TWO TIME DISTRIBUTION IN BROWNIAN DIRECTED PERCOLATION

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Abstract. In the zero temperature Brownian semi-discrete directed polymer we study the joint
distribution of two last-passage times at positions ordered in the time-like direction. This is the situa-
tion when we have the slow de-correlation phenomenon. We compute the limiting joint distribution
function in a scaling limit. This limiting distribution is given by an expansion in determinants which
is not a Fredholm expansion. A somewhat similar looking formula was derived non-rigorously in a
related model by Dotsenko.

1. Introduction and results

Let $B_i(t)$, $t \geq 0$, $i \geq 1$, be independent standard Brownian motions. We consider the zero
temperature Brownian semi-discrete directed polymer, [1], [13], [5], [18]. The last-passage time in
this model is defined by

$$H(\mu,n) = \sup_{0=\tau_0<\tau_1<\cdots<\tau_n=\mu} \sum_{i=1}^{n} B_i(\tau_i) - B_i(\tau_{i-1}).$$

We are interested in the asymptotics of the joint distribution function

$$P[H(\mu_1,n_1) \leq \xi_1, H(\mu_2,n_2) \leq \xi_2]$$

when $(\mu_1,n_1)$ and $(\mu_2,n_2)$ are ordered in the time-like direction, $\mu_1 < \mu_2$, $n_1 < n_2$. The random
variable (1.1) is distributed as the largest eigenvalue of a GUE random matrix, [1]. More precisely,

$$P[H(\mu,n) \leq \xi] = \frac{1}{Z_{\mu,n}} \int_{(-\infty,\xi]^n} \prod_{1 \leq j < k \leq n} (x_k - x_j)^2 \prod_{j=1}^{p} e^{-\frac{x_j^2}{\eta^2}} d^p x.$$

By standard results this leads to the following limit law for $H(\mu,n)$. Let $t, \nu$ and $\eta$ be fixed. Then

$$\lim_{M \to \infty} P\left[H(tM - \nu(tM)^{2/3}, [tM + \nu(tM)^{2/3}]) \leq 2tM + (\eta - \nu^2)(tM)^{1/3}\right] = F_2(\eta),$$

where $F_2$ is the GUE Tracy-Widom distribution,

$$F_2(\eta) = \det(I - K_{Ai})_{L^2(\eta,\infty)}.$$

Here $K_{Ai}$ is the Airy kernel,

$$K_{Ai}(x,y) = \int_{0}^{\infty} \Ai(x + \tau) \Ai(y + \tau) d\tau.$$

When $(\mu_1,n_1)$ and $(\mu_2,n_2)$ have a space-like ordering, $\mu_1 < \mu_2$, $n_1 > n_2$, the asymptotics for (1.2)
analogueous to (1.4) can be computed and expressed in terms of a Fredholm determinant with the
extended Airy kernel. This leads to the possibility of proving convergence to the Airy process along
space-like paths, [3], [12]. However, the case when $(\mu_1,n_1)$ and $(\mu_2,n_2)$ are ordered in the time-like
direction (more precisely along a characteristic, see e.g. [12]) has not been considered previously
except non-rigorously in a related model by Dotsenko, [10], using the replica method. The main
result of this paper is given in the next theorem.

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Theorem 1.1. Let $0 < t_1 < t_2$, $\eta_1, \eta_2, \nu_1, \nu_2 \in \mathbb{R}$ be given. Set
\begin{equation}
\alpha = (t_1/\Delta t)^{1/3},
\end{equation}
where $\Delta t = t_2 - t_1$, and let $F_t(\eta_1, \eta_2; \alpha, \nu_1, \nu_2)$ be given by (1.21) below. Introduce the scaling
\begin{equation}
\mu_i = t_i M - \nu_i (t_i M)^{2/3}, \quad n_i = t_i M + \nu_i (t_i M)^{2/3}, \quad \xi_i = 2t_i M + (\eta_i - \nu_i^2)(t_i M)^{1/3},
\end{equation}
i = 1, 2. With this scaling, define
\begin{equation}
F_M(\eta_1, \eta_2; t_1, t_2, \nu_1, \nu_2) = \mathbb{P}[H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2]
\end{equation}
Then,
\begin{equation}
\lim_{M \to \infty} F_M(\eta_1, \eta_2; t_1, t_2, \nu_1, \nu_2) = F_t(\eta_1, \eta_2; \alpha, \nu_1, \nu_2).
\end{equation}
The theorem will be proved in section 4.

In order to give the formula for the limiting distribution function we first need to define some functions. Set
\begin{equation}
\Delta \nu = \nu_2 \left(\frac{t_2}{\Delta t}\right)^{2/3} - \nu_1 \left(\frac{t_1}{\Delta t}\right)^{2/3},
\end{equation}
\begin{equation}
\Delta \eta = (\eta_2 - \nu_2^2) \left(\frac{t_2}{\Delta t}\right)^{1/3} - (\eta_1 - \nu_1^2) \left(\frac{t_1}{\Delta t}\right)^{1/3} + \Delta \nu^2.
\end{equation}

Let
\begin{equation}
\phi_1(x, y) = -\alpha e^{\Delta \nu x - \nu_1 y} \int_0^\infty e^{(\nu_1 - \Delta \nu)\tau} K_{\lambda_1}(\eta_1 - \tau, \eta_1 - y) K_{\lambda_1}(\Delta \eta + \alpha \tau, \Delta \eta + \alpha x) d\tau,
\end{equation}
\begin{equation}
\psi_1(x, y) = \alpha e^{\Delta \nu x - \nu_1 y} \int_0^\infty e^{-(\nu_1 - \Delta \nu)\tau} K_{\lambda_1}(\eta_1 + \tau, \eta_1 - y) K_{\lambda_1}(\Delta \eta - \alpha \tau, \Delta \eta + \alpha x) d\tau,
\end{equation}
\begin{equation}
\phi_2(x, y) = \alpha e^{\Delta \nu(x-y)} K_{\lambda_1}(\Delta \eta + \alpha x, \Delta \eta + \alpha y),
\end{equation}
and
\begin{equation}
\phi_3(x, y) = e^{\nu_1(x - y)} K_{\lambda_1}(\eta_1 - x, \eta_1 - y).
\end{equation}

Define
\begin{equation}
\phi(x, y) = \phi_1(x, y) + 1(y \geq 0)\phi_2(x, y) - 1(x < 0)\phi_3(x, y),
\end{equation}
and
\begin{equation}
\psi(x, y) = -\phi_1(x, y) - 1(y > 0)\phi_2(x, y) + 1(x \leq 0)\phi_3(x, y),
\end{equation}
where $1(\cdot)$ is the indicator function. We will use the following notation in block matrices. If $f$ is a function of two real variables, $x \in \mathbb{R}^s$ and $y \in \mathbb{R}^t$ we write
\begin{equation}
f(x, y) = (f(x_i, y_j))_{1 \leq i \leq s, 1 \leq j \leq t},
\end{equation}
for a matrix block. Let $r_1, r_2, s, t \geq 0$, $x \in \mathbb{R}^{r_1}$, $x' \in \mathbb{R}^s$, $y \in \mathbb{R}^{r_2}$, $y' \in \mathbb{R}^t$ and $0 \in \mathbb{R}$. Define the determinants
\begin{equation}
W_{r_1, s, r_2, t}(x, x', y, y') = \begin{vmatrix}
\psi(x, x) & \psi(x, x') & \psi(x, 0) & \psi(x, y) & \psi(x, y') \\
\phi(x', x) & \phi(x', x') & \phi(x', 0) & \phi(x', y) & \phi(x', y') \\
\psi(0, x) & \psi(0, x') & \psi(0, 0) & \psi(0, y) & \psi(0, y') \\
\phi(y, x) & \phi(y, x') & \phi(y, 0) & \phi(y, y) & \phi(y, y') \\
\psi(y', x) & \psi(y', x') & \psi(y', 0) & \psi(y', y) & \psi(y', y') \\
\end{vmatrix}_2
\end{equation}
We can now give the expression for the distribution function $F_{tt}(\eta_1, \eta_2; \alpha, \nu_1, \nu_2)$ in theorem 1.1.

Define

(1.21) 

$$F_{tt}(\eta_1, \eta_2; \alpha, \nu_1, \nu_2) = F_2(\eta_2) - \sum_{r,s,t=0}^{\infty} \frac{1}{r! s! t!} \int_{\eta_1}^{\infty} d\eta_1 \int_{(-\infty,0]^r} d\tau_x \int_{(-\infty,0]^s} d\phi_{x'} \int_{[0,\infty)^t} d\psi_{y'} \int_{[0,\infty)^{r-1}} d\phi_{y'} W_{r,s,r,t}^{(1)}(x,x',y,y')$$

where $F_2$ is the Tracy-Widom distribution given by (1.15). Recall that the Tracy-Widom distribution $F_2$ in (1.15) can be written as a Fredholm expansion. The two-time distribution function $F_{tt}$ is not given by a Fredholm expansion although the expansion in (1.21) has some similarities with a block Fredholm expansion.

We will derive the formulas that we will use to prove (1.10) by thinking of $H(\mu, n)$ as a limit of a last-passage time in a discrete model. Let $(w(i,j))_{i,j \geq 1}$ be independent geometric random variables with parameter $q$,

$$\mathbb{P}[w(i,j) = k] = (1-q)q^k, \quad k \geq 0.$$ 

Consider the last-passage times

(1.22) 

$$G(m,n) = \max_{\pi: (1,1) \to (m,n)} \sum_{(i,j) \in \pi} w(i,j),$$

where the maximum is over all up/right paths from $(1,1)$ to $(m,n)$, see [15]. We have the following limit law

(1.23) 

$$\frac{G(\mu T, n) - \frac{q}{1-q} \mu T}{\sqrt{\frac{2}{1-q} \mu T}} \to H(\mu, n)$$

in distribution as $T \to \infty$, see [1]. The distribution function $\mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2]$ will be analyzed using a formula from [17], see section 2 below.

Remark 1.2. As mentioned above Dotsenko has given a non-rigorous derivation of the limiting distribution function $F_{tt}$ given by (1.21). The formula in (1.10) has similarities with (1.21) but we have not attempted to prove that they are the same. Dotsenko also used a similar derivation in the space-like direction, see [11], see also [13].

Remark 1.3. This paper is a contribution to the understanding of models in the so called KPZ universality class, which have been of great interest in the last 15 years. We will not survey this development here, see for example the papers [2], [4], [6], [16], [19] and references therein. In particular the results of this paper could be of interest in understanding the so called Airy sheet, a conjectural limiting object for many models, see [9] and [8]. One aspect about last-passage...
percolation models in the time direction has been studied previously, namely the so called slow decorrelation phenomenon, see [7, 12]. This means that the scaling exponent in the time direction (characteristic direction) is 1; we need $\mu_1$ and $\Delta \mu$ to be of order $M$ above to get a non-trivial limit.

**Remark 1.4.** It is not so hard to check, disregarding technical details, that $F(\eta_1, \eta_2; \alpha) \to F(\eta_1)$ as $\eta_2 \to \infty$ and $F(\eta_1, \eta_2; \alpha) \to F(\eta_1)$ as $\eta_1 \to \infty$. We also expect that $F(\eta_1, \eta_2; \alpha) \to F(\eta_1)F(\eta_2)$ as $\alpha \to 0^+$. This limit can be checked heuristically but appears to be rather subtle and we will not discuss it further.

**Remark 1.5.** Below we will derive formulas for the corresponding problem for the last-passage times $G(m,n)$ before taking the limit to $H(\mu, n)$. It should be possible to carry out the whole proof below but with $G(m,n)$ instead, but some of the computations in section 3 appear to be harder.

The role of the Hermite polynomials there would then be replaced by the Meixner polynomials. The outline of the paper is as follows. In section 2 we will prove a formula for the joint distribution function $\mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2]$ based on results from [17]. By taking a limit this leads to a formula for $\mathbb{P}^{(2.2)}$. This computation involves certain symmetrization identities that will be proved in section 3. In section 3 the formula from section 2 will be rewritten and expanded in terms of determinants. Section 4 gives the proof of theorem 1.1 based on the expansion, certain asymptotic limits and estimates. These limits and estimates are finally proved in section 6.

Throughout this paper $\gamma_r$ will denote a positively oriented circle around the origin with radius $r$, and $\Gamma_d$ will denote a straight line through $d$ parallel to the imaginary axis and oriented upwards.

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## 2. A FORMULA FOR THE JOINT DISTRIBUTION FUNCTION

Let $G(m,n)$, $m, n \geq 1$, be the last-passage times defined by (1.2), and write $G(m) = (G(m,1), \ldots, G(m,n))$.

We put $G(0) = 0$. Introduce the difference operators $\Delta f(x) = f(x+1) - f(x)$, $\Delta^{-1} f(x) = \sum_{x=1}^{\infty} f(y)$, where $f: \mathbb{Z} \to \mathbb{R}$ is a given function. Set, for $m \geq 1$, $x \in \mathbb{Z}$,

$$w_m(x) = (1-q)^m \left( \frac{x+m-1}{x} \right) q^x 1(x \geq 0).$$

Also, we let $W_n = \{ x \in \mathbb{Z}^n; x_1 \leq x_2 \leq \cdots \leq x_n \}$. In [17] the following result was proved, inspired by [21].

**Proposition 2.1.** For $x, y \in W_n$ and $m > \ell$, $x$,

$$\mathbb{P}[G(m) = y \mid G(\ell) = x] = \det (\Delta^j \nu_{m-\ell}(y_j - x_j))_{1 \leq i, j \leq n}. \tag{2.2}$$

In particular

$$\mathbb{P}[G(m) = x] = \det (\Delta^j \nu_m(x_j))_{1 \leq i, j \leq n}. \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\mathbb{P}[G(m_1) = x, G(m_2) = y] = \mathbb{P}[G(m_2) = y \mid G(m_1) = x] \mathbb{P}[G(m_1) = x]$$

$$= \det (\Delta^j \nu_{m_2-m_1}(y_j - x_i))_{1 \leq i, j \leq n} \det (\Delta^j \nu_{m_1}(x_j))_{1 \leq i, j \leq n}.$$
Thus,
\begin{equation}
(2.4) \quad P := \mathbb{P}[G(m_1, n_1) \leq v_1, G(m_2, n_2) \leq v_2] = \sum_{u=-\infty}^{v_1} \sum_{x \in W_{n_1}} \sum_{y \in W_{n_2}} \det \left( \Delta^{-i} w_m(x_j) \right)_{1 \leq i, j \leq n_2} \det \left( \Delta^{-i} w_m(y_j - x_i) \right)_{1 \leq i, j \leq n_2},
\end{equation}
where $1 \leq m_1 \leq m_2$ and $1 \leq n_1 \leq n_2$. This formula is the starting point of our analysis. In order to get a more useful formula we rewrite it in terms of multiple contour integrals. We can write $w_m$ in (2.4) as
\begin{equation}
(2.5) \quad w_m(x) = \frac{(1 - q)^m}{2\pi i} \int_{\gamma_r} \frac{dz}{(1 - qz^{m+1})},
\end{equation}
where $\gamma_r$ is a positively oriented circle around the origin with radius $r$ and $0 < r < 1/q$. This gives
\begin{equation}
(2.6) \quad \Delta^k w_m(x) = \frac{(1 - q)^m}{2\pi i} \int_{\gamma_r} \frac{(1 - z)^k dz}{(1 - qz^{m+k+1})},
\end{equation}
for all $k \in \mathbb{Z}$ if $0 < r < 1$. Inserting (2.6) into (2.4) will after some rather lengthy and non-trivial manipulations lead to the following formula for $P$.

**Proposition 2.2.** Let $P$ be defined by (2.4), and let $0 < s_1 < r_1 < 1$, $0 < r_2 < s_2 < 1$. Then
\begin{equation}
(2.7) \quad P = \sum_{u=-\infty}^{v_1} \frac{(1 - q)^m}{(2\pi i)^{2n_2} n_1!^2 \Delta n!^2} \int_{\gamma_{s_1}} d^{n_1} z \int_{\gamma_{s_2}} d^{n_2} z \int_{\gamma_{s_1}} d^{n_1} w \int_{\gamma_{s_2}} d^{n_2} w \times \det \left( \frac{1}{z_j - z_i} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{w_j - z_i} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{w_j - w_i} \right)_{n_1 < i, j \leq n_2} \prod_{j=n_1+1}^{n_2} \frac{1 - z_j}{1 - w_j} \left( 1 - \prod_{j=1}^{n_1} \frac{z_j}{w_j} \right) \prod_{j=1}^{n_2} \frac{w_j^{u-n_1}}{(1 - z_j)^{\Delta n} (1 - qz_j)^{n_1} (1 - w_j)^{n_1} (1 - qw_j)^{\Delta n}},
\end{equation}
where $\Delta n = n_2 - n_1$, $\Delta m = m_2 - m_1$.

Here we have used the notation
\begin{equation}
(2.8) \quad \int_{\gamma_{s_1}} d^{n_1} z \int_{\gamma_{s_2}} d^{n_2} z = \int_{\gamma_{s_1}} dz_1 \cdots \int_{\gamma_{s_1}} dz_{n_1} \int_{\gamma_{s_2}} dz_{n_1+1} \cdots \int_{\gamma_{s_2}} dz_{n_2}.
\end{equation}

Before we can prove (2.7) we need some preliminary results.

**Lemma 2.3.** We have the following two algebraic symmetrization identities,
\begin{equation}
(2.9) \quad \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{j=1}^{n} \left( 1 - \frac{w_{\sigma(j)}}{w_{\sigma(j)}} \right)^j \frac{1}{(1 - w_{\sigma(1)})(1 - w_{\sigma(2)})(1 - w_{\sigma(1)} \cdot \cdots \cdot w_{\sigma(n)})} = (-1)^{\frac{\binom{n-1}{2}}{2}} \prod_{j=1}^{n} \frac{1}{w_j} \det \left( w_{j}^{i-1} \right)_{1 \leq i, j \leq n},
\end{equation}
and

\begin{equation}
\sum_{\sigma_1, \sigma_2 \in S_n} \text{sgn}(\sigma_1 \sigma_2) \prod_{j=1}^{n} \left( \frac{w_{\sigma_2(j)}(1 - z_{\sigma_1(j)})}{z_{\sigma_1(j)}(1 - w_{\sigma_2(j)})} \right) j \prod_{j=1}^{n} \left( \frac{1}{1 - \frac{z_{\sigma_1(1)}}{w_{\sigma_2(1)}}} \left( 1 - \frac{z_{\sigma_1(1)} z_{\sigma_1(2)}}{w_{\sigma_2(1)} w_{\sigma_2(2)}} \right) \cdots \left( 1 - \frac{z_{\sigma_1(1)} \cdots z_{\sigma_1(n)}}{w_{\sigma_2(1)} \cdots w_{\sigma_2(n)}} \right) \right)
= \prod_{j=1}^{n} \frac{w_j^{n+1}(1 - z_j)^n}{z_j^n(1 - w_j)^n} \det \left( \frac{1}{w_j - z_i} \right)_{1 \leq i, j \leq n}.
\end{equation}

The first identity is a direct consequence of one of the Tracy-Widom ASEP identities, [20]. The other identity is new as far as we know. The lemma will be proved in sect. 5.

We can now give the proof of proposition 2.2.

