Abstract. We introduce new invariants in equivariant birational geometry and study their relation to modular symbols and cohomology of arithmetic groups.

1. Introduction

Let $G$ be a finite abelian group and
\[ A = G^\vee = \text{Hom}(G, \mathbb{C}^\times) \]
the group of characters of $G$. Fix an integer $n \geq 2$. Consider the $\mathbb{Z}$-module
\[ B_n(G) \]
generated by symbols
\[ [a_1, \ldots, a_n], \quad a_i \in A, \]
such that $a_1, \ldots, a_n$ generate $A$, i.e.,
\[ \sum_i \mathbb{Z}a_i = A, \]
and subject to relations:

(S) for all permutations $\sigma \in \mathfrak{S}_n$ and all $a_1, \ldots, a_n \in A$ we have
\[ [a_{\sigma(1)}, \ldots, a_{\sigma(n)}] = [a_1, \ldots, a_n], \]

(B) for all $2 \leq k \leq n$, all $a_1, \ldots, a_k \in A$, and all $b_1, \ldots, b_{n-k} \in A$ such that
\[ \sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A \]
we have
\[ [a_1, \ldots, a_k, b_1, \ldots, b_{n-k}] = \sum_{1 \leq i \leq k, \ a_i \neq a_{i'}, \forall i' < i} [a_1 - a_i, \ldots, a_i(\text{on } i\text{-th place}), \ldots, a_k - a_i, b_1, \ldots, b_{n-k}] \]
For example, for \( n = 4 \) and \( k = 3 \) and \( a_1 = a_2 = a \) and \( a_3 = a' \neq a \) and \( b_1 = b \), the relation translates to
\[
[a, a, a', b] = [a, 0, a' - a, b] + [a - a', a - a', a', b].
\]
When \( n = 2 \) there is only one possibility for \( k \), namely, \( k = 2 \).

**Example 1.** Assume that \( G = \mathbb{Z}/N\mathbb{Z} \simeq A \), \( N \in \mathbb{Z}_{\geq 2} \).

Then \( B_2(G) \) is generated by symbols \([a_1, a_2]\) such that \( a_1, a_2 \in \mathbb{Z}/N\mathbb{Z} \), \( \gcd(a_1, a_2, N) = 1 \), and subject to relations
\[
\begin{align*}
[&a_1, a_2] = [a_2, a_1], \\
[&a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2] \quad \text{where } a_1 \neq a_2, \\
[&a, a] = [a, 0], \quad \text{for all } a \in \mathbb{Z}/N\mathbb{Z}, \; \gcd(a, N) = 1.
\end{align*}
\]

For \( p \geq 5 \) a prime, the \( \mathbb{Q} \)-rank of \( B_2(\mathbb{Z}/p\mathbb{Z}) \) equals
\[
\frac{p^2 + 23}{24}.
\]

For us, this was the first sign that automorphic forms play a role in this theory. We will discuss the connection to modular symbols in Section 8.

**Remark 2.** The group \( B_2(\mathbb{Z}/p\mathbb{Z}) \) can have torsion, e.g., for \( p = 37 \), there is \( \ell \)-torsion for \( \ell = 3 \) and 19.

For \( n \geq 3 \), the system of relations in \( B_n(G) \) is highly overdetermined. Nevertheless, computer experiments show that nontrivial solutions exist, e.g., for \( G = \mathbb{Z}/27\mathbb{Z} \) or \( \mathbb{Z}/43\mathbb{Z} \), the \( \mathbb{Q} \)-rank of \( B_4(G) \) equals 1.

Let \( X \) be a smooth irreducible projective algebraic variety of dimension \( n \geq 2 \), over a fixed algebraically closed field of characteristic zero (e.g., \( \mathbb{C} \)), equipped with a birational, generically free action of \( G \). After \( G \)-equivariant resolution of singularities, we may assume that the action of \( G \) is regular. To such an \( X \) we associate an element of \( B_n(G) \) as follows:
Let
\[
X^G = \prod_{\alpha \in \mathcal{A}} F_\alpha
\]
be the \( G \)-fixed point locus; it is a disjoint union of closed smooth irreducible subvarieties of \( X \). Put
\[
\dim(F_\alpha) = n_\alpha \leq n - 1.
\]

On each irreducible component \( F_\alpha \) we fix a point \( x_\alpha \in F_\alpha \) and consider the action of \( G \) in its tangent space \( T_{x_\alpha}X \) in \( X \); it decomposes
into eigenspaces of characters $a_{1,\alpha}, \ldots, a_{n,\alpha}$, defined up to permutation of indices (here we identify algebraic characters of $G$ with $\mathbb{C}^\times$-valued characters). By the assumption that the action of $G$ is generically free, we have

$$\sum_i Z_{a_{i,\alpha}} = A.$$ 

This does not depend on the choice of $x_{\alpha} \in F_{\alpha}$. The dimension $\dim(F_{\alpha})$ equals the number of zeros among the $a_{i,\alpha}$. Thus we have a symbol, for each $\alpha$,

$$[a_{1,\alpha}, \ldots, a_{n,\alpha}] \in B_n(G).$$

Put

(1.4) \[ \beta(X) := \sum_{\alpha} [a_{1,\alpha}, \ldots, a_{n,\alpha}] \]

One of our main results is that expression (1.4), considered as an element in $B_n(G)$, is invariant under $G$-equivariant blowups.

**Theorem 3.** The class $\beta(X) \in B_n(G)$ is a $G$-equivariant birational invariant.

Now we introduce another $\mathbb{Z}$-module

$$\mathcal{M}_n(G),$$

generated by symbols

$$\langle a_1, \ldots, a_n \rangle,$$

such that $a_1, \ldots, a_n$ generate $A$, and subject to relations which are almost identical to those for $B_n(G)$:

(S) for all $\sigma \in \mathfrak{S}_n$ and all $a_1, \ldots, a_n \in A$ we have

$$\langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle = \langle a_1, \ldots, a_n \rangle,$$

(M) for all $2 \leq k \leq n$, all $a_1, \ldots, a_k \in A$ and all $b_1, \ldots, b_{n-k} \in A$, such that

$$\sum_i Z a_i + \sum_j Z b_j = A,$$

we have

$$\langle a_1, \ldots, a_k, b_1, \ldots, b_{n-k} \rangle =$$

$$= \sum_{1 \leq i \leq k} \langle a_1 - a_i, \ldots, a_i \text{(on } i\text{-th place)}, \ldots, a_k - a_i, b_1, \ldots, b_{n-k} \rangle.$$
Note that we eliminated the constraint \( a_i \neq a_{i'} \), for \( i' < i \), from the sum.

For \( n = 4 \) and \( k = 3 \) and \( a_1 = a_2 = a \) and \( a_3 = a' \neq a \) and \( b_1 = b \), the relation translates to

(1.5) \( \langle a, a, a', b \rangle = \langle a, 0, a' - a, b \rangle + \langle 0, a, a' - a, b \rangle + \langle a - a', a - a', a', b \rangle \).

The right side equals to

\[ 2 \langle a, 0, a' - a, b \rangle + \langle a - a', a - a', a', b \rangle, \]

by symmetry relations. Notice the difference between (1.5) and (1.1).

These groups carry naturally defined, commuting, linear operators

\[ T_{\ell, r} : \mathcal{M}_n(G) \to \mathcal{M}_n(G), \]

for all primes \( \ell \) coprime to the order of \( G \) and all \( 1 \leq r \leq n \). We call these Hecke operators. One can consider their spectrum for

\[ \mathcal{M}_n(G) \otimes \bar{\mathbb{Q}} \text{ or } \mathcal{M}_n(G) \otimes \bar{\mathbb{F}}_p, \]

where \( p \) is any prime not dividing \( \#G \), the order of the group \( G \). We expect that the joint spectrum of \( T_{\ell, r} \) is related to automorphic forms and present evidence for this in Sections 7 and 8.

Consider the map

\[ \mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G) \]

declared on symbols as follows:

- \((\mu_0)\) \[ [a_1, \ldots, a_n] \mapsto \langle a_1, \ldots, a_n \rangle, \text{ if all } a_1, \ldots, a_n \neq 0, \]
- \((\mu_1)\) \[ [0, a_2, \ldots, a_n] \mapsto 2\langle 0, a_2, \ldots, a_n \rangle, \text{ if all } a_2, \ldots, a_n \neq 0, \]
- \((\mu_2)\) \[ [0, 0, a_3, \ldots, a_n] \mapsto 0, \text{ for all } a_3, \ldots, a_n, \]

and extended by \( \mathbb{Z} \)-linearity.

**Theorem 4.** The map \( \mu \) is a well-defined homomorphism, which is a surjection modulo 2-torsion.

Note that

\[ \langle 0, 0, a_3, \ldots, a_n \rangle = 0 \in \mathcal{M}_n(G) \]

which follows from the relations by putting

\[ k = 2, a_1 = a_2 = 0, b_i = a_{i+2}, \text{ for all } i = 1, \ldots, n - 2. \]

We expect that \( \mu \) is an isomorphisms, modulo torsion (see Conjectures 8 and 9).

