Optimizing Reweighted Belief Propagation for Likelihood Consensus Problems

Christopher Lindberg, Julien Hendrickx, and Henk Wymeersch

Abstract—Belief propagation (BP) is a powerful tool to solve distributed inference problems, though it is limited by short cycles in the corresponding factor graph. Such cycles may lead to incorrect solutions or oscillatory behavior. Only for certain types of problems are convergence properties understood. We extend this knowledge by investigating the use of reweighted BP for likelihood consensus problems, which are characterized by equality constraints along possibly short cycles. Through a linear formulation of BP, we are able to analytically derive convergence conditions for certain types of graphs and optimize the convergence speed. We compare with standard belief consensus and observe significantly faster convergence.

I. INTRODUCTION

Belief propagation (BP) [1], [2] is a message-passing algorithm for approximate inference on graphs of problems that arise in many different fields such as statistical physics, computer vision, artificial intelligence, optimization, behavioral modeling in social networks, and wireless communications [3]–[6]. Examples of applications to wireless communications include detection problems, localization and tracking, and decoding [7]–[10]. One of the more notable applications is iterative decoding algorithms for capacity-approaching error-correcting codes, including LDPC and turbo codes. Since BP is a message-passing algorithm, it is also suitable for solving distributed problems in networks of cooperating nodes. Examples include distributed cooperative decision making in cognitive radio [11], distributed cooperative localization and/or tracking [8], [12], network synchronization [13], distributed joint source channel decoding [14], and distributed compressed sensing [15].

While BP generally works well in practice, convergence can in general not be guaranteed. This phenomenon is especially apparent on graphs that have cycles with strong interactions, with extreme case equality constraints, which force variables to maintain the same value along a cycle in the graph. An example of such a setting is the likelihood consensus problem, where nodes in a network must agree on a global likelihood function, based on locally available, mutually independent observations. To mitigate the convergence issues for such problems, one can apply a variation of BP [16]–[21] or apply methods from the field of distributed consensus [22], [23]. In the first class, [16], [17] introduced the tree-reweighted BP (TRW-BP), which optimizes convex combinations of cycle-free graphs (tree-graphs) to represent the original graph problem, leading to promising performance at a cost of solving a high-dimensional optimization problem over spanning trees. The uniformly reweighted BP (URW-BP) algorithm [18], [19] is a special case of TRW-BP that involves optimization over one parameter, lending itself well for implementation in network settings, or where computational efficiency is prioritized. URW-BP variations were applied to improve decoding performance of LDPC codes in [20], [21]. In the second class, likelihood consensus is solved using distributed consensus methods, leading to approaches commonly termed belief consensus: [22] proposes a distributed consensus method, whereby the convergence speed depends on a single scalar parameter, which depends on the maximum node degree. A fast version of such belief consensus was proposed in [23] using Metropolis-type weights, which can be locally computed. However, such consensus methods are generally slow on tree graphs for which BP works well.

In this paper, we cast URW-BP as a linear system (similar to the linear BP expressions in [24]), allowing eigen-analysis. Our contributions are summarized in three parts as follows: (i) We show that for a certain class of network inference problems (i.e., likelihood consensus problems) and certain network topologies (i.e., trees, k-regular graphs, and variations of the latter), both belief consensus and URW-BP can achieve convergence to the correct beliefs; (ii) In such cases, we can analytically optimize the URW-BP parameter to maximize the convergence rate, outperforming belief consensus; (iii) As a side-result, we recover a new way to prove the finite-time convergence of BP on trees.

The remainder of the paper is organized as follows: In Section II we formalize the distributed likelihood consensus problem. Section III introduces the algorithms which are used to solve the problem. Section IV deals with the tools we use to analyze the convergence behavior of these algorithms. In Section V, we present the convergence analysis of the algorithms on tree graphs and k-regular graphs respectively. In Section VI we present results from numerical simulations, and some discussion of those. We conclude the paper in Section VII.

Notation

We use boldface lowercase letters $\mathbf{x}$ for column vectors, and boldface uppercase letters $\mathbf{X}$ for matrices. In particular, $\mathbf{I}_M$
denotes an $M \times M$ identity matrix, $O_M$ denotes an $M \times M$ all zero matrix, $1$ is the all one vector of appropriate size, and $0$ is the all zero vector of appropriate size. Sets are described by calligraphic letters $X$ and the cardinality of a set is denoted by $|X|$. The transpose of a vector is denoted by $[.]^T$. The indicator function of a statement $P$ is written as $\mathbb{1}_{\{P\}} \in \{0,1\}$. We denote by $\sum_{x_i}$ $f(x)$ the summation over all elements in $x$, except $x_i$.  

II. PROBLEM FORMULATION  

We consider a network consisting of $N$ connected nodes which we model by an undirected graph $G = (V,E)$, where $V$ is the set of nodes and $E$ is the set of edges connecting the nodes. Associated with the graph $G$ is the adjacency matrix $A$ with entries $A_{ij} = \mathbb{1}_{\{(i,j) \in E\}}$, the degree matrix $D = \text{diag}(A1)$, and the Laplacian matrix $L = D - A$. For later use, let $\mu_1, \ldots, \mu_N$ be the eigenvalues of $A$ sorted such that $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_N|$. We consider three types of graphs:  

(i) Tree-graphs: The set of edges $E$ connects all the vertices (nodes) in $V$ such that there are no cycles. Nodes connected to exactly one node are called leaves.  

(ii) k-regular graphs: All nodes are connected to exactly $k$ other nodes.  

(iii) General connected graphs: There is no constraint on the edge set, provided the graph is connected.  

The aim of the network is to determine the posterior distribution over a variable $\theta$ given independent local observations $y_n$, at each node $n$. Hence, each node has access to a local likelihood function $p(y_n|\theta)$ where the likelihood functions are conditionally independent given $\theta$, and it is also assumed that each node knows the prior distribution $p(\theta)$. The posterior distribution can be factorized as  

$$p(\theta|y_1, \ldots, y_N) \propto p(\theta) \prod_{m=1}^{N} p(y_m|\theta), \quad (1)$$  

or equivalently in the log-domain as  

$$\log p(\theta|y_1, \ldots, y_N) \propto \log p(\theta) + \sum_{m=1}^{N} \log p(y_m|\theta). \quad (2)$$  

We assume that $\theta$ is a discrete random variable that can only take on $K$ distinct values.  

III. TWO SOLUTION APPROACHES  

In this section, we describe techniques that can be used to compute the posterior distribution from the local likelihood functions at each node in a distributed manner: belief consensus and belief propagation.  

