Determination of the basis of the space of all root functionals of a system of polynomial equations and of the basis of its ideal by the operation of the extension of bounded root functionals

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The notion of a root functional of a system of polynomials or an ideal of polynomials is a generalization of the notion of a root, in particular, for a multiple root. A basis of the space of all root functionals and a basis of the ideal are found by using the operation of extension of bounded root functionals when the number of equations is equal to the number of unknowns and if it is known that the number of roots is finite. The asymptotic complexity of these methods is $dO(n)$ operations, where $n$ is the number of equations and unknowns, $d$ is the maximal degree of polynomials.

Presence of roots at infinity leads to large degrees of polynomials in Buchberger algorithm for construction of a Gröbner basis of the ideal of polynomials [8]. Therefore the complexity of Buchberger algorithm such large, in the case of the 0-dimensional variety of roots it is equal to $dO(n^2)$ for the number of operations [9], where $d$ is the maximal degree of polynomials, $n$ is the number of variables. In the paper [10] it is shown the exactness of this estimation. For a system of polynomial equations, in which the number of polynomials is equal to the number of variables, the application of extension operations to bounded root functionals [6], [7] gradually cuts components of functionals, lying at infinity, not exiting over the limits of degrees $\leq (d_1 - 1) + \ldots + (d_n - 1)$, where $d_1, \ldots, d_n$ are degrees of polynomials. This allows, in the case, if it is known, that the variety of roots is 0-dimensional, to find a basis of the space of all root functionals of the system of polynomials and a basis of the ideal of polynomials in $O(D^4)$ operations, where $D = C^n_{d_1 + \ldots + d_n}$. A similar complexity is had by the method, based on the use of a multivariate resultant, that find all isolated roots of polynomials in $dO(n)$ operations, even in the case of the infinite number of roots at affine domain and at infinity [11].
Let $\mathbf{R}$ be a commutative ring with unity 1 and zero 0.
Let $x = (x_1, \ldots, x_n)$ be variables, $\mathbf{R}[x]$ be a ring of polynomials in variables $x$ with coefficients in $\mathbf{R}$.

In the paper we will use definition and assumption, given in [6,7].

**Lemma 1.** Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials. There holds:
1) a functional $L(x_*)$ annuls $(f(x))_x$ if and only if $\forall i = 1, s : L(x_*) \cdot f_i(x) = 0$;
2) a functional $L(x_*)$ annuls $(f(x))^{\leq d}_x$ if and only if $\forall i = 1, s : L(x_* \cdot f_i(x) = 0$ in $\mathbf{R}[x]^{\leq -\deg(f_i)]}$.

**Proof 1.** $L(x_*)$ annuls $(f(x))_x$ if and only if $\forall i = 1, s : 0 = L(x_*)f_i(x) \cdot g'f(x)$ for any $g'(x) \in \mathbf{R}[x]$. $L(x_* \cdot f_i(x)g'(x) = 0$ for any $g'(x) \in \mathbf{R}[x]^{-\deg(f_i)]}$ means that $L(x_*) \cdot f_i(x) = 0$.

**Proof 2.** $L(x_*)$ annuls $(f(x))^{\leq d}_x$ if and only if $\forall i = 1, s : 0 = L(x_*)f_i(x) \cdot g'f(x)$ for any $g'(x) \in \mathbf{R}[x]^{\leq d-\deg(f_i)]}$. $L(x_* \cdot f_i(x)g'(x) = 0$ for any $g'(x) \in \mathbf{R}[x]^{\leq d-\deg(f_i)]}$ means that $L(x_*) \cdot f_i(x) = 0$ in $\mathbf{R}[x]^{\leq d-\deg(f_i)]}$.

**Definition 1.** Let $\mathcal{V}$ be a module over $\mathbf{R}$, denote by $\mathcal{V}_*$ the set of all linear over $\mathbf{R}$ maps $\mathcal{V} \to \mathbf{R}$. Let $\mathcal{U}$ be a submodule of the module $\mathcal{V}$ over $\mathbf{R}$, denote by $\mathcal{U}^1$ the set of all $l \in \mathcal{V}_*$, annuling $\mathcal{U}$, i. e. such that $\forall F \in \mathcal{U}: l.F = 0$.

