On the regularity of axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions

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Abstract

In this paper, we will discuss the axisymmetric, swirl-free Euler equation in four and higher dimensions. We will show that in four and higher dimensions the axisymmetric, swirl-free Euler equation has properties which could allow finite-time singularity formation of a form that is excluded in three dimensions. We will also consider a model equation that is obtained by taking the infinite-dimensional limit of the vorticity equation in this setup. This model exhibits finite-time blowup of a Burgers shock type. The blowup result for the infinite dimensional model equation heavily suggests that smooth solutions of the Euler equation exhibit finite-time blowup in sufficiently high dimensions.

1 Introduction

The incompressible Euler equation is one of the fundamental equations of fluid dynamics, and is given by

\[ \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \]
\[- \Delta u = 0, \]

where \( u \in \mathbb{R}^d \) is the fluid velocity, and \( p \) is the pressure, which can be determined from the velocity using the divergence free constraint, yielding

\[ - \Delta p = \sum_{i,j=1}^{d} \frac{\partial u_j}{\partial x_i} \frac{\partial u_i}{\partial x_j}. \]

This means that the Euler equation can be expressed in terms of the Helmholtz projection as

\[ \partial_t u + P_{df} ((u \cdot \nabla)u) = 0. \]

It is a classical result that for sufficiently smooth initial data, the Euler equation has global smooth solutions in two dimensions. In three dimensions, it is a major open question in nonlinear PDE whether smooth solutions of the Euler equation can blowup in finite-time. One of the main partial regularity results is the Beale-Kato-Majda criterion, which states that the \( L_t^1 L_x^\infty \) norm of...
the vorticity controls regularity \([1]\), and so if a smooth solution of the Euler equation blows up in finite-time \(T_{\text{max}} < +\infty\), then

\[
\int_0^{T_{\text{max}}} \| \vec{\omega}(\cdot,t) \|_{L^\infty} \, dt = +\infty.
\] (1.5)

A special case of the Euler equation in three dimensions is that of axisymmetric, swirl-free solutions. Axisymmetric, swirl free vector fields have the form

\[ u(x) = u_r(r,z)e_r + u_z(r,z)e_z, \] (1.6)

where

\[
\begin{align*}
  r &= \sqrt{x_1^2 + x_2^2} \quad \text{(1.7)} \\
  z &= x_3 \quad \text{(1.8)} \\
  e_r &= \frac{(x_1, x_2, 0)}{r} \quad \text{(1.9)} \\
  e_z &= e_3 \quad \text{(1.10)}
\end{align*}
\]

and this class of vector fields is preserved by the dynamics of the Euler equation. The scalar vorticity for axisymmetric, swirl-free solutions is \( \omega = \partial_r u_z - \partial_z u_r \), and satisfies the evolution equation

\[
\partial_t \omega + (u \cdot \nabla) \omega - \frac{u_r}{r} \omega = 0.
\] (1.11)

This implies that the quantity \( \frac{\omega}{r} \) is transported by the flow with

\[
(\partial_t + u \cdot \nabla) \frac{\omega}{r} = 0,
\] (1.12)

and this is enough to guarantee global regularity subject to reasonable hypotheses on the initial data. Ladyzhenskaya proved the first global regularity result for axisymmetric, swirl-free fluids in three dimensions, proving global regularity for axisymmetric, swirl-free solutions of the Navier–Stokes equation \([8]\). Ukovskii and Yudovich proved global regularity \([12]\) for axisymmetric, swirl-free solutions of the Euler equation in three dimensions with initial vorticity satisfying \( \omega^0, \frac{\omega^0}{r} \in L^2 \cap L^\infty \). Danchin relaxed this requirement \([2,3]\) to initial data satisfying \( \omega^0 \in L^{3,1} \cap L^\infty, \frac{\omega^0}{r} \in L^{3,1} \).

We will note that in the standard regularity class for well posedness of strong solutions to the three dimensional Euler equation, \( H^s(\mathbb{R}^3), s > \frac{5}{2} \), the conditions by Danchin hold automatically, and so these conditions do not involve any assumptions beyond sufficient regularity of the initial data.

In this paper, we will consider a generalization of axisymmetric, swirl-free solutions of the Euler equation to four and higher dimensions. For \( d \geq 4 \), we will consider solutions of the form

\[ u(x,t) = u_r(r,z,t)e_r + u_z(r,z,t)e_z, \] (1.13)

where

\[
\begin{align*}
  r &= \sqrt{x_1^2 + \ldots + x_{d-1}^2} \quad \text{(1.14)} \\
  z &= x_d \quad \text{(1.15)} \\
  e_r &= \frac{(x_1, \ldots, x_{d-1}, 0)}{r} \quad \text{(1.16)} \\
  e_z &= e_d \quad \text{(1.17)}
\end{align*}
\]
The scalar vorticity is again given by $\omega = \partial_r u_z - \partial_z u_r$, and now satisfies the evolution equation
\begin{equation}
\partial_t \omega + (u \cdot \nabla)\omega - k \frac{u_r}{r} \omega = 0,
\end{equation}
where $k = d - 2$. This implies that the quantity $\frac{\omega}{r^k}$ is transported by the flow with
\begin{equation}
(\partial_t + u \cdot \nabla) \frac{\omega}{r^k} = 0.
\end{equation}

This immediately opens the possibility of finite-time singularity formation when $d \geq 4$, because the transported quantity $\frac{\omega}{r^k}$ is not necessarily bounded when $k \geq 2$. In general, for sufficiently smooth, axisymmetric, swirl-free velocities, the vorticity must vanish linearly at the axis. To be more precise, we will show that if $u \in H^s(\mathbb{R}^d)$, $s > 2 + \frac{d}{2}$, then $\frac{\omega}{r^k} \in L^\infty$. For $d \geq 4$, by contrast, $\frac{\omega}{r^k}$ may be unbounded even for a Schwartz class vector field.

The fact that there is some control on the advected quantity $\frac{\omega}{r}$ of the kind imposed by Ukhovskii and Yudovich or Danchin is essential to the guaranteeing global regularity. In fact, for solutions in which the vorticity is only $C^\epsilon$, for a small $\epsilon > 0$, Elgindi recently proved finite-time blowup for solutions of the axisymmetric, swirl-free Euler equation in three dimensions [1]. Note that for these solutions, $\frac{\omega}{r}$ is singular at the $z$-axis, $r = 0$, even before the blowup time. When $d \geq 4$, the advected quantity $\frac{\omega}{r^k}$ may be singular at the $z$-axis even for smooth solutions, so this gives us a good reason to be optimistic that there will be examples of finite-time blowup for smooth, axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions. If the quantity $\frac{\omega}{r}$ being advected by the flow does not have any reasonable control, then there is no barrier to blowup from axisymmetry, and we can expect blowup for smooth solutions in $d \geq 4$ analogous to the blowup for classical—but not smooth—solutions proven by Elgindi when $d = 3$.

It would seem natural to expect that for $d \geq 4$, in the case where $\frac{\omega}{r^3} \in L^\infty$, then there will be global regularity by similar arguments guaranteeing global regularity in $d = 3$ when $\frac{\omega}{r} \in L^\infty$. Surprisingly, however, this is not the case. The arguments guaranteeing global regularity of sufficiently smooth, axisymmetric, swirl-free solutions involve exponential bounds on the growth of vorticity in time. For instance, Majda and Bertozzi [2] prove that if $\frac{\omega}{r} \in L^\infty$ and $\omega^0$ is compactly supported with maximum distance from the axis $R_0$, then $T_{\text{max}} = +\infty$ and for all $0 \leq t < +\infty$,
\begin{equation}
\|\omega(\cdot, t)\|_{L^\infty} \leq \left\|\frac{\omega^0}{r}\right\|_{L^\infty} R_0 \exp \left(3C_3 \left|\supp (\omega^0)\right|^\frac{1}{4} \left\|\frac{\omega^0}{r}\right\|_{L^\infty} t\right).
\end{equation}
This is based on a differential inequality of the form
\begin{equation}
\frac{dR}{dt} \leq 3C_3 \left|\supp (\omega^0)\right|^\frac{1}{4} \left\|\frac{\omega^0}{r}\right\|_{L^\infty} R.
\end{equation}
This differential inequality can be generalized for compactly supported vorticity with $\frac{\omega^0}{r} \in L^\infty$ when $d \geq 4$, giving
\begin{equation}
\frac{dR}{dt} \leq dC_d \left|\supp (\omega^0)\right|^\frac{1}{2} \left\|\frac{\omega^0}{r}\right\|_{L^\infty} R^{d-2}.
\end{equation}
Note that for $d \geq 4$, this is a nonlinear differential inequality, and can only give a lower bound on the possible blowup time; it cannot prevent finite-time blowup. Estimates on the growth of vorticity based on streamlines moving off towards infinity in the radial direction can also be established for more general vorticities that are not compactly supported. This allows us to give the following regularity criteria for axisymmetric, swirl free solutions of the Euler equation in four and higher dimensions when $\frac{\omega^0}{r}$ is bounded.
Theorem 1.1. Suppose $u \in C\left([0,T_{\text{max}}), H^s_{d} \left(\mathbb{R}^d\right)\right) \cap C^1\left([0,T_{\text{max}}), H^{s-1}_{d} \left(\mathbb{R}^d\right)\right)$, $d \geq 4$, $s > 2 + \frac{d}{2}$ is an axisymmetric, swirl-free solution of the Euler equation with finite-time blowup at $T_{\text{max}} < +\infty$, and that $\frac{u^0}{r^s} \in L^{d,1} \cap L^\infty$. Then for all $0 \leq t < T_{\text{max}},$

$$\|\omega(\cdot, t)\|_{L^{d,1}(\mathbb{R}^d)} \geq \frac{1}{k^{1/(d-3)} C^2_{d} \max_{r,z} \|u\|_{L^{d,1}}} \frac{1}{(T_{\text{max}} - t)^{\frac{d-2}{2}}}. \quad (1.23)$$

Furthermore, we have

$$\int_0^{T_{\text{max}}} \|u_r^+(\cdot, t)\|_{L^\infty} dt = +\infty. \quad (1.24)$$

We will note that in terms of scaling these regularity criteria are substantially stronger than the Beale-Kato-Majda criterion, and so any blowup of this kind must happen in the bulk of the flow, at least to a larger degree than Beale-Kato-Majda on its own would require. These regularity criteria follow from the fact that when $\frac{u^0}{r^s} \in L^\infty$, then there can only be finite-time blowup at some time $T_{\text{max}} < +\infty$ if a fluid trajectory runs off to spatial infinity with $r(t) \to \infty$ as $t \to T_{\text{max}}$.

It heavily appears that blowup for axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions is generic for solutions with vorticity satisfying certain geometric constraints. It appears very likely that whenever the vorticity is odd in $z$ and positive for $z > 0$, then there will be finite-time blowup. We will discuss this potential blowup scenario and the associated conjectures in detail in section 6.

In four and higher dimensions, the vorticity equation can be expressed in terms of a stream function as follows:

$$\partial_t \omega + (u \cdot \nabla) \omega - k \frac{u_r}{r} \omega = 0 \quad (1.25)$$

$$u_r = \partial_z \psi \quad (1.26)$$

$$u_z = -\partial_r \psi - k \frac{\psi}{r} \quad (1.27)$$

$$\left(-\partial^2_z - \partial^2_r - \frac{k}{r} \partial_r + \frac{k}{r^2}\right) \psi = \omega. \quad (1.28)$$

It is possible to take the formal limit of this system as $d \to \infty$, which yields the infinite-dimensional vorticity equation

$$\partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi = 0 \quad (1.29)$$

$$\partial_r \phi = \omega. \quad (1.30)$$

We will prove that smooth solutions of the infinite-dimensional vorticity equation can exhibit finite-time blowup of a Burgers shock type.

Theorem 1.2. Suppose $\omega^0 \in C^{1,2}_{r,z} (\mathbb{R}^+, \mathbb{R})$ and $\omega^0 = \partial_r \phi^0$. Then there exists a unique strong solution $\omega \in C^{1,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{\text{max}}))$ to the infinite-dimensional vorticity equation. If $\partial_z \phi^0(r, z) \geq 0$ for all $r \in \mathbb{R}^+, z \in \mathbb{R}$, then there is a global smooth solution, and so $T_{\text{max}} = +\infty$. If there exists $r_0 \in \mathbb{R}^+, z_0 \in \mathbb{R}$, such that $\partial_z \phi^0(r_0, z_0) < 0$, then there is finite-time blowup with

$$T_{\text{max}} = \frac{1}{\inf_{r \in \mathbb{R}^+, z \in \mathbb{R}} \partial_z \phi^0(r, z)}. \quad (1.31)$$
This solution of the infinite-dimensional vorticity equation is given by

$$\omega(r, z, t) = \frac{\omega^0(r, h(r, z, t))}{1 + t\partial_y\phi^0(r, h(r, z, t))},$$

(1.32)

where

$$h(r, z, t) = g_{r,t}^{-1}(z),$$

(1.33)

is the back-to-labels map of the flow given by

$$g_{r,t}(z) = z + \phi^0(r, z)t.$$

(1.34)

The fact that the infinite-dimensional vorticity equation exhibits finite-time blowup provides yet more evidence that in sufficiently high dimension, axisymmetric, swirl-free solutions of the Euler equation blowup in finite-time. In section 4 we will even discuss a way to write the vorticity equation in very high dimensions as a perturbation of the infinite-dimensional vorticity equation, where the perturbation has size $$\epsilon = \frac{1}{d^2}.$$ This may allow blowup to be proven perturbatively for large enough $$d$$, however the limit of the perturbation as $$\epsilon \to 0$$ is extremely singular, so any perturbative argument would need to be quite subtle in order to overcome the difficulties posed by this singular limit.

The paper will be structured as follows. In section 2 we will go over some key definitions and as well as the notation used in the paper. In section 3 we will show that the class of axisymmetric, swirl-free vector fields is preserved in four and higher dimensions by the dynamics of the Euler equation, derive the vorticity equation, and give a Biot-Savart law for recovering the velocity from the vorticity. In section 4 we will derive the infinite-dimensional vorticity equation, and we will prove the local well-posedness and finite-time blowup result in Theorem 1.2. In section 5 we will discuss the possibility of finite-time blowup in four and higher dimensions when $$\omega^0$$ is bounded, establishing regularity criteria and proving Theorem 1.1. Finally, in section 6 we will derive a Biot-Savart law for the special case where $$\omega$$ is odd in $$z$$, and we will prove certain geometric properties about the flow in the case where $$\omega$$ is odd in $$z$$ and positive for $$z > 0$$ that suggest a plausible path to finite-time blowup when $$d \geq 4$$.

**Remark 1.3.** It must be stated at the outset that the Cauchy problem for the Euler equation in four and higher dimensions is completely unphysical. Real, physical fluid flows are three dimensional. In some cases, the flow of actual physical fluids is close enough to being perfectly two dimensional for the two dimensional equation to be a reasonable model, but it is almost inconceivable that the Euler equation in four or higher dimensions could be a reasonable model for any physical fluids. Even in the case of relativistic fluids, where time is not neatly separable from space, the problem is still fundamentally 3 + 1 dimensional.

Nonetheless, the question of finite-time blowup of the Euler equation in four and higher dimensions is scientifically interesting beyond purely abstract, mathematical curiosity. The dramatic qualitative differences between two and three dimensional fluid mechanics show that turbulence has a very fundamental dependence on dimension. For this reason, any advance on the finite-time blowup of the Euler (or Navier–Stokes) equation in four and higher dimensions—which seems much more within reach than in the three dimensional case—could shed significant light on the possibility of blowup in three dimensions, by allowing the study of the dependence of mechanisms for finite-time blowup on the dimension.
2 Definitions and notation

In this section, we will go over the notation used in the paper, and give some of the main definitions. Throughout this paper, \( d \) will refer to the spatial dimension. We will also let \( k = d - 2 \) throughout, as this quantity frequently becomes important in the analysis.

For axisymmetric solutions, it is useful to have notation for the decomposition \( \mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R} \), so for \( x \in \mathbb{R}^d \), we will write

\[
x = (x', x_d),
\]

where

\[
x' = (x_1, \ldots, x_{d-1}).
\]

We can define the generalized cylindrical coordinates \((r, z)\) in terms of this decomposition by

\[
r = |x'|, \quad z = x_d,
\]

with associated unit vectors

\[
e_r = \left( \frac{x'}{|x'|}, 0 \right), \quad e_z = e_d.
\]

Note that we use \( \{e_1, \ldots, e_d\} \) to denote the standard basis for \( \mathbb{R}^d \). We will say that a vector field \( u : \mathbb{R}^d \to \mathbb{R}^d \) is axisymmetric and swirl-free if

\[
u(x) = u_r(r, z)e_r + u_z(r, z)e_z,
\]

and we will say that a scalar function \( f : \mathbb{R}^d \to \mathbb{R} \) is axisymmetric if

\[
f(x) = \tilde{f}(r, z).
\]

We will sometimes slightly abuse notation by equating \( f \) with \( \tilde{f} \). For derivatives in terms of the spatial variables, we will often use the shorthand

\[
\partial_i = \frac{\partial}{\partial x_i},
\]

and likewise for \( \partial_r \) and \( \partial_z \). We will also define the cylinder \( C_R \in \mathbb{R}^d \) to be the set

\[
C_R = \left\{ x \in \mathbb{R}^d : |x'| < R \right\}.
\]

Note that in cylindrical coordinates this is just the set of points satisfying \( r < R \).

The Lebesgue spaces \( L^p(\mathbb{R}^d) \) will be given the standard definition and norm. For axisymmetric functions, we will take \( L^q = L^q(m_{d-2}r^{d-2} \, dr \, dz) \), where \( m_d \) is the surface area of the \( d \)-sphere embedded in \( \mathbb{R}^{d+1} \). This means that for \( 1 \leq p < +\infty \)

\[
\|\tilde{f}\|_{L^q} = \left( m_{d-2} \int_0^\infty \int_{-\infty}^\infty |\tilde{f}(r, z)|^q r^{d-2} \, dz \, dr \right)^{\frac{1}{q}}.
\]

We define the \( L^q \) this space for functions in terms of cylindrical coordinates so that

\[
\|f\|_{L^q} = \left\| \tilde{f} \right\|_{L^q}.
\]
For $1 \leq p, q < +\infty$, we will take the Lorentz spaces $L^{p,q}(\mathbb{R}^d)$ to be the spaces with the quasinorm
\[
\|f\|_{L^{p,q}} = \left( p \int_0^\infty \alpha^{q-1} \left| \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} \right| \frac{\alpha^{\frac{d}{q}}}{\alpha} \right)^{\frac{1}{q}}. \tag{2.13}
\]
In the case $q = \infty$, we will take the quasinorm to be
\[
\|f\|_{L^{p,\infty}} = \left( \sup_{\alpha > 0} \alpha^p \left| \left\{ x \in \mathbb{R}^d : |f(x)| > \alpha \right\} \right| \right)^{\frac{1}{p}}. \tag{2.14}
\]
Note that $L^{p,q} = L^q$, and that for axisymmetric functions, we will again use the measure $m_{d-2r^{d-2}} \, dr \, dz$ when working with cylindrical coordinates, so that
\[
\|f\|_{L^{p,q}} = \|\hat{f}\|_{L^{p,q}}. \tag{2.15}
\]

We will define both homogeneous and inhomogeneous Sobolev spaces, $\dot{H}^s$ and $H^s$, which we will note are also Hilbert spaces.

**Definition 2.1.** For all $s \in \mathbb{R}, d \geq 1$, we will take $H^s(\mathbb{R}^d)$ to be the space with norm
\[
\|f\|_{H^s} = \left( \int_{\mathbb{R}^d} (1 + 4\pi^2|\xi|^2)^s |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \tag{2.16}
\]
For all $s > -\frac{d}{2}, d \geq 1$, we will take $\dot{H}^s(\mathbb{R}^d)$ to be the space with norm
\[
\|f\|_{\dot{H}^s} = \left( \int_{\mathbb{R}^d} (2\pi|\xi|)^{2s} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \tag{2.17}
\]

It is also necessary to define the set of divergence free vector fields and gradients in $H^s$.

