FERRAND’S PUSHOUTS FOR ALGEBRAIC SPACES

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Abstract. We extend Ferrand’s results about pushouts of schemes to the category of algebraic spaces. We call the corresponding class of pushouts Ferrand’s pushouts. They will be used in our sequel works to extend the notions of valuation rings and Riemann-Zariski spaces to the category of algebraic spaces, and to obtain a new proof of Nagata’s compactification theorem for algebraic spaces.

1. Introduction

In [Fer], D. Ferrand studied schematic pushouts of the form \( Y \coprod_T Z \), where \( f: T \to Y \) is an affine morphism and \( g: T \to Z \) is a closed immersion. When \( f \) is finite such pushout is called pinching or pinching of \( Z \) with respect to \( f \). Although studying pinchings was, probably, Ferrand’s main motivation, he realized that the “right generality”, which allows one to prove all the fundamental results, is obtained by weakening the finiteness assumption on \( f \) and assuming only \( f \) being affine.

The current paper is devoted to the study of pushouts of this type, that we call Ferrand’s pushouts, in the case when \( Y \), \( Z \), and \( T \) are quasi-compact and quasi-separated algebraic spaces. In particular, we give a sufficient condition for a Ferrand’s pushout to be a scheme, and construct Ferrand’s pushouts of schemes that are algebraic spaces but not schemes.

1.1. Motivation. Let \( g: T \to Z \) be a closed immersion. If \( f: T \to Y \) is also a closed immersion then the pinching \( X = Y \coprod_T Z \) can be viewed as the scheme obtained by gluing \( Y \) and \( Z \) along the closed subscheme \( T \).

A more interesting and less intuitive case is the case when \( f \) is an affine open immersion, or, more generally, a pro-open immersion; e.g., the embedding of the generic point. In this case we call the Ferrand’s pushout composition, and say that \( X \) is obtained by composing \( Y \) and \( Z \) along \( T \). At first glance, gluing an open subscheme of \( Y \) to a closed subscheme of \( Z \) may seem unnatural. For example, such pushout is usually not noetherian even if \( Y \) and \( Z \) are. Nevertheless, compositions appear naturally in the theory of valuations, since any valuation ring of non-zero finite height is composed of valuation rings of height one, and on the geometric side this corresponds to the composition of spectra of valuation rings of height one.

Composition of valuations is a basic tool in the theory of valuations, cf. [Tem1, §2]. Thus, compositions of schemes appear naturally in applications of valuation theory to algebraic geometry, such as the study of relative Riemann-Zariski spaces and the proof of Nagata’s compactification theorem by the first author [Tem2, ....

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Similarly, D. Rydh uses compositions in his work [Ryd, §6] on Nagata’s compactification of certain classes of algebraic stacks.

In the sequel papers [TT1, TT2] we define valuation algebraic spaces and use them to study RZ spaces in the category of algebraic spaces, obtaining as an application a new proof of Nagata’s compactification for algebraic spaces. Composing valuation algebraic spaces plays an important technical role in these papers. Although we will only use compositions in our applications, and the proofs are slightly easier when $f$ is a monomorphism, we decided to study arbitrary Ferrand’s pushouts in the category of algebraic spaces because the main results hold true in this generality, and the additional arguments needed for this case are not very involved.

Curiously enough, the only published proof of Nagata’s compactification for algebraic spaces does not involve compositions but makes a serious use of pinchings, see [CLO, Theorem 2.2.2]. Our results, in particular, imply that theorem easily.

1.2. Main results. Theorem 4.6.1 asserts that Ferrand’s pushout $Y \coprod_T Z$ exists if and only if the pushout datum $(T; Y, Z)$ admits an fppf affine presentation $(T'; Y', Z')$, i.e., fppf affine presentations of $Y$, $Z$, and $T$ with identification $T \times_Y Y' = T' = T \times_Z Z'$. Corollary 4.6.2 provides some criteria for existence of such presentation, but the question whether there always exists an fppf affine presentation remains open.

In Theorems 4.4.12 and 4.5.5 we prove that a Ferrand’s pushout $X = Y \coprod_T Z$ is affine (resp. separated) if and only if the spaces $Y, Z, T$ are affine (resp. separated). In particular, an affine Ferrand’s pushout is, actually, the pushout in the category of all algebraic spaces. Theorem 4.6.18 establishes a relation between Ferrand’s results on pushouts of schemes and our theory. In particular, it asserts that the pushout of an effective pushout datum is a scheme if and only if the pushout datum admits an affine Zariski presentation, and this is exactly the case studied by Ferrand in [Fer].

The following properties of Ferrand’s pushout $X = Y \coprod_T Z$ are established in Corollary 4.6.4: (i) the pushout is compatible with topological realizations, i.e., $|X| = |Y| \coprod_ |T| |Z|$, (ii) $T = Y \times_X Z$, (iii) set-theoretically, $X$ is the disjoint union of its closed subscheme $Y$ and open subscheme $Z \setminus T$.

We extend Ferrand’s result on Zariski topologies to étale and flat sites. Roughly speaking, Theorem 4.6.6 asserts that the flat (resp. étale) site of $X$ is equivalent to the flat (resp. étale) site of the pushout datum $(T; Y, Z)$. Furthermore, we show that the pullback and the pushout functors form a pair of essentially inverse equivalences. In particular, this implies various compatibilities of Ferrand’s pushouts with flat morphisms. To be more precise, since we do not know whether any Ferrand’s pushout datum admits an fppf (resp. étale) affine presentation, and any effective pushout datum clearly does, we work with the full subsites of objects that admit such a presentation in the sites on $(T; Y, Z)$. Finally, in Theorem 4.6.14 we show that certain properties of morphisms descent with respect to Ferrand’s pushouts.

1.3. The overview. The paper is written using the language of pushout data and their morphisms (see §3.1). The general plan is as follows: First, we establish the affine case either by referencing to Ferrand’s results or by a straightforward computation. This is done in §§4.2-4.3. Then we study affine presentations of general Ferrand’s pushout data in §4.4, and prove that affine Ferrand’s pushouts
are pushouts in the category of all algebraic spaces. Finally, we use the affine case, the existence of affine presentations $X_0 \to X$, and flat (resp. étale) descent to construct general pushouts. Since $X_1 = X_0 \times_X X_0$ does not have to be affine, we have first to work out the entire theory in the separated case, see §4.5. In particular, this includes the results about existence of pushouts and gluing flat sites. The general case is established in §4.6 by use of the same arguments. The only difference is that $X_1$ can be separated but not affine.

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2. Preliminaries

2.1. Algebraic spaces. A basic reference to algebraic spaces is [Knun] and [Stacks]. Throughout this paper we denote the underlying topological space of an algebraic space $X$ by $|X|$.

2.1.1. Qcqs algebraic spaces. All interesting results will be proved for quasi-compact and quasi-separated (qcqs) algebraic spaces, so the reader can always assume that all geometric objects are qcqs.

2.1.2. Presentations. If not said to the contrary, presentation of an algebraic space $X$ means an étale presentation, i.e., a surjective étale morphism $X_0 \to X$ whose source is a scheme. If $X_0$ is affine then we say that the presentation is affine. Giving a presentation is equivalent to giving an étale equivalence relation $X_1 \to X_0$ with an isomorphism $X_0 / X_1 \to X$. If $X$ is qcqs then we consider only quasi-compact presentations of $X$, so the word quasi-compact will usually be omitted.
2.1.3. **Diagonal.** For the sake of reference we record the following well known result.

**Lemma 2.1.4.** If \( Z \) is a qcqs algebraic space then the diagonal \( \Delta : Z \to Z \times Z \) is quasi-finite.

**Proof.** Choose an affine presentation \( Z = U/R \). Then the base change \( \Delta \times Z (U \times U) \) is the diagonal \( R \to U \times U \) of the equivalence relation. The latter is quasi-finite by [Knu, Proposition I.5.12], hence \( \Delta \) is quasi-finite by étale descent. □

2.1.5. **Zariski points.** An algebraic space \( \eta \) is a point if any monomorphism to \( \eta \) is an isomorphism. It is well known that any point is the spectrum of a field. A point of an algebraic space \( X \) is a morphism \( f : \eta \to X \) from a point. If \( f \) is a monomorphism then we say that \( f \) is a Zariski point. There is a natural bijection between points \( x \in X \) and isomorphism classes of Zariski points \( \text{Spec}(K) \to X \), and one says that \( k(x) := K \) is the residue field of \( x \). We will often use the following two facts [Knu, Proposition II.6.2 and Theorem II.6.4]: (1) any point \( f : \eta \to X \) factors uniquely through a Zariski point \( \eta_0 \to X \), and (2) any point of an algebraic space \( X \) factors through some étale presentation \( X' \to X \).

2.1.6. **A criterion for being a monomorphism.**

**Lemma 2.1.7.** Assume that \( h : Y \to X \) is a morphism of finite type and \( g : X' \to X \) is any surjective morphism. Then \( h \) is a monomorphism if and only if \( h \times_X X' \) is a monomorphism.

**Proof.** By descent, it suffices to prove the lemma in the case of schemes. For any point \( x' \in X' \) the base change \( h \times_X \text{Spec}(k(x')) \) is a monomorphism. It then follows by fpqc descent that the same is true for \( h \times_X \text{Spec}(k(x)) \) for \( x = g(x') \). Since \( g \) is surjective, the restriction of \( h \) over any point of \( X \) is a monomorphism, hence \( h \) is a monomorphism by [EGA, IV 4, 17.2.6]. □

3. **General pushouts**

3.1. **Definitions.** Let \( \mathcal{C} \) be one of the following categories (either absolute or relative): category of affine schemes, category of schemes, or category of algebraic spaces. By a pushout datum in \( \mathcal{C} \) we mean a diagram

\[
\begin{array}{ccc}
T & \xrightarrow{f} & Y \\
\downarrow{g} & & \downarrow \\
Z
\end{array}
\]

where \( T, Y \) and \( Z \) are objects of \( \mathcal{C} \), and \( f, g \) are separated morphisms. Our default category \( \mathcal{C} \) will be the category of algebraic spaces.

We denote pushout data either by triples \( (T; Y, Z) \), where the morphisms are omitted in the notation, or by capital calligraphic letters, e.g., \( \mathcal{P} = (T; Y, Z) \). The colimit of the diagram \( \mathcal{P} \), if exists, is called the pushout of \( Y \) and \( Z \) with respect to \( T \), and is denoted by \( Y \amalg_T^\mathcal{C} Z \) or \( \amalg^\mathcal{C} \mathcal{P} \). If \( \mathcal{C} \) is our default category of algebraic spaces we will omit \( \mathcal{C} \) in the notation \( \amalg^\mathcal{C} \mathcal{P} \).

**Remark 3.1.1.** It often happens that two geometric categories \( \mathcal{C} \subset \mathcal{C}' \) possess different pushouts \( X = Y \amalg_T^\mathcal{C} Z \) and \( X' = Y \amalg_T^\mathcal{C}' Z \), i.e., the natural \( \mathcal{C}' \)-morphism \( X' \to X \) is not an isomorphism. In such case, one can view \( X' \) as a finer or a
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more informative pushout. A classical example would be a quotient stack $X'$ and its coarse moduli space $X$. Another example would be the pushout of Spec$(k[T])$ and Spec$(k[T^{-1}])$ along Spec$(k[T,T^{-1}])$. While in the category of $k$-schemes the pushout is the projective line, in the category of $k$-affine schemes it is just a point. We will see that Ferrand’s pushouts are more stable, and, in particular, an affine Ferrand’s pushout is the pushout in the category of all algebraic spaces.

