CLUSTER-TILTED AND QUASI-TILTED ALGEBRAS

IBRAHIM ASSEM, RALF SCHIFFLER, AND KHRYSTYNA SERHIYENKO

Abstract. In this paper, we prove that relation-extensions of quasi-tilted algebras are 2-Calabi-Yau tilted. With the objective of describing the module category of a cluster-tilted algebra of euclidean type, we define the notion of reflection so that any two local slices can be reached one from the other by a sequence of reflections and coreflections. We then give an algorithmic procedure for constructing the tubes of a cluster-tilted algebra of euclidean type. Our main result characterizes quasi-tilted algebras whose relation-extensions are cluster-tilted of euclidean type.

1. Introduction

Cluster-tilted algebras were introduced by Buan, Marsh and Reiten [BMR] and, independently in [CCS] for type $A$ as a byproduct of the now extensive theory of cluster algebras of Fomin and Zelevinsky [FZ]. Since then, cluster-tilted algebras have been the subject of several investigations, see, for instance, [ABCP, ABS, BFPP, BT, BOW, BMR2, KR, OS, ScSe, ScSe2].

In particular, in [ABS] is given a construction procedure for cluster-tilted algebras: let $C$ be a triangular algebra of global dimension two over an algebraically closed field $k$, and consider the $C$-$C$-bimodule $\text{Ext}^2_C(DC, C)$, where $D = \text{Hom}_k(-, k)$ is the standard duality, with its natural left and right $C$-actions. The trivial extension of $C$ by this bimodule is called the relation-extension $\tilde{C}$ of $C$. It is shown there that, if $C$ is tilted, then its relation-extension is cluster-tilted, and every cluster-tilted algebra occurs in this way.

Our purpose in this paper is to study the relation-extensions of a wider class of triangular algebras of global dimension two, namely the class of quasi-tilted algebras, introduced by Happel, Reiten and Smalø in [HRS]. In general, the relation-extension of a quasi-tilted algebra is not cluster-tilted, however it is 2-Calabi-Yau tilted, see Theorem 3.1 below. We then look more closely at those cluster-tilted algebras which are tame and representation-infinite. According to [BMR], these coincide exactly with the cluster-tilted algebras of euclidean type. We ask then the following question: Given a cluster-tilted algebra $B$ of euclidean type, find all quasi-tilted algebras $C$...
such that $B = \tilde{C}$. A similar question has been asked (and answered) in [ABS2], where, however, $C$ was assumed to be tilted.

For this purpose, we generalize the notion of reflections of [ABS4]. We prove that this operation allows to produce all tilted algebras $C$ such that $B = \tilde{C}$, see Theorem 4.11 in [ABS4] this result was shown only for cluster-tilted algebras of tree type. We also prove that, unlike those of [ABS4], reflections in the sense of the present paper are always defined, that the reflection of a tilted algebra is also tilted of the same type, and that they have the same relation-extension, see Theorem 4.4 and Proposition 4.8 below. Because all tilted algebras having a given cluster-tilted algebra as relation-extension are given by iterated reflections, this gives an algorithmic answer to our question above.

After that, we look at the tubes of a cluster-tilted algebra of euclidean type and give a procedure for constructing those tubes which contain a projective, see Proposition 5.6.

We then return to quasi-tilted algebras in our last section, namely we define a particular two-sided ideal of a cluster-tilted algebra, which we call the partition ideal. Our first result (Theorem 6.1) shows that the quasi-tilted algebras which are not tilted but have a given cluster-tilted algebra $B$ of euclidean type as relation-extension are the quotients of $B$ by a partition ideal. We end the paper with the proof of our main result (Theorem 6.3) which says that if $C$ is quasi-tilted and such that $B = \tilde{C}$, then either $C$ is the quotient of $B$ by the annihilator of a local slice (and then $C$ is tilted) or it is the quotient of $B$ by a partition ideal (and then $C$ is not tilted except in two cases easy to characterize).

2. Preliminaries

2.1. Notation. Throughout this paper, algebras are basic and connected finite dimensional algebras over a fixed algebraically closed field $k$. For an algebra $C$, we denote by $\text{mod } C$ the category of finitely generated right $C$-modules. All subcategories are full, and identified with their object classes. Given a category $\mathcal{C}$, we sometimes write $M \in \mathcal{C}$ to express that $M$ is an object in $\mathcal{C}$. If $\mathcal{C}$ is a full subcategory of $\text{mod } C$, we denote by $\text{add } \mathcal{C}$ the full subcategory of $\text{mod } C$ having as objects the finite direct sums of summands of modules in $\mathcal{C}$.

For a point $x$ in the ordinary quiver of a given algebra $C$, we denote by $P(x)$, $I(x)$, $S(x)$ respectively, the indecomposable projective, injective and simple $C$-modules corresponding to $x$. We denote by $\Gamma(\text{mod } C)$ the Auslander-Reiten quiver of $C$ and by $\tau = D\text{Tr}, \tau^{-1} = \text{Tr}D$ the Auslander-Reiten translations. For further definitions and facts, we refer the reader to [ARS, ASS, S].

2.2. Tilting. Let $Q$ be a finite connected and acyclic quiver. A module $T$ over the path algebra $kQ$ of $Q$ is called tilting if $\text{Ext}^1_{kQ}(T, T) = 0$ and the number of isoclasses (isomorphism classes) of indecomposable summands of
$T$ equals $|Q_0|$, see [ASS]. An algebra $C$ is called \textit{tilted of type} $Q$ if there exists a tilting $kQ$-module $T$ such that $C = \text{End}_{kQ} T$. It is shown in [RI] that an algebra $C$ is tilted if and only if it contains a \textit{complete slice} $\Sigma$, that is, a finite set of indecomposable modules such that

1) $\bigoplus_{U \in \Sigma} U$ is a sincere $C$-module.
2) If $U_0 \to U_1 \to \cdots \to U_t$ is a sequence of nonzero morphisms between indecomposable modules with $U_0, U_t \in \Sigma$ then $U_i \in \Sigma$ for all $i$ (\textit{convexity}).
3) If $0 \to L \to M \to N \to 0$ is an almost split sequence in $\text{mod} C$ and at least one indecomposable summand of $M$ lies in $\Sigma$, then exactly one of $L, N$ belongs to $\Sigma$.

For more on tilting and tilted algebras, we refer the reader to [ASS].

Tilting can also be done within the framework of a hereditary category. Let $\mathcal{H}$ be an abelian $k$-category which is Hom-finite, that is, such that, for all $X, Y \in \mathcal{H}$, the vector space $\text{Hom}_{\mathcal{H}}(X, Y)$ is finite dimensional. We say that $\mathcal{H}$ is \textit{hereditary} if $\text{Ext}^2_{\mathcal{H}}(-, ?) = 0$. An object $T \in \mathcal{H}$ is called a \textit{tilting object} if $\text{Ext}^1_{\mathcal{H}}(T, T) = 0$ and the number of isoclasses of indecomposable objects of $T$ is the rank of the Grothendieck group $K_0(\mathcal{H})$.

The endomorphism algebras of tilting objects in hereditary categories are called \textit{quasi-tilted algebras}. For instance, tilted algebras but also canonical algebras (see [RI]) are quasi-tilted. Quasi-tilted algebras have attracted a lot of attention and played an important role in representation theory, see for instance [HRS, Sk].

2.3. \textbf{Cluster-tilted algebras.} Let $Q$ be a finite, connected and acyclic quiver. The \textit{cluster category} $\mathcal{C}_Q$ of $Q$ is defined as follows, see [BMRRT]. Let $F$ denote the composition $\tau^{-1}_D[1]$, where $\tau^{-1}_D$ denotes the inverse Auslander-Reiten translation in the bounded derived category $D = D^b(\text{mod} kQ)$, and $[1]$ denotes the shift of $D$. Then $\mathcal{C}_Q$ is the orbit category $D/F$: its objects are the $F$-orbits $\widetilde{X} = (F^i X)_{i \in \mathbb{Z}}$ of the objects $X \in D$, and the space of morphisms from $\widetilde{X} = (F^i X)_{i \in \mathbb{Z}}$ to $\widetilde{Y} = (F^i Y)_{i \in \mathbb{Z}}$ is

$$\text{Hom}_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_D(X, F^i Y).$$

Then $\mathcal{C}_Q$ is a triangulated category with almost split triangles and, moreover, for $\widetilde{X}, \widetilde{Y} \in \mathcal{C}_Q$ we have a bifunctorial isomorphism $\text{Ext}^1_{\mathcal{C}_Q}(\widetilde{X}, \widetilde{Y}) \cong D\text{Ext}^1_{\mathcal{C}_Q}(\widetilde{Y}, \widetilde{X})$. This is expressed by saying that the category $\mathcal{C}_Q$ is $2$-\textit{Calabi-Yau}. 
An object $\tilde{T} \in C_Q$ is called tilting if $\text{Ext}^1_{C_Q}(\tilde{T}, \tilde{T}) = 0$ and the number of isoclasses of indecomposable summands of $\tilde{T}$ equals $|Q_0|$. The endomorphism algebra $B = \text{End}_{C_Q} \tilde{T}$ is then called cluster-tilted of type $Q$. More generally, the endomorphism algebra $\text{End}_{C} \tilde{T}$ of a tilting object $\tilde{T}$ in a 2-Calabi-Yau category with finite dimensional Hom-spaces is called a 2-Calabi-Yau tilted algebra, see [Re].

Let now $T$ be a tilting $kQ$-module, and $C = \text{End}_{kQ}T$ the corresponding tilted algebra. Then it is shown in [ABS] that the trivial extension $\widetilde{C}$ of $C$ by the $C$-$C$-bimodule $\text{Ext}^2_C(DC, C)$ with the two natural actions of $C$, the so-called relation-extension of $C$, is cluster-tilted. Conversely, if $B$ is cluster-tilted, then there exists a tilted algebra $C$ such that $B = \widetilde{C}$.

Let now $B$ be a cluster-tilted algebra, then a full subquiver $\Sigma$ of $\Gamma(\text{mod} B)$ is a local slice, see [ABS2], if:

1) $\Sigma$ is a presection, that is, if $X \to Y$ is an arrow then:
   (a) $X \in \Sigma$ implies that either $Y \in \Sigma$ or $\tau Y \in \Sigma$
   (b) $Y \in \Sigma$ implies that either $X \in \Sigma$ or $\tau^{-1}X \in \Sigma$.

