Forecasting Foreground Impact on Cosmic Microwave Background Measurements
Lloyd Knox
University of Chicago

Abstract
We explore the possible impact of galactic and extragalactic foregrounds on measurements of the cosmic microwave background (CMB). We find that, given our present understanding of the foregrounds, they are unlikely to qualitatively affect the ability of the MAP and Planck satellites to determine the angular power spectrum of the CMB, the key statistic for constraining cosmological parameters. Sufficiently far from the galactic plane, the only foregrounds that will affect power spectrum determination with any significance are the extragalactic ones. For MAP we find the most troublesome foregrounds are radio point sources and the thermal Sunyaev-Zeldovich (SZ) effect. For Planck they are these same radio point sources and the Far Infrared Background. Prior knowledge of the statistics of the SZ component (either via theoretical calculation, or higher frequency observations of just a few percent of the sky, as will be done by balloon-borne experiments) may significantly improve MAP’s determination of the CMB power spectrum. We also explore the foreground impact on MAP and Planck polarization power spectrum measurements.

Subject headings: Cosmology, Cosmic Microwave Background

1. Introduction
Much attention has been paid in recent years to the nature of the foregrounds which obscure our view of the background. This attention has resulted in discoveries about the nature of the foregrounds as well as methods for estimating CMB anisotropy from foreground-contaminated data. From studying these developments, we have concluded that for planned large area, multi-frequency experiments, such as MAP and Planck, the foregrounds are unlikely to be responsible for qualitative degradation of the primary cosmological results.

This happy situation is due to a number of factors. First, there is a window in frequency space, where, at high galactic latitude, CMB fluctuations are the brightest diffuse source in the sky. Second, the high-latitude galactic foregrounds are very smooth; they do not have much small-scale fluctuation power. Third, foregrounds, unlike instrument noise and some systematic error sources, are suppressed at small angular scales by the finite angular resolution of the telescope. Fourth, point source count estimates suggest that only a small fraction of pixels in the MAP and Planck maps will be affected—and these can be identified with threshold cuts and removed. Finally, even if uncertainty in a particular mode of the CMB map is dramatically increased by the presence of foregrounds, uncertainty in the CMB power spectrum may not be significantly affected. This is due to the fact that, at least in the foreground free case, the dominant source of power spectrum uncertainty (except at the smallest angular scales) comes from sample variance, not instrument noise.

The primary cosmological results—determination of cosmological parameters—depend mostly on how well the power spectrum is measured. We thus focus on the impact of foregrounds on the determination of this power spectrum. Our method for estimating this impact can be considered to be a generalization of those based on Wiener filtering by Bouchet, Gispert, and Puget (1995, hereafter “BGP95”) and Tegmark and Efstathiou (1996, hereafter “TE96”) as well as that of White (1998, hereafter “W98”). All these approaches take the CMB and foregrounds to be statistically isotropic, Gaussian-distributed fields. Given this assumption, estimation of the power spectrum errors is straightforward, as described below.

As is always the case with parameter estimation, how well the desired parameters can be reconstructed depends on the assumed prior information. The methods of TE96 and W98 essentially assume that the foreground power spectra are known with infinite precision a priori. The most important difference between our method and theirs is that we only assume finite prior information about the foreground power spectra.

Although the method of BGP95 was derived assuming Gaussianity, they have tested it with non-Gaussian simulations of Planck Surveyor maps (see, e.g., Gispert and Bouchet, 1996). Their results lend credibility to the forecasts derived analytically under the Gaussian assumption.

Below we first describe our methods for estimating the power spectrum uncertainties given an experimental configuration and foreground model. In section III we describe our model of the foregrounds, which is based on
that detailed in the Planck Phase A proposal (Bersanelli et al. 1996), and the High Frequency Instrument (HFI) and Low Frequency Instrument (LFI) proposals.

To date, foregrounds have been essentially ignored in estimates of the cosmological parameter uncertainties. We find that they are unlikely to qualitatively change the results. Although for MAP this conclusion depends somewhat on the amplitude of the contribution from the Sunyaev-Zeldovich effect, which is not yet sufficiently well-determined.

Not only does the amplitude of the SZ power spectrum affect the ability of MAP data to constrain the CMB power spectrum, but so does our prior knowledge of it. This is fortunate, because while the amplitude is completely out of our control, we can do something about how well we know it. We emphasize that the prior information we need is not of the actual SZ map, but of the statistics of the map. The statistics can be calculated theoretically, or by actually measuring the SZ map over only a few per cent of the sky.

