Topological categories related to Fredholm operators: II. The analytical index

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1. Introduction

A naive approach to the analytical index of families of self-adjoint Fredholm operators. The classical definition of the analytical index of families of self-adjoint Fredholm operators, due to Atiyah and Singer [AS], [AS5], is indirect. Let $H$ be a separable infinite dimensional Hilbert space, and let $A_x, x \in X$ be a family of self-adjoint Fredholm operators $H \to H$. Atiyah and Singer define the analytical index of $A_x, x \in X$ as the analytical index of a related family of general (not assumed to be self-adjoint) Fredholm operators $H \to H$ parameterized by the suspension $\Sigma X$ of $X$ and defined by a simple formula.

The index of the family $A_x, x \in X$ ought to be the family of kernels $\text{Ker } A_x, x \in X$, or, rather, some equivalence class of this family of kernels. This idea immediately encounters a couple of difficulties. Namely, the kernels usually do not depend continuously on $x$, and it is unclear what kind of an object the family $\text{Ker } A_x, x \in X$ is. Moreover, if the kernels happen to continuously depend on $x$ and hence to define a vector bundle on $X$, then the family can be deformed to a family with zero kernels and its index ought to be zero. In fact, the analytical index of $A_x, x \in X$ can be thought as a measure of the failure of the family of kernels $\text{Ker } A_x, x \in X$ to be a vector bundle.

Nevertheless, there is a definition of the analytical index of families of self-adjoint Fredholm operators making this idea rigorous, and the present paper is devoted to such a definition. It is based on ideas of Segal [S4] as developed by the author [I2].

Enhanced operators. The starting point is the notion of enhanced operators introduced in [I2]. Recall that if $A : H \to H$ is a self-adjoint Fredholm operator, then there exists $\varepsilon > 0$ such that $\varepsilon, -\varepsilon$ do not belong to the spectrum $\sigma(A)$ of $A$, the interval $[-\varepsilon, \varepsilon]$ is disjoint from the essential spectrum of $A$, and hence the image of the spectral projection $P_{[-\varepsilon, \varepsilon]}(A)$ is finitely dimensional. An enhanced operator is defined as a pair $(A, \varepsilon)$ having these properties. By a basic property of self-adjoint Fredholm operators, if $(A, \varepsilon)$ is an enhanced operator and $A' : H \to H$ is a self-adjoint Fredholm operator sufficiently close to $A$ in the norm topology, then $(A', \varepsilon)$ is also an enhanced operator. Moreover, the spectral projection $P_{[-\varepsilon, \varepsilon]}(A')$ and its image continuously depend on $A'$ for $A'$ close to $A$.

Toward a rigorous form of the naive approach. As above, let $A_x, x \in X$ be a family of self-adjoint Fredholm operators in $H$. We would like to turn the family of kernels $\text{Ker } A_x, x \in X$ into something manageable. Let $z \in X$. Since $A_z$ is a self-adjoint Fredholm operator, there exists $\varepsilon = \varepsilon_z > 0$ such that $(A, \varepsilon)$ is an enhanced operator. Clearly, the kernel $\text{Ker } A_z$ is contained in this image of the spectral projection $P_{[-\varepsilon, \varepsilon]}(A_z)$, and if $\varepsilon$ is sufficiently small, $\text{Ker } A_z$ is equal to this image. Even if this is not the case, one may consider the image $\text{Im } P_{[-\varepsilon, \varepsilon]}(A_z)$ as an approximation to the kernel $\text{Ker } A_z$. If the family $A_x, x \in X$ is moderately continuous, there is a neighborhood $U_z$ of $z$ such that $(A_y, \varepsilon)$ is an enhanced
operator for every $y \in U_z$ and the spectral projections $P_{[-\varepsilon, \varepsilon]}(A_y)$ continuously depends on $y$ for $y \in U_z$. The family of images $\text{Im } P_{[-\varepsilon, \varepsilon]}(A_y)$, $y \in U_z$ can be considered as an approximation to the family of kernels $\text{Ker } A_y$, $y \in U_z$. The continuous dependence of $P_{[-\varepsilon, \varepsilon]}(A_y)$ on $y \in U_z$ implies that the family of images $\text{Im } P_{[-\varepsilon, \varepsilon]}(A_y)$, $y \in U_z$ defines a finitely dimensional vector bundle over $U_z$. We will define the analytical index as the homotopy class of a continuous map build from these vector bundles, or, rather, directly from the images $\text{Im } P_{[-\varepsilon, \varepsilon]}(A_y)$ and the restrictions of $A_y$ on these images.

The main problem is in turning a collection of locally defined continuous maps into a globally defined continuous map. The usual tool of partitions of unity does not apply, at least not directly, because the target of our maps is not a vector space, but a Grassmannian, the space of all finitely dimensional subspaces of $H$.

Let us start with a point $x \in U_z$ and move it away from $z$ along a path in $X$. After we move sufficiently far, at some point $y \in X$ we will be forced to replace $\varepsilon = \varepsilon_z$ by another number $\varepsilon' = \varepsilon_{z'}$. This will force us to jump from

$$V = \text{Im } P_{[-\varepsilon, \varepsilon]}(A_y) \quad \text{to} \quad V' = \text{Im } P_{[-\varepsilon', \varepsilon']} (A_y).$$

The key idea is to connect $V$ with $V'$ by a segment of formal convex combinations

$$tV + (1-t)V',$$

where $0 \leq t \leq 1$. This expression is formal in the sense that the multiplication and the addition are purely symbolic and do not correspond to any operations on subspaces of $H$. In fact one needs to be more precise. If, say, $\varepsilon' > \varepsilon$, then the orthogonal complement $V' \oplus V$ splits into two direct summands corresponding the negative and positive parts of the spectrum of $A_y$, and one should introduce a formal connecting segment for each possible splitting. If, say, $y$ belongs to three neighborhoods $U_z$, $U_{z'}$, $U_{z''}$ and $V$, $V'$, $V''$ are the corresponding vector subspaces, then one needs to use formal convex combinations

$$t_0V + t_1V' + t_2V',$$

where $t_0, t_1, t_2 \geq 0$ and $t_0 + t_1 + t_2 = 1$, and, again, to take into account splittings. In general, one needs to use simplices having such subspaces as vertices.

**Topological categories and functors.** The proper language for making this outline rigorous is provided by the notions of topological categories and their classifying spaces introduced by Segal [S_1]. The reader may find a review of definitions and basic properties in the first sections of [I_2]. We will now recall some key constructions from [S_1] and [I_2].

We will identify each topological space $X$ with the topological category having $X$ as the space of objects and only identity morphisms. Every covering $U_a$, $a \in \Sigma$ of a space $X$ defines a topological category $X_U$ together with a functor $\text{pr}: X_U \rightarrow X$. This construction
is due to Segal \([S_1]\) and is recalled in Section 2. If the covering is open and \(X\) is paracompact, then the geometric realization \(|pr|: |X_U| \to X\) is a homotopy equivalence. In Section 2 we also recall definitions of a topological category \(\hat{\mathcal{P}}\) having as objects finitely dimensional subspaces of \(H\), and of Quillen–Segal \([S_4]\) topological category \(Q\), a version of Quillen’s \(Q\)-construction, having as objects finitely dimensional Hilbert spaces. There is a canonical homotopy equivalence \(|\hat{\mathcal{P}}| \to |Q|\).

**Index functors, index map, and the analytical index.** Let us return to the family \(A_x, x \in X\) and choose a subset \(\Sigma \subset X\) such that the neighborhoods \(U_z, z \in \Sigma\) form a covering of \(X\). Let \(\varepsilon\) be the family of the corresponding parameters \(\varepsilon_z\). The category \(\hat{\mathcal{P}}\) is defined in such a way that the covering \(U\) together with the family \(\varepsilon\) define a continuous functor

\[ A_{U, \varepsilon}: X_U \to \hat{\mathcal{P}} \]

which may be called the *index functor*, and hence a continuous map

\[ |A_{U, \varepsilon}|: |X_U| \to |\hat{\mathcal{P}}|. \]

By composing it with a homotopy inverse \(s: X \to |X_U|\) of the map \(|pr|\) we get a map \(X \to |\hat{\mathcal{P}}|\) well defined up to homotopy. One can also combine it with the canonical homotopy equivalence \(|\hat{\mathcal{P}}| \to |Q|\) and get a map \(X \to |Q|\) well defined up to homotopy. The maps \(X \to |\hat{\mathcal{P}}|\) and \(X \to |Q|\) both deserve to be called the *index maps* of the family \(A_x, x \in X\). The *analytical index* of \(A_x, x \in X\) is defined as the homotopy class of either of these two map. Of course, these homotopy classes carry the same information.

If the operators \(A_x\) are bounded and the family \(A_x, x \in X\) is continuous in the norm topology, then the analytical index carries the same information as the homotopy class of the map \(x \mapsto A_x\) from \(X\) to the space \(\hat{\mathcal{F}}\) of bounded self-adjoint Fredholm operators \(H \to H\).

But there are definite advantages of replacing \(\hat{\mathcal{F}}\) by \(|\hat{\mathcal{P}}|\) or \(|Q|\) as the target of the index map even for norm-continuous families of bounded operators. Both index maps \(X \to |\hat{\mathcal{P}}|\) and \(X \to |Q|\) realize the idea of index as an equivalence class of the family of kernels, while the map \(X \to \hat{\mathcal{F}}\) is just another name of the family. The space \(|\hat{\mathcal{P}}|\) is defined in terms of finitely dimensional subspaces of \(H\) and is more accessible than the space \(\hat{\mathcal{F}}\) of operators, while the definition of \(|Q|\) involves only finitely dimensional linear algebra.

**Fredholm families and Hilbert bundles.** These aspects of our definition lead to its main advantages. First, the index maps \(X \to |\hat{\mathcal{P}}|\) and \(X \to |Q|\) are defined even when the family \(A_x, x \in X\) is not continuous in the norm topology. In Section 2 we will define *Fredholm families* essentially as families for which the construction of the index functors \(A_{U, \varepsilon}: X_U \to \hat{\mathcal{P}}\) makes sense. Such families don’t need to be continuous. Second, the construction of these index functors and index maps easily extends to families of operators acting in the fibers of a Hilbert bundle. This is the topic of the second part of Section 2.
Let $H$ be a Hilbert bundle with the base $X$, thought as a family of Hilbert spaces $H_x$, $x \in X$. It is worth to stress that in general there is no analogue of the space $\hat{\mathcal{F}}$ suitable for working with families of operators acting in the fibers. More precisely, there is no reasonable bundle over $X$ having as the fiber over $x \in X$ the space of self-adjoint Fredholm operators $H_x \to H_x$ with the norm topology. The reason is that the transition maps of Hilbert bundles are only rarely continuous in the norm topology.

At the same time one can always define an analogue $\hat{\mathcal{S}}(H)$ of the category $\hat{\mathcal{S}}$. This is a topological category having as objects finitely dimensional subspaces of fibers $H_x$ of $H$. There is also analogue $Q(H)$ of $Q$. It is especially simple and has as objects pairs $(x,V)$, where $x \in X$ and $V$ is an object of $Q$. Both definitions are recalled in Section 2. By the definition the categories $\hat{\mathcal{S}}(H)$ depend only on $X$ up to an isomorphism. In fact, by a well known theorem of Dixmier and Douady [DD] every Hilbert bundle over a paracompact space is isomorphic to the trivial bundle $X \times H \to X$. Moreover, a trivialization of $H$ allows to identify $|\hat{\mathcal{S}}(H)|$ with $X \times |\hat{\mathcal{S}}|$, and such an identification is unique up to homotopy.

With the obvious changes the definition of Fredholm families applies to families of self-adjoint operators $A_x : H_x \to H_x$, $x \in X$. For such families one can define index functors

$$A_{U,\epsilon} : X_U \to \hat{\mathcal{S}}(H), \quad X_U \to Q(H),$$

and index maps

$$X \to |\hat{\mathcal{S}}(H)| = X \times |\hat{\mathcal{S}}|, \quad X \to |Q(H)| = X \times |Q|.$$

The composition of the index maps with the projections to $X$ are equal to the identity map and carry no information. This suggests to define another index maps $X \to |\mathcal{S}|$ and $X \to |Q|$ as the compositions of the original ones with the projections to the second factors. The analytical index is defined as the homotopy class of either of these index maps. Both versions carry the same information, as also the homotopy class of $X \to |\hat{\mathcal{S}}(H)|$.

**Strictly Fredholm families.** In order to go further than the definitions and their correctness, we need to restrict a little the class of considered families and require that spectral projections corresponding to some half-lines continuously depend on parameter.

Suppose first that $H$ is the trivial bundle with the fiber $H$. We will say that the family $A_x : H \to H$, $x \in X$ is strictly Fredholm if it is Fredholm and for every $z \in X$ there exist $\epsilon > 0$ and a neighborhood $U_z$ of $z$ such that $(A_y, \epsilon)$ is an enhanced operator for every $y \in U_z$ and the spectral projections $P_{(\epsilon,\infty)}(A_y)$ continuously depends on $y$ for $y \in U_z$.

In general, $A_x : H_x \to H_x$, $x \in X$ is said to be a strictly Fredholm family if for every $z \in X$ there exists a neighborhood $U_z$ of $z$ and a trivialization of $H$ over $U_z$ turning the restriction $A_x : H_x \to H_x$, $x \in U_z$ into a strictly Fredholm family in the above sense.
Families of elliptic operators. A continuous family of elliptic pseudo-differential operators of order 0 acting on the fibers of locally trivial bundle $\mathbb{M}$ over $X$ with closed manifolds as fibers defines a family of bounded operators acting in the fibers of a Hilbert bundle $\mathbb{H}$. Classical results of Seeley [Se] (namely, the integral presentation in the proof of Theorem 5 in [Se]) imply that the latter family is strictly Fredholm.

Strictly adapted trivializations. A trivialization of $\mathbb{H}$ is said to be strictly adapted to the family $A_x: H_x \to H_x$, $x \in X$ if its restrictions to sufficiently small neighborhoods $U_z$ turn the restrictions $A_x: H_x \to H_x$, $x \in U_z$ into strictly Fredholm families, i.e. turn the families $P_{[\epsilon, \infty)}(A_y)$, $y \in U_z$ with appropriate $\epsilon$ into norm-continuous ones. Strictly adapted trivializations exist for every strictly Fredholm family with paracompact $X$. This is the main result of Section 4. It is proved first for triangulable spaces $X$, when one can use an induction by skeletons, and then for paracompact $X$. See Theorems 4.5 and 4.6. The proofs depend on contractibility theorems of Kuiper [Ku] and of Dixmier–Douady [DD] in Atiyah–Segal [ASe] form. For the rest of the introduction we assume that $X$ is paracompact.

Polarizations. Recall that a polarization of a Hilbert space $H$ is a presentation of $H$ as an orthogonal direct sum $H = H_- \oplus H_+$ of two closed infinitely dimensional subspaces. A polarization leads to the restricted Grassmannian $\text{Gr}$, the space of subspaces $K \subset H$ commensurable with $H_-$, i.e. such that $K \cap H_-$ has finite codimension in both $K$ and $H_-$. The topology of $\text{Gr}$ is defined by the norm topology of orthogonal projections $H \to K$. A strictly Fredholm family leads to a polarization of $\mathbb{H}$, a structure defined in the Appendix. In the main part of the paper these structures are not even mentioned, but in the Appendix Theorems 4.5 and 4.6 and their proofs are translated into the language of polarizations.

In Section 5 we recall from [I2] the definition of a topological category $\mathcal{P}\mathcal{H}$, a version of the category $\mathcal{H}$ involving polarization. There is a forgetting functor $\mathcal{P}\mathcal{H} \to \mathcal{H}$ inducing a homotopy equivalence $|\mathcal{P}\mathcal{H}| \to |\mathcal{H}|$. If we use a strictly adapted trivialization to identify $|\mathcal{H}(\mathbb{H})|$ with $X \times |\mathcal{H}|$, then the index map $X \to |\mathcal{H}|$ naturally lifts to a map $X \to |\mathcal{P}\mathcal{H}|$, which we call the polarized index map. See Section 5.

Grassmannian bundles. Let $A_x: H_x \to H_x$, $x \in X$ be a strictly Fredholm family. We will denote it also by $\mathbb{A}$. If $z \in X$ and $(A_z, \epsilon)$ is an enhanced operator, then

$$H_z = \text{Im } P_{(-\infty, \epsilon]}(A_z) \oplus \text{Im } P_{[\epsilon, \infty)}(A_z)$$

is a polarization of $H_z$. This polarization leads to a restricted Grassmannian, which will denote by $\text{Gr}(z)$. Since the family $\mathbb{A}$ is strictly Fredholm, the family of restricted Grassmannians $\text{Gr}(z)$, $z \in X$ forms a locally trivial bundle

$$\pi(\mathbb{A}): \text{Gr}(\mathbb{A}) \to X.$$

In Theorem 5.1 we prove that this bundle is equal to the bundle induced by the polarized
index map $X \to |\mathcal{P}|$ from the standard bundle $\pi: G \to |\mathcal{P}|$ constructed in $[I_2]$. Motivated by Melrose and Piazza [MP], we introduce another name, *weak spectral sections* of the family $\mathbb{A}$, for continuous sections of the bundle $\pi(\mathbb{A})$. In an agreement with the ideas of Melrose and Piazza [MP], a weak spectral section exists if and only if the analytical index vanishes, i.e. the index map is homotopic to a constant map. See Theorem 5.2. The proof of the “only if” part depends on one of the main results of $[I_2]$, the contractibility of the total space $G$ of the bundle $\pi$. See $[I_2]$, Theorem E or Theorem 13.6.

**Discrete-spectrum families.** Let us say that $A_x: H_x \to H_x$, $x \in X$ is a *discrete-spectrum family* if for every $\lambda \in \mathbb{R}$ the family of operators $A_x - \lambda$, $x \in X$ is a Fredholm family. As usual, here $\lambda$ stands for $\lambda$ times the identity operator. The operators $A_x$ of such a family have discrete spectrum, i.e. their essential spectrum is empty. By this reason these operators cannot be bounded. We assume that they are closed densely defined operators.

Following Melrose and Piazza [MP], we say that a weak spectral section $S: X \to \text{Gr}(\mathbb{A})$ is a *spectral section* if there exists a continuous function $r: X \to \mathbb{R}_{>0}$ such that

$$\text{Im} \, P_{[r(x), \infty)}(A_x) \subset S(x) \subset \text{Im} \, P_{[-r(x), \infty)}(A_x)$$

for every $x \in X$. This notion seems to be useful only for discrete-spectrum strictly Fredholm families. In this case we prove that every weak spectral section can be deformed into a spectral sections, and hence spectral sections exist if and only if the analytical index vanishes. See Theorem 6.1 and Corollary 6.2. These results generalize and clarify Proposition 1 of Melrose and Piazza [MP]. See the discussion at the end of Section 6.

**Families of general Fredholm operators.** For families of general (not assumed to be self-adjoint) Fredholm operators one can develop a theory parallel to the one outlined above. See Section 3. In the present paper we limit the discussion by the results needed to prove the equivalence of our definition of the analytical index with Atiyah–Singer one when the latter or a natural generalization of it applies.

**Comparing two definitions of analytical index.** Since Atiyah–Singer definition of the analytical index of families of self-adjoint operators depends on the definition in the non-self-adjoint case, we have to deal with this case first. The most classical definition applies only to compact $X$. It was extended to paracompact $X$ by Segal $[S_2]$. We prove that two definitions agree for compact $X$ in Theorem 7.1, and for paracompact $X$ in Theorem 7.2.

In Section 8 we deal with families of self-adjoint operators. First, we show that one can extend the approach of Atiyah and Singer to Fredholm families as defined above. See Theorem 8.2. Then we prove that two definitions agree for strictly Fredholm families. See Theorem 8.5. Along the way we prove that every strictly Fredholm family can be replaced by a much better family without affecting its index. See Theorem 8.4. This construction is similar to a spectral deformation used in $[AS]$ and may be of independent interest.
2. Analytical index of self-adjoint Fredholm families

Self-adjoint Fredholm operators. Let us fix a separable infinitely dimensional Hilbert space $H$. By $\hat{F}$ we denote the space of self-adjoint Fredholm operators $H \rightarrow H$. Actually, there are at least two versions of the space $\hat{F}$. In the first version the operators are assumed to be bounded and $\hat{F}$ is equipped with the norm topology. In the second version the operators are allowed to be unbounded closed densely defined operators. In this version $\hat{F}$ is equipped with Riesz topology. Our discussion applies equally well to both versions. Most of our arguments work also for the topology of convergence in the uniform resolvent sense.