**Definition 2.** Let $\mathcal{U}, \mathcal{V}, \mathcal{G}$ be sets, let $l : \mathcal{V} \to \mathcal{G}$ be a map, let $\mathcal{U} \subseteq \mathcal{V}$. Denote by $l|\mathcal{U}$ the restriction of the map $l$ on the set $\mathcal{U}$, i. e. such a map $l' : \mathcal{U} \to \mathcal{G}$, that $\forall F \in \mathcal{U} : l'.F = l.F$.

**Statement 1.** Let $\mathcal{U}$ be a submodule of a module $\mathcal{V}$ over $\mathbf{R}$, let $\mathcal{L}$ be a submodule of the module $\mathcal{V}_*$ over $\mathbf{R}$. If $l_1, l_2 \in \mathcal{L}$, then $l_1 = l_2$ in $\mathcal{U}$ if and only if $l_1 - l_2 \in \mathcal{U}^1$, the last means that $l_1/l^1 = l_2/l^1$. Hence, there is an isomorphism $\mathcal{L}|\mathcal{U} \simeq \mathcal{L}/\mathcal{U}^1$ such that $l_1|\mathcal{U} \leftrightarrow l_2|\mathcal{U}^1$ for any $l \in \mathcal{L}$.

**Theorem 1.** Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1)$.

1. Let $\delta, \delta' \geq 0$. If $L(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta}_x$, $L'(x_*) = L(x_*)$ in $\mathbf{R}[x]^{\leq \delta_f + \delta}$, then $L'(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta'}_x$.
2. Let $0 \leq \delta_f \leq \delta_1, 0 \leq \delta_f + \delta_2 \leq \delta_2, 0 \leq \delta \leq \delta_1 + \delta_2 + 1, 0 \leq \delta' \leq \delta_1' + \delta_2' + 1$. Let $L_1(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta_1}_x$, $L_2(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta_2}_x$, then $L_1(x_*) \ast L_2(x_*)$ annuls $(f(x))^{\leq \delta_f + \delta}_x$. If $L_1'(x_*) = L_1(x_*)$ in $\mathbf{R}[x]^{\leq \delta_f + \delta_1'}$, $L_2'(x_*) = L_2(x_*)$ in $\mathbf{R}[x]^{\leq \delta_f + \delta_2'}$, then $L_1'(x_*) \ast L_2'(x_*) = L_1(x_*) \ast L_2(x_*)$ in $\mathbf{R}[x]^{\leq \delta_f + \delta}$.

From above it follows that the extension map $\ast$ for functionals induces the map

$$\frac{(f(x))^{\leq \delta_f + \delta_1}_x \downarrow}{\mathbf{R}[x]^{\leq \delta_f + \delta_1}} \times \frac{(f(x))^{\leq \delta_f + \delta_2}_x \downarrow}{\mathbf{R}[x]^{\leq \delta_f + \delta_2}} \to \frac{(f(x))^{\leq \delta_f + \delta}_x \downarrow}{\mathbf{R}[x]^{\leq \delta_f + \delta}}.$$

or, in other words, induces the map

$$((f(x))^{\leq \delta_f + \delta_1}_x \downarrow|_{\mathbf{R}[x]^{\leq \delta_f + \delta_1'}} \times ((f(x))^{\leq \delta_f + \delta_2}_x \downarrow|_{\mathbf{R}[x]^{\leq \delta_f + \delta_2'}}) \to (((f(x))^{\leq \delta_f + \delta}_x \downarrow|_{\mathbf{R}[x]^{\leq \delta_f + \delta}}.$$
Proof 1. Let $F(x) \in (f(x))^\leq \delta_f + \delta \cap \mathbb{R}[x^\leq \delta_f + \delta']$, $L'(x) = L(x) \cdot F(x)$, since $L'(x) = L(x) \cdot F(x)$ in $\mathbb{R}[x^\leq \delta_f + \delta'] \ni F(x)$ and $L(x) \cdot F(x) = 0$, so $L(x)$ annihilates $(f(x))^\leq \delta_f + \delta \ni F(x)$. Then, by the arbitrariness of $F(x) \in (f(x))^\leq \delta_f + \delta \cap \mathbb{R}[x^\leq \delta_f + \delta']$, $L'(x)$ annihilates $(f(x))^\leq \delta_f + \delta \cap \mathbb{R}[x^\leq \delta_f + \delta']$.

