Forced vibrations of a two-layered shell in the case of viscous resistance

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Abstract. Forced vibrations of a two-layered orthotropic shell are studied in the case of viscous resistance in the lower layer of the shell. Two versions of spatial boundary conditions on the upper surface of the shell are posed, and the displacement vector is given on the lower surface. An asymptotic method is used to solve the corresponding dynamic equations and relations of the three-dimensional problem of elasticity. The amplitudes of the forced vibrations are determined, and the resonance conditions are established.

1. Basic equations and statement of the boundary-value problems

Forced vibrations of the two-layered orthotropic shell $\Omega = \{\alpha, \beta, \gamma; \alpha, \beta \in \Omega_0, -h_2 \leq \gamma \leq h_1\}$ are considered in the case of viscous resistance in the lower layer. Here $\Omega_0$ is the surface of contact between the layers, $\alpha$ and $\beta$ are the curvature lines of the surface of contact, and $\gamma$ is the rectilinear axis directed perpendicularly to the surface of contact between the layers. In the chosen three-dimensional orthogonal coordinate system, it is required to find nonzero solutions of dynamic equations of elasticity which satisfy the boundary conditions of the second or mixed boundary-value problems of elasticity on the surfaces of the shell. To simplify the computations, the asymmetric stress tensor components $\tau_{ij} [1, 2]$ are denoted by $\tilde{\gamma}_i = 1 + \gamma/R_i \ (i = 1, 2)$, where $R_1$ and $R_2$ are the basic radii of curvature of the surface of contact between the layers.

We have:

the equations of motion

$$\frac{1}{AB} \frac{\partial}{\partial \alpha} (B \tau_{\alpha\alpha}^{(I)}) - k_{\beta} \tilde{\gamma}_{\beta} + \frac{1}{AB} \frac{\partial}{\partial \beta} (A \tau_{\beta\beta}^{(I)}) + k_{\alpha} \tau_{\alpha\beta}^{(I)} + \tilde{\gamma}_{1} \frac{\partial \tau_{\alpha\gamma}^{(I)}}{\partial \gamma} + \frac{2 \tau_{\alpha\gamma}^{(I)}}{R_1} = \rho \tilde{\gamma}_1 \tilde{\gamma}_2 \frac{\partial^2 U^{(I)}}{\partial t^2},$$

$$\frac{1}{AB} \frac{\partial}{\partial \alpha} (B \tau_{\alpha\alpha}^{(II)}) - k_{\beta} \tilde{\gamma}_{\beta} + \frac{1}{AB} \frac{\partial}{\partial \beta} (A \tau_{\beta\beta}^{(II)}) + k_{\alpha} \tau_{\alpha\beta}^{(II)} + \tilde{\gamma}_{1} \frac{\partial \tau_{\alpha\gamma}^{(II)}}{\partial \gamma} + \frac{2 \tau_{\alpha\gamma}^{(II)}}{R_1} = \rho \tilde{\gamma}_1 \tilde{\gamma}_2 \frac{\partial^2 U^{(II)}}{\partial t^2}.$$
\[
\frac{\partial \tau_{\alpha\gamma}^{(I)}}{\partial \gamma} \left( \frac{\tau_{\alpha\alpha}^{(I)}}{R_1} + \frac{\tau_{\beta\beta}^{(I)}}{R_2} \right) + \frac{1}{A} \frac{\partial \tau_{\alpha\gamma}^{(I)}}{\partial \alpha} + \frac{1}{B} \frac{\partial \tau_{\beta\gamma}^{(I)}}{\partial \beta} + k_\beta \tau_{\alpha\gamma}^{(I)} + k_\alpha \tau_{\beta\gamma}^{(I)} = \rho_1 \gamma_{12} \frac{\partial^2 W^{(I)}}{\partial t^2},
\]
\[
\frac{\partial \tau_{\alpha\gamma}^{(II)}}{\partial \gamma} \left( \frac{\tau_{\alpha\alpha}^{(II)}}{R_1} + \frac{\tau_{\beta\beta}^{(II)}}{R_2} \right) + \frac{1}{A} \frac{\partial \tau_{\alpha\gamma}^{(II)}}{\partial \alpha} + \frac{1}{B} \frac{\partial \tau_{\beta\gamma}^{(II)}}{\partial \beta} + k_\beta \tau_{\alpha\gamma}^{(II)} + k_\alpha \tau_{\beta\gamma}^{(II)} - k_1 \gamma_{12} \frac{\partial W^{(II)}}{\partial t} = \rho_1 \gamma_{12} \frac{\partial^2 W^{(II)}}{\partial t^2},
\]
where \(\gamma_{12}\) are elasticity constants, \(A, B\) are coefficients of the first quadratic form, \(\rho^{(j)}\) are the layer densities, \(a^{(j)}_{ik}\) are elasticity constants, and \(j\) is the layer number.

