Dissections of Strange $q$-Series

Dedicated to George E. Andrews on his 80th birthday

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Abstract. In a study of congruences for the Fishburn numbers, Andrews and Sellers observed empirically that certain polynomials appearing in the dissections of the partial sums of the Kontsevich–Zagier series are divisible by a certain $q$-factorial. This was proved by the first two authors. In this paper, we extend this strong divisibility property to two generic families of $q$-hypergeometric series which, like the Kontsevich–Zagier series, agree asymptotically with partial theta functions.

Mathematics Subject Classification. Primary 33D15.

Keywords. Fishburn numbers, Kontsevich–Zagier strange function, $q$-Series, Partial theta functions, Congruences.

1. Introduction

Recall the usual $q$-series notation
\[(a; q)_n := (1 − a)(1 − aq) \cdots (1 − aq^{n−1}), \quad (1.1)\]
and let $\mathcal{F}(q)$ denote the Kontsevich–Zagier “strange” function [13,14],
\[\mathcal{F}(q) := \sum_{n \geq 0} (q; q)_n.\]
This series does not converge on any open subset of $\mathbb{C}$, but it is well defined both at roots of unity and as a power series when $q$ is replaced by $1 − q$. The coefficients $\xi(n)$ of
\[\mathcal{F}(1 − q) = 1 + q + 2q^2 + 5q^3 + 15q^4 + 53q^5 + \cdots\]
were conjectured by Andrews and Sellers [13,14], and have been rigorously verified by the first two authors.

The first author was supported by a grant from the Simons Foundation (#426145 to Scott Ahlgren). Byungchan Kim was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09917344).
are called the Fishburn numbers, and they count a number of different combinatorial objects (see [11] for references).

Andrews and Sellers [4] discovered and proved a wealth of congruences for $\xi(n)$ modulo primes $p$. For example, we have

$$\xi(5n + 4) \equiv \xi(5n + 3) \equiv 0 \pmod{5},$$
$$\xi(7n + 6) \equiv 0 \pmod{7}.$$

(1.2)

In subsequent work of the first two authors [1], Garvan [6], and Straub [12], similar congruences were obtained for prime powers and for generalized Fishburn numbers.

Taking a different approach, Guerzhoy et al. [7] interpreted the coefficients in the asymptotic expansions of the functions $P_{a,b,\chi}^{(1)}(e^{-t})$ defined in (1.8) below in terms of special values of $L$-functions, and proved congruences for these coefficients using divisibility properties of binomial coefficients. These congruences are inherited by any function whose expansion at $q = 1$ agrees with one of these expansions; these include the function $F(q)$ and, more generally, the Kontsevich–Zagier functions described in Sect. 5 below. See [7] for details.

Although the congruences (1.2) bear a passing resemblance to Ramanujan’s congruences for the partition function $p(n)$, it turns out that they arise from a divisibility property of the partial sums of $F(q)$. For positive integers $N$ and $s$, consider the partial sums

$$F(q; N) := \sum_{n=0}^{N} (q; q)_n$$

and the $s$-dissection

$$F(q; N) = \sum_{i=0}^{s-1} q^i A_s(N, i, q^s).$$

Let $S(s) \subseteq \{0, 1, \ldots, s - 1\}$ denote the set of reductions modulo $s$ of the set of pentagonal numbers $m(3m + 1)/2$, where $m \in \mathbb{Z}$. The key step in the proof of Andrews and Sellers is to show that if $p$ is prime and $i \not\in S(p)$ then we have

$$(1 - q)^n \mid A_p(pn - 1, i, q).$$

(1.3)

This divisibility property is also important for the proof of the congruences in [6,12].

Andrews and Sellers [4] observed empirically that $(1 - q)^n$ can be strengthened to $(q; q)_n$ in (1.3). The first two authors showed that this divisibility property holds for any $s$. To be precise, define

$$\lambda(N, s) = \left\lfloor \frac{N + 1}{s} \right\rfloor.$$

(1.4)

Then we have

**Theorem 1.1.** [1] Suppose that $s$ and $N$ are positive integers and that $i \not\in S(s)$. Then

$$(q; q)_{\lambda(N, s)} \mid A_s(N, i, q).$$

(1.5)
The proof of (1.5) relies on the fact that the Kontsevich–Zagier function satisfies the “strange identity”

\[ F(q) = - \frac{1}{2} \sum_{n \geq 1} \frac{(12/n) q^{(n^2 - 1)/24}}{n}. \]

Here the symbol “=“ means that the two sides agree to all orders at every root of unity (this is explained fully in Sections 2 and 5 of [13]). In this paper, we show that an analogue of Theorem 1.1 holds for a wide class of “strange” \( q \)-hypergeometric series—that is, \( q \)-series which agree asymptotically with partial theta functions.

