Parameterized Complexity of Diameter

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Abstract

Diameter—the task of computing the length of a longest shortest path—is a fundamental graph problem. Assuming the Strong Exponential Time Hypothesis, there is no $O(n^{1.99})$-time algorithm even in sparse graphs [Roditty and Williams, 2013]. To circumvent this lower bound we aim for algorithms with running time $f(k)(n + m)$ where $k$ is a parameter and $f$ is a function as small as possible. We investigate which parameters allow for such running times. To this end, we systematically explore a hierarchy of structural graph parameters.

1 Introduction

The diameter is arguably among the most fundamental graph parameters. Most known algorithms for determining the diameter first compute the shortest path between each pair of vertices (APSP: ALL-PAIRS SHORTEST PATHS) and then return the maximum [1]. The currently fastest algorithms for APSP in weighted graphs have a running time of $O(n^{3/2} \Omega(\sqrt{\log n}))$ in dense graphs [12] and $O(nm + n^2 \log n)$ in sparse graphs [25], respectively. In this work, we focus on the unweighted case. Formally, we study the following problem:

**Diameter**

**Input:** An undirected, connected, unweighted graph $G = (V, E)$.

**Task:** Compute the length of a longest shortest path in $G$.

The (theoretically) fastest algorithm for **Diameter** runs in $O(n^{2.373})$ time and is based on fast matrix multiplication [35]. This upper bound can (presumably) not be improved by much as Roditty and Williams [34] showed that solving **Diameter** in $O((n + m)^{2-\varepsilon})$ time for any $\varepsilon > 0$ breaks the SETH (Strong Exponential Time Hypothesis [23, 24]). Seeking for ways to circumvent this lower bound, we follow the line of “parameterization for polynomial-time solvable problems” [20] (also referred to as “FPT in P”). This approach is recently actively studied and sparked a lot of research [1, 4, 10, 15, 16, 18, 26, 27, 29]. Given some parameter $k$ we search for an algorithm with a running time of $f(k)(n + m)^{2-\varepsilon}$ that solves **Diameter**. Starting FPT in P for **Diameter**, Abboud et al. [1] proved that, unless the SETH fails, the function $f$ has to be an exponential function if $k$ is the treewidth of the graph. We extend their research by systematically exploring the parameter space looking for parameters where $f$ can be a polynomial. If this is not possible (due to conditional lower bounds), then we seek for matching upper bounds of the form $f(k)(n + m)^{2-\varepsilon}$ where $f$ is exponential.

In a second step, we combine parameters that are known to be small in many real world graphs. We concentrate on social networks which often have special characteristics, including the “small-world” property and a power-law degree distribution [28, 30–33]. We therefore combine parameters related to the diameter with parameters related to the h-index\(^1\); both parameters can be expected to be magnitudes smaller than the number of vertices.

**Related Work.** Due to its importance, **Diameter** is extensively studied. Algorithms employed in practice have usually a worst-case running time of $O(nm)$, but are much faster in experiments.

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\(^1\)The h-index of a graph $G$ is the largest number $\ell$ such that $G$ contains at least $\ell$ vertices of degree at least $\ell$. 

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Figure 1, we investigate parameter combinations. We prove that a parameterized problem over some finite alphabet $\Sigma$, where $(x, k) \in \Sigma^* \times \mathbb{N}$ denotes an instance of $L$ and $k$ is the parameter. Then $L$ is called fixed-parameter tractable if there is an algorithm that on input $(x, k)$

2 Preliminaries & Basic Observations

For $\ell \in \mathbb{N}$ we set $[\ell] := \{1, 2, \ldots, \ell\}$. We use mostly standard graph notation. Given a graph $G = (V, E)$ set $n := |V|$ and $m := |E|$. A path $P = v_0 \ldots v_a$ is a graph with vertex set $\{v_0, \ldots, v_a\}$ and edge set $\{\{v_i, v_{i+1}\} \mid 0 \leq i < a\}$. For $u, v \in V$, we denote with $\text{dist}_G(u, v)$ the distance between $u$ and $v$ in $G$. If $G$ is clear from the context, then we omit the subscript. For a vertex subset $V' \subseteq V$, we denote with $G[V']$ the graph induced by $V'$. We set $G - V' := G[V \setminus V']$. We denote by $d(G)$ the diameter of $G$.

Parameterized Complexity and GP-hardness. A language $L \subseteq \Sigma^* \times \mathbb{N}$ is a parameterized problem over some finite alphabet $\Sigma$, where $(x, k) \in \Sigma^* \times \mathbb{N}$ denotes an instance of $L$ and $k$ is the parameter. Then $L$ is called fixed-parameter tractable if there is an algorithm that on input $(x, k)$

Concerning worst-case analysis, the theoretically fastest algorithms are based on matrix multiplication and run in $O(n^{2.373})$ time [35] and any $O((n + m)^{2+\varepsilon})$-time algorithm refutes the SETH [34].

The following results on approximating Diameter are known: It is folklore that a simple breadth-first search gives a linear-time 2-approximation. Aingworth et al. [2] improved the approximation factor to $3/2$ at the expense of the higher running time of $O(n^2 \log n + m \sqrt{n \log n})$. The lower bound of Roditty and Williams [34] also implies that approximating Diameter within a factor of $3/2 - \delta$ in $O(n^{2-\varepsilon})$ time refutes the SETH. Moreover, a $3/2 - \delta$-approximation in $O(m^{2-\varepsilon})$ time or a $5/3 - \delta$-approximation in $O(m^{3/2-\varepsilon})$ time also refute the SETH [3, 11]. On planar graphs, there is an approximation scheme with near linear running time [38]; the fastest exact algorithm for Diameter on planar graphs runs in $O(n^{1.667})$ time [19].

Concerning FPT in P, Diameter can be solved in $2^{O(k)}n^{1+o(1)}$ time where $k$ is the treewidth of the graph [10]; however, an $2^{o(k)}n^{1-\varepsilon}$-time algorithm refutes the SETH [1]. In fact, the construction actually proves the same running time lower bound with $k$ being the vertex cover number. The reduction for the lower bound of Roditty and Williams [34] also implicitly implies that the SETH is refuted by any $f(k)(n + m)^{2-\varepsilon}$-time algorithm for Diameter for any computable function $f$ when $k$ is either the (vertex deletion) distance to chordal graphs or the combined parameter $h$-index and domination number. Evald and Dahlggaard [16] adapted the reduction by Roditty and Williams and proved that any $f(k)(n + m)^{2-\varepsilon}$-time algorithm for Diameter parameterized by the maximum degree $k$ for any computable function $f$ refutes the SETH. Our Contribution. We make progress towards systematically classifying the complexity of Diameter parameterized by structural graph parameters. Figure 1 gives an overview of previously known and new results and their implications (see Brandstädt et al. [8] for definitions of the parameters). In Section 3, we follow the “distance from triviality parameterization” [22] aiming to extend known tractability results for special graph classes to graphs with small modulators. For example, Diameter is linear-time solvable on trees. We obtain for the parameter feedback edge number $k$ (edge deletion number to trees) an $O(k \cdot n)$-time algorithm. However, this is our only $k^{O(1)}(n + m)$-time algorithm in this section. For the remaining parameters, it is already known that such algorithms refute the SETH. For the parameter distance $k$ to cographs we therefore provide an $2^{O(k)}(n + m)$-time algorithm. Finally, for the parameter distance $k$ to bipartite graphs, we use the recently introduced notion of General-Problem-hardness [4] to show that if there exists an algorithm with running time $f(k)n^{2-\varepsilon}$, where $f$ is any function, then there also exists an algorithm with running time $n^{2-\varepsilon}$ for Diameter. Hence, instead of searching for parameterized algorithms with respect to distance to bipartite graphs, one should rather look for faster Diameter algorithms in general.

