MATRIX RANDOM PRODUCTS WITH SINGULAR HARMONIC MEASURE

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Abstract. Any Zariski dense countable subgroup of $SL(d, \mathbb{R})$ is shown to carry a non-degenerate finitely supported symmetric random walk such that its harmonic measure on the flag space is singular. The main ingredients of the proof are: (1) a new upper estimate for the Hausdorff dimension of the projections of the harmonic measure onto Grassmannians in $\mathbb{R}^d$ in terms of the associated differential entropies and differences between the Lyapunov exponents; (2) an explicit construction of random walks with uniformly bounded entropy and Lyapunov exponents going to infinity.

Introduction

The notion of harmonic measure (historically first defined in the framework of the theory of potential) has an explicit probabilistic description in terms of the dynamical properties of the associated Markov processes as a hitting distribution. Moreover, “hitting” can be interpreted both as attaining the target set in the usual sense (in finite time) and as converging to it at infinity (when the target is attached as a boundary to the state space).

In concrete situations the target set is usually endowed with additional structures giving rise to other “natural” measures (e.g., smooth, uniform, Haar, Hausdorff, maximal entropy, etc.), which leads to the question about coincidence of the harmonic and these “other” measures (or, in a somewhat weaker form, about coincidence of the respective measure classes). As a general rule, such a coincidence either implies that the considered system has very special symmetry properties or is not possible at all. However, establishing it in a rigorous way is a notoriously difficult problem. See, for example, the cases of the Brownian motion on cocompact negatively curved Riemannian manifolds [Kat88, Led95], of Julia sets of endomorphisms of the Riemann sphere [PUZ89, Zdu91] and of polynomial-like maps [LV95, BPV97, Zdu97], or of Cantor repellers in a Euclidean space [MV86, Vol93].

In the present paper we consider the singularity problem for random matrix products $x_n = h_1 h_2 \ldots h_n$ with Bernoulli increments, or, in other words, for random walks on the group $SL(d, \mathbb{R})$ (or its subgroups). Actually, our results are also valid for general non-compact semi-simple Lie groups, see [LP04], but for the sake of expositional simplicity we restrict ourselves just to the case of matrix groups. Another simplification is that we always assume that the considered subgroups of $SL(d, \mathbb{R})$ are Zariski dense. It guarantees that the harmonic measure of the random walk is concentrated on the space of full flags in $\mathbb{R}^d$ (rather than on its quotient determined by the degeneracies of the Lyapunov spectrum). Modulo an appropriate technical modification our results remain valid without this assumption as well.

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If the distribution $\mu$ of the increments $\{h_n\}$ has a finite first moment $\int \log \|h\| \, d\mu(h)$, then by the famous Oseledec multiplicative ergodic theorem [Ose68] there exists the Lyapunov spectrum $\lambda$ consisting of Lyapunov exponents $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_d$ (they determine the growth of the random products in various directions), and, moreover, a.e. sample path $\{x_n\}$ gives rise to the associated Lyapunov flag (filtration) of subspaces in $\mathbb{R}^d$. The distribution $\nu = \nu(\mu)$ of these Lyapunov flags is then naturally called the harmonic measure of the random product. The geometric interpretation of the Oseledec theorem [Kai89] is that the sequence $x_n o$ asymptotically follows a geodesic in the associated Riemannian symmetric space $S = SL(d, \mathbb{R})/SO(d)$ (here $o = SO(d) \in S$); the Lyapunov spectrum determines the Cartan (“radial”) part of this geodesic, whereas the Lyapunov flag determines its direction.

If $\text{supp} \mu$ generates a Zariski dense subgroup of $SL(d, \mathbb{R})$, then the Lyapunov spectrum is simple (≡ the vector $\lambda$ lies in the interior of the positive Weyl chamber), see [GR85, GM89], so that the associated Lyapunov flags are full (≡ contain subspaces of all the intermediate dimensions). The space $\mathcal{B} = \mathcal{B}(d)$ of full flags in $\mathbb{R}^d$ is also known under the name of the Furstenberg boundary of the associated symmetric space $S$, see [Fur63b] for its definition and [Kai89, GJT98] for its relation with the boundaries of various compactifications of Riemannian symmetric spaces. In the case when the Lyapunov spectrum is simple, a.e. sequence $x_n o$ is convergent in all reasonable compactifications of the symmetric space $S$, and the corresponding hitting distributions can be identified with the harmonic measure $\nu$ on $\mathcal{B}$.

The flag space $\mathcal{B}$ is endowed with a natural smooth structure, therefore it makes sense to compare the harmonic measure class with the smooth measure class (the latter class contains the unique rotation invariant measure on the flag space). The harmonic measure is ergodic, so that it is either absolutely continuous or singular with respect to the smooth (or any other quasi-invariant) measure class. Accordingly, we shall call the jump distribution $\mu$ either absolutely continuous or singular at infinity.

If the measure $\mu$ is absolutely continuous with respect to the Haar measure on the group $SL(d, \mathbb{R})$ (or even weaker: a certain convolution power of $\mu$ contains an absolutely continuous component) then it is absolutely continuous at infinity [Fur63b, Aze70].

As it turns out, there are also measures $\mu$ which are absolutely continuous at infinity in spite of being supported by a discrete subgroup of $SL(d, \mathbb{R})$. Namely, Furstenberg [Fur71] showed that any lattice (for instance, $SL(d, \mathbb{Z})$) carries a probability measure $\mu$ with a finite first moment which is absolutely continuous at infinity. It was used for proving one of the first results on the rigidity of lattices in semi-simple Lie groups.

Furstenberg’s construction of measures absolutely continuous at infinity (based on discretization of the Brownian motion on the associated symmetric space) was further extended and generalized in [LS84, Kai92, BL96]. Another construction of random walks with a given harmonic measure was recently developed by Connell and Muchnik [CM07a, CM07b]. Note that the measures $\mu$ arising from all these constructions are inherently infinitely supported.

Let us now look at the singularity vs. absolute continuity dichotomy for the harmonic measure from the “singularity end”. The first result of this kind was obtained by Chatterji [Cha66] who established singularity of the distribution of infinite continuous fractions with independent digits. This distribution can indeed be viewed as the harmonic measure associated to a certain random walk on $SL(2, \mathbb{Z}) \subset SL(2, \mathbb{R})$ [Fur63a]. See [CLMS1] for an explicit description of the harmonic measure in a similar situation and [KPVWW] for a recent very general result on singularity of distributions of infinite continuous fractions.
Extending the continuous fractions setup Guivarc’h and Le Jan [GLJ90, GLJ93] proved that the measures $\mu$ on non-compact lattices in $SL(2, \mathbb{R})$ satisfying a certain moment condition (in particular, all finitely supported measures) are singular at infinity.

Together with some other circumstantial evidence (as, for instance, pairwise singularity of various natural boundary measure classes on the visibility boundary of the universal cover of compact negatively curved manifolds [Led95] or on the hyperbolic boundary of free products [KLP]) the aforementioned results lead to the following

**Conjecture.** Any finitely supported probability measure on $SL(d, \mathbb{R})$ is singular at infinity.

The principal result of the present paper is the following

**Main Theorem.** Any countable Zariski dense subgroup $\Gamma$ of $SL(d, \mathbb{R})$ carries a non-degenerate symmetric probability measure $\mu$ which is singular at infinity. Moreover, if $\Gamma$ is finitely generated then the measure $\mu$ can be chosen to be finitely supported.

Note here an important difference between the case $d = 2$ and the higher rank case $d \geq 3$. If $\Gamma$ is discrete, then for $d = 2$ the boundary circle can be endowed with different $\Gamma$-invariant smooth structures (parameterized by the points from the Teichmüller space of the quotient surface), so that Furstenberg’s discretization construction readily provides existence of measures $\mu$ (not finitely supported though!) which are singular at infinity. Obviously, this approach does not work in the higher rank case, where our work provides first examples of measures singular at infinity.

Our principal tool for establishing singularity is the notion of the dimension of a measure. Namely, we show that in the setup of our Main Theorem the measure $\mu$ can be chosen in such a way that the Hausdorff dimension of the associated harmonic measure (more precisely, of its projection onto one of the Grassmani ans in $\mathbb{R}^d$) is arbitrarily small, which obviously implies singularity. For doing this we establish an inequality connecting the Hausdorff dimension with the asymptotic entropy and the Lyapunov exponents of the random walk.

