Abstract. We show that the fundamental group of the complement of any irreducible tame torus sextics in $\mathbb{P}^2$ is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_3$ except one class. The exceptional class has the configuration of the singularities $\{C_{3,9}, 3A_2\}$ and the fundamental group is bigger than $\mathbb{Z}_2 * \mathbb{Z}_3$. In fact, the Alexander polynomial is given by $(t^2 - t + 1)^2$. For the proof, we first reduce the assertion to maximal curves and then we compute the fundamental groups for maximal tame torus curves.

1. Introduction

Recall that a sextic curve $C : F(X,Y,Z) = 0$ in $\mathbb{P}^2$ is called a torus curve of type $(2,3)$ if there exists an expression: $F(X,Y,Z) = F_2(X,Y,Z)^3 + F_3(X,Y,Z)^2$ where $F_2(X,Y,Z)$, $F_3(X,Y,Z)$ are polynomials of degree 2 and 3 respectively. Let $C_2$ and $C_3$ be the conic and the cubic defined by $F_2(X,Y,Z) = 0$ and $F_3(X,Y,Z) = 0$ respectively. $C$ is called a tame torus curve if the singularities are only on the intersection $C_2 \cap C_3$.

In [P], the second author classified every possible singularities on tame torus sextics of type $(2,3)$. In particular, he showed that there exist 7 moduli of the maximal tame torus curves and their configurations of the singularities are given as follows.

$$\{C_{3,15}\}, \{C_{9,9}\}, \{B_{3,10}, A_2\}, \{Sp_1, A_2\}, \{B_{3,8}, E_6\}, \{C_{3,7}, A_8\}, \{C_{3,9}, 3A_2\}$$

For a configuration $\Sigma$ in [P] we denote the corresponding moduli space of sextics of torus type by $\mathcal{M}(\Sigma)$. It is also shown that each of the moduli space is connected. Thus the topology of the complement $\mathbb{P}^2 - C$ is independent of the choice of a generic curve $C \in \mathcal{M}(\Sigma)$. Any irreducible tame torus curve can be degenerated into one of them and maximal tame torus curves are rational curves.

The purpose of this paper is to study of the geometry of these curves from the viewpoint of the fundamental group of the complement of curves.

In §2, we prepare a key lemma (Lemma 4) which reduces the computation of the fundamental groups to the case of maximal curves.

In §3, we compute the dual curves of the maximal curves. Some of the maximal curves have a big singularity like $C_{3,7}, C_{3,9}, C_{3,15}$ which are not locally irreducible but have the same tangent cone. We show the self-duality of these singularities.

In §4, we give the main result of this paper (Theorem 12) which state:

$$\pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 * \mathbb{Z}_3$$

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for any irreducible tame torus curve and the Alexander polynomial is given by $t^2 - t + 1$ with one exceptional class $\mathcal{M}((C_{3,9}, 3A_2))$. For a curve $C$ in this exceptional moduli space, the fundamental group is represented by

$$\pi_1(C^2 - C) \cong \langle g_1, g_2, g_3 | \{g_i, g_j\} = e, i \neq j, (g_3g_2g_1)^2 = (g_2g_1g_3)^2 = (g_1g_3g_2)^2 \rangle$$

and the Alexander polynomial is given by $(t^2 - t + 1)^2$.

In §5, we give an example of a non-tame torus curve $C$ with three $E_6$ and an $A_1$ singularities so that the fundamental group of this curve is isomorphic to $B_4(P^1)$ (Theorem [16]).

2. Zariski pencil and Puiseux parametrization

2.1. Zariski-pencil. Let $C$ be a curve of degree $d$ in $\mathbb{P}^2$. To compute the fundamental group $\pi_1(\mathbb{P}^2 - C)$, it is usually most convenient to use the Zariski’s pencil method, which we recall briefly. First choose a line $L_\infty$ and choose a base point $b$ on $L_\infty$. The family of lines $L_\eta$ which pass through $b$ is called the pencil at $b$. Unless otherwise stated, we take $b$ as the base point of the fundamental group as well. A pencil line $L_\eta$ is called singular if $L_\eta \cap C$ contains less than $d$ points. Take homogeneous coordinates $(X, Y, Z)$ so that $L_\infty$ is defined by $Z = 0$ and $b = (0, 1, 0)$ for simplicity. Let $W = \{\eta_1, \ldots, \eta_e\}$ be the set of the parameters corresponding to the singular pencil lines by the correspondence $\eta \leftrightarrow L_\eta = \{X - \eta Z = 0\}$. Choose a generic pencil line $L_{\eta_0}$. Thus $\eta_0 \in C - W$. By the local triviality of the line section $L_\eta \cap C$ over $C - W$, $\pi_1(C - W, \eta_0)$ acts on $\pi_1(L_{\eta_0} - L_{b_0} \cap C)$. This gives the monodromy relations

$$\{g = g^\sigma; g \in \pi_1(L_{\eta_0} - L_{b_0} \cap C), \sigma \in \pi_1(C - W, \eta_0)\}$$

The van Kampen-Zariski theorem says that $\pi_1(\mathbb{P}^2 - C)$ is isomorphic to the quotient group of $\pi_1(L_{\eta_0} - L_{b_0} \cap C)$ by the monodromy relations.

The affine fundamental group $\pi_1(C^2 - C)$ can be computed by the exact same way, replacing $\pi_1(L_{\eta_0} - L_{b_0} \cap C)$ by $\pi_1(L_{\eta_0}^a - L_{\eta_0}^a \cap C)$ where $L_{\eta_0}^a := L_{\eta_0} - L_{b_0} \cap L_\infty$. Moreover for a generic line at infinity, we have a central exact sequence ([O2]):

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(C^2 - C) \xrightarrow{\ell} \pi_1(\mathbb{P}^2 - C) \rightarrow 1$$

The generator $1 \in \mathbb{Z}$ is represented by a lasso of $L_\infty$ and it is homotopic to the “big circle” $\ell \circ S_R \circ \ell^{-1}$ where $S_R$ is the big circle $|y| = R, R \gg 1$ in $L_{\eta_0}^a - L_{\eta_0} \cap C$ which contains all intersection points $C \cap L_{\eta_0}$ and $\ell$ is a path joining $S_R$ and the base point.

Remark 1. In practice, it is extremely difficult to read the monodromy relations for curves which are defined over $\mathbb{C}$. Suppose that we are interested in the fundamental $\pi_1(\mathbb{P}^2 - C)$, where $C$ has a prescribed configuration of the singularities $\Sigma$ and degree $C = n$ is fixed. Let $\mathcal{M}(\Sigma; n)$ be the moduli space of curves of degree $n$ with configuration of the singularities $\Sigma$. As the fundamental group does not change if we move the curve in a connected component of the moduli space $\mathcal{M}(\Sigma)$, we are free to move the original curve in this component. So if possible, it is convenient to choose a curve defined over the real numbers $\mathbb{R}$ which has as many
2.2. Puiseux parametrization. Let \( h(t) = \sum_{i=0}^{\infty} a_i t^i \) be a convergent power series with complex coefficients. Let \( n \) be a given positive integer. The first characteristic power \( P_1(h(t); n) \) is defined by the integer \( \min\{ j > 0; a_j \neq 0, j \not \equiv 0 \pmod{n} \} \) (see [O2]). Let \( \nu_1 := P_1(h(t); n) \) and \( n^{(1)} := \gcd(n, \nu_1) \). If \( \nu_1 < \infty \), we define \( \nu_2 := P_1(h(t); n^{(1)}) \) and \( n^{(2)} := \gcd(n^{(1)}, \nu_2) \) and so on. As the integers \( n, n^{(1)}, \ldots \) are decreasing, they become stationary after a finite steps. So we assume that \( n^{(k-1)} > n^{(k)} = n^{(k+1)} \). We put

\[
\mathcal{P}(h(t); n) := \{\nu_1, \ldots, \nu_k\}, \quad D(h(t); n) := \{n^{(1)}, \ldots, n^{(k)}\}
\]

Let \((C, O)\) be a germ of an irreducible curve with Puiseux pairs \( \{(m_1, n_1), \ldots, (m, n)\} \). Recall that \( \gcd(m_i, n_i) = 1 \) and \( m_i > m_{i-1} n_i \) for \( i = 1, \ldots, k \) with \( m_0 = 1 \). Let \((x, y)\) be coordinates so that \( y = 0 \) defines the tangent cone. Then \( C \) can be parametrized as \( x(t) = t^N \) and \( y = \phi(t) = \sum_{i=S}^{\infty} a_i t^i \) so that

\[
\mathcal{P}(\phi(t); N) = \{m_1 n_2 \cdots n_k, m_2 n_3 \cdots n_k, \ldots, m_k\},
\]

\[
D(\phi(t); N) = \{n_2 \cdots n_k, n_3 \cdots n_k, \ldots, n_k\}
\]

where \( N = n_1 \cdots n_k \). Put \( S = \text{val}_t \phi(t) \). Note that \( S \leq m_1 n_2 \cdots n_k \) and \( S \equiv 0 \pmod{N} \) if and only if \( S < m_1 n_2 \cdots n_k \). The number \( s := S/N \) is called the Puiseux order of \( y(x^{1/N}) \) in [O4]. Recall that \( 2 \leq s \leq m_1/n_1 \) and \( s \) is an integer if \( s < m_1/n_1 \). Consider an irreducible curve \( C \) which is described as above. To see the behavior of the intersection \( C \cap \{y = \eta\} \), we wish to express \( x \) as a function of \( y \). This is the case when we compute the fundamental group \( \pi_1(P^2 - C) \) using the pencil \( \{y = \eta; \eta \in \mathbb{C}\} \). For this purpose, we take the new parameter \( \tau \) which is defined by \( \phi(t) = \tau^S \). Then we have

**Lemma 2.** We can write \( C \) as \( y = \tau^S \) and \( x = \psi(\tau) = \tau^N \psi_0(\tau) \) so that \( \psi_0(0) \neq 0 \) and

\[
\mathcal{P}(\psi(\tau); N) = \{m_1 n_2 \cdots n_k - S, m_2 n_3 \cdots n_k - S, \ldots, m_k - S\}, \quad S < m_1 n_2 \cdots n_k
\]

\[
\mathcal{P}(\psi(\tau); N/n) = \{m_2 n_3 \cdots n_k - S, \ldots, m_k - S\}, \quad S = m_1 n_2 \cdots n_k
\]

**Proof.** First, we can write \( \phi(t) = t^S \phi_0(t) \) with \( \phi_0(0) \neq 0 \). Then

\[
\mathcal{P}(\phi(t); N) = \{m_1 n_2 \cdots n_k - S, \ldots, m_k - S\}, \quad S < m_1 n_2 \cdots n_k
\]

\[
\mathcal{P}(\phi(t); N/n) = \{m_2 n_3 \cdots n_k - S, \ldots, m_k - S\}, \quad S = m_1 n_2 \cdots n_k
\]

Thus by [O2], Lemma 5.1, and Lemma 5.2, the assertion follows immediately. \( \square \)

Suppose that we have to compute the (local or global) fundamental group of a curve which have a reducible singularity at the origin \( O \). The above lemma plays an important role to compute the local or global fundamental group in such a situation. See §4.

