Distributed Computation of the Conditional PCRLB for Quantized Decentralized Particle Filters

Arash Mohammadi*, Amir Asif†, Xionghu Zhong‡, and A. B. Premkumar+

*Computer Science and Engineering, York University, Email: {marash, asif}@cse.yorku.ca.
†Computer Engineering, Nanyang Technological University, Email: {xzhong, asannamalai}@ntu.edu.sg.

Abstract—The conditional posterior Cramér-Rao lower bound (PCRLB) is an effective sensor resource management criteria for large, geographically distributed sensor networks. Existing algorithms for distributed computation of the PCRLB (dPCRLB) are based on raw observations leading to significant communication overhead to the estimation mechanism. This letter derives distributed computational techniques for determining the conditional dPCRLB for quantized, decentralized sensor networks (CQ/dPCRLB). Analytical expressions for the CQ/dPCRLB are derived, which are particularly useful for particle filter-based estimators. The CQ/dPCRLB is compared for accuracy with its centralized counterpart through Monte-Carlo simulations.

Index Terms—Bayesian Estimation, Distributed Signal Processing, Particle Filters, PCRLB, Sensor Resource Management.

I. INTRODUCTION

Recent developments in sensor technologies and advances in communication systems allow a large number of observation nodes (sensors) to be deployed in geographically distributed sensor networks typically configured using the decentralized topology without employing a global fusion centre. To elaborate further, the letter considers a commonly reported decentralized sensor network topology [1] with two types of nodes: (i) Sensor nodes: with limited power used to record measurements, and; (ii) Local processing nodes: responsible for selecting sensors, processing the data locally, and cooperating distributively with other connected processing nodes to reach a consensus tracking estimate for the target. In such geographically distributed sensor networks, limitations in power budget, system bandwidth, and communication capabilities impose two critical restrictions. First, the maximum number of active sensors at a particular time is constrained. Second, only quantized observations are exchanged between the sensors and processing nodes. Within its observation neighbourhood, a local processing node, therefore, activates a small subset of sensors to receive the quantized version of their observations.

Motivation: The Posterior Cramér-Rao Lower Bound (PCRLB) has been used as an effective criteria for adaptive sensor resource management problems [1, 2] because it provides a near-optimal bound of the achievable estimator’s performance and can be computed predictively. Further, it is independent of and not constrained by a specific estimation methodology. Existing PCRLB derivations [3]–[5] using quantized observations are limited to centralized estimation architectures and have not yet been extended to decentralized topologies. The letter addresses this gap and derives the PCRLB for decentralized state estimation in sensor networks with quantized observations. We refer to the distributed computation of the PCRLB as dPCRLB. Our previous work [6] derives distributed expressions for computing the non-conditional dPCRLB for full-order decentralized state estimation. In [7], we extend our derivations to the conditional dPCRLB (where instead of using statistics for the observation model, actual observations from the previous iterations are utilized). Both [6] and [7] consider raw observations in the dPCRLB derivations, as is also the status quo in the decentralized estimation literature [1]. Such a setup leads to a large communication overhead between the sensors and their associated processing node making the system impractical.

Contributions: The letter extends the conditional dPCRLB framework to quantized observations with emphasis on particle filter estimators. Additional contributions of the letter include: (a) Both computational and communication complexity of [7] are reduced in the proposed conditional dPCRLB with quantized observations (CQ/dPCRLB). (b) In [7], the conditional Fisher information matrix (FIM), i.e., the inverse of the conditional dPCRLB, is expressed as a function of the auxiliary FIM which is updated distributively at each iteration. The CQ/dPCRLB updates the conditional dPCRLB directly without the need of computing the auxiliary FIM leading to significant communication savings.

II. SYSTEM DESCRIPTION

We consider the non-linear dynamical system

$$x(k) = f(x(k-1)) + \xi(k),$$

(1)

where the state vector $x = [X_1, X_2, \ldots, X_{n_x}]^T$ and $\xi(k)$ is the global uncertainty in the state model at iteration $k$. Processing node $l$, $(1 \leq l \leq N_t)$, is connected to a set of sensor nodes: a subset of which is active at each iteration. The active sensors connected to node $l$ constitute its local observation neighbourhood $N_{\text{obs}}^{(l)}$. The total number of active sensors in the network is $N_s = \sum_{l=1}^{N_t} |N_{\text{obs}}^{(l)}|$, where $| \cdot |$ denotes the cardinality operator. Sensor $m$ in the observation neighbourhood of node $l$, i.e., $m \in N_{\text{obs}}^{(l)}$, makes observation $Z^{(l,m)}(k)$. Instead of transferring the raw observation, sensor $m$ communicates its quantized version $Y^{(l,m)}(k)$ to the fusion node $l$ based on the following model

$$Y^{(l,m)}(k) = Q^{(l,m)}(g^{(l,m)}(x(k)) + \zeta^{(l,m)}(k),$$

(2)

