Conformal Field Theory Correlators from Classical Scalar Field Theory on $AdS_{d+1}$

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Abstract

We use the correspondence between scalar field theory on $AdS_{d+1}$ and a conformal field theory on $\mathbb{R}^d$ to calculate the 3- and 4-point functions of the latter. The classical scalar field theory action is evaluated at treelevel.

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1 Introduction

Since the suggestion of Maldacena about the equivalence of the large $N$ limit of certain conformal field theories in $d$ dimensions on one hand and supergravity on $AdS_{d+1}$ on the other hand [1], theories on Anti de Sitter spaces seem to have undergone a renaissance. After detailed investigations in the past (see for example [2, 3, 4]), there has been a multitude of papers relating to this subject in various aspects in the last months alone (see [5] for a recent list of references). In particular, the suggested correspondence was made more precise in [6, 7, 8]. According to these references one identifies the partition function of the $AdS$ theory (with suitably prescribed boundary conditions for the fields) with the generating functional of the boundary conformal field theory. Thus, one has schematically

$$Z_{AdS}[\phi_0] = \int_{\phi_0} D\phi \exp(-I[\phi]) \equiv Z_{CFT}[\phi_0] = \langle \exp \left( \int_{\partial \Omega} d^d x \ O \phi_0 \right) \rangle.$$  \hspace{1cm} (1)

The path integral on the l.h.s. is calculated under the restriction that the field $\phi$ asymptotically approaches $\phi_0$ on the boundary. On the other hand, the function $\phi_0$ is considered as a current, which couples to the scalar density operator $O$ in the boundary conformal field theory. Calculating the l.h.s. of (1) thus allows one to explicitly obtain correlation functions of the boundary conformal field theory. Of course, since the 2- and 3-point functions are fixed (up to a constant) by conformal invariance [9], one is especially interested in calculating the cases $n > 3$.

It is not only of pedagogical interest to consider the classical approximation to the $AdS$ partition function, which is obtained by inserting the solutions of the classical field equations into $S[\phi]$. In fact, the suggested $AdS/CFT$ correspondence [1] involves classical supergravity on the $AdS$ side. Moreover, it is instructive to study toy examples in order to better understand this correspondence. A number of examples, including free massive scalar and $U(1)$ gauge fields were studied in [8] and free fermions were considered in [5, 10]. Since a free field theory will inevitably lead to a trivial (i.e. free) boundary CFT, we feel it necessary to consider interactions. A short note on interacting scalar fields is contained in [11]. In this paper we will consider in detail a classical interacting scalar field on $AdS_{d+1}$ at tree level.

We recall here for convenience the formulae necessary for solving the classical scalar field theory with Dirichlet boundary conditions. Let us start with stating the action for a real
scalar field in $d + 1$ dimensions (Riemannian signature) with polynomial interaction,
\[ I[\phi] = \int_{\Omega} d^{d+1}x \sqrt{g} \left[ \frac{1}{2} \left((\nabla \phi)^2 + m^2 \phi^2 \right) + \sum_{n \geq 3} \frac{\lambda_n}{n!} \phi^n \right]. \] (2)

The action (2) yields the equation of motion
\[ (\nabla^2 - m^2)\phi = \sum_{n \geq 3} \frac{\lambda_n}{(n-1)!} \phi^{n-1}. \] (3)

