Relaxation of PLMI in the Form of Double Sum

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Abstract—This letter studies less conservative conditions for a parameterized linear matrix inequality (PLMI) in the form of double convex sum. Without any slack variables, sufficient linear matrix inequalities (LMIs) for the PLMI are derived by using the proposed sum relaxation based on Young’s inequality. It is proved that the derived LMIs are not conservative than those of [5]. An example is given to show the reduced conservatism of the derived LMIs.

Index Terms—Parameterized linear matrix inequality (PLMI), linear matrix inequalities (LMI), fuzzy, linear parameter varying model, stability, stabilization, Lyapunov.

Notations: The relation $P \succ Q (P \prec Q)$ means that the matrix $P-Q$ is positive (negative) definite. $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ means the $n$-dimensional Euclidean space, $\mathbb{R}^{m \times n}$ stands for the field of real matrices of dimension $m \times n$. $\mathbb{I}_r$ indicates the integer set \{1, \ldots, r\} with $r > 1$. For simplicity, $x$ is used in place of $x(t)$ for the continuous-time signal vectors unless otherwise indicated.

I. PROBLEM FORMULATION

Consider the PLMI

$$\Phi(z) = \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij}(z) \Phi_{ij} < 0 \quad (1)$$

in the form of the double convex sum, where $z \in \mathbb{R}^p$, $h_{ij}(z) : \mathcal{G}_z \to \mathbb{R}_{[0,1]}$ satisfying $\sum_{i=1}^{r} h_{ij}(z) = 1$, $\mathcal{G}_z \subset \mathbb{R}^p$ is a domain, and $\Phi_{ij} = \Phi_{ij}^T \in \mathbb{R}^{n \times n}$, $(i,j) \in \mathbb{I}_r \times \mathbb{I}_r$ is a linear combination of decision variables. The PLMI (1) arises often in controller designs or stability analyses of linear parameter varying models [3] and fuzzy systems [3], [4], [7]. The sufficient LMIs for such nonnegativity, $\Phi_{ij} < 0$, $(i,j) \in \mathbb{I}_r \times \mathbb{I}_r$, would be simply derived by the straightforward way, dropping $h_{ij} \Phi_{ij} \in \mathbb{R}^{n \times n}$; there are several efforts to provide less conservative conditions for PLMI (1) in the form of double convex sum. At first, the sum relaxation techniques [3], [4] are introduced by using the properties of the convex double sum such as $\sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij} \Phi_{ij} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij} \Phi_{ij} + \sum_{i=1}^{r} \sum_{j=1}^{r} h_{ij} (\Phi_{ij} + \Phi_{ji})$. These sum relaxations [3], [4] are fundamental for later researches like adding slack variables [5] and extending multidimensional summations [6], [7] (for more details, see [9]). As stated in [3], [5], the following lemma [5] is still one of useful sum relaxations:

Lemma 1 ([5]): The PLMI (1) holds if

$$\frac{2}{r-1} \Phi_{ii} + \Phi_{ij} + \Phi_{ji} < 0, \quad \forall (i,j) \in \{(i,j) \in \mathbb{I}_r \times \mathbb{I}_r \mid i \neq j\}.$$ 

The main interest is addressed as follows.

Problem 1: Consider the PLMI (1) and Lemma 1 i) Derive the less conservative LMIs without any slack variables than those in Lemma 1 ii) Then, prove their less conservatism than those in Lemma 1

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II. MAIN RESULTS

Before proceeding to our main results, the following lemmas and definition will be needed throughout the proof:

Lemma 2 (Young’s inequality [7]): If $a \in \mathbb{R}_{\geq 0}$ and $b \in \mathbb{R}_{\geq 0}$ and if $\lambda_1 \in \mathbb{R}_{>1}$ and $\lambda_2 \in \mathbb{R}_{>1}$ such that $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 1$, then

$$ab \leq \frac{1}{\lambda_1} a^{\lambda_1} + \frac{1}{\lambda_2} b^{\lambda_2}$$

in which equality holds if and only if $a^{\lambda_1} = b^{\lambda_2}$.

Lemma 3: For any matrix $\Xi_{ij}, (i,j) \in \{(i,j) \in \mathbb{I}_r \times \mathbb{I}_r \mid j \neq i\}$, it is true that

$$\sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} h_{ij}^2 (\Xi_{ij} + \Xi_{ji}) = \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} h_{ij}^2 (\Xi_{ij} + \Xi_{ji}).$$

Proof: See Appendix A

Definition 1: Define

$$\begin{bmatrix}
s_{11} & s_{12} & \cdots & s_{1(r-1)} \\
s_{21} & s_{22} & \cdots & s_{2(r-1)} \\
\vdots & \vdots & \ddots & \vdots \\
s_{2r-1} & s_{2r-2} & \cdots & s_{2r-1(r-1)}
\end{bmatrix} := \begin{bmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{bmatrix}$$

that contains all combinations of zeros and ones of $r-1$ like a binary table. For example, if $r = 2$ and $r = 4$, then

$$\begin{bmatrix}
s_{11} & s_{21} \\
0 & 1
\end{bmatrix}$$

and

$$\begin{bmatrix}
s_{11} & s_{12} & s_{13} & s_{14} \\
s_{21} & s_{22} & s_{23} & s_{24} \\
s_{31} & s_{32} & s_{33} & s_{34} \\
s_{41} & s_{42} & s_{43} & s_{44}
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}$$

respectively.

