ON FULL ASYMPTOTICS OF ANALYTIC TORSIONS FOR
COMPACT LOCALLY SYMMETRIC ORBIFOLDS

BINGXIAO LIU

Abstract. In this paper, we consider a certain sequence of homogeneous
flat vector bundles on a compact locally symmetric orbifold, and we compute
explicitly the associated asymptotic Ray-Singer analytic torsion. The basic
idea is converting this question via the Selberg’s trace formula into computing
the semisimple orbital integrals. Then the central part is to evaluate the elliptic
orbital integrals which are not identity orbital integrals. For that purpose, we
deduce a geometric localization formula, so that we can rewrite an elliptic
orbital integral as a sum of certain identity orbital integrals associated with
a smaller Lie group. The explicit geometric formula of Bismut for semisimple
orbital integrals plays an essential role in these computations.

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1. Introduction

Let \((Z, g^{TZ})\) be closed Riemannian manifold of dimension \(m\), and let \((F, \nabla^{F/J}) \to Z\) be a flat complex vector bundle equipped with a Hermitian metric \(h^{F}\). Let \((Ω(Z, F), d^{Z,F})\) be the associated de Rham complex valued in \(F\). It is equipped with an \(L_2\)-metric induced by \(g^{TZ}, h^{F}\). Let \(D^{Z,F,2}\) be the corresponding de Rham-Hodge Laplacian. The real analytic torsion \(T(Z, F)\) is a (graded) spectral invariant of \(D^{Z,F,2}\) introduced by Ray and Singer [35, 36]. When \(Z\) is odd-dimensional, this invariant does not depend on the metric data \(g^{TZ}, h^{F}\). Ray and Singer also conjectured that, for unitarily flat vector bundle \(F\) (i.e., \(\nabla^{F/J} h^{F} = 0\)), this invariant coincides with the Reidemeister torsion, a topological invariant associated with \((F, \nabla^{F/J}) \to Z\). This conjecture was later proved by Cheeger [12] and Müller [31]. Using the Witten deformation, Bismut and Zhang [8, 9] gave an extension of the Cheeger-Müller theorem for arbitrary flat vector bundles.

If \(Z\) is a compact orbifold, and if \(F\) is a flat orbifold vector bundle on \(Z\), the Ray-Singer analytic torsion \(T(Z, F)\) extends naturally to this case (cf. Definition 2.2.4). In particular, if \(F\) is acyclic, and if \(m\) is odd, then \(T(Z, F)\) is independent of the metric data (cf. [41, Corollary 4.9]). We refer to [27, 41], etc for more details.

In this paper, we consider a certain sequence of (acyclic) flat vector bundles \(\{F_d\}_{d \in \mathbb{N}}\) on a compact locally symmetric space \(Z\), and we study the asymptotic behavior of \(T(Z, F_d)\) as \(d \to +\infty\). When \(Z\) is smooth, i.e., \(\Gamma\) is torsion-free, such question was already studied by Müller [32], by Bismut-Ma-Zhang [5, 6] and by Müller-Pfaff [34, 33]. When \(Z\) is a compact hyperbolic orbifold, such question was studied by Ksenia Fedosova in [17] using the method of harmonic analysis. Here, we consider this question for an arbitrary compact locally symmetric orbifold (of noncompact type).

Let \(G\) be a connected linear reductive Lie group, and let \(\theta \in \text{Aut}(G)\) be a Cartan involution. Let \(K \subset G\) be the fixed point set of \(\theta\), which is a maximal compact subgroup of \(G\). Put

\[(1.0.1)\]

\[X = G/K.\]

Then \(X\) is a symmetric space. For convenience, we also assume that \(G\) has a compact center, then \(X\) is of noncompact type.

Now let \(\Gamma \subset G\) be a cocompact discrete subgroup. Set

\[(1.0.2)\]

\[Z = \Gamma \backslash X.\]

Then \(Z\) is a compact locally symmetric space. In general, \(Z\) is an orbifold. Let \(\Sigma Z\) denote the orbifold resolution of the singular points in \(Z\), whose connected components are corresponding to nontrivial elliptic conjugacy classes of \(\Gamma\).

Since \(G\) has compact center, the compact form \(U\) of \(G\) exists and is a connected compact linear Lie group. If \((E, g^{E}, h^{E})\) is a unitary (analytic) representation of \(U\), then it extends uniquely to a representation of \(G\) by unitary trick. This way, \(F = G \times_R E\) is a (unimodular) vector bundle on \(X\) equipped with an invariant flat connection \(\nabla^{F/J}\) and a Hermitian metric \(h^{F}\) induced by \(h^{E}\). Moreover, \((F, \nabla^{F/J}, h^{F})\) descends to a flat orbifold vector bundle on \(Z\). Let \(D^{Z,F,2}\) denote the corresponding de Rham-Hodge Laplacian.

The fundamental rank \(\delta(G)\) (or \(\delta(X)\)) of \(G\) (or \(X\)) is the difference of the complex ranks of \(G\) and of \(K\). As we will see in Theorem 4.1.4, if \(\delta(G) \neq 1\), we always have

\[(1.0.3)\]

\[T(Z, F) = 0.\]

If \(F\) is defined instead by a unitary representation of \(\Gamma\), this result is obtained by Moscovici and Stanton [30, Corollary 2.2]. If \(\Gamma\) is torsion-free, with \(F\) defined via a representation of \(G\) as above, \((1.0.3)\) was proved in [6, Remark 8.7] by using Bismut’s formula for orbital integrals [3, Theorem 6.1.1] (also cf. [28, Theorems 5.4] for more details).
A new proof was given in [33, Proposition 4.2]. Note that in [28, Remark 5.6], Ma has indicated that, using essentially [28, Theorem 5.4], the identity (1.0.3) still holds if \( \Gamma \) is not torsion-free (i.e., \( Z \) is an orbifold), which gives us exactly Theorem 4.1.4 here. Due to this vanishing result, we only need to deal with the case \( \delta(G) = 1 \).

We now describe the sequence of flat vector bundles \( \{F_d\}_{d \in \mathbb{N}} \) which is concerned here. After fixing a root data for \( U \), let \( P_{+}(U) \) denote the set of (real) dominant weights of \( U \). If \( \lambda \in P_{+}(U) \), let \( (E_{\lambda}, \rho^{E_{\lambda}}) \) be the irreducible unitary representation of \( U \) with the highest weight \( \lambda \). We extend it to a representation of \( G \). We require \( \lambda \) to be nondegenerate, i.e., as \( G \)-representations, \( (E_{\lambda}, \rho^{E_{\lambda}}) \) is not isomorphic to \( (E_{\lambda}, \rho^{E_{\lambda}} \circ \theta) \). We also take an arbitrary \( \lambda_0 \in P_{+}(U) \). If \( d \in \mathbb{N} \), let \( (E_d, \rho^{E_d}, h^{E_d}) \) be the unitary representation of \( U \) with highest weight \( d\lambda + \lambda_0 \). By Weyl’s dimension formula, \( \dim E_d \) is a polynomial in \( d \). This way, we get a sequence of (unimodular) flat vector bundles \( \{(F_d, \nabla^{E_d}, h^{F_d})\}_{d \in \mathbb{N}} \) on \( X \) or on \( Z \).

Note that we do not have a canonical choice of \( h^{E_d} \) (or \( h^{F_d} \)) for each \( d \in \mathbb{N} \), but this makes no trouble here. Indeed, since \( \lambda \) is nondegenerate, for \( d \) large enough, we will have

\[
(1.0.4) \quad H(Z, F_d) = 0.
\]

Furthermore, \( \dim Z \) is odd when \( \delta(G) = 1 \). Then \( T(Z, F_d) \) is independent of the different choices of \( h^{E_d} \) (or \( h^{F_d} \)).

Let \( E[\Gamma] \) be the finite set of elliptic classes in \( \Gamma \). Set \( E^{+}[\Gamma] = E[\Gamma] \setminus \{1\} \). The first main result in this paper is the following theorem.

**Theorem 1.0.1.** Assume that \( \delta(G) = 1 \). There exists a (real) polynomial \( P(d) \) in \( d \), and for each \( [\gamma] \in E^{+}[\Gamma] \), there exists a nice pseudopolynomial \( PE^{[\gamma]}(d) \) in \( d \) (i.e., a finite sum of the terms of the form \( ad^l e^{2\pi \sqrt{-1} \theta d} \) with \( a \in \mathbb{C}, l \in \mathbb{N}, \beta \in \mathbb{Q} \), such that there exists a constant \( c > 0 \), for \( d \) large, we have

\[
(1.0.5) \quad T(Z, F_d) = P(d) + \sum_{[\gamma] \in E^{+}[\Gamma]} PE^{[\gamma]}(d) + O(e^{-cd}).
\]

Moreover, the degrees of \( P(d) \), \( PE^{[\gamma]}(d) \) can be determined in terms of \( \lambda, \lambda_0 \).

In [32, Theorem 1.1], for a hyperbolic 3-manifold \( Z \), Müller computed explicitly the leading term of \( T(Z, F_d) \) as \( d \to +\infty \). In [5, 6], under a more general setting for a closed manifold \( Z \), Bismut, Ma and Zhang proved that there exists a constant \( c > 0 \) such that [6, Remark 7.8]

\[
(1.0.6) \quad T(Z, F_d) = \mathcal{T}_{L_2}(Z, F_d) + O(e^{-cd}),
\]

where \( \mathcal{T}_{L_2}(Z, F_d) \) denotes the \( L_2 \)-torsion [26][29] associated with \( F_d \to Z \). Moreover, they constructed universally an element \( W \in \Omega^{ullet}(Z, o(TZ)) \) such that if \( n_0 = \deg E_d \), then

\[
(1.0.7) \quad \mathcal{T}_{L_2}(Z, F_d) = d^{n_0+1} \int_Z W + O(d^{n_0}).
\]

The integral of \( W \) in the right-hand side of (1.0.7) is called a \( W \)-invariant. If we specialize (1.0.7) for a compact locally symmetric manifold \( Z \), we get

\[
(1.0.8) \quad \mathcal{T}_{L_2}(Z, F_d) = d^{n_0+1} \text{Vol}(Z)[W]_{\text{max}} + O(d^{n_0}).
\]

In [6, Subsection 8.7], the explicit computation on \( [W]_{\text{max}} \) was carried out for \( G = \text{SL}_2(\mathbb{C}) \) to recover the result of Müller [32, Theorem 1.1].

We now compare (1.0.5) with (1.0.6). If ignoring that \( \Gamma \) may act on \( X \) non-effectively, we can extend the notion of \( L_2 \)-torsion to the orbifold \( Z \), so that \( \mathcal{T}_{L_2}(Z, F_d) \) is still defined in terms of the \( \Gamma \)-trace of certain heat operators on \( X \). Then the term \( P(d) \) in (1.0.5) will be exactly \( \mathcal{T}_{L_2}(Z, F_d) \). But different from (1.0.6),
we still have the nontrivial terms $PE^{[\gamma]}(d)$, $[\gamma] \in E^+[\Gamma]$ in (1.0.5). We will see, in
a refined version of (1.0.5) stated in Theorem 1.0.2, that the term $PE^{[\gamma]}(d)$ is an
oscillating combination of the $L_2$-torsions for $\Sigma Z$ associated with several smaller
versions of the sequence \{\(F_d\)\} for $d \in \mathbb{N}$. Therefore, we can define a special $L_2$-torsion
for $\Sigma Z$ as follows,
\begin{equation}
\tilde{T}_{L_2}(\Sigma Z, F_d) = \sum_{[\gamma] \in E^+[\Gamma]} PE^{[\gamma]}(d).
\end{equation}
Then, as an analogue to (1.0.6), we restate our Theorem 1.0.1 as follows.

**Theorem 1.0.1**. Assume that $\Gamma$ acts on $X$ effectively. For $Z = \Gamma \setminus X$, as $d \to +\infty$, we have
\begin{equation}
T(Z, F_d) = T_{L_2}(Z, F_d) + \tilde{T}_{L_2}(\Sigma Z, F_d) + O(e^{-cd}).
\end{equation}
Moreover, $T_{L_2}(Z, F_d)$ is a polynomial in $d$, and $\tilde{T}_{L_2}(\Sigma Z, F_d)$ is a nice pseudopolynomial
in $d$. Their leading terms can be determined in terms of $W$-invariants as in
(1.0.8).

In (1.0.10), the term $\tilde{T}_{L_2}(\Sigma Z, F_d)$ represents the nontrivial contribution of $\Sigma Z$. Such term is always expected in the context of orbifold. Note that if $\Gamma$ acts on $X$ non-effectively, we should use Propositions 3.4.1 & 3.5.3 to understand properly the $L_2$-torsions in Theorem 1.0.1.

For a compact locally symmetric manifold $Z$, Müller and Pfaff in [33] (also in
[34] for hyperbolic case) gave a new proof to (1.0.6) and showed that $T_{L_2}(Z, F_d)$ is
a polynomial in $d$. This way, they proved Theorem 1.0.1 with $\Sigma Z = \emptyset$.

Let us give more detail on their results. Let $D^{X,F_d,2}$ be the $G$-invariant operator on $X$ which is the lift of $D^{Z,F_d,2}$. For $t > 0$, let $p_{t}^{X,F_d}(x, x')$ denote the
heat kernel of $D^{X,F_d,2}$ with respect to the Riemannian volume element on $X$. For $t > 0$, the identity orbital integral $\mathcal{I}_X(E_d, t)$ of $p_{t}^{X,F_d}$ is defined as
\begin{equation}
\mathcal{I}_X(F_d, t) = \text{Tr}_{s}^{\Lambda}(T_{X} X) \cdot s_{t}^{d}((N^{\Lambda}(T_{X} X) - \frac{m}{2})p_{t}^{X,F_d}(x, x)),
\end{equation}
where $N^{\Lambda}(T_{X} X)$ is the number operator on $\Lambda(T_{X} X)$, and the right-hand side of
(1.0.11) is independent of the choice of $x \in X$. Let $\mathcal{M}_X(F_d, s)$, $s \in \mathbb{C}$ denote the Mellin transform (cf. (7.3.53)) of $\mathcal{I}_X(F_d, t)$. Then it is holomorphic at 0, and we set
\begin{equation}
\mathcal{P}_X(F_d) = \frac{\partial}{\partial s}_{|s=0} \mathcal{M}_X(F_d, s).
\end{equation}
The $L_2$-torsion is defined as
\begin{equation}
T_{L_2}(Z, F_d) = \text{Vol}(Z) \mathcal{P}_X(F_d).
\end{equation}

Using essentially the Harish-Chandra’s Plancherel theorem for $\mathcal{I}_X(F_d, t)$, Müller-
Pfaff [33] managed to show that $\mathcal{P}_X(F_d)$ is a polynomial in $d$ (for $d$ large enough),
so that there exists a constant $C_{X,\lambda_0} > 0$ such that
\begin{equation}
\mathcal{P}_X(F_d) = C_{X,\lambda_0} d\dim E_d + R(d),
\end{equation}
where $R(d)$ is a polynomial in $d$ of degree no greater than $d\dim E_d$. They also gave concrete formulae for $C_{X,\lambda_0}$ in some model cases [33, Corollaries 1.4 & 1.5].

As we said before, the polynomial $P(d)$ in our Theorem 1.0.1 is essentially the same one in (1.0.13) obtained by Müller and Pfaff. In Subsection 7.3, we use instead an explicit geometric formula of Bismut [3, Theorem 6.1.1] for semisimple orbital integrals to give a different computation on $\mathcal{P}_X(F_d)$. In Subsection 7.4, we verify that our computational results coincide with the ones of Müller-Pfaff [33].

If $Z$ is a hyperbolic orbifold, i.e. $G = \text{Spin}(1, 2n + 1)$, the result in Theorem 1.0.1
(or Theorem 1.0.1') was obtained by Ksenia Fedosova in [17, Theorem 1.1], where
she got the pseudopolynomials $PE^{(d)}(d)$ by evaluating the elliptic orbital integrals associated with nontrivial elliptic elements in $\Gamma$. Correspondingly, for any $Z$ in our setting, a key ingredient to Theorem 1.0.1 is to evaluate explicitly the elliptic orbital integrals associated with $[\gamma] \in E^+T$. For that purpose, we have the full power of Bismut’s formula [3, Theorem 6.1.1].

Instead of proving Theorem 1.0.1, we would like to state a refined version of it, where we give more explicit descriptions on the pseudopolynomials $PE^{(d)}(d)$ or $\tilde{T}_{\lambda_z}(\Sigma Z, F_d)$. For a clear statement of our results, we need introduce some notations and facts. Note that $U$ contains $K$ as a Lie subgroup. Let $T$ be a maximal torus of $K$, and let $T_U$ be the the maximal torus of $U$ containing $T$.

Let $X(k)$ denote the fixed point set of $k$ acting on $X$. Then $X(k)$ is a connected symmetric space with $\delta(X(k)) = 1$. Let $Z(k)^0$ be the identity component of the centralizer $Z(k)$ of $k$ in $G$. Then $X(k) = Z(k)^0/K(k)^0$ with $K(k)^0 = Z(k)^0 \cap K$. Let $U(k)$ denote the centralizer of $k$ in $U$. Then $U(k)^0$ is naturally a compact form of $Z(k)^0$. Then the triple $(X(k), Z(k)^0, U(k)^0)$ becomes a smaller version of $(X, G, U)$, except that $Z(k)^0$ may have noncompact center.

Let $u$ be the Lie algebra of $U$, and let $t_U \subset u$ be the Lie algebra of $T_U$. Let $R(u, t_U)$ be the associated real root system with a system $R^+(u, t_U)$. Then $P_{++}(U) \subset t_U^*$ is taken with respect to the above root data.

Now we fix $k \in T$, and let $u(k)$ denote the Lie algebra of $U(k)^0$. Then $T_U$ is also a maximal torus of $U(k)^0$. We get the following splitting of root systems

$$R(u, t_U) = R(u(k), t_U) \cup R(u^+(k), t_U),$$

where $u^+(k)$ is the orthogonal space of $u(k)$ in $u$ with respect to the Killing form. Let $R^+(u(k), t_U), R^+(u^+(k), t_U)$ be the induced positive root systems, and let $\rho_u, \rho_k$ denote the half of the sum of the roots in $R^+(u, t_U), R^+(u(k), t_U)$ respectively.

Let $W(u(t_U; C))$ be the Weyl group associated with the pair $(u, t_U)$. Put

$$W_0^U(k) = \{ \omega \in W(u(t_U; C) \mid \omega \cdot (R^+(u(k), t_U)) \subset R^+(u, t_U) \}. $$

If $\sigma \in W_0^U(k)$, let $\varepsilon(\sigma)$ denote its sign. If $\mu \in P_{++}(U)$, set

$$\varphi^U_k(\sigma, \mu) = \varepsilon(\sigma) \frac{\xi_{\sigma(\mu + \rho_u) + \rho_k}(k)}{\prod_{\alpha \in R^+(u(k), t_U)}(\xi_{\alpha}(k) - 1)} \in C^*, $$

where $\xi_{\alpha}$ is the character of $T_U$ with (dominant) weight $2\pi\sqrt{-1}\alpha$.

Now we state our second main theorem.

**Theorem 1.0.2.** Assume that $\delta(G) = 1$. For elliptic $\gamma \in G$, there exists a pseudopolynomial $PE_{X, \gamma}(F_d)$ in $d$ (cf. Definition 7.5.1) such that

1. $PE_{X, \gamma}(F_d)$ depends only on the conjugacy class of $\gamma$ in $G$, and

$$ PE_{X, 1}(F_d) = \mathcal{P}I_X(F_d). $$

2. If $\gamma = k \in T$, for $\sigma \in W_0^U(k)$, $\sigma \lambda$ is a nondegenerate dominant weight of $U(k)^0$. Let $E^k_{\sigma, d}$ denote the unitary representations of $U(k)^0$ (up to a finite central extension) with highest weight $d\sigma \lambda + \sigma(\lambda_0 + \rho_u) - \rho_u(k)$, $d \in \mathbb{N}$, and let $\{F_d^{k}\}_{d \in \mathbb{N}}$ be the corresponding sequence of flat vector bundles on $X(k)$.

3. If $\gamma = k \in T$, we have

$$ PE_{X, \gamma}(F_d) = \sum_{\sigma \in W_0^U(k)} \varphi^U_k(\sigma, d\lambda + \lambda_0) \mathcal{P}I_X(F_d^{k}), $$

where $\varphi^U_k(\sigma, d\lambda + \lambda_0)$ is given by (1.0.17), and $\mathcal{P}I_X(F_d^{k})$ is just the (real) polynomial in $d$ defined via (1.0.12) with the sequence $F_d^{k} \to X(k)$. 

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Moreover, there exist constants $C^k_\sigma > 0$, $\sigma \in W_U^1(k)$ such that
\begin{equation}
\mathcal{P}_L X (\sigma) (E^k_{\sigma,d}) = C^k_\sigma d \dim E^k_{\sigma,d} + R^k_\sigma (d),
\end{equation}
where $R^k_\sigma (d)$ is a polynomial in $d \leq \deg \dim E^k_{\sigma,d}$.

(4) For $\sigma \in W_U^1(k)$, $\varphi_k^\sigma (\sigma, d\lambda + \lambda_0)$ is an oscillating term of the form $c_1 e^{2\pi \sqrt{-1} \varepsilon_d d}$ with $c_1 \in \mathbb{C}^*$, $c_2 \in \mathbb{R}$. Moreover, if $\gamma$ is of finite order, $\mathcal{P}_\Sigma X,\gamma (F_d)$ is a nice pseudopolynomial in $d$ (i.e. the parameter $c_2 \in \mathbb{Q}$).

(5) If $\Gamma \subset G$ is a cocompact discrete subgroup, if $\gamma \in \Gamma$ is elliptic, let $S(\gamma)$ denote the finite subgroup of $\Gamma \cap Z(\gamma)$ which acts on $X(\gamma)$ trivially. Then there exists a constant $c > 0$, such that for $Z = \Gamma \setminus X$, as $d \to +\infty$, we have
\begin{equation}
\mathcal{T}(Z,F_d) = \frac{\Vol(Z) \mathcal{P}_L X (F_d)}{|S|} + \sum_{[\gamma] \in E^+ \Gamma} \frac{\Vol(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \mathcal{P}_\Sigma X,\gamma (F_d) + O(e^{-cd}).
\end{equation}

In particular, each $\mathcal{P}_\Sigma X,\gamma (F_d)$ in (1.0.21) is a nice pseudopolynomial in $d$.

Theorem 1.0.1 now is just a consequence of (1.0.21). Note that for $[\gamma] \in E^+ \Gamma$, the (compact) orbifold $\Gamma \cap Z(\gamma) \backslash X(\gamma)$ represents an orbifold stratum in $\Sigma Z$. An important observation on (1.0.21) is that the sequence $\{\mathcal{T}(Z,F_d)\}_{d \in \mathbb{N}}$ encodes the volume information of $Z$ and of $\Sigma Z$. Moreover, combining (1.0.13), (1.0.19) with (1.0.21), we justify that the quantity $\mathcal{T}_{L_2}(\Sigma Z,F_d)$ defined by (1.0.9) is indeed a certain $L_2$-torsion for $\Sigma Z$. For saving notation, in the rest part of this paper, we will not discuss any further about the $L_2$-torsions, but focus on Theorem 1.0.2.

Now we explain our approach to Theorem 1.0.2. Note that the result (2) in it follows from the representation theory, and (4) is an observation on the defining formula (1.0.17) of $\varphi^\sigma_k (\sigma, \lambda_0 + d\lambda)$. Moreover, (1.0.20) is just a consequence of (1.0.14). Therefore, we will concentrate on defining $\mathcal{P}_\Sigma X,\gamma (F_d)$, and on explaining (1.0.19) and (1.0.21).

Let us begin with (1.0.21). In fact, $\mathcal{T}(Z,F_d)$ can be rewritten as the derivative at 0 of the Mellin transform of
\begin{equation}
\Tr_s[\big( N^\lambda (T^* Z) - \frac{m}{2} \exp(-tD^{Z,F_d,2}/2) \big)], \; t > 0,
\end{equation}
where $\Tr_s[\cdot]$ denotes the supertrace with respect to the $\mathbb{Z}_2$-grading on $\Lambda (T^* Z)$.

If $\gamma \in G$ is semisimple, let $\mathcal{E}_{X,\gamma} (F_d, t)$ denote the orbital integral (cf. Subsection 3.3) of the Schwartz kernel of $\big( N^\lambda (T^* X) - \frac{m}{2} \big) \exp(-tD^{X,F_d,2}/2)$ associated with $\gamma$. Note that in $\mathcal{E}_{X,\gamma} (F_d, t)$, we should take the supertrace of the endomorphism on $\Lambda (T^* X) \otimes F$ as in (1.0.22). Moreover, $\mathcal{E}_{X,\gamma} (F_d, t)$ depends only on the conjugacy class of $\gamma$ in $G$. Let $\mathcal{M} \mathcal{E}_{X,\gamma} (F_d, s)$ denote the Mellin transform of $\mathcal{E}_{X,\gamma} (F_d, t)$, $t > 0$ with appropriate $s \in \mathbb{C}$.

We use the notation in Subsection 3.5. Let $[\Gamma]$ denote the set of the conjugacy classes in $\Gamma$. By applying the Selberg’s trace formula to $Z = \Gamma \setminus X$, we get
\begin{equation}
\Tr_s[\big( N^\lambda (T^* Z) - \frac{m}{2} \exp(-tD^{Z,F_d,2}/2) \big)] = \sum_{[\gamma] \in [\Gamma]} \frac{\Vol(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \mathcal{E}_{X,\gamma} (F_d, t).
\end{equation}

Now we compare (1.0.21) with (1.0.23). Then a proof to (1.0.21) mainly includes the following three parts:

1. We show that if $[\gamma] \in E[\Gamma]$, then $\mathcal{M} \mathcal{E}_{X,\gamma} (F_d, s)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is holomorphic at $s = 0$. Thus we define
\begin{equation}
\mathcal{P}_\Sigma X,\gamma (F_d) = \frac{\partial}{\partial s} \big|_{s=0} \mathcal{M} \mathcal{E}_{X,\gamma} (F_d, s).
\end{equation}
Then $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ satisfies well the result (1) in Theorem 1.0.2. Such consideration extends to every elliptic element $\gamma \in G$.

2. If $\gamma \in G$ is elliptic, we show that $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ is a pseudopolynomial in $d$ for $d$ large enough. Moreover, if $\gamma$ is also of finite order, for instance, $[\gamma] \in E[\Gamma]$, then $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ is a nice pseudopolynomial.

3. We prove that all the terms in the big sum of (1.0.23) associated with nonelliptic $[\gamma] \in [\Gamma]$ contribute as $O(e^{-cd})$ in $\mathcal{T}(Z, F_d)$.

Indeed, to handle the contribution of the nonelliptic $[\gamma] \in [\Gamma]$, we use a spectral gap of $D_{Z,F_d}^{*}$ due to the nondegeneracy of $\lambda$. By [5, Théorème 3.2], [6, Theorem 4.4] which holds for a more general setting (cf. also [33, Proposition 7.5, Corollary 7.6] for a proof by using representation theory for symmetric spaces), there exist constants $C > 0$, $c > 0$ such that for $d \in \mathbb{N}$,

$$D_{Z,F_d}^{*} \geq cd^2 - C.$$  

That is why (1.0.4) holds for $d$ large enough. The Part 3 follows from a technical argument which makes good use of (1.0.25) and the fact that nonelliptic elements in $\Gamma$ admit a uniform positive lower bound for their displacement distances on $X$.

For elliptic $\gamma \in \Gamma$, we apply Bismut’s formula [3, Theorem 6.1.1] to evaluate $\mathcal{E}_{X,\gamma}(F_d,t)$. Then we can write $\mathcal{E}_{X,\gamma}(F_d,t)$ as a Gaussian-like integral with the integrand given as a product of an analytic function determined by the adjoint action of $\gamma$ on Lie algebras and the character $\chi_{E_d}$ of the representation $E_d$. By coordinating these two factors, especially using all sorts of character formulae for $\chi_{E_d}$, we can integrate it out. We show that $\mathcal{E}_{X,\gamma}(F_d,t)$ is a finite sum of the terms as follows,

$$t^{-j} \frac{1}{2} e^{-t(cd+b)^2} Q(d),$$

where $j \in \mathbb{N}$, $c \neq 0$, $b$ are real constants, and $Q(d)$ is a (nice) pseudopolynomial in $d$. It is crucial that $c \neq 0$. Indeed, we will see in Subsection 7.1 that this quantity $c$ measures the difference between the representations $(E_{\lambda}, \rho^{E_{\lambda}})$ and $(E_{\lambda}, \rho^{E_{\lambda}} \circ \theta)$.

By (1.0.26), $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$ is well-defined in (1.0.24), which is a (nice) pseudopolynomial in $d$ (for $d$ large enough). The details on these computations are carried out in Subsections 7.3 and 7.5, where we apply the techniques inspired by the computations in Shen’s approach to the Fried conjecture [40, Section 7].

The formula (1.0.19) is a refined version of the above results on $\mathcal{P}\mathcal{E}_{X,\gamma}(F_d)$, since each $\mathcal{P}\mathcal{I}_{X(k)}(F_{d,k})$ is already well understood. For proving it, we apply a geometric localization formula for $\mathcal{E}_{X,\gamma}(F_d,t)$ as follows.

**Theorem 1.0.3.** Assume that $\delta(G) = 1$. We use the same notation as in Theorem 1.0.2. Let $\gamma \in \Gamma$ be semisimple, up to a conjugation, we can write $\gamma$ uniquely as a commuting product of its elliptic part $k \in T$ and its hyperbolic part $\gamma_h \in G$. Then there exists a constant $c(\gamma) > 0$ such that for $t > 0$, $d \in \mathbb{N}$,

$$\mathcal{E}_{X,\gamma}(F_d,t) = c(\gamma) \sum_{\sigma \in W_{\gamma_h}(k)} \varphi_{k}^{U}(\sigma, d\lambda + \lambda_0) \mathcal{E}_{X(k),\gamma_h}(F_{\sigma,d,k}^{k}, t).$$

The above theorem will be restated as Theorem 6.0.1. When $\gamma = k \in T$, then $\gamma_h = 1$ and $c(\gamma) = 1$. Then (1.0.27) reduces to

$$\mathcal{E}_{X,\gamma}(F_d,t) = \sum_{\sigma \in W_{\gamma_h}(k)} \varphi_{k}^{U}(\sigma, d\lambda + \lambda_0) \mathcal{I}_{X(k)}(F_{\sigma,d,k}^{k}, t).$$

After taking the Mellin transform on both sides of (1.0.28), we get exactly (1.0.19).

Our approach to Theorem 1.0.3 is a more delicate application of Bismut’s formula [3, Theorem 6.1.1]. As we said before, $\mathcal{E}_{X,\gamma}(F_d,t)$, $\mathcal{E}_{X(k),\gamma_h}(F_{\sigma,d,k}^{k}, t)$ are equal to...
integrals of some integrands involving $\chi_{E_d}$, $\chi_{E^k_{\sigma,d}}$ respectively. To relate the both sides of (1.0.27), we employ a generalized version of Kirillov character formula (cf. Theorem 5.4.4) which gives an explicit way of decomposing $\chi_{E_d}(U(k)\sigma)$ into a sum of $\chi_{E^k_{\sigma,d}}$, $\sigma \in W_U(k)$. This character formula was proved by Duflo, Heckman and Vergne in [15, Theorem (7)] under a general setting, and we will recall its special case for our need in Subsection 5.4. Then we expand the integral for $E_{X,\gamma}(F_d,t)$ carefully, after replacing $\chi_{E_d}(U(k)\sigma)$ by the sum of $\chi_{E^k_{\sigma,d}}$, we rewrite this big integral as a sum of certain small integrals. Then we verify that each small integral is exactly some $E$ of $\chi_{E_d}(U(k)\sigma)$ by Bismut’s formula. This way, we prove (1.0.27).

Theorem 1.0.3 can be interpreted as follows, the elliptic part $k$ in $\gamma$ could lead to a geometric localization onto its fixed point set $X(k)$ when we evaluate the orbital integrals. Even though we thought only prove it for a very restrictive situation, we still expect such phenomenon in general due to a geometric formulation for the semisimple orbital integrals (cf. [3, Chapter 4], also Subsection 3.3).

Finally, we note that in [6, Section 8], the authors explained well how to use Bismut’s formula for semisimple orbital integrals to study the asymptotic analytic torsion. Here, we go one step further in that direction to get a refined evaluation on it. Bergeron and Venkatesh [2] also studied the asymptotic analytic torsion but under a totally different setting. In [25, Section 7], the asymptotic equivariant analytic torsion for a locally symmetric space was studied, and the oscillating terms also appeared naturally in that case.

This paper is organized as follows. In Section 2, we recall the definition of Ray-Singer analytic torsion for compact orbifolds. We also include a brief introduction to the orbifolds at beginning.

In Section 3, we introduce the explicit geometric formula of Bismut for semisimple orbital integrals and the Selberg’s trace formula for compact locally symmetric orbifolds. They are the main tools to study the analytic torsions in this paper.

In Section 4, we give a vanishing theorem for $T(Z,F)$, so that we only need to focus on the case $\delta(G) = 1$.

In Section 5, we study the Lie algebra of $G$ provided $\delta(G) = 1$. Furthermore, we introduce a generalized Kirillov formula for compact Lie groups.

In Section 6, we prove Theorem 1.0.3.

In Section 7, given the sequence $\{F_d\}_{d \in \mathbb{N}}$, we compute explicitly $E_{X,\gamma}(F_d,t)$ in terms of root data for elliptic $\gamma$, in particular, we prove (1.0.26). Then we give the formulae for $PE_{X,\gamma}(F_d)$.

Finally, in Section 8, we introduce the spectral gap (1.0.25) and we give a proof to Theorem 1.0.2.

In this paper, if $V$ is a real vector spaces and if $E$ is a complex vector space, we will use the symbol $V \otimes E$ to denote the complex vector space $V \otimes_{\mathbb{R}} E$. If both $V$ and $E$ are complex vector spaces, then $V \otimes E$ is just the usual tensor over $\mathbb{C}$.

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2. Ray-Singer analytic torsion

In this section, we recall the definitions of the orbifold and the orbifold vector bundle. We also refer to [37, 38] and [1, Chapter 1] for more details. Then we recall
the definition of Ray-Singer analytic torsion for compact orbifolds, where we refer to [27], [41] for more details.

2.1. Orbifolds and orbifold vector bundles. Let $Z$ be a topological space.

**Definition 2.1.1.** If $U$ is a connected open subset of $Z$, an orbifold chart for $U$ is a triple $(\tilde{U}, \pi_U, G_U)$ such that

- $\tilde{U}$ is a connected open set of some $\mathbb{R}^m$, $G_U$ is a finite group acting smoothly and effectively on $\tilde{U}$ on the left;
- $\pi_U$ is a continuous surjective $\tilde{U} \to U$, which is invariant by $G_\tilde{U}$-action;
- $\pi_U$ induces a homeomorphism between $G_\tilde{U}\backslash\tilde{U}$ and $U$.

If $V \subset U$ is a connected open subset, an embedding of orbifold chart for the inclusion $i : V \to U$ is an orbifold chart $(\tilde{V}, \pi_V, G_V)$ for $V$ and an orbifold chart $(\tilde{U}, \pi_U, G_U)$ for $U$ together with a smooth embedding $\phi_{UV} : \tilde{V} \to \tilde{U}$ such that the following diagram commutes,

$$
\begin{array}{ccc}
\tilde{V} & \xrightarrow{\phi_{UV}} & \tilde{U} \\
\downarrow{\pi_V} & & \downarrow{\pi_U} \\
V & \xrightarrow{i} & U
\end{array}
$$

(2.1.1)

If $U_1, U_2$ are two connected open subsets of $Z$ with the charts $(\tilde{U}_1, \pi_{U_1}, G_{U_1})$, $(\tilde{U}_2, \pi_{U_2}, G_{U_2})$ respectively. We say that these two orbifold charts are compatible if for any point $z \in U_1 \cap U_2$, there exists an open connected neighborhood $V \subset U_1 \cap U_2$ of $z$ with an orbifold chart $(\tilde{V}, \pi_V, G_V)$ such that there exist two embeddings of orbifold charts $\phi_{U_1 V} : (\tilde{V}, \pi_V, G_V) \to (\tilde{U}_1, \pi_{U_1}, G_{U_1})$, $\phi_{U_2 V} : (\tilde{V}, \pi_V, G_V) \to (\tilde{U}_2, \pi_{U_2}, G_{U_2})$. In this case, the diffeomorphism $\phi_{U_2 V} \circ \phi_{U_1 V}^{-1} : \phi_{U_1 V}(\tilde{V}) \to \phi_{U_2 V}(\tilde{V})$ is called a coordinate transformation.

**Definition 2.1.2.** An orbifold atlas on $Z$ is couple $(\mathcal{U}, \tilde{\mathcal{U}})$ consisting of a cover $\mathcal{U}$ of open connected subsets of $Z$ and a family of compatible orbifold charts $\tilde{\mathcal{U}} = \{ (\tilde{U}, \pi_U, G_U) \}_{U \in \mathcal{U}}$.

An orbifold atlas $(\mathcal{V}, \tilde{\mathcal{V}})$ is called a refinement of $(\mathcal{U}, \tilde{\mathcal{U}})$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if every orbifold chart in $\tilde{\mathcal{V}}$ has an embedding into some orbifold chart in $\tilde{\mathcal{U}}$.

Two orbifold atlas are said to be equivalent if they have a common refinement, and the equivalent class of an orbifold atlas is called an orbifold structure on $Z$.

An orbifold is a second countable Hausdorff space equipped with an orbifold structure. It is said to have dimension $m$ if all the orbifold charts which defines the orbifold structure are of dimension $m$.

If $Z, Y$ are two orbifolds, a smooth map $f : Z \to Y$ is a continuous map from $Z$ to $Y$ such that it lifts locally to an equivariant smooth map from an orbifold chart of $Z$ to an orbifold chart of $Y$. In this way, we can define the notion of smooth functions and the smooth action of Lie groups.

