Thermalization in harmonic particle chains with velocity flips

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Abstract

We propose a new mathematical tool for the study of transport properties of models for lattice vibrations in crystalline solids. By replication of dynamical degrees of freedom, we aim at a new dynamical system where the “local” dynamics can be isolated and solved independently from the “global” evolution. The replication procedure is very generic but not unique as it depends on how the original dynamics are split between the local and global dynamics. As an explicit example, we apply the scheme to study thermalization of the pinned harmonic chain with velocity flips. We improve on the previous results about this system by showing that after a relatively short time period the average kinetic temperature profile satisfies the dynamic Fourier’s law in a local microscopic sense without assuming that the initial data is close to a local equilibrium state. The bounds derived here prove that the above thermalization period is at most of the order $L^{2/3}$, where $L$ denotes the number of particles in the chain. In particular, even before the diffusive time scale Fourier’s law becomes a valid approximation of the evolution of the kinetic temperature profile. As a second application of the dynamic replica method, we also briefly discuss replacing the velocity flips by an anharmonic onsite potential.

Dedicated to Herbert Spohn, with sincere gratitude for his support, inspiration and insight.

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1 Introduction

The energy transport properties of lattice vibrations in crystalline solids have recently attracted much research activity. In the simplest case energy is the only relevant conserved quantity, and then it is expected that for a large class of three or higher dimensional systems energy transport is diffusive and the Fourier’s law of heat conduction holds; see for instance [1] for a discussion. In contrast, many one-dimensional systems exhibit anomalous energy transport violating the Fourier’s law. Section 7 of Ref. [2] offers a concise summary of the state of the art in the results and understanding of transport properties of such systems.

If a suitable stochastic noise is added to the Hamiltonian interactions, also one-dimensional particle chains can produce diffusive energy transport. A particularly appealing test case is obtained by taking a harmonic chain, which has ballistic energy transport [3], and endowing each of the particles with its own Poissonian clock whose rings will flip the velocity of the particle. To our knowledge, this velocity flip model was first considered in [4], and it is one of the simplest known particle chain models which has a finite heat conductivity and satisfies the time-dependent Fourier’s law. Its transport properties depend on the harmonic interactions, most importantly on whether the forces have an on-site component (pinning) or not. For nearest neighbor interactions, if there is no pinning, there are two locally conserved fields, while with pinning there is only one, the energy density. In addition, the thermal conductivity, and hence the energy diffusion constant, happens to be independent of temperature, which implies that the Fourier’s law corresponds to a linear heat equation. This allows for explicit representation of its solutions in terms of Fourier transform. It also leads to the useful simplification that for any stochastic initial state also the expectation value of the temperature distribution satisfies the Fourier’s law, even when the initial total energy has macroscopic variation.

The main goal of the present contribution is to introduce a mathematical method which allows splitting the dynamics of the velocity flip model into local and global components in a controlled manner. The local evolution can then be chosen conveniently to simplify the analysis. For instance, here we show how to apply the method to separate the harmonic interactions and a dissipation term generated by the noise into explicitly solvable local dynamics. Although created with perturbation theory in mind, the method itself is non-perturbative. In fact, our main result will assume the exact opposite: we work in the regime in which the noise dominates over the harmonic evolution, as then it will be possible to neglect certain resonant terms which otherwise would require more involved analysis.

The method of splitting is quite generic but in the end it only amounts to reorganization of the original dynamics. Therefore, it is important to demonstrate that it can become a useful tool also in practice. As a test case, we study here the velocity flip model with pinning and in the regime where the noise dominates, i.e., the flipping rate is high enough. The ultimate goal is to prove that this system thermalizes for any sensible initial data: after some initial time \(t_0\) any local correlation function, i.e., an expectation value of a polynomial of positions and velocities of particles microscopically close to some given point, is well approximated by the corresponding expectation taken over some statistical equilibrium (thermal) ensemble. The thermalization time \(t_0\) may depend on the initial data and on the
system size but for systems with “normal” transport this time should be less than diffusive, $t_0 \ll L^2$, where $L$ denotes the length of the chain.

We do not have a proof of such a strong statement yet and we only indicate how the present methods should help in arriving at such conclusions. Instead of the full local statistics, we focus here on the time evolution of the average kinetic temperature profile, the observables $\langle p(t)^2 \rangle_x$ where $p(t)_x$ denotes the momentum of the particle at the site $x$ at time $t$. The momentum is a random variable whose value depends both on the realization of the flips and on the distribution of the initial data at $t = 0$. We use $\langle \cdot \rangle$ to denote the corresponding expectation values. We prove here that for a large class of harmonic interactions with pinning, the average kinetic temperature profile does thermalize and its evolution will follow the time evolution dictated by the dynamic Fourier’s law as soon as a thermalization period $t_0 = O(L^{2/3}) \ll L^2$ has passed.