The main results on Problem 1 are summarized in the following theorems:

Theorem 1: The PLMI (1) holds if

$$\Phi_{ii} + \frac{1}{2} \sum_{j=1, j \neq i}^{r} s_{kl} (\Phi_{ij} + \Phi_{ji}) < 0, \quad \forall (i,k) \in \mathbb{I}_r \times \mathbb{I}_{2r-1} \quad (2)$$

where

$$l = l(j) = \begin{cases} j, & j < i; \\
j - 1, & \text{otherwise}
\end{cases}$$

and $s_{kl}$ is defined in Definition 1.

Proof: It is obvious that

$$\text{PLMI (1)} \Leftrightarrow x^T \Phi(z)x < 0$$

for all non-zero vector $x \in \mathbb{R}^n$. We can equivalently rewrite $x^T \Phi(z)x$ as

$$x^T \Phi(z)x = \sum_{i=1}^{r} h_{ii} x^T \Phi_{ii} x + \frac{1}{2} \sum_{j=1, j \neq i}^{r} h_{ij} x^T \Phi_{ij} + \Phi_{ji}) x$$
If \( x^T (\Phi_{ij} + \Phi_{ji}) x \geq 0 \) on the right-hand side in the foregoing equation, then applying Lemma 2 to \( h_i h_j x^T (\Phi_{ij} + \Phi_{ji}) x \) yields
\[
h_i h_j x^T (\Phi_{ij} + \Phi_{ji}) x \leq \frac{1}{2} \left( h_i^2 + h_j^2 \right) x^T (\Phi_{ij} + \Phi_{ji}) x
\]
Otherwise, if \( x^T (\Phi_{ij} + \Phi_{ji}) x < 0 \), then \( h_i h_j x^T (\Phi_{ij} + \Phi_{ji}) x < 0 \). By considering both cases, we have
\[
h_i h_j x^T (\Phi_{ij} + \Phi_{ji}) x \leq \max \left\{ \frac{1}{2} \left( h_i^2 + h_j^2 \right) x^T (\Phi_{ij} + \Phi_{ji}) x, 0 \right\}
\]
With this inequality, it can be shown that
\[
x^T \Phi_i x < \sum_{i=1}^{r} h_i^2 x^T \Phi_i x + \frac{1}{4} \sum_{i=1}^{r} \sum_{j=1, j \neq i} h_i^2 \max \left\{ x^T (\Phi_{ij} + \Phi_{ji}) x, 0 \right\}
\]
From the fact that by Lemma 3,
\[
\sum_{i=1}^{r} \sum_{j=1, j \neq i} h_i^2 \max \left\{ x^T (\Phi_{ij} + \Phi_{ji}) x, 0 \right\}
\]
we obtain
\[
x^T \Phi_i x + \frac{1}{2} \sum_{j=1, j \neq i} s_{kl} x^T (\Phi_{ij} + \Phi_{ji}) x \leq 0 \Rightarrow x^T \Phi_i x < 0
\]
for all \((i, k) \in \mathbb{I}_3 \times \mathbb{I}_3 - i \). Therefore, we can conclude that
\[
\text{LMIs} 2 \Rightarrow x^T \Phi_i x < 0 \Leftrightarrow \text{PLMI} 1.
\]
for all non-zero vector \( x \).

**Theorem 2:** Consider Lemma 1 and Theorem 1. LMIs 2 of Theorem 1 are not conservative (and can be less conservative) than the LMIs of Lemma 1.

**Proof:** To prove the claim about conservatism, we show that the LMIs of Lemma 1 imply LMIs 2 of Theorem 1 while the converse is not true. First, merging the first and the second LMIs of Lemma 1 into LMIs depending on the discrete variables \( s_{kl} = 0 \) or \( s_{kl} = 1 \), dividing by 2, and summing both sides over \( j \in \mathbb{I}_3, j \neq i \) lead to

