ON FACES OF QUASI-ARITHMETIC COXETER POLYTOPES

NIKOLAY BOGACHEV AND ALEXANDER KOLPAKOV

ABSTRACT. We prove that each lower-dimensional face of a quasi-arithmetic Coxeter polytope, which happens to be itself a Coxeter polytope, is also quasi-arithmetic. We also provide a sufficient condition for a codimension 1 face to be actually arithmetic, as well as a few computed examples.

§ 1. Introduction

Let $X^n$ be one of the three spaces of constant curvature, i.e. either the Euclidean $n$-space $\mathbb{E}^n$, or the $n$-dimensional sphere $S^n$, or the $n$-dimensional hyperbolic (Lobachevsky) space $\mathbb{H}^n$.

Let $P$ be a convex polytope in $X^n$. The group $\Gamma$ generated by reflections in the supporting hyperplanes of the facets (i.e. codimension 1 faces) of $P$ is a discrete reflection group, if the images of $P$ under the action of $\Gamma$ tessellate $X^n$ (i.e. $X^n$ is entirely covered by copies of $P$, such that their interiors do not overlap). The polytope $P$ in this case is the fundamental polytope for $\Gamma$. In particular, $\Gamma$ being discrete implies that any two hyperplanes $H_i$ and $H_j$ bounding $P$ either do not intersect or form a dihedral angle of $\pi/n_{ij}$, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$.

If $P$ is compact, then $\Gamma$ is called a cocompact reflection group, and if $P$ has finite volume (in which case $P$ may or may not be compact), then $\Gamma$ is called cofinite or a discrete group of finite covolume.

Discrete reflection groups of finite covolume acting on spheres and Euclidean spaces were classified by Coxeter in 1933 [16]. Therefore, their fundamental polytopes are called Coxeter polytopes. They belong to the class of so-called acute-angled polytopes, i.e. those whose dihedral angles are less than or equal to $\pi/2$.

Suppose that $F \subset \mathbb{R}$ is a totally real number field, and let $O_F$ denote its ring of integers. Let $G = \text{PO}(n, 1)$ be the isometry group of $\mathbb{H}^n$, and $\tilde{G}$ be an admissible simple algebraic $F$-group, i.e. $\tilde{G}(\mathbb{R}) = G$ and $\tilde{G}^\sigma(\mathbb{R})$ is a compact group for any non-identity embedding $\sigma: F \to \mathbb{R}$.

It is known from the general theory, c.f. [13] and [24], that if $\Gamma \subset G$ is commensurable2 with $\tilde{G}(O_F)$ then $\Gamma$ is a lattice (i.e. a discrete isometry group with a finite volume fundamental polytope). Such groups are called arithmetic.

A lattice $\Gamma \subset G = \tilde{G}(\mathbb{R})$ is called quasi-arithmetic if $\Gamma \subset \tilde{G}(F)$ and properly quasi-arithmetic if it is quasi-arithmetic, but not actually arithmetic.

The history of discrete reflection groups acting on Lobachevsky spaces and, in particular, arithmetic reflection groups goes back to the 19th century, to the works of Poincare, Fricke, and Klein. A systematic study was started by Vinberg in 1967 [31]. Namely, he developed practically efficient methods that allow to determine compactness or volume finiteness of a given Coxeter polytope according to its Coxeter diagram, and provided a (quasi-)arithmeticity

1Here $H^\sigma$ denotes the algebraic group defined over $\sigma(F)$ and obtained from an abstract algebraic group $H$ by applying $\sigma$ to the coefficients of all polynomials that define $H$.

2Two subgroups $\Gamma_1$ and $\Gamma_2$ of some group are called commensurable if the group $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in each of them.
criterion for hyperbolic reflection groups. Later on, Vinberg created an algorithm (colloquially known as “Vinberg’s algorithm”) [32], that constructs a fundamental Coxeter polytope for any hyperbolic reflection group. Practically, it is most efficient for arithmetic reflection groups associated with Lorentzian lattices (see § 2, § 5, § 6).

Due to the further results of Vinberg [34], Long, Maclachlan, Reid [23], Agol [2], Nikulin [25, 26], Agol, Belolipetsky, Storm and Whyte [3], it became known that there are only finitely many maximal arithmetic hyperbolic reflection groups in all dimensions. Moreover, compact Coxeter polytopes in \( \mathbb{H}^n \) can exist only for \( n < 30 \) [34] and finite volume Coxeter polytopes do not exist in \( \mathbb{H}^{>995} \) [22, 27].

Let \( P \subset \mathbb{H}^n \) be a finite volume hyperbolic Coxeter polytope. We say that \( P \) is (quasi-) arithmetic with ground field \( \mathbb{F} \) if it is a fundamental domain for a hyperbolic reflection group \( \Gamma \subset \text{Isom}(\mathbb{H}^n) \) with ground field \( \mathbb{F} \) and \( \Gamma \) is (quasi-)arithmetic. Here, \( \Gamma = \Gamma(P) \) is the group generated by reflections in the bounding hyperplanes of \( P \). By [34] arithmetic Coxeter polytopes with \( \mathbb{F} \neq \mathbb{Q} \) in \( \mathbb{H}^n \), can exist only for \( n < 30 \) (because of compactness), and for \( 14 \leq n < 30 \) only finitely many ground fields \( \mathbb{F} \) are possible. Non-compact arithmetic Coxeter polytopes (i.e. for \( \mathbb{F} = \mathbb{Q} \)) can exist only for \( n \leq 21 \), \( n \neq 20 \) [18].

