CANONICAL VECTOR HEIGHTS ON K3 SURFACES WITH PICARD NUMBER THREE – ADDENDUM

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Abstract. In an earlier paper by the first author, an argument for the nonexistence of canonical vector heights on K3 surfaces of Picard number three was given, based on an explicit surface that was not proved to have Picard number three. In this paper, we fill the gap in the argument by redoing the computations for another explicit surface for which we prove that the Picard number equals three. The conclusion remains unchanged.

1. Introduction

In [1] the first author gave convincing numerical evidence for the nonexistence of canonical vector heights on K3 surfaces of Picard number 3. The intent of this paper is to fill a gap in the argument, which was pointed out by Yuri Tschinkel in the review of the paper, and privately by Bert van Geemen.

As in [1], the Picard number of a surface will always mean the geometric Picard number. The Picard number of the explicit K3 surface $V$ used in [1] is at least 3, but was not proved to equal 3. Since 3 is odd, the only currently known method to prove that this lower bound is sharp requires two primes of good reduction for $V$, such that the Picard number of both reductions equals 4, see [5]. Modulo 2 and 3 the Picard numbers turn out to be 16 and 6 respectively (depending on Tate’s conjecture). The computations required to calculate the Picard number modulo larger primes are currently beyond our ability. This is why in the next section we construct a new example $Y$ for which we can use the primes 2 and 3 to prove that the Picard number equals 3. In the last section we redo the necessary computations for this example $Y$, referring to [1] for details. We compute various canonical heights, which we believe to be correct up to an error of at most 0.0001. Our main theorem states that if the errors are at most 0.1, then a canonical vector height on $Y$ does not exist. This also suggests that, except perhaps in very special cases, a K3 surface with Picard number at least three will not admit a canonical vector height.

2. A K3 surface with Picard number three

Let $k$ be a field with a fixed algebraic closure $\overline{k}$. Let $X$ be a smooth surface over $k$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, given by a $(2,2,2)$-form. Then $X$ is a K3 surface, which implies that linear, algebraic, and numerical equivalence all coincide. This means that the Picard group Pic $\overline{X}$ and the Néron-Severi group NS $\overline{X}$ of $\overline{X} = X_{\overline{k}}$ are naturally isomorphic, finitely generated, and free. Their rank is called the (geometric) Picard number of $X$. By the Hodge Index Theorem, the intersection pairing gives this group the

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structure of a lattice with signature \((1, \text{rk} \text{NS}(\mathcal{X}) - 1)\). For detailed definitions of all these notions, see \[5\].

For \(i = 1, 2, 3\), let \(\pi_i : X \to \mathbb{P}^1\) be the projection from \(X\) to the \(i\)-th copy of \(\mathbb{P}^1\) in \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\). Let \(D_i\) denote the divisor class represented by a fiber of \(\pi_i\). We have \(D_i \cdot D_j = 2\) for \(i \neq j\) and since any two different fibers of \(\pi_i\) are disjoint, we find \(D_i^2 = 0\). It follows that the intersection matrix \((D_i \cdot D_j)_{i,j}\) has rank 3, so the \(D_i\) generate a subgroup of the Néron-Severi group \(\text{NS}(\mathcal{X})\) of rank 3. Our goal is to find an explicit example where the rank of \(\text{NS}(\mathcal{X})\) equals 3.

Let \(x, y,\) and \(z\) denote the affine coordinates of \(\mathbb{A}^1\) inside the three copies of \(\mathbb{P}^1\) in \(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\). Let \(Y/\mathbb{Q}\) be the surface given by \(G_1 x^2 + G_2 x + 3G_3 - 2L_1 L_2 = 0\) with

\[
G_1 = -y^2 z^2 + 3y^2 z + 2y z^2 - 2yz^2 + 3yz + 3z + 2z^2 + 2z - 1,
G_2 = 2y^2 z^2 + 3y^2 z + 2y z^2 + 2yz^2 + 2yz + 3z^2 + z + 2,
G_3 = y^2 z + y^2 + y + z^2 + z,
L_1 = yz - y - z,
L_2 = yz + 1.
\]

**Theorem 2.1.** The surface \(Y\) is smooth. The Picard number of \(\text{NS}(\mathcal{X})\) equals 3.

To bound the Picard number of \(Y\) we use the method described in \[5\]. We first state some results and notation that we will use. Let \(X\) be any smooth surface over a number field \(K\) and let \(p\) be a prime of good reduction with residue field \(k\). Let \(\mathcal{X}\) be an integral model for \(X\) over the localization \(\mathcal{O}_p\) of the ring of integers \(\mathcal{O}\) of \(K\) at \(p\). Let \(k'\) be any extension field of \(k\). Then by abuse of notation we will write \(X_{k'}\) for \(\mathcal{X} \times_{\text{Spec} \mathcal{O}_p} \text{Spec} k'\).

