MEASURABLE SEMIGROUP SELECTION OF THE HEAT FLOW FOR HARMONIC MAPS

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Abstract. J.-M. Coron proved in [5] that the global weak solutions of the heat flow from \( M \) to \( N \), starting at non-stationary weakly harmonic maps, are not unique when \( M = B^3 \) and \( N = S^2 \). Hence, the semigroup property of the solution map does not hold in general. The present short paper uses the techniques developed by J. Cardona and L. Kapitanski to show the existence of infinitely many measurable semigroups solving the heat flow in the same cases where non-uniqueness was shown by J.-M. Coron.

1. Introduction

Let \((M, g)\) and \((N, h)\) be two Riemannian manifolds. Following [8, Chapter 9] and [6, Section 3.4], the energy density of a smooth map \( u : M \to N \) at \( x \in M \) is the quantity

\[
e(u)(x) = \frac{1}{2} |du(x)|^2,
\]

where \( du(x) \) is the differential of \( u \) at \( x \), and the norm is the one in the tensor product \( T_x M \otimes T_{u(x)} N \). In local coordinates \((x^1, \ldots, x^m)\) on \( M \) and \((y^1, \ldots, y^n)\) on \( N \) the energy density is

\[
e(u)(x) = g^{\alpha \beta}(x) h_{ij}(u(x)) \frac{\partial u^i(x)}{\partial x^\alpha} \frac{\partial u^j(x)}{\partial x^\beta},
\]

where the usual summation convention is used, and the Latin indices run over \( 1, \ldots, n \) and the Greek indices over \( 1, \ldots, m \).

The energy of the map \( u \) is the quantity

\[
E(u) = \int_M e(f) \, dM,
\]

where \( dM \) is the volume element in \( M \). The Euler-Lagrange equation associated to the functional \( E \) is

\[
\Delta_M u^i(x) + g^{\alpha \beta}(x) \Gamma^i_{jk}(u(x)) \frac{\partial u^j(x)}{\partial x^\alpha} \frac{\partial u^k(x)}{\partial x^\beta} = 0
\]

where \( \Delta_M \) is the Laplace-Beltrami operator on \( M \), i.e.,

\[
\Delta u^i(x) = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\alpha} \left( \sqrt{|g|} g^{\alpha \beta} \frac{\partial u^i(x)}{\partial x^\beta} \right),
\]

and \( \Gamma^i_{jk}(u(x)) \) are the Christoffel symbols of the metric \( h \) at \( u(x) \).

A map \( u \in C^2(M, N) \) is said to be harmonic if it satisfies (1).
A central question in the study of harmonic maps is the following: Given an arbitrary smooth map \( u_0 : M \to N \), is it possible to deform it into a harmonic map? J. Eells and J.H. Sampson \[7\] proved that, if \( \dim(M) > 2 \) and \( N \) has a non-positive sectional curvature, the heat-flow associated to the functional \( E \), i.e., the Cauchy problem

\[
\begin{align}
\partial_t u - \Delta_M u &= \Gamma_N(u)(\nabla u, \nabla u)_M \\
u(0, x) &= u_0
\end{align}
\] (2a)

has a global and regular solution \( u(t, x) \) which converges to a harmonic map as \( t \to \infty \) (where \( \Gamma_N(u)(\nabla u, \nabla u)_M \) is the second term in (1)). M. Struwe \[10\] considered global weak solutions of (2) to extend the results of Eells and Sampson to arbitrary \( N \) when \( \dim(M) = 2 \).

A map \( u \in H^1_{\text{loc}}(M, N) \) is said to be weakly harmonic if it satisfies (1) in the weak sense, i.e.,

\[
\int_M g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i(x)}{\partial x^\alpha} \left( \frac{\partial \eta^j(x)}{\partial x^\beta} - \eta^k(x) \frac{\partial u^l(x)}{\partial x^\beta} \Gamma^j_{lk}(u(x)) \right) \, dM = 0
\] for every smooth map \( \eta : M \to TN \) such that \( \eta(x) \in T_{u(x)}N \).

