On the maximum size of an anti-chain of linearly separable sets and convex pseudo-discs\footnote{This research was supported by a Grant from the G.I.F., the German-Israeli Foundation for Scientific Research and Development.}

Rom Pinchasi\footnote{Mathematics Dept., Technion—Israel Institute of Technology, Haifa 32000, Israel. room@math.technion.ac.il.} \hspace{1em} Günter Rote\footnote{Institut für Informatik, Freie Universität Berlin, Takustr. 9, 14195 Berlin, Germany. rote@inf.fu-berlin.de}

February 1, 2008

Abstract

We answer a question raised by Walter Morris, and independently by Alon Efrat, about the maximum cardinality of an anti-chain composed of intersections of a given set of \( n \) points in the plane with half-planes. We approach this problem by establishing the equivalence with the problem of the maximum monotone path in an arrangement of \( n \) lines. A related problem on convex pseudo-discs is also discussed in the paper.

1 Introduction

Let \( P \) be a set of \( n \) points in the plane, no three of which are collinear. A subset of \( P \) is called linearly separable if it is the intersection of \( P \) with a closed half-plane. A \( k \)-set of \( P \) is a subset of \( k \) points from \( P \) which is linearly separable. Let \( \mathcal{A}_k = \mathcal{A}_k(P) \) denote the collection of all \( k \)-sets of \( P \). It is a well-known open problem to determine \( f(k) \), the maximum possible cardinality of \( \mathcal{A}_k \), where \( P \) varies over all possible sets of \( n \) points in general position in the plane. The current records are \( f(k) = O(nk^{1/3}) \) by Dey (\cite{D98}) and \( f([n/2]) \geq n e^{\Omega(\sqrt{\log n})} \) by Tóth (\cite{T01}).

Let \( \mathcal{A} = \mathcal{A}(P) = \bigcup_{k=0}^{n} \mathcal{A}_k \) be the family of all linearly separable subsets of \( P \). The family \( \mathcal{A} \) is partially ordered by inclusion. Clearly, each \( \mathcal{A}_k \) is an anti-chain in \( \mathcal{A} \). The following problem was raised by Walter Morris in 2003 in relation with the convex dimension of a point set (see \cite{ES88}) and, as it turns out, it was independently raised by Alon Efrat 10 years before, in 1993:

**Problem 1.** What is the maximum possible cardinality \( g(n) \) of an anti-chain in the poset \( \mathcal{A} \), over all sets \( P \) with \( n \) points?
In Section 2 we show that in fact $g(n)$ can be very large, and in particular much larger than $f(n)$.

**Theorem 1.** $g(n) = \Omega(n^{2 - \frac{d}{\sqrt{\log n}}})$, for some absolute constant $d > 0$.

In an attempt to bound from above the function $g(n)$ one can view linearly separable sets as a special case of a slightly more general concept:

**Definition 1.** Let $P$ be a set of $n$ points in general position in the plane. A family $F$ of subsets of $P$ is called a family of convex pseudo-discs if the following two conditions are satisfied:

1. Every set in $F$ is the intersection of $P$ with a convex set.
2. If $A$ and $B$ are two different sets in $F$, then both sets $\text{conv}(A) \setminus \text{conv}(B)$ and $\text{conv}(B) \setminus \text{conv}(A)$ are connected (or empty).

One natural example for a family of convex pseudo-discs is the family $A(P)$, where $P$ is a set of $n$ points in general position in the plane. To see this, observe that every linearly separable set is the intersection of $P$ with a convex set, namely, a half-plane. It is therefore left to verify that if $A = P \cap H_A$ and $B = P \cap H_B$, where $H_A$ and $H_B$ are two half-planes, then both $\text{conv}(A) \setminus \text{conv}(B)$ and $\text{conv}(B) \setminus \text{conv}(A)$ are connected. Let $A' = A \setminus H_B = A \setminus B = A \setminus \text{conv}(B)$. Since $\text{conv}(A') \cap \text{conv}(B) = \emptyset$, we have $\text{conv}(A) \setminus \text{conv}(B) \supset \text{conv}(A')$. For any $x \in \text{conv}(A) \setminus \text{conv}(B)$, we claim that there is a point $a' \in A'$ such that the line segment $[x, a']$ is fully contained in $\text{conv}(A) \setminus \text{conv}(B)$. This will clearly show that $\text{conv}(A) \setminus \text{conv}(B)$ is connected. Let $a_1, a_2, a_3$ be three points in $A$ such that $x$ is contained in the triangle $a_1a_2a_3$. If each line segment $[x, a_i]$, for $i = 1, 2, 3$, contains a point of $\text{conv}(B)$, it follows that $x \in \text{conv}(B)$, contrary to our assumption. Thus there must be a line segment $[x, a_i]$ that is contained in $A' = A \setminus \text{conv}(B)$, and we are done.

