THE SUPPORT GENUS OF CERTAIN LEGENDRIAN KNOTS

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ABSTRACT. In this paper, the support genus of all Legendrian right-handed trefoil knots and some other Legendrian knots is computed. We give examples of Legendrian knots in the three-sphere with the standard contact structure which have positive support genus with arbitrarily negative Thurston-Bennequin invariant. This answers a question in [009].

1. Introduction

In his seminal paper [G02], Giroux established the surprising one-to-one correspondence between the contact structures up to isotopy and the open book decompositions up to positive stabilizations on a given closed oriented three-manifold. See [E06] for details. It becomes natural and convenient for topologists to study contact structures in the viewpoint of open book decompositions.

For Legendrian knots, Akbulut and Ozbagci [AO01] showed that for any Legendrian link \( L \) in \( S^3 \) with the standard contact structure \( \xi_{std} \), there exists a compatible open book such that \( L \) sits in a page of the open book, furthermore, the framing of \( L \) given by the page of the open book agrees with the contact framing. See also [P04].

In [EO08], Etnyre and Ozbagci introduced the definition of support genus for a contact three-manifold, which is the minimal genus of a page among all open book decompositions of \( M \) supporting \( \xi \) such that \( L \) sits on a page of the open book and the framings given by \( \xi \) and given by the page coincide.

Definition 1.1 ([O09]). Let \( L \) be a Legendrian knot in a contact three-manifold \( (M, \xi) \), the support genus of \( L \), denoted by \( sg(L) \), is the minimal genus of a page among all open book decompositions of \( M \) supporting \( \xi \) such that \( L \) sits on a page of the open book and the framings given by \( \xi \) and given by the page coincide.

Ding and Geiges [DG04] introduced the definition of contact surgeries along a Legendrian link. If a contact three-manifold \( (M, \xi) \) is obtained by a contact \( r \) surgery along a Legendrian knot \( L \) in \( (S^3, \xi_{std}) \), Onaran showed that \( sg(L) \) is greater than or equal to the support genus of \( (M, \xi) \) ([O09, Remark 5.11]). In the same paper, the following question is asked:

Question 1.2. Does every Legendrian knot in \( (S^3, \xi_{std}) \) with negative Thurston-Bennequin invariant have support genus zero?

In the present paper, we study the support genus of certain Legendrian knots in \( (S^3, \xi_{std}) \). First, for Legendrian torus knots, we have the following

Theorem 1.3. Suppose \( k \geq 1 \). Let \( L \) be a Legendrian torus knot \( T(2, 2k + 1) \) in \( (S^3, \xi_{std}) \) with nonnegative Thurston-Bennequin invariant, then \( sg(L) = 1 \).

Next, we study the support genus of Legendrian twist knots \( K_{-2m} \), where \( m \geq 1 \). (See [ENV10] for the meaning of \( K_{-2m} \).) In particular, \( K_{-2} \) is the right-handed trefoil. All Legendrian twist knots are classified in [ENV10]. In fact, \( K_{-2m} \) has \( \lfloor m^2/2 \rfloor \) Legendrian representatives with Thurston-Bennequin invariant one and rotation number zero, and has a unique Legendrian representative with Thurston-Bennequin invariant minus one and rotation number zero ([ENV10, Theorem 1.1 (4)]). We have the following

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Theorem 1.4. Suppose \( m, n_1 \) and \( n_2 \) are natural numbers. Let \( L \) be a Legendrian representative of the twist knot \( K_{-2m} \) in \((S^3, \xi_{\text{std}})\) with Thurston-Bennequin invariant 1. Then \( \text{sg}(S_{+}^{n_1} S_{-}^{n_2}(L)) = 0 \), where \( S_{+} \) and \( S_{-} \) denote the positive and negative stabilizations respectively.

Theorems 1.3 and 1.4 both computed the support genus of some Legendrian right-handed trefoil knots. By the classification results in [EH01], the remaining Legendrian right-handed trefoil knots are \( S_{+}^{n}(L) \) and \( S_{-}^{n}(L) \) for \( n \geq 2 \). The support genus of these Legendrian knots is computed in the following theorem, and thus we computed the support genus of all Legendrian right-handed trefoil knots.

Theorem 1.5. Let \( L \) be a Legendrian right handed trefoil knot in \((S^3, \xi_{\text{std}})\) with Thurston-Bennequin invariant 1. Then for any integer \( n \geq 2 \), both \( S_{+}^{n}(L) \) and \( S_{-}^{n}(L) \) have support genus 1.