Families of self-adjoint Fredholm operators. Let $X$ be a compactly generated topological space. A family of self-adjoint Fredholm operators parameterized by $X$ is a continuous map $A : X \rightarrow \hat{F}$. It is convenient to denote $A(x)$ by $A_x$ and use for $A$ the notation $A_x$, $x \in X$, which reflects the idea of a “family” of operators better.

The analytical index of families. The analytical index of a family $A_x$, $x \in X$ should be an element of the group $K^1(X)$. The group $K^1(X)$ can be defined as the group of homotopy classes of maps from $X$ to a classifying space for the functor $Y \mapsto K^1(Y)$, initially defined only for compact $Y$. The group structure results from a canonical structure of an $H$-space on the classifying space. In the case of bounded operators the space $\hat{F}$ is such a classifying space by the results of Atiyah and Singer [AS]. This allows to define the analytical index of a family $A_x$, $x \in X$ simply as the homotopy class of the corresponding map $A : X \rightarrow \hat{F}$, and this is the most widely, if not exclusively, used definition, even for compact $X$.

This definition seems to be somewhat tautological, in particular if compared with the standard definition of the analytical index of a family of Fredholm operators. At the same time this definition is not quite suitable for generalizations to families of unbounded operators and to families of operators in Hilbert spaces varying with $x \in X$, i.e. to operators in Hilbert bundles over $X$. Such generalizations encounter some subtle continuity problems.

Topological categories related to self-adjoint Fredholm operators. Let us recall some definitions from [I₂]. To begin with, the space $\hat{F}$ can be considered as a topological category having $\hat{F}$ as the space of objects and only identity morphisms. Then the classifying space $|\hat{F}|$ is equal to $\hat{F}$ considered as a topological space.

An enhanced self-adjoint Fredholm operator, or simply an enhanced operator is defined as a pair $(A, \varepsilon)$ consisting of an operator $A \in \hat{F}$ and a real number $\varepsilon > 0$ such that $-\varepsilon, \varepsilon$ do not belong to the spectrum $\sigma(A)$ of $A$ and the spectral projection $P_{[-\varepsilon, \varepsilon]}(A)$ has finite rank. The set $\hat{E}$ of enhanced operators is equipped with the topology defined by the topology of $\hat{F}$ and the discrete topology on the space $\mathbb{R}_{>0}$ of parameters $\varepsilon$. The space $\hat{E}$ is ordered by the relation $\leq$, where $(A, \varepsilon) \leq (A', \varepsilon')$ if $A = A'$ and $\varepsilon \leq \varepsilon'$. This order
defines a structure of a topological category on \( \hat{E} \) having \( \hat{E} \) as the space of objects and a single morphism \((A, \varepsilon) \to (A', \varepsilon')\) if \( A = A' \) and \( \varepsilon \leq \varepsilon' \). The obvious forgetting functor \( \hat{\phi} : \hat{E} \to \hat{F} \) induces a map of classifying spaces \( |\hat{\phi}| : |\hat{E}| \to |\hat{F}| = \hat{F} \).

The topological category \( \hat{F} \) has finitely dimensional subspaces of \( H \) as objects. The topology on the space of such subspaces is the obvious one. Morphisms \( V \to V' \) exist only if \( V \subset V' \), and in this case morphisms correspond to orthogonal decompositions

\[ V' = U_- \oplus V \oplus U_+ . \]

The subspaces \( U_- \) and \( U_+ \) are called the negative and positive parts of the morphism in question. The composition is defined by taking the sum of negative parts and the sum of positive parts to get, respectively, the negative and the positive parts of the composition. The topology on the set of morphisms is the obvious one. There is a forgetting functor \( \hat{\pi} : \hat{S} \to \hat{F} \), defined as follows. The functor \( \hat{\pi} \) takes an enhanced operator \((A, \varepsilon)\) to

\[ \hat{\pi}(A, \varepsilon) = \operatorname{Im} P_{[-\varepsilon, \varepsilon]}(A) \subset H , \]

and a morphism \((A, \varepsilon) \to (A, \varepsilon')\) to the morphism corresponding to the decomposition

\[ \operatorname{Im} P_{[-\varepsilon', \varepsilon]}(A) = \left( \operatorname{Im} P_{[-\varepsilon', -\varepsilon]}(A) \right) \oplus \left( \operatorname{Im} P_{[-\varepsilon, \varepsilon]}(A) \right) \oplus \left( \operatorname{Im} P_{[\varepsilon, -\varepsilon']} (A) \right). \]

Finally, the category \( Q \) is an abstract analogue of \( \hat{F} \). It objects are finitely dimensional Hilbert spaces belonging to a set containing at least one Hilbert space of each finite dimension. The set of objects is equipped with the discrete topology. A morphism \( V \to V' \) is a triple \((U_-, U_+, \iota)\), where \( \iota : V \to V' \) is an isometric embedding and \( U_-, U_+ \) are subspaces of \( V' \) defining an orthogonal decomposition

\[ V' = U_- \oplus \iota(V) \oplus U_+. \]

The composition of morphisms is defined in the same way as in \( \hat{F} \), and the topology on the set of morphisms is the obvious one. The obvious functor \( \hat{F} \to Q \) assigning to a finitely dimensional subspace \( V \subset H \) the space \( V \) considered as an object of \( Q \), is not continuous. But there is an intermediate category \( Q/H \) and continuous functors \( \hat{F} \leftarrow Q/H \to Q \) allowing to compare \( \hat{F} \) and \( Q \). See [I₂], Section 9. By passing to the classifying spaces we get the following two diagrams.

\[
\begin{array}{cccccccccc}
\hat{F} & = & \hat{F} & \leftarrow & \hat{\phi} & \leftarrow & \hat{\phi} & \leftarrow & \hat{\phi} & \leftarrow & \hat{\pi} & \rightarrow & \hat{\pi} & \rightarrow & \hat{\pi} & \rightarrow & \hat{F} \\
|\hat{F}| & \leftarrow & |\hat{\phi}| & \leftarrow & |\hat{\phi}| & \leftarrow & |\hat{\phi}| & \leftarrow & |\hat{\phi}| & \leftarrow & |\hat{\pi}| & \rightarrow & |\hat{\pi}| & \rightarrow & |\hat{\pi}| & \rightarrow & |\hat{\pi}| \\
|Q/H| & \rightarrow & |Q/H| & \rightarrow & |Q/H| & \rightarrow & |Q/H| & \rightarrow & |Q/H| & \rightarrow & |Q| & \rightarrow & |Q| & \rightarrow & |Q| & \rightarrow & |Q| \\
\end{array}
\]

The basic result about these categories and functors is the fact that all maps in these diagrams are homotopy equivalences. See [I₂], Theorems 9.6 and 9.7.
Topological categories defined by coverings. We need a classical construction due to Segal [S₃]. It relates topological categories with coverings of topological spaces. The construction starts with a covering \( U_a, a \in \Sigma \) of a space \( X \). The idea is, to borrow a phrase of Segal [S₃], to "disintegrate" the space \( X \) into pieces \( U_a, a \in \Sigma \) and then assemble these pieces into a "thick" version of \( X \). Let \( \Sigma^{\text{fin}} \) be the set of finite non-empty subsets of \( \Sigma \) and

\[
U_\sigma = \bigcap_{a \in \sigma} U_a
\]

for every \( \sigma \in \Sigma^{\text{fin}} \). Following Segal [S₃], let us consider the following topological category \( X_U \). Its space of objects is the disjoint union of the subspaces \( U_\sigma, \sigma \in \Sigma^{\text{fin}} \). More formally, the objects of \( X_U \) are pairs \( (x, \sigma) \) such that \( \sigma \in \Sigma^{\text{fin}} \) and \( x \in U_\sigma \). The set of such pairs is ordered by the relation \( \leq \), where \( (x, \sigma) \leq (y, \tau) \) if \( x = y \) and \( \tau \subset \sigma \). This order defines a structure of a topological category having this disjoint union as the space of objects and a single morphism \( (x, \sigma) \rightarrow (y, \tau) \) if \( x = y \) and \( \tau \subset \sigma \). If we consider \( X \) as the topological category having \( X \) as the space of objects and only identity morphisms, then the rule \( (x, \sigma) \rightarrow x \) defines a continuous functor \( \text{pr}: X_U \rightarrow X \), and hence defines a map

\[
|\text{pr}|: |X_U| \rightarrow |X| = X.
\]

In fact, the definition of the topological category on \( \hat{\Sigma} \) is a slightly modified version of this construction. The following theorem of Segal [S₃] is the basic result about \( X_U \).

2.1. Theorem. If \( U_a, a \in \Sigma \) is a numerable covering, then the map \(|\text{pr}|: |X_U| \rightarrow X\) is a homotopy equivalence. Moreover, there exists a homotopy inverse \( s: X \rightarrow |X_U| \) such that \(|\text{pr}| \circ s = \text{id}_X\) and the composition \( s \circ |\text{pr}| \) is homotopic to the identity in the class of maps \( f: |X_U| \rightarrow |X_U| \) such that \(|\text{pr}| \circ f = |\text{pr}|\). ■

Enhancing families of operators. Let \( A_x, x \in X \) be a family of self-adjoint Fredholm operators, and let \( A: X \rightarrow \hat{\Sigma} \) be the corresponding map. We would like to lift \( A \) to a family of enhanced operators. This almost never can be done naively, by lifting \( A \) to a map \( X \rightarrow \hat{\Sigma} \). The obvious obstruction is the discrete topology on \( \mathbb{R} \) used in the definition of \( \hat{\Sigma} \), but this wouldn't be possible even if we used the standard topology of \( \mathbb{R} \), because of the condition \(-\varepsilon, \varepsilon \not\in \sigma(A)\) in the definition of enhanced operators. But there are obvious lifts if \( \hat{\Sigma} \) is replaced by \( \mathbb{R} \) and \( X \) is replaced by \( |X_U| \) for an appropriate covering \( U \).

Appropriate coverings \( U \) are defined in terms of the family \( A_x, x \in X \). For every \( x \in X \) the operator \( A_x \) is Fredholm. By a basic property of Fredholm operators, there exists \( \varepsilon_x > 0 \) such that \( (A_x, \varepsilon_x) \) is an enhanced operator. By another basic property of Fredholm operators this implies that \( (A_y, \varepsilon_x) \) is an enhanced operator for every \( y \) in some neighborhood of \( x \). It follows that there exists an open covering \( U_a, a \in \Sigma \) of \( X \) and positive numbers \( \varepsilon_a, a \in \Sigma \) such that \( (A_z, \varepsilon_a) \) is an enhanced operator for every \( z \in U_a \). Any such covering will work for us. For every \( \sigma \in \Sigma^{\text{fin}} \) let \( \varepsilon_\sigma = \min_{a \in \sigma} \varepsilon_a \). Then the pair \( (A_z, \varepsilon_\sigma) \)
is an enhanced operator for every \( z \in U_\sigma \), and \( \varepsilon_\tau \leq \varepsilon_\sigma \) if \( \tau \supset \sigma \). Therefore the rule \((z, \sigma) \mapsto (A_z, \varepsilon_\sigma)\) defines a continuous functor
\[
A_{U, \varepsilon} : X_U \to \hat{\mathcal{E}},
\]
and hence a continuous map
\[
\left| A_{U, \varepsilon} \right| : \left| X_U \right| \to \left| \hat{\mathcal{E}} \right|.
\]

If the map \( \mathcal{A} : X \to \hat{\mathcal{F}} \) is also considered as a functor, we get a commutative diagram

\[
\begin{array}{ccc}
X_U & \xrightarrow{A_{U, \varepsilon}} & \hat{\mathcal{E}} \\
pr \downarrow & & \downarrow \hat{\phi} \\
X & \xrightarrow{\mathcal{A}} & \hat{\mathcal{F}}
\end{array}
\]

of topological categories and functors, and a commutative diagram

\[
\begin{array}{ccc}
\left| X_U \right| & \xrightarrow{\left| A_{U, \varepsilon} \right|} & \left| \hat{\mathcal{E}} \right| \\
\left| pr \right| \downarrow & & \downarrow \left| \hat{\phi} \right| \\
X & \xrightarrow{\mathcal{A}} & \hat{\mathcal{F}}
\end{array}
\]

of topological spaces and continuous maps. The maps \( \left| A_{U, \varepsilon} \right| \) are the promised lifts of \( \mathcal{A} \). If the covering \( U_a, a \in \Sigma \) is numerable, in particular, if the space \( X \) is paracompact, there exist even lifts \( X \to \left| \hat{\mathcal{E}} \right| \) well defined up to homotopy, as we will see now.

Suppose now that \( X \) is paracompact. Let \( s : X \to \left| X_U \right| \) be the homotopy inverse of \( \left| pr \right| : \left| X_U \right| \to X \) such as in Theorem 2.1. Since \( \left| pr \right| \circ s = \text{id}_X \), for every \( x \in X \) the point \( s(x) \) belongs to the geometric realization of the simplex corresponding to a sequence
\[
(x, \sigma_0) \leq (x, \sigma_1) \leq \ldots \leq (x, \sigma_n).
\]

The functor \( A_{U, \varepsilon} \) takes every \((x, \sigma_i)\) to an object of the form \((A_x, \varepsilon_i)\). It follows that
\[
\left| \hat{\phi} \right| \circ \left| A_{U, \varepsilon} \right| \circ s(x) = A_x
\]
for every \( x \in X \) and hence \( \left| A_{U, \varepsilon} \right| \circ s \) is a lift of \( \mathcal{A} \), i.e. \( \left| \hat{\phi} \right| \circ \left| A_{U, \varepsilon} \right| \circ s = \mathcal{A} \).
2.2. **Theorem.** Suppose that $X$ is paracompact. The homotopy class of the lift

$$|\mathbb{A}_{U, \varepsilon}| \circ s : X \to \hat{E}$$

does not depend on the choices involved.

**Proof.** Since $s$ is a homotopy inverse of $|pr|$, the homotopy class of $|\mathbb{A}_{U, \varepsilon}| \circ s$ does not depend on the choice of $s$. Suppose that $V_b, b \in \Sigma'$ is another open covering of $X$ and that the numbers $\delta_b, b \in \Sigma'$ are positive and such that $(A_z, \delta_b)$ is an enhanced operator for every $z \in V_b$. Let us choose a common refinement of the coverings $U_a, a \in \Sigma$ and $V_c, c \in \Sigma'$, i.e. a covering $W_i, i \in \Xi$ such that for every $i \in \Xi$

$$W_i \subset U_{a(i)} \quad \text{and} \quad W_i \subset V_{b(i)},$$

where $a : \Xi \to \Sigma$ and $b : \Xi \to \Sigma'$ are some maps. The rules

$$(x, \sigma) \mapsto (x, a(\sigma)) \quad \text{and} \quad (x, \sigma) \mapsto (x, b(\sigma))$$

define functors

$$a : X_W \to X_U \quad \text{and} \quad b : X_W \to X_V.$$

Clearly, both compositions

$$X_W \xrightarrow{a} X_U \xrightarrow{pr} X \quad \text{and} \quad X_W \xrightarrow{b} X_V \xrightarrow{pr} X$$

are equal to $pr : X_W \to X$ and if $s$ is a homotopy inverse of $|pr| : |X_W| \to |X|$, then $|a| \circ s$ and $|b| \circ s$ are homotopy inverses of, respectively, the maps

$$|pr| : |X_U| \to |X| \quad \text{and} \quad |pr| : |X_V| \to |X|.$$  

For each $i \in \Xi$ let

$$\gamma_i = \min \{ \varepsilon_{a(i)}, \delta_{b(i)} \}.$$  

Then for every $z \in W_i$ the pair $(A_z, \gamma_i)$ is an enhanced operator and

$$(A_z, \gamma_i) \leq (A_z, \varepsilon_{a(i)}), (A_z, \delta_{b(i)})$$

Similar inequalities hold for every $\sigma \in \Xi^{\text{fin}}$ in the role of $i$. It follows that there are canonical natural transformations

$$\mathbb{A}_{W, \gamma} \to \mathbb{A}_{U, \varepsilon} \circ a \quad \text{and} \quad \mathbb{A}_{W, \gamma} \to \mathbb{A}_{V, \delta} \circ b.$$
of functors $X_W \to \hat{E}$. Therefore the map

$$|A_{W,\gamma}| : |X_W| \to \hat{E}$$

is homotopic to each of the maps $|A_{U,\epsilon} \circ a|$ and $|A_{V,\delta} \circ b|$. Therefore the diagram

\[
\begin{array}{ccc}
X_U & \xrightarrow{A_{U,\epsilon}} & \hat{E}, \\
\downarrow{a} & & \\
X_W & \xrightarrow{A_{W,\gamma}} & \hat{E}, \\
\downarrow{b} & & \\
X_V & \xrightarrow{A_{V,\delta}} & \\
\end{array}
\]

turns into a homotopy commutative diagram after passing to classifying spaces. By taking the maps $|a| \circ s$ and $|b| \circ s$ as homotopy inverses, we see the lifts resulting from $A_{U,\epsilon}$ and from $A_{V,\delta}$ are homotopic to the lift resulting from $A_{W,\gamma}$. The theorem follows. ■

**Analytical index.** Now we can define the *analytical index* of the family $A_x, x \in X$ as the homotopy class of the *index map*, defined as the composition

$$X \xrightarrow{s} X_U \xrightarrow{A_{U,\epsilon}} |\hat{E}| \xrightarrow{|\hat{\pi}|} |\hat{\mathcal{P}}|.$$

We say that the analytical index *vanishes* if the index map is homotopic to a constant map.

Of course, the homotopy class of the index map carries the same information as the homotopy class of the map $|A_{U,\epsilon}| \circ s$ and even as the homotopy class of the map $A : X \to \hat{\mathcal{P}}$ defining the family. The point of replacing $A$ and $|A_{U,\epsilon}| \circ s$ by the composition

$$|\hat{\pi}| \circ |A_{U,\epsilon}| \circ s$$

is that the category $\mathcal{P}$ and its classifying space $|\mathcal{P}|$ are defined in terms of only finitely dimensional subspaces of $H$, in contrast with $\hat{E}$ and $|\hat{E}|$. This allows to define the index map under much weaker assumptions about the family $A_x, x \in X$ than the continuity of the corresponding map $A : X \to \hat{\mathcal{P}}$.

One can go one step further and use the canonical homotopy equivalence between $|\mathcal{P}|$ and $|Q|$. It leads to a well defined homotopy class of maps $X \to |Q|$, which can be also considered as the *analytical index* of the family $A_x, x \in X$. Since $Q$ and $|Q|$ are defined in terms of only the finitely dimensional linear algebra and involve no Hilbert spaces, this version has a definite conceptual advantage. At the same time the category $\mathcal{P}$ and the space $|\mathcal{P}|$ are closer to operators in $H$ and by this reason are easier to work with.
Fredholm families. We will say that a family $A_x, x \in X$ of self-adjoint operators $H \rightarrow H$ is a Fredholm family if all operators $A_x$ are Fredholm and for every $x \in X$ there exists $\varepsilon = \varepsilon_x > 0$ and a neighborhood $U_x$ of $x$ with the following properties. First, the pair $(A_y, \varepsilon)$ should be an enhanced operator for every $y \in U_x$. Second, the subspaces $V_y = \text{Im} P_{[-\varepsilon, \varepsilon]}(A_y)$ should depend continuously on $y \in U_x$, and, finally, the operators $V_y \rightarrow V_y$ induced by $A_y$ should also depend continuously on $y \in U_x$. The above construction of the index maps applies without any changes to arbitrary Fredholm families.

We say that families $A_x, x \in X$ and $B_x, x \in X$ are Fredholm homotopic if there exists a Fredholm family $H_{x, u}, (x, u) \in X \times [0, 1]$ such that $H_{x, 0} = A_x$ and $H_{x, 1} = B_x$ for every $x \in X$. Clearly, Fredholm homotopic families have the same analytical index, and if a family is Fredholm homotopic to a constant one, then its analytical index vanishes.

Topological categories related to Hilbert bundles. Let $\mathbb{H}$ be a locally trivial Hilbert bundle with separable fibers over $X$. It is convenient to think that $\mathbb{H}$ is a family $H_x, x \in X$ of Hilbert spaces parameterized by $X$. We are interested in families of self-adjoint Fredholm operators $A_x : H_x \rightarrow H_x$ parameterized by $x \in X$.