where $Q^{(l,m)}(\cdot)$ is the quantization function for sensor $m$, $g^{(l,m)}(x(k))$ is the local sensor measurement, and $\zeta^{(l,m)}(k)$ is the quantization noise.
where \( Q^{(l,m)}(\cdot) \) is the local quantization operator at node \( l \), and \( g^{(l,m)}(\cdot) \) and \( \chi^{(l,m)}(\cdot) \) are, respectively, the local observation model and uncertainty at sensor \( m \) connected to fusion node \( l \). For simplicity and without loss of generality, the quantization operators \( Q^{(l,m)}(\cdot) \) are considered to be the same across the network (i.e., \( Q^{(l,m)}(\cdot) = Q(\cdot) \)). Collectively, the overall quantized observation vector at node \( l \) is denoted by
\[
y^{(l)}(k) = \{Y^{(l,m)}(k) : m \in \mathcal{N}^{(l)}_{\text{obs}}\}, \quad \text{for } (1 \leq l \leq N_f). \tag{3}
\]
Depending on how many sensors are activated by the processing node \( l \), the dimension of observation vector \( y^{(l)}(k) \) is different at each processing node. As for the quantized observations \( y^{(l)}(k) \), vector \( z^{(l)}(k) \) is the collection of all raw observations associated with the processing node \( l \), i.e.,
\[
z^{(l)}(k) = \{Z^{(l,m)}(k) : m \in \mathcal{N}^{(l)}_{\text{obs}}\}, \quad \text{for } (1 \leq l \leq N_f). \tag{4}
\]
We consider an \( N_L \)-bit quantization scheme, where node \( m \)'s quantized observation \( Y^{(l,m)}(k) \) can take any discrete value between 0 and \( 2^{N_L} - 1 \). The set of quantization threshold is denoted by \( q = \{q_0, q_1, \ldots, q_{2^{N_L}}\} \) where for brevity \( q_0 = -\infty \) and \( q_{2^{N_L}} = \infty \). The likelihood that \( Y^{(l,m)}(k) \) is at level \( q_l \) is denoted by \( h^{(l,m)}_i(k) = P(Y^{(l,m)}(k) = q_l | x(k)) \) with
\[
h^{(l,m)}_i(k) = P(q_l \leq Z^{(l,m)}(k) \leq q_{l+1} | x(k)) \tag{5}
\]
\[
= P\left[q_l - g^{(l,m)}(x(k))] \leq \chi^{(l,m)}(k) \leq [q_{l+1} - g^{(l,m)}(x(k))]\right].
\]
Section [III](#) reviews the local conditional dPCRLB for raw observations as presented in [7](#) with one proposed modification.

### III. Conditional dPCRLB for Raw Observations

Based on the conditional PCRLB inequality, the mean square error associated with the local estimate \( \hat{x}^{(l)}(0 : k + 1) \) of the state vector at node \( l \) is lower bounded as follows
\[
\mathbb{E}[P^{(l)}(k+1)] \{e^{(l)}(0 : k + 1) | e^{(l)}(0 : k + 1) \} T \geq \{I^{(l)}(0 : k + 1) \}, \tag{6}
\]
where \( P^{(l)}(k+1) \equiv P(x(0 : k), z^{(l)}(k+1) | z^{(l)}(1 : k)) \) and \( \{e^{(l)}(0 : k + 1) \} \) denotes expectation, and \( e^{(l)}(0 : k + 1) \equiv \chi^{(l)}(0 : k + 1) - \hat{x}^{(l)}(0 : k + 1) \) is the estimation error. Defining the 1st and 2nd order partial derivatives as
\[
\nabla_{x(k)}[\chi^{(l)}(k)] = \left[\frac{\partial}{\partial x_1(k)}, \ldots, \frac{\partial}{\partial x_n(k)}\right]^T \quad \text{and} \quad \Delta_{x(k-1)} = \nabla_{x(k-1)}[\chi^{(l)}(k)] \tag{7}
\]
and \( \Delta_{x(k-1)} = \nabla_{x(k-1)}[\chi^{(l)}(k)] \), the local quantized conditional FIM \( I^{(l)}(0 : k + 1) \) corresponds to the state trajectory \( \hat{x}^{(l)}(0 : k + 1) \) from iteration 0 to \( k + 1 \) and is given by
\[
I^{(l)}(0 : k + 1) = \mathbb{E}[P^{(l)}(k+1)] - \Delta_{x(0 : k + 1)} \log P^{(l)}(k+1)]. \tag{8}
\]
Another local FIM is the local instantaneous conditional FIM \( L^{(l)}(k + 1) \) associated with \( \hat{x}^{(l)}(k + 1) \), which is obtained by taking the inverse of \((n_x \times n_x)\) right-lower block of \( \{I^{(l)}(0 : k + 1) \}. \) Please refer to Appendix A for differences in the two FIMs. Node \( l \) updates \( L^{(l)}(k + 1) \) as follows.

