HOW SMALL HYDRODYNAMICS CAN GO

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Numerous experimental and theoretical results in liquids and plasmas suggest the presence of a critical momentum at which the shear diffusion mode collides with a non-hydrodynamic relaxation mode, giving rise to propagating shear waves. This phenomenon, labelled as "k-gap", could explain the surprising identification of a low-frequency elastic behaviour in confined liquids. More recently, a formal study of the perturbative hydrodynamic expansion showed that critical points in complex space, such as the aforementioned k-gap, determine the radius of convergence of linear hydrodynamics – its regime of applicability. In this Letter, we combine the two new concepts and we study the radius of convergence of linear hydrodynamics in "real liquids" by using several data from simulations and experiments. We generically show that the radius of convergence increases with temperature and it surprisingly decreases with the interactions coupling. More importantly, we find that such radius is universally set by the Wigner–Seitz radius – the characteristic inter-atomic distance of the liquid, which provides a natural microscopic bound.

Introduction

"Πάντα ρεῖ" – everything flows. Hydrodynamics is an effective field theory (EFT) formulated as a perturbative expansion in spatial and time gradients. It governs the dynamics of conserved quantities which in Fourier space can be constructed as an infinite expansion in frequency \( \omega \) and momentum \( k \) – from slow processes and large scales, to fast dynamics and short lengths. It applies to the most disparate systems, from liquids [1] and solids [2] to flocks [3], crowds [4] and even to financial markets [5]. As every EFT, the microscopic physics is hidden in an infinite set of unknown coefficients since it is "irrelevant" (in the RG sense) in the low energy regime of interest, where the EFT applies.

In its linearised version, hydrodynamics is described by a finite set of hydrodynamic modes which can be obtained from the knowledge of the conservation equations and the constitutive relations. These modes display dispersion relations which satisfy the requirement:

\[
\lim_{k \to 0} \omega_i(k) = 0. \tag{1}
\]

Typical examples are diffusive modes \( \omega = -i D k^2 \) and propagating sound modes \( \omega = \pm v k - i \Gamma k^2 \). More broadly, every hydrodynamic mode obeys a dispersion relation:

\[
\omega^{(i)}(k) = \sum_{n=1}^{\infty} \alpha_n^{(i)} k^n, \tag{2}
\]

where \( \omega \in \mathbb{C} \) and \( k \in \mathbb{R} \). These modes are practically obtained from an eigenvalues equation of the form:

\[
\prod_{j=1}^{N} \left( \omega - \omega^{(j)}(k) \right) \equiv F(\omega, k^2) = 0, \tag{3}
\]

which in general not only contains hydrodynamic modes but non-hydrodynamics\(^1\) ones as well.

By staring at Eq.(2), one could immediately ask if such a perturbative series is convergent and if yes which is its radius of convergence. This is tantamount to ask what is the radius of convergence of linearised hydrodynamics (in momentum space)\(^2\) – its regime of applicability. The authors of [8, 9] (see also [10] for previous discussions) recently suggested that in order to answer such question, one has to formally extend the function \( F(\omega, k^2) \) in complex momentum space and treat it as a complex algebraic curve. In this language, series like Eq.(2) are known as Puiseux series and their radius of convergence is fundamentally connected to the so-called critical points \( \{\omega_c, k_c\} \) – points at which:

\[
F(\omega_c, k_c^2) = 0, \quad \frac{\partial F(\omega_c, k_c^2)}{\partial \omega} = 0, \tag{4}
\]

for both \( \omega_c, k_c^2 \in \mathbb{C} \). These critical points signal "irregularities" in the algebraic curve and they are related to QNMs\(^3\) crossing. The radius of convergence \( \mathcal{R} \) of the hydrodynamic expansion is then set by the distance to the

\[^1\] Not satisfying the relation Eq.(1).

\[^2\] In this Letter, we will not consider the question of convergence in real space and for the full non-linear dynamics. For that, see for example [6] and specially [7].

\[^3\] By these we refer simply to all modes with dispersion relation \( \omega(k) \) with \( k \in \mathbb{R} \). In other words, QNMs are all the excitations of a specific system [11].
first of these critical points:
\[ R \equiv |k_c|. \]  
(5)

Moreover, the radius of convergence is intimately connected to the existence of non-hydrodynamic excitations and their interactions with the hydrodynamics modes.

This mathematical machinery is extremely elegant but also rather abstract and hard to digest. So far, it has been applied only to few holographic models [12, 13] whose relevance for more realistic situations is at least disputable. In this Letter, we address the question of the regime of applicability of hydrodynamics in "real" liquids and in particularly we ask the following question: given a specific liquid, until which length-scale are we allowed to trust the hydrodynamic approximation? A naive answer would be that hydrodynamics is a good description of a liquid as far as the full system can be seen as a continuum and not as a set of molecules/particles interacting with each others. Technically, the failure of hydrodynamics is related to the intrusion of non-hydrodynamic modes which cannot be neglected anymore.

