AN INDEX THEOREM ON ANTI-SELF-DUAL ORBIFOLDS

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Abstract. An index theorem for the anti-self-dual deformation complex on anti-self-dual orbifolds with singularities conjugate to ADE-type is proved. In 1988, Claude Lebrun gave examples of scalar-flat Kähler ALE metrics with negative mass, on the total space of the bundle $O(-n)$ over $S^2$. A corollary of this index theorem is that the moduli space of anti-self-dual ALE metrics near each of these metrics has dimension at least $4n - 12$, and thus for $n \geq 4$ the LeBrun metrics admit a plethora of non-trivial anti-self-dual deformations.

1. Introduction

If $(M^4, g)$ is an oriented four-dimensional Riemannian manifold, the Hodge star operator associated to $g$ acting on 2-forms is a mapping $*: \Lambda^2 \mapsto \Lambda^2$ satisfying $*^2 = Id$, and $\Lambda^2$ admits a decomposition of the form

\begin{equation}
\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-,
\end{equation}

where $\Lambda^2_\pm$ are the $\pm 1$ eigenspaces of $*|_{\Lambda^2}$. The Weyl tensor can be viewed as an operator $W_g : \Lambda^2 \to \Lambda^2$, and we define $W^\pm_g = \pi_\pm W_g \pi_\pm$, where $\pi_\pm$ is the projection onto $\Lambda^2_\pm$.

Definition 1.1. Let $(M^4, g)$ be an oriented four-manifold. The metric $g$ is called anti-self-dual if $W^+_g = 0$.

Since Poon’s example of a 1-parameter family of anti-self-dual metrics on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ [Poo86], a large number of examples of anti-self-dual metrics on various four-manifolds have been found. We do not attempt to give a complete history here, and only mention following works [DF89, Flo91, Hon07, Joy95, KS01, LeB91, LS94]. In this paper, we will be concerned with orbifold metrics in dimension four with isolated orbifold points:

Definition 1.2. A Riemannian orbifold $(M^4, g)$ is a topological space which is a smooth manifold of dimension 4 with a smooth Riemannian metric away from finitely many singular points. At a singular point $p$, $M$ is locally diffeomorphic to a cone $C$ on $S^3/\Gamma$, where $\Gamma \subset SO(4)$ is a finite subgroup acting freely on $S^3$. Furthermore, at such a singular point, the metric is locally the quotient of a smooth $\Gamma$-invariant metric on $B^4$ under the orbifold group $\Gamma$.
Given a compact Riemannian orbifold \((\hat{M}, \hat{g})\) with non-negative scalar curvature, one can use the Green’s function for the conformal Laplacian \(G_p\) to associate with any point \(p\) a non-compact scalar-flat orbifold by
\[
(M \setminus \{p\}, g_p = G_p^2 \hat{g}).
\]
A coordinate system at infinity arises from using inverted normal coordinates in the metric \(g\) in a neighborhood of the point \(p\), which gives rise to the following definition:

**Definition 1.3.** A complete Riemannian manifold \((X^4, g)\) is called **asymptotically locally Euclidean** or ALE of order \(\tau\) if there exists a finite subgroup \(\Gamma \subset SO(4)\) acting freely on \(S^3\) and a diffeomorphism \(\psi : X \setminus K \to (\mathbb{R}^4 \setminus B(0, R))/\Gamma\) where \(K\) is a compact subset of \(X\), and such that under this identification,
\[
(\psi_\ast g)_{ij} = \delta_{ij} + O(\rho^{-\tau}),
\]
\[
\partial^k(\psi_\ast g)_{ij} = O(\rho^{-\tau-k}),
\]
for any partial derivative of order \(k\), as \(r \to \infty\), where \(\rho\) is the distance to some fixed basepoint.

By an orbifold compactification of an ALE space \((X, g)\), we mean choosing a conformal factor \(u : X \to \mathbb{R}_+\) such that \(u = O(\rho^{-2})\) as \(\rho \to \infty\). The space \((X, u^2 g)\) then compactifies to a \(C^{1,\alpha}\) orbifold. In the anti-self-dual case, there moreover exists a \(C^\infty\)-orbifold conformal compactification \((\hat{X}, \hat{g})\) with positive Yamabe invariant [CLW08, Proposition 12]. So from the conformal perspective, anti-self-dual ALE spaces are more or less the same as anti-self-dual Riemannian orbifolds.

There are many interesting examples of anti-self-dual ALE spaces. Eguchi-Hanson discovered a Ricci-flat anti-self-dual metric on \(O(-2)\) which is ALE with group \(\mathbb{Z}/2\mathbb{Z}\) at infinity [EH79]. Gibbons-Hawking then wrote down a metric ansatz depending on the choice of \(n\) monopole points in \(\mathbb{R}^3\), giving an anti-self-dual ALE hyperkähler metric with group \(\mathbb{Z}/n\mathbb{Z}\) at infinity, which are called multi-Eguchi-Hanson metrics [GH78, Hit79]. In 1989, Kronheimer then classified all hyperkähler ALE spaces in dimension 4, [Kro89a, Kro89b], which we will describe in Section 2. Using the Joyce construction from [Joy94], Calderbank and Singer wrote down many examples of toric ALE anti-self-dual metrics, which are moreover scalar-flat Kähler, and have cyclic groups \(\mathbb{Z}/n\mathbb{Z}\) at infinity contained in \(U(2)\) [CS04].

### 1.1. Group actions.

We will next consider the following subgroups of SU(2):

- Type \(A_n, n \geq 1\): \(\Gamma\) the cyclic group \(\mathbb{Z}_{n+1}\),
\[
\begin{pmatrix}
\exp^{2\pi ip/(n+1)} & 0 \\
0 & \exp^{-2\pi ip/(n+1)}
\end{pmatrix}, \quad 0 \leq p \leq n.
\]
acting on \(\mathbb{R}^4\), which is identified with \(\mathbb{C}^2\) via the map
\[
(x_1, y_1, x_2, y_2) \mapsto (x_1 + iy_1, x_2 + iy_2) = (z_1, z_2).
\]
Writing a quaternion \(q \in \mathbb{H}\) as \(\alpha + j\beta\) for \(\alpha, \beta \in \mathbb{C}\), we can also describe the action as generated by \(e^{2\pi i/n}\), acting on the left.
• Type \( D_n, n \geq 3 \): \( \Gamma \) the binary dihedral group \( \mathbb{D}_{n-2} \) of order \( 4(n-2) \). This is generated by \( e^{\frac{\pi i}{n-2}} \) and \( \hat{j} \), both acting on the left.