A premorphism of pushout data $(T';Y',Z') \to (T;Y,Z)$ is a commutative diagram

If in addition, both squares are cartesian then the premorphism is called a morphism. The later condition is rather restrictive. For instance, if $P \times P$ is the product defined componentwise, then the natural projections $P \times P \rightrightarrows P$ are usually not morphisms. Furthermore, the category of pushout data admits no products since it admits no final object. However, one can easily check that it does admit fibered products, which are nothing but componentwise fibered products.

Plainly, $U \to (U;U,U)$ defines a fully faithful embedding of the category of algebraic spaces into the category of pushout data. Hence we may identify the algebraic spaces with the corresponding pushout data. Furthermore, any premorphism between two algebraic spaces is a morphism. Note however, that most of the premorphisms between a general pushout datum and an algebraic space are not morphisms.

Unless explicitly said to the contrary, we say that a premorphism of pushout data (resp. a pushout datum) possesses certain property if all its components do. In particular, if $\mathcal{C}$ is the category of algebraic spaces then we define affine pushout data, and étale or affine morphisms via this rule.

An affine (resp. separated) presentation of a pushout datum $\mathcal{P}$ is a surjective étale morphism to $\mathcal{P}$ from an affine (resp. separated) pushout datum. We will also consider flat affine (resp. separated) presentations, in which the covering morphisms are only assumed to be flat rather than étale.

To a separated flat presentation $\mathcal{P}_0 \to \mathcal{P}$, we associate the flat equivalence relation $\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$, where $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$. Observe that $\mathcal{P}_1$ is separated. Furthermore, if $\mathcal{P}$ is separated and the presentation is affine then $\mathcal{P}_1$ is affine.

**Remark 3.1.2.** Since the category of pushout data admits no products one should be careful while defining equivalence relation. So, by equivalence relation we mean two morphisms $\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$ defining equivalence relations on the $T$, $Y$, and $Z$ components.

**Lemma 3.1.3.** Let $\mathcal{P}_1 \rightrightarrows \mathcal{P}_0$ be an fppf equivalence relation. Set $Y := Y_0/Y_1$, $Z := Z_0/Z_1$, $T := T_0/T_1$, and $\mathcal{P} := \mathcal{P}_0/\mathcal{P}_1 = (T;Y,Z)$. Then $Y$, $Z$, and $T$ are algebraic spaces, the natural premorphism $\mathcal{P}_0 \to \mathcal{P}$ is a morphism, and $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$. 
Proof. By [LMB, Corollary 10.4] $Y$, $Z$, and $T$ are algebraic spaces. Thus, the last two assertions follow from [Knu, Proposition I.5.8].

3.2. Quasi-coherent modules, and pullback and pushforward functors. Let $\mathcal{P} = (T; Y, Z)$ be a pushout datum. Set $\mathcal{O}_{\mathcal{P}} := (\mathcal{O}_T; \mathcal{O}_Y, \mathcal{O}_Z)$. By an $\mathcal{O}_{\mathcal{P}}$-premodule $\mathcal{M}$ we mean a triple of modules $(\mathcal{M}_T; \mathcal{M}_Y, \mathcal{M}_Z)$ with homomorphisms $\alpha_\mathcal{M}: f^*(\mathcal{M}_Y) \to \mathcal{M}_T$ and $\beta_\mathcal{M}: g^*(\mathcal{M}_Z) \to \mathcal{M}_T$. If both homomorphisms are isomorphisms we say that $\mathcal{M}$ is an $\mathcal{O}_{\mathcal{P}}$-module. Note that if $\mathcal{P}$ is an algebraic space $X$, i.e., $\mathcal{P} = (X; X, X)$, then an $\mathcal{O}_{\mathcal{P}}$-module is just an $\mathcal{O}_X$-module. An $\mathcal{O}_{\mathcal{P}}$-module (resp. $\mathcal{O}_{\mathcal{P}}$-premodule) is quasi-coherent, flat, etc., if all its components are.

A morphism of (pre-)modules is a triple of morphism compatible with the homomorphisms $\alpha_\bullet$ and $\beta_\bullet$. If $\mathcal{P} = X$ is an algebraic space then we introduce a functor $\xi: (\mathcal{M}_T; \mathcal{M}_Y, \mathcal{M}_Z) \mapsto \mathcal{M}_T \times_{\mathcal{M}_T} \mathcal{M}_Z$ from the category of $\mathcal{O}_{\mathcal{P}}$-premodules to the category of $\mathcal{O}_{\mathcal{P}}$-modules. It is the right adjoint of the forgetful functor from the category of $\mathcal{O}_{\mathcal{P}}$-modules to the category of $\mathcal{O}_{\mathcal{P}}$-premodules.

Let $\phi: \mathcal{P} \to \mathcal{P}'$ be a premorphism of pushout data. Then we have a natural pullback functor $\phi^*$ defined componentwise from the category of $\mathcal{O}_{\mathcal{P}}$-modules (resp. $\mathcal{O}_{\mathcal{P}}$-premodules) to the category of $\mathcal{O}_{\mathcal{P}}$-modules (resp. $\mathcal{O}_{\mathcal{P}}$-premodules). In general, the componentwise pullback defines only a functor $\phi_*$ from the category $\mathcal{O}_{\mathcal{P}}$-premodules to the category of $\mathcal{O}_{\mathcal{P}}$-premodules, which is right adjoint to $\phi^*$. However, if $\phi$ is a flat (resp. affine) morphism then $\phi_*$ takes $\mathcal{O}_{\mathcal{P}}$-modules to $\mathcal{O}_{\mathcal{P}}$-modules by flat (resp. affine) base change, and, as usual, the pushforward functor between the categories of modules is denoted by $\phi_*$.

If $\mathcal{P}' = X$ is an algebraic space then there exists a different pushforward functor from the category of $\mathcal{O}_{\mathcal{P}}$-modules to the category of $\mathcal{O}_X$-modules defined as the composition of the pushforward of premodules followed by $\xi$. This functor is also denoted by $\phi_*$. The usual adjunction property holds:

Lemma 3.2.1. Let $\phi: \mathcal{P}' \to \mathcal{P}$ be a morphism between pushout data and assume that either $\phi$ is flat, or $\phi$ is affine, or $\mathcal{P}$ is an algebraic space. Then the functor $\phi_*$ is right adjoint to $\phi^*$.

Proof. In the flat and affine cases the lemma follows from the standard adjunction for a single morphism between algebraic spaces. If $\mathcal{P}$ is an algebraic space and $\phi$ is arbitrary then the lemma follows from the standard adjunction and the fact that $\xi$ is the right adjoint to the forgetful functor.

Remark 3.2.2. A similar theory applies to the categories of quasi-coherent algebras. In fact, the definitions of $\phi^*$, $\phi_*$, and the lemma about their adjunction, apply to quasi-coherent algebras verbatim, so we omit the details.

3.3. $S$-affine pushouts. It is well known that pushouts exist in the category of affine schemes. If $S$ is an algebraic space, then the existence result can easily be extended to the category of $S$-affine algebraic spaces. We will only need affine pushouts to construct general Ferrand’s pushouts, but $S$-affine pushouts will be used in [TT2]. For this reason, we discuss general $S$-affine pushouts, and view the affine ones as a particular case in which $S = \text{Spec}(\mathbb{Z})$.

By an $S$-affine pushout we mean the pushout in the category of $S$-affine algebraic spaces. Such pushouts are denoted by $Y \bigsqcup^S T Z$ or $\bigsqcup^S T \mathcal{P}$. Since the category of
S-affine spaces is anti-equivalent to the category of quasi-coherent \( \mathcal{O}_S \)-algebras, \( S \)-affine pushouts always exist and correspond to the fibered products in the category of \( \mathcal{O}_S \)-algebras:

**Lemma 3.3.1.** Let \( \mathcal{P} = (T; Y, Z) \) be an \( S \)-affine pushout datum, where \( Y = \text{Spec}(B) \), \( Z = \text{Spec}(C) \), and \( T = \text{Spec}(K) \) for quasi-coherent \( \mathcal{O}_S \)-algebras \( B \), \( C \), and \( K \). Set \( A := B \times_K C \). Then \( X = \text{Spec}(A) \) is the \( S \)-affine pushout of \( \mathcal{P} \).

**3.3.2. Uniformness of \( S \)-affine pushouts.** The construction of \( S \)-affine pushouts is compatible with flat morphisms.

**Lemma 3.3.3.** Let \( S, \mathcal{P}, \) and \( X = \coprod_{S_{\text{aff}}} \mathcal{P} \) be as in Lemma 3.3.1. Then,

(i) If \( X' \to X \) is a flat affine morphism and \( \mathcal{P}' = \mathcal{P} \times_X X' \) then the natural morphism \( \coprod_{S_{\text{aff}}} \mathcal{P}' \to X' \) is an isomorphism.

(ii) If \( S' \to S \) is a flat morphism then the natural morphism \( \coprod_{S_{\text{aff}}} (\mathcal{P} \times_S S') \to (\coprod_{S_{\text{aff}}} \mathcal{P}) \times_S S' \) is an isomorphism.

**Proof.** Let \( Y, Z, T, A, B, C, \) and \( K \) be as in Lemma 3.3.1. Also, let \( \mathcal{P}' = (T'; Y', Z') \) and let \( A', B', C', \) and \( K' \) be the \( \mathcal{O}_S \)-algebras whose spectra are \( X', Y', Z', \) and \( T' \), respectively. The sequence \( 0 \to A \to B \oplus C \to K \to 0 \) is exact by Lemma 3.3.1. Then so is \( 0 \to A' \to B' \oplus C' \to K' \to 0 \), since \( X' \) is flat over \( X \) and the second sequence is obtained from the first by applying \( \otimes_A A' \). Thus, \( A' = B' \times_K C' \), and hence \( X' = \coprod_{S_{\text{aff}}} \mathcal{P}' \) by Lemma 3.3.1. This proves (i), and the proof of (ii) is similar, so we omit it.

**Remark 3.3.4.** As Remark 3.1.1 shows, general affine pushouts are often not very informative. In fact, if an affine pushout datum admits a pushout in the category of schemes (or algebraic spaces) then the affine pushout is its affine hull.

4. Ferrand’s pushouts

The aim of this paper is to study pushouts of algebraic spaces \( Y \coprod_T Z \) in the special case when \( T \to Y \) is affine, \( T \to Z \) is a closed immersion, and \( Y, Z, \) and \( T \) are qcqs. The scheme case was studied extensively by D. Ferrand [Fer], so we call such pushouts Ferrand’s pushouts.

**4.1. Terminology.** A pushout datum \( \mathcal{P} = (T; Y, Z) \) is called **Ferrand’s** if \( T \to Y \) is affine and \( T \to Z \) is a closed immersion. If the pushout \( X = \coprod \mathcal{P} \) exists and the morphisms \( Y \to X \) and \( Z \to X \) are affine then we say that \( X \) is **Ferrand’s pushout** and the Ferrand’s pushout datum \( \mathcal{P} \) is **effective**.

If all components of a Ferrand’s pushout datum \( \mathcal{P} \) are affine over an algebraic space \( S \) then we say that \( \mathcal{P} \) is an **\( S \)-affine Ferrand’s pushout datum** and \( X = \coprod_{S_{\text{aff}}} \mathcal{P} \) is an **\( S \)-affine Ferrand’s pushout**. We will prove in Corollary 4.6.10 that \( X = \coprod \mathcal{P} \), so it is, in fact, Ferrand’s pushout of \( \mathcal{P} \), but we have to distinguish the two notions until then.