In this context finitely dimensional subspaces of the fixed Hilbert space $H$ should be replaced by finitely dimensional subspaces of the fibers $H_x, x \in X$. The set $\hat{S}(\mathbb{H})$ of such subspaces has a natural topology. A local trivialization of $\mathbb{H}$ over an open set $U \subset X$ allows to identify $H_x$ with $x \in U$ with $H$, and hence to identify subspaces contained in these fibers with pairs $(x, V)$, where $x \in U$ and $V$ is a finitely dimensional subspace of $H_x$. This leads to a topology on the set of subspaces of fibers $H_x$ with $x \in U$. Since these subspaces are finitely dimensional, this topology does not depend on the choice of trivialization. Therefore these topologies define a topology on $\hat{S}(\mathbb{H})$.

One can turn $\hat{S}(\mathbb{H})$ into a topological category as follows. We take $\hat{S}(\mathbb{H})$ as the space of objects. Morphisms $V \rightarrow V'$ exists only if $V, V'$ are contained in the same fiber $H_x$, and in this case morphisms are defined exactly as morphisms of $\hat{S}$ with $H$ replaced by $H_x$. If we turn $X$ into topological category having only the identity morphisms, then there is a functor $\eta : \hat{S}(\mathbb{H}) \rightarrow X$ assigning to an object $V$ the point $x \in X$ such that $V \subset H_x$.

There is also an analogue $Q(\mathbb{H})$ of $Q$. The objects of $Q(\mathbb{H})$ are pairs $(x, V)$ such that $x \in X$ and $V$ is a finitely dimensional Hilbert space. The topology on the set of objects is defined by the topology of $X$ and the discrete topology on the set of spaces. Morphisms $(x, V) \rightarrow (x', V')$ exist only if $x = x'$, and in this case they are the same as morphisms $V \rightarrow V'$ in the category $Q$. The topology on the set of morphisms is defined by the topology of $X$ and the topology on the set of morphisms of $Q$. Clearly, $Q(\mathbb{H})$ depends only on $X$ and not on the bundle $\mathbb{H}$. There is a functor $\eta : Q(\mathbb{H}) \rightarrow X$ assigning to
The point $x$, and a functor $Q(\mathbb{H}) \to Q$. These functors lead to a homeomorphism $|Q(\mathbb{H})| \to X \times |Q|$ (this depends on the assumption that $X$ is compactly generated). There is also an intermediate category $Q/\mathbb{H}$ and functors $\hat{\mathcal{S}}(\mathbb{H}) \leftarrow Q/\mathbb{H} \to Q(\mathbb{H})$ relating $\hat{\mathcal{S}}(\mathbb{H})$ with $Q(\mathbb{H})$. The geometric realizations

$$|\hat{\mathcal{S}}(\mathbb{H})| \leftarrow |Q/\mathbb{H}| \to |Q(\mathbb{H})|$$

of these functors are homotopy equivalences. See [I2], Theorem 17.1. The obvious advantage of the classifying space $|Q(\mathbb{H})|$ is its independence on the bundle $\mathbb{H}$. But, as we will see, the classifying space $|\hat{\mathcal{S}}(\mathbb{H})|$ does not depend on $\mathbb{H}$ in any essential way either.

**Fredholm families of operators in Hilbert bundles.** The definitions of *Fredholm families* of operators and of *Fredholm homotopies* apply, with obvious modifications, to families of operators $A_x : H_x \to H_x$, $x \in X$. But the definition of the analytical index should be modified. In this situation there is no good analogue of the space $\hat{\mathcal{S}}$ and a family cannot be considered as a continuous map to some space. There is no good analogue of the category $\hat{\mathcal{E}}$ either. But the construction of the index maps can be easily modified to get a map $X \to |\hat{\mathcal{S}}(\mathbb{H})|$ well defined up to homotopy, and hence a homotopy class $X \to |Q(\mathbb{H})|$. The first steps are the same as for a fixed Hilbert space. Let $A_x : H_x \to H_x$, $x \in X$ be a Fredholm family. Then there exists an open covering $U_a$, $a \in \Sigma$ of $X$ and numbers $\varepsilon_a > 0$, $a \in \Sigma$ such that $(A_z, \varepsilon_a)$ is an enhanced operator in $H_z$ if $z \in U_a$. As before, for every $\sigma \in \Sigma^{\text{fin}}$ let $\varepsilon_\sigma = \min_{a \in \sigma} \varepsilon_a$. Then the pair $(A_z, \varepsilon_\tau)$ is an enhanced operator for every $z \in U_\tau$, and $\varepsilon_\tau \leq \varepsilon_\sigma$ if $\tau \supset \sigma$. Let us construct a continuous functor

$$A_{U, \varepsilon} : X_U \to \hat{\mathcal{S}}(\mathbb{H}).$$

Suppose that $(z, \tau)$ is an object of $X_U$ and let $\varepsilon = \varepsilon_\tau$. The functor $A_{U, \varepsilon}$ takes $(z, \tau)$ to

$$\text{Im } P_{[-\varepsilon, \varepsilon]}(A_z) \in \text{Ob } \hat{\mathcal{S}}(\mathbb{H}).$$

Suppose now that $(z, \tau) \to (z, \sigma)$ is a morphism of $X_U$, and let $\varepsilon = \varepsilon_\tau$ and $\varepsilon' = \varepsilon_\sigma$. The functor $A_{U, \varepsilon}$ takes this morphism to the morphism

$$\text{Im } P_{[-\varepsilon, \varepsilon]}(A_z) \to \text{Im } P_{[-\varepsilon', \varepsilon']} (A_z)$$

of the category $\hat{\mathcal{S}}(\mathbb{H})$ corresponding to the decomposition

$$\text{Im } P_{[-\varepsilon', \varepsilon']} (A_z) = \left( \text{Im } P_{[-\varepsilon', [-\varepsilon, \varepsilon]} (A_z) \right) \oplus \left( \text{Im } P_{[-\varepsilon, \varepsilon]} (A_z) \right) \oplus \left( \text{Im } P_{[\varepsilon, \varepsilon]} (A_z) \right).$$

Clearly, $A_{U, \varepsilon}$ is indeed a functor $X_U \to \hat{\mathcal{S}}(\mathbb{H})$. Since the subspaces

$$V_z = \text{Im } P_{[-\varepsilon, \varepsilon]}(A_z)$$

The obvious advantage of the classifying space $|Q(\mathbb{H})|$ is its independence on the bundle $\mathbb{H}$. But, as we will see, the classifying space $|\hat{\mathcal{S}}(\mathbb{H})|$ does not depend on $\mathbb{H}$ in any essential way either.
depend continuously on \( z \in U_\tau \), it is continuous on objects, and since the induced operators \( V_z \to V_z \) also depend continuously on \( z \), it is continuous on morphisms. By passing to the geometric realizations the index functor \( \mathcal{A}_{U, \varepsilon} \) defines a continuous map

\[
|\mathcal{A}_{U, \varepsilon}| : |X_U| \to |\hat{\mathcal{P}}(\mathbb{H})|.
\]

2.3. Theorem. Suppose that \( X \) is paracompact and \( s : X \to |X_U| \) is a homotopy inverse of \( |\operatorname{pr}| \) as in Theorem 2.1. Then the homotopy class of \( |\mathcal{A}_{U, \varepsilon}| \circ s : X \to |\hat{\mathcal{P}}(\mathbb{H})| \) does not depend on the choices involved.

Proof. The proof is completely similar to the proof of Theorem 2.2. ■

Analytical index of families of operators in Hilbert bundles. Suppose that \( X \) is paracompact. Then any map of the form \( |\mathcal{A}_{U, \varepsilon}| \circ s : X \to |\hat{\mathcal{P}}(\mathbb{H})| \) may be called an index map of the family \( A_x, x \in X \). By Theorem 2.3 the homotopy class of an index map is independent from choices involved in its construction, and we can define the index of the family \( A_x, x \in X \) as the homotopy class of any of its index maps.

A drawback of this definition is that such index maps are analogues not of the index maps \( X \to |\hat{\mathcal{P}}| \) of families of operators in a fixed Hilbert spaces, but of maps \( X \to X \times |\hat{\mathcal{P}}| \) having the identity map \( X \to X \) as the first component and an index map as the second.

A simple way to deal with this issue is to use the homotopy equivalence between \( |\hat{\mathcal{P}}(\mathbb{H})| \) and \( |Q(\mathbb{H})| \), which is well defined up to homotopy. By composing an index map in the above sense with this homotopy equivalence we get a well defined homotopy class of maps \( X \to |Q(\mathbb{H})| \). Since \( |Q(\mathbb{H})| \) is canonically homeomorphic to \( X \times |Q| \), we can consider it as a homotopy class of maps \( X \to X \times |Q| \). The construction of the homotopy equivalence between \( |\hat{\mathcal{P}}(\mathbb{H})| \) and \( |Q(\mathbb{H})| \) shows that the first component of the homotopy class \( X \to X \times |Q| \) is the homotopy class of the identity map \( X \to X \). Therefore the second component \( X \to |Q| \) carries the same information as the homotopy classes

\[
X \to |Q(\mathbb{H})| \quad \text{and} \quad X \to |\hat{\mathcal{P}}|.
\]

This suggests to define the index of the family \( A_x, x \in X \) as the homotopy class \( X \to |Q| \) and to call index maps the maps in this homotopy class. This approach has the advantage of the target space \( |Q| \) being independent from the bundle \( \mathbb{H} \).

Nevertheless, the homotopy class of the index maps \( X \to |\hat{\mathcal{P}}(\mathbb{H})| \) carries the same information as the homotopy class \( X \to |Q| \) and sometimes is easier to work with. Suppose that \( \mathbb{H}, \mathbb{K} \) are two Hilbert bundles over \( X \). Then an isomorphism \( \mathbb{H} \to \mathbb{K} \) covering the identity \( X \to X \) induces an isomorphism of topological categories

\[
\hat{\mathcal{P}}(\mathbb{H}) \to \hat{\mathcal{P}}(\mathbb{K})
\]
and hence a homeomorphism $|\hat{S}(\mathbb{H})| \rightarrow |\hat{S}(\mathbb{K})|$. At the same time it is known that every Hilbert bundle is actually trivial, i.e. is isomorphic to the bundle $X \times H \rightarrow X$. See the discussion in Section 4. It follows that $|\hat{S}(\mathbb{H})|$ is homeomorphic to $X \times |\hat{S}|$. Moreover, the arguments proving the triviality show that any two homeomorphisms obtained in this way are homotopic (in fact, even isotopic). Therefore the index can be defined also as the second component $X \rightarrow |\hat{S}|$ of the homotopy class

$$X \rightarrow |\hat{S}(\mathbb{H})| \rightarrow X \times |\hat{S}|.$$ 

### 3. Analytical index of Fredholm families

**Topological categories related to Fredholm operators.** This section is devoted to families of general Fredholm operators. Let $\mathcal{F}$ be the space of Fredholm operators $H \rightarrow H$ with the norm topology. Our first goal is to define analogues of $\hat{E}$ and $\hat{S}$. Recall that the polar decomposition of an operator $B$ is the unique presentation $B = U|B|$, where

$$|B| = \sqrt{B^*B}$$

and $U$ is a partial isometry of $H$ with $\text{Ker} U = \text{Ker} B$ and $\text{Im} U$ is equal to the closure of $\text{Im} B$. If $B$ is Fredholm, then $\text{Im} B$ is automatically closed.

An enhanced Fredholm operator is a pair $(B, \varepsilon)$, where $B \in \mathcal{F}$ and $\varepsilon \in \mathbb{R}$ are such that $\varepsilon > 0$, the interval $[0, \varepsilon]$ is disjoint from the essential spectrum of $|B|$, and $\varepsilon \not\in \sigma(|B|)$. Let $\mathcal{E}$ be the space of enhanced Fredholm operators. The topology is defined by the topology of $\mathcal{F}$ and the discrete topology on $\mathbb{R}$. The space $\mathcal{E}$ is ordered by the relation

$$(B, \varepsilon) \leq (B', \varepsilon') \text{ if } B = B' \text{ and } \varepsilon \leq \varepsilon'.$$

This order allows to consider $\mathcal{E}$ as a topological category. The obvious forgetting functor $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ induces a homotopy equivalence $|\varphi|: |\mathcal{E}| \rightarrow |\mathcal{F}|$. See [I$_2$], Section 14.

The category $\mathcal{S}$ is an analogue of $\hat{S}$. Its objects are pairs $(E_1, E_2)$ such that $E_1, E_2$ are finitely dimensional subspaces of $H$, with morphisms $(E_1, E_2) \rightarrow (E'_1, E'_2)$ being pairs of subspaces $(F_1, F_2)$ such that

$$E_1 \oplus F_1 = E'_1 \text{ and } E_2 \oplus F_2 = E'_2$$

together with an isometry $f: F_1 \rightarrow F_2$. The composition is defined by taking the direct sums of the corresponding subspaces $F_1, F_2$ and of the isometries. The category $\mathcal{S}$ is a topological category in an obvious way. Like $\hat{S}$, it is not associated with a partial order. Let $\mathcal{S}_0$ be the full subcategory of $\mathcal{S}$ defined by the condition $\dim E_1 = \dim E_2$. 

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There is functor $\pi : \mathcal{E} \rightarrow \mathcal{S}$ taking an object $(B, \varepsilon)$ of $\mathcal{E}$ to the pair $(E_1, E_2)$, where

$$E_1 = P_{[0, \varepsilon]}(|B|), \quad E_2 = P_{[0, \varepsilon]}(|B^*|).$$

The functor $\pi$ takes a morphism $(B, \varepsilon) \rightarrow (B, \varepsilon')$, where $\varepsilon \leq \varepsilon'$, to the morphism $\pi(B, \varepsilon) \rightarrow \pi(B, \varepsilon')$ of $\mathcal{S}$ defined by the pair of subspaces

$$(F_1, F_2) = \left( P_{[\varepsilon, \varepsilon']}(|B|), P_{[\varepsilon, \varepsilon']}(|B^*|) \right)$$

together with the isometry

$$P_{[\varepsilon, \varepsilon']}(|B|) \rightarrow P_{[\varepsilon, \varepsilon']}(|B^*|)$$

induced by $U$, where $B = U|B|$ is the polar decomposition of $B$. A routine check shows that these rules define a functor $\pi : \mathcal{E} \rightarrow \mathcal{S}$. The geometric realization $|\pi| : |\mathcal{E}| \rightarrow |\mathcal{S}|$ is a homotopy equivalence, as it easily follows from the results of [I2], Sections 14 and 15.

There is also an analogue $\mathcal{S}$ of the category $Q$. The objects of $\mathcal{S}$ are pairs $(V_1, V_2)$ of finitely dimensional Hilbert spaces, with morphisms $(V_1, V_2) \rightarrow (W_1, W_2)$ being triples $(i_1, i_2, g)$, where $i_1 : V_1 \rightarrow W_1$ and $i_2 : V_2 \rightarrow W_2$ are isometric embeddings and

$$g : W_1 \ominus i_1(V_1) \rightarrow W_2 \ominus i_2(V_2)$$

is an isometry. The composition is defined in an obvious way and amounts to taking the direct sum of the isometric isomorphisms $g$. As in the case of self-adjoint operators, there is an intermediate category $\mathcal{S}/H$ and continuous functors $\mathcal{S} \leftarrow \mathcal{S}/H \rightarrow \mathcal{S}$ inducing homotopy equivalences $|\mathcal{S}| \leftarrow |\mathcal{S}/H| \rightarrow |\mathcal{S}|$.

**Families of Fredholm operators.** In this section we will say that a family $B_x, x \in X$ of operators $H \rightarrow H$ is a *Fredholm family* if operators $B_x$ are Fredholm (but are not assumed to be self-adjoint) and for every $x \in X$ there exists $\varepsilon = \varepsilon_x > 0$ and a neighborhood $U_x$ of $x$ with the following properties. First, the pair $(B_y, \varepsilon)$ should be an enhanced Fredholm operator for every $y \in U_x$. Second, the subspaces

$$E_1(y) = \text{Im} P_{[0, \varepsilon]}(|B_y|), \quad E_2(y) = \text{Im} P_{[0, \varepsilon]}(|B^*_y|)$$

should depend continuously on $y \in U_x$, and, finally, the operators

$$E_1(y) \rightarrow E_2(y)$$

induced by $B_y$ should also depend continuously on $y \in U_x$. 18
**Index of Fredholm families.** Let $B_x, x \in X$ be a family of Fredholm operators, and let $\mathbb{B}: X \rightarrow \mathcal{F}$ be the corresponding map. As in the self-adjoint case, the map $\mathbb{B}$ can be “lifted” to a map to $|\mathcal{E}|$. In more details, there exists an open covering $U_a, a \in \Sigma$ of $X$ and positive numbers $\varepsilon_a, a \in \Sigma$ such that $(B_z, \varepsilon_a)$ is an enhanced Fredholm operator for every $z \in U_a$. The families $U_a, a \in \Sigma$ and $\varepsilon_a, a \in \Sigma$ define a continuous functor

$$\mathbb{B}_{U, \varepsilon} : X_U \rightarrow \mathcal{E}$$

exactly as in Section 2. The geometric realization of $\mathbb{B}_{U, \varepsilon}$ is a continuous map

$$|\mathbb{B}_{U, \varepsilon}| : |X_U| \rightarrow |\mathcal{E}|.$$

If $X$ is paracompact, then one can take the homotopy inverse $s : X \rightarrow |X_U|$ of the canonical map $|\text{pr}| : |X_U| \rightarrow X$ and consider the composition

$$|\mathbb{B}_{U, \varepsilon}| \circ s : X \rightarrow |\mathcal{E}|.$$

Its homotopy class does not depend on the choices involved. The proof of this fact is completely similar to the proof of Theorem 2.2. Now one can define the *analytical index* of the family $B_x, x \in X$ as the homotopy class of the *index map*, defined as the composition

$$X \xrightarrow{s} X_U \xrightarrow{\mathbb{B}_{U, \varepsilon}} |\mathcal{E}| \xrightarrow{|\pi|} |\mathcal{F}|.$$

As in the case of self-adjoint operators, one can go one step further and use the canonical homotopy equivalence between $|\mathcal{F}|$ and $|S|$. It leads to a homotopy class of maps $X \rightarrow |S|$, which can be also considered as the *analytical index* of the family $B_x, x \in X$.

**Topological categories related to Fredholm operators in Hilbert bundles.** As in Section 2, let $H_x, x \in X$, denoted also by $\mathbb{H}$, be a locally trivial Hilbert bundle with separable fibers over $X$. In this section we are interested in families of Fredholm operators $B_x : H_x \rightarrow H_x$ parameterized by $x \in X$ and not assumed to be self-adjoint.

Let $\mathcal{F}(\mathbb{H})$ be the following category. The objects of $\mathcal{F}(\mathbb{H})$ are pairs $(E_1, E_2)$ such that $E_1, E_2$ are finitely dimensional subspaces of the same fiber $H_x$ of $\mathbb{H}$. The topology on the set of objects is defined in the same way as for $\hat{\mathcal{F}}(\mathbb{H})$. Morphisms $(E_1, E_2) \rightarrow (E'_1, E'_2)$ exists only if the four involved subspaces are subspaces of the same fiber $H_x$, and in this case morphisms are defined exactly as morphisms of $\mathcal{F}$ with $H$ replaced by $H_x$. If we turn $X$ into topological category having only the identity morphisms, then there is a functor $\eta : \mathcal{F}(\mathbb{H}) \rightarrow X$ assigning to an object $(E_1, E_2)$ the point $x \in X$ such that $E_1, E_2 \subset H_x$.

