POSITIVITY IN THE COHOMOLOGY OF FLAG BUNDLES
(AFTER GRAHAM)

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In [Gr], Graham proves that the structure constants of the equivariant cohomology ring of a flag variety are positive combinations of monomials in the roots:

**Theorem 1 ([Gr Cor. 4.1]).** Let $X = G/B$ be the flag variety for a complex semisimple group $G$ with maximal torus $T \subset B$, and let $\{\sigma_w \in H^*_T X \mid w \in W\}$ be the basis of ($B$-invariant) Schubert classes. Let $\{\alpha_i\}$ be the simple roots which are negative on $B$. Then in the expansion

$$\sigma_u \cdot \sigma_v = \sum_w c^w_{uv} \sigma_w,$$

the coefficients $c^w_{uv}$ are in $\mathbb{Z}_{\geq 0}[\alpha]$. Graham deduces this from a more general result about varieties with finitely many unipotent orbits, which is proved using induction and a calculation in the rank-one case.

The goal of this note is to give a short, geometric proof of Graham’s positivity theorem, based on a transversality argument. Here I only discuss type $A$, but other types work as well. (For a type-uniform version, a change of language is needed: one should replace vector bundles with corresponding principal $G$-bundles.)

Throughout, $Fl$ denotes the variety of (complete) flags in $\mathbb{C}^n$, and if $V \to X$ is a vector bundle, $Fl(V) \to X$ is the bundle of flags in $V$.

Recall that for $T' \cong (\mathbb{C}^*)^n$, we have $BT' = (\mathbb{P}^\infty)^\times n$ and $H^*_T Fl = H^*(ET' \times^{T'} Fl) = H^*Fl(E')$, where $E'$ is the sum of the $n$ tautological line bundles on $BT'$. The effective action on $Fl$ is by $T \cong (\mathbb{C}^*)^n/\mathbb{C}^*$, and the classifying space for this torus is $BT = (\mathbb{P}^\infty)^\times n - 1$. We will usually deal with the effective torus.

Let $P = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ ($n - 1$ factors), with $m \gg 0$, and write $H^*P = \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}]$. (We always assume that $m$ is large enough so that there are no relations in the relevant degrees.) Let $M_i = p_i^*(O(-1))$ be the tautological bundle on the $i$th factor, and let $\alpha_i = -c_1(M_i)$. Note that the class of any effective cycle in $H^*P$ is a positive polynomial in the $\alpha$’s.

Let

$$L_i = M_1 \otimes \cdots \otimes M_{i-1}$$

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for \(1 \leq i \leq n\) (so \(L_1 = \mathcal{O}\) is the trivial line bundle), and let \(E_i = L_1 \oplus \cdots \oplus L_i\). Thus we have a flag \(E_\bullet\) in \(E = E_n\). Let \(\tilde{E}_\bullet\) be the opposite flag, with \(\tilde{E}_i = L_n \oplus \cdots \oplus L_{n+1-i}\). In the flag bundle \(p : \mathcal{F}(E) \to \mathbb{P}\), with universal quotient flags \(O_\bullet\), we have Schubert loci \(\Omega_w = \Omega_w(\widetilde{E}_\bullet \to E_\bullet)\), defined by

\[
\Omega_w = \{x \in \mathcal{F}(E) \mid \text{rk}(E_p) - Q_q \leq \#(i \leq q \mid w(i) \leq p)\}.
\]

Opposite Schubert loci \(\Omega_w = \Omega_w(\widetilde{E}_\bullet \to E_\bullet)\) are defined similarly. We also have “Schubert cell bundles” \(\Omega_w^\ast\): these are affine bundles over \(\mathbb{P}\) which are open in the corresponding loci \(\Omega_w\), and are defined by replacing the inequality in (1) with an equality.

The classes \([\Omega_w]\) form a basis for \(H^*\mathcal{F}(E)\) over \(H^*\mathbb{P}\), as \(w\) ranges over \(S_n\). Writing

\[
[\Omega_u] \cdot [\Omega_v] = \sum_w c_{uv}^w[\Omega_w]
\]

with \(c_{uv}^w \in H^*\mathbb{P}\), our main result is the following:

**Proposition 2.** The polynomials \(c_{uv}^w\) are positive, that is, \(c_{uv}^w \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_{n-1}]\).