2) $\Sigma$ is sectionally convex, that is, if $X = X_0 \to X \to \cdots \to X_t = Y$ is a sectional path in $\Gamma(\text{mod} B)$ then $X, Y \in \Sigma$ implies that $X_i \in \Sigma$ for all $i$.

3) $|\Sigma_0| = \text{rk } K_0(B)$.

Let $C$ be tilted, then, under the standard embedding $\text{mod} C \to \text{mod} \widetilde{C}$, any complete slice in the tilted algebra $C$ embeds as a local slice in $\text{mod} \widetilde{C}$, and any local slice in $\text{mod} \widetilde{C}$ occurs in this way. If $B$ is a cluster-tilted algebra, then a tilted algebra $C$ is such that $B = \widetilde{C}$ if and only if there exists a local slice $\Sigma$ in $\Gamma(\text{mod} B)$ such that $C = B/\text{Ann}_B \Sigma$, where $\text{Ann}_B \Sigma = \cap_{X \in \Sigma} \text{Ann}_B X$, see [ABS2].

Let $\Sigma$ be a local slice in the transjective component of $\Gamma(\text{mod} B)$ having the property that all the sources in $\Sigma$ are injective $B$-modules. Then $\Sigma$ is called a rightmost slice of $B$. Let $x$ be a point in the quiver of $B$ such that $I(x)$ is an injective source of the rightmost slice $\Sigma$. Then $x$ is called a strong sink. Leftmost slices and strong sources are defined dually.

### 3. From quasi-tilted to cluster-tilted algebras

We start with a motivating example. Let $C$ be the tilted algebra of type $\tilde{A}$ given by the quiver

```
1 ← β/2 ← α
\downarrow δ \downarrow γ
3 ← γ ← 4
```

An object $\tilde{T} \in C_Q$ is called tilting if $\text{Ext}^1_{C_Q}(\tilde{T}, \tilde{T}) = 0$ and the number of isoclasses of indecomposable summands of $\tilde{T}$ equals $|Q_0|$. The endomorphism algebra $B = \text{End}_{C_Q} \tilde{T}$ is then called cluster-tilted of type $Q$. More generally, the endomorphism algebra $\text{End}_{C} \tilde{T}$ of a tilting object $\tilde{T}$ in a 2-Calabi-Yau category with finite dimensional Hom-spaces is called a 2-Calabi-Yau tilted algebra, see [Re].

Let now $T$ be a tilting $kQ$-module, and $C = \text{End}_{kQ}T$ the corresponding tilted algebra. Then it is shown in [ABS] that the trivial extension $\widetilde{C}$ of $C$ by the $C$-$C$-bimodule $\text{Ext}^2_C(DC, C)$ with the two natural actions of $C$, the so-called relation-extension of $C$, is cluster-tilted. Conversely, if $B$ is cluster-tilted, then there exists a tilted algebra $C$ such that $B = \widetilde{C}$.

Let now $B$ be a cluster-tilted algebra, then a full subquiver $\Sigma$ of $\Gamma(\text{mod} B)$ is a local slice, see [ABS2], if:

1) $\Sigma$ is a presection, that is, if $X \to Y$ is an arrow then:
   (a) $X \in \Sigma$ implies that either $Y \in \Sigma$ or $\tau Y \in \Sigma$
   (b) $Y \in \Sigma$ implies that either $X \in \Sigma$ or $\tau^{-1}X \in \Sigma$.

2) $\Sigma$ is sectionally convex, that is, if $X = X_0 \to X \to \cdots \to X_t = Y$ is a sectional path in $\Gamma(\text{mod} B)$ then $X, Y \in \Sigma$ implies that $X_i \in \Sigma$ for all $i$.

3) $|\Sigma_0| = \text{rk } K_0(B)$.

Let $C$ be tilted, then, under the standard embedding $\text{mod} C \to \text{mod} \widetilde{C}$, any complete slice in the tilted algebra $C$ embeds as a local slice in $\text{mod} \widetilde{C}$, and any local slice in $\text{mod} \widetilde{C}$ occurs in this way. If $B$ is a cluster-tilted algebra, then a tilted algebra $C$ is such that $B = \widetilde{C}$ if and only if there exists a local slice $\Sigma$ in $\Gamma(\text{mod} B)$ such that $C = B/\text{Ann}_B \Sigma$, where $\text{Ann}_B \Sigma = \cap_{X \in \Sigma} \text{Ann}_B X$, see [ABS2].

Let $\Sigma$ be a local slice in the transjective component of $\Gamma(\text{mod} B)$ having the property that all the sources in $\Sigma$ are injective $B$-modules. Then $\Sigma$ is called a rightmost slice of $B$. Let $x$ be a point in the quiver of $B$ such that $I(x)$ is an injective source of the rightmost slice $\Sigma$. Then $x$ is called a strong sink. Leftmost slices and strong sources are defined dually.
bound by $\alpha \beta = 0$, $\gamma \delta = 0$. Its relation-extension is the cluster-tilted algebra $B$ given by the quiver

![Quiver](quiver.png)

bound by $\alpha \beta = 0$, $\beta \lambda = 0$, $\lambda \alpha = 0$, $\gamma \delta = 0$, $\delta \mu = 0$, $\mu \gamma = 0$. However, $B$ is also the relation-extension of the algebra $C'$ given by the quiver

![Quiver](quiver.png)

bound by $\lambda \alpha = 0$, $\delta \mu = 0$. This latter algebra $C'$ is not tilted, but it is quasi-tilted. In particular, it is triangular of global dimension two. Therefore, the question arises naturally whether the relation-extension of a quasi-tilted algebra is always cluster-tilted. This is certainly not true in general, for the relation-extension of a tubular algebra is not cluster-tilted. However, it is 2-Calabi-Yau tilted. In this section, we prove that the relation-extension of a quasi-tilted algebra is always 2-Calabi-Yau tilted.

Let $\mathcal{H}$ be a hereditary category with tilting object $T$. Because of $C$, there exist an algebra $A$, which is hereditary or canonical, and a triangle equivalence $\Phi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\text{mod } A)$. Let $T'$ denote the image of $T$ under this equivalence. Because $\Phi$ preserves the shift and the Auslander-Reiten translation, it induces an equivalence between the cluster categories $\mathcal{C}_\mathcal{H}$ and $\mathcal{C}_A$, see [Am, Section 4.1]. Indeed, because $A$ is canonical or hereditary, it follows that $\mathcal{C}_A \cong \mathcal{D}^b(\text{mod } A)/F$, where $F = \tau^{-1}[1]$. Therefore, we have $\text{End}_{\mathcal{C}_A}T \cong \text{End}_{\mathcal{C}_A}T'$.

We say that a 2-Calabi-Yau tilted algebra $\text{End}_C T$ is of canonical type if the 2-Calabi-Yau category $\mathcal{C}$ is the cluster category of a canonical algebra. The proof of the next theorem follows closely [ABS].

**Theorem 3.1.** Let $C$ be a quasi-tilted algebra. Then its relation-extension $\tilde{C}$ is cluster-tilted or it is 2-Calabi-Yau tilted of canonical type.

**Proof.** Because $C$ is quasi-tilted, there exist a hereditary category $\mathcal{H}$ and a tilting object $T$ in $\mathcal{H}$ such that $C = \text{End}_\mathcal{H} T$. As observed above, there exist an algebra $A$, which is hereditary or canonical, and a triangle equivalence $\Phi : \mathcal{D}^b(\mathcal{H}) \to \mathcal{D}^b(\text{mod } A)$. Let $T' = \Phi(T).$ We have $\mathcal{D}^b(\text{mod } \tilde{C}) \cong \mathcal{D}^b(\text{mod } A) \cong \mathcal{D}^b(\mathcal{H})$, and therefore

\[
\text{Ext}_{\tilde{C}}^2(DC, C) \cong \text{Hom}_{\mathcal{D}^b(\text{mod } C)}(\tau C[1], C[2]) \\
\cong \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(\tau T[1], T[2]) \\
\cong \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, \tau^{-1} T[1]) \\
\cong \text{Hom}_{\mathcal{D}^b(\mathcal{H})}(T, FT).
\]
Thus the additive structure of $C \times \text{Ext}^2_C(DC, C)$ is that of
\[
C \oplus \text{Ext}^2_C(DC, C) \cong \text{End}_{\mathcal{H}}(T) \oplus \text{Hom}_{D^b(\mathcal{H})}(T, FT)
\cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(\mathcal{H})}(T, FT) 
\cong \text{Hom}_{C^h}(T, T)
\cong \text{End}_{C^h}(T).
\]
Then, we check exactly as in \cite{ABS} Section 3.3 that the multiplicative structure is preserved. This completes the proof. □

Let $C$ be a representation-infinite quasi-tilted algebra. Then $C$ is derived equivalent to a hereditary or a canonical algebra $A$. Let $n_A$ denote the tubular type of $A$. We then say that $C$ has canonical type $n_C = n_A$.

\textbf{Lemma 3.2.} Let $C$ be a representation-infinite quasi-tilted. Then its relation-extension $\tilde{C}$ is cluster-tilted of euclidean type if and only if $n_C$ is one of
\((p, q), (2, 2, r), (2, 3, 3), (2, 3, 4), (2, 3, 5),\) with $p \leq q$, $2 \leq r$.

