REMARK ON SINGLE EXPONENTIAL BOUND OF THE VORTICITY GRADIENT FOR THE TWO-DIMENSIONAL EULER FLOW AROUND A CORNER

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Abstract. In this paper, the two dimensional Euler flow under a simple symmetry condition with hyperbolic structure in a unit square

\[ D = \{(x_1, x_2) : 0 < x_1 + x_2 < \sqrt{2}, 0 < -x_1 + x_2 < \sqrt{2}\} \]

is considered. It is shown that the Lipschitz estimate of the vorticity on the boundary is at most single exponential growth near the stagnation point.

1. Introduction

Let \( D \) be a two-dimensional domain. We consider the Euler equation on \( D \), in the vorticity formulation:

\[(1.1) \quad \omega_t + (u \cdot \nabla)\omega = 0, \quad \omega(x, 0) = \omega_0(x).\]

Here \( \omega \) is the fluid vorticity, and \( u \) is the velocity of the flow determined by the Biot-Savart law. We impose a no flow condition for the velocity at the boundary: \( u \cdot n = 0 \) on \( \partial D \), where \( n \) is the unit normal vector on the boundary. This implies the formula:

\[(1.2) \quad u(x, t) = \nabla^\perp \int_D G_D(x, y)\omega(y, t)dy,\]

where \( G_D \) is the Green function for the Dirichlet problem in \( D \) and \( \nabla^\perp = (\partial_{x_2}, -\partial_{x_1}) \). Global regular solutions to (1.1) was proved by Wolibner [11] and Hölder [5] and there are huge literature on this problem. We are concerned with the question how fast the maximum of the gradient of the vorticity can grow as \( t \to \infty \). When \( D \) is a smooth bounded domain, the best known upper bound on the growth is double-exponential [12], but the question whether such upper bound is sharp has been open for a long time. Very recently, Kiselev and Sverak [6] answered the question affirmatively for the case the domain is a ball. They gave an example of the solution growing with double exponential rate. On the other hand, Kiselev and Zlatos [7] considered the 2D Euler flow on some bounded domain with certain cusps. They showed that the gradient of vorticity blows up at the cusps in finite time. These solutions are constructed by imposing certain symmetries on the initial data, which leads to a hyperbolic flow scenario near a stagnation point on the boundary. More precisely, by the hyperbolic flow scenario, particles on the boundary (near the stagnation point) head for the stagnation point for all time. Moreover the relation between this scenario and the geometry of the boundary plays a crucial role in the double exponential growth or the formation of the singularity.

Thus it would be an interesting question to ask how the geometry of the boundary affects the growth of the solution in the hyperbolic flow scenario. In this paper, we consider a unit square \( D = \{(x_1, x_2) : 0 < x_1 + x_2 < \sqrt{2}, 0 < -x_1 + x_2 < \sqrt{2}\} \), and under a simple symmetry condition.
the growth of the Lipschitz norm of the hyperbolic flow on the boundary is shown to be \emph{at most} single exponential near the stagnation point.

To state our result precisely, we rewrite (1.1) into the whole plane $\mathbb{R}^2$, by change of variables and reflection argument. By \cite{9, 10} (see also \cite{8}), we see that the solution uniquely exists in $u \in C([0, \infty); L^\infty), \quad \omega \in L^\infty_{\text{loc}}([0, \infty) : L^\infty)$.

Note in this case, the initial vorticity may not be continuous. However we see that $-\nabla(-\Delta)^{-1} \omega = u \in W^{1,1}_{\text{loc}}$, since the Riesz transform $\nabla(-\Delta)^{-1/2}$ is bounded from $L^\infty$ to $BMO$. Then by the ODE theory of DiPerna-Lions \cite{3}[Theorem III.2], we can define the trajectory $\gamma_X(t)$ corresponding to the 2D Euler equation:

$$\frac{dy_X(t)}{dt} = u(\gamma_X(t), t), \quad \gamma_X(0) = X \in \mathbb{R}^2$$

(c.f. \cite{13}). The following is the main theorem.

