Unified description of intrinsic spin-Hall effect mechanisms

T Fujita\textsuperscript{1,2}, M B A Jalil\textsuperscript{1,3} and S G Tan\textsuperscript{2,3}

\textsuperscript{1} Information Storage Materials Laboratory, Electrical and Computer Engineering Department, National University of Singapore, 4 Engineering Drive 3, Singapore 117576, Singapore
\textsuperscript{2} Data Storage Institute, A*STAR (Agency for Science, Technology and Research) DSI Building, 5 Engineering Drive 1, Singapore 117608, Singapore
\textsuperscript{3} Computational Nanoelectronics and Nano-device Laboratory, Electrical and Computer Engineering Department, National University of Singapore, 4 Engineering Drive 3, Singapore 117576, Singapore

E-mail: elembaj@nus.edu.sg

*New Journal of Physics* 12 (2010) 013016 (15pp)

Received 16 July 2009
Published 19 January 2010
Online at http://www.njp.org/
doi:10.1088/1367-2630/12/1/013016

**Abstract.** The intrinsic spin-Hall effects (SHEs) in p-doped semiconductors (Murakami *et al* Science 301 1348) and two-dimensional electron gases with Rashba spin–orbit coupling (Sinova *et al* 2004 *Phys. Rev. Lett.* 92 126603) have been the subject of many theoretical studies, but their driving mechanisms have yet to be described in a unified manner. The former effect arises from the adiabatic topological curvature of momentum space, from which holes acquire a spin-dependent anomalous velocity. The SHE in Rashba systems, on the other hand, results from momentum-dependent spin dynamics in the presence of an external electric field. Our motivation in this paper is to address the disparity between the two mechanisms and, in particular, to clarify whether there is any underlying link between the two effects. In this endeavor, we consider the explicit time dependence of SHE systems starting with a general spin–orbit model in the presence of an electric field. We find that by performing a gauge transformation of the general model with respect to time, a well-defined gauge field appears in time space which has the physical significance of an effective magnetic field. This magnetic field is shown to precisely account for the SHE in the Rashba system in the adiabatic limit. Remarkably, by applying the same limit to the equations of motion of the general model, this magnetic field is also found to be the underlying origin of the anomalous velocity due to the momentum-space curvature. Thus, our study unifies the two seemingly disparate intrinsic SHEs under a common adiabatic framework.
1. Introduction

The spin-Hall effects (SHEs) are a family of phenomena in which an applied longitudinal electric field gives rise to a transverse spin current. The spin current arises from the transverse separation of spin species in the system, and the physics driving the separation mechanism can be quite distinct across different systems. The earliest prediction of transverse spin separation was made by D’yakonov and Perel’ [1] in the 1970s, who studied the spin-dependent scattering mechanisms of carriers with localized impurities. Such SHEs, which occur as a result of the spin–orbit coupling (SOC) between carriers and impurities, are classified in the literature as extrinsic. Conversely, there are intrinsic forms of the SHE, which have become an active field of research in more recent years [2]–[5], following two seminal papers: the first by Murakami et al [6] which predicts a SHE of holes in p-doped semiconductors described in the Luttinger model, and the other by Sinova et al [7] in two-dimensional electron gases (2DEG) formed in semiconductor heterostructures with Rashba SOC. Both effects are finite in the absence of impurity scattering, and are characterized by a pure transverse spin current (i.e. with no accompanying charge current) generated from the SOC that is built-in to the band structure of the system.

We focus on the intrinsic contributions to the SHE in this paper. It is intriguing that the two SHEs in [6, 7], although both intrinsic in nature, appear to originate from distinct mechanisms. The former effect in [6] is an adiabatic effect described via a gauge potential which arises from the relaxation of hole spins to an effective magnetic field in momentum (\(\vec{k}\)) space. The topological Berry curvature [12] of the gauge potential has the physical significance of a magnetic field in momentum space, and affects the trajectory of carriers in much the same way as a classical magnetic field does in real space. Here, the resulting magnetic Lorentz force

\[ 4 \] Currently, it is still unclear whether any links between the intrinsic and extrinsic SHEs exist, e.g. it has recently been found that the extrinsic ‘side jump’ mechanism can be formally related to the momentum-space Berry curvature in GaAs-type semiconductors [8]. Related discussions of the anomalous Hall effect in ferromagnets have spanned many decades (see recent [9] and reviews in [10, 11]). We focus in this paper not on the relationship between extrinsic and intrinsic mechanisms but on the link between distinct intrinsic mechanisms.

New Journal of Physics 12 (2010) 013016 (http://www.njp.org/)
in momentum space manifests itself as an additional (anomalous) velocity in real space. The semiclassical equations of motion of carriers in the presence of the Berry curvature have been derived previously [13], and will be revisited for the Luttinger system in this article. It is found that the real space trajectory of holes along the transverse direction is spin dependent, thus resulting in a finite SHE. The SHE in Rashba systems [7], on the other hand, was derived originally from a semiclassical analysis of electron spin dynamics in a Rashba 2DEG system, with no apparent relation to the $\vec{k}$-space topology. In the analysis [7], electrons were found to gain a momentum-dependent out-of-plane spin polarization in the presence of Rashba SOC and an external electric field, leading to a transverse separation of spins. It is often stated in the literature that the effect arises from the $\vec{k}$-anisotropic precession of spins [14]–[17]. As part of the motivation for this paper, we will clarify the mechanism and show that the effect is in fact an adiabatic effect in which spins become aligned to momentum-dependent effective magnetic fields.

Thus, from a heuristic viewpoint, the physical mechanism of the two SHEs [6, 7] is clearly distinct: in the former, carriers acquire a spin-dependent anomalous velocity, whereas in the latter they acquire a momentum-dependent spin polarization. Our motivation for this paper is two-fold. Firstly, we ask whether the SHE in Rashba systems can also be formulated within an adiabatic framework and secondly, whether the physical mechanisms of the two SHEs can be unified. In order to describe the Rashba SHE under an adiabatic formulation, it is instructive to make note of several points: (i) the adiabatic Berry curvature of momentum space of the Rashba system vanishes except as a $\delta$-function singularity at $\vec{k} = \vec{0}$ [18]. Therefore, the spin-dependent anomalous velocity in the Rashba system vanishes for electrons with $\vec{k} \neq \vec{0}$ and thus does not contribute any transverse spin current. In contrast, the spin-Hall current in the Luttinger system results from the non-vanishing Dirac monopole curvature ($\sim \vec{k}/|\vec{k}|^3$) of momentum space; (ii) in the Rashba SHE, spins become tilted out-of-plane which appears contradictory to the adiabatic regime whereby they are assumed to follow perfectly the in-plane Rashba field; and (iii) although the Berry curvature in the Rashba system exists only at a singular point in $\vec{k}$-space, the resulting Berry phase is finite, and previous studies have shown that the spin-Hall conductivity in the Rashba system is related to the Berry phase through the Kubo formula [4, 14]. In this paper, we find that the above remarks (i)–(iii) can be consolidated into a consistent adiabatic theory that emerges from a consideration of the explicit time dependence of SHE systems. In particular, a gauge field $A_0(t)$ naturally appears in time space upon applying a unitary transformation to the system, which has the physical significance of a magnetic field in the transformed system. This magnetic field couples to the electron spin, and is shown to precisely account for the SHE in the Rashba system in the adiabatic limit; here, this limit amounts to the spins following the direction of the sum of the Rashba field and the new effective field. Furthermore, the Berry phase can be equivalently expressed in terms of the adiabatic components of the gauge field $A_0(t)$. Thus, a gauge field description can be attributed to both intrinsic SHEs, although their respective gauge fields are defined in different spaces (momentum and time).

