A RELATION BETWEEN MILNOR’S $\mu$-INVARAENTS AND HOMFLYPT POLYNOMIALS

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Abstract. Polyak showed that any Milnor’s $\mu$-invariant of length 3 can be represented as a combination of Conway polynomials of knots obtained by certain band sum of the link components. On the other hand, Habegger and Lin showed that Milnor invariants are also invariants for string links, called $\mu$-invariants. We show that any Milnor’s $\mu$-invariant of length $\leq k + 2$ can be represented as a combination of the HOMFLYPT polynomials of knots obtained from the string link by some operation, if all $\mu$-invariants of length $\leq k$ vanish. Moreover, $\mu$-invariants of length 3 are given by a combination of the Conway polynomials and linking numbers without any vanishing assumption.

1. Introduction

For an ordered oriented link in the 3-sphere, J. Milnor [14, 15] defined a family of invariants, known as Milnor’s $\overline{\mu}$-invariants. For an $n$-component link $L$, Milnor invariant is determined by a sequence $I$ of elements in $\{1, 2, \ldots, n\}$ and denoted by $\overline{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that $J$ is a subsequence of $I$ obtained by removing at least one index or its cyclic permutation. If the sequence is of distinct numbers, then this invariant is also a link-homotopy invariant and we call it Milnor’s link-homotopy invariant. Here, the link-homotopy is an equivalence relation generated by ambient isotopy and self-crossing changes.

In [3], N. Habegger and X. S. Lin showed that Milnor invariants are also invariants for string links, and these invariants are called Milnor’s $\hat{\mu}$-invariants. For any string link $\sigma$, $\mu_\sigma(I)$ coincides with $\overline{\mu}_\sigma(I)$ modulo $\Delta_\sigma(I)$, where $\hat{\sigma}$ is a link obtained by the closure of $\sigma$. Milnor’s $\mu$-invariants of length $k$ are finite type invariants of degree $k − 1$ for any natural integer $k$, as shown by D. Bar-Natan [1] and X. S. Lin [10].

In [16], M. Polyak gave a formula expressing Milnor’s $\bar{\mu}$-invariant of length 3 by the Conway polynomials of knots. His idea was derived from the following relation. Both Milnor’s $\mu$-invariant of length 3 for string link and the second coefficient of the Conway polynomial are finite type invariants of degree 2. He gave this relation by using Gauss diagram formulas.

Then, in [13], J-B. Meilhan and A. Yasuhara generalized it by using the clasper theory introduced by K. Habiro [4]. They showed that general Milnor’s $\bar{\mu}$-invariants can be represented by the HOMFLYPT polynomials of knots under some assumption. Moreover the author and A. Yasuhara improved it in [8].

In this paper, we give a formula expressing Milnor’s $\mu$-invariant by the HOMFLYPT polynomials of knots under some assumption (Theorem 1.1) by using the clasper theory in [4]. The course of proof is similar to that in [13] and [8]. Moreover, Milnor’s $\mu$-invariants of length 3 for any string link are given by the HOMFLYPT polynomial, which is a finite type invariant of degree 2, and the linking number.

A part of this work was supported by Platform for Dynamic Approaches to Living System from the Ministry of Education, Culture, Sports, Science and Technology, Japan.
Because a finite type knot invariant of degree 2 is only the second coefficient of the Conway polynomial essentially, Milnor’s $\mu$-invariants of length 3 are given by the second coefficient of the Conway polynomial and the linking number (Theorem 1.3). It is a string version of Polyak’s result, and by taking modulo $\Delta(I)$, our result coincides with Polyak’s result.

Given a sequence $I$ of elements of $\{1,2,\ldots,n\}$, $J < I$ will be used for any subsequence $J$ of $I$, possibly $I$ itself, and $|J|$ will denote the length of the sequence $J$.

Let $\sigma$ be an $n$-string link. Given a sequence $I = i_1i_2\ldots i_m$ obtained from $12\ldots n$ by deleting some elements and permuting it, and a subsequence $J = j_1j_2\ldots j_k$ of $I$, we define a knot $\sigma_{I,J}$ as the closure of the product $b_I\sigma_J$. Here $\sigma_J$ is the $m$-string link obtained from $\sigma$ by deleting the $i$th string, for all $i \in \{1,2,\ldots,n\} \setminus \{i_1,i_2,\ldots,i_m\}$ and replacing the $i$th string with a trivial string underpassing all other components, for all $i \in \{i_1,i_2,\ldots,i_m\} \setminus \{j_1,j_2,\ldots,j_k\}$, and $b_I$ is the $m$-braid associated with the permutation $b = (i_1i_2\ldots i_m)$. Here $\Delta$ is the second coefficient of the Conway polynomial and the linking number (Theorem 1.3). It is a string version of Polyak’s result, and by taking modulo $\Delta(I)$, our result coincides with Polyak’s result.

**Theorem 1.1.** Let $\sigma$ be an $n$-string link ($n \geq 4$) with vanishing Milnor’s link-homotopy invariants of length $\leq n - 2$. Then for any sequence $I$ obtained from $12\ldots n$ by deleting $n - m$ elements and permuting it, we have

$$\mu_\sigma(I) = \frac{(-1)^{m-1}}{(m-1)!2^{m-1}} \sum_{J \subset I} (-1)^{|J|} P^{(m-1)}(\sigma_{I,J};1),$$

where $P^{(m-1)}(\cdot; 1)$ is the $(m-1)$th derivative of the $0$th coefficient $P_0(\cdot; t)$ of the HOMFLYPT polynomial evaluated at $t = 1$.