• Type \( E_6 \) : \( \Gamma = \mathbb{T}^* \), the binary tetrahedral group of order 24, double cover of \( A(4) \).

• Type \( E_7 \) : \( \Gamma = \mathbb{O}^* \), the binary octahedral group of order 48, double cover of \( S(4) \).

• Type \( E_8 \) : \( \Gamma = \mathbb{I}^* \), the binary icosahedral group of order 120, double cover of \( A(5) \).

The “type” terminology arises from the relation with hyperkähler ALE spaces, which will be discussed in Section 2. Next, we have the notion of conjugate group actions:

**Definition 1.4.** A group action \( \Gamma_1 \subset \text{SO}(4) \) is *conjugate* to another group action \( \Gamma_2 \subset \text{SO}(4) \) if there is an intertwining map between the corresponding representations. That is, writing the \( \Gamma_i \)-action as a map \( F_i : \Gamma_1 \rightarrow \text{SO}(4) \) for \( i = 1, 2 \), then there exists an element \( O \in \text{O}(4) \) such that \( F_1 \circ O = O \circ F_2 \). If \( O \in \text{SO}(4) \), then \( \Gamma_1 \) and \( \Gamma_2 \) are said to be orientation-preserving conjugate, while if \( O \notin \text{SO}(4) \), then \( \Gamma_1 \) and \( \Gamma_2 \) are said to be orientation-reversing conjugate.

When \( \Gamma \) is not a cyclic group, any two subgroups of \( \text{SO}(4) \) which are isomorphic to \( \Gamma \) are in fact conjugate [McC02]. However, in the case of the cyclic group, there can be many conjugacy classes, and in this paper the only cyclic group actions we consider are those conjugate to the \( A_n \)-type. Type \( D_3 \) is in fact orientation-preserving conjugate to type \( A_3 \).

We note the important fact that if \((X, g)\) is an anti-self-dual ALE space with group \( \Gamma \) at infinity, then the conformal compactification \((\hat{X}, \hat{g})\) with the anti-self-dual orientation has group \( \tilde{\Gamma} \) at the orbifold point where \( \tilde{\Gamma} \) is orientation-reversing conjugate to \( \Gamma \).

### 1.2. Orbifold index theorems.

Anti-self-dual metrics have a rich obstruction theory. If \((M, g)\) is an anti-self-dual four-manifold, the deformation complex is given by

\[
\Gamma(T^*M) \xrightarrow{\mathcal{K}_g} \Gamma(S^2_0(T^*M)) \xrightarrow{\mathcal{D}} \Gamma(S^2_0(\Lambda^2_+)),
\]

where \( S^2_0 \) denotes traceless symmetric tensors, \( \mathcal{K}_g \) is the conformal Killing operator defined by

\[
(\mathcal{K}_g(\omega))_{ij} = \nabla_i \omega_j + \nabla_j \omega_i - \frac{1}{2} (\delta \omega) g,
\]

with \( \delta \omega = \nabla^i \omega_i \), and \( \mathcal{D} = (W^+)^g \) is the linearized self-dual Weyl curvature operator.

For a compact smooth closed manifold, there is a formula for the index depending only upon topological quantities. Let us denote by

\[
\text{Ind}(M, g) = \dim(H^0(M, g)) - \dim(H^1(M, g)) + \dim(H^2(M, g)),
\]
where $H^i(M, g)$ is the $i$th cohomology of the complex (1.7), for $i = 0, 1, 2$. For a compact anti-self-dual metric, we have

\begin{equation}
\text{Ind}(M, g) = \frac{1}{2}(15\chi(M) + 29\tau(M)),
\end{equation}

where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the signature of $M$. This formula is proved in [KK92], but was also known to some experts before that paper, see for example [Flo91, equation (1.2)], and [EGH80, page 369] where it is attributed to I.M. Singer in 1978.

Our first result is an index theorem for an anti-self-dual orbifold with a singularity orientation-reversing conjugate to ADE-type:

**Theorem 1.5.** Let $(\hat{M}, \hat{g})$ be a compact anti-self-dual orbifold with a single orbifold point $p$ with orbifold group $\Gamma$. If $\Gamma$ is orientation-reversing conjugate to type $A_1$, then

\begin{equation}
\text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) - 4.
\end{equation}

If $\Gamma$ is orientation-reversing conjugate to type $A_n$ with $n \geq 2$, then

\begin{equation}
\text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) + 4n - 10.
\end{equation}

If $\Gamma$ is orientation-reversing conjugate to type $D_3$, then

\begin{equation}
\text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) + 2.
\end{equation}

If $\Gamma$ is orientation-reversing conjugate to type $D_n$ with $n \geq 4$, or type $E_n$ with $n = 6, 7, 8$, then

\begin{equation}
\text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) + 4n - 11.
\end{equation}

The next result is an index theorem for an anti-self-dual orbifold with a singularity orientation-preserving conjugate to ADE-type:

**Theorem 1.6.** Let $(\hat{M}, \hat{g})$ be a compact anti-self-dual orbifold with a single orbifold point $p$ with orbifold group $\Gamma$ orientation-preserving conjugate to type $A_n$ with $n \geq 1$, or $D_n$ with $n \geq 3$, or $E_n$ with $n = 6, 7, 8$. Then

\begin{equation}
\text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) - 4n.
\end{equation}

The proofs of Theorem 1.5 and Theorem 1.6 use Kawasaki’s orbifold index theorem [Kaw81]. However, we do not compute the correction terms directly, but instead use an analytic method to determine the correction terms using certain examples.