#### Result 1. The instantaneous local FIM \( L^{(l)}(k + 1) \) associated with estimate \( \hat{x}^{(l)}(k + 1) \) at node \( l \) is computed as follows
\[
L^{(l)}(k + 1) \approx \begin{bmatrix} B^{22}(k) \end{bmatrix}^{(l)} \tag{7}
\]
\[
- \begin{bmatrix} B^{11}(k) \end{bmatrix}^{(l)} \begin{bmatrix} L^{(l)}(k) \end{bmatrix}^{(l)} + \begin{bmatrix} B^{11}(k) \end{bmatrix}^{(l)} \begin{bmatrix} B^{22}(k) \end{bmatrix}^{(l)} \tag{8}
\]
where \( \begin{bmatrix} B^{11}(k) \end{bmatrix}^{(l)} = \mathbb{E}\left[-\Delta_{x(k)} \log P(x(k + 1) | x(k))\right], \tag{9} \]
\( \begin{bmatrix} B^{22}(k) \end{bmatrix}^{(l)} = \mathbb{E}\left[-\Delta_{x(k)} \log P(x(k + 1) | x(k))\right] \tag{10} \)
and
\[
\begin{bmatrix} B^{22}(k) \end{bmatrix}^{(l)} = \mathbb{E}\left[-\Delta_{x(k)} \log P(x(k + 1) | x(k))\right] \tag{11} \]

The derivation of Result [I](#) is included in Appendix B. In [7](#), \( L^{(l)}(k + 1) \) is computed recursively from the local instantaneous auxiliary FIM \( J^{(l)}_{\text{aux}}(k) \), which is the inverse of \((n_x \times n_x)\) right-lower block of the accumulated auxiliary FIM \( \{J^{(l)}_{\text{aux}}(0 : k)\}^{-1} \). The latter is defined as
\[
J^{(l)}_{\text{aux}}(0 : k) \triangleq \mathbb{E}[P^{(l)}(k+1)] - \Delta_{x(0 : k)} \log P^{(l)}(k) \tag{12} \]

Having computed the local FIMs \( L^{(l)}(k + 1) \) and the local prediction FIMs \( L^{(l)}(k + k + 1) \) at iteration \( k + 1 \), the next step in the conditional dPCRLB is to fuse these local FIMs to compute the global instantaneous conditional FIM \( L^{(G)}_{\text{aux}}(k + 1) \). Reference [7](#) derives a fusion rule for assimilating local conditional FIMs into the global conditional FIM when raw observations are available at each local node. Section [IV](#) extends our derivations to quantized observations and eliminates the need for fusion of local instantaneous auxiliary FIMs.

### IV. CQ/dPCRLB with Quantized Observations

In Proposition [I](#) raw observations \( Z^{(l,m)}(k) \) are replaced with their quantized version \( Y^{(l,m)}(k) \), which results in the quantized filtering conditional FIM \( L^{(l)}_{\text{Q}}(k + 1) \). Since terms \( B^{11}(k)^{(l)} \), \( B^{22}(k)^{(l)} \), \( B^{21}(k)^{(l)} \) are based on the state model, they remain the same. Term \( B^{22}(k)^{(l)} \) in Eq. (10) is now computed using the quantized observation as follows
\[
B^{22}(k)_{\text{Q}}^{(l)} = \mathbb{E}\left[-\Delta_{x(k)} \log P(x(k + 1) | x(k))\right] \tag{13}
\]

To compute \( J^{(l)}_{\text{aux}}(k + 1) \), likelihood \( P^{(l)}(y(k + 1) | x(k + 1)) \) along with the second derivative of its logarithmic function is needed. Because of quantized observations, \( P^{(l)}(y(k + 1) | x(k + 1)) \) transforms into a probability mass function that is discrete with second derivative replaced by a double summation as described below. Given the state variables, local observations are assumed independent such that
\[
J^{(l)}_{\text{aux}}(k + 1) = \sum_{m \in \mathcal{N}^{(l)}_{\text{obs}}(k)} J(Y^{(l,m)}(k+1)), \tag{14}
\]
where
\[
J(Y^{(l,m)}(k+1)) = \sum_{i=1}^{N_L} -\mathbb{E}\left[\delta(Y^{(l,m)}(k+1) - i) \Delta_{x(k)} \log h^{(l,m)}(k+1)\right] \tag{15}
\]
and \( \delta(\cdot) \) is the delta function. We note that \( \mathbb{E}\{\delta(Y_{l,m}^{(l,m)}(k+1)-i)\}=h_{i}^{(l,m)}(k) \), where \( h_{i}^{(l,m)}(k) \) was defined immediately after Eq. (5) previously and has the second derivative

\[
\Delta \log[h_{i}^{(l,m)}(k)] = \frac{\partial^{2} \log(h_{i}^{(l,m)}(k))}{\partial X_{l,m}(k) \partial X_{u}(k)} = \frac{\partial^{2} \log(h_{i}^{(l,m)}(k))}{\partial X_{l}(k) \partial X_{u}(k)} \cdot \frac{\partial^{2} \log(h_{i}^{(l,m)}(k))}{\partial X_{u}(k) \partial X_{u}(k)}
\]

Under mild regularity conditions, the expected value of (16) is equal to the variance of its first moment, i.e.,

\[
\mathbb{E}\{\frac{\partial^{2} \log(h_{i}^{(l,m)}(k))}{\partial X_{l,m}(k) \partial X_{u}(k)}\} = -\mathbb{E}\left\{\frac{\partial h_{i}^{(l,m)}(k)}{\partial x_{l}(k)} \frac{\partial h_{i}^{(l,m)}(k)}{\partial x_{u}(k)} \left[ h_{i}^{(l,m)}(k) \right]^2 \right\}, \tag{17}
\]

Eqs. (14)-(17) are used to compute \( B_{22}^{(l)}(k) \). Finally, the local quantized filtering FIM is given by

\[
L_{Q}^{(l)}(k+1) \approx B_{22}^{(l)}(k) \tag{18}
\]

\[
-\left[B_{22}^{(l)}(k)\right]^{-1} \left[L_{Q}^{(l)}(k)+B_{11}^{(l)}(k)\right]^{-1} \left[B_{12}^{(l)}(k)\right] \tag{19}
\]

