One-dimensional extended Hubbard model with spin-triplet pairing ground states

Akinori Tanaka

Department of General Education, National Institute of Technology, Ariake College, Omuta, Fukuoka 836-8585, Japan

E-mail: akinori@ariake-nct.ac.jp

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Abstract
We show that the one-dimensional extended Hubbard model has saturated ferromagnetic ground states with the spin-triplet electron pair condensation in a certain range of parameters. The ground state wave functions with fixed electron numbers are explicitly obtained. We also construct two ground states in which both the spin-rotation and the gauge symmetries are broken, and show that these states are transferred from one to the other by applying the edge operators. The edge operators are reduced to the Majorana fermions in a special case. These symmetry breaking ground states are shown to be stabilized by a superconducting mean field Hamiltonian which is related to the Kitaev chain with the charge–charge interaction.

Keywords: extended Hubbard model, spin-triplet pair, ferromagnetism, exact ground state, Majorana edge state

1. Introduction

The extended Hubbard model has been studied extensively to understand phenomena such as charge density wave, spin density wave and unconventional superconductivity which can not be described by the Hubbard model consisting of the electron hopping term and the on-site interaction term [1–3]. The Hamiltonian of the model is obtained by adding interaction terms of electrons on different sites to the Hubbard Hamiltonian. In the case where the added interaction together with the on-site one is dominant and is known to induce a certain ordering state with an energy gap, the model is well understood by considering the electron hopping as a perturbation. On the other hand, in order to understand phenomena which do not arise directly from interactions, we have to face the difficult problem of analyzing the interplay between the electron hopping and some interactions in a convincing way. The unconventional superconductivity corresponds to such a case.
Here we restrict ourselves to the one-dimensional extended Hubbard model with nearest neighbor interactions. Despite the difficulty in analyzing correlated electron systems, there are a few rigorous results associated with superconductivity in this case. Most of these results are obtained through the Bethe ansatz method, and the superconducting ground states so far obtained are related to spin-singlet electron pair condensation [4–6]. In this paper we provide another rigorous result for the model. By using a similar method in [7], we will show that the model exhibits saturated ferromagnetic, spin-triplet electron pair condensation in the ground state over a certain range of interaction parameters.

It is worth noting that in the last decade the Majorana edge state formed on a spinless superconducting wire has attracted much interest both theoretically and experimentally [8–14]. Our model exhibits saturated ferromagnetism where the electrons behave as spinless fermions. We show that a similar edge state is formed in the gauge symmetry breaking ground state of our model.

This paper is organized as follows. In section 2, we give the definition and state the main result. Section 3 is devoted to the proof of the main result. In section 4, we introduce two ground states with broken spin-rotation and gauge symmetries, and show that these ground states exhibit similar properties to those of the spinless superconducting wire in the topological phase. In section 5, we consider mean fields which stabilize the ground states introduced in the previous section. In section 6, we investigate the properties of the ground state with the fixed number of electrons. In section 7, we extend the model to the case of the anisotropic spin–spin interaction. Finally, in section 8, we provide conclusions.

2. Definition of the model and the main result

We consider a one-dimensional array of \( L \) sites, which are labeled as \( 1, 2, \ldots, L \). We write \( \Lambda \) for the set of numbers \( 1, 2, \ldots, L \) and identify \( \Lambda \) with the array of \( L \) sites. We also write \( \bar{\Lambda} \) for \( \Lambda \setminus \{L\} \). In this paper \( L \) is assumed to be an odd integer with \( L \geq 3 \). This condition is adopted only for simplicity, and similar results for even \( L \) are obtained with minor changes.

Let \( c_{x,\sigma} (c_{x,\sigma}^\dagger) \) be the annihilation (creation) operator of an electron at site \( x \in \Lambda \) and with spin \( \sigma = \uparrow, \downarrow \). They satisfy the anticommutation relations

\[
\{ c_{x,\sigma}, c_{y,\tau} \} = \{ c_{x,\sigma}^\dagger, c_{y,\tau}^\dagger \} = 0
\]

and

\[
\{ c_{x,\sigma}, c_{y,\tau} \} = \delta_{x,y} \delta_{\sigma,\tau}
\]

for any sites \( x, y \) and any \( \sigma, \tau = \uparrow, \downarrow \). For each site \( x \), we define the number operators

\[
n_{x,\sigma} = c_{x,\sigma}^\dagger c_{x,\sigma} \quad \text{and} \quad n_x = n_{x,\uparrow} + n_{x,\downarrow},
\]

and the spin operators

\[
S^{(l)}_{x,\sigma,\tau} = \frac{1}{2} \sum_{\gamma,\delta} p^{(l)}_{\gamma,\delta} c_{x,\gamma}^\dagger \sigma c_{x,\delta} \]

with \( l = 1, 2, 3 \), where \( p^{(l)}_{\gamma,\delta} \) are the elements of the Pauli matrices

\[
p^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad p^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad p^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

For each nearest neighbor pair of sites \( x \) and \( x + 1 \), we define local Hamiltonian \( H_x \) by

\[
H_x = H_{i,x} + H_{i',x} + H_{v,x} + H_{j,x} + H_{k,x},
\]

where

\[
H_{i,x} = -t \sum_{\sigma = \uparrow, \downarrow} (c_{x,\sigma}^\dagger c_{x+1,\sigma} + c_{x+1,\sigma}^\dagger c_{x,\sigma}) - \mu_x n_x - \mu_{x+1} n_{x+1},
\]
The term $H_{t,x}$ represents electron hopping, and $H_{U,x}$, $H_{V,x}$, $H_{J,x}$ and $H_{X,x}$ represent electron–electron interactions, usually referred to as the on-site, the charge–charge, the spin–spin and the bond-charge interactions, respectively. In this paper, we assume $0 < 2t \leq V$ and define parameter $\delta$ ranging from 0 to $\pi/2$ by

$$\sin \delta = \frac{2t}{V}. \quad (10)$$

We then consider the Hamiltonian given by

$$H = \sum_{x \in \Lambda} H_x \quad (11)$$

on $\Lambda$ with open boundary conditions.