**Theorem 1.1.** Let $\omega_0$ be a Lipschitz function and odd with respect to the vertical axis, namely, $\omega_0(x_1, x_2) = -\omega_0(-x_1, x_2)$. Then there is a universal constant $C$ such that the following statement holds true: For any $T > 0$, there exist $\delta > 0$ and the solution of (1.1) such that

$$|\omega(x, t)| \leq \|\omega_0\|_{\text{Lip}} |x|^C \|\omega_0\|_{L^\infty_{\text{loc}}} \quad \text{for } x \in \partial D \cap B(0, \delta) \text{ and } t < T.$$  

The above estimate exhibits single exponential bound of the Lipschitz norm at the origin.

**Remark.** (1) We have showed the upper bound of certain hyperbolic flows around the stagnation point. It seems likely that our example grows exactly with the single exponential rate, but we were not able to prove it so far.

(2) For the case when $D$ is the unit square $[0, 1] \times [0, 1]$ with the periodic boundary condition, Zlatos \cite{13} constructed examples of the $C^{1,\gamma}$ solutions which exhibit single exponential growth, where the flows are also of hyperbolic and have odd symmetry in both axes. Moreover Hoang-Radosz \cite{4} considered the same problem and they showed that the compression of the fluid induced by the hyperbolic flow scenario alone is not sufficient for the double exponential growth of the smooth solution (see also \cite{1, 2}).

2. Proof of the main theorem

For $x = (x_1, x_2) \in \mathbb{R}^n$ we let $\tilde{x} = (-x_1, x_2), \quad \overline{x} = (x_1, -x_2), \quad x^* = (x_2, x_1)$. Let $D_+ = \{x \in D : x_1 > 0\}$. The Green function $G_D$ is given explicitly by

$$G_D(x, y) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \left( \log \frac{|x - 2m - y|}{|x^* - 2m - y|} \right),$$

where $m = (m_1, m_2) = (\frac{n_1 - m_2}{\sqrt{2}}, \frac{n_1 + m_2}{\sqrt{2}})$. We can obtain the above $G_D$ just using reflection and $\pi/4$-rotation (c.f. \cite{13}). Since $\omega(\cdot, t) \in L^\infty(D)$ is odd with respect to the vertical axis, then we have

$$(2.1) \int_D G_D(x, y) \omega(y, t) dy = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \int_{D_+} \left( \log \frac{|x - 2m - y - x - 2m - y|}{|x^* - 2m - y|\tilde{x}^* - 2m - y|} \right) \omega(y, t) dy.$$

We now give the key lemma.

**Lemma 2.1.** Let $\omega(\cdot, t) \in L^\infty(D)$ be odd with respect to the vertical axis. Let $a > 1$ and let $D_a = \{x \in D_+ : ax_1 \geq x_2\}$. There exists a constant $C_0$ depending only on $a$ such that
$$\sup_{x \in D_{a} \cap B(0, \frac{1}{2})} \left| \frac{u_j(x, t)}{x_j} \right| \leq C_0 \| \omega(\cdot, t) \|_{\infty} \quad (j = 1, 2).$$

**Proof.** Let us only consider $j = 1$ because $j = 2$ follows by symmetry. Let $x \in D_{a} \cap B(0, \frac{1}{2})$. By (1.2) and (2.1), we have

$$u_1(x, t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \int_{D_n} \frac{x_2 - 2m_2 - y_2}{|x - 2m - y|^2} - \frac{x_2 - 2m_2 - y_2}{|\tilde{x} - 2m - y|^2} - \frac{x_2 - 2m_1 - y_1}{|x^* - 2m - y|^2} + \frac{x_2 - 2m_1 - y_1}{|\tilde{x}^* - 2m - y|^2} \omega(y, t)dy$$

$$= \frac{2x_1}{\pi} \sum_{n \in \mathbb{Z}^2} \int_{D_n} \left[ \frac{(x_2 - 2m_2 - y_2)(2m_1 + y_1)}{|x - 2m - y|^2|\tilde{x} - 2m - y|^2} - \frac{(x_2 - 2m_1 - y_1)(2m_2 + y_2)}{|x^* - 2m - y|^2|\tilde{x} - 2m - y|^2} \right] \omega(y, t)dy.$$

