The spectrum of states with one current acting on the adjoint vacuum of massless $QCD_2$

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Abstract

We consider a “one current” state, which is obtained by the application of a color current on the “adjoint” vacuum. This is done in $QCD_2$, with the underlying quarks in the fundamental representation. The quarks are taken to be massless, in which case the theory on the light-front can be “currentized”, namely, formulated in terms of currents only. The adjoint vacuum is shown to be the application of a current derivative, at zero momentum, on the singlet vacuum. We apply the operator $M^2 = 2P^+P^-$ on these states and find that in general they are not eigenstates of $M^2$ apart from the large $N_f$ limit. Problems with infra-red regularizations are pointed out. We discuss the fermionic structure of these states.

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1 Introduction

The formulation of two dimensional massless QCD in terms of only colored currents (‘currentization’) turns out to be very natural once the system is quantized on the light-front. Both the momentum and the Hamiltonian, and hence also \( M^2 \), are expressed in terms of the light-cone colored currents. In fact only the left (or right) currents are needed [1, 2]. The currentization was shown to hold for multiflavor fundamental quarks, and adjoint quarks [2]. In fact, it can be applied whenever the free fermions energy momentum tensor can be written in term of a Sugawara form. Obviously, the light-cone momentum and Hamiltonian of any CFT that posses an affine Lie algebra and is coupled to non Abelian gauge field associated with the same algebra, can also be described in terms of holomorphic currents [1].

In this approach to QCD, that has been studied in [1, 2, 3], one constructs the Fock space of physical states by applying current creation operators on the vacuum of the system. The lowest physical states constructed by applying current creation operators on the singlet vacuum, are those built from two currents. These type of states were introduced in [2]. In [3] a detailed analysis of their spectra was done. This included writing down a ’t Hooft like equation for the wave function of these two currents states and solving it for the lowest massive state. An excellent agreement with the DLCQ results [4, 5, 6] was found. In addition, the ’t Hooft model and the large \( N_f \) limit spectra were re-derived.

However, it turns out that one can also construct states by using only one current creation operator when applied on the adjoint vacuum [2]. The adjoint vacuum is derived by acting on the singlet vacuum with fermionic zero modes. In the case of adjoint fermions by a single adjoint zero mode and for fundamental fermions by quark anti-quark zero modes. In a scheme where only currents are being used one should be able to express the adjoint vacuum also in terms of currents.

The goal of the present paper is to currentize the adjoint vacuum, compute
the Mass\(^2\) of this state, to check whether there is a domain of parameters where it is an eigenstate and to compare it to the other massive states. This will be carried out in massless QCD with fundamental quarks with \(N_f\) flavors.

The outcome of our analysis is that in the large \(N_f\) limit the one current state applied on the adjoint vacuum is indeed an eigenstate of the mass operator with a mass of \(\sqrt{\frac{e^2 N_f}{\pi}}\). In the large \(N_c\) limit this is not the case and in fact when acting with \(M^2\) on this state, the dominant daughter state is a two current state. We have further analyzed the fermionic structure of these states, especially in the large \(N_c\) limit, to connect to the 't Hooft analysis [7].

On route to these results we were facing two technical obstacles. The first is the normalization of the states built by multi-current creation operators, and the second is IR divergences. In certain parts of the computations we were able to regularize the IR divergences, but in others we left it as an open problem. Assuming that a regularization scheme can be found, we have fully determined the normalizations of the relevant states.

The paper is organized as follows. In section 2 the general setup of the currentization of massless QCD\(^2\) is written down. This includes the fermionic and bosonized action, the light cone gauge fixing, the light-front quantization, the momentum and Hamiltonian operators expressed in terms of holomorphic currents as well as the algebra of the currents. Section 3 is devoted to the construction of the one current states. The adjoint vacuum is derived by applying a current derivative, at zero momentum, on the singlet vacuum. The momentum eigenvalue of the one current state is then the momentum of the applied current on the adjoint vacuum. We act with \(M^2 = 2P^+P^-\) on the one current states in section 4. It is shown that in general these states are not eigenstates of \(M^2\). In the large \(N_f\) limit they are. In the large \(N_c\) limit the two current state is shown to be dominant. The description of the states in terms of their fermionic underlying structure is presented in section 5. We summarize the results and present a short discussion in section 6. In Appendix A we derive some useful formulas for computing traces over generators in the adjoint representation of \(SU(N)\), which appear
in the normalization of the two current state.