**Remark 1.7.** For simplicity, the above theorems are stated in the case of a single orbifold point. However, if there are several orbifold points each of the above types, then a similar formula holds, with the correction term simply the sum of the corresponding correction terms for each type of orbifold point.

\[\text{\footnote{The correction term for any cyclic quotient singularity has recently been computed in [LV12].}}\]
1.3. LeBrun negative mass metrics. In [LeB88], LeBrun presented the first known examples of scalar-flat ALE spaces of negative mass, which gave counterexamples to extending the positive mass theorem to ALE spaces. We briefly describe these as follows. Define
\[ g_{LB} = \frac{dr^2}{1 + Ar^{-2} + Br^{-4}} + r^2 \left[ \sigma_1^2 + \sigma_2^2 + (1 + Ar^{-2} + Br^{-4})\sigma_3^2 \right], \]
where \( r \) is a radial coordinate, and \( \{\sigma_1, \sigma_2, \sigma_3\} \) is a left-invariant coframe on \( S^3 = SU(2) \), and \( A = n - 2, B = 1 - n \). Redefine the radial coordinate to be \( \hat{r}^2 = r^2 - 1 \), and attach a \( \mathbb{CP}^1 \) at \( \hat{r} = 0 \). After taking a quotient by \( \mathbb{Z}_n \), with action given by the diagonal action
\[ (z_1, z_2) \mapsto \exp^{2\pi ip/n}(z_1, z_2), \quad 0 \leq p \leq n - 1, \]
the metric then extends smoothly over the added \( \mathbb{CP}^1 \), is ALE at infinity, and is diffeomorphic to \( \mathcal{O}(-n) \). The mass is computed to be \( -4\pi^2(n-2) \), which is negative when \( n > 2 \). These metrics are scalar-flat Kähler, and satisfy \( b_2 = 1, \tau = -1, \chi = 2 \).

Since the \( \mathbb{Z}/n\mathbb{Z} \)-action in (1.17) is orientation-reversing conjugate to type \( A_{n-1} \) under the intertwining map \( (z_1, z_2) \mapsto (z_1, \overline{z}_2) \), a corollary of Theorem 1.6 is the following:

**Corollary 1.8.** Let \( (\hat{\mathcal{O}(-n)}, \hat{g}_{LB}) \) be a conformally compactified LeBrun metric. Then
\[ \text{Ind}(\hat{\mathcal{O}(-n)}, \hat{g}_{LB}) = 12 - 4n. \]

We briefly recall some details of moduli space theory [Ito93, KK92] (these references deal with the case of smooth manifolds, but the proofs are easily generalized to the setting of orbifolds). Given an anti-self-dual metric \( g \) on a compact orbifold, there is a map \( \Psi : H^1 \to H^2 \), called the Kuranishi map which is equivariant with respect to the action of \( H^0 \), and the moduli space of anti-self-dual conformal structures near \( g \) is locally isomorphic to \( \Psi^{-1}(0)/H^0 \). Therefore, if \( H^2 = 0 \), the moduli space is locally isomorphic to \( H^1/H^0 \).

Our final result is about the moduli space of anti-self-dual metrics nearby the conformally compactified LeBrun negative mass metrics. The case \( n = 1 \) is the Burns metric, which is conformal to Fubini-Study metric on \( \mathbb{CP}^2 \), and is rigid. The case \( n = 2 \) is the Eguchi-Hanson metric which is also rigid. But for \( n \geq 4 \) these metrics are not rigid as anti-self-dual metrics:

**Theorem 1.9.** For \( n \geq 4 \), the dimension of the moduli space of anti-self-dual orbifold metrics near a LeBrun metric \( (\hat{\mathcal{O}(-n)}, \hat{g}_{LB}) \) is at least \( 4n - 12 \).

This is proved in Section 4 using the above index theorems. It is easy to see that any sufficiently close deformed metric has positive orbifold Yamabe invariant, and thus there is an associated anti-self-dual ALE space [AB04, Via10]. Thus the above theorem could equivalently be stated in terms of the moduli space of anti-self-dual ALE metrics near the ALE metric \( (\mathcal{O}(-n), g_{LB}) \).

To exactly determine the dimension of the moduli space near the LeBrun metrics, it would be necessary to explicitly compute the action of \( H^0 \) on \( H^1 \). This does not
follow from the above index theorems, which is why we can only give a lower bound for the dimension of the moduli space.

1.4. Questions. We end the introduction with some interesting questions:

1. The hyperkähler ALE spaces are anti-self-dual spaces with group actions of type $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$ contained in $SU(2)$. The LeBrun negative mass metrics are examples of anti-self-dual ALE spaces with group orientation-reversing conjugate to type $A_n$. Are there in fact non-trivial examples of anti-self-dual ALE spaces with group at infinity orientation-reversing conjugate to the $D_n$ type for $n \geq 3$, and to the types $E_6$, $E_7$ and $E_8$?

2. The paper [PP90] discusses some Kähler scalar-flat deformations of the LeBrun negative mass metrics. But there is no indication given there of the dimension of such deformations; it is not clear what free parameters there are in this family. Of the $4n - 12$ dimensional family found above, how many of these are Kähler scalar-flat deformations?

3. What are the possible conformal automorphism groups of the metrics in the $4n - 12$ dimensional family found above?

4. For the LeBrun negative mass metrics on $\mathcal{O}(-n)$, what is the local dimension of the moduli space of anti-self-dual ALE metrics for $n \geq 3$? Is it equal to $4n - 12$? Are there nontrivial deformations for $n = 3$?

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2. Hyperkähler ALE spaces

The hyperkähler ALE spaces were classified in dimension 4 by Kronheimer [Kro89a, Kro89b]. These are anti-self-dual Ricci flat-ALE metrics of order 4, with groups at infinity of ADE-type described in the introduction. We write the three independent complex structures as $I, J, K$. Using the metric, these are identified with Kähler forms $\omega_I, \omega_J, \omega_K$, which are parallel self-dual 2-forms. The cohomology of these spaces are generated by 2-spheres with self-intersection $-2$, with intersection matrix given by the negative of the corresponding Cartan matrix. We summarize the above in Table 2.1.