Note that the above vanishing assumption for string link is equivalent to that any $(m-2)$-substring link is link-homotopic to the trivial string link.

**Remark 1.2.** Theorem 1.1 remains valid if we use one of the following two alternative definitions of $b_I$. One is that we use “overpasses” instead of “underpasses”. The other is that we use “any $i \in \{i_1,i_2,\ldots,i_m\}$” instead of “$i_1$”.

We also give the case of $\mu$-invariants of length 3 without the assumption.

**Theorem 1.3.** Let $\sigma$ be an $n$-string link and $I = i_1i_2i_3$ be a length 3 sequence with distinct numbers in $\{1,2,\ldots,n\}$. We then have

$$\mu_\sigma(I) = -\sum_{J \subset I} (-1)^{|J|} a_2(\sigma_{I,J}) - lk_\sigma(i_1i_2)lk_\sigma(i_2i_3) + A_I,$$

where $a_2$ is the second coefficient of the Conway polynomial, $lk_\sigma(ij)$ is the linking number of the $i$th component and $j$th component of $\sigma$, and

$$A_I = \begin{cases} lk_\sigma(i_1i_2) & (i_2 < i_3 < i_1) \\ -lk_\sigma(i_1i_2) & (i_1 < i_3 < i_2) \\ 0 & \text{(otherwise)}. \end{cases}$$

**Remark 1.4.** This operation from a string link to a knot corresponds to $Y$-graph sum of links defined by M. Polyak. By taking this formula modulo $\Delta(I)$, we get Polyak’s relation between Milnor’s $\overline{\mu}$-invariants and Conway polynomials [16].

**Remark 1.5.** K. Taniyama gave a formula expressing Milnor’s $\overline{\mu}$-invariants of length 3 for links by the second coefficient of the Conway polynomial assuming that all linking numbers vanish.
Remark 1.6. In [11], J.B. Meilhan showed that all finite type invariants of degree 2 for string link was given a formula by some invariants (Theorem 2.8). So the formula in Theorem 1.3 could also be derived from [11].

This paper is organized as follows. In section 2, we prepare some known results. In section 3, we prove Theorem 1.1 and Theorem 1.3. In section 4, we show examples.

ACKNOWLEDGEMENTS

The author thanks Professor Sadayoshi Kojima for comments and suggestions. She also thanks Professor Akira Yasuhara for discussions and comments. She also thanks Professor Michael Polyak for valuable advices. She also thanks Professor Jean-Baptiste Meilhan for many useful comments. She also thanks Professor Kouki Taniyama for comments.

2. Some known results

2.1. String link. Let $n$ be a positive integer and $D^2 \subset \mathbb{R}^2$ the unit disk equipped with $n$ marked points $x_1, x_2, \ldots, x_n$ in its interior, lying in the diameter on the $x$-axis of $\mathbb{R}^2$ as in Figure 1. Let $I = [0, 1]$. An $n$-string link $\sigma$ is the image of a proper embedding $\bigcup_{i=1}^{n} I_i \to D^2 \times I$ of the disjoint union of $n$ copies of $I$ in $D^2 \times I$, such that $\sigma|_{I_i}(0) = (x_i, 0)$ and $\sigma|_{I_i}(1) = (x_i, 1)$ for each $i$ as in Figure 1. Each string of a string link inherits an orientation from the usual orientation of $I$. The $n$-string link $\{x_1, x_2, \ldots, x_n\} \times I$ in $D^2 \times I$ is called the trivial $n$-string link and denoted by $1_n$ or $1$ simply.

Given two $n$-string links $\sigma$ and $\sigma'$, we denote their product by $\sigma \cdot \sigma'$, which is given by stacking $\sigma'$ on the top of $\sigma$ and reparametrizing the ambient cylinder $D^2 \times I$. By this product, the set of isotopy classes of $n$-string links has a monoid structure with unit given by the trivial string link $1_n$. Moreover, the set of link-homotopy classes of $n$-string links is a group under this product.

2.2. Milnor’s $\mu$-invariant for string links. Let $\sigma = \bigcup_{i=1}^{n} I_i \sigma_i$ in $D^2 \times I$ be an $n$-string link. We consider the fundamental group $\pi_1(D^2 \times I \setminus \sigma)$ of the complement of $\sigma$ in $D^2 \times I$, where we choose a point $b$ as a base point and curves $\alpha_1, \cdots, \alpha_n$ as meridians in Figure 2.

By Stallings’ theorem [17], for any positive integer $q$, the inclusion map $\iota : D^2 \times \{0\} \setminus \{x_1, \cdots, x_n\} \to D^2 \times I \setminus \sigma$
induce an isomorphism of the lower central series quotients of the fundamental groups

\[ \iota_* : \frac{\pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\})}{\pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\})} \longrightarrow \frac{\pi_1(D^2 \times I \setminus \sigma)}{\pi_1(D^2 \times I \setminus \sigma)_q}, \]

where given a group \( G \), \( G_q \) means the \( q \)th lower central subgroup of \( G \). The fundamental group \( \pi_1(D^2 \times \{0\} \setminus \{x_1, \ldots, x_n\}) \) is a free group generated by \( \alpha_1, \ldots, \alpha_n \). We then consider the \( j \)th longitude \( l_j \) of \( \sigma \) in \( D^2 \times I \), where \( l_j \) is the closure of the preferred parallel curve of \( \sigma_j \), whose endpoints lie on the \( x \)-axis in \( D^2 \times \{0,1\} \) as in Figure 2. We then consider the image of the longitude \( \iota_*^{-1}(l_j) \) by the Magnus expansion and denote \( \mu(i_1, \ldots, i_k, j) \) the coefficient of \( X_{i_1}X_{i_2}\cdots X_{i_k} \) in the Magnus expansion.

