On the lower bound of energy functional $E_1 (I)$
—a stability theorem on the Kähler Ricci flow

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In memory of the great mathematician S. S. Chern

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1 Introduction and main results

This is the first of a serious papers aiming to study the lower bound of the energy functional $E_1$ and its relation to the convergence of the Kähler Ricci flow as well as the relation to the existence of Kähler Einstein metrics. The Kähler Ricci flow is known for exists for all times and converges to Kähler Einstein metrics when the first Chern class was negative or zero [4] [5]. When the first Chern class is positive, there is very few known results on global convergence. In the case of positive bisectional curvature, the convergence of the flow is first proved in [7] [8] where the energy functional $E_1$ plays a key role. After the important work of Perelman, there are active researches in this direction and many good works appear, for instance, [9] [20] (more recently, [19]) and references therein.

In the present paper, we prove a stability theorem of the Kähler Ricci flow near the infimum of $E_1$ under the assumption that the initial metric has Ricci $> -1$ and $|\text{Riem}|$ bounded. The underlying moral is: if a Kähler metric is sufficiently closed to a Kähler Einstein metric, then the Kähler Ricci flow shall converges to it. The present work should be viewed as first step in a more ambitious program of deriving the existence of Kähler Einstein metric with arbitrary energy level, provided that this energy functional has a uniform lower bound in this Kähler class.

This stability theorem can be alternatively viewed as generalization of the well known pointwise curvature-pinching theorem proved by method of Ricci flow in the 1980s. The beautiful work of R. Hamilton [10] proved by Ricci flow, that any closed 3-manifold of positive Ricci curvature is diffeomorphic to a spherical space form. This result was generalized by several mathematicians (c.f., [12], [15] [18]) to higher dimensional manifold: if the Riemannian curvature satisfies certain pointwise pinching condition, then the flow will preserve this pinching condition and converges to a spherical space form up re-scaling of the evolved metrics. One wonders if a similar result in the Kähler setting is feasible. Unfortunately, the corresponding pinching condition doesn’t hold in Kähler setting in general. The present work can be viewed as a generalization in this direction with one crucial difference: We didn’t prove that the pinching condition imposed initially is preserved over time. Instead, we prove that, even though the pinching condition might lost immediately after the flow starts, it will be recovered after a fixed period of time (the length of this period is determined by the geometrical condition of the initial metric.). This periodic re-visit of the pinching condition is enough to give us control of $L^\infty$ norm of the curvature of the evolving metrics. In many cases, this will lead to the convergence of the Kähler Ricci flow.

More specifically, let $(M, \omega_0)$ be a polarized $n-$dimensional compact Kähler manifold with positive first Chern class where $[\omega_0]$ is the canonical Kähler class.

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1. This is first introduced in [7]. For the convenience of readers, we will give precise definition of $E_1$ in Section 2 (cf. Definition 9 below).
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2
We always normalize the Kähler class so that the canonical class is the Ricci class. One of the main theorems of this note is to prove the following stability theorem:

**Theorem 1.** For any \( \delta, \Lambda > 0 \), there exists a small positive constant \( \epsilon(\delta, \Lambda) > 0 \) such that if the subspace \( A(\delta, \Lambda, \epsilon) \) of Kähler metrics in \( [\omega] \)

\[
\{ \omega_g \in [\omega_0] \mid \text{Ric}(g) > -1 + \delta, \text{Riem}(g) < \Lambda, E_1(g) \leq \inf_{\text{Ric}(\omega) \geq -1} E_1(\omega) + \epsilon \}
\]

is non-empty, then there exists a Kähler Einstein metrics in this canonical Kähler class if the following integral condition hold

\[
[C_1(M)]^2 \cdot [\omega]^{n-2} - \frac{2(n+1)}{n} [C_2(M)] \cdot [\omega]^{n-1} = 0.
\]

Moreover, for any metric \( g_1 \in A(\delta, \Lambda, \epsilon) \), the Kähler Ricci flow will deform it exponentially fast to a Kähler-Einstein metric in the limit.

**Remark 2.** The reason to assume a \( L^\infty \) bound in the Riemannian curvature of the initial metric is to ensure the inequality \( \text{Ric} > -1 \) is preserved for some fixed amount of time. One should be able to replace the \( L^\infty \) bound on the bisectional curvature by some \( L^p \) estimates for some \( p > \frac{n+2}{2} \). Condition 1.2 is introduced to control the norm of sectional curvature. This condition is removed in Theorem 6 below.

**Remark 3.** In Theorem 1, we can estimate \( \epsilon(\Lambda, \delta) \) explicitly:

\[
\epsilon(\delta, \Lambda) \leq \left( \frac{1}{\Lambda} \right)^{2n} \delta \cdot \epsilon_0(n)^2.
\]

Here \( \epsilon_0(n) \) is some universal constant which depends only on the dimension (cf. Lemma 18). This estimate is not optimal.

This naturally leads to the following question.

**Question 4.** In any canonical Kähler class \( \omega_0 \) where the first Chern class is positive, we can define the following invariant

\[
\beta(M, \omega) = \inf_{\varphi \in \mathcal{P}(M, [\omega])} \max_{x \in M} |\text{Ric}(\omega_\varphi) - \omega_\varphi|_x.
\]

Then this defined a holomorphic invariant which depends only on the underlying complex structure. Apparently, if there is Kähler Einstein metric in \( [\omega_0] \), then this invariant vanishes. What about the manifold without Kähler Einstein metric but this invariant vanishes? How does this invariant change with respect to the deformation of complex structure? An obvious guess is that this is related to the stability of the tangent bundle. When the first Chern class is negative, Aubin, Yau’s solution of Calabi conjecture implies that this invariant (defined accordingly) always vanishes.
Theorem 5. Conditions as stated in Theorem 4 except that the assumption 1.2 is replaced by the following: there exists a constant $C(p)$ such that

$$\int_M |\text{Riem}(g(t))|^p g(t) \, d\text{vol}_g \leq C, \quad \text{for fixed } p > n + 1. \quad (1.4)$$

Here $g(t)$ is the evolved Kähler metrics along the Kähler Ricci flow initiated from $g(0)$. Then the Kähler Ricci flow is non-singular (i.e., the bisectional curvature of $g(t)$ are uniformly bounded). Moreover, the $\text{Ric}(g(t)) - g(t)$ converges to 0 uniformly as $t \to \infty$. If in additional, we assume that the underlying complex structure is stable in the sense that no complex structure in the closure of its orbit of diffeomorphism group contains larger holomorphic automorphism group, then the flow converges to a unique Kähler Einstein metric in the original complex structure exponentially fast.

A natural question is whether condition (1.4) will actually occur?

Theorem 6. Conditions as in Theorem 4 except the assumption 1.2 is removed. In additional, we assume $M$ is complex surface and $X \neq 0$ is a holomorphic vector field in $M$ such that $L_{\text{Im}(X)} g_0 = 0$. Then, the Kähler Ricci flow converges to a Kähler Einstein metrics in the limit exponentially fast.

Remark 7. Tian's solution [23] of Calabi conjecture in complex surface with positive first Chern class certainly include this as a special case. This might be interesting since it demonstrates how to obtain inequality (1.4) in absence of topological condition 1.2.

