FUNCTIONAL EQUATIONS FOR ORBIFOLD WREATH PRODUCTS

CARLA FARSI AND CHRISTOPHER SEATON

Abstract. We present generating functions for extensions of multiplicative invariants of wreath symmetric products of orbifolds presented as the quotient by the locally free action of a compact, connected Lie group in terms of orbifold sector decompositions. Particularly interesting instances of these product formulas occur for the Euler and Euler–Satake characteristics. This generalizes results known for global quotients by finite groups to all closed, effective orbifolds.

1. Introduction

In [17] and [18], Tamanoi introduced a number of orbifold invariants for global quotients, i.e. orbifolds given by the quotient of a manifold by a finite group, generalizing the orbifold Euler characteristics of [2] and [3]. The basic idea behind these invariants is to apply a multiplicative orbifold invariant $\varphi$, e.g. the Euler–Satake characteristic (see [8]), to a sector decomposition of the orbifold, yielding an extension of this invariant. Tamanoi introduced sector decompositions of global quotients associated to an arbitrary group $\Gamma$, a $\Gamma$–set $X$, and a finite covering space $\Sigma' \to \Sigma$ of a connected manifold $\Sigma$. See also [19] for connections between these extensions and the orbifold elliptic genus.

In [6], for a finitely generated discrete group $\Gamma$, the authors introduced the $\Gamma$–sectors associated to a Lie groupoid $\mathcal{G}$, which generalized Tamanoi’s $\Gamma$–sector decomposition to the case of an arbitrary orbifold. The relationship between this construction, quotient presentations of orbifolds, and generalized loop spaces for orbifolds was studied in [7]. In [8], this relationship was used to extend Tamanoi’s generating functions for the extension by free abelian groups of the Euler and Euler–Satake characteristics of wreath symmetric product orbifolds to the case of orbifolds presented by the quotient of a closed manifold by the locally free action of a compact, connected Lie group. Note that this includes the case of all closed, effective orbifolds by [16, Theorem 2.19].

In this paper, we extend Tamanoi’s generating functional equations for the $\Gamma$–extensions of multiplicative invariants to the case of quotients by compact, connected Lie groups acting locally freely; see Theorem 3.3. For specific multiplicative invariants $\varphi$, these formulas relate the values of an extension of $\varphi$ on the wreath symmetric products $MG(S_n) = M^n \rtimes G(S_n)$ to the value of an extension of $\varphi$ on $M \rtimes G$; see Theorems 3.5 and 3.6. This generalization requires the extension of the sector decompositions associated to $\Gamma$–sets to this case. In addition, we

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adapt results of Dress and Müller [5] for decomposable functors to demonstrate a relationship between the Γ–extension of the Euler–Satake characteristic and the $\Gamma/H$–extensions for global quotient orbifolds, see Theorem 4.2.

The outline of this paper is as follows. In Section 2, we collect background material on $K–G$–bundles for groups $K$ and $G$ and review the classifications given in [14], [15], and [18]. In Section 3, we extend Tamanoi’s generating functions for Γ–extensions of multiplicative invariants of wreath symmetric products to orbifolds presented as quotients by compact, connected Lie groups acting locally freely; see Theorem 3.3. We apply these results to the Euler and Euler–Satake characteristics in Subsection 3.2, resulting in generating functions for these invariants given in Theorems 3.5 and 3.6. We then study extensions of invariants associated to arbitrary finite Γ–sets. In Section 4, we prove Theorem 4.2 which demonstrates for global quotients a relationship between extensions of the Euler–Satake characteristic associated to a group Γ and those associated to transitive Γ–sets using a formal functorial functional equation of Dress and Müller [5] for decomposable functors.

By a quotient orbifold, we mean an orbifold that admits a presentation as a translation groupoid $M \rtimes G$ where $M$ is a smooth manifold and $G$ is a Lie group acting locally freely in such a way that $M \rtimes G$ is Morita equivalent to an orbifold groupoid, see [1]. For brevity, we refer to $M \rtimes G$ as a cc–presentation when in addition $G$ is compact and connected and $M$ is closed. Note in particular that only closed orbifolds admit cc-presentations. All manifolds, orbifolds, and group actions are assumed smooth. Unless stated otherwise, we will always use $M$ to denote a smooth, closed manifold, $G$ to denote a compact Lie group, and $Γ$ to denote a finitely generated discrete group.

2. Classification of $K–G$– Bundles, Crossed Products, and Wreath Products

In this section, we collect results on $K–G$–bundles. We assume that $G$ is a compact Lie group and $K$ is a topological group.

**Definition 2.1 ([10], [14], [18]).** Let $X$ be a topological space.

(i) A $K–G$–bundle over $X$ is a locally trivial $G$–bundle $p: P \to X$ with left $K$–actions on $P$ and $X$ such that the projection map $p$ is $K$–equivariant.

(ii) A $K–G$–principal bundle over $X$ is a locally trivial principal $G$–bundle $p: P \to X$ that is also a $K–G$–bundle such that

$$\gamma(eg) = (\gamma e)g \quad \forall \gamma \in K, e \in P, g \in G.$$ 

In particular, as $P$ is a principal $G$–bundle, $G$ acts on $P$ on the right.

Morphisms of $K–G$–bundles and $K–G$–principal bundles are bundle morphisms, respectively principal bundle morphisms, that are $K$–equivariant. For a given $K–G$–bundle or $K–G$–principal bundle $P$, we let $Aut^G_{K–G}$ denote its automorphism group. A $K–G$–bundle or $K–G$–principal bundle is trivial when it is a product.

We refer to a $K–G$–bundle $P$ as a $K$–irreducible $G$–bundle when the $K$–action on $X$ is transitive; similarly, a $K$–irreducible $G$–principal bundle is a $K–G$–principal bundle where $K$ acts transitively on $X$.

By the associated principal bundle construction, every $K–G$–bundle over $X$ induces a $K–G$–principal bundle over $X$. When $K$ and $G$ are compact Lie and $X$ is
completely regular, every $K$–$G$–bundle over $X$ is locally trivial by [14, Corollary 1.5]. The same holds true if the bundle is smooth [14, Corollary 1.6].

As is noted by Tamanoi in [18, pg. 811], wreath products occur as the automorphism groups of $G$–bundles over finite sets. In particular, we have the following generalization of [18, Lemma 3–3].

Proposition 2.2. The automorphism group of the trivial $G$–bundle $X \times G \to X$ over a discrete space $X$ is equal to $\text{Map}(X, G) \rtimes K$, where $K$ is the permutation group of $X$ and the $K$–action is given by $kf(x) = f(k^{-1}x)$.

Proof. By [12, Chapter 5, Theorem 1.1], automorphisms of $X \times G$ as a principal $G$–bundle that restrict to the identity on $X$ are given by $\text{Map}_G(X \times G, G) = \text{Map}(X, G)$; this correspondence is clearly a group isomorphism. Then it is easy to see that every general automorphism is determined by an element of $\text{Map}(X, G)$ and a homeomorphism of $X$. □

Remark 2.3. In the case $X = n = \{1, 2, \ldots, n\}$, $K$ is the symmetric group $S_n$ and we obtain the standard wreath product $G(S_n)$; see [18, Lemma 3–3]. That is, $G(S_n)$ is the semidirect product of $G^n$ by the action of $S_n$ by permuting factors, so that the operation is given by

$$((g_1, \ldots, g_n), s)((h_1, \ldots, h_n), t) = ((g_1 h_s^{-1(1)}, \ldots, g_n h_s^{-1(n)}), st)$$

for $(g_1, \ldots, g_n), (h_1, \ldots, h_n) \in G^n$ and $s, t \in S_n$.