Eq. (18) is derived by applying the following factorization

\[
P(x(0 : k+1), y^{(l)}(1 : k+1)) = P(x(0 : k), y^{(l)}(1 : k)) \times P(x(k+1)|x(k)) P(y^{(l)}(k+1)|x(k+1)) \tag{19}
\]

to the quantized version of Eq. (6) and then taking the inverse of the \((n_{x} \times n_{x})\) right lower block of \([L_{Q}^{(l)}(0 : k+1)]^{-1}\). The similarity between Eqs. (7) and (13) is intuitively pleasing. The local predictive FIM \( L_{Q}^{(l)}(k+1)|k \) is derived in the similar manner as (18) with \([B_{22}^{(l)}(k)]\) replaced by (12).

**Fusing Local FIMs (CQ/dPCRLB):** Result 2 provides a fusion rule for assimilating the local FIMs with quantized observations to compute the global quantized FIM.

**Result 2.** The sequence \( \{L_{Q}^{(G)}(k+1)\} \) corresponding to the global information submatrix (CQ/dPCRLB) with quantized local observations follows the following recursion

\[
L_{Q}^{(G)}(k+1)=C_{22}^{(G)}(k)+C_{12}^{(G)}(k)\left[L_{Q}^{(G)}(k)+C_{11}^{(G)}(k)\right]^{-1}C_{12}^{(G)}(k) \tag{20}
\]

where \( C_{12}^{(G)}(k) = \mathbb{E}\{-\Delta \log h_{i}(x(k)) \log P(x(k+1)|x(k))\} \),

\[
C_{11}^{(G)}(k) = \mathbb{E}\{-\Delta \log h_{i}(x(k)) \log P(x(k+1)|x(k))\} \tag{22}
\]

and \( C_{22}^{(G)}(k) \approx \sum_{l=1}^{n_{l}} L_{Q}^{(l)}(k+1) - \sum_{l=1}^{n_{l}} L_{Q}^{(l)}(k+1|k) \)

\[
+\mathbb{E}\{-\Delta \log h_{i}(x(k+1)) \log P(x(k+1)|x(k))\}. \tag{23}
\]

The proof of Result 2 is included in Appendix C.

**Gaussian Observation Noise:** We derive analytical expressions for the case when local observations \( Z_{l,m}^{(l,m)}(k) \) are zero-mean Gaussian with variance \( R_{l,m}^{(l,m)}(k) \), i.e., \( Z_{l,m}^{(l,m)}(k) \sim N(0, R_{l,m}^{(l,m)}(k)) \). The likelihood that \( Y_{l,m}^{(l,m)}(k) \) is at level \( q_{l} \) is

\[
h_{i}^{(l,m)}(k)=\frac{1}{\sqrt{2\pi R_{l,m}^{(l,m)}(k)}} \int_{q_{l}+g_{l}(x(k))}^{q_{l+1}+g_{l}(x(k))} \exp \left\{-\frac{t}{2R_{l,m}^{(l,m)}(k)}\right\} dt
\]

\[
= \Phi \left( \frac{q_{l}-g_{l}(x(k))}{\sqrt{R_{l,m}^{(l,m)}(k)}} \right) - \Phi \left( \frac{q_{l+1}-g_{l}(x(k))}{\sqrt{R_{l,m}^{(l,m)}(k)}} \right). \tag{24}
\]

where \( \Phi(\cdot) \) is the standard cumulative Gaussian distribution. Based on (24), each derivative term in (17) is represented as

\[
\frac{\partial h_{i}^{(l,m)}(k)}{\partial x_{l}(k)} = -\frac{\partial h_{i}^{(l,m)}(x(k))}{\partial x_{l}(k)} \times \left( \frac{\exp \left(-\frac{(q_{l+1}-g_{l}(x(k)))}{2R_{l,m}^{(l,m)}(k)}\right)}{2R_{l,m}^{(l,m)}(k)} \right) \tag{25}
\]

A. **Computation of The Conditional dPCRLB**

The analytical computation of the expectations in Result 2 is not practical and, therefore, particle filter-based approaches are proposed. If the state estimator is based on distributed particle filters, then the same particle set can be used in the CQ/dPCRLB algorithm. An active sensor communicates its quantized observation to the associated processing node. The processing nodes themselves communicate the local conditional FIMs and statistics of local posteriors (i.e., local state estimates and their corresponding covariance matrices) to the neighbouring processing nodes which are then fused in a distributed fashion to compute the global state estimate and the global conditional FIM. We explain the CQ/dPCRLB algorithm in the context of the consensus based distributed particle filter (CF/DPF) being used as the state estimator. The CF/DPF implements two particle filters at each node: (i) Local filter which approximates the local posterior at node \( l \) with a set of weighted particles \( \{x_{i}^{(l,LF)}(k), W_{i}^{(l,LF)}(k)\} \), and; (ii) Fusion filter which combines the local posteriors to estimate the global posterior with a second set of particles \( \{x_{i}^{(l,FP)}(k), W_{i}^{(l,FP)}(k)\} \). All information regarding the observations collected up to time \( k \) at node \( l \) are presented in the local particles \( x_{i}^{(l,LF)}(k) \), while the information available across the network is provided by the global particles \( x_{i}^{(l,FP)}(k) \). The CQ/dPCRLB comprises of the following steps:

**I. Local FIMs:**

1. Eqs. (6)-(9) are computed at node \( l \) based on Monte-Carlo integration using local particles \( X_{i}^{(l,LF)}(k) \).