There are several notions of dimension of a probability measure $m$ on a compact metric space $(Z, \rho)$ (see [Pes97] and the discussion in Section 6 below). These notions roughly fall into two categories. The global ones are obtained by looking at the dimension of sets which “almost” (up to a piece of small measure $m$) coincide with $Z$; in particular, the Hausdorff dimension $\dim_H m$ of the measure $m$ is $\inf \{\dim_H A : m(A) = 1\}$, where $\dim_H A$ denotes the Hausdorff dimension of a subset $A \subset Z$. On the other hand, the local ones are related to the asymptotic behavior of the measures of concentric balls $B(z, r)$ in $Z$ as the radius $r$ tends to 0. More precisely, the lower $\underline{\dim}_m(z)$ (resp., the upper $\overline{\dim}_m(z)$) pointwise dimension of the measure $m$ at a point $z \in Z$ is defined as the $\lim \inf$ (resp., $\lim \sup$) of the ratio $\log mB(z, r)/\log r$ as $r \to 0$. In these terms $\dim_H m$ coincides with $\ess \sup_z \underline{\dim}_m(z)$. In particular, if $\underline{\dim}_m(z) = \overline{\dim}_m(z) = D$ almost everywhere for a constant $D$, then $\dim_H m = D$. Moreover, in the latter case all the reasonable definitions of dimension of the measure $m$ coincide.

Numerous variants of the formula $\dim m = h/\lambda$ relating the dimension of an invariant measure $m$ of a differentiable map with its entropy $h = h(m)$ and the characteristic exponent(s) $\lambda = \lambda(m)$ have been known since the late 70s – early 80s, see [Led81, Yon82] and the references therein. Ledrappier [Led83] was the first to carry it over to the context of random walks by establishing the formula $\dim \nu = h/2\lambda$ for the dimension of the harmonic measure of random walks on discrete subgroups of $SL(2, \mathbb{C})$. Here $h = h(\mu)$
is the asymptotic entropy of the random walk with the jump distribution $\mu$, and $\lambda = \lambda(\mu)$ is the Lyapunov exponent. However, the dimension appearing in this formula is somewhat different from the classical Hausdorff dimension, because convergence of the ratios $\log mB(z,r)/\log r$ to the value of dimension is only established in measure (rather than almost surely).

Le Prince has recently extended the approach of Ledrappier (also see [Led01, Kai98]) to a general discrete group $G$ of isometries of a Gromov hyperbolic space $X$; in this case the Gromov product induces a natural metric (more rigorously, a gauge) on the hyperbolic boundary $\partial X$. He proved that if a probability measure $\mu$ on $G$ has a finite first moment with respect to the metric on $X$, then the associated harmonic measure $\nu$ on $\partial X$ has the property that $\dim_{\mu}\nu(z) \leq h/\ell$ for $\nu$-a.e. point $z \in \partial X$ (which implies that $\dim_{H}\nu \leq h/\ell$) [LP07], and that the box dimension of the measure $\nu$ is precisely $h/\ell$ [LP08] (here, as before, $h = h(\mu)$ is the asymptotic entropy of the random walk $(G,\mu)$, and $\ell = \ell(\mu)$ is its linear rate of escape with respect to the hyperbolic metric). Note that the question about the pointwise dimension, i.e., about the asymptotic behaviour of the ratios $\log mB(z,r)/\log r$ for $\nu$-a.e. point $z \in \partial X$, is still open in this generality. It was recently proved that these ratios converge to $h/\ell$ almost surely for any symmetric finitely supported random walk on $G$ [BHM08]. This should also be true for finitely supported random walks which are not necessarily symmetric. Namely, the results of Ancona [Anc88] on almost multiplicativity of the Green function imply that in this situation the harmonic measure has the doubling property, which in turn can be used to prove existence of the pointwise dimension.

Yet another example is the inequality $\dim_{H}\nu \leq h/\ell$ for the Hausdorff dimension of the harmonic measure of iterated function systems in the Euclidean space [NSB02] (when is the equality attained?).

In the context of random walks on general countable groups the asymptotic entropy $h(\mu)$, the rate of escape $\ell(\mu)$ and the exponential growth rate of the group $\nu$ satisfy the inequality $h(\mu) \leq \ell(\mu)\nu$ (it was first established by Guivarc’h [Gui80]; recently Vershik [Ver00] revitalized interest in it). As it was explained above, in the “hyperbolic” situations the ratio $h/\ell$ can (under various additional conditions) be interpreted as the dimension of the harmonic measure, whereas $\nu$ is the Hausdorff dimension of the boundary itself.

Unfortunately, the aforementioned formulas of the type $\dim \nu = h/\ell$ can not be directly carried over to the higher rank case. The reason for this is that in this case the boundary is “assembled” of several components which may have different dimension properties. The simplest illustration is provided by the product $\Gamma_1 \times \Gamma_2$ of two discrete subgroups of $SL(2,\mathbb{R})$ with a product jump distribution $\mu = \mu_1 \otimes \mu_2$. The Furstenberg boundary of the bidisk associated with the group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ is the product of the boundary circles of each hyperbolic disc. The harmonic measure of the jump distribution $\mu$ is then the product of the harmonic measures of the jump distributions $\mu_1$ and $\mu_2$ which may have different dimensions (it would be interesting to investigate the dimension properties of the harmonic measure on the boundary of the polydisc for groups and random walks which do not split into a direct product). In the case of $SL(d,\mathbb{R})$ the role of elementary “building blocks” of the flag space $\mathcal{B}$ is played by the Grassmannians $\mathcal{G}_i$ in $\mathbb{R}^d$ (which are the minimal equivariant quotients of $\mathcal{B}$).

In order to deal with spaces with a $\mu$-stationary measure other than the Poisson boundary of the random walk $(\Gamma,\mu)$ it is convenient to use the notion of the differential $\mu$-entropy first introduced by Furstenberg [Fur71]. It measures the “amount of information” about the random walk contained in the action. The differential entropy does not exceed the
asymptotic entropy of the random walk, coincides with it for the Poisson boundary, and is strictly smaller for any proper quotient of the Poisson boundary \([KV83]\). Our point is that in the formulas of the type \(\dim \nu = h/\ell\) (see above) for spaces with a \(\mu\)-stationary measure \(\nu\) other than the Poisson boundary, the asymptotic entropy \(h\) should be replaced with the differential entropy of the space.

In the setup of our Main Theorem let \(\mu\) be a non-degenerate probability measure on \(\Gamma\) with a finite first moment, so that the associated Lyapunov spectrum \(\lambda_1 > \lambda_2 > \cdots > \lambda_d\) is simple. Denote by \(\nu_i\) the image of the harmonic measure \(\nu\) on the flag space \(B\) under the projection onto the rank \(i\) Grassmannian \(G_i\), and let \(E_i\) be the corresponding differential entropy. All the spaces \((G_i, \nu_i)\) are quotients of the Poisson boundary of the random walk \((\Gamma, \mu)\); if \(\Gamma\) is discrete then \((B, \nu)\) is the Poisson boundary of the random walk \((\Gamma, \mu)\) \([Kai85, Led85]\), otherwise the Poisson boundary of the random walk \((\Gamma, \mu)\) may be bigger than the flag space \([KV83, BS06, Bro06]\) (note that it is still unknown whether all the proper quotients of the flag space \(B\) are also always proper measure-theoretically with respect to the corresponding \(\mu\)-stationary measures).

**Theorem 8.5.** The Hausdorff dimensions of the measures \(\nu_i\) satisfy the inequalities

\[
\dim \nu_i \leq \frac{E_i}{\lambda_i - \lambda_{i+1}}.
\]

In the case when \(\Gamma\) is a discrete subgroup of \(SL(2, \mathbb{R})\) the right-hand side of the above inequality precisely coincides with the ratio \(h(\mu)/2\lambda(\mu)\) from Ledrappier’s formula \([Led83]\).

Our Main Theorem follows from a combination of this dimension estimate with the following construction:

**Theorem 9.4.** Let \(\mu\) be a non-degenerate probability measure on \(\Gamma\) whose entropy and the first moment are finite, and let \(\gamma \in \Gamma\) be an \(\mathbb{R}\)-regular element (which always exists in a Zariski dense group). Then the measures

\[
\mu^k = \frac{1}{2} \mu + \frac{1}{4} \left( \delta_{\gamma^k} + \delta_{\gamma^{-k}} \right)
\]

have the property that the entropies \(H(\mu^k)\) (and therefore the corresponding asymptotic entropies as well) are uniformly bounded, whereas the lengths of their Lyapunov vectors \(\lambda(\mu^k)\) (equivalently, the top Lyapunov exponents \(\lambda_1(\mu^k)\)) go to infinity.

