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Liquidity backstops and dynamic debt runs\textsuperscript{☆}

Bin Wei\textsuperscript{a,}*, Vivian Z. Yue\textsuperscript{b}

\textsuperscript{a} Federal Reserve Bank of Atlanta, United States
\textsuperscript{b} Emory University, Federal Reserve Bank of Atlanta, and NBER, United States

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Liquidity backstops can mitigate runs. In this paper we develop a dynamic model of debt runs based on He and Xiong (2012) to identify, both conceptually and quantitatively, the value of a liquidity backstop for its run-mitigating role. For the purpose of identification, we focus on the municipal bond markets for variable rate demand obligations and auction rate securities. Based on the run episodes in these markets during the financial crisis of 2007-09 and the calibrated model, we find that the value of a liquidity backstop is about 14.5 basis points per annum. Our findings have important policy implications regarding the effectiveness of liquidity backstops in ameliorating problems of financial instability.

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1. Introduction

Liquidity may evaporate in periods of crisis when it is most needed. Liquidity backstops provide insurance against such liquidity risk and help stabilize financial markets by mitigating runs. There are both public and private liquidity backstops: deposit insurance and the discount window provided by the government, as well as liquidity commitments (e.g., credit lines) provided by banks.\textsuperscript{1} Much effort has been made to understand the important role of a liquidity backstop in mitigating runs since the seminal paper by Diamond and Dybvig (1983).\textsuperscript{2} However, it remains a challenge to quantify the value of a liquidity backstop for its run-mitigating role. In this paper we propose a way to identify such value both conceptually and quantitatively by using certain run episodes during the recent financial crisis as a laboratory.

To conceptualize the value of a liquidity backstop, consider two otherwise identical money-like bonds A and B issued by the same issuer. The issuer pays a fee to acquire a liquidity backstop for bond A from a liquidity provider who serves as the

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\textsuperscript{1} In this paper we use interchangeably liquidity backstops, liquidity insurance, liquidity commitments, and liquidity facilities.

\textsuperscript{2} See, Covitz et al. (2013) for example, for empirical evidence for the run-mitigating role of liquidity backstops. In particular, the authors show that runs on asset-backed commercial paper (ABCP) are negatively related to the strength of liquidity guarantees.

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"buyer of the last resort." By contrast, bond B has no such liquidity backstop. As a result, all else being equal, the market for bond B is more fragile, subject to a higher run probability. To equalize the run probabilities for both bonds, hypothetically Bond B has to pay a higher interest rate than Bond A. The hypothetical interest rate differential thus measures the value of a liquidity backstop.

It is worthwhile to point out that the value of a liquidity backstop—the focus of this paper—is very different from liquidity premia which typically refer to the extra compensation to investors for holding illiquid assets. One prominent example of liquidity premia is the spread between on-the-run and off-the-run Treasury securities (Krishnamurthy, 2002). In this case, on-the-run securities sell at a premium because they are more liquid relative to off-the-run securities. Moreover, the liquidity difference between these two types of Treasury securities always exists during calm or turbulent periods. By contrast, a liquidity backstop insures against liquidity shortage during market freezes. It enables an asset to be safe at crisis periods or crash-proof liquid (Moreira and Savov, 2017). Broadly speaking, the value of a liquidity backstop can be considered as “safety premium”.

To quantify the value of a liquidity backstop is challenging for the following reasons. First, it is difficult to find such a pair of “twin” securities that are otherwise identical but differ only by whether a liquidity backstop exists or not. Second, severe liquidity shortages such as runs, which necessitate the usage of a liquidity backstop, are rare. The rare occurrence of runs makes it difficult to assess run probabilities in order to identify the value of a liquidity backstop.

In this paper, we overcome the above challenges by using as a laboratory the runs during the financial crisis on the municipal bond markets for variable rate demand obligations (VRDOs) and auction rate securities (ARS). VRDOs and ARS are both municipal bonds with nominal long-term maturities and floating interest rates that are reset typically on a weekly basis. They are both money-like securities and close substitutes, but differ along one important dimension: VRDOs are typically structured with liquidity backstop facilities committed by banks serving as liquidity providers, but there are no such liquidity backstops in the ARS market.

Fig. 1 plots the average interest rates in these markets since May 2006. Before the financial crisis, market participants had mistakenly believed that ARS have the same liquidity backstops as VRDOs, as reflected by the almost identical average interest rates in both markets before 2007. The outbreak of the financial crisis shattered the misperception: the average interest rates in these markets started to diverge since November 2017.

3 For example, Austin (2008) states that “[o]ne survey found that about two thirds of corporate treasurers in firms that held auction-rate securities said that dealers had implied support for auction securities to avoid auction failures, and 17% of treasurers said that dealers had explicitly promised such support.” See the testimony of Linda Chatman Thomsen on September 18, 2008 as another example.
In early 2008 the ARS market suffered a run as banks started to cut uncommitted lending and let auctions fail en masse as a result of their subprime mortgage losses. The run on ARS is evident in Fig. 1 in the spike of its interest rate at around 6.6% in early 2008. By contrast, the existence of the liquidity backstops helped to stabilize the VRDO market and the VRDO rate was stabilized around 2% during that period. Later that year, in September 2008, the Lehman bankruptcy cast into doubt whether the liquidity providers would be able to honor their liquidity commitment. Consequently runs on both VRDO and ARS occurred following the Lehman bankruptcy and the average VRDO and ARS rates jumped in unison to around 8% on September 24, 2008. The next section provides more detail about these markets and their crisis experiences.

These run episodes suggest that whether or not a liquidity backstop exists can lead to dramatically different dynamics in otherwise almost identical markets. This is the basis for our identification of the value of a liquidity backstop in a spirit similar to the difference-in-differences approach. Specifically, in mid February to March 2008, the different experiences in these markets—only the ARS market was under run (not VRDO)—helps identify the probability that an uncommitted liquidity support would fail. Moreover, the different experiences in the VRDO market in 2008—it was under run in September 2008, but not so in early 2008—helps identify the probability that a committed liquidity support in the VRDO market would fail. The value of a liquidity backstop manifests itself in equalizing run probabilities in both markets, should the ARS market possess the same liquidity backstop as in the VRDO market.

To overcome the challenge of evaluating run probabilities in these markets, we develop a continuous time model of dynamic debt runs in the markets for ARS and VRDOs based on He and Xiong (2012, HX hereafter). Our model has two major departures from the HX model: (i) a floating interest rate, and (ii) modeling of committed versus uncommitted liquidity provision. The model is particularly useful. First, it accounts for the “dynamic” nature of the runs in these markets; that is, the fear of possible future runs props up fees for bank runs to run earlier on. Second, the model’s equilibrium is characterized by a unique “rollover threshold” such that creditors’ decision to run or not depends on whether the fundamental falls below the threshold. The distance between the fundamental value and the rollover threshold determines the run probability.

We calibrate the model to the historical interest rate data for both markets. Using calibrated key model parameters, we are able to infer the unobserved fundamental process as well as the model-implied rollover thresholds in both markets. Consistent with our identification assumption that ARS investors have started to recognize the lack of a liquidity backstop since the onset of the financial crisis, the calibrated ARS rollover threshold has since then jumped to a higher level than the VRDO threshold. Based on the calibrated parameter values and the rollover thresholds, we quantify the value of a liquidity backstop by measuring the increase in the ARS rate needed to equalize the rollover thresholds in both markets.

The value of a liquidity backstop is identified to be about 14.5 bps per annum. Interestingly, our estimate seems quite compatible with the FIDIC deposit insurance premiums that range from 1.5 to 40 bps. Furthermore, our notion of the value of a liquidity backstop is closely related to “all-in-spread-undrawn” (AISU) fees for credit lines that include fees paid on the entire (or unused) committed amount, which are about 12 to 21 bps in the data (see, e.g., Bord and Santos, 2014 and Berg et al., 2016). Our estimate of 14.5 basis point is largely in line with the fees on credit lines.\footnote{Feasibility on liquidity facilities for VRDO are, however, not publicly disclosed because municipal securities are exempt from federal securities registration and reporting requirements.}

Our study has several important policy implications. First, the estimation results in this paper shed light on the value of a public liquidity backstop, for instance, the Federal Reserve’s emergency lending facilities established during both the financial crisis and the current COVID–19 pandemic.\footnote{The implicit government guarantee to government-sponsored entities (GSEs) is another example. In the on-going GSE reform, the Treasury department recommends that the implicit government guarantee to the GSEs should be “explicitly defined, tailored, and paid-for.”} Second, the stabilizing role of liquidity backstops studied in this paper helps us better understand the fragility in the shadow banking system due to lack of government guarantees Gorton and Metrick (2010,2012). The value of a liquidity backstop, the key focus of this paper, speaks to the central difference between the shadow banking system and the traditional banking system and gives a direct measure of how shadowy the shadow banking is.

1.1. Literature review

Our paper contributes to the debt-run literature that examines the determinants of runs.\footnote{For empirical studies, please see Carey et al. (2009) and Coutiz, Liang, and Suarez (2012) for the run on ABCP, Gorton and Metrick (2012) for the run on repo, Kacperczyk and Schnabl (2013), Wermers (2012) for the run on money market mutual funds, and Shin (2009) on the run on Northern Rock, and Han and Li (2009) and McConnell and Saretto (2010) for the run on ARS. Foley-Fisher et al. (2015) provide empirical evidence for self-fulfilling expectations in driving the run on life insurers during the sumer of 2007.} Our model is built upon (He and Xiong, 2012), which extends the literature on static bank-run models (Diamond and Dybvig, 1983, Rochet and Vives, 2004, Goldstein and Paunzer, 2005, etc.). The He-Xiong model highlights the dynamic coordination problem between creditors whose contracts mature at different times — fear of future rollover risk could motivate each current maturing creditor to run ahead of others because a run can trigger a premature liquidation and lead to credit losses. In a closely related paper, Schroth et al. (2014) extend and apply the He-Xiong model to the ABCP market. The authors show that an endogenous “dilution risk”, arising from higher yields demanded by maturing creditors, increases the likelihood of runs. By contrast, in our model a run imposes an additional type of negative externalities, because an ARS run in our model may trigger uncommitted liquidity support to fail and its failure makes creditors unable to liquidate their bond holdings. This new channel
explains why the ARS market that lacks liquidity backstops was more susceptible to runs than the VRDO market. In addition, our different focus on empirical identification of the value of a liquidity backstop distinguishes our paper from theirs.

