The comparative analytical (energy method) and numerical study for the circular plates with ribs

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Abstract. The article analyzes the deformations that are created in a circular plate with radial ribs located on one side, under a concentrated force located at its center, \( F = 77000 \text{N} \). To this purpose the deflection obtained by analytically calculating is determined and compares with the displacement obtained by finite element method. To obtain the vertical displacements of a ribbed plate, the results of a similar plane plate, whose vertical displacements were determined, were used and displacements and tensions parameter value, as determined by the analytical method, were introduced.

1. Introduction
The area of practical use of circular plates with reinforcing ribs is wide because:
- ribbing results in superior mechanical properties for strength, stiffness and stability;
- materials with low mechanical properties (involving low price) are used to achieve superior operating characteristics;
- the mass of the plate is substantially reduced with the conservation or increase of the needed bearing capacity of the structure;
- the heat transfer is intensified, increasing the outer surface, in many practical cases revealing itself as an extremely important feature for the performance of structure and its lifetime.

The research done to determine the strain and stress states of the rib stiffened plates can be grouped in:
- approximate methods of calculating the stress and strain states [1-5];
- methods that reduce the study of the behaviour of its components: the plates and ribs being considered under different forms of support [6];
- calculation methods that reduce orthotropic structure to the material structure;
- numerical methods [7-10];
- experimental methods [11-17].

The article examines the state of stress of a circular plate with radial ribs placed on one side (figure 1.a), under a concentrated force applied at its center, \( F = 77000 \text{N} \) [18]. For this two methods are used: the analytical (energy) method and finite element analysis.

2. Analytical calculation (energy method) for ribbed plate displacement
An accurate method of solving the problem of determining the vertical displacements is complicated due to uneven deformation in the circumferential direction. A slightly simple solution, based on the Ritz’s method, was proposed by Boiarsinov [18]. This solution is based on the following assumptions:
plate deflection is considered axial-symmetric (the inequality of circumference deformations is not considered);
- force is applied to similar plates without ribs. Based on this assumption we can assert that:
\[ w_n = A \cdot w \tag{1} \]
in which: \( w_n \) is central deflection for ribbed plate; \( A \)- parameter of displacements and stress, for calculating median surface of ribbed plate; \( w \)- central deflection for analogous plate.
- the displacement of the neutral layer against the surface is constant across the plate. The value of this displacement, \( e \), and of the parameter \( A \), which characterizes the central deflection of the plate, is determined by Ritz's method, from the minimum energy condition. Admitting the non strain perpendiculars assumption and considering the biaxial tension in plaque and uniaxial in ribs, we can start from the expression of strains and stresses:
\[ \varepsilon_r = -\frac{d^2w}{dr^2}(z-e) \tag{2} \]
\[ \varepsilon_i = -\frac{1}{r} \cdot \frac{dw}{dr}(z-e) \tag{3} \]
\[ \sigma_r = -\frac{E(z-e)}{1-\mu^2} \left( \frac{d^2w}{dr^2} + \mu \cdot \frac{1}{r} \cdot \frac{dw}{dr} \right) \tag{4} \]
\[ \sigma_i = -\frac{E(z-e)}{1-\mu^2} \left( \frac{1}{r} \cdot \frac{dw}{dr} + \mu \cdot \frac{d^2w}{dr^2} \right) \tag{5} \]
in which: \( E \)-the modulus of elasticity (Young’s modulus) of plate’s steel; \( \mu \)- Poisson's ratio for the plate’s material; \( w \)-the plate’s central deflection; \( r \)- the current radius of the plate.

The stress in ribs is:
\[ \sigma = E \cdot \varepsilon_r = -E \cdot \frac{d^2w}{dr^2}(z-e) \tag{6} \]
The total energy of the system is:
\[ U = U_{pl} + U_n + Z_n \tag{7} \]
in which: \( U_{pl} \)- the potential energy of deformation in the plate’s volume; \( U_n \)- the potential energy of deformation in the rib’s volume; \( Z_n \)- the potential energy associated to the applied force for the ribbed plate.

