Entanglement String and Spin Liquid with Holographic Duality

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ABSTRACT: We show that the quantum entanglement can be transmuted to a force using the holographic duality. First, we prove that there is an open string in the spectrum of the holographic fermion coupled with scalar. The string ends at two fermions and its tension vanishes in the limit of zero scalar condensation. We associate such string with a dimer and identify the scalar condensation as the degree of the dimerization. Together with divergently large entanglement entropy, the model is expected to describe the Spin Liquid. As a consistency check, we show that there is a Mott transition as the dimerization proceeds. We suggest that the string may be observed in an ARPES experiment of spin liquid or clean Dirac material as a tower of bands.

KEYWORDS: Holography, entanglement string, dimerization, spin liquid
1 Introduction

Strong correlation gives many unexpected phenomena which are very interesting as well as useful. Mott insulator, Strange metal and High Tc superconductivity are some of examples. Most interesting observation is the unreasonably fast equilibration both in strange metal[1] and quark gluon plasma[2], which seems to request non-local force. In such system, the low energy spectrum is very different from that of high energy: QCD string[3, 4] and fractionalized degrees in spin liquid [5–7] are examples. Finding such excitation often gives us a way to analyze a very difficult system in a simple way.

Recently, gravity dual description [8, 9] attracted much attention as a tool [10, 11] for strongly interacting system (SIS). The basic idea is to utilize the universality near the quantum critical point and its similarity with a black hole which shares the same scaling symmetry and the temperature. Since the entropy of black hole is given by its area [12] rather than the volume, we have a holographic relation [13, 14] whose origin can be attributed to the equivalence principle: infinitely large number of metrics can be identified. From the SIS point of view the holographic relation comes because the configurations of SIS at different energy scales are glued together [15–19] by quantum entanglement to form one higher dimensional system. It is very intriguing to ask if there is a mechanism by which entanglements can give a sort of force.

Motivated with these questions, we studied the fermionic particle with a scalar coupling in the dual gravity setup. We found an exact two-point function and found that
mass spectrum is that of open string excitation. Its string tension is given by the scalar condensation, therefore the string is tensionless in the zero condensation limit. The most intriguing part here is the origin of this stringy spectrum. For the hadronic physics, its appearance can be easily attributed to the QCD string, however, it is rather mysterious in the condensed matter because no confinement physics seems to be involved. Nevertheless, the appearance of massive string spectrum indicates that qualitatively similar physics is in action.

The open string should connect two fermions in the boundary which are the only object in our system, to form a dimer, the spin singlet bound state. In the absence of the scalar condensation, the string is massless so that there is no cost in forming a dimer between the far separated fermions. However, in its presence, the string has tension so that it should connect nearest two fermions to minimize its energy. In this picture, we suggest to call the new string as ‘entanglement string’, and the scalar condensation measures the degree of the dimerization of the system, equivalently the string condensation.

There can be a few possibilities with the condensation of the dimers: for example valence bond solid (VBS) or resonating valence bond (RVB) [5], because spin singlet condensation does not form any magnetization. See ref. [1, 20] for review and references. The key is the size of the entanglement entropy. Sometimes ago, Ryu and Takayanagi [21] showed that the presence of dual gravity request infinitely large entanglement entropy. The only system with dimerization with such large entanglement entropy is the RVB state whose fundamental aspect is also the presence of high degree of entanglement entropy [22]. Therefore it is natural to identify our system as the spin liquid rather than spin solid.

As a consistency check, we investigated the metal insulator transition, since the spin liquid is known to be an insulator. Can our holographic model show such transition? We will show that our system indeed has a Mott transition as we increase one or more of following three parameters: scalar condensation, temperature, chemical potential.

Finally we also discuss a few ways to observe the entanglement string, whose direct experimental test is the observation of the stringy spectrum.