Let us mention here several issues naturally arising in connection with our results.

1. Our study of dimension in the present paper is subordinate to proving singularity at infinity, so that we only use rather rudimentary facts about the dimension of the harmonic measure. Under what conditions is the inequality from Theorem 8.5 realized as equality? Is there an exact formula for the dimension of the harmonic measure on the flag space? Of course, in these questions one has to specify precisely which definition of the dimension is used, cf. Section 6.

2. For our purposes it was enough to establish in Theorem 9.4 that the lengths of the Lyapunov vectors \(\lambda(\mu^k)\) go to infinity. Is this also true for all the spectral gaps \(\lambda_i(\mu^k) - \lambda_{i+1}(\mu^k)\) (which would imply that the dimensions of the harmonic measures on \(B\) go to 0)? What happens with the harmonic measures \(\nu^k\) (or with their projection); do they weakly converge, and if so what is the limit? Note that one can ask the latter questions in the hyperbolic (rank 1) situation as well, cf. \([LP07]\).

3. Another interesting issue is the connection of the harmonic measure with the invariant measures of the geodesic flow (more generally, of Weyl chambers in the case of
higher rank locally symmetric spaces). As it was proved by Katok and Spatzier [KS96],
the only invariant measure of the Weyl chamber flow associated to a lattice $\Gamma$ in $SL(d,\mathbb{R})$
with positive entropy along all one-dimensional directions is the Haar measure. In view of
the correspondence between the Radon invariant measures of the Weyl chamber flow and
$\Gamma$-invariant Radon measures on the principal stratum of the product $\mathcal{B} \times \mathcal{B}$ (cf. [Kai90] in
the hyperbolic case), it implies that singular harmonic measures on $\mathcal{B}$ either can not be
lifted to a Radon invariant measure on $\mathcal{B} \times \mathcal{B}$ (contrary to the hyperbolic case [Kai90]) or,
if they can be lifted to a Radon invariant measure on $\mathcal{B} \times \mathcal{B}$, then the associated invariant
measure of the Weyl chamber flow has a vanishing directional entropy (in our setup the
latter option most likely can be eliminated).

The paper has the following structure. In Section 1 we set up the notations and intro-
duce the necessary background information about random products (random walks) on
groups. In Section 2, Section 3 and Section 4 we specialize these notions to the case of
matrix random products, remind the Oseledets multiplicative ergodic theorem, give sev-
eral equivalent descriptions of the limit flags (Lyapunov filtrations) of random products,
and introduce the harmonic measure $\nu$ on the flag space $\mathcal{B}$ and its projections $\nu_i$ to the
Grassmannians $\mathcal{G}_i$ in $\mathbb{R}^d$. In Section 5 we remind the definitions of the asymptotic entropy
of a random walk and of the differential entropy of a $\mu$-stationary measure. In Lemma 5.3
we establish an inequality relating the exponential rate of decay of translates of a sta-
tionary measure along the sample paths of the reversed random walk with the differential
entropy. In Section 6 we discuss the notion of dimension of a probability measure on a
compact metric space. We introduce new lower and upper mean dimensions and estab-
lish inequalities (used in the proof of Theorem 8.5) relating them to other more traditional definitions of dimension (including the classical Hausdorff dimension). After discussing the notion of the limit set in the higher rank case in Section 7, we finally
pass to proving our Main Theorem in Section 8 and Section 9. Namely, in Section 8 we
establish the inequality of Theorem 8.5 and in Section 9 we prove Theorem 9.4 on exis-
tence of jump distributions with uniformly bounded entropy and arbitrarily big Lyapunov
exponents.

1. Random products

We begin by recalling the basic definitions from the theory of random walks on discrete
groups, e.g., see [KV83, Kai00]. The random walk determined by a probability measure $\mu$ on a countable group $\Gamma$ is a Markov chain with the transition probabilities

$$p(g, gh) = \mu(h), \quad g, h \in \Gamma.$$ 

Without loss of generality (passing, if necessary, to the subgroup generated by
the support $\text{supp} \mu$ of the measure $\mu$) we shall always assume that the measure $\mu$
is non-degenerate in the sense that the smallest subgroup of $\Gamma$ containing
$\text{supp} \mu$ is $\Gamma$ itself.

If the position of the random walk at time 0 is a point $x_0 \in \Gamma$, then its position at time
$n > 0$ is the product

$$x_n = x_0 h_1 h_2 \ldots h_n,$$

where $(h_n)_{n \geq 1}$ is the sequence of independent $\mu$-distributed increments of the random
walk. Therefore, provided that $x_0$ is the group identity $e$, the distribution of $x_n$ is the
$n$-fold convolution $\mu^n$ of the measure $\mu$. 
Below it will be convenient to consider bilateral sequences of Bernoulli \( \mu \)-distributed increments \( h = (h_n)_{n \in \mathbb{Z}} \) and the associated bilateral sample paths \( x = (x_n)_{n \in \mathbb{Z}} \) obtained by extending the formula

\[
x_{n+1} = x_n h_{n+1}
\]
to all \( n \in \mathbb{Z} \) under the condition \( x_0 = e \), so that

\[
x_n = \begin{cases} 
  h_0^{-1} h_{-1}^{-1} \ldots h_{n+1}^{-1}, & n < 0; \\
  e, & n = 0; \\
  h_1 h_2 \ldots h_n, & n > 0. 
\end{cases}
\]

(1.1)

Thus, the “negative part” \( \tilde{x}_n = x_{-n} \), \( n \geq 0 \), of the bilateral random walk is the random walk on \( \Gamma \) starting from the group identity at time 0 and governed by the reflected measure \( \tilde{\mu}(g) = \mu(g^{-1}) \).

We shall denote by \( P \) the probability measure in the space \( \mathcal{X} = \Gamma^\mathbb{Z} \) of bilateral sample paths \( x \) which is the image of the Bernoulli measure in the space of bilateral increments \( h \) under the isomorphism \( (1.1) \). It is preserved by the transformation \( T \) induced by the shift in the space of increments:

\[
(Tx)_n = h_1^{-1} x_{n+1}, \quad n \in \mathbb{Z}.
\]

(1.2)

2. Lyaunov exponents

We shall fix a standard basis \( E = (e_i)_{1 \leq i \leq d} \) in \( \mathbb{R}^d \) and identify elements of the Lie group \( G = SL(d, \mathbb{R}) \) with their matrices in this basis. Throughout the paper we shall use the standard Euclidean norms associated with this basis both for vectors and matrices in \( \mathbb{R}^d \).

From now on we shall assume that \( \Gamma \) is a countable subgroup of the group \( G = SL(d, \mathbb{R}) \).

Denote by \( A \) the Cartan subgroup of \( G \) consisting of diagonal matrices \( a = \text{diag}(a_i) \) with positive entries \( a_i \), and by

\[
a = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{R}^d : \sum \alpha_i = 0 \right\}
\]

the associated Cartan subalgebra, so that \( A = \text{exp} a \). Let

\[
a_+ = \left\{ \alpha \in a : \alpha_1 > \alpha_2 > \cdots > \alpha_d \right\},
\]

be the standard positive Weyl chamber in \( a \), and let

\[
A_+ = \text{exp} a_+ = \left\{ a \in A : a_1 > a_2 > \cdots > a_d \right\}.
\]

Denote the closures of \( a_+ \) and \( A_+ \) by \( \overline{a}_+ \) and \( \overline{A}_+ \), respectively.

Any element \( g \in G \) can be presented as \( g = k_1 k_2 \) with \( k_{1,2} \in K = SO(d) \) (which is a maximal compact subgroup in \( G \)) and a uniquely determined \( a = a(g) \in \overline{A}_+ \) (Cartan or polar decomposition).

We shall always assume that the probability measure \( \mu \) on \( \Gamma \) has a finite first moment in the ambient group \( G \), i.e., \( \sum \log \|g\| \mu(g) < \infty \).

