Abstract. Let \( C \) be an irreducible smooth projective curve, of genus at least two, defined over an algebraically closed field of characteristic zero. For a fixed line bundle \( L \) on \( C \), let \( M_C(r, L) \) be the coarse moduli space of semistable vector bundles \( E \) over \( C \) of rank \( r \) with \( \wedge^r E = L \). We show that the Brauer group of any desingularization of \( M_C(r, L) \) is trivial.

1. Introduction

Let \( k \) be an algebraically closed field of characteristic zero. Let \( C/k \) be an irreducible smooth projective curve of genus \( g \), with \( g \geq 2 \). Let \( L \in \text{Pic}(C) \) be a line bundle on \( C \) of degree \( d \); fix an integer \( r \geq 2 \). Let \( M = M_C(r, L) \) denote the moduli space of semistable vector bundles on \( C \) of rank \( r \) and determinant \( L \). It is known that \( M \) is a unirational normal projective variety. Up to isomorphism \( M \) depends only on the class of \( d \) modulo \( r \) and not on the actual line bundle \( L \). This variety \( M \) is known to be rational if \( r \) is coprime to \( d \) [KS, p. 520, Theorem 1.2]; except for the only case of \( g = r = d = 2 \) when \( M \) is known to be \( \mathbb{P}^3_k \), in all other cases, where \( r \) is not coprime to \( d \), it is unknown whether \( M \) is stably rational.

For any projective variety \( X/k \) to be rational (or even stably rational), it is necessary for the Brauer group \( \text{Br}(\tilde{X}) \) to vanish, where \( \tilde{X} \to X \) is a desingularization. This motivated us to study the Brauer group of the desingularization of \( M \).

The following result is proved here:

**Theorem 1.** Let \( \tilde{M} \to M_C(r, L) \) be any desingularization of \( M_C(r, L) \). Then \( \text{Br}(\tilde{M}) \) is trivial.

We now give a brief idea of the proof of it. For any possible nonzero class \( \alpha \in \text{Br}(\tilde{M}) \setminus \{0\} \), we show that there exists a discrete valued field \( K \), and a morphism \( \varphi : \text{Spec}(K) \to \tilde{M} \), such that \( \varphi^*\alpha \in \text{Br}(K) \) is ramified. This morphism \( \varphi \) is constructed explicitly out of a suitable family of vector bundles on \( C \) parameterized by a \( \mathbb{G}_m \)-gerbe over \( \text{Spec}(K) \). On the other hand, since \( \tilde{M} \) is a proper variety, for any \( \xi \in \text{Br}(\tilde{M}) \), the pullback \( \varphi^*\xi \in \text{Br}(K) \) must
be unramified (this is because the morphism \( \text{Spec}(K) \to M \) extends to the discrete valuation ring). Thus \( \alpha \) must be zero.

Theorem 1 was proved earlier in [Ni] under the assumption that \( r = 2 \) (see [Ni, p. 309, Theorem 1]); this theorem of Nitsure was also proved later in [Ba]. When \( r = 2 \), explicit desingularizations of \( M \) are available; these desingularizations are crucially used in [Ni], [Ba].

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2. THE STABLE LOCUS

We continue with the notation of the introduction.

Let \( \mathcal{M} = \mathcal{M}_{c,(r,\mathcal{L})} \) be the moduli stack of semistable vector bundles over \( C \) of rank \( r \) and determinant \( \mathcal{L} \). Let

\[
p : \mathcal{M} \to M
\]

be the natural morphism to the earlier defined moduli space \( M \); this \( M \) is a good moduli space for \( \mathcal{M} \) in the sense of [Ja, p. 10, Definition 4.1].

The following notation and comments will be used.