This paper also contributes to the literature on empirical identification of strategic complementarities in financial markets. As pointed out in Goldstein (2013), it is challenging to empirically identify strategic complementarities due to interactions between fundamentals and strategic behavior. Several recent papers (e.g., Chen et al., 2010, Hertzberg et al., 2011, Foley-Fisher et al., 2015, Schmidt et al., 2016) utilize different sources of variation in strategic complementarities as part of their identification strategies. Different from these papers, we exploit the ARS crisis of early 2008 as a unique natural experiment for identification and can additionally identify the value of a liquidity backstop from studying jointly ARS and VRDO markets. The run-prone behavior of ARS investors is similar to strategic complementarities in withdrawal behavior of uninsured depositors in Egan et al. (2017).

This paper is also related to the literature on the role of banks as liquidity providers. Kashyap et al. (2002) provide a convincing argument that banks have a natural advantage of acting as liquidity providers to provide liquidity on demand. The advantage stems from a synergy between deposit-taking and loan commitments to the extent that both types of activities require banks to hold large balances of liquid assets and are not too highly correlated. However, as Acharya and Mora (2015) argue, the recent financial crisis suggest that both sides of a bank’s balance sheet might be hit simultaneously. In this paper, we further investigate the destabilizing effects when banks as liquidity providers cut back on uncommitted lending (e.g., the wave of auction failures in the ARS market).

The remainder of this paper is structured as follows. In Section 2, we provide an overview of the VRDO and ARS markets and the turmoil in these markets during the financial crisis, and conceptualize the value of liquidity backstop. Section 3 presents the model and contains discussion of key model implications. In Section 4, we present our calibration procedure and results. Section 5 concludes. Most proofs are in the appendix at the end of this paper.

2. Overview of the markets for VRDOs and ARS

In this section, we first provide a description of VRDOs and ARS, and an overview of these markets, followed by a narrative on the disruptions in these markets in 2008 during the recent financial crisis. We also conceptualize the value of liquidity backstop in the end of this section.

2.1. Background

In this subsection we provide some background information on VRDOs and ARS.

**Auction Rate Securities.** ARS are long-term bonds and preferred stocks with interest rates that are periodically reset through a Dutch auction process at short-term intervals, usually 7, 28 or 35 days. We focus on municipal ARS in this paper. Following a successful auction, buyers purchase the bonds at par and receive the market clearing interest rate until the next interest reset date. ARS have nominally long-term maturities that usually range from 20 to 30 years. Nonetheless, the interest rate reset mechanism provides creditors with frequent opportunities to sell their holdings through auctions, and thus makes ARS priced and traded as short-term instruments.

ARS investors are typically high net worth individuals or corporations. At each auction, the auction agent accepts bids from existing bond holders or potential buyers. The auction agent then receives all the bids and submit his/her own order. The market-clearing interest rate is then determined, which is bounded from above by a pre-specified maximum interest rate, often shortened to “max rate” in Wall Street parlance. Fixed max rates are specified for all ARS, in a wide range of 9% to 25%.

An auction fails when there are not sufficient bids to clear the market at a rate less than the max rate. In the case of auction failure, the max rate is imposed, however, importantly, creditors are stuck with the bonds until the next successful auction. Until the ARS market froze in mid-February 2008, auction failures had been extremely rare — there were only 13 failed auctions between 1984 and 2006. However, as described shortly in the next subsection, after the financial crisis broke out, a tidal wave of auction failures hit the market.

**Variable Rate Demand Obligations.** VRDOs are very similar to ARS; they are also long-term floating-rate bonds with periodic interest rate resets. Unlike ARS, VRDO rates are reset periodically through “remarketing agents” so that the securities can be sold at par.

The key distinguishing characteristic of VRDOs is the existence of an explicit liquidity backstop, in the form of Letters of Credit (LOC) or Standby Bond Purchase Agreements (SBPA). VRDO creditors have a “tender” or “put” option that allows them to put the bonds at par value to the remarketing agent who then try to resell (remarket) the tendered bonds to new investors. To make the tender option feasible, VRDOs are usually structured with a liquidity facility provided by a third-party “liquidity

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7 See (Gatev and Strahan, 2006) and (Gatev et al., 2009) for evidence of a negative correlation between deposit withdrawals and commitment drawdowns in the commercial paper market.

8 Typical issuers of municipal ARS include municipalities, non-profit hospitals, utilities, housing finance agencies, student loan finance authorities and universities.

9 “Prolonged disruption of the auction rate market could have negative impact on some ratings,” Special Report, Moody’s Investors Service, February 20, 2008.
provider.” The liquidity provider, usually a large bank, acts as a buyer of last resort; it provides liquidity support by buying the bonds if the remarketing agent is unable to remarket them. In this case, the bonds become the so-called “bank bonds” showing up on the liquidity provider’s balance sheet. The existence of liquidity backstops makes VRDOs eligible for money market funds subject to SEC Rule 2a-7. In fact, the vast majority of VRDO investors are money market funds, which, however, cannot hold ARS.

2.2. The VRDO and ARS markets during the financial crisis

The VRDO and ARS markets are significant components of the $3.7 trillion municipal bond market, with sizes of about $200 billion and $500 billion in 2008 at their peak time, respectively. The markets were an attractive financing venue for municipal issuers because they allow for the issuance of long-term obligations using short-term interest rates that are typically lower than long-term interest rates. For investors, these securities were also attractive because they offered better returns than traditional money market investments. Both markets have existed since 1980s and had functioned well until the financial crisis broke out in 2007. In the aftermath of the financial crisis, the ARS market collapsed afterwards and there have been no new ARS issuances since 2008. Meanwhile, new issuance of VRDOs surged in 2008 as many existing ARS were converted into VRDOs. Fig. 2 below plots the annual amount of issuance in both markets since 1988, calculated using SDC platinum.

The ARS market encountered significant problems in early 2008. Since mid-2007, the disruption in the subprime mortgage market spread to the monoline insurance market where several major municipal bond insurers (e.g., Ambac and MBIA) were downgraded because of their exposure to subprime mortgage debt. These downgrades resulted in increased selling pressure in ARS. On the other hand, the subprime mortgage meltdown also significantly strained balance sheets of auction agents (e.g., Citibank, Goldman Sachs, Lehman Brothers, UBS, Royal Bank of Canada and JP Morgan) to the extent that they decided not to intervene and let the auctions fail in mid-February 2008. Reportedly, about 60% to 80% of auctions failed in the second half of February in 2008. The wave of auction failures drove up the ARS rate to as high as 6.6% around mid-February 2008 as shown in Fig. 1. The sheer volume of failed auctions and fear of future auction failures propelled more investors to run on ARS.

The run on ARS highlighted the implicitness of the liquidity provision in the ARS market: although in less tumultuous times prior to 2007, auction agents had almost always stepped in to buy some of these securities to help keep the market functioning, they had no contractual obligations to do so. During the financial crisis, major auction agents indeed chose...
to no longer be “buyers of last resort.” By contrast, the VRDO market was not affected as much in early 2008 due to the explicit structure of its liquidity facility.

However, later in 2008 the VRDO (as well as ARS) market experienced a run as a result of the bankruptcy of Lehman Brothers declared on September 15, 2008 and the subsequent panic in the market of money market mutual funds (e.g., runs on the Reserve Primary Fund that “broke the buck”, and other money market mutual funds). Investors worried about whether banking institutions that explicitly provided liquidity facility would be able to meet their obligations. The run on VRDO is evident in the spike of 7.96% of the average VRDO rate on September 24, 2008, as shown in Fig. 1.

2.3. Defining the value of a liquidity backstop

The runs on ARS and VRDO in 2008 allow us to define and quantify the value of a liquidity backstop. From our earlier discussion, there is a structural break in the beliefs of ARS investors. Before the wave of auction failures in 2008, ARS were believed to have the same explicit liquidity backstops as VRDOs, implying that, all else being equal, the run probability is the same in both markets. However, following massive auction failures ARS investors started to factor in the possibility of auction failures and are more likely to run.

To define the value of a liquidity backstop, let us consider the following thought experiment. An ARS issuer can pay a certain fee per annum to purchase a liquidity backstop from a liquidity provider, and effectively reduce the run probability to the same level as in the VRDO market. Alternatively, to achieve the same outcome the ARS issuer can raise the level of interest rate by a certain amount. From the perspective of the risk-neutral issuer, the two methods are equivalent as long as the fee to purchase a liquidity backstop is the same as the increase in the ARS interest rate, which thus measures the value of a liquidity backstop.

In the next section we develop a model for both markets to endogenously determine investors’ rollover decision and the run probabilities. The calibrated model is then used to quantify the value of a liquidity backstop.

3. Model

We extend the model of dynamic debt runs in He and Xiong (2012) and apply the extended models to the markets for VRDOs and ARS. Relative to He and Xiong (2012), the extended models allow for (1) a floating interest rate and (2) modeling of committed versus uncommitted liquidity provision. For ease of exposition, we first focus on the model for VRDOs that incorporates a floating interest rate into the He-Xiong model. We then extend the model further to the market for ARS. The model for ARS features uncommitted liquidity provision.11

3.1. The model for VRDOs: Committed liquidity provision

Consider a municipality who issues VRDOs to borrow $1 to finance a long-horizon project that generates cash flow at a constant rate $r$. At a random arrival time $\tau_\phi$ according to a Poisson process with intensity $\phi > 0$, the project is terminated with a final payoff:

$$dy_t = y_t(\mu dt + \sigma dZ_t).$$ (1)

where $\{Z_t\}$ is a standard Brownian motion. The project’s fundamental value under a discount rate $\rho$ is determined as follows:

$$F(y_t) = E_t \left[ \int_t^{\tau_\phi} e^{-\rho(s-t)} r ds + e^{-\rho(\tau_\phi-t)} y_{\tau_\phi} \right] = \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} y_t.$$ (2)

The discount rate $\rho$ equals the after-tax risk-free rate due to tax exemption, i.e., $\rho = r_f (1 - \tau)$, where $r_f$ denotes the taxable Treasury yield and $\tau$ the marginal tax rate. The discount rate $\rho$ is identical for all creditors.