**4.2. Ferrand’s diagrams of rings.**
4.2.1. The definition. Let $B \to K$ and $C \to K$ be homomorphisms of rings and assume that $C \to K$ is surjective. We set $A := B \times_K C$, and say that the left cartesian square below is a Ferrand’s diagram of rings. We draw the corresponding diagram of spectra on the right; as we will see later, $X = Y \coprod_T Z$ in the category of algebraic spaces, so this is the diagram of Ferrand’s pushout.

$$
\begin{array}{ccc}
\Spec(\phi) & \ & \Spec(\phi) \\
\downarrow & & \downarrow \\
Y & \ & X
\end{array}
\quad
\begin{array}{ccc}
K & \xleftarrow{f} & C \\
\downarrow & & \downarrow \\
B & \xleftarrow{g} & A
\end{array}
\quad
\begin{array}{ccc}
T \ar[r] & Z \\
\downarrow & & \downarrow \\
X & \ar[l] & Z
\end{array}
(1)

4.2.2. Conductor. The ideal $I = \ker(A \to B)$ is called the conductor of Ferrand’s diagram. Note that $\phi$ maps $I$ isomorphically onto $\ker(C \to K)$, so by abuse of language, the latter will also be denoted by $I$. Obviously, the diagram is determined by $\phi$ and $I \subseteq A$. Moreover, any homomorphism $\phi : A \to C$ and a conductor ideal $I \subset A$ such that $I \to \phi(I)$ is an ideal of $C$ give rise to Ferrand’s diagram with $B = A/I$ and $K = C/I$.

4.2.3. First properties. We recall few basic properties of Ferrand’s diagrams. By a bicartesian square we mean a square diagram which is both cartesian and cocartesian.

**Lemma 4.2.4.** Let assumptions and notation be as in diagram (1). Then,

(i) $B \otimes_A C = K$, so Ferrand’s diagrams are bicartesian.

(ii) For any flat homomorphism $A \to A'$, the diagram of rings obtained by applying $\cdot \otimes_A A'$ to the left square is Ferrand’s. So, Ferrand’s diagrams are preserved by flat base changes.

(iii) $\Spec(\phi)$ establishes an isomorphism of open subschemes $Z \setminus T = X \setminus Y$.

**Proof.** Let $I$ be the conductor. Then $B \otimes_A C = C/IC = C/I = K$. The second claim is proved precisely as Lemma 3.3.3. To see (iii), note that for any $f \in A$, we have $B_f \times_K C_f = A_f$ by (ii). In particular, if $f \in I$ then $B_f = K_f = 0$, $A_f = C_f$, and so $Z_f = X_f$. Hence $Z \setminus T = \bigcup_{f \in I} Z_f = \bigcup_{f \in I} X_f = X \setminus Y$. $\square$

4.2.5. Liftings of semivaluations. Assume that an integral scheme $X'$ is covered by an open subscheme $Z'$ and a closed subscheme $Y'$, and $T' = Y' \cap Z'$. Usually, there exists a valuation ring $R$ of height one and a morphism $\Spec(R) \to X$ that takes the generic point to $Z' \setminus T'$ and the closed point to $Y' \setminus T'$. Informally speaking, in this case there exists a direct specialization from $Z' \setminus T'$ to $Y' \setminus T'$ that skips $T'$. However, this is not the case for Ferrand’s diagrams, e.g., if $X'$ is composed from $Z'$ and $Y'$ along $T'$.

**Lemma 4.2.6.** Let assumptions and notation be as in diagram (1). Assume that $R$ is a valuation ring of non-zero height, i.e., $R$ is not a field, $S := \Spec(R)$, $s \in S$ the closed point, and $f : S \to X$ a morphism such that $f^{-1}(Y) = \{s\}$. Then $f$ admits a unique lifting to a morphism $g : S \to Z$, and the latter morphism satisfies $g^{-1}(T) = \{s\}$.

**Proof.** It suffices to construct any lifting $g : S \to Z$. Indeed, uniqueness is guaranteed by the separatedness of $Z \to X$, and $g^{-1}(T) = f^{-1}(T) = \{s\}$ because $Z \setminus T = X \setminus Y$. Set $U := Z \setminus T$.

Let $\eta \in X$ be the image of the generic point of $S$. Then $k(\eta) \subseteq \Frac(R)$ and $R' = R \cap k(\eta)$ is a valuation ring. Since $f$ factors through a morphism $f' : \Spec(R') \to X$, this
it suffices to lift \( f' \) to \( Z \). Thus, after replacing \( R \) with \( R' \) we may assume that \( R \) is a valuation ring of \( k(\eta) \). By our assumptions \( \eta \in U \), hence the homomorphism \( A \to k(\eta) \) factors through the homomorphism \( C \to k(\eta) \), and we should only prove that the image \( C(\eta) \subset k(\eta) \) of \( C \) lies in \( R \).

Assume to the contrary that an element \( x \in C \) is mapped to \( \overline{x} = x(\eta) \in k(\eta) \setminus R \). Then \( \overline{x}^{-1} \) lies in the maximal ideal \( m_R \). Let \( I \subset A \) be the conductor of the diagram. Since \( f^{-1}(Y) = \{ s \} \), the radical of \( IR \) coincides with \( m_R \). In particular, there exists \( h \in I \) such that \( \overline{h} \) does not divide \( \overline{h} = h(\eta) \). But then \( \overline{h} \overline{x}^n \notin R \), and hence the element \( hx^n \in IC \) is not mapped to \( R \), which is a contradiction. \( \square \)

The following obvious corollary will be used to establish separatedness of Ferrand’s pushouts. Recall that a ring \( A \) is Prüfer if and only if all its localizations are valuation rings.

**Corollary 4.2.7.** Let assumptions and notation be as in diagram (1). Assume that \( R \) is a Prüfer ring such that \( S := \text{Spec}(R) \) has no isolated points, and \( f : S \to X \) is a morphism such that the preimage of \( Y \) is contained in the set of closed points of \( S \). Then \( f \) admits a unique lifting to a morphism \( g : S \to Z \).

### 4.3. \( S \)-affine Ferrand’s pushouts.

#### 4.3.1. First properties.

General \( S \)-affine pushouts \( \coprod_T^{Saff} P \) may contain almost no information about the components of \( P \), cf. Remark 3.3.4. However, Ferrand’s \( S \)-affine pushouts behave much better:

**Proposition 4.3.2.** Assume that \( X = Y \coprod_T^{Saff} Z \) is a Ferrand’s \( S \)-affine pushout. Then,

(i) The topological pushout \( |Y| \coprod_T |Z| \) is naturally homeomorphic to \( |X| \), 

(ii) \( T = Y \times_X Z \), 

(iii) \( Y \to X \) is a closed immersion, \( U = Z \setminus T \to X \) is an open immersion, and \( |X| = |Y| \coprod |U| \) set-theoretically.

**Proof.** First, assume that \( S \), and hence \( X,T,Y,Z \) are affine, as in §4.2.1. Then (i) follows from [Fer, Theorem 5.1] and [Fer, Scholie 4.3], and (ii) and (iii) follow from Lemma 4.2.4.

Let now \( S \) be arbitrary. Pick an affine presentation \( S_0 \to S \). If \( S \) is separated then \( S_1 := S_0 \times_S S_0 \) is affine, and \( X_1 := X \times_S S_1 \) are the affine Ferrand’s pushouts of \( P_0 = P \times_S S_0 \) by Lemma 3.3.3. Thus, \( X_1 \) satisfy (i)-(iii), and hence so is \( X \) by descent; e.g., in (i) one uses descent of topological realizations, i.e., the homeomorphism \( \text{Coeq}(|X_1| \Rightarrow |X_0|) \cong |X| \). In general, although \( S_1 \) need not be affine it is necessarily separated. So, the \( S_1 \)-affine pushout \( X_1 = Y_1 \coprod_T^{Saff} Z_1 \) satisfies (i)-(iii) by the separated case, and the same descent argument applies. \( \square \)

#### 4.3.3. Descent of properties through \( \phi_* \).

Assume that \( X = Y \coprod_T^{Saff} Z \) is a Ferrand’s \( S \)-affine pushout, and let \( \phi : P \to X \) be the natural premorphism. We will now study the functors \( \phi^* \) and \( \phi_* \). While the pullback preserves all natural properties of modules, the situation with the pushforward is more delicate. When working on \([\text{Tem2}]\) the first author learned from D. Rydh the following example, in which finite presentation is not respected by \( \phi_* \).

**Example 4.3.4.** Let \( k \) be a field. Consider discrete valuation rings \( B = k[x]_{(x)} \) and \( C = k[y]_{(y)} \), where \( K = k(x) \). In particular, \( C/yC = K = \text{Frac}(B) \). Then
their composition $A$ is the preimage of $B$ in $C$, which is a valuation ring of height two. In fact, $A$ is the valuation ring of $k(x, y)$ such that $|y| \ll |x| \ll 1$, where the valuation is written multiplicatively. Note that $C = A_x$ and $B = A/I$, where $I = yC$ is the conductor of the corresponding Ferrand’s diagram.

Set $I_n := x^{-n}yA$, and note that $I = \cup I_n$ is not finitely generated as an ideal of $A$ because $I_n \not\subset I_{n+1}$ for any $n$. In particular, $A' = A/I = B$ is finitely generated but not finitely presented over $A$, while $A_n' = A/I_n$ are finitely presented. Pick a surjective algebra $I_n$ and use that a flat finitely presented module is nothing but a projective module of Proposition 4.3.5. Let $f = \phi_n$ be as above, and let $\phi_n : F_n \to C_n$ be a surjective $A/I_n$-homomorphism, and let us prove that $\phi_n$ is not injective.

Finally, assume that $B'$ and $C'$ are flat and finitely presented. Then the $A$-module $B'$ is flat by (i) and finitely generated by (ii), and we should prove that $B'/A'$ is not injective. In (ii) we prove $B'/A'$ is finitely generated. Since $\phi$ is right exact, this implies that $\phi'(A'/A'') = 0$, and then $A'' = A'$ by [Fer, Theorem 2.2(ii)].

Finally, assume that $B'$ and $C'$ are flat and finitely presented. Then the $A$-algebra $A'$ is flat by (i) and finitely generated by (ii), and we should prove that it is finitely presented. Pick a surjective $A$-homomorphism $f_A : A'' = A[T] \to A'$, where $T = (T_1, \ldots, T_n)$, and let us prove that $I_A = \text{Ker}(f_A)$ is finitely generated. By (ii) for modules, it suffices to show that $I_B = I_A \otimes_A B$ and $I_C = I_A \otimes_A C$ are finitely generated. Since $A'$ is $A$-flat, $I_A$ is flat too. Therefore, $I_B = \text{Ker}(f_B)$ and $I_C = \text{Ker}(f_C)$, where $f_B : B'' \to B'$ and $f_C : C'' \to C'$ are the base changes of $f_A$. Thus, $I_B$ and $I_C$ are finitely generated modules since $B'$ and $C'$ are finitely presented.

4.3.6. The adjunctions. Next we study the adjunction transformations $\text{Id} \to \phi^* \phi_*$ and $\phi^* \phi_* \to \text{Id}$.

Proposition 4.3.7. Let $\phi : \mathcal{P} \to X$ be as above. Then,

(i) The adjunction $\phi^* \phi_* (\mathcal{M}) \to \mathcal{M}$ is an isomorphism for any quasi-coherent $\mathcal{O}_\mathcal{P}$-module (resp. $\mathcal{O}_\mathcal{P}$-algebra) $\mathcal{M}$.

(ii) If $\mathcal{F}$ is a flat quasi-coherent $\mathcal{O}_X$-module (resp. $\mathcal{O}_X$-algebra) then the adjunction $\mathcal{F} \to \phi_* \phi^*(\mathcal{F})$ is an isomorphism.
Proof. Similarly to the proof of Proposition 4.3.2, one uses descent on \( S \) to reduce the assertion to the case when \( S \) is affine. So, we may assume that \( X, T, Y, Z \) are affine, as in \( \S 4.2.1 \). Obviously, it suffices to prove the proposition for modules, and in this case (i) is proved in [Fer, Theorem 2.2(i)]. As for (ii), one should check that any flat \( A \)-module \( M \) satisfies \( M = (M \otimes_A B) \times_{M \otimes_A K} (M \otimes_A C) \), and this is done by tensoring the exact sequence \( 0 \to A \to B \oplus C \to K \to 0 \) with \( M \). \qed

Propositions 4.3.5 and 4.3.7 immediately imply the following result.