**Theorem 2.1** ([3]). For any positive integer \( q \), if \( k < q \), then \( \mu(i_1, \ldots, i_k, j) \) is invariant under isotopy. Moreover, if the sequence \( i_1, \ldots, i_k, j \) is of distinct numbers, then \( \mu(i_1, \ldots, i_k, j) \) is also link-homotopy invariant.

We call this invariant Milnor’s \( \mu \)-invariant.

### 2.3. Claspers

The theory of claspers was introduced by K. Habiro [4]. Here, we define only simple tree clasper. For a general definition, we refer the reader to [4].

Let \( L \) be a (string) link. A disk \( T \) embedded in \( S^3 \) (or \( D^2 \times I \)) is called a simple tree clasper (we will call it tree, simply in this paper) for \( L \) if it satisfies the following four conditions:

1. The disk \( T \) is decomposed into disks and bands. Here, the band connects two distinct disks, and are called edges.
2. The disks attach either 1 or 3 edges. We call a disk attached with only one edge a leaf.
3. The disk \( T \) intersects the (string) link \( L \) transversely and the intersections are contained in the interiors of the leaves.
4. Each of leaves of \( T \) intersects \( L \) at exactly one point.

A simple tree clasper \( T \) with \( k \) leaves is called a \( C_k \)-tree.

Given a \( C_k \)-tree \( T \) for a (string) link \( L \), there exists a procedure to construct a framed link \( \gamma(T) \) in a regular neighborhood of \( T \). We call surgery along \( \gamma(T) \) surgery along \( T \). Because there is an orientation-preserving homeomorphism, fixing the boundary, from the regular neighborhood \( N(T) \) of \( T \) to the manifold obtained from \( N(T) \) by surgery along \( T \), the surgery along \( T \) can be regarded as a local move on \( L \). We denote by \( L_T \) a (string) link obtained from \( L \) by surgery along \( T \). For example, surgery along a \( C_k \)-tree is a local move as illustrated in Figure 3. In this paper, the drawing convention for \( C_k \)-trees are those of [4, Figure 7].
Similarly, let $T_1 \cup \ldots \cup T_m$ be a disjoint union of trees for $L$, we can define $L_{T_1 \cup \ldots \cup T_m}$ as the link obtained by surgery along $T_1 \cup \ldots \cup T_m$.

The $C_k$-equivalence is an equivalence relation on (string) links generated by surgeries along $C_k$-tree claspers and isotopy, and denoted by $\sim_{C_k}$.

**Theorem 2.2** ([4]). A finite type invariant of degree $\leq k - 1$ is an invariant of $C_k$-equivalence.

### 2.4. Moves for claspers.

In [4], the following moves for tree claspers, called a leaf slide and an edge crossing change are given.

**Lemma 2.3** ([4]). Let $T_1$ be a $C_{k_1}$-tree for a (string) link $L$ and $T_2$ a $C_{k_2}$-tree for $L$, where $T_1$ and $T_2$ are disjoint.

1. (Leaf slide) Let $T'_1 \cup T'_2$ be obtained from $T_1 \cup T_2$ by sliding a leaf $f_2$ of $T_2$ over a leaf of $T_1$ (see the left-hand side of Figure 4). Then, $L_{T_1 \cup T_2}$ is $C_{k_1 + k_2 + 1}$-equivalent to $L_{T'_1 \cup T'_2 \cup Y}$, where $Y$ denotes a $C_{k_1 + k_2}$-tree obtained by inserting a vertex $v_1$ in the edge $e_1$ of $T_1$ and connecting $v_1$ to the edge incident to $f_2$ as shown in Figure 4.

2. (Edge crossing change) Let $T'_1 \cup T'_2$ be obtained from $T_1 \cup T_2$ by passing an edge of $T_2$ across an edge of $T_1$ (see the right-hand side of Figure 4). Then, $L_{T_1 \cup T_2}$ is $C_{k_1 + k_2 + 2}$-equivalent to $L_{T'_1 \cup T'_2 \cup H}$, where $H$ denotes a $C_{k_1 + k_2 + 1}$-tree obtained by inserting a vertex in both edges and connecting them by an edge as shown in Figure 4.

### 2.5. Presentation of link-homotopy classes for string links.

Define the set $M_r$ of sequences as follows: Any subsequence $m_0 m_1 \ldots m_r$ of $12 \cdot \ldots \cdot n$ with length $r + 1$ and its permutation $\sigma(m_1) \ldots \sigma(m_r-1)m_r$. That is,

$$M_r = \{m_0 \sigma(m_1) \ldots \sigma(m_{r-1})m_r \mid m_0 m_1 \ldots m_r < 12 \cdot \ldots \cdot n, \sigma \in S_{r-1}\}.$$