A. Belief Consensus  

The problem in (2) can be solved by reaching consensus on the average of the log-likelihood functions, and multiplying the consensus value by the number of nodes. The belief consensus algorithm aims to compute the consensus value by letting the nodes iteratively exchange information with their neighbors and updating their state according to an update rule specified by the algorithm. Let the initial state of the consensus algorithm of node $n$ be its local likelihood function, i.e., $x_n^{(0)}(\theta) = \log p(y_n|\theta)$. The network updating dynamics are described by  

$$x_n^{(t)}(\theta) = W x_n^{(t-1)}(\theta), \quad (3)$$  

where $W$ is an appropriately chosen matrix, with $W_{nm} = 0$ when $(m,n) \notin E$. Examples include Metropolis weighting, where weighting is decided by all nodes determining its outgoing weights and self-weight by  

$$W_{nm} = \begin{cases} 1/(\max\{|N_n|,|N_m|\}+1) & (n,m) \in E \\ 1-\sum_{u \in N_n} W_{nu} & m=n \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$  

or uniform-weight consensus, where  

$$W_{nm} = \begin{cases} \xi & (n,m) \in E \\ 1-\xi |N_n| & m=n \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$  

If $\xi$ is chosen as $0 < \xi < 1/\max m |N_m|$, then $W$ in either (4) or (5) is a doubly stochastic matrix with one eigenvalue 1 (with corresponding normalized eigenvector $1/\sqrt{N}$), while all other eigenvalues are strictly smaller than 1 in absolute value. Hence, the convergence rate of belief consensus is determined by the second largest eigenvalue of $W$. Moreover, it can be shown that for any node $n$  

$$\lim_{t \to \infty} x_n^{(t)}(\theta) = \frac{1}{N} \sum_{m=1}^{N} x_n^{(0)}(\theta) = \frac{1}{N} \sum_{m=1}^{N} \log p(y_m|\theta), \quad (6)$$  

from which after multiplication with $N$, adding $\log p(\theta)$ and taking exponentials, $p(\theta|y)$ can be determined at each node.  

B. Uniformly Reweighted Belief Propagation Consensus  

When expressing (1) as a factor graph, we obtain a graph with a star topology, irrespective of the network graph. This is shown in Fig. 1: Fig. 1–(a) shows a network graph and Fig. 1–(b) shows the corresponding factor graph. Thus the structure of the factor graph does not match the node graph $G$. In order to obtain a factor graph that matches the topology of the graph in Fig. 1–(a), we introduce  

$$f(\theta_1, \ldots, \theta_N) = \prod_{m=1}^{N} p(y_m|\theta_m) \prod_{(m,n) \in E} \mathbb{1}_{\{\theta_m = \theta_n\}}, \quad (7)$$  

which is shown in Fig. 1–(c). The marginal of this function with respect to $\theta_m$ is given by  

$$f_m(\theta_m) = \sum_{\sim \theta_m} f(\theta_1, \ldots, \theta_N) = \prod_{i=1}^{N} p(y_i|\theta_m), \quad (8)$$  

so that for any $\theta$, $f_m(\theta) = f_n(\theta)$, allowing every node to determine the posterior by multiplying $f_m(\theta)$ with $p(\theta)$. The functions $f_m(\theta)$ can be computed using message-passing algorithms, such as BP or URW-BP. The initial belief of node
for \( \ell > 1 \) we find (see Appendix I)

\[
\rho \text{ of UR W-BP (where standard BP corresponds to convergence behavior depends on how the power series described in terms of a matrix-vector multiplication, the UR W-BPC matrix we call the resulting algorithm uniformly reweighted belief resembling a consensus algorithm. Defining the update rule in matrix form for UR W-BPC for } \ell > 1, \text{ where } \rho \in (0, 1) \text{ is the reweighting parameter of UR-BP (where standard BP corresponds to } \rho = 1). \text{ We call the resulting algorithm uniformly reweighted belief propagation consensus (URW-BPC), due to its linear update resembling a consensus algorithm. Defining the } 2N \times 2N \text{ UR-BPC matrix}
\]

\[
P_\rho = \begin{bmatrix}
\rho A & I_N - \rho D \\
I_N & O_N
\end{bmatrix},
\]

the update rule in matrix form for URW-BPC for } \ell > 1 \text{ is}

\[
x^{(\ell)}(\theta) = \begin{bmatrix} I_N & O_N \end{bmatrix} P_\rho^{\ell-1} \begin{bmatrix} \rho A + I_N \\ I_N \end{bmatrix} x^{(0)}(\theta),
\]

for } \ell > 1 \text{}.

The convergence behavior depends on the power series of the update matrix } P_\rho.

Remark 1. We note that, due to (9), it holds that

\[
x^{(\ell)}(\theta) = \begin{bmatrix} I_N & O_N \end{bmatrix} P_\rho^{\ell-1} \begin{bmatrix} \rho A + I_N \\ I_N \end{bmatrix} x^{(0)}(\theta),
\]

which can equivalently be expressed as

\[
x^{(\ell)}(\theta) = \begin{bmatrix} I_N & O_N \end{bmatrix} P_\rho^{\ell-1} (P_\rho + I_{2N}) \begin{bmatrix} x^{(0)}(\theta) \\ 0 \end{bmatrix}.
\]

IV. GENERAL CONVERGENCE RESULTS FOR URW-BPC

Since URW-BPC results in an update rule that can be described in terms of a matrix-vector multiplication, the convergence behavior depends on how the power series } P_\rho \text{ behaves as } \ell \text{ grows large, which will here be analyzed. First, we establish the fact that } \lambda_1 = 1 \text{ is an eigenvalue of any } P_\rho, \text{ and give its corresponding right and left eigenvectors.}

Proposition 2. For any URW-BPC matrix } P_\rho, \text{ there is one eigenvalue } \lambda_1 = 1 \text{ with geometric multiplicity } 1. \text{ Its corresponding right and left eigenvectors are } b_1 = 1, \text{ and } c_1^T = [1^T, 1^T - \rho 1^T D], \text{ respectively.}

Proof: See Appendix II-A.

Hence, if all other eigenvalues are strictly less than } 1, \text{ the convergence of URW-BPC is guaranteed. In contrast to the } W \text{ matrix in belief consensus, the matrix } P_\rho \text{ may not be diagonalizable. Hence, we must consider two cases before providing general convergence conditions.}

A. Case 1: Diagonalizable } P_\rho

If } P_\rho \text{ is a diagonalizable matrix, then by eigendecomposition we have that } P_\rho^\ell = B \Lambda^\ell B^{-1}, \text{ where the columns of } B \text{ form an eigenbasis of } P_\rho \text{ and } \Lambda \text{ is a matrix with the eigenvalues of } P_\rho \text{ on the diagonal. Let}

\[
z^{(0)}(\theta) = \begin{bmatrix} \rho A + I_N \\ I_N \end{bmatrix} x^{(0)}(\theta),
\]

and express } z^{(0)}(\theta) \text{ in the eigenbasis of } P_\rho \text{ as } z^{(0)}(\theta) = B \alpha. \text{ Now we see that}

\[
z^{(\ell)}(\theta) = P_\rho^\ell z^{(0)}(\theta) = B \Lambda^\ell B^{-1} B \alpha = B \Lambda^\ell \alpha.
\]

We can also express this as

\[
z^{(\ell)}(\theta) = \sum_{i=1}^{2N} \lambda_i^\ell b_i \alpha_i,
\]
where $\lambda_i$ is the $i$th eigenvalue of $P_\rho$, $b_i$ is the $i$th eigenvector of $P_\rho$ (and the $i$th column of $B$), and $\alpha_i$ is the $i$th element of $\alpha$. Since according to Proposition 2, $\lambda_1 = 1$, so that
\[
z^{(\ell)}(\theta) = b_1\alpha_1 + \sum_{i=2}^{2N} \lambda_i^\ell b_i\alpha_i \quad (19)
\]
\[
= b_1\alpha_1 + \hat{\varepsilon}.
\]
Later, in Proposition 3 we will establish that $\alpha_1$ is the sought value, therefore we consider $\varepsilon$ to be an error term.

