Inversion of signature for paths of bounded variation

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Abstract
We develop two methods to reconstruct a path of bounded variation from its signature. The first method gives a simple and explicit expression of any axis path in terms of its signature, but it does not apply directly to more general ones. The second method, based on an approximation scheme, recovers any tree-reduced path from its signature as the limit of a uniformly convergent sequence of lattice paths.

1 Introduction
Paths have important roles in many areas of mathematics. A path is a continuous function \( \gamma \) that maps a non-empty interval \( J \subset \mathbb{R} \) into a metric space \((V, d_V)\). A basic property of a path \( \gamma \) on an interval \( J \) is its length, \( |\gamma|_J \), which can be defined as follows:

\[
|\gamma|_J := \sup_{P \subset J} \sum_i d_V(\gamma(u_i), \gamma(u_{i+1})),
\]
where the supremum is taken over all finite partitions \( P = \{s = u_0 < u_1 < \cdots < u_n = t\} \) of the interval \( J = [s, t] \). It is clear that \( |\gamma|_J \) is independent of the choice of parametrization. When there is no possible confusion, we will drop the subscript \( J \), and use \( |\gamma| \) to denote the length of the whole path.

The mesh of a partition \( P, \|P\| \), is defined by \( \|P\| := \max_i |u_{i+1} - u_i| \). By triangle inequality and the continuity of \( \gamma \), one can equivalently define \( |\gamma|_J \) by

\[
|\gamma|_J = \lim_{\|P\| \to 0} \sum_i d_V(\gamma(u_i), \gamma(u_{i+1})),
\]
where the limit exists independent of the actual partition as long as the mesh tends to 0.

If \( V \) is a Banach space, one can rewrite \( d_V(\gamma(u_i), \gamma(u_{i+1})) \) as \( |\gamma(u_{i+1}) - \gamma(u_i)| \), where \( \cdot \) is the Banach space norm.

Paths \( \gamma : J \to V \) of finite length are denoted as elements of \( BV_J(V) \); they are also called paths with bounded variation.

For any path \( \alpha : [0, s] \to V \) and \( \beta : [0, t] \to V \), we can form the concatenation \( \alpha \ast \beta : [0, s + t] \to V \), as follows:

\[
\alpha \ast \beta(u) := \begin{cases} 
\alpha(u), & u \in [0, s] \\
\beta(u - s) + \alpha(s) - \alpha(0), & u \in [s, s + t]
\end{cases}.
\]
and similarly, the decomposition of one path into two can be carried out in the same fashion.

For any path \( \gamma : [s, t] \to V \), the path "\( \gamma \) run backwards", \( \gamma^{-1} \), is defined as:

\[
\gamma^{-1}(u) := \gamma(s + t - u), \quad u \in [s, t],
\]

and the trajectories of \( \gamma \ast \gamma^{-1} \) cancel out each other.

Concatenation and "backwards" of paths of bounded variation are still paths of bounded variation. In fact, we have \( |\alpha \ast \beta| = |\alpha| + |\beta| \), and \( |\gamma^{-1}| = |\gamma| \).

If \( \gamma \in BV(\mathbb{R}^d) \), then one can define a differential equation driven by \( \gamma \) as follows:

\[
dy(t) = f(y(t)) \cdot d\gamma(t), \quad y(0) = a,
\]

where \( f \) is a vector field. If \( f \) is Lipschitz, then equation (2) has a unique solution \( y(t) \).

One may seek the properties of \( \gamma \) that determines the value \( y(t) \) given the initial value \( y(0) = a \), and this leads to the concept of the signature of a path ([8], [6]):

**Definition 1.1.** Let \( \gamma : [s, t] \to V \) be a path of bounded variation, where \( V \) is also a vector space with a countable basis. The signature of \( \gamma \), \( X_{s,t}(\gamma) \), is defined as:

\[
X_{s,t}(\gamma) = 1 + X_{s,t}^1(\gamma) + \cdots + X_{s,t}^n(\gamma) + \cdots,
\]

where

\[
X_{s,t}^n(\gamma) = \int_{s < u_1 < \cdots < u_n < t} d\gamma(u_1) \otimes \cdots \otimes d\gamma(u_n)
\]

as an element in \( V^\otimes n \).

Let \( (e_1, e_2, \cdots) \) be a basis of \( V \), then \( \gamma \) can be written as \( (\gamma_1, \gamma_2, \cdots) \). If \( w = e_{i_1} \cdots e_{i_n} \) be a word of length \( n \), we write

\[
C_{s,t}(w) = C_{s,t}(e_{i_1} \cdots e_{i_n}) = \int_{s < u_1 < \cdots < u_n < t} d\gamma_{i_1}(u_1) \cdots d\gamma_{i_n}(u_n)
\]

as the coefficient of \( w \). As all words of length \( n \) form a basis of \( V^\otimes n \), we can rewrite \( X_{s,t}^n(\gamma) \) as the linear combination of basis elements:

\[
X_{s,t}^n(\gamma) = \sum_{|w| = n} C_{s,t}(w)w,
\]

where the sum is taken over all words of length \( n \).

Reparametrizing \( \gamma \) does not change its signature. The signature contains important information about the path. For example, the first term, \( X_{s,t}^1(\gamma) \), produces the increment of \( \gamma \).

