A Convergence Theorem for Solving Generalized Mixed Equilibrium Problems and Finding Fixed Points of a Weak Bregman Relatively Nonexpansive Mappings in Banach Spaces

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Abstract
In this paper, we study a new iterative method for finding fixed points of a weak Bregman relatively nonexpansive mapping and solutions of generalized mixed equilibrium problems in Banach spaces.

Keywords Banach space · Bregman projection · Bregman distance · Weak Bregman relatively nonexpansive mapping · Fixed point · Generalized mixed equilibrium problem

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1 Introduction
Let $E$ be a real reflexive Banach space and $C$ a nonempty, closed and convex subset of $E$ and $E^*$ be the dual space of $E$ and $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous

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and convex function. We denote by $\text{dom } f$, the domain of $f$, that is the set $\{x \in E : f(x) < +\infty\}$. Let $x \in \text{int}(\text{dom } f)$, the subdifferential of $f$ at $x$ is the convex set defined by

$$
\partial f(x) = \left\{ x^* \in E^* : f(x) + \langle x^*, y - x \rangle \leq f(y), \forall y \in E \right\},
$$

where the Fenchel conjugate of $f$ is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$
f^*(x^*) = \sup \left\{ \langle x^*, x \rangle - f(x) : x \in E \right\}.
$$

Equilibrium problems which were introduced by Blum and Oettli [5] and Noor and Oettli [6] in 1994 have had a great impact and influence in the development of several branches of pure and applied sciences. It has been shown that the equilibrium problem theory provides a novel and unified treatment of a wide class of problems which arise in economics, finance, image reconstruction, ecology, transportation, network, elasticity and optimization. It has been shown [5, 6] that equilibrium problems include variational inequalities, fixed point, Nash equilibrium and game theory as special cases. Hence collectively, equilibrium problems cover a vast range of applications. Due to the nature of the equilibrium problems, it is not possible to extend the projection and its variant forms for solving equilibrium problems. To overcome this drawback, one usually uses the auxiliary principle technique. The main and basic idea in this technique is to consider an auxiliary equilibrium problem related to the original problem and then show that the solution of the auxiliary problems is a solution of the original problem. This technique has been used to suggest and analyze a number of iterative methods for solving various classes of equilibrium problems and variational inequalities (see [4] and the references therein). Related to the equilibrium problems, we also have the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis [18]. It is natural to construct a unified approach for these problems. In this direction, several authors have introduced some iterative schemes for finding a common element of a set of the solutions of the equilibrium problems and a set of the fixed points of finitely many nonexpansive mappings.

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers, $\Psi : X \to X^*$ be a nonlinear operator and $\varphi : C \to \mathbb{R}$ be a real-valued function. The generalized mixed equilibrium problem is to find an element $x \in C$ such that

$$
\Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \forall y \in C.
$$

The set of solutions of the problem (1.1) is denoted by $GMEP(\Theta, \varphi, \Psi)$, that is,

$$
GMEP(\Theta, \varphi, \Psi) = \{ x \in C : \Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \forall y \in C \}.
$$

Let $\Phi_i, i = 1, 2, \ldots, N$ be $N$ bifunctions from $C \times C$ to $\mathbb{R}$, $\varphi_i, i = 1, 2, \ldots, N$ be $N$ real-valued functions from $C$ to $\mathbb{R}$ and $\Psi_i, i = 1, 2, \ldots, N$ be $N$ operators from $X$ to $X^*$. Solving a system of generalized mixed equilibrium problems means finding an element $x \in C$ such that $x \in \cap_{i=1}^N GMEP(\Theta_i, \varphi_i, \Psi_i)$. In particular, if $\Psi = 0$, problem (1.1) is reduced to the following mixed equilibrium problem, which is to find an element $x \in C$ such that

$$
\Theta(x, y) + \varphi(y) \geq \varphi(x), \forall y \in C.
$$

We denote by $MEP(\Theta)$ the set of solutions of problem (1.2). If $\varphi = 0$, problem (1.1) is reduced to the following generalized equilibrium problem, which is to find an element $x \in C$ such that

$$
\Theta(x, y) + \langle \Psi x, y - x \rangle \geq 0, \forall y \in C.
$$
The set of solutions of problem (1.3) is denoted by $GEP(\Theta, \Psi)$. If $\Theta = 0$, problem (1.1) is reduced to the following mixed variational inequality of Browder type, which is to find an element $x \in C$ such that

$$\langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \forall y \in C. \quad (1.4)$$

The set of solutions of the problem (1.4) is denoted by $MVI(C, \varphi, \Psi)$. If $\varphi = 0$ and $\Psi = 0$, problem (1.1) is reduced to the following well known equilibrium problem, which is to find an element $x \in C$ such that

$$\Theta(x, y) \geq 0, \forall y \in C. \quad (1.5)$$

The set of solutions of problem (1.5) is denoted by $EP(\Theta)$.

In [31], Reich and Sabach proposed two algorithms for finding a common fixed point of finitely many Bregman strongly nonexpansive mappings $T_i : C \to C$ ($i = 1, 2, \ldots, N$) satisfying $\cap_{i=1}^N F(T_i) \neq \emptyset$ in a reflexive Banach space $E$ as follows:

$$x_0 \in E, \text{ chosen arbitrarily,}$$

$$y_n^i = T_i \left( x_n + e_n^i \right),$$

$$C_n^i = \left\{ z \in E : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i) \right\},$$

$$C_n = \bigcap_{i=1}^N C_n^i,$$  

$$Q_n^i = \left\{ z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0 \right\},$$

$$x_{n+1} = \text{proj}_{C_n \cap Q_n}(x_0), \forall n \geq 0,$$

and

$$x_0 \in E,$$

$$C_0^i = E, i = 1, 2, \ldots, N,$$

$$y_n^i = T_i \left( v_n + e_n^i \right),$$

$$C_{n+1}^i = \left\{ z \in C_n^i : D_f(z, y_n^i) \leq D_f(z, x_n + e_n^i) \right\},$$

$$C_{n+1} = \bigcap_{i=1}^N C_{n+1}^i,$$  

$$x_{n+1} = \text{proj}_{C_{n+1}}(x_0), \forall n \geq 0,$$

where $\text{proj}_C^f$ is the Bregman projection with respect to $f$ from $E$ onto a closed and convex subset $C$ of $E$. They proved that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$. 