The right above first two terms are essentially the same as those in (2.4) in [13]. Due to the extra symmetry, we have two more terms which bring more cancellations. More precisely, leading terms with singularity as in Lemma 2.1 [13] and Lemma 3.1 [6] do not appear. Let

$$A_n(x, y) = \frac{(x_2 - 2m_2 - y_2)(2m_1 + y_1)}{|x - 2m - y|^2|\tilde{x} - 2m - y|^2}, \quad B_n(x, y) = \frac{(x_2 - 2m_1 - y_1)(2m_2 + y_2)}{|x^* - 2m - y|^2|\tilde{x} - 2m - y|^2},$$

$$C_n(x, y) = \frac{(x_2 + 2m_2 + y_2)(2m_1 + y_1)}{|\tilde{x} - 2m - y|^2 - x - 2m - y|^2}, \quad D_n(x, y) = \frac{(x_2 + 2m_1 + y_1)(2m_2 + y_2)}{|\tilde{x}^* - 2m - y|^2 - x^* - 2m - y|^2}.$$

We need to divide into two parts: $n \neq (0, 0)$ case and $n = (0, 0)$ case. If $n \neq (0, 0)$, we can control each terms for $|n|^{-3}$ order. Thus, summation of these terms converge. More precisely, we have the following calculation for $n \neq (0, 0)$:

$$|A_n(x, y) - B_n(x, y)|$$

$$\leq \left| \frac{-4m_1m_2}{|x - 2m - y|^2|\tilde{x} - 2m - y|^2} + \frac{4m_1m_2}{|x^* - 2m - y|^2|\tilde{x} - 2m - y|^2} \right| + C|n|^{-3}$$

$$= \frac{|4m_1m_2||x|^2 + |2m + y|^2(2m_1 - 2m_2 + y_1 - y_2)x_2 - |x|^2[(2m_1 + y_1)^2 - (2m_2 + y_2)^2]|}{|x - 2m - y|^2|\tilde{x} - 2m - y|^2|x^* - 2m - y|^2} + C|n|^{-3}$$

$$\leq C|n|^{-3}$$

and

$$|C_n(x, y) + D_n(x, y)|$$

$$\leq \left| \frac{-4m_1m_2}{|\tilde{x} - 2m - y|^2 - x - 2m - y|^2} + \frac{4m_1m_2}{|\tilde{x} - 2m - y|^2 - x^* - 2m - y|^2} \right| + C|n|^{-3}$$

$$= \frac{|4m_1m_2||x|^2 + |2m + y|^2(2m_1 + 2m_2 + y_1 + y_2)x_2 - |x|^2[(2m_1 + y_1)^2 - (2m_2 + y_2)^2]|}{|x - 2m - y|^2|x^* - 2m - y|^2 - x^* - 2m - y|^2} + C|n|^{-3}$$

$$\leq C|n|^{-3}.$$
Therefore we obtain
\begin{equation}
\sum_{n \neq 0} \int_{D_n} \left| A_n(x,y) - B_n(x,y) - C_n(x,y) + D_n(x,y) \right| \omega(y,t)dy \leq C \|\omega(\cdot,t)\|_\infty \sum_{n \neq 0} |n|^{-3} \leq C \|\omega(\cdot,t)\|_\infty.
\end{equation}

Next we consider the term \( n = (0,0) \). We show that
\begin{equation}
\int_{D_1} \left( \frac{(x_2 - y_2)y_1}{|x - y|^2} - \frac{(x_2 - y_1)y_2}{|x - y|^2} - \frac{(x_2 + y_1)y_2}{|x - y|^2} + \frac{(x_2 + y_2)y_1}{|x - y|^2} \right) \omega(y,t)dy \leq C \|\omega(\cdot,t)\|_\infty.
\end{equation}