This paper consists of two parts: in Part 1, we present proofs of Theorems 3 and 4 and introduce Hecke operators on \( \mathcal{M}_n(G) \). Our notation \( \mathcal{B}_n(G) \) and \( \mathcal{M}_n(G) \) stands for
In Part 2, we introduce various generalizations of $\mathcal{B}_n(G)$ and $\mathcal{M}_n(G)$, not necessarily related to each other, reflecting a certain divergence of birational and automorphic sides. Our considerations led us to a new question (see Question 11 in Section 7), and a potentially new viewpoint on the Langlands program, based on higher-dimensional generalizations of modular symbols. During the preparation of this paper we discovered the work of Borisov-Gunnels [BG01], who studied constructions related to the modular picture in the case $n = 2$ and raised the question of generalizations to $n \geq 3$ in [BG03, Remark 7.15]. We also explore, in the case $n = 2$, the relation between our groups of symbols and classical Manin symbols for modular forms of weight 2.

In the last section, we present results of computer experiments with equations for new invariants.

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Part 1

2. Invariance under blowups

We use notation and conventions from the Introduction. Let $X$ be a smooth irreducible projective $n$-dimensional variety equipped with generically free regular action of a finite abelian group $G$, and $W \subset X$ a closed smooth irreducible $G$-stable subvariety,

$$0 \leq \dim(W) \leq n - 2.$$ 

Let

$$\pi : \tilde{X} = \text{Bl}_W(X) \to X$$

be the blowup of $X$ in $W$. By the $G$-equivariant Weak Factorization theorem, smooth projective $G$-birational models of $X$ are connected by iterated blowups of such type.

In order to prove Theorem 3 it suffices to show that

$$\beta(\tilde{X}) = \beta(X) \in \mathcal{B}_n(G).$$
Choose an irreducible component $Z \subset W^G$. It suffices to consider the structure of the fixed locus of exceptional divisors in the neighborhood of $Z$. Let

$$F = F(Z) \subseteq X^G$$

be the unique irreducible component containing $Z$, it equals one of the $F_\alpha$ in [1.3]. Let $z \in Z$ be a point and

$$\mathcal{T}_zX = T_1 \oplus T_2 \oplus R_1 \oplus R_2$$

the decomposition of the tangent bundle at $z$, where $T_i$ stand for trivial representations, and $R_1, R_2$ have only nontrivial characters, with

$$\mathcal{T}_zX^G = T_2F = T_1 \oplus T_2, \quad \mathcal{T}_zW = T_2 \oplus R_1.$$

Let

$$d_1 := \dim(T_1), \quad d_2 = \dim(T_2), \quad d_3 = \dim(R_1), \quad d_4 = \dim(R_2).$$

The spectrum of the action of $G$ in $\mathcal{T}_z$ takes the form

$$\begin{array}{cccc}
0, \ldots, 0 & 0, \ldots, 0 & b_1, \ldots, b_d & a^1, \ldots, a^m, \\
\underbrace{\cdots}_{d_1} & \underbrace{\cdots}_{d_2} & \underbrace{\cdots}_{\kappa_1} & \underbrace{\cdots}_{\kappa_m},
\end{array}$$

where $b_j \in A \setminus 0$, and $a^1, \ldots, a^m \in A \setminus 0$, pairwise distinct, with

$$\kappa_1 + \cdots + \kappa_m = d_4, \quad \kappa_i \geq 1, \ m \geq 0.$$

We have

- $d_2 = \dim(Z)$,
- $d_1 + d_2 + d_3 + d_4 = n$,
- $1 \leq d_3 + d_4$, since $\text{codim}(X^G) \geq 1$,
- $2 \leq d_1 + d_4$, since $\text{codim}(W) \geq 2$.

We consider cases, with corresponding geometric configurations:

(I) $d_1 = 0, d_4 \geq 2$, geometrically, this means that $W$ contains a component of $X^G$. Blowing up $W$ we obtain new contributions to formula (1.4). The new fixed locus, with $m$ irreducible components, consists of subvarieties of the exceptional divisor, a projective bundle over $W$. These subvarieties, in turn, are total spaces of projective bundles over $Z$, with fibers

$$\mathbb{P}^{\kappa_i - 1}, \quad i = 1, \ldots, m.$$

The corresponding contribution to $\beta(\tilde{X})$ is given by

$$\sum_{i=1}^m [0, \ldots, b_i, \ldots, b_d, a^1 - a^i, \ldots, a_i, 0, \ldots, a^m - a^i, \ldots].$$
Putting
\[a_1, \ldots, a_k = a_1^{\kappa_1}, \ldots, a_m^{\kappa_m}\]
and
\[b_1, \ldots, b_{n-k} = b_1, \ldots, b_{d_2}, 0, \ldots\]
we find that the formula matches relation (B), in the case when the sequence \(\bar{a} = a_1, \ldots, a_k\) does not contain zeros.

(II) \(d_1, d_4 \geq 1\), geometrically, this means that the tangent spaces of the fixed locus and \(W\) do not span the whole tangent space and, near \(Z\), the component \(F\) is not contained in \(W\). In the blowup, we will have a component of the fixed locus which is birational to \(F\) and new components which are projective bundles \(\mathbb{P}^{\kappa_1-1}, \ldots, \mathbb{P}^{\kappa_m-1}\) over \(Z\). We need to show that the contribution of these \(m\) terms vanishes in \(B_n(G)\). The new components contribute
\[
\sum_{i=1}^{m} \left[ -a^i, a_1^1 - a^i, \ldots, a_i, 0, \ldots, a^m - a^i, \ldots, \bar{b} \right].
\]
We claim that this sum vanishes in \(B_n(G)\). Indeed, consider relation (B) for the sequences
\[\bar{a} = a_1, \ldots, a_k = 0, \ldots, a^1, \ldots, a^m, \ldots\]
and, as before,
\[\bar{b} = b_1, \ldots, b_{n-k} = b_1, \ldots, b_{d_2}, 0, \ldots\]
The left side of (B) equals
\[\left[ \bar{a}, \bar{b} \right] = [a_1, \ldots, a_k, \bar{b}] = [0, \ldots, a^1, \ldots, a^m, \ldots, \bar{b}]\]
The right side is the sum of \((m+1)\) terms. The first summand, corresponding to \(a_i = a_1 = 0\) coincides with the left side. The remaining terms are the same as above.

(III) \(d_1 \geq 2, d_3 \geq 1, d_4 = 0\), in this case, \(F\) is not contained in \(W\), no new contributors to formula (1.4) arise.

This concludes the proof of Theorem 3.
3. Hecke operators

In this section, \( G \) is a finite abelian group, with character group \( A = \text{Hom}(G, \mathbb{C}^\times) \), and \( n \geq 2 \) is an integer. We will define analogs of Hecke operators on \( M_n(G) \).

We start with the following data:

- a (torsion-free) lattice \( L \cong \mathbb{Z}^n \) of rank \( n \),
- an element \( \chi \in L \otimes A \) such that the induced homomorphism \( L^\vee \to A \) is a surjection,
- a basic simplicial cone, i.e., a strictly convex cone \( \Lambda \in L_\mathbb{R} \) spanned by a basis of \( L \). It is isomorphic to the standard octant \( \mathbb{R}_{\geq 0}^n \), for \( L = \mathbb{Z}^n \subset \mathbb{R}^n \).

For every equivalence class of triples \( (L, \chi, \Lambda) \), up to isomorphism, we define a symbol

\[
\psi(L, \chi, \Lambda) \in M_n(G)
\]

as follows: choose a basis \( e_1, \ldots, e_n \) of \( L \), spanning \( \Lambda \), and express

\[
\chi = \sum_{i=1}^n e_i \otimes a_i,
\]

and put

\[
\psi(L, \chi, \Lambda) = (a_1, \ldots, a_n) \in M_n(G).
\]

The ambiguity in the choices is reflected in the action of the symmetric group \( \mathfrak{S}_n \) on the basis elements, hence accounted for by condition (S). Relation (M) has the following geometric meaning: let \( e_1, \ldots, e_n \) be an ordered basis of \( L \) spanning \( \Lambda \). Fix an integer \( 2 \leq k \leq n \). Then

\[
\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k,
\]

where

\[
\Lambda_i := \mathbb{R}_{\geq 0}e_1 + \cdots + \mathbb{R}_{\geq 0}(e_1 + \cdots + e_k) + \cdots + \mathbb{R}_{\geq 0}e_n,
\]
i.e., we are replacing the $i$-th generator $e_i$ by $(e_1 + \cdots + e_k)$. The cones $\Lambda_i$ are also basic simplicial and their interiors are disjoint. Decompose

$$\chi = e_1 \otimes a_1 + \cdots + e_k \otimes a_k + e_{k+1} \otimes b_1 + \cdots + e_n \otimes b_{n-k}$$

as in (3.1), i.e., $a_{k+i} = b_i$, for all $i = 1, \ldots, n-k$. Then, in the basis of $\Lambda_i$, $\chi$ decomposes as

$$e_1 \otimes (a_1 - a_i) + \cdots + (e_1 + \cdots + e_k) \otimes a_i + \cdots e_k \otimes (a_k - a_i) + \sum_{j=1}^{n-k} e_{k+j} \otimes b_j.$$  

We see that relation (M) can be expressed as the following identity

$$(3.2) \quad \psi(L, \chi, \Lambda) = \sum_{i=1}^{k} \psi(L, \chi, \Lambda_i),$$

which we can view as an analog of scissor relations.