As mentioned above, the velocity flip model has been studied before, using several different methods from numerical to mathematically rigorous analysis. In [4], it was proven that every translation invariant stationary state of the infinite chain with a finite entropy density is given by a mixture of canonical Gibbs states, hence with temperature as the sole parameter. This was shown to hold even when fairly generic anharmonic interactions are included. Since the dynamics conserves total energy, this provides strong support to the idea that energy is the only ergodic variable in the velocity flip model with pinning.

Results from numerical analysis of the velocity flip model are described in [5]. There the covariance of the nonequilibrium steady state of a chain with Langevin heat baths attached to both ends was analyzed, and it was observed that the second order correlations in the steady state coincide with those of a similar, albeit more strongly stochastic, model of particles coupled everywhere to self-consistently chosen heat baths [6]. Hence, in its steady state the stationary Fourier’s law is satisfied with an explicit formula for the thermal conductivity; the full mathematical treatment of the case without pinning is given in [7].

It was later proven that, unlike the self-consistent heat bath model, the velocity flip model satisfies also the dynamic Fourier’s law. This was postulated in [8, 9], based on earlier mathematical work on similar models by Bernardin and Olla (see e.g. [10, 11]), and it was later proven by Simon in [12]. (Although the details are only given for the case without pinning, it is mentioned in the Remark after Theorem 1.2 that the proofs can be adapted to include interactions with pinning.) Also the structure of steady state correlations and energy fluctuations are discussed in [9] with supporting numerical evidence presented in [8]. For a more general explanatory discussion about hydrodynamic fluctuation theory, we refer to a recent preprint by Spohn [2].

The strategy for proving the hydrodynamic limit in [12] was based on relative entropy methods introduced by Yau [13] and Varadhan [14]; see also [15] and [16] for more references and details. There one begins by assuming that the initial state is close to a local thermal equilibrium (LTE) state, which allows for unique definition of the initial profiles of the hydrodynamic fields. One considers the relative entropy density (i.e., entropy divided by the volume) of the state evolved up to time $t$ with respect to a local equilibrium state constructed from the hydrodynamic fields at time $t$, and the goal is to show that the entropy density approaches zero in the infinite volume limit.
In the present work we improve on the result proven in [12] in two ways. We prove that it is not necessary to assume that the initial state is close to an LTE state; indeed, our main theorem is applicable for arbitrary deterministic initial data, including those in which just one of the sites carries energy. Instead, we allow for an initial thermalization period—infinitesimal on the diffusive time scale—and only after the period has passed is the evolution of the temperature profile shown to follow a continuum heat equation. In particle systems directly coupled to diffusion processes similar results have been obtained before: for instance, in the references [17, 18] a hydrodynamic limit is proven assuming only convergence of initial data. However, as discussed in Remarks 3.5 of [18], even assuming a convergence restricts the choice of initial data but we do not need to do it here.

Secondly, we show that the Fourier’s law has a version involving a lattice diffusion kernel for which the temperature profile is well approximated by its macroscopic value at every lattice site and at every time after the thermalization period. This improves on the standard estimates which only imply that the macroscopic averages of the two profiles coincide in the limit \( L \to \infty \). As shown in [19], it is sometimes possible to use averaging over smaller regions, of diameter \( O(L^a) \), \( 0 < a < 1 \), but it is not easy to see how relative entropy alone could be used to control local microscopic properties of the solution. The precise statements and assumptions for our main results are given in Theorem 4.4 and Corollary 4.5 in Sec. 4.

However, in two respects our result is less informative than the one in [12]. Firstly, we only describe the evolution of the average temperature profile, whereas the relative entropy methods produce statements which describe the hydrodynamic limit profiles in probability. Secondly, we do not prove here that the full statistics can be locally approximated by equilibrium measures, although the estimate for the temperature profile does indicate that this should be the case. The thermalization of the other degrees of freedom is only briefly discussed in Sec. 5 where we introduce a local version of the dynamic replica method.

The paper is organized as follows: We recall the definition of the velocity flip model in Sec. 2 and introduce various related notations there. The first version of the new tool, called global dynamic replica method, is described in Sec. 3 where we also discuss how it might be applied to prove global equilibration for this model. This discussion, as well as the one involving the local dynamic replica method in Sec. 5, is not completely mathematically rigorous, for instance, due to missing regularity assumptions. We have also included a discussion about applications of the dynamic replica method to other models in Sec. 6. To illustrate the expected differences to the present case, we briefly summarize there the changes occurring when the velocity flips are replaced by an anharmonic onsite potential.

