THE ATTRACTOR-CYCLE NOTATION FOR FINITE TRANSFORMATIONS

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ABSTRACT. We describe a new notation for finite transformations. This attractor-cycle notation extends the orbit-cycle notation for permutations and builds upon existing transformation notations. How the basins of attraction of a finite transformation flow into permuted orbit cycles is visible from the notation. It gives insight into the structure of transformations and reduces the length of expressions without increasing the number of types of symbols.

1. Introduction

What is the right generalization of the orbit-cycle notation for permutations to the case of arbitrary total functions $f : X \rightarrow X$ on a finite set $X$? The discrete dynamical system that $f$ gives on $X$ can be visualized as a digraph with nodes $x \in X$ and directed edges $(x, f(x))$. Iterating $f$ maps a point to $x$ to $f(x)$, to $f(f(x))$, and so on, until eventually some $f^m(x) = f^{m+k}(x)$. Taking the least $m \geq 0$ for which this happens, and the least positive $k$ for that $m$, shows that $x$ eventually must enter a (possibly degenerate) periodic orbit from which it never leaves. Points $x$ and $x'$ eventually entering the same periodic cycle are said to be in the same basin of attraction or generalized cycle. Drawn as digraphs, transformations may have several such disjoint basins of attraction, each consisting of a cycle of points with incoming trees (connected acyclic subgraphs). Figure 1 gives an example with four generalized cycle components in our notation, which is formally introduced in Section 3 with further examples. Unlike the permutation case, the points in a cycle can have incoming edges from outside the cycle, so a notation for transformations has to deal with these tree structures. The trees, which could also be degenerate, are directed toward the cycle, and there may be a tree of points coming into any given point of the cycle.

We call each basin of attraction associated to $f$, a generalized cycle or a component, since it is a connected component of the digraph of $f$. We may restrict our attention to a single component only, since we can write $f$ by concatenating our notation for what $f$ does on each of its components.

The basins of attractions have been studied in the context dynamical systems as they describe the global dynamics of a temporally evolving system [10]. It is important to note that this is the only dynamics in systems where the only action is the unidirectional passage of discrete time.1

Various previous notations have been developed for representing such transformations on a set (Section 2). The aim of these notations is to give useful information about the transformation without drawing the corresponding digraph. Readability

1This is in contrast to more general discrete dynamical systems which are generated by several such transformations on a finite set $X$, as studied in transformation semigroup theory.
Figure 1. A simple discrete dynamical system. A transformation \( g : X \rightarrow X \) with four components, i.e. basins of attraction (generalized cycles), is visualized as a digraph with arrows \((x, g(x))\) for each \( x \in X = \{1, \ldots, 17\} \). In our canonical form of the attractor-cycle notation, \( g \) is denoted \([[[13,2]15,4],6]([[78,9],10],11,[14,13,12])(16,17)\). Note: Singleton components are not written, just as in orbit-cycle notation for permutations. Full, formal details for the notation are given in the text.

is not an easily measurable quantity, but length and the number of distinct symbols used are influencing factors. The real importance of an efficient notation for transformations lies in the growing use of computer algebra systems in finite semigroup theory, as well as in mathematical calculations with transformations.

2. Existing Notations

Several ways for generalizing cyclic notation have been developed based on the particular purposes of each situation in which they were defined\(^1\). Here we give a quick summary of previous suggestions. Our running example, whose digraph is visualized in Figure 2, with just one nontrivial component (basin of attraction) will be the transformation typically written in so-called Cayley notation as

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 2 & 3 & 3
\end{pmatrix}
\]

This common notation denotes the function on the set \( \{1, 2, 3, 4, 5\} \) taking an element \( x \) in the first row to the element written \( f(x) \) immediately below it on the second row.

Caveat: As it often happens in mathematics, the notations surveyed in this section associate different meanings to the same symbols.

2.1. Path Notation for Partial Symmetries and Partial Transformations.

In addition to parentheses (,) for permutation orbits, path notation introduced by S. Lipscomb\(^2\) uses square bracket [ ] following elements with no images in partial

\(^2\)Somewhat ironically, we also developed yet another version (called compact notation), before settling on the attractor-cycle notation presented here.
permutations. For partial transformations the paths connect into other paths and
ultimately into the cycles. This is denoted by the symbol $\triangleright$, may be used as a visual
indication of being funneled into a cycle. The example written in path notation is

$$ (1,2)(4,3,2\triangleright(5,3,2\triangleright, $$

which redundantly represents the paths going into point 2.