LMIs of Lemma 1
\[
\Leftrightarrow \frac{2}{r - 1} \Phi_{ii} + s_{kl} (\Phi_{ij} + \Phi_{ji}) < 0
\]
\[
\Rightarrow \frac{1}{r - 1} \sum_{j=1, j \neq i} \Phi_{ii} + \frac{1}{2} \sum_{j=1, j \neq i} s_{kl} (\Phi_{ij} + \Phi_{ji}) < 0
\]
\[
\Leftrightarrow \text{LMIs} 2 \text{ of Theorem 1}
\]
Next, a counterexample to the converse is that when \( r = 3 \), \( \Phi_{ij}, (i, j) \in \mathbb{I}_3 \times \mathbb{I}_3 \) are given by \( \Phi_{11} = -2, \Phi_{22} = -1, \Phi_{33} = -2, \Phi_{12} = 0, \Phi_{13} = 2, \Phi_{21} = 0, \Phi_{33} = -1, \Phi_{31} = 0 \), and \( \Phi_{32} = 0 \). Then, LMIs 2 of Theorem 1 are satisfied as follows:
\[
\Phi_{11} + \sum_{j=1, j \neq 1}^{3} s_{kl} \frac{\Phi_{11} + \Phi_{11}}{2} = \Phi_{11} = -2
\]
\[
\Phi_{11} + \sum_{j=1, j \neq 1}^{3} s_{kl} \frac{\Phi_{11} + \Phi_{11}}{2} = \Phi_{11} = \frac{\Phi_{13} + \Phi_{31}}{2} = -0.5
\]
\[
\Phi_{11} + \sum_{j=1, j \neq 1}^{3} s_{kl} \frac{\Phi_{11} + \Phi_{11}}{2} = \Phi_{11} = \frac{\Phi_{12} + \Phi_{21}}{2} = -2
\]
\[
\Phi_{11} + \sum_{j=1, j \neq 1}^{3} s_{kl} \frac{\Phi_{11} + \Phi_{11}}{2} = \Phi_{11} + \sum_{j=1, j \neq 1}^{3} s_{kl} \frac{\Phi_{11} + \Phi_{11}}{2} = -0.5
\]
\[
\Phi_{22} + \sum_{j=1, j \neq 2}^{3} s_{kl} \frac{\Phi_{22} + \Phi_{22}}{2} = \Phi_{22} = -1
\]
\[
\Phi_{22} + \sum_{j=1, j \neq 2}^{3} s_{kl} \frac{\Phi_{22} + \Phi_{22}}{2} = \Phi_{22} + \sum_{j=1, j \neq 2}^{3} s_{kl} \frac{\Phi_{22} + \Phi_{22}}{2} = -1.5
\]
\[
\Phi_{22} + \sum_{j=1, j \neq 2}^{3} s_{kl} \frac{\Phi_{22} + \Phi_{22}}{2} = \Phi_{22} + \sum_{j=1, j \neq 2}^{3} s_{kl} \frac{\Phi_{22} + \Phi_{22}}{2} = -1
\]
\[
\Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = \Phi_{33} = -2
\]
\[
\Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = \Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = -2.5
\]
\[
\Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = \Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = -0.5
\]
\[
\Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = \Phi_{33} + \sum_{j=1, j \neq 3}^{3} s_{kl} \frac{\Phi_{33} + \Phi_{33}}{2} = -1
\]
in which we have used
\[
\begin{bmatrix}
    s_{11} & s_{12} \\
    s_{21} & s_{22} \\
    s_{31} & s_{32} \\
    s_{41} & s_{42}
\end{bmatrix}

\begin{bmatrix}
    0 & 0 \\
    0 & 1 \\
    1 & 0 \\
    1 & 1
\end{bmatrix}

\]

**Remark 1:** We emphasize that
1) The number of LMIs of Lemma 1 and Theorem 1 is \( r^2 \) and \( r \times 2^{r-1} \), respectively, in which the size of each LMI is the same as \( \mathbb{R}^{n \times n} \).
2) When \( r = 2 \), LMIs 2 of Theorem 1 are equal to those of Lemma 1.
3) Both Lemma 1 and Theorem 1 need no additional slack variables.
4) Lemma 1 still plays an important role when sufficient LMIs are derived from PLMI 1 in the form of the double convex sum (see, for example, [13], [14], [15], and references therein).

However, Theorem 2 shows that LMIs of Lemma 1 are sufficient to ensure that LMIs 2 of Theorem 1 hold; hence, Theorem 1 could be substituted for Lemma 1.

### III. An Example
Consider the asymptotic stabilization condition 4 in the form of PLMI 1 with
\[
\Phi_{ij} = (A_i Q + B_i F_j)^T + A_i Q + B_j F_j, \quad (i, j) \in \mathbb{I}_3 \times \mathbb{I}_3
\]
where $Q$ provides less conservative results than Lemma 1. As discussed in Theorem 2, one can see that LMIs are feasible, and the mark '$\times$' indicates that LMIs are infeasible. As discussed in Theorem 2, one can see that Theorem 1 provides less conservative results than Lemma 1.

IV. CONCLUSIONS

This letter has provided new sufficient conditions for PLMI in the double convex sum. The sufficient conditions take the LMI form, which are derived without any slack variables. Moreover, their less conservatism than the prior work [3] is proved. All of theoretical claims have been validated successfully through simulation results.

APPENDIX A

PROOF OF LEMMA 3

It is not hard to see that

$$
\sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} h_i^2 (\Xi_{ij} + \Xi_{ji})
= \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} (h_i^2 - h_j^2) (\Xi_{ij} + \Xi_{ji}) + \sum_{i=1}^{r} \sum_{j=1, j \neq i}^{r} h_j^2 (\Xi_{ij} + \Xi_{ji})
$$

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