Higher order moments of the probability distribution may also be of interest if the CMB statistics are non-Gaussian, which they will be to some degree even if the primordial fluctuations are Gaussian. Therefore, we also estimate how well the amplitudes of individual spherical harmonics can be determined. The uncertainty on these amplitudes is much more strongly affected by the presence of foregrounds than is the uncertainty on the power spectrum.

Effects due to contributions not in ones model of the data may be detrimental and our formalism does not address such a problem. Nevertheless, we find it encouraging that for the quite general model we have chosen, where the data are required to simultaneously constrain thousands of foreground parameters, the results look very good.

2. Methodology

We assume that the experiment measures the full sky in \(\nu = 1, ..., n_{\text{ch}}\) channels, and model the (beam-deconvolved, spherical-harmonic transformed) map data as due to the CMB, foregrounds and noise:

\[
\Delta_{\nu lm} = \sum_i g_{\nu i} a_{i lm} + n_{\nu lm} \tag{1}
\]

where \(i\) runs over the components, \((i = 0\) is CMB, \(i > 0\) are the foregrounds) and \(g_{\nu i}\) gives their frequency dependence. In the following we usually suppress all indices and use a notation in which Eq. (1) becomes:

\[
\Delta = g^i a + n. \tag{2}
\]

Throughout we assume that the noise is spatially uniform, Gaussian-distributed, and uncorrelated from channel to channel. Therefore, \(W \equiv < \mathbf{n}^i \mathbf{n}^j > -1\) is given by

\[
W_{\nu lm,\nu' l'm'} = w_\nu \delta_{\nu \nu'} \delta_{\mu \mu'} \delta_{\mu \mu'} \tag{3}
\]

where the weight-per-solid angle for map \(\nu\), \(w_\nu\), equals \(B_{\nu,i}/(\sigma^2_\nu \Omega_{\text{pix}})\). \(\sigma_\nu\) is the standard error in the temperature determination of a map pixel with solid angle \(\Omega_{\text{pix}}\), and \(B_{\nu,l}\) is the beam profile—which for a Gaussian beam with full-width at half-max \(\sqrt{8 \ln 2} \sigma\) is given by \(\exp(-(l\sigma)^2/2)\). The beam-damping of the weight matrix is due to the fact we are describing the noise in the beam-deconvolved maps.

If we make specific assumptions about the statistics of the CMB and foregrounds then we can determine how well we can measure the parameters of those statistical distributions. For simplicity and specificity we assume the CMB and foregrounds to all be statistically isotropic and Gaussian-distributed. In this case a complete statistical description of the data is given by the two-point function:

\[
M \equiv < \Delta \Delta^\dagger > = g^i (\mathbf{a} a^\dagger)_i g + W^{-1}. \tag{4}
\]

If, in addition to statistical isotropy, we assume that each of the foreground components are uncorrelated then we can write

\[
< a^i_{\nu lm} a^j_{\nu' l'm'} > = C_{\nu \nu'} \delta_{\nu \nu'} \delta_{\mu \mu'} \delta_{\mu \mu'} \tag{5}
\]

and Eq. (4) simplifies to (with indices restored):

\[
M_{\nu lm,\nu' l'm'} = \sum_i g_{\nu i} g_{\nu' i} C_{\nu \nu'} \frac{1}{w_\nu} \delta_{\mu \mu'} \delta_{\mu \mu'}. \tag{6}
\]

Given the data, we could write down and calculate the posterior probability distribution of the parameters, \(C_{\nu \nu'}\), or any other parameterization, \(a_p\), of \(M\). The posterior is proportional to the product of the likelihood and the prior. In the limit that the posterior distribution of \(a_p\) is Gaussian, the expectation value for the covariance matrix of the parameters is given by the inverse of the “posterior” Fisher matrix,

\[
F_{pp'} \equiv \left< \frac{\partial^2 \ln P_{\text{posterior}}}{\partial a_p \partial a_{p'}} \right> = \frac{1}{2} \text{Tr} \left( M^{-1} \mathbf{M}_p M^{-1} \mathbf{M}_{p'} \right) + F_{pp'} \tag{7}
\]

4These proposals are not yet publically available. Most of the work referred to here will soon be available as Bouchet and Gispert (1998).

5A notable exception is the “LDB” case in Bond, Efstathiou & Tegmark (1997) which was based on calculations by the TopHat group of CMB pixel errors after pixel-by-pixel subtraction of foregrounds in their Antarctic maps.
Note that the trace is a sum over $\ell s$, $ms$ and $ns$. $M$ is block-diagonal with block size $n_a$ by $n_a$ so its inversion is readily feasible. The matrix $F$, or rather its inverse, is exactly what we want, the expectation value of the covariance matrix of the parameters. We are interested in calculating this parameter covariance matrix for various parameter choices—in particular the $C_{il}$ as well as assumptions about their prior distributions.