The authors of [1] introduced the following algorithm:

\[ x_0 = x \in C \text{ chosen arbitrarily}, \]
\[ z_n = \nabla f^* (\beta_n \nabla f (T(x_n)) + (1 - \beta_n) \nabla f (x_n)), \]
\[ y_n = \nabla f^* (\alpha_n \nabla f (x_0) + (1 - \alpha_n) \nabla f (z_n)), \]
\[ u_n = \text{Res}_f^x(y_n), \]
\[ C_n = \{ z \in C_{n-1} \cap Q_{n-1} : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n) D_f(z, x_n) \}, \]
\[ Q_n = \{ z \in C_{n-1} \cap Q_{n-1} : (\nabla f(x_0) - \nabla f(x_n), z - x_n) \leq 0 \}, \]
\[ x_{n+1} = \text{proj}^f_{C_n \cap Q_n} x_0, \quad \forall n \geq 0, \]

where \( H \) is an equilibrium bifunction and \( T \) is a weak Bregman relatively nonexpansive mapping. They proved that the above sequence converges strongly to the point \( \text{proj}^f_{F(T) \cap \text{EP}(H)} x_0 \).

In this paper, motivated by the above algorithms, we study the following iterative scheme:

\[ z_n = \nabla f^* (\beta_n \nabla f (T(x_n)) + (1 - \beta_n) \nabla f (x_n)), \]
\[ y_n = \nabla f^* (\alpha_n \nabla f (x_0) + (1 - \alpha_n) \nabla f (z_n)), \]
\[ u_n = \text{Res}_f^\Theta(y_n), \]
\[ C_{n+1} = \{ z \in C_n : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n) D_f(z, x_n) \}, \]
\[ Q_{n+1} = \{ z \in Q_n : (\nabla f(x_0) - \nabla f(x_n), z - x_n) \leq 0 \}, \]
\[ x_{n+1} = \text{proj}^f_{C_{n+1} \cap Q_{n+1}} x_0, \quad \forall n \geq 0, \quad (1.6) \]

where \( T \) is a weak Bregman relatively nonexpansive mapping, \( \varphi : C \to \mathbb{R} \) is a real-valued function, \( \Psi : C \to E^* \) is a continuous monotone mapping, \( \Theta : C \times C \to \mathbb{R} \) is an equilibrium bifunction. We will prove that the sequence \( \{x_n\} \) defined in (1.6) converges strongly to the point \( \text{proj}_{F(T) \cap \text{GMEP}(\Theta)} x_0 \).

2 Preliminaries

For any \( x \in \text{int}(\text{dom} f) \), the right-hand derivative of \( f \) at \( x \) in the derivation \( y \in E \) is defined by

\[ f'(x, y) := \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t}. \]

The function \( f \) is called Gâteaux differentiable at \( x \) if \( \lim_{t \searrow 0} \frac{f(x + ty) - f(x)}{t} \) exists for all \( y \in E \). In this case, \( f'(x, y) \) coincides with \( \nabla f(x) \), the value of the gradient \( (\nabla f) \) of \( f \) at \( x \). The function \( f \) is called Gâteaux differentiable if it is Gâteaux differentiable for any \( x \in \text{int}(\text{dom} f) \) and \( f \) is called Fréchet differentiable at \( x \) if this limit is attained uniformly for all \( y \) which satisfy \( \|y\| = 1 \). The function \( f \) is uniformly Fréchet differentiable on a subset \( C \) of \( E \) if the limit is attained uniformly for any \( x \in C \) and \( \|y\| = 1 \). It is known that if \( f \) is Gâteaux differentiable (resp. Fréchet differentiable) on \( \text{int}(\text{dom} f) \), then \( f \) is continuous and its Gâteaux derivative \( \nabla f \) is norm-to-weak* continuous (resp. continuous) on \( \text{int}(\text{dom} f) \) (see [8]).
**Definition 2.1** [10, 24] Let \( f : E \to (-\infty, +\infty] \) be a Gâteaux differentiable function. The function \( D_f : \text{dom } f \times \text{int}(\text{dom } f) \to [0, +\infty) \) defined as follows

\[
D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y \rangle
\]

is called the Bregman distance with respect to \( f \).

**Remark 2.2** [28] The Bregman distance has the following properties:

1. The three-point identity, for any \( x \in \text{dom } f \) and \( y, z \in \text{int}(\text{dom } f) \),
   \[
   D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle;
   \]

2. The four-point identity, for any \( y, w \in \text{dom } f \) and \( x, z \in \text{int}(\text{dom } f) \),
   \[
   D_f(y, x) - D_f(y, z) - D_f(w, x) + D_f(w, z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle.
   \]

The Legendre function \( f : E \to (-\infty, +\infty] \) is defined in [7]. It is well known that in reflexive spaces, \( f \) is a Legendre function if and only if it satisfies the following conditions:

- \((L_1)\) The interior of the domain of \( f \), \( \text{int}(\text{dom } f) \), is nonempty, \( f \) is Gâteaux differentiable on \( \text{int}(\text{dom } f) \) and \( \text{dom } f = \text{int}(\text{dom } f) \);
- \((L_2)\) The interior of the domain of \( f^* \), \( \text{int}(\text{dom } f^*) \), is nonempty, \( f^* \) is Gâteaux differentiable on \( \text{int}(\text{dom } f^*) \) and \( \text{dom } f^* = \text{int}(\text{dom } f^*) \).

Since \( E \) is reflexive, we know that \( (\partial f)^{-1} = \partial f^* \) (see [8]). This, with \((L_1)\) and \((L_2)\), imply the following equalities

\[
\nabla f = (\nabla f^*)^{-1}, \quad \text{ran } \nabla f = \text{dom } \nabla f^* = \text{int}(\text{dom } f^*)
\]

and

\[
\text{ran } \nabla f^* = \text{dom } \nabla f = \text{int}(\text{dom } f),
\]

where \( \text{ran } \nabla f \) denotes the range of \( \nabla f \).