Now we can replace $\Lambda$ by any rational strictly convex polyhedral cone in $L \otimes \mathbb{R}$, by decomposing $\Lambda$ into a finite union of basic simplicial cones with disjoint interiors; the independence of the choice of this decomposition follows from the toric analog of Weak Factorization.

In consequence, $\mathcal{M}_n(G)$ admits an alternative description: as the group generated by symbols 

$$\psi(L, \chi, \Lambda),$$

depending only on the isomorphism classes of triples, where $L$ and $\chi$ are as above, and $\Lambda$ is a finitely generated convex rational polyhedral cone, of full dimension, subject to the relations (3.2), whenever there is a decomposition

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$$

into rational polyhedral subcones of full dimension, with disjoint interiors. This clearly extends to nonconvex cones.

Fix a prime $\ell$ not dividing $\#G$ and an integer $1 \leq r \leq n-1$. Put

$$(3.3) \quad T_{\ell,r}(\psi(L, \chi, A)) := \sum_{L \subset L' \subset L \otimes \mathbb{R}, L' / L \cong (\mathbb{Z} / \ell \mathbb{Z})^r} \psi(L', \chi, A),$$

where $\chi$ is now interpreted as an element of $L' \otimes A$, via inclusion

$$L \otimes A \subset L' \otimes A,$$

the surjectivity property for $\chi \in L' \otimes A$ follows from the surjectivity of $\chi \in L \otimes A$ and the assumption on coprimality of $\ell$ and the order of $G$. 

Proposition 5. The Hecke operators $T_{\ell,r}$ are well-defined on $\mathcal{M}_n(G)$, and commute with each other.

Proof. Follows from the additivity of (3.2) and (3.3). □

Example 6. We consider the case $n = 2$ and $G = \mathbb{Z}/N\mathbb{Z} \simeq A$. Then $\mathcal{M}_n(G)$ is generated by

$$\langle a_1, a_2 \rangle, \quad a_1, a_2 \in \mathbb{Z}/N\mathbb{Z}, \quad \text{gcd}(a_1, a_2, N) = 1,$$

such that

- $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$,
- $\langle a_1, a_2 \rangle = \langle a_1, a_2 - a_1 \rangle + \langle a_1 - a_2, a_2 \rangle$, for all $a_1, a_2$.

We write down an example of a Hecke operator on $\mathcal{M}_2(G)$. For each $\ell$ coprime to $N$ we have only one Hecke operator $T_\ell = T_{\ell,1}$.

Assume that $N$ is odd and $\ell = 2$. Let

$$L = \mathbb{Z}^2, \quad \chi = (1,0) \otimes a_1 + (0,1) \otimes a_2, \quad \Lambda = \mathbb{R}_\geq 0^2,$$

the standard octant. There are three overlattices of $L$ of index 2, corresponding to the three elements of $\mathbb{P}^1(\mathbb{F}_2)$:

- $L'_0 := \mathbb{Z} \cdot (\frac{1}{2}, 0) + \mathbb{Z} \cdot (0,1)$,
- $L'_1 := \mathbb{Z} \cdot (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z} \cdot (0,1) = \mathbb{Z} \cdot (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z} \cdot (1,0)$,
- $L'_{\infty} := \mathbb{Z} \cdot (0,\frac{1}{2}) + \mathbb{Z} \cdot (1,0)$.

The corresponding cones in the first and third case are basic simplicial, whereas in the second case it is not basic and can be decomposed in the union of two basic simplicial cones, with respect to $L'_1$:

$$\Lambda = \Lambda_1 \cup \Lambda_2,$$

$$\Lambda_1 = \mathbb{R}_\geq 0 \cdot (1,0) + \mathbb{R}_\geq 0 \cdot (1,1), \quad \Lambda_2 = \mathbb{R}_\geq 0 \cdot (1,1) + \mathbb{R}_\geq 0 \cdot (0,1).$$

Therefore,

$$T_2(\langle a_1, a_2 \rangle) = \langle 2a_1, a_2 \rangle + (\langle a_1 - a_2, 2a_2 \rangle + \langle 2a_1, a_2 - a_1 \rangle) + \langle a_1, 2a_2 \rangle.$$

The middle term follows from equalities

$$e_1 \otimes a_1 + e_2 \otimes a_2 = e_1 \otimes (a_1 - a_2) + \frac{e_1 + e_2}{2} \otimes 2a_2 = \frac{e_1 + e_2}{2} \otimes 2a_1 + e_2 \otimes (a_2 - a_1).$$

We leave it as an exercise to write down a similar formula for the action of $T_3$ on $\mathcal{M}_2(G)$ and $T_2$ on $\mathcal{M}_3(G)$.
4. Comparison

In this section we study the map

\[ \mu : \mathcal{B}_n(G) \to \mathcal{M}_n(G) \]

defined in Section \[\text{1}\]. The proof that this is a well-defined homomorphism is a long chain of essentially trivial steps.

First we record several corollaries of defining relations for \[\mathcal{M}_n(G)\]:

1. \[\langle 0, 0, \ldots \rangle = 0 \],
2. \[\langle a, a, \ldots \rangle = 2 \langle a, 0, \ldots \rangle \],
3. \[\langle a, a, 0, \ldots \rangle = 0 \],
4. \[\langle a, a, a', a', \ldots \rangle = 0 \],
5. \[\langle a, a, a, \ldots \rangle = 0 \],
6. \[\langle a, -a, \ldots \rangle = 0 \],

here \ldots stands for arbitrary sequences of elements in \[A\], such that the set of all elements of the symbol spans the whole \[A\].

In the proofs below we freely use the symmetry relation (S).

1. We use (M) for \[k = 2 \] and \[a_1 = a_2 = 0\]:
   \[\langle 0, 0, \ldots \rangle = \langle 0, 0, \ldots \rangle + \langle 0, 0, \ldots \rangle \].
2. We use (M) for \[k = 2, a_1 = a_2 = a\].
3. We use (2) and (1):
   \[\langle a, a, 0, \ldots \rangle \overset{(2)}{=} 2 \langle a, 0, 0, \ldots \rangle + \langle 0, 0, \ldots \rangle \overset{(1)}{=} 0 \].
4. We use again (2) and (1):
   \[\langle a, a, a', a', \ldots \rangle \overset{(2)}{=} 4 \langle a, 0, a', 0, \ldots \rangle \overset{(1)}{=} 0 \].
5. We use (M) for \[k = 3 \] and \[a_1 = a_2 = a_3 = a\], and then (1):
   \[\langle a, a, a, \ldots \rangle = 3 \langle a, 0, 0, \ldots \rangle \overset{(1)}{=} 0 \],
6. We use (M) for \[k = 2, a_1 = a, a_2 = 0\]:
   \[\langle a, 0, \ldots \rangle = \langle a, -a, \ldots \rangle + \langle a, 0, \ldots \rangle \].

We proceed to the proof of Theorem \[\text{4}\]. The main point is to check the following compatibility equation

\[ \mu([a_1, \ldots, a_k, b_1, \ldots, b_{n-k}]) = \]

\[ \sum_{i, a_i \neq a_{i'}, \text{ for } i < i'} \mu([a_1 - a_i, \ldots, a_i, \ldots, a_k - a_i, b_1, \ldots, b_{n-k}]) \].
For convenience, we sometimes write
\[ [a_1, \ldots, a_k \mid b_1, \ldots, b_{n-k}] = [a_1, \ldots, a_k, b_1, \ldots, b_{n-k}] \in B_n(G), \]
and similarly, for the symbol in \( M_n(G) \), indicating the position of the separation of \( a \) and \( b \) variables in subsequent relations.

There are three cases, distinguished by the number of zeros in the sequence \( \bar{b} := b_1, \ldots, b_{n-k} \):

(C0) \( \bar{b} \) does not contain zeros.
(C1) \( \bar{b} \) contains exactly one zero.
(C2) \( \bar{b} \) contains at least two zeros.

The case (C2) is obvious, by relation (1), since all terms vanish, by the definition \( (\mu_2) \) (in Section [I]).

The case (C1) splits into subcases

(C10) The sequence \( \bar{a} := a_1, \ldots, a_k \)

contains no zeros,
(C11) \( \bar{a} \) contains at least one zero.

In the case (C11), the left hand side maps to 0, by \( (\mu_2) \):
\[ \mu([0, \ldots \mid 0, \ldots]) = 0. \]

The terms of the right hand side in the relation (B) are of two types, corresponding to \( a_i = 0 \) or \( a_i = a \neq 0 \). If \( a_i = 0 \), then the term has the form
\[ [0, \ldots \mid 0, \ldots], \]
mapping to zero, by \( (\mu_2) \). The underlined 0 indicates that \( a_i \) is left in its place, in the relation (B). If \( a_i = a \neq 0 \), then the corresponding term in the right hand side of (B) has the form
\[ \lbrack -a, \cdots, a, \ldots \mid 0, \ldots \rbrack, \]
mapping to
\[ c \cdot \langle -a, \ldots, a, \ldots 0, \ldots \rangle, \]
where \( c = 0 \) or 2, and the symbol in \( M_n(G) \) equals 0, by (6).