This gives a negative answer to Question 1.2. In fact, Legendrian knots with positive support genus can have arbitrarily negative Thurston-Bennequin invariant.

Our strategy is the following. We will construct a fibered link with a Thurston norm minimizing Seifert surface which contains the interested knot. By computing the Thurston-Bennequin invariant and rotation number of the Legendrian realization of this knot and known classification results, we will be able to determine the Legendrian knot and get an upper bound for the support genus for it. In the other direction, we use the Heegaard Floer contact invariant to get lower bound for the support genus. Combining these results, we will compute the support genus of our interested Legendrian knots.

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2. Proof of Theorem 1.3

The surface \( F \) illustrated in Figure 1 is a punctured torus and contains an embedded torus knot \( T(2, 2k + 1) \) in its interior.

![Figure 1](image_url)

**Figure 1.** A punctured torus containing the torus knot \( T(2, 2k + 1) \).

As illustrated in Figure 2, \( F \) can be obtained by a sequence of positive stabilizations from the disk. By [G02], \( F \) can be viewed as a page of an open book decomposition of \( S^3 \) which supports the standard tight contact structure \( \xi_{\text{std}} \). \( K \) can be realized as a Legendrian knot since \( K \) is
homologically nontrivial (in fact, nonseparating) in $F$, which we denote by $T_m(2, 2k+1)$. Moreover, the contact framing of $T_m(2, 2k+1)$ coincides with the page framing induced by $F$. This is because the contact planes can be arranged to be arbitrarily close to the tangent planes of the pages.

Let $J$ be a push-off of $T_m(2, 2k+1)$ along the surface $F$, then $J$ represents the contact framing of $T_m(2, 2k+1)$. The Thurston-Bennequin invariant of $T_m(2, 2k+1)$ equals to the linking number of $T_m(2, 2k+1)$ and $J$, which is easily seen to be $2k - 1$. By the classification of Legendrian representatives of torus knots in [EH01], $T_m(2, 2k+1)$ has the maximum Thurston-Bennequin invariant over all Legendrian representatives of $T(2, 2k+1)$ and it is the unique one.

So the support genus of the Legendrian knot $T_m(2, 2k+1)$ is at most 1. According to [EH01], all other Legendrian representatives of the torus knot $T(2, 2k+1)$ can be obtained by stabilizing $T_m(2, 2k+1)$. Thus, by [O09, Theorem 5.9], the support genus of any Legendrian representative of $T(2, 2k+1)$ is at most 1.

On the other hand, since a Legendrian knot in a weakly fillable contact structure with positive Thurston-Bennequin invariant has positive support genus ([O09, Lemma 5.4]), the support genus of $T_m(2, 2k+1)$ is at least 1. Therefore the support genus of $T_m(2, 2k+1)$ is 1.

This ends the proof of Theorem 1.3.
3. Proof of Theorem 1.4

By [O09, Theorem 5.9], it suffices to show that the Legendrian representative of the twist knot $K_{-2m}$ with Thurston-Bennequin invariant minus one and rotation number zero has support genus zero.

The punctured sphere illustrated in Figure 3 contains the twist knot $K_{-2m}$ in its interior. It is easy to see that $F$ is a page of an open book decomposition which corresponds to $(S^3, \xi_{\text{std}})$. Since $K$ is not null homologous in $F$, $K$ can be made Legendrian and the contact framing of $K$ coincides with the page framing induced by $F$.

The linking number of $K$ and its push-off along $F$ is $-1$, by the same argument as in the previous section, the Thurston-Bennequin invariant of $K$ is $-1$. Below we shall compute the rotation number of $K$.

We turn the open book decomposition shown in Figure 3 into an abstract open book decomposition $(F', t_{\gamma_1} t_{\gamma_2} \ldots t_{\gamma_{m+2}})$ shown in Figure 4, where $F'$ is a punctured sphere which is homeomorphic to $F$, and $t_{\gamma_i}$ denotes the right handed Dehn twist along $\gamma_i$, $i = 1, 2, \ldots, m + 2$.

If we perform a Legendrian surgery along $K$, then we obtain a Stein fillable tight contact manifold which corresponds to the planar open book decomposition $(F', t_{\gamma_1} t_{\gamma_2} \ldots t_{\gamma_{m+2}} t_K)$. Let $(W, J)$ be the Stein surface obtained from $B^4$ by attaching a two-handle corresponding to the Legendrian
surgery along $K$. Let $c_1(J)$ be the first Chern class of this Stein surface, and $h$ be the generator of $H_2(W; \mathbb{Z}) \cong \mathbb{Z}$ supported on the attached 2-handle, then $\langle c_1(J), h \rangle = \text{rot}(K)$.