**Corollary 4.3.8.** The functors \( \phi_* \) and \( \phi^* \) establish essentially inverse equivalences between the categories of flat modules (resp. algebras) over \( \mathcal{O}_X \) and \( \mathcal{O}_P \). Moreover, these functors establish equivalences of the full subcategories consisting of flat and finitely generated (resp. flat and finitely presented modules).

We would also like to correct a mistake in [Fer, Theorem 2.2(iv)].

**Remark 4.3.9.** The assertion of Corollary 4.3.8 for modules is contained in [Fer, Theorem 2.2(iv)]. In addition, that theorem states that \( \phi_* \) induces an equivalence between the categories of finitely generated modules. The proof contains a gap – essential surjectivity is not checked. In fact, in the situation described in Example 4.3.4, the finitely presented \( \mathcal{O}_* \)-module \( \mathcal{O}_* \) is not contained in the essential image of \( \phi_* \). Indeed, if \( A_n \hookrightarrow A_n(M) \) for a \( P \)-module \( M \) then \( \phi^*(A_n) \hookrightarrow \phi^*(M) = M \) by Proposition 4.3.7(i), and so \( \phi_* \phi^*(A_n) \hookrightarrow \phi_* M = \mathcal{A} \). But we saw in Example 4.3.4 that the cyclic modules \( \phi_* \phi^*(A_n) = A' \) and \( A_n \) are not isomorphic.

4.3.10. **Gluing \( X \)-objects from \( \mathcal{P} \)-objects.** Let \( X_{\mathit{Zar}} \) and \( \mathcal{P}_{\mathit{Zar}} \) denote the Zariski sites of \( X \) and \( \mathcal{P} \), respectively. Then the restriction functor \( \phi_{\mathit{Zar}}^{-1}: X_{\mathit{Zar}} \to \mathcal{P}_{\mathit{Zar}} \) is an equivalence by Proposition 4.3.2(i). We shall extend this to the flat and étale topologies, but at this stage we have to restrict the discussion to \( S \)-affine spaces.

**Definition 4.3.11.** Let \( \mathcal{P} \) be a pushout datum. By \( \mathfrak{fl}/\mathcal{P} \) (resp. \( \mathfrak{\acute{e}t}/\mathcal{P} \)) we denote the category of qcqs flat (resp. étale) pushout data over \( \mathcal{P} \). Note that we allow non-flat morphisms between them. If \( S \) is an algebraic space and \( \mathcal{P} \to S \) is a premorphism then \( \mathfrak{fl}/\mathcal{P} \) (resp. \( \mathfrak{\acute{e}t}/\mathcal{P} \)) denotes the full subcategory of \( S \)-affine objects of \( \mathfrak{fl}/\mathcal{P} \) (resp. \( \mathfrak{\acute{e}t}/\mathcal{P} \)). If \( S = \text{Spec}(\mathbb{Z}) \) then we omit \( S \) in the notation, and denote the latter category by \( \mathfrak{fl}/\mathfrak{\acute{e}t} \). Finally, if \( \mathcal{P} = X \) is an algebraic space then we use the same notation with \( \mathcal{P} \) being replaced by \( X \).

**Lemma 4.3.12.** Assume that \( X = \coprod_{\mathfrak{Saff}}^\mathfrak{Saff} \mathcal{P} \) is a Ferrand’s \( S \)-affine pushout and let \( \phi: \mathcal{P} \to X \) be the natural premorphism. Then,

(i) \( \phi \) induces an equivalence of categories \( \phi_{\mathfrak{Saff}}^{-1}: \mathfrak{Saff}/X \to \mathfrak{Saff}/\mathcal{P} \) whose essential inverse is the \( S \)-affine pushout functor \( \coprod_{\mathfrak{Saff}}^\mathfrak{Saff} \). If no confusion is possible, we will use shorter notation \( \phi^{-1} \) for \( \phi_{\mathfrak{Saff}}^{-1} \).

(ii) Let (i) be one of the following properties: surjective, open immersion, étale, flat, and finitely presented. Then a morphism \( h: X' \to X' \) in \( \mathfrak{Saff}/X \) satisfies (i) if and only if so does \( \phi^{-1}(h) \). In particular, both equivalences \( \phi^{-1} \) and \( \coprod_{\mathfrak{Saff}}^\mathfrak{Saff} \) preserve these properties, and hence respect the fpqc, fppf, and étale topologies on the matched categories.

**Proof.** The category \( \mathfrak{Saff}/X \) is anti-equivalent to the category of flat quasi-coherent \( \mathcal{O}_S \)-algebras, and similarly for \( \mathfrak{Saff}/\mathcal{P} \). So, \( \phi_{\mathfrak{Saff}}^{-1} \) is an equivalence by Corollary 4.3.8. On the level of \( \mathcal{O}_S \)-algebras an essential inverse is given by \( \phi_* \), i.e.,
it is expressed as the fibered product of the pushforwards. By Lemma 3.3.1, this corresponds to the $S$-affine pushout.

Let us prove (ii). Denote $\mathcal{P} = (T; Y, Z)$. Since (i) is preserved under base changes, we shall only prove that it descends from $\phi^{-1}(h)$ to $h$. The claim about surjectivity follows from the fact that the morphism from the disjoint union $Y \amalg Z$ to $X$ is surjective.

Set $\mathcal{P}' := \phi^{-1}(X') = \mathcal{P} \times_X X'$ and $\mathcal{P}'' := \phi^{-1}(X'')$. Then $X'' = \coprod \mathcal{P}''$ by Lemma 3.3.3, and the morphism $g'' : T'' \to Z''$ is a base change of $g : T \to Z$, hence a closed immersion. Thus, $X''$ is Ferrand’s $S$-affine pushout, and Proposition 4.3.5 applies to $\mathcal{P}'' \to X''$. Hence if $\phi^{-1}(h)$ is flat (resp. flat and finitely presented) then so is $h$. Finally, if $\phi^{-1}(h)$ is étale (resp. an open immersion) then its fibers are étale (resp. either empty or isomorphisms). But $Y'' \amalg Z'' \to X''$ is surjective, and hence the fibers of $h$ satisfy the same condition too. Thus, $h$ is étale (resp. an open immersion), and we are done. □

We will extend the above equivalence to the non-affine setting later. At this stage we only have a functor in one direction – the pullback.

**Corollary 4.3.13.** Let $S$ and $X = \coprod \mathcal{P}$ be as in Lemma 4.3.12. Then the pullback functors $\phi_{h, \mathcal{P}}^{-1} : \mathcal{P}/X \to \mathcal{P}/\mathfrak{fl}$ and $\phi_{\mathcal{P}}^{-1} : \mathcal{P}/X \to \mathcal{P}$ are fully faithful.

**Proof.** It is enough to deal with the flat case. By Lemma 4.3.12, $\phi_{h, \mathcal{P}}^{-1}$ is an equivalence respecting the fpqc topology, hence the categories of fpqc sheaves of sets $\text{Sh}(\mathfrak{fl}^{\text{aff}}/X)$ and $\text{Sh}(\mathfrak{fl}^{\text{aff}}/\mathcal{P})$ are equivalent. By locality of sheaves, the latter are equivalent to the categories of fpqc sheaves $\text{Sh}(\mathfrak{fl}/X)$ and $\text{Sh}(\mathfrak{fl}/\mathcal{P})$. On the other hand, by Yoneda’s lemma representable functors provide fully faithful embeddings of $\mathfrak{fl}/X$ and $\mathfrak{fl}/\mathcal{P}$ into the categories of presheaves $\text{Psh}(\mathfrak{fl}/X)$ and $\text{Psh}(\mathfrak{fl}/\mathcal{P})$, respectively. For any algebraic $X$-space $X'$, the presheaf $T \mapsto \text{Hom}_X(T, X')$ is an fpqc sheaf, and the same is true for pushout data, so we obtain fully faithful embeddings $\mathfrak{fl}/X \to \text{Sh}(\mathfrak{fl}^{\text{aff}}/X)$ and $\mathfrak{fl}/\mathcal{P} \to \text{Sh}(\mathfrak{fl}^{\text{aff}}/\mathcal{P})$. It is easy to check that the restriction functor $\phi_{h}^{-1}$ agrees with the equivalence $\text{Sh}(\mathfrak{fl}^{\text{aff}}/X) \cong \text{Sh}(\mathfrak{fl}^{\text{aff}}/\mathcal{P})$ induced by the restriction $\phi^{-1}_{\mathcal{P}}$, and therefore $\phi_{h}^{-1}$ is fully faithful. □

4.3.14. *Fpfp affine equivalence relations.* We conclude our study of affine Ferrand’s pushouts by proving that they respect fpfp equivalence relations. The latter will play a key role in the construction of general Ferrand’s pushouts.

**Lemma 4.3.15.** Let $\mathcal{P}_0, \mathcal{P}_1$ be affine Ferrand’s pushout data, $X_i = \coprod \mathcal{P}_i$, and

$$
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{p_2} & \mathcal{P}_0 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{q_2} & X_0
\end{array}
$$

a commutative diagram, i.e., the two squares with the same indices of $p$ and $q$ are commutative. Then the upper row of the diagram is an fpfp (resp. étale) equivalence relation if and only if so is the lower row.

**Proof.** We prove the lemma in the fpfp case. The étale case is similar and is left to the reader. First, note that if either of the rows is an fpfp equivalence relation then the morphisms in the other row are flat and the diagram is cartesian.
by Lemma 4.3.12. If $X_1 \to X_0$ is an fpff equivalence relation then the quotient $X = X_0/X_1$ is an algebraic space by [LMB, Corollary 10.4]. By fpff descent, applied componentwise to the pushout data, it then follows that $P_0 \to X_0$ is the base change of a premorphism $P \to X$ and $P_1 = P_0 \times_P P_0$. Thus, $P_1 \to P_0$ is an equivalence relation.

Conversely, if $P_1 \to P_0$ is an equivalence relation then it gives rise to a groupoid of pushout data $(p_1, p_2, m, i, \delta)$, where

$$p_{1,2} : P_1 \to P_0, \quad m: P_2 = P_1 \times_{P_2} P_1 \to P_1, \quad i: P_1 \to P_1, \quad \delta: P_0 \to P_1$$

satisfy all usual compatibilities, such as $p_2 \circ \delta = \text{Id}$, etc.; see [Stacks, §35.11, (Tag:0231)]. Since $\prod_{\text{aff}} i$ respects fibered products by Lemma 4.3.12, it takes this groupoid to a groupoid of schemes $(q_1, q_2, \prod_{\text{aff}} m, \prod_{\text{aff}} i, \prod_{\text{aff}} \delta)$. It remains to check that the latter groupoid is, in fact, an equivalence relation, that is, the diagonal $h: X_1 \to X_0 \times X_0$ is a monomorphism.