**B. Case 2. Nondiagonalizable $P_\rho$**

If $P_\rho$ is not diagonalizable, it can be decomposed in its Jordan normal form. Then, $P_\rho = BJB^{-1}$, where the columns of $B$ are the generalized eigenvectors of $P_\rho$ forming a Jordan basis, and $J$ is a Jordan matrix, which is a block diagonal matrix with $M < 2N$ Jordan blocks on its diagonal, i.e.,
\[
J = \begin{bmatrix}
J_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & J_M
\end{bmatrix}.
\]
Each Jordan block corresponds to a certain eigenvalue and its generalized eigenvectors. For example, if the eigenvalue $\lambda_m$ has three generalized eigenvectors, $b_{m,1}, b_{m,2}$ and $b_{m,3}$, then
\[
J_m = \begin{bmatrix}
\lambda_m & 1 & 0 \\
0 & \lambda_m & 1 \\
0 & 0 & \lambda_m
\end{bmatrix}.
\]
Note that if $P_\rho$ is diagonalizable, its Jordan normal form is equal to its eigendecomposition. By expressing $z^{(0)}(\theta) = B\alpha$ in the Jordan basis of $P_\rho$ and decomposing $P_\rho$ in Jordan normal form, we can write
\[
z^{(\ell)}(\theta) = P_\rho^{\ell}z^{(0)}(\theta) = BJ^\ell\alpha, \quad (22)
\]
which we can also express as
\[
z^{(\ell)}(\theta) = \sum_{m=1}^{M} \sum_{r_m=1}^{r_m} \left( \sum_{i=0}^{\min(\ell, r_m-1)} \binom{\ell}{i} \lambda_m^{-i} \right) b_{m,j}\alpha_{m,j}, \quad (23)
\]
where $r_m$ is the size of the $m$th Jordan block, $b_{m,j}$ is the $j$th generalized eigenvector of $\lambda_m$, and $\alpha_{m,j}$ the corresponding entry in $\alpha$. With $\lambda_1 = 1$ and denoting $b_{1,1}\alpha_{1,1}$ by $b_1\alpha_1$, we can break the sum into three parts
\[
z^{(\ell)}(\theta) = b_1\alpha_1 + \sum_{j=2}^{r_1} \sum_{i=0}^{\min(\ell, r_{1,j}-1)} \binom{\ell}{i} b_{1,j}\alpha_{1,j}
\]
\[
+ \sum_{m=2}^{M} \sum_{r_m=1}^{r_m} \left( \sum_{i=0}^{\min(\ell, r_m-1)} \binom{\ell}{i} \lambda_m^{-i} \right) b_{m,j}\alpha_{m,j}
\]
\[
= b_1\alpha_1 + \hat{\varepsilon} + \varepsilon. \quad (24)
\]
Following the reasoning of the case with a diagonalizable $P_\rho$, any quantity that is not $\alpha_1$ is considered an error term. For the nondiagonalizable case, we split it up into $\varepsilon$ and $\hat{\varepsilon}$, since these two terms behave fundamentally different with respect to the eigenvalues of $P_\rho$.

**C. General Convergence Conditions**

We are now able to provide insights into $\alpha_1$ as well as the error terms $\hat{\varepsilon}$ and $\varepsilon$.

**Proposition 3.** The quantity $\alpha_1^{(\ell)} = \frac{c_1^T z^{(\ell)}(\theta)}{(c_1^T b_1)}$ is preserved by the URW-BPC algorithm at each iteration $\ell$. If URW-BPC converges, then the consensus value is the preserved quantity, and it is equal to
\[
\alpha_1 = \frac{2}{c_1^T b_1}1^T \rho(0)(\theta). \quad (26)
\]

**Proof:** See Appendix II-B.

Note according to Proposition 2
\[
c_1^T b_1 = 2N - \rho \text{trace}(D). \quad (27)
\]

Hence, what remains is to establish sufficient conditions for URW-BPC to converge and then to establish the corresponding convergence rate. We note the following:

(i) When $P_\rho$ has an eigenvalue $\lambda = -1$ with equal geometric and algebraic multiplicities, the corresponding value $\alpha_i$ in (19) or $\alpha_{m,j}$ in (23) is zero, since the eigenvector of $\lambda = -1$ is in the null space of $P_\rho + I_{2N}$ and is thus canceled out by the initialization (15).

(ii) When $P_\rho$ is diagonalizable, there is only one eigenvalue $\lambda_1 = 1$. If all other eigenvalues are strictly inside the unit circle, or equal to $-1$ with equal geometric and algebraic multiplicities, then $\varepsilon \to 0$ and convergence of URW-BPC (14) is guaranteed to (26) by Proposition 3.

(iii) When $P_\rho$ is not diagonalizable, if $\lambda_1$ has a Jordan block of size $1 \times 1$ and all other eigenvalues are strictly inside the unit circle, or equal to $-1$ with equal geometric and algebraic multiplicities, then $\varepsilon = 0$, $\varepsilon \to 0$, and convergence of URW-BPC (14) is guaranteed to (26) by Proposition 3.

Finally, the convergence rate is defined as
\[
\gamma(P_\rho) = \sup_{x^{(0)}(\theta) \neq \mathbf{c}} \lim_{\ell \to \infty} \left( \frac{\|x^{(\ell)}(\theta) - \mathbf{c}\|_2}{\|x^{(0)}(\theta) - \mathbf{c}\|_2} \right)^{1/\ell}, \quad (28)
\]
provided that the algorithm is convergent, and at least one eigenvalue strictly inside the unit circle is nonzero. For such cases, we consider the eigenvalues of $P_\rho$ to be sorted such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{2N}|$. The convergence rate is determined by $|\lambda|$ where $\lambda = \max_i |\lambda_i|$ for $i$ such that $|\lambda_i| < 1$, such that a smaller $|\lambda|$ gives a faster convergence.

**V. CONVERGENCE ON SPECIFIC GRAPH TYPES**

In this section, we analyze the convergence properties of URW-BPC for three specific types of graphs. We first consider tree-graphs, recovering the well-known finite-time BP convergence result via the formulation (15). Then, we consider the regular graphs, for which BP is generally not guaranteed to converge. Finally, we consider general connected graphs, for which we can build on the results from regular graphs.
A. Tree Graphs

For trees, the following proposition establishes the possible eigenvalues of $P_i$.