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2 Basic Kähler geometry

2.1 Setup of notations

Let $M$ be an $n$-dimensional compact Kähler manifold. A Kähler metric can be given by its Kähler form $\omega$ on $M$. In local coordinates $z_1, \cdots, z_n$, this $\omega$ is of the form

$$\omega = \sqrt{-1} \sum_{i,j=1}^n g_{\bar{i}j} \, dz^i \wedge d\bar{z}^j > 0,$$

where $\{g_{\bar{i}j}\}$ is a positive definite Hermitian matrix function. The Kähler condition requires that $\omega$ is a closed positive $(1,1)$-form. In other words, the following holds

$$\frac{\partial g_{\bar{i}j}}{\partial z^k} = \frac{\partial g_{k\bar{j}}}{\partial z^i} \quad \text{and} \quad \frac{\partial g_{\bar{i}k}}{\partial z^j} = \frac{\partial g_{\bar{j}k}}{\partial z^i} \quad \forall \, i,j,k = 1,2,\cdots,n.$$
The Kähler metric corresponding to $\omega$ is given by

$$\sqrt{-1} \sum_{i=1}^{n} g_{\alpha \overline{\beta}} \, dz^\alpha \otimes d\overline{z}^\beta.$$ 

For simplicity, in the following, we will often denote by $\omega$ the corresponding Kähler metric. The Kähler class of $\omega$ is its cohomology class $[\omega]$ in $H^2(M, \mathbb{R})$. By the Hodge theorem, any other Kähler metric in the same Kähler class is of the form

$$\omega = \omega + \sqrt{-1} \sum_{i,j=1}^{n} \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} > 0$$

for some real valued function $\varphi$ on $M$. The functional space in which we are interested (often referred as the space of Kähler potentials) is

$$\mathcal{P}(M, \omega) = \{ \varphi \mid \omega = \omega + \sqrt{-1} \frac{\partial^2 \varphi}{\partial \varphi} > 0 \text{ on } M \}.$$ 

Given a Kähler metric $\omega$, its volume form is

$$\omega^n = \frac{1}{n!} (\sqrt{-1})^n \det \left( g_{i\overline{j}} \right) \, dz^1 \wedge d\overline{z}^1 \wedge \cdots \wedge dz^n \wedge d\overline{z}^n.$$ 

Its Christoffel symbols are given by

$$\Gamma^k_{ij} = \sum_{l=1}^{n} g^{kl} \frac{\partial g_{l\overline{j}}}{\partial z^i}, \quad \Gamma^k_{i\overline{j}} = \sum_{l=1}^{n} g^{kl} \frac{\partial g_{l\overline{i}}}{\partial z^j}, \quad \forall \, i, j, k = 1, 2, \cdots, n.$$ 

The curvature tensor is

$$R_{ijkl} = -\frac{\partial^2 g_{k\overline{l}}}{\partial z^i \partial \overline{z}^j} + \sum_{p,q=1}^{n} g^{pq} \frac{\partial g_{k\overline{q}}}{\partial z^i} \frac{\partial g_{l\overline{p}}}{\partial z^j}, \quad \forall \, i, j, k, l = 1, 2, \cdots, n.$$ 

We say that $\omega$ is of nonnegative bisectional curvature if

$$R_{ijkl} v^i \overline{w}^j w^k \overline{v}^l \geq 0$$

for all non-zero vectors $v$ and $w$ in the holomorphic tangent bundle of $M$. The bisectional curvature and the curvature tensor can be mutually determined. The Ricci curvature of $\omega$ is locally given by

$$R_{ij} = -\frac{\partial^2 \log \det (g_{i\overline{j}})}{\partial z^i \partial \overline{z}^j}.$$ 

So its Ricci curvature form is

$$\text{Ric}(\omega) = \sqrt{-1} \sum_{i,j=1}^{n} R_{ij} (\omega) \, dz^i \wedge d\overline{z}^j = -\sqrt{-1} \frac{\partial^2 \log \det (g_{i\overline{j}})}{\partial z^i \partial \overline{z}^j}.$$ 

It is a real, closed $(1,1)$-form. Recall that $[\omega]$ is called a canonical Kähler class if this Ricci form is cohomologous to $\lambda \omega$, for some constant $\lambda$. In our setting, we require $\lambda = 1$. 

5
2.2 The Kähler Ricci flow

Now we assume that the first Chern class $c_1(M)$ is positive. The normalized Ricci flow (c.f. [10] and [11]) on a Kähler manifold $M$ is of the form

$$\frac{\partial g_{ij}}{\partial t} = g_{ij} - R_{ij}, \quad \forall i, j = 1, 2, \cdots, n.$$  (2.1)

If we choose the initial Kähler metric $\omega$ with $c_1(M)$ as its Kähler class. The flow (2.1) preserves the Kähler class $\omega$. It follows that on the level of Kähler potentials, the Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega^n}{\omega^n} + \varphi - h_\omega,$$  (2.2)

where $h_\omega$ is defined by

$$\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} h_\omega, \quad \text{and} \quad \int_M (e^{h_\omega} - 1) \omega^n = 0.$$

Then the evolution equation for bisectional curvature is

$$\frac{\partial R_{ijkl}}{\partial t} = \triangle R_{ijkl} + R_{ijpq} R_{pqkl} - R_{ipq} R_{ipkl} + R_{ljk} R_{ipq} + R_{pj} R_{ijkl}$$

$$- \frac{1}{2} \left( R_{pkl} R_{ijkl} + R_{p} R_{ijkl} + R_{k} R_{ijkl} + R_{l} R_{ijkl} \right).$$  (2.3)

The evolution equation for Ricci curvature and scalar curvature are

$$\frac{\partial R_{ij}}{\partial t} = \triangle R_{ij} + R_{ijpq} R_{qp} - R_{ip} R_{pj},$$  (2.4)

$$\frac{\partial R}{\partial t} = \triangle R + R_{ij} R_{ij} - R.$$  (2.5)

As usual, the flow equation (2.1) or (2.2) is referred as the Kähler Ricci flow on $M$. It is proved by Cao [4], who followed Yau’s celebrated work [24], that the Kähler Ricci flow exists globally for any smooth initial Kähler metric. It was proved by S. Bando [1] in dimension 3 and N. Mok [17] in all dimension that positivity of the bisectional curvature is preserved under the flow. In [7] [8], we prove that the Kähler Ricci flow, in Kähler Einstein manifold, initiated from a metric with positive bisectional curvature converges to a Kähler Einstein metric with constant bisectional curvature.

2.3 The energy functional $E_1$

In this subsection, we introduce the new functionals $E_k = E_k^0 - J_k (k = 0, 1, 2, \cdots, n)$ where $E_k^0$ and $J_k$ are defined below.
Definition 8. For any $k = 0, 1, \ldots, n$, we define a functional $E^0_k$ on $\mathcal{P}(M, \omega)$ by
\[
E^0_{k, \omega}(\varphi) = \frac{1}{V} \int_M \left( \log \frac{\omega_{\varphi}^n}{\omega^n} - h_{\omega} \right) \left( \sum_{i=0}^{k} \text{Ric}(\omega_{\varphi})^i \wedge \omega^{k-i} \right) \wedge \omega_{\varphi}^{n-k} + c_k,
\]
where
\[
c_k = \frac{1}{V} \int_M h_{\omega} \left( \sum_{i=0}^{k} \text{Ric}(\omega)^i \wedge \omega^{k-i} \right) \wedge \omega^n.
\]

Definition 9. For each $k = 0, 1, 2, \ldots, n-1$, we will define $J_{k, \omega}$ as follows:
Let $\varphi(t)$ ($t \in [0, 1]$) be a path from 0 to $\varphi$ in $\mathcal{P}(M, \omega)$, we define
\[
J_{k, \omega}(\varphi) = -\frac{n-k}{V} \int_0^1 \int_M \frac{\partial \varphi}{\partial t} \left( \omega_{\varphi}^{k+1} - \omega^{k+1} \right) \wedge \omega_{\varphi}^{n-k-1} \wedge dt.
\]
Put $J_n = 0$ for convenience in notations.

