the authors use the Hamiltonian method to assign both mass and angular momentum to these data.

This work, in associating a mass with each Delaunay or generalised Delaunay end, is reminiscent of the calculation of Penrose’s quasi-local mass for various 2-spheres in conformally-flat, time-symmetric initial data for the vacuum Einstein equations (so \(\Lambda = 0\)) in \[65\]. In particular, the mass defined can be thought of as a ‘mass at spatial infinity’, given a Delaunay or generalised Delaunay end that extends to spatial infinity, but it is not a mass at null infinity.

4 TBS type total energy-momenta

In contrast to the ADM type energy-momenta, the TBS type expressions are associated with \textit{closed orientable 2-surfaces} \(S\), that is ‘cuts’, rather than points of \(I^+\). Then, to be potentially able to prove a formula analogous to Bondi’s mass-loss for the TBS type expressions, we should have, in fact, a \textit{foliation} of \(I^+\) by such cuts. Thus, the spacelike hypersurfaces \(\Sigma\) that intersect \(I^+\) in the given cuts should be analogous to the hypersurfaces \(\Sigma_\tau\) discussed at the end of subsection 2.1.1.

4.1 Linearization around de Sitter

In \[47\] some general twistor theory of de Sitter spacetime is recorded, sufficient to describe Penrose’s quasi-local mass construction at any space-like 2-sphere for linear gravitational perturbations of de Sitter. This includes 2-spheres at \(I^+\), and it is pointed out that the absence of timelike Killing fields near \(I^+\) entails that this quasi-local mass does not have a positivity property, in contrast to the \(\Lambda = 0\) and \(\Lambda < 0\) cases. It can be the basis of a definition in asymptotically-de Sitter space-times but it won’t have positivity or rigidity properties.

4.2 Three Suggestions of Penrose

In \[46\] Penrose considered the problem of defining a cosmological total mass in the presence of a positive cosmological constant. He was motivated, at least in part, by his Conformal Cyclic Cosmology (or CCC) rather than by a consideration of isolated systems, which was the motivation of Ashtekar and colleagues considered above. Thus Penrose sought a definition at \(I^+\) of one aeon which was suitable for carrying through to the next aeon. Interestingly, Penrose considered but rejected the idea of working in a Poincaré patch and using the Killing vector of the steady-state universe to define energy (section 3 of \[46\]) because galaxies beyond the cosmological event horizon, where this Killing vector is space-like, would be represented as having super-luminal velocities. This could be a
problem in a cosmological setting but one that Ashtekar et. al. avoid by considering only isolated bodies.

Penrose makes three concrete suggestions:

1. To seek an expression motivated by the conserved currents that one constructs from a trace-free energy-momentum tensor and a conformal Killing vector of de Sitter space. In CCC one expects matter to become massless near \( I^+ \), so that \( T_{ab} \) will become trace-free, and in de Sitter space there do exist time-like conformal Killing vectors. This suggestion is not worked out in detail but the insight also underlay the earlier work of [49], [55] and [57].

2. To use his quasi-local energy-momentum construction, as discussed above (see subsection 4.1), at \( I^+ \). This construction is defined for a space-like 2-surface \( S \), usually a topological sphere. One first solves an elliptic system for the 2-surface twistors on \( S \), which are particular 2-component spinor fields, then forms an integrand linear in the spacetime curvature but with the cosmological constant term removed and integrates over \( S \). This is a well-defined procedure but the result does not have any positivity or therefore rigidity property. In particular it can be zero even in the presence of non-trivial curvature [32]: applied to 2-spheres on the \( I^+ \) of the Schwarzschild–de Sitter space-time it gives zero if the 2-sphere \( S \) is homologous to a point on \( I^+ \) and a non-zero constant but of either sign if \( S \) surrounds the source.

