Revisiting the local potential approximation of the exact renormalization group equation

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Abstract

The conventional absence of field renormalization in the local potential approximation (LPA) –implying a zero value of the critical exponent $\eta$– is shown to be incompatible with the logic of the derivative expansion of the exact renormalization group (RG) equation. We present a LPA with $\eta \neq 0$ that strictly does not make reference to any momentum dependence. Emphasis is made on the perfect breaking of the reparametrization invariance in that pure LPA (absence of any vestige of invariance) which is compatible with the observation of a progressive smooth restoration of that invariance on implementing the two first orders of the derivative expansion whereas the conventional requirement ($\eta = 0$ in the LPA) precluded that observation.

Key words: Local potential approximation, Derivative expansion, Exact renormalization group equation, Reparametrization invariance, Anomalous dimension

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1 Introduction

The exact renormalization group (RG) equation \cite{1} (ERGE) –also called non-perturbative or functional RG equation– cannot be concretely used without recourse to approximation (for modern reviews or introductory lectures see, e.g., \cite{2,4}). The best known approximation framework for the ERGE is
the derivative expansion \([5–7]\). The leading order of that expansion, \(O(\partial^0)\)-order, also named the local potential approximation (LPA) \([8–11]\), completely discards any momentum dependence from the study. In principle the LPA amounts to projecting the RG flow of the complete action \(S[\phi]\) (a functional of the field \(\phi(x)\)) onto the space of simple functions \(U(\phi)\) of a uniform field \(\phi\) by assuming that:

\[
S[\phi] = \Omega_D U(\phi)
\]

(1)

where \(\Omega_D\) is the volume of the \(D\)-dimensional space.

Due to its simplicity and because it is thought that it qualitatively involves most of the properties of the complete ERGE in the large distance regime (e.g., stability properties and number of fixed points), the LPA is currently utilized in many studies. Numerically, the LPA is considered as a reasonable approximation because the estimations of the critical properties would only be vitiated by the obligatory zero value of the critical exponent \(\eta\) (characterizing the large distance behavior of the two-point correlation function at the critical point) which, in many circumstances, is actually a small parameter.

In the early studies, the condition \(\eta = 0\) in the LPA has been justified as a consequence of the neglect of the detailed momentum dependence in the RG \([8]\) (the same kind of justification of \(\eta = 0\) may be found in \([11\text{ p. 121}]\)). Though this argumentation by default is sometimes reused \([11, 12]\), it is not very strong. It has been argued that, in the LPA, “it is not possible to consistently determine \(\eta\)”, or \(\eta\) “is set to zero as there is no mechanism to determine” its value \([13]\) (see also \([14]\)). The alert reader could express some surprise and argue that the vanishing of \(\eta\) in the LPA has been clearly demonstrated a long time ago by Hasenfratz and Hasenfratz \([10]\) as often put forward in current studies (see, e.g. \([2, 15, 16]\)). Unfortunately, the arguments are not unassailable because they rely, at least allusively (see section \(3.1\)), on the following truncation of \(S[\phi]\):

\[
S[\phi] = \Omega_D U(\phi) + \bar{z} \frac{\pi}{2} \int d^D x (\partial_x \phi(x))^2
\]

(2)

in which the coefficient \(\bar{z}\) of the kinetic term would be maintained unaltered (equal to unity) along a RG flow of \(U\). A condition which would imply \(\eta = 0\) \([10]\). Pending to show that the argument is actually artificial (see section \(3.1\)), we may already notice that truncation \((2)\) differs in nature from the pure LPA \([1]\) since it refers partly to the \(O(\partial^2)\)-order of the derivative expansion. Consequently, assuming it was correct, this currently accepted argument, basis of what is referred to in the following as the conventional LPA, spoils the logic of the expansion based on a systematic projection of the complete ERGE onto the space of actions successively truncated according to the number of the derivatives of the field \(\phi(x)\). Normally the LPA should correspond to \([1]\) and not to \((2)\) so that the supposedly proof of \([10]\), even true, would be inappropriate. Hence, only remains the poor default argument. It is then
legitimate to wonder whether the condition \( \eta = 0 \) is actually obligatory in the pure LPA.

It is a matter of fact that the conventional value \( \eta = 0 \) in the LPA –not accidentally but associated with a systematic absence of field renormalization– raises some questions:

(1) The absence of any field renormalization precludes in the LPA the eventual setting up of a non-classical power law behavior of correlation functions at criticality \[17\] other than that purely induced by a diverging correlation length. According to the theory of critical phenomena, two critical exponents are necessary to determine all the other critical indices of a second order critical point. With the ERGE considered around a Wilson-Fisher-like (WF) \[18\] fixed point \[18\], these two exponents are \( \eta \) and \( \nu \). The two exponents arise differently in the ERGE. The index \( \nu \) (which characterizes the divergence of the correlation length \( \xi \) when the temperature \( T \) approaches its critical value \( T_c \)) occurs as a positive eigenvalue of the RG equation linearized around a fixed point (the number of such positive values determines the order of the transition). The role of the index \( \eta \) is more subtle. It is associated with the field renormalization allowing for a non-classical power law behavior of the correlation function at criticality. Usually one introduces \( \eta \) in order to reproduce the critical behavior of the correlation function at large distances (momenta going to zero) and \( T = T_c \); this manner of doing tightly links the field renormalization to the momentum dependence and suggests that no field renormalization is required when the momentum dependence is neglected \[18\] implying \( \eta \equiv 0 \). However, the fluctuation-dissipation theorem relates the correlation function to the susceptibility. This allows another introduction of \( \eta \) via the critical exponent \( \gamma = \nu(2 - \eta) \) characterizing the critical behavior of the two-point correlation function at zero momenta and \( T \to T_c \). No reference to an explicit momentum is required in that case but the field should be renormalized nonetheless (to take this eventual non-trivial power law with \( \gamma \neq 2\nu \) into account). In the LPA (conventional or not) \( \nu \) takes on a non-classical value \[33\] when the fixed point is a non-trivial one. Then, there is a priori no reason why \( \eta = 2 - \gamma/\nu \) would take on a classical value at this fixed point (a priori no reason for keeping the field unrenormalized).