The main mathematical content is contained in Sec. 4 where the global replica equations are applied to provide a rigorous analysis of the time evolution of the kinetic temperature profile, with the above mentioned Theorem 4.4 and Corollary 4.5 as the main goals of the section. We have included some related but previously known material in two Appendices. Appendix A concerns the explicit solution of the local dynamic semigroup, and in Appendix B we derive the main properties of the Green’s function solution of the renewal equation describing the evolution of the temperature profile.
2 Velocity flip model on a circle

Mainly for notational simplicity, we consider here only one-dimensional periodic crystals, i.e., particles on a circle. For $L$ particles we parametrize the sites on the circle by

$$\Lambda_L := \left\{ \frac{-L - 1}{2}, \ldots, \frac{L - 1}{2} \right\}, \quad \text{if } L \text{ is odd}, \quad (2.1)$$

$$\Lambda_L := \left\{ \frac{-L}{2} + 1, \ldots, \frac{L}{2} \right\}, \quad \text{if } L \text{ is even}. \quad (2.2)$$

Then always $|\Lambda_L| = L$ and $\Lambda_L \subseteq \Lambda_{L'}$ if $L \leq L'$. In addition, for odd $L$, we have $\Lambda_L = \{ n \in \mathbb{Z} \mid |n| < \frac{L}{2} \}$. We use periodic arithmetic on $\Lambda_L$, setting $x' + x := (x' + x) \mod \Lambda_L$ for $x', x \in \Lambda_L$. Sometimes we will need lattices of several different sizes simultaneously, and to stress the length of the cyclic group, we then employ the notation $[x' + x]_L$ for $x' + x$. Also, we use $-x$ to denote $[0 - x]_L$.

The particles are assumed to interact via linear forces with a finite range. The forces are determined by a map $\Phi : \mathbb{Z} \to \mathbb{R}$ which we assume to be symmetric, $\Phi(-x) = \Phi(x)$. We choose $r_\Phi$ to be odd and assume that $\Phi(x) = 0$ for all $|x| \geq r_\Phi/2$. Then the support of $\Phi$ lies in $\Lambda_{r_\Phi}$. The forces are assumed to be stable and pinning, i.e., the discrete Fourier transform of $\Phi$ is required to be strictly positive. The square root of the Fourier transform determines the dispersion relation $\omega : \mathbb{R} \to \mathbb{R}$ which is a smooth function on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ with $\omega_0 := \min_{k \in \mathbb{T}} \omega(k) > 0$. We define the corresponding periodic interaction matrices $\Phi_L \in \mathbb{R}^{\Lambda_L \times \Lambda_L}$ on $\Lambda_L$ by setting

$$(\Phi_L)_{x',x} := \Phi([x' - x]_L), \quad \text{for all } x', x \in \Lambda_L. \quad (2.3)$$

This clearly results in a real symmetric matrix.

Fourier transform $\mathcal{F}_L$ maps functions $f : \Lambda_L \to \mathbb{C}$ to $\hat{f} : \Lambda_L^* \to \mathbb{C}$, where $\Lambda_L^* := \Lambda_L/L \subset (-\frac{1}{2}, \frac{1}{2}]$ is the dual lattice and for $k \in \Lambda_L^*$ we set

$$\hat{f}(k) = \sum_{x \in \Lambda_L} f(x) e^{-i2\pi k \cdot x}. \quad (2.4)$$

The formula holds in fact for all $k \in \mathbb{Z}/L$, in the sense that the right hand side is then equal to $\hat{f}(k \mod \Lambda_L^*)$, i.e., it coincides with the periodic extension of $\hat{f}$. The inverse transform $\mathcal{F}_L^{-1} : g \mapsto \tilde{g}$ is given by

$$\tilde{g}(x) = \int_{\Lambda_L^*} dk \, g(k) e^{i2\pi k \cdot x}, \quad (2.5)$$

where we use the convenient shorthand notation

$$\int_{\Lambda_L^*} dk \cdots = \frac{1}{|\Lambda_L^*|} \sum_{k \in \Lambda_L^*} \cdots. \quad (2.6)$$

With the above conventions, for any $L \geq r_\Phi$ we have

$$(\mathcal{F}_L \Phi_L \mathcal{F}_L^{-1})_{k',k} := \omega(k)^2 \delta_L(k' - k), \quad \text{for all } k', k \in \Lambda_L^*, \quad (2.7)$$
where $\delta_L$ is a “discrete $\delta$-function” on $\Lambda_L^*$, defined by
\[
\delta_L(k) = |\Lambda_L| \mathbb{1}(k = 0), \quad \text{for } k \in \Lambda_L^*.
\] (2.8)

Here, and in the following, $\mathbb{1}$ denotes the generic characteristic function: $\mathbb{1}(P) = 1$ if the condition $P$ is true, and otherwise $\mathbb{1}(P) = 0$.

