A CHARACTERIZATION OF SEMISTABLE RADIAL SOLUTIONS OF k-HESSIAN EQUATIONS

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Abstract. We characterize semistable radial solutions of the equation

\[ S_k (D^2 u) = g(u) \quad \text{in} \quad B_1, \]

where \( B_1 \) is the unit ball of \( \mathbb{R}^n \), \( D^2 u \) is the Hessian matrix of \( u \), \( g \) is a positive \( C^1 \) nonlinearity and \( S_k (D^2 u) \) denotes the \( k \)-Hessian operator of \( u \). This class of radial solutions has been recently introduced by the authors in [8]. The proofs are new relative to those given in [8] and focus on the structure of the equation directly, thereby improving some previous results.

2010 Mathematics Subject Classification 35J60, 35J25 (primary), 35B35, 35B07 (secondary).

1. Introduction and statement of results

In this work we are concerned with the following nonlinear equation

\[ S_k (D^2 u) = g(u) \quad \text{in} \quad B_1. \tag{1.1} \]

Here \( B_1 \) is the unit ball of \( \mathbb{R}^n \), \( D^2 u \) is the Hessian matrix of \( u \), \( g \) is a positive \( C^1 \) nonlinearity and \( S_k (D^2 u) \) denotes the \( k \)-Hessian operator of \( u \).

In our previous paper [8] we introduced the notion of semistability for solutions of equation (1.1) under a homogeneous Dirichlet boundary condition. In the radial case, sharp pointwise estimates on semistable solutions were obtained, extending some results from the semilinear case, i.e., when \( k = 1 \). However, to work with the new notion of stability for solutions of (1.1) even under rotational symmetry conditions it was necessary to introduce the auxiliary equation

\[ \text{div} \left( |x|^{1-k} |\nabla u|^{k-1} \nabla u \right) = c_{n,k}^{-1} g(u) \quad \text{in} \quad B_1 \setminus \{0\}, \tag{P} \]

\((c_{n,k} = \binom{n}{k}/n)\) and exploit the fact that, for radial solutions, both equations coincide. The main purpose of this paper is to characterize the class of radially symmetric solutions which are semistable, in a suitably-defined sense, for equation (1.1). For this, we use new arguments based on the radial structure of (1.1) in order to remove equation \((P)\) in [8].

For \( k \in \{1, \ldots, n\} \), let \( \sigma_k : \mathbb{R}^n \to \mathbb{R} \) denote the \( k \)-th elementary symmetric function

\[ \sigma_k (\lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \]

and let \( \Gamma_k \) denote the cone \( \Gamma_k = \{ \lambda = (\lambda_1, \ldots, \lambda_n) : \sigma_1 (\lambda) > 0, \ldots, \sigma_k (\lambda) > 0 \} \).

For a twice differentiable function \( u \) defined on a smooth domain \( \Omega \subset \mathbb{R}^n \), the \textit{k-Hessian operator} is defined by \( S_k (D^2 u) = \sigma_k \left( \lambda (D^2 u) \right) \), where \( \lambda (D^2 u) \) are the eigenvalues of \( D^2 u \). Equivalently, \( S_k (D^2 u) \) is the sum of the \( k \)-th principal minors...
of the Hessian matrix. See e.g. [12, 13]. Two relevant examples in this family of operators are the Laplace operator $S_1(D^2u) = \Delta u$ and the Monge-Ampère operator $S_n(D^2u) = \det(D^2u)$. They are fully nonlinear when $k \geq 2$. In particular, $S_2(D^2u) = \frac{1}{2} \left((\Delta u)^2 - |D^2u|^2\right)$. The study of $k$-Hessian equations have many applications in geometry, optimization theory and other related fields. See [12]. These operators have been studied extensively, starting with the seminal work of Caffarelli, Nirenberg and Spruck [2]. See, e.g., [3, 5, 6, 7, 9, 10].

Now consider the class of functions

$$\Phi^k(\Omega) = \{ u \in C^2(\Omega) \cap C(\overline{\Omega}) : \lambda(D^2u) \in \Gamma_i, i = 1, \ldots, k \}. $$

The functions in $\Phi^k(\Omega)$ are called admissible or $k$-convex functions. Further, $S_k(D^2u)$ turns to be elliptic in the class of $k$-convex functions. Denote by $\Phi^k_0(\Omega)$ the set of functions in $\Phi^k(\Omega)$ that vanish on the boundary $\partial \Omega$. Observe that the functions in $\Phi^k_0(\Omega)$ are negative in $\Omega$. For more details we refer the reader to [12].

The following two notions of solutions for problem (1.1) were introduced recently in [8]

**Definition 1.1.** We say that:

i) $u$ is a classical solution of (1.1) if $u \in \Phi^k_0(B_1)$ and equation (1.1) holds;

ii) $u$ is a weak solution of (1.1) if $u \in L^{k+1}(B_1)$, $\int_{B_1} \left\{ \sum u_i u_j S^{ij}_k(D^2u) \right\} < \infty$, $g(u) \in L^1(B_1)$ and

$$\int_{B_1} \left\{ \frac{1}{k} \sum u_i \eta_j S^{ij}_k(D^2u) + g(u) \eta \right\} = 0, \forall \eta \in C^1_c(B_1),$$

where $u_i = u_{x_i}$ indicates the partial derivative of $u$ with respect to the variable $x_i$.

We recall that $S^{ij}_k(D^2u) = \frac{\partial}{\partial u_{ij}} S_k(D^2u)$, where $u_{ij} = u_{x_i x_j}$. This expression is related to the divergence structure of the $k$-Hessians, $S_k(D^2u) = \frac{1}{k} \sum (u_j S^{ij}_k(D^2u))_i$. For instance, when $k = 1$, we have $S^{ij}_1(D^2u) = \delta_{ij}$ and $S_1(D^2u) = \delta_{ij} u_{ij}$, where $\delta_{ij}$ is the Kronecker delta symbol.

**Definition 1.2.** Let $u$ be a solution of (1.1). We say that $u$ is semistable if

(1.3) \[ Q_u(\varphi) = \int_{B_1} \left\{ \sum \varphi_i \varphi_j S^{ij}_k(D^2u) + g'(u) \varphi^2 \right\} \geq 0, \forall \varphi \in C^1_c(B_1 \setminus \{0\}) \]

From a variational point of view, semistable solutions of equation (1.1) in $\Phi^k_0(B_1)$ correspond to critical points of an energy functional with nonnegative second variation (1.3) (in particular, local minimizers of the energy are semistable solutions). See [10, 13].