When the subdifferential of \( f \) is single-valued, it coincides with the gradient \( \partial f = \nabla f \), [23]. By Bauschke et al. [7] the conditions \((L_1)\) and \((L_2)\) also yield that the function \( f \) and \( f^* \) are strictly convex on the interior of their respective domains.

If \( E \) is a smooth and strictly convex Banach space, then an important and interesting Legendre function is \( f(x) := \frac{1}{2p}\|x\|^p \) \((1 < p < \infty)\). In this case the gradient \( \nabla f \) of \( f \) coincides with the generalized duality mapping of \( E \), i.e., \( \nabla f = J_p \) \((1 < p < \infty)\). In particular, \( \nabla f = I \), the identity mapping in Hilbert spaces. From now on we assume that the convex function \( f : E \to (-\infty, \infty] \) is Legendre. In connection with Legendre functions, see also the recent paper [25].

**Definition 2.3** [10] Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The Bregman projection of \( x \in \text{int}(\text{dom } f) \) onto a nonempty, closed and convex subset \( C \subset \text{dom } f \) is the necessary unique vector \( \text{proj}_C^f(x) \in C \) satisfying

\[
D_f \left( \text{proj}_C^f(x), x \right) = \inf \left\{ D_f(y, x) : y \in C \right\}.
\]

**Remark 2.4** If \( E \) is a smooth and strictly convex Banach space and \( f(x) = \|x\|^2 \) for all \( x \in E \), then we have that \( \nabla f(x) = 2Jx \) for all \( x \in E \), where \( J \) is the normalized duality mapping from \( E \) to \( 2^{E*} \), and hence \( D_f(x, y) \) reduces to \( \phi(x, y) = \|x\|^2 - 2\langle Jy, x \rangle + \|y\|^2 \).
for all \( x, y \in E \), which is the Lyapunov function introduced by Alber [3] and Bregman projection \( P_C^f(x) \) reduces to the generalized projection \( \Pi_C(x) \) which is defined by

\[
\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).
\]

If \( E = H \), a Hilbert space, \( J \) is the identity mapping and hence Bregman projection \( P_C^f(x) \) reduces to the metric projection of \( H \) onto \( C, P_C(x) \).

**Definition 2.5** [13, 14] Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. \( f \) is called

1. **Totally convex** at \( x \in \text{int}(\text{dom } f) \) if its modulus of total convexity at \( x \), that is, the function \( \nu_f : \text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty) \) defined by

\[
\nu_f(x, t) := \inf \{ D_f(y, x) : y \in \text{dom } f, \|y - x\| = t \},
\]

is positive whenever \( t > 0 \);
2. **Totally convex** if it is totally convex at every point \( x \in \text{int}(\text{dom } f) \);
3. **Totally convex on bounded sets** if \( \nu_f(B, t) \) is positive for any nonempty bounded subset \( B \) of \( E \) and \( t > 0 \), where the modulus of total convexity of the function \( f \) on the set \( B \) is the function \( \nu_f : \text{int}(\text{dom } f) \times [0, +\infty) \to [0, +\infty) \) defined by

\[
\nu_f(B, t) := \inf \{ \nu_f(x, t) : x \in B \cap \text{dom } f \}.
\]

The set \( \text{lev}^f_r := \{ x \in E : f(x) \leq r \} \) for some \( r \in \mathbb{R} \) is called a sublevel of \( f \).

**Definition 2.6** [14, 31] The function \( f : E \to (-\infty, +\infty] \) is called

1. **Cofinite** if \( \text{dom } f^* = E^* \);
2. **Coercive** [19] if the sublevel set of \( f \) is bounded; equivalently,

\[
\lim_{\|x\| \to +\infty} f(x) = +\infty;
\]
3. **Strongly coercive** if \( \lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty \);
4. **Sequentially consistent** if for any two sequences \( \{x_n\} \) and \( \{y_n\} \) in \( E \) such that \( \{x_n\} \) is bounded,

\[
\lim_{n \to \infty} D_f(y_n, x_n) = 0 \Rightarrow \lim_{n \to \infty} \|y_n - x_n\| = 0.
\]

**Lemma 2.7** [15] The function \( f \) is totally convex on bounded subsets if and only if it is sequentially consistent.

**Lemma 2.8** [31, Proposition 2.3] If \( f : E \to (-\infty, +\infty] \) is Fréchet differentiable and totally convex, then \( f \) is cofinite.

**Lemma 2.9** [15] Let \( f : E \to (-\infty, +\infty] \) be a convex function whose domain contains at least two points. Then the following statements hold:

1. \( f \) is sequentially consistent if and only if it is totally convex on bounded sets;
2. If \( f \) is lower semicontinuous, then \( f \) is sequentially consistent if and only if it is uniformly convex on bounded sets;
3. If \( f \) is uniformly strictly convex on bounded sets, then it is sequentially consistent and the converse implication holds when \( f \) is lower semicontinuous, Fréchet differentiable on its domain and the Fréchet derivative \( \nabla f \) is uniformly continuous on bounded sets.
Lemma 2.10 [27, Proposition 2.1] Let $f : E \to \mathbb{R}$ be uniformly Fréchet differentiable and bounded on bounded subsets of $E$. Then $\nabla f$ is uniformly continuous on bounded subsets of $E$ from the strong topology of $E$ to the strong topology of $E^*$.

Lemma 2.11 [31, Lemma 3.1] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.12 [31, Proposition 2.2] Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let $C$ be a nonempty, closed convex subset of $E$. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to $C$. If $D_f(x_n, x_0) \leq D_f(\text{proj}^f_C x_0, x_0)$ for any $n \in N$, then $\{x_n\}_{n=1}^\infty$ converges to $\text{proj}^f_C x_0$.

Definition 2.13 [31] Let $T : C \to C$ be a nonlinear mapping. The fixed points set of $T$ is denoted by $F(T)$, that is $F(T) = \{x \in C : Tx = x\}$. A mapping $T$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. $T$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - p\| \leq \|x - p\|$, for all $x \in C$ and $p \in F(T)$. A point $p \in C$ is called an asymptotic fixed point of $T$ (see [2]) if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. A point $p \in C$ is called a strong asymptotic fixed point of $T$ (see [2]) if $C$ contains a sequence $\{x_n\}$ which converges strongly to $p$ such that $\lim_{n \to \infty} \|x_n - Tx_n\| = 0$. We denote the sets of asymptotic fixed points and strong asymptotic fixed points of $T$ by $\hat{F}(T)$ and $\tilde{F}(T)$, respectively.