It is straightforward to verify that the definition of $J_k(0 \leq k \leq n)$ is independent of path chosen.

Direct computations lead to

Theorem 10. For any $k = 0, 1, \ldots, n$, we have
\[
\frac{dE_k}{dt} = \frac{k + 1}{V} \int_M \Delta_{\varphi} \left( \frac{\partial \varphi}{\partial t} \right) \text{Ric}(\omega_{\varphi})^k \wedge \omega_{\varphi}^{n-k} - \frac{n-k}{V} \int_M \frac{\partial \varphi}{\partial t} \left( \text{Ric}(\omega_{\varphi})^{k+1} - \omega_{\varphi}^{k+1} \right) \wedge \omega_{\varphi}^{n-k-1}. \tag{2.6}
\]
Here $\{\varphi(t)\}$ is any path in $\mathcal{P}(M, \omega)$ and $V$ is the volume of Kähler metrics in this Kähler class $[\omega]$

Proposition 11. Along the Kähler Ricci flow where $\text{Ric} > -1$ is preserved, we have
\[
\frac{dE_k}{dt} \leq -\frac{k + 1}{V} \int_M (R(\omega_{\varphi}) - r) \text{Ric}(\omega_{\varphi})^k \wedge \omega_{\varphi}^{n-k}. \tag{2.7}
\]
When $k = 0, 1$, we have
\[
\frac{dE_0}{dt} = -\frac{n\sqrt{-1}}{V} \int_M \frac{\partial \varphi}{\partial t} \wedge \frac{\partial \varphi}{\partial t} \wedge \omega_{\varphi}^{n-1} \leq 0, \tag{2.8}
\]
\[
\frac{dE_1}{dt} \leq -2 \frac{1}{V} \int_M (R(\omega_{\varphi}) - r)^2 \omega_{\varphi}^n \leq 0. \tag{2.9}
\]

In particular, both $E_0$ and $E_1$ are decreasing along the Kähler Ricci flow.
3 Proof of Theorem 1

3.1 The average $L^2$ norm of the trace-less Ricci tensor

Lemma 12. Suppose that the curvature of $g_0$ satisfies the following condition

$$\begin{cases}
  |R_{ijkl}(g_0)| & \leq \Lambda > 0, \\
  R_{ij}(g_0) & \geq -1 + \delta > 0.
\end{cases}$$

(3.1)

Then, there exists a constant $T(\delta, \Lambda) > 0$, such that the following bound for the evolved Kähler metric $g(t)(0 \leq t \leq 6T)$

$$\begin{cases}
  |R_{ijkl}(g(t))| & \leq 2\Lambda > 0, \\
  R_{ij}(g(t)) & \geq -1 + \frac{\delta}{2} > 0.
\end{cases}$$

(3.2)

In fact, we can choose $6T$ to be

$$\left(\frac{1}{\Lambda}\right)^{2n} \delta \cdot \epsilon_0(n)^2.$$

Proof. We use $BR$ to denote the bisectional curvature, and $\ast$ to denote the multiplication and contraction of two curvature tensors. Then, the evolution equation 2.3 can be re-written schematically

$$\frac{\partial}{\partial t} BR = \Delta BR + BR \ast BR.$$

If we let $u = |R_{ijkl}|^2$, then

$$\frac{\partial}{\partial t} u \leq \Delta u + c |Riem| \cdot u = \Delta u + c \cdot u^{\frac{3}{2}}.$$

where $c$ is some universal constants (depends on flow equation algebraically). This implies that

$$\max_{M} |R_{ijkl}| \leq \frac{1}{\Lambda - 1 - t}.$$

For $0 \leq t \leq \frac{1}{4\Lambda}$, we have

$$\max_{M} |R_{ijkl}| \leq 2\Lambda.$$

Similarly, re-write the evolution equation 2.4 for the Ricci curvature schematically:

$$\frac{\partial}{\partial t} Ric = \Delta Ric + Ric \ast BR.$$

Applying the estimate of the bisectional curvature for $0 \leq t \leq 6T$, we have

$$\begin{cases}
  |R_{ijkl}(g_0)| & \leq 2\Lambda > 0 \\
  R_{ij}(g_0) & \geq -1 + \frac{\delta}{2} > 0.
\end{cases}$$
Here \( c(n) \) is some universal constant and we can re-choose \( 6T \) (may need to make it smaller if needed) as

\[
6T = \frac{\delta}{(\delta + \Lambda \cdot c(n)) \cdot \Lambda}.
\] (3.3)

Lemma 13. If

\[
E_1(g_0) \leq \inf_{\text{Ric}(\omega) \geq -1, \omega \in [\omega]} E_1(g) + \epsilon(\delta, \Lambda),
\]

then

\[
\frac{1}{6T} \int_0^{6T} \int_M |\text{Ric}(\omega_\varphi) - \omega_\varphi|^2 dt \leq \frac{\epsilon_0(n)^2}{2}.
\] (3.4)

Proof. For \( 0 \leq t \leq 6T \), we have \( \text{Ric}_{\omega(g(t))} + \omega(g(t)) \geq\frac{\delta}{2} \geq 0 \). By Theorem 10, we have

\[
\frac{dE_1}{dt} = \frac{2}{n} \int_M \Delta_\varphi \left( \frac{\partial \varphi}{\partial t} \right) \text{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}
- \frac{n-1}{n} \int_M \left( \text{Ric}(\omega_\varphi)^2 - \omega_\varphi^2 \right) \wedge \omega_\varphi^{n-2}
\]

\[
= \frac{2}{n} \int_M (n - R(\omega(g(t))))(R(\omega(g(t))) - n) \wedge \omega_\varphi^{n-1}
- \frac{n-1}{n} \int_M \sqrt{-1} \theta \frac{\partial \varphi}{\partial t} \wedge \bar{\theta} \frac{\partial \varphi}{\partial t} \wedge (\text{Ric}(\omega_\varphi) + \omega_\varphi) \wedge \omega_\varphi^{n-2}
\]

\[
\leq -2 \int_M (n - R(\omega(g(t))))^2 \omega_\varphi^n \leq 0.
\]

Consequently, we have

\[
2 \int_0^{6T} \int_M |\text{Ric}(\omega_\varphi) - \omega_\varphi|^2 \omega^n dt \leq -\int_0^{6T} \frac{dE_1(g(t))}{dt} dt
\]

\[
\leq E_1(g(0)) - E_1(g(6T))
\]

\[
\leq E_1(g(0)) - \inf_{\omega \in [\omega]} E(g)
\]

\[
\leq \epsilon(\Lambda, \delta).
\]

If

\[
\epsilon(\Lambda, \delta) \leq \frac{1}{2} \epsilon_0(n)^2 \cdot \frac{\delta}{(\delta + \Lambda c(n)) \cdot \Lambda},
\]

and the definition of \( 6T \) (cf. equation 3.3), then

\[
\frac{1}{6T} \int_0^{6T} dt \int_M |\text{Ric}(\omega_\varphi) - \omega_\varphi|^2 d\text{vol}_{g(t)} \leq \frac{\epsilon_0(n)^2}{2}.
\]
3.2 Estimates of Sobolev and Poincare constants

The content of this subsection can be founded in [8]. We reproduce it here for the convenience of readers. In this section, we will prove that for any Kähler metric in the canonical Kähler class, if the scalar curvature is close enough to a constant in $L^2$ sense and if the Ricci curvature is non-negative, then there exists a uniform upper bound for both the Poincaré constant and the Sobolev constant. We first follow an approach taken by C. Sprouse [22] to obtain a uniform upper bound on the diameter.