Let $T_M$ be a $C_r$-tree for $1_n (= 1)$ as illustrated in Figure 5, where $\sigma$ is the unique positive $(r - 1)$-braid defined by the permutation $\sigma$ and such that every pair of strings crosses at most one. Let $T_M^{-1}$ be the $C_r$-tree obtained from $T_M$ by inserting a positive half-twist in the * marked edge in Figure 5.
Theorem 2.4 (cf. [19 Theorem 4.3], [13 Theorem 4.1]). For any $n$-string link $l$, there are canonical string links $l_r$'s such that $l$ is link-homotopic to a string link $l_1 \cdot l_2 \cdot \cdots \cdot l_{n-1}$, where

$$l_r = \prod_{M \in \mathcal{M}_r} (1_{T_M})^{x_M}(M \text{ in the product appears in the lexicographic order of } \mathcal{M}_r),$$

$$x_M = \begin{cases} \mu_l(M) & \text{if } r = 1 \\ (-1)^{r+1} \{\mu_l(M) - \mu_{l_1 \cdots l_{r-1}}(M)\} & \text{if } r \geq 2, \end{cases} \text{ and } \epsilon = \begin{cases} 1 & \text{if } x_M > 0 \\ -1 & \text{if } x_M \leq 0. \end{cases}$$

Remark 2.5. In [13], the statement slightly differs from the original one in [19], because the authors used a different definition for the $C_r$-trees $T_M$ and $T_M^{-1}$ from [19]. But in [13 Theorem 4.1], when $r \geq 2$, a sign $(-1)^{r+1}$ of $\mu_l(M) - \mu_{l_1 \cdots l_{r-1}}(M)$ seems to be missing. In this paper, we use the definition for $C_r$-trees $T_M$ and $T_M^{-1}$ as in [13]. But Theorem 2.4 is different from [13] Theorem 4.1] only by a sign.

Lemma 2.6 (Lemma 3.3 [12]). Let $l$ and $l'$ be $n$-string links. Let $h$ and $h'$ be the integers. If $\mu_l(I) = 0$ for any $I$ with $|I| \leq h$ and $\mu_{l'}(I) = 0$ for any $I$ with $|I| \leq h'$, then for any $J$ with $|J| \leq h + h'$

$$\mu_{l \cdot l'}(J) = \mu_l(J) + \mu_{l'}(J).$$

2.6. HOMFLYPT polynomial. Recall the definition of the HOMFLYPT polynomial, and mention useful properties.

The HOMFLYPT polynomial $P(L; t, z) \in \mathbb{Z}[t^\pm, z^\pm]$ of an oriented link $L$ is defined by the following two formulas:

1. $P(U; t, z) = 1$, and
2. $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z),$

where $U$ denotes the trivial knot and $L_+$, $L_-$ and $L_0$ are link diagrams which are identical everywhere except near one crossing, where they look as follows:

$$L_+ = \quad ; \quad L_- = \quad ; \quad L_0 = \quad .$$

Recall that the HOMFLYPT polynomial of a knot $K$ is of the form $P(K; t, z) = \sum_{k=0}^{N} P_{2k}(K; t)z^{2k}$, where $P_{2k}(K; t) \in \mathbb{Z}[t^\pm]$ is called the $2k$th coefficient polynomial of $K$. Then the following theorem is known.

Theorem 2.7 ([2]). Let $K$ be a knot. $P_{2k}^{(j)}(K; 1)$ is a finite type invariant of degree $2k + j$, where $P_{2k}^{(j)}(K; 1)$ is the $j$th derivative of $P_{2k}(K; t) \in \mathbb{Z}[t^\pm]$ evaluated at $t = 1$.

It is known that the HOMFLYPT polynomial of knots is multiplicative under connected sum. So $P_0(K; t)$ is also multiplicative under connected sum. For any
knots $K$ and $K'$ and any integer $n$, we have

$$P_0^{(n)}(K_2K'; 1) = P_0^{(n)}(K; 1) + P_0^{(n)}(K'; 1) + \sum_{k=1}^{n-1} \binom{n}{k} P_0^{(k)}(K) P_0^{(n-k)}(K'; 1),$$

because $P_0(K; 1) = 1$ for any knot $K$. Moreover, if the knot $K$ is $C_{n+1}$-equivalent to the trivial knot, we have

$$P_0^{(n)}(K_2K'; 1) = P_0^{(n)}(K; 1) + P_0^{(n)}(K'; 1),$$

because finite type invariants of degree less than $n - 1$ are $C_{n+1}$-equivalence invariants and $P_0^{(k)}(K; 1) = 0$ for any knot $K$.

3. Proof of Theorem 1.1 and Theorem 1.3

Proof of Theorem 1.1 Let $n \geq 4$ and $\sigma = \bigcup_{i=1}^{n} \sigma_i$ be an $n$-string link with vanishing Milnor link-homotopy invariants of length $\leq m - 2$ ($\leq n$). We assume that $I$ is a sequence of length $l$ ($4 \leq l \leq m$) with distinct number in \{1, 2, ..., $n$\}. It is sufficient to consider the case $l = m$ because the case $l = m$ contains the case $l = m - 1$ (that is the first non-vanishing case), and the case $m = n$ because we have that $\mu_*(I) = \mu_{\bigcup_{i \in \{t\}} \sigma_i}(I)$ if $m < n$. So we may consider $l = m = n$ and $I = i_1 i_2 \ldots i_n$.

We will prove Theorem 1.1 in three steps. In Step 1, we represent a string link by the trivial string link with tree claspers by Theorem 2.4 and make a knot represented by the trivial knot with the tree claspers of this string link by some closure operation. In Step 2, we transform the tree claspers preserving the value of the HOMFLYPT polynomial by Lemma 2.5. In Step 3, we calculate the $(n - 1)$th derivative of the 0th coefficient of the HOMFLYPT polynomial by Theorem 2.7 and Theorem 2.8. In the proof, we will simply write $\sigma T$ for $\sigma T \cup T$.