To state our result, let \( F \) and \( G \) be functions of the form

\[ F(q) = \sum_{n=0}^{\infty} (q; q)_n f_n(q), \tag{1.6} \]
\[ G(q) = \sum_{n=0}^{\infty} (q; q^2)_n g_n(q), \tag{1.7} \]

where \( f_n(q) \) and \( g_n(q) \) are polynomials. (Functions of the form (1.6) are elements of the Habiro ring, which can be viewed as a ring of analytic functions on the set of roots of unity [8].) Note that \( F(q) \) is not necessarily well defined as a power series in \( q \), but it has a power series expansion at every root of unity \( \zeta \). In other words, \( F(\zeta e^{-t}) \) has a meaningful definition as a formal power series in \( t \) whose coefficients are expressed in the usual way as the “derivatives” of \( F(\zeta e^{-t}) \) at \( t = 0 \). This is explained in detail in the next section. Likewise, \( G(q) \) has a power series expansion at every odd-order root of unity.

We will consider partial theta functions

\[ P^{(\nu)}_{a,b,\chi}(q) := \sum_{n \geq 0} n^\nu \chi(n) q^{n^2/a}, \tag{1.8} \]

where \( \nu \in \{0, 1\} \), \( a \geq 0 \) and \( b > 0 \) are integers, and \( \chi: \mathbb{Z} \to \mathbb{C} \) is a function satisfying the following properties:

\[ \chi(n) \neq 0 \quad \text{only if} \quad \frac{n^2 - a}{b} \in \mathbb{Z}, \tag{1.9} \]

and for each root of unity \( \zeta \),

the function \( n \mapsto \zeta \frac{n^2 - a}{b} \chi(n) \) is periodic and has mean value zero. \( (1.10) \)

These assumptions are enough to ensure that for each root of unity \( \zeta \), the function \( P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \) has an asymptotic expansion as \( t \to 0^+ \) (see Sect. 3 below). We note that (1.10) is satisfied by any odd periodic function. To see this, suppose that \( \chi \) is odd with period \( T \), and let \( \zeta \) be a \( k \)th root of unity. Set \( M = \text{lcm}(T, bk) \). Then we have

\[ \zeta \frac{(M-n)^2-a}{b} \chi(M - n) = -\zeta \frac{n^2-a}{b} \chi(n), \]
and so
\[ \sum_{n=0}^{M-1} \zeta^{\frac{n^2-a}{b}} \chi(n) = 0. \]

For positive integers \( s \) and \( N \), consider the partial sum
\[ F(q; N) := \sum_{n=0}^{N} f_n(q)(q; q)_n \quad (1.11) \]
and its \( s \)-dissection
\[ F(q; N) = \sum_{i=0}^{s-1} q^i A_{F,s}(N, i, q^s). \]
Define \( S_{a,b,\chi}(s) \subseteq \{0, 1, \ldots, s-1\} \) by
\[ S_{a,b,\chi}(s) := \left\{ \frac{n^2 - a}{b} \pmod{s} : \chi(n) \neq 0 \right\}. \]

Our first main result is the following.

**Theorem 1.2.** Suppose that \( F \) is a function as in (1.6) and that \( P^{(\nu)}_{a,b,\chi} \) is a function as in (1.8). Suppose that for each root of unity \( \zeta \) we have the asymptotic expansion
\[ P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim F(\zeta e^{-t}) \quad \text{as } t \to 0^+. \quad (1.12) \]
Suppose that \( s \) and \( N \) are positive integers and that \( i \notin S_{a,b,\chi}(s) \). Then we have
\[ (q; q)_\lambda(N, s) \mid A_{F,s}(N, i, q). \]

Analogously, for positive integers \( s \) and \( N \) with \( s \) odd, consider the partial sum
\[ G(q; N) := \sum_{n=0}^{N} g_n(q)(q^2;q)_n \quad (1.13) \]
and its \( s \)-dissection
\[ G(q; N) = \sum_{i=0}^{s-1} q^i A_{G,s}(N, i, q^s). \]
Then the \( A_{G,s}(N, i, q^s) \) also enjoy strong divisibility properties. Define
\[ \mu(N, k, s) = \left\lfloor \frac{N}{s(2k-1)} + \frac{1}{2} \right\rfloor. \quad (1.14) \]

**Theorem 1.3.** Suppose that \( G \) is a function as in (1.7) and that \( P^{(\nu)}_{a,b,\chi} \) is a function as in (1.8). Suppose that for each root of unity \( \zeta \) of odd order we have
\[ P^{(\nu)}_{a,b,\chi}(\zeta e^{-t}) \sim G(\zeta e^{-t}) \quad \text{as } t \to 0^+. \quad (1.15) \]
Suppose that $s$ and $N$ are positive integers with $s$ odd and that $i \not\in S_{a,b,\chi}(s)$. Then we have

$$(q; q^2)^{\mu(N,1,s)} | A_{G,s}(N, i, q).$$

While a generic $q$-series of the form (1.6) or (1.7) is not expected to be related to a partial theta function as in (1.12) or (1.15), there are a number of examples where this is the case. For example, Hikami [9] introduced a family of quantum modular forms related to torus knots, which we will discuss in Sect. 5. For now, we illustrate Theorem 1.3 with an example from Ramanujan’s lost notebook. Consider the $q$-series