2 A brief review of massless QCD

The theory of QCD with multi-flavor massless fermions in the fundamental representation of SU($N_c$) is described by the following action

$$S = \int d^2 x \, \text{tr} \left( -\frac{1}{2e^2} F_{\mu\nu}^2 + i\bar{\Psi} \not{D} \Psi \right)$$

(1)

where $\Psi = \Psi^i_{\alpha}$, $i = 1 \ldots N_c$, $\alpha = 1 \ldots N_f$ and $D_{\mu} = \partial_{\mu} + iA_{\mu}$.

An alternative description is achieved by bosonizing the theory. The bosonization of multi-flavor massive QCD is complicated since one has to translate the fermions into bosonic variables which are group elements of $U(N_F \times N_c)$ [8]. However, for massless fermions one can use bosonization in the $SU(N_c) \times SU(N_f) \times U_B(1)$ scheme. This scheme is quite useful since it decouples the color and flavor degrees of freedom. The bosonized form of the action in this scheme is given by [9, 10]

$$S_b = N_f S_{WZW}(h) + N_c S_{WZW}(g) + \int d^2 x \, \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \int d^2 x \, \text{tr} \left( -\frac{1}{2e^2} F_{\mu\nu}^2 \right)$$

$$+ \frac{N_f}{2\pi} \int d^2 x \, \text{tr} \left( i h^i \partial_+ h A_- + i h \partial_- h^i A_+ - A_- h A_- h^i + A_+ A_- \right)$$

where $h \in SU(N_c)$, $g \in SU(N_f)$, $\phi$ is the bosonic field for the baryon number and $S_{WZW}$ stands for the Wess-Zumino-Witten action, which reads

$$S_{WZW}(g) = \frac{1}{8\pi} \int_{\Sigma} d^2 x \, \text{tr} \left( \partial_{\mu} g \partial^{\mu} g^{-1} \right) +$$

$$\frac{1}{12\pi} \int_{B} d^3 y e^{ijk} \, \text{tr} \left( g^{-1} \partial_i g \right) \left( g^{-1} \partial_j g \right) \left( g^{-1} \partial_k g \right)$$

$B$ is a three dimensional volume whose boundary is the two dimensional surface $\Sigma$, which in our case is the 1+1 Minkowsky space, of one space and one time.
As was advertised, the flavored sectors are indeed decoupled from the colored one. Moreover the former ones are entirely massless. Hence, since we are interested in the massive spectrum of the theory we will set aside the \( g \) and \( \phi \) fields and analyze only the colored field \( h \). In fact in [1] it is argued that there is a residual interaction of the zero modes of the \( g, h \) and \( \phi \) fields, but this will not be important to our discussion.

Next we choose the light cone gauge \( A_- = 0 \), and quantize the system on the light-front. Upon integrating \( A_+ \) we find the following non local action
\[
S = N_f S_{WZW}(h) - \frac{1}{2} e^2 \int d^2 x \, \text{tr} \left( \frac{1}{\partial_-} J^+ \right)^2,
\]
where \( J^+ = \frac{iN_f}{2\pi} h \partial_- h^\dagger \). The light-front momentum and Hamiltonian take now simplified forms. The momentum
\[
P^+ = \frac{1}{N_c + N_f} \int dx^- : J^a(x^-, x^+ = 0) J^a(x^-, x^+ = 0) : \quad (3)
\]

namely, the Sugawara form, and the energy
\[
P^- = -\frac{e^2}{2\pi} \int dx^- : J^a(x^-, x^+ = 0) \frac{1}{\partial_-^2} J^a(x^-, x^+ = 0) :
\]
where \( J = \sqrt{\pi} J^+ \). In the light-front the mass of any state is given by the eigenvalue equation
\[
2P^+ P^- |\psi\rangle = M^2 |\psi\rangle.
\]

Three remarks are now in order:

- It is clear that we have indeed fully \textit{currentized} the system in the sense that the equations that determines the spectrum are entirely expressed in terms of currents.

