Random Convolutional Coding for Robust and Straggler Resilient Distributed Matrix Computation

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Abstract

Distributed matrix computations (matrix-vector and matrix-matrix multiplications) are at the heart of several tasks within the machine learning pipeline. However, distributed clusters are well-recognized to suffer from the problem of stragglers (slow or failed nodes). Prior work in this area has presented straggler mitigation strategies based on polynomial evaluation/interpolation. However, such approaches suffer from numerical problems (blow up of round-off errors) owing to the high condition numbers of the corresponding Vandermonde matrices. In this work, we introduce a novel solution approach that relies on embedding distributed matrix computations into the structure of a convolutional code. This simple innovation allows us to develop a provably numerically robust and efficient (fast) solution for distributed matrix-vector and matrix-matrix multiplication.

I. INTRODUCTION

Distributed computing clusters are heavily used in domains such as machine learning where datasets are often so large that they cannot be stored in a single computer. The widespread usage of such clusters presents several opportunities and advantages over traditional computing paradigms. However, they also present newer challenges. Large scale clusters which can be heterogeneous in nature suffer from the problem of stragglers (slow or failed workers in the system). The conventional approach [17] to tackle stragglers has been to run multiple copies of tasks on various machines, with the hope that at least one copy finishes on time. However, this is rather inefficient in terms of resource utilization. Coded computation is an emerging research area that deals with the efficient mitigation of stragglers in these systems. It leverage ideas from erasure coding to introduce redundancy in the computation performed by the worker nodes. Roughly speaking, as long as enough worker nodes complete their tasks, the master node can decode the intended result by appropriate postprocessing.

Problem setting: Suppose that we have large matrices $A \in \mathbb{R}^{t \times r}$, $B \in \mathbb{R}^{t \times w}$ and a vector $x \in \mathbb{R}^t$. The goal is to either compute $A^T x$ (matrix-vector multiplication) or $A^T B$ (matrix-matrix multiplication) in a distributed fashion.
using \( n \) worker nodes while being resistant to any \( s \) stragglers. Consider matrix-vector multiplication first. To solve this in a distributed fashion, we partition \( \mathbf{A} \) into block-columns of equal size \( \mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{\Delta-1} \) and distribute a certain number (subject to a storage constraint) of “coded” versions of these block-columns to the \( n \) workers. These are sent to the master node which decodes or recovers \( \mathbf{A}^T \mathbf{x} \) using the output of any \((n-s)\) workers. The assumption here is that some nodes will fail or will be too slow, the maximum number of such nodes is assumed to be \( s \) or less. The goal is to design the coding scheme in such a way that the decoding is possible using the output of any \( k = (n-s) \) workers; is robust to noise (both numerical precision errors and other sources of noise) and is fast. For matrix-matrix multiplication, each worker node receives coded versions of the block columns of \( \mathbf{A} \) and also coded versions of the block columns of \( \mathbf{B} \). It computes all pairwise products of these and sends them to the master node. The master needs to be able to decode \( \mathbf{A}^T \mathbf{B} \) using the output of any \((n-s)\) workers in a robust and fast manner.

Considering such problems is of importance since distributed matrix computations are at the heart of routine computations in the machine learning pipeline.

**Contributions:** In this paper we present an efficient and robust scheme for coded matrix computations that is inspired by convolutional codes. Crucially, our codes operate over the reals, unlike the majority of convolutional codes that are considered over finite fields \([10]\). When operating on finite fields, proving the invertibility of an appropriate sub-matrix of the coding matrix suffices to guarantee correct decoding. However, in decoding a real system of equations, errors in the input can get amplified by the condition number (ratio of maximum and minimum singular values) of the associated matrix; hence, a low condition number is critical. As a thumb rule, a condition number of \( 10^l \) corresponds to the loss of approximately \( l \) bits of precision in the computed solution.

While there has been some previous work on the use of coding theoretic approaches for solving distributed matrix computation problems \([16, 14, 13, 8, 13, 3, 1, 5]\), much of this ignores the key issue of dealing with numerical precision issues or other sources of noise in the worker node computation. The only exceptions are the preprints \([3, 1, 5]\).

We note here that it may be possible to ensure a good condition number for the recovery process (with high probability) if one chooses random linear combinations of the block columns of \( \mathbf{A} \) and \( \mathbf{B} \) where the random coefficients are chosen i.i.d, e.g., from a Gaussian distribution. However, as noted in Remark 8 of \([15]\), this comes at the cost of high decoding complexity. By leveraging ideas from convolutional codes, our work develops a coding scheme that is numerically stable and can be decoded efficiently.

- Our work is among the first to provide an efficient coded computation approach for both matrix-vector and matrix-matrix multiplications that provably works in (i) the essentially noise-free regime where numerical precision issues dominate, and (ii) the noisy regime where noise is significant. To be precise we show the following properties of the relevant sub-matrices of our designed “coding” matrix: (a) these sub-matrices are invertible (Theorem \([1]\)); and (b) upon appropriate random scaling their worst case condition number over all choices of the sub-matrices can be tightly upper bounded by a “computable” upper bound (Theorem \([2]\)). Importantly, Theorem \([2]\) also suggests a natural random sampling algorithm to pick a coding matrix with good performance. The novelty of this work lies in drawing connections to convolutional coding and the asymptotic analysis of large Toeplitz matrices \([8]\) and in using these results to develop practical coded matrix computation algorithms.
• Our codes can be decoded via gradient descent (for a wide range of SNRs), as well as by a simple peeling decoder in the very high SNR regime. As the corresponding matrices in the gradient descent are sparse, the per iteration cost of each gradient step, e.g., the matrix-vector case scales as $O(rs)$. As our matrices are provably well-conditioned, the required number of descent iterations, $\alpha$, to achieve relative accuracy $\epsilon$ is only of order $\frac{\log(1/\epsilon)}{\log(1/(1 - \frac{1}{\kappa^2}))}$. Here $\kappa$ is the condition number of the relevant matrix (see Section V).

**Related Work:** The main finding of several recent works in this area is that it is possible to embed distributed matrix computations into the structure of an equivalent erasure code, where the failed nodes play the role of erasures [9; 16; 15; 4; 13; 14]. For instance, [16; 15] convert distributed matrix computation into polynomial evaluation/interpolation, i.e., the coded submatrices correspond to polynomial evaluation maps. The master node decodes the intended result by polynomial interpolation. It is well recognized that polynomial interpolation over the reals suffers from significant numerical issues since the corresponding Vandermonde matrices have very high condition numbers. In fact even for clusters with around $n = 30$ nodes, the condition number is so large that the decoded result is essentially useless.

Some recent work has highlighted and considered the issue of numerical stability in this context. Preprints [3; 11] presented strategies for distributed matrix-vector multiplication and demonstrated some schemes that empirically have better numerical performance than polynomial based schemes for some values of $n$ and $s$. However, both these approaches work only for the matrix-vector problem and do not provide a computable bound on the condition number of the decoding sub-matrices. We point out that the authors of [3] also use convolutional codes for the matrix-vector problem. However, we show that our proposed approach is superior to theirs across a range of SNRs. Our analysis technique also applies to their codes. We can theoretically demonstrate that some of the codes considered in [3] have a worst case condition number that can be unbounded in the problem parameters. The recent preprint of [5] considers an alternative approach for polynomial based schemes. Their scheme involves an interpolation step that is shown to be more numerically stable and a step that involves the numerical calculation of an integral. The decoding complexity of their scheme appears to be much higher than ours when $k$ is large. It scales as $O(k^3 + rwk)$ for the matrix-matrix case, whereas our decoding complexity per gradient iteration scales as only $O(rws)$. For well-conditioned matrices used in the gradient step, as explained above, our required number of iterations scales as $\log(1/\epsilon)$ for achieving $\epsilon$ level of relative error.