3. To use the original TBS energy expression, given in the Newman–Penrose form as

\[
E = \frac{2}{\kappa} \oint_S (\sigma \sigma' - \psi_2) dS, \tag{4.1}
\]

where \( \psi_2 \) is the \( o^Ao^Bo^Co^D \) component of the Weyl curvature spinor and \( \sigma \) and \( \sigma' \) are the asymptotic shear of the two null geodesic congruences hitting the cut \( S \) of \( I^+ \) orthogonally. This is the basis of the suggestion taken up by Saw [68, 69, 73], following Frauendiener [66, 67].

4.3 Mass-loss and the solution of the NP equations

The key property of the TBS mass in the asymptotically flat context is mass-loss: the mass, as a function of the retarded time coordinate, is a monotonically decreasing function. Thus, accepting (4.1) as the definition of the TBS energy in the \( \Lambda > 0 \) case, the natural question is whether or not the analogue of the mass-loss formula can be derived. Such a formula could be based directly on the analysis of the Bianchi identities, or on the general integral formula of Frauendiener [66, 67]. However, to derive this, one should solve the Newman–Penrose spin coefficient equations to an appropriately high order. This solution, both for the vacuum and electro-vacuum in the physical spacetime, is given by Saw in [68] and [69], respectively (see also [70]). The analogue of the mass-loss formula is derived from the Bianchi identity in [68, 69], using an integral identity in [73]. In [73], with \( E \) as in (4.1), he finds

\[
\dot{E} = -\frac{2}{\kappa} \oint_S \left( |\sigma|^2 + \frac{\Lambda}{2} |\sigma'|^2 - \frac{\Lambda}{6} |\sigma|^2 + \frac{2\Lambda}{9} |\sigma|^4 + \frac{\Lambda^2}{18} \Re(\bar{\sigma}\psi_0) \right) dS, \tag{4.2}
\]

where the dot denotes derivative with respect to the parameter \( u \) which labels the actual cut \( S \) of the foliation of \( I^+ \) by 2-surfaces \( S_u \), \( \Re \) denotes ‘real part’, \( \delta \) and \( \delta' \) are the standard GHP edth and edth-prime operators, respectively, and \( \psi_0 \) is the \( o^Ao^Bo^Co^D \) component of the Weyl curvature spinor. Saw shows that, in the absence of incoming
radiation, so that \( \psi_0 = 0 \), and for purely quadrupole gravitational radiation (which here is taken to mean that \( \sigma = 0 \), which would imply, for a sphere with constant Gauss curvature, that \( \sigma \) is proportional to a combination of the \( Y_{2m} \) spin weighted spherical harmonics) \( \dot{E} \) is nonpositive and vanishes only for zero \( \sigma \). He also notes, in [68], that the vanishing of \( \sigma \) on \( \mathcal{I}^+ \) is equivalent to the intrinsic conformal flatness of \( \mathcal{I}^+ \), in agreement with [21].

Saw also makes use of another expression for \( \dot{E} \), equivalent to (4.2) by an identity for functions on a sphere, namely

\[
\dot{E} = -\frac{2\Lambda}{3\kappa} \int_S K|\sigma|^2 dS - \frac{2}{\kappa} \int_S \left( |\dot{\sigma}|^2 + \frac{\Lambda}{3} |\nabla \sigma|^2 + \frac{2\Lambda}{9} |\sigma|^4 + \frac{\Lambda^2}{18} \Re(\bar{\sigma} \psi_0) \right) dS, \tag{4.3}
\]

where \( K \) is the Gauss curvature of \( S \) (which need not have fixed sign). This is the basis of a suggestion he makes for a different analogue of the TBS mass in the \( \Lambda > 0 \) case, namely

\[
E_\Lambda(u) := E(u) + \frac{2\Lambda}{3\kappa} \int_{u_0}^u \left( \int_{S_\bar{u}} K|\sigma|^2 dS_\bar{u} \right) d\bar{u}
\]

with \( E(u) \) as in (4.1), and for some \( u_0 \). Now the \( u \)-derivative of \( E_\Lambda \) consists of just the second integral in (4.3) and in the absence of incoming radiation, but this time without the restriction to quadrupole radiation, it is therefore negative (strictly speaking it is nonnegative and vanishes only for zero \( \sigma \)). In this mass-loss formula there is a term proportional to \( |\dot{\sigma}|^2 \) that is familiar from the case of \( \Lambda = 0 \), but there are also terms involving the shear and its angular derivatives but undotted. However, this proposed new expression depends not only on the instantaneous state of the physical system at \( u \), but on the whole history of the system until \( u \). A summary of the results of the investigations of Saw prior to [73] is given in [71, 72].

