Structural Relations of Harmonic Sums and Mellin Transforms at Weight $w = 6$

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Abstract

We derive the structural relations between nested harmonic sums and the corresponding Mellin transforms of Nielsen integrals and harmonic polylogarithms at weight $w = 6$. They emerge in the calculations of massless single-scale quantities in QED and QCD, such as anomalous dimensions and Wilson coefficients, to 3- and 4-loop order. We consider the set of the multiple harmonic sums at weight six without index $\{-1\}$. This restriction is sufficient for all known physical cases. The structural relations supplement the algebraic relations, due to the shuffle product between harmonic sums, studied earlier. The original amount of 486 possible harmonic sums contributing at weight $w = 6$ reduces to 99 sums with no index $\{-1\}$. Algebraic and structural relations lead to a further reduction to 20 basic functions. These functions supplement the set of 15 basic functions up to weight $w = 5$ derived formerly. We line out an algorithm to obtain the analytic representation of the basic sums in the complex plane.

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1 Introduction

Inclusive and semi-inclusive scattering cross sections in Quantum Field Theories as Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) at higher loop order can be expressed in terms special classes of fundamental numbers and functions. Zero scale quantities, like the loop-expansion coefficients for renormalized couplings and masses in massless field theories, are given by special numbers, which are the multiple \( \zeta \)-values \([1, 2]\) in the known orders. At higher orders and in the massive case other quantities more will contribute \([3]\). The next class of interest are the single scale quantities to which the anomalous dimensions and Wilson coefficients do belong \([4–6]\), likewise other hard scattering cross sections being differential in one variable \( z = \hat{L}/L \) given by the ratio of two Lorentz invariants with support \( z \in [0, 1] \). A natural way to study these quantities consists in representing them in Mellin-space performing the integral transform

\[
M[f(z)](N) = \int_0^1 dz \, z^N \, f(z) .
\]

In the light-cone expansion \([7]\) these quantities naturally emerge as moments for physical reasons with \( N \in \mathbb{N} \). Their mathematical representation is obtained in terms of nested harmonic sums \([8–10]\)

\[
S_{b,a}(N) = \sum_{k=1}^{N} \frac{\text{sign} b}{k |b|} S_{a}(k), \quad S_{0}(k) = 1 ,
\]

which form a unified language. This is the main reason to adopt this prescription also for other quantities of this kind. The harmonic sums lead to the multiple \( \zeta \)-values in the limit \( N \to \infty \) for \( b \neq 1 \). In the latter case the harmonic sums diverge. \(^2\) To obtain a representation which is as compact as possible we seek to find all relations between the harmonic sums. There are two classes of relations:

i) the algebraic relations, cf. \([11]\). They are due to the index set of the harmonic sums only and result from their quasi–shuffle algebra \([12]\).

ii) the structural relations. These relations depend on the other properties of the harmonic sums. One sub–class refers to relations being obtained considering harmonic sums at \( N \) and integer multiples or fractions of \( N \), which leads to a continuation of \( N \in \mathbb{Q} \). Harmonic sums can be represented in terms of Mellin-integrals of harmonic polylogarithms \( H_{a}(z) \) weighted by \( 1/(1 \pm z) \) \([13]\), which belong to the Poincaré–iterated integrals \([14]\). \(^3\) The Mellin integrals are valid for \( N \in \mathbb{R}, N \geq N_0 \). From these representations integration-by-parts relations can be derived. Furthermore, there is a large number of differentiation relations

\[
\frac{d^l}{dN^l} M[f(z)](N) = M \left[ \ln^l(z)f(z) \right] (N) .
\]

We analyzed a wide class of physical single scale massless processes and those containing a single mass scale at two- and three loops \([4–6]\) in the past, which led to the same set of \textbf{basic harmonic sums} and, related to it, \textbf{basic Mellin transforms}. Like in the case of zero scale quantities, this points to a unique representation, which is widely process independent and rather related

\(^2\)Due to the algebraic relations \([11]\) of the harmonic sums one may show that this divergence is at most of \( O(\ln^m(N)) \), where \( m \) is the number of indices equal to one at the beginning of the index set.

\(^3\)Generalized polylogarithms and \( Z \)-sums were considered in \([15]\).
to the contributing Feynman integrals only. The representation in terms of harmonic sums is usually more compact than a corresponding representation by harmonic polylogarithms, since i) Mellin convolutions emerge as simple products; ii) harmonic polylogarithms are multiple integrals, which are usually not reduced to more compact analytic representations. The latter requires to solve (part of) these integrals analytically. In the case of harmonic sums the analytic continuation of their argument $N$ to complex values has to be performed to apply them in physics problems. As lined out in Ref. [16–18] this is possible since harmonic sums can be represented in terms of factorial series [19] up to known algebraic terms. Harmonic sums turn out to be meromorphic functions with single poles at the non-positive integers. One may derive their asymptotic representation analytically and they obey recursion relations for complex arguments $N$. Due to this their unique representation is given in the complex plane.

In the present paper we derive the structural relations of the weight $w = 6$ harmonic sums extending earlier work on the structural relations of harmonic sums up to weight $w = 5$ [18]. The paper is organized as follows. In Sections 2–6 we derive the structural relations of the harmonic sums of weight $w = 6$ of depth 2 to 6 for the harmonic sums not containing the index $\{-1\}$. The restriction to this class of functions is valid in the massless case at least to three-loop order and the massive case to two-loop order. In Section 7 we summarize the set of basic functions chosen. The principal method to derive the analytic continuation of the harmonic sums to complex values of $N$ is outlined in Section 8 in an example. Section 9 contains the conclusions. Some useful integrals are summarized in the appendix.

2 Twofold Sums

The following $w = 6$ two–fold sums occur: $S_{\pm 5,1}(N), S_{\pm 4,\pm 2}(N), S_{3,3}(N)$ and $S_{3,3}(N), S_{3,-3}(N)$. The latter sums are related to single harmonic sums through Euler’s relation.

$$S_{a,b}(N) + S_{b,a}(N) = S_a(N)S_b(N) + S_{a\wedge b}(N), \quad (2.1)$$

with $a \wedge b = \text{sign}(a) \cdot \text{sign}(b)(|a| + |b|)$. For the former six sums we only consider the algebraically irreducible cases. In Ref. [18] the basic functions, which determine the harmonic sums without index $\{-1\}$ through their Mellin transform, up to $w = 5$ were found:

$$w = 1 : 1/(x − 1) \quad (2.2)$$
$$w = 2 : \ln(1 + x)/(x + 1) \quad (2.3)$$
$$w = 3 : \text{Li}_2(x)/(x ± 1) \quad (2.4)$$
$$w = 4 : \text{Li}_3(x)/(x + 1), \quad S_{1,2}(x)/(x ± 1) \quad (2.5)$$
$$w = 5 : \text{Li}_4(x)/(x ± 1), \quad S_{1,3}(x)/(x ± 1), \quad S_{2,2}(x)/(x ± 1), \quad \text{Li}_2^2(x)/(x ± 1), \quad \ln(x)S_{1,2}(-x) - \text{Li}_2^2(-x)/2]/(x ± 1) \quad (2.6)$$

In the following we determine the corresponding basic functions for $w = 6$.