The study of the signature of a path dates back to Chen [1], where he associated a noncommutative power series to each piecewise smooth path in \( \mathbb{R}^d \), or on a Riemannian manifold \( M \), with iterated integrals along the path as coefficients. This power series, in the form of (4), was obtained by solving (2) using Picard’s iteration for the case when \( f \) is linear. He showed that the logarithm (as a power series) of this power series is a Lie element, and obtained a generalized Campbell-Baker-Hausdorff formula. In his
subsequent work \cite{2}, he proved that two irreducible \footnote{Loosely speaking, a path is irreducible if no part of it can be written as the form $\gamma \ast \gamma^{-1}$.} piecewise smooth paths with the same starting point have the same associated power series if and only if they differ by a parametrization. Later on, he used this power series as a basic tool to compute the homology and cohomology of various path spaces on a manifold. A detailed discussion on the role of such iterated integrals in relating the analysis on a manifold and the homology of its path spaces can be found in \cite{3}.

Half a century later, Lyons \cite{8} formally introduced the notion of signature, and defined this power series as the signature of the path in the form of \eqref{eq:signature}. He also generalized the multiplicativity of signature (first discovered by Chen \cite{1}) to the notion of multiplicative functionals, and showed that any multiplicative functional in $T^{(1)}$ with finite 1-variation must be the signature of some path of finite length.

Hambly and Lyons \cite{6} introduced the notion of tree-like paths, which are generalizations of paths of the form $\gamma_1 \ast \gamma_1^{-1} \ast \cdots \ast \gamma_n \ast \gamma_n^{-1}$. Two paths $\gamma, \tau \in BV(\mathbb{R}^d)$ are equivalent if $\gamma \ast \tau^{-1}$ is tree-like. This equivalence relation defines the quotient group $BV(\mathbb{R}^d)/\sim$. It is a group with concatenation as the multiplication. Within every equivalent class, there is a unique path of minimal length, called the reduced path (or irreducible path). They used various methods in analysis and hyperbolic geometry to get quantitative estimates for Chen’s theorem extended to paths of bounded variation, and obtained as a corollary that two paths in $\mathbb{R}^d$ of bounded variation have the same signature if and only if they are equivalent.

After the work of Hambly and Lyons, a natural question is how to reconstruct the reduced path of bounded variation from its signature. This is known as the inversion problem. Hambly and Lyons \cite{6} gave a formula to recover the length of $C^3$ paths parametrized at unit speed by looking at the asymptotic behavior of $\|X^n(\gamma)\|$. This assumption can also be weakened to paths with continuous derivatives, but the conclusions will also be weaker.

In this paper, we develop two methods to invert the signature. The first method gives an explicit inversion formula for axis paths, which are generalizations of integer lattice paths. But this method depends on the special structure of axis paths, and does not apply directly to more general ones.

The second method is based on approximation of paths of bounded variation by lattice paths. Suppose $X = (1, X^1, \cdots, X^n, \cdots)$ is the signature of a tree-reduced path $\gamma$ with length $L$. We will show that one can construct a sequence of lattice paths $\hat{\gamma}^{(N)}$, with step size $\frac{1}{N}$ and length at most $L$, such that for every $n$, $X^n(\hat{\gamma}^{(N)})$ converges to $X^n$. One can show further that, when properly located and parametrized, the sequence $\hat{\gamma}^{(N)}$ converges uniformly to $\gamma$. Thus, this approximation scheme asymptotically recovers any path of bounded variation from its signature by approaching it uniformly using a sequence of lattice paths. At the heart of our argument is an estimate of the difference of the signatures of two paths in terms of their lengths and uniform distance (theorem 4.4).

The main results in this paper are the following three theorems:

**Theorem 3.4** Fix a finite axis path $\gamma$. Let $w = e_{i_1} \cdots e_{i_k} \cdots e_{i_n}$ be the unique longest square free word such that $C(w) \neq 0$. Let $w_k = e_{i_1} \cdots e_{i_k}^2 \cdots e_{i_n}$, then we can write $\gamma$ as

$$\gamma = r_1 e_{i_1} + \cdots + r_n e_{i_n},$$
where \( r_k = \frac{2C(w_k)}{C(w)} \), and the sum is noncommutative.

**Theorem 4.2** Let \( \gamma : [0, 1] \rightarrow \mathbb{R}^d \) be a path with \( l_1 \) length \( L \). For any integer \( N \), there exists a lattice path \( \hat{\gamma}(N) : [0, 1] \rightarrow \mathbb{R}^d \) with step size \( \frac{1}{2^N} \) and length at most \( L \) such that

\[
|\hat{\gamma}(N)(t) - \gamma(t)| \leq \frac{d}{2^N}
\]

for all \( t \in [0, 1] \).

**Theorem 4.4** Let \( \alpha, \beta : [0, 1] \rightarrow \mathbb{R}^d \) be two paths of finite length with a common control \( \omega \), and \( |\alpha(t) - \beta(t)| < \epsilon \) for all \( t \in [0, 1] \). Then,

\[
\|X^n_{s,t}(\alpha) - X^n_{s,t}(\beta)\| < 2\epsilon \cdot \frac{(4\omega(s,t))^{n-1}}{(n-1)!}
\]

for \( s, t \in (0, 1) \) and all \( n \in \mathbb{N} \).