**Proposition 2.2.** Let \(X\) be a smooth surface over a number field \(K\) and let \(p\) be a prime of good reduction with residue field \(k\). Let \(l\) be a prime not dividing \(q = \# k\). Let \(F\) denote the automorphism on \(H^2_{\text{ét}}(X_{\mathcal{X}}, \mathbb{Q}_l)(1)\) induced by \(q\)-th power Frobenius. Then there are natural injections

\[
\text{NS}(X_{\mathcal{X}}) \otimes \mathbb{Q}_l \hookrightarrow \text{NS}(\mathcal{X}) \otimes \mathbb{Q}_l \hookrightarrow H^2_{\text{ét}}(X_{\mathcal{X}}, \mathbb{Q}_l)(1),
\]

that respect the intersection pairing and the action of Frobenius respectively. The rank of \(\text{NS}(X_{\mathcal{X}})\) is at most the number of eigenvalues of \(F\) that are roots of unity, counted with multiplicity.

**Proof.** See \[4\]. Prop. 6.2 and Cor. 6.4. Note that in the referred corollary, Frobenius acts on the cohomology group \(H^2_{\text{ét}}(X_{\mathcal{X}}, \mathbb{Q}_l)\) without a twist. Therefore, the eigenvalues are scaled by a factor \(q\). \(\Box\)

**Remark 2.3.** Tate’s conjecture (see \[3\]) states that the rank of \(\text{NS}(X_{\mathcal{X}})\) in Proposition 2.2 is in fact equal to the number of eigenvalues of \(F\) that are roots of unity, counted with multiplicity.

**Lemma 2.4.** If \(\Lambda'\) is a sublattice of finite index in a lattice \(\Lambda\), then we have \(\text{disc} \Lambda' = [\Lambda : \Lambda']^2 \text{disc} \Lambda\).

**Proof of Theorem 2.1** We write \(Y_p\) and \(Y'_{p}\) for \(Y_{\mathbb{Q}_p}\) and \(Y_{\mathbb{F}_p}\) respectively. One easily checks that \(Y_p\) is smooth for \(p = 2\) and \(p = 3\), so \(Y\) itself is smooth and \(Y\) has good
Table 1. Number of points over some finite fields.

| n  | \#Y_2(\mathbb{F}_{2^n}) | \#Y_3(\mathbb{F}_{3^n}) |
|----|-------------------------|-------------------------|
| 1  | 13                      | 17                      |
| 2  | 25                      | 107                     |
| 3  | 85                      | 848                     |
| 4  | 289                     | 6719                    |
| 5  | 1153                    | 60632                   |
| 6  | 4273                    | 536564                  |
| 7  | 16897                   | 4793855                 |
| 8  | 65025                   | 43091783                |
| 9  | 266305                  | 387501194               |
| 10 | 1050625                 |                          |

The Lefschetz Trace Formula relates the number of \( \pi_p \)-th power Frobenius we find the traces of the group of \( Y \) reduction at 2 and 3. Both \( Y_2 \) and \( Y_3 \) contain a fourth divisor class that is linearly independent of the earlier described classes \( D_i \) for \( i = 1, 2, 3 \). On \( Y_2 \) we have the curve \( C_2 \) parameterized by \( ([x : 1], [1 : 0], [1 : 1]) \). On \( Y_3 \) we have the curve \( C_3 \) given by \( x = L_1 = 0 \). For \( p = 2, 3 \), let \( \Lambda_p \) denote the sublattice of the Néron-Severi group of \( Y \), generated by \( D_1, D_2, D_3, \) and \( C_p \). The intersection matrices associated to the sequences of classes \( \{D_1, D_2, D_3, C_2\} \) and \( \{D_1, D_2, D_3, C_3\} \) are