Then the asymptotic behaviour of the random walk \( (\Gamma, \mu) \) is described by the famous Oseledets multiplicative ergodic theorem which we shall state in the form due to Kaimanovich [Kai89] (and in the generality suitable for our purposes):
Theorem 2.1. There exists a vector \( \lambda = \lambda(\mu) \in \mathfrak{a}_+ \) (the Lyapunov spectrum of the random walk) such that
\[
\frac{1}{n} \log a(x_n) \longrightarrow \lambda
\]
for \( P \)-a.e. sample path \( x \in \mathcal{X} \). Moreover, for \( P \)-a.e. \( x \) there exists a uniquely determined positive definite symmetric matrix
\[
g = g(x) = k(\exp \lambda)k^{-1}, \quad k \in K,
\]
such that
\[
\log \| g^{-n}x_n \| = o(n) \quad \text{as} \quad n \rightarrow +\infty.
\]

Theorem 2.2 ([GR85, GM89]). If, in addition, the group \( \Gamma \) is Zariski dense in \( G \), then the Lyapunov spectrum \( \lambda(\mu) \) is simple, i.e., it belongs to the Weyl chamber \( \mathfrak{a}_+ \).

Remark 2.3. The Lyapunov spectra \( (\lambda_1, \ldots, \lambda_d) \) and \( (\check{\lambda}_1, \ldots, \check{\lambda}_d) \) of the measure \( \mu \) and of the reflected measure \( \check{\mu} \), respectively, are connected by the formula
\[
\check{\lambda}_i = -\lambda_{d+1-i}, \quad 1 \leq i \leq d.
\]

3. Limit flags

Let \( S = G/K \) be the Riemannian symmetric space associated with the group \( G \) (e.g., see [Be96] for the basic notions). We shall fix a reference point \( o = K \subset S \) (its choice is equivalent to choosing a Euclidean structure on \( \mathbb{R}^d \) such that its rotation group is \( K \)). Being non-positively curved and simply connected, the space \( S \) has a natural visibility compactification \( \overline{S} = S \cup \partial S \) whose boundary \( \partial S \) consists of asymptotic equivalence classes of geodesic rays in \( S \) and can be identified with the unit sphere of the tangent space at the point \( o \) (since any equivalence class contains a unique ray issued from \( o \)).

The action of the group \( G \) extends from \( S \) to \( \partial S \), and the orbits of the latter action are naturally parameterized by unit length vectors \( \alpha \in \mathfrak{a}_+ \): the orbit \( \partial S_\alpha \) consists of the equivalence classes of all the rays \( \gamma(t) = k \exp(t\alpha)o, \ k \in K \). Algebraically the orbits \( \partial S_\alpha \) corresponding to the interior vectors \( \alpha \in \mathfrak{a}_+ \) are isomorphic to the space \( \mathcal{B} \) of full flags
\[
\mathcal{V} = \{V_i\}, \quad \mathcal{V}_0 = \{0\} \subset V_1 \subset \cdots \subset V_{d-1} \subset V_d = \mathbb{R}^d
\]
in \( \mathbb{R}^d \) (also known as the Furstenberg boundary of the symmetric space \( S \)), whereas the orbits corresponding to wall vectors are isomorphic to quotients of \( \mathcal{B} \), i.e., to flag varieties for which certain intermediate dimensions are missing, see [Kai89].

Theorem 2.1 implies that for \( P \)-a.e. sample path \( x \)
\[
d(x_n o, \gamma(t\|\lambda\|)) = o(n) \quad \text{as} \quad n \rightarrow +\infty,
\]
where \( \gamma \) is the geodesic ray \( \gamma(t) = k \exp(t\lambda)o \), hence the sequence \( x_n o \) converges in the visibility compactification to a limit point \( \text{bnd} \ x \in \partial S_{\lambda/\|\lambda\|} \). Moreover, \( \partial S_{\lambda/\|\lambda\|} \cong \mathcal{B} \) by Theorem 2.2, so that below we shall consider the aforementioned boundary map \( \text{bnd} \) as a map from the path space to the flag space \( \mathcal{B} \).

Theorem 2.1 and Theorem 2.2 easily imply the following descriptions of the limit flag \( \text{bnd} \ x \).

Proposition 3.1.

(i) Denote by \( \mathcal{V}_0 \) the standard flag
\[
\{0\} \subset \text{span}\{e_1\} \subset \text{span}\{e_1, e_2\} \subset \cdots \subset \text{span}\{e_1, e_2, \ldots, e_{d-1}\} \subset \mathbb{R}^d
\]
associated with the basis $E$. Then

$$bnd\mathbf{x} = k\nu_0$$

for $k = k(x) \in K$ from Theorem 2.7.

(ii) The spaces $V_i$ from the flag $bnd\mathbf{x}$ are increasing direct sums of the eigenspaces of the matrix $g = g(x)$ from Theorem 2.7 taken in the order of decreasing eigenvalues.

(iii) The flag $bnd\mathbf{x}$ is the Lyapunov flag of the sequence $x_n^{-1}$, i.e.,

$$\lim_{n \to +\infty} \frac{1}{n} \log \|x_n^{-1}v\| = -\lambda_i \quad \forall v \in V_i \setminus V_{i-1}.$$ 

(iv) For any smooth probability measure $\theta$ on $\mathcal{B}$ and $\mathbb{P}$-a.e. sample path $\mathbf{x}$

$$\lim_{n \to +\infty} x_n\theta = \delta_{bnd\mathbf{x}}$$

in the weak* topology of the space of probability measures on $\mathcal{B}$.

Remark 3.2. Equivariance of the boundary map $bnd$ implies that for the transformation $T$ (1.2)

$$bndT^n\mathbf{x} = x_n^{-1}bnd\mathbf{x} \quad \forall n \in \mathbb{Z}.$$ 

In particular,

$$(3.3) \quad bndT^{-n}\mathbf{x} = x_n^{-1}bnd\mathbf{x} \quad \forall n \geq 0.$$ 

4. Harmonic measure

Definition 4.1. The image $\nu = bnd(\mathbb{P})$ of the probability measure $\mathbb{P}$ in the path space $\mathcal{X}$ under the map $bnd : \mathcal{X} \to \mathcal{B}$ is called the harmonic measure of the random walk $(\Gamma, \mu)$. In other words, $\nu$ is the distribution of the limit flag $bnd\mathbf{x}$ under the measure $\mathbb{P}$.

The harmonic measure is $\mu$-stationary in the sense that it is invariant with respect to the convolution with $\mu$:

$$\mu * \nu = \sum_g \mu(g)g\nu = \nu.$$ 

Theorem 4.2 ([GR85, GM89]). Under conditions of Theorem 2.2 $\nu$ is the unique $\mu$-stationary probability measure on $\mathcal{B}$, and any proper algebraic subvariety of $\mathcal{B}$ is $\nu$-negligible.

Theorem 4.3 ([Kai85, Led85]). Under conditions of Theorem 2.2, if the subgroup $\Gamma$ is discrete, then the measure space $(\mathcal{B}, \nu)$ is isomorphic to the Poisson boundary of the random walk $(\Gamma, \mu)$.

Remark 4.4. If $\Gamma$ is not discrete, then the random walk may have limit behaviours other than described just by the limit flags, or, in other words, the Poisson boundary of the random walk $(\Gamma, \mu)$ may be bigger than the flag space, see [KV83] for the first example of this kind (the dyadic-rational affine group) and [BS06, Bro06] for the recent developments.

Denote by $\mathcal{G}_i$ the dimension $i$ Grassmannian (the space of all dimension $i$ subspaces) in $\mathbb{R}^d$. There is a natural projection $\pi_i : \mathcal{B} \to \mathcal{G}_i$ which consists in assigning to any flag in $\mathbb{R}^d$ its dimension $i$ subspace. Let $\nu_i = \pi_i(\nu)$, $1 \leq i \leq d - 1$, be the associated images of the measure $\nu$. Obviously, the measures $\nu_i$ are $\mu$-stationary (along with the measure $\nu$). We shall also use the notation

$$bnd_i\mathbf{x} = \pi_i(bnd\mathbf{x}) \in \mathcal{G}_i,$$

so that $\nu_i = bnd_i(\mathbb{P})$. 
We shall embed each Grassmannian $G_i$ into the projective space $P \bigwedge^i \mathbb{R}^d$ in the usual way and define a $K$-invariant metric $\rho = \rho_i$ on the latter as
\begin{equation}
\rho(\xi, \zeta) = \sin \angle(\xi, \zeta),
\end{equation}
where the angle (varying between 0 and $\pi/2$) is measured with respect to the standard Euclidean structure on $\bigwedge^i \mathbb{R}^d$ determined by the basis $E$ (so that $(e_{j_1} \wedge \cdots \wedge e_{j_i})_{1 \leq j_1 < \cdots < j_i \leq d}$ is an orthonormal basis of $\bigwedge^i \mathbb{R}^d$).

