A sketch of brane dynamics in seven and eight dimension using E theory

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Abstract

Using the general properties that have emerged from E theory we sketch the generic features of the dynamics of branes in seven and eight dimensions. The dynamical equations are a set of duality equations involving the coordinates of the vector representation of $E_{11}$. 
1 Introduction

The dynamics in E theory follows from the non-linear realisation of $E_{11} \otimes s l_1$. The $E_{11}$ part encodes the fields and the vector ($l_1$) representation encodes the coordinates. The non-linear realisation is constructed from a group element $g$ which belongs to the group $E_l$, whose Lie algebra is $E_{11} \otimes s l_1$, and it is subject to the transformations $g \rightarrow g_0g$ where the rigid transformation $g_0 \in E_l$ and $g \rightarrow gh$ where the local transformation $h$ is in the local subgroup which is specified as part of the definition of the non-linear realisation. The dynamics is determined by requiring that it is invariant under these two transformations and so different choices of local subalgebra lead to different dynamics. The symmetries in the local subgroup correspond to symmetries that are preserved and so linearly realised, while the ones not in the local subalgebra are those that are spontaneously broken and these are non-linearly realised.

If we take the fields to depend on the coordinates and the local subalgebra to be the Cartan involution invariant subalgebra of $E_{11}$, denoted $I_c(E_{11})$ then one derives the low energy effective action for strings and branes as conjectured long ago [1,2]. Indeed, once one restricts to the lowest level fields and coordinates, the dynamics contains the maximal supergravity theories. In particular if one takes the decomposition of $E_{11}$ to $GL(11)$ one finds the equations of motion of eleven dimensional supergravity [3,4] and one will inevitably find the other maximal supergravities if one takes the other decompositions corresponding to the algebra that results from deleting the other nodes in the $E_{11}$ Dynkin diagram. For a review see reference [5].

However, if one takes the coordinates to depend on variables that parameterise the branes, the fields to depend on these coordinates and the local subgroup $H$ to be a subalgebra of $I_c(E_{11})$ then one finds the dynamics of branes. The different choices of local subgroup leads to the different branes. However, each brane carries the full $E_{11}$ symmetry but the symmetries that are spontaneously broken vary from brane to brane [6,7].

The decomposition of $E_{11}$ to $GL(11)$ results in the theory in eleven dimensions and the generators of $E_{11}$ can be found, for example, in the book [8], while the generators in the vector representation are given by [2,9,10]

\[ P_\alpha, Z^{a \cdots a_2}, Z^{a_1 \cdots a_2}, Z^{a_1 \cdots a_2 b}, Z^{a_1 \cdots a_2 c}, Z^{(c)(d)c \cdots c}, Z^{cd c \cdots c}, Z^{cd c \cdots c \cdots a_{10}}, Z^{a_1 \cdots a_{11}}, Z^{a_1 \cdots a_{12}}, Z^{a_1 \cdots a_{13}}, Z^{a_1 \cdots a_{14}}, Z^{a_1 \cdots a_{15}}, (2), Z^{a_1 \cdots a_{15} a_{16}}, Z^{a_1 \cdots a_{15} a_{16}}, (2), Z^{a_1 \cdots a_{15} a_{16}}, (3), \ldots \] (1.1)

Each block of indices contain indices that are totally antisymmetrised except when () is present and this indicates that the indices are symmetrised instead. The elements have multiplicity one except when there is a bracket after the object which contains a number that gives the multiplicity. All the generator belong to irreducible representations of $SL(11)$, for example $Z^{a_1 b_2 c_3 \cdots c_s}$ obeys the constraint $Z^{a_1 b_2 c_3 \cdots c_s} = 0$.

The theory in $D$ dimensions results from deleting the node $D$ in the $E_{11}$ Dynkin diagram and the generators in the vector representation which have totally antisymmetrised indices are given in the table below [11, 6,10].
Table 1. The form generators in the $l_1$ representation in D dimensions

| D  | G               | $Z$  | $Z_{a_1a_2}$ | $Z_{a_1...a_3}$ | $Z_{a_1...a_4}$ | $Z_{a_1...a_5}$ | $Z_{a_1...a_6}$ |
|----|-----------------|------|--------------|------------------|------------------|------------------|------------------|
| 8  | $SL(3) \otimes SL(2)$ | (3, 2) | (3, 1)       | (1, 2)           | (3, 1)           | (1, 3)           | (8, 1)           |
|    |                 |      |              |                  |                  | (6, 2)           |                  |
|    |                 |      |              |                  |                  |                  |                  |
| 7  | $SL(5)$         | 10   | 5            | 5                | 24               | 1                | 40               |
|    |                 |      |              |                  |                  |                  | 15               |
|    |                 |      |              |                  |                  |                  | 10               |
|    |                 |      |              |                  |                  |                  | 70               |
|    |                 |      |              |                  |                  |                  | 50               |
|    |                 |      |              |                  |                  |                  | 45               |
|    |                 |      |              |                  |                  |                  | 5                |
| 6  | $SO(5, 5)$      | 16   | 10           | 16               | 45               | 1                | 144              |
|    |                 |      |              |                  |                  |                  | 16               |
|    |                 |      |              |                  |                  |                  | 126              |
|    |                 |      |              |                  |                  |                  | 120              |
|    |                 |      |              |                  |                  |                  |                  |
| 5  | $E_6$           | 27   | 27           | 78               | 351              | 1                | 1728             |
|    |                 |      |              |                  |                  |                  | 27               |
|    |                 |      |              |                  |                  |                  | 351              |
|    |                 |      |              |                  |                  |                  | 27               |
|    |                 |      |              |                  |                  |                  |                  |
| 4  | $E_7$           | 56   | 133          | 912              | 8645             | -                | -                |
|    |                 |      |              |                  |                  |                  |                  |
|    |                 |      |              |                  |                  |                  |                  |
|    |                 |      |              |                  |                  |                  |                  |
| 3  | $E_8$           | 248  | 3875         | 30380            | -                | -                | -                |
|    |                 |      |              |                  |                  |                  |                  |
|    |                 |      |              |                  |                  |                  |                  |
|    |                 |      |              |                  |                  |                  |                  |

We note the presence of the generators at level one that are Lorentz scalars but belong to representations of $E_{11-D}$. Given the one to one relation between generators in the vector representation and coordinates in the non-linear realisation it follows that for each element in the table we have a coordinates in the spacetime through which the brane moves. As such the coordinates in the above table are the main characters in this paper. In fact there are an infinite number of coordinates in the vector representation most of which have indices that can not be written as a single antisymmetrised block. These later coordinates will not play a significant role in this paper as we will be concerned with low level branes. The coordinates in the vector representation at higher levels than the usual coordinates $x^a$ of our familiar spacetime play an essential role in the construction of the low energy effective action of strings and branes even though they must be truncated out to gain the supergravity results we are familiar with. The Lorentz scalar coordinates, in the first column first proposed in the $E_{11}$ papers referenced above, are the starting point for papers on the so called exceptional field theory, see for example reference [12] for an account of this. As we will discover, while some coordinates given the embedding of the
brane in our usual spacetime, some of the coordinates are the world volume fields of the branes [6,7].

