Optimal Order of Encoding for Gaussian MIMO Multi-Receiver Wiretap Channel

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Abstract—The Gaussian multiple-input multiple-output (MIMO) multi-receiver wiretap channel is studied in this paper. The base station broadcasts confidential messages to $K$ intended users while keeping the messages secret from an eavesdropper. The capacity of this channel has already been characterized by applying dirty-paper coding and stochastic encoding. However, $K$ factorial encoding orders may need to be enumerated for that, which makes the problem intractable. We prove that there exists one optimal encoding order and reduced the $K$ factorial times to a one-time encoding. The optimal encoding order is proved by forming a secrecy weighted sum rate (WSR) maximization problem. The optimal order is the same as that for the MIMO broadcast channel without secrecy constraint, that is, the weight of users’ rate in the WSR maximization problem determines the optimal encoding order. Numerical results verify the optimal encoding order.

I. INTRODUCTION

The multi-receiver wiretap channel, depicted in Fig. 1 is a channel model in which a transmitter wants to transmit messages of $K$ legitimate users while keeping them confidential from an external eavesdropper. This model, which is an extension of the well-known wiretap channel [1], is also known as secure broadcasting [4] and wiretap broadcast channel [3]. In the multi-input multiple-output (MIMO) multi-receiver wiretap channel each node can have an arbitrary number of antennas [4]. The secrecy capacity region of the two-receiver channel was characterized in [5], [6] in which secret dirty-paper coding (S-DPC) is proved to be optimal [6].

Ekrem and Ulukus established the secrecy capacity region of the Gaussian MIMO $K$-receiver wiretap channel in [4]. Interestingly, a different order of encoding for users can result in a different achievable region. The capacity-achieving rate region is then characterized by the convex closure of the union of achievable regions obtained by all one-to-one permutations of encoding order. Therefore, to determine the entire capacity region, all possible one-to-one permutations of users, i.e., $K!$ possible encoding orders, may need to be enumerated which makes the problem intractable especially when the number of users becomes large.

In this paper, we prove that such enumerations can be avoided and we find the optimal encoding order. To this end, we form a weighted sum-rate (WSR) maximization problem for the $K$-receiver wiretap channel, convert the objective function to another equivalent function, and prove the optimal order for the new optimization problem. Our work shows that the optimal encoding order for secure broadcasting is the same as that for insecure broadcasting, i.e., the MIMO-BC without secrecy. More specifically, the descending weight ordering in the WSR maximization problem determines the optimal encoding order [7]. The proof is non-trivial because, unlike the MIMO-BC, the associated problem in the multi-receiver MIMO wiretap channel is non-convex and BC-MAC duality [8] cannot be directly applied.

This finding reduces the complexity of evaluating the WSR problem by $K!$ factorial. When all weights are different, there is only one optimal order of encoding. However, when some weights are equal, different encoding orders can give different rate tuples (corner points of the capacity region) while all of them result in the same WSR. Particularly, encoding order is not important when sum-capacity is the concern, i.e., all weights are the same. Nonetheless, the order will determine which corner point of the capacity region will be achieved.

In addition to the order of encoding, we provide a numerical solution for finding the covariance matrices that achieve the capacity region. Previously, the solution existed only for the secrecy sum-capacity of this channel [9], but not the whole capacity region. Simulation results confirm the optimality of the proposed encoding order.

The remainder of this paper is organized as follows. In Section II we describe the system model. We introduce the WSR maximization problem and give the proof for the optimal weight ordering in Section III. We provide a numerical solution to efficiently find the covariance matrices in Section IV. Then we provide examples to verify the results and conclude.

Fig. 1: System model for the $K$-receiver wiretap channel.
the paper in Section \[\text{V}\] and Section \[\text{VI}\].