Proof 2. Since $L_1(x_*)$ annihilates $(f(x))^\leq \delta_f + \delta_1$, $L_2(x_*)$ annihilates $(f(x))^\leq \delta_f + \delta_2$, then by virtue of 2 of theorem 3 in [6] $L_1(x_*) \ast L_2(x_*)$ annihilates $(f(x))^\leq \delta_f + \delta_1 + \delta_2 + 1$ $(f(x))^\leq \delta_f + \delta$, hence, annihil $(f(x))^\leq \delta_f + \delta$.

Since $L_1(x_*)$ annihilates $(f(x))^\leq \delta_f + \delta_1 \geq (f(x))^\leq \delta_f + \delta_1'$, then annihil and $(f(x))^\leq \delta_f + \delta'$, since $L_2(x_*)$ annihil $(f(x))^\leq \delta_f + \delta_2 \geq (f(x))^\leq \delta_f + \delta_2'$, then annihil and $(f(x))^\leq \delta_f + \delta_2'$. Then, since $L_1(x_*) = L_1(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta_1]$ and $L_2(x_*) = L_2(x_*)$, $L_1(x_*) \ast L_2(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta_2]$, by virtue of 3 of theorem 3 in [6] $L_1(x_*) \ast L_2(x_*) = L_1(x_*) \ast L_2(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta_1 + \delta_2 + 1] \supseteq \mathbb{R}[x^\leq \delta_f + \delta]$, hence, $L_1(x_*) \ast L_2(x_*) = L_1(x_*) \ast L_2(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta]'$. Two last statements are obtained by applying of statement 1.

Definition 3. Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $d \geq 0$. Denote by $\mathcal{P}_x^\leq d$ a linear over $\mathbb{R}$ map $\mathbb{R}[x] \to \mathbb{R}[x]$ such that $\mathcal{P}_x^\leq d \cdot x^\alpha = x^\alpha$, if $|\alpha| \leq d$, and $\mathcal{P}_x^\leq d \cdot x^\alpha = 0$, if $|\alpha| > d$.

Statement 2. Let $x = (x_1, \ldots, x_n)$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, let $d \geq 0$.

If $l(x_*)$ is annihil $\mathbb{R}[x]* \mathbb{R}[x]$, then $l(x_*) = L(x_*)$, $\mathcal{P}_x^\leq d \in \mathbb{R}[x^\leq d]$ and $L(x_*) \cdot \mathcal{P}_x^\leq d = 0$ in $\mathbb{R}[x^\geq d]$.

If $l(x_*)$ is annihil $\mathbb{R}[x]^\leq d$, then the functional $l(x_*) = l(x_*)$, $\mathcal{P}_x^\leq d \in \mathbb{R}[x]^\leq d$, and is continuation of $l(x_*)$, $\mathcal{P}_x^\leq d \in \mathbb{R}[x]^\leq d$, moreover, $l(x_*) \cdot \mathcal{P}_x^\leq d = 0$ in $\mathbb{R}[x^\geq d]$.

Proof. Let $|\alpha| \leq d$, then $L(x_*) \cdot \mathcal{P}_x^\leq d \cdot x^\alpha = L(x_*) \cdot x^\alpha$, $L(x_*) \cdot \mathcal{P}_x^\leq d \cdot x^\alpha = L(x_*) \cdot x^\alpha$.

Let $|\alpha| > d$, then $L(x_*) \cdot \mathcal{P}_x^\leq d \cdot x^\alpha = 0$. Since the monoms $x^\alpha$, for which $|\alpha| \leq d$, linearly over $\mathbb{R}$ generate $\mathbb{R}[x^\geq d]$, then $L(x_*) \cdot \mathcal{P}_x^\leq d = 0$ in $\mathbb{R}[x^\geq d]$.

The second part of the statement proved exactly as the first.