We now briefly review some of the main features of how one computes the brane dynamics from the Cartan forms [6,7]. While this paper does not contain any detailed calculations the construction below is required to justify the discussions in this paper. We can write the group element $g$ of the non-linear realisation in the form $g = g_g h g_E$ where $g_E$ is in the Borel subgroup of $E_{11}$, $g_l$ is formed from the generators of the $l_1$ representation and $g_h$ belongs to $I_c(E_{11})$. These group elements can be written in the form

$$g_l = e^{z^A l_A}, \quad g_E = e^{A_\alpha R^\alpha}, \quad g_h = e^{\varphi S}$$  \hspace{1cm} (1.2)

where $R^\alpha$ and $S^\alpha$ are the generators of the Borel subalgebra of $E_{11}$ and $I_c(E_{11})$ respectively. In equation (1.2) the $z^A$ are the coordinates of the background space-time and they depend on the parameters $\xi^\alpha$ of the brane world volume. The fields $A_\alpha$ are the $E_{11}$ background fields, which include those of the maximal supergravity theories, and they depend on the coordinates of the background spacetime $z^A$. The fields $\varphi$ also depend on $\xi^\alpha$. The transformation $h$ of the local subgroup $\mathcal{H}$ depends on the parameters $\xi^\alpha$ in an arbitrary way and we can use this to set some of the fields $\varphi$ to zero. We note that the non-linear realisation used for branes involves the additional fields $\varphi$ which are not present for the non-linear realisation used to derive the low energy effective action of strings and branes.

The Cartan forms are given by

$$V = g^{-1} dg = V_E + V_l + V_h^B,$$  \hspace{1cm} (1.3)

where

$$V_E = g_E^{-1} dg_E, \quad V_l = g_E^{-1} g_h^{-1}(g_l^{-1} dg_l) g_h g_E,$$

$$V_h^B = g_E^{-1} (g_h^{-1} dg_h) g_E = g_E^{-1} V_h g_E = g_E^{-1} g_h^{-1} dg_h g_E$$  \hspace{1cm} (1.4)

The Cartan forms $V_E$ are just the Cartan forms of $E_{11}$ and they only depend on the background fields $A_\alpha$. The Cartan forms associated with the vector representation are given by

$$V_l \equiv \nabla^B z^A l_A = g_E^{-1} g_h^{-1}(dz^A l_A) g_h g_E = g_E^{-1} (\nabla z^A l_A) g_E \equiv \nabla^B \Pi A l_A$$  \hspace{1cm} (1.5)

where $\nabla \equiv d\xi^\alpha \nabla_\alpha$ and $E_{11}^A$ is defined by $g_E^{-1} dz \cdot l g_E \equiv dz^B \Pi A l_A$ which is the vielbein in background spacetime and also only depends on the background fields $A_\alpha$. The fields $\varphi$ only occur in the Cartan forms $\nabla_\alpha z^A$ and $g_h^{-1} dg_h$ which are independent of the background fields $A_\alpha$.

The Cartan forms are inert under the rigid $g_0$ transformations, but under the local $h \in \mathcal{H}$ transformations they transform as

$$V \rightarrow h^{-1} V h + h^{-1} dh$$  \hspace{1cm} (1.6)

and in particular that

$$\nabla^B z^A l_A \rightarrow h^{-1} (\nabla^B z^A l_A) h, \quad V_h^B \rightarrow h^{-1} V_h^B h + h^{-1} dh$$  \hspace{1cm} (1.7)
Using this equation it is straightforward to explicitly compute the local transformations of the individual Cartan forms using the $E_{11} \otimes_{s} l_{1}$ algebra. As the dynamics consists of a set of equations that are invariant under these transformations and that the $\nabla_{\alpha} z^{A}$ transform covariantly, we are looking for equations which relate these Cartan forms to each other. We will also demand that the equations are invariant under arbitrary reparameterisations of the brane world volume.

For simplicity we will consider the brane dynamics in the absence of background fields. In this case we will take $g_{E}$ to be the identity element and instead of taking the algebra $E_{11} \otimes_{s} l_{1}$ we take the non-linear realisation of $I_{c}(E_{11}) \otimes_{s} l_{1}$ with local subalgebra $H$. Then we only have the Cartan forms $\nabla_{\alpha} z^{A}$ and $\mathcal{V}_{h} = g_{h}^{-1} dg_{h}$. In fact the equations of motion are constructed from only the Cartan forms $\nabla_{\alpha} z^{A}$. The brane dynamics in the presence of the background fields can be readily found from the resulting equations by using equation (1.5) to simply reinstate their presence by introducing the vielbein in the way that this equation dictates, that is, make the replacement $\nabla_{\alpha} z^{A} \rightarrow \nabla_{\alpha}^{B} z^{A}$. We will not in this paper consider the construction of the Wess-Zumino term in the brane dynamics.

Although, there are still a number of features of the above construction which are yet to be fully understood, the general features are apparent [6,7].

- The equations of motion are constructed from objects that are first order in derivatives, that is, they involve the Cartan forms $\nabla_{\alpha} z^{A}$ and not derivatives acting on these forms. As a result they are equations which equate the different Cartan forms associated with the vector representation to each other in such a way as to preserve the local subalgebra $H$.

- Some of the equations which follow from the non-linear realisation are ones which can be used to solve analytically for all of the fields $\varphi$ in terms of the coordinates, while others are dynamical equations for the coordinates and these are duality conditions.

In this paper we will use the above guidelines to sketch the dynamics of the low level branes in seven and eight dimensions. The advantage of this approach is that the reader can see what are the general features of branes in $E$ theory without being distracted by the $E_{11}$ formalism and a morass of equations. We will, in particular, focus on finding the generic form of the dynamical equations rather than the equations that are used to solve algebraically for the fields $\varphi$.

As just discussed we are searching for duality relations between the Cartan forms which are invariant under the transformations of the linear local subalgebra $H$. We need not consider the rigid transformations as the Cartan forms are invariant under them. At the linearised level in the fields the equations of motion must be linear in the Cartan forms and as such we begin by searching for equations which relate one Cartan form to another. Indeed we must find relations that pair up the Cartan forms and relate them using the epsilon symbol on the world volume of the brane. For a p-brane we will divide the indices $\underline{a}, \underline{b}, \ldots = 0, 1, \ldots, D - 1$, where $D$ is the dimension of the spacetime in which the brane moves, into those in the brane directions $a, b, \ldots = 0, 1, \ldots, p$ and those transverse to the brane $a', b', \ldots = p + 1, \ldots, D - 1$. As such the epsilon symbol in the brane world volume is denoted by $\epsilon^{a_{1} \cdots a_{p+1}}$. We will adopt this convention throughout this paper.