\textit{Proof.} Indeed, $\tilde{C}$ is cluster-tilted of euclidean type if and only if $C$ is derived equivalent to a tilted algebra of euclidean type, and this is the case if and only if $n_C$ belongs to the above list. □

\textit{Remark 3.3.} It is possible that $C$ is domestic, but yet $\tilde{C}$ is wild. Indeed, we modify the example after Corollary D in \cite{Sk}. Recall from \cite{Sk} that there exists a tame concealed full convex subcategory $K$ such that $C$ is a semiregular branch enlargement of $K$
\[
C = [E_i]K[F_j],
\]
where $E_i, F_j$ are (truncated) branches. Then the representation theory of $C$ is determined by those of $C^- = [E_i]K$ and $C^+ = K[F_j]$. Let $C$ be given by the quiver
\[
\begin{array}{c}
1 \\
\downarrow \beta \\
2 \\
\end{array}
\begin{array}{c}
3 \\
\downarrow \gamma \\
4 \\
\downarrow \nu \\
8 \\
\downarrow \varphi \\
9 \\
\end{array}
\begin{array}{c}
6 \\
\downarrow \delta \\
5 \\
\downarrow \rho \\
7 \\
\end{array}
\begin{array}{c}
10 \\
\end{array}
\end{array}
\]
bound by the relations $\sigma \nu = 0$, $\omega \varphi = 0$, $\zeta \delta \sigma \gamma \beta = 0$. Here $C^-$ is the full subcategory generated by $C_0 \setminus \{11\}$ and $C^+$ the one generated by $C_0 \setminus \{8, 9, 10\}$. Then $C^-$ has domestic tubular type $(2, 2, 7)$ and $C^+$ has domestic tubular type $(2, 3, 4)$. Therefore $C$ is domestic. On the other hand, the canonical type of $C$ is $(2, 3, 7)$, which is wild. In this example, the 2-Calabi-Yau tilted algebra $\tilde{C}$ is not cluster-tilted, because it is not of euclidean type, but the derived category of mod $C$ contains tubes, see \cite{R}.
Remark 3.4. There clearly exist algebras which are not quasi-tilted but whose relation-extension is cluster-tilted of euclidean type. Indeed, let $C$ be given by the quiver

$$
\begin{array}{ccccccc}
6 & \xrightarrow{\alpha} & 5 & \xrightarrow{\beta} & 4 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 2 \\
\end{array}
$$

bound by $\alpha \beta = 0, \delta \lambda = 0$. Then $C$ is iterated tilted of type $\tilde{A}$ of global dimension 2, see [FPT]. Its relation-extension is given by

$$
\begin{array}{ccccccc}
6 & \xrightarrow{\alpha} & 5 & \xrightarrow{\beta} & 4 & \xrightarrow{\gamma} & 3 & \xrightarrow{\delta} & 2 \\
\end{array}
\begin{array}{ccccccc}
\xleftarrow{\sigma} & \xleftarrow{\eta} & \xleftarrow{\lambda} & \xleftarrow{\mu} & 1 \\
\end{array}
$$

bound by $\alpha \beta = 0, \beta \sigma = 0, \sigma \alpha = 0, \delta \lambda = 0, \lambda \eta = 0, \eta \delta = 0$. This algebra is isomorphic to the relation-extension of the tilted algebra of type $\tilde{A}$ given by the quiver

$$
\begin{array}{ccccccc}
6 & \xleftarrow{\sigma} & 4 & \xleftarrow{\beta} & 5 \\
\end{array}
\begin{array}{ccccccc}
\xleftarrow{\gamma} & \xleftarrow{\delta} & \xleftarrow{\lambda} & \xleftarrow{\mu} & 1 \\
\end{array}
$$

bound by $\beta \sigma = 0, \delta \lambda = 0$. Therefore $\tilde{C}$ is cluster-tilted of euclidean type. On the other hand, $C$ is not quasi-tilted, because the uniserial module $\frac{4}{3}$ has both projective and injective dimension 2.

4. Reflections

Let $C$ be a tilted algebra. Let $\Sigma$ be a rightmost slice, and let $I(x)$ be an injective source of $\Sigma$. Thus $x$ is a strong sink in $C$.

Definition 4.1. We define the completion $H_x$ of $x$ by the following three conditions.

(a) $I(x) \in H_x$.

(b) $H_x$ is closed under predecessors in $\Sigma$.

(c) If $L \rightarrow M$ is an arrow in $\Sigma$ with $L \in H_x$ having an injective successor in $H_x$ then $M \in H_x$.

Observe that $H_x$ may be constructed inductively in the following way. We let $H_1 = I(x)$, and $H_2'$ be the closure of $H_1$ with respect to (c) (that is, we simply add the direct successors of $I(x)$ in $\Sigma$, and if a direct successor of $I(x)$ is injective, we also take its direct successor, etc.) We then let $H_2$ be the closure of $H_2'$ with respect to predecessors in $\Sigma$. Then we repeat the procedure: given $H_i$, we let $H_{i+1}'$ be the closure of $H_i$ with respect to (c) and $H_{i+1}$ be the closure of $H_{i+1}'$ with respect to predecessors. This procedure
must stabilize, because the slice $\Sigma$ is finite. If $H_j = H_k$ with $k > j$, we let $H_x = H_j$.

We can decompose $H_x$ as the disjoint union of three sets as follows. Let $\mathcal{J}$ denote the set of injectives in $H_x$, let $\mathcal{J}^-$ be the set of non-injectives in $H_x$ which have an injective successor in $H_x$, and let $\mathcal{E} = H_x \setminus (\mathcal{J} \cup \mathcal{J}^-)$ denote the complement of $(\mathcal{J} \cup \mathcal{J}^-)$ in $H_x$. Thus

$$H_x = \mathcal{J} \cup \mathcal{J}^- \cup \mathcal{E}$$

is a disjoint union.

**Remark 4.2.** If $\mathcal{J}^- = \emptyset$ then $H_x$ reduces to the completion $G_x$ as defined in [ABS4]. Recall that $G_x$ does not always exist, but, as seen above, $H_x$ does. Conversely, if $G_x$ exists, then it follows from its construction in [ABS4] that $\mathcal{J}^- = \emptyset$.

Thus $\mathcal{J}^- = \emptyset$ if and only if $G_x$ exists, and, in this case $G_x = H_x$.

For every module $M$ over a cluster-tilted algebra $B$, we can consider a lift $\tilde{M}$ in the cluster category $\mathcal{C}$. Abusing notation, we sometimes write $\tau M$ to denote the image of $\tau M \tilde{M}$ in mod $B$, and say that the Auslander-Reiten translation is computed in the cluster category.

**Definition 4.3.** Let $x$ be a strong sink in $C$ and let $\Sigma$ be a rightmost local slice with injective source $I(x)$. Recall that $\Sigma$ is also a local slice in mod $B$. Then the reflection of the slice $\Sigma$ in $x$ is

$$\sigma_x^+ \Sigma = \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1} \mathcal{E} \cup (\Sigma \setminus H_x),$$

where $\tau$ is computed in the cluster category. In a similar way, one defines the coreflection $\sigma_y^-$ of leftmost slices with projective sink $P_C(y)$.

**Theorem 4.4.** Let $x$ be a strong sink in $C$ and let $\Sigma$ be a rightmost local slice in mod $B$ with injective source $I(x)$. Then the reflection $\sigma_x^+ \Sigma$ is a local slice as well.

**Proof.** Set $\Sigma' = \sigma_x^+ \Sigma$ and

$$\Sigma'' = \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-) \cup \tau^{-1} \mathcal{E} \cup (\Sigma \setminus H_x) = \tau^{-1}H_x \cup (\Sigma \setminus H_x),$$

where again, $\Sigma''$ and $\tau$ are computed in the cluster category $\mathcal{C}$. We claim that $\Sigma''$ is a local slice in $\mathcal{C}$. Notice that since $H_x$ is closed under predecessors in $\Sigma$, then, if $X \in \Sigma \setminus H_x$ is a neighbor of $Y \in H_x$, we must have an arrow $Y \to X$ in $\Sigma$. This observation being made, $\Sigma''$ is clearly obtained from $\Sigma$ by applying a sequence of APR-tilts. Thus $\Sigma''$ is a local slice in $\mathcal{C}$.

We now claim that $\tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$ is closed under predecessors in $\Sigma''$. Indeed, let $X \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$ and $Y \in \Sigma''$ be such that we have an arrow $Y \to X$. Then, there exists an arrow $\tau X \to Y$ in the cluster category. Because $X \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)$, we have $\tau X \in \mathcal{J} \cup \mathcal{J}^-$. Now if $Y \in \Sigma$, then the arrow $\tau X \to Y$ would imply that $Y \in H_x$, which is impossible, because $Y \in \Sigma''$ and $\Sigma'' \cap H_x = \emptyset$. Thus $Y \not\in \Sigma$, and therefore $Y \in (\Sigma'' \setminus \Sigma) = \tau^{-1}H_x$. Hence $\tau Y \in H_x$. Moreover, there is an arrow $\tau Y \to \tau X$. Using that
\( \tau X \in \mathcal{J} \cup \mathcal{J}^- \), this implies that \( \tau Y \) has an injective successor in \( H_x \) and thus \( Y \in \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-) \). This establishes our claim that \( \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-) \) is closed under predecessors in \( \Sigma'' \).

Thus applying the same reasoning as before, we get that

\[
\Sigma' = (\Sigma'' \setminus \tau^{-1}(\mathcal{J} \cup \mathcal{J}^-)) \cup \tau^{-2}(\mathcal{J} \cup \mathcal{J}^-)
\]

is a local slice in \( \mathcal{C} \). Now we claim that \( \Sigma' \cap \text{add}(\tau T) = \emptyset \).

First, because \( \Sigma \cap \text{add}(\tau T) = \emptyset \), we have \( (\Sigma \setminus H_x) \cap \text{add}(\tau T) = \emptyset \). Next, \( \mathcal{E} \) contains no injectives, by definition. Thus \( \tau^{-1}\mathcal{E} \cap \text{add}(\tau T) = \emptyset \). Assume now that \( X \in \text{add}(\tau T) \) belongs to \( \tau^{-2}\mathcal{J}^- \). Then \( \tau^{-2}X \in H_x \) and there exists an injective predecessor \( I(j) \) of \( \tau^{-2}X \) in \( H_x \), and since \( H_x \) is part of the local slice \( \Sigma \), there exists a sectional path from \( I(j) \) to \( \tau^{-2}X \). Applying \( \tau^{-2} \), we get a sectional path from \( T_j \) to \( X \) in the cluster category. But this means \( \text{Hom}_C(T_j, X) \neq 0 \), which is a contradiction to the hypothesis that \( X \in \text{add}(\tau T) \). Finally, if \( X \in \tau^{-2}\mathcal{J} \) then \( X \) is a summand of \( T \), which, again, is contradicting the hypothesis that \( X \in \text{add}(\tau T) \). \( \square \)

Following \[ABS4\], let \( S_x \) be the full subcategory of \( C \) consisting of those \( y \) such that \( I(y) \in H_x \).

**Lemma 4.5.**

(a) \( S_x \) is hereditary.

(b) \( S_x \) is closed under successors in \( C \).

(c) \( C \) can be written in the form

\[
C = \begin{bmatrix} H & 0 \\ M & C' \end{bmatrix},
\]

where \( H \) is hereditary, \( C' \) is tilted and \( M \) is a \( C' \)-\( H \)-bimodule.

**Proof.**

(a) Let \( H = \text{End}(\oplus_{y \in S_x} I(y)) \). Then \( H \) is a full subcategory of the hereditary endomorphism algebra of \( \Sigma \). Therefore \( H \) is also hereditary, and so \( S_x \) is hereditary.