By definition, $Y_1 \equiv Y_0$ and $Z_1 \equiv Z_0$ are equivalence relations, hence $h_Y: Y_1 \to Y_0 \times Y_0$ and $h_Z: Z_1 \to Z_0 \times Z_0$ are monomorphisms. Note that

$$h_Y': Y_1 \times_{X_1} Y_1 = X_1 \times_{(X_0 \times X_0)} (Y_0 \times Y_0) \to Y_0 \times Y_0$$

is a base change of $h$. Since $Y_1 \to X_1$ is a monomorphism (even a closed immersion), the diagonal $Y_1 \to Y_1 \times X_1, Y_1$ is an isomorphism, and hence $h_Y = h_Y'$ is the base change of $h$. A similar claim fails for $Z$, so for $i = 0, 1$ consider the open subscheme $U_i = Z \setminus T_i$ of $Z_i$. Recall that $U_i \to X_i$ is an open immersion and $|X_i| = |Y_i| \prod_{i} |U_i|$ by Proposition 4.3.2. By the same argument as above, $h_{U_i}: U_1 \to U_0 \times U_0$ is a base change of $h$. On the other hand, $h_U$ is also the base change of $h_Z$, hence a monomorphism. We proved that the base change of $h$ with respect to the surjective morphism $Y_0 \times Y_0 \prod_{U_0 \times U_0} X_0 \times X_0$ is a monomorphism. But $h$ is of finite type by [Knu, Proposition 1.5.12], and hence it is a monomorphism by Lemma 2.1.7. □

### 4.4. Affine presentations.

We do not know if any Ferrand’s pushout datum is effective, but we establish a useful criterion of effectiveness sufficient for our purposes. Note that any effective Ferrand’s pushout datum $P$ possesses an affine presentation $P' \to P$. Indeed, since $Y$, $Z$, and $T$ are affine over $X = \prod_{T} Z$, one can take any affine presentation $X' \to X$ and set $P' = P \times_{X} X'$. One of our main results on Ferrand’s pushouts asserts that existence of an affine presentation implies effectiveness. Moreover, we will show that it suffices to have an affine fpff presentation. For a Ferrand’s pushout datum $P$, we denote by $\text{fl}^{\text{pres}}/P$ (resp. $\text{ét}^{\text{pres}}/P$) the full subcategory of $\text{fl}/P$ (resp. $\text{ét}/P$) of objects that possess an affine fpff (resp. étale) presentation.

#### 4.4.1. A criterion for existence of affine presentations.

The following lemma provides a sufficient condition for the existence.

**Lemma 4.4.2.** Let $P = (T; Y, Z)$ be a Ferrand’s pushout datum, and assume that any étale morphism $T' \to T$ with an affine source lifts to an étale morphism $Z' \to Z$ with an affine source, in the sense that $T' = Z' \times_{Z} T$. Then $P$ admits an affine étale presentation.

**Proof.** Assume that the condition holds true. Pick an affine presentation $Y_0 \to Y$. Then $T_0 := T \times_{Y} Y_0 \to T$ is an affine presentation that can be lifted to an étale morphism $Z_0' \to Z$ with an affine source. Since $T \to Z$ is a closed immersion, its complement is open, and we can pick an affine presentation $Z_0'' \to Z \setminus T$. Then
Case 0. The assertion of the theorem holds true if $Z$ is an affine noetherian scheme henselian along $T$. This is the main case, based on a deep result of R. Elkik. Being a smooth affine $T$-scheme, $T'$ admits a lifting to a smooth affine $Z$-scheme $Z''$ by [Elk, Theorem 6]. Let $Z'$ be the étale locus of the morphism $g': Z'' \to Z$; it is open and closed because the relative dimension of a flat morphism is locally constant. It remains to note that $T' \subset Z'$, and so $g: Z' \to Z$ is an affine étale morphism lifting $f$.

Now, we will gradually remove the additional assumptions made in Case 0. This will be based on the approximation theory of [EGA, IV$_3$, §8] that studies certain filtered limits of schemes. All references to the approximation theory are to [EGA, IV$_3$] and [EGA, IV$_4$].

Case 1. The assertion of the theorem holds true if $Z$ is affine and noetherian. Let $H$ denote the henselization of $Z$ along $T$, so $H$ is the limit of the filtered family \{ $Z_{\alpha}$ \} of affine étale $Z$-schemes strictly étale over $T$. In particular, we can view $T$ as a closed subscheme of $H$ and each $Z_{\alpha}$. Then $f$ lifts to an affine étale morphism $g_\alpha: H' \to H$ by Case 0. By approximation (loc.cit. 8.8.2(ii), 8.10.5(viii), and 17.7.8(ii)) the latter is the base change of an affine étale morphism $Z'_{\alpha} \to Z_{\alpha}$. In particular, $Z'_{\alpha} \to Z_{\alpha} \to Z$ is an affine étale lifting of $f$, and we can set $Z' := Z'_{\alpha}$.

Case 2. The assertion of the theorem holds true if $Z$ is affine and $T$ is finitely presented in $Z$. We can represent $Z$ as the filtered limit of affine noetherian schemes $Z_{\alpha}$, e.g., $Z_{\alpha} = \text{Spec}(A_{\alpha})$, where $A_{\alpha}$ run through the finitely generated subrings of $O_Z(Z)$. By approximation (loc.cit. 8.6.3), $T$ is the preimage of a subscheme $T_{\alpha}$ for some $\alpha$. Set $T_\beta = T_{\alpha} \times_Z Z_{\beta}$ for any $\beta \geq \alpha$. Then $T$ is the filtered limit of $T_\beta$, and by approximation $f$ is induced from an affine étale morphism $f_\beta: T_\beta' \to T_\beta$ for a large enough $\beta$ (we use loc.cit. 8.8.2(ii), 8.10.5(viii), and 17.7.8(ii) again). By
Case 1, the latter lifts to an affine étale morphism \( g_\beta : Z'_\beta \to Z_\beta \), and then the base change \( g = g \times_{Z_\beta} Z \) is as required.

Case 3. The assertion of the theorem holds true if \( Z \) is affine. Let \( Z = \text{Spec}(A) \) and \( T = \text{Spec}(A/I) \). Plainly, \( A/I \) is the filtered colimit of the rings \( A/I_\alpha \), where \( I_\alpha \) run through finitely generated ideals contained in \( I \). Therefore, \( T \) is the filtered limit of the finitely presented closed subschemes \( T_\alpha = \text{Spec}(A/I_\alpha) \) containing \( T \). By approximation, \( f \) lifts to an affine étale morphism \( f_\alpha : T'_\alpha \to T_\alpha \), and the latter lifts to an affine étale morphism \( g : Z' \to Z \) by Case 2.

Case 4. The general case. We will view \( Z \) as an open subscheme of its affine hull \( \overline{Z} \), and let \( \overline{T} \) be the schematic closure of \( T \) in \( \overline{Z} \) (we are aware the reader that the morphism from the affine hull of \( T \) to \( \overline{T} \) does not have to be an open immersion).

By Case 4, the étale morphism \( T \to T' \to \overline{T} \) lifts to an étale morphism \( \overline{g} : \overline{Z} \to \overline{Z} \) with an affine source. The closed set \( V = \overline{Z} \setminus \overline{g}^{-1}(Z) \) is disjoint from \( T' \), hence there exists an element \( a \in \mathcal{O}(\overline{Z}) \) which vanishes on \( V \) and equals to 1 on \( T' \). In particular, the localization \( Z' = \overline{Z}_a \) contains \( T' \) and the étale morphism \( Z' \to \overline{Z} \) factors through \( Z \). Thus, \( g : Z' \to Z \) is a lifting of \( f \) as required. \( \square \)

4.4.6. Some cases when affine presentation exists. Using the criterion of Lemmas 4.4.2 and the lifting results of Lemma 4.4.4 and Theorem 4.4.5 we immediately obtain the following result.

**Corollary 4.4.7.** A Ferrand’s pushout datum \( \mathcal{P} = (T; Y, Z) \) possesses an affine étale presentation \( \mathcal{P}' \to \mathcal{P} \) if one of the following conditions is satisfied:

(i) \( Z \) is quasi-affine,

(ii) \(|T|\) is discrete.

Part (i) of the above corollary can also be modified as follows:

**Corollary 4.4.8.** Assume that a Ferrand’s pushout datum \( \mathcal{P} = (T; Y, Z) \) admits a morphism \( f : \mathcal{P} \to \bar{\mathcal{P}} \) such that \( f_Z : Z \to \bar{Z} \) is quasi-affine and \( \bar{\mathcal{P}} \) possesses an étale (resp. flat) affine presentation \( \bar{\mathcal{P}}' \to \bar{\mathcal{P}} \). Then \( \mathcal{P} \) possesses an étale (resp. flat) affine presentation.

**Proof.** Set \( \mathcal{P}'' = \mathcal{P} \times_{\bar{\mathcal{P}}} \bar{\mathcal{P}}' \) and note that \( \mathcal{P}'' \to \mathcal{P} \) is a surjective étale (resp. flat) morphism. Furthermore, \( Z'' \) is quasi-affine because \( Z'' \to \bar{Z}' \) is quasi-affine. By Corollary 4.4.7(i), \( Z'' \) possesses an affine étale presentation \( \mathcal{P}' \), and the composition \( \mathcal{P}' \to \mathcal{P}'' \to \mathcal{P} \) is an étale (resp. flat) affine presentation of \( \mathcal{P} \). \( \square \)

**Proposition 4.4.9.** Let \( \mathcal{P} \) be a Ferrand’s pushout datum over an algebraic space \( U \). Then,

1. The subcategories \( \text{fl}^\text{pres}/\mathcal{P} \) and \( \text{ét}^\text{pres}/\mathcal{P} \) are closed under fibered products.
2. If \( \phi : \mathcal{P} \to \mathcal{P} \) is a morphism such that \( \phi_Z \) is quasi-affine (e.g., \( \mathcal{P} = \mathcal{P} \times_U \overline{U} \) for a quasi-affine morphism of algebraic spaces \( U \to U \)) then the pullback functor \( \phi^{-1} \) takes \( \text{fl}^\text{pres}/\mathcal{P} \) and \( \text{ét}^\text{pres}/\mathcal{P} \) to \( \text{fl}^\text{pres}/\mathcal{P} \) and \( \text{ét}^\text{pres}/\mathcal{P} \), respectively.

**Proof.** The second assertion follows from Corollary 4.4.8, so, let us prove (1). Assume that \( \mathcal{P}' \), \( \mathcal{P}'' \) are objects of \( \text{fl}^\text{pres}/\mathcal{P} \) (resp. \( \text{ét}^\text{pres}/\mathcal{P} \)), and let \( Q' \to \mathcal{P}' \), \( Q'' \to \mathcal{P}'' \) be their affine presentations. Then \( Q' \times_{\mathcal{P}'} Q'' \to \mathcal{P}' \times_{\mathcal{P}'} \mathcal{P}'' \) is a surjective étale morphism, and hence it is sufficient to prove that \( Q' \times_{\mathcal{P}'} Q'' \) admits an fppf (resp. étale) affine presentation. Thus, we may assume that \( \mathcal{P}' \) and \( \mathcal{P}'' \) are affine.
Let $\mathcal{P} = (T; Y, Z)$, $\mathcal{P}' = (T'; Y', Z')$, $\mathcal{P}'' = (T''; Y'', Z'')$. The diagonal morphism $Z \to Z \times Z$ is quasi-affine by Lemma 2.1.4, hence so is its base change $Z' \times_Z Z'' \to Z' \times Z''$. We obtain that $Z' \times_Z Z''$ is a quasi-affine scheme, and then $\mathcal{P}' \times_{\mathcal{P}} \mathcal{P}''$ admits an affine presentation by Corollary 4.4.7.

4.4.10. Ferrand’s pushouts of affine pushout data. Now, we are in a position to prove that affine Ferrand’s pushouts are, in fact, pushouts in the category of all algebraic spaces.

**Notation 4.4.11.** For an algebraic space $U$ and a pushout datum $\mathcal{R}$ denote by $h_U(\mathcal{R})$ the set $\text{PreMor}(\mathcal{R}, U)$. Since any premorphism of algebraic spaces is a morphism, if $\mathcal{R}$ is an algebraic space then $h_U(\mathcal{R})$ is the set of $\mathcal{R}$-points of $U$ in the usual sense.

If $\mathcal{P}$ is a pushout datum and $\mathcal{P} \to X$ is a premorphism then the morphism $\coprod^\mathcal{P} \mathcal{P} \to X$ is an isomorphism if and only if the natural map $\psi: h_U(X) \to h_U(\mathcal{P})$ is bijective for any $U \in \text{Ob}(\mathcal{C})$.