II. **Convolutio nal Codes for Distributed Matrix Computation**

We explain our key idea by means of the following example. Consider two row vectors in $\mathbb{R}^q$, $\mathbf{u}_0 = [u_{00} u_{01} \ldots u_{0(q-1)}]$ and $\mathbf{u}_1 = [u_{10} u_{11} \ldots u_{1(q-1)}]$. These vectors can also be represented as polynomials $\mathbf{u}_i(D) = \sum_{j=0}^{q-1} u_{ij} D^j$ for $i = 0, 1$. Consider the following encoding of $[\mathbf{u}_0(D) \quad \mathbf{u}_1(D)]$.

$$
\begin{bmatrix}
\mathbf{c}_0(D) & \mathbf{c}_1(D) & \mathbf{c}_2(D) & \mathbf{c}_3(D)
\end{bmatrix} =
\begin{bmatrix}
\mathbf{u}_0(D) & \mathbf{u}_1(D)
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & D
\end{bmatrix}
= [\mathbf{u}_0(D) \quad \mathbf{u}_1(D) \quad (\mathbf{u}_0(D) + \mathbf{u}_1(D)) \quad (\mathbf{u}_0(D) + D\mathbf{u}_1(D))].
$$
It is not too hard to see that the polynomials \( u_0(D) \) and \( u_1(D) \) (equivalently the vectors \( u_0, u_1 \)) can be recovered from any two entries of the vector \([c_0(D) \ c_1(D) \ c_2(D) \ c_3(D)]\). For instance, suppose that we only have \( c_2(D) \) and \( c_3(D) \) where

\[
[c_2(D) \ c_3(D)] = \left[ \sum_{j=0}^{q-1} (u_{0j} + u_{1j})D^j \bigg| \sum_{j=0}^{q-2} (u_{0(j+1)} + u_{1j})D^j + u_{1(q-1)}D^q \right].
\]

Starting with \( u_{00} \) from the constant term of \( c_3(D) \), one can iteratively recover each of the coefficients of \( u_0(D) \) and \( u_1(D) \). A similar argument applies if we consider a different pair of entries from \([c_0(D) \ c_1(D) \ c_2(D) \ c_3(D)]\).

The above argument can also be made in terms of matrices instead of polynomials (see Appendix Section VII-D and Section IV).

This basic example can be significantly generalized. Let \( \bar{a} = [a_0 \ a_1 \ \ldots \ a_{s-1}] \) and \( \bar{b} = [b_0 \ b_1 \ \ldots \ b_{k-1}] \) be vectors of non-negative integers such that \( 0 \leq a_0 < a_1 < \ldots < a_{s-1} \) and \( 0 \leq b_0 < b_1 < \ldots < b_{k-1} \). Let \( Y_{\bar{b},\bar{a}}(D) \) denote a \( k \times s \) matrix whose \((i,j)\)-th entry is given by

\[
[Y_{\bar{b},\bar{a}}(D)]_{i,j} = (D^{a_j})^{b_i}.
\]

(1)

In the sequel, the case \( a_j = j, 0 \leq j \leq s-1 \) and \( b_i = i, 0 \leq i \leq k-1 \) is an important one; we use \( Y_{k,s}(D) \) to specify this matrix, i.e.,

\[
[Y_{k,s}(D)]_{i,j} = (D^j)^i, i = 0, 1, \ldots, k-1, j = 0, 1, \ldots, (s-1).
\]

It can be noted that \( Y_{k,s}(D) \) has the Vandermonde structure (an example appears in (8) in Appendix Section VII-A).

As we show next in Theorem 1 the crucial point is that we can use matrices such as \( Y_{\bar{b},\bar{a}}(D) \) to design a \( k \times n \) matrix \( G(D) \) (with \( n \geq k \)) such that any \( k \times k \) submatrix of \( G(D) \) is nonsingular. This further means that if

\[
[c_0(D) \ c_1(D) \ \ldots \ c_{n-1}(D)] = [u_0(D) \ u_1(D) \ \ldots \ u_{k-1}(D)] G(D),
\]

then \([u_0(D) \ u_1(D) \ \ldots \ u_{k-1}(D)]\) can be recovered from any \( k \) elements of \([c_0(D) \ \ldots \ c_{n-1}(D)]\). The equivalent matrix formulation of this results appears in Section IV. Let \( \circ \) denote the Hadamard product of matrices. The proof of the following theorem uses Lemma 1 in Appendix Section VII-A.

**Theorem 1.** Let \( R \) be a \( k \times (n-k) \) matrix. Suppose that we form the generator matrix \( G(D) \) of size \( k \times n \) which is given by

\[
G(D) = \begin{bmatrix}
I_k & R \circ Y_{\bar{b},\bar{a}}(D)
\end{bmatrix}.
\]

(3)

If either \( R \) is the all-ones matrix or if \( R \) has entries chosen i.i.d. from a continuous p.d.f., then any \( k \times k \) submatrix of \( G(D) \) is nonsingular with probability 1.

**Proof.** The proof is a consequence of the fact that any \( s \times s \) submatrix of \( R \circ Y_{\bar{b},\bar{a}}(D) \) is non-singular by Lemma 1 in Appendix Section VII-A with probability 1.
While nonsingularity by itself does not reveal information about the corresponding condition numbers, Theorem I provides a class of schemes with a specific structure that can be modified and analyzed for condition number using the techniques discussed in Theorem II within Section IV. The structure of $G(D)$ also translates into the desirable decoding properties discussed in Section II.

III. CODED DISTRIBUTED MATRIX COMPUTATION

Following the problem setting in Section I we partition the matrices into block-columns of equal size $A_0, A_1, \ldots, A_{\Delta_A-1}$ and $B_0, B_1, \ldots, B_{\Delta_B-1}$ respectively; the choice of $\Delta_A$ and $\Delta_B$ will be discussed shortly. Each worker node stores at most $\gamma_A\Delta_A$ and $\gamma_B\Delta_B$ block columns where $0 \leq \gamma_A, \gamma_B \leq 1$. These correspond to coded versions of the block rows of $A$ and $B$ respectively. We assume that $q_A = \Delta_A/k_A$ and $q_B = \Delta_B/k_B$ are integers. Consider the following “matrix” polynomials.

$$A_i(D) = A_{i q_A}^T + A_{i q_A+1}^T D^z + \cdots + A_{i(q_A-1)}^T D^{z(q_A-1)} \quad (\text{for } 0 \leq i \leq k_A - 1),$$

$$B_j(D) = B_{jq_B} + B_{jq_B+1} D + \cdots + B_{j(q_B-1)} D^{q_B-1} \quad (\text{for } 0 \leq j \leq k_B - 1),$$

where $z$ is a positive integer whose choice will be discussed shortly. In what follows we explain the general technique for both problems. These are illustrated in Example I.

**Matrix-vector scenario:** We set $z = 1$. The assignment of the coded submatrices to the worker nodes is performed as follows. Let $s_A = n - k_A$. We form

$$[C_0(D) \ldots C_{s_A-1}(D)] = [A_0(D) \ldots A_{k_A-1}(D)] [I_{k_A} | R \circ Y_{k_A,s_A}(D)].$$

(4)

Thus, each $C_i(D)$ is a matrix polynomial whose coefficients correspond to linear combinations of $A_0^T, \ldots, A_{\Delta_A-1}^T$. The $i$-th worker node is assigned the coded submatrices corresponding to the coefficients of $C_i(D)$. Each worker node is also provided $x$ and it computes the matrix-vector product of each of its assigned submatrices and $x$. By Theorem III the master node can decode if any $k_A$ workers complete their tasks. We note that the highest exponent of $D$ in the generator matrix $G(D)$ is $(s_A - 1)(k_A - 1)$; this corresponds to the worker node with the maximum number of assigned submatrices. This translates into a lower bound $\Delta_A \geq \frac{(s_A-1)(k_A-1)}{\gamma_A^{-1} - 1}$ (see Appendix Section VII-D).