Combining the mathematical methods of [8, 9] with data from experiments and molecular dynamics (MD) simulations, we will indeed show that the limiting length-scale for the hydrodynamic framework is given by an \( \mathcal{O}(1) \) fraction of the inter-molecular distance, given formally by the size of Wigner–Seitz cell \( a \). We will also discuss how the regime of applicability depends on the temperature \( T \) and one the interactions strength.

**Convergence of linear hydrodynamics in liquids**

For centuries, the presence of low-frequency propagating shear waves (i.e. transverse phonons) has been considered the fundamental criterion to distinguish solids from liquids. Nevertheless, in the last decade, this definition has been challenged by several experiments observing the presence of shear waves in confined liquids at small frequency [14] and corresponding solid-like elastic effects [15, 16] (see also [17]). This phenomenon indicates that the difference between solids and liquids is only quantitative and it is measured by the relaxation time \( \tau \) – the average time for molecules rearrangement. Within Maxwell theory, such timescale is simply \( \tau \equiv \eta/G \) with \( \eta \) the shear viscosity and \( G \) the shear elastic modulus. It follows that in a solid \( \tau = \infty \) – molecules do not re-arrange but just oscillate around their equilibrium positions.

From a theoretical point of view, this program goes under the name of Maxwell interpolation and its main result is that the dynamics of shear waves in a fluid is governed by the simple equation
\[ \omega^2 + i \omega/\tau - v^2 k^2 = 0, \]  
(6)

which is known for historical reason as the telegraph equation. Here, \( v \) is the asymptotic speed of the shear waves, dictated by the elastic modulus. Solving Eq.(6), one finds a couple of modes
\[ \omega = -\frac{i}{2\tau} \pm \sqrt{v^2 k^2 - \frac{1}{4 \tau^2}}, \]  
(7)

and the existence of a critical momentum \( k_g \), labelled as \( k\)-gap, at which transverse waves start to propagate (\( \text{Re} \omega \neq 0 \)). This mechanism appears in several physical contexts; we refer the Reader to [18] for a complete review of them.

The appearance of propagating waves for \( k > k_g \) is the consequence of the collision between the shear diffusion mode and a non-hydrodynamic excitation, which
happens exactly at $k = k_g$ (see Fig.1). Given Eq.(6), it is straightforward to apply the methods of [8, 9] and determine the critical point of the corresponding algebraic curve using Eq.(4). One immediately obtains that $\omega_c = -i/2\tau$ and more importantly that

$$R \equiv |k_c| = \frac{1}{2v\tau} = k_g.$$  
(8)

This result is confirmed by explicitly computing the set of algebraic curves in the complex plane for complex momentum (see Fig.2).

In summary, in realistic liquids, in which the dynamics of shear waves is well described by the k-gap equation (6), hydrodynamics applies from large distances until a “microscopic” length-scale given by the inverse of the k-gap momentum $L \equiv k_g^{-1}$.

**Data from experiments and simulations**

The validity of the k-gap equation (6) to describe the shear dynamics in liquids has been extensively corroborated by several molecular dynamics simulations (MD) and experiments (EXP) in 2D and 3D liquids and plasmas [19–28]. The dispersion relation of the transverse waves is usually extracted from the transverse current correlation function $\langle J(−k,t)J(k,0) \rangle$ and within the numerical/experimental errors is well fitted by

$$\text{Re} \omega = \sqrt{v^2 k^2 - \frac{1}{4\tau^2}} \quad \text{for} \quad k > k_g,$$  
(9)

It is therefore obvious how to extract the critical momentum from the data.

A fundamental parameter, both in the simulations and in the experiments, is given by the inter-molecular distance $a$, which is extracted from the position of the first maximum in the pair distribution function. Technically, this scale corresponds to the Wigner–Seitz radius $a = \left(\frac{n(D-1)}{D} \rho \right)^{-1/D}$, where $n$ is the number density and $D$ the number of spatial dimensions.

In Table I, we list the value of the product $k_c a$ for a large number of 2D and 3D liquids and plasma and we universally find that:

$$k_c a \approx O(1).$$  
(10)

This result robustly indicates that the radius of convergence of hydrodynamics is universally set by the inter-molecular distance\(^4\). Moreover, it is in agreement with the naive idea that hydrodynamics applies up to the scale at which you can resolve that the continuum of the liquid is actually formed by a collection of particles. It is important to notice that the relation (10) is highly non trivial since the Wigner–Seitz radius can vary from $A (10^{-10})$ m for most of the fluids to mm scale in dusty plasmas (see for example [22]).