2.2. **Factorization Notation.** In order to avoid path redundancy, G. Ayik, H. Ayik,
and J.M. Howie [1] introduced a different notation. Instead of decomposing the
transformation into paths and cycles, this notation decomposes it into a very par-
ticular type of generalized cycle, i.e. into a product of transformations given by the
trajectory of a single point. The example written in path notation is

$$ [4,3,2,1\mid 2][5,3\mid 3] = (\begin{array}{c|c|c|c|c}
1 & 2 & 3 & 4 & 5 \\
2 & 1 & 2 & 3 & 3
\end{array}), $$

where the element after the $|$ shows where the path connects to itself to form a cycle. This notation is excellent for describing factorizations in the full transformation
semigroups, but it introduces maps that might appear to move things in ways that
are not present in the original transformation, since branches connecting to an
already described path are terminated with an artificial self-loop. For instance,
$3 \mapsto 3$. Also, the decomposition is not unique.

2.3. **Linear Total Transformations.** In [5] a new notation was introduced by
O. Ganyushkin and V. Mazorchuk, aiming to provide a natural extension of the
cyclic notation of permutations. Instead of decomposing trees into paths, linear
notation describes the trees explicitly. Trees with one level of branching are denoted
by $[\text{preimages of root} ; \text{root}]$, where preimages are separated by commas and all
map to the root. If a preimage element also has incoming edges from other points,
then the same square bracket structure is applied again recursively to specify the
tree leading to that element. Although called 'linear' as a one-line notation, the
notation describes recursive tree structure.

Parentheses are used to indicate the existence of a nontrivial permutation of the
roots of the trees (which may include degenerate trees consisting of a single point).
Basically the linear notation is the usual cyclic notation used for permutations but
the elements in the cycle describe their incoming tree information. The running
example is written as

$$ ([4,5;3\mid 2],1). $$

One of the most useful features of linear notation is that when restricted to
permutations it is identical to orbit-cycle form. Also, by looking for parentheses
we can easily spot the existence of nontrivial permutations even in large examples.
The only drawback is that describing a transformation given by following a simple
path, a “line” of maps, requires many square brackets. For instance,
becomes $[[[1;2];3];4];5]$. With the attractor-cycle notation we aim to alleviate this particular problem.

2.4. Specialized Notation for Fast Iterations. For special purposes one can develop specialized notations. For example in [9], for the purpose of the efficient calculation of $f^m(x)$, the example would be first decomposed into components $( (1,2), (4,3), (5))$, then the orbits concatenated $(1,2,4,3,5)$ in order to assign indices to points. This of course lacks the information how to put together the orbits, therefore an auxiliary sequence is needed to encode where (which position) the last elements of the orbit components connect to: $f(2)$ is in position 1, $f(3)$ in 2 and $f(5)$ in 4. So the full information is encoded in these strings:

$(((1,2),(4,3),(5)), (1,2,4,3,5), (1,2,4))$.

This illustrates that efficient algorithmic representations do not necessarily provide human readable formats.

3. The Attractor-Cycle Notation

Informally, the attractor-cycle notation makes use of the following conventions and symbols. Formal semantics of the notation are in the next subsection. Here we illustrate the concepts for points:

- **conveyor belt:** Left-to-right comma-separated points. For example, $1,2,3$ reads as $1 \mapsto 2, 2 \mapsto 3$.
- **cycle:** Putting parentheses around a conveyor belt connects the last point to the first, i.e. $(1, \ldots , n)$ adds the map $1 \mapsto n$.
- **tree operator:** Square brackets group points and do not specify the image of the last point. Based on context, this is resolved into one of 3 possibilities.
  1. $[\ldots n], k \ldots$ defines the map $n \mapsto k$, connecting into a conveyor belt;
  2. $(m, \ldots [\ldots n])$ defines $n \mapsto m$, closing a cycle;
  3. $[\ldots n]$ corresponds to $n \mapsto n$, defining a loop edge.

In short, $[\ldots n]$ is a tree “flowing” into point $n$, and the context of $n$ defines what happen to $n$ itself.

