An exact solution to determination of an open orbit

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Abstract

We present an exact solution of the equations for orbit determination of a two body system in a hyperbolic or parabolic motion. In solving this problem, we extend the method employed by Asada, Akasaka and Kasai (AAK) for a binary system in an elliptic orbit.

The solutions applicable to each of elliptic, hyperbolic and parabolic

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orbits are obtained by the new approach, and they are all expressed in an explicit form, remarkably, only in terms of elementary functions. We show also that the solutions for an open orbit are recovered by making a suitable transformation of the AAK solution for an elliptic case.

keywords: astrometry — celestial mechanics — orbit determination

1 Introduction

Two body problems are very classical in celestial mechanics and have been studied thoroughly since Kepler discovered the laws of motion of celestial objects (e.g., Aitken 1964, Goldstein 1980, Danby 1988, Roy 1988, Murray and Dermott 1999, Beutler 2004). The regular orbits of a system of two masses in Newtonian mechanics are of three types: ellipse, parabola and hyperbola. The latter two cases, in which the separation of two bodies will become infinite in the remote future, may be called open orbits. The singular orbit is a linear one corresponding to a head-on collision, which is extremely special. In this paper, we consider regular orbits mentioned above: for a binary system, the orbit determination may bring us some informations about its formation and evolution mechanism. In an open orbit, one may infer an
impact parameter and an initial relative velocity of two masses, one of which may be ejected by some explosive mechanism such as a supernova or by three body scattering. For instance, some observations reveal that kicked pulsars move at unusually high speed (Anderson et al. 1975, Hobbs et al. 2005).

The orbit determination of visual double stars was solved first by Savary in 1827, secondly by Encke 1832, thirdly by Herschel 1833 and by many authors including Kowalsky, Thiele and Innes (Aitken 1964 for a review on earlier works; for the state-of-the-art techniques, e.g, Eichhorn and Xu 1990, Catovic and Olevic 1992, Olevic and Cvetkovic 2004). Here, a visual binary is a system of two stars both of which can be seen. The relative vector from the primary star to the secondary is in an elliptic motion with a focus at the primary. This relative vector is observable because the two stars are seen. On the other hand, an astrometric binary is a system of two objects where one object can be seen but the other cannot like a black hole or a very dim star. In this case, it is impossible to directly measure the relative vector connecting the two objects, because one end of the separation of the binary, namely the secondary, cannot be seen. The measures are made in the position of the primary with respect to unrelated reference objects (e.g., a quasar) whose proper motion is either negligible or known.
As a method to determine the orbital elements of a binary, an analytic solution in an explicit form has been found by Asada, Akasaka and Kasai (2004, henceforth AAK). This solution is given in a closed form by requiring neither iterative nor numerical methods. One may naturally seek an analytic method of orbit determination for open orbits. An extension for open orbits done earlier by Dommanget (1978) used the Thiele-Innes method, namely solved numerically the Kepler equation. As a result, the method by Dommanget does not provide an explicit solution in a closed form. Therefore, let us extend the explicit solution by AAK to open orbits. Then, we would face the following problem. AAK formalism uses a fact that the semimajor and semiminor axes of an ellipse divide it into quarters. This fact plays a crucial role in determining the position of the common center of mass on a celestial sphere; we should note here that the projected common center of mass is not necessarily a focus of an apparent ellipse. The division into quarters is possible for neither a parabola nor a hyperbola.

The purpose of this paper is to generalize AAK approach so that we can treat an open orbit. This paper is organized as follows. Sec. 2 presents a generalized AAK formalism. In Sec. 3, the generalized approach is employed to obtain the method of orbit determination for a hyperbolic orbit. In Sec.
4, the formula for a parabolic orbit is presented. In Sec. 5, we recover these formulae by making a suitable transformation of that for an elliptic orbit with some limiting procedures. Sec. 6 is devoted to Conclusion.