2. For computing Eq. (13), first, node \( l \) computes the predictive particles \( X_{i}^{(l,LF)}(k+1|k) \) through propagating \( X_{i}^{(l,LF)}(k) \) through \( P(x(k+1)|x(k)) \), and then computes Eq. (13) using \( X_{i}^{(l,LF)}(k) \) and \( X_{i}^{(l,LF)}(k+1|k) \).

3. The local FIMs are then computed using Eq. (18).

**II. Global FIM:**

4. The expectations in (21)-(23) are computed using the global particles \( X_{i}^{(l,FP)}(k) \) to derive the FIMs \( C_{G}^{(l)}(k) \). Eq. (23) includes summation of local FIMs across the network typically computed using the average consensus algorithms [7] in a decentralized fashion.

5. Result 2 is used to compute the global FIM based on the local FIMs computed in Step 4.

**B. Communication Savings**

First, the transfer of quantized observation (instead of raw data) between sensors and associated processing nodes leads to significant communication savings. Second, the communication overhead for computing the global auxiliary FIM from the local auxiliary FIMs across the network is eliminated in the proposed CQ/dPCRLB algorithm.
consensus, the second savings is of $O(n_x|N_{\text{time}} I_n| N_c)$ (i.e., the communication complexity reduces by half), where $n_x$ is number of states, $|N_{\text{time}} I_n|$ the number of processing nodes in the neighbourhood of processing node $l$, and $N_c$ the number of consensus iterations. For complexity analysis of the CF/DPF refer to [6]. The CQ/dPCRLB can be further extended to communicate quantized versions of the local state statistics (quantized local tracks [8]) and local FIMs between neighbouring processing nodes during the fusion filter stage which will be considered in future work.

V. SIMULATION

A decentralized bearing-only tracker with nonlinear clockwise coordinate turn state model [6] and observation model

$$z^{(l,m)}(k) = \tan \left[ \frac{X^{(l,m)}(k) - X^{(l,m)}(0)}{Y^{(l,m)}(k) - Y^{(l,m)}(0)} \right] + \zeta^{(l,m)}(k),$$

(26)

is considered. $(X^{(l,m)}, Y^{(l,m)})$ represents the coordinates of sensor $(l,m)$. Both process and observation noises are normally distributed with the observation noise $(\zeta^{(l,m)}(k))$ model assumed to be state dependent such that the bearing noise variance at sensor $(l,m)$ depends on the distance between the observer and target. A sensor network (Fig. 1(a)) consisting of 225 static sensors and $N_f = 9$ processing nodes scattered in a square region of dimension $(1500 \times 1500)\text{m}^2$ is implemented. Our goal is to evaluate the performance of the proposed CQ/dPCRLB, therefore, the activated sensors are selected at random and limited to three sensors per processing node. For simplicity, the sensors are distributed uniformly with the processing node at the centre of its rectangular $(500 \times 500)\text{m}$ neighbourhood. Each processing node communicates only with its activated sensors within its $(500 \times 500)\text{m}$ neighbourhood and other processing nodes within a radius of 550m.

The objective of our Monte Carlo simulations is three folds. The first objective is to validate the effectiveness of the conditional FIM approximation (i.e., to replace the global auxiliary FIM with the global conditional FIM) in Result 2. Fig. 1(b) plots the conditional dPCRLB and CQ/dPCRLB with and without the proposed global conditional FIM approximation. In each case, results for both raw (bottom two plots) and quantized (top two plots) observations are included. Within each set of plots in Fig. 1(b), the bounds virtually overlap verifying the effectiveness of the global conditional FIM approximation. The second objective is to compare the CQ/dPCRLB with quantized observations for accuracy against the conditional dPCRLB computed from raw observations [7]. Comparing bounds across the two sets of plots in Fig. 1(b), we note that the respective plots do not overlap but are fairly close to each other. Despite using quantized observations, the CQ/dPCRLB is a reasonable approximation of the dPCRLB. Illustrated in Fig. 1(c), the third objective is to quantify the potential CQ/dPCRLB performance loss as a function of the number of quantization levels. The CQ/dPCRLB approaches the dPCRLB as the number of quantization levels are increased.

VI. CONCLUSION

The PCRLB has recently been proposed [1] as an effective selection criteria for decentralized sensor resource management in large, geographically distributed sensor networks. Existing decentralized algorithms for computing the PCRLB are typically based on raw observations resulting in a significant communication overhead. The letter derives the PCRLB for decentralized estimators in sensor networks with quantized observations and tests it in a bearing only tracking application.