**Example 3.** Assume that \( s = 2 \) and \( C \) has a single Puiseux pair \( \{(m, n)\} \) with \( m/n > 2 \) and \( y = 0 \) to be the tangent cone. Then \( C \) is parametrized as

\[
x = t^n, \quad y = a_2 t^{2n} + \cdots + a_m t^m + \cdots, \quad a_2, a_m \neq 0
\]
and putting \( y = \tau^{2n} \), we get another parametrization
\[
y = \tau^{2n}, \quad x = \tau^n(b_0 + b_1\tau^n + \cdots + b_k\tau^{kn} + b_{m-2}\tau^{m-2n} + \cdots), \quad k = [m/n] - 2
\]
and \( b_0, b_{m-2} \neq 0 \). The local topological behavior depends only on \( \{s, m, n\} \). We remark that for the line \( x = \eta, |\eta| \ll 1 \) intersects with \( C \) at \( n \) points locally near \( O \), while the line \( y = \eta, |\eta| \ll 1 \) intersects \( C \) at \( 2n \) points.

2.3. **Isomorphism theorem.** Let \( G \) be a group. The commutator subgroup is denoted by \( D(G) \). The first homology of \( G \) is by definition the quotient group \( G/D(G) \) and we denote it by \( H_1(G) \). A free group of rank \( n \) is denoted by \( F(n) \).

**Lemma 4.** Let \( G \) be a group such that \( D(G) \) is a free group \( F(n) \) with \( n < \infty \). Suppose that we have a surjective endomorphism \( \varphi : G \to G \) which induces an isomorphism \( \tilde{\varphi} \) on \( H_1(G) \). Then \( \varphi \) is an isomorphism.

**Proof.** This is observed in [O4]. First we consider the surjective homomorphism \( \varphi' := \varphi|_{D(G)} : D(G) \to D(G) \). By the assumption, we have an isomorphism \( D(G)/\text{Ker}(\varphi') \cong D(G) \). If \( \text{Ker}(\varphi') \) is not trivial, we get an obvious contradiction to the Hopfian property of the free groups (see Theorem 2.13, [MKS]). Now the assertion follows from Five Lemma:

\[
\begin{array}{cccc}
1 & \to & D(G) & \to & G & \to & H_1(G) & \to & 1 \\
\downarrow & \varphi' & \downarrow & \varphi & \downarrow & \tilde{\varphi} & & & \\
1 & \to & D(G) & \to & G & \to & H_1(G) & \to & 1
\end{array}
\]

As is well-known, \( D(\mathbb{Z}_p * \mathbb{Z}_q) \) is a free group of rank \((p - 1)(q - 1)\), we obtain

**Corollary 5.** Assume that \( \varphi : \mathbb{Z}_p * \mathbb{Z}_q \to \mathbb{Z}_p * \mathbb{Z}_q \) is a surjective homomorphism which gives an isomorphism on the first homology. Then \( \varphi \) is an isomorphism.

2.4. **Geometric homomorphism.** Suppose that \( C, C' \) are reduced curves of the same degree and assume that the line at infinity \( L_{\infty} := \{Z = 0\} \) is generic for \( C \) and \( C' \). A homomorphism \( \phi : \pi_1(\mathbb{C}^2 - C) \to \pi_1(\mathbb{C}^2 - C') \) is geometric if it preserves “the big circles” so that it induces an homomorphism \( \tilde{\phi} : \pi_1(\mathbb{P}^2 - C) \to \pi_1(\mathbb{P}^2 - C') \) and the commutative diagrams:

\[
\begin{array}{cccc}
1 & \to & \mathbb{Z} & \to & \pi_1(\mathbb{C}^2 - C) & \overset{\iota}{\to} & \pi_1(\mathbb{P}^2 - C) & \to & 1 \\
\downarrow & \text{Id} & \downarrow & \phi & \downarrow & \tilde{\phi} & & & \\
1 & \to & \mathbb{Z} & \to & \pi_1(\mathbb{C}^2 - C') & \overset{\iota'}{\to} & \pi_1(\mathbb{P}^2 - C') & \to & 1
\end{array}
\]

where \( \iota, \iota' \) are canonical homomorphisms induced by the respective inclusion maps. By the definition, we have

**Proposition 6.** Assume that \( \phi : \pi_1(\mathbb{C}^2 - C) \to \pi_1(\mathbb{C}^2 - C') \) is a geometric homomorphism. In [O4], \( \phi \) is an isomorphism if and only if \( \tilde{\phi} \) is an isomorphism.

Assume that we have a degeneration family of plane curves \( C_t (|t| \leq 1) \) so that \( C_t \) is reduced for any \( t \) and \( C_t, t \neq 0 \), has the same configuration of singularities. In such a situation we have a canonical surjective homomorphism \( \phi : \pi_1(\mathbb{C}^2 - C_0) \to \pi_1(\mathbb{C}^2 - C_t) \) which is geometric.
Corollary 7. Let $C$ be a torus sextic curve. If $\pi_1(C^2 - C)$ is not isomorphic to the braid group $B(3)$, then $\pi_1(\mathbb{P}^2 - C) \not\cong \mathbb{Z}_2 * \mathbb{Z}_3$.

3. Dual of maximal sextics

The information for the dual curves often plays an crucial role for the study of singularities on a plane curve of a given degree. For example, we have used the dual curve information to show the impossibility of the degeneration $\{C_{3,9}, 3A_2\} \to \{B_{3,12}, A_2\}$ in $\mathbb{P}^2$. In this section, we study the dual curves of maximal sextics.

3.1. Dual singularities. We consider the set of germs of irreducible curves $(C, O)$ which has given Puiseux pairs $\mathcal{P} := \{(m_1, n_1), \ldots, (m_k, n_k)\}$ and $y = 0$ is the tangent line of $(C, O)$. This set is denoted by $\sigma(\mathcal{P})$. Let $s$ be the Puiseux order. The subset of $\sigma(\mathcal{P})$ with the Puiseux order $s$ is denoted by $\sigma(\mathcal{P}; s)$ in [O4]. Then $\sigma(\mathcal{P})$ is the union of $\sigma(\mathcal{P}; j)$ for $j = 2, \ldots, [m_1/n_1]$ and $m_1/n_1$. The family $\{\sigma(\mathcal{P}; s); s = 2, \ldots, [m_1/n_1]\}$ and $s = m_1/n_1$ is called the flex stratification of $\sigma(\mathcal{P})$. Take $(C, O) \in \sigma(\mathcal{P}; s)$. Then $(C, O)$ is parametrized as

$$x = \xi(t), \quad y = \phi(t), \quad \xi(t) := t^N, \quad \phi(t) := \sum_{i=sN}^{\infty} a_i t^i, \quad N := n_1 \cdots n_k$$

Though the parameter is local, the coordinates $(x, y)$ is the affine coordinates $x = X/Z, y = Y/Z$, unless otherwise stated. Recall that the defining function of $(C, O)$ is given

$$f(x, y) = \prod_{j=1}^{N}(y - \phi(t \omega^j)), \quad \omega = \exp\left(\frac{2\pi i}{N}\right)$$

where we replace $t^N$ by $x$. Using the dual homogeneous coordinates $(U, V, W)$ of $(X, Y, Z)$ and the affine coordinates $u := U/V$ and $w := W/V$, we can parametrize the dual curves $(C^*, O^*)$ as (see [O4])

$$u(t) := -\frac{\phi'(t)}{\xi'(t)} = -\sum_{i=sN}^{\infty} \frac{ia_i}{N} t^{-i}, \quad w(t) := \frac{\xi(t) \phi'(t) - \xi'(t) \phi(t)}{\xi'(t)} = \sum_{i=sN}^{\infty} \frac{(i-N)a_i}{N} t^i$$

Note that $\text{val}_r u(t) = (s-1)N$. In the homogeneous coordinates, $O^*$ corresponds to $(0, 1, 0)$. Taking the parameter $\tau$ so that $u(t(\tau)) = \tau^{(s-1)N}$, we can write $w = \psi(\tau)$ and $\psi(\tau) := \sum_j b_j \tau^j$ and the characteristic powers $\mathcal{P}(\psi(\tau); (s-1)N)$ is completely described by $s$ and the Puiseux pairs $\mathcal{P}$ (see [O4]). We want to describe the dual singularities of certain reducible singularities. For this purpose, we recall the argument as follows. Put $\ell = [m_1/n_1]$ and write

$$\phi(t) = a_2 t^{2N} + \cdots + a_\ell t^{\ell N} + a_{\ell+1} t^{m_1 n_2 \cdots n_k} + \cdots$$

Put

$$t(\tau) = \tau(c_0 + c_1 \tau^N + \cdots + c_{\ell-2} \tau^{(\ell-2)N} + c_{\ell-1} \tau^{(\ell-1)N} + \cdots)$$
$$\psi(\tau) = b_2 \tau^{2N} + \cdots + b_\ell \tau^{\ell N} + b_{\ell+1} \tau^{m_1 n_2 \cdots n_k} + \cdots$$
Proposition 8. Suppose that \( s = 2 \). Then \( c_0^N = -\frac{1}{2a_2} \) and the following coefficients \( c_0, c_1, \ldots \) are inductively determined by the equality \( u(t(\tau)) = \tau^{(s-1)N} \). The coefficients \( c_1, \ldots, c_p \) of \( t(\tau) \) and thus the coefficients \( b_2, \ldots, b_{2+p} \) of \( \psi(\tau) \) depend only on the first \( p + 1 \) coefficients \( a_2, \ldots, a_{2+p} \) of \( \phi(t) \) and the choice of \( c_0 \) for \( p \leq m_1/n_1 - 2 \). The coefficient \( b_2 \) is given by \( \frac{1}{4a_2} \).

For our later purpose, we recall two special cases. See [D] for detail.

1. **Self-duality of the generic stratum** Suppose that \( s = 2 \) and \((C, O) \in \sigma(\mathcal{P}; 2)\). Then we have \((C^*, O^*) \in \sigma(\mathcal{P}; 2)\). Two special cases which we use later:

   a. \( \mathcal{P} = \emptyset \). In this case, \((C, O)\) is a smooth germ and \( N = 1 \) and the parametrization takes the form: 

   \[
   y = \sum_{i=2}^{\infty} a_i x^i, \quad a_2 \neq 0.
   \]

   The dual curve is parametrized as 

   \[
   w = \sum_{i=2}^{\infty} b_i u^i, \quad b_2 \neq 0
   \]

   and the coefficients \( b_p \) depend only on \( a_2, \ldots, a_p \). Note that \( C^* \) is a regular (not a flex) point of \( C^* \).

   b. \( \mathcal{P} = \{(m, n)\} \) with \( m/n > 2 \). The parametrization takes the following form:

   \[
   x = t^n, \quad y = \sum_{i=2}^{\ell} a_i t^{in} + a_m t^m + \cdots, \quad \ell := [m/n]
   \]

   with \( a_2, a_m/n \neq 0 \). We can easily see that \( f(x, y) \) takes the following form:

   \[
   f(x, y) = y^t - (a_m/n)^n x^m + \text{higher terms}, \quad y_1 = y - (a_2 x^2 + \cdots + a_\ell x^\ell)
   \]

   This implies that \( f \) is non-degenerate with respect to \((x, y_1)\) and \((C, O) = B_{m,n}\). See §3.2 for the definition. The dual curve \((C^*, O^*)\) has a similar parametrization

   \[
   x = t^n, \quad y = \sum_{i=2}^{\ell} b_i t^{in} + b_m t^m + \cdots, \quad b_{2n} = \frac{1}{4a_2}
   \]