Recently, Belolipetsky and Thomson provided infinitely many commensurability classes of properly quasi-arithmetic hyperbolic lattices in any dimension \( n > 2 \) (cf. [5, 29]), as well as Emery [17] proved that the covolume of any quasi-arithmetic hyperbolic lattice is a rational multiple of the covolume of an arithmetic subgroup.

By considering faces of a higher-dimensional Coxeter polytope in \( \mathbb{H}^n \), one can try to find more examples of lower-dimensional Coxeter polytopes in \( \mathbb{H}^k \), \( k < n \). A good indication that such lower-dimensional Coxeter faces should appear is the work by Allcock [4, Theorems 2.1 & 2.2]. Even earlier, Borcherds [12] used a 25-dimensional infinite volume polytope due to Conway in order to build a 21-dimensional Coxeter polytope of finite volume as its face.

In the context of passing to lower-dimensional faces, our first theorem shows that quasi-arithmeticity is inherited from the initial polytope.

**Theorem 1.1.** Let \( P \) be a quasi-arithmetic Coxeter polytope in \( \mathbb{H}^n \) with ground field \( \mathbb{F} \) and \( P' \) be its \( k \)-dimensional face, for \( 2 \leq k \leq n - 1 \). If \( P' \) is itself a Coxeter polytope, then \( P' \) is also quasi-arithmetic with ground field \( \mathbb{F} \).

In the sequel, we enhance the definition of quasi-arithmeticity to all acute-angled polytopes, c.f. Definition 2.4. Then, we have the following easy consequence.

**Corollary 1.** Let \( P \) be a Coxeter polytope in \( \mathbb{H}^n \) with ground field \( \mathbb{F} = \mathbb{F}(P) \), and let \( P' \) be a codimension \( k \geq 1 \) face of \( P \) with ground field \( \mathbb{F}' = \mathbb{F}(P') \). If \( |\mathbb{F} : \mathbb{F}'| > 1 \), then \( P \) cannot be quasi-arithmetic.

Another result shows that arithmetic polytopes can often have arithmetic Coxeter facets.

**Theorem 1.2.** Let \( P \) be an arithmetic Coxeter polytope in \( \mathbb{H}^n \) with ground field \( \mathbb{F} \) and \( P' \) be its facet (i.e. a codimension 1 face). Moreover, assume that \( P' \) meets all its adjacent facets at dihedral angles of the form \( \frac{\pi}{2m} \), for some natural \( m \geq 1 \), not all necessarily equal to each other. Then \( P' \) is itself an arithmetic Coxeter polytope with ground field \( \mathbb{F} \).

The paper is organised as follows. In § 2 some preliminary facts are given. Then, § 3 is devoted to the proof of Theorem 1.1 and § 4 is devoted to the proof of Theorem 1.2.

At the end of the paper, some computed examples are provided (c.f. § 5–§ 6) in order to illustrate the above theorems, as well as to enrich the collection of known Coxeter polytopes. All computations were performed by using SageMath computer algebra system [28].
The closing remarks (§ 7) contain a number of questions that we find quite enticing, although likely hard to approach.

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§ 2. Preliminaries

Let $\mathbb{F} \subset \mathbb{R}$ be a totally real number field, and let $\mathcal{O}_\mathbb{F}$ denote its ring of integers. For convenience, we assume that $\mathcal{O}_\mathbb{F}$ is a principal ideal domain.

Definition 2.1. A free finitely generated $\mathcal{O}_\mathbb{F}$–module $L$ with an inner product of signature $(n, 1)$ is called a Lorentzian lattice if, for any non-identity embedding $\sigma : \mathbb{F} \to \mathbb{R}$, the quadratic space $L \otimes_{\sigma(\mathcal{O}_\mathbb{F})} \mathbb{R}$ is positive definite (we say that the inner product in $L$ is associated with some admissible Lorentzian quadratic form).

Remark 1. As a matter of terminology, Lorentzian lattices, as defined in the present paper, are also called hyperbolic lattices, cf. [8, 9, 10, 34].

Let $L$ be a Lorentzian lattice. Then the vector space $\mathbb{E}^{n, 1} = L \otimes \text{Id}(\mathcal{O}_\mathbb{F}) \mathbb{R}$ is identified with the $(n+1)$-dimensional real Minkowski space. The group $\Gamma = \mathcal{O}'(L)$ of integer (i.e. with coefficients in $\mathcal{O}_\mathbb{F}$) linear transformations preserving the lattice $L$ and mapping each connected component of the cone

$$\mathcal{C} = \{v \in \mathbb{E}^{n, 1} \mid (v, v) < 0\} = \mathcal{C}^+ \cup \mathcal{C}^-$$

onto itself is a discrete group of motions of the Lobachevsky space $\mathbb{H}^n$. Here and below we use the hyperboloid model for $\mathbb{H}^n$, which is the set $\{v \in \mathbb{E}^{n, 1} \cap \mathcal{C}^+ \mid (v, v) = -1\}$. Its isometry group is

$$\text{Isom}(\mathbb{H}^n) = \text{PO}(n, 1) \simeq \text{O}(L \otimes \text{Id}(\mathcal{O}_\mathbb{F}) \mathbb{R}) / \{\pm \text{Id}\},$$

which is the group of orthogonal transformations of the Minkowski space $\mathbb{E}^{n, 1}$ that leaves $\mathcal{C}^+$ invariant.

It is known [13, 24] that if $\mathbb{F} = \mathbb{Q}$ and the lattice $L$ is isotropic (that is, the quadratic form associated with it represents zero), then the quotient space $\mathbb{H}^n / \Gamma$ (the fundamental domain of $\Gamma$) is not compact, but is of finite volume, and in all other cases it is compact. The case $\mathbb{F} = \mathbb{Q}$ was first studied by Venkov [30].