**Matrix-matrix scenario:** In this case, each worker node computes products of coded submatrices. The final goal of the master node is to recover all products of the form $A_i^T B_j$ for $0 \leq i \leq \Delta_A - 1, 0 \leq j \leq \Delta_B - 1$. Let $G_A(D)$ and $G_B(D)$ be $k_A \times n$ and $k_B \times n$ matrices that will be determined shortly. Let $U_A(D) = [A_0(D) \ldots A_{k_A-1}(D)]$ and $U_B(D) = [B_0(D) \ldots B_{k_B-1}(D)]$. Once again by forming

$$[C_0^A(D) \ldots C_{s_A-1}^A(D)] = U_A(D) G_A(D),$$

$$[C_0^B(D) \ldots C_{s_B-1}^B(D)] = U_B(D) G_B(D),$$

we can represent the assignment of coded submatrices of $A$ and $B$ to worker node $i$ by the coefficients of $C_i^A(D)$ and $C_i^B(D)$ respectively. Following this step each worker node computes the pairwise product of each coded submatrix of $A$ and coded submatrix of $B$ assigned to it. We observe here that if $z$ is large enough then the
pairwise product of each coefficient of $C_i^A(D)$ and each coefficient of $C_i^B(D)$ appears in $C_i^A(D) \times C_i^B(D)$, i.e., each worker node equivalently computes $C_i^A(D) \times C_i^B(D)$. Now, using MATLAB notation and Kronecker product properties, for $i = 1, 2, \ldots, n$, we have

$$C_i^A(D) \times C_i^B(D) = (U_A(D)G_A(D)(i,:)) \times (U_B(D)G_B(D)(i,:))$$

$$= (U_A(D) \otimes U_B(D)) \times (G_A(D)(i,:) \otimes G_B(D)(i,:)),$$

where $\otimes$ denotes the Kronecker product. Therefore, the computation performed by the worker nodes can be compactly represented using the Khatri-Rao product \cite{11} (denoted by $\odot$), i.e., column-wise Kronecker products. Moreover, using the properties of the Khatri-Rao product, we have

$$(U_A(D)G_A(D)) \odot (U_B(D)G_B(D)) = (U_A(D) \otimes U_B(D))(G_A(D)) \odot G_B(D))$$

Note here that $(U_A(D) \otimes U_B(D))$ contains all the $\Delta_A \Delta_B$ distinct pairwise products that we are interested in recovering (assuming $z$ large enough). The key idea at this point is to ensure that $G_A(D) \odot G_B(D)$ has the structure of a matrix as in \footnote{3}. Towards this end, we choose

$$G_A(D) = \begin{bmatrix} I_{k_B} & 0 & 0 & \cdots & 0 \\ 0 & 1_{k_B} & 0 & \cdots & 0 \\ 0 & 0 & 1_{k_B} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1_{k_B} \end{bmatrix} \odot \begin{bmatrix} R_A \circ Y_{k_A,s}(D^2) \end{bmatrix},$$

and

$$G_B(D) = \begin{bmatrix} I_{k_B} & I_{k_B} & I_{k_B} & \cdots & I_{k_B} \end{bmatrix} \odot \begin{bmatrix} R_B \circ Y_{k_B,s}(D) \end{bmatrix}$$

where $1_{k_B}$ is an all-ones row vector of length $k_B$, and the total number of rows in $G_A(D)$ and $G_B(D)$ are $k_A$ and $k_B$ respectively. This implies that

$$G_A(D) \odot G_B(D) = [I_{k_A k_B} \mid (R_A \circ Y_{k_A,s}(D^2)) \odot (R_B \circ Y_{k_B,s}(D))].$$

\footnote{2} in the Appendix Section VII-D shows that the RHS of \footnote{3} has the structure of the matrix in \footnote{3}, and Appendix Section VII-D discusses the choice of $z$ and constraints on $\Delta_A$ and $\Delta_B$.

**Example 1.** Consider the computation of $A^T x$ with $n = 4$ workers and $k_A = n - s_A = 2$. Each worker can store $x$ and $\gamma_A = \frac{s}{8}$ fraction of matrix $A$. So we partition $A$ into $\Delta_A = \frac{(2-1)(2-1)}{8-2} = 8$ block-columns of equal size denoted $A_0, A_1, \ldots, A_7$. We have $A_0(D) = A_0^T + A_1^T D^2 + A_2^T D^2z + A_3^T D^3z$ and $A_1(D) = A_4^T + A_5^T D^2 + A_6^T D^2z + A_7^T D^3z$. Setting $z = 1$, the submatrices assigned to the workers are given by

$$[C_0(D) \ C_1(D) \ C_2(D) \ C_3(D)] = [A_0(D) \ A_1(D)] \begin{bmatrix} I_2 & Y_{2,2}(D) \end{bmatrix},$$

where for ease of illustration we did not consider the Hadamard product of $Y_{2,2}(D)$ with a random matrix. The corresponding assignment of the jobs to the workers is shown in Fig. \footnote{11} where we can see that $W0$ and $W1$ are message worker nodes and $W2$ and $W3$ are the parity worker nodes.
workers are assigned more tasks as compared to the message part workers. However, this non-uniformity can be avoided for ease of presentation.

Remark 1. Our proposed scheme results in a slightly non-uniform assignment of tasks whereby the parity part workers are assigned more tasks as compared to the message part workers. However, this non-uniformity can be reduced to any desired level by choosing large enough $\Delta_A$ and $\Delta_B$. There is a tradeoff however with the condition number of the recovery matrices as discussed in Section IV.
IV. CONDITION NUMBER OF RECOVERY MATRICES

A corresponding matrix representation of the transformation induced by a matrix such as \( G(D) \) in (3) can be arrived at in the following manner.

**Definition 1.** Define the \( k \times n \) block matrix \( \tilde{G} \) as follows.

\[
\tilde{G} = \begin{bmatrix}
I_q & 0 & \ldots & 0 & r_{00}^G \tilde{D}^{a_0 b_{k-1}; a_0 b_0} & \ldots & r_{(s-1)0}^G \tilde{D}^{a_s b_{k-1}; a_s b_0} \\
0 & I_q & \ldots & 0 & r_{10}^G \tilde{D}^{a_0 b_{k-1}; a_0 b_1} & \ldots & r_{1(s-1)}^G \tilde{D}^{a_s b_{k-1}; a_s b_1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & I_q & r_{(k-1)0}^G \tilde{D}^{a_0 b_{k-1}; a_0 b_{k-1}} & \ldots & r_{(k-1)(s-1)}^G \tilde{D}^{a_s b_{k-1}; a_s b_{k-1}}
\end{bmatrix}
\]

(6)

where for a positive integer \( h \geq j \) we have \( \tilde{D}^{hj} \triangleq \begin{bmatrix} 0_{q \times j} & I_q & 0_{q \times (h-j)} \end{bmatrix} \). Thus, for the \( \ell \)-th block column of \( \tilde{G} \) denoted \( \tilde{G}_\ell \), where \( \ell \geq k \), the number of columns is given by \( q + a_{\ell-k} b_{k-1} \), i.e., it depends on the maximum exponent of \( D \) in the \( \ell \)-th column of \( G(D) \).