Notations: \(\text{tr}(\cdot)\) and \((\cdot)^{\dagger}\) denote trace and Hermitian of matrices. \(\mathbb{E}\{\cdot\}\) denotes expectation. \(\mathbf{Q} \succeq 0\) represents that \(\mathbf{Q}\) is a positive semidefinite matrix. \(\mathbf{I}_n\) is the identity matrix with size \(n \times n\). \(\log\) represents the natural logarithm and \(|\mathbf{Q}|\) represents the determinant of \(\mathbf{Q}\).

II. SYSTEM MODEL

Consider a \(K\)-receiver wiretap channel, as shown in Fig. 1. The BS transmits independent confidential messages to \(K\) legitimate receivers in the presence of one eavesdropper. Each confidential message \(M_k\) for user \(k, k = 1, 2, \ldots, K\), should be kept secret from the eavesdropper. There is no cooperation among legitimate users. The BS, user \(k\), and the eavesdropper are equipped with \(n_t\), \(n_k\), and \(n_e\) antennas, respectively. The received signals at the legitimate users and at the eavesdropper are given by

\[
y_k = \mathbf{H}_k\mathbf{x} + \mathbf{w}_k, \quad k = 1, 2, \ldots, K, \quad (1a)
\]
\[
y_e = \mathbf{G}\mathbf{x} + \mathbf{w}_e, \quad (1b)
\]

where the channels of \(k\)th legitimate user \(\mathbf{H}_k\) and eavesdropper \(\mathbf{G}\) are \(n_k \times n_t\) and \(n_e \times n_t\) complex matrices in which the elements of the channels are drawn from independent and identically distributed (i.i.d.) complex Gaussian distributions and \(\mathbf{w}_k\) and \(\mathbf{w}_e\) are complex random vectors whose elements are i.i.d., zero-mean, unit-variance Gaussian. The \(K\) independent signals are encoded by Gaussian codebooks and superimposed, and Gaussian signaling is optimal [4]. Thus, the transmit signal is \(\mathbf{x} = \sum_{k=1}^{K} \mathbf{x}_k\) where \(\mathbf{x}_k \sim \mathcal{CN}(0, \mathbf{Q}_k)\).

The channel input is subject to a power constraint. While it is more common to define capacity regions under a total power constraint, i.e., \(\text{tr}(\mathbb{E}\{\mathbf{x}\mathbf{x}^{\dagger}\}) \leq P\), the capacity region of this channel is first proved based on a covariance matrix constraint, i.e., \(\mathbb{E}\{\mathbf{x}\mathbf{x}^{\dagger}\} \preceq \mathbf{S} [3], [4]\). Once the capacity region is obtained under a covariance constraint \(\mathcal{C}((\mathbf{H}_k)_{k=1}^{K}, \mathbf{G}, \mathbf{S})\), the capacity region under more relaxed constraints on the channel inputs \(\mathcal{C}((\mathbf{H}_k)_{k=1}^{K}, \mathbf{G}, P)\) can be obtained by union over compact sets of input covariance matrices as \(\mathcal{C}((\mathbf{H}_k)_{k=1}^{K}, \mathbf{G}, P) = \bigcup_{\mathbf{S}\succ 0, \text{tr}(\mathbf{S}) \leq P} \mathcal{C}((\mathbf{H}_k)_{k=1}^{K}, \mathbf{G}, \mathbf{S})\). In short, the channel input should satisfy

\[
\text{tr}(\mathbb{E}\{\mathbf{x}\mathbf{x}^{\dagger}\}) = \sum_{k=1}^{K} \text{tr}(\mathbf{Q}_k) \leq \text{tr}(\mathbf{S}) \leq P. \quad (2)
\]

