Holographic Duals of Argyres-Douglas Theories

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We propose the first explicit holographic duals for a class of superconformal field theories of Argyres-Douglas type, which are inherently strongly coupled and provide a window onto remarkable non-perturbative phenomena (such as mutually non-local massless dyons and relevant operators of fractional dimension). The theories under examination are realized by a stack of M5-branes wrapped on a sphere with one irregular puncture and one regular puncture. In the dual 11d supergravity solutions, the irregular puncture is realized as an internal M5-brane source.

INTRODUCTION

Strong-coupling phenomena in quantum field theory (QFT) are of crucial importance, both conceptually and phenomenologically, but their study poses considerable theoretical challenges. In the endeavor of exploring the vast and largely uncharted landscape of strongly coupled phases in QFT, valuable lessons can be learned from theories with a higher degree of symmetry. Superconformal field theories (SCFTs) of Argyres-Douglas (AD) type in four dimensions constitute a prominent example. These theories are intrinsically strongly-coupled and describe interactions among mutually non-local massless dyons [1]. Their spectrum contains relevant Coulomb branch operators of fractional dimension. Establishing the existence and surprising properties of these QFTs has been complicated by their lack of an $\mathcal{N}=2$ superconformal IR fixed point. Theories under examination are realized by a stack of M5-branes wrapped over an interval. Σ is supported by a non-constant Killing spinor. Thus, as in [13–15], supersymmetry is not realized in the standard topological twist paradigm.

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SUPERGRAVITY SOLUTIONS

Our $AdS_5$ solutions in 11d supergravity preserve $4d$ $\mathcal{N}=2$ superconformal symmetry. They were obtained in 7d gauged supergravity and uplifted on $S^4$, as will be reported in [12]. The 7d solutions are a warped product of $AdS_5$ and a 2d space $\Sigma$, consisting of a circle fibered over an interval. $\Sigma$ is supported by a $U(1)$ gauge flux, does not have a constant curvature metric, and admits a non-constant Killing spinor. Thus, as in [13–15], supersymmetry is not realized in the standard topological twist paradigm.

The metric of the uplifted 11d solution is

\begin{equation}
\begin{aligned}
ds_{11}^2 &= m^{-2} e^{2\lambda} (ds_{AdS_5}^2 + ds_{M_5}^2) , \\
dw^2 &= 2 w h(w) (1 - w^2)^{3/2} + C^2 h(w) dz^2 , \\
\sqrt{1 - w^2} \left[ \frac{d\mu^2}{w (1 - \mu^2)} + \frac{(1 - \mu^2) D\phi^2}{\mathcal{H}(w, \mu)} + \frac{w^2 ds_{S^2}^2}{\mathcal{H}(w, \mu)} \right] ,
\end{aligned}
\end{equation}

where $m$ is a mass scale, $ds_{AdS_5}^2$ is the metric of the unit-radius $AdS_5$, and $ds_{S^2}$ is the metric of the unit-radius $S^2$. The functions $h(w)$, $\mathcal{H}(w, \mu)$ are defined as

\begin{equation}
h = B - 2 w \sqrt{1 - w^2} , \quad \mathcal{H} = \mu^2 + w^2 (1 - \mu^2) ,
\end{equation}

where $0 < B < 1$ is a constant parameter. The coordinates $\mu$, $w$ have ranges $0 \leq \mu \leq 1$ and $0 \leq w \leq w_1$, with $w_1^2 = \frac{1}{2} (1 - \sqrt{1 - B^2})$. The angular coordinates $\phi$, $z$ have period $2\pi$, and $C$ is a constant. The 1-form $D\phi$ and the warp factor are given by

\begin{equation}
D\phi = d\phi + C (2 w^2 - 1) dz , \quad e^{2\lambda} = \frac{2 B w^{1/3} \mathcal{H}^{1/3}}{\sqrt{1 - w^2}} .
\end{equation}
The directions of the 4d space spanned by $\ell$, and requiring it to be locally an orbifold $\mathbb{R}^4/\mathbb{Z}_\ell$. For $\ell = 1$ we have $\mathcal{B}_3 \cong \mathcal{C}_4$, but $\mathcal{B}_4$ is an independent 4-cycle for $\ell > 1$, with

$$Z \ni \int_{\mathcal{B}_4} \frac{G_4}{(2\pi \ell_p)^3} = \frac{N}{\ell}, \quad \text{hence } \ell \text{ divides } N. \quad (8)$$