The direct applications of this paper are to paths in \( BV(\mathbb{R}^d) \), but many theorems still hold in a general Banach space. We focus on paths of bounded variation. For results and a more general theory of rough paths, we refer to the original work of Lyons [8] and two comprehensive introductions ([9] and [10]) for more details.

We follow the notations in [6] and [8]. We assume the length of a path to be its \( l_1 \) length, unless otherwise specified. If \( \gamma \) is the only path in our concern, we will use \( X_{s,t}(\gamma) \) instead of \( X_{s,t}(\gamma) \) to denote its signature. When we refer to the signature of the whole path, we will drop the subscripts \( s, t \), and use \( X \).

The symbol \( | \cdot | \) has several different meanings in different contexts. If \( w \) is a word, then \( |w| \) denotes the length of the word. If \( E \) is a set, then \( |E| \) denotes the cardinality of \( E \). Finally, if \( \gamma \) is a path in a Banach space with \( l^p \) norm, then \( |\gamma|_p \) denotes the length of the path in the Banach space.

## 2 Preliminary results on the signature

One essential property of the signature is its multiplicativity. It was first discovered by Chen [1]. We state it in the lemma below.

**Lemma 2.1.** Let \( \gamma : [s, t] \rightarrow V \) be a path of bounded variation. Then, for any \( r \in [s, t] \),

\[
X_{s,r} \otimes X_{r,t} = X_{s,t}.
\]

The proof is an application of Fubini’s theorem, after partitioning the domain of integration into \( n \) disjoint parts.

As mentioned in the introduction, the signature characterizes important information about the path. Hambly and Lyons [6] recently proved that paths of bounded variations are completely characterized by their signatures up to tree-like equivalence. To state their theorem precisely, we first need a mathematical characterization of tree-like path, which was first introduced in [5], and then used in [6].

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\(^2\)Path \( \gamma \) has finite length controlled by \( \omega \) if \( |\gamma|(s,t) \leq \omega(s, t) \) for all \( s \leq t \). The control \( \omega \) is assumed to be jointly continuous, and additive in the sense that \( \omega(s, t) = \omega(s, u) + \omega(u, t) \) for all \( s \leq u \leq t \).
Definition 2.2. \( \gamma : [0,1] \to V \) is a tree-like path if there exists a continuous function \( h : [0,1] \to \mathbb{R}^+ \) such that
\[
\| \gamma(t) - \gamma(s) \| \leq h(s) + h(t) - 2 \inf_{u \in [s,t]} h(u).
\]

The function \( h \) is called the height function. Given a metric on the tree, one can think of \( h(t) \) as the distance from \( \gamma(0) \) to \( \gamma(t) \) under that metric. More discussions on paths and trees can be found in [5].

If \( \gamma \) has bounded variation, then \( h \) can be chosen to have bounded variation (Theorem 12 of [6]). With the aid of this characterization, they proved the following main theorem:

Theorem 2.3. Let \( \alpha, \beta \in BV(\mathbb{R}^d) \). Then, \( \alpha \) and \( \beta \) have the same signature if and only if \( \alpha * \beta^{-1} \) is tree-like.

We say two paths \( \alpha \) and \( \beta \) are equivalent if \( \alpha * \beta^{-1} \) is tree-like, denoted by \( \alpha \sim \beta \). It is clear that paths differing by a re-parametrization are equivalent. By the properties of tensors and multiplicativity of the signature, one can immediately conclude the following from the main theorem:

Corollary 2.4. \( \sim \) defines an equivalence relation on \( BV(\mathbb{R}^d) \).

This equivalence relation partitions \( BV(\mathbb{R}^d) \) into equivalent classes. Within each class, there is a unique path with minimal length.

Corollary 2.5. For any \( \gamma \in BV(\mathbb{R}^d) \), there exists a unique path \( \tilde{\gamma} \) such that \( X(\tilde{\gamma}) = X(\gamma) \), and if \( \beta \) is any other path with the same signature, then \( |\tilde{\gamma}| < |\beta| \).

The shortest path \( \tilde{\gamma} \) corresponds to irreducible path in the finite case. We call it the tree-reduced path of \( \gamma \). It is then natural to ask, how can one reconstruct the tree-reduced path of bounded variation from its signature. Hambly and Lyons showed that, one can recover the length of \( \gamma \) by looking at the asymptotic behavior of \( X_{s,t}^n \), provided that \( \gamma \) is smooth enough \( (C^3) \):

Theorem 2.6. Let \( \gamma : J \to \mathbb{R}^d \) be a \( C^3 \) path of length \( L \) parametrized at unit speed. Its signature is \( X = (1, X^1, \cdots, X^n, \cdots) \). If \( T^n \) is given the projective norm, then
\[
\lim_{n \to \infty} \left\| \frac{n! X^n}{L^n} \right\| = 1.
\]

If \( T^n \) is given the Hilbert Schmidt norm, then the limit
\[
\lim_{n \to \infty} \left\| \frac{n! X^n}{L^n} \right\|^2 = \mathbb{E}[\exp(\int_0^1 |W^0_s|^2 \langle \gamma'(s), \gamma'''(s) \rangle ds)] \leq 1,
\]
exists, where \( W^0_s \) is the Brownian bridge in time \([0,1]\) starting and finishing at zero, and \( \langle \cdot, \cdot \rangle \) is the standard inner product on \( \mathbb{R}^d \). The limit is strictly less than 1 unless \( \gamma \) is a straight line.