- Moreover both \( P^+ \) and \( P^- \) depend only on \( J^+ \) and not on \( J^- \) (obviously we could have a dependence only on \( J^- \) by choosing the \( A_+ = 0 \) gauge and the space coordinate to be \( x^+ \)).
Furthermore, it is clear from the derivation of the above that the only condition for currentizing a theory of fermions coupled to non Abelian gauge fields, is that $T^{++}$ of the free fermionic theory could be rewritten in terms of a Sugawara form. This question was analyzed in [11]. In particular in [12, 10] it was shown that this can be done for adjoint fermions and in [13] for fermions in the symmetric and antisymmetric “two box” representations.

Next we write $P^+$ and $P^-$ in terms of the Fourier transform of $J(x^-)$ defined by $J(p^+) = \int \frac{d^4x}{(2\pi)^4} e^{-ip^+x^-} J(x^-,x^+ = 0)$. Normal ordering in the expression of $P^+$ and $P^-$ are naturally with respect to $p$, where $p < 0$ denotes a creation operator. To simplify the notation we write from now on $p$ instead of $p^+$. In terms of these variables the momenta generators are

$$P^+ = \frac{2}{N_f + N_c} \int_0^\infty dp J^b(-p) J^b(p)$$

$$P^- = \frac{e^2}{\pi} \int_0^\infty dp \Phi(p) J^b(-p) J^b(p)$$

$$\Phi(p) = \frac{1}{2} \left( \frac{1}{(p + i\epsilon)^2} + \frac{1}{(p - i\epsilon)^2} \right),$$

$\epsilon$ is (as usual) a small parameter used to regulate the Fourier transform of $\int d^2x Tr \left( \frac{1}{2} J^2 \right)$. $\Phi(p)$ is $(-\frac{\sqrt{\pi}}{2})$ times the Fourier transform of the ‘potential’ $|x - y|$ between the currents.  

An important step in computing the eigenvalues of $M^2 = 2P^+P^-$ is interchanging the locations of current creation and annihilation operators. For this purpose we make use of the level $N_f$, $SU(N_c)$ affine Lie algebra that the light-cone currents $J^a(p)$ obey

$$[J^a(p), J^b(p')] = \frac{1}{2} N_f p \delta^{ab} \delta(p + p') + i f^{abc} J^c(p + p').$$

\footnote{Note the extra 2 factor as compared with eq.(7) of [3], due to an error in the latter.}
The Fock space of physical states is constructed as follows. First, the vacuum \( |0, R \rangle \) is defined as usual by the annihilation property

\[
\forall p > 0, \ J(p) \ |0, R \rangle = 0 \tag{10}
\]

where \( R \) is an “allowed” representation depending on the level. Physical states are built by applying the current creation operators on the vacuum, \( |\psi \rangle = \text{tr} \ J(-p_1) \ldots J(-p_n) |0, R \rangle \). Note that this basis is not orthogonal.

3 "The Adjoint Vacuum and its One-Current State"

We would like to construct a state, which is obtained by the action of a current on the “adjoint vacuum”, in the color singlet combination.

This way we get a physical state, which is in a sense a “one current” state. Such states were considered before, in a different way, and the proof that they are really physical was not complete [14, 1, 2].

The ‘adjoint vacuum’ is created from the singlet vacuum by applying the adjoint zero mode [2, 3], which is taken as the limit \( \epsilon \to 0 \) of the product of a quark and anti-quark creation operators, each one at momentum \( \epsilon \).

Hence in our case

\[
|0, R \rangle = \lim_{\epsilon \to 0} b^\dagger_i \beta(\epsilon) d^\dagger_j \beta(\epsilon) |0 \rangle \tag{11}
\]

where \( b^\dagger_i \beta \) and \( d^\dagger_j \beta \) are the creation operators of a quark and anti-quark respectively (see section 5). We can represent the action of the above adjoint zero mode on the vacuum by the derivative of a creation current taken at zero momentum. Differentiating the current with respect to \( k \), and acting on the vacuum we get:

\[
J'_j(k) |0 \rangle_{k=0^-} = \sqrt{\frac{\pi}{2}} \frac{d}{dk} \int_0^\infty dp \int_0^\infty dq \delta(k + p + q) b^\dagger_i \beta(p) d^\dagger_j \beta(q) |0 \rangle_{k=0^-}
\]
\begin{align*}
= -\sqrt{\frac{\pi}{2}} b^\dagger_\beta (\epsilon) d^\dagger_\beta (\epsilon) \left| 0 \right>_{\epsilon \to 0}.
\end{align*}

As the currents are traceless, we have to subtract the trace part for \( i = j \). The latter can be neglected for large \( N_c \).

