Crossed $S$-matrices and Character Sheaves on Unipotent Groups

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Abstract

Let $k$ be an algebraic closure of a finite field $\mathbb{F}_q$ of characteristic $p$. Let $G$ be an algebraic group over $k$ equipped with an $\mathbb{F}_q$-structure given by a Frobenius map $F : G \rightarrow G$. We will denote the corresponding algebraic group defined over $\mathbb{F}_q$ by $G_0$. Character sheaves on $G$ are supposed to be certain objects in the triangulated braided monoidal category $\mathcal{D}_G(G)$ of bounded conjugation equivariant $\mathbb{Q}_l$-complexes (where $l \neq p$ is a prime number) on $G$. If $C \in \mathcal{D}_G(G)$ is any object equipped with an isomorphism $\psi : F^*(C) \xrightarrow{\sim} C$ and $g \in G$ then using Grothendieck’s sheaf-function correspondence we can define the “trace of Frobenius” class function $t_{g,\psi}^C : G_0[\mathbb{F}_q] \rightarrow \mathbb{Q}_l$ on each pure inner form $G_0^g$ of $G_0$ corresponding to the modified Frobenius, $ad(g) \circ F : G \rightarrow G$.

In [Bo2], Boyarchenko proved that if the neutral connected component $G^o$ is unipotent, then the functions associated with $F$-stable character sheaves on $G$ form an orthonormal basis of the space of class functions on all pure inner forms $G_0^g(\mathbb{F}_q)$ and that the matrix relating this basis to the basis formed by the irreducible characters of the pure inner forms $G_0^g(\mathbb{F}_q)$ is block diagonal with “small” blocks. In this paper we describe these block matrices and interpret them as certain “crossed $S$-matrices”.

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1 Introduction

Let \( k = \mathbb{F}_q \), where \( q \) is a power of a prime \( p \). Let \( G \) be an algebraic group (or perfect quasi-algebraic group, see \([2.1]\)) over \( k \) equipped with an \( \mathbb{F}_q \) structure given by a Frobenius map \( F : G \to G \). Let us denote the corresponding algebraic group defined over \( \mathbb{F}_q \) by \( G_0 \). Let us fix a prime number \( l \neq p \).

One of the main goals of the theory of character sheaves is to study the \( \mathbb{Q}_l \)-valued irreducible characters of the groups \( G_0(\mathbb{F}_q) \) using the machinery of \( \mathbb{Q}_l \)-sheaves and complexes on \( G \).

In the series of papers \([L]\), Lusztig developed a theory of character sheaves on reductive groups and used it to study the characters of finite groups of Lie type. Inspired by Lusztig’s work, Boyarchenko and Drinfeld (in \([BD1]\), \([BD2]\), \([Bo1]\), \([Bo2]\)) developed a theory of character sheaves on unipotent groups and related it to the character theory of finite unipotent groups. In both these cases, character sheaves are isomorphism classes of certain objects in the \( \mathbb{Q}_l \)-linear triangulated braided monoidal category \( D_G(G) \) of conjugation equivariant \( \mathbb{Q}_l \)-complexes on \( G \). We have an action of Frobenius on the set of character sheaves, or in other words pullback by Frobenius of a character sheaf is also a character sheaf.

Let us suppose that \( G \) is a connected unipotent group. In \([Bo2]\), Boyarchenko proves that there is a bijection between the set of irreducible characters of \( G_0(\mathbb{F}_q) \) and the set of isomorphism classes of character sheaves fixed by Frobenius. More precisely he shows that the \( \mathbb{Q}_l \)-valued “trace of Frobenius” functions on \( G_0(\mathbb{F}_q) \) associated with \( F \)-stable character sheaves form an orthonormal basis for the space of \( \mathbb{Q}_l \)-valued class functions on \( G_0(\mathbb{F}_q) \) and that the matrix relating this basis to the basis formed by the irreducible characters of \( G_0(\mathbb{F}_q) \) is block diagonal with “small” blocks labelled by the \( F \)-stable \( \mathbb{Q}_l \)-packets. A similar result is proved by Lusztig in \([L]\) for reductive groups.

Now let \( G \) be (possibly disconnected) such that its neutral connected component \( G^0 \) is unipotent. (The notion of character sheaves on \( G \) is well defined in this case.) In this case, the number of \( F \)-stable character sheaves on \( G \) may be strictly larger than the number of irreducible characters of the group \( G_0(\mathbb{F}_q) \). As observed in \([Bo2]\), in this situation it is more natural to consider all pure inner forms (see \([2.4.1]\) of \( G_0 \)). One reason for this is that if \( C \in D_G(G) \) is such that we have an isomorphism \( \psi : F^*(C) \cong C \), then we can define “trace of Frobenius” function \( t^\psi_{C,F} : G_0^0(\mathbb{F}_q) \to \mathbb{Q}_l \) for the pure inner form \( G_0^0(\mathbb{F}_q) \) corresponding to each \( g \in G \) (see \([2.4.8]\)). The above discrepancy between the number of characters and character sheaves disappears once we consider the irreducible characters of the pure inner forms as well. Once again the “trace of Frobenius functions” of \( F \)-stable character sheaves satisfy similar orthogonality relations and the matrix relating them to the irreducible characters of all inner forms is again block diagonal as in the connected case. (See \([Bo2]\) for more.)

The main goal of this paper is to describe these block matrices. It is shown in \([Bo2]\) that these blocks are labelled by \( F \)-stable minimal idempotents \( e \in D_G(G) \). Now as proved in \([BD2]\), we have the modular category \( M_{G,e} \subset eD_G(G) \) associated with the minimal idempotent \( e \). Since \( e \) is
$F$-stable, $F^*$ induces an autoequivalence of the modular category $\mathcal{M}_{G,e}$. We prove that the block matrix labelled by $e$ is the “crossed $S$-matrix” associated with the modular category $\mathcal{M}_{G,e}^+$ and its autoequivalence $F^*$.

In the first part of the paper, we work with a general algebraic group $G$ and derive an inner product formula (see Theorem 2.14) for the inner product of the “trace of Frobenius” function of a $F$-stable object in $\mathcal{D}_G(G)$ and the character of a representation of an inner form $G^0_0(F_q)$. We then use this formula to describe the block matrices for groups having a unipotent neutral connected component.

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2 Preliminaries and main results

In this section we describe the main definitions and constructions required in this paper, state the main results and describe the organization of the text.

2.1 Notation and conventions

We fix a prime number $p$ and $q$, a power of $p$. The field $k$ will always denote an algebraically closed field of characteristic $p$. Typically we will assume that $k = \overline{F}_q$. All algebraic groups and schemes are assumed to be over $k$ unless explicitly stated otherwise. We also fix a prime number $l$ different from $p$. For a scheme $X$ over $k$, we set $\mathcal{D}(X) := D^b_c(X, \overline{Q}_l)$, the $\overline{Q}_l$-linear triangulated category of bounded constructible $\overline{Q}_l$-complexes on $X$. If an algebraic group $G$ acts on $X$, we let $\mathcal{D}_G(X)$ denote the equivariant derived category. It is defined as the category of $\overline{Q}_l$-complexes on the quotient stack $G \setminus X$. If $G$ is such that $G^0$ is unipotent, then objects of $\mathcal{D}_G(X)$ may be thought of as pairs $(M, \phi)$, where $M \in \mathcal{D}(X)$ and $\phi$ is a $G$-equivariance structure on $M$. We refer to [BD2] for details. We will omit the symbols $L, R$ when talking about Grothendieck’s six functors and all such functors should be interpreted in the derived sense. We denote the Verdier duality functor by $\mathbb{D}$.

If $G$ is an algebraic group, then $\mathcal{D}(G)$ has the structure of a monoidal category under convolution with compact supports. The unit object of this monoidal category is the delta sheaf $\delta_1$ supported at the identity $1 \in G$. The conjugation equivariant category $\mathcal{D}_G(G)$ has the structure of a braided monoidal category. We refer to [BD2] for complete definitions. Let $\iota : G \to G$ be the inversion map. The functor $\mathbb{D}^- := \iota^* \mathbb{D} = \mathbb{D} \iota^*$ defines a duality in the categories $\mathcal{D}(G)$ and $\mathcal{D}_G(G)$ which is

1 This is the same as $\mathcal{M}_{G,e}$, but equipped with the positive spherical structure. Conjecturally the natural spherical structure on $\mathcal{M}_{G,e}$ is itself positive. This is known to be true if $G$ is itself unipotent.
weaker than rigidity. This weak duality makes $\mathcal{D}(G)$ and $\mathcal{D}_G(G)$ $\tau$-categories (see [BD2 §A]). The natural identification $(\mathbb{D}^-)^2 \cong Id$ defines a structure of a ribbon $\tau$-category on $\mathcal{D}_G(G)$. We refer to [BD2 §A] for a detailed exposition.

We will often work in the setting of perfect schemes and perfect group schemes over $k$. We refer to [BD2 §1.9] for more details on the notions of perfect groups, perfect schemes and the perfectization functor. We often abuse notation and use the same symbol to donate a scheme (or algebraic group) and its perfectization. This is not likely to cause any confusion since the categories $\mathcal{D}(X)$ and the groups $G(k), G_0(\mathbb{F}_q)$ do not change after passing to the perfectization. There are many advantages of passing to the perfectization. For example, after passing to the perfectization the Frobenius map $F$ becomes an isomorphism. Also perfectness is needed in the very useful notion of Serre duality of connected unipotent groups. Hence from now on by schemes we in fact mean perfect quasi-algebraic schemes (i.e. perfectizations of schemes of finite type) and by algebraic groups we mean perfect quasi-algebraic groups (i.e. perfectizations of algebraic groups) even if we do not mention this explicitly.

2.2 Sheaf-function correspondence

Let $X$ be a scheme over $k$ equipped with an $\mathbb{F}_q$-structure given by a Frobenius $F : X \to X$. Let $X_0$ be the corresponding scheme defined over $\mathbb{F}_q$. Let $\mathcal{D}^{Wei}(X_0)$ be the category consisting of pairs $(M,\psi)$ such that $M \in \mathcal{D}(X)$ and $\psi : F^*(M) \overset{\cong}{\to} M$ is an isomorphism in $\mathcal{D}(X)$. We call objects of $\mathcal{D}^{Wei}(X_0)$ as Weil complexes on $X$. We have the natural functor $\mathcal{D}(X_0) \to \mathcal{D}^{Wei}(X_0)$. If $(M,\psi) \in \mathcal{D}^{Wei}(X_0)$, then we have the stalk maps $\psi(x) : M_{F(x)} \to M_x$ between complexes of $\mathbb{Q}_l$-vector spaces. Then we define the trace of Frobenius function $t_{M,\psi} : X_0(\mathbb{F}_q) \to \mathbb{Q}_l$ by $x \mapsto tr(\psi(x))$.

If $X, Y$ are schemes over $k$ with an $\mathbb{F}_q$-structure and if $f : X \to Y$ is an $\mathbb{F}_q$-morphism, then we have the induced six functors $f_*, f^!, f^*, f^\sharp, \otimes, Hom$ with all the standard adjunctions in the context of Weil complexes as well. The next lemma says that pullbacks, pushforwards with compact support and tensor product are compatible with the sheaf-function correspondence.

**Lemma 2.1.** (See [Bo1, Lem. 4.4] for more details.) Let $f : X \to Y$ be as in the previous paragraph. Then

(i) If $N \in \mathcal{D}^{Wei}(Y_0)$, then $f^\ast(N) = f^\ast(t_N) := t_N \circ f$.

(ii) If $M, K \in \mathcal{D}^{Wei}(X_0)$, then $t_{M \otimes K} = t_M \cdot t_K$.

(iii) Assume that $f$ is also separated. If $M \in \mathcal{D}^{Wei}(X_0)$, then $t_{f^\ast(M)} = f(t_M)$.

An immediate consequence is the following:

**Lemma 2.2.** Let $G$ be an algebraic group with an $\mathbb{F}_q$-structure. Then if $M, N \in \mathcal{D}^{Wei}(G_0)$, then $t_{M \ast N} = t_M \ast t_N$, where the right hand side is the convolution of functions on the finite group $G_0(\mathbb{F}_q)$.

---

2By a slight abuse of notation we also let $f$ denote the induced map of finite sets $f : X_0(\mathbb{F}_q) \to Y_0(\mathbb{F}_q)$. In this abused context, $f^\ast$ also denotes pullback of functions on $Y_0(\mathbb{F}_q)$ along $f$, and $f_!$ also means summing up a function on $X_0(\mathbb{F}_q)$ along the fibres of $f$. 
2.3 Braided crossed categories and crossed $S$-matrices

Let $\Gamma$ be an abstract group. We briefly recall the notion of a braided $\Gamma$-crossed category. We refer to \cite[§4.4.3]{DGNO} for a precise definition and properties of braided crossed categories and related concepts. A braided $\Gamma$-crossed category $\mathcal{D}$ is an additive monoidal category with the following structure:

- A monoidal grading $\mathcal{D} = \bigoplus_{a \in \Gamma} \mathcal{C}_a$.
- A monoidal action of $\Gamma$ on $\mathcal{D}$ such that $a(\mathcal{C}_b) \subset \mathcal{C}_{aba^{-1}}$ for each $a, b \in \Gamma$.
- For $a \in \Gamma$, $M \in \mathcal{C}_a$ and $C \in \mathcal{D}$ isomorphisms (called crossed braiding isomorphisms)
  $$\beta_{M,C} : M \otimes C \xrightarrow{\cong} a(C) \otimes M$$
  functorial in $M, C$ and satisfying certain compatibility conditions which we do not explicitly recall here. These conditions in particular imply that $\mathcal{C}_1$ is a braided monoidal category and that the induced action of $\Gamma$ on $\mathcal{C}_1$ is a braided action.

We say that such a $\mathcal{D}$ is faithfully graded if $\mathcal{C}_a$ is nonzero for each $a \in \Gamma$. In this paper we will only encounter or consider faithfully graded categories.

Let $\mathcal{D}$ be such a category and let $a \in \Gamma$. Let $M \in \mathcal{C}_a$ and $C \in \mathcal{C}_1$ be such that $a(\mathcal{C}) \cong \mathcal{C}$. Let us fix an isomorphism $\psi : a^{-1}(\mathcal{C}) \cong \mathcal{C}$. We have the following composition

$$C \otimes M \xrightarrow{\psi^{-1} \otimes \text{id}_M} a^{-1}(\mathcal{C}) \otimes M \xrightarrow{\beta_{a^{-1}(\mathcal{C}),M}} M \otimes a^{-1}(\mathcal{C}) \xrightarrow{\beta_{M,a^{-1}(\mathcal{C})}} C \otimes M \text{ in } \mathcal{C}_a. \quad (1)$$

Let $\gamma_{C,\psi,M} \in \text{Aut}_{\mathcal{C}_a}(C \otimes M)$ denote the inverse of the composition above.