A mapping $T : C \to \text{int}(\text{dom} f)$ with $F(T) \neq \emptyset$ is called

1. Quasi-Bregman nonexpansive [31] with respect to $f$ if
   \[ D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T). \]

2. Bregman relatively nonexpansive [12, 31] with respect to $f$ if
   \[ D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T), \text{ and } \tilde{F}(T) = F(T). \]

3. Bregman strongly nonexpansive (see [11, 31]) with respect to $f$ and $\tilde{F}(T)$ if
   \[ D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in \tilde{F}(T) \]
   and if whenever $\{x_n\} \subset C$ is bounded, $p \in \tilde{F}(T)$, and
   \[ \lim_{z \to \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0, \]
   it follows that
   \[ \lim_{n \to \infty} D_f(x_n, Tx_n) = 0. \]

4. Bregman firmly nonexpansive (for short BFNE) [9] with respect to $f$ if for all $x, y \in C$,
   \[ \langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle \]
equivalently,
   \[ D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \]

5. A weak Bregman relatively nonexpansive mapping with $F(T) \neq \emptyset$ if $\tilde{F}(T) = F(T)$ and
   \[ D_f(p, Tx) \leq D_f(p, x), \forall x \in C, p \in F(T). \]

The existence and approximation of Bregman firmly nonexpansive mappings was studied in [26]. It is also known that if $T$ is Bregman firmly nonexpansive and $f$ is Legendre function.
which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of $E$, then $F(T) = \hat{F}(T)$ and $F(T)$ is closed and convex. It also follows that every Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to $F(T) = \hat{F}(T)$.

Let $C$ be a nonempty, closed and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. Let $x \in E$ it is known from [15] that $z = \text{proj}^f_C(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0, \forall y \in C.$$ 

We also know the following

$$D_f(y, \text{proj}^f_C(x)) + D_f(\text{proj}^f_C(x), x) \leq D_f(y, x), \forall x \in E, y \in C.$$ 

Let $f : E \to \mathbb{R}$ be a convex, Legendre and Gâteaux differentiable function. Following [3] and [16], we make use of the function $V_f : E \times E^* \to [0, \infty)$ associated with $f$, which is defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x \rangle + f^*(x^*), \forall x \in E, x^* \in E^*.$$ 

Then $V_f$ is nonexpansive and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$ for all $x \in E$ and $x^* \in E^*$. Moreover, by the subdifferential inequality,

$$V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*)$$ 

for all $x \in E$ and $x^*, y^* \in E^*$ [21]. In addition, if $f : E \to (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper weak* lower semicontinuous and convex function (see [22]). Hence, $V_f$ is convex in the second variable. Thus, for all $z \in E$,

$$D_f(z, \nabla f^* \left( \sum_{i=1}^N t_i \nabla f(x_i) \right)) \leq \sum_{i=1}^N t_i D_f(z, x_i),$$ 

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\}_{i=1}^N \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

**Lemma 2.14** [15] Let $f \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on $\text{int}(\text{dom} f)$. Let $x \in \text{int}(\text{dom} f)$ and $C \subset \text{int}(\text{dom} f)$ be a nonempty, closed convex set. If $\hat{x} \in C$, then the following statements are equivalent:

1. The vector $z$ is the Bregman projection of $x$ onto $C$ with respect to $f$;
2. The vector $z$ is the unique solution of the variational inequality:

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \forall y \in C;$$

3. The vector $z$ is the unique solution of the inequality:

$$D_f(y, z) + D_f(z, x) \leq D_f(y, x), \forall y \in C.$$

**Lemma 2.15** [29] Let $C$ be a nonempty, closed and convex subset of $\text{int}(\text{dom} f)$ and $T : C \to C$ be a quasi-Bregman nonexpansive mappings with respect to $f$. Then $F(T)$ is closed and convex.
For solving the generalized mixed equilibrium problem, let us assume that the bifunction \( \Theta : C \times C \to \mathbb{R} \) satisfies the following conditions:

(A1) \( \Theta(x, x) = 0 \) for all \( x \in C \);

(A2) \( \Theta \) is monotone, i.e., \( \Theta(x, y) + \Theta(y, x) \leq 0 \) for any \( x, y \in C \);

(A3) For each \( y \in C \), \( x \mapsto \Theta(x, y) \) is upper hemicontinuous, i.e., for each \( x, y, z \in C \),

\[
\limsup_{t \searrow 0} \Theta(tz + (1-t)x, y) \leq \Theta(x, y);
\]

(A4) For each \( x \in C \), \( y \mapsto \Theta(x, y) \) is convex and lower semicontinuous.

**Definition 2.16** [17] Let \( C \) be a nonempty, closed and convex subsets of a real reflexive Banach space and let \( \varphi \) be a lower semicontinuous and convex function from \( C \) to \( \mathbb{R} \) and \( \Psi : C \to E^* \) be a continuous monotone mapping. Let \( \Theta : C \times C \to \mathbb{R} \) be a bifunction satisfying (A1)–(A4). The mixed resolvent of \( \Theta \) is the operator

\[
\text{Res}^f_{\Theta, \varphi, \Psi} : E \to 2^C
\]

\[
\text{Res}^f_{\Theta, \varphi, \Psi}(x) = \left\{ z \in C : \Theta(z, y) + \varphi(y) + \langle \Psi x, y - z \rangle + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq \varphi(z), \forall y \in C \right\}.
\] (2.1)

**Lemma 2.17** [17] Let \( f : E \to (-\infty, +\infty] \) be a coercive Legendre function. Let \( C \) be a closed and convex subset of \( E \). If the bifunction \( \Theta : C \times C \to \mathbb{R} \) satisfies conditions (A1)–(A4), then

1. \( \text{Res}^f_{\Theta, \varphi, \Psi} \) is single-valued and \( \text{dom}(\text{Res}^f_{\Theta, \varphi, \Psi}) = E \);
2. \( \text{Res}^f_{\Theta, \varphi, \Psi} \) is a BFNE operator;
3. \( F(\text{Res}^f_{\Theta, \varphi, \Psi}) = \text{GMEP}(\Theta) \);
4. \( \text{GMEP}(\Theta) \) is closed and convex;
5. \( D_f(p, \text{Res}^f_{\Theta, \varphi, \Psi}(x)) + D_f(\text{Res}^f_{\Theta, \varphi, \Psi}(x), x) \leq D_f(p, x), \forall p \in F(\text{Res}^f_{\Theta, \varphi, \Psi}), x \in E \).

