Fractional Moments of Dirichlet $L$-Functions

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1 Introduction

Mean-values of the type

$$I_k(T) := \int_0^T |\zeta(\frac{1}{2} + it)|^{2k} dt,$$

with positive non-integral values of $k$, have been investigated by a number of authors, including Ramachandra [5], [6], Conrey and Ghosh [1] and Heath-Brown [3]. In particular the above papers by Ramachandra show, under the Riemann Hypothesis, that

$$I_k(T) \gg_k T(\log T)^{k^2} \quad (T \geq 2)$$

for all real $k \geq 0$, and that

$$I_k(T) \ll_k T(\log T)^{k^2} \quad (T \geq 2)$$

for all real $k \in [0, 2]$.

It is natural to ask about the corresponding problem for Dirichlet $L$-functions in $q$-aspect, that is to say to investigate

$$M_k(q) := \sum_{\substack{\chi \pmod{q} \\ \chi \neq \chi_0}} |L(\frac{1}{2}, \chi)|^{2k}$$

for positive real $k$. However rather little is known about this in general. The method of Rudnick and Soundararajan [7], enables one to show unconditionally that

$$M_k(q) \gg_k \phi(q)(\log q)^{k^2}$$

for rational $k \geq 1$, at least when $q$ is prime. It is annoying that the range $0 \leq k < 1$ is not covered by this approach.

The present paper will prove results in the reverse direction, motivated by the author’s work [3]. We establish the following theorems.
Theorem 1  Assuming the Generalized Riemann Hypothesis we have
\[ M_k(q) \ll_k \phi(q)(\log q)^{k^2} \]
for all \( k \in (0, 2) \).

Theorem 2  Unconditionally we have
\[ M_k(q) \ll_k \phi(q)(\log q)^{k^2} \]
for any \( k \) of the form \( k = 1/v \), with \( v \in \mathbb{N} \).

Thus taking \( v = 2 \) we have
\[ \sum_{\chi \pmod{q}} |L(\frac{1}{2}, \chi)| \ll \phi(q)(\log q)^{1/4} \]
in particular.

The approach in [3] is based on a convexity theorem for mean-value integrals, which appears to have no analogue for character sums. We therefore work with integrals, and extract the sum \( M_k(q) \) at the end. While we can give lower bounds for the integrals that occur, as well as upper bounds, it is not clear how to give a lower bound for \( M_k(q) \) in terms of an integral.

This work arose from a number of conversations with Dr H.M. Bui, and would not have been undertaken without his prompting. It is a pleasure to acknowledge his contribution.

2 Mean-Value Integrals

Throughout our argument we will write \( v = 1 \) for the proof of Theorem 1 and \( v = k^{-1} \) in handling Theorem 2. In both cases the primary mean-value integral we will work with is
\[ J(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)|^{2k}|W(\sigma + it)|^6 dt, \]
where the weight function \( W(s) \) is defined by
\[ W(s) := \frac{q^{\delta(s-1/2)} - 1}{(s - 1/2) \log q}, \]
with \( \delta > 0 \) to be specified later, see (6) and (7). We emphasize that, for the rest of this paper, all constants implied by the Vinogradov \( \ll \) symbol will
be uniform in \( \sigma \) for the ranges specified. However, they will be allowed to depend on the values of \( k \) and \( \delta \), so that the symbol \( \ll \) should be read as \( \ll_{k, \delta} \) throughout.

In addition to the integral \( J(\sigma, \chi) \) we will use

\[
K(\sigma, \chi) := \int_{-\infty}^{\infty} |S(\sigma + it, \chi)|^2|W(\sigma + it)|^6 dt,
\]

where

\[
S(s) := \sum_{n \leq q} d_k(n) \chi(n)n^{-s}
\]

Notice here that a little care is needed in defining \( d_k(n) \) when \( k \) is not an integer, see [3, §2].

