ON THE UNIQUENESS OF THE NEWTON-WIGNER POSITION OPERATOR
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1. Introduction
The notion of localizability of relativistic particles has attracted considerable attention. It is generally admitted that the proper mathematical setting is the one due to Wightman [1] (see also [2]) which originates from the physical ideas of Newton and Wigner.

Let us outline briefly this framework. We admit that the configuration space of a certain physical system is the Borel space \((Q, \beta)\); here \(\beta\) is the Borel structure on \(Q\). We also consider that the system is a pure quantum one. Then the lattice of the system is of the form \(P(H)\); here \(H\) is some Hilbert space and \(P(H)\) is the lattice of orthogonal projectors in \(H\). (see for instance [2]).

Then one can argue (see [1] pg. 847) that the position observable is a projection-valued measure:

\[ \beta \ni E \rightarrow P_E \in P(H) \]

The physical interpretation is the following: the states in the range of \(P_E\) correspond to localization of our system in \(E \subseteq Q\).

If \(G\) is a (Borel) group of symmetries, the there exists in \(H\) a projective unitary representation of \(G\):

\[ G \ni g \rightarrow U_g \in U(H) \]

Here \(U(H)\) is the set of unitary operators in \(H\).

If \(Q\) is a \(G\)-space, then a natural compatibility condition is:

\[ U_g P_E U_g^{-1} = P_{g \cdot E}. \quad (1.1) \]

Here \(g \cdot E\) is the image of \(E\) under the action of \(G\). The couple \((U, P)\) is called a system of imprimitivity for \(G\) based on \(Q\).

If \(P\) is a position observable on \(Q = \mathbb{R}^3\), then according to the spectral theorem, we can construct three self-adjoint commuting operators \(Q_1, Q_2, Q_3\) namely the position operators appearing in the Newton-Wigner analysis. A convenient way to construct them [2] is to define first the unitary operators:

\[ B(y) \equiv \int \exp(-ix \cdot y)dP(x). \quad (1.2) \]

for any \(y \in \mathbb{R}^3\). Then, \(Q_k \ (k = 1, 2, 3)\) are determined via Stone theorem by:

\[ Q_k f \equiv \frac{d}{dt} B(te_k) f |_{t=0}. \quad (1.3) \]

Here \(e_k \ (k = 1, 2, 3)\) is the canonical basis in \(\mathbb{R}^3\).

A very interesting situation appears when one tries to combine these ideas with relativistic invariance. Suppose that our system admits a group of invariance \(G\) which has as
a subgroup the special Euclidean group $SE(3)$. Then, we must have a projective unitary representation of $G$ in $H$. By restriction, we have a projective unitary representation $U$ of $SE(3)$ in $H$. One says that the system is localisable (in $R^3$) iff there exists a position observable $P$ (based on $R^3$) such that $(U, P)$ is a system of imprimitivity.

According to Newton-Wigner-Wightman analysis the non-zero mass systems are localizable and zero mass systems are not localizable in this sense. The non-zero and zero mass systems are by definition certain irreducible unitary continous representations of the Poincaré (or the Galilei) group. The non-localizability for zero mass systems, especially for the photon, has generated a rather extensive literature [4]-[10].

On the other hand, even in the case of non-zero mass systems (which are localizable) a curious phenomenon appears, namely the position observable is not unique. In some cases this arbitrariness can be explicitely described [2]. Namely, suppose that $P^0$ is a position observable of finite multiplicity and $P$ is an arbitrary position observable. Then, there exists an unitary operator $A$ in $H$ such that $V$ commutes with the representation $U$ of $SE(3)$ and:

$$P = AP^0A^{-1}.$$ (1.4)

This non-unicity problem seems to be considered not very bothersome. In [2] one can fine a side remark to the sense that this problem is “noteworthy” and there exists a particular choise of $P$ which seems to be the “simplest” and which gives the classical relationship between velocity and momentum.

On the other hand, it is known that the similar unicity problem can be solved in the framework of classical Hamiltonian mechanics, imposing the so-called “manifest covari-ance” condition [11]. Namely, if $K_i$ ($i = 1, 2, 3$) are the (Lorentz) boosts generators and $H$ is the Hamiltonian, then one can show that $Q_1, Q_2, Q_3$ behaves as the spatial components of a quadrivector iff one has the relation:

$$\{K_i, Q_j\} = Q_i\{H, Q_j\}. \quad (1.2)$$

for $i = 1, 2, 3$. It is tempting to use something similar in quantum mechanics. In [12], one applies the naive correspondence rules:

$$\{ , \} \rightarrow i[ , ]. \quad (1.3)$$

$$AB \rightarrow 1/2(\hat{A}\hat{B} + \hat{B}\hat{A}). \quad (1.4)$$

and obtains for the quantum operators $K_i, H$ and $Q_i$ the relation:

$$[K_i, Q_j] = 1/2\left(Q_i[H, Q_j] + [H, Q_j]Q_i\right). \quad (1.5)$$

Surprisingly, this equation gives an unique solution for systems of zero-spin and has no solution for systems of non-zero spin.