The Furstenberg formula (see [Fur63a, BL85])
\begin{equation}
\lambda_1 + \lambda_2 + \cdots + \lambda_i = \sum_g \mu(g) \int_{G_i} \log \|gv\| \, d\nu_i(v), \quad 1 \leq i \leq d - 1,
\end{equation}
where $v \in \bigwedge^i \mathbb{R}^d$ is the vector presenting a point $\mathbf{v} \in G_i \subset P \bigwedge^i \mathbb{R}^d$, relates Lyapunov exponents with the harmonic measure.

5. Entropy

Recall that if the measure $\mu$ has a finite entropy $H(\mu)$, then the asymptotic entropy of the random walk $(\Gamma, \mu)$ is defined as
\begin{equation}
h(\Gamma, \mu) = \lim_{n \to +\infty} \frac{H(\mu^\ast n)}{n} \leq H(\mu),
\end{equation}
where $H(\cdot)$ denotes the usual entropy of a discrete probability measure. The asymptotic entropy can also be defined “pointwise” along sample paths of the random walk as
\begin{equation}
h(\Gamma, \mu) = \lim_{n \to +\infty} -\frac{1}{n} \log \mu^\ast n(x_n),
\end{equation}
where the convergence holds both $P$-a.e. and in the space $L^1(\mathcal{X}, P)$, see [KV83, Der86].

The $\mu$-entropy (Furstenberg entropy, differential entropy) of a $\mu$-stationary measure $\theta$ on a $\Gamma$-space $X$ is defined as
\begin{equation}
E_\mu(X, \theta) = -\sum_{g \in \Gamma} \mu(g) \int \log \frac{dg^{-1}}{d\theta}(b) d\theta(b),
\end{equation}
and it satisfies the inequality $E_\mu(X, \theta) \leq h(\Gamma, \mu)$, see [Fur71, KV83, NZ02].

We shall always assume that the probability measure $\mu$ on $\Gamma$ has a finite entropy $H(\mu) < \infty$.

In our context, if the subgroup $\Gamma$ is discrete in $SL(d, \mathbb{R})$, then finiteness of the first moment of the measure $\mu$ easily implies that $H(\mu) < \infty$ (e.g., see [Der86]). Therefore, $E_i = E_\mu(G_i, \nu_i) < \infty$.

Below we shall need the following routine estimate (in fact valid for an arbitrary quotient of the Poisson boundary).

**Lemma 5.3.** For any index $i \in \{1, 2, \ldots, d - 1\}$, any subset $A \subset G_i$ with $\nu_i(A) > 0$ and $P$-a.e. sample path $x$
\begin{equation}
\limsup_{n \to +\infty} \left[ -\frac{1}{n} \log \tilde{x}_n \nu_i(A) \right] \leq E_i.
\end{equation}
Proof. Put
\[ F_i(x) = -\log \frac{d\tilde{x}_i\nu_i}{d\nu_i}(\text{bnd}_i x) = -\log \frac{d\nu_i(\text{bnd}_i T^{-1} x)}{d\nu_i(\text{bnd}_i x)} \]
(see formula (3.3)), so that
\[ E_i = \int F_i(x) \, dP(x). \]
Then
\[ -\log \frac{d\tilde{x}_n \nu_i}{d\nu_i}(\text{bnd}_i x) = -\log \frac{d\nu_i(\text{bnd}_i T^{-n} x)}{d\nu_i(\text{bnd}_i x)} = F_i(x) + F_i(T^{-1} x) + \cdots + F_i(T^{-n+1} x), \]
whence by the ergodic theorem
\[ -\frac{1}{n} \log \frac{d\tilde{x}_n \nu_i}{d\nu_i}(\text{bnd}_i x) \longrightarrow E_i \]
in \( L^1(\mathcal{X}, P) \). It implies that
\[ -\frac{1}{n} \int_A \log \frac{d\tilde{x}_n \nu_i}{d\nu_i}(\xi) \frac{d\nu_i(\xi)}{\nu_i(A)} \longrightarrow E_i, \]
which, by a convexity argument, yields the claim. \( \square \)

6. Dimension of measures

Let us recall several notions of the dimension of a probability measure \( m \) on a compact metric space \( (Z, \rho) \) (all the details, unless otherwise specified, can be found in the book [Pes97]). These notions roughly fall into two categories: the global ones are obtained by looking at the dimension of sets which “almost” coincide with \( Z \) (up to a piece of small measure \( m \)), whereas the local ones are related to the asymptotic behavior of the ratios \( \log mB(z, r)/\log r \) as the radius \( r \) tends to 0.

6.A. Global definitions. The Hausdorff dimension of the measure \( m \) is
\[ \dim_H m = \inf \{ \dim_H A : m(A) = 1 \}, \]
where \( \dim_H A \) denotes the Hausdorff dimension of a subset \( A \subset Z \).

The lower and the upper box dimensions of a subset \( A \subset Z \) are defined, respectively, as
\[ \underline{\dim}_B A = \liminf_{r \to 0} \frac{\log N(A, r)}{\log 1/r} \quad \text{and} \quad \overline{\dim}_B A = \limsup_{r \to 0} \frac{\log N(A, r)}{\log 1/r}, \]
where \( N(A, r) \) is the minimal number of balls of radius \( r \) needed to cover \( A \). Ledrappier [Led81] also considered the minimal number \( N(r, \varepsilon, m) \) of balls of radius \( r \) such that the measure \( m \) of their union is at least \( 1 - \varepsilon \) and defined the “fractional dimension” of the measure \( m \) as
\[ \underline{\dim}_L m = \sup_{\varepsilon \to 0} \limsup_{r \to 0} \frac{\log N(r, \varepsilon, m)}{\log 1/r} \]
(we use the notation from [You82] and below shall call it the upper Ledrappier dimension). As it was noticed by Young [You82], in the same way one can also define what we call the lower Ledrappier dimension of the measure \( m \)
\[ \underline{\dim}_L m = \sup_{\varepsilon \to 0} \liminf_{r \to 0} \frac{\log N(r, \varepsilon, m)}{\log 1/r} \]
as well as, in modern terminology, its lower and the upper box dimensions, respectively,
\[
\dim_B m = \liminf_{m(A)\to 1} \{\dim_B A\} \quad \text{and} \quad \dimover B m = \liminf_{m(A)\to 1} \{\dimover B A\} .
\]

Obviously,
\[
\dimover L m \leq \dimover B m \quad \text{and} \quad \dim L m \leq \dim B m .
\]

The difference between the Ledrappier and the box dimensions is that in the definition of the box dimensions it is the same set \(A\) which has to be covered by balls with varying radii \(r\), unlike in the definition of Ledrappier, so that
\[
\dim L m \leq \dim B m \quad \text{and} \quad \dimover L m \leq \dimover B m .
\]

By \cite{You82}, Proposition 4.1,
\[
(6.1) \quad \dim H m \leq \dimover L m .
\]

6.B. Local definitions. The lower and the upper pointwise dimensions of the measure \(m\) at a point \(z \in Z\) are
\[
\dim P m(z) = \liminf_{r \to 0} \frac{\log mB(z, r)}{\log r} \quad \text{and} \quad \dimover P m(z) = \limsup_{r \to 0} \frac{\log mB(z, r)}{\log r} ,
\]
respectively. Then
\[
(6.2) \quad \dim H m = \text{ess sup}_z \dimover P m(z) .
\]

In particular, if \(m\text{-a.e.}\)
\[
(6.3) \quad \lim_{r \to 0} \frac{\log mB(z, r)}{\log r} = D ,
\]
then \(\dim H m = D\). Moreover, in this case all the reasonable definitions of dimension of the measure \(m\) give the same result \cite[Theorem 4.4]{You82}.

Definition 6.4. In the situation when the convergence in (6.3) holds just in probability we shall say that \(D\) is the mean dimension \(\dim M m\) of the measure \(m\). We shall also introduce the lower and the upper mean dimensions of the measure \(m\) as, respectively,
\[
\dimover M m = \sup \left\{ t : \left[ \frac{\log mB(z, r)}{\log r} - t \right]_+ \xrightarrow{m} 0 \right\} ,
\]
\[
\dim M m = \inf \left\{ t : \left[ \frac{\log mB(z, r)}{\log r} - t \right]_- \xrightarrow{m} 0 \right\} ,
\]
where \([t]_+ = \max\{0, t\}, [t]_- = \min\{0, t\}\), and \(\xrightarrow{m}\) denotes convergence in probability with respect to the measure \(m\).