\( U_{pl} \)- has the following form:
\[ U_{pl} = \int_r \int_z \frac{1}{2} \left( \sigma_r \cdot \varepsilon_r + \sigma_i \cdot \varepsilon_i \right) 2 \cdot \pi \cdot r \, dr \, dz \tag{8} \]

\( U_n \) has the form:
\[ U_n = n \int_S \frac{1}{2} \sigma \cdot \varepsilon_r \, dr \, dS \tag{9} \]
where \( n \) is the number of ribs; \( S \)- area of the rib section.
The potential energy associated to the applied force to the ribbed plate is:

\[ Z_n = A \cdot Z \]  

in which: \( Z \) -potential energy associated to the applied force for the plate without ribs.

\[ \text{Figure 1. Geometry: (a) geometry of the plate with ribs; (b) geometrical elements for analogous plate, without ribs.} \]

For the energy to be minimal, \( U \) ‘s derivates in relation to the two parameters, \( A \) and \( e \), have to be zero. Solving the system of equations will find values for following two unknowns:
where: \( e \) - the neutral layer’s displacement against the surface; \( J_0 \) - the moment of inertia for a radial rib’s cross section, in relation to it’s own axis of symmetry; \( z_c \) - the distance from the center of mass of the rib section to the median plane of the plate; \( t \) - the integral; \( h \) - the plate’s thickness.

If the load is static (speed is very small) [19] all the mechanical work, \( L \), produced by applying force \( F \), turns into potential energy of deformation. The relationship between force \( F \) and the central deflection \( w \) is represented in figure 2 by the OBC line (Hooke's law in the elastic range).

![Figure 2. Relation force \( F \) - (central deflection) displacement \( w \).](image)

For a finite variation of force \( F \), the energy is represented by the area of the OCC'. So:

\[
Z = \frac{1}{2} \cdot F \cdot w_{\text{max}}
\]

In the case of the flat plate in figure 1, the energy is: \( Z = 173721 \text{ W} \)

To find out the plane’s deflection \( w \), we start from the differential equation on the analogous plate[18]:

\[
\frac{d^2 \phi}{dr^2} + \frac{1}{r} \cdot \frac{d \phi}{dr} - \frac{1}{r^2} \cdot \phi = -\frac{T}{D}
\]

in which: \( \phi \) - the slope of the deflection surface; \( T \) - the shear force on a cylindrical section with radius \( r \).

This being the sum of normal projections on the median surface, of the forces within the circle of radius \( r \), and the plate being loaded with a force \( F \) applied in center, we can write:

\[
2 \cdot \pi \cdot r \cdot T = F \Rightarrow T = \frac{F}{2 \cdot \pi \cdot r} ;
\]

\( D \) - the flexural rigidity of the plate, having formula:
\[
D = \frac{E \cdot h^3}{12(1 - \mu^2)}
\]  
(15)

The expression for the central deflection for the plate without ribs is obtained [19].
\[
w = \frac{F \cdot R^2}{8 \cdot \pi \cdot D} (\ln r - 1) - C_1 \frac{r^2}{4} - B \cdot \ln r + C
\]
(16)
in which \(r_c = 285\ mm\) representing the radius of the plate in which the embedding is performed (fig.1).

The maximum deflection of the plate without ribs is in center:
\[
w_{\text{max}} = \frac{F \cdot r_c^2}{16 \cdot \pi \cdot D} = 6,556\ mm
\]
(17)

After considering the boundary conditions for determining constants, the deformation equations become:
\[
w = \frac{F \cdot r^2}{8 \cdot \pi \cdot D} \cdot \ln \frac{r}{r_c} + \frac{F}{16 \cdot \pi \cdot D} \left( \frac{r_c^2}{r^2} - 1 \right)
\]
(18)
\[
\varphi = \frac{dw}{dr} = \frac{F \cdot r}{4 \cdot \pi \cdot D} \cdot \ln \frac{r_c}{r}
\]
(19)

For the ribbed plate the parameter \(A\), containing the following unknown, must be determined:
\[
J_0 = \frac{h(2 \cdot h)^3}{12} = \frac{3}{2} \cdot h^4 = 15000\ mm^4; \ z_c = \frac{3}{2} \cdot h = 15\ mm; \ S = 5 \cdot 20 = 100\ mm;
\]