$$G(q) = \sum_{n\geq 0} (q; q^2)_n q^n.$$ 

From [3, Entry 9.5.2], we have the identity

$$\sum_{n\geq 0} (q; q^2)_n q^n = \sum_{n\geq 0} (-1)^n q^{3n^2+2n}(1 + q^{2n+1}),$$

which may be written as

$$\sum_{n\geq 0} (q; q^2)_n q^n = \sum_{n\geq 0} \chi_6(n)q^{(n^2-1)/3},$$

where

$$\chi_6(n) := \begin{cases} 1, & \text{if } n \equiv 1, 2 \pmod{6}, \\ -1, & \text{if } n \equiv 4, 5 \pmod{6}, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore, for each odd-order root of unity $\zeta$ we find that

$$P_{1,3,\chi_6}^{(0)}(\zeta e^{-t}) \sim G(\zeta e^{-t}) \quad \text{as } t \to 0^+.$$ 

Since $\chi_6$ is odd, it satisfies conditions (1.9) and (1.10). Thus, from Theorem 1.3, we find that for $i \not\in S_{1,3,\chi_6}(s)$ we have

$$(q; q^2)_{\frac{s}{2} + \frac{1}{2}} | A_{G,s}(N, i, q). \quad (1.16)$$

For example, when $s = 5$ we have $S_{1,3,\chi_6}(5) = \{0, 1, 3\}$. For $N = 8$ we have

$$A_{G,5}(8, 2, q) = q^2(q; q^2)_2(1 + q^2 - q^3 + 2q^4 - q^5 + 2q^6 + q^8)$$

and

$$A_{G,5}(8, 4, q) = -q(q; q^2)_2(1 - q + q^2)(1 + q + q^2 + q^4 + q^6),$$

as predicted by (1.16), while the factorizations of $A_{G,5}(8, i, q)$ into irreducible factors for $i \in \{0, 1, 3\}$ are

$$A_{G,5}(8, 0, q) = (1 - q)(1 + q^4 - 2q^5 + \cdots - 2q^{11} + q^{12}),$$

$$A_{G,5}(8, 1, q) = 1 + 2q^3 - q^4 + \cdots + q^{13} - q^{14},$$

$$A_{G,5}(8, 3, q) = q(-1 + q^2 - 2q^3 + 2q^4 - \cdots - 2q^{11} + q^{12}).$$
The rest of the paper is organized as follows. In the next section, we discuss power series expansions of $F$ and $G$ at roots of unity, and in Sect. 3, we discuss the asymptotic expansions of partial theta functions. In Sect. 4, we prove the main theorems. In Sect. 5, we give two further examples—one generalizing (1.5) and one generalizing (1.16). We close with some remarks on congruences for the coefficients of $F(1 - q)$ and $G(1 - q)$.

2. Power Series Expansions of $F$ and $G$

Let $F(q)$ be a function as in (1.6) and $G(q)$ be a function as in (1.7). Here we collect some facts which allow us to meaningfully define $F(\zeta e^{-t})$ and $G(\zeta e^{-t})$ as formal power series.

Lemma 2.1. Let $F(q; N)$ be as in (1.11), and let $G(q; N)$ be as in (1.13). Suppose that $\zeta$ is a $k$th root of unity.

1. The values $(q \frac{d}{dq})^\ell F(q; N)|_{q=\zeta}$ are stable for $N \geq (\ell + 1)k - 1$.

2. If $k$ is odd then the values $(q \frac{d}{dq})^\ell G(q; N)|_{q=\zeta}$ are stable for $2N \geq (2\ell + 1)(2k - 1) + 1$.

Proof. For each positive integer $k$, we have

\[ (1 - q^k)^{\ell+1} | (q; q)_N \quad \text{for} \quad N \geq (\ell + 1)k, \]
\[ (1 - q^{2k-1})^{\ell+1} | (q; q^2)_N \quad \text{for} \quad 2N \geq (2\ell + 1)(2k - 1) + 1. \]

It follows that for $0 \leq j \leq \ell$ we have
\[ (\frac{d}{dq})^j (q; q)_N|_{q=\zeta} = 0 \quad \text{for} \quad N \geq (\ell + 1)k, \]
\[ (\frac{d}{dq})^j (q; q^2)_N|_{q=\zeta} = 0 \quad \text{for odd } k \text{ and } 2N \geq (2\ell + 1)k + 1. \]

The lemma follows since for any polynomial $f(q)$, the polynomial $(q \frac{d}{dq})^\ell f(q)$ is a linear combination (with polynomial coefficients) of $(\frac{d}{dq})^j f(q)$ with $0 \leq j \leq \ell$ (see for example [4, Lemma 2.2]).