(2) Although broken by the derivative expansion, the reparametrization invariance of the complete ERGE is expected to be progressively restored as the order of the expansion grows. When it is satisfied, this invariance specifies, in particular, that a change of normalization of the field by a pure constant \([\text{like the parameter } \bar{z} \text{ in (2)}]\) generates a line of equivalent

\[ A \text{ non-trivial fixed point with one direction of infra-red instability associated with the critical exponent } \nu \text{ of the correlation length.} \]
fixed points characterized by a unique set of critical exponents with the joint existence of a zero eigenvalue mode in the solutions of the ERGE linearized around those fixed points. The breaking of that invariance by the derivative expansion has been concretely observed at next-to-leading order \([O(\partial^2)]\) [5, 7, 19]; it is such that, for a given smooth cutoff function, a line of fixed points is well generated by the change of normalization of \(\phi\) but those fixed points are not equivalent. Nonetheless, in agreement with a remark of Bell and Wilson [20] in such a situation, one observes the existence at \(O(\partial^2)\)-order of a vestige of the invariance via an extremum \([2\] of \(\eta\) [5, 7, 19] accompanied by the presence of a zero mode. This gives a preferred estimate for \(\eta\) (one sometimes also refers to a principle of minimal sensitivity \([3\], see, e.g., [21, 23]). Because the conventional LPA offers no opportunities to look at the state of the invariance \([4\] then no signs of progressive restoration of the invariance may be observed by going from the \(O(\partial^1)\)-order to the \(O(\partial^2)\)-order of the derivative expansion. Thus, considered as the leading order of that expansion, the conventional treatment creates confusion about the convergence property of the derivative expansion. This is a pity because the issue is of some importance.

(3) Having defined the RG-time \(t = -\ln (\Lambda/\Lambda_0)\) (with \(\Lambda\) the running cutoff scale and \(\Lambda_0\) an arbitrary fixed momentum scale) the critical exponent \(\eta\) is actually defined in the RG as the limit of a function \(\eta(t)\) on approaching a given fixed point (when \(t \to +\infty\)). According to Wilson’s prescriptions \([1\], the function \(\eta(t)\) is determined by keeping fixed the coefficient of one “particular” term in the action \(S[\phi, t]\) with the initial condition \(\eta(0) = 0\). It is customary to keep constant the coefficient \(\bar{z}\) of the kinetic term because, due to a symmetry of the action linked to the reparametrization invariance, the flow of such a term is non-essential (redundant) so that it may be constrained without altering the model integrity. When the kinetic term is not part of the approximation, as in the pure LPA, the redundancy still exists at least formally and it seems logical to wonder what the state of the reparametrization process is in the pure LPA. To this end one should introduce a function \(\eta(t)\) that would maintain fixed a “particular” monomial of \(U(\phi)\). The line of fixed points mentioned in point 2 would presumably be generated and this would give back the status of genuine leading order \([O(\partial^0)]\) of the derivative expansion to the LPA (see section 2).

In section 2 we present and discuss a version of the LPA with \(\eta \neq 0\). We show that it satisfies all the required conditions for being a genuine \(O(\partial^0)\)-
order of the derivative expansion. In particular we study, in section 2.1, the structure of the fixed points for any value of the dimension $D$ and show explicitly, for the first time in the LPA, how the reparametrization invariance is broken. We also introduce, in section 2.2.2, a Legendre transformation of the potential adapted to the case studied ($\eta \neq 0$). This allows us to utilize easy quasi-analytic methods (section 2.2.1) of integration of an ordinary differential equation (ODE) well adapted to the obtention of the eigenvalues of the RG flow linearized around a fixed point. It is then shown that the principle of minimal sensitivity (PMS) does not necessarily indicate preferred values of the critical exponents (section 2.2.3). In section 3, we first show that the conventional argument which is generally put forward to justify $\eta = 0$ in the LPA, is actually artificial (section 3.1). We then discuss briefly, according to the RG rules, how the reparametrization invariance could be studied with the partial truncation (2) (section 3.2). In appendix A we illustrate some reason why the quasi-analytic methods of integration of ODE of section 2.2.1 do not work in the case of the potential of the action whereas they work after a Legendre transformation is performed. We conclude in section 4.

2 The revised LPA

Let us consider the RG flow equation of the Polchinski ERGE extended to include the parameter $\eta \neq 0$ in the LPA [12], it reads

$$\dot{U} = U'' - U'^2 - \frac{D - 2 + \eta(t)}{2} \phi U' + D U$$

where $U(\phi, t)$ stands for a simple function of $\phi$ and $t$, $\dot{U} \equiv \partial U/\partial t|_\phi$, $U' = \partial U/\partial \phi|_t$, $U'' = \partial^2 U/\partial \phi^2|_t$, $D$ is the spatial dimension, and $\eta(t)$ the field renormalization parameter which, at a fixed point, takes on the value $\eta^*$. In principle and with the complete ERGE, $\eta^*$ should coincide with the critical index $\eta$.

The flow equation (3) has already been studied by Kubyshin et al [24–26] for the derivative $f(\phi, t) = U'$. But they have considered $\eta(t) \neq 0$ in the LPA for technical reasons exclusively [24]. Hence, in accordance with the conventional LPA, they have left $\eta(t)$ undefined and focused their interest on $\eta^*$ considered as an arbitrarily adjustable parameter while emphasizing that physically $\eta^*$ should be zero at this order of the derivative expansion.

With a view to study the fixed point equation ($\dot{U} = 0$) for any $D$ at one time, we perform the following change of normalization of $\phi$:

$$\phi \rightarrow \frac{\phi}{\sqrt{D}}$$

(4)
then Eq. (3) transforms into [11]:

\[ \dot{U} = D \left[ U'' - U'^2 - \mu(t) \phi U' + U \right] \]

(5)

\[ \mu(t) = \frac{D - 2 + \eta(t)}{2D} \]

(6)

For a given \( D \), \( \mu(t) \) plays the role of \( \eta(t) \) and for any \( D \), the fixed point equation involves only one parameter instead of two in the preceding case of (3).

