Generating Extreme Spin Squeezing

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We propose a novel scheme for the generation of optimal squeezed states for Ramsey interferometry. The scheme consists of an alternating series of one-axis twisting pulses and rotations, both of which are straightforward to implement experimentally. The resulting states show a metrological gain proportional to the Heisenberg limit. We demonstrate that the Heisenberg scaling is maintained even when placing constraints on the amplitude of the pulses implementing the one-axis twisting and when taking into account realistic losses due to photon scattering.

In quantum metrology, an ensemble of \( N \) non-interacting atoms provides a basic statistical scaling of the measurement precision with \( 1/\sqrt{N} \), also known as the standard quantum limit (SQL). However, quantum mechanics also provides for the possibility of entanglement between the atoms. The non-classical correlations can provide an advantage in the scaling of the measurement precision, up to the Heisenberg limit (HL) with a scaling of \( 1/N \), that is, a quadratic advantage \([1]\). Overcoming the SQL is central in the effort of advanced quantum metrology that actively exploits the fundamental quantum properties of the measurement device. Approaching the Heisenberg limit promises dramatic improvements in high-precision sensing with a relatively small amount of atoms. Such high-precision sensing is the ultimate objective of applications that include the search for dark matter \([2]\), tests of the fundamental laws of physics \([3, 4]\), gravitational waves detection \([5]\), timekeeping \([6]\), and geodesy \([7–9]\). Consequently, substantial effort has been devoted to design protocols that allow to achieve the limit.

A well-known strategy to reach a scaling beyond the SQL is the realization of spin squeezed states (SSS) \([10–13]\). These states are characterized by non-classical correlations that lower (squeeze) the variance of one measurement quadrature in the collective state by increasing the variance of the quadrature orthogonal to the measurement. The earliest proposal to generate SSS is based on engineering an effective atom-atom interaction, described as a one-axis twisting (OAT) Hamiltonian \([10]\). The spin-squeezed states that can be realized via one-axis twisting achieve a scaling beyond the SQL, but fall short of achieving the full Heisenberg limit. This is even more true when taking into account realistic losses in a typical experimental setup \([14]\).

A relevant figure of merit for a specific spin-squeezed state is the metrological gain, i.e., the attainable phase sensitivity \( \Delta \varphi \) for the measurement of the accumu-

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ates squeezing Hamiltonian may be used to produce a squeezed state with unequal quadratures of the distribution. Subsequently, the system evolves freely under the Hamiltonian $\delta \hat{S}_z$. This corresponds to a rotation around the $z$-axis of the generalized Bloch sphere, and accumulates a phase $\varphi$ directly proportional to the detuning $\delta$. A final $\pi/2$-pulse maps the phase $\varphi$ into a population difference between spin-up and spin-down states as measured by $\langle \hat{S}_z \rangle$. The measurement allows to estimate $\varphi$ with sensitivity

$$\Delta \varphi = \left| \frac{\Delta \hat{S}_z}{\partial \langle \hat{S}_z \rangle / \partial \varphi} \right|, \quad (1)$$

evaluated for the final state. Thus, squeezing of $\Delta \hat{S}_z$ in favor of a wider distribution along the $x$ and $y$ axes, as in the final state in Fig. 1, improves the precision of the phase estimation. In addition, we also have to consider the signal contrast, which depends on the range of values for $\langle \hat{S}_z \rangle$, with $|\langle \hat{S}_z \rangle| \leq |\langle \hat{S} \rangle|$. Since squeezing also lowers $|\langle \hat{S} \rangle|$, phase sensitivity and signal contrast are generally not independent.