In [6], J. Cheeger and T. Colding proved an interesting and useful inequality which converts integral estimates along geodesic to integral estimates on the whole manifold. In this section, we assume $m = \dim(M)$.

**Lemma 14.** [22] Let $A_1$, $A_2$ and $W$ be open subsets of $M$ such that $A_1$, $A_2 \subset W$, and all minimal geodesics $r_{x,y}$ from $x \in A_1$ to $y \in A_2$ lie in $W$. Let $f$ be any non-negative function. Then

$$
\int_{A_1 \times A_2} \int_{r_{x,y}} f(r(s)) \, ds \, d\text{vol}_{A_1 \times A_2} 
\leq C(m, k, \mathfrak{R})(\text{diam}(A_2)\text{vol}(A_1) + \text{diam}(A_1)\text{vol}(A_2)) \int_W f \, d\text{vol},
$$

where for $k \leq 0$,

$$
C(m, k, \mathfrak{R}) = \frac{\text{area}(\partial B_k(x, \mathfrak{R}))}{\text{area}(\partial B_k(x, \frac{\mathfrak{R}}{2}))},
$$

(3.5)

$$
\mathfrak{R} \geq \sup\{d(x, y) \mid (x, y) \in (A_1 \times A_2)\},
$$

(3.6)

and $B_k(x, r)$ denotes the ball of radius $r$ in the simply connected space of constant sectional curvature $k$.

In this paper, we always assume $\text{Ric} \geq 0$ on $M$, and thus $C(n, k, \mathfrak{R}) = C(n)$. Using this theorem of Cheeger and Colding, C. Sprouse [22] proved an interesting lemma:

**Lemma 15.** [22] Let $(M, g)$ be a compact Riemannian manifold with $\text{Ric} \geq -1$. Then for any $\delta > 0$ there exists $\epsilon = \epsilon(n, \delta)$ such that if

$$
\frac{1}{V} \int_M ((m-1) - \text{Ric}_-) < \epsilon(m, \delta),
$$

(3.7)

then the $\text{diam}(M) < \pi + \delta$. Here $\text{Ric}_-$ denotes the lowest eigenvalue of the Ricci tensor; For any function $f$ on $M$, $f_+(x) = \max\{f(x), 0\}$.

**Remark 16.** Note that the right hand side of equation (3.7) is not scaling correct. A scaling correct version of this lemma should be: For any positive integer $a > 0$, if

$$
\frac{1}{V} \int_M |\text{Ric} - a| \, d\text{vol} < \epsilon(m, \delta) \cdot a,
$$

then the diameter has a uniform upper bound.
Remark 17. It is interesting to see what the optimal constant ε(m, δ) is. Following this idea, the best constant should be

$$\epsilon(m, \delta) = \sup_{N > 2} \frac{N - 2}{8C(m)N^m}.$$  

However, it will be interesting to figure out the best constant here.

Adopting his arguments, we will prove the similar lemma,

Lemma 18. Let (M, ω) be a polarized Kähler manifold and [ω] is the canonical Kähler class. Then there exists a positive constant ε0(n) which only depends on the dimension, such that if the Ricci curvature of ω is non-negative and if

$$\frac{1}{V} \int_M (R - n)^2 \omega^n \leq \epsilon_0(n)^2,$$

then there exists a uniform upper bound on diameter of the Kähler metric ω. Here r is the average of the scalar curvature.

Proof. We first prove that the Ricci form is close to its Kähler form in the $L^1$ sense (after proper rescaling). Note that $\text{Ric}(\omega) - \omega = \sqrt{-1} \partial \bar{\partial} f$ for some real valued function f. Thus

$$\int_M (\text{Ric}(\omega) - \omega)^2 \wedge \omega^{n-2} = \int_M (\sqrt{-1} \partial \bar{\partial} f)^2 \wedge \omega^{n-2} = 0.$$  

On the other hand, we have

$$\int_M (\text{Ric}(\omega) - \omega)^2 \wedge \omega^{n-2} = \frac{1}{n(n-1)} \int_M ((R - n)^2 - |\text{Ric}(\omega) - \omega|^2) \omega^n.$$  

Here we already use the identity $tr \omega (\text{Ric}(\omega) - \omega) = R - n$. Thus

$$\int_M |\text{Ric}(\omega) - \omega|^2 \omega^n = \int_M (R - n)^2 \omega^n.$$  

This implies that

$$\left( \int_M |\text{Ric} - 1| \omega^n \right)^2 \leq \int_M |\text{Ric}(\omega) - \omega|^2 \omega^n \cdot \int_M \omega^n$$

$$= \int_M (R - n)^2 \omega^n \cdot V$$

$$\leq \epsilon_0^2 \cdot V \cdot V = \epsilon_0^2 \cdot V^2;$$

which gives

$$\frac{1}{V} \int_M |\text{Ric} - 1| \omega^n \leq \epsilon_0.$$  

(3.8)
The value of \( \epsilon_0 \) will be determined later.

Using this inequality (3.8), we want to show that the diameter must be bounded from above. Note that in our setting, \( m = \dim(M) = 2n \). Unlike in [22], we are not interested in obtaining a sharp upper bound on the diameter.

Let \( A_1 \) and \( A_2 \) be two balls of small radius and \( W = M \). Let \( f = |\text{Ric} - 1| = \sum_{i=1}^{m} |\lambda_i - 1| \), where \( \lambda_i \) is the eigenvalue of the Ricci tensor. We assume also that all geodesics are parameterized by arc length. By possibly removing a set of measure 0 in \( A_1 \times A_2 \), there is a unique minimal geodesic from \( x \) to \( y \) for all \((x, y) \in A_1 \times A_2 \). Let \( p, q \) be two points on \( M \) such that \( d(p, q) = \text{diam}(M) = D \).

We also used \( d \, \text{vol} \) to denote the volume element in the Riemannian manifold \( M \) and \( V \) denote the total volume of \( M \). For \( r > 0 \), put \( A_1 = B(p, r) \) and \( A_2 = B(q, r) \). Then Lemma [14] implies that

\[
\int_{A_1 \times A_2} \int_{r_{x,y}} |\text{Ric} - 1| \, ds \, d\text{vol}_{A_1 \times A_2} \\
\leq C(n, k, R)(\text{diam}(A_2)\text{vol}(A_1) + \text{diam}(A_1)\text{vol}(A_2)) \int_{W} |\text{Ric} - 1| \, d\text{vol}.
\]

Taking infimum over both sides, we obtain

\[
\inf_{(x, y) \in A_1 \times A_2} \int_{r_{x,y}} |\text{Ric} - 1| \, dt \\
\leq 2r C(n) \left( \frac{1}{\text{vol}(A_1)} + \frac{1}{\text{vol}(A_2)} \right) \int_{W} |\text{Ric} - 1| \, d\text{vol} \\
\leq 4r C(n) \frac{D^n}{V} \int_{M} |\text{Ric} - 1| \, d\text{vol}, \quad (3.9)
\]

where the last inequality follows from the relative volume comparison. We can then find a minimizing unit-speed geodesic \( \gamma \) from \( x \in \overline{A_1} \) and \( y \in \overline{A_2} \) which realizes the infimum, and will show that for \( L = d(x, y) \) much larger than \( \pi \), \( \gamma \) cannot be minimizing if the right hand side of (3.9) is small enough.