Considering exclusively the issue of finding a non-singular solution \( U^*(\phi) \) to the fixed point equation corresponding to (5) [or (3)], gives no possibility for determining a value of \( \mu^* = (D - 2 + \eta^*) / (2D) \). Then \( \mu^* \) may rightly be considered as an extra parameter. Indeed \( U^*(\phi) \) is a solution of the following two-point boundary value problem of a second order non-linear ODE:

\[ U^{***} - U'^2 - \mu^* \phi U^{**} + U^* = 0 \]

(7)

\[ U^{**}(0) = 0 \]

(8)

\[ U^{***}(\infty) = -\frac{1}{2} + \mu^* \]

(9)

Now the whole of the two integration constants are fixed by the property of parity [8] and by the adjustment of \( U^*(0) = k^* \) so as to get a non-singular \( U^*(\phi) \) in the whole range \( \phi \in ]-\infty, +\infty[ \) as prescribed by condition (9); then there is no room for determining \( \mu^* (\eta^* \text{ at fixed } D) \) without a supplementary condition.

In the conventional LPA, the supplementary condition is merely \( \eta(t) \equiv 0 \) which would be obtained (see section 3.1 however) by an explicit reference to a larger space of truncation functions [see Eq. (2)]. However, even correct, this procedure would not be justified because the RG theory gives precise rules to determine both the function \( \eta(t) \) and its fixed point value \( \eta^* \). As recalled in point 3 of the introduction, the function \( \eta(t) \) is determined by keeping one particular term of the action fixed along the RG flows [1]; then \( \eta^* \) is the value reached by \( \eta(t) \) in approaching a given fixed point. This procedure is a direct consequence of the reparametrization invariance of the complete action which induces the redundancy of the flow of one term of the action. In absence of any kinetic term, as in the pure LPA, it is logical, and coherent with the RG theory, to define \( \eta(t) \) by keeping constant the coefficient of the quadratic term \( U''(\phi = 0, t) \).

Let us examine, on a general ground, the approach of the WF fixed point \( U^*_{WF}(\phi) \) with the flow equation (5) starting at \( t = 0 \) with the following simple
potential:

\[ U(\phi, 0) = k_0 + \frac{z}{2} \phi^2 + \frac{g_0}{4!} \phi^4 + \frac{u_0}{6!} \phi^6 \]  

(10)

where the coefficient of the quadratic term has been intentionally noted \( z/2 \).

Because \( U_{WF}(\phi) \) has only one relevant direction, to make the flow approaching \( U_{WF}(\phi) \) only one coefficient of (10), say \( k_0 \), must be fine-tuned (in terms of the other three coefficients) [27]. This adjustment is necessary to place the initial potential on the critical surface (within the domain of attraction) of \( U_{WF}(\phi) \) [27]. In order to follow a RG flow, the function \( \eta(t) \) must be defined. We do it such that \( U''(0, t) = z \) all along the RG flow, with the initial condition \( \eta(0) = 0 \) and \( z \) a constant independent of \( t \). At the fixed point (reached at infinite RG-time provided the initial potential lies in the domain of attraction of the fixed point), \( \eta(\infty) \) takes on the value of \( \eta^*(z) \) and this defines a line of fixed points (parametrized by \( z \)). If it was satisfied, the reparametrization invariance would imply that \( \eta^*(z) \) be independent of \( z \) and equal to \( \eta \). Of course, in the pure LPA one rather expects to observe the breaking of that marvelous property and the true question is: to which extent is that invariance broken in the LPA?

To look at this question, suffices to express the variation of \( \eta^* \) in terms of \( U''(0) \). This is precisely what Kubyshin et al have done in [24,26]: they studied Eq. (3) for \( D = 2 \) and 3 (for \( D = 3 \) in [25]). The purpose of Kubyshin et al was not the status of the reparametrization invariance in the LPA however. In fact, having considered the flow equation for the derivative \( f(\phi, t) = U' \), the connection parameter of their fixed point equation was not \( U^*(0) = k^* \) but instead \( U''(0) \) that they have noted \( \gamma \). Then they have naturally drawn the variation of \( \gamma \) on changing the value of \( \eta^* \) (or the reverse) without relating this variation to the reparametrization process. It is however clear that, with our prescription of keeping \( U''(0) = z \) fixed along a RG flow, the fixed point is reached with \( U''(0) \equiv z \) where obviously \( z \) may be considered as the normalization of the field. Consequently, the evolutions of \( \eta^*(\gamma) \) drawn by Kubyshin et al are nothing but illustrations of the breaking of the reparametrization invariance in the LPA. Let us redo the study of Kubyshin et al using our own conventions.

2.1 Lines of Fixed points

Using a standard numerical shooting method, we have looked for regular solutions (the values of \( k^* = U(0) \)) of the two-point boundary value problem (7,9). Clearly, those solutions are parametrized by \( k^*(\mu^*) \). Since the coefficient of the quadratic term \( U''(0) \equiv z \) is linked to \( k^*(\mu^*) \) via the differential equation
as $z = -k^*$, one easily gets functions $\mu_n^*(z)$ corresponding to the functions $\eta_n(\gamma)$ of Kubyshin et al [26]. The four first solutions are displayed in fig (1) as continuous curved lines; this figure involves simultaneously the two graphs of fig. (1) of Kubyshin et al [26] and displays the same features. Let us discuss them.

