An APTAS for Bin Packing with Clique-graph Conflicts

Ilan Doron-Arad, Ariel Kulik and Hadas Shachnai

Computer Science Department, Technion, Haifa 3200003, Israel.
E-mail: {idoron-arad,kulik,hadas}@cs.technion.ac.il.

Abstract. We study the following variant of the classic bin packing problem. Given a set of items of various sizes, partitioned into groups, find a packing of the items in a minimum number of identical (unit-size) bins, such that no two items of the same group are assigned to the same bin. This problem, known as bin packing with clique-graph conflicts, has natural applications in storing file replicas, security in cloud computing and signal distribution.

Our main result is an asymptotic polynomial time approximation scheme (APTAS) for the problem, improving upon the best known ratio of 2. As a key tool, we apply a novel Shift & Swap technique which generalizes the classic linear shifting technique to scenarios allowing conflicts between items. The major challenge of packing small items using only a small number of extra bins is tackled through an intricate combination of enumeration and a greedy-based approach that utilizes the rounded solution of a linear program.

1 Introduction

In the classic bin packing (BP) problem, we seek a packing of items of various sizes into a minimum number of unit-size bins. This fundamental problem arises in a wide variety of contexts and has been studied extensively since the early 1970’s. In some common scenarios, the input is partitioned into disjoint groups, such that items in the same group are conflicting and therefore cannot be packed together. For example, television and radio stations often assign a set of programs to their channels. Each program falls into a genre such as comedy, documentary or sports on TV, or various musical genres on radio. To maintain a diverse daily schedule of programs, the station would like to avoid broadcasting two programs of the same genre in one channel. Thus, we have a set of items (programs) partitioned into groups (genres) that need to be packed into a set of bins (channels), such that items belonging to the same group cannot be packed together.

We consider this natural variant of the classic bin packing problem that we call group bin packing (GBP). Formally, the input is a set of $N$ items $I = \{1, \ldots, N\}$ with corresponding sizes $s_1, \ldots, s_N \in (0, 1]$, partitioned into $n$ disjoint groups $G_1, \ldots, G_n$, i.e., $I = G_1 \cup G_2 \cup \ldots \cup G_n$. The items need to be packed in unit-size bins. A packing is feasible if the total size of items in each bin does not exceed the bin capacity, and no two items from the same group are packed in the same
bin. We seek a feasible packing of all items in a minimum number of unit-size bins. We give in Appendix A some natural applications of GBP.

Group bin packing can be viewed as a special case of bin packing with conflicts (BPC), in which the input is a set of items $I$, each having size in $(0, 1]$, along with a conflict graph $G = (V, E)$. An item $i \in I$ is represented by a vertex $i \in V$, and there is an edge $(i, j) \in E$ if items $i$ and $j$ cannot be packed in the same bin. The goal is to pack the items in a minimum number of unit-size bins such that items assigned to each bin form an independent set in $G$.

Indeed, GBP is the special case where the conflict graph is a union of cliques. Thus, GBP is also known as bin packing with clique-graph conflicts (see Section 1.2).

1.1 Contribution and Techniques

Our main result (in Section 3) is an APTAS for the group bin packing problem, improving upon the best known ratio of 2 [1].

Existing algorithms for BPC often rely on initial coloring of the instance. This enables to apply in later steps known techniques for bin packing, considering each color class (i.e., a subset of non-conflicting items) separately. In contrast, our approach uses a refined packing of the original instance while eliminating conflicts, thus generalizing techniques for classic BP.

Our first technical contribution is an enhancement of the linear shifting technique of [10]. This enables our scheme to enumerate in polynomial time over packings of relatively large items, while guaranteeing that these packings respect the group constraints. Our Shift & Swap technique considers the set of large items that are associated with many different groups as a classic BP instance, i.e., the group constraints are initially relaxed. Then the scheme applies to these items the linear shifting technique of [10]. In the process, items of the same group may be packed in the same bin. Our Swapping algorithm resolves all conflicts, with no increase in the total number of bins used (see Sections 3.1 and 3.2).

A common approach used for deriving APTASs for BP is to pack in a bounded number of extra bins a set of discarded small items of total size $O(\varepsilon)OPT$, where $OPT = OPT(I)$ is the minimum number of bins required for packing the given instance, $I$, and $\varepsilon \in (0, 1)$ is the accuracy parameter of the scheme. As shown in Appendix B, this approach may fail for GBP, e.g., when the discarded items belong to the same group. Our second contribution is an algorithm that overcomes this hurdle. The crux is to find a set of small items of total size $O(\varepsilon)OPT$ containing $O(\varepsilon)OPT$ items from each group. This would enable to pack these items in a small number of extra bins. Furthermore, the remaining small items should be feasibly assigned to partially packed $OPT$ bins. Our algorithm identifies such sets of small items through an intricate combination of enumeration and a greedy-based approach that utilizes the rounded solution of a linear program.

---

1 We note that 2 is the best known absolute as well as asymptotic approximation ratio for the problem (see Section 1.2). We give formal definitions of absolute/asymptotic ratios in Section 2.
1.2 Related Work

The classic bin packing problem is known to be NP-hard. Furthermore, it cannot be approximated within a ratio better than $\frac{3}{2}$, unless $P=NP$. This ratio is achieved by the simple First-Fit Decreasing algorithm \cite{26}. The paper \cite{10} presents an APTAS for bin packing, which uses at most $(1+\varepsilon)OPT + 1$ bins, for any fixed $\varepsilon \in (0, 1/2)$. The paper \cite{20} gives an approximation algorithm that uses at most $OPT + O(\log^2(OPT))$ bins. The additive factor was improved in \cite{25} to $O(\log OPT \cdot \log \log OPT)$. For comprehensive surveys of known results for BP see, e.g., \cite{6,5}.

The problem of bin packing with conflicts (BPC) was introduced in \cite{19}. As BPC includes as a special case the classic graph coloring problem, it cannot be approximated within factor $N^{1-\varepsilon}$ for an input of $N$ items, for all $\varepsilon > 0$, unless $P = NP$ \cite{28}. Thus, most of the research work focused on obtaining approximation algorithms for BPC on classes of conflict graphs that can be optimally colored in polynomial time. Epstein and Levin \cite{9} presented sophisticated algorithms for two such classes, namely, a $\frac{5}{2}$-approximation for BPC with a perfect conflict graph, and a $\frac{7}{4}$-approximation for a bipartite conflict graph.

The hardness of approximation of GBP (with respect to absolute approximation ratio) follows from the hardness of BP, which is the special case of GBP where the conflict graph is an independent set. A 2.7-approximation algorithm for general instances follows from a result of \cite{19}. Oh and Son \cite{24} showed that a simple algorithm based on First-Fit outputs a packing of any GBP instance $I$ in $1.7OPT + 2.19v_{\text{max}}$ bins, where $v_{\text{max}} = \max_{1 \leq j \leq n} |G_j|$. The paper \cite{23} shows that some special cases of the problem are solvable in polynomial time. The best known ratio for GBP is 2 due to \cite{1}.

Jansen \cite{16} presented an asymptotic fully polynomial time approximation scheme (AFPTAS) for BPC on d-inductive conflict graphs, where $d \geq 1$ is some constant. The scheme of \cite{16} uses for packing a given instance $I$ at most $(1+\varepsilon)OPT + O(d/\varepsilon^2)$ bins. This implies that GBP admits an AFPTAS on instances where the maximum clique size is some constant $d$. Thus, the existence of an asymptotic approximation scheme for general instances remained open.

Das and Wiese \cite{7} introduced the problem of makespan minimization with bag constraints. In this generalization of the classic makespan minimization problem, each job belongs to a bag. The goal is to schedule the jobs on a set of $m$ identical machines, for some $m \geq 1$, such that no two jobs in the same bag are assigned to the same machine, and the makespan is minimized. For the classic problem of makespan minimization with no bag constraints, there are known polynomial time approximation scheme (PTAS) \cite{14,21} as well as efficient polynomial time approximation scheme (EPTAS) \cite{13,2,17,18}. Das and Wiese \cite{7} developed a PTAS for the problem with bag constraints. Later, Grage et al. \cite{11} obtained an EPTAS.

\footnote{For the subclass of interval graphs the paper \cite{9} gives a $\frac{5}{3}$-approximation algorithm.}

\footnote{A graph $G$ is $d$-inductive if the vertices of $G$ can be numbered such that each vertex is connected by an edge to at most $d$ lower numbered vertices.}
2 Preliminaries: Scheduling with Bag Constraints

Our scheme is inspired by the elaborate framework of Das and Wiese [7] for makespan minimization with bag constraints. For completeness, we give below an overview of the scheme of [7]. Given a set of jobs $I$ partitioned into bags and $m$ identical machines, let $p_\ell > 0$ be the processing time of job $\ell \in I$. The instance is scaled such that the optimal makespan is 1. The jobs and bags are then classified using the next lemma.

**Lemma 2.1.** For any instance $I$ and $\varepsilon \in (0, 1)$, there is an integer $k \in \{1, \ldots, \left\lfloor \frac{1}{\varepsilon^2} \right\rfloor\}$ such that $\sum_{\ell \in I: p_\ell \in [\varepsilon^k+1, \varepsilon^k)} p_\ell \leq \varepsilon^2 m$.

A job $\ell$ is small if $p_\ell < \varepsilon^{k+1}$, medium if $p_\ell \in [\varepsilon^{k+1}, \varepsilon^k)$ and large if $p_\ell \geq \varepsilon^k$, where $k$ is the value found in Lemma 2.1. A bag is large if the number of large and medium jobs it contains is at least $\varepsilon m$, and small otherwise.

The scheme of [7] initially enumerates over slot patterns so that large and medium jobs from large bags are optimally assigned to the machines in polynomial time. The enumeration is enhanced by using dynamic programming and a flow network to schedule also the large jobs from small bags. The medium jobs in each small bag are scheduled across the $m$ machines almost evenly, causing only small increase to the makespan. The small jobs are partitioned among machine groups with the same processing time and containing jobs from the same subset of large bags. Then, a greedy approach is used with respect to the bags to schedule the jobs within each machine group, such that the overall makespan is at most $1 + O(\varepsilon)$.

Our scheme classifies the items and groups similar to the classification of jobs and bags in [7]. We then apply enumeration over patterns to pack the large and medium items. Thus, Lemmas 3.2, 3.6 and 3.8 in this paper are adaptations of results obtained in [7]. However, the remaining components of our scheme are different. One crucial difference is our use of a Shift & Swap technique to round the sizes of large and medium items. Indeed, rounding the item sizes using the approach of [7] may cause overflow in the bins, requiring a large number of extra bins to accommodate the excess items. Furthermore, packing the small items using $O(\varepsilon)OPT$ extra bins requires new ideas (see Section 3).

We use standard definitions of approximation ratio and asymptotic approximation ratio. Given a minimization problem $\Pi$, let $\mathcal{A}$ be a polynomial-time algorithm for $\Pi$. For an instance $I$ of $\Pi$, denote by $OPT(I)$ and $\mathcal{A}(I)$ the values of an optimal solution and the solution returned by $\mathcal{A}$ for $I$, respectively. We say that $\mathcal{A}$ is a $\rho$-approximation algorithm for $\Pi$, for some $\rho \geq 1$, if $\mathcal{A}(I) \leq \rho \cdot OPT(I)$ for any instance $I$ of $\Pi$. $\mathcal{A}$ is an asymptotic $\rho$-approximation for $\Pi$ if there is a constant $c \in \mathbb{R}$ such that $\mathcal{A}(I) \leq \rho \cdot OPT(I) + c$ for any instance $I$ of $\Pi$. An APTAS for $\Pi$ is a family of algorithms $(\mathcal{A}_\varepsilon)_{\varepsilon > 0}$ such that $\mathcal{A}_\varepsilon$ is a polynomial-time asymptotic $(1 + \varepsilon)$-approximation for each $\varepsilon > 0$. When clear from the context, we use $OPT = OPT(I)$. 
3 An APTAS for GBP

In this section we present an APTAS for GBP. Let \( OPT \) be the optimal number of bins for an instance \( I \). Our scheme uses as a subroutine a BalancedColoring algorithm proposed in [1] for the group packing problem (see the details in Appendix C). Let \( S(I) \) be the total size of items in \( I \), i.e., \( S(I) = \sum_{\ell \in [N]} s_{\ell} \).

Recall that \( v_{\text{max}} \) is the maximum cardinality of any group. The next lemma follows from a result of [1].

**Lemma 3.1.** Let \( I \) be an instance of GBP. Then BalancedColoring packs \( I \) in at most \( \max\{2S(I), S(I) + v_{\text{max}}\} \) bins.

