STOCHASTIC PERRON FOR STOCHASTIC TARGET GAMES

ERHAN BAYRAKTAR AND JIAQI LI

ABSTRACT. We extend stochastic Perron’s method to analyze the framework of stochastic target games, in which one player tries to find a strategy such that the state process almost-surely reaches a given target no matter which action is chosen by the other player. Within this framework, our method produces a viscosity sub-solution (super-solution) of a Hamilton-Jacobi-Bellman (HJB) equation. Using a comparison result, we characterize the value function as a viscosity solution to the HJB equation.

1. INTRODUCTION

We will extend the stochastic Perron’s method to analyze a stochastic (semi) game where a controller tries to find a strategy such that the controlled state process almost-surely reaches a given target at a given finite time, no matter which control is chosen by an adverse player (nature). More precisely, the controller has access to a filtration generated by a Brownian motion and can observe and react to nature, who may choose a parametrization of the model to be totally adverse to the controller, in a non-anticipative way. This stochastic target game was introduced and analyzed in [8].

In this paper, we will have a fresh look at the problem of [8] with different methodology, namely the stochastic Perron’s method. Using this method we will be able to drop the assumption on the concavity of the Hamiltonian [8] assumes. Stochastic Perron’s method was introduced in [3] for analyzing linear problems, [5] for Dynkin games involving free-boundary games, and in [4] for stochastic control problems. This method is a type of verification theorem, which identifies the value function as the unique solution of a corresponding HJB equation without going through the dynamic programming principle, but does not require the smoothness of the value function. It is a stochastic version of the Perron’s method in [9] in that it creates classes of sub- and super-solutions that envelope the value function and are closed under maximization and minimization respectively. More recently, the stochastic Perron’s method was adjusted to solve exit time problems in [12], state constraint problems in [11], singular control problems in [6], stochastic games in [14], and control problems with model uncertainty in [13] and [1]. In this paper, we show how the main ideas of this method can be modified to analyze the stochastic target games of [8].

The main difficulty in the analysis is identifying the correct collections of stochastic sub- and super-solutions. Once this is established, the technical contribution is in showing that in fact the supremum and the infimum of the respective families are viscosity super- and sub-solutions

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respectively. Then a comparison result establishes the claim since the value function is already enveloped by these two families. The identification of these classes and the technical proofs turn out to be quite different from the previous works cited above because of the different nature of the stochastic target problems from the stochastic control problems. Unlike in the usual stochastic control problems, in the target problems the goal is to beat a stochastic target almost surely by applying the admissible controls. These problems, which are generalizations of the super-hedging problems that appear in Mathematical Finance, were introduced in the seminal papers [16] and [15]. (See the book [17] for a more recent exposition.) Stochastic target games, on the other hand, were considered only recently by [7], but when the target is of controlled loss type. The more difficult case of an almost sure target was then analyzed in [8].

In this paper we achieve the following:

• Give a proof of the result that the value function of the stochastic target game is the unique viscosity solution of the associated HJB equation without going through the geometric dynamic programming principle first. What we have is a new method in analyzing stochastic target problems.

• Give a more elementary proof to the result in [8]. This way we are able to avoid using Krylov’s method of shaken coefficients which requires the concavity of the Hamiltonian.

The rest of the paper is organized as follows: In Section 2 we present the setup of the stochastic target game, introduce the related HJB equation and the definitions of the sets of stochastic super- and sub-solutions (our conceptual contribution). The technical contribution of the paper is given in Section 3 where we characterize the infimum (supremum) of the stochastic super-solutions (sub-solutions) as the viscosity sub-solution (super-solution) of the HJB equation. Finally, a viscosity comparison argument concludes that the value function is the unique viscosity solution of the HJB equation. Some technical results are deferred to the Appendix.

2. STATEMENT OF THE PROBLEM

2.1. The Value Function. Let us denote

\[ D := [0, T] \times \mathbb{R}^d, \quad D_{<T} := [0, T) \times \mathbb{R}^d, \quad D_T := \{T\} \times \mathbb{R}^d. \]

Given \((t, x, y) \in D \times \mathbb{R} \) and \((u, \alpha) \in U^t \times A^t\), consider the stochastic differential equations (SDEs)

\[
\begin{aligned}
    dX(s) &= \mu_X(s, X(s), \alpha_s)ds + \sigma_X(s, X(s), \alpha_s)dW_s, \\
    dY(s) &= \mu_Y(s, X(s), Y(s), u_s, \alpha_s)ds + \sigma_Y(s, X(s), Y(s), u_s, \alpha_s)dW_s,
\end{aligned}
\] (2.1)

with initial data \((X(t), Y(t)) = (x, y)\). Let \(\Omega\) be the space of continuous functions \(\omega : [0, T] \rightarrow \mathbb{R}^d\) and let \(P\) be the Wiener measure on \(\Omega\). We will denote by \(W\) the canonical process on \(\Omega\), i.e. \(W_t(\omega) = \omega_t\), and by \(\mathbb{F} = (\mathcal{F}_s)_{0 \leq s \leq T}\) the augmented filtration generated by \(W\). For \(0 \leq t \leq T\) let \(\mathbb{F}^t = (\mathcal{F}^t_s)_{0 \leq s \leq T}\) be the augmented filtration generated by \((W_s - W_t)_{s \geq t}\). By convention, \(\mathcal{F}^t_s\) is trivial for \(s \leq t\).

We denote by \(U^t\) (resp. \(A^t\)) the collection of all \(\mathbb{F}^t\)-predictable processes in \(L^p(P \otimes dt)\) with values in a given Borel subset \(U\) (resp. compact set \(A\)) of \(\mathbb{R}^d\), where \(p \geq 2\) is fixed.
Assumption 2.1. The coefficients $\mu_X, \mu_Y, \sigma_X$ and $\sigma_Y$ are continuous in all variables and take values in $\mathbb{R}^d$, $\mathbb{R}$, $\mathbb{R}^d$ and $\mathbb{M}^d := \mathbb{R}^{d \times d}$, respectively. There exists $K > 0$ such that

$$\left| \mu_X(\cdot, x, \cdot) - \mu_X(\cdot, x', \cdot) \right| + \left| \sigma_X(\cdot, x, \cdot) - \sigma_X(\cdot, x', \cdot) \right| \leq K |x - x'|,$$

$$\left| \mu_X(\cdot, x, \cdot) \right| + \left| \sigma_X(\cdot, x, \cdot) \right| \leq K,$$

$$\left| \mu_Y(\cdot, y, \cdot) - \mu_Y(\cdot, y', \cdot) \right| + \left| \sigma_Y(\cdot, y, \cdot) - \sigma_Y(\cdot, y', \cdot) \right| \leq K |y - y'|,$$

$$\left| \mu_Y(\cdot, y, u, \cdot) \right| + \left| \sigma_Y(\cdot, y, u, \cdot) \right| \leq K(1 + |u| + |y|),$$

for all $(x, y, (x', y')) \in \mathbb{R}^d \times \mathbb{R}$ and $u \in U$.

This assumption is a bare minimum and ensures that the stochastic differential equations given in (2.1) are well-posed. Denote the solutions to (2.1) by $(X^{\alpha}_{t,x}, Y^{u,\alpha}_{t,x,y})$. Let $t \leq T$. We say that a map $u : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto u[\alpha]$ is a $t$-admissible strategy if it is non-anticipating in the sense that

$$\{ \omega \in \Omega : \alpha(\omega)[|t, s] = \alpha'(\omega)[|t, s] \} \subset \{ \omega \in \Omega : u[\alpha](\omega)[|t, s] = u[\alpha'](\omega)[|t, s] \} \text{ -a.s.}$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, where $|t, s|$ indicates the restriction to the interval $[t, s]$. We denote by $\mathcal{U}(t)$ the collection of all $t$-admissible strategies; moreover, we write $Y^{u,\alpha}_{t,x,y}$ for $(X^{\alpha}_{t,x}, Y^{u,\alpha}_{t,x,y})$. Then we can introduce the value function of the stochastic target game,

$$v(t, x) := \inf \{ y \in \mathbb{R} : \exists u \in \mathcal{U}(t) \text{ s.t. } Y^{u,\alpha}_{t,x,y}(T) \geq g(X^{\alpha}_{t,x}(T)) \text{ -a.s. } \forall \alpha \in \mathcal{A}^t \},$$

(2.2)

where $g : \mathbb{R}^d \to \mathbb{R}$ is a bounded and measurable function. We also need to define strategies starting at a family of stopping times. Let $\mathcal{S}^t$ be the set of $\mathcal{F}^t$-stopping times valued in $[t, T]$.

**Definition 2.1** (Non-anticipating family of stopping times). Let $\{\tau^{\alpha}\}_{\alpha \in \mathcal{A}^t} \subset \mathcal{S}^t$ be a family of stopping times. This family is $t$-non-anticipating if

$$\{ \omega \in \Omega : \alpha(\omega)[|t, s] = \alpha'(\omega)[|t, s] \} \subset$$

$$\{ \omega \in \Omega : t \leq \tau^{\alpha}(\omega) = \tau^{\alpha'}(\omega) \leq s \} \cup \{ \omega \in \Omega : s < \tau^{\alpha}(\omega) , s < \tau^{\alpha'}(\omega) \} \text{ -a.s.}$$

Denote the set of $t$-non-anticipating families of stopping times by $\mathcal{S}^t$.

We will use $\{\tau^{\alpha}\}$ for short to represent $\{\tau^{\alpha}\}_{\alpha \in \mathcal{A}^t}$, which will always denote a $t$-non-anticipating family of stopping times.