By [41, Proposition 2.12], if $\Gamma$ is discrete group acting smoothly and properly discontinuously on the left on an orbifold $X$, then $Z = \Gamma \backslash X$ has a canonical orbifold structure induced from $X$.

In the sequel, let $Z$ be an orbifold with an orbifold structure given by $(\mathcal{U}, \tilde{\mathcal{U}})$. If $z \in Z$, there exists an open connected neighborhood $U_z$ of $z$ with a compatible orbifold chart $(\tilde{U}_z, G_z, \pi_z)$ such that $\pi_z^{-1}(z)$ contains only one point $x \in \tilde{U}_z$. Then $G_z$ does not depend on the choice of such open connected neighborhood (up to canonical isomorphisms compatible with the orbifold structure), then $G_z$ is called the local group at $z$. 

Put

$$(2.1.2) \quad Z_{\text{reg}} = \{ z \in Z : G_z = \{ 1 \} \}, \quad Z_{\text{sing}} = \{ z \in Z : G_z \neq \{ 1 \} \}.$$ 

Then $Z_{\text{reg}}$ is naturally a smooth manifold. But $Z_{\text{sing}}$ is not necessarily an orbifold. In [19, Section 2], the author provided two different methods to view $Z_{\text{sing}}$ as an immersed image of a disjoint union of orbifolds. We just recall that method which appears naturally in Kawasaki's local index theorems for orbifolds [19, 20].

If $z \in Z_{\text{sing}}$, let $1 = (h^0_z), (h^1_z), \ldots, (h^l_z)$ be the conjugacy classes in $G_z$. Put

$$\Sigma Z = \{(z, (h^j_z)) \mid z \in Z_{\text{sing}}, j = 1, \ldots, l_i \}.$$ 

Let $(\tilde{U}_z, G_z, \pi_z)$ be the local orbifold chart for $z \in Z_{\text{sing}}$ such that $\pi_z^{-1}(z)$ contains only one point. For $j = 1, \ldots, l_i$, let $\tilde{U}^{h_j}_z \subset \tilde{U}_z$ be the fixed point set of $h^j_z$, which is a submanifold of $\tilde{U}_z$. Note that $\tilde{U}^{h_j}_z \subset Z_{\text{sing}}$. Let $Z_{G_z}(h^j_z)$ be the centralizer of $h^j_z$ in $G_z$. Then $Z_{G_z}(h^j_z)$ acts smoothly on $\tilde{U}^{h_j}_z$. Put

$$(2.1.3) \quad K^j_z = \ker(Z_{G_z}(h^j_z) \to \text{Aut}(\tilde{U}^{h_j}_z)).$$ 

Then $(\tilde{U}^{h_j}_z, Z_{G_z}(h^j_z)/K^j_z, \pi^j_z : \tilde{U}^{h_j}_z \to \tilde{U}^{h_j}_z/Z_{G_z}(h^j_z))$ defines an orbifold chart near $(z, (h^j_z)) \in \Sigma Z$. They form an orbifold structure for $\Sigma Z$. Let $Z^i, i = 1, \ldots, l$ denote the connected component of the orbifold $\Sigma Z$.

The integer $m_i = |K^j_z|$ is called the multiplicity of $Z^i$ in $Z$. As in [41, Section 2.2], we put

$$(2.1.5) \quad Z^0 = Z, \quad m_0 = 1.$$ 

We say $E$ to be an orbifold vector bundle of rank $r$ on $Z$ if there exists a smooth map of orbifolds $\pi : E \to Z$ such that for any $U \in \mathcal{U}$ and $(\tilde{U}, G_U, \pi_U) \in \tilde{U}$, there exists an orbifold chart $(\tilde{U}^E, G^E_U, \pi^E_U)$ of $E$ such that $\tilde{U}^E$ is a vector bundle on $\tilde{U}$ of rank $r$ equipped an effective action of $G^E_U$ and $\pi^E_U(\tilde{U}^E) = \pi^{-1}(U)$. Moreover, there exists a surjective group morphism $\psi_U : G^E_U \to G_U$ such that the action of $G^E_U$ on $\tilde{U}$ is identified via $\psi_U$ with the action of $G_U$ on $\tilde{U}$. If we have an open embedding $\phi_U^V : (\tilde{V}, \pi_V, G_V) \to (\tilde{U}, \pi_U, G_U)$, we require that it lifts to the open embedding $\phi^E_U : (\tilde{V}^E, \pi^E_V, G^E_V) \to (\tilde{U}^E, \pi^E_U, G^E_U)$ of the orbifold charts of $E$ such that $\phi^E_U : \tilde{V}^E \to \tilde{U}^E$ is a morphism of vector bundles associated with the open embedding $\phi_U^V : \tilde{V} \to \tilde{U}$. If every $\psi_U : G^E_U \to G_U$ is an isomorphism of groups, we call $E$ a proper orbifold vector bundle on $Z$.

Note that if $E$ is proper, then the rank of $E$ can be extended to a locally constant function $\rho$ on $\Sigma Z$. The orbifold chart of $Z^i$ is given by the triples such as $(\tilde{U}^{h^j}_z, Z_{G_z}(h^j_z)/K^j_z, \pi^j_z : \tilde{U}^{h^j}_z \to \tilde{U}^{h^j}_z/Z_{G_z}(h^j_z))$. By the above definition of $E$, we have an orbifold chart $(\tilde{U}^E, G^E_U = G_U, \pi^E_U)$ such that $\tilde{U}^E$ is a $G_U$-equivariant vector bundle on $\tilde{U}$. Then for $x \in \tilde{U}^{h^j}_z, h^j_z$ acts on the fibres $\tilde{U}^E$ linearly, so that we can set $\rho(z, (h^j_z)) = \text{Tr}(\tilde{U}^E[h^j_z])$. One can verify this way, $\rho$ is really a locally constant function on $\Sigma Z$. For $i = 1, \ldots, l$, let $\rho_i$ be the value of $\rho$ on the component $Z^i$. We also put $\rho_0 = r$.

We call $s : Z \to E$ a smooth section of $E$ over $Z$ if it is a smooth map between orbifolds such that $s \circ s = \text{Id}_Z$. We will use $C^\infty(Z, E)$ to denote the vector space of smooth sections of $E$ over $Z$.

Take an orbifold chart $(\tilde{U}, G_U, \pi_U) \in \tilde{U}$ of $Z$, $G_U$ acts canonically on the tangent vector bundle $T\tilde{U}$ of $\tilde{U}$. The open embeddings of orbifold charts of $Z$ also lift to the open embeddings of their tangent vector bundles. This way, we get a proper
orbifold vector bundle $TZ$ on $Z$, and the projection $\pi : TZ \to Z$ is just given by the obvious projection $TU \to U$. We call $TZ$ the tangent vector bundle of $Z$. If we equipped $TZ$ with Euclidean metric $g^{TZ}$, we will call $Z$ a Riemannian orbifold and call $g^{TZ}$ a Riemannian metric of $Z$.

Let $\Omega(Z)$ denote the set of smooth differential forms of $Z$, which has a $Z$-graded structure by degrees. The de Rham operator $d^Z : \Omega(Z) \to \Omega^{+1}(Z)$ is given by the family of de Rham operator $d^U : \Omega(U) \to \Omega^{+1}(U)$. Then we can define the de Rham complex $(\Omega(Z), d^Z)$ of $Z$ and the associated de Rham cohomology $H(Z, \mathbb{R})$.

By [19, Section 1], there is a natural isomorphism between $H(Z, \mathbb{R})$ and the singular cohomology of the underlying topological space $Z$.

The Chern-Weil theory on the characteristic forms extends to orbifolds. We refer to [41, Section 3] for more details. Note that, as in [19, 20], the characteristic forms are not only defined on $Z$ but also defined on $\Sigma Z$. The part $\Sigma Z$ has a nontrivial contribution in Kawasaki’s local index theorems for orbifolds.

Finally, let us recall the integrals on $Z$. Assume that $Z$ is compact. We may take a finite open covering $\{U_i\}_{i \in I}$ of the precompact orbifold charts for $Z$. Since $Z$ is Hausdorff, then there exists a partition of unity subordinate to this open cover.

We find a family of smooth functions $\{\phi_i \in C_c^\infty(Z)\}_{i \in I}$ with values in $[0, 1]$ such that $\text{Supp}(\phi_i) \subset U_i$, and that

$$\sum_{i \in I} \phi_i = 1.$$  

Take $\tilde{\phi}_i = \pi^*_U(\phi_i) \in C_c^\infty(\tilde{U}_i)^{G_{U_i}}$.

If $\alpha \in \Omega^m(Z, o(TZ))$, let $\tilde{\alpha}_{U_i}$ be its lift on the chart $(\tilde{U}_i, \pi_{U_i}, G_{U_i})$. We define

$$\int_Z \alpha = \sum_{i \in I} \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\phi}_i \tilde{\alpha}_{U_i}.$$  

By [41, Section 3.2], if $\alpha \in \Omega^m(Z, o(TZ))$, then $\alpha$ is also integrable on $Z_{\text{reg}}$, so that

$$\int_Z \alpha = \int_{Z_{\text{reg}}} \alpha.$$  

Also if $\alpha \in \Omega(Z, o(TZ))$, we have

$$\int_Z d^Z \alpha = 0.$$  

If $(Z, g^{TZ})$ is a Riemannian orbifold, we can define the integration on $Z$ with respect to the Riemannian volume element. If we have a Hermitian orbifold vector bundle $(F, h^F) \to (Z, g^{TZ})$, one can define the $L^2$ scalar product for the space of continuous sections of $F$ as usual. Then we get the Hilbert space $L^2(Z, F)$. If we also have a connection $\nabla^F$, one also can define the Sobolev spaces of sections of $F$ with respect to $\nabla^F$ and $\nabla^{TZ}$. These constructions are parallel to the case of smooth manifolds. We refer [41, Section 3] for more details.

Now we introduce the orbifold Euler characteristic number of $(Z, g^{TZ})$ [38]. The Euler form $e(TZ, \nabla^{TZ}) \in \Omega^m(Z, o(TZ))$ is given by the family of closed forms

$$\{e(\tilde{U}_i, g^{TZ_{\tilde{U}_i}}) \in \Omega^m(\tilde{U}_i, o(T\tilde{U}_i))^{G_{U_i}}\}_{U_i \in \mathcal{U}}.$$  

If $Z$ is oriented, then we can view $e(TZ, \nabla^{TZ})$ as a differential form on $Z$.

If $Z$ is compact, set

$$\chi_{\text{orb}}(Z) = \int_Z e(TZ, \nabla^{TZ}).$$  

By [38, Section 3], $\chi_{\text{orb}}(Z)$ is a rational number, and it vanishes when $Z$ is odd dimensional.
2.2. Flat vector bundles and analytic torsions of orbifolds. If \((F, \nabla F)\) is an orbifold vector bundle over \(Z\) with a connection \(\nabla F\), we call \((F, \nabla F)\) a flat vector bundle if the curvature \(R^F = \nabla F^2\) vanishes identically on \(Z\). A detailed discussion for the flat vector bundles on \(Z\) is given in [41, Section 2.5].

Let \((Z, g^{TZ})\) be a compact Riemannian orbifold of dimension \(m\). Let \((F, \nabla F)\) be a flat complex orbifold vector bundle of rank \(r\) on \(Z\) with Hermitian metric \(h^F\). Note that we do not assume that \(F\) is proper.

Let \(\Omega \left( Z, F \right)\) be the set of smooth sections of \(\Lambda \left( T^* Z \right) \otimes F\) on \(Z\). Let \(d^Z\) be the exterior differential acting on \(\Omega \left( Z, \mathbb{R} \right)\).

**Definition 2.2.1.** For \(i = 0, 1, \ldots, m\), if \(\alpha \in \Omega^i \left( Z, \mathbb{R} \right)\), \(s \in C^\infty \left( Z, F \right)\), the operator \(d^Z, F\) acting on \(\Omega^i \left( Z, F \right)\) is defined by

\[
\langle d^Z, F \rangle_2 \left( \alpha \otimes s \right) = \langle d^Z \alpha \rangle \otimes s + \langle -1 \rangle^i \alpha \wedge \nabla F s \in \Omega^{i+1} \left( Z, F \right).
\]

Since \(\nabla F^2 = 0\), then \((\Omega \left( Z, F \right), \langle \cdot, \cdot \rangle)\) is a complex, which is called the de Rham vector bundle \((F, \nabla F)\) on \(Z\). Let \(H \left( Z, F \right)\) be its cohomology, which is called the de Rham cohomology of \(Z\) valued in \(F\), as in the case of closed manifolds, \(H \left( Z, F \right)\) is always finite dimensional. Let \(\langle \cdot, \cdot \rangle_{\Lambda \left( T^* Z \right) \otimes F, z}\) be the Hermitian metric on \(\Lambda \left( T^* Z \right) \otimes F\), \(z \in Z\) induced by \(g^{TZ}\) and \(h^F\). Let \(dv\) be the Riemannian volume element on \(Z\) induced by \(g^{TZ}\).

The \(L_2\)-scalar product on \(\Omega \left( Z, F \right)\) is given as follows, if \(s, s' \in \Omega \left( Z, F \right)\), then

\[
\langle s, s' \rangle_{L_2} = \int_Z \langle s(z), s'(z) \rangle_{\Lambda \left( T^* Z \right) \otimes F, z} dv(z).
\]

By (2.1.8), it will be the same if we take the integrals on \(Z\).

Let \(d^Z, F^* \) be the formal adjoint of \(d^Z, F\) with respect to the \(L_2\)-metric on \(\Omega \left( Z, F \right)\), i.e., for \(s, s' \in \Omega \left( Z, F \right)\),

\[
\langle d^Z, F^*, s \rangle_{L_2} = \langle s, d^Z, F s' \rangle_{L_2}.
\]

Then \(d^Z, F^*\) is a first-order differential operator acting \(\Omega \left( Z, F \right)\) on which decreases the degree by 1.

**Definition 2.2.2.** The de Rham - Hodge operator \(D^Z, F\) of \(\Omega \left( Z, F \right)\) is defined as

\[
D^Z, F = d^Z, F + d^Z, F^*.
\]

It is a first-order self-adjoint elliptic differential operator acting on \(\Omega \left( Z, F \right)\).

The Hodge Laplacian is

\[
D^{F, Z, 2} = [d^Z, F, d^Z, F^*] = d^Z, F d^Z, F^* + d^Z, F^* d^Z, F.
\]

Here, \([\cdot, \cdot]\) denotes the supercommutator. Then \(D^{Z, F, 2}\) is a second-order essentially self-adjoint positive elliptic operator, which preserves the degree. Let \(H^2(Z, F)\) be the Sobolev space of the bundle \(\Lambda \left( T^* Z \right) \otimes F\) of order 2 with respect to \(\nabla^{TZ}\), \(\nabla F\).

Then the domain of the self-adjoint extension of \(D^{F, Z, 2}\) is just \(H^2(Z, F)\).

The Hodge decomposition for \(\Omega \left( Z, F \right)\) still holds in this case (cf. [27, Proposition 2.2], [13, Proposition 2.1]). We have the following orthogonal decomposition,

\[
\Omega \left( Z, F \right) = \text{ker} \ D^{Z, F, 2} \oplus \text{Im} \left( d^Z, F \right)_{\Omega \left( Z, F \right)} \oplus \text{Im} \left( d^Z, F^* \right)_{\Omega \left( Z, F \right)}.
\]

Then we have the canonical identification of vector spaces,

\[
\text{ker} \ D^{Z, F, 2} \simeq H \left( Z, F \right).
\]

For \(i = 0, 1, \ldots, m\), let \(D_i^{Z, F, 2}\) denote the restriction of \(D^{Z, F, 2}\) to \(\Omega^i \left( Z, F \right)\). Let \(\mathcal{H}^i \left( Z, F \right)\) be the kernel of \(D_i^{Z, F, 2}\), whose elements are called harmonic forms of degree \(i\). By Hodge theory, we have the canonical isomorphism of finite dimensional vector spaces for \(i = 0, 1, \ldots, m\),

\[
\text{ker} \ D_i^{Z, F, 2} \simeq H^i \left( Z, F \right) \simeq \mathcal{H}^i \left( Z, F \right).
\]
Put $\mathcal{H}(Z, F) = \oplus_{i=0}^m \mathcal{H}^i(Z, F) \subset \Omega(Z, F)$. We have
\begin{equation}
(2.2.9)
\ker D^Z,F = \mathcal{H}(Z, F).
\end{equation}

Put
\begin{equation}
(2.2.10)
\chi(Z, F) = \sum_{j=0}^{m} (-1)^j \dim H^j(Z, F), \quad \chi'(Z, F) = \sum_{j=0}^{m} (-1)^j j \dim H^j(Z, F).
\end{equation}

If $F$ is proper, recall that the numbers $\rho_i, \ i = 0, \ldots, l$ are defined in previous subsection as the extension of the rank of $F$. Then by [41, Theorem 4.3], we have
\begin{equation}
(2.2.11)
\chi(Z, F) = \sum_{i=0}^{1} \rho_i \frac{\chi_{\text{orb}}(Z_i)}{m_i}.
\end{equation}

The right-hand side of (2.2.11) contains the nontrivial contributions from $\Sigma Z$.

The spectrum (with multiplicities) of $D_i^{Z,F,2}$ is of the form
\begin{equation}
(2.2.12)
0 \leq \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots \to +\infty.
\end{equation}

If $s \in \mathbb{C}$, let $\Re(s) \in \mathbb{R}$ denote its real part.

**Definition 2.2.3.** If $i = 0, 1, \ldots, m$, the zeta function of $D_i^{Z,F,2}$ acting on $\Omega^{i}(Z, F)$ is defined as follows, if $s \in \mathbb{C}$ is such that $\Re(s)$ is big enough,
\begin{equation}
(2.2.13)
\vartheta_i(F)(s) = -\sum_{\lambda \in \text{Spec}(D_i^{Z,F,2})} \frac{1}{\lambda^s}.
\end{equation}

Let $\Gamma(s)$ be the Gamma function for $s \in \mathbb{C}$. For $t > 0$, let $\exp(-t D_i^{Z,F,2})$ be the heat operator associated with $D_i^{Z,F,2}$. Then $\exp(-t D_i^{Z,F,2}/2)$ is of trace-class. Let $(D_i^{Z,F,2})^{-1}$ be the inverse of $D_i^{Z,F,2}$ acting on the orthogonal subspace of $\mathcal{H}^i(Z, F)$ in $\Omega^i(Z, F)$. Let $P_i$ denote the orthogonal projection from $\Omega^i(Z, F)$ onto $\mathcal{H}^i(Z, F)$, then we can rewrite (2.2.13) as
\begin{equation}
(2.2.14)
\vartheta_i(F)(s) = -\text{Tr}[\exp(\{D_i^{Z,F,2}\})^{-1}] = \frac{1}{(s)} \int_0^{+\infty} \text{Tr}[\exp(-t D_i^{Z,F,2})(1 - P_i)] t^{s-1} dt.
\end{equation}

By the standard heat equation method, one can see that $\vartheta_i(F)(s)$ is well-defined for $\Re(s) > \frac{m}{2}$, and that $\vartheta_i(F)(s)$ admits a meromorphic extension to $s \in \mathbb{C}$ which is holomorphic at $s = 0$. We also denote by $\vartheta_i(s)$ this meromorphic extension. Then
\begin{equation}
(2.2.15)
\frac{d}{ds}|_{s=0} \vartheta_i(F)(s) \in \mathbb{R}.
\end{equation}

**Definition 2.2.4.** Let $T(g^{TZ}, \nabla^F, h^F) \in \mathbb{R}$ be given by
\begin{equation}
(2.2.16)
T(g^{TZ}, \nabla^F, h^F) = \sum_{i=1}^{m} (-1)^i \frac{d}{ds}|_{s=0} \vartheta_i(F)(s).
\end{equation}

The quantity $T(g^{TZ}, \nabla^F, h^F)$ is called Ray-Singer analytic torsion associated with $(F, \nabla^F, h^F)$.

If $m$ is even and if $Z$ is orientable, then $T(g^{TZ}, \nabla^F, h^F) = 0$. If $m$ is odd, then $T(g^{TZ}, \nabla^F, h^F)$ is independent of the metrics $g^{TZ}$ and $h^F$.

Let $P = \oplus_i P_i$ be the orthogonal projection from $\Omega(Z, F)$ to $\mathcal{H}(Z, F)$. Let $\mathcal{H}^\perp$ denote the orthogonal subspace of $\mathcal{H}(Z, F)$ in $\Omega(Z, F)$, and let $(D^Z,F,2)^{-1}$ be the inverse of $D^Z,F,2$ acting on $\mathcal{H}^\perp$. Let $N^\Lambda(T^*Z)$ be the number operator on $\Lambda(T^*Z)$ which acts on $\Lambda^i(T^*Z)$ by multiplication of $i$. 

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Put
\[(2.2.17) \quad \vartheta(F)(s) = \sum_{i=1}^{m} (-1)^{i} i \vartheta_{i}(F)(s).\]

By (2.2.14), (2.2.17), we get that if \(\Re(s)\) is big enough, then
\[(2.2.18) \quad \vartheta(F)(s) = -\text{Tr}_{s}^{} [N^{\Lambda}(T^{\ast}Z)(D^{Z,F,2})^{-s}] = \int_{0}^{+\infty} \text{Tr}_{s} [N^{\Lambda}(T^{\ast}Z) \exp(-tD^{Z,F,2}) (1 - P)] t^{s-1} dt.\]

By (2.2.13), (2.2.16), we get
\[(2.2.19) \quad T(g^{TZ}, \nabla F, h^{F}) = \frac{d}{ds} \vartheta(F)(s).\]

For \(t > 0\), as in [4, eq.(1.8.5)], put
\[(2.2.20) \quad b_{t}(g^{TZ}, F) = (1 + 2t \frac{\partial}{\partial t}) \text{Tr}_{s} \left( (N^{\Lambda}(T^{\ast}Z) - \frac{m}{2}) \exp(-tD^{Z,F,2}/2) \right).\]

Then \(b_{t}(g^{TZ}, F)\) is a smooth function in \(t > 0\).

By [4, Eqs.(1.8.7) & (1.8.8)] and [41, Subsection 4.3], as \(t \to 0\),
\[(2.2.21) \quad b_{t}(g^{TZ}, F) = O(1/\sqrt{t}).\]

As \(t \to +\infty\),
\[(2.2.22) \quad b_{t}(g^{TZ}, F) = \frac{1}{2} \chi'(Z, F) - \frac{m}{4} \chi(Z, F) + O(1/\sqrt{t}).\]

Set
\[(2.2.23) \quad b_{\infty}(g^{TZ}, F) = \frac{1}{2} \chi'(Z, F) - \frac{m}{4} \chi(Z, F).\]

By [4, Eq.(1.8.11)] and [41, Corollary 4.14], we have
\[(2.2.24) \quad T(g^{TZ}, \nabla F, h^{F}) = -\int_{0}^{1} b_{t}(g^{TZ}, F) \frac{dt}{t} - \int_{1}^{+\infty} (b_{t}(g^{TZ}, F) - b_{\infty}(g^{TZ}, F)) \frac{dt}{t} - (\Gamma'(1) + \log(2) - 2)b_{\infty}(g^{TZ}, F).\]

In particular, if \((F, \nabla F)\) is acyclic, i.e. \(H^{}(Z, F) = 0\), then \(b_{\infty}(g^{TZ}, F) = 0\), and
\[(2.2.25) \quad T(g^{TZ}, \nabla F, h^{F}) = -\int_{0}^{+\infty} b_{t}(g^{TZ}, F) \frac{dt}{t}.\]

3. Orbital integrals and locally symmetric spaces

In this section, we recall some geometric properties of the symmetric space \(X\),
and we recall an explicit geometric formula of Bismut [3, Chapter 6] for semisimple orbital integrals. Then, given a cocompact discrete subgroup \(\Gamma \subset G\), we describe the orbifold structure on \(Z = \Gamma \setminus X\), and we deduce in detail the Selberg’s trace formula for \(Z\).

In this section, \(G\) is taken as a connected linear real reductive Lie group, we do not require that it has a compact center. Then \(X\) is a symmetric space which may have de Rham components of both noncompact type and Euclidean type.
3.1. **Real reductive Lie group.** Let $G$ be a connected linear real reductive Lie group with Lie algebra $\mathfrak{g}$, and let $\theta \in \text{Aut}(G)$ be a Cartan involution. Let $K$ be the fixed point set of $\theta$ in $G$. Then $K$ is a maximal compact subgroup of $G$, and let $\mathfrak{k}$ be its Lie algebra. Let $\mathfrak{p} \subset \mathfrak{g}$ be the eigenspace of $\theta$ associated with the eigenvalue $-1$. The Cartan decomposition of $\mathfrak{g}$ is given by
\begin{equation}
\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}.
\end{equation}

Put $m = \dim \mathfrak{p}$, $n = \dim \mathfrak{k}$.

Let $B$ be a $G$- and $\theta$-invariant nondegenerate symmetric bilinear form on $\mathfrak{g}$, which is positive on $\mathfrak{p}$ and negative on $\mathfrak{k}$. It induces a symmetric bilinear form $B^*$ on $\mathfrak{g}^*$, which extends to a symmetric bilinear form on $\Lambda^2 (\mathfrak{g}^*)$. The $K$-invariant bilinear form $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$ is a scalar product on $\mathfrak{g}$, which extends to a scalar product on $\Lambda (\mathfrak{g}^*)$. We will use $|\cdot|$ to denote the norm under this scalar product.

Let $\mathfrak{u} \mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}$. Let $C^0 \subset U \mathfrak{g}$ be the Casimir element associated with $B$, i.e., if $\{e_i\}_{i=1, \ldots, m+n}$ is a basis of $\mathfrak{g}$, and if $\{e_i^*\}_{i=1, \ldots, m+n}$ is the dual basis of $\mathfrak{g}$ with respect to $B$, then
\begin{equation}
C^0 = - \sum e_i^* e_i.
\end{equation}

We can identify $U \mathfrak{g}$ with the algebra of left-invariant differential operators over $G$, then $C^0$ is a second-order differential operator, which is $\text{Ad}(G)$-invariant.

Let $\mathfrak{z}_G \subset \mathfrak{g}$ be the center of $\mathfrak{g}$. Put
\begin{equation}
\mathfrak{g}_{ss} = [\mathfrak{z}_G, \mathfrak{g}].
\end{equation}

Then
\begin{equation}
\mathfrak{g} = \mathfrak{z}_G \oplus \mathfrak{g}_{ss}.
\end{equation}

They are orthogonal with respect to $B$.

Let $\mathcal{Z}_G$ be the center of $G$, let $\mathcal{G}_{ss}$ be the analytic subgroup of $G$ associated with $\mathfrak{g}_{ss}$. Then $G$ is the commutative product of $\mathcal{Z}_G$ and $\mathcal{G}_{ss}$, in particular,
\begin{equation}
G = \mathcal{Z}_G^0 \mathcal{G}_{ss}.
\end{equation}

Let $i = \sqrt{-1}$ denote one square root of $-1$. Put
\begin{equation}
\mathfrak{u} = \sqrt{-1} \mathfrak{p} \oplus \mathfrak{k}.
\end{equation}

For saving notation, if $a \in \mathfrak{p}$, we use notation $ia \in \sqrt{-1} \mathfrak{p} \subset \mathfrak{u}$ to denote the corresponding vector.

Then $\mathfrak{u}$ is a (real) Lie algebra, which is called the compact form of $\mathfrak{g}$. Then
\begin{equation}
\mathfrak{g}_C = \mathfrak{u}_C.
\end{equation}

Let $G_C$ be the complexification of $G$ with Lie algebra $\mathfrak{g}_C$. Then $G$ is the analytic subgroup of $G_C$ with Lie algebra $\mathfrak{g}$. Let $U \subset G_C$ be the analytic subgroup associated with $\mathfrak{u}$. By [21, Proposition 5.3], if $G$ has compact center, i.e. $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{p} = \{0\}$, then $U$ is a compact Lie group and a maximal compact subgroup of $G_C$.

**Definition 3.1.1.** An element $\gamma \in G$ is said to be semisimple if there exists $g \in G$ such that
\begin{equation}
\gamma = g (e_{\alpha} k) g^{-1}, \quad a \in \mathfrak{p}, k \in K, \text{Ad}(k)a = a.
\end{equation}

We call $\gamma_h = g e_{\alpha} g^{-1}$, $\gamma_e = g k g^{-1}$ the hyperbolic, elliptic parts of $\gamma$. These two parts are uniquely determined by $\gamma$. If $\gamma_h = 1$, we say $\gamma$ to be elliptic, and if $\gamma_e = 1$, we say $\gamma$ to be hyperbolic.
Let $Z(\gamma)$ be the centralizer of $\gamma$ in $G$. If $v \in \mathfrak{g}$, let $Z(v) \subset G$ be the stabilizer of $v$ in $G$ via the adjoint action. Let $\mathfrak{z}(\gamma)$, $\mathfrak{z}(v)$ be the Lie algebras of $Z(\gamma)$, $Z(v)$ respectively.

If $\gamma = \gamma_k \gamma_e$ is semisimple, then

$$Z(\gamma) = Z(\gamma_k) \cap Z(\gamma_e), \quad Z(\gamma_k) = Z(\text{Ad}(g)a).$$

Correspondingly, we have

$$\mathfrak{z}(\gamma) = \mathfrak{z}(\gamma_k) \cap \mathfrak{z}(\gamma_e), \quad \mathfrak{z}(\gamma_k) = \mathfrak{z}(\text{Ad}(g)a).$$

By [22, Proposition 7.25], $Z(\gamma)$ is reductive. Set

$$\theta_g = C(g) \theta C(g^{-1}), \quad B_g(\cdot, \cdot) = B(\text{Ad}(g^{-1}) \cdot, \text{Ad}(g^{-1}) \cdot).$$

Then $\theta_g$ is a Cartan involution of $Z(\gamma)$ and $B_g$ is a nondegenerate symmetric bilinear form on $\mathfrak{z}(\gamma)$. Let $K(\gamma)$ be the fixed point set of $\theta_g$ in $Z(\gamma)$, then

$$K(\gamma) = Z(\gamma) \cap gKg^{-1}.$$

Moreover, $K(\gamma)$ is a maximal compact subgroup of $Z(\gamma)$, which meets every connected components of $Z(\gamma)$.

Let $\mathfrak{k}(\gamma) \subset \mathfrak{z}(\gamma)$ be the Lie algebra of $K(\gamma)$. Put

$$\mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \text{Ad}(g)\mathfrak{p}.$$

Then the Cartan decomposition of $\mathfrak{z}(\gamma)$ with respect to $\theta_g$ is given by

$$\mathfrak{z}(\gamma) = \mathfrak{k}(\gamma) \oplus \mathfrak{p}(\gamma).$$

Moreover, $B_g$ is positive on $\mathfrak{p}(\gamma)$, and negative on $\mathfrak{k}(\gamma)$. The splitting in (3.1.14) is orthogonal with respect to $B_g$.

### 3.2. Symmetric space

Set

$$X = G/K.$$

Then $X$ is a smooth manifold with the differential structure induced by $G$. By definition, $X$ is diffeomorphism to $\mathfrak{p}$.

Let $\omega^\theta \in \Omega^1(G, \mathfrak{g})$ be the canonical left-invariant 1-form on $G$. Then by (3.1.1),

$$\omega^\theta = \omega^\mathfrak{p} + \omega^\mathfrak{f}.$$

Let $p : G \to X$ denote the obvious projection. Then $p$ is a $K$-principal bundle over $X$. Then $\omega^\mathfrak{f}$ is a connection form of this principal bundle. The associated curvature form

$$\Omega^\mathfrak{f} = d\omega^\mathfrak{f} + \frac{1}{2}[[\omega^\mathfrak{f}, \omega^\mathfrak{f}]] = -\frac{1}{2}[[\omega^\mathfrak{p}, \omega^\mathfrak{p}]].$$

If $(E, \rho^E, h^E)$ is a finite dimensional unitary or Euclidean representation of $K$, then $\mathcal{F} = G \times_K E$ is a Hermitian or Euclidean vector bundle over $X$ with the unitary or Euclidean connection $\nabla^\mathcal{F}$ induced by $\omega^\mathfrak{f}$. In particular,

$$TX = G \times_K \mathfrak{p}.$$

The bilinear form $B$ restricting to $\mathfrak{p}$ gives a Riemannian metric $g^TX$, and $\omega^\mathfrak{f}$ induces the associated Levi-Civita connection $\nabla^TX$. Let $d(\cdot, \cdot)$ denote the Riemannian distance on $X$.

Let $C(G, E)$ denote the set of continuous map from $G$ into $E$, if $k \in K$, $s \in C(G, E)$, put

$$(k.s)(g) = \rho^E(k)s(gk).$$

Let $C_K(G, E)$ be the set of $K$-invariant maps in $C(G, E)$. Let $C(X, F)$ denote the continuous sections of $F$ over $X$. Then

$$C_K(G, E) = C(X, F).$$
Let $\text{Ric}$ the adjoint actions. Moreover, we can view not distinguish the heat kernel $p$ (3.2.7).

Then $C\text{an}$ operator $B\text{ismut’s formula for semisimple orbital integrals.}$

Let $C\text{t} \in \text{End}(p), C\text{t}\mathfrak{t} \in \text{End}(\mathfrak{t})$ be the actions of Casimir $C\text{t}$ acting on $p, \mathfrak{t}$ via the adjoint actions. Moreover, we can view $C\text{t} \in \text{End}(\text{End}(T X)).$

Let Ric$^X$ denote the Ricci curvature of $(X, g^{TX})$. By [3, Eq. (2.6.8)],

$$\text{Ric}^X = C\text{t} \in p.$$

If $A \in \text{End}(E)$ commutes with $K$, then it can be viewed a parallel section of $\text{End}(F)$ over $X$. Let $dx$ be the Riemannian volume element of $(X, g^{TX})$.

Definition 3.2.1. Let $L_X^X$ be the Bochner-like Laplacian acting on $C^\infty(X, F)$ given by

$$L_X^X = \frac{1}{2} C\text{g} - \frac{1}{16} \text{Tr}[C\text{t} \mathfrak{t}] = \frac{1}{4} \text{Tr}[C\text{t} \mathfrak{t}] + A.$$

For $t > 0$, $x, x' \in X$, let $p_t^X(x, x')$ denote its heat kernel with respect to $dx'$.

Since $L_X^X$ is $G$-invariant, then $p_t^X(x, x')$ lifts to a function $p_t^X(g, g')$ on $G \times G$ valued in $\text{End}(E)$ such that for $g'' \in G, k, k' \in K$,

$$p_t^X(g'' g, g'') = p_t^X(g, g'), p_t^X(k, g' k') = p_t^X(g, g') \rho_E(k^{-1}) p_t^X(g, g') \rho_E(k').$$

We set

$$p_t^X(g) = p_t^X(1, g).$$

Then $p_t^X$ is a $K \times K$-invariant smooth function on $G$ valued in $\text{End}(E)$. We will not distinguish the heat kernel $p_t^X(x, x')$ and the function $p_t^X(g)$ in the sequel.

3.3. Bismut’s formula for semisimple orbital integrals. The group $G$ acts on $X$ isometrically. If $\gamma \in G$, for $x \in X$, put

$$d_\gamma(x) = d(x, \gamma x).$$

It is called the displacement function associated with $\gamma$, which is a continuous convex function on $X$. Moreover, $d_\gamma^2$ is a smooth convex function on $X$. By [16, Definition 2.19.21] and [3, Theorem 3.1.2], $\gamma$ is semisimple if and only if $d_\gamma$ can reach its minimum $m_\gamma \geq 0$ in $X$. In particular, $\gamma$ is elliptic if and only if $\gamma$ has fixed points in $X$. If $\gamma$ is semisimple, let $X(\gamma)$ be the minimizing set of $d_\gamma$, which is a geodesically convex submanifold of $X$.

If $\gamma \in G$ is semisimple, then there exists $g_\gamma \in G$ such that

$$\gamma = g_\gamma e^a k g_\gamma^{-1}, a \in p, k \in K, \text{Ad}(k)a = a.$$

By [3, Theorem 3.1.2],

$$m_\gamma = |a| = |\text{Ad}(g_\gamma)a|_{B_{p, k}}.$$  

Let $Z(\gamma)^0, K(\gamma)^0$ be the connected components of the identity of $Z(\gamma), K(\gamma)$ respectively. By [3, Theorem 3.3.1], $Z(\gamma)^0$ acts on $X(\gamma)$ isometrically and transitively. Moreover,

$$X(\gamma) \simeq Z(\gamma)/K(\gamma) = Z(\gamma)^0/K(\gamma)^0.$$

We equip the symmetric space $Z(\gamma)/K(\gamma)$ with the Riemannian metric induced from $B_{g_\gamma, p(\gamma)}$, then the above identifications are isometric.
Let $dg$ be the left-invariant Haar measure on $G$ induced by $(g, \langle \cdot, \cdot \rangle)$. Since $G$ is unimodular, then $dg$ is also right-invariant. Let $dk$ be the Haar measure on $K$ induced by $-B_t$, then

$$\tag{3.3.5} dg = dk.$$

Let $dy$ be the Riemannian volume element of $X(\gamma)$, and let $dz$ be the bi-invariant Haar measure on $Z(\gamma)$ induced by $B_g$. Let $dk(\gamma)$ be the Haar measure on $K(\gamma)$ such that

$$\tag{3.3.6} dz = dydk(\gamma).$$

Let $\text{Vol}(K(\gamma) \backslash K)$ be the volume of $K(\gamma) \backslash K$ with respect to $dk, dk(\gamma)$. In particular, we have

$$\tag{3.3.7} \frac{\text{Vol}(K(\gamma) \backslash K)}{\text{Vol}(K)} = \frac{\text{Vol}(K)}{\text{Vol}(K(\gamma))}.$$

Let $dv$ be the $G$-left invariant measure on $Z(\gamma) \backslash G$ such that

$$\tag{3.3.8} dg = dzdv.$$

By [3, Definition 4.2.2, Proposition 4.4.2], for $t > 0$, the orbital integral

$$\tag{3.3.9} \text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_\gamma)] = \frac{1}{\text{Vol}(K(\gamma) \backslash K)} \int_{Z(\gamma) \backslash G} \text{Tr}^F[p_X^Z(\gamma^{-1}\gamma t)]dv$$

is well-defined. As indicated by the notation, it only depends on the conjugacy class $[\gamma]$ of $\gamma$ in $G$.