Each curved line drawn on fig. (1) corresponds to a line of fixed points of a particular nature. It appears as bifurcating from the Gaussian fixed point (full circles) on varying $\mu^*$ each time $\mu^*$ falls below the thresholds $\mu_n^* = 1/[2(1 + n)]$, $n = 1, 2, \ldots$ [horizontal lines on fig. (1), corresponding to the usual dimensional thresholds $D_n = 2 + 2/n$ (for $\eta^* = 0$)]. In particular fig. (1) shows the well-known fact that the Gaussian fixed point is stable for $D > 4$ and $\eta^* = 0$ ($\mu^* > 1/4$) where there is no regular solution to Eq. (7). A new fixed point bifurcates each time the Gaussian fixed point acquires a new direction of instability; then the fixed points belonging to a line $\mu_n^*(z)$ have $n$ directions of instability each. Fig. (1) shows the four first lines of fixed points $\mu_1^*(z)$ to $\mu_4^*(z)$) having respectively one, two, three and four directions of instability. All the lines accumulate at the horizontal line $\mu^* = 0$ as $n \to +\infty$ along which $z$ reaches $-\infty$ or stop at $1/2$ in agreement with the analytical solution found by Kubyshin et al [26] for $\mu^* = 0$.

Let us focus our attention on the line $\mu_1^*(z)$ which is a line of WF fixed points. Because, for a given $D$, $\eta^*$ varies along the line, it is obvious that the reparametrization invariance is broken. Of course, this was expected in the LPA but surprisingly had never been explicitly emphasized before the present study. Because the line is smooth and monotonous, there is no vestige of the invariance, the breaking is perfect except at the limiting value $\mu^* = 0$ where, for $D = 3$, $\eta^*$ takes on the values $-1$. Notice that nothing particular distinguishes the value $\eta^* = 0$ from the other values $0$ except at the limiting cases $\mu^* = 0$, $D = 2$ and $\mu^* = 1/4$, $D = 4$.

One observes also that $z < 0$ on the whole line of fixed points $\mu_n^*(z)$ (except the Gaussian fixed point); this means that the basin of attraction of a non-trivial WF fixed point implies the condition $U''(0, 0) < 0$ on the initial potential $U(\phi, 0)$ (otherwise the RG flow goes away from the critical surface towards the trivial high-temperature fixed point). Notice that the perfect breaking of the reparametrization invariance does not completely spoil the universal character of the critical behavior since the infinite number of initial potentials with a given $U''(0, 0) < 0$ lying on the critical surface are characterized by the same critical behavior governed by a unique value of $\eta^*$ and, subsidiarily, of

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5 Due to (4) the value of $U''(0) = \gamma$ in the study of Kubyshin et al is related to our $z$ as $\gamma = D z$.

6 The usual W-F fixed point of the conventional LPA with $D = 3$ lies at the intersection of $\mu_1^*(z)$ and the horizontal line $D = 3$, $\eta^* = 0$ with $z \simeq -0.07620$ (i.e. $\gamma \simeq -0.2286$).
Fig. 1. Lines of fixed points $\mu^*_1(z)$ to $\mu^*_4(z)$ obtained as regular solutions of Eqs. (7-9). The full circles represent the thresholds of instability of the Gaussian fixed point. The horizontal lines indicate three values of $D$ where such instabilities occur when $\eta^*=0$. See text for more details.

From the line $\mu^*_1(z)$, it is also interesting to notice that non-trivial fixed points may be formally considered as existing for $D = 4$ provided that $\eta^*$ be strictly negative. Usually such fixed points are rejected but in the present study there is no reason to reject them a priori since it is a consequence of the breaking of the reparametrization invariance to generate also negative values of $\eta^*$. It
would be puzzling however if a vestige of that invariance led us to choose such a negative value of $\eta^*$. Fortunately we observe no sign of such a preferred value of $\eta^*$ along the line $\mu_1^*(z)$ except the limit case $\mu^* = 0$.

At this level we conclude that the LPA does not allow one to determine any estimate of $\eta^*$ but only ranges of possible values. For example, if one excludes negative values of $\eta^*$, then for $D = 3$ this range would be $[0, 1/2]$ for the only line $\mu_1^*$, the other lines being excluded. From this example taken alone, one could be inclined to conclude that the conventional LPA, by imposing $\eta^* = 0$, would be merely a reasonable choice since one knows that $\eta^*$ is most often small. However, fig. (1) shows that from $D < 3$ down to $D = 2$, emerges a rich structure of various fixed points with possible different positive and growing values of $\eta^*$ for which the conventional LPA would impose, increasingly poorly as $D$ decreases, the same zero value (one knows that at $D = 2$, $\eta = 1/4$ what is not small).

Notice that, for $D = 3$, we have excluded the limit case $\eta^* = 1/2$ [7] from the range of possible values of $\eta^*$ though it corresponds to the only point on $\mu_1^*(z)$ where a zero eigenvalue exists (see section 2.2.3). Indeed, since $d\eta^*/dz \neq 0$ at this point, this zero mode is not a vestige of the reparametrization invariance; instead it indicates that the nature of the Gaussian fixed point is going to change by losing one direction of instability. Hence, if a direct derivative of the fixed point equation with respect to $z$ shows that an extremum of $\eta^*$ implies the appearance of a zero eigenvalue, the reverse is not true.

In order to better illustrate the role of the zero mode in the process of restoration of the reparametrization invariance, let us look at the critical exponents in the LPA and at their variations on changing the normalization of the field. To this end, we perform a Legendre transformation of the potential which will allow us to make use of user-friendly quasi-analytic methods of “integration” of ODE.

2.2 Eigenvalues: Taylor series, Legendre transformation, principle of minimal sensitivity

2.2.1 Taylor series methods

The interest of using some quasi-analytic methods to solve the RG flow equation in the LPA is the extremely easy access that they offer to estimate:

1. the fixed point value $k^*$ of the connection parameter
2. a set of critical (and subcritical) exponents at one time.

7 Similarly, $\eta^* = 0$ on the line $\mu_2^*$. 
On the contrary, the purely numerical shooting method necessitates a skillful adjustment of an initial guess of the final value of $k^*$ or, independently, of each critical exponent sought.