Definition 2.2. Any group $\Gamma$ obtained in the way described above, and any subgroup of $\text{Isom}(\mathbb{H}^n)$ commensurable to such one, is called an arithmetic group (or lattice) of simplest type. The field $\mathbb{F}$ is called the ground field (or the field of definition) of $\Gamma$.

A primitive vector $e$ of a Lorentzian lattice $L$ is called a root or, more precisely, a $k$-root, where $k = (e, e) \in (\mathcal{O}_\mathbb{F})_{>0}$ if $2 \cdot (e, x) \in k\mathcal{O}_\mathbb{F}$ for all $x \in L$. Every root $e$ defines an orthogonal reflection (called a $k$-reflection if $(e, e) = k$) in $L \otimes \text{Id}(\mathcal{O}_\mathbb{F}) \mathbb{R}$ given by

$$\mathcal{R}_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e,$$

which preserves the lattice $L$. Geometrically speaking, this is a reflection of $\mathbb{H}^n$ with respect to the hyperplane $H_e = \{x \in \mathbb{H}^n \mid (x, e) = 0\}$, called the mirror of $\mathcal{R}_e$.

Let $\mathcal{O}_r(L)$ denote the subgroup of $\mathcal{O}'(L)$ generated by all reflections contained in it.
Definition 2.3. A Lorentzian lattice $L$ is called reflective if the index $[O'(L) : O_r(L)]$ is finite.

Clearly, $L$ is reflective if and only if the group $O_r(L)$ has a finite volume fundamental Coxeter polytope.

The results described in §1 give hope that all reflective Lorentzian lattices, as well as maximal arithmetic hyperbolic reflection groups, can be classified. For more information about the recent progress on the classification problem c.f. [6, 9, 10].

Remark 2. By a result of Vinberg [31, Lemma 7], any arithmetic hyperbolic reflection group is an arithmetic lattice of simplest type with ground field $\mathbb{F}$, and therefore is commensurable with $O_r(L)$, where $L$ is some (necessarily reflective) Lorentzian lattice over a totally real number field $\mathbb{F}$. It also shows that every quasi-arithmetic hyperbolic reflection group is contained in a group $O(f, \mathbb{F})$, where $f$ is some admissible Lorentzian form over $\mathbb{F}$.

Remark 3. As a consequence of Vinberg’s algorithm [32] (its recent software implementations are AlVin [21], for Lorentzian lattices with an orthogonal basis over several ground fields, and VinAl (cf. [7, 8]), for Lorentzian lattices with an arbitrary basis over $\mathbb{Q}$), reflective lattices provide many explicit examples of arithmetic reflection groups and arithmetic Coxeter polytopes. In contrast, finding properly quasi-arithmetic reflection groups appears to be a harder task.

Remark 4. The record example of a compact Coxeter polytope was found by Bugaenko in dimension 8 [15], although the maximal possible dimension is bounded by 30.

The record example of a finite volume Coxeter polytope is due to Borcherds in dimension 21 [12]. It is known that Coxeter polytopes of finite volume can exist only in dimensions smaller than 996 [22, 27].

It is worth mentioning that both these examples come from arithmetic reflection groups.

For every hyperplane $H_e$ with unit normal $e$, let us set $H_e^- = \{x \in \mathbb{E}^{n,1} \mid (x, e) \leq 0\}$. If

$$P = \bigcap_{j=0}^N H_{e_j}^-$$

is an acute-angled polytope of finite volume in $\mathbb{H}^n$, then $G(P) = G(e_0, \ldots, e_N)$ is its Gram matrix, $\mathbb{F}(P) = \mathbb{Q}[\{g_{ij}\}_{1 \leq i, j \leq N}]$, $\mathbb{F}(P)$ denotes the field generated by all possible cyclic products of the entries of $G(P)$ and is called the ground field of $P$. The set of all cyclic products of entries of a matrix $A$ (i.e. the set consisting of all possible products of the form $a_{i_1 j_1}a_{i_2 j_2} \ldots a_{i_k j_k}$) will be denoted by Cyc($A$). Thus, $\mathbb{F}(P) = \mathbb{Q}[\text{Cyc}(G(P))] \subset \mathbb{F}(P)$.

Given a finite covolume hyperbolic reflection group $\Gamma$, the following criterion allows us to determine if $\Gamma$ is arithmetic, quasi-arithmetic, or neither.

Theorem 2.1 (Vinberg’s arithmeticity criterion [31]). Let $\Gamma$ be a cofinite reflection group acting on $\mathbb{H}^n$ with fundamental Coxeter polytope $P$. Then $\Gamma$ is arithmetic if and only if the following conditions hold:

(V1) $\mathbb{F}(P)$ is a totally real algebraic number field;

(V2) for any embedding $\sigma : \mathbb{F}(P) \to \mathbb{R}$, such that $\sigma |_{\mathbb{F}(P)} \neq \text{Id}$, $G^\sigma(P)$ is positive semi-definite;

(V3) $\text{Cyc}(2 \cdot G(P)) \subset O(\mathbb{F}(P))$.

A cofinite reflection group $\Gamma$ acting on $\mathbb{H}^n$ with fundamental polytope $P$ is quasi-arithmetic if and only if it satisfies conditions (V1)–(V2), but not necessarily (V3).
Obviously, every arithmetic reflection group is quasi-arithmetic. For most of our arguments, we shall need a wider definition of quasi-arithmetic polytopes that will allow us to capture their main geometric and algebraic properties without paying attention to whether their dihedral angles are of Coxeter type. This will become important in order to transfer from higher-dimensional faces to lower-dimensional ones in a chain of inclusions, which does not necessarily consist entirely of Coxeter polytopes, c.f. Proposition 3.1.