This is a very strong result obtained by making strong assumptions. One could weaken the \( C^3 \) condition and get a weaker result. This is the following theorem. It is also proved in [6].
Theorem 2.7. Let \( \gamma \) be a path of length \( L \) parametrized at unit speed. Suppose its derivative is continuous. Let \( b_n = \|n!X^n(\gamma)\| \), where \( \| \cdot \| \) is the projective tensor norm. Then, the Poisson averages \( C_\alpha \) of the \( b_n \)'s defined by

\[
C_\alpha := e^{-\alpha} \sum_{n=0}^{+\infty} \frac{\alpha^n}{n!} b_n
\]

satisfy

\[
\lim_{\alpha \to \infty} \frac{1}{\alpha} \log C_\alpha = L - 1.
\]

In the next two sections, we will develop methods to recover the tree reduced path itself from its signature.

3 Inversion for axis paths

In this section, we give an expression of any finite axis path in \( \mathbb{R}^d \) in terms of its signature.

Definition 3.1. \( \gamma : [s, t] \to \mathbb{R}^d \) is a (finite) axis path if its movements are parallel to the Euclidean coordinate axes, has finitely many turns, and each straight line component has finite length.

Any axis path has a unique reduced axis path; integer lattice paths are special cases of axis paths. An \( \mathbb{R}^d \) axis path can move in \( d \) different directions (up to the sign). At time 0, it starts to move along a direction \( e_{i_1} \) for some distance \( r_1 \); then it turns a right angle, and moves along \( e_{i_2} \) for a distance \( r_2 \), and so on, and stops after finitely many turns. Thus, an (unparametrized) axis path \( \gamma \) can be represented as:

\[
\gamma = r_1 e_{i_1} + \cdots + r_n e_{i_n}
\]

(5)

where \( r_i \)'s are real numbers, with the sign denoting the direction. The sum is non-commutative. If \( \gamma \) is already in its reduced form, then it is clear that \( i_k \neq i_{k+1} \), and we call \( (e_{i_1}, \ldots, e_{i_n}) \) the shape of \( \gamma \). We introduce the notion of square free words to characterize the shape of an axis path.

Definition 3.2. Let \( w = e_{i_1} \cdots e_{i_n} \) be a word. We call it a square free word if \( \forall k \leq n - 1, i_k \neq i_{k+1} \).

If a path has shape \( (e_{i_1}, \ldots, e_{i_n}) \), then by multiplicativity, its signature can be written as

\[
X(\gamma) = e^{r_1 e_{i_1}} \cdots e^{r_n e_{i_n}},
\]

(6)

where the product is noncommutative. Let \( w = e_{i_1} \cdots e_{i_n} \), then we have:

\[
C(w) = r_1 \cdots r_k \cdots r_n,
\]

(7)

and in particular, \( C(w) \neq 0 \). Moreover, if \( w' \) is any other square free word with \( C(w') \neq 0 \), then \( w' \) has length at most \( n - 1 \), and we can thus recover the shape of the path by looking at its square free words. We state this observation in the proposition below.

\footnote{We mean \(-re_j = re_j^{-1} \).}
Proposition 3.3. For any finite axis path $\gamma$, there exists a unique square free word $w$ such that $C(w) \neq 0$, and if $w'$ is any other square free word with $C(w') \neq 0$, then $|w'| < |w|$. If $w = e_{i_1} \cdots e_{i_n}$, then $\gamma$ has shape $(e_{i_1}, \cdots, e_{i_n})$.

As a second step, we need to find out the $r_i$'s, assuming that we know the path has shape $(e_{i_1}, \cdots, e_{i_n})$. Let $w = e_{i_1} \cdots e_{i_k} \cdots e_{i_n}$, and $w_k = e_{i_1} \cdots e_{i_k}^2 \cdots e_{i_n}$, then,

$$C(w_k) = \frac{1}{2} r_1 \cdots r_k^2 \cdots r_n,$$

and compare with (7), we immediately get:

$$r_k = \frac{2C(w_k)}{C(w)},$$

thus recovering the length of each straight line component.

Combining the arguments above, we have the following theorem to invert a finite axis path from its signature:

Theorem 3.4. Let $\gamma$ be a finite axis path. Let $w = e_{i_1} \cdots e_{i_k} \cdots e_{i_n}$ be the unique longest square free word such that $C(w) \neq 0$. Let $w_k = e_{i_1} \cdots e_{i_k}^2 \cdots e_{i_n}$, then we can write $\gamma$ as

$$\gamma = r_1 e_{i_1} + \cdots + r_n e_{i_n},$$

where $r_k = \frac{2C(w_k)}{C(w)}$, and the sum is noncommutative.

Thus, if an axis path has $n$ turns, at most $n + 2$ terms in the signature are needed for inversion. For a lattice path with length $L$, it can have at most $L - 1$ turns, so we only need the first $L + 1$ terms in the signature to recover it.