Let \( E_1(t), E_2(t), \ldots E_m(t) \) be a parallel orthonormal basis along the geodesic \( \gamma \) such that \( E_1(t) = \gamma'(t) \). Set now \( Y_i(t) = \sin \left( \frac{\pi t}{L} \right) E_i(t), i = 2, 3, \ldots, m \). Denote by \( L_i(s) \) the length functional of a fixed endpoint variation of curves through \( \gamma \)
with variational direction $Y$, we have the 2nd variation formula

$$
\sum_{i=2}^{m} \frac{d^2 L_i(s)}{ds^2} \bigg|_{s=0}
= \sum_{i=2}^{m} \int_0^L \left( g(\nabla_{\gamma} Y_i, \nabla_{\gamma} Y_i) - R(\gamma', Y_i, \gamma', Y_i) \right) dt
= \int_0^L (m-1) \left( \frac{\pi^2}{L^2} \cos^2 \left( \frac{\pi t}{L} \right) \right) dt
+ \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) \left( 1 - \text{Ric}(\gamma', \gamma') \right) dt
= -\frac{1}{2} \left( 1 - (m-1) \frac{\pi^2}{L^2} \right) \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) \left( 1 - \text{Ric}(\gamma', \gamma') \right) dt.
$$

Note that $1 - \text{Ric}(\gamma', \gamma') \leq |\text{Ric} - 1|$. Combining the above calculation and the inequality (3.9), we obtain

$$
\sum_{i=2}^{m} \frac{d^2 L_i(s)}{ds^2} \bigg|_{s=0}
\leq -\frac{L}{2} \left( 1 - (m-1) \frac{\pi^2}{L^2} \right) + \int_0^L \sin^2 \left( \frac{\pi t}{L} \right) |\text{Ric} - 1| dt
\leq -\frac{L}{2} \left( 1 - (m-1) \frac{\pi^2}{L^2} \right) + 4rC(n) \frac{D^n}{c} \frac{1}{V} \int_M |\text{Ric} - 1| dvol. \quad (3.10)
$$

Here in the last inequality, we have already used the fact that $\gamma$ is a geodesic which realizes the infimum of the left side of inequality (3.9). For any fixed positive larger number $N > 4$, let $D = N \cdot r$. Set $c = \frac{1}{V} \int_M |\text{Ric} - 1| dvol$.

Note that $L = d(x, y) \geq d(p, q) - 2r = D(1 - \frac{2}{N}) \geq \frac{D}{2}$.

Then the above inequality leads to

$$
\sum_{i=2}^{m} \frac{d^2 L_i(s)}{ds^2} \bigg|_{s=0} \leq -\frac{1}{2} \left( 1 - (m-1) \frac{\pi^2}{L^2} \right) + 4C(n) \frac{D^n}{c} \cdot V
= 4C(n) \frac{D^n}{c} \left( c - \frac{(N-2)}{2N} \right) \frac{1}{4C(n)N^{m-1}} + \frac{1}{2} \left( m - 1 \right) \frac{\pi^2}{L^2}.
$$

Note that the second term in the right hand side can be ignored if $L \geq \frac{D}{2}$ is large enough. Set

$$
\epsilon_0 = \frac{(N-2)}{2N} \cdot \frac{1}{4C(n)N^{m-1}} = \frac{N-2}{8C(n)N^m}.
$$
Then if
\[ \frac{1}{V} \int_M (R - n) \omega^n \leq \epsilon_0, \]
by the argument at the beginning of this proof, we have the inequality \(3.8\):
\[ \frac{1}{V} \int_M (R - n) \omega^n \leq \epsilon_0, \]
\[ \frac{1}{V} \int_M |\text{Ric} - 1| \, d\text{vol} < \epsilon_0, \]
which in turns imply
\[ \frac{1}{D} \sum_{i=2}^{n} \frac{d^2 L_i(s)}{ds^2} |_{s=0} < 0, \]
for \(D\) large enough. Thus, if the diameter is too large, \(\gamma\) cannot be a length minimizing geodesic. This contradicts our earlier assumption that \(\gamma\) is a minimizing geodesic. Therefore, the diameter must have a uniform upper bound.

According to the work of C. Croke \[9\], Li-Yau \[14\] and Li \[13\]), we state the following lemma on the upper bound of the Sobolev constant and Poincaré constant:

**Lemma 19.** Let \((M, \omega)\) be any compact polarised Kähler manifold where \([\omega]\) is the canonical class. If \(\text{Ric}(\omega) \geq -1, V = \int_M \omega^n \geq \nu > 0\) and the diameter has a uniform upper bound, then there exists a constant \(\sigma = \sigma(\epsilon_0, \nu)\) such that for all function \(f \in C^\infty(M)\), we have
\[
\left( \int_M |f|^{2\omega^n} \right)^{\frac{2}{2n+4}} \leq \sigma \left( \int_M |\nabla f|^2 \omega^n + \int_M f^2 \omega^n \right).
\]
Furthermore, there exists a uniform Poincaré constant \(c(\epsilon_0)\) such that the Poincaré inequality holds
\[
\int_M \left( f - \frac{1}{V} \int_M f \, \omega^n \right)^2 \omega^n \leq c(\epsilon_0) \int_M |\nabla f|^2 \omega^n.
\]
Here \(\epsilon_0\) is the constant appeared in Lemma \[18\].

**Proof.** Note that \((M, \omega)\) has a uniform upper bound on the diameter. Moreover, it has a lower volume bound and it has non-negative Ricci curvature. Following a proof in \[13\] which is based on a result of C. Croke \[9\], we obtain a uniform upper bound on the Sobolev constant (independent of metric!).

Recall a theorem of Li-Yau \[13\] which gives a positive lower bound of the first eigenvalue in terms of the diameter when Ricci curvature is nonnegative:
\[
\lambda_1(\omega) \geq \frac{\pi^2}{4D},
\]
Here \(\lambda_1(\omega)\) is the first eigenvalue of the Laplacian with respect to the metric \(\omega\).
here \( \lambda_1 \), \( D \) denote the first eigenvalue and the diameter of the Kähler metric \( \omega \). In the same paper, Li-Yau also have a lower bound control of the first eigenvalue in terms of the Diameter when the Ricci curvature is bounded from below.

Now \( D \) has a uniform upper bound according to Lemma 18. Thus the first eigenvalue of \( \omega \) has a uniform positive lower bound; which, in turn, implies that there exists a uniform Poincaré constant.

**Lemma 20.** Along the Kähler Ricci flow, the diameter of the evolving metric is uniformly bounded for \( t \in [1, 4T] \).

**Proof.** Lemma 18 implies implies that

\[
\frac{1}{6T} \int_0^{6T} dt \int_M (R - n) \omega^n \phi \leq \frac{\epsilon_0(n)^2}{2}.
\]

In particular, there exists at least one time \( t_0 < T \) such that

\[
\int_M (R - n)^2 \omega^n \phi(t_0) \leq \frac{\epsilon_0(n)^2}{2}.
\]

Now for this \( t_0 \), applying Lemma 18, we show there exists a uniform constant \( D \) such that the diameters of \( \omega \phi(t_0) \) are uniformly bounded by \( D^2 \).

**Remark 21.** Notice that this diameter constant \( D \) is not related to \( \delta, \Lambda \). As \( \epsilon(\delta, \Lambda) \to 0 \), we have \( D \to \pi \). This should be very useful observation for future application.

Combining this with Lemma 18 we obtain

**Theorem 22.** Along the Kähler Ricci flow, the evolving Kähler metric \( \omega \phi(t)(t \in [T, 4T]) \) has a uniform upper bound on the Sobolev constant and Poincaré constant.