Step1:

By combination of Theorem 2.4 and the assumption that Milnor link-homotopy invariants of length $\leq n - 2$ vanish, $\sigma$ is link-homotopic to $l_{n-2} \cdot l_{n-1}$, where $l_r = \prod_{M \in M_r} (1 T_M)^{|x_{r(M)}|}$. Moreover there is an integer $x_{r(M)}$ such that $l_{n-2} \cdot l_{n-1}$ is $C_n$-equivalent to $l_{n-2} \cdot (c^{-1}_{l_1} \cdot l_{n-1} - c)$, where $l_{n-1} = \prod_{M \in M_{n-1}} (1 T_M)^{|x_{r(M)}|}$ and $c = (i_1 i_2 \ldots i_{n-1} i_n)$, and $c_t$ is the positive permutation braid associated with $c$ (See [5, §4.2]).

Set

$$\mathcal{F} := \left( \bigcup_{M \in M_{n-2}} (T_M)^{|x_M|} \right) \cup \left( \bigcup_{M \in M_{n-1}} (T_M)^{|x_{r(M)}|} \right),$$

where $\epsilon$ is 1 if $x_M > 0$ and otherwise $-1$. We then can regard $l_{n-2} \cdot (c^{-1}_{l_1} \cdot l_{n-1} - c)$ as a string link $(1_n \cdot (c^{-1}_{l_1} \cdot 1_n \cdot c))_T$ obtained from $1_n \cdot (c^{-1}_{l_1} \cdot 1_n \cdot c)$ by surgery along the union of trees $\mathcal{F}$. Because $1_n \cdot (c^{-1}_{l_1} \cdot 1_n \cdot c)$ is ambient isotopic to $1_n$, we can also regard $l_{n-2} \cdot (c^{-1}_{l_1} \cdot l_{n-1} - c)$ as a string link $1_T$. Moreover there is a disjoint union $\mathcal{R}_1$ of $C_t$-trees whose leaves intersect a single component of the string link $l_{n-2} \cdot (c^{-1}_{l_1} \cdot l_{n-1} - c)$ such that

$$\sigma \sim_{C_n} (l_{n-2} \cdot (c^{-1}_{l_1} \cdot l_{n-1} - c))_{\mathcal{R}_1} = (1_T)_{\mathcal{R}_1}.$$

We then can regard $\sigma$ as $C_n$-equivalent to a string link $1_T \cup \mathcal{R}_1$, obtained from $1_n$ by surgery along the union of trees $\mathcal{F} \cup \mathcal{R}_1$.

In the canonical diagram of $1_n$, a tree for $1_n$ is in good position if each component of $1_n$ underpasses all edges of the tree. Note that each tree of $\mathcal{F}$ with $M \in M_{n-2}$
is in good position. On the other hand, a tree of $F$ with $M$ in $\mathcal{M}_{n-1}$ and a tree of $\mathcal{R}_1$ may not be in good position. We now replace $F$ and $\mathcal{R}_1$ with some trees in good position up to $C_n$-equivalence. By Lemma \[2.3\](2) and repeated applications of \[4\] Proposition 4.5, we have

$$1_{F \cup \mathcal{R}_1} \sim_{C_n} 1_{F' \cup \mathcal{R}} ,$$

where

$$F' := \left( \bigcup_{M \in \mathcal{M}_{n-2}} (T'_M)^{|x_M|} \right) \cup \left( \bigcup_{M \in \mathcal{M}_{n-1}} (T'_c)^{|x_c(M)|} \right) ,$$

where $\epsilon$ is 1 if $x_M > 0$ and otherwise $-1$, and $\mathcal{R}$ is a disjoint union of trees for $1_n$ in good position and intersecting some component of $1_n$ more than once. We note that $T'_c(M)$ intersects $1_n$ associated with the order $c(M)$.

It follows from \[13\] Lemma 3.2 and the fact $b_I \cdot 1_n$ is the trivial knot that for any subsequence $J$ of $I$,

$$P_0 \left( \overline{1_{F' \cup \mathcal{R}}} \right)_J = P_0 \left( \overline{1_{\mathcal{G} \cup \mathcal{R}}} \right)_J ,$$

where

$$\mathcal{G} = \left( \bigcup_{M \in \mathcal{M}_{n-2}, M \in I} (T'_M)^{|x_M|} \right) \cup T'_I^{|x_I|} ,$$

meaning that we can ignore the $M$’s in $\mathcal{M}_{n-2}$ and $\mathcal{M}_{n-1}$ such that $M \notin I$ and $c(M) \notin I$ respectively when computing 0th coefficient polynomial $P_0$ of the HOMFLYPT polynomial.

Step 2:

By leaf slides and edge crossing changes, we will separate the union of trees for the trivial knot $U$ up to $C_n$-equivalence. That is, we translate trees preserving a knot obtained from the trivial knot $U$, all intersections with leaves of each of the trees are adjacent and each tree does not link other trees on the standard trivial knot $U$. We then represent the knot as the connected sum of some knots, up to $C_n$-equivalence. We then have to remember to which string each tree belongs and from which each new tree is derived. So we give each tree more information, called a weight. Here, we denote the set of element of $J$ as $\{J\}$.

