LONGTIME BEHAVIOR FOR 3D NAVIER-STOKES EQUATIONS
WITH CONSTANT DELAYS

Dedicated to Tomás Caraballo on his 60th birthday

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Abstract. This paper investigates the longtime behavior of delayed 3D Navier-Stokes equations in terms of attractors. The study will strongly rely on the investigation of the linearized Navier-Stokes system, and the relationship between the discrete dynamical flow for the linearized system and the continuous flow associated to the original system. Assuming the viscosity to be sufficiently large, there exists a unique attractor for the delayed 3D Navier-Stokes equations. Moreover, the attractor reduces to a singleton set.

1. Introduction. The incompressible 3D Navier-Stokes equations are described by time evolution of the velocity $u$ in a bounded or unbounded domain of $\mathbb{R}^3$ and are given by:

$$
\begin{align*}
    u'(t,x) + (u(t,x) \cdot \nabla)u(t,x) - \nu \Delta u(t,x) + \nabla p(t,x) &= 0, \\
    \text{div} \, u(t,x) &= 0, \quad u(0,x) = u_0(x),
\end{align*}
$$

where $\nu > 0$ is the viscosity of the fluid, $p$ denotes the pressure and $u_0(x)$ denotes the initial datum. The uniqueness of global weak solutions is a standing open problem. In order to overcome this challenging difficulty, in a previous paper, see Bessaih et al [2], we introduced a constant delay $\mu > 0$ into the nonlinear term $(u \cdot \nabla)u$. More precisely, we considered the following modified version of the 3D Navier-Stokes equations:

$$
\begin{align*}
    u'(t,x) + (u(t-\mu,x) \cdot \nabla)u(t,x) - \nu \Delta u(t,x) + \nabla p(t,x) &= f(x), \\
    \text{div} \, u(t,x) &= 0, \quad u(0,x) = u_0(x), \quad u(\tau,x) = \phi(\tau,x), \quad \tau \in [-\mu,0).
\end{align*}
$$

This delay introduces a regularizing effect in the equations and allows to prove the uniqueness of global weak solutions if the initial function $(\phi,u_0) \in L_2(-\mu,0,V^{1+\alpha}) \times V^\alpha$ with $\alpha > 1/2$ (for the definition of the spaces $V^\alpha$ see Section 2). In particular, when $\alpha \geq 1$, then our theory can be extended to include strong solutions. The
main ingredient to establish it is to use the regularizing effect of the delay on the convective term by investigating the linearized version of (1). This equation comes naturally when considering the system on the interval \([0,\mu]\). We prove existence and uniqueness of weak solutions, then we establish that these solutions are more regular and are in the spaces \(V^\alpha\). Then, we use a concatenation argument by gluing the solutions obtained on each interval \([0,\mu], [\mu, 2\mu], \ldots\) and so on. Each solution is obtained from the previous step and uses the linearized construction.

As a byproduct, the linearized equation induces a continuous mapping \(U\) on the space \(L_2(0,\mu, V^{1+\alpha}) \times V^\alpha\). The \(n\)th composition of the map \(U\) generates a discrete semigroup \(U(n)\) on the same space \(L_2(0,\mu, V^{1+\alpha}) \times V^\alpha\). Moreover, thanks to the concatenation argument the solution of system (1) generates a continuous semigroup \(S(t)\) on the space \(L_2(-\mu,0, V^{1+\alpha}) \times V^\alpha\) for \(t \geq 0\) given by \(S(t)(\phi,u_0) = (u^\mu(t), u^{\mu}(t))\), where \(u^\mu(t)\) is the segment function defined by \(u^\mu(t+s) = u^\mu(t+s), s \in (-\mu,0)\) \((u^\mu\) denotes the solution of (1)), defined in more details in Section 2.

Our goal in this paper is to study the longtime behavior of (1) in terms of attractors. Let us point out that the existence of a global attractor for (1) is essentially based on getting a positively or forward invariant ball for the map \(U\). The main ingredient that allows to get the invariance of a bounded ball for the discrete semigroup \(U(n)\) and later the semigroup \(S(t)\) is the fact that the unique weak global solution is regular enough, see Lemma 3.1, 3.2 and 3.3. Combined with compactness embeddings, this allows to prove that the positively invariant ball for the discrete semigroup \(U(n)\) is compact in the topology of \(L_2(0,\mu, V^{1+\alpha}) \times V^\alpha\). An interesting feature of this model is that we are able to establish that the attractor \(A\) associated to the discrete semigroup \(U(n)\) is a single point attractor. These properties are transferred to the original delayed 3D Navier-Stokes equations, due to the key relationship between \(U\) and \(S\). In fact, under the same conditions as for the discrete flow \(U\), the continuous flow \(S\) is proved to have an attractor \(A^\mu\), that reduces to a singleton set and is linked to the attractor \(A\) as \(S(t)\) and \(U\) are related on the grid points \(t = n\mu, n \in \mathbb{N}\).

Now, we would like to comment on the previous literature regarding delayed Navier–Stokes equations. In [8], Plana and Hernández considered 2D Navier–Stokes equations with a time–delayed convective term and a forcing term which contains some hereditary features. They proved existence and uniqueness of solutions as well as the asymptotic behavior of solutions, including the exponential stability of stationary solutions. Also for 2D Navier–Stokes equations, García-Luengo et al. [4] dealt with delays in the convective and forcing terms, proving existence and uniqueness of solutions and the existence of pullback attractors in several phase spaces, analyzing the relationships among them. Moving to 3D, as an extension of [8], Guzzo and Planas [5] investigated the existence of weak solutions (but not uniqueness) and the existence and uniqueness of strong solutions with the exponential longtime behavior, with a time variable bounded delay function in the convective term. For infinite delay of convolutional type, Guzzo and Planas also investigated mild solutions in [6], but these solutions are only local in time. In [12], Varnhorn considered a similar delay used in equation (1) and investigated the existence and uniqueness of strong solutions in a bounded and regular domain \(D \subset \mathbb{R}^3\) with Dirichlet boundary conditions. In that article, the initial delay function \(\chi\) is assumed to satisfy that \(\chi \in C([-\mu,0], H^2(D)), \) with \(\frac{\partial \chi}{\partial t} \in C([-\mu,0], L_2(D))\) and \(\div \chi = 0\). With these assumptions, the author proved the existence and uniqueness of strong solutions.
Our setting here is different since we are dealing with weak solutions. More precisely, we consider weaker initial conditions \( \chi = (\phi, u_0) \in L_2(-\mu, 0, V^{1+\alpha}) \times V^\alpha \) with \( \alpha > 1/2 \) (see the functional setting in Section 2), our domain has periodic boundary conditions and we will focus on analyzing the longtime behavior of (1).

The fact that our domain has periodic boundary conditions is crucial in order to use Lemma 2.1 which gives better estimates than in the case of Dirichlet boundary conditions. This lemma allows to improve the regularity of solutions, see Section 3.

The paper is organized as follows. In Section 2 we introduce the abstract setting in which we develop our theory and recall how the construction of the unique weak solution of (1) was carried out in the paper [2], by using a suitable linearization of (1) on \([0, \mu]\). Section 3 addresses the regularization properties of the solution of (1) assuming that the external force \( f \) is in \( V^\alpha \). In Section 4, we first consider a linearized system defined now on any compact interval \([0, T]\) for a given \( T > 0 \) and construct its corresponding unique weak solution. Then we establish a fundamental relationship between the discrete flow \( U \) generated by the solution of the linearized system and the continuous flow \( S \) generated by the solution of (1). Section 5 is devoted to the study of the attractor for the linearized system and finally, in Section 6, we establish the existence of a unique attractor for \( S \) and we study its inner structure.

2. Preliminaries: existence and uniqueness of a weak solution. We introduce in this section the functional setting in which our investigations will be carried out and the existence and uniqueness of solutions of the delayed Navier-Stokes equations as well.

Consider the torus \( \mathbb{T}_L^3 \) in \( \mathbb{R}^3 \) of length \( L \) given by the set

\[
\mathbb{T}_L^3 := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : -L/2 \leq x_i \leq L/2; \quad x_i = -L/2 \text{ is identified with } x_i = L/2, \quad i = 1, 2, 3\}.
\]

Let \( \psi(x) \) be a \( L \)-periodic function that can be expanded into Fourier series

\[
\psi(x) = \sum_{\zeta \in \mathbb{Z}_L^3} e^{i(x, \zeta)} \hat{\psi}(\zeta),
\]

where

\[
\mathbb{Z}_L^3 = \{\zeta = (\zeta_1, \zeta_2, \zeta_3) : \zeta_i = 2\pi k_i/L, \quad k_i \text{ is an integer, } i = 1, 2, 3\},
\]

and

\[
\hat{\psi}(\zeta) = L^{-3} \int_{\mathbb{T}_L^3} e^{-i(y, \zeta)} \psi(y) dy
\]

denote the Fourier coefficients of \( \psi \).