Let \(\pi = [\pi_1, \pi_2, \ldots, \pi_K]\) be a permutation function on the set \(\{1, \ldots, K\}\) and \(\pi_k = j\) represent the \(k\)th element of that arrangement is \(j\). By applying the DPC and stochastic encoding, the achievable non-negative secrecy rate at user \(\pi_k\) under the encoding order \(\pi\) is given by [4, 9]

\[
R_{\pi_k} = \log \left| \frac{\mathbf{I}_{n_k} + \mathbf{H}_{\pi_k} \left( \sum_{j=k}^{K} \mathbf{Q}_{\pi_j} \right) \mathbf{H}_{\pi_k}^{\dagger}}{\mathbf{I}_{n_k} + \mathbf{H}_{\pi_k} \left( \sum_{j=k+1}^{K} \mathbf{Q}_{\pi_j} \right) \mathbf{H}_{\pi_k}^{\dagger}} \right| - \log \left| \frac{\mathbf{I}_{n_e} + \mathbf{G} \left( \sum_{j=k}^{K} \mathbf{Q}_{\pi_j} \right) \mathbf{G}^{\dagger}}{\mathbf{I}_{n_e} + \mathbf{G} \left( \sum_{j=k+1}^{K} \mathbf{Q}_{\pi_j} \right) \mathbf{G}^{\dagger}} \right|, \quad (3)
\]

in which \(k = 1, 2, \ldots, K\). Since \(\log\) represents the natural logarithm, the unite of rates is \(\text{bits}/\text{sec}/\text{Hz}\) in this paper. The secrecy capacity region under the total power constraint (2) is characterized by the convex closure of the union of the DPC rate region over all possible one-to-one permutations \(\pi\) [4, Theorem 4], where \(\pi_k = j\) represents the \(k\)th position in permutation \(\pi\) is user \(j\). In general, \(K\) possible encoding orders may need enumerating to determine the capacity region.

It is worth mentioning that when there is only one legitimate receiver, the problem reduces to the MIMO wiretap channel and (3) becomes the capacity expression of the MIMO Gaussian wiretap channel, established in [10]–[12]. Also, various linear precoding schemes are designed for this case \((K = 1)\), including generalized singular value decomposition [13], alternating optimization and water filling [14], and rotation modeling [15], [16]. In this paper, we are interested in the \(K\)-receiver wiretap channel for \(K \geq 2\), but our solutions apply to the case with \(K = 1\).

III. OPTIMAL ENCODING ORDER

The WSR maximization for the \(K\)-receiver wiretap channel under a total power constraint \(P\) is formulated as

\[
\varphi(P) = \max_{\mathbf{Q}_{\pi_k} \succeq 0} \sum_{k=1}^{K} w_{\pi_k} R_{\pi_k}, \quad k = 1, 2, \ldots, K
\]

s.t. \(\sum_{k=1}^{K} \text{tr}(\mathbf{Q}_{\pi_k}) \leq P, \quad (4)\)

in which \(R_{\pi_k}\) is given in (3). The weights \(w_{\pi_1}, w_{\pi_2}, \ldots, w_{\pi_K}\) are non-negative values adding to one. Solving problem (4) for different permutations \(\pi\) results in different inner bounds (achievable regions) of the secrecy capacity region. The whole capacity region is then determined by enumerating \(K\) possible encoding orders and finding their union, which makes the problem intractable if the number of users is large [4].

In this section, we prove that once the weights are fixed, the encoder order in (4) is determined. Since the proof uses the BC-MAC duality [8], before providing the proof, we first briefly introduced this result in the following.

Proposition 1. (BC-MAC Duality [8]) Suppose that users \(1, \ldots, K\) are encoded sequentially. The achievable rate of user \(k\) for the MIMO-BC can be computed by

\[
R_{\text{BC}}^{k} = \log \left| \frac{\mathbf{I}_{n_k} + \mathbf{H}_k \left( \sum_{j=k}^{K} \mathbf{Q}_j \right) \mathbf{H}_k^{\dagger}}{\mathbf{I}_{n_k} + \mathbf{H}_k \left( \sum_{j=k+1}^{K} \mathbf{Q}_j \right) \mathbf{H}_k^{\dagger}} \right|, \quad (5)
\]

and, in its dual MIMO-MAC, the achievable rate is given by

\[
R_{\text{MAC}}^{k} = \log \left| \frac{\mathbf{I}_{n_e} + \sum_{j=1}^{k} \mathbf{H}_j \mathbf{S}_j \mathbf{H}_j^{\dagger}}{\mathbf{I}_{n_e} + \sum_{j=1}^{k-1} \mathbf{H}_j \mathbf{S}_j \mathbf{H}_j^{\dagger}} \right|. \quad (6)
\]