The $K$–$G$–principal bundles over a finite set $X$ of order $n$ are necessarily trivial, and are classified by conjugacy classes of homomorphisms $\theta: K \to G(S_n)$. Similarly, the $K$–irreducible $G$–principal bundles over $X$ are classified by conjugacy classes of homomorphisms, as explained by the following.

Theorem 2.4 ([18]). Let $K$ and $G$ be any groups, and let $X$ be a finite set of order $n$.

(i) There is a bijective correspondence between the sets

$$\{\text{isomorphism classes of } K\text{–}G\text{–principal bundles over } X \}$$

and

$$\text{HOM}(K, G(S_n)) / G(S_n).$$

(ii) There is a bijective correspondence between the sets

$$\{\text{isomorphism classes of } K\text{–}\text{irreducible } G\text{–principal bundles over } X \}$$

and

$$\bigcup_{(H_n)} \text{HOM}(H_n, G) / (N_K(H_n) \times G)$$

where the union is over $K$–conjugacy classes of subgroups $H_n \leq K$ of index $n$, and the $N_K(H_n) \times G$–action is given by

$$(\rho(u, g))(h) = g^{-1} \rho(u h u^{-1})g$$

for $(u, g) \in N_K(H_n) \times G$. 

Note that if \( \rho: H_n \to G \) is a homomorphism for some \( n \), we use \((\rho)\) to denote the \( G\)-conjugacy class of \( \rho \) and \([\rho]\) to denote the \( N_K(H_n) \times G\)-conjugacy class of \( \rho \).

To conclude this section, we review the classification results for \( K-G\)-bundles given in \[14, 15, and 18\]. Note that \( K-G \)–bundles are equivalent if and only if their associated \( K-G \)–principal bundles are equivalent, see \[13 \text{ pg. 168 and } 14 \text{ pg. 257 and 268}\]. Hence, to classify all \( K-G \)–bundles, it is enough to classify the principal \( K-G \)–bundles.

The following is proved in \[15 \text{ Lemmas 1.1, 1.7, and 1.8}; see also } 19 \text{ Theorem 8.3} \]

**Proposition 2.5** (Classification of \( K-G \)–principal bundles over \( X \)). Let \( X \) be a completely regular topological space, let \( K \) be a compact Lie group, let \( p: P \to X \) be a \( K-G \)–principal bundle, and let \( H \leq K \) be a closed subgroup.

(i) Let \( x \in X^{(H)} \), the fixed point set of \( X \). If \( z \in p^{-1}(x) \), then there is a homomorphism \( \rho: H \to G \) such that \( hz = z\rho(h) \) for each \( h \in H \). We then say that \( z \in (\rho) \), denote by \( X^{(\rho)} \) the set of such \( x \in X \), and \( P^{(\rho)} \) the set of such \( z \). Then \( X^{(\rho)} \) is open in \( X^{(H)} \), and

\[
X^{(H)} = \coprod_{(\rho) \in HOM(H,G)/G} X^{(\rho)}
\]

where \( HOM(H,G)/G \) denotes the set of \( G \)-conjugacy classes of homomorphisms \( \rho: H \to G \).

(ii) We have

\[
X^{(H)} = \coprod_{[\rho] \in HOM(H,G)/(N_K(H) \times G)} (N_K(H)/H) \times_{N_K,\rho(H)} X^{(\rho)},
\]

where \( N_K,\rho(H) \) denotes the subgroup \( \{ n \in N_K(H) \mid nX^{(\rho)} \subseteq X^{(\rho)} \} \) of the normalizer \( N_K(H) \), and the \( N_K(H) \times G \)–action on \( HOM(H,G) \) is that given in Equation \[7\]

(iii) We have

\[
p^{-1}(X^{(H)}) = \coprod_{[\rho] \in HOM(H,G)/(N_K(H) \times G)} N_K(H) \times_{N_K,\rho(H)} P^{(\rho)} \times_{C_G(\rho)} G,
\]

where \( C_G(\rho) \) denotes the centralizer of \( \rho \) in \( G \).

(iv) If \( X \) has a single orbit type \( (H) \), then

\[
X = \coprod_{[\rho] \in HOM(H,G)/(N_K(H) \times G)} K/H \times_{N_K,\rho(H)} X^{(\rho)},
\]

\[
X/K = X^{(H)}/(N_K(H)/H) = \coprod_{[\rho] \in HOM(H,G)/(N_K(H) \times G)} X^{(\rho)}/(N_K,\rho(H)/H),
\]

and

\[
P = \coprod_{[\rho] \in HOM(H,G)/(N_K(H) \times G)} K \times_{N_K,\rho(H)} P^{(\rho)} \times_{C_G(\rho)} G.
\]

**Remark 2.6**. In the case \( X = n \) is finite with the discrete topology, \( X \) is completely regular and splits into a disjoint union of orbit types. Hence, from Proposition 2.5
we obtain Theorem 2.4 for this case. Note that Theorem 2.4 holds for an arbitrary group $K$.

For split $X$ spaces (i.e. spaces that admit a global section $X/K \to X$), more general classification results have been recently proven in [10]. For a generalization to the case of $K$ an $X$–groupoid, see [11]. For the case of $K$ a groupoid and $G$ a group, that is of the Mackey range, see [4, pg. 270].

3. Functional Equations for Quotient Orbifold Wreath Symmetric Products

3.1. Generating Functional Equation for $\Gamma$–Extensions. In this section, we extend the generating functions of extensions of multiplicative invariants for wreath symmetric products in [18] to the case of orbifolds that admit a cc-presentation. In particular, Theorem 3.3 corresponds to [18, Proposition 5–4]. For specific choices of $\Gamma$ and $\varphi$, the formula in Theorem 3.3 specializes to particularly interesting examples; see Section 3.2.

By a multiplicative orbifold invariant, we mean a function $\varphi$ defined on a subclass of Morita equivalence classes of orbifold groupoids such that

$$
\varphi(G \times H) = \varphi(G)\varphi(H)
$$

where $G \times H$ is a product groupoid; see [10, page 123]. Examples include the (usual) Euler characteristic $\chi$ of the orbit space and the Euler–Satake characteristic $\chi_{ES}$, see [8]. We are particularly interested in multiplicative orbifold invariants defined for all orbifolds that admit cc-presentations. We restrict to the case that $\Gamma$ is a finitely generated discrete group to ensure that these extensions are finite.

**Definition 3.1.** Let $\varphi$ be a multiplicative orbifold invariant, and let $\Gamma$ be a finitely generated discrete group.