There is also an analogue $S(\mathbb{H})$ of $S$. The objects of $S(\mathbb{H})$ are triples $(x, V_1, V_2)$ such that $x \in X$ and $(V_1, V_2)$ is an object of $S$. The topology on the set of such triples is defined by the topology of $X$ and the discrete topology on the set of spaces. Morphisms
Fredholm families of operators in Hilbert bundles. The definitions of Fredholm families of operators and of Fredholm homotopies apply, with obvious modifications, to families of operators $B_x : H_x \to H_x$, $x \in X$. But, as in the self-adjoint case, the definition of the analytical index should be modified. Let $B_x, x \in X$ be a family of Fredholm operators, and let $U_a, a \in \Sigma$ of $X$ and $\varepsilon_a, a \in \Sigma$ be as above. Then one can define a continuous functor

$$\mathbb{B}_{U, \varepsilon} : X_U \to \mathcal{S}(\mathbb{H})$$

as follows. Let $(z, \tau)$ be object of $X_U$ and $\varepsilon = \varepsilon_\tau$. The functor $\mathbb{B}_{U, \varepsilon}$ takes $(z, \tau)$ to

$$(E_1, E_2) = \left( \operatorname{Im} P_{[0, \varepsilon]}(|B|), \operatorname{Im} P_{[0, \varepsilon]}(|B^*|) \right),$$

and this rule naturally extends to morphisms. The functor $\mathbb{B}_{U, \varepsilon}$ leads to a continuous map

$$|\mathbb{B}_{U, \varepsilon}| : |X_U| \to |\mathcal{S}(\mathbb{H})|.$$
4. Hilbert bundles and strictly Fredholm families

Compactly generated spaces and the compact-open topology. Recall that we tacitly assume all topological spaces to be compactly generated. In this section this assumption will play a more prominent role than usual, and by this reason it will be sometimes stated explicitly. For two topological spaces $Y, Z$ we will denote by $\text{Map}(Y, Z)$ the space of continuous maps $Y \to Z$ with the compact-open topology. For a compactly generated space $X$ the continuity of a map $X \to \text{Map}(Y, Z)$ is equivalent to the continuity of the corresponding (adjoint) map $X \times Y \to Z$. This is one of the main reasons for working with compactly generated spaces and the compact-open topology in the context of Hilbert bundles.

4.1. Lemma. Suppose that $X$ is a compactly generated space. Let

$$f : X \to \text{Map}(Y, Z) \quad \text{and} \quad g : X \to \text{Map}(Z, W)$$

be continuous maps. Then the map

$$h : X \to \text{Map}(Y, Z)$$

defined by $h(x) = g(x) \circ f(x)$ is continuous.

Proof. The continuity of $f$ is equivalent to the continuity of the map $X \times Y \to X \times Z$ defined by $(x, y) \mapsto (x, f(x)(y))$, and similarly for $g$ and $h$. Therefore the continuity of the composition $X \times Y \to X \times Z \to X \times W$ implies the continuity of $h$. ■

Local trivializations of Hilbert bundles. Suppose that $\mathbb{K}$ is a locally trivial Hilbert bundle over a compactly generated topological space $Y$ with fibers isomorphic to a separable infinitely dimensional Hilbert space $K$. As before, we will treat $\mathbb{K}$ as a family $K_y, y \in Y$ of Hilbert spaces parameterized by $X$. A local trivialization of $\mathbb{K}$ can be considered as a family of Hilbert space isomorphisms $t_U(x) : K_x \to K$, where $x$ runs over an open subset $U \subset Y$. Two trivialization $t_U$ and $t_V$ are related by continuous transition map

$$(U \cap V) \times K \to (U \cap V) \times K$$
given by the formula

$$(x, v) \mapsto (x, t_U(x) \circ t_V(x)^{-1}(v)).$$

Since $Y$ is assumed to be compactly generated, the continuity of the above transition map is equivalent to the continuity of the transition function

$$s_{UV} : x \mapsto t_U(x) \circ t_V(x)^{-1}$$
from $U \cap V$ to the group of isometries $K \to K$ equipped with the compact-open topology. Since $s_{VU} = s_{UV}^{-1}$ is also continuous, the maps $s_{UV}$ will be continuous even if the group of isometries of $K$ is equipped with a stronger topology, namely with the topology induced from the product of the compact-open topologies by the map

$$g \mapsto (g, g^{-1}).$$

We will denote by $\mathcal{U}(K)$ the groups of isometries of $K$ with the latter topology. Actually, this topology coincides with the strong operator topology, but the author prefers to ignore this fact. We will reserve the notation $U(K)$ for the same group with the norm topology.

**Trivializations of Hilbert bundles.** Let $\mathcal{K}$ be a Hilbert bundle as above. A **trivialization** of the bundle $\mathcal{K}$ is an isomorphism between $\mathcal{K}$ and the trivial bundle $Y \times K \to Y$. A trivialization $t$ can be thought as a family of isometries $t_y : K_y \to K$, $y \in Y$ defining a homeomorphism between the total space $\bigcup_{y \in Y} K_y$ of the bundle $\mathcal{K}$ and $Y \times K$. Two trivializations $t, u$ differ by a family

$$t_y \circ u_y^{-1} : K \to K$$

of isometries. Somewhat disappointingly, the map $s : y \mapsto t_y \circ u_y^{-1}$ from $Y$ to the unitary group of $K$ is continuous only in a relatively weak sense. Assuming that $Y$ is compactly generated, $s$ is continuous as a map to $\mathcal{U}(K)$, but usually not as a map to $U(K)$.

In fact, Hilbert bundles with fibers isomorphic to $K$ over compactly generated spaces are bundles with the structure group $\mathcal{U}(K)$. The group $\mathcal{U}(K)$ is known to be contractible. This result is essentially due to Dixmier and Douady [DD], and was put in the present form by Atiyah and Segal [ASe]. The contractibility of $\mathcal{U}(K)$ implies that Hilbert bundles over paracompact compactly generated spaces are trivial. Moreover, given a bundle with the base $Y$ and a closed subset $Z \subset Y$, a trivialization over $Z$ can be extended to a trivialization over $Y$. This implies that any two trivializations are homotopic in an obvious sense.

**Adapted pairs and local trivializations.** As in Section 2, let $\mathcal{H}$ be a locally trivial Hilbert bundle with separable fibers over $X$, thought as a family $H_x, x \in X$ of Hilbert spaces parameterized by $X$. Let $A_x : H_x \to H_x, x \in X$ be a family of self-adjoint Fredholm operators, also denoted by $A$. Let us slightly rephrase the definition of Fredholm families.

Suppose that $U \subset X$ and $\varepsilon > 0$. We will say that the pair $(U, \varepsilon)$ is **adapted to the family** $A$, or simply **adapted**, if for every $x \in U$ the image $\text{Im} P_{[-\varepsilon, \varepsilon]}(A_x)$ is finitely dimensional, $-\varepsilon, \varepsilon \not\in \sigma(A_x)$, the family of the images

$$V_x = \text{Im} P_{[-\varepsilon, \varepsilon]}(A_x), x \in U$$

is continuous, as also the family of operators $V_x \to V_x$ induced by $A_x$. Clearly, $A$ is a
Fredholm family if and only if there exists a family of adapted pairs \((U, \varepsilon)\) such that the sets \(U\) are open and cover \(X\). We will call such a family of pairs \((U, \varepsilon)\) an atlas for \(\mathbb{A}\).

Suppose that the pair \((U, \varepsilon)\) is adapted to \(\mathbb{A}\). A local trivialization \(t_U\) of \(H\) defined over \(U\) is said to be strictly adapted to \((U, \varepsilon)\) (and the family \(\mathbb{A}\)) if the family of projections

\[ t_U(x) \circ P_{\geq \varepsilon}(A_x) \circ t_U(x)^{-1} = P_{\geq \varepsilon}(t_U(x) \circ A_x \circ t_U(x)^{-1}) \]

is norm continuous. Equivalently, the family of subspaces

\[ H_{\geq \varepsilon}^U(x) = t_U(x)\left(\text{Im } P_{\geq \varepsilon}(A_x)\right) \]

is norm continuous. The next lemma shows that the property of being adapted to \((U, \varepsilon)\) is independent of the choice of \(\varepsilon > 0\) such that \((U, \varepsilon)\) is adapted. We will say that a local trivialization is strictly adapted to \(U\) if it is strictly adapted to \((U, \varepsilon)\) for some \(\varepsilon > 0\).

4.2. Lemma. Let \(U \subset X\) and \((U, \varepsilon), (U, \delta)\) are pairs adapted to \(\mathbb{A}\). If a local trivialization \(t_U\) defined over \(U\) is strictly adapted to \((U, \varepsilon)\), then \(t_U\) is strictly adapted to \((U, \delta)\).

Proof. Suppose that \(\delta < \varepsilon\). Then

\[ \text{Im } P_{\geq \varepsilon}(A_x) = \text{Im } P_{\geq \delta}(A_x) \oplus \text{Im } P_{[\delta, \varepsilon]}(A_x) \]

for every \(x \in U\). Since \(\varepsilon, \delta \not\in \sigma(A_x)\), the family of subspaces

\[ t_U(x)\left(\text{Im } P_{[\delta, \varepsilon]}(A_x)\right), \ x \in U \]

is norm continuous. Hence the continuity of the family \(H_{\geq \varepsilon}^U(x)\) is equivalent to the continuity of the family \(H_{\geq \delta}^U(x)\). The case of \(\varepsilon < \delta\) is similar. This proves the lemma. \(\square\)

4.3. Corollary. Let \((U, \varepsilon)\) and \((V, \delta)\) be adapted pairs and \(t_U\) and \(t_V\) be local trivializations strictly adapted to \((U, \varepsilon)\) and \((V, \delta)\) respectively. Then the pair \((U \cap V, \varepsilon)\) is adapted and the restrictions of \(t_U\) and \(t_V\) to \(U \cap V \) are strictly adapted to \((U \cap V, \varepsilon)\).

Proof. The first claim is trivial, and the second one follows from Lemma 4.2. \(\square\)

Strictly Fredholm families and adapted trivializations. We will say that \(\mathbb{A}\) is a strictly Fredholm family if there exists an atlas for \(\mathbb{A}\) such that for each pair \((U, \varepsilon)\) from this atlas there exists a local trivialization \(t_U\) strictly adapted to \((U, \varepsilon)\), or simply to \(U\), and the family \(\mathbb{A}\). We will say that a trivialization \(t\) of the bundle \(H\) is strictly adapted to the family \(\mathbb{A}\) if there exists an atlas for \(\mathbb{A}\) such that for every pair \((U, \varepsilon)\) from this atlas the restriction of \(t\) to \(U\) is strictly adapted to \((U, \varepsilon)\) and \(\mathbb{A}\). Clearly, if there exists a trivialization strictly adapted to \(\mathbb{A}\), then \(\mathbb{A}\) is a strictly Fredholm family.
4.4. Lemma. If $t$ is a trivialization strictly adapted to $\mathcal{A}$ and $(U, \varepsilon)$ is a pair adapted to $\mathcal{A}$, then the restriction of $t$ to $U$ is strictly adapted to $\mathcal{A}$.

**Proof.** Since $t$ is strictly adapted to $\mathcal{A}$, for every $x \in X$ there exists an adapted pair $(V, \delta)$ such that $V$ is open, $x \in V$, and the restriction $t_V$ of $t$ to $V$ is strictly adapted to $(V, \delta)$. By Corollary 4.3 the restriction of $t$ to $U \cap V$ is strictly adapted to $(U \cap V, \varepsilon)$. By the definition, the property of being strictly adapted to $(U, \varepsilon)$ is a continuity property. Therefore it holds if it holds in a neighborhood of each point $x \in V$. The lemma follows. ■

4.5. Theorem. If $\mathcal{A}$ is a strictly Fredholm family and $X$ is a triangulated space, then there exists a trivialization of the bundle $\mathbb{H}$ strictly adapted to $\mathcal{A}$.

**Proof.** Let us choose an atlas of pairs $(U, \varepsilon)$ adapted to $\mathcal{A}$ and local trivializations $t_U$ strictly adapted to $(U, \varepsilon)$ and $\mathcal{A}$. Replacing the given triangulation of $X$ by a subdivision, if necessary, we can assume that every simplex $\sigma$ of this triangulation is contained in the set $U$ for some adapted pair $(U, \varepsilon)$ from our atlas.

Let us begin by choosing isomorphisms $t(v) : H_v \rightarrow H$ for the vertices $v$ of the triangulation. Such isomorphisms define a trivialization of the restriction of the bundle to the 0-skeleton $\text{Sk}_0 X$, which is obviously strictly adapted to the restriction of the family $\mathcal{A}_x$, $x \in X$ to $\text{Sk}_0 X$, i.e. to the family $\mathcal{A}_x$, $x \in \text{Sk}_0 X$.

Suppose that we already found a trivialization $t$ of the restriction of the bundle $\mathbb{H}$ to the $n$-skeleton $\text{Sk}_n X$ which is strictly adapted to the family $\mathcal{A}_x$, $x \in \text{Sk}_n X$ for some $n \geq 0$. Let $\sigma$ be an $(n+1)$-simplex of the triangulation, and let $\partial \sigma$ be its boundary. By our assumptions, $\sigma \subset U$ for an adapted pair $(U, \varepsilon)$. Recall that $t_U$ is strictly adapted to $(U, \varepsilon)$. Let $U_n = \text{Sk}_n X \cap U$. Then $\partial \sigma \subset U_n$, the pair $(U_n, \varepsilon)$ is adapted to $\mathcal{A}_x$, $x \in \text{Sk}_n X$, and the restriction of $t$ to $U_n$ is strictly adapted to $(U_n, \varepsilon)$. Let

$$H_{U_\varepsilon}(x) = H \oplus H_{\geq \varepsilon}(x)$$

for $x \in U$, and for $x \in U_n$ let

$$H_{\geq \varepsilon}(x) = t(x)(\text{Im} P_{\geq \varepsilon}(\mathcal{A}_x)) \quad \text{and} \quad H_{< \varepsilon}(x) = H \oplus H_{\geq \varepsilon}(x).$$

Recall that a polarization of a Hilbert space $K$ is an orthogonal decomposition of the form $K = K_- \oplus K_+$ such that both subspaces $K_-$ and $K_+$ are infinitely dimensional. For each $x \in U_n$ we have two polarizations of $H$, namely

$$H = H_{U_\varepsilon}(x) \oplus H_{\geq \varepsilon}(x) \quad \text{and} \quad H = H_{< \varepsilon}(x) \oplus H_{\geq \varepsilon}(x).$$

Since these polarization arise from strictly adapted trivializations, these polarizations continuously depend on $x \in U_n$ in the norm topology. The contractibility of groups $U(K)$...
with the norm topology implies that the space of polarizations is contractible, and hence the first family is homotopic to the second one by a norm continuous homotopy.

The first family is defined also for \( x \in U \), and we can extend this homotopy to \( U \). By taking the composition of this homotopy with \( t_U \) we get, in particular, a new local trivialization \( t'_U \) strictly adapted to \( (U, \varepsilon) \) and such that for every \( x \in U_n \) the images of

\[
\text{Im } P_{\geq \varepsilon}(A_x)
\]

under \( t'_U(x) \) and \( t(x) \) are equal. The family of these images is a Hilbert subbundle of the trivial bundle \( U_n \times H \). The contractibility of the unitary groups \( \mathcal{U}(K) \) implies first that this subbundle is trivial as a Hilbert bundle, and then that the family \( t'_U(x), x \in U_n \) is homotopic to the family \( t_U(x), x \in U_n \) in the class of families with the same images of subspaces \( \text{Im } P_{\geq \varepsilon}(A_x) \). By extending this homotopy to \( x \in U \) we get a trivialization \( t''_U(x), x \in U \) strictly adapted to \( (U, \varepsilon) \) and such that \( t''_U(x) = t(x) \) for \( x \in U_n \).

Let us extend \( t \) from \( Sk_n X \) to \( Sk_n X \cup \sigma \) by \( t''_U \) and denote the resulting map again by \( t \). Then the restriction of \( t \) to \( U_n \cup \sigma \) is strictly adapted to \((U_n \cup \sigma, \varepsilon)\). Lemma 4.4 implies that the extended \( t \) is strictly adapted. It follows that we can extend the trivialization \( t \) to \( \sigma \). By doing this for all \((n + 1)\)-simplices simultaneously, we can extend \( t \) to the next skeleton \( Sk_{n+1} X \). By continuing in this way we will get a strictly adapted trivialization. \( \blacksquare \)

4.6. Theorem. If \( \mathbb{A} \) is a strictly Fredholm family and \( X \) is a paracompact space, then there exists a trivialization of the bundle \( \mathbb{H} \) strictly adapted to \( \mathbb{A} \).

Proof. Let us choose an atlas \((U_a, \varepsilon_a), a \in \Sigma\) adapted to \( \mathbb{A} \) and and local trivializations \( t_a \) strictly adapted to \((U_a, \varepsilon_a)\) and \( \mathbb{A} \). Let \( \Sigma^\text{fin} \) and \( U_\sigma, \varepsilon_\sigma \) for \( \sigma \in \Sigma^\text{fin} \) be defined as in Section 2, and let \( X_U \) be the topological category from Section 2. The space of objects of \( X_U \) is the disjoint union of subspaces \( U_\sigma \), and the category \( X_U \) is defined by an order \( \leq \) on \( \text{Ob } X_U \). Clearly, the set pairs \((u, u)\), where \( u \in \text{Ob } X_U \), is the union of several components of the set of pairs \((u, v)\) of comparable objects of \( X_U \), i.e. such that either \( u \leq v \) or \( v \leq u \). In other words, the subspace of the identity morphisms is open and closed in the space of all morphisms. In the terminology of \[I_2\] this means that \( \text{Ob } X_U \) is a partially ordered space with \textit{free equalities}, and hence \( X_U \) can be treated as a topological simplicial complex. In particular, the standard geometric realization \( |X_U| \) is equal to the “naive” geometric realization \( |X_U| \). We refer to \[I_2\], Section 5, for the details.

Let \( \mathbb{H}^U \) be the bundle induced from \( \mathbb{H} \) by \(|\text{pr}| : |X_U| \rightarrow X\). The family \( \mathbb{A} \) defines a family \( \mathbb{A}^U \) of operators in the bundle \( \mathbb{H}^U \). If \( s \) is a homotopy inverse of \(|\text{pr}|\) as in Theorem 2.1, then \( \mathbb{H} \) and \( \mathbb{A} \) are equal to the bundle and the family of operators induced by \( s \) from \( \mathbb{H}^U \) and \( \mathbb{A}^U \) respectively. Therefore it is sufficient to prove that there exists a trivialization of \( \mathbb{H}^U \) strictly adapted to \( \mathbb{A}^U \). As in the proof of Theorem 4.5, we will prove this using an induction by skeletons. If \( \sigma \in \Sigma^\text{fin} \) and \( a \in \sigma \), then the restriction \( t_a \) of \( t_a \) to \( U_\sigma \subset U_a \)
is strictly adapted to $A$. Hence the trivializations $t_\sigma$ define a trivialization of the restriction of $H^U$ to the 0th skeleton $[Sk_0 X_U]$ strictly adapted to the restriction of $A^U$.

Suppose that we already constructed a trivialization $t$ of the restriction of $H^U$ to the $n$-skeleton $[Sk_n X_U]$ strictly adapted to the family $A^U$, $u \in [Sk_n X_U]$ for some $n \geq 0$. An $(n+1)$-simplex of $X_U$ is defined by a point $x \in X$ and a strictly decreasing sequence

$$\sigma(n+1) \supseteq \sigma(n) \supseteq \ldots \supseteq \sigma(0)$$

of elements of $\Sigma^{\text{fin}}$. Therefore the space of $(n+1)$-simplices of $X_U$ can be identified with the disjoint union of the sets $U_{\sigma(n+1)}$ over all such sequences. In particular, every component of the space of $(n+1)$-simplices can be identified with $U_{\sigma}$ for some $\sigma \in \Sigma^{\text{fin}}$.

Let us consider such component $U_{\sigma}$ and the canonical map

$$\Delta^{n+1} \times U_{\sigma} \to [X_U] = |X_U|.$$

The bundle $H^U$ and the family $A^U$ induce a bundle $H^\sigma$ and a family of operators $A^\sigma$ over $\Delta^{n+1} \times U_{\sigma}$, and the already constructed trivialization $t$ defines a trivialization $t'$ of $H^\sigma$ over $\partial \Delta^{n+1} \times U_{\sigma}$ strictly adapted $A^\sigma$. Since the map $|pr|$ collapses each simplex to a point, the fibers of $H^U$ over a simplex can be treated as equal, as also the operators $A^U$. Hence the trivialization $t_\sigma$ defines a trivialization of $H^\sigma$ strictly adapted to $A^\sigma$.