Definition 2.4. Let \( P \) be a finite volume acute-angled polytope in \( \mathbb{H}^n \), and let it have well-defined fields \( \mathcal{F} = \mathcal{F}(P) \), \( F = F(P) \subset \mathcal{F} \), as described above. Then \( P \) is called quasi-arithmetic if \( \mathcal{F} \) and \( \mathcal{F} \) satisfy conditions (V1)–(V2).

After presenting the proof of Theorem 1.1, we provide some computed examples of Coxeter polytopes and their faces. For each polytope \( P \subset \mathbb{H}^n \), let its facet tree \( \mathcal{T}(P) \) be a rooted tree with root \( P \), where each vertex of level \( 0 \leq i \leq n - 3 \) represents the isometry type of a codimension \( i \) face \( P^{(i)}_j \) of \( P \), while the level \( i + 1 \) descendants of \( P^{(i)}_j \) represent all distinct isometry types of codimension 1 faces of \( P^{(i)}_j \). Thus, \( \mathcal{T}(P) \) shows all possible isometry types of faces of \( P \) (in codimensions 1 through \( n - 2 \)), as well as the set of mutually non-isometric facets for each lower-dimensional face. Since \( \mathcal{T}(P) \) does not take face adjacency into account, it represents only some features of the geometric structure of \( P \). Mostly, we shall be interested in determining its Coxeter facets, and classifying them into arithmetic and properly quasi-arithmetic ones.

\[ \square \] 3. Proof of Theorem 1.1

3.1. Auxiliary results. Below we formulate a few auxiliary lemmas. Their main point is that the properties (V1)–(V2) are inherited by facets of an acute-angled quasi-arithmetic polytope.

In § 3.1 – § 3.2, \( P \) always denotes a finite volume acute-angled polytope in \( \mathbb{H}^n \).

Lemma 3.1. Let \( P' \subset H_{e_0} \) be a facet of \( P \), bounded by the respective hyperplanes \( H_{e_1}, \ldots, H_{e_k} \) of \( P \) (i.e. \( H_{e_0} \cap H_{e_j} \) are the supporting hyperplanes of \( P' \) in \( \mathbb{H}^{n-1} \)). Let \( G(P) = \{ g_{ij} \} \) be the Gram matrix of \( P \). Then the Gram matrix \( G(P') = \{ g'_{ij} \} \) of \( P' \) has entries

\[
g'_{ij} = \frac{g_{ij} - g_{0i} g_{0j}}{\sqrt{1 - g_{0i}^2} (1 - g_{0j}^2)}.\]

Proof. Let \( e^0_i, e^0_j \) be the projections of \( e_i, e_j \) onto the hyperplane \( H_{e_0} \). Then we obtain \( e^0_j = e_j - (e_j, e_0)e_0 \) and \( (e^0_i, e^0_j) = (e_i, e_j) - (e_0, e_i)(e_0, e_j) \). Therefore, the lemma follows by setting \( e'_i = \frac{e^0_i}{\| e^0_i \|}, e'_j = \frac{e^0_j}{\| e^0_j \|} \) to be the respective unit normals of \( P' \).

Lemma 3.2. In the notation of Lemma 3.1 and its proof, let \( G'_s = G(e'_1, \ldots, e'_s) \) be a corner submatrix of \( G(P) \), and let \( G_s = G(e_0, e_1, \ldots, e_s) \) be the respective corner submatrix of \( G(P) \). Then \( \det G'_s \geq 0 \) if and only if \( \det G_s \geq 0 \).

Proof. Let us consider the matrix \( G_s \). One can perform the following transformation on its rows: the \( j \)-th row \((1 \leq j \leq s)\) is replaced by the difference of itself and the row corresponding to \( e_0 \) multiplied by \( g_{j0} \).

After that, each \( i \)-th column \((1 \leq i \leq s)\) of the resulting matrix is divided by \( \| e^0_i \| = \sqrt{1 - g_{0i}^2} \), and each \( j \)-th row \((1 \leq j \leq s)\) is divided by \( \| e^0_j \| = \sqrt{1 - g_{0j}^2} \).
According to Lemma 3.1, this transformation results in the matrix
\[
G''_s = \begin{pmatrix} 1 & G'_s(e'_1, \ldots, e'_s) \\
0 & \end{pmatrix},
\]
and thus, we have \( \det G_s = \kappa^2 \cdot \det G''_s = \kappa^2 \cdot \det G'_s \), where
\[
\kappa^{-2} = \prod_{i,j=1}^s \|e_i^0\| \cdot \|e_j^0\| = \prod_{i=1}^s \|e_i^0\|^2.
\]

**Lemma 3.3.** Under the above assumptions, \( F(P') = F(P) \).

**Proof.** Let \( F = \mathbb{Q}[\text{Cyc}(G(P))] \) and \( F' = \mathbb{Q}[\text{Cyc}(G(P'))] \). We shall consider a cyclic product
\[
g'_{i_1i_2}g'_{i_2i_3} \cdots g'_{i_hi_1} = \frac{g_{i_1i_2} - g_{0i_1}g_{0i_2}}{\sqrt{(1 - g_{0i_1}^2)(1 - g_{0i_2}^2)}} \cdot \frac{g_{i_2i_3} - g_{0i_2}g_{0i_3}}{\sqrt{(1 - g_{0i_2}^2)(1 - g_{0i_3}^2)}} \cdots \frac{g_{i_hi_1} - g_{0i_h}g_{0i_1}}{\sqrt{(1 - g_{0i_h}^2)(1 - g_{0i_1}^2)}}.
\]
The denominator of the above expression equals
\[(1 - g_{0i_1}^2)(1 - g_{0i_2}^2) \cdots (1 - g_{0i_h}^2) \in F.
\]
Thus, it remains to consider the numerator
\[
(g_{i_1i_2} - g_{0i_1}g_{0i_2})(g_{i_2i_3} - g_{0i_2}g_{0i_3}) \cdots (g_{i_hi_1} - g_{0i_h}g_{0i_1}).
\]
While expanding the above expression, we take from each pair of parentheses either \( g_{im_1m+1} \) or \( g_{0im_0}g_{0i_{m+1}} \), and obtain a sum of cyclic products where each term looks like \( g_{i_1i_2}g_{i_2i_3} \cdots g_{i_hi_1} \) with some terms of the form \( g_{im_1m+1} \) being replaced by the respective products \( g_{0im_0}g_{0i_{m+1}} \).