The open book decomposition in Figure 4 is the same as the one in Figure 5, where each upper horizontal segment is identified with the corresponding lower horizontal segment to form a one-handle. We consider the Kirby diagram in Figure 6, where the framings are labelled with respect to the blackboard framings. According to [E90], this Kirby diagram presents a Stein surface, denoted by $(W', J')$. By handle cancellation, it is not hard to see that $W$ and $W'$ are diffeomorphic. Since the induced contact structures on $\partial W$ by $(W, J)$ and $(W', J')$ are isotopic, we have $c_1(J) = c_1(J')$ by [LM97, Theorem 1.2].
Let $X_1, X_2, \ldots, X_{m+2}$ denote the 1-handles attached to $D^4$ to form $#^{m+2}(S^1 \times D^3)$. Let $S_{\gamma_1}, S_{\gamma_2}, \ldots, S_{\gamma_{m+2}}$ and $S_K$ be the cores of the 2-handles attached to the curves $\gamma_1, \gamma_2, \ldots, \gamma_{m+2}$ and $K$, respectively, and let $C_{\gamma_1}, C_{\gamma_2}, \ldots, C_{\gamma_{m+2}}$ and $C_K$ be the cocores of the 2-handles.

In the tight contact manifold $#^{m+2}(S^1 \times S^2)$ supported by the open book decomposition $(F', id)$, we can Legendrian realize $\gamma_1, \gamma_2, \ldots, \gamma_{m+2}$ and $K$. Moreover, by the argument in Section 3.1 in [EO08] and Figure 5, we can easily compute their rotation numbers as

$$r(\gamma_1) = 0, r(\gamma_2) = r(\gamma_3) = \ldots = r(\gamma_{m+2}) = -1$$

and $r(K) = 0$. Also according to Section 3.1 in [EO08], the class $c_1(J')$ is Poincaré dual to

$$\sum_{i=1}^{3} r(\gamma_i) C_{\gamma_i} + r(K) C_K = -C_{\gamma_2} - C_{\gamma_3} \ldots - C_{\gamma_{m+2}}.$$ 

The cores of the 2-handles form a basis of the 2-chain groups $C_2(W; \mathbb{Z})$; $X_1, X_2, \ldots, X_{m+2}$ together form a basis for the 1-chain groups $C_1(W; \mathbb{Z})$, and the boundary map sends $d_2(S_{\gamma_1}) = X_2, d_2(S_{\gamma_2}) = X_1 - X_2, d_2(S_{\gamma_3}) = X_2 - X_3, d_2(S_{\gamma_{i+1}}) = X_3 - X_4, \ldots, d_2(S_{\gamma_{m+2}}) = X_{m+1} - X_{m+2}, d_2(S_K) = -X_2.$ Thus, $h$, the generator of $H_2(W; \mathbb{Z})$, can be presented by $S_K + S_{\gamma_1}$.

So $\langle c_1(J), h \rangle = \langle c_1(J'), h \rangle = 0$, and hence $\text{rot}(K) = 0$.

**Remark 3.2.** Suppose $F$ is a page of an open book decomposition supporting $(S^3, \xi_{\text{std}})$, and $K$ is a homologically nontrivial simple closed curve in $F$, then we can Legendrian realize $K$. To compute the rotation number of $K$, we perform some necessary positive stabilizations to $F$ so that the resulting surface can be obtained from a disk by some positive stabilizations. Note that there exists open book decomposition supporting $(S^3, \xi_{\text{std}})$ and cannot be obtained from a disk by positive stabilizations, see [BEV10]. On the other hand, the Legendrian knot $K$ does not change in this procedure. So we can factor the monodromy of the open book decomposition or the new one as a product of positive Dehn twists. Therefore we can directly apply the method here to compute the rotation number of $K$.

### 4. Proof of Theorem 1.5

By the classification of Legendrian representatives of torus knots in [EH01], there are $n + 2$ Legendrian right handed trefoil knots $T(2, 3)$ with Thurston-Bennequin invariant $-n$. We order them by $L_1, L_2, \ldots, L_{n+2}$ such that $\text{rot}(L_i) = 2i - n - 3$.

When we perform Legendrian surgery to $(S^3, \xi_{\text{std}})$ along $L_i$, $i = 1, 2, \ldots, n + 2$, we obtain a Stein fillable contact structure $\xi_i$ on the three-manifold $S^3_{n-1}(T(2, 3))$. Let $c^+(\xi_i) \in H^+(S^3_{n+1}(T(-2, 3)))$ and $\hat{c}(\xi_i) \in \widehat{H}^+(S^3_{n+1}(T(-2, 3)))$ be the contact invariants of the contact structure $\xi_i$. By [OSz05, Theorem 1.5], $c^+(\xi_i)$ and $\hat{c}(\xi_i)$ are both nonzero for each $i$.