In Section 3 we bound from above the maximum size of a family of convex pseudo-discs of a set $P$ of $n$ points in the plane, assuming that this family of subsets of $P$ is by itself an anti-chain with respect to inclusion:

**Theorem 2.** Let $F$ be a family of convex pseudo-discs of a set $P$ of $n$ points in general position in the plane. If no member of $F$ is contained in another, then $F$ consists of at most $4 \binom{n}{2} + 1$ members.

Clearly, in view of Theorem 1, the result in Theorem 2 is nearly best possible. We show by a simple construction that Theorem 2 is in fact tight, apart from the constant multiplicative factor of $n^2$.

## 2 Large anti-chains of linearly separable sets

Instead of considering Problem 1 directly, we will consider a related problem.
**Definition 2.** For a pair \( x, y \) of points and a pair \( \ell_1, \ell_2 \) of non-vertical lines, we say that \( x, y \) **strongly separate** \( \ell_1, \ell_2 \) if \( x \) lies strictly above \( \ell_1 \) and strictly below \( \ell_2 \), and \( y \) lies strictly above \( \ell_2 \) and strictly below \( \ell_1 \).

We will also take the dual viewpoint and say that \( \ell_1, \ell_2 \) strongly separate \( x, y \). (In fact, this relation is invariant under the standard point-line duality.)

If we have a set \( L \) of lines, we say that the point pair \( x, y \) is **strongly separated** by \( L \), if \( L \) contains two lines \( \ell_1, \ell_2 \) that strongly separate \( x, y \).

A pair of lines \( \ell_1, \ell_2 \) is said to be strongly separated by a set \( P \) of points if there are two points \( x, y \in P \) that strongly separate \( \ell_1 \) and \( \ell_2 \).

Using the above terminology one can reduce Problem 1 to the following problem:

**Problem 2.** Let \( P \) be a set of \( n \) points in the plane. What is the maximum possible cardinality \( h(n) \) (taken over all possible sets \( P \) of \( n \) points) of a set of lines \( L \) in the plane such that for every two lines \( \ell_1, \ell_2 \in L \), \( P \) strongly separates \( \ell_1 \) and \( \ell_2 \).

![Figure 1: Problem 2](image)

To see the equivalence of Problem 1 and Problem 2, let \( P \) be a set of \( n \) points and \( L \) be a set of \( h(n) \) lines that answer Problem 2. We can assume that none of the points lie on a line of \( L \). Then with each of the lines \( \ell \in L \) we associate the subset of \( P \) which is the intersection of \( P \) with the half-plane below \( \ell \). We thus obtain \( h(n) \) subsets of \( P \) each of which is a linearly separable subset of \( P \). Because of the condition on \( L \) and \( P \), none of these linearly separable sets may contain another. Therefore we obtain \( h(n) \) elements from \( \mathcal{A}(P) \) that form an anti-chain, hence \( g(n) \geq h(n) \).

Conversely, assume we have an anti-chain of size \( g(n) \) in \( \mathcal{A}(P) \) for a set \( P \) of \( n \) points. Each linearly separable set is the intersection of \( P \) with a half-plane,
which is bounded by some line \( \ell \). We can assume without loss of generality that none of these lines is vertical, and at least half of the half-spaces lie below their bounding lines. These lines form a set \( L \) of at least \( \lceil g(n)/2 \rceil \) lines, and each pair of lines is separated by two points from the \( n \)-point set \( P \). Thus, \( h(n) \geq \lceil g(n)/2 \rceil \).

Before reducing Problem 2 to another problem, we need the following simple lemma.