In Section 4, we investigate parameter combinations. We prove that a $k^{O(1)}(n + m)$-time algorithm where $k$ is the combined parameter diameter and maximum degree would refute the SETH. Complementing this lower bound, we provide an $f(k)(n + m)$-time algorithm where $k$ is the combined parameter diameter and $h$-index.
decides whether \((x, k) \in L\) in \(f(k) \cdot |x|^{O(1)}\) time, where \(f\) is some computable function only depending on \(k\) and \(|x|\) denotes the size of \(x\). For a parameterized problem \(L\), the language \(\hat{L} = \{x \in \Sigma^* \mid \exists k: (x, k) \in L\}\) is called the unparameterized problem associated to \(L\). We use the notion of General-Problem-hardness which formalizes the types of reduction that allow us to exclude parameterized algorithms as they would lead to faster algorithms for the general, unparameterized, problem.

**Definition 1** ([41]). Let \(P \subseteq \Sigma^* \times \mathbb{N}\) be a parameterized problem, let \(\hat{P} \subseteq \Sigma^*\) be the unparameterized decision problem associated to \(P\), and let \(g: \mathbb{N} \to \mathbb{N}\) be a polynomial. We call \(P\) \(\ell\)-General-Problem-hard\((g)\) (\(\ell\)-GP-hard\((g)\)) if there exists an algorithm \(A\) transforming any input instance \(I\) of \(\hat{P}\) into a new instance \((I', k')\) of \(P\) such that

\((G1)\) \(A\) runs in \(O(g(|I|))\) time,

\((G2)\) \(I \in \hat{P} \iff (I', k') \in P\),
Lemma 1 ([4]). Let $g : \mathbb{N} \to \mathbb{N}$ be a polynomial, let $P \subseteq \Sigma^* \times \mathbb{N}$ be a parameterized problem that is $\text{GP-hard}(g)$, and let $\hat{P} \subseteq \Sigma^*$ be the unparameterized decision problem associated to $P$. If there is an algorithm solving each instance $(I, k)$ of $P$ in $O(f(k) \cdot g(|I|))$ time, then there is an algorithm solving each instance $I'$ of $\hat{P}$ in $O(g(|I'|))$ time.

Applying Lemma 1 to Diamter yields the following. First, having an $f(k)n^{2.3}$-time algorithm with respect to a parameter $k$ for which Diamter is $\text{GP-hard}$ would yield a faster Diamter algorithm. Moreover, from the known SETH-based hardness results [3, 11, 34] we get the following.

Observation 1. If the SETH is true and Diamter is $\text{GP-hard}(n^{2-\varepsilon})$ with respect to some parameter $k$ for some $\varepsilon > 0$, then there is no $f(k) \cdot n^{2-\varepsilon'}$ time algorithm for any $\varepsilon' > 0$ and any function $f$.

We next present a simple observation that completes the overview in Figure 1.

Observation 2. Diamter parameterized by distance $i$ to interval graphs, distance $c$ to clique, average degree $a$, maximum degree $\Delta$, diameter $d$, and domination number $\gamma$ is solvable

- in $O(i \cdot n^2)$ time provided that the deletion set is given,
- in $O(c \cdot (n + m))$ time,
- in $O(a \cdot n^2)$ time,
- in $O(\Delta^{2d+2})$ time, and,
- in $O(\gamma^2 \cdot \Delta^3)$ time, respectively.

We start with the parameters distance to interval graphs and distance to clique. We first provide a general observation stating that a size $k$ deletion set to some graph class can be used to design a $O(k \cdot n^2)$-time algorithm if All-Pairs Shortest Paths can be solved in $O(n^2)$ time on graphs in the respective graph class. The algorithm is fairly simple: First compute $G'$, that is, the graph without the deletion set $K$, and solve All-Pairs Shortest Paths on it in $O(n^2)$ time. Next, compute a breadth-first search from every vertex in $K$ in $O(k \cdot m)$ time and store all distances found in a table. Lastly, compute for each pair $a, c \in V \setminus K$

$$\text{dist}_G(a, c) := \min\{\text{dist}_{G'}(a, c), \min_{b \in K}\{\text{dist}_G(a, b) + \text{dist}_G(b, c)\}\},$$

that is, the minimum distance in the original graph. Observe that a shortest path either travels through some vertex $b \in K$ or not. In the latter case, $\text{dist}_G(a, c) = \text{dist}_{G'}(a, c)$ and in the former case the distance between $a$ and $c$ in $G$ is $\text{dist}_G(a, b) + \text{dist}_G(b, c)$. This algorithm takes $O(n + m + n^2 + k \cdot m + n^2 \cdot k) = O(k \cdot n^2)$ time.

Observation 3. Let $\Pi$ be a graph class such that All-Pairs Shortest Paths can be solved in $O(n^2)$ time on $\Pi$. If the (vertex) deletion set $K$ to $\Pi$ is given, then All-Pairs Shortest Paths can be solved in $O(|K| \cdot n^2)$ time.

It is known that All-Pairs Shortest Paths can be solved in $O(n^2)$ time on interval graphs [13, 37]. Thus we obtain the following.
Observation 4. **Diameter** parameterized by the distance \(i\) to interval graphs is solvable in \(O(i \cdot n^2)\) time provided that the deletion set is given.

We are not aware of a fast constant factor approximation algorithm to compute the deletion set in the above observation. Finding (or excluding) such an approximation algorithm remains a task for future work.

We next observe that **Diameter** parameterized by distance \(c\) to clique is solvable in \(O(c \cdot (n + m))\) time. Since distance to clique is the vertex cover number in the complement graph, it can be 2-approximated in linear time.

Observation 5. **Diameter** parameterized by distance \(c\) to clique can be solved in \(O(c \cdot (n + m))\) time.

*Proof.* Let \(G = (V, E)\) be the input graph and let \(c\) be its distance to clique. Let \(G'\) be the respective induced clique graph. Compute in linear time the degree of each vertex and the number \(n = |V|\) of vertices. Iteratively check for each vertex \(v\) whether its degree is \(n - 1\). If \(\deg(v) = n - 1\), then \(v\) can be deleted as it is in every largest clique and thus decrease \(n\) by one and the degree of each other vertex by one. If not, then we can find a vertex \(w\) which is not adjacent to \(v\) in \(O(\deg(v))\) time. Put \(v\) and \(w\) in the solution set, delete both vertices and all incident edges and adjust the number of vertices and their degree accordingly. Observe that \(v\) and \(w\) cannot be contained in the same clique and therefore \(v \in K\) or \(w \in K\). Putting both vertices in the solution set results in a 2-approximation. This algorithm takes \(O(\deg(v) + \deg(w))\) time per deleted pair \(v, w\) of vertices. Since \(\sum_{v \in V} \deg(v) \in O(n + m)\) this procedure takes \(O(n + m)\) time. Since \(G'\) is a clique, its diameter is one if there are at least two vertices in the clique. We therefore assume that there is at least one vertex in the deletion set \(K\). Compute for each vertex \(v \in K\) a breadth-first search rooted in \(v\) in linear time and return the largest distance found. The returned value is the diameter of \(G\) as each longest induced path is either of length one or has at least one endpoint in \(K\). The procedure takes \(O(|K| \cdot (n + m) + n + m) = O(c \cdot (n + m))\) time.

We next consider the average degree \(a\). Observe that \(2m = n \cdot a\) and therefore the standard algorithm takes \(O(n \cdot m) = O(a \cdot n^2)\) time.