1. The degree of \( \mathcal{L} \) will be denoted by \( d \), and \( \gcd(r, d) \) will be denoted by \( n \).
2. Let \( M_0 \subseteq M \) (respectively, \( \mathcal{M}_0 \subseteq \mathcal{M} \)) be the Zariski open subset (respectively, sub-stack) parameterizing stable bundles. We note that \( M_0 \) is contained in the smooth locus of \( M \). In fact \( M_0 \) coincides with the smooth locus of \( M \) except for the only case of \( g = r = d = 2 \) (in this case \( M \) is known to be smooth).
3. There is a natural inclusion of \( \mathbb{G}_m \) as a normal subgroup in the isotropy group of any point of \( \mathcal{M} \) corresponding to action of \( \mathbb{G}_m \) on vector bundles given by scalar multiplications. For stable vector bundles, the isotropy group coincides \( \mathbb{G}_m \). Let

\[
F : \mathcal{M} \to \mathcal{N}
\]

be a 1-morphism obtained by rigidifying \( \mathbb{G}_m \) (see [ACV] pp. 3572–3573, Theorem 5.1.5]), meaning \( F \) a 1-morphism defining a \( \mathbb{G}_m \)-gerbe such that for every point \( z \) of \( \mathcal{M} \), the kernel of the homomorphism induced between the isotropy groups at \( z \) and \( F(z) \) is precisely \( \mathbb{G}_m \). \( \mathcal{N} \) is a smooth stack which is generically a scheme.

4. Choose a stable vector bundle \( W \) of rank \( r/n \) and degree \( d/n \) on \( C \). Let \( z'_0 \) be the \( k \)-point of \( \mathcal{N} \) which corresponds to the vector bundle

\[
E_0 = W^\oplus n.
\]

We let \( z_0 \) be the image of \( z'_0 \) in \( M \); this \( z_0 \) is also a \( k \)-point.
(5) Let \( \pi : \tilde{M} \to M \) be a desingularization which is an isomorphism outside the singular locus of \( M \); in particular, it is an isomorphism over \( M_0 \). We thus have the following diagram.

\[
\begin{array}{ccc}
M & \to & N \\
\downarrow & & \downarrow \theta \\
\tilde{M} & \xrightarrow{\pi} & M
\end{array}
\]

**Remark 2.** Since the Brauer group is a birational invariant for smooth projective varieties, proving Theorem 1 for one particular desingularization of \( M \) is equivalent to proving it for all desingularizations of \( M \). Thus it is enough to prove Theorem 1 for the desingularization \( \tilde{M} \) chosen in (5).

**Lemma 3.** Given any \( K \)-point \( x_0 \in N(K) \), where \( K/k \) is a field extension, there exists a smooth curve \( Y/K \), a \( K \)-point \( y_0 \in Y(K) \), and a map \( \psi : Y \to N \), such that \( \psi(y_0) = x_0 \) and \( \psi(Y) \cap M_0 \neq \emptyset \).

**Proof.** By [LM, p. 49, Théorème 6.3], we can choose an atlas \( \pi : U \to N \) such that \( x_0 \) lifts to a \( K \)-point \( \tilde{x}_0 \) of \( U \). Since \( N/k \) is smooth, so is \( U/k \). Thus, a general complete intersection curve in \( U/k \) passing through \( \tilde{x}_0 \) satisfies the conditions. \( \square \)

**Lemma 4.** Fix any integer \( n \geq 2 \). There exists a field extension \( K/k \), a \( k \)-discrete valuation \( v \) on \( K \), and a central division algebra \( D/K \) of index \( n \), such that for any integer \( \ell \), the class \( \ell \cdot [D] \in \text{Br}(K) \) is unramified at \( v \) if and only if \( \ell \) is divisible by \( n \). (Here \( [D] \) denotes the class in \( \text{Br}(K) \) defined by \( D \).)