For any given \( N_c \), our results that follow are the same also after the trace is subtracted.

The adjoint vacuum we have is a bosonic one, constructed from fermion-antifermion zero modes, and as we show can be written as the derivative of the current acting on the singlet vacuum. In the case of adjoint fermions there is another adjoint vacuum, a fermionic one, obtained by applying the adjoint fermion zero mode on the singlet vacuum.

As we showed already, \((J^a)'(0)\left| 0 \right>\) represents the adjoint zero mode \(b^\dagger(0)d^\dagger(0)\left| 0 \right>\) (indices suppressed), for any \( N_f \) and \( N_c \), so in particular also for \( N_f = N_c \). But in the latter case the theory is equivalent to that of adjoint fermions, as stated by the equivalence theorem in [1]. As also stated there, states built on the adjoint vacuum above, cannot be distinguished from those built on the fermionic adjoint vacuum, the latter obtained by applying the adjoint fermions on the singlet vacuum.

The adjoint bosonic vacuum can have also flavor quantum numbers, when the fermion have flavor. This does not change our results about the mass of the new state we have. Our ”currentball” will have flavor too in such a case. In our scheme of bosonization, which is the ”product scheme”, especially convenient when the quarks are massless, the flavor sector is decoupled, and so the flavor multiplets are given by the action of flavor zero modes, not changing the mass values [see next Section].

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Let us introduce the notation

\[ Z^a = -\sqrt{\frac{2}{\pi}}(J^a)'(0). \]

The state we have in mind is

\[ \left| k \right> = J^b(-k)Z^b\left| 0 \right>. \]
This state is obviously a global color singlet, but in our Light Cone gauge \( A_\perp = 0 \) it is also a local color singlet, as the appropriate line integral vanishes.

Now
\[
\sqrt{\frac{\pi}{2}} \left[ J^a(p), Z^b \right] = \frac{1}{2} N_f \delta^{ab} \delta(p) - i f^{abc} (J^c)'(p),
\]
and thus, for \( p > 0 \)
\[
J^a(p) Z^b |0\rangle = Z^b J^a(p) |0\rangle - i \sqrt{\frac{2}{\pi}} f^{abc} (J^c)'(p) |0\rangle = 0.
\]
Hence the state \( Z^b |0\rangle \) is annihilated by all the annihilation currents, and so it is indeed a colored vacuum.

Using
\[
\left[ P^+, J^b(-k) \right] = k J^b(-k)
\]
we get that our state \( |k\rangle \) is indeed of momentum \( k \).

Note that when quantizing on a circle of radius \( R \), the adjoint vacuum would be an eigenstate of \( P^+ \) with eigenvalue \( N_c/R \) [15]. As we work in the continuum limit, we get zero.

4 The action of \( M^2 \) on the one current states

First, we evaluate the commutator of \( P^- \) with a creation current

\[
\left[ \int_0^\infty dp \phi(p) J^a(-p) J^a(p), J^b(-k) \right] =
\frac{1}{2} N_f \frac{1}{k} J^b(-k) + i f^{abc} \int_0^k dp \phi(p) J^a(-p) J^c(p - k)
\]
\[+ i f^{abc} \int_k^\infty dp \left( \phi(p) - \phi(p - k) \right) J^a(-p) J^c(p - k) \]

note that in \( P^- \) (and in \( P^+ \)) we ignore contributions from zero-mode states, that is, we cut the integrals at \( \epsilon \), and then take the limit.