In case of the double sums we show that they all can be related to

$$M \left[ \frac{\text{Li}_5(x)}{1 + x} \right](N) \quad (2.7)$$

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up to derivatives of basic functions of lower degree and polynomials of known harmonic sums. The representation of $S_{\pm,1}(N)$ read:

\[
S_{5,1}(N) = M \left[ \left( \frac{\text{Li}_5(x)}{x-1} \right)_+ \right] (N) - S_1(N)\zeta_5 + S_2(N)\zeta_4 - S_3(N)\zeta_3 + S_4\zeta_2 \tag{2.8}
\]

\[
S_{-5,1}(N) = (-1)^N M \left[ \frac{\text{Li}_5(x)}{1+x} \right] (N) + \frac{15}{16}\zeta_5 \ln(2) - s_6 - S_{-1}(N)\zeta_5 + S_2(N)\zeta_4 - S_3(N)\zeta_3 + S_4\zeta_2, \tag{2.9}
\]

with

\[
\int_0^1 dxg(x)[f(x)]_+ = \int_0^1 dx[g(x) - g(1)]f(x) \tag{2.10}
\]

and

\[
s_6 = \frac{15}{16} \ln(2)\zeta_5 + \int_0^1 dz \frac{\text{Li}_5(z)}{1+z} \tag{2.11}
\]

being one of the basic constants at weight $w = 6$. For the determination of the constants in the alternating case we use the tables associated to Ref. [10]. To express one of the sums given below we also give a second representation of $S_{-5,1}(N)$,

\[
S_{-5,1}(N) = S_{-5}(N)S_1(N) + S_{-6}(N)
\]

\[
+ (-1)^{(N+1)} M \left[ \frac{\text{Li}_5(-x) - \ln(x)\text{Li}_4(-x) + \ln^2(x)\text{Li}_3(-x)/2}{x+1} \right] (N)
\]

\[
+ (-1)^{(N+1)} M \left[ -\frac{\ln^3(x)\text{Li}_2(-x)/6 - \ln^4(x)\ln(1+x)/24}{x+1} \right] (N)
\]

\[
- \frac{15}{16}\zeta_5 [S_{-1}(N) - S_1(N)] - \frac{23}{70}\zeta_2^3 + \frac{3}{4}\zeta_3^2 + \frac{23}{8}\zeta_5 \ln(2) - s_6. \tag{2.12}
\]

The other two-fold sums are

\[
S_{-4,-2}(N) = - M \left[ \left( \frac{4\text{Li}_5(-x) - \ln(x)\text{Li}_4(-x)}{x-1} \right)_+ \right] (N)
\]

\[
+ \frac{1}{2}\zeta_2 [S_4(N) - S_{-4}(N)] - \frac{3}{2}\zeta_3 S_3(N) + \frac{21}{8}\zeta_4 S_2(N) - \frac{15}{4}\zeta_5 S_1(N) \tag{2.13}
\]

\[
S_{-4,2}(N) = (-1)^N M \left[ \frac{4\text{Li}_5(x) - \text{Li}_4(x)\ln(x)}{1+x} \right] (N)
\]

\[
+ 2\zeta_3 S_{-3}(N) - 3\zeta_4 S_{-2}(N) + 4\zeta_5 S_{-1}(N) + \frac{239}{840}\zeta_2^3 - \frac{3}{4}\zeta_3^2 - \frac{15}{4}\zeta_5 \ln(2) + 4s_6 \tag{2.14}
\]

\[
S_{4,-2}(N) = \frac{1}{2}\zeta_2 [S_{-4}(N) - S_4(N)] - \frac{3}{2}\zeta_3 S_{-3}(N) + \frac{21}{8}\zeta_4 S_{-2}(N) - \frac{15}{4}\zeta_5 S_{-1}(N)
\]

\[
+ (-1)^{N+1} M \left[ \frac{4\text{Li}_5(-x) - \ln(x)\text{Li}_4(-x)}{1+x} \right] (N)
\]

\[
- \frac{313}{840}\zeta_2^3 + \frac{15}{16}\zeta_3^2 + 4\zeta_5 \ln(2) - 4s_6 \tag{2.15}
\]

\[
S_{4,2}(N) = - M \left[ \left( \frac{4\text{Li}_5(x) - \ln(x)\text{Li}_4(x)}{x-1} \right)_+ \right] (N) + 2\zeta_3 S_3(N) - 3\zeta_4 S_2(N) + 4\zeta_5 S_1(N) \quad \tag{2.16}
\]
\[ S_{-3,-3}(N) = M \left[ \frac{6 \text{Li}_5(-x) - 3 \ln(x) \text{Li}_1(-x) + \ln^2(x) \text{Li}_3(-x) / 2}{x - 1} \right] (N) \]
\[ - \frac{3}{4} \zeta_3 [S_{-3}(N) - S_3(N)] - \frac{21}{8} \zeta_4 S_2(N) + \frac{45}{8} \zeta_5 S_1(N) \]
\[ = \frac{1}{2} \left[ S_{2,3}^2(N) + S_6(N) \right] \] (2.17)
\[ S_{-3,3}(N) = 3 \zeta_4 S_2(N) - 6 \zeta_5 S_1(N) - 6 \mathbf{M} \left[ \left( \frac{\text{Li}_4(1 - x) - \zeta_3}{x - 1} \right) \right] (N) \]
\[ + \left( -1 \right)^{N+1} \mathbf{M} \left[ \left( \frac{3 \ln(x) \left[ \zeta_3 \left( 1 - x \right) - \zeta_4 \right] + \ln^2(x) \left[ \zeta_2 \left( 1 - x \right) - \zeta_3 \right] / 2}{1 + x} \right) \right] (N) \]
\[ - \frac{271}{280} \zeta_3^2 + \frac{81}{32} \zeta_3^2 + \frac{45}{8} \zeta_5 \ln(2) - 6s_6 \] (2.18)

In the above relations Nielsen integrals, [20], given by
\[ S_{p,n}(x) = \frac{(-1)^{p+n+1}}{(p-1)!n!} \int_0^1 \frac{dz}{z} \ln^{p-1}(z) \ln^n(1 - xz) \] (2.20)

occur. The corresponding functions \( S_{1,k}(1 - x) \) are given by
\[ S_{1,2}(1 - x) = -\text{Li}_3(x) + \log(x) \text{Li}_2(x) + \frac{1}{2} \log(1 - x) \log^2(x) + \zeta(3) \]
\[ S_{1,3}(1 - x) = -\text{Li}_4(x) + \log(x) \text{Li}_3(x) - \frac{1}{2} \log^2(x) \text{Li}_2(x) - \frac{1}{6} \log^3(x) \log(1 - x) + \zeta(4) \]
\[ S_{1,4}(1 - x) = -\text{Li}_5(x) + \ln(x) \text{Li}_4(x) - \frac{1}{2} \ln^2(x) \text{Li}_3(x) + \frac{1}{6} \ln^3(x) \text{Li}_2(x) \]
\[ + \frac{1}{24} \ln^4(x) \ln(1 - x) + \zeta_5 \] (2.21)

They are used to express the respective sums in terms of the Mellin transforms of basic functions and their derivatives w.r.t. \( N \).