2.3.1 Module categories and crossed $S$-matrices

Let $\mathcal{C}$ be a ($\mathbb{Q}_l$-linear) modular category and let $\mathcal{M}$ be an invertible $\mathcal{C}$-module category. Let $\mathcal{O}_\mathcal{C}$ and $\mathcal{O}_\mathcal{M}$ denote the sets of isomorphism classes of simple objects in $\mathcal{C}$ and $\mathcal{M}$ respectively. Let $\mathcal{M}$ be equipped with a $\mathcal{C}$-module trace $\text{tr}_\mathcal{M}$. This means that we can define traces $\text{tr}_\mathcal{M}(f) \in \mathbb{Q}_l$ of endomorphisms $f$ in $\mathcal{M}$ and these are compatible with the spherical structure on $\mathcal{C}$. (We refer to \cite{S} for details.) In particular, using the module trace we can define categorical dimensions of objects of $\mathcal{M}$, namely if $M \in \mathcal{M}$, then $\dim(M) := \text{tr}_\mathcal{M}(\text{id}_M)$. The compatibility with the spherical structure on $\mathcal{C}$ implies that $\dim(C \otimes M) = \dim(C) \cdot \dim(M)$ for each $C \in \mathcal{C}, M \in \mathcal{M}$. We impose an addition condition on the trace, namely we assume that the trace is normalized in such a way that

$$\sum_{M \in \mathcal{O}_\mathcal{M}} \dim(M)^2 = \dim(\mathcal{C}). \quad (2)$$

Since $\mathcal{M}$ is an invertible $\mathcal{C}$-module category that admits a module trace we obtain using \cite[Thm. 5.2]{ENO} the corresponding autoequivalence $a : \mathcal{C} \to \mathcal{C}$ of modular categories which is unique up to natural equivalence of functors. Hence this induces a well defined permutation (which we also
call \( a \) of the set \( \mathcal{O}_C \). For \( C \in \mathcal{O}_C^a \), we choose an isomorphism \( \psi : a^{-1}(C) \xrightarrow{\cong} C \). Then by definition of the autoequivalence \( a \) we have the following composition in \( \mathcal{M} \) analogous to (1):

\[
C \otimes M \xrightarrow{\cong} a^{-1}(C) \otimes M \xrightarrow{\cong} M \otimes a^{-1}(C) \xrightarrow{\cong} C \otimes M \quad \text{for} \quad C \in \mathcal{C}, M \in \mathcal{M}.
\]

We let \( \gamma_{C,\psi, M} \) denote the inverse of this composition.

**Remark 2.3.** Using [ENO] we see that if \( \mathcal{D} \) is a spherical (faithfully graded) braided \( \Gamma \)-crossed category whose trivial component \( C_1 \) is a modular category then for each \( a \in \Gamma \), the component \( C_a \) is an invertible \( C_1 \)-module category (with module trace induced by the spherical structure on \( \mathcal{D} \)) which satisfies the requirements of the first paragraph of this subsection. In particular, the \( C_1 \)-module trace on \( C_a \) obtained from the spherical structure of \( \mathcal{D} \) is normalized in the sense of (2). We note that in this situation, the automorphism \( \gamma_{C,\psi, M} \) defined above matches with the one defined by (1).

**Definition 2.4.** Let \( \mathcal{C} \), \( \mathcal{M} \), and \( a : \mathcal{C} \rightarrow \mathcal{C} \) be as before. For each \( C \in \mathcal{O}_C^a \) let us choose isomorphisms \( \psi_C : a^{-1}(C) \rightarrow C \). We define the crossed \( S \)-matrix associated with the modular category \( C \) and the \( \mathcal{C} \)-module category \( \mathcal{M} \) to be the matrix \( S_{\mathcal{C},\mathcal{M}} = \left( S_{C,M}^{C,M} \right)_{C \in \mathcal{O}_C^a, M \in \mathcal{O}_M} \) whose entries are defined by

\[
S_{C,M}^{C,M} := \text{tr}_\mathcal{M}(\gamma_{C,\psi_C,M}) \quad \text{for each} \quad C \in \mathcal{O}_C^a, M \in \mathcal{O}_M.
\]

Our goal in this paper is to show that the blocks in the block diagonal matrix relating characters and character sheaves are precisely such crossed \( S \)-matrices for suitable choices of the categories \( \mathcal{C} \) and \( \mathcal{M} \) and to describe these categories corresponding to each block.

**Remark 2.5.** The crossed \( S \)-matrix is well defined only up to a rescaling of the rows since the definition involves a choice of the isomorphisms \( \psi_C \).

**Remark 2.6.** Let us consider \( \mathcal{C} \) as an invertible \( \mathcal{C} \)-module category equipped with the trace coming from the spherical structure. We choose the autoequivalence \( a = \text{id}_\mathcal{C} \) and for each \( C \in \mathcal{O}_C \), we choose \( \psi_C = \text{id}_C \). With these choices \( S_{\mathcal{C},\mathcal{C}}^{\mathcal{C},\mathcal{C}} \) equals the usual \( S \)-matrix of the modular category \( \mathcal{C} \).

### 2.4 Arbitrary algebraic groups

We first discuss some generalities for arbitrary (perfect quasi-) algebraic groups \( G \) over \( k \) and state our first main result (Theorem 2.14) which holds for arbitrary algebraic groups \( G \) over \( k \) equipped with an \( \mathbb{F}_q \)-structure. In this subsection we assume that \( G \) is any algebraic group over \( k \) and \( F : G \rightarrow G \) is a Frobenius automorphism which equips \( G \) with an \( \mathbb{F}_q \)-structure. We begin by recalling the notion of pure inner forms and explain their natural role in the theory of character sheaves.

#### 2.4.1 The \( F \)-conjugation action and pure inner forms

We define the \( F \)-conjugation action of \( G \) on itself by \( g : h \mapsto ghF(g^{-1}) \). Let us denote the set of \( F \)-conjugacy classes in \( G \) by \( H^1(F, G) \). By Lang’s theorem, we have a natural bijection \( H^1(F, G) \xrightarrow{\cong} H^1(F, \pi_0(G)) \) and \( H^1(F, G) \) is a finite set.
For each $g \in G$, the map $ad(g) \circ F : G \to G$ is a Frobenius map which defines a new $F_q$-structure on $G$ and we denote the corresponding group defined over $F_q$ by $G_0^q$ and call it the pure inner form of $G_0$ determined by $g \in G$. The isomorphism class of $G_0^q$ only depends on the $F$-conjugacy class of $g$ since we have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{ad(g)\circ F} & G \\
ad(h) \downarrow & & \downarrow ad(h) \\
G & \xrightarrow{ad(hgF(h^{-1}))\circ F} & G.
\end{array}
$$

If $I \subset G$ is a set of representatives of $F$-conjugacy classes in $G$, we consider the finite set of pure inner forms $\{G_0^q\}_{t \in I}$. As we will see, it is more natural to consider all the pure inner forms $\{G_0^q\}$ than to just consider $G_0$.

From the diagram above, we also see that the $F$-conjugacy class $<g> \in H^1(F,G)$ determines $G_0^q$ up to an isomorphism that is unique up to inner automorphisms defined over $F_q$. In particular we see that the commutative $\mathbb{Q}_l$-algebra $\text{Fun}(G_0^q(F_q))^{G_0^q(F_q)}$ of class functions under convolution is canonically determined by $<g> \in H^1(F,G)$. Similarly the set $\text{Irrep}(G_0^q(F_q))$ is also canonically determined by the $F$-conjugacy class of $g$.

Let us define the set

$$\text{Irrep}(G_0) := \bigsqcup_{<t> \in H^1(F,G)} \text{Irrep}(G_0^q(F_q)). \tag{3}$$

**Remark 2.7.** The algebraic groups $G_0^q$ and $G_0^h$ defined over $F_q$ may still be isomorphic even though $g, h \in G$ may lie in different $F$-conjugacy classes. Even so, we regard $G_0^q$ and $G_0^h$ as distinct pure inner forms in this case.

**Remark 2.8.** Note that if $G$ is connected, then there is only one $F$-conjugacy class and we only need to consider the trivial pure inner form $G_0$.

**Proposition 2.9.** (See [Bo2, Cor. 4.10].) For each pure inner form $G_0^q$ there is a natural equivalence $\mathcal{D}_{G_0}(G_0) \cong \mathcal{D}_{G_0^q}(G_0^q)$ of braided monoidal categories. If $C_0 \in \mathcal{D}_{G_0}(G_0)$, we let $C_0^q$ denote the corresponding object in $\mathcal{D}_{G_0^q}(G_0^q)$ obtained using this transport functor.

### 2.4.2 $\mathcal{D}^F_G(G)$ as a triangulated category

Let $\mathcal{D}^F_G(G)$ denote the category of $G$-equivariant $\mathbb{Q}_l$-complexes on $G$ for the $F$-conjugation action of $G$ on itself. By Lang’s theorem each $F$-conjugacy class in $G$ is a union of certain connected components of $G$. Hence we have

$$\mathcal{D}^F_G(G) = \bigoplus_{\mathcal{O} \in H^1(F,G)} \mathcal{D}^F_G(\mathcal{O}). \tag{4}$$

For an $F$-conjugacy class $\mathcal{O} \subset G$, let us describe the category $\mathcal{D}^F_G(\mathcal{O})$. The $F$-conjugation action on $\mathcal{O}$ is transitive, hence we have an equivalence $\mathcal{D}^F_G(\mathcal{O}) \cong \mathcal{D}_{\text{Stab}^F_G(t)}$ where $t$ is some point in $\mathcal{O}$ and $\text{Stab}^F_G(t)$ is its stabilizer for the $F$-conjugation action. But $\text{Stab}^F_G(t) = G^{ad(t)\circ F} = G_0^q(F_q)$ is a finite group. Hence we have

$$\mathcal{D}^F_G(\mathcal{O}) \cong D^b(\text{Rep}(G_0^q(F_q))). \tag{5}$$

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Hence $\mathcal{D}_G^F(G)$ as a $\mathbb{Q}_l$-linear triangulated category encodes the representation theory of the groups of $\mathbb{F}_q$-points of all the pure inner forms of $G_0$. The set $\text{Irrep}(G_0)$ defined by \cite{3} along with all their shifts give all the simple objects of the triangulated category $\mathcal{D}_G^F(G)$ (which also happens to be semisimple abelian). For $W \in \text{Irrep}(G_0(\mathbb{F}_q)) \subset \text{Irrep}(G_0)$, let $W_{\text{loc}} \in \mathcal{D}_G^F(G)$ denote the corresponding object. $W_{\text{loc}}$ is a local system supported on the $F$-conjugacy class $\mathcal{O}$ of $t$.

2.4.3 $\mathcal{D}_G^F(G)$ as a $\mathcal{D}_G(G)$-module category

Let $C \in \mathcal{D}_G(G)$ and $M \in \mathcal{D}_G^F(G)$. Consider the action of $G$ on $G \times G$ given by conjugation of the first coordinate and $F$-conjugation of the second coordinate. Then $C \boxtimes M \in \mathcal{D}_G^{1,F}(G \times G)$, the equivariant derived category for this action of $G$ on $G \times G$. The multiplication map $\mu : G \times G \to G$ is $G$-equivariant where the action of $G$ on the latter $G$ is given by $F$-conjugation. Hence $C \ast M \in \mathcal{D}_G^F(G)$. Hence $\mathcal{D}_G^F(G)$ is a module category over the braided monoidal category $\mathcal{D}_G(G)$. Similarly we can prove that $M \ast C \in \mathcal{D}_G^F(G)$.

Note that if $C \in \mathcal{D}_G(G)$, we have the braiding isomorphism $\beta_{C,M} : C \ast M \to M \ast C$ for each $M \in \mathcal{D}(G)$. Moreover, if $M \in \mathcal{D}_G^F(G)$, we have the crossed braiding isomorphism $\beta_{M,C} : M \ast C \to F^\ast(C) \ast M$.

2.4.4 The semidirect product $G \rtimes \mathbb{Z}$

Sometimes it is more convenient to think of the category $\mathcal{D}_G^F(G)$ in terms of the semidirect product $\tilde{G} := G \rtimes \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on $G$ by the Frobenius $F : G \to G$. We identify $\tilde{G} = \{gF^n | g \in G, n \in \mathbb{Z}\}$ where $gF^n \cdot hF^m = gF^n(h)F^{n+m}$. Now $G$ acts on each coset $GF^n$ by conjugation. We consider the category (see also \cite{1})

$$\mathcal{D}_G(\tilde{G}) := \bigoplus_{n \in \mathbb{Z}} \mathcal{D}_G(GF^n).$$

It has the structure of a braided $\mathbb{Z}$-crossed category. In particular we have a braided monoidal action of $\mathbb{Z}$ on $\mathcal{D}_G(G)$ defined by $F^n(C) := (F^{-n})^\ast(C)$ for each $n \in \mathbb{Z}$ and $C \in \mathcal{D}_G(G)$.

We have an identification

$$\mathcal{D}_G^F(G) \cong \mathcal{D}_G(GF).$$

(6)

Now $\mathcal{D}_G(GF)$ is also a $\mathcal{D}_G(G)$-bimodule category under convolution. For $M \in \mathcal{D}_G^F(G)$, we denote the corresponding object of $\mathcal{D}_G(GF)$ by $\tilde{M}$. Then for $C \in \mathcal{D}_G(G)$ and $M \in \mathcal{D}_G^F(G)$ we have natural isomorphisms

$$\tilde{C} \ast \tilde{M} \cong C \ast M,$$

$$\tilde{M} \ast \tilde{F}(C) \cong \tilde{M} \ast C \text{ or equivalently } \tilde{M} \ast C \cong \tilde{M} \ast F^\ast(C).$$

(7)

(8)

With this identification, we will work interchangeably with the two equivalent categories $\mathcal{D}_G^F(G)$ and $\mathcal{D}_G(GF)$. 

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2.4.5 A trace on $D^F_G(G)$

For an object $M \in D^F_G(G)$ and any endomorphism $f : M \to M$, we will define a trace $\operatorname{tr}_F(f) \in \mathbb{Q}_l$. It suffices to consider $M \in D^F_G(\mathcal{O}) \cong D^F(\operatorname{Rep}(G^0_0(\mathbb{F}_q)))$ for some $\mathcal{O} \in H^1(F,G)$ and a point $t \in \mathcal{O}$. Now the triangulated category $D^F(\operatorname{Rep}(G^0_0(\mathbb{F}_q)))$ has the classical traces $\operatorname{tr}(f)$ for endomorphisms $f : M \to M$. We rescale this classical trace and define

$$\operatorname{tr}_F(f) = \frac{\operatorname{tr}(f)}{|G^0_0(\mathbb{F}_q)|}.$$  

We extend by additivity to define traces in the category $D^F_G(G)$.