This is equivalent to replacing some transpositions \((i_m, i_{m+1})\) in \((i_1, i_2)(i_2, i_3) \cdots (i_{s-1}, i_s) = (i_1, i_2, \ldots, i_s, 0) = (0, i_m)(0, i_{m+1}) \). Obviously, this operation creates another permutation with a different cycle structure, where either new non-trivial cycles of the elements in \( I = \{0, i_1, i_2, \ldots, i_s\} \) will be formed, or some fixed points appear. The latter happens whenever a square of some \( g_{ij}, i, j \in I \), is present. Thus, each term in (2) after expansion is a cyclic product from \( \text{Cyc}(G(P)) \).

Therefore, a cyclic product of the form (1) is a linear combination of cyclic products of \( G(P) \) divided by some elements of the field \( F \). This implies \( g'_{i_1i_2}g'_{i_2i_3} \cdots g'_{i_hi_1} \in F \), and hence \( F' \subset F \).

Now, let us suppose that \( F' \neq F \). This implies that there exists an embedding \( \sigma : F \to \mathbb{R} \) such that \( \sigma|_{F'} = \text{Id} \). Then \( G'^{(P)} \) is positive semi-definite, and so is \( G'^{(P')} \), by Lemma 3.2. However, the latter is impossible, since \( G'^{(P')} = G(P') \) is the Gram matrix of a hyperbolic polytope.

### 3.2. Quasi-arithmeticity of a facet of a quasi-arithmetic acute-angled polytope.

**Proposition 3.1.** Let \( P \) be a quasi-arithmetic acute-angled polytope in \( \mathbb{H}^n \) with ground field \( F = F(P) \), and let \( P' \) be its facet. Then \( P' \) is also a quasi-arithmetic acute-angled polytope with the same ground field \( F' = F(P') = F \).

**Proof.** It is well-known that a face of an acute-angled polytope is also acute-angled. Now the proof follows from verifying conditions (V1)–(V2) of Vinberg’s arithmeticity criterion (Theorem 2.1).

**Verification of (V1).** The field \( \widetilde{F} \) equals \( \widetilde{F}[\{\sqrt{k_i^2}\}_{i,j=1}^k] \), where \( k^{-1} = \|e_i^0\| \cdot \|e_j^0\| \), and thus is a finite extension of \( \tilde{F} \). It remains to show that \( \tilde{F} \) is totally real. Instead, we prove that the larger field \( \widetilde{F}' = \widetilde{F}[\{\sqrt{1 - g_{0i1}^2}\}_{i=1}^k] \) is totally real, so that \( \tilde{F} \subset \widetilde{F}' \) is totally real, as well.
To this end, recall that $|g_{0i}| = \cos \angle(H_0, H_i) \leq 1$, and thus $1 - g_{0i}^2 \geq 0$. Also, $1 - g_{0i}^2 = \det G(e_0, e_i)$ is a corner minor of $G(P)$ and thus remains positive for any non-identity embedding by Theorem 2.1. Thus $\overline{\mathbb{F}}[\sqrt{1 - g_{0i}^2}]$ is totally real and, consequently, so is $\overline{\mathbb{F}}''$.

Verification of (V2). By Lemma 3.2 we have that, up to squares in $\mathbb{F}$, all the corner minors of $G' = G(P')$ coincide with the corresponding corner minors of $G = G(P)$. Since $\mathbb{F}' = \mathbb{F}$ by Lemma 3.3, we have that $\det G_s^\sigma \geq 0$ for every embedding $\sigma : \overline{\mathbb{F}} \to \mathbb{R}$, such that $\sigma|_{\mathbb{F}} \neq \text{Id}$, by Theorem 2.1, and thus $\det(G')_s^\sigma \geq 0$.

3.3. Proof of Theorem 1.1. Let $P$ be a quasi-arithmetic Coxeter polytope in $\mathbb{H}^n$ with ground field $\mathbb{F} = \mathbb{F}(P)$, and let $P'$ be its face of any dimension $\geq 2$ that is also a Coxeter polytope. Then we need to prove that $P'$ is also a quasi-arithmetic Coxeter polytope with the same ground field $\mathbb{F}$.

Clearly, the face $P'$ can be included in the following chain of polytopes, by inclusion:

$$P' = P_1 \subset P_2 \subset \ldots \subset P_t = P,$$

where $P_j$ is a facet of $P_{j+1}$, for every $1 \leq j \leq t - 1$.

Since $P = P_1$ is a quasi-arithmetic Coxeter polytope, then it is an acute-angled one and, by Proposition 3.1, $P_{t-1}$ is also a quasi-arithmetic acute-angled polytope with the same ground field. Thus, clearly, each $P_j$ is a quasi-arithmetic acute-angled polytope with $\mathbb{F}(P_j) = \mathbb{F}(P_{j+1})$ for $1 \leq j \leq t - 1$.