2 Fermion in AdS$_4$ and string at the boundary

In this paper we study the fermion dynamics with a real scalar $\Phi$ so that the fermion action is given by

$$S_D = i \int d^d x \sqrt{-g} \bar{\psi} \left( \Gamma^M \mathcal{D}_M - m - \Phi \right) \psi + S_{bd},$$

(2.1)
where \( \mathcal{D}_M = \partial_M + \frac{1}{4} \omega_{abM} \Gamma^{ab} - iqA_M \) is the covariant derivative in the asymptotically \( AdS_4 \) of radius \( L \) whose explicit form is given in the appendix. \( \Phi \) is a scalar field whose dynamics is given by a free real scalar action with mass \( m_\Phi^2 L^2 = \Delta(\Delta - 3) = -2 \) in the same gravity background. If \( \Phi^{(0)} \) is a scalar field that couples with the boundary fermion bilinear \( \bar{\chi} \chi \) with dimension \( \Delta = 2 \), it is embedded in the bulk field \( \Phi \) such that near boundary behavior of the latter is

\[
\Phi(x, r) = \frac{\Phi^{(0)}}{r} + \frac{M}{r^2} + \cdots \tag{2.2}
\]

Notice that in the zero temperature zero chemical potential limit, the first two terms are two independent exact solutions of the equation of the scalar field equation in the probe limit. Since we are looking for spontaneously generated gap at zero temperature, we set the source \( \Phi^{(0)} = 0 \). Notice that the dimension of \( M \) is that of mass squared. In the appendix, it is shown that the components of the Green function of fermions in the boundary theory are given by

\[
G^{R}_{11}(\omega, k) = \frac{(\omega - k_1)\Gamma(\frac{1}{2} - m)\Gamma(m + \frac{1}{2} + \frac{k^2 - \omega^2}{4M})}{2M^{\frac{1}{2} - m}\Gamma(\frac{1}{2} + m)\Gamma(1 + \frac{k^2 - \omega^2}{4M})}, \quad G^{R}_{22} = G^{R}_{11}(\omega, -k), \tag{2.3}
\]

and \( G^{R}_{12} = G^{R}_{21} = 0 \) in the standard quantization. It can be easily checked that in the limit of \( M \to 0 \), our result is reduced to that in ref. [23]. The spectral function is given by the imaginary part of the Green function, which can be easily recovered from Kramers-Kronig relation or by the prescription \( \omega \to \omega + i0^+ \).

What is remarkable in this result is that the singularities of the Green function are given by those of the Gamma function which has poles at all non-positive integer so that the spectrum of the theory is

\[
\frac{k^2 - \omega^2}{4M} + m + \frac{1}{2} = -n, \quad \text{for} \quad n = 0, 1, 2, \cdots, \tag{2.4}
\]

which is nothing but the spectrum of the relativistic open string

\[
\alpha' m_n^2 = (n + m + \frac{1}{2}), \tag{2.5}
\]

whose string tension \( T = 1/(2\pi \alpha') \) with \( \alpha' = \frac{1}{4M} \). Eq. (2.4) is the Regge trajectory with slope \( \alpha' \) and intercept \( \alpha_0 = -m - 1/2 \). This is an analogous situation of the discovery of the string from the Veneziano amplitude[3], where the origin of the string could be attributed to the confined color flux tube later. In \( M \to 0 \) limit, that is, in the tensionless
limit, the whole tower of string spectrum is reduced to that of a massless particle. Notice that the lowest spectrum given by $n = 0$

$$m_0^2 = 4M(m + \frac{1}{2}),$$  \hspace{1cm} (2.6)

shows that even for $M \neq 0$, we can still have the massless spectrum if $m = -1/2$. The same formula also shows that we get tachyonic spectra for $m < -1/2$. In this paper we consider only $-1/2 \leq m \leq 0$.

2.1 Entanglement string and dimers

The most intriguing question here is about the origin of the string. In fact, the presence of tower of spectrum has been appeared in the numerical plots of earlier literature, although its meaning has not been discussed. From the dual gravity point of view, it is a very special type of Kaluza-Klein (KK) spectrum of the bulk fermion. Usually the KK spectrum is quadratic in $n$, $m_{2n}^2 = c.n^2$, because one of the momentum component is quantized. Here it is linear: the AdS has a character of box drove by the quadratic gravitational potential. Such geometric picture gives an intuition why we have Regge trajectory, nevertheless, it does not help to understand its origin from the view of the original strongly interacting fermions. Let's collect some relevant facts and proceed to identify the origin of the string in a few steps.