For any subset \( \mathcal{I} \subset \{0, \ldots, n-1\} \) such that \( |\mathcal{I}| = k \), we let \( \tilde{G}_\mathcal{I} \) denote the matrix obtained by extracting the corresponding block columns of \( \tilde{G} \) and \( G_\mathcal{I}(D) \) denote the matrix obtained by extracting the corresponding columns of \( G(D) \). Using Theorem 1 any \( \tilde{G}_\mathcal{I} \) has full rank \( kq \) if \( |\mathcal{I}| = k \), however Theorem 1 does not give any information about the condition number of \( \tilde{G}_\mathcal{I} \). This is the focus of the upcoming Theorem 2.

From Lemma 3 it is possible to conclude that \( \tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T \) is a symmetric \( k \times k \) block-matrix with Toeplitz blocks of size \( q \times q \) (see Appendix Section VII-B). This fact is useful since the asymptotics of \( \lambda_{\max}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) \) and \( \lambda_{\min}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) \) when \( q \) is large have been studied in [7]. Let \( i = \sqrt{-1} \). With some abuse of notation, let \( G_\mathcal{I}(e^{i\omega}) \) represent the matrix obtained by extracting \( G_\mathcal{I}(D) \) (from \( G(D) \) in (3)) and then substituting \( D = e^{i\omega} \). By adapting the results of [7] (see Appendix Section VII-C for a detailed description), we have the following theorem.

**Theorem 2.** For \( \mathcal{I} \subset \{0, \ldots, n-1\} \) such that \( |\mathcal{I}| = k \), we have

\[
\lim_{q \to \infty} \lambda_{\min}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) = \min_{\omega \in [-\pi, \pi]} \lambda_{\min}[(G_\mathcal{I}(e^{i\omega}))(G_\mathcal{I}(e^{i\omega}))^*] \quad \text{and} \quad \lim_{q \to \infty} \lambda_{\max}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) = \max_{\omega \in [-\pi, \pi]} \lambda_{\max}[(G_\mathcal{I}(e^{i\omega}))(G_\mathcal{I}(e^{i\omega}))^*].
\]

Moreover, for any \( q \)

\[
\lambda_{\max}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) \leq \max_{\omega \in [-\pi, \pi]} \lambda_{\max}[(G_\mathcal{I}(e^{i\omega}))(G_\mathcal{I}(e^{i\omega}))^*], \quad \text{and} \quad \lambda_{\min}(\tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T) \geq \min_{\omega \in [-\pi, \pi]} \lambda_{\min}[(G_\mathcal{I}(e^{i\omega}))(G_\mathcal{I}(e^{i\omega}))^*].
\]

Theorem 2 shows that we can find an upper bound on the condition number of \( \tilde{G}_\mathcal{I} \) based on a scalar optimization over \( \omega \in [-\pi, \pi] \).

**Remark 2.** Consider the situation when \( R \) in (3) is chosen to be the all-ones matrix; this corresponds to not applying any scaling to the parity part. The characterization of Theorem 2 allows us to conclude that when \( s > 1 \), there exist choices of \( \mathcal{I} \subset \{k, \ldots, n-1\}, |\mathcal{I}| = k \) such that \( \tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T \) has a minimum eigenvalue that will go to
Theorem 2 suggests the following algorithm for choosing worker computation time. Thus, \( \kappa \) of \( n \) the search does not have any dependence on \( n \). We emphasize that this is precisely the issue with some of the constructions in [3].

Algorithm for choosing scaling matrices \( R_A \) and \( R_B \): Consider for instance, the matrix-vector scenario. Theorem 2 suggests the following algorithm for choosing \( R_A \). We proceed by randomly choosing \( R_A \). Let \( \mathcal{I} \subset \{0, \ldots, n-1\} \), \( |\mathcal{I}| = k \) and let \( \Omega = \{0, \pm \frac{\pi}{N}, \pm 2\frac{\pi}{N}, \ldots, \pm \frac{(N-1)\pi}{N}, \pm \pi\} \) denote a fine enough grid of the interval \([0, \pi] \). Let,

\[
\kappa_{R_A} = \max_{0 \leq |\mathcal{I}| \leq k} \sqrt{\frac{\max_{\omega \in \Omega} \lambda_{\max}(\hat{G}_T(e^{i\omega})) \hat{G}_T(e^{i\omega})^* \hat{G}_T(e^{i\omega})^*}{\min_{\omega \in \Omega} \lambda_{\min}(\hat{G}_T(e^{i\omega})) \hat{G}_T(e^{i\omega})^* \hat{G}_T(e^{i\omega})^*}}
\]

Thus, \( \kappa_{R_A} \) indicates the maximum condition number of \( \hat{G}_T \) over all \( \binom{n}{k} \) choices of \( \mathcal{I} \). The algorithm repeatedly generates choices of \( R_A \) and retains the choice that has the lowest value of \( \kappa_{R_A} \); this denoted by \( R_A^* \). The matrix-matrix case is similar, except that we generate \( R_A \) and \( R_B \) and consider the worst case condition number of the appropriate submatrices of \( \hat{G} \). Let \( \kappa^* \) denote the worst case condition number over the choices of \( \mathcal{I} \) when considering the best choice \( R_A^* \).

We emphasize that even though the search requires optimizing over \( \binom{n}{k} \) choices of \( \mathcal{I} \), this is a one-time cost for designing the coding scheme for a system with \( n \) worker nodes which is resilient to \( n-k \) stragglers. Moreover, the search does not have any dependence on \( q \). Thus, it is possible to implement this algorithm for moderate values of \( n \) and \( k \).

V. Decoding Algorithm

We use a gradient descent based decoding algorithm that exploits the sparse nature of the decoding submatrices, i.e., the \( \hat{G}_T \)'s. Suppose that we obtain results from workers in \( \mathcal{I} \subset \{0, \ldots, n-1\} \), where \( |\mathcal{I}| \geq k \). The master node can then solve a set of independent least-squares problems via a low-complexity gradient descent. We describe this in detail below for the matrix-vector case; the discussion is quite similar for the matrix-matrix case.

In the matrix-vector case our unknowns are \( u_i = A_i^T x \), \( i = 0, \ldots, \Delta_A - 1 \); each of these is a vector of length \( r/\Delta_A \). Let row-vector \( z_j = [u_{ij} \quad u_{1j} \ldots \quad u_{(\Delta_A-1)j}] \), for \( j = 0, \ldots, r/\Delta_A - 1 \) denote the collection of the \( j \)-th entries of each of these variables. Let the output of the worker nodes corresponding to \( z_j \) be denoted by \( y_j \). The length of \( y_j \) depends on \( \mathcal{I} \) and is typically larger than \( \Delta_A \); thus we are in the overdetermined least-squares setting. For the \( j \)-th least squares problem the gradient at iteration \( l \) can be computed easily as \( \left(z_j^T \hat{G}_T - y_j \right) \hat{G}_T^T \).

Note that the matrix \( \hat{G}_T \) is a very sparse matrix. In particular, if \( |\mathcal{I}| = k \) then it has at most \( \Delta_A(s_A+1) \) non-zero entries. Thus, each gradient step requires \( O(\Delta_A(s_A+1)) \cdot (r/\Delta_A) = O(r(s_A+1)) \) operations. To reach within an \( \epsilon \)-fraction of the correct value, the gradient descent algorithm takes approximately \( \alpha = \frac{\log(1/\epsilon)}{\log(1/(1-1/\epsilon^2))} \) iterations [2]. On the other hand, each worker node needs at least \( O(\rho t/k_A) \) operations. In a typical setting the underlying matrix \( A \) is large so that \( t \gg k_A(s_A+1)\alpha \). Thus, the decoding time will be much less than the worker computation time.