By the above, given an instance \( I \) of GBP, we can guess \( OPT \) in polynomial time, by iterating over all integer values in \( [1, \max\{2S(I), S(I) + v_{\text{max}}\}] \) and taking the minimal number of bins for which a feasible solution exists.

Similar to Lemma 2.1, we can find a value of \( k \), \( 1 \leq k \leq \lceil \frac{1}{\epsilon^2} \rceil \), satisfying \( \sum_{\ell \in I: s_{\ell} \in \varepsilon^{k+1}, \varepsilon^k} s_{\ell} \leq \varepsilon^2 \cdot OPT \). Now, we classify item \( \ell \) as small if \( s_{\ell} < \varepsilon^{k+1} \), medium if \( s_{\ell} \in \varepsilon^{k+1}, \varepsilon^k \) and large otherwise. A group is large if the number of large and medium items of that group is at least \( \varepsilon^{k+2} \cdot OPT \), and small otherwise. Given an instance \( I \) of GBP and a constant \( \epsilon \in (0, 1) \), we also assume that \( OPT > \frac{3}{\epsilon^{2k+4}} \cdot OPT \) (otherwise, the conflict graph is \( d \)-inductive, where \( d \) is a constant, and the problem admits an AFPTAS [16]).

**Lemma 3.2.** There are at most \( \frac{1}{\epsilon^{2k+3}} \) large groups.

### 3.1 Rounding of Large and Medium Items

We start by reducing the number of distinct sizes for the large and medium items. Recall that in the linear shifting technique we are given a BP instance of \( N \) items and a parameter \( Q \in (0, N] \). The items are sorted in non-increasing order by sizes and then partitioned into classes. Each class (except maybe the last one) contains \( \max\{Q, 1\} \) items. The items in class 1 (i.e., largest items) are discarded (the discarded items are handled in a later stage of the algorithm). The sizes of items in each class are then rounded up to the maximum size of an item in this class. For more details see, e.g., [10].

We apply linear shifting to the large and medium items in each large group with parameter \( Q = [\varepsilon^{2k+4} \cdot OPT] \). Let \( I, I' \) be the instance before and after the shifting over large groups, respectively.

**Lemma 3.3.** \( OPT(I') \leq OPT(I) \).

**Lemma 3.4.** Given a feasible packing of \( I' \) in \( OPT \) bins, we can find a feasible packing of \( I \) in \( (1 + O(\epsilon))OPT \) bins.

Next, we round the sizes of large items in small groups. As the number of these groups may be large, we use the following Shift & Swap technique. We merge all of the large items in small groups into a single group, to which we
apply linear shifting with parameter \( Q = \lfloor 2\varepsilon \cdot OPT \rfloor \). In addition to items in class 1, which are discarded due to linear shifting, we also discard the items in the last size class; these items are packed in a new set of bins (see the proof of Lemma 3.15 in Appendix E).

**Lemma 3.5.** After rounding, there are at most \( O(1) \) distinct sizes of large and medium items from large groups, and large items from small groups.

Relaxing the feasibility requirement for the packing of rounded large items from small groups, the statements of Lemma 3.3 and Lemma 3.4 hold for these items as well. To obtain a feasible packing of these items, we apply a Swapping subroutine which resolves the possible conflicts caused while packing the items.

Our scheme packs in each step a subset of items, using \( OPT \) bins, while discarding some items. The discarded items are packed later in a set of \( O(\varepsilon) \cdot OPT + 1 \) extra bins. In Section 3.2 we pack the large and medium items using enumeration over patterns followed by our Swapping algorithm to resolve conflicts. Section 3.3 presents an algorithm for packing the small items by combining recursive enumeration (for relatively “large” items) with a greedy-based algorithm that utilizes the rounded solution of a linear program (for relatively “small” items). In Section 3.4 we show that the components of our scheme combine together to an APTAS for GBP.

### 3.2 Large and Medium Items

The large items and medium items from large groups are packed in the bins using slot patterns. Let \( G_{i_1}, \ldots, G_{i_L} \) be the large groups, and let ‘\( u \)’ be a label representing all the small groups. Given the modified instance \( I' \), a slot is a pair \((s_\ell, j)\), where \( s_\ell \) is the rounded size of a large or medium item \( \ell \in I' \) and \( j \in \{i_1, \ldots, i_L\} \cup \{u\} \). A pattern is a multiset \( \{t_1, \ldots, t_\beta\} \) for some \( 1 \leq \beta \leq \floor{\frac{1}{\varepsilon + 1}} \), where \( t_i \) is a slot for each \( i \in [\beta] \).

**Lemma 3.6.** By using enumeration over patterns, we find a pattern for each bin for the large and medium items, such that these patterns correspond to an optimal solution. The running time is \( O(N^{O(1)}) \).

Given slot patterns corresponding to an optimal solution, large and medium items from large groups can be packed optimally, since they are identified both by a label and a size. On the other hand, large items from small groups are identified solely by their sizes. A greedy packing of these items, relating only to their corresponding patterns, may result in conflicts (i.e., two large items of the same small group are packed in the same bin). Therefore, we incorporate a process of swapping items of the same (rounded) size between their hosting bins, until there are no conflicts.

Given an item \( \ell \) that conflicts with another item in bin \( b \), for an item \( y \) in bin \( c \) such that \( s_\ell = s_y \), \( \text{swap}(\ell, y) \) is bad if it causes a conflict (either because

\[\text{Recall that the number of medium/large items that fit in a single bin is at most } \floor{\frac{1}{\varepsilon + 1}}.\]
y conflicts with an item in bin b, ℓ conflicts with an item in bin c, or c = b); otherwise, swap(ℓ, y) is good. We now describe our algorithm for packing the large items from small groups.

Let ζ be the given slot patterns for OPT bins. Initially, the items are packed by these patterns, where items from small groups are packed ignoring the group constraints. This can be done simply by placing an arbitrary item of size s from some small group in each slot (s, u). If ζ corresponds to an optimal solution, we meet the capacity constraint of each bin. However, this may result with conflicting items in some bins. Suppose there is a conflict in bin b. Then for one of the conflicting items, ℓ, we find a good swap(ℓ, y) with item y in a different bin, such that sy = sf. We repeat this process until there are no conflicts. We give the pseudocode of Swapping in Algorithm 1.

Algorithm 1 Swapping(ζ, G1, . . . , Gn)
1: Pack the large and medium items from large groups in slots corresponding to their sizes and by labels.
2: Pack large items from small groups in slots corresponding to their sizes.
3: while there is an item ℓ involved in a conflict do
4: Find a good swap(ℓ, y) and resolve the conflict.
5: end while

Theorem 3.7. Given a packing of large and medium items by slot patterns corresponding to an optimal solution, Algorithm 1 resolves all conflicts in polynomial time.

We use the Swapping algorithm for each possible guess of patterns to obtain a feasible packing of the large items and medium items from large groups in OPT bins.

Now, we discard the medium items from small groups and pack them later in a new set of bins with other discarded items. This requires only a small number of extra bins (see the proof of Lemma 3.15 in Appendix E).

3.3 Small Items

Up to this point, all large items and the medium items from large groups are feasibly packed in OPT bins. We proceed to pack the small items. Let I0, B be the set of unpacked items and the set of OPT partially packed bins, respectively. The packing of the small items is done in four phases: an optimal phase, an eviction phase, a partition phase and a greedy phase.

The optimal phase is an iterative process consisting of a constant number of iterations. In each iteration, a subset of bins is packed with a subset of items whose (rounded) sizes are large relative to the free space in each of these bins. As these items belong to a small collection of groups among G1, . . . , Gn, they can
be selected using enumeration. Thus, we obtain a packing of these items which corresponds to an optimal solution. For packing the remaining items, we want each item to be small relative to the free space in its assigned bin. To this end, in the eviction phase we discard from some bins items of non-negligible size (a single item from each bin). Then, in the partition phase, the unpacked items are partitioned into a constant number of sets satisfying certain properties, which guarantee that these items can be feasibly packed in the available free space in the bins. Finally, in the greedy phase, the items in each set are packed in their allotted subset of bins greedily, achieving a feasible packing of all items, except for a small number of items from each group, of small total size. The pseudocode of our algorithm for packing the small items is given in Algorithm 4.

The optimal phase: For any \( b \in B \), denote by \( f_b^0 \) the free capacity in bin \( b \), i.e., \( f_b^0 = 1 - \sum_{t \in \ell} s_t \). We say that item \( t \) is \( b \)-negligible if \( s_t \leq \varepsilon^2 f_b^0 \), and \( t \) is \( b \)-non-negligible otherwise. We start by classifying the bins into two disjoint sets. Let \( E_0 = \{ b \in B \mid 0 < f_b^0 < \varepsilon \} \) and \( D_0 = B \setminus E_0 \).

We now partition \( B \) into types. Each type contains bins having the same total size of packed large/medium items; also, the items packed in each bin type belong to the same set of large groups, and the same number of slots is allocated in these bins to items from small groups. Formally, for each pattern \( p \) we denote by \( t_p \) the subset of bins packed with \( p \).\(^5\) Let \( T \) denote the set of bin types. Then \( |T| = |P| \), where \( P \) is the set of all patterns. The cardinality of type \( t \in T \) is the number of bins of this type. We use for the optimal phase algorithm \textsc{RecursiveEnum} (see the pseudocode in Algorithm 2).

Lemma 3.8. There are \( O(1) \) types before Step 1 of Algorithm 2.

Once we have the classification of bins, each type \( t \) of cardinality smaller than \( 1/\varepsilon^4 \) is padded with empty bins so that \( |t| \geq 1/\varepsilon^4 \). An item \( t \) is \( t \)-negligible if \( t \) is \( b \)-negligible for all bins \( b \) of type \( t \) (all bins in the same type have the same free capacity), and \( t \)-non-negligible otherwise. Denote by \( I_t \) the large/medium items that are packed in the bins of type \( t \), and let \( I_t(g) \) be the set of small items that are packed in \( t \) in some solution \( g \) (in addition to \( I_t \)). For any \( 1 \leq i \leq n \), a group \( G_i \) is \( t(g) \)-significant if \( I_t(g) \) contains at least \( \varepsilon^4 |t| \) \( t \)-non-negligible items from \( G_i \), and \( G_i \) is \( t(g) \)-insignificant otherwise.

\textsc{RecursiveEnum} proceeds in iterations. In the first iteration, it guesses for each type \( t \subseteq E_0 \) a subset of the items \( I_t(g_{opt}) \subseteq I_0 \), where \( g_{opt} \) corresponds to an optimal solution for completing the packing of \( t \). Specifically, \textsc{RecursiveEnum} initially guesses \( L(t, g_{opt}) \) groups that are \( t(g_{opt}) \)-significant: \( G_{i_1}, \ldots, G_{i_{|L(t, g_{opt})|}} \). For each \( G_{i_j}, j \in \{1, \ldots, L(t, g_{opt})\} \), the algorithm guesses which items of \( G_{i_j} \) are added to \( I_t(g_{opt}) \). Since the number of guesses might be exponential, we apply to \( G_{i_j} \) linear shifting as follows. Guess \( \lceil \frac{1}{\varepsilon^4} \rceil \) representatives in \( G_{i_j} \), of sizes \( s_{t_1} \leq s_{t_2} \leq \ldots \leq s_{t_{\lfloor \frac{1}{\varepsilon^4} \rfloor}} \). The \( k \)th representative is the largest item in size class \( k \), \( 1 \leq k \leq \lfloor \frac{1}{\varepsilon^4} \rfloor \) for the linear shifting of \( G_{i_j} \) in type \( t \). Using the parameter \( Q_{i_j}^t = \varepsilon^3 |t| \), the item sizes in class \( k \) are rounded up to \( s_{t_k} \), for \( 1 \leq k \leq \lfloor \frac{1}{\varepsilon^4} \rfloor \).

\(^5\) For the definition of patterns see Section 3.2.
Given a correct guess of the representatives, the actual items in size class $k$ are selected at the end of algorithm RecursiveEnum (in Step 20). Denote the chosen items from $G_i$ to bins of type $t$ by $G_i^t$.

We now extend the definition of patterns for each type $t$. A slot is a pair $(s, t)$, where $s_t$ is the (rounded) size of a $t$-non-negligible item $\ell \in I_t(g_{\text{opt}})$, and there is a label for each $t(g_{\text{opt}})$-significant group $G_i^t$, $j \in \{1, \ldots, L(t, g_{\text{opt}})\}$. A $t$-pattern is a multiset $\{q_1, \ldots, q_{\beta_t}\}$ containing at most $\left\lfloor \frac{1}{\varepsilon} \right\rfloor$ elements, where $q_t$ is a slot for each $i \in \{1, \ldots, \beta_t\}$. Now, for each type $t \in T$ we use enumeration over patterns for assigning $G_i^t, G_j^t(t, g_{\text{opt}})$ to bins in $t$. This completes the first iteration, and the algorithm proceeds recursively.