2 Generalizing AAK formalism for an elliptic motion

2.1 An apparent ellipse

We denote by $({\bar{x}}, {\bar{y}})$ the Cartesian coordinates on a celestial sphere that is perpendicular to the line of sight. A general form of an ellipse on a celestial sphere is

$$a{\bar{x}}^2 + b{\bar{y}}^2 + 2\gamma{\bar{x}}{\bar{y}} + 2\delta{\bar{x}} + 2\varepsilon{\bar{y}} = 1,$$

(1)

which is characterized by five parameters; the position of its center, the length of its semimajor/semiminor axes and the rotational degree of freedom. By at least five measurements of the location of a star, one can determine all the parameters. Henceforth, we adopt the Cartesian coordinates $(x, y)$ such that the apparent ellipse can be reexpressed in the standard form as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

(2)
where we assume $a \geq b$ without loss of generality. The ellipticity, $e$, is \[ \sqrt{1 - b^2/a^2}. \]

### 2.2 The common center of mass

A focus of the original Keplerian ellipse is not always that of the apparent one because of the inclination of the orbital plane. We should note that a focus of the original Keplerian orbit is the common center of mass of a binary, around which a component star moves at the constant-area velocity following the Keplerian second law (the conservation law of the angular momentum in the classical mechanics). This enables us to find out the location of the common center of mass as shown below.

A star is located at $P_j = (x_j, y_j) = (a \cos u_j, b \sin u_j)$ on a celestial sphere at each epoch $t_j$ for $j = 1, \cdots, 4$, where $t_j > t_k$ for $j > k$. Here, $u_j$ denotes the eccentric angle in the apparent ellipse but not the eccentric anomaly in the true one; the eccentric anomaly of the original Keplerian orbit is not observable. We assume anti-clockwise motion, such that $u_j > u_k$ for $j > k$. All we must do in the case of the clockwise motion is to change the signature of the area in Eq. (1) in the following. We define the time interval as $T(j, k) = t_j - t_k$. 

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The common center of mass of the binary is projected onto the celestial sphere at \( P_C = (x_C, y_C) \). Even after the projection, the law of constant-areal velocity still holds, where we should note that the area is swept by the line interval between the projected common center and the star. The area swept during the time interval, \( T(j, k) \), is denoted by \( S(j, k) \). The total area of the observed ellipse is denoted by \( S = \pi ab \). The law of the constant areal velocity on the celestial sphere becomes

\[
\frac{S}{T} = \frac{S(j, k)}{T(j, k)},
\]

where

\[
S(j, k) = \frac{1}{2} ab \left[ u_j - u_k - \frac{x_c}{a} (\sin u_j - \sin u_k) + \frac{y_c}{b} (\cos u_j - \cos u_k) \right].
\]

Equation (3) is rewritten explicitly as

\[
\begin{align*}
\frac{S(2, 1)}{T(2, 1)} &= \frac{S(3, 2)}{T(3, 2)}, \\
\frac{S(3, 2)}{T(3, 2)} &= \frac{S(4, 3)}{T(4, 3)}.
\end{align*}
\]

They are solved for \( x_c \) and \( y_c \) as

\[
\begin{align*}
x_c &= -a \frac{B_1 C_2 - B_2 C_1}{A_1 B_2 - A_2 B_1}, \\
y_c &= b \frac{C_1 A_2 - C_2 A_1}{A_1 B_2 - A_2 B_1}.
\end{align*}
\]
where

\[ A_j = T(j + 1, j) \sin u_{j+2} + T(j + 2, j + 1) \sin u_j - T(j + 2, j) \sin u_{j+1}, \]  
(9)

\[ B_j = T(j + 1, j) \cos u_{j+2} + T(j + 2, j + 1) \cos u_j - T(j + 2, j) \cos u_{j+1}, \]  
(10)

\[ C_j = T(j + 1, j)u_{j+2} + T(j + 2, j + 1)u_j - T(j + 2, j)u_{j+1}. \]  
(11)

The periastron is projected onto the observed ellipse at \( P_A \equiv (x_A, y_A) = (a \cos u_A, b \sin u_A) \). The ratio of the semimajor axis to the distance between the center and the focus of the ellipse remains unchanged, even after the projection. Hence, we find

\[ P_A = \frac{1}{e_K} P_C. \]  
(12)

The positional vector \( P_A \) is still located on the apparent ellipse given by Eq. \((2)\). We thus obtain the ellipticity as

\[ e_K = \sqrt{\frac{x_C^2}{a^2} + \frac{y_C^2}{b^2}}. \]  
(13)

2.3 Projection onto a celestial sphere

In the original derivation of AAK formula, the fact that the semimajor and semiminor axes divide the area of the ellipse in quarters. This still holds
even after the projection. Namely the projected semimajor and semiminor axes divide the area of the apparent ellipse in quarters, though the original semimajor and semiminor axes are not always projected onto the apparent semimajor and semiminor ones. This way of the derivation, however, can be used for neither hyperbolic nor parabolic cases, where there are no counterparts of the semiminor axis. Hence we shall employ another method.

