HOMOTOPY ON SPATIAL GRAPHS AND THE SATO-LEVINE INVARIANT

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Abstract. Edge-homotopy and vertex-homotopy are equivalence relations on spatial graphs which are generalizations of Milnor’s link-homotopy. We introduce some edge (resp. vertex)-homotopy invariants of spatial graphs by applying the Sato-Levine invariant for the 2-component constituent algebraically split links and show examples of non-splitable spatial graphs up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial.

1. Introduction

Throughout this paper we work in the piecewise linear category. Let $G$ be a finite graph which does not have isolated vertices and free vertices. An embedding $f$ of $G$ into the 3-sphere $S^3$ is called a spatial embedding of $G$ or simply a spatial graph. For a spatial embedding $f$ and a subgraph $H$ of $G$ which is homeomorphic to the 1-sphere $S^1$ or a disjoint union of 1-spheres, we call $f(H)$ a constituent knot or a constituent link of $f$, respectively. A graph $G$ is said to be planar if there exists an embedding of $G$ into the 2-sphere $S^2$, and a spatial embedding of a planar graph is said to be trivial if it is ambient isotopic to an embedding of the graph into a 2-sphere in $S^3$. A spatial embedding $f$ of a graph $G$ is said to be split if there exists a 2-sphere $S$ in $S^3$ such that $S \cap f(G) = \emptyset$ and each component of $S^3 - S$ has intersection with $f(G)$, and otherwise $f$ is said to be non-splitable.

Two spatial embeddings of a graph $G$ are said to be edge-homotopic if they are transformed into each other by self crossing changes and ambient isotopies, where a self crossing change is a crossing change on the same spatial edge, and vertex-homotopic if they are transformed into each other by crossing changes on two adjacent spatial edges and ambient isotopies\(^1\). These equivalence relations were introduced by Taniyama \cite{Taniyama} as generalizations of Milnor’s link-homotopy on links \cite{Milnor}, namely if $G$ is homeomorphic to a disjoint union of 1-spheres, then these are none other than link-homotopy. There are many studies about link-homotopy. In particular, the link-homotopy classification was given for 2- and 3-component links by Milnor \cite{Milnor}, for 4-component links by Levine \cite{Levine} and for all links by Habegger and...
lin \[2\]. On the other hand, there are very few studies about edge (resp. vertex)-homotopy on spatial graphs \[18, 9, 13, 11\].

In \[18\], Taniyama defined an edge (resp. vertex)-homotopy invariant of spatial graphs called the \(\alpha\)-invariant by applying the Casson invariant (or equivalently the second coefficient of the Conway polynomial) of the constituent knots and showed that there exists a non-trivial spatial embedding \(f\) of a planar graph up to edge (resp. vertex)-homotopy, even in the case where \(f\) does not contain any constituent link. But the \(\alpha\)-invariant cannot detect a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy. As far as the authors know, an example of a non-splittable spatial embedding of a disconnected graph up to edge (resp. vertex)-homotopy, all of whose constituent links are link-homotopically trivial has not yet been demonstrated.

Our purpose in this paper is to study spatial embeddings of disconnected graphs up to edge (resp. vertex)-homotopy by applying the Sato-Levine invariant \[14\] (or equivalently the third coefficient of the Conway polynomial) for the constituent 2-component algebraically split links and show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge (resp. vertex)-homotopy all of whose constituent links are link-homotopically trivial. These examples show that edge (resp. vertex)-homotopy on spatial graphs behaves quite differently than link-homotopy on links. In the next section we give the definitions of our invariants and state their invariance up to edge (resp. vertex)-homotopy.

### 2. Definitions of invariants

We call a subgraph of a graph \(G\) a cycle if it is homeomorphic to the 1-sphere, and a cycle is called a \(k\)-cycle if it contains exactly \(k\) edges. For a subgraph \(H\) of \(G\), we denote the set of all cycles of \(H\) by \(\Gamma(H)\). We set \(\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}\) for a positive integer \(m\) and \(\mathbb{Z}_0 = \mathbb{Z}\). We regard \(\mathbb{Z}_m\) as an abelian group in the obvious way. We call a map \(\omega : \Gamma(H) \rightarrow \mathbb{Z}_m\) a weight on \(\Gamma(H)\) over \(\mathbb{Z}_m\). For an edge \(e\) of \(H\), we denote the set of all cycles of \(H\) which contain the edge \(e\) by \(\Gamma_e(H)\). For a pair of two adjacent edges \(e_1\) and \(e_2\) of \(H\), we denote the set of all cycles of \(H\) which contain the edges \(e_1\) and \(e_2\) by \(\Gamma_{e_1,e_2}(H)\). Then we say that a weight \(\omega\) on \(\Gamma(H)\) over \(\mathbb{Z}_m\) is weakly balanced\(^2\) on an edge \(e\) if

\[
\sum_{\gamma \in \Gamma_e(H)} \omega(\gamma) = 0
\]

in \(\mathbb{Z}_m\) \[10\], and weakly balanced on a pair of adjacent edges \(e_1\) and \(e_2\) if

\[
\sum_{\gamma \in \Gamma_{e_1,e_2}(H)} \omega(\gamma) = 0
\]

