Density matrix renormalization group study of the Anyon-Hubbard model

J Arcila-Forero¹, R Franco¹ and J Silva-Valencia¹

¹ Universidad Nacional de Colombia, Bogotá, Colombia.
E-mail: jsilvav@unal.edu.co

Abstract. Recently optical lattices allow us to observe phase transition without the uncertainty posed by complex materials, and the simulations of these systems are an excellent bridge between materials-based condensed matter physics and cold atoms. In this way, the computational physics related to many-body problems have increased in importance. Using the density matrix renormalization group method, we studied a Hubbard model for anyons, which is an equivalent to a variant of the Bose-Hubbard model in which the bosonic hopping depends on the local density. This is an exact mapping between anyons and bosons in one dimension. The anyons interloge between bosons and fermions. For two anyons under particle exchange, the wave function acquires a fractional phase $e^{i\theta}$. We conclude that this system exhibits two phases: Mott-insulator and superfluid. We present the phase diagram for some angles. The Mott lobe increases with an increase of the statistical. We observed a reentrance phase transition for all lobs. We showed that the model studied is in the same universality class as the Bose-Hubbard model with two-body interactions.

1. Introduction

Experimental setups of cold atoms using optical lattices have the unique ability to tune interactions and density, and because of that they have emerged as unusual laboratories for the realization of Hubbard boson and fermi models, as well as for observing phase transitions without the intrinsic uncertainty posed by materials. The simulations are an incredible bridge between materials and cold atoms: the scale of modern numerical simulations is able to match the scale of experimental cold atom systems, thereby allowing a direct comparison [1].

We used the density matrix renormalization group (DMRG) method. This method was developed by S. White in 1992 [2] and has become a powerful numerical method that can be applied to low-dimensional strongly-correlated fermionic and bosonic systems. Its field of applicability has now been extended beyond condensed matter, and it is successfully used in statistical mechanics and high-energy physics. The DMRG allows for a systematic truncation of the Hilbert space by keeping the most probable states that describe a wave function [3].

Anyons are the third fundamental category of particles. For two anyons under particle exchange, the wave function acquires a fractional phase $e^{i\theta}$, giving rise to fractional statistics with $0 < \theta < \pi$. Greater interest for the study of anyons emerged when the fractional quantum Hall effect observed experimentally had a natural explanation in term of anyons [4]. Another discovery that reinforced this interest was the evidence of superconductor anyon gas [5]. Various experimental setups have been proposed for the creation and detection of anyons, for example creating bosons with conditional-hopping amplitudes using assisted Raman tunneling.
in an optical lattice [6, 7]. Through other means, anyons in rotating Bose-Einstein condensates consisting of a small number of atoms have been achieved [8].

2. Model and results

Anyons in 1D are defined by the generalized commutation relations $a_j a_k^\dagger - e^{-i\theta \text{sgn}(j-k)} a_k^\dagger a_j = \delta_{jk}$ and $a_j a_k = e^{i\theta \text{sgn}(j-k)} a_k a_j$, where the operators $a_j^\dagger$, $a_j$ create or annihilate an anyon on site $j$.

We study the Anyon-Hubbard model, which can be described by the Hamiltonian

$$
H = -t \sum_j \left( a_j^\dagger a_{j+1} + h.c. \right) + \frac{U}{2} \sum_j n_j (n_j - 1) \tag{1}
$$

Where $t$ is the tunneling amplitude connecting two neighboring sites and $U$ is the on-site interaction energy, h.c. denoting the hermitic conjugate. This Hamiltonian describes an interacting gas of anyons in a one-dimensional optical lattice. Let us define the fractional version of a Jordan-Wigner transformation $a_j = b_j \exp \left( i \theta \sum_{i=1}^{j-1} n_i \right)$, with $n_i = a_i^\dagger a_i = b_i^\dagger b_i$ the number operator for both particle types, provided that the particles of type $b$ are bosons ($b$ are the boson operator) with $[b_j, b_j^\dagger] = \delta_{ji}$ and $[b_j, b_i] = 0$.

By inserting the mapping, the Hamiltonian can be rewritten in terms of bosonic operators.

$$
H = -t \sum_j \left( b_j^\dagger b_{j+1} e^{i\theta n_j} + h.c. \right) + \frac{U}{2} \sum_j n_j (n_j - 1) \tag{2}
$$

The mapped, bosonic Hamiltonian thus describes bosons with an occupation-dependent amplitude $e^{i\theta n_j}$ for hopping processes. If the target site $j$ is unoccupied, the hopping amplitude is simply $t$. If it is occupied by one boson, the amplitude reads $te^{i\theta}$, and so on [6].

We present the results for the Anyon-Hubbard model using the DMRG method. We used the finite-size algorithm for sizes up to $L = 256$; we considered a truncated Hilbert space with five states by site and fixed the density $\rho = N/L = 1$. We kept up to $m = 200$ states per block and obtained a discarded weight around $10^{-8}$ or less. The first aim was to determine the quantum phase. We considered the chemical potential, the energy for adding (removing) a particle to the system, which is given by $\mu^+ = E_0(L, N + 1) - E_0(L, N)$ and $\mu^- = E_0(L, N) - E_0(L, N - 1)$ respectively, where $E_0(L, N)$ is the ground-state energy for $L$ sites and $N$ particles. In this we defined the gap by $\Delta \mu = \mu^+ - \mu^-$.  