APPENDIX A

Below, we highlight the relationship between the local accumulated conditional FIM $I^{(l)}(0:k+1)$ and local instantaneous conditional FIM $L^{(l)}(k+1)$. The local instantaneous conditional FIM $L^{(l)}(k+1)$ is computed using either of the following three approaches: (i) Directly by inverting large matrix $I^{(l)}(0:k+1)$; (ii) Recursively as a function of the previous local instantaneous auxiliary FIM $J_{\text{aux}}(l)(k)$ [7]; and (iii) Recursively as a function of the previous local instantaneous conditional FIM $L^{(l)}(k)$ presented in Result 1. In approach (i), first the local accumulated conditional FIM $I^{(l)}(0:k+1)$ is factorized as follows

$$I^{(l)}(0:k+1) = \begin{bmatrix} [A^{11}(k+1)]^{(l)} & [A^{12}(k+1)]^{(l)} \\ [A^{21}(k+1)]^{(l)} & [A^{22}(k+1)]^{(l)} \end{bmatrix},$$

(27)

$$= \mathbb{E} \left\{ \begin{bmatrix} \Delta \pi^{(0:k)}(k+1) \\ \pi^{(0:k)}(k+1) \end{bmatrix} \begin{bmatrix} \pi^{(0:k)}(k) \\ \Delta \pi^{(0:k)}(k+1) \end{bmatrix} \log P_c^{(l)}(k+1) \right\}.$$
Then, the local instantaneous conditional FIM $L^{(i)}(k+1)$ associated with the estimate $\hat{x}(k+1)$ is obtained by taking the inverse of the $(n_x \times n_x)$ right-lower square block of $[I^{(i)}(0:k+1)]^{-1}$ by applying the following matrix inversion Lemma.

**Lemma 1.** Matrix inversion Lemma:

\[
\begin{bmatrix}
A & B \\
B^T & C
\end{bmatrix}^{-1} = \begin{bmatrix}
\Omega^{-1} & -A^{-1}B\Phi^{-1} \\
-\Phi^{-1}B^TA^{-1} & \Phi^{-1}
\end{bmatrix},
\]

(28)

where subblocks $\{A, B, C\}$ have conformable dimensions, $\Omega = A - BC^{-1}B^T$, and $\Phi = C - B^TA^{-1}B$.

Based on Lemma 1, the local instantaneous conditional FIM is given by

\[
L^{(i)}(k+1) = \begin{bmatrix}
[A^{11}(k+1)]^{(i)} \\
[A^{12}(k+1)]^{(i)} \\
[A^{21}(k+1)]^{(i)} \\
[A^{22}(k+1)]^{(i)}
\end{bmatrix}
\]

(29)

For next iteration $k+1$, we have

\[
L^{(i)}(k+1) = \begin{bmatrix}
[A^{11}(k+1)]^{(i)} + [A^{11}(k)]^{(i)} - [A^{12}(k)]^{(i)} - [A^{21}(k)]^{(i)} \\
[A^{12}(k+1)]^{(i)} - [A^{12}(k)]^{(i)} \\
[A^{22}(k+1)]^{(i)} - [A^{22}(k)]^{(i)}
\end{bmatrix}
\]

(30)

where $P_c^{(i)}(k+1) = P(\hat{x}(0:k+1), z^{(i)}(k+1)|z^{(i)}(1:k))$ which can be factorized as follows

\[
P(\hat{x}(0:k+1), z^{(i)}(k+1)|z^{(i)}(1:k)) = P(\hat{x}(0:k+1)\|z^{(i)}(1:k)) \times P(\hat{x}(0:k+1)|x(k)) \times P(z^{(i)}(k+1)|z^{(i)}(1:k-1))
\]

(31)

Taking logarithm of Eq. (33),

\[
\log P_c^{(i)}(k+1) = \log P(\hat{x}(0:k+1)\|z^{(i)}(1:k)) + \log P(\hat{x}(0:k+1)|x(k)) + \log P(z^{(i)}(k+1)|z^{(i)}(1:k-1)).
\]

(34)

Therefore, Eq. (32) reduces to Eq. (35) on top of the next page where $[B^{11}(k)]^{(i)}, [B^{12}(k)]^{(i)}$, $[B^{21}(k)]^{(i)}$, and $[B^{22}(k)]^{(i)}$ are given by Eqs. (36)-(39). The four blocks on the top left sub-matrix of Eq. (35) are functions of $z^{(i)}(k)$ which make them different from $[A^{**}(k)]^{(i)}$ in Eq. (33). In order to recursively compute $L^{(i)}(k+1)$ from $L^{(i)}(k)$, these four terms are approximated by their expectations with respect to $P(z^{(i)}(k)|z^{(i)}(1:k-1))$, i.e.,

\[
\mathbb{E}[P_c^{(i)}(k+1)] \approx \mathbb{E}[P_c^{(i)}(k)]
\]

Similarly, it can be shown that

\[
\mathbb{E}[P_c^{(i)}(k+1)] \approx \mathbb{E}[P_c^{(i)}(k)]
\]

Finally, Eq. (38) can be approximated as follows

\[
L^{(i)}(k+1) = \begin{bmatrix}
[A^{11}(k)]^{(i)} + [A^{11}(k)]^{(i)} - [A^{12}(k)]^{(i)} - [A^{21}(k)]^{(i)} \\
[A^{12}(k)]^{(i)} - [A^{12}(k)]^{(i)} \\
[A^{22}(k)]^{(i)} - [A^{22}(k)]^{(i)}
\end{bmatrix}
\]

(40)

Going back to complete the proof, we note that the information sub-matrix $L^{(i)}(k+1)$ is given by the inverse of the right bottom $(n_x \times n_x)$ block of $[I^{(i)}(0:k)]^{-1}$ (corresponding to $[B^{22}(k)]^{(i)}$ in Eq. (40)), i.e.,

\[
L^{(i)}(k+1) = \begin{bmatrix}
[B^{22}(k)]^{(i)} & 0 \\
0 & [B^{11}(k)]^{(i)} + [B^{12}(k)]^{(i)}
\end{bmatrix}
\]

(41)

Based on Eq. (31), the middle term in Eq. (42) reduces to $L^{(i)}(k)+[B^{11}(k)]^{(i)}$ which by substituting in Eq. (42) proves Result 1.