2. **Non-generic case with \( s = m_1/n_1 \)**. In this case, \( C \) is parametrized as

   \[
   x = t^{m_1/N}, \quad y = \phi(t), \quad \phi(t) = a_{m_1 n_2} \cdots t^m_{m_1 N/n_1} + \cdots
   \]

   and the dual singularity \((C^*, O^*)\) is given by

   \[
   u = \tau^{(m_1-n_1)/n_1}, \quad w = \psi(\tau), \quad \psi(\tau) = b_{m_1 n_2} \cdots t^m_{m_1 N/n_1} + \cdots
   \]

   and the Puiseux order is \( m_1/n_{m_1-n_1} \). As for the Puiseux pairs, we have

   \[
   \mathcal{P}(\psi(\tau)) = \begin{cases}
   
   \{(m_1, m_1-n_1), (m_2, n_2), \ldots, (m_k, n_k)\}, & m_1 - n_1 > 1 \\
   
   \{(m_2, n_2), \ldots, (m_k, n_k)\}, & m_1 - n_1 = 1
   \end{cases}
   \]

3.2. **Dual singularity of a reducible singularity**. Recall that we have introduced the following topological equivalent classes of curve singularities which appear on irreducible tame torus curves ([E]):

\[
\begin{align*}
B_{p,q} : & \quad y^p + x^q = 0 \quad \text{(Brieskorn-Pham type)} \\
C_{p,q} : & \quad y^p + x^q + x^2 y^2 = 0, \quad \frac{p}{2} + \frac{q}{2} \leq 1 \\
Sp_1 : & \quad (y^2 - x^3)^2 + (xy)^3 = 0
\end{align*}
\]

Now we consider a reducible curve with a common tangent cone. For example, \( B_{p,q}, p < q \) has \( \gcd(p, q) \) irreducible components with \( x = 0 \) as the tangent cone. \( C_{3,q}, q \geq 7 \), has 2
(respectively 3) irreducible components with \( y = 0 \) as the tangent cone if \( q \) is odd (resp.
even).

Assume that two germs of plane curve singularities at the origin \((C_1, O)\) and \((C_2, O)\) have
the same tangent cone. The topological equivalence class of \((C_1 \cup C_2, O)\) is determined by the
respective Puiseux pairs and the intersection number \(I(C_1, C_2; O)\). We assume for simplicity
that they have at most one Puiseux pairs \(((m, n))\), \( m > 2n \) and \(((m', n'))\), \( m' > 2n' \) respectively
and their Puiseux orders are 2. By abuse of notation, we understand that \(C_1\) is smooth if
\( n = 1 \). For our purpose, we only need to consider the case \( n = n' \) or \( n = 1 \). Let \( f_1(x, y), f_2(x, y)\)
be the respective defining functions and suppose that their Puiseux parametrizations be given as

\[
\begin{align*}
C_1 : x &= t^n, \\ y &= \phi_1(t), \\ \phi_1(t) &= a_2 t^{2n} + \cdots + a_{\frac{m}{n}} t^m + \cdots \\
C_2 : x &= t^{n'}, \\ y &= \phi_2(t), \\ \phi_2(t) &= a'_2 t^{2n'} + \cdots + a'_{\frac{m'}{n'}} t^{m'} + \cdots
\end{align*}
\]

Now the intersection number is given as

\[
I(C_1, C_2; O) = \text{val}_t f_2(t^n, \phi_1(t))
\]

Assume that \( \alpha \in \mathbb{Q} \) be the minimum of \( j \) such that \( a_j \neq a'_j \). We consider the case \( \alpha \leq \min\{m/n, m'/n'\} \). As \( f_2(x, y) \) is written as

\[
f_2(x, y) = y_2' - (a_{\frac{m}{n}}') x^{m'} + \text{(higher terms), } y_2 = y - (a_{2n'} x^2 + \cdots + a_{\frac{m'}{n'}} x^{m'/n'})
\]
we can easily compute that

\[
I(C_1, C_2; O) = \alpha n'
\]

Under the same assumption, we consider the parametrization of the dual curves:

\[
\begin{align*}
C'_1 : x &= t^n, \\ y &= \psi_1(t), \\ \psi_1(t) &= b_2 t^{2n} + \cdots + b_{\frac{m}{n}} t^m + \cdots, \\ b_2 &= \frac{1}{4a_2} \\
C'_2 : x &= t^{n'}, \\ y &= \psi_2(t), \\ \psi_2(t) &= b'_2 t^{2n'} + \cdots + b'_{\frac{m'}{n'}} t^{m'} + \cdots, \\ b'_2 &= \frac{1}{4a'_2}
\end{align*}
\]

Assume that \( n = n' \). By Proposition we have \( b_j = b'_j \) for \( j < \alpha \) and \( b_\alpha \neq b'_\alpha \). Thus we get
\( I(C'_1, C'_2; O^*) = I(C_1, C_2; O) = \alpha n' \).

Assume that \( n = 1 \) and \( a_2 \neq a'_2 \). Then by a similar argument, \( I(C_1, C_2; O) = 2n' \) and
\( I(C'_1, C'_2; O^*) = 2n' \).

### 3.3. Dual of singularities on tame torus curve (local)

First we consider reducible germs with a common tangent cone. It is shown that \( C_{3,n}, \ n = 7, 8, 9, 12, 15 \), appears as a
singularity of irreducible tame torus curves (\( F \)). A common nature of these singularities is
the following.

(1): There are a smooth component \( L \) corresponding to the face supporting \( y^3 + x^2 y^2 \). The
face supporting \( x^2 y^2 + x^n \) gives an \( A_{n-3} \) singularity \((K, O)\). \( K \) is irreducible (respectively
\( K \) has two smooth components \( K_1 \) and \( K_2 \)) if \( n \) is odd (resp. \( n \) is even). In any case, the
tangent cone is \( y = 0 \). The intersection number is given by \( I(L, K; O) = 4 \) if \( n \) is odd and
\( I(L, K_i; O) = 2 \) for \( i = 1, 2 \) and \( I(K_1, K_2; O) = n/2 - 1 \) for \( n \) even.
Conversely $C_{3,n}$ is characterized by this property. For example, assume that $n = 2m$ and $(C, O)$ has three smooth components at $O$, which satisfy the above intersection criterion. Take an analytic coordinates $(x_1, y_1)$ so that $K_1 = \{ y_1 = 0 \}$. Then $K_2$ is defined by an analytic function of the form $j_2(x_1, y_1) = y_1 + a_{m-1} x_1^{m-1} + \ldots$ with $a_{m-1} \neq 0$ and $L$ is defined by an analytic function of the form $j_0(x, y) = y + b_2 x_1^2 + \ldots$ with $b_2 \neq 0$. Thus $(C, O)$ is defined, using a new coordinates $(x_1, y)$ with $y_1 = y + \varepsilon x^{m-1}$, $\varepsilon + a_{m-1} \neq 0$ by the function

$$j(x, y) = j_0(x, y + \varepsilon x^{m-1})(y + \varepsilon x^{m-1})j_2(x, y + \varepsilon x^{m-1})$$

$$= y^3 + b_2 x_1^2 y^2 + b_2 \varepsilon (a_{m-1} + \varepsilon) x^n + \text{(higher terms)}$$

Now suppose that we have a tame torus curve $C$ with a singularity at $O$. We consider the dual singularity $(C^*, O^*)$. In the local classification argument, we have shown that the Puiseux order of each component $L, K$ or $L, K_1, K_2$ is 2 with respect to the fixed affine coordinates $(x, y)$.

For example, let us consider a tame torus curve $C$ with $(C, O) \cong C_{3,15}$ and we assume that $y = 0$ is the common tangent cone. We have shown in [P] that $C$ is defined by a polynomial $f(x, y)$ which is written as $f(x, y) = f_2(x, y)^3 + f_3(x, y)^2$ where the conic $f_2(x, y) = 0$ is tangent with $y = 0$ at $O$ and the cubic $f_3(x, y) = 0$ has a node at $O$ and one branch is tangent to $y = 0$ and $I(f_2, f_3; O) = 6$. The component $K$ has a $A_{12}$ singularity and we have the following parametrization.

$$L : y = a_2 x^2 + a_3 x^3 + \ldots$$

$$K : x = t^2, \quad y = \sum_{i=2}^{6} a_i t^{2i} + a_{13} t^{13} + \ldots$$

and $a_2 \neq a_2'$. Thus $\alpha = 2$ and $I(L, K; O) = 4$. Thus $(L^*, O^*)$ is smooth and $(K^*, O^*)$ is again a generic $(2, 13)$ cusp and $I(L^*, K^*; O^*) = 4$. This implies that $(C^*, O^*) \cong C_{3,15}$. In an exact same discussion, we see that $(C_{3,n}, O)$ are self dual for any $n \geq 7$.

Other singularities on irreducible tame torus curves with a common tangent and having $s = 2$ as the Puiseux order are $A_{3n-1}$ $(n = 1, \ldots, 6)$ and $B_{3,n}$ $(n = 6, 8, 10, 12)$. By the same discussion, we can see that these singularities are self dual. Thus in conclusion, we have

**Proposition 9.** The following singularities on tame torus curve are self-dual.

$$C_{3,n}, n = 7, \ldots, 15; \quad A_{3n-1}, n = 1, \ldots, 6; \quad B_{3,2m}, m = 3, \ldots, 6$$

Secondly we consider reducible germs with several tangent cones. Singularities with several tangent cones which we have in mind are $C_{6,6}, C_{6,9}, C_{6,12}, C_{9,9}$. Each of these singularities has two components in the tangent cones. Let $B_x$ (respectively $B_y$) be the union of irreducible components which has $x = 0$ (resp. $y = 0$) as the tangent cone. Each of the irreducible component is generic and therefore on the dual curve and each of them is isomorphic to the original one. Let $O_x := (1, 0, 0)$ and $O_y = (0, 1, 0)$. Then $(C_{m,n}, O)^* = (B_x^*, O_x) \cup (B_y^*, O_y)$ and they are self dual for $m, n \geq 6$; $(B_x^*, O_x) \cong (B_x, O)$ and $(B_y^*, O_y) \cong (B_y, O)$. For example, consider $C_{6,9}$. Note that $(B_x, O_x) \cong A_3$ and $(B_y, O_y) \cong A_6$. On the dual curve $C^*$, the dual singularity splits into a $A_3$ and $A_6$ with a common tangent line.
Finally we consider other singularities on torus curves. Exceptional singularities are $A_2, E_6, B_{4,6}$ and $Sp_1$. As is well-known, the dual of a cusp $B_{n,n+1}$ is a flex point of order $n - 1$. This implies that the dual of $B_{4,6}$ is $A_5$. As two components have a flex point at $O^*$, their Puiseux orders are 3. $Sp_1$ is an irreducible singularity with two Puiseux pairs $\{(2,3), (2,9)\}$ and the Puiseux order is 3. The generic form of $Sp_1$ is given by (up to $\text{PSL}(3, \mathbb{C})$)-action

$$(axy)^3 + (y^2 - x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3)^2 = 0$$

and Puiseux pairs are $\mathcal{P} = \{(3,2), (9,2)\}$. It has the parametrization

$$x = t^4, \quad y = \phi(t), \quad \phi(t) = t^6 + b_8t^8 + b_9t^9 + \cdots, b_9 \neq 0$$

Thus $(Sp_1^*, O^*)$ has the parametrization

$$u = \tau^2, \quad w = \psi(\tau), \quad \psi(\tau) = b_0\tau^6 + b_8\tau^8 + b_9\tau^9 + \cdots$$

Namely $(Sp_1^*, O^*)$ is an $A_8$ singularity which has the Puiseux order 3 and thus does not belong to the generic stratum.