Having identified that both intrinsic SHEs arise from gauge fields in the adiabatic limit, we finally embark on the problem of unifying the physical origin of the two effects. In the presence of an external electric field, the momentum and time spaces become coupled through the usual drift equation of charged carriers. Remarkably, by analyzing carefully the equations of motion of a general SOC model, it is found that the anomalous velocity due to the Berry curvature in momentum space is in fact a direct result of the effective magnetic field component arising from $A_0(t)$. In this sense, the common origin of the two seemingly disparate SHEs is clarified.
We also briefly discuss the important issue of impurity scattering. Previous studies [16, 19] have shown that the SHE described in [7] for infinite Rashba systems vanishes when one includes vertex corrections to model the effects of impurity scattering. In this work, we analyze the SHE in Rashba systems without considering vertex corrections (i.e. in line with the original treatment in [7]). However, we shall provide a phenomenological explanation of the vanishing SHE in the context of our analysis later in the paper. In contrast, the SHE in the Luttinger hole system is robust to vertex corrections [17].

2. Theory

2.1. Carrier dynamics in the presence of Berry’s curvature in momentum space in SOC systems

2.1.1. Holes in the Luttinger system. Let us briefly review the mechanism for the SHE of holes in p-doped semiconductors reported in [6]. The effective Luttinger Hamiltonian for holes in the valence band of conventional semiconductors is given by [20]

$$\hat{H}_{\text{Lutt}} = \frac{\hat{k}^2}{2} \gamma_1 + \frac{5}{2} \gamma - \gamma (\hat{k} \cdot \hat{S})^2 + V(\hat{r}),$$  \hspace{1cm} (1)

where $\gamma_1$, $\gamma$ are valence-band parameters defining the effective hole masses, $\hat{k}$ is the momentum operator, $\hat{S}$ is the vector of spin-3/2 matrices and $V = V(\hat{r})$ is the potential energy (in our notation, a hat ($\hat{\cdot}$) signifies an operator whereas an over-arrow ($\vec{\cdot}$) signifies a vector). The holes described by (1) have a well-defined chirality, $\hat{\chi} = \hbar^{-1} \hat{k} \cdot \hat{S} / |\hat{k}|$. Because of the chirality-squared term in the Hamiltonian, states with opposite signs of chirality (i.e. in line with the original treatment in [7]) are degenerate (they correspond to the light-hole and heavy-hole bands, respectively). In the presence of an external electric field $\vec{E}$, the potential energy term is $V(\hat{r}) = e \vec{E} \cdot \hat{r}$, where $-e$ is the electron charge. Parameterizing the momentum vector $\vec{k} = |\vec{k}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$, we proceed by defining a $4 \times 4$ unitary matrix $U(\vec{k})$,

$$U(\vec{k}) = \exp(i\theta (\vec{k}) S_z) \cdot \exp(i\phi (\vec{k}) S_y),$$  \hspace{1cm} (2)

which aligns the reference spin axis to be along the direction of $\vec{k}$, i.e. it satisfies the relation $U(\vec{k}) (\vec{k} \cdot \vec{S}) U^\dagger(\vec{k}) = |\vec{k}| S_z$. The effective, diagonalized Hamiltonian $\hat{H}_{\text{Lutt}}' = U(\vec{k}) \hat{H}_{\text{Lutt}} U^\dagger(\vec{k})$ then reads

$$\hat{H}_{\text{Lutt}}' = \frac{\hat{k}^2}{2} \left( \gamma_1 + \frac{5}{2} \gamma \right) - \gamma |\vec{k}|^2 S_z^2 + U(\vec{k}) V(\hat{r}) U^\dagger(\vec{k}).$$  \hspace{1cm} (3)

In the last term of (3), the position operator $\hat{r} = i \partial_k$ acts as a partial derivative in momentum space, and we obtain from the $\vec{k}$-dependence of $U$:

$$U(\vec{k})(e \vec{E} \cdot i \partial_k) U^\dagger(\vec{k}) = e \vec{E} \cdot (\hat{r} + i U(\vec{k}) \partial_k U^\dagger(\vec{k})).$$  \hspace{1cm} (4)

Thus, under the local transformation, the position operator transforms into covariant form, $\hat{r} \rightarrow \vec{R} = \hat{r} - A(\vec{k})$, where $A(\vec{k}) = -i U(\vec{k}) \partial_k U^\dagger(\vec{k})$ is a gauge field in reciprocal space. Thus far, the transformation $\hat{H}_{\text{Lutt}} \rightarrow \hat{H}_{\text{Lutt}}'$ is exact. Being a pure gauge field, $A(\vec{k})$ induced by the transformation has no associated curvature. Assuming adiabatic transport, in which we neglect mixing between the light-hole and heavy-hole bands, and applying an Abelian approximation...
within each hole band, we are left with only the diagonal gauge field components of the respective $2 \times 2$ hole band subspaces. Explicitly, the Abelian gauge fields are given by

$$A_{\text{ad}}^\text{ad}(\vec{k}) = -\lambda \cos \theta \nabla_\vec{k} \phi,$$

where the superscript $\text{ad}$ denotes adiabatic transport and $\lambda$ is the hole chirality. The corresponding gauge invariant quantity (which is thus related to a real physical effect) is the curvature tensor $\Omega_\vec{k}(\vec{k})$, defined by

$$\Omega_\vec{k}(\vec{k}) = \partial_\vec{k} A_{\text{ad}}^\text{ad}(\vec{k}) - \partial_\vec{k} A_{\text{ad}}^\text{ad}(\vec{k}).$$