**Theorem 4.4.12.** Let $X = \coprod^\mathcal{P} \mathcal{P}$ be an affine Ferrand’s pushout. Then $X = \coprod \mathcal{P}$ in the category of all algebraic spaces.

**Proof.** Let us show that the natural map $\psi: h_U(X) \to h_U(\mathcal{P})$ is bijective for any algebraic space $U$. Fix an affine presentation $U_0 \to U$.

**Injectivity of $\psi$:** Pick $h_1, h_2 \in h_U(X)$ and assume that $\psi(h_1) = \psi(h_2)$. Set $\mathcal{P}'_0 := \mathcal{P} \times_U U_0$ and $X'_0 := X \times_{h_U} U_0$. Since $X'_0 \times_X \mathcal{P} \simeq \mathcal{P}'_0 \simeq X'_0 \times_X \mathcal{P}$, it follows from Corollary 4.3.13 that $X'_0 \simeq X'_0$, which we denote simply by $X'_0$. Pick an affine covering $X_0 \to X'_0$, and set $\mathcal{P}_0 := \mathcal{P} \times_X X_0 = \mathcal{P}'_0 \times_{X'_0} X_0$. Then $X_0 = \coprod^\mathcal{P}_0$ by Lemma 4.3.12. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_0 & \to & \mathcal{P}'_0 \\
\downarrow \psi(h_1) = \psi(h_2) & & \downarrow \psi \\\n\mathcal{P} & \to & \mathcal{P}
\end{array}
\]

It is sufficient to check the equality of $h_i$-s étale locally on the base, i.e., it is enough to prove that the two morphisms $X'_0 \to X_0$ coincide, or, equivalently, that the two composed morphisms $X_0 \to X'_0 \to U_0$ coincide. But, the composed premorphism $\mathcal{P}_0 \to \mathcal{P}'_0 \to U_0$ factors uniquely through $X_0 = \coprod^\mathcal{P}_0$ since $U_0$ is affine, and hence $h_1 = h_2$.

**Surjectivity of $\psi$:** Let $\alpha \in h_U(\mathcal{P})$ be arbitrary, and set $\mathcal{P}'_0 := \mathcal{P} \times_U U_0$. Then $\mathcal{P}'_0$ admits an affine presentation $\mathcal{P}_0 \to \mathcal{P}'_0$ by Proposition 4.4.9, and hence $\mathcal{P}_0 \to \mathcal{P}$ is an affine presentation. Set $\mathcal{P}_i = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$ and $X_i := \coprod^\mathcal{P}_i$ for $i = 0, 1$. Note that $\mathcal{P}_0 \to U_0$ factors through a morphism $h_0: X_0 = \coprod^\mathcal{P}_0 \to U_0$. By Lemma 4.3.12, $X_1 = X_0 \times X_0$, so $X_1 \to X_0$ is an étale equivalence relation with quotient $X$. Consider the commutative diagram

\[
\begin{array}{ccc}
\mathcal{P}_0 & \to & \mathcal{P}'_0 \\
\downarrow \psi(h_1) = \psi(h_2) & & \downarrow \psi \\\n\mathcal{P} & \to & \mathcal{P}
\end{array}
\]
Indeed, if $U \vdash X_1 \rightarrow X_0 \rightarrow U_0 \rightarrow U$ coincide. Thus, the two morphisms $X_1 \rightarrow X_0$ coincide by the injectivity of $\psi_1: h_U(X_1) \rightarrow h_U(P_1)$. Hence the morphism $X_0 \rightarrow U$ induces a morphism $h: X \rightarrow U$ since $X = X_0/X_1$. It remains to verify that $\psi(h) = \alpha$. By the construction $\alpha \circ \pi = h \circ \phi \circ \pi$, where $\pi: P_0 \rightarrow P$ is the étale covering. Thus $\alpha = h \circ \phi = \psi(h)$, and we are done. \qed

4.5. **Ferrand's pushouts of separated pushout data.**

4.5.1. **A criterion of effectiveness.**

**Proposition 4.5.2.** Let $P = (T; Y, Z)$ be a separated Ferrand’s pushout datum. Then,

(i) The following conditions are equivalent: (a) $P$ is effective, (b) $P$ admits an affine étale presentation, (c) $P$ admits an affine fppf presentation.

(ii) If $P$ is effective, $P_0 \rightarrow P$ is an affine fppf presentation, $P_1 = P_0 \times_P P_0$, $X = \bigsqcup P$, and $X_1 = \bigsqcup P_i$, then $q_{1,2}: X_1 \Rightarrow X_0$ is an fppf equivalence relation, $X = X_0/X_1$, and $P_0 = P \times_X X_0$.

**Proof.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. So assume that $P_0 \rightarrow P$ is an affine fppf presentation and $P_1 = P_0 \times_P P_0$. Then $P_1$ is affine since $P$ is separated. Set $X_i := \bigsqcup^\text{aff} P_i$. Then $X_i = \bigsqcup P_i$ by Theorem 4.4.12, and $X_1 \Rightarrow X_0$ is an fppf equivalence relation by Lemma 4.3.15. We claim that, $X_0/X_1 = X$. Indeed, if $U$ is an algebraic space then $h_U(X_0/X_1)$ (resp. $h_U(P)$) is the equalizer of $h_U(X_0) \rightrightarrows h_U(X_1)$ (resp. $h_U(P_0) \rightrightarrows h_U(P_1)$), and the two equalizers coincide by Theorem 4.4.12. Thus, $h_U(P) = h_U(X_0/X_1)$ for any $U$, and hence $X_0/X_1 = \bigsqcup P = X$. Finally, if $i = 1, 2$ and $p_i: P_1 \rightarrow P_0$ denote the projections then $p_i = q_i \times_{X_0} X_1$ by Lemma 4.3.12. The equality $P_0 = P \times_X X_0$ follows by descent. \qed

4.5.3. **Properties.**

**Corollary 4.5.4.** Let $P = (T; Y, Z)$ be an effective separated Ferrand’s pushout datum, $X = \bigsqcup P$, and let $\phi: P \rightarrow X$ be the natural premorphism. Then,

(i) The topological pushout $|Y| \bigsqcup_{|T|} |Z|$ is naturally homeomorphic to $|X|$.

(ii) $T = Y \times_X Z$.

(iii) $Y \rightarrow X$ is a closed immersion, $U = Z \setminus T \rightarrow X$ is an open immersion, and $|X| = |Y| \bigsqcup |Z|$ set-theoretically.

**Proof.** By Proposition 4.5.2, there exists an affine presentation $P_0 \rightarrow P$ such that $P_0 = P \times_X X_0$, where $X = \bigsqcup P$, and $X_0 = \bigsqcup P_0 = \bigsqcup^\text{aff} P_0$. Thus, the Corollary follows from Proposition 4.3.2 by descent. \qed

**Theorem 4.5.5.** Let $P$ be an effective separated Ferrand’s pushout datum. Then $\bigsqcup P$ is separated.
Proof. Consider a valuation ring $R$ with fraction field $K$, and let $i \colon \eta = \text{Spec}(K) \to X = \coprod P$ be a morphism. By the valuative criterion of separatedness it suffices to prove that $i$ admits at most one extension to a morphism $S = \text{Spec}(R) \to X$. Let $f, g : S \to X$ be two extensions of $i$. Recall that $|X| = |Y| \coprod |U|$ and $Y \to X$ is a closed immersion by Corollary 4.5.4, where $\mathcal{P} = (T; Y, Z)$, and $U = Z \setminus T$. 

If $i(\eta) \in Y$ then the morphisms $f, g$ factor through $Y$ since $Y \to X$ is a closed immersion, and hence $f = g$ by the separatedness of $Y$. So, we may assume that $i(\eta) \in U$.

Since $f^{-1}(Y)$ and $g^{-1}(Y)$ are closed subschemes of $S$, one of them contains the other (we use here the fact that the ideals of $R$ are linearly ordered with respect to inclusion). Without loss of generality we assume that $f^{-1}(Y)$ is the larger subscheme and denote its generic point by $\varepsilon$. Note that $\varepsilon \neq \eta$, hence the localization $S_\varepsilon$ is a valuation ring of non-zero height. We claim that $f(\varepsilon) \in T$ and $g(\varepsilon) \in T$. Consider an affine presentation $\mathcal{P}' \to \mathcal{P}$, and set $X' := \coprod \text{aff} \mathcal{P}'$. Then $X' = \coprod \mathcal{P}'$ by Theorem 4.4.12, and $\mathcal{P}' = \mathcal{P} \times_X X'$ by Proposition 4.5.2. Set $S' := S_\varepsilon \times_{f, X} X'$, and $S'' := S_\varepsilon \times_{g, X} X'$. Both $S'$ and $S''$ are affine and étale over $S$, hence they are spectra of Prüfer rings. By our construction, the preimages of $Y'$ in both $S'$ and $S''$ are contained in the preimage of $\varepsilon$. Hence the morphisms $S' \to X'$ and $S'' \to X'$ factor uniquely through $Z'$ by Corollary 4.2.7. In particular, $f(\varepsilon) \in T$ and $g(\varepsilon) \in T$.

We proved that $f$ and $g$ take $S_\varepsilon$ to $Z$, hence their restrictions onto $S_\varepsilon$ coincide by the separatedness of $Z$. Let $W$ be the closed subscheme of $S$ with generic point $\varepsilon$. Since $f$ and $g$ take $W$ to the separated subspace $Y \hookrightarrow X$ and coincide on the generic point, we have that $f|_W = g|_W$. It remains to note that $S$ is the composition of $S_\varepsilon$ and $W$ along $\varepsilon$. So, $W \coprod S_\varepsilon = S$ by Theorem 4.4.12, and hence $f = g$. 

Lemma 4.5.6. Let $\mathcal{P}$ be an effective separated Ferrand’s pushout datum, $X = \coprod \mathcal{P}$, and $\phi : \mathcal{P} \to X$ the natural premorphism. Then,

(i) $\phi_1 : \text{fl}_{\text{sep}}/X \to \text{fl}_{\text{sep}}^\text{pres}/\mathcal{P}$ (resp. $\phi_\text{et}^{-1} : \text{ét}_{\text{sep}}/X \to \text{ét}_{\text{sep}}^\text{pres}/\mathcal{P}$) is an equivalence of categories, whose essential inverse is $\coprod$, where $\mathcal{C}_{\text{sep}}$ denotes the full subcategory of separated objects in $\mathcal{C}$.

(ii) If (i) is one of the following properties: surjective, open immersion, étale, flat, flat and finitely presented, then a morphism $h : X' \to X''$ in $\text{fl}_{\text{sep}}/X$ satisfies (i) if and only if so does $\phi_1^{-1}(h)$. In particular, both equivalences $\phi^{-1}$ and $\coprod$ preserve these properties, and hence respect the fpqc, fppf, and étale topologies on the matched categories.

Proof. We start with (i). The proofs are the same for $\phi^{-1}$ and $\phi_\text{et}^{-1}$, so we will only deal with the categories $\mathcal{C} = \text{fl}_{\text{sep}}/X$ and $\mathcal{D} = \text{fl}_{\text{sep}}^\text{pres}/\mathcal{P}$. Note that the functors $\coprod : \mathcal{D} \to \mathcal{C}$ and $\phi^{-1} : \mathcal{C} \to \mathcal{D}$ are adjoint, since $\text{Hom}_D(\mathcal{P}', \phi^{-1} W) = \text{Hom}_C(\coprod \mathcal{P}', W)$ by the universal property of pushouts. So we should only prove that $\phi^{-1}$ is fully faithful and essentially surjective, or, equivalently, that the functor $\phi^{-1}$ is faithful and the adjunction transformation $\varepsilon : \text{Id}_D \to \phi^{-1} \circ \coprod$ is an isomorphism, i.e., the morphism $\varepsilon_{\mathcal{P}'} : \mathcal{P}' \to \mathcal{P} \times_X (\coprod \mathcal{P}')$ is an isomorphism for any $\mathcal{P}' \in \text{Ob}(\mathcal{D})$.