First of all, we assign to each tree of $\mathcal{G} \cup \mathcal{R}$ the collection of all integers $i$ such that the tree intersects the $i$th component of $1_n$ as weight. Then, since $\mathcal{G} \cup \mathcal{R}$ is in good position for $1_n$, it is obvious that for any subsequence $J$ of $I$,

$$\overline{(1_{\mathcal{G} \cup \mathcal{R}})}_J = T_I(\bigcup_{w(T) \subseteq \{J\}} T) = U(\bigcup_{w(T) \subseteq \{J\}} T) ,$$

where $\bigcup_{w(T) \subseteq \{J\}} T$ means the union of trees $T$’s of $\mathcal{G} \cup \mathcal{R}$ whose weight is contained in $\{J\}$.

Secondly, we define the weight of new trees obtained by leaf slides or edge crossing changes. Now, we consider a knot $U_{T_1 \cup T_2 \cup X}$, where $T_1$ is a $C_{k_1}$-tree with weight $w(T_1)$, $T_2$ a $C_{k_2}$-tree with $w(T_2)$ and $X$ is the union of some trees. When we perform a leaf slide or an edge crossing change between $T_1$ and $T_2$ (see Lemma \[2.3\]), we assign the weight $w(T_1)$ and $w(T_2)$ to $T'_1$ and $T'_2$ respectively, and the union $w(T_1) \cup w(T_2)$ of weights as the weight of $Y$ (resp. of $H$).

So for any subsequence $J$ of $I$, a knot obtained from the trivial knot $U$ by surgery along the union of trees of $T_1 \cup T_2 \cup X$ with weight in $\{J\}$ is $C_{k_1+k_2+1}$-equivalent.
to the knot obtained by surgery along the union of trees $T'_1 \cup T'_2 \cup X$ with weight in $\{J\}$. The case of edge crossing changes is similar to leaf slides.

We separate some unions of trees for the trivial knot $U$ up to $C_n$-equivalence, by using leaf slides and edge crossing changes. We separate $|x_I|$ copies of the tree $T'_I$ and then each $T'_M$ ($M \in M_{n-2}$, $M < I$). We move $T'_M$ through the only strings corresponding to $\{M\}$, that is, we must not do a leaf slide between $T'_M$ and a tree whose weight is not contained in $\{M\}$. We need to consider the following two types of move.

(1) Leaf slide or edge crossing change between $T'_M$ $(M \in M_{n-2}$, $M < I)$ and a repeated tree $R$.
(2) Leaf slide or edge crossing change between $T'_M$ and $T'_{M'}$ $(M, M' \in M_{n-2}, M < I, M' < I)$.

(1) If $w(R) \cap w(T'_M) = \emptyset$ (i.e. $|w(R)| = 1$), we only have to consider the edge crossing change. The new tree obtained by the edge crossing change is a $C_1$-tree ($l = n + 1$). Therefore we do not have to consider it, up to $C_n$-equivalence. If $w(R) \subset w(T'_M)$ and $w(R) \cap w(T'_M) \neq \emptyset$, then $R$ is not a $C_1$-tree. So the new tree obtained by the leaf slide or edge crossing change is a $C_l$-tree ($l = n$). Therefore we do not have to consider it, up to $C_n$-equivalence. If $w(R) \supset w(T'_M)$, then the new tree obtained by the leaf slide or edge crossing change is a repeated tree.

(2) New trees obtained by a leaf slide or edge crossing change are $C_1$-trees ($l \geq n$). So we do not have to consider it, up to $C_n$-equivalence.

By separating trees, the knot $U_{G \cup R} = (\overline{1_{G \cup R}})_I$ is $C_n$-equivalent to the connected sum as follows.

\[
\sharp_{x_I, T'_I} \sharp_{M \in M_{n-2}, M < I}( \sharp_{x_M, T'_M} ) \sharp_{R'} U_{R'},
\]

where $U_{T'_I}$ means $\overline{(1_{T'_I})_I}$, $U_{T'_M}$ means $\overline{(1_{T'_M})_I}$ and $R'$ is a disjoint union of some repeated trees for the trivial knot $U$. We note that $\overline{(1_{G \cup R})}_J$ is also $C_n$-equivalent to the above connected sum whose weights are in $\{J\}$.

Step3:
We calculate $\sum_{J \subset I} (-1)^{|J|} P_0^{n-1}(\overline{1_{T'_I}}, 1)$. It follows from Theorem 2.2 and Theorem 2.4 that $P_0^{n-1}(\overline{1_{T'_I}}, 1)$ is invariant under $C_n$-equivalence. Moreover by using Equation (2.1), for any subsequence $J$ of $I$, we have

\[
\sum_{J \subset I} (-1)^{|J|} P_0^{n-1}(\overline{1_{T'_I}}, 1) = \sum_{J \subset I} (-1)^{|J|} P_0^{n-1}(\overline{1_{G \cup R}})_J (\overline{1_{T'_I}}); 1)
\]

\[
= (-1)^{|I|} |x_I| P_0^{n-1}(U_{T'_I}; 1) + \sum_{J \subset I} (-1)^{|J|} \sum_{M < J, |M| = n-1} |x_M| P_0^{n-1}(U_{T'_M}; 1) + \sum_{J \subset I} (-1)^{|J|} P_0^{n-1}(U_{R'_J}; 1),
\]

where $R'_J$ is the subset of $R'$ whose index is in $J$. We then can cancel the second term because for each $M$ ($M < J$, $|M| = n - 1$)

\[
\sum_{M < J \subset I} (-1)^{|J|} = (-1)^{|I|} + (-1)^{|M|} = 0,
\]

and by Claim 6.1, the third term is also 0. So we have

\[
\sum_{J \subset I} (-1)^{|J|} P_0^{n-1}(\overline{1_{T'_I}}, 1) = (-1)^{|I|} |x_I| P_0^{n-1}(U_{T'_I}; 1).
\]
Let $K_m^\delta$ be the knot as illustrated in Figure 6 with $\delta = (\delta_0, \delta_1, \ldots, \delta_{m+1})$. Here, an edge has a negative half-twist if $\delta = -1$ and an edge is a non-twist edge if $\delta = 1$.