Before stating our main result, we have to introduce some more notations. Let us define $\tilde{a}$ operators by

$$\tilde{a}_{x,\sigma} = \begin{cases} \sin \delta \left( \sum_{y=1}^{x} w_y c_{y,\sigma} - \sum_{y=x+1}^{L} w_y c_{y,\sigma} \right) & \text{if } x \in \tilde{\Lambda}; \\ \sin \delta \sum_{y=1}^{L} w_y c_{y,\sigma} & \text{if } x = L, \end{cases} \quad (12)$$

where

$$w_x = \begin{cases} \sin(\delta/2) & \text{if } x \text{ is odd}; \\ \cos(\delta/2) & \text{otherwise}. \end{cases} \quad (13)$$

By using the $\tilde{a}$ operators, we define pair operators $\zeta_{\sigma,\tau}^\dagger$ with $\sigma, \tau = \uparrow, \downarrow$ by

$$\zeta_{\sigma,\tau}^\dagger = \sum_{x,y \in \Lambda} F_{x,y} \tilde{a}_{x,\sigma}^\dagger \tilde{a}_{y,\tau}, \quad (14)$$

where $F_{x,y}$ is given by

$$F_{x,y} = \begin{cases} -\frac{1}{2} \sin \delta & \text{if } y - x = 1; \\ -\frac{1}{2} \sin \delta & \text{if } x = 1, y = L; \\ \frac{1}{2} \sin \delta & \text{if } x = L, y = 1; \\ 0 & \text{otherwise}. \end{cases} \quad (15)$$

It is noted that $F_{x,y} = -F_{y,x}$.
We denote by $F_0$ the state with no electrons on $L$. The total number of electrons on $L$ is denoted by $N_e$. We assume $0 \leq N_e \leq L$ and define the number $N_p$ of electron pairs by

$$N_p = \begin{cases} \frac{N_e}{2} & \text{for even } N_e; \\ \frac{N_e - 1}{2} & \text{for odd } N_e. \end{cases}$$

(16)

With the values of the parameters given by

$$U_0 = V \sin^2 \delta = \frac{4t^2}{V},$$

(17)

$$J_0 = V (2 - \sin^2 \delta) = 2V - \frac{4t^2}{V},$$

(18)

$$X_0 = \frac{V}{2} \sin \delta \cos \delta = t \sqrt{1 - \left(\frac{2t}{V}\right)^2},$$

(19)

our main result is summarized as follows:

**Proposition 2.1.** Suppose that both $U > U_0$ and $J \geq J_0$ are satisfied. Then, the ground state energy of $H$ with

$$X_e = (-1)^{x+1}X_0$$

(20)

and

$$\mu_s = -\frac{V}{2} \{1 - (-1)^x \cos \delta\}$$

(21)

is zero for $0 \leq N_e \leq L$. For fixed $N_e$, the ground state is unique apart from the degeneracy due to the spin-rotation symmetry, and is given by

$$\Phi_G = \begin{cases} (\zeta_{L,1}^\dagger)^N \Phi_0 & \text{for even } N_e \\ \tilde{a}_{L,1}^\dagger (\zeta_{L,1}^\dagger)^N \Phi_0 & \text{for odd } N_e. \end{cases}$$

(22)

and its SU(2) rotations.

The parameters in the Hamiltonian $H$ must satisfy several conditions so that $\Phi_G$ can become the ground state of $H$. Here we briefly comment on the physical feasibility of these conditions. Firstly we need sufficiently large on-site repulsion and nearest neighbor ferromagnetic interaction. These are necessary mainly to stabilize the ferromagnetic state. (Note that the Hamiltonian $H$ is proved to exhibit metallic ferromagnetism for $J > 0$ in the limit $U \to \infty$ [15].) Ferromagnetic materials are expected to satisfy these conditions. As for the nearest neighbor charge–charge interaction, it must be attractive. Although the charge–charge interaction arising directly from the Coulomb interaction is repulsive, it may become an effective attractive interaction such as a phonon-mediated interaction. The strength $V$ of the charge–charge interaction also needs to be $2t \leq V$. This condition will hold in systems with narrow conduction bands. We furthermore need to fine-tune $\mu_s$ and $X_e$. Note, however, that the anisotropic spin–spin interaction removes the condition on $X_e$ (see proposition 7.1). Although it will be difficult to fine-tune $\mu_s$ and $X_e$, the ground state $\Phi_G$ or a
state which has a large overlap with $\Phi_G$ may be realized in one-dimensional ferromagnetic materials with narrow conduction bands if an effective attraction between electrons can be generated in the system. See also section 5 where we treat the superconducting paring field.

3. Proof

Proof of proposition 2.1. In the following, we assume that the conditions (20) and (21) are satisfied. We also assume that the electron number $N_e$ is fixed.

Firstly we shall show that the Hamiltonian $H$ can be expressed as a sum of positive semi-definite operators. We define $a$ operators by

$$a_{x,\sigma} = w_{x+1,\sigma} c_{x,\sigma} - w_x c_{x+1,\sigma}$$

(23)

for $x \in \bar{\Lambda}$ and

$$a_{L,\sigma} = w_2 c_{1,\sigma} + w_{L-1} c_{L,\sigma}.$$  

(24)

We also define $b$ operators by

$$b_{x,\sigma} = w_x c_{x+1,\sigma} + c_{x,\sigma}$$

(25)

for $x \in \bar{\Lambda}$ and

$$b_{L,\sigma} = -w_1 c_{1,\sigma} + w_L c_{L,\sigma}.$$ 

(26)

By using the $a$ operators and the $b$ operators we define

$$H_{0,x} = V (a_{x,y}^\dagger b_{x,1}^\dagger + a_{x,1} b_{x,y}^\dagger + b_{x,1}^\dagger a_{x,y}) (b_{x,y}^\dagger a_{x,1} + b_{x,1}^\dagger a_{x,y}).$$

(27)

It is noted that $H_{0,0}$ is positive semi-definite. Then, after a lengthy but straightforward calculation, one finds that $H_x$ is rewritten as

$$H_x = H_{0,x} + H_{U',x} + H_{J',x} + H_{W,x},$$

(28)

where $H_{U',x}$ and $H_{J',x}$ are, respectively, defined by (6) and (8) with $U$ and $J$ replaced by $U' = U - U_0$ and $J' = J - J_0$, and $H_{W,x}$ is defined by

$$H_{W,x} = W (c_{x,1}^\dagger c_{x,1} + c_{x+1,1}^\dagger c_{x+1,1} + c_{x,1}^\dagger c_{x+1,1})$$

(29)

with $W = U_0/2$. For $U \geq U_0$ and $J \geq J_0$, all the terms in the right-hand side of (28) are positive semi-definite. This proves that $H$ is the sum of the positive semi-definite operators for $U > U_0$, $J > J_0$. Therefore, a zero energy state of $H$, if it exists, is a ground state.