The curvature $\Omega(\vec{k})$ above is frequently called the Berry curvature in momentum space. In the present case, a simple calculation reveals that the Berry curvature is

$$\Omega(\vec{k}) = \lambda \vec{k} / |\vec{k}|^3,$$

i.e. it is a Dirac monopole with strength $\epsilon g = \lambda$. It turns out that the $\vec{k}$-space curvature (7) has important implications on carrier dynamics. In particular, $\Omega(\vec{k})$ can be regarded as a magnetic field in $\vec{k}$-space, which gives rise to a $\vec{k}$-space Lorentz-type force. The modified semiclassical equations of motion for carriers in the presence of a nontrivial curvature in $\vec{k}$-space have been derived elsewhere to be [13]

$$\hat{\hbar} \dot{\vec{k}} = -e \vec{E},$$
$$\dot{\vec{r}} = \frac{1}{\hbar} \nabla_\vec{k} \epsilon - \vec{k} \times \Omega(\vec{k}),$$

where the over-dot signifies time differentiation and $\epsilon$ is the energy eigenvalue of the system. The final term in (9) is the Lorentz-type force in $\vec{k}$-space, and is equivalent to an additional velocity of electrons corresponding to the so-called anomalous Karplus–Luttinger [21] term. Substituting the expression for the curvature (7) into the equation of motion, the anomalous velocity component is given by

$$v_{\text{anom}}^\text{anom} = -\lambda \vec{k} \times \frac{\vec{k}}{|\vec{k}|^3},$$

which is perpendicular to both the applied electric fields $\vec{E}$ and $\vec{k}$. Since the chirality of the holes has sign $\lambda > 0$ ($\lambda < 0$) for hole spins (anti-) parallel to the electron momentum, the anomalous velocity is also perpendicular to the spin $\vec{S}$, and points along opposite directions depending on the sign of the chirality. This transverse separation of the spins gives rise to the SHE of holes in the Luttinger system.

2.1.2. Conduction electrons in the Rashba system. We now analyze the Berry curvature in momentum space for the case of the linear Rashba SOC [22, 23], which is present in 2DEG formed in semiconductor heterostructures. We begin with the generalized spin–orbit Hamiltonian

$$\hat{\mathcal{H}} = \frac{\hat{\vec{p}}^2}{2m} - \gamma \vec{\sigma} \cdot \vec{B}(\vec{k}) + V(\vec{r}),$$

where $m$ is the effective electron mass, $\gamma$ is the SOC strength, $\vec{\sigma} = \{\sigma_i\}$ is the vector of Pauli spin matrices, $\vec{B}(\vec{k})$ is a momentum-dependent effective magnetic field and $V(\vec{r}) = e \vec{E} \cdot \vec{r}$ in
the presence of an external electric field $\tilde{E}$. The above Hamiltonian captures the physics of many other types of SOC, including the linear and cubic Dresselhaus [24] and strain-induced [25, 26] SOC systems. The Luttinger Hamiltonian (1) can also be transformed to be of this general form when re-cast in terms of the SO(5) Clifford algebra as was done in [27], although this representation is in a five-dimensional space rather than the usual spin-1/2 space. The single particle eigenstates of the Hamiltonian are of the form $|\psi_\pm\rangle = \exp(ik \cdot r)\chi_\pm(\vec{k})$, i.e. a product of the spatial plane wave state and the spinor part which encodes the electron spin state. For any $\vec{k}$, the spin degeneracy is lifted between the two subbands $|\psi_\pm\rangle$, which have corresponding spin–orbit energy eigenvalues of $\epsilon_\pm = \pm \gamma |\vec{B}(\vec{k})|$. Let us rotate the reference spin axis such that it points along the direction of the spin–orbit field $\vec{B}(\vec{k})$, i.e. we diagonalize the Hamiltonian with respect to $\vec{B}(\vec{k})$. By parameterizing the spin–orbit field in terms of spherical angles, $\vec{B} = |\vec{B}|(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \equiv |\vec{B}|\vec{n}$, where $\theta$ and $\phi$ are explicit functions of $\vec{k}$, the diagonalization may be achieved through the SU(2) rotation matrix $U = U(\vec{k})$ given by equation (2) but with the replacements $S_\gamma \rightarrow \sigma_\gamma/2$, $S_z \rightarrow \sigma_z/2$. However, the choice of $U$ for the diagonalization is not unique: for convenience we shall adopt another rotation matrix given by [28]

$$U(\vec{k}) = \vec{m}(\vec{k}) \cdot \vec{\sigma},$$

where $\vec{m} = (\sin \theta/2 \cos \phi, \sin \theta/2 \sin \phi, \cos \theta/2)$. The effective, diagonalized Hamiltonian is given by

$$\hat{H}' = U \hat{H} U^\dagger = \frac{\hat{p}^2}{2m} - \gamma \sigma_i |\vec{B}(\vec{k})| + UV(\vec{r})U^\dagger.$$

The $\sigma_i$ Pauli matrix in the diagonalized spin–orbit term represents the two spin states either parallel (the majority band) or anti-parallel (minority band) to the spin–orbit field $\vec{B}(\vec{k})$. In the last term of (13), the position operator $\hat{r} = i\partial_\vec{k}$ is transformed into the covariant form of equation (4), i.e. $\hat{r} \rightarrow \hat{R} = \hat{r} - \hat{A}(\vec{k})$, where $\hat{A}(\vec{k})$ is an SU(2) gauge field in reciprocal space. From equations (4) and (12), the gauge field components can be represented in terms of the $\vec{m}$-vector and the Pauli spin matrices, i.e.

$$\hat{A}_k(\vec{k}) = (\vec{m} \times \partial_k \vec{m}) \cdot \vec{\sigma} = \vec{A}_k \cdot \vec{\sigma},$$

and in terms of the $\vec{n}$-vector by replacing $\vec{A}_k$ in the above equation with

$$\vec{A}_k = \frac{1}{2} (\vec{n} \times \partial_k \vec{n}) + (\vec{A}_k \cdot \vec{n})\vec{n},$$

where $i = x, y, z$ are real space coordinates. Up to this point, the transformation of the Hamiltonian is general. We now impose the adiabatic approximation, in which mixing between the two eigenstates of the diagonalized Hamiltonian is neglected. Mathematically, this corresponds to retaining only the diagonal terms of $\hat{A}(\vec{k})$, i.e. the $\sigma_i$ coefficients in equation (14), from which we obtain an Abelian gauge field known as the Berry connection, $\hat{A}_{ad}(\vec{k}, s)$. $\hat{A}_{ad}(\vec{k}, s)$ has two values, representing the two spin states, $s = \pm 1$, of the diagonalized Hamiltonian (we denote the majority band as $s = +1$), and which correspond to the diagonal terms of $\hat{A}_k(\vec{k})$. Explicitly, the Abelian gauge field is given by

$$\hat{A}_{ad}(\vec{k}, s) = \frac{s}{2} (1 - \cos \theta) \nabla_k \phi.$$

The curvature tensor $\Omega(\vec{k})$ of this connection, defined by equation (6), is invariant with respect to the gauge transformation $U(\vec{k})$. From the definition, it is clear that $\Omega(\vec{k})$ respects the same symmetry in $s$ as the connection, i.e. $\Omega(\vec{k}, s) = -\Omega(\vec{k}, -s)$.
In principle, one can define the curvature $\Omega$ in any arbitrary space. For example, in the special case of the magnetic field space $\vec{B}$, the Berry curvature has the classic form of Dirac’s monopole [12],