### 4.4 A general TBS-type spinorial expression

As noted above, a key property of the TBS energy in the asymptotically flat case is its positivity and rigidity. Since in the \( \Lambda = 0 \) and \( \Lambda < 0 \) cases the total energy-momenta could be recovered from a unified form based on the integral of the Nester–Witten form (1.2), and moreover the simplest energy positivity proof is arguably the one based on the use of 2-component spinors and Witten-type arguments [28, 30], it seems natural to try to formulate the TBS energy-momentum in the presence of a positive \( \Lambda \) in this framework, too. This was done in [32]. However, since it is not \textit{a priori} clear what the ‘asymptotic time translations’ near \( \mathcal{I}^+ \) should be, in the investigations of [32] the spinor fields at the cut are initially left unspecified. Surprisingly enough, the requirement of the finiteness of the resulting integral together with the desire to have a Witten-type energy positivity proof determine the spinor fields in (1.2) at the cut: \textit{They should solve Penrose’s 2-surface twistor equations.} Thus, the \textit{boundary conditions} for the Witten spinors, i.e. the spinor constituents of what one could considered to be the ‘asymptotic translations’, \textit{come out of the formalism.}

The integral of the Nester–Witten form with these boundary conditions defines a \( 4 \times 4 \) complex Hermitian matrix with the structure

\[
H = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}.
\]
given an origin for sections choices are possible, the authors obtain an expansion:

\[ \Lambda \]

varieties of examples, with the for which a corresponding calculation was presented in [76], and, by consideration of a \( N \) at a renormalised volume \( \Lambda = 0 \) as required. The paper proposes a definition of mass from a consideration of the TBS making assumptions about asymptotic decay rates, for example of any matter content, \( a \) for coefficients \( \Lambda \) all values of \( n \) (compactifiable fields satisfy the dominant energy condition, then it has been shown that \( H \) is positive definite (positivity), and it is vanishing precisely when the past domain of dependence of the (asymptotically hyperboloidal) spacelike hypersurface \( \Sigma \) for which \( \Sigma \cap S^+ \) is just the given cut \( S \) is locally isometric to the de Sitter spacetime (rigidity). The symmetry group of the 2-surface twistor space \( T \) on \( S \) is \( (0, \infty) \) times the spin group of \( SO(1, 5) \). If there exists a volume 4-form on \( T \), then the TBS-type mass can be defined as the determinant of \( H \). For example, if there is a scalar product on \( T \) (which with \((+, +, -, -)\) signature is guaranteed e.g. when \( S^+ \) is intrinsically conformally flat), then such a volume form exists. In these special cases the symmetry group reduces to the spin group of \( SO(1, 5) \) and to the spin group of the de Sitter group \( SO(1, 4) \), respectively.

The same general analysis can be repeated in the \( \Lambda < 0 \) and \( \Lambda = 0 \) cases, too [74]. If \( \Lambda < 0 \), then the result is similar but the symmetry group of \( T \) is \( (0, \infty) \) times the spin group of \( SO(3, 3) \), which, in the presence of the reflective boundary condition, reduces to the spin group of the anti-de Sitter group \( SO(2, 3) \). If \( \Lambda = 0 \), then \( T \) splits to the direct sum \( T_0 \oplus T_0 \), where \( T_0 \) is the space of the spinor constituents of the BMS translations, \( Q \) in \( H \) is zero, \( P \) reduces to the TBS 4-momentum, and the symmetry group of \( T \) is \( SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \). Therefore, the above construction is a natural generalization of the TBS 4-momentum of asymptotically flat spacetimes to the \( \Lambda > 0 \) case. It has positivity and rigidity, at the cost of being a matrix with six independent real components rather than a Lorentz scalar or vector.