In [3, Section 4.2], a geometric formula for $\text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_\gamma)]$ is established. We explain it as follows. Recall that $X(\gamma)$ is a totally geodesic submanifold of $X$ on which $Z(\gamma)$ acts isometrically and transitively. Let $N_{X(\gamma)/X}$ be the orthogonal normal bundle of $X(\gamma)$ in $X$, and let $N_{X(\gamma)/X}$ denote its total space. Then $N_{X(\gamma)/X} \cong X$ via the normal geodesics.

For $x \in X(\gamma)$, let $df$ be the Euclidean volume element on $N_{X(\gamma)/X,x}$. Then there exists a positive function $r(f)$ on $N_{X(\gamma)/X,x}$ such that $dx = r(f)dydf$. We have

$$\tag{3.3.10} \text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_\gamma)] = \int_{N_{X(\gamma)/X,x}} \text{Tr}^F[p_X^Z(\exp_x(f), \gamma \exp_x(f))\gamma]r(f)df.$$

It is clear that the right-hand side of (3.3.10) does not depend on the choice of $x \in X(\gamma)$. Because of this geometric interpretation for $\text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_\gamma)]$, we also call it a geometric orbital integral.

An explicit formula for $\text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_\gamma)]$ is given in [3, Theorem 6.1.1], and an extension to the wave operators of $\mathcal{L}^X_\gamma$ is given in [3, Section 6.3]. Now we describe in detail this formula of Bismut. We assume that

$$\tag{3.3.11} \gamma = e^a k, \ a \in \mathfrak{p}, \ k \in K, \ \text{Ad}(k)a = a.$$

Recall that $\mathfrak{z}(\gamma)$ is the Lie algebra of $Z(\gamma)^0 \subset G$. Put

$$\tag{3.3.12} \mathfrak{p}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{p}, \ \mathfrak{k}(\gamma) = \mathfrak{z}(\gamma) \cap \mathfrak{k}.$$

Then

$$\tag{3.3.13} \mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma).$$

Put

$$\tag{3.3.14} \mathfrak{z}_0 = \mathfrak{z}(a), \ \mathfrak{p}_0 = \ker \text{ad}(a) \cap \mathfrak{p}, \ \mathfrak{k}_0 = \ker \text{ad}(a) \cap \mathfrak{k}.$$

Let $\mathfrak{z}_0^\perp, \mathfrak{p}_0^\perp, \mathfrak{k}_0^\perp$ be the orthogonal vector spaces to $\mathfrak{z}_0, \mathfrak{p}_0, \mathfrak{k}_0$ in $\mathfrak{g}, \mathfrak{p}, \mathfrak{k}$ with respect to $B$. Then

$$\tag{3.3.15} \mathfrak{z}_0 = \mathfrak{p}_0 \oplus \mathfrak{k}_0, \ \mathfrak{z}_0^\perp = \mathfrak{p}_0^\perp \oplus \mathfrak{k}_0^\perp.$$. 
By [3, Eq.(3.3.6)],
\begin{equation}
\hat{g}(\gamma) = \mathfrak{g}_0 \cap \mathfrak{g}(k).
\end{equation}
Also \( p(\gamma), \mathfrak{t}(\gamma) \) are subspaces of \( \mathfrak{p}_0, \mathfrak{t}_0 \) respectively. Let \( \mathfrak{z}_0^\perp(\gamma), \mathfrak{p}_0^\perp(\gamma), \mathfrak{t}_0^\perp(\gamma) \) be the orthogonal spaces to \( \hat{g}(\gamma), p(\gamma), t(\gamma) \) in \( \mathfrak{g}_0, \mathfrak{p}_0, \mathfrak{t}_0 \). Then
\begin{equation}
\mathfrak{z}_0^0(\gamma) = \mathfrak{p}_0^0(\gamma) \oplus \mathfrak{t}_0^0(\gamma).
\end{equation}
Also the action \( \text{ad}(a) \) gives an isomorphism between \( \mathfrak{p}_0^0 \) and \( \mathfrak{t}_0^0 \).

For \( Y_0^4 \in \mathfrak{t}(\gamma) \), \( \text{ad}(Y_0^4) \) preserves \( p(\gamma), t(\gamma), \mathfrak{p}_0^0(\gamma), \mathfrak{t}_0^0(\gamma) \), and it is an antisymmetric endomorphism with respect to the scalar product.

Recall that the function \( \hat{A} \) is given by
\begin{equation}
\hat{A}(x) = \frac{x/2}{\sinh(x/2)}.
\end{equation}
Let \( H \) be a finite-dimensional Hermitian vector space. If \( B \in \text{End}(H) \) is self-adjoint, then \( \frac{B/2}{\sinh(B/2)} \) is a self-adjoint positive endomorphism. Put
\begin{equation}
\hat{A}(B) = \det^{1/2}\left[ \frac{B/2}{\sinh(B/2)} \right].
\end{equation}
In (3.3.19), the square root is taken to be the positive square root.

If \( Y_0^4 \in \mathfrak{t}(\gamma) \), as explained in [3, pp. 105], the following function \( A(Y_0^4) \) has a natural square root that is analytic in \( Y_0^4 \in \mathfrak{t}(\gamma) \).
\begin{equation}
A(Y_0^4) = \frac{1}{\det(1 - \text{Ad}(k))_{\mathfrak{z}_0^0(\gamma)}} \frac{\det(1 - \exp(-i\text{ad}(Y_0^4)\text{Ad}(k)))_{\mathfrak{t}_0^0(\gamma)}}{\det(1 - \exp(-i\text{ad}(Y_0^4)\text{Ad}(k)))_{\mathfrak{p}_0^0(\gamma)}}.
\end{equation}
Its square root is denoted by
\begin{equation}
\frac{1}{\det(1 - \text{Ad}(k^{-1}))_{\mathfrak{p}_0^0(\gamma)}}.
\end{equation}
The value of (3.3.21) at \( Y_0^4 = 0 \) is taken to be such that
\begin{equation}
\frac{1}{\det(1 - \text{Ad}(k^{-1}))_{\mathfrak{p}_0^0(\gamma)}}.
\end{equation}

We recall the definition of the function \( J_y \) in [3, eq. (5.5.5)].

**Definition 3.3.1.** Let \( J_y(Y_0^4) \) be the analytic function of \( Y_0^4 \in \mathfrak{t}(\gamma) \) given by
\begin{equation}
J_y(Y_0^4) = \frac{1}{\det(1 - \text{Ad}(k))_{\mathfrak{z}_0^0(\gamma)}} \frac{\hat{A}(\text{ad}(Y_0^4)\text{Ad}(k))_{\mathfrak{t}(\gamma)}}{\det(1 - \exp(-i\text{ad}(Y_0^4)\text{Ad}(k)))_{\mathfrak{t}_0^0(\gamma)}}^{1/2}.
\end{equation}
By [3, Eq. (6.1.1)], there exists \( C_\gamma > 0, c_\gamma > 0 \) such that if \( Y_0^4 \in \mathfrak{t}(\gamma) \),
\begin{equation}
|J_y(Y_0^4)| \leq C_\gamma e^{c_\gamma |Y_0^4|}.
\end{equation}
Put \( p = \dim \mathfrak{p}(\gamma), q = \dim \mathfrak{t}(\gamma) \). Then \( r = \dim \hat{g}(\gamma) = p + q \). By [3, Theorem 6.1.1], for \( t > 0 \), we have
\begin{equation}
\text{Tr}^{\gamma}[\exp(-tL^X_{A})] = \frac{1}{(2\pi t)^{p/2}} \int_{\hat{g}(\gamma)} J_y(Y_0^4) \text{Tr}^{E}[\rho^E(k) \exp(-i\rho^E(Y_0^4) - tA)] e^{-|Y_0^4|^2/2t} \frac{dY_0^4}{(2\pi t)^{q/2}}.
\end{equation}
Remark 3.3.2. A generalization of Bismut’s formula (3.3.25) to the twisted case is obtained in [24, 25]. An extension of this formula for considering arbitrary elements in the center of enveloping algebra instead of Casimir operator (3.2.7) was obtained in [7] by Bismut and Shen.

3.4. Compact locally symmetric spaces. Let $\Gamma$ be a cocompact discrete subgroup of $G$. Then $\Gamma$ acts on $X$ isometrically and properly discontinuously. Then $Z = \Gamma \setminus X$ is compact second countable Hausdorff space.

If $x \in X$, put
\begin{equation}
(3.4.1)
\Gamma_x = \{ \gamma \in \Gamma : \gamma x = x \}.
\end{equation}
Then $\Gamma_x$ is a finite subgroup of $\Gamma$. Put
\begin{equation}
(3.4.2)
r_x = \inf_{\gamma \in \Gamma_x} d(x, \gamma x).
\end{equation}
Then we always have $r_x > 0$. Set
\begin{equation}
(3.4.3)
U_x = B(x, \frac{r_x}{4}) \subset X.
\end{equation}
If $x \in X$, $\gamma \in \Gamma$, we have
\begin{equation}
(3.4.4)
r_{\gamma x} = r_x, \quad U_{\gamma x} = \gamma U_x.
\end{equation}
It is clear that $\Gamma_x \setminus U_x$ can identified with a connected open subset of $Z$.

Set
\begin{equation}
(3.4.5)
S = \ker(\Gamma \to \text{Diffeo}(X)) = \Gamma \cap \ker(K \overset{\text{Ad}}{\to} \text{Aut}(\mathfrak{p})).
\end{equation}
Then $S$ is a finite subgroup of $\Gamma \cap K$, and a normal subgroup of $\Gamma$.

Put
\begin{equation}
(3.4.6)
\Gamma' = \Gamma / S.
\end{equation}
Then $\Gamma'$ acts on $X$ effectively and we have $Z = \Gamma' \setminus X$.

If $x \in X$, we have
\begin{equation}
(3.4.7)
S \subset \Gamma_x, \quad \Gamma'_x = \Gamma_x / S.
\end{equation}
Then the orbifold charts $(U_x, \Gamma'_x, \pi_x : U_x \to \Gamma'_x \setminus U_x)_{x \in X}$ together with the action of $\Gamma'$ on these charts give an orbifold structure for $Z$, so that $Z = \Gamma \setminus X$ is a compact orbifold with a Riemannian metric $g^{TX}$ induced by $g^X$.

By [39, Lemma 1], if $\gamma \in \Gamma$, then $\gamma$ is semisimple. Let $[\Gamma]$ denote the set of the conjugacy classes of $\Gamma$. If $\gamma \in \Gamma$, we say $[\gamma] \in [\Gamma]$ to be an elliptic class if $\gamma$ is elliptic. Let $E[\Gamma] \subset [\Gamma]$ be the set of elliptic classes, then $E[\Gamma]$ is always a finite set. If $E[\Gamma] = \emptyset$, i.e. $\Gamma$ is torsion free, then $Z$ is compact smooth manifold.

Let $[\Gamma']$ be the set of conjugacy classes in $\Gamma'$, and let $E[\Gamma']$ denote the set of elliptic classes in $[\Gamma']$. If $\gamma' \in \Gamma'$, let $Z_{\Gamma'}(\gamma')$ denote the centralizer of $\gamma'$ in $\Gamma'$, and let $[\gamma']'$ denote the conjugacy class of $\gamma'$ in $\Gamma'$. If $\gamma' \in \Gamma'$ is elliptic, let $X(\gamma')$ be its fixed point set in $X$ on which $Z_{\Gamma'}(\gamma')$ acts isometrically and properly discontinuously (cf. [39, Lemma 2]). Note that if $\gamma \in \Gamma$ is a lift of $\gamma' \in \Gamma'$, then $X(\gamma) = X(\gamma')$, and $\gamma$ is elliptic if and only if $\gamma'$ is elliptic.

Proposition 3.4.1. We have
\begin{equation}
(3.4.8)
Z_{\text{sing}} = \Gamma' \setminus \bigcup_{[\gamma'] \in E[\Gamma'] \setminus \{ 1 \}} X(\gamma') \subset Z.
\end{equation}
Moreover, we have
\begin{equation}
(3.4.9)
\Sigma Z = \bigcup_{[\gamma'] \in E[\Gamma'] \setminus \{ 1 \}} Z_{\Gamma'}(\gamma') \setminus X(\gamma').
\end{equation}
Note that the right-hand side of (3.4.9) is a disjoint union of compact orbifolds.

If $\gamma' \in \Gamma'$, put
\begin{equation}
(3.4.10)
S'(\gamma') = \ker(Z_{\Gamma'}(\gamma') \to \text{Diffeo}(X(\gamma'))).
\end{equation}
Then \(|S'(\gamma')|\) is the multiplicity of the connected component \(Z_{\Gamma'}(\gamma') \setminus X(\gamma')\) in \(\Sigma Z\).

**Proof.** Note that \(z \in Z\) with a lift \(x \in X\) belongs to \(Z_{\text{sing}}\) if and only if the stabilizer \(\Gamma_{z}\) is nontrivial. Thus \(x\) is a fixed point of some \(\gamma' \in \Gamma'\), from which (3.4.8) follows. By definition in Subsection 2.1, we get the rest part of this proposition. This completes the proof. \(\square\)

Note that \(\Gamma \setminus G\) is a compact smooth homogeneous space equipped with a right action of \(K\). Moreover, the action of \(K\) is almost free, i.e., for each \(\tilde{g} \in \Gamma \setminus G\), the stabilizer \(K_{\tilde{g}}\) is finite. Then the quotient space \((\Gamma \setminus G)/K\) also have a natural orbifold structure, which is equivalent to \(Z\). Indeed, given \(\tilde{g} \in \Gamma \setminus G\), we get a unique Cartan decomposition of \(g = e^{f}k\) with \(f \in \mathfrak{p}, k \in K\), then \(C(k^{-1})S \subset K_{\tilde{g}}\). One can verify that \(S\) represents the isotropy group of the principal orbit type for the right action of \(K\) on \(\Gamma \setminus G\). If \(B^{p}(0, \varepsilon)\) is a small enough open ball in \(p\) centered at 0, then \(\tilde{g}B^{p}(0, \varepsilon)\) is an equivariant slice at \(\tilde{g}\) with respect to the action of \(K\). Then the orbifold charts of \((\Gamma \setminus G)/K\) are given by the effective right action of \(C(k^{-1})S \setminus K_{\tilde{g}}\) on \(\tilde{g}B^{p}(0, \varepsilon)\). If \(x = pg \in X\), then the adjoint action of \(K_{\tilde{g}}\) on \(B^{p}(0, \varepsilon)\) is equivalent to the action of \(\Gamma_{x}\) on the corresponding open ball in \(X\) centered \(x\). This implies exactly that \(Z = (\Gamma \setminus G)/K\).

Set
\[
\Delta = \ker(K \to \text{Diffeo}(\Gamma \setminus G)).
\]
Recall that \(Z_{G}\) is the center of \(G\). Using the fact that \(\Gamma\) is discrete, we get
\[
\Delta = Z_{G} \cap \Gamma \cap K.
\]
Moreover,
\[
\Delta \subset S, \quad \Delta = \cap_{g \in G} K_{g} = \cap_{k \in K} C(k)S.
\]
Let \(dg\) be the volume element on \(\Gamma \setminus G\) induced by \(dg\). By (3.3.5), we get
\[
\text{Vol}(\Gamma \setminus G) = \frac{\text{Vol}(K)}{|S|} \text{Vol}(Z).
\]

**Example 3.4.2** (An non-natural example of the case \(S \neq \Delta\)). Let \(G, K, \theta, \Gamma\) be as before. Take \(K'\) a connected compact linear group which is center free, and let \(\Sigma \subset K'\) be any nontrivial finite subgroup. Put \(\hat{G} = G \times K', \hat{K} = K \times K',\) the Cartan involution \(\hat{\theta}\) is just the extension of \(\theta\) with trivial action on \(K'\). Then \(X = G/K = \hat{G}/\hat{K}\).

Put \(\hat{\Gamma} = \Gamma \times \Sigma,\) which is a cocompact lattice in \(\hat{G}\). Then the corresponding \(\hat{S} = S \times \Sigma\). But the group \(\hat{\Delta} = \Delta\). This way, we can construct many examples such that \(S \neq \Delta\).

**Lemma 3.4.3.** If \(G_{ss}\) is a connected noncompact simple linear Lie group, then
\[
\Delta = Z_{G} \cap \Gamma \cap K.
\]

**Proof.** We only need to prove that in this case, \(S \subset Z_{G} \). Since \(G\) is connected, it is enough to prove that if \(s \in S\), the adjoint action of \(s\) on \(\mathfrak{g}\) is trivial.

The action of \(\theta\) preserves the splitting in (3.1.4). Let \(\mathfrak{g}_{ss} = \mathfrak{p}_{ss} \oplus \mathfrak{t}_{ss}\) be the Cartan decomposition of \(\mathfrak{g}_{ss}\) with respect to \(\theta\), where \(\mathfrak{t}_{ss} \subset \mathfrak{t}, \mathfrak{p}_{ss} \subset \mathfrak{p}\). Moreover, since \(G_{ss}\) is noncompact, then \(\mathfrak{p}_{ss}\) is nonzero.

Since \(\mathfrak{g}_{ss}\) is simple, then we have
\[
[\mathfrak{t}_{ss}, \mathfrak{p}_{ss}] = \mathfrak{p}_{ss}, \quad [\mathfrak{p}_{ss}, \mathfrak{p}_{ss}] = \mathfrak{t}_{ss}.
\]
If \(s \in S\), then \(\text{Ad}(s)\) acts trivially on \(\mathfrak{p}_{ss}\), thus it acts trivially on \(\mathfrak{g}_{ss}\) and \(\mathfrak{g}\). This completes the proof of our lemma. \(\square\)
Corollary 3.4.4. If \( g_{\infty} \) has no compact simple factor, then
\[
S = \Delta = Z_G \cap \Gamma \cap K.
\] (3.4.17)

We note that in many interesting cases, we can reduce to the case of \( S = \{1\} \). For instance, given a Riemannian symmetric space \((X, g^{TX})\) of noncompact type, let \( G = \text{Isom}(X)^0 \) be the connected component of identity of the Lie group of isometries of \( X \). By [16, Proposition 2.1.1], \( G \) is a semisimple Lie group with trivial center. We refer to [16, Chapter 2] and [3, Chapter 3] for more details. This way, any subgroup of \( G \) acts on \( X \) effectively. In particular, if \( \Gamma \) is a cocompact discrete subgroup of \( G \), then \( Z = \Gamma \backslash X \) is a compact good orbifold with the orbifold fundamental group \( \Gamma \). By (3.4.9), we have
\[
\Sigma Z = \bigcup_{\gamma \in E \setminus \{1\}} \Gamma \cap Z(\gamma) \backslash X(\gamma).
\] (3.4.18)

In general, by [18, Ch.V §4, Theorem 4.1], \( G = \text{Isom}(X = G/K)^0 \) if and only if \( K \) acts on \( p \) effectively. Another particular case is that \( S = \Delta \). Then after replacing \( G, K \) by their quotients \( G/S, K/S \), we also go back to the case of \( S = \{1\} \).

If \( \rho : \Gamma \to \text{GL}(C^k) \) is a representation of \( \Gamma \), which can be viewed as a representation of \( \Gamma \) via the projection \( \Gamma \to \Gamma' = \Gamma/S \), then \( F = X_{\Gamma'} \times C^k \) is a proper flat orbifold vector bundle on \( Z \) with the flat connection \( \nabla^{F,J} \) induced from the exterior differential \( dX \) on \( C^k \)-valued functions. By [41, Theorem 2.31], all the proper orbifold vector bundle on \( Z \) of rank \( k \) comes from this way.

Now let \( \rho : \Gamma \to \text{GL}(C^k) \) be a representation of \( \Gamma \), we do not assume that it comes from a representation of \( \Gamma' \). We still have a flat orbifold vector bundle \( (F, \nabla^{F,J}) \) on \( Z \), which is not proper in general. Note that \( \Gamma \) acts on \( C^\infty(X, C^k) \) so that if \( \varphi \in C^\infty(X, C^k), \gamma \in \Gamma \), then
\[
(\gamma \varphi)(x) = \rho(\gamma) \varphi(\gamma^{-1}x).
\] (3.4.19)

Let \( C^\infty(X, C^k)^\Gamma \) denote the \( \Gamma \)-invariant sections in \( C^\infty(X, C^k) \). Then
\[
C^\infty(Z, F) = C^\infty(X, C^k)^\Gamma.
\] (3.4.20)

We have the following results.

Proposition 3.4.5. Let \( (V, \rho^V) \) be the isotypic component of \( (C^k, \rho|_S) \) corresponding to the trivial representation of \( S \) on \( C \), i.e. the maximal \( S \)-invariant subspace of \( C^k \) via \( \rho \). Set \( F^{pp} = X_{\Gamma} \times V \), then \( F^{pp} \) is a proper flat orbifold vector bundle on \( Z \). Moreover,
\[
C^\infty(Z, F) = C^\infty(Z, F^{pp}).
\] (3.4.21)

In particular, if \( \rho|_S : S \to \text{GL}(C^k) \) does not have the isotypic component of the trivial representation of \( S \) on \( C \), then
\[
C^\infty(Z, F) = \{0\}.
\] (3.4.22)

Let \( (E, \rho^E) \) be a finite dimensional complex representation of \( G \). When restricting to \( \Gamma, K \), we get the corresponding representations of \( \Gamma, K \) respectively, which are still denoted by \( \rho^E \).

Set \( F = G \times_K E \). Then \( F \) is a homogeneous vector bundle on \( X \) discussed in Subsection 3.2. Moreover, \( G \) acts on \( F \). Then it descends to an orbifold vector bundle on \( Z \).

The map \( (g, v) \in G \times_K E \to (pg, \rho^E(g)v) \in X \times E \) gives a canonical trivialization of \( F \) over \( X \). This identification gives a flat connection \( \nabla^{F,J} \) for \( F \). Recall that the connection \( \nabla^F \) is induced from \( \omega^F \). Then
\[
\nabla^{F,J} = \nabla^F + \rho^E(\omega^F).
\] (3.4.23)

This flat connection is \( G \)-invariant on \( X \), then it descends to a flat connection on the orbifold vector bundle \( F \) on \( Z \). Moreover, the above trivialization of \( F \to X \)
implies that the flat orbifold vector bundle \((F, \nabla_{F,F})\) is exactly the one given by \(X_\Gamma \times E\) with \(\nabla_{F,F}\) induced by \(d^X\). By (3.2.6), (3.4.20), we get
\[
(3.4.24) \quad C^\infty(Z, F) = C^\infty_{Z,F}(G, E)^F.
\]

3.5. Selberg’s trace formula. Let \(Z\) be the compact locally symmetric space discussed in Subsection 3.4, and let \((F, \nabla^F)\) be a Hermitian vector bundle on \(X\) defined by a unitary representation \((E, \rho^E)\) of \(K\). As said before, \((F, \nabla^F)\) descends to a Hermitian orbifold vector bundle on \(Z\). Recall the Bochner-like Laplacian \(L_A^Z\) is defined by (3.2.9). Since it commutes with \(G\), then it descends to a Bochner-like Laplacian \(L_A^Z\) acting on \(C^\infty(Z, F)\).

Here the convergences of the integrals and infinite sums are already guaranteed by the results in [3, Chapters 2 & 4] and in [40, Section 4D].

For \(t > 0\), let \(p^Z_t(z, z')\), \(z, z' \in Z\) be the heat kernel of \(L_A^Z\) over \(Z\) with respect to \(dz'\). If \(z, z'\) are identified with their lifts in \(X\), then
\[
(3.5.1) \quad p^Z_t(z, z') = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma p^X_t(\gamma^{-1}z, z') = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} p^X_t(z, \gamma z')\gamma.
\]

If \(F\) is a proper orbifold vector bundle on \(Z\), i.e., \(S\) acts trivially on \(E\), then \(\Gamma'\) acts on \(F \to X\), and (3.5.1) can be rewritten as
\[
(3.5.2) \quad p^Z_t(z, z') = \sum_{\gamma' \in \Gamma'} \gamma' p^X_t((\gamma')^{-1}z, z') = \sum_{\gamma' \in \Gamma'} p^X_t(z, \gamma' z')\gamma'.
\]

For the case where \(F\) is not a proper orbifold vector bundle, it will be more complicated. If \(g \in G\), then \(g^{-1}Sg\) is also a subgroup of \(K\). Put \(E^F_g\) be the maximal subspace of \(E\) on which \(g^{-1}Sg\) acts trivially via \(\rho^E\). Then if \(k \in K\), we have
\[
(3.5.3) \quad E^F_g = \rho^E(k^{-1})E^F_g \subset E.
\]

Then all the pairs \((g, E^F_g), g \in G\) defines a subbundle of \(F\) on \(Z\), which is just the corresponding proper orbifold bundle \(F^\nu\) of \(F\). As in Proposition 3.4.5, we have
\[
(3.5.4) \quad C^\infty(Z, F) = C^\infty(Z, F^\nu).
\]

If we write \((E, \rho^E)|_S\) as a direct sum of the trivial \(S\)-representation \(E_1^\nu\) and the nontrivial part \((E^\nu, \rho^E)\). Note that
\[
(3.5.5) \quad \sum_{s \in S} \rho^E(s) = 0 \in \text{End}(E^\nu).
\]

Combining (3.5.1), (3.5.4) and (3.5.5), we see that (3.5.2) still holds as kernels of integral operators acting on \(C^\infty(Z, F)\).

Remark 3.5.1. If \(S = \Delta\), then \(E_1^\nu\) is a \(K\)-subrepresentation of \((E, \rho^E)\). Then \(F^\nu = G \times_K E_1^\nu\).

Since \(Z\) is compact, then for \(t > 0\), \(\exp(-tL_A^Z)\) is trace class. We have
\[
(3.5.6) \quad \text{Tr}[\exp(-tL_A^Z)] = \int_Z \text{Tr}^F[p^Z_t(z, z)]dz.
\]

Combining (3.2.10), (3.2.11), (3.4.14) and (3.5.1), (3.5.6), we get
\[
(3.5.7) \quad \text{Tr}[\exp(-tL_A^Z)] = \frac{1}{\text{Vol}(K)} \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} \text{Tr}^F[p^X_t(\bar{\gamma}^{-1}\bar{g})]dg.
\]

Take \(\gamma \in [\Gamma]\). Set
\[
(3.5.8) \quad p^X_{\gamma}([\gamma])(g, g') = \sum_{\gamma \in [\gamma]} p^X_{\gamma}(g^{-1}\gamma g').
\]
Then \( p_t^{\gamma}(g, g) \) descends to a function on \( \Gamma \backslash G \).

Set

\[
(3.5.9) \quad \text{Tr}[\exp(-tL^Z_A)^{\gamma}] = \frac{1}{\text{Vol}(K)} \int_{\Gamma \backslash G} \text{Tr} E[p_t^{\gamma}(g, g)]dg.
\]

By (3.5.7), (3.5.9), we have

\[
(3.5.10) \quad \text{Tr}[\exp(-tL^Z_A)] = \sum_{\gamma \in \Gamma} \text{Tr}[\exp(-tL^Z_A)^{\gamma}].
\]

By (3.5.8), (3.5.9), we get

\[
(3.5.11) \quad \text{Tr}[\exp(-tL^Z_A)^{\gamma}] = \frac{1}{\text{Vol}(K)} \int_{\Gamma \cap Z(\gamma) \backslash G} \text{Tr} E[p_t^{\gamma}(g^{-1} g)]dg.
\]

Recall that the measures \( dz, dv \) on \( Z(\gamma), Z(\gamma) \backslash G \) are defined in (3.3.8). Then

\[
(3.5.12) \quad \text{Tr}[\exp(-tL^Z_A)^{\gamma}] = \frac{\text{Vol}(\Gamma \cap Z(\gamma) \backslash Z(\gamma))}{\text{Vol}(K)} \int_{Z(\gamma) \backslash G} \text{Tr} E[p_t^{\gamma}(v^{-1} g v)]dv.
\]

By (3.3.7), (3.3.9), we get

\[
(3.5.13) \quad \text{Tr}[\exp(-tL^Z_A)^{\gamma}] = \frac{\text{Vol}(\Gamma \cap Z(\gamma) \backslash Z(\gamma))}{\text{Vol}(K)} \cdot \text{Tr}^{[\gamma]}[\exp(-tL^X_A)]
\]

Take \( \gamma \in \Gamma \). Let \( K(\gamma) \) be a maximal compact subgroup of \( Z(\gamma) \) so that \( X(\gamma) = Z(\gamma)/K(\gamma) \). Then \( K(\gamma) \) acts on \( Z(\gamma) \) on the right, which induces an action on \( \Gamma \cap Z(\gamma) \backslash Z(\gamma) \) on the right. Set

\[
(3.5.14) \quad \Delta(\gamma) = \ker(K(\gamma) \to \text{Diffeo}(\Gamma \cap Z(\gamma) \backslash Z(\gamma))).
\]

Then \( \Delta(\gamma) \) is a finite subgroup of \( \Gamma \cap K(\gamma) \).

Set

\[
(3.5.15) \quad S(\gamma) = \ker(\Gamma \cap Z(\gamma) \to \text{Diffeo}(X(\gamma))).
\]

Then \( \Delta(\gamma) \subset S(\gamma) \) and \( S(\gamma) \) represents the isotropy group of the principal orbit type for the right action of \( K(\gamma) \) on \( \Gamma \cap Z(\gamma) \backslash Z(\gamma) \). As in (3.4.14), we have

\[
(3.5.16) \quad \text{Vol}(\Gamma \cap Z(\gamma) \backslash Z(\gamma)) = \frac{\text{Vol}(K(\gamma))}{|S(\gamma)|} \cdot \text{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma)).
\]

**Theorem 3.5.2.** For \( t > 0 \), we have the following identity,

\[
(3.5.17) \quad \text{Tr}[\exp(-tL^Z_A)] = \sum_{\gamma \in \Gamma} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \cdot \text{Tr}^{[\gamma]}[\exp(-tL^X_A)].
\]

**Proof.** This is a direct consequence of (3.5.10), (3.5.13) and (3.5.16). \( \square \)

As explained before, many interesting cases have the property \( S = 1 \) or can be reduced to \( S = 1 \). In these cases, the trace formula (3.5.17) shows clearly the different contributions from \( Z \) and from each components of \( \Sigma Z \). In the sequel, we give a precise description of the contribution of \( \Sigma Z \) when \( S \) is not such trivial.

The orbifold resolution \( \Sigma Z \) of \( Z_{sing} \) is described by (3.4.9). Recall that \( |S(\gamma')| \) is the multiplicity of the connected component \( Z_{\gamma'} \backslash X(\gamma') \) in \( \Sigma Z \). Let \( \pi_T : \Gamma \to \Gamma' \) denote the obvious projection. If \( \gamma' \in \Gamma' \), put

\[
(3.5.18) \quad \{\gamma'\} = \{[\gamma] \in [\Gamma] : \pi_T(\gamma) \in [\gamma']\}.
\]

In particular, we have \( \{1\} = [S]_\Gamma \).

If \( f \) is a function on \( S \), put

\[
(3.5.19) \quad (f)_S = \frac{1}{|S|} \sum_{s \in S} f(s).
\]
Proposition 3.5.3. If $s \in S$, then

\[(3.5.20) \quad \text{Tr}^{[s]}[\exp(-tL_A X)] = \text{Tr}^{E}[p_t X (1) \rho^E(s)].\]

Then

\[(3.5.21) \quad \sum_{[\gamma] \in [S]} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus X)}{|S(\gamma)|} \text{Tr}^{[\gamma]}[\exp(-tL_A X)] = \text{Vol}(Z)\langle \text{Tr}^{[1]}[\exp(-tL_A X)] \rangle_S.\]

Fix $\gamma_0 \in \Gamma$ such that $\pi_\Gamma(\gamma_0) \in [\gamma']'$. Then

\[(3.5.22) \quad \sum_{[\gamma] \in [\gamma']'} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus X)}{|S'(\gamma)'|} \text{Tr}^{[\gamma]}[\exp(-tL_A X)] = \text{Vol}(Z')\langle \text{Tr}^{[\gamma_0]}[\exp(-tL_A X)] \rangle_{S'}.\]

Proof. It is clear that (3.5.21) is just a special case of (3.5.22). But for a better understanding, we also include an easy proof to this special case, then we will prove the general case.

If $s \in S$, then $\text{Ad}(s)$ acts trivially on $p$. Then we have the identification

\[(3.5.23) \quad Z(s) = \exp(p)K(s).\]

Therefore, we get the following identification together with their volume elements,

\[(3.5.24) \quad Z(s) \setminus G \simeq K(s) \setminus K.\]

Then the identity (3.5.20) follows from (3.2.10), (3.3.9) and the fact that $S$ acts trivially both on $X$.

Now we prove (3.5.21). By (3.5.16), if $[\gamma] \in [S]_\Gamma$ we have

\[(3.5.25) \quad \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus X)}{|S(\gamma)|} = \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus Z(\gamma))}{\text{Vol}(K(\gamma))}.\]

Then by (3.5.24),

\[(3.5.26) \quad \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus X)}{|S(\gamma)|} = \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus G)}{\text{Vol}(K)} = \frac{[\Gamma \cap Z(\gamma)]}{[\Gamma]} \frac{\text{Vol}(\Gamma \setminus G)}{\text{Vol}(K)}.\]

Then (3.5.21) follows from (3.4.14), (3.5.20), (3.5.26), and the fact that if $\gamma \in \Gamma$,

\[(3.5.27) \quad \Gamma / \Gamma \cap Z(\gamma) \simeq [\gamma].\]

Now we prove the second part. Let $\gamma_0 \in \Gamma$ be such that $\pi_\Gamma(\gamma_0) \in [\gamma']'$. Then

\[(3.5.28) \quad \cup_{[\gamma] \in [\gamma']'} [\gamma] = [\gamma_0]S \subset \Gamma.\]

Also note that the quantity $\langle \text{Tr}^{[\gamma_0]}[\exp(-tL_A X)] \rangle_S$ is independent of the choice of such $\gamma_0$, which is determined uniquely by $[\gamma]'$. 

By (3.5.10), (3.5.13) and (3.5.16), we get
\[
\sum_{[\gamma] \in \{\gamma\}} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_A)] = \sum_{[\gamma] \in \{\gamma\}} \int_{\Gamma \setminus G} \sum_{\gamma \in [\gamma]} \text{Tr}^E[p_t^X(\bar{g}^{-1}\gamma \bar{g})]d\bar{g} \\
= \int_{\Gamma \setminus G} \sum_{\gamma \in [\gamma]} \sum_{\gamma' \in [\gamma']} \text{Tr}^E[p_t^X(\bar{g}^{-1}\gamma \bar{g})]d\bar{g} \\
= \int_{\Gamma \setminus G} \sum_{\gamma' \in [\gamma']} \sum_{s \in S} \text{Tr}^E[p_t^X(\bar{g}^{-1}\gamma' \bar{g})]d\bar{g} \\
= \int_{Z=\Gamma \setminus X} \sum_{\gamma' \in [\gamma']} \frac{1}{|S|} \sum_{s \in S} \text{Tr}^E[p_t^X(s^{-1}\gamma' sz)]dz \\
= \int_{Z_{\Gamma^r(\gamma')} \setminus X} \frac{1}{|S|} \sum_{s \in S} \text{Tr}^E[p_t^X(s^{-1}x_0sx)]dz.
\]

(3.5.29)

Note that $X$ can be identified with the total space of the orthogonal normal bundle of $X(\gamma') = X(\gamma_0) = X(\gamma_0s)$. Moreover $Z_{\Gamma^r}(\gamma')$ acts on $X(\gamma')$ isometrically, and $Z(\gamma_0s)$ acts on $X(\gamma')$ transitively. Then we get
\[
\sum_{[\gamma] \in \{\gamma\}} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \text{Tr}^{[\gamma]}[\exp(-t\mathcal{L}^X_A)] = \frac{1}{|S'(\gamma')|} \frac{1}{|S|} \sum_{s \in S} \int_{Z_{\Gamma^r(\gamma')} \setminus X(\gamma')} \int_{f \in N_{X(\gamma')} \setminus X,z} \text{Tr}^E[p_t^X(\exp_s(f), \gamma_0 s \exp_s(f))\gamma_0 s^r(f)]dfdz \\
= \frac{\text{Vol}(Z_{\Gamma^r(\gamma')} \setminus X(\gamma'))}{|S'(\gamma')|} \frac{1}{|S(\gamma')|} \text{Tr}^{[\gamma_0^r]}[\exp(-t\mathcal{L}^X_A)]|_{S}.
\]

(3.5.30)

Note that we use the geometric formula (3.3.10) for the orbital integrals in the last step of (3.5.30). The proof of our proposition is completed. \qed

The operation $\langle \cdot \rangle_S$ in the right-hand sides of equations (3.5.21), (3.5.22) indicates only the part $E^{pr}$ contributes to the final results. Combining (3.4.9), (3.5.17) with the results in Proposition 3.5.3 and [25, Theorem 6.4.1], we can recover (2.2.11) for $Z$. If we use the same settings as in [3, Sections 7.1, 7.2] and we use instead the results in [3, Theorem 7.7.1], then we can recover the Kawasaki’s local index theorem [20] for $Z$.

4. Analytic Torsions for Compact Locally Symmetric Spaces

In this section, we explain how to make use of Bismut’s formula (3.3.25) and Selberg’s trace formula (3.5.17) to study the analytic torsions of $Z$. We continue using the same settings as in Section 3.