On the surface \(\gamma = h_1\), the boundary conditions are
\[
\tau_{\alpha\gamma}^{(I)}(h_1) = 0, \quad \tau_{\beta\gamma}^{(I)}(h_1) = 0, \quad \tau_{\gamma\gamma}^{(I)}(h_1) = 0, \quad J = I, II
\]
\[
\tau_{\alpha\gamma}^{(II)}(h_1) = 0, \quad \tau_{\beta\gamma}^{(II)}(h_1) = 0, \quad \tau_{\gamma\gamma}^{(II)}(h_1) = 0, \quad J = I, II
\]
(2)

and on the surface \(\gamma = -h_2\), the conditions are
\[
U^I(h_1) = 0, \quad V^I(h_1) = 0, \quad W^I(h_1) = 0, \quad J = I, II
\]
\[
U^II(h_2) = u^-(\alpha, \beta) \sin(\Omega t), \quad V^II(h_2) = v^-(\alpha, \beta) \sin(\Omega t), \quad W^II(h_2) = w^-(\alpha, \beta) \sin(\Omega t).
\]
(4)

On the surface of the contact between the layers, the conditions of complete contact must be fulfilled:
\[
\tau_{\alpha\gamma}^{(I)}(\gamma = 0) = \tau_{\alpha\gamma}^{(II)}(\gamma = 0), \quad \tau_{\beta\gamma}^{(I)}(\gamma = 0) = \tau_{\beta\gamma}^{(II)}(\gamma = 0), \quad \tau_{\gamma\gamma}^{(I)}(\gamma = 0) = \tau_{\gamma\gamma}^{(II)}(\gamma = 0), \quad J = I, II
\]
\[
U^I(\gamma = 0) = U^II(\gamma = 0), \quad V^I(\gamma = 0) = V^II(\gamma = 0), \quad W^I(\gamma = 0) = W^II(\gamma = 0).
\]
(5)

The conditions on the lateral surface are not specified, and in the problems of this class, they imply origination of a boundary layer [1-3].

2. Solution of the outer problem
To solve above-formulated boundary-value problems (1)–(6), we pass to dimensionless coordinates and displacements in equations (1) by the formulas
\[
\alpha = R\xi, \quad \beta = R\eta, \quad \gamma = \varepsilon R\zeta = h\zeta, \quad U = Ru, \quad V = Rv, \quad W = Rw,
\]
\[
h = \max\{h_1, h_2\}, \quad h \ll R,
\]
2
where $R$ is the characteristic dimension of the shell (the smallest value of the radii of curvature and the linear dimensions of the surface of contact between the layers) and $\varepsilon = h/R$ is a small parameter. The solution of the transformed equations [4, 5] is sought in the form

$$Q^{(j)}_{\alpha\beta}(x, y, z, t) = Q^{(j)}_{1}(\xi, \eta, \zeta) \sin(\Omega t) + Q^{(j)}_{2}(\xi, \eta, \zeta) \cos(\Omega t) \quad (\alpha, \beta, \gamma), \ j = I, II, \ (7)$$

where $Q^{(j)}_{\alpha\beta}$ are any values of stresses and displacements and $\Omega$ is the frequency of the external forcing effect. As a result, a system of equations for $Q^{(j)}_{1}$, $Q^{(j)}_{2}$ singularly perturbed by the small parameter $\varepsilon$ is obtained, and its solution is the sum of solutions of the outer problem and boundary layers: $I = Q^{\text{out}} + R_b [1, 3, 6]$.

The solutions of the outer problem are sought as the asymptotic expansion [7–12]

$$\tau^{\text{out}(j)}_{mk,i}(\xi, \eta, \zeta) = \varepsilon^{1+s} \tau^{(j,s)}_{mk,i}(\xi, \eta, \zeta), \ m, k = 1, 2, 3, \ s = 0, N, \ j = I, II, \ i = 1, 2,$$

$$(u^{\text{out}(j)}_{i}(\xi, \eta, \zeta), v^{\text{out}(j)}_{i}(\xi, \eta, \zeta), w^{\text{out}(j)}_{i}(\xi, \eta, \zeta)) = \varepsilon^{s}(u^{(j,s)}_{i}(\xi, \eta, \zeta), v^{(j,s)}_{i}(\xi, \eta, \zeta), w^{(j,s)}_{i}(\xi, \eta, \zeta)). \quad (8)$$

Here and further, $s = 0, N$ means that by the mute (repeated) summation index $s$ ranges between 0 and $N$.