The use of quasi-analytic methods based on Taylor series, in solving a two-point boundary value problem like (7.9), has been recently reviewed and illustrated in [28]. Among such methods is an extremely simple procedure [15, 29, 30] (named the simplistic method in [28]) that merely consists of imposing the vanishing of the last term $a_M(k)$ of the Maclaurin series of the truncated solution:

$$U_M(\phi) = k + \sum_{i=1}^{M} a_i(k) \phi^{2i}$$

the coefficients $a_i(k)$ being determined such that the EDO considered be satisfied order by order in powers of $\phi^2$. The auxiliary condition $a_M(k) = 0$ gives a condition from which one tries to extract an estimate of the connection parameter $k^* = U^* (0)$ corresponding to the only regular solution of (7.9). Of course, because it is too simple, this simplistic method is not always (most often never) efficient. Firstly the finite character of the radius of convergence of the series limits the accuracy of the method [15,30]. Secondly the method may simply not work at all (in the sense that even a rough estimate of $k^*$ may not be approachable). Indeed, in trying to solve (7.9), the issue we are faced with amounts to pushing a movable singularity to infinity. The efficiency of the simplistic method then depends on whether or not that singularity lies within the circle of convergence of the Maclaurin series or not (see appendix A).

A variant of the simplistic method, referred to below as the Taylor method, is frequently used which is based on a Taylor expansion around the minimum of the potential, as proposed in [31,32] (see also [15,30]). The solution of the ODE is thus expressed as:

$$U_M(\phi) = k_0 + \sum_{i=2}^{M} b_i(k_0, x_0) (x_0 - x)^i$$

$$x = \phi^2$$

$$x_0 = \phi_0^2$$

where $\phi_0$ is the expansion point chosen to coincide with the minimum of the potential since $b_1(k_0, x_0) = 0$, and $k_0 = U_M(\phi_0)$, whereas the original

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8 This choice is not obligatory. One could have fixed $x_0$ arbitrarily and the two unknowns would have been $k_0$ and $b_1$. This would offer the possibility of improving the apparent convergence of the Taylor method by varying $x_0$, see [30].
connection parameter is, by definition, given by:

\[ k = k_0 + \sum_{i=2}^{M} b_i (k_0, x_0) x_0^i \]

There are two unknowns \((k_0 \text{ and } x_0)\) to be determined. The Taylor method consists of imposing the vanishing of the two last terms of the Taylor series to get two auxiliary conditions on \(k_0\) and \(x_0\). This method may improve considerably the simplistic method (it has provided excellent estimates of the critical exponents in the LPA \([33]\)). The reason is due to the fact that, other things being equal compared to the simplistic method, one starts closer to the movable singularity. But the Taylor method requires that the expansion point (the minimum of the potential) lies within the circle of convergence of the Maclaurin series (otherwise one could not get a reliable estimate of \(k^*\) by summing the series back to the origin). Also, the accuracy of the method is naturally limited by the finite range of convergence of the Taylor expansion.

It is a matter of fact that, in the conventional LPA with \(\eta(t) \equiv 0\), the two quasi-analytic methods\(^9\) presented just above do not work when they are applied to the Polchinski RG flow equation of \(U\) but they work if one first performs a Legendre transformation \((\{U, \phi\} \rightarrow \{V, \varphi\})\) as that defined in \([34]\).

With a view to make use of these user-friendly quasi-analytic methods—that allow anyone to easily verify the content of the present paper—, let us introduce a Legendre transformation appropriate to the case \(\eta \neq 0\).

### 2.2.2 Legendre transformation for \(\eta \neq 0\)

To begin with, we consider the Legendre transformation originally introduced for \(\eta(t) \equiv 0\) in \([34]\) and we apply it to the flow equation of \(V\) extended to include \(\eta \neq 0\), namely:

\[ \dot{V} = \frac{V''}{1+V''} - \frac{D-2+\eta(t)}{2}\varphi V' + D V \]  
(12)

According to \([34]\), the Legendre transformation reads

\[ U(\phi, t) = V(\varphi, t) + \frac{1}{2}(\phi - \varphi)^2 \]  
(13)
\[ \varphi = \phi - U''(\phi, t) \]  
(14)
\[ \dot{U} = \dot{V} \]  
(15)

\(^9\) The methods are not completely analytic because the determination of the solution of the auxiliary condition is finally numerically performed.
from which we have:

\[ U' = V' \]
\[ U'' = \frac{V''}{1 + V''} \]

Thus applied to (12) we get the following flow equation for \( U \):

\[ \dot{U} = U'' - \varpi(t) U'^2 - \frac{D - 2 + \eta(t)}{2} \phi U' + DU \] (16)
\[ \varpi(t) = 1 - \frac{\eta(t)}{2} \] (17)

which differs from the usual Polchinski equation (3) when \( \varpi(t) \neq 1 \). The appearance of this coefficient may be seen as the consequence of a non-linear introduction \(^{10}\) of \( \eta(t) \) in the ERGE \(^{35}\) instead of the linear introduction of (12) that corresponds to (3). Though, near a fixed point, the coefficient \( \varpi^* = 1 - \eta^*/2 \) may be removed from (16) through the change \( U^* \to U^*/\varpi^* \) to get the same equation as (3), we have numerically studied \(^{11}\) Eq. (16) explicitly for \( D = 3 \) (using a shooting method). We have, this way, verified explicitly (in the case of W F fixed points) both that we get the same kind of line of fixed points as previously (monotonous function \( \eta^*(z) \)) and that the simplistic and Taylor methods applied to (12) work well also for \( \eta \neq 0 \) [at least for the values of \( z \) shown in fig (2)].

### 2.2.3 Eigenvalues

Let us focus our interest on the eigenvalue problem corresponding to Eq. (12) linearized around a fixed point \( V^* \) [solutions of (12) such as \( \dot{V}^* = 0 \)]. We get the following second order linear ODE (once \( V^* \) is known):

\[- \frac{v''}{(1 + V^*)^2} - \frac{D - 2 + \eta^*}{2} \phi v' + (D - \lambda) v = 0 \] (18)

where \( v(\phi) \) is the eigenfunction and \( \lambda \) the eigenvalue parameter.