We next have a proposition regarding infinitesimal deformations of hyperkähler ALE spaces. Notice that these spaces are anti-self-dual with the complex orientation.

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2Questions 2–4 have recently been answered by Nobuhiro Honda using arguments from twistor theory. We refer the reader to [Hon12] for the complete statement of his result.
Proposition 2.1. Let $(X, g)$ be a hyperkähler ALE space of type $A_n$ for $n \geq 1$, type $D_{n}$ for $n \geq 3$, or type $E_6$ for $n = 6, 7, 8$. For $-4 \leq \epsilon < 0$, let $H^1_{\epsilon}(X, g)$ denote the space of traceless symmetric 2-tensors $h \in S_0^2((T^*X))$ satisfying
\begin{equation}
(W^+)'_g(h) = 0, \quad \delta_g(h) = 0,
\end{equation}
where $h = O(\rho^\epsilon)$ as $\rho \to \infty$, where $\rho$ is the distance to some fixed basepoint, and $(\delta_g h)_i = \nabla^j h_{ij}$ is the divergence. Then $H^1_{\epsilon}(X, g) = H^1_{\epsilon}(\mathbb{C}P^n, g)$, and using the isomorphism $S_0^2(T^*M) = \Lambda^2 \otimes \Lambda^2$, $H^1_{\epsilon}(X, g)$ has a basis
\begin{equation}
\{\omega_1 \otimes \omega_j^- , \omega_j \otimes \omega_j^- , \omega_K \otimes \omega_j^- \},
\end{equation}
where $\{\omega_j^-, j = 1, \ldots, n = \dim(H^2(M))\}$ is a basis of the space of $L^2$ harmonic 2-forms. Consequently,
\begin{equation}
\dim(H^1_{\epsilon}(X, g)) = 3n.
\end{equation}
Proof. We begin by observing that
\begin{equation}
B'_g(h) = \Delta_L \Delta_L h = D^*_g D_g h,
\end{equation}
where $B_g$ is the Bach tensor. The first identity is proved in [GV11, Section 3] and the second identity is proved in [Itô95, Section 4]. Consequently, $h \in \text{Ker}(B'_g)$ and $\delta_g h = 0$. In the traceless divergence-free gauge, $B'_g$ is asymptotic to $\Delta^2$ as $\rho \to \infty$, so [AVis, Proposition 2.2] implies that there is no $O(\rho^{-1})$ term in the asymptotic expansion of $h$ and therefore $h = O(\rho^{-2})$ as $\rho \to \infty$. Integrating by parts, we have that $\Delta_L h = 0$, and consequently, $h$ is an infinitesimal Einstein deformation. It follows from [CT94, Section 5], that $h = O(\rho^{-4})$ as $\rho \to \infty$. We also see that if $\Delta_L h = 0$ and $\delta_g h = 0$ then $D(h) = 0$. This shows that decaying infinitesimal Einstein deformations are equivalent to decaying infinitesimal anti-self-dual deformations on these spaces.

The identification of the kernel is then given by the argument in [Biq11, Proposition 1.1]. Briefly, the operator $\Delta_L$ acting on traceless divergence free tensors can be identified with the operator $d_- d^*_-$ where
\begin{equation}
d_+: \Omega^1 \otimes \Omega^2_+ \to \Omega^2 \otimes \Omega^1_+ \cong \Gamma(S_0^2(T^*X))
\end{equation}
is the exterior derivative. Since $\Omega^2_+$ has a basis of parallel sections $\{\omega_1, \omega_j, \omega_K\}$, the proposition follows since the $L^2$-cohomology $H^2_{(2)}(X)$ is isomorphic to the usual cohomology $H^2(X)$ [Car98].
We next write down the index on the conformal compactifications of the hyperkähler ALE spaces:

**Theorem 2.2.** Let \((\hat{X}, \hat{g})\) be the conformal compactification of a hyperkähler ALE space \((X, g)\) with group \(\Gamma\) at infinity. If \(\Gamma\) is type \(A_1\), then
\[
\text{Ind}(\hat{X}, \hat{g}) = 4.
\]
(2.6)

If \(\Gamma\) of type \(A_n\) for \(n \geq 2\), then
\[
\text{Ind}(\hat{X}, \hat{g}) = -3n + 5.
\]
(2.7)

If \(\Gamma\) is of type \(D_3\), then
\[
\text{Ind}(\hat{X}, \hat{g}) = -4.
\]
(2.8)

If \(\Gamma\) is of type \(D_n\) with \(n \geq 4\), or \(E_n\) with \(n = 6, 7, 8\), then
\[
\text{Ind}(\hat{X}, \hat{g}) = -3n + 4.
\]
(2.9)

This will be proved in the following section.

### 3. Index comparison

We next have a proposition relating the index on an ALE space \((X, g)\) and the index on the compactification \((\hat{X}, \hat{g})\). We let \(\{z\}\) denote coordinates at infinity for \((X, g)\), let \(\rho = |z|\), let \(\{x\}\) denote coordinate at the orbifold point \(p\) of \((\hat{X}, \hat{g})\), and let \(r = |x|\). These satisfy \(z = x/|x|^2\) and \(\rho = r^{-1}\). We write the metric as \(g = G_p^2 \hat{g}\) where \(G_p = O(r^{-2})\) as \(x \to 0\). Similarly to \(H^1(X, g)\) defined in Proposition 2.1, we define \(H^2(X, g)\) to be the space of solutions of \(D^*_g Z = 0\) satisfying \(Z = O(\rho^s)\) as \(\rho \to \infty\). We also let \(\dim(H^0(\mathbb{R}^4/\Gamma))\) denote the dimension of the space of conformal Killing fields on \(\mathbb{R}^4/\Gamma\) with respect to the Euclidean metric.