**Definition 2.2** (Strategies starting at a non-anticipating family of stopping times). Fix $t$ and let $\{\tau^{\alpha}\} \in \mathcal{S}^t$. We say that a map $u : \mathcal{A}^t \to \mathcal{U}^t$, $\alpha \mapsto u[\alpha]$ is a $(t, \{\tau^{\alpha}\})$-admissible strategy if it is non-anticipating in the sense that

$$\{ \omega \in \Omega : \alpha(\omega)[|t, s] = \alpha'(\omega)[|t, s] \} \subset \{ \omega \in \Omega : s < \tau^{\alpha}(\omega) , s < \tau^{\alpha'}(\omega) \} \cup$$

$$\{ \omega \in \Omega : t \leq \tau^{\alpha}(\omega) = \tau^{\alpha'}(\omega) \leq s, u[\alpha](\omega)[|\tau^{\alpha}(\omega), s] = u[\alpha'](\omega)[|\tau^{\alpha'}(\omega), s] \} \text{ -a.s.}$$

for all $s \in [t, T]$ and $\alpha, \alpha' \in \mathcal{A}^t$, denoted by $u \in \mathcal{U}(t, \{\tau^{\alpha}\})$.

It is clear that, in the Definition 2.2 if we set $\tau^{\alpha} = t$ for all $\alpha$, then $\mathcal{U}(t, \{\tau^{\alpha}\})$ is then same as $\mathcal{U}(t)$. Hence, the above definitions are consistent.
Let us define for \(\alpha_1, \alpha_2 \in A^t\) a stopping time. The concatenation of \(\alpha_1, \alpha_2\) is defined as follows:

\[
\alpha_1 \otimes_\tau \alpha_2 := \alpha_1 1_{[t, \tau)} + \alpha_2 1_{[\tau, T]}.
\]

The concatenation of elements in \(U^t\) is defined in the similar fashion.

**Lemma 2.1.** Fix \(t \in \mathbb{R}\) and let \(\{\tau^\alpha\} \in \mathbb{S}^t\). For \(u \in U\) and \(u \in U(t, \{\tau^\alpha\})\), define \(u_*[\alpha] := u[\alpha] \otimes_{\tau^\alpha} u[\alpha]\). Then \(u_* \in U(t)\). For the rest of the paper, we will use \(\otimes_{\tau^\alpha} \) to represent \(u[\alpha] \otimes_{\tau^\alpha} u[\alpha]\).

**Proof.** It is obvious that \(u_*\) maps \(A^t\) to \(U^t\). Let us check the non-anticipatory of the map. For any fixed \(s \in [t, T]\) and \(\alpha, \alpha' \in A^t\), \(\omega' \in \{\omega \in \Omega : \alpha(\omega)|_{[t, s]} = \alpha'(\omega)|_{[t, s]}\}\), by Definition 2.1

\[
\omega' \in \{t \leq \tau^\alpha = \tau'^{\alpha'} \leq s\} \cup \{s < \tau^\alpha, s < \tau'^{\alpha'}\} -a.s. \quad (2.3)
\]

(i) If \(\omega' \in \{t \leq \tau^\alpha = \tau'^{\alpha'} \leq s\}\), by definition of \(u_*\),

\[
\begin{align*}
\alpha_*[\alpha](\omega')|_{[t, s]} &= \alpha[\alpha](\omega')1_{[t, \tau^\alpha(\omega'))}|_{[t, s]} + \tilde{u}[\alpha](\omega')1_{[\tau^\alpha(\omega'), T]}|_{[t, s]}, \\
\alpha_*[\alpha'](\omega')|_{[t, s]} &= \alpha[\alpha'](\omega')1_{[t, \tau'^{\alpha'}(\omega'))}|_{[t, s]} + \tilde{u}[\alpha'](\omega')1_{[\tau'^{\alpha'}(\omega'), T]}|_{[t, s]}.
\end{align*}
\]

Since \(\tau^\alpha(\omega') = \tau'^{\alpha'}(\omega')\), \(u \in U(t)\) and by Definition 2.2 we know

\[
\omega' \in \{\omega \in \Omega : \alpha[\alpha](\omega)|_{[t, s]} = \alpha[\alpha'](\omega)|_{[t, s]}\} -a.s.
\]

(ii) If \(\omega' \in \{s < \tau^\alpha, s < \tau'^{\alpha'}\}\), using definition of \(u_*\),

\[
\begin{align*}
\alpha_*[\alpha](\omega')|_{[t, s]} &= \alpha[\alpha](\omega')|_{[t, s]}, \\
\alpha_*[\alpha'](\omega')|_{[t, s]} &= \alpha[\alpha'](\omega')|_{[t, s]}.
\end{align*}
\]

Since \(\omega' \in \{\omega \in \Omega : \alpha(\omega)|_{[t, s]} = \alpha'(\omega)|_{[t, s]}\}\) and \(u \in U(t)\), then \(\omega' \in \{\omega \in \Omega : \alpha[\alpha](\omega)|_{[t, s]} = \alpha[\alpha'](\omega)|_{[t, s]}\} -a.s. \)

\[
\square
\]

**2.2. The HJB Equation.** Before giving the HJB equation, we will introduce some notations and an assumption, which was also assumed by [3]. Given \((t, x, y, z, a) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times A\), define the set

\[
N(t, x, y, z, a) := \{u \in U : \sigma_Y(t, x, y, u, a) = z\}.
\]

**Assumption 2.2.** \(u \mapsto \sigma_Y(t, x, y, u, a)\) is invertible. More precisely, there exists a measurable map \(\hat{u} : \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times A \rightarrow U\) such that \(N = \{\hat{u}\}\). Moreover, the map \(\hat{u}\) is continuous.

Let us define for \((t, x, y, p, M) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{M}^d\),

\[
H(t, x, y, p, M) := \sup_{a \in A} \left\{ -\mu_Y(t, x, y, \sigma_X(t, x, a)p, a) + \mu_X(t, x, a)^\top p + \frac{1}{2} \text{Tr}[\sigma_X \sigma_X^\top(t, x, a)M] \right\},
\]

where \(\mu^\hat{u}_Y(t, x, y, z, a) := \mu_Y(t, x, y, \hat{u}(t, x, y, z, a), a), \ z \in \mathbb{R}^d\).

Consider the equation

\[
\phi_t + H(t, x, \phi, D\phi, D^2\phi) = 0 \quad \text{on} \quad \mathcal{D}_{<\tau},
\]

\[
\phi = g \quad \text{on} \quad \mathcal{D}_\tau,
\]

(2.4)
2.3. **Stochastic Solutions.** We will introduce weak solution concepts to the HJB equation that are stable under minimization and maximization respectively and envelope the value function \( v \) of the stochastic target game.

**Definition 2.4 (Stochastic super-solutions).** A function \( w : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is called a stochastic super-solution of (2.4) if

1. it is bounded, continuous and \( w(T, \cdot) \geq g(\cdot) \),
2. for fixed \((t, x, y) \in D \times \mathbb{R} \) and \( \{\tau^\alpha \} \in \mathcal{G}^t \), there exists a strategy \( \tilde{u} \in \mathcal{U}(t, \{\tau^\alpha \}) \) such that, for any \( u \in \mathcal{U}(t) \), \( \alpha \in \mathcal{A}^t \) and each stopping time \( \rho \in \mathcal{S}^t \), \( \tau^\alpha \leq \rho \leq T \) with the simplifying notation \( X := X^\alpha_{t,x}, Y := Y^\alpha_{t,x,y} \tilde{u}[\cdot,\cdot] \), we have
   \[
   Y(\rho) \geq w(\rho, X(\rho)) \quad \mathbb{P} - \text{a.s. on } \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}.
   \]

The set of stochastic super-solutions is denoted by \( \mathcal{U}^+ \). Assume it is nonempty and \( v^+ := \inf_{w \in \mathcal{U}^+} w \). For any stochastic super-solution \( w \), choose \( \tau^\alpha = t \) for all \( \alpha \) and \( \rho = T \), then there exists \( \tilde{u} \in \mathcal{U}(t) \) such that, for any \( \alpha \in \mathcal{A}^t \),

\[
Y^\alpha_{t,x,y}(T) \geq w \left(T, X^\alpha_{t,x}(T) \right) \geq g \left( X^\alpha_{t,x}(T) \right) \quad \mathbb{P} - \text{a.s. on } \{y > w(t, x)\}.
\]

Hence, \( y > w(t, x) \) implies \( y \geq v(t, x) \) from (2.2). This gives \( w \geq v \) and \( v^+ \geq v \). Similarly, we could define the stochastic sub-solutions.

**Definition 2.5 (Stochastic sub-solutions).** A function \( w : [0, T] \times \mathbb{R}^d \to \mathbb{R} \) is called a stochastic sub-solution of (2.4) if

1. it is bounded, continuous and \( w(T, \cdot) \leq g(\cdot) \),
2. for fixed \((t, x, y) \in D \times \mathbb{R} \) and \( \{\tau^\alpha \} \in \mathcal{G}^t \), for any \( u \in \mathcal{U}(t) \), \( \alpha \in \mathcal{A}^t \) (may depend on \( u \), \( \alpha \) and \( \tau^\alpha \)) such that for each stopping time \( \rho \in \mathcal{S}^t \), \( \tau^\alpha \leq \rho \leq T \) with the simplifying notation \( X := X^\alpha_{t,x}, Y := Y^\alpha_{t,x,y} \tilde{u}[\cdot,\cdot] \), we have
   \[
   \mathbb{P} \left( Y(\rho) < w(\rho, X(\rho)) \mid B \right) > 0,
   \]
   for any \( B \subset \{Y(\tau^\alpha) < w(\tau^\alpha, X(\tau^\alpha))\}, B \in \mathcal{F}^t_{\tau^\alpha} \) and \( \mathbb{P}(B) > 0 \).