(i) The $\Gamma$–extension $\varphi_\Gamma$ of $\varphi$ is defined by

$$
\varphi_\Gamma(M \rtimes G) := \sum_{(\theta) \in \text{HOM}(\Gamma, G)/G} \varphi\left(M^{(\theta)} \rtimes C_G(\theta)\right)
$$

where $(\theta)$ ranges over conjugacy classes of homomorphisms from $\Gamma$ to $G$ and $\varphi\left(M^{(\theta)} \rtimes C_G(\theta)\right)$ is taken to be zero when $M^{(\theta)} = \emptyset$.

(ii) Let $H \leq \Gamma$ be a subgroup of finite index, and let $(\Gamma/H)$ denote the isomorphism class of the $\Gamma$–set $\Gamma/H$. The $(\Gamma/H)$–extension of a multiplicative orbifold invariant $\varphi$ is defined by

$$
\varphi_{(\Gamma/H)}(M \rtimes G) := \sum_{[\rho] \in \text{HOM}(H,G)/(N_H(H) \times G)} \varphi\left(M^{(\rho)} \rtimes \text{Aut}_{\Gamma/G}^P\right).
$$

Here, $[\rho]$ ranges over $(N_H(H) \times G)$–orbits of homomorphisms from $H$ to $G$ where the action on $\rho \in \text{HOM}(H,G)$ is that of Equation 1. As well, $p_\rho: P_\rho = \Gamma \times_\rho G \to \Gamma/H$ is a $\Gamma$–irreducible $G$–principal bundle, and $\text{Aut}_{\Gamma/G}^P$ is the automorphism group of $P_\rho$ described in [18, Theorem 4–4] and recalled below.

If $M \rtimes G$ is a cc–presentation of the orbifold $Q$, it follows from [7, Theorem 3.5] that

$$
\bigsqcup_{(\theta) \in \text{HOM}(\Gamma, G)/G} M^{(\theta)} \rtimes C_G(\theta)
$$
is a presentation of the orbifold of $\Gamma$–sectors of $Q$ defined in [6, Definition 2.3]. As the $\Gamma$–sectors of a closed orbifold consist of a finite disjoint union of closed orbifolds by [6, Lemma 2.9], $M^{(\theta)} \times C_G(\theta) = \emptyset$ for all but finitely many elements of $HOM(\Gamma, G)/G$ so that $\varphi_T$ is finite. That $\varphi_T$ is multiplicative is a consequence of [8, Proposition 3.2]. In particular, when $\varphi$ is equal to the Euler or Euler–Satake characteristic, the definition of $\chi_T$ and $\chi_T^{ES}$ given above coincides with that of [8].

Note that by [13, Theorem 4–4], $Aut^P_{\Gamma-G}$ is isomorphic to the quotient $H \backslash T_\rho$ where $T_\rho$ is the isotropy group of $\rho$ in $N_T(H) \times G$ with respect to the action given in Equation 1. Using this identification, the action of $Aut^P_{\Gamma-G}$ on $M(\rho)$ is given by $H(u, g)x = gx$ as in [13, Proposition 5–3]. In particular, as $H$ has finite index in $\Gamma$ and $G$ acts locally freely, the action of each $Aut^P_{\Gamma-G}$ on $M(\rho)$ is clearly locally free, and hence presents an orbifold. As in the case of $G$ finite, when $H = \Gamma$, the $N_T(\Gamma)$–action on $\rho$: $\Gamma \to G$ is absorbed by conjugation by $G$. It follows that $HOM(H, G)/(N_T(H) \times G) = HOM(H, G)/G$ and $Aut^P_{\Gamma-G} = C_G(\rho)$, so that $\varphi(\Gamma/H) = \varphi_T$. That $\varphi(\Gamma/H)$ is in general finite follows from the following.

**Proposition 3.2.** Let $\Gamma$ be a finitely generated discrete group, and let $M \rtimes G$ be a cc–presentation of the orbifold $Q$. Let $H \leq \Gamma$ be a subgroup of index $n$. Then for a multiplicative orbifold invariant $\varphi$, we have

$$\varphi(\Gamma/H)(M \rtimes G) = \sum_{(\tau) \in \pi^{-1}(\Gamma/H)} \varphi\left((M^n)^{(\tau)} \rtimes C_G(S_n)((\tau))\right),$$

where $\pi$: $HOM(\Gamma, G(S_n))/G(S_n) \to HOM(\Gamma, S_n)/S_n$ denotes composition with the obvious homomorphism $G(S_n) \to S_n$ and $HOM(\Gamma, S_n)/S_n$ is identified with the set of isomorphism classes of $\Gamma$–sets of order $n$.

The proof is identical to [13, Proposition 6–1] and hence omitted.

When $M \rtimes G$ is a cc–presentation of the orbifold $Q$, the $n$th wreath symmetric product $MG(S_n)$ of $Q$ is the orbifold presented by $M^n \rtimes G(S_n)$ where $G(S_n)$ is the wreath product as in Remark 2.3 and the action of $((g_1, \ldots, g_n), s) \in G(S_n)$ on $(x_1, \ldots, x_n) \in M^n$ is given by

$$(g_1, \ldots, g_n, s)(x_1, \ldots, x_n) = (g_1x_{s^{-1}(1)}, \ldots, g_nx_{s^{-1}(n)}).$$

The proof of [13, Proposition 5–4] extends directly to this case; we recall it briefly.

**Theorem 3.3.** Let $\Gamma$ be a finitely generated discrete group, and let $M \rtimes G$ be a cc–presentation of the orbifold $Q$. For a multiplicative orbifold invariant $\varphi$,

$$\sum_{n \geq 0} q^n \varphi_T(M \rtimes G(S_n)) = \prod_{(\tau) \in \pi^{-1}(\Gamma/H)} \sum_{n \geq 0} q^{([\tau]/H \rtimes n)} \varphi\left((M^n)^{(\tau)} \rtimes Aut^P_{\Gamma-G}(S_n)\right),$$

where the product runs over all conjugacy classes ($H$) of subgroups of $\Gamma$ of finite index and all elements $[\rho]$ of $HOM(H, G)/(N_T(H) \times G)$, and $P_\rho$ is the $\Gamma$–principal bundle corresponding to $\rho$ by Theorem 2.4(ii).

**Proof.** A homomorphism $\theta$: $\Gamma \to G(S_n)$ corresponds by Theorem 2.4(i) to a $\Gamma$–principal bundle $P_\theta$ over $n = \{1, 2, \ldots, n\}$. Such a bundle decomposes into a finite collection of $\Gamma$–irreducible $G$–principal bundles over the $\Gamma$–orbits in $n$, each identified with $\Gamma/H$ for some $H \leq \Gamma$ with finite index. Each irreducible bundle then corresponds to an element of $HOM(H, G)/(N_T(H) \times G)$ by Theorem 2.4(ii).
Let \( r(H, \rho) \) denote the number of \( \Gamma \)-irreducible \( G \)-principal bundles whose isomorphism class corresponds to \( \rho: H \to G \) in the decomposition of \( P_{\theta} \). Note that \( \sum (\rho) [\Gamma / H] r(H, \rho) = n \).