In this paragraph, we use the Cartesian coordinates \((X, Y)\) on the original orbital plane. Let \(i\) be the inclination angle between the original orbital plane and the celestial sphere. We define as \(\omega\) the angular distance of the periastron, namely the angle between the periastron and the ascending node. Let us express the original Keplerian ellipse as

\[
\frac{X^2}{a^2_K} + \frac{Y^2}{b^2_K} = 1. \tag{14}
\]

We consider the line that is perpendicular to the semimajor axis at a focus \((a_Ke_K, 0)\). This line intersects the original ellipse at points \(Q = (a_Ke_K, a_K(1-e^2_K))\) and \(R = (a_Ke_K, -a_K(1-e^2_K))\). For later convenience, we adopt the coordinates \((\bar{X}, \bar{Y})\) whose origin is located at the focus, by making a translation as \(\bar{X} = X - a_Ke_K\). Then, one rewrites \(Q = (0, a_K(1-e^2_K))\) and \(R = (0, -a_K(1-e^2_K))\).

Only in this paragraph, we adopt other Cartesian coordinates \((x', y')\) so
that the ascending node can be located on the $x'$-axis and the origin can be the common center of mass. The true periastron of the original ellipse is projected at $\mathbf{P}_A \equiv (x'_A, y'_A) = a_K(1 - e_K)(\cos \omega, \sin \omega \cos i)$. The point denoted by $Q$ is projected at $\mathbf{P}_Q \equiv (x'_Q, y'_Q) = a_K(1 - e^2_K)(- \sin \omega, \cos \omega \cos i)$. It is useful to consider the following invariants because the components of a vector depend on the adopted coordinates.

$$|\mathbf{P}_A|^2 = a^2_K (1 - e_K)^2 (\cos^2 \omega + \sin^2 \omega \cos^2 i), \quad (15)$$

$$|\mathbf{P}_Q|^2 = a^2_K (1 - e^2_K)^2 (\sin^2 \omega + \cos^2 \omega \cos^2 i), \quad (16)$$

$$|\mathbf{P}_A \times \mathbf{P}_Q| = a^2_K (1 + e_K)(1 - e_K)^2 \cos i. \quad (17)$$

We consider the area surrounded by the ellipse and the line interval between $Q$ and $R$. This area is divided into equal halves by the semimajor axis. Even after the projection, the divided areas are still equal halves. Hence one can determine the location of the projected $Q$ as

$$\mathbf{P}_Q = \left( x_c - \frac{aycs}{b}, y_c + \frac{bxc s}{a} \right), \quad (18)$$

in the apparent ellipse coordinates, where we defined

$$s = \frac{\sqrt{1 - e^2_K}}{e_K}. \quad (19)$$

In this computation, it is useful to stretch the apparent ellipse along its semiminor axis by $a/b$ so that one can consider a circle with radius $a$. In
this stretching, importantly, the areal division into equal halves still holds.

We make a translation as $x \to x - x_c$ and $y \to y - y_c$ so that the projected
common center of mass can become the origin of the new coordinates. Then,
we have

$$ P_A = \frac{1 - e_K}{e_K} (x_c, y_c), \quad (20) $$

$$ P_Q = \left(-\frac{ay_c s}{b}, \frac{bx_c s}{a}\right). \quad (21) $$

Hence, we obtain the invariants from these vectors as

$$ |P_A|^2 = \left(\frac{1 - e_K}{e_K}\right)^2 (x_c^2 + y_c^2), \quad (22) $$

$$ |P_Q|^2 = \frac{(a^4 y_c^2 + b^4 x_c^2)(1 - e_K^2)}{a^2 b^2 e_K^2}, \quad (23) $$

$$ |P_A \times P_Q| = ab(1 - e_K) \sqrt{1 - e_K^2}. \quad (24) $$

whose values can be estimated because $a, b, x_c, y_c$ and $e_K$ have been already
all determined up to this point.