The metrological gain is given by the Wineland squeezing parameter $\xi^2$,

$$\xi^2 = \frac{\Delta \varphi^2}{\Delta \varphi_{\text{CSS}}} = \min_{\hat{u}_\perp} \left( \frac{\langle \hat{S}_z \rangle}{\langle \hat{S} \rangle} \right) \sqrt{\frac{N}{\langle \hat{S} \rangle^2}}, \quad (2)$$

which relates the phase sensitivity without squeezing to the phase sensitivity with squeezing in Fig. 1. On the right-hand-side, the minimum is taken over all directions perpendicular to the mean spin direction $\langle \hat{S} \rangle$. For $\xi^2 = 1$, the measurement precision scales as the SQL with $1/\sqrt{N}$. For $\xi^2 < 1$, we have a squeezed state. In particular, a sensitivity scaling of the form $\xi^2 = N^{-1}$ would be proportional to the Heisenberg limit, and reach the exact Heisenberg limit for $a = 1$.

We define an extreme spin-squeezed state as the states that minimizes $\xi^2$, Eq. 2, under the condition that the total wavefunction is normalized and that $|\langle \hat{S} \rangle|$ and thus the signal contrast is fixed. Consequently, these states allow to achieve the maximum sensitivity of the Ramsey interferometric measurements [15,19]. To find these states, we apply variational calculus to minimize the functional

$$\mathcal{L}[|\Psi\rangle] = N \frac{\langle \hat{S}_z^2 \rangle}{\langle \hat{S}_z \rangle^2} - \lambda_1 \langle \hat{S} \rangle \langle \hat{S}_z \rangle - \lambda_2 \langle \hat{S}_z \rangle, \quad (3)$$

where $\lambda_{1,2}$ are Lagrange multipliers and the average is taken over the state $|\Psi\rangle$. Without loss of generality, we have chosen the mean spin direction to be $\hat{S}_z$, and the minimum perpendicular variance to be along the $z$ axis. Note that this is a different choice than in Fig. 1.

Using the Euler-Lagrange formalism, we obtain

$$N \left[ \frac{1}{\langle \hat{S}_z \rangle^2} \dot{\hat{S}}_z^2 - 2 \frac{\langle \hat{S}_z \rangle^2}{\langle \hat{S}_z \rangle^3} \dot{\hat{S}}_z - \frac{\lambda_2}{\langle \hat{S}_z \rangle} \right] |\Psi\rangle = \lambda_1 |\Psi\rangle, \quad (4)$$

which is an eigenvalue equation that characterizes extreme spin-squeezed states. Specifically, we recognize extreme-spin-squeezed states as eigenfunctions of a Hamiltonian of the form

$$\hat{H} = \chi \dot{\hat{S}}_z^2 - \Omega_\chi \hat{S}_z,$$  

where the ratio $\Omega/\chi$ is a function of $\langle \hat{S}_z \rangle$. The extreme spin-squeezed state $|\Psi\rangle$ has to be determined iteratively by solving for the $\Omega/\chi$ that gives the desired value of $\langle \hat{S}_z \rangle$ when solving the eigenvalue equation with the Hamiltonian in Eq. 4.

Figure 2 (a) shows the solution of the eigenvalue problem for different values of $\langle \hat{S}_z \rangle$. As $\langle \hat{S}_z \rangle \to 0$, the extreme spin-squeezed state converges to a ring on the equator of the Bloch sphere. In principle, this state provides the greatest metrological gain (minimal $\xi^2$). However, a very short spin length also implies a minimal signal in the experiment, which renders Ramsey interferometric measurements too susceptible to noise [15].