The algebraic relation for \( S_{3,3}(N) \) can be used to express \( \mathbf{M}[(\text{Li}_5(x)/(x-1))]_+(N) \). The Mellin transform in \( S_{-3,-3}(N) \) allows to express \( S_{-4,-2}(N) \) and \( S_{-4,2}(N) \) through (2.12). \( S_{4,2}(N) \) and \( S_{-3,3}(N) \) do not contain new Mellin transforms. Therefore the only non-trivial Mellin transform needed to express the double sums at \( w = 6 \) is \( \mathbf{M}[\text{Li}_5(x)/(1 + x)]_+(N) \).

In some of the harmonic sums Mellin transforms of the type
\[ \frac{\text{Li}_k(-x)}{x \pm 1} \] (2.22)

contribute. For odd values of \( k = 2l + 1 \) the harmonic sums \( S_{1, -(k-1)}(N), S_{2, -(k-1), 1}(N) \) and \( S_{4, -(k-1)}(N) \) allow to substitute the Mellin transforms of these functions in terms of Mellin transforms of basic functions and derivatives thereof.
For even values of $k$ this argument applies to $M[\text{Li}_k(-x)/(1 + x)](N)$ but not to $M[\text{Li}_k(-x)/(1 + x)](N)$. In the latter case one may use the relation
\[
\frac{1}{2^{k-2}} \frac{\text{Li}_k(x^2)}{1 - x^2} = \frac{\text{Li}_k(x)}{1 - x} + \frac{\text{Li}_k(x)}{1 + x} + \frac{\text{Li}_k(-x)}{1 - x} + \frac{\text{Li}_k(x)}{1 + x}.
\] (2.23)

Since in massless quantum field-theoretic calculations both denominators occur, one may apply this decomposition based on the first two cyclotomic polynomials, cf. [21], and the relation between $\text{Li}_k(x^2)$ and $\text{Li}_k(\pm x)$, [22]. The corresponding Mellin transforms also require half-integer arguments. In more general situations other cyclotomic polynomials might emerge. The relation
\[
\frac{1}{2^{k-1}} M \left[(\frac{\text{Li}_k(x^2)}{x^2 - 1})^+ \right] \left(\frac{N - 1}{2}\right) = M \left[(\frac{\text{Li}_k(x)}{x - 1})^+ \right] (N) + M \left[(\frac{\text{Li}_k(x)}{x + 1})^+ \right] (N)
+ M \left[(\frac{\text{Li}_k(-x)}{x - 1})^+ \right] (N) + M \left[(\frac{\text{Li}_k(-x)}{x + 1})^+ \right] (N)
- \int_0^1 dx \frac{\text{Li}_k(x^2)}{1 + x}
\] (2.24)
determines $M[\text{Li}_k(-x)/(1 + x)](N)$. For $k = 2, 4$ the last integral in (2.24) is given by
\[
\int_0^1 dx \frac{\text{Li}_2(x^2)}{1 + x} = \zeta_2 \ln(2) - \frac{3}{4} \zeta_3
\] (2.25)
\[
\int_0^1 dx \frac{\text{Li}_4(x^2)}{1 + x} = \frac{2}{5} \ln(2) \zeta_2^2 + 3 \zeta_2 \zeta_3 - \frac{25}{4} \zeta_5.
\] (2.26)

The corresponding relations for $M[\text{Li}_k(-x)/(1 + x)](N)$ are:
\[
M \left[\frac{\text{Li}_2(-x)}{x + 1}\right] (N) = -\frac{1}{2} M \left[(\frac{\text{Li}_2(x)}{x - 1})^+ \right] \left(\frac{N - 1}{2}\right) + M \left[(\frac{\text{Li}_2(x)}{x - 1})^+ \right] (N)
+ M \left[(\frac{\text{Li}_2(-x)}{x - 1})^+ \right] (N) - M \left[\frac{\text{Li}_2(x)}{x + 1}\right] (N)
+ \frac{3}{8} \zeta_3 - \frac{1}{2} \zeta_2 \ln(2)
\] (2.27)
\[
M \left[\frac{\text{Li}_4(-x)}{x + 1}\right] (N) = -\frac{1}{8} M \left[(\frac{\text{Li}_4(x)}{x - 1})^+ \right] \left(\frac{N - 1}{2}\right) + M \left[(\frac{\text{Li}_4(x)}{x - 1})^+ \right] (N)
+ M \left[(\frac{\text{Li}_4(-x)}{x - 1})^+ \right] (N) - M \left[\frac{\text{Li}_4(x)}{x + 1}\right] (N)
- \frac{1}{20} \zeta_2^2 \ln(2) - \frac{3}{8} \zeta_2 \zeta_3 + \frac{25}{32} \zeta_5.
\] (2.28)

In the case of $w = 6$ these relations do not lead to a further reduction of basic functions but are required at lower weights, cf. [18].
3 Threefold Sums

The triple sums are:

\[ S_{4,1,1}(N) = -M \left[ \frac{S_{3,2}(x)}{x - 1} \right]_+ (N) + S_1(N)(2\zeta_5 - \zeta_2\zeta_3) - \frac{\zeta_4}{4} S_2(N) + \zeta_3 S_3(N) \]  

(3.1)

\[ S_{-4,1,1}(N) = (-1)^{N+1} M \left[ \frac{S_{3,2}(x)}{1 + x} \right]_+ (N) + (2\zeta_5 - \zeta_2\zeta_3) S_{-1}(N) - \frac{\zeta_4}{4} S_{-2}(N) + \zeta_3 S_{-3}(N) + \frac{71}{840} \zeta_2^3 + \frac{1}{8} \zeta_3^2 - \frac{29}{32} \zeta_5 \ln(2) - \zeta_2 \zeta_3 \ln(2) + \frac{3}{2} s_6 \]  

(3.2)

\[ S_{-3,-2,1}(N) = M \left[ \frac{H_{0.0,-1.0,1}(x)}{x - 1} \right]_+ (N) + \zeta_2 S_{-3,-1}(N) + [S_{-3}(N) - S_3(N)] \left[ \zeta_2 \ln(2) - \frac{5}{8} \zeta_3 \right] + \frac{3}{40} \zeta_2^3 S_2(N) - \frac{9}{4} \zeta_2 \zeta_3 - \frac{67}{16} \zeta_5 S_1(N) \]  

(3.3)

\[ S_{-2,-3,1}(N) = S_{-2}(N) S_{-3,1}(N) + S_{5,1}(N) + S_{-3,-3}(N) - S_{-3,1,-2} - S_{-3,-2,1} \]  