Observation 6. **Diameter** parameterized by the average degree \(a\) is solvable in \(O(a \cdot n^2)\) time.

Lastly, we look at two parameter combinations related to both maximum degree and diameter. Usually, this parameter is not interesting as the graph size can be upper-bounded by this parameter and thus fixed-parameter tractability with respect to this combined parameter is trivial. The input size is, however, only exponentially bounded in the parameter, so it might be tempting to search for fully polynomial algorithms. In Section 4.2 we exclude such a fully polynomial algorithm. Thus, the subsequent algorithm is basically optimal.

Observation 7. **Diameter** parameterized by diameter \(d\) and maximum degree \(\Delta\) is solvable in \(O(\Delta^{2d+3})\) time.

*Proof.* Since we may assume that the input graph only consists of one connected component, every vertex is found by any breadth-first search. Any breadth-first search may only reach depth \(d\), where \(d\) is the diameter of the input graph, and as each vertex may only have \(\Delta\) neighbors there are at most \(\sum_{i=0}^{d} \Delta^i < \Delta^{d+1}\) vertices. Since \(m \leq n \cdot \Delta\) the \(O(n \cdot m)\)-time algorithm runs in \(O(\Delta^{2d+3})\) time.

Next, observe that for any graph of \(n\) vertices, domination number \(\gamma\), and maximum degree \(\Delta\) it holds that \(n \leq \gamma \cdot (\Delta + 1)\) as each vertex is in a dominating set or is a neighbor of at least one vertex in it. The next observation follows from \(m \leq n \cdot \Delta\).

Observation 8. **Diameter** parameterized by domination number \(\gamma\) and maximum degree \(\Delta\) is solvable in \(O(\gamma^2 \cdot \Delta^3)\) time.
hold. It is easy to verify that the reduction can be implemented in linear time and

\[ \text{Definition 1} \]

\[ \text{Figure 1} \]

Let \( G \) be a graph and \( g \) be the girth of \( G \). Theorem 1.

\[ \text{Result. The parameter} \]

\[ \text{function} \]

\[ v \]

\[ d \]

\[ a \]

\[ \text{path of length} \]

\[ (\text{measures the edge deletion distance to trees. Note that the lower bound of Abboud et al.} \]

\[ \text{is bipartite there is no induced cycle of length three in} \]

\[ \{v, v_i, v_j\} \in E \} \cup \{v, v_i, v_j, v_k\} \in V \} \}

\[ \text{of} \]

\[ \text{two parameters measure the vertex deletion distance to some graph class. Feedback edge number} \]

\[ \text{measures the edge deletion distance to trees. Note that the lower bound of Abboud et al.} \]

\[ \text{is bipartite and thus} \]

\[ \text{Distance to cographs.} \]

Providing an algorithm that matches the lower bound of Abboud et al. [1], we will show that \( \text{DIAMETER parameterized by} \]

\[ k \]

\[ \text{to} \]

\[ \text{graphs} \]

\[ \text{is} \]

\[ \text{hypothesis} \]

\[ \text{in this order:} \]

\[ \text{Distance to bipartite graphs.} \]

\[ \text{We show that} \]

\[ \text{parameterized by} \]

\[ k \]

\[ \text{already implies that there is no} \]

\[ \text{Diameter} \]

\[ \text{in} \]

\[ \text{almost} \]

\[ \text{parameter} \]

\[ \text{vertex cover number} \]

\[ \text{cographs.} \]

Figure 2: Example for the construction in the proof of Theorem 1. The input graph given on the left side has diameter two and the constructed graph on the right side has diameter three. In each graph one longest shortest path is highlighted.

3 Deletion Distance to Special Graph Classes

In this section we investigate parameterizations that measure the distance to special graph classes. The hope is that when \( \text{DIAMETER} \) can be solved efficiently in a special graph class, then \( \text{DIAMETER} \) can be solved if the input graph is “almost” in II. We study the following parameters in this order: \( \text{distance to bipartite graphs}, \text{distance to cographs}, \) and \( \text{feedback edge number} \). The first two parameters measure the vertex deletion distance to some graph class. \( \text{Feedback edge number} \) measures the edge deletion distance to trees. Note that the lower bound of Abboud et al. [1] for the parameter \( \text{vertex cover number} \) already implies that there is no \( k^{O(1)}(n + m)^2 - \varepsilon \)-time algorithm for \( k \) being one of the first two parameters in our list unless the SETH breaks, since each of these parameters is smaller than \( \text{vertex cover number} \) (see Figure 1).

\[ \text{Distance to bipartite graphs.} \]

We show that \( \text{DIAMETER parameterized by} \)

\[ \text{graphs} \]

\[ \text{is} \]

\[ \text{4-GP-hard.} \]

\[ \text{Consequently solving} \]

\[ \text{Diameter} \]

\[ \text{in} \]

\[ \text{for any computable} \]

\[ d \]

\[ a \]

\[ \text{is} \]

\[ \text{Distance to bipartite graphs.} \]

\[ \text{We show that} \]

\[ \text{parameterized by} \]

\[ \text{graphs} \]

\[ \text{is} \]

\[ \text{4-GP-hard} \]

\[ \text{with respect to the combined parameter} \]

\[ \text{distance to bipartite graphs} \]

\[ \text{and} \]

\[ \text{girth.} \]

\[ \text{Proof.} \]

\[ \text{Let} \]

\[ \text{new graph} \]

\[ \text{as follows:} \]

\[ \text{and} \]

\[ \text{are both independent sets and therefore} \]

\[ \text{is bipartite. Notice further that for any} \]

\[ \text{there is an induced cycle in} \]

\[ \text{containing the vertices} \]

\[ \text{Since} \]

\[ \text{is bipartite there is no induced cycle of length three in} \]

\[ \text{and thus the girth of} \]

\[ \text{four.} \]

Lastly, we show that \( d(G') = d(G) + 1 \) by proving that if \( \text{dist}(v_i, v_j) \) is odd, then \( \text{dist}(u_i, u_j) = \text{dist}(v_i, v_j) \) and \( \text{dist}(u_i, u_j) = \text{dist}(v_i, v_j) + 1 \), and if \( \text{dist}(v_i, v_j) \) is even, then \( \text{dist}(u_i, u_j) = \text{dist}(v_i, v_j) \) and \( \text{dist}(u_i, u_j) = \text{dist}(v_i, v_j) + 1 \). As \( \text{dist}(u_i, u_j) = 1 \) and \( \text{dist}(u_i, u_j) = \text{dist}(u_i, u_j) \), this will conclude the proof.

Let \( P = v_{a_0}v_{a_1} \ldots v_{a_d} \) be a shortest path from \( v_i \) to \( v_j \) where \( v_{a_0} = v_i \) and \( v_{a_d} = v_j \). Let \( P' = u_{a_0}w_{a_1}u_{a_2}w_{a_3} \ldots \) be a path in \( G' \). Clearly, \( P' \) is also a shortest path as there are no edges \( \{u_i, w_j\} \in E' \) where \( \{v_i, v_j\} \notin E \).