For a given set of initial conditions, such as \( v(0) = 1, v'(0) = 0 \) in the even case, one expects to obtain an infinite set of discrete couples \( \{v_n(\phi), \lambda_n\} \) ordered according to the magnitude of \( \lambda_n \). The number of positive values

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\(^{10}\) As done originally in the historic first version \(^{1}\).

\(^{11}\) We could have considered instead a modified version of Eq. (12) corresponding to applying the Legendre transformation \(^{13}\) \(^{15}\) on (3), but this would have made the quasi-analytic methods heavier and thus less attractive.
depends on the fixed point considered. Except the trivial eigenvalue \( \lambda_0 = D \), the WF fixed point is characterized by the existence of only one positive value \( \lambda_1 \) corresponding to the critical exponent \( \nu = 1/\lambda_1 \), the next eigenvalue \( \lambda_2 \) is negative and corresponds to the leading subcritical exponent \( \Delta_1 = \omega_1 \nu \) characterizing the leading correction-to-scaling with \( \omega_1 = -\lambda_2 \). These two exponents have been estimated, for \( D = 3 \), with a very high accuracy in the conventional LPA (with \( \eta \equiv 0 \)) to get \([33, 36, 37]\):

\[
\nu = 0.6495617738806480176 \cdots \\
\omega_1 = 0.6557459391933387407 \cdots
\]

Of course, due to the Legendre transformation, the same set of critical exponents is obtained in both cases of the flow equations of \( V \) and \( U \) \([33]\). It is clear that this is also the case in the present study with \( \eta \neq 0 \), provided the methods used converge.

We have determined the evolution in terms of \( z = U''(0) = V''(0)/(1 + V''(0)) \) of the two first exponents \( \nu \) and \( \omega_1 \) using both the simplistic and Taylor methods and obtained the curves shown in figures (2, 3).

Fig. (2) shows that \( \nu(z) \) undergoes a minimum at \( \nu_{\text{min}} \simeq 0.64496 \) corresponding to \( \eta^* \simeq 0.16 \) whereas, at this point we get \( \omega_1 \simeq 0.47731 \) [see fig. (3)]. According to a principle of minimal sensitivity (PMS) \([21]\) – sometimes used in calculations at higher orders of the derivative expansion of the ERGE (see, e.g, \([22, 23]\) – one could be inclined to propose those values as being the preferred estimates of the critical exponents in the LPA for \( D = 3 \). However, one may observe that those values are not designated as the consequence of a vestige of the reparametrization invariance which is the only reason that fundamentally led us to vary \( z \). Indeed no zero eigenvalue is obtained at this point as shown in table 1. Fig. (3) clearly shows that the only point where a zero mode occurs corresponds to the Gaussian fixed point which is losing one direction of instability (\( \lambda_2 = 0, \lambda_1 = 3/2, \eta^* = 1/2, \) for \( D = 3 \); or \( \lambda_3 = 0, \lambda_1 = 2, \eta^* = 0, \) for \( D = 4 \)) but at this point \( d\eta^*/dz \) does not vanish [see fig. (1)].

Notice that this is only a confirmation of the absence of any extrema in the function \( \eta^*(z) \). Indeed, if one performs a derivation with respect to \( z \) of the fixed point equation corresponding to (12), assuming \( d\eta^*(z)/dz = 0 \), then one gets (18) with \( \lambda = 0 \). The reverse is not true however: the presence of a zero mode may reveal instead the change of the stability properties of the fixed point.

We may thus conclude that, because it has no link with the reparametrization invariance, the observed minimum of \( \nu \) occurs accidentally and that the PMS cannot be utilized in the circumstances as a tool to determine a preferred set
Fig. 2. Evolution, for $D = 3$, of the critical exponent $\nu$ as function of the field-normalization $z$. The full line corresponds to calculations done using the Taylor method, open circles correspond to results obtained with the simplistic method. The point located at $z = 0$ corresponds to the Gaussian fixed point with $\nu = \frac{2}{3}$.

3 Conventional LPA versus pseudo-LPA

In this section, we first show that the argument of Hasenfratz-Hasenfratz [10], by which the RG flow projected on (2) would imply $\eta = 0$ if $\bar{z}$ is kept unaltered by the flow of $U$, is artificial and reduces to a triviality that poorly “justifies” the default argument. Then we briefly illustrate that, correctly treated, the projection of the ERGE on (2) gets an “intermediate order” [between the $O(\bar{\partial}^0)$ and $O(\bar{\partial}^2)$] of the derivative expansion that we name pseudo-LPA. This partial $O(\bar{\partial}^2)$-order differs from the approximation introduced in [32] in...
Fig. 3. Evolution, for $D = 3$, of the subcritical exponent $\omega_1$ as function of the field-normalization $z$ (full line) obtained using the Taylor method. The point located at $z = 0$, $\omega_1 = 0$ is the only possibility of having a zero mode; it is located far from the value corresponding to $\eta^* = 0.16$ for which $\nu(z)$ undergoes a minimum [see fig. 2].

that we try to account for the reparametrization invariance.

3.1 Invalidity of the conventional argument

To get the RG flow equations of the Wilson-Polchinski ERGE correctly projected on [2], suffices to consider the complete $O(\partial^2)$-order equations available in the literature as, e.g., in [7,12], and to impose within them that the kinetic term is a pure number that remains constant along a RG flow of the potential.