Let us next show that $\Phi_G$ is a zero energy state of all the terms in (28) for any $x \in \bar{\Lambda}$.

Note that the $\tilde{a}$ operators form a basis for fermion operators on $\Lambda$, since $[\tilde{a}_{x,\sigma}^\dagger, a_{y,\tau}] = \delta_{x,y}$ for $x, y \in \Lambda$ by our definition. So we expand $b_{x,\sigma}$ with $x \in \Lambda$ in terms of $\tilde{a}_{x,\sigma}$ as

$$b_{x,\sigma} = \sum_{y \in \Lambda} \{a_{y,\sigma}^\dagger, b_{x,\sigma}\} \tilde{a}_{y,\sigma}. $$

(30)
From this expression of the $b$ operators we obtain

$$
\begin{align*}
  a_{\sigma \bar{\sigma}} \left( \sum_{x \in \Lambda} F_{x, z} \tilde{a}_{y, 1} \tilde{a}_{z, 1} \right) = & \left( \sum_{x \in \Lambda} F_{x, z} a_{\sigma \bar{\sigma}} + \sum_{y \in \Lambda} F_{y, z} a_{\sigma \bar{\sigma}} a_{\pi \bar{\pi}} \right) \\
  = & -b_{\pi \bar{\pi}} + \left( \sum_{x \in \Lambda} F_{x, z} a_{\sigma \bar{\sigma}} a_{\pi \bar{\pi}} \right) \\
  = & -2b_{\pi \bar{\pi}} + \left( \sum_{x \in \Lambda} F_{x, z} a_{\sigma \bar{\sigma}} a_{\pi \bar{\pi}} \right)
\end{align*}
$$

(31)

Since $(b_{\pi \bar{\pi}})^2 = 0$, (31) implies that $\zeta_{\pi \bar{\pi}}$ commutes with $(b_{\pi \bar{\pi}} a_{\pi \bar{\pi}} + b_{\pi \bar{\pi}} a_{\pi \bar{\pi}})$ for $x \in \bar{\Lambda}$. The creation operator $a_{\sigma \bar{\sigma}}$ anticommutes with $a_{\pi \bar{\pi}}$, i.e., it also commutes with $(b_{\pi \bar{\pi}} a_{\pi \bar{\pi}} + b_{\pi \bar{\pi}} a_{\pi \bar{\pi}})$ for $x \in \bar{\Lambda}$. Therefore, we have $H_0 \Phi_G = 0$. This together with the fact that there is no creation operator with the $\pi$-spin in $\Phi_G$ leads to $H_0 \Phi_G = 0$ for any $x \in \bar{\Lambda}$. This proves that $\Phi_G$ is a zero energy state of $H$. From the $c$ operator representation of $\Phi_G$ (see appendix A), we find that the ground state is not the null state.

Finally we shall show the uniqueness of the zero energy state.

Let $M$ be the eigenvalue of the third component of the total spin. Since the Hamiltonian $H$ has the spin-rotation symmetry, it is convenient to decompose the Hilbert space $\mathcal{H}$ of states into the subspaces $\mathcal{H}_M$ each of which has the fixed eigenvalue $M$. Let $\Phi_M$ be a lowest-energy state in $\mathcal{H}_M$. Since the representative of $\Phi_G$ in $\mathcal{H}_M$ is also a zero energy state of $H$, the lowest energy in $\mathcal{H}_M$ is guaranteed to be zero. This implies that $\Phi_M$ must satisfy $H_0 \Phi_M = 0$ for $x \in \bar{\Lambda}$. In particular, for $U > U_0$, $c_{\pi \bar{\pi}}^\dagger \Phi_M$ must be zero for any $x \in \Lambda$. Now we represent $\Phi_M$ by using the $c$ operators. As mentioned above, since each site is forbidden to be doubly occupied by electrons in $\Phi_M$, it can be expanded in terms of normalized basis states in the form

$$
\left( \prod_{x \in \Lambda} c_{\pi \bar{\pi}}^\dagger \right) \Phi_0,
$$

(32)

where $\Lambda$ is a subset of $\Lambda$ with $|\Lambda| = N_c$, $\sigma_\pi = \uparrow \downarrow$, and $\sum_{\pi \in \Lambda} \sigma_\pi = M$. In the product, the $c$ operators are ordered in such a way that the site indexes $x$ increase from left to right.

Let us consider the matrix representation $\mathcal{H}$ of the Hamiltonian $H$ with respect to the basis states in the form (32). We assume that the basis states are ordered in an arbitrary manner and denote by $H_{ij}$ the matrix element corresponding to $i$th and $j$th basis states. Then one easily finds that any non-zero off-diagonal matrix element is $-t$ or $-J/2$, which is negative. It is also easy to see that for any $i$, $j$ there is a sequence $i_1, i_2, \ldots, i_k$ such that $H_{i_1} H_{i_2} \cdots H_{i_k} \neq 0$. Therefore it follows from the Perron–Frobenius theorem that the lowest energy state of $H$ is unique [15], which implies that the lowest energy state of $H$ in $\mathcal{H}_M$ is also unique and is given by the representative of $\Phi_G$ in $\mathcal{H}_M$. This completes the proof of proposition 2.1.

Before ending this section, we make a remark on the related exact results of the extended Hubbard model. In the above proof we have rewritten the Hamiltonian as a sum of the positive semi-definite operators, and then have shown that the ground state attains the lowest eigenvalue, zero, of these operators. This strategy was used in [16] to determine a parameter range for which the extended Hubbard model has the ferromagnetic ground states at half-filling (where the number of electrons is equal to that of sites). Our result corresponds to an extension of [16] to the case away from half-filling. Note, however, that our method for the
construction of the exact ground state away from half-filling is quite different from that at half-filling.

4. Ground states with broken spin-rotation and gauge symmetries

In this section, we assume that the parameters \( U, J, \chi, \) and \( \mu \) satisfy the conditions in proposition 2.1, and hence the ground states of \( H \) with the fixed electron number are given by (2) and its \( \text{SU}(2) \) rotations. Since the ground states are saturated ferromagnetic, we furthermore assume that the third component of the total spin is fixed to \( \mathcal{N}/2 \). In the following, since all the electrons are assumed to have the \( \uparrow \)-spin, we omit the spin indexes in the fermion operators for notational simplicity.