4.5 Characteristic data: the work of Chruściel–Ifsits

In [75] Chruściel and Ifsits define a mass from data for the Einstein equations given on an outgoing null hypersurface \( \mathcal{N} \), which could be the null cone of a point, in a conformally-compactifiable \((n+1)\)-dimensional space-time. The main interest in [76] is with \( \Lambda > 0 \) but all values of \( \Lambda \) are allowed, and the method is to construct a Bondi coordinate system at \( \mathcal{N} \) and solve for the space-time metric and connection at \( \mathcal{N} \) in terms of free data, making assumptions about asymptotic decay rates, for example of any matter content, as required. The paper proposes a definition of mass from a consideration of the TBS mass defined for \( \Lambda = 0 \). The generalised definition is checked against the \( \Lambda = 0 \) case, for which a corresponding calculation was presented in [76], and, by consideration of a variety of examples, with the \( \Lambda < 0 \) case, which is fairly well understood.

The calculation leading to the definition of mass naturally leads to the definition of a renormalised volume for \( \mathcal{N} \). This is defined from the integral \( V(r) \) of the area \( A(r) \) of section\(^7\) of \( \mathcal{N} \) of constant \( r \) where \( r \) is an affine parameter along the generators of \( \mathcal{N} \). Given an origin for \( r \), which would be the vertex of \( \mathcal{N} \) if \( \mathcal{N} \) were a light-cone but other choices are possible, the authors obtain an expansion:

\[
V(r) = \int_0^r A(r')dr' = a_3 r^3 + a_2 r^2 + a_1 r + a_0 \log r + a_0 + a_{-1} r^{-1} + o(r^{-1}),
\]

for coefficients \( a_k \) given in terms of the data and quantities obtained from the data. Then the coefficient \( a_0 = V_{\text{ren}} \) is the renormalised volume.

In discussion of the result in the last section of the paper it is observed that the mass defined is geometric and gauge-invariant and coincides in cases of \( \Lambda \leq 0 \) with other

\(^7\)In space-time dimension \( n + 1 \), \( A \) is the \((n - 1)\)-dimensional volume of these sections.
accepted definitions, but that it is not obviously non-negative in general, nor rigid in the sense that vanishing mass implies that the space-time is exactly de Sitter inside the cone \( \mathcal{N} \).

5 Total mass in closed universes

The construction of the total mass for closed universes, introduced first for \( \Lambda = 0 \) (and mentioned at the end of subsection 1.1.3), can be generalized in a straightforward way to closed universes with positive \( \Lambda \) \([77]\). The basis of this construction is the observation that, in the Witten type gauge, the \( \lambda^A \bar{\lambda}^A \) -component of the hypersurface integral form of the spinorial expression of all the energy-momentum expressions above (independently of the sign of \( \Lambda \)) takes the manifestly positive definite expression

\[
P(\lambda) := \frac{\sqrt{2}}{\kappa} \| \mathcal{D}_{(AB}\lambda_C) \|_{L^2}^2 + \int_{\Sigma} t^a T_{ab} \lambda^B \bar{\lambda}^B d\Sigma.
\]

Here \( \Sigma \) is the asymptotically flat/hyperboloidal hypersurface whose boundary ‘at infinity’ is just the 2-surface \( S \), the \( L^2 \)-norm is defined on this \( \Sigma \) with \( \sqrt{2}t_{AA'} \) as the pointwise positive definite Hermitian scalar product on the spinor spaces, and \( \mathcal{D}_{AB} \) is the unitary spinor form of the Sen derivative operator on \( \Sigma \) (see \([38]\)).