Commentary to theorem 1. In 2 of theorem 1 computation of $L_1(x_*) \ast L_2(x_*)$ on $\mathbb{R}[x^\leq \delta_f + \delta']$ is used values of $L_1(x_*)$ outside $\mathbb{R}[x^\leq \delta_f + \delta']$ and values of $L_2(x_*)$ outside $\mathbb{R}[x^\leq \delta_f + \delta']$, therefore necessary to determine values of $L_1(x_*)$ outside $\mathbb{R}[x^\leq \delta_f + \delta']$ and values of $L_2(x_*)$ outside $\mathbb{R}[x^\leq \delta_f + \delta']$. With computational point of view it is convenient to determine the action of $L_1(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta']$, and the action of $L_2(x_*)$ in $\mathbb{R}[x^\leq \delta_f + \delta']$ as zeroes. This holds in the case, if we set $L_1(x_*) = L_1(x_*) \cdot \mathcal{P}_x^\leq \delta_f + \delta'$, $L_2(x_*) = L_2(x_*) \cdot \mathcal{P}_x^\leq \delta_f + \delta'$. It is enough to compute values of the functional $L_1(x_*) \ast L_2(x_*)$ only on $\mathbb{R}[x^\leq \delta_f + \delta']$.

Definition 4. Let $x = (x_1, \ldots, x_n)$, $y = x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials. A functional $E(x_*)$ we call a unit root functional of polynomials $f(x_*)$, if it annihilates $(f(x))^\leq \delta_f + x$, and $E(x_*) = E(y_*)$. den $\|\nabla f(x, y)\| = 1 + f(x) \cdot g(x)$. A functional $E'(x_*)$ we call a unit bounded root functional of polynomials $f(x_*)$, if it annihilates $(f(x))^\leq \delta_f + \varepsilon$, where $\varepsilon \geq 0$, and $E'(x_*) = E'(y_*)$, den $\|\nabla f(x, y)\| = 1 + f(x) \cdot g(x)$.

Theorem 2. Let $R$ be a field. Let $x = (x_1, \ldots, x_n)$ be variables, let $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials, $\delta_f = \sum_{i=1}^n (\deg(f_i) - 1)$. Let $R[x]/(f(x))_x$ be a finite-dimensional space over $R$, in this case there exist a unit root functional $E(x_*)$ of polynomials $f(x_*)$. 

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Let $\varepsilon \geq 0$, $\mathcal{A}(x_*)$ be the set of all functionals annulling $(f(x))^\leq_{x^2} \delta^f + \varepsilon$, $\mathcal{L}(x_*)$ be the set of all functionals annulling $(f(x))_x$, $\mathcal{U}(x) = \mathbb{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. Then:

1) $\mathcal{A}(x_*)|_{\mathcal{U}(x)}$ with the extension operations for functionals is an associative and commutative algebra over $\mathbb{R}$;

2) there exists $d$ such that $\mathcal{A}(x_*)^d|_{\mathcal{U}(x)} = \mathcal{A}(x_*)^{d+1}|_{\mathcal{U}(x)}$, and for any such $d$ there holds $\mathcal{L}(x_*)|_{\mathcal{U}(x)} = \mathcal{A}(x_*)^d|_{\mathcal{U}(x)} = (\mathcal{A}(x_*)|_{\mathcal{U}(x)})^d$.

**Proof 1.** By virtue of 2 of theorem 1 the extension operation for functionals induces the map $((f(x))_x^{\leq_{x^2} \delta^f + \varepsilon})_x^{-1}|_{\mathbb{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]} \times ((f(x))_x^{\leq_{x^2} \delta^f + \varepsilon})_x^{-1}|_{\mathbb{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]} \to ((f(x))_x^{\leq_{x^2} \delta^f + \varepsilon})_x^{-1}|_{\mathbb{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]}$.

since for $\delta^1 = \varepsilon$, $\delta^2 = \varepsilon$, $\delta^3 = \varepsilon$, $\delta^4 = \varepsilon$, $\delta^5 = \varepsilon$ there holds the condition 2 of this theorem. Hence, $\mathcal{A}(x_*)|_{\mathcal{U}(x)} = ((f(x))_x^{\leq_{x^2} \delta^f + \varepsilon})_x^{-1}|_{\mathbb{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]}$ is an algebra with the extension operation for functionals.