- **parallel paths:** A vertical bar $|$ suggestive of logical OR is written to separate two or more points (respectively, subtrees) that go to the same point. For instance, $1|2|3,4$ means $1 \mapsto 4$, $2 \mapsto 4$, and $3 \mapsto 4$.

3.1. Examples. Any permutation in the attractor-cycle notation is the same as the cyclic notation. In particular, the identity is simply $()$.

A constant map to $n$ is written as $[1|2|\ldots |n-1,n]$, while a simple path trajectory from 1 to $n$ is denoted by $[1,2,\ldots,n]$. This compact notation for a path is also better matching the cycle notation for permutations.

The transformation whose digraph is in Figure 3 on 9 points is written as $(((8|9,5]|6|7,1], [4,3,2])$.

The parentheses immediately tell that this transformation has a non-trivial permutation component. Collapsing the tree operators to their roots, we can see that the cycle is $(1,2)$. 
3.2. Syntax. The following context-free grammar defines the language of attractor-cycle notations. The terminal symbols are \([, ], ( )\) and the symbols for the \(n\) points. The nonterminal symbols are \(C\) for components, \(F\) for in-flows, \(T\) for trees and \(P\) for points. One has to be careful with \(|\|\) which is part of the grammar, while \(|\) is on the meta level.

\[
\begin{align*}
S &\rightarrow \ C^+ \ | \ ( ) \\
C &\rightarrow ( ( T, )^+ T ) \ | \ F \\
F &\rightarrow [ ( T, )^+ P ] \ | \ [ ( T )^+ T, P ] \\
T &\rightarrow F \ | \ P \\
P &\rightarrow 1 \ | 2 \ | 3 \ | \ldots \ | n
\end{align*}
\]

Rule (1) states that the attractor-cycle notation denotes a positive number nontrivial components (generalized cycles, i.e. basins of attractions), or we can have just an empty cycle for the identity transformation. Rule (2) says that a component can be a in-flow (where something flows into a point) or a permutation cycle (with incoming trees to its members). Rule (3) describes the two cases of an in-flow: either a tree with multiple branches leading to the same root (parallel branches) or a path (conveyor belt). Rule (4) defines a tree to be an in-flow. It allows a point to be a tree. Rule (5) specifies the points.

An important additional constraint in the notation is that each point can only occur at most once.\textsuperscript{3}

3.3. Semantics. We define the semantics by recursively interpreting the valid attractor-cycle notation words as a collection of individual maps of points. We write such a map \(p \mapsto q\) as a pair \((p, q)\). We will put the total function \(f\) together from these pieces. Using the parse tree of a well-formed word \(w\) in the notation, we denote its interpretation by \(I(w)\). This determines a unique function \(f\) from

\textsuperscript{3}As the intersection of a regular language with a context-free one is still context-free \cite{2}, we may constrain the strings generated by the above context-free grammar to have no repeated points (i.e., by intersecting with the regular language of all strings \(w\) over all the symbols we use such that \(w\) does not contain any repeated point symbol from among \(1, 2, 3, \ldots, n\) ) and obtain a context-free language for the attractor-cycle notation without any repeated points.
\{1, \ldots, n\} to itself, by identifying \( f \) with the set of pairs \( I(w) \). If \( w = \emptyset \), then \( I(w) \) is the identity transformation. Otherwise \( w \) is derived using \( S \rightarrow C_1 \ldots C_k \), and we define

\[
I(S) := \bigcup_{i=1}^{k} I(C_i) \cup \{(p, p) \mid f(p) \text{ is not defined in any } I(C_i)\}.
\]

An auxiliary function \( r \) gives the root of a tree, so we let

\[
r(p) = r([T_1 | \ldots | T_k, p]) = p \in \{1, \ldots, n\}.\]

and \( r([T_1, \ldots, T_k]) = r(T_k) \).