If \( d \) is odd, then \( u_{a_0}w_{a_1} \ldots w_{a_d} \) is a path of length \( d \) from \( u_i \) to \( w_j \) and \( u_{a_0}w_{a_1} \ldots w_{a_d}u_{a_d} \) is a path of length \( d + 1 \) from \( u_i \) to \( w_j \). If \( d \) is even, then \( u_{a_0}w_{a_1} \ldots w_{a_d}u_{a_d} \) is a path of length \( d \) from \( u_i \) to \( u_j \) and \( u_{a_0}w_{a_1} \ldots w_{a_d}u_{a_d} \) is a path of length \( d + 1 \) from \( u_i \) to \( w_j \). Notice that \( G' \) is bipartite and thus \( \text{dist}(u_i, u_j) \) must be even and \( \text{dist}(u_i, w_j) \) must be odd.

\[ \square \]

\[ \text{Distance to cographs.} \]

Providing an algorithm that matches the lower bound of Abboud et al. [1], we will show that \( \text{DIAMETER parameterized by} \)

\[ k \]

\[ \text{to} \]

\[ \text{graphs} \]

\[ \text{can be solved in} \]

\[ O(k \cdot (n + m) + 2^{O(k)}) \] time.
A graph is a cograph if and only if it is $P_4$-free. Given a graph $G$ one can determine in linear time whether $G$ is a cograph and can return an induced $P_4$ if this is not the case \cite{Bodlaender98,Bodlaender2000}. This implies that in $O(k \cdot (n + m))$ time one can compute a set $K \subseteq V$ with $|K| \leq 4k$ such that $G - K$ is a cograph: Iteratively add all four vertices of a returned $P_4$ into the solution set and delete those vertices from $G$ until it is $P_4$-free. In the following, we hence assume that such a set $K$ is given. Notice that every cograph has diameter at most two as any graph with diameter at least three contains an induced $P_4$.

**Theorem 2. Diameter** can be solved in $O(k \cdot (n + m) + 2^{16k} k)$ time when parameterized by distance $k$ to cographs.

**Proof.** Let $G = (V, E)$ be the input graph with distance $k$ to cograph. Let $K = \{x_1, \ldots, x_j\}$ be a set of vertices such that $G' = G - K$ is a cograph with $j \leq 4k$. Recall that $K$ can be computed in $O(k \cdot (n + m))$ time.

We first compute all connected components and their diameter in $G'$ in linear time and store for each vertex the information in which connected component it is. Notice that we only need to check for each connected component $C$, whether $C$ induces a clique in $G'$ since otherwise its diameter is two. In a second step, we perform in $O(k \cdot (n + m))$ time a breadth-first search in $G$ from each vertex $v \in K$ and store the distance between $v$ and every other vertex $w$ in a table.

Next we introduce some notation. The type of a vertex $u \in V \setminus K$ is a vector of length $d$ where the $i$th entry describes the distance from $u$ to $x_i$ with the addition that any value above three is set to 4. We say a type is non-empty, if there is at least one vertex with this type. We compute for each vertex $u \in V \setminus K$ its type. Additionally we store for each non-empty type the connected component its vertex is in or that there are at least two different connected components containing a vertex of that type. This takes $O(n \cdot k)$ time and there are at most $4^k$ many different types.

Lastly, we iterate over all pairs of types (including the pairs where both types are the same) and compute the largest distance between vertices of these types. Let $y, z$ be the vertices of the respective types with maximum pairwise distance. We will first show how to find $y$ and $z$ and then show how to correctly compute their distance in $O(k)$ time. If both types only appear in the same connected component, then the distance between the two vertices of these types is at most two.

If two types appear in different connected components, then a longest shortest path between vertices of the respective type contain at least one vertex in $K$. Observe that since each cograph has diameter at most two, each third vertex in any longest shortest path must be in $K$.

Thus a shortest $y$-$z$-path contains at least one vertex $x_i \in K$ with $\text{dist}(x_i, y) < 3$. By definition, each vertex with the same type as $y$ has the same distance to $x_i$ and therefore the same distance to $z$ unless there is no shortest path from it to $z$ that passes through $x_i$, that is, it is in a same connected component as $z$. Thus, we can choose two arbitrary vertices of the respective types in different connected components. We can compute the distance between $y$ and $z$ in $O(k)$ time by computing $\min_{x \in K} \text{dist}(y, x) + \text{dist}(x, z)$. Observe that the shortest path from $y$ to $z$ contains $x_i$, and therefore $\text{dist}(y, x_i) + \text{dist}(x_i, z) = \text{dist}(y, z)$. We can compute the diameter of $G$ this way in $O(k \cdot (n + m) + 4^{16} k)$ time.

**Feedback edge number.** We will prove that Diameter parameterized by feedback edge number $k$ can be solved in $O(k \cdot n)$ time. One can compute a minimum feedback edge set $K$ (with $|K| = k$) in linear time by taking all edges not in a spanning tree. Recently, this parameter was used to speed up algorithms computing maximum matchings \cite{Fleischner2021}. In the remainder of this section we will prove the following adaptive algorithm.

**Theorem 3. Diameter** parameterized by feedback edge number $k$ can be solved in $O(k \cdot n)$ time.

We will first present a data reduction rule that iteratively removes all vertices of degree one in linear time from the graph and therein introduces a weight function (for pending) on the remaining vertices. Intuitively, pen stores the length of a longest shortest path to a vertex in a “pending tree” from a vertex outside, that is, to a vertex which is removed by the reduction rule. After we removed all vertices of degree one, we will compute the diameter of the remaining vertex-weighted graph in $O(k \cdot n)$ time. This weighted problem is formally defined as follows.
**Weighted Diameter**

**Input:** An undirected, connected graph $G = (V, E)$, a weight function $\text{pen} : V \rightarrow \mathbb{N}$, and an integer $s$.

**Task:** Compute $\max\{d^{\text{pen}}(G), s\}$, where

$$d^{\text{pen}}(G) := \max_{v, w \in V}\{\text{dist}_{G}^{\text{pen}}(v, w)\} := \max_{v, w \in V}\{\text{pen}(v) + \text{dist}_{G}(v, w) + \text{pen}(w)\}.$$  

Notice that if all weights are set to 0, then the problem is the same as Diameter. We therefore start with initializing all weights to 0 and applying our reduction rule that removes degree-one vertices from the graph. The main idea of the reduction rule is simple: If a degree-one vertex $u$ is removed, then the value $\text{pen}(v)$ ($v$ is the unique neighbor of $u$) is adjusted and we store in an additional variable $s$ the length of a longest shortest path that cannot be recovered from the reduced graph. This addresses the case that a longest shortest path has both its endpoints in pending trees (trees removed by our reduction rule) that are connected to the same vertex. Initially, $s$ is set to zero. The reduction rule is defined as follows.

**Reduction Rule 1.** Let $u$ be a vertex of degree one and let $v$ be its neighbor. Delete $u$ and the incident edge from $G$, set $s = \max\{s, \text{pen}(u) + \text{pen}(v) + 1\}$ and $\text{pen}(v) = \max\{\text{pen}(u) + 1, \text{pen}(v)\}$.

We first prove the correctness of this data reduction rule.

**Lemma 2.** Reduction Rule 1 is correct, that is, $d(G) = \max\{s, d(G')\}$, where $G$ is the input graph and $G'$ is the resulting graph after Reduction Rule 1 is applied.

**Proof.** Observe that a degree-one vertex has only one neighbor so any path ending in the degree-one vertex must pass through the unique neighbor. If we delete $u$, then the pen-value of its unique neighbor $v$ is set to $\max\{\text{pen}(u) + 1, \text{pen}(v)\}$ since any longest shortest path from a pending tree that ends in $u$ now ends in $v$ and its length increases by one. We will first show that $\max(s, d^{\text{pen}}(G'))$ is at least the diameter of the original graph and afterwards show that it’s at most the diameter of the original graph.