Note that the condition number of \( \hat{G}_T \) improves as \( |\mathcal{I}| \) increases beyond \( k \). Thus, if more than \( k \) workers return the results of their computation, the gradient descent will converge even faster.
Remark 3. As alluded to in the toy example considered in Section II, our scheme can be decoded in a so-called peeling decoding fashion whereby at each step the algorithm is guaranteed to find an equation with only one unknown. We demonstrate this by means of an example in Appendix Section VII-D2. This algorithm provides good performance in somewhat restrictive settings (high SNR, trivial scaling of the parity part and small $\Delta_A$). Nevertheless, it has the advantage of being very fast.

VI. NUMERICAL EXPERIMENTS

Our first experiment demonstrates the tightness of the bound on the worst-case condition number $\kappa^*$ computed by our algorithm in Section IV. We considered a system with $n = 9$ worker nodes and $s_A = 3$ stragglers. We conducted 50 random trials where each entry of $R_A$ was chosen from a uniform distribution on the interval $[-1, 1]$ and determined the best choice $R^*_A$. Once $R^*_A$ is chosen, we can define the “best” coding matrix $\bar{G}$ for any value of $q$. Our algorithm also returns the asymptotic (in $q$) upper bound on $\kappa^*$. By sweeping over values of $q$, we can also compute the actual worst-case condition number for each particular chosen value of $q$. As can be seen from

[Fig. 2: Obtaining superior condition numbers for distributed matrix-multiplication]

[Fig. 3: Recovery Error Comparison: Decoding time at SNR = 80dB is shown in legend in (a). All Simulations are done with CPU Speed 3.5 GHz and RAM 16 GB.]
Fig. 2a (i) when using the best random matrix, the actual worst-case condition number is below its computed upper bound of 238 for all values of $q$. (ii) Moreover, our computed bound is quite tight even for moderate values of $q$, and hence picking $R_A$ using our algorithm indeed returns a good choice of the coding matrix for any choice of $q$.

We also computed the worst-case condition number for many other values of $n, s_A$ and compared with that of other schemes. For [3], this was computed for a specific value of $q$ (minimum allowed by [3, eq. (6)]), while for [15], $q = 1$ is the only assumed choice in their scheme. We compare the values in for $n = 12$ and $s = 2, 3$ in Fig. 2b observe that the condition number for both the existing approaches is 10-100 times higher than ours.

Next, we ran experiments for comparing the recovery performance (normalized mean-squared error) of our approach with that of existing work for both matrix-vector and matrix-matrix multiplications. We considered both high signal to noise ratio (SNR), and low SNR settings. High SNR simulates “noise” due to numerical precision and round-off errors on regular 64-bit machines; while low SNR simulates either low precision workers or noisy channels between the worker and master nodes. As is expected based on the condition numbers’ comparison, our proposed approach significantly outperforms both the existing schemes in all four scenarios. Due to lack of space, we show high SNR matrix-vector case plots in Fig 3a and low SNR matrix-matrix case plots in Fig. 3b. The other two plots are provided in Appendix Section VII-E. In the high SNR setting, our approach (with the best random matrix choice $R_{\star}A$) has the smallest recovery error, and our approach (with $R_A$ set as all-ones) is only slightly worse. However, in the more difficult lower SNR setting, only our approach with $R_A$ picked as the best random matrix has small error. Our approach with $R_A$ being the all-ones matrix has very large error in this case and so do the methods of [15, 3]. We summarize the time taken by all the approaches in the legend. As can be seen, our approach is a little slower than [15] (but [15] has much larger errors even in the high SNR setting); and is, in fact, faster than [3]. In conclusion, based on these experiments, our approach (with $R_A$ picked as the best random matrix) is the best approach that works in all SNR regimes. It is marginally slower than our approach with the all-ones case (because it involves a few extra multiplications with real numbers), so if SNR is known to be high and speed is an important concern, the all-ones case of our approach could be used.

The above experiments used the following parameters. For the matrix-vector case, the matrix $A$ was of size $25000 \times 21000$. We assumed the availability of $n = 10$ workers and $s_A = 3$ stragglers. Each worker could store at most $\gamma_A = \frac{4}{25}$ fraction of the full matrix. In this case, for our approach, we set $\Delta = \Delta_A$ as the smallest value allowed (cf. Appendix Section VII-D), this turned out to be $\Delta_A = 700$. The same was done for the two existing approaches: for [3], [3, eq.(6)] dictated this choice, while the scheme of [15] uses $\Delta_A = k_A$. For distributed matrix-matrix multiplication, $A$ was of size $6000 \times 4500$ and $B$ was of size $6000 \times 4000$. Our goal is to calculate $A^T B$ in a distributed fashion on $n = 10$ workers, such that the scheme is resilient to $s = 2$ stragglers. Also, each worker node can store a maximum $\gamma_A = \frac{1}{10}$ fraction of $A$ and a maximum $\gamma_B = \frac{3}{5}$ fraction of matrix $B$. The approach of [3] does not provide a solution for matrix-matrix case and hence we can only compare with [15].

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VII. APPENDIX

A. Invertibility of relevant matrices

We begin by a formal description of the field in which the polynomials in variable $D$ lie. Consider the set of real infinite sequences $\{u_r, u_{r+1}, \ldots\}$ for $r \in \mathbb{Z}$ that start at some finite integer index and continue thereafter. These sequences can be treated as elements of the formal Laurent series \[ u \in \mathbb{R} \] in indeterminate $D$ and can be added, multiplied and divided to obtain other members of $\mathbb{R}((D))$ and can be added, multiplied and divided to obtain other members of $\mathbb{R}((D))$. The zero element and identity elements are precisely the real number 0 and the real number 1 within this field.

With this background, the following result can be shown.

**Lemma 1.** Consider a square matrix $X(D)$ such that

$$X(D) = \begin{bmatrix}
(D^{a_0})^{b_0} & (D^{a_1})^{b_0} & \cdots & (D^{a_{v-1}})^{b_0} \\
(D^{a_0})^{b_1} & (D^{a_1})^{b_1} & \cdots & (D^{a_{v-1}})^{b_1} \\
\vdots & \vdots & \ddots & \vdots \\
(D^{a_0})^{b_{v-1}} & (D^{a_1})^{b_{v-1}} & \cdots & (D^{a_{v-1}})^{b_{v-1}}
\end{bmatrix}$$

where positive integers $0 \leq a_0 < a_1 < \cdots < a_{v-1}$ and $0 \leq b_0 < b_1 < \cdots < b_{v-1}$. Then $X(D)$ is nonsingular, i.e., its determinant is a non-zero polynomial in $D$. Furthermore, if $R$ is a $v \times v$ matrices with entries chosen i.i.d. from a continuous distribution, then $R \circ X(D)$ (where $\circ$ denotes the Hadamard product) is nonsingular with probability 1.

The proof of Lemma 1 involves Schur polynomials that are defined next.

**Definition 2.** Let $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_{v-1}$ be non-negative integers and let $\lambda = (\lambda_0, \ldots, \lambda_{v-1})$. Then,

$$S_\lambda(x_0, \ldots, x_{v-1}) = \sum_{T} x_0^{t_0} x_1^{t_1} \ldots x_{v-1}^{t_{v-1}}$$

where the summation is over all semistandard Young tableaux $T$ of shape $\lambda$ [12].