### 3.4. Generic dual curves (global).

First we recall that irreducible tame torus curves can be degenerated into a rational curve with one of the following configurations $\left[\mathbb{P}\right]$.

(17) $\{C_{3,15}, \{C_{9,9}\}, \{C_{3,7}, A_8\}, \{B_{3,10}, A_2\}, \{Sp_1, A_2\}, \{B_{3,8}, E_6\}, \{C_{3,9}, 3A_2\}$

Let $\Sigma$ be one of the above configuration and let $\mathcal{M}(\Sigma)$ be the set of tame torus curves with configuration $\Sigma$. We say that $C \in \mathcal{M}(\Sigma)$ is generic if $C^*$ has only $A_2$ and $A_1$ singularities besides the singularities which are dual to those of $C$. For a generic curve $C \in \mathcal{M}(\Sigma)$, the number of cusps and nodes are constant, we have observed that $\mathcal{M}(\Sigma)$ is connected for each of the above maximal configuration. The following is immediate from the previous consideration and by computing the dual curve of an explicit generic curve.

**Proposition 10.** Let $\Sigma$ be one of the configuration in (17) and let $C \in \mathcal{M}(\Sigma)$ be a generic curve. Let $n^*$ be the degree of $C^*$ and let $\Sigma^*$ be the configuration of the singularities of $C^*$. Then the dual curves are described by the following table.

| No | $\Sigma$ | $n^*$ | $\Sigma^*$ |
|----|-----------|-------|-----------|
| 1  | $\{C_{3,15}\}$ | 9     | $\{C_{3,15}, 9A_2, 9A_1\}$ |
| 2  | $\{C_{9,9}\}$   | 8     | $\{C_{3,7}, A_8, 6A_2, 5A_1\}$ |
| 3  | $\{C_{3,7}, A_8\}$ | 8     | $\{C_{3,7}, A_8, 6A_2, 5A_1\}$ |
| 4  | $\{B_{3,10}, A_2\}$ | 6     | $\{B_{3,10}, A_2\}$ |
| 5  | $\{Sp_1, A_2\}$ | 6     | $\{A_8, 3A_2, 3A_1\}$ |
| 6  | $\{B_{3,8}, E_6\}$ | 6     | $\{B_{3,8}, 2A_2, A_1\}$ |
| 7  | $\{C_{3,9}, 3A_2\}$ | 6     | $\{C_{3,9}, 3A_2\}$ |

**Remark 11.** It is interesting to observe that in the last four cases, the dual curves are also sextics. It is possible to show that they are torus curves. In the case 5 and 6, the dual torus curves are not tame as it has $A_1$ singularity.
4. FUNDAMENTAL GROUP OF TAME TORUS CURVES

The fundamental group of the complement of a generic sextics of torus type is isomorphic to \( \mathbb{Z}_2 * \mathbb{Z}_3 \) by Zariski [2]. The same assertion is true for a certain class of non-generic sextics of torus curves (\([3]\)). The main result of this paper is:

**Theorem 12.** Let \( C \) be an irreducible tame torus sextic of type \((2,3)\) defined by the homogeneous polynomial \( F(X,Y,Z) \) and let \( M \) be the Milnor fiber of \( F(X,Y,Z) \).

1. If \( C \notin \mathcal{M}(\{C_{3,9},3A_2\}) \), then \( \pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 * \mathbb{Z}_3 \) and the generic Alexander polynomial is \( t^2 - t + 1 \) and the first Betti number \( b_1(M) \) of \( M \) is 2.

2. (Exceptional moduli) For \( C \in \mathcal{M}(\{C_{3,9},3A_2\}) \), we have

\[
\begin{align*}
\pi_1(C^2 - C) & \cong \langle g_1, g_2, g_3 \mid \{g_i, g_j\} = e, i \neq j, (g_3g_2g_1)^2 = (g_2g_1g_3)^2 = (g_1g_3g_2)^2 \rangle \\
\pi_1(\mathbb{P}^2 - C) & \cong \langle g_1, g_2, g_3 \mid \{g_i, g_j\} = e, i \neq j, (g_3g_2g_1)^2 = e \rangle
\end{align*}
\]

and the generic Alexander polynomial is given by \( (t^2 - t + 1)^2 \) and the first Betti number \( b_1(M) \) is 4. This moduli space \( \mathcal{M}(\{C_{3,9},3A_2\}) \) is self dual.

Recall that the **generic Alexander polynomial** of a curve \( C \) is defined by the Alexander polynomial of \( \pi_1(C^2 - C) \) with respect to a generic line at infinity. The Milnor fiber \( M \) is defined by the affine surface \( \{(X,Y,Z) \in \mathbb{C}^3; F(X,Y,Z) = 1\} \) and it is the total space of the cyclic covering of order 6, \( p: M \to \mathbb{P}^2 - C \), which is defined by the quotient map of the \( \mathbb{Z}/6\mathbb{Z} \)-action induced by the monodromy map \( h: M \to M \). We have seen that there are 7 configurations of the possible maximal tame torus curves, as is listed in \([7]\). In \([4]\), it is observed that each of the moduli space of the maximal configuration is an Zariski-open subspace of an affine space and thus it is connected. Thus the topology of the complement is independent of the choice of a curve. We have seen in \([3]\) that there are degenerations of curves of tame torus curves corresponding to

\[ \{C_{3,7},3A_2\} \to \{C_{3,8},3A_2\} \to \{C_{3,9},3A_2\} \]

Any non-maximal tame torus curve \( C \) can be degenerated into a maximal curve with configuration in \([7]\) and if \( C \) degenerates into \( \{C_{3,9},3A_2\} \), it can be factored with a degeneration into \( \{C_{3,8},3A_2\} \).

For the proof of the assertion (1) of Theorem 12 we prepare the following.

**Lemma 13.** Let \( C \) be an irreducible tame torus curve and let \( C_t \) be a degenerating family such that \( C_1 = C \) and \( C_t \), \( t \neq 0 \) has the same configuration of singularities with that of \( C_1 \). Assume that \( \pi_1(\mathbb{P}^2 - C_0) \cong \mathbb{Z}_2 * \mathbb{Z}_3 \). Then \( \pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 * \mathbb{Z}_3 \).

**Proof.** In fact, the degeneration family gives a surjective homomorphism \( \alpha: \pi_1(\mathbb{P}^2 - C_0) \to \pi_1(\mathbb{P}^2 - C) \) which is an isomorphism on the first homology. Now take another degeneration families \( D_t \) such that \( D_t, t \neq 0 \) is a generic torus curve with 6 \( A_2 \) and \( D_0 = C \). Such a family always exists. We get a surjective homomorphism \( \beta: \pi_1(\mathbb{P}^2 - C) \to \pi_1(\mathbb{P}^2 - D_t) \) which also induces an isomorphism on the first homology group. Note that \( \pi_1(\mathbb{P}^2 - D_t) \cong \mathbb{Z}_2 * \mathbb{Z}_3 \) by...
Now we apply Lemma 4 to conclude that the composition
\[ \mathbb{Z}_2 \ast \mathbb{Z}_3 \cong \pi_1(\mathbb{P}^2 - C_0) \xrightarrow{\alpha} \pi_1(\mathbb{P}^2 - C) \xrightarrow{\beta} \pi_1(\mathbb{P}^2 - D_t) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \]
is an isomorphism. This implies both of \( \alpha, \beta \) are isomorphisms.

Thus the proof of Theorem 12 is reduced to the assertion for the maximal curves and a curve \( C \in \mathcal{M}(\{C_{3,8}, 3A_2\}) \). The assertion for these curves will be proved by direct computations choosing good members from respective moduli spaces. To show that \( \pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \) for \( C \in \mathcal{M}(\Sigma) \) where \( \Sigma \) is in (17) and \( \Sigma \neq \{C_{3,9}, 3A_2\} \) or \( \Sigma = \{C_{3,8}, 3A_2\} \), we use the following lemma.

Lemma 14. The following conditions are equivalent for a tame torus curve \( C \) and a generic line at infinity.
1. \( \pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \).
2. \( \pi_1(C^2 - C) \equiv B(3) \).
3. There is a surjective homomorphism \( \psi : \mathbb{Z}_2 \ast \mathbb{Z}_3 \to \pi_1(\mathbb{P}^2 - C) \) which gives an isomorphism on the first homology.
4. There exists a surjective homomorphism \( \phi : F(2) \to D(\pi_1(\mathbb{P}^2 - C)) \).

Proof. Using a degeneration family \( D_t \) from the generic torus curves as in the proof of Lemma 13, we have always a surjective homomorphism \( \beta : \pi_1(\mathbb{P}^2 - C) \to \mathbb{Z}_2 \ast \mathbb{Z}_3 \). Thus the assertion follows from Lemma 4.

By Lemma 14, to show \( \pi_1(\mathbb{P}^2 - C) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \) for a given sextic curve \( C \) of torus type, we only need to show the existence of generators \( g_1, g_2 \) given by lassos which satisfy the braid relation \( \{g_1, g_2\} = e \) and the torsion relation \( (g_1 g_2)^3 = e \). Usually we need a few monodromy relations to get these relations. Once we get these relations, we can ignore the other monodromy relations and we do not need any further computation.

Hereafter the base point of the fundamental group is the base point of the pencil lines which we use. The local singularities in the three configurations \( \{B_{3,10}, A_2\}, \{Sp_1, A_2\}, \{B_{3,8}, E_6\} \) are irreducible.