The set of stochastic sub-solutions is denoted by \( \mathcal{U}^- \). Assume it is nonempty and let \( v^- := \sup_{w \in \mathcal{U}^-} w \). For any stochastic sub-solution \( w \), choose \( \tau^\alpha = t \) for all \( \alpha \) and \( \rho = T \). Hence for any \( u \in \mathcal{U}(t) \), there exists \( \tilde{\alpha} \in \mathcal{A}^t \), such that

\[
\mathbb{P} \left( Y^\alpha_{t,x,y}(T) < w \left(T, X^\tilde{\alpha}_{t,x}(T) \right) \leq g(X^\tilde{\alpha}_{t,x}(T)) \mid y < w(t, x) \right) > 0.
\]

Hence, \( y < w(t, x) \) implies \( y \leq v(t, x) \) from (2.2). This gives \( w \leq v \) and \( v^- \leq v \). As a result we have,

\[
v^- \triangleq \sup_{w \in \mathcal{U}^-} w \leq v \leq \inf_{w \in \mathcal{U}^+} w \triangleq v^+ \tag{2.5}
\]

We will show in Section 3 that under some suitable assumptions, \( v^+ \) and \( v^- \) are viscosity sub- and super-solutions of (2.4), respectively.
2.4. Additional Technical Assumptions. We will need to make some more technical assumptions as in \[8\].

**Assumption 2.3.** The map \((y, z) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mu_Y(t, x, y, z, a)\) is Lipschitz continuous and has linear growth, uniformly in \((t, x, a) \in \mathcal{D} \times \mathcal{A}\).

For the derivation of the super-solution property of \(v^-\), we will impose a condition on the growth of \(\mu_Y\) relative to \(\sigma_Y\).

**Assumption 2.4.**

\[
\sup_{u \in U} \frac{|\mu_Y(\cdot, u, \cdot)|}{1 + \|\sigma_Y(\cdot, u, \cdot)\|} \text{ is locally bounded,}
\]

where \(\| \cdot \|\) is the Euclidean norm.

To characterize \(v\) as the unique viscosity solution of \((2.4)\), we need a comparison principle.

**Assumption 2.5 (Comparison principle).** Let \(\underline{v}\) (resp. \(\overline{v}\)) be a LSC (resp. USC) bounded viscosity super-solution (resp. sub-solution) of \((2.4)\). Then, \(\underline{v} \geq \overline{v}\) on \(\mathcal{D}\).

In \((2.5)\) we implicitly assumed that the sets \(\mathcal{U}^+\) and \(\mathcal{U}^-\) are nonempty. The assumptions we made already imply that \(\mathcal{U}^+\) is not empty, but the same may not be true for \(\mathcal{U}^-\) is not empty.

**Assumption 2.6.** The collection \(\mathcal{U}^-\) is not empty.

2.5. When are \(\mathcal{U}^+\) and \(\mathcal{U}^-\) not empty. As the next result shows, the assumptions above already guarantee that \(\mathcal{U}^+\) is not empty.

**Proposition 2.1.** Under Assumptions 2.1, 2.2 and 2.3 the collection \(\mathcal{U}^+\) is not empty.

**Proof.** See the Appendix. \(\square\)

In the above proposition the assumptions made can be replaced by the following natural assumption (although this is not the route we will take):

**Assumption 2.7.** There exists \(u \in U\) such that \(\mu_Y(t, x, y, u, a) = 0\), \(\sigma_Y(t, x, y, u, a) = 0\) for all \((t, x, y, a) \in \mathcal{D}_{<T} \times \mathbb{R} \times \mathcal{A}\). (In these equations the right-hand-sides are denoted by just 0 for simplicity, but they in fact are collections of 0’s matching the dimension on the left-hand-side.)

In the context of super-hedging in mathematical finance, in which \(Y\) represents the wealth of an investor and \(X\) the stock price, and \(g(X_T)\) a financial contract, the last assumption is equivalent to allowing the investor not to trade in the risky assets.

**Proposition 2.2.** Under Assumptions 2.4 and 2.7 the collection \(\mathcal{U}^+\) is not empty.

**Proof.** Choose the strategy \(\tilde{u}[\alpha] = u\). For any given \(\{\tau^\alpha\} \in \mathcal{S}^t\), we have \(\tilde{u} \in U(t, \{\tau^\alpha\})\) and from Assumption 2.7, it holds for any \(u \in U(t)\) that

\[
Y_{t, x, y}^{u \otimes \alpha, \tilde{u}[\alpha], \alpha}(\rho) = Y_{t, x, y}^{u \otimes \alpha, \tilde{u}[\alpha], \alpha}(\tau^\alpha), \forall \alpha \in \mathcal{A}^t \text{ and } \rho \in \mathcal{S}^t \text{ such that } \tau^\alpha \leq \rho \leq T.
\]
From the boundedness of \( g \), there exists an \( C \), such that \( g(x) < C \). Now take \( w(t, x) \equiv C \), which clearly satisfies the first condition in Definition 2.4. On the other hand, on the set \( \{ Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha)) \} \), we clearly have that \( \{ Y(\rho) > w(\rho, X(\rho)) \} \) for any \( \rho \) such that \( \tau^\alpha \leq \rho \leq T \), which gives the second condition in Definition 2.4.

**Proposition 2.3.** If in addition to Assumptions 2.1 there exists \( a \in A \) such that \( \mu_Y(t, x, y, u, a) = 0, \sigma_Y(t, x, y, u, a) = 0 \) for all \( (t, x, y, u) \in D_{<T} \times \mathbb{R} \times U \), then \( U^- \) is not empty.

**Proof.** The proof is similar to that of Proposition 2.2. \( \square \)

The additional assumption in the latter proposition is not very reasonable. Below we introduce an alternative assumption.

**Assumption 2.8.** \( \frac{|\mu_Y|}{|\sigma_Y|} \) is bounded on \( N = \{(t, x, y, u, a) : \sigma_Y(t, x, y, u, a) \neq 0\} \).

**Proposition 2.4.** Under Assumptions 2.1, 2.2, 2.7, and 2.8, the collection \( U^- \) is not empty.

**Proof.** See the Appendix. \( \square \)

3. The Main result and its proof

To prove the main theorem, we need some preparatory lemmas.

**Lemma 3.1.** The set of stochastic super/sub solutions is upwards/downwards directed, i.e.,

1. if \( w_1, w_2 \in U^+ \), then \( w_1 \wedge w_2 \in U^+ \);
2. if \( w_1, w_2 \in U^- \), then \( w_1 \lor w_2 \in U^- \).

**Proof.** This lemma is in the spirit of Lemma 3.7 in [14]. Here we only sketch the proof for (1). For \( w_1, w_2 \in U^+ \), let \( w = w_1 \wedge w_2 \). Clearly \( w \) is bounded, continuous and \( w(T, x) \geq g(x) \). For fixed \( (t, x, y) \in D_{<T} \times \mathbb{R} \) and \( \{ \tau^\alpha \} \in \mathcal{S}^t \), let \( u_1 \) and \( u_2 \) are the strategies starting at \( \{ \tau^\alpha \} \) for \( w_1 \) and \( w_2 \), respectively. Let

\[
    u[\alpha] = u_1[\alpha] 1_{\{w_1(\tau^\alpha, X(\tau^\alpha)) < w_2(\tau^\alpha, X(\tau^\alpha))\}} + u_2[\alpha] 1_{\{w_1(\tau^\alpha, X(\tau^\alpha)) \geq w_2(\tau^\alpha, X(\tau^\alpha))\}}.
\]

It is easy to show that \( u \) works for \( w \) in the definition of stochastic super-solutions. \( \square \)

Next, we restate Lemma 3.8 in [14] for readers’ convenience, since the result will be used a few times.

**Lemma 3.2.** There exists a non-increasing sequence \( U^+ \ni w_n \searrow v^+ \) and a non-decreasing sequence \( U^- \ni v_n \nearrow v^- \).

Let us also state the following well-known result without proof.

**Lemma 3.3.** \( f(x, a) \) is defined on \( X \times A \subseteq \mathbb{R}^n \times \mathbb{R}^m \) and \( f(x, a) \) is uniformly continuous. Assume \( F(x) := \sup_{a \in A} f(x, a) < \infty \), then \( F(x) \) is continuous.

**Theorem 3.1** (Stochastic Perron for stochastic target games). Let Assumptions 2.1 and 2.2 hold.
(1) If in addition $g$ is USC and Assumption 2.3 holds, the function $v^+$ is a bounded upper semi-continuous (USC) viscosity sub-solution of (2.4).

(2) On the other hand if $g$ is LSC and Assumptions 2.4 and 2.6 hold in addition to the main assumptions, the function $v^-$ is a bounded lower semi-continuous (LSC) viscosity super-solution of (2.4).

**Corollary 3.1.** If $g$ is continuous and Assumptions 2.1-2.6 hold, then $v$ is continuous and is the unique bounded viscosity solution of (2.4).

**Proof.** The statement follows directly from Assumption 2.5 and Theorem 3.1. □

**Proof of the Theorem 3.1.**

**Step 1.** ($v^+$ is the viscosity sub-solution). First due to Proposition 2.1, $v^+$ is well-defined. We will first show the interior viscosity sub-solution property and then demonstrate the boundary condition.