Unlike this, in the MIMO-BC without security, there exists an optimal encoding order \(\pi\) to maximize the WSR problem [7], [17]. The proof is determined by the well-known duality between BC and MAC [8].
Given a set of BC covariance matrices $Q_k$, $k = 1, \ldots, K$, there are MAC covariance matrices $\Sigma_k$, $k = 1, \ldots, K$, such that $R_k^{MAC} = R_k^{BC}$ $\forall k$, and $K \sum_{k=1}^{K} \text{tr}(Q_k) = \sum_{k=1}^{K} \text{tr}(\Sigma_k)$ via the BC-MAC transformation, and vice versa. Let us define

$$C_k \triangleq I_{n_k} + H_k (\sum_{j=k+1}^{K} Q_j) H_k^\dagger,$$

$$D_k \triangleq I_{n_k} + \sum_{j=1}^{k-1} H_j^\dagger \Sigma_j H_j,$$  \hspace{1cm} (7)

Then, the BC and MAC covariance matrices can be expressed as [8]

$$Q_k = D_k^{-1/2} E_k F_k^\dagger C_k^{1/2} \Sigma_k C_k^{1/2} F_k^\dagger E_k D_k^{-1/2},$$

$$\Sigma_k = C_k^{-1/2} E_k F_k^\dagger D_k^{-1/2} Q_k D_k^{-1/2} F_k^\dagger E_k C_k^{-1/2},$$  \hspace{1cm} (8)

in which $E_k$ and $F_k$ are obtained by singular value decomposition of $D_k^{-1/2} H_k^\dagger C_k^{1/2} D_k^{-1/2} = E_k A_k F_k$, where $A_k$ is a square and diagonal matrix.

**Theorem 1.** The WSR problem in (4) can be solved by the following equivalent optimization problem

$$\max_{\pi_k \geq 0} \sum_{k=1}^{K} (w_{\pi_k} - w_{\pi_{k-1}}) \times$$

$$\left( \log |I_{n_k} + \sum_{j=k}^{K} H_j^\dagger \Sigma_j H_j| - \log |I_{n_k} + \sum_{j=k}^{K} G_j^\dagger \Sigma_j G_j| \right),$$

s.t. $\sum_{k=1}^{K} \text{tr}(\Sigma_k) \leq P$, $k = 1, 2, \ldots, K$ \hspace{1cm} (9a)

where $w_{\pi_0} \triangleq 0$ and $G_j \triangleq C_j^{1/2} F_j^\dagger E_j^\dagger D_j^{-1/2} G_j$. Further, the optimal decoding order $\pi$ is a permutation of the set $\{1, \ldots, K\}$ such that the weights satisfy $w_{\pi_1} \leq w_{\pi_2} \leq \cdots \leq w_{\pi_K}$. The optimal encoding order is the reverse.

**Proof.** First, by expanding and rewriting the WSR maximization problem (4), we have

$$\max_{Q_k \geq 0} \sum_{k=1}^{K} w_{\pi_k} R_{\pi_k}$$

$$= \max_{Q_k \geq 0} \left( \sum_{k=1}^{K} w_{\pi_k} \log \left| I_{n_k} + H_k \left( \sum_{j=k+1}^{K} Q_j \right) H_k^\dagger \right| \right)$$

$$- \sum_{k=1}^{K} w_{\pi_k} \log \left| I_{n_k} + G \left( \sum_{j=k+1}^{K} Q_j \right) G^\dagger \right|$$

$$= \max_{\pi_k \geq 0} \left( \sum_{k=1}^{K} (w_{\pi_k} - w_{\pi_{k-1}}) \times \right)$$

$$\left( \log |I_{n_k} + \sum_{j=k}^{K} H_j^\dagger \Sigma_j H_j| - \log |I_{n_k} + \sum_{j=k}^{K} G_j^\dagger \Sigma_j G_j| \right),$$

where the (a) follows because $\sum_{j=1}^{k-1} H_j^\dagger \Sigma_j H_j \geq 0$ which results in $|I + \sum_{j=1}^{k-1} H_j^\dagger \Sigma_j H_j| \geq 1$. (b) and (c) follow from Proposition 1, and (d) follows with the same argument we had moving from (10a) to (10b).