By the results detailed in \cite{18} Sections 3 and 4, which hold for all groups \( G \),

\[ \varphi_{\Gamma}(M^n \rtimes G(S_n)) = \sum \prod \prod \varphi_{\Gamma} \left( (M^{(\rho)})^{r(H, \rho)} \rtimes Aut_{\Gamma-H}^G(S_{r(H, \rho)}) \right), \]

where the sum over \( r(H, \rho) \) is over all sets of non-negative integers such that \( \sum (\rho) [\Gamma / H] r(H, \rho) = n \). Taking the sum over \( n \) and rearranging terms yields

\[ \sum_{n \geq 0} q^n \varphi_{\Gamma}(M^n \rtimes G(S_n)) = \prod (H) \prod \sum q^{r/H} \varphi_{\Gamma} \left( (M^{(\rho)})^{r} \rtimes Aut_{\Gamma-H}^G(S_{r}) \right). \]

\[ \square \]

3.2. Examples: The Euler and Euler–Satake Characteristics. In this section, we detail some examples of Theorem 3.3 for orbifold invariants. In particular, we consider the standard Euler characteristic \( \chi(M \times G) \) and the Euler–Satake characteristic \( \chi_{ES}(M \rtimes G) \), extending \cite{18} Theorems 5–5 and 6–3 to the case of orbifolds that admit a cc–presentation.

We first consider the \( \Gamma \)-extension \( \chi_{\text{E}} \) of the usual Euler characteristic \( \chi(M \times G) = \chi(M / G) \) given in Definition 3.1. The following is needed for the case of \( \Gamma \) abelian, see \cite{18} Lemma 6–2.

**Lemma 3.4.** Let \( \Gamma \) be a finitely generated abelian discrete group, and let \( M \rtimes G \) be a cc–presentation of the orbifold \( Q \). For any subgroup \( H \leq \Gamma \) of finite index in \( \Gamma \) we have

\[ \chi_{(\Gamma/H)}(M \rtimes G) = \sum_{(\rho) \in HOM(H,G)/G} \chi(M^{(\rho)} \rtimes C_G(\rho)) = \chi_{H}(M \times G), \]

**Proof.** By Definition 3.1

\[ \chi_{(\Gamma/H)}(M \rtimes G) = \sum_{(\rho) \in HOM(H,G)/\left\{ N_{\Gamma}(H) \times G \right\}} \chi(M^{(\rho)} \rtimes Aut_{\Gamma-H}^G). \]

As \( \Gamma \) is abelian, \( HOM(H,G) / (N_{\Gamma}(H) \times G) = HOM(H,G)/G \), and

\[ Aut_{\Gamma-H}^G \equiv \Gamma \times C_G(\rho), \]

for \( \rho \in HOM(H,G) \) by \cite{18} Equation 4–4. Thus we have

\[ \chi_{(\Gamma/H)}(M \rtimes G) = \sum_{(\rho) \in HOM(H,G)/G} \chi(M^{(\rho)} \rtimes Aut_{\Gamma-H}^G) \]

\[ = \sum_{(\rho) \in HOM(H,G)/G} \chi(M^{(\rho)} \rtimes (\Gamma \times C_G(\rho))). \]

Hence, if \( \{ \gamma_j \} \) is a set of representatives for the cosets \( H \backslash \Gamma \), we have

\[ \Gamma \times C_G(\rho) = \bigcap \{(i(\gamma_j))C_G(\rho)\}. \]
Recalling that the action of $\text{Aut}_{\Gamma \times C_G}^P$ on $M^{(\rho)}$ is given by $H(u, g)x = gx$, the image of the natural injection $i : \Gamma \rightarrow (\Gamma \times \rho \text{C}_G(\rho))$ acts trivially on $M^{(\rho)}$. Therefore,

$$M^{(\rho)} / (\Gamma \times \rho \text{C}_G(\rho)) = M^{(\rho)} / \text{C}_G(\rho),$$

and so

$$\chi \left( M^{(\rho)} \times (\Gamma \times \rho \text{C}_G(\rho)) \right) = \chi \left( M^{(\rho)} \times \text{C}_G(\rho) \right),$$

from which the result follows.

With this, we have the following.

**Theorem 3.5 (Γ–Extensions of the Euler Characteristic).** Let $\Gamma$ be a finitely generated discrete group, and let $M \times G$ be a cc–presentation of the orbifold $Q$. For the multiplicative orbifold invariant $\chi$, we have

$$(2) \quad \sum_{n \geq 0} q^n \chi_\Gamma(M^n \times G(S_n)) = \prod_{r \geq 1} \left[ (1 - q^r)^{-\sum_{(H_r)} \chi_{\Gamma/H_r}(M \times G)} \right],$$

where $(H_r)$ runs over the $\Gamma$–conjugacy classes of subgroups of $\Gamma$ of finite index $r$. If $\Gamma$ is abelian, then

$$(3) \quad \sum_{n \geq 0} q^n \chi_\Gamma(M^n \times G(S_n)) = \prod_{r \geq 1} \left[ (1 - q^r)^{-\sum_{H_r} \chi_{H_r}(M \times G)} \right],$$

where $H_r$ runs over all subgroups of $\Gamma$ of finite index $r$.

**Proof.** By Theorem 3.3, we have that $\sum_{n \geq 0} q^n \chi_\Gamma(M^n \times G(S_n))$ is given by

$$\sum_{n \geq 0} q^n \chi_\Gamma(M^n \times G(S_n)) = \prod_{(H_r), [\rho]} \sum_{r \geq 0} q^{(1/H_r)\rho} \chi \left( (M^{(\rho)})^r \times \text{Aut}_{\Gamma \times G}^P \right),$$

where the product ranges over $\Gamma$–conjugacy classes $(H)$ of subgroups $H \leq \Gamma$ of finite index as well as $(N_\Gamma(H) \times G)$–orbits $[\rho]$ of homomorphisms $\rho \in \text{HOM}(H, G)$. By MacDonald’s formula [8, Theorem 5.2], we have that this is equal to

$$\prod_{(H_r), [\rho]} \left( 1 - q^{(1/H_r)\rho} \right) = \prod_{r \geq 1} \left( 1 - q^r \right)^{-\sum_{(H_r)} \chi \left( M^{(\rho)} \times \text{Aut}_{\Gamma \times G}^P \right)},$$

where $(H_r)$ ranges over the $\Gamma$–conjugacy classes of subgroups of finite index $r$ in $\Gamma$ and $[\rho]$ ranges over $(N_\Gamma(H_r) \times G)$–conjugacy classes of homomorphisms. Noting that the last summation over $[\rho]$ yields exactly $\chi_{(\Gamma/H_r)}(M \times G)$, Equation 2 follows. Then Equation 3 follows from Lemma 3.3. 

For the Euler–Satake characteristic $\chi_{\text{ES}}(M \times G)$ (see [3]), define $\chi_{\text{ES}}^{\Gamma}(M \times G)$ to be the corresponding $\Gamma$–extension defined in Definition 3.1. Then we have the following.

**Theorem 3.6 (Γ–Extension of the Euler–Satake Characteristic).** Let $\Gamma$ be a finitely generated discrete group, and let $M \times G$ be a cc–presentation of the orbifold $Q$. For the multiplicative orbifold invariant $\chi_{\text{ES}}(M \times G)$, we have

$$(4) \quad \sum_{n \geq 0} q^n \chi_{\text{ES}}^{\Gamma}(M^n \times G(S_n)) = \exp \left( \sum_{n \geq 1} \frac{q^n}{n} \sum_{H \leq \Gamma / H | H \times \rho} \chi_{\text{ES}}^{\Gamma}(M \times G) \right).$$
Proof. We follow the proof of [13] Theorem 5–6: the main modification is in using orbifold covers to avoid dealing with infinite orders of $G$ and its subgroups.