(b) Let \( y \in S_x \) and \( y \to z \) in \( C \). Then there exists a morphism \( I(z) \to I(y) \). Because \( I(z) \) is an injective \( C \)-module and \( \Sigma \) is sincere, there exist a module \( N \in \Sigma \) and a non-zero morphism \( N \to I(z) \). Then we have a path \( N \to I(z) \to I(y) \), and since \( N, I(y) \in \Sigma \), we get that \( I(z) \in \Sigma \) by convexity of the slice \( \Sigma \) in \( \text{mod} \ C \). Moreover, since \( I(y) \in H_x \) and \( H_x \) is closed under predecessors in \( \Sigma \), it follows that \( I(z) \in H_x \). Thus \( z \in S_x \) and this shows (b).

(c) This follows from (a) and (b). \( \square \)

We recall that the cluster duplicated algebra was introduced in \[ABS3\].
Corollary 4.6. The cluster duplicated algebra \( \overline{C} \) of \( C \) is of the form

\[
\overline{C} = \begin{bmatrix}
H & 0 & 0 & 0 \\
M & C' & 0 & 0 \\
0 & E_0 & H & 0 \\
0 & E_1 & M & C'
\end{bmatrix}
\]

where \( E_0 = \text{Ext}^2_C(\text{DC}', H) \) and \( E_1 = \text{Ext}^2_C(\text{DC}', C') \).

Proof. We start by writing \( C \) in the matrix form of the lemma. By definition, \( H \) consists of those \( y \in C_0 \) such that the corresponding injective \( I(y) \) lies in \( H_x \) inside the slice \( \Sigma \). In particular, the projective dimension of these injectives is at most 1, hence \( \text{Ext}_C^2(DC, C) = \text{Ext}_C^2(DC', C) \). The result now follows upon multiplying by idempotents. \( \square \)

Definition 4.7. Let \( x \) be a strong sink in \( C \). The reflection at \( x \) of the algebra \( C \) is

\[
\sigma_x^+ C = \begin{bmatrix}
C' & 0 \\
E_0 & H
\end{bmatrix}
\]

where \( E_0 = \text{Ext}^2_C(\text{DC}', H) \).

Proposition 4.8. The reflection \( \sigma_x^+ C \) of \( C \) is a tilted algebra having \( \sigma_x^+ \Sigma \) as a complete slice. Moreover the relation-extensions of \( C \) and \( \sigma_x^+ \Sigma \) are isomorphic.

Proof. We first claim that the support \( \text{supp}(\sigma_x^+ \Sigma) \) of \( \sigma_x^+ \Sigma \) is contained in \( \sigma_x^+ C \). Let \( X \in \sigma_x^+ \Sigma \). Recall that \( \sigma_x^+ \Sigma = \tau^{-2}(J \cup J^-) \cup \tau^{-1} \mathcal{E} \cup (\Sigma \setminus H_x) \). If \( X \in \tau^{-2} \mathcal{J} \), then \( X = P(y') \) is projective corresponding to a point \( y' \in H \). Thus \( I(y') \in H_x \) and the radical of \( P(y) \) has no non-zero morphism into \( I(y) \). Therefore \( \text{supp}(X) \subseteq \sigma_x^+ C \).

Assume next that \( X \in \tau^{-2} \mathcal{J} \), that is, \( X = \tau^{-2} Y \), where \( Y \in J^- \) has an injective successor \( I(z) \) in \( H_x \). Because all sources in \( \Sigma \) are injective, there is an injective \( I(y') \in \Sigma \) and a sectional path \( I(y') \to \ldots \to Y \to \ldots \to I(z) \). Applying \( \tau^{-2} \), we obtain a sectional path \( P(y') \to \ldots \to X \to \ldots \to P(z) \). In particular the point \( y' \) belongs to the support of \( X \). Assume that there is a point \( h \) in \( H \) that is in the support of \( X \). Then there exists a nonzero morphism \( X \to I(h) \). But \( I(h) \in \Sigma \) and there is no morphism from \( X \in \tau^{-2} \Sigma \) to \( \Sigma \). Therefore \( \text{supp}(X) \subseteq \sigma_x^+ C \).

By the same argument, we show that if \( X \in \tau^{-1} \mathcal{E} \), then \( \text{supp}(X) \subseteq \sigma_x^+ C \).

Finally, all modules of \( \Sigma \setminus H_x \) are supported in \( C' \). This establishes our claim.

Now, by Theorem 4.4, \( \sigma_x^+ \Sigma \) is a local slice in mod \( \overline{C} \). Therefore \( \overline{C} / \text{Ann} \sigma_x^+ \Sigma \) is a tilted algebra in which \( \sigma_x^+ \Sigma \) is a complete slice. Since the support of \( \sigma_x^+ \Sigma \) is the same as the support of \( \sigma_x^+ C \), we are done. \( \square \)

We now come to the main result of this section, which states that any two tilted algebras that have the same relation-extension are linked to each other by a sequence of reflections and coreflections.
Definition 4.9. Let $\mathcal{B}$ be a cluster-tilted algebra and let $\Sigma$ and $\Sigma'$ be two local slices in $\text{mod} \, \mathcal{B}$. We write $\Sigma \sim \Sigma'$ whenever $\mathcal{B}/\text{Ann} \, \Sigma = \mathcal{B}/\text{Ann} \, \Sigma'$.

Lemma 4.10. Let $\mathcal{B}$ be a cluster-tilted algebra, and $\Sigma_1, \Sigma_2 \in \text{mod} \, \mathcal{B}$. Then there exists a sequence of reflections and coreflections $\sigma$ such that

$$\sigma \Sigma_1 \sim \Sigma_2.$$  

Proof. Given a local slice $\Sigma$ in $\text{mod} \, \mathcal{B}$ such that $\Sigma$ has injective successors in the transjective component $\mathcal{T}$ of $\Gamma(\text{mod} \, \mathcal{B})$, let $\Sigma^+$ be the rightmost local slice such that $\Sigma \sim \Sigma^+$. Then $\Sigma^+$ contains a strong sink $x$, thus reflecting in $x$ we obtain a local slice $\sigma^+_x \Sigma^+$ that has fewer injective successors in $\mathcal{T}$ than $\Sigma$. To simplify the notation we define $\sigma^+_y \Sigma = \sigma^+_y \Sigma^+$, where $\Sigma^-$ is the leftmost local slice containing a strong source $y$ and $\Sigma \sim \Sigma^-$.

Since we can always reflect in a strong sink, there exist sequences of reflections such that

$$\begin{align*}
\sigma^+_x \cdots \sigma^+_x \sigma^+_y \Sigma_1 &= \Sigma_1^1 \\
\sigma^+_y \cdots \sigma^+_y \sigma^+_x \Sigma_2 &= \Sigma_2^1 
\end{align*}$$

and $\Sigma_1^1, \Sigma_2^1$ have no injective successors in $\mathcal{T}$. This implies that $\Sigma_1^1 \sim \Sigma_2^1$. Let

$$\sigma = \sigma^-_{y_1} \sigma^-_{y_2} \cdots \sigma^-_{y_s} \sigma^+_{x_r} \cdots \sigma^+_{x_2} \sigma^+_{x_1}$$

thus $\sigma \Sigma_1 \sim \Sigma_2$. □

Theorem 4.11. Let $C_1$ and $C_2$ be two tilted algebras that have the same relation-extension. Then there exists a sequence of reflections and coreflections $\sigma$ such that $\sigma C_1 \cong C_2$.

Proof. Let $\mathcal{B}$ be the common relation-extension of the tilted algebras $C_1$ and $C_2$. By [ABS2], there exist local slices $\Sigma_i$ in $\text{mod} \, \mathcal{B}$ such that $C_i = \mathcal{B}/\text{Ann} \, \Sigma_i$, for $i = 1, 2$. Now the result follows from Lemma 4.10 and Theorem [4.4]. □

Example 4.12. Let $A$ be the path algebra of the quiver

$$\begin{align*}
&1 \\
3 &\longrightarrow 2 \\
&4 \\
&5 \\
&6 \\
&\downarrow \\
&\begin{array}{c}
\begin{array}{c}
\text{Mutating at the vertices 4, 5, and 2 yields the cluster-tilted algebra } \mathcal{B} \text{ with quiver}
\end{array}
\end{array}
\end{align*}$$

$$\begin{align*}
\end{align*}$$
In the Auslander-Reiten quiver of \( \text{mod} B \) we have the following local configuration.

\[
\begin{array}{c}
I(1) \rightarrow I(3) \rightarrow I(6) \leftarrow \cdots \leftarrow \sigma_6^+ \Sigma \\
1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6
\end{array}
\]

where

\[
I(1) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad I(3) = \begin{pmatrix} 2 \\ 11 \\ 44 \\ 44 \\ 44 \end{pmatrix}, \quad I(6) = \begin{pmatrix} 555 \\ 44 \\ 555 \end{pmatrix}
\]

The 6 modules on the left form a rightmost local slice \( \Sigma \) in which both \( I(3) \) and \( I(6) \) are sources, so 3 and 6 are strong sinks. For both strong sinks the subset \( J^- \) of the completion consists of the simple module 1. The simple module 2 = \( \tau^{-1} \) does not lie on a local slice.

The completion \( H_6 \) is the whole local slice \( \Sigma \) and therefore the reflection \( \sigma_6^+ \Sigma \) is the local slice consisting of the 6 modules on the right containing both \( P(1) \) and \( P(6) \).

On the other hand, the completion \( H_3 \) consists of the four modules \( I(3), S(1), I(1) \) and \( 5555_{4444} \), and therefore the reflection \( \Sigma' = \sigma_3^+ \Sigma \) is the local slice consisting of the 6 modules on the straight line from \( I(6) \) to \( P(1) \). This local slice admits the strong sink 6 and the completion \( H_6' \) in \( \Sigma' \) consists of the two modules \( I(6) \) and \( 555_{44} \). Therefore the reflection \( \sigma_6^+ \Sigma' \) is equal to \( \sigma_6^+ \Sigma \). Thus

\[
\sigma_6^+ \Sigma = \sigma_6^+ ( \sigma_3^+ \Sigma ).
\]

This example raises the question which indecomposable modules over a cluster-tilted algebra do not lie on a local slice. We answer this question in a forthcoming publication [AsScSe].
5. Tubes

The objective of this section is to show how to construct those tubes of a tame cluster-tilted algebra which contain projectives. Let $B$ be a cluster-tilted algebra of euclidean type, and let $T$ be a tube in $\Gamma(\text{mod } B)$ containing at least one projective. First, consider the transjective component of $\Gamma(\text{mod } B)$. Denote by $\Sigma_L$ a local slice in the transjective component that precedes all indecomposable injective $B$-modules lying in the transjective component. Then $B/\text{Ann}_B\Sigma_L = C_1$ is a tilted algebra having a complete slice in the preinjective component. Define $\Sigma_R$ to be a local slice which is a successor of all indecomposable projectives lying in the transjective component. Then $B/\text{Ann}_B\Sigma_R = C_2$ is a tilted algebra having a complete slice in the postprojective component. Also, $C_1$ (respectively, $C_2$) has a tube $T_1$ (respectively, $T_2$) containing the indecomposable projective $B$-modules (respectively, injective $C_2$-modules) corresponding to the projective $B$-modules in $T$ (respectively, injective $B$-modules in $T$).