Recently, in [14], the authors gave a definition of (classical) stable radial solutions of the $k$-Hessian equation $F_k(D^2V) = (-V)^p$ in $\mathbb{R}^n$. They established connections between certain critical exponents of Joseph-Lundgren type and stability. We point out that our definition of semistability was motivated by the variational structure of equation in (1.1) and the fact that the $k$-Hessians can be written in divergence form. Note that our semistability condition (1.3) agrees (with the obvious changes) with the one given in [14] if $u$ is radial. See (1.5) below. Furthermore, we obtain explicitly $S^{ij}_k(D^2u)$ in terms of the eigenvalues of the Hessian matrix of $u$, which is key for characterizing the semistable solutions (classical or weak solutions).
We recall that, for a radially symmetric $C^2$ function $u$, the $k$-Hessian operator is given by

$$S_k(D^2u) = c_{n,k}r^{1-n}(r^{n-k}(u')^k)' = c_{n,k} \left( \frac{u'}{r} \right)^{k-1} \left( n \left( \frac{u'}{r} \right) + k \left( u'' - \frac{u'}{r} \right) \right),$$

where $u(x) = u(r)$, $r = |x|, x \in \mathbb{R}^n \setminus \{0\}$ and $c_{n,k} = \binom{n}{k}/n$. Here $u'$ denote the radial derivative of the radial function $u$. This formula is well known. For self-containment, we include a proof of \eqref{Radial:Hess} in Section 5 (the reader who is familiar with \eqref{Radial:Hess} may certainly skip this proof).

\textit{Remark 1.} Using the sign condition on $g$, it is easy to see that the following statements are equivalent: (a) $u$ is a classical radial solution of \eqref{eq:radial}; (b) $u$ is a $C^2$ solution of $c_{n,k}r^{1-n}(r^{n-k}(u')^k)' = g(u)$, $r \in (0,1)$ satisfying $u'(0) = u(1) = 0$. In particular, for a classical radial solution $u$ of \eqref{eq:radial}, the above equivalence implies that $u(r) > 0$ for all $r \in (0,1)$.

Our main results are the following two theorems which characterize weak and semistable radial solutions of \eqref{eq:radial}

\textbf{Theorem 1.3.} Let $g(u) \in L^1(B_1)$. A function $u \in W^{1,k+1}(B_1, |x|^{1-k})$ is a weak radial solution of \eqref{eq:radial} if and only if

$$\int_{B_1} \left\{ c_{n,k} |x|^{-k}(u')^k(x, \nabla u) + g(u) \xi \right\} = 0,$$

for every radially symmetric function $\xi \in C_0^1(B_1)$.

\textbf{Theorem 1.4.} A function $u$ is a semistable radial solution of \eqref{eq:radial} if and only if

\textbf{Definition 1.5.} An absolutely continuous function $u$ defined on $(0,1)$ is an integral radial solution of \eqref{eq:radial} if $r^{n-1}g(u) \in L^1(0,1)$, $\int_0^1 r^{n-1} |u|^{k+1} < \infty$, $\int_0^1 r^{n-k}(u')^{k+1}dr < \infty$ and

$$r^{n-k}(u')^k = c_{n,k}^{-1} \int_0^r s^{n-1}(u') ds a.e. \text{ in } (0,1).$$

Thus, in the context of radial solutions, we can use Theorem 1.3 to show that the definitions of weak solution and integral solution are equivalent.

\textbf{Lemma 1.6.} Let $u$ be a weak radial solution of \eqref{eq:radial}. Then $u$ is an integral radial solution, and conversely. A consequence of the previous lemma is the following statement concerning regularity of the solutions.

\textbf{Corollary 1.7.} Let $u$ be an integral radial solution of \eqref{eq:radial}. Then $u \in C^2(0,1)$.

In the following statement, the semistability inequality \eqref{eq:semistable} is rewriting in a form that makes it independent of $g$.\[\text{(ineq:propstable)}\]
Corollary 1.8. A function $u$ is a semistable radial solution of (1.1) if and only if

$$\int_{B_1} \left( \frac{u'}{|x|} \right)^{k+1} \left\{ |x| \nabla \eta |^2 + \left( \frac{k-1}{k+1} \right) (x, \nabla \eta)^2 - \left( \frac{2n-k-1}{k+1} \right) \eta^2 \right\} \geq 0,$$

for every radially symmetric function $\eta \in H^1_c(B_1) \cap L^\infty_{loc}(B_1)$.

The preceding result is necessary for characterizing the semistable solutions.

The key point in the proof of Theorems 1.3 and 1.4 is to obtain explicitly $S_k(D^2u)$ in terms of the eigenvalues of $D^2u$. To this end, we take advantage of the radial structure of equation (1.1) and thus avoid to use of equation $(P)$, which was a technical requirement in [8]. We also employ some ideas from [1] and [4]. It is worth noting that the main arguments in our proofs appear to be new.

As a consequence of our results, we obtain the following pointwise estimates for solutions of equation (1.1), which are analogues of the corresponding results in [8, Section 2].

Theorem 1.9. Let $n \geq 2$, $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative locally Lipschitz function and $u \in W^{1,k+1}(B_1, |x|^{1-k})$ be a semistable radial solution of equation (1.1). Then the following holds:

i) If $n < 2k+8$, then

$$|u(r)| \leq C, \forall r \in (0,1].$$

ii) If $n = 2k+8$, then

$$|u(r)| \leq C (\log r + 1), \forall r \in (0,1].$$

iii) If $n > 2k+8$, then

$$|u(r)| \leq Cr^{-(k+1)n+2\sqrt{(2k+1)n-4k+2k^2+4k}} (k+1)^2, \forall r \in (0,1].$$

Here $C = D_{n,k} \|u\|_{W^{1,k+1}(B_1, |x|^{1-k})}$ where $D_{n,k}$ is a constant depending only on $n$ and $k$.

Theorem 1.10. Let $n \geq 2k+8$, $g : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function and $u \in W^{1,k+1}(B_1, |x|^{1-k})$ be a semistable radial solution of equation (1.1) in $B_1$. Then the following holds:

i) If $g$ is a nonnegative function, then

$$u'(r) \leq D_{n,k} \|\nabla u\|_{L^{k+1}(B_1, |x|^{1-k})} r^{-(k+1)n+2\sqrt{(2k+1)n-4k+2k^2+4k-1}} (k+1)^2, \forall r \in (0,1/2].$$

ii) If $g$ is a nonnegative and nonincreasing function, then

$$|u^{(i)}(r)| \leq D'_{n,k} \left( \min_{t \in [1/2,1]} u'(t) \right) r^{-(k+1)n+2\sqrt{(2k+1)n-4k+2k^2+2(3-i)k-2i}} (k+1)^2, \forall r \in (0,1], i \in \{1,2\}. $$

iii) If $g$ is a nonnegative, nonincreasing and convex function, then

$$|u^{(3)}(r)| \leq D''_{n,k} \left( \min_{t \in [1/2,1]} u'(t) \right) r^{-(k+1)n+2\sqrt{(2k+1)n-4k+2k^2-3}} (k+1)^2, \forall r \in (0,1].$$

Here $D_{n,k}$ and $D'_{n,k}$ are constants depending only on $n$ and $k$.