Assume that a longest shortest path in the original graph ended in $u$ (or a pending tree adjacent to $u$). There are two possible scenarios: Either the path ended in another pending tree adjacent to $v$ that was already deleted by Reduction Rule 1 or it ends in some other part of the graph. In the latter case, the path still passes through $v$ and since it is a longest shortest path in the original graph, it holds that $\text{pen}(u) + 1 \geq \text{pen}(v)$. Hence, the length of the pass is still represented by $\text{pen}(v)$. In the former case, $\text{pen}(v)$ represents the length to the other end of the path and therefore $s = \text{pen}(u) + \text{pen}(v) + 1$ stores the diameter of the original graph.

We will now show the existence of a shortest path of length $\max(s, d^{\text{pen}}(G'))$ in $G$. If $s > 0$, then $\text{pen}(v)$ was increased before and there is therefore a path to some vertex in another pending tree adjacent to $v$ of length $\text{pen}(v)$. Hence, the distance from $u$ to this vertex is $\text{pen}(v) + 1$ and if $\text{pen}(u) > 0$, then the same argument applies to $u$ and therefore there are two vertices in the original graph that have distance $s$ in $G$. Assume that $v$ is an endpoint of a longest shortest path (with respect to $\text{dist}_{G}^{\text{pen}}$) as otherwise the longest shortest path is not affected by Reduction Rule 1. Let $w$ be the other endpoint of this path. By definition, this path is of length $\text{dist}_{G}^{\text{pen}}(v, w) = \text{pen}(v) + \text{dist}_{G}(v, w) + \text{pen}(w)$. Since $\text{dist}_{G}(v, w) = \text{dist}_{G}(v, w)$, it holds that there are two vertices in the original graph, that have distance $\text{pen}(v) + \text{dist}_{G}(v, w) + \text{pen}(w) = \text{dist}_{G}^{\text{pen}}(v, w)$ in the original graph.

Before we analyze the running time, we first present a second reduction rule that we apply after Reduction Rule 1 is not applicable anymore. Since the resulting graph has no degree-one vertices we can partition the vertex set of the remaining graph into vertices $V^{\geq 3}$ of degree exactly two and vertices $V^{\geq 3}$ of degree at least three. Using standard argumentation we can show that $|V^{\geq 3}| \in O(\min\{k, n\})$ and all vertices in $V^{\geq 2}$ are either in pending cycles or in maximal paths [5, Lemma 1]. A maximal path is an induced subgraph $P = x_0x_1 \ldots x_a$ where $\{x_i, x_{i+1}\} \in E$ for all $0 \leq i < a$, where
Let \( x_0, x_a \in V^\geq 3 \), \( x_i \in V^\leq 2 \) for all \( 0 < i < a \), and \( x_0 \neq x_a \). A pending cycle is basically the same except \( x_0 = x_a \) and \( \deg (x_0) \) may possibly be two. The set \( C \) of all pending cycles and \( P \) of maximal paths can be computed in \( O(n + m) \) time \cite{5, Lemma 2}. The second reduction rule works similar to Reduction Rule 1, but instead of deleting degree-one vertices, it removes pending cycles.

**Reduction Rule 2.** Let \( C = x_0 x_1 \ldots x_a \) be a pending cycle. Let \( x_k \) be the vertex that maximizes \( \text{pen}(x_k) + \text{dist}(x_0, x_k) \) in \( C \). Delete all vertices in \( C \) except for \( x_0 \) (and all incident edges) from \( G \), set \( s = \max \{ s, \text{dpen}(C) \} \) and \( \text{pen}(x_0) = \max \{ \text{pen}(x_k) + \text{dist}(x_0, x_k), \text{pen}(x_0) \} \).

We first prove the correctness of this data reduction rule and then analyze the running time of Reduction Rules 1 and 2.

**Lemma 3.** Reduction Rule 2 is correct, that is, \( \text{dpen}(G) = \max(s, \text{dpen}(G')) \), where \( G \) is the input graph and \( G' \) is the resulting graph after Reduction Rule 2 is applied and given a pending cycle \( C = x_0 x_1 \ldots x_a \), Reduction Rule 2 can be applied in \( O(a) \) time.

**Proof.** Let \( C = x_0 x_1 \ldots x_a \) be a pending cycle. If \( s > \text{dpen}(G') \), then there is no shortest path of length \( s \) in \( G' \). Since \( G \) and \( G' \) only differ in \( C \), it suffices to show that there is a shortest path of length \( s \) in \( G \) and that there is no longer path that starts in \( C \). By construction, there is a pair of vertices \( x_i, x_j \) in \( C \) such that \( \text{dpen}(x_i, x_j) = s \). Now assume that there is a shortest path of length at least \( s + 1 \) in \( G \) that starts in \( C \). By construction the path has to end outside of \( C \) as otherwise \( s \) would be larger. Let \( v \) be the other endpoint of the path. Then, \( \text{dpen}(G') \geq \text{dpen}(x_0, v) > s \) — a contradiction. We will show that if \( s \leq \text{dpen}(G') \), then \( \text{dpen}(G') = \text{dpen}(G') \). By the construction of \( \text{pen}(x_0) \), the length of any path in \( G \) that starts in \( C \) and ends outside in some vertex \( v \) is upper-bounded by \( \text{dpen}(x_0, v) \) and for every path in \( G' \) that ends in \( x_0 \), there is a path of the same length in \( G \). Lastly, there is no shortest path in \( G \) both starting and ending in \( C \) of length at least \( \text{dpen}(G') + 1 \) as otherwise \( s > \text{dpen}(G') \).

We will now show how to apply Reduction Rule 2 in \( O(a) \) time. We first start with those paths that start and end in \( C \) and compute their maximum length to compute \( s \). To this end in \( O(a) \) time we compute \( i \) such that \( \text{dist}(x_i, x_0) + \text{pen}(x_i) \) is maximized and if \( i \) \( \neq 0 \), then we set \( s = \max \{ s, \text{pen}(x_0) + \text{dist}(x_i, x_0) + \text{pen}(x_i) \} \). (For \( i = 0 \) we do not update \( s \).) Next we use a dynamic program that given the furthest vertex \( x_j \) from some vertex \( x \) (with respect to \( \text{dist}(x_j, x_i) + \text{pen}(x_j) \) computes the furthest vertex \( x_j \) from \( x_{i+1} \). To this end, we only consider the shortest paths \( x_{i+1} x_{i+2} \ldots x_r \) for \( \ell + 1 \leq r \leq \ell + 1 + [a/2] \) mod \( a \), that is, we only consider shortest paths to half of the vertices in \( C \). Observe that all paths that are ignored by this restriction will be considered in the iteration where \( x_j \) is considered and \( x_{i+1} \) is the vertex furthest away. Notice that the furthest vertex \( x_j \) from \( x_{i+1} \) is either the furthest vertex from \( x_i \) or some vertex that is ignored by \( x_j \) since all distances that are considered are exactly one larger for \( x_j \) than they are for \( x_{i+1} \). The only possible vertex that is ignored by \( x_j \) but not by \( x_{i+1} \) is \( x_{i+1} + [a/2] \) mod \( a \). Thus we can compute the furthest vertex from \( x_{i+1} \) in constant time and update \( s \) accordingly. (Note that if the furthest vertex is \( x_{i+1} \), then we do not update \( s \)) The whole pending cycle can be checked in \( O(a) \) time in this way.

Notice that all paths that leave \( C \) contain \( x_0 \) and therefore only the distance to \( x_0 \) and the pen-value matter. Thus we can use \( i \) (recall, this is the index that maximizes \( \text{dist}(x_i, x_0) + \text{pen}(x_i) \), possibly \( i = 0 \)) to update \( \text{pen}(x_0) = \text{dist}(x_i, x_0) + \text{pen}(x_i) \) and delete \( C \) from \( G \).