2.4.6 The Frobenius algebra $\operatorname{Fun}([G_0][\mathbb{F}_q])$

Consider the subset $R \subset G \times G$ defined by

$$R = \{(x,g) | F(x) = g^{-1}xg\} = \{(x,g) | x \in G^0_0(\mathbb{F}_q) = \operatorname{Stab}_G^F(g)\}.$$  

Clearly $(x,g) \in R$ if and only if $(x, xg) \in R$.

If $(x,g) \in R$ and $h \in G$ then $h(x,g) := (h^{-1}xg, h^t) \in R$. Hence $R \subset G \times G$ is $G$-stable for the left action of $G$ on $G \times G$ described in [2.4.3]. The number of $G$-orbits in $R$ is finite and the orbits correspond bijectively to the union of sets of conjugacy classes of the groups of $\mathbb{F}_q$-points of all inner forms of $G_0$ corresponding to $H^1(F,G)$. More precisely, each $G$ orbit in $R$ can be uniquely represented by a pair $(h,t)$ where $t \in G$ is a representative of an $F$-conjugacy class and $h \in G^t_0(\mathbb{F}_q)$ represents a conjugacy class in the inner form $G^t_0(\mathbb{F}_q)$ associated with $t \in G$.

Note that for $(x,g) \in R$

$$\operatorname{Stab}_G(x,g) = Z_{G^0_0(\mathbb{F}_q)}(x),$$

the centralizer of $x$ in $G^0_0(\mathbb{F}_q)$.

Remark 2.10. The map $(p_1, \mu) : (x,g) \mapsto (x,xg)$ is $G$-equivariant. Hence for $(x,g) \in R$, we have

$$\operatorname{Stab}_G(x,g) = \operatorname{Stab}_G(x,xg).$$

Also, a subset $\{(h,t)\} \subset R$ is a complete set of representatives of $G$-orbits in $R$ if and only if the subset $\{(h, h^{-1}t)\} \subset R$ is a complete set of representatives.

Consider the space of $G$-invariant $\mathbb{Q}_l$-valued functions defined on the set $R$. We can identify this space with the space $\operatorname{Fun}([G_0][\mathbb{F}_q])$ defined in [Bo2, §2.2]. In the terminology of op. cit., $[G_0]$ denotes the quotient stack of $G_0$ by the conjugation action and the groupoid $[G_0][\mathbb{F}_q]$ can be identified with the disjoint union of the groupoids $[G^t_0(\mathbb{F}_q)]$ (see [Bo2, Example 2.8]) as $t$ ranges over a set of representatives of $F$-conjugacy classes in $G$.

The space $\operatorname{Fun}([G_0][\mathbb{F}_q]) = \operatorname{Fun}_G(R)$ is an algebra under convolution. As an algebra it can be identified with the product of the convolution algebras of class functions on the pure inner forms $G^t_0(\mathbb{F}_q)$:

$$\operatorname{Fun}_G(R) = \operatorname{Fun}([G_0][\mathbb{F}_q]) = \prod_{(t) \in H^1(F,G)} \operatorname{Fun}(G^t_0(\mathbb{F}_q))^{G^0_0(\mathbb{F}_q)},$$

where $\operatorname{Fun}(G^t_0(\mathbb{F}_q))$ is the category of $G^t_0(\mathbb{F}_q)$-equivariant class functions on $G^t_0(\mathbb{F}_q)$. 

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Moreover it is a Frobenius algebra with bilinear form defined by

\[ (f_1, f_2) := \sum_{<h, t> \in G \setminus R} \frac{f_1(h, t) f_2(h^{-1}, t)}{|\text{Stab}_G(h, t)|}. \quad (14) \]

By Remark 2.10 we see that

\[ (f_1, f_2) = \sum_{<h, t> \in G \setminus R} \frac{f_1(h, h^{-1}t) f_2(h^{-1}, h^{-1}t)}{|\text{Stab}_G(h, t)|}. \quad (15) \]

The corresponding linear functional \( \lambda : \text{Fun}([G_0](\mathbb{F}_q)) \rightarrow \overline{\mathbb{Q}}_l \) is given by

\[ \lambda(f) = \sum_{<t> \in H^1(F, G)} \frac{f(1, t)}{|G_0^t(\mathbb{F}_q)|}. \quad (16) \]

We also define a Hermitian inner form \( \langle \cdot, \cdot \rangle \) on the space \( \text{Fun}([G_0](\mathbb{F}_q)) \). For this we fix an isomorphism \( \overline{\mathbb{Q}}_l \rightarrow \mathbb{C} \), whence we get a conjugation involution \( (\cdot)^* \) of the field \( \overline{\mathbb{Q}}_l \) which maps roots of unity in \( \overline{\mathbb{Q}}_l \) to their inverses. We define a Hermitian inner product on \( \text{Fun}([G_0](\mathbb{F}_q)) \) by

\[ \langle f_1, f_2 \rangle := \sum_{<h, t> \in G \setminus R} \frac{f_1(h, t) f_2(h, t)}{|\text{Stab}_G(h, t)|} = \sum_{<h, t> \in G \setminus R} \frac{f_1(h, h^{-1}t) f_2(h^{-1}, h^{-1}t)}{|\text{Stab}_G(h, t)|}. \quad (17) \]

By taking characters of irreducible representations, we consider the set \( \text{Irrep}(G_0) \) defined by (3) as a subset of \( \text{Fun}([G_0](\mathbb{F}_q)) \). Using the orthogonality relations for irreducible characters we get:

**Proposition 2.11.** The set \( \text{Irrep}(G_0) \subset \text{Fun}([G_0](\mathbb{F}_q)) \) is an orthonormal basis of \( \text{Fun}([G_0](\mathbb{F}_q)) \) with respect to both the Hermitian inner product \( \langle \cdot, \cdot \rangle \) as well as the bilinear inner product \( (\cdot, \cdot) \).

### 2.4.7 The character of an object of \( \mathcal{D}_G^F(G) \)

If \( M \in \mathcal{D}_G^F(G) \), we will define its character \( \chi_M \) as a \( G \)-invariant function from the set \( R \) to \( \overline{\mathbb{Q}}_l \). If \( M \in \mathcal{D}_G^F(G) \) and \( g \in G \), the stalk \( M_g \) can be considered as an object of \( D^b(\text{Rep}(G_0^g(\mathbb{F}_q))) \) and we have its character \( \chi_{M_g} : G_0^g(\mathbb{F}_q) \rightarrow \overline{\mathbb{Q}}_l \). If \( (x, g) \in R \), we define \( \chi_{M}(x, g) := \chi_{M_g}(x) \).

Equivalently, for \( M \in \mathcal{D}_G^F(G) \) and \( (x, g) \in G \times G \), we have an isomorphism \( \phi_M(x, g) : M_g \rightarrow M_{xgF(x)^{-1}} \), where \( \phi_M \) is the equivariant structure associated to \( M \in \mathcal{D}_G^F(G) \). For \( (x, g) \in R \), \( \phi_M(x, g) \) is an automorphism of \( M_g \) and we have

\[ \chi_M(x, g) := \text{tr}(\phi_M(x, g)). \quad (18) \]

It is easy to check that \( \chi_M : R \rightarrow \overline{\mathbb{Q}}_l \) is \( G \)-invariant.

Hence given any object \( M \in \mathcal{D}_G^F(G) \), we have its character \( \chi_M \in \text{Fun}([G_0](\mathbb{F}_q)) \).

Now if \( W \in \text{Irrep}(G_0) \), we can take its character \( \chi_W \in \text{Fun}([G_0](\mathbb{F}_q)) \). It is clear from the definition that \( \chi_W \) is the same as \( \chi_{W_{loc}} \) as defined in this subsection.
2.4.8 The “trace of Frobenius” function of an equivariant Weil sheaf

Consider the category $\mathcal{D}_G^{W,0}(G_0)$ whose objects are pairs $(C, \psi)$ consisting of an object $C \in \mathcal{D}_G(G)$ and an isomorphism $\psi : F^*(C) \to C$ in $\mathcal{D}_G(G)$. Let $(C, \psi) \in \mathcal{D}_G^{W,0}(G_0)$. Let $g \in G$ and consider the inner form $G^0_0$ defined over $\mathbb{F}_q$. Using the sheaf-function correspondence we define the associated trace function

$$t^g_{C,\psi} : G^0_0(\mathbb{F}_q) \to \overline{\mathbb{Q}}_l.$$

We define the function $T_{C,\psi} : R \to \overline{\mathbb{Q}}_l$ by $T_{C,\psi}(x, g) := t^g_{C,\psi}(x)$.

More precisely, for $C \in \mathcal{D}_G(G)$, $x, g \in G$, we have isomorphisms

$$\phi_C(g^{-1}, gF(x)g^{-1}) : C_gF(x)g^{-1} \to C_F(x).$$

If $\psi : F^*M \cong M$ is a Weil structure, we have isomorphisms $\psi(x) : C_{F(x)} \to C_x$. If $(x, g) \in R$, then $\psi(x) \circ \phi_C(g^{-1}, gF(x)g^{-1})$ is an automorphism of $C_x$ and we define

$$T_{C,\psi}(x, g) := tr(\psi(x) \circ \phi_C(g^{-1}, gF(x)g^{-1))).$$

(19)

The function $T_{C,\psi} : R \to \overline{\mathbb{Q}}_l$ is also $G$-invariant.

Hence given an object $(C, \psi) \in \mathcal{D}_G^{W,0}(G_0)$, we have its trace function $T_{C,\psi} \in \text{Fun}([G_0](\mathbb{F}_q))$.

For each $g \in G$, we have natural functors $\mathcal{D}_G^0(G_0^0) \to \mathcal{D}_G^{W,0}(G_0)$ that are compatible with the identification in Proposition 2.14. If $C_0 \in \mathcal{D}_G^0(G_0)$, we let $T_{C_0} \in \text{Fun}([G_0](\mathbb{F}_q))$ denote the corresponding function.

Using Lemma 2.12 we obtain:

Lemma 2.12. Let $C, D \in \mathcal{D}_G^{W,0}(G_0)$. Then $T_{C*} \ast T_D = T_{C \ast D}$.

2.4.9 The inner product formula

Suppose that $M \in \mathcal{D}_G^F(G)$ and $(C, \psi) \in \mathcal{D}_G^{W,0}(G_0)$. Then we have the automorphism

$$\gamma_{C,\psi,M} : C \ast M \xrightarrow{\beta_{C,M}} M \ast C \xrightarrow{\beta_{M,C}} F^*(C) \ast M \xrightarrow{\psi \ast id_M} C \ast M$$

(20)

in $\mathcal{D}_G^F(G)$.

Remark 2.13. If we use the identification $\mathcal{D}_G^F(G) \cong \mathcal{D}_G(GF)$ (see 2.4.4), then using (7), (8) we see that $\gamma_{C,\psi,M}$ as defined above gets identified with $\gamma_{C,\psi,M}$ as defined in 2.3.

In our first main result, we describe the inner product of the functions associated with $(C, \psi)$ and $M$.

Theorem 2.14. For $(C, \psi) \in \mathcal{D}_G^{W,0}(G_0)$ and $M \in \mathcal{D}_G^F(G)$ the inner product of the functions $T_{C,\psi}, \chi_{M} \in \text{Fun}([G_0](\mathbb{F}_q))$ is given by

$$(T_{C,\psi}, \chi_{M}) = \langle T_{C,\psi}, \chi_{M} \rangle = tr_F(\gamma_{C,\psi,M}).$$

(21)

We will prove this result in 3 using the Grothendieck-Lefschetz trace formula.
2.5 Neutrally unipotent groups

We now restrict our attention to neutrally unipotent groups $G$, namely groups whose neutral connected component $G^0$ is unipotent. The theory of character sheaves for such groups was developed by Boyarchenko and Drinfeld in [BD1, BD2, Bo1, Bo2]. We briefly recall some aspects of this theory in this subsection and state our main results and conjectures.

2.5.1 Character sheaves on neutrally unipotent groups

Let $G$ be a neutrally unipotent group defined over $k$. Character sheaves on $G$ are defined in terms of minimal idempotents in the braided monoidal category $\mathcal{D}_G(G)$. To define character sheaves we do not need an $\mathbb{F}_q$-structure on $G$. We recall the definition of character sheaves in Definition 2.16 below.

An idempotent in $\mathcal{D}_G(G)$ is an object $e \in \mathcal{D}_G(G)$ such that $e \ast e \cong e$. An idempotent $e$ is said to be minimal if $e \neq 0$ and for any idempotent $e'$ in $\mathcal{D}_G(G)$, either $e \ast e' = 0$ or $e \ast e' \cong e$. An idempotent $e$ is said to be closed if there exists an arrow $\delta_1 \to e$ in $\mathcal{D}_G(G)$ which becomes an isomorphism after convolution with $e$.

The following statements are proved in [BD2, De1] and [De2]:

**Theorem 2.15.** Let $G$ be a neutrally unipotent group and let $e$ be a minimal idempotent in $\mathcal{D}_G(G)$. Then we have the following:

(i) The idempotent $e$ is in fact a closed idempotent and hence the Hecke subcategory $e\mathcal{D}_G(G)$ is a monoidal category with unit object $e$.

(ii) Let $\mathcal{M}^{\text{perv}}_{G,e} \subset e\mathcal{D}_G(G)$ be the full subcategory of those objects whose underlying $\mathbb{Q}_l$-complex is a perverse sheaf on $G$. Then $\mathcal{M}^{\text{perv}}_{G,e}$ is a semisimple abelian category with finitely many simple objects and $e\mathcal{D}_G(G) \cong D^b(\mathcal{M}^{\text{perv}}_{G,e})$.

(iii) There exists an integer $n_e \in \{0, 1, \ldots, \dim G\}$ such that $e[-n_e] \in \mathcal{M}^{\text{perv}}_{G,e}$. This means that every minimal idempotent $e \in \mathcal{D}_G(G)$ is a perverse sheaf up to a shift.

(iv) The category $\mathcal{M}_{G,e} := \mathcal{M}^{\text{perv}}_{G,e}[n_e]$ is closed under convolution. In fact $\mathcal{M}_{G,e}$ is a modular category with unit object $e$. The ribbon structure on $\mathcal{M}_{G,e}$ comes from the natural ribbon $\mathcal{r}$-category structure on $\mathcal{D}_G(G)$.

(v) We have $\dim \mathcal{M}_{G,e} = \text{FPdim} \mathcal{M}_{G,e} = m^2$ for some integer $m \in \mathbb{Z}$, where $\dim$ and $\text{FPdim}$ denote the categorical dimension and the Frobenius Perron dimension respectively. The modular category $\mathcal{M}_{G,e}$ is integral i.e. the categorical dimensions of all objects are integers.

**Definition 2.16.** (i) For a minimal idempotent $e \in \mathcal{D}_G(G)$, the number $d_e := \frac{\dim G - n_e}{2}$ is called the functional dimension of $e$.