1. We assumed the duality between the original fermion $\chi$ and the bulk fermion $\psi$ in AdS$_4$. One should notice that the spectral function is that of $\chi$, and so is the string. The original fermion lost its character as an elementary excitation due to the interaction and became a D0 brane[24], an object whose dynamics is given by an open string attached to it. Summarizing, the fermion particle in AdS is the holographic image of the string in the strongly interacting fermion system at the boundary. See the figure 1(a).

2. What is the physical reality of the string in the original system? Since the open string should end somewhere, our string should connect two fermions. Such fermion-string-fermion system can be identified as a dimer, which is a spin singlet Einstein-Podolski-Rosen (EPR) pair that is used as a building block of RVB (resonating valence bond) state [5, 6]. See figure 1(b). Also it is natural to interpret the $\Phi$ as the low energy description of the dimer where the separation of the two fermions can be neglected. Then, the condensation of the scalar should describe the degree of dimerization of the system.
appears naturally in many models, including the simple models discussed here. Fermi volume different from the Luttinger volume [\[\text{Fig.}\] 9]. However, with the availability of the low energy topological states discussed in Fig. 9, the only low energy excitations on the torus are the quasiparticles around the Fermi surface. Of adiabatically inserting a fluxoid presented a proof of the Luttinger volume in a Fermi liquid by considering the consequences of adiabatically inserting a fluxoid. The numbers of associated near-degeneracies on the torus, apply equally to the FL* wavefunction after counting in Fig. 9 by applying them to wavefunctions like Fig. 9. Liquids, and we can transfer all of the arguments of Section 3. In the limit of zero scalar condensation $M \to 0$, namely when there is no other dimers around the one we consider, the string tension is zero and there is no energy cost in connecting arbitrarily far separated EPR pair. Therefore it is natural to call our string as entanglement string. See figure 1(c). As scalar condensation $M$ increases, the dimers are more and more tightly bound by the string tension so that the system certainly prefers dimers formed between nearest fermions to minimize the energy. Long strings should be suppressed for such case.

4. $\bar{\chi}\chi(x)$ is a spin singlet and charge neutral regardless of $\chi$ being Dirac or Majorana fermion. Since the open string is connecting two fermions not fermion anti-fermion, they are better to be Majorana fermions or a spinons [5, 6]. This also confirms that the condensation of the scalar can encode only the degree of dimerization.

5. Half of the origin of the stringy spectrum is the presence of the dual space itself, because fermion in a AdS space has Kaluza-Klein spectrum. The other half is the specific power 2 in the Eq. (2.2). It depends on the spacetime dimension $d$, the spin of the condensed field $s$ and the conformal dimension $\Delta$ of the operator the bulk field is coupled to. Namely the string spectrum is the consequence of $\Delta - s = 2$. For the fermion bilinear, $\Delta = d - 1$.

6. There still are a few options for configuration of dimers: either it can have some order in their alignment called valence bond solid (VBS), or it can be symmetry unbroken state called resonating valence bond (RVB). Their main difference is the amount of the entanglement entropy [22]. The VBS can not be described by

![Figure 1](image1.png)
a scalar condensation because the rotation symmetry is broken. See figure 2(a,b). According to Ryu-Takayanagi[26], any system with holographic description has divergently large entanglement entropy proportional to the interface area. Therefore we can identify the system of holographic fermions with scalar condensation as **spin liquid** or resonating valence bond (RVB), the system of dimer condensation with maximal entanglement. See figure 2(b). The identification is consistent both symmetry and entanglement entropic point of views.

![Figure 2](image.png)

**Figure 2.** (a) Valence bond solid phase: The rotation symmetry is broken and can not be described by a scalar condensation. (b) The rotation symmetry is restored and huge entanglement entropy is guaranteed by the superposition of all possible dimer condensed states. Such system is possible holographically without lattice. The figures are minor adaptation from ref. [27, 28].

The message is that the presence of the holography which by itself is a consequence of entanglement, gives a set up where the quantum entanglement (QE) acts on a system as a force, the string tension, enforcing the system to realize the dimerization by nearest fermions. It also suggests that even without a strong correlation, a string is associated with EPR pair which resembles the statement ‘**ER=EPR**’ [16, 29, 30]. This may be useful to explain the non-local forces that seem to be responsible for the rapid thermalization of the strange metal [1] and quark gluon plasma [2].