Using the covariant Green’s function, which satisfies
\[ (\nabla^2 - m^2)G(x, y) = \frac{\delta(x - y)}{\sqrt{g(x)}} \] (4)
and the boundary condition
\[ G(x, y)|_{x \in \partial \Omega} = 0, \]
the classical field $\phi$ satisfying the equation of motion (3) and a Dirichlet boundary condition on $\partial \Omega$ satisfies the integral equation
\[ \phi(x) = \int_{\partial \Omega} d^d y \, \sqrt{h} \, n^\mu \frac{\partial}{\partial y^\mu} G(x, y) \phi(y) + \int_{\Omega} d^{d+1}y \sqrt{g} G(x, y) \sum_{n \geq 3} \frac{\lambda_n}{(n-1)!} \phi(0)^{n-1}, \] (5)
where $h$ is the determinant of the induced metric on $\partial \Omega$ and $n^\mu$ the unit vector normal to $\partial \Omega$ and pointing outwards. A perturbative expansion in the couplings $\lambda_n$ is obtained by using (5) recursively. We shall denote the surface term in (5) by $\phi^{(0)}$ and the remainder by $\phi^{(1)}$. Then, substituting the classical solution (5) into (2), integrating by parts and using the properties of the Green’s function one obtains to tree level
\[ I[\phi] = \frac{1}{2} \int_{\partial \Omega} d^d x \sqrt{h} \, n^\mu \phi^{(0)} \partial_{\rho} \phi^{(0)} + \sum_{n \geq 3} \frac{\lambda_n}{n!} \int_{\Omega} d^{d+1}x \sqrt{g} \, (\phi^{(0)})^n. \] (6)

A short outline of the remainder of this paper is as follows. In Sec. 2 we consider the free field on $AdS_{d+1}$. We explicitly calculate the solutions to the wave equation, the Green’s function, solve the Dirichlet boundary problem and find the 2-point function of the boundary conformal field theory. In Sec. 3 we perform the calculations at tree level. An explicit closed formula for the $n$-point function does not seem attainable for $n > 3$. However, we will stay general as far as possible and only then specialize in the cases $n = 3$ and $n = 4$. Finally, Sec. 4 contains conclusions.
2 Free Field Theory on $AdS_{d+1}$

We will use the representation of $AdS_{d+1}$ as the upper half space $(x_0 > 0)$ with the metric

$$ds^2 = \frac{1}{x_0^2} \sum_{i=0}^{d} dx_i^2,$$

(7)

which possesses the constant curvature scalar $R = -d(d + 1)$. The boundary $\partial \Omega$ is given by the space $\mathbb{R}^d$ with $x_0 = 0$ plus the single point $x_0 = \infty$. In the sequel we shall adopt the notations $x = (x_0, \mathbf{x})$, $x^* = (-x_0, \mathbf{x})$ and $x^2 = x_0^2 + \mathbf{x}^2$.

Let us first solve the “massive” wave equation

$$(\nabla^2 - m^2) \phi = \left( x_0^2 \sum_{i=0}^{d} \partial^2_i - x_0(d-1)\partial_0 - m^2 \right) \phi = 0. \tag{8}$$

The linearly independent solutions of (8) are found to be

$$x_0^\frac{d}{2} e^{-ik(x_0)} \begin{cases} I_\alpha(kx_0) \\ K_\alpha(kx_0) \end{cases}, \quad \text{where} \quad \alpha = \sqrt{\frac{d^2}{4} + m^2}, \tag{9}$$

$k$ is a momentum $d$-vector and $k = |k|$. It is easy to check that these modes are not square integrable, if $m^2 \geq -d^2/4$.

The modes can now be used to calculate the Green’s function on (4). Making the ansatz

$$G(x, y) = \int \frac{d^d k}{(2\pi)^d} x_0^\frac{d}{2} e^{-ik(x-y)} f(k, y) \begin{cases} I_\alpha(kx_0)K_\alpha(ky_0) \quad \text{for} \quad x_0 < y_0, \\ K_\alpha(kx_0)I_\alpha(ky_0) \quad \text{for} \quad x_0 > y_0, \end{cases} \tag{10}$$

we explicitly satisfy the boundary condition at $x_0 = 0$ and $\infty$ and ensure continuity at $x_0 = y_0$. Matching the two regions at the discontinuity yields $f = -y_0^\frac{d}{2}$. The ansatz (10) can be integrated and gives

$$G(x, y) = -\frac{c}{2\alpha} \xi^{-\Delta} F \left( \frac{d}{2}, \Delta; \alpha + 1; \frac{1}{\xi^2} \right), \tag{11}$$

where $F$ denotes the hypergeometric function [12],

$$\xi = \frac{1}{2x_0 y_0} \left[ \frac{1}{2} \left( (x - y)^2 + (x - y^*)^2 \right) + \sqrt{(x - y)^2(x - y^*)^2} \right]$$