For \( s \in \mathbb{R} \), we denote by \( H^s(\mathbb{T}_L^3) \) the Sobolev space of \( L \)-periodic functions such that \( \hat{\psi}(\zeta) = \overline{\hat{\psi}(-\zeta)} \) equipped with the norm

\[
\|\psi\|_s = \left( \sum_{\zeta \in \mathbb{Z}_L^3} (1 + |\zeta|^2)^s |\hat{\psi}(\zeta)|^2 \right)^{1/2}.
\]

When \( \hat{\psi}(0) = 0 \) the corresponding subspace is denoted by \( \dot{H}^s(\mathbb{T}_L^3) \) with equivalent norm

\[
\left( \sum_{\zeta \in \mathbb{Z}_L^3 \setminus \{0\}} |\zeta|^2s |\hat{\psi}(\zeta)|^2 \right)^{1/2}.
\]
These spaces are Hilbert–spaces with the inner product
\[
\langle \psi_1, \psi_2 \rangle_s = \sum_{\zeta \in \mathbb{Z}^3_+ \setminus \{0\}} |\zeta|^{2s} \psi_1(\zeta) \overline{\psi_2(\zeta)}.
\]
We denote \(\dot{H}^s(T^3_3) = \dot{H}^s(T^3_1)^3\) and, for \(s = -1, 0, 1\), we introduce the spaces
\[
V^s = \{u \in \dot{H}^s(T^3_3), \text{div } u = 0\}.
\]
Then \(V^{-1}\) is the dual space of \(V^1\) and \(V^1 \subset V^0 \subset V^{-1}\) where the injections are continuous and each space is dense in the following one. We shall denote by \(<\cdot, \cdot>\) the scalar product in \(V^0\).

We introduce the Stokes operator \(A\) as in [10], Section 2.2, page 9, with domain given by
\[
D(A) = \{u \in V^0, \Delta u \in V^0\}.
\]
For the periodic boundary conditions we know that
\[
Au = -\Delta u.
\]
The operator \(A\) can be seen as an unbounded positive linear selfadjoint operator on \(V^0\), and we can define the powers \(A^s, s \in \mathbb{R}\) with domain \(D(A^s)\). We set \(V^s = D(A^{s/2})\), that is a closed subspace of \(\dot{H}^s(T^3_3)\), then for any \(s \in \mathbb{R}\)
\[
V^s = \{u \in \dot{H}^s(T^3_3), \text{div } u = 0\}
\]
and the norms \(\|A^{s/2}u\|_0\) and \(\|u\|_s\) are equivalent on \(V^s\). The operator \(A\) defines an isomorphism from \(V^s\) to \(V^{s-2}\), and has a positive countable spectrum of finite multiplicity \(0 < \lambda_1 \leq \lambda_2 \leq \cdots, \lambda_j \to \infty\), where the associated eigenvectors \(e_1, e_2, \cdots\) form a complete orthogonal system in \(V^s\).

When \(s_1 < s_2\), the embedding \(V^{s_2} \subset V^{s_1}\) is compact and dense. The space \(V^{-s}\) is the dual space of \(V^s\) for \(s \in \mathbb{R}\), see Temam [10], from page 9. We shall denote by \(<\cdot, \cdot>\) the duality product between \(V^s\) and \(V^{-s}\) no matter the value of \(s \in \mathbb{R}\).

Let us introduce the trilinear form \(b\) given by
\[
b(u, v, w) = \sum_{i,j=1}^3 \int_{T^3_3} u_i \frac{\partial v_j}{\partial x_j} w_j dx.
\]

The following result is essential in our estimates. For the proof, we refer to [2].

**Lemma 2.1.** The trilinear form \(b\) can be continuously extended to \(V^{s_1} \times V^{s_2+1} \times V^{s_3}\) for \(s_i \in \mathbb{R}\) if either \(s_i + s_j \geq 0\) for \(i \neq j\), \(s_1 + s_2 + s_3 > 3/2\) or \(s_i + s_j > 0\) for \(i \neq j\), \(s_1 + s_2 + s_3 \geq 3/2\). Therefore, under any of the previous settings, there exists a constant \(c\) depending only on \(s_i\) such that
\[
|b(u, v, w)| \leq c \|u\|_{s_1} \|v\|_{s_2+1} \|w\|_{s_3}
\]
for \(u \in V^{s_1}, v \in V^{s_2+1}, w \in V^{s_3}\).

Notice that similar results were proved by Fursikov [3] when considering a bounded domain \(\Omega \subset \mathbb{R}^3\), \(\partial \Omega \in C^\infty\) with homogeneous Dirichlet conditions, but with more restrictive assumptions. In the periodic boundary setting, for a similar result as Lemma 2.1 above see also Temam [10], Lemma 2.1, page 12, which holds true under the additional assumptions \(s_i \geq 0\).
When \( u, v, w \in V^1 \) it is known that \( b(u, v, w) = -b(u, w, v) \), which implies \( b(u, v, v) = 0 \). Furthermore, from the trilinear form \( b \) we can derive a bilinear operator \( B: V^{s_1} \times V^{s_2+1} \to V^{-s_3} \) given by

\[
\langle B(u, v), w \rangle = b(u, v, w),
\]
such that

\[
\|B(u, v)\|_{-s_3} \leq c \|u\|_{s_1} \|v\|_{s_2+1}
\]

with \( s_1, s_2, s_3 \) satisfying the conditions of Lemma 2.1.

Finally we mention that for \( \mu > 0 \) and \( s \in \mathbb{R} \) the spaces \( L_\infty(0, \mu, V^s) \), \( L_2(0, \mu, V^s) \), \( C([0, \mu], V^s) \) and \( C^\beta([0, \mu], V^s) \), \( \beta \in (0, 1) \), have the usual meanings.

We are interested in studying the dynamics of the following version of the 3D Navier-Stokes equations with constant delay \( \mu \):

\[
\begin{align*}
&u'(t, x) + (u(t - \mu), \nabla)u(t) - \nu \Delta u(t) + \nabla p(t, x) = f(x), \\
&\text{div } u(t, x) = 0, \quad u(0) = u_0(x), \quad u(t) = \phi(t), \quad t \in [-\mu, 0).
\end{align*}
\]

Denote the solution of this equation depending on the time shift by \( u^\mu \). On account of the Helmholtz-projection, we can formulate the equation as

\[
\begin{cases}
&\frac{d}{dt}u^\mu(t) + (\nu Au^\mu(t) + B(u^\mu(t - \mu), u^\mu(t)))dt = f dt, \quad t \geq 0, \\
&u^\mu(0) = u_0, \\
&u^\mu(t) = \phi(t), \quad t \in [-\mu, 0).
\end{cases}
\]

**Definition 2.2.** Let \( \mu > 0 \) and \( \alpha > 1/2 \). We are given \( u_0 \in V^\alpha, \phi \in L^2(-\mu, 0, V^{1+\alpha}), f \in V^{\alpha-1} \) and \( T > 0 \). We say that \( u^\mu \) is a weak solution to system (4) on the time interval \( [-\mu, T] \) if

\[
u u^\mu \in L^2(-\mu, T, V^{1+\alpha}),
\]

with \( u^\mu(0) = u_0, \ u^\mu(t) = \phi(t) \) for \( t \in [-\mu, 0) \), and, given any \( v \in V^{\alpha+1} \) and any test function \( \varphi \in C_0^\infty([0, T]) \),

\[
\begin{align*}
&- \int_0^T \langle u^\mu(r), v \varphi'(r) \rangle dr + \nu \int_0^T \langle A^{1/2}u^\mu(r), A^{1/2}v \rangle \varphi(r) dr \\
&+ \int_0^T \langle B(u^\mu(r - \mu), u^\mu(r)), v \rangle \varphi(r) dr = \int_0^T \langle f, v \rangle \varphi(r) dr.
\end{align*}
\]

In order to prove the existence and uniqueness of solutions to (5), for \( t \in [0, \mu] \) and \( \psi \in L_2(0, \mu, V^{1+\alpha}) \), we introduce the following 3D linearized Navier–Stokes equations with periodic boundary conditions over the torus \( T^3_L \) in \( \mathbb{R}^3 \)

\[
\begin{cases}
&\frac{du(t) + (\nu Au(t) + B(\psi(t), u(t)))dt = f dt, \quad t \in [0, \mu], \\
&u(0) = u_0.
\end{cases}
\]

These equations are a simpler version of the 3D Navier–Stokes equations, since the term \( (u, \nabla)u \) has been replaced by \( (\psi, \nabla)u \). The existence and uniqueness of solutions to (4) and (6) can be summarized as follows.

For the sake of readability, the solutions of (4) will be denoted by \( u^\mu \) while the corresponding solutions to (6) are denoted by \( u \).