**Proof.** (Proposition 2.2) Inserting (2.6) into (2.4) gives

\begin{equation}
P = \sum_{u=-\infty}^{v_1} \sum_{x_1 = \eta_{n_1}} \sum_{x_2 = \eta_{n_2}} \left( \frac{1 - q)^{m_1}}{2\pi i} \int_{\gamma_{n_1}} \left( \frac{1 - z}{z} \right)^{j-i} \prod_{j=1}^{n} \frac{dz_j}{(1 - qz_j)^{m_1} z_j^{x_j+1}} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1 - q)^{m_1}}{2\pi i} \int_{\gamma_{n_2}} \left( \frac{1 - w}{w} \right)^{j-i} \prod_{j=1}^{n} \frac{dw_i}{(1 - qw_i)^{m_2} w_i^{y_j-x_j+1}} \right)_{1 \leq i, j \leq n_2},
\end{equation}

where $0 < r_1, r_2 < 1$. Now, the first determinant in (2.11) can be rewritten as

\begin{equation}
\det \left( \frac{1 - q)^{m_1}}{2\pi i} \int_{\gamma_{n_1}} \left( \frac{1 - z}{z} \right)^{j-i} \prod_{j=1}^{n} \frac{dz_j}{(1 - qz_j)^{m_1} z_j^{x_j+1}} \right)_{1 \leq i, j \leq n_2} = \frac{(1 - q)^{m_1 n_2}}{(2\pi i)^{n_2}} \int_{\gamma_{n_1}} d^{n_2} z \prod_{j=1}^{n_2} \left( \frac{1 - z_j}{z_j} \right)^{j-i} \prod_{j=1}^{n_2} \frac{1}{(1 - qz_j)^{m_1} z_j^{x_j+1}} \det \left( \frac{z_j^i}{1 - z_j} \right)_{1 \leq i, j \leq n_2}
\end{equation}

\begin{equation}
= \frac{(1 - q)^{m_1 n_2}}{(2\pi i)^{n_2}} \int_{\gamma_{n_1}} d^{n_2} z \prod_{j=1}^{n_2} \left( \frac{1 - z_j}{z_j} \right)^{j-i} \prod_{j=1}^{n_2} \frac{1}{(1 - qz_j)^{m_1} z_j^{x_j+1}} \det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2}.
\end{equation}
Consider now the second determinant in (2.11) together with the \( y \)-summation. We get

(2.13)

\[
\sum_{x \in W_{n_2}} \det \left( \frac{(1-q)^{\Delta m}}{2\pi i} \int_{\gamma_{r_2}} \left( 1 - \frac{w_i}{w_j} \right)^{j-i} \frac{dw_i}{(1 - qw_i)^{\Delta m} w_i^{y_j-x_i+1}} \right)_{1 \leq i,j \leq n_2}
\]

\[
= \frac{(1-q)^{\Delta m n_2}}{(2\pi i)^{n_2}} \int_{\gamma_{r_2}} d^{n_2} w \sum_{x \in W_{n_2}} \sum_{\sigma \in S_{n_2}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n_2} \left( 1 - \frac{w_{\sigma(j)}}{w_{\sigma(j)}} \right)^{j-\sigma(j)} \frac{1}{(1 - qw_{\sigma(j)})^{\Delta m} w_{\sigma(j)}^{y_j-x_{\sigma(j)}+1}}
\]

\[
= \frac{(1-q)^{\Delta m n_2}}{(2\pi i)^{n_2}} \int_{\gamma_{r_2}} d^{n_2} w \prod_{j=1}^{n_2} \left( \frac{w_j}{1 - w_j} \right)^{j} \frac{x_j^{-1}}{(1 - qw_j)^{\Delta m}}
\]

\[
\times \sum_{\sigma \in S_{n_2}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n_2} \left( \frac{1 - w_{\sigma(j)}}{w_{\sigma(j)}} \right)^{j} \left( \sum_{x \in W_{n_2}} \sum_{j=1}^{n_2} \frac{1}{w_{\sigma(j)}} \right).
\]

Since \( 0 < r_2 < 1 \) we see, by summing the geometric series, that

(2.14)

\[
\sum_{x \in W_{n_2}} \prod_{j=1}^{n_2} \frac{1}{w_{\sigma(j)}} = \prod_{j=1}^{n_2} \frac{1}{w_j} \frac{1}{(1-w_{\sigma(1)})(1-w_{\sigma(1)}w_{\sigma(2)}) \cdots (1-w_{\sigma(1)} \cdots w_{\sigma(n)})}.
\]

Combining (2.14) with the identity (2.9) we get

\[
\sum_{\sigma \in S_{n_2}} \operatorname{sgn}(\sigma) \prod_{j=1}^{n_2} \left( \frac{1 - w_{\sigma(j)}}{w_{\sigma(j)}} \right)^{j} \left( \sum_{x \in W_{n_2}} \sum_{j=1}^{n_2} \frac{1}{w_{\sigma(j)}} \right) = (-1)^{\frac{n_2(n_2-1)}{2}} \prod_{j=1}^{n_2} \frac{1}{w_j} \det \left( w_j^{-1} \right)_{1 \leq i,j \leq n_2}.
\]

We can now use this identity in (2.13) and obtain

(2.15)

\[
\sum_{x \in W_{n_2}} \det \left( \frac{(1-q)^{\Delta m}}{2\pi i} \int_{\gamma_{r_2}} \left( 1 - \frac{w_i}{w_j} \right)^{j-i} \frac{dw_i}{(1 - qw_i)^{\Delta m} w_i^{y_j-x_i+1}} \right)_{1 \leq i,j \leq n_2}
\]

\[
= \frac{(1-q)^{\Delta m n_2}}{(2\pi i)^{n_2}} \int_{\gamma_{r_2}} d^{n_2} w \prod_{j=1}^{n_2} \left( \frac{w_j}{1 - w_j} \right)^{j} \frac{x_j^{-1}}{(1 - qw_j)^{\Delta m}} \det \left( w_j^{-1} \right)_{1 \leq i,j \leq n_2}.
\]

Next, insert (2.12) and (2.15) into (2.11) to get

(2.16)

\[
P = \sum_{u=-\infty}^{v_1} \sum_{x \in W_{n_2}} \left( \frac{1-q}{(2\pi i)^{2n_2}} \int_{\gamma_{r_1}} d^{n_2} z \int_{\gamma_{r_2}} d^{n_2} w \det \left( z_j^{-1} \right)_{1 \leq i,j \leq n_2} \det \left( w_j^{-1} \right)_{1 \leq i,j \leq n_2} \right)
\]

\[
\times \prod_{j=1}^{n_2} \left( \frac{1 - z_j}{z_j} \right)^{j} \left( \frac{w_j}{1 - w_j} \right)^{j} \frac{1}{(1-qz_j)^{n_1} z_j^{y_j-x_j}} \frac{w_j^{x_j-y_j}}{(1 - qw_j)^{\Delta m}}.
\]
In this expression we symmetrize in \( \{z_j\} \) and \( \{w_j\} \). We find

\[
(2.17) \quad P = \sum_{u=-\infty}^{v_1} \sum_{x \in W_{n_2} \, \text{if } x_{n_1} = u} \frac{(1-q)^{m_2n_2}(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2m_2(n_2)!}} \int_{\gamma^*_i} d^{n_2}z \int_{\gamma^*_j} d^{n_2}w \det \left( z_j^{i-1} \right)_{1 \leq i,j \leq n_2} \det \left( w_j^{i-1} \right)_{1 \leq i,j \leq n_2}
\]

\[
\times \prod_{j=1}^{n_2} \frac{1}{(1-qz_j)^{m_1} (1-z_j)^{n_2} w_j^{-n_2+1}(1-qw_j)\Delta m} \times \left( \sum_{\sigma_1, \sigma_2 \in S_n} \text{sgn}(\sigma_1 \sigma_2) \prod_{j=1}^{n_2} \left( \frac{1-z_{\sigma_1(j)}}{z_{\sigma_1(j)}} \right)^j \left( \frac{w_{\sigma_2(j)}}{1-w_{\sigma_2(j)}} \right)^j \left( \frac{w_{\sigma_2(j)}}{z_{\sigma_1(j)}} \right)^{x_j} \right).
\]

Let \( (S_j^-, S_j^+) \), \( j = 1, 2 \), be two partitions of \([1, n_2] = \{1, \ldots, n_2\}\), such that \( |S_j^-| = n_1 \) and \( |S_j^+| = \Delta n \). For \( \sigma_1, \sigma_2 \in S_{n_2} \), we say that \( \sigma_j \in S_{n_2}(S_j^-, S_j^+) \) if \( \sigma_j([1, n_1]) = S_j^- \) and consequently \( \sigma_j([n_1 + 1, n_2]) = S_j^+ \), \( j = 1, 2 \). Write

\[
\sigma_j^- = \sigma_j \big|_{[1, n_1]}, \quad \sigma_j^+ = \sigma_j \big|_{[n_1+1, n_2]}, \quad j = 1, 2,
\]

for the restricted permutations. Given \( S_j^- \) (\( S_j^+ \)) we can identify \( \sigma_j^- \) (\( \sigma_j^+ \)) with a permutation in \( S_{n_1} \) (\( S_{\Delta n} \)) and we have

\[
(2.18) \quad \text{sgn}(\sigma_j) = (-1)^{\kappa(S_j^-, S_j^+)} \text{sgn}(\sigma_j^-) \text{sgn}(\sigma_j^+),
\]

where

\[
\kappa(U, V) = |\{(i, j) ; i \in U, j \in V, i > j\}|.
\]

We will now choose our radii in the circles in the contour integrals depending on \( S_j^\pm \). Recall that

\[
0 < s_1 < r_1 < 1, \quad 0 < r_2 < s_2 < 1,
\]

which we assumed in the proposition. Given \( S_j^- \), \( j = 1, 2 \), we take

\[
(2.19) \quad |z_k| = s_1, \quad k \in S_1^-, \quad |w_k| = r_1, \quad k \in S_2^-;
\]

\[
|z_k| = s_2, \quad k \in S_1^+, \quad |w_k| = r_2, \quad k \in S_2^+.
\]

We can write

\[
(2.20) \quad \prod_{j=1}^{n_2} \left( \frac{1-z_{\sigma_1(j)}}{z_{\sigma_1(j)}} \right)^j \left( \frac{w_{\sigma_2(j)}}{1-w_{\sigma_2(j)}} \right)^j \left( \frac{w_{\sigma_2(j)}}{z_{\sigma_1(j)}} \right)^{x_j} = \prod_{j=1}^{n_1} \left( \frac{1-z_{\sigma_1^-(j)}}{z_{\sigma_1^-(j)}} \right)^j \left( \frac{w_{\sigma_1^-(j)}}{1-w_{\sigma_1^-(j)}} \right)^j \left( \frac{w_{\sigma_1^-(j)}}{z_{\sigma_1^+(j)}} \right)^{x_j} \prod_{j=n_1+1}^{n_2} \left( \frac{1-z_{\sigma_1^+(j)}}{z_{\sigma_1^+(j)}} \right)^j \left( \frac{w_{\sigma_1^+(j)}}{1-w_{\sigma_1^+(j)}} \right)^j \left( \frac{w_{\sigma_1^+(j)}}{z_{\sigma_1^+(j)}} \right)^{x_j}
\]
From (2.17), (2.18) and (2.20) we find

\[(2.21) \quad P = \sum_{S_1^- \in S_1^-} \sum_{S_2^+ \in S_2^+} (-1)^{n(S_1^- \times S_2^+)} \sum_{x_{n1} = \infty}^{v_1} \frac{(1 - q)^{n(S_1^- \times S_2^+)}(1 - \frac{n(S_1^- \times S_2^+)}{2})}{(2\pi i)^{2n(S_1^- \times S_2^+)}}, \]

\[\times \int_{\sigma_1^-} \prod_{S_1^-} \int_{\Delta_1} \prod_{j \in S_1^+} dz_j \int_{\sigma_1^+} \prod_{j \in S_1^+} \prod_{j \in S_2^+} dw_j \int_{\sigma_2^+} \prod_{j \in S_2^+} dw_j \]

\[\times \det \left( z_j^{-1} \right)_{1 \leq i, j \leq n_2} \det \left( w_j^{-1} \right)_{1 \leq i, j \leq n_2} \prod_{j = 1}^{n_2} \frac{1}{(1 - q z_j)^{n_1(1 - z_j)^{n_2} w_j^{n_2 + n_2 + 1}(1 - q w_j)^{\Delta m}}\]

\[\times \sum_{x \in W_{n_2}} \left( \sum_{\sigma_1^-} \sum_{\sigma_2^-} \left( \frac{1 - z_{\sigma_1^-}(j)}{z_{\sigma_1^-}(j)} \right)^j \left( \frac{w_{\sigma_2}(j)}{1 - w_{\sigma_2}(j)} \right)^j \left( \frac{w_{\sigma_2}(j)}{z_{\sigma_2^+}(j)} \right)^{x_j} \right) \]

\[\times \left( \sum_{\sigma_1^+} \sum_{\sigma_2^+} \left( \frac{1 - z_{\sigma_1^+}(j)}{z_{\sigma_1^+}(j)} \right)^j \left( \frac{w_{\sigma_2}(j)}{1 - w_{\sigma_2}(j)} \right)^j \left( \frac{w_{\sigma_2}(j)}{z_{\sigma_2^+}(j)} \right)^{x_j} \right). \]

The next step is to do the \( x \)-summations,

\[(2.22) \quad \sum_{x_1 \leq \cdots \leq x_{n1 - 1}} \sum_{\sigma_1^-} \prod_{j = 1}^{n_1} \left( \frac{w_{\sigma_2}(j)}{z_{\sigma_1^-}(j)} \right)^{x_j} \]

\[= \sum_{j = 1}^{n_1} \frac{w_{\sigma_2}(j)}{z_{\sigma_1^-}(j)} \frac{1}{\left( 1 - \frac{z_{\sigma_1^-}}{w_{\sigma_2}} \right) \left( 1 - \frac{z_{\sigma_1^-}(2)}{w_{\sigma_2}(2)} \right) \cdots \left( 1 - \frac{z_{\sigma_1^-}(n_1 - 1)}{w_{\sigma_2}(n_1 - 1)} \right)}, \]

since \( |w_{\sigma_2}(j)/z_{\sigma_1^-}(j)| = r_1/s_1 > 1 \). Also,

\[(2.23) \quad \sum_{u \leq x_{n1 + 1} \leq \cdots \leq x_{n2}} \prod_{j = n_1 + 1}^{n_2} \left( \frac{w_{\sigma_2}(j)}{z_{\sigma_1^+}(j)} \right)^{x_j} \]

\[= \prod_{j = n_1 + 1}^{n_2} \frac{w_{\sigma_2}(j)}{z_{\sigma_1^+}(j)} \frac{1}{\left( 1 - \frac{w_{\sigma_2}^+}{w_{\sigma_2}(n_2)} \right) \left( 1 - \frac{w_{\sigma_2}^+ \cdots w_{\sigma_2}^+}{w_{\sigma_2}(n_2) \cdots w_{\sigma_2}(n_2 - 1)} \right) \cdots \left( 1 - \frac{w_{\sigma_2}^+ \cdots w_{\sigma_2}^+}{w_{\sigma_2}(n_2) \cdots w_{\sigma_2}(n_2 + 1)} \right)}, \]

since \( |w_{\sigma_2}(j)/z_{\sigma_1^+}(j)| = r_2/s_2 < 1 \).

We can now apply (2.10) in lemma 2.3 to see that

\[(2.24) \quad \sum_{\sigma_1^+, \sigma_2^-} \sum_{\sigma_1^-} \sum_{\sigma_2^+} \prod_{j = 1}^{n_1} \left( \frac{w_{\sigma_2}(j)}{z_{\sigma_1^-}(j)} \right)^j \frac{1}{\left( 1 - \frac{z_{\sigma_1^-}(1)}{w_{\sigma_2}} \right) \left( 1 - \frac{z_{\sigma_1^-}(2)}{w_{\sigma_2}^+} \right) \cdots \left( 1 - \frac{z_{\sigma_1^-}(n_1 - 1)}{w_{\sigma_2}(n_1 - 1)} \right)}. \]

\[= \left( 1 - \prod_{j \in S_1^-} z_j \prod_{j \in S_2^+} w_j \right) \prod_{j \in S_1^-} \left( 1 - z_j \right)^{n_1} \prod_{j \in S_2^+} \left( 1 - w_j \right)^{n_1} \det \left( \frac{1}{w_j - z_i} \right)_{i \in S_1^-}, \]
By reversing the permutations it also follows from (2.10) that

\[
(2.25) \quad \sum_{\sigma^+_1, \sigma^+_2} \text{sgn}(\sigma^+_1) \text{sgn}(\sigma^+_2) \prod_{j=n_1+1}^{n_2} \left( \frac{w_{\sigma^+_1(j)}(1-z_{\sigma^+_1(j)})}{z_{\sigma^+_1(j)}(1-w_{\sigma^+_2(j)})} \right)^j \times \frac{1}{\prod_{\nu=1}^{n_2} (1 - w_{\sigma^+_2(\nu)}) (1 - w_{\sigma^+_1(\nu)}) (1 - w_{\sigma^+_1(\nu-1)})} \ldots (1 - w_{\sigma^+_1(n_2-1)}) \prod_{\nu=1}^{n_2} \frac{1}{(1 - w_{\nu})^{n_1+1}} \det \left( \frac{1}{z_{\nu} - w_j} \right)_{j \in S^+_2}.
\]

Using (2.22) - (2.25) in (2.21) we find

\[
(2.26) \quad P = \sum_{S^+_1, S^+_2} (-1)^{\kappa(S^-_1, S^+_1) + \kappa(S^-_2, S^+_2)} \sum_{u=-\infty}^{v_1} \frac{(1 - q)^{m_2} (1 - q^{n_2-1})^u}{(2\pi i)^{2m_2(n_2)!}} \prod_{j \in S^-_1} \prod_{j \in S^-_2} \frac{dz_j}{\gamma_{\Delta_0}} \prod_{j \in S^-_1} \prod_{j \in S^-_2} \frac{dz_j}{\gamma_{\Delta_0}} \prod_{j \in S^-_1} \prod_{j \in S^-_2} \frac{dw_j}{\gamma_{\Delta_0}} \times \det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} \det \left( w_j^{i-1} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{w_j - z_i} \right)_{i \in S^-_1, j \in S^-_2} \det \left( \frac{1}{z_i - w_j} \right)_{i \in S^-_1, j \in S^-_2} \times \prod_{j=1}^{n_2} \left( 1 - \prod_{z_j \in S^-_1} \frac{1}{w_j} \right) \prod_{j=1}^{n_2} \left( 1 - q z_j \right)^{m_1} \left( 1 - z_j \right)^{n_2} \left( 1 - w_j \right)^{m_1} \left( 1 - q^{n_2-1} w_j \right)^{n_2+1} \Delta^m.
\]

To see that the summation over \(S^-_1, S^-_2\) in (2.26) is actually trivial we use the following observation. Write

\[
\Delta_S(z) = \prod_{j < k, j, k \in S} (z_k - z_j)
\]

for \(S \subseteq [1,n_2]\). Then,

\[
\det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} = \Delta_{S^-_1}(z) \Delta_{S^+_1}(z) \prod_{j \in S^-_1, k \in S^+_1} (z_k - z_j) (-1)^{\kappa(S^-_1, S^+_1)}.
\]

If we insert this into (2.26) for both \(z\) and \(w\) we see that we can relabel the indices

\[
(z_j)_{j \in S^-_1} \rightarrow (z_j)_{j=1}^{n_1}, \quad (z_j)_{j \in S^+_1} \rightarrow (z_j)_{j=n_1+1}^{n_2+1},
\]

\[
(w_j)_{j \in S^-_2} \rightarrow (w_j)_{j=1}^{n_1}, \quad (w_j)_{j \in S^+_2} \rightarrow (w_j)_{j=n_1+1}^{n_2+1},
\]

and then the sums over \(S^-_1, S^-_2\) become trivial. Note that

\[
\sum_{S^-_1} 1 = \binom{n_2}{n_1},
\]

\(i = 1, 2\). Formula (2.26) then reduces to (2.24) and we have proved the proposition. \(\square\)
From proposition 2.2 we can, by a limiting procedure, obtain a corresponding formula in the Brownian directed polymer model.