Remark 2. Similar to [11], in the last section we show that, without making assumptions on the sign of $g'$ or $g''$, it is impossible to obtain any pointwise estimates for $|u'|$ and $|u''|$ (see Corollaries 4.4 and 4.6).
Theorem 1.11. Let $n \geq 2$, $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative and nonincreasing locally Lipschitz function and $u \in W^{1,k+1}(B_1, |x|^{1-k})$ be a semistable radial solution of equation (1.1). Then the following holds:

i) If $n < 2k + 8$, then
$$|u(r) - u(1)| \leq C(1 - r), \forall r \in (0, 1].$$

ii) If $n = 2k + 8$, then
$$|u(r) - u(1)| \leq C \log r, \forall r \in (0, 1].$$

iii) If $n > 2k + 8$, then
$$|u(r) - u(1)| \leq C \left(r^{-(k+1)n+2\sqrt{2(k+1)n-4k+2k^2+6k}} - 1\right), \forall r \in (0, 1].$$

Here $C = D_{n,k}\min_{t \in [1/2, 1]} \{u'(t)\}$ where $D_{n,k}$ is a constant depending only on $n$ and $k$.

Remark 3. Note that in the above statements no mention is made of equation (P). Compare with Theorems 2.5, 2.6 and 2.8 in [8].

The rest of this paper is organized as follows: in Section 2 we will state preliminaries which include new insights into the radial structure of equation (1.1), which are key to proving our main results. Section 3 will then be devoted to the proof of Theorems 1.3 and 1.4, Lemma 1.6 and Corollaries 1.7 and 1.8. In Section 4 we provide a large family of semistable radially increasing unbounded solutions of problem (1.1) in the punctured unit ball. Section 5, which concludes the paper, contain a derivation of the radial form of the $k$-Hessians operators.

2. Preliminaires

In this section we compute $S^i_j(D^2u)$ for a radially symmetric $C^2$ function $u$. As observed above, we exploit the radial structure of equation (1.1) to obtain explicitly $S^i_j(D^2u)$ in terms of the eigenvalues of $D^2u$, which is fundamental for characterizing the solutions. This is one of the novelties of this paper. To this end, set $x = (x_1, \ldots, x_n) \in \mathbb{R}^n \setminus \{0\}$. Then, for a radial function $u = u(r)$, $r = |x|$, we have

$$u_i = x_i \lambda_2 \text{ and } u_{ij} = \delta_{ij} \lambda_2 + \frac{x_i x_j}{|x|^2} (\lambda_1 - \lambda_2),$$

where $\lambda_1 = u''$ and $\lambda_2 = u'/|x|$ are the eigenvalues of $D^2u$ at the point $x = (r, 0, \ldots, 0)$ with multiplicities 1 and $(n - 1)$, respectively. Now, as we saw in the Introduction, $S^i_j(D^2u) = \delta_{ij}$. Note that in the case $k = 2$ we can use the formula $S_2(D^2u) = \frac{1}{2} \left( (\Delta u)^2 - |D^2u|^2 \right)$ to obtain $S^i_j(D^2u)$. Indeed, we have

$$S_2(D^2u) = \frac{1}{2} \left( (\Delta u)^2 - |D^2u|^2 \right) = \frac{1}{2} \left( \left( \sum u_{ij} \right)^2 - \sum u_{ij}^2 \right),$$

$$S^i_j(D^2u) = S_1(D^2u) \delta_{ij} - u_{ij} = (n \lambda_2 + (\lambda_1 - \lambda_2)) \delta_{ij} - \delta_{ij} \lambda_2 - \frac{x_i x_j}{|x|^2} (\lambda_1 - \lambda_2)$$

$$= (n - 1) \lambda_2 \delta_{ij} + (\lambda_1 - \lambda_2) \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right).$$
To unify all the cases for $k \in \{1, \ldots, n\}$, we use the radial form of the $k$-Hessian operator (1.4). For simplicity we denote $S_k = S_k(D^2 u)$ and $S_{ij}^k = S_{ij}^k(D^2 u)$. Thus, we rewrite $S_k = c_{n,k} \lambda_2^{k-1}(n \lambda_2 + k(\lambda_1 - \lambda_2))$ depending on the eigenvalues of $D^2 u$.

Then

\[ S_{ij}^k = \frac{\partial S_k}{\partial u_{ij}} = \frac{\partial S_k}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial u_{ij}} + \frac{\partial S_k}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial u_{ij}}, \tag{2.2} \]

where

\[ \frac{\partial S_k}{\partial \lambda_1} = kc_{n,k} \lambda_2^{k-1} \text{ and } \frac{\partial S_k}{\partial \lambda_2} = kc_{n,k} \lambda_2^{k-2}((n - 1) \lambda_2 + (k - 1)(\lambda_1 - \lambda_2)). \tag{2.3} \]

From the equality $\Delta u = n \lambda_2 + (\lambda_1 - \lambda_2)$ we deduce that

\[ (n - 1) \frac{\partial \lambda_2}{\partial u_{ij}} + \frac{\partial \lambda_1}{\partial u_{ij}} = \delta_{ij}. \tag{2.4} \]

Thus, from (2.2)-(2.4), we have

\[ S_{ij}^k = kc_{n,k} \lambda_2^{k-1} \left( \frac{\partial \lambda_1}{\partial u_{ij}} + (n - 1) \frac{\partial \lambda_2}{\partial u_{ij}} \right) \lambda_2 + (k - 1)(\lambda_1 - \lambda_2) \frac{\partial \lambda_2}{\partial u_{ij}} \]

\[ = kc_{n,k} \lambda_2^{k-2} \left( \lambda_2 \delta_{ij} + (k - 1)(\lambda_1 - \lambda_2) \frac{\partial \lambda_2}{\partial u_{ij}} \right). \tag{2.5} \]

In particular

\[ S_k = kc_{n,k} \lambda_2^{k-1} \delta_{ij}, \tag{2.6} \]

when $\lambda_1 = \lambda_2$.

Now, from (2.1) it follows that

\[ \sum u_{ij}^2 = n \lambda_2^2 + 2 \lambda_2(\lambda_1 - \lambda_2) + (\lambda_1 - \lambda_2)^2. \]

Differentiating the preceding equality with respect to $u_{ij}$ and using (2.4), we have

\[ \frac{\partial \lambda_1}{\partial u_{ij}} = \frac{x_i x_j}{|x|^2}, \tag{2.7} \]

provided that $\lambda_1 \neq \lambda_2$.