We deal with $\varepsilon_{\mathcal{P}'}$ first. If $\mathcal{P}_0$ is an affine presentation of $\mathcal{P}$ then $\varepsilon_{\mathcal{P}_0}$ is an isomorphism by Proposition 4.5.2(ii). More generally, if $\mathcal{P}'$ is affine and $\mathcal{P}_0 \to \mathcal{P}$ is any affine presentation then $\mathcal{P}' \coprod \mathcal{P}_0$ is an affine presentation of $\mathcal{P}$. Thus, $\varepsilon_{\mathcal{P}'}$ is an isomorphism since the functors $\phi^{-1}$ and $\coprod$ are compatible with disjoint unions.
Assume now that $\mathcal{P}' \in \text{Ob}(\mathcal{D})$ is arbitrary. Pick an affine presentation $\mathcal{P}_0 \to \mathcal{P}'$. Set $\mathcal{P}_1 := \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$, $X_i := \bigsqcup \mathcal{P}_i$, and $X' := \bigsqcup \mathcal{P}'$. By Proposition 4.5.2, $X_0 \to X'$ is an affine presentation, and by the affine case we have just proved $\mathcal{P}_i = \mathcal{P} \times_X X_i$ and $\mathcal{P}_i = \mathcal{P}' \times_{X'} X_i$. Now it follows by descent that $\mathcal{P}' = \mathcal{P} \times_X X'$.

Let us prove that $\phi^{-1}$ is faithful. Assume to the contrary that $f_{1,2}: W \to W'$ are two different morphisms in $\mathcal{C}$, whose images in $\mathcal{D}$ coincide. After replacing $X$ with an affine presentation $X_0$, and replacing $W$, $W'$, and $f_i$ with their base changes with respect to $X_0 \to X$, we may assume that $X$ is affine. But in this case $\phi^{-1}$ is faithful by Corollary 4.3.13, and we get a contradiction.

Let us prove (ii). All properties (†) are stable under arbitrary base changes. So, if $h: X' \to X''$ satisfies (†) then so does $\phi^{-1}(h)$. Conversely, assume that $\phi^{-1}(h)$ satisfies (†), and let us show that so does $h$. Pick an affine presentation $W \to X$. Then, by étale descent, after replacing $X, X', X'', P, P', P''$ with $W, X' \times_X W, X'' \times_X W, P \times_X W, P' \times_X W, P'' \times_X W$ we may assume that $X$ and $P$ are affine. Similarly, after replacing $X''$ with an étale presentation, and $X', P', P''$ with the corresponding base changes, we may assume that $X''$ and $P''$ are affine too. Now, if (†) is any of the properties, but the property of being an open immersion then it can be checked étale locally on the source, and hence the assertion reduces to the case when $X'$ and $P'$ are also affine. The latter case follows from Lemma 4.3.12(ii).

Finally, an open immersion is nothing but an étale monomorphism, hence it remains to show that if $\phi^{-1}(h)$ is a monomorphism then so is $h$. This follows from Lemma 2.1.7 and the surjectivity of $Y \bigsqcup Z \to X$. 

\textbf{Lemma 4.5.7.} Let $\mathcal{P}_0, \mathcal{P}_1$ be separated Ferrand’s pushout data, $X_i = \bigsqcup \mathcal{P}_i$, and

$$
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{p_2} & \mathcal{P}_0 \\
\downarrow{p_1} & & \downarrow{p_1} \\
X_1 & \xrightarrow{q_1} & X_0
\end{array}
$$

a commutative diagram. Then the upper row of the diagram is an fpfp (resp. étale) equivalence relation if and only if so is the lower row. Furthermore, if both rows are equivalence relations then $X_0/X_1 = \bigsqcup(\mathcal{P}_0/\mathcal{P}_1)$.

\textbf{Proof.} The proof of the first claim is identical to the proof of Lemma 4.3.15, but one should use Lemma 4.5.6 instead of Lemma 4.3.12, and Corollary 4.5.4 instead of Proposition 4.3.2.

To prove the second claim, set $X := X_0/X_1$ and $\mathcal{P} := \mathcal{P}_0/\mathcal{P}_1$, and note that for any algebraic space $U$, the set $h_U(X)$ (resp. $h_U(\mathcal{P})$) is the equalizer of $h_U(X_0) \rightrightarrows h_U(X_1)$ (resp. $h_U(\mathcal{P}_0) \rightrightarrows h_U(\mathcal{P}_1)$). The two equalizers are functorially isomorphic since $X_i = \bigsqcup \mathcal{P}_i$. Thus, $h_U(\mathcal{P}) = h_U(X)$, and hence $X = \bigsqcup \mathcal{P}$. 

\textbf{4.6. General Ferrand’s pushouts.}

\textbf{Theorem 4.6.1.} Let $\mathcal{P} = (T; Y, Z)$ be a Ferrand’s pushout datum. Then,

(i) The following conditions are equivalent: (a) $\mathcal{P}$ is effective, (b) $\mathcal{P}$ admits an affine étale presentation, (c) $\mathcal{P}$ admits an affine fpfp presentation.

(ii) If $\mathcal{P}$ is effective, $\mathcal{P}_0 \to \mathcal{P}$ is an effective fpfp separated presentation, $\mathcal{P}_1 = \mathcal{P}_0 \times_{\mathcal{P}} \mathcal{P}_0$, $X = \bigsqcup \mathcal{P}$, and $X_1 = \bigsqcup \mathcal{P}_1$, then $X_1 \rightrightarrows X_0$ is an fpfp equivalence relation, $X = X_0/X_1$, and $\mathcal{P}_i = \mathcal{P} \times_X X_i$. 

\textbf{Proof.}
Proof. The only implication that requires a proof in (i) is \((c) \Rightarrow (a)\). Pick a separated \(fppf\) presentation \(P_0 \to P\) such that \(P_0\) itself admits an affine presentation, e.g., \(P_0\) is affine. Set \(P_i := P_0 \times_P P_0\). Then \(P_i\) are separated, admit affine presentation by Proposition 4.4.9, and hence effective by Proposition 4.5.2. Set \(X_i := \coprod P_i\). Then, by Lemma 4.5.7, \(X_1 \to X_0\) is an \(fppf\) equivalence relation, \(P = P_0 / P_1\) is effective, and \(\coprod P = X_0 / X_1\). Finally, since \(P_1 = P_0 \times_X X_0\) by Lemma 4.5.6, it follows that \(P_0 = P \times_X X_0\) by flat descent. Since \(P_i \to P\) is an \(fppf\) presentation for any \(i\), the same argument shows that \(P_i = P \times_X X_i\). \(\square\)

Corollary 4.6.2. Let \(P = (T; Y, Z)\) be a Ferrand’s pushout datum. Then \(P\) is effective in each one of the following cases: (a) \(|T|\) is discrete, (b) \(P\) admits a morphism \(f\) into an effective Ferrand’s pushout datum \((T'; Y', Z')\) such that \(f_Z: Z \to Z'\) is quasi-affine.

Proof. Follows immediately from Theorem 4.6.1 and Corollaries 4.4.7 and 4.4.8. \(\square\)

4.6.3. Properties.

Corollary 4.6.4. Let \(P = (T; Y, Z)\) be an effective Ferrand’s pushout datum, \(X = \coprod P\), and \(\phi: P \to X\) the natural premorphism. Then,

(i) The topological pushout \(|Y| \coprod |T| |Z|\) is naturally homeomorphic to \(|X|\).

(ii) \(T = Y \times_X Z\).

(iii) \(Y \to X\) is a closed immersion, \(U = Z \setminus T \to X\) is an open immersion, and \(|X| = |Y| \coprod |U|\) set-theoretically.

Proof. The proof is identical to the proof of Corollary 4.5.4, but one uses Theorem 4.6.1 instead of Proposition 4.5.2. \(\square\)

4.6.5. Gluing of flat sites. Now, we can establish a gluing result for the categories of flat objects over Ferrand’s pushouts.

Theorem 4.6.6. Let \(P\) be an effective Ferrand’s pushout datum, \(X = \coprod P\), and \(\phi: P \to X\) the natural premorphism. Then,

(i) \(\phi^{-1}: \text{fl}/X \to \text{fl}^{\text{pres}}/P\) (resp. \(\phi^{-1}: \text{ét}/X \to \text{ét}^{\text{pres}}/P\)) is an equivalence of categories, whose essential inverse is \(\coprod\).

(ii) If (i) is one of the following properties: surjective, open immersion, étale, flat, flat and finitely presented, then a morphism \(h: X' \to X''\) in \(\text{fl}/X\) satisfies (i) if and only if so does \(\phi^{-1}(h)\). In particular, both equivalences \(\phi^{-1}\) and \(\coprod\) preserve these properties, and hence respect the \(\text{fpqc}, \text{fppf},\) and étale topologies on the matched categories.

Proof. Recall that the fibered product of effective separated pushout data flat over a given datum is effective and separated by Propositions 4.4.9 and 4.5.2. Therefore, the proof of the theorem is identical to the proof of Lemma 4.5.6, but one should consider effective separated presentations instead of affine presentations, and use Theorem 4.6.1 instead of Proposition 4.5.2, and Lemma 4.5.6 instead of Lemma 4.3.12 and Corollary 4.3.13. \(\square\)
Lemma 4.6.7. Let \( \mathcal{P}_0, \mathcal{P}_1 \) be effective Ferrand’s pushout data, \( X_1 = \coprod \mathcal{P}_i \), and

\[
\begin{array}{ccc}
\mathcal{P}_1 & \xrightarrow{p_2} & \mathcal{P}_0 \\
\downarrow & & \downarrow \\
X_1 & \xrightarrow{q_2} & X_0
\end{array}
\]

a commutative diagram. Then the upper row of the diagram is an fpqc (resp. étale) equivalence relation if and only if so is the lower row. Furthermore, if both rows are equivalence relations then \( X_0/X_1 = \coprod (\mathcal{P}_0/\mathcal{P}_1) \).

Proof. The proof is identical to the proof of Lemma 4.5.7, but one should use Theorem 4.6.6 instead of Lemma 4.5.6, and Corollary 4.6.4 instead of Corollary 4.5.4. \( \square \)

4.6.8. Some consequences.

Corollary 4.6.9. Let \( \mathcal{P}_0 \to \mathcal{P} \) be a surjective flat finitely presented morphism of Ferrand’s pushout data, and assume that \( \mathcal{P}_0 \) is effective. Then \( \mathcal{P} \) is effective and the pushout \( X = \coprod \mathcal{P} \) can be described as \( X_0/X_1 \), where \( X_1 = \coprod \mathcal{P}_i \) and \( \mathcal{P}_1 = \mathcal{P}_0 \times \mathcal{P} \). Furthermore, \( \mathcal{P}_0 = \mathcal{P} \times X X_0 \).

Proof. Any affine fpqc presentation \( \overline{\mathcal{P}} \to \mathcal{P}_0 \) induces an affine fpqc presentation \( \overline{\mathcal{P}} \to \mathcal{P} \). Hence \( \mathcal{P} \) is effective by Theorem 4.6.1. Furthermore, \( \coprod \mathcal{P} = X_0/X_1 \) by Lemma 4.6.7. The last assertion follows from Theorem 4.6.6. \( \square \)

Now it is easy to show that any \( S \)-affine Ferrand’s pushout is Ferrand’s pushout in the category of algebraic spaces.