In practice, lattice paths are often generated by drawing uniformly randomly $n$ letters and their inverses from an alphabet, and putting them in a row in the order they are drawn. It is then interesting to ask about the number of turns in its reduced path. We give uniform counting measure on all lattice paths with $n$ steps, and let $T_n$ denote the number of turns in the reduced path. The asymptotics of $T_n$ for large $n$ was computed by Jiang and Xu in [7].

4 Inversion for paths of bounded variation

Given the signature $X = (1, X^1, \cdots)$ of a tree-reduced path of bounded variation, we will find a sequence of lattice paths $\{\hat{\gamma}^{(N)}\}$ whose signatures converge to $X$. We show that $\{\hat{\gamma}^{(N)}\}$ also necessarily converges uniformly, and thus obtain the original path as the uniform limit of this sequence of lattice paths. In this approach, the first and most important problem is that it is not at all clear whether there exists such a sequence of lattice paths. But once the existence is verified, one can find this sequence by checking all the possibilities, as there are only finitely many candidates for each $\gamma^{(N)}$. Therefore, we will devote most of our efforts in proving the existence of such a sequence. Our procedure is as follows.
1. Let $\gamma$ be a path of length $L$. For every integer $N$, we construct a lattice path $\hat{\gamma}^{(N)}$ on the same time interval with $\gamma$ with step size $\frac{1}{2N}$ and length at most $L$ such that
\[
\left\| \hat{\gamma}^{(N)}(t) - \gamma(t) \right\| \leq \frac{d}{2N}
\]
for all $t$. In particular, $\{\hat{\gamma}^{(N)}\}$ converges to $\gamma$ uniformly.

2. We will show that each $\hat{\gamma}^{(N)}$ constructed above satisfies
\[
\left\| X^{(n)}(\hat{\gamma}^{(N)}) - X^n \right\| < \frac{d}{2N-1} \cdot \frac{(4L)^{n-1}}{(n-1)!}
\]
for all $n$.

3. Suppose $X$ is the signature of a tree-reduced path $\gamma$ with length $L$. Then, in light of the previous two steps, one can find a sequence of lattice paths satisfying (9). This sequence, if located and parametrized properly, converges uniformly to $\gamma$. Thus, we are able to asymptotically recover any tree-reduced path from its signature by approximating it uniformly using lattice paths.

Note that the first two steps provides a verification for the existence of the sequence of lattice paths mentioned at the beginning of this section. Step 3 is a simple consequence of the previous two. We now start with the first step.

### 4.1 Lattice path approximation to paths of bounded variation

Let $\gamma : [0, 1] \rightarrow \mathbb{R}^d$ be a one dimensional path with length $L$. For any $x \in \mathbb{R}$, let $c(x)$ denote the cardinality of the level set $\{t | \gamma(t) = x\}$. $c(x)$ is also the 0-dimensional Hausdorff measure of the level set. Then, by the coarea formula (theorem 1 of section 3.4 in [4]), we have
\[
\int_{-\infty}^{+\infty} c(x) dx = L,
\]
or equivalently,
\[
\sum_{k=-\infty}^{+\infty} \int_{0}^{+\infty} c(x + \frac{k}{2N}) dx = L
\]
for all $N$. Since $c$ takes values on positive integers, monotone convergence theorem implies
\[
\int_{0}^{2N} \sum_{k=-\infty}^{+\infty} c(x + \frac{k}{2N}) dx = L
\]
for all $N$. We then have the following lemma:
Lemma 4.1. Let \( \gamma \) be a path in \( \mathbb{R} \) with length \( L \). Then, for any \( N \), there exists an \( x \in \mathbb{R} \) such that

\[
\sum_{k=-\infty}^{+\infty} c(x + \frac{k}{2^N}) \leq L \cdot 2^N.
\]  

(10)

Now we let \( \gamma \) live in \( \mathbb{R}^d \), and write \( \gamma(t) = (\gamma_1(t), \cdots, \gamma_d(t)) \). Since (10) is true for all one dimensional paths, one can choose for each \( i \) an \( x_i \in \mathbb{R} \) such that

\[
\sum_{k=-\infty}^{+\infty} c(x_i + \frac{k}{2^N}) \leq l_i \cdot 2^N
\]  

(11)

for all \( i \), where \( l_i = |\gamma_i| \). This implies that, for each \( i \), the number of times \( \gamma \) hits the parallel hyperplanes \( \bar{x}_i = \{x_i + \frac{k}{2^N}\}_{k=-\infty}^{+\infty} \) is at most \( l_i \cdot 2^N \). Let \( x = (x_1, \cdots, x_d) \), then the set of such \( x \)'s has positive Lebesgue measure, and one can choose an \( x \) such that there is a unique \( k = (k_1, \cdots, k_d) \in \mathbb{Z}^d \) satisfying

\[
x + \frac{k_i}{2^N} < \gamma_i(0) < x + \frac{k_i + 1}{2^N},
\]

for all \( i = 1, \cdots, d \). That is, the starting point of \( \gamma \) is in the interior of a hypercube formed by the parallel hyperplanes \( \bar{x}_i = \{x_i + \frac{k}{2^N}\}_{k=-\infty}^{+\infty} \).