Finally we note that Result 1 is valid with the following approximation:

The top left four blocks of the accumulated conditional FIM given by Eq. (35) are replaced by their expectations with respect to $P(z^{(i)}(k)|z^{(i)}(1:k-1))$.

As shown above, this leads to Eqs. (7)-(10) of Result 1. Comparing Eqs. (7)-(10) with our earlier result [7], we note...
that the instantaneous auxiliary FIM $J_{\text{AUX}}^{(l)}(k)$ is replaced with the instantaneous conditional FIM $L^{(l)}(k)$. Consequently, the CQ/dPCRLB updates the conditional dPCRLB directly without the need of computing the auxiliary FIM leading to significant communication savings (by a factor of 2).

### APPENDIX C

Below, Result 2 is proved. First, we derive Lemma 2 which provides a factorization of the global quantized conditional posterior distribution $P_{Q,c}(k+1)$ at iteration $k+1$ as a function of the local quantized conditional posterior distribution $P_{Q,c}^{(l)}(k+1)$ at iteration $k+1$ and the global quantized conditional posterior distribution $P_{Q,c}(k)$ at iteration $k$.

**Lemma 2.** Assuming that the quantized observations conditioned on the state variables are independent, the global posterior for a network with $N_f$ processing nodes is factorized as follows

$$P_{Q,c}(k+1) \triangleq P(x(0:k+1), y(k+1)|y(1:k))$$

$$= \prod_{l=1}^{N_f} P_{Q,c}^{(l)}(k+1) \prod_{l=1}^{N_f} P(x(k+1)|y(1:k)) P(x(k+1)|x(k)) P_{Q,c}(k),$$

where $P_{Q,c}^{(l)}(k+1) \triangleq P(x(0:k), y(k)|y(1:k-1))$, and

$P_{Q,c}(k) \triangleq P(x(0:k), y(k)|y(1:k-1)).$

**Proof of Lemma 2** Using the Markovian property

$$P_{Q,c}(k+1) = P(y(k+1)|x(0:k+1))$$

$$\times P(x(k+1)|x(k)) P(x(0:k)|y(1:k)).$$

Comparing Eq. (43) with (44), we need to prove: (i) $P(y(k+1)|x(0:k+1)) \propto \prod_{l=1}^{N_f} P_{Q,c}^{(l)}(k+1)/P(x(k+1)|y(1:k))$, and; (ii) $P_{Q,c}(k) \propto P(x(0:k), y(k)|y(1:k-1)).$

Relationship (i): Given the state variables, the observations are assumed to be independent as is the case in most Bayesian estimators. Then, the first term on the right hand side (RHS) of (43) is given by

$$P(y(k+1)|x(k+1)) = \prod_{l=1}^{N_f} P(y^{(l)}(k+1)|x(k+1)).$$

We also factorize the local conditional distribution at node $l$, for $(1 \leq l \leq N_f)$, as follows

$$P(x(k+1), y^{(l)}(k+1)|y^{(l)}(1:k))$$

$$= P(y^{(l)}(k+1)|x(k+1)) P(x(k+1)|y^{(l)}(1:k)).$$

In terms of the local likelihood $P(y^{(l)}(k+1)|x(k+1))$, Eq. (46) can be expressed as follows

$$P(y^{(l)}(k+1)|x(k+1)) = \frac{P(x(k+1), y^{(l)}(k+1)|y^{(l)}(1:k))}{P(x(k+1)|y^{(l)}(1:k))}.$$

Substituting Eq. (47) in Eq. (45), we have

$$P(y(k+1)|x(k+1)) = \prod_{l=1}^{N_f} \frac{P(x(k+1), y^{(l)}(k+1)|y^{(l)}(1:k))}{P(x(k+1)|y^{(l)}(1:k))},$$

which proves Relation (i).

Relationship (ii): Term $P_{Q,c}(k)$ can be factorized as follows

$$P_{Q,c}(k) = P(x(0:k), y(k)|y(1:k-1)).$$

Since $P(y(k)|y(1:k-1))$ is independent of the state variables, Eq. (48) can be expressed as follows

$$P_{Q,c}(k) \propto P(x(0:k)|y(1:k)),$$

which proves Relation (ii).

This completes the proof for Lemma 1.