We show that by a vanishing result on the analytic torsion, only the case $\delta(G) = 1$ remains interesting. Then if $G$ has noncompact center, we can get very explicit information for evaluating $T(Z, F)$ due to the Euclidean component in $X$. If $G$ has compact center, we need more tools, which will be carried out in Sections 5 & 6.
4.1. A vanishing result on the analytic torsions. Recall that $G$ is a connected linear real reductive Lie group. Recall that $\mathfrak{z}_G$ is the center of $\mathfrak{g}$. Set
\begin{equation}
\mathfrak{z}_p = \mathfrak{z}_G \cap \mathfrak{p}, \quad \mathfrak{z}_t = \mathfrak{z}_G \cap \mathfrak{t}.
\end{equation}
Then
\begin{equation}
\mathfrak{z}_G = \mathfrak{z}_p \oplus \mathfrak{z}_t, \quad Z_G = \exp(\mathfrak{z}_p)(Z_G \cap K).
\end{equation}

Let $T$ be a maximal torus of $K$ with Lie algebra $\mathfrak{t}$, put
\begin{equation}
b = \{ f \in \mathfrak{p} : [f, \mathfrak{t}] = 0 \}.
\end{equation}
It is clear that
\begin{equation}
\mathfrak{z}_p \subset b.
\end{equation}

Put $\mathfrak{h} = b \oplus \mathfrak{t}$, then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $H$ be analytic subgroup of $G$ associated with $\mathfrak{h}$, then it is also a Cartan subgroup of $G$. Moreover, $\dim \mathfrak{t}$ is just the complex rank of $K$, and $\dim \mathfrak{h}$ is the complex rank of $G$.

Definition 4.1.1. Using the above notations, the deficiency of $G$, or the fundamental rank of $G$ is defined as
\begin{equation}
\delta(G) = \text{rk}_{\mathbb{C}}G - \text{rk}_{\mathbb{C}}K = \dim \mathbb{R} b.
\end{equation}
The number $m - \delta(G)$ is even.

The following result was proved in [40, Proposition 3.3].

Proposition 4.1.2. If $\gamma \in G$ is semisimple, then
\begin{equation}
\delta(G) \leq \delta(Z(\gamma)^0).
\end{equation}
The two sides of (4.1.6) are equal if and only if $\gamma$ can be conjugated into $H$.

Recall that $u = \sqrt{-1} \mathfrak{p} \oplus \mathfrak{t}$ is the compact form of $G$, and that $U \subset G_{\mathbb{C}}$ is the analytic subgroup with Lie algebra $u$. Let $U_u, U_{g_{\mathbb{C}}}$ be the enveloping algebras of $u, g_{\mathbb{C}}$ respectively. Then $U_{g_{\mathbb{C}}}$ can be identified with the left-invariant holomorphic differential operators on $G_{\mathbb{C}}$. Let $C^u \in U_u$ be the Casimir operator of $u$ associated with $B$, then
\begin{equation}
C^u = C^g \in U_{g_{\mathbb{C}}} \cap U_u \subset U_{g_{\mathbb{C}}}.
\end{equation}

In the sequel, we always assume that $U$ is compact, this is the case when $G$ has compact center.

Proposition 4.1.3 (Unitary trick). Assume that $U$ is compact. Then any irreducible finite dimensional (analytic) complex representation of $U$ extends uniquely to an irreducible finite dimensional complex representation of $G$ such that their induced representations of Lie algebras are compatible.

We now fix a unitary representation $(E, \rho^E, h^E)$ of $U$, and we extend it to a representation of $G$, whose restriction to $K$ is still unitary.

Put $F = G \times_K E$ with the Hermitian metric $h^F$ induced by $h^E$. Let $\nabla^F$ be the Hermitian connection induced by the connection form $\omega^F$. If $G$ has compact center, then $F$ is a unimodular.

Furthermore, as explained in the last part of Subsection 3.4, $F$ is equipped with a canonical flat connection $\nabla^{F_F}$ as follows,
\begin{equation}
\nabla^{F_F} = \nabla^F + \rho^E(\omega^p).
\end{equation}

Let $(\Omega_\cdot(X, F), d^{X,F})$ be the (compactly supported) de Rham complex twisted by $F$. Let $d^{X,F,*}$ be the adjoint operator of $d^{X,F}$ with respect to the $L_2$ metric on $\Omega_\cdot(X, F)$. The Dirac operator $D^{X,F}$ of this de Rham complex is given by
\begin{equation}
D^{X,F} = d^{X,F} + d^{X,F,*}.
\end{equation}
The Clifford algebras $c(TX), \tilde{c}(TX)$ act on $\Lambda(T^*X)$. We still use $e_1, \ldots, e_m$ to denote an orthonormal basis of $p$ or $TX$, and let $e^1, \ldots, e^m$ be the corresponding dual basis of $p^*$ or $T^*X$.

Let $∇^{(T^*X) \otimes F}$ be the unitary connection on $\Lambda(T^*X) \otimes F$ induced by $∇^X$ and $∇^F$. Then the standard Dirac operator is given by

$$\frac{D^X,F}{2} = \sum_{j=1}^{m} \delta(e_j)∇_{e_j}(T^*X) \otimes F,a.$$  

(4.1.10)

By [6, eq.(8.42)], we have

$$D^X,F = D^X,F + \sum_{j=1}^{m} \delta(e_j)\rho^F(e_j).$$  

(4.1.11)

In the same time, $C^g$ descends to an elliptic differential operator $C^g,X$ acting on $C^\infty(X, \Lambda(T^*X) \otimes F)$. Let $k^g \in \Lambda^3(g^*)$ be such that if $a, b, c \in g$,

$$\kappa^g(a, b, c) = B([a, b], c).$$  

(4.1.12)

Then $\kappa^g$ is a $G$-invariant closed 3-form on $G$. The bilinear form $B$ induces a corresponding bilinear form $B^*$ on $\Lambda(g^*)$. Let $C^t$ be the Casimir operator associated with $(\mathfrak{t}, B_\mathfrak{t}, \mathfrak{g})$, and let $C^\mathfrak{t}, C^\mathfrak{g}$ be the the actions of $C^g$ on $\mathfrak{t}, \mathfrak{g}$ via adjoint actions. By [3, Eq.(2.6.11)], we have

$$B^*(\kappa^g, \kappa^g) = \frac{1}{2} \text{Tr}^\mathfrak{p}[C^\mathfrak{g}, \mathfrak{p}] + \frac{1}{6} \text{Tr}^\mathfrak{f}[C^\mathfrak{t}, \mathfrak{f}].$$  

(4.1.13)

Set

$$\mathcal{L}^X,F = \frac{1}{2} C^g,X + \frac{1}{8} B^*(\kappa^g, \kappa^g).$$  

(4.1.14)

By [6, Proposition 8.4], we have

$$\frac{D^X,F,2}{2} = \mathcal{L}^X,F - \frac{1}{2} C^g,E - \frac{1}{8} B^*(\kappa^g, \kappa^g).$$  

(4.1.15)

Let $\gamma \in G$ be a semisimple element. In the sequel, we may assume that

$$\gamma = e^a k, a \in \mathfrak{p}, k \in K, \text{Ad}(k)a = a.$$  

(4.1.16)

We also use the same notation as in Subsection 3.3.

Recall that $p = \dim p(\gamma), q = \dim \mathfrak{t}(\gamma)$. By (3.3.25) and (4.1.15), as in [6, Section 8], we have

$$\text{Tr}_{e}^{\mathfrak{g}}[(N^\Lambda(T^*X) - \frac{m}{2}) \exp(-iD^X,F,2/2)]$$  

$$\begin{align*} 
= & \frac{e^{\frac{-t|Y_0|^2}{2}}}{(2\pi t)^{p/2}} \exp\left(\frac{t}{8} B^*(\kappa^g, \kappa^g)\right) \cdot 
\int_{0(\gamma)} J_{\gamma}(Y_0^t) \text{Tr}_{\mathfrak{g}}^{\Lambda}(\mathfrak{g}^*) \left[(N^\Lambda(\mathfrak{g}^*) - \frac{m}{2}) \text{Ad}(k) \exp(-iad(Y_0^t))\right] \cdot 
\text{Tr}^E(\rho^F(k) \exp(-i\rho^F(Y_0^t) + \frac{t}{2} C^E,F)] \frac{dy_{\gamma}^t}{(2\pi t)^{q/2}}. 
\end{align*}$$  

(4.1.17)

Now we take a cocompact discrete subgroup $\Gamma \subseteq G$. Then $Z = \Gamma \setminus X$ is a compact locally symmetric orbifold. We use the same notation as in Subsections 3.4 & 3.5.

Then we get a flat orbifold vector bundle $(F, \nabla^{\mathfrak{g}, F}, h^F)$ on $Z$. Furthermore, $D^Z,F$ descends to the corresponding Hodge-de Rham operator $D^Z,F$ acting on $Ω(Z,F)$. Let $\mathcal{T}(Z,F)$ denote the associated analytic torsion as in Definition 2.2.4, i.e.,

$$\mathcal{T}(Z,F) = \mathcal{T}(g^Z, \nabla^{\mathfrak{g}, F}, h^F).$$  

(4.1.18)
As explained in Subsection 2.2, for computing $\mathcal{T}(Z, F)$, it is enough to evaluate

\[(4.1.19) \quad \text{Tr}_s[(N^A(T^*Z) - \frac{m}{2}) \exp(-tD^{Z,F,2}/2)], \quad t > 0.\]

Then we apply the Selberg’s trace formula in Theorem 3.5.2. We get

\[(4.1.20) \quad \text{Tr}_s[(N^A(T^*Z) - \frac{m}{2}) \exp(-tD^{Z,F,2}/2)] = \sum_{\gamma \in \mathcal{F}} \frac{\text{Vol}(\Gamma \cap \mathbb{Z}(\gamma) \backslash X(\gamma))}{|S(\gamma)|} \text{Tr}_{\gamma}[(N^A(T^*X) - \frac{m}{2}) \exp(-tD^{X,F,2}/2)].\]

As in [6, Remark 8.7], by [28, Theorems 5.4 & 5.5, Remark 5.6], we have the following vanishing theorem on $\mathcal{T}(Z, F)$.

**Theorem 4.1.4.** If $m$ is even, or if $m$ is odd and $\delta(G) \geq 3$, then

\[(4.1.21) \quad \mathcal{T}(Z, F) = 0.\]

**Proof.** By [3, Theorem 7.9.1], [28, Theorem 5.4], and use instead (4.1.20), we get that under the assumptions in this theorem, for $t > 0$,

\[(4.1.22) \quad \text{Tr}_s[(N^A(T^*Z) - \frac{m}{2}) \exp(-tD^{Z,F,2})] = 0.\]

The proof of (4.1.21) is identical to the proof of [25, Proposition 6.5.3]. \[\square\]

Therefore, the only nontrivial case is that $\delta(G) = 1$, so that $m$ is odd. If $\gamma \in G$ is of the form (4.1.16). Let $t(\gamma) \subset \mathfrak{t}(\gamma)$ be a Cartan subalgebra. Put

\[(4.1.23) \quad \mathfrak{b}(\gamma) = \{v \in \mathfrak{p}(\gamma) : [v, t(\gamma)] = 0\}.\]

In particular, $a \in \mathfrak{b}(\gamma)$. Then $\mathfrak{b}(\gamma) = \mathfrak{b}(\gamma) \oplus \mathfrak{t}(\gamma)$ is a Cartan subalgebra of $\mathfrak{z}(\gamma)$.

Recall that $H$ is a maximal compact Cartan subgroup of $G$. The following result is just an analogue of [40, Theorem 4.12] and [3, Theorem 7.9.1].

**Proposition 4.1.5.** If $\delta(G) = 1$, if $\gamma$ is semisimple and can not be conjugated into $H$ by an element in $G$, then

\[(4.1.24) \quad \text{Tr}_s^{\gamma}[(N^A(T^*X) - \frac{m}{2}) \exp(-tD^{X,F,2}/2)] = 0.\]

**Proof.** Let $t$ be a Cartan subalgebra of $\mathfrak{t}$ containing $t(\gamma)$. Then $\mathfrak{b} \subset \mathfrak{b}(\gamma)$. If $a \notin \mathfrak{b}$, then dim $\mathfrak{b}(\gamma) \geq 2$. Therefore, by [40, Eq.(44)], for $Y_0^t \in \mathfrak{t}(\gamma)$, we have

\[(4.1.25) \quad \text{Tr}_s^{\gamma}(\mathfrak{p}^+) [(N^A(\mathfrak{p}^+) - \frac{m}{2}) \text{Ad}(k^{-1}) \exp(-i\text{ad}(Y_0^t))] = 0.\]

This implies (4.1.24). The proof is completed. \[\square\]

Set

\[(4.1.26) \quad \mathfrak{g}' = \mathfrak{z}_\mathfrak{t} \oplus \mathfrak{g}_{\text{ad}}.\]

Then $\mathfrak{g}'$ is an ideal of $\mathfrak{g}$. Let $G'$ be the analytic subgroup of $G$ associated with $\mathfrak{g}'$, which has a compact center. The group $K$ is still a maximal subgroup of $G'$. Let $U' \subset U$ be the compact form of $G'$ with Lie algebra $\mathfrak{u}'$, then

\[(4.1.27) \quad \mathfrak{u} = \sqrt{-1} \mathfrak{z}_\mathfrak{t} \oplus \mathfrak{u}'.\]

Now we assume that $\delta(G) = 1$ and that $G$ has noncompact center, so that $\mathfrak{b} = \mathfrak{z}_\mathfrak{t}$ has dimension 1. Then $\delta(G') = 0$. Under the hypothesis that $U$ is compact, then up to a finite cover, we may write

\[(4.1.28) \quad U \simeq S^1 \times U'.\]

We take $a_1 \in \mathfrak{b}$ with $|a_1| = 1$. If $(E, \rho^E)$ is an irreducible unitary representation of $U$, then $\rho^E(a_1)$ acts on $E$ by a real scalar operator. Let $\alpha_E \in \mathbb{R}$ be such that

\[(4.1.29) \quad \rho^E(a_1) = \alpha_E \text{id}_E.\]
Put $X' = G'/K$. Then $X'$ is an even-dimensional symmetric space (of noncompact type). We identify $\mathfrak{g}_p$ with a real line $\mathbb{R}$, then
\begin{equation}
G = \mathbb{R} \times G', \quad X = \mathbb{R} \times X'.
\end{equation}

In this case, the evaluation for analytic torsions can be made more explicit. If $\gamma \in G'$, let $X'(\gamma)$ denote the minimizing set of $d\gamma(\cdot)$ in $X'$, so that
\begin{equation}
X(\gamma) = \mathbb{R} \times X'(\gamma).
\end{equation}

Let $\left[\cdot\right]^{\max}$ denote the coefficient of a differential form on $X'$ of the (oriented) Riemannian volume form. Similarly, for $k \in T$, let $\left[\cdot\right]^{\max(k)}$ denote the coefficient of the Riemannian volume form in a differential form on $X'(k)$.

The following results are the analogue of [40, Proposition 4.14].

**Proposition 4.1.6.** Assume that $G$ has noncompact center with $\delta(G) = 1$ and that $(E, \rho^E)$ is irreducible. Then
\begin{equation}
\Tr_s[\left[N^X \left(T^*X\right) - \frac{m}{2}\right] \exp(-tD_{X,F,2}/2)] = -\frac{e^{-t\delta^2/2}}{\sqrt{2\pi t}} \left[\epsilon(TX', \nabla TX')\right]^{\max} \dim E.
\end{equation}

If $\gamma = e^{a}k$ is such that $a \in \mathfrak{b}$, $k \in T$, then
\begin{equation}
\Tr_s[\gamma][\left[N^X \left(T^*X\right) - \frac{m}{2}\right] \exp(-tD_{X,F,2}/2)]
= -\frac{1}{\sqrt{2\pi t}} e^{-\frac{ia^2}{t} - \frac{t\delta^2}{2}} \left[\epsilon(TX'(k), \nabla TX'(k))\right]^{\max(k)} \Tr_{\rho^E(k)}[\rho^E(k)].
\end{equation}

If $\gamma$ can not be conjugated into $H$, then
\begin{equation}
\Tr_s[\gamma][\left[N^X \left(T^*X\right) - \frac{m}{2}\right] \exp(-tD_{X,F,2}/2)] = 0.
\end{equation}

**Proof.** Let $C^{u'}$ denote the Casimir operator of $u'$ associated with $B|_{u'}$. Then we have
\begin{equation}
C^u = -a_1^2 + C^{u'}.
\end{equation}

Since $(E, \rho^E)$ is an irreducible representation, by (4.1.29) and (4.1.35), we get
\begin{equation}
C^{u,E} = -a_2^2 + C^{u',E}.
\end{equation}

Then by (4.1.36) and [6, Theorem 8.5], an modification of the proof to [40, Proposition 4.14] proves the identities in our proposition. Note that (4.1.34) is just a special case of (4.1.24). \hfill \Box

If we assembly the results in Proposition 4.1.6, it is enough to study the corresponding analytic torsions. We will get back to this point in Corollary 7.3.7 for asymptotic analytic torsions.

### 4.2. Symmetric spaces of noncompact type with rank 1

In this subsection, we focus on the case where $\delta(G) = 1$ and $G$ has compact center (i.e. $\mathfrak{g}_p = 0$), so that $X$ is a symmetric space of noncompact type [40, Proposition 6.18].

Note that the rank $\delta(X)$ of $X$ (cf. [16, Section 2.7]) is the same as $\delta(G)$, then $\delta(X) = 1$. By the de Rham decomposition, we can write
\begin{equation}
X = X_1 \times X_2,
\end{equation}
where $X_1$ is an irreducible symmetric space of noncompact type with $\delta(X_1) = 1$, and $X_2$ is a symmetric space of noncompact type with $\delta(X_2) = 0$.

As in [3, Remark 7.9.2], among the noncompact simple connected real linear groups such that $m$ is odd and $\dim \mathfrak{b} = 1$, there are only $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{H})$, and $\mathrm{SO}^0(p, q)$ with $pq$ odd $> 1$. Also, we have $\mathfrak{sl}_2(\mathbb{R}) = \mathfrak{so}(3, 3)$ and $\mathfrak{sl}_2(\mathbb{H}) = \mathfrak{so}(5, 1)$. 
Therefore, the above list can be reduced to $\text{SL}_2(\mathbb{R})$ and $\text{SO}^0(p,q)$ with $pq$ odd $> 1$. Therefore, $X_1$ is one of the following cases

\begin{equation}
X_1 = \text{SL}_2(\mathbb{R})/\text{SO}(3) \text{ or } \text{SO}^0(p,q)/\text{SO}(p+q) \text{ with } pq > 1 \text{ odd}.
\end{equation}

Since $\delta(G) = 1$, we have the following decomposition of Lie algebras,

\begin{equation}
\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2,
\end{equation}

where

\begin{equation}
\mathfrak{g}_1 = \mathfrak{sl}_3(\mathbb{R}) \text{ or } \mathfrak{so}(p,q)
\end{equation}

with $pq > 1$ odd, and $\mathfrak{g}_2$ is real reductive with $\delta(\mathfrak{g}_2) = 0$.

By (4.2.3), up to finite (central) extension, we may assume that $(G, \theta, K) = (G_1, \theta_1, K_1) \times (G_2, \theta_2, K_2)$, where $(G_2, \theta_2, K_2)$ is a connected linear reductive Lie group with compact center and Lie algebra $\mathfrak{g}_2$, and $G_1$ is just a finite covering group of $\text{SL}_2(\mathbb{R})$ or $\text{SO}^0(p,q)$ with $pq > 1$ odd. In particular, $\delta(G_1) = 1$, $\delta(G_2) = 0$, and

\begin{equation}
X_j = G_j/K_j, j = 1, 2.
\end{equation}

Let $U_1, U_2$ be (connected linear) compact forms of $G_1, G_2$. Then we may take $U = U_1 \times U_2$. Let $(E, \rho^E)$ be an irreducible unitary representation of $U$, then

\begin{equation}
(E, \rho^E) = (E_1, \rho^{E_1}) \otimes (E_2, \rho^{E_2}),
\end{equation}

where $(E_1, \rho^{E_1})$ is an irreducible unitary representation of $U_1, j = 1, 2$. Let $F, F_1, F_2$ be the homogeneous flat vector bundles on $X, X_1, X_2$ associated with these representations. Then we have

\begin{equation}
F = F_1 \boxtimes F_2 := \pi_1^*(F_1) \otimes \pi_2^*(F_2),
\end{equation}

where $\pi_i$ denote the projection $X \to X_i, i = 1, 2$ induced from (4.2.1).

Take $\gamma = (\gamma_1, \gamma_2) \subset G = G_1 \times G_2$. For $i = 1, 2$, let $Z_{G_i}(\gamma_i) \subset G_i$ denote the centralizer of $\gamma_i$. Then

\begin{equation}
Z(\gamma) = Z_{G_1}(\gamma_1) \times Z_{G_2}(\gamma_2).
\end{equation}

Furthermore, $\gamma$ is semisimple (resp. elliptic) if and only if both $\gamma_1, \gamma_2$ are semisimple (resp. elliptic). Set $m_i = \dim X_i$, then $m_2$ is even.

**Proposition 4.2.1.** If $\gamma = (\gamma_1, \gamma_2) \in G$ is semisimple, for $t > 0$, we have

\begin{equation}
\text{Tr}_{s^{[\gamma]}}[(N^\Lambda (T^*X) - \frac{m}{2})\exp(-tD^{X,F_2;2}/2)] = \text{Tr}_{s^{[\gamma_1]}}[(N^\Lambda (T^*X_1) - \frac{m_1}{2})\exp(-tD^{X_1,F_1;2}/2)] \cdot \text{Tr}_{s^{[\gamma_2]}}[\exp(-tD^{X_2,F_2;2}/2)].
\end{equation}

Then if $\gamma_2$ is nonelliptic,

\begin{equation}
\text{Tr}_{s^{[\gamma]}}[(N^\Lambda (T^*X) - \frac{m}{2})\exp(-tD^{X,F_2;2}/2)] = 0.
\end{equation}

If $\gamma_2$ is elliptic, then

\begin{equation}
\text{Tr}_{s^{[\gamma]}}[(N^\Lambda (T^*X) - \frac{m}{2})\exp(-tD^{X,F_2;2}/2)] = \exp(TX_2(\gamma_2), \nabla^TX_2(\gamma_2))^\text{max}_2(\gamma_2)\gamma_2] \cdot \text{Tr}_{s^{[\gamma_1]}}[(N^\Lambda (T^*X_1) - \frac{m_1}{2})\exp(-tD^{X_1,F_1;2}/2)],
\end{equation}

where $[^\text{max}_2(\gamma_2)]$ is taking the coefficient of the Riemannian volume form on $X_2(\gamma_2)$. 
Proof. Note that the orbital integrals here are multiplicative with respect to the products of the underlying Lie groups.

We write
\begin{equation}
\tag{4.2.12}
N^\lambda(T^*X) = \frac{m}{2} = \left( N^\lambda(T^*X_1) - \frac{m_1}{2} \right) + \left( N^\lambda(T^*X_2) - \frac{m_2}{2} \right).
\end{equation}

Note that, since \( \delta(G_1) = 1 \), then by [3, Theorem 7.8.2], we always have
\begin{equation}
\tag{4.2.13}
\text{Tr}_{s}\left[ \exp(-tD_{X_1,F_1,2}/2) \right] = 0.
\end{equation}

Combining together (4.2.12) and (4.2.13), we get (4.2.9).

The identities (4.2.10), (4.2.11) follow from applying the results in [3, Theorem 7.8.2], [25, Theorem 6.4.1] to \( \text{Tr}_{s}\left[ \exp(-tD_{X_2,F_2,2}/2) \right] \). This completes the proof of our proposition. \( \square \)

For studying \( T(Z,F) \), Proposition 4.2.1 helps us to reduce the computations on \( \text{Tr}_{s}\left[ (N^\lambda(T^*Z) - \frac{m}{2}) \exp(-tD_{Z,1,F_2,2}/2) \right] \) to the model cases listed in (4.2.2). But it is far from enough to get explicit evaluations. In Sections 5 & 6, we will carry out more tools, which allows us work out a proof to Theorem 1.0.2.

5. Cartan subalgebra and root system of \( G \) when \( \delta(G) = 1 \)

We use the same notation as in previous sections. In Subsections 5.1 - 5.3, we assume that \( G \) has compact center with \( \delta(G) = 1 \). But, as we will see in Remark 5.3.3, the constructions and results in these subsections are still true (most of them are trivial) if \( U \) is compact and if \( G \) has noncompact center with \( \delta(G) = 1 \).

Subsection 5.4 is independent from other subsections, where we introduce a generalized Kirillov formula for compact Lie groups by specializing a result of Duflo, Heckman and Vergne [15, Theorem (7)].

Recall that \( T \) is a maximal torus of \( K \) with Lie algebra \( t \subset \mathfrak{t} \), and that \( b \subset \mathfrak{p} \) is defined in (4.1.3). Since \( \delta(G) = 1 \), then \( b \) is 1-dimensional. We now fix a vector \( \alpha_1 \in \mathfrak{b} \), \( |\alpha_1| = 1 \). Recall that \( b = \mathfrak{b} \oplus \mathfrak{t} \) is a Cartan subalgebra of \( \mathfrak{g} \). Let \( h^{\mathfrak{c}} \) be the Hermitian product on \( \mathfrak{g}^{\mathfrak{c}} \) induced by the scalar product \(-B(\cdot,\cdot)\) on \( \mathfrak{g} \).

5.1. Reductive Lie algebra with fundamental rank 1. Since \( G \) has compact center, then \( \mathfrak{b} \subset \mathfrak{z}_g \).

Let \( Z(\mathfrak{b}) \) be the centralizer of \( \mathfrak{b} \) in \( G \), and let \( Z(\mathfrak{b})^0 \) be its identity component with Lie algebra \( \mathfrak{z}(\mathfrak{b}) = \mathfrak{p}(\mathfrak{b}) \oplus \mathfrak{t}(\mathfrak{b}) \subset \mathfrak{g} \). Let \( \mathfrak{m} \) be the orthogonal subspace of \( \mathfrak{b} \) in \( \mathfrak{z}(\mathfrak{b}) \) (with respect to \( B \)) such that
\begin{equation}
\tag{5.1.1}
\mathfrak{z}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{m}.
\end{equation}

Then \( \mathfrak{m} \) is a Lie subalgebra of \( \mathfrak{z}(\mathfrak{b}) \), which is invariant by \( \theta \).

Put
\begin{equation}
\tag{5.1.2}
\mathfrak{p}_m = \mathfrak{m} \cap \mathfrak{p}, \quad t_m = \mathfrak{m} \cap \mathfrak{t}.
\end{equation}

Then
\begin{equation}
\tag{5.1.3}
\mathfrak{m} = \mathfrak{p}_m \oplus t_m, \quad \mathfrak{p}(\mathfrak{b}) = \mathfrak{b} \oplus \mathfrak{p}_m, \quad \mathfrak{t}(\mathfrak{b}) = t_m.
\end{equation}

Let \( \mathfrak{z}^\bot(\mathfrak{b}), \mathfrak{p}^\bot(\mathfrak{b}), \mathfrak{t}^\bot(\mathfrak{b}) \) be the orthogonal subspaces of \( \mathfrak{z}(\mathfrak{b}), \mathfrak{p}(\mathfrak{b}), \mathfrak{t}(\mathfrak{b}) \) in \( \mathfrak{g}, \mathfrak{p}, \mathfrak{t} \) respectively with respect to \( B \). Then
\begin{equation}
\tag{5.1.4}
\mathfrak{z}^\bot(\mathfrak{b}) = \mathfrak{p}^\bot(\mathfrak{b}) \oplus \mathfrak{t}^\bot(\mathfrak{b}).
\end{equation}

Moreover,
\begin{equation}
\tag{5.1.5}
\mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_m \oplus \mathfrak{p}^\bot(\mathfrak{b}), \quad \mathfrak{t} = \mathfrak{t}(\mathfrak{b}) \oplus \mathfrak{t}^\bot(\mathfrak{b}).
\end{equation}
Let $M \subset Z(b)^0$ be the analytic subgroup associated with $m$. If we identify $b$ with $\mathbb{R}$, then

$$Z(b)^0 = \mathbb{R} \times M. \quad (5.1.6)$$

Then $M$ is a Lie subgroup of $Z(b)^0$, i.e., it is closed in $Z(b)^0$. Let $K_M$ be the analytic subgroup of $M$ associated with the Lie subalgebra $\mathfrak{t}_m$. Since $M$ is reductive, $K_M$ is a maximal compact subgroup of $M$. Then the splittings in $(5.1.3)$, $(5.1.4)$, $(5.1.5)$ are invariant by the adjoint action of $K_M$.

Then $\mathfrak{t}$ is Cartan subalgebra of $\mathfrak{t}$, of $\mathfrak{t}_m$, and of $m$. Recall that $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. We fix $a_1 \in \mathfrak{b}$ such that $B(a_1, a_1) = 1$. The choice of $a_1$ fixes an orientation of $\mathfrak{b}$. Let $\mathfrak{n} \subset \mathfrak{z}^+(\mathfrak{b})$ be the direct sum of the eigenspaces of $\text{ad}(a_1)$ with the positive eigenvalues. Set $\overline{\mathfrak{n}} = \theta \mathfrak{n}$. Then

$$\mathfrak{z}^+(\mathfrak{b}) = \mathfrak{n} \oplus \overline{\mathfrak{n}}. \quad (5.1.7)$$

By [40, Subsection 6A], $\dim \mathfrak{n} = \dim \mathfrak{p} - \dim \mathfrak{p}_m - 1$. Then $\dim \mathfrak{n}$ is even under our assumption $\delta(G) = 1$. Put

$$l = \frac{1}{2} \dim \mathfrak{n}. \quad (5.1.8)$$

By [40, Proposition 6.2], there exists $\beta \in \mathfrak{b}^*$ such that if $a \in \mathfrak{b}$, $f \in \mathfrak{n}$, then

$$[a, f] = \beta(a)f, [a, \theta(f)] = -\beta(a)\theta(f). \quad (5.1.9)$$

The map $f \in \mathfrak{n} \mapsto f - \theta(f) \in \mathfrak{p}^+(\mathfrak{b})$ is an isomorphism of $K_M$-modules. Similarly, $f \in \mathfrak{n} \mapsto f + \theta(f) \in \mathfrak{z}^+(\mathfrak{b})$ is also an isomorphism of $K_M$-modules. Since $\theta$ fixes $K_M$, $\mathfrak{n} \simeq \overline{\mathfrak{n}}$ as $K_M$-modules via $\theta$.

By [40, Proposition 6.3], we have

$$[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{z}(\mathfrak{b}), [\mathfrak{n}, \overline{\mathfrak{n}}] = [\mathfrak{n}, \overline{\mathfrak{n}}] = 0. \quad (5.1.10)$$

Also

$$B|_{\mathfrak{n} \times \mathfrak{n}} = 0, B|_{\mathfrak{n} \times \overline{\mathfrak{n}}} = 0. \quad (5.1.11)$$

Then the bilinear form $B$ induces an isomorphism of $\mathfrak{n}^*$ and $\overline{\mathfrak{n}}$ as $K_M$-modules. Therefore, as $K_M$-modules, $\mathfrak{n}$ is isomorphic to $\mathfrak{n}^*$.

As a consequence of $(5.1.10)$, we get

$$[\mathfrak{z}(\mathfrak{b}), \mathfrak{z}(\mathfrak{b})], [\mathfrak{z}^+(\mathfrak{b}), \mathfrak{z}^+(\mathfrak{b})] \subset \mathfrak{z}(\mathfrak{b}), [\mathfrak{z}(\mathfrak{b}), \mathfrak{z}^+(\mathfrak{b})] \subset \mathfrak{z}^+(\mathfrak{b}). \quad (5.1.12)$$

Then $(\mathfrak{g}, \mathfrak{z}(\mathfrak{b}))$ is a symmetric pair.

If $k \in K_M$, let $M(k)$ be the centralizer of $k$ in $M$, and let $m(k)$ be its Lie algebra. Let $M(k)^0$ be the identity component of $M(k)$. The Cartan involution $\theta$ acts on $M(k)$. The associated Cartan decomposition is

$$m(k) = \mathfrak{p}_m(k) \oplus \mathfrak{t}_m(k), \quad (5.1.13)$$

where $\mathfrak{p}_m(k) = \mathfrak{p}_m \cap m(k), \mathfrak{t}_m(k) = \mathfrak{t}_m \cap m(k)$.

Recall that $Z(k)$ be the centralizer of $k$ in $G$ and that $Z(k)^0$ is the identity component of $Z(k)$ with Lie algebra $\mathfrak{z}(k) \subset \mathfrak{g}$. Then

$$M(k) = M \cap Z(k), \quad m(k) = m \cap \mathfrak{z}(k). \quad (5.1.14)$$

Note that $Z(k)^0$ is still a reductive Lie group equipped with the Cartan involution induced by the action of $\theta$. By the assumption that $\delta(G) = 1$, we have

$$\delta(Z(k)^0) = 1. \quad (5.1.15)$$

In particular,

$$\mathfrak{b} \subset p(k). \quad (5.1.16)$$

Set

$$\mathfrak{z}_b(k) = \mathfrak{z}(\mathfrak{b}) \cap \mathfrak{z}(k), \quad \mathfrak{p}_b(k) = \mathfrak{p}(\mathfrak{b}) \cap \mathfrak{p}(k), \quad \mathfrak{t}_b(k) = \mathfrak{t}(\mathfrak{b}) \cap \mathfrak{t}(k). \quad (5.1.17)$$
Then
\begin{equation}
(5.1.18)
\mathfrak{z}(k) = b \oplus m(k) = \mathfrak{p}_b(k) \oplus \mathfrak{t}_b(k).
\end{equation}
We also have the following identities,
\begin{equation}
(5.1.19)
\mathfrak{p}_b(k) = b \oplus \mathfrak{p}_m(k), \quad \mathfrak{t}_b(k) = \mathfrak{t}_m(k).
\end{equation}

Let $\mathfrak{p}^\perp_b(k)$, $\mathfrak{t}^\perp_b(k)$, $\mathfrak{z}^\perp_b(k)$ be the orthogonal spaces of $\mathfrak{p}_b(k)$, $\mathfrak{t}_b(k)$, $\mathfrak{z}_b(k)$ in $\mathfrak{p}(k)$, $\mathfrak{t}(k)$, $\mathfrak{z}(k)$ with respect to $B$, so that
\begin{equation}
(5.1.20)
\mathfrak{p}(k) = \mathfrak{p}_b(k) \oplus \mathfrak{p}^\perp_b(k), \quad \mathfrak{t}(k) = \mathfrak{t}_b(k) \oplus \mathfrak{t}^\perp_b(k), \quad \mathfrak{z}(k) = \mathfrak{z}_b(k) \oplus \mathfrak{z}^\perp_b(k).
\end{equation}
Then
\begin{equation}
(5.1.21)
\mathfrak{z}^\perp_b(k) = \mathfrak{p}^\perp_b(k) \oplus \mathfrak{t}^\perp_b(k) = \mathfrak{z}^\perp(k) \cap \mathfrak{z}(k).
\end{equation}
Put
\begin{equation}
(5.1.22)
n(k) = \mathfrak{z}(k) \cap \mathfrak{n}, \quad \bar{n}(k) = \mathfrak{z}(k) \cap \bar{\mathfrak{n}}.
\end{equation}
Then
\begin{equation}
(5.1.23)
\mathfrak{z}^\perp_b(k) = n(k) \oplus \bar{n}(k).
\end{equation}
By (5.1.17), (5.1.23), we get
\begin{equation}
(5.1.24)
\mathfrak{z}(k) = \mathfrak{p}_b(k) \oplus \mathfrak{t}_b(k) \oplus n(k) \oplus \bar{n}(k).
\end{equation}
Since $\delta(m(k)) = 0$, $\dim n(k)$ is even. We set
\begin{equation}
(5.1.25)
l(k) = \frac{1}{2} \dim n(k).
\end{equation}

Let $K_M(k)$ denote the centralizer of $k$ in $K_M$. The map $f \in n(k) \mapsto f - \theta(f) \in \mathfrak{p}^\perp_b(k)$ is an isomorphism of $K_M(k)$-modules, similar for $\mathfrak{t}^\perp_b(k)$. Since $\theta$ fixes $K_M(k)$, $n(k) \cong \bar{n}(k)$ as $K_M(k)$-modules via $\theta$.

5.2. A compact Hermitian symmetric space $Y$. Recall that $u = \sqrt{-1}p \oplus t$ is the compact form of $g$.

Let $u(b) \subset u$, $u_m \subset u$ be the compact forms of $\mathfrak{g}(b)$, $\mathfrak{m}$. Then
\begin{equation}
(5.2.1)
u(b) = \sqrt{-1}b \oplus u_m, \quad u_m = \sqrt{-1}p_m \oplus t_m.
\end{equation}
Since $M$ has compact center, let $U_M$ be the analytic subgroup of $U$ associated with $u_m$. Then $U_M$ is the compact form of $M$. Let $U(b) \subset U$, $A_0 \subset U$ be the connected subgroups of $U$ associated with Lie algebras $u(b)$, $\sqrt{-1}b$. Then $A_0$ is in the center of $U(b)$. By [40, Proposition 6.6], $A_0$ is closed in $U$ and is diffeomorphic to a circle $S^1$. Moreover, we have
\begin{equation}
(5.2.2)
U(b) = A_0 U_M.
\end{equation}
The bilinear form $-B$ induces an $\text{Ad}(U)$-invariant metric on $u$. Let $u^\perp(b) \subset u$ be the orthogonal subspace of $u(b)$. Then
\begin{equation}
(5.2.3)
u^\perp(b) = \sqrt{-1}p^\perp(b) \oplus t^\perp(b).
\end{equation}
By (5.1.12), we get
\begin{equation}
(5.2.4)[u(b), u(b)], [u^\perp(b), u^\perp(b)] \subset u(b), [u(b), u^\perp(b)] \subset u^\perp(b).
\end{equation}
Then $(u, u(b))$ is a symmetric pair.