It has been shown that all extreme spin squeezed states scale proportionally to the HL [18,24], and the constant of proportionality depends on the mean spin length, in
implementations for an OAT setup. Figure 3 (a) shows any fundamental alterations to existing experimental im-
tation in the extended Bloch sphere. Hence, we propose
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iment. The first term is the Hamiltonian that imple-
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fortunately, this process is too slow to be practical. A
Remarkably, up to one extra term, this Hamiltonian has
The key to creating extreme spin-squeezed states is in
the Hamiltonian Eq. (5) of which they are eigenfunctions.
Remarkably, up to one extra term, this Hamiltonian has
the form of a one-axis twisting Hamiltonian [10] which is
easily implemented experimentally [14, 20–22]. The extra
term is a rotation of the collective spin in a perpendicular
direction.
Since the extreme spin-squeezed state is the ground
state of the Hamiltonian in Eq. (5), one possibility for
generating the state is by adiabatic time evolution. This
process would start from a coherent spin state pointing
along the x axis [19]. Considering the parameters $\chi$ and $\Omega$ as time-dependent control function would allow
to slowly turn on the one-axis twisting, tuning the
Hamiltonian from $\hat{H} = -\Omega S_z$ to $\hat{H} = \chi S_x^2 - \Omega S_z$. Un-
fortunately, this process is too slow to be practical. A
possible approach to overcome the slow adiabatic pro-
cess is to implement the well-known “shortcuts to adia-
baticity” (STA) [25], which would allow to quickly reach
the target state, at least approximately [26, 27]. How-
ever, an extra complication is that some experimental
sets generate the OAT evolution by a series of pulses,
which includes additional steps to cancel additional detri-
tial effects in the effective Hamiltonian. Incorporating
these refocusing pulses in an STA approach would not be
straightforward.
Here, we take an entirely different approach based on
the realization that the two terms in the Hamiltonian
in Eq. (5) are individually easy to realize in an exper-
iment. The first term is the Hamiltonian that imple-
ments OAT, whereas the second term is a simple rota-
tion in the extended Bloch sphere. Hence, we propose
a sequence of $n$ alternating OAT and $\hat{S}_x$ pulses that al-
low the creation of extreme spin-squeezed states without
any fundamental alterations to existing experimental im-
plementations for an OAT setup. Figure 3 (a) shows
the proposed pulse sequence for an example of $N = 60$
atoms initially in a CSS pointing along the x-axis. As
shown, only $n = 4$ pulses are sufficient to drive the CSS
into an extreme spin-squeezed states with $\langle \hat{S}_x \rangle = 0.9S$.
The extreme spin-squeezed state is reached with error
$\epsilon = 1 - |\eta|^2 \approx 1.3 \times 10^{-4}$, where $\eta = \langle \Psi \rangle$ is the
overlap between the final and target state $|\Psi_\text{CSS}\rangle$. To
find the optimal pulse sequence, we propagate the initial
state $|\Psi_\text{CSS}\rangle$ as
\[
|\Psi_\text{CSS}\rangle = \exp \left( -i\mu_n \hat{S}_z \right) \exp \left( -iQ_n \hat{S}_x^2 \right) \ldots \exp \left( -i\mu_2 \hat{S}_z \right) \exp \left( -iQ_1 \hat{S}_x^2 \right) |\Psi_\text{CSS}\rangle,
\]
where $Q_k = \chi A t$ is the shearing strength of $k$th OAT
pulse and $\mu_k = \Omega A t$ is the rotation angle produced
by the $k$th $S_z$ pulse, $A t$ is the pulse duration. We use
SciPy’s minimization routine [28] with the L-BFGS-
B method [29, 30] to iteratively find the parameters
$(Q_k, \mu_k)$ that minimize $\epsilon$.
In Fig. 3 (b), we show the achieved error $\epsilon$ as a function
of the number of atoms $N$ for different values of the con-
trast $\langle \hat{S}_x \rangle / S = 0.5$ and $\langle \hat{S}_x \rangle / S = 0.9$, using a sequence of
two and six pulses, respectively. Unsurprisingly, higher
contrast allows for higher fidelities. This is because the
extreme spin-squeezed state is closer to the initial state.
The fidelity also improves by increasing the number of
pulses. Remarkably, we find that when generating the
extreme spin-squeezed state with $\langle \hat{S}_z \rangle = 0.9S$ with four
pulses, the optimization error introduces a metrologi-
ugal loss of less than 0.05 dB compared to the result in
Fig. 2 (b). The loss of precision due to the optimization
error is higher for $\langle \hat{S}_x \rangle = 0.5S$ (between 3 and 6 dB).
Note that the four-pulse sequence $(n = 4)$ is the first
non-trivial case, since the two-pulse sequence is equiva-
ient to the OAT scheme, where the noise reduction scales
proportionally to $N^{-2/3}$ [10].
Optical methods to generate collective entanglement rely
on the interaction of the atomic ensemble with a
cavity light field. Cavity feedback squeezing [20, 23] is
a deterministic technique to generate a control Hamilton-
ian $\chi S_x^2$. The spin quantum noise tunes the atom-
cavity resonance such that the intracavity light inten-
sity is proportional to the $z$ component of the collective
atomic spin. Thus, the light induces an $\hat{S}_z$-dependent
light-shift which generates $\hat{S}_y$-$\hat{S}_z$ quantum correlations and
atomic entanglement in the process. Due to the
cavity-enhanced atom-light interaction, any information
contained in the light field results in the non-unitary evolu-
tion of the atomic system [31]. Moreover, any photon
scattered in the free space projects one atom into either
spin-up or spin-down, thus reducing the spin coherence.
However, as it has been recently demonstrated [22, 23]
and modeled [31, 32], decoherence can be minimized to
reach a near-unitary evolution by tuning the entangling-
light frequency. The sign of the $\chi S_x^2$ Hamiltonian can
be reversed simply by switching the sign of the detuning
between the atom-cavity resonance and the light [29, 31].
Note that this is essentially the OAT Hamiltonian plus the $z$-rotation term $\alpha \hat{S}_z$; the $\alpha S$ term produces only a global phase and thus can be neglected.