We construct the lattice path \( \hat{\gamma}(N) \) as follows. Start at the center of the cube where \( \gamma(0) \) lives; when \( \gamma \) crosses a surface and moves to another (neighboring) cube, move \( \hat{\gamma}(N) \) one step of size \( \frac{1}{2^N} \), from the current position to the center of that neighboring cube. When \( \gamma \) does not move out of a cube, keep \( \hat{\gamma}(N) \) stayed at the center of a cube. Since \( \gamma \) crosses the surfaces at most \( L \cdot 2^N \) times, \( \hat{\gamma}(N) \) has at most \( L \cdot 2^N \) steps.

From that construction, one can parametrize \( \hat{\gamma}(N) \) such that \( \hat{\gamma}(N)(t) \) and \( \gamma(t) \) are in the same hypercube of side length \( \frac{1}{2^N} \) for all \( t \). This leads to the following theorem.

Theorem 4.2. Let \( \gamma : [0,1] \rightarrow \mathbb{R}^d \) be a path with \( l_1 \) length \( L \). For any integer \( N \), there exists a lattice path \( \hat{\gamma}(N) : [0,1] \rightarrow \mathbb{R}^d \) with step size \( \frac{1}{2^N} \) and length at most \( L \) such that

\[
|\hat{\gamma}(N)(t) - \gamma(t)| \leq \frac{d}{2^N}
\]

for all \( t \in [0,1] \).

4.2 Convergence of the signatures

In this subsection, we will show that the signatures of the lattice paths constructed in the previous section converge to that of the original path. One crucial ingredient of the proof is an estimate of the difference of the signatures of two paths in terms of their lengths and uniform distance (theorem 4.3).

Let \( \alpha, \beta : [0,1] \rightarrow \mathbb{R}^d \) be two paths of bounded variation with a common control \( \omega \), and \(|\alpha(t) - \beta(t)| < \epsilon \) for all \( t \in [0,1] \). For any \( N \), there exists a partition \( \{0 = t_0 < t_1 < \cdots < t_{N+1}\} \) of \([0,1]\) such that \( \omega(t_i, t_{i+1}) = \delta \leq \frac{L}{N} \) for all \( i \), and one can decompose the paths as

\[
\alpha = \alpha_1 \ast \cdots \ast \alpha_N, \quad \beta = \beta_1 \ast \cdots \ast \beta_N,
\]
where $\alpha_i$ and $\beta_i$ are restrictions of $\alpha$ and $\beta$ to time interval $[t_{i-1}, t_i]$.

By multiplicativity of the signature, we have

$$X(\alpha) = X(\theta_1 * \beta_1 * \cdots * \theta_N * \beta_N),$$

where $\theta_i = \alpha_i * \beta_i^{-1}$. Then, $\theta = \theta_1 * \cdots * \theta_N$ satisfies

$$\sup_t |\theta_{k+1} * \cdots * \theta_{k+p}(t)| < 2\epsilon \quad (12)$$

for all $k$ and $l$ with $k + l \leq N$. Indeed, we have

$$X^n(\alpha) = \sum_{\sum_k (i_k + j_k) = n} X^{i_1}(\theta_1) \otimes X^{j_1}(\beta_1) \otimes \cdots \otimes X^{i_N}(\theta_N) \otimes X^{j_N}(\beta_N)$$

Similar as before, we let

$$A^{(1)}_{N,n} = \{(i_k, j_k)_{k=1}^N | i_k = 0 \text{ or } 1 \text{ for all } k, \text{ and } i_k = 1 \text{ for some } k\},$$

$$A^{(2)}_{N,n} = \{(i_k, j_k)_{k=1}^N | i_k = 0 \text{ or } 1 \text{ for all } k, \text{ and } i_k = 1 \text{ for some } k\},$$

and

$$S^{(1)}_{N,n} = \sum_{A^{(1)}_{N,n}} X^{i_1}(\theta_1) \otimes X^{j_1}(\beta_1) \otimes \cdots \otimes X^{i_N}(\theta_N) \otimes X^{j_N}(\beta_N),$$

$$S^{(2)}_{N,n} = \sum_{A^{(2)}_{N,n}} X^{i_1}(\theta_1) \otimes X^{j_1}(\beta_1) \otimes \cdots \otimes X^{i_N}(\theta_N) \otimes X^{j_N}(\beta_N),$$

then

$$X^n(\alpha) = X^n(\beta) + S^{(1)}_{N,n} + S^{(2)}_{N,n}.$$
On the other hand, we have

$$|A^{(1)}_{N,n}| = \binom{2N}{n} - \binom{N}{n}$$

Combining them together, we get

$$|A^{(2)}_{N,n}| = |A_{N,n}| - |A^{(1)}_{N,n}|$$

$$\leq \binom{2N + n - 1}{n} - \binom{2N}{n}$$

$$\leq \frac{1}{n!}[(2N + n - 1)^n - (2N)^n]$$

$$\leq \frac{1}{n!}(n-1)n(2N + n - 1)^{n-1}$$

$$= \frac{(2N + n - 1)^{n-1}}{(n-2)!}.$$ 

Thus, we have

$$\|S^{(2)}_{N,n}\| \leq 2\delta \frac{(4L + 2\delta(n - 1))^{n-1}}{(n-2)!}$$

Now we are ready to prove our main estimate.