**Proposition 4.** The URW-BPC matrix $P_i$ of any tree-graph has three distinct eigenvalues: $\lambda_1 = 1$, $\lambda_2 = -1$, and $\lambda_i = 0$ for $i = 3, \ldots, 2N$.

**Proof:** See Appendix II-C.

We then immediately find the following well-known results for trees, that BP converges in a finite number of iterations.

**Theorem 5.** If $G$ is a tree graph of $N$ nodes, then URW-BPC with $P_i$ converges to consensus after at most $2N - 3$ iterations. Moreover, with the initialization as in (14), the consensus value after $\kappa$ iterations ($\kappa$ such that consensus is reached) is

$$x^{(\kappa)}(\theta) = \sum_{m=1}^{N} \hat{x}_m^{(0)}(\theta) 1.$$  

(29)

**Proof:** Due to Proposition 4, $P_i$ has $2N - 2$ eigenvalues $\lambda = 0$. Hence, the largest possible size of its corresponding Jordan block, denoted by $J_0$, is $2N - 2$. Since $J_0^{2N-2} = O$, the error contribution from the eigenvalues equal to zero is zero after at most $2N - 3$ iterations. Furthermore, applying the results from Propositions 2 and 3, and using the fact that the sum of the degrees of $G = 2N - 2$ for undirected tree-graphs in (27), the consensus value $\alpha_1$ is given by (29).

B. Regular Graphs

In order to understand when URW-BPC converges, we first show how to choose the weighting parameter $\rho$ in order to guarantee convergence. Then we proceed to optimize $\rho$ for a given graph $G$ such that the magnitude of the largest eigenvalue inside the unit circle, $|\lambda|$, is minimized. We recall that for $k$-regular graphs, the largest eigenvalue of the adjacency matrix is $\mu_1 = k$ for non-bipartite graphs, while for bipartite graphs, eigenvalues come in symmetric pairs, so that both $\mu_1 = k$ and $\mu_2 = -k$ are eigenvalues [25, Prop.2.3].

1) Convergence: To find for which $\rho$ URW-BPC converges on $k$-regular graphs, we first show how the eigenvalues of $P_\rho$ and $A$ are connected in terms of magnitudes. Note, that the eigenvalues of $P_\rho$ and $A$ are sorted such that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{2N}|$ and $|\mu_1| \geq |\mu_2| \geq \cdots \geq |\mu_N|$.

**Lemma 6.** Let $\rho \in (0,1)$. Then the eigenvalue $\lambda_i$ of $P_\rho$, $\lambda_i \neq 0$, can be expressed in terms of $\mu_i$, and its magnitude is

$$|\lambda_i| = \frac{1}{2} \sqrt{|\mu_i|\rho + \sqrt{|\mu_i|^2 \rho^2 - 4k\rho + 4}}.$$  

(30)

**Proof:** See Appendix III-A.

Note that for $\lambda_i = 0$ with eigenvector $[v^T, w^T]^T$, we have that

$$\rho Av + w - \rho Dw = 0$$  

$$v = 0.$$  

(31) (32)

We conclude that for $\lambda_i = 0$, we must have that $k\rho = 1$. Note however, that this does not mean that all eigenvalues are equal to 0 for $\rho = 1/k$.

Now, since $\mu_1 = k$, the magnitude of $\lambda_1$ of $P_\rho$ is either 1 or $|pk - 1|$. Thus, we can prove the following result regarding the convergence conditions of URW-BPC.

**Theorem 7.** For any $k$-regular graph, URW-BPC is convergent if and only if $\rho \in (0, 2/k)$, and the asymptotic consensus value is

$$\lim_{t \to \infty} x^{(t)} (\theta) = \frac{1}{N(1 - \rho k/2)} \sum_{m=1}^{N} x_m^{(0)}(\theta) 1.$$  

(33)

**Proof:** See Appendix III-B.

This result provides the interval for $\rho$ within which we can guarantee convergence, and to which value the algorithm converges.

2) Optimizing the Convergence Rate: In order to maximize convergence rate, we show which $\rho$ minimizes the largest eigenvalue within the unit circle, denoted by $|\lambda| < 1$.

**Theorem 8.** The choice of $\rho$ that minimizes $|\lambda|$ is

$$\rho_{\text{opt}} = \frac{2}{\mu^2} \left( k - \sqrt{k^2 - \mu^2} \right),$$  

(34)

where $\mu = \max_{i} |\mu_i|$ for $i$ such that $|\mu_i| < k$. The magnitude of the second largest eigenvalue of $P_{\rho_{\text{opt}}}$ is

$$|\lambda| = \frac{1}{\mu} \left( k - \sqrt{k^2 - \mu^2} \right).$$  

(35)

**Proof:** See Appendix III-C.

**Remark 9.** For any $k$-regular non-bipartite graph $G$, $\mu = \mu_2$. However, for a $k$-regular bipartite graph we have that $\mu_2 = -\mu_1 = -k$. Hence, choosing $\rho_{\text{opt}}$ with $\mu_2$ instead of $\mu_1$ in this case would yield $\rho_{\text{opt}} = 2/k$, which in turn gives (see (30)),

$$|\lambda_1|^2 = 1/k^2 \left( \mu_1 \pm \sqrt{\mu_1^2 - k^2} \right)^2 = 1,$$

for every eigenvalue $\lambda_1$ of $P_\rho$. Hence, the optimal reweighting for $k$-regular bipartite graphs is achieved with $\tilde{\mu} = \mu_3$. Another consequence of $\mu_2 = -k$ is that there is always an eigenvalue $\lambda = -1$ for bipartite graphs. This remark also applies to tree-graphs, which is a class of bipartite graphs, where $P_1$ of a tree-graph has an eigenvalue $\lambda = -1$. However, the component associated with this eigenvalue is irrelevant, as it is removed by the initialization procedure. This relies on the following result.

**Proposition 10.** For a URW-BPC matrix $P_\rho$, the algebraic and geometric multiplicities of $\lambda = \pm 1$ are equal.

**Proof:** See Appendix III-D.

3) Limit Results for $k$-regular Graphs: Due to their structure, the eigenvalue distribution of $A$ for large $k$-regular graphs is given by [26] (with $G$ satisfying certain properties regarding the number of cycles in the graph, for details see [26])

$$f(\mu) = \begin{cases} \frac{k(4(k-1)-\mu^2)^{1/2}}{2\pi(k^2-\mu^2)}, & |\mu| \leq 2\sqrt{k-1} \\ 0, & |\mu| > 2\sqrt{k-1}. \end{cases}$$  

(36)
This means that \( \lim_{N \to \infty} |\mu_2| = 2\sqrt{k-1} \), and thus

\[
\lim_{N \to \infty} \rho_{\text{opt}} = \frac{1}{2(k-1)} \left( k - \sqrt{k^2 - 4(k-1)} \right),
\]

so that the second largest eigenvalue of \( P_{\rho} \) for non-bipartite graphs tends to

\[
\lim_{N \to \infty} |\lambda_{2,\text{BPC}}| = \frac{1}{2\sqrt{k-1}} \left( k - \sqrt{k^2 - 4(k-1)} \right).
\]

For the belief consensus, \( |\lambda_{2,\text{Metr}}| \) of \( W = I_N - \xi L \) tends to

\[
\lim_{N \to \infty} |\lambda_{2,\text{Metr}}| = \frac{1 + 2\sqrt{k-1}}{k+1} > \lim_{N \to \infty} |\lambda_{2,\text{BPC}}|,
\]

so that BPC always converges faster than belief consensus on large \( k \)-regular graph.