**Lemma 2.18** [20, Proposition 5] [30] Let \( f : E \to \mathbb{R} \) be a Legendre function such that \( \nabla f^* \) is bounded on bounded subsets of \( \text{int}(\text{dom} f^*) \). For \( x \in E \), if \( \{D_f(x, x_n)\} \) is bounded, then the sequence \( \{x_n\} \) is bounded.

### 3 Main Result

In this section, we prove our main theorem.

**Theorem 3.1** Let \( E \) be a real reflexive Banach space, \( C \) be a nonempty, closed and convex subset of \( E \). Let \( f : E \to \mathbb{R} \) be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of \( E \), and \( \nabla f^* \) be bounded on bounded subsets of \( E^* \). Let \( T : C \to C \) be a weak Bregman relatively nonexpansive mapping, \( \Theta : C \times C \to \mathbb{R} \) satisfies conditions (A1)–(A4), \( \varphi : C \to \mathbb{R} \) is real-valued
convex function and $\Psi : C \to E^*$ is continuous monotone mapping. Assume that $F(T) \cap GMEP(\Theta)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by
\[
\begin{align*}
  z_n &= \nabla f^*(\beta_n \nabla f(T(x_n)) + (1 - \beta_n) \nabla f(x_n)), \\
  y_n &= \nabla f^*(\alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(z_n)), \\
  u_n &= \text{Res}_{\Theta,\varphi,\Psi}(y_n), \\
  C_{n+1} &= \{z \in C : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n) D_f(z, x_n)\}, \\
  Q_{n+1} &= \{z \in Q_n : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\
  x_{n+1} &= \text{proj}_{C_{n+1} \cap Q_{n+1}} x_0, \quad \forall n \geq 0,
\end{align*}
\] (3.1)
where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$ and $\liminf_{n \to \infty} (1 - \alpha_n) \beta_n > 0$. Let $x_0 \in C$ be chosen arbitrarily, $Q_0 = C$ and $C_0 = \{z \in C : D_f(z, u_0) \leq D_f(z, x_0)\}$. Then, $\{x_n\}$ converges strongly to $\text{proj}^f_{F(T) \cap GMEP(\Theta)} x_0$.

Proof. We prove our theorem by several claims:

Claim 1 The sequence $\{x_n\}$ in (3.1) is well defined.

We note from Lemmas 2.15 and 2.17 that $F(T)$ and $GMEP(\Theta)$ are closed and convex.

First, we show that $C_n$ and $Q_n$ are closed and convex subsets of $E$. It is clear that $C_0$ and $Q_0$ are closed and convex subsets. Suppose that $C_n$ and $Q_n$ are closed and convex subsets of $E$ for some $n \geq 0$. We rewrite the set $C_{n+1}$ in the following form
\[
C_{n+1} = C_n \cap \{z \in E : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n) D_f(z, x_n)\}
\]
\[
= C_n \cap \{z \in E : \langle \alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(x_n), z - x_n \rangle \leq \alpha_n \langle \nabla f(x_0), x_0 \rangle
\]
\[
+ (1 - \alpha_n) \langle \nabla f(x_n), x_n \rangle - \alpha_n f(x_0) - (1 - \alpha_n) f(x_n) + f(u_n) - \langle \nabla f(u_n), u_n \rangle.
\]
Thus, $C_{n+1}$ is a closed and convex subset of $E$.

Next, it follows from
\[
Q_{n+1} = Q_n \cap \{z \in E : \langle \nabla f(x_0) - \nabla f(x_n), z \rangle \leq \langle \nabla f(x_0) - \nabla f(x_n), x_n \rangle
\]
that $Q_{n+1}$ is also a closed and convex subset of $E$.

Now, in order to finish the proof of this claim, we will prove that $F(T) \cap GMEP(\Theta) \subset C_n \cap Q_n$ for all $n \geq 0$. Indeed, obviously $F(T) \cap GMEP(\Theta) \subset C_0 \cap Q_0$. We suppose that $F(T) \cap GMEP(\Theta) \subset C_n \cap Q_n$ for some $n \geq 0$.

Let $p \in F(T) \cap GMEP(\Theta)$, from (3.1) and Lemma 2.17, we have
\[
D_f(p, u_n) = D_f(p, \text{Res}_{\Theta,\varphi,\Psi}^f(y_n)) \leq D_f(p, y_n) - D_f\left(\text{Res}_{\Theta,\varphi,\Psi}^f(y_n), y_n\right).
\] (3.2)

Next, we have
\[
D_f(p, y_n) = D_f(p, \nabla f^*(\alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(z_n)))
\]
\[
\leq \alpha_n D_f(p, x_0) + (1 - \alpha_n) D_f(p, z_n).
\] (3.3)

We now estimate $D_f(p, z_n)$. It follows from (3.1), and the property of $T$ that
\[
D_f(p, z_n) = D_f(p, \beta_n \nabla f(T(x_n)) + (1 - \beta_n) \nabla f(x_n))
\]
\[
\leq \beta_n D_f(p, T(x_n)) + (1 - \beta_n) D_f(p, x_n) \leq D_f(p, x_n).
\] (3.4)

From (3.2)–(3.4), we get that
\[
D_f(p, u_n) \leq \alpha_n D_f(p, x_0) + (1 - \alpha_n) D_f(p, x_n).
\]
This implies that $p \in C_{n+1}$ and hence $F(T) \cap GMEP(\Theta) \subset C_{n+1}$. 
Since \( x_n = \text{proj}_{C_n \cap Q_n}^f (x_0) \), it follows from Lemma 2.14 that
\[
\langle \nabla f(x_0) - \nabla f(x_n), x_n - v \rangle \geq 0, \quad \forall v \in C_n \cap Q_n.
\]
Thus, from \( p \in F(T) \cap GMEP(\Theta) \subset C_n \cap Q_n \), we obtain that
\[
\langle \nabla f(x_0) - \nabla f(x_n), x_n - p \rangle \geq 0,
\]
that is, \( p \in Q_{n+1} \) and hence \( F(T) \cap GMEP(\Theta) \subset Q_{n+1} \). So, we deduce that \( F(T) \cap GMEP(\Theta) \subset C_{n+1} \cap Q_{n+1} \). By mathematical induction, we get that \( F(T) \cap GMEP(\Theta) \subset C_n \cap Q_n \) for all \( n \geq 0 \).