We now update $D_0, E_0$ for the next iteration by removing from $E_0$ bins $b$ that have a considerably large free capacity with respect to $f_b^0$. For each $b \in B$, let $f_b^1$ be the capacity available in $b$ after iteration 1. Then $E_1 = \{b \in E_0 | 0 < f_b^1 < \varepsilon f_b^0 \}$ and $D_1 = B \setminus E_1$.

Now, each type $t \in T$ is partitioned into sub-types that differ by the packing of $I_t(g_{\text{opt}})$ in the first iteration. The set of types $T$ is updated to contain these sub-types. At this point, a recursive call to RecursiveEnum computes for each bin type $t \subseteq E_1$ a guessing and a packing of its $t$-non-negligible items. We repeat this recursive process $\alpha = \frac{1}{\varepsilon} + 5 = O(1)$ times.

Let $G_i^t$ be the subset of items (of rounded sizes) assigned from $G_i$ to bins of type $t$ at the end of RecursiveEnum, for $1 \leq i \leq n$ and $t \in T$. Recall that the algorithm did not select specific items in $G_i^t$; that is, we only have their rounded sizes and the number of items in each size class. The algorithm proceeds to pack items from $G_i$ in all types $t$ for which $G_i$ was $t(g_{\text{opt}})$-significant in some iteration. Let $T_{G_i}$ be the set of these types. The algorithm considers first the type $t \in T_{G_i}$ for which the class $C$ of largest size items contains the item of maximal size, where the maximum is taken over all types $t \in T_{G_i}$. The algorithm packs in bins of type $t$ the $Q_i^t$ largest remaining items in $G_i$ in the slots allocated to items in $C$; it then proceeds similarly to the remaining size classes in types $t \in T_{G_i}$, and the remaining items in $G_i^t$.

**Lemma 3.9.** The following hold for RecursiveEnum: (i) the running time is polynomial; (ii) the increase in the number of bins in Step 7 is at most $\varepsilon$OPT; (iii) In Step 12 we discard at most $\varepsilon$OPT items from each group of total size at most $\varepsilon$OPT. (iv) One of the guesses in Steps 9, 12 corresponds to an optimal solution.

**The eviction phase:** One of the guesses in the optimal phase corresponds to an optimal solution. For simplicity, henceforth assume that we have this guess. Recall that $E_\alpha$ is the set of all bins $b$ for which $0 < f_b^\alpha < \varepsilon f_b^{\alpha-1}$. In Step 3 of PackSmallItems (Algorithm 4) we evict an item from each $b \in E_\alpha$ such that $f_b^\alpha$ in this iteration, or w.r.t $f_b^h$ in iteration $h+1$, $h \in \{0, \ldots, \alpha - 1\}$.

---

6. Note that we do not need a label for the $t(g_{\text{opt}})$-insignificant groups, because their items are packed separately.

7. An item is $b$-non-negligible w.r.t $f_b^h$ in this iteration, or w.r.t $f_b^h$ in iteration $h+1$, $h \in \{0, \ldots, \alpha - 1\}$. 

---

9
Algorithm 2 \textit{RecursiveEnum}(I_0, B)

1: Let $f^0_b$ be the remaining free capacity in bin $b \in B$.
2: Let $E_0 = \{b \in B|0 < f^0_b < \varepsilon\}$ and $D_0 = B \setminus E_0$.
3: Denote by $T$ the collection of bin types.
4: for $h = 0, \ldots, \alpha$ do
5: \hspace{1em} for all types $t \subseteq E_h$ do
6: \hspace{2em} if $|t| < \frac{1}{\sqrt{\alpha}}$ then
7: \hspace{3em} increase the cardinality of $t$ to $\frac{1}{\sqrt{\alpha}}$.
8: \hspace{1em} end if
9: \hspace{1em} Guess $t(g_{\text{opt}})$-significant groups: $G_{i1}, \ldots, G_{iL(t,g_{\text{opt}})}$
10: \hspace{1em} for $j = 1, \ldots, L(t,g_{\text{opt}})$ do
11: \hspace{2em} Guess the number of items from $G_{ij}$ to be added to bins of type $t$.
12: \hspace{2em} Guess a representative for each size class of $t$-non-negligible items of $G_{ij}$ for linear shifting.
13: \hspace{1em} end for
14: \hspace{1em} Replace type $t$ in $T$ by all of the sub-types of $t$.
15: \hspace{1em} end for
16: Let $f^h_b$ be the remaining free capacity in bin $b \in B$.
17: Let $E_{h+1} = \{b \in E_h|0 < f^h_b < \varepsilon f^h_b\}$ and $D_{h+1} = B \setminus E_{h+1}$.
18: end for
19: Complete the packing of all size classes by assigning items greedily.

the available capacity of $b$ increases to at least $\frac{\alpha}{2}$.
This is done greedily: consider the bins in $E_\alpha$ one by one in arbitrary order.
From each bin discard a small item $\ell \in G_i$, for some $G_i$, $1 \leq i \leq n$,
such that the following hold: (i) $s_\ell \geq \frac{\alpha}{2}$, and (ii) less than $\varepsilon OP$ items were discarded from $G_i$ in this phase.
Since $\alpha$ is large enough, this phase can be completed successfully, as shown below.
Let $T = \{t_1, \ldots, t_\mu, t'\}$ be the types after the optimal phase, where $t'$ is a new type such that $|t'| = \varepsilon OP$.
Bins of type $t'$ are empty, i.e., each bin $b$ of type $t'$ has free space 1.
Denote by $f(t)$ the free space in each bin $b$ of type $t$ after the eviction phase, and let $I_L$ be the large items from small groups (already packed in the bins).

Lemma 3.10. After Step 5 of PackSmallItems there exists a partition of $I_\alpha$ into types $I_{t1}, \ldots, I_{t_\mu}, I_L$, for which the following hold.
For each $t \in T$, (i) $|G'_t| = |G_j \cap I_t| \leq |t| - |(I'_t \setminus I_L) \cap G_j|$, for all $1 \leq j \leq n$.
(ii) for any $t \in I_L : s_\ell \leq \varepsilon f(t)$, and (iii) $S(I_L) \leq f(t)|t|$.

We explain the conditions of the lemma below.

The partition phase: Let $T$ be the set of types after Step 5 of Algorithm 4, and $I_\alpha$ the remaining unpacked items.$^8$ We seek a partition of $I_\alpha$ into subsets

$^8$ Recall that we consider only items that were not discarded in previous steps, as discarded items are packed in a separate set of bins.
associated with bin types such that the items assigned to each type \( t \) are relatively tiny; also, the total size and the cardinality of the set of items assigned to \( t \) allow to feasibly pack these items in bins of this type. This is done by proving that a polytope representing the conditions in Lemma 3.10 has vertices at points which are integral up to a constant number of coordinates. Each such coordinate, \( x_{\ell,t} \), corresponds to a fractional selection of some item \( \ell \in I_{\alpha} \) to type \( t \in T \). We use \( G_j \) to denote the subset of remaining items in \( G_j \), \( 1 \leq j \leq n \).

Formally, we define a polytope \( P \) as the set of all points \( x \in [0,1]^{I_{\alpha} \times T} \) which satisfy the following constraints.

\[
\forall \ell \in I_{\alpha}, t \in T \text{ s.t. } s_\ell > \varepsilon f(t) : x_{\ell,t} = 0
\]
\[
\forall t \in T : \sum_{\ell \in I_{\alpha}} x_{\ell,t} s_\ell \leq f(t)|t|
\]
\[
\forall \ell \in I_{\alpha} : \sum_{t \in T} x_{\ell,t} = 1
\]
\[
\forall 1 \leq j \leq n, t \in T : \sum_{\ell \in G_j} x_{\ell,t} \leq |t| - |(I'_t \setminus I_L) \cap G_j|
\]

The first constraint refers to condition (ii) in Lemma 3.10, which implies that items assigned to type \( t \) need to be tiny w.r.t the free space in the bins of this type. The second constraint reflects condition (iii) in the lemma, which guarantees that the items in \( I_t \) can be feasibly packed in the bins of type \( t \). The third constraint ensures that overall each item \( \ell \in I_{\alpha} \) is (fractionally) assigned exactly once.

The last constraint reflects condition (i) in Lemma 3.10. Overall, we want to have at most \( |t| \) items of \( G_j \) assigned to bins of type \( t \). Recall that these bins may already contain large/medium items from \( G_j \) packed in previous steps. While large/medium items from large groups are packed optimally, the packing of large items from small groups, i.e., \( I_L \), is not necessarily optimal. In particular, the items in \( I_L \) packed by our scheme in bins of type \( t \) may not appear in these bins in the optimal solution \( g_{opt} \) to which our packing corresponds. Thus, we exclude these items and only require that the number of items assigned from \( G_j \) to bins of type \( t \) is bounded by \( |t| - |(I'_t \setminus I_L) \cap G_j| \).

**Theorem 3.11.** Let \( x \in P \) be a vertex of \( P \). Then,

\[
|\{ \ell \in I_{\alpha} \mid \exists t \in T : x_{\ell,t} \in (0,1) \}| = O(1).
\]

By Theorem 3.11, we can find a feasible partition (with respect to the constraints of the polytope) by finding a vertex of the polytope, and then discarding the \( O(1) \) fractional items. These items can be packed in \( O(1) \) extra bins. By Lemma 3.10 we have that \( P \neq \emptyset \); thus, a vertex of \( P \) exists and the partition can be found in polynomial time.

**The greedy phase:** In this phase we pack the remaining items using algorithm GreedyPack (see the pseudocode in Algorithm 3). Let \( G_1^t, \ldots, G_n^t \) be the items
in $I_t$ from each group, and let $S(I_t)$ be the total size of these items, i.e., $S(I_t) = \sum_{j=1}^{n} \sum_{\ell \in G_{tj}} s_{\ell}$.

Algorithm 3  

| GreedyPack($I_t = \{G_{i1}^t, \ldots, G_{in}^t\}$, $t = \{b_1, \ldots, b_{|t|}\}$) |
|---|
| 1: for $j = 1, \ldots, H$ do |
| 2: Sort $G_{ij}$ in a non-increasing order by sizes. |
| 3: end for |
| 4: Let $y_{ij}$ be the largest remaining item in $G_{ij}$, $j = 1, \ldots, H$. |
| 5: for each bin $b \in t$ do |
| 6: Add to bin $b$ the items $y_{i1}, \ldots, y_{iH}$. |
| 7: while total size of items packed in bin $b > 1$ do |
| 8: Select a group $G_{ij}^t \in \{G_{i1}^t, \ldots, G_{in}^t\}$ such that $y_{ij}$ is not last in $G_{ij}$. |
| 9: if cannot complete last step then |
| 10: return failure |
| 11: end if |
| 12: Return $y_{ij}$ to $G_{ij}^t$. |
| 13: Let $y'_{ij}$ be the next largest item in $G_{ij}$. |
| 14: Add $y'_{ij}$ to bin $b$. |
| 15: end while |
| 16: for $j = 1, \ldots, H$ do |
| 17: if $G_{ij}^t$ has a large item in bin $b$ then |
| 18: discard the small item. |
| 19: end if |
| 20: end for |
| 21: end for |

We now describe the packing of the remaining items in $I_t$ in bins of type $t$. First, we add $2|t|$ extra bins to $t$. The extra bins are empty and thus have capacity 1; however, we assume that they have capacity $f(t) \leq 1$. This increases the overall number of bins in the solution by $2\varepsilon OPT$. Consider the items in each group in non-increasing order by sizes. For each bin $b \in t$ in an arbitrary order, GreedyPack assigns to $b$ the largest remaining item in each group $G_{i1}^t, \ldots, G_{in}^t$. If an overflow occurs, replace an item from some group $G_{ij}^t$ by the next item in $G_{ij}^t$. This is repeated until there is no overflow in $b$. W.l.o.g., we may assume that $|G_j| = OPT$ for all $1 \leq j \leq n$; thus, $b$ contains one item from each group (otherwise, we can add to $G_j$ dummy items of size 0, with no increase to the number of bins in an optimal solution).