We parameterize the (diagonal) prior as zero for $i = 0$ and

$$
F^{\text{prior}}_{il,il} = (\alpha/C_{il})^2
$$

for $i > 0$ where $C_{il}$ are the assumed actual power spectra, to be discussed in the next section. Note that if we take the foreground $C_{il}$ as to be a priori perfectly known ($\alpha \to \infty$), then Eq. [8] gives the Fisher matrix for the Wiener filter method of foreground removal (TE96, BGP95), an explicit expression for which is in W98. In the absence of foregrounds it is equivalent to that given by Knox 1995 and by Jungman et al. (1996).

Below we vary $\alpha$ to see quantitatively how the strength of our prior assumptions determines the ability to measure $C_{il}$.

It is straightforward to generalize the above to include polarization information. Maps of the Q and U Stokes parameters can be decomposed into two components, $a^Q_{il}$ and $a^U_{il}$ (Kamionkowski et al. 1997; Zaldarriaga & Seljak 1997), which are now in addition to the temperature component $a^T_{il}$. In general, we can write the contribution from each component as $a^{b}_{il}$ and the data in each channel as $\Delta^{b}_{l,\nu} where the superscript is either $T$, $E$ or $B$. Then the covariance matrix for the data (Eq. [8]) becomes

$$
M^{b,\nu_1,\nu_2}_{11} = \left[ \sum_i g_{il} g_{i,\nu_1} C^{bb}_{il} + \frac{1}{w^{b}_{\nu_1}} \delta_{b}^{b} \delta_{\nu_1}^{\nu_2} \right] \delta_{\nu_1}^{\nu_2} \delta_{mm'} (9)
$$

where $C_{il}^{bb'}$ equals $C_{il}^{T} \equiv \langle a^T_{il} a^T_{il} \rangle$ for $b = b' = T$, $C_{il}^{EE} \equiv \langle a^E_{il} a^E_{il} \rangle$ for $b = b' = E$, $C_{il}^{TT} \equiv \langle a^T_{il} a^T_{il} \rangle$ for $b = b' = B$, and $C_{il}^{TT} \equiv \langle a^{BB}_{il} a^{BB}_{il} \rangle$ for $b = T, b' = E$. All other elements vanish. Thus, while the matrix of Eq. [8] is block-diagonal in blocks of dimension $n_a$, this matrix is block-diagonal in blocks of dimension $3n_a$. This approach generalizes the multi-frequency Wiener filter error forecasting of Bouchet et al. (1998, hereafter “BGS”), who generalized the single-frequency, no foreground, treatment of Zaldarriaga et al. (1997).

We may also be interested in how well an individual mode can be measured. The covariance matrix for the error in the minimum variance estimate of $a$ is

$$
\langle \delta a_{il} \rangle = \left( g^{T} W g + W^{\text{prior}} \right)^{-1}
$$

where we have assumed a prior probability for $a$ that is Gaussian-distributed with weight matrix $W^{\text{prior}}$. For example, we may wish to assume that foreground $i$ has variance $C_{il} \delta_{j} \delta_{mm'}$ in which case $W^{\text{prior}}_{\nu_1,\nu_2} = 1/C_{il} \delta_{\nu_1}^{\nu_2} \delta_{mm'}$. With this prior, this is the variance given by the Wiener filter procedure. Without the prior it is the variance given by the pixel-by-pixel subtraction procedure of Dodelson (1996) and also of Brandt et al. (1994) (except for their non-linear parameter dependences). When there are more foregrounds than channels, $g^{T} W g$ is singular and therefore addition of a prior is necessary to make $\langle \delta a_{il} \rangle$ finite. For more flexibility in the prior choice later, we define $\beta$ so that $W^{\text{prior}}_{\nu_1,\nu_2} = \beta/C_{il}$. Note that Eq. [10] does not assume anything about the statistical properties of the foregrounds and CMB—except through the prior, which we have explicitly assumed to be Gaussian.

3. Foreground Models

Our foreground model is based on that developed for the Planck Phase A proposal (Bersanelli et al. 1996) and updated in the HFI and LFI instrument proposals. We refer the interested reader to these proposals and to Bouchet and Gispert (1998, hereafter “BG98”). Below we briefly describe our model, with an emphasis on the modifications and additions we have made. In all cases, these alterations make the model more pessimistic.

3.1. Galactic

Analyses of the DIRBE (Diffuse Infrared Background Explorer) and IRAS (Infrared Astronomy Satellite) Sky Survey Atlas maps have determined the shape of the dust power spectrum to be $C_{l} \propto \ell^{-2.5}$ (Gautier et al. 1997) or $C_{l} \propto \ell^{-3}$ (Schlegel et al. 1998). We assume $C_{l} \propto \ell^{-2.5}$ since it is the more pessimistic choice, given that we normalize at large angles.