**1.1 The interior sub-solution property:** Let $(t_0, x_0)$ be in the parabolic interior $[0, T) \times \mathbb{R}^d$ such that a smooth function $\varphi$ strictly touches $v^+$ from above at $(t_0, x_0)$. Assume, by contradiction, that

$$\varphi_t + H(t, x, \varphi, D\varphi, D^2\varphi) < 0 \quad \text{at} \quad (t_0, x_0).$$

From the continuity of $\mu_Y$ and $\sigma_X$ in Assumption 2.1 and the continuity of $\hat{u}$ in Assumption 2.2, the map $(t, x, y, a) \rightarrow -\mu_Y^2(t, x, y, \sigma_X(t, x, a)D\varphi(t, x), a)$ is continuous, hence uniformly continuous in $B_0 \times C_0 \times A$, where $B_0$ and $C_0$ are neighborhoods of $(t_0, x_0)$ and $\varphi(t_0, x_0)$, respectively. Similar analysis applies to $\mu_X(t, x, a)^T D\varphi(t, x)$ and $\frac{1}{2}Tr[\sigma_X^T(\sigma_X(t, x, a)D^2\varphi(t, x))]$. Hence the map $(t, x, y) \rightarrow H(t, x, y, D\varphi(t, x), D^2\varphi(t, x))$ is continuous in $B_0 \times C_0$ due to Lemma 3.3. This implies that there exists a $\varepsilon > 0$ and $\delta > 0$ such that

$$\varphi_t + H(t, x, y, D\varphi, D^2\varphi) < 0, \quad \forall (t, x) \in B(t_0, x_0, \varepsilon) \quad \text{and} \quad |y - \varphi(t, x)| \leq \delta,$$

where $B(t_0, x_0, \varepsilon) = \{(t, x) \in D : \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}$. Now, on the compact torus $T = B(t_0, x_0, \varepsilon) - B(t_0, x_0, \varepsilon/2)$, we have that $\varphi > v^+$ and the min of $\varphi - v^+$ is attained since $v^+$ is USC. Therefore, $\varphi > v^+ + \eta$ on $T$ for some $\eta > 0$. Since $w_n \searrow v^+$, a Dini type argument shows that, for large enough $n$ we have $\varphi > w_n + \eta/2$ on $T$ and $\varphi > w_n - \delta$ on $\overline{B(t_0, x_0, \varepsilon/2)}$. For simplicity, fix such an $n$ and denote $w = w_n$. Now, define, for small $\kappa < \frac{\eta}{2} \wedge \delta$

$$w^\kappa \equiv \begin{cases} 
(\varphi - \kappa) \wedge w & \text{on} \ B(t_0, x_0, \varepsilon), \\
 w & \text{outside} \ B(t_0, x_0, \varepsilon).
\end{cases}$$

Since $w^\kappa(t_0, x_0) < v^+(t_0, x_0)$, we would obtain a contradiction if we can show $w^\kappa \in U^+$. Fix $t$ and $\{\tau^\alpha\} \in \mathcal{S}^t$. We need to construct a strategy $\hat{u} \in \mathcal{U}(t, \{\tau^\alpha\})$ in the definition of stochastic super-solutions for $w^\kappa$. This can be done as follows: since $w$ is a stochastic super-solution, there exists an "optimal" strategy $\hat{u}_1$ in the Definition 2.4 for $w$ starting at $\{\tau^\alpha\}$. We will construct $\hat{u}$ in two steps:

(i) $w^\kappa(\tau^\alpha, X(\tau^\alpha)) = w(\tau^\alpha, X(\tau^\alpha))$: set $\hat{u} = \hat{u}_1;$
(ii) $w^\kappa(\tau^\alpha, X(\tau^\alpha)) < w(\tau^\alpha, X(\tau^\alpha))$: Let $Y$ be the unique strong solution (which is thanks in particular to Assumption 2.3) of the equation

$$Y(t) = Y^u(t, \alpha) + \int_0^t \sigma_X(s, \alpha) d\varphi(s, \tau_{\alpha}) ds + \int_0^t \sigma_X(s, \alpha) dW_s, \quad l \geq \tau^\alpha,$$

for any $u \in \mathcal{U}(t)$ and $\alpha \in \mathcal{L}$, and set $\tilde{Y}(s) = Y^u(s, \alpha)$ for $s < \tau^\alpha$. Let $\tau_{\alpha}$ be the first exit time of $(s, X(s))$ after $\tau^\alpha$ from $B(t_0, x_0; \varepsilon/2)$ and $\theta_\alpha$ be the first time after $\tau^\alpha$ when $|\tilde{Y}(s) - \varphi(s, X(s))| \geq \delta$. Let $\theta^{\alpha} = \tau_{\alpha}^{\alpha} \wedge \theta_\alpha^{\alpha}$. We know that $\{\theta^{\alpha}\} \in \mathcal{S}^i$ from Example 1 in [2]. We will set $\tilde{u}$ to be

$$\tilde{u}_0 := \tilde{u}_0[\alpha](s) = \tilde{u}(s, \alpha, \Gamma_{\alpha, T}(s), \sigma_X(s, \alpha)) d\varphi(s, \alpha, \Gamma_{\alpha, T}(s)), \quad \tilde{u}_0 \geq \tilde{u},$$

until $\theta^{\alpha}$. Starting at $\theta^{\alpha}$, we will then follow the strategy $u^{\alpha} \in \mathcal{U}(t, \{\theta^{\alpha}\})$ which is "optimal" for $w$.

In summary, (i) and (ii) together gave us the following strategy:

$$Y(\rho) = w(\rho, X(\rho)) \quad \text{P - a.s. on } \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\},$$

where

$$\tilde{u}[\alpha] := 1_{A^i} \tilde{u}_1[\alpha] + 1_{A^i} \tilde{u}_0[\alpha][\theta, T),$$

and

$$X := \tilde{X}^\alpha \quad \text{and} \quad Y := \tilde{Y}^\alpha \tilde{u}[\alpha].$$

Note that $Y(s) = Y^u(t, \alpha) \tilde{u}[\alpha](s)$ for $s \geq \tau_{\alpha}$ and

$$Y = 1_{A^i} Y^u(t, \alpha) \tilde{u}[\alpha] + 1_{A^i} Y^u(t, \alpha) \tilde{u}[\alpha] \quad \text{for } \tau_{\alpha} \leq s \leq \theta^{\alpha}. \quad (3.2)$$

We will carry out the proof in two steps:

(i) **On the set** $A \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha))\}$, we have

$$Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha)),$$

From (3.2) and the "optimality" of $\tilde{u}_1$ (for $w$), we know

$$Y(\rho) = Y^u(t, \alpha) \tilde{u}[\alpha](\rho) \geq w(\rho, X(\rho)) \geq w^\kappa(\rho, X(\rho)) \quad \text{P - a.s on the above set.}$$

(ii) **On the set** $A^e \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X(\tau^\alpha))\}$, by the definition of $\tilde{u}_0$ and (3.2), using Itô’s Formula,

$$Y(\cdot \wedge \theta^{\alpha}) - \varphi(\cdot \wedge \theta^{\alpha}) = Y(\tau^\alpha) - \varphi(\tau^\alpha) + \int_{\tau^\alpha}^{\tau^{\alpha}} \gamma(s) ds,$$
where
\[
\gamma := \mu^\nabla_Y(\cdot, X, \sigma_X(\cdot, X, \alpha)D\varphi(\cdot, X, \alpha), \alpha) - \mu_X(\cdot, X, \alpha)D\varphi(\cdot, X) - \frac{1}{2} \text{Tr}[\sigma_X\sigma_X^\nabla(\cdot, X, \alpha)D^2\varphi(\cdot, X)] - \varphi_t(\cdot, X),
\]
since the definition of \( \hat{u} \) allows us to cancel the Brownian motion terms on the right-hand-side. On \([\tau^\alpha, \theta^\alpha], (t, X) \in B(t_0, x_0, \varepsilon) \) and \(|Y(t) - \varphi(t, X(t))| \leq \delta \), therefore from (3.1) we have that \( \gamma > 0 \). This implies that \( Y(\cdot \land \theta^\alpha) - \varphi(\cdot \land \theta^\alpha, X(\cdot \land \theta^\alpha)) \) is non-decreasing on \([\tau^\alpha, T]\) and
\[
Y(\theta^\alpha) - \varphi(\theta^\alpha, X(\theta^\alpha)) + \kappa > Y(\tau^\alpha) - \varphi(\tau^\alpha, X(\tau^\alpha)) + \kappa > 0.
\]
As a result, on the one hand, we have
\[
0 < (Y(\theta^\alpha_1) - \varphi(\theta^\alpha_1, X(\theta^\alpha_1)) + \kappa) \leq (Y(\theta^\alpha_1) - w(\theta^\alpha_1, X(\theta^\alpha_1))) \quad \text{on} \quad \{\theta^\alpha_1 < \theta^\alpha_2\}.
\]
On the other hand,
\[
Y(\theta^\alpha_2) - \varphi(\theta^\alpha_2, X(\theta^\alpha_2)) = \delta \quad \text{on} \quad \{\theta^\alpha_1 \geq \theta^\alpha_2\}.
\]
In fact, the right-hand-side of the above expression cannot be \(-\delta \) due to (3.3). Therefore,
\[
(Y(\theta^\alpha_2) - w(\theta^\alpha_2, X(\theta^\alpha_2))) = (\delta + \varphi(\theta^\alpha_2, X(\theta^\alpha_2)) - w(\theta^\alpha_2, X(\theta^\alpha_2))) > 0, \quad \text{on} \quad \{\theta^\alpha_1 \geq \theta^\alpha_2\}
\]
since \( \varphi > w - \delta \) on \( B(t_0, x_0, \varepsilon/2) \). Combining (3.3) and (3.5) we obtain
\[
Y(\theta^\alpha) - w(\theta^\alpha, X(\theta^\alpha)) > 0 \quad \text{on} \quad A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}.
\]
It follows from this conclusion and the "optimality" of \( u^\theta \) starting at \( \{\theta^\alpha\} \) that
\[
\left(Y(\rho \lor \theta^\alpha) - w^\kappa(\rho \lor \theta^\alpha, X(\rho \lor \theta^\alpha))\right) \geq \left(Y(\rho \lor \theta^\alpha) - w(\rho \lor \theta^\alpha, X(\rho \lor \theta^\alpha))\right) \geq 0,
\]
on \( A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\} \).