Let \( \Gamma_d \) denote the vertical straight line contour through \( d \in \mathbb{R} \) oriented upwards, \( \Gamma_d : t \to d + it, \ t \in \mathbb{R} \). Define

\[
Q(h) = \left( -1 \right)^{\frac{n_2(n_2-1)}{2}} \sqrt{2\pi n_1! \Delta n!} \int_{\Gamma_d^{n_1}} d^{n_1} z \int_{\Gamma_d^{\Delta n}} d^{\Delta n} z \int_{\Gamma_d^{n_2}} d^{n_2} w \int_{\Gamma_d^{\Delta n}} d^{\Delta n} w \det \left( z_j^{i-1} \right)_{1 \leq i,j \leq n_2} \det \left( w_j^{i-1} \right)_{1 \leq i,j \leq n_2}
\]

\[
\times \prod_{j=1}^{n_1} e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j} \left( \frac{1}{z_j - w_j} \right) \prod_{j=n_1+1}^{n_2} e^{\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j} \left( \frac{1}{z_j - w_j} \right)
\]

where

\[
d_1 < d_3 < 0, \ d_4 < d_2 < 0.
\]

Here, we have written

\[
\Delta \xi = \xi_2 - \xi_1, \ \Delta \mu = \mu_2 - \mu_1.
\]

We can now state a proposition concerning the joint distribution function that we are interested in in theorem 1.1.

**Proposition 2.4.** Let \( H(\mu, n) \) be defined by (1.1). Then

\[
\frac{\partial}{\partial \xi_1} \mathbb{P} \left[ H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2 \right] = \frac{\partial}{\partial h} \bigg|_{h=0} Q(h).
\]

**Proof.** Just as in (1.23) we have the formula

\[
\mathbb{P} \left[ H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2 \right] = \lim_{T \to \infty} \mathbb{P} \left[ G([\mu_1 T], n_1) \leq \frac{q}{1 - q} [\mu_1 T] + \xi_1 \sqrt{\frac{q}{T}}, G([\mu_2 T], n_2) \leq \frac{q}{1 - q} [\mu_2 T] + \xi_2 \sqrt{\frac{q}{T}} \right].
\]

In the formula (2.7) we assume that we have chosen \( r_1, r_2, s_1, s_2 \) so that

\[
(r_1/s_1)^{n_1} > (s_2/r_2)^{\Delta n},
\]

which can always be done for fixed \( n_1, n_2 \). We can then do the \( u \)-summation in (2.7) to get

\[
\sum_{u=-\infty}^{v_1} \left( \prod_{j=1}^{n_2} \frac{w_j^{v_1}}{z_j^{v_1}} \right)^u = \frac{\prod_{j=1}^{n_2} w_j^{v_1}/z_j^{v_1}}{1 - \prod_{j=1}^{n_2} z_j/w_j}.
\]
Insert this into (2.7), expand the Cauchy determinants and symmetrize. This gives

\[ P = \frac{(1 - q)^{m_2 n_2}(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2 n_1}!(\Delta n)!} \int d^{n_1} z \int d\Delta z \int d^{n_1} w \int d\Delta w 1 - n_1 \sum_{j=1}^{n_1} z_j / w_j \]

\[
\times \det \left( (z_j - 1)^{-1} \right)_{1 \leq i, j \leq n_2} \det \left( (w_j - 1)^{-1} \right)_{1 \leq i, j \leq n_2} \prod_{j=1}^{n_2} 1 - \prod_{j=n_1+1}^{n_2} \frac{1 - z_j / w_j}{1 - w_j} \]

\[
\times \prod_{j=n_1+1}^{n_2} \frac{1 - z_j}{1 - w_j} \prod_{j=1}^{n_2} z_j^{n_1+n_1} (1 - z_j)^{\Delta n} (1 - q z_j)^{m_1} w_j^{n_1+\Delta n} (1 - w_j)^{n_1} (1 - q w_j)^{\Delta n}.
\]

Here, we have also used the fact that \( \det (z_j^{-1}) = \det ((z_j - 1)^{-1}) \). We now want to take the limit in (2.31) using the formula (2.34), i.e. we let

\[ m_i = [\mu_i], v_i = \frac{q}{1 - q} [\mu_i T] + \xi_i \frac{\sqrt{q}}{1 - q} \sqrt{T}, \]

\[ i = 1, 2. \] Let \( \Gamma_d^{(T)} \) be given by \( t \to d + it, |t| \leq \pi \sqrt{q}(1 - q)^{-1} \sqrt{T} \). In (2.34) we make the change of variables

\[ z_j = e^{(1-q)z_j'/T\sqrt{q}}, z_j' \in \Gamma_d^{(T)}, 1 \leq j \leq n_1, \]

\[ z_j = e^{(1-q)z_j'/T\sqrt{q}}, z_j' \in \Gamma_d^{(T)}, n_1 + 1 \leq j \leq n_2, \]

\[ w_j = e^{(1-q)w_j'/T\sqrt{q}}, w_j' \in \Gamma_d^{(T)}, 1 \leq j \leq n_1, \]

\[ w_j = e^{(1-q)w_j'/T\sqrt{q}}, w_j' \in \Gamma_d^{(T)}, n_1 + 1 \leq j \leq n_2, \]

where \( d_i \) satisfy (2.28).

The condition (2.32) becomes

\[ n_1 d_3 + \Delta n d_4 > n_1 d_1 + \Delta n d_2. \]

If we insert (2.36) into (2.34) it is straightforward to take the limit \( T \to \infty \). After some computation we find, using (2.31), that (we have dropped the primes on the \( z \)- and \( w \)-variables)

\[ \mathbb{P} [H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2] \]

\[ = \frac{(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2 n_1}!(\Delta n)!} \int d^{n_1} z \int d\Delta z \int d^{n_1} w \int d\Delta w \sum_{j=1}^{n_1} w_j - z_j \prod_{j=1}^{n_1} \frac{1}{z_j^{n_1} w_j^{n_1} (z_j - w_j)} \prod_{j=n_1+1}^{n_2} \frac{1}{z_j^{\Delta n-1} w_j^{n_1+1} (w_j - z_j)} \]

\[
\times \sum_{j=1}^{n_2} w_j - z_j \int d^{n_1} z \int d\Delta z \int d^{n_1} w \int d\Delta w \det \left( z_j^{-1} \right)_{1 \leq i, j \leq n_2} \det \left( w_j^{-1} \right)_{1 \leq i, j \leq n_2} \]

\[ \frac{e^{-\frac{1}{2} \mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j}}{z_j^{n_1} w_j^{n_1} (z_j - w_j)} \frac{e^{-\frac{1}{2} \mu_2 z_j^2 - \xi_2 z_j + \frac{1}{2} \Delta \mu w_j^2 - \Delta \xi w_j}}{z_j^{n_1} w_j^{n_1} (z_j - w_j)}.
\]

From (2.27) and (2.38) we see that (2.30) follows (recall that \( \Delta \xi = \xi_2 - \xi_1 \)). Note that in \( Q \) the condition (2.37) is no longer important. This completes the proof.
3. Expansion

In order to use the formula (2.30) to prove theorem 1.1 we must rewrite \( Q(h) \) given in (2.27) further so that we can expand it in a way appropriate for the asymptotic analysis. This expansion is similar in some ways to writing a distribution function like (1.3) as a Fredholm expansion. Behind this expansion there is a certain orthogonality related to the orthogonality of the Hermite polynomials. However, this orthogonality is seen at the level of the generating function for the Hermite polynomials. We will prove a lemma which is the first step towards the expansion and which uses an integral formula for the Hermite polynomials.

**Lemma 3.1.** The function \( Q(h) \) defined by (2.27) is also given by

\[
(3.1) \quad Q(h) = \frac{1}{(2\pi i)^{4n_2 n_1} (\Delta n)} \int_{\Gamma_{d_1}} d^{n_1} z \int_{\Gamma_{d_2}} d^{\Delta n} z \int_{\Gamma_{d_3}} d^{n_1} w \int_{\Gamma_{d_4}} d^{\Delta n} w \int_{\gamma_{r_1}} d^{n_2} \zeta \int_{\gamma_{r_2}} d^{n_2} \omega \\
\times \det \left( \frac{1}{\zeta_j^2} \right)_{1 \leq i, j \leq n_2} \det \left( \frac{1}{\omega^{n_2+1-i}} \right)_{1 \leq i, j \leq n_2} \\
\times \prod_{j=1}^{n_1} \frac{z_j^{n_1} w_j^{\Delta n_1} e^{\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu_1 w_j^2 - \Delta \xi w_j} (z_j - w_j)}{z_j^{n_1+1} w_j^{\Delta n_1-1} e^{\mu_1 z_j^2 - \xi_1 z_j + \frac{1}{2} \Delta \mu_1 w_j^2 - \Delta \xi w_j} (w_j - z_j)} \\
\times \prod_{j=n_1+1}^{n_2} \frac{z_j^{n_2} w_j^{\Delta n_2} e^{\mu_2 z_j^2 - \xi_2 z_j + \frac{1}{2} \Delta \mu_2 w_j^2 - \Delta \xi w_j} (z_j - w_j)}{z_j^{n_2+1} w_j^{\Delta n_2-1} e^{\mu_2 z_j^2 - \xi_2 z_j + \frac{1}{2} \Delta \mu_2 w_j^2 - \Delta \xi w_j} (w_j - z_j)},
\]

where

\[
(3.2) \quad d_1 < d_3 < -\max(\tau_1, \tau_2) < 0, \quad d_4 < d_2 < -\max(\tau_1, \tau_2) < 0.
\]

**Proof.** Using the identities

\[
\det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} = (-1)^{n_2 (n_2-1)/2} \prod_{j=1}^{n_2} z_j^{n_2} \det \left( \frac{1}{z_j^i} \right)_{1 \leq i, j \leq n_2}
\]

and

\[
\int_0^{\infty} \frac{u_j^{i-1} e^{u_j z_j}}{(i-1)!} \, du_j = \frac{1}{z_j^i},
\]

\( i \geq 1, \) provided \( \Re z_j < 0, \) we see that

\[
\det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} = (-1)^{n_2 (n_2-1)/2} \prod_{j=1}^{n_2} z_j^{n_2} \det \left( \frac{(-1)^i}{(i-1)!} \int_0^{\infty} u_j^{i-1} e^{u_j z_j} \, du_j \right)_{1 \leq i, j \leq n_2}
\]

\[
= (-1)^{n_2 (n_2-1)/2} \prod_{j=1}^{n_2} z_j^{n_2} \int_{-\infty}^{\infty} \prod_{j=1}^{n_2} e^{(a-x_j) z_j} \det \left( \frac{(x_j - a)^{i-1}}{(i-1)!} \right)_{1 \leq i, j \leq n_2} \, d^n x,
\]

for any \( a \in \mathbb{R}. \) From this we see that

\[
(3.3) \quad \det \left( z_j^{i-1} \right)_{1 \leq i, j \leq n_2} = (-1)^{n_2 (n_2+1)/2} \prod_{j=1}^{n_2} z_j^{n_2} \int_{-\infty}^{\infty} \prod_{j=1}^{n_2} e^{(a-x_j) z_j} \det \left( \frac{x_j^{i-1}}{(i-1)!} \right)_{1 \leq i, j \leq n_2} \, d^n x.
\]
Similarly, we get

\begin{equation}
(3.4) \quad \det \left( w_{ij}^{-1} \right)_{1 \leq i,j \leq n_2} = (-1)^{n_2(n_2+1)/2} \prod_{j=1}^{n_2} w_{n_2}^{n_2} \int_{(-\infty,a]^{n_2}} \prod_{j=1}^{n_2} e^{(b-y_j)w_j} \det \left( \frac{y_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2} d^n w_j,
\end{equation}

for any \( b \in \mathbb{R} \). Choose \( a = \xi_1 \) and \( b = \Delta \xi \). Using the identities \((3.3)\) and \((3.4)\) in \( (2.27)\) we obtain

\begin{equation}
(3.5) \quad Q(h) = \frac{(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{2n_2} n_2! (\Delta n)!} \int_{\Gamma_{1/2}}^{\Gamma_{1/2}} d^n z \int_{\Gamma_{1/2}}^{\Gamma_{1/2}} d^n w \int_{\Gamma_{1/2}}^{\Gamma_{1/2}} d^n v \int_{\Gamma_{1/2}}^{\Gamma_{1/2}} d^n y \det \left( \frac{x_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2} \det \left( \frac{y_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2}
\times \prod_{j=1}^{n_1} z_j^{n_1} w_j^{\Delta n} e^{\Delta y_j z_j^2 - x_j z_j + \Delta \mu w_j^2} - y_j w_j \left( \frac{1}{z_j - w_j} - h \right)
\times \prod_{j=n_1+1}^{n_2} w_j - z_j.
\end{equation}

Let \( H_k(x) = 2^k x^k + \ldots, k \geq 0 \), be the standard Hermite polynomials so that, for any \( a > 0 \),

\begin{equation}
\det \left( \frac{x_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2} = \frac{1}{a^{n_2(n_2-1)/2}} \det \left( (ax_j)^{i-1} \right)_{1 \leq i,j \leq n_2} = \det \left( \frac{H_{i-1}(ax_j)}{(2i-1)(i-1)!} \right)_{1 \leq i,j \leq n_2} = \frac{1}{(a\sqrt{2\mu})^{n_2(n_2-1)/2}} \det \left( \frac{1}{2\pi i} \int_{\gamma_1} e^{ax_j z_j^2} \frac{\xi_j^2}{\xi_j^2} d\xi_j \right)_{1 \leq i,j \leq n_2},
\end{equation}

where we have chosen \( \gamma_1 \) so that \((3.2)\) holds. Take \( a = 1/\sqrt{2\mu_1} \). We have shown that

\begin{equation}
(3.6) \quad \det \left( \frac{x_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2} = \frac{1}{(2\pi i)^{n_2}} \int_{\gamma_1}^{\gamma_1} d^n z \prod_{j=1}^{n_2} e^{x_j z_j - \frac{1}{2} \mu_1 z_j^2} \det \left( \frac{1}{z_j} \right)_{1 \leq i,j \leq n_2}.
\end{equation}

Similarly,

\begin{equation}
(3.7) \quad \det \left( \frac{y_j^{i-1}}{(i-1)!} \right)_{1 \leq i,j \leq n_2} = \frac{(-1)^{n_2(n_2-1)/2}}{(2\pi i)^{n_2}} \int_{\gamma_2}^{\gamma_2} d^n \omega \prod_{j=1}^{n_2} e^{y_j \omega_j - \frac{1}{2} \mu_1 \omega_j^2} \det \left( \frac{1}{\omega_j^{n_2+1}} \right)_{1 \leq i,j \leq n_2},
\end{equation}

where \( \gamma_2 \) satisfies \((3.2)\). If we insert \((3.6)\) and \((3.7)\) into \((3.5)\) the \( x_j \)- and \( y_j \)-integrations become

\[
\int_{-\infty}^{\xi_1} e^{x_j (\zeta_j - z_j)} dx_j = \frac{e^{\xi_1 (\zeta_j - z_j)}}{\zeta_j - z_j}, \quad \int_{-\infty}^{\Delta \xi} e^{y_j (\omega_j - w_j)} dy_j = \frac{e^{\Delta \xi (\omega_j - w_j)}}{\omega_j - w_j},
\]

where the integrals converge because of \((3.2)\). The resulting formula is \((3.1)\).

Write

\begin{equation}
(3.8) \quad G_{n,\mu,\xi}(z) = z^n e^{\mu z^2 - \xi z}
\end{equation}
Recall (3.2). For \(1 \leq k, \ell \leq n_2\), we define \(A_h(\ell, k)\) by

\[
\delta_{\ell k}1(\ell \leq n_1) + A_h(\ell, k)
= \frac{1}{(2\pi i)^4} \int_{\Gamma_{d_1}} dz \int_{\Gamma_{d_3}} dw \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\omega \frac{G_{n_1, \mu_1, \xi_1}(z)G_{\Delta n, \Delta \mu, \Delta \xi}(w)}{G_{k, \mu_1, \xi_1}(\zeta)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(\omega)} \frac{1}{(z - \zeta)(w - \omega)} \left(\frac{1}{z - w} - h\right),
\]

and \(B(\ell, k)\) by

\[
\delta_{\ell k}1(\ell > n_1) + B(\ell, k)
= -\frac{1}{(2\pi i)^4} \int_{\Gamma_{d_2}} dz \int_{\Gamma_{d_4}} dw \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\omega \frac{G_{n_1+1, \mu_1, \xi_1}(z)G_{\Delta n-1, \Delta \mu, \Delta \xi}(w)}{G_{k, \mu_1, \xi_1}(\zeta)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(\omega)} \frac{1}{(z - w)(z - \zeta)(w - \omega)}.
\]

Expand the determinants in (3.11),

\[
\det\left(\frac{1}{\omega^j}\right) = \sum_{\sigma \in S_{n_2}} \text{sgn}(\sigma) \prod_{j=1}^{n_2} \frac{1}{\zeta^{\sigma(j)}},
\]

\[
\det\left(\frac{1}{\omega^{j+1}}\right) = \sum_{\tau \in S_{n_2}} \text{sgn}(\tau) \prod_{j=1}^{n_2} \frac{1}{\omega^{j+1-\tau(j)}}.
\]

From (3.9) and (3.10) it then follows that we can write

\[
Q(h) = \frac{1}{n_1!\Delta n} \sum_{\sigma, \tau \in S_{n_2}} \text{sgn}(\sigma) \prod_{j=1}^{n_1} (\delta_{\tau(j)}\sigma(j)1(\tau(j) \leq n_1) + A_h(\tau(j), \sigma(j)))
\times \prod_{j=n_1+1}^{n_2} (\delta_{\tau(j)}\sigma(j)1(\tau(j) > n_1) + B(\tau(j), \sigma(j))).
\]

This way of writing \(Q(h)\) is useful because (3.11) leads to a determinant expansion of \(Q(h)\), and \(A_h\) and \(B\) can be rewritten in a way that is useful for taking limits, see lemma 4.1.