When \( \chi \) is a non-principal character the function \( L(s, \chi) \) is entire. Moreover, if we assume the Generalized Riemann Hypothesis then there are no zeros for \( \sigma > \frac{1}{2} \), so that one can define a holomorphic extension of

\[
L(s, \chi)^k = \sum_{m=1}^{\infty} d_k(m) \chi(m)m^{-s} \quad (\sigma > 1)
\]

in the half-plane \( \sigma > \frac{1}{2} \). Having defined \( L(s, \chi)^k \) in this way we now set

\[
G(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)^k - S(\sigma + it, \chi)|^2|W(\sigma + it)|^6 dt, \quad (\sigma > \frac{1}{2}).
\]

This integral will be used in the proof of Theorem 1 while for the unconditional Theorem 2 we will employ

\[
H(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)^v - S(\sigma + it, \chi)^v|^2|W(\sigma + it)|^6 dt.
\]

In addition to \( J(\sigma, \chi), K(\sigma, \chi), G(\sigma, \chi) \) and \( H(\sigma, \chi) \) we will consider their averages over non-principal characters,

\[
J(\sigma) := \sum_{\substack{\chi \pmod{q} \\
\chi \neq \chi_0}} J(\sigma, \chi), \quad K(\sigma) := \sum_{\substack{\chi \pmod{q} \\
\chi \neq \chi_0}} K(\sigma, \chi)
\]

\[
G(\sigma) := \sum_{\substack{\chi \pmod{q} \\
\chi \neq \chi_0}} G(\sigma, \chi), \quad \text{and} \quad H(\sigma) := \sum_{\substack{\chi \pmod{q} \\
\chi \neq \chi_0}} H(\sigma, \chi).
\]

To derive estimates relating values of these integrals we begin with the following convexity estimate of Gabriel [2, Theorem 2].
Lemma 1 Let $F$ be a complex-valued function which is regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \leq \Re(z) \leq \beta$. Suppose that $|F(z)|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \leq \Re(z) \leq \beta$. Then for any $\gamma \in [\alpha, \beta]$ and any $a > 0$ we have

$$I(\gamma) \leq I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}$$

where

$$I(\eta) := \int_{-\infty}^{\infty} |F(\eta + it)|^a dt.$$

The inequality should be interpreted appropriately if any of the integrals diverge. From Lemma 1 we will deduce the following variant.

Lemma 2 Let $f$ and $g$ be complex-valued functions which are regular in the strip $\alpha < \Re(z) < \beta$, and continuous for $\alpha \leq \Re(z) \leq \beta$. Let $b$ and $c$ be positive real numbers. Suppose that $|f(z)|^b |g(z)|^c$ and $|g(z)|$ tend to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \leq \Re(z) \leq \beta$. Set

$$I(\eta) := \int_{-\infty}^{\infty} |f(\eta + it)|^b |g(\eta + it)|^c dt.$$

Then for any $\gamma \in [\alpha, \beta]$ we have

$$I(\gamma) \leq I(\alpha)^{(\beta-\gamma)/(\beta-\alpha)} I(\beta)^{(\gamma-\alpha)/(\beta-\alpha)}.$$

To deduce Lemma 2 from Lemma 1 we choose a rational number $p/q > c/b$, and apply Lemma 1 with $F = f^q g^p$ and $a = b/q$. Since

$$|F| = (|f|^b |g|^c)^{p/q} |g|^{p-cq/b}$$

with $p-cq/b > 0$, we deduce that $|F|$ tends to zero as $|\Im(z)| \to \infty$, uniformly for $\alpha \leq \Re(z) \leq \beta$. We then obtain an inequality of the same shape as (1), but with the exponent $c$ replaced by $bp/q$. Lemma 2 then follows on choosing a sequence of rationals $p_n/q_n$ tending downwards to $c/b$.

We now apply Lemma 2 to $J(\sigma, \chi)$. When $\sigma = 3/2$ we have

$$W(s) \ll q^\delta/(1 + |t|)$$

whence we trivially obtain

$$J(\frac{3}{2}, \chi) \ll q^{6\delta}.$$

An immediate application of Lemma 2 therefore yields

$$J(\sigma, \chi) \ll J(\frac{1}{2}, \chi)^{3/2-\sigma} q^{6\delta(\sigma-1/2)}$$
for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$, whence we trivially deduce that

$$J(\sigma) \ll J(\frac{1}{2})^{3/2-\sigma}q^{6\delta(\sigma-1/2)},$$

by Hölder’s inequality. Since

$$J^f \leq \left(\frac{\log q}{q}\right)^{1-f}\left(\frac{q}{\log q} + J\right) \ll q^{-1-\delta(1-f)}\left(\frac{q}{\log q} + J\right)$$

for any $J \geq 0$ and any $f \in [0, 1]$, we conclude as follows.

**Lemma 3** We have

$$J(\sigma) \ll q^{-1-7\delta(\sigma-1/2)}\left(\frac{q}{\log q} + J(\frac{1}{2})\right)$$

for $\frac{1}{2} \leq \sigma \leq \frac{3}{2}$.