**Theorem 2.3.** Assume that \( u_0 \in V^\alpha \) and \( f \in V^{\alpha-1} \). Then

1. If \( \psi \in L_2(0, \mu, V^{1+\alpha}) \), (6) has a weak solution \( u \in L_\infty(0, \mu, V^\alpha) \cap L_2(0, \mu, V^{1+\alpha}) \cap C([0, \mu], V^\alpha) \).
Moreover, the following expression

\[ u \in C([0,T], V^{\alpha}) \] and if \( 0 \leq \gamma \leq 1/2 \) and \( s \geq 1 \), we also obtain \( u^\mu |_{[0,T]} \in L_\infty(0,T,V^\alpha) \cap C^1([0,T],V^{-s}) \), and \( \frac{du^\mu}{dt} \in L_2(0,T,V^{\alpha-1}). \)

**Proof.** Although the proof of this theorem is in the paper [2], for the sake of completeness we would like to give here some explanations of how to prove this result. The existence of a weak solution for the linearized problem (6) is obtained thanks to the use of Galerkin approximations, while the uniqueness relies on an energy inequality, based on the fact that \( u^\mu \in L_2(0,\mu,V^{-1}) \).

To prove existence and uniqueness of a weak solution of (4) on \([-\mu,T]\), the strategy followed in [2] consists in solving the problem (4) step by step, in intervals of length \( \mu \), where in each step the fact that for (6) there exists a unique weak solution is used. As a result, a sequence \( \{u^\mu_k\}_{k \in \mathbb{N}} \subset L_2(-\mu,\mu, V^{1+\alpha}) \) is built, with \( u^\mu_k(\mu) \in V^\alpha \) and \( u^\mu_k(\cdot-\mu) = u^\mu_{k-1}(\cdot) \in L_2(0,\mu,V^{1+\alpha}) \), for any \( k \in \mathbb{N} \). Concatenating the elements of the sequence the global solution of (4) is constructed, having the following expression

\[
u^\mu(t) = \begin{cases} 
\phi(t) & \text{if } t \in [-\mu,0), \\
u_0 & \text{if } t = 0, \\
u_1(t) & \text{if } t \in [0,\mu], \\
u_2(t-\mu) & \text{if } t \in [\mu,2\mu], \\
 & \vdots \\
u_k(t-(k-1)\mu) & \text{if } t \in [(k-1)\mu,T],
\end{cases}
\]

assuming that \( t \in [0,T] \), with \( T \in ((k-1)\mu,k\mu] \).

To finish this section, we give the definitions of a dynamical system and a global attractor, concepts that we will use later throughout the text.

**Definition 2.4.** Let \( \mathcal{X}, d \) be a metric space. A (semi)dynamical system is a continuous mapping \( \Phi : T^+_0 \times \mathcal{X} \to \mathcal{X} \) such that

\[ \Phi(0,x) = x, \quad \text{for all } x \in \mathcal{X}, \]

\[ \Phi(t+\tau,x) = \Phi(t,\Phi(\tau,x)), \quad \text{for all } t,\tau \in T^+_0, x \in \mathcal{X}. \]

Note that above \( T = \mathbb{Z} \) for discrete time while that \( T = \mathbb{R} \) for continuous time. Moreover, \( T^+_0 = \{ t \in T : t \geq 0 \} \).

**Definition 2.5.** A nonempty compact subset \( \mathcal{A} \) is called a global attractor for the dynamical system \( \Phi \) on \( \mathcal{O} \subset \mathcal{X} \) if

1. \( \mathcal{A} \) is \( \Phi \)-invariant, that is, \( \Phi(t,A) = A \) for all \( t \in T^+_0 \).
2. \( \mathcal{A} \) attracts all bounded sets of \( \mathcal{O} \), i.e.,

\[ \lim_{t \to \infty} \text{dist}(\Phi(t,B),\mathcal{A}) = 0 \]

for any bounded subset \( B \subset \mathcal{O} \), where \( \text{dist} \) denotes the Hausdorff semi-distance given by

\[ \text{dist}(A,B) = \sup_{a \in A} \inf_{b \in B} d(a,b). \]

The interested reader is referred to the classical monographs [1], [7] and [11] for sufficient conditions ensuring the existence and further properties of global attractors.
3. Regularization of weak solutions. In this section, we are going to show that assuming \( f \in V^\alpha \) the solution \( u^\mu \) to (4) is more regular. Thanks to this regularity, we will obtain a suitable compact property that will be further necessary to establish the existence of an attractor for the delayed Navier–Stokes equations.

From now on, we denote \( Y_\alpha^\mu = L_2(-\mu, 0, V^{1+\alpha}) \times V^\alpha \).

**Lemma 3.1.** Assume that \((\phi, u_0) \in Y_\alpha^\mu \) and \( f \in V^\alpha \). Then \( u^\mu(t) \in V^{1+\alpha} \), for \( t > 0 \).

**Proof.** Let us first assume that \( t \in (0, \mu] \). Considering the scalar product with \( A^{1+\alpha}u^\mu(t) \) in \( V^\alpha \), it can be derived that

\[
\frac{d}{dt}(t\|u^\mu(t)\|_{1+\alpha}^2) = \|u^\mu(t)\|_{1+\alpha}^2 + 2t(t\frac{du^\mu}{dt}(t), A^{1+\alpha}u^\mu(t))
\]

\[
\leq \|u^\mu(t)\|_{1+\alpha}^2 - 2t(Au^\mu(t), A^{1+\alpha}u^\mu(t))
\]

\[
- 2t\langle B(\phi(t - \mu), u^\mu(t)), A^{1+\alpha}u^\mu(t) \rangle + 2t\langle f, A^{1+\alpha}u^\mu(t) \rangle
\]

\[
\leq \|u^\mu(t)\|_{1+\alpha}^2 - 2t\|u^\mu(t)\|_{2+\alpha}^2
\]

\[
+ 2t\|\phi(\sigma - \mu)\|_{s_1+1}\|u^\mu(t)\|_{s_2+1}\|u^\mu(t)\|_{2+2s_3} + 2t\|f\|_\alpha \|u^\mu(t)\|_{2+\alpha}
\]

\[
\leq \|u^\mu(t)\|_{1+\alpha}^2 + t\|\phi(t - \mu)\|_{1+\alpha}\|u^\mu(t)\|_{2+\alpha} + t\|f\|_\alpha^2,
\]

where we have applied Lemma 2.1 with \( s_1 = 1 + \alpha, s_2 = \alpha \) and \( s_3 = -\alpha \) (we remind here that in all the paper \( \alpha > 1/2 \)). As a consequence,

\[
t\|u^\mu(t)\|_{1+\alpha}^2 \leq \|u^\mu\|_{L_2(0, \mu, V^{1+\alpha})}^2 + \int_0^t s\|f\|_\alpha^2 ds + \int_0^t \|\phi(s - \mu)\|_{1+\alpha}\|u^\mu(s)\|_{2+\alpha}^2 ds,
\]

and in virtue of Gronwall’s lemma,

\[
t\|u^\mu(t)\|_{1+\alpha}^2 \leq (\|u^\mu\|_{L_2(0, \mu, V^{1+\alpha})}^2 + t\|f\|_\alpha^2)e^{t\|\phi\|_{L_2(-\mu, 0, V^{1+\alpha})}^2}.
\]

If now \( t \in [\mu, 2\mu] \), we can repeat similar steps than before to arrive at

\[
\frac{d}{dt}(t\|u^\mu(t)\|_{1+\alpha}^2) \leq \|u^\mu(t)\|_{1+\alpha}^2 - 2t(Au^\mu(t), A^{1+\alpha}u^\mu(t))
\]

\[
- 2t\langle B(\mu u^\mu(t - \mu), u^\mu(t)), A^{1+\alpha}u^\mu(t) \rangle + 2t\langle f, A^{1+\alpha}u^\mu(t) \rangle
\]

\[
\leq \|u^\mu(t)\|_{1+\alpha}^2 + t\|u^\mu(t - \mu)\|_{1+\alpha}\|u^\mu(t)\|_{2+\alpha} + t\|f\|_\alpha^2,
\]

and integrating

\[
t\|u^\mu(t)\|_{1+\alpha}^2 \leq \mu\|u^\mu(\mu)\|_{1+\alpha}^2 + \|u^\mu\|_{L_2(\mu, 2\mu, V^{1+\alpha})}^2 + (\mu - t)\|f\|_\alpha^2
\]

\[
+ \int_\mu^t \|u^\mu(s - \mu)\|_{1+\alpha}\|u^\mu(s)\|_{1+\alpha} ds,
\]

hence

\[
t\|u^\mu(t)\|_{1+\alpha}^2 \leq (\mu\|u^\mu(\mu)\|_{1+\alpha}^2 + \|f\|_\alpha^2) + \|u^\mu\|_{L_2(\mu, 2\mu, V^{1+\alpha})}^2 e^{\int_\mu^t \|u^\mu(s - \mu)\|_{1+\alpha} ds ds}
\]

\[
= (\mu\|u^\mu(\mu)\|_{1+\alpha}^2 + \|f\|_\alpha^2) + \|u^\mu\|_{L_2(\mu, 2\mu, V^{1+\alpha})}^2 e^{\int_0^t \|u^\mu(t)\|_{1+\alpha} ds ds}.
\]

It is clear that due to the regularity of the weak solution \( u^\mu \) we can repeat this procedure in any interval. This completes the proof.