We now analyze the running time of both reduction rules.

**Lemma 4.** Reduction Rules 1 and 2 can be exhaustively applied in \( O(n + m) \) time.

**Proof.** Notice that we can sort all vertices by their degree in linear time using bucket sort. Applying Reduction Rule 1 or Reduction Rule 2 takes constant time per deleted vertex. After applying a reduction rule, we adjust the degree of the remaining vertex (either the unique neighbor of a degree-one vertex or the high-degree vertex in a pending cycle) in constant time by moving it to the appropriate bucket. Note that applying Reduction Rule 2 can lead to a new vertex of degree one and an application of Reduction Rule 1 can lead to two maximal paths merging to either
Figure 3: An illustration of the different subcases in Case 3 in the proof of Theorem 3. For \( x_1 \) the third subcase applies, that is, shortest paths to \( y_3 \) pass through \( x_0 \) (solid blue path) and shortest paths to \( y_5 \) pass through \( x_a \) (dashed brown paths). Note that for \( x_3 \) the second subcase applies as both the shortest path to \( y_0 \) and shortest paths to \( y_6 \) pass through \( x_a \).

a longer maximal path or a pending cycle. Since these cases can be detected in constant time and each vertex is only removed once, the overall running time to apply Reduction Rules 1 and 2 exhaustively is in \( O(n + m) \).

We now present the algorithm that computes the maximum \( \text{dist}^{\text{pen}}(u, v) \) over all pairs of remaining vertices \( u, v \) after applying Reduction Rules 1 and 2 exhaustively. This algorithm distinguishes between three different cases: The longest shortest path has at least one endpoint in \( V^{\geq 3} \) (Case 1), its two endpoints are in the same maximal path (Case 2), or its endpoints are in two different maximal paths (Case 3).

**of Theorem 3.** Let \( G = (V, E) \) be the input graph with feedback edge number \( k \) and let \( K \) be a feedback edge set with \( |K| = k \).

**Case 1:** First we perform a breadth-first search (BFS) from each vertex \( v \in V^{\geq 3} \) and store for each vertex \( u \in V \setminus \{v\} \) the distance \( \text{dist}(v, u) \) and update \( s = \max\{s, \text{pen}(v) + \text{pen}(u) + \text{dist}(v, u)\} \). This way we find all shortest paths that start or end in a vertex in \( V^{\geq 3} \) (or a pendant tree connected to such a vertex).

**Case 2:** This case is similar to the case of pending cycles (see Reduction Rule 2). The only adjustment is the computation of the index that is considered by \( x_{\ell+1} \) but not by \( x_\ell \). For a maximal path \( P = x_0x_1\ldots x_\ell \) the respective index for \( x_{\ell+1} \) is \( \ell + 1 + \left\lfloor \frac{\text{dist}(x_0, x_\ell)}{2} \right\rfloor \) if \( \ell + 1 < a/2 \) and \( \ell + 1 - \left\lfloor \frac{\text{dist}(x_0, x_\ell)}{2} \right\rfloor \) otherwise. If this index is not within \([1, a-1]\), then it can safely be ignored as there exists a vertex which is further away than any vertex within the maximal path and therefore the diameter of the graph is still computed correctly.

**Case 3:** The last remaining case is where \( v \) and \( u \) are in two different maximal paths. Let \( P_1 = x_0x_1\ldots x_a \) and \( P_2 = y_0y_1\ldots y_b \) be two different maximal paths. We compute the longest shortest path between \( v \) and \( u \) where \( v \in \{x_1, \ldots, x_{a-1}\} \) and \( u \in \{y_1, \ldots, y_{b-1}\} \) in \( O(a+b) \) time as follows: We use two dynamic programs to sweep over \( P_1 \). One computes the furthest distance of each vertex with the additional requirement, that a shortest path passes through \( x_a \) and the other one with the requirement that a shortest path passes through \( x_0 \). Since they are completely symmetric, we will only describe the first one here. To this end, we first compute the index \( i \) that maximizes \( \text{pen}(y_i) + \text{dist}(x_i, y_i) \) where a shortest path passes through \( x_a \). This can be done by computing for each vertex \( y_i \) the distances \( \text{dist}(x_a, y_i) = \min(\text{dist}(x_a, y_0) + c, \text{dist}(x_a, y_b) + b - c) \).

For each vertex \( x_i \in V(P_1) \) there are three cases (see Figure 3): There exist shortest paths to all vertices in \( P_2 \) that pass through \( x_0 \), there exists shortest paths to all vertices in \( P_2 \) that pass through \( x_a \) or shortest paths to some vertices pass through \( x_0 \) and to other vertices they...
Lemma 4

Definition 1

is the furthest vertex. Analogously, for all vertices to which the second case applies, the anchor of this index where the distance through $x_0$ is larger than through $a$. The vertex $y_j$ is the anchor for our dynamic program. Now the main idea is still the same as for pending cycles—the furthest vertex either stays the same or is ignored by $x_t$. The distance from $x_t$ to this vertex $y_j$ through $x_0$ is larger than through $x_0$ and for $x_{t+1}$ it is at most as large as through $x_0$. There are two possible cases: The shortest path from $x_t$ to $y_0$ passes through $x_0$ or through $x_0$. Since both cases are symmetric, we will only consider the first one. The index $j$ for $x_{t+1}$ is then computed by $j = \ell + 1 + \left\lfloor (b + \text{dist}(x_0, y_0) - a - \text{dist}(x_0, y_0))/2 \right\rfloor$ since this is the index where the distance through $x_0$ becomes smaller than the distance through $x_0$. We again also consider the pen-values and update $s$ accordingly.

It remains to analyze the running time of this algorithm. All of the preprocessing steps can be done in $O(n + m) = O(n + k)$ time. To deal with vertices in $V^{\geq 3}$ we run a BFS from $O(\min\{n, k\})$ vertices which takes $O(\min\{n, k\} \cdot (2n + k)) = O(n \cdot k)$ time. The time to compute all the dynamic programs for the cases where $u$ and $v$ are in the same maximal path or pending cycle is $O(\sum_{x_i \in P} |S_i|) \subseteq O(|V^2|) \subseteq O(n)$. The time to compute length of longest shortest path between vertices in pairs of maximal paths is in $O(\sum_{P_1 \in P} \sum_{P_2 \in P} |P_1| + |P_2|) = O(\sum_{P_1 \in P} (|P| |P_1| + \sum_{P_2 \in P} |P_2|)) \subseteq O((\sum_{P_1 \in P} |P||P_1| + \sum_{P_2 \in P} n)) \subseteq O(n \cdot k)$. Combining this with Lemma 4 concludes the proof.

4 Parameters for Social Networks

Here, we study parameters that we expect to be small in social networks. Recall that social networks have the “small-world” property and a power-law degree distribution [28, 30–33]. The “small-world” property directly transfers to the diameter. We capture the power-law degree distribution by the $h$-index as only few high-degree exist in the network. Thus, we investigate parameters related to the diameter and to the $h$-index starting with degree-related parameters.

4.1 Degree Related Parameters

We next investigate the parameter minimum degree. Unsurprisingly, the minimum degree is not helpful for parameterized algorithms. In fact, we show that DIAMETER is 2-GP-hard with respect to the combined parameter bisection width and minimum degree. The bisection width of a graph $G$ is the minimum number of edges to delete from $G$ in order to partition $G$ into two connected components whose number of vertices differ by at most one.

Theorem 4. DIAMETER is 2-GP-hard with respect to bisection width and minimum degree.