We take the same $C_{l}$ shape for the free-free power spectrum because both the dust intensity and free-free are expected to be from the same warm interstellar medium. Indeed, there is strong observational evidence for a correlation (Kogut et al. 1996, Leitch et al. 1997, de Oliveira-Costa et al. 1997, Jaffe et al. 1998). Note, however, that we assume no cross-correlation between free-free and dust, because any correlation makes the foreground separation easier. The same shape is also taken for synchrotron radiation.

We choose amplitudes and frequency dependences for the galactic foregrounds consistent with the Kogut et
The analysis of DMR, DIRBE and Haslam maps. We take the antenna temperatures of the free-free and synchrotron to vary with power-law indices -2.9 and -2.16, respectively. For the dust we assume a $\nu^2$ emissivity dependence and a single component with $T = 18\,K$.

Draine and Lazarian (1997) have proposed an alternative explanation to the observed correlation between dust and 30 GHz to 90 GHz radiation. They propose that the rotational emission from spinning dust grains, greatly increases the emission in the 10 GHz to 100 GHz range above what one expects from the vibrational emission. We have not included this component of dust emission in our model. Instead, we include something worse – a component with spectral shape similar to the “anomalous” emission, but which has no correlations with the dust. Again, this is more pessimistic than the strong correlation expected in a realistic model.

![Image](image_url)

Fig. 1.— The frequency-dependent rms antenna temperature, $g_i\sqrt{l(l+1)C_{ll}^{i}/(2\pi)}$ evaluated at $l = 500$ (top panel) and $l = 1500$ (lower panel) for standard cdm CMB (black), dipole and thermal emission dust (both red), free-free (green), synchrotron (blue), SZ (cyan), and radio and FIR point sources (both magenta).

3.2. Extragalactic

Extragalactic contributions to the microwave sky include inverse Compton scattering of CMB photons by hot gas in clusters (the thermal Sunyaev-Zeldovich (SZ) effect), the Far Infrared Background (FIRB) and radio point sources.

Following Tegmark and Efstathiou, we model the contribution from a diffuse background of unresolved radio point sources as having an intensity, $I(\nu) \propto \nu^{-\alpha}$ with a white noise angular power spectrum ($C_l$ independent of $l$). Deviations from white noise due to clustering are observed to be negligible at 1.5GHz (TE96; Tofollati et al. 1998, herafter “To98”). Below 200 GHz, we take $\alpha = 0$ but above 200 GHz we introduce a break to $\alpha = 0.7$ as suggested by To98. We adopt this break in our spirit of pessimism because, despite decreasing the brightness of this contaminant, it actually makes determination of the CMB more difficult. This is due to the fact that with the break, the spectral shape more closely resembles that of the CMB.

We are actually considering the power spectrum of the sources which remain after some cut is done of clearly contaminated pixels, e.g., those above a $5\sigma$ threshold where $\sigma^2$ is the variance in the map. Thus the amplitude depends both on the number-count distribution and on the level of flux cut that is used. Although this flux cut will vary for maps at different frequencies and from different instruments, we choose to fix it at 1 Jy. We view this as quite conservative since the typical level for all the Planck maps is about $\sigma = 0.1\,\text{Jy}$. This is according to Tegmark & de Oliveira-Costa (1998) who used the point-source model of To98 and included the effect of reduction in $\sigma$ that one can achieve by applying a Wiener filter. The values of $\sigma$ for the MAP maps should not differ by more than a factor of 2 from those for the LFI.

For the amplitude of the FIRB we rely on the estimates of BG98 which are derived from the model of FIR point sources of Guiderdoni et al. (1998). This model has successfully predicted source counts in a wide wavelength range, from 15 to 850 microns (see BG98 and references there in). The mean frequency dependence of the model is shown in Fig. 1. Bouchet and Gispert (1998) have shown that this frequency dependence has only slight spatial variations, lending credence to our modeling of it as a frequency dependence times a fixed brightness spatial template. We assume clustering is unimportant and therefore the spatial power spectrum has the same shape as we have assumed for radio point sources: $C_l$ is a constant.