3.4. A Coxeter prism and its ground field. The 3-dimensional compact prism $P \subset \mathbb{H}^3$ with Coxeter diagram depicted in Figure 1 has one of its bases $P'$ (namely, facet 1) orthogonal to all neighbours (namely, facets 3, 4, and 5). The Coxeter diagram of $P'$ is the sub-diagram in the diagram of $P$ spanned by vertices 3, 4, and 5.

From the Coxeter diagram in Figure 1, one easily gets that $\mathbb{F}(P) = \mathbb{Q}[\sqrt{2}, \sqrt{5}]$, while $\mathbb{F}(P') = \mathbb{Q}[\sqrt{5}]$. Thus, $P$ cannot be quasi-arithmetic by Corollary 1. Another obstruction to quasi-arithmeticity is the fact that $\cosh^2 \ell$ is not totally positive. However, we do not need to make this computation in order to come up with our conclusion.

§ 4. Proof of Theorem 1.2

We start with an auxiliary lemma, that will become useful in the computations below.
4.1. Even algebraic integers. Let \( \rho_m = \sin^{-2} \frac{\pi}{2m} \), for \( m \geq 2 \). We claim that \( \rho_m \) is even, meaning that \( \frac{\rho_m}{2} \) is an algebraic integer.

**Lemma 4.1.** For all \( m \geq 2 \), we have that \( \frac{\rho_m}{2} \) is an algebraic integer.

**Proof.** Let \( p_m(z) \) be the following function:

\[
p_m(z) = \begin{cases} 
T_m \left( \frac{z}{\sqrt{2}} \right) \cdot T_m \left( -\frac{z}{\sqrt{2}} \right), & \text{if } m \text{ is even}, \\
U_m \left( \frac{z}{\sqrt{2}} \right) \cdot U_m \left( -\frac{z}{\sqrt{2}} \right) + 1, & \text{if } m \text{ is odd},
\end{cases}
\]

where \( T_m, \) resp. \( U_m, \) is the \( m \)-th Chebyshev polynomial of the 1st, resp. 2nd, kind. Their basic properties and some relations that will be used below are collected in [1, Chapter 22].

From the relation \( T_m(z) = T_{m/2}(2z^2 - 1), \) for even \( m, \) we see that \( T_m \left( \frac{\sqrt{2}}{z} \right) \) is a polynomial. From the recurrence \( U_{m+1}(z) = 2zU_m(z) - U_0(z), \) with \( U_0(z) = 1 \) and \( U_1(z) = 2z, \) we get that \( U_m(z) \) is a polynomial in \( 2z. \) This, together with the fact that the odd powers of \( \frac{\sqrt{2}}{z} \) in the above expression for \( p(z) \) cancel out, means that \( p(z) \) is a polynomial with integer coefficients.

Moreover, \( T_m(z) \) has constant term \( \pm 1 \) for even \( m, \) and \( U_m(z) \) has constant term 0 for odd \( m. \) Thus, \( p_m(z) \) has constant term 1. By using the trigonometric definitions of \( T_m(z) \) and \( U_m(z), \) let us observe that \( \tau_m = \sqrt{\frac{m}{2}} \cdot \sin \frac{\pi}{2m} \) is a root of \( p_m(z). \)

Let \( \tilde{p}_m(z) = z^\deg p_m \cdot p_m(z^{-1}) \) be the reciprocal of \( p_m(z). \) Then \( \tilde{p}_m(z) \) is a unitary polynomial having \( \tau_m^{-1} = \sqrt{\frac{m}{2}} \) among its roots. Hence, \( \frac{\rho_m}{2} \) is an algebraic integer.  

4.2. Proof of Theorem 1.2. Let \( P' \) be a facet of an arithmetic Coxeter polytope with ground field \( \mathbb{F}. \) Let \( H_{e_0} \) be the supporting hyperplane of \( P' \) as a facet of \( P, \) and let \( F_i, i = 1, \ldots, s, \) be the facets of \( P'. \) Let \( H_{e_i}, i = 1, \ldots, s \) be the hyperplanes of \( P \) such that \( H_{e_i} \cap H_{e_0} \) is the supporting hyperplane of \( F_i \) in \( H_{e_0}. \) Since the stabiliser of \( H_{e_i} \cap H_{e_0} \) in the reflection group of \( P \) is the dihedral group of order \( 4m, \) and \( H_{e_0} \) is one of its mirrors, there exists another mirror \( H'_i \) (not necessarily a supporting hyperplane for \( P \)), such that \( H'_i \) and \( H_{e_0} \) are orthogonal, while \( F_i \subset H'_i \cap H_{e_0}. \) Thus, the facets of \( P' \) come from orthogonal projections of some of the mirrors of the reflection group of \( P \) onto the hyperplane \( H_{e_0}. \) Therefore, \( P' \) is a Coxeter polytope.

By Theorem 1.1, \( P' \) is quasi-arithmetic with ground field \( \mathbb{F}, \) and it remains to verify condition (V3) of Vinberg’s arithmeticity criterion.