Finally, we construct the 4-cycle $\mathcal{D}_4$ by combining $P_3P_4$ with $S^2$—which shrinks at $P_4$—and the combination of $S^2_\phi$ and $S^2_\chi$ that does not shrink in the interior of $P_3P_4$. Integrating $G_4$ on $\mathcal{D}_4$ defines

$$K = \int_{\mathcal{D}_4} \frac{G_4}{(2\pi \ell_p)^3} = \frac{N(1 - \sqrt{1 - B^2})}{\ell \sqrt{1 - B^2}}, \quad K \in \mathbb{N}. \quad (9)$$

In the vicinity of $P_1P_4$, the geometry is singular and $e^{2\phi}$ vanishes. We interpret this in terms of a smeared M5-brane source, as inferred from $G_4$ near $w = 0$,

$$G_4 = -\frac{\text{vol}_{S^2} \wedge d\left[\frac{k^3 D\phi}{\mathcal{H}}\right]}{m^3}, \quad (5)$$

where $\text{vol}_{S^2}$ is the volume form of the $S^2$.

The space $M_6$ is an $S^2_\phi \times S^1_\chi \times S^2$ fiberation over the rectangle $[0, w_1] \times [0, 1]$ in the $(w, \mu)$ plane, see Figure 1. The directions $w, S^2_\phi$ are identified with $\Sigma$ in the 7d solution, while $\mu, S^1_\chi, S^2$, span the $S^4$ used in the uplift.

### Regularity and Flux Quantization

As we approach a point in the interior of the $P_1P_2$ segment in the $(w, \mu)$ plane (see Figure 1), the $S^2$ shrinks smoothly. The Killing vector $\partial_\phi$ shrinks smoothly in the interior of $P_3P_4$, and $S^2_\phi$ shrinks smoothly along $P_3P_4$, where $\ell$ is given as

$$\ell = \frac{1}{C \sqrt{1 - B^2}} , \quad \ell \in \mathbb{N} . \quad (6)$$

The quantization of $\ell$ stems from analyzing the local geometry of the 4d space spanned by $w, \mu, \phi, \gamma$ near $P_4$, and requiring it to be locally an orbifold $\mathbb{R}^4/\mathbb{Z}_\ell$.

The internal space $M_6$ admits non-trivial 4-cycles which lead to flux quantization conditions for $G_4$. The 4-cycle $\mathcal{C}_4$ is obtained by combining the segment $Q_1Q_2$, $S^1_\chi$, and $S^2$. $\mathcal{C}_4$ has the topology of a 4-sphere because the $S^2$ shrinks at $Q_1$ and the $S^1_\chi$ shrinks at $Q_2$. The flux of $G_4$ through $\mathcal{C}_4$ with suitable orientation defines

$$N = \int_{\mathcal{C}_4} \frac{G_4}{(2\pi \ell_p)^3} = \frac{1}{\pi m^3 \ell_p^3} , \quad N \in \mathbb{N} , \quad (7)$$

where $\ell_p$ is the 11d Planck length. Next, we define the 4-cycle $\mathcal{B}_4$ by combining $S^2$, the segment $P_2P_3$, and the linear combination of $S^1_\phi$ and $S^2_\chi$ that does not shrink in the interior of $P_2P_3$. $\mathcal{B}_4$ is topologically a 4-sphere, because the $S^2$ shrinks at $P_2$ and both $S^1_\phi$ and $S^1_\gamma$ shrink at the orbifold point $P_3$. For $\ell = 1$ we have $\mathcal{B}_3 \cong \mathcal{C}_4$, but $\mathcal{B}_4$ is an independent 4-cycle for $\ell > 1$.