We recall that the theory in $D$ dimensions arises when we delete the node labelled $D$ in the $E_{11}$ Dynkin diagram which leaves the algebra $GL(D) \otimes E_{11-D}$. We then decompose
$E_{11}$ into representations of this later algebra. These representations are labelled by a level which depends on the node being deleted; the level zero representation is just the algebra that remains when we delete the node labelled $D$ in the $E_{11}$ Dynkin, namely $GL(D) \otimes E_{11-D}$. The Cartan involution invariant subalgebra of $E_{11}$, denoted $I_c(E_{11})$, is of the form $R^\alpha - R^{-\alpha}$ where $\alpha$ is a positive root which has a positive level. Thus the generators in $I_c(E_{11})$ are made up of two generators of opposite level. The exception is when these generators have level zero. We will refer to this as the level zero part of $I_c(E_{11})$ and it is the Cartan involution invariant subalgebra of $GL(D) \otimes E_{11-D}$ which contains the Lorentz algebra $SO(1,D-1)$ and $I_c(E_{11-D})$ and is just $SO(1,D-1) \otimes I_c(E_{11-D})$.

The non-linear realisation used to construct the low energy effective action for string and branes requires a local subalgebra $H$ which is a subalgebra of $I_c(E_{11})$. Indeed, a p-brane breaks the $SO(1,D-1)$ Lorentz symmetry of the background spacetime to $SO(1,p) \otimes SO(D-p-1)$ which is part of $H$. A brane charge also generically belongs to a representation of the U duality algebra, $E_{11-D}$, and so also to $I_c(E_{11-D})$. As such the particular brane charge which is active will also break $I_c(E_{11-D})$ into an algebra that is part of $H$ and which we denote by $H_0^c$. If we denote the level zero part of $H$, in the above sense, by $H_0$, then $H_0$ will contain $SO(1,p) \otimes SO(D-p-1)$ and $H_0^c$; indeed $H_0 = SO(1,p) \otimes SO(D-p-1) \otimes H_0^c$.

A brane moves through a spacetime with the coordinates in the vector representation. In any dimensions the level zero coordinate in the vector representation is $x^\alpha$ which is the usual coordinate of our familiar spacetime. As we have just explained this must satisfy a duality relation with one of the other coordinates. If we consider a simple p-brane, that is, a brane whose charge in totally antisymmetric in its indices, then its charge will be of the form $Z^{a_1 \ldots a_{p+1}}$ where $\bullet$ denotes the indices of the representation of the U duality group $E_{11-D}$ to which it belongs. This representation breaks into representations of $I_c(E_{11-D})$. The particular brane charge that is active will break this latter group to $H_0^c$ under which the active charge is a singlet. Corresponding to this singlet charge the non-linear realisation possess a coordinate which we denote by $y_{a_1 \ldots a_{p+1}}$. We can write down a duality relation between this coordinate and the coordinate $x^\alpha$ [7]

$$\nabla_\alpha x^a = -e_1 \epsilon^{a b_1 \ldots b_p} \nabla_\alpha y_{b_1 \ldots b_p} \tag{1.8}$$

where $e_1$ is a constant. In all the cases studied so far one also finds that the non-linear realisation implies the condition $\nabla_\alpha x^a = 0$. Part of this equation can be solved for a certain field $\varphi$ and the other part is a dynamical equation for the transverse coordinates $x^a$ [7]. As a result one can derive the equation

$$\sqrt{-\gamma} \gamma^{\alpha \beta} \nabla_\beta x^a = e_1 \epsilon^{\alpha \beta \gamma_1 \ldots \gamma_{p-1}} \nabla_\beta x^{ab_1 \ldots b_{p-1}} \nabla_{\gamma_1} x_{b_1} \ldots \nabla_{\gamma_{p-1}} x_{b_{p-1}} \tag{1.9}$$

where $\gamma_{\alpha \beta} = \nabla_\alpha x^a \nabla_\beta x^b \eta_{ab}$. This last equation can be shown to be the familiar equations for brane dynamics and we refer the reader to reference [7] for an account. These equations hold for all the branes considered in this paper and so we will in what follows concentrate on the other dynamical equations.

The higher level symmetries of the non-linear realisation will transform the above equation (1.9) into equations that are duality relations between the Cartan forms of some the other coordinates in the vector representation. However, in this paper, rather than
carry out such transformations we will simply search for generic duality relations that are
invariant under the local transformations of level zero, \( \mathcal{H}^0 \), which consists of the groups
\( SO(1,p) \otimes SO(D-p-1) \) and \( \mathcal{H}^U_0 \). As it is a duality relation, it will contain the epsilon
symbol \( \epsilon_{a_1...a_{p+1}} \). Let us consider the coordinates \( x_{a_1...a_n} \) which will have the Cartan form
\( \nabla_\alpha x_{a_2...a_{n+1}} \). It will prove advantageous to rewrite this Cartan as \( \nabla_{a_1} x_{a_2...a_{n+1}} \) where
\( \nabla_a = (s^{-1})_a^\alpha \nabla_\alpha \) and \( s^{a}_{\alpha} = \nabla_\alpha x^a \). We will also use this definition throughout this paper.
We expect that it will be related to a coordinate with \( m \) indices, that is \( x_{a_1...a_m} \) where
\( m = p - n - 1 \) through an equation whose generic form is given by

\[
\nabla_{a_1} x_{a_2...a_{n+1}} = \epsilon_{a_1a_2...a_{n+1}} b_1 b_2 ... b_{m+1} \nabla_{b_1} x_{b_2...b_{m+1}} \tag{1.10}
\]

where \( \epsilon_2 \) is a constant.

The above discussion has neglected the fact that the coordinates \( x_{a_1...a_n} \) and \( x_{a_1...a_m} \)
belong to the representations, denoted by \( R^0_n \) and \( R^0_m \) respectively of the \( U \) duality group.
However, these representations decompose into a sum of representations of the algebra \( \mathcal{H}^U_0 \)
preserved by the brane. To find a consistent equation one must select representations in
the two decompositions that can be related by an \( \mathcal{H}^U_0 \) invariant tensor and use this tensor
in the above duality relation. If \( m = n \), that is when \( 2n = p - 1 \) the duality relation can
become a self duality relation.