To obtain a second estimate involving $J(\sigma, \chi)$ we use Lemma 2 to show that if $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ and $1 - \sigma \leq \gamma \leq \sigma$ then

$$J(\gamma, \chi) \leq J(\sigma, \chi)^{(\gamma-1+\sigma)/(2\sigma-1)}J(1-\sigma, \chi)^{(\sigma-\gamma)/(2\sigma-1)}.$$

An application of Hölder’s inequality then shows that

$$J(\gamma) \leq J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)}J(1-\sigma)^{(\sigma-\gamma)/(2\sigma-1)}.$$

To handle $J(1-\sigma, \chi)$ we will use the functional equation for $L(s, \chi)$. If $\psi$ is primitive, with conductor $q_1$, this yields

$$L(1-\sigma + it, \psi) \ll (1 + |t|)^{\sigma-1/2}q_1^{-1/2}\rho L(1-\sigma + it, \psi)$$

for $\frac{1}{2} \leq \sigma \leq \frac{3}{4}$ say. Thus if $\psi$ induces a character $\chi$ modulo $q$ we will have

$$L(1-\sigma + it, \chi) \ll (1 + |t|)^{\sigma-1/2}q^{-1/2}\rho|L(1-\sigma + it, \chi)|$$

with

$$\rho = \prod_{p|q_2} \left(\frac{|1-\chi(p)p^{-\sigma-it}|}{|1-\chi(p)p^{1-\sigma-it}|}\right),$$

where $q_2 = q/q_1$. Thus

$$\log \rho \leq (2\sigma - 1) \sum_{p|q_2} \frac{\log p}{p^{1-\sigma}-1}.$$
However
\[ \sum_{p|m} \frac{\log p}{p^{1/4} - 1} \leq \frac{1}{2} \log m \]
for all sufficiently large \( m \), whence \( \rho \ll q^\sigma^{-1/2} \). We therefore conclude that
\[ L(1 - \sigma + it, \chi) \ll (1 + |t|)^{\sigma-1/2} q^{\sigma-1/2} |L(\sigma + it, \chi)| \]
when \( \frac{1}{2} \leq \sigma \leq \frac{3}{4} \), for any character \( \chi \) modulo \( q \), whether primitive or not.

We now deduce that
\[ L(1 + \sigma + it, \chi) \ll (1 + |t|)^{\sigma-1/2} q^{\sigma-1/2} |W(1 - \sigma + it)|^6 dt. \]

The presence of the factor \( (1 + |t|)^{2k(\sigma-1/2)} \) is inconvenient. However, since \( 0 < k < 2 \) we have
\[ (1 + |t|)^{2k(\sigma-1/2)} |W(1 - \sigma + it)|^6 \ll (\log q)^{-6} |t|^{-2}, \]
for \( |t| \geq 1 \) and \( \frac{1}{2} \leq \sigma \leq 1 \). It follows that
\[ J(1 - \sigma, \chi) \ll q^{2k(\sigma-1/2)} \left( J(\sigma, \chi) + (\log q)^{-6} J^*(\sigma, \chi) \right), \]
where
\[ J^*(\sigma, \chi) := \int_{-\infty}^{\infty} |L(\sigma + it, \chi)|^{2k} \frac{dt}{1 + t^2}. \]

Thus
\[ J(1 - \sigma) \ll q^{2k(\sigma-1/2)} \left( J(\sigma) + (\log q)^{-6} J^*(\sigma) \right) \]
with
\[ J^*(\sigma) := \sum_{\chi \not\equiv \chi_0 \pmod{q}} \int_{-\infty}^{\infty} |L(\sigma + it, \chi)|^{2k} \frac{dt}{1 + t^2}. \]

Finally we observe that
\[ J(\sigma)^{(\gamma-1+\sigma)/(2\sigma-1)} \left\{ J(\sigma) + (\log q)^{-6} J^*(\sigma) \right\}^{(\sigma-\gamma)/(2\sigma-1)} \leq J(\sigma) + (\log q)^{-6} J^*(\sigma). \]

On comparing our results we therefore conclude that
\[ J(\gamma) \ll q^{k(\sigma-\gamma)} \left( J(\sigma) + (\log q)^{-6} J^*(\sigma) \right). \]