Consequently, from (2.4), (2.5) and (2.7) we obtain

\[ S_k = kc_{n,k} \lambda_2^{k-2} \left( \lambda_2 \delta_{ij} + \frac{k - 1}{n - 1} (\lambda_1 - \lambda_2) \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) \right). \tag{2.8} \]

Let $S$ be the matrix whose entries are $S_{ij}^k$. Then using (2.6) and (2.8) we can write $S$ as

\[ S = kc_{n,k} \lambda_2^{k-2} \left( \lambda_2 I_n + \frac{k - 1}{n - 1} (\lambda_1 - \lambda_2) \left( I_n - \frac{x^T x}{|x|^2} \right) \right), \tag{2.9} \]

where $I_n$ denotes the identity matrix of order $n$ and $x^T x$ denotes a square matrix whose entries are given by $(x^T x)_{ij} = x_i x_j$. 
Finally, from (2.9) we have

\[
\begin{align*}
ws^Tv^T &= kc_{n,k} \lambda_2^{-2} \left( \lambda_2 \left( \frac{x}{|x|}, w \right) \left( \frac{x}{|x|}, v \right) + 
\right. \\
&+ \left. \left( \lambda_2 + \left( \frac{k-1}{n-1} \right) (\lambda_1 - \lambda_2) \right) \left( (w, v) - \left( \frac{x}{|x|}, w \right) \left( \frac{x}{|x|}, v \right) \right) \right),
\end{align*}
\]  
\tag{2.10}
\]

(2.10)

which completes the proof. □

3. Proof of the main results

In this section we will prove the Theorems 1.3 and 1.4, Lemma 1.6 and Corollaries 1.7 and 1.8.

Proof of Theorem 1.3. Let \( \eta \in C^1_0(B_1) \) (not necessarily radial). We consider the spherical averages of \( \eta \), i.e., the radial function

\[ \xi(r) := \frac{1}{|\partial B_1|} \int_{\partial B_1} \eta(r\theta) \, d\theta = \int_{\partial B_1} \eta(r\theta) \, d\theta, \]

where, using polar coordinates, for \( x \in \mathbb{R}^n \setminus \{0\} \) we have set \( x = r\theta \), with \( r = |x| > 0 \) and \( \theta = x/r \in \partial B_1 = \{ \theta \in \mathbb{R}^n : |\theta| = 1 \}. \)

Differentiating with respect to \( r \), we get

\[ \xi_r = \int_{\partial B_1} (\theta, \nabla \eta) \, d\theta. \]

Replacing \( w \) by \( x \) and \( v \) by \( \nabla \eta \) in (2.10) and using \( \nabla u = \lambda_2 r \theta \) (see (2.1)), we have

\[
\int_{B_1} c_{n,k} |x|^{-k} (u')^k (x, \nabla \xi) = \int_{B_1} \lambda_2^k (x, \nabla \xi) \\
= |\partial B_1| c_{n,k} \int_0^1 r^n \lambda_2^k \xi_r \, dr \\
= c_{n,k} \int_0^1 \int_{\partial B_1} \left\{ r^n \lambda_2^k (\theta, \nabla \eta) \right\} \, d\theta \, dr \\
= \int_{B_1} \frac{1}{k} \sum u_i \eta_j S^{ij}_k(D^2u). \]

Thus, as \( \int_{B_1} g(u) \xi = \int_{B_1} g(u) \eta \), we have

\[
\int_{B_1} \left\{ c_{n,k} |x|^{-k} (u')^k (x, \nabla \xi) + g(u) \xi \right\} = \int_{B_1} \left\{ \frac{1}{k} \sum u_i \eta_j S^{ij}_k(D^2u) + g(u) \eta \right\}. 
\]

Finally, replacing \( w \) and \( v \) by \( \nabla u \) in (2.10), we obtain

\[
\int_{B_1} \left\{ \sum u_i u_j S^{ij}_k(D^2u) \right\} = \int_{B_1} \left\{ kc_{n,k} \lambda_2^{k-1} |\nabla u|^2 \right\} = kc_{n,k} \int_{B_1} |x|^{1-k} |\nabla u|^{k+1}. 
\]

Therefore

\[
 u \in W^{1,k+1}(B_1, |x|^{-k}) \Leftrightarrow u \in L^{k+1}(B_1) \quad \text{and} \quad \int_{B_1} \left\{ \sum u_i u_j S^{ij}_k(D^2u) \right\} < +\infty,
\]

which completes the proof.
Proof of Theorem 1.4. Let \( u \) be a semistable radial solution of (1.1) according to Definition 1.2. Then \( u \) satisfies (1.5), since for radial perturbations \( \varphi \), (1.3) reduces to (1.5) (see (3.2) below).

Following [1, Remark 1.7] and [4, Lemma 2.5], for any \( \varphi \in C^1_0(B_1 \setminus \{0\}) \) (not necessarily radial) we consider the spherical averages of \( \varphi^2 \), i.e., the radial function

\[
\psi(r) = \frac{1}{|\partial B_1|} \int_{\partial B_1} \varphi^2(r\theta) \, d\theta = \frac{1}{2} \int_{\partial B_1} \varphi^2(r\theta) \, d\theta,
\]

where, as before, we write \( x \in \mathbb{R}_n \setminus \{0\} \) also as \( r\theta \), where \( \theta \in \partial B_1 \).

A short computation shows that \( \sqrt{\psi} \) is a Lipschitz continuous function with compact support contained in \( (0, +\infty) \), whence \( \xi(x) := \sqrt{\psi(|x|)} \) can be used as a test function in (1.5).

Differentiating the last expression with respect to \( r \) and using the Cauchy-Schwarz inequality, we obtain

\[
\frac{\xi^2}{2} \leq \int_{\partial B_1} (\theta, \nabla \varphi)^2 \, d\theta.
\]

Now, from equality (2.10) applied with \( w = v = \nabla \varphi \), we obtain

\[
\sum_{i,j} \varphi_i \varphi_j S_{ij}^k(D^2 u) = \nabla \varphi S(\nabla \varphi)^T = k c_n, k \lambda_2^{k-2} \left( \lambda_2 (\theta, \nabla \varphi)^2 + \frac{k - 1}{n - 1} (\lambda_1 - \lambda_2) \right) \left( |\nabla \varphi|^2 - (\theta, \nabla \varphi)^2 \right).
\]

In view of Remark 1 and Corollaries 1.6 and 1.7, \( \lambda_2 \) and \( n \lambda_2 + k(\lambda_1 - \lambda_2) \) are positive in the range \( 0 < r < 1 \). Moreover, \( 0 \leq n(k - 1)/(n - 1) \leq k \) for all \( k \in \{1, \ldots, n\} \). Therefore, regardless of the order of the eigenvalues \( \lambda_1 \) and \( \lambda_2 \), we conclude that

\[
\lambda_2 + \left( \frac{k - 1}{n - 1} \right) (\lambda_1 - \lambda_2) = \frac{1}{n} \left( n \lambda_2 + \frac{n(k - 1)}{n - 1} (\lambda_1 - \lambda_2) \right) > 0.
\]

Combining the above with the Cauchy-Schwarz inequality, namely \(|\nabla \varphi|^2 - (\theta, \nabla \varphi)^2 \geq 0\), we deduce from (3.2) that

\[
\sum_{i,j} \varphi_i \varphi_j S_{ij}^k(D^2 u) \geq k c_n, k \lambda_2^{k-1} (\theta, \nabla \varphi)^2.
\]