(ii) Let $e$ be a minimal idempotent in $\mathcal{D}_G(G)$. We define the $\mathbb{L}$-packet of character sheaves on $G$ associated with the minimal idempotent $e$ to be the following (finite) set

$$CS_{e}(G) := \{\text{isom. classes of simple objects of } \mathcal{M}_{G,e} \subset \mathcal{D}_G(G)\}.$$ (22)

(iii) If $e' \in \mathcal{D}_G(G)$ is a minimal idempotent which is not isomorphic to $e$, then the sets $CS_{e}(G)$ and $CS_{e'}(G)$ are disjoint. Let $\hat{G}$ denote the set of isomorphism classes of minimal idempotents in $\mathcal{D}_G(G)$. The set of character sheaves on $G$ is defined to be the disjoint union of all $\mathbb{L}$-packets of...
character sheaves:

\[ CS(G) := \bigoplus_{e \in \hat{G}} CS_e(G). \]  

(23)

So far we have defined character sheaves using the rather abstract notion of minimal idempotents in \( \mathcal{D}_G(G) \). We now give a more explicit description of minimal idempotents in \( \mathcal{D}_G(G) \) using the notion of admissible pairs for \( G \).

We first briefly recall the notion of Serre duality of connected unipotent groups. Our assumption of perfectness is essential in the theory of Serre duality. We refer to [Bo1 §A] for a detailed exposition. Let \( U \) be a connected unipotent group. Then we have its Serre dual \( U^\ast \) which is a (perfect) commutative (possibly disconnected) unipotent group. Roughly speaking, \( U^\ast \) is the moduli space of central extensions of \( U \) by the discrete group scheme \( \mathbb{Q}_p/\mathbb{Z}_p \). Let us fix an identification of \( \mathbb{Q}_p/\mathbb{Z}_p \) with the group \( \mu_p^\infty(\mathbb{Q}_l) \) of the \( p \)-power-th roots of unity in \( \mathbb{Q}_l^\times \). Then by [Bo1, Lemma 7.3], we may think of \( U^\ast \) as the moduli space of multiplicative local systems on \( U \).

Remark 2.17. We have a natural identification \( \mathbb{Q}_p/\mathbb{Z}_p \cong \mu_p^\infty(\mathbb{C}) \) where we identify \( \frac{1}{p^n} \in \mathbb{Q}_p/\mathbb{Z}_p \) with \( e^{2\pi i/n} \). Since we have already chosen an identification \( \mathbb{Q}_l \cong \mathbb{C} \) in (2.1.6), this also determines the identification \( \mathbb{Q}_p/\mathbb{Z}_p \cong \mu_p^\infty(\mathbb{Q}_l) \). Henceforth we assume that these two identifications are compatible in this sense.

Definition 2.18. ([Bo1 Def. 7.4]) Let \( G \) be a neutrally unipotent group. Let \((H, \mathcal{L})\) be a pair consisting of a connected subgroup \( H \subset G \) and a multiplicative local system \( \mathcal{L} \) on \( H \). We say that the pair \((H, \mathcal{L})\) is admissible for \( G \) if the following conditions hold:

(i) Let \( G' \subset G \) denote the normalizer of the pair \((H, \mathcal{L})\) (see [Bo1 §7.3]) and let \( G'^\circ \) denote its neutral connected component. Then \( G'^\circ/H \) should be commutative.

(ii) The group morphism \( \phi_\mathcal{L} : G'^\circ/H \to (G'^\circ/H)^\ast \) that is defined in this situation (see [Bo1 §A.13]) should be an isogeny.

(iii) (Geometric Mackey condition) For every \( g \in G - G' \), we should have

\[ \mathcal{L}|_{(H \cap gH)^\circ} \cong g \mathcal{L}|_{(H \cap gH)^\circ}, \]

where \( gH = gHg^{-1} \) and \( g \mathcal{L} \) is the multiplicative local system on \( gH \) obtained from \( \mathcal{L} \) by transport of structure.

Remark 2.19. In the situation above, let \( e_{H, \mathcal{L}} := \mathcal{L} \otimes \mathbb{K}_H \in \mathcal{D}_G(G') \). Then \( e_{H, \mathcal{L}} \in \mathcal{D}_G(G') \) is in fact a minimal idempotent (see [BD2]).

Definition 2.20. If \((H, \mathcal{L})\) is an admissible pair for \( G \) such that \( G' = G \), we say that \((H, \mathcal{L})\) is a Heisenberg admissible pair for \( G \). In this case the minimal idempotent \( e_{H, \mathcal{L}} \in \mathcal{D}_G(G) \) is said to be a Heisenberg idempotent.

In [BD2], the induction (with compact support) functor \( \text{ind}^G_{G'} : \mathcal{D}_G(G') \to \mathcal{D}_G(G) \) is defined for closed subgroups \( G' \subset G \). In the following theorem, Boyarchenko and Drinfeld prove that every minimal idempotent in \( \mathcal{D}_G(G) \) comes from an admissible pair and in particular from a Heisenberg idempotent on a subgroup by induction.

Theorem 2.21. ([BD2]) (i) Let \((H, \mathcal{L})\) be an admissible pair for a neutrally unipotent group \( G \) and let \( e_{H, \mathcal{L}} \in \mathcal{D}_G(G') \) be the corresponding Heisenberg idempotent on \( G' \) as defined above. Then

\[ 3 \text{Note that in this situation, condition (iii) of the definition is vacuous.} \]
\[ f_{H,L} := \text{ind}_{G_H}^G e_{H,L} \in D_G(G) \] is a minimal idempotent. The functional dimensions of \( e_{H,L} \in D_G(G') \) and \( f_{H,L} \in D_G(G) \) are related as follows:

\[ d_{G,f_{H,L}} = d_{G',e_{H,L}} + \dim(G/G'). \]

(ii) In the situation of (i), the induction functor induces an equivalence of modular categories

\[ \text{ind}^G_{G'} : \mathcal{M}_{G',e_{H,L}} \xrightarrow{\sim} \mathcal{M}_{G,f_{H,L}}. \]

(iii) Every minimal idempotent \( f \in D_G(G) \) comes from an admissible pair by the procedure described in (i). Hence every minimal idempotent \( f \in D_G(G) \) comes from induction from a Heisenberg idempotent \( e \) on some subgroup \( G' \).

This result reduces many problems about general minimal idempotents to the case of Heisenberg idempotents.

### 2.5.2 \( L \)-packets of characters of neutrally unipotent groups

From now on we will need the \( \mathbb{F}_q \)-structure on \( G \) provided by a Frobenius map \( F : G \to G \). The functor \( F := F^{-1} : D_G(G) \to D_G(G) \) is an autoequivalence of ribbon \( \tau \)-categories as we have already mentioned in [2.4.4]. This functor gives an action of \( F \) on the set \( \hat{G} \) of minimal idempotents as well as on the set \( CS(G) \) of character sheaves. We see that \( F \) maps the set \( CS_e(G) \) to the set \( CS_{F(e)}(G) \). We are interested in the set \( CS(G)^F \) of \( F \)-stable character sheaves. Clearly we have

\[ CS(G)^F = \coprod_{e \in \hat{G}^F} CS_e(G)^F. \]  \hspace{1cm} (24)

The following result from [Bo2] describes the set \( \hat{G}^F \).

**Theorem 2.22.** (See [Bo2, Thm. 2.17].) Let \( e \in \hat{G}^F \), i.e. \( e \) is a minimal idempotent in \( D_G(G) \) such that \( F^*e \equiv e \). Then:

(i) Consider a pure inner form \( G_0^0 \) of \( G_0 \). There exists a unique (up to isomorphism) weak idempotent \( e_0^0 \in D_{G_0^0}(G_0^0) \) such that \( e \) is obtained from \( e_0^0 \) by base change. This idempotent \( e_0^0 \) is in fact a closed idempotent in \( D_{G_0^0}(G_0^0) \).

(ii) There exists an inner form \( G_0 \) such that the closed idempotent \( e_0^0 \in D_{G_0^0}(G_0^0) \) is obtained from an admissible pair defined over \( \mathbb{F}_q \) for the inner form \( G_0^0 \).

**Remark 2.23.** In general in the situation of the theorem, \( e_0 \in D_{G_0}(G_0) \) may not be defined by an admissible pair for \( G_0 \) and passing to an inner form is necessary.

**Definition 2.24.** An idempotent \( e_0 \in D_{G_0}(G_0) \) is said to be a geometrically minimal idempotent if the idempotent \( e \in D_G(G) \) obtained by base change is a minimal idempotent. It is clear that if \( e_0 \in D_{G_0}(G_0) \) is a geometrically minimal idempotent then for each \( g \in G \), \( e_0^g \in D_{G_0^g}(G_0^g) \) is also a geometrically minimal idempotent. By Theorem 2.22 there is a natural bijection between the set \( \hat{G}^F \) and the set of isomorphism classes of geometrically minimal idempotents in \( D_{G_0}(G_0) \).
Definition 2.25. (See \[Bo2\] Defn. 2.10.) Let \(e_0 \in \mathcal{D}_{G_0}(G_0)\) be a geometrically minimal idempotent. For each \(g \in G\), let \(\text{Irrep}_{e_0}(G_0^g(\mathbb{F}_q))\) be the set of isomorphism classes of irreducible representations of \(G_0^g(\mathbb{F}_q)\) in which the idempotent \(t_{e_0}^g \in \text{Fun}(G_0^g(\mathbb{F}_q))\) acts as the identity. The \(L\)-packet of irreducible representations of \(G_0\) defined by \(e_0\) is defined to be the set

\[
\text{Irrep}_{e_0}(G_0) := \bigcap_{<t> \in H^1(F, G)} \text{Irrep}_{e_0^L}(G_0^g(\mathbb{F}_q)) \subset \text{Irrep}(G_0).
\] (25)

By taking the characters of representations, we consider the set \(\text{Irrep}_{e_0}(G_0)\) as a subset of the Frobenius algebra \(\text{Fun}([G_0][\mathbb{F}_q])\). Then the set \(\text{Irrep}_{e_0}(G_0) \subset \text{Fun}([G_0][\mathbb{F}_q])\) forms an orthonormal basis for the subspace \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q]) \subset \text{Fun}([G_0][\mathbb{F}_q])\) corresponding to the idempotent \(T_{e_0} \in \text{Fun}([G_0][\mathbb{F}_q])\).

2.5.3 \textbf{F}-stable character sheaves on neutrally unipotent groups

Let \(e \in \tilde{G}^F\). Then the equivalence \(F : \mathcal{D}_G(G) \rightarrow \mathcal{D}_G(G)\) induces an autoequivalence of the modular category \(F : \mathcal{M}_{G,e} \rightarrow \mathcal{M}_{G,e}\) and a permutation of the \(L\)-packet \(CS_{e_0}(G)\) of character sheaves associated with \(e\). The following is proved in \[Bo2\].

Theorem 2.26. (See \[Bo2\] Thm. 2.17.) (i) Let \(C \in CS_{e_0}(G)^F\) and let \(\psi : F^*C \rightarrow C\) be any Weil structure. Then \(T_{C,\psi} \in T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\). (ii) For each \(C \in CS_{e_0}(G)^F\) choose a Weil structure \(\psi_C\) so that \(<T_{C,\psi_C}, T_{C,\psi_C}>=1\). Then the finite set \(\{T_{C,\psi_C}\}_{C \in CS_{e_0}(G)^F}\) is an orthonormal basis of \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\) (with respect to the Hermitian inner product \(<\cdot, \cdot>\)).

This theorem proves that the trace functions \(\{T_{C,\psi_C}\}_{C \in CS(G)^F}\) (suitably normalized) also form an orthonormal basis for \(\text{Fun}([G_0][\mathbb{F}_q])\), that the matrix relating this basis to the basis (formed by characters of) \(\text{Irrep}(G_0)\) is block diagonal (see also Definition 2.25) and that these blocks correspond to \(F\)-stable minimal idempotents \(e \in \mathcal{D}_G(G)\) or equivalently \(F\)-stable \(L\)-packets of \(G\).

2.5.4 Main results and conjectures

As before, let \(G\) be a neutrally unipotent group equipped with an \(\mathbb{F}_q\)-structure. Let \(e \in \tilde{G}^F\) and let \(e_0 \in \mathcal{D}_{G_0}(G_0)\) be the corresponding geometrically minimal idempotent. Note that the subspace \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q]) \subset \text{Fun}([G_0][\mathbb{F}_q])\) is an algebra with unit \(T_{e_0}\). As seen before, the set \(\text{Irrep}_{e_0}(G_0)\) is an orthonormal basis of \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\) with respect to the bilinear form \((\cdot, \cdot)\) (as well as the Hermitian form \(<\cdot, \cdot>\)). Hence the restriction of the bilinear form to \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\) is non-degenerate and hence \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\) is a Frobenius algebra. Moreover the suitably normalized trace of Frobenius functions \(\{T_{C,\psi_C}\}_{C \in CS_{e_0}(G)^F}\) form another orthonormal basis of \(T_{e_0} \text{Fun}([G_0][\mathbb{F}_q])\) with respect to the Hermitian inner product \(<\cdot, \cdot>\). Our goal is to describe the unitary matrix \(\tilde{S}^{G_0,e_0}\) which relates these two orthonormal bases. Since the basis \(\text{Irrep}_{e_0}(G_0)\) is orthonormal, this matrix is obtained by taking inner products between the bases:

\[
(\tilde{S}^{G_0,e_0})_{C \in CS_{e_0}(G)^F, W \in \text{Irrep}_{e_0}(G_0)} = (<T_{C,\psi_C}, \chi W>)_{C \in CS_{e_0}(G)^F, W \in \text{Irrep}_{e_0}(G_0)}.
\] (26)

\footnote{In fact it is easy to see that if \(A\) is a Frobenius algebra and \(e \in A\) is a central idempotent, then \(eA\) is again a Frobenius algebra.}
First we give another characterization of the set \( \text{Irrep}_{e_0}(G_0) \) of irreducible representations of the various inner forms lying in the \( l \)-packet defined by \( e_0 \).

**Theorem 2.27.** Let \( \mathcal{M}_{GF,e} \subset \mathcal{D}_G(GF) \) be the full subcategory formed by objects whose underlying \( \mathbb{Q}_l \)-complex is a perverse sheaf shifted by \( n_e \). (Recall the definition of \( n_e, d_e \) from \((2.5.7)\))

(i) Then \( \mathcal{M}_{GF,e} \) has a natural structure of an invertible \( \mathcal{M}_{G,e} \)-module category equipped with a natural normalized \( \mathcal{M}_{G,e} \)-module trace \( \text{tr}_{F,e} \) and we have \( D^b(\mathcal{M}_{GF,e}) \cong \mathcal{D}_G(GF) \).

(ii) Let \( W \in \text{Irrep}_{e_0}(G_0) \) i.e. \( W \in \text{Irrep}_{e_0}(G(t)(\mathbb{F}_q)) \) for some \( t \in G \). Then by \((3.1)\), \( W \) corresponds to a \( G \)-equivariant local system \( W_{\text{loc}} \in \mathcal{D}_G(GF) \) supported on the \( G \)-conjugacy class of \( tF \subset GF \).