**Definition 2.2.** For all $s \in \mathbb{R}, d \geq 2$, we will take $H^s_{df}(\mathbb{R}^d)$ to be the space of divergence free vector fields:
\[
H^s_{df}(\mathbb{R}^d; \mathbb{R}^d) = \left\{ v \in H^s_{df}(\mathbb{R}^d; \mathbb{R}^d) : \nabla \cdot v = 0 \right\}. \tag{2.18}
\]
Note that the divergence is only a continuous function when $s > 1 + \frac{d}{2}$, and so in general we will say that $\nabla \cdot v = 0$ if
\[
\xi \cdot \hat{v}(\xi) = 0, \tag{2.19}
\]
almost everywhere $\xi \in \mathbb{R}^d$. We will also take $H^s_{gr}(\mathbb{R}^d)$ to be the space of gradients,
\[
H^s_{gr}(\mathbb{R}^d; \mathbb{R}^d) = \left\{ \nabla f : f \in \dot{H}^1 \cap \dot{H}^{s+1} \right\}. \tag{2.20}
\]

**Remark 2.3.** We should note that $H^s_{gr}$ also has a Fourier space characterization. In particular $v \in H^s_{gr}$ if and only if $\hat{v}(\xi)$ is co-linear with $\xi$ almost everywhere $\xi \in \mathbb{R}^d$. In particular this means that $v \in H^s_{gr}$ if and only if $\hat{v}(\xi) \in \span(\xi)$ almost everywhere $\xi \in \mathbb{R}^d$, and $v \in H^s_{df}$ if and only if $\hat{v}(\xi) \in \span(\xi)^\perp$ almost everywhere $\xi \in \mathbb{R}^d$. This leads to the Helmholtz decomposition
\[
H^s = H^s_{df} \oplus H^s_{gr}. \tag{2.21}
\]

We will also define the set of axisymmetric, swirl-free vector fields in Sobolev spaces.
Definition 2.4. We will define the Sobolev space of axisymmetric, swirl-free vector fields as

$$H^s_{as} \left( \mathbb{R}^d; \mathbb{R}^d \right) = \{ v \in H^s : v(x) = v_r(r,z)e_r + v_z(r,z)e_z \}. \quad (2.22)$$

We will define the space of axisymmetric, swirl-free, divergence free vector fields and axisymmetric, swirl-free gradients as

$$H^s_{as\&df} = H^s_{as} \cap H^s_{df} \quad (2.23)$$
$$H^s_{as\&gr} = H^s_{as} \cap H^s_{gr}. \quad (2.24)$$

Definition 2.5. We will define $C^k(\mathbb{R}^d)$ to be the space of $k$ times continuously differentiable functions, with bounded derivatives up to order $k$, and we will give this space the norm

$$\| f \|_{C^k} = \sup_{\alpha_1, \ldots, \alpha_d \geq 0, \alpha_1 + \ldots + \alpha_d \leq k} \| \partial^{\alpha_1} \ldots \partial^{\alpha_d} f \|_{L^\infty} \quad (2.25)$$

We will take the analogous definition when dealing with $C^k$ functions that also depend on a time variable.

Remark 2.6. Note that for all $s > k + \frac{d}{2}$, we have the continuous embedding

$$H^s(\mathbb{R}^d) \hookrightarrow C^k(\mathbb{R}^d), \quad (2.26)$$

and there exists a constant $C_{s,k,d} > 0$ such that

$$\| f \|_{C^k} \leq C_{s,k,d} \| f \|_{H^s}. \quad (2.27)$$

We will refer to the anti-symmetric and symmetric parts of the gradient of a vector field $u$ as $A$ and $S$ with

$$A_{ij} = \frac{1}{2} (\partial_i u_j - \partial_j u_i) \quad (2.28)$$
$$S_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i). \quad (2.29)$$

We will also define the anti-symmetric gradient operator and symmetric gradient operator likewise with

$$\nabla_{asym} u = A \quad (2.30)$$
$$\nabla_{sym} u = S. \quad (2.31)$$

In three dimensions $A$ can be represented in terms of the vorticity vector $\vec{\omega} = \nabla \times u$ by

$$A = \begin{pmatrix} 0 & \vec{\omega}_3 & -\vec{\omega}_2 \\ -\vec{\omega}_3 & 0 & \vec{\omega}_1 \\ \vec{\omega}_2 & -\vec{\omega}_1 & 0 \end{pmatrix}. \quad (2.32)$$

For axisymmetric, swirl-free vector fields

$$A(x) = \frac{1}{2} \omega(r,z) \left( e_r \otimes e_z - e_z \otimes e_r \right), \quad (2.33)$$

and in three dimensions

$$\vec{\omega}(x) = -\omega(r,z) e_\theta. \quad (2.34)$$
3 The evolution of the axisymmetric, swirl-free Euler equation in higher dimensions

In this section, we will show that the axisymmetric, swirl-free Euler equation in four and higher dimensions can be expressed in terms of the evolution of the scalar vorticity, just as it can in three dimensions.

3.1 Preliminaries

We will begin by going over some of the vector calculus results necessary for the analysis of axisymmetric, swirl-free solutions. The computations are mostly routine exercises, however they are very important to the analysis.

**Proposition 3.1.** Suppose \( u \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}, \) is axisymmetric and swirl-free. Then the gradient of \( u \) can be expressed as

\[
\nabla u(x) = \partial_r u_r(r, z)e_r \otimes e_r + \frac{u_r(r, z)}{r} \tilde{I}_d + \partial_z u_z(r, z)e_z \otimes e_z + \partial_r u_z(r, z)e_r \otimes e_z + \partial_z u_r(r, z)e_z \otimes e_r, \tag{3.1}
\]

where

\[
\tilde{I}_d = I_d - e_r \otimes e_r - e_z \otimes e_z. \tag{3.2}
\]

**Proof.** First we know that

\[
u(x) = u_r(r, z)e_r + u_z(r, z)e_z \tag{3.3}
\]

We know that

\[
\nabla r = e_r, \tag{3.4}
\]

and therefore we can see that

\[
\nabla u(x) = \partial_r u_r(r, z)e_r \otimes e_r + \partial_z u_z(r, z)e_z \otimes e_z + \partial_r u_z(r, z)e_r \otimes e_z + \partial_z u_r(r, z)e_z \otimes e_r + u_r(r, z)\nabla e_r. \tag{3.5}
\]

It remains only to show that

\[
\nabla e_r = \frac{1}{r} \tilde{I}_d \tag{3.6}
\]

Recall that \( e_r = \frac{x'}{|x'|}, \) where \( x' = (x_1, ..., x_{d-1}, 0). \) Clearly we have

\[
\nabla x' = I_d - e_z \otimes e_z, \tag{3.7}
\]

and

\[
\nabla \frac{1}{|x'|} = \nabla \frac{1}{r} = -\frac{1}{r^2} e_r. \tag{3.8}
\]

Therefore we can conclude that

\[
\nabla e_r = \frac{1}{r} \nabla x' + \frac{1}{|x'|} \otimes x' \tag{3.9}
\]

\[
= \frac{1}{r} (I_d - e_z \otimes e_z) - \frac{1}{r^2} e_r \otimes x' \tag{3.10}
\]

\[
= \frac{1}{r} (I_d - e_z \otimes e_z - e_r \otimes e_r). \tag{3.11}
\]

This completes the proof. \( \square \)
Proposition 3.2. Suppose \( u \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}, \) is axisymmetric and swirl-free. Then the divergence of \( u \) can be expressed as
\[
\nabla \cdot u(x) = \partial_r u_r(r, z) + k \frac{u_r(r, z)}{r} + \partial_z u_z(r, z),
\]
where \( k = d - 2 \).

Proof. Taking the identity from Proposition 3.1, we can see that
\[
\nabla \cdot u(x) = \text{tr}(\nabla u(x))
\]
\[
= \partial_r u_r + \partial_z u_z + \frac{u_r(r, z)}{r} \text{tr}(\tilde{I}_d)
\]
\[
= \partial_r u_r(r, z) + \partial_z u_z(r, z) + (d - 2) \frac{u_r(r, z)}{r}.
\]
This completes the proof.

Proposition 3.3. Suppose \( u \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2}, \) is axisymmetric and swirl-free. Then \( (u \cdot \nabla)u \) is axisymmetric and swirl free.

Proof. We know by hypothesis that
\[
u(x) = u_r(r, z)e_r + u_z(r, z)e_z
\]
and by Proposition 3.1 that
\[
\nabla u(x) = \partial_r u_r(r, z)e_r \otimes e_r + \frac{u_r(r, z)}{r} \tilde{I}_d + \partial_z u_z(r, z)e_z \otimes e_z
\]
\[
+ \partial_r u_z(r, z)e_r \otimes e_z + \partial_z u_r(r, z)e_z \otimes e_r.
\]
We can see that \( \tilde{I}_d \) is the matrix for the projection onto the subspace \( \text{span}(e_r, e_z)^\perp \), and so
\[
(\nabla \cdot u)u = (\nabla u)^{\text{tr}} u
\]
\[
= u_r \partial_r u_r e_r + u_z \partial_z u_z e_z + u_r \partial_r u_z e_z + u_z \partial_z u_r e_r
\]
\[
= (u \cdot \nabla u_r)e_r + (u \cdot \nabla u_z)e_z.
\]
Recalling the definition of an axisymmetric vector field, this completes the proof.

Proposition 3.4. Suppose \( u \in H^s(\mathbb{R}^d), s > 1 + \frac{d}{2} \), is axisymmetric and swirl-free. Then
\[
A(x) = \frac{1}{2} \omega(r, z)(e_r \otimes e_z - e_z \otimes e_r),
\]
where \( \omega(r, z) = \partial_r u_z - \partial_z u_r \).

Proof. Taking the formula for \( \nabla u \) from Proposition 3.1, we can see that
\[
A(x) = \frac{1}{2} (\partial_r u_z e_r \otimes e_z + \partial_z u_z e_z \otimes e_r - \partial_r u_z e_z \otimes e_r - \partial_z u_r e_r \otimes e_z)
\]
\[
= \frac{1}{2} (\partial_r u_z - \partial_z u_r)(e_r \otimes e_z - e_z \otimes e_r).
\]
This completes the proof.
**Proposition 3.5.** For all \(d \geq 3\), the divergence of the radial direction \(e_r \in \mathbb{R}^d\) is
\[
\nabla \cdot e_r = \frac{k}{r}.
\] (3.24)

**Proof.** Recall that
\[
e_r = \frac{x'}{|x'|}.
\] (3.25)
It then follows that
\[
\nabla \cdot e_r = \frac{\nabla \cdot x'}{|x'|} + \frac{x'}{|x'|^3}.
\] (3.26)
\[
= \frac{d - 1}{r} - \frac{x'}{|x'|^3}.
\] (3.27)
\[
= \frac{d - 2}{r}.
\] (3.28)
This completes the proof. \(\square\)

**Proposition 3.6.** Suppose \(u \in H^s(\mathbb{R}^d; \mathbb{R}^d), s > 2 + \frac{d}{2}\) is an axisymmetric, swirl-free vector field. Then the vector Laplacian can be expressed as
\[
-\Delta u(x) = \left(-\partial_r^2 - \frac{k}{r}\partial_r + \frac{k}{r^2} - \partial_z^2\right) u_r(r, z)e_r + \left(-\partial_r^2 - \frac{k}{r^2}\partial_r - \partial_z^2\right) u_z(r, z)e_z
\] (3.29)

**Proof.** Taking the divergence of the gradient formula in Proposition 3.1, using the fact that \(-\Delta u = -\text{div} \nabla u\), and applying Proposition 3.5, we find that
\[
-\Delta u = -\partial_z^2 u_r e_r - \partial_z^2 u_z e_z - \partial_z^2 u_z e_z - \frac{k}{r}\partial_r u_z e_z - \text{div} (\partial_r u_r e_r \otimes e_r) - \text{div} \left(\frac{u_r}{r}I_d\right).
\] (3.30)
Again applying Proposition 3.5, we can see that
\[
-\text{div} (\partial_r u_r (e_r \otimes e_r)) = \left(-\partial_r^2 u_r - \frac{k}{r}\partial_r u_r\right) e_r,
\] (3.31)
and that
\[
-\text{div} \left(\frac{u_r}{r}I_d\right) = -\left(\partial_r \frac{u_r}{r}\right) I_d e_r - \frac{u_r}{r} \text{div} (I_d - e_r \otimes e_r - e_z \otimes e_z)
\] (3.32)
\[
= \frac{k}{r^2} u_r e_r.
\] (3.33)
This completes the proof. \(\square\)

**Proposition 3.7.** Suppose \(f \in H^s, s > 2 + \frac{d}{2}\), is a scalar axisymmetric function with \(f(x) = \tilde{f}(r, z)\). Then
\[
\nabla f(x) = \partial_r \tilde{f}(r, z)e_r + \partial_z \tilde{f}(r, z)e_z,
\] (3.34)
and
\[
-\Delta f(x) = -\partial_r^2 \tilde{f}(r, z) - \frac{k}{r}\partial_r \tilde{f}(r, z) - \partial_z^2 \tilde{f}(r, z).
\] (3.35)
Taking the divergence of this equation and applying Proposition 3.2, this completes the proof.

Proof. Observe that
\[ \nabla r = e_r, \]  
and so applying the chain rule,
\[ \nabla f(x) = \partial_r \bar{f}(r, z)e_r + \partial_z \bar{f}(r, z)e_z. \]  
(3.37)

Taking the divergence of this equation and applying Proposition 3.2, this completes the proof. \( \square \)

**Proposition 3.8.** Suppose \( u \in H^s_{\text{as}}(\mathbb{R}^d), s > 2 + \frac{d}{2}. \) Then the vector Laplacian can be expressed as
\[ -\Delta u = -2 \operatorname{div} \nabla_{\text{asym}} u - \nabla \nabla \cdot u. \]  
(3.38)

**Proof.** First note that for \( s > 2 + \frac{d}{2}, H^s_{\text{as}}(\mathbb{R}^d) \hookrightarrow C^2(\mathbb{R}^d), \) so the both sides of the expression are well defined for all \( x \in \mathbb{R}^d. \) Let
\[ u(x) = u_r(r, z)e_r + u_z(r, z)e_z \]  
(3.39)

Recalling that
\[ \nabla \cdot u = \partial_r u_r + \frac{k}{r} u_r + \partial_z u_z, \]  
(3.40)
we can compute that
\[ \nabla \nabla \cdot u = \left( \frac{\partial^2}{r^2} + \frac{k}{r} \partial_r - \frac{k}{r^2} \right) u_r e_r + \partial^2_z u_z + \partial_z \left( \partial_r u_r + \frac{k}{r} u_r \right) e_z + \partial_r \partial_z u_z e_r. \]  
(3.41)

Likewise recalling that
\[ 2 \nabla_{\text{asym}} u = (\partial_r u_z - \partial_z u_r)(e_r \otimes e_z - e_z \otimes e_r), \]  
(3.42)
and applying the product rule
\[ \operatorname{div} (f M) = f \operatorname{div}(M) + M^\ell \nabla f, \]  
(3.43)
we find that
\[ 2 \operatorname{div} \nabla_{\text{asym}} u = \frac{k}{r} (\partial_r u_z - \partial_z u_r) e_z + \partial_r (\partial_r u_z - \partial_z u_r) e_z - \partial_z (\partial_r u_z - \partial_z u_r) e_r \]  
(3.44)
\[ = \partial^2_z u_r e_r + \left( \frac{\partial^2}{r^2} + \frac{k}{r} \partial_r \right) u_z e_z - \frac{k}{r} \partial_z u_r e_z - \partial_r \partial_z u_z e_z - \partial_z \partial_z u_z e_z. \]  
(3.45)

Adding together (3.41) and (3.45), this completes the proof. \( \square \)

**Remark 3.9.** Note that the operators \(-\Delta, \operatorname{div} \nabla_{\text{asym}}, \nabla \nabla \cdot, (-\Delta)^{-1}\) all preserve the class of axisymmetric, swirl-free vector fields. For the first three operators, this can be observed directly from the computations above. For \((-\Delta)^{-1}\), the result follows in any space where the kernel of the Laplace operator is trivial from the fact that \(-\Delta\) preserves this class.

**Proposition 3.10.** Axisymmetric, swirl free vector fields in dimension \( d \geq 3 \) have the following Helmholtz decomposition:
\[ H^s_{\text{as}}(\mathbb{R}^d) = H^s_{\text{as}kdf}(\mathbb{R}^d) \oplus H^s_{\text{as}kgr}(\mathbb{R}^d), \]  
(3.46)
for all \( s \in \mathbb{R}. \) In particular, for all \( v \in H^s_{\text{as}}, \)
\[ v = -2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1} v - \nabla \nabla \cdot (-\Delta)^{-1} v, \]  
(3.47)
with \(-2 \operatorname{div} \nabla_{\text{asym}}(-\Delta)^{-1} v \in H^s_{\text{as}kdf}(\mathbb{R}^d)\) and \(-\nabla \nabla \cdot (-\Delta)^{-1} v \in H^s_{\text{as}kgr}(\mathbb{R}^d)\).
Proof. We know immediately from the classical Helmholtz decomposition that for all \( u \in H^s_{as\&df} \), \( \nabla f \in H^s_{as\&gr} \)
\[
\langle u, \nabla f \rangle_s = 0,
\]
and so the spaces are orthogonal. It remains only to show that
\[
H^s_{as}(\mathbb{R}^d) = H^s_{as\&df}(\mathbb{R}^d) + H^s_{as\&gr}(\mathbb{R}^d)
\] (3.49)
We will first consider this result for \( s > \frac{3}{2} \). Applying Proposition 3.8 to \((-\Delta)^{-1} v\), we find that
\[
v = -2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v - \nabla \nabla \cdot (-\Delta)^{-1} v.
\] (3.50)
It is obvious that \(-\nabla \nabla \cdot (-\Delta)^{-1} v \in H^s_{as\&gr}\), so it remains only to show that \(-2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v \in H^s_{as\&df}\). Letting \( A = \nabla_{asym}(-\Delta)^{-1} v \), we can see that
\[
\nabla \cdot (-2 \text{ div } A) = -2 \text{ div}^2 A
\]
\[
= -2 \sum_{i,j=1}^d \partial_i \partial_j A_{ij}
\]
\[
= 0,
\]
(3.53)
due to the anti-symmetry of \( A \). This completes the proof for \( s > \frac{d}{2} \).

We should note that the formula for the Laplacian in Proposition 3.8 holds pointwise only for \( C^2 \) vector fields, so \((-\Delta)^{-1} v\) must be at least \( C^2 \) for this identity to hold classically. This is why we require that \( s > \frac{d}{2} \) for the identity to hold pointwise. However, \(-2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v\) and \(-\nabla \nabla \cdot (-\Delta)^{-1} v\) are well defined, bounded, linear operators mapping \( H^s \to H^s \), for all \( s \in \mathbb{R} \), and so the decomposition is still valid and well defined for \( s \leq \frac{d}{2} \).

In order to prove that the identity
\[
v = -2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v - \nabla \nabla \cdot (-\Delta)^{-1} v
\] (3.54)
holds in \( H^s_{as} \) for all \( s \in \mathbb{R} \), we will need to work in Fourier space. We will show that
\[
\mathcal{F}(-2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v)(\xi) = P_{\text{span}\{\xi\}} \hat{v}(\xi)
\] (3.55)
\[
\mathcal{F}(-\nabla \nabla \cdot (-\Delta)^{-1} v)(\xi) = P_{\text{span}\{\xi\}} \hat{v}(\xi),
\] (3.56)
almost everywhere \( \xi \in \mathbb{R}^d \). This establishes the decomposition, because it proves both the identity (3.37) and also shows that \(-2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v \in H^s_{as\&df} \), \(-\nabla \nabla \cdot (-\Delta)^{-1} v \in H^s_{as\&gr}\). It is simple to compute that
\[
\mathcal{F}(-\nabla \nabla \cdot (-\Delta)^{-1} v)(\xi) = \left( \frac{\xi \otimes \xi}{|\xi|^2} \right) \hat{v}(\xi)
\]
\[
= P_{\text{span}\{\xi\}} \hat{v}(\xi),
\] (3.58)
almost everywhere \( \xi \in \mathbb{R}^d \). We also compute that
\[
\mathcal{F}(-2 \text{ div } \nabla_{asym}(-\Delta)^{-1} v)(\xi) = \frac{1}{|\xi|^2} \xi \cdot (\xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi).
\] (3.59)
For all $\xi \in \mathbb{R}^d$, let
\begin{align}
P_{\text{span}(\xi) \perp} \hat{v}(\xi) &= w(\xi) \quad (3.60) \\
P_{\text{span}(\xi) \perp} \hat{v}(\xi) &= \lambda(\xi)\xi. \quad (3.61)
\end{align}
We can immediately see that almost everywhere $\xi \in \mathbb{R}^d$,
\[ \xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi = \xi \otimes w(\xi) - w(\xi) \otimes \xi. \quad (3.62) \]
By definition, we have $\xi \cdot w(\xi) = 0$, almost everywhere $\xi \in \mathbb{R}^d$, and so we can conclude that
\[ \xi \cdot (\xi \otimes \hat{v}(\xi) - \hat{v}(\xi) \otimes \xi) = \xi \cdot (\xi \otimes w(\xi) - w(\xi) \otimes \xi) \\
= \|\xi\|^2 w(\xi), \quad (3.63) \]
and that therefore
\[ \mathcal{F} \left(-2 \, \text{div} \, \nabla_{\text{asym}} (-\Delta)^{-1} v\right)(\xi) = P_{\text{span}(\xi) \perp} \hat{v}(\xi). \quad (3.65) \]
This completes the proof. \qed

\textbf{Remark 3.11.} We note here that the above proposition shows that for all $v \in H^s_{\text{as}}, P_{df}(v) \in H^s_{\text{as}}$.