By Theorem 3.3, we have

$$\sum_{n \geq 0} q^n \chi^{ES}_G(M^n \rtimes G(S_n)) = \prod_{(H), [\rho]} \left[ \sum_{n \geq 0} q^{\Gamma/H |n} \chi^{ES}_E \left( (M^{(\rho)})^n \rtimes Aut_{T-G}^{P_0}(S_n) \right) \right].$$

By [13] Theorem 4–2, the index $[Aut_{T-G}^{P_0} : C_G(\rho)]$ is given by $|N_{\Gamma}^G(H)/H|$, which is finite, so that

$$[Aut_{T-G}^{P_0}(S_n) : C_G(\rho)(S_n)] = |N_{\Gamma}^G(H)/H|^n.$$

It follows that

$$(M^{(\rho)})^n \rtimes C_G(\rho)(S_n) \longrightarrow (M^{(\rho)})^n \rtimes Aut_{T-G}^{P_0}(S_n)$$

is an orbifold cover of $|N_{\Gamma}(H)/H|^n$ sheets, so that by [20] Proposition 13.3.4 and [8] Lemma 2.2, theorems 2.3 and 5.11, we have

$$\prod_{(H), [\rho]} \left[ \sum_{n \geq 0} q^{\Gamma/H |n} \chi^{ES}_E \left( (M^{(\rho)})^n \rtimes Aut_{T-G}^{P_0}(S_n) \right) \right] = \prod_{(H), [\rho]} \exp \left[ \frac{q^{\Gamma/H}}{|N_{\Gamma}(H)/H|} \chi^{ES}_E \left( M^{(\rho)} \rtimes C_G(\rho) \right) \right],$$

$$= \exp \left[ \sum_{n \geq 1} q^n \sum_{(H): [\Gamma/H] = n} \sum_{[\rho]} \frac{|\Gamma/N_{\Gamma}(H)| \chi^{ES}_E(\rho) \times C_G(\rho)}{|N_{\Gamma}(H)/N_{\Gamma}(H)| \chi^{ES}_E(\rho) \times C_G(\rho)} \chi^{ES}_E(\rho) \right].$$

In the last equation, note that we switch from summing over $N_{\Gamma}(H) \rtimes G$–conjugacy classes $[\rho]$ of $\rho: H \rightarrow G$ to $G$–conjugacy classes $(\rho)$, and the $N_{\Gamma}(H)$–orbit of $\rho$ has $|N_{\Gamma}(H)/N_{\Gamma}(H)|$ elements. Then as the final sum in the last expression is the definition of $\chi^{ES}_H(M^{(\rho)} \rtimes C_G(\rho))$, and each conjugacy class $(H)$ contains $|\Gamma/N_{\Gamma}(H)|$ elements, this is equal to

$$\exp \left[ \sum_{n \geq 1} q^n \sum_{(H): [\Gamma/H] = n} \frac{|\Gamma/N_{\Gamma}(H)| \chi^{ES}_E(\rho) \times C_G(\rho)}{|N_{\Gamma}(H)/N_{\Gamma}(H)| \chi^{ES}_E(\rho) \times C_G(\rho)} \chi^{ES}_E(M \rtimes G) \right].$$

$$= \exp \left[ \sum_{n \geq 1} q^n \sum_{(H): [\Gamma/H] = n} \chi^{ES}_E(M \rtimes G) \right].$$

□
3.3. Extensions Associated to General $\Gamma$–Sets. In this section, we extend Definition 3.1 to include extensions of multiplicative orbifold invariants associated to arbitrary finite $\Gamma$–sets where $\Gamma$ is a finitely generated discrete group. This extends the definition given by [18, Equation 6–13].

Definition 3.7. Let $X$ be a finite $\Gamma$–set of order $n$ and $\varphi$ a multiplicative orbifold invariant. The extension of $\varphi$ associated to the $\Gamma$–isomorphism class $[X]$ of $X$ is defined by

$$
\varphi_{[X]}(M \rtimes G) = \sum_{[P \to X]} \varphi \left( \mathcal{S}[P \rtimes_G M]^{\Gamma \rtimes \text{Aut}_{P_{\Gamma}} G} \right) = \sum_{[\theta] \in \pi^{-1}[X]} \varphi \left( (M^n)^{\theta} \rtimes C_{G(S_n)}(\theta) \right),
$$

where the first sum is over all isomorphism classes of $\Gamma$–$G$–principal bundles over $X$, $\pi$ is as in Proposition 3.2, and $\mathcal{S}[P \rtimes_G M]^{\Gamma}$ denotes the $\Gamma$–invariant sections.

As in the case of $G$ finite, Theorem 2.4(i) implies

$$
\sum_{X} q^{|X|} \varphi_{[X]}(M \rtimes G) = \sum_{n \geq 0} q^n \varphi_{\Gamma}(M^n \rtimes G(S_n)),
$$

where the first summation is over all isomorphism classes of $\Gamma$–$G$–principal bundles over finite $\Gamma$–sets $X$. For a finite $\Gamma$–set $X$, let $X = \bigsqcup (H)^{r(H)}$ be its decomposition into $\Gamma$–orbits where $(H)$ ranges over conjugacy classes of isotropy groups, $r(H)$ is the number of $\Gamma$–orbits which are isomorphic to $\Gamma/H$, and $r(H)\Gamma/H$ denotes the disjoint union of these $r(H)$ isomorphic $\Gamma$–orbits. Then for a multiplicative orbifold invariant $\varphi$, we have

$$
\varphi_{[X]}(M \rtimes G) = \prod_{(H)} \varphi_{r(H)\Gamma/H}(M \rtimes G).
$$

Combining this with Equation 5 yields the following interpretation of Theorem 3.3 in terms of $\Gamma$–sets, which coincides with [18, Proposition 6–9] for the case of $G$ finite and follows its proof.

Theorem 3.8. Let $\Gamma$ be a finitely generated discrete group, and let $M \rtimes G$ be a cc–presentation of the orbifold $Q$. Let $\varphi$ be a multiplicative orbifold invariant and let $X$ be a finite $\Gamma$–set. With the notation as above,

$$
\sum_{n \geq 0} q^n \varphi_{\Gamma}(M^n \rtimes G(S_n)) = \prod_{(H)} \left( \sum_{r \geq 0} q^r \varphi_{r(\Gamma/H)}(M \rtimes G) \right).
$$

The generating function of $\varphi_{r(\Gamma/H)}$ is given by

$$
\sum_{n \geq 0} q^n \varphi_{r(\Gamma/H)}(M \rtimes G) = \prod_{[\rho]} \sum_{r \geq 0} q^r \varphi \left( (M^{(|\rho|)})^r \rtimes \text{Aut}_{\Gamma}(\rho) \right)
$$

where the product over $[\rho]$ again ranges over $\text{HOM}(H,G)/(N_{\Gamma}(H) \times G)$. 