The spin-triplet electron pairing ground state of \( H \) is regarded as the pairing state of spinless fermions. The ground state of our model is thus expected to have some similar aspects to that of the Kitaev chain model in which there appears the Majorana edge state at the ends of the chain. We will show that it is the case.

Let us define the zero energy ground states with the broken gauge symmetry

\[
\Phi_{G,0} = \exp\left(-\frac{\eta}{2} e^{-i\theta} \zeta^\dagger\right) \Phi_0
\]

and

\[
\Phi_{G,1} = \sqrt{2\eta \sin\delta} \tilde{a}_x^\dagger \exp\left(-\frac{\eta}{2} e^{-i\theta} \zeta^\dagger\right) \Phi_0 = \sqrt{2\eta \sin\delta} \tilde{a}_x^\dagger \Phi_{G,0},
\]

where \( \eta \) is a positive parameter and \( \theta \) is a phase parameter (note that \( \zeta^\dagger = \zeta^\dagger \)). The state \( \Phi_{G,0}(\Phi_{G,1}) \) is a superposition of the zero energy states of \( H \) with even(odd) numbers of electrons. The states \( \Phi_{G,0} \) and \( \Phi_{G,1} \) have the different fermionic parities, and, as we shall see in the next section, these states are stabilized by superconducting pairing fields.

As usual, let us define the Majorana fermion operators

\[
\gamma_{A,x} = e^{i\beta} c_x + e^{-i\beta} c_x^\dagger,
\]

\[
\gamma_{B,x} = -i e^{i\beta} c_x + i e^{-i\beta} c_x^\dagger,
\]

which satisfy \( \gamma_{A,x}^\dagger = \gamma_{A,x} \) and \( \{\gamma_{\alpha,x}, \gamma_{\beta,y}\} = 2\delta_{\alpha,\beta}\delta_{x,y} \) for any \( \alpha, \beta \in \{A, B\} \) and \( x, y \in \Lambda \). By using \( \gamma_{A,1} \) and \( \gamma_{B,L} \), with \( \alpha = A, B \), we introduce new edge operators as

\[
\Gamma_1 = \frac{1}{2\eta \sin\delta} \{(w_1 + \eta w_1)\gamma_{A,1} + i(w_2 - \eta w_1)\gamma_{B,1}\},
\]

\[
\Gamma_2 = \frac{1}{2\eta \sin\delta} \{(w_1 + \eta w_1)\gamma_{B,L} - i(w_2 - \eta w_1)\gamma_{A,L}\}.
\]

(Recall that \( w_1 = \sin(\delta/2) \) and \( w_2 = \cos(\delta/2) \).) The edge operators \( \Gamma_1 \) and \( \Gamma_2 \) are rewritten as

\[
\Gamma_1 = \frac{1}{\eta \sin\delta} \left(\sqrt{\frac{2}{\eta \sin\delta}} \right) \left( w_2 e^{i\beta} c_1 + \eta w_1 e^{-i\beta} c_1^\dagger \right),
\]

\[
\Gamma_2 = i \frac{1}{\eta \sin\delta} \left(\sqrt{\frac{2}{\eta \sin\delta}} \right) \left( -w_2 e^{i\beta} c_L + \eta w_1 e^{-i\beta} c_L^\dagger \right).
\]
with the \( c \) operators. Then, we find that
\[
\Gamma_l \Phi_{G,0} = -i \Gamma_l \Phi_{G,1} = e^{-i \frac{\eta}{2} \xi^+} \Phi_{G,1}, \tag{41}
\]
\[
\Gamma_l \Phi_{G,1} = i \Gamma_l \Phi_{G,0} = e^{i \frac{\eta}{2} \xi} \Phi_{G,0}. \tag{42}
\]
Furthermore, from the above relations, we obtain
\[
-i \Gamma_l \Phi_{G,0} = \Phi_{G,0}, \tag{43}
\]
\[
-i \Gamma_l \Phi_{G,1} = -\Phi_{G,1}. \tag{44}
\]

The relations \((41)\) and \((42)\) are obtained as follows. For \( x \in \Lambda \) we have from \((31)\) that
\[
e^{i \frac{\eta}{2} \alpha} \left( e^{-i \eta \theta \xi} \right)^{n} \Phi_0 = e^{-i \frac{\eta}{2} b_y^\dagger n} \left( e^{-i \eta \theta \xi} \right)^{-1} \Phi_0, \tag{45}
\]
which yields
\[
e^{i \frac{\eta}{2} \alpha} \Phi_{G,0} = e^{-i \frac{\eta}{2} b_y^\dagger} \Phi_{G,0}. \tag{46}
\]
Here we used \( b_y^\dagger \left( \xi^+ \right)^{n} \Phi_0 = 0 \). By \((46)\), we also have
\[
e^{i \frac{\eta}{2} \alpha} \Phi_{G,1} = e^{i \frac{\eta}{2} a_x^\dagger (2\eta \sin \delta \tilde{a}_L^\dagger) \Phi_{G,0} = \eta e^{i \frac{\eta}{2} b_y^\dagger} \Phi_{G,1} + \delta_{L,L} \sqrt{2\eta \sin \delta} \ e^{i \frac{\eta}{2} \Phi_{G,0}}. \tag{47}
\]

By representing \((46)\) and \((47)\) with the \( c \) operators and setting \( x = L \), one finds
\[
\left\{ w_2 e^{i \frac{\eta}{2} (c_1^+ + c_L^+)} + \eta w_2 e^{-i \frac{\eta}{2} (c_1^+ - c_L^+)} \right\} \Phi_{G,L} = \delta_{L,1} \sqrt{2\eta \sin \delta} \ e^{i \frac{\eta}{2} \Phi_{G,0}} \tag{48}
\]
with \( l = 0, 1 \). On the other hand, \((46)\) and \((47)\) combined with
\[
\sum_{x \in \Lambda} b_x^\dagger = -w_1 c_1^+ + w_1 c_L^+ + 2 \sin \delta \tilde{a}_L^\dagger, \tag{49}
\]
\[
\sum_{x \in \Lambda} a_x = w_2 c_1 - w_2 c_L, \tag{50}
\]
which follow from the definition, yield
\[
\left\{ w_2 e^{i \frac{\eta}{2} (c_1^+ - c_L^+)} + \eta w_2 e^{-i \frac{\eta}{2} (c_1^+ + c_L^+)} \right\} \Phi_{G,L} = \delta_{L,0} \sqrt{2\eta \sin \delta} \ e^{-i \frac{\eta}{2} \Phi_{G,1}} \tag{51}
\]
with \( l = 0, 1 \). From \((48)\) and \((51)\) we obtain \((41)\), and \((42)\).