\textbf{Theorem 3.12.} For all $u^0 \in H^s_{df} (\mathbb{R}^d), s > 1 + \frac{d}{2}, d \geq 3$, there exists a unique solution of the Euler equation $u \in C \left([0, T_{\max}), H^s_{df} (\mathbb{R}^d) \right) \cap C^1 \left([0, T_{\max}), H^{s-1}_{df} (\mathbb{R}^d) \right)$. Furthermore, if $T_{\max} < +\infty$,
\[ \int_0^{T_{\max}} \| \mathcal{A}(\cdot, t) \|_{L^\infty} \, dt = +\infty. \quad (3.66) \]

\textbf{Remark 3.13.} This is the classic Beale-Kato-Majda criterion, and was proven in the case $d = 3, s \geq 3$ in [1]. Kato and Ponce proved the general case where $d \geq 3, s > 1 + \frac{d}{2}$ in [7]. They also proved an analogous regularity criterion in terms of the strain, showing that if $T_{\max} < +\infty$, then
\[ \int_0^{T_{\max}} \| \mathcal{S}(\cdot, t) \|_{L^\infty} \, dt = +\infty. \quad (3.67) \]
See also the work of Kato [6] for details on the local wellposedness theory.

\textbf{Theorem 3.14.} For all $u^0 \in H^{s}_{\text{askdf}} (\mathbb{R}^d), s > 1 + \frac{d}{2}$, there exists a unique solution of the Euler equation $u \in C \left([0, T_{\max}), H^{s}_{\text{askdf}} (\mathbb{R}^d) \right) \cap C^1 \left([0, T_{\max}), H^{s-1}_{\text{askdf}} (\mathbb{R}^d) \right)$. Furthermore, if $T_{\max} < +\infty$,
\[ \int_0^{T_{\max}} \| \mathcal{\omega}(\cdot, t) \|_{L^\infty} \, dt = +\infty. \quad (3.68) \]

\textbf{Proof.} We know from Theorem 3.12 that there must be a smooth solution of the Euler equation $u \in C \left([0, T_{\max}), H^s_{df} (\mathbb{R}^d) \right) \cap C^1 \left([0, T_{\max}), H^{s-1}_{df} (\mathbb{R}^d) \right)$. Applying Propositions 3.3 and 3.10, we can see that if $u(\cdot, t) \in H^{s}_{\text{askdf}}$, then $P_{df}((u \cdot \nabla)u)(\cdot, t) \in H^{s}_{\text{askdf}}$. Recall that $\partial_t u + P_{df}((u \cdot \nabla)u) = 0$, we can conclude that if $u(\cdot, t) \in H^{s}_{\text{askdf}}$, then $\partial_t u(\cdot, t) \in H^{s-1}_{\text{askdf}}$. This clearly implies that the class of axisymmetric, swirl-free vector fields is preserved by the dynamics of the Euler equation, and therefore $u^0 \in H^{s}_{\text{askdf}}$ implies that $u \in C \left([0, T_{\max}), H^{s}_{\text{askdf}} (\mathbb{R}^d) \right) \cap C^1 \left([0, T_{\max}), H^{s-1}_{\text{askdf}} (\mathbb{R}^d) \right)$. This completes the proof. \qed
3.2 The vorticity equation

Now that the necessary preliminaries are out of the way, we can derive the evolution equation for the vorticity in four and higher dimensions.

**Proposition 3.15.** Suppose \( u \in C \left( [0, T_{\text{max}}); H^s_{\text{as}}(\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H^{s-1}_{\text{as}}(\mathbb{R}^d) \right), d \geq 3, s > 2 + \frac{d}{2}, \) is a solution of the Euler equation. Then for all \( 0 \leq t < T_{\text{max}} \), the vorticity satisfies the evolution equation

\[
\partial_t \omega + (u \cdot \nabla) \omega - k \frac{u_r}{r} \omega = 0.
\]

**Proof.** We begin by observing that due to Sobolev embedding, we have \( u(\cdot, t) \in C^2 \) and consequently \( \omega(\cdot, t) \in C^1 \), for all \( 0 \leq t < T_{\text{max}} \). Next we write the axisymmetric Euler equation in terms of the components \( u_r \) and \( u_z \), yielding

\[
\partial_t u_r + (u \cdot \nabla) u_r + \partial_r p = 0 \quad (3.70)
\]

\[
\partial_t u_z + (u \cdot \nabla) u_z + \partial_z p = 0 \quad (3.71)
\]

Recalling that \( \omega = \partial_r u_z - \partial_z u_r \), we differentiate the equation for \( u_r \) with respect to \( z \) and the equation for \( u_z \) with respect to \( r \), yielding

\[
\partial_t \partial_z u_r + (u \cdot \nabla) \partial_z u_r + (\partial_z u \cdot \nabla) u_r + \partial_r \partial_z p = 0 \quad (3.72)
\]

\[
\partial_t \partial_r u_z + (u \cdot \nabla) \partial_r u_z + (\partial_r u \cdot \nabla) u_z + \partial_r \partial_z p = 0 \quad (3.73)
\]

Subtracting (3.72) from (3.73), we find that

\[
\partial_t \omega + (u \cdot \nabla) \omega - (\partial_z u \cdot \nabla) u_r + (\partial_r u \cdot \nabla) u_z = 0. \quad (3.74)
\]

It remains only to compute the difference term. Plugging in we find that

\[
-(\partial_z u \cdot \nabla) u_r + (\partial_r u \cdot \nabla) u_z = -\partial_z u_r \partial_r u_r - \partial_z u_z \partial_z u_r + \partial_r u_r \partial_z u_z + \partial_r u_z \partial_z u_z = (\partial_r u_r + \partial_z u_z)(\partial_r u_z - \partial_z u_r) = -k \frac{u_r}{r} \omega, \quad (3.75)
\]

\[
\Rightarrow \partial_t u_r + \partial_z u_z + k \frac{u_r}{r} = 0. \quad (3.78)
\]

This completes the proof. \( \square \)

**Remark 3.16.** We will show in the next subsection that for axisymmetric, swirl-free vector fields, the velocity \( u \) is uniquely determined by the the scalar vorticity \( \omega \), so the vorticity equation completely determines the dynamics of the axisymmetric, swirl-free Euler equation in four and higher dimensions, just as it does in three dimensions.

One of the most important features of the axisymmetric Euler equation in three dimensions is that the quantity \( \frac{\omega}{r} \) is advected by the flow, and therefore remains bounded in a number of important spaces. There is an analogous result in four and higher dimensions: the quantity \( \frac{\omega}{r} \) is advected by the flow.
Proposition 3.17. Suppose \( u \in C\left([0,T_{\text{max}}); H^s_{as,\text{df}}(\mathbb{R}^d)\right) \cap C^1\left([0,T_{\text{max}}), H^{s-1}_{as,\text{df}}(\mathbb{R}^d)\right), d \geq 3, s > 2 + \frac{d}{2}, \) is a solution of the Euler equation. Then for all \( 0 \leq t < T_{\text{max}}, \)

\[
(\partial_t + u \cdot \nabla) \frac{\omega}{r^k} = 0. \tag{3.79}
\]

If we let \( \xi = \frac{\omega}{r^k} \), then this can be expressed as

\[
(\partial_t + u \cdot \nabla) \xi = 0. \tag{3.80}
\]

Proof. Applying the product rule and plugging into the vorticity equation we can see that

\[
(\partial_t + u \cdot \nabla) \frac{\omega}{r^k} = \frac{1}{r^k} \partial_t \omega + \frac{1}{r^k} (u \cdot \nabla) \omega - u_r \frac{\omega}{r^{k+1}} \tag{3.81}
\]

\[
= \frac{1}{r^k} (\partial_t \omega + (u \cdot \nabla) \omega - u_r \frac{\omega}{r}) \tag{3.82}
\]

\[
= 0. \tag{3.83}
\]

This completes the proof. \( \square \)

Proposition 3.18. Suppose \( u \in C\left([0,T_{\text{max}}); H^s_{as,\text{df}}(\mathbb{R}^d)\right) \cap C^1\left([0,T_{\text{max}}), H^{s-1}_{as,\text{df}}(\mathbb{R}^d)\right), d \geq 3, s > 2 + \frac{d}{2}, \) is a solution of the Euler equation, and let \( X(r,z,t) \) be the associated Lagrangian flow map satisfying

\[
\frac{dX}{dt}(r,z,t) = u(X(r,z,t),t) \tag{3.84}
\]

\[
X(r,z,0) = re_r + ze_z. \tag{3.85}
\]

Then for all \( 0 \leq t < T_{\text{max}}, r \in \mathbb{R}^+, z \in \mathbb{R}, \)

\[
\frac{\omega(X(r,z,t),t)}{X_r(r,z,t)^k} = \frac{\omega^0(r,z)}{r^k} \tag{3.86}
\]

Proof. We will begin by letting

\[
\xi(r,z,t) = \frac{\omega(r,z,t)}{r^k}. \tag{3.87}
\]

We have already shown that \( \xi \) is advected by the flow and so

\[
(\partial_t + u \cdot \nabla) \xi = 0. \tag{3.88}
\]

Define \( \tilde{\xi} \) by

\[
\tilde{\xi}(r,z,t) = \xi(X(r,z,t),t). \tag{3.89}
\]

Applying the chain rule, we can see that for all \( 0 \leq t < T_{\text{max}}, r \in \mathbb{R}^+, z \in \mathbb{R}, \)

\[
\partial_t \tilde{\xi}(r,z,t) = (\partial_t \xi + \frac{dX}{dt} \cdot \nabla \xi)(X(r,z,t),t) \tag{3.90}
\]

\[
= (\partial_t \xi + u \cdot \nabla \xi)(X(r,z,t),t) \tag{3.91}
\]

\[
= 0. \tag{3.92}
\]
Therefore, we can conclude that for all \(0 \leq t < T_{\text{max}}\), \(r \in \mathbb{R}^+, z \in \mathbb{R}\),

\[
\xi(X(r, z, t), t) = \xi(X(r, z, 0), 0)
\]

and that consequently

\[
\frac{\omega(X(r, z, t), t)}{X_r(r, z, t)^k} = \frac{\omega^0(r, z)}{r^k}.
\]

This completes the proof.

**Corollary 3.19.** Suppose \(u \in C\left([0, T_{\text{max}}); H^s_{\text{as}k}\left(\mathbb{R}^d\right)\right] \cap C^1\left([0, T_{\text{max}}), H^{s-1}_{\text{as}k}\left(\mathbb{R}^d\right)\right), d \geq 3, s > 2 + \frac{d}{2}\), is a solution of the Euler equation, and that \(\frac{\omega^0}{r^k} \in L^\infty\). Then for all \(r \in \mathbb{R}^+, z \in \mathbb{R}, 0 \leq t < T_{\text{max}}\),

\[
|\omega(r, z, t)| \leq \left\|\frac{\omega^0}{r^k}\right\|_{L^\infty} r^k.
\]

**Proof.** We can see immediately from Proposition 3.17 that for all \(0 \leq t < T_{\text{max}}\),

\[
\left\|\xi(\cdot, t)\right\|_{L^\infty} = \left\|\xi^0\right\|_{L^\infty},
\]

and therefore for all \(r \in \mathbb{R}^+, z \in \mathbb{R}, 0 \leq t < T_{\text{max}}\),

\[
\frac{\omega(r, z, t)}{r^k} \leq \left\|\xi^0\right\|_{L^\infty}.
\]

This completes the proof.

**Remark 3.20.** We will note that there is a fundamental difference between the case \(d = 3\) and the case \(d \geq 4\). When we take \(d = 3\), must have \(\frac{\omega^0}{r^k} \in L^\infty\) for sufficiently smooth initial data, while in \(d \geq 4\), there are Schwartz class initial data where \(\frac{\omega^0}{r^k} \notin L^\infty\). We will show this now.

**Proposition 3.21.** Suppose \(u \in H^s_{\text{as}k}(\mathbb{R}^d), s > 2 + \frac{d}{2}\). Then \(\frac{\omega}{r^k} \in L^\infty\) and

\[
\left\|\frac{\omega}{r^k}\right\|_{L^\infty} \leq C_{s, d}\|u\|_{H^s}.
\]

**Proof.** We will begin by observing that if \(u \in H^s(\mathbb{R}^d)\), then clearly \(\nabla A \in H^{s-2}(\mathbb{R}^d), s - 2 > \frac{d}{2}\), so by Sobolev embedding

\[
\|\nabla A\|_{L^\infty} \leq C\|\nabla A\|_{H^{s-2}} \leq C\|u\|_{H^s}.
\]

Now it remains only to control magnitude of \(\frac{\omega}{r^k}\) by the magnitude of \(\nabla A\). By definition \(\nabla A\) is a three tensor with

\[
(\nabla A)_{ijk} = \partial_i A_{jk}.
\]

Recall that

\[
A(x) = \frac{1}{2}\omega(r, z)(e_r \otimes e_z - e_z \otimes e_r),
\]

\[
\frac{\omega(r, z, t)}{r^k} \leq \left\|\xi^0\right\|_{L^\infty}.
\]
and we can see that
\[
\nabla A = \frac{1}{2} \nabla \omega \otimes (e_r \otimes e_z - e_z \otimes e_r) + \frac{1}{2} \omega (\nabla e_r \otimes e_z - (\nabla e_r \otimes e_z)^*) \tag{3.104}
\]
\[
= \frac{1}{2} \nabla \omega \otimes (e_r \otimes e_z - e_z \otimes e_r) + \frac{1}{2} \omega \left( \tilde{I}_d \otimes e_z - \left( \tilde{I}_d \otimes e_z \right)^* \right), \tag{3.105}
\]
where \( M^* \) for a three tensor is transpose of the last two elements
\[
M^*_{ijk} = M_{ikj}. \tag{3.106}
\]
Recalling that \( \nabla \omega = \partial_r \omega e_r + \partial_z \omega e_z \), and observing that the two terms in (3.105) are orthogonal, we can see that
\[
|\nabla A|^2 = \frac{1}{2} |\nabla \omega|^2 + \frac{d - 2}{2} \frac{\omega^2}{r} \tag{3.107}
\]
This clearly implies that
\[
\left\| \frac{\omega}{r} \right\|_{L^\infty} \leq \sqrt{\frac{2}{d-2}} \|\nabla A\|_{L^\infty}, \tag{3.108}
\]
and this completes the proof. □

**Proposition 3.22.** For all \( d \geq 4 \), there exists a Schwartz class divergence free vector field \( u \in S(\mathbb{R}^d) \) such that \( \frac{\omega}{r^k} \notin L^\infty \). In particular,
\[
u(x) = r(1 - 2z^2) \exp(-r^2 - z^2) e_r - z(k + 1 - 2r^2) \exp(-r^2 - z^2) e_z, \tag{3.109}
\]
satisfies these conditions.

**Proof.** We will begin by showing that \( u \) is Schwartz class and divergence free. Using that \( x' = re_r \), we can see that
\[
u(x) = \left( x'(1 - 2x_d^2) - x_d(k + 1 - 2|x'|^2) \right) \exp(-|x|^2), \tag{3.110}
\]
and so we can see that \( u \in S(\mathbb{R}^d) \). Next we compute that
\[
\nabla \cdot u = \partial_r u_r + \frac{k}{r} u_r + \partial_z u_z \tag{3.111}
\]
\[
= \left( (k + 1 - 2r^2)(1 - 2z^2) + (-1 + 2z^2)(k + 1 - 2r^2) \right) \exp(-r^2 - z^2) \tag{3.112}
\]
\[
= 0. \tag{3.113}
\]
Now that we have shown that \( u \in S_{df} \), it remains only to show that \( \frac{\omega}{r^k} \) is unbounded. Computing the vorticity we find that
\[
\omega = \partial_r u_z - \partial_z u_r \tag{3.114}
\]
\[
= (4rz + 2(k + 1)rz - 4r^3z + 4rz + 2rz - 4rz^3) \exp(-r^2 - z^2) \tag{3.115}
\]
\[
= rz (2k + 12 - 4r^2 - 4z^2) \exp(-r^2 - z^2), \tag{3.116}
\]
and so we can conclude that
\[
\frac{\omega}{r^k} = \frac{z}{r^{d-3}} \left( 2k + 12 - 4r^2 - 4z^2 \right) \exp(-r^2 - z^2). \tag{3.117}
\]
By hypothesis, \( d \geq 4 \), and so \( \frac{\omega}{r^k} \notin L^\infty \), and this completes the proof. □
We should note that the vorticity equation can also be expressed in “divergence form”, by making use of the coordinate divergence \( \nabla \cdot \mathbf{v} = \partial_r v_r + \partial_z v_z \).

**Proposition 3.23.** Suppose \( u \in C \left( [0, T_{\text{max}}); H^s_{\text{as}}(\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H^{s-1}_{\text{as}}(\mathbb{R}^d) \right), d \geq 3, s > 2 + \frac{d}{2}, \) is a solution of the Euler equation. Then for all \( 0 \leq t < T_{\text{max}}, \)

\[
\partial_t \omega + \nabla \cdot (\omega u) = 0 \quad (3.118)
\]

*Proof.* We begin by applying the product rule, and observing that

\[
\nabla \cdot (\omega u) = (u \cdot \nabla) \omega + (\nabla \cdot u) \omega. \quad (3.119)
\]

Recalling the identity (3.76) from Proposition 3.15, we already shown that

\[
\partial_t \omega + (u \cdot \nabla) \omega + (\nabla \cdot u) \omega = 0, \quad (3.120)
\]

and this completes the proof. \( \square \)

### 3.3 The Biot-Savart law in higher dimensions

The scalar vorticity uniquely determines the velocity for axisymmetric, swirl-free vector fields. In this section, we will derive integral formulae for recovering the velocity and velocity component functions from the vorticity.

**Proposition 3.24.** Suppose \( u \in H^s_{\text{as}}(\mathbb{R}^d), s > \frac{d}{2}, d \geq 3, \) is axisymmetric and swirl-free. Then \( u \) can be recovered from the scalar vorticity using the formula,

\[
u(x) = -\alpha_d \int_{\mathbb{R}^d} \omega(\rho, s)(e_\rho \otimes e_z - e_z \otimes e_\rho) \frac{x - y}{|x - y|^d} \, dy, \quad (3.121)
\]

where \( \rho = |y|, s = y_d, \) and \( \alpha_d = \frac{(d-2)\Gamma\left(\frac{d-1}{2}\right)}{4\pi^\frac{d}{2}}. \)

*Proof.* Recalling that \( A = \frac{1}{2}(\nabla u - (\nabla u)^{tr}) \), we can see that

\[-2 \text{div}(A) = -\Delta u + \nabla (\nabla \cdot u) = -\Delta u, \quad (3.122)\]

and so \( u \) can be obtained from \( A \) by convolution with the Poisson kernel,

\[
u = -2 \text{div}(\Delta)^{-1} A. \quad (3.123)
\]

We know that

\[
(-\Delta)^{-1} A(x) = \frac{\alpha_d}{d - 2} \int_{\mathbb{R}^d} \frac{1}{|x - y|^{d-2}} A(y) \, dy. \quad (3.124)
\]

Taking the divergence of both sides of this equation, we find that

\[
-2 \text{div}(\Delta)^{-1} A(x) = -2 \frac{\alpha_d}{d - 2} \sum_{i=1}^{d} \int_{\mathbb{R}^d} \partial_{x_i} \frac{1}{|x - y|^{d-2}} A_{ij}(y) \, dy \quad (3.125)
\]

\[
= -2 \frac{\alpha_d}{d - 2} \int_{\mathbb{R}^d} A(y)^{tr} \nabla_x \frac{1}{|x - y|^{d-2}} \quad (3.126)
\]

\[
= -\alpha_d \int_{\mathbb{R}^d} \omega(\rho, s)(e_\rho \otimes e_z - e_z \otimes e_\rho) \frac{x - y}{|x - y|^d} \, dy. \quad (3.127)
\]

This completes the proof. \( \square \)
Lemma 3.25. For all $d \geq 2$, and for all $f \in L^1([-1,1])$,
\[
\int_{S^{d-1}} f(y_1) \, dS(y) = m_{d-2} \int_{-1}^{1} (1 - y_1^2)^{\frac{d-1}{2}} f(y_1) \, dy_1,
\]
(3.128)
where $m_d = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}$, is the surface area of the $d$-sphere embedded in $\mathbb{R}^{d+1}$.