(12)
and the new constants are defined by \( \Delta = d/2 + \alpha \) and \( c = \Gamma(\Delta)/(\pi^{\frac{d}{2}} \Gamma(\alpha)) \). The Green’s function \( G_0 \) coincides with the one found by Burgess and Lütken \([4]\) after using a transformation formula for the hypergeometric function \([12, \text{formula 9.134 2.}]\). Our form has the advantage that for even \( d \) the result can, using either special value formulae or the definition as a series, be expressed in terms of rational functions. For example, for \( d = 2 \) we can use

\[
F(1, 1 + \alpha; 1 + \alpha; z) = \frac{1}{1 - z}.
\]

We shall in this paper make use only of the boundary behaviour of the Green’s function. Since the induced metric diverges on the boundary of \( AdS_{d+1} (x_0 = 0) \), one has to consider the standard formalism described in Sec. 1 on a near-boundary surface \( x_0 = \epsilon > 0 \) and then take the limit \( \epsilon \to 0 \). It has been pointed out recently by Freedman et al. \([13]\) that this limit has to be taken carefully, in particular at the very end of those calculations, which involve only the boundary behaviour of the classical solution. It is therefore necessary to find the Green’s function, which vanishes not at \( x_0 = 0 \), but at \( x_0 = \epsilon \). One can easily change \( G_0 \) to accommodate this. Denoting the new Green’s function by \( G_\epsilon \), we find

\[
G_\epsilon(x, y) = G_0(x, y) + \int \frac{d^d k}{(2\pi)^d} (x_0 y_0)^{\frac{d}{2}} e^{-i k \cdot (x - y)} K_\alpha(k x_0) K_\alpha(k y_0) \frac{I_\alpha(k \epsilon)}{K_\alpha(k \epsilon)},
\]

where \( G_0 \) is given by \( G_0 \) and \( G_0 \). It does not seem possible to perform the momentum integral, but this is not necessary in order to obtain the desired boundary behaviour. In particular, we find the normal derivative on the boundary as

\[
\frac{\partial}{\partial y_0} G_\epsilon(x, y) \bigg|_{y_0 = \epsilon} = -x_0^d \epsilon^{d-1} \int \frac{d^d k}{(2\pi)^d} e^{-i k \cdot (x - y)} K_\alpha(k x_0) K_\alpha(k y_0) \frac{I_\alpha(k \epsilon)}{K_\alpha(k \epsilon)},
\]

giving \(-\epsilon^{d-1}\delta(x - y)\) for \( x_0 = \epsilon \).

The bulk behaviour of the free field can be obtained from \( G_\epsilon \) using the asymptotic behaviour of the Bessel function in the denominator of \( G_0 \) for \( \epsilon \to 0 \). We note that for \( AdS_{d+1} \) one has \( \sqrt{h(y)} = \epsilon^{-d} \) and \( n^\mu = (-\epsilon, 0) \). The minus sign comes from \( n^\mu \) pointing outward. One finds

\[
\phi^{(0) bulk}(x) = c \epsilon^{\Delta - d} \int d^d y \phi_\epsilon(y) \left( \frac{x_0}{x_0^2 + |x - y|^2} \right)^\Delta,
\]

where \( \phi_\epsilon \) denotes the Dirichlet boundary value at \( x_0 = \epsilon \). We define

\[
\phi_0(x) = \epsilon^{\Delta - d} \phi_\epsilon(x)
\]
in order to make contact with the conformal field theory on the boundary of $AdS_{d+1}$.