It is noted that in the case \( \eta = w_2 / w_1 = 1 / \tan(\delta/2) \) we have \( \Gamma_l = \gamma_{L,1} \) and \( \Gamma_l = \gamma_{L,L} \) which are the Majorana fermion operators. In this case, we can reconstruct the edge fermion operator by combining \( \gamma_{A,1} \) and \( \gamma_{B,L} \) as
\[
d_{\text{edge}} = \frac{1}{2} \ e^{-i \frac{\eta}{2} (\gamma_{A,1} + i \gamma_{B,L})}. \tag{52}
\]
The fermion operator \( d_{\text{edge}} \) satisfies \( \{ d_{\text{edge}}, d_{\text{edge}}^\dagger \} = \{ d_{\text{edge}}^\dagger, d_{\text{edge}} \} = 0 \) and \( \{ d_{\text{edge}}^\dagger, d_{\text{edge}} \} = 1 \).

From \((41)\) and \((42)\) we also have
\[
d_{\text{edge}} \Phi_{G,0} = \Phi_{G,1}, \tag{53}
\]
\[
d_{\text{edge}} \Phi_{G,1} = \Phi_{G,0}. \tag{54}
\]

\[\text{1 Since } b_y^\dagger \left( \xi^+ \right)^{n} \Phi_0 \text{ is a state with } L \text{ electrons, we have } b_y^\dagger \left( \xi^+ \right)^{n} \Phi_0 = C \Gamma_{L,L} a_y^\dagger \Phi_0 \text{ with a certain constant number } C. \text{ Note that } \{ b_y^\dagger, a_y \} = 0. \text{ Then, by applying } a_y \text{ on the both sides of this equation, we find that } C = 0.\]
The above relations yield \( n_{\text{edge}} \Phi_{G,1} = \Phi_{G,1} \) and \( n_{\text{edge}} \Phi_{G,0} = 0 \) with \( n_{\text{edge}} = d^{\dagger}_{\text{edge}} d_{\text{edge}} \), which imply that the Majorana edge state is formed at the ends of the chain.

5. Mean field Hamiltonian

In this section we consider external fields (or mean fields) which remove the ground state degeneracy and select \( \Phi_{G,0} \) and \( \Phi_{G,1} \) as the two ground states.

It is well known that the external magnetic field can remove the degeneracy due to the spin-rotation symmetry. So we assume that the system is in a magnetic field, and fix the third component of the total spin to \( N_2/2 \). (As in the previous section, the spin indexes are omitted in this and the next sections under this assumption.)

In order to remove the degeneracy due to the electron pair condensation, we shall consider the Hamiltonian which does not conserve the electron number. More precisely, we will introduce Hamiltonian \( H' \) of spinless fermions with a superconducting pairing field, and show that the ground states of \( H + H' \) are given by \( \Phi_{G,0} \) and \( \Phi_{G,1} \).

Let us define

\[
H' = \sum_{x \in \Lambda} H'_x,
\]

\[
H'_x = \frac{\Delta}{\eta} (e^{-i\delta} a^+_x - \eta e^{i\delta} a_x) (1 - (1 - \alpha)a^+_x a_x) (e^{i\delta} a_x - \eta e^{-i\delta} a^+_x),
\]

where \( \alpha \) and \( |\Delta| \) are non-negative parameters. As we will see below, \( \Delta = |\Delta| e^{i\theta} \) corresponds to the superconducting pairing field. Since \( \{a^+_x, a_x\} = 1 \) for \( x \in \Lambda \), we have

\[
H'_x = \frac{\Delta}{\eta} (e^{-i\delta} a^+_x - \eta e^{i\delta} a_x) (1 - (1 - \alpha)a^+_x a_x) (e^{i\delta} a_x - \eta e^{-i\delta} a^+_x),
\]

and hence \( H'_x \) is a positive semi-definite operator for \( \alpha \geq 0 \). From (46) and (47) we find that \( \Phi_{G,0} \) and \( \Phi_{G,1} \) are zero energy states of \( H'_x \) for \( x \in \Lambda \). Therefore \( \Phi_{G,0} \) and \( \Phi_{G,1} \) are ground states of \( H + H' \). It is easy to see that there is no other ground state. The Hamiltonian \( H' \) removes the ground state degeneracy of \( H \) and stabilizes the states \( \Phi_{G,0} \) and \( \Phi_{G,1} \).

After some lengthy but straightforward calculations, \( H' \) is rewritten as

\[
H' = -s \sum_{x \in \Lambda} (c^+_x c_{x+1} + c^+_x c_{x+1} c_x c_{x+1}) - \sum_{x \in \Lambda} (\nu_x c^+_x c_x + \nu_{x+1} c^+_x c_{x+1} c_x c_{x+1})
- V' \sum_{x \in \Lambda} c^+_x c_x c_{x+1} c_{x+1} + \sum_{x \in \Lambda} (\Delta c_x c_{x+1} + \Delta' c^+_x c_{x+1}) + \eta|\Delta|(L - 1)
\]

with

\[
s = \frac{|\Delta|}{2\eta} (1 + \alpha \eta^2) \sin \delta,
\]

\[
\nu_x = -\frac{|\Delta|}{2\eta} \left[ (1 + \alpha \eta^2 - 2 \eta^2) - (-1)^x (1 + \alpha \eta^2) \cos \delta \right],
\]

\[
V' = (\alpha - 1) \eta |\Delta|.
\]

From the above representation of \( H' \), one immediately realizes that \( \Delta \) corresponds to the superconducting pairing field, which may be induced from a nearby superconductor. This field term essentially removes the degeneracy. It is noted that, in the case where \( \delta = \pi/2 \), \( \eta = 1 \) and \( \alpha = 1 \), \( H' \) is reduced to the Hamiltonian of the Kitaev chain of the spinless
fermions in the topological phase. Thus our model can be also regarded as an extension of the Kitaev chain to the spinful system with the electron–electron interactions.