Equations (25)-(27) for $\cos i$, $a_K$ and $\cos 2\omega$ are solved as

$$ \cos i = \frac{1}{2}(\xi - \sqrt{\xi^2 - 4}), \quad (25) $$

$$ a_K = \frac{1}{1 - e_K^2} \sqrt{\frac{(1 + e_K)^2 |P_A|^2 + |P_Q|^2}{1 + \cos^2 i}}, \quad (26) $$

$$ \cos 2\omega = \frac{(1 + e_K)^2 |P_A|^2 - |P_Q|^2}{a_K^2 (1 - e_K^2)^2 \sin^2 i}, \quad (27) $$
where we define
\[ \xi = \frac{(1 + e_K)^2 \left| P_A \right|^2 + \left| P_Q \right|^2}{(1 + e_K) \left| P_A \times Q_Q \right|}. \tag{28} \]

One can show
\[ \xi \geq 2, \tag{29} \]

because the arithmetic mean is not smaller than the geometric one. It is worthwhile to mention that Eq. (25) is obtained by solving a quadratic equation for \( \cos \, \iota \) as
\[ \cos^2 \iota - \xi \cos \iota + 1 = 0, \tag{30} \]

which can be obtained from Eqs. (15)-(17) by eliminating \( \omega \) and \( a_K \). Furthermore, one can prove that a root of \( \cos \, \iota = (\xi + \sqrt{\xi^2 - 4})/2 \) must be abandoned because Eq. (29) implies that it is always larger than the unity. Only in the case of \( \iota = 0 \), the apparent ellipse coincides with the true orbit. Hence, the ascending node and consequently the angular distance of the periastron make no sense. As a result, the denominator of R. H. S. of Eq. (27) vanishes.

Equations (13), (25), (26) and (27) agree with those of AAK, where different notations were employed. In this paper, the semiminor axis is not used for areal divisions. Therefore, this formalism can be generalized straightforwardly to an open orbit, as shown below.
3 The solution for a hyperbolic orbit

3.1 An apparent hyperbola

Let a star move in a hyperbola on a celestial sphere. Without loss of gener-
ality, we can assume that the hyperbola is expressed as

\[ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{31} \]

and the orbit is the left-hand side of the hyperbola, \( x < 0 \). Then the position
of the star at each epoch \( t_j \) is denoted by

\[ P_j = (x_j, y_j) \]

\[ = (-a \cosh u_j, b \sinh u_j). \tag{32} \]

The projected common center of mass \( P_C \equiv (x_C, y_C) \) is not necessarily
a focus of the apparent hyperbola but the projected focus of the original
Keplerian hyperbola.

The projected areal velocity with respect to the projected common center
of mass is denoted by \( dS/dt \). The law of the constant areal velocity on the
observed plane is written as

\[ \frac{dS}{dt} = \frac{S(j, k)}{T(j, k)}, \tag{33} \]
where for $t_j > t_k$ we obtain

$$S(j, k) = -\frac{1}{2} ab \left[ u_j - u_k + \frac{x_c}{a} (\sinh u_j - \sinh u_k) + \frac{y_c}{b} (\cosh u_j - \cosh u_k) \right].$$  \(34\)

Equation (33) is rewritten explicitly as

$$\frac{S(2, 1)}{T(2, 1)} = \frac{S(3, 2)}{T(3, 2)}, \quad (35)$$

$$\frac{S(3, 2)}{T(3, 2)} = \frac{S(4, 3)}{T(4, 3)}. \quad (36)$$

They are solved for $x_c$ and $y_c$ as

$$x_c = a \frac{E_1 F_2 - E_2 F_1}{D_1 E_2 - D_2 E_1}, \quad (37)$$

$$y_c = b \frac{F_1 D_2 - F_2 D_1}{D_1 E_2 - D_2 E_1}, \quad (38)$$

where

$$D_j = T(j + 1, j) \sinh u_{j+2} + T(j + 2, j + 1) \sinh u_j - T(j + 2, j) \sinh u_{j+1}, \quad (39)$$

$$E_j = T(j + 1, j) \cosh u_{j+2} + T(j + 2, j + 1) \cosh u_j - T(j + 2, j) \cosh u_{j+1}, \quad (40)$$

$$F_j = T(j + 1, j) u_{j+2} + T(j + 2, j + 1) u_j - T(j + 2, j) u_{j+1}. \quad (41)$$

The periastron is projected onto the observed hyperbola at $P_A \equiv (x_A, y_A)$.
focus of the hyperbola remains the same, even after the projection. Hence, we find

\[ \mathbf{P}_A = \frac{1}{e_K} \mathbf{P}_C. \]  

(42)

