CLASSIFYING THE CLOSED IDEALS OF BOUNDED OPERATORS ON TWO FAMILIES OF NON-SEPARABLE CLASSICAL BANACH SPACES

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Abstract. We classify the closed ideals of bounded operators acting on the Banach spaces \((\bigoplus_{n \in \mathbb{N}} \ell_2^n) \oplus c_0(\Gamma)\) and \((\bigoplus_{n \in \mathbb{N}} \ell_1^n) \oplus \ell_1(\Gamma)\) for every uncountable cardinal \(\Gamma\).

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1. Introduction

Very few Banach spaces \(X\) are known for which the lattice of closed ideals of the Banach algebra \(\mathcal{B}(X)\) of all bounded operators on \(X\) is fully understood. When \(X\) is finite-dimensional, \(\mathcal{B}(X)\) is simple, meaning that it contains no non-zero, proper ideals, so we shall henceforth discuss infinite-dimensional Banach spaces only.

Our focus is on the “classical” case, that is, Banach spaces that can be defined by elementary means and/or were known to Banach and his contemporaries. There are currently only two families of Banach spaces of this kind whose lattices of closed operator ideals are fully understood:

(i) Daws \[5, \text{Theorem 7.4}\] has shown that for \(X = c_0(\Gamma)\) or \(X = \ell_p(\Gamma)\), where \(\Gamma\) is an infinite cardinal and \(1 \leq p < \infty\), the lattice of closed ideals of \(\mathcal{B}(X)\) is

\[
\{0\} \subsetneq \mathcal{K}(X) \subsetneq \mathcal{K}(X) \subsetneq \mathcal{K}(X) \subsetneq \cdots \subsetneq \mathcal{K}(X) \subsetneq \mathcal{K}(X) = \mathcal{B}(X). \tag{1.1}
\]

Here, \(\mathcal{K}(X)\) denotes the ideal of \(\kappa\)-compact operators for an uncountable cardinal \(\kappa\) (see page 4 for the precise definition), and \(\kappa^+\) denotes the cardinal successor of \(\kappa\). An alternative description of the closed ideals of \(\mathcal{B}(X)\) is given in \[12, \text{Theorem 1.5}\]; see also \[10, \text{Theorem 3.7}\].

Daws’ theorem generalises and unifies previous results of Calkin \[3\] for \(X = \ell_2\), Gohberg, Markus and Feldman \[8\] for \(X = c_0\) or \(X = \ell_p\), \(1 \leq p < \infty\), and Gramsch \[9\] and Luft \[21\] independently for \(X = \ell_2(\Gamma)\), where \(\Gamma\) is an arbitrary infinite cardinal.
(ii) Let $E = \bigoplus_{n \in \mathbb{N}} \ell_2^n_D$ for $D = c_0$ or $D = \ell_1$. Then it was shown in [19] and [20], respectively, that the lattice of closed ideals of $\mathcal{B}(E)$ is

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \overline{\mathcal{T}_D(E)} \subsetneq \mathcal{B}(E),$$

where $\overline{\mathcal{T}_D(E)}$ denotes the closure of the ideal of operators on $E$ that factor through the space $D$.

We shall combine the above results to obtain two new "hybrid" families of Banach spaces, namely $(\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{c_0} \oplus c_0(\Gamma)$ and its dual space $(\bigoplus_{n \in \mathbb{N}} \ell_2^n)_{\ell_1} \oplus \ell_1(\Gamma)$, for any uncountable cardinal $\Gamma$, whose closed ideals of operators we classify. The precise statement is as follows.

**Theorem 1.1.** Let $(D, D_\Gamma) = (c_0, c_0(\Gamma))$ or $(D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))$ for an uncountable cardinal $\Gamma$, and set $E = \bigoplus_{n \in \mathbb{N}} \ell_2^n_D$ and $X = E \oplus D_\Gamma$. Then the lattice of closed ideals of $\mathcal{B}(X)$ is

$$\{0\} \downarrow \mathcal{K}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{K}_{\mathcal{R}_1}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{K}_{\mathcal{R}_2}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{K}_{\mathcal{R}_3}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{K}_{\mathcal{R}_4}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{K}_{\Gamma}(X) \downarrow \overline{\mathcal{T}_D(X)} \mathcal{B}(X),$$

where

$$\mathcal{I}_\kappa(X) = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \in \mathcal{B}(X) : T_{1,1} \in \overline{\mathcal{T}_D(E)}, T_{1,2} \in \mathcal{B}(D_\Gamma, E), T_{2,1} \in \mathcal{B}(E, D_\Gamma), T_{2,2} \in \mathcal{K}_{\kappa}(D_\Gamma) \right\}$$

(1.3)
for each cardinal $\aleph_2 \leq \kappa \leq \Gamma^+$, where an arrow from an ideal $\mathcal{I}$ pointing to an ideal $\mathcal{J}$ denotes that $\mathcal{I} \subsetneq \mathcal{J}$ and there are no closed ideals of $\mathcal{B}(X)$ strictly contained in between $\mathcal{I}$ and $\mathcal{J}$.