Y. Chen \[4\] proved the existence of global weak solution of (2) when \( N \) is a sphere and \( M \) is a compact and smooth manifold without boundary and with \( \dim(M) > 2 \). J.-M. Coron \[3\] and later F. Béthuel, J.-M. Coron, J.-M. Ghidaglia, and A. Soyeur \[2\] proved the existence of infinitely many global weak solutions for some initial conditions in the case \( M = B^3 \) and \( N = S^2 \). In particular, the initial conditions they consider are weakly harmonic maps that fail to be stationary points of the energy functional. A map \( u \) is stationary if it is invariant under variations on the spatial domain, i.e.,

\[
\frac{d}{d\varepsilon} E(u_{\varepsilon\eta}) \bigg|_{\varepsilon=0} = 0
\] for every smooth vector field \( \eta \).

The non-uniqueness result in \[2\] begs the question: Is there a semigroup solving the heat-flow? In general, if the solutions of an evolution equation are unique the existence of a semigroup is warranted. Due to the non-uniqueness result in the case \( M = B^3 \) and \( N = S^2 \), is not immediately obvious that such a semigroup exists. The main result in this paper is the existence of an infinite number of semigroups solving the heat flow in the same case considered in \[2\]. The main tool is the measurable semigroup selection thereom developed by J. Cardona and L. Kapitanski \[3\] in the same spirit of the Markov selection theorem of N.V. Krylov \[9\].

1.1. Notation. In what follows, \( M = B^3 := \{ x \in \mathbb{R}^3 : |x| < 1 \} \) and \( N = \partial B^3 \subset \mathbb{R}^3 \), both are considered as sub-manifolds of \( \mathbb{R}^3 \). A map \( u : M \to N \) is simply a map \( u : B^3 \to \mathbb{R}^3 \) with the constraint \( |u(x)| = 1 \) for every \( x \in B^3 \). The energy of a smooth map \( u : M \to N \) is the quantity

\[
E(u) = \frac{1}{2} ||u||_{H^1}^2 = \frac{1}{2} \sum_{i=1}^3 \int_{B^3} |\nabla u^i(x)|^2 \, dx,
\]
and the space of measurable functions with finite energy is denoted by $H^1(M, N)$ (or simply $H^1$) endowed with the norm $||| \cdot |||_{H^1}$. Since $\partial B^3$ is a $C^1$ surface, the trace operator $T : H^1 \to L^2(\partial B^3)$ is bounded and linear.

A map $u \in C^2$ is harmonic if and only if it satisfies
(3) $\Delta u + u |\nabla u|^2 = 0$.

A map $u \in H^1$ is weakly harmonic if and only if (3) holds in the weak sense; and a weakly harmonic map $u$ is stationary if and only if
$$\sum_{j=1}^3 \frac{\partial}{\partial x^j} \left( \frac{\partial u(x)}{\partial x^j} \right)^2 - 2 \frac{\partial}{\partial x^j} \left( \frac{\partial u(x)}{\partial x^j} \cdot \frac{\partial u(x)}{\partial x^k} \right) = 0 \ \forall k = 1, 2, 3.$$ 

Functions from $[0, \infty) \times B^3$ to $S^2$ are seen as paths from $[0, \infty)$ taking values in some functional space. The space of paths taking values in $H^1$ with uniform bounds is denoted by $L^\infty([0, \infty); H^1)$. The space of paths taking values in $L^2$ and continuous with respect to its strong topology is denoted by $C([0, \infty); L^2)$. Finally, the space of paths taking values in $H^1$ and continuous with respect to its weak topology is denoted by $C_w([0, \infty); H^1)$.

For the definitions and notation of set-valued analysis we refer to [3] and reference therein.

2. Measurable semigroups of the heat-flow

The heat flow reads
(4a) $\partial_t u - \Delta u = u |\nabla u|^2$
(4b) $u(0) = u_0$

Following [4] and [2], a global weak solution of (4), started at $u_0 \in H^1$, is a measurable map $u$ from $[0, \infty) \times B^3$ to $\mathbb{R}^3$ that satisfies the following
i) $u$ takes values in $S^2$, i.e. $|u(t, x)| = 1$ for a.e. $(t, x) \in [0, \infty) \times M$;
ii) $u$ satisfies the initial condition $u(0) = u_0$ and the boundary condition $Tu(t) = Tu_0$ for every $t \geq 0$;
iii) $u \in L^\infty([0, \infty); H^1) \cap C([0, \infty); L^2) \cap C_w([0, \infty); H^1)$;
iv) the equation (4) holds in the weak sense; and
v) the following energy inequality holds
$$E(u(t+s)) + \int_t^{t+s} \int_M |\partial_t u(\tau)|^2 \, dM \, d\tau \leq E(u(t))$$
for every $s \geq 0$ and almost every $t \geq 0$ including $t = 0$.