Write

\[
[a, b]^n = \{(x_1, \ldots, x_n) \in [a, b] ; x_1 < \cdots < x_n\}.
\]

and recall the notation (1.18). By expanding (3.11) we can prove

**Proposition 3.2.** We have the formula

\[
Q(h) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-\Delta n} \det \begin{pmatrix}
B(\mathbf{c}, \mathbf{c}) & B(\mathbf{c}, \mathbf{c}') & B(\mathbf{c}, \mathbf{d}) & B(\mathbf{c}, \mathbf{d}') \\
A_h(\mathbf{c}', \mathbf{c}) & A_h(\mathbf{c}', \mathbf{c}') & A_h(\mathbf{c}', \mathbf{d}) & A_h(\mathbf{c}', \mathbf{d}') \\
A_h(\mathbf{d}, \mathbf{c}) & A_h(\mathbf{d}, \mathbf{c}') & A_h(\mathbf{d}, \mathbf{d}) & A_h(\mathbf{d}, \mathbf{d}') \\
B(\mathbf{d}', \mathbf{c}) & B(\mathbf{d}', \mathbf{c}') & B(\mathbf{d}', \mathbf{d}) & B(\mathbf{d}', \mathbf{d}')
\end{pmatrix}.
\]

**Proof.** Set

\[
E_h(j; \ell, k) = \begin{cases} 
\delta_{\ell k}1(\ell \leq n_1) + A_h(\ell, k) & \text{if } 1 \leq j \leq n_1 \\
\delta_{\ell k}1(\ell > n_1) + B(\ell, k) & \text{if } n_1 < j \leq n_2
\end{cases}
\]

Then, by (3.11),

\[
Q(h) = \frac{1}{n_1!\Delta n} \sum_{\sigma, \tau \in S_{n_2}} \text{sgn}(\sigma) \prod_{j=1}^{n_2} E_h(j; \tau(j), \sigma(j)).
\]
We can write \((\tau \to \tau^{-1})\)

\[
(3.15) \quad Q(h) = \frac{1}{n_1!\Delta n!} \sum_{\sigma, \tau \in S_{n_2}} \text{sgn}(\sigma \tau^{-1}) \prod_{j=1}^{n_2} E_h(\tau(j); j, \sigma(\tau(j)))
\]

\[
= \frac{1}{n_1!\Delta n!} \sum_{\sigma, \tau \in S_{n_2}} \text{sgn}(\sigma) \prod_{j=1}^{n_2} E_h(\tau(j); j, \sigma(j)),
\]

since \(\text{sgn}(\sigma \tau^{-1}) = \text{sgn}(\sigma \tau)\). Let \(J_- \subseteq [1, n_2]\), \(|J_-| = n_1\) and \(J_+ = [1, n_2] \setminus J_-\). Then, by (3.15),

\[
(3.16) \quad Q(h) = \frac{1}{n_1!\Delta n!} \sum_{J_- \sigma \in S_{n_2}} \text{sgn}(\sigma) \sum_{\tau \in S_{n_2}; \tau(J_-) = [1, n_1]} \prod_{j \in J_-} E_h(\tau(j); j, \sigma(\tau(j))) \prod_{j \in J_+} E_h(\tau(j); j, \sigma(\tau(j)))
\]

\[
= \sum_{J_- \sigma \in S_{n_2}} \text{sgn}(\sigma) \prod_{j \in J_-} (\delta_{j, \sigma(j)} 1(j \leq n_1) + A_h(j, \sigma(j))) \prod_{j \in J_+} (\delta_{j, \sigma(j)} 1(j > n_1) + B(j, \sigma(j))),
\]

since

\[
\sum_{\tau \in S_{n_2}; \tau(J_-) = [1, n_1]} 1 = n_1!\Delta n!.
\]

We can rewrite (3.16) as

\[
(3.17) \quad Q(h) = \sum_{J_- \sigma \in S_{n_2}} \text{sgn}(\sigma) \prod_{j \in J_- \cap [1, n_1]} (\delta_{j, \sigma(j)} + A_h(j, \sigma(j))) \prod_{j \in J_- \cap [n_1+1, n_2]} A_h(j, \sigma(j))
\]

\[
\times \prod_{j \in J_+ \cap [1, n_1]} B(j, \sigma(j)) \prod_{j \in J_+ \cap [n_1+1, n_2]} (\delta_{j, \sigma(j)} + B(j, \sigma(j))).
\]

We want to expand the products involving the Kronecker deltas. Let

\[
\gamma = J_+ \cap [1, n_1], \quad \delta = J_- \cap [n_1+1, n_2].
\]

Set \(r = |J_+ \cap [1, n_1]|\). Then, \(|J_- \cap [1, n_1]| = n_1 - r\) and we see that \(0 \leq r \leq n_1\). Since \(|J_-| = n_1\), we get

\[
|J_- \cap [n_1+1, n_2]| = n_1 - |J_- \cap [1, n_1]| = r.
\]

Thus, \(|\gamma| = |\delta| = r\). Given \(\gamma, \delta\) we see that \(J_- = \delta \cup ([1, n_1] \setminus \gamma)\), so \(J_-\) is uniquely determined by \(\gamma, \delta\). Hence, (3.17) can be written as

\[
(3.18) \quad Q(h) = \sum_{r=0}^{\min(n_1, n_2)} \sum_{\gamma, \delta \in S_{n_2}} \text{sgn}(\sigma) \prod_{j \in [1, n_1] \setminus \gamma} (\delta_{j, \sigma(j)} + A_h(j, \sigma(j))) \prod_{j \in \delta} A_h(j, \sigma(j))
\]

\[
\times \prod_{j \in \gamma} B(j, \sigma(j)) \prod_{j \in [n_1+1, n_2] \setminus \delta} (\delta_{j, \sigma(j)} + B(j, \sigma(j))),
\]

where we sum over all \(\gamma, \delta\) such that \(\gamma \subseteq [1, n_1], \delta \subseteq [n_1+1, n_2], |\gamma| = |\delta| = r\).

Now,

\[
(3.19) \quad \prod_{j \in [1, n_1] \setminus \gamma} (\delta_{j, \sigma(j)} + A_h(j, \sigma(j))) = \sum_{\gamma' \subseteq [1, n_1] \setminus \gamma} \prod_{j \in [1, n_1] \setminus (\gamma \cup \gamma')} \delta_{j, \sigma(j)} \prod_{j \in \gamma} A_h(j, \sigma(j))
\]

and

\[
(3.20) \quad \prod_{j \in [n_1+1, n_2] \setminus \delta} (\delta_{j, \sigma(j)} + B(j, \sigma(j))) = \sum_{\delta' \subseteq [n_1+1, n_2] \setminus \delta} \prod_{j \in [n_1+1, n_2] \setminus (\delta \cup \delta')} \delta_{j, \sigma(j)} \prod_{j \in \gamma} B(j, \sigma(j)).
\]
Inserting (3.19) and (3.20) into (3.18) yields

\(Q(h) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-r} \sum_{t=0}^{\frac{\Delta n-r}{n_1-r}} \sum_{\gamma, \gamma', \delta, \delta' \sigma \in S_{n_2}} \text{sgn}(\sigma)\)
\[\times \prod_{j \in [1,n_1] \setminus (\gamma \cup \gamma')} \prod_{j \in [n_1+1,n_2] \setminus (\delta \cup \delta')} \delta_{j,\sigma(j)} B(j, \sigma(j)) A_h(j, \sigma(j)) A_h(j, \sigma(j)) B(j, \sigma(j))\]

where we sum over all \(\gamma, \gamma', \delta, \delta'\) such that

(3.22) \(\gamma, \gamma' \subseteq [1, n_1], \delta, \delta' \subseteq [n_1 + 1, n_2], \gamma \cap \gamma' = \emptyset, \delta \cap \delta' = \emptyset,\)
\(|\gamma| = |\delta| = r, |\gamma'| = s, |\delta'| = t.\)

Let \(\Lambda = \gamma \cup \gamma' \cup \delta \cup \delta'\) and \(L = |\Delta| = 2r + t + s.\) Terms in (3.23) are \(\neq 0\) only if \(\sigma(j) = j\) for \(j \in [1, n_2] \setminus \Lambda.\) The permutation \(\sigma\) is then reduced to a bijection \(\tilde{\sigma} : \Lambda \to \Lambda.\) Let \(\Lambda = \{\lambda_1, \ldots, \lambda_L\},\) where \(\lambda_1 < \cdots < \lambda_L.\) Then \(\tilde{\sigma}\) is a permutation of \(\Lambda\) and we have \(\text{sgn}(\tilde{\sigma}) = \text{sgn}(\sigma).\) Thus,

(3.23) \(Q(h) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-r} \sum_{t=0}^{\frac{\Delta n-r}{n_1-r}} \sum_{\gamma, \gamma', \delta, \delta' \sigma : \Lambda \to \Lambda} \text{sgn}(\tilde{\sigma})\)
\[\times \prod_{j \in \gamma} B(j, \tilde{\sigma}(j)) A_h(j, \tilde{\sigma}(j)) A_h(j, \tilde{\sigma}(j)) B(j, \tilde{\sigma}(j)).\]

Define,

(3.24) \(T_h(j; \ell, k) = \begin{cases} B(\ell, k) & \text{if } j \in \gamma \\ A_h(\ell, k) & \text{if } j \in \gamma' \\ A_h(\ell, k) & \text{if } j \in \delta \\ B(\ell, k) & \text{if } j \in \delta'. \end{cases}\)

Then, by (3.23),

(3.25) \(Q(h) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-r} \sum_{t=0}^{\frac{\Delta n-r}{n_1-r}} \sum_{\gamma, \gamma', \delta, \delta' \sigma : \Lambda \to \Lambda} \text{sgn}(\tilde{\sigma}) \prod_{j \in \Lambda} T_h(j; \tilde{\sigma}(j)).\)

Define \(\tau \in S_L\) by \(\tilde{\sigma}(\lambda_j) = \lambda_{\tau(j)}\), \(1 \leq j \leq L.\) Then \(\text{sgn}(\tilde{\sigma}) = \text{sgn}(\tau)\) and we find

(3.26) \(Q(h) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-r} \sum_{t=0}^{\frac{\Delta n-r}{n_1-r}} \text{sgn}(\tau) \prod_{i=1}^{L} T_h(\lambda_i; \lambda_i, \lambda_{\tau(i)}).
= \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1-r} \sum_{t=0}^{\frac{\Delta n-r}{n_1-r}} \text{det}(T_h(\lambda_i; \lambda_i, \lambda_j)).\)

Let
\[
\gamma = \{c_1, \ldots, c_r\}, \ c = (c_1, \ldots, c_r) \in [1, n_1]^r <,
\gamma' = \{c'_1, \ldots, c'_s\}, \ c' = (c'_1, \ldots, c'_s) \in [1, n_1]^s,<
\delta = \{d_1, \ldots, d_r\}, \ d = (d_1, \ldots, d_r) \in [n_1 + 1, n_2]^r <,
\delta' = \{d'_1, \ldots, d'_t\}, \ d' = (d'_1, \ldots, d'_t) \in [n_1 + 1, n_2]^t <.
\]

Notice that the determinant in (3.26) is unchanged under permutations of the \(\lambda_i\)'s. Thus we can reorder the \(\lambda_i\)'s in (3.26) so that we get the order \(c_1, \ldots, c_r, c'_1, \ldots, c'_s, d_1, \ldots, d_r, d'_1, \ldots, d'_t.\) Also,
notice that if \( c_i = c_j' \) or \( d_i = d_j' \) for some \( i, j \), then the determinant is \( = 0 \). Hence, we can remove
the restrictions \( \gamma \cap \gamma' = \emptyset \) and \( \delta \cap \delta' = \emptyset \) in (3.22). Note that if e.g. \( s > n_1 - r \), then we must have
\( c_i = c_j' \) for some \( i, j \). Thus, the right side in (3.26) equals the right side in (3.12). \( \square \)

We now want to give expressions for \( A_h \) and \( B \) that will be useful in the asymptotic analysis. First, we need some definitions. Recall the notation (3.8).

Let

\[
0 < \tau_1, \tau_2 < D_1 < D_2 \text{ and define (3.27)}
\]

\[
a_{0,1}(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\Gamma_D} dz \int_{\Gamma_D} dw \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1, \mu_1, \xi_1}(z)G_{\Delta n, \Delta \mu, \Delta \xi}(w)}{G_{n_1, \mu_1, \xi_1}(\zeta)G_{n_2 + 1 - \ell, \Delta \mu, \Delta \xi}(\omega)(z - w)(z - \zeta)(w - \omega)},
\]

(3.28)

\[
b_1(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\Gamma_D} dz \int_{\Gamma_D} dw \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1 + 1, \mu_1, \xi_1}(\zeta)G_{\Delta n - 1, \Delta \mu, \Delta \xi}(w)}{G_{n_1 + 1, \mu_1, \xi_1}(\zeta)G_{n_2 + 1 - \ell, \Delta \mu, \Delta \xi}(\omega)(z - w)(z - \zeta)(w - \omega)}.
\]

Let \( 0 < \tau < D \) and define

\[
c_2(\ell, k) = \frac{1}{(2\pi i)^2} \int_{\Gamma_D} dw \int_{\gamma_r} d\omega \frac{G_{n_2 - k, \Delta \mu, \Delta \xi}(w)}{G_{n_2 + 1 - \ell, \Delta \mu, \Delta \xi}(\omega)(w - \omega)},
\]

(3.29)

\[
c_3(\ell, k) = \frac{1}{(2\pi i)^2} \int_{\Gamma_D} dz \int_{\gamma_r} d\zeta \frac{G_{\ell - 1, \mu_1, \xi_1}(\zeta)}{G_{k_1, \mu_1, \xi_1}(\zeta)(z - \zeta)}.
\]

(3.30)

We now set,

\[
(3.31a) \quad a_{0,2}(\ell, k) = -1(k > n_1)c_2(\ell, k)
\]

(3.31b) \quad a_{0,3}(\ell, k) = 1(\ell \leq n_1)c_3(\ell, k)

(3.31c) \quad b_2(\ell, k) = -1(k > n_1 + 1)c_2(\ell, k)

(3.31d) \quad b_3(\ell, k) = 1(\ell \leq n_1 + 1)c_3(\ell, k)

(3.31e) \quad a_2^*(\ell) = c_2(\ell, n_1)

(3.31f) \quad a_3^*(\ell) = c_3(n_1 + 1, k),

and finally, we define

\[
(3.32a) \quad a_0(\ell, k) = a_{0,1}(\ell, k) - a_{0,2}(\ell, k) - a_{0,3}(\ell, k)
\]

(3.32b) \quad b(\ell, k) = -b_1(\ell, k) + b_2(\ell, k) + b_3(\ell, k)

(3.32c) \quad A_0^*(\ell, k) = -(\delta_{k,n_1 + 1} - a_3^*(k))\delta_{\ell,n_1} - a_2^*(\ell)).

With this notation we can formulate our next lemma.

**Lemma 3.3.** If \( A_h(\ell, k) \) and \( B(\ell, k) \), \( 1 \leq \ell, k \leq n_2 \), are defined by (3.9) and (3.11) then

\[
(3.33) \quad A_0(\ell, k) = a_0(\ell, k),
\]

(3.34) \quad B(\ell, k) = -\delta_{k,n_1 + 1}\delta_{\ell,n_1 + 1} + b(\ell, k)

and

\[
(3.35) \quad \frac{\partial}{\partial h} \bigg|_{h=0} A_h(\ell, k) = A_0^*(\ell, k).
\]

**Proof.** Recall the condition (3.2).

\[
(3.36) \quad d_1 < d_3 < -\max(\tau_1, \tau_2) < 0, \quad d_4 < d_2 < -\max(\tau_1, \tau_2) < 0.
\]

Choose \( D_1, D_2, r_1, r_2, \tau_1, \tau_2 \) so that

\[
(3.37) \quad 0 < \tau_2 < \tau_1 < r_1 < r_2 < D_1 < D_2.
\]
In the integral in the right side of (3.9) we first move $\Gamma_d$ to $\Gamma_{D_2} - \gamma_r$, and then $\Gamma_2$ to $\Gamma_{D_1} - \gamma_r$. This gives
\begin{equation}
\delta_{\ell k}1(\ell \leq n_1) + A_h(\ell, k) = \frac{1}{(2\pi i)^4} \left( \int_{\Gamma_{D_1}} dz \int_{\Gamma_{D_2}} dw \int_{\Gamma_D} dz \int_{\gamma_r} dw \right. \left. - \int_{\gamma_r} dw \int_{\gamma_r} dz \int_{\Gamma_{D_2}} dw \right)
\times \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)G_{n_1,\mu_1,\xi}(\zeta)}{(z-\zeta)(w-\zeta)} \left( \frac{1}{z-w} - h \right) \frac{1}{w-\omega} \left( \frac{1}{z-w} - h \right).
\end{equation}
Consider the last integral in (3.38). The $w$-integral has its only pole in $w = \omega$ and hence it equals
\begin{equation}
\frac{1}{(2\pi i)^3} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)}{G_{n_1,\mu_1,\xi_1}(\zeta)} \omega^{n_1+1-\ell}(z-\zeta) \left( \frac{1}{z-\omega} - h \right).
\end{equation}
In this integral the $z$-integral has poles at $z = \zeta$ and at $z = \omega$, which gives
\begin{equation}
\frac{1}{(2\pi i)^2} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)}{G_{n_1,\mu_1,\xi_1}(\zeta)} \omega^{n_1+1-\ell}(z-\zeta) \left( \frac{1}{z-\omega} - h \right) + \frac{1}{(2\pi i)^2} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(\zeta)}{G_{n_1,\mu_1,\xi_1}(\omega)} \left( \frac{1}{z-\omega} - h \right).
\end{equation}
Since $\tau_2 < \tau_1$, the $\zeta$-integral in the second integral in (3.40) is $= 0$. The first integral in (3.40) equals $\delta_{\ell k}1(\ell \leq n_1) - h\delta_{\ell n_1+1-\ell,n_1}$. Combined with (3.38) this gives,
\begin{equation}
A_h(\ell, k) = -h\delta_{k,n_1+1}\delta_{\ell,n_1} + a_{h,1}(\ell, k) - a_{h,2}(\ell, k) - a_{h,3}(\ell, k),
\end{equation}
where
\begin{equation}
a_{h,1}(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\Gamma_{D_1}} dz \int_{\Gamma_{D_2}} dw \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)G_{n_1,\mu_1,\xi}(\zeta)}{(z-\zeta)(w-\zeta) \left( \frac{1}{z-w} - h \right)}.
\end{equation}
\begin{equation}
a_{h,2}(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)G_{n_1,\mu_1,\xi}(\zeta)}{(z-\zeta)(w-\zeta)} \left( \frac{1}{z-w} - h \right).
\end{equation}
and
\begin{equation}
a_{h,3}(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\Gamma_{D_1}} dz \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)G_{n_1,\mu_1,\xi}(\zeta)}{(z-\zeta)(w-\zeta)} \left( \frac{1}{z-w} - h \right).
\end{equation}
We see that $a_{0,1}(\ell, k)$ in (3.42) agrees with (3.27). Also
\begin{equation}
a_{0,2}(\ell, k) = \frac{1}{(2\pi i)^4} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)G_{n_1,\mu_1,\xi}(\zeta)}{(z-\zeta)(w-\zeta)} \left( \frac{1}{z-w} - h \right).
\end{equation}
The $z$-integral in (3.45) has its only pole in $z = \zeta$ and hence
\begin{equation}
a_{0,2}(\ell, k) = \frac{1}{(2\pi i)^3} \int_{\gamma_r} d\zeta \int_{\gamma_r} d\omega \frac{G_{n_1,\mu_1,\xi_1}(z)G_{n_2+1-\ell,\mu_1,\xi}(w)}{\zeta^{n_1}G_{n_2+1-\ell,\mu_1,\xi}(\zeta)G_{n_1,\mu_1,\xi}(\zeta-w)(w-\zeta)}.
\end{equation}
The $\zeta$-integral is $= 0$ unless $k > n_1$, and if $k > n_1$ the $\zeta$-integral has $\zeta = w$ as its only pole outside $\gamma_{r_1}$. Thus,

$$a_{0,2}(\ell, k) = -\frac{1(k > n_1)}{(2\pi i)^2} \int_{\Gamma D_2} dw \int_{\gamma_{r_2}} d\omega \frac{G_{n_2-\ell, \Delta \mu, \Delta \xi}(w)}{G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w - \omega)} = -1(k > n_1)c_2(\ell, k).$$

This proves (3.33). Similarly, we can show that

$$a_{0,3}(\ell, k) = \frac{1(\ell \leq n_1)}{(2\pi i)^2} \int_{\Gamma D_1} dz \int_{\gamma_{1a1}} d\zeta \frac{G_{\ell-1, \mu_1, \xi}(z)}{G_{k, \mu_1, \xi}(\zeta)(z - \zeta)} = 1(\ell \leq n_1)c_3(\ell, k).$$

Now,

(3.46)

$$\frac{\partial}{\partial h}\bigg|_{h=0} a_{h,1}(\ell, k) = -\left(\frac{1}{(2\pi i)^2} \int_{\Gamma_{r_1}} dz \int_{\gamma_{r_1}} d\zeta \frac{G_{n_2, \mu_2, \xi}(z)}{G_{k, \mu_1, \xi}(\zeta)(z - \zeta)}\right) \left(\frac{1}{(2\pi i)^2} \int_{\Gamma_{r_2}} dw \int_{\gamma_{r_2}} d\omega \frac{G_{\Delta n, \Delta \mu, \Delta \xi}(w)}{G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w - \omega)}\right) = -c_3(n_1 + 1, k)c_2(\ell, n_1) = -a_3^{*}(k)a_2^{*}(\ell).$$

Similarly

(3.47)

$$\frac{\partial}{\partial h}\bigg|_{h=0} a_{h,2}(\ell, k) = -\left(\frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} \zeta n_1 - k d\zeta\right) c_2(\ell, n_1) = -\delta_{k, n_2 + 1}a_2^{*}(\ell),$$

and

(3.48)

$$\frac{\partial}{\partial h}\bigg|_{h=0} a_{h,3}(\ell, k) = -\delta_{\ell, n_1}a_3^{*}(k).$$

If we use (3.46) - (3.48) in (3.41) we see that we have proved (3.35).