For any polynomial $f(q)$, any $\zeta$ and any $\ell \geq 0$, we have [4, Lemma 2.3]
\[ \left( \frac{d}{dt} \right)^\ell f(\zeta e^{-t})|_{t=0} = (-1)^\ell \left( q \frac{d}{dq} \right)^\ell f(q)|_{q=\zeta}. \quad (2.1) \]

Let $F(q)$ be as in (1.6) and let $\zeta$ be a $k$th root of unity. The last fact together with Lemma 2.1 allows us to define
\[ \left( \frac{d}{dt} \right)^\ell F(\zeta e^{-t})|_{t=0} := \left( \frac{d}{dt} \right)^\ell F(\zeta e^{-t}; N)|_{t=0} \quad \text{for any} \quad N \geq k(\ell + 1) - 1. \]

We, therefore, have a formal power series expansion
\[ F(\zeta e^{-t}) = \sum_{\ell=0}^\infty \frac{\left( \frac{d}{dt} \right)^\ell F(\zeta e^{-t})|_{t=0}}{\ell!} t^\ell. \quad (2.2) \]
Similarly, if \( G(q) \) is a function as in (1.7) and \( \zeta \) is a \( k \)th root of unity with odd \( k \), then we can define
\[
\left( \frac{d}{dt} \right)^\ell G(\zeta e^{-t}) \bigg|_{t=0} := \left( \frac{d}{dt} \right)^\ell G(\zeta e^{-t}; N) \bigg|_{t=0} \text{ for any } 2N \geq k(2\ell + 1),
\]
using (2.1) and Lemma 2.1. Thus, we have a formal power series expansion
\[
G(\zeta e^{-t}) = \sum_{\ell=0}^{\infty} \left( \frac{d}{dt} \right)^\ell G(\zeta e^{-t}) \bigg|_{t=0} \frac{t^\ell}{\ell!}.
\]

3. The Asymptotics of \( P_{a,b,\chi}^{(\nu)} \)

In this section, we discuss the asymptotic expansion of the partial theta functions \( P_{a,b,\chi}^{(\nu)}(q) \) defined in (1.8). Recall that
\[
P_{a,b,\chi}^{(\nu)}(q) := \sum_{n \geq 0} \nu \chi(n) q^{n^2 - ab},
\]
where \( \nu \in \{0, 1\} \), \( a \geq 0 \) and \( b > 0 \) are integers, and \( \chi: \mathbb{Z} \rightarrow \mathbb{C} \) is a function satisfying properties (1.9) and (1.10).

The properties which we describe in the next proposition are more or less standard (see, for example, [10, p. 98]). For convenience and completeness we sketch a proof of the following:

**Proposition 3.1.** Suppose that \( P_{a,b,\chi}^{(\nu)}(q) \) is as in (1.8). Let \( \zeta \) be a root of unity and let \( N \) be a period of the function \( n \mapsto \zeta \frac{n^2 - a}{b} \chi(n) \). Then we have the asymptotic expansion
\[
P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) \sim \sum_{n=0}^{\infty} \gamma_n(\zeta) t^n, \quad t \to 0^+,
\]
where
\[
\gamma_n(\zeta) = \sum_{\substack{1 \leq m \leq N \\chi(m) \neq 0}} a(m, n, N) \zeta^{\frac{m^2 - a}{b}}
\]
with certain complex numbers \( a(m, n, N) \).

We begin with a lemma. For \( n \geq 0 \), let \( B_n(x) \) denote the \( n \)th Bernoulli polynomial. In the rest of this section, we use \( s \) for a complex variable since there can be no confusion with the parameter \( s \) used above.

**Lemma 3.2.** Let \( C: \mathbb{Z} \rightarrow \mathbb{C} \) be a function with period \( N \) and mean value zero, and let
\[
L(s, C) := \sum_{n=1}^{\infty} \frac{C(n)}{n^s}, \quad \text{Re}(s) > 0.
\]
Then $L(s, C)$ has an analytic continuation to $\mathbb{C}$, and we have

$$L(-n, C) = -N^n \frac{1}{n+1} \sum_{m=1}^{N} C(m) B_{n+1} \left( \frac{m}{N} \right) \quad \text{for } n \geq 0. \quad (3.2)$$

**Proof.** Let $\zeta(s, \alpha)$ denote the Hurwitz zeta function, whose properties are described, for example, in [5, Chapter 12]. We have

$$L(s, C) = N^{-s} \sum_{m=1}^{N} C(m) \zeta \left( s, \frac{m}{N} \right). \quad (3.3)$$

The lemma follows using the fact that each Hurwitz zeta function has only a simple pole with residue 1 at $s = 1$ and the formula for the value of each function at $s = -n$ [5, Theorem 12.13]. \qed

**Proof of Proposition 3.1.** It is enough to prove the proposition for the function

$$f(t) := e^{-\frac{at}{b}} P_{a,b,\chi}^{(\nu)}(\zeta e^{-t}) = \sum_{n \geq 1} n^{\nu} \chi(n) \zeta^{\frac{n^2-a}{b}} e^{-\frac{n^2 t}{b}}, \quad t > 0.$$  

Setting

$$C(n) := \zeta^{\frac{n^2-a}{b}} \chi(n), \quad (3.4)$$

we have the Mellin transform

$$\int_0^\infty f(t) t^{s-1} dt = b^s \Gamma(s) L(2s - \nu, C), \quad \text{Re}(s) > \frac{1}{2}.$$  