(3.4)

\[ S_{1,-2,-3}(N) = S_{-3}(N) S_{-1,-2}(N) + S_{1,5}(N) - S_1(N) S_{-3,-2}(N) - S_{-3,3}(N) + S_{-3,-2,1}(N) \]  

(3.5)

\[ S_{1,-3,-2}(N) = S_{-3,-3}(N) - S_{-3}(N) S_{1,-2}(N) - S_{1,5}(N) + S_1(N) S_{-2,-3}(N) - S_{-2}(N) S_{-3,1}(N) - S_{5,1}(N) + S_{-2,-4}(N) + S_1(N) S_{-3,-2}(N) + S_{-3,1,-2}(N) \]  

(3.6)

\[ S_{-3,1,-2}(N) = M \left[ \frac{A_1(-x)/2 + S_{3,2}(-x) - S_{2,2}(-x) \ln(x)}{x - 1} \right]_+ (N) - \frac{1}{2} \zeta_2 [S_{-3,1}(N) - S_{-3,-1}(N)] - \frac{1}{8} \zeta_3 - \frac{1}{2} \zeta_2 \ln(2) \right] [S_{-3}(N) - S_3(N)] + \frac{1}{8} \zeta_2 S_2(N) + \frac{23}{16} \zeta_5 - \frac{7}{8} \zeta_2 \zeta_3 S_1(N) \]  

(3.7)

\[ S_{-3,2,1}(N) = (-1)^N M \left[ \frac{2S_{3,2}(x) - A_1(x)/2}{x + 1} \right]_+ (N) + \zeta_2 S_{-3,1}(N) - \frac{3}{4} \zeta_4 S_{-2}(N) - \frac{11}{2} \zeta_5 - 3 \zeta_2 \zeta_3 \right) S_{-1}(N) + \frac{23}{168} \zeta_5^3 + \frac{59}{64} \zeta_3^2 + \frac{41}{32} \zeta_5 \ln(2) + \frac{1}{2} \zeta_2^2 \ln^2(2) + \frac{5}{4} \zeta_2 \zeta_3 \ln(2) - \frac{1}{12} \zeta_2 \ln^4(2) - 2 \zeta_2 \zeta_5 \ln(2) + \frac{1}{2} \zeta_2^2 \ln^2(2) + \frac{1}{2} \zeta_2^2 \ln^2(2) + \frac{1}{2} \zeta_2^2 \ln^2(2) + \frac{11}{12} \zeta_2 + \frac{87}{64} \zeta_3^2 \]  

(3.8)

\[ S_{2,-3,1}(N) = (-1)^N M \left[ \frac{H_{0.0,-1.0,1}(x)}{1 + x} \right]_+ (N) - \frac{83}{16} \zeta_5 - \frac{21}{8} \zeta_2 \zeta_3 \right) S_{-1}(N) + \zeta_2 S_{-2}(N) - \zeta_3 S_{-1}(N) + \frac{3}{5} \zeta_2^2 + 2 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{3}{4} \zeta_3 \ln(2) - \frac{1}{2} \zeta_2 \ln^2(2) + \frac{1}{12} \ln^4(2) \right] [S_2(N) - S_2(N)] + \frac{7}{8} \zeta_2 \zeta_3 \ln(2) - \frac{1}{6} \zeta_2 \ln^4(2) - 4 \zeta_2 \text{Li}_4 \left( \frac{1}{2} \right) + \frac{11}{12} \zeta_2^3 + \frac{87}{64} \zeta_3^2 \]  

(3.9)

\[ S_{1,-2,3}(N) = S_{-3}(N) S_{1,2}(N) + S_{1,-5}(N) - S_1(N) S_{-3,2}(N) - S_{-3,3}(N) + S_{-3,2,1}(N) \]  

(3.10)

\[ S_{1,-3,2}(N) = -S_2(N) S_{-3,1}(N) - S_{-5,1}(N) + S_{2,-3,1}(N) + S_1(N) S_{-3,2}(N) + S_{-4,2}(N) \]  

(3.11)
\( S_{2,1,-3}(N) = S_{3,-3}(N) - S_{-3}(N)S_{1,2}(N) - S_{1,-5}(N) + S_{1}(N)S_{2,-3}(N) - S_{2,-3,1}(N) \\
+ S_{2,-4}(N) + S_{1}(N)S_{3,2}(N) + S_{3,3}(N) - S_{-3,2,1}(N) \) \( (3.13) \)

\( S_{-3,1,2}(N) = S_{2}(N)S_{-3,1}(N) + S_{-5,1}(N) + S_{-3,3}(N) - S_{2,-3,1}(N) - S_{-3,2,1}(N) \) \( (3.14) \)

\( S_{-2,3,1}(N) = (-1)^N \mathcal{M} \left[ \frac{(3/2)A_1(x) - \text{Li}_2(x)\text{Li}_3(x)}{x+1} \right] \) \( (N) \) + \( \zeta_2 S_{2,2}(N) - \zeta_3 S_{2,1}(N) \) \( (3.15) \)

\( S_{3,-2,1}(N) = S_{3,-3}(N) - S_{3,1,-2}(N) - S_{-2,3,1}(N) + S_{-2}(N)S_{3,1}(N) + S_{-5,1}(N) \) \( (3.16) \)

\( S_{1,-2,3}(N) = S_{-3,3}(N) - S_{3}(N)S_{2,1}(N) - S_{-2,4}(N) + S_{2}(N)S_{1,3}(N) + S_{1,-5}(N) \\
+ 2S_{-2}(N)S_{3,1}(N) + S_{4,-3}(N) - S_{1}(N)S_{3,-2}(N) - S_{4,-2}(N) + S_{-5,1}(N) \\
- S_{3,1,-2}(N) - S_{-2,3,1}(N) \) \( (3.17) \)

\( S_{1,3,-2}(N) = -S_{2,-2}(N)S_{3,1}(N) - S_{-5,1}(N) + S_{-2,3,1}(N) + S_{1}(N)S_{3,-2}(N) \\
+ S_{4,-2}(N) \) \( (3.18) \)

\( S_{-2,1,3}(N) = S_{3}(N)S_{-2,1}(N) + S_{-2,4}(N) - S_{2}(N)S_{3,1}(N) - S_{-3,3}(N) + S_{3,1,-2}(N) \) \( (3.19) \)

\( S_{3,1,-2}(N) = (-1)^N \mathcal{M} \left[ \frac{A_1(-x)/2 + S_{3,2}(-x) - \ln(x)S_{2,2}(-x)}{1 + x} \right] \) \( (N) \) \\
- \( \frac{1}{2} \zeta_2 [S_{3,1}(N) - S_{3,-1}(N)] - \left[ \frac{1}{8} \zeta_3 - \frac{1}{2} \zeta_2 \ln(2) \right] [S_3(N) - S_{-3}(N)] \\
+ \frac{1}{8} \zeta_2^2 S_{-2}(N) + \left[ \frac{23}{18} \zeta_5 - \frac{7}{8} \zeta_2 \zeta_3 \right] S_{-1}(N) \\
+ \frac{113}{560} \zeta_2 \zeta_3 - \frac{17}{64} \zeta_2^2 - \frac{1}{2} \zeta_3 \ln(2) - \frac{7}{8} \zeta_2 \zeta_3 \ln(2) + s_6 \) \( (3.20) \)