**Lemma 1.** Let \( \ell_1, \ldots, \ell_n \) be \( n \) non-vertical lines arranged in increasing order of slopes. Let \( P \) be a set of points. Assume that for every \( 1 \leq i < n \), \( P \) strongly separates \( \ell_i \) and \( \ell_{i+1} \). Then for every \( 1 \leq i < j \leq n \), \( P \) strongly separates \( \ell_i \) and \( \ell_j \).

**Proof.** We prove the lemma by induction on \( j - i \). For \( j = i + 1 \) there is nothing to prove. Assume \( j - i \geq 2 \). We first show the existence of a point \( x \in P \) that lies above \( \ell_i \) and below \( \ell_{i+1} \). Let \( B \) denote the intersection point of \( \ell_i \) and \( \ell_{i+1} \). Let \( r_i \) denote the ray whose apex is \( B \), included in \( \ell_i \), and points to the right. Similarly, let \( r_j \) denote the ray whose apex is \( B \), included in \( \ell_j \), and points to the right.

Since the slope of \( \ell_{i+1} \) is between the slope of \( \ell_i \) and the slope of \( \ell_j \), \( \ell_{i+1} \) must intersect either \( r_i \) or \( r_j \) (or both, in case it goes through \( B \)).

**Case 1.** \( \ell_{i+1} \) intersects \( r_i \). Then there is a point \( x \in P \) that lies above \( \ell_i \) and below \( \ell_{i+1} \). This point \( x \) must also lie below \( \ell_j \).

**Case 2.** \( \ell_{i+1} \) intersects \( r_j \). Then, by the induction hypothesis, there is a point \( x \in P \) that lies above \( \ell_{i+1} \) and below \( \ell_j \). This point \( x \) must also lie above \( \ell_i \).

The existence of a point \( y \) that lies above \( \ell_j \) and below \( \ell_i \) is symmetric. \( \square \)

By Lemma 1 Problem 2 is equivalent to following problem.

**Problem 3.** What is the maximum cardinality \( h(n) \) of a collection of lines \( L = \{ \ell_1, \ldots, \ell_{h(n)} \} \) in the plane, indexed so that the slope of \( \ell_i \) is smaller than the slope of \( \ell_j \) whenever \( i < j \), such that there exists a set \( P \) of \( n \) points that strongly separates \( \ell_i \) and \( \ell_{i+1} \), for every \( 1 \leq i < h(n) \)?

We will consider the dual problem of Problem 3.

**Problem 4.** What is the maximum cardinality \( h(n) \) of a set of points \( P = \{ p_1, \ldots, p_{h(n)} \} \) in the plane, indexed so that the \( x \)-coordinate of \( p_i \) is smaller than the \( x \)-coordinate of \( p_j \), whenever \( i < j \), such that there exists a set \( L \) of \( n \) lines that strongly separates \( p_{i+1} \) and \( p_i \), for every \( 1 \leq i < h(n) \)?

We will relate Problem 4 to another well-known problem: the question of the longest monotone path in an arrangement of lines.

Consider an \( x \)-monotone path in a line arrangement in the plane. The **length** of such a path is the number of different line segments that constitute the path, assuming that consecutive line segments on the path belong to different lines in the arrangement. (In other words, if the path passes through a vertex of the arrangement without making a turn, this does not count as a new edge.)
Problem 5. What is the maximum possible length $\lambda(n)$ of an $x$-monotone path in an arrangement of $n$ lines?

A construction of [BRSSS04] gives a simple line arrangement in the plane which consists of $n$ lines and which contains an $x$-monotone path of length $\Omega(n^{2 - \frac{d}{\log n}})$ for some absolute constant $d > 0$. No upper bound that is asymptotically better than the trivial bound of $O(n^2)$ is known.

Problem 5 is closely related to Problem 4, and hence also to the other problems:

Proposition 1.

\[
\begin{align*}
h(n) &\geq \left\lceil \frac{\lambda(n) + 1}{2} \right\rceil, \\
\lambda(n) &\geq h(n) - 2
\end{align*}
\]

Proof. We first prove $h(n) \geq \lceil (\lambda(n) + 1)/2 \rceil$. Let $L$ be a simple arrangement of $n$ lines that admits an $x$-monotone path of length $m = \lambda(n)$. Denote by $x_0, x_1, \ldots, x_m$ the vertices of a monotone path arranged in increasing order of $x$-coordinates. In this notation $x_1, \ldots, x_{m-1}$ are vertices of the line arrangement $L$, while $x_0$ and $x_m$ are chosen arbitrarily on the corresponding two rays which constitute the first and last edges, respectively, of the path. For each $1 \leq i < m$ let $s_i$ denote the line that contains the segment $x_{i-1}x_i$, and let $r_i$ denote the line through the segment $x_ix_{i+1}$.