However, if \( \Sigma \) is a compact Cauchy surface in a closed universe, e.g. when \( \Sigma \) is analogous to the leaves \( \Sigma_t \) of the global foliation of the de Sitter spacetime given in subsection 2.1.1, then the above quantity can be formed even when \( T_{ab} \) is replaced by \( T_{ab} + g_{ab} \Lambda / \kappa \) with positive \( \Lambda \) and even for any spinor field \( \lambda^A \) with the normalization \( \| \lambda \|_{L^2}^2 = \sqrt{2} \). The total mass \( M \) (in fact, mass density) associated with \( \Sigma \) is defined to be just the infimum of \( P(\lambda) \) on the set of spinor fields satisfying the normalization above. Then, the spinor fields \( \lambda^A \) for which \( P(\lambda) = M \) holds are precisely the eigenspinors in the eigenvalue equation \( \mathcal{D}^{AA'} \mathcal{D}_{A'B} \lambda^B = (3\kappa/8) M \lambda^A \). Thus, the mass \( M \) could have been defined as the first eigenvalue in this eigenvalue problem. Clearly, \( \kappa M \geq \Lambda \) by construction, but it has the non-trivial rigidity property: \( \kappa M = \Lambda \) if and only if the whole spacetime is locally isometric to the de Sitter spacetime and the Cauchy surface \( \Sigma \) is homeomorphic to \( S^3/G \), where \( G \) is a discrete subgroup of \( SU(2) \approx S^3 \).

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References

[1] A. Trautman, Conservation laws in general relativity, in Gravitation: An Introduction to Current Research, Ed: L. Witten, pp. 169-198, Wiley, New York 1962

A. Trautman, F. A. E. Pirani, H. Bondi, Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics, New Jersey 1964

[2] C. Misner, K. Thorne, J. A. Wheeler, Gravitation, Freeman, San Francisco 1973

[3] J. Goldberg, Invariant transformations, conservation laws and energy-momentum, in General Relativity and Gravitation, Ed.: A. Held, vol 1, pp 469-489, Plenum, New York 1980
[4] G. A. J. Sparling, Twistor, spinors and the Einstein vacuum equations, Pittsburgh preprint, 1982; also in Further Advances in Twistor Theory, III: Curved Twistor Spaces, Ed.: L. Mason et al., pp. 179-186, Chapman and Hill, London 2001

[5] M. Dubois-Violette, J. Madore, Conservation laws and integrability conditions for gravitational and Yang-Mills equations, Commun. Math. Phys. 108 213-223 (1987)

[6] J. Frauendiener, Geometric description of energy-momentum pseudotensors, Class. Quantum Grav. 6 L237-L241 (1989)

[7] L. B. Szabados, On canonical pseudotensors, Sparling’s form and Noether currents, Class. Quantum Grav. 9 2521-2541 (1992); preprint KFKI-1991-29/B

[8] L. B. Szabados, Quasi-local energy-momentum and angular momentum in GR, Living Rev. Relativity 12 (2009) No 4.

[9] S.W. Hawking, G. F. R. Ellis, The Large Scale Structure of Spacetime, Cambridge University Press, Cambridge 1973

[10] R. Arnowitt, S. Deser, C. W. Misner, The dynamics of general relativity, in Gravitation: An Introduction to Current Research, pp. 227–265, Ed.: Witten, L., Wiley, New York, London, 1962, arXiv: gr-qc/0405109

[11] A. Trautman, Lectures on general relativity (Lectures at King’s College in London, May-June 1958), Gen. Relativity Grav. 34 721-762 (2002)

[12] H. Bondi, Gravitational waves in general relativity, Nature, 186 535 (1960)

H. Bondi, M. G. J. van der Burg, A. W. K. Metzner, Gravitational waves in general relativity. VII. Waves from axi-symmetric isolated systems, Proc. R. Soc. London, Ser. A 269 21-52 (1962)

[13] R. K. Sachs, Asymptotic symmetries in gravitational theory, Phys. Rev. 128 2851–2864 (1962)

