Financial markets as adaptive ecosystems

Marc Potters*, Rama Cont*,# and Jean-Philippe Bouchaud†,*

*Science & Finance, 109–111 rue Victor Hugo,
92532 Levallois Cedex, France

†Service de Physique de l’État Condensé, Centre d’études de Saclay,
Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

#Laboratoire de Physique de la Matière Condensée CNRS URA 190
Université de Nice- Sophia Antipolis B.P. 70, 06108 Nice, France

January 13, 2022

First version: Sept. 1996. This version: June 1997.

02.50 Probability theory, stochastic processes and statistics
05.40.+j Fluctuation phenomena, stochastic processes and statistics
89.90+n Other areas of general interest to physicists

Abstract

We show, by studying in detail the market prices of options on liquid markets, that the market has empirically corrected the simple, but inadequate Black-Scholes formula to account for two important statistical features of asset fluctuations: ‘fat tails’ and correlations in the scale of fluctuations. These aspects, although not included in the pricing models, are very precisely reflected in the price fixed by the market as a whole. Financial markets thus behave as rather efficient adaptive systems.

Options markets offer an interesting example of the adaptation of a population (the traders) to a complex environment, through trial and errors and
natural selection (inefficient traders disappear quickly). The problem is the following: an ‘option’ is an insurance contract protecting its owner against the rise (or fall) of financial assets, such as stocks, currencies, etc. The problem of knowing the value of such contracts has become extremely acute ever since organized option markets opened twenty five years ago, allowing one to buy or sell options much like stocks. Almost simultaneously, Black and Scholes (BS) proposed their famous option pricing theory, based on a simplified model for stock fluctuations, namely the (geometrical) continuous time Brownian motion model. The most important parameter of the model is the ‘volatility’ $\sigma$, which is the standard deviation of the market price’s fluctuations. The Black-Scholes model is known to be based on unrealistic assumptions but is nevertheless used as a benchmark by market participants. Guided by the Black-Scholes theory, but constrained by the fact that ‘bad’ prices lead to arbitrage opportunities, the option market fixes prices which are close, but significantly and systematically different from the BS formula. Surprisingly, a detailed study of the observed market prices clearly shows that, despite the lack of an appropriate model, traders have empirically adapted to incorporate some subtle information on the real statistics of price changes. Although this ability to price financial assets correctly is often assumed in the literature (the ‘efficient market’ hypothesis), it is in general difficult to assess quantitatively, because the ‘true’ value of a stock, if it exists, is difficult to determine. The case of option markets is interesting in that respect, because the ‘true’ value of an option is, in principle, calculable.

More precisely, a ‘call’ option is such that if the price $x(T)$ of a given asset at time $T$ (the ‘maturity’) exceeds a certain level $x_s$ (the ‘strike’ price), the owner of the option receives the difference $x(T) - x_s$. Conversely, if $x(T) < x_s$, the contract is lost. To make a long story short [1, 2, 3, 4], if $T$ is small enough (a few months) so that interest rate effects and average returns are negligible compared to fluctuations, the ‘fair’ price $C$ of the option today ($T = 0$), knowing that the price of the asset now is $x_0$ is simply given by [1]:

$$C(x_0, x_s, T) = \int_{x_s}^{\infty} dx' (x' - x_s) P(x', T|x_0, 0)$$  \hspace{1cm} (1)

where $P(x', T|x_0, 0)$ is the conditional probability density that the stock price at time $T$ will be equal to $x'$, knowing its present value is $x_0$. Eq. (1) means that the option price is such that on average, there is no winning party. Pricing correctly an option is thus tantamount to having a good model for
the probability density \( P(x', T|x_0, 0) \).