Consider next $B(\ell, k)$. In the integral in the right side of (3.10) we first move $\Gamma D_2$ to $\Gamma D_2 - \gamma_{r_1}$ and the $\Gamma D_4$ to $\Gamma D_1 - \gamma_{r_2}$. We obtain

(3.49)

$$\delta_{0k}(\ell > n_1) + B(\ell, k) = \frac{1}{(2\pi i)^4} \left(-\int_{\Gamma D_2} dz \int_{\Gamma D_1} dw + \int_{\Gamma D_2} dw \int_{\gamma_{r_2}} dz + \int_{\gamma_{r_1}} dz \int_{\Gamma D_1} dw\right)$$

$$\times \left(\int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\zeta \frac{G_{n_1+1, \mu_1, \xi}(z)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w)}{G_{k, \mu_1, \xi}(\zeta)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(\zeta)(z - \zeta)(w - \omega)}\right)$$

$$- \frac{1}{(2\pi i)^4} \left(\int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\zeta \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\zeta \frac{G_{n_1+1, \mu_1, \xi}(z)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w)}{G_{k, \mu_1, \xi}(\zeta)G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(\zeta)(z - \zeta)(w - \omega)}\right).$$

Consider the last integral in (3.49). The $z$-integral has its only pole at $z = \zeta$ and hence it equals

(3.50)

$$\frac{1}{(2\pi i)^2} \int_{\gamma_{r_2}} dw \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\zeta \frac{G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w)}{\zeta^{n_1+1}G_{n_2+1-\ell, \Delta \mu, \Delta \xi}(w - \omega)(w - \zeta)}.$$
The \(w\)-integral has poles at \(w = \omega\) and \(w = \zeta\) and consequently (3.50) equals

\[
(3.51) \quad \frac{1}{(2\pi i)^2} \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\omega \frac{\ell^{1-n-k} \omega^{-(n+2)}}{\omega - \zeta} + \frac{1}{(2\pi i)^2} \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\omega \frac{G_{n-2-k, \Delta, \Delta}(\zeta)}{G_{n+1-\ell, \Delta, \Delta}(\omega)(\zeta - \omega)}.
\]

The first integral in (3.51) equals \(-\delta_{\ell,k}(\ell \leq n_1 + 1)\) and in the second one the \(\zeta\)-integral has its only pole at \(\zeta = \omega\) and hence equals \(\delta_{\ell,k}\). Thus, the integral in (3.51) equals \(\delta_{\ell,k}(\ell > n_1 + 1)\) and we see from (3.49) that

\[
B(\ell, k) = -\delta_{k,n_1+1} \delta_{\ell,n_1+1} + \frac{1}{(2\pi i)^4} \left( - \int_{\Gamma_{D_2}} dz \int_{\Gamma_{D_1}} dw + \int_{\Gamma_{D_2}} dz \int_{\Gamma_{D_1}} dw + \int_{\Gamma_{D_2}} dz \int_{\Gamma_{D_1}} dw \right)
\times \int_{\gamma_{r_1}} d\zeta \int_{\gamma_{r_2}} d\omega \frac{G_{n+1, \mu_1, \xi_1}(\zeta) G_{\Delta n-1, \Delta, \Delta}(w)}{G_{\mu_1, \xi_1}(\zeta) G_{\Delta n+1-\ell, \Delta, \Delta}(w)} \frac{1}{(z - w)(z - \zeta)(w - \omega)}.
\]

This leads to the formula (3.34) by using an argument that is analogous to how we proved (3.33). \(\square\)

Before we can carry out the asymptotic analysis of the expression for \(Q(h)\) in (3.12) we have to rewrite it further. Define

\[
(3.52) \quad a_0(\ell, n_1) = a_0(\ell, n_1) + a_2(\ell) = a_0(\ell, n_1) + c_2(\ell, n_1) - 1(\ell \leq n_1)c_3(\ell, n_1).
\]

Set

\[
(3.53) \quad V(\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}') = \begin{pmatrix}
    b(\mathbf{c}, \mathbf{c}) & b(\mathbf{c}, \mathbf{c}') & b(\mathbf{c}, n_1) & b(\mathbf{c}, \mathbf{d}) & b(\mathbf{c}, \mathbf{d}') \\
    a_0(\mathbf{c}', \mathbf{c}) & a_0(\mathbf{c}', \mathbf{c}') & a_0(\mathbf{c}', n_1) & a_0(\mathbf{c}', \mathbf{d}) & a_0(\mathbf{c}', \mathbf{d}') \\
    b(n_{1} + 1, \mathbf{c}) & b(n_{1} + 1, \mathbf{c}') & b(n_{1} + 1, n_1) & b(n_{1} + 1, \mathbf{d}) & b(n_{1} + 1, \mathbf{d}') \\
    a_0(\mathbf{d}, \mathbf{c}) & a_0(\mathbf{d}, \mathbf{c}') & a_0(\mathbf{d}, n_1) & a_0(\mathbf{d}, \mathbf{d}) & a_0(\mathbf{d}, \mathbf{d}') \\
    b(\mathbf{d}', \mathbf{c}) & b(\mathbf{d}', \mathbf{c}') & b(\mathbf{d}', n_1) & b(\mathbf{d}', \mathbf{d}) & b(\mathbf{d}', \mathbf{d}')
\end{pmatrix},
\]

\[
(3.54) \quad U(\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}') = \begin{pmatrix}
    b(\mathbf{c}, \mathbf{c}) & b(\mathbf{c}, \mathbf{c}') & b(\mathbf{c}, n_1) & b(\mathbf{c}, \mathbf{d}) & b(\mathbf{c}, \mathbf{d}') \\
    a_0(\mathbf{c}', \mathbf{c}) & a_0(\mathbf{c}', \mathbf{c}') & a_0(\mathbf{c}', n_1) & a_0(\mathbf{c}', \mathbf{d}) & a_0(\mathbf{c}', \mathbf{d}') \\
    a_0(n_{1} + 1, \mathbf{c}) & a_0(n_{1} + 1, \mathbf{c}') & a_0(n_{1} + 1, n_1) & a_0(n_{1} + 1, \mathbf{d}) & a_0(n_{1} + 1, \mathbf{d}') \\
    a_0(\mathbf{d}, \mathbf{c}) & a_0(\mathbf{d}, \mathbf{c}') & a_0(\mathbf{d}, n_1) & a_0(\mathbf{d}, \mathbf{d}) & a_0(\mathbf{d}, \mathbf{d}') \\
    b(\mathbf{d}', \mathbf{c}) & b(\mathbf{d}', \mathbf{c}') & b(\mathbf{d}', n_1) & b(\mathbf{d}', \mathbf{d}) & b(\mathbf{d}', \mathbf{d}')
\end{pmatrix},
\]

and

\[
(3.55) \quad M_h(\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}') = \begin{pmatrix}
    b(\mathbf{c}, \mathbf{c}) & b(\mathbf{c}, \mathbf{c}') & b(\mathbf{c}, \mathbf{d}) & b(\mathbf{c}, \mathbf{d}') \\
    A_h(\mathbf{c}', \mathbf{c}) & A_h(\mathbf{c}', \mathbf{c}') & A_h(\mathbf{c}', \mathbf{d}) & A_h(\mathbf{c}', \mathbf{d}') \\
    A_h(\mathbf{d}, \mathbf{c}) & A_h(\mathbf{d}, \mathbf{c}') & A_h(\mathbf{d}, \mathbf{d}) & A_h(\mathbf{d}, \mathbf{d}') \\
    b(\mathbf{d}', \mathbf{c}) & b(\mathbf{d}', \mathbf{c}') & b(\mathbf{d}', \mathbf{d}) & b(\mathbf{d}', \mathbf{d}')
\end{pmatrix},
\]

Define

\[
(3.56) \quad Q_1'(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n - 1} \sum_{\mathbf{c} \in [1, n_1]^r \subset 1} \sum_{\mathbf{d} \in [n_1 + 2, n_2]^r \subset 1} \det V(\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}')
\]

and

\[
(3.57) \quad Q_2'(0) = \sum_{r=1}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n - 1} \sum_{\mathbf{c} \in [1, n_1]^r \subset 1} \sum_{\mathbf{d} \in [n_1 + 2, n_2]^r \subset 1} \det U(\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}')
\]
If \( A \) is an \( n \times n \)-matrix and \( 1 \leq i, j \leq n \), we let \( A\{i\}' \), \( \{j\}' \) denote the matrix \( A \) with row \( i \) and column \( j \) removed. Set (recall \( L = 2r + s + t \))

\[
Q_3'(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1]^{j}}^{|c|} \sum_{c' \in [n_1]^i}^{|c'|} \sum_{d \in [n_1]^{e^{i+1}}}^{d_{i+1}-1} \sum_{j=1}^{L} (-1)^{r+s+j} a_3'(f_j) \det M_0(\{r + s\}', \{j\}'),
\]

where we use the notation

\[
f_j = \begin{cases} 
    c_j & \text{if } 1 \leq j \leq r \\
    d_{j-r} & \text{if } r < j \leq r + s \\
    d_{j-r-s} & \text{if } r + s < j \leq 2r + s \\
    d'_{j-2r-s} & \text{if } 2r + s < j \leq L
\end{cases}
\]

Also, set

\[
Q_4'(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1]^{j}}^{|c|} \sum_{c' \in [n_1]^i}^{|c'|} \sum_{d \in [n_1]^{e^{i+1}}}^{d_{i+1}-1} \sum_{j=1}^{L} (-1)^{i+j} a_2'(f_i) a_3'(f_j) \det M_0(\{i\}', \{j\}').
\]

Similarly, we define

\[
Q_5'(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n} \sum_{c \in [n_1]^{j}}^{|c|} \sum_{c' \in [n_1]^i}^{|c'|} \sum_{d \in [n_1]^{e^{i+1}}}^{d_{i+1}-1} \sum_{j=1}^{L} (-1)^{r+s+j} a_3'(f_j) \det M_0(\{r + s\}', \{j\}'),
\]

and

\[
Q_6'(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n} \sum_{c \in [n_1]^{j}}^{|c|} \sum_{c' \in [n_1]^i}^{|c'|} \sum_{d \in [n_1]^{e^{i+1}}}^{d_{i+1}-1} \sum_{j=1}^{L} (-1)^{i+j} a_2'(f_i) a_3'(f_j) \det M_0(\{i\}', \{j\}').
\]

In section 4 we will compute the asymptotics of \( Q_k'(0) \), \( 1 \leq k \leq 6 \), which is all we need because of the next lemma and proposition 2.4.

**Lemma 3.4.** We have the formula

\[
\frac{\partial}{\partial h} \bigg|_{h=0} Q(h) = \sum_{k=1}^{6} Q_k'(0),
\]

with \( Q_k'(0) \) as defined above.

**Proof.** From (3.34) we see that \( B(\ell, k) = b(\ell, k) \) unless \( k = \ell = n_1 + 1 \) in which case \( B(n_1 + 1, n_1 + 1) = -1 + b(n_1 + 1, n_1 + 1) \). This case can occur in the formula (3.12) for \( Q(h) \) if \( d'_{i} = n_1 + 1 \), which requires \( t \geq 1 \). Let \( E_{2r+s+1} \) be the matrix which is zero everywhere except at position \( (2r + s + 1, 2r + s + 1) \) where it is \( 1 \). In the sum in (3.12) we can assume that \( d'_{i} \neq d''_{i} \) since otherwise the determinant is \( 0 \). Hence, by (3.12) we can write

\[
Q(h) = q_0(h) + q_1(h) + q_2(h),
\]
We see from (3.66) that
\[
q_3(h) = \sum_{r=0}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det M_h,
\]
where \(M_h\) is given by (3.55),
\[
q_1(h) = \sum_{r=0}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det (-E_{2r+s+1} + M_h),
\]
and
\[
q_2(h) = \sum_{r=1}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det M_h.
\]
We see from (3.66) that
\[
q_1(h) = \sum_{r=0}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det M_h
- \sum_{r=0}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det M_h := q_3(h) - q_4(h).
\]
Note that
\[
q_0(h) - q_4(h) = \sum_{r=0}^{\min(n_1,\Delta n)} \sum_{s=0}^{\min(n_1,\Delta n)} \sum_{t=0}^{\Delta n} \sum_{d' = n_1 + 1}^{\Delta n} \det M_h = 0
\]
since \([n_1 + 2, n_2]^{\Delta n} = \emptyset\). Thus, by (3.64),
\[
Q(h) = q_2(h) + q_3(h).
\]
If \(A\) is a matrix and \(v\) a row vector, \((A|v)_{\text{row}(i)}\) will denote the matrix obtained by replacing row \(i\) in \(A\) with \(v\). Similarly, if \(v\) is a column vector, \((A|v)_{\text{col}(j)}\) will denote the matrix obtained by replacing column \(j\) in \(A\) with \(v\). Let
\[
v_i = (A^*_0(f_i, c), A^*_0(f_i, c'), A^*_0(f_i, d), A^*_0(f_i, d'))
\]
where $A_0^*$ is given by \[3.32\]; recall \[3.35\]. We see then that

$$
(3.71) \quad q'_3(0) = \frac{\partial}{\partial h} \bigg|_{h=0} q_3(h)
= \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1, n_2]}^r \sum_{d \in [n_1+2, n_2]}^r \sum_{i=r+1}^{2r+s} \det(M_0|v_i)_{row(i)}.
$$

We have to have $r + s \geq 1$ to get a non-zero contribution when taking the $h$-derivative. Similarly,

$$
(3.72) \quad q'_2(0) = \frac{\partial}{\partial h} \bigg|_{h=0} q_2(h)
= \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1, n_2]}^r \sum_{d \in [n_1+2, n_2]}^r \sum_{i=r+1}^{2r+s} \det(M_0|v_i)_{row(i)}.
$$

Expand the determinant in \[3.71\] along row $i$. This gives

$$
(3.73) \quad q'_3(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1, n_2]}^r \sum_{d \in [n_1+2, n_2]}^r \sum_{i=r+1}^{2r+s} \sum_{j=1}^{L} (-1)^{i+j+1} \times (\delta_{f_i, n_1} - a_2^*(f_i)) \left(\delta_{f_i, n_1+1} - a_3^*(f_j)\right) \det(M_0(\{i\}', \{j\}')),
$$

where we have used \[3.32\] and \[3.70\]. Now,

$$
(\delta_{f_i, n_1} - a_2^*(f_i)) \left(\delta_{f_i, n_1+1} - a_3^*(f_j)\right) = \delta_{f_i, n_1} \delta_{f_i, n_1+1} - \delta_{f_i, n_1} a_2^*(f_i) - \delta_{f_i, n_1+1} a_3^*(f_j) + a_2^*(f_i) a_3^*(f_j)
$$

leads to a corresponding decomposition

$$
(3.74) \quad q'_3(0) = q'_{3,1}(0) + q'_{3,2}(0) + q'_{3,3}(0) + q'_{3,4}(0).
$$

The term $\delta_{f_i, n_1} \delta_{f_i, n_1+1}$ requires $j = 2r + s + 1$ and $f_{2r+s+1} = d'_i = n_1 + 1$, and $i = r + s$ and $f_{r+s} = c'_i = n_1$. Hence, $s \geq 1$ and we obtain

$$
(3.75) \quad q'_{3,1}(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=1}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1, n_2]}^r \sum_{d \in [n_1+2, n_2]}^r (-1)^r \det(M_0(\{r+s\}', \{2r+s+1\}')).
$$

The term $-\delta_{f_i, n_1} a_3^*(f_j)$ gives

$$
(3.76) \quad q'_{3,2}(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=1}^{n_1} \sum_{t=1}^{\Delta n} \sum_{c \in [n_1, n_2]}^r \sum_{d \in [n_1+2, n_2]}^r (-1)^r \sum_{j=1}^{L} (-1)^{r+s+1} a_3^*(f_j) \det(M_0(\{r+s\}', \{j\}')).
$$
which is equal to $Q'_3(0)$ as defined by (3.58). The term $-\delta_{f_i,n_1+1}a_2^*(f_i)$ gives

$$q'_3(0) = \min(n_1,\Delta n) \sum_{r=0}^{n_1} \sum_{s=0}^{\Delta n} \sum_{t=1}^{\Delta n} \sum_{i=1}^{2r+s} \sum_{j=1}^{L} (-1)^{i+j+1} a_2^*(f_i) a_3^*(f_j) \det M_0(\{i\}',\{j\}').$$

If we write

$$a_2^* = \begin{pmatrix} 0 \\ a_2^*(c) \\ a_2^*(d) \\ 0 \end{pmatrix},$$

where the blocks have length $r, s, r$ and $t$ respectively, we see that

$$q'_3(0) = \min(n_1,\Delta n) \sum_{r=0}^{n_1} \sum_{s=0}^{\Delta n} \sum_{t=1}^{\Delta n} \sum_{i=1}^{2r+s} \sum_{j=1}^{L} \det(M_0|a_2^*)_{col}(2r+s+1).$$

Finally, we get

$$q'_3(0) = \sum_{r=0}^{n_1} \sum_{s=0}^{\Delta n} \sum_{t=1}^{\Delta n} \sum_{i=1}^{2r+s} \sum_{j=1}^{L} (-1)^{i+j+1} a_2^*(f_i) a_3^*(f_j) \det M_0(\{i\}',\{j\}'),$$

which is $Q'_4(0)$. We can now split (3.72) in the same way,

$$q'_2(0) = q'_{2,1}(0) + q'_{2,2}(0) + q'_{2,3}(0) + q'_{2,4}(0),$$

where

$$q'_{2,1}(0) = \min(n_1,\Delta n) \sum_{r=1}^{n_1} \sum_{s=1}^{\Delta n} \sum_{t=0}^{s} \sum_{i=1}^{2r+s} \sum_{j=1}^{L} \det M_0(\{r+s\}',\{2r+s+1\}'),$$

$q'_{2,2}(0) = Q'_5(0)$, with $Q'_5(0)$ given by (3.61),

$$q'_{2,3}(0) = \min(n_1,\Delta n) \sum_{r=1}^{n_1} \sum_{s=0}^{\Delta n} \sum_{t=0}^{\Delta n} \sum_{i=1}^{2r+s} \sum_{j=1}^{L} \det(M_0|a_2^*)_{col}(r+s+1),$$

and, with $Q'_6(0)$ given by (3.62), $q'_{2,4}(0) = Q'_6(0)$.