Inverting, we find that

$$f(t) = \frac{1}{2\pi i} \int_{x=c} b^s \Gamma(s) L(2s - \nu, C) t^{-s} ds,$$  

for $c > \frac{1}{2}$, where we write $s = x + iy$. Using (3.3), the functional equation for the Hurwitz zeta functions, and the asymptotics of the Gamma function, we find that, for fixed $x$, the function $L(s, C)$ has at most polynomial growth in $|y|$ as $|y| \to \infty$. Shifting the contour to the line $x = -R - \frac{1}{2}$, we find that for each $R \geq 0$ we have

$$f(t) = \sum_{n=0}^{R} \frac{(-1)^n}{b^n n!} L(-2n - \nu, C) t^n + O \left( t^{R+\frac{1}{2}} \right),$$

from which

$$f(t) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{b^n n!} L(-2n - \nu, C) t^n.$$  

The proposition follows from (3.4) and (3.2). \qed
4. Proofs of Theorems 1.2 and 1.3

We begin with a lemma. The first assertion is proved in [4, Lemma 2.4], and the second, which is basically equation (2.4) in [1], follows by extracting an arithmetic progression using orthogonality. (We note that there is an error in the published version of [1] which is corrected below; in that version the operators $\frac{d}{dq}$ and $q\frac{d}{dq}$ are conflated in the statements of (2.3) and (2.4). This does not affect the truth of the rest of the results.)

Let $C_{\ell,i,j}(s)$ be the array of integers defined recursively as follows:

1. $C_{0,0,0}(s) = 1$,
2. $C_{\ell,i,0}(s) = i\ell$ and $C_{\ell,i,j}(s) = 0$ for $j \geq \ell + 1$ or $j < 0$,
3. $C_{\ell+1,i,j}(s) = (i + js)C_{\ell,i,j}(s) + sC_{\ell,i,j-1}(s)$ for $1 \leq j \leq \ell$.

Lemma 4.1. Suppose that $s$ is a positive integer and that

$$h(q) = \sum_{i=0}^{s-1} q^i A_s(i, q^s)$$

with polynomials $A_s(i, q)$. Then the following are true:

1. For all $\ell \geq 0$, we have

$$\left(q \frac{d}{dq}\right)^{\ell} h(q) = \sum_{j=0}^{\ell} \sum_{i=0}^{s-1} C_{\ell,i,j}(s)q^{i+j} A_s^{(j)}(i, q^s).$$

2. Let $\zeta_s$ be a primitive $s$th root of unity. Then for $\ell \geq 0$ and $i_0 \in \{0, \ldots, s-1\}$, we have

$$\sum_{j=0}^{\ell} C_{\ell,i_0,j}(s)q^{i_0+j} A_s^{(j)}(i_0, q^s) = \frac{1}{s} \sum_{k=0}^{s-1} \zeta_s^{-k-i_0} \left. \left(q \frac{d}{dq}\right)^{\ell} h(q) \right|_{q \rightarrow \zeta_k^k}. \quad (4.1)$$

Proof of Theorem 1.2. Suppose that $F(q)$ and $P_{a,b,\chi}(q)$ are as in the statement of the theorem. Suppose that $s$ and $k$ are positive integers, that $i \notin S_{a,b,\chi}(s)$ and that $\zeta_k$ is a primitive $k$th root of unity. Let $\Phi_k(q)$ be the $k$th cyclotomic polynomial. Recall the definition (1.4) of $\lambda(N,s)$ and note that since

$$\left(q; q\right)_n = \pm \prod_{k=1}^{n} \Phi_k(q)^{\left\lfloor \frac{n}{k} \right\rfloor} \quad (4.2)$$

and

$$\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{x}{ks} \right\rfloor,$$

we have

$$(q; q)_{\lambda(N,s)} = \pm \prod_{k=1}^{\lambda(N,s)} \Phi_k(q)^{\lambda(N,ks)}.$$ 

Therefore, Theorem 1.2 will follow once we show for each $\ell \geq 0$ that

$$A_{F,s}^{(\ell)}(N, i, \zeta_k) = 0 \quad \text{for} \quad N \geq (\ell + 1)ks - 1$$
since this implies that $\Phi_k(q)^{\lambda(N,ks)} \mid A_{F,s}(N,i,q)$ for $1 \leq k \leq \lambda(N,s)$.