### 2.2 More on experimental implementation

The most direct prediction of the theory is the existence of the string. Finding string means observing its spectrum given in figure 3. We expect that such tower of string spectrum may be found in the ARPES experiment of some Dirac materials with strong correlation.

Any Dirac material with small fermi surface, which is realized when the material is clean enough, is strongly interacting. Therefore we expect that the stringy spectrum can be observed in the experiment with clean Dirac material. Notice that the higher spectral tower with large $n$ is suppressed only by $1/\pi n$ instead of $1/n!$, which is naively
(a) gapless $m = -\frac{1}{2}, M = 1$

(b) gaped $m = -\frac{1}{4}, M = 1$

**Figure 3.** Stringy spectrums (a) without and (b) with gap. White peaks are contour plots of the numerical results and red lines are plots of the analytical result Eq. (2.4). Finite widths of white peaks are effects due to the interaction and small but finite temperature.

expected from the Gamma function residue. This can be seen easily from Eq. (2.4) using the formula $\Gamma(1 + z)\Gamma(1 - z) = \pi z/\sin \pi z$. The role of another Gamma function in the denominator of Eq. (2.4) is crucial for enhancing the spectral weight of higher order spectrum. Using a non-Dirac spin liquid material will also give a chance to observe it, because the difference between different quantum critical points is only quantitative one.

By figuring the elementary excitation as a dimer we may visualize how we can observe a stringy tower of spectrum more easily. When we shine a photon into a dimer, one of the end point fermion is affected if it is Dirac particle and it excites the string between two fermions. Such excitation will be detected in ARPES experiment. If we consider the end point fermions as neutral Majorana particles which exist due to the spin charge separation, similar stringy spectrum may be observed in the susceptibility measurement of the neutron scattering with spin liquid target material. We expect that there should be maximum of $M$ corresponding to the saturation of dimerization.

Finding such stringy spectrum would be extremely interesting, since it is an evidence of the holographic dual space as well as the entanglement string in real matter.

### 3 Mott transition

Now lets do some consistency check. The spin liquid is known to be a Mott insulator. Is our system a Mott insulator for full dimerization or equivalently for large enough scalar condensation $M$? In this section, we will see that in fact our system has Mott transition as we increase the scalar condensation.

The study of the spectral function shows that there are three easily distinguishable phases: gapless, pseudo-gap and gapped phases as one can see in the $(M, m)$ phase space
diagram given in figure 4. Here, gapless means the existence of degrees of freedom at the

Figure 4. $m$-$M$ Phase Diagrams. In all phase boundaries, the changes are smooth crossover. Notice that for all values of $m$, there is a transition from gapless to gapful phase through the pseudogap regime. Notice that $m$ in the vertical axis means its absolute value $|m|$.

Fermi sea $\omega = 0$ and gapped phase does not have any, pseudo gap means the depletion of DOS at the central peak. We choose the onset of pseudo gap as the 10% depletion.

We remark that a similar model with the Pauli interaction instead of scalar coupling [31] it was shown that the model exhibits the Mott transition [32] connecting the free fermion point [33–35] and gapped phase [36, 37] under increasing the coupling strength. To compare with our phase diagram with that of the Pauli interaction, we divide the gapless region into 4 subclasses: Fermi Liquid like (FL), Bad Metal (BM), Bad Metal prime (BM') and half Metal (hM). The characters of these phases are qualitatively similar to those of Pauli case and we refer the interested readers to the ref. [31]. From the fact that $m = -1/2, M = 0$ is the free fermion point and there is a large gapped phase region, any line connecting these two can realize the Mott transition. Furthermore, increasing $M$ induces the transfer of the degree of freedom from the central peak to shoulder peak. These features are common with the model with Pauli interaction studied in [31].

There is an important difference, however: while there is a Mott transition for any value of $m$ as $M$ varies for the scalar interaction, there is none for fixed $m$ if $|m| > 0.35$ in the Pauli interaction case.