\( S_{3,2,1}(N) = \mathcal{M} \left[ \frac{2S_{3,2}(x) - A_1(x)/2}{x-1} \right] \) \( (N) \) \\
+ \( \zeta_2 S_{3,1}(N) - \frac{3}{4} \zeta_4 S_2(N) - \left( \frac{11}{2} \zeta_5 - 3 \zeta_2 \zeta_3 \right) S_1(N) \) \( (3.21) \)

\( S_{2,3,1}(N) = \mathcal{M} \left[ \frac{(3/2)A_1(x) - \text{Li}_2(x)\text{Li}_3(x)}{x-1} \right] \) \( (N) \) \\
+ \( \zeta_2 S_{2,2}(N) - \zeta_3 S_{2,1}(N) + \left( \frac{9}{2} \zeta_5 - 2 \zeta_2 \zeta_3 \right) S_1(N) \) \( (3.22) \)

\( S_{1,2,3}(N) = S_3(N)S_{1,2}(N) + S_{1,5}(N) - S_{1}(N)S_{3,3}(N) - S_{-3,3}(N) + S_{3,2,1}(N) \) \( (3.23) \)

\( S_{2,1,3}(N) = 2S_{3,3}(N) - S_3(N)S_{1,2}(N) - S_{1,5}(N) + S_1(N)S_{2,3}(N) - S_{2,3,1}(N) \\
+ S_{2,4}(N) + S_1(N)S_{3,2}(N) - S_{3,2,1}(N) \) \( (3.24) \)

\( S_{1,3,2}(N) = -S_{2}(N)S_{3,1}(N) - S_{3,1}(N) + S_{2,3,1}(N) + S_1(N)S_{3,2}(N) + S_{4,2}(N) \) \( (3.25) \)

\( S_{3,1,2}(N) = S_{2}(N)S_{3,1}(N) + S_{3,1}(N) + S_{3,3}(N) - S_{2,3,1}(N) - S_{3,2,1}(N) \) \( (3.26) \)

\( S_{2,2,2}(N) = -\mathcal{M} \left[ \frac{2A_1(x) + \text{Li}_2^2(x) \ln(x)/2 - 2S_{2,2}(x) \ln(x)}{x-1} \right] \) \( (N) \) \\
- \( \mathcal{M} \left[ \frac{4S_{3,2}(x) + 2\text{Li}_2(x)\text{Li}_3(x)}{x-1} \right] \) \( (N) \) \\
+ 2\zeta_3 S_{2,1}(N) + 2 \left( \zeta_5 - \zeta_2 \zeta_3 \right) S_1(N) \)
which cannot be reduced significantly further. Harmonic polylogarithms are Poincaré–iterated numerator functions are given by harmonic polylogarithms

\[ H \]

reduces to single harmonic sums and known Mellin transforms algebraically. Furthermore, some

\[ S \]

There emerge numerator functions, which do not belong to the class of Nielsen integrals

\[ \text{Li} \]

Note a misprint in Eq. (14), [17]. \( \text{Li} \)
integrals [14] over the alphabet \([f_0, f_1, f_{-1}] = [1/x, 1/(x-1), 1/(x+1)], [13]\), with

\[
H_0(x) = \ln(x) \\
H_1(x) = -\ln(1-x) \\
H_{-1}(x) = \ln(1+x)
\]

and

\[
H_{a,b}(x) = \int_0^x dy \ f_a(y) H_b(y) .
\]

4 Fourfold Sums

The quadruple–index sums are:

\[
S_{-3,1,1,1}(N) = (-1)^N M \left[ \frac{S_{2,3}(x)}{x+1} \right] (N) + \zeta_4 S_{-2}(N) - (2\zeta_5 - \zeta_2 \zeta_3) S_{-1}(N)
\]

\[
+ \frac{1}{8} \zeta_2 \zeta_3 \ln(2) - \frac{1}{6} \zeta_2 \ln^4(2) - 2 \zeta_2 \ln \left(\frac{1}{2}\right) + \frac{1}{4} \zeta_2^2 \ln^2(2) - \frac{257}{840} \zeta_3 + \frac{7}{24} \zeta_3 \ln^3(2)
\]

\[
- \frac{41}{64} \zeta_3^2 - \frac{33}{32} \zeta_3 \ln(2) + 2 \ln(2) \ln \left(\frac{1}{2}\right) + \ln^2(2) \ln \left(\frac{1}{2}\right)
\]

\[
+ \frac{1}{36} \ln^6(2) + 2 \zeta_6 \ln \left(\frac{1}{2}\right) - \frac{s_6}{2}
\]

\[
S_{3,1,1,1}(N) = - M \left[ \frac{S_{2,3}(x)}{x-1} \right] + (N) + \zeta_4 S_{2}(N) - (2\zeta_5 - \zeta_2 \zeta_3) S_{1}(N)
\]

\[
S_{-2,1,1,1}(N) = (-1)^{N+1} M \left[ \frac{3S_{2,3}(x) + A_2(x)}{x+1} \right] (N) + \zeta_3 S_{-2,1}(N) + \left( \frac{11}{2} \zeta_5 - 3 \zeta_2 \zeta_3 \right) S_{-1}(N)
\]

\[
- \frac{5}{4} \zeta_3 \ln(2) + \frac{1}{3} \zeta_2 \ln^4(2) + 2 \zeta_2 \ln \left(\frac{1}{2}\right) - \frac{1}{2} \zeta_2^2 \ln^2(2) + \frac{411}{560} \zeta_3^2 - \frac{7}{12} \zeta_3 \ln^3(2)
\]

\[
- \frac{9}{8} \zeta_3^2 + \frac{73}{64} \zeta_3 \ln(2) - 4 \ln(2) \ln \left(\frac{1}{2}\right) - 2 \ln^2(2) \ln \left(\frac{1}{2}\right) - \frac{1}{18} \ln^6(2)
\]

\[
- 4 \zeta_6 \ln \left(\frac{1}{2}\right) + \frac{9}{4} s_6
\]

\[
S_{2,-2,1,1}(N) = (-1)^{N+1} M \left[ \frac{H_{0,-1,0,1,1}(x)}{1+x} \right] (N) + \zeta_3 S_{2,-1}(N) + \left( \frac{11}{16} \zeta_2 \zeta_3 - \frac{43}{32} \zeta_5 \right) S_{-1}(N)
\]

\[
+ \left[ - \zeta_4 \ln \left(\frac{1}{2}\right) + \frac{1}{8} \zeta_2^2 + \frac{1}{8} \zeta_3 \ln(2) + \frac{1}{4} \zeta_2 \ln^2(2) - \frac{1}{24} \ln^4(2) \right] [S_{2}(N) - S_{-2}(N)]
\]