If \( y \in D_+ \) and \(|y| \geq 2|x|\), then
\[
\frac{x_2y_1}{|x - y|^2} \leq C \frac{|x|}{|y|^3}, \quad \frac{x_2y_2}{|x^* - y|^2} \leq C \frac{|x|}{|y|^3},
\]
and
\[
\left| -\frac{y_1y_2}{|x - y|^2} - \frac{y_1y_2}{|x^* - y|^2} \right| \leq \frac{y_1y_2[4(|x|^2 + |y|^2)(y_1 - y_2)x_2 - 4|x|^2(y_1^2 - y_2^2)]}{|x - y|^2|x - y|^2|x^* - y^2| - |x^* - y^2|^2} \leq C \frac{|x|}{|y|^3}.
\]

We obtain the above estimates by the extra symmetry, and these show that leading terms with singularity as in Lemma 2.1 [13] and Lemma 3.1 [6] do not appear. The rest calculation are essentially the same as in [6, 13]. These inequalities imply that (away from the origin)
\[
\int_{D_1 \setminus B(0,2|x|)} \left| \frac{(x_2 - y_2)y_1}{|x - y|^2} - \frac{(x_2 - y_1)y_2}{|x^* - y|^2} \right| \omega(y,t)dy \leq C \|\omega(\cdot,t)\|_\infty \int_{2|x|}^{\sqrt{r}} \frac{|x|}{r^2} dr \leq C \|\omega(\cdot,t)\|_\infty.
\]

In a similar way, we have
\[
\int_{D_1 \setminus B(0,2|x|)} \left| -\frac{(x_2 + y_1)y_1}{|x - y|^2} - \frac{(x_2 + y_1)y_2}{|x^* - y|^2} \right| \omega(y,t)dy \leq C \|\omega(\cdot,t)\|_\infty.
\]

Observe that if \( y \in D_+ \cap B(0,2|x|) \), then
\[
|x - y|, |x - y|, |x^* - y|, |x^* - y| \geq C|x|.
\]

Therefore, we obtain that (near the origin)
\[
\int_{D_1 \cap B(0,2|x|)} \left| \frac{(x_2 - y_1)y_1}{|x - y|^2} + \frac{(x_2 + y_1)y_2}{|x^* - y|^2} \right| \omega(y,t)dy \leq C \|\omega(\cdot,t)\|_{\infty} |x|^{-2} \int_{D_1 \cap B(0,2|x|)} dy \leq C \|\omega(\cdot,t)\|_{\infty}.
\]

and
\[
\int_{D_1 \cap B(0,2|x|)} \left| \frac{(x_2 - y_2)y_2}{|x^* - y|^2} \right| dy \leq C \|\omega(\cdot,t)\|_{\infty} \int_{0}^{2|x|} \frac{|x_2 - y_1|}{|x - y|^2} dy_1dy_2 \leq C \|\omega(\cdot,t)\|_{\infty} \int_{0}^{2|x|} \frac{z_1}{z_1^4 + z_2^4} dz_1 dz_2 \leq C \|\omega(\cdot,t)\|_{\infty} \int_{0}^{2|x|} \arctan \frac{2|x|}{z_1} dz_1 \leq C \|\omega(\cdot,t)\|_{\infty}.
\]

Finally we prove that
\[
\int_{D_1 \cap B(0,2|x|)} \left| \frac{(x_2 - y_2)y_1}{|x - y|^2} \right| dy \leq C \|\omega(\cdot,t)\|_{\infty}.
\]
We separate the integral into two regions. Let $D_1 = D_+ \cap B(0, 2|x|) \cap [0, 2x_1] \times [0, 1]$ and $D_2 = D_+ \cap B(0, 2|x|) \cap [2x_1, 1] \times [0, 2x_2]$. We have
\[
\left| \int_{D_1} \frac{(x_2 - y_2)y_1}{|x - y|^2} \omega(y, t) dy \right| \leq C\|\omega(\cdot, t)\|_\infty \int_{D_1} \frac{|x_2 - y_2|y_1}{|x - y|^4} dy
\]
\[
\leq C\|\omega(\cdot, t)\|_\infty \int_0^{2x_1} \int_0^1 \frac{|z_2x_1|}{(z_1^2 + z_2^2)(z_1^2 + z_2^2)} dz_1 dz_2
\]
\[
= C\|\omega(\cdot, t)\|_\infty \int_0^{2x_1} \frac{x_1}{(x_1^2 + z_2^2)} \arctan \frac{x_1}{z_2} dz_2
\]
\[
\leq C\|\omega(\cdot, t)\|_\infty \frac{1}{x_1} \leq C\|\omega(\cdot, t)\|_\infty.
\]
Since $x_2 \leq ax_1$, we obtain
\[
\left| \int_{D_2} \frac{(x_2 - y_2)y_1}{|x - y|^2} \omega(y, t) dy \right| \leq \|\omega(\cdot, t)\|_\infty \int_{D_2} \frac{|x_2 - y_2|y_1}{|x - y|^4} dy
\]
\[
\leq C\|\omega(\cdot, t)\|_\infty \int_0^{2x_2} \int_0^1 \frac{z_1 z_2}{(z_1^2 + z_2^2)(z_1^2 + z_2^2)} dz_1 dz_2
\]
\[
\leq C\|\omega(\cdot, t)\|_\infty \int_0^{2x_2} \frac{z_2}{x_1^2 + z_2^2} dz_2
\]
\[
\leq C\|\omega(\cdot, t)\|_\infty \log(1 + \frac{x_2^2}{x_1^2}) \leq C\log(1 + a^2)\|\omega(\cdot, t)\|_\infty.
\]
Thus we have (2.4). \[\square\]