Remark 1.2. In addition to the “classical” Banach spaces listed above, there are a number of “exotic”, or purpose-built, Banach spaces whose closed ideals of operators can be classified. They occur in two classes:

- The first class contains the famous Banach space of Argyros and Haydon [1] solving the scalar-plus-compact problem, as well as several variants and descendants of it obtained in [26, 22, 16].
- The other class consists of the Banach space $C(K)$ of continuous, scalar-valued functions defined on Koszmider’s Mrówka space $K$, as shown in [15, Theorem 5.5]. Koszmider’s original construction of $K$ in [17] assumed the Continuum Hypothesis. A construction within ZFC is given in [18].

A possible explanation for the scarcity of Banach spaces $X$ whose closed ideals of operators have been classified, especially among classical spaces, is that recent research has shown that in many cases $\mathcal{B}(X)$ has $2^\mathfrak{c}$ closed ideals, where $\mathfrak{c}$ denotes the cardinality of the continuum. Note that this is the largest possible number of closed ideals for separable $X$.

Spaces for which $\mathcal{B}(X)$ has $2^\mathfrak{c}$ closed ideals include $X = L^p[0, 1]$ for $p \in (1, \infty) \setminus \{2\}$ (see [14]), $X = \ell_p \oplus \ell_q$ for $1 \leq p < q \leq \infty$ with $(p, q) \neq (1, \infty)$ and $X = \ell_p \oplus c_0$ for $1 < p < \infty$ (see [6, 7]). For several other spaces $X$, it is known that $\mathcal{B}(X)$ contains at least continuum many closed ideals. This includes $X = L_1[0, 1], X = C[0, 1]$ and $X = L_\infty[0, 1]$ (see [13]; note that these results also cover $X = \ell_\infty$ because $\ell_\infty$ and $L_\infty[0, 1]$ are isomorphic as Banach spaces by [23]), as well as the Tsirelson space and the Schreier space of order $n \in \mathbb{N}$ (see [2]). For $X = \ell_1 \oplus c_0$, the best known result is that $\mathcal{B}(X)$ has at least uncountably many closed ideals (see [25]).

2. Preliminaries

Below we explain this paper’s most important notation, which is mostly standard. All of our results are valid for both real and complex Banach spaces; we write $\mathbb{K}$ for the scalar field. By an operator we always mean a bounded and linear map between normed spaces.

**Operator ideals.** Following Pietsch [24], an operator ideal is an assignment $\mathcal{I}$ which designates to each pair $(X, Y)$ of Banach spaces a subspace $\mathcal{I}(X, Y)$ of the Banach space $\mathcal{B}(X, Y)$ of operators from $X$ to $Y$ such that:

(i) there exists a pair $(X, Y)$ of Banach spaces for which $\mathcal{I}(X, Y) \neq \{0\}$;

(ii) for any quadruple $(W, X, Y, Z)$ of Banach spaces and any operators $S \in \mathcal{B}(W, X), T \in \mathcal{I}(X, Y), U \in \mathcal{B}(Y, Z)$, we have that $UTS \in \mathcal{I}(W, Z)$.

As usual, we write $\mathcal{I}(X)$ to abbreviate $\mathcal{I}(X, X)$. For any operator ideal $\mathcal{I}$, the map $\overline{\mathcal{I}}$ sending a pair of Banach spaces $(X, Y)$ to the norm closure of $\mathcal{I}(X, Y)$ in $\mathcal{B}(X, Y)$ is also an operator ideal. If $\mathcal{I} = \overline{\mathcal{I}}$, then we call $\mathcal{I}$ a closed operator ideal.

We shall consider the following three operator ideals:
K, the ideal of compact operators.

Kₖ, the ideal of k-compact operators, defined for any infinite cardinal k.

The precise definition is as follows. An operator T ∈ B(X, Y) is k-compact if, for each ε > 0, the closed unit ball Bₓ of X contains a subset Xₑ with |Xₑ| < k such that

$$\inf\{\|T(x - y)\| : y \in Xₑ\} \leq \varepsilon$$

for every x ∈ Bₓ. Writing Kₖ(X, Y) for the set of k-compact operators from X to Y, we obtain a closed operator ideal Kₖ.