in \(\mathbb{Z}_m\). Let \(G = G_1 \cup G_2\) be a disjoint union of two connected graphs and \(\omega_i : \Gamma(G_i) \rightarrow \mathbb{Z}_m\) a weight on \(\Gamma(G_i)\) over \(\mathbb{Z}_m\) \((i = 1, 2)\). Let \(f\) be a spatial embedding of \(G\) such that

\[
\omega_1(\gamma)\omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0
\]

in \(\mathbb{Z}\) for any \(\gamma \in \Gamma(G_1)\) and \(\gamma' \in \Gamma(G_2)\), where \(\text{lk}(L) = \text{lk}(K_1, K_2)\) denotes the linking number of a 2-component oriented link \(L = K_1 \cup K_2\). Then we define

\[\text{balanced on an edge } e \text{ of } H \text{ if } \sum_{\gamma \in \Gamma_e(H)} \omega(\gamma) = 0 \text{ in } H_1(H; \mathbb{Z}_m), \]

where the orientation of \(\gamma\) is induced by the one of \(e\) \[18\].
\( \beta_{\omega_1, \omega_2}(f) \in \mathbb{Z}_m \) by
\[
\beta_{\omega_1, \omega_2}(f) \equiv \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \omega_2(\gamma') a_3(f(\gamma), f(\gamma')) \pmod{m},
\]
where \( a_3(L) = a_3(K_1, K_2) \) denotes the third coefficient of the Conway polynomial of a 2-component oriented link \( L = K_1 \cup K_2 \). We remark here that \( a_3(L) \) coincides with the Sato-Levine invariant \( \beta(L) \) of \( L \) if \( L \) is algebraically split, namely \( \text{lk}(K_1, K_2) = 0 \) \cite{11, 17}. Thus our \( \beta_{\omega_1, \omega_2}(f) \) is also the modulo \( m \) reduction of the summation of Sato-Levine invariants for the constituent 2-component algebraically split links of \( f \).

**Remark 2.1.** For a 2-component algebraically split link \( L = K_1 \cup K_2 \),

1. The value of \( a_3(L) \) does not depend on the orientations of \( K_1 \) and \( K_2 \).
   Actually we can check it easily by the original definition of the Sato-Levine invariant.

2. The value of \( a_3(L) \) is not a link-homotopy invariant of \( L \) (see also Lemma \ref{thm:2.1}). For example, the Whitehead link \( L \) is link-homotopically trivial but \( a_3(L) = 1 \).

Now we state the invariance of \( \beta_{\omega_1, \omega_2} \) up to edge (resp. vertex)-homotopy under some conditions on the graphs.

**Theorem 2.2.** Let \( G = G_1 \cup G_2 \) be a disjoint union of two connected graphs and \( \omega_i \) a weight on \( \Gamma(G_i) \) over \( \mathbb{Z}_m \) \((i = 1, 2)\). Let \( f \) be a spatial embedding of \( G \) such that
\[
\omega_1(\gamma) \omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0
\]
in \( \mathbb{Z} \) for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \). Then we have the following:

1. If \( \omega_i \) is weakly balanced on any edge of \( G_i \) \((i = 1, 2)\), then \( \beta_{\omega_1, \omega_2}(f) \) is an edge-homotopy invariant of \( f \).

2. If \( \omega_i \) is weakly balanced on any pair of adjacent edges of \( G_i \) \((i = 1, 2)\), then \( \beta_{\omega_1, \omega_2}(f) \) is a vertex-homotopy invariant of \( f \).

We prove Theorem 2.2 in the next section. In addition, by using an integer-valued invariant (Theorem 2.1), we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to edge-homotopy all of whose constituent links are link-homotopically trivial (Example 4.3). We also exhibit an infinite family of non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy which can be distinguished by our integer-valued invariant (Example 4.4).

We note that if a graph \( G \) contains a connected component which is homeomorphic to the 1-sphere, then our invariants in Theorem 2.2 are useless. For such cases, we can define edge (vertex)-homotopy invariants that take values in \( \mathbb{Z}_2 \) on weaker condition for weights than the one stated in Theorem 2.2. For a subgraph \( H \) of a graph \( G \), we say that a weight \( \omega \) on \( \Gamma(H) \) over \( \mathbb{Z}_2 \) is *totally balanced* if
\[
\sum_{\gamma \in \Gamma(H)} \omega(\gamma)[\gamma] = 0
\]
in \( H_1(H; \mathbb{Z}_2) \). We note that if a weight \( \omega \) on \( \Gamma(H) \) over \( \mathbb{Z}_2 \) is totally balanced, then it is weakly balanced on any edge \( e \) of \( H \) (Lemma 3.2), but not always weakly
balanced on any pair of adjacent edges of $H$ (Remark 3.3). Then we have the following.

**Theorem 2.3.** Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs and \( \omega_i \) a weight on $\Gamma(G_i)$ over $\mathbb{Z}_2$ \((i = 1, 2)\). Let $f$ be a spatial embedding of $G$ such that

\[
\omega_1(\gamma) \omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0
\]

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then we have the following:

1. If either $\omega_1$ is totally balanced on $\Gamma(G_1)$ or $\omega_2$ is totally balanced on $\Gamma(G_2)$, then $\beta_{\omega_1, \omega_2}(f)$ is an edge-homotopy invariant of $f$.
2. If either $\omega_1$ is totally balanced on $\Gamma(G_1)$ and weakly balanced on any pair of adjacent edges of $G_1$, or $\omega_2$ is totally balanced on $\Gamma(G_2)$ and weakly balanced on any pair of adjacent edges of $G_2$, then $\beta_{\omega_1, \omega_2}(f)$ is a vertex-homotopy invariant of $f$.