As \( P^+ \) and \( P^- \) act on a singlet state, and as \( J^a(0) \), being the color charge, annihilates this state, the contribution from the zero modes in both \( P^+ \) and
$P^-$ is zero. Therefore it is legitimate to cut the integration limit above the zero mode and then take the cutoff to zero, as we have done. Note also that the integral of $\phi(p)$ around $p = 0$ is finite, and in fact zero when integrating over the whole line, therefore there are no divergences when we take the limit.

It is important, however, to remember that the zero mode does contribute when we act upon non singlet states, like the adjoint vacuum $Z^b|0\rangle$ itself. As mentioned above, when quantizing on a circle of radius $R$ one gets that $P^+$ is of order $1/R$. And then, with $P^-$ of order $e^2 R$, $M^2$ is $R$ independent, and so remains finite in the continuum limit. However, this is subtle, as $P^-$ becomes IR divergent in the continuum and needs to be regularized. This subtlety does not affect our calculation as we work in the singlet sector only.

Actually, the argument connected with $P^-$ acting on singlets should be somewhat sharpened. Let us put the lower limit at $\epsilon$, and let it go to zero at the end. Then $J(\epsilon)$, when acting on a singlet, would go like $\epsilon^2$. We have two currents in the integral, so we get $\epsilon^2$. But then we have $1/\epsilon^2$ from the denominator, so a finite integrant. But the region of integration for is of order $\epsilon$, so indeed total contribution goes to zero.

Now apply $P^-$ on our state

$$P^− J^b(−k)Z^b|0\rangle = [P^−, J^b(−k)] Z^b|0\rangle \quad (15)$$

as the Hamiltonian annihilates the color vacuum as well.

Using the commutator of the Hamiltonian with a current, which we evaluated before, we get

$$\frac{\pi}{e^2} P^− J^b(−k)Z^b|0\rangle = \frac{1}{2} N_f \frac{1}{k} J^b(−k)Z^b|0\rangle$$

$$+ \ i f^{abc} \int_0^k dp\phi(p)J^a(−p)J^c(p−k)Z^b|0\rangle.$$  

Note that we used the fact that annihilation currents do annihilate also the colored vacuum.

Let us apply the operator $M^2$ to our one-current state
\[ M^2 J^b(-k)Z^b|0\rangle = 2P^-P^+ J^b(-k)Z^b|0\rangle = 2kP^- J^b(-k)Z^b|0\rangle = \\
\left(\frac{e^2 N_f}{\pi}\right) J^b(-k)Z^b|0\rangle + \left(\frac{2e^2}{\pi}k\right) \int_0^k dp \phi(p) J^a(-p)J^c(p - k)Z^b|0\rangle. \] (16)

So it seems that, in the large \( N_f \) limit, the state \( J^b(-k)Z^b|0\rangle \) is an (approximate) eigenstate, with eigenvalue \( e^2 N_f \pi \).

To see the exact dependence of the two terms in the equation above (the one and two current states) on \( N_f \) and \( N_c \), we should normalize them. The normalization of \( J^b(-k)Z^b|0\rangle \) is

\[ \langle 0| Z^a J^a(k) J^b(-k)Z^b |0\rangle = \langle 0| Z^a [J^a(k), J^b(-k)] Z^b |0\rangle = \\
\frac{1}{2} N_f k \delta(0) \langle 0| Z^b Z^b |0\rangle + i f^{abc} \langle 0| Z^a J^c(0)Z^b |0\rangle = \\
\frac{1}{2} N_f k \delta(0) \langle 0| Z^b Z^b |0\rangle + N_c \langle 0| Z^b Z^b |0\rangle \] (17)

The second term in the last line can be neglected compared to the first, as it is a constant to be compared with \( \delta(0) \) [the space volume divided by \( 2\pi \)].

Now \( \langle 0| Z^b Z^b |0\rangle = (N_c^2 - 1) \langle 0| Z^1 Z^1 |0\rangle \) and the factor \( k \delta(0) \) is the normalization of a plane wave of momentum \( k \).

So the normalized state is, for \( N_c \gg 1 \),

\[ \frac{1}{N_c \sqrt{2N_f}} J^b(-k)Z^b|0\rangle \] (18)

relative to \( \langle 0| Z^1 Z^1 |0\rangle \).