From the definition we find that

$$A_{F,s}(N,i,q) = \sum_{j=0}^{k-1} q^j A_{F,ks}(N,i + js, q^k).$$

If $i \notin S_{a,b,\chi}(s)$, then $i + js \notin S_{a,b,\chi}(ks)$. It is, therefore, enough to show that for all $s$, $k$, and $\ell$, and for $i \notin S_{a,b,\chi}(ks)$, we have

$$A_{F,ks}(N,i,1) = 0 \text{ for } N \geq (\ell + 1)ks - 1.$$

After replacing $ks$ by $s$, it is enough to show that for all $s$ and $\ell$, and for $i \notin S_{a,b,\chi}(s)$, we have

$$A_{F,s}(N,i,1) = 0 \text{ for } N \geq (\ell + 1)s - 1. \tag{4.3}$$

We prove (4.3) by induction on $\ell$. For the base case $\ell = 0$, assume that $N \geq s - 1$. Using (4.1) with $q = 1$ gives

$$A_{F,s}(N,i,1) = \frac{1}{s} \sum_{j=1}^{s-1} \zeta_s^{-ji} F(\zeta_s^j; N).$$

By (1.12), (2.1), Lemma 2.1, and Proposition 3.1, we find that

$$A_{F,s}(N,i,1) = \frac{1}{s} \sum_{j=1}^{s} \zeta_s^{-ji} \gamma_0(\zeta_s^j).$$

By (3.1) and orthogonality (recalling that $i \notin S_{a,b,\chi}(s)$), we find that

$$A_{F,s}(N,i,1) = 0.$$

For the induction step, suppose that $N \geq (\ell + 1)s - 1$, that $i \notin S_{a,b,\chi}(s)$, and that (4.3) holds with $\ell$ replaced by $j$ for $1 \leq j \leq \ell - 1$. By (4.1) and the induction hypothesis we have

$$C_{\ell,i,\ell}(s) A_{F,s}^{(\ell)}(N,i,1) = \frac{1}{s} \sum_{j=1}^{s} \zeta_s^{-ji} \left( q \frac{d}{dq} \right)^\ell F(q; N)|_{q = \zeta_s^j}.$$

Using Proposition 3.1, (2.2), (3.1), and orthogonality, we find as above that

$$C_{\ell,i,\ell}(t) A_{F,s}^{(\ell)}(N,i,1) = 0.$$

This establishes (4.3) since $C_{\ell,i,\ell}(s) > 0$. Theorem 1.2 follows. \qed

**Proof of Theorem 1.3.** Suppose that $s$ and $k$ are positive integers with $s$ odd, that $i \notin S_{a,b,\chi}(s)$ and that $\zeta_{2k-1}$ is a $(2k - 1)$th root of unity. Recall the definition (1.14) of $\mu(N,k,s)$. In analogy with (4.2), we have

$$(q; q^2)_n = \pm \prod_{k=1}^{n} \Phi_{2k-1}(q)|_{\frac{(2n-1)}{2(2k-1)+1} + \frac{1}{2}},$$

and as above we obtain

$$(q; q^2)_{\mu(N,1,s)} = \pm \prod_{k=1}^{\mu(N,1,s)} \Phi_{2k-1}(q)^{\mu(N,k,s)}.$$
Therefore, Theorem 1.3 follows once we show for each $\ell \geq 0$ that
\[
A_{G,s}^{(\ell)}(N, i, \zeta_{2k-1}) = 0 \quad \text{for} \quad 2N \geq (2\ell + 1)(2k - 1)s.
\]

The rest of the proof is similar to that of Theorem 1.2 (we require $s$ to be odd because $G(q)$ has a series expansion only at odd-order roots of unity). Arguing as above, we show that for each odd $s$ we have
\[
A_{G,s}^{(\ell)}(N, i, 1) = 0 \quad \text{for} \quad 2N \geq (2\ell + 1)s,
\]
and the result follows. ∎

5. Examples

In this section, we illustrate Theorems 1.2 and 1.3 with two families of examples.

5.1. The Generalized Kontsevich–Zagier Functions

In a study of quantum modular forms related to torus knots and the Andrews–Gordon identities, Hikami [9] defined the functions

\[
X_{\alpha}^{(\alpha)}(q) := \sum_{k_1, k_2, \ldots, k_m \geq 0} (q; q)_{k_m} q^{k_1^2 + \cdots + k_{m-1}^2 + k_{m+1} + \cdots + k_{m-1}} \prod_{i=1}^{m-1} \left[ \begin{array}{c} k_{i+1} \\ k_i \end{array} \right] \left[ \begin{array}{c} k_{\alpha+1} + 1 \\ k_{\alpha} \end{array} \right], \tag{5.1}
\]

where $m$ is a positive integer and $\alpha \in \{0, 1, \ldots, m-1\}$. Here we have used the usual $q$-binomial coefficient (or Gaussian polynomial)

\[
\left[ \begin{array}{c} n \\ k \end{array} \right] := \left[ \begin{array}{c} n \\ k \end{array} \right]_q := \begin{cases} (q; q)_n \\ (q; q)_k(q; q)_{n-k}, \quad \text{if} \ 0 \leq k \leq n, \\ 0, \quad \text{otherwise}. \end{cases}
\]

The simplest example

\[
X_{1}^{(0)}(q) = \sum_{n \geq 0} (q; q)_n
\]

is the Kontsevich–Zagier function. From (5.1) we can write

\[
X_{\alpha}^{(\alpha)}(q) = \sum_{k_m \geq 0} (q; q)_{k_m} f_{k_m}^{(\alpha)}(q),
\]

with polynomials $f_{k_m}^{(\alpha)}(q)$.