For $1 \leq i < m$, we say that the path bends downward at the vertex $x_i$ if the slope of $s_i$ is greater than the slope of $r_i$, and it bends upward if the slope of $s_i$ is smaller than the slope of $r_i$. Without loss of generality we may assume that at least half of the vertices $x_1, \ldots, x_{m-1}$ of the monotone path are downward bends.

![Figure 2: Constructing a solution for Problem 4](image)
Let $i_1 < i_2 < \cdots < i_k$ be all indices such that $x_{i_j}$ is a downward bend, where $k \geq (m - 1)/2$. Observe that for every $1 \leq j < k$, the monotone path between $x_{i_j}$ and $x_{i_{j+1}}$ is an upward-bending convex polygonal path.

We will now define $k + 1$ points $p_0, p_1, \ldots, p_k$ such that for every $0 \leq j < k$ the $x$-coordinate of $p_j$ is smaller than the $x$-coordinate of $p_{j+1}$, and the line $r_{i_j}$ lies above $p_j$ and below $p_{j+1}$ while the line $s_{i_j}$ lies below $p_j$ and above $p_{j+1}$. This construction will thus show that $h(n) \geq \lceil \lambda(n+1)/2 \rceil$.

For every $1 \leq j \leq k$ let $U_j$ and $W_j$ denote the left and respectively the right wedges delimited by $r_{i_j}$ and $s_{i_j}$. That is, $U_j$ is the set of all points that lie below $r_{i_j}$ and above $s_{i_j}$. Similarly, $W_j$ is the set of all points that lie above $r_{i_j}$ and below $s_{i_j}$.

**Claim 1.** For every $1 \leq j < k$, $W_j$ and $U_{j+1}$ have a nonempty intersection.

*Proof.* We consider two possible cases:

**Case 1.** $i_{j+1} - i_j = 1$. In this case $r_{i_j} = s_{i_{j+1}}$. Therefore any point above the line segment $[x_{i_j}, x_{i_{j+1}}]$ that is close enough to that segment lies both below $s_{i_j}$ and below $r_{i_{j+1}}$, and hence $W_j \cap U_{j+1} \neq \emptyset$.

**Case 2.** $i_{j+1} - i_j > 1$. In this case, as we observed earlier, the monotone path between $x_{i_j}$ and $x_{i_{j+1}}$ is a convex polygonal path. Therefore, $r_{i_j}$ and $s_{i_{j+1}}$ are different lines that meet at a point $B$ whose $x$-coordinate is between the $x$-coordinates of $x_{i_j}$ and $x_{i_{j+1}}$. Any point that lies vertically above $B$ and close enough to $B$ belongs to both $W_j$ and $U_{j+1}$. □

Now it is very easy to construct $p_0, p_1, \ldots, p_k$, see Figure 2. Simply take $p_0$ to be any point in $U_1$, and for every $1 \leq j < k$ let $p_j$ be any point in $W_j \cap U_{j+1}$. Finally, let $p_k$ be any point in $W_k$. It follows from the definition of $U_1, \ldots, U_k$ and $W_1, \ldots, W_k$ that for every $0 \leq j < k$, $r_{i_{j+1}}$ lies above $p_j$ and below $p_{j+1}$ and the line $s_{i_{j+1}}$ lies below $p_j$ and above $p_{j+1}$.

We now prove the opposite direction: $\lambda(n) \geq h(n) - 2$.

Assume we are given $h(n)$ points $p_1, \ldots, p_{h(n)}$ sorted by $x$-coordinate and a set of $n$ lines $L$ such that every pair $p_i, p_{i+1}$ is strongly separated by $L$. By perturbing the lines if necessary, we can assume that none of the lines goes through a point, and no three lines are concurrent. For $1 < i < h(n)$, let $f_i$ be the face of the arrangement that contains $p_i$, and let $A_i$ and $B_i$ be, respectively, the left-most and right-most vertex in this face. (The faces $f_i$ are bounded, and therefore $A_i$ and $B_i$ are well-defined.) The monotone path will follow the upper boundary of each face $f_i$ from $A_i$ to $B_i$.