### Solutions in Canonical $\mathcal{N} = 2$ Form

The general form of an $AdS_5$ M-theory solution preserving 4d $\mathcal{N} = 2$ superconformal symmetry was determined in [10] by Lin, Lunin, and Maldacena (LLM). The 11d metric and flux are summarized in [17]. In LLM form, the internal space $M_6$ is an $S^1 \times S^2$ fibration over a 3d space with local coordinates $\Sigma = (x_1, x_2, y)$. The Killing vector $\partial_\chi$ that does not shrink near $w = 0$ with the standard M5-brane solution, we see that the M5-branes are extended along $AdS_5$ and the $S^1_\gamma$, smeared along the $\mu, S^1_\chi$ directions, and sitting at the origin $w = 0$ of the local 3d space $dw^2 + w^2 ds^2_{S^2}$.

$$\partial_\chi = \partial_\phi + \frac{N \ell}{N + K \ell} \partial_\gamma , \quad \partial_\beta = \partial_\phi + \left[1 + \frac{N \ell}{N + K \ell}\right] \partial_\gamma . \quad (12)$$

With reference to the uplift from 7d, the isometry $\partial_\chi$ mixes the $\Sigma$ and $S^4$ directions. This is in contrast to the solutions of [17] [18], in which $\partial_\chi = \partial_\phi$. 

FIG. 1. The internal space is an $S^2_\phi \times S^1_\chi \times S^2$ fibration over $[0, w_1] \times [0, 1]$ in the $(w, \mu)$ plane. The $S^2$ shrinks smoothly along $P_1P_2$. Different linear combinations of $\partial_\phi, \partial_\chi$ shrink smoothly along $P_2P_3$ and $P_3P_4$, as indicated. At $P_3$ the 4d space parametrized by $w, \mu, \phi, \gamma$ is locally $\mathbb{R}^4/\mathbb{Z}_\ell$. The region near $P_1P_4$ enters in terms of smeared M5-branes. The segment $Q_1Q_2$ enters the definition of the 4-cycle $\mathcal{C}_4$. 

The $G_4$ flux supporting the solution reads

$$G_4 = -\frac{1}{m^3} \text{vol}_{S^2} \wedge d\left[\frac{k^3 D\phi}{\mathcal{H}}\right], \quad (5)$$

where $\text{vol}_{S^2}$ is the volume form of the $S^2$.
The function $D$ and the map between the LLM coordinates $y$, $r$ and the coordinates $\mu$, $w$, are

$$y = \frac{4Bw\mu}{\sqrt{1-w^2}}, \quad r = (1-\mu^2)^{-\frac{1}{B}} G(w),$$

$$e^D = \frac{16B^{2} (1-\mu^2)^{1+\frac{1}{B}}}{(1-w^2)} \frac{G'(w)}{G(w)} = \frac{-Bw}{C(1-w^2)h}.$$  

This determines a class of exact solutions $D$ to the Toda equation (11) which are separable in the variables $\mu$, $w$. Crucially, in our setup $D$ does not describe a constant curvature Riemann surface, in contrast to the 4d $\mathcal{N} = 2$ Maldacena-Nuñez solutions [18].

Holographic Central Charge, Flavor Central Charge, and Probe M2-Branes

The holographic central charge is extracted from the warped volume of the internal space [19],

$$c = \frac{1}{2^2 \pi^6 m_6^9 r_3^3} \int_{M_6} e^{\lambda_0} \text{vol}_{M_6} = \frac{\ell N^2 K^2}{12 (N + K \ell)},$$

where $\text{vol}_{M_6}$ is the volume form of $ds^2_{M_6}$ in (1).