We also prove Theorem 2.3 in the next section and give some examples in Section 5. In particular, we show that there exist infinitely many non-splittable spatial embeddings of a certain disconnected graph up to vertex-homotopy, all of whose constituent links are link-homotopically trivial (Example 5.4). We remark here that the $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one (Remark 5.5).

Theorems 2.2 and 2.3 do not work for spatial graphs as illustrated in Figure 2.1, for instance. In Section 6, we state a method to detect such non-splittable spatial graphs up to edge-homotopy by using a planar surface having a graph as a spine (Theorem 6.1). Actually we show that each of the spatial graphs as illustrated in Figure 2.1 is non-splittable up to edge-homotopy (Example 6.2).

![Figure 2.1.](image)

**Figure 2.1.**

3. **Proofs of Theorems 2.2 and 2.3**

We first calculate the change in the third coefficient of the Conway polynomial of 2-component algebraically split links which differ by a single self crossing change.

**Lemma 3.1.** Let $L_+$ and $L_-$ be two 2-component oriented links and $L_0 = J_1 \cup J_2 \cup K$ a 3-component oriented link which are identical except inside the depicted regions as illustrated in Figure 3.1. Suppose that $\text{lk}(L_+) = \text{lk}(L_-) = 0$. Then it holds that

\[
a_3(L_+) - a_3(L_-) = -\text{lk}(J_1, K)^2 = -\text{lk}(J_2, K)^2.
\]
Proof. By the skein relation of the Conway polynomial and a well-known formula for the second coefficient of the Conway polynomial of a 3-component oriented link (cf. [4], [3], [5]), we have that

\[ a_3(L_+) - a_3(L_-) = \text{lk}(J_1, J_2)\text{lk}(J_2, K) + \text{lk}(J_2, K)\text{lk}(J_1, K) + \text{lk}(J_1, K)\text{lk}(J_1, J_2). \]

We note that

\[ \text{lk}(J_1, K) + \text{lk}(J_2, K) = 0 \]

by the condition \( \text{lk}(L_+) = \text{lk}(L_-) = 0 \). Thus by (3.1) and (3.2), we have that

\[ a_3(L_+) - a_3(L_-) = \text{lk}(J_2, K)\text{lk}(J_1, K). \]

Therefore by (3.2) we have the result.

Proof of Theorem 2.2. (1) Let \( f \) and \( g \) be two spatial embeddings of \( G \) such that

\[ \omega_1(\gamma) \omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = 0 \]

in \( \mathbb{Z} \) for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \) and \( g \) is edge-homotopic to \( f \). Then it also holds that

\[ \omega_1(\gamma) \omega_2(\gamma') \text{lk}(g(\gamma), g(\gamma')) = 0 \]

in \( \mathbb{Z} \) for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(G_2) \) because the linking number of a 2-component constituent link of a spatial graph is an edge-homotopy invariant. First we show that if \( f \) is transformed into \( g \) by self crossing changes on \( f(G_1) \) and ambient isotopies, then \( \beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g) \). It is clear that any link invariant of a constituent link of a spatial graph is also an ambient isotopy invariant of the spatial graph. Thus we may assume that \( g \) is obtained from \( f \) by a single crossing change on \( f(e) \) for an edge \( e \) of \( G_1 \) as illustrated in Figure 3.2. Moreover, by smoothing this crossing point we can obtain the spatial embedding \( h \) of \( G \) and the knot \( J_h \) as illustrated in Figure 3.2. Then by (3.3), (3.4), Lemma 3.1 and the assumption for \( \omega_1 \) we have
that
\[ \beta_{\omega_1, \omega_2}(f) - \beta_{\omega_1, \omega_2}(g) = \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \omega_2(\gamma') \left\{ a_3(f(\gamma), f(\gamma')) - a_3(g(\gamma), g(\gamma')) \right\} \]
\[ = \sum_{\gamma \in \Gamma_*(G_1)} \omega_1(\gamma) \omega_2(\gamma') \left\{ a_3(f(\gamma), f(\gamma')) - a_3(g(\gamma), g(\gamma')) \right\} \]
\[ = - \sum_{\gamma \in \Gamma_*(G_1)} \omega_1(\gamma) \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)^2 \]
\[ = - \left( \sum_{\gamma \in \Gamma_*(G_1)} \omega_1(\gamma) \right) \sum_{\gamma' \in \Gamma(G_2)} \omega_2(\gamma') \text{lk}(h(\gamma'), J_h)^2 \]
\[ = 0. \]
Therefore we have that \( \beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g) \). In the same way we can show that if \( f \) is transformed into \( g \) by self crossing changes on \( f(G_2) \) and ambient isotopies, then \( \beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g) \). Thus we have that \( \beta_{\omega_1, \omega_2} \) is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3, we can prove (2) in a similar way as the proof of (1). We omit the details. \qed

Next we prove Theorem 2.3. For a subgraph \( H \) of a graph \( G \), we have the following.