Proof. We begin by applying the Biot-Savart law from Proposition 3.24, and using axisymmetry, 
\[
\int_{S^{d-1}} f(y_1) \, dS(y) = \int_{-1}^{1} \frac{1}{\sqrt{1 - y_1^2}} f(y_1) \, dy_1 \, dS(\tilde{y}).
\]
(3.131)

Therefore we may conclude that
\[
\int_{S^{d-1}} f(y_1) \, dS(y) = \int_{-1}^{1} \frac{1}{\sqrt{1 - y_1^2}} \left| S^{d-2} \sqrt{1 - y_1^2} \right| dy_1.
\]
(3.132)

Observe that
\[
\left| S^{d-2} \sqrt{1 - y_1^2} \right| = m_{d-2} \left(\frac{1}{\sqrt{1 - y_1^2}}\right)^{d-2},
\]
(3.133)
and this completes the proof.

\[\square\]

Proposition 3.26. Suppose $u \in H^s_d(\mathbb{R}^d)$, $s > \frac{d}{2}$, $d \geq 3$, is axisymmetric and swirl-free. Then the components $u_r$ and $u_z$ can be recovered from the vorticity using the formulas
\[
u = -\alpha_d m_{d-2} m_d \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho^{d-2} (z-s) \omega(\rho, s) \int_{-1}^{1} \frac{\tilde{y}_1 (1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r_1^2 + \rho^2 - 2 r_1 \rho \tilde{y}_1 + (z-s)^2)^{\frac{d+2}{2}}} \, d\tilde{y}_1 \, d\rho \, ds \quad (3.134)
\]

and
\[
u_z = \alpha_d m_{d-2} m_d \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho^{d-2} \omega(\rho, s) \int_{-1}^{1} \frac{(r_1 \tilde{y} - \rho)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r_1^2 + \rho^2 - 2 r_1 \rho \tilde{y}_1 + (z-s)^2)^{\frac{d+2}{2}}} \, d\tilde{y}_1 \, d\rho \, ds. \quad (3.135)
\]

Proof. We begin by applying the Biot-Savart law from Proposition 3.24 and using axisymmetry, we set $x = re_1 + ze_d$. Note that this means that $e_r = e_1$. Therefore we can compute,
\[
u_r = e_1 \cdot u(re_1 + ze_d) \quad (3.136)
\]
\[
u_r = -\alpha_d e_1 \cdot \int_{\mathbb{R}^d} \mu(\rho, s)(e_1 \otimes e_z - e_z \otimes e_1) \frac{r_1 - \tilde{y}_1^2 + (z-s)e_z}{(r_1^2 + |\tilde{y}_1^2| - 2 r_1 \tilde{y}_1 + (z-s)^2)^{\frac{d+2}{2}}} \, dy \quad (3.137)
\]
\[
u_r = -\alpha_d \int_{\mathbb{R}^d} \omega(\rho, s) \frac{y_1}{\rho} \frac{z-s}{(r_1^2 + |\tilde{y}_1^2| - 2 r_1 \tilde{y}_1 + (z-s)^2)^{\frac{d+2}{2}}} \, dy. \quad (3.138)
\]
Letting \( \tilde{y} = \frac{y}{|y|} \), parameterize the \( d - 2 \) dimensional sphere, we find that

\[
 u_r(r, z) = -\alpha_d \int_{-\infty}^{\infty} \int_{0}^{\infty} m_{d-2} r^{d-2} (z - s) \omega(\rho, s) \int_{S^{d-2}} \frac{\tilde{y}_1}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d-4}{2}}} dS(\tilde{y}) d\rho ds
\]

(3.139)

Applying Lemma 3.25, we find that

\[
 u_r(r, z) = -\alpha_d m_{d-2} m_{d-3} \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho^{d-2} (z - s) \omega(\rho, s) \int_{1}^{1} \frac{\tilde{y}_1 (1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} dy d\rho ds.
\]

(3.140)

We will follow a similar approach for computing \( u_z \) in terms of the vorticity. Again exploiting axisymmetry, we find that

\[
 u_z(r, z) = e_d \cdot u(re_1 + ze_d)
\]

(3.141)

\[
 = -\alpha_d e_z \cdot \int_{\mathbb{R}^d} \omega(\rho, s) (e_\rho \otimes e_z - e_z \otimes e_\rho) \frac{re_1 - y' + (z - s)e_z}{(r^2 + |y'|^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} dy
\]

(3.142)

\[
 = \alpha_d \int_{\mathbb{R}^d} \omega(\rho, s) \frac{r \tilde{y}_1 - \rho}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} dy
\]

(3.143)

\[
 = \alpha_d \int_{-\infty}^{\infty} \int_{0}^{\infty} m_{d-2} r^{d-2} \omega(\rho, s) \int_{S^{d-2}} \frac{r \tilde{y}_1 - \rho}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} dS(\tilde{y}) d\rho ds
\]

(3.144)

Again applying Lemma 3.25, we find that

\[
 u_z(r, z) = \alpha_d m_{d-2} m_{d-3} \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho^{d-2} \omega(\rho, s) \int_{1}^{1} \frac{(r \tilde{y}_1 - \rho)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} dy d\rho ds.
\]

(3.145)

This completes the proof.

We will conclude this section by deriving a stream function formulation of the axisymmetric, swirl free Euler equation.

**Proposition 3.27.** Suppose \( u \in H^s_{\text{ask,df}} (\mathbb{R}^d), d \geq 3, s > 1 + \frac{d}{2} \). Then there exists a stream function \( \psi \in \mathcal{C}^2_{r,z} \) satisfying

\[
 -\partial_r \psi - k \frac{\psi}{r} = u_z
\]

(3.146)

\[
 \partial_z \psi = u_r
\]

(3.147)

\[
 \left( -\partial_z^2 - \partial_r^2 - k \frac{\partial_r}{r} + \frac{k}{r^2} \right) \psi = \omega.
\]

(3.148)

Furthermore this stream function has decay at infinity and is unique among the class of \( \mathcal{C}^2_{r,z} \) functions with decay at infinity.

**Proof.** We will begin by letting

\[
 M = (-\Delta)^{-1} A.
\]

(3.149)

Observe that \( A = \nabla_{\text{asymp}} u \), and so if we let \( v = (-\Delta)^{-1} u \) we can see that

\[
 M = \nabla_{\text{asymp}} v.
\]

(3.150)
This implies that
\[ M(x) = \frac{1}{2}\psi(r, z)(e_r \otimes e_z - e_z \otimes e_r), \tag{3.151} \]
where \( \psi \) is the scalar vorticity associated with \( v \). Recalling that \( u = -2 \text{div} M \), we can see that
\[ u = \partial_z \psi e_r - \partial_r \psi e_z - \frac{1}{r} \psi e_z. \tag{3.152} \]

Next we will write
\[ M = \frac{1}{2}(\psi e_r \otimes e_z - e_z \otimes \psi e_r). \tag{3.153} \]
The vector \( e_z \) is constant, so applying the vector Laplacian to the vector \( \psi e_r \) and applying Proposition 3.6 we find that
\[ -\Delta M = \frac{1}{2}\left(\frac{1}{2}(\psi e_r \otimes e_z - e_z \otimes \psi e_r)\right) \tag{3.154} \]
\[ = \frac{1}{2} \left(-\frac{\partial^2}{r^2} - \frac{k}{r} \partial_r + \frac{k}{r^2} \right) \psi (e_r \otimes e_z - e_z \otimes e_r). \tag{3.155} \]
We also know by hypothesis that
\[ -\Delta M = A \tag{3.156} \]
\[ = \frac{1}{2} \omega (e_r \otimes e_z - e_z \otimes e_r), \tag{3.157} \]
and so we may conclude that
\[ \left(-\frac{\partial^2}{r^2} - \frac{k}{r} \partial_r + \frac{k}{r^2} \right) \psi = \omega. \tag{3.158} \]

We have now proven that all the partial differential equations for the stream function hold. It remains only to show that \( \psi \in C^2_{r,z} \) and decays at infinity, and is unique in that class. First, observe that \( M \in H^1(\mathbb{R}^d) \cap H^{s+1}(\mathbb{R}^d) \). By Sobolev embedding this implies that \( M \in C^2(\mathbb{R}^d) \), and that consequently \( \psi \in C^2_{r,z} \). We can also see from the regularity assumptions that \( \tilde{M} \in L^1(\mathbb{R}^d) \). This implies that \( M \) has decay at infinity, and therefore \( \psi \) must also decay at infinity.

With this, we have established that \( \psi \in C^2_{r,z} \) and decays at infinity. Now we must now show that it is unique in that class. Suppose \( \tilde{\psi} \in C^2_{r,z} \) with decay at infinity and satisfies
\[ \left(-\frac{\partial^2}{r^2} - \frac{k}{r} \partial_r + \frac{k}{r^2} \right) \tilde{\psi} = \omega. \tag{3.159} \]
Let
\[ \tilde{M}(x) = \frac{1}{2}\tilde{\psi}(r, z)(e_r \otimes e_z - e_z \otimes e_r). \tag{3.160} \]
By the same calculations as above, we can see that
\[ -\Delta \tilde{M} = A, \tag{3.161} \]
and it is also immediately apparent that \( \tilde{M} \) must have decay at infinity. Let
\[ Q = M - \tilde{M}. \tag{3.162} \]
We can see that
\[-\Delta Q = 0,\]
and \(Q\) has decay at infinity. There are no nontrivial harmonic functions with decay at infinity, and so we can conclude that
\[Q = 0,\]
and that therefore
\[\tilde{M} = M\]
and
\[\psi = \tilde{\psi}.\]
This completes the proof.

Remark 3.28. Proposition 3.27 allows us to give a stream function formulation for the axisymmetric, swirl-free Euler equation generically in three and higher dimensions. We have the evolution equation for vorticity,
\[\partial_t \omega + (u \cdot \nabla) \omega - ku_r \omega = 0,\]
where the velocity \(u\) is determined from the stream function by
\[u_z = -\partial_r \psi - \frac{k \psi}{r},\]
\[u_r = \partial_z \psi,\]
and the stream function is determined from the vorticity by
\[\left(-\partial_z^2 - \partial_r^2 - \frac{k}{r} \partial_r + \frac{k}{r^2}\right) \psi = \omega.\]

Note that the dimension \(d = k + 2\) enters these equations as a parameter; the qualitative structure of the equations is the same for any \(d \geq 3\). Therefore, it is possible to study how the dynamics of this equation change as this parameter changes, and in particular to consider the infinite dimensional limit \(k \to \infty\). We will do this in the next section.

4 The infinite dimensional limit of the vorticity equation

In this section, we will derive a formal limit for the vorticity equation in the infinite dimensional limit. We will then show that the resulting infinite dimensional vorticity equation exhibits finite-time blowup of a Burgers shock type, and in fact can be reduced to the one dimensional Burgers equation. Because \(d = k - 2\), taking the infinite dimensional limit is equivalent to taking the formal limit \(k \to \infty\) in the stream function formulation. If we take the stream function formulation from the end of the last section, there are terms of order \(k\), which makes taking the limit \(k \to \infty\) difficult.

We will deal with this issue by taking the re-normalization
\[\tilde{\psi} = k \psi.\]
The stream function formulation of the vorticity equation in terms of $\tilde{\psi}$ can then be given as follows: the velocity is determined in terms of $\tilde{\psi}$

$$u_z = -\frac{1}{k} \partial_r \tilde{\psi} - \frac{\tilde{\psi}}{r}$$  \hspace{1cm} (4.2)

$$u_r = \frac{1}{k} \partial_z \tilde{\psi},$$  \hspace{1cm} (4.3)

and $\tilde{\psi}$ is determined by the vorticity from the elliptic equation

$$\left( -\frac{1}{k} \partial_z^2 - \frac{1}{k} \partial_r^2 - \frac{1}{r} \partial_r + \frac{1}{r^2} \right) \tilde{\psi} = \omega.$$ \hspace{1cm} (4.4)

The evolution equation remains

$$\partial_t \omega + u_r \partial_r \omega + u_z \partial_z \omega - \frac{k}{r} u_r \omega = 0.$$ \hspace{1cm} (4.5)

Plugging in the velocity given by $\tilde{\psi}$ to this evolution equation, we find that

$$\partial_t \omega - \left( \frac{1}{k} \partial_r \tilde{\psi} + \frac{\tilde{\psi}}{r} \right) \partial_z \omega + \frac{1}{k} \partial_z \tilde{\psi} \partial_r \omega - \frac{\partial_z \tilde{\psi}}{r} \omega = 0.$$ \hspace{1cm} (4.6)

Taking the formal limit $k \to +\infty$, we obtain the equations

$$\omega = -\frac{1}{r} \partial_r \tilde{\psi} + \frac{1}{r^2} \tilde{\psi} $$ \hspace{1cm} (4.7)

$$= -\partial_r \left( \frac{\tilde{\psi}}{r} \right),$$ \hspace{1cm} (4.8)

and

$$\partial_t \omega - \frac{\tilde{\psi}}{r} \partial_z \omega - \omega \partial_z \left( \frac{\tilde{\psi}}{r} \right) = 0.$$ \hspace{1cm} (4.9)

If we make the substitution

$$\phi = -\frac{\tilde{\psi}}{r},$$ \hspace{1cm} (4.10)

then our equation reduces to

$$\partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi = 0,$$ \hspace{1cm} (4.11)

$$\partial_r \phi = \omega.$$ \hspace{1cm} (4.12)

Taking the boundary condition

$$\lim_{r \to +\infty} \phi(r, z) = 0, $$ \hspace{1cm} (4.13)

for all $z \in \mathbb{R}$, and integrating the equation (4.12) from infinity, we find that

$$\phi(r, z) = -\int_{r}^{+\infty} \omega(\rho, z) \, d\rho$$ \hspace{1cm} (4.14)

We will use this integral to define the operator $\partial_r^{-1}$ as

$$(\partial_r^{-1} \omega) (r, z) = - \int_{r}^{+\infty} \omega(\rho, z) \, d\rho, $$ \hspace{1cm} (4.15)
noting that
\[ \phi = \partial_r^{-1} \omega, \]  
(4.16)
and that this is the unique solution of the equation
\[ \partial_r \phi = \omega, \]  
(4.17)
with decay as \( r \to \infty \).

**Remark 4.1.** We will note that the infinite dimensional vorticity equation (4.11) can, shockingly enough, be further simplified to the one dimensional Burgers equation. Recalling that \( \omega = \partial_r \phi \), we can see that (4.11) can be rewritten as
\[ \partial_t \partial_r \phi + \phi \partial_z \partial_r \phi + \partial_r \phi \partial_z \phi = 0, \]  
(4.18)
which can be in turn expressed as
\[ \partial_t \partial_r \phi + \partial_r (\phi \partial_z \phi) = 0. \]  
(4.19)
Applying the operator \( \partial_r^{-1} \), we can see that \( \phi \) must satisfy Burgers equation,
\[ \partial_t \phi + \phi \partial_z \phi = 0. \]  
(4.20)
This formally establishes the reduction from the infinite dimensional vorticity equation to the one dimensional Burgers equation. We will give a more rigorous proof shortly.

First it is necessary to define strong solutions to both of these equations and recall some classical results for Burgers equation. Our definition of the one dimensional Burgers equation will vary slightly from the standard one, because for our purposes \( \phi \) will depend on \( r \) in addition to \( z \), even though only derivatives with respect to \( z \) appear in Burgers equation. First we will define the time-independent spaces \( C^2_0 (\mathbb{R}^+, \mathbb{R}) \) and \( C^{1,2}_{r,z} (\mathbb{R}^+, \mathbb{R}) \), and the time-dependent spaces \( C^2_0 (\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \) and \( C^{1,2,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \), which will be the natural spaces for initial data and strong solutions of Burgers equation and the infinite dimensional vorticity equation respectively.

**Definition 4.2.** We will say that \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}) \) if and only if \( \phi \in C^2 (\mathbb{R}^+, \mathbb{R}) \), and for all \( z \in \mathbb{R} \),
\[ \partial_r \phi (0, z) = 0 \]  
(4.21)
and
\[ \lim_{r \to \infty} \phi (r, z) = 0, \]  
(4.22)
uniformly in \( z \).

**Definition 4.3.** Let
\[ C^{1,2}_{r,z} (\mathbb{R}^+, \mathbb{R}) = \partial_r C^2_0 (\mathbb{R}^+, \mathbb{R}); \]  
(4.23)
that is, \( \omega \in C^{1,2}_{r,z} (\mathbb{R}^+, \mathbb{R}) \) if and only if \( \omega = \partial_r \phi \) for some \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}) \).

**Definition 4.4.** We will say that \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \) if and only if \( \phi \in C^2 (\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \), and for all \( z \in \mathbb{R} \), and for all \( 0 \leq t < T_{\text{max}} \),
\[ \partial_r \phi (0, z, t) = 0 \]  
(4.24)
and
\[ \lim_{r \to \infty} \phi (r, z, t) = 0, \]  
(4.25)
with uniformly in \( z \) and \( t \).
Definition 4.5. Let
\[ C^{1,2,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{max})) = \partial_t C^2_0 (\mathbb{R}^+, \mathbb{R}, [0,T_{max})); \] (4.26)
that is, \( \omega \in C^{1,2,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) if and only if \( \omega = \partial_t \phi \) for some \( \phi \in C^1_0 (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \).

Definition 4.6. We will say that \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) is a strong solution of Burgers equation if and only if for all \( (r,z,t) \in (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \),
\[ \partial_t \phi + \phi \partial_z \phi = 0. \] (4.27)

Definition 4.7. We will say that \( \omega \in C^{1,2,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) is a strong solution of the infinite dimensional vorticity equation if and only if for all \( (r,z,t) \in (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \),
\[ \partial_t \omega + \phi \partial_z \omega + \omega \partial_x \phi = 0, \] (4.28)
where \( \omega = \partial_r \phi \).

Proposition 4.8. Suppose \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) is a strong solution of Burgers equation, and let \( \omega = \partial_t \phi \). Then \( \omega \in C^1_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) is a strong solution of the infinite dimensional vorticity equation.

Proof. We know that \( \phi \) is a strong solution to Burger’s equation, and so for all \( (r,z,t) \in (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \)
\[ \partial_t \phi + \phi \partial_z \phi = 0. \] (4.29)
Differentiating this equation with respect to \( r \), we find that for all \( (r,z,t) \in (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \),
\begin{align*}
\partial_r \partial_t \phi + \partial_t (\phi \partial_z \phi) &= 0 \quad (4.30) \\
\partial_t \partial_r \phi + \phi \partial_z \partial_r \phi + \partial_r \phi + \partial_z \partial_r \phi &= 0 \quad (4.31) \\
\partial_t \omega + \phi \partial_z \omega + \omega \partial_x \phi &= 0. \quad (4.32)
\end{align*}
This completes the proof. \( \square \)

Remark 4.9. The requirement that \( \phi \in C^2 \) is stronger than is strictly necessary. We really only need the equality of the mixed partials, \( \partial_t \partial_r \phi = \partial_r \partial_t \phi \), and \( \partial_r \partial_z \phi = \partial_z \partial_r \phi \). Requiring that \( \phi \) be continuously twice differentiable is a convenient way to guarantee the equality of the mixed partials will hold.

Proposition 4.10. Suppose \( \omega \in C^{1,2,2}_{r,z,t} (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \) is a strong solution of the infinite dimensional vorticity equation, and \( \omega = \partial_t \phi \), where \( \phi \in C^2_0 (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \). Then \( \phi \) is a strong solution of Burgers equation.