Then the interpretation of a component derived from a nonterminal symbol \( C \) is:

\[
I(C) := \begin{cases} 
I(F) & \text{if } C \rightarrow F \\
\bigcup_{i=1}^{k} I(T_i) \cup \{(r(T_i), r(T_{i+1})) : 1 \leq i < k\} & \text{if } C \rightarrow [T_1, \ldots, T_k] \\
\bigcup_{i=1}^{k} I(T_i) \cup \{(r(T_i), r(T_{i+1})) : 1 \leq i < k\} & \text{if } F \rightarrow [T_i | [T_1 | \ldots | T_k, p]] \\
\bigcup_{i=1}^{k} I(T_i) \cup \{(r(T_i), r(T_{i+1})) : 1 \leq i < k\} & \text{if } F \rightarrow [T_1, \ldots, T_k]
\end{cases}
\]

A tree gives a nonempty set of pairs, if it is a nontrivial in-flow. A point (degenerate tree) does not yield a map.

\[
I(T) := \begin{cases} 
I(F) & \text{if } T \rightarrow F \\
\emptyset & \text{if } T \rightarrow P
\end{cases}
\]

A nontrivial tree is an in-flow and it gives a set of pairs as follows:

\[
I(F) := \begin{cases} 
\bigcup_{i=1}^{k} I(T_i) \cup \{(r(T_i), p) : 1 \leq i \leq k\} & \text{if } F \rightarrow [T_1 | \ldots | T_k, p] \\
\bigcup_{i=1}^{k} I(T_i) \cup \{(r(T_i), r(T_{i+1})) : 1 \leq i < k\} & \text{if } F \rightarrow [T_1, \ldots, T_k]
\end{cases}
\]

The notation \( w \) thus clearly determines a unique well-defined transformation \( f = I(w) \) on \( \{1, \ldots, n\} \) to itself, since each element \( p \) appears at most once in \( w \). The transformation is total (i.e., not partial, but fully defined) by Equation 6. Moreover, every \( f \) can be written in this attractor-cycle notation in a canonical form.

3.4. Canonical Form. The price to pay for the short length is the loss of uniqueness. Both \([1, 2, [3|4, 6]]\) and \([[1, 2] | 3|4, 6]\) denote the same transformation. One can simply choose between a conveyor belt (comma-separated list of elements in a trajectory) or the parallel branches (using |). However, a simple recursive algorithm that starts from the point(s) of each component’s cycle can produce a canonical form. All we need to do is to examine the cardinality of the preimage set from outside the cycle, i.e. the number of incoming arrows.

1. If there is no incoming arrow, then we have a leaf of the tree, only the point needs to be printed.
2. If there is only one preimage, then a conveyor belt is built by traversing the tree as long as there is only one preimage. Then recursion is done on the first element of the conveyor belt.
3. In case there are more than one element in the preimage then parallel branches need to be used and recursion on all parallel elements.
For each transformation \( f \) on \( n \) points one can now obtain a completely canonical expression in this notation by additionally requiring the \( C \)'s to appear in order according to their least elements, and that the trees preceding a \( | \) are ordered by their least element, and the cycles start with their least point (or tree root) first.

For instance, here are the conjugacy class representatives of the full transformation semigroup \( T_4 \) on four points in canonical form:

\[
[1|2|3,4] \\
[[1,2]|3,4] \\
[1|2,3] \\
[[1|2,3],4] \\
[1,2,3,4] \\
[1,2,3] \\
[1,2]|3,4] \\
[1,2] \\
[1,2](3,4) \\
() \\
(1,2) \\
([1,2],3) \\
(1,2,3) \\
([1|2,3],4) \\
([1|2],[3,4]) \\
([1,2,3],4) \\
(1,2)(3,4) \\
([1,2],3,4) \\
(1,2,3,4) \\
\]

Looking at the list, it is easy to spot the existence of nontrivial cycles and idempotents. Idempotents, other than the identity (\( () \)), are exactly those transformations given by concatenating a number of conveyor belts of length 2, \([x, y]\), and single-level parallel branches \([x_1]|...|x_k, x_{k+1}]\). Also, a common feature of all the notations discussed here is that conjugation by a permutation is just relabelling of points according to the permutation, e.g. \((1,2,3)^{-1}([1,2],3,4) (1,2,3) = ([2,3],1,4)\).

4. Conclusion

By the spreading use of computer algebra systems for investigating transformations and discrete dynamical systems, efficient notation has become a necessity. The orbit-cycle has been described in numerous group theory textbooks. Transformations can have the same status as well.

In the attractor-cycle notation we tried to blend the best features of previous notations and also considered the computational experience to derive what we have found to be a useful and readable notation, giving insight into the structure of transformations. We implemented this notation in our semigroup decomposition computer algebra package \texttt{SgpDec} \cite{4,5} available for \texttt{GAP} \cite{6}.
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