Then in fact \( W_{\text{loc}} \in e \mathcal{D}_G(GF) \) and \( M_W := W_{\text{loc}}[\dim G + n_e] \in \mathcal{M}_{GF,e} \) is a simple object.

(iii) There is a natural bijection between the set \( \text{Irrep}_{e_0}(G_0) \) and the set \( \mathcal{O}_{\mathcal{M}_{GF,e}} \) of (isomorphism classes of) simple objects of \( \mathcal{M}_{GF,e} \). This bijection is induced by the map

\[
\text{Irrep}_{e_0}(G_0) \ni W \mapsto M_W = W_{\text{loc}}[\dim G + n_e] \in \mathcal{O}_{\mathcal{M}_{GF,e}}
\]

from (ii).

**Remark 2.28.** By taking characters (\( \text{Irrep}(G_0) \ni W \mapsto \chi_W \in \text{Fun}([G_0](\mathbb{F}_q)) \)), we have identified the set \( \text{Irrep}(G_0) \) as a subset of \( \text{Fun}([G_0](\mathbb{F}_q)) \). We see that \( \chi_W = (-1)^{\dim G + n_e} \chi_{M_W} = (-1)^{2d_e} \chi_{M_W} \).

Using Theorem 2.22(ii) we will first reduce Theorem 2.27 to the case of Heisenberg idempotents in \((5.1)\). We then prove the theorem in the Heisenberg case in \((6.1)\).

Combining the Remark 2.28 with \((26)\) and Theorem 2.14 we obtain:

**Corollary 2.29.** The entries of the matrix \( S_{G_0,e_0}^G \) are

\[
\tilde{s}_{G_0,e_0}^{C,W} = (-1)^{2d_e} \cdot \text{tr}_F(\gamma_{C,\psi_C,M_W})
\]

where \( \gamma_{C,\psi_C,M_W} \) is the automorphism of \( C \ast M_W \in \mathcal{D}_G(GF) \) defined in \((2.4.9)\) and \( \text{tr}_F \) is the trace on \( \mathcal{D}_G(GF) \) defined in \((2.4.5)\).

Now the full subcategory \( \mathcal{M}_{GF,e} \subset \mathcal{D}_G(GF) \) has the natural module trace \( \text{tr}_{F,e} \) from Theorem 2.27(i). We would like to compare this module trace with the trace \( \text{tr}_F \) on \( \mathcal{D}_G(GF) \) restricted to \( \mathcal{M}_{GF,e} \).

If \( G \) is a neutrally unipotent group, it is known that the braided fusion category \( \mathcal{M}_{G,e} \) is integral. Conjecturally the modular category \( \mathcal{M}_{G,e} \) is positive integral, but this is not yet proved. Instead let us consider the modular category \( \mathcal{M}_{G,e}^+ \) which has the same underlying integral braided category as \( \mathcal{M}_{G,e} \) but which is equipped with the (unique) positive spherical structure. With this spherical structure, for any object \( C \in \mathcal{M}_{G,e}^+ \) we have \( \dim(C) = \text{FPdim}(C) \in \mathbb{Z} \). Also we let \( \mathcal{M}_{GF,e}^+ \) denote the category which is same as \( \mathcal{M}_{GF,e} \) as a module category but which is equipped with the positive module trace (normalized according to \((29)\)) \( \text{tr}_{F,e}^+ \), i.e. any object in \( \mathcal{M}_{GF,e}^+ \) has positive categorical dimension. (Recall that we have fixed an identification \( \mathbb{Q}_l \cong \mathbb{C} \).) This means that for any \( M \in \mathcal{M}_{G,e}^+ \) we have \( \dim(M) = \text{FPdim}(M) \). Note that we have \( \dim(\mathcal{M}_{G,e}) = \dim(\mathcal{M}_{G,e}^+) = \text{FPdim}(\mathcal{M}_{G,e}) \).

The next result compares the \( \mathcal{M}_{G,e}^+ \)-module trace \( \text{tr}_{F,e}^+ \) on \( \mathcal{M}_{GF,e}^+ \) with the trace \( \text{tr}_F \) restricted to \( \mathcal{M}_{GF,e} \subset \mathcal{D}_G(GF) \) and gives an interpretation of the matrix \( S_{G_0,e_0}^G \) defined in \((26)\) as a crossed \( S \)-matrix up to scaling.\(^5\)

\(^5\)See \((2)\).

\(^6\)See \([ENO, \S 2.5]\) for the notion of Frobenius-Perron dimension in module categories. By \([ENO, \text{Prop. 3.2}]\), the positive trace \( \text{tr}_{F,e}^+ \) defined this way satisfies \((2)\).
Theorem 2.30. (i) On the subcategory $\mathcal{M}_{GF,e} \subset \mathcal{D}_G(GF)$ we have the following equality:

$$
\text{tr}^+_{F,e} = (-1)^{2d_e} \cdot \frac{q^{\dim G} \cdot \sqrt{\dim \mathcal{M}_{G,e}^-}}{q^{d_e}} \cdot \text{tr}_F. 
$$

(ii) The matrix $\tilde{S}^{G_0,e_0}$ relating the two orthonormal bases $\{T_{C,eC}\}_{C \in \text{CS}(G)}$ and $\text{Irrep}_{e_0}(G_0)$ of $T_{e_0} \text{Fun}([G_0](\mathbb{F}_q))$ is equal to the crossed $S$-matrix corresponding to the modular category $\mathcal{M}_{G,e}^+$ and the $\mathcal{M}_{G,e}^+$-module category $\mathcal{M}_{GF,e}^+$ up to a scaling:

$$
\tilde{S}^{G_0,e_0} = \frac{q^{d_e}}{q^{\dim G} \cdot \sqrt{\dim \mathcal{M}_{G,e}}} \cdot S_{\mathcal{M}_{G,e}^+,\mathcal{M}_{GF,e}^+}. 
$$

Remark 2.31. By Theorem 2.15(v), the categorical dimension $\dim \mathcal{M}_{G,e}$ is the square of an integer and we extract the positive integral square root in the theorem above. Also $d_e \in \frac{1}{2} \mathbb{Z}$ and we again extract the positive square root in the formulas above if $d_e \in \frac{1}{2} + \mathbb{Z}$ according to our chosen identification $\mathbb{Q}_l \cong \mathbb{C}$.

Once again we will first reduce Theorem 2.30 to the case of Heisenberg idempotents in §5.2 and prove it in the Heisenberg case in §6.2.

So far we have only assumed that $G$ is neutrally unipotent. If $G$ is itself also unipotent, then it is known that the modular category $\mathcal{M}_{G,e}$ is positive integral (see [De2, Prop. 2.21]).

Corollary 2.32. Let us keep all our previous notation and also assume that $G$ is a unipotent group. Then

$$
\tilde{S}^{G_0,e_0} = \pm \frac{q^{d_e}}{q^{\dim G} \cdot \sqrt{\dim \mathcal{M}_{G,e}}} \cdot S_{\mathcal{M}_{G,e}^+,\mathcal{M}_{GF,e}^+}. 
$$

Finally we make the following conjecture.

Conjecture 2.33. (i) If $G$ is a neutrally unipotent group and $e \in \mathcal{D}_G(G)$ is any minimal idempotent, then the modular category $\mathcal{M}_{G,e}$ is positive integral.

(ii) Now assume as before that we have an $\mathbb{F}_q$-structure on $G$ and that $e \in \mathcal{D}_G(G)$ is an $F$-stable minimal idempotent. Then we can choose an identification $\mathbb{Q}_l \cong \mathbb{C}$ (compatible with any chosen identification $\mathbb{Q}_p/\mathbb{Z}_p \cong \mu_{p^n}(\mathbb{Q}_l)$ in the sense of Remark 2.17) in such a way that the $\mathcal{M}_{G,e}$-module trace $\text{tr}_{F,e}$ on $\mathcal{M}_{GF,e}^+$ is positive.

This says that the natural structures themselves are already positive and that for example we may drop the ‘+’ symbols in (28). In §5.2 we reduce this conjecture to the Heisenberg case.

3 Proof of Theorem 2.14

In this section we prove Theorem 2.14 for an arbitrary algebraic group $G$. The proof is an application of the Grothendieck-Lefschetz trace formula. For notational convenience we use the integral symbol ($\int$) to denote pushforward with compact supports. If $\mathcal{E} \in \mathcal{D}(X)$, we use $\int_X \mathcal{E}$ to denote $R\Gamma(X, \mathcal{E})$.

\footnote{Here we do not need an $\mathbb{F}_q$-structure and we may assume the base field $k$ is any algebraically closed field of characteristic $p$.}
The first equality \( (T_{C,\psi, \chi_M}) = (T_{C,\psi, \chi_M}) \) is clear since \( \chi_M(h, t) = \chi_M(h^{-1}, t) \) for each \((h, t) \in R\). Next we compute the stalk of the automorphism \( \gamma_{C,\psi, M} \) at a point \( t \in G \). The stalk

\[
\gamma_{C,\psi, M}(t) : (C * M)_t \rightarrow (C * M)_t
\]

is given by

\[
(C * M)_t = \int_{h_1 h_2 = t} C_{h_1} \otimes M_{h_2}
\]

\[
= \int_{h_1 h_2 = t} \frac{(\psi(F^{-1}(h_2^{-1} h_1 h_2)) \otimes \phi_{C}(h_2^{-1}, h_1)) \otimes \phi_{M}(F^{-1}(h_2^{-1} h_1^{-1} h_2), h_2)}{C_{F^{-1}(h_2^{-1} h_1 h_2)} \otimes M_{F^{-1}(h_2^{-1} h_1^{-1} h_2) h_1 h_2}}
\]

Consider the automorphism of the antidiagonal \( \Delta_t := \{(h_1, h_2) \in G \times G | h_1 h_2 = t\} \) defined by

\[
(h_1, h_2) = (h_1, h_1^{-1} t) \mapsto (F^{-1}(h_2^{-1} h_1 h_2), F^{-1}(h_2^{-1} h_1^{-1} h_2) h_1 h_2) = (F^{-1}(t^{-1} h_1 t), F^{-1}(t^{-1} h_1^{-1} t) t).
\]

The fixed point set of this automorphism is precisely the finite set \( \Delta_t \cap R \). The inverse of this automorphism corresponds to the Frobenius automorphism \( \text{ad}(t) \circ F \) of \( G \) under the identification \( \Delta_t \overset{\text{p}_1}{\rightarrow} G \). Hence by the Grothendieck-Lefschetz trace formula (trace of the induced map on cohomology equals the sum of traces of the stalks of the map over all fixed points, see also [Bo2, Lemma 4.4(iii)]) we deduce that

\[
\text{tr}(\gamma_{C,\psi, M}(t)) = \sum_{(h_1, h_2) \in R \atop h_1 h_2 = t} T_{C,\psi}(h_1, h_2) \chi_M(h_1^{-1}, h_2).
\]

Now \( C * M \) is an object of \( \mathcal{D}_G^F(G) \cong \bigoplus_{<t> \in H^1(F, G)} \mathcal{D}^b(\text{Rep}(G_0^F(F_q))). \) Hence by definition (9),

\[
\text{tr}_F(\gamma_{C,\psi, M}) = \sum_{<t> \in H^1(F, G) \atop (h, t) \in R} \frac{\text{tr}(\gamma_{C,\psi, M}(t))}{|G_0^F(F_q)|}
\]

\[
= \sum_{<t> \in H^1(F, G) \atop (h, t) \in R} \sum_{(h_1, h_2) \in R \atop h_1 h_2 = t} \frac{T_{C,\psi}(h_1, h_2) \chi_M(h_1^{-1}, h_2)}{|G_0^F(F_q)|}
\]

\[
= \sum_{<t> \in H^1(F, G) \atop (h, t) \in R} \sum_{(h_1, h_2) \in R \atop h_1 h_2 = t} \frac{T_{C,\psi}(h, h^{-1} t) \chi_M(h^{-1}, h^{-1} t)}{|G_0^F(F_q)|}
\]

\[
= \sum_{<t> \in H^1(F, G) \atop (h, t) \in R} \frac{\text{tr}(\gamma_{C,\psi, M}(t))}{|G_0^F(F_q)|}
\]

\[
= \sum_{<t> \in H^1(F, G) \atop (h, t) \in R} \frac{T_{C,\psi}(h, h^{-1} t) \chi_M(h, h^{-1} t)}{|Z_{G_0^F(F_q)}(h)|}
\]

\[
= \sum_{(h, t) \in G \setminus R} \frac{T_{C,\psi}(h, h^{-1} t) \chi_M(h, h^{-1} t)}{|Z_{G_0^F(F_q)}(h)|}
\]
\[\sum_{<h,t> \in G \setminus R} \frac{T_{C,\psi}(h, h^{-1}t) \chi_M(h, h^{-1}t)}{|\text{Stab}_G(h, t)|} \]

(35)

as desired. Here \(\text{Conj}_{G_0}(\mathbb{F}_q)(h)\) denotes the conjugacy class of \(h\) in \(G_0(\mathbb{F}_q)\).

4 Induction functors for braided crossed categories

4.1 Braided crossed categories associated with group extensions

Let \(G\) be a perfect quasi-algebraic group. Let \(\Gamma\) be a discrete (possibly infinite) group and consider the extension

\[0 \to G \to \widetilde{G} \to \Gamma \to 0.\]

(37)

Consider the category

\[\mathcal{D}_G(\widetilde{G}) := \bigoplus_{\tilde{g} \in \Gamma = G/G} \mathcal{D}_G(G\tilde{g}).\]

This is a \(\Gamma\)-graded monoidal category. For each \(\gamma \in \Gamma\), let \(\tilde{g} \in \widetilde{G}\) be a lift. Consider the conjugation by \(\tilde{g}^{-1}\) automorphism \(\tilde{g}^{-1} : \widetilde{G} \to \widetilde{G}\) and define the functor \(\tilde{g} : \mathcal{D}_G(\widetilde{G}) \to \mathcal{D}_G(\widetilde{G})\) as pullback by the conjugation automorphism \(\tilde{g}^{-1}\). This defines a monoidal action of \(\Gamma\) on \(\mathcal{D}_G(\widetilde{G})\).