Therefore, from (3.1), we obtain

\[
\int_{B_1} \left\{ k c_n, k \lambda_2^{k-1} |\xi_r|^2 + g'(u) \xi^2 \right\} \leq \int_{B_1} \left\{ k c_n, k \lambda_2^{k-1} (\theta, \nabla \varphi)^2 + g'(u) \varphi^2 \right\} \leq \int_{B_1} \left\{ \sum_{i,j} \varphi_i \varphi_j S_{ij}^k(D^2 u) + g'(u) \varphi^2 \right\} = Q_u(\varphi).
\]

Finally, the semistability of \( u \) follows from a standard density argument. The proof is now complete. \( \square \)
Proof of Lemma 1.6. Let \( u \) be a weak radial solution of (1.1). By Theorem 1.3, \( u \) satisfies

\[
\text{(weak radial)} \quad c_{n,k} \int_0^1 r^{n-k}(u')^k \xi' \, dr = - \int_0^1 r^{n-1} g(u) \xi \, dr,
\]

for all \( \xi \in C^1_c(0,1) \). Take \( \xi(r) = \int_0^1 \eta(s) \, ds \), where \( \eta \) is smooth and has compact support in \((0,1)\) as a test function in (3.3). After an integration by parts, we have

\[
c_{n,k} \int_0^1 r^{n-k}(u')^k (-\eta) \, dr = - \int_0^1 \left( \int_0^1 \eta(s) \, ds \right) r^{n-1} g(u) \, dr
= \int_0^1 \left( \int_0^r s^{n-1} g(u) \, ds \right) (-\eta) \, dr,
\]

which leads to

\[
c_{n,k} r^{n-k}(u')^k = \int_0^r s^{n-1} g(u) \, ds \text{ a.e.}
\]

Conversely, applying Hölder’s inequality to \( r^{n-k}(u')^k |\xi'| \) with exponents \( p = (k+1)/k \) and \( q = k+1 \), we obtain

\[
\left| \int_0^1 r^{n-k}(u')^k \xi' \, dr \right| \leq \int_0^1 r^{n-k}(u')^k |\xi'| \, dr
\leq \left( \int_0^1 r^{n-k}(u')^{k+1} \, dr \right)^{\frac{k}{k+1}} \left( \int_0^1 r^{n-k} |\xi'|^{k+1} \, dr \right)^{\frac{1}{k+1}} < \infty.
\]

Thus, if \( u \) is an integral radial solution of (1.1), we can integrate from 0 to 1 the integral solution condition multiplied by \( \xi'(r) \), where \( \xi \in C^1_c(0,1) \), thus obtaining

\[
\text{(on weak)} \quad \int_0^1 \left( \int_0^r s^{n-1} g(u) \, ds \right) \xi' \, dr = c_{n,k} \int_0^1 r^{n-k}(u')^k \xi' \, dr < \infty.
\]

Finally, integrating by parts the left-hand side of (3.4), we conclude that \( u \) is a weak radial solution of (1.1). \( \square \)

Proof of Corollary 1.7. Let \( u \) be an integral radial solution of (1.1). By Lemma 1.6, \( u \) is a weak radial solution of (1.1), and by the Sobolev embedding in one dimension, \( u \) is a continuous function of \( r = |x| \in [\delta,1] \) for every \( \delta \in (0,1) \). Thus \( u \) is continuous on \((0,1)\) and therefore so also is \( s^{n-1} g(u) \). Now from (1.6) we have

\[
u' = c_{n,k}^{-\frac{1}{k}} \left( \int_0^r \frac{s^{n-1} g(u) \, ds}{r^{n-k}} \right)^{\frac{1}{k}}, \quad r \in (0,1].
\]

It follows directly that

\[
u'' = c_{n,k}^{-\frac{1}{k}} \left( \int_0^r \frac{s^{n-1} g(u) \, ds}{r^{n-k}} \right)^{\frac{1}{k} - 1} \left( r^k g(u) + \frac{(k-n) \int_0^r s^{n-1} g(u) \, ds}{r^{n-k+1}} \right), \quad r \in (0,1],
\]

which concludes the proof. \( \square \)

We claim that if \( u \) is a \( C^2 \) radial function on \( B_1 \), then \( u' \in C^{0,1}_{\text{loc}}(B_1) \). To prove our claim, we first observe that \( u' \in C^0(B_1) \cap C^1(B_1 \setminus \{0\}) \) with \( u'(0) = 0 \) and
Lemma 3.1. Let $u$ be any radial solution of (1.1). Then

$$Q_u(u) = k c_{n,k} \int_{B_1} \left( \frac{u'}{|x|} \right)^{k+1} \left( \frac{u''}{|x|} \right)^{k-1} \left( \frac{k-1}{k+1} \right) \left( \frac{u'}{|x|} \right) + g'(u)(\eta)^2,$$

for every radially symmetric function $\eta \in (H^1 \cap L^\infty_{loc}) (B_1)$.

Proof. Let $\eta \in H^1_c(B_1) \cap L^\infty_{loc}(B_1)$ and $\xi \in C^0(B_1)$ be radial functions. Then, by a standard density argument, we can take $\phi = \zeta \eta \in H^1_c(B_1) \cap L^\infty_{loc}(B_1)$ in (3.5) to obtain

$$Q_u(\zeta \eta) = \int_{B_1} k c_{n,k} \left( \frac{|\nabla (\zeta \eta)|}{|x|} \right)^{k-1} + g'(u)(\zeta \eta)^2.$$

Thus, as $u$ and $\zeta$ are radial functions, we get

$$|u'|^k = \left| \frac{u'}{|x|} \right|^k |\nabla (\zeta \eta)|^2 + \left( \frac{2}{2} \left| \nabla \zeta \eta \nabla \eta \right| + \eta^2 |\nabla \zeta|^2 \right),$$

where $\lambda_2 = u'/|x|$.
From (1.4) and differentiating (1.1) with respect to \( r \), we obtain
\[
-c_{n,k}^{-1}g'(u)u' = -\frac{n-1}{r^2} \left( \frac{u'^r}{r^{k-1}} \right) + \Delta \left( \frac{u'^r}{r^{k-1}} \right)
\]
\[
= -\frac{(n-1)\lambda_k^2}{r} + \Delta \left( r \lambda_k^2 \right), \quad r > 0.
\]

Then, multiplying the latter equation by \( \eta^2 u' \), integrating by parts and taking into account that \( \lambda_2 \in L^\infty_{\text{loc}}(B_1) \), we have
\[
c_{n,k}^{-1} \int_{B_1} g'(u)(u' \eta)^2 = -(n-1) \int_{B_1} \lambda_k^{k+1} \eta^2 - \int_{B_1} (\nabla (r \lambda_k^2), \nabla (\eta^2 u'))
\]
\[
= -\frac{k(2n-k-1)}{k+1} \int_{B_1} \lambda_k^{k+1} \eta^2 + \frac{k(k-1)}{k+1} \int_{B_1} \lambda_k^{k+1} \nabla \cdot \nabla \eta^2
\]
\[
- k \int_{B_1} \lambda_k^{k-1} \left( \|\nabla u' \|^2 \eta^2 + 2u' \eta \nabla u', \nabla \eta \right).
\]

Finally, using \( \zeta = u' \) in (3.7), (3.8) and the previous equation, we obtain (3.6), which completes the proof. \( \square \)

Proof of Corollary 1.8. If \( u \) is a semistable radial solution of (1.1), then from (3.6) it follows that \( Q_{u}(u' \eta) \geq 0 \) for every radially symmetric function \( \eta \in (H_1^1 \cap L^\infty_{\text{loc}})(B_1) \), that is, (1.7) holds.