Also, since \( Y(\cdot \land \theta^\alpha) - \varphi(\cdot \land \theta^\alpha, X(\cdot \land \theta^\alpha)) \) is non-decreasing on \([\tau^\alpha, T]\) it follows that \( (Y(\rho \land \theta^\alpha) - \varphi(\rho \land \theta^\alpha, X(\rho \land \theta^\alpha)) + \kappa) > 0 \), which further gives
\[
(Y(\rho \land \theta^\alpha) - w^\kappa(\rho \land \theta^\alpha, X(\rho \land \theta^\alpha))) > 0, \quad \text{on} \quad A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}.
\]
From (3.6) and (3.7) we have
\[
Y(\rho) - w^\kappa(\rho, X(\rho)) \geq 0, \quad \text{on} \quad A^c \cap \{Y(\tau^\alpha) > w^\kappa(\tau^\alpha, X^\alpha)\}.
\]

1.2 The boundary condition:

**Step A:** In this step we will assume that \( \mu^\alpha_Y \) is non-decreasing in its \( y \)-variable. Assume to the contrary that for some \( x_0 \in \mathbb{R}^d \), we have
\[
v^+(T, x_0) > g(x_0).
\]
Since \( g \) is USC, then from (3.8) there exists \( \varepsilon > 0 \) such that
\[
v^+(T, x_0) > g(x) + \varepsilon \quad \text{for} \quad |x - x_0| \leq \varepsilon.
\]
Since \( v^+ \) is USC, then \( v^+ \) is bounded above on the compact (rectangular) torus \( T = B(T, x_0; \varepsilon) - B(T, x_0; \varepsilon/2) \), where \( B(T, x_0; \varepsilon) = \{ (t, x) \in D : \max \{|T - t|, |x - x_0|\} < \varepsilon \} \). Choose \( \beta > 0 \) small
enough, such that
\[ v^+(T, x_0) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_T v^+(t, x). \]

By a Dini type argument there exists a \( w \in \mathcal{U}^+ \) such that
\[ v^+(T, x_0) + \frac{\varepsilon^2}{4\beta} > \varepsilon + \sup_T w(t, x). \]  \hspace{1cm} (3.10)

For \( C > 0 \) let us denote
\[ \varphi^{\beta, \varepsilon, C}(t, x) = v^+(T, x_0) + \frac{|x - x_0|^2}{\beta} + C(T - t). \]

The map
\[ (t, x, a) \rightarrow -\mu^\beta_Y(t, x, \varphi^{\beta, \varepsilon, C} - \varepsilon, \sigma_X(t, x, a)D\varphi^{\beta, \varepsilon, C}, a) + \mu_X(t, x, a)^\top D\varphi^{\beta, \varepsilon, C} + \frac{1}{2} \text{Tr} \left[ \sigma_X\sigma_X^\top(t, x, a)D^2\varphi^{\beta, \varepsilon, C} \right] \]

is continuous, hence uniformly continuous on \( B(T, x_0; \varepsilon) \times A \). From Lemma 3.3 \( H(\cdot, \varphi^{\beta, \varepsilon, C} - \varepsilon, D\varphi^{\beta, \varepsilon, C}, D^2\varphi^{\beta, \varepsilon, C})(t, x) \) is continuous on \( B(T, x_0; \varepsilon) \) and is therefore bounded from above. As a result for a large enough \( C \) we have that
\[ \varphi^{\beta, \varepsilon, C}_i + H(\cdot, y, D\varphi^{\beta, \varepsilon, C}, D^2\varphi^{\beta, \varepsilon, C})(t, x) < 0, \forall (t, x, y) \in B(T, x_0; \varepsilon) \times \mathbb{R} \text{ s.t. } y \geq \varphi^{\beta, \varepsilon, C}(t, x) - \varepsilon, \]

where we used the monotonicity assumption of \( \mu^\beta_Y \). Making sure that \( C \geq \varepsilon/2\beta \), we obtain from 3.10 that
\[ \varphi^{\beta, \varepsilon, C} \geq \varepsilon + w \text{ on } T. \]

Also,
\[ \varphi^{\beta, \varepsilon, C}(T, x) \geq v^+(T, x_0) > g(x) + \varepsilon \text{ for } |x - x_0| \leq \varepsilon. \] \hspace{1cm} (3.11)

Now we can choose \( \kappa < \varepsilon \) and define
\[ w^{\beta, \varepsilon, C, \kappa}(t, x) = \begin{cases} (\varphi^{\beta, \varepsilon, C} - \kappa) \land w & \text{on } B(T, x_0, \varepsilon), \\
w & \text{outside } B(T, x_0, \varepsilon). \end{cases} \] \hspace{1cm} (3.12)

From 3.11 and 3.12 it is easy to see that \( w^{\beta, \varepsilon, C, \kappa}(T, x) \geq g(x) \). By applying similar arguments as in Step 1.1, we can show that \( w^{\beta, \varepsilon, C, \kappa} \) is a stochastic super-solution with \( w^{\beta, \varepsilon, C, \kappa}(T, x_0) < v^+(T, x_0) \). This contradicts the definition of \( v^+ \).

**Step B:** We now turn to showing the same result for more general \( \tilde{\mu}^\beta_Y \) and follow a proof similar to that in [8]. Fix \( c > 0 \) and define \( \tilde{Y}^{u, \alpha}_{t, x, y} \) as the strong solution of
\[ d\tilde{Y}(s) = \tilde{\mu}_Y(s, X^\alpha_{t, x}(s), \tilde{Y}(s), u[\alpha]_s, \alpha_s)ds + \tilde{\sigma}_Y(s, X^\alpha_{t, x}(s), \tilde{Y}(s), u[\alpha]_s, \alpha_s)dW_s \]

with initial data \( \tilde{Y}(t) = y \), where
\[ \tilde{\mu}_Y(t, x, y, u, a) := cy + e^{ct} \mu_Y(t, x, e^{-ct} y, u, a), \]
\[ \tilde{\sigma}_Y(t, x, y, u, a) := e^{ct} \sigma_Y(t, x, e^{-ct} y, u, a). \]
Hence, \( \bar{Y}_{t,x,y}^{u,\alpha}(s) e^{-cs} = Y_{t,x,y}^{u,\alpha}(s) \) for any \( s \in [t,T] \) by the strong uniqueness. Set \( \tilde{g}(x) := e^{Tg(x)} \) and define

\[
\tilde{v}(t, x) := \inf \{ y \in \mathbb{R} : \exists u \in \mathcal{U} \text{ s.t. } \bar{Y}_{t,x,y}^{u,\alpha}(T) \geq \tilde{g}(X_{t,x}(T)) \text{ a.s. } \forall \alpha \in \mathcal{A} \}.
\]

Therefore, \( \tilde{v}(t, x) = e^{ct}v(t, x) \). Since \( \mu_Y \tilde{g} \) has linear growth in its second argument \( y \), one can choose large enough \( c > 0 \) so that

\[
\tilde{\mu}_Y^\tilde{g} : (t, x, y, z, a) \mapsto cy + e^{ct}\tilde{\mu}_Y^\tilde{g}(t, x, e^{-ct}y, e^{-ct}z, a)
\]

is non-decreasing in its \( y \)-variable. This means that these dynamics satisfy the monotonicity assumption used in Step A above. Moreover, all the assumptions needed to apply Step A to this new problem are also satisfied. Let

\[
\tilde{H}(t, x, y, p, M) := \sup_a \left\{ -cy - e^{ct}\tilde{\mu}_Y^\tilde{g}(t, x, e^{-ct}y, e^{-ct}\sigma_X(t, x, a)p, a) + \mu_X(t, x, a)\top p + \frac{1}{2} \text{Tr} \left[ \sigma_X\sigma_X\top(t, x, a)M \right] \right\},
\]

where \( \tilde{u} \) is defined like \( u \) but now in terms of \( \tilde{\sigma}_Y \). We will denote by \( \tilde{U}^+ \) be the set of stochastic super-solutions of

\[
\varphi_t + \tilde{H}(\cdot, \varphi, D\varphi, D^2\varphi) = 0 \quad \text{on} \quad \mathcal{D}_{<T},
\]

\[
\varphi = \tilde{g} \quad \text{on} \quad \mathcal{D}_T,
\]

and \( \tilde{v}^+(t, x) := \inf_{w \in \tilde{U}^+} w(t, x) \).

From step A, we know that \( \tilde{v}^+ \) is a viscosity sub-solution of the above PDE. Since any function \( w(t, x) \) is a stochastic super-solution of (2.4) if and only if \( \tilde{w}(t, x) = e^{ct}w(t, x) \) is a stochastic super-solution of (3.13), it follows that \( \tilde{v}^+(t, x) = e^{ct}v^+(t, x) \). Now it is easy to conclude that \( v^+ \) is a viscosity sub-solution of (2.4).

**Step 2.** (\( v^- \) is the viscosity super-solution) Due to Assumption 2.6, \( v^- \) is well-defined. Next we will show that it satisfies the interior viscosity super-solution property followed by the boundary condition.