There is fairly strong evidence that beyond a time scale \( \tau \) of the order of ten minutes, the fluctuations of prices in liquid markets are uncorrelated, but not independent variables \([1, 4, 8, 9, 10]\). In particular, it has been observed that although the signs of successive price movements seem to be independent, their magnitude - as represented by the absolute value or square of the price increments - is correlated in time \([3, 10]\): this is related to the so-called ‘volatility clustering’ effect \([11, 7]\). More precisely one can represent the price \( x(T) \) of the asset as

\[
x(T) = x_0 + \sum_{k=0}^{T-1} \delta x_k
\]

where the increments \( \delta x_k \) are obtained as the product of two random variables:

\[
\delta x_k = \epsilon_k \gamma_k
\]

where \((\epsilon_k)_{k \geq 0}\) is a sequence of independent, identically distributed random variables of mean zero and unit variance, and \(\gamma_k\) is a stochastic scale factor independent from the \(\epsilon_k\)s. The sequence \((\gamma_k)_{k \geq 0}\) is considered to be a stationary random process but allowed to exhibit non-trivial correlations (see below). Under these hypotheses, the conditional distribution of \(\delta x_k\), conditioned on \(\gamma_k\), may be written as:

\[
P(\delta x_k) \equiv \frac{1}{\gamma_k} P_0 \left( \frac{\delta x_k}{\gamma_k} \right)
\]

where \(P_0\) is independent of \(k\). Models with conditionally Gaussian increments - i.e. where \(P_0\) is a Gaussian – have been extensively studied \([11]\) both in discrete time (ARCH models) and continuous time (stochastic volatility models) settings. The present model is more general since we do not assume that \(P_0\) is Gaussian.

Let us first consider the case where \(\gamma_k = \gamma_0\) is independent of \(k\), which corresponds to the classical problem of a sum of independent, identically distributed variables. Although \(P(\delta x)\) is strongly non Gaussian (see, e.g. \([13]\)), it has a finite variance \([3]\) and the Central Limit Theorem \([14]\) tells us that for large \(N = T/\tau\), \(P(x', T|x_0, 0)\) will be close to a Gaussian. Using then Eq. \([1]\) essentially leads back to the BS formula \([12]\). For finite \(N\),
however, there are corrections to the Gaussian, and thus corrections to the BS price. More precisely, the difference between $P(x', T|x_0, 0)$ and the limiting Gaussian distribution $G_{x_0, \sigma^2}$ may be calculated using a cumulant expansion \[14\]. To a very good approximation, the distribution $P_0(\delta x)$ is symmetric \[13, 4\] for time scales less than a month i.e. drift effects are negligible compared to fluctuations. This in turn implies that the third cumulant, which measures the skewness of the distribution, is small, in which case the leading correction in the cumulant expansion mentioned above is, for large $N$, proportional to the kurtosis $\kappa$, defined as $\kappa = \langle \delta x^4 \rangle / \langle \delta x^2 \rangle^2 - 3$ \[14\]. $\kappa$ vanishes if the increments $\delta x$ are Gaussian random variables, and measures the ‘fatness’ of the tails of the distribution as compared to a Gaussian.

Neglecting higher order cumulants, the expansion takes the following form:

\[
\int_{-\infty}^{x} \left\{ P(x', T|x_0, 0) - G_{x_0, \sigma^2_T}(x') \right\} dx' = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \left[ \frac{\kappa_T}{24} (u^3 - 3u) + \ldots \right]
\]  

where $u = (x - x_0)^2 / \sigma^2_T$, $\sigma^2_T$ and $\kappa_T$ being the variance and kurtosis corresponding to the scale $T$. $G_{x_0, \sigma^2_T}$ is the gaussian centered at $x_0$ of variance $\sigma^2_T$.

It is then easy to show, using Eq. (1), that the leading correction to the BS price can be reproduced by using the BS formula, but with a modified value for the volatility $\sigma = \sqrt{\langle \delta x^2 \rangle}$ (which traders call the ‘implied volatility’ $\Sigma$), which depends both on the strike price $x_s$ and on the maturity $T$ through:

\[
\Sigma(x_s, T) = \sigma \left[ 1 + \frac{\kappa_T}{24} \left( \frac{(x_s - x_0)^2}{\sigma^2_T} - 1 \right) \right]
\]  

The fact that implied volatility depends on the strike price $x_s$ is known as the ‘smile effect’, because the plot of $\Sigma$ versus $x_s$, for a given value of $T = N\tau$ has the shape of a smile (see Fig 1).

That the volatility had to be smiled up was realized long ago by traders – this reflects the well known fact that the elementary increments have fat-tailed distributions: large fluctuations occur much more often than for a Gaussian random walk.