4.1. Notations and choice of the pencil. Let \( C \) be an irreducible torus curve of degree 6. A lasso is a loop represented as \( \ell \circ \sigma \circ \ell^{-1} \) where \( \sigma \) is a loop, given by the boundary of a small normal disk of a regular point of \( C \) and \( \ell \) is a path joining \( \sigma \) and the base point. Thus through the Hurewicz homomorphism \( \xi : \pi_1(\mathbb{P}^2 - C) \to H_1(\mathbb{P}^2 - C) \cong \mathbb{Z}/6\mathbb{Z} \), a lasso is mapped to the canonical generator of the first homology. Recall that \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) has two representations:
\[
\mathbb{Z}_2 \ast \mathbb{Z}_3 = \langle \rho, \xi \mid \{\rho \xi\} = e, (\rho \xi)^3 = e \rangle = \langle a, b \mid a^2 = b^3 = e \rangle
\]
where \( e \) is the unit element and \( \{\rho, \xi\} := \rho \xi \rho (\xi \rho \xi)^{-1} \). Thus it implies the braid relation \( \rho \xi \rho = \xi \rho \xi \). In the first representation, \( \rho, \xi \) can be represented by lassos. In the following figures, we denote, for simplicity of drawing pictures, a small lasso oriented in the counter clockwise direction by a bullet with a path as in [O3] [O4]. Thus ⚫ indicates ⬅️.
For the computation of $\pi_1(P^2 - C)$ of a maximal tame torus curve $C$, we note that it has a big singularity $\xi \in C$ and it is usually better (by experiment) to choose Zariski pencil so that the pencil line, say $L_\rho$, passing through $\xi$ is the tangent cone of the singularity. In this way, the monodromy relation around $L_\rho$ contains more relations. Unless otherwise stated, we use $y = t$ as pencil lines in this section. We explain how to read the monodromy relation by Example 3 in §2 with one Puiseux pair putting $(m, n) = (13, 2)$ which appears as a component of $C_{3,15}$.

$$y = \tau^4, \quad x = b_1 \tau^2 + b_2 \tau^4 + \cdots + b_5 \tau^{10} + b_{11} \tau^{11} + \cdots$$

(20)

The local topology of $C$ is determined by the two terms $b_1 \tau^2, b_{11} \tau^{11}$ and the other terms does not change the topology. For brevity, we denote the other terms by nn-terms. Then the parametrization (20) is simply written as

$$y = \tau^4, \quad x = b_1 \tau^2 + b_{11} \tau^{11} + \text{(nn-terms)}$$

Let us consider a generic pencil $L_\xi, 0 < \varepsilon \ll 1$. Assume $g_1, g_2, \ldots, g_6$ are generators of the fundamental group $\pi_1(L_{\eta_0} - C)$. We may assume that $g_1, g_2, \ldots, g_4$ are the generators which correspond to those points bifurcated from $O$ of $C$. When $y = \varepsilon \exp(\theta i)$ moves around the origin once, each branch of $\tau = \varepsilon^{1/4} \exp(\theta i/4)$ moves an arc of angle $\pi/2$. The topological behavior of 4 points among $C \cap \{y = \varepsilon \exp(\theta i)\}$ looks like the movement of the two planets which accompany two respective satellites. Two planets (which correspond to the term $b_1 \tau^2$), do the half turn around the sun (=the origin). For a fixed $\sqrt[4]{y} = \tau^2$, there exists 2 roots of $\tau = \sqrt[4]{y}$ which correspond to 2 satellites. They do $\frac{11}{4}$-turns around the respective planet. This interpretation is useful when the local singularity is reducible.

Given a polynomial $f(x,y)$ we will denote by $\Delta_x(f)$ the discriminant of $f$ as a polynomial in $x$. Then $\Delta_x(f)$ is a polynomial in $y$, we denote $P_y$ is the set of roots of $\Delta_x(f) = 0$ in $y$. Thus $\lambda \in P_y$ corresponds to the singular pencil line $y = \lambda$.

4.2. Irreducible singularities. We first consider 3 moduli spaces, $\{B_{3,10}, A_2\}, \{Sp_1, A_2\}, \{B_{3,8}, E_6\}$, which contain only irreducible singularities. Hereafter $\varepsilon$ is assumed to be a sufficiently small generic positive number throughout the paper.

(I) Moduli space $\mathcal{M}(\{B_{3,10}, A_2\})$. Let us consider the following curve $C_I \in \mathcal{M}(\{B_{3,10}, A_2\})$ which is defined by

$$C_I : f = (y(1 - y) - x^2)^3 + \left(y^2(1 - y) - x^2y + xy^2 + \frac{18}{25}y^3\right)^2$$

$$\Delta_x(f) = cy^{23}(949y^3 - 625)^3h_4(y), \quad P_y = \{0, 625/949, \text{roots of } h_4(y) = 0\}$$

where

$$h_4(y) = 135771071y^4 - 676592025y^3 + 1149291875y^2 - 820856875y + 284765625.$$  

This curve $C_I$ has a $B_{3,10}$ singularity at $O$ and an $A_2$ singularity at $(-450/949, 625/949)$. We consider the pencils $L_t = \{y = t\}, t \in \mathbb{C}$. We take generators $g_1, g_2, \ldots, g_6$ of the fundamental group $\pi_1(L_{\eta_0} - C_I)$ as in Figure 3 where $\eta_0 = \varepsilon$. Put $\omega_1 := g_3g_2g_1$ and $\omega_2 := g_6g_5g_4$. To see
the monodromy relation, we look at the Puiseux parametrization at the origin. It is given by

\[ x = t^3 + \frac{1}{2}t^7 + \text{(nn-terms)}, \quad y = t^6 \]

Therefore, at \( y = 0 \) we have the following monodromy relations:

\[ g_1 = g_6, \quad g_2 = \omega_2 g_4 \omega_1^{-1}, \quad g_3 = \omega_2 g_5 \omega_1^{-1} \]

(21)

\[ g_4 = (\omega_2 \omega_1) g_3 (\omega_2 \omega_1)^{-1}, \quad g_5 = (\omega_2 \omega_1^2) g_1 (\omega_2 \omega_1^2)^{-1}, \quad g_6 = (\omega_2 \omega_1^2) g_2 (\omega_2 \omega_1^2)^{-1} \]

(22)

By taking the product of the relations in (21), we get

\[ \omega_1 = \omega_2 \]

(23)

Then the big circle relation (=vanishing relation at infinity) \( \omega_2 \omega_1 = e \) reduces to:

\[ \omega_1^2 = e \]

(24)

From (22), (23) and (24) we have

\[ g_4 = g_3, \quad g_5 = \omega_1 g_1 \omega_1^{-1}, \quad g_6 = \omega_1 g_2 \omega_1^{-1} \]

(25)

From (21), (23) and (24) we obtain \( g_6 = g_4 = g_3 = g_1 \). Thus the generators are reduced to \( g_1, g_2 \). Rewriting (21) as a relation for \( g_1, g_2 \), we get the braid relation \( \{ g_1, g_2 \} = e \). On the other hand \( e = \omega_1^2 = (g_1 g_2 g_1)^2 \). Thus the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Lemma 14.

(II) **Moduli space** \( \mathcal{M}(\{ S p_1, A_2 \}) \). Let us consider the following curve \( C_{II} \in \mathcal{M}(\{ S p_1, A_2 \}) \) which is defined by

\[ C_{II} : f(x, y) = (y^2 - y^3 - x^3)^2 - x^3 y^3 \]

\[ \Delta_x(f) = -729y^{23}(y - 1)^4(3y - 4)^3, \quad P_y = \{ 0, 1, 4/3 \} \]

This curve \( C_{II} \) has a \( S p_1 \) singularity at the origin and an \( A_2 \) singularity at \((0,1)\). We consider the pencils \( L_t = \{ y = t \}, t \in \mathbb{C} \). We take generators \( g_1, g_2, \ldots, g_6 \) of the fundamental group \( \pi_1(L_{\eta_0} - C_{II}) \) as in Figure 3 where \( \eta_0 = \varepsilon \). Put \( \omega_1 := g_2 g_1, \omega_2 := g_4 g_3, \omega_3 := g_6 g_5 \) and \( \Omega := \omega_3 \omega_2 \omega_1 \).
To see the monodromy relation at \( y = 0 \), we look at the Puiseux parametrization at the origin. It is given by
\[
x = -t^4 + \frac{1}{3}t^7 + \text{(nn-terms)}, \quad y = t^6
\]
The monodromy relations at \( y = 0 \) are given by
\[
\begin{align*}
g_1 &= g_6, \quad g_2 = \omega_3 g_5 \omega_3^{-1} \\
g_3 &= \Omega g_2 \Omega^{-1}, \quad g_4 = (\Omega \omega_1) g_1 (\Omega \omega_1)^{-1} \\
g_5 &= \Omega g_4 \Omega^{-1}, \quad g_6 = (\Omega \omega_2) g_3 (\Omega \omega_2)^{-1}
\end{align*}
\]
Since the big circle relation is \( \Omega = e \), from the relations (26), (27), (28) we get
\[
\begin{align*}
g_6 &= g_1, \quad g_3 = g_2, \quad g_5 = g_4, \\
g_2 &= g_1 g_4 g_1^{-1}, \quad g_4 = g_2 g_1 g_2^{-1}, \quad g_1 = g_4 g_2 g_4^{-1}
\end{align*}
\]
From (29) and the second relation of (30), we can reduce the generators to \( g_1, g_2 \) and we obtain the braid relation \( \{g_1, g_2\} = e \) from (30). On the other hand, the big circle relation gives \( (g_1 g_2)^3 = e \). Thus the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Lemma 14.

(III) **Moduli space** \( \mathcal{M}(\{B_{3,8}, E_6\}) \). Let us consider the curve which defined by
\[
C_{III} : f = ((y - 1)^2 + x^2 - 1)^3 + x^4 y^2
\]
\[
\Delta_x(f) = -64 y^{19} (y - 2)^9 (31 y - 54)^2, \quad P_y = \{0, 54/31, 2\}
\]
This curve \( C_{III} \) has a \( B_{3,8} \) singularity at the origin and an \( E_6 \) singularity at \((0,2)\).
To see the monodromy relation, we look at the Puiseux parametrization at the origin. It is given by
\[
x = -\sqrt{2} t^3 + \frac{\sqrt{2}}{2} t^5 + \text{(nn-terms)}, \quad y = t^6
\]
We take generators \( g_1, g_2, \ldots, g_6 \) of the fundamental group \( \pi_1(L_{\eta_0} - C_{III}) \) as in Figure 3 where \( \eta_0 = \varepsilon \). Put \( \omega_1 := g_3 g_2 g_1 \) and \( \omega_2 := g_6 g_5 g_4 \).
At the singular pencil line \( y = 0 \), we have the monodromy relations:

\[
g_1 = \omega_2 g_4 \omega_2^{-1}, \quad g_2 = \omega_2 g_5 \omega_2^{-1}, \quad g_3 = \omega_2 g_6 \omega_2^{-1}
\]

\[
g_4 = \omega_2 g_3 \omega_2^{-1}, \quad g_5 = (\omega_2 \omega_1) g_1 (\omega_2 \omega_1)^{-1}, \quad g_6 = (\omega_2 \omega_1) g_2 (\omega_2 \omega_1)^{-1}
\]

By taking the product of (31), we get \( \omega_2^2 = \omega_1^2 \). Hence the big circle relation is

\[
\omega_2 \omega_1 = \omega_1 \omega_2 = e
\]

From (32) and (33) we get \( g_5 = g_1 \) and \( g_6 = g_2 \). Substituting these equalities to (31), we get

\[
g_1 g_2 g_1 = g_2 g_1 g_4, \quad g_1 g_4 g_1 = g_2 g_1 g_4.
\]

Thus we have \( g_4 = g_2 \) and therefore the braid relation \( \{g_1, g_2\} = e \) follows. Finally, from (32) we get \( (g_2 g_1)^3 = e \). Thus the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Lemma 14.

Remark 15. Recently, Uludağ studied fundamental group of rational cuspidal curves. Especially for sextics his results (see [U, Theorem 3.1.5]) is related to our result. Namely, our cases \( \{B_{3,10}, A_2\}, \{B_{3,8}, E_6\} \) and \( \{Sp_1, A_2\} \) correspond to his cases \( \{[3], [2]\}, \{[3,2], [3]\} \) and \( \{[4,2], [2]\} \). He showed the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \), without assuming the curve to be a torus type.

4.3. Reducible singularities. The last 4 moduli spaces contain reducible singularities, these singularities have 2 or 3 analytic branches. The monodromy relation at these singularities is more complicate, but the method we use is the same as before. For the computation of the fundamental groups for the curves in \( \mathcal{M}(\{C_{9,9}\}) \) and in \( \mathcal{M}(\{C_{3,15}\}) \), we use the existence of \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) actions respectively on chosen curves. We assume that \( \varepsilon \) is a positive, sufficiently small number as before.