**2.1 The interior super-solution property:** Let \((t_0, x_0)\) in the parabolic interior \([0, T) \times \mathbb{R}^d\) such that a smooth function \( \varphi \) strictly touches \( v^- \) from below at \((t_0, x_0)\). Assume, by contradiction, that

\[
\varphi_t + H(\cdot, \varphi, D\varphi, D^2\varphi) > 0 \quad \text{at} \quad (t_0, x_0).
\]

Hence there exists \( a_0 \in A \), such that

\[
\varphi_t + H_{u_0,a_0}(\cdot, \varphi, D\varphi, D^2\varphi) > 0 \quad \text{at} \quad (t_0, x_0),
\]

where \( u_0 = \hat{u}(t_0, x_0, \varphi(t_0, x_0), \sigma_X(t_0, x_0, a_0)D\varphi(t_0, x_0), D^2\varphi(t_0, x_0)) \) and

\[
H_{u,a}(t, x, y, p, M) := -\mu_Y(t, x, y, u, a) + \mu_X(t, x, a)\top p + \frac{1}{2} \text{Tr} \left[ \sigma_X\sigma_X\top(t, x, a)M \right].
\]

From the continuity assumption on the coefficients in Assumption 2.3 and the continuity of \( \hat{u} \) in Assumption 2.2, there exists \( \varepsilon, \delta > 0 \) such that

\[
|y - \varphi(t, x)| \leq \delta \quad \text{and} \quad \|\sigma_Y(t, x, y, u, a_0) - \sigma_X(t, x, a_0)D\varphi(t, x)\| \leq \delta.
\]
Now, on the compact torus $T = \overline{B(t_0, x_0, \varepsilon)} - B(t_0, x_0, \varepsilon/2)$, we have that $\varphi < v^-$ and the max of $\varphi - v^-$ is attained since $v^-$ is LSC. Therefore, $\varphi + \eta < v^-$ on $T$ for some $\eta > 0$. Since $w_n \not\geq v^-$, a Dini type argument shows that, for large enough $n$ we have $\varphi + \eta/2 < w_n$ on $T$ and $\varphi < w_n + \delta$ on $\overline{B(t_0, x_0, \varepsilon/2)}$. For simplicity, fix such an $n$ and denote $w = w_n$. Now, define, for small $\kappa << \frac{\eta}{2} \land \delta$

$$w^\kappa \triangleq \begin{cases} (\varphi + \kappa) \lor w & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ w & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Since $w^\kappa(t_0, x_0) > v^-(t_0, x_0)$, we obtain a contradiction if we can show that $w^\kappa \in \mathcal{U}^\ominus$.

In order to do so, fix $t$ and $\{\tau^\alpha\} \in \mathcal{S}^t$. For a given $u \in \mathcal{U}^t$ and $\alpha \in \mathcal{A}^t$, we will construct an "optimal" $\tilde{\alpha} \in \mathcal{A}^t$ in the definition of stochastic sub-solutions for $w^\kappa$. We will divide the construction into two cases:

(i) $w(\tau^\alpha, X(\tau^\alpha)) = w^\kappa(\tau^\alpha, X(\tau^\alpha))$: Since $w$ is a stochastic sub-solution, there exists an $\tilde{\alpha}_1$ for $w$ in the definition which is "optimal" for the nature given $u$, $\alpha$ and $\tau^\alpha$. Let $\tilde{\alpha} = \tilde{\alpha}_1$.

(ii) $w(\tau^\alpha, X(\tau^\alpha)) < w^\kappa(\tau^\alpha, X(\tau^\alpha))$: Let

$$\theta_1^\alpha := \inf \left\{ t \in [\tau^\alpha, T] : (t, X_{t,x}^{\alpha \otimes \tau^\alpha}(t)) \notin B(t_0, x_0, \varepsilon/2) \right\},$$

and

$$\theta_2^\alpha := \inf \left\{ t \in [\tau^\alpha, T] : \left| Y_{t,x}^{u,\alpha \otimes \tau^\alpha}(t) - \varphi(t, X_{t,x}^{\alpha \otimes \tau^\alpha}(t)) \right| \geq \delta \right\},$$

with the convention that $\inf \emptyset = T$. Denote $\theta^\alpha = \theta_1^\alpha \land \theta_2^\alpha$. Then let $\tilde{\alpha} = a_0$ until $\theta^\alpha$. Starting from $\theta^\alpha$, choose $\tilde{\alpha} = \alpha^\ast$, where the latter is "optimal" for the nature given $\alpha$, $u$ this time onward.

In summary, the above construction yields a candidate "optimal" control for $w^\kappa$ given by

$$\tilde{\alpha} = 1_A \tilde{\alpha}_1 + 1_{A^c} (a_0 \mathbb{1}_{[\theta^\alpha, T]} + \alpha^\ast \mathbb{1}_{[\theta^\alpha, T]}),$$

where

$$A = \{ w(\tau^\alpha, X(\tau^\alpha)) = w^\kappa(\tau^\alpha, X(\tau^\alpha)) \}. $$

Let us check what we constructed actually works: Let us abbreviate

$$(X, Y) = (X_{t,x}^{\alpha \otimes \tau^\alpha \tilde{\alpha}}, Y_{t,x,y}^{u,\alpha \otimes \tau^\alpha \tilde{\alpha}}).$$

Note that

$$X = 1_A X_{t,x}^{\alpha \otimes \tau^\alpha \tilde{\alpha}_1} + 1_{A^c} X_{t,x}^{\alpha \otimes \tau^\alpha a_0} \text{ for } \tau^\alpha \leq s \leq \theta^\alpha, \quad Y_{t,x,y}^{u,\alpha \otimes \tau^\alpha \tilde{\alpha}_1} + 1_{A^c} Y_{t,x,y}^{u,\alpha \otimes \tau^\alpha a_0} \text{ for } \tau^\alpha \leq s \leq \theta^\alpha. \quad (3.18)$$

Again for brevity, let us introduce the following sets

$$E = \{ Y(\tau^\alpha) < w^\kappa(\tau^\alpha, X(\tau^\alpha)) \}, \quad E_0 = \{ Y(\tau^\alpha) < w(\tau^\alpha, X(\tau^\alpha)) \},$$

$$E_1 = \{ w(\tau^\alpha, X(\tau^\alpha)) \leq Y(\tau^\alpha) < w^\kappa(\tau^\alpha, X(\tau^\alpha)) \},$$

$$G = \{ Y(\rho) < w^\kappa(\rho, X(\rho)) \}, \quad G_0 = \{ Y(\rho) < w(\rho, X(\rho)) \}.$$ 

Observe that

$$E = E_0 \cup E_1, \quad E_0 \cap E_1 = \emptyset \text{ and } G_0 \subset G.$$
The proof will be complete if we can show that \( P(G|B) > 0 \) for any non-null set \( B \subset E \). In fact, it suffices to show that \( P(G \cap B) > 0 \). Relying on the decomposition \( P(G \cap B) = P(G \cap B \cap E_0) + P(G \cap B \cap E_1) \) (recall that \( B \subset E \)), we will divide the proof into two steps:

(i) \( P(B \cap E_0) > 0 \): Directly from the way \( \tilde{\alpha}_1 \) is defined and the definition of the stochastic sub-solutions, we get

\[
P(G_0|B \cap E_0) = P(Y_{t,x,y}^{\alpha \otimes \alpha, \beta}(\rho) < w(\rho, X_{t,x}^{\alpha \otimes \alpha, \beta}(\rho))|B \cap E_0) > 0.
\]

This further implies that \( P(G \cap B \cap E_0) \geq P(G_0 \cap B \cap E_0) > 0 \).

(ii) \( P(B \cap E_1) > 0 \): From \( [3,18] \),

\[
P(Y(\theta^\alpha) < w(\theta^\alpha, X(\theta^\alpha))|B \cap E_1) = P(Y_{t,x,y}^{\alpha \otimes \alpha, \delta_0}(\theta^\alpha) < w(\theta^\alpha, X_{t,x}^{\alpha \otimes \alpha, \delta_0}(\theta^\alpha))|B \cap E_1).
\]

The analysis in \( [3] \) show that

\[
\Delta(s) = Y(s \wedge \theta^\alpha) - (\varphi(s \wedge \theta^\alpha, X(s \wedge \theta^\alpha)) + \kappa).
\]

is a super-martingale up to a change of measure. We will summarize these arguments here: Let

\[
\lambda(s) := \sigma_Y(s, X(s), Y(s), u[a_0], a_0) - \sigma_X(s, X(s), a_0)D\varphi(s, X(s)),
\]

\[
\beta(s) := \left(\varphi_t(s, X(s)) + H^{a[a_0], 0}(s, X(s), Y(s), D\varphi(s, X(s)), D^2\varphi(s, X(s)))\right) ||\lambda(s)||^{-2} \lambda(s) 1_{\{||\lambda(s)|| > \delta\}}.
\]

From the definition of \( \theta^\alpha \) and the regularity and growth conditions in Assumptions \( 2.1 \) and \( 2.4 \) \( \beta \) is uniformly bounded on \([\tau^\alpha, \theta^\alpha] \). This ensures that the positive exponential local martingale \( M \) defined by the SDE

\[
M(\cdot) = 1 + \int_{\tau^\alpha}^{\cdot} M(s)\beta_s dW_s
\]

is a true martingale. An application of Itô’s formula immediately implies that \( M\Delta \) is a local super-martingale. By the definition of \( \theta^\alpha \), \( \Delta \) is bounded by \(-\delta - \kappa\) from below and by \( \delta - \kappa \) from above on \([\tau^\alpha, \theta^\alpha] \). Therefore, \( M\Delta \) is bounded above by a martingale \( 2M\delta \), and below by another martingale \(-2M\delta \). An application of Fatou’s Lemma implies that \( M\Delta \) is a super-martingale.