VRDOs are issued to a continuum of risk-neutral creditors with measure one. For tractability, we assume that in the time interval $[t, t + dt]$ a creditor is chosen with probability $\delta dt$ to trade with the remarketing agent. If the creditor decides to keep holding the bond or if he decides to sell it to a new creditor via the remarketing agent, the bond is rolled over. Whoever holds the bond (i.e., the original or new creditor) will wait for the next time to be chosen to trade. In the meantime, the bondholder gets paid a floating interest rate $r_t$. If the creditor decides to sell the bond to the remarketing agent and the agent cannot remit it to another buyer, then a run occurs and the remarketing agent is obligated to provide liquidity support to buy the bonds at par.

We assume that a run on VRDOs has a deadweight loss in the form of a premature liquidation of the asset or the eventual bankruptcy of the issuer. Specifically, even though the remarketing agent is committed to provide liquidity upon a run, its liquidity provision may not be perfectly reliable: with probability $\theta \delta dt$, the committed liquidity support may fail, and, once it fails, the asset will be forced into premature liquidation, sold at a fraction $\alpha$ of its fundamental value. That is, the liquidation value is

$$L(y_t) = \alpha F(y_t) = \frac{\alpha r}{\rho + \phi} + \frac{\alpha \phi}{\rho + \phi - \mu} y_t = L + ly_t.$$ (3)

11 We use “floating-rate bonds” or simply “bonds” to refer to both VRDOs and ARS when describing the model setup that applies to both.
If the liquidation value is not enough to pay off all the creditors, a bankruptcy occurs. Therefore, a run in the future will expose creditors to possible bankruptcy losses.

As a result a coordination problem between current and future creditors arises in the model: current creditors are exposed to possible bankruptcy losses in the future if future creditors choose not to roll over their debt. This gives rise to a so-called rollover risk that would propel creditors to refuse to roll over the debt earlier on.

We now turn to the characterization of monotone equilibria in which creditors choose to roll over if and only if the fundamental is above a threshold. Consider an individual creditor who is making his rollover decision. Suppose all the other creditors choose a rollover threshold $y_{RDO}^*$ and the remarketing agent resets the interest rate by $r_{t} = R^{RDO}(y_{t}; y_{RDO}^*)$. Let $V^{RDO}(y_{t}; y_{RDO}^*)$ denote the creditor’s value function, which is given by

$$V^{RDO}(y_{t}; y_{RDO}^*) = E_t\left[\int_{t}^{\tau} e^{-\rho(t-s)} R^{RDO}(y_{s}; y_{RDO}^*) ds + e^{-\rho(\tau-t)} \min (1, y_{\tau}) 1_{\{\tau = \tau_{y}\}} \right. $$

$$\left. + e^{-\rho(\tau-t)} \min (1, L + ly_{\tau}) 1_{\{\tau = \tau_{L}\}} + e^{-\rho(\tau-t)} \max \left( V^{RDO}(y_{\tau}; y_{RDO}^*), 1 \right) 1_{\{\tau = \tau_{1}\}} \right],$$

(4)

where $\tau_{y}$ denotes the stopping time when the asset matures and the creditor gets a final payoff of $\min (1, y_{\tau})$, $\tau_{L}$ denotes the stopping time when the creditor gets the opportunity to decide whether to roll over the debt, $\tau_{1}$ denotes the stopping time when the project is forced to premature liquidation with payoff $\min (1, L + ly_{\tau})$, and finally the stopping time $\tau = \min (\tau_{y}, \tau_{L}, \tau_{1})$ is the earliest time among these three stopping times. By the same argument as in He and Xiong (2012), whether or not the creditor decides to roll over depends on whether or not the continuation value $V^{RDO}(y_{\tau}; y_{RDO}^*)$ exceeds the one-dollar par value.

The Hamilton-Jacobi-Bellman (HJB) equation is given below:

$$\rho V^{RDO}(y_{t}; y_{RDO}^*) = R^{RDO}(y_{t}; y_{RDO}^*) + \mu y_{t} V_{y}^{RDO}(y_{t}; y_{RDO}^*) + \frac{\sigma^2}{2} V_{y}^{RDO}(y_{t}; y_{RDO}^*) + \phi \left[ \min (1, y_{t}) - V^{RDO}(y_{t}; y_{RDO}^*) \right] + \theta \delta \left[ \min (1, L + ly_{t}) - V^{RDO}(y_{t}; y_{RDO}^*) \right]$$

$$+ \delta \left[ \max \left( 0, 1 - V^{RDO}(y_{t}; y_{RDO}^*) \right) \right].$$

It shows that the creditor’s required return on the left hand side, $\rho V^{RDO}(y_{t}; y_{RDO}^*)$, is equal to the expected increase in his continuation value as summarized by the terms on the right hand side.

Based on the HJB Eq. (5), it is straightforward to show that the value function is always equal to one if the floating interest rate schedule is chosen as follows

$$R^{RDO}(y_{t}; y_{RDO}^*) = \rho + \phi (1 - y_{t})^{+} + 1_{\{y_{t} \leq y_{RDO}^*\}} \theta \delta (1 - [L + ly_{t}])^{+},$$

(6)

where $(x)^{+}$ denotes $x$ if $x > 0$, or zero otherwise. The lowest possible interest rate, or the rate floor, is given by $\rho$. The interest rate schedule in Eq. (6) can be decomposed into three components: a risk-free component $\rho$, a component related to losses at maturity $\phi (1 - y_{t})^{+}$, and the last component associated with possible credit losses. Intuitively, the interest rate decreases with the fundamental $y_{t}$. When the fundamental deteriorates creditors are generally paid by a higher interest rate as compensation for possible losses at maturity or bankruptcy. The interest rate is adjusted so as to guarantee the value of debt is always equal to one. As a result, under this interest rate schedule creditors are indifferent between rolling over and running.

In reality, the interest rates of VRDOs are subject to a max rate or interest rate cap, denoted by $\bar{r}$. Imposing the interest rate cap $\bar{r}$ would leave creditors under-compensated when $\bar{r}$ falls short of the market rate. Therefore, as long as the interest rate cap is possibly binding, then creditors’ continuation value is strictly less than 1 and thus they always prefer to run (i.e., $y_{t}^{RDO} = \infty$). To avoid such a degenerate case and to keep tractability, throughout the rest of the paper, we add a new component $\Delta > 0$ to the unconstrained interest rate to increase the interest rate floor from $\rho$ to $\rho + \Delta$, implying the following interest rate schedule:

$$R^{RDO}(y_{t}; y_{RDO}^*) = \min (R^{RDO}(y_{t}; y_{RDO}^*) + \Delta, \bar{r}).$$

(7)

In a symmetric equilibrium, each creditor’s optimal threshold choice must coincide with other creditors’ threshold $y_{t}$. Thus the optimality condition is $V^{RDO}(y_{t}; y_{RDO}^*) = 1$. The threshold $y_{RDO}^*$ is defined as the minimum value at which $V^{RDO}(y_{t}; y_{RDO}^*) \geq 1$, i.e., $y_{RDO}^* = \min (y_{t}; V^{RDO}(y_{t}; y_{RDO}^*) \geq 1)$. When $y_{t}$ falls below the threshold $y_{RDO}^*$, due to monotonicity of the value function, the decision to run is strictly preferable since $V(y_{t}; y_{RDO}^*) < 1$ for $y_{t} < y_{RDO}^*$. Theorem 1 below proves the existence of a unique symmetric monotone equilibrium.

**Theorem 1.** Under some parameter restrictions, there exists a unique symmetric monotone equilibrium for VRDOs in which the rollover threshold $y_{RDO}^*$ is uniquely determined – each maturing creditor chooses to roll over his debt if $y_{t} > y_{RDO}^*$, and to run otherwise.

**Proof.** See Appendix C. □
We show in Proposition 1 below that the equilibrium rollover threshold $y^{VRDO}_r$ decreases with the maximum interest rate $\bar{r}$ or the liquidity premium $\Delta$, which highlights the role of the floating interest rate in driving a run.\footnote{The proofs of Propositions 1 and 2 can be found in the working paper version of this paper (see Wei and Yue, 2015).}

**Proposition 1.** The equilibrium rollover threshold $y^{VRDO}_r$ decreases with the maximum interest rate $\bar{r}$ or the liquidity premium $\Delta$.

The former result that $y^{VRDO}_r$ decreases with the maximum interest rate is consistent with empirical findings in McConnell and Saretto (2010). The intuition is straightforward: a higher maximum interest rate $\bar{r}$ allows the interest rate to increase further in a severely adverse environment; therefore, it increases the expected interest income for creditors and they will roll over more frequently. In the extreme case where the maximum interest rate $\bar{r}$ is sufficiently high, then the rollover threshold is zero (i.e., $y^{VRDO}_r = 0$), that is, the likelihood of runs is zero. The similar intuition explains the latter result that $y^{VRDO}_r$ decreases with the floor $\Delta$, which is very important when we define and measure the value of a liquidity backstop later.

### 3.2. The model for ARS: uncommitted liquidity provision

Unlike VRDOs that are structured with an explicit liquidity backstop committed by a liquidity provider, ARS have only uncommitted liquidity support. Without a liquidity commitment, the auction agent can choose not to participate in an auction when there are not enough buyers and simply let the auction fail. As a result, ARS creditors face an additional risk of auction failures.

To capture this layer of uncertainty in the ARS market due to the lack of a liquidity backstop, we assume that upon a run, with probability $\kappa \delta dt$, the auction agent will not step in to intervene in the market and the auctions will fail; with probability $1 - \kappa \delta dt$, the auction agent will intervene to keep the auctions functioning. For tractability, we further assume that once an auction fails, all the following auctions, including those for other creditors, continue to fail. In the event of successful auctions, the market-clearing interest rate $r_I$ prevails and premature liquidation occurs with probability $\theta \delta dt$. In the autarkic event of failed auctions, the max rate $\bar{r}$ is imposed and premature liquidation occurs with probability $\bar{r} \delta dt$.