The normalization of the second term is more complicated. A lengthy but straightforward calculation gives

\[ \left\| \left(i f^{def} k \int_0^k dq \Phi(q) J^d(-q)J^f(q - k) \right) Z^c |0\rangle \right\|^2 = \]
\[
N_c \left( N_c^2 - 1 \right) \left( \frac{1}{2} N_f \right)^2 k \delta(0) \langle 0 | Z^1 Z^1 | 0 \rangle \times 
\]
\[
k \left( \int_0^k dp (k - p) \Phi(p) \left( \Phi(p) - \Phi(k - p) \right) - \frac{N_c}{N_f} \int_0^k dp \Phi(p) \int_0^{k - p} dq \Phi(q) \right)
\]

We have written only the terms proportional to \( \delta(0) \) as they are the dominant ones. Useful formulae for the evaluation, involving sums of products of structure functions of \( SU(N) \), are given in the Appendix.

The various momentum integrals (including the ones for the non dominant terms) are divergent for \( \epsilon \to 0 \), thus they should be regulated. We leave this problem for now, and assume henceforth that they are regulated and finite. For simplicity the integrals (including the factor \( k \)) appearing in the two dominant terms will be notated \( R_1 \) and \( -R_2 \) in the following expressions. Note that we have \( \frac{1}{\epsilon^2} \) and \( \frac{1}{\epsilon} \) divergences and also \( \ln(\frac{k^2}{\epsilon^2}) \). It seems that these are cancelled in \( R_2 \).

Define now the normalized states

\[
| S_1 \rangle = C_1 \left( J^b (-k) Z^b | 0 \rangle \right)
\]
\[
| S_2 \rangle = i C_2 k f^{abc} \int_0^k dp \Phi(p) J^a (-p) J^c (p - k) Z^b | 0 \rangle
\]

where

\[
C_1 = \frac{1}{N_c \sqrt{\frac{1}{2} N_f}}, \quad C_2 = \frac{N_f^{2 N_f}}{\sqrt{R_1 + R_2 N_f}}.
\]

The mass eigenvalue equation of the normalized states is

\[
M^2 | S_1 \rangle = \frac{e^2}{\pi} N_f | S_1 \rangle + \frac{e^2 N_c}{\pi} \sqrt{2} \sqrt{\frac{N_f}{N_c} + R_2} | S_2 \rangle,
\]

thus, we see that in the large flavor limit, our state \( | S_1 \rangle \) is an eigenstate with mass

\[
M = \sqrt{\frac{e^2 N_f}{2 \pi}}.
\]
In the large color limit, however, we actually get that the second term dominates by a factor of $N_c$. Moreover, while the first term goes to zero in the large $N_c$ limit, due to the factor of $e^2$, the second term survives in that limit.

5 Fermionic structure of the states

The analysis of the spectrum of $QCD_2$ in the fermionic formulation was addressed in [16, 17, 18, 19, 20]. Here we do not intend to perform the full process of determining the spectrum of the theory with fundamental fermions but rather only to find out the fermion structure of our state. To this end we write it as

$$J_i^j(-k)b_{ji}^\beta(\epsilon)d_{ij}^{\beta\epsilon}(\epsilon)|0\rangle$$

(25)

for $\epsilon > 0$ tending to zero and where

$$J_i^j(-k) = \frac{1}{2\pi}\int_0^\infty dp[b_{ji}^\beta(p+k)b_{ji}^\beta(p) + \theta(k-p)b_{ji}^\beta(k-p)d_{ij}^{\beta\epsilon}(p) - d_{ij}^{\beta\epsilon}(p+k)d_{ji}^\epsilon(p)].$$

(26)

So the 4-quarks part in (25) has a coefficient which is independent of $N_c$.

As for the 2-quark part, it involves an anti-commutator of creation with annihilation, yielding a state which is a combination of

$$b_{ji}^\beta(k)d_{ij}^{\beta\epsilon}(\epsilon) \text{ and } b_{ji}^\beta(\epsilon)d_{ij}^{\beta\epsilon}(k)$$

(27)

with a coefficient that is proportional to $N_c$. Thus for large $N_c$, we have a quark-antiquark combination of momenta $(k, 0)$ and $(0, k)$.