The OAT Hamiltonian can clearly be implemented by turning on the non-linear interaction shown in Eq. (8) for a specific time, then applying a $\pi$-pulse, and finally turning on the non-linear interaction again. Effectively, the result of this sequence is identical to the OAT Hamiltonian $\hat{H} = \chi \hat{S}_z^2$, since the $\pi$-pulse cancels the other terms, similar to a spin echo sequence. Applying $\hat{S}_z$ rotation simultaneously with the OAT Hamiltonian may be challenging, as it affects the spin echo effect. The optimized squeezing sequence of our proposal is not susceptible to this issue.

Another relevant experimental consideration is the contrast loss due to photon scattering. The contrast loss, $C_{\text{sc}}$, depends on both the total shearing strength $\hat{Q} = \sum_k Q_k$ and the number of atoms $N$, and can be parametrized as $C_{\text{sc}} = \exp(-\gamma \hat{Q})$, where $\hat{Q} = \sqrt{N} Q$ is the normalized shearing strength and $\gamma$ is the scaling parameter which we choose to be equal to 0.36 addressing the experimental conditions reported in [22]. This reduces the above calculated metrological gain to $\xi^2 = \xi^2/C_{\text{sc}}$, affecting the precision scaling.

A primary advantage of our scheme is that it is robust in the presence of such photon scattering. This is in contrast to other approaches that have been proposed, e.g., the combination of OAT interactions and rotations for the creation of effective two-axis twisting [31, 34–36], or modulating the OAT interaction and rotations to implement extreme-spin-squeezed states [19, 26, 35, 37]. Neither of these proposals consider the contrast loss due to photon scattering during the OAT process. Moreover, our scheme involves only minimal modifications to an experimental setup realizing OAT and uses only a small number of OAT applications, reducing the OAT-associated contrast loss.