This proves that in (10d), the term inside the parentheses is non-negative. Hence, $w_{\pi_1} \leq w_{\pi_2} \leq \cdots \leq w_{\pi_K}$, and

We assume that not all the weights are identical, otherwise, the WSR maximization reduces to a scaled sum-rate problem whose optimal solution is obtained at any of the $K!$ corner points of the capacity region [9].
the decoding order is a permutation $[\pi_1, \pi_2, \ldots, \pi_K]$ satisfying $w_{\pi_1} \leq w_{\pi_2} \leq \cdots \leq w_{\pi_K}$. This completes the proof.

Remark 1. The secrecy capacity region of the general Gaussian MIMO multi-receiver wiretap channel is characterized by the convex closure of the union of the DPC rate region over all possible one-to-one permutations $\pi$ \cite[Theorem 4]{4}. Based on Theorem 1 only one encoding order is enough to achieve each point on the boundary of the secrecy capacity region, and $K!$ encoding orders are reduced to one order. The optimal order is the same as that for the MIMO-BC, that is, a user with a higher weight in the WSR should be encoded earlier.

Corollary 1. When all weights are equal, the secrecy sum-rate is obtained. Different encoding orders may result in different rate tuples $(R_{\pi_1}, R_{\pi_2}, \ldots, R_{\pi_K})$, but all encoding orders yield the same secrecy sum-rate.

IV. SOLVING THE WSR PROBLEM

To reach the border of the secrecy capacity region, we need to solve the WSR problem in \cite{4}. Unfortunately, this problem is non-convex and challenging to solve–even after finding the optimal encoding order. In this section, we propose a numerical algorithm to solve \cite{4}. For simplicity of derivations and notation, without loss of generality, we assume the encoding order is ascending order, that is, $\pi_k = k$. The Lagrangian of the problem \cite{4} is

$$L(Q_1, \ldots, Q_K, \lambda) = \sum_{k=1}^{K} w_k R_k - \lambda \left( \sum_{k=1}^{K} \text{tr}(Q_k) - P \right), \quad (13)$$

where $\lambda$ is the Lagrange multiplier related to the total power constraint.

To solve this, similar to \cite{18} and \cite{19} we can then apply the block coordinate descent (BCD) algorithm. The BCD algorithm periodically finds optimal solutions for a single block of variables while maintaining other blocks of variables fixed at each iteration \cite{20}. An example of BCD is the block successive maximization method (BSMM) \cite{18} which updates the covariance matrices by successively optimizing a lower bound of local approximation of $f(Q_1, \ldots, Q_K) = L(Q_1, \ldots, Q_K, \lambda)$. At iteration $i$ of the algorithm, the variables $Q_k^{(i)}$, $k = 1, 2, \ldots, K$, are updated by solving the following problem \cite{18}, \cite{20}

$$Q_k^{(i)} = \arg \max_{Q_k \geq 0} f(Q_1^{(i)}, \ldots, Q_{k-1}^{(i)}, Q_k, Q_{k+1}^{(i-1)}, \ldots, Q_K^{(i-1)}), \quad (14)$$

The function $f(Q_1, \ldots, Q_K)$ can be written into the summation of convex and concave functions

$$f(Q_1, \ldots, Q_K) = f_{\text{cvx}}(Q_k, Q_k) + f_{\text{cvx}}(Q_k, Q_k), \quad (15)$$

in which $Q_k = (Q_1^{(i)}, \ldots, Q_{k-1}^{(i)}, Q_{k+1}^{(i)}, \ldots, Q_K^{(i)})$ represents all covariance matrices excluding $Q_k$, $f_{\text{cvx}}(Q_k, Q_k)$ is the concave function of $Q_k$ by fixing $Q_k$, and $f_{\text{cvx}}(Q_k, Q_k)$ denotes the convex function of $Q_k$. The details of the functions are omitted for space limitations. We write the functions in \cite{16} shown at the bottom of the page.