In this note, we propose a different solution. Namely, we will impose a “manifest covariance” condition by a purely quantum argument, without invoking the correspondence rules (1.3) and (1.4). We will find an unique position operator for any spin, which is exactly the one considered in [2] as the “simplest” one. Moreover, we will show that the relation
(1.5) is valid up to terms of order $\hbar c^{-2}$. This quantum correction is zero iff the spin is zero. This completely explains the result of [12]. Moreover, this indicates that for the Galilei group, this quantum correction must dissapear.

We mention that these exists a number of attempts in the literature to define a Lorentz covariant quadrivector position operator (see e.g. [13]) but they seem to be different from ours as motivation and outcome.

In Section 2 and 3 we analyse the case of Poincaré and respectively Galilei invariance and in Section 4 we formulate some conclusions.

2. The Position Operator for Non-zero Mass Poincaré Systems

2.1 In the notations of [2], the system $[m, s]$ of mass $m$ and spin $s$ corresponds to the irreducible projective representation $W^{m,+,:s}$ of the proper orthochronous Poincaré group $P^+_+$. We can realize this representation in the Hilbert space $H = L^2(R^3, C^{2j+1}, \frac{dp}{E(p)})$ where:

$$E(p) \equiv \sqrt{p^2 + m^2}.$$  \hspace{1cm} (2.1)

as follows. Let $X^+_m$ be the positive energy hyperboloid:

$$X^+_m \equiv \{ p \in R^4 | p_0^2 - p^2 = m^2, p_0 > 0 \}.$$  \hspace{1cm} (2.2)

We define $\tau : R^3 \rightarrow X^+_m$ by:

$$\tau(p) \equiv (E(p), p).$$  \hspace{1cm} (2.3)

and $\eta : X^+_m \rightarrow R^3$ by:

$$\eta(p) \equiv p.$$  \hspace{1cm} (2.4)

Then we have for any $a \in R^4$ and any $A \in SL(2, C)$

$$(W^{m,+,:s}_{a, a} f)(p) = e^{ia \cdot \eta(p)} D^{(s)}(U(A, p)) f(\eta(\delta(L)^{-1}(\tau(p)))).$$  \hspace{1cm} (2.5)

for any $f \in H$. Here $U(A, p) \in SU(2)$ is the so-called Wigner rotation, $D^{(s)}$ is the representation of weight $s$ of $SU(2)$ and $\delta : SL(2, C) \rightarrow L^1_+$ is the covering homomorphism.

2.2 As we have said in the Introduction, the system $[m, s]$ is localizable in the sense of Newton and Wigner. Moreover, one can describe the most general expression of the position observable. More conveniently, one can describe the most general expression for the operators $B(y)$ defined by (1.2). Namely, one finds out (see [2]) that:

$$(B(y)f)(p) = A(p) A(y^{-1} \left( \frac{E(p)}{E(p + y)} \right)^{1/2} f(p + y).$$  \hspace{1cm} (2.6)

There $A : R^3 \rightarrow C^{2s+1}$ is a Borel function verifying for any $U \in SU(2)$ almost everywhere:

$$D^{(s)}(U) A(p) = A(\delta(U)p) D^{(s)}(U).$$  \hspace{1cm} (2.7)
Here $\delta : SU(2) \rightarrow SO(3)$ is the canonical homomorphism. Then, according to (1.3), one finds the following expression for the position operators $Q_k$:

$$(Q_k f)(p) = \frac{i}{2} \frac{\partial f}{\partial p_k}(p) - i \frac{p_k}{E(p)} \left[ \frac{1}{2E(p)} + \frac{\partial A}{\partial p_k}(p)A(p)^{-1} \right] f(p). \quad (2.8)$$

2.3 We formulate now the condition of relativistic covariance for the position operator. First, we start with the kinematic classical picture. Suppose we have two observers $O$ and $O'$ and they are connected by a boost of velocity $v = th(\chi)$ in the direction $e_3$. If $O'$ sees the system at $t' = 0$ in the position $x' = (x'_1, x'_2, x'_3)$ then, according to the Lorentz rules of transformation, $O$ sees the system localized at $x = (x_1, x_2, x_3)$ at time $t$ where:

$$t = th(\chi)x_3, \quad x'_1 = x_1, \quad x'_2 = x_2, \quad x'_3 = \frac{1}{ch(\chi)} x_3. \quad (2.9)$$

When we try to formulate this result in the quantum framework we are faced with the well-known problem that there are no states in $H$ strictly localized in $x$. More precisely, the eigenvalue equation:

$$Q_k \psi_x = x_k \psi_x \quad (k = 1, 2, 3). \quad (2.10)$$

has solution of the form:

$$\psi_x(p) = E(p)A(p)ve^{-ipx}. \quad (2.11)$$

which is not an element of $H$. Here $v$ is an arbitrary vector from $C^{2s+1}$.