From (3.69), (3.71) and (3.72) we see that

$$\frac{\partial}{\partial h} \bigg|_{h=0} Q(h) = q'_{3,1}(0) + q'_{3,3}(0) + q'_{2,1}(0) + q'_{2,3}(0) + \sum_{k=3}^{6} Q'_k(0).$$

In order to prove the lemma it remains to show that

$$Q'_1(0) = q'_{3,1}(0) + q'_{3,3}(0),\quad Q'_2(0) = q'_{2,1}(0) + q'_{2,3}(0).$$
In the expression \((3.75)\) for \(q'_{3,1}(0)\) we move row \(2r + s + 1\) to row \(r + s + 1\). This gives a sign change \((-1)^r\). We then shift the \(s\)-and \(t\)-summations by 1, using the fact that \(d'_1 = n_1 + 1\) and \(c'_s = n_1\) are fixed. This gives

\[
q'_{3,1}(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1 - 1} \sum_{t=0}^{\Delta n - 1} \sum_{\substack{c' \in [1, n_1]_< \times d' \in [n_1 + 2, n_2]_< \times n \geq 1 \times \sum}} \left( \begin{array}{cccc}
\frac{b(c, c)}{a_0(c', c)} & \frac{b(c, c')}{a_0(c', c')}, & \frac{b(c, n_1)}{a_0(c', n_1)} & \frac{b(c, d)}{a_0(c', d)} & \frac{b(c, d')}{a_0(c', d')} \\
\frac{b(n_1 + 1, c)}{a_0(d, c)} & \frac{b(n_1 + 1, c')}{a_0(d, c')} & \frac{b(n_1 + 1, d)}{a_0(d, d)} & \frac{b(n_1 + 1, d')}{a_0(d, d')} \\
\frac{b(d', c)}{a_0(d', c')} & \frac{b(d', d)}{a_0(d', d')} \end{array} \right)
\]

In the expression \((3.78)\) for \(q'_{3,3}(0)\) we move row \(2r + s + 1\) to row \(r + s + 1\) and column \(2r + s + 1\) to column \(r + s + 1\). This gives no net sign change. Note that if \(r + s = 0\) then \(a'_2 = 0\) so we can remove the condition \(r + s \geq 1\) in the summation in \((3.78)\). Also, we shift the \(t\)-summation by 1. We obtain

\[
q'_{3,3}(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1 - 1} \sum_{t=0}^{\Delta n - 1} \sum_{\substack{c' \in [1, n_1]_< \times d' \in [n_1 + 2, n_2]_< \times n \geq 1 \times \sum}} \left( \begin{array}{cccc}
\frac{b(c, c)}{a_0(c', c)} & \frac{b(c, c')}{a_0(c', c')}, & \frac{b(c, d)}{a_0(c', d)} & \frac{b(c, d')}{a_0(c', d')} \\
\frac{b(n_1 + 1, c)}{a_0(d, c)} & \frac{b(n_1 + 1, c')}{a_0(d, c')} & \frac{b(n_1 + 1, d)}{a_0(d, d)} & \frac{b(n_1 + 1, d')}{a_0(d, d')} \\
\frac{b(d', c)}{a_0(d', c')} & \frac{b(d', d)}{a_0(d', d')} \end{array} \right)
\]

Note that the \(c'\)-summation in \((3.84)\) can be extended to \([1, n_1]_<\), since if \(c'_s = n_1\), then two columns in the determinant are equal. Also, we can extend the summation to \(s = n_1\) since in that case we must have \(c'_s = n_1\). We can thus add the two formulas \((3.84)\) and \((3.85)\) and this gives the first formula in \((3.83)\) with \(a_0(\ell, n_1) = a_0(\ell, n_1) + a'_2(\ell)\), which agrees with \((3.52)\). The proof of the second formula in \((3.83)\) is analogous.

\(\square\)

4. Asymptotics and proof of the main theorem

We begin by recalling some notation from section 1. Let \(\lambda_i = \eta_i - \nu_i^2\), \(i = 1, 2\) and write

\[
\Delta \lambda = \lambda_2 \left( \frac{t_2}{\Delta t} \right)^{1/3} - \lambda_1 \left( \frac{t_1}{\Delta t} \right)^{1/3}.
\]

Then,

\[
\Delta \eta = \Delta \lambda + \Delta \nu^2,
\]

where \(\Delta \nu\) is given by \((1.1)\). We will write

\[
N_1 = t_1 M, \quad N_2 = \Delta t M,
\]
where we will let $M \to \infty$ as in theorem 1.1. The scalings in (1.8) and in the arguments $\ell, k$ can then be written

\begin{align}
(4.3) \quad n_1 &= N_1 + \nu_1 N_1^{2/3}, \quad n_2 = t_2 M + \nu_2 (t_2 M)^{2/3}, \quad \Delta n = n_2 - n_1 = N_2 + \Delta \nu N_2^{2/3} \\
\mu_1 &= N_1 - \nu_1 N_1^{2/3}, \quad \mu_2 = t_2 M - \nu_2 (t_2 M)^{2/3}, \quad \Delta \mu = \mu_2 - \mu_1 = N_2 - \Delta \nu N_2^{2/3} \\
\xi_1 &= 2N_1 + \lambda_1 N_1^{1/3}, \quad \xi_2 = 2t_2 M + \lambda_2 (t_2 M)^{1/3}, \quad \Delta \xi = \xi_2 - \xi_1 = 2N_2 + \Delta \lambda N_2^{2/3} \\
\ell &= n_1 + 1 + xN_1^{1/3}, \quad k = n_1 + yN_1^{1/3},
\end{align}

where we have ignored integer parts.

We will now state two lemmas that we will need in order to prove theorem 1.1 from proposition 3.2 and lemma 3.4. The proofs of the lemmas is postponed to section 6.

**Lemma 4.1.** Recall (3.27) to (3.30). Under the scalings (4.3) with $N_1, N_2$ given by (4.2) we have the following limits, uniformly for $\nu, \eta, x, y$ in compact sets,

\begin{align}
(4.4) \quad &\lim_{M \to \infty} N_1^{1/3} a_{0,1}(\ell, k) = \phi_1(x, y), \\
(4.5) \quad &\lim_{M \to \infty} N_1^{1/3} b_1(\ell, k) = \psi_1(x, y), \\
(4.6) \quad &\lim_{M \to \infty} N_1^{1/3} c_2(\ell, k) = \phi_2(x, y), \\
(4.7) \quad &\lim_{M \to \infty} N_1^{1/3} c_3(\ell, k) = \phi_3(x, y),
\end{align}

where $\phi_i, \psi_i$ are given by (1.13) to (1.16).

We will also need some estimates in order to control the convergence of the whole expansion.

**Lemma 4.2.** Assume that we have the scalings (4.3) with $N_1, N_2$ given by (4.2). There are constants $c, C > 0$, which depend on $t, \nu, \eta$, such that for all $M \geq 1$,

\begin{align}
(4.8) \quad &|N_1^{1/3} a_{0,1}(\ell, k)| \leq Ce^{-c(x_+^{3/2} + (-y)^{3/2}) + C(y_+ + (-x_+))}, \\
(4.9) \quad &|N_1^{1/3} b_1(\ell, k)| \leq Ce^{-c(x_+^{3/2} + (-y)^{3/2}) + C(y_+ + (-x_+))}, \\
(4.10) \quad &|N_1^{1/3} c_2(\ell, k)| \leq Ce^{-c(x_+^{3/2} + y_+^{3/2}) + C(y_+ + x_+)}, \\
(4.11) \quad &|N_1^{1/3} c_3(\ell, k)| \leq Ce^{-c((-x)^{3/2} + (-y)^{3/2}) + C(y_+ + x_+)},
\end{align}

for all $1 \leq \ell, k \leq n_2$, where $a_+ = \max(0, a)$.

As an immediate corollary of this lemma and the definitions (3.31), (3.32) and (3.35), we obtain
Corollary 4.3. Assume that we have the scalings (4.3) with $N_1, N_2$ given by (4.2). There are constants $c, C > 0$, which depend on $t_1, v_1, \eta_1$, such that for all $M \geq 1$, and all $1 \leq \ell, k \leq n_2$,

\begin{align*}
(4.12a) & \quad |N_1^{1/3}a_0(\ell, k)| \leq Ce^{-c(x_+^{3/2}+(-y)_+^{3/2})+C(y_+(-x)_+)}, \\
(4.12b) & \quad |N_1^{1/3}b(\ell, k)| \leq Ce^{-c(x_+^{3/2}+(-y)_+^{3/2})+C(y_+(-x)_+)}, \\
(4.12c) & \quad |N_1^{1/3}a_0(\ell, n_1)| \leq Ce^{-cx_+^{3/2}+C(-x)_+}, \\
(4.12d) & \quad |N_1^{1/3}a_2(\ell)| \leq Ce^{-cx_+^{3/2}+C(-x)_+}, \\
(4.12e) & \quad |N_1^{1/3}a_3(\ell)| \leq Ce^{-c(-y)_+^{3/2}+Cy_+}.
\end{align*}

Recall the formula (3.63) in lemma 3.4. We want to control the terms $Q_k(0)$ asymptotically as $M \to \infty$.

Lemma 4.4. We have the following limits.

\begin{align*}
(4.13) & \quad \lim_{M \to \infty} N_1^{1/3}Q_k(0) = 0 \\
& \text{for } 3 \leq k \leq 6,
\end{align*}

\begin{align*}
(4.14) & \quad \Psi^{(1)}(\eta_1, \eta_2) := \lim_{M \to \infty} N_1^{1/3}Q_1(0) \\
& = \sum_{r, s, t=0}^{\infty} \frac{1}{(r!)^2s!t!} \int_{(-\infty, 0)^r} d^r x \int_{(-\infty, 0)^s} d^s x' \int_{[0, \infty)^t} d^t y W^{(1)}_{r,s,r,t}(x, x', y, y'),
\end{align*}

where $W^{(1)}_{r,s,r,t}$ is given by (1.19), and

\begin{align*}
(4.15) & \quad \Psi^{(2)}(\eta_1, \eta_2) := \lim_{M \to \infty} N_1^{1/3}Q_2(0) \\
& = \sum_{r=1, s, t=0}^{\infty} \frac{1}{r!(r-1)!s!t!} \int_{(-\infty, 0)^r} d^r x \int_{(-\infty, 0)^s} d^s x' \int_{[0, \infty)^{r-1}} d^{r-1} y \int_{[0, \infty)^t} d^t y W^{(2)}_{r,s,r-1,t}(x, x', y, y'),
\end{align*}

where $W^{(2)}_{r,s,r,t}$ is given by (1.20).

Proof. Consider $M_0$ given by (3.55) with $h = 0$. Recall (3.38). Let $[M_0]_i$ denote the $i$:th row in $M_0$, and $[M_0]^j$ the $j$:th column. We will use the following scalings

\begin{align*}
(4.16) & \quad c_i = n_1 + x_iN_1^{1/3}, \quad c_i' = n_1 + x_i'N_1^{1/3} \\
& \quad d_i = n_1 + 1 + y_iN_1^{1/3}, \quad d_i' = n_1 + 1 + y_i'N_1^{1/3}
\end{align*}

so that $x_i \leq 0$, $x_i' \leq 0$, $y_i \geq 0$, and $y_i' \geq 0$. Set

$$Y_{\max} = \max_{1 \leq j \leq r} y_j + \max_{1 \leq j \leq t} y_j', \quad X_{\max} = \max_{1 \leq j \leq r} (-x_j) + \max_{1 \leq j \leq s} (-x_j').$$
It follows from corollary 4.3 that under the scaling (4.16) there exist constants $c, C > 0$ such that

\[
\begin{cases}
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{C(Y_{\text{max}} - x_i)} & \text{if } 1 \leq i \leq r \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{C(Y_{\text{max}} - x'_{i-r})} & \text{if } r < i \leq r + s \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{-c(3/2)(-x_{i-r})+C} & \text{if } r + s < i \leq 2r + s \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{-c(3/2)x_{i-(2r+s)}+C} & \text{if } 2r + s < i \leq L
\end{cases}
\]

where $L = 2r + s + t$, and

\[
\begin{cases}
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{(-x_j)^{3/2}+C} & \text{if } 1 \leq j \leq r \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{(-x_{j-r})^{3/2}+C} & \text{if } r < j \leq r + s \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{C(X_{\text{max}} - y_j)} & \text{if } r + s < j \leq 2r + s \\
\|N_1^{1/3}[M_0]\|_2 \leq CL^{1/2}e^{C(X_{\text{max}} - y'_{j-(r+s)})} & \text{if } 2r + s < j \leq L
\end{cases}
\]

From Hadamard’s inequality it follows that

\[
\left|\det(N_1^{1/3}M_0)\right| \leq \prod_{i=1}^{L} \|N_1^{1/3}[M_0]\|_2 \prod_{j=1}^{L} \|N_1^{1/3}[M_0]\|_2.
\]

If we use the estimates (4.17) and (4.18) in (4.19) we see that there are constants $c, C > 0$ such that

\[
\left|\det(N_1^{1/3}M_0)\right| \leq C^LL^{L/2} \prod_{j=1}^{r} e^{(-x_j)^{3/2}} \prod_{j=1}^{s} e^{(-x'_{j})^{3/2}} \prod_{j=1}^{r} e^{-cy_j^{3/2}} \prod_{j=1}^{t} e^{-cy_j^{3/2}}.
\]

Consider now the expression for $Q'_3(0)$ in (3.58). If we use the estimate

\[
\left|N_1^{1/3}a^*(n_1 + yN_1^{1/3})\right| \leq Ce^{(-y)^{3/2}+Cy}
\]

from (4.12) and the same estimates and arguments as above we see that

\[
\left|N_1^{L/3} \sum_{j=1}^{L} (-1)^{r+s+j}a^*_3(f_j) \det M_0(\{r+s\}, \{j\})\right|
\]

\[
\leq C^LL^{L/2} \prod_{j=1}^{r} e^{(-x_j)^{3/2}} \prod_{j=1}^{s} e^{(-x'_{j})^{3/2}} \prod_{j=1}^{r} e^{-cy_j^{3/2}} \prod_{j=1}^{t} e^{-cy_j^{3/2}}.
\]

Note that in (3.58), $y'_1 = 0$ and $x'_1 = 0$, so if we write

\[
N_1^{1/3}Q'_3(0) = \frac{1}{N_1^{1/3}} \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=1}^{n_1} \sum_{t=1}^{\Delta n} \frac{1}{N_1^{(r+s-1)/3}} \sum_{d \in [n_1+2, n_2]^{r-1}} \sum_{c \in [n_1+2, n_2]^{s}} \sum_{c' \in [n_1+2, n_2]^{t}} \sum_{d' \in [n_1+2, n_2]^{r-t}} \left[ \det M_0(\{r+s\}, \{j\}) \right]
\]

we see that we can control the convergence of the Riemann sum using (4.21) (note ordered variables instead of factorials), but since we have the factor $1/N_1^{1/3}$ in front of the whole expression we see
that it → 0 as $M \to \infty$. From (3.60) we can write

$$N_1^{1/3} Q'_3(0) = \frac{1}{N_1^{1/3}} \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n} \frac{1}{N_1^{(r+s)/3}} \sum_{d \in [n_1+2, n_2]^2} \frac{1}{N_1^{(r+t-1)/3}} \times N_1^{(L+1)/3} \sum_{i=r+1}^{2r+s+ L} \sum_{j=1}^{L} (-1)^{i+j+1} a^*_3(f_i) a^*_3(f_j) \det M_0(\{i\}', \{j\}').$$

Using the estimates of $a^*_2$ and $a^*_3$ from corollary 4.3 it follows that we can prove an estimate analogous to (4.21) and again we see that $N_1^{1/3} Q'_3(0) \to 0$ as $M \to \infty$. This proves (4.13) for $k = 3, 4$. The proof for $k = 5, 6$ is a analogous.

Using the estimates in corollary 4.3 we see that in analogy with the proof of (4.20) we can prove

$$(4.22) \quad \left| \det \left( N_1^{1/3} V \right) \right| \leq C L + 1 (L + 1)^{(L+1)/2} \prod_{j=1}^{r} e^{-c(-x_j)^{3/2}} \prod_{j=1}^{s} e^{-c(-x'_j)^{3/2}} \prod_{j=1}^{t} e^{-c y_j^{3/2}}.$$

where $V$ is given by (3.52). From (3.56) we can write

$$N_1^{1/3} Q'_1(0) = \sum_{r=0}^{\min(n_1, \Delta n)} \sum_{s=0}^{n_1} \sum_{t=0}^{\Delta n} \frac{1}{N_1^{(r+s)/3}} \sum_{d \in [n_1+2, n_2]^2} \frac{1}{N_1^{(r+t-1)/3}} \det \left( N_1^{1/3} V \right).$$

It follows from lemma 4.1, 4.31, 3.32 and 3.52 that

$$\lim_{M \to \infty} \det \left( N_1^{1/3} V \right) = W_{r,s,r,t}^{(1)}(x, x', y, y').$$

From the estimate (4.22) we see that we can take the limit in (4.23) and obtain (4.14). The proof of (4.14) is completely analogous.

We now have all the results that we need to prove theorem 1.1.