$$\Omega(\vec{B}) = s \frac{\vec{B}}{2|\vec{B}|^3}. \quad (17)$$

The above relation is general and applies to any system that is Zeeman-coupled to either a real or effective magnetic field. For SOC, one can transform the curvature from $\vec{B}$-space to $\vec{k}$-space by using the relation [29, 30]

$$\Omega_\vec{k}(\vec{k}) = \epsilon_{ijk}\Omega(\vec{B}) \cdot \left( \frac{\partial \vec{B}}{\partial k_i} \times \frac{\partial \vec{B}}{\partial k_j} \right), \quad (18)$$

where $\epsilon_{ijk}$ is the Levi–Civita symbol. Generally, the curvature in momentum space $\Omega(\vec{k})$ is not the Dirac monopole field (although it still is for the case of the Luttinger Hamiltonian). The actual form of $\Omega(\vec{k})$ depends on the $\vec{k}$-dependence of the effective magnetic field. We saw in equation (9) how this curvature $\Omega(\vec{k})$ gave rise to an anomalous velocity which resulted in the SHE in p-doped semiconductors. However, the same reasoning cannot be applied to the SHE in the Rashba system, as $\Omega(\vec{k})$ in this system is vanishing (for $\vec{k} \neq 0$) as we outline below.

The Hamiltonian in the presence of the Rashba SOC is given by [22, 23]

$$\hat{H}_R = \frac{\hat{p}_k^2}{2m} + \alpha(\hat{k}_x\sigma_y - \hat{k}_y\sigma_x), \quad (19)$$

where $\alpha$ is the Rashba SOC parameter expressed in units of eV m. The effective magnetic field is given by $\vec{B}_R(\vec{k}) = (\vec{k}_y, -\vec{k}_x)$, and the eigenvectors are $|\vec{k}, \pm\rangle = 1/\sqrt{2}\exp(\pm ik_x\cdot \vec{r})(\pm k_y, k_x, 1)^T$, with the corresponding energy eigenvalues of $\epsilon_{\pm} = \pm \alpha k$ where $k = |\vec{k}| = \sqrt{k_x^2 + k_y^2}$ is the in-plane wave-vector magnitude. In momentum space, the effective magnetic field $\vec{B}_R$ is directed along $\theta = \pi/2$ and $\phi = \tan^{-1}(-k_x/k_y)$, and from equation (16) the Berry connection is given by $A^{\alpha\beta}(\vec{k}, \pm) = \pm (1/2k^2)(-k_y, k_x, 0)$. Evidently, curvature (6) of this connection is trivial, i.e. $\Omega(\vec{k}) = 0$ over the entire $\vec{k}$-space, except at the singularity point at $\vec{k} = 0$ where the $k_x$-component of the curvature has non-vanishing value $\pm \pi$, i.e. the curvature is of the form $\Omega(\vec{k}, \pm) = (0, 0, \pm \pi \delta^z(\vec{k}))$ [18]. Thus, conduction electrons having a finite momentum in the 2DEG plane cannot be perceived to experience any Lorentz-type force in $\vec{k}$-space, as is the case for holes in the Luttinger system. Furthermore, even if this force existed, it would only separate the Rashba SOC eigenstates (whose spins lie entirely in-plane) in the transverse direction, and thus cannot explain the out-of-plane spin polarization acquired by the electrons in the SHE. Now the question arises as to whether the SHE in Rashba systems can be described within a gauge field framework. In particular, the out-of-plane spin polarization seems to suggest the presence of an additional magnetic field in the system. It turns out that such a gauge formulation does exist, but one must turn to another parameter space, namely the time space.

2.2. Time component of the gauge field in SOC systems with an electric field

When considering the temporal evolution of a quantum system, the unitary transformation is explicitly time dependent, i.e. $U = U(t)$. In SHE systems, the $t$-dependence of the unitary transformations naturally arises due to the acceleration of carriers in the presence of an electric field: the electron wave-vector $\langle \vec{k} \rangle$ changes linearly in $t$, and consequently $\langle \vec{B}(\vec{k}) \rangle$ acquires a time
dependence. To incorporate the explicit time dependence of the system quantum mechanically, we switch to the interaction picture [31]. In this picture, the original Hamiltonian (11) is split into two parts, $\hat{H} = \hat{H}_0 + \hat{H}_1$, where

$$\hat{H}_0 = e\vec{E} \cdot \hat{r}$$

(20)
governs the time evolution of the operators, and

$$\hat{H}_1 = \frac{\hat{p}^2}{2m} - \gamma \vec{\sigma} \cdot \vec{B}(\hat{k})$$

(21)
governs the time evolution of the states. In the usual sense, an operator $\hat{A}$ in the Schrödinger picture is transformed to the interaction picture (subscript I) as $\hat{A}_I(t) = e^{i\hat{H}_0 t/\hbar} \hat{A} e^{-i\hat{H}_0 t/\hbar}$ and carries an explicit time dependence by satisfying the Heisenberg relation, $\dot{\hat{A}}_I = (i\hbar)^{-1}[\hat{A}_I, \hat{H}_0]$. In particular, the momentum operator in the new picture is found to be $\dot{\hat{p}}_I(t) = \hat{p} - e\vec{E} t$, i.e. with the expected linear time dependence due to the electric field. The state vectors $|\psi(t)\rangle$ in the Schrödinger picture correspondingly transform as $|\psi_I(t)\rangle = e^{i\hat{H}_0 t/\hbar} |\psi(t)\rangle$, and evolve according to the new ‘Schrödinger equation’

$$\dot{\hat{H}}_I(t)|\psi_I(t)\rangle = i\hbar \partial_t |\psi_I(t)\rangle,$$

(22)

where $\dot{\hat{H}}_I(t) = e^{i\hat{H}_0 t/\hbar} \dot{\hat{H}}_I e^{-i\hat{H}_0 t/\hbar}$. For the case of linear (e.g. Rashba) SOC, $\dot{\hat{H}}_I(t)$ is evaluated to be

$$\dot{\hat{H}}_I(t) = \frac{\hat{p}_I^2}{2m} - \gamma \vec{\sigma} \cdot \vec{B}(\hat{k}) = \frac{\hat{p}_I^2}{2m} - \gamma \vec{\sigma} \cdot \vec{B}(t),$$

(23)

where summation over repeated indices is implied. Higher order spin–orbit terms ($\sim k^n, n \geq 2$) generally lead to correspondingly higher order partial derivatives of the spin–orbit field in $\dot{\hat{H}}_I(t)$. Hamiltonian (23) governing the state vector evolution in the interaction picture is that of an electron subject to an explicitly time-dependent magnetic field, which we denote as $\vec{B}(t)$. Analogous to our previous treatment, we proceed to diagonalize the Schrödinger equation (22) at time $t$, by applying a unitary rotation $U(t)$ (defined as in equation (12) but with $\theta$ and $\phi$ in $\vec{m}$ carrying an explicit time dependence). This transformation aligns the $\vec{z}$-axis to be parallel to the instantaneous magnetic field $\vec{B}(t)$, i.e.