**Lemma 3.2.** A totally balanced weight \( \omega \) on \( \Gamma(H) \) over \( \mathbb{Z}_2 \) is weakly balanced on any edge \( e \) of \( H \).

**Proof.** For an edge \( e \) of \( H \), we can represent any \( \gamma \in \Gamma_*(H) \) as \( e + c_\gamma \in \mathbb{Z}_1(H; \mathbb{Z}_2) \), where \( c_\gamma \) is a 1-chain in \( C_1(H \setminus e; \mathbb{Z}_2) \). Then we have that
\[ 0 = \sum_{\gamma \in \Gamma(H)} \omega(\gamma)[\gamma] \]
\[ = \sum_{\gamma \in \Gamma_*(H)} \omega(\gamma)[e + c_\gamma] + \sum_{\gamma' \in \Gamma(H \setminus \Gamma_*(H))} \omega(\gamma')[\gamma'] \]
in $H_1(H;\mathbb{Z}_2)$. This implies that if $\omega$ is not weakly balanced on $e$, then $\omega$ is not totally balanced on $\Gamma(H)$ over $\mathbb{Z}_2$.

**Remark 3.3.** A totally balanced weight $\omega$ on $\Gamma(H)$ over $\mathbb{Z}_2$ is not always weakly balanced on any pair of adjacent edges of $H$. For example, let $\omega$ be a weight on $\Theta_3$ (see Example 4.3) over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$ for any cycle $\gamma \in \Gamma(\Theta_3)$. It is easy to see that $\omega$ is totally balanced, but not weakly balanced, on each pair of adjacent edges of $\Theta_3$.

**Proof of Theorem 2.3.** (1) Let $f$ and $g$ be two spatial embeddings of $G$ which are edge-homotopic such that

$$\omega_1(\gamma)\omega_2(\gamma') \text{lk}(f(\gamma), f(\gamma')) = \omega_1(\gamma)\omega_2(\gamma') \text{lk}(g(\gamma), g(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. First we show that if $f$ is transformed into $g$ by self crossing changes on $f(G_1)$ and ambient isotopies, then $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. In the same way as the proof of Theorem 2.2, we may consider three spatial embeddings $f, g$ and $h$ of $G$ and the knot $J_h$ as illustrated in Figure 3.2. Then, by the same calculation in the proof of Theorem 2.2 we have that

$$\beta_{\omega_1,\omega_2}(f) - \beta_{\omega_1,\omega_2}(g) = \left( \sum_{\gamma \in \Gamma(G_1)} \omega_1(\gamma) \right) \left( \sum_{\gamma' \in \Gamma(G_2)} \omega_2(\gamma') \text{lk}(h(\gamma'), J_h) \right)^2$$

If $\omega_1$ is totally balanced on $\Gamma(G_1)$, then by Lemma 3.2 it is weakly balanced on any edge $e$ of $G_1$. This implies that $\beta_{\omega_1,\omega_2}(f) = \beta_{\omega_1,\omega_2}(g)$. If $\omega_2$ is totally balanced on
\( \Gamma(G_1) \), then we have that
\[
\text{lk} \left( \sum_{\gamma' \in \Gamma(G_2)} \omega_2(\gamma') h(\gamma'), J_h \right) \equiv \text{lk} (0, J_h) = 0.
\]
Therefore this also implies that \( \beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g) \). In the same way we can show that if \( f \) is transformed into \( g \) by self crossing changes on \( f(G_2) \) and ambient isotopies, then \( \beta_{\omega_1, \omega_2}(f) = \beta_{\omega_1, \omega_2}(g) \). Thus we have that \( \beta_{\omega_1, \omega_2} \) is an edge-homotopy invariant.

(2) By considering the triple of spatial embeddings as illustrated in Figure 3.3, we can prove (2) in a similar way as the proof of (1). We also omit the details.

Since the Conway polynomial of a split link is zero, our invariants take the value zero for any split (2-component) spatial graph. Therefore if the value of our invariant of a spatial graph is not zero, then it is non-splittable up to edge (resp. vertex)-homotopy.

4. Integer-valued invariants

Let \( G \) be a planar graph. An embedding \( p : G \to S^2 \) is said to be cellular if the closure of each of the connected components of \( S^2 - p(G) \) is homeomorphic to the disk. Then we regard the set of the boundaries of all of the connected components of \( S^2 - p(G) \) as a subset of \( \Gamma(G) \) and denote it by \( \Gamma_p(G) \). We say that \( G \) admits a checkerboard coloring on \( S^2 \) if there exists a cellular embedding \( p : G \to S^2 \) such that we can color all of the connected components of \( S^2 - p(G) \) by two colors (black and white) so that any of the two components which are adjacent by an edge have distinct colors; see Figure 4.1. We denote the subset of \( \Gamma_p(G) \) which corresponds to the black (resp. white) colored components by \( \Gamma^b_p(G) \) (resp. \( \Gamma^w_p(G) \)).

**Figure 4.1.**

**Proposition 4.1.** Let \( G \) be a planar graph which is not homeomorphic to \( S^1 \) and admits a checkerboard coloring on \( S^2 \) with respect to a cellular embedding \( p : G \to S^2 \). Let \( \omega_p \) be a weight on \( \Gamma(G) \) over \( \mathbb{Z} \) defined by
\[
\omega_p(\gamma) = \begin{cases} 
1 & (\gamma \in \Gamma^b_p(G)), \\
-1 & (\gamma \in \Gamma^w_p(G)), \\
0 & (\gamma \in \Gamma(G) \setminus \Gamma_p(G)).
\end{cases}
\]

Then \( \omega_p \) is weakly balanced on any edge of \( G \).