We see that we have the following crossed braiding isomorphisms:

**Lemma 4.1.** Let \(\tilde{g} \in \widetilde{G}\). Let \(M \in \mathcal{D}_G(\widetilde{G})\) and \(N \in \mathcal{D}_G(G\tilde{g})\). Then we have a natural isomorphism

\[\beta_{N,M} : N \ast M \xrightarrow{\cong} \tilde{g}(M) \ast N \text{ in } \mathcal{D}(\widetilde{G}).\]

(38)

In fact we can define the structure of a braided \(\Gamma\)-crossed category on \(\mathcal{D}_G(\widetilde{G})\). It is also an \(\tau\)-category with the duality functor given by \(\mathbb{D}^- = \mathbb{D} \circ \iota^* = \iota^* \circ \mathbb{D}\), where \(\mathbb{D} : \mathcal{D}_G(\widetilde{G}) \to \mathcal{D}_G(\widetilde{G})\) is the Verdier duality functor and \(\iota : G \to \widetilde{G}\) is the inversion map. Moreover it has a natural pivotal structure coming from the natural monoidal isomorphism \(\text{Id} \cong (\mathbb{D}^-)^2\) and the monoidal action of \(\Gamma\) preserves this pivotal structure. Hence \(\mathcal{D}_G(\widetilde{G})\) is a braided \(\Gamma\)-crossed pivotal \(\tau\)-category.

4.2 Induction functors

In this section we define induction functors and study their properties in the setting of braided crossed categories which is slightly more general than the setting in [De3, §4]. However the proofs from loc. cit. and [BD2] can be readily adapted to our current set up.

Consider an extension \(0 \to G \to \widetilde{G} \to \Gamma \to 0\) as in §1.1. Let \(\widetilde{G}' \subset \widetilde{G}\) be a subgroup that surjects onto \(\Gamma\). Let \(G' := G \cap \widetilde{G}'\). Then we have the extension

\[0 \to G' \to \widetilde{G}' \to \Gamma \to 0.\]

(39)
We can define the induction with compact supports functor
\[ \text{ind}^G_{\mathcal{G}'} : \mathcal{D}_{\mathcal{G}'}(\tilde{G}') \to \mathcal{D}_G(\tilde{G}) \] (40)
as the composition \( \mathcal{D}_{\mathcal{G}'}(\tilde{G}') \to \mathcal{D}_{\mathcal{G}'}(\tilde{G}) \xrightarrow{\text{av}_{\mathcal{G}',\mathcal{G}}} \mathcal{D}_G(\tilde{G}) \). By [De3, Prop. 3.4], we have the structure of a weak semigroupal functor on \( \text{ind}^G_{\mathcal{G}'} \). Moreover we can show that it is compatible with the \( \Gamma \)-actions and the braided \( \Gamma \)-crossed structure as well as the pivotal \( \tau \)-category structure.

**Definition 4.2.** Let \( G' \subset G \) be any closed subgroup in an algebraic group \( G \). Let \( e \in \mathcal{D}_{\mathcal{G}'}(G') \) be any idempotent. We say that \( e \) satisfies the Mackey condition above for the closed idempotent \( \widetilde{G} \) if the functors \( e \) are isomorphic as objects of \( \mathcal{D}_{\mathcal{G}'}(G') \) and \( \mathcal{D}_G(\tilde{G}) \).

**Remark 4.3.** Note that the condition (iii) from Definition 2.18 of admissible pairs is equivalent to the Mackey condition above for the closed idempotent \( e_{H',L} \in \mathcal{D}_{\mathcal{G}'}(G') \) associated with the pair \((H,L)\).

Now let \( e \in \mathcal{D}_{\mathcal{G}'}(G) \subset \mathcal{D}_{\mathcal{G}'}(\tilde{G}') \) be an idempotent satisfying the Mackey condition with respect to \( G \) such that \( \tilde{g}(e) \cong e \) for each \( \tilde{g} \in \tilde{G}' \).

**Proposition 4.4.** (See [De3, Prop. 4.3].) In the situation above we have:

(i) The induced object \( f := \text{ind}^G_{\mathcal{G}'} e \) is an idempotent in \( \mathcal{D}_G(G) \subset \mathcal{D}_G(\tilde{G}) \) such that \( \tilde{g}(f) \cong f \) for each \( \tilde{g} \in \tilde{G} \).

(ii) If \( M \in e \mathcal{D}_{\mathcal{G}'}(G) \), then \( \text{ind}^G_{\mathcal{G}'} M \in f \mathcal{D}_G(\tilde{G}) \).

(iii) If \( N \in \mathcal{D}_G(G) \), then \( e \ast N \ast e \cong e \ast N \ast e \) and hence \( e \ast N \ast e \cong e \ast N \ast e \). Hence we can consider \( e \ast N \ast e \) as an object of \( \mathcal{D}(G) \).

(iv) Let \( \tilde{g} \in \tilde{G}' \) and \( N \in \mathcal{D}_G(\tilde{G}) \). Then \( e \ast N \cong e \ast N \ast e \cong e \ast N \ast \delta_{\tilde{g}} \ast e \ast \delta_{\tilde{g}} \) is supported on \( G' \tilde{g} \subset \tilde{G}' \). Hence for each \( N \in \mathcal{D}_G(\tilde{G}) \) we may consider \( e \ast N \cong e \ast N \mid_{\tilde{G}'} \) as objects of \( e \mathcal{D}_{\mathcal{G}'}(G') \).

(v) For each \( N \in \mathcal{D}_G(\tilde{G}) \), there is an isomorphism
\[ f \ast N \xrightarrow{\cong} \text{ind}^G_{\mathcal{G}'}(e \ast N) \] (41)
functorial with respect to \( N \).

(vi) For each \( M \in e \mathcal{D}_{\mathcal{G}'}(G) \) we obtain functorial isomorphisms
\[ e \ast (\text{ind}^G_{\mathcal{G}'} M) \xrightarrow{\cong} e \ast M \text{ in } \mathcal{D}_{\mathcal{G}'}(G) \] (42)
\[ e \ast (\text{Res}^G_{\mathcal{G}'} \circ \text{ind}^G_{\mathcal{G}'} M) \xrightarrow{\cong} e \ast M \text{ in } \mathcal{D}_{\mathcal{G}'}(G'), \] (43)
by taking the convolution of \( e \) with the canonical morphism \( \text{ind}^G_{\mathcal{G}'} M \to M \) in \( \mathcal{D}_{\mathcal{G}'}(G') \) and \( \text{Res}^G_{\mathcal{G}'} \circ \text{ind}^G_{\mathcal{G}'} M \to M \) in \( \mathcal{D}_{\mathcal{G}'}(G') \). In particular we see that \( e \ast f \cong e \ast e \).

(vii) The restriction
\[ \text{ind}^G_{\mathcal{G}'} |_{e \mathcal{D}_{\mathcal{G}'}(G')} : e \mathcal{D}_{\mathcal{G}'}(G') \to f \mathcal{D}_G(\tilde{G}) \] (44)
is strong semigroupal, compatible with the pivotal and braided \( \Gamma \)-crossed structures and induces a bijection on isomorphism classes of objects.

(viii) If the functor \( M \mapsto e \ast M \) is isomorphic to the identity functor on \( e \mathcal{D}_{\mathcal{G}'}(G') \), the functor (44) is faithful.

(ix) If the functors \( M \mapsto e \ast M \) and \( N \mapsto f \ast N \) are isomorphic to the identity functors on \( e \mathcal{D}_{\mathcal{G}'}(\tilde{G}') \) and \( f \mathcal{D}_G(\tilde{G}) \) respectively, the functor (44) is an equivalence of braided \( \Gamma \)-crossed pivotal \( \tau \)-categories, a quasi-inverse to which is provided by the functor (also see (vi) above)
\[ f \mathcal{D}_G(\tilde{G}) \ni N \mapsto e \ast N \in e \mathcal{D}_{\mathcal{G}'}(G'). \] (45)
4.3 Induction and sheaf-function correspondence

Let $G$ be an algebraic group equipped with a Frobenius $F : G \to G$ which corresponds to the form $G_0$ over $\mathbb{F}_q$. As in [2.4.2], we form the semidirect product $\tilde{G} = G \rtimes \mathbb{Z}$, where $1 \in \mathbb{Z}$ acts on $G$ by $F$. Suppose $G' \subset G$ is an $F$-stable subgroup or in other words $G'$ is obtained by base change from a closed subgroup $G'_0 \subset G_0$ over $\mathbb{F}_q$. We have the subgroup $G := G' \rtimes \mathbb{Z} \subset \tilde{G}$ and by [4.2] we have the weak semigroupal functor

$$\text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_G(\tilde{G}).$$

In particular we get the functors

$$\text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G'F) \to \mathcal{D}_G(GF) \quad (47)$$

$$\text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_G(G) \quad (48)$$

as well as

$$\text{ind}_{G'}^{G} : \mathcal{D}_{G'}^{\text{Weld}}(G'_0) \to \mathcal{D}_G^{\text{Weld}}(G_0). \quad (49)$$

Next we interpret the functor [16] in terms of the usual notion of induction of representations. Note that the inclusion $G' \subset G$ induces a map $H^1(F, G') \to H^1(F, G)$. By [2.4.2] we have

$$\mathcal{D}_{G'}(G'F) \cong \bigoplus_{<t> \in H^1(F, G')} \mathcal{D}^b \text{Rep}(G'_0(\mathbb{F}_q)), \quad (46)$$

$$\mathcal{D}_G(GF) \cong \bigoplus_{<t> \in H^1(F, G)} \mathcal{D}^b \text{Rep}(G_0(\mathbb{F}_q)).$$

**Lemma 4.5.** Let $t \in G'$ and let $\mathcal{O}' \subset G'F$ be the $G'$-conjugacy class of $tF \in G'F$ and let $\mathcal{O} \subset GF$ be the $G$-conjugacy class of $tF$. Then the induction functor $\text{ind}^{G}_{G'}$ takes the full subcategory $\mathcal{D}^b \text{Rep}(G'_0(\mathbb{F}_q)) \cong \mathcal{D}_{G'}(\mathcal{O}') \subset \mathcal{D}_{G'}(G'F)$ to the full subcategory $\mathcal{D}^b \text{Rep}(G'_0(\mathbb{F}_q)) \cong \mathcal{D}_{G}(\mathcal{O}) \subset \mathcal{D}_{G}(GF)$ and the restriction of the functor $\text{ind}^{G}_{G'}$ to $\mathcal{D}_{G'}(\mathcal{O}')$ can be identified with the functor

$$\text{ind}^{G}_{G'}(\mathcal{O}') : \mathcal{D}^b \text{Rep}(G'_0(\mathbb{F}_q)) \to \mathcal{D}^b \text{Rep}(G_0(\mathbb{F}_q)).$$

**Proof.** This follows from the equivalences $\mathcal{D}_{G'}(\mathcal{O}') \cong \mathcal{D}^{\text{Weld}}_{G'_0}(tF)$ and $\mathcal{D}_{G}(\mathcal{O}) \cong \mathcal{D}^{\text{Weld}}_{G_0}(tF)$ and the definition of induction functors using the averaging functors. \qed

In this situation, we also have an induction map between the function spaces

$$\text{ind}_{G'_0}^{G_0} : \text{Fun}(G'_0(\mathbb{F}_q)) \to \text{Fun}(G_0(\mathbb{F}_q)) \quad (50)$$

obtained from induction of class functions

$$\text{ind}_{G'_0}^{G_0} : \text{Fun}(G'_0(\mathbb{F}_q)) \to \text{Fun}(G_0(\mathbb{F}_q)).$$

Using Lemma 4.5 we get
Lemma 4.6. Let $M \in \mathcal{D}_{G'}(G'F)$. Then we have $\text{ind}^{G_0}_{G_0'}(\chi_M) = \chi_{\text{ind}^{G_0}_{G_0'}(M)}$.

We recall the following from [Bo2]:

Proposition 4.7. ([Bo2, Prop. 4.12]) Let $(C, \psi) \in \mathcal{D}^{W_{el}}(G_0')$. Then $T_{\text{ind}^{G_0}_{G_0'}(C, \psi)} = \text{ind}^{G_0}_{G_0'}(T_{C, \psi})$, i.e. induction of conjugation equivariant Weil sheaves is compatible with the sheaf-function correspondence (provided we take all pure inner forms into account).

5 Reduction process for neutrally unipotent groups

In this section we assume that $G$ is a neutrally unipotent group equipped with an $\mathbb{F}_q$-structure given by a Frobenius $F : G \to G$. In this section we reduce the proofs of Theorems 2.27 and 2.30 to the case of Heisenberg idempotents. In [Bo] we will prove these theorems in the Heisenberg case.

Let $f \in \widehat{G'}$ and let $f_0 \in \mathcal{D}_{G_0}(G_0)$ be the corresponding geometrically minimal idempotent. Now without loss of generality (by passing to a pure inner form of $G_0$ if necessary) by Theorem 2.22 we may assume that $f_0$ comes from an admissible pair $(H_0, L_0)$ for $G_0$. Let $G'_0$ be the normalizer of the admissible pair. Let $e_0 \in \mathcal{D}_{G'_0}(G'_0)$ be the corresponding Heisenberg idempotent. By extension of scalars we get an $F$-stable admissible pair $(H, L)$ for $G$ with normalizer $G'$. Let $e \in \mathcal{D}_{G'}(G')$ be the corresponding Heisenberg idempotent on $G'$.

The idempotent $e \in \mathcal{D}_{G'}(G')$ satisfies the Mackey condition with respect to $G$. As before, let $\widetilde{G} = G \rtimes \mathbb{Z}$ and $\widetilde{G'} = G' \rtimes \mathbb{Z}$. Now the subgroup $\widetilde{G'} \subset \widetilde{G}$ and $e \in \mathcal{D}_{G'}(\widetilde{G'})$ satisfy the conditions of Proposition 4.4. Moreover since $e \in \mathcal{D}_{G'}(G')$ and $f \in \mathcal{D}_{G}(G)$ are closed idempotents, we conclude using Proposition 4.4(ix) that we have an equivalence

$$\text{ind}^{G'}_{G_0'} : e \mathcal{D}_{G'}(\widetilde{G'}) \cong f \mathcal{D}_{\widetilde{G}}(\widetilde{G})$$

(51)

of braided $\mathbb{Z}$-crossed pivotal $\tau$-categories. Moreover by [BD2, Thm. 1.52] the induction functor $\text{ind}^{G'}_{G_0'}$ preserves perverse sheaves in $e \mathcal{D}_{G'}(G')$ up to a shift by $\text{dim}(G/G')$. In particular this induces an equivalence of modular categories

$$\text{ind}^{G'}_{G_0'} : \mathcal{M}_{G',e} \cong \mathcal{M}_{G,f}$$

(52)

as well as their respective module categories

$$\text{ind}^{G'}_{G_0'} : \mathcal{M}_{G',F,e} \cong \mathcal{M}_{G,F,f}$$

(53)

and identifies their natural module traces. It is clear that we also have an equivalence of the modular categories

$$\text{ind}^{G'}_{G_0'} : \mathcal{M}^+_{G',e} \cong \mathcal{M}^+_{G,f}$$

(54)

and their respective module categories

$$\text{ind}^{G'}_{G_0'} : \mathcal{M}^+_{G',F,e} \cong \mathcal{M}^+_{G,F,f}$$

(55)

equipped with the positive traces.