\[
\begin{bmatrix}
0 & 2 & 2 & 1 \\
2 & 0 & 2 & 0 \\
2 & 2 & 0 & 0 \\
1 & 0 & 0 & -2
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
0 & 2 & 2 & 0 \\
2 & 0 & 2 & 1 \\
2 & 2 & 0 & 1 \\
0 & 1 & 1 & -2
\end{bmatrix},
\]

so \( \Lambda_2 \) and \( \Lambda_3 \) have discriminants \(-28\) and \(-32\) respectively. We will now show that the Picard numbers of \( Y_2 \) and \( Y_3 \) both equal 4. Almost all fibers of the fibration \( \pi_1 \) are smooth curves of genus 1. Using Magma we counted the number of points over small fields fiber by fiber. The total numbers of points are given in Table 1. The Lefschetz Trace Formula relates the number of \( \mathbb{F}_{p^n} \)-rational points on \( Y_p \) to the traces of the \( p^n \)-th power Frobenius acting on \( H_{\text{et}}^2(Y_p, \mathbb{Q}_l)(1) \) for \( i = 0, \ldots, 4 \) by

\[
\#Y_p(\mathbb{F}_{p^n}) = \sum_{i=0}^{4} (-p^{n/2})^i \cdot (\text{trace of } p^n\text{-th power Frobenius on } H_{\text{et}}^2(Y_p, \mathbb{Q}_l)(1)).
\]

Normally this is phrased in terms of the cohomology groups without the twist. For K3 surfaces we have \( \dim H^i = 1, 0, 22, 0, 1 \) for \( i = 0, 1, 2, 3, 4 \) respectively. Since the action for \( i \neq 2 \) is trivial, from the numbers in Table 1 we can compute the traces of powers of the automorphism \( F_p \) on \( H_{\text{et}}^2(Y_p, \mathbb{Q}_l)(1) \) that is induced by \( p \)-th power Frobenius. We find \( p^n \cdot \text{Tr } F_p^n = \#Y_p(\mathbb{F}_{p^n}) - p^{2n} - 1 \). For \( p = 2, 3 \), let \( W_p \) denote the quotient of \( H_{\text{et}}^2(Y_p, \mathbb{Q}_l)(1) \) by the image \( V_p \) of \( \Lambda_p \otimes \mathbb{Q}_l \) under the second homomorphism in Proposition 2.2, and let \( \Phi_p \) denote the action of Frobenius on \( W_p \). Since \( F_p \) acts trivially on \( V_p \), we have \( \text{Tr } \Phi_p^n = \text{Tr } F_p^n - \text{Tr } F_p^n|_{V_p} = \text{Tr } F_p^n - 4 \) for all \( n \geq 0 \), and \( f_{F_p} = \int_{F_p|_{V_p}} \cdot f_{\Phi_p} = (t - 1)^4 f_{\Phi_p} \), where \( f_T \) stands for the characteristic polynomial of the linear operator \( T \). From the traces of the first \( s > 0 \) powers of a linear operator one can derive the first \( s \) coefficients of its characteristic polynomial, see [5], Lemma 2.4. Once enough coefficients of \( f_{\Phi_p} \) are...
computed, the full polynomial \( f_{\Phi} \) follows from the functional equation \( f_{\Phi}(1/x) = \pm x^{-\dim W_p} f_{\Phi}(x) \). Putting all this together, we find \( f_{\Phi} = \frac{1}{p}(t - 1)^4 f_{\Phi} \), with

\[
\begin{align*}
{\Phi}_2 &= 2t^{18} + 2t^{16} + 2t^{15} + 2t^{14} + 2t^{13} + 2t^{12} + t^{11} + 3t^{10} + \\
&\quad + 3t^8 + 2t^6 + t^5 + 2t^4 + t^3 + 2t^2 + 2, \\
{\Phi}_3 &= 3t^{18} + 5t^{17} + 6t^{16} + 5t^{15} + 5t^{14} + 6t^{13} - 6t^{11} - 5t^{10} + \\
&\quad - 6t^9 - 5t^8 - 6t^7 + 6t^5 + 5t^4 + 5t^3 + 6t^2 + 5t + 3.
\end{align*}
\]