C. General Graphs

For general graphs, it is not obvious how to render BPC convergent. A possible approach is to determine a spanning tree of the network graph and then running BPC with finite-convergent. A possible approach is to determine a spanning tree of the network graph and then running BPC with finite-time convergence [27]. However, we can also build on the results from regular graphs. We outline two procedures to convert a general graph to a regular graph.

(i) Edge addition: The simplest way to make a graph into a \( k \)-regular graph, is to first determine the maximum node degree \( d_{\text{max}} \) (this can be done through max-consensus). Then a node \( i \) with degree \( d_i \) adds \( d_{\text{max}} - d_i \) self-loops. Then BPC with \( \rho_{\text{opt}} \) set based on \( k = d_{\text{max}} \) and \( \tilde{\mu} \) of the new \( A \), is applied.

(ii) Edge deletion: A more complex way to create a \( k \)-regular graph is by selectively deleting edges from those nodes with maximum degree, while maintaining connectivity. This procedure can be applied until a certain minimal value for \( d_{\text{max}} \) is attained.

VI. NUMERICAL RESULTS AND DISCUSSION

A. Simulation Parameters

We present numerical results comparing the URW-BPC algorithms with Metropolis weighted belief consensus. The simulations were performed with the number of nodes \( N = 100 \), with a fixed \( G \) for tree-graphs and random \( G \) for \( k \)-regular graphs. The node degree for the \( k \)-regular graphs was fixed to \( k = 4 \). The elements of the initial data \( x(0)(\theta) \) were generated according to a standard normal distribution. We calculated the averaged (over \( n \) instances of \( x(0)(\theta) \) for tree-graphs, and over both \( x(0)(\theta) \) and \( G \) for \( k \)-regular graphs) and normalized mean squared error (MSE), with the MSE being normalized with respect to the initial consensus error. The simulations were performed over 100 Monte Carlo runs.

B. Results for Tree-graphs

The simulated error of URW-BPC on a tree-graph is shown in Fig. 2. We observe that the algorithm indeed reaches consensus in a finite number of steps. However, before reaching consensus, the error of URW-BPC behaves differently from that of the other consensus algorithm. The increasing error we see can be explained by \( \varepsilon_1 \) in (25). It takes a few iterations for the Jordan blocks of the eigenvalues \( \lambda_i = 0 \) to become zero, and until they do, the error they contribute with increases as \( k \) increases.

C. Results for \( k \)-regular Graphs

To illustrate the benefit of URW-BPC on \( k \)-regular graphs, we provide the following example of URW-BPC on a so called small-world graph [28]. Let \( G \) be of the type detailed in [28, Appendix A]. For this type of graph, there exist closed-form expressions for the eigenvalues of \( A \). In particular, if we let \( G \) be such a graph with \( N = 10 \) and \( k = 4 \), we have that \( \mu_2 \approx 2.23 \). Hence, using (34) to calculate the optimal \( \rho \), we get from (35) that \( |\lambda_2| \approx 0.31 \). On the other hand, using BC on this graph with step-size \( \varepsilon = 0.25 \) yields \( |\lambda_2| \approx 0.56 \).
In Fig. 4 we show how the average error of optimally weighted URW-BPC and belief consensus compare. The error is averaged over instances of the graph $\mathcal{G}$ as well as the initial data $x^{(0)}(\theta)$. Clearly, URW-BPC outperforms belief consensus in terms of convergence rate.

In Fig. 5 we compare the magnitude of $\lambda_2$ for the two consensus algorithms by taking the ratio $r = |\lambda_{2,BPC}| / |\lambda_{2,Metr}|$ over 10,000 Monte Carlo runs. Clearly, URW-BPC always outperforms belief consensus, since the ratio stays well below one. Moreover, as we increase the network size, we see that the ratio converges to the specific limit value discussed in Section V-B3.

**D. Results for a General Graph**

To illustrate the method for general graphs discussed in Section V-C, we perform numerical simulations for the graph shown in Fig. 6, both without self-loops and Metropolis weight consensus, and with added self-loops and optimally weighted (according to the result of Theorem 8) URW-BPC. The results are shown in Fig. 7. We see that the strategy of adding self-loops and running URW-BPC on the resulting $k = 3$ regular graph indeed works well, and asymptotically outperforms belief consensus on the original graph.

**VII. Conclusion**

We studied the uniformly reweighted belief propagation algorithm for likelihood consensus, which was described by a factor graph with strong interactions, generally considered a challenging case for belief propagation. The belief propagation consensus algorithm resulted in a linear update rule much like a consensus algorithm with memory. By eigenvalue analysis we were able to prove a collection of results on several types of graphs: (i) we recovered the classical finite-time convergence of belief propagation on tree graphs for the likelihood consensus problem; (ii) we provided conditions on the reweighting parameter necessary and sufficient for convergence, and (iii) we found an analytical expression for the reweighting parameter optimizing convergence rate, on $k$-regular graphs, and on general graphs artificially transformed.
into $k$-regular graphs by adding self-loops or removing edges. Based on both numerical results, and eigenvalue limits on large $k$-regular graphs, belief propagation consensus outperformed consensus with Metropolis-type weights. Open issues include analytically comparing the performance of belief propagation consensus to other algorithms for likelihood consensus, and to investigate how it compares to consensus algorithms with memory.