**Proposition 3.1.** Let \((X, g)\) be an anti-self-dual ALE metric with group \(\Gamma\) at infinity, and let \((\hat{X}, \hat{g})\) be the orbifold conformal compactification. Then for \(-2 < \delta < 0\), we have
\[
-\dim(H^1_\delta(X, g)) + \dim(H^2_{2-\delta}(X, g)) + \dim(H^0(\mathbb{R}^4/\Gamma)) = \text{Ind}(\hat{X}, \hat{g}).
\]
(3.1)

**Proof.** Letting \(D_g\) denote the linearized self-dual Weyl curvature (viewed as a (1,3)-tensor), we have the conformal transformation formulas
\[
D_g(h) = D_{\hat{g}}(\hat{h}),
\]
where \(\hat{h} = G_p^{-2} h\),
(3.2)
\[
D_g^*(Z) = G_p^{-2} D_{\hat{g}}^*(\hat{Z}),
\]
where \(\hat{Z} = Z\), and
(3.3)
\[
K_g(\omega) = G_p^2 K_{\hat{g}}(\hat{\omega}),
\]
where \(\hat{\omega} = G_p^{-2} \omega\).
We note that elementary Fredholm theory shows that if (\(\hat{X}, \hat{g}\)) is a compact anti-self-dual orbifold then the cohomology groups of the complex (1.7) are isomorphic to the following:

\[
H^1(X, g) \cong \{ h \in S_0^2(T^*\hat{X}) \mid \mathcal{D}_g(h) = 0, \delta_g(h) = 0 \},
\]

and

\[
H^2(X, g) \cong \{ Z \in S_0^2(\Lambda^2) \mid \mathcal{D}_g^*Z = 0 \}.
\]

We first claim that

\[
dim(H^2_{2-\delta}(X, g)) = dim(H^2(\hat{X}, \hat{g})).
\]

To see this, from [AV12 Theorem 1.11], if \( Z \in H^2_\epsilon(X, g) \) for \( \epsilon < 0 \), then \( Z \in H^2_{-\epsilon}(X, g) \). Using the formula

\[
|Z|_g = G_\rho^{-2}|\hat{Z}|_{\hat{g}},
\]

the tensor \( \hat{Z} \) is then a bounded solution of \( \mathcal{D}_g^*\hat{Z} = 0 \) on \( \hat{X} \setminus \{p\} \). Since \( \mathcal{D}_g\mathcal{D}_g^* \)

is an elliptic fourth order operator with leading term \( \Delta^2 \), \( \hat{Z} \) extends to a smooth solution on all of \( \hat{X} \). Conversely, any bounded solution of \( \mathcal{D}_g^*\hat{Z} = 0 \) yields an element \( Z \in H^2_{-\epsilon}(X, g) \) satisfying \( \mathcal{D}_g^*Z = 0 \), and (3.7) is proved.

Using the identity (3.7), (3.1) is then equivalent to

\[
- \dim(H^2(\hat{X}, \hat{g})) + \dim(H^0(\mathbb{R}^4/\Gamma)) = \dim(H^0(\hat{X}, \hat{g})) - \dim(H^1(\hat{X}, \hat{g})),
\]

which we rewrite as

\[
\dim(H^3(\hat{X}, \hat{g})) = \dim(H^1(\hat{X}, \hat{g})) + \{ \dim(H^0(\mathbb{R}^4/\Gamma)) - \dim(H^0(\hat{X}, \hat{g})) \}.
\]

We next claim that there is a surjection

\[
F : H^1(\hat{X}, \hat{g}) \to H^1(\hat{X}, \hat{g}).
\]

To define this mapping, if \( h \in H^1(\hat{X}, \hat{g}) \) then \( h = O(\rho^{-2}) \) as \( \rho \to \infty \) by [AV12 Theorem 1.11]. From (3.2), and the formula

\[
|h|_\hat{g} = |h|_g,
\]

it follows that \( \hat{h} \) is a solution of \( \mathcal{D}_{\hat{g}}(\hat{h}) = 0 \) on \( \hat{X} \setminus \{p\} \) satisfying \( \hat{h} = O(r^2) \) as \( r \to 0 \). This implies that \( \hat{h} \in C^{1,\alpha}(S_0^2(T^*\hat{X})) \), so \( \delta_{\hat{g}}(\hat{h}) \in C^{0,\alpha}(T^*\hat{X}) \). Next, consider the operator \( \Box_{\hat{g}} = \delta_{\hat{g}}K_{\hat{g}} \) mapping from

\[
\Box_{\hat{g}} : C^{2,\alpha}(T^*\hat{X}) \to C^{0,\alpha}(T^*\hat{X}).
\]

This operator is elliptic and self adjoint, with kernel exactly the space of conformal Killing fields. Since \( \delta_{\hat{g}}(\hat{h}) \) is orthogonal to this kernel, by Fredholm theory there exists a solution \( \hat{\omega} \in C^{2,\alpha}(T^*\hat{X}) \) of the equation \( \Box_{\hat{g}}\hat{\omega} = \delta_{\hat{g}}\hat{h} \). We therefore have the decomposition

\[
\hat{h} = K_{\hat{g}}\hat{\omega} + \hat{h}_0,
\]
with \( \hat{\omega} \in C^{2,\alpha}(T^*\hat{X}) \) and \( \hat{h}_0 \in C^{1,\alpha}(S^2_0(T^*\hat{X})) \) satisfying \( \delta \hat{h}_0 = 0 \). Since \( \mathcal{D}_g \mathcal{K}_g \omega = 0 \), we have \( \mathcal{D}_g \hat{h}_0 = 0 \). As mentioned in the proof of Theorem 2.1, \( \mathcal{D}_g^* \mathcal{D}_g = B_g' \), has leading term \( \Delta^2 \) in the traceless divergence-free gauge, so the singularity is removable and therefore \( \hat{h}_0 \in H^1(\hat{X}, \hat{g}) \). The mapping \( F : h \mapsto \hat{h}_0 \) is the required mapping in (3.11).