We assume all particles to have the same mass, and choose units in which the mass is equal to one. The linear forces on the circle are then generated by the Hamiltonian
\[
H_L(X) := \sum_{x \in \Lambda_L} \frac{1}{2} (X_x^2)^2 + \sum_{x', x \in \Lambda_L} \frac{1}{2} X_{x'} X_x^1 \Phi([x' - x]_L) = \frac{1}{2} X^T \mathcal{G}_L X,
\] (2.9)
\[
\mathcal{G}_L := \begin{pmatrix} \Phi_L & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{(2\Lambda_L) \times (2\Lambda_L)},
\] (2.10)
on the phase space $X \in \Omega := \mathbb{R}^{\Lambda_L} \times \mathbb{R}^{\Lambda_L}$. The canonical pair of variables for the site $x$ are the position $q_x := X_x^1$, and the momentum $p_x := X_x^2$. The Hamiltonian evolution is combined with a velocity-flip noise. The resulting system can be identified with a Markov process $X(t)$ and the process generates a Feller semigroup on the space of observables vanishing at infinity, see [7, 12] for mathematical details. Then for $t > 0$ and any $F$ in the domain of the generator $\mathcal{L}$ of the Feller process the expectation values of $F(X(t))$ satisfy an evolution equation
\[
\partial_t \langle F(X(t)) \rangle = \langle (\mathcal{L} F)(X(t)) \rangle,
\] (2.11)
where $\mathcal{L} := \mathcal{A} + \mathcal{S}$, with
\[
\mathcal{A} := \sum_{x \in \Lambda_L} \left( X_x^2 \partial X_x^1 - (\Phi_L X_x^1) \partial X_x^2 \right),
\] (2.12)
\[
(SF)(X) := \frac{\gamma}{2} \sum_{x \in \Lambda_L} (F(S_{x,0} X) - F(X)), \quad \gamma > 0,
\] (2.13)
\[
(S_{x,0} X)^i_x := \begin{cases} -X_x^i & \text{if } i = 2 \text{ and } x = x_0, \\ X_x^i & \text{otherwise}. \end{cases}
\] (2.14)

We consider the time evolution of the moment generating function
\[
f_t(\xi) := \langle e^{i\xi \cdot X(t)} \rangle,
\] (2.15)
where $\xi$ belongs to some fixed neighborhood of 0. Although the observable $X \mapsto e^{i\xi \cdot X}$ does not vanish at infinity, $f_t(\xi)$ is always well defined, and we assume that it satisfies the evolution equation dictated by (2.11). This will require some additional constraints on the distribution of initial data, but for instance it should suffice that all second moments of $X(0)$ are finite. (The existence of initial second moments will also be an assumption for our main theorem.)
Ultimately, the goal is to prove thermalization, i.e., the appearance of local thermal equilibrium. More precisely, we would like to prove that after a thermalization period the local restrictions of the generating functional are well approximated by mixtures of equilibrium expectations. To do such a comparison, the first step is to classify the generating functions of equilibrium states. We start with a heuristic argument based on ergodicity which gives a particularly appealing formulation for the present case in which the canonical Gibbs states are Gaussian measures.

Suppose that the evolution of our finite system is ergodic, with energy as the only ergodic variable. Then for any invariant measure \( \bar{\mu} \) there is a Borel probability measure \( \nu \) on \( \mathbb{R} \) such that for any \( g \in L^1(\bar{\mu}) \) we have

\[
\int \bar{\mu}(dX)g(X) = \int \nu(dE) \int \frac{dX}{Z^\text{me}_E} \delta(E - H_L(X))g(X),
\]

where \( Z^\text{me}_E := \int dX \delta(E - H_L(X)) \) denotes the microcanonical partition function. (Details about mathematical ergodic theory can be found for instance from \[20\].) Hence, if \( \mu_0 \) is an initial state which converges towards a steady state \( \bar{\mu} \), we have for any observable \( g \)

\[
\lim_{t \to \infty} \int \mu_0(dX)g(X) = \int \nu(dE) \int \frac{dX}{Z^\text{me}_E} \delta(E - H_L(X))g(X).
\]

Applying this for \( g(X) = \varphi(H_L(X)) \), \( \varphi \) continuous with a compact support, we find by conservation of \( H_L \) that \( \int \mu_0(dX)\varphi(H_L(X)) = \int \mu_t(dX)\varphi(H_L(X)) = \int \nu(dE)\varphi(E) \). Therefore, we can formally identify \( \nu(dE) = dE \int \mu_0(dX)\delta(E - H_L(X)) \).

Finally, we can rewrite the somewhat unwieldy microcanonical expectations in terms of the canonical Gaussian measures by using the representation \( \int dX \delta(E - H_L(X))g(X) = \int^{\beta + i\infty} d\xi e^{\beta \xi} \int dX e^{-zH_L(X)}g(X) \) which should be valid for all sufficiently nice \( g \) and \( \beta > 0 \). Applying this representation to \( g(X) = e^{\xi X} \) yields \( \int^{\beta + i\infty} d\xi e^{\beta \xi} Z^\text{can}_z e^{-\frac{1}{2} \xi^T \xi\bar{G}^{-1}} \), where the canonical partition function is \( Z^\text{can}_z := \int dX e^{-zH_L(X)} \). Hence, we arrive at the conjecture that for all sufficiently nice initial data \( \mu_0 \)

\[
\lim_{t \to \infty} f_t(\xi) = \int^{\beta + i\infty} d\xi e^{-\frac{1}{2} \xi^T \xi\bar{G}^{-1}} \int \mu_0(dX) \frac{\int dX' e^{z(H_L(X) - H_L(X'))}}{\int dX' \delta(H_L(X) - H_L(X'))}.
\]

By a change of variables to \( z^{-1} \) and using the fact that \( f_t(0) = 1 \), the limit function can also be represented in the form \( \int \bar{\nu}(d\lambda) e^{-\lambda \frac{1}{2} \xi^T \xi\bar{G}^{-1}} \) where the integral is taken around a circle in the right half of the complex plane and \( \bar{\nu} \) is a complex measure satisfying \( \int \bar{\nu}(d\lambda) = 1 \).