Put $a_0 = a_1/\beta(a_1) \in b$. Set
\begin{equation}
(5.2.5)J = \sqrt{-1}\text{ad}(a_0) |_{u^\perp(b)} \in \text{End}(u^\perp(b)).
\end{equation}
By (5.1.9), $J$ is an $U(b)$-invariant complex structure on $u^\perp(b)$ which preserves $B_{u^\perp(b)}$. The spaces $\mathfrak{n}_C = n \otimes \mathbb{C}$, $\bar{n}_C = \bar{n} \otimes \mathbb{C}$ are exactly the eigenspaces of $J$ associated with eigenvalues $\sqrt{-1}$, $-\sqrt{-1}$.

The following proposition is just the summary of the results in [40, Section 6B].
Proposition 5.2.1. Set
(5.2.6) \[ Y_b = U/U(b). \]
Then \( Y_b \) is a compact symmetric space, and \( J \) induces an integrable complex structure on \( Y_b \) such that
(5.2.7) \[ T^{(1,0)} Y_b = U \times_{U(b)} {\mathfrak n}_C, \quad T^{(0,1)} Y_b = U \times_{U(b)} {\mathfrak n}_C. \]
The form \( -B(\cdot, J \cdot) \) induces a Kähler form \( \omega_{Y_b} \) on \( Y_b \).

Remark 5.2.2. By [40, Proposition 6.20], if \( G \) has compact center, then as symmetric spaces, the Kähler manifold \( Y_b \) is isomorphic either to \( SU(3)/U(2) \) or to \( SO(p + q)/SO(p + q - 2) \times SO(2) \) with \( pq > 1 \) odd. This way, the computations on \( Y_b \) can be made explicitly.

Let \( \omega^u \) be the canonical left-invariant 1-form on \( U \) with values in \( u \). Let \( \omega^{u(b)} \) and \( \omega^{u^+(b)} \) be the \( u(b) \) and \( u^+(b) \) components of \( \omega^u \), so that
(5.2.8) \[ \omega^u = \omega^{u(b)} + \omega^{u^+(b)}. \]
Moreover, \( \omega^{u(b)} \) defines a connection form on the principal \( U(b) \)-bundle \( U \to Y_b \). Let \( \Omega^{u(b)} \) be the curvature form, then
(5.2.9) \[ \Omega^{u(b)} = \frac{1}{2}[\omega^{u^+(b)}, \omega^{u^+(b)}]. \]
Note that the real tangent bundle of \( Y_b \) is given
(5.2.10) \[ TY_b = U \times_{U(b)} u^-(b). \]
Then \( -B|_{u^-(b)} \) induces a Riemannian metric \( g_{TY_b} \) on \( Y_b \). The corresponding Levi-Civita connection is induced by \( \omega^{u(b)} \).

Recall that the first splitting in (5.2.1) is orthogonal with respect to \( -B \). Let \( \Omega^{u_m} \) be the \( u_m \)-component of \( \Omega^{u(b)} \). By [40, eq. (6.48)],
(5.2.11) \[ \Omega^{u(b)} = \beta(a_1) \omega_{Y_b} \otimes \sqrt{-1} \Omega^{u_m}. \]
Moreover, by [40, Proposition 6.9], we have
(5.2.12) \[ B(\Omega^{u(b)}, \Omega^{u(b)}) = 0, \quad B(\Omega^{u_m}, \Omega^{u_m}) = \beta(a_1)^2 \omega_{Y_b} \otimes \Omega^{u_m}. \]

Now we fix \( k \in K_M \). Let \( U(k) \) be the centralizer of \( k \) in \( U \), and let \( U(k)^0 \) be its identity component. If \( k \) is not of finite order, then \( U(k) = U(k)^0 \). Let \( u(k) \) be the Lie algebra of \( U(k)^0 \). Then \( u(k) \) is the compact form of \( \mathfrak z(k) \), and \( U(k)^0 \) is the compact form of \( Z(k)^0 \).

We will use the same notation as in Subsection 5.1. Then the compact form of \( m(k) \) is given by
(5.2.13) \[ u_{m}(k) = \sqrt{-1} p_{m}(k) \oplus {\mathfrak r}_m(k). \]
Let \( u_b(k) \) be the compact form of \( \mathfrak z_b(k) \). Then
(5.2.14) \[ u_b(k) = \sqrt{-1} b \oplus u_m(k). \]
Let \( U_b(k) \) be the analytic subgroup associated with \( u_b(k) \). Then
(5.2.15) \[ U_b(k) = U(b) \cap U(k)^0. \]
Set
(5.2.16) \[ Y_b(k) = U(k)^0/U_b(k). \]
As in Proposition 5.2.1, \( Y_b(k) \) is a connected complex manifold equipped with a Kähler form \( \omega_{Y_b(k)} \).

Let \( u_b^+(k) \) be the orthogonal space of \( u_b(k) \) in \( u(k) \) with respect to \( B \). Then
(5.2.17) \[ u_b^+(k) = \sqrt{-1} p_b^+(k) \oplus t_b^+(k). \]
Then the real tangent bundle of $Y_b(k)$ is given by
\[(5.2.18)\quad TY_b(k) = U(k)^0 \times U_b(k) u^+_{b}(k).\]
Moreover,
\[(5.2.19)\quad T^{(1,0)} Y_b(k) = U(k)^0 \times U_b(k) n(k)C, \quad T^{(0,1)} Y_b(k) = U(k)^0 \times U_b(k) n(k)C.\]

Since $\omega Y_b(k)$ is invariant under the left action of $U(k)^0$ on $Y_b(k)$, we also can view $\omega Y_b(k)$ as an element in $A^2(u^+_{b}(k))^*$. Let $\Omega^u_{m}(k)$ be the curvature form as in (5.2.9) for the pair $(U(k)^0, U_b(k))$, which can be viewed as an element in $A^2(u^+_{b}(k))^* \otimes u_b(k)$. Using the splitting (5.2.14), let $\Omega^u_{m}(k)$ be the $u_m(k)$-component of $\Omega^u_{m}(k)$. Then as in (5.2.11) and (5.2.12), we have
\[(5.2.20)\quad \Omega^u_{ab}(k) = \beta(a_1)\omega Y(k) \otimes \sqrt{-1}a_1 + \Omega^u_{m}(k),\]
and
\[(5.2.21)\quad B(\Omega^u_{ab}(k), \Omega^u_{ab}(k)) = 0, \quad B(\Omega^u_{m}(k), \Omega^u_{m}(k)) = \beta(a_1)^2 \omega Y_b(k),.\]

5.3. Positive root system and character formula. Recall that $t$ is Cartan subalgebra of $\mathfrak{t}$, of $\mathfrak{t}_m$, and of $\mathfrak{m}$. Recall that $\mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $R(\mathfrak{g}_C, \mathfrak{h}_C)$, $R(\mathfrak{m}_C, \mathfrak{t}_C)$, $R(\mathfrak{t}_m)_C$, and $R(\mathfrak{h}_C, \mathfrak{h}_C)$ be the corresponding root systems. Then $R(\mathfrak{m}_C, \mathfrak{t}_C) \hookrightarrow R(\mathfrak{h}_C, \mathfrak{h}_C)$. Moreover, if we view $\mathfrak{t}_C$ as the subspace of $\mathfrak{h}_C$, which consists of the $C$-linear map vanishing on $\mathfrak{b}$, then $R(\mathfrak{m}_C, \mathfrak{t}_C) = R(\mathfrak{z}(\mathfrak{b})_C, \mathfrak{h}_C)$.

Let $H \subset G$ be the analytic subgroup associated with $\mathfrak{h}$. Then
\[(5.3.1)\quad H = T \exp(\mathfrak{b}).\]
The group $H$ is a maximally compact Cartan subgroup of $G$. Put
\[(5.3.2)\quad t_U = \sqrt{-1}t \subset u.\]
Then $t_U$ is a Cartan subalgebra of $u$. Let $T_U \subset U$ be the corresponding maximal torus. Then $T_U$ is a circle in $T_U$. Then $t$ is a Cartan subalgebra of $u_m$, and the corresponding maximal torus is $T$.

We have $t_{U_C} = \mathfrak{h}_C$. Let $R(u_C, t_{U_C})$ be the associated root system. Then
\[(5.3.3)\quad R(u_C, t_{U_C}) = R(\mathfrak{g}_C, \mathfrak{h}_C).\]
Similarly, we have
\[(5.3.4)\quad R(u_m, t_{C}) = R(u, \mathfrak{b}_C, \mathfrak{h}_C) = R(\mathfrak{m}_C, t_{C}).\]

For a root $\alpha \in R(\mathfrak{g}_C, \mathfrak{h}_C)$, if $\alpha(a_1) = 0$, then $\alpha \in R(\mathfrak{m}_C, t_{C})$. Fix a positive root system $R^+(\mathfrak{m}_C, t_{C})$, we get a positive root system $R^+(\mathfrak{g}_C, \mathfrak{h}_C)$ consisting of elements $\alpha$ such that $\alpha(a_1) > 0$ and the elements in $R^+(\mathfrak{m}_C, t_{C})$.

Put $W_\mathfrak{g} = W(\mathfrak{g}_C, \mathfrak{h}_C) = W(u_C, t_{U_C})$ the associated Weyl group. If $\omega \in W_\mathfrak{g}$, let $l(\omega)$ denote the length of $\omega$ with respect to $R^+(\mathfrak{g}_C, \mathfrak{h}_C)$. Set
\[(5.3.5)\quad \varepsilon(\omega) = (-1)^{l(\omega)}.\]
Let $W(U, T_U)$ be the analytic Weyl group, then $W_\mathfrak{g} = W(U, T_U)$.
Put
\[(5.3.6)\quad w_a = \{\omega \in W_\mathfrak{g} : \omega^{-1} \cdot \alpha > 0, \text{for all } \alpha \in R^+(\mathfrak{m}_C, t_{C})\}.\]
Let $R(u, t_U)$ be the real root system for the pair $(U, T_U)$ [11, Chapter V]. Then if $\alpha^0 \in R(u, t_U)$, after tensoring with $C$, we view it as an element in $t_{U_C}$, then $2\pi i a_0 \in R(u_C, t_{U_C})$. If $\alpha \in R(u_C, t_{U_C})$, then $\alpha^0 = \frac{\pi i}{2\pi i} a_0 \in R(u, t_U)$. Similarly,
The relations of them to the complex root systems are the same as above. 

The positive root systems \( R^+(u, t_U) \), \( R^+(u, b(t_U), t_U) = R^+(u, m, t) \) are induced by \( R^+(g_{\omega}, h_{\omega}) \), \( R^+(m_{\omega}, t_U) \). More precisely, \( \alpha^0 \in R^+(u, t_U) \) if \( \alpha^0(\sqrt{-\theta_1}) > 0 \) or if \( \alpha^0(\sqrt{-\theta_1}) = 0 \) and \( \alpha^0 \in R^+(u, m, t) \).

Put

\[
\rho_u = \frac{1}{2} \sum_{\alpha \in R^+(u, t_U)} \alpha^0 \in t^*_U, \quad \rho_{u_m} = \frac{1}{2} \sum_{\alpha \in R^+(u_m, t)} \alpha^0 \in t^*.
\]

Then \( \rho_u |_U = \rho_{u_m} \).

Let \( P_{U+}(U) \subset t_U^* \) be the set of dominant weights of \((U, t_U)\) with respect to \( R^+(u, t_U) \). If \( \lambda \in P_{U+}(U) \), let \( (E_{\lambda}, \rho^{E_{\lambda}}) \) be the irreducible unitary representation of \( U \) with the highest weight \( \lambda \), which by the unitary trick extends to an irreducible representation of \( G \).

By [42, Lemmas 1.1.2.15 & 2.4.2.1], if \( \omega \in W_u \), then \( \omega(\lambda + \rho_u) - \rho_u \) is a dominant weight for \( R^+(u, b(t_U), t_U) \). Let \( V_{\lambda, \omega} \) denote the representation of \( U(b) \) with the highest weight \( \omega(\lambda + \rho_u) - \rho_u \).

Recall that \( U(b) \) acts on \( n_C \). Let \( H(n_C, E_{\lambda}) \) be the Lie algebra cohomology of \( n_C \) with coefficients in \( E_{\lambda} \) (cf. [23]). By [42, Theorem 2.5.1.3], for \( i = 0, \ldots, 2l \), we have the identification of \( U(b) \)-modules,

\[
H^i(n_C, E_{\lambda}) \cong \bigoplus_{\omega \in W_u} V_{\Lambda^i \omega}.
\]

By (5.3.9) and the Poincaré duality, we get the following identifications as \( U(b) \)-modules,

\[
\bigoplus_{i=0}^{2l} (-1)^i \Lambda^i n_C^* \otimes E_{\lambda} = \bigoplus_{\omega \in W_u} \epsilon(\omega) V_{\Lambda^i \omega}.
\]

Note that if we apply the unitary trick, the above identification also holds as \( Z(b)^0 \)-modules.

**Definition 5.3.1.** Let \( P_0 : t^*_U \to t^* \) denote the orthogonal projection with respect to \( R^+|_{t_U^*} \). For \( \omega \in W_u \), \( \lambda \in P_{U+}(U) \), put

\[
\eta_\omega(\lambda) = P_0(\omega(\lambda + \rho_u) - \rho_u) \in t^*.
\]

Note that

\[
P_0 \rho_u = \rho_{u_m}.
\]

Then

\[
\eta_\omega(\lambda) = P_0(\omega(\lambda + \rho_u)) - \rho_{u_m}.
\]

**Proposition 5.3.2.** If \( \lambda \in P_{U+}(U) \), for \( \omega \in W_u \), \( \eta_\omega(\lambda) \) is a dominant weight of \((U_M, T)\) with respect to \( R^+(u_m, t) \). Moreover, the restriction of the \( U(b) \)-representation \( V_{\lambda, \omega} \) to the subgroup \( U_M \) is irreducible, which has the highest weight \( \eta_\omega(\lambda) \).

**Proof.** Since \( \omega(\lambda + \rho_u) - \rho_u \) is analytically integrable, then \( \eta_\omega(\lambda) \) is also analytically integrable as a weight associated with \((U_M, T)\). By (5.3.7) and the corresponding identification of positive root systems, we know that \( \eta_\omega(\lambda) \) is dominant with respect to \( R^+(u_m, t) \).

Recall that \( A_0 \cong S^1 \) is defined in Subsection 5.2. The group \( U(b) \) has a finite extension \( A_0 \times U_M \), then we view \( V_{\lambda, \omega} \) as an irreducible unitary representation of \( A_0 \times U_M \), whose restriction to \( U_M \) is clearly irreducible with highest weight \( \eta_\omega(\lambda) \).

This completes the proof of our proposition. \( \square \)
Remark 5.3.3. In general, $U$ is just the analytic subgroup of $G_C$ with Lie algebra $u$. If $U$ is compact but $G$ has noncompact center, i.e., $\mathfrak{z}_u = \mathfrak{b}$, then $\mathfrak{n} = \mathfrak{a} = 0$, so that $l = 0$. Recall that in this case, $G'$, $U'$ are defined in Subsection 4.1. Then we have
\begin{equation}
M = G', U_M = U'.
\end{equation}
The compact symmetric space $Y_b$ now reduces to one point.
Moreover, in (5.3.6), the set $W_u = \{1\}$, so that $V_{\lambda, \omega}$ becomes just $E_\lambda$ itself. The identities (5.3.9), (5.3.10) are trivially true, so is Proposition 5.3.2.

5.4. Kirillov character formula for compact Lie groups. In this subsection, we recall the Kirillov character formula for compact Lie groups. We only use the group $U_M$ as an explanatory example. We fix the maximal torus $T$ and the positive (real) root system $R^+(u_m, t)$.

Let $\lambda \in t^*$ be a dominant (analytically integrable) weight of $U_M$. Let $(V_\lambda, \rho^{V_\lambda})$ be the irreducible unitary representation of $U_M$ with the highest weight $\lambda$.

Put
\begin{equation}
\mathcal{O}_{\lambda + \rho_{u_m}} = \text{Ad}^*(U_M)(\lambda + \rho_{u_m}) \subset u_m^*.
\end{equation}
Then $\mathcal{O}_{\lambda + \rho_{u_m}}$ is an even-dimensional closed manifold.

Since $\lambda + \rho_{u_m}$ is regular, then we have the following identifications of $U_M$-manifolds,
\begin{equation}
\mathcal{O}_{\lambda + \rho_{u_m}} \simeq U_M/T.
\end{equation}

If $a \in u_m$, if $f \in \mathcal{O}_{\lambda + \rho_{u_m}}$, put
\begin{equation}
a_L(f) = -\text{ad}^*(a)f \in T_f \mathcal{O}_{\lambda + \rho_{u_m}}.
\end{equation}
Then $a_L$ is a tangent vector field on $\mathcal{O}_{\lambda + \rho_{u_m}}$. Such vector fields span the whole tangent space at each point.

If $a, b \in u_m$, $f \in \mathcal{O}_{\lambda + \rho_{u_m}}$, put
\begin{equation}
\omega_L(a_L, b_L)f = -(f, [a, b]).
\end{equation}
Then $\omega_L$ is a $U_M$-invariant symplectic form on $\mathcal{O}_{\lambda + \rho_{u_m}}$. Put $r^+ = \frac{1}{2} \dim u_m/t$.

In fact, if we can define an almost complex structure on $T\mathcal{O}_{\lambda + \rho_{u_m}}$ such that the holomorphic tangent bundle is given by the positive root system $R^+(u_m, t)$. Then $(\mathcal{O}_{\lambda + \rho_{u_m}}, \omega_L)$ become a closed Kähler manifold, and $r^+$ is just its complex dimension.

The Liouville measure on $\mathcal{O}_{\lambda + \rho_{u_m}}$ is given as follows,
\begin{equation}
d\mu_\lambda = (\omega_L)^{r^+}/(r^+)!.
\end{equation}
It is invariant by the left action of $U_M$.

By the Kirillov formula, if $y \in u_m$, we have
\begin{equation}
\hhat{A}^{-1}(\text{ad}(y)|_{u_m}) \text{Tr}^{V_\lambda}(\rho^{V_\lambda}(e^y)) = \int_{f \in \mathcal{O}_{\lambda + \rho_{u_m}}} e^{2\pi i(f, y)} d\mu_\lambda.
\end{equation}
If $k \in T$, put $Z = U_M(k)^0$, then $T \subset Z$. Then $T$ is also a maximal torus of $Z$.

In the sequel, we will give a generalized version of (5.4.6) for describing the function $\text{Tr}^{V_\lambda}(\rho^{V_\lambda}(ke^y))$ with $y \in \mathfrak{z}$.

Let $\mathfrak{q}$ be the orthogonal space of $\mathfrak{z}$ in $u_m$ with respect to $B$, so that
\begin{equation}
u_m = \mathfrak{z} \oplus \mathfrak{q}.
\end{equation}
Let $R(\mathfrak{z}, t)$ be the real root system associated with the pair $(\mathfrak{z}, t)$. Since the adjoint action of $T$ preserves the splitting in (5.4.7), then $R(u_m, t)$ splits into two disjoint parts

$$R(u_m, t) = R(\mathfrak{z}, t) \cup R(\mathfrak{q}, t),$$

where $R(\mathfrak{q}, t)$ is just the set of real roots for the adjoint action of $t$ on $\mathfrak{q}_C$.

The positive root system $R^+(u_m, t)$ induces a positive root system $R^+(\mathfrak{z}, t)$. Set

$$R^+(\mathfrak{q}, t) = R^+(u_m, t) \cap R(\mathfrak{q}, t).$$

Then we have the disjoint union as follows,

$$R^+(u_m, t) = R^+(\mathfrak{z}, t) \cup R^+(\mathfrak{q}, t).$$

Put

$$\rho_z = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{z}, t)} \alpha^0, \quad \rho_q = \frac{1}{2} \sum_{\alpha^0 \in R^+(\mathfrak{q}, t)} \alpha^0.$$  

Then

$$\rho_{u_m} = \rho_z + \rho_q \in t^*.$$  

Let $C \subset t^*$ denote the Weyl chamber corresponding to $R^+(u_m, t)$, and let $C_0 \subset t^*$ denote the Weyl chamber corresponding to $R^+(\mathfrak{z}, t)$. Then $C \subset C_0$.

Let $W(u_m, t), W(\mathfrak{q}, t)$ be the Weyl groups associated with the pair $(u_m, t), (\mathfrak{z}, t)$ respectively. Then $W(\mathfrak{q}, t)$ is canonically a subgroup of $W(u_m, t), t_C$.

Put

$$W^1(k) = \{ \omega \in W(u_m, t) \mid \omega(C) \subset C_0 \}.$$  

Note that the set $W^1(k)$ is similar to the set $W_u$ defined in (5.3.6).

**Lemma 5.4.1.** The inclusion $W^1(k) \hookrightarrow W(u_m, t)$ induces a bijection between $W^1(k)$ and the quotient $W(\mathfrak{q}, t) \backslash W(u_m, t), t_C$.

**Proof.** This lemma follows from that $W(\mathfrak{q}, t)$ acts simply transitively on the Weyl chambers associated with $(\mathfrak{z}, t)$.

Let $\lambda \in t^*$ be the dominant weight of $U_M$ as before. Then $\lambda + \rho_{u_m} \in C$. Set

$$\mathcal{O} = O_{\lambda + \rho_{u_m}} \subset u_m^*.$$  

Let $\mathcal{O}^k$ denote the fixed point set of the holomorphic action of $k$ on $\mathcal{O}$. We embed $\mathcal{O}^* \subset u_m^*$ by the splitting (5.4.7). Then

$$\mathcal{O}^k = \mathcal{O} \cap \mathcal{O}^*.$$  

The following lemma can be found in [15, Lemma (7)], [10, Lemmas 6.1.1, 7.2.2].

**Lemma 5.4.2.** As subsets of $\mathcal{O}^*$, we have the following identification,

$$\mathcal{O}^k = \cup_{\sigma \in W^1(k)} \text{Ad}^*(Z)(\sigma(\lambda + \rho_{u_m})) \subset \mathcal{O}^*,$$

where the union is disjoint.

For each $\sigma \in W^1(k)$, put

$$\mathcal{O}^k_{\sigma(\lambda + \rho_{u_m})} = \text{Ad}^*(Z)(\sigma(\lambda + \rho_{u_m})) \subset \mathcal{O}^*.$$  

Let $d\mu^k_\delta$ denote the Liouville measure on $\mathcal{O}^k_{\sigma(\lambda + \rho_{u_m})}$ as defined in (5.4.5). If $\delta \in t^*$ is (real) analytically integrable, let $\xi_\delta$ denote the character of $T$ with differential $2\pi i \delta$. Note that for $\sigma \in W^1(k)$, $\sigma \rho_{u_m} + \rho_{u_m}$ is analytically integrable even $\rho_{u_m}$ may not be analytically integrable.
Definition 5.4.3. For $\sigma \in W^1(k)$, set

$$
\varphi_\sigma(\lambda) = \epsilon(\sigma) \frac{\xi_{\omega(\lambda + r_m) + r_m}(k)}{\Pi_{\alpha^\vee \in R^+(q, t)}(\xi_{\alpha^\vee}(k) - 1)}.
$$

Note that if $y \in \mathfrak{g}$, the following analytic function

$$
\frac{\det(1 - e^{ad(y)}\text{Ad}(k))|_q}{\det(1 - \text{Ad}(k))|_q}
$$

has a square root which is analytic in $y \in \mathfrak{g}$ and equals to 1 at $y = 0$. We denote this square root by

$$
\left[\frac{\det(1 - e^{ad(y)}\text{Ad}(k))|_q}{\det(1 - \text{Ad}(k))|_q}\right]^{\frac{1}{2}}.
$$

The following theorem is a special case of a generalized Kirillov formula obtained by Duflot, Heckman and Vergne [15, Theorem (7)]. We will also include a proof for the sake of completeness.

Theorem 5.4.4 (Generalized Kirillov formula). For $y \in \mathfrak{g}$, we have the following identity of analytic functions,

$$
\widehat{A}^{-1}(\text{ad}(y))|_q \frac{\det(1 - e^{ad(y)}\text{Ad}(k))|_q}{\det(1 - \text{Ad}(k))|_q} \text{Tr}^{V_n}[\rho^{V_n}(ke^y)]
$$

(5.4.21)

$$
= \sum_{\sigma \in W^1(k)} \varphi_\sigma(\lambda) \int_{f \in \mathcal{O}_n^{\tau_0}(\lambda + r_m)} e^{2\pi i(f, y)} d\mu^y.
$$

If $k = 1$, (5.4.21) is reduced to (5.4.6).

Proof. Let $t'$ denote the set of regular element in $\mathfrak{t}$ associated with the root $R(u_m, t)$, which is an open dense subset of $t$. Since both sides of (5.4.21) are invariant by adjoint action of $Z$, then we only need to prove (5.4.21) for $y \in t'$.

We firstly compute the left-hand side of (5.4.21).

For $y \in t'$, then

$$
\Pi_{\alpha^\vee \in R^+(u_m, t)}(\alpha^0, y) \neq 0.
$$

We have

$$
\widehat{A}^{-1}(\text{ad}(y))|_q = \Pi_{\alpha^\vee \in R^+(q, t)} e^{\pi i(\alpha^0, y) - e^{-\pi i(\alpha^0, y)}}.
$$

(5.4.23)

Let $y_0 \in t$ be such that $k = \exp(y_0)$. Then

$$
\frac{\det(1 - e^{ad(y)}\text{Ad}(k))|_q}{\det(1 - \text{Ad}(k))|_q}^{\frac{1}{2}}
$$

(5.4.24)

$$
= \Pi_{\alpha^\vee \in R^+(q, t)} e^{\pi i(\alpha^0, y+y_0) - e^{-\pi i(\alpha^0, y+y_0)}} e^{\pi i(\alpha^0, y_0) - e^{-\pi i(\alpha^0, y_0)}}
$$

By the Weyl character formula for $(U_M, T)$, we get

$$
\text{Tr}^{V_n}[\rho^{V_n}(ke^y)] = \text{Tr}^{V_n}[\rho^{V_n}(e^{y+y_0})]
$$

(5.4.25)

$$
= \sum_{\omega \in W(u_m, \mathfrak{t}, t)} \epsilon(\omega) e^{2\pi i(\omega(\lambda + r_m), y+y_0)}
$$

$$
= \sum_{\alpha^\vee \in R^+(u_m, t)} \epsilon(\alpha^\vee) e^{\pi i(\alpha^0, y+y_0) - e^{-\pi i(\alpha^0, y+y_0)}}
$$

Note that

$$
\Pi_{\alpha^\vee \in R^+(q, t)} e^{\pi i(\alpha^0, y+y_0) - e^{-\pi i(\alpha^0, y+y_0)}} = e^{2\pi i(\rho_0, y_0)}.
$$

(5.4.26)
Note that since $2\rho_3$ is analytically integrable, then
\begin{equation}
(5.4.27) \quad \xi_{2\rho_3}(k) = 1.
\end{equation}
This implies that
\begin{equation}
(5.4.28) \quad e^{-2\pi i (\rho_3, y_0)} = e^{2\pi i (\rho_3, y_0)}.
\end{equation}
Combining (5.4.23) - (5.4.28), we get the left-hand side of (5.4.21) is equal to the following function,
\begin{equation}
(5.4.29) \quad \frac{e^{2\pi i (\rho_3, y_0)}}{\Pi_{\omega \in \mathbb{R}^+ (\lambda, \mu)} (2\pi i \alpha^0, y)} \sum_{\omega \in W(\mathbb{C}, t)} \frac{\varepsilon(\omega) e^{2\pi i (\omega (\lambda + \rho_{\mu_m}), y + y_0)}}{\Pi_{\alpha^0 \in \mathbb{R}^+ (\lambda, \mu)} (e^{\pi i (\alpha^0, y)} - e^{-\pi i (\alpha^0, y)})}.
\end{equation}
Now we show that the right-hand side of (5.4.21) is also equal to (5.4.29).
Note that for $\omega \in W(\mathbb{C}, t)$, $\omega \rho_{\mu_m} - \rho_{\mu_m}$ is analytically integrable. We claim that if $\omega \in W(\mathbb{C}, t)$, then
\begin{equation}
(5.4.30) \quad \xi_{\omega \rho_{\mu_m} - \rho_{\mu_m}}(k) = e^{2\pi i (\omega \rho_{\mu_m} - \rho_{\mu_m}, y_0)} = 1.
\end{equation}
We now prove (5.4.30). Let $e$ be the center of $u_m$, and put
\begin{equation}
(5.4.31) \quad u_{m, ss} = [u_m, u_m].
\end{equation}
As in (3.1.4), we have
\begin{equation}
(5.4.32) \quad u_m = e \oplus u_{m, ss}.
\end{equation}
Let $C^0$ denote the identity component of the center of $U_M$, and let $U_{M, ss}$ be the analytic subgroup of $U_M$ associated with $u_{m, ss}$. Then
\begin{equation}
(5.4.33) \quad U_M = C^0 U_{M, ss}.
\end{equation}
By the Weyl’s theorem [21, Theorem 4.26], the universal covering group of $U_{M, ss}$ is compact, which we denote by $\tilde{U}_{M, ss}$. Put
\begin{equation}
(5.4.34) \quad \tilde{U}_M = C^0 \times \tilde{U}_{M, ss}.
\end{equation}
Then $\tilde{U}_M$ is canonically a finite covering of $U_M$. Let $\tilde{T}$ be the maximal torus of $\tilde{U}_M$ associated with the Cartan subalgebra $t$.
Let $k = \exp(y_0) \in \tilde{T}$ be a lift of $k \in T$. Let $\tilde{Z}$ be the analytic subgroup of $\tilde{U}_M$ associated with $\tilde{t}$.
Note that $\text{Ad}(k)|_{u_m} = \text{Ad}(k)|_{u_m}$, then $\tilde{Z}$ is a finite cover of $Z$. Let $N_{\tilde{Z}}(\tilde{T})$ be the normalizer of $\tilde{T}$ in $\tilde{Z}$, then
\begin{equation}
(5.4.35) \quad W(\mathbb{C}, t) \simeq N_{\tilde{Z}}(\tilde{T})/\tilde{T}.
\end{equation}
The weight $\rho_{\mu_m}$ is analytically integrable with respect to $\tilde{T}$. Then by (5.4.35), if $\omega \in W(\mathbb{C}, t)$,
\begin{equation}
(5.4.36) \quad \xi_{\rho_{\mu_m}}(k) = \xi_{\rho_{\mu_m}}(\tilde{k}).
\end{equation}
The equation (5.4.30) follows exactly from (5.4.36).
As a consequence of (5.4.30), we get that for $\sigma \in W^1 (k)$, if $\omega \in W(\mathbb{C}, t)$, then
\begin{equation}
(5.4.37) \quad e^{2\pi i (\omega \sigma (\lambda + \rho_{\mu_m}), y_0)} = e^{2\pi i (\omega (\lambda + \rho_{\mu_m}), y_0)}.
\end{equation}
For $\sigma \in W^1 (k)$, since $\sigma (\lambda + \rho_{\mu_m}) \in C_0$ and $y$ is regular, we have
\begin{equation}
(5.4.38) \quad \int_{f \in C^\infty_{\alpha^0 (\lambda + \rho_{\mu_m})}} \frac{e^{2\pi i (f, y)} d\mu_f^k}{\Pi_{\alpha^0 \in \mathbb{R}^+ (\lambda, \mu)} (2\pi i \alpha^0, y)} \sum_{\omega \in W(\mathbb{C}, t)} \varepsilon(\omega) e^{2\pi i (\omega (\lambda + \rho_{\mu_m}), y_0)}.
\end{equation}
We rewrite $\varphi_k(\sigma, \lambda)$ as follows,

\begin{equation}
(5.4.39) \\
\varepsilon(\sigma) \prod_{\alpha^0 \in R^+ (\mathfrak{g}, \mathfrak{h})} (e^{i\pi(\alpha^0, y_0)} - e^{-i\pi(\alpha^0, y_0)})^{a_{\alpha^0}(\sigma(\lambda + \rho_{u_m}), y_0)}.
\end{equation}

Combining together (5.4.37) - (5.4.39), a direct computation shows that the right-hand side of (5.4.21) is given exactly by (5.4.29). This completes the proof of our theorem. □

**Remark 5.4.5.** Note that for $\sigma \in W^1(k)$, the regular positive weight $\sigma(\lambda + \rho_{u_m})$ is analytically integrable with respect to $(\tilde{Z}, \tilde{T})$. If $\rho_3$ is also analytically integrable with respect to $\tilde{T}$, then $\sigma(\lambda + \rho_{u_m}) - \rho_3$ is a dominant weight for $(\tilde{Z}, \tilde{T})$ with respect to $R^+(\mathfrak{t}, \mathfrak{h})$. In this case, let $E^k_{\sigma} = E_{\sigma(\lambda + \rho_{u_m}) - \rho_3}$ be the irreducible unitary representation of $\tilde{Z}$ with highest weight $\sigma(\lambda + \rho_{u_m}) - \rho_3$. Then by (5.4.6), (5.4.21), we get that for $y \in \mathfrak{h}$,

\begin{equation}
(5.4.40) \\
\frac{[\det(1 - e^{ad(y)}) \text{Ad}(k)]_Q}{\det(1 - \text{Ad}(k))_Q} \cdot \text{Tr} V_\lambda \left[ \mu^{V_\lambda} (k e^y) \right] = \sum_{\sigma \in W^1(k)} \varphi_k(\sigma, \lambda) \text{Tr} E^k_{\sigma} \left[ \mu^{E^k_{\sigma}} (e^y) \right].
\end{equation}

Let $\text{Vol}_L(O^k_{\sigma(\lambda + \rho_{u_m})})$ denote its symplectic volume with respect to the Liouville measure. Then

\begin{equation}
(5.4.41) \\
\text{Vol}_L(O^k_{\sigma(\lambda + \rho_{u_m})}) = \prod_{\alpha^0 \in R^+ (\mathfrak{t}, \mathfrak{h})} \frac{\langle \alpha^0, \sigma(\lambda + \rho_{u_m}) \rangle}{\langle \alpha^0, \rho_3 \rangle} = \dim E^k_{\sigma}.
\end{equation}

Note that the first equality of (5.4.41) still holds even $\rho_3$ is not analytically integrable with respect to $\tilde{T}$.

6. A GEOMETRIC LOCALIZATION FORMULA FOR ORBITAL INTEGRALS

Recall that $G_C$ is the complexification of $G$ with Lie algebra $\mathfrak{g}_C$, and that $G$, $U$ are the analytic subgroups of $G_C$ with Lie algebra $\mathfrak{g}$, $\mathfrak{u}$ respectively. In this section, we always assume that $U$ is compact, we do not require that $G$ has compact center. We need not to assume $\delta(G) = 1$ either.

Under the settings in Subsection 4.1, for $t > 0$ and semisimple $\gamma \in G$, we set

\begin{equation}
(6.0.1) \\
\mathcal{E}_{X, \gamma}(F, t) = \text{Tr} V_\gamma \left[ (N^X (T^X) - \frac{m}{2}) \exp(-t D X, F, 2) \right].
\end{equation}

The indice $X$, $F$ in this notation indicate precisely the symmetric space and the flat vector bundle which are concerned for defining the orbital integrals.

If $\gamma \in G$ is semisimple, then there exists a unique elliptic element $\gamma_e$ and a unique hyperbolic element $\gamma_h$ in $G$, such that $\gamma = \gamma_e \gamma_h = \gamma_h \gamma_e$. Here, we will show that $\mathcal{E}_{X, \gamma}(F, t)$ becomes a sum of the orbital integrals associated with $\gamma_h$, but defined for the centralizer of $\gamma_e$ instead of $G$. This suggests that the elliptic part of $\gamma$ should lead to a localization for the geometric orbital integrals.

We still fix a maximal torus $T$ of $K$ with Lie algebra $\mathfrak{t}$. For simplicity, if $\gamma \in G$ is semisimple, we may and we will assume that

\begin{equation}
(6.0.2) \\
\gamma = e^a k, k \in T, a \in \mathfrak{p}, \text{Ad}(k^{-1}) a = a.
\end{equation}

In this case,

\begin{equation}
(6.0.3) \\
\gamma_e = k \in T, \gamma_h = e^a.
\end{equation}

Recall that $Z(\gamma_e)^0$ is the identity component of the centralizer of $\gamma_e$ in $G$. Then

\begin{equation}
(6.0.4) \\
\gamma_h \in Z(\gamma_e)^0.
\end{equation}
The Cartan involution $\theta$ preserves $Z(\gamma_\epsilon)^0$ such that $Z(\gamma_\epsilon)^0$ is a connected linear reductive Lie group. Then we have the following diffeomorphism

\[(6.0.5)\quad Z(\gamma_\epsilon)^0 = K(\gamma_\epsilon)^0 \exp(p(\gamma_\epsilon)).\]

It is clear that $\delta(Z(\gamma_\epsilon)^0) = \delta(G)$. Moreover, $H$ is still a maximally compact Cartan subgroup of $Z(\gamma_\epsilon)^0$.

Recall that $T_U$ is a maximal torus of $U$ with Lie algebra $t_U = \sqrt{-1}b \oplus t \subset u$. Let $R^+(u, t_U)$ be a positive root system for $R(u, t_U)$, which is not necessarily the same as in Subsection 5.3 when $\delta(G) = 1$.

Since $U$ is the compact form of $G$, then $U(\gamma_\epsilon)^0$ is the compact form for $Z(\gamma_\epsilon)^0$. Moreover, $T_U$ is also a maximal torus of $U(\gamma_\epsilon)^0$. Let $R(u(\gamma_\epsilon), t_U)$ be the corresponding real root system with the positive root system $R^+(u(\gamma_\epsilon), t_U) = R(u(\gamma_\epsilon), t_U) \cap R^+(u, t_U)$. Let $\rho_u, \rho_u(\gamma_\epsilon)$ be the corresponding half sums of positive roots.