Proof. Letting $P_H$ denote the restriction of a $\Gamma$–$G$–principal bundle $P \to X$ to $r(H)(\Gamma/H)$, Definition 5.7 becomes

$$\varphi_{[X]}(M \rtimes G) = \sum_{[P \to X]} \prod_{(H)} \varphi \left( S[P_H \times_G M]^G \rtimes \text{Aut}_{\Gamma}(P_H) \right)$$

$$= \prod_{(H)} \sum_{[P_H]} \varphi \left( S[P_H \times_G M]^G \rtimes \text{Aut}_{\Gamma}(P_H) \right)$$

$$= \prod_{(H)} \varphi_{r(H)(\Gamma/H)}(M \rtimes G),$$

where the sum over $[P_H]$ ranges over the set of all isomorphism classes of $\Gamma$–$G$–principal bundles over the $\Gamma$–set $r(H)(\Gamma/H)$. Equation 6 follows. The proof of Equation 7 is analogous to the proof of Theorem 3.3, and continues to follow [18, Proposition 6–9].

3.4. The $H$–Inertia. In this section, we give a generalized sector construction interpreting the extensions of multiplicative orbifold invariants associated to transitive $\Gamma$–sets $\Gamma$ as evaluation on sectors.

Recall from Equation 11 that the action of $N_{\Gamma}(H) \times G$ on $\text{HOM}(H,G)$ is given by

$$[\rho(u,g)](h) = g^{-1} \rho(uh\omega^{-1}g), \ \forall \rho \in \text{HOM}(H,G).$$

As noted in Subsection 3.3 if $T_\rho$ denotes the stabilizer of $\rho \in \text{HOM}(H,G)$ in $N_{\Gamma}(H) \times G$, then $\text{Aut}_{\Gamma}(\rho)$ is isomorphic to $H \backslash T_\rho$, and the action of $\text{Aut}_{\Gamma}(\rho)$ on $M^{(\rho)}$ depends only on the $G$–factor. Let

$$\mathcal{A} = \prod_{[\rho] \in \text{HOM}(H,G)/(N_{\Gamma}(H) \times G)} [M^{(\rho)} \rtimes \text{Aut}_{\Gamma}(\rho)].$$

Then $\mathcal{A}$ is an orbifold groupoid, and $\varphi_{(\Gamma/H)}$ is the application of $\varphi$ to the groupoid $\mathcal{A}$.

As noted in the proof of Lemma 3.3 when $\Gamma$ is abelian, $N_{\Gamma}(H) = \Gamma$ and $T_\rho = \Gamma \times_{\rho} C_G(\rho)$, so that $\text{Aut}_{\Gamma}(\rho) = H \backslash (\Gamma \times_{\rho} C_G(\rho))$ and $\text{HOM}(H,G)/(N_{\Gamma}(H) \times G) = \text{HOM}(H,G)/G$. Therefore the associated groupoid $\mathcal{A}$ reduces to the product $(M \rtimes G)^G \times H \backslash \Gamma$, where $(M \rtimes G)^G$ is the groupoid of $\Gamma$–sectors. When in addition $H = \Gamma$, $\text{Aut}_{\Gamma}(\rho) = C_G(\rho)$ so that $\mathcal{A}$ reduces to the groupoid of $\Gamma$–sectors.

In general, if $H = \Gamma$ we claim that each connected component of $\mathcal{A}$ is isomorphic to a $\Gamma$–sector of $M \rtimes G$, possibly with different multiplicities. In this case we have $N_{\Gamma}(H) = \Gamma$, $T_\rho = \Gamma \times_{\rho} C_G(\rho)$, and $\text{Aut}_{\Gamma}(\rho) = C_G(\rho)$. Then

$$\mathcal{A} = \prod_{[\rho] \in \text{HOM}(H,G)/(N_{\Gamma}(H) \times G)} [M^{(\rho)} \rtimes C_G(\rho)],$$

and hence that each $[\rho]$ corresponds to a union of $\Gamma$–sectors; see [11]. As

$$\prod_{[\rho] \in \text{HOM}(H,G)/G} [M^{(\rho)} \rtimes C_G(\rho)]$$

presents the orbifold of $\Gamma$–sectors, we see that $\mathcal{A}$ simply identifies $\Gamma$–sectors that are isomorphic via an element of $N_{\Gamma}(H)$.
4. Decomposable Functors and Wreath Products

In this section, we assume the orbifold $Q$ is a global quotient orbifold of the form $M \times G$ where $G$ is a finite group. We use a modification of a formal functorial functional equation of Dress and Müller [5] for decomposable functors to determine a relationship between $\chi_{ES}$ and $\chi_{ES}(H)$. Note that we modify their approach to replace counting functions for finite sets with functions related to the invariant $\chi_{ES}$ as defined below. We follow the notation of [5, Section 1].

Fix a global quotient orbifold $M \times G$ and a finitely generated discrete group $\Gamma$. By Theorem 2.4, there is a bijection between the isomorphism classes of $\Gamma$–principal bundles over $\Gamma$–sets of order $n$ and the conjugacy classes of homomorphisms into the wreath product $G(S_n)$, i.e. elements of $HOM(\Gamma, G(S_n))/G(S_n)$. Given a homomorphism $\theta: \Gamma \to G(S_n)$, we let $P_{[\theta]}$ denote a representative of the corresponding isomorphism class $[P_{[\theta]}]$ of $\Gamma$–principal bundles.

**Definition 4.1.** Let $\text{Ens}$ denote the category with finite sets as objects and injective mappings as morphisms, and let $\tilde{\text{Ens}}$ denote the category with finite sets as objects and injective mappings as morphisms. For a fixed finitely generated discrete group $\Gamma$ and finite group $G$, define the covariant functor

$$\mathcal{F}_{\Gamma, G}: \text{Ens} \longrightarrow \tilde{\text{Ens}}$$

by assigning to the finite set $\Omega$ (which is given the discrete topology) the finite set of $\Gamma$–principal bundles over $\Omega$. We adopt the convention that $\mathcal{F}_{\Gamma, G}(\emptyset)$ consists of a single “empty bundle” corresponding to the trivial homomorphism from $\Gamma$ into the trivial group.

Consider the functors

$$\mathcal{F}_{\Gamma, G} \times \mathcal{F}_{\Gamma, G}: \text{Ens}^2 \xrightarrow{\mathcal{F}_{\Gamma, G}} \text{Ens} \xrightarrow{\times} \tilde{\text{Ens}}$$

and

$$\mathcal{F}_{\Gamma, G} \times \sqcup: \text{Ens}^2 \xrightarrow{\mathcal{F}_{\Gamma, G}} \text{Ens} \xrightarrow{\sqcup} \tilde{\text{Ens}}$$

where $\times$ is the Cartesian product functor, $\sqcup$ is the (disjoint) union functor, and $\iota$ is the natural inclusion functor. That is, $\mathcal{F}_{\Gamma, G} \times \mathcal{F}_{\Gamma, G}(\Omega_1, \Omega_2)$ is the finite set of pairs $(P_1, P_2)$ where $P_1 \in \mathcal{F}_{\Gamma, G}(\Omega_1)$ is a $\Gamma$–principal bundle over $\Omega_1$ and $P_2 \in \mathcal{F}_{\Gamma, G}(\Omega_2)$ is a $\Gamma$–principal bundle over $\Omega_2$, and $(\mathcal{F}_{\Gamma, G} \times \sqcup)(\Omega_1, \Omega_2)$ is the finite set of $\Gamma$–principal bundles in $\mathcal{F}_{\Gamma, G}(\Omega_1 \sqcup \Omega_2)$.