Each cyclic product from \( \text{Cyc}(2G(P')) \) has the form \( 2^s g_{i_1i_2}g_{i_2i_3} \ldots g_{i_si_1}, \) which is similar to (1). Taking into account that the denominator in (1) has each \( g_{0i_k} = \cos \frac{\pi}{2mi_k} \), it can be written as

\[
2^s \prod_{k=1}^{s} \sin^{-2} \left( \frac{\pi}{2mi_k} \right) \cdot (g_{i_1i_2} - g_{0i_1}g_{0i_2})(g_{i_2i_3} - g_{0i_2}g_{0i_3}) \ldots (g_{i_si_1} - g_{0i_s}g_{0i_1}).
\]

(3)

Same as in (2), \( g_{ij} \)’s form some cyclic products from \( \text{Cyc}(G(P)). \) Let \( t \) be the cardinality of the set \( I = \{ i_k \mid m_{ik} = 1 \}. \) Since \( \rho_{m_{ik}} = \sin^{-2} \left( \frac{\pi}{2m_{ik}} \right) \in 2\mathcal{O}_F \) by Lemma 4.1 for each \( i_k \notin I, \) we can rewrite (3) as

\[
2^{2s-t} \cdot \rho \cdot (g_{i_1i_2} - g_{0i_1}g_{0i_2})(g_{i_2i_3} - g_{0i_2}g_{0i_3}) \ldots (g_{i_si_1} - g_{0i_s}g_{0i_1}).
\]

(4)

with \( \rho \in \mathcal{O}_F. \) Let \( \mu = 2s - t. \) The longest cyclic product that appears in the above expression has length \( \lambda = 2s, \) if \( t = 0, \) or \( \lambda = 2s - t - 1, \) if \( t \geq 1 \) (which happens whenever we have \( g_{0i_j} = \ldots = g_{0i_{j+1-1}} = 0 \) for \( t \) consecutive terms). Thus, each term in (4) of length \( \lambda \) is multiplied by \( 2^n \) with \( \mu \geq \lambda. \) Due to the arithmeticity of \( P, \) each such product belongs to \( \mathcal{O}_F \) by Vinberg’s criterion, c.f. Theorem 2.1.
§ 5. A polytope by Bugaenko

In this section we consider the polytope $P$ with Coxeter diagram depicted in Figure 2. This is a compact polytope in $\mathbb{H}^7$ first described by Bugaenko [14], and later on included into a larger census by Felikson and Tumarkin [19]. This is an arithmetic polytope: $P$ is the fundamental polytope for $\mathcal{O}_r(L)$, where the lattice $L$ is associated with the quadratic form $q(x) = -\frac{1+\sqrt{5}}{2} x_0^2 + x_1^2 + \ldots + x_7^2$.

Figure 2. A polytope $P$ with ground field $\mathbb{Q}[\sqrt{5}]$ from [14]

The outer normals to the facets of $P$ can be easily computed by using AlVin [20, 21] and are included in the PLoF\textsuperscript{3} worksheet [11].

The geometric characteristics of its lower-dimensional faces can be found by using our PLoF SageMath worksheet [11]. These include Coxeter diagrams and Gram matrices of all isometry types of faces from dimension 7 down to 2.

Furthermore, PLoF builds the facet tree $\mathcal{T}(P)$ for $P$. Here, we would like to stress the fact that on each level of the tree only isometry types of faces are given, and thus the adjacency structure of $P$ is not entirely revealed. Another feature is that each isometry type of non-Coxeter faces happens only once in $\mathcal{T}(P)$, since we prune the tree by removing duplicates. However, Coxeter faces are always preserved, so that their inclusion chains can be easily observed.

Figure 3. A 3-dimensional Coxeter face $P'$ of $P$

Let us remark that $P$ has properly quasi-arithmetic faces only in dimension 2. One of them, the complete list being computed by PLoF [11], belongs to a 3-dimensional Coxeter face $P'$ with Coxeter diagram in Figure 3. The subdiagram of $P$ giving rise to $P'$ is generated by the two

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triple bonds in Figure 2. Then, the diagram of $P'$ can be easily computed by using the method of [4].

The face $P''_1$ in question has label 2 in the Coxeter diagram of $P'$. This is a right-angled hexagon with Gram matrix

$$G(P''_1) = 
\begin{pmatrix}
1 & -2\sqrt{5} - 5 & 0 & 0 & -\frac{1}{2}(2\sqrt{5} + 5) & -2\sqrt{5} - 4 \\
-2\sqrt{5} - 5 & 0 & -2\sqrt{5} - 4 & 0 & -\frac{1}{2}(2\sqrt{5} + 5) & 0 \\
0 & -2\sqrt{5} - 4 & 1 & -\frac{1}{2}\sqrt{5 + 5} & 0 & -\frac{1}{2}(3\sqrt{5} + 5) \\
-\frac{1}{2}(2\sqrt{5} + 5) & 0 & 0 & -\frac{1}{2}\sqrt{5 + 5} & 1 & 0 \\
-2\sqrt{5} - 4 & 0 & -\frac{1}{2}(3\sqrt{5} + 5) & -\frac{1}{2}\sqrt{5 + 5} & 0 & 1 \\
\end{pmatrix}.$$ 

One can easily verify that $P''_1$ is not arithmetic. Let us point out that $P''_1$ makes angles of $\frac{\pi}{7}$, $\frac{2\pi}{7}$, and $\frac{3\pi}{10}$ with its neighbours. Thus, the “evenness of the dihedral angles” condition in Theorem 1.2 seems fairly reasonable.

Another face $P''_2$ of $P'$ is labelled 1 in Figure 3, and makes angles of $\frac{\pi}{2}$ and $\frac{3\pi}{10}$ only with its neighbours. This face is therefore arithmetic by Theorem 1.2. This is a right-angled pentagon with Gram matrix

$$G(P''_2) = 
\begin{pmatrix}
1 & -\sqrt{2\sqrt{5} + 5} & 0 & 0 & -\frac{1}{2}(\sqrt{5} + 1) \\
-\sqrt{2\sqrt{5} + 5} & 1 & -\frac{1}{2}(\sqrt{5} + 3) & 0 & 0 \\
0 & -\frac{1}{2}(\sqrt{5} + 3) & 1 & -\frac{1}{2}\sqrt{5 + 5} & 0 \\
0 & 0 & -\frac{1}{2}\sqrt{5 + 5} & 1 & -\frac{\sqrt{5} + 1}{2\sqrt{2}} \\
-\frac{1}{2}(\sqrt{5} + 1) & 0 & 0 & -\frac{\sqrt{5} + 1}{2\sqrt{2}} & 1 \\
\end{pmatrix}.$$ 

**Remark 5.** We also computed the faces of Bugaenko’s compact polytope in $\mathbb{H}^8$, which did not give very informative outcome. Indeed, it happens to have only few Coxeter faces, all of which are arithmetic. The complete computation can be found in [11].