Now we study the temperature and chemical potential dependence of the phase diagram. The figure 5(a,b) show how the $m$-$M$ phase diagram (PD) changes as we change those parameters respectively. We can easily see that as we increase temperature, the boundary of PG and Gapped phases is moving to the higher $M$ region significantly, while the boundary of Gapless and PG phase does not move much. The chemical potential evolution shows similar behavior qualitatively. Phase boundaries move to the higher $\mu$ region as we increase $\mu$. These are consistent with our expectations that Gap formation is harder at higher temperature or higher chemical potentials. We also draw $\mu$-$T$ phase
diagram explicit in figure 5(c). It would be interesting if these diagrams can be compared with experimental data.

![Diagram](image)

(a) m-M PD for various $T$  (b) m-M PD for various $\mu$  (c) $\mu$-$T$ PD

Figure 5. $|m|$-$M$ Phase Diagram (PD) for (a) $T$-evolution (b) $\mu$-evolution. Dotted lines are the phase boundaries for Gapless-PG phases and real lines are those for PG-Gapped phases. The Phase boundaries move to the higher $T$ or $\mu$ region as we increase $T$ or $\mu$. (c) $\mu$-$T$ Phase Diagram. Here $m = -0.3$ and $M = 4$.

4 Summary and Discussion

In this paper we investigated the scalar coupling of the fermion. First, it is shown that two entangled fermions are connected by a string and the string tension is proportional to the scalar condensation, therefore for vanishing condensation, the string becomes tensionless and all spectrum becomes massless so that the string is invisible. Secondly, the scalar condensation is identified as the dimerization and the system is suggested to be describing the spin liquid. Thirdly, we found that there is a Mott transition as we change the scalar coupling.

The scalar field condensation in this paper is just one possible mechanism of gap generation and it is identified as dimerization of the system. It would be interesting if we can similarly identify other gap generating interactions as physical phenomena. In this regard, we mention that in ref. [31], we suggested that when the mass generation is associated with the Pauli term, such instability can be related to the “nesting” for the charge density wave. This means that the gap generated by the Pauli term is associated with the CDW order just as the gap generated by the Yukawa term is associated with dimerization.

We expect that for all gap generating mechanism, similar trajectory should appear although it may not be strictly a linear one. Also, notice that in this paper the scalar field is considered as a probe. It would be interesting to couple it with the gravity so that the condensation is determined including the gravity back-reaction.
The appearance of the stringy spectrum can be more universal than we expect because matter can become a SIS in many ways: first, by having a small Fermi surface [38–40] as in the Dirac materials which form rather wide class, second by having small Fermi velocity as in the case of all transition metal oxides. This is because the effective coupling inside matter is $\sim \alpha/\epsilon v_F$, where $\epsilon$ is a screening constant. We wish that the entanglement string can be found in the spectrum of diverse material.

Acknowledgments

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References

[1] S. Sachdev, *Quantum Phase Transitions*. Cambridge University Press. (2nd ed.), 2011.

[2] J. Adams, M. Aggarwal, Z. Ahammed, J. Amonett, B. Anderson, D. Arkhipkin et al., *Experimental and theoretical challenges in the search for the quark–gluon plasma: The star collaboration's critical assessment of the evidence from rhic collisions*, Nuclear Physics A 757 (2005) 102–183.

[3] G. Veneziano, *Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories*, Nuovo Cim. A57 (1968) 190–197.

[4] S. Mandelstam, *Vortices and Quark Confinement in Nonabelian Gauge Theories*, Phys. Rept. 23 (1976) 245–249.

[5] P. W. ANDERSON, *The resonating valence bond state in la2cuo4 and superconductivity*, Science 235 (1987) 1196–1198, [http://science.sciencemag.org/content/235/4793/1196.full.pdf].

[6] S. A. Kivelson, D. S. Rokhsar and J. P. Sethna, *Topology of the resonating valence-bond state: Solitons and high-t c superconductivity*, Physical Review B 35 (1987) 8865.

[7] P. Fulde and F. Pollmann, *Strings in strongly correlated electron systems*, Annalen Phys. 17 (1997) 441–449, [0711.2129].

[8] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Int.J.Theor.Phys. 38 (1999) 1113–1133, [hep-th/9711200].