If $L_1(x_1), L_2(x_2) \in \mathcal{A}(x_*)$, then they annul $(f(x))^\leq_{x^2} \delta^f + \varepsilon$. Then by virtue of theorem 1 in [7] $L_1(x_1)* L_2(x_2) = L_2(x_2)* L_1(x_1) \in \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon} + 1] \supseteq \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$, and so, in $\mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. This implies the commutativity of $\mathcal{A}(x_*)|_{\mathcal{U}(x)}$.

If $L_1(x_1), L_2(x_2), L_3(x_3) \in \mathcal{A}(x_*)$, then they annul $(f(x))^\leq_{x^2} \delta^f + \varepsilon$. Then by virtue of 1 of theorem 2 in [7] $(L_1(x_1)* L_2(x_2) + L_3(x_3)) = L_1(x_1) + (L_2(x_2)* L_3(x_3)) \in \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon} + 2] \supseteq \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$, and so, in $\mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. This implies the associativity of $\mathcal{A}(x_*)|_{\mathcal{U}(x)}$.

**Proof 2.** In papers [1,3,4,5] there is the theorem about existence of a unit root functional of polynomials $f(x)$ in the case, when $\mathcal{R}[x]/(f(x))_x$ be a finite-dimensional space over $\mathbb{R}$.

An functionals in $\mathcal{A}(x_*)$ annul $(f(x))^\leq_{x^2} \delta^f + \varepsilon$, then by virtue of 2 of theorem 3 in [6] any functional $L'_{x_0} \in \mathcal{A}(x_*)^p$ annuls $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1)$, and so, annuls $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. By the finite dimensionality of $\mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$ over $\mathbb{R}$, there exists such $p$, that $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}] = (f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. Hence, any functional $L'_{x_0} \in \mathcal{A}(x_*)^d$ annuls $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. Then by virtue of 4 of theorem 6 in [7] the functional $L_{x_0} = L'_{x_0} + E(x_0)$ annuls $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$, then $L_{x_0} = L'_{x_0} + E(x_0)$ in $\mathcal{U}(x) = \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. Since $L'_{x_0}$ is an arbitrary element in $\mathcal{A}(x_*)^d$, and $L(x_0) \in \mathcal{L}(x_0)$, then $\mathcal{A}(x_*)^d|_{\mathcal{U}(x)} \subseteq \mathcal{L}(x_0)|_{\mathcal{U}(x)}$.

Let $L_{x_0}$ is an arbitrary element in $\mathcal{L}(x_0)$, then $L_{x_0}$ annuls $(f(x))_x$. By virtue of 2 of theorem 6 in [7] $L_{x_0} + E(x_0) = L_{x_0}$. Since and $E(x_0) \in \mathcal{L}(x_0)$, then $L(x_0) + \mathcal{L}(x_0) = \mathcal{L}(x_0)$, and so, $\mathcal{L}(x_0)^d = \mathcal{L}(x_0)$. There holds $\mathcal{L}(x_0) \subseteq \mathcal{A}(x_0)$, since any functional, annulling $(f(x))_x$, annuls $(f(x))^\leq_{x^2} \delta^f + p\varepsilon + (p-1) \cap \mathcal{R}[x^{\leq_{x^2} \delta^f + \varepsilon}]$. Hence, $\mathcal{L}(x_0) = \mathcal{A}(x_0)^d$, and so, $\mathcal{L}(x_0)|_{\mathcal{U}(x)} \subseteq \mathcal{A}(x_0)^d|_{\mathcal{U}(x)}$.

From above it follows that $\mathcal{A}(x_0)^d|_{\mathcal{U}(x)} = \mathcal{L}(x_0)|_{\mathcal{U}(x)}$.