---

This $F$ may be different from the original one we started with since we may have to choose a different pure inner form.
5.1 Reduction of Theorem 2.27 to the Heisenberg case

We continue to use all the notation from 5.1. We will now prove Theorem 2.27 for the general minimal idempotent $f \in \mathcal{R}_G(G)$ assuming that it holds for the Heisenberg idempotent $e \in \mathcal{R}_{G'}(G')$.

Theorem 2.27(i) for $f$ follows from (52) and (53) assuming the corresponding statement for $e$. To prove the remaining statements, we first prove two auxiliary lemmas.

Lemma 5.1. (See [Bo2, Lem. 6.24]) Let $f_0 \in \mathcal{R}_{G_0}(G_0)$ be any idempotent. Then:

(i) The “trace of Frobenius” functions associated with the objects of $\mathcal{R}_{G_0}(G_0)$ span the space $\text{Fun}([G_0](\mathbb{F}_q))$.

(ii) The “trace of Frobenius” functions associated with the objects of $f_0 \mathcal{R}_{G_0}(G_0)$ span the space $T_{f_0} \text{Fun}([G_0](\mathbb{F}_q))$.

Proof. The statement (i) is precisely Lemma 6.24 in [Bo2]. Now if $C_0 \in \mathcal{R}_{G_0}(G_0)$, then $T_{f_0 * C_0} = T_{f_0} * T_{C_0}$ by Lemma 2.12. Hence (ii) follows from (i).

Lemma 5.2. The induction of class functions on the pure inner forms induces an isomorphism of Frobenius algebras

\[ \text{ind}_{G_0}^{G'} : T_{e_0} \text{Fun}([G_0'](\mathbb{F}_q)) \xrightarrow{\cong} T_{f_0} \text{Fun}([G_0](\mathbb{F}_q)). \]

Proof. It is clear that the induction map $\text{ind}_{G_0}^{G'} : \text{Fun}([G_0'](\mathbb{F}_q)) \xrightarrow{\cong} \text{Fun}([G_0](\mathbb{F}_q))$ commutes with the linear functionals $\lambda' : \text{Fun}([G_0'](\mathbb{F}_q)) \to \mathbb{Q}$ and $\lambda : \text{Fun}([G_0](\mathbb{F}_q)) \to \mathbb{Q}$ defined by (16). By Proposition 4.7, $\text{ind}_{G_0}^{G'}(T_{e_0}) = T_{f_0}$. We have an equivalence of braided categories

\[ \text{ind}_{G'}^{G_0} : e \mathcal{R}_{G_0}(G_0') \xrightarrow{\cong} f \mathcal{R}_G(G) \]

compatible with the action of $F$ on both sides and hence an equivalence

\[ \text{ind}_{G'}^{G_0} : e_0 \mathcal{R}_{G_0}^{\text{Weil}}(G_0') \xrightarrow{\cong} f_0 \mathcal{R}_{G_0}^{\text{Weil}}(G_0). \]

The lemma then follows from Proposition 4.7 and Lemma 5.1.

\[ \square \]

Now by (53) $\text{ind}_{G'}^{G_0}$ induces a bijection between the sets of simples $O_{M_{G',e}}$ and $O_{M_{G,F}}$. Now we use the assumption that Theorem 2.27(ii) and (iii) hold for $e \in \mathcal{R}_{G'}(G')$. With this assumption we have the bijection $O_{M_{G',e}} \cong \text{Irrep}_{e_0}(G_0')$ and hence $\{\chi_{M'}\}_{M' \in O_{M_{G',e}}}$ is a basis of $T_{e_0} \text{Fun}([G_0'](\mathbb{F}_q))$ (see Remark 2.28). For $M' \in O_{M_{G',e}}$, let $M = \text{ind}_{G'}^{G_0}(M') \in O_{M_{G,F}}$. Then by Lemmas 4.6 and 5.2 we see that $\chi_M \in T_{f_0} \text{Fun}([G_0](\mathbb{F}_q))$ and $\{\chi_M\}_{M \in O_{M_{G,F},f}}$ is a basis of $T_{f_0} \text{Fun}([G_0](\mathbb{F}_q))$. Hence the irreducible representation (of some pure inner form $G_0'(\mathbb{F}_q)$) corresponding to $M$ indeed lies in $\text{Irrep}_{f_0}(G_0)$ and this sets up a bijection between $O_{M_{G,F}}$ and $\text{Irrep}_{f_0}(G_0)$ proving Theorem 2.27(ii) and (iii) for the idempotent $f \in \mathcal{R}_G(G)$.
5.2 Reduction of Theorem 2.30 and Conjecture 2.33 to the Heisenberg case

Statement (ii) of Theorem 2.30 is an immediate consequence of (i) using Corollary 2.29. We now prove the statement (i) for \( \mathcal{F} \) assuming it for \( e \).

Note that for an endomorphism \( \mathcal{F} \) in \( \mathcal{D}_G(G'F) \) we have \( \text{tr}_{G,F}(\text{ind}_{G'}^G(a)) = \text{tr}_{G,F}(a) \) for the traces on \( \mathcal{D}_G(G'F) \) and \( \mathcal{D}_G(GF) \) defined in (2.4.5). Also by [BD2 Thm. 1.52], \( \dim G - d_f = \dim G' - d_e \) and \( 2d_f \equiv 2d_e \mod 2 \). Moreover by (52) \( \dim(\mathcal{M}_G,e) = \dim(\mathcal{M}_G,f) \) and by (53) the traces \( \text{tr}_{F,e} \) and \( \text{tr}_{F,f} \) are compatible with the induction functor and so are the positive traces \( \text{tr}^+_F,e \) and \( \text{tr}^+_F,f \). Hence we see that Theorem 2.30 holds for \( \mathcal{F} \) if it holds for \( e \).

This argument also proves that Conjecture 2.33 for \( f \in \mathcal{D}_G(G) \) is equivalent to the same statement for \( e \in \mathcal{D}_G(G') \). Hence the conjecture is reduced to the case of Heisenberg idempotents.

6 The Heisenberg case

In this section we complete the proofs of Theorems 2.27 and 2.30 in the case of Heisenberg idempotents.

We begin by recalling the setup in the Heisenberg case. As before we assume that \( G \) is a neutrally unipotent group equipped with an \( \mathbb{F}_q \)-structure given by a Frobenius \( \mathcal{F} : G \to G \). We assume that we have a Heisenberg admissible pair \( (H_0, L_0) \) for the group \( G_0 \) over \( \mathbb{F}_q \). Let \( e_0 \in \mathcal{D}_{G_0}(G_0) \) be the corresponding geometrically minimal (Heisenberg) idempotent and \( e \in \mathcal{D}_G(G) \) the Heisenberg idempotent on \( G \) obtained by extension of scalars from \( e_0 \). Note that in this case \( n_e = \dim H \).

Let \( U = G^\circ \). Then \( U \) is an \( \mathcal{F} \)-stable connected unipotent group and let \( U_0 \) be the corresponding unipotent group defined over \( \mathbb{F}_q \). Let \( \Gamma := \Pi_0(G) = G/U \).

By definition \( U/H \) is commutative and the homomorphism \( \phi_L : U/H \to (U/H)^\ast \) (from condition (ii) of Definition 2.15) is an isogeny. The finite group \( \Gamma \) acts on \( U/H \) and \( \phi_L \) is \( \Gamma \)-equivariant. Let \( K_L := \ker(\phi_L) \). Let \( \theta : K_L \to \mathbb{Q}^\ast \) be the corresponding quadratic form defined in [BD2 §A.10]. By [De1] we have \( e\mathcal{D}_U(U) \cong D^b\mathcal{M}(K_L, \theta) \), where \( \mathcal{M}(K_L, \theta) \) is the pointed modular category defined by the metric group \( (K_L, \theta) \).

Now by definition the pair \( (H, L) \) is normalized by \( G \) and is also \( \mathcal{F} \)-stable. Note that we have the induced automorphism \( \mathcal{F} : \Gamma \to \Gamma \). Using the Frobenius action, we form the semidirect products \( \widetilde{\Gamma} := \Gamma \rtimes \mathbb{Z} \) and \( \widetilde{G} := G \rtimes \mathbb{Z} \). Thus we have an extension

\[
0 \to U \to \widetilde{G} \to \widetilde{\Gamma} \to 0
\]

(56)
of the discrete group \( \widetilde{\Gamma} \) by \( U \). The admissible pair \( (H, L) \) is normalized the group \( \widetilde{G} \). By §4.1 the category \( \mathcal{D}_U(\widetilde{G}) \) is a braided \( \widetilde{\Gamma} \)-crossed pivotal \( \tau \)-category and so is the Hecke subcategory \( e\mathcal{D}_U(\widetilde{G}) \).

Let \( \widetilde{\mathcal{M}}_{\widetilde{G},e} \subset e\mathcal{D}_U(\widetilde{G}) \) denote the full subcategory of those objects whose underlying complexes are perverse sheaves shifted by \( \dim H \). For \( \tilde{g} \in \widetilde{G} \) we let \( \widetilde{\mathcal{M}}_{\tilde{g},e} \subset e\mathcal{D}_U(\tilde{g}) \) be defined similarly. Then we have

\[
\widetilde{\mathcal{M}}_{\widetilde{G},e} = \bigoplus_{\tilde{g} \in \tilde{G}/U} \widetilde{\mathcal{M}}_{\tilde{g},e}.
\]

(57)
The following result is proved in [De1]:

**Theorem 6.1.** (i) For each \( \widetilde{Ug} \in \widetilde{G}/U \), \( \widetilde{M}_{\widetilde{Ug},e} \) is a semisimple abelian category with finitely many simple objects and we have \( D^b\widetilde{M}_{\widetilde{Ug},e} \cong e\mathcal{D}_{U}(\widetilde{G}) \).
(ii) The category \( e\mathcal{M}_{\widetilde{G},e} \subset e\mathcal{D}_{U}(\widetilde{G}) \) is closed under convolution. It is a braided \( \tilde{\Gamma} \)-crossed spherical fusion category with trivial component \( \widetilde{M}_{U,e} = M_{U,e} \cong M(K_L, \theta) \).

The category \( e\mathcal{D}(\widetilde{G}) \) is a braided \( \mathbb{Z} \)-crossed pivotal \( \Gamma \)-category. We have \( D^b\mathcal{M}_{\widetilde{G},e} \cong e\mathcal{D}_{U}(\widetilde{G}) \).

Let \( \mathcal{M}_{\widetilde{G},e} \subset e\mathcal{D}_{U}(\widetilde{G}) \) be the full subcategory of objects whose underlying complex is a perverse sheaf shifted by \( \dim H \). Similarly for an integer \( n \) we define \( \mathcal{M}_{GF^n,e} \subset e\mathcal{D}_{U}(GF^n) \subset e\mathcal{D}_{U}(\widetilde{G}) \). We have

\[
\mathcal{M}_{\widetilde{G},e} = \bigoplus_{n \in \mathbb{Z}} \mathcal{M}_{GF^n,e}.
\]

(58)

Now using Theorem 6.1 we obtain:

**Lemma 6.2.** The category \( \mathcal{M}_{\widetilde{G},e} \subset e\mathcal{D}_{U}(\widetilde{G}) \) is closed under convolution. In fact \( \mathcal{M}_{\widetilde{G},e} \) is a braided \( \mathbb{Z} \)-crossed spherical fusion category with trivial component \( \mathcal{M}_{G,e} \) and can be obtained as a \( \Gamma \)-equivariantization:

\[
\mathcal{M}_{\widetilde{G},e} \cong \left( \mathcal{M}_{G,e} \right)^\Gamma.
\]

(59)

We have \( D^b\mathcal{M}_{\widetilde{G},e} \cong e\mathcal{D}_{U}(\widetilde{G}) \).

### 6.1 Proof of Theorem 2.27

We now prove Theorem 2.27 in the Heisenberg case.

Using (58), Lemma 6.2 along with [ENO, §6] we see that each \( \mathcal{M}_{GF^n,e} \) (and in particular \( \mathcal{M}_{GF,e} \)) is an invertible \( \mathcal{M}_{G,e} \)-module category equipped with an \( \mathcal{M}_{G,e} \)-module trace obtained from the natural spherical structure on the braided \( \mathbb{Z} \)-crossed category \( \mathcal{M}_{\widetilde{G},e} \). This trace is normalized in the sense of (2) by Remark 2.3. This completes the proof of Theorem 2.27(i).

We make the following general observation:

**Lemma 6.3.** Let \( H \) be any connected algebraic group equipped with a Frobenius \( F : H \to H \). We have a functor

\[
\alpha : \mathcal{D}_{F}^H(H) \to \mathcal{D}_{Weil}(H_0)
\]

such that for each \( M \in \mathcal{D}_{F}^H(H) \) the underlying \( \mathbb{Q}_l \)-complexes of \( M \) and \( \alpha(M) \) are the same and we have \( \chi_M = \chi_{\alpha(M)} \) i.e. the character of \( M \) is equal to the “trace of Frobenius” function associated with \( \alpha(M) \).

**Proof.** If \( (M, \phi_M) \in \mathcal{D}_{F}^H(H) \) where \( M \in \mathcal{D}(H) \) and \( \phi_M \) is the equivariance structure of \( M \), then we have isomorphisms of stalks

\[
\phi_M(h, F(h)) : M_{F(h)} \cong M_h \text{ for each } h \in H.
\]

(61)

In other words we obtain a Weil structure on \( M \in \mathcal{D}(H) \). To be precise, let \( \Delta_F \subset H \times H \) denote the graph of the Frobenius \( F : H \to H \). Then we set \( \alpha(M, \phi_M) = (M, \phi_M|\Delta_F) \in \mathcal{D}_{Weil}(H_0) \).
Now suppose that \( h \in H_0(\mathbb{F}_q) \). Then by (18) we have \( \chi_{M,\phi_M}(h) = tr(\phi_M(h,1)) \) where

\[
\phi_M(h,1) : M_1 \xrightarrow{\cong} M_1.
\] (62)

On the other hand by (61) for the “trace of Frobenius function” associated with \( \alpha(M) \) we have \( t_{\alpha(M),\phi_M} = tr(\phi_M(h,h)) \). Now since \( H \) is connected, there exists \( v \in H \) such that \( h = vF(v)^{-1} \) by Lang’s theorem. Then we have a commutative diagram

\[
\begin{array}{ccc}
M_1 & \xrightarrow{\phi_M(h,1)} & M_1 \\
\downarrow{\phi_M(v,1)} & & \downarrow{\phi_M(v,1)} \\
M_h & \xrightarrow{\phi_M(h,h)} & M_h.
\end{array}
\]

Hence \( tr(\phi_M(h,1)) = tr(\phi_M(h,h)) \). Hence we see that \( \chi_M = t_{\alpha(M)} \) for each \( M \in \mathcal{D}_H^F(\mathcal{H}) \). \( \square \)

Let us now return to our original setting where we have the admissible pair \((H_0,L_0)\) for the neutrally unipotent group \( G_0 \) and the corresponding geometrically minimal idempotent \( e_0 \in \mathcal{D}_{G_0}(G_0) \).