Nevertheless, according to the quantum point of view, we can consider states which are localized at $x$ in an approximative sense. For instance, let us define the states $\psi_{x, \alpha, v} \in H$ for any $\alpha \in R_+$ and any $v \in C^{2s+1}$, by:

$$\psi_{x, \alpha, v}(p) = \sqrt{2} \left( \frac{2\alpha}{\pi} \right)^{3/4} E(p)^{1/2}e^{-\alpha p^2}A(p)ve^{-ipx}. \quad (2.12)$$

for $v \in C^{2s+1}$ of norm 1. Then it is easy to prove that:

$$\|\psi_{x, \alpha, v}\|^2 = 1. \quad (2.13)$$

and:

$$<q_k>_{\psi_{x, \alpha, v}} = x_k. \quad (2.14)$$

$$\Delta q_k = \alpha^{1/2}. \quad (2.15)$$

So, in the limit $\alpha \rightarrow 0$, $\psi_{x, \alpha, v}$ becomes a state more and more localized at $x \in R^3$.

2.4 Now, let $\psi_{x', \alpha, v}$ be a state “approximatively” localized at $x'$. We consider that this measurement is done by the observer $O'$ at time $t' = 0$. (See Section 2.3 for notations). According to the physical interpretation of the representation $W^{m,+}_{A \circ B(-\chi,e_3),0}$, from the point of view of the observer $O$, the state of the system at $t = 0$ is $W^{m,+}_{A \circ B(-\chi,e_3),0} \psi_{x', \alpha, v}$ where $A_B(\chi, n) \in SL(2, C)$ corresponds to the boost in the direction $n$ of velocity $v = th(\chi)$. The discussion in Section 2.3 leads us to require that the observer $O$ sees the particle
localized around $x$ at time $t = \hbar(\chi)x_3$. But the state of the system at time $t = \hbar(\chi)x_3$ is from the point of view of $O$:

$$
\psi_{x,\alpha,v,\chi} \equiv W_{m,+,s}^{1,\hbar(\chi)x_3\epsilon_0} W_{A_B(-\chi,e_3),0}^{m,+,s} \psi'_{x',\alpha,v}.
$$

(2.16)

So, our condition of manifest covariance is that $\psi_{x,\alpha,v,\chi}$ describes a state localized around $x$, where the connection between $x$ and $x'$ is given by (2.9). Namely, we must have:

$$
<q_k>_{\psi_{x,\alpha,v,\chi}} = x_k.
$$

(2.17)

and:

$$
\lim_{\alpha \to 0} \Delta q_k = 0.
$$

(2.18)

for $k = 1, 2, 3$ and any $v \in C^{2s+1}$.

Let us note that if (2.17),(2.18) are true, then because of the rotation covariance, a similar statement will be true for boosts in any direction.

Now, it is not hard to prove that (2.17) is true for $k = 1, 2$ and is also true for $k = 3$ iff $A = \text{const.}$ almost everywhere. So, we can take $A = 1$. In this case it follows easily that $\Delta q_k$ behaves as $\alpha^{1/2}$ for small $\alpha$, so (2.18) follows.

In conclusion, our main result is that the manifest covariance condition formulated above is compatible with an unique expression for the position operator, namely:

$$
(Q_k f)(p) = i \frac{\partial f}{\partial p_k}(p) - \frac{p_k}{2E(p)} f(p).
$$

(2.19)

2.5 To study the validity of (1.4) it is more easy to work in the vector bundle representation for $W_{m,+,s}$ (see [2], pg.365).

Then the operators $Q_k$ are given by the formula (VIII.230) of [2]. One can easily compute the infinitesimal generators $K_l$ ($l = 1, 2, 3$) in this representation. Then, inserting appropriately the factors $\hbar$ and $c$, one can show that we have:

$$
[K_i, Q_j] = 1/2 (Q_i[H, Q_j] + [H, Q_j]Q_i) + o(\hbar c^{-2}).
$$

(2.20)

where $o(\hbar c^{-2})$ is an expression containing only spin terms. This is the quantum version of the “manifest covariance” condition of Currie, Jordan and Sudarshan [12].