Eqn. (13) is the solution to the Dirichlet problem with the boundary at $x_0 = 0$ \[8\]. However, for the two-point function we need to calculate the surface integral in (8), i.e. we need the near-boundary behaviour for a boundary at $x_0 = \epsilon$. Using the exact expression (14) we find

$$\partial_0 \phi |_{x_0 = \epsilon} = \frac{1}{\epsilon} \int d^d y \phi_\epsilon (y) \int \frac{d^d k}{(2\pi)^d} e^{-ik(x-y)} \left[ \frac{d}{2} - \alpha + k \frac{\partial}{\partial k} \ln \left( (ke)^\alpha K_\alpha(ke) \right) \right].$$

The first two terms in the squared bracket yield \(\delta\) function contact terms in the two-point function, which are of no interest to us. In the third term, the divergence of the Bessel function for $\epsilon \to 0$ is exactly cancelled by the power of $\epsilon$ in front of it. Using the series expansion

$$z^\alpha K_\alpha(z) = 2^{\alpha-1} \Gamma(\alpha) \left[ 1 - \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} \left( \frac{z}{2} \right)^{2\alpha} + \cdots \right],$$

where the dots denote terms of order $z^n$ and $z^{2\alpha + n}$, one can approximate the logarithm and then evaluate the integral to obtain

$$\partial_0 \phi |_{x_0 = \epsilon} = 2\alpha c \epsilon^{2\alpha-1} \int d^d y \frac{\phi_\epsilon (y)}{|x-y|^{2\Delta}} + \cdots. \tag{17}$$

Inserting (17) into (8) we find the value of the free field action as

$$I^{(0)} = -\frac{1}{2} \int d^d x d^d y 2\alpha c (\Delta - d) \frac{\phi_\epsilon (x) \phi_\epsilon (y)}{|x-y|^{2\Delta}} + \cdots. \tag{18}$$

Taking the limit $\epsilon \to 0$ with the definition (16) we hence obtain, in agreement with (13), the two-point function for the boundary conformal operators,

$$\langle O(x) O(y) \rangle = \frac{2\alpha c}{|x-y|^{2\Delta}}. \tag{19}$$

### 3 Calculations at Tree Level

At tree level, one can take the limit $\epsilon \to 0$ beforehand, which makes the considerations somewhat easier. The reason is that the tree level calculations involve bulk integrals over $AdS_{d+1}$, as in the second term of (11). Hence only the bulk behaviour of the free field will be
needed, which was obtained in Sec. 2. Inserting (15) (with \( \epsilon \to 0 \)) into the interaction term of the action one obtains

\[
I^{(1)}[\phi_0] = \sum_{n \geq 3} c^n \lambda_n \int d^d x_1 \ldots d^d x_n \phi_0(x_1) \ldots \phi_0(x_n) I_n(x_1, \ldots, x_n), \tag{20}
\]

with

\[
I_n(x_1, \ldots, x_n) = \int d^{d+1} y \frac{y_0^{-(d+1)+n\Delta}}{[(y_0^2 + |y - x_1|^2) \ldots (y_0^2 + |y - x_n|^2)]^{\frac{1}{2}\Delta}}. \tag{21}
\]

We can read off the connected part of the tree level \( n \)-point functions \((n \geq 3)\) for the operator \( \mathcal{O} \) from (20),

\[
\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle_{\text{conn.}} = -\lambda_n c^n I_n(x_1, \ldots, x_n). \tag{22}
\]