Analyzing the extreme spin-squeezed states and the optimal pulse sequences that generate them, we observe that the states with lower $\langle \hat{S}_z \rangle / S$ provide higher metrological gain and creating them requires higher shearing strength. However, a reduction of the metrological gain due to the photon-scattering contrast loss is higher for the extreme SSSs with low $\langle \hat{S}_z \rangle / S$. Therefore, as an example of a good trade-off between the level of squeezing and shearing strength we consider the case $\langle \hat{S}_z \rangle = 0.9 S$ in the following analysis. In this case, applying only $n = 4$ pulses, the optimized extreme SSSs result in an error on the order of $10^{-4}$, see Fig. 3(b).

One way to minimize the effect of the contrast loss in the precision scaling is to fix $\hat{Q}$ during the optimization of the pulse sequence. As a result, the contrast loss $C_{\text{sc}}$ affects only the proportionality factor, but not the scaling. However, this reduces the metrological gain due to a reduction in the fidelity of the prepared state, so a scaling proportional to the Heisenberg limit is achievable only for sufficiently high values of $\hat{Q}$. Fortunately, the convergence to the extreme spin-squeezed state is fast with an increase of $\hat{Q}$, as we show in Fig. 4(a). The necessary shearing strength is of the order of one, $\hat{Q} \sim 1$, as demonstrated experimentally in previous work [22]. In Fig. 4(b), we plot the metrological gain corrected by the contrast loss due to photon scattering as a function of $\hat{Q}$ for different values of $N$. We observe a peak around $\sqrt{N} Q = 0.5$ where the trade-off between contrast loss and precision is optimal. The position of the optimal precision is moving to larger values of shearing strength $Q$ as we increase the number of atoms, $N$. We obtain 14.9 dB of metrological gain for $N = 350$ atoms using $Q = 0.55$, which surpasses the result reported in Ref. [22]. Finally, in Fig. 4(c), we show the scaling of the metrological gain as a function of $\hat{Q}$. As expected, it approaches the Heisenberg limit as we increase $\hat{Q}$. Thus, the proposed optimized squeezing sequence achieves a scaling proportional to the HL.

To conclude, we have shown that a sequence of one-axis twisting and rotations can overcome the precision scal-
ing of a single application of a one-axis twisting Hamiltonian, and reach the fundamental Heisenberg scaling. The approach uses the same elements as existing OAT implementations. Thus, it does not require significant modification to an existing experimental setup.

Fundamentally, the extreme spin squeezed states generated by our pulse scheme maximize metrological gain for Ramsey interferometry at a fixed signal output amplitude. They always attain Heisenberg scaling up to a constant factor, giving us additional flexibility, e.g., to limit the number of OAT pulses. This feature is useful when employing light-mediated interactions in an optical cavity to create an effective OAT as the photon scattering reduces contrast. To account for the contrast reduction due to photon scattering, we specifically target extreme SSSs that yield a high contrast signal, requiring a reduced shearing strength $Q$.

We have found that a single non-linear interaction, such as OAT, already provides a remarkable degree of control when combined with suitable non-entangling operations. This flexibility could be related to the results in Ref. [28], which shows that alternating applications of different Hamiltonians are often capable of approximating any quantum gate, or at least a relevant subset, as required here. We expect that similar strategies could be used to create other entangled states that optimize other figures of merit, such as the Fisher information or planar squeezing [39] [40]. The latter corresponds to a modification of the squeezing parameter that quantifies the squeezed states’ fitness over an extensive range of phases acquired during free evolution.