A Young diagram of shape $\lambda$ consists of a collection of boxes arranged in left-justified rows. The $i$-th row has $\lambda_i$ boxes. A semistandard Young tableau $T$ is obtained by filling the boxes with the integers $0, \ldots, v-1$ such that entries are in ascending order from left to right in the rows and in strictly increasing order from top to bottom in the columns. The $t_i$ values in (7) are obtained the counting the occurrences of the number $i$ in tableau $T$.

**Proof.** Matrix $X(D)$ can be written upon permuting some rows as

$$\hat{X}(D) = \begin{bmatrix}
(D^{a_0})^{\lambda_0 + v-1} & (D^{a_1})^{\lambda_0 + v-1} & \cdots & (D^{a_{v-1}})^{\lambda_0 + v-1} \\
(D^{a_0})^{\lambda_1 + v-2} & (D^{a_1})^{\lambda_1 + v-2} & \cdots & (D^{a_{v-1}})^{\lambda_1 + v-2} \\
\vdots & \vdots & \ddots & \vdots \\
(D^{a_0})^{\lambda_{v-1}} & (D^{a_1})^{\lambda_{v-1}} & \cdots & (D^{a_{v-1}})^{\lambda_{v-1}}
\end{bmatrix}$$
where we can assume that $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{v-1}$. We need to prove that the determinant of $\hat{X}$ is non-zero.

According to [12],

$$\det(\hat{X}) = \det(Y_{v,v}(D)) \times S_\lambda(D^{a_0}, D^{a_1}, \ldots, D^{a_{v-1}})$$

But $\det(Y_{v,v})$ is a non-zero polynomial in $D$ as it is the Vandermonde matrix shown below.

$$Y_{v,v}(D) = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & D & \ldots & D^{v-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & D^{v-1} & \ldots & D^{(v-1)(v-1)}
\end{bmatrix} \quad (8)$$

Furthermore, based on Definition 2, $S_\lambda(D^{a_0}, D^{a_1}, \ldots, D^{a_{v-1}})$ consists of the sum of terms of the form

$$(D^{a_0})^f_0 (D^{a_1})^f_1 \ldots (D^{a_{v-1}})^f_{v-1},$$

all of which have positive coefficients. Thus, it follows that $S_\lambda(D^{a_0}, D^{a_1}, \ldots, D^{a_{v-1}})$ is not the zero-polynomial.

To see the second part, we note that $\det(R \odot X(D))$ is a polynomial in $D$ whose coefficients in turn are multivariate polynomials in the elements of $R$, i.e., $\{r_{i,j}\}, 0 \leq i, j \leq v - 1$. Based on the proof above, it is clear that setting $R$ to be a matrix of all-ones results in a nonsingular matrix. This implies that $\det(R \odot X(D))$ is not identically zero. Next, the elements of $R$ are chosen i.i.d. from a continuous distribution. Therefore the probability that all the coefficients evaluate to zero over the random choice is also zero.

**Example 2.** For example, we consider $v = 3$ nodes, $D, D^2$ and $D^4$. So, we consider a square submatrix of size $3 \times 3$ as

$$E = \begin{bmatrix}
D^4 & D^8 & D^{16} \\
D^2 & D^4 & D^8 \\
D & D^2 & D^4
\end{bmatrix}$$

where $\lambda_0 = 2, \lambda_1 = 1$ and $\lambda_2 = 1$, so $\lambda = (2, 1, 1)$. The determinant of $E$ is given by

$$\det(E) = S_\lambda(D, D^2, D^4) \times \det(Y_{3,3})$$

The Schur polynomial can be obtained from Fig. 4 as

$$S_\lambda(D, D^2, D^4) = (D^4)^2 (D^2)^1 (D)^1 + (D^4)^1 (D^2)^2 (D)^1 + (D^4)^1 (D^2)^1 (D)^2 = D^{11} + D^9 + D^8$$

**B. Auxiliary Proofs**

**Lemma 2.** The Khatri-Rao product $Y_{k_A,s}(D^2) \odot Y_{k_B,s}(D)$ is a matrix in the form of (I)
Fig. 4: Young tableaux of shape $\lambda = (2, 1, 1)$ leads to three different distribution for $T = \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$ which helps to obtain $S_\lambda (D, D^2, D^4)$

**Proof.** Note that the Kronecker product of the $l$-th column of $Y_{kA,s}(D^z)$ and the $\ell$-th column of $Y_{kB,s}(D)$ can be expressed as

$$
\begin{bmatrix}
1 \\
D^{zl} \\
D^{2zl} \\
\vdots \\
D^{(kA-1)zl}
\end{bmatrix}
\otimes
\begin{bmatrix}
1 \\
D^l \\
D^{2l} \\
\vdots \\
D^{(kB-1)l}
\end{bmatrix}
= 
\begin{bmatrix}
1 & D^{(kB-1)l} & \cdots \\
& D^{zl} & \cdots \\
& & \ddots & \vdots \\
& & & D^{(kB-1+z)l} & \cdots \\
& & & & D^{(kA-1)zl} & \cdots \\
& & & & & D^{(kB-1+(kA-1))zl}
\end{bmatrix}
$$

The vector on the RHS above consists of powers of $D^l$ and can be seen to be in the form of $\textbf{1}$.

Let $U$ and $L = U^T$ denote square upper and lower shift matrices respectively, i.e., $U$ is a $q \times q$ matrix such that

$$
U_{ij} = \begin{cases} 
1 & \text{if } j = i + 1 \\
0 & \text{otherwise.}
\end{cases}
$$

Thus, for instance if $q = 5$, then

$$
U = 
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

We define $U^0 = L^0 = I_q$. 

Lemma 3. Let \( h \geq \max(i, j) \). Then
\[
(\tilde{D}^{h,i})(\tilde{D}^{h,j})^T = \begin{cases} U^{i-j} & \text{if } i > j, \text{ and,} \\ I^{j-i} & \text{if } i \leq j \end{cases},
\]
where we observe that the matrices on the RHS are Toeplitz.

Proof. We only prove the case when \( i > j \) as the other part is very similar. The product \((\tilde{D}^{h,i})(\tilde{D}^{h,j})^T\) can be expressed as
\[
\begin{bmatrix} 0_{q \times i} & I_q & 0_{q \times (h-i)} \end{bmatrix} \begin{bmatrix} 0_{j \times q} \\ I_q \\ 0_{(h-j) \times q} \end{bmatrix} = \begin{bmatrix} 0_{(q-(i-j))(i-j)} & I_{q-(i-j)} \\ 0_{(i-j) \times (i-j)} & 0_{(i-j) \times (q-(i-j))} \end{bmatrix} = U^{i-j}.
\]

Proof that \( \tilde{G}_x \tilde{G}_x^T \) has Toeplitz blocks.
Recall that the matrix \( \tilde{G} \) is such that
\[
\tilde{G} = \begin{bmatrix} \begin{array}{c} \begin{array}{cccc} I_q & 0 & \ldots & 0 \\ 0 & I_q & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & I_q \end{array} & \begin{array}{c} r_{00} D_{a_0 b_{k-1};a_0 b_0} & \ldots & r_{(s-1)0} D_{a_{s-1} b_{k-1};a_{s-1} b_0} \\ r_{10} D_{a_0 b_{k-1};a_0 b_1} & \ldots & r_{1(s-1)} D_{a_{s-1} b_{k-1};a_{s-1} b_1} \\ \vdots & \vdots & \ddots & \vdots \\ r_{(k-1)0} D_{a_0 b_{k-1};a_0 b_{k-1}} & \ldots & r_{(k-1)(s-1)} D_{a_{s-1} b_{k-1};a_{s-1} b_{k-1}} \end{array} \end{array} \end{bmatrix}.
\]

Lemma 4. Let \( k \leq \ell \leq n-1 \), and \( \tilde{\ell} = \ell - k \). Then, for \( i \geq j \)
\[
[\tilde{G}_\ell \tilde{G}_\ell^T]_{i,j} = \begin{cases} r_{i\ell}^2 I_q & \text{if } i = j, \\ r_{i\ell} r_{\ell j} U^{a_j(b_i-b_j)} & \text{if } i > j. \end{cases}
\]

Proof. This follows directly by using Lemma 3 and the definition of \( \tilde{G}_\ell \) and the \( \ell \)-th block column of \( \tilde{G} \).