CMB photons moving through a hot gas of electrons have their frequency shifted as they Compton scatter, leading to the generation of anisotropy with a non-thermal spectrum. This Sunyaev-Zeldovich (SZ) effect can also be treated as an additive foreground, with the frequency-dependence of a Compton $y$ distortion. Calculations of the power spectrum of this foreground component, assuming a Press-Schechter distribution of clusters with masses greater than some cutoff have been done (Aghanim et al. 1996, herafter A96; Atrio-Barandella & Mucket 1998). We use the results of A96 for the $\Omega = 1$
cosmology. Their power spectrum is well-fit in the range of interest by $C_l = a(1 + l_c/l)$ where $l_c = 1190$ and $a$ is such that $l(l+1)C_l/(2\pi) = 5.3\mu K^2$ at $l = 1500$ in the Rayleigh-Jeans (low frequency) limit (see Fig. 2). Modelling of this contribution will soon be improved by replacement of the use of Press-Schechter with N-body/hydro simulations.

3.3. Spectral Shape Uncertainty

Implicit in our formalism is that the frequency dependence of the foregrounds is known perfectly and has no spatial variation. However, we can allow for some degree of spatial dependence of the spectrum as follows. Consider foreground $i$ with mean power-law frequency dependence, $\beta$, and deviation $\delta \beta_i$. Then, the signal contribution to the data, $\Delta \nu_{lm}$, from component $i$ is

$$a_{ilm} (\nu/\nu_0)^\beta + \delta \beta_i \simeq a_{ilm} (\nu/\nu_0)^\beta + a_{ilm} \delta \beta_i (\nu/\nu_0)^\beta \ln(\nu/\nu_0).$$

(11)

Thus we can treat radiation from a component with spatially varying spectral index as due to two components with amplitudes $a_{ilm}$ and $a_{ilm} \delta \beta_i$, which will, in general, be correlated. For simplicity we have modeled these additional components as uncorrelated with the parent component and taken $\langle a_{ilm} \delta \beta_i a_{ilm}', \beta_i \rangle = C_{il} \langle \delta \beta^2 \rangle$. We have assumed $\langle \delta \beta^2 \rangle = 0.25$ for the rotating small dust grains, dust, and synchrotron with the same prior as used on other foregrounds. TE96 also considered using extra components to model spatial dependence of the spectral shape. For an alternative approach, see Tegmark (1997).

3.4. Foreground Polarization

Precious little is known about the polarization of foregrounds. For a review, see Keating et al. (1998). Extrapolation from observations at low-frequency ($\lesssim 1$ GHz) are complicated by Faraday rotation along the line-of-sight, which is negligible at higher frequencies. Measurements at higher frequencies are in the galactic plane in dense star-forming regions (Hildebrand & Drago van 1995) and are not expected to be representative of the statistics at high latitude. We make the same assumptions about foreground polarization as BPS. They neglect polarization in all foregrounds except for synchrotron and dust. For the synchrotron, they take $C_l^E = 0.2C_l^T$ and for the dust they take the model of Primet et al. (1998, hereafter “PSB”) (see also Sethi et al. (1998)). It must be kept in mind that the PSB calculation relies on indirect arguments and is therefore quite uncertain, as is the synchrotron model, as the authors readily admit.

4. Application to Planned Experiments

4.1. Temperature

In Fig. 2 one can see that MAP’s ability to measure the power spectrum is not significantly affected by the foregrounds below $\ell \simeq 500$. Going to smaller values of $\ell$ we have greater frequency coverage, and greater ratio of signal to instrument noise. The only thing that gets slightly worse as $\ell$ decreases is the relative amplitude of the galactic foreground power spectra, but this effect is overwhelmed by the others. Of course going to higher $\ell$ we have less frequency coverage and a smaller ratio of signal to instrument noise. The galactic foregrounds still do not become a problem though since their relative power continues to decrease.

What does become a concern at higher $\ell$ are foregrounds with rising angular power spectra: radio point sources and the thermal Sunyaev-Zeldovich effect from galaxy clusters. These alone are responsible for the deviation of $\Delta C_l$ from the no foreground case, visible in Fig. 3.

The impact of the Sunyaev-Zeldovich component is worth exploring more. It is quite possible that the actual amplitude is ten times larger than in our baseline model. The A96 calculation ignores the contribution from filaments—which may actually dominate the contribution from the clusters, and it ignores the clustering of the clusters. If we increase the power by a factor of 10, and relax the prior on it to $\alpha = 0.1$ from $\alpha = 1.1$, $\Delta C_l$ doubles in the range from $l = 400$ to $l = 700$. On the other hand, if we increase the power by a factor of 10, and do not relax the prior, $\Delta C_l$ only increases by a few per cent. What we learn from this is that having some constraints on the power spectrum of the SZ component can be just as important as the actual amplitude.

The usefulness of prior knowledge of the SZ $C_l$ is encouraging. It suggests that the analysis of MAP data can profit significantly from accurate theoretical predictions of the statistical properties of the SZ component. It also suggests that measurements of the SZ component in much smaller regions of the sky, which roughly constrain the power spectrum, can be beneficial to the analysis of the full-sky MAP data. Such analyses should be possible from combining MAP data with datasets from higher frequency instruments such as TopHat6 and BOOMERANG7, which by themselves will be extremely interesting CMB datasets.