The positional vector \( \mathbf{P}_A \) is still located on the apparent hyperbola given by Eq. (31). We thus obtain the ellipticity as

\[ e_K = \sqrt{\frac{x^2}{a^2} - \frac{y^2}{b^2}} \]  

(43)

3.2 Projection onto a celestial sphere

In this paragraph, we use the Cartesian coordinates \((X, Y)\) on the original orbital plane. Let us express an original Keplerian hyperbola as

\[ \frac{X^2}{a_K^2} - \frac{Y^2}{b_K^2} = 1. \]  

(44)

We consider the line that is perpendicular to the semimajor axis at a focus \((a_K e_K, 0)\). This line intersects the original hyperbola at \(Q = (a_K e_K, a_K (e_K^2 - 1))\) and \(R = (a_K e_K, -a_K (e_K^2 - 1))\).

Only in this paragraph, we shall employ other Cartesian coordinates \((x', y')\) so that the ascending node can be located on the \(x'\)-axis and the origin can be the common center of mass. The true periastron of the original hyperbola is projected at \(\mathbf{P}_A \equiv (x'_A, y'_A) = a_K (e_K - 1)(\cos \omega, \sin \omega \cos i), \)
where $i$ and $\omega$ are the inclination angle and the angular distance of the periastron, respectively. The point denoted by $Q$ is projected at $P_Q \equiv (x'_Q, y'_Q) = a_K(e_K^2 - 1)(-\sin \omega, \cos \omega \cos i)$. We shall use the following invariants as

\begin{align}
|P_A|^2 &= a_K^2(e_K - 1)^2(\cos^2 \omega + \sin^2 \omega \cos^2 i), \\
|P_Q|^2 &= a_K^2(e_K^2 - 1)^2(\sin^2 \omega + \cos^2 \omega \cos^2 i), \\
|P_A \times P_Q| &= a_K^2(e_K + 1)(e_K - 1)^2 \cos i.
\end{align}

We consider the area surrounded by the hyperbola and the line interval between $Q$ and $R$. This area is divided into equal halves by the semimajor axis. Even after the projection, the divided areas are still equal halves. Hence one can determine the location of the projected $Q$ as

$$P_Q = \left( x_c - \frac{ay_cs_h}{b}, y_c + \frac{bx_cs_h}{a} \right),$$

in the apparent hyperbola coordinates, where we defined

$$s_h = \sqrt{\frac{e_K^2 - 1}{e_K}}. $$

We make a translation as $x \rightarrow x - x_c$ and $y \rightarrow y - y_c$ so that the center of the coordinates can move to the projected common center of mass. Then, we have

$$P_A = \frac{1}{e_K}(x_c, y_c).$$
$$P_Q = \left( -\frac{aycsh}{b}, \frac{bxcsinh}{a} \right),$$

(51)

where we used Eqs. (42) and (48). Hence, we obtain the invariants from these vectors as

$$|P_A|^2 = \left( \frac{eK - 1}{eK} \right)^2 (x_c^2 + y_c^2),$$

(52)

$$|P_Q|^2 = \frac{(a^4 y_c^2 + b^4 x_c^2)(eK^2 - 1)}{a^2 b^2 eK^2},$$

(53)

$$|P_A \times P_Q| = ab(eK - 1)\sqrt{eK^2 - 1},$$

(54)

whose values can be estimated because $a, b, x_c, y_c$ and $eK$ have been all determined up to this point.

Equations (45)-(47) for $\cos i$, $a_K$ and $\cos 2\omega$ are solved as

$$\cos i = \frac{1}{2} \left( \xi - \sqrt{\xi^2 - 4} \right),$$

(55)

$$a_K = \frac{1}{eK^2 - 1} \sqrt{\frac{(eK + 1)^2 |P_A|^2 + |P_Q|^2}{1 + \cos^2 i}},$$

(56)

$$\cos 2\omega = \frac{(eK + 1)^2 |P_A|^2 - |P_Q|^2}{a_K^2 (eK^2 - 1) \sin^2 i},$$

(57)

where we define

$$\xi = \frac{(eK + 1)^2 |P_A|^2 + |P_Q|^2}{(eK + 1)|P_A \times P_Q|}.$$  

(58)

In the similar manner to the elliptic case, one can show

$$\xi \geq 2.$$  

(59)
Hence, for a quadratic equation for \( \cos i \) as

\[
\cos^2 i - \xi \cos i + 1 = 0, \tag{60}
\]

which is derived from Eqs. (45)-(47). One can prove that a root of \( \cos i = (\xi + \sqrt{\xi^2 - 4})/2 \) is larger than the unity according to Eq. (59) and thus must be abandoned.