### Appendix I

**Derivation of the URW-BPC Algorithm**

According to the message-passing equations of the uniformly reweighted BP [19], we can write the marginal belief of some variable $\theta_n$ of node $n$ at iteration $\ell$ as

$$b_n^{(\ell)}(\theta_n) \propto p(y_n | \theta_n) \prod_{m \in \mathcal{N}_n} \left( \mu_{m \rightarrow n}^{(\ell)}(\theta_n) \right)\rho,$$

(40)

for $\rho \in (0, 1]$, where the message from node $m$ to node $n$ at iteration $\ell$ is computed by

$$\mu_{m \rightarrow n}^{(\ell)}(\theta_n) \propto \sum_{\theta_m} I_{\theta_m = \theta_n} p(y_m | \theta_m) \prod_{u \in \mathcal{N}_m \setminus n} \left( \mu_{u \rightarrow m}^{(\ell-1)}(\theta_m) \right)^{1-\rho} \mu_{m \rightarrow n}^{(\ell-1)}(\theta_m) \rho,$$

(41)

$$= \sum_{\theta_m} I_{\theta_m = \theta_n} \frac{b_m^{(\ell-1)}(\theta_m)}{\mu_{m \rightarrow n}^{(\ell-1)}(\theta_m)} \mu_{m \rightarrow n}^{(\ell-1)}(\theta_m) \mu_{m \rightarrow n}^{(\ell-1)}(\theta_m) \rho,$$

(42)

$$= \frac{b_m^{(\ell-1)}(\theta_n)}{\mu_{m \rightarrow n}^{(\ell-1)}(\theta_n)}.$$

We note that $\theta_n = \theta_m = \theta$, and plug (43) into (40)

$$b_n^{(\ell)}(\theta) \propto p(y_n | \theta) \prod_{m \in \mathcal{N}_n} \left( \frac{b_m^{(\ell-1)}(\theta)}{\mu_{m \rightarrow n}^{(\ell-1)}(\theta)} \right)^{\rho},$$

(44)

$$= p(y_n | \theta) \prod_{m \in \mathcal{N}_n} \left( \frac{b_m^{(\ell-1)}(\theta)}{b_m^{(\ell-2)}(\theta)} \frac{\mu_{m \rightarrow n}^{(\ell-2)}(\theta)}{\mu_{m \rightarrow n}^{(\ell-2)}(\theta)} \right)^{\rho},$$

(45)

$$= b_n^{(\ell-2)}(\theta) \prod_{m \in \mathcal{N}_n} \left( \frac{b_m^{(\ell-1)}(\theta)}{b_m^{(\ell-2)}(\theta)} \right)^{\rho}. $$

(46)

For the initial values of the marginals, we assume that

$$\mu_{m \rightarrow n}^{(0)}(\theta) = 1$$

for all nodes $n$, all $m \in \mathcal{N}_n$. Hence, by (40) we have that $b_n^{(0)}(\theta) = p(y_n | \theta).$ Now we can compute the marginals at iteration $\ell = 1$ by using (44), which gives

$$b_n^{(1)}(\theta) = p(y_n | \theta) \prod_{m \in \mathcal{N}_n} (p(y_m | \theta))^{\rho}.$$

### Appendix II

**Proofs of Propositions**

A. Proof of Proposition 2

**Proof:** Let $b_1 = [v^T, w^T]^T$ be a left eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. Then, it holds that

$$v = \rho_1 Av + (I_N - \rho_1 D)w$$

(47)

$$w = v,$$

(48)

which boils down to $Lv = 0$, where $L = D - A$ is the graph Laplacian. The graph Laplacian $L$ of a connected graph has an eigenvalue $\nu = 0$ with algebraic multiplicity equal to 1, and its right eigenvector is $v = 1$. Thus, $\nu_1 = 1$ has geometric multiplicity equal to 1. Since (48) states that $w = v$, we see that $b_1 = 1$. Now, let $c_1^T = [v^T, w^T]$ be a left eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. Then, we know that

$$v^T = \rho_1 v^T A + w^T$$

(49)

$$w^T = v^T - \rho_1 v^T D.$$  

(50)

Plugging (50) into (49) gives us that

$$\rho_1 v^T A + v^T - \rho_1 v^T D = v^T,$$

(51)

which implies that $v^T L = 0$, and in turn that $v^T = 1^T$. Using this result in (50) we immediately get that $c_1^T = [1^T, 1^T - \rho 1^T D].$

B. Proof of Proposition 3

**Proof:** Denote by $\tilde{C}^T = B^{-1}$ the matrix whose rows are the scaled left eigenvectors, such that $c_1^T b_1 = 1$. In particular, this means that $c_1^T = c_1^T / (c_1^T b_1)$. Now, since

$$\alpha_1 = B^{-1} z^{(0)}(\theta),$$

(52)

$$= \tilde{C}^T z^{(0)}(\theta),$$

(53)

and the first row of $\tilde{C}^T$ is $c_1^T$, then clearly $\alpha_1^{(0)} = c_1^T z^{(0)}(\theta) / (c_1^T b_1)$. Furthermore, since $\alpha_1^{(\ell)}$ is the coordinate of $z^{(\ell)}(\theta)$ in the basis $B$ corresponding the eigenvalue $\lambda_1 = 1$, the part of $z^{(\ell)}(\theta)$ (in $B$) preserved at each iteration is $\alpha_1^{(\ell)}$. Moreover, if URW-BPC converges then from (19) and (24) we observe that $\alpha_1$ (which is the preserved value) is the consensus value, and since $c_1^T z^{(0)} = 0$ $x^{(0)}(\theta)$ the consensus value is given by

$$\alpha_1 = \frac{2}{c_1^T b_1} 1^T x^{(0)}(\theta).$$

(54)

C. Proof of Proposition 4

**Proof:** The eigenvalues of $P_1$ are given by the roots of the polynomial

$$\det(P_1 - \lambda I_{2N}) \overset{(a)}{=} \det(\lambda^2 I_N - \lambda A + D - I_N) = 0,$$

(55)

where the equality $(a)$ holds due to the four $N \times N$-blocks of $P_1 - \lambda I_{2N}$ being mutually commutative [29, Theorem 3]. For brevity, denote $\Psi = \lambda^2 I_N - \lambda A + D - I_N$. Consider now the case where we add a leaf node to $\mathscr{G}$. Without loss of generality, we assume that the leaf node is node 1, and its
Hence, adding a leaf node only adds two extra roots \( \lambda = 0 \) to the eigenvalue generating polynomial. By exchanging \( \mathcal{G} \) for \( \hat{\mathcal{G}} \) and vice versa, we see that by removing a leaf node, we remove two roots \( \lambda = 0 \) instead. Consequently, the nonzero eigenvalues of \( \hat{P}_1 \) are the same as those of \( P_1 \). Starting from a graph with only one node, with URW-BPC matrix

\[
P_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

and thus eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \), we see that any tree graph with \( N \geq 2 \) has eigenvalues \( \lambda_1 = 1, \lambda_2 = -1, \) and \( \lambda_i = 0 \) for \( i = 3, \ldots, 2N \).

### Appendix III

**Proofs related to Section V-B**

First, we prove a few results regarding the behavior of the eigenvalues of the URW-BPC matrix of a \( k \)-regular graph, with respect to the eigenvalues of the adjacency matrix. After these useful results are obtained, we proceed to prove the main results.

**Lemma 11.** The roots of the polynomial \( \lambda^2 - \mu \rho \lambda + \rho k - 1 = 0 \) are given by

\[
\lambda_{a,b}(\mu) = \frac{1}{2} \left( \mu \rho \pm \sqrt{\mu^2 \rho^2 - 4 \rho k + 4} \right).
\]

For \( k \geq 2, \mu \in [-k,k] \) and \( \rho > 0 \) they have the following properties:

(i) If \( \lambda_{a}(\mu) \) and \( \lambda_{b}(\mu) \) are complex-valued, then \( |\lambda_{a}(\mu)| = |\lambda_{b}(\mu)| = \sqrt{|\rho k - 1|} \).