From the definition of \( E_1 \) and \( w^\kappa \), \( \Delta(\tau^\alpha) < 0 \) on \( B \cap E_1 \). The super-martingale property of \( M\Delta \) implies that there exists a non-null \( H \subset B \cap E_1, H \in F_{\tau^\alpha}^I \) such that \( \Delta(\theta^\alpha \wedge \rho) < 0 \) on \( H \). Therefore, from the decomposition

\[
\Delta(\theta^\alpha \wedge \rho) 1_H = \left(Y(\theta_1^\alpha) - (\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa)\right) 1_{H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}} + \left(Y(\theta_2^\alpha) - (\varphi(\theta_2^\alpha, X(\theta_2^\alpha)) + \kappa)\right) 1_{H \cap \{\theta_2^\alpha < \theta_1^\alpha \wedge \rho\}} + \left(Y(\rho) - (\varphi(\rho, X(\rho)) + \kappa)\right) 1_{H \cap \{\rho < \theta^\alpha\}}.
\]

we see that

\[
Y(\theta_1^\alpha) - (\varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\theta_1^\alpha < \theta_2^\alpha \wedge \rho\}, \tag{3.19}
\]

\[
Y(\theta_2^\alpha) - (\varphi(\theta_2^\alpha, X(\theta_2^\alpha)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\theta_2^\alpha < \theta_1^\alpha \wedge \rho\}, \tag{3.20}
\]

and that

\[
Y(\rho) - (\varphi(\rho, X(\rho)) + \kappa) < 0 \quad \text{on} \quad H \cap \{\rho < \theta^\alpha\}. \tag{3.21}
\]
On the one hand, on $H \cap \{ \theta_1^\alpha < \theta_2^\alpha \land \rho \}, \varphi(\theta_1^\alpha, X(\theta_1^\alpha)) + \kappa < w(\theta_1^\alpha, X(\theta_1^\alpha))$. Then from (3.19), we will have
\[ Y(\theta_1^\alpha) < w(\theta_1^\alpha, X(\theta_1^\alpha)) \quad \text{on} \quad H \cap \{ \theta_1^\alpha < \theta_2^\alpha \land \rho \}. \tag{3.22} \]

On the other hand, on $H \cap \{ \theta_2^\alpha \leq \theta_1^\alpha \land \rho \}$, we get $Y(\theta_2^\alpha) - \varphi(\theta_2^\alpha, X(\theta_2^\alpha)) = -\delta$. (The right-hand-side can not be equal to $\delta$, otherwise (3.20) would be contradicted.) Recalling the fact that $\varphi < w + \delta$ on $B(t_0, x_0; \varepsilon/2)$, this observation gives that
\[ Y(\theta_2^\alpha) - w(\theta_2^\alpha, X(\theta_2^\alpha)) = (\varphi - w)(\theta_2^\alpha, X(\theta_2^\alpha)) - \delta < 0 \quad \text{on} \quad H \cap \{ \theta_2^\alpha \leq \theta_1^\alpha \land \rho \}. \tag{3.23} \]

We have obtained in (3.22) and (3.23) that
\[ Y(\theta^\alpha) < w(\theta^\alpha, X(\theta^\alpha)) \quad \text{on} \quad H \cap \{ \theta^\alpha \leq \rho \}. \]

Now from the definition of stochastic sub-solutions and of $\alpha^*$, we have that
\[ \mathbb{P}(G_0 \cap \{ \theta^\alpha \leq \rho \}) > 0 \quad \text{if} \quad \mathbb{P}(H \cap \{ \theta^\alpha \leq \rho \}) > 0. \tag{3.24} \]

On the other hand, (3.21) implies that
\[ \mathbb{P}(G \cap \{ \theta^\alpha \geq \rho \}) > 0 \quad \text{if} \quad \mathbb{P}(H \cap \{ \theta^\alpha > \rho \}) > 0. \tag{3.25} \]

Since $\mathbb{P}(H) > 0, G_0 \subset G$, and $H \subset E_1 \cap B$, (3.24) and (3.25) imply $\mathbb{P}(G \cap E_1 \cap B) > 0$.

### 2.2 The boundary condition:
Assume that for some $x_0 \in \mathbb{R}^d$, we have
\[ v^-(T, x_0) < g(x_0). \tag{3.26} \]

Since $g$ is LSC, then from (3.20) there exists $\varepsilon > 0$ such that
\[ v^-(T, x_0) < g(x) - \varepsilon \quad \text{for} \quad |x - x_0| \leq \varepsilon. \tag{3.27} \]

Since $v^-$ is LSC, then $v^-$ is bounded below on the compact (rectangular) torus $T = B(T, x_0; \varepsilon) - B(T, x_0; \varepsilon/2)$. Choose $\beta > 0$ small enough, such that
\[ v^-(T, x_0) - \frac{\varepsilon^2}{4\beta} < \inf_T v^-(t, x) - \varepsilon. \]

By a Dini type argument, there exists a $w \in \mathcal{U}^-$, such that
\[ v^-(T, x_0) - \frac{\varepsilon^2}{4\beta} < \inf_T w(t, x) - \varepsilon. \tag{3.28} \]

We now define for $C > 0$,
\[ \varphi_{\beta, \varepsilon, C} = v^-(T, x_0) - \frac{|x - x_0|^2}{\beta} - C(T - t). \]

For any $a_0$ we can choose large enough $C$
\[ \varphi_t + H_{u_0, a_0}(\cdot, \varphi, D\varphi, D^2\varphi) > 0 \quad \text{on} \quad \overline{B(T, x_0; \varepsilon)}, \]

where $H_{u, a}$ is the same as that in (3.14), $u_0 = \hat{u}(T, x_0, \varphi(T, x_0), \sigma_X(T, x_0, a_0)D\varphi(T, x_0), a_0)$. Then from the continuity of the coefficients in Assumption 2.1 and the continuity of $\hat{u}$ in Assumption 2.2.
for any \( a_0 \), and there exists a small enough \( \delta > 0 \) such that

\[
\varphi^{\beta,\varepsilon,C}_t + H^{u,a_0}(\cdot, y, D\varphi^{\beta,\varepsilon,C}, D^2\varphi^{\beta,\varepsilon,C}) \geq 0 \quad \forall \ (t, x) \in B(T, x_0, \varepsilon) \quad \text{and} \quad (y, u) \in R \times U \ 	ext{s.t.} \quad |y - \varphi^{\beta,\varepsilon,C}(t, x)| \leq \delta \quad \text{and} \quad \|\sigma_Y(t, x, y, u, a_0) - \sigma_X(t, x, a_0)D\varphi^{\beta,\varepsilon,C}(t, x)\| \leq \delta.
\]

Choosing \( C \) at least as large as \( \varepsilon/2\beta \), we obtain from (3.28) that

\[
\varphi^{\beta,\varepsilon,C} \leq w - \varepsilon \quad \text{on} \ T.
\]

Also we have that,

\[
\varphi^{\beta,\varepsilon,C}(T, x) \leq v^{-}(T, x_0) < g(x) - \varepsilon \quad \text{for} \ |x - x_0| \leq \varepsilon.
\]

Now for \( \kappa < \varepsilon \wedge \delta \) define

\[
w^{\beta,\varepsilon,C,\kappa}(t, x, \phi, D\phi, D^2\phi) \triangleq \begin{cases} 
(\varphi^{\beta,\varepsilon,C} + \kappa) \vee w \text{ on } B(T, x_0, \varepsilon), \\
w \text{ outside } B(T, x_0, \varepsilon).
\end{cases}
\]

From (3.29) and (3.30) it is easy to see that \( w^{\beta,\varepsilon,C,\kappa}(t, x) \leq g(x) \). By applying arguments similar to Step 2.1, we can show that \( w^{\beta,\varepsilon,C,\kappa} \) is a stochastic sub-solution with \( w^{\beta,\varepsilon,C,\kappa}(T, x_0) > v^{-}(T, x_0) \). This contradicts the definition of \( v^{-} \).

### 4. Appendix

#### 4.1. Proof of Proposition 2.1

We carry out the proof in two steps. First under Assumptions 2.2 and 2.3, we will show that there exists a classical solution to (2.4). Next, we will show that, if we additionally have Assumption 2.1 then every classical super-solution is a stochastic super-solution, which implies in particular that \( U^{+} \) is not empty.

**Step 1. Existence of a classical super-solution to (2.4).**

1-A. In this step we will assume that \( \mu_0 \) is non-decreasing in its \( y \)-variable. Letting \( \phi(t, x) = -e^{\lambda t} \) we have that

\[
\phi_t + H(t, x, \phi, D\phi, D^2\phi) = -\lambda e^{\lambda t} + \sup_{a \in A} \{-\mu_0(t, x, \phi(t, x), 0, a)\}.
\]

From the linear growth condition of \( \mu_0 \) in Assumption 2.3 we know there exists an \( L > 0 \), such that \( -\mu_0(t, x, \phi(t, x), 0, a) \leq L(1 + |\phi(t, x)|) = L(1 + e^{\lambda t}). \) Therefore, from (4.1),

\[
\phi_t + H(t, x, \phi, D\phi, D^2\phi) \leq -\lambda e^{\lambda t} + L(1 + e^{\lambda t}) \leq 0, \quad \text{in} \ D, \quad \text{for} \ \lambda > 2L.
\]

Fix \( \lambda > 2L \) and choose \( N_2 \) such that \(-e^{\lambda T} + N_2 \geq \|g\|_{\infty} \). Then \( \phi'(T, x) = \phi(T, x) + N_2 \geq g(x) \). From the assumption that \( \mu_0 \) is non-decreasing in its \( y \)-variable, it holds that

\[
\phi'_t + H(t, x, \phi', D\phi', D^2\phi') \leq 0 \quad \text{on} \ D.
\]

Therefore, \( \phi' \) is a classical super-solution.

1-B. We now turn to showing the same result for more general \( \mu_0 \). This follows the same reparametrization argument outlined in Step 1.2-B in the proof of the main theorem.