Notice that K₀₀(X, Y) = K(X, Y), so the notion of k-compactness is indeed a generalisation of compactness.

Gᵩ, the ideal of operators factoring through a certain Banach space D.

Here, we say that an operator T ∈ B(X, Y) factors through D if there are operators U: X → D and V: D → Y such that T = VU, and we write Gᵩ(X, Y) for the set of operators factoring through D. This defines an operator ideal Gᵩ provided that D contains a complemented subspace isomorphic to D ⊕ D, which is true in the cases that we shall consider, namely D = c₀ and D = ℓ₂ for some p ∈ [1, ∞).

Operators on the direct sum of a pair of Banach spaces. Let X₁ and X₂ be Banach spaces, and endow their direct sum X₁ ⊕ X₂ with any norm satisfying

$$\max\{\|x₁\|, \|x₂\|\} \leq \|(x₁, x₂)\| \leq \|x₁\| + \|x₂\| \quad (x₁ ∈ X₁, x₂ ∈ X₂).$$

For i ∈ {1, 2}, we write Qᵢ: X₁ ⊕ X₂ → Xᵢ and Jᵢ: Xᵢ → Xᵢ ⊕ Xᵢ for the i-th coordinate projection and embedding, respectively. For T ∈ B(X₁ ⊕ X₂) and i, j ∈ {1, 2}, set Tᵢⱼ = QᵢTJⱼ ∈ B(Xⱼ, Xᵢ). Then we have

$$T = \sum_{i,j=1}^{2} J_i T_{i,j} Q_j. \quad (2.1)$$

It follows that, for any operator ideal I,

$$T ∈ I(X₁ ⊕ X₂) ⇐⇒ Tᵢⱼ ∈ I(Xⱼ, Xᵢ) \text{ for } i, j ∈ \{1, 2\}. \quad (2.2)$$

We shall identify the operator T ∈ B(X₁ ⊕ X₂) with the matrix

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}. \quad (2.3)$$

Then the action of T on an element (x₁, x₂) ∈ X₁ ⊕ X₂ is given by

$$\begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} T_{1,1}x_1 + T_{1,2}x_2 \\ T_{2,1}x_1 + T_{2,2}x_2 \end{pmatrix}. \quad (2.4)$$

Composition of operators is given by matrix multiplication, addition is entrywise, and the operator norm satisfies

$$\max_{i,j\in\{1,2\}} \|Tᵢⱼ\| \leq \|T\| \leq \sum_{i,j=1}^{2} \|Tᵢⱼ\|. \quad (2.3)$$
For a subset $\mathcal{I}$ of $\mathcal{B}(X_1 \oplus X_2)$ and $i, j \in \{1, 2\}$, we define the $(i, j)^{\text{th}}$ quadrant of $\mathcal{I}$ by

$$\mathcal{I}_{i,j} = \{ Q_i T J_j : T \in \mathcal{I} \} \subseteq \mathcal{B}(X_j, X_i).$$

On the other hand, given subsets $\mathcal{I}_{i,j}$ of $\mathcal{B}(X_j, X_i)$ for $i, j \in \{1, 2\}$, we define

$$\begin{pmatrix} \mathcal{I}_{1,1} & \mathcal{I}_{1,2} \\ \mathcal{I}_{2,1} & \mathcal{I}_{2,2} \end{pmatrix} = \left\{ \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix} : T_{i,j} \in \mathcal{I}_{i,j} \ (i, j \in \{1, 2\}) \right\} \subseteq \mathcal{B}(X_1 \oplus X_2).$$

(2.5)

The first part of the following lemma can be seen as a generalisation of (2.2) to the case where the ideal $\mathcal{I}$ does not come from an operator ideal. It says that if we decompose $\mathcal{I}$ into its quadrants according to (2.4) and then reassemble the quadrants according to (2.5), we obtain $\mathcal{I}$.

**Lemma 2.1.** Let $\mathcal{I}$ be an ideal of $\mathcal{B}(X_1 \oplus X_2)$ for some Banach spaces $X_1$ and $X_2$. Then

$$\mathcal{I} = \begin{pmatrix} \mathcal{I}_{1,1} & \mathcal{I}_{1,2} \\ \mathcal{I}_{2,1} & \mathcal{I}_{2,2} \end{pmatrix},$$

(2.6)

and $\mathcal{I}_{i,i}$ is an ideal of $\mathcal{B}(X_i)$ for $i \in \{1, 2\}$. Moreover, $\mathcal{I}_{i,j}$ is closed in $\mathcal{B}(X_j, X_i)$ for each $i, j \in \{1, 2\}$ if and only if $\mathcal{I}$ is closed in $\mathcal{B}(X_1 \oplus X_2)$.