An indecomposable projective $P(x)$ (respectively, injective $I(x)$) $B$-module that lies in a tube, is said to be a root projective (respectively, a root injective) if there exists an arrow in $B$ between $x$ and $y$, where the corresponding indecomposable projective $P(y)$ lies in the transjective component of $\Gamma(\text{mod } B)$.

Let $S_1$ be the coray in $T_1$ passing through the projective $C_1$-module that corresponds to the root projective $P_B(i)$ in $T$. Similarly, let $S_2$ be the ray in $T_2$ passing through the injective that corresponds to the root injective $I_B(i)$ in $T$.

Recall that if $A$ is hereditary and $T \in \text{mod } A$ is a tilting module, then there exists an associated torsion pair $(\mathcal{T}(T), \mathcal{F}(T))$ in $\text{mod } A$, where

\[
\mathcal{T}(T) = \{ M \in \text{mod } A \mid \text{Ext}_A^1(T, M) = 0 \}
\]

\[
\mathcal{F}(T) = \{ M \in \text{mod } A \mid \text{Hom}_A(T, M) = 0 \}.
\]

Lemma 5.1. With the above notation
(a) $S_1 \otimes_{C_1} B$ is a coray in $T$ passing through $P_B(i)$.
(b) $\text{Hom}_{C_2}(B, S_2)$ is a ray in $T$ passing through $I_B(i)$.

Proof. Since $C_1$ is tilted, we have $C_1 = \text{End}_A T$ where $T$ is a tilting module over a hereditary algebra $A$. As seen in the proof of Theorem 5.1 in [SeSe], we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{F}(T) & \xrightarrow{\text{Hom}_A(T, -)} & \mathcal{Y}(T) \\
\downarrow & & \downarrow \otimes_{C_1} B \\
\mathcal{C}_A & \xrightarrow{\text{Hom}_{C_2}(T, -)} & \text{mod } B
\end{array}
\]

where $\mathcal{Y}(T) = \{ N \in \text{mod } C \mid \text{Tor}_1^C(N, T) = 0 \}$. 
Let $\mathcal{T}_A$ be the tube in $\text{mod } A$ corresponding to the tube $\mathcal{T}$ in $\text{mod } B$. By what has been seen above, we have a commutative diagram

$$
\begin{array}{ccc}
\mathcal{T}_A \cap \mathcal{F}(T) & \xrightarrow{\text{Hom}_{A}(T,-)} & \mathcal{T}_1 \\
\downarrow \text{Hom}_{C_A}(T,-) & & \downarrow - \otimes_{C_1} B \\
\mathcal{T}_1 \otimes_{C_1} B & \subset & \mathcal{T}
\end{array}
$$

Let $S$ be any coray in $\mathcal{T}_1$, so it can be lifted to a coray $S_A$ in $\mathcal{T}_A \cap \mathcal{F}(T)$ via the functor $\text{Hom}_{A}(T,-)$. If we apply $\text{Hom}_{C_A}(T,-)$ to this lift, we obtain a coray in $\mathcal{T}_1 \otimes_{C_1} B$. Thus, any coray in $\mathcal{T}_1$ induces a coray in $\mathcal{T}$. Let $S_1$ be the coray passing through the root projective $P_{C_1}(i)$. Then $S_1 \otimes_{C_1} B$ is the coray passing through $P_{C_1}(i) \otimes_{C_1} B = P_B(i)$. This proves (a) and part (b) is proved dually.

However, we must still justify that the ray $S_1 \otimes_{C_1} B$ and the coray $\text{Hom}_{C_2}(B,S_2)$ actually intersect (and thus lie in the same tube of $\Gamma(\text{mod } B)$). Because $P_{C_1}(i) \in S_1$, we have $P_{C_1}(i) \otimes B \cong P_B(i) \in S_1 \otimes_{C_1} B$, and $P_B(i)$ lies in a tube $\mathcal{T}$. It is well-known that the injective $I_B(i)$ also lies in $\mathcal{T}$. In particular, we have the following local configuration in $\mathcal{T}$, where $R$ is an indecomposable summand of the radical of $P_B(i)$ and $J$ an indecomposable summand of the quotient of $I_B(i)$ by its socle.

$$
\begin{array}{ccc}
I_B(i) & \xrightarrow{\circ} & P_B(i) \\
\downarrow J & & \downarrow R \\
N & & N \\
\end{array}
$$

Now $I_B(i) = \text{Hom}_{C_2}(B,I_C(i))$ is coinduced, and we have shown above that the ray containing it is also coinduced. Because $I_C(i) \in S_2$, this is the ray $\text{Hom}_{C_2}(B,S_2)$. Therefore, this ray and this coray lie in the same tube, so must intersect in a module $N$, where there exists an almost split sequence

$$0 \longrightarrow J \longrightarrow N \longrightarrow R \longrightarrow 0.
$$

\begin{remark}
Knowing the ray $\text{Hom}_{C_2}(B,S_2)$ and the coray $S_1 \otimes_{C_1} B$ for every root projective $P_B(i)$ in $\mathcal{T}$, one may apply the knitting procedure to construct the whole of $\mathcal{T}$. In this way, $\mathcal{T}$ can be determined completely.
\end{remark}

Next we show that all modules over a tilted algebra lying on the same coray change in the same way under the induction functor.

\begin{lemma}
Let $A$ be a hereditary algebra of euclidean type, $T$ a tilting $A$-module without preinjective summands and let $C = \text{End}_{A} T$ be the corresponding tilted algebra. Let $\mathcal{T}_A$ be a tube in $\text{mod } A$ and $T_i \in \mathcal{T}_A$ an indecomposable summand of $T$, such that $\text{pd } I_C(i) = 2$.
\end{lemma}
Then there exists an $A$-module $M$ on the mouth of $\mathcal{T}_A$ such that we have
\[ \tau_C \Omega_C I_C(i) = \text{Hom}_A(T, M) \]
in mod $C$. In particular, the module $\tau_C \Omega_C I_C(i)$ lies on the mouth of the tube $\text{Hom}_A(T, T \cap \mathcal{T}(T))$ in mod $C$.

Proof. The injective $C$-module $I_C(i)$ is given by
\[ I_C(i) \cong \text{Ext}_A^1(T, \tau T_i) \cong D\text{Hom}_A(T, T), \]
where the first identity holds by [ASS Proposition VI 5.8] and the second identity is the Auslander-Reiten formula. Moreover, since $T_i$ lies in the tube $\mathcal{T}_A$ and $T$ has no preinjective summands, we have $\text{Hom}(T_i, T_j) \neq 0$ only if $T_j$ lies in the hammock starting at $T_i$. Furthermore, if $T_j$ is a summand of $T$ then it must lie on a sectional path starting from $T_i$ because $\text{Ext}_A^1(T_j, T_i) = 0$. This shows that a point $j$ is in the support of $I_C(i)$ if and only if there is a sectional path $T_i \rightarrow \cdots \rightarrow T_j$ in $\mathcal{T}_A$. We shall distinguish two cases.

Case 1. If $T_i$ lies on the mouth of $\mathcal{T}_A$ then let $\omega$ be the ray starting at $T_i$ and denote by $T_1$ the last summand of $T$ on this ray. Let $L_1$ be the direct predecessor of $T_1$ not on the ray $\omega$. Thus we have the following local configuration in $\mathcal{T}_A$.

\[ \tau T_i \rightarrow T_i \rightarrow \tau T_1 \rightarrow T_1 \rightarrow \tau L_1 \rightarrow L_1 \rightarrow L_1 \rightarrow \tau^{-1} L_1 \rightarrow E_1 \]

Then $I_C(i)$ is uniserial with simple top $S(1)$. Moreover there is a short exact sequence
\[ 0 \rightarrow \tau T_i \rightarrow L_1 \rightarrow T_1 \rightarrow 0 \]
and applying $\text{Hom}_A(T, -)$ yields
\[ 0 \rightarrow \text{Hom}_A(T, L_1) \rightarrow P_C(1) \xrightarrow{f} I_C(i) \rightarrow \text{Ext}_A^1(T, L_1) \rightarrow 0 \]

By the Auslander-Reiten formula, we have $\text{Ext}_A^1(T, L_1) \cong D\text{Hom}(\tau^{-1} L_1, T)$ and this is zero because $T_1$ is the last summand of $T$ on the ray $\omega$. Thus the
sequence (5.1) is short exact, the morphism \( f \) is a projective cover, because \( IC(i) \) is uniserial, and hence

\[
\Omega C IC(i) \cong \text{Hom}_A(T, L_1).
\]

Applying \( \tau_C \) yields

\[
\tau_C \Omega C IC(i) \cong \tau_C \text{Hom}_A(T, L_1).
\]

Let \( E_1 \) be the indecomposable direct predecessor of \( L_1 \) such that the almost split sequence ending at \( L_1 \) is of the form

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \tau L_1 & \rightarrow & E_1 \oplus \tau T_1 & \rightarrow & L_1 & \rightarrow & 0 \\
& & \downarrow \tau T_1 & & \downarrow & & \downarrow \tau L_1 & & \\
& & \tau L_1 & & E_1 & & \tau^{-1} L_1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \tau^{-1} E_1 & & \tau^{-1} E_1 & & \tau^{-1} E_1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & N & & T^1 & & \tau \omega & & \\
& & \downarrow & & \rightarrow & & \rightarrow & & \\
& & T^1 & & \rightarrow & & \rightarrow & &
\end{array}
\]

Then we have a short exact sequence

\[
0 \rightarrow L_1 \xrightarrow{h} T_1 \oplus T^1 \rightarrow N \rightarrow 0
\]

with \( h \) an add-\( T \)-approximation. Applying \( \text{Hom}_A(\cdot, T) \) yields

\[
0 \rightarrow \text{Hom}_A(N, T) \rightarrow \text{Hom}_A(T_1 \oplus T^1, T) \xrightarrow{h^*} \text{Hom}_A(L_1, T) \rightarrow \text{Ext}^1_A(N, T) \rightarrow 0
\]
and since \( h \) is an add \( T \)-approximation, the morphism \( h^* \) is surjective. Thus \( \Ext^1_A(N, T) = 0 \).