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[1] C. L. Degen, F. Reinhard, and P. Cappellaro, Quantum sensing, Rev. Mod. Phys. 89, 035002 (2017)
[2] A. Derevianko and M. Pospelov, Hunting for topological dark matter with atomic clocks, Nat. Phys. 10, 933 (2014)
[3] M. Safronova, D. Budker, D. DeMille, D. F. J. Kimball, A. Derevianko, and C. W. Clark, Search for new physics with atoms and molecules, Rev. Mod. Phys. 90, 025008 (2018)
[4] M. S. Safronova, Atomic clocks: The search for variation of fundamental constants with clocks (Ann. Phys. 5/2019), Ann. Phys. 531, 1970023 (2019)
[5] S. Kolkowitz, I. Pikovski, N. Langellier, M. D. Lukin, R. L. Walsworth, and J. Ye, Gravitational wave detection with optical lattice atomic clocks, Phys. Rev. D 94, 124043 (2016)
[6] B. Guinot and F. A. Arias, Atomic time-keeping from 1955 to the present, Metrologia 42, S20 (2005)
[7] T. E. Mehlstäubler, G. Groche, C. Lisdat, P. O. Schmidt, and H. Denker, Atomic clocks for geodesy, Rep. Prog. Phys. 81, 064401 (2018)
[8] J. Grotti, S. Koller, S. Vogt, S. Häfner, U. Sterr, C. Lisdat, H. Denker, C. Voigt, L. Timmen, A. Rolland, et al., Geodesy and metrology with a transportable optical clock, Nat. Phys. 14, 437 (2018)
[9] M. Takamoto, T. Ushijima, N. Ohmae, T. Yahagi, K. Kokado, H. Shinaki, and H. Katori, Test of general relativity by a pair of fundamental optical lattice clocks, Nat. Photon. 14, 411 (2020)
[10] M. Kitagawa and M. Ueda, Squeezed spin states, Phys. Rev. A 47, 5138 (1993)
[11] D. J. Wineland, J. J. Bollinger, W. M. Itano, F. L. Moore, and D. J. Heinzen, Spin squeezing and reduced quantum noise in spectroscopy, Phys. Rev. A 46, R6797 (1992)
[12] D. J. Wineland, J. J. Bollinger, W. M. Itano, and D. J. Heinzen, Squeezed atomic states and projection noise in spectroscopy, Phys. Rev. A 50, 67 (1994)
[13] J. Ma, X. Wang, C. Sun, and F. Nori, Quantum spin squeezing, Phys. Rep. 509, 89 (2011)
[14] B. Braverman, A. Kawasaki, E. Pedrozo-Peñaíel, S. Colombo, C. Shu, Z. Li, E. Mendez, M. Yamash, L. Salvi, D. Akamatsu, Y. Xiao, and V. Vuletíć, Near-unity spin squeezing in $^{171}$Yb, Phys. Rev. Lett. 122, 223203 (2019)
[15] P. R. Berman, Atom Interferometry (Elsevier, 1997)
[16] A. D. Cronin, J. Schmiedmayer, and D. E. Pritchard, Optics and interferometry with atoms and molecules, Rev. Mod. Phys. 81, 1051 (2009)
[17] A. M. Bloch and A. G. Rojo, Control of squeezed phonon and spin states, Eur. J. Control 10, 469 (2004)
[18] G. Tóth and I. Apellaniz, Quantum metrology from a quantum information science perspective, J. Phys. A: Math. Theor. 47, 424006 (2014)
[19] A. S. Sørensen and K. Mølmer, Entanglement and extreme spin squeezing, Phys. Rev. Lett. 86, 4431 (2001)
[20] I. D. Leroux, M. H. Schliefer-Smith, and V. Vuletíć, Implementation of cavity squeezing of a collective atomic spin, Phys. Rev. Lett. 104, 073602 (2010)
[21] I. D. Leroux, M. H. Schliefer-Smith, H. Zhang, and V. Vuletíć, Unitary cavity spin squeezing by quantum erasure, Phys. Rev. A 85, 013803 (2012)
[22] S. Colombo, E. Pedrozo-Peñaíel, A. F. Adiyatullin, Z. Li, E. Mendez, C. Shu, and V. Vuletíć, Time-reversal-based quantum metrology with many-body entangled states, arXiv:2106.03754 (2021)
[23] Z. Li, B. Braverman, S. Colombo, C. Shu, A. Kawasaki, A. Adiyatullin, E. Pedrozo-Peñaíel, E. Mendez, and V. Vuletíć, Collective spin-light and light-mediated spin-spin interactions in an optical cavity, arXiv:2106.13234 (2021).
[24] A. G. Rojo, Optimally squeezed spin states, Phys. Rev. A 68, 013807 (2003)
[25] A. del Campo, Shortcuts to adiabaticity by counterdiabatic driving, Phys. Rev. Lett. 111, 100502 (2013)
[26] T. Opatrný, H. Saberi, E. Brion, and K. Mølmer, Counterdiabatic driving in spin squeezing and Dicke-state preparation, Phys. Rev. A 93, 023815 (2016).