**Proof.** Set \( K = k(x,y) \) to be the purely transcendental extension. Let \( v \) be the valuation given by the height one prime ideal

\[(x) \subset k[x,y]\]

and let \( L = k(y) \) denote the residue field of \( K \) at \( v \). Set \( D \) to be the cyclic algebra \( (x,y)_\zeta \), where \( \zeta \) is any chosen primitive \( n \)-th root of unity. The obstruction for an \( n \)-torsion class in \( \text{Br}(K) \) to be unramified is measured by the tame symbol

\[H^2(K, \mu_n) \to H^1(L, \mathbb{Z}/n) \cong L^*/(L^*)^n.\]

Here we identify \( H^2(K, \mu_n) \) with the \( n \)-torsion subgroup of \( \text{Br}(K) \). Note that the isomorphism \( H^1(L, \mathbb{Z}/n) \cong L^*/L^{*n} \) depends on the choice of \( \zeta \). The image of \([D]\) in \( L^*/L^{*n} \) is the class defined by \( y^{-1} \) (see [GS, Example 7.1.5 and Corollary 7.5.3]) which has order \( n \). Hence \( \ell \cdot [D] \) is unramified at \( v \) if and only if \( \ell \) is
divisible by \( n \). Moreover, since the order of \([D]\) is equal to its index, \( D \) is a division algebra. \(\square\)

**Lemma 5.** There is a natural inclusion \( \text{Br}(\widetilde{\mathcal{M}}) \hookrightarrow \text{Br}(M_0) \), where \( M_0 \) is defined in (2).

**Proof.** Since \( \widetilde{\mathcal{M}} \) is smooth, the homomorphism \( \text{Br}(\widetilde{\mathcal{M}}) \rightarrow \text{Br}(\pi^{-1}(M_0)) \) induced by the open embedding \( \pi^{-1}(M_0) \hookrightarrow \widetilde{\mathcal{M}} \) is injective. Now the lemma follows immediately from the assumption that the morphism

\[
\pi|_{\pi^{-1}(M_0)} : \pi^{-1}(M_0) \rightarrow M_0
\]

is an isomorphism. \(\square\)

### 3. Proof of Theorem 1

If \( g = 2 = r \), and \( d \) is even, then \( M = \mathbb{P}^d_k \) [NR, pp. 33–34, Theorem 2]; hence Theorem 1 holds in this case. If \( g = 2 = r \), and \( d \) is odd, then \( M \) is a smooth projective rational variety as \( n = 1 \) (see (1) of Section 2), so Theorem 1 holds in this case also. Hence we assume that \( g \geq 3 \) if \( r = 2 \).

Consider the \( \mathbb{G}_m \)-gerbe \( \mathcal{M} \rightarrow \mathcal{N} \) in (3) of Section 2. Let

\[
\alpha \in \text{Br}(\mathcal{N})
\]

be the class defined by it. Since \( \theta : \mathcal{N} \rightarrow M \) is an isomorphism over \( M_0 \), we consider \( M_0 \) also as an open subset of \( \mathcal{N} \). Thus

\[
(3.1) \quad \alpha' := \alpha|_{M_0}
\]

defines an element of \( \text{Br}(M_0) \). This class \( \alpha' \in \text{Br}(M_0) \) generates \( \text{Br}(M_0) \), and its order is precisely \( n \) [BBGN, p. 267, Theorem 1.8].

Since \( M_0 \) can also be identified with an open subset of \( \widetilde{\mathcal{M}} \) (see (5) of Section 2), it makes sense to ask whether a class in \( \text{Br}(M_0) \) extends to a class in \( \text{Br}(\widetilde{\mathcal{M}}) \).

In view of Lemma 5 and the above description of \( \text{Br}(M_0) \), in order to prove Theorem 1 it suffices to show the following:

**Statement A.** For a given integer \( \ell \), the class \( \ell \alpha' \) extends to an element of \( \text{Br}(\widetilde{\mathcal{M}}) \) only if \( \ell \) is a multiple of \( n \) (or equivalently if \( \ell \alpha' \) vanishes).

We will prove Statement A in three steps.