Figs. 3A and 3B summarize the sequence of events in the models for VRDOs and ARS, respectively. All participants observe the fundamental $y_1$ and the max rate $\bar{r}$. At the beginning, the (remarketing or auction) agent announces and commits to an interest rate formula $r^ARS = R^ARS(y_1; y^ARS)$. In the case of VRDOs (Fig. 3A), at each time $\bar{r}$, a fraction $\delta dt$ of creditors decide whether to roll over their debt or to run. If they decide to roll over, the game continues to the next instant. If they decide to run, the liquidity facility is drawn upon to purchase the tendered bonds, but the facility may fail with probability $\theta \delta dt$. If it fails, the game ends and the project is liquidated to pay off all the creditors. If it succeeds, the game continues to
the next instant. The case of ARS (Fig. 3B) has a similar timeline as VRDOs, except that when the creditors decide to run, with probability $\kappa \delta dt$ the auction agent may decide not to intervene and then the auctions would continue to fail until the project fails eventually. This additional layer of uncertainty, highlighted by the flowchart within the dashed circle in Fig. 3B, captures the central distinction between VRDOs and ARS in terms of the existence of a liquidity backstop.

Let $U(y_t)$ denote the value function when auctions have continued to fail. Under the assumption that the auctions, once failed, would continue to fail, creditors’ rollover decision becomes irrelevant and thus the value function $U(y_t)$ does not depend on their rollover threshold $y_{ARS}$. In this autarkic scenario, the max rate $\tau$ is imposed until the asset matures at the stopping time $\tau_\phi$ or the project is prematurely liquidated at the stopping time $\tau_\Pi$. As a result, the value function $U(y_t)$ is given by

$$U(y_t) = E_t \left[ \int_t^{\tau_{\phi}} e^{-\rho(s-t)} RdS + e^{-\rho(\tau_{\phi}-t)} \min(1, y_t) \mathbb{1}_{[\tau_\phi \leq \tau_{\phi}]} + e^{-\rho(\tau_{\Pi}-t)} \min(1, L + \mu y_{\tau_{\Pi}}) \mathbb{1}_{[\tau_{\Pi} > \tau_{\phi}]} \right].$$

(8)

In Lemma 1 in the appendix, we derive the value function in closed form and prove that it is strictly monotonically increasing.

In contrast to the model for VRDOs, the HJB equation in the case of ARS has an additional term that reflects the loss from inability to unload bond holdings during auction failures:

$$
\rho V_{ARS}(y_t; y_{ARS}^*) = R_{ARS}(y_t; y_{ARS}^*) + \mu y_t V_{ARS}(y_t; y_{ARS}^*) + \frac{\sigma^2}{2} V_{yy}^{ARS}(y_t; y_{ARS}^*) + \phi \left[ \min(1, y_t) - V_{ARS}(y_t; y_{ARS}^*) \right] + \theta \delta \mathbb{1}_{[y_{ARS} < y_{ARS}]} \left[ \min(1, L + \mu y_t) - V_{ARS}(y_t; y_{ARS}^*) \right] + \frac{\delta}{\max \{ 0, 1 - V_{ARS}(y_t; y_{ARS}^*) \}} + \kappa \delta \mathbb{1}_{[y_{ARS} < y_{ARS}]} \left[ U(y_t) - V_{ARS}(y_t; y_{ARS}^*) \right].
$$

As a result, the unconstrained interest rate schedule has an additional term as well:

$$R_{ARS}(y_t; y_{ARS}^*) = \rho + \phi(1 - y_t)^+ + \theta \delta \left[ 1 - (1 - U(y_t)) \right]^{+} + \kappa \delta \left( 1 - U(y_t) \right).$$

(9)

We can also prove that under certain parameter restrictions, there exists a unique monotone equilibrium.

Next, we examine how the lack of a liquidity backstop in the ARS market affects equilibrium outcomes. To study the role of (lack of) a liquidity backstop in isolation, we assume that the interest rate is always fixed at $\tau$, the max rate, regardless of auction success or failure.

In Proposition 2 below, we prove that when the max rate is low enough, increasing $\kappa$ from zero to a positive value makes creditors more likely to run. Intuitively, a low enough max rate leads to a very low continuation value $U(y_t)$ in the event of failed auctions and thus, ex ante, creditors choose to run more often.

**Proposition 2.** If $\tau$ is sufficiently low, the equilibrium rollover threshold $y_{ARS}^*$ increases as the arrival intensity of auction failures $\kappa$ increases from zero.

**Proposition 2** illustrates how the lack of a liquidity backstop may exacerbate runs, which provides an explanation for the turmoil in the ARS market in early 2008 when investors started to factor in the possibility of auctions failures. As we show below, the destabilizing effect of the lack of liquidity backstops results from a new type of externality. The running decision of current creditors accelerates the issuer’s default probability and may also trigger auction failures. Therefore, their decision to run affects payoffs of future creditors. Table 1 summarizes the current and future creditors’ payoffs in different scenarios depending on whether the current creditors run or not.

![](https://example.com/table1.png)

Table 1: Run-Induced Externalities This table summarizes the payoffs of the current creditors and future creditors depending on the decision of the current creditors to run or rollover as well as the resulting liquidity event.

| Choice of current creditors | Run | Rollover |
|-----------------------------|-----|----------|
| Liquidity Provision | NO | YES | Failed | Survived | Failed | Survived |
| Probability | $\kappa \delta dt$ | $\theta \delta dt$ | $1 - \kappa \delta dt - \theta \delta dt$ | 1 | $U(y)$ | $U(y)$ | $V(y)$ | $V(y)$ |
| Payoff of current creditors | $U(y)$ | $U(y)$ | $1 - \kappa \delta dt - \theta \delta dt$ | 1 | $U(y)$ | $U(y)$ | $V(y)$ | $V(y)$ |
| Payoff of future creditors | $U(y)$ | $U(y)$ | $V(y)$ | $V(y)$ | $U(y)$ | $U(y)$ | $V(y)$ | $V(y)$ |

From Table 1, we can see that the current creditors will choose to run if and only if $1 \cdot (1 - \kappa \delta dt - \theta \delta dt) + L(y) \cdot \theta \delta dt + U(y) \cdot \kappa \delta dt > V(y) \cdot 1$, or $V(y) < 1$ after ignoring higher order terms. Furthermore, because of the lack of a committed liquidity facility, a run on ARS may lead to auction failure when the auction agent stops providing liquidity, which imposes an additional implicit cost on future maturing creditors. Specifically, a run by the current creditors reduce the future creditors’ value function by...
\[ V(y) - [V(y) \cdot (1 - \kappa \delta dt - \theta \delta dt) + L(y) \cdot \theta \delta dt + U(y) \cdot \kappa \delta dt] \]
\[ = [V(y) - L(y)]\theta \delta dt + [V(y) - U(y)]\kappa \delta dt. \]

Besides the implicit cost of default loss as studied in HX, a run in our model also induces an additional cost in the event of auction failure. This additional externality, absent in the VRDO market, makes the ARS market more susceptible to runs: in anticipation of possible auction failures and the associated losses as a result of runs by future creditors, the current creditors have less incentive to roll over their debt.

### 3.3. Quantifying the value of a liquidity backstop

We are now in position to quantify the value of a liquidity backstop. From earlier discussion in Section 2.3, because of the structural change in ARS investors’ beliefs, the perceived probability of auction failure switched from zero to \( \kappa > 0 \). Consequently, the estimated rollover threshold \( y^\text{ARS}_*(\Delta, \kappa) \) in the ARS market increases relative to \( y^\text{VRDO}_*(\Delta, 0) \) in the VRDO market (see Proposition 2).\(^\text{13}\) To reduce the ARS rollover threshold to the same level as \( y^\text{VRDO}_*(\Delta, 0) \), the ARS issuer can raise the level of interest rate by a constant amount \( \Gamma > 0 \) (i.e., increases \( \Delta \) to \( \Delta + \Gamma \)) such that

\[ y^\text{ARS}_*(\Delta + \Gamma, \kappa) = y^\text{VRDO}_*(\Delta, 0). \]

The value of a liquidity backstop is thus measured by \( \Gamma \). In the next section, we calibrate the model and quantitatively measure \( \Gamma \).

### 4. Quantitative analysis

The markets for VRDOs and ARS provide an ideal laboratory for us to identify the value of a liquidity backstop. The identification scheme hinges on the structural change in the belief of ARS investors following the wave of auction failures in mid-February 2008. We first describe the data and our empirical methodology, and then report calibration results.

#### 4.1. Data

The weekly data of 1-week tax-exempt VRDO and ARS rates are obtained directly from the Securities Industry and Financial Markets Association (SIFMA) website.\(^\text{14}\) The historical data for the VRDO rate is available for the period from May 22, 1991 to October 24, 2012, while the ARS historical rate is only available for a shorter period from May 31, 2006 to December 30, 2009. We also obtain the 1-week Treasury repo rate from Bloomberg for the same period as the VRDO sample period.

We obtain information about characteristics of VRDOs or ARS (e.g., the max rate) from the Municipal Securities Rulemaking Board (MSRB)’s SHORT database from its inception date of April 1, 2009 through November 8, 2012.\(^\text{15}\) The SHORT database provides a centralized source of information about municipal ARS and VRDOs that was previously unavailable. Starting from May 2011, MSRB rules require VRDO remarketing agents to report to the MSRB the aggregate amount of par value of bonds held by investors or remarketing agents. There are 20,547 distinct VRDOs in the SHORT database during our sample period. We focus on the VRDOs with weekly interest resets, which accounts for 90.7% of the whole sample (i.e., 18,630). The SHORT database does not contain maturity information. Therefore, we merge it with the Mergent Municipal Bond database to collect information on maturities.

#### 4.2. Calibration

There are eleven primitive parameters in the model: \( \bar{r}, r, \phi, \rho, \delta, \alpha, \mu, \sigma, \Delta, \theta, \kappa \). Using the SHORT database and the Mergent Municipal Bond database, we first calibrate the parameters \( \bar{r}, r, \phi, \rho, \delta, \alpha \). We then use the MLE method to estimate the remaining parameters.

The contractual maximum interest rate \( \bar{r} \) is calibrated to be 12% using the SHORT database. Among the 18,630 VRDOs with weekly interest resets in the SHORT database, 53.42% of them have the max rate of 12%, 26.13% of them have the max rate of 10%, and 10.37% of them have the max rate of 15%. The weighted average of these three rates is 11.76%. Therefore, we set \( \bar{r} \) as 12%. The cash flow rate from the project \( r \) is set to be equal to the average VRDO interest rate, or \( r = 2.39% \). That is, the municipality issuers have balanced budgets.