**Theorem 4.4.** Let $\alpha, \beta : [0, 1] \to \mathbb{R}^d$ be two paths of finite $l^1$ length with a common control $\omega$, and $|\alpha(t) - \beta(t)| < \epsilon$ for all $t \in [0, 1]$. Then, we have

$$\|X^n_{s,t}(\alpha) - X^n_{s,t}(\beta)\| < 3\epsilon \cdot \frac{(3\omega(s,t))^{n-1}}{(n-1)!}$$

for $s, t \in (0, 1)$ and all $n \in \mathbb{N}$.

**Proof.** In light of the previous lemma, by sending $\delta$ to 0, it suffices to prove the bound for $\|S^{(1)}_{N,n}\|$. A typical term in the sum of $S^{(1)}_{N,n}$, denoted by $W^n$, is the tensor product of $n$ ordered terms of $X^1(\theta_i)$ and $X^1(\beta_j)$'s, and consists of at least one $X^1(\theta_i)$. Now fix $W^n$. Suppose the rightmost $X^1(\theta_i)$ in the product appears at position $k$, then $W^n$ has the form

$$W^n = W^n(Y^{k-1}, Z^{n-k}) = Y^{k-1}(\theta, \beta) \otimes X^1(\theta_i) \otimes Z^{n-k}(\beta),$$

where $Y^{k-1}(\theta, \beta)$ is the tensor product of the first $k-1$ ordered terms, each term being $X^1(\theta_i)$ or $X^1(\beta_j)$, and $Z^{n-k}(\beta)$ is the tensor product of the last $n-k$ terms, which by assumption are all of the form $X^1(\beta_j)$.

We first restrict to all $W^n$'s such that the rightmost $X^1(\theta_i)$ appears at position $k$, where $k$ is any integer between 1 and $n$. Denote this set as $E_k$. The collections of $W^n$'s with the same feasible choice $(Y, Z) = (Y^{k-1}, Z^{n-k})$ as a subset of $E_k$, denoted by $E_k(Y, Z)$. For every feasible choice of $(Y^{k-1}, Z^{n-k})$, we can sum up all $W^n(Y, Z)$'s
in $E_k$ with the same realization $(Y^{k-1}, Z^{n-k})$. By linearity, we can sum together the terms $X^1(\theta_i)$ on the position $k$ to get

$$\sum_{E_k(Y,Z)} W^n(Y, Z) = Y^{n-1}(\theta, \beta) \otimes \left( \sum_{i=l+1}^{l+p} X^1(\theta_i) \right) \otimes Z^{n-k}(\beta),$$

where the sum is taken over $E_k(Y, Z)$. Now suppose the product $Y^{k-1}(\theta, \beta)$ consists of $p$ terms of $X^1(\theta_i)$’s and $k-1-p$ terms of $X^1(\beta_j)$’s. Since $\|X^1(\theta_i)\| \leq 2\delta$, $\|X^1(\beta_j)\| < \delta$, and also by assumption

$$\left\| \sum_{i=l+1}^{l+p} X^1(\theta_i) \right\| < 2\varepsilon,$$

it is straightforward that

$$\left\| W^n(Y^{k-1}, Z^{n-k}) \right\| < 2\varepsilon \cdot 2^p \cdot \delta^{n-1}.$$

We first consider the case $k = n$. In this case, if there are $p$ terms of $X^1(\theta_i)$’s and $n-1-p$ terms of $X^1(\beta_j)$’s in the product of the first $n-1$ terms, the number of such choices are

$$\binom{N-1}{p} \cdot \binom{N-1}{n-1-p} \leq \frac{N^{n-1}}{p!(n-1-p)!}.$$

Thus, we have

$$\left\| \sum_{E_n} W^n(Y^{n-1}, Z^0) \right\| \leq \sum_{E_n} \left\| W^n(Y^{n-1}, Z^0) \right\|$$

$$\leq \sum_{p=0}^{n-1} 2\varepsilon \cdot 2^p \cdot \delta^{n-1} \cdot \frac{N^{n-1}}{p!(n-1-p)!}$$

$$= 2\varepsilon \cdot (3L)^{n-1} \cdot \frac{N^{n-1}}{(n-1)!}.$$

Now we consider the case $k \leq n-1$. Same as before, we assume there are $p$ terms of $X^1(\theta_i)$’s and $k-1-p$ terms of $X^1(\beta_j)$’s in the first $k-1$ terms. In addition, we let call the $k+1$-th term $X^1(\beta_l)$. The above setting implicit assumes the following relationships:

$$\frac{k-1}{2} \leq l \leq N - (n-k), \quad \max\{0, k-1-l\} \leq p \leq \min\{k-1, l\}.$$
With this setting, we have
\[
\left\| \sum_{k=1}^{n-1} \sum_{E_k} W^n(Y^{k-1}, Z^{n-k}) \right\| \leq 2\epsilon \cdot \delta^{n-1} \sum_{k=1}^{n-1} \sum_{l=\left\lfloor \frac{k-1}{n-1} \right\rfloor} \sum_{p=\max\{0, k-1-1\}}^{n-1} 2^p \cdot \frac{1}{p!(k-1-p)!} \cdot \frac{(N-l)^{n-k-1}}{(n-k-1)!}
\]
\[
\leq 2\epsilon \cdot \delta^{n-1} \sum_{k=1}^{n-1} \sum_{l=0}^{n-1} \sum_{p=0}^{l} 2^p \cdot \frac{l^{k-1}}{p!(k-1-p)!} \cdot \frac{(N-l)^{n-k-1}}{(n-k-1)!}
\]
\[
= 2\epsilon \cdot \delta^{n-1} \sum_{k=1}^{n-1} (3l)^{k-1} \cdot \frac{(N-l)^{n-2-(k-1)}}{(k-1)!} \cdot \frac{1}{(n-k-1)!}
\]
\[
= 2\epsilon \cdot \delta^{n-1} \frac{(n-2)!}{(n-2)!} \sum_{l=0}^{n-1} (N+2l)^{n-2}
\]
\[
\leq 2\epsilon \cdot \delta^{n-1} \frac{(n-2)!}{(n-2)!} \int_1^N (N+2x)^{n-2}dx
\]
\[
\leq \epsilon \cdot (3L)^{n-1} \frac{(n-1)!}{(n-1)!}
\]

Combining the above estimate with the case \(k = n\), we get
\[
\left\| S_{N,n}^{(1)} \right\| \leq 3\epsilon \cdot (3L)^{n-1} \frac{(n-1)!}{(n-1)!}.
\]

Sending \(N \to +\infty\), by the previous lemma, we conclude that
\[
\left\| X^n - X^n(\beta) \right\| \leq 3\epsilon \cdot (3L)^{n-1} \frac{(n-1)!}{(n-1)!}.
\]

The right hand side of (13) appears like the bound for signature of degree \(n\). This is because one needs to compensate the uniform distance \(\epsilon\) with one degree of signature. It is worthy to note that, the scale of \(n-1\) in the bound cannot be both improved to \(n\). To see this, consider the one dimensional path \(\gamma(t) = et, t \in [0, 1]\). Then,
\[
\left\| X^n_{s,t}(\gamma) \right\| = e^n \frac{(t-s)^n}{n!}
\]

If the bound were at the scale of \(n\), then it would be \(e^{n+1} \frac{C(t-s)^n}{n!}\) for some constant \(C\) independent of \(n\). One can choose \(\epsilon\) small enough so that \(\left\| X^n_{s,t}(\gamma) \right\|\) exceeds this bound. So, it is not possible to improve the right hand side of (13) to the scale of \(n\).

For the lattice path approximation in the previous section, we have \(|\hat{\gamma}^N(t) - \gamma(t)| \leq \frac{1}{2^n}\), so we immediately get the following corollary.

**Corollary 4.5.** Let \(\gamma\) be a path in \(\mathbb{R}^d\) of length \(L\), and \(\{\hat{\gamma}^N\}\) be a sequence of lattice paths, each with step size \(\frac{1}{2^n}\) and length at most \(L\), and satisfying
\[
|\hat{\gamma}^N(t) - \gamma(t)| < \frac{d}{2^n}
\]
for all \(t\), as constructed in the previous section. Then,
\[
\left\| X^n(\hat{\gamma}^N) - X^n(\gamma) \right\| < 3d \cdot (3L)^{n-1} \frac{2^n}{(n-1)!}.
\]
4.3 Inversion

Let \( X = (1, X^1, X^2, \ldots) \) be the signature of a path \( \gamma \). Suppose further that \( \gamma \) is tree-reduced and has \( L \). By the construction and estimates previous two sections, for every \( N \), there exists a lattice path \( \hat{\gamma}^{(N)} \) with step size \( \frac{1}{2N} \) and length at most \( L \) such that (9) holds. The uniform bound on the lengths implies that there are only finitely many candidates for each \( \hat{\gamma}^{(N)} \), so it guarantees one can find such a sequence by checking all the possibilities at each level \( N \).

This sequence \( \{\hat{\gamma}^{(N)}\} \), if all starting from the origin, and parametrized at unit speed, is uniformly bounded and equicontinuous. Thus, it contains a uniformly convergent subsequence. Denote its limit by \( \hat{\gamma} \). By theorem 4.4, for each \( n \), we have

\[
X^n(\hat{\gamma}^{(N)}) \to X^n(\hat{\gamma})
\]
as \( N \to +\infty \), and thus we get

\[
X(\hat{\gamma}) = X(\gamma),
\]
so \( \hat{\gamma} \sim \gamma \) by the uniqueness theorem of Hambly and Lyons.

On the other hand, as a uniform limit, we have

\[
|\hat{\gamma}| \leq L,
\]
but \( L \) is the length of the tree reduced path \( \gamma \), so we must have

\[
\hat{\gamma} = \gamma.
\]

Thus, every uniformly convergent subsequence of \( \{\hat{\gamma}^{(N)}\} \) has the same limit \( \gamma \), and so the whole sequence must also converge uniformly to \( \gamma \). In this way, we can approach \( \gamma \) arbitrarily close in uniform distance, and invert the signature asymptotically.

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\(^4\)Note that in the approximation scheme, we require the lengths of all lattice paths to be bounded by a real number \( L' \). If \( L' > L \), we can lower the bound \( L' \) until it reaches \( L \). So in fact we can approach \( L \) arbitrarily closely.
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