### 3.3 The pointwise norm of traceless Ricci

We first state a parabolic version of Moser iteration argument

**Lemma 23.** If the Poincaré constant and the Sobolev constant of the evolving Kähler metrics \( g(t) \) are both uniformly, and if a non-negative function \( u \) satisfying the following inequality

\[
\frac{\partial}{\partial t} u \leq \Delta u + f(t, x)u, \quad \forall a < t < b,
\]

...
where \( f = \vert Riem(g(t)) \vert \) and \( f \vert_{L^p(M, g(t))} \) is uniformly bounded by some constant \( c \), then for any \( \lambda \in (0, 1) \) fixed, we have

\[
\max_{(1-\lambda)a + \lambda b \leq t \leq b} u \leq C(c, b - a, \lambda) \int_a^b \int_M u.
\]

**Remark 24.** This is more or less standard and we refer readers to literatures of evolution equations (c.f. Lemma 4.7 [8]).

**Theorem 25.** For any \( \delta, \Lambda > 0 \), there exists a small positive constant \( \epsilon(\delta, \Lambda) > 0 \) such that if the initial metric \( g_0 \) of the Kähler Ricci flow satisfies the following condition:

\[
Ric(g_0) > -1 + \delta, \quad \vert Riem(g_0) \vert < \Lambda, \quad E_1(g_0) \leq \inf E_1 + \epsilon,
\]

then there exists an interval length \( T \) which depends on \( \delta, \Lambda \) such that, after time \( 2T \) along the Kähler Ricci flow, we have

\[
\max_{t \in [2T, 6T]} \max_{x \in M} \vert Ric(g(t)) - \omega(g(t)) \vert \leq \epsilon(n).
\]

**Proof.** Recalled that the evolution equation for Ricci curvature is:

\[
\frac{\partial R_{ij}(g(t))}{\partial t} = \Delta g(t) R_{ij}(g(t)) + R_{pq}(g(t)) \cdot R_{qpij}(g(t)) - R_{ij}, \quad \forall \, i, j = 1, 2, \ldots, n.
\]

Set \( Ric^0 = Ric - \omega \). Then \( u = \vert Ric^0(g(t)) \vert^2 \) satisfies this parabolic inequality

\[
\frac{\partial}{\partial t} u \leq \Delta u + c_3 \vert Riem(g(t)) \vert \cdot g \cdot u, \quad \forall 0 < t < 4T.
\]

where \( c_3 \) is some universal constant. In this section, we use \( \vert Riem(g(t)) \vert \leq 2\Lambda \) for \( 0 \leq t \leq 6T \). Then

\[
\max_{2T \leq t \leq 4T} \max_{x \in M} \vert Ric^0(g(t)) \vert = \max_{2T \leq t \leq 4T} \max_{x \in M} u(x, t)
\]

\[
\leq C(\Lambda, 4T, \frac{1}{4}) \int_T^{4T} dt \int_M u \, dvol_g
\]

\[
= C(\Lambda, 4T, \frac{1}{4}) \int_T^{4T} dt \int_M \vert Ric(g(t)) - g(t) \vert^2 \, dvol_g
\]

\[
\leq C(\Lambda, 4T, \frac{1}{4}) \frac{\epsilon(n)}{2} \cdot 4T \leq \epsilon(n).
\]

It is easy to see that we can extend this to proof the claim of this lemma. \( \square \)

In other words, the metric is nearly Kähler-Einstein at \( t = 4T \). If the norm of the bisectional curvature is still bounded by \( \Lambda \), then we can continue to flow the metric from \( 4T \) to \( 8T \) to obtain a better estimate on ricci curvature, etc. Unfortunately, we only have that the norm of the bisectional curvature bounded from above by \( 2\Lambda \).
3.4 The proof of Theorem 1.

We called a Kähler class satisfies condition (*) if it satisfies the following condition:

\[
[C_1(M)]^2 \cdot [\omega]^{n-2} - \frac{2(n+1)}{n} [C_2(M)] \cdot [\omega]^{n-1} = 0.
\] (3.12)

We actually only need the left hand side to be small enough for the present paper. Note that the left side is always an integer and the only manifold we know of which satisfies this condition is \( CP^n \). Consider its evolution equation:

\[
\frac{\partial R_{i\bar{j}k\bar{l}}}{\partial t} = \triangle \varphi(t) R_{i\bar{j}k\bar{l}} + R_{i\bar{p}q\bar{r}} R_{\bar{q}p\bar{r}j} - R_{p\bar{q}r} R_{i\bar{p}q\bar{r}j} - R_{\bar{q}p\bar{r}j} R_{i\bar{p}q\bar{r}j} - R_{i\bar{j}k\bar{l}}.
\]

Set

\[
Q^0_{i\bar{j}k\bar{l}} = R_{i\bar{j}k\bar{l}} - \frac{1}{n+1} (g_{ij} g_{kl} + g_{il} g_{jk})
\]

Then, modulo \( Ric^0 \) and \( R - n \) (both can be controlled by \(| Q^0_{i\bar{j}k\bar{l}} | \)), we have

\[
\frac{\partial}{\partial t} v = (\frac{\partial}{\partial t} - \triangle) v = Q_{i\bar{j}k\bar{l}} - Q_{ijkl} - Q_{ij\bar{k}l} - Q_{i\bar{j}kl} - Q_{i\bar{j}k\bar{l}}.
\]

Let \( v = | Q^0_{i\bar{j}k\bar{l}} | \). Then \( v \) satisfies this parabolic inequality

\[
\frac{\partial}{\partial t} v \leq \triangle v + \Lambda v, \quad \forall 0 < t < 4T.
\]

A calculation of E. Calabi [2] shows that in the Canonical Kähler class which satisfies condition 3.12 we have

\[
\int_M | Q_{i\bar{j}k\bar{l}} |^2 = \int_M | Ric^0 |^2.
\]

Then

\[
\max_{3T \leq t \leq 4T} \max_{x \in M} | Q_{i\bar{j}k\bar{l}} (g(t)) | = \max_{3T \leq t \leq 4T} \max_{x \in M} v(x, t)
\]

\[
\leq C(\Lambda, 4T, 2T) \int_{2T}^{4T} \int_M v(x, t) \leq C(\Lambda, 4T, 2T) \int_{2T}^{4T} \frac{1}{V} \int_M v^2
\]

\[
= (\Lambda, 4T, 2T) \int_{2T}^{4T} \frac{1}{V} \int_M | Q_{i\bar{j}k\bar{l}} |^2
\]

\[
= (\Lambda, 4T, 2T) \int_{2T}^{4T} \frac{1}{V} \int_M | Ric^0 |^2
\]

\[
\leq C(\Lambda, 4T, 2T) \int_{2T}^{4T} \frac{1}{V} \frac{\epsilon(n)^2}{2} \cdot 4T \leq \epsilon(n).
\]

Consequently, we obtain
Proposition 26. The bisectional curvature has a pointwise pinching estimate for \( t \in [3T, 4T] \)

\[
\max_{3T}^{4T} \max_{x \in M} | R_{ijkl} - \frac{1}{n+1}(g_{ij}g_{kl} + g_{il}g_{kj}) | < \epsilon(n)\nu.
\]

Combining Theorems 29 and Proposition 30, noticed that the pinching condition for bisectional curvature, Ricci curvature as well the \( E_1 \) norm all improved at time \( t = 4T \) (comparing to time \( t = 0 \)), repeatedly applying this procedure, we have

Theorem 27. Along the Kähler Ricci flow, the bisectional curvature becomes positive after finite time. Moreover, there exists a sequence of time \( t_i \to \infty \) and \( t_{i+1} - t_i \leq 1 \) such that

\[
0 = \lim_{i \to \infty} \max_{x \in M} | Ric_{g(t)} - \omega_{g(t)} |_{t_i},
\]

\[
= \lim_{i \to \infty} \max_{x \in M} | R_{ijkl} - \frac{1}{n+1}(g_{ij}g_{kl} + g_{il}g_{kj}) |_{t_i}.
\]

Appealing to a theorem in [7], the flow converges exponentially fast to a Kähler Einstein metric with constant positive bisectional curvature.