Planck’s ability to measure the power spectrum is not significantly affected by the foregrounds below $\ell \simeq 1200$.  

---

6 TopHat home page: [http://topweb.gsfc.nasa.gov](http://topweb.gsfc.nasa.gov)
7 BOOMERANG home page: [http://astro.caltech.edu/~mc/boom/boom.html](http://astro.caltech.edu/~mc/boom/boom.html)
At higher $\ell$, the frequency coverage reduces, the noise in each channel increases and the SZ, radio and FIRB components increase in amplitude. Unlike for MAP, SZ is not important because in the HFI frequency range, SZ is easily distinguished from CMB; there is even the null at 217 GHz. However, the radio point sources and FIRB are a concern. There is strong dependence on the prior. Even with moderate prior information ($\alpha = 1.1$ on these two components), $\Delta C_{\ell}$ is 3 times larger than the no foreground case. With an infinite prior this reduces to a much less significant factor of about 1.2. The situation is greatly improved if the flux from the two backgrounds of unresolved sources is a factor of 4 less in amplitude (16 in $C_{\ell}$) than we have assumed. This is not unlikely since our assumed flux cut of 1 Jy is about 20 times the level of confusing noise, calculated by Tegmark & de Oliveira-Costa (1998), in the (post-Wiener filtering) 143 GHz, 217 GHz and 353 GHz HFI maps, and is therefore an extremely conservative 20$\sigma$ cut. Thus, we also show the results with our input power spectrum for point sources, and the FIRB each reduced by a factor of 16 as the dashed line in Fig. 2.

We see that with only the use of a moderate amount of prior information, the errors on the $C_{\ell}$s here are not qualitatively different from the no-foreground results. The conclusions of those forecasting cosmological parameter errors would not be qualitatively changed by including the effect of the foregrounds as modelled here.

If galactic foregrounds are well-described by the model used here, then they will not have significant impact on the primary science goals of MAP and Planck. That is perhaps the most robust conclusion to draw from the above. This is not to say that these foregrounds do not have their impact on how well the CMB can be measured. The left side of Fig. 2 shows how the foregrounds affect the uncertainties in $\ell_{\text{CMB}}$. As long as $\delta_{\ell_{\text{CMB}}} / \sqrt{C_{\ell}} < 1$ then sample-variance dominates the errors in $C_{\ell}$. As can be seen in the figure, this inequality holds out to at least $l = 500$, except for MAP in the case of pixel-by-pixel subtraction ($\beta = 0$, or no use of prior information).

4.2. Polarization

The CMB is expected to be polarized at a level of about 10% of the anisotropy. The polarization foregrounds are no where near as well-understood and explored as the temperature foregrounds. However, taking some initial guesses at the polarization foregrounds we find the outlook for CMB polarization measurement by MAP and Planck to be fairly bright. The reason being that, once again, there is a window in frequency space where the CMB is the dominant contributor to spatial variations in polarization.
This window does not necessarily exist across the entire polarization power spectrum, and in particular may disappear at low $l$. This is unfortunate since the two most interesting features in the polarization power spectra are the bump at $l \approx 2\sqrt{z_\text{re}}$ where $z_\text{re}$ is the redshift of reionization and the B-mode power spectrum due to tensor and vector modes (Kamionkowski et al. 1997, Seljak & Zaldarriaga 1997) which also peaks at low $l$.

Here we focus on the reionization bump. To study sensitivity to it we have not implemented Eq. 9. Instead we have ignored cross-correlations between temperature and polarization so that Eq. 6 is applicable with appropriate substitutions (e.g., $C_{il} \rightarrow C_{il}^E$). In general the cross-correlations improve the constraints on the polarization power spectrum (BPS) but that shouldn’t be the case here since the reionization bump is a feature solely of the polarization maps and does not show up in cross-correlation with temperature maps.

For standard CDM with an optical depth to Thomson scattering of $\tau = 0.1$, Planck measures the reionization feature with cosmic variance precision (although HFI alone does not and neither does MAP). At larger $l$, where the signal is large, the foregrounds, as modelled here, have no significant impact on the ability of either of the satellites to measure the CMB polarization power spectrum. Our infinite foreground prior (or Wiener filter) results are in agreement with the Wiener filter results of BPS.