Recall that the large items from small groups are packed using the Swapping algorithm, that yields a feasible packing. Yet, it does not guarantee that the small items can be added without causing conflicts. Hence, GreedyPack may output a packing in which a small and large item from the same small group are packed in the same bin. Such conflicts are resolved by discarding the small item in each.
Lemma 3.12. The total size of items discarded in GreedyPack in Step 18 due to conflicts is at most $\varepsilon OPT$, and at most $\varepsilon^{k+2} \cdot OPT$ items are discarded from each group.

Proof. The number of items discarded from each group is at most $\varepsilon^{k+2} \cdot OPT$, since all groups are small. Assume that the total size of these items is strictly larger than $\varepsilon OPT$. Since each discarded item is coupled with a large conflicting item from the same group, whose size is at least $1/\varepsilon$ times larger (recall that the medium items are discarded), this implies that the total size of large conflicting items is greater than $OPT$. Contradiction. \qed

Algorithm 4 PackSmallItems($I_0, B$)

1: for each guess of RecursiveEnum($I_0, B$) do
2:   for $b \in E_\alpha$ do
3:     evict from $b$ the largest item $\ell$ satisfying: $\ell$ is small, and less than $\varepsilon OPT$ items where evicted from $G_i$, where $\ell \in G_i$.
4:   end for
5: Add to $T$ a new type $t'$ consisting of $\varepsilon OPT$ empty bins.
6: Compute a feasible partition of $I_\alpha$ into the types in $T$.
7: for $t \in T$ do
8:   Add $2\varepsilon|t|$ extra bins to $t$.
9: Assign $I_t$ to bins of type $t$ using GreedyPack($I_t$, $t$).
10: end for
11: end for

Lemma 3.13. For any $t \in T$, given a parameter $0 < \delta < \frac{1}{2}$ and a set of items $I_t$ such that (i) $|G^t_i| \leq |t| - |(I^t_i \setminus I_L) \cap G^t_j|$; (ii) for all $\ell \in I_t: s_\ell \leq \delta f(t)$, and (iii) $S(I_t) \leq (1 - \delta)f(t)|t|$, GreedyPack finds a feasible packing of $I_t$ in bins of type $t$.

Lemma 3.14. Algorithm 4 assigns in Step 9 to OPT bins all items except for $O(\varepsilon)OPT$ items from each group, of total size $O(\varepsilon)OPT$.

3.4 Putting it all Together

It remains to show that the items discarded throughout the execution of the scheme can be packed in a small number of extra bins.

Lemma 3.15. The medium items from small groups and all discarded items can be packed in $O(\varepsilon) \cdot OPT$ extra bins.

Algorithm 6 summarizes the steps of our scheme (see Appendix D).

Theorem 3.16. There is an APTAS for the group bin packing problem.
References

1. Adany, R., Feldman, M., Haramaty, E., Khandekar, R., Schieber, B., Schwartz, R., Shachnai, H., Tamir, T.: All-or-nothing generalized assignment with application to scheduling advertising campaigns. ACM Transactions on Algorithms (TALG) 12(3), 1–25 (2016)
2. Alon, N., Azar, Y., Woeginger, G.J., Yadid, T.: Approximation schemes for scheduling on parallel machines. Journal of Scheduling 1(1), 55–66 (1998)
3. Anderson, D.P.: Boinc: A system for public-resource computing and storage. In: Fifth IEEE/ACM international workshop on grid computing. pp. 4–10. IEEE (2004)
4. Anderson, D.P.: BOINC: A platform for volunteer computing. Journal of Grid Computing 18, 99 – 122 (2017)
5. Christensen, H.I., Khan, A., Pokutta, S., Tetali, P.: Approximation and online algorithms for multidimensional bin packing: A survey. Computer Science Review 24, 63–79 (2017)
6. Coffman, E.G., Csisrik, J., Galambos, G., Martello, S., Vigo, D.: Bin packing approximation algorithms: survey and classification. In: Handbook of combinatorial optimization, pp. 455–531 (2013)
7. Das, S., Wiese, A.: On minimizing the makespan when some jobs cannot be assigned on the same machine. In: 25th Annual European Symposium on Algorithms, ESA. pp. 31:1–31:14 (2017)
8. Ennajjar, I., Tabii, Y., Benkaddour, A.: Securing data in cloud computing by classification. In: Proceedings of the 2nd international Conference on Big Data, Cloud and Applications. pp. 1–5 (2017)
9. Epstein, L., Levin, A.: On bin packing with conflicts. SIAM Journal on Optimization 19(3), 1270–1298 (2008)
10. Fernandez de la Vega, W., Lueker, G.S.: Bin packing can be solved within 1 + ε in linear time. Combinatorica 1, 349–355 (1981)
11. Grage, K., Jansen, K., Klein, K.M.: An EPTAS for machine scheduling with bag-constraints. In: The 31st ACM Symposium on Parallelism in Algorithms and Architectures. pp. 135–144 (2019)
12. Guerine, M., Stockinger, M.B., Rosseti, I., Simonetti, L.G., Ocaña, K.A., Plastino, A., de Oliveira, D.: A provenance-based heuristic for preserving results confidentiality in cloud-based scientific workflows. Future Generation Computer Systems 97, 697–713 (2019)
13. Hochbaum, D.S. (ed.): Approximation Algorithms for NP-Hard Problems. PWS Publishing Co., USA (1996)
14. Hochbaum, D.S., Shmoys, D.B.: Using dual approximation algorithms for scheduling problems theoretical and practical results. Journal of the ACM 34(1), 144–162 (1987)
15. Hoffman, A.J., Kruskal, J.B.: Integral boundary points of convex polyhedra. In: Linear Inequalities and Related Systems.(AM-38), Volume 38, pp. 223–246. Princeton University Press (1956)
16. Jansen, K.: An approximation scheme for bin packing with conflicts. Journal of combinatorial optimization 3(4), 363–377 (1999)
17. Jansen, K.: An EPTAS for scheduling jobs on uniform processors: using an MILP relaxation with a constant number of integral variables. SIAM Journal on Discrete Mathematics 24(2), 457–485 (2010)
18. Jansen, K., Klein, K., Verschae, J.: Closing the gap for makespan scheduling via sparsification techniques. In: 43rd International Colloquium on Automata, Languages, and Programming (ICALP)., pp. 72:1–72:13 (2016)
19. Jansen, K., Øhring, S.R.: Approximation algorithms for time constrained scheduling. Inf. Comput. 132(2), 85–108 (1997)
20. Karmarkar, N., Karp, R.M.: An efficient approximation scheme for the one-dimensional bin-packing problem. In: 23rd Annual Symposium on Foundations of Computer Science. pp. 312–320. IEEE (1982)
21. Leung, J.Y.: Bin packing with restricted piece sizes. Information Processing Letters 31(3), 145–149 (1989)
22. Lin, Y., Shen, H.: Eafr: An energy-efficient adaptive file replication system in data-intensive clusters. IEEE Transactions on Parallel and Distributed Systems 28(4), 1017–1030 (2017)
23. McCloskey, B., Shankar, A.: Approaches to bin packing with clique-graph conflicts. Computer Science Division, University of California (2005)
24. Oh, Y., Son, S.: On a constrained bin-packing problem. Technical Report CS-95-14 (1995)
25. Rothvoß, T.: Approximating bin packing within $O(\log OPT \times \log \log OPT)$ bins. In: 54th Annual IEEE Symposium on Foundations of Computer Science. pp. 20–29. IEEE Computer Society (2013)
26. Simchi-Levi, D.: New worst-case results for the bin-packing problem. Naval Research Logistics (NRL) 41(4), 579–585 (1994)
27. Vazirani, V.V.: Approximation Algorithms. Springer-Verlag, Berlin, Heidelberg (2001)
28. Zuckerman, D.: Linear degree extractors and the inapproximability of max clique and chromatic number. Theory of Computing 3(1), 103–128 (2007)

A Applications of Group Bin Packing

A.1 Storing File Replicas

Different versions (or replicas) of critical data files are distributed to servers around the network [22]. Each server has its storage capacity and can thus be viewed as a bin. Each data file is an item. The set of replicas of each data file forms a group. To ensure better fault tolerance, replicas of the same data file must be stored on different servers. The problem of storing a given set of file replicas on a minimal number of servers in the network can be cast as an instance of GBP.

A.2 Security in Cloud Computing

Computational projects of large data scale, such as scientific experiments or simulations, often rely on cloud computing. Commonly, the project data is also stored in the cloud. In this setting, a main concern is that a malicious entity might gain access to confidential data [8]. To strengthen security, data is dispersed among multiple cloud storage services [12]. Projects are fragmented into critical tasks, so that no single task can reveal substantial information about the entire project. Then, each task is stored on a different storage service. Viewing a cloud storage service as a bin and each project as a group containing a collection of critical tasks (items), the problem of storing a set of projects on a minimal number of (identical) storage services yields an instance of GBP.
A.3 Signal Distribution

Volunteer computing allows researchers and organizations to harvest computing capacity from volunteers, e.g., donors among the general public. The principal framework for volunteer computing is the Berkeley Open Infrastructure for Network Computing, popularly known as BOINC [3,4]. Such distributed systems must dispense work items to clients. The clients can be viewed as bins containing work items. Each item requires some amount of processing time. A client will contribute only a fixed number of processor cycles per day. Assume that work items that are correlated (such as signals from the same region of the sky) can be verified against each other. To avoid tampering, signals are distributed so that no client processes more than one signal from the same region. Viewing signals from the same region as groups, we have an instance of GBP.

B Group Bin Packing vs. BP

A common approach in developing asymptotic approximation schemes for the bin packing problem is to distinguish between large and small items. Initially, the small items are discarded from the instance, and an (almost) optimal packing is obtained for the large items. The small items are then added in the remaining free space, with possible use of a small number of extra bins (see, e.g., [10] and the comprehensive survey in [27]). Unfortunately, when handling a GBP instance, this approach may not lead to an APTAS. Indeed, it may be the case that all of the small items belong to a single group. Thus, a large number of extra bins may be required to accommodate small items which cannot be added to previously packed bins. We give a detailed example below.

Figure 1 (a) and (b) illustrate two different packings of a given GBP instance $I$ with $N$ items and $n$ groups, where $I = \{G_1, \ldots, G_n\}$. The content of each group is given as a multiset, where each item is represented by a number (its size) in the range $(0, 1]$. The instance consists of $n_1$ groups of a single item of
size $\frac{1}{5}$: $G_1 = \{\frac{1}{5}\}, G_2 = \{\frac{1}{5}\}, \ldots, G_{n_1} = \{\frac{1}{5}\}$; $n_2$ groups with a single item of size $\frac{\varepsilon}{5}$: $G_{n_1+1} = \{\frac{\varepsilon}{5}\}, \ldots, G_{n_1+n_2} = \{\frac{\varepsilon}{5}\}$, and one group with $\hat{N}$ items of size $\frac{\varepsilon}{5}$; that is, $G_n = \{\frac{\varepsilon}{5}, \ldots, \frac{\varepsilon}{5}\}$, where $n = n_1 + n_2 + 1$ and $|G_n| = \hat{N}$. In addition, $n_1 = 4\hat{N}$ and $n_2 = \hat{N} \cdot (1 - \varepsilon)^9$. The checkered boxes (the upper rectangles on each column) represent items in $G_n$. All other groups contain a single item. Each of these other groups is represented by a box of a different color, of size corresponding to the size of the single item in the group, which can be either $\frac{1}{5}$ or $\frac{\varepsilon}{5}$. Each column represents a bin of unit capacity.

Figure 1(a) shows a packing where each bin contains 4 items of size $\frac{1}{5}$ and $\frac{1}{5}$ items of size $\frac{\varepsilon}{5}$; among these items, exactly one item belongs to $G_n$. Thus, the total number of bins is $\hat{N}$. Since each bin is full, this packing is optimal, i.e., $OPT = \hat{N}$.

Now, suppose that, initially, all items of sizes larger than $\delta$, for some $\delta \in (0, \frac{1}{5})$, are packed optimally. The small items are then added in the free space in a greedy manner. Specifically, starting from the first bin, small items are added until the bin is full, or until it contains an item from each group. We then proceed to the next bin. Figure 1(b) shows a packing in which the large items (each of size $\frac{1}{5}$) are packed first optimally, 4 items in each bin, using $n_1 = 4\hat{N}$ bins. Then, the items in $G_{n_1+1}, \ldots, G_{n_1+n_2}$ are added greedily so that the first $(1 - \varepsilon)\hat{N}$ bins are full. Then, the first $\varepsilon\hat{N}$ items in $G_n$ are packed in the remaining bins and a set of $(1 - \varepsilon)\hat{N}$ bins is added for the remaining items. Overall, the number of bins used is $(2 - \varepsilon)\hat{N} = (2 - \varepsilon)OPT$.