4 The proof of Theorem 5 and 6

4.1 A compactness lemma

In this section, we specialize in the complex surface (4 real dimensional). Consider the compactness of the following space

\[ \mathcal{A}(c, C) = \{(M, g) | \text{vol}(M, g) = 1, (n-1)c \leq \text{Ric}(g) \leq (n-1)Cg, \int_M |\text{Riem}(g)|^2 \leq C \}, \]

here \( c, C \) are two positive constants. Then this is a compact space in \( W^{2,p}(M, g) \) space for any \( p > 1 \). In particular,

Theorem 28. Suppose \( X \) is a fixed holomorphic vector field in \( M \). Consider all the Kähler metrics in \( \mathcal{A}(c, C) \) where \( \text{Im}(X) \) is a Killing vector field. For any \( p > 1 \), there exists a universal constant \( \Lambda(n, c, C, p) \) such that for all such Kähler metrics \( g \), we have

\[
\left( \int_M |\text{Riem}(g)|^p \, d\text{vol}_g \right)^{\frac{1}{p}} \leq \Lambda(n, c, C, p).
\]

Proof. For any metric \( g \in \mathcal{A}(c, C) \), we have \( \text{Ric}(g) > (n-1)c \). By Meyer’s theorem, the diameter \( \text{Diam}(g) \) of \( g \) has a uniform upper bound

\[
\text{Diam}(g) \leq \frac{\pi}{\sqrt{c}}.
\]
Suppose the volume of Euclidean ball of radius \( \rho \) is \( \omega_n \rho^n \). By Bishop-Gromov relative volume comparison theorem, for any \( p \in M \), the volume ratio \( \frac{\text{vol}(B_\rho(p))}{\omega_n \rho^n} \) is non-increasing function in \( \rho > 0 \). In particular, this means the volume of geodesic ball of \( g \) has a uniform upper bound (in terms of Euclidean ball). On the other hand, we have \( \text{vol}(B_{\text{Diam}(g)})(p) = \text{vol}(M, g) = 1 \). In other words, we have the volume ratio of geodesic ball has a uniform lower bound

\[
\frac{\pi^{\frac{n}{2}}}{\omega_n \rho^n} \leq \frac{\text{vol}(B_\rho(p))}{\omega_n \rho^n} \leq 1, \quad \forall \ p \in M, \text{ and } 0 \leq \rho \leq \text{Diam}(g).
\]

The fact \( \text{vol}(M, g) = 1 \) also implies that \( \text{Diam}(g) \) has a uniform positive lower bound. These together with the uniform Ricci bound implies a uniform positive lower bound of injectivity radius for metrics in \( A(c, C) \) which are invariant under action of the fixed Killing vector field. Otherwise, there exists a sequence of metrics \( g_k \in A(c, C), \ k \in \mathbb{N} \) such that the injectivity radius of \( g_k \) approaches to 0 and \( \mathcal{L}_{\text{Im}(X)} g_k = 0 \). We can re-scale this sequence of metrics to be \( \hat{g}_k \) such that harmonic radius is 1 and the \( L^\infty \) norm of Ricci curvature tends to 0. Consequently, there is a subsequence of \( \hat{g}_k \) such that it converges to some metric \( g_\infty \) in a manifold \( M_\infty \). The convergence in each harmonic ball of radius 1 is uniformly \( C^{1, \alpha} \). It is easy to see that \( (M_\infty, g_\infty) \) is complete Kähler manifold with flat Ricci curvature. Moreover, there is a smooth holomorphic vector field in \( M_\infty \) whose imaginary part acting isometrically on the metric \( g_\infty \). The orbit of this isometric action is either all closed or all open. In the case when the orbit is closed, the length of each orbit is finite and the limit metric must admit a circle action. If the orbit of isometric action is open, then each orbit must have infinite length. In the second case, since the \( L^2 \) norm of the Riemannian curvature of \( g_\infty \) is uniformly bound, the full Riemannian curvature must vanish completely. Thus \( g_\infty \) is Euclidean and \( M_\infty \) is either \( \mathbb{R}^4 \) or \( \mathbb{R}^3 \times S^1 \). The cylinder case is ruled out because \( g_\infty \) has maximum Euclidean volume growth. Consequently, the harmonic radius of the limit metric is infinity. Since the Harmonic radius is lower semi-continuous under \( C^{1, \alpha} \) convergence, the harmonic radius of \( \hat{g}_k \) for \( k \) large enough must be bigger than 2. This is a contradiction since we assume initially that the harmonic radius is 1 for all \( \hat{g}_k, k \in \mathbb{N} \). Consequently, the limit metric must admit an invariant circle action. According to Hitchin's theorem, in the global holomorphic coordinate of \( M_\infty \), the limit metric \( g_\infty \) can be expressed as

\[
w^{-1}(d x^2 + d y^2) + w d z^2 + w \theta^2, \quad w > 0.
\]

Here \( \theta \) represents the connection 1-form of the circle action. The fact of Ricci flat means

\[
\triangle w = 0.
\]

Note that there is no positive harmonic function in \( \mathbb{R}^3 \), which follows that \( w = \text{constant} > 0 \) and the limit metric is Euclidean again. In particular, \( M_\infty = \mathbb{C}^2 \) and the harmonic radius is again \( \infty \). As before, this contradicts with our earlier assumption that the harmonic radius of \( \hat{g}_k \) is only 1. Thus, there is a uniform bound on the injectivity radius for this sequence of metrics
{g_k, k ∈ N}. Consequently, the metric g_k, k ∈ N is in W^{2,p}(M, g) for any p > 1. In other words, we have
\[
\left( \int_M |\operatorname{Riem}(g)|^p \, d\text{vol}_g \right)^{\frac{1}{p}} \leq \Lambda(c, C, p)
\]
for some uniform constant \(\Lambda(c, C, p)\).

### 4.2 Moser iteration in the “move”

In this subsection, we want to use Theorem 25 repeatedly to obtain global pinching of Ricci tensor over time \(t \in [T, \infty)\). By Theorem 25, we have
\[
1 - \epsilon_n < \text{Ric}(g(t)) \leq 1 + \epsilon_n, \quad \forall t \in [2T, 6T].
\]

**Lemma 29.** Fix a constant \(p > \frac{2n+2}{2n} = n + 1\). Condition as in Theorem 25.

Along the Kähler Ricci flow, if \(A > 6T\) is any positive number such that
\[
1 - \epsilon_n < \text{Ric}(g(t)) \leq 1 + \epsilon_n, \quad \forall t \in [2T, A]
\]
and
\[
\int_M |\operatorname{Riem}(g(t))|^p \leq \Lambda(\frac{1}{2}, 2, p), \quad \forall t \in [2T, A],
\]
then there exists a \(\epsilon_1 > 0\), such that both inequalities hold for \(t \in [2T, A + \epsilon_1]\).