The solution of the problem must satisfy conditions (2)–(6). If the above-mentioned structure of the general solution is known, then to determine $\tau^{(j,s)}_{mk,i}$, $u^{(j,s)}_{i}$, and $(u, v, w)$ in the outer problem from conditions (2)–(4), we must satisfy the following boundary conditions at $\zeta = \zeta_1 \ (\zeta_1 = h_1/h)$:

$$\tau^{(I,s)}_{13,i}(\zeta_1) = -\tau^{(I,s)}_{13,ib}(\zeta = \zeta_1), \ (13, 23, 33), \ i = 1, 2, \ (9)$$

or

$$u^{(I,s)}_{i}(\zeta_1) = u^{(I,s)}_{ib}(\zeta = \zeta_1), \ (u, v, w), \ i = 1, 2, \ (10)$$

and the conditions at $\zeta = -\zeta_2 \ (\zeta_2 = h_2/h)$

$$u^{-(II,s)}_{i}(\zeta, \eta) = u^{-(II,s)}_{ib}(\zeta = -\zeta_2), \ (u, v, w), \ (11)$$

where

$$u^{-(II,0)}_{1} = \frac{u}{R}, \ u^{-(II,0)}_{2} = 0, \ \tau^{(I,0)}_{m3,ib} = 0, \ u^{(I,0)}_{ib} = 0, \ u^{-(II,s)}_{i} = -u^{(II,s)}_{ib}(\zeta = -\zeta_2), \ (12)$$

$s \neq 0, \ (u, v, w), \ m = 1, 2, 3, \ i = 1, 2.$

The values $\tau^{(I,s)}_{13,ib}$, $u^{(I,s)}_{ib}$, $u^{(II,s)}_{ib}$, and $(u, v, w)$ are determined after constructing the boundary layer solution.

Standardly substituting expressions (8) into the obtained singularly perturbed system of
equations, we obtain

\[
\frac{1}{AB} \frac{\partial}{\partial \xi} \left( B^{(I,s-1)}_{11,i} \right) - k_B R^{(I,s-1)}_{22,i} + \frac{1}{AB} \frac{\partial}{\partial \eta} \left( A R^{(I,s-1)}_{21,i} \right) + k_A R^{(I,s-1)}_{12,i} + \frac{\partial^{(I,s)}}{\partial \xi} + r_1 \frac{\partial^{(I,s)}}{\partial \eta} - 2\sigma_1^{(I,s-1)} + \rho_2^{(I,s-1)} + (r_1 + r_2)\zeta^{(I,s-1)} + r_1 r_2 \frac{\partial^{(I,s-2)}}{\partial \eta} = 0,
\]

\[
\left( A \leftrightarrow B; \quad \alpha \leftrightarrow \beta; \quad r_1, r_2; \quad \xi \leftrightarrow \eta; \quad u, v; \quad \tau_{12} \leftrightarrow \tau_{21}; \quad \tau_{11} \leftrightarrow \tau_{22}; \quad \tau_{13}, \tau_{23}, \right)
\]

\[
\frac{\partial^{(I,s-1)}}{\partial \eta} - \sigma_1^{(I,s-1)} - r_2 \sigma_2^{(I,s-1)} + \frac{1}{A} \frac{\partial^{(I,s-1)}}{\partial \xi} + \frac{1}{B} \frac{\partial^{(I,s-1)}}{\partial \eta} + k_B R^{(I,s-1)}_{13,i} + k_\alpha R^{(I,s-1)}_{23,i} + \rho_2^{(I,s-1)} + (r_1 + r_2)\zeta^{(I,s-1)} + r_1 r_2 \frac{\partial^{(I,s-2)}}{\partial \eta} = 0,
\]

\[
\frac{1}{AB} \frac{\partial}{\partial \xi} \left( B^{(I,s-1)}_{11,i} \right) - k_B R^{(I,s-1)}_{22,i} + \frac{1}{AB} \frac{\partial}{\partial \eta} \left( A R^{(I,s-1)}_{21,i} \right) + k_A R^{(I,s-1)}_{12,i} + \frac{\partial^{(I,s)}}{\partial \xi} + r_1 \frac{\partial^{(I,s)}}{\partial \eta} + 2\sigma_1^{(I,s-1)} - 2K_\eta u^{(I,s-2)} + \rho_2^{(I,s-2)} u^{(I,s-1)} - (r_1 + r_2)\zeta \frac{2K_\eta u^{(I,s-2)}}{\partial \eta} = 0,
\]

\[
\left( A \leftrightarrow B; \quad \alpha \leftrightarrow \beta; \quad r_1, r_2; \quad \xi \leftrightarrow \eta; \quad u, v; \quad \tau_{12} \leftrightarrow \tau_{21}; \quad \tau_{11} \leftrightarrow \tau_{22}; \quad \tau_{13}, \tau_{23}, \right)
\]
\[
\frac{\partial u_{i}^{(j,s)}}{\partial \zeta} + \zeta (r_1 + r_2) \frac{\partial u_{i}^{(j,s-1)}}{\partial \zeta} + \zeta^2 r_1 r_2 \frac{\partial w_{i}^{(j,s-2)}}{\partial \zeta} = \sum_{3i} + r_1 \zeta a_{13}^{j} \tau_{11,i}^{(j,s-1)} + r_2 \zeta a_{23}^{j} \tau_{22,i}^{(j,s-1)},
\]