We can also establish the following regularity result:

**Lemma 3.2.** Assume that \((\phi, u_0) \in Y_\alpha^\mu \) and \( f \in V^\alpha \). Then for every \( \epsilon > 0 \), the solution of (4) satisfies \( u^\mu \in L_\infty(\epsilon, T, V^{1+\alpha}) \cap L_2(\epsilon, T, V^{2+\alpha}) \).
Proof. The proof is based on the regularity properties of the weak solution together with the fact that, as a consequence of Lemma 3.1, we know that for any $\epsilon > 0$
\[
\sup_{t \in [\epsilon, T]} \|u^\mu (t)\|_{1+\alpha}^2 < \infty.
\]
Indeed, as in the previous proof, assume first that $t \in [0, \mu]$. Then
\[
\frac{d}{dt}\|u^\mu (t)\|_{1+\alpha}^2 + 2\nu\|u^\mu (t)\|_{2+\alpha}^2
\]
\[
\leq 2c\|u^\mu (t-\mu)\|_{1+\alpha} \|u^\mu (t)\|_{1+\alpha} A^{1+\alpha} \|u^\mu (t)\|_{-\alpha} + \frac{2}{\nu} \|f\|_{\alpha}^2 + \frac{\nu}{2} \|u^\mu (t)\|_{2+\alpha}^2
\]
\[
\leq \frac{c^2}{\nu} \|u^\mu (t-\mu)\|_{1+\alpha}^2 \|u^\mu (t)\|_{1+\alpha}^2 + \nu \|u^\mu (t)\|_{2+\alpha}^2 + \frac{2}{\nu} \|f\|_{\alpha}^2 + \frac{\nu}{2} \|u^\mu (t)\|_{2+\alpha}^2.
\]
Above, to estimate the trilinear form, we have taken in Lemma 2.1 the parameters $s_1 = 1 + \alpha$, $s_2 = \alpha$ and $s_3 = -\alpha$.

Hence, by integration,
\[
\|u^\mu (t)\|_{1+\alpha}^2 + \frac{\nu}{2} \int_{\epsilon}^t \|u^\mu (s)\|_{2+\alpha}^2 ds
\]
\[
\leq \|u^\mu (\epsilon)\|_{1+\alpha}^2 + \frac{c^2}{\nu} \sup_{t \in [\epsilon, \mu]} \|u^\mu (t)\|_{1+\alpha}^2 \|\phi\|_{L^2(\mu, 0, V^{1+\alpha})}^2 + \frac{2}{\nu} \|f\|_{\alpha}^2
\]
which implies $u^\mu \in L^2(\epsilon, \mu, V^{2+\alpha})$. Reasoning in a similar way, when $t \in [\mu, 2\mu]$, we have
\[
\|u^\mu (t)\|_{1+\alpha}^2 + \frac{\nu}{2} \int_{\mu}^t \|u^\mu (s)\|_{2+\alpha}^2 ds
\]
\[
\leq \|u^\mu (\mu)\|_{1+\alpha}^2 + \frac{c^2}{\nu} \sup_{t \in [\mu, 2\mu]} \|u^\mu (t)\|_{1+\alpha}^2 \|u^\mu\|_{L^2(0, \mu, V^{1+\alpha})}^2 + \frac{2}{\nu} \|f\|_{\alpha}^2
\]
hence $u^\mu \in L^2(\mu, 2\mu, V^{2+\alpha})$. Repeating the same argument we conclude the proof.

We can also establish the Hölder regularity of the solution.

Lemma 3.3. Assume that $(\phi, u_0) \in \mathcal{V}^\mu_\alpha$ and $f \in V^\alpha$. Then for every $\epsilon > 0$, the solution of (4) satisfies $u^\mu \in C^{\beta}([\epsilon, T], V^\alpha)$ for $\beta \in (0, 1/2]$.

Proof. Consider $\epsilon \leq s < t \leq T$. Then
\[
\|u^\mu (t) - u^\mu (s)\|_{\alpha} \leq \nu (t - s)^{1/2} \left( \int_{\epsilon}^T \|A^\mu u(r)\|_{\alpha}^2 dr \right)^{1/2}
\]
\[
+ (t - s)^{1/2} \left( \int_{\epsilon}^T \|B(u^\mu (r - \mu), u(r))\|_{\alpha}^2 dr \right)^{1/2}.
\]
On the one hand,
\[
\int_{\epsilon}^T \|A^\mu u(r)\|_{\alpha}^2 dr = \int_{\epsilon}^T \|A^{1/2} A\mu u(r)\|_{\alpha}^2 dr = \int_{\epsilon}^T \|u(r)\|_{2+\alpha}^2 dr < \infty,
\]
thanks to Lemma 3.2. On the other hand,
\[
\int_{\epsilon}^T \|B(u^\mu (r - \mu), u(r))\|_{\alpha}^2 dr \leq \sup_{t \in [\epsilon, T]} \|u(t)\|_{1+\alpha}^2 \|u\|_{L^2(\epsilon, T, V^{1+\alpha})}^2 < \infty,
\]
just taking $s_3 = -\alpha$, $s_1 = 1 + \alpha$ and $s_2 = 1 + \alpha$ in Lemma 2.1. 

\qed
4. **Discrete and continuous dynamical flows.** As pointed out above in the Introduction, in this paper we are interested in investigating the longtime behavior of the delayed Navier-Stokes equations (4). As in the study of the existence and uniqueness of solutions for (4), the analysis of its longtime behavior is based on the study of its corresponding linearized system. Hence, we first consider the linearized uniqueness of solutions for (4), the analysis of its longtime behavior is based on the study of its corresponding linearized system. Therefore, we establish a crucial relationship between $U$ and the continuous flow $S$ related to (4), see (12) below.

We consider the solution of (6) on any compact interval. We can rewrite (6) as

$$
\begin{aligned}
\begin{cases}
  du_1(t) + (\nu A u_1(t) + B(\psi(t), u_1(t)))dt &= f dt, \\
  u_1(0) &= u_0,
\end{cases}
\end{aligned}
$$

and consider generalizations of the above problem given for $k = 2, 3, \ldots$ by

$$
\begin{aligned}
\begin{cases}
  du_k(t) + (\nu A u_k(t) + B(u_{k-1}(t), u_k(t)))dt &= f dt, \\
  u_k(0) &= u_{k-1}(\mu),
\end{cases}
\end{aligned}
$$

It turns out that we can construct a sequence $\{ (u_k) \}_{k \in \mathbb{N}} \subset L_2(0, \mu, V^{1+\alpha})$ such that, for any $k \in \mathbb{N}$, $u_k(\mu) \in V^\alpha$. Concatenating the elements of this sequence we can define the function $u$ given by

$$
u = \begin{cases}
  u_0 & \text{if } t = 0, \\
  u_1(t) & \text{if } t \in [0, \mu], \\
  u_2(t - \mu) & \text{if } t \in [\mu, 2\mu], \\
  \vdots & \text{if } t \in [(k-1)\mu, T], \\
  u_k(t - (k-1)\mu) & \text{if } t \in [(k-1)\mu, T],
\end{cases}
$$

assuming that $t \in [0, T]$, with $T \in ((k-1)\mu, k\mu]$. Therefore, we have constructed $u$ to be the solution of the linearized Navier-Stokes equations (8)-(9) for $t \geq 0$. Due to the above construction, Lemma 3.1, Lemma 3.2 and Lemma 3.3 can be also established for the solution $u$ given by (10).

Furthermore, we consider the (continuous) dynamical system defined by $u^\mu$ and the (discrete) dynamical system defined by $u$, and analyze the relationship between them. To be more precise, from now on we consider the two Hilbert spaces

$$
\mathcal{X}_\alpha^\mu = L_2(0, \mu, V^{1+\alpha}) \times V^\alpha, \quad \mathcal{Y}_\alpha^\mu = L_2(-\mu, 0, V^{1+\alpha}) \times V^\alpha.
$$

If $(x_1, x_2) \in \mathcal{X}_\alpha^\mu$ and $(y_1, y_2) \in \mathcal{Y}_\alpha^\mu$, the symbol

$$(x_1, x_2) \cong (y_1, y_2)$$

means that

$$
x_1(\cdot) = y_1(\cdot - \mu) \text{ on } [0, \mu],
$$

$$
x_2 = y_2.
$$

Therefore, $(\psi, u_0) \cong (\phi, u_0)$ means that $\psi(\cdot) := \phi(\cdot - \mu)$.