(3)
We have now to consider $J^*(\sigma)$. It was shown by Montgomery [4, Theorem 10.1] that

$$\sum_{\chi \pmod{q}}^* \int_{-T}^{T} |L(\frac{1}{2} + it, \chi)|^4 dt \ll \phi(q)T(\log qT)^4$$

for $T \geq 2$, where $\Sigma^*$ indicates that only primitive characters are to be considered. (It should be noted that there is a misprint in the statement of [4, Theorem 10.1], in that $L(\frac{1}{2} + it, \chi)$ should be replaced by $L(\sigma + it, \chi)$. However we are only interested in the case $\sigma = \frac{1}{2}$. moreover, in the proof of [4, Theorem 10.1], at the top of page 83, the reference to Theorem 6.3 should be to Theorem 6.5.)

If $\chi$ is an imprimitive character modulo $q$, induced by a primitive character $\psi$ with conductor $q_1$, then

$$|L(\frac{1}{2} + it, \chi)|^4 \leq |L(\frac{1}{2} + it, \psi)|^4 \prod_{p | q, p \not| q_1} (1 + p^{-1/2})^4.$$ 

Thus if $\Sigma^{(1)}$ indicates summation over all characters $\chi$ modulo $q$ for which the conductor has a given value $q_1$, we will have

$$\sum_{\chi}^{(1)} \int_{-T}^{T} |L(\frac{1}{2} + it, \chi)|^4 dt \ll \phi(q_1)T(\log q_1 T)^4 \prod_{p | q, p \not| q_1} (1 + p^{-1/2})^4.$$ 

If we now sum for $q_1 | q$ we obtain

$$\sum_{\chi \pmod{q}} \int_{-T}^{T} |L(\frac{1}{2} + it, \chi)|^4 dt \ll T(\log qT)^4 f(q),$$

where

$$f(q) = \sum_{q_1 | q} \phi(q_1) \prod_{p | q, p \not| q_1} (1 + p^{-1/2})^4.$$ 

The function $f$ is multiplicative, with

$$f(p^e) = (1 + p^{-1/2})^4 + \phi(p) + \phi(p^2) + \ldots + \phi(p^e) = p^e (1 + O(p^{-3/2})).$$

Thus $f(q) \ll q$ and we conclude that

$$\sum_{\chi \pmod{q}} \int_{-T}^{T} |L(\frac{1}{2} + it, \chi)|^4 dt \ll qT(\log qT)^4.$$ 

7
We may now deduce that if \( f(s) = L(s, \chi)^2 s^{-1} \) then
\[
\sum_{\chi \pmod{q}} \int_{-\infty}^{\infty} |f(\frac{1}{2} + it)|^2 dt \ll q(\log q)^4.
\]
Moreover the trivial bound \( L(s, \chi) \ll 1 \) for \( \sigma = 3/2 \) shows that
\[
\sum_{\chi \pmod{q}} \int_{-\infty}^{\infty} |f(\frac{3}{2} + it)|^2 dt \ll q.
\]
We can therefore apply Lemma 4 together with Hölder’s inequality, to deduce that
\[
\sum_{\chi \pmod{q}} \int_{-\infty}^{\infty} |f(\sigma + it)|^2 dt \ll q(\log q)^4
\]
uniformly for \( \frac{1}{2} \leq \sigma \leq \frac{3}{2} \). A final application of Hölder’s inequality then implies that
\[
J^*(\sigma) \ll q(\log q)^4.
\]
We can now insert this into (3) and deduce as follows.

**Lemma 4** We have
\[
J(\gamma) \ll q^{k(\sigma-\gamma)} \left( \frac{q}{\log q} + J(\sigma) \right)
\]
for \( \frac{1}{2} \leq \sigma \leq 1 \) and \( 1 - \sigma \leq \gamma \leq \sigma \).