3. The Position Operator for Non-Zero Mass Galilei System

3.1 In the notations of [2], the system of mass $m$ and spin $s$ corresponds to the representation $V^{\tau,s}$ of the covering group of the proper orthochronous Galilei group $G^\uparrow_+$. They can be realized in the Hilbert space $H = L^2(R^3, C^{2s+1}, dp)$, according to the following formula:

$$
(V^{\tau,s}_{U,u,\eta,a}f)(p) = \exp \left( i a \cdot p + \frac{i}{2} \tau a \cdot u - i \eta \frac{p^2}{2\tau} \right) D^{(s)}(U)f(\delta(U)^{-1}(p + \tau u)).
$$

(3.1)
Here $U \in SU(2), u \in R^3$ is the velocity, $a$ is the space translation, and $\eta \in R$ is the time translation.

As in Section 2, we can describe explicitly the most general position observable. Namely, the operators $B(x)$ must have the following generic form:

$$B(x)f(x) = A(p)A(p + x)^{-1}f(p + x).$$ (3.2)

where $A : R^3 \to C^{2s+1}$ is a Borel function verifying (2.7). It follows that the position operators are:

$$(Q_kf)(p) = i\frac{\partial f}{\partial p_k}(p) - i\frac{\partial A}{\partial p_k}(p)A(p)^{-1}f(p).$$ (3.3)

3.2 The condition of manifest covariance with respect to Galilei boosts is simpler than in the Poincaré case. Let us suppose that the two observers $O$ and $O'$ are connected by a boost of velocity $u$. If $O$ sees the system at $t = 0$ in the position $x$, then according to the formula for the Galilei boosts, the observer $O'$ sees the system at $t' = 0$ in the position $x' = x - tu = x$.

As in the Section 2, the solution of the eigenvalue equation:

$$Q_k\psi_x = x_k\psi_x.$$ (3.4)

$(k = 1, 2, 3)$ is of the following form:

$$\psi_{x,v}(p) = e^{-i x \cdot p}A(p)v.$$ (3.5)

for any $v \in C^{2s+1}$, so is not an element of the Hilbert space $H$. Nevertheless, we can consider the elements

$$\psi_{x,\alpha,v}(p) = \left(\frac{\pi}{2\alpha}\right)^{3/4}e^{-i x \cdot p}e^{-\alpha p^2}A(p)v.$$ (3.6)

for $v \in C^{2s+1}$ of norm 1 and we have:

$$<q_k>_{\psi_{x,\alpha,v}} = x_k.$$ (3.7)

and:

$$\Delta q_k = \alpha^{1/2}.$$ (3.8)

for $k = 1, 2, 3$.

3.3 Now, let $\psi_{x,\alpha,v}$ the state of the system from the point of view of the observer $O'$. As in Section 2 the state of the system from the point of view of the observer $O$ is:

$$\psi_{x,\alpha,u,v} \equiv V_{1,-u,0,0}^r \psi_{x,\alpha,v}.$$ (3.9)

According to the discussion in Section 3.2, our condition of manifest covariance is that $\psi_{x,\alpha,u,v}$ describes a system localized around $x$ at $t = 0$ i.e.:

$$<q_k>_{\psi_{x,\alpha,u,v}} = x_k.$$ (3.10)
and:

\[ \lim_{\alpha \to 0} \Delta q_k = 0. \]  \hspace{1cm} (3.11)

for \( k = 1, 2, 3 \).

It is elementary to prove that (3.10) is equivalent to \( A = \text{const.} \) so, as in Section 2, we can take \( A = 1 \). So, in this case also, the manifest covariance condition gives an unique position operator, namely:

\[ (Q_k f)(p) = i \frac{\partial f}{\partial p_k}(p). \]  \hspace{1cm} (3.12)

It is easy to see that in this case we have:

\[ [K_i, Q_j] = 0. \]  \hspace{1cm} (3.13)

Again we note that (3.13) is the quantum counterpart of a similar classical relation [12]. In this case there are no quantum corrections. This can be explained by the presence of \( c^{-2} \) in the expression of the relativistic quantum correction in (2.20).

4. Conclusions

The relation (2.20) shows that the naive quantization rules (1.3) and (1.4) do not work always. This explains in part the difficulties of all quantization schemes used in the literature.

It is plausible that the same unicity result hold also for the photon if the localizability of this system is described in the sense of Wightman framework [14].

It is interesting to try to generalize along these lines the well known results of the “non-interaction” type theorems [11,15].

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