### C Balanced Coloring

Our scheme uses as a subroutine an algorithm proposed in [1] for the group packing problem. For completeness, we include an outline of the algorithm adapted to handle instances of our problem. Given a GBP instance $I$ with a set of groups $G_1, \ldots, G_n$, denote by $S(I)$ the total size of items in $I$, i.e., $S(I) = \sum_{t \in |N|} s_t$, where $|N| = \{1, \ldots, N\}$. Let $v_j$ be the number of items in group $G_j$. Consider the following balanced coloring of the groups. Color the items of $G_1$ in arbitrary order, using $v_1$ colors (so that each item is assigned a distinct color). Now, sort the items in $G_2$ in non-increasing order by size. Scanning the sorted list, add the next item in $G_2$ to the color class of minimum total size which does not contain an item in $G_2$. We handle similarly the items in $G_3, \ldots, G_n$. Then, each color class can be packed, using First-Fit, as a bin packing instance, i.e., with no group constraints. Algorithm 5 is the pseudocode of BalancedColoring.

### D Approximation Scheme for GBP

Algorithm 6 gives the pseudocode of our APTAS for GBP.

---

9 For simplicity, assume that $1/\varepsilon$ and $\varepsilon\hat{N}$ are integers.
Algorithm 5 BalancedColoring($G_1, \ldots, G_n$)

1: Let $v_j = |G_j|$ be the cardinality of $G_j$ and $v_{max} = \max_{1 \leq j \leq n} v_j$ be the maximum cardinality of any group.
2: Partition arbitrarily the items of $G_1$ into $v_1$ color classes, such that each item is assigned a distinct color.
3: Add $v_{max} - v_1$ empty color classes.
4: \hspace{1em} for $j = 2, \ldots, n$ do
5: \hspace{2em} Sort $G_j$ in a non-increasing order by item sizes.
6: \hspace{2em} Let $i_1, \ldots, i_{v_j}$ be the items in $G_j$ in the sorted order.
7: \hspace{2em} for $\ell = 1, \ldots, v_j$ do
8: \hspace{3em} Add $i_\ell$ to a color class of minimum total size with no items from $G_j$.
9: \hspace{2em} end for
10: \hspace{2em} end for
11: \hspace{1em} for $k = 1, \ldots, v_{max}$ do
12: \hspace{2em} Pack the items in color class $c_k$ in a new set of bins using First-Fit.
13: \hspace{1em} end for

E Omitted Proofs

Proof of Lemma 3.2: Each large group contains at least $\varepsilon^{k+2}OPT$ items that are large or medium; thus, the total size of a large group is at least $(\varepsilon^{k+2} \cdot OPT)^{k+1} = \varepsilon^{2k+3}OPT$. Since $OPT$ is an upper bound on the total size of the instance, there are at most $\frac{1}{\varepsilon^{k+1}}$ large groups. \qed

Proof of Lemma 3.3: Given a feasible packing $\Pi$ of the instance $I$, we define a feasible packing $\Pi'$ of $I'$ as follows. For each large group $G_i$, pack items of class 2 in bins where items of class 1 of $I$ are packed in $\Pi$, items of class 3 where items of class 2 of $I$ are packed in $\Pi$, etc. The items of class $r$ in $I'$ are no smaller than the items of the corresponding class in $I$; thus, the capacity constraint is satisfied. Moreover, no conflict can occur since $\Pi'$ is a feasible packing for $I$. \qed

Proof of Lemma 3.4: Given a feasible packing $\Pi'$ of the instance $I'$, we define a feasible packing $\Pi$ of $I$ as follows. For each large group $G_i$, pack items of class 2 where items of class 2 of $I'$ are packed in $\Pi'$, items of class 3 where items of class 3 of $I'$ are packed in $\Pi'$, etc. The items of class $r$ in $I'$ are no smaller than the items of the corresponding class in $I$; thus, the capacity constraint is satisfied. Moreover, no conflict can occur since $\Pi'$ is a feasible packing for $I'$.

The discarded items can be packed in $O(\varepsilon)OPT$ extra bins. The number of discarded items from each large group is at most $\varepsilon^{2k+4}OPT$. By Lemma 3.2, there are $\frac{1}{\varepsilon^{k+1}}$ large groups; thus, the number of discarded items is at most $\frac{1}{\varepsilon^{k+1}} \cdot \varepsilon^{2k+4}OPT = \varepsilon OPT$. It follows that these items fit in at most $O(\varepsilon)OPT$ extra bins. Hence, the resulting packing of $I$ is feasible and uses at most $(1 + O(\varepsilon))OPT$ bins. \qed

Proof of Lemma 3.5: Clearly, $OPT$ is an upper bound on the total size of the large and medium items. Since each of these items has a size at least $\varepsilon^{k+1}$, the
Algorithm 6 ApproximationScheme($I, G_1, \ldots, G_n, \varepsilon$)

1: Guess $OPT$.
2: Let $b_1, \ldots, b_{OPT}$ be $OPT$ empty bins.
3: Find an integer $k \in \{1, \ldots, \frac{1}{\varepsilon^2}\}$ such that $\sum_{\ell \in I: s_\ell \in [\varepsilon^{k+1}, \varepsilon^k]} s_\ell \leq \varepsilon^{k+2} \cdot OPT$.
4: An item $\ell$ is small if $s_\ell < \varepsilon^{k+1}$, medium if $s_\ell \in [\varepsilon^{k+1}, \varepsilon^k)$ and large otherwise.
5: Define a group as large if it contains at least $\varepsilon^{k+2} OPT$ large and medium items, and as small otherwise.
6: for large groups $G_i$ do
7: Apply linear shifting to medium and large items from $G_i$ with parameter $Q = \lceil \varepsilon^{k+4} \cdot OPT \rceil$.
8: end for
9: Apply linear shifting jointly to all large items from small groups with parameter $Q = \lceil 2 \varepsilon \cdot OPT \rceil$ and discard the last size class.
10: for each guess $\zeta$ of slot patterns for the bins do
11: Discard medium items from small groups.
12: Pack large and medium items by Swapping($\zeta, G_1, \ldots, G_n$).
13: pack the remaining items using PackSmallItems($I_0, B$).
14: end for

overall number of large and medium items is at most $\frac{OPT}{2 \varepsilon^{k+4}}$. Hence, after shifting, the number of distinct sizes of large items from small groups is at most

$$\frac{OPT}{2 \varepsilon^{k+4}} \leq \frac{|cOPT|+1}{2 \varepsilon^{k+2}} \leq \frac{1}{\varepsilon^{k+2}} + \frac{1}{\varepsilon^{k+2}[\varepsilon \cdot OPT]} \leq \frac{2}{\varepsilon^{k+3}} = O(1).$$

The second inequality holds since $|\varepsilon \cdot OPT| \geq 1$. Using a similar calculation for each large group, we conclude that after shifting of these groups, there can be at most $\frac{OPT}{2 \varepsilon^{k+3}}$ distinct sizes for each group. By Lemma 3.2, there are at most $\frac{1}{\varepsilon^{k+3}}$ large groups. Hence, there can be at most $\frac{2}{\varepsilon^{k+2}} \cdot \frac{1}{\varepsilon^{k+3}} = \frac{2}{e^{k+3}}$ distinct sizes for all large and medium items in large groups. In addition, there are $\frac{2}{\varepsilon^{k+3}} = O(1)$ distinct sizes for large items from small groups. Thus, overall there are at most $\frac{2}{e^{k+3}} + \frac{2}{e^{k+3}} = O(1)$ distinct sizes for large items from small groups and large and medium items from large groups.

Proof of Lemma 3.6: Let $L$ be the number of large groups. Denote by $G_{i_1}, \ldots, G_{i_L}$ the large and medium items in the large groups. Our scheme enumerates over all slot patterns for packing the medium and large items from large groups, and the large items from small groups. The slot patterns indicate how many items of each size are assigned to each bin from each large group.

Denote by $T$ the set of slots for an instance $I$, and let $P$ be the set of patterns. Recall that a slot is a 2-tuple $(s_\ell, j)$, where $s_\ell$ is the size of an item, and $j \in \{i_1, \ldots, i_L\} \cup \{u\}$ labels one of the $L$ large groups, or any of the small groups, represented by a single label $u$. Let $\beta$ be the number of slots in a pattern $p \in P$. We note that $1 \leq \beta \leq \lceil \frac{1}{\varepsilon^{k+3}} \rceil$ since the number of medium/large items
that fit in a single bin is at most \( \lfloor \frac{1}{1+\varepsilon} \rfloor \). Then, \( p \) is defined as a multi-set, i.e., 
\[ p = \{t_1, \ldots, t_\beta\} \]
where \( t_i \in T \), for all \( 1 \leq i \leq \beta \).

By Lemma 3.2, there are at most \( \frac{1}{2+\varepsilon} \) large groups; thus, the number of distinct labels is at most \( \frac{1}{2+\varepsilon} + 1 \). By Lemma 3.5, after rounding the sizes of the large and medium items, there are at most \( O(1) \) distinct sizes of these items. Therefore, \( |T| = O(1) \). We conclude that \( |P| \leq |T|^\beta = O(1) \).

We proceed to enumerate over the number of bins packed by each pattern. The number of possible packings is \( OPT^{O(1)} = O(N^{O(1)}) \). One of these packings corresponds to an optimal solution for the given instance \( I \). At some iteration, this packing will be considered and used in later steps for packing the remaining items. This gives the statement of the lemma.

\( \square \)

**Proof of Theorem 3.7:** We prove that for each conflict involving an item \( \ell \in G_i \) of size \( s_\ell \) in bin \( b \), there is an item \( y \in G_j \neq G_i \) of size \( s_y = s_\ell \) in bin \( c \neq b \), such that \( swap(\ell, y) \) is good. Consider a packing of large and medium items by a slot pattern corresponding to an optimal solution. Then, the items are packed in \( OPT \) bins with no overflow, and the only conflicts may occur among items from small groups.

Due to shifting with parameter \( Q = \lfloor 2\varepsilon \cdot OPT \rfloor \) for large items from small groups, there are \( \lceil 2\varepsilon \cdot OPT \rceil - 1 \) items of size \( s_\ell \) in addition to \( \ell \) (recall that the last size class, which may contain less items, is discarded). We prove that the number of items \( y \) for which \( swap(\ell, y) \) is bad is at most \( \lfloor 2\varepsilon \cdot OPT \rfloor - 2 \); therefore, there exists an item \( y \) of size \( s_\ell \), for which \( swap(\ell, y) \) is good. We note that \( swap(\ell, y) \) is bad if (at least) one of the following holds: (i) \( y \) belongs to a group \( G_i \) which has an item in bin \( b \), or (ii) there is an item from \( G_i \) in bin \( c \).

We handle (i) and (ii) separately. (i) The number of items of size \( s_\ell \) from groups \( G_j \) that have an item in bin \( b \) is at most \( \frac{1}{2\varepsilon} \cdot \varepsilon^{k+2} OPT = \varepsilon^2 OPT \), since at most \( \frac{1}{\varepsilon} \) small groups can have a large item in \( b \), and each such group has at most \( \varepsilon^{k+2} OPT \) large items. (ii) We now bound the number of items of size \( s_\ell \) in bins that contain items in \( G_i \). Note that the total number of bins containing large items in \( G_i \) is at most \( \varepsilon^{k+2} OPT \), since \( G_i \) is small. Also, in each such bin, the total number of large items is at most \( \frac{1}{\varepsilon} \). Thus, the total number of items of size \( s_\ell \) in bins containing items in \( G_i \) is at most \( \frac{1}{\varepsilon} \cdot \varepsilon^{k+2} OPT = \varepsilon^2 OPT \). Using the union bound, the number of bad swaps for \( \ell \), i.e., \( swap(\ell, y) \) for some item \( y \), is at most \( 2\varepsilon^2 OPT \). We have \( 2\varepsilon^2 OPT < \varepsilon OPT < \varepsilon OPT + OPT - 3 \leq \lfloor 2\varepsilon \cdot OPT \rfloor - 2 \). The first inequality holds since we may assume that \( \varepsilon < \frac{1}{2} \). For the second inequality, we note that \( OPT > \frac{3}{\varepsilon^2} > \frac{3}{\varepsilon^2} \). We conclude that there is an item \( y \) in the size class of \( \ell \) such that \( swap(\ell, y) \) is good.