\[
\frac{1}{B} \frac{\partial u_{i}^{(j,s-1)}}{\partial \eta} - k_\beta R \tau_{i}^{(j,s-1)} + r_1 (1 - k_\beta R \tau_{i}^{(j,s-2)}) + \frac{1}{A} \frac{\partial v_{i}^{(j,s-1)}}{\partial \xi} - k_\alpha Ru_{i}^{(j,s-1)}
\]

\[
\gamma_{j,s}^{(s-2)} + r_2 \zeta \frac{\partial w_{i}^{(j,s-2)}}{\partial \xi} = a_{55}^{j} \tau_{13,i}^{(j,s)} + r_1 \zeta a_{35}^{j} \tau_{13,i}^{(j,s-1)},
\]

\[
(\zeta(r_1 + r_2) \frac{\partial u_{i}^{(j,s)}}{\partial \zeta} + \zeta^2 r_1 r_2 \frac{\partial w_{i}^{(j,s-2)}}{\partial \zeta} - r_1 u_{i}^{(j,s-1)} - r_2 \zeta \tau_{12,i}^{(j,s-1)}),
\]

\[
\sum_{ki} = a_{k1}^{j} \gamma_{11,i}^{(j,s)} + a_{k2}^{j} \gamma_{22,i}^{(j,s)} + a_{k3}^{j} \gamma_{33,i}^{(j,s)}, \quad i = 1, 2, \quad k = 1, 2, 3, \quad j = I, II.
\]

Taking into account that \(Q^{(j,m)} \equiv 0\) for \(m < 0\), from system (13) we derive the following equations for determining the decomposition coefficients (8):

\[
\frac{\partial \gamma_{i}^{(1,s)}}{\partial \zeta} + \rho \Omega_{u_{i}^{(1,s)}} = P_{6r_{1}}^{(1,s-1)}, \quad (13, 23, 33; \quad u, v, w; \quad 6\tau, 5\tau, 4\tau),
\]

\[
\frac{\partial u_{i}^{(1,s)}}{\partial \zeta} + \Omega_{a_{55}^{(1,s)}} = F_{u_{i}^{(1,s-1)}},
\]

\[
\frac{\partial \gamma_{i}^{(1,s)}}{\partial \zeta} + \rho \Omega_{a_{55}^{(1,s)}} = F_{u_{i}^{(1,s-1)}},
\]

\[
\frac{\partial \omega_{i}^{(1,s)}}{\partial \zeta} = - \sum_{3i} = F_{u_{i}^{(1,s-1)}},
\]

\[
\frac{\partial \tau_{13,i}^{(1,s)}}{\partial \zeta} + 2 K \Omega_{u_{2}^{(1,s)}} + \rho \Omega_{u_{1}^{(1,s)}} = P_{6r_{1}}^{(1,s-1)}, \quad (13, 23, 33; \quad u, v, w; \quad 6\tau, 5\tau, 4\tau),
\]

\[
\frac{\partial \tau_{13,i}^{(2,s)}}{\partial \zeta} - 2 K \Omega_{u_{1}^{(2,s)}} + \rho \Omega_{u_{2}^{(2,s)}} = P_{6r_{1}}^{(2,s-1)}, \quad (13, 23, 33; \quad u, v, w; \quad 6\tau, 5\tau, 4\tau),
\]

\[
\gamma_{j,s}^{(j,s)} = r_{1i} \tau_{1r_{1}}^{(j,s-1)} + r_{2i} \tau_{21}^{(j,s-1)} - r_{2} \zeta \tau_{12,i}^{(j,s-1)},
\]

\[
\sum_{j,s}^{P_{2r_{i}}^{(j,s-1)}}, \quad \sum_{j,s}^{P_{3r_{i}}^{(j,s-1)}}, \quad j = I, II, \quad i = 1, 2,
\]

\[
\frac{\partial \gamma_{i}^{(1,s)}}{\partial \zeta} - \Omega_{u_{i}^{(1,s)}} = F_{u_{i}^{(1,s-1)}}, \quad \frac{\partial \gamma_{i}^{(2,s)}}{\partial \zeta} - \Omega_{u_{i}^{(2,s)}} = F_{u_{i}^{(2,s-1)}}, \quad \frac{\partial \omega_{i}^{(1,s)}}{\partial \zeta} - \sum_{3i} = F_{u_{i}^{(1,s-1)}},
\]
where

\[ P_{3r_1}^{(I,s-1)} = r_1 \tau_{11,i,x}^{(I,s-1)} + r_2 \tau_{22,i,x}^{(I,s-1)} - \frac{1}{A} \frac{\partial \tau_{13,i,x}^{(I,s-1)}}{\partial \xi} - \frac{1}{B} \frac{\partial \tau_{23,i,x}^{(I,s-1)}}{\partial \eta} \]