Notice that if $(\psi, u_0) \cong (\phi, u_0)$ then $u^\mu_1(t) = u_1(t)$, $t \in [0, \mu]$, which also implies

$$
(u^\mu_1)_\mu(t - \mu) = u^\mu_1(t) = u_1(t), \quad t \in [0, \mu],
$$

that is,

$$(\psi, u_0) \cong (\phi, u_0) \Rightarrow (u_1, u_1(\mu)) \cong ((u^\mu_1)_\mu, u^\mu_1(\mu)).
$$

By induction, for any $k \in \mathbb{N}$ we obtain

$$(\psi, u_0) \cong (\phi, u_0) \Rightarrow (u_k, u_k(\mu)) \cong ((u^\mu_k)_\mu, u^\mu_k(\mu)).$$

11
Define for $n \in \mathbb{N}$ and $t \in [0,T]$ the mappings $U(n,\cdot) : \mathcal{X}^\mu_n \to \mathcal{X}^\mu_n$ and $S(t,\cdot) : \mathcal{Y}^\mu_n \to \mathcal{Y}^\mu_n$, given respectively by

$$U(n,(\psi,u_0)) = (u|_{(n-1)\mu,n\mu}], u(n\mu)), \quad S(t,(\phi,u_0)) = ((u^\mu)_t, u^\mu(t)),$$

where $u$ is defined by (10) for $(\psi,u_0) \in \mathcal{X}^\mu_n$, and $u^\mu_t$ is the segment function defined by $(u^\mu_t)(s) = u^\mu_t(t+s)$, $s \in (-\mu,0)$, where $u^\mu$ is the weak solution to (4) corresponding to the initial data $(\phi,u_0) \in \mathcal{Y}^\mu_n$. Note that $U$ can be defined simply considering compositions of $U(1,\cdot)$ with itself. In fact, we can consider the one-step function $U(1,(\psi,u_0)) = (u_1, u_1(\mu))$ and compose it with itself $n$ times, getting

$$U(n,(\psi,u_0)) = U(1,\cdot) \circ U(1,\cdot) \circ \cdots \circ U(1,(\psi,u_0)).$$

It was proven, see [2], that the discrete dynamical system $U(n,\cdot)$ is a continuous mapping on $\mathcal{X}^\mu_n$ while $S(t,\cdot)$ is continuous on $\mathcal{Y}^\mu_n$.

Now we can rewrite (11) to establish the relationship between the discrete and continuous dynamical systems $U$ and $S$. If $(\psi,u_0) \equiv (\phi,u_0),

$$U(n,(\psi,u_0)) = (u_n,u_n(\mu)) \equiv ((u^\mu_n)_\mu, u^\mu_n(\mu)) = ((u^\mu)_n\mu, u^\mu(n\mu)) = S(n\mu,(\phi,u_0)),$$

or, in other words,

$$(\psi,u_0) \equiv (\phi,u_0) \Rightarrow U(n,(\psi,u_0)) \equiv S(n\mu,(\phi,u_0)). \quad (12)$$

**Remark 4.1.** Our aim in the next section is to study the longtime behavior for (4). In particular, we will prove that there is a positively invariant ball $B^{n\mu}_\xi$ for the continuous semigroup $S(t)$, for $t \geq 0$. Notice that, given $t \geq 0$ there exists $n^* \in \mathbb{N}$ such that $t \in [n^*\mu, (n^*+1)\mu]$, hence defining $\tau = t - n^*\mu \in [0,\mu]$, by the semigroup property

$$S(t,(\phi,u_0)) = S(t - n^*\mu, S(n^*\mu,(\phi,u_0))) = S(\tau, ((u^\mu_{n^*})_\mu, u^\mu_{n^*}(\mu))). \quad (13)$$

Therefore, combining (12) and (13), it is clear that it is enough to restrict the investigation of the positively invariant ball for the continuous dynamical system $S$ to the interval $[0,\mu]$. More precisely, the invariance of the ball $B^{n\mu}_\xi$ will follow in two steps: first, we will find a positively invariant ball $B^{n\mu}_\xi$ for the discrete dynamical flow $U$, and then we will study the invariance of a ball for $S$ on $[0,\mu]$. 

5. **Longtime behavior for the linearized equation.** To investigate the existence of an attractor for the delayed Navier-Stokes equations (4) we are going to use the relationship (12). To be more precise, we will first consider the discrete dynamical system $U$ and look for the existence of a discrete attractor associated to $U$. The existence of this discrete attractor rests upon the invariance of a ball $B \in \mathcal{X}^\mu_n$ for $U$ (see Lemma 5.1 below) and suitable compact embeddings of some spaces (see Lemma 5.3).

To simplify the presentation, we identify $U(\psi,u_0)$ with $U(1,(\psi,u_0))$.

Note that for an arbitrary positive real number $R$, we can choose $\nu := \nu(R)$ big enough such that

$$-\frac{\nu \lambda \mu}{2} + \frac{\epsilon^2 R^2}{\nu} < -\ln(2), \quad (14)$$

and

$$\frac{8}{\nu^2 \lambda} \|f\|_\alpha^2 e^{\frac{\epsilon^2 R^2}{\nu}} \left(\frac{2}{\nu} + \frac{2 \epsilon^2 R^2}{\nu^2} (e^{\frac{\epsilon^2 \mu^2}{2}} + 1 + \frac{\lambda \mu e^{-\frac{\epsilon^2 \mu^2}{2}}}{2}) \right) \leq \frac{R^2}{2}, \quad (15)$$
where $c$ is the positive constant determined in Lemma 2.1 and $\lambda$ denotes the first eigenvalue of $A$.

**Lemma 5.1.** Consider $(\psi, u_0) \in \mathcal{X}^\mu_\alpha$ and $f \in V^\alpha$. For $R > 0$ we take a large enough viscosity $\nu > 0$ such that (14) and (15) hold true. Then for $U(\psi, u_0) = (v_t, u_1(\mu))$ defined in Section 4 we have

$$U(B_{\mathcal{X}^\mu_\alpha}(R; \rho)) \subset B_{\mathcal{X}^\mu_\alpha}(R; \rho),$$

where $B_{\mathcal{X}^\mu_\alpha}(R; \rho) := B_{L_2(0, \mu, V^{1+\alpha})}(0, R) \times B_{V^\alpha}(0, \rho)$, being

$$\rho^2 = \frac{8}{\nu^2 \lambda} \|f\|_{\alpha-1}^2 e^{c^2 \frac{\nu^2}{\lambda^2}}.$$ (16)

**Proof.** Since we believe that confusion is not possible, we drop the subindex and represent the solution by $u$ instead of by $u_1$.

We denote by $\text{pr}_i(\cdot)$, $i = 1, 2$, the projection into the corresponding component. We start proving that

$$\text{pr}_2 U(B_{\mathcal{X}^\mu_\alpha}(R; \rho)) \subset B_{V^\alpha}(0, \rho),$$ (17)

for which we need to prove that if $\|\psi\|_{L_2(0, \mu, V^{1+\alpha})}^2 \leq R^2$ and $\|u_0\|_{\alpha}^2 \leq \rho^2$, then $\|\mu(\mu)\|_{\alpha}^2 \leq \rho^2$. For $t \in [0, \mu]$, considering the scalar product with $A^\alpha u(t)$ in $V^0$, we have

$$\frac{d}{dt} \|u(t)\|_{\alpha}^2 + 2\nu \|u(t)\|_{1+\alpha}^2 \leq 2c\|\psi(t)\|_{1+\alpha} \|u(t)\|_{\alpha} \|u(t)\|_{1+\alpha} + \frac{2}{\nu} \|f\|_{\alpha-1}^2 + \frac{2}{\nu} \|u(t)\|_{1+\alpha}^2,$$

where we have applied Lemma 2.1 taking $s_3 = -\alpha$, $s_1 = 1 + \alpha$ and $s_2 = \alpha$. Hence, applying Gronwall’s lemma,

$$\|u(t)\|_{\alpha}^2 \leq \|u_0\|_{\alpha}^2 e^{-\frac{\nu}{2} t + \frac{c}{\nu}} e^{\frac{c}{\nu}} \int_0^t \|\psi(r)\|_{1+\alpha}^2 dr + \frac{2}{\nu} \|f\|_{\alpha-1}^2 \int_0^t e^{-\frac{\nu}{2} (t-s) + \frac{c}{\nu}} \|\psi(r)\|_{1+\alpha}^2 dr ds,$$

and in particular we obtain

$$\|u(\mu)\|_{\alpha}^2 \leq \|u_0\|_{\alpha}^2 e^{-\frac{\nu}{2} \mu + \frac{c}{\nu}} + \frac{4}{\nu^2 \lambda} \|f\|_{\alpha-1}^2 e^{\frac{c^2 \nu}{\lambda}} (1 - e^{-\frac{\nu}{2} \mu}).$$

Since $-\frac{\nu}{2} + \frac{c^2 \nu}{\lambda} < -\ln(2) < 0$, we have that

$$\|u(\mu)\|_{\alpha}^2 \leq \frac{1}{2} \|u_0\|_{\alpha}^2 + \frac{4}{\nu^2 \lambda} \|f\|_{\alpha-1}^2 e^{\frac{c^2 \nu}{\lambda}};$$

thus, if $\|u_0\|_{\alpha}^2 \leq \rho^2$ with $\rho$ defined by (16), we get that $\|u(\mu)\|_{\alpha}^2 \leq \rho^2$, thus (17) is proved.