We claim that the map \( F \) is surjective. To see this, let \( \hat{h}_0 \) satisfy \( \delta \hat{h}_0 = 0 \) and \( \mathcal{D}_g(\hat{h}_0) = 0 \). Then \( h_0 = G_p^2 \hat{h}_0 \) satisfies \( \mathcal{D}_g(h_0) = 0 \). From (3.12), we have that \( h_0 = O(1) \) as \( \rho \to \infty \), so \( \delta \rho h_0 = O(\rho^{-1}) \) as \( \rho \to \infty \). Consider

\[
\Box_g : C^{k,\alpha}(T^*X) \to C^{k-2,\alpha}(T^*X),
\]

for \( \epsilon > 0 \) small, and where the spaces are weighted Hölder spaces (see [Bar86]). The adjoint mapping has domain weight \(-4 - (1 + \epsilon) = -3 - \epsilon \). An integration by parts shows that the kernel of the adjoint therefore consists of decaying conformal Killing fields, which are necessarily trivial. Consequently, the mapping in (3.15) is surjective.

So there exists a solution \( \omega \in C^{k,\alpha}_\Gamma(T^*X) \) to the equation \( \Box_g(\omega) = \delta \rho h_0 \). Defining \( \tilde{h}_0 = h_0 - \mathcal{K}_g \omega \), we have \( \mathcal{D}_g(\tilde{h}_0) = 0 \) and \( \delta \rho (\tilde{h}_0) = 0 \), and therefore \( \tilde{h}_0 \in H^1_{\delta - 2}(X, g) \). Finally, we have that

\[
\tilde{h}_0 = G_p^{-2}\hat{h}_0 = G_p^{-2}(h_0 - \mathcal{K}_g \omega) = \hat{h}_0 - \mathcal{K}_g \hat{\omega},
\]

and therefore \( F(\tilde{h}_0) = \hat{h}_0 \).

We next identify the kernel of the map \( F \). If \( \hat{h}_0 = 0 \), then \( h = \mathcal{K}_g \hat{\omega} \) for some \( \hat{\omega} \in C^{2,\alpha}(T^*\hat{X}) \). The transformation formula (3.4) implies that \( h = \mathcal{K}_g(\omega) \), where \( \omega = G_p^2 \hat{\omega} \). Since \( \hat{\omega} \) satisfies \( \hat{\omega} = O(1) \) as \( r \to 0 \), from the formula

\[
|\omega|_g = G_p|\hat{\omega}|_{\hat{g}},
\]

it follows that \( \omega = O(\rho^2) \) as \( \rho \to \infty \). Since \( \delta \rho h = \Box_g \omega = 0 \), and \( \Box_g \) is an elliptic operator, \( \omega \) admits an asymptotic expansion at infinity with leading term a solution of \( \Box \omega = 0 \) in \( \mathbb{R}^4/\Gamma \). To count these solutions, we use the relative index theorem of [LM85], which says that for non-exceptional weights \( \delta_1 < \delta_2 \),

\[
\text{Ind}(\Box_g, \delta_2) - \text{Ind}(\Box_g, \delta_1) = N(\delta_1, \delta_2),
\]

where \( N(\delta_1, \delta_2) \) counts the dimension of the space of homogeneous solutions in \( \mathbb{R}^4/\Gamma \) with growth rate between \( \delta_1 \) and \( \delta_2 \). It is easy to see that this implies the following.

First,

\[
\dim(Ker(\Box_g, \epsilon)) = \begin{cases} 4 & \text{if } \Gamma = \{e\} \\ 0 & \text{if } \Gamma \text{ is nontrivial,} \end{cases}
\]

and \( \dim(Ker(\Box_g, 1+\epsilon)) - \dim(Ker(\Box_g, 1-\epsilon)) \) is the dimension of the space of 1-forms with linear coefficients on \( \mathbb{R}^4 \) which descend to \( \mathbb{R}^4/\Gamma \). Finally, \( \dim(Ker(\Box_g, 2+\epsilon)) - \dim(Ker(\Box_g, 2-\epsilon)) \) is the dimension of the space of 1-forms \( \omega_2 \) on \( \mathbb{R}^4 \) with quadratic coefficients which descend to \( \mathbb{R}^4/\Gamma \) and which solve \( \Box \omega_2 = 0 \).

Consequently, we have the following statements: Given any 1-form on \( \mathbb{R}^4/\Gamma \) with constant coefficients \( \omega_0 \), there is a unique solution of \( \Box_g \omega = 0 \) on \( (X, g) \) with \( \omega = \).
\[ \omega_0 + O(\rho^{-1}) \] as \( \rho \to \infty \). Given any 1-form on \( \mathbb{R}^4/\Gamma \) with linear coefficients \( \omega_1 \), there is a unique solution of \( \Box_g \omega = 0 \) on \( (X, g) \) with \( \omega = \omega_1 + O(1) \) as \( \rho \to \infty \). Finally, given any 1-form on \( \mathbb{R}^4/\Gamma \) with quadratic coefficients \( \omega_2 \) satisfying \( \Box \omega_2 = 0 \), there is a unique solution of \( \Box_g \omega = 0 \) on \( (X, g) \) with \( \omega = \omega_2 + O(\rho) \) as \( \rho \to \infty \).

However, since \( h \) is decaying and \( h = K_\omega \), the leading terms in the asymptotic expansion of \( \omega \) must be a conformal Killing field in \( \mathbb{R}^4/\Gamma \). If the group \( \Gamma = \{ e \} \), then there is a 15-dimensional space of such solutions. We are only interested in counting such solutions which do not extend to global conformal Killing fields on \( (X, g) \). Note that from (3.24), conformal Killing fields on \( (\hat{X}, \hat{g}) \) correspond exactly to the conformal Killing fields on \( (X, g) \), so the kernel of the map \( F \) is of dimension \( 15 - \dim(H^0(\hat{X}, \hat{g})) = \dim(H^0(\mathbb{R}^4)) - \dim(H^0(\hat{X}, \hat{g})) \). Since \( F \) is surjective, the theorem follows in this case. If the group \( \Gamma \) is non-trivial, then the leading term in the asymptotic expansion of \( \omega \) is of the form \( c_1 \rho d\rho + \omega_0 \), where \( \omega_0 \) is a rotational Killing field on \( \mathbb{R}^4/\Gamma \), for some constant \( c_1 \). The dimension of the space of such leading terms is given by \( \dim(H^0(\mathbb{R}^4/\Gamma)) \). Again, we are only interested in counting such solutions which do not extend to conformal Killing fields on \( (X, g) \), so the kernel of \( F \) is of dimension \( \dim(H^0(\mathbb{R}^4/\Gamma)) - \dim(H^0(\hat{X}, \hat{g})) \), and the proof is complete. \( \square \)