We have to show that we can connect $B_i$ to $A_{i+1}$ by a monotone path. This follows from the separation property of $L$. Let $s_i, r_i$ be a pair of lines that strongly separates $p_i$ and $p_{i+1}$ in such a way that $r_i$ lies above $p_i$ and below $p_{i+1}$ and $s_i$ lies below $p_i$ and above $p_{i+1}$. Since $B_i$ lies on the boundary of the face $f_i$ that contains $p_i$, $B_i$ lies also between $r_i$ and $s_i$, including the possibility of lying on these lines. We can thus walk on the arrangement from $B_i$ to the right until we hit $r_i$ or $s_i$, and from there we proceed straight to the intersection point $Q_i$ of $r_i$ and $s_i$. Similarly, there is a path in the arrangement
from $A_{i+1}$ to the left that reaches $Q_i$. and these two paths together link $B_i$ with $A_{i+1}$.

To count the number of edges of this path, we claim that there must be at least one bend between $B_i$ and $A_{i+1}$ (including the boundary points $B_i$ and $A_{i+1}$). If there is no bend at $Q_i$, the path must go straight through $Q_i$, say, on $r_i$. But then the path must leave $r_i$ at some point when going to the right: if the path has not left $r_i$ by the time it reaches $A_{i+1}$ and $A_{i+1}$ lies on $r_i$, then the path must bend upward at this point, since it proceeds on the upper boundary of the face $f_{i+1}$ that lies above $r_i$.

Thus, the path makes at least $h(n) - 3$ bends (between $B_i$ and $A_{i+1}$, for $1 < i < h(n) - 1$) and contains at least $h(n) - 2$ edges.

Now it is very easy to give a lower bound for $g(n)$, and prove Theorem 1. Indeed, this follows because $g(n) \geq h(n)$ and $h(n) \geq \lceil \frac{\lambda(n)+1}{2} \rceil = \Omega(n^{2-\frac{d}{\sqrt{\log n}}})$.

The close relation between Problems 1 and 5 comes probably as no big surprise if one considers the close connection between $k$-sets and levels in arrangements of lines (see [ES7, Section 3.2]). For a given set of $n$ points $P$, the $k$-sets are in one-to-one correspondence with the faces of the dual arrangements of lines which have $k$ lines passing below them and $n-k$ lines passing above them (or vice versa). The lower boundaries of these cells form the $k$-th level in the arrangement, and the upper boundaries form the $(k+1)$-st level.

Our chain of equivalence from Problem 1 to Problem 5 extends this relation between $k$-sets and levels in a way that is not entirely trivial: for example, establishing that we get sets that form an antichain requires some work, whereas for $k$-sets this property is fulfilled automatically.

3 Proof of Theorem 2

The heart of our argument uses a linear algebra approach first applied by Tverberg [T82] in his elegant proof for a theorem of Graham and Pollak [GP72] on decomposition of the complete graph into bipartite graphs.

Let $F$ be a collection of convex pseudo-discs of a set $P$ of $n$ points in general position in the plane. We wish to bound from above the size of $F$ assuming that no set in $F$ contains another. For every directed line $L = \overrightarrow{xy}$ passing through two points $x$ and $y$ in $P$ we denote by $L_x$ the collection of all sets $A \in F$ that lie in the closed half-plane to the left of $L$ such that $L$ touches $\text{conv}(A)$ at the point $x$ only. Similarly, let $L_y$ be the collection of all sets $A \in F$ that lie in the closed half-plane to the left of $L$ such that $L$ touches $\text{conv}(A)$ at the point $y$ only. Finally, let $L_{xy}$ be those sets $A \in F$ that lie in the closed half-plane to the left of $L$ such that $L$ supports $\text{conv}(A)$ at the edge $xy$.

Definition 3. Let $A$ and $B$ be two sets in $F$. Let $L$ be a directed line through two points $x$ and $y$ in $P$. We say that $L$ is a common tangent of the first kind
with respect the pair \((A, B)\) if \(A \in L_x\) and \(B \in L_y\).