Let us use this trivialization to identify $H^\sigma$ with the trivial bundle over $\Delta^{n+1} \times U_{\sigma}$. Now we can use the contractibility of the space of polarizations and the contractibility of the unitary groups $U(K)$ as in the proof of Theorem 4.5 to extend $t'$ to a trivialization $t''$ of $H^\sigma$ over $\Delta^{n+1} \times U_{\sigma}$ strictly adapted to $A^\sigma$. Since we are dealing with a topological simplicial complex, the map (1) is injective and we can extend $t$ to the image of the map (1) by using $t''$. By doing this simultaneously for every component of the space of $(n+1)$-simplices we can extend $t$ to next skeleton $[Sk_{n+1} X_U]$. The topology of $|X_U| = |X_U|$ is, by the definition, the direct limit of the topologies of the skeletons. Therefore, by continuing in this way we will get a trivialization of $H^U$ strictly adapted to $A^U$.

\[\square\]

**4.7. Theorem.** Under the assumptions of Theorem 4.5 or 4.6, suppose that $Y$ is a subcomplex or closed subset of $X$ respectively. If $t(y)$, $y \in Y$ is a strictly adapted trivialization for $A_y$, $y \in Y$, then $t$ can be extended to a strictly adapted trivialization for $A_x$, $x \in X$.

**Proof.** The proof is a standard modification of the proofs of Theorems 4.5 and 4.6. $\square$

**4.8. Corollary.** Under the assumptions of Theorem 4.5, every two strictly adapted trivializations are homotopic in the class of strictly adapted trivializations.

**Proof.** It is sufficient to apply Theorem 4.7 to the subset $X \times \{0, 1\}$ of $X \times [0, 1]$. $\square$
5. Analytical index of strictly Fredholm families

The topological category \( \mathcal{P} \hat{\mathcal{S}} \). Let us recall the definition of \( \mathcal{P} \hat{\mathcal{S}} \) from \([I_2]\). This is a topological category having as objects triples \((V, H_-, H_+)\), where \( V \) is a finitely dimensional subspace of \( H \), \( H_- \) and \( H_+ \) are infinitely dimensional, and

\[
H = H_- \oplus V \oplus H_+.
\]

In other terms, \( V \) is an object of \( \hat{\mathcal{S}} \) and \( H \ominus V = H_- \oplus H_+ \) is a polarization of \( H \ominus V \). The topology on the space of such triples is defined by the usual topology on subspaces \( V \) and the norm topology on subspaces \( H_- \), \( H_+ \). This space is ordered by the relation \( \leq \), where

\[
(V, H_-, H_+) \leq (V', H'_-, H'_+)
\]

if \( V \subset V' \) and \( H'_- \subset H_- \), \( H'_+ \subset H_+ \).

The morphisms of \( \mathcal{P} \hat{\mathcal{S}} \) are defined by this order, i.e. a morphism \( P \rightarrow P' \) exists if and only if \( P \leq P' \), and in this case it is unique. Morphisms have the form

\[
(V, H_-, H_+) \rightarrow (U_- \oplus V \oplus U_+, H_- \oplus U_-, H_+ \oplus U_+),
\]

where \( U_-, U_+ \) are finitely dimensional subspaces of \( H_- \), \( H_+ \) There is a forgetting functor \( \pi: \mathcal{P} \hat{\mathcal{S}} \rightarrow \hat{\mathcal{S}} \) taking \((V, H_-, H_+)\) to \( V \) and taking a morphism

\[
(V, H_-, H_+) \rightarrow (V', H'_-, H'_+)
\]

to the morphism \( V \rightarrow V' \) of \( \hat{\mathcal{S}} \) defined by the orthogonal decomposition

\[
V' = U_- \oplus V \oplus U_+,
\]

where \( U_- = H_- \oplus H'_- \) and \( U_+ = H_+ \oplus H'_+ \). The geometric realization

\[
|\pi|: |\mathcal{P} \hat{\mathcal{S}}| \rightarrow |\hat{\mathcal{S}}|
\]

is a homotopy equivalence. See \([I_2]\), Theorem 10.3.

The index map of strictly Fredholm families. As before, let \( \mathcal{H} \) be a Hilbert bundle over \( X \). Let \( \mathcal{A} \) be a strictly Fredholm family of operators in the fibers of \( \mathcal{H} \). As before, we think about \( \mathcal{H} \) as about a family \( H_x, x \in X \) of Hilbert spaces parameterized by \( X \), and denote \( \mathcal{A} \) also as \( A_x: H_x \rightarrow H_x, x \in X \). We will assume that \( X \) is a paracompact space. By Theorem 4.6 there exists a trivialization of the bundle \( \mathcal{H} \) adapted to the family \( \mathcal{A} \). Let us fix such a trivialization. Then we may consider \( \mathcal{A} \) as a family of operators in the fibers of the trivial bundle \( X \times H \rightarrow X \) and hence as a family \( A_x: H \rightarrow H, x \in X \) of operators in \( H \).
The fixed trivialization of the bundle $H$ allows to identify the topological category $\hat{\mathcal{P}}(H)$ with the product $X \times \hat{\mathcal{P}}$, where $X$ is considered as a category having $X$ as the space of objects and only the identity morphisms. Moreover, the functor

$$\mathcal{A}_{U,\varepsilon}: X_U \to \hat{\mathcal{P}}(H) = X \times \hat{\mathcal{P}}$$

constructed in Section 2 for $\mathcal{A}$ as a family of operators in the fibers of $H$ can be identified with the product of the projection functor $pr: X_U \to X$ with the functor

$$\mathcal{A}_{U,\varepsilon}: X_U \to \hat{\mathcal{P}}$$

constructed for $\mathcal{A}$ as a family of operators in $H$. Up to homotopy the geometric realization

$$|\mathcal{A}_{U,\varepsilon}|: |X_U| \to |\hat{\mathcal{P}}|$$

does not depend on the choice of the adapted trivialization, because any two choices are homotopic in the class of trivializations of $H$. This fact follows from the contractibility of the unitary group $\mathcal{U}(H)$ and does not depend on Corollary 4.8.

The strictly Fredholm property allows to lift the functor $\mathcal{A}_{U,\varepsilon}: X_U \to \hat{\mathcal{P}}$ to a functor to the category $\mathcal{P}\hat{\mathcal{P}}$. In more details, suppose that $U_a, a \in \Sigma$ is an open covering of $X$, and $\varepsilon_a > 0, a \in \Sigma$ are such that $(A_z, \varepsilon_a)$ is an enhanced operator for $z \in U_a$, i.e. such that the pairs $(U_a, \varepsilon_a)$ are adapted to $\mathcal{A}$. As in Section 2, for each $\sigma \in \Sigma_{\text{fin}}$ let

$$\varepsilon_\sigma = \min_{a \in \sigma} \varepsilon_a .$$

Let us assign to each object $(z, \sigma)$ of $X_U$ the object $(V, H_-, H_+)$ of $\mathcal{P}\hat{\mathcal{P}}$, where

$$V = \text{Im} P_{[-\varepsilon, \varepsilon]}(A_z) \quad \text{and} \quad H_- = \text{Im} P_{\varepsilon-\varepsilon}(A_z), \quad H_+ = \text{Im} P_{\varepsilon}(A_z).$$

This assignment preserves the orders and hence defines a functor

$$\mathcal{P}\mathcal{A}_{U,\varepsilon}: X_U \to \mathcal{P}\hat{\mathcal{P}}.$$
In other words, the functor $\mathcal{P} A_{U, \varepsilon}$ is a lift of the functor $A_{U, \varepsilon}$. By passing to the geometric realizations we get a continuous map

$$|\mathcal{P} A_{U, \varepsilon}| : |X_U| \to |\mathcal{P} \hat{S}|.$$  

Corollary 4.8 implies that the homotopy class of this map is independent from the choice of adapted trivialization. Since $X$ is triangulable and hence paracompact, there is a homotopy inverse $s$ of $|pr|$ and get a map

$$|\mathcal{P} A_{U, \varepsilon}| \circ s : X \to |\mathcal{P} \hat{S}|,$$

which we call a polarized index map of $A$. Its homotopy class carries the same information as the index, but the map $|\mathcal{P} A_{U, \varepsilon}| \circ s$ keeps information about subspaces $\text{Im } P_{\geq \varepsilon}(A_x)$.

The Grassmannian bundle of $A$. Recall that two closed subspaces $K, K' \subset H$ are said to be commensurable if the intersection $K \cap K'$ has finite codimension in both $K$ and $K'$. Given a subspace $K \subset H$ of infinite dimension and codimension in $H$, the restricted Grassmannian $\text{Gr}(K)$ is the space of closed subspaces of $H$ commensurable with $K$ with the norm topology. Up to a homeomorphism $\text{Gr}(K)$ do not depend on $K$. For each $x \in X$ let

$$\text{Gr}(x) = \text{Gr}\left(\text{Im } P_{\geq \varepsilon}(A_x)\right)$$

for some $\varepsilon > 0$ such that $(A_x, \varepsilon)$ is an enhanced operator. Clearly, $\text{Gr}(x)$ does not depend on the choice of $\varepsilon$. Since the family $A$ is strictly Fredholm, the family $\text{Gr}(x), x \in X$ forms a locally trivial bundle having as the fiber the restricted Grassmannian $\text{Gr}(x)$. We will denote the total space of this bundle by $\text{Gr}(A)$ and the bundle itself by

$$\pi(A) : \text{Gr}(A) \to X.$$  

We will call sections of this bundle weak spectral sections for the family $A$.

A universal Grassmannian bundle. Given an object $P = (V, H_-, H_+)$ of $\mathcal{P} \hat{S}$, let

$$\text{Gr}(P) = \text{Gr}(H_+).$$

The points of the geometric realization $|\mathcal{P} \hat{S}|$ can be represented by weighted sums

$$(2) \quad t_0 P_0 + t_1 P_1 + \ldots + t_n P_n,$$

where $t_0, t_1, \ldots, t_n \geq 0$, $t_0 + t_1 + \ldots + t_n = 1$, and $P_0, P_1, \ldots, P_n$ are objects of $\mathcal{P} \hat{S}$ such that $P_0 < P_1 < \ldots < P_n$. See [I2], Section 13. One can easily check that if $P, P'$ are objects of $\mathcal{P} \hat{S}$ and $P \preceq P'$, then $\text{Gr}(P) = \text{Gr}(P')$. It follows that the restricted Grassmannian $\text{Gr}(P_i)$ does not depend on $i$ and hence depends only on the point $p \in |\mathcal{P} \hat{S}|$.  

represented by the weighted sum (2). In particular, we can set \( \text{Gr}(p) = \text{Gr}(P_i) \). Let \( G \) be the space of pairs \((p, K)\) such that \( p \in |\mathcal{P}| \) and \( K \in \text{Gr}(p) \), and let

\[
\pi : G \longrightarrow |\mathcal{P}|,
\]
be the projection \((p, K) \longrightarrow p\). The map \( \pi \) is a locally trivial bundle having restricted Grassmannians \( \text{Gr}(P) \) as fibers. See \([I_2]\), the beginning of Section 13.

The polarized index maps and Grassmannian bundles. As above, let \( X \) be a paracompact space, \( \mathbb{H} \) be a Hilbert bundle over \( X \), and \( \mathbb{A} \) be a strictly Fredholm family of operators in the fibers of \( \mathbb{H} \). Then we can construct a functor

\[
\mathcal{P}\mathbb{A}_{U, \varepsilon} : X_U \longrightarrow \mathcal{P}\mathcal{I}.
\]

By choosing a homotopy inverse \( s : X \longrightarrow |X_U| \) of \( \text{pr} \) we get the polarized index map

\[
|\mathcal{P}\mathbb{A}_{U, \varepsilon}| \circ s : X \longrightarrow |\mathcal{P}\mathcal{I}|.
\]

By Theorem 2.1 we can assume that \( \text{pr} \circ s = \text{id}_X \).

5.1. Theorem. Suppose that \( X \) is paracompact. The bundle \( \pi(\mathbb{A}) : \text{Gr}(\mathbb{A}) \longrightarrow X \) is equal to the bundle induced from \( \pi : G \longrightarrow |\mathcal{P}| \) by the polarized index map \( |\mathcal{P}\mathbb{A}_{U, \varepsilon}| \circ s \).

Proof. Since \( \text{pr} \circ s = \text{id}_X \), for every \( x \in X \) the point

\[
|\mathcal{P}\mathbb{A}_{U, \varepsilon}| \circ s(x)
\]

is equal to a weighted sum (2) such that for each \( P_i = (V_i, H_{i-}, H_{i+}) \) we have

\[
H_{i+} = \text{Im} P_{\geq \varepsilon(i)}(A_x)
\]

for some \( \varepsilon(i) > 0 \). It follows that the fiber of the induced bundle over \( x \) is equal to \( \text{Gr}(x) \), i.e. to the fiber of \( \pi(\mathbb{A}) \) over \( x \). Therefore the induced bundle is equal to \( \pi(\mathbb{A}) \).

5.2. Theorem. Suppose that \( X \) is paracompact. The analytical index of a strictly Fredholm family \( \mathbb{A} \) vanishes if and only if there exists a weak spectral section for \( \mathbb{A} \).

Proof. Suppose that the analytical index vanishes. Since \( |\pi| : |\mathcal{P}| \longrightarrow |\mathcal{I}| \) is a homotopy equivalence, the polarized analytical index also vanishes. Together with Theorem 5.1 this implies that the bundle \( \pi(\mathbb{A}) : \text{Gr}(\mathbb{A}) \longrightarrow X \) is trivial. Hence it admits a section. Conversely, if there exists a weak spectral section for \( \mathbb{A} \), then the polarized index map can be lifted to a map \( X \longrightarrow G \). By \([I_2]\), Theorem 13.6, the space \( G \) is contractible, and hence the polarized index map and index map are homotopic to constant maps.
6. Discrete-spectrum families

**Discrete-spectrum families.** As before, $\mathcal{H}$ is a Hilbert bundle over $X$, thought also a family $H_x$, $x \in X$ of its fibers. Let $\mathcal{A}$ be a family of self-adjoint operators in the fibers of $\mathcal{H}$, denoted also by $A_x : H_x \rightarrow H_x$, $x \in X$. We will say that $\mathcal{A}$ is a discrete-spectrum family if for every $\lambda \in \mathbb{R}$ the family $A - \lambda$ is a Fredholm family. Of course, in the last formula $\lambda$ stands for $\lambda$ times the family of the identity operators. As usual, we assume that operators $A_x$ are neither essentially positive, nor essentially negative.

If $\mathcal{A}$ is a discrete-spectrum family, then every operator $A_x$ has discrete spectrum, i.e. its essential spectrum is empty. Indeed, for every $\lambda \in \mathbb{R}$ the operator $A_x - \lambda$ is Fredholm and hence 0 does not belong to its essential spectrum. It follows that $\lambda$ does not belong to its essential spectrum of $A_x$. Since this is true for every $\lambda \in \mathbb{R}$, the essential spectrum of $A_x$ is empty, i.e. $A_x$ indeed has discrete spectrum. But being a discrete-spectrum family is a much stronger property than being a Fredholm family of operators with discrete spectrum.

Since operators $A_x$ have discrete spectrum, they cannot be bounded. By this reason in this section we work in the framework of closed densely defined operators.

For the rest of this section $\mathcal{A}$ is assumed to be a discrete-spectrum family. For each $x \in X$ the spectrum $\sigma(A_x)$ consists of a double infinite sequence of eigenvalues

$$\ldots < \lambda_{-1}(x) < \lambda_0(x) < \lambda_1(x) < \ldots$$

and the eigenspaces $H_n(x)$ corresponding to each $\lambda_n(x)$, $n \in \mathbb{Z}$, are finitely dimensional. Moreover, the spectrum $\sigma(A_x)$ has no accumulation points in $\mathbb{R}$ and hence

$$\lim_{n \rightarrow -\infty} \lambda_n(x) = -\infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \lambda_n(x) = \infty.$$ 

The eigenspaces $H_n(x)$ define a decomposition of $H_x$ into an orthogonal direct sum

$$H_x = \bigoplus_{n \in \mathbb{Z}} H_n(x) \quad \text{(3)}$$

Strictly speaking, the numberings of eigenvalues $\lambda_n(x)$ and of subspaces $H_n(x)$ are well defined only up to shifts $n \mapsto n + a$ of subscripts, where $a$ is an integer. But, in fact, only the order $n < m$ on $\mathbb{Z}$ matters, and the order is well defined.

The decomposition (3) depends continuously on $x$ only in a relatively weak sense. Let $x \in X$ and $a, b \in \mathbb{R}$. Suppose that $a < b$ and that $a, b \not\in \sigma(A_x)$. Then there exists a neighborhood $U_x \subset X$ of $x$ such that $a, b \not\in \sigma(A_y)$ for every $y \in U_x$. Let

$$H_y(a, b) = \bigoplus H_n(y),$$

where the direct sum is taken over $n$ such that $a < \lambda_n(y) < b$. Clearly, $H_y(a, b)$ does
not depend on the choice of numbering of the eigenvalues. Since the operators $A_y - \lambda$ are Fredholm for all $\lambda \in [a, b]$, this sum is finite and the spaces $H_y(a, b)$ are finitely dimensional. The decomposition (3) continuously depends on the parameter in the sense that the subspace $H_y(a, b)$ continuously depends on $y \in U_x$. This follows from the assumption that the families $A - \lambda$ are Fredholm. Since spaces $H_y(a, b)$ are finitely dimensional, there is only one natural notion of continuous dependence of these subspaces on $y$.

The definition of subspaces $H_y(a, b)$ makes sense also for $a = -\infty$ and $b = \infty$. Clearly, $H_y(-\infty, \infty) = H_y$, and the fibers $H_y$ should be thought as continuously depending on $y$. For general bundles $\mathbb{H}$ and $a, b \in \mathbb{R}$ there is even no suitable notion of the continuous dependence of $H_y(-\infty, b)$ and $H_y(a, \infty)$ on $y$. But if $A$ is strictly Fredholm, then these subspaces continuously depend on $y$ almost by the definition. In more details, we can assume that for some $\varepsilon > 0$ the pair $(U_x, \varepsilon)$ is adapted to $A$. If $\varepsilon < a$, then

$$\text{Im} P_{\geq \varepsilon}(A_y) = \text{Im} P_{[\varepsilon, a]}(A_y) \oplus H_y(a, \infty).$$

In an adapted trivialization the left hand side of this equality continuously depends on $y$ in the norm topology, and the summand $\text{Im} P_{[\varepsilon, a]}(A_y) = H_y(\varepsilon, a)$ also continuously depends on $y$. Hence $H_y(a, \infty)$ continuously depends on $y$. The case of $a < \varepsilon$ is similar, and the case of $a = \varepsilon$ is trivial.

**Spectral sections.** Let us assume that $A$ is not only a discrete-spectrum family, but also is a strictly Fredholm family. Then the Grassmannian bundle $\pi(A) : \text{Gr}(A) \to X$ is defined. A section $S : X \to \pi(A)$ of this bundle can be considered as a family of subspaces $S_x \subset H_x$, $x \in X$ such that $S_x$ is commensurable with

$$H_{\geq k}(x) = \bigoplus_{n \geq k} H_n(x),$$

for some $k \in \mathbb{Z}$ and every $x$, or, equivalently, with

$$H_y(a, \infty)$$

for some $a \not\in \sigma(A_x)$. Clearly, these condition do not depend on the choice of $k$, the numberings of eigenvalues, or of $a$. Of course, we are interested in continuous sections. Since $A$ is strictly Fredholm, the continuity of a family $S_x, x \in X$ is well defined.