We shall now elaborate on a detailed calculation of the 3- and 4-point functions. After a Feynman parametrization the \( y \) integral in (21) can be done yielding

\[
I_n = \pi \frac{\frac{1}{2} \Gamma \left( \frac{d}{2} \Delta - \frac{d}{2} \right) \Gamma \left( \frac{n}{2} \Delta \right)}{2 \Gamma(\Delta)^n} \int_0^\infty d\alpha_1 \ldots d\alpha_n \frac{\prod \alpha_i^{\Delta-1}}{(\sum \alpha_i \alpha_j x_{ij}^2)^{\frac{1}{2}\Delta}},
\]

where \( x_{ij} = |x_i - x_j| \). Now we can introduce new integration variables \( \beta_i \) by \( \alpha_1 = \beta_1 \) and \( \alpha_i = \beta_1 \beta_i \) \((i \geq 2)\). The integration over \( \beta_1 \) is then trivial and leads to

\[
I_n = \pi \frac{\frac{1}{2} \Gamma \left( \frac{d}{2} \Delta - \frac{d}{2} \right) \Gamma \left( \frac{n}{2} \Delta \right)}{2 \Gamma(\Delta)^n} \int_0^\infty d\beta_2 \ldots d\beta_n \frac{\prod \beta_i^{\Delta-1}}{\left[\sum \beta_i \left( x_{1i}^2 + \sum \beta_j x_{ij}^2 \right)\right]^\frac{1}{2}\Delta}. \tag{23}
\]

We shall not try to perform the remaining integration in the general formula, but consider the cases \( n = 3 \) and \( n = 4 \). For \( n = 3 \) the integrations can be carried out straightforwardly. Inserting the result in (22) gives

\[
\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3) \rangle = \frac{-\lambda_3 \Gamma \left( \frac{1}{2} \Delta + \alpha \right)}{2\pi^d} \left[ \frac{\Gamma \left( \frac{1}{2} \Delta \right)}{\Gamma(\alpha)} \right]^3 \frac{1}{(x_{12}x_{13}x_{23})^\Delta}. \tag{24}
\]
For $n = 3$ there is no disconnected contribution, hence (24) describes the full 3-point function at tree level.

For $n = 4$, we obtain after integration over $\beta_4$ and $\beta_3$

$$I_4 = \frac{\Gamma (2\Delta - \frac{d}{2})}{2\Gamma (2\Delta)} \frac{\pi^{\frac{d}{2}}}{(x_{12}x_{34})^{2\Delta}} \int_0^\infty \frac{d\beta_2}{\beta_2} F \left( \Delta, \Delta; 2\Delta; 1 - \frac{(x_{13}^2 + \beta_2 x_{23}^2)(x_{14}^2 + \beta_2 x_{24}^2)}{\beta_2 x_{12}^2 x_{34}^2} \right).$$

A change of integration variables and the introduction of the conformal invariants (harmonic ratios) then yields

$$\beta_2 = x_{13} x_{14} e^{2z}, \quad \eta = \frac{x_{12} x_{34}}{x_{14} x_{23}}, \quad \zeta = \frac{x_{12} x_{34}}{x_{13} x_{24}},$$

then yields

$$I_4 = \frac{\Gamma (2\Delta - \frac{d}{2})}{\Gamma (2\Delta)} \frac{2\pi^{\frac{d}{2}}}{\pi^{\frac{d}{2}}} \int_0^\infty dz F \left( \Delta, \Delta; 2\Delta; 1 - \frac{(\eta + \zeta)^2}{(\eta \zeta)^2} - \frac{4}{\eta \zeta} \sinh^2 z \right). \quad (25)$$

Obviously, Eqn. (25) is of exactly the form dictated for a four point function by conformal invariance.

## 4 Conclusions

We have considered an example of the correspondence between field theories on an $AdS$ space and CFTs on its boundary. The classical interacting scalar field has been treated at tree level and a nontrivial conformal field theory of boundary operators has been obtained. We calculated a non-trivial coefficient of the 3-point function and, for the first time with this method, found an expression for the function $f(\eta, \zeta)$ contained in the 4-point function (24). We believe that the obtained results will also be helpful for studying more complicated field theories containing fermions and gauge fields.

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