The dynamics of the two and five brane in eleven dimensions, from the view point of
E theory, was given in references [6,7]. It will be instructive to illustrate how the above
procedures apply to these branes. The brane charges in eleven dimensions are given in
equation (1.2) and as a result the non-linear realisation has the following coordinates

\[
x_{a_1...a_n}, x_{a_1...a_{n+1}}, x_{a_1...a_{n+2}}, x_{a_1...a_{n+3}}, x_{a_1...a_{n+4}}, \ldots \tag{1.11}
\]

The Cartan involution invariant subalgebra of \( E_{11} \), denoted \( I_c(E_{11}) \), has the Lorentz algebra
SO(1,10) at level zero.

The charge of the M2 brane is \( Z^{a_1a_2} \) and selecting a particular charge breaks the
SO(1,10) Lorentz algebra down to \( SO(1,2) \otimes SO(8) \) which is the local subalgebra \( \mathcal{H} \) at
level zero, that is \( \mathcal{H}_0 = SO(1,2) \otimes SO(8) \). Corresponding to the M2 brane charge we
have the coordinate \( x_{a_1a_2} \). This coordinate, together with the coordinate \( x^a \) will obey
the duality relation of equation of equation (1.9). This does indeed correctly describe the
motion of the M2 brane.

The five brane has the brane charge \( Z^{a_1...a_5} \) and this brane breaks the Lorentz symmetry down to \( SO(1,5) \otimes SO(5) \). The corresponding coordinate \( x_{a_1...a_5} \), together with the coordinate \( x^a \) obeys equation (1.9). At a lower level we have the two form coordinate
\( x_{a_1a_2} \) which, following the above discussion, should satisfy a self-duality relation which is
invariant under the level zero symmetries, that is,

\[
\nabla_{[a_1} x_{a_2a_3]} = \frac{1}{3!} \epsilon_{a_1a_2a_3} b_1 b_2 b_3 \nabla_{b_1} x_{b_2a_3} \tag{1.12}
\]
where \( \nabla_a \) was defined above equation (1.10). These equations we derived in reference [7] using the higher level symmetries in addition to those at level zero. They reproduce the correct dynamics to all orders in the usual embedding coordinate \( x^a \) and up to the linear level in the world volume field \( x_{a_1 a_2} \); the situation for non-linear terms in the later field is discussed in reference [7]. These branes illustrate the general pattern, the coordinates in the vector representation contain the usual embedding coordinate as well as the world volume fields and they satisfy duality relations.

2 Branes in seven dimensions

The seven dimensional theory emerges when we decompose \( E_{11} \) into \( GL(7) \otimes SL(5) \) which is the algebra that emerges when we delete node seven in the \( E_{11} \) Dynkin diagram. The generators in the vector representation are given by [7]

\[
P_a; \quad Z^{MN}; \quad Z^a_M; \quad Z^{a_1 a_2}_M; \quad Z^{a_1 a_2 a_3}_M; \quad Z^{a_1 a_2 a_3}_M; \quad Z^{a_1 a_2 a_3}_M; \quad Z^{a_1 a_2 a_3}_M; \quad Z^{a_1 a_2 a_3}_M; \quad Z^{a_1 a_2 a_3}_M; \quad ...
\]

where \( a, b, \ldots = 0, 1, \ldots, 6 \) and the indices \( M, N, \ldots = 1, \ldots, 5 \) are those of \( SL(5) \).

As a result the brane moves through a spacetime with the coordinates

\[
x^a; \quad x^{MN}; \quad x_a^M; \quad x_{a_1 a_2}^M; \quad x_{a_1 a_2 a_3}^M; \quad x_{a_1 a_2 a_3 b}^M; \quad x_{a_1 a_2 a_3}^M; \quad ...
\]

and the Cartan forms which belong to the vector representation of the \( E_{11} \) algebra can be written in the form

\[
\mathcal{V}_l = \nabla x^a P_a + \nabla x^{PQ} Z^{PQ} + \nabla x^a Z^a - M + \nabla x_{a_1 a_2}^M Z^{a_1 a_2}_M + \nabla x_{a_1 a_2 a_3}^M Z^{a_1 a_2 a_3}_M
\]

\[
+ \nabla x_{a_1 a_2 a_3 b} Z^{a_1 a_2 a_3} + \nabla x_{a_1 a_2 a_3}^M Z^{a_1 a_2 a_3}_M + \ldots
\]

The Cartan forms transform under the local subalgebra \( \mathcal{H} \) which is a subalgebra of \( I_c(E_{11}) \). At level zero the latter contains the Lorentz algebra \( SO(1,6) \) and the U duality algebra \( SO(5) \) and as such at level zero \( \mathcal{H} \) is a subalgebra of this. We have already constructed the dynamics of the one and two branes in seven dimensions but it will be useful to illustrate the methods of this paper to find their generic form.

2.1 The one brane

The one brane has a two dimensional world volume and, as explained above, we take the indices in the directions of the world volume of the string to take the values \( a, b, \ldots = 0, 1 \) and the reminder to be given by \( a', b', \ldots = 2, \ldots, 6 \). The brane will preserve \( SO(1,1) \otimes SO(5) \) of the \( SO(1,6) \) Lorentz symmetry and so this symmetry belongs to the local subaglebra \( \mathcal{H} \). The one brane charge is given by \( Z_a^a \). A given string has a given charge and making this selection we break the internal \( SO(5) \) symmetry down to \( SO(4) \) which will also belong to \( \mathcal{H} \). Indeed the level zero part is given by \( \mathcal{H}_0 = SO(1,1) \otimes SO(5) \otimes SO(4) \). As explained at the end of section one we adopt the duality relation (1.9) between the corresponding coordinate \( y_a \) and the usual spacetime coordinate \( x^a \).
Examining the coordinates of equation (2.1) we find the Lorentz scalar coordinates $x_{PQ}$ which under the decomposition to SO(4) leads to a coordinates in the $10 = 6 \oplus 4$ representations of SO(4). We can denote these by $x_{ij}$ and $x_i$, where $i, j = 1, \ldots, 4$ respectively. Taking the former coordinate, we can write down equations that are first order in the Cartan forms and are invariant under the local subalgebra $H$ and in particular its level zero part, given above. Such equations are of the generic form

$$\nabla_a x_{ij} = -\frac{1}{2} \epsilon_a^b \epsilon_{ijkl} \nabla_b x^{kl} \quad \text{or equivalently} \quad \sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_{\beta} x_{ij} = -\frac{1}{2} \epsilon^{\alpha\beta} \epsilon_{ijkl} \nabla_{\beta} x^{kl} \quad (2.1.1)$$

where $\nabla_a = (s^{-1})_a^\alpha \nabla_\alpha$ and $s_\alpha^a = \nabla_\alpha x^a$. Of course to find the full equations of motion one has to find a set of equations that are invariant under the full symmetries of the non-linear realisation and not just the level zero symmetries. This was been done in reference [7] where the choice of local subalgebra $H$ and the corresponding transformations of the Cartan forms can be found.