For the converse, let \( 0 < \epsilon < 2 \), let \( \varphi \in C_0^1(B_2) \) be a radially symmetric function and \( \varphi_{\epsilon, \sigma} = \varphi_{\epsilon, \sigma}/u' \), where
\[
\varphi_{\epsilon, \sigma}(r) := \begin{cases} 
0 & \text{if } 0 \leq r < \epsilon^2, \\
1 - \frac{\log r}{\log \epsilon} & \text{if } \epsilon^2 \leq r < \epsilon, \quad \text{if } n = 2\sigma, \\
1 & \text{if } r \geq \epsilon, \\
0 & \text{if } 0 \leq r < \epsilon, \\
\xi \left( \frac{r}{\epsilon} \right) & \text{if } \epsilon \leq r < 2\epsilon, \quad \text{if } n \geq 2\sigma + 1, \\
1 & \text{if } r \geq 2\epsilon.
\end{cases}
\]

where \( \sigma \in \mathbb{N} \) and \( \xi(t) = 2(1-t)^2 \left( \frac{2}{t} - t \right) \).

By applying (3.7) with \( \eta = \varphi_{\epsilon, \sigma} \), we conclude that
\[
0 \leq Q_{u}(u' \varphi_{\epsilon, \sigma}) = \int_{B_1} \left( k_{n,k} \lambda_k^{k-1} \left| \nabla (u' \varphi_{\epsilon, \sigma}) \right| + g'(u)(u' \varphi_{\epsilon, \sigma})^2 \right)
\]
\[
= \int_{B_1} \left( k_{n,k} \lambda_k^{k-1} \left| \nabla (\varphi_{\epsilon, \sigma}) \right| + g'(u)(\varphi_{\epsilon, \sigma})^2 \right)
\]
\[
= \int_{B_1} \left( \lambda_k \lambda_k^{k-1} \left| \nabla \varphi \right| + g'(u) \varphi^2 \right) \varphi_{\epsilon, \sigma} + k_{n,k} \lambda_k \lambda_k^{k-1} \left( 2\varphi \varphi' \varphi_{\epsilon, \sigma} + \varphi^2 \varphi_{\epsilon, \sigma} \right)
\]

Now let
\[
I_{\epsilon, \sigma} := \int_{B_1} \left\{ \lambda_k^{k-1} \left( 2\varphi \varphi' \varphi_{\epsilon, \sigma} + \varphi^2 \varphi_{\epsilon, \sigma} \right) \right\}
\]
\[
= \int_0^1 \left\{ (u')^{k-1} \left( 2\varphi \varphi' \varphi_{\epsilon, \sigma} + \varphi^2 \varphi_{\epsilon, \sigma} \right) \right\} r^{n-k} dr.
\]
\begin{itemize}
  \item If \( u \in C^2(B_1) \) or \( u \in W^{1,k+1}(B_1, |x|^{-k}) \) and \( n < 2k \), we take \( \sigma = 1 \) in (3.9) (Note that, if \( n < 2k \) and \( u \in W^{1,k+1}(B_1, |x|^{-k}) \), then \( u'(0) = 0 \) by [8, Lemma 3.3]). Then

\begin{equation}
|I_{\epsilon,1}| \leq \begin{cases}
\frac{k-1}{\log \epsilon} \int_\epsilon^\infty (16 |\varphi |^r + \left( \frac{64 \varphi^2}{-\log \epsilon} \right)) r^{n-k-2} dr & \text{if } n = 2, \\
\epsilon^{-3} \int_\epsilon^\infty (16 |\varphi |^r + 64 \varphi^2) r^{n-k} dr & \text{if } n \geq 3,
\end{cases}
\end{equation}

where \( I_{\epsilon,1} \) is given by (3.11).

\begin{itemize}
  \item If \( u \in W^{1,k+1}(B_1, |x|^{-k}) \) and \( n \geq 2k \), we take \( \sigma = k \) in (3.9). Then

\begin{equation}
|I_{\epsilon,k}| \leq \begin{cases}
\frac{1}{\log \epsilon} \int_\epsilon^\infty r^{n-k}(u')^{k-1} \left( 16 |\varphi |^r + \left( \frac{64 \varphi^2}{-\log \epsilon} \right) \right) dr & \text{if } n = 2k, \\
\epsilon^{-2} \frac{n-2k}{2} (u')^{k+1-2k} & \text{otherwise},
\end{cases}
\end{equation}

\end{itemize}

Hence, from (3.10)-(3.13), for \( \sigma \in \{1,k\} \), we have

\[
\int_{B_1} (\kappa c_n \kappa^2 |\nabla \varphi | + g'(u) \varphi^2) \zeta^2_{\epsilon,\sigma} + o(1) \geq 0.
\]

Letting \( \epsilon \to 0 \), we obtain, using dominated convergence theorem and Theorem 1.4, that \( u \) is semistable, as desired. \qed

**Proof of theorems 1.9 to 1.11.** The proofs of theorems 1.9 to 1.11 follow by direct applications of Corollary 1.8 (which is fundamental in our characterization of semistable solutions) and the corresponding results in [8, Theorems 2.5, 2.6 and 2.8]. \qed

4. A FAMILY OF SEMISTABLE RADIAL SOLUTIONS

In this section we provide, for \( n \geq 2k + 8 \), a large family of semistable radially increasing unbounded solutions of problems of the type (1.1) on \( B_1 \setminus \{0\} \). This family is similar to that constructed in [8] for problem (P).

We start with the following statement, which was proved in [8].