Proof. We know that \( \omega \) is a strong solution to the infinite-dimensional vorticity equation, and so by Definition [4.5], we know that \( \omega = \partial_t \phi \) for some \( \phi \in C^2_0 \) and that
\[ \partial_t \phi + \phi \partial_z \phi = 0. \] (4.33)
Substituting in \( \omega = \partial_t \phi \), and using the fact that \( \phi \in C^2 \), we can see that for all \( (r,z,t) \in (\mathbb{R}^+, \mathbb{R}, [0,T_{max}) \),
\begin{align*}
\partial_r (\partial_t \phi + \phi \partial_z \phi) &= \partial_r \partial_t \phi + \phi \partial_z \partial_r \phi + \partial_r \phi \partial_z \phi \\
&= 0. \quad (4.34)
\end{align*}
This allows us to conclude that

\[ \partial_t \phi + \phi \partial_z \phi = h(z, t) \]  

(4.36)

for some \( h \in C^1(\mathbb{R}, [0, T_{max}]) \).

It remains to prove that \( h = 0 \). The key fact driving this will be that

\[ \lim_{r \to \infty} \phi(r, z, t) = 0, \]  

(4.37)

and that this convergence is uniform with respect to \( z \) and \( t \). Using this fact we may observe that for all \( z \in \mathbb{R}, 0 \leq t < T_{max} \),

\[ \lim_{r \to \infty} \partial_t \phi(r, z, t) = h(z, t), \]  

(4.38)

and that again this convergence is uniform with respect to \( z \) and \( t \) as long as \( t \) is bounded away from \( T_{max} \).

Suppose towards contradiction that \( h \) is not identically zero. We may assume without loss of generality, that there is some \( (z_0, t_0) \in \mathbb{R} \times [0, T_{max}) \), such that \( h(z_0, t_0) > 0 \), as the arguments will work exactly the same way in the less than zero case. By continuity we may assume there is an interval, \([t_1, t_2]\) (with \( t_2 < T_{max} \)), and \( C > 0 \) such that for all \( t_1 < t < t_2 \),

\[ h(z_0, t) \geq C. \]  

(4.39)

We can then see that for all \( r \in \mathbb{R}^+ \),

\[ \phi(r, z_0, t_2) - \phi(r, z_0, t_1) = \int_{t_1}^{t_2} (\phi \partial_z \phi)(r, z_0, \tau) + h(z_0, \tau) \, d\tau, \]  

(4.40)

which we can then re-write this equation in the form

\[ \int_{t_1}^{t_2} h(z_0, \tau) \, d\tau = \phi(r, z_0, t_2) - \phi(r, z_0, t_1) - \int_{t_1}^{t_2} (\phi \partial_z \phi)(r, z, \tau) \, d\tau. \]  

(4.41)

Using the fact that \( \phi \in C^2 \), we can conclude that

\[ \sup_{0 \leq r < \infty, t_1 < t < t_2} |\partial_z \phi(r, z_0, t)| = M < +\infty, \]  

(4.42)

and so we can conclude for all \( r \in \mathbb{R}^+ \)

\[ \int_{t_1}^{t_2} h(z_0, \tau) \, d\tau \leq 2 \sup_{t_1 < t < t_2} \left| \phi(r, z_0, t) \right| + M(t_2 - t_1) \sup_{t_1 < t < t_2} \left| \phi(r, z_0, t) \right|. \]  

(4.43)

Applying (4.37), we can see that

\[ \lim_{r \to \infty} 2 \sup_{t_1 < t < t_2} \left| \phi(r, z_0, t) \right| + M(t_2 - t_1) \sup_{t_1 < t < t_2} \left| \phi(r, z_0, t) \right| = 0, \]  

(4.44)

and therefore

\[ \int_{t_1}^{t_2} h(z_0, \tau) \, d\tau \leq 0. \]  

(4.45)

Applying the inequality (4.37), we can see that

\[ \int_{t_1}^{t_2} h(z_0, \tau) \, d\tau \geq C(t_2 - t_1). \]  

(4.46)

This contradicts our assumption that \( h \) is not identically zero, and completes the proof. \( \Box \)
We will now recall the standard theory for the existence, uniqueness, and finite-time blowup of solutions of Burgers equation, with proofs provided for the sake of completeness, as elements of the proof will be important in discussing the solutions to the infinite dimensional vorticity equation that are generated by solutions of Burgers equation.

**Theorem 4.11.** Suppose $\phi^0 \in C^2(R^+, R)$. Then there exists a unique, strong solution of Burgers equation $\phi \in C^2(R^+, R, [0, T_{\max})$. If $\partial_z \phi^0(r, z) \geq 0$ for all $r \in R^+, z \in R$, then there is a global smooth solution, and so $T_{\max} = +\infty$. If there exists $r_0 \in R^+, z_0 \in R$, such that $\partial_z \phi^0(r_0, z_0) < 0$, then there is finite-time blowup with

$$T_{\max} = \frac{1}{-\inf_{r \in R^+, z \in R} \partial_z \phi(r, z)}.$$  

(4.47)

This solution is given by

$$\phi(r, z, t) = \phi^0(r, h(r, z, t), t),$$

(4.48)

where

$$h(r, z, t) = g_{r,t}^{-1}(z),$$

(4.49)

and

$$g_{r,t}(z) = z + \phi^0(r, z)t.$$  

(4.50)

Proof. The Burgers equation involves the advection of $\phi$ by itself, resulting in the flow map

$$z \rightarrow z + t\phi^0(r, z)$$

(4.51)

and the back to labels map $h(r, z, t)$. We will begin by computing the derivatives of $h$ and introducing the variable $y = h(r, z, t)$.

First observe that $g_{r,t}(z)$ is a twice continuously differentiable function of $r$ and $z$—although technically not in $C^2$ because it is unbounded in the $z$ variable—and that

$$\partial_z g_{r,t}(z) = 1 + t\partial_z \phi^0(r, z).$$

(4.52)

This means that if for all $r \in R^+, z \in R$,

$$\partial_z \phi^0(r, z) \geq 0,$$  

(4.53)

then for all $t > 0$,

$$\partial_z g_{r,t}(z) \geq 1.$$  

(4.54)

Furthermore, if this derivative is negative at some point, then for all $0 < t < T_{\max}$,

$$\partial_z g_{r,t}(z) \geq 1 - \frac{t}{T_{\max}}.$$  

(4.55)

In either case, for all $0 < t < T_{\max}$, we can conclude that $g_{r,t}(z)$ is a twice continuously differentiable function, strictly increasing in $z$ with a derivative bounded below by a positive constant. Consequently it has an inverse that is also continuously twice differentiable, and therefore

$$\phi(r, z, t) = \phi^0(r, h(r, z, t)).$$

(4.56)
is a well defined function \( \phi \in C^2(\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \). In order to establish that \( \phi \in C^2_0(\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \), we must also deal with uniform convergence as \( r \to \infty \). It is simple to observe that the fact that
\[
\lim_{r \to \infty} \phi^0(r, z) = 0 \quad (4.57)
\]
with uniform in \( z \) convergence clearly implies that
\[
\lim_{r \to \infty} \phi^0(r, h(r, z, t)) = 0 \quad (4.58)
\]
uniformly in \( h \) and consequently uniformly in \( z \) and \( t \). Therefore, we can conclude that \( \phi \in C^2_0(\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}]) \).

It remains to show that \( \phi \) is in fact a solution of Burgers equation. Observe that by definition,
\[
z = g_{r,t}(h(r, z, t)) \quad (4.59)
\]
\[
= g_{r,t}(y) \quad (4.60)
\]
\[
= y + t \phi^0(r, y), \quad (4.61)
\]
and also
\[
h(r, y + t \phi^0(r, y), t) = y. \quad (4.62)
\]
Differentiating (4.62) with respect to \( y \), we find that
\[
(1 + t \frac{\partial y}{\partial \phi^0(r, y)}) \frac{\partial z}{\partial h(r, z, t)} = 1. \quad (4.63)
\]
Plugging for \( y \) and dividing over a term, we find that
\[
\frac{\partial z}{\partial h(r, z, t)} = \frac{1}{1 + t \frac{\partial y}{\partial \phi^0(r, h(r, z, t))}}. \quad (4.64)
\]
Meanwhile, differentiating (4.62) with respect to \( t \), we find that
\[
\frac{\partial t}{\partial h(r, z, t)} + \phi^0(r, y) \frac{\partial z}{\partial h(r, z, t)} = 0. \quad (4.65)
\]
Rearranging terms, we find that
\[
\frac{\partial t}{\partial h(r, z, t)} = -\frac{\phi^0(r, h(r, z, t))}{1 + t \frac{\partial y}{\partial \phi^0(r, h(r, z, t))}}. \quad (4.66)
\]
Now we can show that
\[
\phi(r, z, t) = \phi^0(r, h(r, z, t)) \quad (4.67)
\]
is a solution to Burgers equation. First we compute that
\[
\frac{\partial t}{\partial \phi}(r, z, t) = \frac{\partial y}{\partial \phi^0(r, h(r, z, t))} \frac{\partial t}{\partial h(r, z, t)}
\]
\[
= -\frac{\phi^0(r, y) \partial y}{1 + t \partial y \phi^0(r, y)}. \quad (4.68)
\]
Likewise we compute that
\[
\frac{\partial z}{\partial \phi}(r, z, t) = \frac{\partial y}{\partial \phi^0(r, h(r, z, t))} \frac{\partial z}{\partial h(r, z, t)}
\]
\[
= \frac{\partial \phi^0(r, y)}{1 + t \partial y \phi^0(r, y)}. \quad (4.69)
\]
Putting these computations together with the fact that $\phi(r, z, t) = \phi^0(r, y)$, we can conclude that
\[
\partial_t \phi(r, z, t) + \phi(r, z, t) \partial_z \phi(r, z, t) = 0,
\]
and so we have a solution to Burgers equation.

We have now proven that there is a strong solution of Burgers equation on the interval $[0, T_{\text{max}})$. We still need to show that there is in fact blowup as $t \to T_{\text{max}}$ when $T_{\text{max}} < +\infty$. The key point is that when there exists $r_0 \in \mathbb{R}^+, z_0 \in \mathbb{R}$ such that $\partial_z \phi(r_0, z_0) < 0$, then for $t > T_{\text{max}}$ the map
\[
z \to z + t \phi^0(r, z)
\]
is no longer invertible. The derivative with respect to $z$ of the inverse of this map becomes singular as $t \to T_{\text{max}}$, which produces the Burgers shock. To proceed rigorously, recall that
\[
\partial_z \phi(r, z, t) = \frac{\partial_y \phi^0(r, y)}{1 + t \partial_y \phi^0(r, y)}.
\]
This clearly implies that
\[
\lim_{t \to T_{\text{max}}} \inf z \partial_z \phi(r, z, t) = -\infty,
\]
as the denominator is clearly going to 0 as $t \to \frac{-1}{\inf \partial_y \phi^0(r, y)}$. This completes the proof of finite-time blowup in the form of the Burgers shock.

It now only remains to prove the uniqueness of classical solutions. Suppose $\phi \in C^2 (\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}))$ with initial data
\[
\phi(\cdot, 0) = \phi^0.
\]
We want to show that $\phi$ must satisfy (4.48). We will show by the method of characteristics that this is the only possible solutions, and will introduce an error function $\sigma$ to establish this. Let
\[
\sigma(r, y, t) = \left(\phi(r, y + \phi^0(r, y)t, t) - \phi^0(r, y)\right)^2,
\]
and again let
\[
z = y + \phi^0(r, y)t.
\]
We begin by computing that
\[
\partial_t \sigma(r, y, t) = 2 \left(\phi(r, y + \phi^0(r, y)t, t) - \phi^0(r, y)\right) \left(\partial_t \phi + \phi^0(r, y) \partial_z \phi\right)(r, y + \phi^0(r, y), t).
\]
Observe that because $\phi$ is a solution of Burgers equation,
\[
\partial_t \phi(r, y + \phi^0(r, y)t, t) = - (\phi \partial_z \phi)(r, y + \phi^0(r, y)t, t),
\]
and so we can conclude that
\[
\left(\partial_t \phi + \phi^0(r, y) \partial_z \phi\right)(r, y + \phi^0(r, y), t) = \left(\phi(r, y) - \phi(r, u + \phi^0(r, y)t)\right) \partial_z \phi(r, y + \phi^0(r, y)t, t).
\]
Plugging this back into (4.79), we find that
\[
\partial_t \sigma(r, y, t) = -2 \partial_z \sigma(r, y + \phi^0(r, y)t, t) \left(\phi(r, y + \phi^0(r, y)t, t) - \phi^0(r, y)\right)^2
\]
\[
= -2 \partial_z \phi(r, y + \phi^0(r, y)t, t)\sigma(r, z, t)
\]
Applying Grönwall’s inequality, we can note that for all $r \in \mathbb{R}^+, z \in \mathbb{R}, 0 \leq t < T_{\text{max}}$,

$$\sigma(r, y, t) \leq \sigma(r, y, 0) \exp\left(2 \int_0^t \|\partial_z \phi(\cdot, \tau)\|_{L^\infty} \, d\tau \right).$$

(4.84)

It is trivial to observe that for all $r \in \mathbb{R}^+, y \in \mathbb{R},$

$$\sigma(r, y, 0) = 0,$$

(4.85)

and so consequently for all $r \in \mathbb{R}^+, z \in \mathbb{R}, 0 \leq t < T_{\text{max}},$

$$\sigma(r, y, t) = 0$$

(4.86)

and

$$\phi(r, y + \phi^0(r, y)y, t) = \phi^0(r, y)$$

(4.87)

We can recall that by definition,

$$z = y + \phi^0(r, y)t,$$

(4.88)

and we have already shown that

$$y = h(r, z, t)$$

(4.89)

follows immediately from the definition of $h$. Therefore we may conclude that for all $r \in \mathbb{R}^+, z \in \mathbb{R}, 0 \leq t < T_{\text{max}},$

$$\phi(r, z, t) = \phi^0(r, h(r, z, t)).$$

(4.90)

This shows that the solution to Burgers equation given in (4.48) is unique in the class of strong solutions, and this completes the proof.

With the classical well-posedness theory of the one dimensional Burgers equation worked out, we can now immediately lift this theory to the infinite-dimensional vorticity equation. We will prove Theorem 1.2 which is restated for the readers convenience.

**Theorem 4.12.** Suppose $\omega^0 \in C^{1,2}_{r,z,t}(\mathbb{R}^+, \mathbb{R})$. Note that this implies $\omega^0 = \partial_t \phi^0$ with $\phi^0 \in C^2_0(\mathbb{R}^+, \mathbb{R})$. Then there exists a unique strong solution $\omega \in C^{1,2,2}_{r,z,t}(\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}])$ to the infinite-dimensional vorticity equation. If $\partial_z \phi^0(r, z) \geq 0$ for all $r \in \mathbb{R}^+, z \in \mathbb{R}$, then there is a global smooth solution, and so $T_{\text{max}} = +\infty$. If there exists $r_0 \in \mathbb{R}^+, z_0 \in \mathbb{R}$, such that $\partial_z \phi^0(r_0, z_0) < 0$, then there is finite-time blowup with

$$T_{\text{max}} = \frac{1}{-\inf_{r \in \mathbb{R}^+, z \in \mathbb{R}} \partial_z \phi^0(r, z)}.$$  

(4.91)

This solution of the infinite-dimensional vorticity equation is given by

$$\omega(r, z, t) = \frac{\omega^0(r, h(r, z, t))}{1 + t\partial_y \phi^0(r, h(r, z, t))},$$

(4.92)

where the back-to-labels map $h(r, z, t)$ is defined as in Theorem 4.11.

**Proof.** Applying Propositions 4.8 and 4.10 to the solutions of Burgers equation in Theorem 4.11 the local existence and uniqueness of strong solutions, as well as the specified blowup time follows immediately. It only remains to show that the equation 4.92 holds for this unique solution.

Recall that

$$\phi(r, z, t) = \phi^0(r, h(r, z, t)),$$

(4.93)
and that $\omega = \partial_r \phi$. Applying the chain rule we find that
\[
\omega(r, z, t) = \partial_r \phi^0(r, y) + \partial_y \phi^0(r, y) \partial_r h(r, z, t).
\]
(4.94)

Also recall that
\[
h(r, y + t \phi^0(r, y), t) = y,
\]
and that therefore differentiating with respect to $r$, we find that
\[
\partial_r h(r, z, t) + \partial_z h(r, z, t) t \partial_r \phi^0(r, y) = 0.
\]
(4.96)

Subtracting over the latter term and plugging into our expression for $\partial_z h$,
\[
\partial_z h(r, z, t) = \frac{1}{1 + t \partial_y \phi^0(r, y)},
\]
(4.97)

we find that
\[
\partial_r h(r, z, t) = \frac{-t \omega^0(r, y)}{1 + t \partial_y \phi^0(r, y)}.
\]
(4.98)

Plugging (4.98) back into our equation (4.94), we find that for all $0 \leq t < T_{\text{max}}$,
\[
\omega(r, z, t) = \omega^0(r, y) + \frac{-t \partial_y \phi^0(r, y) \omega^0(r, y)}{1 + t \partial_y \phi^0(r, y)}
\]
(4.99)
\[
= \frac{\omega^0(r, y)}{1 + t \partial_y \phi^0(r, y)}
\]
(4.100)
\[
= \frac{\omega^0(r, h(r, z, t))}{1 + t \partial_y \phi^0(r, h(t, z, t))}.
\]
(4.101)

This completes the proof. $\square$

**Proposition 4.13.** Suppose $\omega \in C^{1,2}_{r,z,t}(\mathbb{R}^+, \mathbb{R}, [0, T_{\text{max}}])$ is a strong solution to the infinite-dimensional vorticity equation with finite-time blowup at $T_{\text{max}} < +\infty$. Suppose that
\[
\inf_{r \in \mathbb{R}^+, z \in \mathbb{R}} \partial_z \phi(r, z) = \partial_z \phi^0(r_0, z_0),
\]
(4.102)
and that
\[
\omega^0(r_0, z_0) \neq 0,
\]
(4.103)
then
\[
\int_0^{T_{\text{max}}} \|\omega(\cdot, t)\|_{L^\infty} \, dt = +\infty.
\]
(4.104)

In particular, for all $0 \leq t < T_{\text{max}}$
\[
\|\omega(\cdot, t)\|_{L^\infty} \geq \frac{\|\omega^0(r_0, z_0)\|}{1 - \frac{t}{T_{\text{max}}}},
\]
(4.105)

and consequently for all $0 \leq t < T_{\text{max}}$
\[
\int_0^t \|\omega(\cdot, \tau)\| \, d\tau \geq -\|\omega^0(r_0, z_0)\| \log \left( 1 - \frac{t}{T_{\text{max}}} \right),
\]
(4.106)
Proof. Observe that for all $0 \leq t < T_{max}$,
\[
\|\omega(\cdot, t)\|_{L^\infty} \geq \frac{\|\omega(0, z_0 + t\phi^0(0, z_0), t)\|}{1 + t\partial_y\phi^0(0, z_0)}
\]
\[
= \frac{\|\omega^0(0, z_0)\|}{1 - \frac{1}{T_{max}}}
\]
(4.107)
(4.108)
(4.109)
Integrating this lower bound immediately gives the lower bound (4.106), and this completes the proof.

Remark 4.14. It can easily be seen from the formula for the evolution of the vorticity that singularities form at a rate $\sim \frac{1}{1 - \frac{1}{T_{max}}}$, and that furthermore, the vorticity blows up in the $L^\infty$ norm at this rate whenever the initial vorticity is nonzero at a point where $\partial_y\phi^0$ is minimized. It is not clear, however, that a Beale-Kato-Majda criterion must hold in the case where the initial vorticity is zero at all points where $\partial_y\phi^0$ is minimized. In fact, there is a simple counterexample showing that the Beale-Kato-Majda criterion does not hold for all solutions of the infinite-dimensional vorticity equation that blowup in finite-time. Indeed, there are solutions of the infinite-dimensional vorticity equation that are uniformly bounded up until blowup time.