**Proof.** Because that the Kähler Ricci flow exists globally and the metrics evolved smoothly, there is a small constant \(\epsilon_1\) such that
\[
0 < (n - 1)c < \text{Ric}(g(t)) \leq (n - 1)C, \quad \forall t \in [2T, A + \epsilon_1].
\]

Note that \(\text{Ric}(g(t)) > -1\) for any \(t \leq A + \epsilon_1\). Thus, \(E_1\) is decreasing and
\[
\int_0^{A+\epsilon_1} dt \int_M |\text{Ric}(\omega_g(t)) - \omega_g(t)|^2_{\omega_g(t)} \, d\text{vol}_{\omega_g(t)} \leq E_1(0) - E_1(A + \epsilon_1) \leq E_1(0) - \inf_{[\omega]} E_1(g) < \epsilon(\delta, A).
\]

On the other hand, applying the compactness Theorem 28 to metric \(g(t)(2T \leq t \leq A + \epsilon_1)\), we have
\[
|f(x, t)|_{L^p(M, g(t))} \leq \Lambda(\frac{1}{2}, 2, p), \quad \text{here } f(x, t) = |\operatorname{Riem}(g(t))|.
\]

For \(t \in [A - 4T + \epsilon_1, A + \epsilon_1]\), applying Lemma 28, we obtain
\[
\max_{t \in [A - 4T + \epsilon_1, A + \epsilon_1]} \max_{x \in M} |\text{Ric}_g(t) - \omega_g(t)|_g \leq \epsilon(n).
\]

Here we already use the fact that
\[
\int_{A - 4T + \epsilon_1}^{A + \epsilon_1} dt \int_M |\text{Ric}_g(t) - \omega_g(t)|^2_{\omega_g(t)} \, d\text{vol}_g \leq E_1(A - 4T + \epsilon_1) - E_1(A + \epsilon_1) < E_1(0) - \inf_{[\omega]} E_1(g) \leq \epsilon(\delta, A) \leq \frac{\epsilon_0(n)^2}{2} \cdot 5T.
\]
Hence, we can repeatedly applied this iteration Lemma to itself. This Lemma essentially says that the time \( t \) where both conditions 4.1 and 4.2 to hold is relatively open in \([2T, \infty)\). It is easy to show that it is relatively closed as well. Therefore, we have the following direct corollary.

**Corollary 30.** Along the Kähler Ricci flow, we have

\[
\max_{t \in [2T, \infty)} \max_{x \in M} |\text{Ric}_{g(t)} - \omega_{g(t)}| \leq \epsilon(t) \to 0.
\]

Moreover, we have a uniform bound of \( \text{Riem}(g) \) for any \( t \in [0, \infty) \).

**Proof.** It is easy to see that the pinching condition for Ricci curvature holds. Using the compactness theorem 28, we show that \( L^p \) norm of the Riemannian curvature is uniformly bounded over the entire flow. Using the iteration scheme as in section 3.4, but working on a longer interval, we can show that the full Riemannian curvature is uniformly bounded.

Now we are ready to wrap up the proof to Theorem 5 and 6.

**Proof.** According to the preceding corollary, the Riemannian curvature is uniformly bounded and the Ricci curvature uniformly pinched towards a Kähler Einstein metric. Therefore, for any sequence \( t_i \to \infty \), there exists a subsequence such that \( g(t_i) \) converges to a Kähler Einstein metric, perhaps in a different complex manifold. However, the stability assumption in general dimension will enable us to derive the exponential convergence of the flow, similar to what is done in [7]. This completes the proof of Theorem 5. In complex surface, we can bypass this stability conditions because the classification theorem: Only Kähler surface with positive first Chern class are \( \mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1 \) and \( \mathbb{C}P^2 \) blowup \( k \) points in generic position (means no three points in one line and no six points in a quadratic). To have a non-trivial reductive automorphism group, we need further restricted to \( k = 3, 4 \) or \( \mathbb{C}P^2, \mathbb{C}P^1 \times \mathbb{C}P^1 \). To have a Killing vector field, then the case \( k = 4 \) is ruled out. In either of the remaining cases, the orbit of complex structure under the diffeomorphism group is well known. We can show that the complex structure is stable in the sense described in Theorem 5. This complete the proof for Theorem 6.

## 5 Future problems

The following problems are interesting and will be investigated in the sequel of this paper.

1. Can we prove a general convergence theorem without assuming small energy on \( E_1 \)? This is still interesting even if we assume the condition 3.12.

2. In Kähler Einstein manifold, is \( E_1 \) necessary has a lower bound or proper? Is the critical point of \( E_1 \) unique (up to holomorphic transformation)? In other words, is the critical points necessary Kähler Einstein?
3. Under what geometric condition, can we derive the existence of lower bound of $E_1$ in absence of Kähler Einstein metric? In particular, what about the special case where the bisectional curvature is positive?

References

[1] S. Bando. On the three dimensional compact Kähler manifolds of nonnegative bisectional curvature. *J. D. G.*, 19:283–297, 1984.

[2] E. Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 16 of 102, pages 259–290. Ann. of Math. Studies, University Press, 1982.

[3] B. L. Chen Cao, H. D. and X. P. Zhu. Ricci flow on kähler manifold of positive bisectional curvature, 2003. [math.DG/0302087](http://arxiv.org/abs/math.DG/0302087).

[4] H. D. Cao. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81:359–372, 1985.

[5] H. D. Cao. On Harnack’s inequalities for the Kähler-Ricci flow. *Invent. Math.*, 109:247–263, 1992.

[6] J. Cheeger and T. H. Colding. Lower bounds onRicci curvature and almost rigidity of wrapped products. *Ann. Math.*, 144:189–237, 1996.

[7] X. X. Chen and G. Tian. Ricci flow on complex surfaces. *Inventiones mathematicae*, 147(3):487–544, 2002.

[8] X. X. Chen and G. Tian. Ricci flow on Kähler-Einstein manifolds, 2000. to appear in Duke.

[9] C. Croke. Some Isoperimetric Inequalities and Consequences. *Ann. Sci. E. N. S.*, Paris, 13:419–435, 1980.

[10] R. Hamilton. Three-manifolds with positive Ricci curvature. *J. Diff. Geom.*, 17:255–306, 1982.

[11] R. Hamilton. Four-manifolds with positive curvature operator. *J. Diff. Geom.*, 24:153–179, 1986.

[12] G. Huisken. Ricci deformation of the metric on a Riemannian manifold. *J. Diff. Geom.*, 21:47–62, 1985.

[13] P. Li. On the sobolev constant and the $p$– spectrum of a compact Riemannian manifold. *Ann. Sci. E. N. S.*, Paris, 13:451–468, 1980.

[14] P. Li and S. T. Yau. Estimates of eigenvalues of a compact riemannian manifold. In *Proceedings of Symposia in Pure Mathematics*, volume 36, pages 205–239, 1979.
[15] C. Margein. Positive pinched manifolds are space forms. volume 4 of Proc. Sympos. Pure Math., pages 307–328, 1986.

[16] C. Margein. A sharp characterization of the smooth 4-sphere in curvature terms. volume 6 of Comm. Anal. Geom., pages 21–65, 1998.

[17] N. Mok. The uniformization theorem for compact Kähler manifolds of non-negative holomorphic bisectional curvature. J. Differential Geom., 27:179–214, 1988.

[18] S. Nishikawa. Deformation of Riemannian metrics and manifolds with bounded curvature ratios. Proceedings of Symp. in Pure Math., 44:345–352, 1986.

[19] D. H. Phong and J. Sturm. On the Kähler-Ricci flow on complex surfaces, 2004. priprint, math.DG/0407232.

[20] Natasa Sesum. Convergence of the kähler ricci flow note math.DG/0402238, 2004.

[21] W.D. Ruan On the Convergence and Collapsing of Kähler metrics. J. Diff. Geom., 52:1-40, 1999

[22] C. Sprouse. Integral curvature bounds and bounded diameter. Communication of Analysis and Geometry, 8(3):531–543, 2000.

[23] G. Tian. On kähler-Einstein metrics on certain kähler manifolds with $c_1(m) > 0$. Invent. Math., 89:225–246, 1987.

[24] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampere equation, I*. Comm. Pure Appl. Math., 31:339–441, 1978.

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