Expanding the M-theory 3-form $C_3$ onto the resolution cycles of the $\mathbb{R}^4/\mathbb{Z}_2$ orbifold singularity at $P_3$, one obtains $\ell - 1$ Abelian gauge fields. The gauge group enhances to $SU(\ell)$ by virtue of states from M2-branes wrapping the resolution cycles [17]. We compute the associated flavor central charge $k_{SU(\ell)}$ using the ’t Hooft anomaly inflow methods of [20], yielding

$$k_{SU(\ell)} = \frac{2N K \ell}{N + K \ell}.$$  

M2-brane probes wrapping calibrated 2-cycles in the internal space are dual to BPS operators in the SCFT. The calibration condition was written in [19] for a generic solution preserving 4d $\mathcal{N} = 1$ superconformal symmetry and can be adapted to the $\mathcal{N} = 2$ solutions at hand. The conformal dimension $\Delta$ of the operator dual to an M2-brane wrapping the calibrated 2-cycle $C_2$ is [19]

$$\Delta = \frac{1}{4 \pi^2 m_6^4 r_3^3} \int_{C_2} e^{3\lambda} \text{vol}_{C_2},$$

where $\text{vol}_{C_2}$ is the volume form on $C_2$ induced from $ds^2_{M_6}$.

We identify two supersymmetric M2-brane probes in our setup. Firstly, we can wrap an M2-brane on the $S^2$ on top of the orbifold point $P_3$ in the $(w, \mu)$ plane. We denote the corresponding operator as $O_1$. Secondly, we can wrap an M2-brane on the 2d subspace consisting of the segment $P_3P_4$ and the combination of $S_0^1$ and $S_1^1$ that does not shrink in the interior of $P_3P_4$. This subspace corresponds to an open M2-brane ending on the M5-branes at $w = 0$. We denote the associated operator as $O_2$. The dimensions of $O_1$, $O_2$ from (16) are

$$\Delta(O_1) = \frac{NK\ell}{N + K\ell}, \quad \Delta(O_2) = K.$$  

The $U(1)_r \times SU(2)_R$ charges of $O_1$, $O_2$ can be computed from the M2-brane coupling to the 11d 3-form $C_3$ [19],

$$\langle r, R_0 \rangle (O_1) = (2\Delta(O_1), 0), \quad \langle r, R_0 \rangle (O_2) = (0, \Delta(O_2)),$$

with $R$ the Cartan generator of $SU(2)_R$. Thus $O_1$ and $O_2$ have the R-charges of $\mathcal{N} = 2$ Coulomb branch and Higgs branch operators, respectively.

A NOVEL STÜCkelberg MECHANISM

The Killing vector $\partial_\beta$ in (12) is a symmetry of the 11d metric and flux, but it does not correspond to a continuous flavor symmetry of the dual SCFT. This is due to a Stuckelberg mechanism in the 5d low-energy effective action of M-theory on $M_6$. The components of the 11d metric with one external leg and one leg along $\partial_\beta$ yield a $U(1)$ gauge field $A^\beta$. When $A^\beta$ is turned on, the 1-form $d\beta$ must be replaced by the gauge invariant combination $d\beta + A^\beta$. This replacement affects the closure of $G_4$, which is restored by adding suitable terms, including

$$G_4^{\text{tot}} = G_4|_{d\beta \to d\beta + A^\beta} + D_{a0} \wedge \omega_3 + \ldots.$$  

The improved $G_4^{\text{tot}}$ is built with the closed but not exact 3-form $\omega_3 \propto \iota_{\partial_\beta} G_4$, whose non-exactness hinges on the M5-brane source at $w = 0$. The 1-form $D_{a0}$ is the field strength of an external axion $a_0$.