Following Melrose and Piazza [MP] we say that a continuous family $S_x, x \in X$ is a *spectral section* of $A_x, x \in X$ if there exists a continuous function $r : X \to \mathbb{R}_{\geq 0}$ such that

$$\text{P}_{\{r(x), \infty\}}(A_x) \subset S_x \subset \text{Im} \text{P}_{\{-r(x), \infty\}}(A_x)$$

for every $x \in X$. If $X$ is compact, the function $x \mapsto r(x)$ can be replaced by a constant. Note that $r(x)$ is allowed to be an eigenvalue of $A_x$. Locally, one can always ensure that $r(x)$ is not an eigenvalue, but, in general, one cannot do this globally.
**Convex combinations of projections.** Our next goal is to prove that every weak spectral section can be deformed into a spectral section. To this end we need the following elementary tool, a subspaces version of convex combinations of points. Let

\[ H_0 \supset H_1 \supset \ldots \supset H_n \]

be a finite sequence of closed subspaces of \( H \). Let \( p_i : H \to H_i \) be the orthogonal projection onto \( H_i \), and let \( q_i : H \to H_i \oplus H_{i+1} \) be the orthogonal projection onto \( H_i \oplus H_{i+1} \). Clearly, \( p_i = q_i + p_{i+1} \). Let \( t_0, t_1, \ldots, t_n \) be non-negative real numbers such that

\[ t_0 + t_1 + \ldots + t_n = 1, \]

and let \( s_i = t_0 + t_1 + \ldots + t_i \). The convex combination of \( p_i \) with coefficients \( t_i \) is

\[ t_0 p_0 + t_1 p_1 + \ldots + t_n p_n = s_0 q_0 + s_1 q_1 + \ldots + s_{n-1} q_{n-1} + p_n. \]

If \( K \subset H \) is a subspace such that \( p_n \) is injective on \( K \), then the above identity implies that

\[ t_0 p_0 + t_1 p_1 + \ldots + t_n p_n \]

is also injective on \( K \). The image

\[ L = \left( t_0 p_0 + t_1 p_1 + \ldots + t_n p_n \right)(K) \]

is contained in \( H_0 \) and can be thought as the **convex combination**

\[ L = t_0 p_0(K) + t_1 p_1(K) + \ldots + t_n p_n(K) \]

of subspaces \( p_i(K) \). Let \( p_K : H \to K \) be the orthogonal projection. The maps \( p_K \) and

\[ \left( t_0 p_0 + t_1 p_1 + \ldots + t_n p_n \right) \circ p_K \]

are connected by the standard linear homotopy, and this homotopy defines a canonical path of subspaces connecting \( K \) and \( L \).

**6.1. Theorem.** Suppose that \( X \) is a paracompact space and \( \mathbb{A} \) is a discrete-spectrum and strictly Fredholm family. Then every weak spectral section \( S \) of \( \mathbb{A} \) is homotopic to a spectral section of \( \mathbb{A} \) in the class of sections of the bundle

\[ \pi(\mathbb{A}) : \text{Gr}(\mathbb{A}) \to X. \]

If the restriction of the section \( S \) to a subspace \( Y \subset X \) is a spectral section of the family \( \mathbb{A}_Y, y \in Y \), then the homotopy can be chosen in the class of sections with this property.
Proof. Let us write the section $S$ as a family $S_x, x \in X$. Let us define the subspaces

$$H_{<k}(x) = \bigoplus_{n < k} H_n(x)$$

in terms of the decomposition (3). Let us temporarily fix a point $z \in X$. The subspace $S_z$ is commensurable with $H_{\geq 0}(z)$ and hence the intersection $S_z \cap H_{<0}(z)$ is finitely dimensional. By a compactness argument this implies that

$$(S_z \cap H_{<0}(z)) \cap H_{<-n}(z) = 0$$

for sufficiently large $n$. See the beginning of the proof of Lemma 11.1 in [I2] for the details. It follows that $S_z \cap H_{<-n}(z) = 0$ for sufficiently large $n$. By the definition,

$$H_{<-n}(z) = \text{Im } P_{<v(z)}(A_x) = \text{Ker } P_{\geq v(z)}(A_x)$$

for some $v(z) \in \mathbb{R}$ different from the eigenvalues of $A_x$. Therefore the restriction

$$P_{\geq v(z)}(A_x) \vert S_z$$

is injective. We claim that, moreover,

$$P_{\geq v(z)}(A_y) \vert S_y$$

is injective for every $y$ in some neighborhood $U$ of $z$. In order to prove this, let us choose a pair $(U, \varepsilon)$ adapted to $A$ and such that $z \in U$ and a local trivialization $t_U$ of $H$ defined over $U$ and strictly adapted to $(U, \varepsilon)$ and $A$. Then the families $t_U(S_y), y \in U$ and

$$t_U(\text{Im } P_{\geq v(z)}(A_y)), y \in U$$

are norm continuous families of subspaces of $H$. This reduces our claim to the following one. Let $T_y, y \in U$ be a norm continuous family of closed subspaces of $H$, and let $P_y, y \in U$ be a norm continuous family of projections. If the restriction $P_y \vert T_y$ is injective and $P_y(T_y)$ is closed, then $P_y \vert T_y$ is injective for all $y$ sufficiently close to $z$. In order to prove this, consider the orthogonal complement $F$ of $P_y(T_y)$ in $\text{Im } P_y$. Then the map $F \oplus T_y \rightarrow \text{Im } P_y$ equal to the identity on $F$ and to $P_y$ on $T_y$ is an isomorphism. By the open mapping theorem similar maps $F \oplus T_y \rightarrow \text{Im } P_y$ are isomorphisms for all $y$ sufficiently close to $z$. Therefore $P_y \vert T_y$ is injective for all $y$ sufficiently close to $z$. This proves our claim.

Since $X$ is paracompact, there exist a locally finite open covering $G_i, i \in I$ of $X$ refining the covering $U_z, z \in X$ and a partition of unity $t_i, i \in I$ subordinated to this covering. Since $G_i, i \in I$ refines $U_z, z \in X$, for every $i \in I$ there is a number $v(i) \in \mathbb{R}$ such that $v(i)$ is not an eigenvalue of $A_x$ and

$$P_{\geq v(z)}(A_x) \vert S_x$$

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is injective for every \( x \in G_i \). Let us consider the convex combinations

\[
L_x = \sum_{i \in I} t_i(x) \left( P_{\geq \nu(i)}(A_x) \right) (S_x)
\]

interpreted as explained before the theorem. These convex combinations make sense because for every \( x \) only a finite number of coefficients \( t_i(x) \) is non-zero, and the images \( \text{Im } P_{\geq \nu(i)}(A_x) \) are linearly ordered by the inclusion. It follows that

\[
(5) \quad L_x \subset \text{Im } P_{\geq \mu(x)}(A_x),
\]

where \( \mu(x) \) is the minimum of the numbers \( \nu(i) \) over \( i \) such that \( t_i(x) \neq 0 \). Moreover, \( S_x \) is connected with \( L_x \) by a canonical path of subspaces. These canonical paths form a homotopy between the sections \( x \mapsto S_x \) and \( x \mapsto L_x \).

We would like the inclusions (5) to hold for some continuous function \( \mu(x) \). This can be done if we carefully choose the partition of unity. First, given a locally finite open covering \( G_i, i \in I \) as above, there exists a closed covering \( F_i, i \in I \) such that \( F_i \subset G_i \) for every \( i \). See, for example, [I1], Theorem 7.5. Since \( X \), being a paracompact Hausdorff space, is normal, there exist open sets \( U_i, i \in I \) such that \( F_i \subset U_i \) and \( \overline{U_i} \subset G_i \) for every \( i \), where \( \overline{U_i} \) is the closure of \( U_i \). Using the fact that \( X \) is normal once more, we see that there exist continuous functions \( s_i: X \rightarrow \mathbb{R} \) equal to 1 on \( \overline{U_i} \) and to 0 on \( X \setminus G_i \). We may assume that the partition of unity \( t_i, i \in I \) is subordinated to the covering \( U_i, i \in I \). Then \( s_i(x) = 1 \) if \( t_i(x) \neq 0 \). Let us redefine \( \mu(x) \) as

\[
\mu(x) = \sum_{i \in I} s_i(x) \nu(i).
\]

Since this is a locally finite sum and \( \nu(i) \) are constant, the function \( \mu(x) \) is continuous. At the same time \( \mu(x) \) is greater or equal than the sum of of the numbers \( \nu(i) \) over \( i \) such that \( t_i(x) \neq 0 \). Hence \( \mu(x) \) is greater or equal than the maximum of these numbers. It follows that (5) holds for this \( \mu \).

Let \( L_x^\perp = H \cap L_x \). Clearly, \( L_x^\perp \) is commensurable with \( P_{\leq 0}(A_x) \) and

\[
(6) \quad L_x^\perp \supset \text{Im } P_{< \mu(x)}(A_x).
\]

By interchanging the roles of positive and negative numbers in the above construction, we can deform the map \( L_x^\perp \) to a map \( M_x^\perp \) such that

\[
M_x^\perp \subset \text{Im } P_{< \mu^\perp(x)}(A_x)
\]

for some continuous function \( \mu^\perp(x) \). Moreover, it is clear from the construction that the inclusion (6) holds with \( M_x^\perp \) in the role of \( L_x^\perp \). By passing from the subspaces \( M_x^\perp \) to their orthogonal complements \( M_x \) we get a section \( M_x, x \in X \) homotopic to the section \( S_x, x \in X \) and such that \( P_{> \mu^\perp(x)}(A_x) \subset M_x \subset \text{Im } P_{> \mu(x)}(A_x) \). It follows that the fam-
ily $M_x, x \in X$ is a spectral section of $A_x, x \in X$. This proves the first claim of the theorem. In order to prove the second claim, it is sufficient to observe that if the subspaces $S_x$ satisfy (4) for $x \in Y$, then this remains the case during the whole deformation from $S_x$ to $M_x$, except that one may need to replace the function $r(x)$ by a larger one. 

6.2. Corollary. Suppose that $X$ is paracompact and $A$ is a discrete-spectrum and strictly Fredholm family. Then the analytical index of $A$ vanishes if and only if there exists a spectral section of $A$.

Proof. By Theorem 5.2, if the analytical index of $A$ vanishes, then there exists a weak spectral section for $A$. Since $X$ is paracompact, Theorem 6.1 implies that in this case there exists a spectral section of $A$. The other implication follows from Theorem 5.2.

Remark. The really important part is the “only if” one. It is independent from the “if” part of Theorem 5.2, and hence from the theorem [I$_2$] about the contractibility of $G$.

Remark. The proof of Theorem 6.1 resulted from an attempt to understand the last paragraph in the proof of Proposition 1 of Melrose and Piazza [MP]. Melrose and Piazza [MP] claim that a weak spectral section can be transformed into a spectral section “simply by smoothly truncating the eigenfunction expansion”. Since the eigenvalues change with $x$, it is only natural to truncate expansions between the eigenvalues. This can be always done locally, but if we can do this globally, then we already know that there exists a spectral section. The proof of Theorem 6.1 provides a natural way to overcome this difficulty.

A theorem of Melrose–Piazza. The classical analytical index is defined only under much stronger assumptions than that of Corollary 6.2. As we will see in Section 8, which is independent of the present one, when the classical analytical index is defined, it agrees with the analytical index as defined in Section 2. See Theorem 8.5. For such families the analytical index in Corollary 6.2 can be understood in the classical sense, and this turns its conclusion into the conclusion of Proposition 1 of Melrose and Piazza [MP]. The assumptions are still weaker than the assumptions of Melrose and Piazza [MP], who required that the space $X$ to be compact and $A$ to be a continuous family of differential operators of order 1. The last assumption is not really used in Melrose–Piazza proof, but the proof depends on continuity properties of the family which are much stronger than being strictly Fredholm.

Melrose and Piazza [MP] gloss over the definition of analytical index and the meaning of trivializations of Hilbert bundles, and a straightforward interpretation of their proof works only for operators in a fixed Hilbert space. The author learned about this issue from M. Prokhorova [P$_1$]. The above proof overcomes these difficulties thanks to Theorem 4.6 and the new definition of analytical index. There are also a less general versions of Melrose–Piazza theorem, which are still more general than the original one, and which can be proved using only Theorem 4.5 or 4.6 and the classical definition of analytical index. See [I$_3$].

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7. The classical index of Fredholm families

Two definitions of the analytical index. The goal of this section is to prove that the definition of analytical index of Fredholm families from Section 3 is equivalent to the classical one when the latter applies. Let $B_x : H_x \to H_x$, $x \in X$, be such a family of operators acting in the fibers of a Hilbert bundle $H_x$, $x \in X$. The most classical definition of Atiyah–Singer [AS4] applies only when $X$ is compact, and we will discuss this case first.

The classical analytical index of the family $B_x$, $x \in X$, is an element of the group $K(X)$, which can be defined either in terms of finitely dimensional vector bundles on $X$, or as the group of homotopy classes of maps to a classifying space for $K$-theory. The space $\mathcal{F}$ of Fredholm operators is a well known such classifying space. Since there are canonical homotopy equivalences $\mathcal{F} \to |\mathcal{F}| \to |S|$, we can interpret our analytical index as a map to $\mathcal{F}$. This almost turns the problem of comparing two definition into a triviality when the Hilbert bundle $H_x$, $x \in X$, is trivial. But there is something to discuss in the general case.

The classical analytical index of Fredholm families. As usual, we will denote the Hilbert bundle $H_x$, $x \in X$, and the family $B_x$, $x \in X$, also by $\mathbb{H}$ and $\mathbb{B}$ respectively. Classically, the analytical index of $\mathbb{B}$ is defined only under very strong assumptions about $\mathbb{B}$. See [AS4]. But the classical definition works without any changes also for general Fredholm families.

Suppose that $X$ is compact. Then a classical argument due to Atiyah shows that there exists a finitely dimensional Hilbert space $V$ and a continuous family of maps $q_x : V \to H_x$, $x \in X$, such that the map $Q_x : V \oplus H_x \to H_x$ defined by the formula

$$Q_x(v \oplus u) = q_x(v) + B_x(u)$$

is surjective for every $x \in X$. Moreover, the standard construction leads to a family of maps $q_x$, $x \in X$, having the following property. For every $x \in X$ there exists a neighborhood $U_x$ of $x$ and $\varepsilon = \varepsilon_x > 0$ such that for $y \in U_x$ the subspaces $\text{Im} P_{[0,\varepsilon]}(B^*_y)$ are finitely dimensional, continuously depend on $y$, and contain the image $\text{Im} q_x$. Then the family of finitely dimensional vector spaces $\ker Q_x$, $x \in X$, is a finitely dimensional vector bundle, which we denote by $\mathcal{K}$. Let $\mathcal{V}$ be the trivial bundle $V \times X \to X$. The classical analytical index of $\mathbb{B}$ is defined as the element of $K(X)$ represented by the difference $[\mathcal{K}] - [\mathcal{V}]$.

7.1. Theorem. If $X$ is compact, then two definitions of the analytical index of $\mathbb{B}$ agree.

Proof. Let $P_x : V \oplus H_x \to V \oplus H_x$ be the composition of $Q_x$ with the inclusion of $H_x$ into $V \oplus H_x$ as the second summand. The family $P_x$, $x \in X$, can be considered as a bundle map $P : \mathcal{V} \oplus \mathcal{H} \to \mathcal{V} \oplus \mathcal{H}$. Clearly, its cokernel is $\mathcal{V}$, and, by the definition, its kernel is $\mathcal{K}$. At the same time $P_x$, $x \in X$, is a Fredholm family.
Let us define a family of operators $R_x : V \oplus H_x \to V \oplus H_x$, $x \in X$, by the formula

$$R_x(v \oplus u) = v \oplus B_x(u).$$

Clearly, the family $R_x$, $x \in X$, is also a Fredholm family. The index maps to $|\mathcal{S}(V \oplus H)|$ of the families $R_x$, $x \in X$, and $B_x$, $x \in X$, can be arranged to be not only homotopic, but equal. At the same time the family $R_x$, $x \in X$, is obviously Fredholm homotopic to the family $P_x$, $x \in X$, and hence their index maps are homotopic. Therefore it is sufficient to prove that the index map of the family $P_x$, $x \in X$, corresponds to $[\mathcal{K}] - [\mathcal{V}]$.

Let $x \in X$. Since $\operatorname{Ker} P_x^* P_x = \operatorname{Ker} P_x$ and $P_x$ is Fredholm, there exists $\epsilon > 0$ such that

$$\operatorname{Ker} Q_x = \operatorname{Ker} P_x = \operatorname{Im} P_{[0, \epsilon]} \left( |P_x| \right).$$

Moreover, we can assume that $(P_x, \epsilon)$ is an enhanced operator. Clearly,

$$\operatorname{Ker} Q_z = \operatorname{Ker} P_z \subset \operatorname{Im} P_{[0, \epsilon]} \left( |P_z| \right)$$

for every $z \in X$. Since $P_x$, $x \in X$, is a Fredholm family, the spaces $\operatorname{Im} P_{[0, \epsilon]} \left( |P_z| \right)$ continuously depend on $z$ for $z$ sufficiently close to $x$. Since the spaces $\operatorname{Ker} Q_z$ also continuously depend on $z$, it follows that

$$\operatorname{Ker} Q_z = \operatorname{Ker} P_z = \operatorname{Im} P_{[0, \epsilon]} \left( |P_z| \right)$$

for $z$ sufficiently close to $x$. Since the space $X$ is compact, for sufficiently small $\epsilon > 0$ these equalities will hold for every $z \in X$. Similarly, for sufficiently small $\epsilon > 0$

$$V \oplus 0 = \operatorname{Ker} Q_z^* = \operatorname{Ker} P_z^* = \operatorname{Im} P_{[0, \epsilon]} \left( |P_z^*| \right)$$

for every $z \in X$. Then $(Q_z, \epsilon)$ is an enhanced operator for every $z \in X$.

Now we can construct an index map for $P_x$, $x \in X$, using the covering $U$ of $X$ by single open set $X$ and $\epsilon$ as the corresponding number. Then $|X_U| = X$ and the homotopy inverse $s : X \to |X_U|$ is the identity map. The index map $X \to |\mathcal{S}(\mathbb{H})|$ takes point $x \in X$ to the point of $|\mathcal{S}|$ represented by the object

$$(\operatorname{Ker} P_x, V) = (\operatorname{Ker} Q_x, V \times x)$$

of $\mathcal{S}(\mathbb{H})$. Now we can either compose this index map with $|\mathcal{S}(\mathbb{H})| \to |\mathcal{S}(\mathbb{H})| \to |\mathcal{S}|$ to get an index map $X \to |\mathcal{S}|$, or use a trivialization of the bundle $\mathbb{H}$ to get an index map $X \to |\mathcal{S}|$. Either way, the resulting index map represents the virtual bundle $\mathcal{K} - \mathcal{V}$ at least on the intuitive level. In fact, it is easy to construct a map $X \to \mathcal{F}$, i.e. a family of Fredholm operators in a fixed Hilbert space, with this index map and representing $[\mathcal{K}] - [\mathcal{V}] \in K(X)$. We leave this task to the reader. ■
Families parameterized by paracompact spaces. When X is not compact, the index is still an element of $K(X)$, but it is not reasonable to define $K(X)$ in terms of finitely dimensional vector bundles on X. Instead, $K(X)$ is defined in terms of homotopy classes to a classifying space, with $\mathcal{F}$ being a natural choice. The corresponding definition of index was given by Segal [S$_2$] under the assumption that X is paracompact.

Sigal’s definition applies to families of operators $D_x$, $x \in X$ which he called Fredholm operators on X. They are defined as families invertible modulo compact families, i.e. such that there exists a family of operators $C_x: H_x \longrightarrow H_x$, $x \in X$ such that operators

$$id - D_x \circ C_x, \ x \in X, \text{ and } id - C_x \circ D_x, \ x \in X,$$

are compact and, moreover, define a compact operator $\mathbb{H} \longrightarrow \mathbb{H}$. We will not need the definition of compact operator, but note that the family of zero operators is a compact operator. Hence a family of invertible operators is a Fredholm operator in Segal’s sense. But, in general, such a family is not a Fredholm family in our sense.

Segal considers families of operators $D_x: H_x \longrightarrow H_x$, $x \in X$ as families of chain complexes $0 \longrightarrow H_x \longrightarrow H_x \longrightarrow 0$, $x \in X$ (in fact, his theory applies to general chain complexes). Segal [S$_2$] calls a family of operators $\varphi_x: H \longrightarrow H_x$, $x \in X$ finite if locally, over open subsets $U \subset X$ covering X, it can be factored as the composition of the projection to a finitely dimensional subbundles of $U \times H \longrightarrow U$ and a continuous map from this subbundle to $\mathbb{H}$. Suppose that $D_x$, $x \in X$ is a Fredholm operator on X. Then there exists a family of Fredholm operators $E_x: H \longrightarrow H$, $x \in X$ continuous as a map $E: X \longrightarrow \mathcal{F}$ to $\mathcal{F}$ with the norm topology, and finite families of operators $\theta_x, \phi_x: H \longrightarrow H_x$, $x \in X$ such that

$$\begin{array}{ccc}
0 & \longrightarrow & H \\
\downarrow \theta_x & & \downarrow \phi_x \\
0 & \longrightarrow & H_x \\
\downarrow & & \downarrow \\
0 & \longrightarrow & H_x \\
\downarrow E_x & & \downarrow \\
0 & \longrightarrow & H \\
\end{array}$$

(7)

is a chain homotopy equivalence. The homotopy class of the family $E_x$, $x \in X$ considered as a map $E: X \longrightarrow \mathcal{F}$ depends only on $D_x$, $x \in X$. Therefore one can define the analytical index of Fredholm operator $D_x$, $x \in X$ on X as the homotopy class of $E: X \longrightarrow \mathcal{F}$.