For any fixed initial data \( X_0 \) with energy \( E = H_L(X_0) \), there is a natural choice for the parameter \( \beta \) as the unique solution to the equation \( E = \langle H_L \rangle_\beta^\text{can} \). This choice coincides with the unique saddle point for \( \beta = 0 \) on the positive real axis, i.e., it is the only \( \beta > 0 \) for which \( \partial_\beta \ln g(\beta) \big|_{\beta = 0} = 0 \) with \( g(\beta) := \int dX e^{z(E - H_L(X'))} / \int dX' \delta(E - H_L(X')) \). Then also \( \partial_\beta^2 \ln g(\beta) = \text{Var}_\beta(H_L) > 0 \) and hence the integration path in (2.18) follows the path of steepest descent through the saddle point. As the energy variance typically is proportional to the volume, the integrand should be concentrated to the real axis, with a standard deviation \( O(L^{1/2}) \). Hence, for fixed initial data and large \( L \) we would expect to have here equivalence of ensembles in the form \( \lim_{t \to \infty} f_t(\xi) \approx e^{-\frac{1}{2} \xi^T \xi\bar{G}^{-1}} \).
3 Global dynamic replica method

In order to treat the local dynamics independently, we replicate the whole chain at each lattice site, and transform the evolution equation into a new form by selecting some terms to act on the replicated direction. We use a generating function with variables $\zeta \in \mathbb{R}^{2\times \Lambda_L \times \Lambda_L}$, where each $\zeta_{x,y}^i$ controls the random variable $X_{x+y}(t)$, and we think of $x$ as the original site and $y$ as the position in its “replica”. Explicitly, we study the dynamics of the generating function

$$h_t(\zeta) := \langle e^{i\sum_{x,y} \zeta_{x,y}^i X(t)^{x+y}} \rangle$$  \hspace{1cm} (3.1)$$

where the mean is taken over the distribution of $X(t)$ at some given initial distribution $\mu_0$ of $X(0)$. Clearly, $h_t$ depends on $\zeta$ only via the combinations $\sum_y \zeta_{x-y,y}^i$, $x \in \Lambda_L$. If $h_t$ is known, the local statistics at $x_0 \in \Lambda_L$ for some given time $t$ can be obtained directly from its restriction $f_{x_0}(\xi) := h_t(\zeta[\xi, x_0])$ where we set $\zeta[\xi, x_0] := 1(x=x_0, y\in \Lambda_{R_0})\zeta_y^i$ for $\xi \in \mathbb{R}^{2\times \Lambda_{R_0}}$. We assume $R_0 \leq L$ but otherwise it can be chosen independently of $L$. The parameter $R_0$ determines which neighboring particles are chosen to belong to the same “local” neighborhood.

By (2.11), the generating function $h_t$ satisfies the evolution equation

$$\partial_t h_t(\zeta) = \sum_{x,y \in \Lambda_L} \zeta_{x,y}^i (e^{iX_{x+y}^2 Y_t}) - \sum_{x,y,z \in \Lambda_L} \zeta_{x,y}^i (e^{iX_{x+y+z}^2 Y_t})$$

$$+ \frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} (h_t(\sigma_{x_0} \zeta) - h_t(\zeta)), \hspace{1cm} \text{where we use the random variable } Y_t := \sum_{x,y,i} \zeta_{x,y}^i X(t)^{x+y} \text{ and have defined for } x_0 \in \Lambda_L$$

$$\sigma_{x_0} \zeta_{x,y}^i := \begin{cases} -\zeta_{x,y}^i, & \text{if } i = 2 \text{ and } [x+y]_L = x_0; \\ \zeta_{x,y}^i, & \text{otherwise.} \end{cases} \hspace{1cm} \text{(3.3)}$$

The equation can be closed by using the identity

$$\partial_{\zeta_{x,y}^i} h_t(\zeta) = (e^{iX_{x+y}^2 Y_t}). \hspace{1cm} (3.4)$$