Hikami’s identity [9, eq. (70)] implies that for each root of unity $\zeta$, we have

\[
P_{(2m-2\alpha-1)^2,8(2m+1)}^{(1)}(\zeta e^{-t}) \sim X_{\alpha}^{(\alpha)}(\zeta e^{-t})
\]

\]
as \( t \to 0^+ \), where \( \chi_{8m+4}^{(\alpha)}(n) \) is defined by

\[
\chi_{8m+4}^{(\alpha)}(n) = \begin{cases} 
-1/2, & \text{if } n \equiv 2m - 2\alpha - 1 \text{ or } 6m + 2\alpha + 5 \pmod{8m+4}, \\
1/2, & \text{if } n \equiv 2m + 2\alpha + 3 \text{ or } 6m - 2\alpha + 1 \pmod{8m+4}, \\
0, & \text{otherwise.}
\end{cases}
\]

(5.2)

The function \( \chi_{8m+4}^{(\alpha)}(n) \) satisfies condition (1.9). For (1.10) we record a short lemma.

**Lemma 5.1.** Suppose that \( \chi_{8m+4}^{(\alpha)}(n) \) is as defined in (5.2) and that \( \zeta \) is a root of unity of order \( M \). Define

\[
\psi(n) = \zeta^{n^2 - (2m - 2\alpha - 1)^2 / 8(2m+1)} \chi_{8m+4}^{(\alpha)}(n).
\]

Then

\[
\sum_{n=1}^{M(8m+4)} \psi(n) = 0.
\]

**Proof.** Note that \( \psi \) is supported on odd integers, so we assume in what follows that \( n \) is odd. From the definition, we have

\[
\chi_{8m+4}^{(\alpha)}(n + M(4m + 2)) = (-1)^M \chi_{8m+4}^{(\alpha)}(n). \tag{5.3}
\]

The exponent in the ratio of the corresponding powers of \( \zeta \) is \( mM^2 + \frac{M^2 + Mn}{2} \). So the ratio of these powers of \( \zeta \) is

\[
\zeta^{\frac{M^2 + Mn}{2}}.
\]

If \( M \) is odd then this becomes \( \zeta^{M(\frac{M^2 + Mn}{2})} = 1 \), while if \( M \) is even then this becomes \( \zeta^{\frac{M^2}{2}} \zeta^{\frac{Mn}{2}} = -1 \) (since \( M \) is the order of \( \zeta \) and \( n \) is odd). Therefore, the ratio in either case is \((-1)^{M+1}\). Combining this with (5.3) gives

\[
\psi(n + M(4m + 2)) = -\psi(n),
\]

from which the lemma follows. \( \square \)

Therefore, \( X_m^{(\alpha)}(q) \) satisfies the conditions of Theorem 1.2, and we obtain the following.

**Corollary 5.2.** If \( s \) is a positive integer and \( i \notin S_{(2m - 2\alpha - 1)^2, 8(2m+1), \chi_{8m+4}^{(\alpha)}}(s) \), then

\[
(q; q)_{\lambda(N,s)} | A_{X_m^{(\alpha)},s}(N, i, q),
\]

where \( A_{X_m^{(\alpha)},s}(N, i, q) \) are the coefficients in the \( s \)-dissection of the partial sums (in \( k_m \)) of \( X_m^{(\alpha)}(q) \).

For example, when \( s = 3 \) we have \( S_{9,40,\chi_{20}^{(o)}}(3) = \{0, 1\} \) and \( S_{1,40,\chi_{20}^{(l)}}(3) = \{0, 2\} \). For \( N = 8 \) we have

\[
A_{X_2^{(o)},3}(8, 2, q) = (q; q)_{3}(1 + q)(1 + q + q^2)(1 - q + \cdots - q^{25} + q^{26})
\]
and
\[ A_{X_2^{(1)},3}(8, 1, q) = (q; q)_3(1 + q)(1 − q + q^2) \]
\[ \times (1 + q + q^2)(1 + 2q + \cdots q^{26} + q^{27}), \]
as predicted by Corollary 5.2, while
\[ A_{X_2^{(0)},3}(8, 0, q) = (1 − q + q^2)(9 + 9q + \cdots + q^{33} + q^{34}), \]
\[ A_{X_2^{(0)},3}(8, 1, q) = −8 − 7q + \cdots + q^{34} − q^{35}, \]
\[ A_{X_2^{(1)},3}(8, 0, q) = 9 − 7q + \cdots + 2q^{36} + q^{39}, \]

and
\[ A_{X_2^{(1)},3}(8, 2, q) = −7 + 3q^3 − \cdots + q^{36} − q^{38} \]
are not divisible by \((q; q)_3\).