$$U(t) \dot{\hat{H}}_I(t) U^\dagger(t) = U(t) (i\hbar \partial_t) U^\dagger(t),$$

$$\dot{\hat{p}}_I^2 - \gamma \sigma_i \vec{B}(t) = i\hbar \partial_t + i\hbar U(t) \partial_t U^\dagger(t)$$

$$\equiv \dot{\hat{\epsilon}} - \hbar \vec{A}_0(t),$$

(24)

where $\dot{\hat{\epsilon}} = i\hbar \partial_t$ is the energy operator. On the left-hand side, the local transformation diagonalizes the time-dependent Zeeman term as required. On the right-hand side, we obtain from the time dependence of $U$ a gauge field $\vec{A}_0(t) \equiv -iU(t) \partial_t U^\dagger(t)$ related to the temporal evolution of the system [32]. From the relations in equations (14) and (15), we can express the gauge field as $\vec{A}_0(t) = \vec{A}_I \cdot \vec{\sigma}$, where $\vec{A}_I = \vec{m} \times \vec{m} = \frac{m}{2} \vec{n} \times \vec{n} + (\vec{A}_I \cdot \vec{n})\vec{n}$. Thus, the term $\hbar \vec{A}_0(t)$ represents an additional Zeeman-like term, indicating the presence of an effective magnetic field in the rotating frame. We elucidate the origin of this field in more detail below.

The unitary transformation $U(t)$ defines the instantaneous angular velocity $\vec{\omega}^I = \vec{\omega}^I(t)$ of the coordinates (in the laboratory frame, l) as it follows the time-dependent magnetic field. In the rotating frame $r$, this vector is given by $\vec{\omega}^r$, where $\vec{\sigma} \cdot \vec{\omega}^r = U(\vec{\sigma} \cdot \vec{\omega}^l) U^\dagger$. Since
\( \hat{U} = \frac{i}{\hbar} U \hat{\sigma} \cdot \hat{\omega} \) [33], the Zeeman-like term \( \hbar A_0(t) \) in (24) thus yields \(-\hbar/2(\hat{\sigma} \cdot \hat{\omega})\), which corresponds to an effective magnetic field of \(-\hat{\omega}\) (omitting a scaling factor) in the rotating frame. This translates into an effective magnetic field \( \hat{B}_\parallel = -\hat{\omega} \) in the laboratory frame. If we now denote by \( \hat{n} = \hat{n}(t) \) the unit vector pointing along the direction of the magnetic field at time \( t \), we have the equation of motion \( \dot{\hat{n}} = \hat{\omega} \times \hat{n} \). Performing a post cross product on both sides by \( \hat{n} \), one arrives at the expression for the angular velocity \( \dot{\omega} = \hat{n} \times \hat{n} + (\hat{\omega} \cdot \hat{n}) \hat{n} \), or, in terms of the effective magnetic field,

\[
\hat{B}_\parallel = \hat{n} \times \hat{n} + (\hat{\omega} \cdot \hat{n}) \hat{n}.
\]

Thus, the effective magnetic field arising from the gauge field \( \mathcal{A}_0(t) \) of the unitary transformation has a component along \( \hat{n} \times \hat{n} \) and along \( \hat{n} \). Note that it does not have any component along \( \hat{n} \). As we noted previously, the unitary rotation matrix used by us (12) is not unique. Specifically, different rotation matrices \( U_i \), each specifying distinct angular velocities \( \hat{\omega}_i \), can be used to align the reference \( z \)-axis with the instantaneous magnetic field \( \hat{B}(t) \); the freedom of choice here lies in determining the trajectory of the remaining \( \hat{x}-\hat{y} \) axes, that is, the rotation about \( \hat{n} \) itself. The second term on the right-hand side of equation (25) reflects the particular choice of the gauge transformation \( U_i \). It is not an invariant of the gauge transformation (its magnitude being dependent on the particular gauge choice), and does not represent a physical field. However, the first component \( \hat{n} \times \hat{n} \) of the effective magnetic field is invariant with respect to the gauge transformation, depending only on the time dependence of the magnetic field \( \hat{B}(t) \). This term can be understood to be a direct consequence of the time-dependent rotation of the axes [34, 35]. The same expression can be derived classically by directly comparing the spin vector in adjacent time frames [34]—as a complement to the quantum derivation, the classical treatment is shown in detail in the appendix. The \( \hat{n} \times \hat{n} \) component represents a physical magnetic field which couples to the electron spins [34, 35], and, as we show below, is precisely the component which leads to the SHE in Rashba 2DEG systems.

3. Results and analysis

3.1. The intrinsic SHE due to Rashba SOC

The Hamiltonian of conduction electrons in the Rashba system is given by equation (19). Following our analysis above, the time dependence of the effective Rashba field \( \hat{B}_R \) due to the electrons’ motion in momentum space necessarily gives rise to a secondary component \( \hat{B}_\perp = \hat{n} \times \hat{n} \), where \( \hat{n} = p^{-1}(p_x, -p_y, 0) \) is the unit vector in the direction of \( \hat{B}_R \). We assume a longitudinally applied electric field along the \( \hat{x} \)-direction \( \vec{E} = E_x \hat{x} \), so that \( \hat{n} = p^{-1}(0, eE_x, 0) \). Because \( \hat{B}_R \) is strictly in-plane (i.e. it lies in the \( \hat{x}, \hat{y} \)-plane of the 2DEG), the term \( \hat{n} \times \hat{n} \) represents an out-of-plane magnetic field component which is along the \( \hat{z} \)-direction by convention. Next, we apply the adiabatic condition for the electron spins. In the ideal adiabatic limit, the magnetic field \(|\hat{B}_R|\) is infinitely strong, so the spins always remain aligned to it as it varies with time. In reality, \(|\hat{B}_R|\) is finite and there is a nonzero secondary component \( \hat{B}_\perp \), and the relevant condition is \(|\hat{B}_R| \gg |\hat{B}_\perp| \), i.e. the electron spin is primarily aligned to \( \hat{B}_R \), but with a small deviation along \( \hat{B}_\perp \). In terms of the parameters of the Rashba system, the adiabatic condition reads as

\[
\frac{\alpha k^2}{e} \gg E_x.
\]
Inserting typical values for the Rashba parameter \( \alpha = 10^{-11} \text{ eV m} \) and the Fermi wave-vector \( k = 10^8 \text{ m}^{-1} \), we arrive at the condition \( E_x \ll \sim 10^5 \text{ V m}^{-1} \), which usually holds true in experiments. Assuming that the spin of electrons follow the direction of the net effective magnetic field, \( \vec{B}_\Sigma \), which is the sum of the spin–orbit field \( \vec{B}_R \) and the secondary component, the classical spin vector is given by

\[
\vec{s} = \pm \frac{\hbar}{2 |B_\Sigma|} (\hat{n} \times \vec{n}) \cdot \hat{z},
\]