Assuming that the equations that follow from the non-linear realisation imply that these are the only dynamical field we can count the number of bosonic degrees of freedom. We have $7-2 = 5$ degrees of freedom in $x_{i}^a'$, taking into account the world volume reparametrisation symmetry, and $\frac{4 \cdot 3}{2} = 3$ from $x_{ij}$ which gives us 8 bosonic degrees of freedom. This is the number required for a half BPS brane that is maximally supersymmetric. A brane with these degrees of freedom would arise from the dimensional reduction of the IIA string.

2.2 The two brane

The two brane has a three dimensional world volume and it breaks the Lorentz symmetry SO(1,6) into $SO(1,2) \otimes SO(4)$ which is in the local subalgebra. The charge for the two brane is $Z^{a_1a_2M}$ which transforms in the 5 of the internal SO(5) symmetry. Selecting a particular charge breaks SO(5) to SO(4). As a result the local subalgebra at level zero is $H_0 = SO(1,2) \otimes SO(4) \otimes SO(4)$. The active two brane charge corresponds to a coordinate which we denote by $y^{a_1a_2}$ and this, together with the coordinate $x_{ij}$, will satisfy the duality relation of equation (1.9).

Examining the other coordinates in equation (2.2) we find the coordinates $x_{MN}$ and $x_{a}^{M}$ which, under the decomposition to SO(4), decompose into a $10 = 6 \oplus 4$ and a $5 = 4 \oplus 1$ respectively. To find a duality relation between the corresponding Cartan forms we must choose the 4 from each coordinate and then we can write down the relation

$$\nabla_a x_{i} + \frac{1}{4} \epsilon_a^{c_1c_2} \nabla_{c_1} x_{c_2i} = 0 \quad \text{or equivalently} \quad \sqrt{-\gamma} \gamma^{\alpha\beta} \nabla_{\beta} x_{i} + \frac{1}{4} \epsilon^{\alpha\beta_1\beta_2} \nabla_{\beta_1} x_{d_1} \nabla_{\beta_2} x_{d} = 0 \quad (2.2.1)$$

Assuming that these are the only active fields the number of bosonic degrees of freedom are 4 for $x_{i}^{a'}$ and 4 for $x_i$ giving a total of 8 bosonic degrees of freedom. This is the correct number for a maximally supersymmetric brane in a type II theory. This is the same content as the dimensional reduced M2 brane of eleven dimensions. The fully non-linear equations of motion, the choice of local subalgebra $H$ and the corresponding transformations of the Cartan forms were given in reference [7].

2.3 The three brane
The three brane has a four dimensional world volume and the part of the Lorentz symmetry $SO(1,6)$ which is in the local subalgebra $\mathcal{H}$ is $SO(1,3) \otimes SO(3)$. The charge for the three brane is $Z_{a_1 a_2 M N}$ which belongs to the 10 dimensional representation of the internal symmetry $SO(5)$. Choosing a particular charge for the three brane preserves only $SO(3) \otimes SO(2)$ which belongs to the local subalgebra $\mathcal{H}$ at level zero. Decomposing the 10 into this group we find that it consists of $10 = (3, 1) \oplus (3, 2) \oplus (1, 1)$, the last component being the active three brane charge. Let us denote the coordinate corresponding to this charge by $y_{a_1 a_2}$, and this, together with the coordinate $x^a$, can be taken to obey the duality equation of equation (1.9).

Examining equation (2.2) we find the coordinate $x^M_{a_1 a_2}$ which belongs to the 5 of $SO(5)$ which can only be dual to itself given that the world volume epsilon symbol has four spacetime indices. The 5 decomposes into the $5 = (3, 1) \oplus (1, 2)$ representations of $SO(3) \otimes SO(2)$. If we consider the latter we can write down the duality equation

$$\nabla_{[a_1} x^{a_2]} = \pm \frac{1}{2} \epsilon_{a_1 a_2 b_1 b_2} \epsilon^{i' j'} \nabla_{b_1} x_{b_2 j'}$$

where $i', j', \ldots = 1, 2$ are the $SO(2)$ indices. There are no consistent duality relations one can write down for the $(3, 1)$.

The coordinate $x_{MN}$ of equation (2.1) could be dual to the coordinate $x_{a_1 a_2 M}$ which belong to the 10 and 5 of $SO(5)$ respectively. Using the decompositions of these representations given above we find that they have $(3, 1)$ in common which we denote by $x_{ij}$ and $x_{a_1 a_2 i}$ where $i, j = 1, 2, 3$ respectively. Using these coordinates we can write down the duality relation

$$\nabla_{a_1} x_{ij} = \epsilon_{a_1 b_1 b_2 b_3} \epsilon_{ij k} \nabla_{b_1} x_{b_2 b_3 k}$$

As with all the duality relations in this paper one can rewrite them so that they contain $\nabla_{a}$ rather than $\nabla_{a}$ using the techniques given at the end of section one. For equation (2.3.2) we find that it can be rewritten as

$$\sqrt{-\gamma} \gamma^{\alpha \beta} \nabla_{\beta} x_{ij} = \epsilon^{\alpha \beta_1 \beta_2 \beta_3} \epsilon_{ij k} \nabla_{\beta_1} x_{d_1 d_2} \nabla_{\beta_2} x_{d_1} \nabla_{\beta_3} x_{d_2}$$

Assuming that the other coordinates do not contribute to the dynamics then the number of bosonic degrees of freedom is $7 - 4 = 3$ for $x^{a'}$, 3 for $x_{ij}$ and 2 for $x^k_a$ making 8 in all. Thus it contains 6 scalars and one vector as does $N = 4$ supersymmetric Yang-Mills theory. We might consider this theory to be a dimensional reduction of the IIB D3 brane.

2.4 The four brane

The four brane has a five dimensional world volume and the part of the Lorentz symmetry $SO(1,6)$ which is in the local subalgebra $\mathcal{H}$ is $SO(1,4) \otimes SO(2)$. Examining equation (2.1) we find that there are three brane charges with four indices only two of which have four antisymmetric indices. We consider the case that the four brane charge arrises from the charge $Z\bar{a}_1 \cdots \bar{a}_i M N$ which belongs to the $24 = 10 \oplus 14$-dimensional representation of $SO(5)$. We will choose the active brane charge to be $Z\bar{a}_1 \cdots \bar{a}_1 1 \cdots 2$ which breaks $SO(5)$ down to $SO(3)$. Under which the 24 contains the four singlets and one of these leads to a coordinate $y_{\bar{a}_1 \cdots \bar{a}_i}$ which, together with $x^a$, obeys equation (1.9).
The Lorentz scalar coordinates are dual to the three form coordinates both of which belong to the 10 of SO(5) which decomposes into \(10 = 3 \oplus 3 \oplus 3 \oplus 1\) of SO(3). We choose one of the 3’s and then write down the generic duality equation

\[
\nabla_a x_i = \epsilon_a^{b_1 \ldots b_4} \nabla_{b_1} x_{b_2 b_3 b_4} \quad i = 1, 2, 3
\]

(2.4.1)

The one form coordinates are dual to the two form coordinates and these both belong to the 5 of SO(5). We choose one of the singlets under SO(3) and then we can write down the generic equation

\[
\nabla_{[a_1} x_{a_2]} = \epsilon_{a_1 a_2}^{b_1 b_2} \nabla_{b_1} x_{b_2}
\]

(2.4.2)

We have \(7 - 5 = 2\) degrees of freedom from the transverse coordinates, 3 from the Lorentz scalars and \(5 - 2 = 3\) from the one form, which makes a count of 8 bosonic degrees of freedom.