**Proof.** The inclusion $\subseteq$ in (2.6) holds true by the definitions.

Conversely, suppose that $T = (Ti,j)_{i,j=1}^2$ with $Ti,j \in \mathcal{I}_{i,j}$ for each $i, j \in \{1, 2\}$, say $Ti,j = Qi,S^{i,j}J_j$, where $S^{i,j} \in \mathcal{I}$. Then, by (2.1), we have

$$T = \sum_{i,j=1}^2 J_i Ti,j Q_j = \sum_{i,j=1}^2 (Qi,Q_j)S^{i,j}(J_jQ_j) \in \mathcal{I}$$

because $\mathcal{I}$ is an ideal of $\mathcal{B}(X_1 \oplus X_2)$ and $J_k Q_k \in \mathcal{B}(X_1 \oplus X_2)$ for $k \in \{1, 2\}$.

Next, we verify that $\mathcal{I}_{i,i}$ is an ideal of $\mathcal{B}(X_i)$ for $i \in \{1, 2\}$. It is clear that $\mathcal{I}_{i,i}$ is a subspace. Suppose that $S \in \mathcal{I}_{i,i}$ and $T \in \mathcal{B}(X_i)$, say $S = U_{i,i}$, where $U \in \mathcal{I}$. Then $U_{i,i}TQ_i \in \mathcal{I}$ because $\mathcal{I}$ is an ideal of $\mathcal{B}(X_1 \oplus X_2)$ and $J_i T Q_i \in \mathcal{B}(X_1 \oplus X_2)$, and hence

$$\mathcal{I}_{i,i} \ni (U_{i,i}TQ_i)_{i,i} = Q_i U_{i,i}T Q_i = ST.$$ 

The proof that $TS \in \mathcal{I}_{i,i}$ is similar.

The final clause follows easily from (2.3) and (2.6). \qed

**Long sequence spaces.** For a non-empty set $\Gamma$ and $p \in [1, \infty)$, we consider the Banach spaces

$$c_0(\Gamma) = \{ x \in \mathbb{K}^\Gamma : \{ \gamma \in \Gamma : |x(\gamma)| > \epsilon \} \text{ is finite for every } \epsilon > 0 \}$$

and

$$\ell_p(\Gamma) = \left\{ x \in \mathbb{K}^\Gamma : \sum_{\gamma \in \Gamma} |x(\gamma)|^p < \infty \right\}.$$

The norm of $c_0(\Gamma)$ is the supremum norm, and the norm of $\ell_p(\Gamma)$ is $\|x\| = (\sum_{\gamma \in \Gamma} |x(\gamma)|^p)^{\frac{1}{p}}$. As usual, we write $c_0$ and $\ell_p$ instead of $c_0(\mathbb{N})$ and $\ell_p(\mathbb{N})$, respectively. For notational
convenience, we use the convention that \( D_\Gamma \) will denote either \( c_0(\Gamma) \) or \( \ell_p(\Gamma) \) for some \( p \in [1, \infty) \), unless otherwise specified. Only the cardinality of the index set \( \Gamma \) matters, in the sense that \( D_\Gamma \) is isometrically isomorphic to \( D_{[\Gamma]} \), and \( D_\Gamma \) is not isomorphic to \( D_\Delta \) for any index set \( \Delta \) of cardinality other than \( |\Gamma| \).

The support of an element \( x \in D_\Gamma \) is \( \text{supp} \, x = \{ \gamma \in \Gamma : x(\gamma) \neq 0 \} \), which is always a countable set.

For \( \gamma \in \Gamma \), \( e_\gamma \in D_\Gamma \) denotes the element given by \( e_\gamma(\beta) = 1 \) if \( \beta = \gamma \) and \( e_\gamma(\beta) = 0 \) otherwise. The (transfinite) sequence \( (e_\gamma)_{\gamma \in \Gamma} \) is the (long) unit vector basis for \( D_\Gamma \). The only facts about it that we shall use are that \( (e_\gamma)_{\gamma \in \Gamma} \) spans a dense subspace of \( D_\Gamma \), and that the existence of such a basis implies that \( D_\Gamma \) has the approximation property.

For a subset \( \Delta \) of \( \Gamma \), \( P_\Delta \in \mathcal{B}(D_\Gamma) \) denotes the basis projection given by \( (P_\Delta x)(\gamma) = x(\gamma) \) if \( \gamma \in \Delta \) and \( (P_\Delta x)(\gamma) = 0 \) otherwise, for every \( x \in D_\Gamma \).