4 The solution for a parabolic orbit

4.1 An apparent parabola

Let a star move in a parabola on a celestial sphere as

\[
y^2 + 4qx = 0. \tag{61}
\]

Then the position of the star at each epoch \( t_j \) is denoted by

\[
P_j = (x_j, y_j) = (-\frac{1}{2}qu_j^2, \sqrt{2}qu_j). \tag{62}
\]

The projected common center of mass \( P_C \equiv (x_C, y_C) \) is not necessarily a focus of the apparent parabola but the projected focus of the original Keplerian parabola.
The law of the constant areal velocity on the observed plane is written as

\[
\frac{dS}{dt} = \frac{S(j, k)}{T(j, k)},
\]  

(63)

where for \( t_j > t_k \) we obtain

\[
S(j, k) = -\frac{1}{3}(\sqrt{-x_j} - \sqrt{-x_k})\left[\sqrt{q}(x_j - \sqrt{x_j}x_k + x_k) + \frac{3}{2}(\sqrt{-x_j} + \sqrt{-x_k})y_C + 3\sqrt{q}x_C\right].
\]  

(64)

Equation (68) is rewritten explicitly as

\[
\frac{S(2, 1)}{T(2, 1)} = \frac{S(3, 2)}{T(3, 2)},
\]  

(65)

\[
\frac{S(3, 2)}{T(3, 2)} = \frac{S(4, 3)}{T(4, 3)}.
\]  

(66)

They are solved for \( x_c \) and \( y_c \) as

\[
x_c = -\frac{H_1 I_2 - H_2 I_1}{G_1 H_2 - G_2 H_1},
\]  

(67)

\[
y_c = -\sqrt{q} \frac{G_1 I_2 - G_2 I_1}{G_1 H_2 - G_2 H_1},
\]  

(68)

where

\[
G_j = 3[T(j + 1, j)\sqrt{-x_{j+2}} + T(j + 2, j + 1)\sqrt{-x_j} - T(j + 2, j)\sqrt{-x_{j+1}}],
\]  

(69)

\[
H_j = \frac{3}{2}[T(j + 1, j)x_{j+2} + T(j + 2, j + 1)x_j - T(j + 2, j)x_{j+1}],
\]  

(70)

\[
I_j = -[T(j + 1, j)x_{j+2}\sqrt{-x_{j+2}} + T(j + 2, j + 1)x_j\sqrt{-x_j} - T(j + 2, j)x_{j+1}\sqrt{-x_{j+1}}].
\]  

(71)
The periastron is projected onto the observed parabola at \( P_A \equiv (x_A, y_A) \). The semimajor axis is projected onto a line, which may be expressed as 
\[
y = Kx + L.
\]
This line intersects the apparent parabola only at the projected periastron. Therefore, we find \( K = 0 \) because \( |x| \) must be larger than \( \sqrt{-x} \) for a sufficient large \( |x| \). In addition, the projected semimajor axis goes through the projected common center. This implies \( L = y_c \). Hence we obtain

\[
P_A = \left( -\frac{y_c^2}{4q}, y_c \right).
\]

(72)

4.2 Projection onto a celestial sphere

In this paragraph, we use the Cartesian coordinates \((X, Y)\) on the original orbital plane. Let a Keplerian parabola be

\[
Y^2 + 4q_K X = 0.
\]

(73)

We consider the line that is perpendicular to the semimajor axis at a focus \((-q_K, 0)\). This line intercepts the original parabola at \( Q = (-q_K, 2q_K) \) and \( R = (-q_K, -2q_K) \).