The case (C10) splits into two cases:
(C10\( \neq \)) all terms in \( \bar{a} \) are pairwise distinct,
(C10\( = \)) there exists at least two equal terms in \( \bar{a} \).
In case (C10≠), in the left and in the right hand side of the relation (B), all symbols contain exactly one zero. Thus, they are mapped to similar symbols in $\mathcal{M}_n(G)$, but multiplied by 2, by $(\mu_1)$. Since every element in $\bar{a}$ occurs only once, the expressions on the right side of (B) and (M) consist of matching terms.

In case (C10=), the left hand side of (B) equals

$$[a,a,\ldots | 0,\ldots] \in \mathcal{B}_n(G).$$

Its image under $\mu$ equals

$$2\langle a,a,\ldots,0,\ldots \rangle \in \mathcal{M}_n(G),$$

which vanishes, by (3). We claim that all terms on the right side of (B) map to zero as well. Indeed, they are either of the form

$$[a,0,\ldots | 0,\ldots] \quad \text{or} \quad [a-a',a-a',\ldots,a',\ldots | 0,\ldots], \quad a' \neq a.$$

The image of this symbol is proportional to

$$\langle a,0,\ldots,0,\ldots \rangle \quad \text{or} \quad \langle a-a',a-a',\ldots,a',\ldots,0,\ldots \rangle,$$

vanishing by (1) or (3), respectively.

The case (C0) splits into three cases:

(C00) $\bar{a}$ does not contain zeros,
(C01) $\bar{a}$ contains exactly one zero,
(C02) $\bar{a}$ contains at least two zeros.

Recall that $\bar{b}$ does not contain zeros, in case (C0). We start with (C02). The left hand side in (B) has the form

$$[0,0,\ldots | \ldots],$$

hence maps to 0, by $(\mu_2)$. We check that all terms on the right hand side of (B) map to 0 as well. These symbols have the form

$$[0,0,\ldots | \ldots] \quad \text{or} \quad [-a,-a,\ldots,a,\ldots | \ldots], \quad a \neq 0,$$

mapping to elements in $\mathcal{M}_n(G)$ which are proportional to either

$$\langle 0,0,\ldots \rangle \quad \text{or} \quad \langle -a,-a,\ldots,a,\ldots \rangle,$$

vanishing by (1) or (6), respectively.

The case (C01) splits into two cases:

(C01≠) all terms in $\bar{a}$ are pairwise distinct,
(C01=) there exists at least two equal terms in $\bar{a}$. 

In case (C01\(=\)), the left side in (B) has the form
\[ [0, a, a, \ldots | \ldots], \quad \text{for } a \neq 0, \]
mapping to 0, by relation (3). The right side contains terms of the form
\[ [0, a, a, \ldots | \ldots] \quad \text{or} \quad [-a, a, 0, \ldots | \ldots], \]
or
\[ [-a', a - a', a - a', \ldots, a' | \ldots], \quad a' \neq a, 0. \]
Their images under \(\mu\) are proportional to
\[ \langle 0, a, a, \ldots \rangle, \quad \text{or} \quad \langle -a, -a, 0, \ldots \rangle, \]
or
\[ \langle -a', a - a', a - a', \ldots \rangle, \]
which vanish by (3), (6), and (6), respectively.

Consider the case (C01\(\neq\)). The left side of (B) has the form
\[ [0, a_2, \ldots, a_k | \ldots], \quad \text{for } a_i \neq 0, i \geq 2, \text{ pairwise distinct, } b_j \neq 0. \]
Its image under \(\mu\) equals, by (\(\mu_1\)), to
\[ 2\langle 0, a_2, \ldots, a_k, \ldots \rangle. \]
The right side of (B) is the sum
\[ [0, a_2, \ldots, a_k | \ldots] + [-a_2, a_2, \ldots, a_k - a_2 | \ldots] + [-a_3, a_2 - a_3, a_3, \ldots | \ldots] + \cdots \]
where the first summand maps, by (\(\mu_1\)), to
\[ 2\langle 0, a_2, \ldots, a_k, \ldots \rangle \]
and all the other terms map to 0, by relation (6). This proves (C01\(\neq\)).

We are left with the case (C00), i.e., all elements of the sequences \(\bar{a}\) and \(\bar{b}\) are nonzero. We have two cases:

(C00\(\neq\)) all terms in \(\bar{a}\) are pairwise distinct,

(C00\(=\)) at least two terms in \(\bar{a}\) are equal.

In case (C00\(\neq\)), the left and the right side of (B) do not contain symbols with zeroes, hence we use (\(\mu_0\)) and the relation (B) is mapped precisely to the corresponding relation (M).

The case (C00\(=\)) splits into three subcases:

(C00\(=\) 2) \(\bar{a}\) has only one pair of equal terms, i.e.,
\[ \bar{a} = a, a, a_3, \ldots, a_k, \]
where \(a_3, \ldots, a_k\) are pairwise distinct and different from \(a\),
(C00= 2, 2) \( \bar{a} \) has the form

\[ \bar{a} = a, a, a', a', a_5, \ldots, a_k, \]

where \( a \neq a' \) and \( a_5, \ldots, a_k \) are pairwise distinct and different from \( a, a' \).

(C00= 3) \( \bar{a} \) has the form

\[ \bar{a} = a, a, a, \ldots \]

We start with (C00= 3). The left side is mapped to 0, by relation (5). The right side has terms of the form

\[ [a, 0, 0, \ldots | \ldots] \quad \text{or} \quad [a - a', a - a', a - a', a'_5, \ldots | \ldots], \quad a \neq a'. \]

They are mapped to terms proportional to

\[ \langle a, 0, 0, \ldots \rangle \quad \text{or} \quad \langle a - a', a - a', a - a', \ldots \rangle, \]

vanishing by (1) or (5), respectively.

We consider (C00= 2, 2). The left side is mapped to

\[ \langle a, a', a', \ldots \rangle \]

which vanishes by relation (4). The right side has terms of three shapes

\[ [a, 0, a' - a, a' - a, \ldots | \ldots] \quad \text{or} \quad [a - a', a - a', a'_3, 0, \ldots | \ldots], \quad a \neq a'. \]

or

\[ [a - a'', a - a'', a' - a'', a' - a'', a''_5, \ldots | \ldots], \quad a, a', a'' \text{ pairwise distinct.} \]

Their images are proportional to

\[ \langle a, 0, a' - a, a' - a, \ldots \rangle \quad \text{or} \quad \langle a - a', a - a', a'_3, 0, \ldots \rangle, \quad a \neq a'. \]

or

\[ \langle a - a'', a - a'', a' - a'', a' - a'', a''_3, \ldots \rangle, \quad a, a', a'' \text{ pairwise distinct,} \]

which vanish by (3), (3), and (4), respectively.

In the last case (C00= 2), relation (B) has the form

\[ [a, a, a_3, \ldots, a_k | \ldots] = [a, 0, a_3 - a, \ldots, a_k - a | \ldots] + \]

\[ + [a - a_3, a - a_3, a_3, \ldots, a_k - a_3 | \ldots] + [a - a_4, a - a_4, a_3 - a_4, a_4, \ldots | \ldots] + \cdots \]

The left side maps to

\[ \langle a, a, a_3, \ldots \rangle \]

and the right side to

\[ 2\langle a, 0, a_3 - a, \ldots, a_k - a | \ldots \rangle + \langle a - a_3, a - a_3, a_3, \ldots, a_k - a_3 | \ldots \rangle + \cdots. \]

Here the first summand is obtained by (\( \mu_1 \)) and the other summands by (\( \mu_0 \)). We see that, modulo relation (S), the image of the right hand
side of (B) coincides with the right hand side of the relation (M) in $\mathcal{M}_n(G)$.

This concludes the proof of Theorem 4.

**Proposition 7.** The homomorphism
\begin{equation}
\mu : B_2(G) \rightarrow M_2(G)
\end{equation}
is injective, with cokernel isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{\phi(N)}$, if $G \simeq \mathbb{Z}/N\mathbb{Z}$ is a cyclic group, and is an isomorphism otherwise.

**Proof.** One can write the generators and relations for $B_2(G)$ and $M_2(G)$ as follows:

- **Generators:**
  - ("non-degenerate") symbols $[a_1, a_2]$ (resp., $\langle a_1, a_2 \rangle$), where $a_1, a_2 \in A \setminus 0$ are such that $Za_1 + Za_2 = A$, and
  - ("degenerate") symbols $[a, 0]$ (resp., $\langle a, 0 \rangle$), where $a \in A \setminus 0$ is such that $Za = A$,

- **Relations:**
  1. $[a_1, a_2] = [a_2, a_1]$ (resp. $\langle a_1, a_2 \rangle = \langle a_2, a_1 \rangle$) for $a_1, a_2 \in A \setminus 0$,
  2. $[a_1, a_2] = [a_1, a_2 - a_1] + [a_1 - a_2, a_2]$ (and correspondingly $\langle a_1, a_2 \rangle = \langle a_1, a_2 - a_1 \rangle + \langle a_1 - a_2, a_2 \rangle$) for $a_1, a_2 \in A \setminus 0$ and $a_1 \neq a_2$,
  3. $[a, a] = [a, 0]$ (resp. $\langle a, a \rangle = 2\langle a, 0 \rangle$) for $a \neq 0$.