Let us prove now that

$$\text{pr}_1 U(B_{\mathcal{X}^\mu_\alpha}(R; \rho)) \subset B_{L_2(0, \mu, V^{1+\alpha})}(0, R).$$ (19)

Notice that from the previous estimates we also obtain

$$\frac{\nu}{2} \int_0^\mu \|u(r)\|_{1+\alpha}^2 dr \leq \|u_0\|_{\alpha}^2 + \frac{2}{\nu} \int_0^\mu \|\psi(r)\|_{1+\alpha}^2 \|u(r)\|_{\alpha}^2 dr + \frac{2\mu}{\nu} \|f\|_{\alpha-1}^2,$$
and from (18) we also know that

\[
\sup_{t \in [0,\mu]} \|u(t)\|^2_{\alpha} \leq \|u_0\|^2_{\alpha} e^{\frac{c_2 R^2}{\nu}} \sup_{t \in [0,\mu]} e^{-\frac{\nu \lambda t}{2}} + \frac{2}{\nu} \|f\|^2_{\alpha-1} e^{\frac{c_2 R^2}{\nu}} \sup_{t \in [0,\mu]} \int_0^t e^{-\frac{\nu \lambda (t-s)}{2}} ds \\
\leq \|u_0\|^2_{\alpha} e^{\frac{c_2 R^2}{\nu}} + \frac{4}{\nu^2 \lambda} \|f\|^2_{\alpha-1} e^{\frac{c_2 R^2}{\nu}} \sup_{t \in [0,\mu]} (1 - e^{-\frac{\nu \lambda t}{2}}) \\
\leq \|u_0\|^2_{\alpha} e^{\frac{c_2 R^2}{\nu}} + \frac{4}{\nu^2 \lambda} \|f\|^2_{\alpha-1} e^{\frac{c_2 R^2}{\nu}} (1 - e^{-\frac{\nu \lambda t}{2}}) \\
\leq \rho^2 (e^{\frac{c_2 R^2}{\nu}} + \frac{1}{2}).
\]

Therefore, by (16),

\[
\int_0^\mu \|u(r)\|^2_{1+\alpha} dr \leq \frac{2}{\nu} \rho^2 + \frac{2c_2 R^2}{\nu^2} \rho^2 (e^{\frac{c_2 R^2}{\nu}} + \frac{1}{2}) + \frac{4\mu}{\nu^2} \|f\|^2_{\alpha-1} \\
\leq \frac{2}{\nu} \rho^2 + \frac{2c_2 R^2}{\nu^2} \rho^2 (e^{\frac{c_2 R^2}{\nu}} + \frac{1}{2}) + \frac{\lambda \mu e^{\frac{c_2 R^2}{\nu}}}{2} \rho^2 \\
\leq \rho^2 \left( \frac{2}{\nu} + \frac{2c_2 R^2}{\nu^2} (e^{\frac{c_2 R^2}{\nu}} + \frac{1}{2}) + \frac{\lambda \mu e^{\frac{c_2 R^2}{\nu}}}{2} \right). 
\]

(20)

We conclude by (15) that

\[
\|u\|^2_{L^2(0,\mu; V^{1+\alpha})} dr \leq \frac{R^2}{2} \leq R^2,
\]

hence (19) is proved and the proof is finished.

\[\Box\]

**Remark 5.2.** In the expression (20), we take the left hand side to be smaller than \(\frac{R^2}{2}\) but not directly smaller than \(R^2\). The reason is that this choice will help us later to show the invariance of a ball for the continuous dynamical system \(S\), see Section 6. Anyway, \(B_{x_{0}^{\alpha}}(R; \rho)\) is positively invariant for \(U\) because starting in \((\psi, u_0) \in B_{x_{0}^{\alpha}}(R; \rho)\) we know that

\[
U(\psi, u_0) \in B_{x_{0}^{\alpha}} \left( \frac{R}{\sqrt{\nu}}, \rho \right) \subset B_{x_{0}^{\alpha}}(R; \rho),
\]

where \(B_{x_{0}^{\alpha}} \left( \frac{R}{\sqrt{\nu}}, \rho \right) := B_{L^2(0,\mu; V^{1+\alpha})} \left( 0, \frac{R}{\sqrt{\nu}} \right) \times B_{V^{\alpha}}(0, \rho)\).

Now we want to establish the existence of a unique discrete attractor associated to \(U\). To do that, we are using the following lemma, whose proof can be found in Vishik and Fursikov [13] Chapter IV Theorem 4.1.

**Lemma 5.3.** The space \(L^2(s, t; V^{2+\alpha}) \cap C^0([s, t], V^{\alpha})\) is compactly embedded into \(L^2(s, t; V^{1+\alpha}) \cap C([s, t], V^{\alpha})\).

As a consequence of the previous results, we can establish one of the main theorems of this article:

**Theorem 5.4.** Let \(f \in V^{\alpha}\). Under the assumptions (14) and (15), the discrete dynamical system \(U\) associated to the linearized 3D Navier-Stokes equations possesses a unique global attractor \(A\).
Proof. According to Lemma 5.1 and Remark 5.2, since the viscosity is large enough we can find \( R \) such that (14) and (15) hold true, which imply that \( U(n, B_{X^\alpha}^{n}(R; \rho)) \subset B_{X^\alpha}^{n}(R; \rho) \), that is, \( B_{X^\alpha}^{n}(R; \rho) \) is a forward invariant ball, with \( \rho \) given by (16).

On the other hand, due to the extra regularity of the solution given by Lemma 3.1, Lemma 3.2 and Lemma 3.3 (for the solution of the linearized equation), in virtue of Lemma 5.3, we know that \( U(n, B_{X^\alpha}^{n}(R; \rho)) \) is relatively compact for \( n \geq 2 \). Now defining

\[ K := \overline{U(2, B_{X^\alpha}^{n}(R; \rho))}^{\mathbb{V}^\alpha} \]

we have that \( K \) is a forward invariant compact set, hence \( U \) possesses a unique global attractor \( \mathcal{A} \) (for a comprehensive presentation of the concept of attractors we refer to the monographs by Babin and Vishik [1], Hale [7] or Temam [11]). \( \square \)

### 5.1. Single point attractor.

Now we are interested in finding sufficient conditions that ensure that the global attractor \( \mathcal{A} \) associated to the discrete dynamical system \( U \) is a single point.

For \( R > 0 \), choose \( \nu > 0 \) such that (14) and (15) hold true. Moreover, we assume the following two inequalities

\[
\left( \frac{2\nu^2 R^2}{\nu^2} + 1 \right) e^{-\lambda \nu} + \frac{1}{\nu} < \frac{1}{2},
\]

(21)

\[
\left( \frac{1}{\lambda} \left( \frac{2\nu^2 R^2}{\nu^2} + 1 \right) (1-e^{-\lambda \nu}) e^{\frac{2\nu^2 R^2}{\nu^2}} + 1 \right) \frac{2\nu^2 R^2}{\nu^2} < \frac{1}{2}.
\]

(22)

**Theorem 5.5.** Let \( f \in \mathbb{V}^\alpha \). Assume (14), (15), (21) and (22). Then the global attractor \( \mathcal{A} \) of Theorem 5.4 consists of a single point.