This implies Graham’s positivity theorem (in this context), since \(\mathbb{P}\) approximates \(BT\) for \(m\) sufficiently large, and \(\mathcal{F}(E)\) approximates \(ET \times T \mathcal{F}(E)\), with \([\Omega_w]\) corresponding to the equivariant class \(\sigma_w\). (See \([Fu2]\) §9.)

Proposition 2 is a consequence of a transversality statement:

**Proposition 3.** For any \(u, v, w \in S_n\), there is a translate \(\Omega'_v\) of \(\Omega_v\) by the action of a connected algebraic group such that \(\Omega'_v\) intersects \(\Omega_u\) and \(\tilde{\Omega}_{w_0w}\) properly and generically transversally.

To deduce Proposition 3, first note that the intersection \(\Omega_u \cap \tilde{\Omega}_{w_0w}\) is always proper and generically transverse. Thus Proposition 3 says that \(\Omega'_v \cap (\Omega_u \cap \tilde{\Omega}_{w_0w})\) is proper and generically transverse. By \([Fu1]\) Ex. (8.1.11)], this says that

\[
[\Omega_v] \cdot [\Omega_u] \cdot [\tilde{\Omega}_{w_0w}] = [\Omega'_v \cap \Omega_u \cap \tilde{\Omega}_{w_0w}].
\]

(Since \(\Omega'_v = g \cdot \Omega_v\) for some \(g\) in a connected algebraic group, \([\Omega'_v]\) = \([\Omega_v]\).)

Using relative Poincaré duality (see e.g. \([Fu2\) §A.6]), we have

\[
c_{uv}^w = p_\ast([\Omega_u] \cdot [\Omega_v] \cdot [\tilde{\Omega}_{w_0w}]) = p_\ast([\Omega_u \cap \Omega'_v \cap \tilde{\Omega}_{w_0w}]).
\]

This is an effective class in \(H^*\mathbb{P}\), so Proposition 2 follows.

**Proof of Proposition 3.** This is essentially an application of Kleiman’s theorem. The endomorphism bundle

\[
\text{End}(E) = \bigoplus_{i,j} L_i^{-1} \otimes L_j
\]

\[
= \left( \bigoplus_{i<j} M_i \otimes \cdots \otimes M_{j-1} \right) \oplus O^{\oplus n} \oplus \left( \bigoplus_{i>j} M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1} \right)
\]
has global sections in lower-triangular matrices, so the group $B$ of (invertible) lower-triangular matrices acts on $\text{Fl}(E)$, fixing the flag $\widetilde{E}_\bullet$ and stabilizing $\Omega_{\omega_0,w}$. (Note that the entries of a matrix in $B$ are global sections of the line bundles $M_j^{-1} \otimes \cdots \otimes M_{i-1}^{-1}$, i.e., multi-homogeneous polynomials. This is a connected group over $\mathbb{C}$, acting on a fiber $p^{-1}(x) \subset \text{Fl}(E)$ by first evaluating the sections at $x$.)

Now let $H = (GL_{m+1})^{\times(n-1)}$, and for $b \in B$, let $b_x$ be the evaluation at $x \in \mathbb{P}$ (so the action of $b$ on $p^{-1}(x)$ is by $b_x$). Consider the semidirect product $\Gamma = B \rtimes H$, given by $(h \cdot b \cdot h^{-1})_x = b_{h^{-1}.x}$. (This action of $H$ on $B$ is just the usual action of $H$ on global sections of the equivariant vector bundle $\text{End}(E)$.) As a semidirect product of connected groups, $\Gamma$ is a connected algebraic group. We claim that the locus $\widetilde{\Omega}_{\omega_0,w}$ is homogeneous for the action of $\Gamma$. Indeed, $B$ acts transitively on each fiber of $\Omega_{\omega_0,w}$, and the action of $H$ on $\text{Fl}(E)$ induces a transitive action on the set of fibers of $\Omega_{\omega_0,w}$. (The line bundles $L_i$ are equivariant for $H$, so $H$ preserves the flag $\widetilde{E}_\bullet$, and therefore acts on $\Omega_{\omega_0,w}$.)