The average debt maturity of our merged VRDO sample from the SHORT and Mergent databases is 25.2 years (and the median is 25.96 years). We therefore set \( 1/\phi \), the expected asset maturity, to 25 based on the assumption that the average maturity coincides with the average asset maturity; that is, \( \phi = 0.04 \). The tax-adjusted risk-free rate \( \rho \) is set to the average

\(^\text{13}\) Note that we explicitly express the rollover threshold as a function of the arguments \( \Delta \) and \( \kappa \) (in the case of VRDOs, \( \kappa \) is always zero).

\(^\text{14}\) The website’s URL is [http://archives.sifma.org/swapdata.html](http://archives.sifma.org/swapdata.html).

\(^\text{15}\) The SHORT database has been built from the Short-term Obligation Rate Transparency (SHORT) System and the Real-Time Transaction Reporting System (RTRS), which the MSRB launched in early 2009 to collect and disseminate interest rates and important descriptive information about these ARS and VRDOs.
value of the tax-adjusted repo rate, or $\rho = 0.0195$, during the sample period between 1991 and 2012 using a tax rate of 40% following (Longstaff, 2011).

The parameter $\delta$ represents the arrival intensity of creditors who make the running decision. In the model, once a run occurs, the proportion of creditors who decide not to roll over the debt is $\delta \Delta t$, where we set $\Delta t = 7/365.25$ to reflect the weekly frequency of the interest rate reset for the constituent VRDOs/ARS in the SIFMA indexes. In reality, VRDO/ARS creditors come to the remarketing or auction agent to buy or sell the securities on the interest rate reset dates. A run is considered to occur if a significant number of creditors decide not to roll over the debt. As a result, we set $\delta = 12$, meaning that on average creditors make the running decision on a monthly basis, and upon a run, about $\delta \Delta t = 23\%$ of the securities are not rolled over. Furthermore, we set the recovery rate $\alpha = 50\%$.16 The calibration results are reported in Table 2 Panel A.

The rest of the parameters, $\mu$, $\sigma$, $\Delta$, $\theta$, $\kappa$, are estimated using the Maximum Likelihood Estimation (MLE) method. Furthermore, under one identification assumption that both VRDO and ARS markets share a common fundamental process, we are able to infer about the fundamental from one market and apply the inferred process to the other market. Therefore, instead of simulation-based methods (such as, simulated method of moments or simulated maximum likelihood estimation), we use the simpler MLE method to estimate these parameters. Details of our estimation methodology can be found in Appendix C.

We report the estimated values for parameters $\mu$, $\sigma$, $\Delta$, $\theta$, $\kappa$ in Panel B of Table 2. The default intensity $\theta$ is estimated to be 0.0111 so that the average time from a run to eventual bankruptcy is equal to $1/(\theta \delta) = 7$ years, which is roughly in line with the bankruptcy experience of Jefferson County, AL (Woodley, 2012). Furthermore, the parameter $\kappa$ is estimated to be 0.003, under which the fraction of auctions that have failed within the 14-week window between November 14, 2007 and February 20, 2008 is about 1%.17 Based on the formula $L = \frac{\sigma \theta}{\rho + \sigma}$ and $I = \frac{\rho \phi}{\rho + \phi - \mu}$ and the estimated parameter values, the parameters governing the recovery rate of the asset in the worst case scenario are calibrated as: $(L, I) = (20.2\%, 55.8\%)$. The estimated value of $\Delta$ equal to 1 bps accounts for about 0.4% of the average VRDO interest rate (i.e., 2.4%), or 0.3% of the average ARS interest rate (i.e., 3.02%) in the data.

4.3 Results

Model Fit. Now we turn to the goodness of fit of our model. First, we back out the fundamental process using the VRDO historical rate, which is plotted by solid line in Fig. 4 Panel A below. In plotting the figure, we focus on the period between May 2006 and December 2009 when both ARS and VRDO data are available. In Panel A, we also plot the VRDO rollover threshold (dashed line) and the ARS rollover threshold (dash-pointed line). The latter is plotted only after November 2007 when the structural change took place. The estimation results confirm that once a positive probability of auction failures is taken into account, ARS investors face a higher threshold and are more likely to run. Moreover, in February and March 2008, it is only the higher ARS rollover threshold that is crossed, not the VRDO threshold. This is consistent with the differential crisis experiences in these markets in early 2008 when there was a run in the ARS market, but not in the VRDO market. Moreover, in late 2008 following the Lehman’s bankruptcy, both rollover thresholds were crossed, indicating runs in both markets. This is consistent with the market commentary that Lehman’s bankruptcy put in doubt the ability of liquidity providers to honor their commitments.

In Panels B and C of Fig. 4, we plot the actual and model-implied excess interest rates in both markets. Because the actual VRDO excess interest rate is used to exactly fit the model-implied one in order to back out the fundamental process, these two series coincide with each other as shown in Panel B of Fig. 4. We then estimate the model using MLE to best fit the actual ARS excess interest rate. Panel C of Fig. 4 shows a reasonably good fit between the model and the data. In particular, consistent with the data our model is able to generate spikes in the ARS excess interest rate in both run episodes in 2008.

Rollover Thresholds. The equilibrium thresholds are reported in Table 2 Panel C. First, the estimation results confirm that with a positive probability of auction failures (i.e., $\kappa > 0$), the rollover threshold for ARS investors is indeed higher than that for VRDO investors: $y^\text{ARS} > y^\text{VRDO}$. This is consistent with the economic intuition discussed in Section 5.2 that the fear of getting stuck when future auctions fail propels ARS creditors more likely to run, ex ante, relative to VRDO creditors. The higher rollover threshold for ARS creditors reflects the lack of a liquidity backstop in the ARS market.

We also study the effect of the floating interest rate on the run behavior. To compare with the HX model which has a fixed interest rate, we consider the case with the interest rate fixed at $r$. The equilibrium rollover threshold in this case, labeled as $y^\text{Hx}$, turns out to be higher than the rollover thresholds in either VRDO or ARS markets (see Panel C of Table 2). This result suggests that floating interest rates tend to mitigate runs in both markets. Intuitively, as the issuer’s fundamental

16 The recovery rate of municipal bonds is not readily available given municipal bankruptcy is rare. See, for instance, Coval and Stafford (2007) for the estimates of the recovery rate for stocks; and Andrade and Kaplan (1998), Hennessy and Whited (2007), Ellul, Jotikashala, and Lundblad (2010), for the estimates of the recovery rate for corporate bonds.

17 By definition, following a run, a fraction $(1-\kappa dt)^n$ of auctions will fail in the first week, or $(1-\kappa dt)^n$ of auctions will survive the first week. Similarly, among the ARS whose auctions succeeded in the first week, a fraction of them, $(1-\kappa dt)^2$, will continue to survive in the second week, etc. The cumulative fraction of auctions that have failed with $N$ weeks equals $1-(1-\kappa dt)^N$. Plugging in $\kappa = 0.003$, $\delta = 12$, $dt = 7/365$ leads to a failure rate of 1% in a 14-week window.
deteriorates the interest rate increases to compensate investors for the higher default risk, and thus makes them more willing to roll over.

**Run Likelihood.** The calibration results also allow us to compute the likelihood of a run within the following week. Fig. 5 plots the run likelihood for both VRDOs (Panel A) and ARS (Panel B). From Panel A the model implies a 50% chance of a VRDO run within a week following the Lehman’s bankruptcy. As shown in Panel B the run probability increases to about 80% during the first ARS run and to about 100% during the second ARS run in 2008. Except these run episodes, the run probabilities are close to zero. In summary, our model is able to reproduce the differential crisis experiences for both markets.

**The Value of a Liquidity Backstop.** We estimate the value of a liquidity backstop (denoted by \(\Gamma\)) and the estimation result is reported in Table 2 Panel D. Recall that the value of a liquidity backstop \(\Gamma\) is defined as the interest rate increase needed so that the ARS rollover threshold can be reduced to the same level of the VRDO rollover threshold; that is, 

\[
\text{V}_{\text{ARS}}(\Delta + \Gamma, \kappa) = \text{V}_{\text{VRDO}}(\Delta, 0).
\]

We find that the value of a liquidity backstop is estimated to be about 14.5 bps. In present value terms (i.e., \(\frac{\Gamma}{1+\phi}\)), a liquidity backstop is evaluated to be about 2.4% of par value. Therefore, the implied value (or cost) of providing liquidity backstops for the ARS market is about $4.7 billion for the $200 billion ARS market at the peak level before its collapse.

Our estimated value of a liquidity backstop seems quite reasonable. First, it is compatible with the FDIC deposit insurance premiums that range from 1.5 to 40 bps. Second, its value is also similar to “all-in-spread-undrawn” (AISU) fees for bank-issued credit lines that include fees paid on the entire committed amount (Sufi, 2009). For example, Berg et al. (2016) find that AISU fees range from 12 to 21 bps for investment-grade firms using the DealScan data on the U.S. syndicated loan market from 1986 to 2011. Similarly, Bord and Santos (2014) reports that the average all-in undrawn fee on lines of credit is about 20 bps for the same market from 2005 to 2007. Our estimate of 14.5 basis point for the liquidity backstops is largely in line with the fees on credit lines as reported in these papers.

**Time Variation in the Value of a Liquidity Backstop.** The above estimate of a liquidity backstop’s value is based on the whole sample period. Presumably, how valuable a liquidity backstop may vary with market conditions. To investigate this issue, we estimate the parameters \(\kappa\) and \(\theta\) in a pseudo real time manner and then back out the implied value for a liquidity backstop. Specifically, keeping all other parameters fixed as in Table 2, we re-estimate parameters \((\kappa, \theta)\) by expanding the
Panel A: Estimated Run Probability for VRDOs

Panel B: Estimated Run Probability for ARS

Fig. 5. Model-Implied Run Probabilities This figure plots the model-implied run probabilities based on the calibration results during the period between May 2006 and December 2009, for the VRDO market in Panel A and for the ARS market in Panel B.