**Theorem 4.1.** Let \( h \in (C^1 \cap L^1)[0,1] \) be a nonnegative function and consider

\[
V(r) = r^{(k+1)(\delta_n,\kappa-2)+2} \left( 1 + \int_0^r h(s) ds \right), \quad \forall r \in (0,1],
\]

\textbf{thm:familysemistable}
where

\[ \delta_{n,k} = \frac{-(k+1)n + 2\sqrt{2(k+1)n - 4k + 2k^2 + 6k}}{(k+1)^2}. \]

Now define

\[ u'(r) := \frac{\hat{r} + 1}{r^{n+1}} (V(r))^{\frac{1}{n+1}}, \quad \forall r \in (0,1]. \]

Then, for all \( n \geq 2k + 8 \), \( u \) is a semistable radially increasing unbounded \( W^{1,k+1}(B_1, |x|^{1-k}) \)-solution of a problem of the type (P) on \( B_1 \setminus \{0\} \) with \( g \geq 0 \), where \( u \) is any function with radial derivative \( u' \).

We prove the above statement for problem (1.1). To this end, we need the following lemma

**Lemma 4.2.** Let \( \alpha, \beta \in \mathbb{R} \), and let \( V \in C^1(B_1 \setminus \{0\}) \) be a nonnegative radial function such that

\[ \alpha (rV' + (n-2\beta - \alpha - 2)V) \geq 0, \quad \forall r \in (0,1]. \]

Further, assume that

\[ \lim_{r \to 0} r^{n-2}V = 0. \]

Then, for every radially symmetric function \( \eta \in C^1_c(B_1) \), we have

\[ \int_0^1 r^{n-3}V \left( (r\eta' + \beta\eta)^2 - \frac{\alpha^2 \eta^2}{4} \right) \, dr \geq 0. \]

**Remark 4.** If \( \alpha = n - 2 \), \( \beta = 0 \), \( V \equiv 1 \) and \( n \geq 3 \) in Lemma 4.2, then we obtain a Hardy-type inequality on the unit ball.

**Proof.** Let \( \eta \in C^1_c(B_1) \) be a radially symmetric function. Then

\[ \int_0^1 r^{n-3}V (\alpha \eta - t(r\eta' + \beta\eta))^2 \, dr \geq 0 \]

for all \( t \in \mathbb{R} \). Extending the above expression, we get the following quadratic inequality for \( t \):

\[ \alpha^2 \int_0^1 r^{n-3}V \eta'^2 \, dr - 2\alpha t \int_0^1 r^{n-3}V (r\eta' + \beta\eta^2) \, dr + t^2 \int_0^1 r^{n-3}V (r\eta' + \beta\eta)^2 \, dr \geq 0. \]

Integrating by parts above and using (4.3), we obtain

\[ \alpha^2 \int_0^1 r^{n-3}V \eta'^2 \, dr + \alpha t \int_0^1 r^{n-3} (rV' + (n-2\beta - 2)V) \eta^2 \, dr + t^2 \int_0^1 r^{n-3}V (r\eta' + \beta\eta)^2 \, dr \geq 0. \]

Thus, the above quadratic inequality is equivalent to

\[ \left( \alpha \int_0^1 r^{n-3} (rV' + (n-2\beta - 2)V) \eta^2 \, dr \right)^2 \leq 4\alpha^2 \left( \int_0^1 r^{n-3}V \eta^2 \, dr \right) \times \left( \int_0^1 r^{n-3}V (r\eta' + \beta\eta)^2 \, dr \right). \]

From this and (4.2), we get the desired inequality (4.4). The proof is now complete. \( \square \)
Proof of Theorem 4.1. The proof presented here follows the same lines of the proof of [8, Theorem 4.1], but except for some minor changes due to the condition (4.3) that gives a control of \( V \) near 0.

First, since \( n \geq 2k + 8 \), from (4.1) we obtain

\[
\text{(ineq:nlamb)} \quad n + (k + 1)(S_{n,k} - 2) = \frac{2\sqrt{2(k+1)n - 4k + 4k}}{k+1} - 2 \geq 6,
\]

which shows that \( V \in L^1(B_1) \) and hence that \( u \in W^{1,k+1}(B_1, |x|^{-k}) \).

Thus, by the definition of \( u' \), we have

\[
\text{(eq:deru)} \quad u'(r) = r^{S_{n,k} - 1} \left( 1 + \int_0^r h(s) \, ds \right)^{\frac{1}{S_{n,k}}} \geq r^{S_{n,k} - 1}, \quad \forall r \in (0,1].
\]

We now rewrite (4.1) as

\[
\delta_{n,k} = -\frac{\sqrt{2(k+1)n - 4k + 4k}}{(k+1)(\sqrt{2(k+1)n - 4k + 4k})} (n - 2k - 8).
\]

Then

\[
u(1) - u(r) \geq \begin{cases} \frac{-\log r}{1 - r^{\delta_{n,k}}}, & \text{if } \delta_{n,k} < 0 \iff n > 2k + 8, \\ \frac{-\log r}{1 - r^{\delta_{n,k}}}, & \text{if } \delta_{n,k} = 0 \iff n = 2k + 8, \\ \frac{-\log r}{1 - r^{\delta_{n,k}}}, & \text{if } \delta_{n,k} > 0 \iff n < 2k + 8. \end{cases}
\]

Therefore

\[
\lim_{r \to 0} u(r) = -\infty.
\]

Thus, from (4.6) and (1.1) in their radial form, we obtain

\[
c_{n,k}^{-1}S_k(D^2u) = r^{1-n} (r^{n-k}(u')^k)' = r^{k(\delta_{n,k} - 2)} \left( 1 + \int_0^r h(s) \, ds \right)^{\frac{k}{k+1}} \times
\]

\[
\left( n + k(\delta_{n,k} - 2) + \frac{krh(r)}{(k+1)(\int_0^r h(s) \, ds)} \right).
\]

Next, since \( h \in C^1([0,1]) \), we conclude that \( u' \in C^2([0,1]) \). Thus \( 0 < S_k(D^2u) \in C^1(B_1 \setminus \{0\}) \) (Note that \( n + k(\delta_{n,k} - 2) = 8 - \delta_{n,k} \geq 8 \) by (4.5)). Hence, taking a nonnegative function \( g \in C^1 \) such that \( g(s) = c_{n,k}^{-1}S_k(D^2u(u^{-1}(s))) \), for \( s \in (-\infty, u(1)] \), we conclude that \( u \) is a solution of a problem of type (1.1) on \( B_1 \setminus \{0\} \).