Since by virtue of 1 of the theorem the extension map * for functionals induces the map $\mathcal{A}(x_0)|_{\mathcal{U}(x)} \times \mathcal{A}(x_0)|_{\mathcal{U}(x)} \to \mathcal{A}(x_0)|_{\mathcal{U}(x)}$, then $(\mathcal{A}(x_0)|_{\mathcal{U}(x)})^d = \mathcal{A}(x_0)^d|_{\mathcal{U}(x)}$. 
Algorithm. (Finding a basis of all root functionals and a basis of the ideal of polynomials, and also the unit root functional.) Let $\mathbf{R}$ be a field, let $x = (x_1, \ldots, x_n)$, $y \sim x$ be variables, $f(x) = (f_1(x), \ldots, f_n(x))$ be polynomials. Let $\mathbf{R}[x]/(f(x))_x$ be a finite-dimensional space over $\mathbf{R}$. Denote by $\delta_f = \sum_{i=1}^{n}(\deg(f_i) - 1)$, $\mathcal{L}(x_\ast) = ((f(x))_x)^\perp$, $\mathcal{U}(x) = \mathbf{R}[x^{\lessdot \delta_f}]$. Here and below by space we shall mean a linear space over $\mathbf{R}$.

The algorithm finding a basis of the space $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)} = ((f(x))_x)^\perp|_{\mathbf{R}[x^{\lessdot \delta_f}]}$ of restrictions of all root functionals on $\mathbf{R}[x^{\lessdot \delta_f}]$ and a basis of the space $(f(x))_x \cap \mathbf{R}[x^{\lessdot \delta_f}]$, and also the restriction of the unit root functional on $\mathbf{R}[x^{\lessdot \delta_f}]$ consists of the following steps:

1. Construct by Gauss elimination method a basis of the space $(f(x))_x^{\lessdot \delta_f}$.
2. From Gauss basis of the space $(f(x))_x^{\lessdot \delta_f}$ construct Gauss basis of the space of functionals defined on $\mathbf{R}[x^{\lessdot \delta_f}]$ and annulling $(f(x))_x^{\lessdot \delta_f}$, this space coincide with $\mathcal{A}(x_\ast)|_{\mathcal{U}(x)} = ((f(x))_x^{\lessdot \delta_f})^{\perp}|_{\mathbf{R}[x^{\lessdot \delta_f}]}$. Let obtained basis be $L_1(x_\ast), \ldots, L_d(x_\ast)$.
3. Compute the restriction of operators $[L_1(x_\ast)], \ldots, [L_d(x_\ast)]$ on $\mathbf{R}[x^{\lessdot \delta_f}]$.
4. Compute the restriction of functionals $(L_1(x_\ast))^d, \ldots, (L_d(x_\ast))^d$ on $\mathbf{R}[x^{\lessdot \delta_f}]$ by

$$\forall p = 1, d : \forall \delta = 2, d : (L_p(x_\ast))^{\delta} = (L_p(x_\ast))^{\delta - 1}. [L_p(x_\ast)] \text{ in } \mathbf{R}[x^{\lessdot \delta_f}].$$

5. Compute the following generators:

$$\{(L_p(x_\ast))^d \ast L_q(x_\ast)|p = 1, d \& q = 1, d\}$$

of the space $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)} = (\mathcal{A}(x_\ast)|_{\mathcal{U}(x)})^{d+1}$.

6. By Gauss elimination method construct a basis of the space $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)}$ from its system of generators.

7. From Gauss basis of the space $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)}$ construct Gauss basis of the space of polynomials $\in \mathcal{U}(x) = \mathbf{R}[x^{\lessdot \delta_f}]$ annulling by $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)}$. This space of polynomials coincide with $(f(x))_x \cap \mathbf{R}[x^{\lessdot \delta_f}]$.

8. Let $h_1(x), \ldots, h_d'(x)$ be a basis of the space $(f(x))_x \cap \mathbf{R}[x^{\lessdot \delta_f}]$, let $l_1(x), \ldots, l_d''(x)$ be a basis of the space $\mathcal{L}(x_\ast)|_{\mathcal{U}(x)}$. From $\{l_p(y_\ast), \det \|\nabla f(x, y)\| | p = 1, d''\}$ and $\{h_q(x)|q = 1, d'\}$ by Gauss elimination method find the decomposition

$$\sum_{p=1}^{d''} a_p \cdot (l_p(y_\ast), \det \|\nabla f(x, y)\|) + \sum_{q=1}^{d'} h_q \cdot h_q(x) = 1,$$

$E'(x_\ast) = \sum_{p=1}^{d''} a_p \cdot l_p(x_\ast)$ is the restriction of the unit root functional of polynomials $f(x)$ on $\mathbf{R}[x^{\lessdot \delta_f}]$, since $E'(y_\ast), \det \|\nabla f(x, y)\| - 1 \in (f(x))_x$. 