For this case the duality equations are not uniquely determined by the level zero transformations of the local subalgebra and it is quite likely that there are other possible branes corresponding to the different choices of brane charge, the local subalgebra and the selection of different representations of the internal symmetry. What branes actually exist is determined by the full symmetries of the local algebra.

2.5 The five brane

The five brane has a six dimensional world volume and the part of the Lorentz symmetry SO(1,6) which is in the local subalgebra \(\mathcal{H}\) is SO(1,5). Examining equation (2.1) we find that there are five charges with five Lorentz indices which could be the brane charge. We will consider that the active charge is \(Z^{a_1 \ldots a_5}_{(MN)}\) and in particular the component \(Z^{a_1 \ldots a_5}_{(11)}\) with corresponding coordinate \(y_{a_1 \ldots a_5}\). As a result the internal SO(5) symmetry is broken to SO(4) which belongs to the local subgroup. As a result \(\mathcal{H}_0 = SO(1,5) \otimes SO(4)\). The coordinates \(x^a\) and \(y_{a_1 \ldots a_5}\) satisfies the duality relation of equation (1.9).

The Lorentz scalar coordinates \(x_{MN}\) belong to the 10 of SO(5) which decomposes into the \(10 = 6 \oplus 4\) of SO(4). It is dual to one of the coordinates with four Lorentz indices. Two of these are SO(5) singlets and the remaining one belongs to the \(24 = 10 \oplus 14\) of SO(5) which under SO(4) decomposes as \(24 = 6 \oplus 9 \oplus 4 \oplus 4 \oplus 1\). From these we take one of the 4’s and the 4 from the Lorentz scalar coordinates and then we can write down the generic duality relation

\[
\nabla_a x_i = \epsilon_a^{b_1 \ldots b_5} \nabla_{b_1} x_{b_2 b_3 b_4 i} \quad i, j = 1, 2, 3, 4
\]

(2.5.1)

The two form coordinate \(x_{a_1 a_2 M}\) belongs to the 5 of SO(5) which decomposes into \(5 = 4 \oplus 1\) under SO(4). Taking the singlet we find the self-duality relation which takes the generic form

\[
\nabla_{[a_1} x_{a_2 a_3]} = \frac{1}{3!} \epsilon_{a_1 a_2 a_3}^{b_1 b_2 b_3} \nabla_{b_1} x_{b_2 b_3}
\]

(2.5.2)

We have \(7 - 6 = 1\) transverse bosonic degrees of freedom, 4 from the scalars and \(\frac{4 \cdot 3}{2} = 3\) from the two form making 8 in all.

For the case of the five brane the level zero transformations of the local subalgebra do not uniquely determine the duality relations. For example we could instead select the
6-dimensional representation of SO(4) for the $x_{MN}$ and $x_{b_1\ldots b_4}{}^{MN}$ and then write down a duality relation. This would contribute 6 bosonic degrees of freedom. We have also assumed that the $x_a^M$ and $x_{a_1 a_2 a_3}{}^{MN}$ do not satisfy a duality relation that leads to degrees of freedom. Indeed, if we had chosen each to belong to the 4-dimensional representation of SO(4) then we could have written down a duality relation that contributes $4.4 = 16$ degrees of freedom. However, if one wants to get only 8 bosonic degrees of freedom then one must adopt the possibilities we first gave. It is likely that there exist other branes corresponding to different choices and this will be resolved by see which putative brane dynamics carries the full symmetries.

3 Branes in eight dimensions

We will now sketch the dynamics of some of the branes in eight dimensions; the pattern is similar to that in seven dimensions and so we will be brief. The eight dimensional theory emerges when we decompose $E_{11}$ into $GL(8)$ and the duality symmetry $SL(3) \otimes SL(2)$ which is the algebra that emerges when we delete node eight in the $E_{11}$ Dynkin diagram. The generators in the vector representation are given by

$$P_a (1,1); \ Z^{i,j'} (3,2); \ Z^{a_i \otimes i} (1,2); \ Z^{\bar{a}_i \otimes i} (1,1); \ Z^{a_1 \otimes a_4 \otimes i} (3,2),$$

$$Z^{a_1 \otimes a_3 \otimes i} (1,3); \ Z^{a_i \otimes i} (1,1); \ Z^{a_1 \otimes a_3 \otimes i} (3,2); \ Z^{a_1 \otimes a_2 (ij), k'} (6,2), \ldots (3.1)$$

where $a, b, \ldots = 0, 1, \ldots, 6$, the numbers in the brackets refer to the representations of $SL(3) \otimes SL(2)$ that the generators belong to and $i, j, \ldots = 1, 2, 3$ and $i', j', \ldots = 1, 2$ are the indices of $SL(3)$ and $SL(2)$ respectively.

As a result the brane moves through a spacetime with the coordinates

$$x^a (1,1); \ x_{i,j'} (3,2); \ x^i (3,1); \ x_{a_i \otimes i} (1,2); \ x_{a_1 \otimes a_3 \otimes i} (3,1); \ x_{a_1 \otimes a_3 i} (3,2),$$

$$x_{a_1 \otimes a_3 i} (1,3); \ x_{a_1 \otimes a_3 i} (1,1); \ x_{a_1 \otimes a_3 i} (8,1); \ x_{a_1 \otimes a_3 i} (3,2); \ x_{a_1 \otimes a_3 i} (6,2), \ldots (3.2)$$

The Cartan forms which belong to the vector representation of the $E_{11}$ algebra can be written in the form

$$\mathcal{V}_l = \nabla x^a P_a (1,1) + \nabla x_{i,j'} Z^{i,j'} + \nabla x^i Z^{a_i}_{i} + \nabla x_{a_1 a_2 i} Z^{a_1 a_2 i} + \nabla x_{a_1 a_3 i} Z^{a_1 a_3 i} + \nabla x_{a_1 a_3 i} Z^{a_1 a_3 i} + \nabla Z^{a_1 a_3 i} x_{a_1 a_3 i} \ldots (3.3)$$

The Cartan forms transform under the local subalgebra $\mathcal{H}$ which is a subalgebra of $I_c(E_{11})$ which at level zero contains the Lorentz algebra $SO(1,7)$ and the Cartan invariant subalgebra of the $U$ duality algebra, that is, $SO(3) \times SO(2)$.