The definition of \(\dim M\) first appeared in Ledrappier’s paper \cite{Led83} (also see \cite{Led84}), whereas \(\dim M\) and \(\dimover M m\) (although obvious generalizations of \(\dim M\)) are, apparently, new. Clearly, \(\dimover M m \leq \dim M m\). In slightly different terms, \([\dim M m, \dimover M m]\) is the minimal closed subinterval of \(\mathbb{R}\) with the property that for any closed subset \(I\) of its complement
\[
m \left\{ z \in Z : \frac{\log mB(z, r)}{\log r} \in I \right\} \xrightarrow{r \to 0} 0 .
\]

In particular, if \(\dim M m\) exists, then \(\dimover M m = \dimover M m = \dim M m\).
6.C. Mean, box and Ledrappier dimensions. We shall now establish simple inequalities between these dimensions.

Proposition 6.5. For any probability measure \( m \) on a compact metric space \( (Z, \rho) \)

\[
\dim_m m \leq \dim_L m .
\]

Proof. Fix a number \( D < \dim_m m \). Then for any \( \varepsilon > 0 \) there exist \( r_0 > 0 \) and a set \( A \subset Z \) with \( m(A) > 1 - \varepsilon \) and such that \( mB(z, r) \leq r^D \) for all \( z \in A \) and \( r \leq r_0 \). Suppose now that

\[
m \left( \bigcup_i B(z_i, r) \right) \geq 1 - \varepsilon
\]

for a certain set of points \( \{z_i\} \) of cardinality \( N \) and a certain \( r \leq r_0/2 \). If \( B(z_i, r) \) intersects \( A \), then \( B(z_i, r) \subset B(z, 2r) \) for some \( z \in A \), so that

\[
m (A \cap B(z_i, r)) \leq mB(z_i, r) \leq mB(z, 2r) \leq (2r)^D .
\]

Thus,

\[
1 - 2\varepsilon \leq m \left( A \cap \bigcup_i B(z_i, r) \right) \leq N(2r)^D ,
\]

whence the claim. \( \square \)

For establishing the inverse inequality we shall additionally require that the metric space \( (Z, \rho) \) has the Besicovitch covering property, i.e., that for any precompact subset \( A \subset Z \) and any bounded function \( r : A \to \mathbb{R}_+ \) (important particular case: \( r \) is constant) the cover \( \{B(z, r(z)), z \in A\} \) of \( A \) contains a countable subcover whose multiplicity is bounded from above by a universal constant \( M = M(Z, \rho) \) (recall that the multiplicity of a cover is the maximal number of elements of this cover to which a single point may belong). The Besicovitch property is, for instance, satisfied for the Euclidean space, hence, for all its compact subsets endowed with a metric which is Lipschitz equivalent to the Euclidean one. Therefore, it is satisfied for each of the Grassmannians \( G_i \) endowed with the metric (4.5).

Proposition 6.6. For any probability measure \( m \) on a compact space \( (Z, \rho) \) satisfying the Besicovitch property

\[
\overline{\dim}_B m \leq \overline{\dim}_M m .
\]

Proof. Take a number \( D > \overline{\dim}_M m \), let

\[
A_r = \{ z \in Z : mB(z, r) \geq r^D \} ,
\]

and consider a cover of \( A_r \) by the balls \( B(z_i, r) \), \( z_i \in A_r \) obtained from applying the Besicovitch property. The cardinality of this cover is at least \( N(A_r, r) \), whereas its multiplicity is at most \( M \), whence

\[
N(A_r, r)r^D \leq \sum_i mB(z_i, r) \leq M ,
\]

so that

\[
\frac{\log N(A_r, r)}{\log 1/r} \leq D + \frac{\log M}{\log 1/r} .
\]

For \( r \to 0 \) the right-hand side of the above inequality tends to \( D \), whereas \( m(A_r) \to 1 \) by the choice of \( D \), whence \( \overline{\dim}_B m \leq D \). \( \square \)
6.D. Final conclusions. Taking stock of the above discussion we obtain

**Theorem 6.7.** For any probability measure \( m \) on a compact metric space

\[
\dim_H m \leq \dim_M m \leq \dim_B m .
\]

If, in addition, the space has the Besicovitch property, then

\[
\overline{\dim}_M m \leq \overline{\dim}_B m \leq \overline{\dim}_M m .
\]

**Remark 6.8.** There is no general inequality between \( \dim_H m \) and \( \dim_M m \). For instance, take two singular measures \( m_1, m_2 \) for which the dimensions (6.3) exist and are different, say, \( D_1 < D_2 \), and let \( m \) be their convex combination. Then \( \dim_M m = D_1 \), whereas \( \dim_H m = D_2 > \dim_M m \) by (6.2).

On the other hand, by exploiting the difference between the convergence in probability and the convergence almost everywhere one can also construct examples with \( \dim_H m < \dim_M m \). We shall briefly describe one such example. Let \( Z' \) be the space of unilateral binary sequences \( \alpha = (\alpha_1, \alpha_2, \ldots) \) with the uniform measure \( m' \) and the usual metric \( \rho' \) for which \( -\log \rho'(\alpha, \beta) \) is the length of the initial common segment of the sequences \( \alpha \) and \( \beta \). Take a sequence of cylinder sets \( A_n \) with the property that \( m' A_n \to 0 \), but any point of \( Y' \) belongs to infinitely many sets \( A_n \). Also take a sequence of integers \( s_n \) (to be specified later), and let \( Z \) be the image of the space \( Z' \) under the following map: given a sequence \( \alpha \in Z' \) take the set \( I = \{ n : \alpha \in A_n \} \) and replace with 0 all the symbols \( \alpha_k \) with \( s_n \leq k \leq 2s_n \) for a certain \( n \in I \). The space \( Z \) is endowed with the quotient measure \( m \) and the quotient metric \( \rho \). If the sequence \( s_n \) is very rapidly increasing, then (6.2) can be used to show that \( \dim_H m = \log 2/2 \), whereas \( \overline{\dim}_M m = \dim_M m = \log 2 \).

7. Limit set

Denote by \( \mathcal{P}(\mathcal{B}) \) the compact space of probability measures on the flag space \( \mathcal{B} \) endowed with the weak* topology, and let \( m \) be the unique \( K \)-invariant probability measure on \( \mathcal{B} \). Then the map \( g \mapsto gm \) determines an embedding of the symmetric space \( S = G/K \) into \( \mathcal{P}(\mathcal{B}) \), and gives rise to the Satake–Furstenberg compactification of \( S \). Its boundary \( \overline{S} \setminus S \) contains the space \( \mathcal{B} \) under the identification of its points with the associated delta-measures (but, unless the rank of \( G = SL(d, \mathbb{R}) \) is 1, i.e., \( d = 2 \), it also contains other limit measures, see [GJT98]). The limit set \( L_\Gamma \) of a subgroup \( \Gamma \subset G \) in this compactification is then defined (see [Gm90]) as

\[
L_\Gamma = \overline{\Gamma o} \cap \mathcal{B} \subset \mathcal{B} .
\]

The limit set is obviously \( \Gamma \)-invariant and closed. Moreover,

**Theorem 7.1** ([Gm90]). If the group \( \Gamma \) is Zariski dense in \( G \), then its action on the limit set \( L_\Gamma \) is minimal (i.e., \( L_\Gamma \) has no proper closed \( \Gamma \)-invariant subsets).

Below we shall need the following elementary property.

**Proposition 7.2.** Under the conditions of Theorem 7.1 let \( U \subset \mathcal{B} \) be an open set with \( U \cap L_\Gamma \neq \emptyset \). Then there exists finitely many elements \( \gamma_1, \ldots, \gamma_r \in \Gamma \) such that

\[
L_\Gamma \subset \bigcup_i \gamma_i U .
\]
Proof. Minimality of $L_\Gamma$ means that any closed $\Gamma$-invariant subset of $\mathcal{B}$ either contains $L_\Gamma$ or does not intersect it. Since the set $\mathcal{B} \setminus \bigcup_{\gamma \in \Gamma} \gamma U$ does not contain $L_\Gamma$ (as $U \cap L_\Gamma \neq \emptyset$), it does not intersect $L_\Gamma$, so that $L_\Gamma \subset \bigcup_{\gamma \in \Gamma} \gamma U$. Finally, since $L_\Gamma$ is compact, the above cover contains a finite subcover. \hfill $\square$

Remark 7.3. Theorem 7.1 and Proposition 7.2 obviously carry over to the projections $L_\Gamma^i = \pi_i(L_\Gamma) \subset \mathcal{G}_i$ of the limit set $L_\Gamma$ to the Grassmannians $\mathcal{G}_i$.