Table 2
Calibration and Estimation Results Panel A of this table reports calibrated values for some of the model’s primitive parameters. In Panel B, the estimated values for the other primitive parameters are reported. Standard errors, reported in parentheses, are constructed from Monte Carlo simulations. Each simulation begins by randomly generating 119 weekly data of the underlying fundamental process. In Panel C, we compute the equilibrium rollover threshold in our model as well as the one in the benchmark HX model. In Panel D, the estimated value of a liquidity backstop is reported.

| Panel A: Calibration | VRDO | ARS |
|----------------------|------|-----|
| max rate             | r    | 0.12 same |
| cash flow rate       | r    | 0.0239 same |
| avg. maturity        | 1/φ  | 25   same |
| tax-adj. riskless rate | ρ    | 0.0195 same |
| avg. duration        | 1/δ  | 1/12 same |
| recovery rate        | α    | 0.5   same |

Panel B: Estimation

| drift | μ   | 0.024 same |
|-------|-----|-----|
| volatility | σ  | 0.217 same |
| liquidity premium | Δ | 0.0001 same |
| default intensity | θ | 0.0111 same |
| auction failure intensity | κ | 0 0.0030 |
| pricing error | ν | 0.0107 same |

Panel C: Equilibrium Rollover Threshold (y∗)

| Eqm. threshold | y∗ | 0.403 |
|                |     | 0.569 |

Panel D: Value of a Liquidity Backstop (Γ)

| Value of a liq. backstop | Γ | 14.5 b.p. |
|---------------------------|---|---|

max rate, cash flow rate, avg. maturity, tax-adj. riskless rate, avg. duration, recovery rate, drift, volatility, liquidity premium, default intensity, auction failure intensity, pricing error, Eqm. threshold, Value of a liq. backstop.
estimation period. The estimation results are plotted in Fig. 6. The historical rates for both markets are plotted in Panel A. In Panel B (respectively, Panel C), We plot the estimated values for $\kappa$ and $\theta$ in Panel B and $\Gamma$ in Panel C.

Fig. 6 shows that the value of a liquidity backstop is estimated to range between 7 to 23 bps, with the maximum value achieved in early 2008 during the ARS crisis. The maximum value of a liquidity backstop value (i.e., 23 bps) is almost entirely attributable to the increase in the parameter $\kappa$. The estimate of $\theta$ is very stable, moving within a narrow range of 0.010 to 0.012. By contrast, the estimate of $\kappa$ sharply increases from almost zero to its maximum value of 0.0045 at the peak of the ARS crisis in early 2008, and stays elevated ever since.

Alternatively, we can interpret the above results as follows. We assume that investors believe the parameter $\kappa$ to have two possible values, 0 or $\kappa > 0$ and follows a degenerate Markov process. Conditional on the belief of $\kappa = 0$, the parameter $\kappa$ in the next period takes the value of $\kappa$ with probability $p$, or 0 with probability $1 - p$. Similarly, once the belief $\kappa$ is realized, it will continue to hold with probability $q$, or switch to 0 with probability $1 - q$. In the model we have essentially assumed $p = 0$ until mid November 2007 when the belief permanently switches to $\kappa$ (i.e., $q = 1$). Under this alternative interpretation, the estimated value of $\kappa$ in Panel B of Fig. 6 can be reinterpreted as the expected value of the above belief process, which allows us to infer the potential time-varying transition probabilities $p$ and $q$. Specifically, we set $\kappa = 0.0045$ since the parameter $\kappa$ takes the maximum value 0.0045 in early April 2008 (see Panel B). Our results suggest that until the end of January 2008 the probability $p$ is believed to be relatively low, around 37%. Since then the massive auction failures during February and March resulted in a sharp increase in the probability $p$ that reached 100% in early April 2008. Put differently, the beliefs of ARS investors dramatically changed amid auction failures. Afterwards, the belief of $\kappa$ continues to hold with a high probability $q$ above 70%. With this alternative interpretation, our results provide evidence for a dramatic change in beliefs during the ARS crisis in early 2008, which is consistent with a structural break in beliefs assumed in the model.

**Investor Base.** We now discuss the possible implications of the difference in investor base on our estimation. On the one hand, the vast majority of VRDO investors are money market funds. On the other hand, ARS are mainly held by high net worth individuals and corporations, and cannot be held by money market funds. These two different classes of investors may differ in terms of their preferences or perhaps even the level of sophistication. This may explain why VRDO rate were in general slightly higher than ARS rate prior to the crisis, with a difference of about 7.5 bps in our data. If we adjust for this difference and redo the estimation, we find a slightly higher value for a liquidity backstop at around 16.7 bps per annum. So taking into account the difference in investor base attaches a slightly higher value to a liquidity backstop.
5. Concluding remarks

In this paper, we develop a model of dynamic debt runs to study the important role of liquidity backstops in mitigating runs. We focus on the municipal bond markets for ARS and VRDOs, which provide an idea laboratory to identify the value of a liquidity backstop in terms of its run-mitigating role. As discussed in the paper, ARS were considered almost identical to VRDOs prior to the financial crisis, however, investors started to recognize the lack of a liquidity backstop in the ARS market at the onset of the crisis. The structural change in investors’ beliefs drove a wedge in the experiences of these two markets during the crisis: the liquidity-backstop-lacking ARS market was more susceptible to runs and collapsed, while the liquidity-backstop-possessing VRDO market survived. Such structural change is also the key in identifying the value of a liquidity backstop. Based on the calibrated model, we find that a liquidity backstop is valued at about 14.5 bps per annum.

Our paper has broader applications beyond these municipal bond markets studied here. Similar to the run on ARS in early 2008, a wide-spread run on money market funds was also triggered when investors started to realize that the implicit guarantee by fund sponsors may fail and thus a money market fund may “break the buck.” Furthermore, our paper also sheds light on the on-going GSE reform on the implicit government guarantee to the GSEs.

As another application, the value of a liquidity backstop identified in this paper speaks to the central difference between the shadow banking system and the traditional banking system in terms of their differential access to public liquidity backstops. Consistent with the literature on the “neglected risk” view of shadow banking, “shadow money” can stop being liquid or safe once investors take account of tail risks that are previously neglected (see Gennaioli et al., 2013). The key model implication in this paper is general and can be applied to shadow banking in the sense that the possibility of shadow money becoming illiquid in the future prompts investors to run more often, ex ante. Having a public liquidity backstop (e.g., deposit insurance) effectively mitigates (or even eliminates) runs induced by such liquidity risk. We leave this interesting application to future research.

Appendix A. Notation

We denote by $-\gamma_i$ and $\eta_i$ two real roots of the quadratic equation $\frac{1}{2}\sigma^2 x(x - 1) + \mu x - (\rho + \phi + \delta_i) = 0$, $i = 1, 2, 3$,

$$-\gamma_i = -\frac{\mu - \frac{1}{2}\sigma^2 + \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2[\rho + \phi + \delta_i]}}{\sigma^2} < 0,$$

$$\eta_i = \frac{\mu - \frac{1}{2}\sigma^2 - \sqrt{(\frac{1}{2}\sigma^2 - \mu)^2 + 2\sigma^2[\rho + \phi + \delta_i]}}{\sigma^2} > 0.$$

where $\delta_1 = \delta(1 + \theta + \kappa)$, $\delta_2 = 0$, $\delta_3 = \tilde{\delta}$.

The following notation is used in determining equilibrium threshold

\begin{align*}
K_1 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) + \mu} & \bar{K}_1 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} - \delta} \\
K_2 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) - \mu} & \bar{K}_2 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} + \delta} \\
K_3 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) - \mu} & \bar{K}_3 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} - \delta} \\
K_4 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) + \mu} & \bar{K}_4 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} + \delta} \\
K_5 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \mu} & \bar{K}_5 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} - \mu} \\
K_6 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \mu} & \bar{K}_6 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} + \mu} \\
K_7 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) + \mu} & \bar{K}_7 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} - \delta} \\
K_8 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) + \mu} & \bar{K}_8 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} + \delta} \\
K_9 &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \delta(1 + \theta) - \mu} & \bar{K}_9 &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} - \mu} \\
K_{10} &= \frac{\tau + \delta(1 + \theta)}{\rho + \phi + \mu} & \bar{K}_{10} &= \frac{\tau + \delta\tilde{L}}{\rho + \phi + \tilde{L} + \mu}
\end{align*}

(A.1)

Appendix B. Proofs

Lemma 1. $U(y)$ is strictly monotonically increasing.

Proof of Lemma 1. The HJB equation for $U(y)$ is the following

$$\rho U(y_t) = \mu y_t U_y(y_t) + \frac{\sigma^2}{2} U_{yy}(y_t) + \tau + \phi [\min (1, y_t) - U(y_t)] + \tilde{\delta} [\min (1, L + ly_t) - U(y_t)].$$
Depending on the value of \( y \), the HJB equation can be re-expressed as

\[
(\rho + \phi + \bar{\theta} \delta) U - \mu y U_y - \frac{\sigma^2}{2} y^2 U_{yy} = \begin{cases} \tau + \phi y + \bar{\theta} \delta (L + Ly), & \text{if } y \in (0, 1]; \\ \tau + \phi + \bar{\theta} \delta (L + Ly), & \text{if } y \in (1, \frac{1}{L}); \\ \tau + \phi + \bar{\theta} \delta, & \text{if } y \in (\frac{1}{L}, \infty). \end{cases}
\]

Therefore, the solution has the following functional form

\[
U(y) = \begin{cases} K_1 + K_2 y_1 + U_1 y_1^{\eta_1}, & \text{if } y \in (0, 1] \\ K_3 + K_4 y_1 + U_2 y_1^{\eta_2} + U_3 y_1^{\eta_3}, & \text{if } y \in (1, \frac{1}{L}) \\ K_5 + U_4 y_1^{\eta_4}, & \text{if } y \in (\frac{1}{L}, \infty) \end{cases}
\]

We determine the unknown coefficients \( U_1, \ldots, U_4 \) from the value-matching and smooth-pasting conditions:

\[
\begin{align*}
U_1 &= (K_3 + K_4) - (K_1 + K_2) + U_2 + U_3, \\
U_2 &= -\frac{\eta_3 (K_3 - K_1) + (\eta_3 - 1) (K_4 - K_2)}{\eta_3 + \gamma_3}, \\
U_3 &= -\gamma_3 (K_3 - K_5) - (\gamma_3 + 1) K_4 \left(\frac{L}{1 - L}\right)^{\eta_3}, \\
U_4 &= -\frac{K_4 \left(\frac{L}{1 - L}\right)^{\eta_4} - \gamma_4 U_2 \left(\frac{L}{1 - L}\right)^{\gamma_3} + \gamma_4 U_3 \left(\frac{L}{1 - L}\right)^{\gamma_3}}{\gamma_3}.
\end{align*}
\]