Only in this paragraph, we employ other Cartesian coordinates \((x', y')\) so that the ascending node can be located on the \( x'\)-axis and the origin can be the common center of mass. The true periastron of the original
parabola is projected at \( P_A \equiv (x'_A, y'_A) = q_K (\cos \omega, \sin \omega \cos i) \), where \( i \) and \( \omega \) are the inclination angle and the angular distance of the periastron, respectively. The point denoted by \( Q \) is projected at \( P_Q \equiv (x'_Q, y'_Q) = 2q_K (-\sin \omega, \cos \omega \cos i) \). We shall use the following invariants as
\[
|P_A|^2 = q_K^2 (\cos^2 \omega + \sin^2 \omega \cos^2 i), \tag{74}
\]
\[
|P_Q|^2 = 4q_K^2 (\sin^2 \omega + \cos^2 \omega \cos^2 i), \tag{75}
\]
\[
|P_A \times P_Q| = 2q_K^2 \cos i. \tag{76}
\]

We consider the area surrounded by the parabola and the line interval between \( Q \) and \( R \). This area is divided into equal halves by the semimajor axis. Even after the projection, the divided areas are still equal halves. Hence one can determine the location of the projected \( Q \) as
\[
P_Q = \left( x_c - \frac{y_c s_p}{2q}, y_c + s_p \right), \tag{77}
\]
in the apparent parabola coordinates, where we defined
\[
s_p = \sqrt{-y_c^2 + 4q x_c}. \tag{78}
\]
We make a translation as \( x \to x - x_c \) and \( y \to y - y_c \) so that the origin of the coordinates can be the projected common center of mass. Then, we obtain
\[
P_A = \left( \frac{y_c^2 + 4q x_c}{4q}, 0 \right), \tag{79}
\]
\[ \mathbf{P}_Q = \left( -\frac{y_c s_p}{2q}, s_p \right). \]  

Hence, we obtain the invariants from these vectors as

\[ |\mathbf{P}_A|^2 = \frac{(y_c^2 + 4qx_c)^2}{16q^2}, \quad (81) \]

\[ |\mathbf{P}_Q|^2 = -\frac{(y_c^2 + 4qx_c)(y_c^2 + 4q^2)}{4q^2}, \quad (82) \]

\[ |\mathbf{P}_A \times \mathbf{P}_Q| = \frac{[-(y_c^2 + 4qx_c)]^{3/2}}{4q}, \quad (83) \]

whose values can be estimated because \( q, x_c \) and \( y_c \) have been all determined up to this point.

Equations (74)-(76) for \( \cos i, a_K \) and \( \cos 2\omega \) are solved as

\[ \cos i = \frac{1}{2} \left( \xi - \sqrt{\xi^2 - 4} \right), \quad (84) \]

\[ q_K = \frac{1}{2} \sqrt{\frac{4|\mathbf{P}_A|^2 + |\mathbf{P}_Q|^2}{1 + \cos^2 i}}, \quad (85) \]

\[ \cos 2\omega = \frac{4|\mathbf{P}_A|^2 - |\mathbf{P}_Q|^2}{4q_K^2 \sin^2 i}, \quad (86) \]

where we define

\[ \xi = \frac{4|\mathbf{P}_A|^2 + |\mathbf{P}_Q|^2}{2|\mathbf{P}_A \times \mathbf{P}_Q|}. \quad (87) \]

In the similar manner to the above two cases, one can show

\[ \xi \geq 2. \quad (88) \]

Hence, for a quadratic equation for \( \cos i \) as

\[ \cos^2 i - \xi \cos i + 1 = 0, \quad (89) \]
which is derived from Eqs. (74)-(76). One can prove that a root of \( \cos i = (\xi + \sqrt{\xi^2 - 4})/2 \) is larger than the unity according to Eq. (88) and thus must be abandoned.

5 Transformations from an elliptic case to hyperbolic/parabolic ones

5.1 To a hyperbolic case

Let us rederive the formula for a hyperbolic case from that for an elliptic one by making a transformation as

\[
\begin{align*}
  u &\rightarrow \hat{iu}, \quad (90) \\
  a &\rightarrow -a, \quad (91) \\
  b &\rightarrow -\hat{ib}, \quad (92)
\end{align*}
\]

which imply

\[
\begin{align*}
  \cos u &\rightarrow \cosh u, \quad (93) \\
  \sin u &\rightarrow \hat{i} \sinh u, \quad (94)
\end{align*}
\]
where \( \hat{i} = \sqrt{-1} \). Then, from Eqs. (39)-(41) and (39)-(41) we find

\[
A_j \rightarrow \hat{i} D_j, \quad (95)
\]

\[
B_j \rightarrow E_j, \quad (96)
\]

\[
C_j \rightarrow \hat{i} F_j. \quad (97)
\]

We can thus show that the location of the common center is transformed from Eqs. (7) and (8) to Eqs. (37) and (38). Equation (13) is transformed into Eq. (43), Eqs. (22)-(24) into Eqs. (52)-(54), Eqs. (25)-(27) into Eqs. (55)-(57), because \( \xi \) remains unchanged.