Proof. (Proof of theorem 1.1) In the scaling (4.3) we see that

$$(4.23) \quad \frac{\partial}{\partial \eta^*_1} P \left[ H(\mu_1, n_1) \leq \xi_1, H(\mu_2, n_2) \leq \xi_2 \right] = \frac{\partial}{\partial h} \bigg|_{h=0} N_1^{1/3} Q(h).$$

From lemma 3.4 and lemma 4.4 we see that

$$(4.24) \quad \lim_{M \to \infty} \frac{\partial}{\partial h} \bigg|_{h=0} N_1^{1/3} Q(h) = \Psi(1)(\eta_1, \eta_2) + \Psi(2)(\eta_1, \eta_2) := \Psi(\eta_1, \eta_2)$$

uniformly for $\eta_1, \eta_2$ in a compact set. Let

$$X_M = \frac{H(\mu_1, n_1) - 2t_1 M}{(t_1 M)^{1/3}} + \nu^*_1, \quad Y_M = \frac{H(\mu_2, n_2) - 2t_2 M}{(t_2 M)^{1/3}} + \nu^*_2.$$

Then (4.23) can be written

$$\frac{\partial}{\partial \eta^*_1} P \left[ X_M \leq \eta_1, Y_M \leq \eta_2 \right] = \frac{\partial}{\partial h} \bigg|_{h=0} N_1^{1/3} Q(h)$$

and for fixed $\eta^*_1$ and $\tilde{\eta}_1$ we see that

$$P \left[ \eta^*_1 < X_M \leq \tilde{\eta}_1, Y_M \leq \eta_2 \right] = \int_{\eta^*_1}^{\tilde{\eta}_1} \frac{\partial}{\partial h} \bigg|_{h=0} N_1^{1/3} Q(h) \, d\eta_1.$$
From (4.24) it follows that

\[ (4.25) \quad \lim_{M \to \infty} \mathbb{P} [\eta_1^* < X_M \leq \tilde{\eta}_1, Y_M \leq \eta_2] = \int_{\eta_1^*}^{\tilde{\eta}_1} \Psi(\eta_1, \eta_2), d\eta. \]

Now,

\[ (4.26) \quad \mathbb{P} [\eta_1^* < X_M \leq \tilde{\eta}_1, Y_M \leq \eta_2] \leq \mathbb{P} [\eta_1^* < X_M \leq \tilde{\eta}_1, Y_M \leq \eta_2] + \mathbb{P} [X_M > \tilde{\eta}_1, Y_M \leq \eta_2] \]

From (1.4), (4.25) and (4.26) we see that

\[ (4.27) \quad \int_{\eta_1^*}^{\tilde{\eta}_1} \Psi(\eta_1, \eta_2), d\eta_1 \leq \lim \inf_{M \to \infty} \mathbb{P} [\eta_1^* < X_M, Y_M \leq \eta_2] \leq \lim \sup_{M \to \infty} \mathbb{P} [\eta_1^* < X_M, Y_M \leq \eta_2] \leq \int_{\eta_1^*}^{\infty} \Psi(\eta_1, \eta_2), d\eta_1 + 1 - F_2(\tilde{\eta}_1). \]

If we let \( \tilde{\eta}_1 \to \infty \) in (4.27) we see that

\[ (4.28) \quad \int_{\eta_1^*}^{\infty} \Psi(\eta_1, \eta_2), d\eta_1, \]

which is what we wanted to prove.

Note that in order for this last argument to work we need an estimate of \( \Psi(\eta_1, \eta_2) \) in terms of \( \eta_1 \). In fact, there are constants \( c, C > 0 \) such that

\[ (4.29) \quad |\Psi(\eta_1, \eta_2)| \leq C e^{-c(\eta_1)^{3/2}}. \]

We will only sketch the argument for (4.29). Note that \( \phi_1, \psi_1 \) and \( \phi_3 \) all have a decay of the form \( e^{-c(\eta_1)^{3/2}} \) in \( \eta_1 \) by known estimates for the Airy function. Hence, the difficulty is in the presence of \( \phi_2 \). If \( r \geq 1 \), the first column in \( W_{r,s,r,t}^{(1)} \) does not depend on \( \phi_2 \) (we can assume \( x_1 < 0 \)) and hence the first column (in a Hadamard estimate) will give the right \( \eta_1 \)-decay. If \( r = 0 \), but \( s \geq 1 \), we can again consider the first column (\( x'_1 < 0 \)), and get the right \( \eta_1 \)-decay. If \( r = s = 0 \),

\[
W_{0,0,0,t}^{(1)}(x, x', y, y') = \begin{vmatrix} \psi(0, 0) & \psi(0, y') \\ \psi(y', 0) & \psi(y', y') \end{vmatrix}
\]

and again the first column does not depend on \( \phi_2 \). The argument for \( W_{r,s,r-1,t}^{(2)} \) is easier since we now always have \( r \geq 1 \).

\[ \square \]

5. PROOF OF COMBINATORIAL IDENTITIES

In this section we will prove lemma 2.3.

Proof. Consider first the identity (2.9). Note that

\[ \prod_{i<j} \left( \frac{1}{w_{\sigma(i)} w_{\sigma(j)}} - \frac{1}{w_{\sigma(i)}} \right) = \prod_{j=1}^{n} \frac{1}{(1 - w_j) w_{\sigma(j)}^{n-j}} \left( \frac{1 - w_{\sigma(j)}}{w_{\sigma(j)}} \right)^j. \]
Hence, the left side of (2.9) can be written

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i<j} \left( \frac{1}{w_{\sigma(i)} w_{\sigma(j)}} - \frac{1}{w_{\sigma(i)}} \right) \frac{\prod_{j=1}^{n} (1 - w_j) w_{\sigma(j)}^{n-1-j}}{(1 - w_{\sigma(1)}) \cdots (1 - w_{\sigma(1)} \cdots w_{\sigma(n)})}
\]

\[
= \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i<j} \left( \frac{1}{w_{\sigma(i)} w_{\sigma(j)}} - \frac{1}{w_{\sigma(i)}} \right) \frac{\prod_{j=1}^{n} w_j^{-2} (1 - w_j)}{(1/w_{\sigma(1)} - 1) \cdots (1/w_{\sigma(1)} \cdots w_{\sigma(n)} - 1)}.
\]

By the identity (1.7) in [20] with \( p = 0, q = 1 \) the last expression equals

\[
\prod_{j=1}^{n} w_j^{-1} \left( \frac{1}{w_j} - 1 \right) \frac{1}{w_j - 1} \det \left( \frac{1}{w_j^{-1}} \right) = (-1)^{n(n-1)/2} \prod_{j=1}^{n} \frac{1}{w_j} \det \left( w_j^{j-1} \right).
\]

This proves (2.9).

We now turn to the proof of (2.10). Denote the left side of (2.10) by \( \omega_n(z,w) \). We will use induction on \( n \). It is easy to see that the identity is true for \( n = 1 \). Fix \( \sigma_1(n) = k \) and \( \sigma_2(n) = \ell \). Then

\[
\sum_{k,\ell=1}^{n} \sum_{\sigma_1, \sigma_2 \in S_n} \text{sgn}(\sigma_1) \text{sgn}(\sigma_2) \left( \frac{1 - z_k}{z_k} \right)^{n} \left( \frac{1 - w_\ell}{w_\ell} \right)^{n} \prod_{j=1}^{n-1} \left( \frac{1 - z_{\sigma_1(j)}}{z_{\sigma_1(j)}} \right) \left( \frac{1 - w_{\sigma_2(j)}}{w_{\sigma_2(j)}} \right) \prod_{j=1}^{n} \left( 1 - \frac{w_{\sigma(j)}}{z_{\sigma(j)}} \right)
\]

\[
= \sum_{k,\ell=1}^{n} \frac{(-1)^{n-k+n-\ell}}{1 - \frac{z_1 \cdots z_n}{w_1 \cdots w_n}} \left( \frac{1 - z_k}{z_k} \right)^{n} \left( \frac{1 - w_\ell}{w_\ell} \right)^{n} \omega_{n-1}(z_1, \ldots, z_k, \ldots, z_n, w_1, \ldots, w_\ell, \ldots, w_n),
\]

where \( \omega_{n-1}(\hat{w}_1, \ldots, \hat{w}_\ell) \) means that we leave out \( z_k(w_\ell) \). By the induction hypothesis the last expression in (5.1) equals

\[
\sum_{k,\ell=1}^{n} \frac{(-1)^{k+\ell}}{1 - \frac{z_1 \cdots z_n}{w_1 \cdots w_n}} \left( \frac{1 - z_k}{z_k} \right)^{n} \left( \frac{1 - w_\ell}{w_\ell} \right)^{n} \prod_{j \neq k} \frac{(z_j - 1)^{n-1}}{w_j} \prod_{j \neq \ell} \frac{(w_j - z_j)^{n-1}}{(1 - w_j)^{n-1}}
\]

\[
\times \det \left( \frac{1}{w_j - z_j} \right)_{1 \leq j,k \leq n} \prod_{j=1}^{\ell - 1} (w_\ell - z_j) \prod_{j=1}^{n} (w_\ell - w_j) \prod_{j=\ell+1}^{n} (w_\ell - w_j) \prod_{j=1}^{k-1} (z_k - z_j) \prod_{j=k+1}^{n} (z_j - z_k)
\]

where we also used the Cauchy determinant formula. The expression in (5.2) can be written

\[
\det \left( \frac{1}{w_j - z_j} \right)_{1 \leq k,l \leq n} \prod_{j=1}^{n} (z_j - 1)^{n-1} (1 - w_j)^{n-1} \sum_{k,\ell=1}^{n} \frac{(1 - z_k) \prod_{j=1}^{n} (w_\ell - z_j)(w_j - z_k) \prod_{j=1}^{\ell - 1} (w_\ell - z_j) \prod_{j=\ell+1}^{n} (w_\ell - w_j) \prod_{j=1}^{k-1} (z_k - z_j) \prod_{j=k+1}^{n} (z_j - z_k)}{1 - \frac{z_1 \cdots z_n}{w_1 \cdots w_n}}
\]
We see from (5.3) and the final formula (2.10) that in order to complete the proof we have to show

\[ \sum_{k,\ell=1}^{n} \frac{(1-z_k)}{z_k(1-w_k)(w_k-z_k)} \prod_{j=1}^{n} \frac{(w_j-z_j)(w_j-z_k)}{(w_j-w_j)(z_j-z_k)} \]

\[ = \prod_{j=1}^{n} \frac{w_j(1-z_j)}{z_j(1-w_j)} \left( 1 - \frac{z_1 \ldots z_n}{w_1 \ldots w_n} \right) = \prod_{j=1}^{n} \frac{w_j(1-z_j)}{z_j(1-w_j)} - \prod_{j=1}^{n} \frac{1-z_j}{1-w_j}. \]

We can assume that \(|z_i|, |w_i| < 1, 1 \leq i \leq n\). Take \(0 < r_1 < r_2 < 1\) such that \(|z_i| < r_1, |w_i| < r_2\) for \(1 \leq i \leq n\). Consider the contour integral

\[ \frac{(-1)^n}{(2\pi i)^2} \int_{\gamma_{r_1}} dz \int_{\gamma_{r_2}} dw \frac{1-z}{z(1-w)(w-z)} \prod_{j=1}^{n} \frac{(w-z_j)(w_j-z)}{(w-w_j)(z-z_j)} = \frac{1}{2\pi i} \int_{\gamma_{r_2}} dw \frac{1}{(1-w)w} \prod_{j=1}^{n} \frac{(w-z_j)w_j}{w-w_j} \]

\[ + \sum_{k=1}^{n} \frac{(-1)^n}{2\pi i} \int_{\gamma_{r_2}} dw \frac{1-z_k}{z_k(1-w)(w-z_k)} \prod_{j=1}^{n} \frac{(w-z_j)(w_j-z_k)}{(w-w_j)(z-z_k)} \prod_{j \neq k} \frac{1}{z_k-z_j}, \]

where we have computed the z-integral. The first expression in the right side of (5.5) can be computed by noticing that the only pole outside \(\gamma_{r_2}\) (including \(\infty\)) is at \(w = 1\) and this gives

\[ \prod_{j=1}^{n} \frac{w_j(1-z_j)}{z_j(1-w_j)} \]

The second expression in the right side of (5.5) equals

\[ - \sum_{k,\ell}^{n} \frac{(1-z_k)}{z_k(1-w_\ell)(w_\ell-z_k)} \prod_{j=1}^{n} \frac{(w_j-z_j)(w_j-z_k)}{(w_j-w_j)(z_j-z_k)} \prod_{j \neq k} \frac{1}{z_k-z_j} \]

and thus by comparing (5.4) and (5.5) we see that it remains to show

\[ \frac{1}{(2\pi i)^2} \int_{\gamma_{r_1}} dz \int_{\gamma_{r_2}} dw \frac{1-z}{z(1-w)(w-z)} \prod_{j=1}^{n} \frac{(w-z_j)(w_j-z)}{(w-w_j)(z-z_j)} = \prod_{j=1}^{n} \frac{1-z_j}{1-w_j}. \]

The w-integral in (5.6) has its only pole outside \(\gamma_{r_2}\) at \(w = 1\) which gives

\[ \frac{1}{2\pi i} \int_{\gamma_{r_1}} dz \prod_{j=1}^{n} \frac{w_j-z_j}{z_j-z_j} \prod_{j=1}^{n} \frac{1-z_j}{1-w_j} = \prod_{j=1}^{n} \frac{1-z_j}{1-w_j} \int_{\gamma_{r_1}} dz \prod_{j=1}^{n} \frac{z_jw_j}{w_jz_j} = \prod_{j=1}^{n} \frac{1-z_j}{1-w_j}, \]

since the only pole in the last z-integral is at \(z = 0\). \(\square\)

### 6. Asymptotic analysis

In this section we will prove lemma (4.1) and lemma (4.2). Recall the notations and scalings (4.1) to (4.3). Define, with \(k\) and \(\ell\) as in (4.3),

\[ f_1(z; x) = (\ell - 1) \log z + \frac{1}{2} \mu_1 z^2 - \xi_1 z \]

\[ f_2(z; y) = (n_2 - k) \log z + \frac{1}{2} \Delta \mu z^2 - \Delta \xi z \]

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and note that \( n_2 - k = \Delta n - y N_1^{1/3} \). Recall the notation (3.8) and the definitions (3.27) - (3.30). We have that

\[
G_{n_1, \mu_1, \xi_1}(z) = e^{f_1(z;0)} \quad G_{\Delta n, \Delta \mu, \Delta \xi}(w) = e^{f_2(w;0)} \\
G_{n_1, \mu_1, \xi_1}(\zeta) = e^{f_1(\zeta;y)} \quad G_{n_2+1-\nu, \Delta \mu, \Delta \xi}(\omega) = e^{f_2(\omega;x)} \\
G_{n_2-k, \mu_2, \Delta \xi}(w) = e^{f_2(w;y)} \quad G_{\nu-1, \mu_1, \xi_1}(z) = e^{f_1(z;x)}
\]

Let \( d_i, 1 \leq i \leq 4, \) be some positive parameters that will be chosen later. Introduce the following contour parametrizations

\[
(6.2) \quad z(t_1) = 1 + (d_1 + it_1)N_1^{-1/3}, \quad t_1 \in \mathbb{R}, \\
\zeta(s_1) = (1 - d_2 N_1^{-1/3}) e^{i s_1 N_1^{1/3}}, \quad s_1 \in I_1 = [-\pi N_1^{1/3}, \pi N_1^{1/3}], \\
w(t_2) = 1 + (d_3 + it_2)N_2^{-1/3}, \quad t_2 \in \mathbb{R}, \\
\omega(s_2) = (1 - d_4 N_2^{-1/3}) e^{i s_2 N_2^{1/3}}, \quad s_2 \in I_1 = [-\pi N_2^{1/3}, \pi N_2^{1/3}].
\]

Define

\[
(6.3) \quad g_1(t_1; x) = \text{Re} f_1(z(t_1); x), \quad h_1(s_1; x) = \text{Re} f_1(\zeta(s_1); x) \\
g_2(t_2; x) = \text{Re} f_2(w(t_2); y), \quad h_2(s_2; y) = \text{Re} f_1(\omega(s_2); y).
\]

Let

\[
(6.4) \quad \Delta_1 = d_1 - \nu_1 + \frac{1}{2}(d_1^2 - 2 \nu_1 d_1 - x) N_1^{-1/3}, \quad \Delta_2 = 2(d_2 + \nu_1) + (\eta_1 - \nu_1^2 - \nu_1 d_2) N_1^{-1/3}, \\
\Delta_3 = d_3 - \Delta \nu + \frac{1}{2}(d_3^2 - 2 \Delta \nu d_3 + y) N_2^{-1/3}, \quad \Delta_4 = 2(d_4 + \Delta \nu) + (\Delta \eta - \Delta \nu^2 - \Delta \nu d_4) N_2^{-1/3}.
\]

**Lemma 6.1.** Assume that, for \( M \) large,

\[
(6.5) \quad 1 \leq d_1 \leq N_1^{1/3}, \quad 1 \leq \Delta_1 \leq N_1^{1/3},
\]

\[
(6.6) \quad 1 \leq d_2 \leq \frac{1}{2} N_1^{1/3}, \quad \Delta_1 \geq 1,
\]

\[
(6.7) \quad 1 \leq d_3 \leq N_3^{1/3}, \quad 1 \leq \Delta_3 \leq N_3^{1/3},
\]

\[
(6.8) \quad 1 \leq d_4 \leq \frac{1}{2} N_2^{1/3}, \quad \Delta_4 \geq 1.
\]

Then,

\[
(6.9) \quad g_1(t_1; x) - g_1(0; x) \leq -\frac{\Delta_1}{20} t_1^2
\]

for all \( t_1 \in \mathbb{R}, \) and

\[
(6.10) \quad h_1(s_1; x) - h_1(0; x) \leq -\frac{\Delta_2}{20} s_1^2
\]

for all \( s_1 \in I_1. \) Furthermore

\[
(6.11) \quad g_2(t_2; y) - g_2(0; y) \leq -\frac{\Delta_3}{20} t_2^2
\]

for all \( t_2 \in \mathbb{R}, \) and

\[
(6.12) \quad h_2(s_2; y) - h_2(0; y) \leq -\frac{\Delta_4}{20} s_2^2
\]

for all \( s_2 \in I_2. \)
Proof. By (6.2) and (6.3),
\[ g_1(t_1; x) = \frac{\ell}{2} \log \left( (1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} \right) + \frac{1}{2} \mu_1 \left( (1 + d_1 N_1^{-1/3})^2 - N_1^{-2/3} t_1^2 \right) - \xi_1 (1 + d_1 N_1^{-1/3}). \]
Thus,
\[ g_1'(t_1; x) = N_1^{-2/3} t_1 \left( \frac{\ell - 1 - \mu_1 \left( (1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} t_1^2 \right)}{(1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} t_1^2} \right), \]
and by introducing the scalings (4.3) we obtain
\[ g_1'(t_1; x) = -t_1 \left( \frac{2 \Delta_1 + \left( N_1^{-1/3} - \nu_1 N_1^{-2/3} \right) t_1^2}{(1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} t_1^2} \right). \]
If \( 0 \leq t_1 \leq N_1^{1/3} \), then \((1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} t_1^2 \leq 5 \) by (6.5), and if \( M \) is large enough \( N_1^{-1/3} - \nu_1 N_1^{-2/3} \geq 0 \), so (6.14) gives
\[ g_1'(t_1; x) \leq -\frac{\Delta_1}{5} t_1. \]
If \( t_1 \geq N_1^{1/3} \) and \( M \) is sufficiently large, then (6.14) gives
\[ g_1'(t_1; x) \leq -t_1 \frac{\frac{1}{2} N_1^{-1/3} t_1^2}{(1 + d_1 N_1^{-1/3})^2 + N_1^{-2/3} t_1^2} \leq -t_1 \frac{\frac{1}{2} N_1^{-1/3} t_1^2}{5 N_1^{-2/3} t_1^2} \leq -\frac{1}{10} N_1^{1/3} \leq -\frac{\Delta_1}{10} t_1, \]
by (6.5). Hence, (6.15) holds for all \( t_1 \geq 0 \), and we have proved (6.9) for \( t_1 \geq 0 \). The case \( t_1 \leq 0 \) follows by symmetry.