Let $\tilde{U}(\gamma_\epsilon)$ be a connected finite covering group of $U(\gamma_\epsilon)^0$ such that $\rho_u, \rho_u(\gamma_\epsilon)$ are analytically integrable with respect to the maximal torus $\tilde{T}_U$ of $\tilde{U}(\gamma_\epsilon)$ associated with $t_U$. It always exists by a similar construction as in the proof to Theorem 5.4.4.

Let $\tilde{K}(\gamma_\epsilon)$ be the analytic subgroup of $\tilde{U}(\gamma_\epsilon)$ associated with Lie algebra $t(\gamma_\epsilon)$. By [22, Proposition 7.12], $\tilde{U}(\gamma_\epsilon)$ has a unique complexification $\tilde{U}(\gamma_\epsilon)_{\mathbb{C}}$ which is a connected linear reductive Lie group. Let $\tilde{Z}(\gamma_\epsilon)$ be the analytic subgroup of $\tilde{U}(\gamma_\epsilon)_{\mathbb{C}}$ associated with $\tilde{\gamma}(\gamma_\epsilon) \subset u(\gamma_\epsilon)_{\mathbb{C}} = \tilde{\gamma}(\gamma_\epsilon)_{\mathbb{C}}$. Then we have the following Cartan decomposition

\[(6.0.6)\quad \tilde{Z}(\gamma_\epsilon) = \tilde{K}(\gamma_\epsilon) \exp(p(\gamma_\epsilon)).\]

We still denote by $\theta$ the corresponding Cartan involution on $\tilde{Z}(\gamma_\epsilon)$.

The Lie group $\tilde{Z}(\gamma_\epsilon)$ is a finite covering group of $Z(\gamma_\epsilon)^0$. Moreover, we have the identification of symmetric spaces

\[(6.0.7)\quad X(\gamma_\epsilon) \simeq \tilde{Z}(\gamma_\epsilon)/\tilde{K}(\gamma_\epsilon).\]

Note that even under an additional assumption that $G$ has compact center, $\tilde{Z}(\gamma_\epsilon)$ may still have noncompact center.

Let $\lambda$ be a dominant weight for $(U, T_U)$ with respect to $R^+(u, t_U)$. Let $(E_\lambda, \rho_\lambda)$ be the irreducible unitary representation of $U$ with highest weight $\lambda$. As before, let $(F_\lambda, \nabla_{F_\lambda}^X, h_f)$ be the corresponding homogeneous flat vector bundle on $X$ with flat connection $\nabla_{F_\lambda}^X$. Let $D^{X, F_{\lambda}, 2}$ denote the associated de Rham-Hodge Laplacian.

Let $W_U^2(\gamma_\epsilon) \subset W(u, t_U, \mathbb{C})$ be the set defined as in (5.4.13) but with respect to the group $U$ and to $\gamma_\epsilon = k \in T \subset T_U$. As in Definition 5.4.3, for $\sigma \in W_U^2(\gamma_\epsilon)$, set

\[(6.0.8)\quad \varphi_{\gamma_\epsilon}^U(\sigma, \lambda) = \varepsilon(\sigma) \frac{\xi_{\sigma(\lambda + p_\lambda)}^\gamma(\gamma_\epsilon)}{\prod_{\sigma(\lambda + p_\lambda) - \rho_\lambda(\gamma_\epsilon) \in R^+(u^\gamma(\gamma_\epsilon), t_U)} (\xi_{\sigma(\lambda + p_\lambda)}^\gamma(\gamma_\epsilon) - 1)}.

As explained in Remark 5.4.5, if $\sigma \in W_U^2(\gamma_\epsilon)$, then $\sigma(\lambda + p_\lambda) - \rho_\lambda(\gamma_\epsilon)$ is a dominant weight of $\tilde{U}(\gamma_\epsilon)$ with respect to $R^+(u(\gamma_\epsilon), t_U)$. Let $E_{\sigma, \lambda}$ be the irreducible unitary representation of $\tilde{U}(\gamma_\epsilon)$ with highest weight $\sigma(\lambda + p_\lambda) - \rho_\lambda(\gamma_\epsilon)$.

We extend $E_{\sigma, \lambda}$ to an irreducible representation of $\tilde{Z}(\gamma_\epsilon)$ by the unitary trick. Then $F_{\sigma, \lambda} = \tilde{Z}(\gamma_\epsilon) \times_{R(\gamma_\epsilon)} E_{\sigma, \lambda}$ is a homogeneous vector bundle on $X(\gamma_\epsilon)$ with an invariant flat connection $\nabla_{F_{\sigma, \lambda}}^X$ as explained in Subsection 4. Let $D^{X(\gamma_\epsilon), F_{\sigma, \lambda}, 2}$ denote the associated de Rham-Hodge Laplacian acting on $\Omega(X(\gamma_\epsilon), F_{\sigma, \lambda})$.

We also view $\gamma_\epsilon = e^\sigma$ as a hyperbolic element in $\tilde{Z}(\gamma_\epsilon)$ for $\sigma \in W_U^2(\gamma_\epsilon)$. As in (6.0.1), we set

\[(6.0.9)\quad \mathcal{E}_{X(\gamma_\epsilon), \gamma_\epsilon}(F_{\sigma, \lambda}, t) = \text{Tr}_{e^\sigma} [(N^\lambda (T^*)^X(\gamma_\epsilon)) - \frac{\lambda' \gamma_\epsilon}{2}] \exp(-tD^{X(\gamma_\epsilon), F_{\sigma, \lambda}, 2}/2)].\]
Note that we use $B|_{\mathfrak{z}(\gamma_e)}$ on $\mathfrak{z}(\gamma_e)$ to define this orbital integral for $\mathcal{Z}(\gamma_e)$.

Set

$$
(6.0.10) \quad c(\gamma) = \left| \frac{\det(1 - \text{Ad}(\gamma))|_{\mathfrak{z}(\gamma_e)}}{\det(1 - \text{Ad}(\gamma))|_{\mathfrak{z}(\gamma_e)}} \right|^{1/2} > 0.
$$

In particular, $c(\gamma_e) = 1$.

The following theorem is essentially a consequence of the generalized Kirillov formula in Theorem 5.4.4.

**Theorem 6.0.1.** Let $\gamma \in G$ be given as in (6.0.2). For $t > 0$, we have the following identity,

$$
(6.0.11) \quad \mathcal{E}_{X,\gamma}(F_{\lambda}, t) = c(\gamma) \sum_{\sigma \in W_{\lambda}^+} \varphi_{\gamma_e}^{U}(\sigma, \lambda) \mathcal{E}_{X(\gamma_e),\gamma}(F_{\sigma,\lambda}, t).
$$

We call (6.0.11) a localization formula for the geometric orbital integral.

**Proof.** Set $p' = \dim \mathfrak{p}(\gamma_e) = \text{dim } X(\gamma_e)$. At first, if $m$ is even, then $p'$ is even. Then the both sides of (6.0.11) are $0$ by [3, Theorem 7.9.1].

If $m$ is odd, then $p'$ is odd, and $\delta(G) = \delta(\mathcal{Z}(\gamma_e)^0) = \text{odd}$. If $\delta(G) \geq 3$, then the both sides of (6.0.11) are $0$ by [3, Theorem 7.9.1].

Now we consider the case where $\delta(G) = \delta(\mathcal{Z}(\gamma_e)^0) = 1$. If $\gamma$ can not be conjugated into $H$ by an element in $G$, then $\gamma_h$ can not be conjugated into $H$ by an element in $\mathcal{Z}(\gamma_e)^0$. Then the both sides of (6.0.11) are $0$ by Proposition 4.1.5.

Now we assume that $\delta(G) = 1$ and $a \in \frak{a}$. Note that $\hat{\mathcal{z}}(\gamma)$ is the centralizer of $\gamma_h$ in $\mathfrak{z}(\gamma_e)$. We will prove (6.0.11) using (4.1.17).

For $y \in \mathfrak{z}(\gamma)$, let $J_{\gamma_h}^{-}(y)$ be the function defined in (3.3.1) for $\gamma_h = e^a \in \mathfrak{z}(\gamma_e)$,

$$
(6.0.12) \quad J_{\gamma_h}^{-}(y) = \frac{1}{|\det(1 - \text{Ad}(\gamma_h))|_{\mathfrak{z}(\gamma_e)}|^{1/2}} \mathcal{A}(i\mathfrak{ad}(y))_{\mathfrak{p}(\gamma_e)} \mathcal{A}(i\mathfrak{ad}(y))_{\mathfrak{r}(\gamma_e)}.
$$

Let $\kappa^\mathfrak{z}(\gamma_e) \in \Lambda^3(\gamma_e)^*$ be the analogue of $\kappa^\mathfrak{g}$ as in (4.1.12), i.e., if $u, v, w \in \mathfrak{z}(\gamma_e),

$$
(6.0.13) \quad \kappa^\mathfrak{z}(\gamma_e)(u, v, w) = B([u, v], w).
$$

Let $\kappa^\mathfrak{u}(\gamma_e) \in \Lambda^3(\gamma_e)^*$ be defined in a similar way.

Then

$$
(6.0.14) \quad B^*(\kappa^\mathfrak{z}(\gamma_e), \kappa^\mathfrak{z}(\gamma_e)) = B^*(\kappa^\mathfrak{u}(\gamma_e), \kappa^\mathfrak{u}(\gamma_e)).
$$

The Casimir operator $C^\mathfrak{u}(\gamma_e).E_{\sigma, \lambda}$ acts on $E_{\sigma, \lambda}$ by the scalar given

$$
(6.0.15) \quad -4\pi^2(|\lambda + \rho_u|^2 - |\rho_u(\gamma_e)|^2).
$$

Then by [3, Proposition 7.5.1],

$$
(6.0.16) \quad 2\pi^2|\rho_u(\gamma_e)|^2 = -\frac{1}{8} B^*(\kappa^\mathfrak{u}(\gamma_e), \kappa^\mathfrak{u}(\gamma_e)).
$$

By (4.1.17), (6.0.15), (6.0.16), for $\sigma \in W_{\lambda}^+(\gamma_e)$, we get

$$
(6.0.17) \quad \mathcal{E}_{X(\gamma_e),\gamma}(F_{\sigma,\lambda}, t) = \frac{e^{-|\gamma|^2}}{(2\pi t)^{q/2}} \exp \left( -2\pi^2 t|\lambda + \rho_u|^2 \right) \cdot \int_{\mathfrak{z}(\gamma)} J_{\gamma_h}^{-}(y) \text{Tr}_{A^\lambda} \left( (N^\lambda(\varphi(\gamma_e)^*) - \frac{p'}{2}) \exp(-i\lambda(\gamma_e)) \right) \text{Tr}_{E_{\sigma, \lambda}} \left( \exp(-i\rho_{E_{\sigma, \lambda}}(y)) \right) e^{-|y|^2/2t} \frac{dy}{(2\pi t)^{q/2}}.
$$
Note that \( \dim p^\perp(\gamma_c) \) is even. We claim that if \( y \in \mathfrak{f}(\gamma) \), then
\[
(6.0.18) \quad \text{Tr}^\Lambda_{\gamma}(p^*)([\Lambda^\Lambda(p^*) - \frac{m}{2} \exp(-iad(y))\text{Ad}(k^{-1})] \\
= \text{Tr}^\Lambda_{\gamma}(p(\gamma_c)^*) \left[ (\Lambda^\Lambda(p(\gamma_c)^*) - \frac{p'}{2} e^{-iad(y)}) \det(1 - e^{-iad(y)}\text{Ad}(k^{-1})) \right]_{|p^\perp(\gamma_c)}. \\

\]
Indeed, we can verify \((6.0.18)\) for \( y \in \mathfrak{t} \). Since both sides of \((6.0.18)\) are invariant by adjoint action of \( K(\gamma_c)^0 \), then \((6.0.18)\) holds in full generality.

Also \( K(\gamma)^0 \) preserves the splitting
\[
(6.0.19) \quad p^\perp(\gamma_c) = p^\perp_\mathfrak{k}(\gamma) \oplus (p^\perp(\gamma_c) \cap p^\perp_\mathfrak{t}).
\]
The action \( ad(a) \) gives an isomorphism between \( p^\perp(\gamma_c) \cap p^\perp_\mathfrak{t} \) and \( p^\perp(\gamma_c) \cap p^\perp_\mathfrak{t} \) as \( K(\gamma) \)-vector spaces.

Note that
\[
(6.0.20) \quad j^\perp(\gamma_c) \cap j^\perp_\mathfrak{t} = (p^\perp(\gamma_c) \cap p^\perp_\mathfrak{t}) \oplus (\mathfrak{t}^\perp(\gamma_c) \cap \mathfrak{t}^\perp_\mathfrak{t}).
\]
Then
\[
(6.0.21) \quad \det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))_{|p^\perp(\gamma_c)} \quad = \det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)[\det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))]^{1/2}}_{|j^\perp(\gamma_c) \cap j^\perp_\mathfrak{t}}.
\]
Here the square root is taken to be positive at \( y = 0 \).

By \((3.3.1), (6.0.12)\), for \( y \in \mathfrak{f}(\gamma) \),
\[
(6.0.22) \quad J_\gamma(y) = J_{\gamma_c}(y) \frac{1}{\sqrt{\det(1 - \text{Ad}(\gamma_c))_{|j^\perp(\gamma_c) \cap j^\perp_\mathfrak{t}}}} \quad \frac{1}{\sqrt{\det(1 - \exp(-iad(y))\text{Ad}(\gamma_c))_{|p^\perp(\gamma_c)}}}.
\]

Combining \((6.0.18), (6.0.21)\) and \((6.0.22)\), we get
\[
(6.0.23) \quad \text{Tr}^\Lambda_{\gamma}(p^*)([\Lambda^\Lambda(p^*) - \frac{m}{2} \exp(-iad(y))\text{Ad}(\gamma_c)] \\
= \exp(-2\pi^2 t |\lambda + \rho_u|^2) \quad \frac{1}{\sqrt{\det(1 - \exp(-iad(y))\text{Ad}(\gamma_c))_{|j^\perp(\gamma_c) \cap j^\perp_\mathfrak{t}}}} \quad \frac{1}{\sqrt{\det(1 - \text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}}}.
\]

Note that for \( y \in \mathfrak{f}(\gamma) \),
\[
(6.0.24) \quad \frac{\det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))_{|j^\perp(\gamma_c) \cap j^\perp_\mathfrak{t}}}{\det(1 - \text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}}^{1/2} \quad = \frac{\det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}}{\det(1 - \text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}}^{1/2}.
\]

By \((4.1.17), (6.0.15), (6.0.16), (6.0.23)\) and \((6.0.24)\), we get
\[
(6.0.25) \quad \mathcal{E}_{X,\lambda}(F_{\lambda, t}) = c(\gamma) e^{-\frac{\rho_u^2}{2(2\pi t)^{p/2}}} \exp(-2\pi^2 t |\lambda + \rho_u|^2) \\
\cdot \int_{\mathfrak{f}(\gamma)} J_{\gamma_c}(y) \text{Tr}^\Lambda_{\gamma}(p(\gamma_c)^*) \left[ (\Lambda^\Lambda(p(\gamma_c)^*) - \frac{p'}{2} e^{-iad(y)}) \right] \\
\left[ \frac{\det(1 - e^{-iad(y)}\text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}}{\det(1 - \text{Ad}(\gamma_c))_{|\mathfrak{p}^\perp(\gamma_c)}} \right]^{1/2} \text{Tr}^\Lambda_{\gamma} [\rho^\Lambda_{\gamma}(\gamma_c) e^{-ip\xi_{\gamma}}] |y|^2 e^{-t|y|^2/2t} \frac{dy}{(2\pi t)^{n/2}}.
\]
Then (6.0.11) follows from the (5.4.40), (6.0.17) and (6.0.25). This completes the proof of our theorem. 

\( \square \)

Remark 6.0.2. (1) If \( \gamma \) is hyperbolic, then (6.0.11) is trivial.

(2) A similar consideration can be made for \( \text{Tr}_\gamma \left[ \exp \left( -t X_{F, \lambda} \right) \right] \), where (6.0.11) will become an analogue of the index theorem for orbifolds as in (2.2.11). The related computation can be found in [7, Subsection 10.4].

7. Full asymptotics of elliptic orbital integrals

In this section, we always assume that \( \delta(G) = 1 \) and that \( U \) is compact. We also use the notation and settings as in Subsections 5.1, 5.2 and 5.3.

In this section, we are concerned with a sequence of flat vector bundles \( \{ F_d \}_{d \in \mathbb{N}} \) on \( X \) defined by a nondegenerate dominant weight \( \lambda \). For elliptic \( \gamma \), we will compute explicitly \( \mathcal{E}_{\gamma, \lambda}(F_d, t) \) and its Mellin transform in terms of the root data.

Note that when \( \gamma = 1 \), \( \mathcal{E}_{X, \gamma}(F_d, t) \) is already computed by Müller-Pfaff [33] using the Plancherel formula for identity orbital integral. We here give a different approach via Bismut’s formula as in (4.1.17), which inspires an analogue computation for general elliptic orbital integrals.

7.1. A family of representations of \( G \). Recall that \( T \) is a maximal torus of \( K \), and \( T_U \) is a maximal torus of \( U \). Let \( W(U, T_U) \) denote the (analytic) Weyl group of \( (U, T_U) \), so that \( W(U, T_U) = W(U, T_U) \).

The positive root system \( R^+(u, t_U) \) is given in Subsection 5.3. Recall that \( P_{++}(U) \) is the set of dominant weights of \( (U, T_U) \) with respect to \( R^+(u, t_U) \). By [6, Definition 1.13 & Proposition 8.12], we take a definition of nondegeneracy of \( \lambda \) as follows.

Definition 7.1.1. A dominant weight \( \lambda \in P_{++}(U) \) is said to be nondegenerate with respect to the Cartan involution \( \theta \) if

\[
(7.1.1) \quad W(U, T_U) \cdot \lambda \cap t^* = \emptyset.
\]

It is equivalent to

\[
(7.1.2) \quad \text{Ad}^+(U) \lambda \cap t^* = \emptyset.
\]

Note that if such dominant weight exists, we must have \( \delta(G) > 0 \).

Let \((E_\lambda, \rho^{E_\lambda})\) be the irreducible unitary representation of \( U \) with highest weight \( \lambda \). By the unitary trick, it extends to an irreducible representation of \( G \), which we still denote by \((E_\lambda, \rho^{E_\lambda})\). Then \( \lambda \) being nondegenerate is equivalent to say that \((E_\lambda, \rho^{E_\lambda} \circ \theta)\) as \( G \)-representation (as in [33]).

Recall that \( a_1 \in \mathfrak{b} \) is such that \( B(a_1, a_1) = 1 \).

Definition 7.1.2. If \( \lambda \in t_U^* \), for \( \omega \in W(U, T_U) \), put

\[
(7.1.3) \quad a_{\lambda, \omega} = \langle \omega \cdot \lambda, \sqrt{-1}a_1 \rangle \in \mathbb{R},
\]

\[
b_{\lambda, \omega} = a_{\lambda, \omega} + a_{\rho_\omega, \omega} = \langle \omega \cdot (\lambda + \rho_\omega), \sqrt{-1}a_1 \rangle \in \mathbb{R}
\]

In particular, we simply put \( a_\lambda = a_{\lambda, 1} \), \( b_\lambda = b_{\lambda, 1} \).

Lemma 7.1.3. If \( \lambda \in P_{++}(U) \) is nondegenerate, then for \( \omega \in W(U, T_U) \), \( a_{\lambda, \omega} \neq 0 \).

Now we fix two dominant weights \( \lambda, \lambda_0 \in P_{++}(U) \). Let \( \{(E_d, \rho^{E_d})\}_{d \in \mathbb{N}} \) be the sequence of representations of \( G \) given by the irreducible unitary representations of \( U \) with the highest weights \( d\lambda + \lambda_0 \), \( d \in \mathbb{N} \).

Put \( F_d = G \times_K E_d \). Let \( D_{X, F_d}^{\lambda_0 \omega} \) denote the associated de Rham-Hodge Laplacian. For \( t > 0 \), let \( \exp(-tD_{X, F_d}^{\lambda_0 \omega}/2) \) denote the heat operator associated with \( D_{X, F_d}^{\lambda_0 \omega}/2 \).
For $t > 0$, $d \in \mathbb{N}$, if $\gamma \in G$ is semisimple, as in (6.0.1), set

\begin{equation}
(7.1.4) \quad E_{X, \gamma}(F_d, t) = \text{Tr}_s [\gamma] \left[ (N^\Lambda (T^* X) - \frac{m}{2}) \exp(\frac{t}{2} D_{X,F_d}^2) \right].
\end{equation}

It is clear that $E_{X, \gamma}(F_d, t)$ only depends on the conjugacy class $[\gamma]$ in $G$. If $\gamma = 1$, we also write

\begin{equation}
(7.1.5) \quad \mathcal{I}_X(F_d, t) = E_{X,1}(F_d, t).
\end{equation}

7.2. Estimates for $t$ small. By (4.1.17), (6.0.15), (6.0.16), if $\gamma = k \in K$, we have

\begin{equation}
(7.2.1) \quad \text{Tr}_s [\gamma] \left[ (N^\Lambda (T^* X) - \frac{m}{2}) \exp(\frac{t}{2} D_{X,F_d}^2) \right] = \frac{1}{(2\pi t)^{p/2}} \exp \left( -2\pi^2 t |d\lambda + \lambda_0 + \rho_u|^2 \right)
\end{equation}

\begin{align*}
&\int_{t(k)} J_{\gamma}(\sqrt{7} Y_0^t) \text{Tr}_s^\Lambda (p^r) \left[ (N^\Lambda (p^r) - \frac{m}{2}) \text{Ad}(k) \exp(-i\text{ad}(Y_0^t)) \right] \\
&\cdot \text{Tr} \left[ \rho E_d(k) \exp(-i\rho E_d(\sqrt{7} Y_0^t)) e^{-|Y_0^t|^2/2} \right] dY_0^t \\
&\text{Tr} \left[ \rho E_d(k) \exp(-i\rho E_d(\sqrt{7} Y_0^t)) e^{-|Y_0^t|^2/2} \right] dY_0^t
\end{align*}

Proposition 7.2.1. For $d \in \mathbb{N}$, an elliptic $\gamma \in G$, there exists a constant $C_d, \gamma > 0$ such that for $t \in [0, 1]$

\begin{equation}
(7.2.2) \quad |\sqrt{7} E_{X, \gamma}(F_d, t)| \leq C_d, \gamma,
\end{equation}

\begin{equation}
(7.2.3) \quad (1 + 2t \frac{\partial}{\partial t}) E_{X, \gamma}(F_d, t) \leq C_d, \gamma \sqrt{t}.
\end{equation}

As $t \to 0$, $E_{X, \gamma}(E_d, t)$ has the asymptotic expansion in the form of

\begin{equation}
(7.2.4) \quad \frac{1}{\sqrt{7}} \sum_{j=0}^{+\infty} a_j^\gamma(d) t^j.
\end{equation}

Where $a_j^\gamma(d)$ are functions in $d$.

Proof. If $\gamma$ is elliptic, up to a conjugation, we assume that $\gamma = k \in T$. Thus $H$ is also a Cartan subgroup of $Z(\gamma)^0$, then $\mathfrak{b}(\gamma) = \mathfrak{b}$. Let $\mathfrak{b}^\perp(\gamma)$ be the orthogonal complementary space of $\mathfrak{b}(\gamma)$ in $\mathfrak{p}(\gamma)$, whose dimension is $p - 1$.

If $\lambda_0 = 0$, we can use the conclusions in [25, Theorem 7.4.1], which give the estimates in (7.2.2).

We now include a direct proof to (7.2.2) in general case. By (7.2.1), we have

\begin{align*}
\text{Tr}_s [\gamma] &\left[ (N^\Lambda (T^* X) - \frac{m}{2}) \exp(\frac{t}{2} D_{X,F_d}^2) \right] \\
&\int_{t(k)} J_{\gamma}(\sqrt{7} Y_0^t) \text{Tr}_s^\Lambda (p^r) \left[ (N^\Lambda (p^r) - \frac{m}{2}) \text{Ad}(k) \exp(-i\text{ad}(Y_0^t)) \right] \\
&\cdot \text{Tr} \left[ \rho E_d(k) \exp(-i\rho E_d(\sqrt{7} Y_0^t)) e^{-|Y_0^t|^2/2} \right] dY_0^t \\
&\text{Tr} \left[ \rho E_d(k) \exp(-i\rho E_d(\sqrt{7} Y_0^t)) e^{-|Y_0^t|^2/2} \right] dY_0^t
\end{align*}

where the integral is rescaled by $\sqrt{t}$.

In this proof, we denote by $C$ or $c$ a positive constant independent of the variables $t$ and $Y_0^t$. We use the symbol $O_{\text{ind}}$ to denote the big-O convention which does not depend on $t$ and $Y_0^t$.
The same computations as in [25, Eqs. (7.4.8) - (7.4.10)] shows that for $Y^t_0 \in \mathfrak{t}(k)$,
\[
J_k(\sqrt{7}Y^t_0) = \frac{1}{\det(1 - \text{Ad}(k))|_{p^1(k)}} + O_{\text{ind}}(\sqrt{7}Y^t_0^t|e^{C\sqrt{7}Y^t_0^t}),
\]
(7.2.5)
\[
\frac{1}{
\exp(-ipF_d(m)\exp(-ipF_d(\sqrt{7}Y^t_0^t)))}
\]
(7.2.6)
\[
\text{Vol}(\mathbb{H},\eta_0) = \frac{1}{\nu^p(m|_\eta_0)} \cdot \frac{|W(U_M,T)|}{|W(K,T)|}.
\]
(7.3.1)
\[
\text{Vol}(U_M,T) = \prod_{\alpha^\prime \in R^+(\rho_{\text{ad}})} \frac{1}{2\pi \cdot (\alpha^\prime,\rho_{\text{ad}})}.
\]
(7.3.2)
\[
\sqrt{7}E_{X,\gamma}(F_d, t) = C_{d,k} \int_{(k)} (1 + |Y^t_0|)^N \exp(C|Y^t_0| - |Y^t_0|^2/2) dy^t_0.
\]
(7.3.7)
\[
\text{Vol}(K/T) = \frac{\text{Vol}(K/T)}{|W(U_M,T)|}.
\]
(7.3.3)
\[
\nu = [\nu^{\max} + \omega_0, 2l/(2l)!].
\]
(7.3.4)
\[
Q_{\omega_0}^{\lambda_0}(d) = \frac{\omega^{\max}}{(2l)!} \cdot \frac{\dim V^{(d)}}{(2l)!}.
\]
7.3. Identity orbital integrals for Hodge Laplacians. In this subsection, we compute $I_X(F_d, t)$ using Bismut’s formula (7.2.1). In next subsection, we connect our computational results to the ones obtained by Müller-Pfaff [33]. We will give in detail the main points in the computation, which are also applicable for computing $E_{X,\gamma}(F_d, t)$ with any elliptic $\gamma$.

Let $\text{Vol}(K/T), \text{Vol}(U_M/T)$ be the Riemannian volumes of $K/T, U_M/T$ with respect to the restriction of $-B$ to $\mathfrak{t}, \mathfrak{u}$ respectively. We have explicit formulae for them in terms of root data, for example,

\[
\text{Vol}(U_M,T) = \prod_{\alpha^\prime \in R^+(\rho_{\text{ad},m})} \frac{1}{2\pi \cdot (\alpha^\prime,\rho_{\text{ad},m})}.
\]

Set

\[
e_G = \frac{(-1)^{\frac{m-1}{2} + 1}}{\text{Vol}(K/T)/|W(U_M,T)|}.
\]

Recall that $\omega Y^t_0 \in \Lambda^2(u^+\langle b \rangle^*)$ is defined in Proposition 5.2.1 and that $\Omega^{\max} \in \Lambda^2(u^+\langle b \rangle^*) \cap \mathfrak{u}_m$ is defined in (5.2.11).

Note that $\dim u^+\langle b \rangle = 4l$. If $\nu \in \Lambda^2(u^+\langle b \rangle^*)$, let $[\nu]^{\max} \in \mathbb{R}$ be such that

\[
\nu = [\nu^{\max}, \omega_0, 2l/(2l)!].
\]

is of degree strictly smaller than $4l$.

We use the notation in Subsection 5.3. In particular, the positive root systems $R^+(\mathfrak{u},\mathfrak{g})$ and $R^+(\mathfrak{u}_m,\mathfrak{t})$ are fixed in Subsection 5.3. The set $W_u \subset W(U,T_U)$ is given by (5.3.6). As in Proposition 5.3.2, for $\omega \in W_u V^{d\lambda_0}$, it is an irreducible unitary representation of $U_M$ with highest weight $\eta_\omega(d\lambda + \lambda_0)$ given by (5.3.11).

Definition 7.3.1. For $j = 0, 1, \ldots, l$, $\omega \in W_u$, set

\[
Q_{\omega_0}^{d\lambda_0}(d) = \frac{(-1)^{\frac{m-1}{2} + 1}}{j!(2l - 2j)!} \dim V^{d\lambda_0}(\omega + \lambda_0 + \rho_\omega, \Omega^{\max})^{2i - 2j]^{\max}}.
\]

Since $\dim V^{d\lambda_0}$ is a polynomial in $d$ by the Weyl dimension formula, then $Q_{\omega_0}^{d\lambda_0}(d)$ is a polynomial in $d$ of degree $\leq \dim(u/\mathfrak{h}) - 2j$. 

\[
Q_{\omega_0}^{d\lambda_0}(d) = \frac{(-1)^{\frac{m-1}{2} + 1}}{j!(2l - 2j)!} \dim V^{d\lambda_0}(\omega + \lambda_0 + \rho_\omega, \Omega^{\max})^{2i - 2j]^{\max}}.
\]
In particular, if $l \geq 1$, we have
\begin{align}
Q_{0,\omega}^{\lambda,\lambda_0}(d) &= \frac{1}{(2t)!} \dim V_{d\lambda + \lambda_0,\omega} \left[ (\omega(d\lambda + \lambda_0 + \rho_u), \Omega^{\mu_m}) \right]_{\text{max}}, \\
Q_{t,\omega}^{\lambda,\lambda_0}(d) &= \frac{(-1)^t \beta(a_1)^2(2t - 1)!}{(4t^2)^{t/2}} \dim V_{d\lambda + \lambda_0,\omega}.
\end{align}
(7.3.5)

Recall that for $\omega \in W_u$, $a_{\lambda,\omega}$, $b_{\lambda_0,\omega}$ are defined in Definition 7.1.2.

Theorem 7.3.2. For $\omega \in W_u$
\begin{equation}
|\eta_\omega(d\lambda + \lambda_0) + \rho_u|^2 - |d\lambda + \lambda_0 + \rho_u|^2 = -(d\alpha_{\lambda,\omega} + b_{\lambda_0,\omega})^2.
\end{equation}
For $t > 0$, we have the following identity
\begin{equation}
I_X(F_d, t) = \frac{CG}{\sqrt{2\pi t}} \sum_{j=0}^I t^{-j} \sum_{\omega \in W_u} \varepsilon(\omega) e^{-2\pi i t (d\alpha_{\lambda,\omega} + b_{\lambda_0,\omega})^2} Q_{x,\omega}^{\lambda,\lambda_0}(d).
\end{equation}
(7.3.7)

Remark 7.3.3. The formula (7.3.7) is compatible with the estimate (7.2.2). For example, if we take the asymptotic expansion of the right-hand side of (7.3.7) as $t \to 0$, the coefficient of $t^{-l-1/2}$ is given by
\begin{equation}
\frac{CG}{\sqrt{2\pi}} \sum_{\omega \in W_u} \varepsilon(\omega) Q_{x,\omega}^{\lambda,\lambda_0}(d).
\end{equation}
(7.3.8)

Then by (5.3.10), if $l \geq 1$, we get
\begin{equation}
\sum_{\omega \in W_u} \varepsilon(\omega) \dim V_{d\lambda + \lambda_0,\omega} = \text{Tr}_s^\lambda(\alpha_\omega^d)[1] \dim E_\lambda = 0.
\end{equation}
(7.3.9)

By (7.3.5) and (7.3.9), the quantity in (7.3.8) is 0 (provided $l \geq 1$).

Before proving Theorem 7.3.2, we need some preparation work.

Note that $R(u_m, t)$ contains $R(\mathfrak{t}_m, t)$ as a sub root system. Since the adjoint action of $t$ preserves the splitting $u_m = \sqrt{-1} \mathfrak{p}_m \oplus \mathfrak{t}_m$, then we can write
\begin{equation}
R(u_m, t) = R(\sqrt{-1} \mathfrak{p}_m, t) \cup R(\mathfrak{t}_m, t),
\end{equation}
(7.3.10)

where the two subsets are disjoint and $R(\sqrt{-1} \mathfrak{p}_m, t)$ is the set of roots associated with the adjoint action of $t$ on $\mathfrak{p}_m$.

We have fixed a positive root system $R^+(u_m, t)$ in Subsection 5.3, which induces a positive root system $R^+(\mathfrak{t}_m, t) \subset R(\mathfrak{t}_m, t)$. We also put
\begin{equation}
R^+(\sqrt{-1} \mathfrak{p}_m, t) = R^+(u_m, t) \cap R(\sqrt{-1} \mathfrak{p}_m, t).
\end{equation}
(7.3.11)

Then
\begin{equation}
R^+(u_m, t) = R^+(\sqrt{-1} \mathfrak{p}_m, t) \cup R^+(\mathfrak{t}_m, t),
\end{equation}
(7.3.12)

Definition 7.3.4. For $y \in t$, put
\begin{align}
\pi_{u_m/t}(y) &= \prod_{\alpha^0 \in R^+(u_m, t)} \langle 2\pi \sqrt{-1} \alpha^0, y \rangle, \\
\pi_{\sqrt{-1} \mathfrak{p}_m/t}(y) &= \prod_{\alpha^0 \in R^+(\sqrt{-1} \mathfrak{p}_m, t)} \langle 2\pi \sqrt{-1} \alpha^0, y \rangle, \\
\pi_{\mathfrak{t}_m/t}(y) &= \prod_{\alpha^0 \in R^+(\mathfrak{t}_m, t)} \langle 2\pi \sqrt{-1} \alpha^0, y \rangle.
\end{align}
(7.3.13)
For $y \in \mathfrak{t}$, put
\begin{equation}
(7.3.14)
\sigma_{u/y}(y) = \prod_{\alpha \in R^+(u, y)} \left( \exp(\langle \pi \sqrt{-1} \alpha^0, y \rangle) - \exp(-\langle \pi \sqrt{-1} \alpha^0, y \rangle) \right).
\end{equation}
\begin{equation}
(7.3.15)
\sigma_{\sqrt{t}p_m/y}(y) = \prod_{\alpha \in R^+(\sqrt{t}p_m, y)} \left( \exp(\langle \pi \sqrt{-1} \alpha^0, y \rangle) - \exp(-\langle \pi \sqrt{-1} \alpha^0, y \rangle) \right),
\end{equation}
\begin{equation}
(7.3.16)
\sigma_{t/y}(y) = \prod_{\alpha \in R^+(t, y)} \left( \exp(\langle \pi \sqrt{-1} \alpha^0, y \rangle) - \exp(-\langle \pi \sqrt{-1} \alpha^0, y \rangle) \right).
\end{equation}

We can always extend analytically the above functions to $y \in \mathfrak{c}$.

It is clear that if $y \in \mathfrak{c}$,
\begin{equation}
(7.3.17)
\sigma_{u/y}(y) = \sigma_{\sqrt{t}p_m/y}(y)\sigma_{t/y}(y).
\end{equation}

Recall that $\mathfrak{t}_U$ is a Cartan subalgebra of $\mathfrak{u}$. Then using $R^+(\mathfrak{u}, \mathfrak{t}_U)$ defined in Subsection 5.3, we can define the associated functions $\pi_{u/y}(y)$, $\sigma_{u/y}(y)$ for $y \in \mathfrak{t}_U$ as in Definition 7.3.4. Let $R^+(\mathfrak{t}, \mathfrak{t})$ be a positive root system of $R(\mathfrak{t}, \mathfrak{t})$ such that it induces the same $R^+(\mathfrak{t}_m, \mathfrak{t})$. Then we define the functions $\pi_{\mathfrak{t}/y}(y)$, $\sigma_{\mathfrak{t}/y}(y)$ as in Definition 7.3.4.