Define a natural transformation $\eta: \mathcal{F}_{\Gamma, G} \times \mathcal{F}_{\Gamma, G} \to \mathcal{F}_{\Gamma, G} \times \sqcup$ as follows. To each pair $(\Omega_1, \Omega_2)$ of finite sets, we assign the morphism

$$(\mathcal{F}_{\Gamma, G} \times \mathcal{F}_{\Gamma, G})(\Omega_1, \Omega_2) \xrightarrow{\eta(\Omega_1, \Omega_2)} (\mathcal{F}_{\Gamma, G} \times \sqcup)(\Omega_1, \Omega_2)$$

such that $\eta(\Omega_1, \Omega_2)([P_1, P_2]) = [P_1 \sqcup P_2]$ as a $\Gamma$–principal bundle over $\Omega_1 \sqcup \Omega_2$. It is straightforward to show that $\eta$ is a weak decomposition of the functor $\mathcal{F}_{\Gamma, G}$ as defined in [5, page 192]. As well, define

$$\mathcal{F}^0_{\Gamma, G}(\Omega) = \left\{ \begin{array}{ll} \mathcal{F}^\eta_{\Gamma, G}(\Omega) \setminus \bigcup_{I \cup J = \Omega; I \neq J} \eta(\mathcal{F}_{\Gamma, G}(I) \times \mathcal{F}_{\Gamma, G}(J)), & \Omega \neq \emptyset \\ \emptyset, & \Omega = \emptyset \end{array} \right.$$ 

Then $\mathcal{F}^0_{\Gamma, G}(\Omega)$ is the collection of irreducible $\Gamma$–principal bundles over $\Omega$.

Recall that $n$ denotes the set $\{1, 2, \ldots, n\}$. Given a pair $(P_1, P_2)$ of $\Gamma$–principal bundles over $r$ and $s$, respectively, choose homomorphisms $\theta_1 \in HOM(\Gamma, G(S_r))$
and \( \theta_2 \in \text{HOM}(\Gamma, G(S_n)) \) such that \( P_1 \cong P_{(\theta_1)} \) and \( P_2 \cong P_{(\theta_2)} \). Then the \( \Gamma \)-principal bundle \( \eta_{(\theta_1, \Omega)}(P_{(\theta_1)}) \) corresponds to the homomorphism \( \theta: \Gamma \to G(S_r) \times G(S_s) \leq G(S_{r+s}) \) given by \( \theta(\gamma) = \theta_1(\gamma)\theta_2(\gamma) \), where \( \theta_1(\gamma) \) and \( \theta_2(\gamma) \) are considered elements of the first and second factors, respectively, of \( G(S_r) \times G(S_s) \leq G(S_{r+s}) \). Note that the fixed point set of \( \theta \) in \( M^{r+s} \) is
\[
(M^{r+s})^{(\theta)} = (M^r)^{(\theta_1)} 	imes (M^s)^{(\theta_2)}.
\]
The sets in Equation 8 of course depend on the choice of \( \theta_1 \) and \( \theta_2 \), but their diffeomorphism–type depends only on the corresponding conjugacy classes and hence on the isomorphism type of the bundles \( P_1 \) and \( P_2 \).

For a global quotient orbifold \( M \times G \) where \( M \) is a closed manifold, define the functions
\[
\phi_{\Gamma, M \times G}^n: \mathbb{Z}_{\geq 0} \to \mathbb{Q}, \quad \psi_{\Gamma, M \times G}^n: \mathbb{Z}_{\geq 0} \to \mathbb{Q}
\]
by setting
\[
\phi_{\Gamma, M \times G}^n(n) = \sum_{P(\theta) \in \mathcal{F}_{\Gamma, G}(\mathbb{N})} \frac{1}{|\Gamma|} \chi((M^n)^{(\theta)}) \quad \text{and} \quad \psi_{\Gamma, M \times G}^n(n) = \sum_{P(\theta) \in \mathcal{F}_{\Gamma, G}(\mathbb{N})} \frac{1}{|\Gamma|} \chi((M^n)^{(\theta)}).
\]

Of course, if \((M^n)^{(\theta)} = \emptyset\), we let the associated term be 0. Note that \( \chi((M^n)^{(\theta)}) \) depends only on the conjugacy class of \( \theta \) so that both functions are well defined. As a convention, we set \( \phi_{\Gamma, M \times G}^n(0) = 0 \) and \( \psi_{\Gamma, M \times G}(0) = 1 \).

Define in addition the formal power series
\[
\Phi(q) = \sum_{n \geq 0} \phi_{\Gamma, M \times G}^n(n) \frac{q^n}{n!} \quad \text{and} \quad \Psi(q) = \sum_{n \geq 0} \psi_{\Gamma, M \times G}^n(n) \frac{q^n}{n!}.
\]
The \( G(S_n) \)-conjugacy class \((\theta): \Gamma \to G(S_n)\) contains \( \frac{|G(S_n)|}{|C_{G(S_n)}(\theta)|} \) elements. Hence,
\[
\Psi(q) = \sum_{n \geq 0} \frac{q^n}{n!} \sum_{P(\theta) \in \mathcal{F}_{\Gamma, G}(\mathbb{N})} \frac{1}{|\Gamma|} \chi((M^n)^{(\theta)}) = \sum_{n \geq 0} \frac{q^n}{n!} \sum_{(\theta) \in \text{HOM}(\Gamma, G(S_n))/G(S_n)} \sum_{\tau \in \mathcal{G}^{(\theta)}} \chi((M^n)^{(\tau)}) = \sum_{n \geq 0} \frac{q^n}{n!} \sum_{(\theta) \in \text{HOM}(\Gamma, G(S_n))/G(S_n)} \left( \frac{|G(S_n)|}{|C_{G(S_n)}(\theta)|} \right) \frac{1}{|\Gamma|} \chi((M^n)^{(\theta)})(9)
\]
\[
= \sum_{n \geq 0} q^n \sum_{(\theta) \in \text{HOM}(\Gamma, G(S_n))/G(S_n)} \frac{\chi((M^n)^{(\theta)})}{|C_{G(S_n)}(\theta)|} = \sum_{n \geq 0} q^n \sum_{(\theta) \in \text{HOM}(\Gamma, G(S_n))/G(S_n)} \chi_{ES}(M_{G(S_n)}) \chi_{ES}(\theta).
\]
The same computation shows that

$$\Phi(q) = \sum_{n \geq 0} q^n \sum_{(\theta) \in \text{HOM}(\Gamma, G(S_n))/G(S_n)} \chi_{ES} \left( (M^n)^{(\theta)} \rtimes C_G(S_n)(\theta) \right),$$

where $\text{HOM}(\Gamma, G(S_n))/G(S_n)$ denotes the conjugacy classes $(\theta)$ of homomorphisms such that the image of $\pi(\theta)$ acts transitively on $n$, or equivalently such that $P(\theta)$ is irreducible. As isomorphism classes of finite, transitive $\Gamma$–sets correspond to conjugacy classes of subgroups $H \leq \Gamma$ of finite index, this becomes

$$\sum_{n \geq 0} q^n \sum_{(H_n)} \sum_{(\theta) \in \pi^{-1}(\Gamma/H)} \chi_{ES} \left( (M^n)^{(\theta)} \rtimes C_G(S_n)(\theta) \right),$$

where the second sum ranges over all $\Gamma$–conjugacy classes $(H_n)$ of subgroups $H_n \leq \Gamma$ of index $n$ and $\pi: \text{HOM}(\Gamma, G(S_n))/G(S_n) \rightarrow \text{HOM}(\Gamma, S_n)/S_n$ is as in Proposition 3.2; by the same Proposition, this is equal to

$$\Phi(q) = \sum_{n \geq 1} q^n \sum_{(H):[\Gamma:H]=n} \chi_{ES}^{\Gamma/H} (MG(S_n)).$$

We are now ready to prove the following.