The Heegaard Floer homology groups of $S^3_{n}(T(-2, 3))$ ($n > 6$) are ([OSz04, Proposition 3.2])

$$HF^+(S^3_{n+1}(T(-2, 3)), s_0) = \mathcal{T}^+ \oplus \mathbb{Z},$$

$$HF^+(S^3_{n+1}(T(-2, 3)), s_j) = \mathcal{T}^+, 1 \leq j \leq n,$$

where $s_0, s_1, \ldots, s_n$ denote the $n + 1$ Spin$^c$ structures of $S^3_{n+1}(T(-2, 3))$ and $\mathcal{T}^+ = \mathbb{Z}[U^{-1}]$, while

$$\widehat{HF}(S^3_{n+1}(T(-2, 3)), s_0) = \mathbb{Z}(\mathcal{T}^+) \oplus \mathbb{Z} \oplus \mathbb{Z},$$

$$\widehat{HF}(S^3_{n+1}(T(-2, 3)), s_j) = \mathbb{Z}(\mathcal{T}^+), 1 \leq j \leq n,$$

where $\mathbb{Z}(\mathcal{T}^+) \cong \mathbb{Z}$ comes from the kernel of the map $U : \mathcal{T}^+ \rightarrow \mathcal{T}^+$.

Each contact manifold $(S^3_{n-1}(T(2, 3)), \xi_i)$ bounds a Stein surface $(X, J_i)$, and $\langle c_1(X, J_i), H \rangle = \text{rot}(L_i)$, where $H$ is the capped Seifert surface of $L_i$. Since $\text{rot}(L_i)$ are pairwise distinct, $(X, J_i)$ ($i = 1, 2, \ldots, n + 2$) are pairwise non-isomorphic. By Theorems 2 and 4 in [P03], $\hat{c}(\xi_i)$, $i = 1, 2, \ldots, n + 2$,
are distinct primitive elements in $\widehat{HF}(S_{n+1}^3(T(-2,3)))$. Each $\hat{c}(\xi_i)$ lies in $\widehat{HF}(S_{n+1}^3(T(-2,3)), s_j)$ for some $j = 0, 1, \ldots, n$. On the other hand, there are exactly two distinct primitive elements in $\mathbb{Z}(T^+)$ and there are exactly $n+1$ $\mathbb{Z}(T^+)$ summands in $\widehat{HF}(S_{n+1}^3(T(-2,3)))$. Moreover, also by Theorem 5 in [P04], for $i_1 \neq i_2$, $\hat{c}(\xi_{i_1})$ and $\hat{c}(\xi_{i_2})$ cannot both belong to a $\mathbb{Z}(T^+)$ summand. So at least one of $\hat{c}(\xi_i), i = 1, 2, \ldots, n+2$, does not belong to the $n+1$ $\mathbb{Z}(T^+)$ summands in $\widehat{HF}(S_{n+1}^3(T(-2,3)))$. We denote this element as $\hat{c}(\xi_{i_0})$. Therefore, $c^+(\xi_{i_0})$ does not belong to the $n+1$ $T^+$-summands in $\widehat{HF}^+(S_{n+1}^3(T(-2,3)))$. Hence $c^+(\xi_{i_0})$ does not vanish in $\widehat{HF}_{\text{red}}(S_{n+1}^3(T(-2,3))) \cong \mathbb{Z}$.

By Theorem 1.2 in [OSSz05], the contact manifold $\xi_{i_0}$ does not admit a planar open book decomposition. By Theorem 5.10 in [O09], the support genus of $L_{i_0}$ is positive.

When $1 \leq n \leq 6$, by Theorem 5.9 in [O09], among the Legendrian right handed trefoil knots with Thurston-Bennequin invariant $-n$, there must be one whose support genus is positive.

Let $L$ be a Legendrian right handed trefoil with Thurston-Bennequin invariant $1$, then, by Theorem 1.3, the support genus of $S_{n+1}^n S_{n-2}^n (L)$ is $0$, where $n_i \geq 1$ ($i = 1, 2$). On the other hand, for $n \geq 2$, the support genus of $S_{n+1}^n (L)$ and $S_{n-2}^n (L)$ are equal, since they only differ in the orientation. So the support genus of both $S_{n+1}^n (L)$ and $S_{n-2}^n (L)$ are 1.

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