There is some arbitrariness in the resulting equation: (3.4) is true for all $x, y \in \Lambda_L$, but the right hand side depends only on $x+y$. We choose here to use it as indicated by the choice of summation variables in (3.2). Since in the summand always $(\Phi_L)_{x+y,x+z} = (\Phi_L)_{yz}$ this results in the evolution equation

$$\partial_t h_t(\zeta) = - (\mathcal{M}_0 \zeta) \cdot \nabla h_t(\zeta) + \frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} (h_t(\sigma_{x_0} \zeta) - h_t(\zeta)), \hspace{1cm} \text{(3.5)}$$

where $\nabla h$ denotes the standard gradient, i.e., it is a vector whose $(i, x, y)$-component is $\partial_{\zeta_{x,y}^i} h$, and

$$\mathcal{M}_\gamma := \bigoplus_{x_0 \in \Lambda_L} M_{\gamma}^{(x_0)} \gamma, \quad \text{(3.6)}$$

$$\mathcal{M}_\gamma^{(x_0)} \zeta_{x,y}^i := 1(x=x_0) (M_{\gamma} \zeta_{x_0})_y^i$$

$$M_{\gamma} := \begin{pmatrix} 0 & \Phi_L \\ -1 & \gamma I \end{pmatrix}.$$
Since $\frac{1}{2} \sum_x (1 - \sigma_x) = P^{(2)} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, this can also be written as

$$
\partial_t h_t(\zeta) = -(M_\gamma \zeta) \cdot \nabla h_t(\zeta) + \frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} (h_t(\sigma_{x_0} \zeta) - h_t(\zeta) - (\sigma_{x_0} - 1)\zeta \cdot \nabla h_t(\zeta)).
$$

(3.7)

For any $h_t$ resulting from the replication procedure, we obviously have $\partial_{z_y} h_t(\zeta) = \partial_{z'_y} h_t(\zeta)$ whenever $x + y = x' + y'$. Hence, by Taylor expansion with remainder up to second order

$$
\frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} (h_t(\sigma_{x_0} \zeta) - h_t(\zeta) - (\sigma_{x_0} - 1)\zeta \cdot \nabla h_t(\zeta))
$$

$$
= 2\gamma \sum_{x_0 \in \Lambda_L} \int_0^1 dr (1 - r) \sum_{x' y' \in \Lambda_L} \mathbb{1}(x_0 = x + y) \mathbb{1}(x_0 = x' + y')
$$

$$
\times \zeta^2 (\partial_{z_y}^2 \partial_{z'_y}^2 h_t)(\zeta - r(1 - \sigma_{x_0})\zeta)
$$

$$
= 2\gamma \int_0^1 dr (1 - r) \sum_{x_0 \in \Lambda_L} (\partial_{z_y}^2 h_t)(\zeta - r(1 - \sigma_{x_0})\zeta) \left( \sum_{x y} \mathbb{1}(x_0 = x + y) \zeta^2 \right) \frac{1}{2} (3.8)
$$

Now for any continuously differentiable function $t \mapsto \zeta_t$

$$
h_t(\zeta_0) - h_0(\zeta_t) = -\int_0^t ds \partial_s (h_{t-s}(\zeta_s)) = \int_0^t ds \left( \dot{h}_{t-s}(\zeta_s) - \dot{\zeta}_s \cdot \nabla h_{t-s}(\zeta_s) \right).
$$

(3.9)

Setting $\zeta_s := e^{-sM_\gamma} \zeta$ and $Q_{r,x_0} := 1 - r(1 - \sigma_{x_0})$ thus yields the “Duhamel formula”

$$
h_t(\zeta) = h_0(\zeta_t) + \int_0^t ds \frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} (h_{t-s}(\sigma_{x_0} \zeta_s) - h_{t-s}(\zeta_s) - (\sigma_{x_0} - 1)\zeta_s \cdot \nabla h_{t-s}(\zeta_s))
$$

$$
= h_0(\zeta_t) + 2\gamma \int_0^t ds \int_0^1 dr (1 - r) \sum_{x_0 \in \Lambda_L} \left( \sum_{y \in \Lambda_L} \zeta_s^2 (x_0 - y,y)^2 \right) (\partial_{z_y}^2 h_{t-s})(Q_{r,x_0} \zeta_s).
$$

(3.10)

In this formula, the replica dynamics has been exponentiated in the operator semigroup $e^{-sM_\gamma}$. No approximations have been made in the derivation of the formula, but to show that its solutions, under some natural assumptions, are unique and correspond to LTE states does not look straightforward. We do not attempt to do it here. Instead, the formula will be used in the next section to derive a closed evolution equation for the temperature profile.