### 5.2 To a parabolic case

To rederive the formula for a parabolic case, we perform a transformation from an elliptic case with a limiting procedure as

\[
e \rightarrow 1, \quad (98)
\]

\[
q = \lim_{e \to 1} a(1 - e), \quad (99)
\]

\[
x' = x - a, \quad (100)
\]

where the finite \( q \) implies \( a \to \infty \) and \( b^2 = a(1 + e) \times a(1 - e) \to 2aq \). Then, we find

\[
A_j \rightarrow \frac{2\sqrt{q}}{3b} G_j, \quad (101)
\]
\[ B_j \rightarrow \frac{2}{3a} H_j, \quad (102) \]
\[ C_j \rightarrow \frac{2\sqrt{q}}{3b} G_j + \frac{4q^{3/2}}{3b^3} I_j. \quad (103) \]

We can transform the location of the common center from Eqs. (7) and (8) to

\[ x_c \rightarrow a - \frac{H_1 I_2 - H_2 I_1}{G_1 H_2 - G_2 H_1}, \quad (104) \]
\[ y_c \rightarrow -\sqrt{q} \frac{G_1 I_2 - G_2 I_1}{G_1 H_2 - G_2 H_1}, \quad (105) \]

which agrees with Eq. (68). We thus recover Eq. (67) as

\[ x_c' \rightarrow -\frac{H_1 I_2 - H_2 I_1}{G_1 H_2 - G_2 H_1}. \quad (106) \]

Equation (13) is transformed as

\[
e_K = 1 + \frac{y_c^2 + 4qx_c'}{4aq} + O\left(\frac{1}{a^2}\right)
\rightarrow 1, \quad (107)
\]

where we used \( a \rightarrow \infty \) and \( b^2 \rightarrow 2aq \). By using Eq. (107) and \( b^2 \rightarrow 2aq \), we obtain

\[
\|P_A\|^2 \rightarrow \frac{(y_c^2 + 4qx_c')^2}{16q^2}, \quad (108)
\]
\[
\|P_Q\|^2 \rightarrow \frac{2[a^4y_c^2 + 4a^2q^2(a + x_c')^2](1 - e_K)}{2a^3q}
\rightarrow \frac{-(y_c^2 + 4q^2)(y_c^2 + 4qx_c')}{4q^2}, \quad (109)
\]
\[ |\mathbf{P}_A \times \mathbf{P}_Q| = 2a^{3/2} \sqrt{q}(1 - e_K)^{3/2} \]
\[ \rightarrow \frac{[-(y_c^2 + 4qx_c')^{3/2}}{4q}, \quad (110) \]

which agree with Eqs. (81)-(83).

Equations (25)-(27) are transformed as

\[ \cos i = \frac{1}{2}(\xi - \sqrt{\xi^2 - 4}), \quad (111) \]

\[ q_K = \lim_{e_K \rightarrow 1} a_K(1 - e_K) \]
\[ = \lim_{e_K \rightarrow 1} \frac{1}{1 + e_K} \sqrt{\frac{(1 + e_K)^2|\mathbf{P}_A|^2 + |\mathbf{P}_Q|^2}{1 + \cos^2 i}} \]
\[ = \frac{1}{2} \sqrt{\frac{4|\mathbf{P}_A|^2 + |\mathbf{P}_Q|^2}{1 + \cos^2 i}}, \quad (112) \]

\[ \cos 2\omega = \frac{(1 + e_K)^2|\mathbf{P}_A|^2 - |\mathbf{P}_Q|^2}{a_K^2(1 - e_K)^2(1 + e_K)^2 \sin^2 i} \]
\[ \rightarrow \frac{4|\mathbf{P}_A|^2 - |\mathbf{P}_Q|^2}{4q_K^2 \sin^2 i}, \quad (113) \]

where \(\xi\) remains unchanged. They agree with Eqs. (84)-(86).

6 Conclusion

The formulae for orbit determination of elliptic, hyperbolic and parabolic orbits are obtained in a unified manner by generalizing AAK approach, which originally needed a fact of the areal divisions by the semimajor and semiminor
axes of an ellipse. We show also that the present formulae are recovered from AAK result by a suitable transformation among an ellipse, hyperbola and parabola.

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