[9] E. Witten, *Anti-de Sitter space and holography*, Adv. Theor. Math. Phys. 2 (1998) 253–291, [hep-th/9802150].

[10] J. Zaanen, Y.-W. Sun, Y. Liu and K. Schalm, *Holographic Duality in Condensed Matter Physics*. Cambridge Univ. Press, 2015.
[11] S. A. Hartnoll, A. Lucas and S. Sachdev, *Holographic quantum matter*, 1612.07324.

[12] J. D. Bekenstein, *Generalized second law of thermodynamics in black hole physics*, *Phys. Rev.* **D9** (1974) 3292–3300.

[13] G. ’t Hooft, *Dimensional reduction in quantum gravity*, *Conf. Proc.* **C930308** (1993) 284–296, [gr-qc/9310026].

[14] L. Susskind, *The World as a hologram*, *J. Math. Phys.* **36** (1995) 6377–6396, [hep-th/9409089].

[15] M. Van Raamsdonk, *Building up spacetime with quantum entanglement*, *Gen. Rel. Grav.* **42** (2010) 2323–2329, [1005.3035].

[16] J. Maldacena and L. Susskind, *Cool horizons for entangled black holes*, *Fortsch. Phys.* **61** (2013) 781–811, [1306.0533].

[17] T. Faulkner, M. Guica, T. Hartman, R. C. Myers and M. Van Raamsdonk, *Gravitation from Entanglement in Holographic CFTs*, *JHEP* **03** (2014) 051, [1312.7856].

[18] T. Hartman, *Entanglement Entropy at Large Central Charge*, 1303.6955.

[19] E. Oh, I. Y. Park and S.-J. Sin, *Complete Einstein equations from the generalized First Law of Entanglement*, *Phys. Rev.* **D98** (2018) 026020, [1709.05752].

[20] L. Balents, *Spin liquids in frustrated magnets*, *Nature* **464** (03, 2010) 199 EP –.

[21] S. Ryu and T. Takayanagi, *Holographic derivation of entanglement entropy from AdS/CFT*, *Phys. Rev. Lett.* **96** (2006) 181602, [hep-th/0603001].

[22] L. Savary and L. Balents, *Quantum spin liquids: a review*, *Reports on Progress in Physics* **80** (2016) 016502.

[23] N. Iqbal and H. Liu, *Real-time response in AdS/CFT with application to spinors*, *Fortsch. Phys.* **57** (2009) 367–384, [0903.2596].

[24] J. Polchinski, S. Chaudhuri and C. V. Johnson, *Notes on D-branes*, hep-th/9602052.

[25] A. Walia, https://www.collective-evolution.com/2016/03/15/this-new-experiment-will-allow-us-to-see-quantum-entanglement-with-the-naked-eye/.

[26] S. Ryu and T. Takayanagi, *Aspects of Holographic Entanglement Entropy*, *JHEP* **08** (2006) 045, [hep-th/0605073].

[27] S. Sachdev, *The Quantum phases of matter*, in *Proceedings, 25th Solvay Conference on Physics: The Theory of the Quantum World: Brussels, Belgium, October 19-25, 2011*, 2012. 1203.4585.

[28] S. Sachdev, *Emergent gauge fields and the high temperature superconductors*, 1512.00465.
A Femion spectral function

The basic idea of fermion correlation function is that we take the dual of the fundamental fermion as the bulk fermion whose action is given by Eq. (2.1) in the bulk background given by RN-AdS4

\[ ds^2 = -\frac{r^2}{L^2}f(r)dt^2 + \frac{L^2}{r^2f(r)}dr^2 + \frac{r^2}{L^2}dx^2, \quad f(r) = 1 - \frac{r_+}{r^3} + \frac{r_+\mu^2}{r^6} + \frac{r_+^2\mu^2}{r^8}. \]  

(A.1)
Gauge potential is given by
\[ A = \mu (1 - r^2)dt \] and the horizon of this metric \( r_+ = \frac{1}{3}(2\pi T + \sqrt{4\pi^2 T^2 + 3\mu^2}) \). In this paper, we treat both the scalar \( \Phi \) and the fermion \( \psi \) as probes to get analytic solution. Near the boundary, the real scalar \( \Phi \) has usual expansion given by
\[ \Phi = \frac{M_0}{r^2} + \frac{M}{r^4} \cdots . \tag{A.2} \]