We now turn our attention to \( G(\sigma, \chi) \) and \( H(\sigma, \chi) \). By Lemma 2 we have
\[
G(\sigma, \chi) \leq G(\frac{1}{2}, \chi)^{3/2-\sigma} G(\frac{3}{2}, \chi)^{\sigma-1/2} \quad (\frac{1}{2} \leq \sigma \leq \frac{3}{2})
\]
for non-principal characters \( \chi \) modulo \( q \). We then find via Hölder’s inequality that
\[
G(\sigma) \leq G(\frac{1}{2})^{3/2-\sigma} G(\frac{3}{2})^{\sigma-1/2}
\]
(4)
Since
\[
W(\frac{3}{2} + it) \ll q^6 (1 + |t|)^{-1}
\]
we see that
\[
G(\frac{3}{2}, \chi) \ll q^6 \int_{-\infty}^{\infty} |L(\frac{3}{2} + it, \chi)^k - S(\frac{3}{2} + it, \chi)|^2 \frac{dt}{1 + |t|^2}.
\]
However
\[
L(\frac{3}{2} + it, \chi)^k - S(\frac{3}{2} + it, \chi) = \sum_{n > q} d_k(n) \chi(n)n^{-3/2-it}
\]
for non-principal \( \chi \).
whence
\[
\int_{-\infty}^{\infty} |L(\frac{3}{2} + it, \chi) - S(\frac{3}{2} + it, \chi)|^2 \frac{dt}{1 + |t|^2}
\]
\[
= \pi \sum_{m,n>q} d_k(m)d_k(n)\chi(m)\overline{\chi(n)} \min \left\{m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}\right\}.
\]

It follows that
\[
\sum_{\chi \pmod{q}} \int_{-\infty}^{\infty} |L(\frac{3}{2} + it, \chi) - S(\frac{3}{2} + it, \chi)|^2 \frac{dt}{1 + |t|^2}
\]
\[
= \pi \phi(q) \sum_{q|m,n>q \atop q|m-n, (mn, q) = 1} d_k(m)d_k(n) \min \left\{m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}\right\}
\]

To estimate this double sum we use that fact that \(d_k(n) \ll \varepsilon n^\varepsilon\) for any fixed \(\varepsilon > 0\). This leads to the bound
\[
\sum_{q|m,n>q \atop q|m-n} d_k(m)d_k(n) \min \left\{m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}\right\} \ll \varepsilon q^{2\varepsilon - 2}.
\]

It therefore follows that
\[
\sum_{\chi \pmod{q}} \int_{-\infty}^{\infty} |L(\frac{3}{2} + it, \chi) - S(\frac{3}{2} + it, \chi)|^2 \frac{dt}{1 + |t|^2} \ll \varepsilon q^{2\varepsilon - 1}.
\]

Inserting this bound into (4) we obtain
\[
G(\sigma) \ll \varepsilon q^{(\frac{1}{2})^{3/2 - \sigma}q^{(\sigma - 1/2)(6\delta + 2\varepsilon - 1)}}.
\]

Using (2) again, we see that
\[
G(\sigma) \ll \varepsilon q^{1 - 2\sigma + (7\delta + 2\varepsilon)(\sigma - 1/2)} \left( \frac{q}{\log q} + G(\frac{1}{2}) \right)
\]

for \(\sigma \in [\frac{1}{2}, \frac{3}{2}]\). The positive number \(\varepsilon\) is at our disposal, and we choose it to be \(\varepsilon = \delta/2\), whence
\[
G(\sigma) \ll q^{-(1-4\delta)(2\sigma-1)} \left( \frac{q}{\log q} + G(\frac{1}{2}) \right).
\]

The treatment of \(H(\sigma, \chi)\) is similar. This time, since \(k = 1/v\), we have
\[
H(\frac{3}{2}, \chi) \leq \left\{ \int_{-\infty}^{\infty} |W(\frac{3}{2} + it)|^6 dt \right\}^{1-k}
\]
\[
\times \left\{ \int_{-\infty}^{\infty} |L\left(\frac{3}{2} + it, \chi\right) - S\left(\frac{3}{2} + it, \chi\right)|^2 |W\left(\frac{3}{2} + it\right)|^6 dt \right\}^k
\]
by Hölder’s inequality. The first integral on the right is trivially \(O(q^{6\delta})\).

Moreover
\[
L\left(\frac{3}{2} + it, \chi\right) - S\left(\frac{3}{2} + it, \chi\right) \sim \sum_{n \geq q} a_k(n) \chi(n) n^{-3/2 - it}
\]
with certain coefficients \(a_k(n) \ll \varepsilon n^\varepsilon\). The argument then proceeds as before, noting that
\[
\sum_{\substack{m,n \geq q \atop q|m-n}} a_k(m)a_k(n) \min\left(m^{-1/2}n^{-5/2}, n^{-1/2}m^{-5/2}\right) \ll \varepsilon q^{2\varepsilon - 2}.
\]

It follows that
\[
\sum_{\chi(\mod q)} \int_{-\infty}^{\infty} |L\left(\frac{3}{2} + it, \chi\right) - S\left(\frac{3}{2} + it, \chi\right)|^2 |W\left(\frac{3}{2} + it\right)|^6 dt \ll q^{3\delta + \varepsilon - 1}.
\]
we then deduce, by the same line of argument as before, that
\[
H(\sigma) \ll q^{-\left(1 - 4\delta\right)(2\sigma - 1)} \left( \frac{q}{\log q} + H\left(\frac{1}{2}\right) \right)
\]
for \(\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]\).