6. Electron number conserving case

In the previous two sections we considered the case where the number of electrons is not conserved. From the expressions (33) and (34) of the symmetry breaking ground states, one finds that the edge state is closely related to the zero energy mode corresponding to $a^\dagger_L$. Indeed, we have shown that the occupation of $a^\dagger_L$ by an electron is reflected as an eigenvalue of the number operator $n_{edge}$ of the edge fermion operator.

For the fixed electron number, the ground state $\Phi_G$ can not be the eigenstate of $n_{edge}$, since we have

$$n_{edge} = \frac{1}{2}(1 + i\gamma_{A,L})(62)$$

with

$$i\gamma_{A,L} = e^{i\phi}c_L + e^{-i\phi}c_L^\dagger + c_L^\dagger c_L.$$ (63)

Instead, we can expect that there is a difference between the expectation values of $n_{edge}$ for $\Phi_G$ with $N_e$ even and odd.

Let $\langle \cdots \rangle_0$ and $\langle \cdots \rangle_1$ be the expectation values $\langle \Phi_G, \cdots \Phi_G\rangle/\langle \Phi_G, \Phi_G\rangle$ for $\Phi_G$ with $N_e$ even and odd, respectively. We will estimate $\langle n_{edge}\rangle_0$ and $\langle n_{edge}\rangle_1$. Clearly, we have $\langle c_Lc_L\rangle = \langle c_L^\dagger c_L^\dagger\rangle = 0$ and $\langle c_L^\dagger c_L\rangle = \langle c_L^\dagger c_L\rangle$ with $l = 0, 1$. Let us consider $\langle c_L^\dagger c_L\rangle$. By using the $c$ operator representation of $\Phi_G$ (see appendix A), we obtain

$$\langle c_L^\dagger c_L\rangle = (-1)^{l+1}\sin^2\left(\frac{\delta}{2}\right)\sum_{A:|A|=N_e-1}^{A:|A|=N_e-1} \chi[1, L \not\in A]W_A,$$ (64)

where $W_A = \prod_{x \in A} w_x^2$, and $\chi[E]$ takes the value 1 if $E$ is true and 0 otherwise. Since we have

$$\sum\limits_{A:|A|=N_e} W_A \leq \frac{L(L-1)}{N_e(L-N_e)}\cos^2\left(\frac{\delta}{2}\right)\sum\limits_{A:|A|=N_e-1} \chi[1, L \not\in A]W_A,$$ (65)

(see appendix B), $\langle c_L^\dagger c_L\rangle$ is bounded from below as

$$\langle c_L^\dagger c_L\rangle \geq \tan^2\left(\frac{\delta}{2}\right)\rho(1 - \rho)$$ (66)

with $\rho = N_e/L$. Therefore, we obtain

$$\langle n_{edge}\rangle_0 \leq \frac{1}{2} - \tan^2\left(\frac{\delta}{2}\right)\rho(1 - \rho),$$ (67)

$$\langle n_{edge}\rangle_1 \geq \frac{1}{2} + \tan^2\left(\frac{\delta}{2}\right)\rho(1 - \rho).$$ (68)

The inequalities obtained above show that the ground state expectation value of the occupation number $n_{edge}$ corresponding to the edge fermion reconstructed by the Majorana fermions depends on the fermionic parity, regardless of the chain length $L$. Let $N_e$ be even. Since we have $\rho = N_e/L \approx (N_e + 1)/L \approx (N_e - 1)/L$ for sufficiently large $L$, these inequalities imply that the expectation value of $n_{edge}$ for $N_e$ decreases by at least
2\tan^2(\delta/2)\rho(1 - \rho) compared with that for \(N_\text{e} - 1\), while the expectation value of \(n_\text{edge}\) for \(N_\text{e} + 1\) increases by at least 2\tan^2(\delta/2)\rho(1 - \rho) compared with that for \(N_\text{e}\). This behavior in the edge fermion number indicates the formation of an edge state in the electron number conserving setting. In the following, we propose a concrete example of a system having the two-fold degenerate ground states each of which is characterized by a zero energy mode related to the Majorana edge state. Very recently, a similar model has been investigated in [17] and [18].

Firstly we prepare a copy of \(H\). The operators in the copied system are denoted by the underline as \(\underline{\xi}\). We then consider the Hamiltonian \(H = H + H_\text{e} + H_\text{f}\) on the two chains, where

\[
H_\text{e} = \epsilon \left\{ \sum_{\sigma = \uparrow, \downarrow} (a_\downarrow^{\dagger} b_{\downarrow,\sigma} + a_{\uparrow,\sigma}^{\dagger} b_{\uparrow,\sigma}) \right\} \left\{ \sum_{\sigma = \uparrow, \downarrow} (b_\downarrow^{\dagger} a_{\downarrow,\sigma} + b_{\uparrow,\sigma}^{\dagger} a_{\uparrow,\sigma}) \right\}
\]

with \(\epsilon > 0\) is an interchain interaction. The number of electrons on the whole system is fixed to \(N_\text{e}\). We suppose that the values of the parameters in \(H\) and \(H_\text{e}\) are taken so that each Hamiltonian is positive semi-definite and has the zero energy ground states (see proposition 2.1). Under the assumption that the system is in a magnetic field, one finds that the two states

\[
\Phi'_{G,0} = \hat{a}_{\downarrow,\uparrow}^\dagger (\xi^\dagger + \xi) N_\text{e} \Phi_0, \quad \Phi'_{G,1} = \hat{a}_{\uparrow,\downarrow}^\dagger (\xi^\dagger + \xi) N_\text{e} \Phi_0
\]

for odd \(N_\text{e}\), and

\[
\Phi'_{G,0} = (\xi^\dagger + \xi) N_\text{e} \Phi_0, \quad \Phi'_{G,1} = \hat{a}_{\downarrow,\uparrow}^\dagger \hat{a}_{\uparrow,\downarrow}^\dagger (\xi^\dagger + \xi) N_\text{e}^{-1} \Phi_0
\]

for even \(N_\text{e}\) are the only ground states of this system. In fact, \(H, H_\text{e}\) and \(H_\text{f}\) are positive semi-definite, and \(\Phi'_{G,0}\) and \(\Phi'_{G,1}\) are the only zero energy states for these Hamiltonians. It is expected that similar inequalities corresponding to (67) and (68) hold for the ground state expectation values of the number operators \(n_\text{edge}\) and \(n_\text{edge}\) of the edge fermions on the chains, although explicit analytical expressions are difficult to obtain.