Corollary 4.6.10. If \( \mathcal{P} \) is an \( S \)-affine Ferrand’s pushout then \( \coprod^{\text{aff}} \mathcal{P} = \coprod \mathcal{P} \).

Proof. If \( S_0 \to S \) is an affine presentation then \( \mathcal{P} \times_S S_0 \) is an affine presentation of \( \mathcal{P} \), so the latter is effective by Theorem 4.6.1. Second, let \( \coprod \mathcal{P} \to \coprod^{\text{aff}} \mathcal{P} \) be the natural morphism of \( S \)-spaces. To prove that it is an isomorphism, it is sufficient to do so étale locally on \( S \). Thus, by Lemmas 3.3.3 and 4.6.7, we may assume that \( S \) is affine. But in this case \( \coprod^{\text{aff}} \mathcal{P} = \coprod^{\text{aff}} \mathcal{P} = \coprod \mathcal{P} \) by Theorem 4.4.12, and we are done. \( \square \)

Corollary 4.6.11. If \( \mathcal{P} \) is an effective Ferrand’s pushout datum and \( X = \coprod \mathcal{P} \), then the categories of flat \( \mathcal{O}_X \)-modules and flat \( \mathcal{O}_\mathcal{P} \)-modules are naturally equivalent. Furthermore, the equivalence preserves the finite presentation property.

Proof. Pick an affine presentation \( \mathcal{P}_0 \to \mathcal{P} \), and set \( \mathcal{P}_1 := \mathcal{P}_0 \times \mathcal{P} \). Assume, first, that \( \mathcal{P} \) is separated. Then \( \mathcal{P}_i \) are affine, and \( \coprod \mathcal{P}_i = \coprod^{\text{aff}} \mathcal{P} \) by Theorem 4.4.12. Thus, the assertion of the corollary holds true for \( (\mathcal{P}_i, X_i) \) for all \( i \) by Lemma 4.3.8, and hence it holds true for the pair \( (\mathcal{P}, X) \) by Lemma 4.6.7 and descent. In general, \( \mathcal{P}_1 \) need not be affine, but it is separated. Hence the assertion of the corollary holds true for \( (\mathcal{P}_i, X_i) \) for all \( i \) by the separated case, and as before, it holds true for \( (\mathcal{P}, X) \) by descent. \( \square \)

Remark 4.6.12. (i) It is an interesting question if one can give an explicit criterion for a Ferrand’s pushout datum to be effective. We even do not know if there exist non-effective data. Note that Ferrand describes in [Fer, Theorem 7.1] such a criterion in the category of schemes, which essentially reduces to existence of an
affine Zariski presentation, i.e., presentation, in which the coverings are disjoint unions of open immersions. In particular, one obtains a source of examples which are non-effective in the category of schemes. However, many of these examples are effective in the category of algebraic spaces (cf. Example 4.6.17), and it is not clear whether one can use this to construct a Ferrand’s pushout datum, which is not effective in the algebraic spaces.

(ii) It seems that the question of existence of a non-effective Ferrand’s pushout datum is closely related to the affineness conjecture for henselian schemes, see Conjecture B in [GS, Remark 1.23(ii)].

4.6.13. Descent of properties. Various properties of morphisms descend through Ferrand’s pushouts. We do not try to make a complete list but only establish some cases that are easy to prove. A morphism of Ferrand’s pushout data \( f: \mathcal{P}' \to \mathcal{P} \) is a pro-open immersion if it is isomorphic to the filtered projective limit of a family of open immersions (at least \( a \text{ priori} \), this condition is stronger than the componentwise condition).

**Theorem 4.6.14.** (i) Let \( X = Y \bigsqcup_T Z \) be Ferrand’s pushout, and \( (\dagger) \) be one of the following properties: open immersion, pro-open immersion, schematically dominant, finite, quasi-finite, finite type. Then \( g': Z \to X \) satisfies \( (\dagger) \) if and only if so does \( g: T \to Y \).

(ii) Assume that \( \psi: \mathcal{P}' \to \mathcal{P} \) is a morphism between effective Ferrand’s pushout data, and \( h: X' \to X \) is the corresponding morphism between the pushouts. Let \( (\dagger) \) be one of the following properties: surjective, flat, flat and finitely presented, étale, open immersion, pro-open immersion. If \( \psi \) satisfies \( (\dagger) \) then so does \( h \).

**Proof.** We prove (ii) first. If \( \psi \) is surjective then so is \( h \) by Corollary 4.6.4. For the next four properties the assertion follows from Theorem 4.6.6. Assume now that \( \psi \) is a pro-open immersion: \( \mathcal{P}' = \lim_{\alpha} \mathcal{P}_\alpha \) where \( \mathcal{P}_\alpha \) are open in \( \mathcal{P} \). Then \( X' = \bigsqcup \mathcal{P}' \) is the limit of \( X_\alpha := \bigsqcup \mathcal{P}_\alpha \) in the category \( \text{fl}/X \) by Theorem 4.6.6. On the other hand, \( X_\alpha \to X \) are open immersions since \( \mathcal{P}_\alpha \to \mathcal{P} \) are so. Thus, \( X' \to X \) is a pro-open immersion.

Only inverse implications in (i) need a proof because \( g \) is a base change of \( g' \). If \( g \) is a (pro-)open immersion then \( \mathcal{P}' = (T; T, Z) \to \mathcal{P} \) is a (pro-)open immersion, and hence so is \( g': Z = \bigsqcup \mathcal{P}' \to X \) by (ii). The remaining four properties can be checked étale locally on \( X \), so we may assume that \( \mathcal{P} \) and \( X \) are affine by Theorems 4.6.1 and 4.4.12. The case of schematic dominance follows from [Fer, Proposition 5.6(2)] and the remaining three cases follow from [Fer, Proposition 5.6(3)]. \( \Box \)

**Remark 4.6.15.** (i) In the case of pro-open immersions, Theorem 4.6.14(i) actually asserts that if \( X = Y \bigsqcup_T Z \) is a composition then \( Z \to X \) is a pro-open immersion. This justifies the terminology, since \( X \) is “composed” from a closed subspace \( Y \) and a pro-open subspace \( Z \).

(ii) The assertion of Theorem 4.6.14(i) fails for the properties of being étale or flat. In fact, this happens for typical pinchings. For example, if \( Z \) is a smooth curve over \( k, T = \text{Spec}(k \times k) \) consists of two \( k \)-points, and \( g': T \to Z = \text{Spec}(k) \) is the projection, then \( X \) is a nodal curve. So, \( g' \) is split étale, but \( g: Z \to X \) is a non-flat normalization morphism.

4.6.16. Ferrand’s pushouts of schemes. By Theorem 4.4.12, Ferrand’s pushout of an affine scheme is affine. On the other hand, it is easy to give examples of Ferrand’s pushouts of schemes which are not schemes.
Example 4.6.17. Assume that \( Z \) is a scheme with a closed subscheme \( T = \{ t_1, t_2 \} \) consisting of two points and not contained in any affine open subscheme, and \( f: T \to Y \) is a morphism such that \( f(t_1) \) and \( f(t_2) \) have a common specialization \( y \in Y \). The Ferrand's pushout \( X = Y \amalg_T Z \) exists by Corollary 4.6.2, and for any neighborhood \( U \subseteq X \) of the image of \( y \), its preimage in \( Z \) contains \( T \). In particular, \( U \) is not affine, and hence \( X \) is not a scheme. Obviously, there exist compositions \( (f \text{ is an open immersion}) \) and pinchings \( (f \text{ is finite}) \) of this kind.

On the positive side, assume that the following condition is satisfied: \( (\dagger) \) \( \mathcal{P} = (T; Y, Z) \) is a schematic Ferrand’s pushout datum such that: (a) the pushout \( X \) of \( \mathcal{P} \) in the category of locally ringed spaces is a scheme, (b) the morphism \( Z \to X \) is affine, and (c) the morphism \( Y \to X \) is a closed immersion. Ferrand proved in [Fer, Theorem 7.1] that in this case \( T = Y \times_X Z \). We conclude the paper by relating this condition to general Ferrand’s pushouts. We will need the following definition:

by a Zariski presentation of \( \mathcal{P} \) we mean a morphism \( \mathcal{P}' \to \mathcal{P} \), where \( \mathcal{P}' = \coprod \mathcal{P}'_i \) and each morphism \( \mathcal{P}'_i \to \mathcal{P} \) is an open immersion.

Theorem 4.6.18. Let \( \mathcal{P} = (T; Y, Z) \) be a Ferrand’s pushout datum. Then the following conditions are equivalent:

(i) \( \mathcal{P} \) is schematic and satisfies Ferrand’s condition \( (\dagger) \).

(ii) \( \mathcal{P} \) is effective in the category of algebraic spaces and \( \coprod \mathcal{P} \) is a scheme.

(iii) \( \mathcal{P} \) possesses a Zariski affine presentation.

Proof. Assume (i) holds, and let \( X \) be the schematic pushout of \( \mathcal{P} \). Since the premorphism \( \mathcal{P} \to X \) is affine, any affine open covering of \( X \) induces an open affine covering of \( \mathcal{P} \), and we obtain (ii).

Assume (iii) holds, and fix such a presentation \( \mathcal{P}' = \coprod \mathcal{P}'_i \to \mathcal{P} \). Recall that \( \mathcal{P} \) is effective by Theorem 4.6.1, so set \( X := \coprod \mathcal{P} \). By Theorem 4.6.14(ii), the morphisms \( X'_i := \coprod \mathcal{P}'_i \to X \) are open immersions. Since each \( X'_i \) is affine by Theorem 4.4.12, we obtain that \( X \) is covered by open affine subschemes \( X'_i \). Hence \( X \) is a scheme.

Finally, assume that (ii) holds and \( X = \coprod \mathcal{P} \). By definition, \( Z \to X \) is affine, and by Corollary 4.3.2(iii), \( Y \to X \) is a closed immersion. It remains to show that \( X = \coprod \mathcal{P} \) in the category of locally ringed spaces. The latter can be checked Zariski locally on \( X \) by Theorem 4.6.6. Thus, we may assume that \( \mathcal{P} \) and \( X \) are affine, and the assertion follows from Theorem 4.4.12 and [Fer, Theorem 5.1].

In [TT1] we will need the following result providing a criterion for a composition to be schematic at a point.

Lemma 4.6.19. Let \( \mathcal{P} = (T; Y, Z) \) be an effective Ferrand’s pushout datum, and \( X = \coprod \mathcal{P} \). Assume that \( T \to Y \) is a pro-open immersion, and \( Y \) is schematic at a point \( y \in Y \) that possesses a unique generalization \( t \in T \). Further, assume that \( t \in T \) is isolated, and \( Z \) is schematic at \( t \). Then \( X \) is schematic at \( y \in Y \to X \), and the localization \( X_y \) is isomorphic to the Ferrand’s pushout \( Y_y \coprod_{T_y} Z_t \).

Proof. Since \( t \in T \) is isolated and \( T \to Y \) is a pro-open immersion, there exist fundamental families of schematic neighborhoods \( y \in Y_\alpha \subseteq Y \) and \( t \in Z_\alpha \subseteq Z \) such that \( Y_\alpha \cap T = Z_\alpha \cap T = \{ t \} \). Thus, the natural premorphisms \( (T_t; Y_\alpha, Z_\alpha) \to \mathcal{P} \) are morphisms, and moreover, open immersions. Hence \( y \in X \) is a schematic point, and \( (T_t; Y_y, Z_t) = \text{lim}(T_t; Y_\alpha, Z_\alpha) \to \mathcal{P} \) is a pro-open immersion. Then so is \( Y_y \coprod_{T_y} Z_t \to X \) by Theorem 4.6.14(ii). It remains to observe that on the level of sets \( |Y_y \coprod_{T_y} Z_t| = |Y_y| \cup |Z_t| = |X_y| \).

\[ \square \]
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