7.2. Theorem. If X is paracompact and $D_x$, $x \in X$ is a Fredholm family and a Fredholm operator in Segal’s sense, then two definitions of the analytical index agree for $D_x$, $x \in X$.

Proof. It is sufficient to prove that, given a chain homotopy equivalence of the form (7), the analytical index of $D_x$, $x \in X$ in the sense of Section 3 is equal to that of $E_x$, $x \in X$.  

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Since (7) is a chain homotopy equivalence, the maps $\theta_x, \phi_x$ induce isomorphisms of the homology groups of the row complexes. In particular, $\theta_x$ induces an isomorphism of kernels $\text{Ker} E_x \rightarrow \text{Ker} D_x$ for every $x$. Let us temporarily fix an arbitrary $x \in X$. Since $E_x$ and $D_x$ are Fredholm operators, there exist $\varepsilon, \varepsilon' > 0$ such that

$$\text{Ker} E_x = \text{Im} P_{[0, \varepsilon]}(\|E_x\|) \quad \text{and} \quad \text{Ker} D_x = \text{Im} P_{[0, \varepsilon']}(|D_x|).$$

Moreover, one can assume that $(E_z, \varepsilon)$ and $(D_z, \varepsilon')$ are enhanced Fredholm operators for $z \in U_x$ for a neighborhood $U_x$ of $x$.

We claim that the parameters $\varepsilon, \varepsilon' > 0$ and the neighborhood $U_x$ can be chosen in such a way that the composition $P_{[0, \varepsilon']}(|D_z|) \circ \theta_z$ induces an isomorphism

(8) $\text{Im} P_{[0, \varepsilon]}(\|E_z\|) \rightarrow \text{Im} P_{[0, \varepsilon']}(|D_z|)$

for every $z \in U_x$. Let us consider the family of restrictions

$$\theta'_z: \text{Im} P_{[0, \varepsilon]}(|d_z|) \rightarrow H_z, \quad z \in U_x$$

induced by $\theta_z$. Since the family $\text{Im} P_{[0, \varepsilon]}(|E_z|), z \in U_x$ is a finitely dimensional vector bundle, the family $\theta'_z, z \in U_x$ is continuous.

Since $\theta'_x$ is injective, after replacing $U_x$ by a smaller neighborhood we can assume that $\theta'_z$ is injective for every $z \in U_x$. Moreover, we can assume that

$$c \|v\| \leq \|\theta'_z(v)\|$$

for some $c > 0$ and every $z \in U_x, v \in \text{Im} P_{[0, \varepsilon]}(|d_z|)$. Similarly, we can assume that

$$\|\phi_z(E_z(v))\| \leq c' \|E_z(v)\|$$

for some $c' > 0$ and every $z \in U_x, v \in \text{Im} P_{[0, \varepsilon]}(|E_z|)$. Then

$$\|D_z(\theta'_z(v))\| = \|D_z \circ \theta'_z(v)\| = \|\phi_z \circ E_z(v)\| \leq c' \|E_z(v)\|$$

$$\leq c' \varepsilon \|v\| \leq c' \varepsilon c^{-1} \|\theta'_z(v)\|.$$
Similarly, the map $\phi_x$ induces an isomorphism of cokernels $\text{coker } E_x \to \text{coker } D_x$, or, equivalently, an isomorphism of kernels of the adjoint operators $\text{Ker } E^*_x \to \text{Ker } D^*_x$. As above, there exist $\delta, \delta' > 0$ such that

$$\text{Ker } E^*_x = \text{Im } P_{[0, \delta]}(|E_x^*|) \quad \text{and} \quad \text{Ker } D^*_x = \text{Im } P_{[0, \delta']}(|D_x^*|),$$

and, moreover, $(E^*_x, \delta)$ and $(D^*_x, \delta')$ are enhanced Fredholm operators. Then $(E^*_z, \delta)$ and $(D^*_z, \delta')$ are enhanced Fredholm operators for $z \in U^*_x$ for a neighborhood $U^*_x$ of $x$. As above, the parameters $\delta, \delta' > 0$ and the neighborhood $U^*_x$ can be chosen in such a way that for every $z \in U^*_x$ the composition $P_{[0, \delta']}(|D^*_z|) \circ \phi_z$ induces an isomorphism

$$(9) \quad \text{Im } P_{[0, \delta]}(|E_z^*|) \to \text{Im } P_{[0, \delta']}(|D_z^*|).$$

In the definition of index the same parameter $\varepsilon > 0$ is used both for the operator and its adjoint, although this is largely a matter of convenience. By this reason we would like to have $\varepsilon = \delta$ and $\varepsilon' = \delta'$. This can be achieved by choosing first a sufficiently small positive numbers as $\varepsilon' = \delta'$. Then a sufficiently small positive number will work as both $\varepsilon$ and $\delta$. We should also replace both $U_x$ and $U^*_x$ by their intersection $U_x \cap U^*_x$.

Now we can use $U_x$ and $\varepsilon = \varepsilon_x$ and $\varepsilon' = \varepsilon'_x$ for $x \in X$ to construct continuous functors

$$E_{U, \varepsilon} : X_U \to \mathcal{S} \quad \text{and} \quad D_{U, \varepsilon'} : X_U \to \mathcal{S}(\mathbb{H})$$

and then the corresponding geometric realizations

$$|E_{U, \varepsilon}| : |X_U| \to |\mathcal{S}| \quad \text{and} \quad |D_{U, \varepsilon'}| : |X_U| \to |\mathcal{S}(\mathbb{H})|.$$

But the functors and maps in these pairs have different targets. Let $\mathbb{T}$ be the trivial bundle $\text{pr} : X \times \mathbb{H} \to \mathbb{H}$. By using a trivialization of the bundle $\mathbb{H}$ we can identify it with $\mathbb{T}$. Then we can reinterpret $E_{U, \varepsilon}$ and $D_{U, \varepsilon'}$ as continuous functors

$$E_{U, \varepsilon} : X_U \to \mathcal{S}(\mathbb{T}) \quad \text{and} \quad D_{U, \varepsilon'} : X_U \to \mathcal{S}(\mathbb{T}).$$

One can hope that the isomorphisms (8) and (9) define isomorphisms of these functors, but these isomorphisms are not morphisms of $\mathcal{S}(\mathbb{T})$. The solution of this problem is to lift these functors to $S/\mathbb{T}$. If we choose sufficiently small neighborhoods $U_x$, the bundles

$$\text{Im } P_{[0, \varepsilon]}(|E_z|), \ z \in U_x \quad \text{and} \quad \text{Im } P_{[0, \varepsilon]}(|E_z^*|), \ z \in U_x$$

will be trivial. By choosing trivializations of these bundles we can lift $E_{U, \varepsilon}$ to a functor $\overline{E}_{U, \varepsilon} : X_U \to S/\mathbb{T}$. Let use these trivializations and the isomorphisms (8) and (9) to trivialize the corresponding bundles for $D$. Then the resulting functor $\overline{D}_{U, \varepsilon} : X_U \to S/\mathbb{T}$ will be not only isomorphic, but equal to $\overline{E}_{U, \varepsilon}$. By composing it with $S/\mathbb{T} \to S$ and then taking the geometric realizations, we will get equal index maps. ■
8. The classical index of self-adjoint Fredholm families

The Atiyah–Singer definition of the analytical index. In the papers of Atiyah, Patodi, and Singer [AS\textsubscript{5}], [AS], [APS] the analytical index of families of self-adjoint Fredholm operators is defined only for the families of bounded operators in a fixed Hilbert space. The approach of these papers naturally extends to operators in Hilbert bundles. Let \( A_x : H_x \to H_x \), \( x \in X \), denote also by \( A \) be a Fredholm family of self-adjoint operators in the fibers of a Hilbert bundle \( H_x \), \( x \in X \), denoted also by \( H \). Following [AS], [APS], let us consider the family \( B \) of Fredholm operators defined by

\[
B_{x,t} = \text{id}_H \cos t + i A_x \sin t, \quad x \in X, \quad t \in [0, \pi],
\]

where \( \text{id}_H \) is the identity operator \( H \to H \) and \( i = \sqrt{-1} \). By Theorem 8.2 below, this is a Fredholm family. By a homotopy of such a family, or, more generally, a family parameterized by \( X \times [0, \pi] \), we will understand a homotopy fixed on \( X \times 0 \) and \( X \times \pi \).

By the classical analytical index of \( A \) we will understand the analytical index of \( B \) either in the sense of Atiyah and Singer [AS\textsubscript{4}], or in the sense of Section 3. The latter is the homotopy class of the index map \( X \times [0, \pi] \to |\mathcal{F}| \) with respect to homotopies fixed on \( X \times 0 \) and \( X \times \pi \). Classically the operators \( A_x \) are bounded, \( H_x = H \) for every \( x \), and the target of index maps is \( \mathcal{F} \). Then one can take the map \( (x, t) \to B_{x,t} \) as the index map.

8.1. Lemma. Let \((A, \varepsilon)\) be an enhanced self-adjoint Fredholm operator in \( H \), and let

\[
B_t = \text{id}_H \cos t + i A \sin t,
\]

where \( t \in [0, \pi] \). Let \( \delta > 0 \). If either \( \sin t = 0 \) and \( \delta < 1 \), or \( \delta^2 < \cos^2 t \), then

\[
\text{Im} \left. \mathcal{P}_{[0, \delta]} \right| B_t \right) = 0.
\]

If \( \sin t \neq 0 \) and \( \delta^2 = \cos^2 t + \varepsilon^2 \sin^2 t \), then \( \delta \notin \sigma(B_t) \) and

\[
\text{Im} \left. \mathcal{P}_{[0, \delta]} \right| B_t \right) = \text{Im} \left. \mathcal{P}_{[0, \varepsilon]} \right| A \right).
\]

Proof. The case of \( \sin t = 0 \) is trivial. Suppose that \( \sin t \neq 0 \). Since \( A^* = A \),

\[
|B_t|^2 = B_t B_t^* = B_t^* B_t = \text{id}_H \cos^2 t + A^2 \sin^2 t.
\]

If \( \lambda^2 \in \sigma(B_t B_t^*) \), then the operator

\[
\text{id}_H \lambda^2 - B_t B_t^* = \text{id}_H (\lambda^2 - \cos^2 t) - A^2 \sin^2 t
\]

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is not invertible. Since $\sin t \neq 0$, this implies that

\[
\frac{\lambda^2 - \cos^2 t}{\sin^2 t} \in \sigma(A^2).
\]

Since $A^2 = A^* A$ is a positive operator, this implies that $\lambda^2 \geq \cos^2 t$. Therefore

\[
\text{Im } P_{[0, \delta]}(|B_t|) = \text{Im } P_{[0, \delta^2]}(|B_t|^2) = 0
\]

if $\delta^2 < \cos^2 t$. If $\lambda^2 \leq \cos^2 t + \varepsilon^2 \sin^2 t$, then

\[
\frac{\lambda^2 - \cos^2 t}{\sin^2 t} \leq \varepsilon^2.
\]

Since $(A, \varepsilon)$ is an enhanced self-adjoint operator, in this case

\[
\frac{\lambda^2 - \cos^2 t}{\sin^2 t} = \mu^2
\]

for some eigenvalue $\mu < \varepsilon$ of $A$ of finite multiplicity. It follows that $\delta \not\in \sigma(|B_t|)$ and

\[
\text{Im } P_{[0, \delta]}(|B_t|) = \text{Im } P_{[0, \varepsilon]}(|A|)
\]

if $\delta^2 = \cos^2 t + \varepsilon^2 \sin^2 t$. This completes the proof. ■

8.2. Theorem. If $\mathbb{A}$ is a self-adjoint Fredholm family, then $\mathbb{B}$ is a Fredholm family.

Proof. Let $x \in X$. Then there exists $\alpha = \alpha_x > 0$ such that $(A_x, \alpha)$ is an enhanced self-adjoint operator. Moreover, there exists $\beta = \beta_x \in (0, \alpha)$ such that $(A_x, \varepsilon)$ is an enhanced self-adjoint operator for every $\varepsilon \in [\beta, \alpha]$. Since $\mathbb{A}$ is a self-adjoint Fredholm family, there exists a neighborhood $U = U_x$ of $x$ such that $(A_z, \varepsilon)$ is an enhanced self-adjoint operator for every $\varepsilon \in [\beta, \alpha]$ and $z \in U$. Since $\beta < \alpha$, there exists $\delta = \delta_x > 0$ such that

\[
\cos^2 t + \beta^2 \sin^2 t < \delta^2 < \cos^2 t + \alpha^2 \sin^2 t
\]

for all $t$ sufficiently close to $\pi/2$, say, for $t \in (u, \pi - u)$ for some $u = u_x \in (0, \pi/2)$. For every such $t$ the number $\delta^2$ can be written in the form

\[
\delta^2 = \cos^2 t + \varepsilon(t)^2 \sin^2 t
\]

for some $\varepsilon(t) \in (\beta, \alpha)$. Lemma 8.1 implies that $\delta \not\in \sigma(|B_z, t|)$ and

\[
\text{Im } P_{[0, \delta]}(|B_z, t|) = \text{Im } P_{[0, \varepsilon(t)]}(|A_z|)
\]
for every \( z \in U \) and \( t \in (u, \pi - u) \). Also, since \((A_z, \varepsilon)\) is an enhanced self-adjoint operator for every \( \varepsilon \in [\beta, \alpha] \), the spectrum \( \sigma(A_z) \) is disjoint from \([\beta, \alpha]\) and hence
\[
\text{Im } P_{[0, \varepsilon(t)]}(|A_z|) = \text{Im } P_{[0, \alpha]}(|A_z|).
\]
The last two equalities imply that
\[
\text{Im } P_{[0, \delta]}(|B_z, t|) = \text{Im } P_{[0, \alpha]}(|A_z|)
\]
for every \( z \in U \) and \( t \in (u, \pi - u) \). In turn, this implies that the pair \((B_z, t, \delta)\) is an enhanced operator for every \( z \in U \) and \( t \in (u, \pi - u) \), and hence the family
\[
B_z, t, z \in U, \ t \in (u, \pi - u)
\]
is a Fredholm family. Now, let us choose some \( w = w_x \in (u, \pi/2) \). Lemma 8.1 implies that \((B_z, t, \cos w)\) is an enhanced operator if \( z \in U \) and \( \cos^2 w < \cos^2 t \), i.e. if \( t < w \) or \( t > \pi - w \). It follows that the family \( B_z, t, z \in U, \ t \in [0, w) \cup (\pi - w, \pi] \) is a Fredholm family. It follows that \( B_z, t, z \in U, \ t \in [0, \pi] \) is a Fredholm family. Since \( x \in U \) and \( x \in X \) is arbitrary, we see that \( B_x, t, x \in X \) is a Fredholm family. 

8.3. Theorem. There is a canonical homotopy equivalence \(|\hat{\mathcal{I}}| \to \Omega|\mathcal{I}|\), where \( \Omega|\mathcal{I}| \) is the loop space of \( |\mathcal{I}| \).

Proof. This immediately follows from \([I_2], \) Theorems 9.8 and 15.4. This also follows from Theorem 15.6 there, and is implicit in its proof.

Comparing two definitions of the analytical index. Since \( B_{x, 0} = \text{id}_H \) and \( B_{x, \pi} = -\text{id}_H \) for every \( x \in X \), one can consider \( \mathbb{B} \) as a family parameterized by the space \( \Sigma X \) obtained by collapsing in \( X \times [0, \pi] \) the subspace \( X \times 0 \) into a point, and the subspace \( X \times \pi \) into another point. This turns the homotopies fixed on \( X \times 0 \) and \( X \times \pi \) into the homotopies of maps with the domain \( \Sigma X \). In view of this the analytical index of the family \( \mathbb{B} \) can be defined as the homotopy class of an index map \( \Sigma X \to |\mathcal{I}| \). By Theorem 7.1 this definition is equivalent to the classical one.

So, the classical analytical index of the family \( \mathbb{A} \) is the homotopy class of the index map \( \Sigma X \to |\mathcal{I}| \) of the family \( \mathbb{B} \). The analytical index of \( \mathbb{A} \) in the sense of Section 2 is the homotopy class of an index map \( X \to |\hat{\mathcal{I}}| \). In order to compare these definitions, recall that there is a canonical one-to-one correspondence between maps \( \Sigma X \to |\mathcal{I}| \) and maps \( X \to \Omega|\mathcal{I}| \). The same is true for homotopies, and we can consider the classical analytical index of \( \mathbb{A} \) as the homotopy class of a map \( X \to \Omega|\mathcal{I}| \) deserving to be called the classical index map. Theorem 8.3 allows us to interpret the analytical index in the sense of Section 2 as the homotopy class of a map \( X \to \Omega|\mathcal{I}| \), which also may be called an index map.
 Finite-polarized operators. We will call a self-adjoint operator \( A : K \to K \) in a Hilbert space \( K \) finite-polarized if \( \| A \| = 1 \), the essential spectrum of \( A \) is consists of two points \(-1, 1\), and the spectral projection \( P_{(-1, 1)}(A) \) is an operator of finite rank. If we omit the last property, we will get exactly the operators from the space \( \hat{F} \) from [AS], Section 2.

 Finite-polarized replacements. We would like to replace the family \( \mathbb{A} \) by a family \( \mathbb{A}' \) of finite-polarized operators \( A'_x : H_x \to H_x \) without changing neither the classical analytical index, nor the analytical index in the sense of Section 2. This can be done by a spectral deformation similar to spectral deformations used in [AS].

We will assume that \( X \) is paracompact. Let us choose an atlas for \( \mathbb{A} \), i.e. a family of pairs \( (U, \varepsilon_U) \) adapted to \( \mathbb{A} \) such that the sets \( U \) form an open covering of \( X \). We can assume that \( \varepsilon_U < 1 \) for every \( U \). Since \( X \) is paracompact, we can further assume that this covering is locally finite and there exists a partition of unity subordinated to this covering. Let \( r_U : X \to R_{\geq 0} \) be the function from this partition of unity corresponding to \( U \). Let

\[
 r(x) = \sum_U r_U(x) \varepsilon_U.
\]

Then \( r(x) \in (0, 1) \) and \( r(x) \leq \max_{U \ni x} \varepsilon_U \), where the maximum is taken over \( U \) such that \( x \in U \), for every \( x \in X \). It follows that for every \( x \in X \) the essential spectrum of \( A_x \) is disjoint from \((-r(x), r(x))\). For \( r > 0 \) let \( \chi_r : R \to R \) be the function defined by

\[
\chi_r(u) = u \quad \text{for} \quad 0 \leq u \leq r/2, \\
\chi_r(u) = (2/r - 1) u + r - 1 \quad \text{for} \quad r/2 \leq u \leq r, \\
\chi_r(u) = 1 \quad \text{for} \quad r \leq u, \\
\text{and} \quad \chi_r(u) = -\chi_r(-u) \quad \text{for every} \quad u.
\]

In particular, \( \chi_1 \) is the identity map \( R \to R \). For \( x \in X \) let

\[
 A'_x = \chi_{r(x)}(A_x) : H_x \to H_x
\]

Clearly, operators \( A'_x \) are finite-polarized and \( A'_x, x \in X \) is a Fredholm family. This is our finite-polarized replacement \( \mathbb{A}' \). Clearly, if \( 0 < \varepsilon < r(x)/2 \), then

\[
 \text{Im} \ P_{[\varepsilon, \infty)}(A_x) = \text{Im} \ P_{[\varepsilon, \infty)}(A'_x).
\]

It follows that if \( \mathbb{A} \) is strictly Fredholm, then \( \mathbb{A}' \) is also strictly Fredholm.