Proof. Let $G = (V, E)$ be an arbitrary input graph for DIAMETER where $V = \{v_1, v_2, \ldots, v_n\}$ and let $d$ be the diameter of $G$. We construct a new graph $G' = (V', E')$ with diameter $d$ as follows: Let $V' = \{s_i, t_i, u_i | i \in \{n\}\} \cup \{w_i | i \in \{3n\}\}$ and $E' = T \cup W \cup E''$, where $T = \{\{s_i, t_i\}, \{t_i, u_i\} | i \in \{n\}\}$, $W = \{u_1, w_1\} \cup \{\{w_i, w_{i+1}\} | i \in \{3n\} \setminus \{1\}\}$, and $E'' = \{\{u_i, u_j\} | \{v_i, v_j\} \in E\}$.

An example of this construction can be seen in Figure 4. We will now prove that all properties of Definition 1 hold. It is easy to verify that the reduction runs in linear time and that there
are 6n vertices and 5n + m edges in $G'$. Notice that $\{s_i, t_i, u_i \mid i \in [n]\}$ and $\{w_i \mid i \in [3n]\}$ are both of size 3n and that there is only one edge ($\{u_1, w_1\}$) between these two sets of vertices. The bisection width of $G'$ is therefore one and the minimum degree is also one as $s_1$ is only adjacent to $t_1$.

It remains to show that $G'$ has diameter $d + 4$. First, notice that the subgraph of $G'$ induced by $\{u_i \mid i \in [n]\}$ is isomorphic to $G$. Note that $\text{dist}(s_i, u_i) = 2$ and $\text{dist}(s_i, s_j) = \text{dist}(u_i, u_j) + 4 = \text{dist}(v_i, v_j) + 4$ and therefore the diameter of $G'$ is at least $d + 4$. Third, notice that for all vertices $x \in V' \setminus \{s_i\}$ it holds that $\text{dist}(s_i, x) > \text{dist}(t_i, x)$. Lastly, observe that for all $i \in [3n]$ and all vertices $x \in V'$ it holds that $\text{dist}(v_i, x) \leq \max\{\text{dist}(s_1, x), 4\}$. Thus the longest shortest path in $G'$ is between two vertices $s_i, s_j$ and is of distance $\text{dist}(u_i, u_j) + 4 = \text{dist}(v_i, v_j) + 4 \leq d + 4$. □

We mention in passing, that the constructed graph in the proof of Theorem 4 contains the original graph as an induced subgraph and if the original graph is bipartite, then so is the constructed graph. Thus, first applying the construction in the proof of Theorem 1 (see also Figure 2) and then the construction in the proof of Theorem 4 proves that DIA METER is GP-hard even parameterized by the sum of girth, bisection width, minimum degree, and distance to bipartite graphs.

4.2 Parameters related to both diameter and $h$-index

Here, we will study combinations of two parameters where the first one is related to diameter and the second to $h$-index (see Figure 1 for an overview of closely related parameters). We start with the combination maximum degree and diameter.

Interestingly, although the parameter is quite large, the naive algorithm behind Observation 7 cannot be improved to a fully polynomial running time.

**Theorem 5.** There is no $(d + \Delta)^{O(1)}(n + m)^{2-\epsilon}$-time algorithm that solves DIA METER parameterized by maximum degree $\Delta$ and diameter $d$ unless the SETH is false.

**Proof.** We prove a slightly stronger statement excluding $2^{O(\sqrt{d+\Delta})}(n + m)^{2-\epsilon}$-time algorithms for some constant $c$. Assume towards a contradiction that for each constant $r$ there is a $2^{O(\sqrt{d+\Delta})}(n + m)^{2-\epsilon}$-time algorithm that solves DIA METER parameterized by maximum degree $\Delta$ and diameter $d$. Evald and Dahlgaard [16] have shown a reduction from CNF-SAT to DIA METER where the resulting graph has maximum degree three such that for any constant $\epsilon > 0$ an $O((n + m)^{2-\epsilon})$-time algorithm (for DIA METER) would refute the SETH. A closer look reveals that there is some constant $c$ such that the diameter $d$ in their constructed graph is in $O(\log^n(n + m))$. By assumption we can solve DIA METER parameterized by maximum degree and diameter in $2^{O(\sqrt{d+\Delta})}(n + m)^{2-\epsilon}$.
time. Observe that
\[
2^{o\left(\sqrt{n+m}\right)} \cdot (n + m)^{2-\varepsilon} = 2^{o\left(\sqrt{\log^2(n+m)}\right)} \cdot (n + m)^{2-\varepsilon} \\
= (n + m)^{o(1)} \cdot (n + m)^{2-\varepsilon} \subseteq O((n + m)^{2-\varepsilon}) \text{ for some } \varepsilon' > 0.
\]

Since we constructed for some \(\varepsilon' > 0\) an \(O((n + m)^{2-\varepsilon'})\)-time algorithm for \textsc{Diameter} the SETH fails and thus we reached a contradiction. Finally, notice that \((d + \Delta)^{O(1)} \subseteq 2^{o\left(\sqrt{n+m}\right)}\) for any constant \(c\).

\subsection*{h-index and diameter}

We next investigate in the combined parameter \(h\)-index and diameter. The reduction by Roditty and Williams \cite{RodW18} produces instances with constant domination number and logarithmic vertex cover number (in the input size). Since the diameter \(d\) is upper-bounded by the domination number and the \(h\)-index \(h\) is upper-bounded by the vertex cover number, any algorithm that solves \textsc{Diameter} parameterized by the combined parameter \((d+h)\) in \(2^{o(d+h)} \cdot (n + m)^{2-\varepsilon}\) time disproves the SETH. We will now present an algorithm for \textsc{Diameter} parameterized by \(h\)-index and diameter that almost matches the lower bound.

\begin{theorem}
\textsc{Diameter} parameterized by diameter \(d\) and \(h\)-Index \(h\) is solvable in \(O(h \cdot (m + n) + n \cdot d \cdot 2^{h \log d + d \log h})\) time.
\end{theorem}

\begin{proof}
Let \(H = \{x_1, \ldots, x_h\}\) be a set of vertices such that all vertices in \(V \setminus H\) have degree at most \(h\) in \(G\). Clearly, \(H\) can be computed in linear time. We will describe a two-phase algorithm with the following basic idea: In the first phase it performs a breadth-first search from each vertex \(v \in H\), stores the distance to each other vertex and uses this to compute the “type” of each vertex, that is, a characterization by the distance to each vertex in \(H\). In the second phase it iteratively increases a value \(e\) and verifies whether there is a vertex pair of distance at least \(e\). If at any point no vertex pair is found, then the diameter of \(G\) is \(e - 1\).

The first phase is straightforward: Compute a BFS from each vertex \(v \in H\) and store the distance from \(v\) to every other vertex \(w\) in a table. Then iterate over each vertex \(w \in V \setminus H\) and compute a vector of length \(h\) where the \(i\)th entry represents the distance from \(w\) to \(x_i\). Also store the number of vertices of each type. Since the distance to any vertex is at most \(d\), there are at most \(d^h\) different types. This first phase takes \(O(h \cdot (m + n))\) time.