Thus, \( C_n \cap Q_n \) is a nonempty, closed and convex subset of \( E \) for all \( n \geq 0 \) and hence the sequence \( \{x_n\} \) is well defined.

**Claim 2** In (3.1), the sequence \( \{x_n\} \) is bounded.

Since \( \langle \nabla f(x_0) - \nabla f(x_n), v - x_n \rangle \leq 0 \) for all \( v \in Q_{n+1} \), it follows from Lemma 2.14 that \( x_n = \text{proj}_{Q_{n+1}}^f x_0 \) and by \( x_{n+1} = \text{proj}_{C_n \cap Q_{n+1}}^f x_0 \in Q_{n+1} \), we have
\[
D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0).
\]
(3.5)

Let \( p \in F(T) \cap GMEP(\Theta) \in Q_{n+1} \). It follows from Lemma 2.14 that
\[
D_f\left(p, \text{proj}_{Q_{n+1}}^f x_0\right) + D_f\left(\text{proj}_{Q_{n+1}}^f x_0, x_0\right) \leq D_f(p, x_0)
\]
and so
\[
D_f(x_n, x_0) \leq D_f(p, x_0) - D_f(p, x_n) \leq D_f(p, x_0).
\]

Therefore, \( \{D_f(x_n, x_0)\} \) is bounded. By Lemma 2.11 \( \{x_n\} \) is bounded and so are \( \{T(x_n)\}, \{y_n\}, \{z_n\}. \)

**Claim 3** In (3.1), the sequence \( \{x_n\} \) is a Cauchy sequence.

By the proof of Claim 2, we know that \( \{D_f(x_n, x_0)\} \) is bounded. It follows from (3.5) that \( \lim_{n \to \infty} D_f(x_n, x_0) \) exists. From \( x_m \in Q_m \subset Q_{n+1} \) for all \( m > n \) and Lemma 2.14, we have
\[
D_f\left(x_m, \text{proj}_{Q_{n+1}}^f x_0\right) + D_f\left(\text{proj}_{Q_{n+1}}^f x_0, x_0\right) \leq D_f(x_m, x_0)
\]
and hence \( D_f(x_m, x_n) \leq D_f(x_m, x_0) - D_f(x_n, x_0) \). Therefore, we have
\[
\lim_{n \to \infty} D_f(x_m, x_n) \leq \lim_{n, m \to \infty} (D_f(x_m, x_0) - D_f(x_n, x_0)) = 0.
\]
(3.6)

Since \( f \) is totally convex on bounded subsets of \( E \), by Definition 2.6, Lemma 2.9 and (3.6) we obtain
\[
\lim_{n \to \infty} \|x_m - x_n\| = 0.
\]
Thus \( \{x_n\} \) is a Cauchy sequence and so \( \lim_{n \to \infty} \|x_{n+1} - x_n\| = 0 \).

Now, we prove that the sequence \( \{x_n\} \) generated by (3.1) converges strongly to \( x^\dagger = \text{proj}_{F(T) \cap GMEP(\Theta)}^f x_0 \).

From the proof of Claim 2, the sequence \( \{x_n\} \) is a Cauchy sequence. Without loss of generality, let \( x_n \to q \in C \). Since \( f \) is uniformly Fréchet differentiable on bounded subsets of \( E \), it follows from Lemma 2.9 that \( \nabla f \) is norm-to-norm uniformly continuous on bounded subsets of \( E \). Hence, by \( \|x_{n+1} - x_n\| \to 0 \), we have
\[
\lim_{n \to \infty} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| = 0.
\]
Since \( x_{n+1} \in C_{n+1} \subset C_n \), we have
\[
D_f(x_{n+1}, u_n) \leq \alpha_n D_f(x_{n+1}, x_0) + (1 - \alpha_n) D_f(x_{n+1}, x_n).
\]
It follows from $\lim_{n \to \infty} \alpha_n = 0$ and $\lim_{n \to \infty} D_f(x_{n+1}, x_n) = 0$ that $\{D_f(x_{n+1}, u_n)\}$ is bounded and

$$\lim_{n \to \infty} D_f(x_{n+1}, u_n) = 0.$$ 

By Lemma 2.7, we obtain

$$\lim_{n \to \infty} \|x_{n+1} - u_n\| = 0. \quad (3.7)$$

So,

$$\lim_{n \to \infty} \|\nabla f(x_{n+1}) - \nabla f(u_n)\| = 0. \quad (3.8)$$

Taking into account that $\|x_n - u_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - u_n\|$, we obtain

$$\lim_{n \to \infty} \|x_n - u_n\| = 0$$

so, $u_n \to q$ as $n \to \infty$.