For the $i$th iteration, the convex function for $f_{\text{cvx}}(Q_k, Q_k)$ can be lower-bounded by its gradient \cite{20}

$$f_{\text{cvx}}(Q_k, Q_k) \geq f_{\text{cvx}}(Q_k, Q_k) + \text{tr}[A_k^{(i)}(Q_k - Q_k^{(i)})], \quad (17)$$

in which $Q_k^{(i)} = (Q_1^{(i)}, \ldots, Q_{k-1}^{(i)}, Q_{k+1}^{(i)}, \ldots, Q_K^{(i-1)})$, the first $k - 1$ covariance matrices have been optimized in the $i$ iteration, while the $k + 1$ to $K$ covariance matrices are from the previous $i - 1$ iteration waiting to be optimized. The power price matrix is a partial derivative with respect to $Q_k$

$$A_k^{(i)} = \nabla Q_k f_{\text{cvx}}(Q_k, Q_k)|_{Q_k^{(i)}, Q_k^{(i)}}. \quad (18)$$

A lower bound of the function $f(Q_1, \ldots, Q_K)$ at $i$th iteration by substituting the right terms in \cite{17} into \cite{15} can be re-written as

$$f(Q_1^{(i)}, \ldots, Q_{k-1}^{(i)}, Q_k, Q_{k+1}^{(i)}, \ldots, Q_K^{(i-1)}) \geq f_{\text{cvx}}(Q_k, Q_k^{(i)}) + f_{\text{cvx}}(Q_k^{(i)}, Q_k^{(i)}) + \text{tr}[A_k^{(i)}(Q_k - Q_k^{(i)})]. \quad (19)$$

Then, we optimize the right-hand side of the inequality in \cite{19} by omitting the constant terms and obtaining the general iteration formula. The general iteration formula is a convex problem and any convex tool can be applied. It is worth mentioning that the solution of maximizing the right-hand side of \cite{19} is unilaterally optimal following Nash equilibrium \cite{21} Proposition 2]. The details of the algorithm and its complexity analysis are omitted, interested readers can refer to \cite{18}, \cite{19}, for examples.

\begin{align}
 f_{\text{cvx}}^+(Q_k, Q_k) &= w_k \log |I + (I + H_k) \sum_{j=k+1}^{K} Q_j H_j^\dagger|^{-1} H_k Q_k H_k^\dagger| + \sum_{j=1}^{k-1} w_j \log |I + G \sum_{i=j+1}^{K} Q_i G_i^\dagger| - \lambda \text{tr}(Q_k), \quad (16a) \\
 f_{\text{cvx}}^-(Q_k, Q_k) &= -w_k \log |I + (I + G) \sum_{j=k+1}^{K} Q_j G_j^\dagger|^{-1} G Q_k G_k^\dagger| + \sum_{j=1}^{k-1} w_j \log |I + H_j \sum_{i=j+1}^{K} Q_i H_i^\dagger|^{-1} H_k Q_k H_k^\dagger| \\
 &- \sum_{j=1}^{k-1} w_j \log |I + G \sum_{i=j}^{K} Q_i G_i^\dagger| + \sum_{j=k+1}^{K} w_j R_j - \lambda \text{tr}(\sum_{j=1}^{K} Q_j - Q_k - P). \quad (16b)
\end{align}
V. SIMULATIONS

In this section, we present some simulation results through two examples.