**Step 2. Classical super-solutions are stochastic super-solutions.** Let \( w \) be a classical super-solution. Fix \( (t, x, y) \in D \times R \) and \( \{x^a\} \in G^t \). Let \( \overline{Y} \) be the unique strong solution (which is
thanks to Assumption 2.3 of the equation
\[ Y(l) = Y_{l,x,y}^{u,\alpha}(\tau^\alpha) + \int_{\tau^\alpha}^{\infty} \mu_\hat{Y}(s, X_{l,x}^\alpha(s), Y(s), \sigma_X(s, X_{l,x}^\alpha(s), \alpha_s)Dw(s, X_{l,x}^\alpha(s)), \alpha_s)ds \]
\[ + \int_{\tau^\alpha}^{\infty} \sigma_X(s, X_{l,x}^\alpha(s), \alpha_s) Dw(s, X_{l,x}^\alpha(s))dW_s, \quad l \geq \tau^\alpha, \]
for any \( u \in \mathcal{U}(t) \) and \( \alpha \in \mathcal{A}^t \) and set \( \overline{Y}(s) = Y_{l,x,y}^{u,\alpha}(s) \) for \( s < \tau^\alpha \). We will set \( \tilde{u} \) to be
\[ \tilde{u} := \tilde{u}[\alpha](s) = \hat{u}(s, X_{l,x}^\alpha(s), \overline{Y}(s), \sigma_X(s, X_{l,x}^\alpha(s), \alpha_s))Dw(s, X_{l,x}^\alpha(s)), \alpha_s). \]
It is not difficult to check that \( \tilde{u} \in \mathbb{U}(t, \{\tau^\alpha\}) \). We will show that for any \( u \in \mathbb{U}(t), \alpha \in \mathcal{A}^t \) and each stopping time \( \rho \in \mathbb{S} \), \( \tau^\alpha \leq \rho \leq T \) with the simplifying notation \( X := X_{l,x}^\alpha, Y := Y_{l,x,y}^{u,\alpha,\tilde{u}[\alpha],\alpha} \), we have
\[ Y(\rho) \geq w(\rho, X(\rho)) \quad \mathbb{P} - \text{a.s. on } \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}. \]
Note that \( \overline{Y} = Y_{l,x,y}^{u,\alpha,\tilde{u}[\alpha],\alpha} \) for \( s \geq \tau^\alpha \). We will carry out the rest of the proof in two steps.

**2-A.** In this step we will assume that \( \mu_\hat{Y} \) is non-decreasing in its \( y \)-variable. Let
\[ A = \{Y(\tau^\alpha) > w(\tau^\alpha, X(\tau^\alpha))\}, \quad Z(s) = w(s, X(s)), \quad \Gamma(s) = (Z(s) - Y(s)) \mathbb{1}_A. \]
Therefore, for \( s \geq \tau^\alpha \),
\[ dY = \mu_\hat{Y}(s, X(s), Y(s), \sigma_X(s, X(s), \alpha_s))Dw(s, X(s)), \sigma_X(s, X(s), \alpha_s))Dw(s, X(s))dW_s, \]
\[ dZ = \left\{ w_l(s, X(s)) + \mu_X(s, X(s), \alpha_s)^\top Dw(s, X(s)) + \frac{1}{2} Tr[\sigma_X \sigma_X^\top(s, X(s), \alpha_s)D^2 w(s, X(s))] \right\} ds \]
\[ + \sigma_X(s, X(s), \alpha_s) Dw(s, X(s))dW_s. \]
From above equations,
\[ \Gamma(s) = \mathbb{1}_A \int_{\tau^\alpha}^{s} (\xi(u) - \gamma'(u))du \quad \text{for } s \geq \tau^\alpha, \quad (4.2) \]
where \( \gamma' := \mu_\hat{Y}(\cdot, X, w(\cdot, X), \sigma_X(\cdot, X, \alpha))Dw(\cdot, X, \alpha) - \mu_X(\cdot, X, \alpha)^\top Dw(\cdot, X) - \frac{1}{2} Tr[\sigma_X \sigma_X^\top(\cdot, X, \alpha)D^2 w(\cdot, X)] - w_l(\cdot, X), \)
and
\[ \xi := \mu_\hat{Y}(\cdot, X, Z, \sigma_X(\cdot, X, \alpha))Dw(\cdot, X, \alpha) - \mu_\hat{Y}(\cdot, Y, \sigma_X(\cdot, X, \alpha))Dw(\cdot, X, \alpha). \]
Since \( w \) is a classical super-solution \( \gamma' \geq 0 \). Then from (4.4) it follows that
\[ \Gamma(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^{s} \xi(u)du \quad \text{and} \quad \Gamma^+(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^{s} \xi^+(u)du, \quad \text{for } s \geq \tau^\alpha. \]
From the Lipschitz continuity of \( \mu_\hat{Y} \) in Assumption 2.3 Therefore,
\[ \Gamma^+(s) \leq \mathbb{1}_A \int_{\tau^\alpha}^{s} \xi^+(u)du \leq \int_{\tau^\alpha}^{s} L \Gamma^+(u)du \quad \text{for } s \geq \tau^\alpha, \]
where we use the assumption that \( \mu_\hat{Y} \) is non-decreasing in its \( y \)-variable to obtain the second inequality. Since \( E \Gamma^+(\tau^\alpha) = 0 \), an application of Gronwall’s Inequality implies that \( E \Gamma^+(\rho) \leq 0 \).
\textbf{Step B:} Now let us turn to showing same result for more general \( \mu^\alpha_Y \). However, this again follows the same reparametrization argument outlined in Step 1.2-B in the proof of the main theorem. \( \square \)

4.2. Proof of Proposition 2.3. Take \( w(t,x) = m \) for any \( (t,x) \in \mathcal{D} \), where the constant \( m \) is a lower bound of \( g \). For any given \( u \in \mathcal{U}(t) \), \( \alpha \in \mathcal{A}_t \), choose any \( \tilde{\alpha} \in \mathcal{A}_t \). Let \( B \subset \{ Y(\tau^\alpha) < w(\tau, X(\tau^\alpha)) \} \) and \( \mathbb{P}(B > 0) \). Set

\[
\theta_s = \begin{cases} \frac{\mu_Y^\alpha}{\| \sigma_Y \|}(s,X(s),Y(s),u[\alpha \otimes \tau^\alpha \tilde{\alpha}]_s,\sigma_Y(s),\tilde{\alpha})_s, & \text{if } \sigma_Y(s,X(s),Y(s),u[\alpha \otimes \tau^\alpha \tilde{\alpha}]_s,\sigma_Y(s),\tilde{\alpha})_s \neq 0 \\ C, & \text{otherwise,} \end{cases}
\]

for some constant \( C \). Therefore, \( \theta_s \) satisfies the Novikov’s condition due to Assumption 2.8 and \( \hat{W}(s) = W(s) - \int_0^s \theta_u du \) is a Brownian motion under the probability measure \( \mathbb{Q} \), where

\[
\mathbb{Q}(A) = \mathbb{E}_\mathbb{P}(Z_T \mathbb{1}_A) \quad \text{for all } A \subset \mathcal{F}, \quad \text{and } Z_s := \exp \left( \int_0^s \theta_u dW_u - \frac{1}{2} \int_0^s \| \theta_u \|^2 du \right).
\]

\( Z_T \in \mathbb{L}^q(\mathbb{P}) \) for any \( q \geq 1 \) since \( \theta \) is a bounded. From Assumption 2.7 and assumption that \( \sigma_Y \) is invertible in its \( u \)-variable (Assumption 2.2), it follows that \( \sigma_Y = 0 \) implies \( \mu_Y = 0 \). Therefore, under \( \mathbb{Q} \)

\[
dY(s) = \sigma_Y(s,X(s),Y(s),u[\tilde{\alpha}]_s,\tilde{\alpha})_s d\hat{W}_s \quad \text{for } s \geq \tau^\alpha,
\]

where \( Y := Y^{u,\alpha \otimes \tau^\alpha \tilde{\alpha}}_{L,x,y} \). We will show that the \( \mathbb{Q} \)-local martingale \( Y \) is actually a \( \mathbb{Q} \)-martingale. Assumption 2.1 implies that

\[
\mathbb{E}_\mathbb{P} \left[ \sup_{0 \leq s \leq T} |Y(s)|^2 \right] < \infty,
\]

see e.g. Theorem 1.3.5 in [10] or Theorem 2.2 in [17]. As a result an application of Hölder’s inequality yields

\[
\mathbb{E}_\mathbb{Q} \left[ \sup_{0 \leq s \leq T} |Y(s)| \right] = \mathbb{E}_\mathbb{P} \left[ \sup_{0 \leq s \leq T} |Y(s)| \cdot Z_T \right] \leq \mathbb{E}_\mathbb{P} \left[ \sup_{0 \leq s \leq T} |Y(s)|^2 \right] \mathbb{E}_\mathbb{P}[Z_T^2] < \infty
\]

From (4.4), \( Y \) is a martingale on \( [\tau^\alpha, T] \) under \( \mathbb{Q} \). Moreover, since \( \mathbb{Q} \) is equivalent to \( \mathbb{P} \) we have \( \mathbb{Q}(B) > 0 \). As a result of the latter two statements

\[
Y(\rho) \leq Y(\tau^\alpha) \quad \text{on some set } H \subset B \quad \text{with } \mathbb{Q}(H) > 0.
\]

Since \( H \subset B \)

\[
Y(\rho) \leq Y(\tau^\alpha) < m = w(t,x) \quad \text{on } H.
\]

This implies \( \mathbb{Q}(Y(\rho) < m|B) > 0 \) and by equivalence of the measures \( \mathbb{P}(Y(\rho) < m|B) > 0 \). Therefore, \( w(t,x) = m \) is a stochastic sub-solution. \( \square \)

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Department of Mathematics, University of Michigan
E-mail address: erhan@umich.edu

Department of Mathematics, University of Michigan
E-mail address: lijiaqi@umich.edu