§ 6. A curious reflective lattice

In this example, we consider the reflective lattice $L$ associated with the Lorentzian quadratic form $f(x) = -15x_0^2 + x_1^2 + \ldots + x_3^2$. Let $P$ be the fundamental polytope for $O_r(L)$. An apparently interesting fact is that $P$ has a descending chain entirely of Coxeter faces starting from the polytope itself that ends up with two 2-dimensional faces: one arithmetic, and the other properly quasi-arithmetic. None of the previous examples has such a long chain of Coxeter faces. This can be observed by using PLoF [11]. Namely, this chain ends in a 2-dimensional face $P'_1$ with Gram matrix

$$G(P'_1) = 
\begin{pmatrix}
1 & -\frac{1}{2}\sqrt{2} & -1 \\
-\frac{1}{2}\sqrt{2} & 1 & 0 \\
-1 & 0 & 1 \\
\end{pmatrix},$$

and another 2-dimensional face $P'_2$ with Gram matrix

$$G(P'_2) = 
\begin{pmatrix}
1 & -\frac{1}{2}\sqrt{2} & -\frac{2}{3}\sqrt{3} & 0 \\
-\frac{1}{2}\sqrt{2} & 1 & 0 & -\frac{1}{2}\sqrt{10} \\
-\frac{2}{3}\sqrt{3} & 0 & 1 & 0 \\
0 & -\frac{1}{2}\sqrt{10} & 0 & 1 \\
\end{pmatrix}.$$ 

It is easy to check that $P'_1$ is arithmetic, while $P'_2$ is properly quasi-arithmetic.
Moreover, since 15 is not a sum of three rational squares, \( P \) has a descending chain of faces corresponding to the restrictions of \( f \) onto the subspaces \( x_5 = \ldots = x_{5-i} = 0 \), for \( i = 0, 1, 2, 3 \), that starts with two non-compact finite volume faces and ends with two compact ones. All of them are obviously arithmetic (cf. [15, Theorem 2.1]).

§ 7. Some open questions

Below we list some, to the best of our knowledge, open problems, that seems interesting to us and are related to the above discussion of (quasi-)arithmeticty.

**Question 1.** Do there exist compact Coxeter polytopes in \( \mathbb{H}^{\geq 9} \)?

Let us recall that the record example is due to Bugaenko in \( \mathbb{H}^8 \), and there have been no compact polytopes found in higher dimensions since almost 30 years to date.

**Question 2.** Do there exist properly quasi-arithmetic Coxeter polytopes in \( \mathbb{H}^{\geq 6} \), compact or non-compact?

Some related work was done by Vinberg in the non-compact case [36] for all dimensions \( n \leq 12 \) and \( n = 14, 18 \) by constructing analogues of the non-arithmetic hybrids due to Gromov — Pyatetski–Shapiro (later on, Thomson [29] proved that these are never quasi-arithmetic). There are also some examples among Coxeter prisms, including properly quasi-arithmetic ones [31]. If more Coxeter polytopes become available, one can use the same idea and try to fill in the gaps for \( n = 13, 15, 16, 17 \), as well as to answer the above question.

**Question 3.** Is it true that a quasi-arithmetic Coxeter polytope in \( \mathbb{H}^{\geq 4} \) with ground field \( \mathbb{Q} \) cannot be compact?

This question is motivated by the examples of cocompact quasi-arithmetic subgroups of \( \text{SL}_2(\mathbb{Q}) \) constructed by Vinberg in [35] that preserve an isotropic quadratic form.

If Question 3 has affirmative answer, then Theorem 1.1 limits us only to non-compact polytopes appearing as Coxeter faces of arithmetic polytopes in \( \mathbb{H}^{\geq 4} \) with ground field \( \mathbb{Q} \).

**Question 4.** Does there exist an arithmetic Coxeter polytope in \( \mathbb{H}^{\geq 4} \) that has a properly quasi-arithmetic Coxeter face of small codimension?

The fact that the only properly quasi-arithmetic faces we found in Coxeter polytopes from § 5 – § 6 have dimension 2 is the main motivation for Question 4.

**Question 5.** Are there only finitely many maximal quasi-arithmetic hyperbolic reflection groups in all dimensions? Is it true at least for the fixed dimension and degree \( d = [F : \mathbb{Q}] \) of the ground field \( F \)?

This question is naturally motivated by the affirmative answer for arithmetic groups. It is also interesting, whether Nikulin’s methods (cf. [25, 26]) or the spectral method (cf. [3, 6]) can be applied.

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Skolkovo Institute of Science and Technology, Skolkovo, Russia

Moscow Institute of Physics and Technology (State University), Dolgoprudny, Moscow region, Russia

Caucasus Mathematical Centre, Adyghe State University, Maikop, Russia

E-mail address, N.Bogachev: nvbogach@mail.ru

Institut de Mathématiques, Université de Neuchâtel, 2000 Neuchâtel, Suisse/Switzerland

E-mail address: kolpakov.alexander@gmail.com