By Proposition 3.1(iv) (see [GR85] for a more general argument) a.e. sample path $x$ converges as $n \to +\infty$ to the limit flag $\text{bnd}^{-} x$ in the Satake–Furstenberg compactification as well. Therefore, $\text{supp} \nu \subset L_\Gamma$. If $\text{supp} \mu$ generates $\Gamma$ as a semigroup, then $\mu$-stationarity of the harmonic measure $\nu$ implies its quasi-invariance, so that in this case $\text{supp} \nu$ is $\Gamma$-invariant, and, if $\Gamma$ is Zariski dense, $\text{supp} \nu = L_\Gamma$ by Theorem 7.1.

Remark 7.4. If $\Gamma$ is a lattice, then $L_\Gamma = \mathcal{B}$. On the other hand, if $\Gamma$ is Zariski dense, then, as it is shown in [Gui90], $L_\Gamma$ has positive Hausdorff dimension, which is deduced from the positivity of the dimension of the harmonic measure (under the assumption that $\mu$ has an exponential moment). Also see [Lin04] for recent results on the Hausdorff dimension of the radial limit set. It would be interesting to investigate existence of random walks such that their harmonic measure has the maximal Hausdorff dimension (equal to the dimension of the limit set), cf. [CM07a].

8. Dimension of the harmonic measure

8.A. Rate of contraction estimate. Recall (see Section 1) that the negative part $(\hat{x}_n)_{n \geq 0} = (x_{-n})_{n \geq 0}$ of a bilateral sample path $x$ performs the random walk on $\Gamma$ governed by the reflected measure $\hat{\mu}$. Denote by $\text{bnd}^{-} x \in \mathcal{B}$ the corresponding limit flag, and for $i \in \{1, 2, \ldots, d-1\}$ let

$$\xi_x = (\text{bnd}_{d-i} x)^\perp \in \mathcal{G}_i$$

be the orthogonal complement in $\mathbb{R}^d$ of the $(d-i)$-dimensional subspace of the flag $\text{bnd}^{-} x$ (for simplicity we omit the index $i$ in the notation for $\xi_x$; the Grassmannian $\mathcal{G}_i$ to which it belongs should always be clear from the context).

Theorem 8.1. For any Grassmannian $\mathcal{G}_i$, any $r < 1$, and $P$-a.e. $x \in \mathcal{X}$,\n
$$\liminf_n \left[ -\frac{1}{n} \log \text{diam} \hat{x}_n^{-1} B(\xi_x, r) \right] \geq \lambda_i - \lambda_{i+1}.$$\n
Proof. Let us consider first the case when $i = 1$. Then by the definition of the metric $\rho$ ([1.5]) on $\mathcal{G}_1 \cong P\mathbb{R}^d$ the ball $B(\xi_x, r)$ consists of the projective classes of all the vectors $v + w$, where $v \in \xi_x \setminus \{0\}$ and $w \perp \xi_x$ with

$$\frac{\|w\|}{\|v\|} \leq C$$

for a certain constant $C = C(r)$.

By Proposition 3.1(iii) applied to the random walk $(\Gamma, \hat{\mu})$ we have that\n
$$\lim_n \frac{1}{n} \log \frac{\|\hat{x}_n^{-1} w\|}{\|\hat{x}_n^{-1} v\|} \leq \lambda_2 - \lambda_1$$

uniformly under condition (8.2), whence the claim.

The general case argument follows the same lines with the only difference that now instead of the action on $P\mathbb{R}^d$ one has to consider the action on $P \bigwedge^i \mathbb{R}^d$. The sequence of matrices $\bigwedge^i \hat{x}_n^{-1}$ (which are the images of $\hat{x}_n^{-1}$ in the $i$-th external power representation) is also Lyapunov regular with the Lyapunov spectrum consisting of all the sums of the
form \(\lambda_1 + \lambda_2 + \cdots + \lambda_i\) with \(1 \leq j_1 < \cdots < j_i \leq d\). In particular, the top of the spectrum (corresponding precisely to the eigenspace \(\xi_\mathbf{x} \in P \bigwedge^i \mathbb{R}^d\) as orthogonal to the highest dimension proper subspace of the Lyapunov flag) is \(\lambda_1 + \lambda_2 + \cdots + \lambda_i\), whereas the second point in the spectrum is \(\lambda_1 + \lambda_2 + \cdots + \lambda_{i-1} + \lambda_{i+1}\), the spectral gap being \(\lambda_i - \lambda_{i+1}\).

**Proposition 8.3.** For any index \(i \in \{1, 2, \ldots, d-1\}\), any \(\varepsilon > 0\) and \(\mathbf{P}\)-a.e. sample path \(\mathbf{x}\) the sets

\[
A_N = \bigcap_{n \geq N} \bar{x}_n B (\text{bnd}_i, T^{-n} \mathbf{x}, e^{-n(\lambda_i - \lambda_{i+1} - \varepsilon)}) \subset \mathcal{G}_i
\]

have positive measure \(\nu_i\) for all sufficiently large \(N\).

**Proof.** By definition of the metric \(\rho (4.5)\) its values do not exceed 1, and for any \(\xi \in \mathcal{G}_i\) the radius 1 sphere \(S(\xi, 1)\) centered at \(\xi\) consists precisely of those \(\zeta \in \mathcal{G}_i\) for which the vectors from \(\bigwedge^i \mathbb{R}^d\) associated with \(\xi\) and \(\zeta\) are orthogonal. The latter condition on \(\zeta\) is algebraic, so that by Theorem 4.2 \(\nu_i S(\xi, 1) = 0\) for any \(\xi \in \mathcal{G}_i\) (the measure \(\nu_i\) being the projection of the measure \(\nu\)). Therefore, for a.e. sample path \(\mathbf{x}\) there exists \(r < 1\) such that \(\nu_i B(\xi_\mathbf{x}, r) > 0\). On the other hand, by Theorem 8.1 and (3.3) \(B(\xi_\mathbf{x}, r) \subset A_N\) for all sufficiently large \(N\). \(\square\)

**8.B. Dimension estimate.** Recall that \(\text{bnd}_i, T^{-n} \mathbf{x} = \bar{x}_n^{-1} \text{bnd}_i \mathbf{x}\) (3.3). By Lemma 5.3 and Proposition 8.3 for \(\mathbf{P}\)-a.e. sample path \(\mathbf{x}\)

\[
\limsup_{n \to \infty} \left[ \frac{1}{n} \log \nu_i B (\text{bnd}_i, T^{-n} \mathbf{x}, e^{-n(\lambda_i - \lambda_{i+1} - \varepsilon)}) \right] \leq \limsup_{n \to \infty} \left[ \frac{1}{n} \log \bar{x}_n \nu_i(A_N) \right] \leq E_i.
\]

(8.4)

The left-hand side of this inequality looks almost like in the definition of the pointwise dimension, with the only difference that the ball centers vary. We shall take care of this difference by switching to the mean dimension and using \(\mu\)-stationarity of the measures \(\nu_i\).