Note that the coefficient of \( t^8 \) in \( f_{\Phi} \) is zero, so we used the number of points over \( \mathbb{F}_2^{10} \) to compute the coefficient of \( t^8 \), from which we determined the sign of the functional equation to be positive. Both \( f_{\Phi} \) are irreducible. Their roots are not integral and therefore not roots of unity. By Proposition 2.2 we find that the Picard numbers of \( \mathbb{Y}_2 \) and \( \mathbb{Y}_3 \) are both bounded by 4, so they are equal to 4 and \( \Lambda_p \) has finite index in \( \text{NS}(\mathbb{Y}_p) \) for \( p = 2, 3 \). From Lemma 2.4 we conclude that up to a square factor the discriminants of \( \text{NS}(\mathbb{Y}_2) \) and \( \text{NS}(\mathbb{Y}_3) \) are equal to \(-28\) and \(-32\) respectively. From the first injection of Proposition 2.2 we find \( \text{rk} \text{NS}(\mathbb{Y}) \leq 4 \). Suppose we had equality. Then the lattice \( \text{NS}(\mathbb{Y}) \) would be isomorphic to a sublattice of finite index in \( \text{NS}(\mathbb{Y}_p) \) for both \( p = 2 \) and \( p = 3 \). By Lemma 2.4 this implies that up to a square factor, the discriminant of \( \text{NS}(\mathbb{Y}) \) is equal to both \(-28\) and \(-32\). This contradicts the fact that \(-28\) and \(-32\) do not differ by a square factor. We therefore conclude that equality does not hold and we have \( \text{rk} \text{NS}(\mathbb{Y}) \leq 3 \). Since the classes \( D_1, D_2, \) and \( D_3 \) are linearly independent, we deduce \( \text{rk} \text{NS}(\mathbb{Y}) = 3 \). \( \square \)

3. Nonexistence of canonical vector heights

As in [1], we let \( \sigma_i \) denote the involution associated to the 2-to-1 projection \( Y \to \mathbb{P}_1 \times \mathbb{P}_1 \) along the \( i \)-th copy of \( \mathbb{P}_1 \) in \( \mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1 \), and for \( i, j, k \in \{1, 2, 3\} \), we set \( \sigma_{ijk} = \sigma_i \sigma_j \sigma_k \). Let \( D^*_i = \{ D^*_1, D^*_2, D^*_3 \} \) be the basis that is dual to the basis \( D = \{ D_1, D_2, D_3 \} \) of \( \text{NS}(\mathbb{Y}) \otimes \mathbb{R} \). Then

\[
\mathbf{h} = \sum_{i=1}^{3} h_{D_i} D^*_i,
\]

is a vector height, so for every divisor class \( E \in \text{NS}(\mathbb{Y}) \otimes \mathbb{R} \), a Weil height \( h_E \) associated to \( E \) is up to \( O(1) \) given by \( P \mapsto \mathbf{h}(P) \cdot E \). In our computations, we use the heights \( h_{D_i} \) defined by \( \pi_i \) and the usual logarithmic height on \( \mathbb{P}_1(\mathbb{Q}) \).

Suppose \( \sigma \) is an automorphism of \( Y \) and that the pullback \( \sigma^* \) acting on \( \text{NS}(\mathbb{Y}) \otimes \mathbb{R} \) has a real eigenvalue \( \omega > 1 \) with associated eigenvector \( E \). Silverman [2] defined the canonical height (with respect to \( \sigma \)) to be

\[
\hat{h}_E(P) = \lim_{n \to \infty} \omega^{-n} h_{\sigma^n E}(P).
\]

This height is canonical with respect to the automorphism \( \sigma \), since \( \hat{h}_E(\sigma P) = \omega \hat{h}_E(P) \).

Set \( \gamma = \frac{1}{2}(1 + \sqrt{5}) \). Then \( \alpha \) and \( \omega \) in [1] are equal to \( \gamma^2 \) and \( \gamma^6 \) respectively. Suppose \( (i, j, k) \) is a permutation of \( (1, 2, 3) \). The eigenvector \( E_{ijk} \) of \( \sigma_i^* \sigma_j^* \sigma_k^* = \sigma_{kji}^* \) associated to the eigenvalue \( \omega \), as defined in [1], equals \( \frac{1}{2} \gamma(-D_i + \gamma D_j + \gamma^2 D_k) \). Set \( P_0 = ([0 : 1], [0 : 1], [0 : 1]) \). Table 2 contains the estimates \( \omega^{-n} \mathbf{h}(\sigma_{kji}^* P_0) \cdot E_{ijk} \) to
the canonical height $\hat{h}_{E_{ijk}}(P_0)$ (canonical with respect to $\sigma_{kji}$) for all permutations $(i,j,k)$ and $n \in \{1, \ldots, 5\}$.