**Step 1:** Let \( K/k \) be a field extension, and let \( D/K \) be a central division algebra of index \( n \), given by Lemma 4. Let \( X \rightarrow \text{Spec}(K) \) be the \( \mathbb{G}_m \)-gerbe defined by the class \([D] \in H^2(K, \mathbb{G}_m)\). Then there exists a twisted bundle \( V \) of rank \( n \) on \( X \) such that \( \mathcal{E}nd_{\mathcal{O}_X}(V) \) descends to the coherent sheaf on \( \text{Spec}(K) \) corresponding to \( D \). Consider the vector bundle

\[
F := V \otimes W
\]
on $X \times_k C$, where $W$ is the vector bundle in (4) of Section 2. This $F$ can be thought of as a family of semistable vector bundles on $C$ parameterized by $X$. Hence we get a morphism

$$f : X \rightarrow \mathcal{M}$$

representing this family. This morphism $f$ induces a morphism

(3.2) \hspace{1cm} x_0 : \text{Spec}(K) \rightarrow \mathcal{N}.

By the construction of $x_0$, it has the following properties:

1. $x_0^*\alpha = [D]$, and
2. the image of $x_0$ is equivalent to the point $z'_0$ that corresponds to $E_0$ in (4) of Section 2.

Step 2: Using Lemma 3 we find a smooth curve $Y/K$, a 1-morphism $\psi : Y \rightarrow \mathcal{N}$ and $y_0 \in Y(K)$, such that

$$\psi(y_0) = x_0.$$

Consider the map from $Y \rightarrow M$ given by the composition $Y \rightarrow \mathcal{N} \xrightarrow{\theta} M$. By Lemma 3, the generic point of $Y$ maps into $M_0 \subset M$ and hence lifts to $\widetilde{M}$. Since

$$\pi : \widetilde{M} \rightarrow M$$

is proper, by using the valuative criterion of properness we see that the map $Y \rightarrow M$ factors through $\widetilde{M}$.

We thus have a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\psi} & \mathcal{N} \\
\downarrow{h} & & \downarrow{\theta} \\
\widetilde{M} & \xrightarrow{\pi} & M
\end{array}
$$

Step 3: Now suppose for some integer $\ell$, the class $\ell \cdot \alpha' \in \text{Br}(M_0)$ (see 3.1) extends to a class $\beta$ on entire $\widetilde{M}$. The two classes $\ell \cdot \alpha \in \text{Br}(\mathcal{N})$ and $\beta \in \text{Br}(\widetilde{M})$ coincide when restricted to $M_0$, which is canonically identified with an open subset of $\mathcal{N}$ as well as of $\widetilde{M}$.

We claim that

(3.3) \hspace{1cm} \psi^*(\ell \cdot \alpha) = h^*\beta \in \text{Br}(Y).

The claim follows since $Y$ is a regular integral scheme and the above Brauer classes coincide on the dense open subset $h^{-1}(M_0)$ of $Y$.

Thus, restricting the above classes in $Br(Y)$ to the $K$-point $y_0 : \text{Spec}(K) \rightarrow Y$, and using $\psi \circ y_0 = x_0$, we get that

$$x_0^*(\ell \cdot \alpha) = y_0^*(h^*\beta) \in \text{Br}(K).$$
However, $\tilde{M}$ being a proper variety, the morphism

$$y_0 \circ h : \text{Spec} (K) \to \tilde{M}$$

extends to a morphism

$$\text{Spec} (R) \to \tilde{M},$$

where $R$ is the discrete valuation ring corresponding to the valuation $v$ on $K$. Thus the class $y_0^*(h^*\beta) \in \text{Br}(K)$ is unramified at $v$. But by construction of $x_0$ in Step 1, the class $x_0^*(\ell \cdot \alpha)$ coincides with $\ell \cdot [D]$, which, by Lemma 4, is ramified at $v$ unless $\ell$ is divisible by $n$. This implies that $\ell$ is divisible by $n$. This proves Statement A, and hence the proof of Theorem 1 is complete.

Remark 6. Since the moduli space $M$ is locally factorial, it follows that the natural homomorphism $\text{Br}(M) \to \text{Br}(M_0)$ is injective. From Statement A we know that no nonzero class $\ell \alpha' \in \text{Br}(M_0)$ extends to $\text{Br}(\tilde{M})$, hence such a class cannot extend to $\text{Br}(M)$. Consequently, the Brauer group of the moduli space $M$ vanishes.

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