\[
- \frac{17}{16} \zeta_2 \zeta_3 \ln(2) - \frac{1}{3} \zeta_2 \ln^4(2) - 2 \zeta_2 \ln \left(\frac{1}{2}\right) + \frac{1}{2} \zeta_2^2 \ln^2(2) - \frac{87}{280} \zeta_3^3 + \frac{7}{12} \zeta_3 \ln^3(2)
\]

\[
+ \frac{105}{128} \zeta_3^2 - \frac{103}{32} \zeta_3 \ln(2) + 4 \ln(2) \ln \left(\frac{1}{2}\right) + 2 \ln^2(2) \ln \left(\frac{1}{2}\right) + \frac{1}{18} \ln^6(2)
\]

\[
+ 4 \zeta_6 \ln \left(\frac{1}{2}\right) + s_6
\]

\[
S_{-2,1,1,2}(N) = (-1)^N M \left[ \frac{A_2(x)}{1+x} \right] (N) - (\zeta_2 \zeta_3 - 3 \zeta_5) S_{-1}(N) - \zeta_3 S_{-2,1}(N) + \zeta_2 S_{-2,1,1}(N)
\]

\[
+ \frac{5}{16} \zeta_2 \zeta_3 \ln(2) + \frac{1}{16} \zeta_2 \ln^4(2) + \frac{3}{2} \zeta_2 \ln \left(\frac{1}{2}\right) - \frac{3}{8} \zeta_2^2 \ln^2(2) + \frac{11}{120} \zeta_3^2 - \frac{27}{16} \zeta_3^2
\]
Only one sixfold sum contributes at $w = 6$, $S_{1,1,1,1,1,1}(N)$. This sum is completely reducible into a polynomial of single harmonic sums, cf.\cite{tayl2002},

$$S_{1,\ldots,1}^{(6)} = \frac{1}{720} S_1^6 + \frac{1}{48} S_2 S_1^4 + \frac{1}{18} S_3 S_1^3 + \frac{1}{8} S_4 S_1^2 + \frac{1}{5} S_5 S_1 + \frac{1}{16} S_1^2 S_2^2$$

$$+ \frac{1}{6} S_1 S_2 S_3 + \frac{1}{48} S_3^3 + \frac{1}{8} S_2 S_4 + \frac{1}{18} S_3^2 + \frac{1}{6} S_6$$

(6.1)

$S_{-2,-2,1,1}(N) = -\mathcal{M} \left[ \left( \frac{H_{0,1,0,1,1}(x)}{x-1} \right)_+ \right] (N) + \zeta_3 S_{-2,-1}(N) + \left( \frac{11}{16} \zeta_2 \zeta_3 - \frac{41}{32} \zeta_5 \right) S_1(N)$

$$+ \left( -\text{Li}_4 \left( \frac{1}{2} \right) + \frac{1}{8} \zeta_2^2 + \frac{1}{8} \zeta_3 \ln(2) + \frac{1}{4} \zeta_2 \ln^2(2) - \frac{1}{24} \ln^4(2) \right) [S_{-2}(N) - S_2(N)]$$

(4.6)

$$S_{2,2,1,1}(N) = -\mathcal{M} \left[ \left( 3S_{2,3}(x) + A_2(x) \right)_+ \right] (N) + \zeta_3 S_{2,1}(N) + \left( \frac{11}{2} \zeta_5 - 3 \zeta_2 \zeta_3 \right) S_1(N) .$$

(4.7)

Here, the harmonic polylogarithm $H_{0,1,0,1,1}(x)$ is given by

$$H_{0,1,0,1,1}(x) = \int_0^x \frac{dy}{y} \int_0^y \frac{dz}{1+z} .$$

(4.8)

We tested the above sum-relations containing harmonic polylogarithms in the Mellin transforms numerically using the code of Ref.\cite{two2002}.

5 Fivefold Sums

Two 5–fold sums contribute:

$$S_{2,1,1,1,1}(N) = -\mathcal{M} \left[ \left( \frac{S_{1,4}(x)}{x-1} \right)_+ \right] (N) + \zeta_5 S_1(N)$$

(5.1)

$$S_{-2,1,1,1,1}(N) = (-1)^{N+1} \mathcal{M} \left[ \left( \frac{S_{1,4}(x)}{1+x} \right)_+ \right] (N) + \zeta_5 S_1(N)$$

$$+ \frac{7}{16} \zeta_2 \zeta_3 \ln(2) + \frac{1}{12} \zeta_2 \ln^2(2) + \frac{1}{4} \zeta_2 \text{Li}_4 \left( \frac{1}{2} \right) - \frac{1}{8} \zeta_2^2 \ln^2(2) - \frac{7}{48} \zeta_3 \ln(2)$$

$$- \frac{49}{128} \zeta_3^2 - \ln(2) \text{Li}_5 \left( \frac{1}{2} \right) - \frac{1}{2} \ln^2(2) \text{Li}_4 \left( \frac{1}{2} \right) - \frac{1}{72} \ln^6(2) - \text{Li}_6 \left( \frac{1}{2} \right) + \ln(2) \zeta_5 .$$

(5.2)

All other sums can be traced back to these sums using algebraic relations \cite{algeb}. The other Mellin transforms emerging in their representation were all calculated in Refs.\cite{tayl2002,ohler2002} before.

6 Sixfold Sums

Only one sixfold sum contributes at $w = 6$, $S_{1,1,1,1,1,1}(N)$. This sum is completely reducible into a polynomial of single harmonic sums, cf.\cite{tayl2002},
7 The Basic Functions

In the following we summarize the basic functions the Mellin transforms of which represents the harmonic sums up to weight \( w = 6 \) without those carrying an index \(-1\). The corresponding sums of lower weight were determined in Refs. [8, 18, 24]. The new 20 functions are given by

\[
\begin{align*}
\text{w} = 6 : & \, \text{Li}_5(x)/(x+1), \quad S_{1,4}(x)/(x \pm 1), \quad S_{2,3}(x)/(x \pm 1), \\
& \text{S}_{3,2}(x)/(x \pm 1), \quad \text{Li}_2(x)\text{Li}_3(x)/(x \pm 1), \\
& A_1(x)/(x+1), \quad A_2(x)/(x \pm 1), \quad A_3(x)/(x+1) \\
& H_{0,-1,0,1,1}(x)/(x \pm 1), \quad H_{0,0,-1,0,1}(x)/(x \pm 1) \\
& [A_1(-x) + 2S_{3,2}(-x) - 2S_{2,2}(-x) \ln(x)]/(x \pm 1) \\
& [A_1(-x) + 2S_{3,2}(-x) - S_{2,2}(-x) \ln(x) + \text{Li}_2^2(-x) \ln(x)/4 - \text{Li}_3(-x)\text{Li}_2(-x)]/(x - 1)
\end{align*}
\]

and extend the set Eqs. (2.2–2.6). The algebraic relations allow to express the initial set of 99 functions by 30 functions and the structural relations reduce the basis further to 20 functions.