3.1 The one brane

The one brane will preserve $SO(1,1) \otimes SO(6)$ of the $SO(1,7)$ Lorentz symmetry. The one brane charge is given by $Z^a_i$ which belong to the $(3,1)$ representation of the $SO(3) \times SO(2)$. Choosing a given charge we break the internal symmetry $SO(3) \times SO(2)$ down
to \(SO(2) \times SO(2)\) and we have the decomposition \((3, 1) = (2, 1) \oplus (1, 1)\). Denoting the coordinate associated with the later representation by \(y_{2\al}\), we adopt the duality relation of equation (1.9) between this coordinate and the usual spacetime coordinate \(x_{\al}\).

Examining the coordinates of equation (3.2) we find the Lorentz scalar coordinates \(x_{i,j'}\) in the \((3, 2)\) representation of the internal symmetry \(SO(3) \times SO(2)\); this decomposes into the \((3, 2) = (2, 2) \oplus (1, 2)\) representations of \(SO(2) \times SO(2)\). These must satisfy a self-duality equation and we choose this to hold for the \((2, 2)\) representation;

\[
\nabla_a x_{ij'} = -\frac{1}{2} \epsilon_a^b \epsilon_i^k \epsilon_{j'}^{l'} \nabla_b x_{kl'}
\]

where we define, as before, \(\nabla_a = (s^{-1})_a^\al \nabla_\al\) and \(s_a^\al = \nabla_\al x^a\). In fact this is not the only equation we can write down which is invariant under the level zero symmetries. We can write the above equation but with no \(\epsilon\)'s in the internal indices and we could also write an equation for the \((1, 2)\) representation, also with no \(\epsilon\)'s in the internal indices Using the techniques discussed around equation (1.8) we can rewrite equation (3.1.1) in the form

\[
\sqrt{-\gamma} \gamma^\alpha\beta \nabla_\beta x_{ij'} = -\frac{1}{2} \epsilon^\alpha\beta \epsilon_i^k \epsilon_{j'}^{l'} \nabla_\beta x_{kl'}
\]

Assuming that these are the only dynamical field we can count the number of bosonic degrees of freedom.

We have 8 - 2 = 6 degrees of freedom in \(x_{a'}\) and \(\frac{10 \times 2}{2} = 2\) from \(x_{ij'}\) which gives us 8 bosonic degrees of freedom.

3.2 The two brane

The two brane preserves only \(SO(1, 2) \otimes SO(5)\) of the \(SO(1, 7)\) Lorentz symmetry. The charge \(Z_{a_1 a_2 j'}\) of the two brane is which transforms in the \((1, 2)\) representation of the internal \(SO(3) \times SO(2)\) symmetry and choosing a particular charge, say \(Z_{a_1 a_2 1}\), breaks the internal symmetry down to \(SO(3)\). As a result \(H_0 = SO(1, 2) \otimes SO(4) \otimes SO(3)\). We denote the coordinate associated with the active charge by \(y_{\al a_2}\) and take this, together with the coordinate \(x_{\al}\), to satisfy the duality relation of equation (1.9).

Examining the other coordinates in equation (3.2) we find the Lorentz scalar coordinates \(x_{ii'}\) in the \((3, 2)\) representation of \(SO(3) \times SO(2)\). Under the decomposition to \(SO(3)\) it breaks into \((3, 2) = 3 \oplus 3\) while the dual coordinate \(x^i_a\) belongs to the \((3, 1)\) representation of \(SO(3) \times SO(2)\) and it, of course, belongs to the 3 of \(SO(3)\). As a result we can write the duality equation

\[
\nabla_a x_i = \epsilon_a^{b_1 b_2} \nabla_{b_1} x_{b_2 i}, \quad i = 1, 2, 3
\]

Assuming that these are the only active fields the number of bosonic degrees of freedom are \(8 - 3 = 5\) for \(x^a_{a'}\) and \(3\) for \(x_i\) giving a total of 8 bosonic degrees of freedom.

The two brane in eight dimensions was discussed in reference [13] and it would be interesting to see what is the relation to this work.

3.3 The three brane

The three brane has a four dimensional world volume and it breaks the Lorentz symmetry \(SO(1, 7)\) into \(SO(1, 3) \otimes SO(4)\). The charge for the three brane is \(Z_{a_1 a_2 a_3 i}\) which
belongs to the \((3, 1)\) representation of the internal symmetry \(SO(3) \times SO(2)\). Choosing a particular charge preserves only \(SO(2) \otimes SO(2)\) and the \((3, 1)\) representation becomes the representations \((3, 1) = (2, 1) \oplus (1, 1)\). Denoting the coordinate associated with the last representation by \(y_\alpha \), it, together with the coordinate \(x^a\), obeys the duality equation (1.9).

Examining equation (3.2) we find the coordinate \(x_a^i\). This belongs to the \((3, 1)\) of \(SO(3) \otimes SO(2)\) which decomposes into \((3, 1) = (2, 1) \oplus (1, 1)\). This field is self-dual and choosing the \((2, 1)\) representation we can write down the equation

\[
\nabla_{[a_1 x_{a_2}]} i = \frac{1}{2} \epsilon_{a_1 a_2} b_1 b_2 b_4 \epsilon_i j \nabla_{b_1} x_{b_2 b_3 j} \tag{3.3.1}
\]

There is no such consistent equation for the \((1, 1)\) representation.

The Lorentz scalar coordinates \(x_{ij}'\) belong to the \((3, 2)\) representation of \(SO(3) \times SO(2)\) and which decomposes as given above. This coordinate is dual to the three form coordinate \(x_{a_1 a_2 a_3 i}\) which belongs to the \((3, 1)\) representation of \(SO(3) \times SO(2)\). The two representations have in common the 2 and 1 representations If we choose the former representation we can write down the duality relation

\[
\nabla_a x_i = \epsilon_a b_1 b_2 b_3 b_4 \epsilon^{ij} \nabla_{b_1} x_{b_2 b_3 b_4 j}, \quad i, j = 1, 2 \tag{3.4.1}
\]

In fact one could omit the epsilon in the internal indices and we could also taken instead the singlet at the symmetry level at which we are working.

The one form coordinate \(x_{a i}\) is dual to the two form coordinate \(x_{a_1 a_2 i'}\) which belong to the \((3, 1)\) and \((1, 2)\) representation of \(SO(3) \times SO(2)\) respectively. When decomposed
into $SO(2)$ they have in common only a singlet of $SO(2)$ and we can write down the generic equation

$$\nabla_{[a_1} x_{a_2]} = \epsilon_{a_1 a_2} b^1 b^2 b^3 \nabla_{b_1} x_{b_2 b_3}$$ (3.4.2)

We have $8 - 5 = 3$ transverse bosonic degrees of freedom, 2 in $x_i$ and $(5 - 2) = 3$ in $x_a$, making 8 in all.