We see that the first two relations are identical and deal only with non-degenerate symbols $[a_1, a_2]$ (resp., $\langle a_1, a_2 \rangle$), when both $a_1, a_2$ are nonzero. In the case $B_2(G)$, relation (3) just identifies the degenerate symbol $[a, 0]$ via the nondegenerate symbol $[a, a]$, whereas in the case of $M_2(G)$ it adds one half of the nondegenerate symbol $\langle a, a \rangle$. Obviously, if we add to any abelian group an extra generator which is one half of any given element of this group, then the new group contains the initial one, and the quotient is $\mathbb{Z}/2\mathbb{Z}$. The statement of the Proposition immediately follows from these considerations, as the Euler function $\phi(N)$ is the number of degenerate elements $[a, 0]$ in the case $G \simeq A \simeq \mathbb{Z}/N\mathbb{Z}$. \qed

**Conjecture 8.** For $n \geq 3$ the homomorphism
\begin{equation}
\mu : B_n(G) \rightarrow M_n(G)
\end{equation}
is an isomorphism, modulo torsion.

This statement reduces to the following: For any integer $N \geq 2$,
\begin{equation}
[0, 0, 1] \in B_3(\mathbb{Z}/N\mathbb{Z})
\end{equation}
is a torsion element. Indeed, if this were the case, then any symbol \([0, 0, \ldots]\) would vanish modulo torsion, and then one could repeat all the steps in the proof of Theorem 4 and construct an inverse morphism from \(\mathcal{M}_2(G) \otimes \mathbb{Q}\) to \(\mathcal{B}_2(G) \otimes \mathbb{Q}\).

Computer experiments for \(N \leq 23\) support the following:

**Conjecture 9.** For \(N \geq 2\), the element
\[
[0, 0, 1] \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})
\]
has order 1, i.e., \([0, 0, 1] = 0 \in \mathcal{B}_3(\mathbb{Z}/N\mathbb{Z})\), if \(N\) is composite or \(N = 2, 3, 5\), and order exactly equal to
\[
\frac{p^2 - 1}{24}, \quad \text{if } N = p \geq 7 \text{ is a prime.}
\]

**Part 2**

5. **Refined birational invariants**

There is a refinement of \(\mathcal{B}_n(G)\), connecting it to the Burnside group of varieties considered in [KT17]. Let \(K\) be an algebraically closed field of characteristic zero. Let
\[
\text{Bir}_{n-1,m}(K), \quad 0 \leq m \leq n - 1,
\]
be the set of equivalence classes of \((n-1)\)-dimensional irreducible varieties over \(K\), modulo \(K\)-birational equivalence, which are \(K\)-birational to products \(W \times \mathbb{A}^m\), and not to \(W' \times \mathbb{A}^{m+1}\), for any \(W'\). Let
\[
\mathcal{B}_n(G, K) := \bigoplus_{m=0}^{n-1} \bigoplus_{[\gamma] \in \text{Bir}_{n-1,m}(K)} \mathcal{B}_m+1(G),
\]
with
\[
\mathcal{B}_1(G) = \begin{cases} 
\bigoplus_{a \in (\mathbb{Z}/N\mathbb{Z})^*} \mathbb{Z} & \text{if } G = \mathbb{Z}/N\mathbb{Z}, \ N \geq 2, \\
0 & \text{if } G \text{ is not cyclic.}
\end{cases}
\]
Let \(X\) be an irreducible \(K\)-variety with a generically free action of \(G\). As in Section 1, we may assume that \(G\) acts regularly; let
\[
X^G = \bigsqcup_{\alpha} F_{\alpha}
\]
be the decomposition of the fixed point locus into irreducible, disjoint, components. The spectrum for the \(G\)-action in the tangent space to \(X\) at any point \(x_\alpha \in F_\alpha\) is given by
\[
a_1, \ldots, a_{n-\dim(F_\alpha)}, 0, \ldots, a_i \neq 0.
\]
Define
\[ \beta_K(X) \in B_n(G, K) \]
by taking into account the birational types of fixed loci under \( G \), as follows: write
\[ Y_\alpha := F_\alpha \times \mathbb{A}^{n-1-\dim(F_\alpha)} \]
and let \( m_\alpha \in \mathbb{Z}_{>0} \) be the maximal integer such that
\[ Y_\alpha \sim Z_\alpha \times \mathbb{A}^{m_\alpha}, \]
clearly,
\[ m_\alpha \geq n - 1 - \dim(F_\alpha). \]
Then
\[ \beta_K(X) = \sum_\alpha \beta_\alpha(X), \]
where
\[ \beta_\alpha(X) = [a_1, \ldots, a_{n-\dim(F_\alpha)}, 0, \ldots] \in \text{copy of } B_{m_\alpha+1}(G), \]
labeled by the birational type of \( Y_\alpha \).

The invariance under blowups follows from the fact that all \((n-1)\)-dimensional birational types arising as labels in each particular subcase of the proof of Theorem 3 coincide with each other.

### 6. Hecke operators: variants

Let \( G \) be a finite abelian group and \( A \) its group of characters. Here we introduce variations of previous constructions: instead of
\[ \chi \in \mathbf{L} \otimes A = \text{Hom}(\mathbf{L}^\vee, A) \]
we can consider
\[ \chi^* \in \text{Hom}(\mathbf{L}, A), \]
again assuming that \( \chi^* \) is surjective. In a similar fashion, we can introduce the group \( \mathcal{M}_n^*(G) \), which we call the co-vector version of (the vector version) \( \mathcal{M}_n(G) \). This group is generated by symbols,
\[ \langle a_1, \ldots, a_n \rangle^*, \]
subject to relations
\[ (S^*) \text{ for all } \sigma \in \mathfrak{S}_n \text{ and all } a_1, \ldots, a_n \in A \text{ we have} \]
\[ \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)} \rangle^* = \langle a_1, \ldots, a_n \rangle^*, \]
(M*) for all $2 \leq k \leq n$, all $a_1, \ldots, a_k \in A$ and all $b_1, \ldots, b_{n-k} \in A$ such that

$$\sum_i \mathbb{Z}a_i + \sum_j \mathbb{Z}b_j = A$$

we have

$$\langle a_1, \ldots, a_k, b_1, \ldots b_{n-k} \rangle^* = \sum_{1 \leq i \leq k} \langle a_1, \ldots, a_i, b_1, \ldots b_{n-k} \rangle^*$$

It is not hard to show that the $\mathbb{Q}$-ranks of $\mathcal{M}_n(G)$ and $\mathcal{M}_n^*(G)$ are the same. Indeed, by Möbius-type inversion formula, one can reduce the question to the extended versions of groups $\mathcal{M}_n(G)$ and $\mathcal{M}_n^*(G)$ omitting the condition that the map $\chi : L^\vee \to A$, resp. $\chi^* : L \to A$, is surjective. Then the finite Fourier transform (after a choice of an identification $G \simeq A$) identifies two complex vector spaces consisting of homomorphisms from two extended groups to $\mathbb{C}$.

To define Hecke operators $T^*_\ell, r$, we consider sublattices $L' \subset L$, of index $\ell^r$, such that the quotient is isomorphic to $(\mathbb{Z}/\ell\mathbb{Z})^r$. In particular, $T^*_2 = T^*_{2,1}$ on $\mathcal{M}_2^*(G)$ is given by

$$T^*_2([a_1, a_2]^*) = [2a_1, a_2]^* + [2a_1, a_1 + a_2]^* + [a_1 + a_2, 2a_2]^* + [a_1, 2a_2]^*$$

and $T^*_3$ on $\mathcal{M}_3(G)$ by

$$T^*_3([a_1, a_2, a_3]^*) = [2a_1, a_2, a_3]^* + [a_1, 2a_2, a_3]^* + [a_1, a_2, 2a_3]^* + [a_1, a_1 + a_2, a_3]^* + [a_1 + a_2, 2a_2, a_3]^* + [a_1, 2a_2, a_2 + a_3]^* + [a_1 + a_2, a_2 + a_3, a_3]^* + [a_1 + a_2 + a_3, a_1 + a_3]^* + [a_1 + a_2, a_2 + a_3]^* + [a_1 + a_2 + a_3, a_1 + a_3]^*.$$

Another variant concerns coefficients. It works both for the vector and co-vector versions. For simplicity, we consider symbols with coefficients in $\mathbb{Q}$. Consider an irreducible algebraic representation

$$\rho_\lambda : \text{GL}_n(\mathbb{Q}) \to \text{Aut}(V_\lambda),$$

with highest weight

$$\lambda = (\lambda_1 \leq \ldots \leq \lambda_n), \quad \lambda_i \in \mathbb{Z}.$$
The representation $\rho_\lambda$ defines a functor from the groupoid of $n$-dimensional $\mathbb{Q}$-vector spaces to the category $\text{Vect}_\mathbb{Q}$ of all $\mathbb{Q}$-vector spaces, which we denote by the same letter. In particular, for any lattice $L$ of rank $n$ we can speak of

$$\rho_\lambda(L \otimes \mathbb{Q}) \in \text{Vect}_\mathbb{Q}.$$  

For example, if $\rho_\lambda$ is the $m$-th symmetric power $\text{Sym}^m(V)$ of the standard representation, i.e., $\lambda = (0, \ldots, 0, m)$, then

$$\rho_\lambda(L \otimes \mathbb{Q}) = \text{Sym}^m(L \otimes \mathbb{Q}).$$

Consider the $\mathbb{Q}$-vector space

$$\mathcal{M}_n(G, \rho_\lambda)$$

generated by symbols

$$\psi(L, \chi, \Lambda, v),$$

on isomorphism classes of quadrupels, where $L, \chi, \Lambda$ are as in Section 3 and

$$v \in \rho_\lambda(L \otimes \mathbb{Q}),$$

subject to relations

- $\psi(L, \chi, \Lambda, v_1 + v_2) = \psi(L, \chi, \Lambda, v_1) + \psi(L, \chi, \Lambda, v_2),$

- $\psi(L, \chi, \Lambda, v) = \sum_{i=1}^k \psi(L, \chi, \Lambda_i, v)$, for any decomposition

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k.$$  

Here, one can assume that subcones $\Lambda_i$ are basic simplicial and that the decomposition is standard, as in Section 3 or simply that $\Lambda_i$ are finitely-generated rational subcones of full dimension, with disjoint interiors. The action of Hecke operators on $\mathcal{M}_n(G, \rho_\lambda)$ is defined as in (3.3).