We say that \(L\) is a common tangent of the second kind with respect to \((A, B)\) if \(A \in L_{xy}\) and \(B \in L_y\), or if \(A \in L_x\) and \(B \in L_{xy}\).

The crucial observation about any two sets \(A\) and \(B\) in \(F\) is stated in the following lemma.

**Lemma 2.** Let \(A\) and \(B\) be two sets in \(F\). Then exactly one of the following conditions is true.

1. There is precisely one common tangent of the first kind with respect to \((A, B)\) and no common tangent of the second kind with respect to \((A, B)\), or

2. there is no common tangent of the first kind with respect to \((A, B)\), and there are precisely two common tangents of the second kind with respect \((A, B)\).

**Proof.** The idea is that because \(A\) and \(B\) are two pseudo-discs and none of \(\text{conv}(A)\) and \(\text{conv}(B)\) contains the other, then as we roll a tangent around \(C = \text{conv}(A \cup B)\), there is precisely one transition between \(A\) and \(B\), and this is where the situation described in the lemma occurs (see Figure 3).

Formally, by our assumption on \(F\), none of \(A\) and \(B\) contains the other. Any directed line \(L\) that is a common tangent of the first or second kind with respect to \(A\) and \(B\) must be a line supporting \(\text{conv}(A \cup B)\) at an edge.

Let \(x_0, \ldots, x_{k-1}\) denote the vertices of \(C = \text{conv}(A \cup B)\) arranged in counterclockwise order on the boundary of \(C\). In what follows, arithmetic on indices is done modulo \(k\).

There must be an index \(i\) such that \(x_i \in A \setminus B\), for otherwise every \(x_i\) belongs to \(B\) and therefore \(\text{conv}(B) = \text{conv}(A \cup B) \supset \text{conv}(A)\) and therefore \(B \supset A\) (because both \(A\) and \(B\) are intersections of \(P\) with convex sets) in contrast to our assumption. Similarly, there must be an index \(i\) such that \(x_i \in B \setminus A\).
Let \( I_A \) be the set of all indices \( i \) such that \( x_i \in A \setminus B \), and let \( I_B \) be the set of all indices \( i \) such that \( x_i \in B \setminus A \).

We claim that \( I_A \) (and similarly \( I_B \)) is a set of consecutive indices. To see this, assume to the contrary that there are indices \( i, j, i', j' \) arranged in a cyclic order modulo \( k \) such that \( x_i, x_{i'} \in A \setminus B \) and \( x_j, x_{j'} \in B \). Then it is easy to see that \( \text{conv}(A) \setminus \text{conv}(B) \) is not a connected set because \( x_i \) and \( x_{i'} \) are in different connected components of this set.

We have therefore two disjoint intervals \( I_A = \{i_A, i_A + 1, \ldots, j_A\} \) and \( I_B = \{i_B, i_B + 1, \ldots, j_B\} \). It is possible that \( i_A = j_A \) or \( i_B = j_B \).

Observe that \( x_{i_A}, x_{j_A}, x_{i_B}, x_{j_B} \) are arranged in this counterclockwise cyclic order on the boundary of \( C \), and for every index \( i \notin I_A \cup I_B \), \( x_i \in A \cap B \). The only candidates for common tangents of the first kind or of the second kind with respect to \( A \) and \( B \) are of the form \( x_i x_{i+1} \), that is, they must pass through two consecutive vertices of \( C \).

We distinguish two possible cases:

1. \( i_B = j_A + 1 \). In this case, the line through \( x_{j_A} \) and \( x_{i_B} \) is the only common tangent of the first kind with respect to \( (A, B) \) and there are no common tangents of the second kind with respect to \( (A, B) \).

2. \( i_B \neq j_A + 1 \). In this case, there is no common tangent of the first kind with respect to \( (A, B) \). The line through \( x_{i_B-1} \) and \( x_{i_B} \) and the line through \( x_{j_A} \) and \( x_{j_A+1} \) are the only common tangents of the second kind with respect to \( (A, B) \).