4 The one brane in four dimensions

We now briefly discuss the one brane in four dimensions. Decomposing $E_{11}$ into representations of $GL(4) \otimes E_7$ we find the theory in four dimensions. The level zero part of $I_c(E_{11})$ is $SO(1,3) \otimes SU(8)$. The coordinates at low levels can be read off from the table given earlier in this paper. It will be useful to present the coordinates in terms of representations of $SU(8)$

$$x^{\mathbf{28}}(1), x^{ij}(28), x_{ij}(28), x^{a^i j}(63), x^{a^i i_1 \ldots i_4}(70), x^a, \ldots (1), \ i, j \ldots = 1, \ldots, 8$$ (4.1)

where the number in the brackets gives the dimensions of the $SU(8)$ representations.

The one brane preserves $SO(1,1) \otimes SO(2)$ of the $SO(1,3)$ Lorentz symmetry. The brane charge belongs to the $63 \oplus 70 \oplus 1$ representations of $SU(8)$. Let us choose the it to belong to the 70 dimensional representation, that is, $Z^{a_1 i_1 \ldots i_4}$ and take the charge $Z^{a_{1234}}$ to be the active charge. As a result the internal symmetry $SU(8)$ gets broken to $SU(4) \otimes SU(4)$ with the decomposition $70 = (1,1) \oplus (1,1) \oplus (4,4) \oplus (6,6) \oplus (4,\bar{4})$ with $Z^{a_{1234}}$ being one of the singlets. We denote the corresponding coordinate by $y_a$ and it, together with $x^{\mathbf{28}}$ will obey equation (1.9).

The Lorentz scalar coordinates decompose into representations of $SU(4) \otimes SU(4)$ as $28 = (6,1) \oplus (4,4) \oplus (1,6)$ with similar results for the $\bar{28}$. These coordinates must obey a self-duality condition, namely

$$\nabla_a x^{ij} = \frac{1}{2} \epsilon_a^b \epsilon^{ijkl} \nabla_b x_{kl}, \ i, j, k, l = 1, 2, 3, 4$$ (4.1)

Counting the bosonic degrees of freedom we have $4 - 2 = 2$ from the transverse coordinates $x^{a^i}$ and $\frac{4 \cdot 3}{2} = 6$ from the $x^{ij}$ making 8 in all. There may well be other possible one branes one could construct using the full symmetries of the non-linear realisation.

5 Discussion

In two previous papers [6,7] we discussed how to construct brane dynamics as a non-linear realisation of $E_{11} \otimes s_1$ and we have shown that it leads to many features of the brane dynamics that we know. While the construction of the dynamics of a given brane is rather complicated there have emerged a number of generic features; for example the dynamical equations when constructed from the Cartan forms are a set of duality equations. By construction these equations are invariant under the symmetries of the non-linear realisation and in particular the lowest level such symmetries. In this paper we have applied these general features to find in outline only the dynamics of the low level branes in seven and eight dimensions. We find that the coordinates of the vector representation do indeed provided the fields for a set of duality relations that look to be of the right type in that
they contain the expected coordinates describing how the brane moves through the usual spacetime as well as the required world volume fields. Unlike the superficial impression that might be gained by studying the familiar branes in eleven and ten dimensions, world volume fields are generically present as long as they are consistent with the duality relations of equation (1.10). Hence although this paper contains no calculations of any length we hope that it does provide an insight into the general form of the dynamics of branes in E theory that is not obscured by the formalism or lengthy equations.

We find that the generic form of the dynamical equations are generally determined by the lowest level symmetries although for some of the higher level branes considered here are several possibilities. It would be interesting to see how the generic dynamical equations become completely determined once they are required to be invariant under the higher level symmetries.

While references [6,7] set out the general method to determine the brane dynamics from the non-linear realisation there are a number of steps where a very systematic path is absent. The situation is not unlike that which occurred when the $E_{11} \otimes s l_1$ non-linear realisation was used to find the low energy effective action for strings and branes, indeed it took quite a few years before the unique path became clear and the dynamics constructed. Three of the outstanding issues are as follows

- The non-linear realisation requires for its construction a choice of local subalgebra $H$ which for the branes is a subalgebra of the Cartan involution invariant subalgebra of $E_{11}$. Studies of particular branes have shown that the local subalgebras are much more subtle than one might naively expect [7]. It would be good to have a systematic way of choosing the local subalgebras that lead to brane dynamics; indeed such an understanding may lead to a classification of all branes from a purely algebraic viewpoint. We hope to report on progress in this direction elsewhere.

- The brane dynamics, as usually formulated, consists of equations that contain second order derivatives acting on some of the fields. However, the brane dynamics that emerges from E theory is a set of duality equations that are first order in derivatives acting on fields. This is like the derivation of the low energy effective action of strings and branes from E theory. In the later case one can act on the duality equations with a spacetime derivative to eliminate field certain fields and obtain equations which are second order in derivatives and are those of maximal supergravity once one restricts the equations to the lowest level fields and coordinates. One would expect that a similar pattern will hold for the brane dynamics derived from E theory and it would be good to see that this is the case. In fact this is the case for branes with no world volume fields and at the linearised level in world volume fields when they are present. However, it would be good to see the known features of the world volume fields emerge at the non-linear level.

The equations that follow from the non-linear realisation contain dynamical equations, as studied in this paper, and algebraic equations that solve for some of the $\phi$ fields, associated with the breaking of $I_c(E_{11})$ to $H$, in terms of derivatives acting on the coordinates. However, the dynamical equations also contain the $\phi$ fields and so only once one has solved for these fields can one see the final form of the dynamical equation in terms of the usual fields. A systematic way to do this has yet to be found.
- The non-linear realisation for branes is formulated so that the fields and coordinates depend on the brane parameters which label points in the world volume swept out by the brane. However, in E theory the spacetime is infinite dimensional and so one can wonder what is the brane world volume in this very large spacetime?

The vector representations at low levels contains the brane charges of all branes that we are familiar with. However, it contains an infinite number of branes and it is reasonable to suppose that it encodes all brane charges. As a result E theory contains an infinite number of new degrees of freedom which might be very useful when studying problem such as black hole entropy. Once the above issues are resolved it would be very interesting to determine the dynamics of the exotic branes which occur at higher levels in the vector representation.

Acknowledgements

I wish to thank Michaella Pettit and Paul Cook for discussions. We wish to thank the SFTC for support from Consolidated grants number ST/J002798/1 and ST/P000258/1.

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