To prove the monotonicity of \( U(y) \), we first prove that \( U_i < 0 \), for \( i = 1, \ldots, 4 \). Substituting the expressions of \( K_1, \ldots, K_5 \) into \( U_1, U_2, U_3 \), we have

\[
\begin{align*}
U_1 &= \frac{\phi + \bar{\theta} \delta (1 - L) \left(\frac{L}{1 - L}\right)^{\eta_3}}{(\eta_3 + \gamma_3) \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right]} < 0; \\
U_2 &= -\frac{\phi}{\eta_3 + \gamma_3} \left[ \frac{\eta_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\eta_3 - 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right] < 0; \\
U_3 &= \frac{\bar{\theta} \delta (1 - L) \left(\frac{L}{1 - L}\right)^{\eta_3}}{(\eta_3 + \gamma_3) \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right]} < 0.
\end{align*}
\]

Lastly, from the above expression of \( U_3 \) and the result \( U_2 < 0 \), we have

\[
\gamma_3 \left(\frac{L}{1 - L}\right)^{-\gamma_3} U_4 = -\left[ \frac{\gamma_3}{\eta_3} \left(\frac{L}{1 - L}\right)^{\eta_3} - \gamma_3 U_2 \left(\frac{L}{1 - L}\right)^{-\gamma_3} \right] < 0.
\]

Next, we prove \( U(y) \) is monotonically increasing for \( y > 0 \). Note that \( U'(y) = U_4 (\gamma_3 y^{\gamma_3 - 1}) > 0 \) for \( y > \frac{1}{L} \) since \( U_4 < 0 \). Therefore, we only need to establish the monotonicity for \( 0 < y \leq \frac{1}{L} \). We prove it for the cases of \( 0 < y \leq 1 \) and \( 1 < y \leq \frac{1}{L} \), respectively. For \( 0 < y \leq 1 \), because \( \left(\frac{L}{1 - L}\right)^{-\eta_3} < \frac{1}{L^2} \) (or equivalently, \( \left(\frac{1-L}{L}\right)^{\eta_3 - 1} > 1 \))

\[
U'(y) = K_2 + \gamma_3 U_1 y^{\eta_3 - 1} \geq K_2 + \gamma_3 U_1 \\
= \frac{\phi + \bar{\theta} \delta}{\rho + \phi + \bar{\theta} \delta - \mu} + \gamma_3 \frac{\phi + \bar{\theta} \delta (1 - L) \left(\frac{L}{1 - L}\right)^{\eta_3}}{(\eta_3 + \gamma_3) \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right]} \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right] \\
> \gamma_3 \frac{\phi + \bar{\theta} \delta}{\rho + \phi + \bar{\theta} \delta - \mu} + \gamma_3 \frac{\phi + \bar{\theta} \delta (1 - L) \left(\frac{L}{1 - L}\right)^{\eta_3}}{(\eta_3 + \gamma_3) \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right]} \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right] \\
> \gamma_3 \frac{\phi + \bar{\theta} \delta}{\rho + \phi + \bar{\theta} \delta - \mu} \left[ \frac{\eta_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\eta_3 - 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right] \left[ \frac{\gamma_3}{\rho + \phi + \bar{\theta} \delta} - \frac{\gamma_3 + 1}{\rho + \phi + \bar{\theta} \delta - \mu} \right] \\
> 0;
\]

and for \( 1 < y \leq \frac{1}{L} \),

\[
U'(y) = K_4 + (-\gamma_3) U_2 y^{\gamma_3 - 1} + \gamma_3 U_3 y^{\eta_3 - 1} \\
> K_4 - \gamma_3 U_2 + \gamma_3 U_3 \left(\frac{L}{1 - L}\right)^{\eta_3 - 1}.
\]
\[
\begin{align*}
&= \frac{\theta \delta l}{\rho + \phi + \theta \delta - \mu} + \gamma_2 \frac{\phi}{\eta_3 + \gamma_2} \left[ \frac{\eta_2 - \eta_3 - 1}{\rho + \phi + \theta \delta - \mu} \right] + \eta_3 \left( \frac{\gamma_2}{\eta_3 + \gamma_2} \right) \left[ \frac{\gamma_2}{\rho + \phi + \theta \delta - \mu} - \frac{\gamma_2 + 1}{\rho + \phi + \theta \delta - \mu} \right] + \gamma_2 \left( \frac{\phi + \theta \delta}{\eta_3 + \gamma_2} \right) \left[ \frac{\eta_3}{\rho + \phi + \theta \delta - \mu} - \frac{\eta_3 - 1}{\rho + \phi + \theta \delta - \mu} \right] > 0.
\end{align*}
\]

**Parameter Restrictions.** To ensure the monotonicity of the function \(W(y)\), we impose a few parameter restrictions. First, we keep the same parameter restrictions as in He and Xiong (2012):

\[
\mu < \rho + \phi, \tag{B.11}
\]

\[
\phi < \theta \delta (1 - L - l) + \kappa \delta (1 - U(1)), \tag{B.12}
\]

\[
\alpha < \left[ \frac{r}{\rho + \phi} + \frac{\phi}{\rho + \phi - \mu} \right]^{-1}. \tag{B.13}
\]

In addition, we impose the following restriction for the additional parameters in our model, namely, \(\overline{r}\), \(\Delta\), and \(\kappa\):

\[
\overline{r} > \rho + \Delta + \kappa \delta \left(1 - U(\left(\frac{1-L}{1-l}\right))\right), \tag{B.14}
\]

\[
0 \leq \Delta < \frac{\gamma_1 (\rho + \phi) \left(\rho + \phi + \theta \delta (1 - L) - r\right)}{\gamma_2 (\rho + \phi + \theta (1 + \theta + \kappa))}, \tag{B.15}
\]

and

\[
(\eta_1 - 1) (K_2 + \overline{r}) + \eta_3 (\eta_1 - \eta_3) U_1 > 0, \tag{B.16}
\]

\[
(\eta_1 - 1) (K_4 + \overline{r}) + \eta_3 (\eta_1 - \eta_3) U_2 ((1 - L)/l)^{\gamma_1 - 1} > 0. \tag{B.17}
\]

The restriction (B.14) ensures that the max rate is sufficiently high for the model to be meaningful. The restriction (B.15) rules out the degenerate case where the liquidity component \(\Delta\) is too large and it is thus always profitable to hold the bonds, implying that the equilibrium threshold \(y^*\) is zero. Note that this restriction also implies \(U(0) < 1\). Lastly, to simplify exposition, we assume \(\overline{\theta} = 1 + \theta\) throughout the paper.

**Proof of Theorem 1.** The equilibrium threshold \(y^*\) is determined by the condition \(V(y^*; y^*) = 1\). Define \(W(y^*) \equiv V(y^*; y^*)\). Here we prove that there always exists a unique \(y^*\) such that \(W(y^*) = 1\). To simplify notation, we replace \(y^*\) by \(y\) and express \(W(y)\) as \(W(y)\) throughout the proof. It is easy to show that under the parameter restriction (B.15), \(W_C(0) < W_B(0) < 1\), \(W_A(\infty) > 1\), and \(W_B(\infty) > 1\).

Denote by \(y_{\text{ss}} = \max \{y : R(y; y_{\text{ss}}) = \overline{r}\}\) the maximum fundamental value that is associated with the max rate. That is, the constraint of the max rate is binding if and only if \(y \leq y_{\text{ss}}\). It is straightforward to see that in Case B or Case F, \(y_{\text{ss}}\) coincides with \(y^*\) (i.e., \(y_{\text{ss}} = y^*\)), and in Case C, \(y_{\text{ss}}^* = \frac{r + \phi - \overline{r}}{\phi} \leq 1\). For the other cases, \(y_{\text{ss}}\) is determined by \(f(y_{\text{ss}}) = 0\) where the function \(f(y)\) is defined as

\[
 f(y) = \overline{r} + \phi (1 - y)^{+} + \theta \delta (1 - L - ly)^{+} + \kappa \delta (1 - U(y)) - \overline{r}.
\]

Then from Lemma 1, \(f(y)\) is continuous and strictly decreasing. Furthermore, under the parameter restrictions (B.14) and (B.15), we have \(f(0) > 0\) and \(f\left(\frac{1-L}{1-l}\right) \leq 0\), implying that \(f(y_{\text{ss}}) = 0\) has a unique solution \(y_{\text{ss}} \in (0, 1-\frac{L}{l})\). It is straightforward to check that \(W_B(y_{\text{ss}}) = W_C(y_{\text{ss}}) \equiv W_B(y_{\text{ss}}) = W_F(y_{\text{ss}}) = W_F(y_{\text{ss}})\), and \(W_B(1) = W_F(1)\).

Now we prove the existence of the unique threshold \(y^*\) by considering all the possible max rates \(\overline{r}\). Under the restriction (B.12), \(\overline{r} > \rho + \theta \delta (1 - L - l) + \kappa \delta (1 - U(1))\). There are three possibilities.

1. Consider the possibility where \(\overline{r} \geq \overline{r} + \theta \delta (1 - L - l) + \kappa \delta (1 - U(1))\). implying \(f(1) \leq 0\) and \(y_{\text{ss}} \in (0, 1]\). Based on the strict monotonicity of \(W_A\) and \(W_B\), as well as \(y_{\text{ss}} \leq 1\), we have

\[
W_B(0) < W_A(y_{\text{ss}}) = W_B(y_{\text{ss}}) \leq W_A(1) < W_A(\infty).
\]

2. If \(W_A(1) < 1\), then Case D or Case G holds (note \(W_B(\infty) > 1\)) where \(W_A(y) = 1\) has a unique root \(y > 1\), depending on whether \(W_A(\frac{1-L}{1-l}) \geq 1\) or not. Otherwise, if \(W_A(1) \geq 1\), depending on whether \(W_A(y_{\text{ss}}) = W_B(y_{\text{ss}}) < 1\) or not, either Case A holds where \(W_A(y) = 1\) has a unique root \(y \in (y_{\text{ss}}, 1]\), or Case B holds where \(W_B(y) = 1\) has a unique root \(y \in (0, y_{\text{ss}}]\).
(ii) Consider the possibility where \( \overline{\sigma} + \phi \leq r < \overline{\sigma} + \theta \delta (1 - L - I) + \kappa \delta (1 - U(1)) \), implying \( f(1) > 0 \) and \( y_{m} \in (1, \frac{1}{1-L}] \). Based on the strict monotonicity of \( W_{B} \), \( W_{F} \), and \( y_{m} \), we have
\[
W_{B}(1) = W_{F}(1) < W_{F}(y_{m}) = W_{F}(y_{m}).
\]
If \( W_{F}(y_{m}) < 1 \), then Case E or Case H holds (note \( W_{E}(\infty) > 1 \)) where \( W_{E}(y) = 1 \) has a unique root \( y > y_{m} \), depending on whether \( W_{E}(\frac{1}{1-L}) \geq 1 \) or not. Otherwise, if \( W_{E}(y_{m}) = W_{F}(y_{m}) \geq 1 \), depending on whether \( W_{B}(1) = 1 \) or not, either Case F holds where \( W_{F}(y) = 1 \) has a unique root \( y \in (1, y_{m}] \), or Case B holds where \( W_{B}(y) = 1 \) has a unique root \( y \in (0, 1] \).