**Remark 3.2.** Formula (3.10) implies that, apart from the case of \((S^4, g_S)\), the space \( H^1_1(X, g) \) is always strictly larger than \( H^1(\hat{X}, \hat{g}) \) for \(-2 < \delta < 0\). The additional kernel elements are of the form \( h = K_\omega \) where \( \omega \) is a solution of \( \Box_g \omega = 0 \), which is not a conformal Killing field. Using the 1-parameter group of diffeomorphisms generated by \( \omega \) it is possible to identify this extra kernel with a subspace of “gluing parameters” in gluing theory, but we do not do this here.

**Proof of Theorem 2.2.** Since these spaces are scalar-flat Kähler, from [AV12, Theorem 1.11] and [LM08, Theorem 4.2], we have that \( \dim(H^2_{2-\delta}(X, g)) = 0 \). Consequently, Proposition 3.1 takes the form

\[
\text{Ind}(\hat{X}, \hat{g}) = -\dim(H^1(X, g)) + \dim(H^0(\mathbb{R}^4/\Gamma)).
\]

For type \( A_n \), if \( n = 1 \), then \( \dim(H^0(\mathbb{R}^4/\Gamma)) = 7 \) (this is the dimension of the isometry group plus one for the radial scaling), so Propositions 2.1 and 3.1 yield that

\[
\text{Ind}(\hat{X}, \hat{g}) = -3 + 7 = 4.
\]

If \( n \geq 2 \), then from [McC02, Section 1.3] we have \( \dim(H^0(\mathbb{R}^4/\Gamma)) = 4 + 1 = 5 \), so Propositions 2.1 and 3.1 yield that

\[
\text{Ind}(\hat{X}, \hat{g}) = -3n + 5.
\]

For type \( D_3 \), [McC02, Section 1.3] we have \( \dim(H^0(\mathbb{R}^4/\Gamma)) = 4 + 1 = 5 \), so Propositions 2.1 and 3.1 yield that

\[
\text{Ind}(\hat{X}, \hat{g}) = -4.
\]

For type \( D_m \), \( m \geq 4 \), and type \( E_n \), \( n = 6, 7, 8 \), from [McC02, Section 1.3] we have \( \dim(H^0(\mathbb{R}^4/\Gamma)) = 3 + 1 = 1 \), so Propositions 2.1 and 3.1 yield that

\[
\text{Ind}(\hat{X}, \hat{g}) = -3n + 4.
\]
4. Completion of proofs

In this section, we complete the proofs of the results stated in the Introduction.

Proof of Theorem 1.2. From Kawasaki’s orbifold index theorem [Kaw81], it follows that there is an index formula of the form

\[ \text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi_{\text{orb}}(\hat{M}) + 29\tau_{\text{orb}}(\hat{M})) + N'_\Gamma, \]

where \( N'_\Gamma \) is a correction term depending only upon the oriented conjugacy class of the group action. The quantity \( \chi_{\text{orb}} \) is the orbifold Euler characteristic defined by

\[ \chi_{\text{orb}} = \frac{1}{8\pi^2} \int_{\hat{M}} \left( |W|^2 - \frac{1}{2} |\text{Ric}|^2 + \frac{1}{6} R^2 \right) dV_{\hat{g}}, \]

where \( W \) is the Weyl tensor, \( \text{Ric} \) is the Ricci tensor, and \( R \) is the scalar curvature. The quantity \( \tau_{\text{orb}} \) is the orbifold signature defined by

\[ \tau_{\text{orb}} = \frac{1}{12\pi^2} \int_{\hat{M}} \left( |W^+_g|^2 - |W^-_g|^2 \right) dV_{\hat{g}}. \]

We have the orbifold signature formula

\[ \tau(\hat{M}) = \tau_{\text{orb}}(\hat{M}) - \eta(S^3/\Gamma), \]

where \( \Gamma \subset \text{SO}(4) \) is the orbifold group at \( p \) and \( \eta(S^3/\Gamma) \) is the \( \eta \)-invariant. The \( \eta \)-invariant for the ADE-type singularities is written down in [Nak90], but we do not require this. The Gauss-Bonnet formula in this context is

\[ \chi(\hat{M}) = \chi_{\text{orb}}(\hat{M}) + 1 - \frac{1}{|\Gamma|}. \]

See [Hit97] for a nice discussion of the formulas (4.1) and (4.5).

The quantity \( 15\chi_{\text{orb}}(\hat{M}) + 29\tau_{\text{orb}}(\hat{M}) \) may then be written as follows

\[ 15\chi_{\text{orb}}(\hat{M}) + 29\tau_{\text{orb}}(\hat{M}) = 15\chi(\hat{M}) + 29\tau(\hat{M}) - 15\left(1 - \frac{1}{|\Gamma|}\right) + 29\eta(S^3/\Gamma). \]

Consequently, Kawasaki’s formula (4.1) becomes

\[ \text{Ind}(\hat{M}, \hat{g}) = \frac{1}{2}(15\chi(\hat{M}) + 29\tau(\hat{M})) + N_\Gamma, \]

where \( N_\Gamma \) is a correction term depending only upon the oriented conjugacy class of the group action. The important point is that (4.6) holds on any anti-self-dual orbifold with one singular point orientation-preserving conjugate to type \( \Gamma \). Consequently, we can simply plug in the examples in Theorem 2.2 to determine the correction term.