[14] R. Penrose, Zero rest-mass fields including gravitation: asymptotic behaviour, Proc. Roy. Soc. (London) A 284 159-203 (1965)

[15] R. Penrose, W. Rindler, Spinors and Spacetime, vol 2, Cambridge University Press, Cambridge 1986

[16] R. P. A. C. Newman, The global structure of simple space-times, Commun. Math. Phys., 123 17–52 (1989)

[17] R. Geroch, Asymptotic structure of spacetime, in Asymptotic structure of spacetime, Ed. F. P. Esposito, L. Witten, Plenum Press, New York 1977

[18] E. T. Newman, K. P. Tod, Asymptotically flat spacetimes, in General Relativity and Gravitation, vol 2, Ed. A. Held (New York, Plenum) 1980

[19] J. Frauendiener, Conformal infinity, Living Rev. Relativ. (2004) 7: 1. https://doi.org/10.12942/lrr-2004-1
[20] R. Geroch, J. Winicour, Linkages in general relativity, J. Math. Phys. 22 803-812 (1981)

[21] A. Ashtekar, B. Bonga, A. Kesavan, Asymptotics with a positive cosmological constant: I. Basic framework, Class. Quant. Grav. 32 025004 (41pp) (2014), arXiv: 1409.3816 [gr-qc]

[22] L. Andersson, P. T. Chruściel, Hyperboloidal Cauchy data for vacuum Einstein equations and obstructions to smoothness of null infinity, Phys. Rev. Lett. 70 2829-2832 (1993), arXiv: gr-qc/9304019

J. A. Valiente-Kroon, A new class of obstructions to the smoothness of null infinity, Commun. Math. Phys. 244 133-156 (2004)

[23] L. B. Szabados, On certain quasi-local spin-angular-momentum expressions for large spheres near the null infinity, Class. Quantum Grav. 18 5487-5510 (2001), arXiv: gr-qc/0109047

[24] J. Winicour, L. Tamburino, Lorentz-covariant gravitational energy-momentum linkages, Phys. Rev. Lett. 15 601-605 (1965)

[25] J. M. Nester, A new gravitational energy expression with a simple positivity proof, Phys. Lett. A 83 241-241 (1981)

[26] E. Witten, A new proof of the positive energy theorem, Commun. Math. Phys. 80 381-402 (1981)

[27] R. Schoen, S.-T. Yau, Proof of the positive mass theorem, II. Commun. Math. Phys. 79 231-260 (1981)

[28] G. Horowitz, K. P. Tod, A relation between local and total energy in general relativity, Commun. Math. Phys. 85 429–447 (1982)

[29] O. Reula, K. P. Tod, Positivity of the Bondi energy, J. Math. Phys. 25 1004-1008 (1984)

[30] G. W. Gibbons, S. W. Hawking, G. T. Horowitz, M. J. Perry, Positive mass theorems for black holes, Commun. Math. Phys. 88 295-308 (1983)

[31] Y. Choquet-Bruhat, D. Christodoulou, Elliptic systems in $H_{s,\delta}$ spaces on manifolds which are Euclidean at infinity, Acta. Math. 146 129-150 (1981)

[32] L. B. Szabados, P. Tod, A positive Bondi-type mass in asymptotically de Sitter spacetimes, Class. Quantum Grav. 32 (2015) 205011 (pp 51), arXiv: 1505.06637 [gr-qc]

[33] L. B. Szabados, Mass, gauge conditions and spectral properties of the Sen–Witten and 3-surface twistor operators in closed universes, Class. Quantum Grav. 29 095001 (2012) (30pp), arXiv: 1112.2966 [gr-qc]

[34] L. B. Szabados, On the total mass of closed universes, Gen. Rel. Grav. 45 2325-2339 (2013), arXiv:1212.0147 [gr-qc]
35] G. Abbott, S. Deser, Stability of gravity with a cosmological constant, Nucl. Phys. B 195 76-96 (1982)