On the other hand, \( T_1 \oplus T^1 \) generates \( N \), so \( N \in \Gen T = \mathcal{F}(T) \), and thus \( \Ext^1_A(T, N) = 0 \). But then both \( \Ext^1_A(T, N) = \Ext^1_A(N, T) = 0 \) and we see that \( N \) is a summand of \( T \). This is a contradiction to the assumption that \( T_1 \) is the last summand of \( T \) on the ray \( \omega \). Thus \( E_1 \in \mathcal{F}(T) \).

Therefore, in the almost split sequence \((5.2)\), we have \( L_1, E_1 \in \mathcal{F}(T) \) and \( \tau T_1 \in \mathcal{F}(T) \). Moreover, all predecessors of \( \tau T_1 \) on the ray \( \tau \omega \) are also in \( \mathcal{F}(T) \) because the morphisms on the ray are injective. Since \( \Hom_A(T, -) : \mathcal{F}(T) \to \mathcal{Y}(T) \) is an equivalence of categories, it follows that \( \Hom_A(T, L_1) \) has only one direct predecessor

\[
\Hom_A(T, E_1) \to \Hom_A(T, L_1)
\]

in \( \mod C \) and this irreducible morphism is surjective. The kernel of this morphism is \( \Hom_A(T, t(\tau A L_1)) \) where \( t \) is the torsion radical. Thus we get

\[
\tau_C \Omega_C I_C(i) = \tau_C \Hom_A(T, L_1) = \Hom_A(T, t(\tau A L_1)).
\]

We will show that \( t(\tau A L_1) \) lies on the mouth of \( \mathcal{T}_A \) and this will complete the proof in case 1.

Let \( M \) be the indecomposable \( A \)-module on the mouth of \( \mathcal{T}_A \) such that the ray starting at \( M \) passes through \( \tau A L_1 \). Thus \( M \) is the starting point of the ray \( \tau^2 \omega \). Then there is a short exact sequence of the form

\[
(5.3) \quad 0 \longrightarrow M \longrightarrow \tau A L_1 \longrightarrow \tau A T_1 \longrightarrow 0
\]

with \( \tau A T_1 \in \mathcal{F}(T) \). We claim that \( M \in \mathcal{F}(T) \).

Suppose to the contrary that \( 0 \neq \Ext^1_A(T, M) = D \Hom_A(\tau^{-1} M, T) \). Since \( \tau^{-1} M \) lies on the mouth of \( \mathcal{T}_A \), this implies that there is a direct summand \( T^1 \) of \( T \) which lies on the ray \( \tau \omega \) starting at \( \tau^{-1} M \). Since \( T \) is tilting, \( T^1 \) cannot be a predecessor of \( \tau T_1 \) on this ray and since \( L_1 \) is not a summand of \( T^1 \), we also have \( L_1 \neq T^1 \). Thus \( T^1 \) is a successor of \( L_1 \) on the ray \( \tau \omega \). This is impossible since such a \( T^1 \) would satisfy \( \Ext^1_A(T^1, E_1) \neq 0 \) contradicting the fact that \( E_1 \in \mathcal{F}(T) \).

Therefore, \( M \in \mathcal{F}(T) \) and the sequence \((5.3)\) is the canonical sequence for \( \tau A L_1 \) in the torsion pair \((\mathcal{F}(T), \mathcal{F}(T))\). This shows that \( t(\tau A L_1) = M \) and hence \( \tau_C \Omega_C I_C(i) = \Hom_A(T, M) \) as desired.

Case 2. Now suppose that \( T_1 \) does not lie on the mouth of \( \mathcal{T}_A \). Let \( \omega_1 \) denote the ray passing through \( T_1 \) and \( \omega_2 \) the coray passing through \( T_1 \). Denote by \( T_1 \) the last summand of \( T \) on \( \omega_1 \), by \( T_2 \) the last summand of \( T \) on \( \omega_2 \), and by \( L_j \) the direct predecessor of \( T_j \) which does not lie on \( \omega_j \). Note that \( L_2 \) does not exist if \( T_2 \) lies on the mouth of \( \mathcal{T}_A \), and in this case we let \( L_2 = 0 \). Thus we have the following local configuration in \( \mathcal{T}_A \).
The injective $C$-module $I_C(i) = \Ext^1_A(T, \tau T_i)$ is biserial with top $S(1) \oplus S(2)$. Moreover, there is a short exact sequence
\[ 0 \to \tau T_i \to L_1 \oplus L_2 \oplus T_i \to T_1 \oplus T_2 \to 0. \]
Applying $\Hom_A(T, -)$ yields the following exact sequence.

\[ \begin{array}{ccccccccc}
0 & \to & \Hom_A(T, L_1 \oplus L_2) \oplus P_C(i) & \to & P_C(1) \oplus P_C(2) & \overset{f}{\to} & I_C(i) & \to & 0.
\end{array} \]

(5.4)

By the same argument as in case 1, using that $T_1$ and $T_2$ are the last summands of $T$ on $\omega_1$ and $\omega_2$ respectively, we see that $\Ext^1_A(T, L_1 \oplus L_2) = 0$. Therefore, the sequence (5.4) is short exact. Moreover, the morphism $f$ is a projective cover and thus
\[ \Omega_C I_C(i) = \Hom_A(T, L_1 \oplus L_2) \oplus P_C(i). \]
Applying $\tau_C$ yields
\[ \tau_C \Omega_C I_C(i) = \tau_C \Hom_A(T, L_1) \oplus \tau_C \Hom_A(T, L_2). \]

By the same argument as in case 1 we see that
\[ \tau_C \Hom_A(T, L_1) = \Hom_A(T, t(\tau A L_1)) = \Hom_A(T, M) \]
where $M$ is the indecomposable $A$-module on the mouth of $T_A$ such that the ray starting at $M$ passes through $\tau L_1$. In other words, $M$ is the starting point of the ray $\tau^2 \omega$. 
Therefore, it only remains to show that $\tau_C \text{Hom}_A(T, L_2) = 0$. To do so, it suffices to show that $L_2$ is a summand of $T$.

We have already seen that $\text{Ext}_A^1(T, L_2) = 0$. We show now that we also have $\text{Ext}_A^1(L_2, T) = 0$. Suppose the contrary. Then there exists a non-zero morphism $u : T \to \tau_A L_2$. Composing it with the irreducible injective morphism $\tau_A L_2 \to \tau_A T_2$ yields a non-zero morphism in $\text{Hom}_A(T, \tau_A T_2)$. But this is impossible since $T$ is tilting.

Thus we have $\text{Ext}_A^1(T, L_2) = \text{Ext}_A^1(L_2, T) = 0$ and thus $L_2$ is a summand of $T$, the module $\text{Hom}_A(T, L_2)$ is projective and $\tau_C \text{Hom}_A(T, L_2) = 0$. This completes the proof. □

Remark 5.4. The module $M$ in the statement of the lemma is the starting point of the ray passing through $\tau^2 T_i$.

Corollary 5.5. Let $A, T, C, T_A$ be as in Lemma 5.3, and let $B = C \ltimes E$, with $E = \text{Ext}_C^2(DC, C)$. Let $X, Y$ be two modules lying on the same coray in the tube $\text{Hom}_A(T, T_A \cap \mathcal{T}(T))$ in mod $C$. Then $X \otimes C E \cong Y \otimes C E$ and thus the two projections $X \otimes C B \to X \to 0$ and $Y \otimes C B \to Y \to 0$ have isomorphic kernels.

Proof. For all $C$-modules $X$ we have

$$X \otimes_B E \cong D\text{Hom}(X, DE) \cong D\text{Hom}(X, \tau_C \Omega_C DC)$$

where the first isomorphism is [SeSe, Proposition 3.3] and the second is [SeSe, Proposition 4.1]. Since $T$ has no preinjective summands, and $X$ is regular, the only summand of $\tau \Omega DC$ for which $\text{Hom}(X, \tau \Omega DC)$ can be nonzero, must lie in the same tube as $X$. By the lemma, the only summands of $\tau \Omega DC$ in the tube lie on the mouth of the tube. Let $M$ denote an indecomposable $C$-module on the mouth of a tube. Then

$$\text{Hom}_C(X, M) \cong \text{Hom}_C(Y, M) \cong \begin{cases} k & \text{if } M \text{ lies on the coray passing through } X \text{ and } Y, \\ 0 & \text{otherwise}. \end{cases}$$

We summarize the results of this section in the following proposition.

Proposition 5.6. (a) Let $S_1$ be the coray in $\Gamma(\text{mod } C_1)$ passing through the projective $C_1$-module corresponding to the root projective $P_B(i)$. Then $S_1 \otimes_{C_1} B$ is a coray in $\Gamma(\text{mod } B)$ passing through $P_B(i)$. Furthermore all modules in $S_1 \otimes_{C_1} B$ are extensions of modules of $S_1$ by the same module $P_{C_1}(i) \otimes E$.