(IV) Moduli space \( \mathcal{M}(\{C_{3,7}, A_8\}) \). Let us consider the curve \( C_{IV} \in \mathcal{M}(\{C_{3,7}, A_8\}) \) defined by

\[
C_{IV} : \quad f(x,y) = (y(2y-2) - x^2)^3 + \left( \frac{46079}{54000} x^3 + \frac{152279}{18000} x^2 y + \left( \frac{311579}{18000} y^2 - \frac{59}{10} y \right) x - \frac{2351327}{54000} y^3 + \frac{178829}{4500} y^2 \right)^2
\]
The discriminant polynomial is given by $\Delta_x(f) = cy^{16}(7y - 6)^2(y - 2)^9h_3(y)$ where $h_3(y)$ is a polynomial of degree 3 which has 3 real solutions, they are approximately $\beta_1 \approx 0.053$, $\beta_2 \approx 0.059$ and $\beta_3 \approx 0.831$. This curve $C_{IV}$ has two singularities, a $C_{3,7}$ singularity at the origin and an $A_8$ singularity at $(2, 2)$. Figure 4 shows the global topological (not numerical) situation of $C_{IV}$.

To see the monodromy relation at the origin, we look at the Puiseux expansion of $(C_{IV}, O) \cong C_{3,7}$. It has two components, a smooth component $L$ and a component $K$ of the $(2,5)$-cusp, which are parametrized as follows.

$$L : \quad x = -\frac{60\sqrt{238}}{119}i\tau + (\text{nn-terms}), \quad y = \tau^2$$

$$K : \quad x = -\frac{30\sqrt{2}}{\sqrt{2581}}t^2 + \frac{3481\sqrt{1161450}}{38715}t^3 + (\text{nn-terms}), \quad y = t^4$$

We take generators $g_1, g_2, \ldots, g_6$ of the fundamental group $\pi_1(L_{\eta_0} - C_{IV})$ as in Figure 4 where $\eta_0 = \varepsilon$ and $0 < \varepsilon \ll 1$. $g_5, g_6$ correspond to the points of $L$ and $g_1, \ldots, g_4$ correspond to the points of $K$. Put $\omega_1 := g_2g_1$, $\omega_2 := g_4g_3$. The monodromy relations at the origin are given by:

(34) \hspace{1cm} g_5 = g_6 \quad (\text{relation for L})

(35) \hspace{1cm} g_1 = g_5g_4g_5^{-1}, \quad g_2 = (g_5\omega_2)g_3(g_5\omega_2)^{-1}

(36) \hspace{1cm} g_3 = (g_5\omega_2\omega_1)g_1(g_5\omega_2\omega_1)^{-1}, \quad g_4 = (g_5\omega_2\omega_1)g_2(g_5\omega_2\omega_1)^{-1}

Putting $h := g_5 = g_6$, the big circle relation is given by

(37) \hspace{1cm} h^2\omega_2\omega_1 = e

Then (35) and (36) can be rewritten as follows.

(38) \hspace{1cm} g_1 = hg_4h^{-1}, \quad g_2 = (h\omega_2)g_3(h\omega_2)^{-1}

(39) \hspace{1cm} g_3 = h^{-1}g_1h, \quad g_4 = h^{-1}g_2h
From (38) and (39) we get \( \rho := g_2 = g_1, \xi := g_4 = g_3 \). We rewrite (38) and (37) as follows.

\[
\rho = h_1 \xi h_1^{-1} \tag{40}
\]

\[
h_2^2 \xi^2 \rho^2 = e \tag{41}
\]

To read the monodromy at \( y = \beta_1 \) we need to know how the generators move, when the pencil line \( y = \eta \) move from \( \eta = \varepsilon \to \beta_1 := \beta_1 - \varepsilon \). We can show see the generators are deformed as in Figure 5, using a similar argument as in \([O4]\).

\[\text{Figure 5. The movement of the generators } y : \varepsilon \to \beta_1^{-} \]

Now the monodromy relations at \( y = \beta_1 \) and \( y = \beta_2 \) are simple tangent relations, which are given by

\[
h = h' = \xi' \tag{42}
\]

where \( h' = h \) (because of the big circle relation) and \( \xi' = \rho^{-1} \xi \rho \). Thus \( h = \rho^{-1} \xi \rho \). Thus we can take \( \rho, \xi \) as the generators. Now apply to (38) and (37), we obtain \( \rho \xi \rho = \xi \rho \xi \) and \( (\rho \xi \rho)^2 = e \). Thus the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Lemma 14.

(V) **Moduli space** \( \mathcal{M}(\{C_{9,9}\}) \). We will use the technique in \([O3]\). Let us consider the affine curve

\[
C_g : g = (y - x^2)^3 + \left(2y - 2x^2 + \frac{32}{27}x^3\right)^2 = 0
\]

and let \( f(x, y) = g(x, y^2) \) and let \( C_V \) be the curve defined by \( f(x, y) = 0 \). We can easily observe that \( (C_g, O) \cong A_8 \) and \( (C_V, O) \cong C_{9,9} \). The discriminant polynomial of \( g \) is given by

\[
\Delta_x g = cy^{11}(y + 4)(295y^3 + 208y^2 - 3456y + 729)
\]

Thus \( \Delta_x g \) has 5 real solutions, they are approximately \( \gamma_0 = 0, \gamma_1 \approx 0.21, \gamma_2 \approx 2.96, \gamma_3 \approx -3.88 \) and \( \gamma_4 = -4 \). We denote \( \delta_i = \sqrt{\gamma_i} \) for \( i = 1, 2, 3, 4 \) and \( \delta_0 := \gamma_0 \). An easy observation that \( P_y(f) = \{\delta_0, \pm \delta_i\} \), for \( i = 1, 2, 3, 4 \). See Figure 4. Since \( C_V \) is a double covering of \( C_g \) along the \( x \)-axis, we are able to read the monodromy relations of \( C_V \) via \( C_g \). We consider the pencils \( L_t = \{y = t\}, t \in \mathbb{C} \).

The monodromy relations at \( y = \delta_i \) are enough to compute the fundamental group. We first observe that the monodromy relation at \( y = \delta_i \) for the curve \( C_V \) is nothing but the monodromy relation for the curve \( C_g \) at \( y = \gamma_i \) for \( i \neq 0 \).
To see the monodromy relation at \( y = 0 \), we look at the Puiseux parametrization of \( C_g \) at the origin. It is given by

\[
x = \phi(t), \quad \phi(t) := t^2 + \frac{16\sqrt{3}}{243} t^7 + (\text{nn-terms}), \quad y = t^4
\]

We take generators \( g_1, g_2, \ldots, g_6 \) of the fundamental group \( \pi_1(L_\varepsilon - C_V) \) as in Figure 8. Put \( \omega_1 := g_2g_1 \) and \( \omega_2 := g_4g_3 \). Note that \( C_V \) is parametrized at the origin by \( x(t) = \phi(t), \ y(t) = t^2 \) and \((C_V, O)\) has two irreducible components. Thus the monodromy relations at \( y = 0 \) for \( C_V \)
are given by
\begin{equation}
(43) \quad g_1 = \omega_2 \omega_1^3 g_2 (\omega_2 \omega_1^3)^{-1}, \quad g_2 = \omega_2 \omega_1^4 g_1 (\omega_2 \omega_1^4)^{-1}
\end{equation}
\begin{equation}
(44) \quad g_3 = \omega_2 \omega_1^2 g_1 (\omega_2 \omega_1^2)^{-1}, \quad g_4 = \omega_2 \omega_1^3 g_3 (\omega_2 \omega_1^3)^{-1}
\end{equation}
The tangent relation at \( y = \delta_1 \):
\begin{equation}
(45) \quad g_6 = g_1
\end{equation}
The tangent relation at \( y = \delta_2 \):
\begin{equation}
(46) \quad g_5 = g_1^{-1} g_2 g_1
\end{equation}
From (43) and (46), the big circle relation \( \omega_2 \omega_1 g_6 g_5 = e \) and (43) reduce to:
\begin{equation}
(47) \quad \omega_2 \omega_1^2 = e, \quad g_1 g_2 g_1 = g_2 g_1 g_2
\end{equation}
Also the relation (44) can be written as
\begin{equation}
(48) \quad \omega_1^5 g_3 = g_4 \omega_1^5
\end{equation}
The tangent relation at \( y = \delta_3 \) (when 2 points which have indices 2 and 4 in Figure 8 coincide):
\begin{equation}
(49) \quad g_3^{-1} g_4 g_1 = g_1^{-1} g_2 g_1
\end{equation}
From (43) and (48) we can have
\begin{equation}
(50) \quad g_3 = \omega_1^{-3} g_1, \quad g_4 = \omega_1^2 g_1 \omega_1^{-5}
\end{equation}
Finally we rewrite (47) and obtain
\begin{equation}
(51) \quad \omega_1^2 g_1 \omega_1^{-8} g_1 \omega_1^2 = e
\end{equation}
Using the second relation of (47) we get \((g_2 g_1)^3 = e \) and we get a canonical surjection from \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) to \( \pi_1(\mathbb{P}^2 - C_V) \). Thus the fundamental group is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Lemma 14.

(VI) **Moduli space** \( \mathcal{M}(\{C_{3,15}\}) \). We consider the following family of tame sextics curve \( C_{VI,t} \in \mathcal{M}(\{C_{3,15}\}) \) with a \( C_{3,15} \)-singularity at the origin which is defined by
\begin{equation}
C_{VI,t} : f(x,y) = f_2(x,y)^3 + f_3(x,y,t)^2 = 0
\end{equation}
\begin{equation}
f_2(x,y) = y - x^2, \quad f_3(x,y,t) = tyx + 2y^3 - tx^3, \quad 1 \leq t \leq \frac{3\sqrt{2}}{4}
\end{equation}
This family enjoys the following properties:

1. For any \( t, C_{VI,t} \) has a unique \( C_{3,15} \) singularity at the origin \( O \). For \( t \neq 1 \), \( C_{VI,t} \) is irreducible but \( C_{VI,1} \) is reducible and it consists of a line \( y = 0 \) and a quintic with an \( A_{12} \)-singularity at the origin. However the local singularity at the origin is still \( C_{3,15} \).

2. \( C_{VI,t} \) is stable under the \( \mathbb{Z}_3 \) action which is induced by \( (x,y) \mapsto (x\alpha, y\alpha^2) \) where \( \alpha := \exp(2\pi i/3) \).

For a practical computation, we take \( t = t_0 \) with \( t_0 := 3\sqrt{2}/4 \). We consider the pencil \( L_t = \{y = t\}, \ t \in \mathbb{C} \). The discriminant polynomial \( \Delta_x(f) \) is given as
\begin{equation}
\Delta_x(f) = 11664y^{24}(1 + (2y^3)^2)
\end{equation}
Thus the singular singular pencil lines $y = -\frac{\alpha^j}{2}, j = 0, 1, 2$ which are the tangent lines at the flexes.