On the other hand for \( 0 \leq \ell \leq k - 1 \)
\[
[\tilde{G}_\ell \tilde{G}_\ell^T]_{i,j} = \begin{cases} I_q & \text{if } i = j = \ell, \\ 0 & \text{otherwise}. \end{cases}
\]

Furthermore, using the property that the sum of Toeplitz matrices is Toeplitz, we can conclude that for any subset \( \mathcal{I} \subset \{0, \ldots, n-1\} \) such that \( |\mathcal{I}| = k \), we have that the matrix \( \tilde{G}_\mathcal{I} \tilde{G}_\mathcal{I}^T \) is a matrix with Toeplitz blocks.
For ease of presentation let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ where $\mathcal{I}_1 \subseteq \{0, \ldots, k - 1\}$, $\mathcal{I}_2 \subseteq \{k, \ldots, n - 1\}$ and $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$ and $\ell = \ell - k$. Then, for $0 \leq i, j \leq k - 1$ and $i \geq j$ we can express the $(i, j)$-th block of $\mathcal{G}_T \mathcal{G}^T_T$ as follows.

$$[(\mathcal{G}_T)(\mathcal{G}_T)^T]_{i,j} = \begin{cases} \sum_{\ell \in \mathcal{I}_2} T^{\ell} \delta^{\ell}_{ij} + 1_{i \in \mathcal{I}_1} \delta_{ij}, & \text{if } i = j, \\ \sum_{\ell \in \mathcal{I}_2} T^{\ell} U^{\ell}_{i,j}(b_i - b_j), & \text{if } i > j. \end{cases}$$

(9)

where $\mathbb{I}$ denotes the indicator function. By symmetry it suffices to specify $[(\mathcal{G}_T)(\mathcal{G}_T)^T]_{i,j}$ for $i \geq j$. Each of the blocks is of dimension $q \times q$.

C. Proof of Theorem 2

Let $b^{u,v}$ for $0 \leq u, v \leq k-1$ be a sequence of length $2q-1$ indexed as $b^{u,v}_{-(q-1)}, b^{u,v}_{-(q-2)}, \ldots, b^{u,v}_0, b^{u,v}_1, b^{u,v}_{(q-2)}, b^{u,v}_{(q-1)}$ and let

$$\tilde{B}(b^{u,v}) = \begin{bmatrix} b^{u,v}_{-1} & b^{u,v}_{-2} & \cdots & b^{u,v}_{-(q-1)} \\ b^{u,v}_0 & b^{u,v}_{-1} & \cdots & b^{u,v}_{-(q-2)} \\ \vdots & \vdots & \ddots & \vdots \\ b^{u,v}_{q-2} & b^{u,v}_{q-3} & \cdots & b^{u,v}_0 \end{bmatrix}$$

be a $q \times q$ Toeplitz matrix whose entries are specified by $b^{u,v}$. Furthermore let $b^{u,v}(e^{-i\omega}) = \sum_{p=-q}^{q-1} b^{u,v}_p e^{-ip\omega}$ for $\omega \in [-\pi, \pi]$ be the DFT of the sequence $b^{u,v}$ and let

$$B(e^{-i\omega}) = \begin{bmatrix} b^{0,0}(e^{-i\omega}) & b^{0,1}(e^{-i\omega}) & \cdots & b^{0,k-1}(e^{-i\omega}) \\ b^{1,0}(e^{-i\omega}) & b^{1,1}(e^{-i\omega}) & \cdots & b^{1,k-1}(e^{-i\omega}) \\ \vdots & \vdots & \ddots & \vdots \\ b^{k-1,0}(e^{-i\omega}) & b^{k-1,1}(e^{-i\omega}) & \cdots & b^{k-1,k-1}(e^{-i\omega}) \end{bmatrix}.$$ 

Consider a matrix $\tilde{B}$ (that has Toeplitz blocks) defined as follows.

$$\tilde{B} = \begin{bmatrix} \tilde{B}(b^{0,0}) & \tilde{B}(b^{0,1}) & \cdots & \tilde{B}(b^{0,k-1}) \\ \tilde{B}(b^{1,0}) & \tilde{B}(b^{1,1}) & \cdots & \tilde{B}(b^{1,k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{B}(b^{k-1,0}) & \tilde{B}(b^{k-1,1}) & \cdots & \tilde{B}(b^{k-1,k-1}) \end{bmatrix}.$$ 

(10)

The work of [7] shows the following result.

Lemma 5. For all $q$,

(i) The eigenvalues of $\tilde{B}$ lie in $[\min_{\omega \in [-\pi, \pi]} \lambda_{\min}(B(e^{-i\omega})), \max_{\omega \in [-\pi, \pi]} \lambda_{\max}(B(e^{-i\omega}))].$

(ii) Furthermore,

$$\lim_{q \to \infty} \lambda_{\min}(\tilde{B}) = \min_{\omega \in [-\pi, \pi]} \lambda_{\min}(B(e^{-i\omega})).$$

(11)

$$\lim_{q \to \infty} \lambda_{\max}(\tilde{B}) = \max_{\omega \in [-\pi, \pi]} \lambda_{\max}(B(e^{-i\omega})).$$

(12)

In other words, the behavior of the eigenvalues of $\tilde{B}$ which is a $kq \times kq$ matrix can be studied instead by computing the eigenvalues of the $k \times k$ matrix $B(e^{-i\omega})$ and finding its minimum and maximum eigenvalues over the range $\omega \in [-\pi, \pi]$. 

We emphasize that our matrix \([\mathcal{Z}(\mathcal{Z}^T)]\) in (9) is in the form of the matrix in (10), so that we can set \(\mathbf{B} = (\mathcal{Z}(\mathcal{Z}^T))^T\). Furthermore, based on (9), we have

\[
\mathbf{B}_{i,j} = [(\mathcal{Z}(\mathcal{Z}^T)]_{i,j} = \begin{cases} 
(\sum_{\ell \in \mathcal{I}} r_{i\ell} r_{j\ell}) \mathbf{L}_{q} + \mathbf{I}_{q}, & \text{if } i = j, \\
(\sum_{\ell \in \mathcal{I}} r_{i\ell} r_{j\ell}) \mathbf{L}_{q}, & \text{if } i > j.
\end{cases}
\]

Therefore, the corresponding sequence \(b_{i,j}\) for \(i > j\) is given by

\[
b_{k}^{i,j} = \begin{cases} 
r_{i\ell} r_{j\ell} & \text{if } k = -a_{\ell}(b_{i} - b_{j}) \\
0 & \text{otherwise}
\end{cases}
\]

so that

\[
b^{i,j}(e^{-j\omega}) = \sum_{\ell \in \mathcal{I}} r_{i\ell} r_{j\ell} \exp(j\omega a_{\ell}(b_{i} - b_{j}))
\]

The expressions above can equivalently be expressed as replacing \(D\) with \(e^{i\omega}\) and computing the inner product of \(\mathcal{Z}(e^{i\omega})(i,:)\) with \((\mathcal{Z}(e^{i\omega})(j,:))^*\). Therefore, we can compactly represent

\[
\mathbf{B}(e^{-i\omega}) = \mathcal{Z}(e^{i\omega})\mathcal{Z}(e^{i\omega})^*.
\]

This concludes the proof.