where \( \pm \) represents spin aligned parallel (+) or anti-parallel (−) to the net field. The component of the spin along the \( \hat{z} \)-direction is

\[
s_z = \pm \frac{\hbar}{|B_\Sigma|} \left( \hat{n} \times \vec{n} \right) \cdot \hat{z},
\]

where, to be consistent in units, the magnetic field in the denominator is defined in terms of its equivalent angular velocity. Note that in the convention above the + corresponds to the majority band \( \epsilon_- \), whereas − corresponds to the eigenstate \( \epsilon_+ \). In the adiabatic limit, the magnitude of \( \vec{B}_\Sigma \) approaches that of \( \vec{B}_R \), and applying this limit to equation (28), we obtain for the out-of-plane spin polarization

\[
s_z \approx \pm \frac{1}{|B_\Sigma|} \left( \hat{n} \times \vec{n} \right) \cdot \hat{z}
\]

\[
= \pm \frac{\hbar^2}{2 \alpha \mu} \left( -\frac{1}{p^2} e E_x p_y \right)
\]

\[
= \pm \frac{e \hbar^3 p_y E_x}{4 \alpha \mu^3}.
\]

Equation (29) above describes a transverse separation of spins in the Rashba system. Let us consider the case for the majority subband. Since the spin \( s_z \propto -p_y \), we find that electrons moving in the +\( \hat{y} \)-direction are polarized out-of-plane along the −\( \hat{z} \)-direction, whereas those moving in the −\( \hat{y} \)-direction are polarized along +\( \hat{z} \). For the other eigenstate, the direction of the polarization is reversed, and hence there is a certain degree of canceling of the polarization if both eigenstates are present. However, at the Fermi level, there are more electrons in the majority band \( \epsilon_- \), giving rise to a net transverse spin separation and hence the SHE described in [7] (see equations (5)–(7) there). Summing over the Fermi surfaces of the two eigenstates yields an intrinsic spin-Hall (sH) conductivity of \( \sigma_{\text{sH}} \equiv j_z^s/E_x = -e/8\pi \), where \( j_y^s = \hbar/4(s_z, v_y) \) is the transverse spin current. From our analysis above, we have clarified that the SHE in Rashba systems occurs as a result of an adiabatic process, in which electrons’ spins become aligned to momentum-dependent magnetic fields that arise from the time dependence of the system. The effect is therefore not due to the precessional behavior of spins, as is often stated in the literature.

**3.1.1. Berry’s phase.** We alluded earlier to previous work which related the intrinsic spin-Hall conductivity in Rashba systems to the \( \vec{k} \)-space Berry phase of electrons through the Kubo formula [4]. It was found there that \( \sigma_{\text{sH}} = e\varphi_{\pm}/8\pi^2 \), where \( \varphi_{\pm} \) is the Berry phase of electrons,

\[
\varphi_{\pm} = \oint A_0^{\text{ad}}(\vec{k}) \cdot d\vec{k} = \frac{s}{2} \oint (1 - \cos \theta) \nabla_k \phi \cdot d\vec{k}.
\]
The natural parameterization for the vector $\vec{k}$ is the time variable $t$, and rewriting the line integral above in terms of $t$ we obtain
\[
\varphi_{\pm} = \frac{s}{2} \int (1 - \cos \theta) \nabla \phi \cdot \frac{\dot{\vec{k}}}{\dot{k}} dt \\
= \frac{s}{2} \int (1 - \cos \theta) \phi \, dt \\
\equiv \int A_{0}^{ad}(t) \, dt.
\]
(31)

Thus the Berry phase and hence the intrinsic spin-Hall conductivity of the Rashba system can be written equivalently in terms of the time component of the adiabatic gauge field, $A_{0}^{ad}(t)$.

3.1.2. Effects of disorder. The effect of impurities is an important topic in the discussion of the intrinsic SHE. Our analysis above for the Rashba SOC system follows for a perfectly ballistic system in which scattering is not considered. This reproduces the spin-Hall current obtained in [7]. Previous studies have investigated the effects of disorder on this spin current by introducing vertex corrections. Remarkably, the vertex correction was shown to exactly cancel the intrinsic conductivity of $-e/8\pi$ even in the limit of arbitrarily weak disorder [16, 19], [36]–[40] for infinite systems. We provide a simple, heuristic argument based on our analysis for the vanishing SHE. In the presence of disorder, the scattering provides a braking effect which cancels the acceleration of carriers on average in the steady state [41]. This implies that in the steady state we have $\langle \dot{\vec{k}} \rangle = 0$, i.e. there is no net change in the momentum and thus the magnetic field component $\vec{B}_\perp = \hat{n} \times \vec{n}$ averages out to zero. Note, however, that this picture is an oversimplification and that the SHE in Rashba systems does not vanish in general. For example, the SHE persists in finite-sized systems [19], [41]–[44] and in the presence of spin-dependent impurities [45].

3.2. The intrinsic SHE due to linear Dresselhaus SOC

The case for the linear Dresselhaus SOC is also easily verified by our analysis. The Dresselhaus spin–orbit Hamiltonian is given by
\[
\mathcal{H}_D = \beta (\hat{k}_x \sigma_y - \hat{k}_y \sigma_x) \equiv -\beta \vec{\sigma} \cdot \vec{B}_D,
\]
(32)
where $\beta$ is the Dresselhaus SOC strength and $\vec{B}_D$ is the effective Dresselhaus SOC field. Here we have $\vec{n} = p^{-1}(p_x, -p_y, 0)$, and we find that $(\hat{n} \times \vec{n})_z = +eE_x p_y / p^2$. Consequently, the out-of-plane spin polarization $s_z$ has the same magnitude but opposite sign compared to the Rashba SOC case. This is in agreement with previous theoretical studies [4] which predicts the spin-Hall conductivity in this system to be $\sigma_{sH} = e/8\pi$.

4. Discussions

Having established the Rashba SHE as an adiabatic effect, we now identify two common traits of the two intrinsic SHEs: adiabaticity and time dependence. For the Rashba system, we found that

\[\text{This argument was stated previously in the context of spin precession in [17]: in the presence of impurities, the scattering scrambles the spin precession sufficiently such that no net SHE results.}\]
the time-dependent spin–orbit field \( \vec{B}_R(t) = |\vec{B}_R(t)|\vec{n} \) is always accompanied by an additional effective magnetic field, \( \vec{B}_\perp = \vec{n} \times \vec{n} \). This correction to the magnetic field results in a net field \( \vec{B}_\Sigma = \vec{B}_R + \vec{B}_\perp \) which is different to \( \vec{B}_R \). Considering the adiabatic limit, where \( |\vec{B}_R| \gg |\vec{B}_\perp| \), we recovered exactly the results of Sinova et al describing the SHE in the Rashba system. The field \( \vec{B}_\perp \) was shown to be described quantum mechanically by a gauge field in time space. In the Luttinger system, the adiabatic assumption results in a nontrivial momentum-space curvature which enters the equation of motion as the spin-dependent anomalous velocity component (10).