To see the monodromy relations at $y = 0$, we use at the Puiseux parametrizations of $C_{VI,t_0}$ at the origin. It has a smooth component $L$ and a component $K$ of the $(2, 13)$-cusp. They have the following parametrizations by Lemma 2:

$L: \quad x = -2\sqrt{2}i\tau + (\text{higher terms}), \quad y = \tau^2$

$K: \quad x = s^2 + \frac{2\sqrt{2}}{3}s^8 + \frac{11\sqrt{3}\sqrt{2}}{27}s^{11} + (\text{higher terms}), \quad y = s^4$

We take generators $g_0, g_1, \ldots, g_5$ of the fundamental group $\pi_1(L_{\eta_0} - C)$ with $\eta_0 = -\varepsilon$ as in Figure 9. Note that $g_0, g_5$ correspond to the points of $L$ and $g_1, \ldots, g_4$ correspond to the points of $K$. We can show $\pi_1(P^2 - C_{VI,t_0}) \cong \mathbb{Z}_2 * \mathbb{Z}_3$ by a similar computation as before.

Instead of giving a boring computation, we give a simpler proof by considering the degenerated curve $C_{VI} := C_{VI,1}$. As $C_{VI}$ is reducible and has the line $y = 0$ as a component, we take $y = 0$ as the line at infinity. Then

\[ \pi_1(P^2 - C_{VI}) \cong \pi_1(C_y^2 - Q) \]

where $C_y^2$ is the affine chart with $(x, z)$ as coordinates and $Q$ is a quintic which is defined by

\[ Q: \quad z^3 - 2z^2x^2 + zx^4 + 4xz + 4 - 4x^3 = 0 \]

This quintic is a rational curve with an $A_{12}$ singularity at the infinity. We consider the pencil $x = \eta, \eta \in \mathbb{C}$. It has three simple tangents defined by $32x^3 - 432 = 0$. We take $O = (0, 0)$ to be the base point of the fundamental group and take three generators $g_0, g_1, g_3$ on the pencil line $x = 0$ as in Figure 10. As $Q$ has also $\mathbb{Z}_3$ symmetry defined by $(x, z) \mapsto (x\alpha, z\alpha^2)$, the monodromy relations are given by

\[ g_0 = g_1g_2g_1^{-1}, \quad g_1 = g_2g_0g_2^{-1}, \quad g_2 = g_0g_1g_0^{-1} \]

We can immediately see that

\[ \pi_1(C^2 - Q) \cong \langle g_0, g_1; g_0g_1g_0 = g_1g_0g_1 \rangle \cong B(3) \]
As $D(B(3)) \cong F(2)$, we get the surjective homomorphism:

$$F(2) \cong D(\pi_1(C^2 - Q)) = D(\pi_1(P^2 - CVI)) \to D(\pi_1(P^2 - C_{VI,t_0}))$$

this implies that $\pi_1(P^2 - C_{VI,t_0}) \cong \mathbb{Z}_2 \ast \mathbb{Z}_3$ by Lemma 14.

\[\begin{align*}
\text{Figure 10. Quintics and generators at } x = 0
\end{align*}\]

\textbf{(VII) Moduli space } \mathcal{M}([C_{3,8}, 3A_2]). \text{ Let us consider the curve } C_{VII} \in \mathcal{M}([C_{3,8}, 3A_2]) \text{ which is defined by}

$$C_{VII} : f(x, y) = (y - x^2)^3 + \left(\frac{\sqrt[4]{17} \cdot 27}{423} y^3 + \left(\frac{\sqrt[3]{3}}{4} + \frac{1}{72} x\right)y^2 + \left(\frac{3\sqrt[3]{3}}{2} x + \frac{\sqrt[3]{3}}{2} x^2\right)y\right)^2$$

$$\Delta_x(f) = cy^{17}(239y^2 + 10800\sqrt{3}y - 139968)(5y + 108\sqrt{3})^2(y - 4\sqrt{3})^3(y + 108\sqrt{3})^6$$

Note that $P_y = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ where $\alpha_1 := -108\sqrt{3}$, $\alpha_2 := -\frac{5400\sqrt{3}}{239} - \frac{7776\sqrt{3}}{239}$, $\alpha_3 := -\frac{108\sqrt{3}}{5}$, $\alpha_4 := 0$, $\alpha_5 := -\frac{5400\sqrt{3}}{239} + \frac{7776\sqrt{3}}{239} \text{ and } \alpha_6 := 4\sqrt{3}$. We observe that $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 = 0 < \alpha_5 < \alpha_6$. This curve $C_{VII}$ has a $C_{3,8}$ singularity at the origin and three $A_2$ singularities, namely one $A_2$ in the level $y = \alpha_6$ and two $A_2$ singularities in the level $y = \alpha_1$. The other singular pencils are $y = \alpha_2, \alpha_5$ (simple tangents) and $y = \alpha_3$ (double tangent). We

\[\begin{align*}
\text{Figure 11. Graph of } C_{VII} \text{ and its generators at } y = -\varepsilon
\end{align*}\]

are going to show that the monodromy relations at $y = 0$, $y = -\frac{108\sqrt{3}}{5}$ and $y = \alpha_2$ are enough to compute the fundamental group.
We take generators $g_1, g_2, \ldots, g_6$ of the fundamental group $\pi_1(L_{\eta_0} - C_{VII})$ as in Figure 11 where $\eta_0 = -\varepsilon$. Since $y = \alpha_3$ is a double tangent, we obtain the relations $g_5 = g_6$ and $g_3 = g_4$. Put $\rho := g_5 = g_6, \xi := g_3 = g_4$. The tangent relation at $y = \alpha_2$ is given by

\begin{equation}
    g_2 = \xi^{-1}\rho\xi
\end{equation}

To see the monodromy relation at the origin, we look at the Puiseux expansion of $(C_{VII}, O) \cong C_{3,8}$. It has three components, a smooth component $L$ and two smooth components $K_1, K_2$. Moreover we can compute explicitly their parametrization,

- $L: x = 2t^2 + (\text{nn-terms}), \quad y = \tau^2$
- $K_i: x = -\frac{i\sqrt{2}}{2}t^2 + z_it^4 + (\text{nn-terms}), \quad y = t^4, \quad z_i \neq 0, \ i = 1, 2$

Position of the generators at $y = -\varepsilon$ are showed in Figure 11. Thus the monodromy relations at $y = 0$ are given by:

\begin{equation}
    g_2 = g_1 = \rho\xi^{-1}
\end{equation}

Finally, using the big loop relation $\rho^2\xi^2g_2g_1 = e$, and together with (53) and (54), we obtain $\{\rho, \xi\} = e$ and $(\xi\rho\xi)^2 = e$. Thus the fundamental group is isomorphic to $\mathbb{Z}_2 \ast \mathbb{Z}_3$ by Lemma 14.

### 4.4. Exceptional moduli space $\mathcal{M}(\{C_{3,9}, 3A_2\})$

We take the curve $C_{ex} \in \mathcal{M}(\{C_{3,9}, 3A_2\})$ which is defined by

\[ C_{ex}: f(x, y) = (y - x^2)^3 + \left(y^2 + \frac{4}{3}y^3 + \frac{3\sqrt{3}}{2}xy + \frac{2\sqrt{3}}{3}xy^2 + 2x^2y\right)^2 \]

\[ \Delta_x(f) = \frac{8192}{19683}y^{18}(2y + 3)^2(4y - 3)^4(4y + 9)^6 \]

Thus the singular pencil lines correspond to $P_y = \{-9/4, -3/2, 0, 3/4\}$. This curve $C_{ex}$ has a $C_{3,9}$ singularity at the origin and three $A_2$ singularities (where one $A_2$ in the level $y = 3/4$, two $A_2$ in the level $y = -9/4$).

**Figure 12.** Graph of $C_{ex}$ and its generators at $y = -\varepsilon$
To see the monodromy relation at the origin, we look at the Puiseux expansion of \((C_{\text{ex}}, O) \cong C_{3,9}\). It has two components, a smooth component \(L\) and a component \(K\) of the \((2,7)\)-cusp, and they are parametrized as follows.

\[
L: \quad x = \sqrt{4\tau} + (\text{nn-terms}), \quad y = \tau^2
\]

\[
K: \quad x = -\frac{i\sqrt{2}}{2} t^2 - \frac{4\sqrt{-6}}{9} t^5 + (\text{nn-terms}), \quad y = t^4
\]

We take generators \(g_1, g_2, \ldots, g_6\) of the fundamental group \(\pi_1(L_{\eta_0} - C_{\text{ex}})\) as in Figure 12 where \(\eta_0 = -\varepsilon\) and \(0 < \varepsilon \ll 1\). \(g_5, g_6\) correspond to the points of \(L\) and \(g_1, \ldots, g_4\) correspond to the points of \(K\). Put \(\omega_1 := g_2 g_1, \omega_2 := g_4 g_3\). The monodromy relations at \(O\) are given by:

\[
(55) \quad g_5 = g_6, \quad g_6 = (\omega_2 \omega_1) g_5 (\omega_2 \omega_1)^{-1} \quad \text{(relation for } L) \]

\[
(56) \quad g_1 = (g_5 \omega_2) g_3 (g_5 \omega_2)^{-1}, \quad g_2 = (g_5 \omega_2) g_4 (g_5 \omega_2)^{-1}
\]

\[
(57) \quad g_3 = (g_5 \omega_2 \omega_1) g_2 (g_5 \omega_2 \omega_1)^{-1}, \quad g_4 = (g_5 \omega_2 \omega_1^2) g_1 (g_5 \omega_2 \omega_1^2)^{-1}
\]

By (56), we reduce the generators to \(g_5, g_3, g_4\). Taking product of (56), we get

\[
(58) \quad \omega_1 = g_5 \omega_2 g_5^{-1}
\]

The second relation of (55) is equivalent to

\[
(59) \quad (g_5 \omega_2)^2 = (\omega_2 g_5)^2, \quad \text{or} \quad (g_5 g_4 g_3)^2 = (g_4 g_3 g_5)^2
\]

From (56) and (58), (57) is rewritten as

\[
(60) \quad g_3 = (g_5 \omega_2)^2 \omega_2 g_4 g_2^{-1} (g_5 \omega_2)^{-2}, \quad g_4 = (g_5 \omega_2)^2 \omega_2^2 g_3 \omega_2^{-2} (g_5 \omega_2)^{-2}
\]

To read the monodromy relation at \(y = -3/2\), which is a flex tangent relation (with respect to \(g_5, g_6, g_2\) in Figure 12),

\[
(61) \quad g_3 g_2 g_3^{-1} = g_5
\]

The singular pencil line \(y = -9/4\) passes through two \(A_2\) singularities, thus the monodromy relations are braid relations, they are given by

\[
(62) \quad \{g_5, g_3\} = e, \quad \{g_5, g_4\} = e
\]

Under these braid relations, the relation (61) reduces to the braid relation

\[
(63) \quad \{g_3, g_4\} = e
\]

To see the monodromy relation at \(y = 3/4\) first we show the position of the generators after a half turn, see the right hand side of Figure 13. The local equation of the \(A_2\) singularity at this level is \(x^3 + y^2 = 0\). Thus the monodromy relations are given by

\[
(64) \quad g_1 = g_4', \quad \{g_5, g_1\} = e \quad \text{where} \quad g_4' := g_5 (g_3^{-1} g_4 g_3) g_5^{-1}
\]
Figure 13. Generators position at \( y = (-9/4)^+ \) (left) and at \( y = 0^+ \) (right). See Figure 13. The first relation of (64) follows from the braid relations. The relations (60) and the second relation of (64) reduces to the commuting relations: \([g_5 g_4 g_3]^2; g_4] = e\) which is equivalent to

\[(g_5 g_4 g_3)^2 = (g_4 g_3 g_5)^2 = (g_3 g_5 g_4)^2\]