**Example 3.** Consider the \((n,k) = (4,2)\) example with \(G(D) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & D \end{bmatrix}\). Suppose that \(\mathcal{I} = \{2,3\}\). This implies that

\[
\hat{\mathcal{Z}}(\mathcal{Z}^T) = \hat{\mathcal{Z}}_2 \hat{\mathcal{Z}}_2^T + \hat{\mathcal{Z}}_3 \hat{\mathcal{Z}}_3^T
\]

\[
= \begin{bmatrix} 2I_q & I_q + L \\ I_q + U & 2I_q \end{bmatrix}.
\]

The corresponding \(\mathcal{Z}(e^{i\omega})\mathcal{Z}(e^{i\omega})^*\) can be obtained as

\[
\mathcal{Z}(e^{i\omega})\mathcal{Z}(e^{i\omega})^* = \begin{bmatrix} 2 & 1 + e^{-i\omega} \\ 1 + e^{i\omega} & 2 \end{bmatrix}.
\]

Using Theorem[2] we can conclude therefore that \(\lim_{q \to \infty} \lambda_{max} [T] = 2\) (achieved at \(\omega = \pi\)) and \(\lim_{q \to \infty} \lambda_{min} [T] = 0\) (achieved at \(\omega = 0\)). This implies therefore that as \(q\) becomes larger and larger, the matrix \(\hat{\mathcal{Z}}\) becomes more and more ill-conditioned, though it is nonsingular for any fixed \(q\).

**D. Auxiliary Observations**

**Example 4.** For the polynomial encoding of \([u_0(D) \ u_1(D)]\)

\[
[c_0(D) \ c_1(D) \ c_2(D) \ c_3(D)] = [u_0(D) \ u_1(D)] \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & D \end{bmatrix}
\]

\[
= [u_0(D) \ u_1(D) \ (u_0(D) + u_1(D)) \ (u_0(D) + Du_1(D))].
\]
It is not too hard to see that the polynomials \( u_0(D) \) and \( u_1(D) \) (equivalently the vectors \( u_0, u_1 \)) can be recovered from any two entries of the vector \([c_0(D) c_1(D) c_2(D) c_3(D)]\). The above argument can also be made in terms of matrices instead of polynomials.

In particular, let \( c_0, c_1, c_2 \) be row vectors of length \( q \) and \( c_3 \) be a row vector of length \( q + 1 \). Then,

\[
\begin{bmatrix}
  c_0 & c_1 & c_2 & c_3
\end{bmatrix} = \begin{bmatrix}
  u_0 & u_1
\end{bmatrix} \begin{bmatrix}
  I_q & 0_{q \times q} & I_q & I_q \\
  0_{q \times q} & I_q & I_q & 0_{q \times 1}
\end{bmatrix}^{q+1}.
\]

where the subscripts of the submatrices denote their size. It is important to note that in going from the polynomial representation to the matrix representation, we replaced the element \( D \) by a \( q \times (q + 1) \) matrix that transforms the vector \( u_1 \) to \([0 \ u_1]\).

1) Storage capacity constraint and choice of \( z \) in Section 3: We note that the highest exponent of \( D \) in the generator matrix in (4) is \((s_A - 1)(k_A - 1)\); this corresponds to the worker node with the maximum number of assigned submatrices. Thus, to satisfy the storage constraints, we need

\[
\frac{\Delta_A}{k_A} + (s_A - 1)(k_A - 1) \leq \gamma_A \Delta_A
\]

and \( k_A \) to divide \( \Delta_A \).

For the matrix-matrix case the choice of \( z \) needs to be large enough so that all-pairwise products of the coefficients appear when multiplying two polynomials. By examining the structure of (5) we can conclude that

\[
z = \frac{\Delta_B}{k_B} + (s - 1)(k_B - 1).
\]

The values of \( \Delta_A \) and \( \Delta_B \) can be obtained enforcing the storage constraints and divisibility requirements (\( k_A \) divides \( \Delta_A \), \( k_B \) divides \( \Delta_B \)) as

\[
\Delta_A \geq \frac{(s - 1)(k_A - 1)}{\gamma - \frac{1}{k_A}} \quad \text{and} \quad \Delta_B \geq \frac{(s - 1)(k_B - 1)}{\gamma - \frac{1}{k_B}}.
\]

2) Example of peeling decoder:

**Example 5.** Consider the matrix-matrix example in Fig. 1B and suppose that workers W0 and W1 are stragglers. We need to recover a total of \( \Delta_A \Delta_B = 48 \) block products. Note that we can directly obtain \( 4 \times 6 = 24 \) block products from workers W2 and W3. So it remains to recover \( A_i^T B_j \)'s for \( i = 0, 1, 2, 3 \) and \( j = 0, 1, \ldots, 5 \) from workers W4 and W5. First, concentrate on block products of the form \( A_0^T B_j \). The first block product of W5, recovers \( A_0^T B_0 \), following which we examine the first block product of W4, which is \((A_0 + A_4)^T(B_0 + B_3)\); the only unknown in this equation is \( A_0^T B_3 \) which can therefore be decoded. We can keep moving back and forth between W4 and W5 and recover all the block products \( A_0^T B_j \). In a similar fashion it can be verified that all other missing block products can be decoded.

**E. More numerical experiments**

In this section we present more numerical experiments demonstrating the superior performance of our techniques. First we run an experiment to tabulate the time needed to find a good random matrix \( R \). We run the 50 trials to
find the best $R$ for $n = 7, 8, \ldots, 15$ with $s = 2, 3, 4, 5$ each. It should be noted that the choice of $R$ depends on all $\binom{n}{k}$ choices of stragglers. Fig. 5 shows the corresponding time for different pairs of $n$ and $k$. From the figure, it can be seen that our system (a processor with CPU speed 3.5GHz and 16GB RAM) needs around 15 minutes to find a good choice of $R$ for even $n = 15$ and $s = 5$. In other cases, the required amount of time is even lesser. This indicates that for a reasonable system size, we do not need to wait too long to obtain an optimum $R$.

Next we show more performance results comparisons in terms of recovery error for the master node. For the matrix-vector case, we need to compute $A^T x$ in a distributed fashion where the matrix $A$ is of size $25000 \times 21000$. We assumed a system with $n = 10$ distributed workers with $\gamma = \frac{4}{25}$ where any $s = 3$ workers can be stragglers. In this case, for our approach, we set $\Delta_A = 700$. For the approach in [3] based on [3, eq. (6)], we used $\Delta_A = 840$ which is the lowest choice possible for their approach. We can see in Fig. 6a that in a low SNR setting, our approach based on best choice random matrix $R_A$ only has around 2.5% error at $10dB$ SNR whereas approach [3] has around 12% error. The performance of [15] is not reported as it had a very high error (more than 50% at $20dB$ and even higher at lower SNR’s).
For distributed matrix-matrix multiplication, \( \mathbf{A} \) is of size \( 6000 \times 4500 \) and \( \mathbf{B} \) is of size \( 6000 \times 4000 \). The system has \( n = 10 \) workers and needs to be resilient to \( s = 2 \) stragglers. Each worker node can store a maximum \( \gamma_A = \frac{3}{10} \) portion of \( \mathbf{A} \) and a maximum \( \gamma_B = \frac{3}{5} \) portion of matrix \( \mathbf{A} \). The approach of [3] does not provide a solution for matrix-matrix case and hence we can only compare with [15]. From Fig. 6b, we can see than even in high SNR around 65dB, our proposed approach performs significantly better than the approach in [15].