\[ - k_{13} R_{13,i,x}^{(I,s-1)} - k_{12} R_{23,i,x}^{(I,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,x}^{(I,s-1)} - r_1 r_2 \zeta^2 \partial R_{22,i}^{(I,s-2)}, \]

\[ P_{5r_1}^{(I,s-1)} = - \frac{1}{AB} \frac{\partial \tau_{22,i,x}^{(I,s-1)}}{\partial \eta} + k_{12} R_{12,i,x}^{(I,s-1)} - \frac{1}{AB} \frac{\partial B}{\partial \xi} (B \tau_{12,i,x}^{(I,s-1)}) - k_{23} R_{21,i,x}^{(I,s-1)} \]

\[ - r_2 \zeta \frac{\partial \tau_{23,i,x}^{(I,s-1)}}{\partial \xi} - 2r_2 \tau_{23,i,x}^{(I,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,x}^{(I,s-1)} - r_1 r_2 \zeta^2 \partial R_{22,i}^{(I,s-2)}, \]

\[ \left(57, 67; A, B, u, v, \alpha, \beta; r_2, r_1; \xi, \eta; \tau_{11,2}, \tau_{21,2}, \tau_{23}, \tau_{13}\right), \]

\[ P_{4r_1}^{(II,s-1)} = r_1 \tau_{11,i,x}^{(II,s-1)} + r_2 \tau_{22,i,x}^{(II,s-1)} - \frac{1}{A} \frac{\partial \tau_{13,i,x}^{(II,s-1)}}{\partial \xi} - \frac{1}{B} \frac{\partial \tau_{23,i,x}^{(II,s-1)}}{\partial \eta} \]

\[ - k_{13} R_{13,i,x}^{(II,s-1)} - k_{12} R_{23,i,x}^{(II,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,x}^{(II,s-1)} - r_1 r_2 \zeta^2 \partial R_{22,i}^{(II,s-2)}, \]

\[ \left(57, 67; A, B, u, v, \alpha, \beta; r_2, r_1; \xi, \eta; \tau_{11,2}, \tau_{21,2}, \tau_{23}, \tau_{13}\right), \]

\[ P_{5r_1}^{(II,s-1)} = - \frac{1}{AB} \frac{\partial \tau_{22,i,x}^{(II,s-1)}}{\partial \eta} + k_{12} R_{12,i,x}^{(II,s-1)} - \frac{1}{AB} \frac{\partial B}{\partial \xi} (B \tau_{12,i,x}^{(II,s-1)}) \]

\[ - k_{23} R_{21,i,x}^{(II,s-1)} - r_2 \zeta \frac{\partial \tau_{23,i,x}^{(II,s-1)}}{\partial \xi} - 2r_2 \tau_{23,i,x}^{(II,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,x}^{(II,s-1)} - r_1 r_2 \zeta^2 \partial R_{22,i}^{(II,s-2)}, \]

\[ \left(57, 67; A, B, u, v, \alpha, \beta; r_2, r_1; \xi, \eta; \tau_{11,2}, \tau_{21,2}, \tau_{23}, \tau_{13}\right), \]

\[ P_{4r_2}^{(II,s-1)} = r_1 \tau_{11,i,y}^{(II,s-1)} + r_2 \tau_{22,i,y}^{(II,s-1)} - \frac{1}{A} \frac{\partial \tau_{13,i,y}^{(II,s-1)}}{\partial \xi} - \frac{1}{B} \frac{\partial \tau_{23,i,y}^{(II,s-1)}}{\partial \eta} \]

\[ - k_{13} R_{13,i,y}^{(II,s-1)} - k_{12} R_{23,i,y}^{(II,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,y}^{(II,s-1)} + r_1 r_2 \zeta^2 \partial R_{22,i}^{(II,s-2)}, \]

\[ \left(57, 67; A, B, u, v, \alpha, \beta; r_2, r_1; \xi, \eta; \tau_{11,2}, \tau_{21,2}, \tau_{23}, \tau_{13}\right), \]

\[ P_{5r_2}^{(II,s-1)} = - \frac{1}{AB} \frac{\partial \tau_{22,i,y}^{(II,s-1)}}{\partial \eta} + k_{12} R_{12,i,y}^{(II,s-1)} - \frac{1}{AB} \frac{\partial B}{\partial \xi} (B \tau_{12,i,y}^{(II,s-1)}) \]

\[ - k_{23} R_{21,i,y}^{(II,s-1)} - r_2 \zeta \frac{\partial \tau_{23,i,y}^{(II,s-1)}}{\partial \xi} - 2r_2 \tau_{23,i,y}^{(II,s-1)} - (r_1 + r_2) \zeta \partial R_{11,i,y}^{(II,s-1)} + r_1 r_2 \zeta^2 \partial R_{22,i}^{(II,s-2)}, \]