We now show that the Swapping algorithm is polynomial in \( N \). We note that items of some group are in conflict only if they are placed in the same bin. As these are only large items, an item may conflict with at most \( \frac{1}{\varepsilon} \) items. Hence, there are at most \( \frac{N}{\varepsilon} = O(N) \) conflicts. As finding a good swap takes at most \( O(N) \), the overall running time of Swapping is \( O(N^2) \).

\( \square \)

**Proof of Lemma 3.8:** Note that at most \( \frac{1}{1+\varepsilon} \) large or medium items can be packed together in a single bin. By Lemma 3.5, after rounding there are at most
distinct sizes for these items. Therefore, the number of distinct total sizes for bins is at most \((\frac{2}{\varepsilon k} + \frac{2}{\varepsilon k+3})^\frac{2}{3}\). By Lemma 3.2, there are at most \(\frac{1}{\varepsilon k+1}\) large groups. Thus, the number of subsets of large groups is bounded by \(2\sqrt{k}\). Each bin can also contain at most \(\lceil \frac{1}{\varepsilon k+1} \rceil\) slots assigned to items from small groups. It follows that the total number of bin types is bounded by
\[
\left(\frac{2}{\varepsilon k+3} + \frac{2}{\varepsilon k+11}\right)^\frac{2}{3} \cdot 2\sqrt{k} \cdot \frac{1}{\varepsilon k+1} = O(1)
\]

\[\Box\]

**Proof of Lemma 3.9**: We prove that *RecursiveEnum* satisfies all the conditions.

(i) We first show that the running time is polynomial. Using arguments similar to the proof of Lemma 3.2, there is a constant number of \(t(g_{opt})\)-significant groups for each \(t \in T\) and \(h = 0, \ldots, \alpha\) (Step 9). These groups can be guessed in polynomial time. Furthermore, guessing the largest item in each of these groups (Step 12), for each size class in the linear shifting, is also done in polynomial time, because there is a constant number of size classes. Then, the enumeration of patterns in Step 14 can be done in polynomial time, for each type \(t\).

By Lemma 3.8, the initial number of types is a constant. It follows that the number of sub-types is a constant, for each type. Therefore, the overall number of types is a constant. Thus, Step 15 increases the number of types by a constant, and because there are \(O(1)\) iterations, the overall running time is polynomial.

(ii) We show that the increase in the number of bins due to padding types of small cardinality (Step 7) is bounded by a constant. The number of types in each iteration is a constant, and the number of iterations is a constant. To each type we add (at most) a constant number of bins. Thus, the total number of bins added is a constant.

(iii) Recall that we apply linear shifting to each \(t(g_{opt})\)-significant group, for each type \(t\), in each iteration (Step 12). Thus, we discard from \(G_{ij}\) at most \(\varepsilon^3|t|\) items (the largest size class). This holds for any type \(t\) and any iteration in which \(G_{ij}\) is \(t(g_{opt})\)-significant. Thus, in \(\alpha\) iterations we discard from \(G_{ij}\) at most \(\varepsilon|t|\) items. We conclude that the number of items discarded from \(G_{ij}\) in all types in all iterations is at most \(\varepsilon OPT\).

We now bound the total size of discarded items. For each \(b \in E_0\) \(f_b^0 \leq \varepsilon\). Therefore, \(\sum_{b \in E_0} f_b^0 \leq \varepsilon OPT\). Thus, the total size discarded in *RecursiveEnum* is at most \(\varepsilon OPT\), because the linear shifting is done solely on items that are guessed to be packed in bins in \(E_0\). The latter is true if our guess corresponds to an optimal solution, as we prove below.

(iv) For each type \(t\) we enumerate over all possible selections of its \(t(g_{opt})\)-significant groups in Step 9. One of the guesses corresponds to an optimal solution. Let \(G_{ij}\) be a group that is \(t\)-significant in an optimal solution \(g_{opt}\). Then, applying linear shifting and discarding the largest size class will preserve the feasibility of the packing of the remaining size classes. Hence, our use of shifting preserves the optimal guess. We prove next that our greedy algorithm for choosing the items to each size class yields a feasible subset, given an optimal guess for the
Proof of Lemma 3.10: We construct a partition $I_t, t \in T$ satisfying conditions (i)-(iii) for all $t \in T$. Let $g_{\text{opt}}$ be a guess corresponding to an optimal solution, i.e., $I_{a}$ can be added to the bins resulting in an optimal solution. Let $I_{t(\text{opt})} \subseteq I_{a}, t \in T \setminus \{t'\}$ be a packing of items in $I_{a}$ in the type that yields $g_{\text{opt}}$. Recall that in Step 5 of Algorithm 4 we add $\varepsilon \text{OPT}$ empty bins as a new type $t'$. The partition is defined as follows. $I_{t'}$ consists of all $\ell \in I_{a}$ such that $\ell$ is a $t$-non-negligible item from a $t(\text{opt})$-insignificant group in some iteration $h \in [a]$, where $t \in T \setminus \{t'\}$. Also, for $t \in T \setminus \{t'\}$, $I_{t} = I_{t(\text{opt})} \setminus I_{t'}$.

We first prove that $I_{t'}$ satisfies conditions (i)-(iii). There are $\alpha$ iterations. For some group $G_i$, $1 \leq i \leq n$, the number of non-negligible items from $G_i$ in types $t$ for which $G_i$ is $t(\text{opt})$-insignificant is at most $\sum_{t \in T} \varepsilon^d|t| = \varepsilon^d\text{OPT}$. Thus, after $\alpha$ iterations the number of bins of type $t'$ required to avoid conflicts in $G_i$ is at most $\varepsilon \text{OPT}$ and (i) holds for $I_{t'}$. The bins in $t'$ are of capacity 1 and for all items in $\ell \in I_{t'} : s_\ell \leq \varepsilon$ because $I_{a}$ contains only small items. Therefore, condition (ii) also holds for $I_{t'}$. Finally, the total size of these items is at most $\varepsilon \text{OPT}$, as these items are packed in bins that are in $E_0$ and condition (iii) holds.\footnote{Recall that we do not use guessing on bins in $D_0$.}

Next, we prove that $I_{t}, t \in T \setminus \{t'\}$ satisfies conditions (i)-(iii). If condition (i) does not hold for $I_{t}$ for some group $G_i$, then $I_{t(\text{opt})}$ cannot be packed into bins of type $t$ without causing a conflict (there must be a bin with two items from the same group by the pigeonhole principle) and this is a contradiction. In addition, there must be a partition in which condition (iii) holds, or some $I_{t(\text{opt})}$ cannot be packed in bins of type $t$ contradicting the optimality of the packing up to this point, and the correct guess of $I_{t(\text{opt})}$.

Now, consider condition (ii). The items in $I_{t}$ are either packed according to $g_{\text{opt}}$ such that they are $t$-negligible after iteration $\alpha$, or $t \subseteq E_\alpha$. First, for all types $t \subseteq D_\alpha$, by the definition of $D_\alpha$, $I_{t(\text{opt})}$ contains only $t$-negligible items; thus, condition (ii) holds in this case.

Note that condition (ii) does not necessarily hold if we omit Step 3 in Algorithm 4. We show that this step is completed successfully. Note that any bin $b \in E_0$ contains at least $\alpha$ items, since in each iteration at least one item is added to $b$ (else, $b$ is moved to $D_b$ in some iteration $1 \leq h < \alpha$, and $b \notin E_\alpha$). Now, if we omit from bin $b$ and item $\ell$ that was added to $b$ in iteration $h \leq \alpha - 3$ then the free space in $b$ increases at least by $\frac{\alpha}{2}$. Indeed, item $\ell$ was $t$-non-negligible in iteration $h \leq \alpha - 3$; thus, $s_\ell \geq \varepsilon^2 f_b^{\alpha - 3}$. For $t \in E_\alpha$, it holds that $f_b^{\alpha - 3} \geq \frac{\alpha}{2}$. Hence, $s_\ell \geq \varepsilon^2 f_b^{\alpha - 3} = \frac{\alpha}{2}$.
We now show that we can evict in the process at most \( \varepsilon \text{OPT} \) items from each group \( G_i, 1 \leq i \leq n \). Recall that for each \( t \in E_\alpha \) the bins of type \( t \) contain items from distinct groups. Since there are at least \( \alpha \) items in each bin, and \( \alpha > \frac{1}{\varepsilon} + 4 \), the bins of type \( t \) can be partitioned into \( 1/\varepsilon \) subsets, \( B^t_1, \ldots, B^t_{1/\varepsilon} \), each consists of \( \varepsilon |t| \) bins. In the worst case, all the bins of type \( t \) contain items from the same set of groups, \( G_{K_1}, \ldots, G_{K_R} \), where \( R \geq 1/\varepsilon \). We can now evict from each bin in \( B^t_r \), an item from \( G_{K_r} \), \( 1 \leq r \leq 1/\varepsilon \). Thus, we omit from each group at most \( \varepsilon |t| \leq \varepsilon \text{OPT} \) items.

Let \( I_b(g_{opt}) \) be the set of items packed in bin \( b \in t \subseteq E_\alpha \) in \( g_{opt} \). For any \( \ell \in I_b(g_{opt}) : s_\ell \leq f(t) \) because \( g_{opt} \) is a feasible packing. After Step 3, the capacity available in each bin in \( t \) and in particular in \( b \) is at least \( \frac{f(t)}{\varepsilon} \). Hence, condition (ii) holds for each \( t \subseteq E_\alpha \) after the eviction phase.

**Proof of Lemma 3.13**: We prove the claim by induction on \( |t| \). For the base case, let \( |t| = 1 \). Since there is only one bin in \( t \), \( |G^t_i| \leq 1 \) by Condition (i). Thus, \( I_t \) can be packed in \( t \) without conflicts. Also, the total size of all items is at most the free capacity of the bin, by Condition (iii). Hence, we can pack all items feasibly.

For the induction step, assume the claim holds for \( |t| - 1 \) bins. Now, suppose there is a type \( t \) with \( |t| \) bins. Recall that GreedyPack initially assigns items to the bin of maximum total size. Now, consider two cases.

(1) After the packing of the first bin, the total size of items from \( I_t \) in this bin is strictly less than \( (1 - \delta)f(t) \). Then, we prove in this case that the first bin contains the largest item left in each small group. In any two consecutive attempts in GreedyPack of packing the first bin, the difference in the total size packed is by the size of one item. By Condition (ii), this difference is at most \( \delta f(t) \). Therefore, if packing the largest item from each group overflows, GreedyPack would continue until the first iteration that it finds a feasible packing. Observe the last attempt, and assume towards a contradiction that it is not the largest item of each group. Then, the previous attempt overflowed, therefore, the current attempt must be with total size at least \( (1 - \delta)f(t) \), which is a contradiction to (1).

The remaining items can be feasibly packed in the remaining \( |t| - 1 \) bins, since taking an arbitrary remaining item from each group cannot overflow because the packing of the first bin does not overflow and contains the largest item from each group.

(2) The first bin is packed with total size at least \( (1 - \delta)f(t) \). We prove that all conditions hold for applying the induction hypothesis for the last \( |t| - 1 \) bins. First, GreedyPack packs in the first bin an item from each group. Assume towards a contradiction that GreedyPack fails in doing so in the first bin. Therefore, the sum of the smallest item of each group in \( I_t \) overflows from \( f(t) \). Let \( s^{\min}_{j,t} \) be the size of the smallest item in group \( G^t_j \). Since there are exactly \( |t| \) item from each group in \( I_t \), it follows that \( S(I_t) > |t| \sum_{G^t_j} s^{\min}_{j,t} > |t|/f(t) \) in contradiction to Condition (iii). Thus, we are left with \(|t| - 1 \) item from each group after the packing of the first bin.
Second, Condition (ii) is trivially satisfied for any number of bins in $t$. Third, by Case (2) we pack in the first bin total size of at least $(1 - \delta)f(t)$. Therefore, the residual size of $I_t$ after the packing of the first bin is at most $(1 - \delta)f(t)|t| - (1 - \delta)f(t) \leq (1 - \delta)f(t)(|t| - 1)$ and Condition (iii) holds.