As shown in Fig. 2, the smile formula \[6\] reproduces correctly the observed option prices on the ‘Bund’ market provided the kurtosis $\kappa_T$ in formula
Figure 1: Example of a smile curve: Implied volatility $\Sigma(x_s, T)$ vs distance from strike price $(x - x_s)$ for a given $T$. The data shown correspond to all 227 transactions of December options on the German Bund future (LIFFE) on November 13, 1995. This is a very ‘liquid’ market, meaning that price anomalies are expected to be small, in particular for short maturities $T$. Both call and put options are included with put options transformed into call options using the put-call parity $[2]$. Volatilities are expressed as annualized standard deviation of price differences. According to Eq. (6) the data should fall on a parabola. From a fit of the average curvature of this parabola, we extract the ‘implied kurtosis’ $\kappa_{imp}$ for a given $N = \frac{T}{\tau}$. In this particular case we find $\kappa_{imp} = 1.92$ at $N = 144$ (9 trading days).
Figure 2: Plot (in log-log coordinates) of the average implied kurtosis $\kappa_{\text{imp}}$ (determined as in Fig. 1) and of the empirical kurtosis $\kappa_T$ (determined directly from the historical movements of the Bund contract), as a function of the reduced time scale $N = T/\tau$, $\tau = 30$ minutes. All transactions of options on the Bund future from 1993 to 1995 were analyzed along with 5 minute tick data of the Bund future for the same period. The growth of the error bars for the latter quantity comes from the fact that less data is available for larger $N$. Finally, we show for comparison a fit with formula (8), with $g(\ell) \simeq \ell^{-0.6}$, which leads to $\kappa_T \simeq T^{-0.6}$ (dark line). A fit with an exponentially decaying $g(\ell)$ is however also acceptable (dotted line).
becomes itself $T$-dependent. The shape of the ‘implied’ kurtosis $\kappa_{\text{imp}}(T)$ as a function of $T$ is given in Fig. 2; $\kappa_{\text{imp}}(T)$ is seen to decrease more slowly than $T^{-1}$. However, if the increments $\delta x$ were independent and identically distributed (i.e. $\gamma_k \equiv \gamma_0$), one should observe that $\kappa_T = \kappa/N$.

Let us then study directly the kurtosis of the distribution of the underlying stock, $P(x,T|x_0,0)$, as a function of $N \equiv T/\tau$. In Fig. 2, we have also shown $\kappa_T$ as a function of $N$. One can notice that not only $T \kappa_T$ is not constant (as it should if $\delta x$ were identically distributed), but actually $\kappa_T$ matches quantitatively (at least for $N \leq 200$) the evolution of the implied kurtosis $\kappa_{\text{imp}}$! (Note that there is no adjustable overall factor.) In other words, the price over which traders agree capture rather precisely the anomalous evolution of $\kappa_T$. A similar agreement has been found on other liquid option markets, where bid-ask spreads are sufficiently small to ascertain that the quoted prices should indeed be set by a fair game condition. For ‘over the counter’ options, this is likely not to be the case, since a rather high risk premium is generally included in the price.

As we shall show now, the non trivial behaviour of $\kappa_T$ is related to the fact that the scale of the fluctuations $\gamma_k$ is itself a time dependent random variable \[11, 6\], with rather long range correlations. The random character of $\gamma_k$ could come from the fact that $\gamma_k$ is related to the level of market activity, which fluctuates with time.

We define the correlation function of the scale of fluctuations as:

$$g(\ell) = \frac{\langle \delta x_k^2 \delta x_{k-\ell}^2 \rangle - \langle \delta x_k^2 \rangle^2}{\langle \delta x_k^2 \rangle^2}$$