(iii) Consider the possibility where \( r < \overline{\sigma} + \phi \), implying \( 0 < y_{m}^{*} \leq 1 \) and \( y_{m} \in (1, \frac{1}{1-L}] \). Based on the strict monotonicity of \( W_{B} \) and \( W_{F} \), as well as \( y_{m} \), we have
\[
W_{C}(0) = W_{B}(y_{m}^{*}) = W_{C}(y_{m}^{*}) < W_{B}(1) = W_{F}(1) < W_{F}(y_{m}) = W_{F}(y_{m}).
\]
If \( W_{C}(y_{m}^{*}) \geq 1 \), then Case C holds (note \( W_{C}(0) < 1 \)) where \( W_{C}(y) = 1 \) has a unique solution \( y \in (0, y_{m}^{*}] \). Otherwise, if \( W_{C}(y_{m}^{*}) < 1 \), by the same argument used in Possibility (ii), we can prove that Case B holds if \( W_{B}(1) \geq 1 \), or Case E or Case H holds if \( W_{B}(1) < 1 \) and \( W_{F}(y_{m}) \leq 1 \). □

Appendix C. Estimation Methodology

The key equation used in the MLE estimation is the following equation:
\[
RX_{t}^{ARS} = RX_{t}^{ARS}(y_{1}, y_{m}(\Theta_{t})) + w_{t}, \quad \text{where } w_{t} \sim N(0, \sigma^{2}).
\]
(18)
where \( RX_{t}^{ARS} \) is the ARS interest rate in excess of the repo rate at time \( t \), and \( RX_{t}^{ARS}(y_{1}, y_{m}(\Theta_{t})) = R(y_{1}, y_{m}(\Theta_{t})) - \rho \) is the model-implied excess interest rate for the ARS market (see Eq. (7) for the expression of \( R^{ARS}(y_{1}, y_{m}(\Theta_{t})) \)), and \( w_{t} \) denotes the pricing error that is assumed to follow a normal distribution with mean zero and standard deviation \( \sigma \). Note that the risk-free rate \( \rho \) is assumed to be constant in the model for tractability. We thus work directly with the excess rates in estimation.

Note that the model-implied ARS excess rate depends on the equilibrium rollover threshold \( y_{1}(\Theta_{t}) \), which in turn depends on the vector of the parameters to estimate \( \Theta_{t} = (\mu, \sigma, \Delta, \theta, \kappa_{t}, \nu) \), as well as the other parameters calibrated. The subscript \( t \) reflects one important identification assumption that there is a structural change in the beliefs of ARS investors. Denote by \( \tau \) the date of structural change. The probability of auction failures is considered to be zero before time \( \tau \), and becomes positive and equal to \( \kappa \delta dt > 0 \) at time \( \tau \) and onwards once investors realize that auction agents have no contractual obligations to provide liquidity (i.e., no explicit liquidity backstops). That is,
\[
k_{t} = \begin{cases} 
0, & t \in [T_{1}, \tau) \\
\kappa > 0, & t \in [\tau, T_{2}] 
\end{cases}
\]
(19)
where \( T_{1} \) and \( T_{2} \) denote the start and end time periods of the ARS sample period, respectively. As we will describe shortly, we choose the date of structural change \( \tau \) as November 14, 2007 when the VRDO and ARS rates start to diverge, and the ARS sample period is between May 31, 2006 and December 30, 2009.

The model-implied ARS excess rate also depends on the fundamental value \( y_{1} \), which, however, is unobservable in the data. Based on the other identification assumption that both VRDO and ARS markets have the same fundamental process, we can infer \( y_{1} \) from the VRDO market. As mentioned before, this identification assumption is realistic since municipality issuers of VRDOs or ARS are very similar (and in fact the same in many cases). To infer \( y_{1} \), we assume that there are no measurement errors in the VRDO market
\[
RX_{t}^{VRDO} = RX_{t}^{VRDO}(y_{1}, y_{m}^{VRDO}(\Theta_{0})) = \phi(1 - y_{1})^{\gamma} + 1_{\{y_{1} \leq y_{m}^{VRDO}(\Theta_{0})\}} \delta(1 - [L + Ly_{1}])^{\gamma}.
\]
Note that the explicit arrangement of liquidity backstops in the VRDO market implies \( \kappa = 0 \). Therefore, the VRDO rollover threshold \( y_{m}^{VRDO}(\Theta_{0}) \) remains unchanged throughout the sample period, and \( \Theta_{0} = (\mu, \sigma, \Delta, \theta) \) denotes the set of constant parameters to estimate. Also note that the model-implied VRDO excess rate is always non-negative while the VRDO excess rate in the data are sometimes (but very infrequently) negative. In this case, we set \( y_{1} = max\{1, y_{m}^{VRDO}(\Theta_{0})\} \). We denote by \( y_{*}^{VRDO} \) the value of \( y_{1} \) inferred from the VRDO data.

The above method of assuming zero measurement error to extract latent factors in one market segment and then applying the extracted factors to the other market segment assuming non-zero measurement errors is widely used in the literature of affine term-structure models (ATSMs).\(^{18}\) In this literature, it is very common to assume that the Treasury yield curve is driven mainly by a finite number of latent factors (e.g., level, slope, and curvature, etc.). The usual way to estimate such ATSMs is to extract the latent factors by assuming zero measurement errors for the same number of Treasury securities, and then to estimate the model using the extracted factors together with the rest of the yield curve. Our method is similar, but has a major difference: when we extract the unobservable fundamental process \( y_{*}^{VRDO} \) and apply it to the ARS

\(^{18}\) This commonly adopted approach was first used Chen and Scott (1993) and by Pearson and Sun (1994).
market, our estimation takes into consideration the structural change in November 2007 when the probability of auction failures is (correctly) perceived to be positive.

Lastly, we can estimate the parameters $\Theta = (\mu, \sigma, \Delta, \theta, \kappa, \nu)$ using the MLE method. Specifically, the MLE estimator is the maximizer of log-likelihood function $\ln L(\Theta; r_{x|t_1:T_2}, y_{y|t_1:T_2})$. \(^{19}\)

$$\hat{\Theta} = \arg \max_\Theta \ln L(\Theta; r_{x|t_1:T_2}, y_{y|t_1:T_2}).$$

where $T_1, T_2$ denotes the sequence of time periods $\{T_1, T_1 + 1, \ldots, T_2\}$. The log-likelihood function is constructed as follows:

$$\ln L(\Theta; r_{x|t_1:T_2}, y_{y|t_1:T_2}) = \sum_{t=T_1}^{T_2} \ln f(0)(r_{x|t_1:T_2}, y_{y|t_1:T_2}, \Theta) + \sum_{t=T_1}^{T_2} \ln f(1)(r_{x|t_1:T_2}, y_{y|t_1:T_2}, \Theta)$$

where

$$f(0)(r_{x|t_1:T_2}, y_{y|t_1:T_2}, \Theta) = \frac{1}{\sqrt{2\pi \nu}} \exp \left\{ -\frac{(r_{x|t_1:T_2} - RX_{y|t_1:T_2}(\nu y_{y|t_1:T_2}(\Theta_0)))^2}{2\nu^2} \right\}.$$ 

$$f(1)(r_{x|t_1:T_2}, y_{y|t_1:T_2}, \Theta) = \frac{1}{\sqrt{2\pi \nu}} \exp \left\{ -\frac{(r_{x|t_1:T_2} - RX_{y|t_1:T_2}(\nu y_{y|t_1:T_2}(\Theta)))^2}{2\nu^2} \right\}.$$ 

As discussed above, in the subperiod $[T_1, T]$ the probability of auction failures is assumed to be zero by ARS investors. Therefore, in this subperiod, the rollover threshold is the same as in the VRDO market, i.e., $y_{ARS}(\Theta_0)$, which is used in the density function $f(0)(\cdot)$. However, in the subperiod $[T, T_2]$ following the structural change, the ARS rollover threshold $y_{ARS}(\Theta)$ jumps to a higher level as a result of positive probability of auction failures (or $\kappa > 0$). The density function $f(1)(\cdot)$ captures the structural change by using the higher threshold $y_{ARS}(\Theta)$.

We apply the above estimation methodology to the SIFMA historical interest rate indexes for the VRDO and ARS markets. The VRDO sample period ranges between May 22, 1991 and October 24, 2012, while the ARS sample period ranges between May 31, 2006 and December 30, 2009 when the SIFMA stopped producing the ARS index. Recall that from Fig. 1 the ARS rate had largely moved in lockstep with the VRDO rate until November 14, 2007, and has diverged since then. Based on this observation, we set $\kappa$ to zero for the pre-crisis period when using the ARS data, but allow for a positive $\kappa$ between November 14, 2007 and December 30, 2009 as a reflection the structural change in investors’ beliefs. For the VRDO data, we restrict $\kappa$ to zero for the entire sample period. In addition, following Schrot et al. (2014) we set $\mu = \sigma^2/2$ so that the (log) fundamental process has zero expected growth rate.

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\(^{19}\) To obtain the global maximizers, we randomly generate starting values for the parameters from a multivariate normal distribution whose mean and variances are set to plausible values, and then use the randomly generated starting values in “minsearch” to determine the parameter vector that maximizes the likelihood. We repeat this procedure 1000 times and choose the parameter estimates with the maximum possible likelihood value as our final estimates.
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