**Proof.** For any edge \( e \) of \( G \), there exist exactly two cycles \( \gamma \in \Gamma^b_p(G) \) and \( \gamma' \in \Gamma^w_p(G) \) such that \( e \subset \gamma \) and \( e \subset \gamma' \). Thus we have the result.
We call the weight $\omega_p$ in Proposition 4.1 a checkerboard weight. Thus by Proposition 4.1 and Theorem 2.2 (1), we can obtain an integer-valued edge-homotopy invariant as follows.

**Theorem 4.2.** Let $G = G_1 \cup G_2$ be a disjoint union of two connected planar graphs such that $G_i$ is not homeomorphic to $S^1$ and admits a checkerboard coloring on $S^2$ with respect to a cellular embedding $p_i : G_i \rightarrow S^2$ $(i = 1, 2)$. Let $\omega_{p_i}$ be a checkerboard weight on $\Gamma(G_i)$ over $\mathbb{Z}$ $(i = 1, 2)$ and $f$ a spatial embedding of $G$ such that

$$\omega_{p_1}(\gamma)\omega_{p_2}(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{\omega_{p_1}, \omega_{p_2}}(f)$ is an integer-valued edge-homotopy invariant of $f$.\[\square\]

**Example 4.3.** Let $\Theta_n$ be a graph with two vertices $u$ and $v$ and $n$ edges $e_1, e_2, \ldots, e_n$, each of which joins $u$ and $v$. A spatial embedding of $\Theta_n$ is called a (spatial) theta $n$-curve or simply a theta curve if $n = 3$. For $n \geq 2$, we denote that a cycle of $\Theta_n$ consists of two edges $e_i$ and $e_j$ by $\gamma_{ij}$ $(i < j)$. Then it is clear that $\Theta_n$ admits a cellular embedding $p : \Theta_n \rightarrow S^2$ so that

$$\Gamma_p(\Theta_n) = \{\gamma_{12}, \gamma_{23}, \ldots, \gamma_{n-1,n}, \gamma_{1n}\}.$$ Moreover, for $m \geq 1$, $\Theta_{2m}$ admits a checkerboard coloring on $S^2$ so that

$$\Gamma_p^c(\Theta_{2m}) = \{\gamma_{12}, \gamma_{24}, \ldots, \gamma_{2m-1,2m}\},$$

$$\Gamma_p^w(\Theta_{2m}) = \{\gamma_{23}, \gamma_{45}, \ldots, \gamma_{2m-2,2m-1}, \gamma_{1,2m}\}.$$ Now let $G$ be a disjoint union of two copies of $\Theta_4$, each of which admits a checkerboard coloring on $S^2$ with respect to the cellular embedding $p$ as above. Let $\omega_p$ be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ and $g_1$ a spatial embedding of $G$ as illustrated in Figure 4.2. We can see that any of the 2-component constituent links of $g_1$ has a zero linking number. More precisely, $g_1$ contains exactly one non-trivial 2-component link $L = g_1(\gamma_{14}) \cup g_1(\gamma_{14}')$ whose linking number is zero. Thus by Theorem 1.2 we have that $\beta_{\omega_p, \omega_p}(g_1)$ is an integer-valued edge-homotopy invariant of $g_1$. Then, by a direct calculation we have that $a_3(L) = 2$, namely $\beta_{\omega_p, \omega_p}(g_1) = 2$. Note that a 2-component link is link-homotopically trivial if and only if its linking number is zero [8]. This implies that $g_1$ is non-splittable up to edge-homotopy despite the fact that any of the constituent links of $g_1$ is link-homotopically trivial.

\textbf{Figure 4.2.}
Moreover, for an integer \( m \), let \( g_m \) be a spatial embedding of \( G \) as illustrated in Figure 4.3. If \( m \neq 0 \), we can see that \( g_m \) contains exactly one non-trivial 2-component link \( L = g_m(\gamma_{14}) \cup g_m(\gamma'_{14}) \) whose linking number is zero. Thus we also have that \( \beta_{\omega_1, \omega_p}(g_m) \) is an integer-valued edge-homotopy invariant of \( g_m \). Then, by a calculation we have that \( a_3(L) = 2m \), namely \( \beta_{\omega_1, \omega_p}(g_m) = 2m \). This implies that there exist infinitely many non-splitting spatial embeddings of \( G \) up to edge-homotopy, all of whose constituent links are link-homotopically trivial.

![Figure 4.3](image)

**Example 4.4.** Let \( H \) be a graph as illustrated in Figure 4.4. We denote the cycle of \( H \) which contains \( e_i \) and \( e_j \) by \( \gamma_{ij} \) \((i < j)\). Let \( G \) be a disjoint union of two copies of \( H \) and \( g_1 \) a spatial embedding of \( G \) as illustrated in Figure 4.5. This spatial embedding \( g_1 \) contains exactly one 4-component constituent link \( L = g_1(\gamma_{12} \cup \gamma_{34} \cup \gamma'_{12} \cup \gamma'_{34}) \). Note that if \( g_1 \) is split up to vertex-homotopy, then \( L \) is split up to link-homotopy. Since \( |\mu_{1234}(L)| = 1 \), where \( \mu_{1234} \) denotes Milnor’s \( \mu \)-invariant of length 4 of 4-component links [8], we have that \( L \) is non-splittable up to link-homotopy. Therefore we have that \( g_1 \) is non-splittable up to vertex-homotopy.