Remark. Note that v) is usually written only with $t = 0$. In general, it is not possible to obtain an inequality valid for every $t \geq 0$ due to the mode of convergence of the limit arguments in the constructions of solutions in [4, Eqs. 2.19-2.22] or [2, Eqs. 1.8, 1.9, and 1.12]. See [3, Section 4.4] for a similar situation with the Navier-Stokes system.

Theorem 2.1. There exist infinitely many measurable maps $u : H^1 \to C_w([0, \infty), H^1)$ such that $u(a)$ is a global weak solution of (4) started at $a \in H^1$, such that $u(a, 0) = a$ and $u(a, t+s) = u(u(a, t), s)$ for every $s \geq 0$ and almost every $t \geq 0$ (including $t = 0$).
Proof. For every $a \in H^1$, let $S_a \subset C_w([0, \infty); H^1)$ be the set of all the weak global solutions of (1). We know this set is not empty from Theorem 1 in [2]. The arguments to pass to the limits in Theorem 1 in [2] ensure that the set-valued map $a \mapsto S_a$ is upper-semicontinuous, hence measurable. Moreover, the same arguments ensure that the set $S_a$ is compact in the topology of $C_w([0, \infty); H^1)$. So, the set-valued map $a \mapsto S_a$ is valued in the non-empty compact subsets of $C_w([0, \infty); H^1)$.

Let $u \in S_a$, and let $t$ be such that the energy inequality is valid. Is not hard to see that for every $v \in S_u(t)$ the map

$$w(s) = \begin{cases} u(s) & \text{for } s \leq t \\ v(s-t) & \text{for } s > t \end{cases}$$

is an element in $S_a$. Theorem 2.5 in [3] ensures the existence of a measurable semigroup.

Finally, we need to show that Theorem 2.5 in [3] yield infinitely many measurable semigroup. Let $a \in H^1$ be a weakly harmonic map that is not stationary. It was shown in [2] that (1) has at least two global weak solutions $u_1$ and $u_2$, moreover, one of them is constant $u_1(t) = a$. Recall that the weak topology of $H^1$ is metrizable on the closed ball of radius $2\|a\|$, hence, there is a weakly continuous function $\varphi : H^1 \to \mathbb{R}$ satisfying $\varphi(a) = \max \{ \varphi(v) : v \in H^1 \text{ with } \|v\| \leq \|a\| \}$. Since the two continuous maps $t \mapsto \varphi(u_1(t))$ and $t \mapsto \varphi(u_2(t))$ are different, there exists $\lambda > 0$ such that $I_{\lambda,\varphi}[u_1] > I_{\lambda,\varphi}[u_2]$ where

$$I_{\lambda,\varphi}[u] = \int_0^\infty e^{-\lambda t} \varphi(u(t)) \, dt.$$ 

By ensuring that the first functional used to refine the set-valued maps in the proof of Theorem 2.5 in [3] is either $I_{\lambda,\varphi}$ or $I_{\lambda,-\varphi}$ the resulting semigroup would be different. Hence, different enumerations of the family of separating functions used in [2] results in different semigroups. □

2.1. An Example: Landau-Lifshitz equations. The system of Landau-Lifshitz equations

$$\partial_t u = u \times \Delta u - \lambda u \times (u \times \Delta u) \quad |u| = 1$$

describing the evolution of spin fields in continuum ferromagnetism enjoys similar bounds and it was proven by F. Alouges and A. Soyeur [1] that weak solutions are also non-unique using the same method as J.-M. Coron [3]. Theorem 2.1 ensures the existence of infinitely many measurable semigroups solving the Landau-Lifshitz equations.

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