5.2. An Example with \(\nu = 0\)

For \(k \geq 1\), let \(G_k(q)\) denote the \(q\)-series
\[ G_k(q) = \sum_{n_k \geq n_{k−1} \geq \cdots \geq n_1 \geq 0} \frac{q^{n_k+2n_{k−1}+2n_{k−2}+\cdots+2n_1^2+2n_1}}{(q; q^2)^{n_k} \cdots \left(n_2 \atop n_1\right)^2}. \]

Then we have the identity
\[ G_k(q) = \sum_{n \geq 0} (-1)^n q^{(2k+1)n^2+2kn} (1 + q^{2n+1}), \tag{5.4} \]
which follows from Andrews’ generalization \([2]\) of the Watson–Whipple transformation
\[ \sum_{m=0}^{N} \frac{(1 − aq^{2m})}{(1 − a)} \frac{(a, b_1, c_1, \ldots, b_k, c_k; q^N)_m}{(aq/b_1, aq/c_1, \ldots, aq/b_k, aq/c_k, q^{-N})_m} \frac{(a^k q^{k+N})_{m}}{(b_1 c_1 \cdots b_k c_k)_m} \]
\[ = \frac{(aq, aq/b_k c_k)_N}{(aq/b_k, aq/c_k)_N} \sum_{N \geq n_k \geq \cdots \geq n_1 \geq 0} \frac{(b_k, c_k)_{n_k-1} \cdots (b_2, c_2)_{n_1}}{(q; q)_{n_k-1-n_{k-2}} \cdots (q; q)_{n_2-n_1} (q; q)_{n_1}} \]
\[ \times \frac{(aq/b_k c_k-1)_{n_k-1-n_{k-2}} \cdots (aq/b_2 c_2)_{n_2-n_1} (aq/b_1 c_1)_{n_1}}{(aq/b_k-1, aq/c_k-1)_{n_k-1} \cdots (aq/b_2 c_2)_{n_1}} \]
\[ \times \frac{(q^{-N})_{n_k-1} (aq)_{n_k-2+\cdots+n_1} q^{n_k-1}}{(b_k c_k q^{-N}/a)_{n_k-1} (b_k-1 c_k-1)_{n_k-2} \cdots (b_2 c_2)^{n_1}}. \]

Here we have extended the notation in \((1.1)\) to
\[ (a_1, a_2, \ldots, a_k)_n := (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n. \]

To deduce \((5.4)\), we set \(q = q^2, a = q^2, b_k = q, \) and \(c_k = q^2\) and then let \(N \to \infty\) along with all other \(b_i, c_i\).
The identity (5.4) may be written as

$$G_k(q) = \sum_{n \geq 0} \chi_{4k+2}(n)q^{\frac{n^2-k^2}{4k+2}},$$

where

$$\chi_{4k+2}(n) := \begin{cases} 
1, & \text{if } n \equiv k, k+1 \pmod{4k+2}, \\
-1, & \text{if } n \equiv -k, -k-1 \pmod{4k+2}, \\
0, & \text{otherwise}.
\end{cases}$$

This implies that for each odd-order root of unity $\zeta$, we have

$$P_{k^2,2k+1,\chi_{4k+2}}^{(0)}(\zeta e^{-t}) \sim G_k(\zeta e^{-t}) \text{ as } t \to 0^+.$$ 

The function $\chi_{4k+2}(n)$ satisfies conditions (1.9) and (1.10) (see the remark following (1.10)), so Theorem 1.3 gives

**Corollary 5.3.** Suppose that $k$ and $N$ are positive integers, that $s$ is a positive odd integer, and that $i \not\in S_{k^2,2k+1,\chi_{4k+2}}(s)$. Then

$$(q; q^2)^{\lfloor \frac{N}{2} + \frac{s}{2} \rfloor} | A_{G_k,s}(N, i, q).$$

### 6. Remarks on Congruences

Congruences for the coefficients of the functions $F(q)$ and $G(q)$ in Theorems 1.2 and 1.3 can be deduced from the results of [7]. In closing we mention another approach. Theorems 1.2 and 1.3 guarantee that many of the coefficients in the $s$-dissection are divisible by high powers of $1 - q$, and the congruences follow from this fact when $s = p^r$ together with an argument as in [1, Section 3].

For example, let $G_k$ be the function defined in the last section and define $\xi_{G_k}(n)$ by

$$G_k(1 - q) = \sum_{n \geq 0} \xi_{G_k}(n)q^n.$$ 

Consider the expansions

$$G_1(1 - q) = \sum_{n \geq 0} \xi_{G_1}(n)q^n = 1 + q + 2q^2 + 6q^3 + 25q^4 + 135q^5 + \cdots,$$

$$G_2(1 - q) = \sum_{n \geq 0} \xi_{G_2}(n)q^n = 1 + 2q + 6q^2 + 28q^3 + 189q^4 + 1680q^5 + \cdots.$$ 

Then we have such congruences as

$$\xi_{G_1}(5^r n - 1) \equiv 0 \pmod{5^r},$$

$$\xi_{G_1}(7^r n - 1) \equiv 0 \pmod{7^r},$$

$$\xi_{G_1}(13^r n - \beta) \equiv 0 \pmod{13^r}$$

for $\beta \in \{1, 2, 3, 4\}$, and
\[ \xi_{G_2}(7^n \cdot n - 1) \equiv 0 \pmod{7^n}, \]
\[ \xi_{G_2}(11^n \cdot n - 1) \equiv 0 \pmod{11^n}. \]

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Received: 30 November 2018.  
Accepted: 9 April 2019.