As ’t Hooft found all meson states for large $N_c$, and each has a well defined momentum distribution [7], it is clear that our state is not a mass eigenstate of large $N_c$. This is of course part of our explicit calculation in the previous section.
6 Discussion

In this note we have continued the investigation of two dimensional massless multi flavor \(QCD\) in its “currentized” form. The currentization formalism serves as a universal description [1] of systems of matter fields coupled to non Abelian gauge fields. In particular fermions in various representations and WZW actions at any level. The currentization also sets a common framework for the dual pairs of fermionic theories and their bosonized partners (or vice versa). The hope has been of course that this method will also be a useful tool to solve those \(QCD_2\) systems.

The concrete question that has been addressed in this paper is the spectrum of states constructed by applying a single current creation operator on the adjoint vacuum. For that purpose we had to write down the adjoint vacuum in terms of currents. We proved that it is given in terms of the derivative with respect to \(k\), at \(k = 0\), of the current acting on the singlet vacuum. We then applied the operator \(M^2\) on one current states built on the adjoint vacuum. We found that in general, and in particular also in the large \(N_f\) limit they are eigenstates of \(M^2\). However, in the large \(N_f\) limit we have indeed found an eigenstate of \(M^2\) with exactly this mass. The state we have found is a color singlet. In fact it is easy to see that in the large \(N_f\) limit, there are \(N_c^2 - 1\) colored eigensates of \(M^2\) with the same mass. This can be interpreted as follows. In the large \(N_f\) \(QCD_2\) is transfered into a set of \(N_c^2 - 1\) Abelian systems. This can be easily seen in (9) where in the large \(N_f\) limit the non Abelian term of the commutator is sub-leading. Now in \(QED_2\) it is well known that the Schwinger mechanism yields a massive state of mass \(e\sqrt{\frac{N_f}{\pi}}\). This Schwinger state is often considered...
as a bound state of an electron-positron. In the large $N_f$ the $M^2$ eigenstates are therefore just the Schwinger states appearing in a multiplicity of $N_c^2 - 1$.

For any finite $N_f$ only the singlet state remains in the spectrum, of course. However, at least formally, in the infinite $N_f$ limit, we get that the current algebra is like of $N_c^2 - 1$ currents, each one a U(1).

In [14], using a different currentization approach to $QCD_2$, it was conjectured that there should be also non-Abelian Schwinger mechanism that will yield massive physical states in the color “adjoint” representation, the analogs of the massive particle of the Abelian Schwinger model. Since we have detected such eigenstates of $M^2$ in the large $N_f$ limit, and we found that without this limit they are not eigenstates, we conclude that if the non-Abelian Schwinger particles indeed exist, they mix with other states constructed by applying more than one current creation operator on a vacuum state.

Note, that when having adjoint fermions, there is also a fermionic adjoint vacuum, as we mentioned already in Section 3. Trittmann [21] discussed this case.

There are certain open questions that follow from the analysis of this paper:

- As mentioned in section 4 the computation of the spectrum of the states requires a regularization prescription that we have only partially found.

- The “ultimate” goal of this research work is to diagonalize the “currentball” states created by applying any number of current creation operators on the various vacua of the theory.

- On top of everything stands the question what have we learned from the currentization procedure and from the two dimensional spectrum of states about four dimensional $QCD$. In particular a challenging question is to investigate the possibility of a Schwinger like mechanism also in four dimensions.
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7 Appendix

The generators $T^a$ of $SU(N)$, in the adjoint representation, are related to the structure constants $f^{abc}$ as

$$(T^a)_{bc} = -if^{abc}.$$ 

Thus

$$f^{abc}f^{abd} = Tr(T^cT^d) = N\delta^{cd}$$

and

$$f^{abc}f^{a'd'}f^{ad'd'}f^{cc'd'} = Tr(T^{b'd}T^{d'b}T^{d'b}T^{d'b}) =$$

$$if^{bde}Tr(T^eT^{b'd}T^{d'}) + Tr(T^{b'd}T^{d'b}T^{d'b}T^{d']} = \frac{1}{2}N^2(N^2 - 1).$$

where we used

$$\sum_a T^aT^a = NI_{adj}$$

with $I_{adj}$ the unit matrix in the adjoint representation.

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A historical survey and more complete list of references can be found here.

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