Taking the functions associated with sheaves, we get the character \( t_{L_0} : H_0(\mathbb{F}_q) \to \mathbb{Q}_l^\times \) and the corresponding idempotent function \( t_{e_0} = \frac{1}{q^{dim_H}} \cdot t_{L_0} \in \text{Fun}(G_0(\mathbb{F}_q))\text{Fun}(G_0(\mathbb{F}_q)) \) which is supported on the normal subgroup \( H_0(\mathbb{F}_q) \triangleleft G_0(\mathbb{F}_q) \).

Let \( \text{Rep}_{e_0}(H_0(\mathbb{F}_q)) \subset \text{Rep}(H_0(\mathbb{F}_q)) \) denote the full subcategory of representations of \( H_0(\mathbb{F}_q) \) where the group acts by the character \( t_{L_0} \). Note that we have \( D^b \text{Rep}(H_0(\mathbb{F}_q)) \cong \mathcal{D}_H^F(\mathcal{H}) \) the equivariant derived category for the \( F \)-conjugation action\(^9\) of \( H \) on itself.

**Lemma 6.4.** Let \( W \in \text{Rep}(H_0(\mathbb{F}_q)) \). Let \( W_{\text{loc}} \) be the corresponding local system in \( \mathcal{D}_H^F(\mathcal{H}) \). Then \( W \in \text{Rep}_{e_0}(H_0(\mathbb{F}_q)) \) if and only if \( W_{\text{loc}} \in e\mathcal{D}_H^F(\mathcal{H}) \).

**Proof.** Let \( W \) be the 1-dimensional representation of \( H_0(\mathbb{F}_q) \) corresponding to the character \( t_{L_0} \), i.e. \( W \) is the unique simple object in the category \( \text{Rep}_{e_0}(H_0(\mathbb{F}_q)) \) (which is equivalent to \( \text{Vec} \) as an abelian category). It is clear that \( \mathcal{D}_H^F(\mathcal{H}) \supset W_{\text{loc}} \cong \mathcal{L} \) (see also Lemma 6.3). Since of \( e = \mathcal{L} \otimes \mathbb{K}_H \) and \( \mathcal{L} \) is a multiplicative local system, it is clear that \( e \ast \mathcal{L} \cong \mathcal{L} \), i.e. \( \mathcal{L} \cong W_{\text{loc}} \in e\mathcal{D}_H^F(\mathcal{H}) \).

Now since \( H \) is connected and unipotent, \( \mathcal{D}_H^F(\mathcal{H}) \subset \mathcal{D}(\mathcal{H}) \) is a full subcategory and hence \( e\mathcal{D}_H^F(\mathcal{H}) \subset e\mathcal{D}(\mathcal{H}) \) is a full subcategory. But \( e\mathcal{D}(\mathcal{H}) \cong D^b \text{Vec} \) and all objects of \( e\mathcal{D}(\mathcal{H}) \) are of the form \( V \otimes \mathcal{L} \) where \( V \in \text{Vec} \) (thought of as a constant local system on \( H \)). Since \( \mathcal{L} \in e\mathcal{D}_H^F(\mathcal{H}) \), we see that \( e\mathcal{D}_H^F(\mathcal{H}) = e\mathcal{D}(\mathcal{H}) \). The lemma now follows. \( \square \)

To prove Theorem 2.27(ii) and (iii) we have to compare the set of simple objects in the category \( \mathcal{M}_{GF,e} \subset e\mathcal{D}_{G}(GF) \) and the set \( \text{Irrep}_{e_0}(G_0) \). Let \( O \subset G \) be an orbit of the \( F \)-conjugation action. Without any loss of generality, for convenience of notation (after possibly modifying the Frobenius \( F \) by an inner automorphism of \( G \)) we assume that \( 1 \in O \). Hence we have \( \mathcal{D}_O^F(\mathcal{O}) \cong D^b \text{Rep}(G_0(\mathbb{F}_q)) \).

---

\(^9\)Since \( H \) is connected, this action is transitive by Lang’s theorem.
Let $\text{Rep}_{\text{et}}(G_0(\mathbb{F}_q)) \subset \text{Rep}(G_0(\mathbb{F}_q))$ denote the full subcategory of representations of $G_0(\mathbb{F}_q)$ in which the normal subgroup $H_0(\mathbb{F}_q)$ acts by the character $t_{\mathcal{L}_0}$.

We now prove:

**Proposition 6.5.** Let $W \in \text{Rep}(G_0(\mathbb{F}_q))$. Let $W_{\text{loc}}$ be the corresponding local system in $\mathcal{D}_G^F(\mathcal{O}) \subset \mathcal{D}_G^F(G)$. Then $W \in \text{Rep}_{\text{et}}(G_0(\mathbb{F}_q))$ if and only if $W_{\text{loc}} \in e\mathcal{D}_G^F(G)$. We have an equivalence $e\mathcal{D}_G^F(\mathcal{O}) \cong D^b \text{Rep}_{\text{et}}(G_0(\mathbb{F}_q))$.

**Proof.** Let $Q$ denote the algebraic group $G/H$. Then we have the induced Frobenius $F : Q \to Q$ and the algebraic group $Q_0$ over $\mathbb{F}_q$. Since $H$ is connected, we have $Q_0(\mathbb{F}_q) = G_0(\mathbb{F}_q)/H_0(\mathbb{F}_q)$. Let us define the group $Q_H := \{g \in G|gF(g)^{-1} \in H\}$. This is the subgroup of $G$ that takes $H \subset O$ to itself under the $F$-conjugation action of $G$ on $O$. We have that $H = Q_H^0$ and $Q_H/H \cong Q_0(\mathbb{F}_q)$. We have $G_0(\mathbb{F}_q) \subset Q_H$. We have an equivalence $\mathcal{D}_G^F(\mathcal{O}) \cong \mathcal{D}_{Q_H}^F(H)$, where the latter is the category of $Q_H$-equivariant sheaves on $H$ for the $F$-conjugation action of $Q_H$ on $H$. Hence we have $\mathcal{D}_G^F(\mathcal{O}) \cong \mathcal{D}_{Q_H}^F(H)^Q_0(\mathbb{F}_q)$. The proposition now follows from Lemma 6.4.

Finally to prove Theorem 2.27(ii) and (iii) we apply Proposition 6.5 to each $F$-conjugacy class $O \in H^1(F,G)$.

### 6.2 Proof of Theorem 2.30

It suffices to prove Theorem 2.30(i) since (ii) is an immediate consequence.

Let $W \in \text{Rep}_{\text{et}}(G_0(\mathbb{F}_q))$ for some $g \in G$ and let $M_W := W_{\text{loc}}[\dim G + \dim H] \in \mathcal{M}_{GF,e}$ be the corresponding object obtained by Theorem 2.27. We have

$$\text{tr}_{F,e}(\text{id}_{M_W}) = \text{FPdim}(M_W) \quad \text{and} \quad \text{tr}_F(\text{id}_{M_W}) = (-1)^{\dim G + \dim H} \cdot \frac{\dim(W)}{|G_0(\mathbb{F}_q)|}.$$ \hspace{1cm} (64)

Hence to prove Theorem 2.30(i) it suffices to prove that

$$\text{FPdim}(M_W) = \frac{q^{\dim G} \cdot \sqrt{\dim M_G,e}}{q^{d_e}} \cdot \frac{\dim(W)}{|G_0(\mathbb{F}_q)|}.$$ \hspace{1cm} (65)

for each such $W$. As before for ease of notation, without loss of generality we may assume $W \in \text{Rep}_{\text{et}}(G_0(\mathbb{F}_q))$ corresponding to the trivial inner form. As before, let $O \subset G$ denote the $F$-conjugacy class of $1 \in G$. Then by our assumption, $M_W \in \mathcal{M}_{GF,e} \subset \mathcal{D}_G(GF)$ is supported on $OF$.

Recall that we have defined $U = G^0$ and $\Gamma = G/U$. Also recall that by (57) and Theorem 6.1 we have the braided $\Gamma$-crossed spherical fusion category $\tilde{\mathcal{M}}_{G,e}$. Set

$$\tilde{\mathcal{M}}_{G,e} := \bigoplus_{Ug \subset G/U \subset \tilde{\Gamma}} \tilde{\mathcal{M}}_{Ug,e};$$

$$\tilde{\mathcal{M}}_{GF,e} := \bigoplus_{Ug \subset G/U \subset \tilde{\Gamma}} \tilde{\mathcal{M}}_{Ugf,e}.$$ \hspace{1cm} (67)
Then \( \tilde{\mathcal{M}}_{G,e} \) is an \( \tilde{\mathcal{M}}_{G,e} \) bimodule category and by Lemma 6.2, we have

\[
\mathcal{M}_{G,e} = (\tilde{\mathcal{M}}_{G,e})^\Gamma, \tag{68}
\]

\[
\mathcal{M}_{G,F,e} = (\tilde{\mathcal{M}}_{G,F,e})^\Gamma. \tag{69}
\]

\( \tilde{\mathcal{M}}_{G,e} \) is a (faithfully graded) braided \( \Gamma \)-crossed spherical fusion category with trivial component \( \tilde{\mathcal{M}}_{U,e} = \mathcal{M}_{U,e} \). Hence \( \dim(\tilde{\mathcal{M}}_{G,e}) = |\Gamma| \cdot \dim(\mathcal{M}_{U,e}) \) and hence after taking the \( \Gamma \)-equivariantization we conclude that (see also [De2, Prop. 2.17])

\[
\dim(\mathcal{M}_{G,e}) = |\Gamma|^2 \cdot \dim(\mathcal{M}_{U,e}). \tag{70}
\]

The object \( M_W \in \mathcal{M}_{G,F,e} \) is supported on \( \mathcal{O}F \). Let \( \mathcal{O}' \subset \Gamma \) be the \( F \)-conjugacy class of \( 1 \in \Gamma \). By Lang’s theorem

\[
\mathcal{O} = \bigsqcup_{Ug \in \mathcal{O}' \subset \Gamma} Ug. \tag{71}
\]

Using (69) we think of \( M_W \) as an object of \( (\tilde{\mathcal{M}}_{G,F,e})^\Gamma \). Let \( M' \in \tilde{\mathcal{M}}_{G,F,e} \) denote the underlying object. Then \( M' \in \bigoplus_{Ug \in \mathcal{O}' \subset \Gamma} \tilde{\mathcal{M}}_{Ug,F,e} \). Let \( M'_{UF} \in \tilde{\mathcal{M}}_{UF,e} \) be the projection of \( M' \) in \( \tilde{\mathcal{M}}_{UF,e} \). Then we have

\[
\text{FPdim}(M_W) = |\mathcal{O}'| \cdot \text{FPdim}(M'_{UF}) = \frac{|\Gamma|}{|\Gamma_0(F_q)|} \cdot \text{FPdim}(M'_{UF}). \tag{72}
\]

The object \( M'_{UF} \in \tilde{\mathcal{M}}_{UF,e} \subset eD(UF) \) corresponds (in the sense of Theorem 2.30 applied to \( U \)) to the restriction \( W' = \text{Res}^{G_0(F_q)}_{U_0(F_q)}(W) \). Hence by (70) and (72), to prove (65) we are reduced to showing that\(^{10}\)

\[
\text{FPdim}(M'_{W'}) = \frac{\sqrt{\dim(\mathcal{M}_{U,e})}}{q^{de}} \cdot \dim(W') \tag{73}
\]

for each \( W' \in \text{Rep}_{e_0}(U_0(F_q)) \), where \( M'_{W'} \in \tilde{\mathcal{M}}_{UF,e} \subset eD(UF) \) is the object corresponding to \( W' \) according to Theorem 2.27. Now, \( \tilde{\mathcal{M}}_{U,e} = \mathcal{M}_{U,e} \) is a pointed modular category. Hence all simple objects in the invertible \( \tilde{\mathcal{M}}_{U,e} \)-module category \( \tilde{\mathcal{M}}_{UF,e} \) have equal Frobenius-Perron dimension. Hence

\[
|\text{Irrep}_{e_0}(U_0(F_q))| \cdot \text{FPdim}(M'_{W'})^2 = \dim(\mathcal{M}_{U,e}) \tag{74}
\]

Also \( (U_0(F_q)/H_0(F_q)) \) is commutative, so all irreducible representations in \( \text{Rep}_{e_0}(U_0(F_q)) \) have the same dimension and we have

\[
|\text{Irrep}_{e_0}(U_0(F_q))| \cdot \dim(W')^2 = |U_0(F_q)/H_0(F_q)| = \frac{q^{\dim G}}{q^{\dim H}} = q^{2de}. \tag{75}
\]

The equality (73) follows from equations (74) and (75). This completes the proof of Theorem 2.30.

\(^{10}\)We have \( \Gamma_0(F_q) = G_0(F_q)/U_0(F_q) \) and \( |U_0(F_q)| = q^{\dim U} = q^{\dim G} \) since \( U \) is a connected and unipotent group.
References

[BD1] M. Boyarchenko and V. Drinfeld. *A motivated introduction to character sheaves on unipotent groups in positive characteristic*, September 2006, arXiv:math/0609769v2.

[BD2] M. Boyarchenko and V. Drinfeld. *Character Sheaves on Unipotent Groups in Positive Characteristic: Foundations*, October 2008, Selecta Mathematica, DOI: 10.1007/s00029-013-0133-7, arXiv:0810.0794v1.

[Bo1] M. Boyarchenko. *Characters of Unipotent Groups over Finite Fields*, Selecta Mathematica, Vol. 16 (2010), No. 4, pp. 857–933, arXiv:0712.2614v4.

[Bo2] M. Boyarchenko. *Character sheaves and characters of unipotent groups over finite fields*, American Journal of Mathematics, Volume 135, Number 3, June 2013 pp. 663-719, 10.1353/ajm.2013.0023

[De1] T. Deshpande. *Heisenberg Idempotents on Unipotent Groups*, Math. Res. Lett. 17 (2010), no. 3, 415 - 434, arxiv:0907.3344.

[De2] T. Deshpande. *Modular Categories Associated with Unipotent Groups*, Selecta Mathematica (2013), DOI: 10.1007/s00029-013-0126-6, arXiv:1201.6473.

[De3] T. Deshpande. *Minimal Idempotents on Solvable Groups*, arXiv:1312.4257.

[DGNO] V. Drinfeld, S. Gelaki, D. Nikshych and V. Ostrik. *On Braided Fusion Categories I*, June 2009, arXiv:0906.0620v1.

[ENO] P. Etingof, D. Nikshych, V. Ostrik. *Fusion Categories and Homotopy Theory*, September 2009, arXiv:0909.3140v2.

[L] G. Lusztig. *Character sheaves I-V*, Adv. in Math. 56, 57, 59, 61 (1985,1986).

[S] G. Schaumann. *Traces on module categories over fusion categories*, J. Algebra, 379 (2013), arXiv:1206.5716.