3. The proof of Theorem 1.1

To aid the presentation, we split the proof of Theorem 1.1 into a series of lemmas, some of which may essentially be known. However, to keep the presentation as self-contained as possible, we include full proofs, except for references to the ideal classifications [5, 19] that we will ultimately need anyway. The proof of Theorem 1.1 itself requires results about the transfinite sequence spaces \( c_0(\Gamma) \) and \( \ell_1(\Gamma) \) only, not \( \ell_p(\Gamma) \) for \( 1 < p < \infty \). However, our first few results hold true also for the latter spaces and with identical proofs, so we give these more general results.

**Lemma 3.1.** Let \( D_\Gamma = c_0(\Gamma) \) or \( D_\Gamma = \ell_p(\Gamma) \) for some \( p \in [1, \infty) \) and some set \( \Gamma \neq \emptyset \).

(i) Every separable subspace of \( D_\Gamma \) is contained in the image of the basis projection \( P_\Delta \) for some countable subset \( \Delta \) of \( \Gamma \).

(ii) Suppose that \( D_\Gamma \neq \ell_1(\Gamma) \). Then, for every Banach space \( E \) and every operator \( T : D_\Gamma \to E \) for which there exists an injective operator from the image of \( T \) into \( \ell_\infty \), there is a countable subset \( \Delta \) of \( \Gamma \) such that \( T = TP_\Delta \).

**Proof.** (i). Every separable subspace \( E \) of \( D_\Gamma \) has the form \( E = \overline{\text{span}} \, W \) for some countable subset \( W \) of \( D_\Gamma \). Define \( \Delta = \bigcup_{w \in W} \text{supp} \, w \), which is a countable union of countable sets and is thus countable. The continuity of the projection \( P_\Delta \) implies that \( x = P_\Delta x \) for every \( x \in E \). Hence the image of \( P_\Delta \) contains \( E \).

(ii). Let \( U : T(D_\Gamma) \to \ell_\infty \) be an injective operator. Assume towards a contradiction that the set

\[
\Delta_{k,m} = \left\{ \gamma \in \Gamma : |UTe_\gamma(m)| \geq \frac{1}{k} \right\}
\]

is infinite for some \( k, m \in \mathbb{N} \), so that it contains an infinite sequence \( (\gamma_n)_{n \in \mathbb{N}} \) of distinct elements. For each \( n \in \mathbb{N} \), take a scalar \( \sigma_n \) of modulus one such that \( \sigma_n \cdot (UTe_\gamma)(m) \geq 1/k \). Since \( D_\Gamma \neq \ell_1(\Gamma) \), we have \( x = \sum_{n \in \mathbb{N}} \sigma_n e_{\gamma_n} \in D_\Gamma \), but

\[
(UTx)(m) = \sum_{n \in \mathbb{N}} \sigma_n (UTe_{\gamma_n})(m) \geq \frac{1}{k} \sum_{n \in \mathbb{N}} \frac{1}{n} = \infty,
\]
Lemma 3.4. Let $(D, D_{\Gamma}) = (c_0, c_0(\Gamma))$ or $(D, D_{\Gamma}) = (\ell_p, \ell_p(\Gamma))$ for some $p \in [1, \infty)$ and some uncountable set $\Gamma$, and let $E$ be any separable Banach space. Then 
\[ \mathcal{B}(D_{\Gamma}, E) = \mathcal{G}_D(D_{\Gamma}, E) \quad \text{and} \quad \mathcal{B}(E, D_{\Gamma}) = \mathcal{G}_D(E, D_{\Gamma}). \]

Proof. The first identity for $D_{\Gamma} \neq \ell_1(\Gamma)$, and the second identity in full generality, both follow easily from Lemma 3.1 because the image of the projection $P_\Delta$ for $\Delta$ countable is either finite-dimensional or isomorphic to $D$, and $E$, being separable, embeds isometrically into $\ell_\infty$.

It remains to show that every operator $T: \ell_1(\Gamma) \to E$ factors through $\ell_1$. We use the lifting property of $\ell_1$ (see for instance [4, Theorem 5.1]) to verify this. Indeed, since $E$ is separable, we can take a surjective operator $S: \ell_1 \to E$. By the open mapping theorem, there is a constant $c > 0$ such that, for every $y \in E$, there is $x \in \ell_1$ with $Sx = y$ and $\|x\| \leq c\|y\|$. Hence, for each $\gamma \in \Gamma$, we can find $x_{\gamma} \in \ell_1$ such that $Sx_{\gamma} = Te_\gamma$ and $\|x_{\gamma}\| \leq c\|Te_\gamma\| \leq c\|T\|$. It follows that we can define an operator $R: \ell_1(\Gamma) \to \ell_1$ by $Re_\gamma = x_{\gamma}$ for each $\gamma \in \Gamma$, and clearly $T = SR$. 