(ii) Let \( \mu = k \), then \( \lambda_{b}(k) = 1 \) and \( \lambda_{a}(k) = \rho k - 1 \).

(iii) \( |\lambda_{a}(\mu)| = |\lambda_{b}(\mu)| \).

(iv) \( |\lambda_{a}(\mu)| \) is a nondecreasing function of \( \mu \); \( |\lambda_{b}(\mu)| \) is a nonincreasing function of \( \mu \).

**Proof:** When the roots \( \lambda_{a}(\mu), \lambda_{b}(\mu) \) are complex-valued, the squared absolute values are given by

\[
|\lambda_{a,b}(\mu)|^2 = \frac{1}{4} \left( \mu \rho \pm \sqrt{\mu^2 \rho^2 - 4 \rho k + 4} \right)^2.
\]

For property (ii), suppose first that \( k \rho \geq 2 \). We plug in \( \mu = k \)

\[
|\lambda_{a}(k)| = \frac{1}{2} |k \rho + \sqrt{k^2 \rho^2 - 4 k \rho + 4}|
\]

\[
= \frac{1}{2} |k \rho + (k \rho - 2)|
\]

\[
= k \rho - 1.
\]

\[
|\lambda_{b}(k)| = \frac{1}{2} |k \rho - k \rho + 2| = 1.
\]

For \( k \rho < 2 \), the roots are interchanged.

For (iii) we have that

\[
|\lambda_{b}(-\mu)| = -|\mu \rho - \sqrt{\mu^2 \rho^2 - 4 \rho k + 4}|
\]

\[
= -\left( \mu \rho + \sqrt{\mu^2 \rho^2 - 4 \rho k + 4} \right)
\]

\[
= |\mu \rho + \sqrt{\mu^2 \rho^2 - 4 \rho k + 4}|
\]

\[
= |\lambda_{a}(\mu)|.
\]

To show (iv), we focus on the case when \( \lambda_{a}(\mu) \) is real, since for complex \( \lambda_{a}(\mu), |\lambda_{a}(\mu)| \) is constant in \( \mu \). We check that the derivative of \( \lambda_{a}^2(\mu) \) wrt. \( \mu \) is positive, and since \( \lambda_{a}(\mu) \) is assumed to be real this holds for \( |\lambda_{a}(\mu)| \) as well. The derivative of \( \lambda_{a}^2(\mu) \) wrt. \( \mu \) is given by

\[
\frac{\partial}{\partial \mu} \lambda_{a}^2(\mu) = 2 \lambda_{a}(\mu) \frac{\partial}{\partial \mu} \lambda_{a}(\mu)
\]

\[
= \frac{2 \lambda_{a}^2(\mu) \rho}{\sqrt{\mu^2 \rho^2 - 4 \rho k + 4}}.
\]

Since \( \rho > 0 \), the derivative of \( \lambda_{a}^2(\mu) \) is clearly positive, and thus so is the derivative of \( |\lambda_{a}(\mu)| \). We conclude that \( |\lambda_{a}(\mu)| \) is nondecreasing in \( \mu \), and due to (iii) that \( |\lambda_{b}(\mu)| \) is nonincreasing in \( \mu \). Note that the functions are not necessarily monotonic since they are constant for complex eigenvalues.
This means that the eigenvalues of $P_\rho$ are given by the roots of the polynomial

$$\lambda = \frac{1}{2} \left( \mu \rho \pm \sqrt{\mu^2 \rho^2 - 4k \rho + 4} \right),$$  \hspace{1cm} (82)

where $\mu$ is an eigenvalue of $A$ with eigenvector $w$. Suppose that $\mu_i > 0$. Then, Lemma 11, which says that $\lambda_\alpha(\mu)$ is a nondecreasing function in $\mu$, implies that $\lambda_i = \lambda_\alpha(\mu_i)$ and thus its magnitude is given by

$$|\lambda_i| = \frac{1}{2} |\mu_i \rho + \sqrt{\mu_i \rho - 4k \rho + 4}|.$$  \hspace{1cm} (83)

Since $|\lambda_\alpha(\mu)| = |\lambda_\beta(-\mu)|$ by Lemma 11, we observe that we achieve the same result should $\mu_i < 0$.

### B. Proof of Theorem 7

**Proof**: Let $\tilde{\mu} = \max_i |\mu_i|$ for $i$ such that $|\mu_i| < k$. Note that for a non-bipartite $G$, $\tilde{\mu} = \mu_2$, whereas for a bipartite $G$ we have that $\mu_2 = -k$, and hence $\tilde{\mu} = \mu_3$. Since $|\lambda_\alpha(\mu)|$ is a nondecreasing function in $\mu$ (due to Lemma 11) and $\tilde{\mu} < k$, and if $k \rho < 2$ we have that

$$|\tilde{\lambda}| = \frac{1}{2} \left| \tilde{\mu} \rho + \sqrt{\tilde{\mu}^2 \rho^2 - 4k \rho + 4} \right|$$  \hspace{1cm} (84)

$$< \frac{1}{2} \left| k \rho + \sqrt{k^2 \rho^2 - 4k \rho + 4} \right|$$  \hspace{1cm} (85)

$$= \frac{1}{2} \left| k \rho + (2 - k \rho) \right|$$  \hspace{1cm} (86)

$$= 1.$$  \hspace{1cm} (87)

Hence, $|\lambda_i| < 1$ for $i = 2, \ldots, 2N$. On the other hand, if $k \rho > 2$, we have that

$$|\lambda_1| = \frac{1}{2} \left| k \rho + \sqrt{k^2 \rho^2 - 4k \rho + 4} \right|$$  \hspace{1cm} (88)

$$= \frac{1}{2} |k \rho + k \rho - 2|$$  \hspace{1cm} (89)

$$= k \rho - 1.$$  \hspace{1cm} (90)

$$> 1.$$  \hspace{1cm} (91)

So, in that case URW-BPC is not convergent. In particular for $\rho = 2/k$ we have that

$$\lambda = \frac{1}{2} \left( \frac{2}{k} \rho \pm \sqrt{\frac{4}{k^2} \rho^2 - 4} \right),$$  \hspace{1cm} (92)

so all eigenvalues are complex-valued except the ones generated from $\mu = k$ or $\mu = -k$ (the smallest eigenvalue of $A$ for bipartite $G$ is $\mu = -k$ [25]), which are equal to $\lambda = 1$ or $\lambda = -1$. Thus, using property (i) we find that $|\lambda_i| = 1$ for all $i = 1, \ldots, 2N$. Moreover, for $\rho = 0$ we clearly see that $|\lambda_1| = 1$ for all $i = 1, \ldots, 2N$. For $\rho < 0$ it is obvious that $|\lambda_1| > 1$.