\[ \left(57, 67; A, B, u, v, \alpha, \beta; r_2, r_1; \xi, \eta; \tau_{11,2}, \tau_{21,2}, \tau_{23}, \tau_{13}\right), \]

\[ P_{1r_1}^{(j,s-1)} = \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-1)}}{\partial \xi} - k_{13} R_{13,i}^{(j,s-1)} + r_2 \zeta \left( \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-2)}}{\partial \xi} - k_{13} R_{13,i}^{(j,s-2)} \right) \]

\[ + \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-1)}}{\partial \eta} - k_{12} R_{12,i}^{(j,s-1)} + 2r_2 \zeta \left( \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-2)}}{\partial \eta} - k_{12} R_{12,i}^{(j,s-2)} \right) - r_1 \zeta a_{66} \tau_{12,i}^{(j,s-1)} \]

\[ P_{2r_1}^{(j,s-1)} = \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-1)}}{\partial \xi} + k_{12} R_{12,i}^{(j,s-1)} + r_1 u_{i,j}^{(j,s-1)} \]

\[ + r_2 \zeta \left( \frac{1}{A} \frac{\partial u_{i,j}^{(j,s-2)}}{\partial \xi} + k_{12} R_{12,i}^{(j,s-2)} + r_1 u_{i,j}^{(j,s-2)} \right) - r_1 \zeta a_{66} \tau_{11,i}^{(j,s-1)} - r_2 \zeta a_{66} \tau_{12,i}^{(j,s-1)}, \]

\[ \left(27, 37; A, B; \alpha, \beta; r_1 \leftrightarrow r_2; \xi, \eta; u \leftrightarrow v; \tau_{11} \leftrightarrow \tau_{22}; a_{11}, a_{22}\right). \]
whose solutions have the form

\[ u_i^{(s)} = C_1 u_i^{(s)} \sin(\chi^{(1, i)} \zeta) + C_2 u_i^{(s)} \cos(\chi^{(1, i)} \zeta) + \bar{u}_i^{(s)} \zeta, \]

\[ v_i^{(s)} = C_3 v_i^{(s)} \sin(\chi^{(1, i)} \zeta) + C_4 v_i^{(s)} \cos(\chi^{(1, i)} \zeta) + \bar{v}_i^{(s)} \zeta, \]

\[ w_i^{(s)} = C_5 w_i^{(s)} \sin(\chi^{(1, i)} \zeta) + C_6 w_i^{(s)} \cos(\chi^{(1, i)} \zeta) + \bar{w}_i^{(s)} \zeta, \quad i = 1, 2, \]

where

\[ \chi^{(1, i)} = \sqrt{a_{55}^1 \rho_1 \Omega_s}, \quad \chi^{(1, v)} = \sqrt{a_{44}^1 \rho_1 \Omega_s}, \quad \chi^{(1, w)} = \sqrt{\frac{\Delta_2^1}{\Delta_{12}^1} \Omega_s}, \]

and \( \bar{u}_i^{(s)}, \bar{v}_i^{(s)}, \bar{w}_i^{(s)} \) are particular solutions of equations (17). For determining the
displacement vector components in the second layer, we obtain the equations

\[ \frac{\partial^2 u_{1}^{(II,s)}}{\partial \zeta^2} + a_{55}^{II} (\rho \Omega_2^2 u_{1}^{(II,s)} + 2 K \Omega_3 u_{2}^{(II,s)}) = a_{55}^{II} (\rho \Omega_2^2 u_{1}^{(II,s)} - 2 K \Omega_3 u_{1}^{(II,s)}) + \frac{\partial P_{u1}^{(II,s-1)}}{\partial \zeta}, \quad (u, v; \ a_{55}, a_{44}; \ 6\tau, 5\tau), \]

\[ \frac{\partial^2 u_{2}^{(II,s)}}{\partial \zeta^2} + a_{55}^{II} (\rho \Omega_2^2 u_{2}^{(II,s)} - 2 K \Omega_3 u_{1}^{(II,s)}) = a_{55}^{II} (\rho \Omega_2^2 u_{2}^{(II,s)} - 2 K \Omega_3 u_{2}^{(II,s)}) + \frac{\partial P_{u2}^{(II,s-1)}}{\partial \zeta}, \quad (u, v; \ a_{55}, a_{44}; \ 6\tau, 5\tau), \]

\[ \frac{\partial^2 w_{1}^{(II,s)}}{\partial \zeta^2} + \frac{\Delta^{(II)}}{\Delta^{(II)}_{12}} (\rho \Omega_2^2 w_{1}^{(II,s)} + 2 K \Omega_3 w_{2}^{(II,s)}) = F_{w1}^{(II,s-1)}, \]