**Table I.** A summary of the available data for the momentum cutoff of shear waves $k_c$ in liquids and plasma. EXP stands for experiments while MD for molecular dynamics simulation. The inter-atomic distance is defined as $a$.

| Liquid                        | $k_c a$ |
|-------------------------------|---------|
| 2D Yukawa (MD) [19]           | 0.25    |
| dusty plasma (MD) [20–21]     | 0.3–1.2 |
| 2D Yukawa (EXP) [22]          | 0.16–0.31 |
| Liquid Fe (EXP) [23, 24]      | 0.3     |
| Liquid Cu (EXP) [23, 24]      | 0.4     |
| Liquid Zn (EXP) [23, 24]      | 0.3     |
| 3D LJ fluid (MD) [25]         | 0.2–0.7 |
| Liquid Fe (MD) [25]           | 0.2–0.7 |
| IPI8-IPL12 fluid (MD) [25]    | 0.2–0.7 |
| Liquid Hg (MD) [25]           | 0.15–0.55 |
| Supercritical Ar (MD) [26]    | 0.05–0.8 |
| Subcritical liquid Ar (MD) [26] | 0.2–0.7 |
| Supercritical CO\(_2\) [26]  | 0.1–0.5 |
| Liquid Ga (EXP, MD) [27]      | 0.25–0.6 |
| 2D Coulomb classical fluids (MD) [28] | 0.3–2 |
| Quark Gluon Plasma [29]       | 3.3     |

\(^4\) In principle, the critical scale $k_c$ could be pushed all the way down to zero by achieving a very large relaxation time $\tau$ (i.e. in glasses or solids). Nevertheless, in that situation, the hydrodynamic window $\omega \tau \ll 1$ would shrink completely and therefore hydrodynamics would not be applicable at all. We thank Kostya Trachenko for pointing this out.
Interestingly, the critical momentum for Quark Gluon Plasma was estimated in [29] around $k_c^{-1} \approx 0.15$ fm. Assuming a inter-parton distance of the order of $a \approx 0.5$ fm [30], we again obtain a value $k_c a = \mathcal{O}(1)$. This fact is reminiscent of the universality found in the values of the kinematic viscosity in [31].

Given the abundance of available data, we can do one step further and investigate the temperature dependence of the hydrodynamic convergence radius. In Fig.3 we have plotted the data for nine different liquids in a large range of temperatures. In all the cases, the regime of applicability increases with temperature and it is consistent with the idea that $k_g \sim 1/v\tau$ and $\tau$ decreases monotonically with temperature (in standard liquids following the well-known Arrhenius law). It also connects with the idea that at $T = 0$ (or equivalently $\tau = \infty$) hydrodynamics is not applicable\(^5\).

Finally, using the data for Coulomb fluids and plasmas we can also investigate the radius of convergence in terms of the effective coupling parameter:

$$\Gamma = \frac{Q^2}{4\pi \epsilon_0 a k_B T}, \quad (11)$$

which determines the strength of the Coulomb interactions in the plasma. Here $Q$ is the charge and $\epsilon_0$ the dielectric constant. The data from simulations universally show (see Fig.4) a power-like decrease of the critical momentum in function of $\Gamma$, which is further confirmed by experimental data [22]. The dusty plasma considered is a suspension of highly-charged micro-size particles which repel each others via a nearly-Coulomb potential:

$$U(r) = \frac{Q}{4\pi \epsilon_0 r} e^{-r/\lambda_D}, \quad (12)$$

in which the exponential correction takes into account the screening effects via the screening length $\lambda_D$. The results therefore demonstrate that stronger Coulomb repulsion decreases the regime of applicability of hydrodynamics. This observation seems to contradict the common wisdom that hydrodynamics works better for strongly coupled systems in which the mean free path becomes smaller (see for example Fig.4 in [29] and the Supplemental material). Nevertheless, it is important to notice that the coupling therein refers to the nuclear interactions – the strength of the bound states (corresponding to the $SU(N)$ gauge group) – and not to the inter-molecular Coulomb force\(^6\). Therefore, no evident discordance is present.

\(^5\) This follows simply by the violation of the requirements $\omega/T \ll 1$ or $\omega\tau \ll 1$.

\(^6\) We thanks Saso Grozdanov for suggesting this point.

![Figure 4](image-url)

**Figure 4.** The dimensionless combination $k_c a$ with $k_c = k_g$ determining the radius of convergence and $a$ the intermolecular characteristic distance. The empty squares indicate data from MD simulations taken from [20, 24, 28]. The filled squares are experimental data from [22].

**Outlook**

In this Letter, we have combined the theoretical methods proposed by [8, 9] with a large set of data from molecular dynamics simulations and experiments in 2D and 3D liquids and plasmas to determine the regime of applicability of hydrodynamics in realistic systems – the hydrodynamic convergence radius. Our main result is that the convergence radius is universally set in terms of the inter-molecular distance, namely the length-scale at which one can resolve the independent set of particles within the fluid continuum.