**Example 1:** We consider the two-user case \((K = 2)\) and set the channels and power the same as \(\text{[2]}\), which are

\[
H_1 = \begin{bmatrix} 1 & -0.5 \\ 0.5 & 2 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.3 & 1 \\ 2.0 & -0.4 \end{bmatrix},
\]

\[
G = \begin{bmatrix} 0.8 & -1.6 \end{bmatrix}, \quad P = 1.
\]

We set \(w_1 = w_2 = 0.5\) to verify the subcase of the weighted sum rate, i.e., sum-rate. If user 1 is encoded first and then user 2, i.e., \(\pi_1 = 1, \pi_2 = 2\), the optimal secrecy rates are \((R_1, R_2) = (0.8334, 0.7643)\) nats/sec/Hz and the secrecy sum-rate is 1.5977. If user 2 is encoded first and then user 1, i.e., \(\pi_1 = 2, \pi_2 = 1\), the optimal secrecy rates are \((R_1, R_2) = (0.5324, 1.065)\) which yields the same security sum-rate which is 1.5977. If we set weights \(w_1\) and \(w_2\) from 0 to 1 with a step 0.01, the results of the rate pairs are shown in Fig. 2.

The blue circle curve denotes the secure broadcasting with ascending coding order, while the yellow circle curve denotes the descending coding order. Different encoding orders result in different rates. The purple and green curves represent the achievable rate region of MIMO BC \([22]\), i.e., no eavesdropper or \(G = 0\).

The ordering of the users weight in the WSR maximization problem determines the optimal encoding order. An example is shown in Fig. 3. If \(w_1 > w_2\), the optimal encoding order is to encode message \(M_1\) first and message \(M_2\) next. If \(w_1 \leq w_2\), the optimal encoding order is to encode \(M_2\) first and \(M_1\) is encoded next. If \(w_1 = w_2\), the two orders both will give the same sum-rate but each order will give a different corner point of the capacity region. See Fig. 2 and Fig. 3 for illustrative representations.

**Example 2:** We consider \(K = 3\), in which the channels are

\[
H_1 = \begin{bmatrix} -0.4332 + 0.7954i & -0.3152 - 1.8835i \\ -1.0443 + 1.2282i & -0.2614 + 0.2198i \end{bmatrix},
\]

\[
H_2 = \begin{bmatrix} 1.3389 - 0.5995i & -0.6924 - 0.4542i \\ -1.2542 + 0.1338i & -2.1644 + 0.6520i \end{bmatrix},
\]

\[
H_3 = \begin{bmatrix} 1.0291 - 0.0212i & -0.3016 - 0.3662i \\ 0.1646 + 0.5179i & 0.3075 + 0.2919i \end{bmatrix},
\]

\[
G = \begin{bmatrix} -0.0875 - 0.9443i & -0.4637 + 0.7799i \end{bmatrix}.
\]

Figure 4 shows two different weights allocations with all six encoding orders. Each bar reflects the secure WSR under a specific weighting coefficient. The maximum WSR is achieved with the encoding order \(w_{\pi_1} > w_{\pi_2} > \cdots > w_{\pi_K}\). For example, for \((w_1, w_2, w_3) = (0.15, 0.2, 0.65)\) we get \(\pi_1 = 3, \pi_2 = 2, \pi_3 = 1\), whereas for \((w_1, w_2, w_3) = (0.2, 0.1, 0.7)\) we have \(\pi_1 = 3, \pi_2 = 1, \pi_3 = 2\). That is, the optimal encoding order is \(\pi = [3, 2, 1]\) in the former and \(\pi = [3, 1, 2]\) in the latter.

VI. CONCLUSIONS

Optimal encoding order for \(K\)-receiver wiretap channel is established in this paper. The proof indicates that the optimal order depends on the weight ordered. A receiver having a higher weight should be encoded earlier. The proof is characterized by translating the original WSR maximization problem of the MIMO-BC wiretap channel into an equivalent problem. This finding reduces the complexity of determining the capacity region from \(K!\) encoding orders to only one-time encoding. We have also solved the WSR maximization problem using BSMM. Numerical results verify the optimal encoding order.
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