\[ \frac{\partial^2 w_{2}^{(II,s)}}{\partial \zeta^2} + \frac{\Delta^{(II)}}{\Delta^{(II)}_{12}} (\rho \Omega_2^2 w_{2}^{(II,s)} - 2 K \Omega_3 w_{1}^{(II,s)}) = F_{w2}^{(II,s-1)}, \]

\[ F_{w1}^{(II,s-1)} = \frac{1}{\Delta^{(II)}_{12}} \left[ \Delta^{(II)}_{41} P_{41}^{(II,s-1)} - \Delta^{(II)}_{21} \frac{\partial P_{21}^{(II,s-1)}}{\partial \zeta} - \Delta^{(II)}_{31} \frac{\partial P_{31}^{(II,s-1)}}{\partial \zeta} + \Delta^{(II)}_{12} \frac{\partial P_{12}^{(II,s-1)}}{\partial \zeta} \right], \quad i = 1, 2. \]

From (19),

\[ u_{2}^{(II,s)} = \frac{1}{2 K \Omega_3 a_{55}^{II}} \left[ a_{55}^{II} P_{61}^{(II,s-1)} + \partial u_{1}^{(II,s-1)} - \partial^2 u_{1}^{(II,s-1)} - a_{55} \rho \Omega_2^2 u_{1}^{(II,s)} \right], \]

\[ (u, v; \ a_{55}, a_{44}; \ 6\tau, 5\tau); \]

\[ w_{2}^{(II,s)} = \frac{\Delta^{(II)}_{12}}{2 \Delta^{(II)}_{12} K \Omega_3} \left[ F_{w1}^{(II,s-1)} - \frac{\partial^2 w_{1}^{(II,s-1)}}{\partial \zeta^2} - \frac{\Delta^{(II)}_{12}}{\Delta^{(II)}_{12}} \rho \Omega_2^2 w_{1}^{(II,s)} \right], \]

we derive the equations

\[ \frac{\partial^4 u_{1}^{(II,s)}}{\partial \zeta^4} + 2 a_{55} \rho \Omega_2^2 \frac{\partial^2 u_{1}^{(II,s)}}{\partial \zeta^2} + a_{55} \Omega_2^2 (\rho \Omega_2^2 + 4 K^2) u_{1}^{(II,s)} = Q_{u1}^{(II,s-1)}, \]

\[ (u, v; \ a_{55}, a_{44}, \Delta/\Delta_{12}), \]

\[ Q_{u1}^{(II,s-1)} = \Omega_2 \frac{\partial^3 F_{u1}^{(II,s-1)}}{\partial \zeta^3} + \rho \Omega_2 a_{55} \Delta \frac{\partial^2 F_{u1}^{(II,s-1)}}{\partial \zeta^2} + \Omega_2 \rho \frac{\partial^2 P_{u1}^{(II,s-1)}}{\partial \zeta}, \]

\[ (u, v; \ a_{55}, a_{44}; \ 6\tau, 5\tau), \]

\[ Q_{w1}^{(II,s-1)} = \frac{\Delta^{(II)}_{12} \Delta^{(II)}_{12}}{\Delta^{(II)}_{12}} \frac{\partial^2 F_{w1}^{(II,s-1)}}{\partial \zeta^2} + \Omega_2 \rho \frac{\partial^2 F_{w1}^{(II,s-1)}}{\partial \zeta} - 2 K F_{w2}^{(II,s-1)}, \]

The solutions of equations (21) are

\[ u_{1}^{(II,s)} = u_{10}^{(II,s)}(\xi, \eta, \zeta) + u_{1p}^{(II,s)}(\xi, \eta, \zeta), \quad (u, v, w), \]

where the index “0” denotes the general solution of the corresponding homogeneous equation, and the index “p” denotes a particular solution of the inhomogeneous equation.

The solutions of the corresponding homogeneous equations are

\[ u_{10}^{(II,s)}(\xi, \eta, \zeta) = C_{1}^{(u, II,s)}(\xi, \eta, \varphi_{1u}(\zeta)) + C_{2}^{(u, II,s)}(\xi, \eta, \varphi_{2u}(\zeta)) + C_{3}^{(u, II,s)}(\xi, \eta, \varphi_{3u}(\zeta)) + C_{4}^{(u, II,s)}(\xi, \eta, \varphi_{4u}(\zeta), \quad (u, v, w), \]

\[ (u, v, w), \]
where

\[ \varphi_{1u}(\zeta) = \cosh(\gamma_a \zeta) \cos(\delta_u \zeta), \quad \varphi_{2u}(\zeta) = \sinh(\gamma_a \zeta) \sin(\delta_u \zeta), \]

\[ \varphi_{3u}(\zeta) = \cosh(\gamma_a \zeta) \sin(\delta_u \zeta), \quad \varphi_{4u}(\zeta) = \sinh(\gamma_a \zeta) \cos(\delta_u \zeta), \]

\[ \gamma_u = \sqrt{\frac{a_{55}}{2} \Omega_s (\sqrt{\rho \Omega_s^2 + 4K^2 - \Omega_s})}, \quad \delta_u = \sqrt{\frac{a_{55}}{2} \Omega_s (\sqrt{\rho \Omega_s^2 + 4K^2 + \Omega_s})}, \]