![Figure 6. The knot $K_m^\delta$](image)

By using [13, Lemma 3.1], we have

$$P_0^{(m+1)}(K_m^\delta; 1) = (-1)^{m+1}2^{m+1}(m+1)! \prod_{i=0}^{m+1} \delta_i.$$  \hspace{1cm} (3.2)

Now $U_I$ is $K_{n-2}^\delta$ with $\delta = (-\epsilon, 1, \ldots, 1)$. So we have

$$(-1)^{|I|}x_I P_0^{(n-1)}(U_I; 1) = -x_I 2^{(n-1)(n-1)!}.$$  \hspace{1cm} (3.1)

By Theorem 2.4 and Lemma 2.6, for $n \geq 4$,

$$x_I = (-1)^{|I|} \mu_\sigma(I).$$

Therefore, we have

$$\mu_\sigma(I) = (-1)^{|I|}x_I = \frac{(-1)^{n-1}}{(n-1)!2^{n-1}} \sum_{J < I} (-1)^{|J|} P_0^{(n-1)}(\sigma_J; 1).$$

\[ \square \]

**Proof of Theorem 1.3** We consider the case $n = 3$. First of all, we assume that $I = 123, 231$ or $312$. This proof is similar to that of Theorem 1.1. In the Step 2 of the proof of Theorem 1.1, if $x_{12}, x_{13}, x_{23} > 0$, we obtain that $\sigma_I$ is $C_3$-equivalent to the following connected sum.

$$\sharp x_{12} U_{T_I} \sharp x_{13} U_{T_I} \sharp x_{23} U_{T_I} K_{1}^\delta \sharp U_{R'},$$

where $\delta_1 = (-1,1,-1)$ and $\delta_2 = (-1,1,-1)$, and $R'$ is a disjoint union of some repeated trees.

By using Equations (2.1) and (3.2), we have

$$\sum_{J \leq I} (-1)^{|J|} P_0^{(2)}(\sigma_J; 1) = \sum_{J \leq I} (-1)^{|J|} P_0^{(2)}((1 \cap \sigma_R) J; 1)$$

$$= (-1)^{|I|} \{ x_{12} P_0^{(2)}(U_I; 1) + x_{13} P_0^{(2)}(K_{1}^\delta; 1) + x_{23} P_0^{(2)}(K_{1}^\delta; 1) \}$$

$$= (-1)^3 \{ x_{12} (-1)^2 2^2 2! + x_{13} x_{13} (-1)^3 2^2 2! + x_{23} x_{23} (-1)^3 2^2 2! \}$$

$$= -8 \{ x_{12} x_{13} x_{23} - x_{13} x_{13} x_{23} \}.$$

If the case of other signs of $x_{12}, x_{13}, x_{23}$, we have the same formula.

By the definition of $x_{ij}$ ($1 \leq i < j \leq 3$) and $x_{123}$, we have

$$x_{ij} = lk_\sigma(ij) (1 \leq i < j \leq 3),$$

$$x_{123} = -\{ \mu_\sigma(123) - lk_\sigma(12) lk_\sigma(13) - lk_\sigma(13) lk_\sigma(23) + lk_\sigma(12) lk_\sigma(23) \}.$$
Therefore, we have
\[ \sum_{J < I} (-1)^{|J|} P_0^{(2)}(\sigma_J, 1) = 8\{\mu_\sigma(123) + lk_\sigma(12)lk_\sigma(23)\}. \]

Therefore, we have
\[ \mu_\sigma(123) = \frac{1}{8} \sum_{J < I} (-1)^{|J|} P_0^{(2)}(\sigma_J, 1) - lk_\sigma(12)lk_\sigma(23). \]

Because \(-\frac{1}{8}P_0^{(2)}\) coincides with the second coefficient \(a_2\) of the Conway polynomial, this equation can be rewritten as
\[ \mu_\sigma(123) = -\sum_{J < I} (-1)^{|J|}a_2(\sigma_J) - lk_\sigma(i_1i_2)lk_\sigma(i_2i_3). \]

Similarly, we have
\[
\begin{cases}
  x_{231} = -\mu_\sigma(231) + lk_\sigma(12)lk_\sigma(13) \\
  x_{312} = -\mu_\sigma(312) + lk_\sigma(13)lk_\sigma(23) + lk_\sigma(13).
\end{cases}
\]

Therefore we have the formulas in Theorem 1.3.

Next, we consider the other case; \(I = 132, 213\) or \(321\). If \(x_{12}, x_{23} > 0\), we obtain that \(\sigma_J\) is \(C_3\)-equivalent to
\[ \hat{\sigma}_{[x_1]}U_{[J]}\hat{\sigma}_{[x_2]x_{23}}K^{d_3}_I\hat{U}_R', \]
where \(d_3 = (-1, 1, -1)\) and \(R'\) is a disjoint union of some repeated trees.