Closure of $G_4$ requires $dD_{a0} \propto dA^\beta$, signaling a non-trivial Stuckelberg coupling between $A^\beta$ and $a_0$. As a result, $A^\beta$ is massive and is dual to a spontaneously broken $U(1)$ symmetry in the SCFT. As discussed in detail in [24], this mechanism provides a non-trivial physical realization of a mathematical obstruction to promoting $G_4$ to an equivariant cohomology class [22]. In contrast, $\iota_{\partial_\beta} G_4$ is exact, and the $U(1)$ gauge field $A^\chi$ (originating from the components of the 11d metric with one external leg and one leg along $\partial_\chi$) does not participate in any Stuckelberg coupling to $a_0$ and remains massless. This is expected since $\partial_\chi$ is dual to the $U(1)_r$ R-symmetry of the SCFT. Similar versions of the Stuckelberg mechanism for isometries in flux backgrounds are known for flat internal spaces (see e.g. [23]). The internal geometry discussed in this letter is richer, and this Stuckelberg mechanism is novel in the context of holographic M-theory solutions.

FIELD THEORY DUALS

We claim that the 11d supergravity solutions presented above are holographically dual to 4d $\mathcal{N} = 2$ SCFTs that
arise from the low energy limit of $N$ M5-branes wrapping a sphere with an irregular puncture of type $A_{N-1}^{(N)}[k]$, labeled by the integer $k > -N$. For $\ell = 1$ the irregular puncture is the only puncture on the sphere, and the 4d field theories coincide with the Type I theories with $b = N$ and $J = A_{N-1}$ in the classification of [24, 25] (also called $I_{N,k}$ in [26]). These are the AD theories of type $(A_{N-1}, A_{k-1})$, obtained in Type IIB in [27] (generalizing the $N = 2$ cases obtained in [11, 28, 29]). For $\ell > 1$ there is an additional regular puncture at the opposite pole of the sphere that is labeled by a box Young diagram with $\ell$ columns and $N/\ell$ rows, contributing an $SU(\ell)$ non-Abelian flavor symmetry [30]. We label the resulting 4d theories by $(A_{N-1}^{(N)}[k], Y_\ell)$, which belong to the class labeled Type IV in [24, 25]. For $\ell = N$ the regular puncture is of maximal type and these are the AD theories studied in [31–33]. The case $\ell > 1$ is the “non-puncture”, equivalent to the $(A_{N-1}, A_{k-1})$ theories.

The irregular puncture is identified with the M5-brane source in the gravity dual. Due to the irregular puncture, the $U(1)_r$-symmetry of the SCFT is given as the combination $r = R_0 + k/N R_2$, where $R_0$ is the generator of the $U(1)$-symmetry that would be preserved in the absence of the irregular defect and $R_2$ is the generator of the global $U(1)_2$ isometry of the sphere [24, 25]. Comparison with [12] gives the map between $k$ in the SCFT and the flux quantum $K$,

$$K = k + N \left( 1 - \frac{1}{\ell} \right).$$

(20)

The central charges of the $(A_{N-1}^{(N)}[k], Y_\ell)$ theories are summarized in Table 1. They are computed in the literature [24, 32, 35] using useful formulae from [36]. For $\ell > 1$, an especially simple way to compute the central charges as a function of $\ell$ is to apply the results of [33, 37] for the partial closure of a maximal puncture, initiated by a nilpotent VEV for the moment map operator of the maximal puncture’s flavor symmetry. The third row of Table 1 gives the central charge in the limit $N, k \to \infty$ with $k/N$ finite. Using (20), we get a perfect match with the holographic central charge (14).

The dimensions of the Coulomb branch operators $u_i$ of the theory $(A_{N-1}^{(N)}[k], Y_\ell)$ are conveniently captured by a Newton polygon [24] and obey the bounds

$$1 < \Delta(u_i) \leq N - \frac{N^2}{\ell(N+k)}.$$ 

(21)

The upper bound is saturated by exactly one $u_i$, which has the correct dimension and R-charges to be identified with the M2-brane operator $O_1$ in [17, 18, 38]. Using (20), the $k_{SU(\ell)}$ central charge (15) reads

$$k_{SU(\ell)} = 2N - \frac{2N^2}{\ell(N+k)}.$$ 

(22)

For $\ell = N$ it matches the field theory computation of [32]. For generic $\ell$, it matches the conjecture of [20] that the flavor central charge is equal to twice the maximal Coulomb branch operator dimension—see (21).