8 Complex Analysis of Harmonic Sums

The anomalous dimensions and Wilson coefficients expressed in Mellin space allow simple representations of the scale evolution of single-scale observables, which are given by ordinary differential equations. The experimental measurement of the observables requires the representation in \( z \)-space. Therefore, one has to perform the analytic continuation of harmonic sums to complex values of \( N \). Precise numerical representations for the analytic continuation of the basic functions up to weight \( w = 5 \) were derived in [25] based on the MINIMAX-method [26]. One may even obtain corresponding representations for quite general functions \( \Phi(z) \), \( z \in [0,1] \), as worked out for the heavy flavor Wilson coefficients to 2-loop order in [27]. For other effective parameterizations see [28].

Here we aim on an exact representations. The inverse Mellin transforms are obtained by a contour integral around the singularities of the respective functions in the complex plane.

We traced back all the harmonic sums to Mellin transforms of basic functions \( f_i(z) \),

\[
\begin{align*}
F_i^-(N) &= \int_0^1 dz f_i(z) \frac{z^N - 1}{z - 1}, \quad F_i^+(N) = \int_0^1 dz f_i(z) \frac{(-z)^N - 1}{z + 1} .
\end{align*}
\]

Eqs. (8.1) imply the recursion relations

\[
\begin{align*}
F_i^-(N + 1) &= -F_i^-(N) + \int_0^1 dz z^N f_i(z) , \\
F_i^+(N + 1) &= F_i^+(N) + (-1)^{N+1} \int_0^1 dz z^N f_i(z) .
\end{align*}
\]

The remaining integrals are simpler Mellin transforms, which correspond to harmonic sums of lower weight.

\footnote{For another proposal for the analytic continuation of harmonic sums to \( N \in \mathbb{R} \), for which some simple examples were presented, cf. [29].}
If the functions \( f_i(z)/(z - 1) \), \( f_i(z)/(z + 1) \) are analytic at \( z = 1 \) the Mellin transforms (8.1) can be represented in terms of factorial series [19]. Not all basic functions chosen above have this property. A corresponding analytic relation replacing

\[
f_i(z) \rightarrow f_i(1 - z)
\]  

always exists. The additional terms are lower weight functions in \( N \) or are related to these by differentiation. We use this representation and consider the factorial series. Due to this both the pole–structure and the asymptotic relation for \( |N| \rightarrow \infty \) are known. \(^6\) The poles are located at the integers below a fixed value \( N_0 \). The recursion relations (8.1) are used to express the respective harmonic sums at any value \( N \in \mathbb{C} \) except the poles.

Let us illustrate this representation in an example for the harmonic sum \( S_{2,1,1,1,1}(N) \). The corresponding basic function is

\[
\left( \frac{S_{1,4}(z)}{z - 1} \right) _+(8.5)
\]

The recursion relation is given by

\[
M\left[ \frac{S_{1,4}(z)}{z - 1} \right](N + 1) = M\left[ \left( \frac{S_{1,4}(z)}{z - 1} \right)_+ \right](N) + M[S_{1,4}(z)](N), 
\]

with

\[
M[S_{1,4}(z)](N) = \frac{1}{N+1}\left[ \zeta_5 - \frac{1}{N+1}S_{1,1,1,1}(N) \right], 
\]

cf. [33].

The numerator function possesses a branch–point at \( z = 1 \). The contributions related to terms \( \ln^k(1 - z)/(z \pm 1) \) contained have to be subtracted explicitly due to there logarithmic growth (to a power) for \( |N| \rightarrow \infty \). This is either possible using the relation \( S_{1,4}(z) \) to \( \text{Li}_5(1 - z) \)

\[
S_{1,4}(z) = -\text{Li}_5(1 - z) + \ln(1 - z)\text{Li}_4(1 - z) - \frac{1}{2}\ln^2(1 - z)\text{Li}_3(1 - z) + \frac{1}{6}\ln^3(1 - z)\text{Li}_2(1 - z)
\]

\[
+ \frac{1}{24}\ln^4(1 - z)\ln(z) + \zeta_5
\]  

or considering harmonic sums, which are algebraic equivalent to the above and are related to a basic function which is regular at \( z \rightarrow 1 \). We will follow the latter way and use the algebraic relations [11] to express \( S_{2,1,1,1,1}(N) \) afterwards,

\[
S_{2,1,1,1,1} = S_{1,1,1,1,2} + \frac{1}{4}\left[ S_1S_{2,1,1,1} + S_{3,1,1,1} + S_{2,2,1,1} + S_{2,1,2,1} + S_{2,1,1,2} \right]
\]

\[
- \frac{1}{12}\left[ S_1S_{1,2,1,1} + S_{2,1,1,1} + S_{1,3,1,1} + S_{1,2,1,2} - S_1S_{1,1,2,1} - S_{2,1,2,1} - S_{1,1,3,1} - S_{1,1,2,2} \right]
\]

\[
- \frac{1}{3}\left[ S_1S_{1,1,1,2} + S_{2,1,1,2} + S_{1,2,1,2} + S_{1,1,2,2} + S_{1,1,1,3} \right]
\]

\(^6\)In [30] asymptotic relations for non-alternating harmonic sums to low orders in \( 1/N^k \) were derived. Our algorithm given below is free of these restrictions. The main ideas were presented in January 2004 [31], see also [32].
through known harmonic sums of lower weight. The latter sum obeys the representation

$$
S_{1,1,1,2}(N) = -M \left[ \frac{\text{Li}_5(1-x)}{1-x} \right](N) + \zeta_2 S_{1,1,1}(N) - \zeta_3 S_{1,1}(N) + \zeta_4 S_{1}(N) + \zeta_5 \ .
$$

(8.10)

The function in the remaining Mellin transform is regular at $z = 1$ and can be represented in terms of a factorial series. The remainder terms in (8.10) are polynomials of single harmonic sums. Therefore the poles of $S_{1,1,1,2}(N)$, resp. $S_{2,1,1,1}(N)$, are located at the non-positive integers. Finally we need the asymptotic representations of $M[\text{Li}_5(1-x)/(1-x)](N)$,

$$
M \left[ \frac{\text{Li}_5(1-x)}{1-x} \right](z) \sim \frac{1}{z} + \frac{1}{32z^2} - \frac{179}{776} \frac{1}{z^3} + \frac{515}{41472} \frac{1}{z^4} - \frac{216383}{19440000} \frac{1}{z^5} - \frac{183781}{25920000} \frac{1}{z^6}
$$

$$
+ \frac{4644828197}{z^7} + \frac{153375307}{49787136000} \frac{1}{z^8} - \frac{371224706507}{252047376000} \frac{1}{z^9}
$$

$$
+ \frac{959290541}{z^{10}} + \frac{575134377343021}{14855426650259} \frac{1}{z^{11}} - \frac{1691353414674000}{312400053504000} \frac{1}{z^{12}}
$$

$$
+ \frac{29106619674489691525729}{1025456132289801603} \frac{1}{z^{13}} + \frac{15601079506240000}{78860107950624000} \frac{1}{z^{14}}
$$

$$
+ \frac{26356770271300558681}{3550619453958701} \frac{1}{z^{15}} - \frac{10536534276008976000}{18718443205862400} \frac{1}{z^{16}}
$$

$$
+ \frac{143247755854737705445843733}{104882668989177092212440000} \frac{1}{z^{17}}
$$

$$
+ \frac{19214070284092335916939}{13028192458306945920000} \frac{1}{z^{18}}
$$

$$
+ \frac{2027981189268747465011536794768001}{254294408120596135866406712880000} \frac{1}{z^{19}} + O\left(\frac{1}{z^{20}}\right)
$$

(8.11)

The corresponding representations for all other harmonic sums of weight $w = 6$ will be given in a forthcoming paper.