This window does not necessarily exist across the entire polarization power spectrum, and in particular may disappear at low $l$. This is unfortunate since the two most interesting features in the polarization power spectra are the bump at $l \approx 2\sqrt{z_\text{re}}$ where $z_\text{re}$ is the redshift of reionization and the B-mode power spectrum due to tensor and vector modes (Kamionkowski et al. 1997, Seljak & Zaldarriaga 1997) which also peaks at low $l$.

Here we focus on the reionization bump. To study sensitivity to it we have not implemented Eq. 9. Instead we have ignored cross-correlations between temperature and polarization so that Eq. 6 is applicable with appropriate substitutions (e.g., $C_{il} \rightarrow C_{il}^E$). In general the cross-correlations improve the constraints on the polarization power spectrum (BPS) but that shouldn’t be the case here since the reionization bump is a feature solely of the polarization maps and does not show up in cross-correlation with temperature maps.

For standard CDM with an optical depth to Thomson scattering of $\tau = 0.1$, Planck measures the reionization feature with cosmic variance precision (although HFI alone does not and neither does MAP). At larger $l$, where the signal is large, the foregrounds, as modelled here, have no significant impact on the ability of either of the satellites to measure the CMB polarization power spectrum. Our infinite foreground prior (or Wiener filter) results are in agreement with the Wiener filter results of BPS.
5. Discussion

We have presented a method to calculate the sensitivity to the CMB and its power spectrum given multi-resolution, multi-wavelength observations of a sky that consists of multiple foreground contributions. The applications to MAP and Planck have allowed for much greater freedom in the behavior of the foregrounds than did previous analyses (TE96, BG98). Despite this extra freedom, the conclusions are similar—that foregrounds are not likely to qualitatively affect the uncertainties that can be achieved on cosmological parameters. Similar conclusions have been reached by de Oliveira-Costa et al. (1998).

Our approach has not fully taken into account the non-Gaussianity of the foregrounds, spatial dependence of the spectrum of each component, uncertainty in the spectral shapes, and unknown components (e.g., a population of points sources whose spectra peak at 90 GHz). For these reasons it is difficult to conclude with certainty that the foregrounds will not qualitatively affect the determination of cosmological parameters. However, a very important reason for our rosy conclusions is a very simple fact: for most of the multipole moments measured by a given experiment, the quality of the CMB map can be highly degraded, without having any impact on the quality of the power spectrum. Thus, any effect we have not included here has to overcome this hurdle in order to be important.

Non-gaussianity is both a friend and a foe. We have already exploited it here in assuming that the brightest points sources could be identified with threshold cuts and removed. However, it can present a challenge to the above sample-variance argument if it resulted in the errors in each $a_{lm}$, in such a way that they did not beat down with many averagings. One can think of this as an effective reduction in the number of independent modes in the foreground (Eisenstein 1998). However, we expect that small-scale behavior in patches of the sky sufficiently separated to be decorrelated. Hence we do not expect the mode number reduction to be large, though further investigation of effects of non-Gaussianity is clearly warranted.

We have also neglected things that will improve estimation of the CMB from MAP and Planck data, such as the use of maps at other frequencies, e.g., DIRBE, IRAS and FIRST (which will fly with Planck). Assumptions about the smoothness of foreground power spectra are also reasonable and could significantly reduce our error forecasts at high $l$, by extending the information gained at lower $l$ where there is greater frequency coverage.

It is clear though, that even if foregrounds do not do anything more than double the errors on cosmological parameters, the determination of the exact size of the error bars will probably be dominated by foreground considerations. Small patches of the sky will be analyzed separately, with those appearing the cleanest given more weight. Foreground model residuals will be aggressively sought. Thus the study of foregrounds remains very important. We close by listing the following improvements in our understanding of foregrounds which could prove to be extremely beneficial:

- More accurate theoretical calculation of the statistics of the SZ component. Our positive conclusions for MAP depend on the amplitude of the SZ power spectrum and on how well that power spectrum can be determined a priori. We have shown that having a prediction of $C_l^{SZ}$ good to about a factor of 2 (which would justify our use of $\alpha = 1.1$) is enough to keep $\Delta C_l$ within about ten per cent of the no foreground case, even if $C_l^{SZ}$ is ten times larger than the A96 calculation.
- Higher frequency complements to MAP, such as are coming from balloon flights (e.g., TopHat and BOOMERANG). Even coverage of just a few per cent of the sky, can be used to characterize the level of contamination in the rest of the sky.
- A point source survey near 90 GHz (see Gawiser et al. 1998).
- Further development of methods for removing non-Gaussian foregrounds.
- Measurements of high galactic latitude dust and synchrotron polarization.

I thank K. Ganga, A. Jaffe and J. Ruhl for useful conversations, as well as the organizers and participants of the Sloan Summit on Microwave Foregrounds, who have informed the above discussion. I acknowledge the use of CMBFAST (Seljak & Zaldarriaga 1996).