We then have
\[ \sum_{J < I} (-1)^{|J|} P_0^{(2)}(\sigma_J, 1) = -8\{x_I - x_{12}x_{23}\} \]

If the case of other signs of \(x_{12}, x_{23}\), we have the same formula.

On the other hand, we have
\[
\begin{cases}
  x_{132} = -\mu_\sigma(132) + lk_\sigma(12)lk_\sigma(23) - lk_\sigma(13)lk_\sigma(23) + lk_\sigma(13) \\
  x_{213} = -\mu_\sigma(213) - lk_\sigma(12)lk_\sigma(13) + lk_\sigma(12)lk_\sigma(23) \\
  x_{321} = -\mu_\sigma(321).
\end{cases}
\]

Therefore we have the formulas in Theorem 1.3.

\[ \square \]

4. Examples

Example 4.1. Let \(\sigma\) be a 3-string link showed by Figure 4. Then \(\mu_{123}(\sigma) = -1, \mu_{122}(\sigma) = \mu_{213}(\sigma) = 1\) and \(\mu_{231}(\sigma) = \mu_{312}(\sigma) = \mu_{321}(\sigma) = 0\). And \(lk_\sigma(12) = \mu_\sigma(12) = 13, 23) = 1\) and \(lk_\sigma(13) = 0\).

On the other hand, \(\sigma_{123,123}\) and \(\sigma_{123,23}\) are the figure-eight knot, and \(\sigma_{123,J}\) \((J \neq 123, 23)\) is the trivial knot. Therefore we obtain
\[ -\sum_{J < 123} (-1)^{|J|} a_2(\sigma_{123,J}) - lk_\sigma(12)lk_\sigma(23) = a_2(41) - a_2(41) - 1 \cdot 1 = -1. \]

Similarly, we have
\[
\begin{align*}
-\sum_{J < 12} (-1)^{|J|} a_2(\sigma_{231,J}) - lk_\sigma(23)lk_\sigma(31) & = a_2(31) - a_2(31) - a_2(41) - 1 \cdot 0 = 0, \\
-\sum_{J < 312} (-1)^{|J|} a_2(\sigma_{312,J}) - lk_\sigma(31)lk_\sigma(12) + lk_\sigma(13) & = a_2(31) - a_2(31) - 0 \cdot 1 + 0 = 0.
\end{align*}
\]
Moreover, $\sigma_{132,32}$ is the figure-eight knot and $\sigma_{132,J}$ ($J \neq 32$) is the trivial knot. Therefore we obtain

$$- \sum_{J < 132} (-1)^{|J|} a_2(\sigma_{132,J}) - lk_{\sigma}(13)lk_{\sigma}(32) - lk_{\sigma}(13) = -a_2(41) - 0 \cdot 1 - 0 = 1.$$ 

Similarly, we have

$$- \sum_{J < 213} (-1)^{|J|} a_2(\sigma_{213,J}) - lk_{\sigma}(21)lk_{\sigma}(13) = a_2(7_6) - a_2(3_1) - a_2(41) - 0 \cdot 1 = 1,$n

$$- \sum_{J < 321} (-1)^{|J|} a_2(\sigma_{321,J}) - lk_{\sigma}(32)lk_{\sigma}(21) = a_2(5_2) - a_2(3_1) - 1 \cdot 1 = 0.$$

\[ \text{Figure 7.} \]

Example 4.2. Let $\sigma$ be a 3-string link showed by Figure 8. Then $\mu_{132}(\sigma) = -1$, $\mu_{312}(\sigma) = 1$ and $\mu_{231}(\sigma) = \mu_{123}(\sigma) = \mu_{321}(\sigma) = \mu_{213}(\sigma) = 0$. And $lk_{\sigma}(12) = lk_{\sigma}(23) = 0$ and $lk_{\sigma}(13) = 1$.

On the other hand, any $\sigma_{132,J}$ is the trivial knot. Therefore we obtain

$$- \sum_{J < 132} (-1)^{|J|} a_2(\sigma_{132,J}) - lk_{\sigma}(12)lk_{\sigma}(23) = 0 \cdot 0 = 0.$$ 

Similarly, we have

$$- \sum_{J < 213} (-1)^{|J|} a_2(\sigma_{213,J}) - lk_{\sigma}(21)lk_{\sigma}(13) = 0,$n

$$- \sum_{J < 321} (-1)^{|J|} a_2(\sigma_{321,J}) - lk_{\sigma}(32)lk_{\sigma}(21) = 0,$$ 

Moreover, $\sigma_{132,32} = \sigma_{132,13}$ (In fact, both $\sigma_{132,32}$ and $\sigma_{132,13}$ are the figure-eight knot) and $\sigma_{132,J}$ ($J \neq 132, 13$) is the trivial knot. Therefore we obtain

$$- \sum_{J < 132} (-1)^{|J|} a_2(\sigma_{132,J}) - lk_{\sigma}(13)lk_{\sigma}(32) - lk_{\sigma}(13) = -1.$$ 

Similarly, we have

$$- \sum_{J < 213} (-1)^{|J|} a_2(\sigma_{213,J}) - lk_{\sigma}(21)lk_{\sigma}(13) = 0,$n

$$- \sum_{J < 321} (-1)^{|J|} a_2(\sigma_{321,J}) - lk_{\sigma}(32)lk_{\sigma}(21) = 0.$$
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Figure 8.

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