(b) Let $S_2$ be the ray in $\Gamma(\text{mod } C_2)$ passing through the injective $C_2$-module corresponding to the root injective $I_B(i)$. Then $\text{Hom}_{C_2}(B, S_2)$ is a ray in $\Gamma(\text{mod } B)$ passing through $I_B(i)$. Furthermore all modules in $\text{Hom}_{C_2}(B, S_2)$ are extensions of modules of $S_2$ by the same module $\text{Hom}_{C_2}(E, I_{C_2}(i))$. 
Proof. (a) The first statement is Lemma 5.1 and the second statement is a restatement of Corollary 5.5. □

Example 5.7. Let $B$ be the cluster-tilted algebra given by the quiver

```
1 → λ → 5
    ↘  α  ↘
    ↗  β  ↗
3 → 3
    ↘  δ  ↘
    ↗  γ  ↗
2 → 2
    ↘  σ  ↘
    ↗  σ  ↗
4
```

bound by $\alpha\beta = 0$, $\beta\epsilon = 0$, $\epsilon\alpha = 0$, $\gamma\delta = 0$, $\sigma\gamma = 0$, $\delta\sigma = 0$. The algebras $C_1$ and $C_2$ are respectively given by the quivers

```
1 → λ → 5
    ↘  α  ↘
    ↗  β  ↗
3 → 3
    ↘  δ  ↘
    ↗  γ  ↗
2 → 2
    ↘  σ  ↘
    ↗  σ  ↗
4
```

and

```
1 → λ → 5
    ↘  β  ↘
    ↗  ε  ↗
3 → 3
    ↘  δ  ↘
    ↗  γ  ↗
2 → 2
    ↘  σ  ↘
    ↗  σ  ↗
4
```

with the inherited relations. We can see the tube in $\Gamma(\text{mod } C_1)$ below and the coray passing through the root projective $P_{C_1}(3) = 4 \frac{1}{5}$ is given by

$$S_1: \ldots \rightarrow 1 \rightarrow 5 \rightarrow 4 \frac{1}{5} \rightarrow 3 \frac{1}{5} \rightarrow 5 \rightarrow 3 \frac{1}{5} \rightarrow 2 \frac{1}{5}.$$
Dually, the ray in $\Gamma(\text{mod } C_2)$ passing through the root injective $I_{C_2}(3) = \begin{pmatrix} 1&5 \\ 3 \end{pmatrix}$ is given by

$$S_2 : \begin{array}{c}
\begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 3 \\ 5 \\ 4 \end{pmatrix} \\
\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} \\
\cdots
\end{array}$$

The root projective $P_B(3)$ lies on the coray

$$S_1 \otimes C_1 B : \begin{array}{c}
\begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
\cdots
\end{array}$$

and the root injective $I_B(3)$ lies on the ray

$$\text{Hom}_{C_2}(B, S_2) : \begin{array}{c}
\begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix} \\
\begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
\cdots
\end{array}$$

Note that by Proposition 5.6, every module in $S_1 \otimes C_1 B$ is an extension of a module in $S_1$ by $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Similarly, every module in $\text{Hom}_{C_2}(B, S_2)$ is an extension of a module in $S_2$ by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

Applying the knitting algorithm we obtain the tube in $\Gamma(\text{mod } B)$ containing both $S_1 \otimes C_1 B$ and $\text{Hom}_{C_2}(B, S_2)$.

6. From Cluster-Tilted Algebras to Quasi-Tilted Algebras

Let $B$ be cluster-tilted of euclidean type $Q$ and let $A = kQ$. Then there exists $T \in \mathcal{C}_A$ tilting such that $B = \text{End}_{C_A} T$. 

Because $Q$ is euclidean, $\mathcal{C}_A$ contains at most 3 exceptional tubes. Denote by $T_0, T_1, T_2, T_3$ the direct sums of those summands of $T$ that respectively lie in the transjective component and in the three exceptional tubes.

In the derived category $\mathcal{D}^b(\mod A)$, we can choose a lift of $T$ such that we have the following local configuration.

\[
\begin{array}{cccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
T_1 & T_2 & T_3 & \vdots & T_0 & FT_1 & FT_2 & FT_3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\]

Let $\mathcal{H}$ be a hereditary category that is derived equivalent to $\mod A$ and such that $\mathcal{H}$ is not the module category of a hereditary algebra. Then $\mathcal{H}$ is of the form $\mathcal{H} = \mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$, where $\mathcal{T}^-$, $\mathcal{T}^+$ consist of tubes, and $\mathcal{C}$ is a transjective component, see [LS]. Let $T_-, T_+$ be the direct sum of all indecomposable summands of $T$ lying in $\mathcal{T}^-$, $\mathcal{T}^+$ respectively. We define two subspaces $L$ and $R$ of $B$ as follows.

\[
L = \text{Hom}_{\mathcal{D}^b(\mod A)}(F^{-1}T_+, T_0) \quad \text{and} \quad R = \text{Hom}_{\mathcal{D}^b(\mod A)}(T_0, FT_-).
\]

The transjective component of $\mod B$ contains a left section $\Sigma_L$ and a right section $\Sigma_R$, see [A]. Thus $\Sigma_L, \Sigma_R$ are local slices, $\Sigma_L$ has no projective predecessors, and $\Sigma_R$ has no projective successors in the transjective component. Define $K$ to be the two-sided ideal of $B$ generated by $\text{Ann } \Sigma_L \cap \text{Ann } \Sigma_R$ and the two subspaces $L$ and $R$. Thus

\[
K = \langle \text{Ann } \Sigma_L \cap \text{Ann } \Sigma_R, L, R \rangle.
\]

We call $K$ the partition ideal induced by the partition $\mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$.

**Theorem 6.1.** The algebra $C = B/K$ is quasi-tilted and such that $B = \bar{C}$. Moreover $C$ is tilted if and only if $L = 0$ or $R = 0$.

**Proof.** We have $B = \text{End}_A T = \oplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\mod A)}(T, F^iT)$, where the last equality is as $k$-vector spaces. Using the decomposition $T = T_- \oplus T_0 \oplus T_+$, we see that $B$ is equal to

\[
\begin{align*}
\text{Hom}_\mathcal{D}(T_-, T_-) & \oplus \text{Hom}_\mathcal{D}(T_-, T_0) \oplus \text{Hom}_\mathcal{D}(T_-, FT_-) \\
\oplus \text{Hom}_\mathcal{D}(T_0, T_-) & \oplus \text{Hom}_\mathcal{D}(T_0, T_0) \oplus \text{Hom}_\mathcal{D}(T_0, FT_-) \\
\oplus \text{Hom}_\mathcal{D}(T_0, FT_0) & \oplus \text{Hom}_\mathcal{D}(F^{-1}T_+, FT_0) \oplus \text{Hom}_\mathcal{D}(F^{-1}T_+, T_+) \\
\oplus \text{Hom}_\mathcal{D}(T_+, T_+) & 
\end{align*}
\]

where all Hom spaces are taken in $\mathcal{D}^b(\mod A)$. On the other hand,

\[
\text{End}_\mathcal{H} T = \text{Hom}_\mathcal{H}(T_-, T_-) \oplus \text{Hom}_\mathcal{H}(T_-, T_0) \oplus \text{Hom}_\mathcal{H}(T_0, T_0) \\
\oplus \text{Hom}_\mathcal{H}(T_0, T_+) \oplus \text{Hom}_\mathcal{H}(T_+, T_+)
\]

is a quasi-tilted algebra. Thus in order to prove that $C$ is quasi-tilted it suffices to show that $K$ is the ideal generated by

\[
\text{Hom}_\mathcal{D}(T_-, FT_-) \oplus \text{Hom}_\mathcal{D}(T_0, FT_- \oplus FT_0) \oplus \text{Hom}_\mathcal{D}(F^{-1}T_+, T_0 \oplus T_+).
\]
But this follows from the definition of $L$ and $R$ and the fact that the annihilators of the local slices $\Sigma_L$ and $\Sigma_R$ are given by the morphisms in $\text{End}_{C^*}T$ that factor through the lifts of the corresponding local slice in the cluster category. More precisely,

$$\text{Ann } \Sigma_L \cong \text{Hom}_D(F^{-1}T_0 \oplus F^{-1}T_+ \oplus T_- \oplus T_0 \oplus T_+ \oplus FT_-),$$

$$\text{Ann } \Sigma_R \cong \text{Hom}_D(F^{-1}T_+ \oplus T_- \oplus T_0 \oplus T_+ \oplus FT_- \oplus FT_0),$$

and thus

$$\text{Ann } \Sigma_L \cap \text{Ann } \Sigma_R \cong \text{Hom}_D(T_0, FT_0) \oplus \text{Hom}_D(T_-, FT_-) \oplus \text{Hom}_D(F^{-1}T_+, T_+),$$

where we used the fact that $\text{Hom}_D(T_-, T_+) = \text{Hom}_D(T_+, T_-) = 0$. This completes the proof that $C$ is quasi-tilted.

Since $C = \text{End}_H T$, we have $\tilde{C} = \text{End}_{C^*}T \cong \text{End}_{C^*}T = B$.

Now assume that $R = 0$. Then $T_- = 0$ and thus $K$ is generated by $(\text{Ann } \Sigma_L \cap \text{Ann } \Sigma_R) \oplus L$, and this is equal to

$$(6.1) \quad \text{Hom}_D(T_0, FT_0) \oplus \text{Hom}_D(F^{-1}T_+, T_+) \oplus \text{Hom}_D(F^{-1}T_+, FT_0).$$

On the other hand, $T_- = 0$ implies that

$$\text{Ann } \Sigma_L = \text{Hom}_D(F^{-1}T_0 \oplus F^{-1}T_+, T_0 \oplus T_+),$$

and since $\text{Hom}_D(F^{-1}T_0, T_+) = 0$, this implies that $K = \text{Ann } \Sigma_L$ is the annihilator of a local slice. Therefore $C = B/K$ is tilted by \cite{ABS2}. The case where $L = 0$ is proved in a similar way.

Conversely, assume $C$ is tilted. Then $K = \text{Ann } \Sigma'$ for some local slice $\Sigma'$ in $\text{mod } B$. We show that $K = \text{Ann } \Sigma_L$ or $K = \text{Ann } \Sigma_R$. Suppose to the contrary that $\Sigma'$ has both a predecessor and a successor in $\text{add } T_0$. Then there exists an arrow $\alpha$ in the quiver of $B$ such that $\alpha \in \text{Hom}_D(T_0, T_0)$ and $\alpha \in \text{Ann } \Sigma' = K$. But by definition of $\Sigma_L, \Sigma_R, L$ and $R$, we see that this is impossible.

Thus $K = \text{Ann } \Sigma_L$ or $K = \text{Ann } \Sigma_R$. In the former case, we have $R = 0$, by the computation (6.1), and in the latter case, we have $L = 0$.

**Theorem 6.2.** If $C$ is quasi-tilted of euclidean type and $B = \tilde{C}$ then

$$C = B/\text{Ann}(\Sigma^- \oplus \Sigma^+),$$

where $\Sigma^-$ is a right section in the postprojective component of $C$ and $\Sigma^+$ is a left section in the preinjective component.