We conclude the section by showing that Eq. (3.10) is consistent with the discussion in Sec. 2. Suppose $\nu$ is a complex bounded measure on $\beta + i\mathbb{R}$, for some $\beta > 0$. Then $h_t(\zeta) := \int \nu(dz) e^{-\frac{\beta}{2} \zeta^T G_\xi^{-1} \zeta}$, with $G_\xi = \sum_y \zeta^2$, solves (3.10). To see this, first note that the first two terms in the integrand cancel, since $(\sum_y \zeta^2_{x-y,y})^2 = (\sum_y \zeta^2_{x-y,y})^2$ and thus $h_t(\sigma_{x_0} \zeta) = h_t(\zeta)$. Therefore, the value of the integral is equal to $\int_0^t ds \gamma P^{(2)} \zeta_s \cdot \nabla h_0(\zeta_s)$. Here $P^{(2)} \zeta_s \cdot \nabla h_0(\zeta_s) = -\int \nu(dz) \frac{\beta}{2} e^{-\frac{\beta}{2} \zeta^T G_\xi^{-1} \zeta} \sum_x (\xi_s^2)^2$, with $(\xi_s^2) = \sum_{t',y'} (e^{-sM_\gamma})_{t'y'} \zeta^2_{x-y,y'}$.
(e^{-sM_\gamma} \xi_x^i) where in the second equality we have used the periodicity of $M_\gamma$. However, then $\partial_s (\xi_x^i G^{-1}_L \xi_x^i) = -\xi_x^T (M_\gamma G^{-1}_L M_\gamma) \xi_x = -2\gamma \sum_x ((\xi_x^i)^2)^2$, and thus $\int_0^t ds \gamma P^{(2)}(\xi_s) \cdot \nabla h_0(\xi_s) = -\int_0^t ds \partial_s h_0(\xi_s) = -h_0(\xi_t) + h_t(\xi)$. Hence, the functions of $\xi$ defined by setting $\xi_x^i = \sum_y \xi_x^{i-y,y}$ on the right hand side of (2.18) are solutions to the equation (3.10). To check that energy is the only ergodic variable one would need to prove that there are no other time-independent solutions. We postpone the analysis of this question to a future work, although by the results proven in [4] it would seem to be a plausible conjecture.

4 Thermalization of the temperature profile

Since the “replicated” generating function satisfies (3.10), a direct differentiation results in an evolution equation for the kinetic temperature profile $T_{t,x} := \langle (X(t)^2)^2 \rangle = -\partial^2_{\xi^2_0} h_t(0)$. We obtain

$$T_{t,x} = (e^{-tM_\gamma} \Gamma_x e^{-tM_\gamma} x_{00}^2) + 2\gamma \int_0^t ds \sum_{y \in \Lambda_L} ((e^{-sM_\gamma} y_{00}^2) T_{t-s,x+y} , (4.1)$$

where each $(\Gamma_x y^{i'_y}_y) := -\partial_{\xi^{i'_y} y} \partial_{\xi^i y} h_0(0) = \langle X(0)^y_{x+y,x} X(0)^{i'}_{x+y,y} \rangle$ is a symmetric matrix obtained by a periodic translation of the matrix of the initial second moments. The final sum can be transformed into a standard convolution form by changing the summation variable $y \rightarrow -y$. This yields

$$T_{t,x} = g_{t,x} + \int_0^t ds \sum_{y \in \Lambda_L} p_{s,y} T_{t-s,x-y} , (4.2)$$

where for $t \geq 0$, $x \in \Lambda_L$, the “source term” $g$ and the “memory kernel” $p$ are given by

$$g_{t,x} := (e^{-tM_\gamma} \Gamma_x e^{-tM_\gamma} x_{00}^2) = \sum_{i' y' y} A_{t,x}^{i'} A_{t,y'}^{i'} (\Gamma_x y^{i'_y}_y) , (4.3)$$

$$p_{t,x} := 2\gamma (A_{t,x}^2)^2 , (4.4)$$

with the following shorthand

$$A_{t,x}^{i} := (e^{-tM_\gamma} i_{x0}^2) . (4.5)$$

We will prove later that $p_{t,x} = p_{t,x} \geq 0$ and that $\int_0^\infty dt \sum_x p_{t,x} = 1$. Hence, mathematically the equation (4.2) has the structure of a generalized renewal equation. Renewal equations have bounded solutions in great generality [21, Theorem 9.15]. The problem is closely connected to Tauberian theory; the classical paper by Karlin [22] contains a discussion and detailed analysis of the standard case. Unfortunately, most of these results are not of direct use here since they do not give estimates for the speed of convergence towards the asymptotic value and thus cannot be used for estimating the $L$-dependence of the asymptotics. Nevertheless, the standard methods can be applied to an extent also
in the present case: in Appendix B we give the details for the existence and uniqueness of solutions to (4.2) and derive an explicit representation of the solutions using Laplace transforms.

The analysis relies on upper and lower bounds for the tail behavior of \( p_{t,x} \). These follow from explicit formulae for the solutions of the semigroup \( e^{-tM_\gamma} \) derived in Appendix A. In particular, we have for any \( k \in \Lambda^*_L \)

\[
\hat{A}_1^j(k) = \frac{1}{2u(k)} \sum_{\sigma = \pm 1} \sigma \omega(k)^2 e^{-t\mu_\sigma(k)}, \quad \hat{A}_2^j(k) = \frac{1}{2u(k)} \sum_{\sigma = \pm 1} \sigma \mu_\sigma(k) e^{-t\mu_\sigma(k)}, \quad (4.6)
\]

where

\[
u(k) := \sqrt{(\gamma/2)^2 - \omega(k)^2}, \quad \mu_\sigma(k) := \frac{\gamma}{2} + \sigma u(k). \quad (4.7)
\]

In principle, the formulae should only be used if \( \omega(k) < \gamma/2 \) which implies \( u(k) > 0 \). However, they also hold for all other values of \( k \) if extended using the following “analytic continuation”: if \( \omega(k) > \gamma/2 \), we set \( u(k) = i\sqrt{\omega(k)^2 - (\gamma/2)^2} \) and the values for case \( \omega(k) = \gamma/2 \) agree with the limit \( u(k) \to 0^+ \). Since we consider here the case in which the noise dominates, only the expressions in (4.6) will be needed in the following.