Proposition 4.15. Take the initial data
\[
\omega^0(r, z) = 2rz \exp(-r^2 - z^2),
\]
(4.110)
and let $\omega \in C^1_{z,t}(\mathbb{R}^+, \mathbb{R}, [0, T_{max}))$ be the unique strong solution to the infinite-dimensional vorticity equation. Then $T_{max} = 1$, and for all $0 \leq t < 1$,
\[
|\omega(r, z, t)| \leq 1.
\]
(4.111)

Proof. We will again take $y = h(r, z, t)$ where $h$ is the back-to-labels map used in Theorems 4.11 and 4.12. We know that
\[
\omega(r, z, t) = \frac{\omega^0(r, y)}{1 + t\partial_y\phi^0(r, y)}.
\]
(4.112)
It is simple to check that for the initial vorticity chosen, the corresponding initial potential is
\[
\phi^0(r, y) = -y \exp(-r^2 - y^2),
\]
(4.113)
and so
\[
\partial_y\phi^0(r, y) = -(1 - 2y^2) \exp(-r^2 - y^2).
\]
(4.114)
Clearly
\[
\inf_{r \in \mathbb{R}^+, y \in \mathbb{R}} \partial_y\phi^0(r, y) = \partial_y\phi^0(0, 0) = -1,
\]
(4.115)
is the sole minimum, and $\omega^0(0, 0) = 0$. This implies that $T_{max} = 1$ and that Proposition 4.13 does not apply to this initial data.
In fact, it is straightforward to compute that the vorticity remains uniformly bounded. We compute that for all \(0 \leq t < 1\)

\[ |\omega(r, z, t)| = \frac{|\omega(0, r, y)|}{1 + t \partial \omega \phi(0, r, y)} \] (4.116)

\[ = \frac{2r|y| \exp(-r^2 - y^2)}{1 - t(1 - 2y^2) \exp(-r^2 - y^2)} \] (4.117)

\[ = \frac{2r|y| \exp(r^2 + y^2) - t(1 - 2y^2)}{1 + r^2 + y^2 - t(1 - 2y^2)} \] (4.118)

\[ \leq 1 \] (4.119)

where we have used Young’s inequality and the fact that \(\exp(s) \geq 1 + s\). This completes the proof. \(\Box\)

**Remark 4.16.** The separation of variables that occurs in the infinite dimensional vorticity equation, with the vorticity potential satisfying the one dimensional Burgers equation involving only the \(z\)-direction is a consequence of the structure of the divergence free constraint. It is not possible to take the formal limit of divergence free constraint written as

\[ \nabla \cdot u = \partial_z u_z + \partial_r u_r + \frac{u_r}{r} = 0 \] (4.123)

however if we re-normalize this equation, writing the constraint as

\[ \frac{1}{k} \partial_z u_z + \frac{1}{k} \partial_r u_r + \frac{u_r}{r} = 0 \] (4.124)

then it is very clear that taking the formal limit \(k \to +\infty\) yields

\[ u_r = 0 \] (4.125)

Recalling that \(\omega = \partial_r u_z - \partial_z u_r\), the constraint \(u_r = 0\), then implies that

\[ u_z = \partial_r^{-1} \omega \] (4.126)

Likewise, taking the divergence form of the vorticity equation, we find that

\[ \partial_t \omega + \nabla \cdot (\omega u) = \partial_t \omega + \nabla \cdot (\omega u_z e_z) \] (4.127)

\[ = \partial_t \omega + u_z \partial_z \phi + \phi \partial_z u_z \] (4.128)

This means that \(\phi\) in the infinite dimensional vorticity equation is exactly \(u_z\), and that axial compression is the dominant factor in singularity formation in very high dimension. It is well known that axial compression/planar stretching is a dominant factor in singularity formation for the three dimensional Navier–Stokes equation, as can be expressed by regularity criteria in terms of
λ₂⁺, the positive part of the middle eigenvalue of the strain matrix [10][11]. This result suggests that for very high dimensions axial compression/hyperplanar stretching leads to singularity formation, and axial compression becomes stronger relative to hyperplanar stretching in very high dimensions. This is because more directions to stretch out means that rate of hyperplanar stretching will go like a factor of \( \frac{1}{d-1} \) the rate of axial compression due to the divergence free constraint, which requires that \( \text{tr}(S) = 0. \)

The last observation we will make in this section is that axisymmetric, swirl-free solutions of the Euler equation in higher dimensions can be treated as a perturbation of the infinite-dimensional vorticity equation, where the size of the perturbation is \( \epsilon = \frac{1}{d-2} \).

**Proposition 4.17.** Suppose \( u \in C \left( 0, T_{\text{max}}; H^s_{\text{as\&df}}(\mathbb{R}^d) \right) \) is an axisymmetric swirl-free solution of the Euler equation in \( d \)-dimensional space, with \( s > \frac{d}{2} + 2, d \geq 3. \) Then the vorticity satisfies the equation

\[
\partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi + \epsilon Q_\epsilon \omega = 0,
\]

where

\[
\phi = \partial_r^{-1} \omega,
\]

\[
\epsilon = \frac{1}{d-2},
\]

and the operator \( Q_\epsilon \) is defined by

\[
Q_\epsilon \omega = (-r \partial_z \phi + \epsilon \partial_z \sigma) \partial_r \omega + \left( \phi + r \omega - \frac{\sigma}{r} - \epsilon \partial_r \sigma \right) \partial_z \omega - \frac{\partial_z \sigma}{r} \omega,
\]

with \( \sigma \) is defined by a Poisson equation

\[
-\Delta_\epsilon \sigma = - \left( \partial_r^2 + \partial_z^2 \right) (r \phi),
\]

where

\[
-\Delta_\epsilon = -\epsilon \left( \partial_r^2 + \partial_z^2 \right) - \frac{1}{r} \partial_r + \frac{1}{r^2}.
\]

**Proof.** We begin by recalling the formulation of the vorticity equation in terms of the re-normalized stream function at the beginning of this section,

\[
\partial_t \omega + u_r \partial_r \omega + u_z \partial_z \omega - \frac{k}{r} u_r \omega = 0
\]

\[
u_r = \epsilon \partial_z \tilde{\psi}
\]

\[
u_z = -\epsilon \partial_r \tilde{\psi} - \frac{\tilde{\psi}}{r}
\]

\[-\Delta_\epsilon \tilde{\psi} = \omega,
\]

where we have made the substitution \( \epsilon = \frac{1}{k}. \) We wish to invert the operator \(-\Delta_\epsilon\) in the small epsilon limit. We can see that if \( \epsilon = 0 \), then

\[
-\partial_r \left( \frac{\tilde{\psi}}{r} \right) = \left( -\frac{1}{r} \partial_r + \frac{1}{r^2} \right) \tilde{\psi}
\]

\[
= \omega
\]

\[
= \partial_r \phi,
\]

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and so
\[ \tilde{\psi} = -r\phi. \] (4.143)

We will write \( \tilde{\psi} \) as an \( \epsilon \) perturbation of this function, with
\[ \tilde{\psi} = -r\phi + \epsilon\sigma, \] (4.144)
where \( \sigma \) is defined as above. It is straightforward to verify that this stream function satisfies the equation \(-\Delta \tilde{\psi} = \omega\). We can see that
\[ -\Delta \epsilon \tilde{\psi} = -\epsilon \Delta \epsilon \sigma + \epsilon \left( \partial_r^2 + \partial_z^2 \right) (r\phi) + \partial_r \phi \] (4.145)
\[ = \omega. \] (4.146)

Computing the velocity from the stream function then yields
\[ u_r = -\epsilon r \partial_z \phi + \epsilon^2 \partial_z \sigma \] (4.147)
\[ u_z = \epsilon r \partial_r \phi - \epsilon^2 \partial_r \sigma - \frac{\epsilon \sigma}{r} + (1 + \epsilon) \phi. \] (4.148)

We can also see that
\[ k \frac{u_r}{r} = -\partial_z \phi + \epsilon \frac{\partial_z \sigma}{r}. \] (4.149)

Plugging this into the vorticity evolution equation, we find that
\[ \partial_t \omega + \left( -\epsilon r \partial_z \phi - \epsilon^2 \partial_z \phi \right) \partial_r \omega + \left( \epsilon r \partial_r \phi - \epsilon^2 \partial_r \sigma - \epsilon \frac{\sigma}{r} + (1 + \epsilon) \phi \right) \partial_z \omega \]
\[ - \left( -\partial_z \phi + \epsilon \frac{\partial_z \phi}{r} \right) \omega = 0. \] (4.150)

Rearranging terms, we find that
\[ \partial_t \omega + \phi \partial_z \omega + \omega \partial_z \phi \]
\[ + \epsilon \left( -\epsilon \partial_z \phi + \epsilon \partial_z \sigma \right) \partial_r \omega + \left( \epsilon \omega - \epsilon \partial_r \sigma - \frac{\sigma}{r} + \phi \right) \partial_z \omega - \frac{\partial_z \sigma}{r} \omega \] = 0. (4.151)

This completes the proof. \( \square \)

**Remark 4.18.** Note that although the size of the perturbation tends to zero as \( d \to \infty \), there is no immediately natural way to lift finite-time blowup for the infinite-dimensional vorticity equation to blowup for the Euler equation in very high dimensions. This is because the operator \( Q_\epsilon \) is also becoming singular as \( \epsilon \to 0 \). If there is a way to show that the Euler equation blows up in finite-time based on a perturbative argument of the Burgers-type blowup of the infinite dimensional limit, then it will require very subtle estimates on this operator in the small \( \epsilon \) limit.

## 5 Conditional regularity in four and higher dimensions

In this section, we will derive regularity criteria for axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions when \( \frac{\omega}{\rho^2} \) is bounded, and we will prove Theorem 1.1. We will begin by proving growth estimates for the vorticity in terms of the maximal velocity by looking at the transport of \( \frac{\omega}{\rho^2} \) over fluid trajectories.
Proposition 5.1. Suppose \( u \in C \left( [0, T_{\text{max}}), H^s_{df} (\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H^{s-1}_{df} (\mathbb{R}^d) \right), d \geq 3, s > \frac{d}{2} \), is an axisymmetric, swirl-free solution of the Euler equation, and that \( \frac{\omega^0}{r} \in L^\infty \). Then for all \( R > 0 \) and for all \( 0 \leq t < T_{\text{max}} \),

\[
\| \omega(\cdot, t) \|_{L^\infty} \leq \max \left( \| \omega^0 \|_{L^\infty(C^R)} , R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^\infty(C^R)} \right) \left( 1 + \frac{1}{R} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau \right)^k .
\]  

(5.1)

Recall that \( C_R \) is the cylinder \( \{ (r, z) : r < R \} \).

Proof. Again letting \( X(r, z, t) \) be the associated flow map, and recalling Proposition 3.8, we let

\[
\omega(X(r, z, t), t) = \frac{\omega^0(r, z)}{r^k} X_r(r, z, t)^k .
\]  

(5.2)

We know that

\[
X_r(r, z, t) = r + \int_0^t u_r(X(r, z, \tau), \tau) \, d\tau ,
\]  

(5.3)

and so we can conclude that for all \( 0 \leq t < T_{\text{max}} \), \( r \in \mathbb{R}^+, z \in \mathbb{R} \),

\[
X_r(r, z, t) \leq r + \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau .
\]  

(5.4)

This clearly implies that

\[
|\omega(X(r, z, t), t)| \leq \frac{|\omega^0(r, z)|}{r^k} \left( r + \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \right)^k .
\]  

(5.5)

Now fix \( R > 0 \). We can clearly see that for all \( 0 \leq r \leq R \),

\[
|\omega(X(r, z, t), t)| \leq \frac{|\omega^0(r, z)|}{r^k} \left( R + \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \right)^k
\]  

= \frac{|\omega^0(r, z)|}{r^k} R^k \left( 1 + \frac{1}{R} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau \right)^k .
\]  

(5.6)

Likewise, for all \( r \geq R \),

\[
|\omega(X(r, z, t), t)| \leq |\omega^0(r, z)| \left( 1 + \frac{1}{r} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau \right)^k
\]  

\leq |\omega^0(r, z)| \left( 1 + \frac{1}{R} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau \right)^k .
\]  

(5.7)

Therefore, we can clearly see that for all \( 0 \leq t < T_{\text{max}} \),

\[
\| \omega \circ X(\cdot, t) \|_{L^\infty} \leq \max \left( \| \omega^0 \|_{L^\infty(C^R)} , R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^\infty(C^R)} \right) \exp \left( 1 + \frac{1}{R} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau \right)^k .
\]  

(5.10)

Recall that \( X(\cdot, t) \) is a volume-preserving diffeomorphism, we see that

\[
\| \omega \circ X(\cdot, t) \|_{L^\infty} = \| \omega(\cdot, t) \|_{L^\infty} ,
\]  

(5.11)

and this completes the proof.
Corollary 5.2. Suppose \( u \in C\left([0, T_{\max}), H^s_{df}(\mathbb{R}^d)\right) \cap C^1\left([0, T_{\max}), H^{s-1}_{df}(\mathbb{R}^d)\right)\), \( d \geq 4, s > 2 + \frac{d}{2}\) is an axisymmetric, swirl-free solution of the Euler equation, that \( \frac{\omega_0}{r} \in L^\infty\). If \( T_{\max} < +\infty\), then
\[
\int_0^{T_{\max}} \|u(\cdot, t)\|_{L^\infty} \, dt = +\infty.
\]
(5.12)

Proof. This follows immediately from Proposition 5.1 and the Beale-Kato-Majda theorem. \(\square\)

Another key result will be a Sobolev-type inequality that allows us bound the maximum velocity in terms of a Lorentz norm of the vorticity of the same scaling.

Lemma 5.3. Suppose \( d \geq 3, s > \frac{d}{2}\). Then for all \( u \in H^s_{df}(\mathbb{R}^d)\),
\[
\|u\|_{L^\infty} \leq C_d \|\omega\|_{L^{d,1}},
\]
where \( C_d \) is an absolute constant depending only on the dimension.

Proof. First observe that \( u \in H^s_{df}(\mathbb{R}^d), s > \frac{d}{2}\) implies \( \omega \in L^{d,1}\) by Sobolev embedding. Next we will use the Biot-Savart law in Proposition 3.24:
\[
\omega(x) = \mathcal{B}(\omega) = \alpha_d \int_{\mathbb{R}^d} \omega(\rho, s) \left( e_\rho \otimes e_\rho - e_z \otimes e_z \right) \frac{y - x}{|y - x|^d} \, dy.
\]
(5.14)

For all \( x \in \mathbb{R}^d\), we will let
\[
\mathcal{H}_x(y) = \frac{1}{|y - x|^{d-1}}.
\]
(5.15)

Observe that for all \( x \in \mathbb{R}^d, \mathcal{H}_x \in L^{\frac{d}{d-1},\infty}(\mathbb{R}^d)\), and by translation invariance
\[
\|\mathcal{H}_x\|_{L^{\frac{d}{d-1},\infty}} = \|\mathcal{H}_0\|_{L^{\frac{d}{d-1},\infty}}.
\]
(5.16)

Applying Hölder’s inequality for Lorentz spaces, we can then see that for all \( x \in \mathbb{R}^d\),
\[
|u(x)| \leq \sqrt{2} \alpha_d \int_{\mathbb{R}^d} |\omega(\rho, s)| \frac{1}{|y - x|^{d-1}} \, dy
\]
(5.17)
\[
= \sqrt{2} \alpha_d \int_{\mathbb{R}^d} |\omega(\rho, s)| \mathcal{H}_x(y) \, dy
\]
(5.18)
\[
\leq \sqrt{2} \alpha_d \|\omega\|_{L^{d,1}} \|\mathcal{H}_0\|_{L^{\frac{d}{d-1},\infty}}.
\]
(5.19)

This completes the proof. \(\square\)

Remark 5.4. Throughout this section, the constant \( C_d \) will always refer specifically to the constant in Lemma 5.3.

We will now put this Sobolev-type inequality together with our earlier bounds on the growth of vorticity in terms of the velocity in order to give regularity criteria for the finite-time blowup of axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions that are stronger than the standard Beale-Kato-Majda criterion.
**Proposition 5.5.** Suppose $u \in C \left( [0, T_{\text{max}}), H^{s}_{\text{df}} (\mathbb{R}^d) \right) \cap C^{1} \left( [0, T_{\text{max}}), H^{s-1}_{\text{df}} (\mathbb{R}^d) \right)$, $d \geq 3, s > 2 + \frac{d}{2}$, is an axisymmetric, swirl-free solution of the Euler equation, and that $\omega^{0}_{\tau} \in L^{d,1}$. Then for all $R > 0$ and for all $0 \leq t < T_{\text{max}}$

\[
\|u(\cdot, t)\|_{L^{\infty}} \leq C_{d} \left( \|\omega^{0}\|_{L^{d,1}(C_{R}^{c})} + R^{k} \left\| \frac{\omega^{0}}{r^{k}} \right\|_{L^{d,1}(C_{R})} \right) \left( 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k}.
\]  

(5.20)

Furthermore, letting

\[
f(t) = 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau,
\]

we can conclude that for all $0 \leq t < T_{\text{max}}$,

\[
\frac{df}{dt} \leq \mu f^{k},
\]

where

\[
\mu = \frac{C_{d}}{R} \left( \|\omega^{0}\|_{L^{d,1}(C_{R}^{c})} + R^{k} \left\| \frac{\omega^{0}}{r^{k}} \right\|_{L^{d,1}(C_{R})} \right).
\]

(5.23)

**Proof.** We begin the proof with two bounds from Proposition 5.1 for all $0 \leq r \leq R$

\[
|\omega \circ X(r, z, t)| \leq \frac{\|\omega^{0}(r, z)\|}{r^{k}} R^{k} \left( 1 + \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k},
\]

and for all $r \geq R$,

\[
|\omega \circ X(r, z, t)| \leq \|\omega^{0}(r, z)\| \left( 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k}.
\]

(5.24)

(5.25)

Taking the $L^{d,1}$ norms of these bounds over the unit ball and its complement respectively, it immediately follows that

\[
\|\omega \circ X\|_{L^{d,1}(C_{R})} \leq R^{k} \left\| \frac{\omega^{0}}{r^{k}} \right\|_{L^{d,1}(C_{R})} \left( 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k},
\]

(5.26)

\[
\|\omega \circ X\|_{L^{d,1}(C_{R}^{c})} \leq \|\omega^{0}\|_{L^{d,1}(C_{R}^{c})} \left( 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k}.
\]

(5.27)

Applying the triangle inequality, we can see that

\[
\|\omega \circ X\|_{L^{d,1}(\mathbb{R}^{d})} \leq \|\omega \circ X\|_{L^{d,1}(C_{R})} + \|\omega \circ X\|_{L^{d,1}(C_{R}^{c})}
\]

\[
\leq \left( \|\omega^{0}\|_{L^{d,1}(C_{R}^{c})} + R^{k} \left\| \frac{\omega^{0}}{r^{k}} \right\|_{L^{d,1}(C_{R})} \right) \left( 1 + \frac{1}{R} \int_{0}^{t} \|u(\cdot, \tau)\|_{L^{\infty}} \, d\tau \right)^{k}.
\]

(5.28)

(5.29)

Next we observe that $X(\cdot, t)$ is a volume preserving diffeomorphism. Because the map $X(\cdot, t)$ preserves measure, it leaves the distribution function invariant when used as a change of variables with

\[
\lambda_{\omega} = \lambda_{\omega \circ X},
\]

(5.30)

and so it immediately follows that

\[
\|\omega \circ X\|_{L^{d,1}} = \|\omega\|_{L^{d,1}}.
\]

(5.31)
Applying Lemma 5.3, this yields the bound (5.20).

It remains to show that \( \frac{df}{dt} \leq \mu f^k \). The fundamental theorem of calculus immediately implies that
\[
\frac{df}{dt}(t) = \frac{1}{R} \|u(\cdot, t)\|_{L^\infty}.
\]
Plugging into the bound (5.20), we find that for all \( 0 \leq t < T_{\text{max}} \),
\[
\frac{df}{dt}(t) \leq \mu f(t)^k,
\]
and this completes the proof.