Remark 3.2. (i) The case $D_{\Gamma} = \ell_1(\Gamma)$ must be excluded in Lemma 3.1(ii) because, for every non-zero Banach space $E$, there is an operator $T: \ell_1(\Gamma) \to E$ which has one-dimensional image and satisfies $Te_\gamma \neq 0$ for every $\gamma \in \Gamma$, namely the summation operator $T$ given by $Tx = \sum_{\gamma \in \Gamma} x(\gamma) y$ for every $x \in \ell_1(\Gamma)$, where $y$ is any fixed non-zero element of $E$.

(ii) We refer to [11] for a detailed discussion of the condition in Lemma 3.1(ii) that the image of $T$ admits an injective operator into $\ell_\infty$.

Corollary 3.3. Let $(D, D_{\Gamma}) = (c_0, c_0(\Gamma))$ or $(D, D_{\Gamma}) = (\ell_p, \ell_p(\Gamma))$ for some $p \in [1, \infty)$ and some uncountable set $\Gamma$, and let $E$ be any separable Banach space. Then 
\[ \mathcal{B}(D_{\Gamma}, E) = \mathcal{G}_D(D_{\Gamma}, E) \quad \text{and} \quad \mathcal{B}(E, D_{\Gamma}) = \mathcal{G}_D(E, D_{\Gamma}). \]

Proof. The first identity for $D_{\Gamma} \neq \ell_1(\Gamma)$, and the second identity in full generality, both follow easily from Lemma 3.1 because the image of the projection $P_\Delta$ for $\Delta$ countable is either finite-dimensional or isomorphic to $D$, and $E$, being separable, embeds isometrically into $\ell_\infty$.

It remains to show that every operator $T: \ell_1(\Gamma) \to E$ factors through $\ell_1$. We use the lifting property of $\ell_1$ (see for instance [4, Theorem 5.1]) to verify this. Indeed, since $E$ is separable, we can take a surjective operator $S: \ell_1 \to E$. By the open mapping theorem, there is a constant $c > 0$ such that, for every $y \in E$, there is $x \in \ell_1$ with $Sx = y$ and $\|x\| \leq c\|y\|$. Hence, for each $\gamma \in \Gamma$, we can find $x_{\gamma} \in \ell_1$ such that $Sx_{\gamma} = Te_\gamma$ and $\|x_{\gamma}\| \leq c\|Te_\gamma\| \leq c\|T\|$. It follows that we can define an operator $R: \ell_1(\Gamma) \to \ell_1$ by $Re_\gamma = x_{\gamma}$ for each $\gamma \in \Gamma$, and clearly $T = SR$. 

Lemma 3.4. Let $D = c_0$ or $D = \ell_p$ for some $p \in [1, \infty)$, and let $(E_n)_{n \in \mathbb{N}}$ be a sequence of non-zero Banach spaces. Then there are operators $R: D \to \left( \bigoplus_{n \in \mathbb{N}} E_n \right)_D$ and $S: \left( \bigoplus_{n \in \mathbb{N}} E_n \right)_D \to D$ such that $SR = I_D$.

Proof. For every $n \in \mathbb{N}$, choose $y_n \in E_n$ and $f_n \in E_n^*$ with $\|y_n\| = \|f_n\| = \langle y_n, f_n \rangle = 1$, and define $R: (\lambda_n) \mapsto (\lambda_n y_n)$ for $(\lambda_n) \in D$ and $S: (x_n) \mapsto (\langle x_n, f_n \rangle)$ for $(x_n) \in \left( \bigoplus_{n \in \mathbb{N}} E_n \right)_D$. 

Lemma 3.5. Let $D = c_0$ or $D = \ell_1$. Then $D$ contains a subspace which is isomorphic to $\left( \bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_D$.

Proof. This follows by combining the fact that $D$ contains almost isometric copies of $\ell_2^n$ for every $n \in \mathbb{N}$ with the fact that $D$ is isomorphic to the $D$-direct sum of countably many copies of itself.