We can also prove this fact by our integer-valued vertex-homotopy invariant as follows. Let \( \omega \) be a weight on \( \Gamma(H) \) over \( \mathbb{Z} \) defined by \( \omega(\gamma_{14}) = \omega(\gamma_{23}) = 1 \),
\( \omega(\gamma_{13}) = \omega(\gamma_{24}) = -1 \) and \( \omega(\gamma) = 0 \) if \( \gamma \) is a 2-cycle. Then it is easy to see that \( \omega \) is weakly balanced on any pair of adjacent edges of \( H \). We can see that \( g_1 \) contains exactly one non-trivial 2-component constituent link \( M = g_1(\gamma_{14} \cup \gamma'_{14}) \) with \( \text{lk}(M) = 0 \) and \( a_3(M) = 2 \). Thus by Theorem 2.2 (2) we have that \( \beta_{\omega,\omega}(g_1) \) is an integer-valued vertex-homotopy invariant of \( g_1 \) and \( \beta_{\omega,\omega}(g_1) = 2 \). This implies that \( g_1 \) is non-splittable up to vertex-homotopy.

\[ \text{Figure 4.4.} \]

Moreover, let \( g_m \) be a spatial embedding of \( G \) as illustrated in Figure 4.5, which can be constructed in the same way as in Example 4.3. Then we can see that \( \beta_{\omega,\omega}(g_m) \) is an integer-valued vertex-homotopy invariant of \( g_m \) and \( \beta_{\omega,\omega}(g_m) = 2m \). This implies that \( g_m \) is non-splittable up to vertex-homotopy for any integer \( m \neq 0 \) and \( g_i \) and \( g_j \) are not vertex-homotopic for any \( i \neq j \).

5. Modulo Two Invariants

**Proposition 5.1.** Let \( G \) be a planar graph which is not homeomorphic to \( S^1 \) and \( p: G \to S^2 \) a cellular embedding. Let \( \omega_p : \Gamma(G) \to \mathbb{Z}_2 \) be a weight on \( \Gamma(G) \) over \( \mathbb{Z}_2 \) defined by

\[
\omega_p(\gamma) = \begin{cases} 
1 & (\gamma \in \Gamma_p(G)), \\
0 & (\gamma \in \Gamma(G) \setminus \Gamma_p(G)).
\end{cases}
\]

Then \( \omega_p \) is totally balanced.

**Proof.** It holds that

\[
\sum_{\gamma \in \Gamma(G)} \omega_p(\gamma) [\gamma] = \sum_{\gamma \in \Gamma_p(G)} [\gamma] = 2 \left[ \sum_{e \in E(G)} e \right] = 0
\]
in $H_1(G; \mathbb{Z}_2)$, where $E(G)$ denotes the set of all edges of $G$. Thus we have the result.

Thus by Proposition 5.1 and Theorem 2.3 (1), we can obtain an edge-homotopy invariant as follows.

**Theorem 5.2.** Let $G = G_1 \cup G_2$ be a disjoint union of two connected graphs such that $G_1$ is planar, not homeomorphic to $S^1$ and admits a cellular embedding $p_1 : G_1 \to S^2$. Let $\omega_{p_1}$ be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ as in Proposition 5.1, $\omega_2$ a weight on $\Gamma(G_2)$ over $\mathbb{Z}_2$ and $f$ a spatial embedding of $G$ such that

$$\omega_{p_1}(\gamma)\omega_2(\gamma')\text{lk}(f(\gamma), f(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(G_2)$. Then $\beta_{\omega_{p_1}, \omega_2}(f)$ is an edge-homotopy invariant of $f$. \hfill $\square$

**Example 5.3.** Let $G$ be a disjoint union of $\Theta_3$ and a circle $\gamma$. Let $\omega_p$ be a weight on $\Gamma(\Theta_3)$ over $\mathbb{Z}_2$ as in Proposition 5.1 with respect to a cellular embedding $p : \Theta_3 \to S^2$ as in Example 4.3 and $\omega$ a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. Let $g$ be a spatial embedding of $G$ as illustrated in Figure 5.1 (1). We can see that $g$ contains exactly one non-trivial 2-component link $L = g(\gamma_3) \cup g(\gamma)$ which is the Whitehead link, so $\text{lk}(L) = 0$ and $a_3(L) = 1$. Thus by Theorem 5.2 we have that $\beta_{\omega_p, \omega}(g)$ is an edge-homotopy invariant of $g$ and $\beta_{\omega_p, \omega}(g) = 1$. Namely $g$ is non-splittable up to edge-homotopy despite the fact that any of the constituent links of $g$ is link-homotopically trivial.