Thus, at first glance it appears that the two effects are rather independent phenomena. However, an interesting duality exists between the two effects. In the former Luttinger case, a \textit{spin-dependent anomalous velocity} pushes opposite spin species to opposite lateral sides of the sample. The magnetic field responsible for this effect is the Berry curvature defined by the \( k \)-space gauge field. On the other hand, in the Rashba system, a \textit{momentum-dependent magnetic field} polarizes electrons along opposite directions out-of-the-plane depending on their transverse propagation direction. The magnetic field responsible for this effect is defined by the \( t \)-space gauge field. Given this duality, it would be tempting to ask whether there is any underlying relation between the two pictures.

We proceed to consolidate the link between the two effects by investigating the connection between the anomalous velocity due to the Berry curvature, and the presence of the \( \vec{B}_\perp \) term. In this endeavor, we employ the reciprocal space analogue of the analysis by Aharanov and Stern [34] of the origin of the Berry’s curvature in real space. We consider again the general spin–orbit Hamiltonian in equation (11). It is worthwhile to note that the full Luttinger Hamiltonian can be decomposed into two such copies of the general Hamiltonian, after applying the adiabatic and Abelian approximations mentioned in section 2.1.1 (the magnitude of the spin in each copy equals the helicity, \( |\lambda| = 1/2 \) and 3/2). The velocity along the \( i \)th coordinate within the general model is given by Hamilton’s equation

\[
v_i = \frac{1}{\hbar} [r_i, \mathcal{H}]
= \frac{p_i}{m} - \gamma \frac{\partial \vec{B}(\vec{k})}{\partial p_i} \cdot \vec{\sigma}.
\] (33)

When the magnetic field is time dependent, the spins see an additional magnetic field \( \vec{B}_\perp \). Assuming that spins align parallel (anti-parallel) to \( \vec{B}_\Sigma = \vec{B} + \vec{B}_\perp \), the unit spin vector is given by \( \vec{s} = \langle \vec{\sigma} \rangle = s \vec{B}_\Sigma / |\vec{B}_\Sigma| \) where \( s = +1 \) (\( s = -1 \)). Writing the spin–orbit field \( \vec{B} = |\vec{B}|\vec{n} \), the partial derivative in equation (33) can be expanded into its magnitude and directional parts as \( (\partial |\vec{B}| / \partial p_i)\vec{n} + |\vec{B}| \partial \vec{n} / \partial p_i \). Taking the adiabatic limit \( |\vec{B}_\Sigma| / |\vec{B}| \to 1 \), the velocity expression becomes

\[
v_i = \frac{p_i}{m} - s \gamma \frac{\partial |\vec{B}|}{\partial p_i} - s \frac{\hbar}{2} \left( \hat{n} \times \vec{n} \right) \cdot \frac{\partial \vec{n}}{\partial p_i}.
\] (34)

Writing \( \hat{n} = \vec{k}_j \partial \vec{n} / \partial k_j \) in the final term, where summation over \( j \) is implicit, and rearranging the terms we then get

\[
v_i = \frac{p_i}{m} - s \gamma \frac{\partial |\vec{B}|}{\partial p_i} - s \frac{k_j}{2} \left( \frac{\partial \vec{n}}{\partial k_i} \times \frac{\partial \vec{n}}{\partial k_j} \right) \cdot \vec{n}.
\] (35)

The second term in the above equation represents a velocity term that is due to the inhomogeneity of the spin–orbit field \( \vec{B} \) in momentum space, i.e. it is the reciprocal space
analogue of the Stern–Gerlach force. Remarkably, the final term in equation (35) is the anomalous velocity of electrons due to Berry’s curvature in $\vec{k}$-space. This becomes clearer when written in terms of the magnetic field vector $\vec{B} = |\vec{B}|\vec{n}$,

$$v_{t}^{\text{anom}} = -\hat{k}_j s \frac{\vec{B}}{2|\vec{B}|^3} \cdot \left( \frac{\partial \vec{B}}{\partial k_i} \times \frac{\partial \vec{B}}{\partial k_j} \right)$$

$$= -\epsilon_{ijk} \hat{k}_j \Omega_k (\vec{k}),$$

where the last line follows from equation (18). We find that this is exactly the anomalous velocity component in equation (9), which arises from the Berry curvature in $\vec{k}$-space. Thus, we have shown that the anomalous velocity due to the Berry curvature actually arises because of the $\vec{B}_\perp$ magnetic field component, which in turn is related to the time component of the gauge field. This gauge field component therefore plays an equally important role in the SHE in the Luttinger system, as it does in the Rashba one, and acts as the unifying bridge between the two effects.

We finally note that in general, both intrinsic mechanisms coexist and would inevitably couple with each other. This study, however, is beyond the scope of the present paper and is a subject for further investigations.

5. Summary

The primary motivation in this paper was to establish the link between the two intrinsic SHEs reported in [6, 7], which had not been clarified hitherto. We first considered the intrinsic SHE in the Luttinger system, which is driven by the spin-dependent anomalous velocity due to the nontrivial curvature of momentum space. However, this theoretical picture is not applicable in the Rashba system. Instead, the SHE in the Rashba system was shown to arise from spins acquiring a component (in the adiabatic sense) along an additional effective magnetic field $\vec{B}_\perp$, arising from the time dependence of the system. This field component was shown to be described by a gauge field in time space. Finally, we showed that in the adiabatic limit, $\vec{B}_\perp$ is also the origin of the anomalous velocity due to the momentum-space Berry curvature. Thus, we conclude that the intrinsic SHEs in the two systems are simply different manifestations of $\vec{B}_\perp$, and that this field provides a unifying link between the two effects.

Acknowledgments

The authors would like to thank the Agency for Science, Technology and Research (A*STAR) of Singapore, the SERC Grant No. 092 101 0060 (R-398-000-061-305) and the National University of Singapore (NUS) Nanoscience and Nanotechnology Initiative for financially supporting their work.

Appendix. Classical derivation of effective magnetic field component, $\hat{n} \times \vec{n}$

Consider the dynamics of the spin vector $\vec{s}(t)$ in a time-dependent magnetic field $\vec{B}(t)$,

$$\dot{\vec{s}}(t) = g(\vec{s}(t) \times \vec{B}(t)),$$

where $g$ is the coupling factor. To solve the above equation, we freeze the time dependence by transforming to a rotated coordinate frame at each point in time, such that the $\hat{z}$-axis is aligned.
The classical spin vector $\vec{s}(t)$ precesses about the magnetic field which is along the $\hat{z}$-direction at some instant $t$. Because of the time dependence of the magnetic field, the spin is also subject to a rotation about $\vec{\omega}(t) = \hat{z} \times \hat{z}$ (see text) which transforms it from the frame at time $t$ (left) to the frame at time $t + dt$ (right). The $\vec{\omega}(t)$ acts as an additional magnetic field which governs the overall spin dynamics.

with the magnetic field. A spin vector $\vec{s}$ defined relative to the coordinate frame at time $t$, is expressed as the vector $\vec{s}' = \vec{s} + \vec{s} \times \vec{\omega}(t) \, dt$ in the coordinate frame at time $t + dt$, where $\vec{\omega}(t)$ is the generator of infinitesimal rotations (see figure A.1).

