Scaling and Persistence in the
Two-Dimensional Ising Model

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The spatial distribution of persistent spins at zero-temperature in the pure two-dimensional Ising model is investigated numerically. A persistence correlation length, $\xi(t) \sim t^Z$ is identified such that for length scales $r \ll \xi(t)$ the persistent spins form a fractal with dimension $d_f$; for length scales $r \gg \xi(t)$ the distribution of persistent spins is homogeneous. The zero-temperature persistence exponent, $\theta$, is found to satisfy the scaling relation $\theta = Z(2 - d_f)$ with $\theta = 0.209 \pm 0.002$, $Z = 1/2$ and $d_f \sim 1.58$. 
The ‘persistence’ problem has attracted considerable interest in recent years [1-9]. In its most general form, it is concerned with the fraction of space which persists in its initial state up to some later time.

Hence, in the non-equilibrium dynamics of spin systems at zero-temperature we are interested in the fraction of spins, \( P(t) \), that persist in the same state as at \( t = 0 \) up to some later time \( t \). For the pure ferromagnetic two-dimensional Ising model, \( P(t) \) has been found to decay algebraically [1-4]

\[
P(t) \sim t^{-\theta}
\]

where \( \theta = 0.209 \pm 0.002 \) [5]. Similar algebraic decay has been found in numerous other systems displaying persistence [9]. Most of the recent theoretical effort has gone into obtaining the numerical value of \( \theta \) for different models.

Very recently, Manoj and Ray [10] have studied the spatial correlation of persistent sites in the 1d \( A + A \rightarrow 0 \) model. They found that the set of persistent sites in their 1d model forms a fractal over sufficiently small length scales.

In this letter we present the results of an extensive numerical study of the spatial distribution of persistent spins in the pure 2d Ising model at zero-temperature. As we will see, the 2d Ising model exhibits behaviour very similar to that found by Manoj and Ray [10] in their simple 1d model.

The Hamiltonian for our model is given by

\[
H = - \sum_{<ij>} S_i S_j
\]

where \( S_i = \pm 1 \) are Ising spins situated on every site of a square lattice with periodic boundary conditions; the summation in Eqn. (2) runs over all nearest-neighbour pairs only.

The data presented in this work were obtained for a lattice with dimensions 1000 × 1000 (\( = N \)).
Each simulation run begins at $t = 0$ with a random ($\pm 1$) starting configuration of the spins and then we update the lattice via single spin flip zero-temperature Glauber dynamics [5]. The rule we use is: always flip if the energy change is negative, never flip if the energy change is positive and flip at random if the energy change is zero.

For each spin $S_i$ we define

$$n_i(t) = (S_i(t)S_i(0) + 1)/2.$$  \hspace{1cm} (3)

Hence, if $n_i(t) = 1$ for all $t \geq 0$ spin $S_i$ is persistent at time $t$; $n_i(t) = 0$ otherwise.

The total number, $n(t)$, of spins which have never flipped until time $t$ is then given by $n(t) = \sum_i n_i(t)$, and the persistence probability by [1]

$$P(t) = \sum_i < n_i(t) > /N$$  \hspace{1cm} (4)

where $< \ldots >$ indicates averages over different initial conditions and histories. We averaged over at least 100 different initial conditions and histories for each run.

To investigate the spatial correlations in this model, we follow Manoj and Ray [10] and study the 2-point correlator defined by

$$C(r, t) = < n_i(t)n_{i+r}(t) > / < n_i(t) >,$$  \hspace{1cm} (5)

where $< \ldots >$ now also includes the average over the lattice shown explicitly in Eqn (4). $C(r, t)$ is simply the probability that spin $n_{i+r}(t)$ is persistent given that $n_i(t)$ is persistent, averaged over the entire lattice. According to [10], the 2-point correlator satisfies the following dynamic scaling relation

$$C(r, t) = P(t)f(r/\xi(t))$$  \hspace{1cm} (6)

where $\xi(t)$ is the persistence correlation length and $f(x)$ is a scaling function such that

$$f(x) \sim \begin{cases} x^{-\alpha}, & \text{for } x << 1; \\ 1, & \text{for } x >> 1. \end{cases}$$  \hspace{1cm} (7)
As a consequence, the expected behaviour of $C(r, t)$ in the two limits is given by

$$C(r, t) \sim \begin{cases} r^{-\alpha} & \text{for } r << \xi(t); \\ t^{-\theta} & \text{for } r >> \xi(t). \end{cases}$$ \hspace{1cm} (8)

Clearly, as $P(t) \sim t^{-\theta}$, we must also have $\xi^{-\alpha} \sim t^{-\theta}$ to satisfy Eqn (8) in the limit $r << \xi(t)$. Assuming a power-law divergence for the persistence correlation length with $t$ i.e. $\xi(t) \sim t^Z$ then leads to the scaling relation $Z\alpha = \theta$. As we are working with the pure $2d$ Ising model at zero-temperature, we expect [11] $Z = 1/2$; our results are completely consistent with this assumption.

To examine the correlated region $(r << \xi(t))$ we study the average number of persistent spins, $n(l, t)$, in a square grid with dimensions $l \times l$. As

$$n(l, t) = \int_0^l C(r, t) r dr$$ \hspace{1cm} (9)

we have that

$$n(l, t) \sim \begin{cases} l^{2-\alpha} & \text{for } l << \xi(t); \\ l^2 P(t) & \text{for } l >> \xi(t). \end{cases}$$ \hspace{1cm} (10)

Hence, we expect the persistent spins to form a fractal with dimension $d_f = 2 - \alpha$ for length scales $l << \xi(t)$; the distribution is homogeneous on longer length scales, namely for $l >> \xi(t)$. We expect the crossover to occur at $l \approx \xi(t) \sim t^{1/2}$. The scaling form for $n(l, t)$ is given by

$$n(l, t) = l^2 P(t) g(l/\xi(t)),$$ \hspace{1cm} (11)

where $g(x)$ is a scaling function satisfying

$$g(x) \sim \begin{cases} x^{-\alpha} & \text{for } x << 1; \\ 1 & \text{for } x >> 1. \end{cases}$$ \hspace{1cm} (12)

We now discuss our results.