The co-vector version of this construction is straightforward.

**Remark 10.** We expect that for $n = 2$, $G = \mathbb{Z}/N\mathbb{Z}$, and $\rho_\lambda$ given by the $m$-th symmetric power, the $\mathbb{Q}$-vector spaces $\mathcal{M}_n(G, \rho_\lambda)$, endowed with the action of Hecke operators $T_{\ell, r}$, are related to modular forms of weight $(m + 2)$, for the congruence subgroup $\Gamma_1(N)$.
7. Algebraic versions of automorphic forms

A further generalization of results in Section 6 takes place in the following context. Let $G$ be a connected reductive group over $\mathbb{Q}$. There is a notion of admissible Harish-Chandra modules $E$ for $G(\mathbb{R})$: these are $\mathbb{C}$-vector spaces of countable dimension, endowed with an action of the maximal compact subgroup $K \subset G(\mathbb{R})$ and a compatible action of the complexified Lie algebra $g_\mathbb{C} = \mathrm{Lie}(G) \otimes \mathbb{C}$. The action of $K$ decomposes $E$ as a countable sum of finite-dimensional representations of $K$, each appearing with finite multiplicity. We assume that the center $z \subset U(g)$ acts by scalars, called the central character of $E$. The group $G(\mathbb{R})$ acts on the Schwartz completion of $S(E)$. Let $S(E)'$ be the continuous dual space, it is a subspace of the algebraic dual space $E'$. The congruence subgroups of $G(\mathbb{Q})$ have finite-dimensional invariants in $S(E)'$. One can view the theory of automorphic forms as the study of these finite-dimensional spaces of invariants, together with the action of a Hecke algebra. Note that in the last step we consider $S(E)'$ only as a $G(\mathbb{Q})$-module, and not as a $G(\mathbb{R})$-module.

Almost all automorphic forms are not related to motives or Galois representations; the part relevant for number theory (called algebraic automorphic forms) is specified by a certain integrality constraint on the central character.

Returning to considerations above, we see that we can imitate the theory of automorphic forms, with representations of $G(\mathbb{Q})$ in $S(E)'$, by a different class of representations of $G(\mathbb{Q})$, defined over $\mathbb{Q}$. Assume that $G = \mathrm{GL}_n$, over $\mathbb{Q}$. Let

\begin{equation}
F_n = \langle \mathcal{X}_\Lambda \rangle \otimes_\mathbb{Q},
\end{equation}

be the $\mathbb{Q}$-vector space generated by characteristic functions $\mathcal{X}_\Lambda$ of convex finitely generated rational polyhedral cones $\Lambda \subset \mathbb{R}^n$, modulo functions with support of dimension $\leq (n - 1)$. Note that

$$F_n \subset L_\infty(\mathbb{R}^n),$$

the space of bounded measurable functions. Clearly, $G(\mathbb{Q}) = \mathrm{GL}_n(\mathbb{Q})$ acts on $F_n$. Let

$$\rho = \rho_\lambda : \mathrm{GL}_n(\mathbb{Q}) \to \mathrm{Aut}(V)$$

be a finite-dimensional irreducible representation as above. Let

$$\Gamma \subset \mathrm{GL}_n(\mathbb{Q})$$
be an arithmetic subgroup. The spaces of invariants, respectively, coinvariants

\[(7.2) \quad H^0(\Gamma, \mathcal{F}_n^\vee \otimes V^\vee) = (\mathcal{F}_n^\vee \otimes V^\vee)^\Gamma, \quad H_0(\Gamma, \mathcal{F}_n \otimes V) = (\mathcal{F}_n \otimes V)_\Gamma, \]

are finite-dimensional, since the module of characteristic functions is finitely-generated over the group ring of the arithmetic subgroup $\Gamma$.

For example, if $\rho$ is the trivial representation, and $\Gamma \subset \text{GL}_n(\mathbb{Z}) = \text{Aut}(L)$ is the stabilizer of the vector $\chi = (1, 0, 0, \ldots) \in L \otimes \mathbb{Z}/N\mathbb{Z}$

then the group of coinvariants is (up to torsion) our group $\mathcal{M}_n^*(\mathbb{Z}/N\mathbb{Z})$.

Similarly, by taking the stabilizer of the coordinate co-vector modulo $N$, we obtain the co-vector version $\mathcal{M}_n^*(\mathbb{Z}/N\mathbb{Z})$.

The key point is that $\mathcal{F}_n$ is a $\text{GL}_n(\mathbb{Q})$-module which is finitely generated as $\text{GL}_n(\mathbb{Z})$-module; moreover,

\[(7.3) \quad \text{Res}_{\text{GL}_n(\mathbb{Q})}^{\text{GL}_n(\mathbb{Z})}(\mathcal{F}_n) \in \text{Perf}(\mathbb{Q}[\text{GL}_n(\mathbb{Z})] - \text{mod}), \]

i.e., $\mathcal{F}_n$, considered as a $\text{GL}_n(\mathbb{Z})$-module, admits a finite-length resolution by finitely-generated projective modules over the group ring of $\text{GL}_n(\mathbb{Z})$ (see Proposition 12).

**Question 11.** Are there other interesting $\text{GL}_n(\mathbb{Q})$-modules which are finitely-generated as $\text{GL}_n(\mathbb{Z})$-modules, or more strongly, belong to $\text{Perf}(\mathbb{Q}[\text{GL}_n(\mathbb{Z})] - \text{mod})$?

An even more general question would be to find a bounded from above complex of representations of $G(\mathbb{Q})$ which, after restriction to $G(\mathbb{Z})$, is quasi-isomorphic to a complex of finitely-generated projective modules over the group ring.

Both $\mathbb{Q}$-vector spaces in (7.2) carry actions of Hecke operators, which have algebraic eigenvalues in these spaces. By (7.3),

\[\dim(H_i(\Gamma, \mathcal{F}_n \otimes V)) < \infty, \quad \text{for all } i \geq 0,\]

and the spaces, for $i \geq 1$, also carry actions of Hecke operators with algebraic eigenvalues.

We will see below that our representation $\mathcal{F}_n$ falls into a well-studied subclass of cohomological automorphic forms, i.e., those realized in cohomology of arithmetic groups with coefficients in finite-dimensional representations $\rho$.

Recall the definition of Steinberg modules: Let $V/\mathbb{Q}$ be a $\mathbb{Q}$-vector space of dimension $n \geq 0$, and $\mathcal{T}_n$ the simplicial complex of flags of
Q-vector subspaces of \( V \), i.e., the geometric realization of the poset of nontrivial subspaces in \( V \). Put

\[
\text{St}(V) := \begin{cases} 
H_{n-2}(\mathcal{T}_n, \mathbb{Z}) & n \geq 3 \\
\mathbb{Z}\text{-combinations of lines in } V \text{ with total weight } 0 & n = 2 \\
\mathbb{Z} & n = 0, 1.
\end{cases}
\]

This is a representation of \( \text{Aut}(V) \), which we denote by \( \text{St}_n \) for \( V = \mathbb{Q}^n \). One of the roles of the Steinberg module is as a dualizing module, in the sense that

\[
H^i(\text{SL}_n(\mathbb{Z}), \text{St}_n \otimes M) = H^{n(n-1)/2-i}(\text{SL}_n(\mathbb{Z}), M),
\]

for any representation \( M \) of \( \text{SL}_n(\mathbb{Z}) \) with coefficients in \( \mathbb{Q} \).

The module

\[
\mathcal{F}(V) = \mathcal{F}_n(V)
\]

defined in (7.1) has a filtration by submodules

\[
0 \subset \mathcal{F}^{\leq 0}(V) \subset \mathcal{F}^{\leq 1}(V) \subset \cdots \subset \mathcal{F}^{\leq n}(V) = \mathcal{F}(V),
\]

where \( \mathcal{F}^{\leq i}(V) \) are generated by functions pulled back from quotient spaces of dimension \( i \). In particular,

\[
\mathcal{F}^{\leq 0}(V) = \mathbb{Z} = \{ \text{constant } \mathbb{Z}\text{-valued functions on } V \}.
\]

The following fact is presumably well-known:

**Proposition 12.**

\[
\text{gr}^i(\mathcal{F}(V)) = \oplus_{V \to V', \dim(V') = i} \text{St}(V') \otimes \text{or}(V'),
\]

where \( \text{or}(V') \) is the 1-dimensional \( \mathbb{Z} \)-module of orientations of \( V' \), i.e., \( \text{GL}(V') \) acts via the sign of the determinant.