**Proof of Theorem 1.1.** It is well known that $\|\omega(\cdot, t)\|_\infty = \|\omega_0\|_\infty$ and recall that if $\omega_0$ is odd with respect to vertical axis, then $\omega(x, t)$ is also odd with respect to vertical axis. Also recall that $\gamma_X(t) = (\gamma_X(t), \gamma_X(2)(t))$ is the flow map corresponding to the 2D Euler evolution:

\begin{equation}
\frac{d\gamma_X(t)}{dt} = u(\gamma_X(t), t), \quad \gamma_X(0) = X.
\end{equation}

Due to the boundary condition on $u$, the trajectories which start at the boundary stay on the boundary for all times. Due to Lemma 2.1, the following observation holds true: For any $T > 0$, there is $\delta > 0$ such that the trajectory starting a point $X \in \partial D \cap B(0, \delta)$ stays in $\gamma_X(t) \in B(0, \frac{1}{2})$ for $t < T$. Let $x = \gamma_X(t)$. By Lemma 2.1 and (2.5), we have

\[
\frac{d\gamma_{X,1}(t)}{dt} \geq -C_0\|\omega_0\|_\infty \gamma_{X,1}(t) \quad \text{for all } t > 0.
\]

By Gronwall’s lemma we have $\gamma_{X,1}(t) \geq X_t e^{-C_0\|\omega_0\|_\infty t}$, so that $\gamma_{X,1}^{-1}(t) \leq x_t e^{C_0\|\omega_0\|_\infty t}$. In a similar way, we obtain $\gamma_{X,2}^{-1}(t) \leq x_t e^{C_0\|\omega_0\|_\infty t}$. Hence we have

\[
|\gamma_{X,1}^{-1}(t)| \leq |x| e^{C_0\|\omega_0\|_\infty t}.
\]

Since $\omega(x, t) = \omega_0(\gamma_{X,1}^{-1}(t))$ by the 2D Euler flow in the Lagrangian form, and $\omega_0$ is Lipschitz, we obtain

\[
|\omega(x, t)| = |\omega_0(\gamma_{X,1}^{-1}(t))| \leq \|\omega_0\|_{Lip} |\gamma_{X,1}^{-1}(t)| \leq \|\omega_0\|_{Lip} |x| e^{C_0\|\omega_0\|_\infty t}.
\]

\[\square\]
3. Conclusion

We have seen that under some symmetry conditions of the initial data for the Euler equations in a unit square, the vorticity gradient grows at most single exponential rate along the boundary near the stagnation point on the corner. Taking account of the results [6, 7] concerning growing solutions in other domains, we observe that the angle of the corner of the boundary is related to the growth of the flows. It seems likely that the sharper the angle of the corner becomes, the slower the growth rate of the solutions under the hyperbolic flow scenario gets. We will pursue this issue in a future work. Note that we could not deal with the solution on the corner with general angle, since the symmetry of the domain plays an important role in our construction of the solutions.

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