9 Conclusions

We derived the basic functions spanning the nested harmonic (alternating) sums up to weight $w = 6$ with no index $\{-1\}$. This sub-class governs the functions contributing to the massless single-scale quantities, like the anomalous dimensions and Wilson coefficients to 3-loop order in QED and QCD. There are first indications, that in the massive case, even in the limit $Q^2 \gg m^2$ this class needs to be extended at 3-loop order, cf. [34]. Up to weight $w = 5$ all basic functions were given by polynomials of Nielsen integrals, Eq. (2.20), of argument $x$ or $-x$ weighted by $1/(x \pm 1)$. Although most of the basic functions at $w = 6$ share this property, some contain 1-dimensional integrals over polynomials of Nielsen integrals $A_i(\pm x)|_{1\ldots 3}$ and more dimensional integrals, which are not reducible. This is generally expected and the cases up to $w = 5$ form an exception.

We lined out how the exact the representation of the Mellin transforms of the basic functions can be obtained, generalizing effective numerical high-precision representations [25, 27]. Up to terms which can be determined algebraically the Mellin transforms of the basic functions are factorial series. The singularities of the Mellin transforms are located at the non-positive integers. They obey recursion relations for $N \to N + 1$. The asymptotic representation of
the Mellin transforms can be determined analytically. The basic Mellin transforms are thus
generalizations of Euler’s $\psi$-function and their derivatives, which describe the single harmonic
sums.

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10 Appendix A: Useful Integrals

In this appendix we list useful constants and integrals.

\[ \text{Li}_k(1) = \zeta_k \]  \hspace{1cm} (10.1)
\[ S_{1,k}(1) = \zeta_{k+1} \]  \hspace{1cm} (10.2)
\[ S_{2,2}(1) = \frac{1}{10} \zeta_2 \]  \hspace{1cm} (10.3)
\[ S_{3,2}(1) = 2\zeta_5 - \zeta_2 \zeta_3 \]  \hspace{1cm} (10.4)
\[ S_{3,2}(-1) = -\frac{29}{32} \zeta_5 + \frac{1}{2} \zeta_2 \zeta_3 \]  \hspace{1cm} (10.5)
\[ S_{2,3}(1) = 2\zeta_5 - \zeta_2 \zeta_3 \]  \hspace{1cm} (10.6)
\[ A_{1}(1) = -3\zeta_5 + 3 \zeta_2 \zeta_3 \]  \hspace{1cm} (10.7)
\[ A_{1}(-1) = -\frac{17}{16} \zeta_5 + \frac{3}{2} \zeta_2 \zeta_3 \]  \hspace{1cm} (10.8)
\[ A_{2}(1) = -\frac{1}{2} \zeta_5 \]  \hspace{1cm} (10.9)
\[ A_{3}(1) = -3\zeta_5 + \zeta_2 \zeta_3 \]  \hspace{1cm} (10.10)

\[ \int_0^x \frac{\text{Li}_3(y)}{1+y} \, dy = \ln(1+x)\text{Li}_3(-x) + \frac{1}{2} \text{Li}_2^2(-x) \]  \hspace{1cm} (10.11)
\[ \int_0^x \frac{\ln(y)\text{Li}_2(-y)}{1+y} \, dy = \ln(1+x) \ln(x)\text{Li}_2(-x) + \frac{1}{2} \text{Li}_2^2(-x) \]  
\hspace{2cm} -2S_{2,2}(-x) + 2 \ln(x)S_{1,2}(-x) \]  \hspace{1cm} (10.12)
\[ \int_0^x \frac{S_{1,2}(y)}{y-1} \, dy = \ln(1-x)S_{1,2}(x) + 3S_{1,3}(x) \]  \hspace{1cm} (10.13)
\[ \int_0^x \frac{\text{Li}_3(y) - \zeta_3}{y} \, dy = \frac{1}{2} \text{Li}_2^2(x) + \ln(1-x)\text{Li}_3(x) \]  \hspace{1cm} (10.14)
\[ \int_0^x \frac{\ln(y)\text{Li}_2(y)}{y-1} \, dy = \frac{1}{2} \text{Li}_2^2(x) + \ln(x) \ln(1-x)\text{Li}_2(x) \]  
\hspace{2cm} -2S_{2,2}(x) + 2 \ln(x)S_{1,2}(x) \]  \hspace{1cm} (10.15)
\[ \int_0^x \frac{\ln(1-y)\text{Li}_3(y)}{y} \, dy = -\text{Li}_2(x)\text{Li}_3(x) + A_1(x) \]  \hspace{1cm} (10.16)
\[ \int_0^x \frac{\ln(y)\text{Li}_2(y)\ln(1-y)}{y} \, dy = -\frac{1}{2} \ln(x)\text{Li}_2^2(x) + \frac{1}{2} A_1(x) \]  \hspace{1cm} (10.17)
\[ \int_0^x \frac{\text{Li}_2(-y)\ln(y)\ln(1+y)}{y} \, dy = -\frac{1}{2} \ln(x)\text{Li}_2^2(-x) + \frac{1}{2} A_1(-x) \]  \hspace{1cm} (10.18)
\[ \int_0^x \frac{\ln(y)\text{Li}_3(-y)}{y} \, dy = -\text{Li}_2(-x)\text{Li}_3(-x) + A_1(-x) \]  \hspace{1cm} (10.19)
\[ \int_0^x \frac{\text{Li}_3(-y)}{y} \, dy \ln(1+y) = -\text{Li}_2(-x)\text{Li}_3(-x) + A_1(-x) \]  \hspace{1cm} (10.20)
\[ \int_0^x \frac{\ln(y)\text{Li}_3(1-x)}{y} \, dy = -S_{1,2}(x) \ln(x) + \frac{1}{2} \text{Li}_2^2(x) - \zeta_2 \text{Li}_2(x) \]  \hspace{1cm} (10.21)
\begin{align}
\int_0^x \frac{dy}{y} S_{1,2}(y) \ln(y) &= S_{2,2}(x) \ln(x) - S_{3,2}(x) \quad (10.22) \\
\int_0^x \frac{dy}{y} S_{1,2}(-y) \ln(y) &= S_{2,2}(-x) \ln(x) - S_{3,2}(-x) \quad (10.23)
\end{align}
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