Thus the affine fundamental group \( \pi_1(C^2 - C_{ex}) \) is isomorphic to

\[\langle g_3, g_4, g_5; \{g_5, g_3\} = \{g_5, g_4\} = \{g_3, g_4\} = e, (g_5 g_4 g_3)^2 = (g_4 g_3 g_5)^2 = (g_3 g_5 g_4)^2 \rangle\]

Since \( C_{ex} \) intersects transversely with the line at infinity \( L_\infty \), the generic Alexander polynomial \( \Delta(t) \) can be computed by Fox calculus [F]. Let \( p: \tilde{X} \to \mathbb{P}^2 - (C_{ex} \cup L_\infty) \) be the infinite cyclic covering corresponding to the kernel of the Hurewicz homomorphism

\[\xi: \pi_1(\mathbb{P}^2 - (C_{ex} \cup L_\infty)) \to H_1(\mathbb{P}^2 - (C_{ex} \cup L_\infty); \mathbb{Z}) \cong \mathbb{Z}\]

and put \( \Lambda := \mathbb{Q}[t, t^{-1}] \). Then the knot polynomials are given by \( \Delta(t) = \Delta_1(t) = (t^2 - t + 1)^2 \) and \( \Delta_2(t) = t^2 - t + 1 \). Thus as \( \Lambda \)-module, we have

\[H_1(\tilde{X}, \mathbb{Q}) \cong \Lambda/(t^2 - t + 1) \oplus \Lambda/(t^2 - t + 1)\]

Thus this implies \( H_1(M; \mathbb{Q}) \cong \mathbb{Q}^4 \). We can also show that \( H_1(M; \mathbb{Z}) \cong \mathbb{Z}^4 \) by computing the commutator subgroup \( D(\pi_1(C^2 - C_{ex})) \) using Reidmeister-Schreier method ([MKS]).

To obtain the projective fundamental group, we add to (66) the big circle relation, which is given by

\[(g_5 g_4 g_3)^2 = e\]

Relations \((g_4 g_3 g_5)^2 = (g_3 g_5 g_4)^2 = e\) follows from (67). Thus we get

\[\pi_1(\mathbb{P}^2 - C_{ex}) \cong \langle g_3, g_4, g_5; \{g_5, g_3\} = \{g_5, g_4\} = \{g_3, g_4\} = e, (g_5 g_4 g_3)^2 = e \rangle\]

This completes the proof of Theorem 12. We remark here that \( \pi_1(\mathbb{P}^2 - C_{ex}) \) is much bigger than \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \) by Corollary 7 and the canonical surjection is given by identifying \( g_3 = g_5 \).
5. Non-tame torus curves

5.1. Braid group $B_4(\mathbf{P}^1)$. First we recall that the braid group of $n$ strings in $\mathbf{P}^1$, which is denoted by $B_n(\mathbf{P}^1)$, has the usual generators $g_1, \ldots, g_{n-1}$ and it has the representation (see for example [13]):

\begin{align}
(68) \quad g_ig_j &= g_jg_i \quad |i - j| \geq 2 \\
(69) \quad g_ig_{i+1}g_i &= g_{i+1}g_ig_{i+1}, \quad 1 \leq i \leq n - 2 \\
(70) \quad g_1g_2 \cdots g_{n-2}g_{n-1}g_{n-2} \cdots g_1 &= e
\end{align}

In particular, $B_4(\mathbf{P}^1)$ is generated by three elements $g_1, g_2, g_3$ with relations:

\begin{align}
(71) \quad g_1g_3 &= g_3g_1, \quad \{g_1, g_2\} = \{g_2, g_3\} = e \\
(72) \quad g_1g_2g_3 &= e
\end{align}

Recall that $\mathbb{Z}_2 \times \mathbb{Z}_3$ is generated by two elements $\rho, \xi$ which satisfies the relations $\{\rho, \xi\} = e, (\rho \xi)^3 = e$. The correspondence $g_1, g_3 \mapsto \xi$ and $g_2 \mapsto \rho$ defines a surjective homomorphism $\Psi : B_4(\mathbf{P}^1) \to \mathbb{Z}_2 \times \mathbb{Z}_3$. It is known (and easy to show) that $\text{Ker} \Psi$ is not trivial.

5.2. Example. We are ready to give an example which show that the property $\pi_1(\mathbf{P}^2 - C) \neq \mathbb{Z}_2 \times \mathbb{Z}_3$ is not so exceptional for non-tame torus curves. As an example, we consider a tame torus sextics with three $E_6$ singularities. Note that such a curve is elliptic. This curve can be degenerated into rational curves in two ways. First degeneration is to combine two $E_6$ to make $B_{3,8}$ so that the configuration of singularities is $\{B_{3,8}, E_6\}$ [13]. This degeneration can be done in the tame torus curves. As we have seen in (III), the fundamental groups is unchanged by this degeneration. Another degeneration is to put one $A_1$ singularity outside of the conic $C_2$. In [14], it is shown that the moduli of the sextics of torus type with three $E_6$ is one-dimensional and it can be parametrized as

$$C_s : f(x, y, s) = (y^2 + x^2 - 2x)^3 + s(y^2 - x^2)^2(x - 1)^2 = 0$$

for $s \in \mathbb{C}^*$ and $\pi_1(\mathbf{P}^2 - C_s) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ for $s \neq 27, 0$ and $C_{27}$ is the unique sextics which obtain an $A_1$ singularity. The fundamental group $\pi_1(\mathbf{P}^2 - C_{27})$ changes by this degeneration. In fact, we have

**Theorem 16.** $\pi_1(\mathbf{P}^2 - C_{27}) \cong B_4(\mathbf{P}^1)$ but the generic Alexander polynomial remain unchanged.

**Proof.** It is H. Tokunaga who has first observed that $\pi_1(\mathbf{P}^2 - C_{27}) \neq \pi_1(\mathbf{P}^2 - C_s) \cong \mathbb{Z}_2 \times \mathbb{Z}_3$ with $s \neq 27$. His method is to use the existence of a certain finite covering [1].

For the proof, we use the pencil $x = \eta, \eta \in \mathbb{C}$. First the discriminant polynomial of $\Delta_yf$ is given by

$$\Delta_yf(x, s) = cx^9((1 + s)x^3 - (6 + 2s)x^2 + (12 + s)x - 8)(x - 1)^6s^4((27 + 2s)x - 2s)^2$$

$$\Delta_yf(x, 27) = cx^9(7x - 8)(2x - 1)^2(3x - 2)^2(x - 1)^6$$

Observe that $(1 + s)x^3 - (6 + 2s)x^2 + (12 + s)x - 8 = 0$ has three real roots for $s \geq 27$, where two of them make the multiple root $x = 1/2$ for $s = 27$. Thus we have 5 (respectively 6).
Figure 14. Graphs of \( f(x, y) = 0 \), \( f(x, yi) = 0 \) and generators at \( x = 1 - \varepsilon \)

singular pencil lines for \( s = 27 \) (resp. for \( s > 27 \)): \( x = 0, 1/2, 2/3, 1, 8/7 \). In \( C_s \), \( s > 27 \), the node disappears and the singular line \( x = 1/2 \) splits into two simple tangent lines \( x = \alpha, \beta \) with \( \alpha < 1/2 < \beta \).

Hereafter we consider the case \( s = 27 \). The line \( x = 1/2 \) is passing the node \((1/2, 0) \in C_{27}, x = 2/3 \) is a bitangent line and \( x = 8/7 \) is a simple tangent line. The graphs of \( f(x, y, 27) = 0 \) and \( f(x, yi, 27) = 0 \) provide us the necessary informations. We take 6 generators \( g_1, g_2, g_3, g_4, g_5, g_6 \) on the pencil line \( x = 1 - \varepsilon, 0 < \varepsilon \ll 1 \) as in Figure 14.

Put \( \omega_1 := g_3g_2g_1 \) and \( \omega_2 := g_6g_5g_4 \). The big circle relation is \( \omega_2\omega_1 = e \). The monodromy relation at \( x = 1 \) is given by

\[
\begin{align*}
(73) & \quad g_1 = \omega_1 g_2 \omega_1^{-1}, \quad g_2 = \omega_1 g_3 \omega_1^{-1} \\
(74) & \quad g_4 = \omega_2 g_5 \omega_2^{-1}, \quad g_5 = \omega_2 g_6 \omega_2^{-1}
\end{align*}
\]

At \( x = 8/7 \), we get a simple tangent relation:

\[
(75) \quad \omega_1^{-1}g_3\omega_1 = \omega_2^{-1}g_6\omega_2
\]

The line \( x = 2/3 \) is a bi-tangent line and the relation is:

\[
(76) \quad g_1 = g_6, \quad g_3 = g_4
\]

Thus we can eliminate \( g_4, g_6 \) from generators. At \( x = 1/2 \), we get the commuting relation:

\[
(77) \quad g_1g_4 = g_4g_1 \quad \text{or} \quad g_1g_3 = g_3g_1
\]

Putting \( \Omega := g_5g_3g_2g_1 \), the monodromy relations at \( x = 0 \) is given by

\[
\begin{align*}
(78) & \quad (g_2g_1)^{-1}g_1(g_2g_1) = g_5, \\
(79) & \quad g_1^{-1}g_2g_1 = \Omega(g_2g_1)^{-1}g_1(g_2g_1)\Omega^{-1}, \quad g_3 = \Omega g_1^{-1}g_2g_1\Omega^{-1}
\end{align*}
\]

Now we can rewrite (73) using (77) as the braid relations:

\[
(80) \quad g_1g_2g_1 = g_2g_1g_2, \quad g_2g_3g_2 = g_3g_2g_3
\]

The second relation of (78) reduces to \( g_5 = g_2 \). Thus we can eliminate \( g_5 \) and we take \( g_1, g_2, g_3 \) as generators. They satisfy (80) and (77). The relation \( \omega_2\omega_1 = e \) reduces to

\[
(81) \quad g_1g_2g_3g_2g_1 = e
\]
and we can see easily that other relations follow from (80), (77) and (81). Thus we have proved
\[ \pi_1(\mathbb{P}^2 - C_{27}) \cong \langle g_1, g_2, g_3 \mid \{g_1, g_2\} = \{g_2, g_3\} = e, g_1 g_3 = g_3 g_1, (81) \rangle \cong B_4(\mathbb{P}^1) \]
The Alexander polynomial of \( C_{27} \) is equal to \( t^2 - t + 1 \). This can be shown by the exact same computation as in [O4] or a direct Fox calculus \([F]\) from the above relations. This completes the proof.

Consider the degeneration \( C_s, s \to 27 + 0 \). Then we can see immediately that the monodromy relation at \( x = 1/2 \) splits into two simple tangent relations \( x = \alpha \) and \( x = \beta \) which gives the relation \( g_3 = g_1 \) in the place of \( g_1 g_3 = g_3 g_1 \). This is the geometrical interpretation of the surjective homomorphism: \( \Psi : B_4(\mathbb{P}^1) \to \mathbb{Z}_2 * \mathbb{Z}_3 \).

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