![Figure 1. System size dependence on the chemical potential. The upper set of data points corresponds to the particle excitation energy, and the lower one to the hole excitation energy. Points are DMRG results.](image)
The Mott insulating phase has a finite gap for single-particle excitations. At the thermodynamic limit, \( N/L \to \infty \) and \( \rho = N/L \) integer. A finite gap is expected, i.e., \( \Delta \mu = \lim_{L,N \to \infty} \Delta \mu(L) > 0 \). On the other hand, the superfluid phase is gapless. This is a criterion for determining the type of phase, given the parameters \( t, U \) and \( \theta \).

We show in Figure 1 (left panel) the energy for adding and removing particles with \( \rho = 1, \theta = \pi/4 \) and \( t/U = 0.1 \). We can see that the particle (hole) excitation energy decreases (increases) as the lattice size increases; we also observe that the size functional dependence is quadratic. At the thermodynamic limit \( (L \to \infty) \), we determine their values \( \Delta \mu(L) = \mu^+(L \to \infty) - \mu^-(L \to \infty) = 0.63 - 0.15 = 0.48 \) considering that the gap is nonzero.

We can conclude that the state of the system is Mott insulator. In Figure 1 (right panel), we show the particle and hole excitation energies for the same angle and density but now with \( t/U = 0.4 \); in this case the functional dependence is lineal and we observe that both quantities coincide at the thermodynamic limit, i.e. the gap disappears, and the ground state is superfluid. In this context, we observe the closing of the gap with an increase of \( t/U \).

We show the phase diagram for different angles in Figure 2. The values of the particle and hole excitation energies were extrapolated at the thermodynamic limit. We observe a Mott insulator phase surrounded by a superfluid phase and we reproduce the well-known phase diagram of the Bose-Hubbard model \( (\theta = 0) \) [9]. The Mott lobe grows in both directions with an increase of the angle (leading to anyonization of the gas), implying that the critical point position is shifted to the right. Thus the state-dependent hopping helps to localize the particles. This presents the possibility of inducing a quantum phase transition from the superfluid into the Mott insulator phase by changing the statistical angle and not just by the competition between \( t \) and \( U \).

![Figure 2. Phase diagram of anyons with statistical angle \( \theta = \pi/2 \) (Left) and \( \theta = 3\pi/4 \) (right) for the densities \( \rho = 1 \) and \( \rho = 2 \). The lines are visual guides and the points are DMRG results.](image)

For first time using DMRG, we calculate the lobes with \( \rho = 2 \) for angles \( \theta = \pi/2 \) and \( \theta = 3\pi/4 \) (Figure 2). We show the comparison with the bosonic case in the same phase diagram. In order to ensure the high accuracy needed for the considerations below, we truncate the local Hilbert space at \( \rho + 5 \) bosons in local operators [10]. When we consider a density greater than one, the Mott region increases and the critical point position is shifted to the left for both angles considered. The increase of the density favors the emergence of the superfluid phase. This result contrasts with the previous mean field calculations [6].

We observe a reentrance phase transition in Figure 3. Reentrance is an unusual feature where the phase boundaries of a system exhibit a succession of transitions between two phases \( A \) and \( B \), such as \( A \rightarrow B \rightarrow A \rightarrow B \), when just one parameter is varied monotonically [11]. In our model, at some suitable constant chemical potential (for example \( \mu = 0.1 \)), the model displays a sequence of quantum phase transitions between the Mott insulator and superfluid phases. In the sequence, we have \( A: \) Mott Insulator and \( B: \) Superfluid \( \) change from an insulator region.
with $\rho = 1$ to a superfluid region with $\rho < 1$ and then again to an insulator region, and finally the system remains in a superfluid state.

We discussed how the quantum phase transition happens in a similar study of Silva-Valencia and Souza [12]. We can use the critical point and try to fit the gap to a special function. One possibility is the function that describes the Kosterlitz-Thouless transition, for which the gap follows $\Delta \mu = A \exp \left[-k/\sqrt{t_c - t}\right]$, where $A$ and $k$ are constants.

In Figure 4, we show the gap as a function of $t_c - t$ with density $\rho = 1$ and statistical angle $\theta = \pi/4$. In the inset, we observe a linear tendency, which indicates that the Kosterlitz-Thouless behavior is suitable for describing the closing of the gap. We conclude that the model studied is in the same universality class as the usual Bose-Hubbard model.

3. Conclusions
We used the density matrix renormalization group method to study the Anyon-Hubbard model. This model exhibits a Mott insulator and a superfluid phase, and we determined the phase diagram for different angles, which increases the Mott insulator region in both directions when increasing the statistical. We observed a reentrance phase transition. The position of the critical point decreases with an increase in the density, in contradiction to previous mean-field results. We show that the gap of the model closes, following the Kosterlitz-Thouless function.

Acknowledgments
The authors are thankful for the support of DIB-Universidad Nacional de Colombia.

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