Hence, by the induction hypothesis, the remaining items in $I_t$ can be packed in bins $2, \ldots, |t|$.

\[\□\]

Proof of Lemma 3.14: Given $g_{opt}$, we can find a partition $I_t, t \in T$ of the remaining items, not violating constraints 1,2,3. Furthermore, in Step 8 we add $2 \cdot \varepsilon|t|$ extra bins to $t$, thus $S(I_t) \leq (1 - \varepsilon)|t|$ ($|t|$ refers to the cardinality of $t$ after Step 8). Therefore, conditions (i), (ii) for Lemma 3.13 hold for each type $t \in T$ with parameter $\delta = \varepsilon$ (except for maybe too many items from some group because of large items from small groups, which GreedyPack discards and by Lemma 3.12 this results with a small number of extra bins). For each $t \in T$ GreedyPack discards items of total size at least $\varepsilon|t|f(t)$. Therefore, the remaining items in $I_t$ are with total size at most $(1 - \varepsilon)|t|f(t)$ because of constraint (1). We conclude that besides the discarded items, all items $I_t$ are packed feasibly in $t$. The discarded items are at most $\varepsilon|t|$ from each group in each type $t$ by GreedyPack. Thus, combined, the overall total size discarded is $O(\varepsilon)OPT$ since these are small items, and $O(\varepsilon)OPT$ items are discarded from each group.

\[\□\]

Proof of Lemma 3.15: We prove the claim by deriving a bound on the number of extra bins required for packing the items discarded throughout the execution of the scheme. During the scheme, we discard $O(\varepsilon)OPT$ items from each group, and discard a total size $O(\varepsilon)OPT$ overall. These items are packed by a 2-approximation algorithm (BalancedColoring), and thus, the number of extra bins needed for packing these items is $O(\varepsilon)OPT + 1$.

\[\□\]

Proof of Theorem 3.16: The feasibility of the packing follows from the way algorithms RecursiveEnum, GreedyPack and SmallGroups assign items to the bins. We now bound the total number of bins used by the scheme. As shown in the proof of Lemma 3.15, given the parameter $\varepsilon \in (0,1)$, the total number of extra bins used for packing the medium items from small groups and the discarded items is at most $O(\varepsilon)OPT + 1$. By scaling $\varepsilon$ by a constant (that does not depend on $\varepsilon$), we have that the total number of bins used by the scheme is $ALG(I) \leq (1 + \varepsilon)OPT + 1$. As shown above, each step of the scheme has running time polynomial in $N$.

\[\□\]

F Proof of Theorem 3.11

We prove the theorem using the next lemmas. Let $\bar{x} \in P$ be a vertex of $P$. For a group $G_j, j \in [n]$, a movement is a vector $\bar{m}^{ij} \in \mathbb{R}^{I_n \times T}$ such that:
\[\forall \ell \in I_\alpha, t \in T, \ell \notin G_j \text{ or } x_{\ell, t} \in \{0, 1\} : \quad \bar{m}_j^{\ell, t} = 0 \quad (1)\]
\[\forall \ell \in I_\alpha : \quad \sum_{t \in T} \bar{m}_j^{\ell, t} = 0 \quad (2)\]
\[\forall t \in T \text{ s.t } \sum_{\ell \in G_j} x_{\ell, t} = L_{t,j} : \quad \sum_{\ell \in G_j} \bar{m}_j^{\ell, t} = 0 \quad (3)\]

where \(L_{t,j} = |t| - |(I'_t) \setminus I_L) \cap G_j|\) for \(t \in T\) and \(j \in \{1, \ldots, N\}\). Additionally, define the sets \(X_j = \{(\ell, t) \in G_j \times T \mid s_t > \varepsilon \cdot f(t)\}\) for \(j \in [n]\).

We say that group \(G_j\) is fractional if there are \(\ell \in G_j\) and \(t \in T\) such that \(\bar{x}_{\ell, t} \in (0, 1)\).

**Lemma F.1.** If Group \(G_j\) is fractional then \(G_j\) has a movement \(\bar{m}_j \neq 0\).

The proof of Lemma F.1 utilizes properties of *totally unimodular* matrices. A matrix \(A\) is totally unimodular if every square submatrix of \(A\) has a determinant 1, \(-1\) or 0. If \(A \in \mathbb{R}^{n \times m}\) is totally unimodular and \(b \in \mathbb{Z}^m\) is an integral vector, it holds that the vertices of the polytope \(P_A = \{\bar{x} \in \mathbb{R}^n_{\geq 0} \mid A\bar{x} \leq b\}\) are integral. That is, if \(\bar{x} \in P_A\) is a vertex of \(P_A\) then \(\bar{x} \in \mathbb{Z}^n\) [15]. We use the following criteria for total unimodularity, which is a simplified version of a theorem from [15].

**Lemma F.2.** Let \(A \in \mathbb{R}^{n \times m}\) be a matrix which satisfies the following properties.

- All the entries of \(A\) are in \(\{-1, 1, 0\}\).
- Every column of \(A\) has up to two non-zero entries.
- If a column of \(A\) has two non-zero entries, then these entries have opposite signs.

Then \(A\) is totally unimodular.

**Proof of Lemma F.1:** We show the existence of the movement \(\bar{m}_j\) using the polytope \(P_j\) defined as follows.

\[
P_j = \left\{ \bar{y} \in \mathbb{R}^{G_j \times T}_{\geq 0} \mid \begin{array}{l}
\forall (l, t) \in X_j : \quad \bar{y}_{l, t} \leq 0 \\
\forall l \in G_j : \quad \sum_{t \in T \text{ s.t. } (l, t) \notin X_j} -\bar{y}_{l, t} \leq -1 \\
\forall t \in T : \quad \sum_{l \in G_j \text{ s.t. } (l, t) \notin X_j} \bar{y}_{l, t} \leq L_{t,j}
\end{array} \right\}. \quad (4)
\]

Define a vector \(\bar{y}^* \in \mathbb{R}^{G_j \times T}\) by \(\bar{y}^*_{l, t} = \bar{x}_{l, t}\). It follows from the definition of \(P\) that \(\bar{y}^* \in P_j\). We can represent the inequalities in (4) using a matrix notation as \(P_j = \{\bar{y} \in \mathbb{R}^{G_j \times T}_{\geq 0} \mid A\bar{y} \leq b\}\). It follows that \(A\) contains only entries in \(\{-1, 0, 1\}\) and the entries in \(b\) are all integral. Furthermore, every column of \(A\) contains at most 2 non-zero entries, and if there are two non-zero entries in a column then they are of a different sign. By Lemma F.2, it follows that \(A\) is totally unimodular. It thus holds that all the vertices of the polytope \(P_j\) are integral. As
\( \bar{y}^* \in P_j \) is non-integral (since \( G_j \) is fractional), it follows that \( \bar{y}^* \) is not a vertex of \( P_j \). Hence, there is a vertex \( \check{y} \in \mathbb{R}^{G_j \times T} \), \( \check{y} \neq 0 \) such that \( \bar{y}^* + \check{y}, \bar{y}^* - \check{y} \in P_j \).

We define \( \bar{m}^j \in \mathbb{R}^{I^2 \times T} \) by \( \bar{m}^j_{t,j} = \check{y}_{t,j} \) for \( (t,j) \in G_j \times T \) and \( \bar{m}^j_{t,j} = 0 \) otherwise. Clearly, \( \bar{m}^j \neq 0 \) as \( \check{y} \neq 0 \). Observe that for \( \ell \in I_a \setminus G_j \) and \( t \in T \) it holds that \( \bar{m}^j_{\ell,t} = 0 \) by definition. For \( \ell \in G_j \) and \( t \in T \) such that \( \bar{x}_{\ell,t} = 0 \), as \( \bar{y}_{\ell,t} + \check{y}_{\ell,t} - \gamma_{\ell,t} \geq 0 \) and \( \bar{y}_{\ell,t} = \bar{x}_{\ell,t} = 0 \), it follows that \( \check{y}_{\ell,t} = 0 \). For \( \ell \in G_j \) and \( t \in T \) such that \( \bar{x}_{\ell,t} = 1 \), it follows that \( \bar{y}_{\ell,t} - \gamma_{\ell,t} = 0 \) for \( t' \in T \setminus \{ t \} \) by the definition of \( P \) as well as \( (t,t) \notin X_j \). Thus, by the previous argument, we have \( \check{y}_{t,t'} = 0 \) for every \( t' \in T \setminus \{ t \} \). Therefore,

\[ -1 - \gamma_{\ell,t} = - \sum_{t' \in T \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} + \check{y}_{t,t'}) \leq -1, \quad (5) \]

where the last inequality is due to \( \bar{y}^* + \check{y} \in P_j \). Similarly, as \( \bar{y}^* - \check{y} \in P_j \), we have

\[ -1 + \check{y}_{\ell,t} = - \sum_{t' \in T \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} - \check{y}_{t,t'}) \leq -1, \quad (6) \]

By (5) and (6) we have \( \check{y}_{\ell,t} = 0 \). Overall, we have that \( \bar{m}^j \) satisfies (1).

For any \( \ell \in I_a \setminus G_j \) it holds that \( \sum_{t \in T} \bar{m}^j_{\ell,t} = 0 \). For \( \ell \in G_j \), since \( \bar{y}^* + \check{y} \in P_j \) we have

\[ -1 - \sum_{t \in T} \gamma_{\ell,t} = \sum_{t \in T} (\bar{x}_{\ell,t} + \check{y}_{\ell,t}) = \sum_{t \in T \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} - \check{y}_{t,t'}) \leq -1, \quad (7) \]

where the first equality holds since \( \bar{x} \in P \) and the second equality uses \( \bar{y}_{\ell,t} + \check{y}_{\ell,t} = 0 \) for \( (t,t) \in X_j \). Similarly, since \( \bar{y}^* - \check{y} \in P_j \), we have

\[ -1 + \sum_{t \in T} \gamma_{\ell,t} = \sum_{t \in T} (\bar{x}_{\ell,t} - \check{y}_{\ell,t}) = \sum_{t \in T \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} - \check{y}_{t,t'}) \leq -1. \quad (8) \]

By (7) and (8) we have \( \sum_{t \in T} \bar{m}^j_{\ell,t} = \sum_{t \in T} \check{y}_{\ell,t} = 0 \). Thus, \( \bar{m}^j \) satisfies (2).

Finally, let \( t' \in T \) such that \( \sum_{t \in G_j} \bar{x}_{t,t'} = L_{t,t'} \). As before,

\[ L_{t,j} + \sum_{t \in G_j} \gamma_{\ell,t} = \sum_{t \in G_j} (\bar{y}_{t,t} - \check{y}_{t,t}) = \sum_{t \in G_j \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} + \check{y}_{t,t'}) \leq L_{t,j}, \quad (9) \]

where the inequality follows from \( \bar{y}^* + \check{y} \in P_j \). Using a similar argument,

\[ L_{t,j} - \sum_{t \in G_j} \gamma_{\ell,t} = \sum_{t \in G_j} (\bar{y}_{t,t} - \check{y}_{t,t}) = \sum_{t \in G_j \text{ s.t. } (t,t') \notin X_j} (\bar{y}_{t,t'} - \check{y}_{t,t'}) \leq L_{t,j}. \quad (10) \]

By (9) and (10), we have \( \sum_{t \in G_j} \bar{m}^j_{\ell,t} = \sum_{t \in G_j} \check{y}_{\ell,t} = 0 \). Thus, \( \bar{m}^j \) satisfies (3).

Overall, we show that \( \bar{m}^j \neq 0 \) is a movement of \( G_j \).

\[ \square \]

**Lemma F.3.** There are at most \( |T| \) fractional groups.
Proof. Assume towards a contradiction that there are $|T| + 1$ fractional groups. W.l.o.g., assume that these groups are $G_1, \ldots, G_{|T|+1}$. By Lemma F.1, $G_j, j \in [|T| + 1]$ has a movement $\bar{m}^j \neq 0$. Consider the following set of equalities over $\lambda_1, \ldots, \lambda_{|T|+1}$:

$$\forall t \in T : \sum_{j=1}^{|T|+1} \lambda_j \sum_{\ell \in I_a} \bar{m}^j_{\ell,t} \cdot s_{\ell} = 0. \tag{11}$$

These are $|T|$ homogeneous linear equalities in $|T| + 1$ variables. Thus, there exist $\lambda_1, \ldots, \lambda_{|T|+1}$, not all zeros, for which (11) holds.

Define

$$K_1 = \min_{(\ell,t) \in I_a \times T} \min \{ \bar{x}_{\ell,t}, 1 - \bar{x}_{\ell,t} \mid 0 < \bar{x}_{\ell,t} < 1 \},$$

$$K_2 = \min \left\{ L_{t,j} - \sum_{\ell \in G_j} \bar{x}_{\ell,t} \mid 1 \leq j \leq |T| + 1, \sum_{\ell \in G_j} \bar{x}_{\ell,t} < L_{t,j} \right\},$$

and $K = \min \{K_1, K_2\}$. Additionally, define

$$M = \max \left\{ |\bar{m}^j_{\ell,t}| \mid 1 \leq j \leq |T| + 1, (\ell,t) \in I_a \times T \right\}.$$

Observe that $M, K > 0$. Using a scaling argument, we may assume that $\lambda_j \leq \frac{K}{|I_a| \cdot M}$ for any $1 \leq j \leq |T| + 1$.