**Proof.** Assume that \( u_1 \) and \( u_2 \) are two weak solutions to (6) with initial conditions given, respectively, by \( (\psi_1, u_{0,1}) \), \( (\psi_2, u_{0,2}) \) \( \in B_{X^\alpha}^{n}(R; \rho) \), the forward invariant ball of Lemma 5.1. Then, the difference \( u_1 - u_2 \) verifies

\[
\frac{d}{dt} (u_1(t) - u_2(t)) + \nu A(u_1(t) - u_2(t)) = B(\psi_1(t), u_1(t)) - B(\psi_2(t), u_2(t))
\]

\[= B(\psi_1(t) - \psi_2(t), u_1(t)) + B(\psi_2(t), u_1(t) - u_2(t)).\]

Multiplying it by \( A^\alpha(u_1(t) - u_2(t)) \) in \( \mathbb{V}^\alpha \) we have

\[
\frac{d}{dt} \| u_1(t) - u_2(t) \|^2_{\alpha} + 2\nu \| u_1(t) - u_2(t) \|^2_{1+\alpha}
\]

\[\leq 2\nu \| \psi_1(t) - \psi_2(t) \|^2_{1+\alpha} \| u_1(t) - u_2(t) \|^2_{1+\alpha}
\]

\[+ 2\nu \| \psi_2(t) \|^2_{1+\alpha} \| u_1(t) - u_2(t) \|^2_{1+\alpha}
\]

\[\leq \frac{2\nu^2}{\nu} \| \psi_1(t) - \psi_2(t) \|^2_{1+\alpha} \| u_1(t) - u_2(t) \|^2_{1+\alpha}
\]

\[+ \frac{2\nu^2}{\nu} \| \psi_2(t) \|^2_{1+\alpha} \| u_1(t) - u_2(t) \|^2_{1+\alpha} + \frac{\nu}{2} \| u_1(t) - u_2(t) \|^2_{1+\alpha}.
\]

(considering in Lemma 2.1 \( s_1 = 1 + \alpha \), \( s_2 = \alpha - 1 = -s_3 \)), which gives

\[
\frac{d}{dt} \| u_1(t) - u_2(t) \|^2_{\alpha} + \nu \| u_1(t) - u_2(t) \|^2_{1+\alpha}
\]

\[\leq \frac{2\nu^2}{\nu} \| \psi_1(t) - \psi_2(t) \|^2_{1+\alpha} \| u_1(t) \|^2_{\alpha} + \frac{2\nu^2}{\nu} \| \psi_2(t) \|^2_{1+\alpha} \| u_1(t) - u_2(t) \|^2_{\alpha}.
\]
Hence, adding the two estimates yields
\[ \psi \] and since \( \psi \in B_{L^2(0,\mu,V^{1+\alpha})}(0,R) \), applying Gronwall’s lemma,
\[
\|u_1(t) - u_2(t)\|_{1+\alpha}^2 \leq e^{-\lambda \nu + \frac{2c^2\rho^2}{\nu}} \|u_{0,1} - u_{0,2}\|_{\alpha}^2 \\
+ \frac{2c^2}{\nu} \int_0^\mu e^{-\lambda \nu(s) + \frac{2c^2\rho^2}{\nu}} \|\psi_1(s) - \psi_2(s)\|_{1+\alpha}^2 \|u_1(s)\|_{\alpha}^2 ds,
\]
and because we know that \( U(1,B) \subset B \), where \( B = B_{X_0}(R;\rho) \), then
\[
\sup_{s \in [0,\mu]} \|u_1(s)\|_{\alpha}^2 \leq \rho^2
\]
and this implies
\[
\|u_1(t) - u_2(t)\|_{1+\alpha}^2 \leq e^{-\lambda \nu + \frac{2c^2\rho^2}{\nu}} \|u_{0,1} - u_{0,2}\|_{\alpha}^2 \\
+ \frac{2c^2\rho^2}{\lambda \nu^2} (1 - e^{-\lambda \nu}) e^{\frac{2c^2\rho^2}{\nu}} \|\psi_1 - \psi_2\|_{L^2(0,\mu,V^{1+\alpha})}^2.
\]
Furthermore,
\[
\nu \|u_1(t) - u_2(t)\|_{1+\alpha}^2 \leq \frac{2c^2}{\nu} \|\psi_1(t) - \psi_2(t)\|_{1+\alpha}^2 \|u_1(t)\|_{\alpha}^2 \\
+ \frac{2c^2}{\nu} \|\psi_2(t)\|_{1+\alpha}^2 \|u_1(t) - u_2(t)\|_{\alpha}^2,
\]
and thus
\[
\nu \int_0^\mu \|u_1(r) - u_2(r)\|_{1+\alpha}^2 dr \\
\leq \|u_{0,1} - u_{0,2}\|_{\alpha}^2 + \frac{2c^2}{\nu} \int_0^\mu \|\psi_1(r) - \psi_2(r)\|_{1+\alpha}^2 \|u_1(r)\|_{\alpha}^2 dr \\
+ \frac{2c^2}{\nu} \int_0^\mu \|\psi_2(r)\|_{1+\alpha}^2 \|u_1(r) - u_2(r)\|_{\alpha}^2 dr \\
\leq \|u_{0,1} - u_{0,2}\|_{\alpha}^2 + \frac{2c^2\rho^2}{\nu} \|\psi_1 - \psi_2\|_{L^2(0,\mu,V^{1+\alpha})}^2 \\
+ \frac{2c^2}{\nu} \sup_{s \in [0,\mu]} \|u_1(s) - u_2(s)\|_{\alpha}^2 \int_0^\mu \|\psi_2(r)\|_{1+\alpha}^2 dr.
\]
Now, if we divide by \( \nu \) we obtain that
\[
\int_0^\mu \|u_1(r) - u_2(r)\|_{1+\alpha}^2 dr \leq \frac{1}{\nu} \|u_{0,1} - u_{0,2}\|_{\alpha}^2 + \frac{2c^2\rho^2}{\nu^2} \|\psi_1 - \psi_2\|_{L^2(0,\mu,V^{1+\alpha})}^2 \\
+ \frac{2c^2\rho^2}{\nu^2} \sup_{s \in [0,\mu]} \|u_1(s) - u_2(s)\|_{\alpha}^2.
\]
Hence, adding the two estimates yields
\[
\|U(1,(\psi_1,u_{0,1})) - U(1,(\psi_2,u_{0,2}))\|_{X_0}^2 \\
\leq \left( \frac{2c^2\rho^2}{\nu^2} + 1 \right) e^{-\lambda \nu + \frac{2c^2\rho^2}{\nu}} \|u_{0,1} - u_{0,2}\|_{\alpha}^2 \\
+ \left( \frac{1}{\lambda} \left( \frac{2c^2\rho^2}{\nu^2} + 1 \right) (1 - e^{-\lambda \nu}) e^{\frac{2c^2\rho^2}{\nu}} + 1 \right) \frac{2c^2\rho^2}{\nu^2} \|\psi_1 - \psi_2\|_{L^2(0,\mu,V^{1+\alpha})}^2.
\]
Now, due to the choice of \( \nu \) satisfying (21) and (22), the above inequality in turn implies

\[
\|U(1, (\psi_1, u_{0,1})) - U(1, (\psi_2, u_{0,2}))\|_{\mathcal{X}_n^w} < \frac{1}{2} (\|u_{0,1} - u_{0,2}\|_n^2 + \|\psi_1 - \psi_2\|_{L^2(0, \mu, V^{1+\alpha})}^2)
\]

provided that \( \nu \) is sufficiently large. By repeating the same arguments, for any \( n \in \mathbb{N} \) we obtain

\[
\|U(n, (\psi_1, u_{0,1}))-U(n, (\psi_2, u_{0,2}))\|_{\mathcal{X}_n^w} < \frac{1}{2^n} (\|u_{0,1} - u_{0,2}\|_n^2 + \|\psi_1 - \psi_2\|_{L^2(0, \mu, V^{1+\alpha})}^2).
\]

In other words, due to the invariance property of the attractor \( \mathcal{A} \) we have obtained

\[
\sup_{y_1, y_2 \in \mathcal{A}} \|y_1 - y_2\|_{\mathcal{X}_n^w} < \frac{1}{2^n} \sup_{x_1, x_2 \in \mathcal{A}} \|x_1 - x_2\|_{\mathcal{X}_n^w},
\]

and, as the right–hand side tends to zero, this implies that \( \mathcal{A} \) is a single point attractor for \( U \).

\[\square\]

6. **Longtime behavior for the delayed Navier-Stokes equations.** Now we are in position to establish our main result: the existence and uniqueness of an attractor for the continuous dynamical system \( S \). As mentioned before, the results will be based on the relationship (12) and the fact that \( U \) has a unique attractor \( \mathcal{A} \).

Let us recall that

\[
B_{\mathcal{X}_n^w}(R; \rho) := B_{L^2(0, \mu, V^{1+\alpha})}(0, R) \times B_{V^w}(0, \rho),
\]

\[
B_{\mathcal{X}_n^w}\left(\frac{R}{\sqrt{2}}; \rho\right) := B_{L^2(0, \mu, V^{1+\alpha})}\left(0, \frac{R}{\sqrt{2}}\right) \times B_{V^w}(0, \rho).
\]

Define now

\[
B_{Y_n^w}(R; \rho) := B_{L^2(-\mu, 0, V^{1+\alpha})}(0, R) \times B_{V^w}(0, \rho),
\]

\[
B_{Y_n^w}\left(\frac{R}{\sqrt{2}}; \rho\right) := B_{L^2(-\mu, 0, V^{1+\alpha})}\left(0, \frac{R}{\sqrt{2}}\right) \times B_{V^w}(0, \rho).
\]

We would like to show that \( B_{Y_n^w}(R; \rho) \) is a forward invariant ball for \( S \). First of all, we prove the following result for the time interval \([0, \mu]\).