For example, let us consider Eqs. (12) of [7] for the derivative $f(\phi, t) = U'(\phi, t)$
Table 1
Comparison of the six first eigenvalues of Eq. (18) obtained for \( \eta^* = 0.16 \), \( D = 3 \) and with the two quasi-analytic methods considered in the study. The first line of numbers corresponds to \( \lambda_1 (1/\nu) \), the second to \( \lambda_2 (-\omega_1) \), etc. No zero eigenvalue is present.

| Simplistic \((M = 20)\) | Taylor \((M = 10)\) |
|-------------------------|----------------------|
| 1.55050                 | 1.55049              |
| -0.47729                | -0.47731             |
| -2.7753                 | -2.7773              |
| -5.243                  | -5.231               |
| -7.91                   | -8.49                |
| -10.4                   | -13.4                |

and a function \( Z(\phi, t) \) reduced to \( \bar{z}(t) \), it comes:

\[
\dot{f} = f'' - 2f f' - \frac{D - 2 + \eta(t)}{2} \phi f' + \frac{D + 2 - \eta(t)}{2} f \\
\dot{\bar{z}}(t) = -\eta(t) \bar{z}(t) + 2B f'(0, t)^2 - 4 [\bar{z}(t) - 1] f'(0, t)
\]

(19)

(20)

where \( B \) is a constant parameter depending on the choice of cutoff function.

Up to inessential changes, Eq. (19) is the same flow equation as (5) of the pure potential \( U \) discussed previously in section 2. Eq. (20) shows that the flow of \( U \) induces a flow of \( \bar{z} \) so that keeping it constant, i.e. imposing \( \dot{\bar{z}}(t) = 0 \), yields:

\[
\eta(t) \bar{z}(t) - 2B f'(0, t)^2 + 4 [\bar{z}(t) - 1] f'(0, t) = 0
\]

(21)

this condition considered at a fixed point of (19) may be rewritten as:

\[
\eta^* \bar{z} - 2B \gamma^2 + 4 [\bar{z} - 1] \gamma = 0
\]

(22)

where, as seen in section 2 (footnote 5), \( \gamma = f'''(0) = U'''(0) \) is a function of \( \eta^* \) as that given implicitly by the lines of fixed points \( \mu_\eta^* \) drawn in fig. 11 where \( U'''(0) \) was playing the role of \( \bar{z} \). For \( \eta^* = 0 \) and \( \bar{z} = 1 \) (the conventional values), Eq. (22) implies \( U'''(0) = 0 \). Fig. 11 shows that this is not possible along the line of WF fixed points \( \mu_1^* \) except trivially at \( D = 4 \) where the fixed point is Gaussian.

The conventional argument of [10] actually relies upon the arbitrary requirement that no contribution of \( U \) must alter the flow of \( \bar{z} \) so that the right-hand-side of (20) would be reduced to the first term exclusively, thus implying \( \eta(t) \equiv 0 \) for a constant \( \bar{z} \). Clearly, this is only an illustration of an obvious
fact: the non-necessity of renormalizing the field (here forced by the obligatory absence of contribution coming from $U$), induces $\eta(t) \equiv 0$ (what is true by definition). We add that, not only truncation (2) is incompatible with a pure $O(\partial^0)$-order, it is also in contradiction with the default argument -by which $\eta(t)$ is absent because there is no momenta in the LPA. Actually, the conventional argument is merely artificial. As shown in section (2) it is the basis of a conventional LPA which is misleading concerning the concept of reparametrization invariance and in contradiction with the logic of the derivative expansion.

3.2 Pseudo-LPA

Considered as an actual truncation of the action $S[\phi]$, Eq. (2) gives access to an intermediate approximate order of the ERGE [between $O(\partial^0)$ and $O(\partial^2)$]. By allowing the coefficient of the kinetic term to flow (whereas it remains independent of $\phi$) one obtains the partial truncation first used by Tetradis and Wetterich [32] in order to easily have $\eta^* \neq 0$ in an “improved” (conventional) LPA. But the way of determining $\eta^*$, in the original proposal, is limited to the ERGE for the effective average action $\Gamma[\phi]$ (for a review see [3]). This is because one utilizes the available momentum dependence of the exact propagator $\Gamma^{(2)}$ to determine the function $\bar{z}(t)$ yielding a value for $\eta^*$. That way of doing is not convenient to the Polchinski ERGE, with which an easy access to $\Gamma^{(2)}$ is not possible. Moreover, the Tetradis and Wetterich approach does not give a clear account of the reparametrization invariance (being in the spirit of the conventional LPA criticized in the present paper).

Let us look at truncation (2) for the Polchinski ERGE by strictly applying the basic rules of the RG theory.

The RG rules prescribe a field renormalization in order to maintain constant one term of the action. With truncation (2), we choose it to be the kinetic term (12) (the only momentum-dependent monomial of the approximation). Hence we get Eq. (21) and, at a fixed point, Eq. (22). Considering $\bar{z}$ as a free parameter at hand, $\eta^*$ and $\gamma$ appear to be functions of $\bar{z}$. We thus get lines of fixed points parametrized by $\bar{z}$. The relation between $\eta^*$ and $\gamma$ being unchanged compared to the LPA [Eq. (19) is the derivative with respect to $\phi$ of (3)], we have:

$$\gamma(\bar{z}) = \gamma_{\text{LPA}}[\eta^*(\bar{z})]$$

With a complete $O(\partial^2)$-order we could have pursued the process of maintaining constant the quadratic term of the action (instead of the kinetic term). In the present pseudo-LPA this procedure would give nothing new compared to the pure LPA. This underlines the particular character of that truncation.
where $\gamma_{LPA}[\eta^*]$ has been determined at leading order in section 2.1 [implicitly through the lines of fixed points displayed by fig. (1)].

It is then easy to seek for a vestige of the reparametrization invariance eventually displayed by the new lines of fixed points (parametrized by $\bar{z}$). Suffices to look at possible values of $\eta^*$ where $d\eta^*/d\bar{z} = 0$.

Differentiating (22) with respect to $\bar{z}$, we get:

\[
\frac{d\eta^*}{d\bar{z}} \left\{ \bar{z} - 4 B \gamma_{LPA}\gamma'_{LPA} + 4 [\bar{z} - 1] \gamma'_{LPA} \right\} + \eta^* + 4 \gamma_{LPA} = 0
\]

where $\gamma'_{LPA} = d\gamma_{LPA}/d\eta^*$. Finally, imposing the required condition gives a preferred value of $\eta^*$ defined by:

\[
\eta^*_\text{opt} = -4 \gamma_{LPA} \left( \eta^*_\text{opt} \right)
\]

Notice that this condition is independent of the choice of the cutoff function, contrary to what is observed with the complete $O(\partial^2)$-order [7,12] (for another difference with the complete order, see footnote 12).