For the displacements with index “II” we have

\[ u_{2}^{(II)} = u_{20}^{(II)}(\xi, \eta, \zeta) + u_{2p}^{(II)}(\xi, \eta, \zeta), \quad (u, v, w), \]

\[ u_{20}^{(II)}(\xi, \eta, \zeta) = -C_1^{(u,II)}(\xi, \eta)\varphi_{2u} + C_2^{(u,II)}(\xi, \eta)\varphi_{1u} + C_3^{(u,II)}(\xi, \eta)\varphi_{3u} - C_4^{(u,II)}(\xi, \eta)\varphi_{4u}, \quad (u, v, w), \]

\[ u_{2p}^{(II)}(\xi, \eta, \zeta) = \frac{1}{a_{55}2K\Omega_s} \left( a_{55}P_{61}^{(II,-1)} + \frac{\partial P_{61}^{(II)}}{\partial \zeta} - \frac{\partial^2 P_{61}^{(II)}}{\partial \zeta^2} - a_{55}P^{(II)}_{92}u_{1p}^{(II)} \right). \]

Satisfying conditions (9), (11), (5), (6) and (10), (11), (5), (6), we correspondingly obtain three algebraic systems of equations for \( C_{ki}^{(I)}(\xi, \eta), C_{m}^{(u,II)}(k = \frac{1}{2}, m = \frac{1}{2}, i = 1, 2; u, v, w) \). The obtained systems have solutions if their determinants are different from zero. So the following conditions corresponding to boundary conditions (9), (11), (5), (6) must be satisfied:

\[ \Delta_{u\tau} = \frac{\delta_u^2 + \gamma_u^2}{2a_{55}^2} \cos^2(\chi^{(I)}(\zeta)) \left[ \cosh(2\gamma_u \zeta_2) + \cos(2\delta_u \zeta_2) \right] + \frac{(\chi^{(I)}(\zeta))^2}{2a_{55}^2} \sin^2(\chi^{(I)}(\zeta)) \left[ \cosh(2\gamma_u \zeta_2) - \cosh(2\delta_u \zeta_2) \right] \]

\[ - \cos(2\delta_u \zeta_2) - \frac{\chi^{(I)}(\zeta)}{2a_{55}} \sin(2\chi^{(I)}(\zeta)) \left[ \delta_u \sin(2\delta_u \zeta_2) + \gamma_u \sin(2\gamma_u \zeta_2) \right] \neq 0, \quad (u, v, w), \]

(25)

and the conditions corresponding to boundary conditions (10), (11), (5), (6)

\[ \Delta_{uu} = \frac{\delta_u^2 + \gamma_u^2}{2a_{55}^2} \sin^2(\chi^{(I)}(\zeta)) \left[ \cosh(2\gamma_u \zeta_2) + \cos(2\delta_u \zeta_2) \right] + \frac{(\chi^{(I)}(\zeta))^2}{2a_{55}^2} \cos^2(\chi^{(I)}(\zeta)) \left[ \cosh(2\gamma_u \zeta_2) - \cos(2\delta_u \zeta_2) \right] \]

\[ + \frac{\chi^{(I)}(\zeta)}{2a_{55}} \sin(2\chi^{(I)}(\zeta)) \left[ \delta_u \sin(2\delta_u \zeta_2) + \gamma_u \sin(2\gamma_u \zeta_2) \right] \neq 0, \quad (u, v, w). \]

(26)

In the case of one-layered shell [11] without viscous resistance, resonance (infinite amplitude) always occurs, and in the presence of viscous resistance [4], at resonance in the shell, the amplitude of forced vibrations is finite. In the case considered here, the presence of viscous resistance only in the lower layer of the shell results in resonance with an unlimited amplitude of forced vibrations, as the equations \( \Delta_{u\tau} = 0, \Delta_{uu} = 0, (u, v, w) \) have real roots which are the basic frequencies of natural vibrations, and resonance occurs when the imparted frequency of the forced effect concurs with the frequency of natural vibrations.

**Conclusion**

Dynamical three-dimensional problems for a two-layered orthotropic shell are solved asymptotically in the case of viscous resistance in one of the layers for two versions of the boundary conditions on the face surfaces. The amplitudes of forced vibrations are determined. It is shown that, for such a configuration, resonance with an infinite amplitude always occurs in the shell, in contrast to the case of a single-layered shell with viscosity, when the amplitude is finite.
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