Recall that if $Y_0^t \in \mathfrak{t}$, we have
\begin{equation}
(7.3.18)
J_1(Y_0^t) = \frac{\hat{A}(\text{ad}(Y_0^t))}{\hat{A}(\text{ad}(Y_0^t))}.\quad \text{Put}
\end{equation}
\begin{equation}
(7.3.19)
F(d, t) = \frac{1}{(2\pi)^{m/2}} \int_{\mathfrak{t}} J_1(Y_0^t) \text{Tr}_s^\Lambda(p) \left[(N^\Lambda(p) - m/2) \exp(-i\text{ad}(Y_0^t)) \right] \cdot \text{Tr}_{E^d} \left[ \exp(-i\rho_{E^d}(Y_0^t)) \right] e^{-|Y_0^t|^2/2t} \frac{dY_0^t}{(2\pi)^{m/2}}.
\end{equation}

Then by (7.2.1), we have
\begin{equation}
(7.3.20)
\mathcal{I}_X(F_d, t) = \exp \left(-\frac{1}{2\pi^2 t} |d\lambda + \lambda_0 + \rho_{E^d}|^2 \right) F(d, t).
\end{equation}

By Weyl integration formula, we have
\begin{equation}
(7.3.21)
F(d, t) = \frac{\text{Vol}(K/T)}{(2\pi)^{(m+n)/2}|W(K, T)|} \int_{\mathfrak{t}} |\pi_{\mathfrak{t}/y}(y)|^2 J_1(y) \text{Tr}_s^\Lambda(p) \left[(N^\Lambda(p) - m/2) \exp(-i\text{ad}(y)) \right] \cdot \text{Tr}_{E^d} \left[ \exp(-i\rho_{E^d}(y)) \right] e^{-|y|^2/2t} dy\cdot
\end{equation}

**Lemma 7.3.5.** If $y \in \mathfrak{t}$,
\begin{equation}
(7.3.22)
\text{Tr}_s^\Lambda(p) \left[(N^\Lambda(p) - m/2) \exp(-i\text{ad}(y)) \right] = (-1)^{\frac{m+n+1}{2}} \sigma_{\sqrt{t}p_m/y}(iy)^2 \cdot \text{Tr}_s^\Lambda(\text{ad}(y))\cdot
\end{equation}

Moreover, we have the following identity
\begin{equation}
(7.3.23)
\frac{\pi_{\mathfrak{t}/y}(iy)^2}{\pi_{u/y}(iy)^2} J_1(y) \sigma_{\sqrt{t}p_m/y}(iy)^2 = (-1)^{\frac{m+n+1}{2}} \hat{A}^{-1}(\text{ad}(y)|_{\mathfrak{u}_m}) \det(\text{ad}(y))|_{\mathfrak{c}}.
\end{equation}

**Proof.** Recall that as $K_M$-modules, we have
\begin{equation}
(7.3.24) p^\perp(b) \simeq \mathfrak{t}^\perp(b) \simeq \mathfrak{n} \simeq \mathfrak{n}.
\end{equation}
By (5.1.5) and (7.3.22), we get the following identification of $K_M$-modules,

\[
(7.3.23) \quad p \simeq b \oplus p_m \oplus n.
\]

Then if $y \in t$,

\[
(7.3.24) \quad \text{Tr}_s^\Lambda \left( (N_s^\Lambda (p^s) - \frac{m}{2}) e^{-\text{id}(y)} \right) = \text{Tr}_s^\Lambda \left( b^s \oplus p_m^s \right) [N_s^\Lambda (b^s \oplus p_m^s) e^{-\text{id}(y)}] \text{Tr}_s^\Lambda (n^s_\pi) [e^{-\text{id}(y)}].
\]

Note that

\[
(7.3.25) \quad \Lambda \left( b^s \oplus p_m^s \right) = \Lambda \left( p_m^s \right) + b^s \wedge \Lambda \left( p_m^s \right).
\]

Then by (7.3.24), we get

\[
(7.3.26) \quad \text{Tr}_s^\Lambda \left( (N_s^\Lambda (p^s) - \frac{m}{2}) e^{-\text{id}(y)} \right) = -\text{Tr}_s^\Lambda \left( p_m^s \right) [e^{-\text{id}(y)}] \text{Tr}_s^\Lambda (n^s_\pi) [e^{-\text{id}(y)}] = -\det(1 - e^{-\text{id}(y)}) |_{p_m} \text{Tr}_s^\Lambda (n^s_\pi) [e^{-\text{id}(y)}].
\]

Then (7.3.20) follows from the equality

\[
(7.3.27) \quad \det(1 - e^{-\text{id}(y)}) |_{p_m} = (-1)^{(\dim p_m)/2} \sigma \sqrt{-1}_{p_m/1}(iy)^2.
\]

The proof to (7.3.21) is similar to the proof to [40, Eq.(7-22)]. We include the details as follows. By (5.1.5) and (7.3.22), using [3, Eq.(7.5.24)], if $y \in t$, we have

\[
(7.3.28) \quad \tilde{\Lambda}(\text{id}(y)|_{t}) = \frac{\pi \sqrt{-1}_{p_m/1}(iy)}{\sigma \sqrt{-1}_{p_m/1}(iy)} \tilde{A}(\text{id}(y)|_{n}),
\]

\[
(7.3.29) \quad \pi_{t/1}(iy)^2 = (-1)^l \pi_{t/1}(iy)^2 \det(\text{id}(y)|_{n_c}).
\]

By (7.3.16) and (7.3.28), if $y \in t$,

\[
(7.3.30) \quad J_1(y) = \frac{\sigma_{t/1}(iy)}{\sigma \sqrt{-1}_{p_m/1}(iy)} \frac{\pi \sqrt{-1}_{p_m/1}(iy)}{\pi_{t/1}(iy)}.
\]

Then by (5.1.5), (7.3.15), (7.3.23) and (7.3.29), for $y \in t$, we get

\[
(7.3.31) \quad \frac{\pi_{t/1}(iy)^2}{\pi_{u_n/1}(iy)^2} J_1(y) \sigma \sqrt{-1}_{p_m/1}(iy)^2 = (-1)^l \frac{\sigma_{u_n/1}(iy)}{\pi_{u_n/1}(iy)} \det(\text{id}(y)|_{n_c}).
\]

Then (7.3.21) follows. This completes the proof of our lemma.

\[
\square
\]

**Proof to Theorem 7.3.2.** The equality (7.3.6) follows from Definitions 5.3.1 & 7.1.2, and from (5.3.13). We now prove (7.3.7).

If $y \in t$ is such that $\pi_{u_n/1}(iy) \neq 0$,

\[
(7.3.32) \quad \frac{|\pi_{t/1}(iy)|^2}{|\pi_{u_n/1}(iy)|^2} = \frac{\pi_{t/1}(iy)^2}{\pi_{u_n/1}(iy)^2}.
\]

Recall that $m = \dim p$. Then

\[
(7.3.33) \quad \dim p_m + 2l = m - 1.
\]
By (7.3.19), (7.3.32), (7.3.33) and using the results in Lemma 7.3.5, we get

\[
F(d, t) = \frac{(-1)^{n-1+n} \text{Vol}(K/T)}{(2\pi t)^{(m+n)/2}|W(K, T)|} \int_{y \in u_m} |\pi_{u_m/T}(y)|^2 \widetilde{A}^{-1}(\text{iad}(y)|_{u_m}) \det(\text{iad}(y)) |_{u_c} \mathcal{T}_\text{tr}^A(n_c^c) \otimes E_d[\exp(-i\rho^A (n_c^c) \otimes E_d(y))] e^{-|y|^2/2t} dy.
\]

(7.3.34)

Note that the function in \( y \in \mathfrak{t} \)

\[
\widetilde{A}^{-1}(\text{iad}(y)|_{u_m}) \det(\text{iad}(y)) |_{u_c} \mathcal{T}_\text{tr}^A(n_c^c) \otimes E_d[\exp(-i\rho^A (n_c^c) \otimes E_d(y))]
\]

can be extended directly to an \( U_M \)-invariant function in \( y \in \mathfrak{u}_m \). Since \( \mathfrak{t} \) is a Cartan subalgebra of \( \mathfrak{u}_m \), we can apply the Weyl integration formula again for the pair \( (\mathfrak{u}_m, \mathfrak{t}) \), then we rewrite (7.3.34) as

\[
F(d, t) = \frac{(-1)^{m-1+n} \text{Vol}(K/T)|W(U_M, T)|}{(2\pi t)^{(m+n)/2}\text{Vol}(U_M/T)|W(K, T)|} \int_{y \in \mathfrak{u}_m} \widetilde{A}^{-1}(\text{iad}(y)|_{u_m}) \det(\text{iad}(y)) |_{u_c} \mathcal{T}_\text{tr}^A(n_c^c) \otimes E_d[\exp(-i\rho^A (n_c^c) \otimes E_d(y))] e^{-|y|^2/2t} dy.
\]

(7.3.36)

If \( y \in \mathfrak{u}_m \), then

\[
B(y, \frac{\Omega^\mathfrak{u}_m}{2\pi}) \in \Lambda^2(\mathfrak{u}^+(\mathfrak{b})^*)
\]

(7.3.37)

If \( y \in \mathfrak{u}_m \), by [40, Eq.(7.27)], we have

\[
\frac{\det(\text{iad}(y)) |_{u_c}}{(2\pi t)^{n-1}} = |\exp\left(\frac{1}{t} B(y, \frac{\Omega^\mathfrak{u}_m}{2\pi})\right)|^{\max}.
\]

Note that

\[
m+n = \dim \mathfrak{u}_m + 4l + 1.
\]

The quantity \( c_G \) is defined in (7.3.2). Combining (7.3.36) - (7.3.39), we get

\[
F(d, t) = \frac{c_G}{\sqrt{2\pi t}} \left[ \int_{y \in \mathfrak{u}_m} \widetilde{A}^{-1}(\text{iad}(y)|_{u_m}) \mathcal{T}_\text{tr}^A(n_c^c) \otimes E_d[\exp(-i\rho^A (n_c^c) \otimes E_d(y))] \mathcal{E}(\frac{1}{t} B(y, \frac{\Omega^\mathfrak{u}_m}{2\pi})) e^{-|y|^2/2t} dy \right]^{\max}.
\]

(7.3.40)

If \( y \in \mathfrak{u}_m \), then

\[
-|y|^2 = B(y, y).
\]

(7.3.41)

By (5.2.12), if \( y \in \mathfrak{u}_m \), then

\[
B(y, \frac{\Omega^\mathfrak{u}_m}{2\pi}) = \frac{|y|^2}{2} = \frac{1}{2} B(y + \frac{\Omega^\mathfrak{u}_m}{2\pi}, y + \frac{\Omega^\mathfrak{u}_m}{2\pi}) - \frac{\beta(a_1)^2}{8\pi^2} \mathcal{Y}_s^2.
\]

(7.3.42)

Let \( \Delta^\mathfrak{u}_m \) be the standard negative Laplace operator on the Euclidean space \( (\mathfrak{u}_m, -B|_{\mathfrak{u}_m \times \mathfrak{u}_m}) \). Then we can rewrite (7.3.40) as follows,

\[
F(d, t) = \frac{c_G}{\sqrt{2\pi t}} \left[ \left| \exp\left(-\frac{\beta(a_1)^2}{8\pi^2 t} \mathcal{Y}_s^2\right) \right| \right.\left. \cdot \exp\left(\frac{t}{2} \Delta^\mathfrak{u}_m \right) \left| \mathcal{T}_\text{tr}^A(n_c^c) \otimes E_d[\exp(-i\rho^A (n_c^c) \otimes E_d(y))] \right|_{y=-\frac{\Omega^\mathfrak{u}_m}{2\pi}} \right]^{\max}.
\]

(7.3.43)
Recall that if \( \omega \in W_u \), \( \eta_u(d\lambda + \lambda_0) \in \mathfrak{t}^* \) is defined in (5.3.11). As in Proposition 5.3.2, \( V_{d\lambda + \lambda_0, \omega} \) is an irreducible unitary representation of \( U_M \) with highest weight \( \eta_u(d\lambda + \lambda_0) \). By (5.3.10), if \( y \in u_m \), then
\[
(7.3.44) \quad \text{Tr}_\Lambda^{(n_2)} \otimes E_d[\exp(-i\rho^{\Lambda}(n_2)\otimes E_d(y))] = \sum_{\omega \in W_u} \varepsilon(\omega) \text{Tr}^{V_{d\lambda + \lambda_0, \omega}} \left[ \exp \left( -i\rho V_{d\lambda + \lambda_0, \omega}(y) \right) \right].
\]

As explained in [3, Eqs. (7.5.22) - (7.5.26)], the function
\[
(7.3.45) \quad y \in u_m \mapsto \hat{A}^{-1}(\text{iad}(y)|_{u_m}) \text{Tr}_{V_{d\lambda + \lambda_0, \omega}} \left[ \exp \left( -i\rho V_{d\lambda + \lambda_0, \omega}(y) \right) \right] \in \mathbb{C}
\]
is an eigenfunction of \( \Delta^{u_m} \) associated with the eigenvalue \( 4\pi^2|\eta_u(d\lambda + \lambda_0) + \rho_{u_m}|^2 \).

By (5.3.11), (5.3.12), for \( \omega \in W_u \), we get
\[
(7.3.46) \quad \eta_u(d\lambda + \lambda_0) + \rho_{u_m} = P_0(\omega(d\lambda + \lambda_0 + \rho_u)).
\]

Recall that \( \text{Vol}_U(O_{\eta_u(d\lambda + \lambda_0) + \rho_{u_m}}) \) is the volume of \( O_{\eta_u(d\lambda + \lambda_0) + \rho_{u_m}} \) with respect to the Liouville measure. By the Kirillov formula, we have
\[
(7.3.47) \quad \text{Vol}_U(O_{\eta_u(d\lambda + \lambda_0) + \rho_{u_m}}) = \text{dim} V_{d\lambda + \lambda_0, \omega}.
\]

We claim the following identity,
\[
(7.3.48) \quad \left[ \exp \left( -\frac{\beta(a_1)^2 \omega Y_{2} - 2}{8\pi^2 t} \right) \cdot \{ \hat{A}^{-1}(\text{iad}(y)|_{u_m}) \text{Tr}_{V_{d\lambda + \lambda_0, \omega}} \left[ \exp \left( -i\rho V_{d\lambda + \lambda_0, \omega}(y) \right) \right] \} \right]_{y = -\omega^{-1} \mu_m}^{\text{max}}
\]
\[
= \text{dim} V_{d\lambda + \lambda_0, \omega} \left[ \exp \left( -\frac{\beta(a_1)^2 \omega Y_{2} - 2}{8\pi^2 t} - \langle \eta_u(d\lambda + \lambda_0) + \rho_{u_m}, \Omega^{u_m} \rangle \right) \right]_{y = -\omega^{-1} \mu_m}^{\text{max}}.
\]

Indeed, by (5.4.6), we have the following identity as elements in \( \Lambda(u^+(b)^*), \)
\[
(7.3.49) \quad \{ \hat{A}^{-1}(\text{iad}(y)|_{u_m}) \text{Tr}_{V_{d\lambda + \lambda_0, \omega}} \left[ \exp \left( -i\rho V_{d\lambda + \lambda_0, \omega}(y) \right) \right] \} \}_{y = -\omega^{-1} \mu_m}^{\text{max}}
\]
\[
= \int_{f \in O_{\lambda^{-1} + \mu_m}} e^{-\langle f, \Omega^{u_m} \rangle} d\mu_{\Lambda}. \]

Recall that the curvature form \( \Omega^{u_m}(b) \) is invariant by the action of \( U_M \) on \( Y_b \).
Since \( a_1 \) and \( \omega Y_{2} \) are invariant by \( U_M \)-action, so is \( \Omega^{u_m} \). Therefore, for \( f \in u_m^* \), \( u \in U_M \), then
\[
(7.3.50) \quad \left[ \exp \left( -\frac{\beta(a_1)^2 \omega Y_{2} - 2}{8\pi^2 t} \right) \exp \left( -\langle Ad^*(u)f, \Omega^{u_m} \rangle \right) \right]_{y = -\omega^{-1} \mu_m}^{\text{max}}
\]
\[
= \det Ad(u)|_{u^+(b)} \left[ \exp \left( -\frac{\beta(a_1)^2 \omega Y_{2} - 2}{8\pi^2 t} \right) \exp \left( -\langle f, \Omega^{u_m} \rangle \right) \right]_{y = -\omega^{-1} \mu_m}^{\text{max}}.
\]

Since \( U_M \) acts on \( u^+(b) \) isometrically with respect to \(-B\), then
\[
(7.3.51) \quad \det Ad(u)|_{u^+(b)} = 1. \]

Then (7.3.48) follows from (7.3.47) and from (7.3.49) - (7.3.51).

The right-hand side of (7.3.48) is a polynomial in \( d \) and in \( t^{-1} \). Recall that \( \text{dim} u^+(b) = 4t \). Then we can rewrite the right-hand side of (7.3.48) as follows,
\[
(7.3.52) \quad \text{dim} V_{d\lambda + \lambda_0, \omega} \sum_{j=0}^{l} \frac{1}{j!} \left[ (-1)^j \beta(a_1)^{2j} \right] \left[ \omega Y_{2} - 2j \right] \left[ \omega(d\lambda + \lambda_0 + \rho_u), \Omega^{u_m} \right]_{2j-2j}^{\text{max}}
\]
\[
(7.3.7) \quad \text{Then (7.3.7) follows from (7.3.4), (7.3.52). This completes the proof.} \]
The Mellin transform of \( I_X(F_d, t) \) (if applicable) is defined by the following formula as a function in \( s \in \mathbb{C} \),
\[
(7.3.53) \quad \mathcal{M} I_X(F_d, s) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} I_X(F_d, t) t^{s-1} dt.
\]

**Theorem 7.3.6.** Suppose that \( \lambda \) is nondegenerate with respect to \( \theta \). For \( d \in \mathbb{N} \) large enough and for \( s \in \mathbb{C} \) with \( \Re(s) \gg 0 \), \( \mathcal{M} I_X(F_d, s) \) is well-defined and holomorphic, which admits a meromorphic extension to \( s \in \mathbb{C} \).

Moreover, we have the following identity,
\[
(7.3.54) \quad \mathcal{M} I_X(F_d, s) = -\frac{c_G}{\sqrt{2\pi}} \sum_{j=0}^{\delta} \frac{\Gamma(s-j-\frac{1}{2})}{\Gamma(s)} \cdot \left[ \sum_{\omega \in W_0} \varepsilon(\omega) Q_{j,\omega}^\lambda(s) (d) (2\pi^2 (d a_{\lambda,\omega} + b_{\lambda,\omega})^2)^{j+\frac{1}{2}-s} \right].
\]

Then \( \mathcal{M} I_X(F_d, s) \) is holomorphic at \( s = 0 \).

Set
\[
(7.3.55) \quad \mathcal{P} I_X(F_d) = \frac{\partial}{\partial s} \bigg|_{s=0} \mathcal{M} I_X(F_d, s),
\]
then we have
\[
(7.3.56) \quad \mathcal{P} I_X(F_d) = -\frac{c_G}{\sqrt{2\pi}} \sum_{j=0}^{\delta} \frac{(-4)^{j+1} (j+1)!}{(2j+2)!} \cdot \left[ \sum_{\omega \in W_0} \varepsilon(\omega) Q_{j,\omega}^\lambda(s) (d) (2\pi^2 (d a_{\lambda,\omega} + b_{\lambda,\omega})^2)^{j+\frac{1}{2}} \right].
\]

The quantity \( \mathcal{P} I_X(F_d) \) is a polynomial in \( d \) for \( d \) large enough, whose coefficients depend only on the root data and \( \lambda, \lambda_0 \).

**Proof.** Since \( \lambda \) is nondegenerate, by Lemma 7.1.3, \( a_{\lambda,\omega} \neq 0, \omega \in W_\omega \). Then there exists \( d_0 \in \mathbb{N} \) such that for \( d \geq d_0 \), \( (d a_{\lambda,\omega} + b_{\lambda,\omega})^2 > 0 \). By (7.2.3) and Theorem 7.3.2, we get (7.3.54). This proves the first part of this theorem.

The equation (7.3.56) is a direct consequence of (7.3.54) by taking its derivative at 0. Note that \( (d a_{\lambda,\omega} + b_{\lambda,\omega})^2 \) is a polynomial in \( d \) for \( d \) large enough. This proves the second part of this theorem. \( \square \)

As explained in Remark 5.3.3, when \( G \) has noncompact center with \( \delta(G) = 1 \) (but \( U \) is still assumed to be compact), most of the above computations can be reduce into very simple ones. Recall that \( a_{\lambda,0} \in \mathbb{R} \) are defined in Definition 7.1.2. When \( \lambda \) is nondegenerate, \( a_{\lambda,0} \neq 0 \).

**Corollary 7.3.7.** Assume that \( U \) is compact and that \( G \) has noncompact center with \( \delta(G) = 1 \), and assume that \( \lambda \) is nondegenerate. Then for \( t > 0, s \in \mathbb{C} \),
\[
I_X(F_d, t) = \frac{c_G}{\sqrt{2\pi t}} e^{-2\pi^2 t (d a_{\lambda} + b_{\lambda_0})^2} \dim E_d,
\]
\[
(7.3.57) \quad \mathcal{M} I_X(F_d, s) = -\frac{c_G}{\sqrt{2\pi}} \frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \left( (2\pi^2 (d a_{\lambda} + b_{\lambda_0})^2)^{1/2-s} \right) \dim E_d.
\]

Furthermore,
\[
(7.3.58) \quad \mathcal{P} I_X(F_d) = 2\pi c_G |d a_{\lambda} + b_{\lambda_0}| \dim E_d.
\]
Proof. By the hypothesis, we get that \( l = 0, \mathcal{W}_\omega = \{1\} \) and \( Q_{\lambda, \omega}^{\lambda, \omega}(d) = \dim E_d \). Then (7.3.57), (7.3.58) are just special cases of (7.3.7), (7.3.54) and (7.3.56).

However, we can prove them more directly using a result of Proposition 4.1.6. It is enough to prove the first identity in (7.3.57). Note that by (5.3.14), we have (7.3.59)

\[ X' = M/K, \]

with \( \delta(X') = 0 \).

By [33, Proposition 5.2] or [40, Proposition 4.1], we have

\[
|e(TX', \nabla TX')|_{\text{max}}^{(7.60)} = (-1)^{\frac{\dim U}{2}} \frac{|W(U_M, T)/|W(K, T)|}{\text{Vol}(U_M/K)}.
\]

Then by (7.3.2), we have

\[
|e(TX', \nabla TX')|_{\text{max}}^{(7.61)} = -c_G.
\]

By (4.1.29) and (7.1.3), we have

\[
\alpha_E = -2\pi (d\alpha + b_{\lambda_0}).
\]

Combining (4.1.32) and (7.60) - (7.62), we get the first identity in (7.3.57), and hence the other identities. This gives a second proof to this corollary. \( \Box \)

7.4. Connection to Müller-Pfaff’s results. In this subsection, we assume that \( G \) has compact center with \( \delta(G) = 1 \). We explain here how to connect our computations in previous subsection to Müller-Pfaff’s results in [33].

If \( \Lambda \in \mathcal{P}_+(U) \), by Proposition 5.3.2, if \( \omega \in \mathcal{W}_\omega \), then \( \eta_\omega(\Lambda) \) is a dominant weight for \( U_M \). Let \( V_{\Lambda, \omega} \) denote the corresponding irreducible representation of \( U_M \). Moreover, by the Weyl dimension formula, \( \dim V_{\Lambda, \omega} \) is a polynomial in \( \eta_\omega(\Lambda) \).

Let \( a^1 \in b^* \) be which takes value \(-1\) at \( a_1 \).

**Definition 7.4.1.** For \( \omega \in \mathcal{W}_\omega \) and \( \Lambda \in \mathcal{P}_+(U) \), for \( z \in \mathbb{C} \), set

\[
P_{\omega, \Lambda}(z) = \dim V_{\Lambda, \omega} \left[ \exp \left( (\eta_\omega(\Lambda) + \rho_{u_m} + z\sqrt{-1}a^1, \Omega^{u(b)}) \right) \right]_{\text{max}}^{\max}.
\]

Since \( \theta \) fix \( \Omega^{u(b)} \), by the fact that \( \det \theta|_{a^1} = 1 \), we get that \( P_{\omega, \Lambda}(z) \) is an even polynomial in \( z \).

We can verify directly that for \( \omega \in W^1, \Lambda \in \mathcal{P}_+(U) \), we have

\[
P_{\omega, \Lambda}(z) = \frac{\text{Vol}(U_M/T)}{\text{Vol}(U/T)} \Pi^a_{a^1 \in R^+,u \in U} \langle a^0, \eta_\omega(\Lambda) + \rho_{u_m} + z\sqrt{-1}a^1 \rangle_{\langle a^0, a^1 \rangle}.
\]

The scalar product in (7.4.2) is taken with respect to \(-B|_U\). Up to a universal constant, \( P_{\omega, \Lambda}(z) \) is just the polynomial related to the Plancherel measure of representation \( V_{\Lambda, \omega} \) as given in [33, Eq.(6.10)]. Note that there is no factor \((2\pi)^{2l}\) appeared in (7.4.2) because of our normalization for \([\cdot]_{\text{max}}^{\max} \).

**Lemma 7.4.2.** We have the following identity

\[
\sum_{j=0}^l \frac{(-4)^{j+1}(j+1)!}{\sqrt{2(2j+2)!}} Q_{j\lambda, \omega}^{\lambda, \omega}(d)(2\pi^2 (d\lambda + b_{\lambda_0} \omega)^2)^{j+\frac{1}{2}}
\]

\[
= -2\pi \int_0^{[d\lambda + b_{\lambda_0} \omega]} P_{\omega, d\lambda + b_{\lambda_0} \omega}(t) dt.
\]

**Proof.** We have

\[
(\eta_\omega(\Lambda) + \rho_{u_m} + z\sqrt{-1}a^1, \Omega^{u(b)}) = z\beta(a_1)\omega^{Y_b} + \langle \omega(d\lambda + \lambda_0 + \rho_u), \Omega^{u_m} \rangle.
\]

\[
\int_0^{[d\lambda + b_{\lambda_0} \omega]} P_{\omega, d\lambda + b_{\lambda_0} \omega}(t) dt.
\]

By [33, Proposition 5.2] or [40, Proposition 4.1], we have

\[
|e(TX', \nabla TX')|_{\text{max}}^{(7.60)} = (-1)^{\frac{\dim U}{2}} \frac{|W(U_M, T)/|W(K, T)|}{\text{Vol}(U_M/K)}.
\]

Then by (7.3.2), we have

\[
|e(TX', \nabla TX')|_{\text{max}}^{(7.61)} = -c_G.
\]

By (4.1.29) and (7.1.3), we have

\[
\alpha_E = -2\pi (d\alpha + b_{\lambda_0}).
\]

Combining (4.1.32) and (7.60) - (7.62), we get the first identity in (7.3.57), and hence the other identities. This gives a second proof to this corollary. \( \Box \)
Since $P_{\omega,d\lambda+\lambda_0}(z)$ is an even function in $z$, then

\begin{equation}
\tag{7.4.5}
P_{\omega,d\lambda+\lambda_0}(z) = \dim V_{d\lambda+\lambda_0,\omega} \left. \frac{1}{(2l)!} \left[ \left. \left( z\beta(a_1)\omega^{Y_z} + \langle \omega(d\lambda + \lambda_0 + \rho_0), \Omega^{u_0} \rangle \right)^{2l} \right|_{\max} \right. \right. \\
= \dim V_{d\lambda+\lambda_0,\omega} \sum_{j=0}^{l} \frac{\beta(a_1)^{2j} z^{2j}}{(2l-2j)! (2j)!} \cdot \left. \left[ \omega^{Y_z} + \langle \omega(d\lambda + \lambda_0 + \rho_0), \Omega^{u_0} \rangle^{2l-2j} \right|_{\max} \right. 
\end{equation}

Note that for $j = 0, 1, \ldots, l$,

\begin{equation}
\tag{7.4.6}
\int_0^{\left| da_{\lambda,\omega} + b_{\lambda_0,\omega} \right|^2} t^{2j} \frac{1}{2j+1} \left| da_{\lambda,\omega} + b_{\lambda_0,\omega} \right|^{2j+1}.
\end{equation}

The equality in (7.4.3) is a consequence of (7.3.4), (7.4.5) and (7.4.6).

As a consequence of (7.3.56) and Lemma 7.4.2, we have the following result.

**Corollary 7.4.3.** Suppose that $\lambda$ is nondegenerate with respect to $\theta$. Then

\begin{equation}
\tag{7.4.7}
\mathcal{PI}_X(F_d) = 2\pi c_G \sum_{\omega \in W_u} \varepsilon(\omega) \int_0^{\left| da_{\lambda,\omega} + b_{\lambda_0,\omega} \right|^2} P_{\omega,d\lambda+\lambda_0}(t) dt.
\end{equation}

By [33, Lemma 6.1], we can get the following identity,

\begin{equation}
\tag{7.4.8}
[W(K,T)] = 2[W(K_M,T)].
\end{equation}

Combining (7.3.2), (7.4.2), (7.4.8), we see that the formula in Corollary 7.4.3, is exactly the same formula of Müller-Pfaff [33, Proposition 6.6] for $\mathcal{PI}_X(F_d)$.

Recall that the $U$-representation $E_d$ has highest weight $d\lambda + \lambda_0 \in P_+$. Then by Weyl dimension formula, $\dim E_d$ is a polynomial in $d$. If $\lambda$ is regular, then the degree (in $d$) of $\dim E_d$ is $\frac{\dim_{sl(n)\mathbb{Z}}}{2}$.

For determining the leading term of $\mathcal{PI}_X(F_d)$, as mentioned in the introduction part, we can specialize the result of Bismut-Ma-Zhang [6, Theorem 0.1] as in [6, Section 8] for the symmetric space $X$. Here to emphasize $\mathcal{PI}_X(F_d)$ being a polynomial in $d$, we prefer to state a result of Müller-Pfaff [33, Proposition 1.3] as follows.

**Proposition 7.4.4.** Suppose that $\lambda$ is nondegenerate. Then there exists a constant $C_{X,\lambda,\lambda_0} \neq 0$ such that

\begin{equation}
\tag{7.4.9}
\mathcal{PI}_X(F_d) = C_{X,\lambda,\lambda_0} d \dim E_d + R(d),
\end{equation}

where $R(d)$ is a polynomial whose degree is no greater than the degree of $\dim E_d$.

In [33, Propositions 6.7 & 6.8], when $G = SL_3(\mathbb{R})$ or $SO^0(p,q)$ with $pq > 1$ odd, they gave very explicit computations for $\mathcal{PI}_X(F_d)$ which verify (7.4.9). In particular, for certain nondegenerate dominant weight $\lambda$, they also worked out explicitly the constant $C_{X,\lambda,0}$ [33, Corollaries 1.4 & 1.5].

An important step in Müller-Pfaff’s proof to Proposition 7.4.4 is reducing the computation of $\mathcal{PI}_X(F_d)$ to the cases where $G = SL_3(\mathbb{R})$ or $SO^0(p,q)$ with $pq > 1$ odd. Such reduction is already explained in Subsection 4.2. More precisely, we have

\begin{equation}
\tag{7.4.10}
X = X_1 \times X_2,
\end{equation}

where $X_1$ is one case listed in (4.2.1), and $X_2$ is a symmetric space rank 0.

We use the notation in Subsection 4.2. Let $\lambda_i, \lambda_{0,i}$ be dominant weights of $U_i, i = 1, 2$ such that

\begin{equation}
\tag{7.4.11}
\lambda = \lambda_1 + \lambda_2, \lambda_0 = \lambda_{0,1} + \lambda_{0,2}.
\end{equation}
Now we consider the sequence \( d\lambda + \lambda_0 \), \( d \in \mathbb{N} \). Then
\[
(7.4.12) \quad E_d = E_{d\lambda_1 + \lambda_0,1} \otimes E_{d\lambda_2 + \lambda_0,2}.
\]

Since \( G_2 \) is equal rank, the nondegeneracy of \( \lambda \) with respect to \( \theta \) is equivalent to the nondegeneracy of \( \lambda_1 \) with respect to \( \theta_1 \). Then by Proposition 4.2.1, after taking the Mellin transform, we have
\[
(7.4.13) \quad \mathcal{M}I_X(F_d, s) = [e(TX_2, \nabla^{TX_2})]^{\max_2} \dim E_{d\lambda_2 + \lambda_0,2} \mathcal{M}I_{X_1}(F_{d\lambda_1 + \lambda_0,1}, s).
\]

Then
\[
(7.4.14) \quad \mathcal{P}I_X(F_d) = [e(TX_2, \nabla^{TX_2})]^{\max_2} \dim E_{d\lambda_2 + \lambda_0,2} \mathcal{P}I_{X_1}(F_{d\lambda_1 + \lambda_0,1}).
\]

Then we only need to evaluate \( \mathcal{P}I_{X_1}(F_{d\lambda_1 + \lambda_0,1}) \) explicitly, which has been done in [33, Section 6].

Remark 7.4.5. Note that in [33, Proposition 1.3], the authors proved Proposition 7.4.4 for \( \lambda_0 = 0 \). Actually, their results still hold if we take a nonzero \( \lambda_0 \in P_{\pm}(U) \) via repeating their computations for \( G = \text{SL}_3(\mathbb{R}) \) and \( \text{SO}^0(p,q) \) with \( pq > 1 \) odd.

7.5. Asymptotic elliptic orbital integrals.

Definition 7.5.1. A function \( f(d) \) in \( d \) is called a pseudopolynomial in \( d \) if it is a finite sum of the term \( c_{j,s} e^{2\pi \sqrt{-1} d j} \) with \( j \in \mathbb{N}, s \in \mathbb{R}, c_{j,s} \in \mathbb{C} \). The largest \( j \geq 0 \) such that \( c_{j,s} \neq 0 \) in \( f(d) \) is called the degree of \( f(d) \).

We say that the oscillating term \( e^{2\pi \sqrt{-1} d j} \) is nice if \( s \in \mathbb{Q} \). We say that a pseudopolynomial \( f(d) \) in \( d \) is nice if all its oscillating terms are nice.

Remark 7.5.2. If \( f(d) \) is a nice pseudopolynomial in \( d \), then there exists a \( N_0 \in \mathbb{N}_{>0} \) such that the function \( f(d N_0) \) is a polynomial in \( d \).

We will use the same notation as in Section 6. The following theorem is a consequence of the geometric localization formula obtained in Theorem 6.0.1.

Theorem 7.5.3. Suppose that \( \gamma \in G \) is elliptic and that \( \lambda \) is nondegenerate with respect to \( \theta \). If \( s \in \mathbb{C} \) is with \( \Re(s) > 0 \), the Mellin transform \( \mathcal{M}E_{X,\gamma}(F_d, s) \) of \( E_{X,\gamma}(F_d, t) \), \( t > 0 \) is well-defined and holomorphic. It admits a meromorphic extension to \( s \in \mathbb{C} \) which is holomorphic at \( s = 0 \).

Set
\[
(7.5.1) \quad \mathcal{P}E_{X,\gamma}(F_d) = \frac{\partial}{\partial s}|_{s=0} \mathcal{M}E_{X,\gamma}(F_d, s).
\]

Then \( \mathcal{P}E_{X,\gamma}(F_d) \) is a pseudopolynomial in \( d \) (for \( d \) large). If \( \gamma \) is of finite order, then \( \mathcal{P}E_{X,\gamma}(F_d) \) is a nice pseudopolynomial in \( d \).

More precisely, let \( k \in T \) be an element conjugate to \( \gamma \) in \( G \). Let \( W_0^k(k) \subset W(U,T_U) \) be defined as in (5.4.13) with respect to \( R^+(u,t_U) \). Then for \( \sigma \in W_0^k(k) \), \( \sigma \lambda \in P_{\pm}(U(k)) \) is nondegenerate with respect to the Cartan involution \( \theta \) on \( (k) \).

For \( d \in \mathbb{N} \), let \( E_{\sigma,d}^k \) be the irreducible unitary representation of \( U(k) \) with highest weight \( d \lambda + \sigma(\lambda_0 + \rho_u) - \rho_u(k) \). This way, we get a sequence of flat vector bundles \( \{F_{\sigma,d}^k\}_{d \in \mathbb{N}} \) on \( X(k) \). Then we have
\[
(7.5.2) \quad \mathcal{P}E_{X,\gamma}(F_d) = \sum_{\sigma \in W_0^k(k)} \varphi_U^k(\sigma, d\lambda + \lambda_0) \mathcal{P}I_X(k)(F_{\sigma,d}^k).
\]

For \( \sigma \in W_0^k(k) \), the term \( \varphi_U^k(\sigma, d\lambda + \lambda_0) \) defined as in (6.0.8) is an oscillating term, which is nice if \( \gamma \) is of finite order.
Proof. For proving this theorem, we only need to prove (7.5.2). Actually, by Theorem 6.0.1, for \( t > 0 \), we get

\[
\mathcal{E}_{X,\gamma}(F_d, t) = \sum_{\sigma \in W^0_k(k)} \varphi^U_k(\sigma, d\lambda + \lambda_0)\mathcal{I}_{X,k}(F_{\sigma,d}, t),
\]

Then (7.5.2) follows from the linearity of Mellin transform.

For \( \sigma \in W^0_k(k) \), it is clear that \( \varphi^U_k(\sigma, d\lambda + \lambda_0) \) is an oscillating term by its defining formula (6.0.8). If \( \gamma \) is of finite order, so is \( k \in T \), then there exists \( N \in \mathbb{N} \) such that \( k^N = 1 \). Since \( \sigma(d\lambda + \lambda_0 + \rho_u) + \rho_u \) is analytically integrable, then

\[
\xi_{\sigma(d\lambda + \lambda_0 + \rho_u) + \rho_u}(k)^N = 1.
\]

Therefore, by (7.5.4), \( \varphi^U_k(\sigma, d\lambda + \lambda_0) \) is a nice oscillating term in \( d \). The rest part follows from the fact each \( \mathcal{PI}_{X,k}(F_{\sigma,d}) \) is a polynomial in \( d \), where we can use Theorem 7.3.6, Corollaries 7.3.7 & 7.4.3 to compute them. This completes the proof of our theorem.

If we write down each polynomial \( \mathcal{PI}_{X,k}(F_{\sigma,d}) \) by the formulae as in (7.3.50), then we can get an explicit formula for \( \mathcal{PE}_{X,\gamma}(F_d) \) in terms of root data and \( \lambda, \lambda_0 \).

In the sequel, we give a different way to evaluate \( \mathcal{PE}_{X,\gamma}(F_d) \) inspired by the computations in Subsection 7.3. Let \( \gamma \in G \) be elliptic, after conjugation, we may assume that \( \gamma = k \in T \). Then \( T \) is also a maximal torus for \( K(\gamma)^0 \), and \( b(\gamma) = b \).

Recall that \( \omega_{Y_k(\gamma)}, \Omega_{\mu_k(\gamma)}, \Omega_{\mu_m(\gamma)} \) are defined in Subsection 5.2. Note that \( \dim u^0_k(\gamma) = 4l(\gamma) \). If \( \nu \in \Lambda(u^0_k(\gamma)^*) \), let \( \nu^{\max(\gamma)} \in \mathbb{R} \) be such that

\[
\nu - \nu^{\max(\gamma)} \omega_{Y_k(\gamma), 2l(\gamma)} \geq (2l(\gamma))!.
\]

is of degree strictly smaller than \( 4l(\gamma) \).