It remains to prove the semistability of \( u \). First, from the definition of \( V \) and (4.5), it is easily seen that

\[
\text{(eq:Vcond2proof)} \quad r^{n-2}V = r^{n+(k+1)(\delta_{n,k} - 2)} \left( 1 + \int_0^r h(s) \, ds \right) \to 0 \quad \text{as } r \to 0,
\]

and

\[
\text{(eq:Vcond1proof)} \quad rV'(r) - ((k+1)(\delta_{n,k} - 2) + 2)V(r) = r^{(k+1)(\delta_{n,k} - 2) + 3}h(r) \geq 0, \quad \forall r \in (0,1].
\]

Next define

\[
\alpha = \frac{2\sqrt{2(k+1)n - 4k}}{k+1} \quad \text{and} \quad \beta = \frac{k - 1}{k+1}
\]

Then we can write the left-hand side of (4.5) in the form

\[
n + (k+1)(\delta_{n,k} - 2) = \alpha + 2\beta.
\]
Since \( n \geq 2k + 8 \), we have \( \alpha \geq 4(k + 2)/(k + 1) > 0 \). Then from (4.8), (4.9) and the last equality, we obtain

\[
\alpha (rV' + (n - 2\beta - \alpha - 2)V) \geq \alpha (n + (k + 1)(\delta_{n,k} - 2) - 2\beta - \alpha) V = 0.
\]

Now, by a simple calculation, we have

\[
\alpha^2 \frac{k^2}{4} - \beta^2 = \frac{2n - k - 1}{k + 1}.
\]

Finally, from the above, (4.2), (4.7), (4.10), Lemma 4.2 and Corollary 1.8, we conclude that \( u \) is semistable. The proof of the theorem is now complete. \( \square \)

**Proposition 4.3.** Let \( \{r_m\} \subset (0, 1), \{M_m\} \subset \mathbb{R}^+ \) be two sequences with \( r_m \downarrow 0 \). Then, for \( n \geq 2k + 8 \), there exists a \( u \in W^{1,k+1}(B_1, |x|^{1-k}) \), which is a semistable radially increasing unbounded solution of a problem of type (1.1) on \( B_1 \setminus \{0\} \), satisfying

\[
|u''(r_m)| \geq M_m, \quad \forall m \in \mathbb{N}.
\]

**Corollary 4.4.** Let \( n \geq 2k + 8 \). Then there is no function \( \varphi : (0, 1] \to \mathbb{R}^+ \) having the following property: for every \( u \in W^{1,k+1}(B_1, |x|^{1-k}) \) semistable radially increasing solution of a problem of type (1.1) on \( B_1 \setminus \{0\} \) with \( g \geq 0 \), there exist \( C > 0 \) and \( \epsilon \in (0, 1) \) such that \( |u''(r)| \leq C\varphi(r) \) for every \( r \in (0, \epsilon] \).

**Proposition 4.5.** Let \( \{r_m\} \subset (0, 1), \{M_m\} \subset \mathbb{R}^+ \) be two sequences with \( r_m \downarrow 0 \). Then, for \( n \geq 2k + 8 \), there exists a \( u \in W^{1,k+1}(B_1, |x|^{1-k}) \), which is a semistable radially increasing unbounded solution of a problem of type (1.1) on \( B_1 \setminus \{0\} \) with \( g' \leq 0 \), satisfying

\[
|u''(r_m)| \geq M_m, \quad \forall m \in \mathbb{N}.
\]

**Corollary 4.6.** Let \( n \geq 2k + 8 \). Then there is no function \( \varphi : (0, 1] \to \mathbb{R}^+ \) having the following property: for every \( u \in W^{1,k+1}(B_1, |x|^{1-k}) \) semistable radially increasing solution of a problem of type (1.1) on \( B_1 \setminus \{0\} \) with \( g' \leq 0 \), there exist \( C > 0 \) and \( \epsilon \in (0, 1) \) such that \( |u''(r)| \leq C\varphi(r) \) for every \( r \in (0, \epsilon] \).

**Proof.** The proofs of propositions 4.3 and 4.5 and of corollaries 4.4 and 4.6 follow from Theorem 4.1, Corollary 1.8 and [8, propositions 4.3 and 4.4, corollaries 4.4 and 4.5]. \( \square \)

5. Additional comments

For the convenience of the reader, and in order to make explicit the constant \( c_{n,k} \) which intervenes in the radial form of the \( k \)-Hessians, we give here a detailed proof of (1.4). Consider a radial function \( u \) on the unit ball \( B_1 \) of \( \mathbb{R}^n \), that is, \( u(x) = v(r) \) where \( r = |x| = \sqrt{x_1^2 + \ldots + x_n^2} \). We compute the Hessian of \( v \), rotating \( B_1 \) so that \( x_1 = r \):

\[
D^2 v = \begin{pmatrix}
v'' & 0 & \cdots & 0 \\
0 & \frac{v'}{r} & \cdots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \frac{v'}{r}
\end{pmatrix}.
\]
Since the Hessian operators are invariant under rotations of coordinates, we have $S_k(D^2 u) = S_k(D^2 v)$. Then we compute the minors of order $k$ of $D^2 v$ using an ingenious argument from [15]. Let

$$I = \{ \alpha \subset \{1, \ldots, n\} : |\alpha| = k \},$$

and let $A = \{ \alpha \in I : 1 \in \alpha \}$ and $B = \{ \alpha \in I : n \in \alpha \}$.

We express $I$ as a disjoint union

$$I = (A \cap B) \cup (B \setminus A) \cup (I \setminus B).$$

Now define

$$S_k^{(\alpha)} = \det \begin{pmatrix} k \times k \text{ matrix of (5.1)} \end{pmatrix} \text{ with row and columns chosen from } \alpha.$$

For $\alpha \in I$, we need to consider three cases:

Case 1: If $\alpha \in (A \cap B)$,

$$S_k^{(\alpha)} = v'' \left( \frac{v'}{r} \right)^{k-1} \text{ and } |A \cap B| = \binom{n-2}{k-2}.$$

Case 2: If $\alpha \in (B \setminus A)$,

$$S_k^{(\alpha)} = \left( \frac{v'}{r} \right)^k \text{ and } |B \setminus A| = \binom{n-1}{k}.$$

Case 3: If $\alpha \in (I \setminus B)$,

$$S_k^{(\alpha)} = v'' \left( \frac{v'}{r} \right)^{k-1} \text{ and } |I \setminus B| = \binom{n-2}{k-1}.$$

From (5.2) and the binomial identities $\binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1} = \frac{k}{n} \binom{n}{k}$ and $\binom{n-1}{k-1} = \frac{n-k}{n} \binom{n}{k}$, we have

$$S_k(D^2 v) = \sum_{\alpha \in I} S_k^{(\alpha)} = \binom{n-2}{k-2} v'' \left( \frac{v'}{r} \right)^{k-1} + \binom{n-1}{k} \left( \frac{v'}{r} \right)^k + \binom{n-2}{k-1} v'' \left( \frac{v'}{r} \right)^{k-1}$$

$$= c_{n,k} \left( \frac{v'}{r} \right)^{k-1} \left( n - k \right) \left( \frac{v'}{r} \right) + kv'' = c_{n,k} r^{1-n} (v^{n-k}(v')^k)' ,$$

where the last equality shows that we must have $c_{n,k} = \binom{n}{k}/n$. This completes the proof of the formula in (1.4).

**Acknowledgements**

M. Navarro was supported by XUNTA de Galicia under Grant Axudas á etapa de formación posdoutoral 2017 and partially supported by AEI of Spain under Grant MTM2016-75140-P and co-financed by European Community fund FEDER and XUNTA de Galicia under grants GRC2015-004 and R2016/022.
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