These estimates appear to converge geometrically, as expected. We believe, without rigorous proof, that the estimates of the canonical heights for $n = 5$ are correct up to an error of at most 0.0001, and are probably correct up to 0.00001. The following theorem therefore gives evidence against the existence of a canonical vector height on $Y$.

**Theorem 3.1.** If the estimates $\omega^{-5} \hat{h}_{E_{ijk}}(\sigma_{kji}^5 P_0)$ in Table 2 are equal to the canonical heights $\hat{h}_{E_{ijk}}(P_0)$ up to an absolute error of at most 0.1, then the surface $Y$ does not admit a canonical vector height.

**Proof.** Suppose a canonical vector height $\hat{h}$ exists on $Y$. Then for every permutation $(i,j,k)$ of $(1,2,3)$ we get a linear equation (see [1])

$$\hat{h}(P_0) \cdot E_{ijk} = \hat{h}_{E_{ijk}}(P_0).$$

The three permutations $(3,2,1)$, $(2,3,1)$, and $(3,1,2)$ give three linearly independent equations from which we can compute the coefficients $a_i$ in $\hat{h}(P_0) = \sum_{i=1}^3 a_i E_i$. We get

$$[a_1 \quad a_2 \quad a_3] A = [\hat{h}_{E_{321}}(P_0) \quad \hat{h}_{E_{231}}(P_0) \quad \hat{h}_{E_{312}}(P_0)],$$

where $A$ is the matrix whose columns contain the coefficients with respect to the basis $D$ for $E_{321}$, $E_{231}$, and $E_{312}$, respectively. That is,

$$A = \frac{\gamma}{2} \begin{bmatrix} \gamma^2 & \gamma^2 & \gamma \\ \gamma & -1 & \gamma^2 \\ -1 & \gamma & -1 \end{bmatrix}.$$ 

The absolute values of the entries of $A^{-1}$ are bounded by $2 \gamma^{-1}$, so when we use the estimates of $\hat{h}_{E_{ijk}}(P_0)$ for $n = 5$ in Table 2, the solution

$$(a_1, a_2, a_3) = (0.719498, 0.805119, 0.963093)$$

to (1) is accurate up to $\varepsilon = 3(2 \gamma^{-1})(0.1)$. From $E_{123} = \frac{1}{3} \gamma(-D_1 + \gamma D_2 + \gamma^2 D_3)$ we find that, up to an absolute error of at most $\frac{1}{2} \gamma(1 + \gamma + \gamma^2) \varepsilon \approx 1.571$, the canonical height $\hat{h}_{E_{123}}(P_0)$ equals

$$(0.719498 D_1^* + 0.805119 D_2^* + 0.963093 D_3^*) \cdot E_{123} \approx 2.51169.$$ 

This contradicts the estimate in the first column of Table 2, so we conclude that $Y$ does not admit a canonical vector height.

$\Box$

| $n$ | $(1, 2, 3)$ | $(1, 3, 2)$ | $(2, 1, 3)$ | $(2, 3, 1)$ | $(3, 1, 2)$ | $(3, 2, 1)$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 0.47111022  | 1.0364196   | 1.8329385   | 2.1332594   | 1.8679417   | 1.7986626   |
| 2   | 0.3438678   | 1.0306631   | 1.7914641   | 2.0624775   | 1.7723601   | 1.6340533   |
| 3   | 0.4745990   | 1.0365615   | 1.8328000   | 2.1330968   | 1.8675712   | 1.7982461   |
| 4   | 0.4747015   | 1.0364020   | 1.8329385   | 2.1332594   | 1.8679417   | 1.7986626   |
| 5   | 0.4746928   | 1.0364196   | 1.8329385   | 2.1332721   | 1.8679467   | 1.7986781   |

Table 2. Estimates for $\hat{h}_{E_{ijk}}(P_0)$ for the permutations $(i,j,k)$ of $(1,2,3)$. 

This contradicts the estimate in the first column of Table 2, so we conclude that $Y$ does not admit a canonical vector height.
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