This completes the proof of the lemma. \( \square \)

Let \( A_1, \ldots, A_m \) be all the sets in \( F \), and for every \( 1 \leq i \leq m \) let \( z_i \) be an indeterminate associated with \( A_i \). For each directed line \( L = \overline{xy} \), define the following polynomial \( P_L \):

\[
P_L(z_1, \ldots, z_m) = \left( \sum_{A_i \in L_x} z_i \right) \left( \sum_{A_j \in L_y} z_j \right) + \frac{1}{2} \left( \sum_{A_i \in L_x} z_i \right) \left( \sum_{A_j \in L_{xy}} z_j \right) + \frac{1}{2} \left( \sum_{A_i \in L_y} z_i \right) \left( \sum_{A_j \in L_{xy}} z_j \right)
\]

This polynomial contains a term \( z_u z_v \) whenever \( L \) is a tangent line for the pair \((A_u, A_v)\) or for the pair \((A_v, A_u)\) (of the first or of the second kind, and with coefficient 1 or \( \frac{1}{2} \), accordingly). If we sum this equation over all directed lines \( L \), it follows by Lemma 2 that every term \( z_u z_v \) with \( u \neq v \) appears with coefficient 2:

\[
\sum_L P_L(z_1, \ldots, z_m) = \sum_{u < v} 2z_u z_v = (z_1 + \cdots + z_m)^2 - (z_1^2 + \cdots + z_m^2) \quad (3)
\]

Consider the system of linear equations \( \sum_{A_i \in L_x} z_i = 0 \) and \( \sum_{A_i \in L_y} z_i = 0 \), where \( L = \overline{xy} \) varies over all directed lines determined by \( P \). Add to this
system the equation $z_1 + \cdots + z_m = 0$. There are $4\binom{n}{2} + 1$ equations in this system and if $m > 4\binom{n}{2} + 1$, there must be a nontrivial solution. However, it is easily seen that a nontrivial solution $(z_1, \ldots, z_m)$ will result in a contradiction to (3). This is because the left-hand side of (3) vanishes, while the right-hand side equals $-(z_1^2 + \cdots + z_m^2) \neq 0$. We conclude that $|F| = m \leq 4\binom{n}{2} + 1$. 

We now show by a simple construction that Theorem 2 is tight apart from the multiplicative constant factor of $n^2$. Fix three rays $r_1, r_2, r_3$ emanating from the origin such that the angle between two rays is 120 degrees. For each $i = 1, 2, 3$, let $p_{i1}, \ldots, p_{in}$ be $n$ points on $r_i$, indexed according to their increasing distance from the origin. Slightly perturb the points to get a set $P$ of $3n$ points in general position in the plane. For every $1 \leq j, k, l \leq n$ define

$$F_{jkl} = \{p_{11}, \ldots, p_{1j}\} \cup \{p_{21}, \ldots, p_{2k}\} \cup \{p_{31}, \ldots, p_{3l}\}.$$ 

It can easily be checked that the collection of all $F_{jkl}$ such that $1 \leq j, k, l \leq n$ and $j + k + l = n + 2$ is an anti-chain of convex pseudo-discs of $P$. This collection consists of $\binom{n+1}{2}$ sets.

References

[BRSS04] J. Balogh, O. Regev, C. Smyth, W. Steiger, and M. Szegedy, Long monotone paths in line arrangements. Discrete Comput. Geom. 32 (2004), no. 2, 167–176.

[D98] T. K. Dey, Improved bounds for planar $k$-sets and related problems. Discrete Comput. Geom. 19 (1998), no. 3, 373–382.

[ES88] P. H. Edelman and M. E. Saks, Combinatorial representation and convex dimension of convex geometries. Order 5 (1988), no. 1, 23–32.

[E87] H. Edelsbrunner, Algorithms in Combinatorial Geometry, EATCS Monographs on Theoret. Comput. Sci., vol. 10, Springer-Verlag, Berlin, 1987.

[GP72] R. L. Graham and H. O. Pollak, On embedding graphs in squashed cubes. In Proc. Conf. Graph Theory Appl., Western Michigan Univ., May 10–13, 1972, ed. Y. Alavi, D. R. Lick, and A. T. White, Lecture Notes in Mathematics, vol. 303, Springer-Verlag, Berlin, 1972, pp. 99–110.

[T01] G. Tóth, Point sets with many $k$-sets, Discrete Comput. Geom. 26 (2001) no. 2, 187–194.

[T82] H. Tverberg, On the decomposition of $K_n$ into complete bipartite graphs. J. Graph Theory 6 (1982), no. 4, 493–494.