Moreover, the data indicate that the regime of validity of hydrodynamics increases with the temperature $T$ and decreases with the effective coupling parameter $\Gamma$.

Interestingly, the critical momentum for plasmon modes\(^7\) in Dirac fluids has been recently computed in [32] and it reads:

$$k_c = \frac{\tau_V}{4v\tau_{e,1}}, \quad (13)$$

where $\tau_{e,1}$ is the relaxation time, $v$ the electrons group velocity and $\tau_V$ is the time scale characterizing electrostatic interaction. In the near future, comparing better (13) with our results would be desirable.

Taking into account the fundamental role of non-hydrodynamic modes, it would be extremely helpful...
to build experimental setups able to pinpoint them on the lines of [33]. Moreover, it would be interesting to understand which are the experimental and physical consequences of the non-convergence of the hydrodynamic series in realistic systems.

Finally, the k-gap dynamics (6) universally appears in the context of diffusive Goldstone bosons [34, 35] in dissipative systems (e.g. in quasicrystals [36, 37]). It would be interesting to connect such mechanism with dissipative systems (e.g. in quasicrystals [36, 37]). It would be interesting to connect such mechanism with dissipative systems (e.g. in quasicrystals [36, 37]).

In conclusion, given the extremely wide usage of hydrodynamic methods, it is mandatory to understand until which length-scale those can be trusted. In this Letter, we provided a direct and pragmatic answer to this question in a large set of realistic liquids and plasmas. We hope that our results will boost the efforts to deeply understand hydrodynamics and its regime of applicability and to connect more closely theory with experiments.

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HOLOGRAPHIC EXAMPLES

In this Appendix, we summarize some of the known results concerning the critical momentum within the k-gap dynamics in several holographic models.

A common playground where the k-gap equation (6) appears is in the context of holographic models with higher-from global symmetries [38–41]. These models are used to described both magneto-hydrodynamics in charged plasmas and the elasto-dynamics in viscoelastic media. In the first scenario, the emergent propagating degree of freedom at $k = k_{g}$ is the photon. In the neutral plasma approximation, the corresponding critical momentum is given in terms of [42]

$$k_{c} \approx \frac{\pi T}{2 \log(M/\pi T)}, \quad (14)$$

where $T$ is the temperature and $M$ is an UV cutoff of the theory.

In the second scenario, the emergent mode is a propagating phonon and the critical momentum reads [43]

$$k_{c} = 8\pi^{2} \frac{T^{4}}{(3M-4\pi T)(27M-2\pi T(18+\sqrt{3}\pi-9 \log(3)))}, \quad (15)$$

where the same notations are used. It is simple to see that in both cases the critical momentum, determining the radius of convergence of hydrodynamics, grows monotonically with temperature independently of the value of the UV cutoff $M$.

Another situation where the k-gap equation (6) is at work is in models with dynamical Coulomb interactions and emergent plasmonic modes [44–46]. In those models, it is possible to obtain numerically the dispersion relations in function of the electromagnetic coupling $\lambda$. The numerical results are shown in Fig.6 and they clearly indicate a power-law fall-off of the critical momentum in function of $\lambda$. This is in agreement with the data obtained from 2D plasmas and presented in the main text (see Fig.4) under identifying $\lambda$ with $\Gamma$.

Finally, a well-known place where the k-gap appears is in the Israel-Stewart theory for linearised relativistic hydrodynamics [47]. In this model, the critical momentum is given by

$$k_{c} = \frac{1}{\sqrt{4D\tau}} = \sqrt{\frac{\epsilon + p}{4\eta \tau_{\pi}}}. \quad (16)$$

The gravity dual of the Israel-Stewart theory is provided by a Gauss-Bonnet model [40] , where the k-gap momen-
\[ k_c = -\frac{\sqrt{2\pi T}}{\gamma(k+2)+2\log\left(\frac{4\pi T}{\gamma}\right)-3} \lambda_{GB}, \quad (17) \]

with \( \gamma \equiv \sqrt{1-4\lambda_{GB}} \) and \( \lambda_{GB} \in [-\infty, 1/4] \).

From Eq. (17) two facts are evident. (I) The critical momentum grows linearly with temperature in agreement with the data shown in Fig. 3. (II) The critical momentum decreases by increasing the value of the Gauss-Bonnet coupling on the negative axes. Given that such direction corresponds to decrease the coupling of the dual field theory from infinity, the results shown in Fig. 7 suggest that hydrodynamics works better at strong coupling. This outcome is in agreement with [29]. Notice that the disagreement with the behaviour with respect to \( \Gamma \) is certainly puzzling but it does not constitute a discordance since \( \lambda \) refers to the \( SU(N) \) coupling while \( \Gamma \) to the intermolecular Coulomb potential.

![Figure 7](image-url)