Define $\bar{x}^+ = \bar{x} + \sum_{j=1}^{|T|+1} \lambda_j \cdot \bar{m}^j$. In the following we show that $\bar{x}^+ \in P$.

For every $\ell \in I_a \setminus (G_1 \cup \ldots \cup G_{|T|+1})$ and $t \in T$ it holds that $\bar{m}^j_{\ell,t} = \ldots = \bar{m}^{|T|+1}_{\ell,t} = 0$ due to (1). Thus, $\bar{x}^+_{\ell,t} = \bar{x}_{\ell,t} \in \{0, 1\}$. For $\ell \in G_j$ with $1 \leq j \leq |T| + 1$ and $t \in T$, using (1) once more we have $\bar{x}^+_{\ell,t} = \bar{x}_{\ell,t} + \lambda_j \cdot \bar{m}^j_{\ell,t}$. Following the definitions of $K$ and $M$, we have

$$0 \leq K - M \cdot \frac{K}{|I_a| \cdot M} \leq \bar{x}_{\ell,t} + \lambda_j \cdot \bar{m}^j_{\ell,t} \leq 1 - K + M \cdot \frac{K}{|I_a| \cdot M} \leq 1.$$

Thus, $\bar{x}^+ \in [0,1]^I_a \times T$.

For every $(\ell,t) \in T$ such that $s_{\ell} > \varepsilon f(t)$, it holds that $\bar{x}_{\ell,t} = 0$. Thus by (1) we have $\bar{m}^j_{\ell,t} = 0$ for all $1 \leq j \leq |T| + 1$. Therefore, $\bar{x}^+_{\ell,t} = 0$.

For every $t \in T$ it holds that

$$\sum_{\ell \in I_a} \bar{x}^+_{\ell,t} \cdot s_{\ell} = \sum_{\ell \in I_a} \bar{x}_{\ell,t} \cdot s_{\ell} + \sum_{j=1}^{|T|+1} \lambda_j \sum_{\ell \in I_a} \bar{m}^j_{\ell,t} s_{\ell} = \sum_{\ell \in I_a} \bar{x}_{\ell,t} \cdot s_{\ell} \leq f(t)|t|,$$

where the second equality is by (11), and the inequality is due to $\bar{x} \in P$.

For every $\ell \in I_a$, we have

$$\sum_{t \in T} \bar{x}^+_{\ell,t} = \sum_{t \in T} \bar{x}_{\ell,t} + \sum_{j=1}^{|T|+1} \lambda_j \sum_{t \in T} \bar{m}^j_{\ell,t} = \sum_{t \in T} \bar{x}_{\ell,t} = 1,$$
We showed that \( \sum x_{\ell,t} = \bar{x}_{\ell,t} \) for every \( \ell \in G_j \), and thus \( \sum_{\ell \in G_j} \bar{x}_{\ell,t} = \sum_{\ell \in G_j} \bar{x}_{\ell,t} \leq L_{t,j} \), as \( \bar{x} \in P \). Otherwise \( 1 \leq j \leq |T| + 1 \).

Observe that \( \bar{x}_{\ell,t} = \bar{x}_{\ell,t} + \lambda_j \bar{m}_{\ell,t} \) for every \( \ell \in G_j \), and consider the following cases.

- \( \sum_{\ell \in G_j} \bar{x}_{\ell,t} < L_{t,j} \). Using the definitions of \( K \) and \( M \), we have
  \[
  \sum_{\ell \in G_j} \bar{x}_{\ell,t} = \sum_{\ell \in G_j} \bar{x}_{\ell,t} + \lambda_j \sum_{\ell \in G_j} \bar{m}_{\ell,t} \leq L_{t,j} - K + |I_a| \cdot \frac{K}{|I_a|} \cdot M = L_{t,j}.
  \]

- \( \sum_{\ell \in G_j} \bar{x}_{\ell,t} = L_{t,j} \). By (3) we have
  \[
  \sum_{\ell \in G_j} \bar{x}_{\ell,t} = \sum_{\ell \in G_j} \bar{x}_{\ell,t} + \lambda_j \sum_{\ell \in G_j} \bar{m}_{\ell,t} = L_{t,j}.
  \]

We showed that \( \sum_{\ell \in G_j} \bar{x}_{\ell,t} \leq L_{t,j} \) in all cases. Overall, we have that \( \bar{x} \in P \).

We can also define \( \bar{x}^- = \bar{x} - \sum_{j=1}^{|T|+1} \lambda_j \cdot m^j \). By a symmetric argument we can show that \( \bar{x}^- \in P \) as well. It also holds that \( \bar{x}^+, \bar{x}^- \notin \bar{x} \) as there is \( 1 \leq j^* \leq |T| + 1 \) such that \( \lambda_{j^*} \neq 0 \); also, there are \( \ell^* \in G_j \) and \( t^* \in T \) such that \( \bar{m}_{\ell^*,t^*} \neq 0 \).

Thus, \( \bar{x}_{\ell^*,t^*} = \bar{x}_{\ell^*,t^*} + \lambda_{j^*} \bar{m}_{\ell^*,t^*} \neq \bar{x}_{\ell^*,t^*} \). Furthermore, \( \bar{x} = \frac{1}{2} \cdot \bar{x}^+ + \frac{1}{2} \cdot \bar{x}^- \), and we conclude that \( \bar{x} \) is not a vertex of \( P \). Contradiction. \( \square \)

**Lemma F.4.** If Group \( G_j \) is fractional then

\[
|\{ \ell \in G_j \mid \exists t \in T : \bar{x}_{\ell,t} \in (0,1) \}| \leq 2|T|.
\]

**Proof.** Define \( R_j = (G_j \times T) \setminus X_j \), and let \( Q_j \subseteq \mathbb{R}^{R_j} \) be the set (polytope) of all the vectors \( \bar{y} \in \mathbb{R}^{R_j} \) which satisfy the following inequalities:

\[
\forall (\ell,t) \in R_j : \bar{y}_{\ell,t} \geq 0 \tag{12}
\]

\[
\forall \ell \in G_j : \sum_{t \in T \text{ s.t. } (\ell,t) \in R_j} \bar{y}_{\ell,t} \geq 1 \tag{13}
\]

\[
\forall t \in T : \sum_{\ell \in G_j \text{ s.t. } (\ell,t) \in R_j} \bar{y}_{\ell,t} \cdot s_t + \sum_{\ell \in I_a \setminus G_j} \bar{x}_{\ell,t} \cdot s_t \leq f(t)|t| \tag{14}
\]

\[
\forall t \in T : \sum_{\ell \in G_j \text{ s.t. } (\ell,t) \in R_j} \bar{y}_{\ell,t} \leq L_{t,j} \tag{15}
\]

Also, define \( \bar{y}^* \in \mathbb{R}^{R_j} \) by \( \bar{y}_{\ell,t}^* = \bar{x}_{\ell,t} \) for \( (\ell,t) \in R_j \). As \( \bar{x} \in P \), it holds that \( \bar{y}^* \in Q_j \) (recall \( \bar{x}_{\ell,t} = 0 \) for all \( (\ell,t) \in X_j \)).

Assume towards contradiction that \( \bar{y}^* \) is not a vertex of \( Q_j \). Thus, there is a vector \( \bar{\gamma} \in \mathbb{R}^{R_j}, \bar{\gamma} \neq 0 \) such that \( \bar{y}^* + \bar{\gamma}, \bar{y} - \bar{\gamma} \in Q_j \). Thus, by (13), for all \( \ell \in G_j \) it holds that,

\[
1 + \sum_{t \in T \text{ s.t. } (\ell,t) \in R_j} \bar{\gamma}_{\ell,t} = \sum_{t \in T \text{ s.t. } (\ell,t) \in R_j} (\bar{y}_{\ell,t}^* + \bar{\gamma}_{\ell,t}) \geq 1,
\]

28
and
\[
1 - \sum_{t \in T \text{ s.t. } (t, t) \in R_j} \gamma_{t, t} = \sum_{t \in T \text{ s.t. } (t, t) \in R_j} (\tilde{y}_{t, t}^* - \gamma_{t, t}) \geq 1.
\]
Hence,
\[
\sum_{t \in T \text{ s.t. } (t, t) \in R_j} \gamma_{t, t} = 0.
\] (16)

Define \( \bar{x}^+, \bar{x}^- \in \mathbb{R}^{I_\alpha \times T} \) by \( \bar{x}_{t, t}^+ = \bar{x}_{t, t}^- = \bar{x}_{t, t} \) for \( (t, t) \notin R_j \), and \( \bar{x}_{t, t}^+ = \bar{x}_{t, t} + \tilde{\gamma}_{t, t} \), \( \bar{x}_{t, t}^- = \bar{x}_{t, t} - \tilde{\gamma}_{t, t} \) for \( (t, t) \in R_j \). Since \( \tilde{\gamma} \neq 0 \) it follows that \( \bar{x}^+ \neq \bar{x}^- \).

Let \( (t, t) \in I_\alpha \times T \). If \( (t, t) \in R_j \) then \( \bar{x}_{t, t}^+ = \tilde{y}_{t, t}^* + \tilde{\gamma}_{t, t} \geq 0 \), since \( \tilde{y}^* + \tilde{\gamma}_{t, t} \in Q_j \) and due to (12). If \( (t, t) \notin R_j \) then \( \bar{x}_{t, t}^- = \bar{x}_{t, t} \geq 0 \). Overall, we have that \( \bar{x}_{t, t}^+ \geq 0 \) in both cases. Furthermore, if \( s \neq \varepsilon f(t) \) then \( \bar{x}_{t, t}^+ = \bar{x}_{t, t}^- = 0 \) as \( (t, t) \notin R_j \).

For any \( t \in T \), it holds that
\[
\sum_{t \in I_\alpha} \bar{x}_{t, t}^+ s_t = \sum_{t \in I_\alpha \setminus G_j} \bar{x}_{t, t} s_t + \sum_{t \in G_j \text{ s.t. } (t, t) \in R_j} (\tilde{y}_{t, t}^* + \tilde{\gamma}_{t, t}) s_t \leq f(t)|t|,
\]
where the last equality is due to (14) and \( \tilde{y}^* + \tilde{\gamma} \in Q_j \).

Finally, for any \( t \in I_\alpha \), if \( t \notin G_j \) then
\[
\sum_{t \in T} \bar{x}_{t, t}^+ = \sum_{t \in T} \bar{x}_{t, t} = 1.
\]

Also, if \( \ell \in G_j \) then using (16), we have
\[
\sum_{t \in T} \bar{x}_{t, t}^+ = \sum_{t \in T} \bar{x}_{t, t} = \sum_{t \in T} \gamma_{t, t} = 1.
\]

Overall, we have that \( \bar{x}^+ \in P \). Using a symmetric argument it follows that \( \bar{x}^- \in P \) as well. Since \( \bar{x} = \frac{\bar{x}^+ + \bar{x}^-}{2} \), it follows that \( \bar{x} \) is not a vertex of \( P \). A contradiction. Thus, \( \tilde{y}^* \) is a vertex of \( Q_j \).

Let \( F = \{ \ell \in G_j \mid \exists t \in T : \tilde{y}_{t, t}^* \in (0, 1) \} \). By the definition of \( \tilde{y}^* \), it suffices to show that \( |F| \leq 2|T| \). We say that an inequality from (12), (13), (14) and (15) is \textit{tight} if it holds with equality with respect to \( \tilde{y}^* \). For any \( \ell \in G_j \), define \( \xi_{\ell} = |\{ t \in T \mid (t, t) \in R_j \}| \). It follows that \( |R_j| = \sum_{\ell \in G_j} \xi_{\ell} \). For \( \ell \in F \) up to \( \xi_{\ell} - 1 \) of the inequalities in (12) and (13) are tight. Also, for \( \ell \in G_j \setminus F \) up to \( \xi_{\ell} \) of inequalities in (12) and (13) are tight. As the number of inequalities in (15) and (14) is \( 2|T| \), it follows that the number of tight equalities is at most
\[
\sum_{\ell \in F} (\xi_{\ell} - 1) + \sum_{\ell \in G_j \setminus F} \xi_{\ell} + 2|T| \leq |R_j| - |F| + 2|T|.
\]
As \( \tilde{y}^* \) is a vertex, there are at least \( |R_j| \) tight inequalities. Thus, \( |F| \leq 2|T| \) as required. \( \square \)

Since \( |T| = O(1) \), Theorem 3.11 follows from Lemmas F.3 and F.4.

29