Lemma 3.6. Let $(D, D_{\Gamma}) = (c_0, c_0(\Gamma))$ or $(D, D_{\Gamma}) = (\ell_1, \ell_1(\Gamma))$ for some infinite set $\Gamma$, and set $E = \left( \bigoplus_{n \in \mathbb{N}} \ell_2^n \right)_D$. Then the identity operator on $D$ factors through every non-compact operator belonging to either $\mathcal{B}(E)$, $\mathcal{B}(D_{\Gamma})$, $\mathcal{B}(E, D_{\Gamma})$ or $\mathcal{B}(D_{\Gamma}, E)$.
Remark 3.7. Lemma 3.6 is also true for \((D, D_\Gamma) = (\ell_p, \ell_p(\Gamma))\) when \(1 < p < \infty\). However, \(E = (\bigoplus_{n \in \mathbb{N}} \ell^p_n)_p\) is isomorphic to \(\ell_p\) in these cases, so only case (ii) above would be non-trivial.

Corollary 3.8. Let \(X = \left( \bigoplus_{n \in \mathbb{N}} \ell^p_n \right)_D \oplus D_\Gamma\), where \((D, D_\Gamma) = (c_0, c_0(\Gamma))\) or \((D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))\) for some infinite set \(\Gamma\), and let \(\mathcal{I}\) be an ideal of \(\mathcal{B}(X)\). Then either \(\mathcal{I} \subseteq \mathcal{H}(X)\) or \(\mathcal{G}_D(X) \subseteq \mathcal{I}\).

Proof. For notational convenience, write \(X = X_1 \oplus X_2\), where \(X_1 = \left( \bigoplus_{n \in \mathbb{N}} \ell^p_n \right)_D\) and \(X_2 = D_\Gamma\). Suppose that \(\mathcal{I} \not\subseteq \mathcal{H}(X)\), and choose \(T \in \mathcal{I} \setminus \mathcal{H}(X)\). Then (2.2) shows that \(T_{i,j} \notin \mathcal{H}(X_j, X_i)\) for some \(i, j \in \{1, 2\}\). Lemma 3.6 implies that there are operators \(U: D \to X_j\) and \(V: X_i \to D\) such that \(VT_{i,j}U = I_D\). Hence, for each \(S = R_2R_1 \in \mathcal{G}_D(X)\), where \(R_1 \in \mathcal{B}(X, D)\) and \(R_2 \in \mathcal{B}(D, X)\), we have
\[
S = R_2VT_{i,j}UR_1 = (R_2VQ_i)T(J_iUR_1) \in \mathcal{I}
\]
because \(\mathcal{I}\) is an ideal of \(\mathcal{B}(X)\). This shows that \(\mathcal{G}_D(X) \subseteq \mathcal{I}\), as desired. \(\square\)

For a Banach space \(X\), define
\[
\Xi(X) = \{ \mathcal{I} : \mathcal{I}\ is\ a\ closed\ ideal\ of\ \mathcal{B}(X)\ and\ \mathcal{I} \supseteq \mathcal{H}(X) \},
\]
and order \(\Xi(X)\) by inclusion. For a pair of Banach spaces \(X_1\) and \(X_2\), we endow the set \(\Xi(X_1) \times \Xi(X_2)\) with the product order; that is,
\[
(\mathcal{I}_1, \mathcal{I}_2) \preceq (\mathcal{J}_1, \mathcal{J}_2) \iff [\mathcal{I}_1 \subseteq \mathcal{J}_1] \wedge [\mathcal{I}_2 \subseteq \mathcal{J}_2].
\]
Proposition 3.9. Let \( X = E \oplus D_\Gamma \), where \( E = (\bigoplus_{n \in \mathbb{N}} \ell_2^n)^c \) and either \((D, D_\Gamma) = (c_0, c_0(\Gamma))\) or \((D, D_\Gamma) = (\ell_1, \ell_1(\Gamma))\) for some infinite set \( \Gamma \). The map

\[
\xi : \Xi(E) \times \Xi(D_\Gamma) \to \Xi(X), \quad (\mathcal{I}, \mathcal{J}) \mapsto \left( \mathcal{I} \to \mathcal{B}(E, D_\Gamma), \mathcal{J} \to \mathcal{B}(D_\Gamma, E) \right),
\]

is an order isomorphism.

Proof. Recall from Corollary 3.3 that \( \mathcal{B}(E, D_\Gamma) = \mathcal{G}_D(E, D_\Gamma) \) and \( \mathcal{B}(D_\Gamma, E) = \mathcal{G}_D(D_\Gamma, E) \), and that \( \mathcal{G}_D(E) \subseteq \mathcal{I} \) and \( \mathcal{G}_D(D_\Gamma) \subseteq \mathcal{J} \) for every \((\mathcal{I}, \mathcal{J}) \in \Xi(E) \times \Xi(D_\Gamma)\) by the ideal classifications (1.2) and (1.1), respectively. Using these facts, one can easily verify that \( \xi(\mathcal{I}, \mathcal{J}) \) is an ideal of \( \mathcal{B}(X) \) with \( \mathcal{K}(X) \varsubsetneq \xi(\mathcal{I}, \mathcal{J}) \). Moreover, \( \xi(\mathcal{I}, \mathcal{J}) \) is closed by Lemma 2.1, so it belongs to \( \Xi(X) \).