36] A. Ashtekar, A. Magnon, Asymptotically anti-de Sitter spacetimes, Class. Quantum Grav. 1 L39-L44 (1984)

37] S. W. Hawking, The boundary conditions for gauged supergravity, Phys. Lett. 126 B 175-177 (1983)

38] K. P. Tod, Three-surface twistors and conformal embedding, Gen. Rel. Grav. 16 435-43 (1984)

39] H. Friedrich, G. Nagy, The initial boundary value problem for Einstein’s vacuum field equation, Commun. Math. Phys. 201 619-655 (1999)

H. Friedrich, Initial boundary value problems for Einstein’s field equations and geometric uniqueness, Gen. Relativ. Grav. 41 1947-1966 (2009), arXiv: 0903.5160

40] H. Friedrich, On the AdS stability problem, Class. Quantum Grav. 31 (2014) 105001, arXiv: 1401.7172

41] P. T. Chruściel, G. Nagy, The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times, Adv. Theor. Math. Phys. 5 697-754 (2002), arXiv: gr-qc/0110014

42] R. Kelly, Asymptotically anti-de Sitter spacetimes, Twistor Newsletter, No 20, 11–23 (1985)

R. Kelly, P. Tod, Penrose’s quasi-local mass for asymptotically anti-de Sitter space-times, arXiv: 1505.00214

43] A. G. Riess etal, Observational evidence from supernovae for an accelerating Universe and a cosmological constant, Astron. J. 116 1009-1038 (1998), arXiv: astro-ph/9805201

44] S. Perlmutter etal, Measurements of Omega and Lambda from 42 high-redshift supernovae, Astrophys. J. 517 565-586 (1999), arXiv: astro-ph/9812133

45] R. Penrose, Cycles of Time, The Bodley Head, London 2010

46] R. Penrose, On cosmological mass with positive \( \Lambda \), Gen. Relat. Grav. 43 3355-3366 (2011)

47] P. Tod, Some geometry of de Sitter space, arXiv: 1505.06123 [gr-qc]

48] M. Luo, N. Xie, X. Zhang, Positive mass theorems for asymptotically de Sitter spacetimes, Nucl. Phys. B 825 98-118 (2010), arXiv: 0712.4113v3 [math.DG]

49] D. Kastor, J. Traschen, A positive energy theorem for asymptotically de Sitter spacetimes, Class. Quantum Grav. 19 5901-5920 (2002)

50] A. Ashtekar, B. Bonga, A. Kesavan, Asymptotics with a positive cosmological constant: II. Linear fields on de Sitter spacetime, Phys. Rev. D 92 044011 (2015), arXiv: 1506.06152 [gr-qc]
[51] A. Ashtekar, B. Bonga, A. Kesavan, Asymptotics with a positive cosmological constant: III. The quadruple formula, Phys. Rev. D 92 10432 (2015), arXiv: 1510.05593 [gr-qc]

[52] A. Ashtekar, B. Bonga, A. Kesavan, Gravitational waves from isolated systems: Surprising consequences of a positive cosmological constant, Phys. Rev. Lett. 116 051101 (2016), arXiv: 1510.04990 [gr-qc]

[53] A. Ashtekar, Implications of a positive cosmological constant for general relativity, Rep. Prog. Phys. 80 102901 (2017), arXiv: 1706.07482 [gr-qc]

[54] H. Friedrich, Geometric asymptotics and beyond, in *Surveys in Differential Geometry*, Vol.20. Ed.: L. Bieri, S.-T. Yau, International Press, Boston 2015; arXiv: 1411.3854
H. Friedrich, Smooth non-zero rest-mass evolution across timelike infinity, arXiv: 1311.0700 [gr-qc]

[55] T. Shiromizu, Positivity of gravitational mass in asymptotically de Sitter spacetimes, Phys. Rev. D 49 5026-5029 (1994)

[56] K. Nakao, T. Shiromizu, K. Maeda, Gravitational mass in asymptotically de Sitter spacetimes, Class. Quantum Grav. 11 2059-2071 (1994)