**Proof of Result 2** Given the quantized observations up to and including time $k$, the global accumulated conditional FIM can be decomposed as follows

$$I_Q^{(G)}(0:k) = \mathbb{E} \left\{ \begin{bmatrix} \Delta x(0:k-1) & \Delta x(0:k-1) \\ \Delta x(0:k-1) & \Delta x(0:k-1) \end{bmatrix} \log P_{Q,c}(k) \right\}$$

$$\approx \begin{bmatrix} E^{11}(k) E^{12}(k) \\ E^{21}(k) E^{22}(k) \end{bmatrix}.$$

As stated previously in Appendix A, the instantaneous conditional FIM $L_Q^{(G)}(k)$ is obtained by taking the inverse of the right lower block of $[I_Q^{(G)}(0:k)]^{-1}$. Using Lemma 2 we get

$$L_Q^{(G)}(k) = E^{11}(k) - E^{21}(k) E^{11}(k)^{-1} E^{12}(k).$$

For iteration $k+1$, we decompose $x(0:k+1) = [x^T(0:k-1), x^T(k), x^T(k+1)]^T$. As for Eq. (50), the global accumulated conditional FIM for iteration $k+1$ is then given by

$$I_Q^{(G)}(0 : k+1) = \mathbb{E} \left\{ \begin{bmatrix} \Delta x(0:k-1) & \Delta x(0:k-1) & \Delta x(0:k-1) \\ \Delta x(0:k-1) & \Delta x(0:k-1) & \Delta x(0:k-1) \\ \Delta x(0:k-1) & \Delta x(0:k-1) & \Delta x(0:k-1) \end{bmatrix} \log P_{Q,c}(k+1) \right\}.$$
\[ I_Q^{(G)}(0; k+1) = \begin{bmatrix} -\mathbb{E}_{P_{Q,c}(k+1)}[\Delta_{\pi(0:k-1)}^{x} \log P_{Q,c}(k)] & -\mathbb{E}_{P_{Q,c}(k+1)}[\Delta_{\pi(0:k-1)}^{x} \log P_{Q,c}(k)] & 0 \\ -\mathbb{E}_{P_{Q,c}(k+1)}[\Delta_{\pi(0:k-1)}^{x} \log P_{Q,c}(k)] & -\mathbb{E}_{P_{Q,c}(k+1)}[\Delta_{\pi(0:k-1)}^{x} \log P_{Q,c}(k)] & C_{21}^{12}(k) \\ 0 & C_{21}^{12}(k) & C_{22}^{12}(k) \end{bmatrix}, \] (53)

where \( P_{Q,c}(k+1) \equiv P(x(0:k+1), y(k+1)|y(1:k)). \)

Using Lemma 2, Eq. (52) reduces to Eq. (53) given at the top of the page. Similar to our discussion in Appendix B, the four blocks on the top left sub-matrix of Eq. (53) are functions of \( y(k) \), which make them different from \( E^{\star \star}(k) \) in Eq. (50). In order to recursively compute \( L_Q^{(G)}(k+1) \) from \( L_Q^{(G)}(k) \), these four blocks are approximated by taking their expectations with respect to \( P(y(k)|y(1:k-1)) \) resulting in

\[
I_Q^{(G)}(0; k+1) \approx \begin{bmatrix} E_{11}^{(k)} & E_{12}^{(k)} \\ E_{21}^{(k)} & E_{22}^{(k)} + C_{21}^{12}(k) \\ 0 & C_{21}^{12}(k) \end{bmatrix}, \] (54)

where \( 0 \) denotes a block of all zeros. Terms \( C_{11}^{12}(k) \), \( C_{12}^{12}(k) \) and \( C_{21}^{12}(k) \) were defined previously in Eqs. (22)-(23). Next, using Lemma 2 term \( C_{22}^{12}(k) = \mathbb{E}\{-\Delta_{\pi(x(k-1))}^{x} \log P_{Q,c}(k+1)\} \) in Eq. (54) is expressed as

\[
C_{22}^{12}(k) = \mathbb{E}_{P_{Q,c}(k+1)}\left\{-\Delta_{\pi(x(k-1))}^{x} \log \left(P \left( x(k+1)|x(k) \right) \right) \right\} + \sum_{l=1}^{N_f} \mathbb{E}_{P_{Q,c}(k+1)}\left\{-\Delta_{\pi(x(k-1))}^{x} \log \left(P \left( x(k+1), y_l^{(k+1)}|x(k) \right) \right) \right\} \] (55)

Finally, we note that the two summation terms in Eq. (55) are individual sums of the local instantaneous conditional FIMs at iteration \( k+1 \), i.e.,

\[
\sum_{l=1}^{N_f} \mathbb{E}_{P_{Q,c}(k+1)}\left\{-\Delta_{\pi(x(k-1))}^{x} \log \left(P \left( x(k+1), y_l^{(k+1)}|x(k) \right) \right) \right\} \approx \sum_{l=1}^{N_f} L_Q^{(l)}(k+1) \]

and

\[
\sum_{l=1}^{N_f} \mathbb{E}_{P_{Q,c}(k+1)}\left\{-\Delta_{\pi(x(k-1))}^{x} \log \left(P \left( x(k+1), y_l^{(k+1)}|x(k) \right) \right) \right\} \approx \sum_{l=1}^{N_f} L_Q^{(l)}(k+1) \times k \].

Term \( C_{22}^{12}(k) \) in Eq. (55), therefore, reduces to

\[
C_{22}^{12}(k) \approx \sum_{l=1}^{N_f} L_Q^{(l)}(k+1) - \sum_{l=1}^{N_f} L_Q^{(l)}(k+1) \times k \]

where \( -\Delta_{\pi(x(k-1))}^{x} \log \left(P \left( x(k+1)|x(k) \right) \right) \).