We end this section with the remark that the fermion operator defined by \(a_\sigma = \sum_{\xi \in L} (-1)^{x_\xi + 1} a_\sigma\) plays an interesting role in manipulating the zero energy mode in the condensate. More precisely, \(a_\sigma\) satisfies \(a_\sigma^\dagger \xi = \xi a_\sigma\) since \(\sum_{\xi \in L} (-1)^{x_\xi + 1} b_\xi = 0\) and \(\{a_\downarrow^\dagger, a_\downarrow\} = \{a_\uparrow^\dagger, a_\uparrow\} = 1\). Therefore, we have the relations \((\sqrt{2} \eta \sin \delta)^{-1} a_\sigma \Phi_{G,1} = \Phi_{G,0}\), \(\hat{a}_{\downarrow,\uparrow}^\dagger a_\sigma \Phi_{G,0} = \Phi_{G,1}\) and \(\hat{a}_{\uparrow,\downarrow}^\dagger a_\sigma \Phi_{G,0} = \Phi_{G,1}\) for the symmetry breaking ground states. Similar relations are also found for the electron number conserving system. Namely, we have \(\hat{a}_{\downarrow,\uparrow}^\dagger a_\sigma \Phi_{G,0} = \Phi'_{G,0}\), \(\hat{a}_{\uparrow,\downarrow}^\dagger a_\sigma \Phi_{G,0} = \Phi'_{G,1}\) and \(\hat{a}_{\downarrow,\uparrow}^\dagger a_\sigma \Phi_{G,0} = \Phi'_{G,1}\) for odd \(N_\text{e}\), and \(\hat{a}_{\downarrow,\uparrow}^\dagger a_\sigma \Phi_{G,0} = \Phi'_G,1\) and \(\hat{a}_{\uparrow,\downarrow}^\dagger a_\sigma \Phi_{G,0} = \Phi'_G,1\) for even \(N_\text{e}\).

7. Spin–spin interaction with Ising-like anisotropy

In this section, we treat the case of the spin–spin interaction with an Ising-like anisotropy.

Let us define

\[
(S_x \cdot S_{x+1})_j = S^{(1)}_x S^{(1)}_{x+1} + \beta (S^{(1)}_x S^{(2)}_{x+1} + S^{(2)}_x S^{(1)}_{x+1}),
\]

where \(\beta\) is a non-negative parameter and denote by \(H_{J,\beta,s}\) the Hamiltonian obtained by replacing \(S_\cdot S_{s+1}\) with \((S_\cdot S_{s+1})_j\) in \(H_{s,s}\). Then we consider the Hamiltonian

\[
H_{J,\beta} = \sum_{s \in L} H_{J,\beta,s}.
\]
Note that the bond-charge interaction $H_{X,x}$ is omitted in $H_{\beta,x}$. For $H_{\beta}$, we have the following result:

**Proposition 7.1.** Suppose that both $U > U_0 + 2X_0$ and $J > J_0 + 4X_0$ are satisfied. We furthermore suppose that $\mu_\epsilon$ is given by (21). Then, the ground state energy of $H_{\beta}$ with

$$J_0 \leq \beta < 1 - \frac{4X_0}{J},$$

is zero. For fixed $N_\epsilon$, the ground state is two-fold degenerate and is given by

$$\Phi_G = \begin{cases} (\xi_{\sigma}^\dagger \phi_0)^{N_\epsilon}_\epsilon & \text{for even } N_\epsilon \\ \delta^{\dagger N_\epsilon}_\epsilon (\xi_{\sigma}^\dagger \phi_0)^{N_\epsilon}_\epsilon & \text{for odd } N_\epsilon \end{cases}$$

(76)

with $\sigma = \uparrow, \downarrow$.

The outline of the proof is as follows. As in the isotropic spin–spin interaction case, we rewrite $H_{\beta,x}$ as

$$H_{\beta,x} = H_{0,x} + H_{U^\epsilon,\beta^\epsilon,x} + H_{X_0,x} + H_{W,x},$$

(77)

where $H_{X_0,x}$ is given by

$$H_{X_0,x} = X_0 \sum_{\sigma = \uparrow, \downarrow} \{ c_{x,\sigma}^\dagger + (-1)^x c_{x+1,\sigma}^\dagger \} n_{x,-\sigma} \{ c_{x,\sigma} + (-1)^x c_{x+1,\sigma} \}$$

$$+ X_0 \sum_{\sigma = \uparrow, \downarrow} \{ c_{x,\sigma}^\dagger - (-1)^x c_{x+1,\sigma}^\dagger \} n_{x+1,-\sigma} \{ c_{x,\sigma} - (-1)^x c_{x+1,\sigma} \},$$

(78)

$H_{U^\epsilon,\beta^\epsilon,x}$ is obtained by replacing $U$ with $U^\epsilon = U - U_0 - 2X_0$ in $H_{U,x}$, and $H_{X_0,x}$ is obtained by replacing $J$ and $\beta$ with $J^\epsilon = J - J_0 - 4X_0$ and $\beta^\epsilon = (J\beta - J_0)/(J - J_0 - 4X_0)$, respectively, in $H_{\beta,x}$. When $U^\epsilon > 0$, $J^\epsilon > 0$ and $0 \leq \beta^\epsilon < 1$, all the terms in (77) are positive semi-definite and $\Phi_G$ in (76) is their zero energy state. The fact that there is no other zero energy state follows from the application of the Perron–Frobenius theorem.

In the case of the isotropic spin–spin interaction, the bond-charge interaction whose strength parameter is fixed must be included in the Hamiltonian to obtain the exact ground states. On the other hand, the Hamiltonian with the anisotropic spin–spin interaction has the exact ground states even if the bond-charge interaction is absent. Although the on-site potentials still have to be adjusted to certain values, the model with the anisotropic spin–spin interaction exhibits the spin-triplet electron pair condensation over the wide range of parameters.