Figure 1 shows a plot of the scaling function $f(x)(= C(r, t)/P(t))$ against $x = r/\xi(t)$ for various different values of $t$. We have assumed that $\xi(t) \sim t^{1/2}$. The data in Fig 1 ranges over almost three orders of magnitude and is clearly consistent with this assumption. The large $x$ behaviour of $f(x)$ clearly follows the expected behaviour given in Eqn (7).
To extract a value for $\alpha$ we re-plot the data shown in Fig 1 on a log-log scale in Fig 2. The algebraic behaviour for $x \ll 1$ of the scaling function is confirmed by the linear fit. The slope of the straight line implies a value of $\alpha = 0.428 \pm 0.007$. Hence, the scaling relation would suggest that $\theta = Z\alpha = 0.214 \pm 0.004$. This is, of course, consistent with value $(0.209 \pm 0.002)$ quoted above for $\theta$ [5].

We investigate the correlated regions by obtaining a direct estimate of the fractal dimension $d_f$. This is undertaken by first partitioning the lattice into square grids of size $l \times l$ with $l$ ranging from 4 to 250. The average number of persistent spins in each $l \times l$ square is then obtained.

In Figure 3 we plot $\ln n(l, t)$ versus $\ln l$ for $t = 10^2, 10^3, 5 \times 10^3$ and $10^4$. We notice that for each of the values of $t$, the behaviour over sufficiently small (typically, $l \ll \sqrt{t}$) length scales is consistent with a fractal dimension $d_f = 2 - \alpha \sim 1.58$; over longer length scales (typically, $l \gg \sqrt{t}$) we retrieve homogeneous behaviour ($d_f = d = 2$). Actual values of $d_f$ range from $d_f(t = 10^2) \sim 1.62$ to $d_f(t = 10^4) \sim 1.58$. The straight lines, with slopes 1.58 and 2.00, shown in Fig 3 are linear fits to the behaviour in the two respective regimes for $t = 10^4$.

We obtain an independent estimate for the exponent $\alpha$ by re-plotting the data for $t = 5 \times 10^3$ and $10^4$ in scaling form. Figure 4 shows a log-log plot of the scaling function $g(x) = n(l, t)/l^2P(t)$ against $x$ where $x = l/\sqrt{t}$. We see that the data clearly fall onto a single scaling curve consistent with the expected behaviour given in Eqn (12). On fitting all of the data for $\ln x < -0.5$ we get a value of $\alpha \sim 0.438$. However, restricting the linear fit to $\ln x < -1$, as indicated by the straight line in Fig 4, would imply a value of $\alpha \sim 0.50$. Although this is slightly higher than the value we obtained from the analysis of the scaling behaviour of the 2-point correlator (see Eqn (8)), it is, nevertheless, consistent with our value of the fractal dimension in the correlated regime.

To conclude, we have investigated the spatial distribution of persistent spins at zero-temperature in the pure two-dimensional Ising model. We find that the persistent spins
form a fractal with dimension $d_f \sim 1.58$ for length scales $r << \xi(t)$, where $\xi(t) \sim t^Z$ is the persistence correlation length. Furthermore, the persistence exponent satisfies the scaling relation $\theta = Z(2 - d_f)$ with $Z = 1/2$.

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FIGURE CAPTIONS

Fig. 1
A plot of the scaling function \( f(x) = C(r,t)/P(t) \) against \( x \) where \( x = r/\sqrt{t} \) for \( t \) ranging over approximately three orders of magnitude.

Fig. 2
A re-plot of the data shown in Figure 1 on a log-log scale. The straight line implies a value of \( \alpha = 0.428 \pm 0.007 \).

Fig. 3
A log-log plot of \( n(l,t) \) against \( l \). Here, \( n(l,t) \) is the average number of persistent spins in a square \((l \times l)\) grid at time \( t \). The data is shown for (top) \( t = 10^2 \), \( 10^3 \), \(+\), \( 5 \times 10^3 \) and \( 10^4 \) \((bottom)\). There is a clear crossover at \( l \approx \sqrt{t} \) from a fractal distribution with dimension \( d_f \sim 1.58 \) to a homogeneous one with \( d_f = d = 2 \). The two straight lines (with slopes 1.58 and 2.00) are fits of the data in the two extreme cases for \( t = 10^4 \).

Fig. 4
A plot of \( \ln g(x) \) against \( \ln x = \ln l/\xi(t) \). Here the scaling function \( g(x) = n(l,t)/l^2 P(t) \). The straight line has slope = -0.50 and implies a value of \( \alpha \sim 0.50 \).
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Figure 1

\[ f(x) = \frac{r}{t^{1/2}} \]

\[ N = 1000 \times 1000; \ t = 32 \quad \Diamond \]
\[ t = 64 \quad + \]
\[ t = 128 \quad \Box \]
\[ t = 256 \quad \times \]
\[ t = 512 \quad \triangle \]
\[ t = 1024 \quad * \]
\[ t = 2048 \quad \Diamond \]
Figure 2

$N = 1000 \times 1000; t = 32$  
$t = 64$  
$t = 128$  
$t = 256$  
$t = 512$  
$t = 1024$  
$t = 2048$ 

$\ln f(x)$ vs $\ln x$
Figure 3

\[ \ln n(l, t) \] vs. \[ \ln l \]