**Lemma 6.1.** Assume (14) and (15) and consider \((\phi, u_0) \in B_{Y_n^w}\left(\frac{R}{\sqrt{2}}; \rho\right)\), for \( \rho \) given by (16). Then for \( t \in [0, \mu] \) yields

\[
S(t, (\phi, u_0)) = ((u^t_1), u^t_1(t)) \in B_{Y_n^w}(R; \rho).
\]

**Proof.** We only sketch the proof since it is very similar to the one of Lemma 5.1. Indeed, following the same steps than the ones of Lemma 5.1 we can obtain that

\[
\|u^t_1\|_{L^2(0, \mu, V^{1+\alpha})} \leq \frac{R^2}{2}.
\]

Hence if \((\phi, u_0) \in B_{Y_n^w}\left(\frac{R}{\sqrt{2}}; \rho\right)\) we obtain that

\[
\|u^t_1(t)\|_{L^2(-\mu, 0, V^{1+\alpha})} \leq \|\phi\|_{L^2(-\mu, 0, V^{1+\alpha})} + \|u^\mu_1\|_{L^2(0, \mu, V^{1+\alpha})} \leq \frac{R^2}{2} + \frac{R^2}{2} = R^2.
\]

For the second component, we should prove that

\[
\|u^t_1(t)\|_{\mathcal{X}_n^w}^2 \leq \rho^2.
\]
We know that
\[
\frac{d}{dt} \|u^n(t)\|_1^2 + 2\nu \|u^n(t)\|_1^{2+\alpha}
\]
\[
\leq 2c \|\phi(t-\mu)\|_1 \|u^n(t)\|_1 \|u^n(t)\|_1^{2+\alpha} + \frac{2}{\nu} \|f\|_1^{2+\alpha} + \frac{\nu}{2} \|u^n(t)\|_1^{2+\alpha}
\]
\[
\leq \frac{c^2}{\nu} \|\phi(t-\mu)\|_1 \|u^n(t)\|_1^{2+\alpha} + \nu \|u^n(t)\|_1^{2+\alpha} + \frac{2}{\nu} \|f\|_1^{2+\alpha} + \frac{\nu}{2} \|u^n(t)\|_1^{2+\alpha},
\]
hence, applying Gronwall’s lemma,
\[
\|u^n(t)\|_1^{2+\alpha} \leq \|u_0\|_1^{2+\alpha} e^{-\frac{\nu t}{\nu} + \frac{c^2}{\nu} \int_0^t \|\phi(r-\mu)\|_1^{2+\alpha} dr}
\]
\[
+ \frac{2}{\nu} \|f\|_1^{2+\alpha} \int_0^t e^{-\frac{\nu s}{\nu} - \frac{c^2 s}{\nu} + \frac{2}{\nu} \int_0^s |\phi(r-\mu)|_1^{2+\alpha} ds} ds
\]
\[
\leq \rho^2 e^{-\frac{\nu t}{\nu} + \frac{c^2 t}{\nu} + \frac{2}{\nu} \int_0^t |\phi(s-\mu)|_1^{2+\alpha} ds} + \frac{4}{\nu^2} \|f\|_1^{2+\alpha} e^{-\frac{\nu t}{\nu} - \frac{c^2 t}{\nu} + \frac{2}{\nu} \int_0^t |\phi(s-\mu)|_1^{2+\alpha} ds}.
\]
Using the definition of \(\rho\) given by (16) and assumption (14), we get that
\[
\|u^n(t)\|_1^{2+\alpha} \leq \rho^2.
\]

In the next result we prove that \(B_{\mathcal{Y}^\alpha}(R; \rho)\) is a positively invariant ball for \(S\).

**Lemma 6.2.** Assume (14) and (15). Then \(B_{\mathcal{Y}^\alpha}(R; \rho)\) defined by (23) with \(\rho\) defined by (16) is forward invariant for \(S\), that is, for every \(t \geq 0\) and \((\phi, u_0) \in B_{\mathcal{Y}^\alpha}(R; \rho)\), we have
\[
S(t, (\phi, u_0)) \in B_{\mathcal{Y}^\alpha}(R; \rho).
\]

**Proof.** First of all, notice that if \((\psi, u_0) \equiv (\phi, u_0)\) and \((\phi, u_0) \in B_{\mathcal{Y}^\alpha}(R; \rho)\) then trivially \((\psi, u_0) \in B_{\mathcal{X}^\alpha}(R; \rho)\), the positively invariant ball obtained in Lemma 5.1.

Given \(t \geq 0\) there exists \(n^* \in \mathbb{N}\) such that \(t \in [n^* \mu, (n^* + 1)\mu]\) and (13) holds true, that is
\[
S(t, (\phi, u_0)) = S(t - n^* \mu, S(n^* \mu, (\phi, u_0))) = S(\tau, ((u^{n*}_n)_\mu, u^{n*}_n, (\mu))_\mu),
\]
with \(\tau = t - n^* \mu \in [0, \mu]\).

On the other hand, (12) reads as
\[
U(n^*, (\psi, u_0)) = (u^{n*}_n, u^{n*}_n, (\mu)) \equiv ((u^{n*}_n)_\mu, u^{n*}_n, (\mu)) = S(n^* \mu, (\phi, u_0)).
\]

The invariance of \(B_{\mathcal{X}^\alpha}(R; \rho)\) under \(U\), \((\psi, u_0) \in B_{\mathcal{X}^\alpha}(R; \rho)\) implies \((u^{n*}_n, u^{n*}_n, (\mu)) \in B_{\mathcal{X}^\alpha}(R; \rho)\), see Remark 5.2. This statement together with (13) imply that, in order to see the invariance of \(B_{\mathcal{Y}^\alpha}(R; \rho)\) under \(S\) it is enough to prove that given \((\phi, u_0) \in B_{\mathcal{Y}^\alpha}(R; \rho)\) then \(S(t, (\phi, u_0)) = ((u^n_t)_\mu, u^n_t, (\mu)) \in B_{\mathcal{Y}^\alpha}(R; \rho)\), when \(t \in [0, \mu]\).

Now, in virtue of Lemma 6.1, the result is proven.

As an immediate consequence, we obtain the existence of the attractor \(\mathcal{A}^\mu\) for \(S\):

**Theorem 6.3.** Let \(f \in V^\alpha\). Assume (14), (15), (21) and (22). Then the continuous dynamical system \(S\) possesses a unique attractor \(\mathcal{A}^\mu\). Furthermore, under the same choice of the viscosity as in Theorem 5.5, the attractor \(\mathcal{A}^\mu\) of \(S\) consists of a single point.
Theorem 5.5 we know that $A_{\alpha}^\mu \subseteq \mathcal{A}^\mu$, that is, $A_{\alpha}^\mu$ is a single point attractor that in addition is linked to the single point attractor $\mathcal{A}$ by the relation $(\hat{\psi}, \hat{u}_0) \Leftrightarrow (\hat{\phi}, \hat{u}_0)$.

Consider precisely $(\hat{\psi}, \hat{u}_0) \in X_{\alpha}^\mu$ and its corresponding pair $(\hat{\phi}, \hat{u}_0) \in Y_{\alpha}^\mu$. Then

$$S(n\mu, (\hat{\phi}, \hat{u}_0)) \cong U(n, (\hat{\psi}, \hat{u}_0)) = (\hat{\psi}, \hat{u}_0) \cong (\hat{\phi}, \hat{u}_0),$$

therefore $(\hat{\phi}, \hat{u}_0) \in \mathcal{A}^\mu$. Conversely, take any pair $(\hat{\phi}, \hat{u}_0) \in \mathcal{A}^\mu$ and the corresponding $(\hat{\psi}, \hat{u}_0) \in X_{\alpha}^\mu$ such that $(\hat{\psi}, \hat{u}_0) \cong (\hat{\phi}, \hat{u}_0)$. Then

$$U(n, (\hat{\psi}, \hat{u}_0)) \cong S(n\mu, (\hat{\phi}, \hat{u}_0)) = (\hat{\psi}, \hat{u}_0) \cong (\hat{\psi}, \hat{u}_0)$$

thus $(\hat{\psi}, \hat{u}_0) \in \mathcal{A}$, that is to say, $(\hat{\psi}, \hat{u}_0) = (\hat{\psi}, \hat{u}_0)$, and then $\mathcal{A}^\mu = \{(\hat{\phi}, \hat{u}_0)\}$. □

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