Finally, note that $\Omega_{\omega_0,w}$ and $\omega_0 \cdot \Omega_{\omega_0,w}$ intersect transversally, as do $\omega_0 \cdot \Omega_{\omega_0,w}$ and $\Omega_{\omega_0,w}$. The proposition follows from Lemma 3 below, taking $U = \Omega_{\omega_0,w}$, $V = \omega_0 \cdot \Omega_{\omega_0,w}$, and $W = \Omega_{\omega_0,w}$, with their stratifications by Schubert loci. 

**Lemma 4.** Let $X$ be a nonsingular variety over a field of characteristic 0, with an action of a connected algebraic group $\Gamma$. Let $U,V,W \subset X$ be subvarieties with stratifications

\[
U_0 \subset \cdots \subset U_k = U,
V_0 \subset \cdots \subset V_m = V,
W_0 \subset \cdots \subset W_n = W,
\]

with each stratum $U_i \setminus U_{i-1}$ nonsingular. Assume also that $\Gamma$ acts on $W$, with each stratum $W_i \setminus W_{i-1}$ a disjoint union of homogeneous spaces.

If $U_i \setminus U_{i-1}$ meets $W_k \setminus W_{k-1}$ transversally for all $i,k$, and similarly for $V_j \setminus V_{j-1}$ and $W_k \setminus W_{k-1}$, then there is an element $g \in \Gamma$ such that $g \cdot V$ meets $U \cap W$ properly and generically transversally.

This can be deduced from results found in [Sp]; see also [Si] for a vast generalization. The proof of this version is quite short, so we give it here.

**Proof.** Applying Kleiman’s theorem (cf. [Ha] III.10.8) to the pairs $(U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1})$ and $(V_j \setminus V_{j-1} \cap W_k \setminus W_{k-1})$ inside the homogeneous space $W_k \setminus W_{k-1}$, we can choose $g \in \Gamma$ such that each intersection

\[
(U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1}) \cap g \cdot (V_j \setminus V_{j-1} \cap W_k \setminus W_{k-1})
= (U_i \setminus U_{i-1} \cap W_k \setminus W_{k-1}) \cap (g \cdot V_j \setminus g \cdot V_{j-1} \cap W_k \setminus W_{k-1})
\]

Alternatively, one could take $\Gamma$ to be the subgroup of $\text{Aut}(\text{Fl}(E))$ generated by the images of $B$ and $G$ via the homomorphisms corresponding to their respective actions.
is transverse, so the intersection $U \cap W \cap g \cdot V$ is proper and generically transverse.

\textbf{Remark 5.} All that is required in the proof of Proposition 3 are the facts that $P$ is homogeneous for the action of an algebraic group $H$, and $L_i$ are $H$-equivariant line bundles such that $L_i^{-1} \otimes L_j$ is globally generated for $i > j$.

\textbf{Remark 6.} To recover the result that for (type $A$) equivariant Schubert calculus, the structure constants $c^w_{uv}$ are in $\mathbb{Z}_{\geq 0}[t_2 - t_1, \ldots, t_n - t_{n-1}]$, let $\mathbb{P}' = (\mathbb{P}^n)^{\times n}$ and choose a map $\varphi: \mathbb{P}' \to \mathbb{P}$ such that $\varphi^* M_i = M'_i \otimes (M'_{i+1})^{-1}$, where $M'_i$ is the tautological bundle on the $i$th factor of $\mathbb{P}'$, with $t_i = c_1(M'_i)$.

(Note that $\varphi$ will not be holomorphic!)

The $T'$-equivariant class of a Schubert variety (for $T' = (\mathbb{C}^*)^n$) can be identified with the class of the locus $\Omega_w(E'_i \to Q_i) \subset \text{Fl}(E')$, where $E'_i = M'_1 \oplus \cdots \oplus M'_i$ is a flag of bundles on $\mathbb{P}'$. Since this is $\varphi^{-1} \Omega_w$, the equivariant structure constants are $\varphi^* c^w_{uv}$, which are positive in the variables $\varphi^* \alpha_i = t_{i+1} - t_i$.

\textbf{Remark 7.} The naive choice of flag, with $F_i = M_1 \oplus \cdots \oplus M_i$, does not work: The bundle $\text{End}(F)$ has only diagonal global sections, so the corresponding loci $\Omega^o_w$ are not homogeneous. This explains why one does not see positivity over $\mathbb{P}'$.

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