Recall that \(-B(\cdot, \theta^*)\) is an Euclidean product on \( g \). Let \( n^+(\gamma), n^-(\gamma) \) be the orthogonal spaces of \( n(\gamma), n(\gamma) \) in \( n, n \) respectively. As \( T \)-modules, \( n^+(\gamma) \simeq n^-(\gamma) \).

For \( \gamma = k \in T \), set

\[
c_G(\gamma) = \frac{(-1)^{\frac{\nu^{\max(\gamma)} - 1}{2}} \text{Vol}(K(\gamma)^0/T) |W(U_M(\gamma)^0, T)|}{\text{Vol}(U_M(\gamma)^0/T) |W(K(\gamma)^0, T)|} \frac{1}{\det(1 - Ad(\gamma))|n^+(\gamma)|}.
\]

Then \( c_G(1) \) is just the constant \( c_G \) defined in (7.3.2).

We will use the same notation as in Subsections 5.3 & 5.4. In particular, \( W_u \) is defined by (5.3.6) as a subset of \( W(u_{\mathbb{C}}, t_{\mathbb{C}}, \gamma) \), and \( W^1(\gamma) \) is defined by (5.4.13) as a subset of \( W(u_{m, \mathbb{C}}, t_{\mathbb{C}}) \). Now we extend Definition 7.4.1 for \( \gamma \in T \).

**Definition 7.5.4.** For \( \omega \in W_u \), \( \sigma \in W^1(\gamma) \), if \( \Lambda \in P_+(U) \), for \( z \in \mathbb{C} \), set

\[
P^\gamma_{\omega,\sigma,\Lambda}(z) = \text{Vol}_L(O^\gamma_{\sigma(\eta_{\omega}(\Lambda) + \rho_m)}) \cdot \left[ \exp \left( (\Omega_{\mu_k(\gamma)}, \sigma(\eta_{\omega}(\Lambda) + \rho_m)) + z\sqrt{-1}n^1(\gamma) \right) \right]^{\max(\gamma)}.
\]

As in Subsection 7.4, the coefficients of \( z^j \) in \( P^\gamma_{\omega,\sigma,\Lambda}(z) \) are polynomials in \( \Lambda \).

**Theorem 7.5.5.** Suppose that \( \lambda \) is nondegenerate, and that \( \gamma = k \in T \). Recall that \( \varphi_\gamma(\sigma, \eta_{\omega}(d\lambda + \lambda_0)) \) is an oscillating term defined by (5.4.18). Then

\[
\mathcal{PE}_{X,\gamma}(F_d) = 2\pi r c_G(\gamma) \sum_{\omega \in W^1(\gamma)} \varepsilon(\omega) \varphi_\gamma(\sigma, \eta_{\omega}(d\lambda + \lambda_0)) \int_0^{|d\alpha_\omega + \beta_{\lambda_0,\omega}|} P^\gamma_{\omega,\sigma,d\lambda + \lambda_0}(t) dt.
\]
If we consider $G = \text{Spin}(1, 2n + 1)$, $n \geq 1$ as in [17], then up to a constant, the pseudopolynomial $\sum_{\sigma \in W^+(\gamma)} \varphi_\gamma(\sigma, \eta_\nu(d\lambda + \lambda_0))P^{\gamma}_{\sigma, d\lambda + \lambda_0}(t)$ is just the one defined by Ksenia Fedosova in [17, Proposition 5.1]. This way, our results are compatible with her results in [17, Theorem 1.1] for hyperbolic orbifolds.

Remark 7.5.6. Let $\text{Char}(A)$ denote the character ring of the complex representations of a compact Lie group $A$. One key ingredient in both (7.5.2) and (7.5.8) is an explicit decomposition of characters of $U$ into characters of $U_M(\gamma)^0$. In the diagram (7.5.9), we give two ways of this decomposition. The formula in (7.5.2) is obtained by the computations along the upper path in (7.5.9), and a proof to (7.5.8) will follow the lower path as in Subsection 7.3.

(7.5.9)

Now we focus on (7.5.8). Since $\gamma = k \in T$, for $Y^e_k \in \mathfrak{t}(\gamma)$, $J_\gamma(Y^e_k)$ is given by

$$
\frac{\hat{A}(\text{ad}(Y^e_k)|_{\mathfrak{p}(\gamma)})}{\hat{A}(\text{ad}(Y^e_k)|_{\mathfrak{t}(\gamma)})} \left[ \frac{1}{\det(1 - e^{-\text{ad}(Y^e_k)\text{Ad}(k)})|_{\mathfrak{k}^+(\gamma)}} \right]^{1/2}.
$$

Since $\mathfrak{t} \subset \mathfrak{t}(\gamma) \subset \mathfrak{k}$, then $R(\mathfrak{t}(\gamma), \mathfrak{t})$ is a subset of $R(\mathfrak{t}, \mathfrak{t})$. Let $R^+(\mathfrak{t}(\gamma), \mathfrak{t})$ be the positive root system for $(\mathfrak{t}(\gamma), \mathfrak{t})$ induced by $R^+(\mathfrak{t}, \mathfrak{t})$. We use the notation in Subsections 5.1, 5.2. Then $\mathfrak{t}$ is a Cartan subalgebra for $\mathfrak{t}_m(\gamma)$, $\mathfrak{u}_m(\gamma)$, $\mathfrak{m}(\gamma)$. Let $R(\mathfrak{t}_m(\gamma), \mathfrak{t})$, $R(\mathfrak{u}_m(\gamma), \mathfrak{t})$ be the corresponding root systems.

As in (7.3.10), we have the following disjoint union

$$
R(\mathfrak{u}_m(\gamma), \mathfrak{t}) = R(\sqrt{-1}\mathfrak{p}_m(\gamma), \mathfrak{t}) \cup R(\mathfrak{t}_m(\gamma), \mathfrak{t}).
$$

Since $R(\mathfrak{u}_m(\gamma), \mathfrak{t}) \subset R(\mathfrak{u}_m, \mathfrak{t})$, then by intersecting with $R^+(\mathfrak{u}_m, \mathfrak{t})$, we get a positive root system $R^+(\mathfrak{u}_m(\gamma), \mathfrak{t})$. Moreover,

$$
R^+(\mathfrak{u}_m(\gamma), \mathfrak{t}) = R^+(\sqrt{-1}\mathfrak{p}_m(\gamma), \mathfrak{t}) \cup R^+(\mathfrak{t}_m(\gamma), \mathfrak{t}).
$$

If $y \in \mathfrak{t}_C$, let $\pi_\mathfrak{t}_m(\gamma)/t(y)$, $\pi_\sqrt{-1}\mathfrak{p}_m(\gamma)/t(y)$, $\pi_\mathfrak{t}_m(\gamma)/t(y)$ be the analogues as in (7.3.13). We also define $\sigma_\mathfrak{t}_m(\gamma)/t(y)$, $\sigma_\sqrt{-1}\mathfrak{p}_m(\gamma)/t(y)$, $\sigma_\mathfrak{t}_m(\gamma)/t(y)$ as in (7.3.14).

Set

$$
\begin{align*}
\mathfrak{t}_m(\gamma) &= \mathfrak{t}(\gamma) \cap \mathfrak{t}_m, \\
\mathfrak{p}_m(\gamma) &= \mathfrak{p}(\gamma) \cap \mathfrak{p}_m; \\
\mathfrak{t}_m(\gamma) &= \mathfrak{t}(\gamma) \cap \mathfrak{t}(b), \\
\mathfrak{p}_m(\gamma) &= \mathfrak{p}(\gamma) \cap \mathfrak{p}(b).
\end{align*}
$$

Let $\mathfrak{m}(\gamma)$ be the orthogonal space of $\mathfrak{m}(\gamma)$ in $\mathfrak{m}$ with respect to $B$. Then

$$
\mathfrak{m}(\gamma) = \mathfrak{p}_m(\gamma) \oplus \mathfrak{t}_m(\gamma).
$$

We also have

$$
\mathfrak{t}_m = \mathfrak{t}_m(\gamma) \oplus \mathfrak{t}_m(\gamma), \\
\mathfrak{p}_m = \mathfrak{p}_m(\gamma) \oplus \mathfrak{p}_m(\gamma), \\
\mathfrak{t}(\gamma) = \mathfrak{t}_m(\gamma) \oplus \mathfrak{t}_m(\gamma), \\
\mathfrak{p}(\gamma) = \mathfrak{p}_m(\gamma) \oplus \mathfrak{p}_m(\gamma).
$$

Set

$$
\mathfrak{u}_m(\gamma) = \sqrt{-1}\mathfrak{p}_m(\gamma) \oplus \mathfrak{t}_m(\gamma).
$$

Then it is the orthogonal space of $\mathfrak{u}_m(\gamma)$ in $\mathfrak{u}_m$ with respect to $B$. 

Lemma 7.5.7. The following spaces are isomorphic to each other as modules of $T$ by the adjoint actions,

\[(7.5.18) \quad n^+ (\gamma) \simeq \bar{n}^+ (\gamma) \simeq \mathfrak{p}''_m (\gamma) \simeq p''_m (\gamma).\]

Proof. Note that

\[(7.5.19) \quad \dim n = \dim \mathfrak{t} - \dim \mathfrak{f}_m, \quad \dim n(\gamma) = \dim \mathfrak{t}(\gamma) - \dim \mathfrak{f}_m (\gamma).\]

Together with the splittings (7.5.15), (7.5.16), we get

\[(7.5.20) \quad \dim \mathfrak{p}''_m (\gamma) = \dim n^+ (\gamma).\]

Similarly, $\dim p''_m (\gamma) = \dim n^+ (\gamma)$.

If $f \in n^+ (\gamma)$, then $f + \theta (f) \in \mathfrak{f}$, we can verify directly that $f + \theta (f) \in \mathfrak{p}''_m (\gamma)$.

Then the map $f \in n^+ (\gamma) \to f + \theta (f) \in \mathfrak{p}''_m (\gamma)$ defines an isomorphism of $T$-modules. Similar for $n^+ (\gamma) \simeq p''_m (\gamma)$. □

Since $\gamma = k \in T$, let $y_0 \in \mathfrak{t}$ be such that $\exp(y_0) = \gamma$. Note that $y_0$ is not unique.

Lemma 7.5.8. If $y \in \mathfrak{t}$ is regular with respect to $R(\mathfrak{f}_m, \mathfrak{t})$, then we have

\[(7.5.21) \quad J_\gamma (y) \text{Tr}_n^\Lambda (\mathfrak{p}^*) \left[ (N^\Lambda (\mathfrak{p}^* - \frac{m}{2}) \text{Ad}(k) \exp(-i \text{ad}(y)) \right]
= (-1)^{\dim \mathfrak{p}_m (\gamma)/2 + 1} \det(1 - \text{Ad}(k))|_{n^+ (\gamma)} \text{Tr}_n^\Lambda (\mathfrak{p}_m (\gamma)) [e^{-i \text{ad}(y)} \text{Ad}(k)]
\]

\[
\frac{\pi \sqrt{-\text{Tr}_n (\gamma) / (iy)} \sigma_{n^+ (\gamma)} / (iy) \sigma_u (\gamma) / (iy)}{\pi \text{Tr}_n (\gamma) / (iy)} \left[ \frac{1}{\det(1 - e^{-i \text{ad}(y)} \text{Ad}(k))|_{\mathfrak{p}''_m (\gamma)}} \right]^{1/2}
\]

\[(7.5.22) \quad \left[ \frac{1}{\det(1 - e^{-i \text{ad}(y)} \text{Ad}(k))|_{\mathfrak{p}''_m (\gamma)}} \right]^{1/2} \frac{1}{\det(1 - \text{Ad}(k))|_{n^+ (\gamma)}} \frac{1}{\det(1 - e^{-i \text{ad}(y)} \text{Ad}(k))|_{\mathfrak{p}''_m (\gamma)}}
\]

\[
\frac{\pi \sqrt{-\text{Tr}_n (\gamma) / (iy)} \sigma_{n^+ (\gamma)} / (iy) \sigma_u (\gamma) / (iy)}{\pi \text{Tr}_n (\gamma) / (iy)} \left[ \frac{1}{\det(1 - e^{-i \text{ad}(y)} \text{Ad}(k))|_{\mathfrak{p}''_m (\gamma)}} \right]^{1/2}
\]

Recall that as $K_M$-modules, we have the following isomorphism

\[(7.5.23) \quad \mathfrak{p} \simeq \mathfrak{b} \oplus \mathfrak{p}_m \oplus \mathfrak{n}.
\]

Note that

\[(7.5.24) \quad \text{Ad}(k) = e^{i \text{ad}(y_0)}.
\]

If $y \in \mathfrak{t}$, when acting on $\mathfrak{p}$, we have

\[(7.5.25) \quad \text{Ad}(k) \exp(-i \text{ad}(y)) = \exp(\text{ad}(-iy + y_0)).
\]

By Lemma 7.3.5, if $y \in \mathfrak{t}$, then

\[(7.5.26) \quad \text{Tr}_n^\Lambda (\mathfrak{p}^*) \left[ (N^\Lambda (\mathfrak{p}^* - \frac{m}{2}) \text{Ad}(k) \exp(-i \text{ad}(y)) \right] = (-1)^{\dim \mathfrak{p}_m + 1} \sigma \sqrt{-\text{Tr}_n (\gamma) / (iy)} \text{Tr}_n^\Lambda (\mathfrak{p}_m (\gamma)) [e^{-i \text{ad}(y)} \text{Ad}(k)].
\]

Combining (7.3.30), (7.5.22) and (7.5.26), we get (7.5.21). □

We now prove Theorem 7.5.5.

Proof to Theorem 7.5.5. Put

\[(7.5.27) \quad F_\gamma (d, t) = \frac{1}{(2\pi t)^{1/2}} \int_{\mathfrak{t}(\gamma)} J_\gamma (Y_0^d) \text{Tr}_n^\Lambda (\mathfrak{p}^*) \left[ (N^\Lambda (\mathfrak{p}^* - \frac{m}{2}) \text{Ad}(k) e^{-i \text{ad}(Y_0^d)} \right.
\]

\[
\left. \text{Tr}_n^\epsilon_d [\rho^d (k) e^{-i \epsilon_d (Y_0^d)}] e^{-Y_0^d/2t} \right] \frac{dY_0^d}{(2\pi t)^{1/2}}.
\]
By (7.2.1), we have

\[(7.5.28)\]
\[E_{\gamma}(F_{d}, t) = \exp \left( -2\pi^{2} t|d\lambda + \lambda_{0} + \rho u|^{2} \right) F_{\gamma}(d, t). \]

Recall that \(r = p + q = \dim_{\mathbb{R}} \gamma \). By Weyl integration formula, then

\[(7.5.29)\]
\[F_{\gamma}(d, t) = \frac{\text{Vol}(K(\gamma)^{0}/T)}{(2\pi t)^{r/2}} \left[ \int_{\pi t(\gamma)} \int (y)^{2} J_{\gamma}(y) \text{Tr}_{\Lambda}(\gamma) \left[ (\Lambda^{N}(\gamma) - \frac{m}{2}) \text{Ad}(k) e^{-i\lambda(y)} \right] \right. \]
\[\left. \cdot \text{Tr}_{E_{d}[\rho]E_{d}(k) \exp(-i\rho E_{d}(y))]} e^{-|y|^{2}/2t} \right] dy. \]

Recall that \(t(\gamma) = \frac{1}{2} \dim n(\gamma) \). As in (7.3.29), if \(y \in t, \) then

\[(7.5.30)\]
\[\pi t(\gamma) t_{t}(iy)^{2} = (-1)^{k(\gamma)} \pi t_{t}(\gamma) t_{t}(iy)^{2} \det(\text{Ad}(y))|_{n(\gamma)_{\mathbb{C}}}. \]

By the argument as (7.3.34) - (7.3.36), using Lemma 7.5.8 and (7.3.30), we get

\[(7.5.31)\]
\[F_{\gamma}(d, t) = \frac{c_{G}(\gamma)}{(2\pi t)^{r/2}} \int_{y \in u_{m}(\gamma)} \det(\text{Ad}(y))|_{u_{m}(\gamma)_{\mathbb{C}}} \frac{\tilde{A}^{-1}(\text{Ad}(y)|_{u_{m}(\gamma)_{\mathbb{C}}})}{\det(1 - \text{Ad}(k))|_{u_{m}(\gamma)_{\mathbb{C}}}} \]
\[\cdot \text{Tr}_{u_{m}(\gamma)_{\mathbb{C}}} \left[ e^{-|y|^{2}/2t} \right] dy. \]

The constant \(c_{G}(\gamma) \) is defined by (7.5.6).

If \(y \in u_{m}(\gamma), \) then

\[(7.5.32)\]
\[B(y, \frac{\Omega u_{m}(\gamma)}{2\pi}) \in \Lambda^{2}(u_{m}(\gamma)^{*}). \]

If \(y \in u_{m}(\gamma), \) as in (7.3.38), we have

\[(7.5.33)\]
\[\frac{\det(\text{Ad}(y))|_{u_{m}(\gamma)_{\mathbb{C}}}}{(2\pi t)^{2}} = \left[ \exp \left( \frac{\pi}{t} B(y, \frac{\Omega u_{m}(\gamma)}{2\pi}) \right) \right]^{\max(\gamma)}. \]

Note that

\[(7.5.34)\]
\[r = \dim u_{m}(\gamma) + 4l(\gamma) + 1. \]

Combining (7.5.31) - (7.5.34), we get

\[(7.5.35)\]
\[F_{\gamma}(d, t) = \frac{c_{G}(\gamma)}{(2\pi t)^{r/2}} \left[ \int_{u_{m}(\gamma)} \tilde{A}^{-1}(\text{Ad}(y)|_{u_{m}(\gamma)_{\mathbb{C}}}) \left[ \frac{\det(1 - e^{-\lambda(y)} \text{Ad}(k))|_{u_{m}(\gamma)_{\mathbb{C}}}}{\det(1 - \text{Ad}(k))|_{u_{m}(\gamma)_{\mathbb{C}}}} \right] \right. \]
\[\left. \cdot \text{Tr}_{u_{m}(\gamma)_{\mathbb{C}}} \left[ e^{-|y|^{2}/2t} \right] dy \right] \left[ \frac{\max(\gamma)}{2 \pi} \right]. \]

Let \(\Delta^{u_{m}(\gamma)} \) be the standard negative Laplace operator on \(u_{m}(\gamma). \) Using the same argument as in (7.3.42). Then we can rewrite (7.3.5) as follows,

\[(7.5.36)\]
\[F_{\gamma}(d, t) = \frac{c_{G}(\gamma)}{(2\pi t)^{r/2}} \left[ \exp \left( \frac{\pi}{t} B(y, \frac{\Omega u_{m}(\gamma)}{2\pi}) \right) \right. \]
\[\exp \left( \frac{\pi}{t} \Delta^{u_{m}(\gamma)} \right) \left[ \tilde{A}^{-1}(\text{Ad}(y)|_{u_{m}(\gamma)_{\mathbb{C}}}) \right] \left[ \frac{\det(1 - e^{-\lambda(y)} \text{Ad}(k))|_{u_{m}(\gamma)_{\mathbb{C}}}}{\det(1 - \text{Ad}(k))|_{u_{m}(\gamma)_{\mathbb{C}}}} \right] \right. \]
\[\left. \cdot \text{Tr}_{u_{m}(\gamma)_{\mathbb{C}}} \left[ e^{-|y|^{2}/2t} \right] dy \right] \left[ \frac{\max(\gamma)}{2 \pi} \right]. \]
As explained in (7.3.44), $\operatorname{Tr}_{A}^\Lambda (\pi_k^2) \otimes \operatorname{End}[\pi^\Lambda (\pi_k^2) \otimes \operatorname{End}(e^{-iyk})]$ is an alternative sum of characters of $V_{\lambda + \lambda_0, \omega}$, $\omega \in W_u$ of $U_M$. Then we apply the generalized Kirillov formula (5.4.21) to each character.

For $\omega \in W_u$, the function in $y \in u_m(\gamma)$

$$\tilde{A}^{-1}(\lambda_1)|_{u_m(\gamma)} \{ \frac{\det(1 - e^{-i\lambda_1(\lambda_1)}Ad(k))}{\det(1 - Ad(k))} \} \operatorname{Tr}_{A}^\Lambda V_{\lambda + \lambda_0, \omega} \rho V_{\lambda + \lambda_0, \omega}(e^{-iyk})$$

is an eigenfunction of $\Delta_{u_m(\gamma)}$ associated with the eigenvalue $4\pi^2|\eta_\omega(d\lambda + \lambda_0) + \rho_{u_m}|^2$.

We will use the same notation as in Subsection 5.4. The orbit $O^\gamma_{\sigma(\eta_\omega(d\lambda + \lambda_0) + \rho_{u_m})}$ is defined in (5.4.17) equipped with a Liouville measure $d\mu_\gamma$. Finally, as (7.3.48), we get the following identity,

$$\left[ \exp \left( -\frac{\beta(a_1)^2(\omega Y_k(\gamma))^2}{8\pi^2t} \right) \right] \left[ \tilde{A}^{-1}(\lambda_1)|_{u_m(\gamma)} \{ \frac{\det(1 - e^{-i\lambda_1(\lambda_1)}Ad(k))}{\det(1 - Ad(k))} \} \right]_{\max(\gamma)}$$

$$\operatorname{Tr}_{A}^\Lambda V_{\lambda + \lambda_0, \omega} \rho V_{\lambda + \lambda_0, \omega}(e^{-iyk})$$

$$= \sum_{\sigma \in W^1(\gamma)} \varphi_\gamma(\sigma, \eta_\omega(d\lambda + \lambda_0)) \operatorname{Vol}_L(O^\gamma_{\sigma(\eta_\omega(d\lambda + \lambda_0) + \rho_{u_m})})$$

$$\left[ \exp \left( -\frac{\beta(a_1)^2(\omega Y_k(\gamma))^2}{8\pi^2t} \right) \right] \left[ \{ \sigma(\eta_\omega(d\lambda + \lambda_0) + \rho_{u_m}, \Omega^\gamma_{u_m(\gamma)}) \} \right]_{\max(\gamma)}.$$

Then we proceed as in (7.3.52), Theorem 7.3.6, by the same arguments as in the proof of Lemma 7.4.2, we get (7.5.38). This completes the proof of our theorem. □

Remark 7.5.9. Note that by (7.5.28), (7.5.36) - (7.5.38), we can give formulae for $E_{X, \gamma}(F_d, t)$, $\mathcal{M}E_{X, \gamma}(F_d, s)$ in terms of the root data as in (7.3.7), (7.3.54).

8. A proof to Theorem 1.0.2

In this section, we give a complete proof to Theorem 1.0.2, then Theorem 1.0.1 (and Theorem 1.0.1') follows as a consequence. We assume that $G$ is a connected linear reductive Lie group with $\delta(G) = 1$ and compact center, so that $U$ is a compact Lie group.

8.1. A lower bound for the Hodge Laplacian on $X$. We use the notation as in Subsection 4. Recall that $e_1, \cdots, e_m$ is an orthogonal basis of $TX$ or $p$. Put

$$C^{g, H} = -\sum_{j=1}^{m} e_j^2 \in U \mathfrak{g}.$$ 

Let $C^{g, H, E}$ be its action on $E$ via $\rho^E$. Then

$$C^{g, E} = C^{g, H, E} + C^{g, E}$$

Let $\Delta^{H, X}$ be the Bochner-Laplace operator on bundle $\Lambda^1(T^*X) \otimes F$ associated with the unitary connection $\nabla^\Lambda (T^*X) \otimes F, u$. Put

$$\Theta(F) = \frac{SX}{4} - \frac{1}{8}(\langle R^T(e_1, e_2)\tilde{c}(e_1)\tilde{c}(e_2)\rangle - \langle c(e_1)\tilde{c}(e_2)\rangle)$$

$$-C^{g, H, E} + \frac{1}{2}(\langle c(e_1)\tilde{c}(e_2)\rangle - \langle \tilde{c}(e_1)\tilde{c}(e_2)\rangle)R^F(e_1, e_2),$$

where $R^F$ is the curvature of the unitary connection $\nabla^F$ on $F$.
Then $\Theta(F)$ is a self-adjoint section of $\text{End}(\Lambda(T^*X) \otimes F)$, which is parallel with respect to $\nabla^\Lambda(T^*X) \otimes F$. By [6, eq.(8.39)], we have
\begin{equation}
D_{X,F,2} = -D_{H,X} + \Theta(F).
\end{equation}

Let $\Omega_c(X,F)$ be the set of smooth sections of $\Lambda(T^*X) \otimes F$ on $X$ with compact support. Let $\langle \cdot, \cdot \rangle_{L_2}$ be the $L_2$ scalar product on it. If $s \in \Omega_c(X,F)$, we have
\begin{equation}
(D_{X,F,2} s, s)_{L_2} \geq \langle \Theta(F)s, s \rangle_{L_2}.
\end{equation}

Let $D_{H,X}^i$ denote the Bochner-Laplace operator acting on $\Omega_i(X,F)$, and let $p_{t}^{H,i}(x,x')$ be the kernel of $\exp(tD_{H,X}^i/2)$ on $X$ with respect to $dx'$. We will denote by $p_{t}^{H,i}(g) \in \text{End}(\Lambda^i(p^*) \otimes E)$ its lift to $G$ explained in Subsection 3.2. Let $\Delta_0^X$ be the scalar Laplacian on $X$ with the heat kernel $p_0^X$.

Let $\|p_t^{H,i}(g)\|$ be the operator norm of $p_t^{H,i}(g)$ in $\text{End}(\Lambda^i(p^*) \otimes E)$. By [34, Proposition 3.1], if $g \in G$, then
\begin{equation}
||p_t^{H,i}(g)|| \leq p_{t=0}^{X,0}(g).
\end{equation}

Let $p_t^H$ be the kernel of $\exp(t\Delta_{H,X}/2)$, then
\begin{equation}
p_t^H = \bigoplus_{i=1}^p p_t^{H,i}.
\end{equation}

Let $q_t^{X,F}$ be the heat kernel associated with $D_{X,F,2}$, by (8.1.4), for $x,x' \in X$,
\begin{equation}
q_t^{X,F}(x,x') = \exp(-t\Theta(F)/2)p_t^H(x,x').
\end{equation}

Recall that $P^+_+(U)$ is the set of dominant weights of $U$ with respect to $R^+(u,t_U)$ defined in Subsection 5.3. As in Subsection 7.1, we fix $\lambda, \lambda_0 \in P^+_+(U)$ such that $\lambda$ is nondegenerate with respect to $\theta$. Recall that for $d \in \mathbb{N}$, $(E_d, \rho^E_d)$ is the irreducible unitary representation of $U$ with highest weight $d\lambda + \lambda_0$, which extends uniquely to a representation of $G$. By [5, Théorème 3.2] [6, Theorem 4.4 and Remark 4.5] and [33, Proposition 7.5], there exist $c > 0$, $C > 0$ such that, for $d \in \mathbb{N}$,
\begin{equation}
\Theta(F_d) \geq cd^2 - C,
\end{equation}
where the estimate $d^2$ comes from the positive operator $C^\theta_{H,E_d}$. By (8.1.4), (8.1.5), (8.1.9), we get
\begin{equation}
D_{X,F,d,2} \geq cd^2 - C.
\end{equation}

\textbf{Lemma 8.1.1.} There exists $d_0 \in \mathbb{N}$ and $c_0 > 0$ such that if $d \geq d_0$, $x,x' \in X$
\begin{equation}
\|q_t^{X,F_d}(x,x')\| \leq e^{-c_0 d^2/2} p_{t=0}^{X,0}(x,x').
\end{equation}

\textbf{Proof.} By (8.1.9), there exist $d_0 \in \mathbb{N}$, $c' > 0$ such that if $d \geq d_0$,
\begin{equation}
\Theta(F_d) \geq c'd^2.
\end{equation}
Then if $t > 0$,
\begin{equation}
\|\exp(-t\Theta(F_d)/2)\| \leq e^{-c'd^2t/2}.
\end{equation}
By (8.1.6), (8.1.7), (8.1.8), (8.1.13), we get (8.1.11). This completes the proof of our lemma. \hfill \Box

Our locally symmetric orbifold $Z$ is defined as $\Gamma \backslash X$, where $\Gamma$ is a cocompact discrete subgroup of $G$. For $\gamma \in \Gamma$, the number $m_\gamma \geq 0$ is given by (3.3.3), which depends only on the conjugacy class of $\gamma$ (in $G$ or $\Gamma$). Recall that $E[\Gamma]$ is the finite set of elliptic conjugacy classes in $\Gamma$.

For $t > 0$, $x \in X$, $\gamma \in \Gamma$, set
\begin{equation}
v_t(F_d, \gamma, x) = Tr_{\gamma}((T^*X) \otimes F_d \left( (N^\Lambda(T^*X) - \frac{m_\gamma}{2})q_t^{X,F_d}(x, \gamma(x)) \right).
\end{equation}
By Lemma 8.1.1 and (8.1.14), we have the following result.
Lemma 8.1.2. There exist $C_0 > 0$, $c_0 > 0$ such that if $d$ is large enough, for $t > 0$, $x \in X$, $\gamma \in \Gamma$,

\[(8.1.15) \quad |v_t(F_d, \gamma, x)| \leq C_0 (\dim E_d) e^{-c_0 d^2 t} p_t^{X,0}(x, \gamma(x)).\]

Set

\[(8.1.16) \quad m_\Gamma = \inf_{|\gamma| \in [\Gamma] \setminus \Gamma} m_\gamma.\]

By [25, Proposition 1.8.4], $m_\Gamma > 0$.

The following proposition is a special case of [25, Proposition 7.5.3].

Proposition 8.1.3. There exist constants $C > 0$, $c > 0$ such that if $x \in X$, $t \in [0, 1]$, then

\[(8.1.17) \quad \sum_{\gamma \in \Gamma \setminus \Gamma \text{ nonelliptic}} p_t^{X,0}(x, \gamma(x)) \leq C \exp(-c/t).\]

Proof. By [14, Theorem 3.3], there exists $C_0 > 0$ such that when $0 < t \leq 1$,

\[(8.1.18) \quad p_t^{X,0}(x, x') \leq C_0 t^{-m/2} \exp\left(-\frac{d^2(x, x')}{4t}\right).\]

By [25, Proposition 1.8.5], there exist $c > 0, C > 0$ such that for $R > 0, x \in X$,

\[(8.1.19) \quad \#\{\gamma \in \Gamma \mid \gamma \text{ nonelliptic}, d_\gamma(x) \leq R\} \leq C \exp(cR).\]

By (8.1.16), (8.1.18), (8.1.19), and using the same arguments as in the proof of [34, Proposition 3.2], we get (8.1.17). \(\square\)

8.2. A proof to Theorem 1.0.2. In this subsection, we give a proof to Theorem 1.0.2. As explained in the introduction, the result (1) in Theorem 1.0.2 is trivial by the definition of $\mathcal{P}E_{X,\gamma}(F_d)$ in (7.5.1). The results (2) - (4) are already proved in Theorem 7.5.3. We only need to prove the result (5). We restate it as follows.

Proposition 8.2.1. If $\Gamma \subset G$ is a cocompact discrete subgroup, set $Z = \Gamma \setminus X$, then there exists $c > 0$ such that for $d$ large enough,

\[(8.2.1) \quad T(Z, F_d) = \frac{\text{Vol}(Z)}{|S|} \mathcal{P}E_{X}(F_d) + \sum_{[\gamma] \in E^+ [\Gamma]} \frac{\text{Vol}(\Gamma \cap Z(\gamma) \setminus X(\gamma))}{|S(\gamma)|} \mathcal{P}E_{X,\gamma}(F_d) + O(e^{-cd}),\]

where $E^+ [\Gamma] = E^+ [\Gamma] \setminus \{[1]\}$ is the finite set of nontrivial elliptic classes in $[\Gamma]$.

We do some preparations before giving the proof. By (8.1.10), we have

\[(8.2.2) \quad D^{Z,F_d,2} \geq c d^2 - C.\]

Then if $d$ is large enough, we have

\[(8.2.3) \quad H'(Z, F_d) = 0.\]

By (2.2.10), (8.2.3), if $d$ is large enough, we have

\[(8.2.4) \quad \chi(Z, F_d) = 0, \quad \chi'(Z, F_d) = 0.\]

As in (2.2.20), for $t > 0$, set

\[(8.2.5) \quad b(F_d, t) = (1 + 2t \frac{\partial}{\partial t}) \text{Tr}_s \left( N^A (T^+ Z) - \frac{m}{2} \right) \exp(-t D^{Z,F_d,2}/2).\]

As in [6, Subsection 7.2], by (8.2.2), there exist constants $\tilde{c} > 0, \tilde{C} > 0$ such that for $d$ large enough and for $t > 1/d$,

\[(8.2.6) \quad |b(F_d, t)| \leq \tilde{C} \exp(-\tilde{c}d - \tilde{c}t).\]
By (2.2.25), (8.2.4), we have

\[ T(Z, F_d) = - \int_0^{+\infty} b(F_d, t) \frac{dt}{T}. \]  

**Proof to Proposition 8.2.1.** We rewrite (8.2.7) as follows,

\[ T(Z, F_d) = - \int_{1/d}^{+\infty} b(F_d, t) \frac{dt}{t} - \int_0^d b(F_d, t/d^2) \frac{dt}{t}. \]

By (8.2.6), there exists \( c > 0 \) such that for \( d \) large enough,

\[ \int_{1/d}^{+\infty} b(F_d, t) \frac{dt}{t} = \mathcal{O}(e^{-cd}). \]

By (3.5.1), (8.1.14), (8.2.5), we get

\[ b(F_d, t) = (1 + 2t \frac{\partial}{\partial t}) \int_{|z|} \frac{1}{|S|} \sum_{\gamma \in \Gamma} v_t(F_d, \gamma, z) dz. \]

We split the sum in (8.2.10) into two parts,

\[ \sum_{\gamma \in \Gamma, \gamma \text{ elliptic}} + \sum_{\gamma \in \Gamma, \gamma \text{ nonelliptic}} \]

so that we write

\[ b(F_d, t) = b_{\text{ell}}(F_d, t) + b_{\text{nonell}}(F_d, t). \]

By (3.5.7) - (3.5.13), we get

\[ b_{\text{ell}}(F_d, t) = \sum_{[\gamma] \in \ell \Gamma} \frac{\text{Vol}([\gamma] \cap Z(\gamma) \setminus X(\gamma))}{|S(\gamma)|} (1 + 2t \frac{\partial}{\partial t}) \mathcal{E}_{X, \gamma}(F_d, t). \]

By (7.3.7) and (7.5.3), the terms in \( \mathcal{E}_{X, \gamma}(F_d, t) \) are of the form

\[ t^{-j+1/2} \exp(-2\pi^2 t (dd' + b')^2) Q(d), \]

where \( Q(d) \) is a pseudopolynomial in \( d \), and \( a', b' \in \mathbb{R} \) with \( a' \neq 0 \) due to the nondegeneracy of \( \lambda \). By (8.2.14), there exists \( c > 0 \) such that for \( d \) large enough,

\[ \int_0^d b_{\text{ell}}(F, t/d^2) \frac{dt}{T} = \int_0^{+\infty} b_{\text{ell}}(F_d, t) \frac{dt}{t} + \mathcal{O}(e^{-cd}). \]

Using Proposition 7.2.1 and by (8.2.14), we get

\[ \mathcal{P} \mathcal{E}_{X, \gamma}(F_d) = - \int_0^{+\infty} (1 + 2t \frac{\partial}{\partial t}) \mathcal{E}_{X, \gamma}(F_d, t) \frac{dt}{T}. \]

Now we consider the contribution from the nonelliptic elements. If \( x \in X \), put

\[ h_t(F_d, x) = \frac{1}{|S|} \sum_{\gamma \in \Gamma, \gamma \text{ nonelliptic}} v_t(F_d, \gamma, x). \]

Then

\[ b_{\text{nonell}}(F_d, t) = (1 + 2t \frac{\partial}{\partial t}) \int_{|z|} h_t(F_d, z) dz. \]

Now we prove the following uniform estimates for \( x \in X \),

\[ \int_0^d (1 + 2t \frac{\partial}{\partial t}) h_{t/d^2}(F_d, x) \frac{dt}{t} = \mathcal{O}(e^{-cd}). \]

Indeed, using Lemma 8.1.2 and Proposition 8.1.3, there exists \( C > 0, c' > 0, c'' > 0 \) such that if \( d \) is large enough, \( 0 < t \leq d \), then

\[ |h_{t/d^2}(F_d, x)| \leq C \dim(E_d) e^{-c't} \exp(-c''d^2/t). \]
Recall that \( \dim E_d \) is a polynomial in \( d \). Then by (8.2.20), we have
\[
(8.2.21) \quad \left| \int_0^1 h_{t,x}(F_d,x) \frac{dt}{t} \right| \leq C e^{-c_1 d^2/2} \dim(E_d) \int_0^1 e^{-c_2 d^2/2} \frac{dt}{t} = O(e^{-c_3 d}),
\]
and
\[
(8.2.22) \quad \left| \int_1^d h_{t,x}(F_d,x) \frac{dt}{t} \right| \leq C e^{-c_4 d} \dim(E_d) \int_1^d e^{-c_5 x} \frac{dx}{x} = O(e^{-c_6 d}).
\]
By (8.2.20) - (8.2.21), we get (8.2.19).

At last, we assemble together (8.2.8), (8.2.9), (8.2.12), (8.2.15), (8.2.16), (8.2.18), and (8.2.19), we get exactly (8.2.1). This completes the proof of our proposition. \( \square \)

In Proposition 8.2.1, each elliptic \( \gamma \in \Gamma \) is of finite order, therefore \( \mathcal{PE}_{X,\gamma}(F_d) \) is a nice pseudopolynomial. Since \( \Gamma(Z,F_d) \) is always real number, then (8.2.1) still holds if we take the real part of \( \mathcal{PE}_{X,\gamma}(F_d) \) instead.

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