To consider the scalar condensation that arises spontaneously, we set the source \( M_0 = 0 \) with positive condensation \( M > 0 \). Notice that \( M = \langle \bar{\chi}\chi \rangle \) has dimension 2, since the boundary fermion \( \chi \) has the dimension 1 in 2+1. \(^1\) We choose Dirac matrices as follows:
\[
\Gamma^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \Gamma^t = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \Gamma^x = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \Gamma^y = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}. \tag{A.3} \]

Following \cite{41}, we Fourier transform followed by introducing \( \phi_\pm \)
\[
\psi_\pm = (-gg^r)^{-\frac{1}{2}}\phi_\pm, \quad \phi_\pm = \begin{pmatrix} \phi_\pm \\ \bar{\phi}_\pm \end{pmatrix}, \tag{A.4} \]

and \( \xi_\pm \) by \( \xi_+ = i\frac{\psi_+}{\bar{\psi}_+} \), and \( \xi_- = -i\frac{\bar{\psi}_-}{\psi_-} \). Then the equation for \( \xi_\pm \) can be written as
\[
\sqrt{g_{ii}} g_{rr} \left( \partial_r + 2(m + \frac{M}{r\alpha})\sqrt{g_{ii}} \right) \xi_\pm = \pm \left( k_1 \pm \sqrt{\frac{g_{ii}}{g_{rt}} (w + A_t)} \right) \xi_{\pm} \mp \left( k_1 \mp \sqrt{\frac{g_{ii}}{g_{rt}} (w + A_t)} \right). \tag{A.5} \]

If we set \( \mu \to 0, T \to 0 \) and \( \alpha = 2 \), the Eq.(A.5) for \( \xi_- \) is reduced to
\[
r^2 \partial_r \xi_- + (2mr + 2\frac{M}{r})\xi_- + (k_1 - w)\xi_- + (k_1 - w) = 0. \tag{A.5} \]

The solution to Eq. (A.5) is given by \( \xi_- = N/D \), with
\[
N = C_0 \frac{r^{2m}}{m - \frac{1}{2}} (k_1 + w) \frac{1}{F_1(h_1; \frac{3}{2} - m; \frac{M}{r^2})} \tag{A.6} \]
\[
- \frac{4ie^{im\pi}M^{m+\frac{1}{2}}}{(k_1 - w)} \left[ ((m + \frac{1}{2})r + \frac{M}{r}) \frac{1}{F_1(h_1; \frac{3}{2} + m; \frac{M}{r^2})} + \frac{k_1^2 - w^2 - 4M}{4(\frac{3}{2} + m)r} \frac{1}{F_1(h_2; \frac{5}{2} + m; \frac{M}{r^2})} \right], \]
\[
D = 2C_0 r^{1+2m} \frac{1}{F_1(h_0; \frac{1}{2} - m; \frac{M}{r^2})} + ie^{im\pi}M^{\frac{1}{2}+m} \frac{1}{F_1(h_2; \frac{3}{2} + m; \frac{M}{r^2})}, \tag{A.7} \]
\[
h_0 = \frac{k_1^2 - w^2}{4M}, \quad h_1 = 1 + \frac{k_1^2 - w^2}{4M}, \quad h_2 = m + \frac{1}{2} + \frac{k_1^2 - w^2}{4M}. \tag{A.8} \]

\(^1\)Notice that similar conclusion can be drawn for Majorana fermion \( \chi \). In that case, one should replace the mass term to the Majorana one.
For the retarded Green function $G_R = \lim_{r \to \infty} r^{2m} \text{diag}(\xi_+, \xi_-)$,

$$G_{22} = -ie^{im\pi}(1 + 2m)M^{\frac{1}{2} + m}_{1} \frac{1}{(k_1 - w)C_0(k_1, w, M)}, \quad \text{if} \quad m > -\frac{1}{2}$$

(A.9)

can be determined with $C_0$ determined by the in-falling condition $\xi_-(r)|_{r \to 0} = -i$ to be given by Eq. (2.3).