To see that \( \xi \) is surjective, let \( \mathcal{L} \in \Xi(X) \). Lemma 2.1 shows that

\[
\mathcal{L} = \left( \begin{array}{cc}
\mathcal{L}_{1,1} & \mathcal{L}_{1,2} \\
\mathcal{L}_{2,1} & \mathcal{L}_{2,2}
\end{array} \right),
\]

where \( \mathcal{L}_{1,1} \) and \( \mathcal{L}_{2,2} \) are closed ideals of \( \mathcal{B}(E) \) and \( \mathcal{B}(D_\Gamma) \), respectively. Moreover, Corollary 3.8 implies that \( \mathcal{G}_D(X) \subseteq \mathcal{L} \), so by (2.2), we have:

- \( \mathcal{L}_{1,1} \supseteq \mathcal{G}_D(E) \), so \( \mathcal{L}_{1,1} \in \Xi(E) \);
- \( \mathcal{L}_{2,2} \supseteq \mathcal{G}_D(D_\Gamma) = \mathcal{B}(D_\Gamma, E) \), so \( \mathcal{L}_{2,2} = \mathcal{B}(E, D_\Gamma) \), and similarly \( \mathcal{L}_{2,1} = \mathcal{B}(E, D_\Gamma) \);
- \( \mathcal{L}_{2,2} \supseteq \mathcal{G}_D(D_\Gamma) \), so \( \mathcal{L}_{2,2} \in \Xi(D_\Gamma) \).

This verifies that \( \mathcal{L} = \xi(\mathcal{L}_{1,1}, \mathcal{L}_{2,2}) \).

Finally, working straight from the definitions, we see that \( (\mathcal{I}_1, \mathcal{J}_1) \leq (\mathcal{I}_2, \mathcal{J}_2) \) if and only if \( \xi(\mathcal{I}_1, \mathcal{J}_1) \leq \xi(\mathcal{I}_2, \mathcal{J}_2) \) for \( (\mathcal{I}_1, \mathcal{J}_1), (\mathcal{I}_2, \mathcal{J}_2) \in \Xi(E) \times \Xi(D_\Gamma) \). This shows first that \( \xi \) is injective and thus a bijection, and secondly that both \( \xi \) and its inverse are order-preserving.

We can now prove Theorem 1.1 easily.

Proof of Theorem 1.1. Both \( E \) and \( D_\Gamma \) have the approximation property, so the same is true for their direct sum \( X \). Therefore \( \mathcal{K}(X) \) is the smallest non-zero closed ideal of \( \mathcal{B}(X) \). Proposition 3.9 shows that any other non-zero closed ideal \( \mathcal{L} \) of \( \mathcal{B}(X) \) has the form \( \mathcal{L} = \xi(\mathcal{I}, \mathcal{J}) \) for unique closed ideals \( \mathcal{I} \in \Xi(E) \) and \( \mathcal{J} \in \Xi(D_\Gamma) \). By the ideal classifications (1.2) and (1.1), either \( \mathcal{I} = \mathcal{G}_D(E) \) or \( \mathcal{J} = \mathcal{B}(E) \), while \( \mathcal{J} = \mathcal{K}(D_\Gamma) \) for a unique cardinal \( \kappa_1 \leq \kappa \leq \Gamma^+ \).

Suppose first that \( \mathcal{I} = \mathcal{G}_D(E) \). If \( \kappa = \kappa_1 \), then \( \mathcal{J} = \mathcal{G}_D(D_\Gamma) \), so \( \mathcal{L} = \mathcal{G}_D(X) \). Otherwise \( \kappa \geq \kappa_2 \) and \( \mathcal{L} = \mathcal{J}_\kappa(X) \) in the notation of (1.3).

Next, suppose that \( \mathcal{J} = \mathcal{B}(E) \), which is equal to \( \mathcal{K}_\kappa(E) \) because \( E \) has density \( \aleph_0 < \kappa \). Hence we have \( \mathcal{L} = \mathcal{K}_\kappa(X) \). (Note that this is equal to \( \mathcal{B}(X) \) if \( \kappa = \Gamma^+ \).) \( \square \)

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