**Example 5.4.** Let $G$ be a disjoint union of the complete bipartite graph on $3 + 3$ vertices $K_{3,3}$ and a circle $\gamma$. Let $\omega_{3,3}$ be a weight on $K_{3,3}$ over $\mathbb{Z}_2$ defined by $\omega_{3,3}(\gamma') = 1$ if $\gamma'$ is a 4-cycle and 0 if $\gamma'$ is a 6-cycle. Let $\omega$ be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. Then it is not hard to see that $\omega_{3,3}$ is totally balanced and weakly balanced on any pair of adjacent edges of $K_{3,3}$. For a positive integer $m$, let $g_m$ be a spatial embedding of $G$ as illustrated in Figure 5.1 (2). Note that $g_i(K_{3,3})$ and $g_j(K_{3,3})$ are not vertex-homotopic for any $i \neq j$ [9], namely $g_i$ and $g_j$ are not vertex-homotopic for any $i \neq j$. Since all of the 2-component constituent links of $g_m$ are algebraically split, by Theorem 2.3 (2) we have that $\beta_{\omega_3, \omega}(g)$ is a vertex-homotopy invariant of $g_m$. Moreover we can see that there exists exactly one 4-cycle $\gamma'$ of $K_{3,3}$ so that $L = g_m(\gamma \cup \gamma')$ is non-trivial. Since $L$ is the Whitehead link, we have that $\beta_{\omega_3, \omega}(g_m) = 1$. Therefore $g_m$ is non-splittable up to vertex-homotopy despite the fact that any of the constituent links of $g$ is link-homotopically trivial.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig51}
\caption{Figure 5.1.}
\end{figure}
Remark 5.5. The $\mathbb{Z}_2$-valued invariant in Theorem 2.3 cannot always be extended to an integer-valued one. For example,

1. Let us consider the graph $G$ and the invariant $\beta_{\omega_p, \omega}$ as in Example 5.3. Let $f$ be a spatial embedding of $G$ as illustrated in Figure 5.2. We can see that $f$ is edge-homotopic to the trivial spatial embedding $h$ of $G$. But by a calculation we have that $\sum_{1 \leq i < j \leq 3} a_3(f(\gamma_{ij}), f(\gamma)) = -2$.

![Figure 5.2.](#)

2. Let $G$ be a disjoint union of $\Theta_4$ and a circle $\gamma$. Let $\omega_p$ be a checkerboard weight on $\Gamma(\Theta_4)$ over $\mathbb{Z}$ as in Example 4.3. Note that the modulo two reduction of a checkerboard weight is totally balanced. So by Theorem 2.3 (1), the modulo two reduction of $\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(f(\gamma_{ij} \cup \gamma))$ is an edge-homotopy invariant of a spatial embedding $f$ of $G$. Moreover, we can see that the integer-value $\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(f(\gamma_{ij} \cup \gamma))$ is invariant under the self crossing change on $f(\Theta_4)$ in the same way as in the proof of Theorem 2.2 (1). But this value may change under a self crossing change on $f(\gamma)$. For example, let $f$ and $g$ be two spatial embeddings of $G$ as illustrated in Figure 5.3. We can see that $f$ is edge-homotopic to $g$. But by a calculation we have that

$$\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(f(\gamma_{ij}), f(\gamma)) = -1,$$

$$\sum_{\gamma_{ij} \in \Gamma(\Theta_4)} \omega_p(\gamma_{ij}) a_3(g(\gamma_{ij}), g(\gamma)) = 1.$$
6. Applying the Boundary of a Planar Surface

Let $X$ be a disjoint union of a graph $G$ and a planar surface $F$ with boundary. Let $\omega$ be a weight on $\Gamma(G)$ over $\mathbb{Z}_2$ and $\varphi$ an embedding of $X$ into $S^3$ such that

$$\omega(\gamma)\text{lk}(\varphi(\gamma), \varphi(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G)$ and $\gamma' \in \Gamma(\partial F)$. Then we define $\beta_\omega(\varphi) \in \mathbb{Z}_2$ by

$$\beta_\omega(\varphi) \equiv \sum_{\gamma \in \Gamma(G) \setminus \Gamma(\partial F)} \omega(\gamma)\text{a}_3(\varphi(\gamma), \varphi(\gamma')) \pmod{2}.$$ 

Let $G$ be a disjoint union of a connected graph $G_1$ and a connected planar graph $G_2$. Let $f$ be a spatial embedding of $G$ and $p$ an embedding of $G_2$ into $S^2$. We denote the regular neighborhood of $p(G_2)$ in $S^2$ by $F(G_2; p)$, which is a planar surface having $p(G_2)$ as a spine. Then the spatial embedding $f$ induces an embedding $\tilde{f}_p$ of the disjoint union $G_1 \cup F(G_2; p)$ into $S^3$, so that $\tilde{f}_p(G_1) = f(G_1)$ and $\tilde{f}_p(F(G_2; p))$ has $f(G_2)$ as a spine in the natural way. Note that such an induced embedding $\tilde{f}_p$ is not unique up to ambient isotopy. Let $\omega$ be a weight on $\Gamma(G_1)$ over $\mathbb{Z}_2$ so that

$$\omega(\gamma)\text{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$. Then we have the following.

**Theorem 6.1.** If $f$ is split up to edge-homotopy, then $\beta_\omega(\tilde{f}_p) = 0$ for any induced embedding $\tilde{f}_p$ of $G_1 \cup F(G_2; p)$.