**Theorem 8.5.** For any \(i \in \{1, 2, \ldots, d-1\}\),

\[
\dim_H(\nu_i) \leq \dim_B(\nu_i) \leq \dim_M \nu_i \leq \frac{E_i}{\lambda_i - \lambda_{i+1}}.
\]

**Proof.** Let

\[
A = \bigcap_{\eta \in \mathcal{N}} \bigcap_{n \geq N} \left\{ \mathbf{x} : -\frac{1}{n} \log \nu^i B (\text{bnd}_i, T^{-n} \mathbf{x}, e^{-n(\lambda_i - \lambda_{i+1} - \varepsilon)}) \leq E_i + \eta \right\}
\]

be the set of sample paths satisfying condition (8.4). Since \(\mathbf{P}(A) = 1\), for any \(\eta, \chi > 0\)

\[
\mathbf{P} \left\{ \mathbf{x} : -\frac{1}{n} \log \nu^i B (\text{bnd}_i, T^{-n} \mathbf{x}, e^{-n(\lambda_i - \lambda_{i+1} - \varepsilon)}) \leq E_i + \eta \right\} \geq 1 - \chi
\]

for all sufficiently large \(n\). The transformation \(T\) preserves the measure \(\mathbf{P}\), so that its image under the map \(\mathbf{x} \mapsto \text{bnd}_i, T^{-n} \mathbf{x}\) is \(\nu_i\), whence the rightmost inequality. The other inequalities follow from Theorem 6.7 because the metric \(\rho\) has the Besicovitch property (see Section 6.C). \(\square\)
9. The construction

For the rest of this Section we shall fix a probability measure $\mu$ on a subgroup $\Gamma \subset G$ satisfying our standing assumptions from Section 1 and Section 2.

In addition we shall assume in this Section that $\text{supp}\, \mu$ generates $\Gamma$ as a semi-group.

Recall that an element of the group $G = SL(d, \mathbb{R})$ is called $\mathbb{R}$-regular if it is diagonalizable over $\mathbb{R}$ and the absolute values of its eigenvalues are pairwise distinct. By [BL93] any Zariski dense subgroup of $G$ contains such an element. Let us fix an $\mathbb{R}$-regular element $\gamma \in \Gamma$ and consider the sequence of probability measures on $\Gamma$

\begin{equation}
\mu^k = \frac{1}{2} \mu + \frac{1}{4} \left( \delta_{\gamma^k} + \delta_{\gamma^{-k}} \right).
\end{equation}

where $\delta_g$ denotes the Dirac measure at $g$.

Remark 9.2. Actually, we just need that $\gamma$ be diagonalizable over $\mathbb{R}$ with the absolute value of some of its eigenvalues being not 1.

Denote by $\nu^k$ the harmonic measures on the flag space $\mathcal{B}$ of the random walks $(\Gamma, \mu^k)$, and by $\nu^k_i$ their quotients on the Grassmannians $\mathcal{G}_i$, $1 \leq i \leq d - 1$. Our Main Theorem follows from

**Theorem 9.3.**

\[ \min_i \{ \dim_H \nu^k_i \} \longrightarrow 0. \]

In its turn, Theorem 9.3 is an immediate consequence of a combination of the inequality from Theorem 8.5 and

**Theorem 9.4.** The measures $\mu^k$ (9.1) have the property that the entropies $H(\mu^k)$ are uniformly bounded, whereas the lengths of their Lyapunov vectors $\lambda(\mu^k)$ (equivalently, the top Lyapunov exponents $\lambda_1(\mu^k)$) go to infinity.

The rest of this Section is devoted to a proof of Theorem 9.4 split into a number of separate claims. The first one is obvious:

**Claim 9.5.** The measures $\mu^k$ have uniformly bounded entropies $H(\mu^k)$.

In view of the discussion from Section 5 it immediately implies

**Corollary 9.6.** The asymptotic entropies $h(\Gamma, \mu^k)$ and therefore all the differential entropies $E^k_i = E_{\mu^k}(\mathcal{G}_i, \nu^k_i)$ are uniformly bounded.

For estimating the top Lyapunov exponent $\lambda_1(\mu^k)$ we shall use the Furstenberg formula (4.6), by which

\begin{equation}
\lambda_1(\mu^k) = \frac{1}{2} \sum_g \mu(g) \int_{\mathbb{R}^d} \log \frac{\|gv\|}{\|v\|} \, d\nu^k_1(v) \bigg( \log \frac{\|\gamma^k v\|}{\|v\|} + \log \frac{\|\gamma^{-k} v\|}{\|v\|} \bigg) \, dv^k_1(v).
\end{equation}

The absolute value of the first term of this sum is uniformly bounded because $\mu$ has a finite first moment. For dealing with the second term of the sum (9.7) we need the following claims.
Claim 9.8. The sum
\[
\left( \log \frac{\|\gamma^k v\|}{\|v\|} + \log \frac{\|\gamma^{-k} v\|}{\|v\|} \right)
\]
is bounded from below uniformly on \(v \in \mathbb{R}^d \setminus \{0\}\) and \(k \geq 0\).

Proof. If \(\delta\) is a diagonal matrix, then obviously,
\[
\|\delta^k v\|\|\delta^{-k} v\| \geq \langle \delta^k v, \delta^{-k} v \rangle = \|v\|^2.
\]
Now, \(\gamma = h^{-1} \delta h\) is diagonalizable, whence
\[
\|\gamma^k v\|\|\gamma^{-k} v\| \geq \|h\|^{-2}\|\delta^k hv\|\|\delta^{-k} hv\| \geq \|h\|^{-2}\|hv\|^2
\]
\[
\geq \|h\|^{-2}\|h^{-1}\|^2\|v\|^2.
\]

\(\square\)

Claim 9.9. For any open subset \(U \subset P\mathbb{R}^d\) which intersects the limit set \(L_1^1\) (see Section 7), the measures \(\nu_k^1(U)\) are bounded away from zero uniformly on \(k\).

Proof. By Proposition 7.2 there exists a finite set \(A \subset \Gamma\) such that \(L_1^1 \subset \bigcup_{g \in A} g^{-1}U\).

Since supp \(\mu\) generates \(\Gamma\) as a semigroup, for each \(g \in A\) there exists an integer \(s = s(g)\) such that \(g \in \text{supp} \mu^s\). Then by \(\mu^k\)-stationarity of \(\nu_k^1\), for each \(g \in A\)
\[
\nu_k^1(U) \geq \frac{1}{2s}\mu^s(g)\nu_k^1(U) = \frac{1}{2s}\mu^s(g)\nu_k^1(g^{-1}U) \geq \varepsilon \nu_k^1(g^{-1}U)
\]
for a certain \(\varepsilon = \varepsilon(A) > 0\). Summing up the above inequalities over all \(g \in A\) we obtain
\[
|A|\nu_k^1(U) \geq \varepsilon \sum_{g \in A} \nu_k^1(g^{-1}U) \geq \varepsilon \nu_k^1(L_1^1) = \varepsilon,
\]
whence the claim. \(\square\)

Now we are ready to prove

Claim 9.10.
\[
\lim_{k \to \infty} \lambda_1(\mu^k) = +\infty.
\]

Proof. We shall fix a diagonalization \(\gamma = h^{-1} \delta h\) with \(\delta = \text{diag}(\delta_1, \ldots, \delta_d)\) in such a way that \(|\delta_1| > 1\) and \(|\delta_d| < 1\), and define the open set \(U\) as
\[
U = \left\{ \mathbf{v} \in P\mathbb{R}^d : \frac{\langle hv, e_1 \rangle}{\|hv\|} > \beta \text{ and } \frac{\langle hv, e_d \rangle}{\|hv\|} > \beta \right\},
\]
i.e., by requiring that the first and the last coordinates (with respect to the standard basis \(E\)) of the normalized vector \(hv\) be greater than \(\beta\). The value of \(\beta\) is chosen to make sure that \(U\) is non-empty (for instance, one can take \(\beta = 1/2\)).

If \(\mathbf{v} \in U\), then
\[
\|\gamma^k v\| = \|h^{-1} \delta^k hv\| \geq \|h\|^{-1}\|\delta_1|^k \langle hv, e_1 \rangle \geq \|h\|^{-1}\|\delta_1|^k \beta \|hv\|,
\]
and thus
\[
\frac{\|\gamma^k v\|}{\|v\|} \geq \|h^{-1}\|^{-1} \|h\|^{-1} \beta |\delta_1|^k.
\]
In the same way,
\[ \|\gamma^{-k}v\| / \|v\| \geq \|h^{-1}\|^{-1} \|h\|^{-1} \beta |\delta|^{-k}, \]
so that
\[ \log \|\gamma^k v\| / \|v\| + \log \|\gamma^{-k}v\| / \|v\| \to \infty \]
as \( k \to \infty \) uniformly on \( \bar{v} \in U \), which in view of (9.7) in combination with Claim 9.8 and Claim 9.9 finishes the argument.

Remark 9.11. The measure \( \mu \) in our construction can clearly be chosen symmetric and, if the group \( \Gamma \) is finitely generated, finitely supported. Obviously, the measures \( \mu^k \) then also have these properties, so that singular harmonic measures can be produced by symmetric finitely supported measures on the group.

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