8.4. Theorem. For both definitions of the analytical index, the analytical index of \( \mathbb{A} \) is equal to the analytical index of its finite-polarized replacements \( \mathbb{A}' \).
Proof. The open covering $U_a$, $a \in \Sigma$ of $X$ and the family of positive numbers $\varepsilon_a$, $a \in \Sigma$ used to construct the index functor

$$A_{U, \varepsilon} : X_U \to \hat{\mathcal{F}}(\mathbb{H})$$

don’t need to be equal to the covering by the sets $U$ and the numbers $\varepsilon_U$ used to construct the finite-polarized replacement $A'$. By choosing sufficiently small open sets $U_a$ and numbers $\varepsilon_a$ we can ensure that $\varepsilon_a < r(x)/2$ for every $x \in U_a$. Then the same covering and the same numbers will work for both $A$ and $A'$. Moreover, the index functors will be equal,

$$A_{U, \varepsilon} = A'_{U, \varepsilon} : X_U \to \hat{\mathcal{F}}(\mathbb{H}).$$

Therefore the index maps for $A$ and $A'$ can be assumed to be equal. It follows that the analytical index of $A$ in the sense of Section 2 is equal to that of $A'$. This proves the theorem for the analytical index in the sense of Section 2.

Let us consider now the classical analytical index. We may consider the classical analytical index of $A$ as the homotopy class of an index map $\Sigma X \to |\mathcal{F}|$, or, equivalently, of the index map $X \times [0, \pi] \to |\mathcal{F}|$, for the family $\mathbb{B}$. Similarly, the classical analytical index of $A'$ is the homotopy class of an index map of the family

$$B'_{x, t} = \text{id}_H \cos t + i A' x \sin t, \quad x \in X, \quad t \in [0, \pi],$$

which we will denote also by $\mathbb{B}'$. The proof of Theorem 8.2 includes a construction of open subsets $U_x \subset X$ and numbers $\delta_x$, $u_x$, $w_x$ such that the open subsets

$$U_x \times (u_x, \pi - u_x) \quad \text{and} \quad U_x \times ([0, w_x] \cup (\pi - w_x, \pi])$$

of $X \times [0, \pi]$ with parameters $\delta_x$ and $\cos w_x$ respectively can be used for constructing an index map for $\mathbb{B}$. The starting point of this construction is a choice of parameters $\alpha_x > 0$. One can always choose parameters $\alpha_x < r(x)/2$. Then Lemma 8.1 implies that the same open subsets $U_x$ and numbers $\delta_x$, $u_x$, $w_x$ work also for $\mathbb{B}'$, and the resulting index maps for $\mathbb{B}$ and $\mathbb{B}'$ are equal. This proves the theorem for the classical analytical index. ■

8.5. Theorem. Suppose that $X$ is paracompact. If $A$ is a strictly Fredholm family, then the two index maps $X \to \Omega |\mathcal{F}|$ are homotopic.

Proof. Theorem 8.4 implies that it is sufficient to prove the theorem for a finite-polarized replacement $A'$ in the role of $A$. The family $A'$ is a strictly Fredholm. By Theorem 4.6 there exists a strictly adapted to $A'$ trivialization of the bundle $\mathbb{H}$. As explained in Section 5, this allows to consider $A'$ as a family of self-adjoint Fredholm operators in the Hilbert space $H$, i.e. as a map $A' : X \to \hat{\mathcal{F}}$. We claim that this map is a continuous map to space $\hat{\mathcal{F}}$ considered with the norm topology.
We may assume that the pairs \((U, \epsilon_U)\) used for the construction of \(A'_x\) are strictly adapted to \(\mathbb{A}\). Let \(z \in X\), and let \(\epsilon = \max_{U \ni z} \epsilon_U\), where the maximum is taken over \(U\) such that \(x \in U\). Then \(\epsilon = \epsilon_U(z)\) for some set \(U(z)\) from the covering, and \((A_x, \epsilon)\) is an enhanced operator for \(x \in U(z)\). Moreover, \((U(z), \epsilon)\) is strictly adapted to \(\mathbb{A}\).

If \(z \not\in U\) for some set \(U\) form the covering, then \(z\) does not belong to the support of \(r_U\). Since \(r_U\) is strictly adapted to \(\mathbb{A}\), the projection \(P_{[-\epsilon, \epsilon]}(A_x)\) has finite rank and continuously depends on \(x\) for \(x \in V\). Therefore \(A'_x \circ P_{[-\epsilon, \epsilon]}(A_x)\) also continuously depends on \(x\) for \(x \in V\). Since \((V, \epsilon)\) is adapted to \(\mathbb{A}\), the projections

\[
P_{[-\epsilon, \epsilon]}(A_x) \quad \text{and} \quad P_{[\epsilon, \infty]}(A_x)
\]

continuously depend on \(x \in V\) in the norm topology. It follows that \(A'_x\) continuously depend on \(x \in V\) in the norm topology. Since the point \(x \in X\) was arbitrary, this proves that the map \(\mathbb{A}' : x \mapsto A'_x\) is continuous as a map to \(\hat{\mathbb{F}}\) with the norm topology.

The classical analytical index of \(\mathbb{A}'\) can be considered as the homotopy class of the composition of \(\mathbb{A}'\) with the Atiyah–Singer map \(\alpha : \hat{\mathbb{F}} \rightarrow \Omega \mathbb{F}\) defined by the formula (10) with omitted parameter \(x\), and then with the canonical homotopy equivalence \(\Omega \mathbb{F} \rightarrow \Omega |\mathcal{S}|\). The analytical index of \(\mathbb{A}'\) in the sense of Section 2 can be considered as the homotopy class of the composition of \(\mathbb{A}'\) with the homotopy equivalences \(\hat{\mathbb{F}} \rightarrow |\hat{\mathbb{F}}| \rightarrow \Omega |\mathcal{S}|\) from Section 2 and Theorem 8.3. Using the homotopy equivalence \(\Omega \mathbb{F} \rightarrow \Omega |\mathcal{S}|\) one can interpret this analytical index as the homotopy class of the composition of \(\mathbb{A}'\) with a canonical homotopy equivalence \(\alpha : \hat{\mathbb{F}} \rightarrow \Omega \mathbb{F}\). By \([12]\), Theorem 16.6, the maps \(\alpha\) and \(\alpha'\) are homotopic. It follows that the maps \(\alpha \circ \mathbb{A}'\) and \(\alpha' \circ \mathbb{A}'\) are homotopic, as also their compositions with \(\Omega \mathbb{F} \rightarrow \Omega |\mathcal{S}|\). It follows that the two index maps are homotopic. ■
A. Polarizations and trivializations of Hilbert bundles

Hilbert bundles and local trivializations. Let $X$ be a paracompact space and let $H$ be a separable infinite dimensional Hilbert space. Let $H$ be a locally trivial Hilbert bundle over $X$ with fibers isomorphic to $H$, thought as a family $H_x, x \in X$ of Hilbert spaces parameterized by $X$. Recall that a local trivialization of $H$ can be considered as a family of Hilbert space isomorphisms $t_U(x) : H_x \to H$, where $x$ runs over an open subset $U \subset X$. One can speak also about trivializations over an arbitrary subset of $X$.

Polarizations. A polarization of a Hilbert space $K$ is a presentation of $K$ as an orthogonal direct sum $K = K_- \oplus K_+$ of two infinitely dimensional closed subspaces $K_-, K_+$. Clearly, a polarization $K = K_- \oplus K_+$ is uniquely determined by $K_+$, and a polarization of $K$ can be also defined as closed infinitely dimensional subspace $P \subset K$ such that the orthogonal complement $K \ominus P$ is also infinitely dimensional. The set of polarizations of $K$ is equipped by the the topology defined by the norm topology of orthogonal projections $K \to P$. By a well known theorem of Atiyah and Singer [AS] the space of polarizations is contractible.

A local polarization of $H$ over an open set $U \subset X$ is a family of polarizations $P_x \subset H_x, x \in U$ such that for some trivialization $t_U$ of $H$ over $U$ the family $t_U(x)(P_x), x \in U$ of polarizations of $H$ is norm continuous. Two local polarizations $P_x \subset H_x, x \in U$ and $P'_x \subset H_x, x \in U'$ are said to be compatible at a point $z \in U \cap U'$ if there exists a neighborhood $V \subset U \cap U'$ of $z$ such that either $P_x$ is a subspace of finite codimension in $P'_x$ for every $x \in V$ and the family of orthogonal complements $P'_x \ominus P_x, x \in V$ is continuous, or this condition holds with the roles of $P_x$ and $P'_x$ interchanged. Two local polarizations $P_x \subset H_x, x \in U$ and $P'_x \subset H_x, x \in U'$ are said to be compatible if they are compatible at every point $z \in U \cap U'$.

A polarization atlas is a collection of pair-wise compatible local polarizations over some open subsets covering $X$. Two polarization atlases are said to be equivalent if their union is also a polarization atlas. A polarization of $H$ is an equivalence class of polarization atlases. Clearly, every polarization is represented by unique maximal atlas. A local polarization is said to be a chart of a polarization $P$ if it belongs to the maximal atlas of $P$. If $P$ is a polarization of $H$ and $Y \subset X$, then $P$ defines, in an obvious way, a polarization of the restriction $H|Y$ of $H$ to $Y$, which we will denote by $P|Y$.

Adapted trivializations. Let $P$ be a polarization of $H$, and $P_x, x \in U$ be a chart of $P$. A trivialization $t_V$ of $H$ defined over a subset $V \subset X$ is said to be adapted to the chart $P_x, x \in U$ if the family of polarizations $t_V(x)(P_x), x \in U \cap V$ of $H$ is norm continuous. A trivialization $t_V$ is said to be adapted to the polarization $P$ if $t_V$ is adapted to every chart of $P$. The following properties of adapted trivializations immediately follow from the definitions and will be used without references.
A.1. Lemma. Let \( P \) be a polarization of \( H \), and \( P_x, x \in U \) and \( P'_x, x \in U' \) be some charts of \( P \). If \( V \subset U \) and a trivialization \( t_V(x), x \in V \) is adapted to \( P_x, x \in U \), then it is adapted to \( P'_x, x \in U' \). If a trivialization \( t_V(x), x \in V \) is adapted to every chart from some atlas of \( P \), then \( t_V \) is adapted to \( P \). ■

Existence of adapted trivializations. Recall that every Hilbert bundle over a paracompact space is trivial. In particular, there exists a trivialization of \( H \), i.e. a local trivialization defined over the whole space \( X \). We will strengthen this result by proving that for every polarization \( P \) of \( H \) there exists a trivialization of \( H \) adapted to \( P \). We will consider first the simpler case of triangulable spaces \( X \).

A.2. Theorem. Let \( P \) be a polarization of \( H \). If \( X \) is a triangulable space, then there exists a trivialization of \( H \) adapted to \( P \).

Proof. Let us choose an atlas of charts \( P^U_x \subset H_x, x \in U \) defining \( P \). Replacing a triangulation of \( X \) by a subdivision we can assume that every simplex \( \sigma \) of a fixed triangulation of \( X \) is contained in one of the sets \( U \). Let \( S_k \) be the \( n \)th skeleton of this triangulation.

Let us begin by choosing isomorphisms \( t(v): H_v \to H \) for the vertices \( v \) of the triangulation. Such isomorphisms define a trivialization of the restriction \( H|S_0 \) of \( H \). This trivialization is obviously adapted to \( P|S_0 \).

Suppose that we already found a trivialization \( t \) of \( H|S_n \) adapted to \( P|S_n \). Let \( \sigma \) be an \((n + 1)\)-simplex of our triangulation, and let \( \partial \sigma \) be its boundary. By our assumptions there exists a chart \( P^U_x, x \in U \) such that \( \sigma \subset U \). By the definition there exists a local trivialization \( t_U \) of \( H \) over \( U \) adapted to \( P^U_x, x \in U \). Then the families

\[
t(P^U_x), x \in \partial \sigma \quad \text{and} \quad t_U(P^U_x), x \in \partial \sigma
\]

of polarizations of \( H \) are norm continuous. The contractibility of the space of polarizations of \( H \) implies that the second family is homotopic to the first one. The second family is defined also for \( x \in U \), and we can extend this homotopy to \( U \). The last family in the extended homotopy is a new local trivialization \( t'_U \) adapted to \( P^U_x, x \in U \) and such that

\[
t(x)(P^U_x) = t'_U(x)(P^U_x)
\]

for every \( x \in \partial \sigma \). The family \( P^U_x, x \in \partial \sigma \) is a Hilbert subbundle of \( H|\partial \sigma \). It is trivial as a Hilbert bundle, and its orthogonal complement \( H_x \ominus P^U_x, x \in \partial \sigma \) is also trivial. By using some trivializations of these bundles we can consider the maps

\[
t'_U(x)^{-1} \circ t(x), x \in \partial \sigma
\]

as isomorphisms of a Hilbert space \( K \) respecting a polarization \( K = K_- \oplus K_+ \). The con-
tractibility of the unitary groups of Hilbert spaces in the compact-open topology (see Atiyah and Segal [AS]) implies that this family is homotopic by families of maps respecting the polarization to the family of identity maps. By taking the compositions with \( t'_U(x) \), \( x \in \partial \sigma \) we get a homotopy from the trivialization \( t'_U(x) \), \( x \in \partial \sigma \) to the trivialization \( t(x) \), \( x \in \partial \sigma \) by trivializations adapted to \( P_U \), \( x \in U \). By extending this homotopy to \( x \in \sigma \) we get a trivialization \( t''(x) \), \( x \in \sigma \) adapted to \( P_U \), \( x \in U \) and such that

\[
t''(x) = t(x)
\]

for every \( x \in \partial \sigma \). Let us extend \( t \) from \( \text{Sk}_n X \) to \( \text{Sk}_n X \cup \sigma \) by \( t'' \) and denote the resulting trivialization again by \( t \). Then the restriction of \( t \) to \( \sigma \) is adapted to \( P_U \), \( x \in U \). It follows that \( t \) is adapted to \( P|\text{Sk}_n X \cup \sigma \). By doing this for all \((n+1)\)-simplices simultaneously, we can extend \( t \) to a trivialization of \( H|\text{Sk}_{n+1} X \) adapted to \( P|\text{Sk}_{n+1} X \). By continuing in this way we will get an adapted trivialization of \( H \).

\[\Box\]

**A.3. Theorem.** Let \( P \) be a polarization of \( H \). If \( X \) is a paracompact space, then there exists a trivialization of \( H \) adapted to \( P \).

**Proof.** Let us choose an atlas of charts \( P^a \), \( x \in U_a \) defining \( P \), where \( a \) runs over some set \( \Sigma \), and let us choose for every \( a \in \Sigma \) a local trivializations \( t_a \) of \( H \) over \( U_a \) such that the family \( t_a(x)(P^a) \), \( x \in U_a \) is norm continuous.

Let \( \Sigma^{\text{fin}} \) and \( U_\sigma, \varepsilon_\sigma \) for \( \sigma \in \Sigma^{\text{fin}} \) be defined as in Section 2, and let \( X_U \) be the topological category from Section 2. The space of objects of \( X_U \) is the disjoint union of subspaces \( U_\sigma \), and the category \( X_U \) is defined by an order \( \leq \) on \( \text{Ob} X_U \). Clearly, the set pairs \((u, u)\), where \( u \in \text{Ob} X_U \), is the union of several components of the set of pairs \((u, v)\) of comparable objects of \( X_U \), i.e. such that either \( u \leq v \) or \( v \leq u \). In the terminology of [I_2] this means that \( \text{Ob} X_U \) is a partially ordered space with free equalities, and hence \( X_U \) can be treated as a topological simplicial complex. In particular, \(|X_U|\) is equal to the “naive” geometric realization \(|X_U|\). We refer to [I_2], Section 5, for the definition of the geometric realization \([\bullet]\) and its basic properties.

Let \( H^U \) be the bundle induced from \( H \) by \([\text{pr}] : [X_U] \longrightarrow X\). The polarization \( P \) induces a polarization \( P^U \) of of \( H^U \). If \( s \) is a homotopy inverse of \(|\text{pr}| \) as above, then \( H \) and \( P \) are equal to the bundle and the family of operators induced by \( s \) from \( H^U \) and \( P^U \) respectively. Therefore it is sufficient to prove that there exists a trivialization of \( H^U \) adapted to \( P^U \). As in the proof of Theorem A.2, we will prove this using an induction by skeletons. If \( \sigma \in \Sigma^{\text{fin}} \) and \( a \in \sigma \), then the restriction \( t_\sigma \) of \( t_a \) to \( U_\sigma \subset U_a \) is adapted to \( P \). Hence the trivializations \( t_\sigma \) define a trivialization of the restriction of \( H^U \) to the 0th skeleton \([\text{Sk}_0 X_U]\) adapted to the restriction of \( P^U \).

Suppose that we constructed a trivialization \( t \) of the restriction \( H^U|\text{[Sk}_n X_U| \) adapted to \( P^U \), \( u \in \text{[Sk}_n X_U| \) for some \( n \geq 0 \). An \((n+1)\)-simplex of \( X_U \) is defined by a point \( x \in X \)
and a strictly decreasing sequence
\[ \sigma(n + 1) \supseteq \sigma(n) \supseteq \ldots \supseteq \sigma(0) \]
of elements of \( \Sigma^\text{fin} \). Therefore the space of \((n + 1)\)-simplices of \( X_U \) can be identified with the disjoint union of the sets \( U_{\sigma(n+1)} \) over all such sequences. In particular, every component of the space of \((n + 1)\)-simplices can be identified with \( U_{\sigma} \) for some \( \sigma \in \Sigma^\text{fin} \). Let us consider such a component \( U_{\sigma} \) and the canonical map
\[ e_\sigma : \Delta^{n+1} \times U_{\sigma} \longrightarrow [X_U] = |X_U|. \]
The bundle \( H^U \) and the polarization \( \mathbb{P}^U \) induce a bundle \( H^\sigma \) and a polarization \( \mathbb{P}^\sigma \) over \( \Delta^{n+1} \times U_{\sigma} \), and the already constructed trivialization \( t \) defines a trivialization \( t' \) of \( H^\sigma \) over \( \partial \Delta^{n+1} \times U_{\sigma} \) adapted \( \mathbb{P}^\sigma \). Since the map \( |\text{pr}| \) collapses each simplex to a point, the fibers of \( H^U \) over a simplex can be treated as equal, as also the polarizations \( \mathbb{P}^U \). Hence the trivialization \( t_\sigma \) defines a trivialization of \( H^\sigma \) adapted to \( \mathbb{P}^\sigma \).

Let us use this trivialization to identify \( H^\sigma \) with the trivial bundle over \( \Delta^{n+1} \times U_{\sigma} \). Now we can use the contractibility of the space of polarizations and the contractibility of the unitary groups in the compact-open topology as in the proof of Theorem 4.5 to extend \( t' \) to a trivialization \( t'' \) of \( H^\sigma \) over \( \Delta^{n+1} \times U_{\sigma} \) adapted to \( \mathbb{P}^\sigma \). Since we are dealing with a topological simplicial complex, the map \( e_\sigma \) is injective and we can extend \( t \) to the image of the map \( e_\sigma \) by using \( t'' \). By doing this simultaneously for every component of the space of \((n + 1)\)-simplices we can extend \( t \) to next skeleton \([\text{Sk}_{n+1} X_U]\). The topology of \( |X_U| = |X_U| \) is, by the definition, the direct limit of the topologies of the skeletons. Therefore, by continuing in this way we will get a trivialization of \( H^U \) adapted to \( \mathbb{P}^U \).

\textbf{A.4. Theorem.} \textit{Under the assumptions of Theorem A.2 or A.3, suppose that \( Y \) is a subcomplex or closed subset of \( X \) respectively. If \( t(y), y \in Y \) is a trivialization of \( H|Y \) adapted to \( \mathbb{P}|Y \), then \( t \) can be extended to an adapted trivialization of \( \mathbb{P} \).}

\textbf{Proof.} The proof is a standard modification of the proofs of Theorems A.2 and A.3.

\textbf{A.5. Corollary.} \textit{Under the assumptions of Theorems A.2 or A.3, every two trivializations adapted to \( \mathbb{P} \) are homotopic in the class of trivializations adapted to \( \mathbb{P} \).}

\textbf{Proof.} It is sufficient to apply Theorem A.4 to the subset \( X \times \{0, 1\} \) of \( X \times [0, 1] \).

\textbf{Remark.} M. Prokhorova [P2], influenced by the first version of the present paper, considered families of local polarizations compatible in a weaker sense, related to ours as compact operators are related to operators of finite rank. Namely, the orthogonal projections onto \( P_x \) and \( P'_x \) are required to differ by compact operators continuously depending on \( x \).
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