[27] T. Pichler, T. Caneva, S. Montangero, M. D. Lukin, and T. Calarco, Noise-resistant optimal spin squeezing via quantum control, Phys. Rev. A 93, 013851 (2016).

[28] P. Virtanen, R. Gommers, T. E. Oliphant, M. Haberland, T. Reddy, D. Cournapeau, E. Burovski, P. Peterson, W. Weckesser, J. Bright, S. J. van der Walt, M. Brett, J. Wilson, K. J. Millman, N. Mayorov, A. R. J. Nelson, E. Jones, R. Kern, E. Larson, C. J. Carey, I. Polat, Y. Feng, E. W. Moore, J. VanderPlas, D. Laxalde, J. Perktold, R. Cimrman, I. Henriksen, E. A. Quintero, C. R. Harris, A. M. Archibald, A. H. Ribeiro, F. Pedregosa, P. van Mulbregt, and SciPy 1.0 Contributors, SciPy 1.0: Fundamental algorithms for scientific computing in Python, Nat. Methods 17, 261 (2020).

[29] C. Zhu, R. H. Byrd, P. Lu, and J. Nocedal, Algorithm 778: L-BFGS-B: Fortran subroutines for large-scale bound-constrained optimization, ACM Trans. Math. Softw. 23, 550 (1997).

[30] R. H. Byrd, P. Lu, J. Nocedal, and C. Zhu, A limited memory algorithm for bound constrained optimization, SIAM J. Sci. Comput. 16, 1190 (1995).

[31] Y.-L. Zhang, C.-L. Zou, X.-B. Zou, L. Jiang, and G.-C. Guo, Detuning-enhanced cavity spin squeezing, Phys. Rev. A 91, 033625 (2015).

[32] M. H. Schleier-Smith, I. D. Leroux, and V. Vuletić, Squeezing the collective spin of a dilute atomic ensemble by cavity feedback, Phys. Rev. A 81, 021804 (2010).

[33] C. Shen and L.-M. Duan, Efficient spin squeezing with optimized pulse sequences, Phys. Rev. A 87, 051801 (2013).

[34] Z. Zhang and L. M. Duan, Quantum metrology with Dicke squeezed states, New J. Phys. 16, 103037 (2014).

[35] W. Huang, Y.-L. Zhang, C.-L. Zou, X.-B. Zou, and G.-C. Guo, Two-axis spin squeezing of two-component Bose-Einstein condensates via continuous driving, Phys. Rev. A 91, 043642 (2015).

[36] T. Opatrný and K. Mølmer, Spin squeezing and Schrödinger-cat-state generation in atomic samples with Rydberg blockade, Phys. Rev. A 86, 023845 (2012).

[37] S. Lloyd, Almost any quantum logic gate is universal, Phys. Rev. Lett. 75, 346 (1995).

[38] G. Vitagliano, G. Colangelo, F. Martin Ciurana, M. W. Mitchell, R. J. Sewell, and G. Tóth, Entanglement and extreme planar spin squeezing, Phys. Rev. A 97, 020301 (2018).

[39] R. J. Birrittella, J. Ziskind, E. E. Hach, P. M. Alsing, and C. C. Gerry, Optimal spin- and planar-quantum squeezing in superpositions of spin coherent states, J. Opt. Soc. Am. B 38, 3448 (2021).