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Proof of the algorithm.
5. The dimension of the space \( A(x_*)|_{U(x)} \) is equal to \( d \). Therefore the chain

\[
(A(x_*)|_{U(x)})^1 \supseteq \ldots \supseteq (A(x_*)|_{U(x)})^d \supseteq (A(x_*)|_{U(x)})^{d+1} \supseteq \ldots
\]
is stabilized for some \( \delta \leq d + 1 \), i.e. \( (A(x_*)|_{U(x)})^{d'} = (A(x_*)|_{U(x)})^{d'+1} \) for any \( d' \geq \delta \). Then

\[
(A(x_*)|_{U(x)})^{d+1} = (A(x_*)|_{U(x)})^{d+2} = \ldots = (A(x_*)|_{U(x)})^{d+\delta'} = \ldots
\]

Any element in \( (A(x_*)|_{U(x)})^{d+1} = (A(x_*)|_{U(x)})^{d+(d-1)+2} \) generated by elements of the form

\[
(L_1(x_*))^\alpha_1 \ast \ldots \ast (L_d(x_*))^\alpha_d = (L_p(x_*))^d \ast \left( (L_p(x_*))^\alpha_p - \left( \prod_{q \neq p} (L_q(x_*))^\alpha_q \right) \right) = (L_p(x_*))^d \ast L(x_*).
\]

Here \( L(x_*) \in A(x_*)|_{U(x)} \), since \( \alpha_1 + \ldots + \alpha_d - d = d \cdot (d-1)+2-d = d \cdot (d-2)+2 = (d-1)^2+1 \geq 1 

Then \( L(x_*) \) is expressed via \( L_1(x_*) \), \ldots, \( L_d(x_*) \) linearly over \( \mathbb{R} \). Hence, the space \( (A(x_*)|_{U(x)})^{d+1} \) is generated by generators

\[
\{(L_p(x_*))^d \ast L_q(x_*)| p = 1, d & q = 1, d\}.
\]

That \( L(x_*)|_{U(x)} = (A(x_*)|_{U(x)})^{d+1} \) is stated in 2 of theorem 2.

7. Any functional in \( L'(x_*) = L(x_*)|_{U(x)} = \left( (f(x_*)|_{x})\right)_x |_{R[x \leq \delta]} \) annihilates the space \( \mathcal{M}(x) = (f(x_*)|_{x}) \subseteq \mathcal{M}(x) \). Here we consider the annule of the space \( \mathcal{M}(x) \) as a subspace of the space \( U(x) \), and the annulet of the space \( L'(x_*) \) as a subspace of the space \( U(x) \). Let a functional \( l(x_*) \), determined on \( U(x) = R[x \leq \delta] \), annihilates \( \mathcal{M}(x) = (f(x_*)|_{x}) \subseteq \mathcal{M}(x) \). Then by virtue of statement 2 functional \( L'(x_*) = l(x_*) \), \( P^{<\delta} \), is determined on \( R[x] \), and \( L'(x_*) \) annihilates \( \mathcal{M}(x) = (f(x_*)|_{x}) \subseteq \mathcal{M}(x) \). Then by virtue of 4 of theorem 6 in [7] there exists \( L(x_*), \) annihilating \( (f(x_*)|_{x}) \), such that \( L(x_*)|_{U(x)} = L(x_*) \). Since \( L(x_*) \subseteq \mathcal{M}(x) \), then \( l(x_*) \subseteq L(x_*) \). Hence, \( \mathcal{M}(x)^{\perp} \subseteq L'(x_*) \). Thus \( \mathcal{M}(x) = \mathcal{M}(x)^{\perp} \). Then by virtue of the finite dimensionality of the space \( U(x) = R[x \leq \delta] \) there holds \( \mathcal{M}(x) = L'(x_*)^{\perp} \). Here we identify \( (U(x_*), \ast) \) with \( U(x) \).