The choice of $\vec{\omega}(t)$ is not unique; however, specifically choosing $\vec{\omega}(t) = \dot{\hat{z}} \times \hat{z}$ where $\hat{z}$ is the unit vector $\vec{n} = \vec{B} / |\vec{B}|$ as seen in the rotated frame, coincides with the parallel transport of the coordinate frames [34, 46]. Suppose we have a vector representing the spin, $\vec{s}(t)$, in the rotated frame at time $t$. At time $t + dt$, this vector becomes (relative to frame $t + dt$) $\vec{s}(t + dt) + \vec{s}(t + dt) \times \vec{\omega}(t) \, dt$ where $\vec{s}(t + dt) \approx \vec{s}(t) + g(\vec{s}(t) \times |\vec{B}|\hat{z}) \, dt$. For infinitesimally small $dt$, we may write $\vec{s}(t + dt)$ [in frame $t$] $\approx \vec{s}(t) + \vec{s}(t + dt) \times \vec{\omega}(t) \, dt$ [in frame $t + dt$]. The right-hand side of the resulting equation becomes $\vec{s}(t) + g(\vec{s} \times |\vec{B}|\hat{z}) \, dt + \vec{s} \times \vec{\omega}(t) \, dt + O(dt^2)$. Rearranging and taking the limit $dt \to 0$, we have

$$\lim_{dt \to 0} \frac{\vec{s}(t + dt) - \vec{s}(t)}{dt} = \dot{s} \approx g \left( \vec{s} \times |\vec{B}(t)|\hat{z} \right) + \vec{s} \times \vec{\omega}(t) \equiv \vec{s} \times \left( g|\vec{B}(t)|\hat{z} + \dot{\hat{z}} \times \hat{z} \right).$$

Therefore, as seen in the laboratory frame, there is an additional effective magnetic field $\vec{B}_{\text{eff}} = \vec{B}(t) + g^{-1}\dot{\vec{n}} \times \vec{n}$.

References

[1] D’yakonov M I and Perel’ V I 1971 JETP Lett. 13 467
D’yakonov M I and Perel’ V I 1971 Phys. Lett. A 35 459
[2] Dai X, Fang Z, Yao Y-G and Zhang F-C 2006 Phys. Rev. Lett. 96 086802
[3] Kontani H, Naito M, Hirashima D S, Yamada K and Inoue J-I 2007 J. Phys. Soc. Japan 76 103702
[4] Shen S-Q 2004 Phys. Rev. B 70 081311
[5] Tan S G, Jalil M B A, Liu X-J and Fujita T 2008 Phys. Rev. B 78 245321
[6] Murakami S, Nagaosa N and Zhang S-C 2003 Science 301 1348

New Journal of Physics 12 (2010) 013016 (http://www.njp.org/)
[7] Sinova J, Culcer D, Niu Q, Sinitsyn N A, Jungwirth T and MacDonald A H 2004 Phys. Rev. Lett. 92 126603
[8] Rashba E I 2008 Semiconductors 42 905
[9] Onoda S, Sugimoto N and Nagaosa N 2006 Phys. Rev. Lett. 97 126602
[10] Sinitsyn N A 2008 J. Phys.: Condens. Matter 20 023201
[11] Nagaosa N 2009 arxiv:0904.4154
[12] Berry M V 1984 Proc. R. Soc. A 392 45
[13] Sundaram G and Niu Q 1999 Phys. Rev. B 59 14915
[14] Chen T-W, Huang C-M and Guo G Y 2006 Phys. Rev. B 73 235309
[15] Engel H-A, Rashba E I and Halperin B I 2007 Theory of Spin Hall Effects in Semiconductors in Handbook of Magnetism and Advanced Magnetic Materials (Chichester: Wiley) pp 2858–77
[16] Inoue J, Bauer G E and Molenkamp L W 2004 Phys. Rev. B 70 041303
[17] Murakami S 2004 Phys. Rev. B 69 241202
[18] Chang M-C 2005 Phys. Rev. B 71 085315
[19] Mishchenko E G, Shytov A V and Halperin B I 2004 Phys. Rev. Lett. 93 226602
[20] Luttinger J M 1956 Phys. Rev. 102 1030
[21] Karplus R and Luttinger J M 1954 Phys. Rev. 95 1154
[22] Rashba E I 1960 Fiz. Tverd. Tela (Leningrad) 2 1224
[23] Bychkov Y A and Rashba E I 1984 J. Phys. C: Solid State Phys. 17 6039
[24] Dresselhaus G 1955 Phys. Rev. 100 580
[25] Mal’shukov A G, Tang C S, Chu C S and Chao K A 2005 Phys. Rev. Lett. 95 107203
[26] Jiang L and Wu M W 2005 Phys. Rev. B 72 033311
[27] Murakami S, Nagaosa N and Zhang S-C 2004 Phys. Rev. B 69 235206
[28] Tatara G, Kohno H and Shibata J 2008 Phys. Rep. 468 213
[29] Blokh K Y and Blokh Y P 2005 Ann. Phys. 319 13
[30] Tan S G, Jalil M B A and Fujita T 2008 unpublished
[31] Townsend J S 1992 A Modern Approach to Quantum Mechanics (New York: McGraw-Hill)
[32] Serebrennikov Y A 2006 Phys. Rev. B 73 195317
[33] Wagh A G and Rakhecha V C 1993 Phys. Rev. A 48 1729
[34] Aharonov Y and Stern A 1992 Phys. Rev. Lett. 69 3593
[35] Xiao J, Zangwill A and Stiles M D 2006 Phys. Rev. B 73 054428
[36] Rashba E I 2004 Phys. Rev. B 70 210309
[37] Raimondi R and Schwab P 2005 Phys. Rev. B 71 033311
[38] Chalaev O and Loss D 2005 Phys. Rev. B 71 245318
[39] Dimitrova O V 2005 Phys. Rev. B 71 245327
[40] Khaetskii A 2006 Phys. Rev. Lett. 96 056602
[41] Adagideli I and Bauer G E W 2005 Phys. Rev. Lett. 95 256602
[42] Xing Y X, Sun Q and Wang J 2006 Phys. Rev. B 73 205339
[43] Wang J, Chan K S and Xing D Y 2006 Phys. Rev. B 73 033316
[44] Liu S Y and Lei X L 2006 Phys. Rev. B 73 205327
[45] Inoue J et al 2006 Phys. Rev. Lett. 97 046604
[46] Anandan J and Stodolsky L 1987 Phys. Rev. D 35 2597