**Theorem 4.2.** Let $\Gamma$ be a finitely generated discrete group, let $M$ be a compact manifold, and let $M \rtimes G$ be a presentation of the global quotient orbifold $Q$ so that $G$ is finite. Then with the definitions given above, we have

$$\Psi(q) = \exp(\Phi(q)),$$

i.e., applying Equations 10 and 4

$$\sum_{n \geq 0} q^n \chi_{ES}^{\Gamma}(MG(S_n)) = \exp \left( \sum_{n \geq 1} q^n \sum_{(H):[\Gamma:H]=n} \chi_{ES}^{\Gamma/H} (MG(S_n)) \right).$$

**Proof.** This result is an analog of [5, Equation (1.6)], which is proven for an arbitrary decomposable functor $F$, but defining the functions $\phi_{\Gamma,M \rtimes G}$ and $\psi_{\Gamma,M \rtimes G}$ to be the counting functions for finite sets. Here, we illustrate that Dress and Müller’s arguments apply to extensions of the Euler–Satake characteristic as defined above. Their proof of this result is separated into parts (i), (iii), (iv), (v), (vii), (viii), and (xi); only (xi) refers to the counting functions (note that (ii), (vi), and (ix) are used to prove a separate result). As our functors $F_{\Gamma,G}$ and $F_{\Gamma,G}^n$ are special cases of theirs, their results apply and we need only verify (xi).

By [5] (vii) and (viii)], for any finite set $\Omega$ and any fixed element $\omega \in \Omega$, we have

$$F_{\Gamma,G}(\Omega) = \bigcup_{\Omega_1: \omega \in \Omega_1 \subseteq \Omega} \eta(F_{\Gamma,G}(\Omega_1) \times F_{\Gamma,G}(\Omega \setminus \Omega_1)),$$

and the right-side of this equation is a disjoint union. As each $\eta(\Omega_1, \Omega \setminus \Omega_1)$ is injective, we can use this decomposition, Equation 8 and the multiplicativity of $\chi$ to rewrite
\[ \psi_{\Gamma, M \rtimes G}(n) = \sum_{P(\theta) \in \mathcal{F}_{\Gamma, G}(n)} \frac{1}{|G|} \chi \left( (M^n)^{(\theta)} \right) \]

\[ = \sum_{\Omega_1 : \Omega \subseteq n} \sum_{P(\theta_1) \in \mathcal{F}_{\Gamma, G}(\Omega_1)} \sum_{P(\theta_2) \in \mathcal{F}_{\Gamma, G}(n \setminus \Omega_1)} \frac{1}{|G|} \chi \left( (M^n)^{(\theta_1, \theta_2)} \right) \]

\[ = \sum_{\Omega_1 : \Omega \subseteq n} \sum_{P(\theta_1) \in \mathcal{F}_{\Gamma, G}(\Omega_1)} \sum_{P(\theta_2) \in \mathcal{F}_{\Gamma, G}(n \setminus \Omega_1)} \frac{1}{|G|} \chi \left( (M^n)^{(\theta_1, \theta_2)} \right) \]

(\text{where } \mu \text{ is the cardinality of } \Omega_1)

\[ = \sum_{\Omega_1 : \Omega \subseteq n} \phi^n(\mu) \psi(n - \mu). \]

In particular, the expression \( \phi^n_{\Gamma, M \rtimes G}(\mu) \psi_{\Gamma, M \rtimes G}(n - \mu) \) depends only on the cardinality of \( \Omega_1 \). For each \( \mu \) with \( 1 \leq \mu \leq n \), the \( \binom{n-1}{\mu-1} \) subsets of \( n \) of cardinality \( \mu \) containing the element 1 contribute \( \binom{n-1}{\mu-1} \phi^n_{\Gamma, M \rtimes G}(\mu) \psi_{\Gamma, M \rtimes G}(n - \mu) \), and

\[ \psi_{\Gamma, M \rtimes G}(n) = \sum_{\mu=1}^{n} \left( \frac{n-1}{\mu-1} \right) \phi^n_{\Gamma, M \rtimes G}(\mu) \psi_{\Gamma, M \rtimes G}(n - \mu) \]

for \( n \geq 1 \). Multiplying both sides by \( q^{n-1}/(n-1)! \) and summing over \( n \geq 1 \) yields

\[ \sum_{n \geq 1} \frac{q^{n-1}}{(n-1)!} \psi_{\Gamma, M \rtimes G}(n) = \sum_{n \geq 1} \frac{q^{n-1}}{(n-1)!} \sum_{\mu=1}^{n} \frac{1}{\mu-1} \phi^n_{\Gamma, M \rtimes G}(\mu) \psi_{\Gamma, M \rtimes G}(n - \mu) \]

\[ = \sum_{n \geq 1} \frac{q^{n-1}}{(n-1)!} \sum_{\mu=1}^{n} \frac{1}{(n-\mu-1)!} \phi^n_{\Gamma, M \rtimes G}(\mu) \psi_{\Gamma, M \rtimes G}(n - \mu), \]

and hence

\[ \Psi'(q) = \Phi'(q) \Psi(q). \]

By \( \Psi'(q) \) and \( \Phi'(q) \), we mean the formal derivatives of the corresponding power series. Recalling that \( \psi_{\Gamma, M \rtimes G}(0) = 1 \) and \( \phi^n_{\Gamma, M \rtimes G}(0) = 0 \), the claim follows.

For example, note that if \( \Gamma \) is trivial, then the only irreducible \( \Gamma - G \)-principal bundles occur over singletons. Hence, the only nonzero term of the sum over \( n \) on the right-hand side of Equation (11) is that corresponding to \( n = 1 \). As \( \chi_{ES}^{ES}(1_{(1/1)}) = \chi_{ES}^{ES} \) is clear, Equation (11) becomes

\[ \sum_{n \geq 0} q^n \chi_{ES}(MG(S_n)) = \exp(q \chi_{ES}(M \rtimes G)), \]

yielding Tamanoi’s expression of MacDonald’s formula [13, Equation 2–10] for the Euler–Satake characteristic.
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Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309-0395
E-mail address: farsi@euclid.colorado.edu

Department of Mathematics and Computer Science, Rhodes College, 2000 N. Parkway, Memphis, TN 38112
E-mail address: seatonc@rhodes.edu