Estimation of the complexity of the algorithm. Let \( D \) be a dimension of the space \( R[x \leq \delta] \), then \( D = C^{\delta}_{D} + \ldots + C^{\delta}_{D} \), where \( d_i = \deg(f_i) \). Let us estimate the complexity of steps of the algorithm.

1. The number of polynomials in system of polynomials \{\( f_i(x) \cdot x^{\alpha(i)} \in R[x \leq \delta] \mid |i| = 1, n \} \) not exceed \( n \cdot D \). Construction of a basis from this system of polynomials by Gauss elimination method requires \( \leq (n \cdot D) \cdot O(D^2) = n \cdot O(D^3) \) operations.

2. The step requires \( \leq O(D^2) \) operations.
3. Computation of all minors of the matrix
\[
\begin{vmatrix}
\nabla f(x, y) \\
\quad f(x)
\end{vmatrix}
\]
of order \(n\) without divisions requires \(\leq (\delta f \cdot n^2 + n^4) \cdot O(D^3)\) operations. Within this it computing and det \(|\nabla f(x, y)|\). Computation of the operator
\[
[L_p(x_s)] = L_p(y_s) \cdot \det \begin{vmatrix}
\nabla f(x, y) \\
\quad f(x) \\
\quad \nabla_x(x, y) \\
\quad 1_x(x)
\end{vmatrix}
\]
on \(R[x^{\leq\delta_f}]\) requires \(O(D^3)\) operations. Computation of such operators for all \(p = 1, d\) requires \(\leq d \cdot O(D^3) \leq D \cdot O(D^3) = O(D^4)\) operations.

4. Computation of \((L_p(x_s))^d = (L_p(x_s))^{d-1}, [L_p(x_s)]\) requires \(O(D^2)\) operations, and for all \(\delta = 2, d\) and \(p = 1, d\) requires \(\leq d^2 \cdot O(D^2) \leq D^2 \cdot O(D^2) = O(D^4)\) operations.

5. Computation of \((L_p(x_s))^d \ast L_q(x_s) = (L_p(x_s))^d, [L_q(x_s)]\) requires \(O(D^2)\) operations. Since this computation necessary to perform for all \(p = 1, d\) and of all \(q = 1, d\), then in all performed \(\leq d^2 \cdot O(D^2) \leq D^2 \cdot O(D^2) = O(D^4)\) operations.

6. Computation of a basis of the space \(L(x_s)\) from \(d^2\) generators by Gauss elimination method requires \(\leq d^2 \cdot O(D^2) \leq D^2 \cdot O(D^2) = O(D^4)\) operations.

7. The step requires \(\leq O(D^2)\) operations.

8. Computation of \(L_p(y_s) \cdot \det |\nabla f(x, y)|\) for single \(p\) requires \(O(D^2)\) operations, then computation for all \(p = 1, d\) and \(d'\) requires \(\leq d'' \cdot O(D^2) \leq D \cdot O(D^2) = O(D^3)\) operations. Decomposition of 1 by Gauss elimination method requires \(\leq O(D^3)\) operations, and computation of \(E'(x_s) = \sum_{p=1}^{d'} a_p \cdot L_p(x_s)\) requires \(\leq d'' \cdot O(D) \leq D \cdot O(D) \leq O(D^2)\) operations.

If to regard \(n\) as constant, then the summarized number of operations performed in the algorithm is \(\leq O(D^4)\).

**Theorem 3.** Let \(x = (x_1, \ldots, x_n)\) be variables, \(f(x) = (f_1(x), \ldots, f_n(x))\) be polynomials, \(\delta_f = \sum_{i=1}^{n} (\deg(f_i) - 1)\). Let \(R[x]/f(x)_{x}\) be a finite generated as module over \(R\), then
\[
(f(x))_x \cap R[x^{\leq\delta_f + \delta}] = ((f(x))_x \cap R[x^{\leq\delta_f}]) \cdot R[x^{\leq\delta_f + \delta}],
\]
\[
(f(x))_x \cap R[x^{\leq\delta_f + \delta + \varepsilon + 1}] = ((f(x))_x \cap R[x^{\leq\delta_f + \delta + 1}]) \cdot R[x^{\leq\delta}],
\]
where \(\delta \geq 0\) and \(\varepsilon \geq 0\).

Proof of theorem 3 will be given in the subsequent papers.

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