REFERENCES

Aghanim, N. et al. 1997, A&A, 325, 9 (“A96”)
Atrio-Barandela, F., & Muecket, J. P. 1998, preprint astro-ph/9811158

Bersanelli, M., et al. 1996, COBRAS/SAMBA, Phase A Study for an ESA M3 Mission, ESA Report D/SCI(96)3, http://astro.estec.esa.nl/Planck/

Bond, J. R., Efstathiou, G., & Tegmark, M. 1997, MNRAS, 291, L33

Bouchet, F. R., & Gispert, R. 1998, preprint in preparation (“BG98”)

Bouchet, F. R., Gispert, R., & Puget, J. The mm/sub-mm Foregrounds and Future CMB Space Missions, in Unveiling the Cosmic Infrared Background AIP Conference Proceedings 348, ed. E. Dwek (Baltimore, MD, USA: pages 255-268, 1995)

Bouchet, F. R., Prunet, S., & Sethi, S. K. 1998, preprint astro-ph/9809353 (“BSP”)

Brandt, W. N. et al. 1994, ApJ, 424, 1

Dodelson, S. 1996, ApJ, 482, 577

Draine, B. T., Lazarian, A. 1997, preprint astro-ph/9710152

de Oliveira-Costa, A. et al. 1997, ApJ, 482, L17

de Oliveira-Costa, A., Kogut, A., Devlin, M. J., Netterfield, C. B., Page, L. A., & Wollack E J 1997, ApJ, 482, L17

de Oliveira-Costa, A., Eisenstein, D., Hu, W., & Tegmark, M. 1998, preprint in preparation

Eisenstein, D., personal communication, 1998

Gawiser, E., Jaffe, A. H., & Silk, J. 1998, preprint astro-ph/9811148

Gispert, R., & Bouchet, F. R. Microwave Background Anisotropies, in Proceedings of the XVth Moriond Astrophysics Meeting, ed. F. R. Bouchet, R. Gispert, B. Guiderdoni & J. T. T. Van (Les Arcs, Savoie, France: pages 503-509, 1996)

Guiderdoni, B., Hivon, E., Bouchet, F. R., & Maffei, B. 1998, preprint astro-ph/9710340

Hildebrand, R., & Dragovan M 1995, ApJ, 450, 663

Jaffe, A. H., Finkbeiner, D., & Bond, J. R. 1998, preprint in preparation

Jungman, G., Kamionkowski, M., Kosowsky, A., & Spergel, D. N. 1996, Phys. Rev. Lett., 76, 1007

Kamionkowski, M., Kosowsky, A., & Stebbins A 1997a, Phys. Rev., D55, 7368

Kamionkowski, M., Kosowsky, A., & Stebbins A 1997b, Phys. Rev. Lett., 78, 2058

Keating, B., Timbie, P., Polnarev, A., & Steinberger, J. 1997, preprint astro-ph/9710087

Kogut, A. et al. 1996, ApJ, 464, L5 (“K96”)

Knox, L. 1995, Phys. Rev. D, 48, 3502

Leitch, E. M. Readhead, A. C. S., Pearson, T. J., & Myers, S. T. 1997, preprint astro-ph/9705241

McCullough, et al., 1998, preprint in preparation

Refriger A, Spergel, D. N., & Herbig, T. 1998, in preparation

Reynolds, R. J., Haffner, L. M., & Tufte, S. L. 1998, preprint astro-ph/9811315

Schlegel, D. J., Finkbeiner, D. P., & Davis, M. 1998, preprint astro-ph/9710327

Seljak, U., & Zaldarriaga, M. 1996, ApJ, 469, 437

Seljak, U., & Zaldarriaga, M. 1997, Phys. Rev. Lett., 78, 2054

Prunet, S., Sethi, S. K., & Bouchet, F. R.; 1998; submitted to MNRAS (“PSB”)

Sethi, S. K., Prunet, S., Bouchet, F. R. 1998, preprint astro-ph/9803158

Tegmark, M. 1997, preprint astro-ph/9712038

Tegmark, M., & Efstathiou, G. 1996, MNRAS, 281, 1297 (“TE96”)

Tegmark, M., & de Oliveira-Costa, A. 1998, preprint astro-ph/9802123

Toffolatti, M. et al. 1998, preprint astro-ph/9711082 (“To98”)

White, M. 1998, Phys. Rev., D57, 5273

Wright, E. L. 1998, preprint astro-ph/9711261

Zaldarriaga, M., Spergel, D. N., & Seljak, U. 1997, ApJ, 488, 1

Zaldarriaga, M., & Seljak, U. 1997, Phys. Rev., D55, 1830