**8. Conclusion**

We have introduced the one-dimensional extended Hubbard model whose ground state simultaneously exhibits saturated ferromagnetism and spin-triplet electron pair condensation under certain conditions. Recently, the extended Hubbard chain with charge–charge and spin–spin interactions at low filling was studied by means of mean field and numerical methods in [19]. The results showed that the ground state is in the spin-triplet pairing phase for strong ferromagnetic coupling, even if there are no fine-tuned bond-charge interactions.
and on-site potentials which are necessary to get our exact results. These results together with ours indicate that the model exhibits spin-triplet pairing over a wide range of parameters. We have constructed two ground states in which both of the spin-rotation symmetry and the gauge symmetry are broken. It has been shown that these ground states are transferred from one to the other by applying the edge operators. The edge operators become the Majorana fermions in a certain case, and, in this sense, the Majorana state is formed on the edges of a chain in our model. We have introduced the mean field Hamiltonian with the pairing field which stabilizes the gauge symmetry breaking ground states. Here we remark that the spin-triplet pair condensation found in the ground state of $H$ is unstable against the thermal fluctuation since $H$ is constituted of short-range interactions and is defined on a chain. However we can expect that the spin-triplet pair condensate survives at non-zero temperatures in the strong pairing field. The mean field Hamiltonian has been shown to be regarded as the Kitaev chain with the nearest neighbor charge–charge interaction. It is noted that a similar spinless fermion model has been studied by Katsura, Schuricht, and Takahashi recently [20].

Our extended Hubbard model together with the mean field is an extension of the Kitaev chain to the spinful electron model. We have also estimated the expectation values of the edge fermion number operator for the ground states with fixed even and odd numbers of electrons, and found that there is the difference between them. Furthermore, we have proposed the model on the two chains in the electron number conserving setting and have shown that the model has the two-fold degenerate ground states which are characterized by the zero modes on the chains.

To conclude, it is interesting to note that Nadj-Perge et al reported the observation of Majorana fermions in a chain of Fe atoms, which intrinsically have ferromagnetic nature, on a superconducting Pb substrate [13, 14]. It is also noted that the recent developments in the field of cold atoms open a route to the experimental realization of one-dimensional interacting fermion systems [21]. We hope that our results stimulate these fields.

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Appendix A. $c$ operator representation of $\Phi_G$

Here we express $\Phi_G$ in terms of the $c$ operators. For the notational simplicity, we omit the spin indexes of the fermion operators.

From (15) and (14), one obtains

$$\zeta^\dagger = -\sin \delta \left( \sum_{x \in \mathbb{N}} a_x^\dagger a_{x+1}^\dagger + a_1^\dagger a_L^\dagger \right).$$

(A.1)

Substituting (12) into the right-hand side of (A.1), we have

$$\zeta^\dagger = -\frac{4}{\sin \delta} \sum_{x,y \in \mathbb{N}, x < y} (w_x w_y c_x^\dagger c_y^\dagger).$$

(A.2)

Then, taking into account the sign factor arising from the exchange of fermion operators, we have
\[(\zeta^+) \gamma \Phi_0 = \left( -\frac{4}{\sin \delta} \right)^N \gamma \left( N_p \right)^N \left( \sum_{A \subset \Lambda; |A| = N_p+1} \prod_{1 \leq i \leq N_p+1} (w_i c_i) \right) \Phi_0 \] (A.3)

and

\[\tilde{a}^i_\gamma (\zeta^+) \gamma \Phi_0 = \left( -\frac{4}{\sin \delta} \right)^N \gamma \left( N_p \right)^N \left( \sum_{A \subset \Lambda; |A| = N_p+1} \prod_{1 \leq i \leq N_p+1} (w_i c_i) \right) \Phi_0, \] (A.4)

where \(N_p! = N_p (N_p - 1) \cdots 1\), and \(|A|\) denotes the number of elements in a set \(A\).

**Appendix B. Proof of the inequality (65)**

Let us prove the inequality (65). Firstly we rewrite the left-hand side as

\[
\sum_{A \subset \Lambda; |A| = N_c} W_A = \frac{1}{N_c} \sum_{x \in A} w_x^2 \sum_{A \subset \Lambda; |A| = N_c-1} \chi [x \not\in A] W_A.
\] (B.1)

Then, by using \(w_L = \sin(\delta/2) \leq w_x \leq \cos(\delta/2)\) (recall that \(0 < \delta \leq \pi/2\)), we obtain

\[
\sum_{A \subset \Lambda; |A| = N_c} W_A \leq \frac{1}{N_c} \cos^2 \left( \frac{\delta}{2} \right) \sum_{x \in A} w_x^2 \sum_{A \subset \Lambda; |A| = N_c-1} \chi [x \not\in A] W_A
\]

\[
\leq \frac{L}{N_c} \cos^2 \left( \frac{\delta}{2} \right) \sum_{A \subset \Lambda; |A| = N_c-1} \chi [L \not\in A] W_A.
\] (B.2)

Here, note that \(\chi [L \not\in A] = \chi [1, L \not\in A] + \chi [1 \in A, L \not\in A]\). Since the sum related to \(\chi [1 \in A, L \not\in A]\) is bounded as

\[
\sum_{A \subset \Lambda; |A| = N_c-1} \chi [1 \in A, L \not\in A] W_A
\]

\[
= \sin^2 \left( \frac{\delta}{2} \right) \sum_{A \subset \Lambda; |A| = N_c-2} \chi [1, L \not\in A] W_A
\]

\[
= \sin^2 \left( \frac{\delta}{2} \right) \sum_{A \subset \Lambda; |A| = N_c-2} \chi [1, L \not\in A] W_A \sum_{x \in A} w_x^2 (L - N_c) \chi [x \not\in A \cup \{1, L\}]
\]

\[
\leq \frac{1}{L - N_c} \sum_{A \subset \Lambda; |A| = N_c-2} \chi [1, L \not\in A] \chi [x \not\in A \cup \{1, L\}] W_A w_x^2
\]

\[
= \frac{N_c - 1}{L - N_c} \sum_{A \subset \Lambda; |A| = N_c-1} \chi [1, L \not\in A] W_A,
\] (B.3)

we conclude that

\[
\sum_{A \subset \Lambda; |A| = N_c} W_A \leq \frac{L}{N_c} \left( 1 + \frac{N_c - 1}{L - N_c} \right) \cos^2 \left( \frac{\delta}{2} \right) \sum_{A \subset \Lambda; |A| = N_c-1} \chi [1, L \not\in A] W_A
\]

\[
= \frac{L}{N_c} \left( \frac{L - 1}{L - N_c} \right) \cos^2 \left( \frac{\delta}{2} \right) \sum_{A \subset \Lambda; |A| = N_c-1} \chi [1, L \not\in A] W_A.
\] (B.4)
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