**Proof.** $C$ being quasi-tilted implies that there is a hereditary category $H$ with a tilting object $T$ such that $C = \text{End}_H T$. Moreover, $B = \text{End}_{C^*}T$ is the corresponding cluster-tilted algebra. As before we use the decomposition $T = T_- \oplus T_0 \oplus T_+$. Then the algebras

$$C^- = \text{End}_H(T_- \oplus T_0) \quad \text{and} \quad C^+ = \text{End}_H(T_0 \oplus T_+)$$

are tilted. Let $\Sigma^-$ and $\Sigma^+$ be complete slices in $\text{mod } C^-$ and $\text{mod } C^+$ respectively. Note that $\Sigma^-$ lies in the postprojective component and $\Sigma^+$ lies in the preinjective component of their respective module categories.
Then $C$ is a branch extension of $C^-$ by the module

$$M^+ = \text{Hom}_H(T_+, T_+) \oplus \text{Hom}_H(T_0, T_+).$$

Similarly $C$ is a branch coextension of $C^+$ by the module

$$M^- = \text{Hom}_H(T_-, T_-) \oplus \text{Hom}_H(T_-, T_0).$$

Observe that the postprojective component of $C^-$ does not change under the branch extension, and the preinjective component of $C^+$ does not change under the branch coextension. Therefore $\Sigma^-$ is a right section in the postprojective component of $C$ and $\Sigma^+$ is a left section in the preinjective component. Moreover, by construction, we have

$$\text{Ann}_B\Sigma^- = M^+ \oplus \text{Ext}_C^2(DC, C) \quad \text{and} \quad \text{Ann}_B\Sigma^+ = M^- \oplus \text{Ext}_C^2(DC, C),$$

and therefore

$$\text{Ann}_B(\Sigma^- \oplus \Sigma^+) = \text{Ann}_B\Sigma^- \cap \text{Ann}_B\Sigma^+ = \text{Ext}_C^2(DC, C).$$

This completes the proof.

The main theorem of this section is the following.

**Theorem 6.3.** Let $C$ be a quasi-tilted algebra whose relation-extension $B$ is cluster-tilted of euclidean type. Then $C$ is one of the following.

(a) $C = B/\text{Ann} \Sigma$ for some local slice $\Sigma$ in $\Gamma(\text{mod} \ B)$.

(b) $C = B/K$ for some partition ideal $K$.

**Proof.** Assume first that $C$ is tilted. Then, because of [ABS2], there exists a local slice $\Sigma$ in the transjective component of $\Gamma(\text{mod} \ B)$ such that $B/\text{Ann} \Sigma = C$. Otherwise, assume that $C$ is quasi-tilted but not tilted. Then, because of [LS], there exists a hereditary category $\mathcal{H}$ of the form

$$\mathcal{H} = \mathcal{T}^- \vee \mathcal{C} \vee \mathcal{T}^+$$

and a tilting object $T$ in $\mathcal{H}$ such that $C = \text{End}_\mathcal{H}T$. Because of Theorem 6.1 we get $C = B/K$ where $K$ is the partition ideal induced by the given partition of $\mathcal{H}$. \qed
Example 6.4. Let $B$ be the cluster-tilted algebra of type $\tilde{E}_7$ given by the quiver

```
1 --\alpha_1\rightarrow 3
\downarrow\beta_1\downarrow\beta_2\downarrow\beta_3
\downarrow\alpha_2\downarrow\alpha_3
\downarrow\epsilon\downarrow\epsilon\downarrow\epsilon
2 --\alpha_1\rightarrow 8
\downarrow\beta_1\downarrow\beta_2\downarrow\beta_3
\downarrow\alpha_2\downarrow\alpha_3
\downarrow\epsilon\downarrow\epsilon\downarrow\epsilon
5
\downarrow\beta_1\downarrow\beta_2\downarrow\beta_3
\downarrow\alpha_2\downarrow\alpha_3
\downarrow\epsilon\downarrow\epsilon\downarrow\epsilon
6
\downarrow\beta_1\downarrow\beta_2\downarrow\beta_3
\downarrow\alpha_2\downarrow\alpha_3
\downarrow\epsilon\downarrow\epsilon\downarrow\epsilon
7
```

As usual let $T_i$ denote the indecomposable summand of $T$ corresponding to the vertex $i$ of the quiver. In this example $T$ has two transjective summands $T_1, T_2$, and the other summands lie in three different tubes. $T_3, T_4$ lie in a tube $T_1$, $T_5$ lies in a tube $T_2$ and $T_6, T_7, T_8$ lie in a tube $T_3$.

Choosing a partition ideal corresponds to choosing a subset of tubes to be predecessors of the transjective component. Thus there are 8 different partition ideals corresponding to the 8 subsets of $\{T_1, T_2, T_3\}$. If the tube $T_i$ is chosen to be a predecessor of the transjective component, then the arrow $\beta_i$ is in the partition ideal. And if $T_i$ is not chosen to be a predecessor of the transjective component, then it is a successor and consequently the arrow $\alpha_i$ is in the partition ideal. The arrow $\epsilon$ is always in the partition ideal since it corresponds to a morphism from $T_8$ to $FT_7$ in the derived category.

Summarizing, the 8 partition ideals $K$ are the ideals generated by the following sets of arrows.

$$\{\alpha_i, \beta_j, \epsilon \mid i \notin I, j \in I, I \subset \{1, 2, 3\}\}.$$  

The quiver of the corresponding quasi-tilted algebra $B/K$ is obtained by removing the generating arrows from the quiver of $B$. Exactly 2 of these 8 algebras are tilted, and these correspond to cutting $\alpha_1, \alpha_2, \alpha_3, \epsilon$, respectively $\beta_1, \beta_2, \beta_3, \epsilon$.

References

[Am] C. Amiot, Cluster categories for algebras of global dimension 2 and quivers with potential, *Ann. Inst. Fourier* 59 no 6, (2009), 2525–2590.

[A] I. Assem, Left sections and the left part of an Artin algebra, *Colloq. Math.* 116 (2009), no. 2, 273–300.
[ABCP] I. Assem, T. Brüstle, G. Charbonneau-Jodoin and P. G. Plamondon, Gentle algebras arising from surface triangulations. *Algebra Number Theory* 4 (2010), no. 2, 201–229.

[ABS] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras as trivial extensions, *Bull. Lond. Math. Soc.* 40 (2008), 151–162.

[ABS2] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras and slices, *J. of Algebra* 319 (2008), 3464–3479.

[ABS3] I. Assem, T. Brüstle and R. Schiffler, On the Galois covering of a cluster-tilted algebra, *J. Pure Appl. Alg.* 213 (7) (2009) 1450–1463.

[ABS4] I. Assem, T. Brüstle and R. Schiffler, Cluster-tilted algebras without clusters, *J. Algebra* 324, (2010), 2475–2502.

[AsScSe] I. Assem, R. Schiffler and K. Serhiyenko, Modules that do not lie on local slices, in preparation.

[ASS] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, 1: Techniques of Representation Theory*, London Mathematical Society Student Texts 65, Cambridge University Press, 2006.

[ARS] M. Auslander, I. Reiten and S.O. Smalø, *Representation Theory of Artin Algebras* Cambridge Studies in Advanced Math. 36, (Cambridge University Press, Cambridge, 1995).

[BT] M. Barot and S. Trepode, Cluster tilted algebras with a cyclically oriented quiver. *Comm. Algebra* 41 (2013), no. 10, 3613–3628.

[BFPT] M. Barot, E. Fernandez, I. Pratti, M. I. Platzeck and S. Trepode, From iterated tilted to cluster-tilted algebras, *Adv. Math.* 223 (2010), no. 4, 1468–1494.

[BOW] M. A. Bertani-Økland, S. Oppermann and A Wralsen, Constructing tilted algebras from cluster-tilted algebras, *J. Algebra* 323 (2010), no. 9, 2408–2428.

[BMRRT] A. B. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), no. 2, 572-618.

[BMR] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras, *Trans. Amer. Math. Soc.* 359 (2007), no. 1, 323–332 (electronic).

[BMR2] A. B. Buan, R. Marsh and I. Reiten, Cluster-tilted algebras of finite representation type, *J. Algebra* 306 (2006), no. 2, 412–431.

[CCS] P. Caldero, F. Chapoton and R. Schiffler, Quivers with relations arising from clusters (A_n case), *Trans. Amer. Math. Soc.* 358 (2006), no. 3, 1347–1364.

[FPT] E. Fernández, N. I. Pratti and S. Trepode, On m-cluster tilted algebras and trivial extensions, *J. Algebra* 393 (2013), 132–141.

[FZ] S. Fomin and A. Zelevinsky, Cluster algebras I: Foundations, *J. Amer. Math. Soc.* 15 (2002), 497–529.

[H] D. Happel, A characterization of hereditary categories with tilting object. *Invent. Math.* 144 (2001), no. 2, 381–398.

[HRS] D. Happel, I. Reiten and S. Smalø, Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.* 120 (1996), no. 575.

[LS] H. Lenzing and A. Skowroński, Tilting in abelian categories and quasitilted algebras. *Mem. Amer. Math. Soc.* 120 (1996), no. 575.

[LR] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, *Adv. Math.* 211 (2007), no. 1, 123–151.

[OS] M. Oryu and R. Schiffler, On one-point extensions of cluster-tilted algebras, *J. Algebra* 357 (2012), 168–182.

[Re] I. Reiten, Cluster categories. *Proceedings of the International Congress of Mathematicians. Volume I, 558–594*, Hindustan Book Agency, New Delhi, 2010. 16-02

[Ri] C.M. Ringel, The regular components of the Auslander-Reiten quiver of a tilted algebra. *Chinese Ann. Math. Ser. B* 9 (1988), no. 1, 1–18.
[R] C. M. Ringel, Representation theory of finite-dimensional algebras. Representations of algebras (Durham, 1985), 7–79, London Math. Soc. Lecture Note Ser., 116, Cambridge Univ. Press, Cambridge, 1986.

[S] R. Schiffler, *Quiver Representations*, CMS Books in Mathematics, Springer International Publishing, 2014.

[ScSe] R. Schiffler and K. Serhiyenko, Induced and coinduced modules over cluster-tilted algebras, preprint, arXiv:1410.1732.

[ScSe2] R. Schiffler and K. Serhiyenko,Injective presentations of induced modules over cluster-tilted algebras, preprint, arXiv:1604.06907.

[Sk] A. Skowroński, Tame quasi-tilted algebras, *J. Algebra* 203 (1998), no. 2, 470–490.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ DE SHERBROOKE, SHERBROOKE, QUÉBEC, CANADA J1K 2R1

E-mail address: ibrahim.assem@usherbrooke.ca

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CT 06269-3009, USA

E-mail address: schiffler@math.uconn.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840, USA

E-mail address: khrystyna.serhiyenko@berkeley.edu