If the orbifold point is of type \( A_4 \), then from Table 2.1 \( \chi(X) = 2 \), so \( \chi(\hat{X}) = 3 \), and \( \tau(\hat{X}) = -1 \) (with the anti-self-dual orientation), so from Theorem 2.2 we have

\[ 4 = \frac{1}{2}(15\chi(\hat{X}) + 29\tau(\hat{X})) + N_\Gamma = 8 + N_\Gamma, \]

so \( N_\Gamma = -4 \).
If the orbifold point is type \( A_n \) for \( n \geq 2 \), then from Table 2.1 \( \chi(\hat{X}) = n + 2 \), and \( \tau(\hat{X}) = -n \), so from Theorem 2.2 we have
\[
-3n + 5 = \frac{1}{2}(15\chi(\hat{X}) + 29\tau(\hat{X})) + N_\Gamma = \frac{1}{2}(15(n + 2) - 29n) + N_\Gamma,
\]
which implies that \( N_\Gamma = 4n - 10 \).

If the orbifold point is type \( D_3 \), then from Table 2.1 \( \chi(\hat{X}) = 5 \), and \( \tau(\hat{X}) = -3 \), so from Theorem 2.2 we have
\[
-4 = \frac{1}{2}(15\chi(\hat{X}) + 29\tau(\hat{X})) + N_\Gamma = \frac{1}{2}(15 \cdot 5 - 29 \cdot 3) + N_\Gamma,
\]
which implies that \( N_\Gamma = 2 \).

Finally, if the orbifold point is of type \( D_n \), \( n \geq 4 \), or type \( E_n \), \( n = 6, 7, 8 \), again from Table 2.1 we have \( \chi(\hat{X}) = n + 2 \), and \( \tau(\hat{X}) = -n \), so from Theorem 2.2 we have
\[
-3n + 4 = \frac{1}{2}(15\chi(\hat{X}) + 29\tau(\hat{X})) + N_\Gamma = \frac{1}{2}(15(n + 2) - 29n) + N_\Gamma,
\]
which implies that \( N_\Gamma = 4n - 11 \).

□

Proof of Theorem 1.6. If \( \Gamma \subset \text{SO}(4) \) is a finite subgroup acting freely on \( S^3 \), then we let \( \Gamma \) act on \( S^4 \subset \mathbb{R}^5 \) acting as rotations around the \( x_5 \)-axis. The quotient \( S^4/\Gamma \) is an orbifold with two singular points, and the spherical metric \( g_S \) descends to this orbifold. The north pole has orbifold group \( \Gamma \), while the south pole has orbifold group \( \tilde{\Gamma} \) orientation-reversing conjugate to \( \Gamma \). As in the proof of Theorem 1.6, and as mentioned in Remark 1.7, Kawasaki’s formula yields
\[
\text{Ind}(S^4/\Gamma, g_S) = \frac{1}{2}(15\chi(S^4/\Gamma) + 29\tau(S^4/\Gamma)) + N_\Gamma + N_{\tilde{\Gamma}}, \tag{4.8}
\]

It is easy to see that \( \text{Ind}(S^4/\Gamma, g_S) = \dim(H^0(S^4/\Gamma, g_S)) \), \( \chi(S^4/\Gamma) = 2 \), and that \( \tau(S^4/\Gamma) = 0 \), so we have
\[
\dim(H^0(S^4/\Gamma, g_S)) = 15 + N_\Gamma + N_{\tilde{\Gamma}}. \tag{4.9}
\]

We first consider \( \Gamma \) of type \( A_n \). We need only consider \( n \geq 2 \) since \( n = 1 \) is already covered in Theorem 1.5 (the \( \mathbb{Z}/2\mathbb{Z} \)-action is orientation-reversing conjugate to itself). For \( n > 1 \), (4.9) and Theorem 1.5 yield
\[
5 = 15 + 4n - 10 + N_{\tilde{\Gamma}}, \tag{4.10}
\]
which yields \( N_{\tilde{\Gamma}} = -4n \).

For \( \Gamma \) of type \( D_3 \), (4.9) and Theorem 1.5 yield
\[
5 = 15 + 2 + N_{\tilde{\Gamma}}, \tag{4.11}
\]
which yields \( N_{\tilde{\Gamma}} = -12 \) which is \( -4n \) for \( n = 3 \).

For \( \Gamma \) of type \( D_n \), \( n \geq 4 \), or type \( E_n \), \( n = 6, 7, 8 \), we have
\[
4 = 15 + 4n - 11 + N_{\tilde{\Gamma}}, \tag{4.12}
\]
which again yields \( N_{\tilde{\Gamma}} = -4n \). □
Proof of Corollary 1.8. As mentioned in the Introduction, as an ALE space, the group at infinity of the metric \((\mathcal{O}(-n), g_{LB})\) is orientation-reversing conjugate to type \(A_{n-1}\). Consequently, the group at the orbifold point of \(\hat{\mathcal{O}}(-n)\) is orientation-preserving conjugate to type \(A_{n-1}\). Theorem 1.6 yields the index
\[
\text{Ind}(\hat{\mathcal{O}}(-n), \hat{g}_{LB}) = \frac{1}{2} \left( 15\chi(\hat{\mathcal{O}}(-n)) + 29\tau(\hat{\mathcal{O}}(-n)) - 4(n-1) \right).
\]
(4.13)
Using that \(\chi(\hat{\mathcal{O}}(-n)) = 3\) and \(\tau(\hat{\mathcal{O}}(-n)) = -1\), (1.18) follows. \qed

Proof of Theorem 1.9. Since \((\mathcal{O}(-n), g_{LB})\) is scalar-flat Kähler, by the same argument as given above in the proof of Theorem 2.1, and (3.7), we have that
\[
\dim(H^2(\hat{\mathcal{O}}(-n), \hat{g}_{LB})) = 0.
\]
(4.14)
The conformal automorphism group of these metrics is \(U(2)\), which has real dimension 4, so Corollary 1.8 implies that \(\dim(H^1(\hat{\mathcal{O}}(-n), \hat{g}_{LB})) = 4n - 8\). From (4.14), as mentioned above in the Introduction, it follows that the moduli space is locally isomorphic to \(H^1/H^0\). Finally, since \(H^0\) is of dimension 4, the moduli space therefore has dimension at least \(4n - 12\). \qed

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