Consider now \( h_1 \). We have that
\[ h_1(s_1; x) = (\ell - 1) \log (1 - d_2 N_1^{-1/3}) + \frac{1}{2} \mu_1 (1 - d_2 N_1^{-1/3})^2 \cos 2 N_1^{-1/3} s_1 - \xi_1 (1 - d_2 N_1^{-1/3}) \cos N_1^{-1/3} s_1 \]
and hence
\[ h_1'(s_1; x) = N_1^{-1/3} (1 - d_2 N_1^{-1/3}) \sin N_1^{-1/3} s_1 \left( \xi_1 - 2 \mu_1 (1 - d_2 N_1^{-1/3}) \cos N_1^{-1/3} s_1 \right). \]
From the scaling (4.3) we see that if \( M \) is sufficiently large then \( \xi_1 - 2 \mu_1 (1 - d_2 N_1^{-1/3}) \cos N_1^{-1/3} s_1 \geq N_1^{2/3} \Delta_2 \) and hence,
\[ h_1(s_1; x) - h_1(0; x) \geq N_1^{1/3} (1 - d_2 N_1^{-1/3}) \Delta_2 \int_0^{s_1} \sin N_1^{-1/3} t \, dt \geq \frac{\Delta_2}{2} N_1^{2/3} (1 - \cos N_1^{-1/3} s_1) \]
by (6.6) for all \( s_1 \in I_1 \). If \( |N_1^{-1/3} s_1| \in [0, \pi/2] \), then
\[ \frac{1}{2} (1 - \cos N_1^{-1/3} s_1) = \sin^2 \left( \frac{1}{2} N_1^{-1/3} s_1 \right) \geq \frac{1}{4} N_1^{-2/3} s_1^2 \]
and hence \( h_1(s_1; x) - h_1(0; x) \geq \frac{1}{4} \Delta_2 s_1^2 \). If \( |N_1^{-1/3} s_1| \in [\pi/2, \pi] \), then \( 1 - \cos N_1^{-1/3} s_1 \geq 1 \), and
\[ h_1(s_1; x) - h_1(0; x) \geq \frac{1}{2} \Delta_2 N_1^{2/3} \geq \frac{1}{2 \pi^2} \Delta_2 s_1^2 \geq \frac{\Delta_2}{20} s_1^2. \]
Exactly the same argument gives (6.11) and (6.12). \( \square \)

We will now prove lemma (4.1).
Proof. (Proof of lemma 4.1) All the limits below will be uniform for \( \nu, \eta, x, y \) in compact sets. Write

\[
u d = \text{large that (6.5), (6.6), (6.7) and (6.8) hold for all sufficiently large М. In (3.27) we will use the parametrizations (6.2) and we choose \( d_1 \) and \( d_3 \) so that}
\]

\[
(6.18) \quad \alpha d_3 - d_1 \geq 1
\]

which ensures that the \( z- \) and \( w- \) contours have the right ordering. If we let

\[
J(t_1, s_1, t_2, s_2) = \frac{(1 - d_2 N_1^{-1/3})(1 - d_4 N_1^{-1/3})e^{is_1 N_1^{-1/3} + is_2 N_2^{-1/3}}}{N_1^{2/3} N_2^{1/3} (w(t_1) - w(t_2))(w(t_1) - \zeta(t_1))(w(t_2) - \omega(s_2))}
\]

then

\[
(6.19) \quad \frac{N_1^{1/3} d\zeta dt \omega d\omega}{(z - w)(z - \zeta)(w - \omega)} = \alpha J(t_1, s_1, t_2, s_2) dt_1 ds_1 dt_2 ds_2
\]

and

\[
(6.20) \quad J(t_1, s_1, t_2, s_2) \to \frac{1}{(u_1(t_1) - \nu u_2(t_2))(u_1(t_1) - v_1(s_1))(v_2(t_2) - v_2(s_2))}
\]

as \( M \to \infty \); also \( J \) is bounded. Furthermore,

\[
(6.21) \quad f_1(z(t_1); x) - f_1(1; x) \to \frac{1}{3} u_1(t_1)^3 - \nu u_1(t_1)^2 - (\lambda - x) u_1(t_1),
\]

\[
(6.22) \quad f_1(\zeta(s_1); x) - f_1(1; x) \to \frac{1}{3} v_1(s_1)^3 - \nu v_1(s_1)^2 - (\lambda - x) v_1(s_1),
\]

\[
(6.23) \quad f_2(w(t_2); y) - f_2(1; y) \to \frac{1}{3} u_2(t_2)^3 - \nu u_2(t_2)^2 - (\lambda + y) u_2(t_2),
\]

\[
f_2(\omega(s_2); y) - f_2(1; y) \to \frac{1}{3} v_2(s_2)^3 - \nu v_2(s_2)^2 - (\lambda + y) v_2(s_2)
\]

as \( M \to \infty \).

It follows from (3.27) and (6.1) that

\[
(6.24) \quad N_1^{1/3} a_{0,1}(\ell, k) = \frac{\alpha}{(2\pi)^4} \int_{\mathbb{R}} dt_1 \int_{I_1} ds_1 \int_{\mathbb{R}} dt_2 \int_{I_2} ds_2 J(t_1, s_1, t_2, s_2) \frac{e^{f_1(z(t_1); 0) + f_2(w(t_2); 0)}}{e^{f_1(\zeta(s_1); y) + f_2(\omega(s_2); x)}}.
\]

The integrand in (6.24) is bounded by

\[
C e^{g_1(t_1; 0) + g_2(t_2; 0) - h_1(s_1; y) - h_2(s_2; x)} \leq C e^{g_1(0; 0) + g_2(0; 0) - h_1(0; y) - h_2(0; x) + \frac{1}{20}(t_1^2 + s_1^2 + t_2^2 + s_2^2)} \leq C e^{-\frac{1}{20}(t_1^2 + s_1^2 + t_2^2 + s_2^2)},
\]

where the first inequality follows from lemma 6.1 since \( \Delta \geq 1 \), and the second inequality follows from (6.21) by letting \( t_1 = s_1 = t_2 = s_2 = 0 \) and taking real parts. Thus, by the dominated
convergence theorem we can take the limit $M \to \infty$ in (6.24) and get
(6.25)
\[
\lim_{M \to \infty} N_{1/3}^{1/3} a_{0,1}(\ell, k) = \frac{\alpha}{(2\pi i)^2 \Gamma(d_1) \Gamma(d_3) \Gamma(d_4)} \int_{\Gamma_{d_1}} dz \int_{\Gamma_{d_2}} d\zeta \int_{\Gamma_{d_3}} d\omega \frac{e^{(\frac{1}{3} w^3 - \nu w^2 - (\lambda_1 z)x)}}{(w - \zeta) e^{\frac{1}{3} \nu^2 - \nu \zeta^2 - (\lambda_1 - y)\zeta}}
\]
(6.31)
\[
\lim_{M \to \infty} N_{1/3}^{1/3} c_3(\ell, k) = \frac{\alpha}{(2\pi i)^2 \Gamma(d_1) \Gamma(d_2)} \int_{\Gamma_{d_1}} dz \int_{\Gamma_{d_2}} d\zeta \int_{\Gamma_{d_3}} d\omega \frac{e^{(\frac{1}{3} w^3 - \nu w^2 - (\lambda_1 z)x)}}{(w - \zeta) e^{\frac{1}{3} \nu^2 - \nu \zeta^2 - (\lambda_1 - y)\zeta}}
\]
where $\phi_1$ is given by (1.12). Recall the condition in (6.25). The last equality is a straightforward rewriting of the contour integral in terms of Airy functions, see the end of this section. This proves (4.3). The limit of $N_{1/3}^{1/3} b_1(\ell, k)$ is the same as the right side of (6.25), but we have the condition $d_1 > 2d_3$ instead. For $c_2$ we get
(6.26)
\[
\lim_{M \to \infty} N_{1/3}^{1/3} c_2(\ell, k) = \frac{\alpha}{(2\pi i)^2 \Gamma(d_1) \Gamma(d_3) \Gamma(d_4)} \int_{\Gamma_{d_1}} dz \int_{\Gamma_{d_2}} d\zeta \int_{\Gamma_{d_3}} d\omega \frac{e^{(\frac{1}{3} w^3 - \nu w^2 - (\lambda_1 z)x)}}{(w - \zeta) e^{\frac{1}{3} \nu^2 - \nu \zeta^2 - (\lambda_1 - y)\zeta}}
\]
(6.27)
\[
\lim_{M \to \infty} N_{1/3}^{1/3} c_2(\ell, k) = \frac{\alpha}{(2\pi i)^2 \Gamma(d_1) \Gamma(d_3) \Gamma(d_4)} \int_{\Gamma_{d_1}} dz \int_{\Gamma_{d_2}} d\zeta \int_{\Gamma_{d_3}} d\omega \frac{e^{(\frac{1}{3} w^3 - \nu w^2 - (\lambda_1 z)x)}}{(w - \zeta) e^{\frac{1}{3} \nu^2 - \nu \zeta^2 - (\lambda_1 - y)\zeta}}
\]
\[
\phi_2(x, y),
\]
\[
\phi_3(x, y).
\]

We turn now to the proof of lemma 4.2.

Proof. (Proof of lemma 4.2) To prove the estimate (4.8) we will use (6.24) but we will make appropriate choices of the $d_i$’s in order to get the estimate. From (6.24) we find
(6.28)
\[
N_{1/3}^{1/3} a_{0,1}(\ell, k) \leq \frac{C}{|d_1 - \alpha d_3|(d_1 + d_2)(d_3 + d_4)} \int_{\Gamma_{d_1}} dt_1 \int_{\Gamma_{d_2}} ds_1 \int_{\Gamma_{d_3}} dt_2 \int_{\Gamma_{d_4}} ds_2 e^{g_1(t_1; 0) - h_1(s_1; y) + g_2(t_2; 0) - h_2(s_2; x)}.
\]
We will choose $d_i$ so that the conditions (6.5), (6.6), (6.7), (6.8) and (6.18) are satisfied. Hence, it follows from (6.28) and lemma 6.1 that
(6.29)
\[
N_{1/3}^{1/3} a_{0,1}(\ell, k) \leq e^{g_1(0; 0) - h_1(0; y) + g_2(0; 0) - h_2(0; x)}.
\]
From (6.13), (6.16) and the scalings (4.3) we see that
(6.30)
\[
g_1(0; x) = (N_1 + \nu_1 N_1^{2/3} + x N_1^{1/3}) \log(1 + d_1 N_1^{-1/3}) + \frac{1}{2}(N_1 - \nu_1 N_1^{2/3})(1 + d_1 N_1^{-1/3})^2 - (2 N_1 + \lambda_1 N_1^{1/3})(1 + d_1 N_1^{-1/3})
\]
and
(6.31)
\[
h_1(0; y) = (N_1 + \nu_1 N_1^{2/3} + y N_1^{1/3}) \log(1 - d_2 N_1^{-1/3}) + \frac{1}{2}(N_1 - \nu_1 N_1^{2/3})(1 - d_2 N_1^{-1/3})^2 - (2 N_1 + \lambda_1 N_1^{1/3})(1 - d_2 N_1^{-1/3}).
\]
It is straightforward to show that

\[(6.32)\]
\[\log(1 + x) \leq x - \frac{x^2}{2} + \frac{x^3}{3}\]

for all \(x \geq 0\), and

\[(6.33)\]
\[\log(1 - x) \geq -x - \frac{x^2}{2} - \frac{x^3}{3(1 - x)^3}\]

if \(0 \leq x < 1\). If we use the estimate \((6.32)\) in \((6.30)\) we get

\[(6.34)\]
\[g_1(0; x) \leq -\frac{3}{2} N_1 - \frac{1}{2} \nu_1 N_1^{2/3} - \lambda_1 N_1^{1/3} + \frac{1}{3} d_1^3 \left(1 + \nu_1 N_1^{-1/3} + x N_1^{2/3}\right) - \nu_1 d_1^2 - \lambda_1 d_1 + x \left(d_1 - \frac{1}{2} d_1^2 N_1^{-1/3}\right)\]

Similarly, using \((6.33)\) in \((6.31)\) we find

\[(6.35)\]
\[h_1(0; y) \geq -\frac{3}{2} N_1 - \frac{1}{2} \nu_1 N_1^{2/3} - \lambda_1 N_1^{1/3} + \frac{1}{3} d_2^3 \left(1 + \nu_1 N_1^{-1/3} + y N_1^{2/3}\right) - \nu_1 d_2^2 - \lambda_1 d_2 - y \left(d_2 - \frac{1}{2} d_2^2 N_1^{-1/3}\right)\]

Combining \((6.34)\) and \((6.35)\) we obtain

\[(6.36)\]
\[g_1(0; x) - h_1(0; y) \leq \frac{1}{3} d_1^3 \left(1 + \nu_1 N_1^{-1/3} + x N_1^{2/3}\right) - \nu_1 d_1^2 - \lambda_1 d_1 + x \left(d_1 - \frac{1}{2} d_1^2 N_1^{-1/3}\right) + \frac{1}{3} d_2^3 \left(1 + \nu_1 N_1^{-1/3} + y N_1^{2/3}\right) - \nu_1 d_2^2 - \lambda_1 d_2 + y \left(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}\right)\]

In an analogous way, we obtain

\[(6.37)\]
\[g_2(0; y) - h_2(0; x) \leq \frac{1}{3} d_3^3 \left(1 + \Delta \nu N_2^{-1/3} - y N_2^{2/3}\right) - \Delta \nu d_3^2 - \Delta \lambda d_3 - y \left(d_3 - \frac{1}{2} d_3^2 N_2^{-1/3}\right) + \frac{1}{3} d_4^3 \left(1 + \Delta \nu N_2^{-1/3} - x N_2^{2/3}\right) - \Delta \nu d_4^2 - \Delta \lambda d_4 - x \left(d_4 + \frac{1}{2} d_4^2 N_2^{-1/3}\right)\]

We will use the estimates \((6.36)\) and \((6.37)\) in \((6.29)\). Take

\[(6.38)\]
\[d_1 = k_1, \quad d_2 = k_2 + \delta_2 (-y)^{1/2}, \quad d_3 = k_3, \quad d_4 = k_4 + \delta_4 x^{1/2},\]

where \(k_i\) and \(\delta_i\) are to be specified.

Note that since \(1 \leq l, k \leq n_2\), there is a constant \(k_0\) so that \(|x| \leq k_0 N_1^{1/3}\) and \(|y| \leq k_0 N_1^{1/3}\). First choose \(k_1\) large enough so that \(\Delta_1 \geq 1\) holds. Then \((6.5)\) will hold if \(M\) is large enough. We can choose \(k_2\) so that \(\Delta_2 \geq 1\) and \(d_2 \geq 1\) hold provided that \(d_2 \leq \frac{1}{2} N_1^{1/3}\). Now,

\[d_2 = k_2 + \delta_2 (-y)^{1/2} \leq k_2 + k_0^{1/2} \delta_2 N_1^{1/3} \leq \frac{1}{2} N_1^{1/3}\]

for large \(M\) if we choose \(\delta_2\) small enough. With these choices \((6.5)\) and \((6.6)\) are satisfied for large \(M\). In a similar way we can choose \(k_3, k_4\) and \(\delta_4\) so that \((6.7)\) and \((6.8)\) hold, and we can also choose \(k_3\) so large that \((6.18)\) holds. Note that there is a constant \(C\) so that

\[
\frac{1 + \nu_1 N_1^{-1/3} + y N_1^{-2/3}}{(1 - d_2 N_1^{-1/3})^3} \leq C, \quad \frac{1 + \Delta \nu N_2^{-1/3} - x N_2^{-2/3}}{(1 - d_4 N_2^{-1/3})^3} \leq C
\]
and consequently we see from (6.36) and (6.37) that

(6.39)
\[ g_1(0;0) - h_1(0;y) + g_2(0;0) - h_2(0;x) \leq \frac{1}{3} d_1^3(1 + \nu_1 N_1^{-1/3}) - \nu_1 d_1^2 - \lambda_1 d_1 + + C d_3^2 + \nu_1 d_2^2 - \lambda_1 d_2 \]
\[ + y(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}) + \frac{1}{3} d_3^2(1 + \Delta \nu N_2^{-1/3}) - \Delta \nu d_3^2 - \Delta \lambda d_3 + C d_4^2 + \Delta \nu d_4^2 - \Delta \lambda d_4 - x(d_4 + \frac{1}{2} d_4^2 N_2^{-1/3}) \]
\[ \leq C(1 + d_2 + d_4 + d_4^2 + d_2 + d_4) + y(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}) - x(d_4 + \frac{1}{2} d_4^2 N_2^{-1/3}), \]

since \( d_1 \) and \( d_3 \) are constants. From (6.38) we see that \( d_2^2 \leq 4(k_2^2 + \delta_2^2(-y)^{3/2}) \), \( d_4^2 \leq 2(k_2^2 + \delta_2^2(-y)_+^3) \) and similarly for \( d_4 \). If \( y \geq 0 \), then
\[ y(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}) \leq C y = C y_+, \]

since \( d_2 = k_2 \), and if \( y < 0 \), then
\[ y(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}) \leq d_2 y = k_2 y - \delta_2 (-y)^{3/2} \leq -\delta_2 (-y)^{3/2}. \]

Thus
\[ y(d_2 + \frac{1}{2} d_2^2 N_1^{-1/3}) \leq -\delta_2 (-y)^{3/2} + C y_+ \]

for all \( y \). Similarly,
\[ -x(d_4 + \frac{1}{2} d_4^2 N_2^{-1/3}) \leq -\delta_4 x_+^3 + C(-x)_+. \]

We can pick \( \delta_2 \) so small that
\[ C(\delta_2 (-y)^{3/2} + \delta_2^2 (-y)_+) - \delta_2 (-y)^{3/2} \leq -c(-y)^{3/2}, \]

\( c > 0 \) is a small constant. A similar argument can be done for \( \delta_4 \). Using these estimates in (6.39) it follows from (6.29) that
\[ \left| N_1^{1/3} a_{0,1}(\ell, k) \right| \leq C e^{-c(x^3_+ + (-y)^{3/2}) + C(y_+ + (-x)_+)}, \]

which is what we wanted to prove. The estimates (4.9), (4.10) and (4.11) can be proved in a similar way using (6.36) and (6.37). We will not go into the details.

Let us briefly indicate how we can go from the contour integral form of \( \phi_1(x,y) \) in (6.25) to the Airy form in (1.12). We use the fact that if \( D > 0 \), then

(6.40)
\[ \frac{1}{2\pi i} \int_{\Gamma_D} e^{1/3 z^3 + A z^2 + B z} dz = \text{Ai}(-B + A^2) e^{-AB + \frac{2}{3} A^3} \]

and

(6.41)
\[ \frac{1}{2\pi i} \int_{\Gamma_D} e^{-1/3 z^3 + A z^2 + B z} d\zeta = \text{Ai}(B + A^2) e^{AB + \frac{2}{3} A^3}. \]

Also, we write

(6.42)
\[ \frac{1}{z - \alpha w} = - \int_0^\infty e^{\tau_1(z - \alpha w)} d\tau_1, \quad \frac{1}{z - \zeta} = \int_0^\infty e^{-\tau_2(z - \zeta)} d\tau_2, \quad \frac{1}{w - \omega} = \int_0^\infty e^{-\tau_3(w - \omega)} d\tau_3. \]

If we insert (6.42) into (6.25) and use (1.6), (6.40) and (6.41) we get (1.12) after some manipulations.