**Proof.** Let us first prove that

\[
\text{gr}^n(\mathcal{F}(V)) = \mathcal{F}(V)/\mathcal{F}^{\leq n-1}(V)
\]

is isomorphic to

\[
\text{St}(V) \otimes \text{or}(V).
\]

We apply the Fourier transform to elements of \( \mathcal{F}(V) \) viewed as distributions with moderate growth on \( V \otimes \mathbb{R} \simeq \mathbb{R}^n \).

For example, the Fourier transform of the characteristic function of the standard coordinate octant \( (\mathbb{R}_{\geq 0})^n \) is equal to the distribution

\[
\prod_{i=1}^n (\sqrt{-1} v.p.(1/x_i) + \pi \delta(x_i)) \prod_{i=1}^n |dx_i|
\]
with values in volume forms, where \( v.p. (1/x) \) is the unique odd distribution of homogeneity degree \(-1\) on \( \mathbb{R}^1 \) equal to \( 1/x \) on \( \mathbb{R} \setminus 0 \).

The image of \( F^{\leq n-1}(V) \) is characterized by the property that the support of the distribution is contained in a finite union of hyperplanes. Therefore, the quotient group \( F(V)/F^{\leq n-1}(V) \) is identified with the abelian group generated by volume elements on the dual space \((V \otimes \mathbb{R})^\vee\), of the form

\[
(\sqrt{-1})^n |dx_1 \wedge \cdots \wedge dx_n|/(x_1 \cdots x_n),
\]

where \( x_1, \ldots, x_n \) are coordinates in \((V \otimes \mathbb{R})^\vee\) in a rational basis. Choosing an orientation of \( V \) (or, equivalently, of \( V^\vee \)) and dividing by \((\sqrt{-1})^n\), we identify the latter space with top-degree meromorphic differential forms on the vector space \( V^\vee \) considered as an algebraic variety \( \mathbb{A}^n_\mathbb{Q} \) over \( \mathbb{Q} \) spanned by forms of type \( \wedge^n_{i=1} (dx_i/x_i) \) for coordinates in a rational basis. This is an alternative description of the Steinberg module. The case of deeper terms of the dimension filtration is similar.

This implies that the computation of cohomology with coefficients in \( F(V) \), tensored with finite-dimensional modules, and, in particular, of coinvariants, would reduce to the computation of cohomology for St-modules and their pullbacks from parabolic subgroups. There is extensive literature on the cohomology of St-modules (see, e.g., [APS18] and the references therein), but these computations do not capture the potentially interesting extension data in \( F(V) \).

In the application to our initial problem, we found that the groups \( B_n(\mathbb{Z}/N\mathbb{Z}) \), up to torsion, are quite big for \( n = 2 \) (see the next section), have smaller but similar size for \( n = 3 \), and are quite small but not zero, even for \( n \geq 5 \), e.g., \( \dim(B_5(\mathbb{Z}/81\mathbb{Z}) \otimes \mathbb{Q}) = 1 \).

The cases when \( \dim(B_4(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) > 0 \) do not correspond (to our knowledge) to anything remarkable on the motivic side, (although we have to admit that we did not yet calculate the Hecke spectrum in \( n = 4 \) cases in order to nail down the corresponding motives).

In the following section, we will see that, for \( n = 2 \), the main actors are modular forms of weight 2, and sums of two Tate motives twisted by characters.

Among other variants in the definition of \( F \) are:

- using \( \mathbb{Z} \) or finite fields as coefficients, instead of \( \mathbb{Q} \)-coefficients,
- one can study torsion effects.
• one can omit the condition of factoring by characteristic functions with support in dimension \( \leq (n - 1) \).
• when the representation \( \rho \) is on the space of degree-\( d \) polynomials, one can consider *polynomial splines*, with respect to some complete rational fan \( \Sigma \) on \( \mathbb{R}^n \), i.e., functions on \( \mathbb{R}^n \) which are piecewise polynomial on the cones of \( \Sigma \), with \( \mathbb{Q} \)-coefficients, and with continuous derivatives up to some fixed \( d' < d \).

The last example is especially interesting as such representations are realized as submodules of extensions of Steinberg modules, and coinvariants with values in such modules could capture higher homology groups of Steinberg modules, thus making them computationally much more accessible.

We finish this section with a challenge concerning the possibility, in the framework of Question 11, to go beyond the realm of cohomological (but still algebraic) automorphic forms.

**Question 13.** Can one find a representation of \( \text{SL}_2(\mathbb{Q}) \) whose restriction to \( \text{SL}_2(\mathbb{Z}) \) is finitely-generated, and whose Hecke spectrum captures modular forms of weight 1 and Maass forms with Laplace eigenvalue \( 1/4 \)?

Morally, such modules should be realized in a class of odd/even functions on \( \mathbb{R}^2 \) of homogeneity degree \(-1\).

8. **Case \( n = 2 \): Modular Symbols**

We recall the definition of modular symbols of weight 2 for

\[
\Gamma_1(N) := \left\{ \gamma \in \text{SL}_2(\mathbb{Z}) : \gamma = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}, \quad N \in \mathbb{Z}_{\geq 2}.
\]

Let \( \mathbb{M}_2(\Gamma_1(N)) \) be the \( \mathbb{Q} \)-vector space generated by pairs \( (c, d) \) with

\[
c, d \in \mathbb{Z}/N, \quad \gcd(c, d, N) = 1,
\]

and subject to relations

(1) \( (c, d) = -(d, -c) \) (and hence \( = (-c, -d) = -(-d, c) \)),

(2) \( (c, d) + (d, -c - d) + (-c - d, c) = 0 \).

It is known that \( \mathbb{M}_2(\Gamma_1(N)) \) is naturally identified with Borel-Moore homology group \( H_1^{BM}(X_1(N), \mathbb{Q}) \) of the complex modular curve

\[
X_1(N) := \mathcal{H}/\Gamma_1(N),
\]
where $\mathcal{H}$ is the upper half-plane. The symbol $(c, d)$ corresponds to the image in $X_1(N)$ of the geodesic path from $a/c$ to $b/d$, where
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)
\]
is any element with $c, d = c, d \mod N$.

Using (1) we can rewrite (2) as
\[
(2') (d, c) = (d, c - d) + (d - c, c).
\]
Indeed, substituting $c \mapsto -c$ into (2), we obtain
\[
0 (2) = (-c, d) + (d, c - d) + (c, -d) - (1)
\]
\[
= -(d, c) + (d, c - d) + (c - d, c) - (1)
\]
\[
= -(d, c) + (d, c - d) + (d, c - d) + (d, c - c) - (1)
\]

There is an involution on $M_2(\Gamma_1(N))$
\[
\iota : (c, d) \mapsto -(c, d) \overset{(1)}{=} (d, c).
\]
Written in the form $(c, d) \mapsto (d, c)$ it obviously preserves relations $(2')$ and cyclic anti-symmetry (1). It corresponds (up to sign) to the automorphism of the first homology group coming from the anti-holomorphic involution on $X_1(N)$ associated with the map $\tau \mapsto -\bar{\tau}$, $\tau \in \mathcal{H}$, on the universal cover. Let $M_2^+(\Gamma_1(N))$ be the $(+)$-eigenspace for the involution $\iota$. This subspace (or, better, quotient space) can be described in terms of generators and relations as
\[
\begin{align*}
(R1) & \quad (a_1, a_2)^+ = (a_2, a_1)^+ \\
(R2) & \quad (a_1, a_2)^+ = (a_1, a_2 - a_1)^+ + (a_1 - a_2, a_2)^+ \\
(R3) & \quad (a_1, a_2)^+ = -(a_2, -a_1)^+
\end{align*}
\]
Here (R3) is the same as (1), (R2) is the same as $(2')$, and (R1) is $\iota$-invariance. We see exactly two defining relations for $M_2(\mathbb{Z}/N\mathbb{Z})$, and an additional relation (R3). Hence, we have an epimorphism
\[
\begin{equation}
\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q} \twoheadrightarrow M_2^+(\Gamma_1(N)), \quad \langle a_1, a_2 \rangle \mapsto (a_1, a_2)^+.
\end{equation}
\]
Incidentally, relation (R2) can be replaced by the co-vector version
\[
(R2^*) (a_1, a_2)^+ = (a_1 + a_2, a_2)^+ + (a_1, a_1 + a_2)^+
\]
Indeed, substitute $a_1 \mapsto a_1, a_2 \mapsto a_1 + a_2$ into relation (R2) and use dihedral symmetry by (R1) and (R3).

The dimensions are given by
\[
\dim(M_2(\Gamma_1(N))) = 2g + C(N) - 1, \quad \dim(M_2^+(\Gamma_1(N))) = g + \frac{C(N) - C_2(N)}{2}.
\]
where

- \( g = g(N) \) is the genus of \( \overline{X_1(N)} \), which is the same as the dimension of the space of cusp forms of weight 2 for \( \Gamma_1(N) \),
- \( C(N) \) is the number of cusps (elements of \( \mathbb{P}^1(\mathbb{Q})/\Gamma_1(N) \)), and
- \( C_2(N) \) is the number of cusps fixed by the anti-holomorphic involution described above.