In terms of the quantity $\mu$ defined by (6) and taking into account the change of field variable (4), this condition writes, for $D = 3$ and $\gamma_{LPA} = Dz$ (see footnote 5)

\[
\mu^* = \frac{1 - 12z}{6}
\]

From the calculations done in section 2.1 we obtain:

\[
\eta^*_\text{opt} \simeq 0.269 \\
\nu \simeq 0.649
\]

which are not excellent values. This result may however be considered as an improvement compared to the LPA for which no vestige of the reparametrization invariance was observed.

### 4 Summary and conclusion

We have justified the presence of a non-vanishing value of $\eta$ in the LPA as a strict consequence of the general principles of the RG theory. Without field renormalization, as usually prescribed in the conventional LPA (with $\eta = 0$), the approximation could not be considered as the genuine $O(\partial^2)$-order of the derivative expansion. If no particular estimate of $\eta$ can actually be proposed
at this order, $\eta$ is not an arbitrary parameter, it varies monotonously (within some limits) on changing the normalization of the field as a consequence of a perfect breaking of the reparametrization invariance. The situation is coherent with the idea that, if the derivative expansion converges then, at least, a progressive and smooth restoration of the invariance must be observed from the few first terms of the expansion (this was not possible with the conventional view). We have done explicit calculations of the lines of fixed points generated by the change of the normalization of the field by a constant $z$ using both purely numerical and quasi-analytic methods in order to offer the possibility to anyone to easily redo the calculations. We also emphasize that, despite the minimum observed (for $D = 3$) in the evolution of the critical exponent $\nu$ on varying $z$, the principle of minimal sensitivity cannot be applied being not compatible with a possible vestige of the reparametrization invariance in the LPA. We have shown (in section 3.1) that it is purely artificial the conventional argument stating that $\eta$ should vanish if one keeps the kinetic term unchanged along a RG flow of the potential. We have illustrated (in appendix A) the respective roles of the movable and fixed singularities of the fixed point solutions in the convergence property of the quasi-analytic methods of integration utilized in the study.

A Movable singularity and convergence of the simplistic method

In this appendix we illustrate the role of the movable singularity of the solution of the fixed point equation of the LPA in the convergence and efficiency of the simplistic method (see section 2.2.1).

Let us consider the fixed point equations of respectively the RG flow (3) of $U$ and the RG flow (12) of the Legendre transformed potential $V$ [see (13-15)]. For $D = 3$ and $\eta (t) = 0$ the common value of the connection parameter $k^* = U^* (0) = V^* (0)$, corresponding to the respective regular fixed point solutions, is known with a huge number of digits [37] to be:

$$k^* = 0.07619940081234 \cdots$$  \hfill (A.1)

That value corresponds precisely to the only solution of the two-point boundary problem (19) with $\mu^* = 1/6$. If one forgets about the condition at infinity, then it exists a solution involving a singularity for each value $^{13}$ of $k = U (0)$ different from $k^*$. Hence getting the value (A.1) may be viewed as the consequence of pushing that movable singularity to infinity.

$^{13}$That singularity having a location which depends on the value of $k$, is named movable singularity.
In addition to a movable singularity, a solution of (7-8) displays also fixed singularities. Potentially, those singularities control the convergence properties of the Maclaurin series of the ultimate $U^*$.

We have performed Padé approximants on the Maclaurin series of solutions for various $k$ of the fixed point equations in the two cases of $U$ and $V$. The complex zeros of the denominators of the corresponding rational fractions give an approximate image of the location of the singularities in the complex plane of the variables $x = \phi^2$ and $x = \varphi^2$ of respectively $U$ and $V$.

Figure (A.1) shows the singularity structure of $V$ for two values of $k$ on approaching $k^*$. One clearly sees that the movable singularity still lies within the circle of convergence of the Maclaurin series though $k$ is already close to $k^*$. On approaching closer to $k^*$ the movable singularity is pushed to the right and the simplistic method ceases to converge when the singularity comes out of the disc of convergence of the series. Concretely the method provides an estimate of $k^*$ with 9 accurate figures.

On the contrary, fig. (A.2) shows that the movable singularity is already well outside the circle of convergence of the Maclaurin series of $U$ when $k$ is still far from $k^*$. In that case the simplistic method cannot even give a poor estimate of $k^*$.

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14 The locations of which vary very slowly with $k$. 

21
Fig. A.1. Evolution of the disposition, in the complex plane of the variable $x = \varphi^2$, of the singularities of the solutions $V^*(\varphi)$ of the fixed point equation, for $\eta^* = 0$ and $D = 3$, of the RG flow Eq. (12) in the process of determining $k^* = V^*(0)$ the value of which is given by Eq. (A.1). Two steps are shown: $k = k^* + 8 \times 10^{-4}$ (top) and $k = k^* - 8 \times 10^{-12}$ (bottom). As one approaches $k^*$, the movable singularity is pushed on the right (ideally up to infinity). The efficiency of the simplistic method ceases when the movable singularity is about to leave the disc of convergence of the Maclaurin series (determined by the location of the fixed singularity the closest to the origin). The method yields a rather high accuracy on the estimate of $k^*$ with 9 accurate figures. The radius of convergence of the Maclaurin series for $k = k^*$ is $R = 11.449$.

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Fig. A.2. Disposition, in the complex plane of the variable $x = \phi^2$, of the singularities of the solution of Eq. (7), for $\eta^* = 0$ and $D = 3$, in the process of determining $k^* = U^*(0)$ given by Eq. (A.1). Same presentation as in fig. (A.1). The step corresponds to $k = k^* + 4 \times 10^{-3}$ which is relatively far from $k^*$ whereas the movable singularity is already outside the disc of convergence of the Maclaurin series. The simplistic method does not work in that case. The radius of convergence of the Maclaurin series for $k = k^*$ is $R = 5.2719$.

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