But
\[
\frac{d^2 w}{dr^2} = -\frac{F}{4 \cdot \pi \cdot D} + \frac{F \cdot \ln \frac{r_c}{r}}{4 \cdot \pi \cdot D},
\]
from which it follows that:
\[
t = \int_0^{r_c} \left( \frac{d^2 w}{dr^2} \right)^2 dr = \int_0^{r_c} \left( -\frac{F}{4 \cdot \pi \cdot D} + \frac{F \cdot \ln \frac{r_c}{r}}{4 \cdot \pi \cdot D} \right)^2 dr = 0,0000297
\]
(21)

This way the values of the unknowns are obtained \(c=1,603\) and \(A=0,166\). So deflection of ribbed plate will be: \(w_n=1,08532\ mm\).

3. Calculation of displacement of ribbed plate with finite element method

Finite element method has had great development in the last years, having broad applicability in various fields. The basic concept of the method, when applied to a structural analysis problem, lies in the fact that a body can be modelled analytically by dividing it into regions (finite elements) for which the behaviour is described by functions which represent the displacements and the stresses. These functions are chosen so as to ensure continuity of the body’s behaviour.

To study the circular flat plate and the plate with radial ribs, in this work, the ANSYS program was used. The mesh method was tetrahedrons, algorithm- patch conforming, element midside nodes-
program controlled, element size 4 mm. The thirty-two holes flat plate, provided on the bearing contour, was divided into a total of 127704 elements, with nodes 70892, and the ribbed plate was divided into a total of 149669 elements, to 83109 nodes.

The materials from which the plates are made have the following characteristics:
- modulus of elasticity (Young’s modulus): \( E = E_n = 2.1 \times 10^{11} \text{[Pa]} \);
- Poisson's ratio: \( v = v_n = 0.28 \);
- shear modulus: \( G = 7.9 \times 10^{10} \text{[Pa]} \);
- density: \( \rho = 7800 \text{[kg/m}^3\text{]} \);
- tensile strength (min): \( \sigma_R = 3.99826 \times 10^8 \text{[Pa]} \);
- yield strength: \( \sigma_c = 2.20594 \times 10^8 \text{[N/m}^2\text{]} \);
- thermal expansion coefficients: \( \alpha_T = 1.3 \times 10^{-5} \text{[1/K Kelvin]} \).

![Central displacements of the plain plate, fixed on holes.](image)

**Figure 3.** Central displacements of the plain plate, fixed on holes.

The analogous plate (figure 2.b) was analyzed with the ANSYS program and a maximum central displacement value \( w = 6.626 \text{ mm} \) was obtained.

The ribbed plate in figure 1.a. was analyzed using the same software, the same load conditions, obtaining the result shown in figure 4. The maximum central displacement is \( w_n = 1.1792 \text{ mm} \). In figure 5 the displacements of the ribbed plate, recessed, after a plane passes through a rib is shown. The central deflection for flat plate is given in table 1.

**Table 1.** The central deflection for flat plate.

|          | mm   |
|----------|------|
| \( w_{MA} \) | 6.556 |
| \( W_{MEF} \) | 6.626 |

The central deflection for ribbed plate is given in table 2.
Table 2. The central deflection for ribbed plate.

|                  |        |
|------------------|--------|
| $w_{n_{MA}}$ [mm]| 1,085  |
| $w_{n_{MEF}}$ [mm]| 1,1792 |

Figure 4. Central displacements of the ribbed plate, fixed on holes.

Figure 5. Displacements in the ribbed plate, recessed, after a plane passes through a rib.
4. Results comparison
From the conducted analysis, it is observed that the values obtained by analytical study ($w_{MA} = 6,556$ mm) and finite elements study ($w_{MEF} = 6,626$ mm) of the flat plate, are closed.

Similarly, for the ribbed plate, when loaded in the same conditions, similar results were obtained: $w_{MA} = 1,085$ mm and $w_{MEF} = 1,1792$ mm.

From the values of central deflection obtained, the advantage of the stiffening of the plate that was subject to loads is observed. Thus, stiffening of plate decreases the deflection about 6 times. This way the mass of the plate is still low, but the bearing capacity of the structure increases.

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