For \( N = 1, 2, 3, 4 \), \( C(N) = C_2(N) = 1, 2, 2, 3 \), respectively; and for \( N \geq 5 \), the cardinalities \( C(N), C_2(N) \) are given by

\[
C(N) = \frac{1}{2} \sum_{d|N} \phi(d) \phi(N/d),
\]

\[
C_2(N) = \begin{cases} \phi(N) + \phi(N/2) & \text{if } N \text{ is even}, \\ \phi(N) & \text{if } N \text{ is odd}. \end{cases}
\]

**Conjecture 14.** (V. Pestun) For any \( N \geq 5 \) we have

\[
\dim(\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = g(N) + C(N) - \frac{C_2(N)}{2}.
\]

Notice that the last formula can be rewritten as

\[
\dim(\mathcal{M}_2(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = g(N) + \frac{1}{2} \sum_{d|N, d \geq 3} \phi(d) \phi(N/d).
\]

The kernel of the map (8.1) seems to be spanned by elements of the form \( \langle a, -a \rangle \).

Presumably, one can deduce the above formula using the relation between the Steinberg module and module \( F_2 \) (see Proposition 12).

The formulas for dimensions simplify when \( N = p \geq 5 \) is a prime:

\[
g(p) = \frac{(p - 5)(p - 7)}{24}, \quad C(p) = C_2(p) = p - 1,
\]

\[
\dim(\mathcal{M}_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = \frac{p^2 + 23}{24} = g(p) + \frac{p - 1}{2}, \quad \dim(\mathcal{M}_2^+(\Gamma_1(p))) = g(p).
\]

9. **Experiments**

Here we present results of numerical experiments, performed using a fast linear algebra solver \[gro17\]. We computed dimensions of

\( \mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}), \mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \)

over \( \mathbb{Q} \) and various finite fields. The size of the (very sparse) matrices grows as \( \sim N^n \). For example, for \( n = 5 \) and \( N = 81 \), the part of constraints corresponding to \( k = 2 \) in (B) or (M), gives \( \sim 3 \cdot 10^8 \).
equations on \( \sim 3 \cdot 10^7 \) variables, with \( \sim 10^9 \) non-zero coefficients. This overdetermined system has a unique (up to scalar) nontrivial solution in \( \mathbb{Q} \). The calculation takes about 4 hours.

Numerically, we found:

- For each \( n \), the first \( N \) such that \( \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) > 0 \) is:

\[
\begin{array}{c|c|c|c|c}
 n & 2 & 3 & 4 & 5 \\
 N & 3 & 9 & 27 & 81
\end{array}
\]

We guess that, in general, the minimal such \( N \) is \( 3^{n-1} \).
- For each \( n \), the first \( N \) such that \( \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2) > 0 \) is:

\[
\begin{array}{c|c|c|c|c}
 n & 2 & 3 & 4 & 5 \\
 N & 2 & 4 & 8 & 16
\end{array}
\]

We guess that the minimal such \( N \) is \( 2^{n-1} \).
- For \( p \) a prime,

\[
\dim(B_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = \frac{p^2 - 1}{24} + 1 = \frac{p^2 + 23}{24},
\]

while the difference

\[
\Delta_{2,\ell}(\mathbb{Z}/p\mathbb{Z}) := \dim(B_2(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_\ell) - \frac{p^2 + 23}{24}
\]

varies significantly; there are frequent jumps when \( \ell \mid (p \pm 1) \), e.g.,

\[
\Delta_{2,31}(\mathbb{Z}/61\mathbb{Z}) = 1.
\]
- For \( p \) a prime,

\[
\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) := \dim(B_3(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) - \frac{(p-5)(p-7)}{24} = 0
\]

for all primes up to 41, but

\[
\Delta_{3,\mathbb{Q}}(\mathbb{Z}/p\mathbb{Z}) = 1, \quad \text{for } p = 43, 59, \ldots.
\]
- The difference

\[
\Delta_{3,\ell}(\mathbb{Z}/p\mathbb{Z}) := \dim(B_3(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{F}_\ell) - \frac{(p-5)(p-7)}{24}
\]

also jumps for many \( \ell \mid (p \pm 1) \).
- For all primes \( p \) up to 41 we have \( \dim(B_4(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = 0 \), but

\[
\dim(B_4(\mathbb{Z}/p\mathbb{Z}) \otimes \mathbb{Q}) = 1, \quad \text{for } p = 43, 59, \ldots.
\]

On the next page we present a more systematic table of dimensions. The items in bold indicate the smallest \( N \) for which the rank is positive.
• \( \dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) \) for \( n = 2, 3 \):

| \( N \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|------|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| \( n=2 \) | 0 | 1 | 1 | 2 | 3 | 3 | 5 | 4 | 6 | 7 | 8 | 7 | 13 | 10 | 13 | 12 |
| \( n=3 \) | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 2 | 2 | 1 | 5 | 3 | 5 | 5 |

| \( N \) | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | ... | 180 | 181 |
|------|----|----|----|----|----|----|----|----|----|----|----|-----|-----|-----|
| \( n=2 \) | 16 | 17 | 23 | 16 | 23 | 23 | 30 | 22 | 34 | 31 | 36 | ... | 989 | 1366 |
| \( n=3 \) | 7 | 7 | 11 | 7 | 12 | 13 | 16 | 12 | 21 | 17 | 22 | ... | 1740 | 1276 |

• \( \dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{Q}) \) for \( n = 4 \):

| \( N \) | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | ... | 105 | 106 | 107 |
|------|----|----|----|----|----|----|----|----|----|----|-----|-----|-----|
| \( n=4 \) | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 3 | ... | 114 | 0 | 3 |

• \( \dim(\mathcal{B}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2) = \dim(\mathcal{M}_n(\mathbb{Z}/N\mathbb{Z}) \otimes \mathbb{F}_2) \) for \( n = 2, 3, 4, 5 \):

| \( N \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | ... | 32 |
|------|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| \( \mathcal{B}_2 \) | 0 | 1 | 1 | 2 | 3 | 4 | 4 | ... | 13 | ... | 44 |
| \( \mathcal{M}_2 \) | 1 | 2 | 3 | 5 | 5 | 8 | 8 | ... | 21 | ... | 60 |
| \( \mathcal{B}_3 \) | 0 | 0 | 0 | 0 | 0 | 1 | 1 | ... | 8 | ... | 43 |
| \( \mathcal{M}_3 \) | 0 | 0 | 1 | 1 | 3 | 2 | 5 | ... | 21 | ... | 87 |
| \( \mathcal{B}_4 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... | 1 | ... | 12 |
| \( \mathcal{M}_4 \) | 0 | 0 | 0 | 0 | 0 | 1 | ... | 9 | ... | 55 |
| \( \mathcal{B}_5 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... | 0 | ... | 1 |
| \( \mathcal{M}_5 \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | ... | 1 | ... | 13 |
Equations (B) in Section 1 are labeled by pairs of positive integers \( n, k \), where \( n \) is the dimension and \( 2 \leq k \leq n \). Computer experiments show a remarkable property of our equations: for given \( n \) and \( k \), the highly overdetermined subsystem of linear equations (B) or (M) (and assuming implicitly (S), the symmetry property) has a very large space of solutions, usually much larger than the whole system for given \( n \), which is the conjunction of subsystems for \( k = 2, \ldots, n \). We have no explanation for this striking fact. There are no obvious actions of Hecke operators on the solution spaces \( n, k \) individually, and it is very surprising that the highly overdetermined systems admit any nontrivial solution at all.

It seems that equations for \( k = 2 \) imply all other equations for \( 3 \leq k \leq n \), hence the \( \mathbb{Q} \)-rank is the same as for the total system (listed on the previous page).

- \( \mathbb{Q} \)-ranks of partial systems \( B_{n,k} \) and \( M_{n,k} \) for \( k \geq 3 \), and for some primes and composite numbers \( N \):

| \( N \) | 2 | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 9 | 12 | 27 | 36 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( B_{3,3} \) | 1 | 2 | 4 | 6 | 12 | 15 | 22 | 27 | 35 | 11 | 36 | 87 | 468 |
| \( M_{3,3} \) | 0 | 1 | 3 | 3 | 7 | 10 | 15 | 18 | 24 | 9 | 40 | 78 | 480 |
| \( B_{4,3} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 63 |
| \( M_{4,3} \) | 0 | 0 | 0 | 0 | 1 | 2 | 5 | 7 | 12 | 1 | 5 | 24 | 121 |
| \( B_{4,4} \) | 0 | 3 | 6 | 9 | 17 | 20 | 29 | 35 | 45 | 42 | 101 | 620 | 2515 |
| \( M_{4,4} \) | 0 | 3 | 2 | 3 | 7 | 8 | 13 | 17 | 23 | 45 | 123 | 649 | 2716 |
| \( B_{5,3} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| \( M_{5,3} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 7 |
| \( B_{5,4} \) | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 4 | 55 | 267 |
| \( M_{5,4} \) | 0 | 0 | 0 | 0 | 1 | 2 | 5 | 7 | 12 | 5 | 12 | 122 | ? |
| \( B_{5,5} \) | 1 | 3 | 9 | 12 | 22 | 26 | 37 | 44 | 56 | 30 | 161 | 572 | ? |
| \( M_{5,5} \) | 0 | 1 | 3 | 3 | 7 | 8 | 13 | 17 | 23 | 17 | 212 | ? | ? |

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