By the properties of the Bregman distance we have

$$D_f(p, x_{n+1}) - D_f(p, u_n) = f(p) - f(x_{n+1}) - \langle \nabla f(x_{n+1}), p - x_{n+1} \rangle - f(p) + f(u_n) + \langle \nabla f(u_n), p - u_n \rangle$$

$$= f(u_n) - f(x_{n+1}) + \langle \nabla f(u_n), p - u_n \rangle - \langle \nabla f(x_{n+1}), p - x_{n+1} \rangle$$

for each $p \in F(T)$. By (3.7)–(3.8), we obtain

$$\lim_{n \to \infty} (D_f(p, x_{n+1}) - D_f(p, u_n)) = 0. \quad (3.9)$$

On the other hand, for any $p \in F(T) \cap GMEP(\Theta)$ by Lemma 2.17, we have

$$D_f(u_n, y_n) \leq D_f(p, y_n) - D_f(p, u_n)$$

$$\leq D_f(p, \nabla f^\ast (\alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(z_n))) - D_f(p, u_n)$$

$$\leq \alpha_n D_f(p, x_0) + (1 - \alpha_n) D_f(p, \nabla f^\ast (\beta_n \nabla f(T(x_n)) + (1 - \beta_n) \nabla f(x_n))) - D_f(p, u_n)$$

$$\leq \alpha_n D_f(p, x_0) + (1 - \alpha_n) D_f(p, x_n) - D_f(p, u_n)$$

for each $p \in F(T)$. By (3.7)–(3.8), we obtain

$$\lim_{n \to \infty} (D_f(p, x_{n+1}) - D_f(p, u_n)) = 0. \quad (3.9)$$

So, by (3.9) and (3.10) we have $D_f(u_n, y_n) = 0$ and $D_f(p, y_n) - D_f(p, u_n) \to 0$ as $n \to \infty$. Moreover, $\lim_{n \to \infty} \|u_n - y_n\| = 0$ and thus $\lim_{n \to \infty} \|\nabla f(u_n) - \nabla f(y_n)\| = 0$. Since $u_n \to q$ as $n \to \infty$, we have $y_n \to q$ as $n \to \infty$.

Here, we prove that $q \in GMEP(\Theta)$. It follows from (3.2) and

$$D_f(p, y_n) - D_f(p, u_n) \to 0$$

that

$$D_f(u_n, y_n) \to 0. \quad (3.11)$$

Moreover, from (3.11), we also have that

$$\|\nabla f(u_n) - \nabla f(y_n)\| \to 0. \quad (3.12)$$

Also, consider that $u_n = \text{Res}_T^f \Theta, \varphi, \psi(y_n)$, so we have

$$\Theta(u_n, y) + \langle \Psi y_n, y - u_n \rangle + \varphi(y) + \langle \nabla u_n - \nabla y_n, y - u_n \rangle \geq \varphi(u_n)$$

for all $y \in C$. 

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From (A2), we have
\[
\Theta(y, u_n) \leq -\Theta(u_n, y) \leq \langle \Psi y_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \langle \nabla u_n - \nabla y_n, y - u_n \rangle
\]
for all \( y \in C \). Hence,
\[
\Theta(y, u_n) \leq \langle \Psi y_n, y - u_n \rangle + \varphi(y) - \varphi(u_n) + \langle \nabla u_n - \nabla y_n, y - u_n \rangle
\]
for \( y \in C \). Since \( u_n \to q \) and (3.12), from continuity of \( \Psi \) and weak lower semicontinuity of \( \varphi \) and \( \Theta(\cdot, \cdot) \) in the second variable \( y \), we also have
\[
\Theta(y, q) + \langle \Psi q, q - y \rangle + \varphi(q) - \varphi(y) \leq 0
\]
for all \( y \in C \).

For \( t \) with \( 0 \leq t \leq 1 \) and \( y \in C \), let \( y_t = ty + (1-t)q \). Since \( y \in C \) and \( q \in C \) we have \( y_t \in C \) and hence \( \Theta(y_t, q) + \langle \Psi q, q - y_t \rangle + \varphi(q) - \varphi(y_t) \leq 0 \). So, we have
\[
0 = \Theta(y_t, y_t) + \langle \Psi q, y_t - y_t \rangle + \varphi(y_t) - \varphi(y_t)
\]
\[
\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, q) + t\langle \Psi q, y - y_t \rangle + (1-t)\langle \Psi q, q - y_t \rangle
\]
\[
+ t\varphi(y) + (1-t)\varphi(q) - \varphi(y_t)
\]
\[
\leq t[\Theta(y_t, y) + \langle \Psi q, y - y_t \rangle + \varphi(y) - \varphi(y_t)].
\]

Therefore, \( \Theta(y_t, y) + \langle \Psi q, y - y_t \rangle + \varphi(y) - \varphi(y_t) \geq 0 \). Then, we have
\[
\Theta(q, q) + \langle \Psi q, q - q \rangle + \varphi(q) - \varphi(q) \geq 0
\]
for all \( y \in C \). Hence, we have \( q \in GMEP(\Theta) \).

Now, we prove that \( q \in F(T) \). Note that
\[
\|\nabla f(x_n) - \nabla f(y_n)\| = \|\nabla f(x_n) - \nabla f(\nabla f^*(\alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(z_n)))\|
\]
\[
= \|\nabla f(x_n) - (\alpha_n \nabla f(x_0) + (1 - \alpha_n) \nabla f(z_n))\|
\]
\[
= \|\alpha_n \nabla f(x_n) - \nabla f(x_0) + (1 - \alpha_n) (\nabla f(x_n) - \nabla f(z_n))\|
\]
\[
= \|\alpha_n \nabla f(x_n) - \nabla f(x_0) + (1 - \alpha_n) (\nabla f(x_n) - \nabla f(z_n)) - \nabla f(\nabla f^*(\beta_n \nabla f(T(x_n))) + (1 - \beta_n) \nabla f(x_n)))\|
\]
\[
\geq (1 - \alpha_n) \beta_n \|\nabla f(x_n) - \nabla f(T(x_n))\| - \alpha_n \|\nabla f(x_n) - \nabla f(x_0)\|.
\]

This implies that
\[
(1 - \alpha_n) \beta_n \|\nabla f(x_n) - \nabla f(T(x_n))\|
\]
\[
\leq \alpha_n \|\nabla f(x_n) - \nabla f(x_0)\| + \|\nabla f(x_n) - \nabla f(y_n)\|.
\]

Letting \( n \to \infty \) in the above inequality, it follows from \( \liminf_{n \to \infty} (1 - \alpha_n) \beta_n > 0 \) and \( \lim_{n \to \infty} \alpha_n = 0 \) that
\[
\lim_{n \to \infty} \|\nabla f(x_n) - \nabla f(T(x_n))\| = 0.
\]

So, we have \( \lim_{n \to \infty} |x_n - T(x_n)| = 0 \). This together with \( x_n \to q \) implies that \( q \in \tilde{F}(T) \).