$g(\ell)$ is normalized such that $g(0) = 1$. In this case, one can show that Eq. (6) holds, with $\kappa_T$ given by:

$$\kappa_T = \frac{T}{\tau} \left[ \kappa_T + 6(\kappa_T + 2) \sum_{\ell=1}^{N} (1 - \frac{\ell}{N}) g(\ell) \right]$$

where $\kappa_T$ is the kurtosis of $\delta x = x(t + \tau) - x(t)$. We have computed from historical data the correlation function $g(\ell)$, which we show in Fig. 3. Interestingly, $g(\ell)$ decreases rather slowly, as $\ell^{-\lambda}$, with $\lambda \simeq 0.6 \pm 0.1$, from minutes to several days. A similar decay of $g(\ell)$ was observed on other markets as well, with rather close values for $\lambda$, such as the S&P500 (for which $\lambda \simeq 0.37$) \[10]\ and the DEM/$\$ market (for which $\lambda \simeq 0.57$). Remarkably, Eq. (8) with
$g(N) \propto N^{-0.6}$ leads to $\kappa_T \propto T^{-0.6}$, in good agreement with both the direct determination of $\kappa_T$ and the one deduced from the volatility smile, $\kappa_{\text{imp}}$. Note that the effect of a non zero kurtosis on Black-Scholes prices was previously investigated in [15, 16]. However, the relation between $\kappa_T$ and $\kappa_{\text{imp}}$, and their anomalous $T$ dependence, were not, to our knowledge, previously reported.

In conclusion, we have shown by studying in detail the market prices of options that traders have evolved from the simple, but inadequate BS formula to an empirical know-how which encodes two important statistical features of asset fluctuations: ‘fat tails’ (i.e. a rather large kurtosis) and the fact that the scale of fluctuations exhibits slowly decaying correlations. These features, although not explicitly included in the theoretical pricing models used by traders, are nevertheless reflected rather precisely in the price fixed by the market as a whole. Option markets offer an interesting ground where
‘theoretical’ and ‘experimental’ prices can be systematically compared, and were found to agree rather well \[17\]. This has enabled us to test quantitatively the idea that the trader population behaves as an efficient adaptive system.

References

[1] F. Black, M. Scholes, *Journal of Political Economy*, 81 (1973) 637.

[2] J.C. Hull *Futures, Options and Other Derivative Securities*, Prentice Hall (1997).

[3] J.P. Bouchaud, D. Sornette, *Journal de Physique I* (France) 4 (1994) 863.

[4] J.P. Bouchaud, M. Potters *Theory of Financial Risk: Portfolios and Options*, C. Godrèche Editor, Aléa-Saclay (1997) (in french).

[5] In the presence of a non zero average return of the stock \(m \neq 0\), the probability distribution \(P\) in Eq. (1) must be replaced by another probability distribution \(Q\), called the ‘pricing kernel’. The formula relating \(Q\) to \(P\) to first order in \(m\) has been obtained in [4]. See also E. Aurell, S. Simdyankin, ”Pricing risky options simply”, submitted to *Int. J. Theo. Appl. Finance*.

[6] C. W. J. Granger, Z. X. Ding, ”Stylized facts on the temporal distributional properties of daily data from speculative markets”, Working Paper, University of California, San Diego (1994).

[7] D. M. Guillaume et al., ”From the bird’s eye to the microscope”, *Finance and Stochastics* 1, 2 (1997).

[8] A. Arneodo, J.P. Bouchaud, R. Cont, J.F. Muzy, M. Potters, D. Sornette, *cond-mat/9607120* Preprint, 1996.

[9] R. Cont, M. Potters & J.P. Bouchaud ”Scaling in stock market data: stable laws and beyond” in *Scale invariance and beyond* (Proceedings of the CNRS Workshop on Scale Invariance, Les Houches, March 1997).
[10] R. Cont “Scaling and correlation in financial data” cond-mat/9705075. Preprint, 1997.

[11] R. Engle, *Econometrica* 50 (1982) 987; T. Bollerslev, Journal of Econometrics, 31 (1986) 307. C. Gourieroux, *Modèles ARCH et Applications financières*, Economica, Paris (1992).

[12] Although in principle BS use a log-normal, rather than a normal distribution, the difference is not relevant for the present discussion. See [4] for a detailed discussion of this point.

[13] R. Mantegna and H.E. Stanley, Nature 376, 46-49 (1995).

[14] W. Feller, *An Introduction to Probability Theory and its Applications*, Wiley (1971).

[15] R. Jarrow, A. Rudd, Journal of Financial Economics, 10 (1982) 347.

[16] C.J. Corrado and T. Su, Journal of Financial Research, XIX (1996) 175.

[17] Note that small systematic differences have in some cases been found. These differences are however within transaction costs.