Using the results from Propositions 2 and 3, and that the sum of the degrees for undirected $k$-regular graphs is $\sum_{i=1}^N D_{ii} = Nk$, the consensus value $\alpha_1$ is given

$$\alpha_1 = \frac{2}{2N - \rho N k} \sum_{m=1}^N x_m^{(0)}(\theta).$$  \hspace{1cm} (93)

### C. Proof of Theorem 8

**Proof**: We want to find the $\rho$ that minimizes the magnitude of the largest eigenvalue inside the unit circle, i.e., $|\tilde{\lambda}|$. Let $\tilde{\mu}$ be the eigenvalue of $A$ that generates $\tilde{\lambda}$. Then, we minimize $|\tilde{\lambda}|$ by

$$\min_{\rho \in (0, 1]} \frac{1}{2} \left( \tilde{\mu} \rho + \sqrt{\tilde{\mu}^2 \rho^2 - 4k \rho + 4} \right).$$  \hspace{1cm} (94)

First we get the roots with respect to $\rho$ of the polynomial under the square-root

$$\rho = \frac{2k}{\tilde{\mu}^2} \pm \sqrt{\frac{4k^2}{\tilde{\mu}^4} - 4}$$  \hspace{1cm} (95)

$$= \frac{2}{\tilde{\mu}^2} \left( k \pm \sqrt{k^2 - \tilde{\mu}^2} \right).$$  \hspace{1cm} (96)

Since $\sqrt{k^2 - \tilde{\mu}^2} > 0$ and $k > \sqrt{k^2 - \tilde{\mu}^2}$, we see that the smallest $\rho$ is given by

$$\rho^* = \frac{2}{\tilde{\mu}^2} \left( k - \sqrt{k^2 - \tilde{\mu}^2} \right).$$  \hspace{1cm} (97)

This value of $\rho$ will make the second term inside the absolute value in (94) equal to zero, yielding

$$|\tilde{\lambda}| = |\tilde{\mu} \rho^*|.$$  \hspace{1cm} (98)

However, it is still not clear that this is the global minimum, since there is a linear term in the expression too. First, since $\tilde{\mu}$ is positive, $\rho > \rho^*$ cannot give smaller $|\tilde{\lambda}|$ than the one given by $\rho^*$. But, there might be a $\rho < \rho^*$ that gives a smaller $|\tilde{\lambda}|$. So, consider using $\rho_\epsilon = \rho^* - \epsilon$, where $\epsilon > 0$. Then we get

$$|\tilde{\lambda}_\epsilon| = \frac{1}{2} \left| \tilde{\mu} \rho^* - \tilde{\mu} \epsilon + \sqrt{\tilde{\mu}^2 \epsilon^2 + 4k \epsilon \sqrt{k^2 - \tilde{\mu}^2}} \right|.$$  \hspace{1cm} (99)

Since $4k \epsilon \sqrt{k^2 - \tilde{\mu}^2} > 0$, we have that

$$\sqrt{\tilde{\mu}^2 \epsilon^2 + 4k \epsilon \sqrt{k^2 - \tilde{\mu}^2}} > \tilde{\mu} \epsilon,$$  \hspace{1cm} (100)

and hence $|\tilde{\lambda}_\epsilon| > |\tilde{\lambda}|$. Consequently, the optimal $\rho$ is

$$\rho_{opt} = \frac{2}{\tilde{\mu}^2} \left( k - \sqrt{k^2 - \tilde{\mu}^2} \right),$$  \hspace{1cm} (101)

and, plugging this value into (98) gives

$$|\tilde{\lambda}| = \frac{1}{2} \left( \frac{2}{\tilde{\mu}} \left( k - \sqrt{k^2 - \tilde{\mu}^2} \right) \right),$$  \hspace{1cm} (102)

and

$$= \frac{1}{2} \left( \frac{2}{\tilde{\mu}} \left( k - \sqrt{k^2 - \tilde{\mu}^2} \right) \right).$$  \hspace{1cm} (103)

### D. Proof of Proposition 10

**Proof**: Denote by $\alpha(\lambda)$ and $\gamma(\lambda)$ the algebraic and geometric multiplicities of an eigenvalue $\lambda$ of a URW-BPC matrix $P_\rho$. Suppose that $\alpha(\lambda) \neq \gamma(\lambda)$, i.e., $\alpha(\lambda) > \gamma(\lambda)$. Then, there exist vectors $v$ and $w$ such that

$$P_\rho \begin{bmatrix} v \\ w \end{bmatrix} = \lambda \begin{bmatrix} v \\ w \end{bmatrix} + \begin{bmatrix} \tilde{v} \\ \tilde{w} \end{bmatrix},$$  \hspace{1cm} (104)
where \( \begin{bmatrix} \hat{v}^T, \hat{w}^T \end{bmatrix}^T \) is an eigenvector of \( P_\rho \) with eigenvalue \( \lambda \). As established in (78)–(81), if \( \lambda \neq 0 \),

\[
\begin{bmatrix}
\hat{v} \\
\hat{w}
\end{bmatrix} = \begin{bmatrix}
\lambda z \\
z
\end{bmatrix}, 
\tag{105}
\]

for some \( z \) such that

\[
A z = \mu z, 
\tag{106}
\]

\[
\mu = \frac{\lambda^2 + \rho k - 1}{\rho \lambda}.
\tag{107}
\]

Using (105) in (104), we get that

\[
\rho A v + (1 - \rho k) w = \lambda v + \lambda z, 
\tag{108}
\]

\[
v = \frac{\lambda}{2} w + z.
\tag{109}
\]

Substituting \( v \) in (108), we have

\[
\rho A \lambda w + \rho A z + (1 - \rho k) w = \lambda^2 w + 2\lambda z,
\tag{110}
\]

which in turn, using (106), becomes

\[
\rho A \mu w = -\mu z + (\rho k - 1) w + \lambda^2 w + 2\lambda z.
\tag{111}
\]

Rearranging the terms, we have that

\[
A w = \frac{\lambda^2 + \rho k - 1}{\rho \lambda} w + \frac{2\lambda - \rho \mu}{\rho \lambda} z
\tag{112}
\]

\[
= \mu z^T w + \frac{2\lambda - \rho \mu}{\rho \lambda} \| z \|^2.
\tag{113}
\]

Left-multiplying by \( z^T \) and using the symmetry of \( A \) (so that \( z^T A = z^T \mu \)), we get

\[
\mu z^T w = \mu z^T w + \frac{2\lambda - \rho \mu}{\rho \lambda} \| z \|^2.
\tag{114}
\]

This implies that \( \frac{(2\lambda - \rho \mu)}{\rho \lambda} = 0 \), and thus that

\[
2\lambda = \frac{\rho \mu}{\lambda} = \frac{\lambda^2 + \rho k - 1}{\lambda}.
\tag{115}
\]

Hence, we have that \( \lambda^2 = \rho k - 1 \). For \( \lambda = \pm 1 \), this implies that \( \rho k = 2 \). But, \( \rho \in (0, 2/k) \), hence the original claim is false. We conclude that \( \alpha(\lambda) = \gamma(\lambda) \) for \( \lambda = \pm 1 \). ■

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