We record these results formally in the following lemma.

**Lemma 5** For \(\sigma \in \left[\frac{1}{2}, \frac{3}{2}\right]\) we have
\[
G(\sigma) \ll q^{-\left(1 - 4\delta\right)(2\sigma - 1)} \left( \frac{q}{\log q} + G\left(\frac{1}{2}\right) \right)
\]
and
\[
H(\sigma) \ll q^{-\left(1 - 4\delta\right)(2\sigma - 1)} \left( \frac{q}{\log q} + H\left(\frac{1}{2}\right) \right).
\]

We end this section by considering \(K(\sigma)\). We have
\[
K(\sigma) \leq \sum_{\chi(\mod q)} K(\sigma, \chi) = \sum_{m,n \leq q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m,n)I(m,n),
\]
where
\[
S(m,n) = \sum_{\chi(\mod q)} \chi(m)\overline{\chi(n)}
\]
and
\[ I(m, n) = \int_{-\infty}^{\infty} \left( \frac{n}{m} \right)^{it} |W(\sigma + it)|^6 dt. \]

Evaluating the sum \( S(m, n) \) we find that
\[
\sum_{m,n \leq q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} S(m, n) I(m, n)
= \phi(q) \sum_{m,n \leq q} \frac{d_k(m)d_k(n)}{(mn)^\sigma} I(m, n)
= \phi(q) \sum_{n \leq q} \frac{d_k(n)^2}{n^{2\sigma}} \int_{-\infty}^{\infty} |W(\sigma + it)|^6 dt.
\]

We then observe that
\[
\sum_{n \leq q} \frac{d_k(n)^2}{n^{2\sigma}} \leq \sum_{n \leq q} \frac{d_k(n)^2}{n} \ll (\log q)^{k^2},
\]
and that
\[
\int_{-\infty}^{\infty} |W(\sigma + it)|^6 dt \ll q^{3(2\sigma - 1)}(\log q)^{-1}.
\]

These bounds allow us to conclude as follows.

**Lemma 6** For \( \frac{1}{2} \leq \sigma \leq \frac{3}{2} \) we have
\[
K(\sigma) \ll \phi(q) q^{3(2\sigma - 1)}(\log q)^{k^2 - 1}.
\]

### 3 Proof of the Theorems

By definition of \( G(\sigma, \chi) \) and \( H(\sigma, \chi) \) we have
\[
J(\sigma) \ll K(\sigma) + G(\sigma)
\]
under the Generalized Riemann Hypothesis, and
\[
J(\sigma) \ll K(\sigma) + H(\sigma)
\]
unconditionally. In view of Lemma 5 these produce
\[
J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma - 1)} \left( \frac{q}{\log q} + G(\frac{1}{2}) \right).
\]
and

\[ J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)} \left( \frac{q}{\log q} + H(\frac{1}{2}) \right) \]

respectively. However we also have

\[ G(\frac{1}{2}) \ll K(\frac{1}{2}) + J(\frac{1}{2}) \]

and

\[ H(\frac{1}{2}) \ll K(\frac{1}{2}) + J(\frac{1}{2}) \]

from the definitions again, so that

\[ J(\sigma) \ll K(\sigma) + q^{-(1-4\delta)(2\sigma-1)} \left( \frac{q}{\log q} + K(\frac{1}{2}) + J(\frac{1}{2}) \right) \]

and

\[ J(\sigma) \ll K(\sigma) + q^{-(k-4\delta)(2\sigma-1)} \left( \frac{q}{\log q} + K(\frac{1}{2}) + J(\frac{1}{2}) \right) \]

in the two cases respectively.