**Theorem 5.6.** Suppose \( u \in C \left( [0, T_{\text{max}}), H^s_\alpha (\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H^{s-1}_\alpha (\mathbb{R}^d) \right), d \geq 4, s > 2 + \frac{d}{2}, \)
is an axisymmetric, swirl-free solution of the Euler equation, and that \( \frac{\omega^0}{r^k} \in L^{d,1} \cap L^\infty \). Then for all \( R > 0 \) and for all \( 0 \leq t < T_{\text{max}} \),
\[
\| \omega(\cdot, t) \|_{L^\infty} \leq \max \left( \| \omega^0 \|_{L^\infty(C_R^c)}, R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^\infty(C_R)} \right) \left( 1 - \frac{C_d}{R} \left( \| \omega^0 \|_{L^d,1(C_R^c)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^d,1(C_R)} \right) \right)^{\frac{d-2}{d-3}}.
\]
In particular, this implies that
\[
T_{\text{max}} \geq \frac{R}{(d-3)C_d \left( \| \omega^0 \|_{L^d,1(C_R^c)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^d,1(C_R)} \right)}.
\]

**Proof.** We know from the Beale-Kato-Majda theorem that if \( T_{\text{max}} < +\infty \), then
\[
\int_0^{T_{\text{max}}} \| \omega(\cdot, t) \|_{L^\infty} \, dt = +\infty,
\]
and so it suffices to prove the bound (5.31), which immediately then implies the lower bound on \( T_{\text{max}} \). We will begin by defining \( f \) and \( \mu \) as in Proposition 5.5, letting
\[
f(t) = 1 + \frac{1}{R} \int_0^t \| u(\cdot, \tau) \|_{L^\infty} \, d\tau,
\]
and by letting
\[
\mu = \frac{C_d}{R} \left( \| \omega^0 \|_{L^d,1(C_R^c)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^d,1(C_R)} \right).
\]
Applying Proposition 5.1, we have for all \( 0 \leq t < T_{\text{max}} \),
\[
\| \omega(\cdot, t) \|_{L^\infty} \leq \max \left( \| \omega^0 \|_{L^\infty}, R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^\infty} \right) f(t)^k,
\]
and so it suffices to prove that for all \( 0 \leq t < T_{\text{max}} \),
\[
f(t)^k \leq \frac{1}{(1 - \frac{C_d}{R} \mu t)^{\frac{d-2}{d-3}}}.
\]
Recall from Proposition 5.5 that for all $0 \leq t < T_{\text{max}}$,
\[
\frac{d}{dt} f(t) \leq \mu f(t)^k.
\] (5.41)

We will need to integrate this differential inequality. First observe that
\[
\frac{d}{dt} \left( -f(t)^{-k+1} \right) = (k-1) f(t)^{-k} \frac{d}{dt} f(t)
\] (5.42)
\[
\leq (k-1) \mu.
\] (5.43)

Integrating with respect to time and observing that $f(0) = 1$, we find that for all $0 \leq t < T_{\text{max}}$,
\[
1 - f(t)^{-k+1} \leq (k-1) \mu t.
\] (5.44)

Rearranging this inequality, we find that for all $0 \leq t < T_{\text{max}},$
\[
f(t) \leq \frac{1}{(1 - (k-1) \mu t)^{k-1}}.
\] (5.45)

Recalling that $k = d - 2$, we can see that for all $0 \leq t < T_{\text{max}},$
\[
f(t)^k \leq \frac{1}{(1 - (d-3) \mu t)^{\frac{d}{d-2}}}.
\] (5.46)

and this completes the proof.

**Corollary 5.7.** Suppose $u \in C\left([0, T_{\text{max}}), H^s_{\text{df}}(\mathbb{R}^d)\right) \cap C^1\left([0, T_{\text{max}}), H^{s-1}_{\text{df}}(\mathbb{R}^d)\right)$, $d \geq 4, s > 2 + \frac{d}{2}$ is an axisymmetric, swirl-free solution of the Euler equation, with finite-time blowup at $T_{\text{max}} < +\infty$, and that $\frac{\omega^0}{r^k} \in L^{d,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then for all $R > 0$ and for all $0 \leq t < T_{\text{max}},$
\[
\|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)} \geq \frac{R}{(d-3)C_d(T_{\text{max}} - t)^{\frac{d}{d-2}}}. \] (5.47)

Note that this implies that $\|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)}$ controls regularity and has a minimal blowup rate
\[
\lim_{t \to T_{\text{max}}} \frac{(T_{\text{max}} - t)\|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)}}{(d-3)C_d} \geq \frac{R}{(d-3)C_d}. \] (5.48)

**Proof.** First we observe that if $\frac{\omega^0}{r^k} \in L^{d,1}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then for all $0 \leq t < T_{\text{max}}, \frac{\omega(\cdot, t)}{r^k} \in L^{d,1}(\mathbb{R}^d)$, and
\[
\left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)} = \left\| \frac{\omega^0}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)}.
\] (5.49)

Fix $0 \leq t < T_{\text{max}}$. Applying Theorem 5.6 and treating $\omega(\cdot, t)$ as initial data, we find that
\[
T_{\text{max}} - t \geq \frac{R}{(d-3)C_d \left( \|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)} + R^k \left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)} \right)}.
\] (5.50)
\[
\geq \frac{R}{(d-3)C_d \left( \|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)} + R^k \left\| \frac{\omega(\cdot, t)}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)} \right)}.
\] (5.51)
\[
= \frac{R}{(d-3)C_d \left( \|\omega(\cdot, t)\|_{L^{d,1}(C^0_R)} + R^k \left\| \frac{\omega^0}{r^k} \right\|_{L^{d,1}(\mathbb{R}^d)} \right)}.
\] (5.52)

This completes the proof.
Corollary 5.8. Suppose \(u \in C \left( [0, T_{\text{max}}), H_{df}^s (\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H_{df}^{s-1} (\mathbb{R}^d) \right), d \geq 4, s > 2 + \frac{d}{2}\)
is an axisymmetric, swirl-free solution of the Euler equation with finite-time blowup at \(T_{\text{max}} < +\infty\), and that \(\frac{\omega^0}{r^3} \in L^{d,1} \cap L^\infty\). Then for all \(0 \leq t < T_{\text{max}}\),

\[
\| \omega(\cdot, t) \|_{L^{d,1}(\mathbb{R}^d)} \geq \left( \frac{1}{k^{\frac{1}{k-1}} (d-3)^{\frac{1}{d-3}} C_d^\frac{d}{d-3}} \right) \left( \frac{1}{\| \omega^0 \|_{L^{d,1}}} \right) \left( \frac{1}{(T_{\text{max}} - t)^{\frac{d-3}{d-4}}} \right).
\]

(5.53)

Proof. From Corollary 5.7, we can see that for all \(R > 0\) and for all \(0 \leq t < T_{\text{max}}\),

\[
\| \omega(\cdot, t) \|_{L^{d,1}} \geq \left( \frac{R}{(d-3)C_d(T_{\text{max}} - t)} \right) - R^k \left( \frac{1}{\| \omega^0 \|_{L^{d,1}}} \right).
\]

(5.54)

Taking the maximum of the right hand side, we find that for all \(0 \leq t < T_{\text{max}}\),

\[
\| \omega(\cdot, t) \|_{L^{d,1}} \geq \max_{R > 0} \left( \frac{R}{(d-3)C_d(T_{\text{max}} - t)} - R^k \left( \frac{1}{\| \omega^0 \|_{L^{d,1}}} \right) \right).
\]

(5.55)

Let

\[
M = (d - 3)C_d(T_{\text{max}} - t)
\]

(5.56)

\[
B = \| \omega^0 \|_{L^{d,1}}
\]

(5.57)

\[
f(R) = \frac{R}{M} - BR^k.
\]

(5.58)

It suffices to compute the maximum of \(f\) over \(\mathbb{R}^+\). Observe that

\[
f'(R) = \frac{1}{M} - kBR^{k-1},
\]

(5.59)

and we can see that the maximum value of \(f\) will be attained at

\[
R_0 = \frac{1}{(kB)^{\frac{1}{k-1}}}.
\]

(5.60)

Therefore, after a little tedious arithmetic, we find that

\[
\max_{R > 0} f(R) = F(R_0)
\]

(5.61)

\[
= \frac{R_0}{M} - BR_0^k
\]

(5.62)

\[
= \frac{1}{(kB)^{\frac{1}{k-1}} M^{\frac{1}{k-1}}} - \frac{B}{(kB)^{\frac{1}{k-1}} M^{\frac{1}{k-1}}}
\]

(5.63)

\[
= \left( \frac{1}{k^{\frac{1}{k-1}}} - \frac{1}{k^{\frac{1}{k-1}}} \right) \frac{1}{B^{\frac{1}{k-1}} M^{\frac{1}{k-1}}}
\]

(5.64)

\[
= \frac{1}{k^{\frac{1}{k-1}}} (k - 1) \frac{1}{B^{\frac{1}{k-1}} M^{\frac{1}{k-1}}}
\]

(5.65)

\[
= \left( \frac{1}{k^{\frac{1}{k-1}} (d-3)^{\frac{1}{d-3}} C_d^\frac{d}{d-3}} \right) \left( \frac{1}{\| \omega^0 \|_{L^{d,1}}} \right) \left( \frac{1}{(T_{\text{max}} - t)^{\frac{d-3}{d-4}}} \right).
\]

(5.66)

This completes the proof. \(\Box\)
Remark 5.9. This is stronger than the Beale-Kato-Majda criterion in two respects. First, we have a lower order norm, requiring that \( \omega(\cdot,t) \) must blowup in \( L^{d,1} \) in addition to \( L^{\infty} \). Second, we have a faster rate. The Beale-Kato-Majda criterion requires the vorticity to blowup in \( L^{1}\), which corresponds to a minimal blowup rate of the \( L^{\infty} \) norm like
\[
\| \omega(\cdot,t) \| \sim \frac{1}{T_{\text{max}} - t},
\] (5.67)
which is slower than the blowup rate
\[
\| \omega(\cdot,t) \|_{L^{d,1}} \sim \frac{1}{(T_{\text{max}} - t)^{\frac{d-2}{d-3}}}. \quad (5.68)
\]

Corollary 5.8 provides an explicit lower bound, and a faster rate of blowup of a lower order norm than the Beale-Kato-Majda criterion. While it is possible—in fact very likely—that there may be finite-time blowup for smooth axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions even when \( \frac{\omega_0}{r} \) is bounded, such blowup must occur in a very particular way.

The same bounds that give the regularity criteria in Corollary 5.8 in four and higher dimensions guarantee global regularity in three dimensions, and provide an exponential bound on the growth of vorticity.

**Theorem 5.10.** Suppose the initial data \( u^0 \in H^{s}_d(\mathbb{R}^3) \), \( s > \frac{7}{2} \) is axisymmetric and swirl-free. Then there exists a global smooth solution of the Euler equation \( u \in C\left([0, +\infty), H^{s}_d(\mathbb{R}^3)\right) \cap C^1\left([0, +\infty), H^{s-1}_d(\mathbb{R}^3)\right) \). Furthermore, we have a bound on vorticity for all \( R > 0 \) and for all \( 0 \leq t < +\infty \),
\[
\| \omega(\cdot,t) \|_{L^{\infty}} \leq \max \left( \| \omega^0 \|_{L^{\infty}(C R)}, R \| \frac{\omega^0}{r} \|_{L^{\infty}(C R)} \right) \exp \left( \frac{C_d}{R} \left( \| \omega^0 \|_{L^{d,1}(C R)} + R \| \frac{\omega^0}{r} \|_{L^{d,1}(C R)} \right) t \right).
\] (5.69)

**Proof.** We will begin by letting
\[
f(t) = 1 + \frac{1}{R} \int_0^t \| u(\tau) \|_{L^{\infty}} \, d\tau,
\] (5.70)
and letting
\[
\mu = \frac{C_d}{R} \left( \| \omega^0 \|_{L^{d,1}(C R)} + R^k \| \frac{\omega^0}{r} \|_{L^{d,1}(C R)} \right). \quad (5.71)
\]
We know from the local wellposedness result in Theorem 3.14 that there is an axisymmetric, swirl-free solution of the Euler equation \( u \in C\left([0, T_{\text{max}}), H^{s}_d(\mathbb{R}^3)\right) \cap C^1\left([0, T_{\text{max}}), H^{s-1}_d(\mathbb{R}^3)\right) \) for some \( T_{\text{max}} > 0 \).

We can see from Proposition 5.1 that for all \( 0 \leq t < T_{\text{max}} \),
\[
\| \omega(\cdot,t) \|_{L^{\infty}} \leq \max \left( \| \omega^0 \|_{L^{\infty}(C R)}, R^k \| \frac{\omega^0}{r} \|_{L^{\infty}(C R)} \right) f(t).
\] (5.72)
Furthermore, we can see from Proposition 5.10 that for all $0 \leq t < T_{\text{max}}$,
\[
\frac{d}{dt} f(t) \leq \mu f(t).
\] (5.73)
We clearly have $f(0) = 1$ by definition, so applying Grönwall’s inequality, we find that for all $0 \leq t < T_{\text{max}}$,
\[
f(t) \leq \exp(\mu t),
\] (5.74)
and consequently
\[
\|\omega(\cdot, t)\|_{L^\infty} \leq \max \left( \|\omega^0\|_{L^\infty(C_R)}, R \right) \|\frac{\omega^0}{r}\|_{L^\infty(C_R)} \exp(\mu t). \] (5.75)
This completes the proof of the bound (5.69). Finally, we will observe that if $T_{\text{max}} < +\infty$, then by the Beale-Kato-Majda criterion we must have
\[
\int_0^{T_{\text{max}}} \|\omega(\cdot, t)\|_{L^\infty} \, dt = +\infty,
\] (5.76)
and so this bound also implies that $T_{\text{max}} = +\infty$. This completes the proof.□

**Remark 5.11.** This is a relatively standard approach to the classical result that for sufficiently smooth axisymmetric, swirl-free initial data, a smooth solution of the Euler equation in three dimensions exists globally in time. The conditions on the vorticity in Theorem 5.10 is of a very similar form to the conditions in the global regularity result of Ukhovskii and Yudovich in [12], although the methods are quite different, as that paper involves the vanishing viscosity limit. In fact this result can be strengthened. Danchin proved [2,3] that if $u \in C \left( [0, T_{\text{max}}), H^s(\mathbb{R}^3) \right)$, $s > \frac{5}{2}$ is an axisymmetric, swirl-free solution of the Euler equation, then for all $0 \leq t < +\infty$,
\[
\|\omega(\cdot, t)\|_{L^\infty} \leq \|\omega^0\|_{L^\infty} \exp \left( \frac{C}{r} \left\| \frac{\omega^0}{r} \right\|_{L^{3,1}} t \right),
\] (5.77)
where $C > 0$ is an absolute constant independent of $\omega^0$. Note that $u^0 \in H^s(\mathbb{R}^3), s > \frac{5}{2}$ is sufficient to guarantee both that $\omega^0 \in L^\infty$ and that $\frac{\omega^0}{r} \in L^{3,1}$, by Sobolev embedding.

Danchin actually proves this exponential upper bound in a class slightly larger than $u^0 \in H^s(\mathbb{R}^3), s > \frac{5}{2}$, but this is the standard regularity class for strong solutions of the three dimensional Euler equation when there is no assumption of axisymmetry, and this regularity class is good enough to guarantee both norms involving the initial vorticity are bounded, so we will not concern ourselves with further relaxing these assumptions here. The key element of the proof is the inequality
\[
\left\| \frac{u_r}{r} \right\|_{L^\infty} \leq C \left\| \frac{\omega}{r} \right\|_{L^{3,1}}. \] (5.78)
We have already proven one of the regularity criteria in Theorem 1.1. Now we will complete the proof of this theorem.

**Theorem 5.12.** Suppose $u \in C \left( [0, T_{\text{max}}), H^s_{df}(\mathbb{R}^d) \right) \cap C^1 \left( [0, T_{\text{max}}), H^{s-1}_{df}(\mathbb{R}^d) \right), d \geq 3, s > 2 + \frac{d}{2}$, is an axisymmetric, swirl-free solution of the Euler equation, and that $\frac{\omega^0}{r} \in L^\infty$. Then for all $R > 0$ and for all $0 \leq t < T_{\text{max}}$,
\[
\|\omega(\cdot, t)\|_{L^\infty} \leq \max \left( \|\omega^0\|_{L^\infty(C_R)}, \mu^k \right) \left( 1 + \frac{1}{R} \int_0^t \left\| u_r^+(\cdot, \tau) \right\|_{L^\infty} \, d\tau \right)^k.
\] (5.79)
In particular, if $T_{\text{max}} < +\infty$, then
\[
\int_0^{T_{\text{max}}} \|u_r^+(\cdot, t)\|_{L^\infty} dt = +\infty. \tag{5.80}
\]

**Proof.** The Beale-Kato-Majda theorem implies that it is sufficient to prove the bound \(5.79\). The proof will follow along the lines of Proposition 5.1. Again let $X(r, z, t)$ be the associated flow map and recall that
\[
\omega(X(r, z, t), t) = \frac{\omega^0(r, z)}{r^k} X_r(r, z, t)^k. \tag{5.81}
\]
We therefore know that
\[
X_r(r, z, t) = r + \int_0^t u_r(X(r, z, \tau), \tau) d\tau, \tag{5.82}
\]
and so we can conclude that for all $0 \leq t < T_{\text{max}}$, $r \in \mathbb{R}^+$, $z \in \mathbb{R}$,
\[
X_r(r, z, t) \leq r + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau. \tag{5.83}
\]
This clearly implies that
\[
|\omega(X(r, z, t), t)| \leq \frac{|\omega^0(r, z)|}{r^k} \left( r + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} \right)^k. \tag{5.84}
\]

Now fix $R > 0$. We can clearly see that for all $0 \leq r \leq R$,
\[
|\omega(X(r, z, t), t)| \leq \frac{|\omega^0(r, z)|}{r^k} \left( R + \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} \right)^k \tag{5.85}
\]
\[
= \frac{|\omega^0(r, z)|}{r^k} R^k \left( 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} \right)^k. \tag{5.86}
\]
Likewise, for all $r \geq R$,
\[
|\omega(X(r, z, t), t)| \leq |\omega^0(r, z)| \left( 1 + \frac{1}{r} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k \tag{5.87}
\]
\[
\leq |\omega^0(r, z)| \left( 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k. \tag{5.88}
\]
Therefore, we can clearly see that for all $0 \leq t < T_{\text{max}},$
\[
\|\omega \circ X(\cdot, t)\|_{L^\infty} \leq \max \left( \|\omega^0\|_{L^\infty(C_R)}, R^k \|\frac{\omega^0}{r^k}\|_{L^\infty(C_R)} \right) \exp \left( 1 + \frac{1}{R} \int_0^t \|u_r^+(\cdot, \tau)\|_{L^\infty} d\tau \right)^k. \tag{5.89}
\]
Recall that $X(\cdot, t)$ is a volume-preserving diffeomorphism, we see that
\[
\|\omega \circ X(\cdot, t)\|_{L^\infty} = \|\omega(\cdot, t)\|_{L^\infty}, \tag{5.90}
\]
and this completes the proof. \(\square\)

Finally, we will consider the case of the evolution of compactly supported vorticities in four and higher dimensions when $\frac{\omega^0}{r^k}$ is bounded. The arguments which lead to global regularity in the three dimensional case have analogues in four and higher dimensions that are too singular to prevent finite-time blowup via the support running off to infinity in the radial direction.
Proposition 5.13. Suppose \( u \in C \left( [0, T_{\max}), H_{d}^{s} (\mathbb{R}^{d}) \right) \cap C^{1} \left( [0, T_{\max}), H_{d}^{s-1} (\mathbb{R}^{d}) \right), d \geq 3, s > 2 + \frac{d}{2}, \) is an axisymmetric, swirl-free solution of the Euler equation, that \( \frac{\omega_{0}}{r^{k}} \in L^{\infty}, \) and that \( \omega_{0} \) has compact support. Let \( R(t) \) be the maximal radial value in the support at a given time:

\[
R(t) = \sup_{(r,z) \in \text{supp}(\omega(\cdot,t))} r
\]

Then for all \( 0 \leq t < T_{\max}, \)

\[
\|\omega(\cdot,t)\|_{L^{\infty}} \leq \frac{\|\omega_{0}\|_{L^{\infty}}}{r^{k}} R(t)^{k}
\]

and

\[
\frac{d}{dt} R(t) \leq dC_{d} \|\omega_{0}\| \frac{d}{dt} \left( \|\omega_{0}\|_{L^{\infty}} \right) \frac{d}{dt} R(t)^{k},
\]

where \( C_{d} > 0 \) is a constant independent of \( \omega_{0} \) depending only on the dimension.