For the second phase, we initialize \(e\) with the largest distance found so far, that is, the maximum value stored in the table and compute \(G' = G - H\). Iteratively check whether there is a pair of vertices in \(V \setminus H\) of distance at least \(e + 1\) as follows. We check for each vertex \(v \in V \setminus H\) and each type whether there is a path of length at most \(e\) from \(v\) to each vertex of this type through a vertex in \(H\). This can be done by computing the sum of the two type-vectors in \(O(h)\) time and comparing the minimum entry in this sum with \(e\). If all entries are larger than \(e\), then no shortest path from \(v\) to some vertex \(w\) of the respective type of length at most \(e\) can contain any vertex in \(H\). Thus we compute a BFS from \(v\) in \(G'\) up to depth \(e\) and count the number of vertices of the respective type we found. If this number equals the total number of vertices of the respective type, then for all vertices \(w\) of this type it holds that \(\text{dist}(v, w) \leq e\). If the two numbers do not match, then there is a vertex pair of distance at least \(e + 1\) so we can increase \(e\) by one and start the process again.

There are at most \(d\) iterations in which \(e\) is increased and the check is done. Recall that the maximum degree in \(G'\) is \(h\) and therefore each of these iterations takes \(O(n \cdot d^h \cdot (h^e + h))\) time as each BFS to depth \(e\) takes \(O(h^e)\) time. Thus, the overall running time is in \(O(h \cdot (m + n) + n \cdot d \cdot 2^{h \log d + d \log h})\).
\end{proof}

\subsection*{Acyclic chromatic number and domination number}

We next analyze the parameterized complexity of \textsc{Diameter} parameterized by acyclic chromatic number \(a\) and domination number \(d\). Since the acyclic chromatic number upper-bounds the average degree the standard \(O(n \cdot m)\)-time algorithm runs in \(O(n^2 \cdot a)\) time. We will show that this is essentially the best one can hope for as we can exclude \(f(a, d) \cdot (n + m)^{2-\varepsilon}\)-time algorithms under SETH. Our result is based on the
Figure 5: A schematic illustration of the construction in the proof of Theorem 7. Note that the resulting graph has acyclic chromatic number four ($V_1 \cup V_2 \cup B, S_1 \cup S_2 \cup \{t_1, t_2\}, \{t_3\}$ and \{t_4\}, also represented by colors) and a dominating number four ($\{t_1, t_2, t_3, t_4\}$).

reduction by Roditty and Williams [34] and is modified such that the acyclic chromatic number and domination number are both four in the resulting graph.

**Theorem 7.** There is no $f(a, d) \cdot (n + m)^{2-\varepsilon}$-time algorithm for any computable function $f$ that solves Diameter parameterized by acyclic chromatic number $a$ and domination number $d$ unless the SETH is false.

**Proof.** We provide a reduction from CNF-SAT to DIAMETER where the input instance has constant acyclic chromatic number and domination number and such that an $O((n + m)^{2-\varepsilon})$-time algorithm refutes the SETH. Since the idea is the same as in Roditty and Williams [34] we refer the reader to their work for more details. Let $\phi$ be a CNF-SAT instance with variable set $W$ and clause set $C$. Assume without loss of generality that $|W|$ is even. We construct an instance $(G = (V, E), k)$ for DIAMETER as follows:

Randomly partition $W$ into two set $W_1, W_2$ of equal size. Add three sets $V_1, V_2$ and $B$ of vertices to $G$ where each vertex in $V_1$ (in $V_2$) represents one of $2^{|W_1|} = 2^{|W_2|}$ possible assignments of the variables in $W_1$ (in $W_2$) and each vertex in $B$ represents a clause in $C$. Clearly $|V_1| + |V_2| = 2 \cdot 2^{|W|/2}$ and $|B| = |C|$. For each $v_i \in V_1$ and each $u_j \in B$ we add a new vertex $s_{ij}$ and the two edges $\{v_i, s_{ij}\}$ and $\{u_j, s_{ij}\}$ to $G$ if the respective variable assignment does not satisfy the respective clause. We call the set of all these newly introduced vertices $S_1$. Now repeat the process for all vertices $w_i \in V_2$ and all $u_j \in B$ and call the newly introduced vertices $q_{ij}$ and the set $S_2$.

Finally we add four new vertices $t_1, t_2, t_3, t_4$ and the following sets of edges to $G$: $\{\{t_1, v\} \mid v \in V_1\}, \{\{t_2, s\} \mid s \in S_1\}, \{\{t_3, q\} \mid q \in S_2\}, \{\{t_4, w\} \mid w \in V_2\}, \{\{t_2, b\}, \{t_3, b\} \mid b \in B\}$, and $\{\{t_1, t_2\}, \{t_2, t_3\}, \{t_3, t_4\}\}$. See Figure 5 for a schematic illustration of the construction.

We will first show that $\phi$ is satisfiable if and only if $G$ has diameter five and then show that both the domination number and acyclic chromatic number of $G$ are four. First assume that $\phi$ is satisfiable. Then, there exists some assignment $\beta$ of the variables such that all clauses are satisfied, that is, the two assignments of $\beta$ with respect to the variables in $W_1$ and $W_2$ satisfy all clauses. Let $v_1 \in V_1$ and $v_2 \in V_2$ be the vertices corresponding to $\beta$. Thus for each $b \in B$ we
have dist(v_1, b) + dist(v_2, b) ≥ 5. Observe that all paths from a vertex in V_1 to a vertex in V_2 that do not pass a vertex in B pass through t_2 and t_3. Since dist(v_1, t_3) = 3 and dist(v_2, t_3) = 3 it follows that dist(v_1, v_2) = 5. Observe that the diameter of G is at most five since each vertex is connected to some vertex in \{t_1, t_2, t_3, t_4\} and these four are of pairwise distance at most three.

Next assume that the diameter of G is five. Clearly there is a shortest path between a vertex \(v_i \in V_1\) and \(v_j \in V_2\) of length five. Thus there is no path of the form \(v_i, u_h, q_h, w_j\) for any \(u_h \in B\). This corresponds to the statement that the variable assignment of \(v_i\) and \(w_j\) satisfy all clauses and therefore \(\phi\) is satisfiable.

The domination number of G is four since \{t_1, t_2, t_3, t_4\} is a dominating set. The acyclic chromatic number of G is at most four as \(V_1 \cup V_2 \cup B, S_1 \cup S_2 \cup \{t_1, t_2\}, \{t_3\}\) and \{t_4\} each induce an independent set.

Now assume that we have an \(O(f(k) \cdot (n+m)^{2-\epsilon})\)-time algorithm for DIAMETER parameterized by domination number and acyclic chromatic number. Since the constructed graph has \(O(2^{|W|/2} \cdot |C|)\) vertices and edges, this would imply an algorithm with running time

\[
O(f(8) \cdot (2^{|W|/2} \cdot |C|)^{2-\epsilon}) = O(2^{(|W|/2)(2-\epsilon)} \cdot |C|(2-\epsilon)) = O(2^{|W|(1-\epsilon/2)} \cdot |C|(2-\epsilon)) = 2^{|W|(1-\epsilon')} \cdot (|C| + |W|)^{O(1)}
\]

for some \(\epsilon' > 0\).

Hence, such an algorithm for DIAMETER would refute the SETH.

\[\square\]

5 Conclusion

We have resolved the complexity status of DIAMETER for most of the parameters in the complexity landscape shown in Figure 1. Several open questions remain. For example, is there an \(f(k)n^2\)-time algorithm with respect to the parameter diameter? Moreover, our algorithms working with parameter combinations have mostly impractical running times which, assuming SETH, cannot be improved by much. So the question arises, whether there are parameters \(k_1, \ldots, k_\ell\) that allow for practically relevant running times like \(\prod_{i=1}^{\ell} k_i \cdot (n + m)\) or even \((n + m) \cdot \sum_{i=1}^{\ell} k_i\)? The list of parameters displayed in Figure 1 is by no means exhaustive. Hence, the question arises which other parameters are small in typical scenarios? For example, what is a good parameter capturing the special community structures of social networks [21].

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