**Proof.** By the assumption we have that $f$ is transformed into a split spatial embedding $u$ of $G$ by self crossing changes and ambient isotopies. Then each of the self crossing changes induces a self crossing change on $\tilde{f}_p(G_1)$ or a band-pass move [6] (see Figure 6.1 on $\tilde{f}_p(F(G_2; p))$). Namely $\tilde{f}_p$ can be transformed into an induced embedding $\tilde{u}_p$ of $G_1 \cup F(G_2; p)$ by such moves and ambient isotopies. Let $\tilde{g}_p$ be an embedding of $G_1 \cup F(G_2; p)$ into $S^3$ obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$ or a single band-pass move on $\tilde{f}_p(F(G_2; p))$. Then it still holds that

$$\omega(\gamma)\text{lk}(\tilde{g}_p(\gamma), \tilde{g}_p(\gamma')) = 0$$

in $\mathbb{Z}$ for any $\gamma \in \Gamma(G_1)$ and $\gamma' \in \Gamma(\partial F(G_2; p))$.

**Claim.** $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$.

Assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single self crossing change on $\tilde{f}_p(G_1)$. Since it holds that

$$\sum_{\gamma' \in \Gamma(\partial F(G_2; p))} [\gamma'] = 0$$

in $H_1(F(G_2; p); \mathbb{Z}_2)$, we can see that $\beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p)$ in a similar way as the proof of Theorem 2.3 (1). Next we assume that $\tilde{g}_p$ is obtained from $\tilde{f}_p$ by a single band-pass move on $\tilde{f}_p(F(G_2; p))$. Then $\tilde{g}_p|_{G_1 \cup \partial F(G_2; p)}$ is obtained from $\tilde{f}_p|_{G_1 \cup \partial F(G_2; p)}$ by a single pass move [6] (see Figure 6.1 on $\tilde{f}_p(\partial F(G_2; p))$). We divide our situation into the following two cases.

**Case 1.** Four strings in the pass move belong to $\tilde{f}_p(\gamma_1')$ and $\tilde{f}_p(\gamma_2')$ for exactly two cycles $\gamma_1'$ and $\gamma_2'$ in $\Gamma(\partial F(G_2; p))$. 

This pass move causes a single self crossing change on \( \tilde{f}_p(\gamma'_1) \) and a single self crossing change on \( \tilde{f}_p(\gamma'_2) \). Then the separated components that result from smoothing each of the self crossings are orientation-reversing parallel knots; see Figure 6.2. So the difference between \( \beta_\omega(\tilde{f}_p) \) and \( \beta_\omega(\tilde{g}_p) \) is cancelled out in a similar way as in the proof of Theorem 2.2 (1). Thus we have that \( \beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p) \).

Case 2. Four strings in the pass move belong to \( \tilde{f}_p(\gamma') \) for a cycle \( \gamma' \) in \( \Gamma(\partial F(G_2; p)) \).

It is known that a pass move on the same component of a proper link \( L = J_1 \cup J_2 \cup \cdots \cup J_n \) preserves \( \overline{\operatorname{Arf}}(L) \equiv \operatorname{Arf}(L) - \sum_{i=1}^n \operatorname{Arf}(J_i) \in \mathbb{Z}_2 \) (cf. [16]).\(^3\) Especially, if \( n = 2 \) then \( a_3(L) \equiv \overline{\operatorname{Arf}}(L) \pmod{2} \) [12 Lemma 3.5 (ii)]. Therefore in this case the pass move preserves \( \omega(\gamma)a_3(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) \) for any cycle \( \gamma \in \Gamma(G_1) \). This implies that \( \beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{g}_p) \).

Now by the argument above, we have that \( \beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p) \). Then, each 2-component link \( \tilde{u}_p(\gamma \cup \gamma') \) is split for any \( \gamma \in \Gamma(G_1) \) and \( \gamma' \in \Gamma(\partial F(G_2; p)) \) because \( u \) is split. Therefore we have that \( \beta_\omega(\tilde{f}_p) = \beta_\omega(\tilde{u}_p) = 0 \). This completes the proof.

\[\text{Figure 6.1.}\]

\[\text{Figure 6.2.}\]

\(^3\)The value of \( \overline{\operatorname{Arf}}(L) \) is called the reduced Arf invariant of \( L \) [15].
Example 6.2. Let $G$ be a disjoint union of a circle $\gamma$ and the \textit{handcuff graph} (resp. \textit{2-bouquet}) $G_2$. Let $\omega$ be a weight on $\Gamma(\gamma)$ over $\mathbb{Z}_2$ defined by $\omega(\gamma) = 1$. We fix an embedding $p : G_2 \to S^2$ and take a regular neighborhood $F(G_2; p)$ as illustrated in Figure 6.3 (1) (resp. (2)).

Let $f$ be a spatial embedding of $G$ as illustrated in Figure 6.1 (1) (resp. (2)). Let us take an induced embedding $\tilde{f}_p : \gamma \cup F(G_2; p) \to S^3$ as illustrated in Figure 6.4 (1) (resp. (2)). Note that $\text{lk}(\tilde{f}_p(\gamma), \tilde{f}_p(\gamma')) = 0$ for any $\gamma' \in \Gamma(\partial F(G_2; p))$. Then it can be calculated that $\beta_\omega(\tilde{f}_p) = 1$. Thus by Theorem 6.1 we have that $f$ is non-splittable up to edge-homotopy.

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