Since \( \tilde{F}(T) = F(T) \), we have \( q \in F(T) \cap GMEP(\Theta) \). Therefore, the sequence \( \{x_n\} \) converges strongly to a point \( q \in F(T) \cap GMEP(\Theta) \).

Finally, we prove that \( q = x^\dagger = \text{proj}_{F(T) \cap GMEP(\Theta)}^f(x_0) \). Since
\[
x^\dagger = \text{proj}_{F(T) \cap GMEP(\Theta)}^f(x_0) \in F(T) \cap GMEP(\Theta)
\]
it follows from $x_{n+1} = \text{proj}_{C_{n+1}\cap Q_{n+1}}^f x_0$ and $x^\dagger \in F(T) \cap \text{GMEP}^{\Theta} \subset C_{n+1}\cap Q_{n+1}$ that
\[ D_f(x_{n+1}, x_0) \leq D_f(x^\dagger, x_0). \]

Hence by Lemma 2.12, we have $x_n \to x^\dagger$ as $n \to \infty$. Thus $q = x^\dagger$. Therefore, the sequence $\{x_n\}$ converges strongly to the point $x^\dagger = \text{proj}_{F(T)\cap \text{GMEP}(\Theta)}^f x_0$.

This completes the proof. 

Let $\varphi = \Psi = 0$, then we have the result of [1] as follows:

**Corollary 3.2** Let $E$ be a real reflexive Banach space, $C$ be a nonempty, closed and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$, and $\nabla f^*$ be bounded on bounded subsets of $E^*$. Let $T : C \to C$ be a weak Bregman relatively nonexpansive mapping, $\Theta : C \times C \to \mathbb{R}$ satisfies conditions (A1)–(A4). Assume that $F(T) \cap EP(\Theta)$ is nonempty and bounded. Let $\{x_n\}$ be a sequence generated by
\[
z_n = \nabla f^*(\beta_n \nabla f(T(x_n)) + (1 - \beta_n)\nabla f(x_n)), \\
y_n = \nabla f^*(\alpha_n \nabla f(x_0) + (1 - \alpha_n)\nabla f(z_n)), \\
u_n = \text{Res}_\varphi^f(y_n), \\
C_{n+1} = \{z \in C_n : D_f(z, u_n) \leq \alpha_n D_f(z, x_0) + (1 - \alpha_n)D_f(z, x_n)\}, \\
Q_{n+1} = \{z \in Q_n : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n \rangle \leq 0\}, \\
x_{n+1} = \text{proj}_{C_{n+1}\cap Q_{n+1}}^f x_0, \forall n \geq 0,
\]
where $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$ and $\lim \inf_{n \to \infty} (1 - \alpha_n)\beta_n > 0$. Let $x_0 \in C$ be chosen arbitrarily, $Q_0 = C$ and $C_0 = \{z \in C : D_f(z, u_0) \leq D_f(z, x_0)\}$. Then, $\{x_n\}$ converges strongly to $\text{proj}_{F(T)\cap EP(\Theta)}^f x_0$.

**4 Numerical Example**

In this section, we present an example illustrating the behavior of the iterative algorithm presented in this paper.

**Example 4.1** Let $E = \mathbb{R}^N$, $C = \{x \in \mathbb{R}^N : \frac{-N}{2} \leq x_i \leq \frac{N}{2}\}$, and $f : \mathbb{R}^N \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}\|x\|^2$. Let $T : C \to C$ be defined by $Tx = \frac{2}{N}x$, and the bifunction $\Theta : C \times C \to \mathbb{R}$ be defined by $\Theta(x, y) = x(y - x)$ for all $x, y \in C$, $\varphi : C \to \mathbb{R}$ be defined by $\varphi(x) = x^2$ for all $x \in C$ and $\Psi : C \to E^*$ such that $\Psi(x) = \sin(x)$ for all $x \in C$.

We observe that $f$ is a coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $\mathbb{R}^N$ and $\nabla f(x) = x$. Since $f^*(x^*) = \sup\{(x^*, x) - f(x) : x \in \mathbb{R}^N\}$, we obtain that $f^*(u) = \frac{1}{2}\|u\|^2$ and $\nabla f^*(u) = u$. Further, we observe that $T$ is a weak Bregman relatively nonexpansive mapping with $F(T) = \{0\} = F(T)$. We also observe that $\Theta$ satisfies conditions (A1)–(A4) and $\varphi$ is a convex function and $\Psi$ is continuous monotone mapping. Moreover, we have $\text{GMEP}(\Theta) = \{0\} = F(T)$. Let $\{x_n\}$ be generated by the algorithm (3.1). Then the sequence $\{x_n\}$ converges strongly to $\text{proj}_{F(T)\cap \text{GMEP}(\Theta)}^f x_0 = 0$. The algorithm (3.1) is
checked by using the stopping criterion $||x_n - x_{n+1}|| < 10^{-3}$. Set $\{\alpha_n\} = \{\frac{1}{n+1}\}$ and the following four cases of the control parameter $\{\beta_n\}$ are considered:

Case 1 $\beta_n = 0.99 - \frac{1}{n+1}$.

Case 2 $\beta_n = 10^{-10} + \frac{1}{n+1}$.

Case 3 $\beta_n = 0.5 + \frac{1}{n+2}$.

Case 4 $\beta_n = 0.5 - \frac{1}{n+2}$.

We generate the starting points $x_0$ by 10 randomly starting points and present results on average which are showed in Table 1 and Fig. 1.

| No. | Average iterations |
|-----|-------------------|
|     | Case 1 | Case 2 | Case 3 | Case 4 |
| 5   | 354    | 213    | 499    | 295    |
| 10  | 917    | 384    | 779    | 322    |
| 100 | 1237   | 1460   | 1271   | 1662   |
| 500 | 3070   | 2615   | 3779   | 1741   |

Fig. 1 The numerical results for the generalized mixed equilibrium problem for each case of $\{\beta_n\}$
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