[57] T. Shiromizu, D. Ida, T. Torii, Gravitational energy, dS/CFT correspondence and cosmic no-hair, JHEP 0111 010, arXiv: hep-th/0109057

[58] Z. Liang, X. Zhang, Spacelike hypersurfaces with negative total energy in de Sitter spacetime, J. Math. Phys. 53 022502 (2012), arXiv: 1105.1213v2 [gr-qc]

[59] P. T. Chruściel, J. Jezierski, J. Kijowski, *Hamiltonian field theory in the radiating regime*. Lecture Notes in Physics. Monographs, 70 Springer-Verlag, Berlin, 2002.

[60] P. T. Chruściel, J. Jezierski, J. Kijowski, The Hamiltonian mass of asymptotically Schwarzschild-de Sitter space-times, Phys. Rev. D 87 (2013) 124015, arXiv: 1305.1014

[61] P. T. Chruściel, J. Jezierski, J. Kijowski, Hamiltonian dynamics in the space of asymptotically Kerr-de Sitter spacetimes, Phys. Rev. D 92 (2015) 084030, arXiv: 1507.03868

[62] P. T. Chruściel, D. Pollack, Singular Yamabe metrics and initial data with exactly Kottler-Schwarzschild-de Sitter ends, Ann. Henri Poincaré 9 (2008) 639-654, arXiv: 0710.3365

[63] J. Cortier, Gluing construction of initial data with Kerr–de Sitter ends, Ann. Henri Poincaré 14 (2013) 1109–1134, arXiv: 1202.3688

[64] D. Birmingham, Topological black holes in anti-de Sitter space, Class. Quantum Grav. 16 (1999) 1197–1205, arXiv: hep-th/9808032

[65] K. P. Tod, Some examples of Penrose’s quasi-local mass construction, Proc. Roy. Soc. London A388 457–477 (1983)
[66] J. Frauendiener, On an integral formula on hypersurfaces in general relativity, Class. Quantum Grav. **14** 2413-2423 (1997)

[67] J. Frauendiener, Talk given at the workshop *Mathematics of CCC: Mathematical Physics with Positive Lambda*, The Clay Mathematics Institute, University of Oxford, September 11-13, 2013

[68] V.-L. Saw, Mass-loss of an isolated gravitating system due to energy carried away by gravitational waves, with cosmological constant, Phys. Rev. D **94** 104004 (2016), arXiv: 1605.05151 [gr-qc]

[69] V.-L. Saw, Behaviour of asymptotically electro-Λ spacetimes, Phys. Rev. D **95** 084038 (2017), arXiv: 1608.06886 [gr-qc]

[70] V.-L. Saw, Peeling property as a consequence of the cosmological constant, arXiv: 1705.00435 [gr-qc]

[71] V.-L. Saw, Asymptotically simple spacetimes and mass loss due to gravitational waves, Int. J. Mod. Phys. D **26** (2017) 1730027, arXiv: 1706.00160 [gr-qc]

[72] V.-L. Saw, Mass-loss due to gravitational waves with Λ > 0, Mod. Phys. Lett. A **32** 1730020 (2017), arXiv: 1704.07514 [gr-qc]

[73] V.-L. Saw, Bondi mass with a cosmological constant, Phys. Rev. D **97** 084017 (2018), arXiv: 1711.01808 [gr-qc]

[74] L. B. Szabados, P. Tod, (unpublished)

[75] P. T. Chruściel, L. Ifsits, The cosmological constant and the energy of gravitational radiation, Phys. Rev. D **93** 124075 (2016), arXiv: 1603.07018 [gr-qc]

[76] P. T. Chruściel, T.-T. Paetz, The mass of light-cones, Class. Quantum Grav. **31** (2014) 102001, arXiv:1401.3789

[77] L. B. Szabados, On the total mass of closed universes with a positive cosmological constant, Class. Quantum Grav. **30** 165013 (2013), arXiv: 1306.3863 [gr-qc]