Proof. We will begin by observing that for all \( 0 \leq t < T_{\max}, \)

\[
\frac{d}{dt} R(t) \leq \|u(\cdot,t)\|_{L^{\infty}} \leq C_{d} \|\omega(\cdot,t)\|_{L^{d,1}}.
\]

It is also straightforward to observe that for all \( 0 \leq t < T_{\max}, \)

\[
\|\omega(\cdot,t)\|_{L^{\infty}} \leq \frac{\|\omega_{0}\|_{L^{\infty}}}{r^{k}} R(t)^{k}.
\]

Recall that

\[
\|\omega(\cdot,t)\|_{L^{d,1}} = d \int_{0}^{\infty} \left\{ x \in \mathbb{R}^{d} : |\omega(x,t)| > \alpha \right\} \frac{1}{\alpha} d\alpha.
\]

We can clearly see from (5.96), that for all \( \alpha > \frac{\|\omega_{0}\|}{r^{k}} R(t)^{k}, \)

\[
\left\{ x \in \mathbb{R}^{d} : |\omega(x,t)| > \alpha \right\} = 0
\]

Because we have \( \text{supp}(\omega) = \text{supp} \left( \frac{\omega}{r^{k}} \right) \), and \( \frac{\omega}{r^{k}} \) is transported by a divergence free vector field, we can see that for all \( 0 \leq t < T_{\max} \)

\[
|\text{supp}(\omega(\cdot,t))| = |\text{supp} \left( \frac{\omega}{r^{k}}(\cdot,t) \right)|
\]

\[
= |\text{supp} \left( \frac{\omega_{0}}{r^{k}} \right)|
\]

\[
= |\text{supp} \left( \omega_{0} \right)|
\]

Therefore, we can see that for all \( 0 \leq \alpha \leq \frac{\|\omega_{0}\|}{r^{k}} R(t)^{k} \)

\[
\left\{ x \in \mathbb{R}^{d} : |\omega(x,t)| > \alpha \right\} \leq |\text{supp} \left( \omega_{0} \right)|
\]

Plugging in these bounds in we find that for all \( 0 \leq t < T_{\max}, \)

\[
\|\omega(\cdot,t)\|_{L^{d,1}} \leq d \|\omega_{0}\| \frac{1}{\alpha} \frac{\|\omega_{0}\|}{r^{k}} R(t)^{k}.
\]
Remark 5.14. In the case where $d = 3$, this is the standard proof of global regularity for the axisymmetric, swirl-free Euler equation in three dimensions given by Majda and Bertozzi [9]. In this case, Grönwall’s inequality gives an exponential bound on the radius of the support of the vorticity,

$$ R(t) \leq R_0 \exp \left( 3C_3 \| \text{supp}(\omega^0) \|_t \right), $$

and consequently on the size of the vorticity,

$$ \| \omega(.,t) \|_{L^\infty} \leq \| \frac{\omega^0}{r} \|_{L^\infty} R_0 \exp \left( C_3 \| \text{supp}(\omega^0) \|_t \right), $$

This in turn yields global regularity. We can see that even in the case where $\frac{\omega^0}{r} \in L^\infty$, this approach cannot guarantee global regularity in four and higher dimensions, as the bound from the differential inequality in Proposition 5.13 will become singular in finite-time.

Remark 5.15. It is an interesting question whether the inequality in Proposition 5.13 is sharp for $d \geq 3$. In the case of $d = 3$, this would mean exponential growth of the maximum vorticity, and in the case $d \geq 4$, this would mean finite-time blowup.

In the two dimensional case, where $k = 0$, there is an analogous result, even without assuming axisymmetry. In two dimensions the scalar vorticity is given by

$$ \omega = \partial_1 u_2 - \partial_2 u_1, $$

and this vorticity is transported by the flow,

$$ \partial_t \omega + u \cdot \nabla \omega = 0. $$

The velocity can be recovered from the vorticity by the convolution

$$ u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x-y)\perp}{|x-y|^2} \omega(y) \, dy, $$

where $x\perp = (-x_2, x_1)$ From this it follows that

$$ \| u \|_{L^\infty} \leq C \| \omega \|_{L^{2,1}}. $$

If $\omega^0$ has compact support and

$$ R(t) = \sup_{x \in \text{supp}(\omega(.,t))} x_1, $$

then for all $0 \leq t < +\infty$,

$$ \frac{d}{dt} R(t) \leq \| u(.,t) \|_{L^\infty} \leq C \| \omega^0 \|_{L^{2,1}}. $$

Iftimie, Gamblin, and Sideris showed that this inequality is in fact sharp [5]. They showed that for a broad set of vorticities satisfying anti-symmetry and sign conditions, that for all $0 \leq t < +\infty$,

$$ \frac{d}{dt} R(t) \geq \kappa (\omega^0), $$

where $\kappa (\omega^0) > 0$, is a constant depending on the initial vorticity. The fact that the differential inequality governing the radius of the support of the vorticity (5.112) is sharp up to a constant in two dimensions suggests that it is likely that the analogous differential inequality in Proposition 5.13 will also be sharp up to a constant in three and higher dimensions, but this remains an open question.
A geometric setup for possible finite-time blowup

In this section, we will discuss a geometric scenario for the finite-time blowup of axisymmetric, swirl-free solutions of the Euler equation in four and higher dimensions. This scenario will involve vorticities which are odd with respect to \( z \) and nonnegative for \( z > 0 \). We will begin by deriving a special form of the Biot-Savart law that applies to vorticities which are odd with respect to \( z \).

**Proposition 6.1.** Suppose \( u \in H^s_d \left( \mathbb{R}^d \right), s > \frac{d}{2}, d \geq 3 \), is axisymmetric and swirl-free. Further suppose that \( \omega(r, z) \) is odd in \( z \), with for all \( r \geq 0, z \in \mathbb{R} \),

\[
\omega(r, -z) = -\omega(r, z) \quad (6.1)
\]

Then the velocity can be recovered from the vorticity in the upper half plane using the following formulas:

\[
u_r(r, z) = \alpha_d m_{d-2} m_{d-3} \int_0^\infty \int_0^\infty H(r, z, \rho, s) \rho^{d-2} \omega(\rho, s) \, d\rho \, ds,
\]

where

\[
H(r, z, \rho, s) = \int_0^1 (1 - \tau^2)^\frac{d-4}{2} \left( \frac{z + s}{(r^2 + \rho^2 - 2r \rho \tau + (z + s)^2)^\frac{d}{2}} + \frac{z - s}{(r^2 + \rho^2 + 2r \rho \tau + (z - s)^2)^\frac{d}{2}} \right)
\]

\[+ \frac{z + s}{(r^2 + \rho^2 + 2r \rho \tau + (z + s)^2)^\frac{d}{2}} - \frac{z - s}{(r^2 + \rho^2 - 2r \rho \tau + (z - s)^2)^\frac{d}{2}} \right) \, d\tau, \quad (6.3)
\]

and

\[
u_z(r, z) = -\alpha_d m_{d-2} m_{d-3} \int_0^\infty \int_0^\infty G(r, z, \rho, s) \rho^{d-2} \omega(\rho, s) \, d\rho \, ds,
\]

where

\[
G(r, z, \rho, s) = \int_0^1 (1 - \tau^2)^\frac{d-4}{2} \left( \frac{\rho - r \tau}{(r^2 + \rho^2 - 2r \rho \tau + (z - s)^2)^\frac{d}{2}} + \frac{\rho + r \tau}{(r^2 + \rho^2 + 2r \rho \tau + (z - s)^2)^\frac{d}{2}} \right)
\]

\[+ \frac{\rho - r \tau}{(r^2 + \rho^2 + 2r \rho \tau + (z - s)^2)^\frac{d}{2}} - \frac{\rho + r \tau}{(r^2 + \rho^2 - 2r \rho \tau + (z - s)^2)^\frac{d}{2}} \right) \, d\tau. \quad (6.5)
\]

Further note that \( \nu_r(r, z) \) is even in \( z \), while \( \nu_z(r, z) \) is odd in \( z \).

**Proof.** We begin by recalling the Biot-Savart law in Proposition 3.2:

\[
u_r(r, z) = -\alpha_d m_{d-2} m_{d-3} \int_{-\infty}^\infty \int_0^\infty \rho^{d-2}(z - s) \omega(\rho, s) \int_0^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^\frac{d-4}{2}}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^\frac{d}{2}} \, d\tilde{y}_1 \, d\rho \, ds.
\]

Using the odd symmetry of \( \omega \) in \( s \), and the change of variables \( \tilde{s} = -s \),

\[
- \int_{-\infty}^0 \int_0^\infty \rho^{d-2}(z - s) \omega(\rho, s) \int_0^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^\frac{d-4}{2}}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z - s)^2)^\frac{d}{2}} \, d\tilde{y}_1 \, d\rho \, ds =
\]

\[
\int_0^\infty \int_0^\infty \rho^{d-2}(z + s) \omega(\rho, s) \int_0^1 \frac{\tilde{y}_1(1 - \tilde{y}_1^2)^\frac{d-4}{2}}{(r^2 + \rho^2 - 2r \rho \tilde{y}_1 + (z + s)^2)^\frac{d}{2}} \, d\tilde{y}_1 \, d\rho \, ds \quad (6.7)
\]
Therefore, the identity for \( u_r \) can be rewritten as
\[
 u_r(r, z) = \int_{0}^{\infty} \int_{0}^{\infty} \rho^{d-2} \omega(\rho, s) \int_{-1}^{1} \left( \frac{(z + s)\tilde{y}_1(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z + s)^2)^{\frac{d}{2}}} - \frac{(z - s)\tilde{y}_1(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} \right) d\tilde{y}_1 d\rho ds. \quad (6.8)
\]
Making the substitution \( \tau = -\tilde{y}_1 \), for \(-1 \leq \tilde{y} \leq 0\), we can see that
\[
 \int_{-1}^{0} \left( \frac{(z + s)\tilde{y}_1(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z + s)^2)^{\frac{d}{2}}} - \frac{(z - s)\tilde{y}_1(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} \right) d\tilde{y}_1 = \int_{0}^{1} \left( \frac{(z + s)\tau(1 - \tau^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 + 2r\rho\tau + (z + s)^2)^{\frac{d}{2}}} + \frac{(z - s)\tau(1 - \tau^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 + 2r\rho\tau + (z - s)^2)^{\frac{d}{2}}} \right) d\tau. \quad (6.9)
\]
Making the trivial substitution \( \tau = \tilde{y} \) for \(0 \leq \tilde{y} \leq 1\), this completes the proof of the identity \(6.2\).

We proceed similarly for \( u_z \), starting with the identity from Proposition\(3.26\)
\[
 u_z(r, z) = \alpha_d m_{d-2} m_{d-3} \int_{-\infty}^{\infty} \int_{0}^{\infty} \rho^{d-2} \omega(\rho, s) \int_{-1}^{1} \left( \frac{(r\tilde{y}_1 - \rho)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} \right) d\tilde{y}_1 d\rho ds. \quad (6.10)
\]
Again applying the odd symmetry of \( \omega \) in \( z \), we find that
\[
 u_z(r, z) = -\alpha_d m_{d-2} m_{d-3} \int_{0}^{\infty} \int_{0}^{\infty} \rho^{d-2} \omega(\rho, s) \int_{-1}^{1} \left( \frac{(\rho - r\tilde{y}_1)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} - \frac{(\rho - r\tilde{y}_1)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z + s)^2)^{\frac{d}{2}}} \right) d\tilde{y}_1 d\rho ds. \quad (6.11)
\]
Again taking the substitution \( \tau = -\tilde{y}_1 \), we find that
\[
 \int_{-1}^{0} \left( \frac{(\rho - r\tilde{y}_1)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z - s)^2)^{\frac{d}{2}}} - \frac{(\rho - r\tilde{y}_1)(1 - \tilde{y}_1^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 - 2r\rho\tilde{y}_1 + (z + s)^2)^{\frac{d}{2}}} \right) d\tilde{y}_1 = \int_{0}^{1} \left( \frac{(\rho + r\tau)(1 - \tau^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 + 2r\rho\tau + (z - s)^2)^{\frac{d}{2}}} - \frac{(\rho + r\tau)(1 - \tau^2)^{\frac{d-4}{2}}}{(r^2 + \rho^2 + 2r\rho\tau + (z + s)^2)^{\frac{d}{2}}} \right) d\tau. \quad (6.12)
\]
Making the trivial substitution \( \tau = \tilde{y}_1 \) for \(0 \leq \tilde{y}_1 \leq 1\), this completes the proof of the identity \(6.4\).

Finally, we observe that
\[
 H(r, -z, \rho, s) = H(r, z, \rho, s), \quad (6.13)
\]
which implies that
\[
 u_r(r, z) = u_r(r, -z). \quad (6.14)
\]
Likewise, we can observe that
\[
 G(r, -z, \rho, s) = -G(r, z, \rho, s), \quad (6.15)
\]
which implies that
\[
 u_z(r, -z) = -u_z(r, z). \quad (6.16)
\]
This completes the proof.
\[\square\]
It is clear from the analysis in Section 5, that blowup for solutions of the axisymmetric, swirl-free Euler equation in four and higher dimensions requires compression along the $z$-axis and stretching in the $r$-hyperplane. We will show that this is precisely what is afforded to us by vorticities which are odd in $z$, and nonnegative for $z > 0$.

**Proposition 6.2.** Suppose $u \in H^s_{as} \left( \mathbb{R}^d \right)$, $s > 1 + \frac{d}{2}$, $d \geq 3$, is axisymmetric and swirl-free. Further suppose that $\omega$ is odd in $z$, that for all $r, z > 0, \omega(r, z) \geq 0$, and that $\omega$ is not identically zero. Then for all $z > 0$,

$$u_z(0, z) < 0,$$

and for all $r > 0$,

$$u_r(r, 0) > 0.$$

**Proof.** Taking the identity (6.4) from Proposition 6.1, we see that

$$u_z(0, z) = -2\alpha d m_d - 2 m_d - 3 \int_0^\infty \int_0^\infty \rho^{d-1} \omega(\rho, s) \int_0^1 (1 - \tau^2) \frac{d^{d-4}}{2} \left( \frac{1}{(\rho^2 + (z-s)^2)^{\frac{d}{2}}} - \frac{1}{(\rho^2 + (z+s)^2)^{\frac{d}{2}}} \right) d\tau d\rho ds. \quad (6.19)$$

Observe that for all $\rho, s, z > 0$,

$$\int_0^1 (1 - \tau^2)^\frac{d^{d-4}}{2} \left( \frac{1}{(\rho^2 + (z-s)^2)^{\frac{d}{2}}} - \frac{1}{(\rho^2 + (z+s)^2)^{\frac{d}{2}}} \right) d\tau > 0. \quad (6.20)$$

This implies that for all $z > 0$,

$$u_z(0, z) < 0. \quad (6.21)$$

Likewise, applying the identity (6.2) from Proposition 6.1, we can see that for all $r > 0$,

$$u_r(r, 0) = 2\alpha d m_d - 2 m_d - 3 \int_0^\infty \int_0^\infty s \rho^{d-2} \omega(\rho, s) \int_0^1 \tau (1 - \tau^2) \frac{d^{d-4}}{2} \left( \frac{1}{(r^2 + \rho^2 - 2r \rho \tau + s^2)^{\frac{d}{2}}} - \frac{1}{(r^2 + \rho^2 + 2r \rho \tau + s^2)^{\frac{d}{2}}} \right) d\tau d\rho ds. \quad (6.22)$$

Observe that for all $\rho, s, r > 0$,

$$\int_0^1 \tau(1 - \tau^2) \frac{d^{d-4}}{2} \left( \frac{1}{(r^2 + \rho^2 - 2r \rho \tau + s^2)^{\frac{d}{2}}} - \frac{1}{(r^2 + \rho^2 + 2r \rho \tau + s^2)^{\frac{d}{2}}} \right) d\tau > 0, \quad (6.23)$$

and so for all $r > 0$,

$$u_r(r, 0) > 0. \quad (6.24)$$

**Conjecture 6.3.** Suppose $u^0 \in H^s_{as} \left( \mathbb{R}^d \right)$, $d \geq 4, s > 1 + \frac{d}{2}$ is not identically zero that furthermore the associated vorticity $\omega^0$ is odd in $z$, and that for all $z > 0, \omega^0(r, z) \geq 0$. Then the smooth solution of the Euler equation $u \in C \left( [0, T_{max}); H^s_{as} \left( \mathbb{R}^d \right) \right) \cap C^1 \left( [0, T_{max}), H^{s-1}_{as} \left( \mathbb{R}^d \right) \right)$ with initial data $u^0$ blows up in finite-time, $T_{max} < +\infty$.  

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Remark 6.4. This is precisely the geometric setup given in Elgindi’s recent blowup result for axisymmetric, swirl-free $C^{1,0}$ solutions of the Euler equation in three dimensions [4]. Similar to the geometric setup proposed for finite-time blowup here, the vorticity Elgindi’s blowup solutions are odd in $z$ and positive in the upper half plane, and involve blowup driven by compression along the $z$ axis, and stretching in the horizontal plane. Furthermore, a key aspect of Elgindi’s blowup result is that $\frac{\omega_0}{r} \notin L^\infty$. This suggests that a similar approach may be able to yield smooth solutions of the Euler equation in four and higher dimensions when $\frac{\omega_0}{r} \notin L^\infty$.

Remark 6.5. Iftimie, Gamblin, and Sideris proved [5] that in two dimensions, if $\omega$ is compactly supported, odd with respect to both $x_1$ and $x_2$, $\omega(-x_1,x_2) = -\omega(x_1,x_2)$ \hspace{1cm} (6.25) \\
$\omega(x_1,-x_2) = -\omega(x_1,x_2)$ \hspace{1cm} (6.26)
and $\omega(x_1,x_2) \geq 0$ in the first quadrant $x_1,x_2 > 0$, and not identically zero, then the radius of the support in the $x_1$ direction grows linearly in time with $\frac{dR}{dt} \geq \kappa (\omega^0)$, \hspace{1cm} (6.27)
where $\kappa (\omega^0) > 0$ is a constant depending on the initial vorticity.

We should note that this is precisely the geometric setup proposed for finite-time blowup in four and higher dimensions in Conjecture [6.3]. Note that in two dimensions $r = |x_1|$ and $e_r = \text{sgn}(x_1)e_1$, and so $u$ is axisymmetric if and only if $u_1$ is odd in $x_1$ and $u_2$ is even in $x_1$. Recall that $\omega = \partial_1 u_2 - \partial_2 u_1$, and we can see that if $u$ is axisymmetric implies that $\omega$ is odd with respect to $x_1$. Using the Biot-Savart law, it can be seen that in fact $u$ is axisymmetric in two dimensions if and only if $\omega$ is odd with respect to $x_1$. Furthermore, in two dimensions any axisymmetric vector field is automatically swirl-free, as $\mathbb{R}^2 = \text{span}\{e_r,e_z\}$. Therefore, the requirement that $\omega$ is odd with respect to $x_1$ in two dimensions is analogous to the requirement that $u$ is axisymmetric, swirl-free in three and higher dimensions. Likewise the requirement that $\omega$ is odd with respect to $x_2$ in two dimensions is analogous to the requirement that $\omega$ is odd with respect to $z$ in three and higher dimensions. Finally the requirement that $\omega$ is nonnegative in the first quadrant in two dimensions is analogous to the requirement that $\omega$ is nonnegative for $z > 0$ in three and higher dimensions. This shows that the geometric setup proposed for finite-time blowup in four and higher dimensions in Conjecture [6.3] is entirely analogous to the geometric setup for the linear growth of support in two dimensions in [5].

We know that for all $d \geq 2$, the radial size of a compactly supported vorticity can grow at most like $\frac{dR}{dt} \leq \tilde{\kappa} (\omega^0) R^{d-2}$. \hspace{1cm} (6.28)
The fact that this inequality is sharp up to a constant in two dimensions, suggests it may very well be sharp up to a constant in the same geometric setup in three and higher dimensions, which would lead directly to the finite-time blowup result in four and higher dimensions conjectured above.

In addition to expecting blowup in four and higher dimensions for smooth, axisymmetric, swirl-free solutions of the Euler equation, it also seems very likely that under comparable geometric constraints the vorticity will grow exponentially in time in three dimensions, and the exponential bound proven by Danchin in [2] will be sharp. In particular, if (6.28) is sharp up to a constant when $d = 3$, this implies both exponential growth of the support of the vorticity and the $L^\infty$ norm of the vorticity.
Conjecture 6.6. Suppose \( u^0 \in H^{s,d+f}_{ax} (\mathbb{R}^3) \), \( s > \frac{5}{2} \) is not identically zero that furthermore the associated vorticity \( \omega^0 \) is odd in \( z \), and that for all \( z > 0 \), \( \omega^0(r,z) \geq 0 \). Then there exist constants \( \kappa_1, \kappa_2 > 0 \), depending on \( \omega^0 \) such that for the solution of the Euler equation \( u \in C \left( [0, +\infty), H^{s,d_f}_{df} (\mathbb{R}^3) \right) \cap C^1 \left( [0, +\infty), H^{s-1,d_f}_{df} (\mathbb{R}^3) \right) \) with initial data \( u^0 \), for all \( 0 \leq t < +\infty \),
\[
\| \omega(\cdot, t) \|_{L^\infty} \geq \kappa_1 \exp (\kappa_2 t). \tag{6.29}
\]

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