For $\ell = 1$, the rank of the global symmetry of the $(A_{N-1}, A_{k-1})$ theories is $\text{GCD}(k, N) - 1$ [33]. The maximal rank $N - 1$ on the SCFT side matches with the maximal rank that can be achieved via the M5-brane source on the gravity side. It would be interesting to establish a precise match with the SCFT formula for generic $k, N$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$a$ & $4k^2(N^2 - 2 - 5(k+N))(\frac{18 - 4\ell}{48(k+N)}) + N^{-1} \sum_{j=1}^{\text{GCD}(k, N)} \left( \frac{j(k+N)}{N} \right) \left( 1 - \left\{ \frac{j(k+N)}{N} \right\} \right) + 4N^3(1 - \frac{1}{N})(2k + N(1 - \frac{1}{N}))$ & 48(k+N) \\
$c$ & $k^2(N^2 - 2 - (k+N)(N - \ell - 2 + 2\text{GCD}(k, N)))$ & 12(2k+N) \\
\hline
$N, k \to \infty$ & $\frac{k^2(N^2 - 2 - (k+N)(N - \ell - 2 + 2\text{GCD}(k, N)))}{12(2k+N)}$ & $rac{4n^2}{12(2k+N)}$; $n = c = \frac{N^2}{2} - \frac{(k+N)(N - \ell - 2 + 2\text{GCD}(k, N))}{12(2k+N)}$ \tabularnewline
\hline
\end{tabular}
\caption{The central charges of the $(A_{N-1}^{(N)}[k], Y_\ell)$ theories. $[x] = x - [x]$ denotes the fractional part.}
\end{table}

At large $N$, this exactly matches with the dimension of the wrapped M2-brane operators $O_1$ in [17, 18]. We expect that the field-theory degeneracy factor $2^{N-2}$ could be understood on the gravity side by studying the possible ways in which the M2-brane can end on the M5-brane source. Heuristically, we can picture the M2-brane worldvolume, which has a disk topology, as the collapsed version of a multi-pronged configuration that can have a boundary component on each of the $N$ M5-branes independently. Since the M2-brane must end on at least one of them, the degeneracy is $2^N - 1$. Notice the mismatch by one between the degeneracy in field theory and in gravity. It would be interesting to sharpen this argument and to understand the origin of the additional decoupled mode, which we expect is associated to the center-of-mass mode of the M5-brane source stack.

\textbf{DISCUSSION}

We have proposed gravity duals for the 4d $\mathcal{N} = 2$ SCFTs $(A_{N-1}^{(N)}[k], Y_\ell)$ of AD type, performing checks on the central charge, the $SU(\ell)$ flavor central charge, and the dimensions of suitable Coulomb branch and Higgs branches.
branch operators. Our $AdS_5$ solutions contain internal M5-brane sources. They admit an isometry algebra $\mathfrak{su}(2)_R \oplus u(1)_\gamma \oplus u(1)_\beta$. The $\mathfrak{su}(2)_R \oplus u(1)_\gamma$ is dual to the SCFT $R$-symmetry, while $u(1)_\beta$ does not yield a continuous flavor symmetry thanks to a Stückelberg mechanism in which the $u(1)_\beta$ vector eats an axion originating from the expansion of the M-theory 3-form. There could be still a discrete symmetry remnant of $u(1)_\beta$, which we plan to study elsewhere.

We expect our 11d solutions to admit generalizations corresponding to a regular puncture labeled by an arbitrary Young diagram. Constructing Lagrangian descriptions for these cases would yield further insights into SCFTs of AD type and allow for precision tests of the holographic duality.

It would be interesting to investigate whether the classification of irregular punctures in field theory can be recovered by a systematic study of exact solutions to the Toda equation of the class we discovered. The supergravity construction can be generalized to obtain $N = 1$ systems. More interestingly, our solutions can be used to study the holographic dual of the supersymmetry enhancing flows observed in the Lagrangian realizations of AD theories.

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