If we now call on Lemma [6] we then find that

\[ J(\sigma) \ll \phi(q)q^{3\delta(2\sigma-1)}(\log q)^{k^2-1} + q^{-(1-4\delta)(2\sigma-1)} \left( \frac{q}{\log q} + J(\frac{1}{2}) \right) \]

\[ \ll q^{4\delta(2\sigma-1)} \left( \phi(q)(\log q)^{k^2-1} + q^{1-2\sigma} J(\frac{1}{2}) \right) \]

under the Generalized Riemann Hypothesis, since

\[ \frac{q}{\log q} \ll \phi(q)(\log q)^{k^2-1} \] (5)

for 0 < k < 2. Similarly we have

\[ J(\sigma) \ll q^{4\delta(2\sigma-1)} \left( \phi(q)(\log q)^{k^2-1} + q^{k(1-2\sigma)} J(\frac{1}{2}) \right) \]

unconditionally.

Finally we apply Lemma [4] with \( \gamma = \frac{1}{2} \) and use (5) again, to deduce that

\[ J(\sigma) \ll q^{4\delta(2\sigma-1)} \left( \phi(q)(\log q)^{k^2-1} + q^{-(2-k)(\sigma-1/2)} J(\sigma) \right) \]

under the Generalized Riemann Hypothesis. Similarly we may derive the unconditional bound

\[ J(\sigma) \ll q^{4\delta(2\sigma-1)} \left( \phi(q)(\log q)^{k^2-1} + q^{-k(\sigma-1/2)} J(\sigma) \right). \]
We are now ready to choose our value of $\delta$. For Theorem 1 we take
\[ \delta = \frac{2-k}{10}, \tag{6} \]
and for Theorem 2 we choose
\[ \delta = \frac{k}{10}. \tag{7} \]
Then in either case we will have
\[ J(\sigma) \ll q^{4\delta(2\sigma - 1)} \phi(q)(\log q)^{k^2 - 1} + q^{-\delta(2\sigma - 1)} J(\sigma). \]
We write $c_k$ for the implied constant in this last estimate, and note that $c_k$ depends only on $k$. We then take
\[ \sigma = \sigma_0 := \frac{1}{2} + \frac{\kappa}{\log q} \]
with
\[ \kappa = (2\delta)^{-1} \max(1, \log 2c_k). \]
These choices ensure that
\[ c_k q^{-\delta(2\sigma_0 - 1)} \leq \frac{1}{2}, \]
and hence imply that
\[ J(\sigma_0) \ll q^{4\delta(2\sigma_0 - 1)} \phi(q)(\log q)^{k^2 - 1} \ll \phi(q)(\log q)^{k^2 - 1}. \]
Finally, we may apply Lemma 4 to deduce the following

**Lemma 7** With $\sigma_0$ as above we have
\[ J(\gamma) \ll \phi(q)(\log q)^{k^2 - 1} \]
uniformly for $1 - \sigma_0 \leq \gamma \leq \sigma_0$.

All that remains is to bound $M_k(q)$ from above, using averages of $J(\gamma)$. Since $|L(s, \chi)|^{2k}$ is subharmonic we have
\[ |L(\frac{1}{2}, \chi)|^{2k} \leq \frac{1}{2\pi} \int_0^{2\pi} |L(\frac{1}{2} + re^{i\theta}, \chi)|^{2k} d\theta. \]
We now multiply by $r$ and integrate for $0 \leq r \leq R$ to show that
\[ |L(\frac{1}{2}, \chi)|^{2k} \leq \frac{1}{\text{Meas}(D)} \int_D |L(\frac{1}{2} + z, \chi)|^{2k} dA, \]
where \( D = D(0, R) \) is the disc of radius \( R \) about the origin, and \( dA \) is the measure of area. We take

\[
R = \frac{\min(\kappa, \delta^{-1})}{\log q},
\]

so that if \( z \in D \) then \( 1 - \sigma_0 \leq \Re(\frac{1}{2} + z) \leq \sigma_0 \) and \( |W(\frac{1}{2} + z)| \gg 1 \). It follows that

\[
\int_D |L(\frac{1}{2} + z, \chi)|^{2k} dA \ll \int_{1-\sigma_0}^{\sigma_0} J(\gamma, \chi) d\gamma
\]

whence

\[
M_k(q) \ll \frac{1}{\text{Meas}(D)} \int_{1-\sigma_0}^{\sigma_0} J(\gamma) d\gamma.
\]

Since \( \text{Meas}(D) \gg (\log q)^{-2} \) we now deduce from Lemma \( \text{[7]} \) that

\[
M_k(q) \ll \phi(q)(\log q)^{k^2},
\]

as required.

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