We begin by summarizing the regularity assumptions about the free evolution, already discussed in Sec. 2. Without additional effort, we can relax the assumption of \( \Phi \) having a finite support to mere exponential decay. There are then several possibilities for fixing the finite volume dynamics; here, we set \( \omega(k; L) := \sqrt{\hat{\Phi}(k)} \) for \( k \in \Lambda^*_L \). Then (4.6) is still pointwise valid for the Fourier transform of the semigroup generated by \( M_\gamma \).

**Assumption 4.1** We assume that the map \( \Phi : \mathbb{Z} \to \mathbb{R} \) has all of the following properties.

1. (exponential decay) There are \( C, \delta > 0 \) such \( |\Phi(x)| \leq Ce^{-\delta|x|} \) for all \( x \in \mathbb{Z} \).
2. (symmetry) \( \Phi(-x) = \Phi(x) \) for all \( x \in \mathbb{Z} \).
3. (pinning) There is \( \omega_0 > 0 \) such that \( \hat{\Phi}(k) \geq \omega_0^2 \) for all \( k \in \mathbb{T} \).

The assumptions imply that the Fourier transform of \( \Phi \) can be extended to an analytic map \( z \mapsto \sum_x e^{-i2\pi x z} \Phi(x) \) on the strip \( \mathbb{R} + i(-\delta, \delta)/(2\pi) \), and the extension is 1-periodic. By continuity and periodicity of \( \hat{\Phi} \), we can then find \( \varepsilon > 0 \) such that \( \text{Re} \hat{\Phi}(z) > 0 \) on the strip \( \mathbb{R} + i(-\varepsilon, \varepsilon) \). Therefore, the infinite volume dispersion relation has the following regularity properties:

**Corollary 4.2** Assume \( \Phi \) satisfies the assumptions in 4.1. Then \( \omega(k) := \sqrt{\hat{\Phi}(k)} \), \( k \in \mathbb{T} \), defines a smooth function on \( \mathbb{T} \) which satisfies \( \omega(k) \geq \omega_0 \) and \( \omega(-k) = \omega(k) \) for all \( k \in \mathbb{T} \). In addition, there is \( \varepsilon > 0 \) such that \( \omega \) has an analytic, 1-periodic continuation to the region \( \mathbb{R} + i(-\varepsilon, \varepsilon) \).
I(k₀)

Figure 1: Plots of the function $I(k₀)$ defined by the left hand side of the nondegeneracy condition in (4.8) for $γ = 6$ and two different dispersion relations. The left plot is for the standard nearest neighbor case, $ω(k) = \sqrt{1 + 4 \sin²(πk)}$, while the right one depicts a degenerate next-to-nearest neighbor case, with $ω(k) = \sqrt{1 + 4 \sin²(2πk)}$. The plots have been generated by numerical integration using Mathematica.

From now on we assume that $Φ$ satisfies Assumption 4.1 and $γ > 0$ is some fixed flipping rate. This already fixes the functions $\hat{A}^i$ defined above. However, here we aim at convenient exponential bounds for the errors from diffusive evolution of the temperature profile. This requires to rule out resonant behavior, which can be achieved if the noise flipping rate is high enough and the dispersion relation satisfies a certain integral bound excluding degenerate behavior. Explicitly, we only consider $γ$ and $Φ$ satisfying the following:

**Assumption 4.3** Suppose that $Φ$ satisfies the assumptions in 4.1 and $γ > 0$ is given such that:

1. (noise dominates) $γ² > 4 \max_k \hat{Φ}(k)$.
2. (harmonic forces are nondegenerate) For any $ε > 0$ there is $C_ε > 0$ such that

$$\int_0^∞ dt \int d k \left( \hat{A}²_t(k + k₀/2) - \hat{A}²_t(k - k₀/2) \right)^2 ≥ C_ε, \quad \text{whenever } ε ≤ |k₀| ≤ 1/2.$$ (4.8)

The nondegeneracy condition is satisfied by the nearest neighbor interactions, for which $ω(k) = \sqrt{ω₀² + 4 \sin²(πk)}$ with $ω₀ > 0$. This can be proven for instance by relying on the estimate $\partial_k \hat{A}²_t(k) ≥ Cτe^{−tγ} \sin(2πk)$ valid for all $|k| < 1/2$ and large enough $t$. However, the condition fails for the degenerate next-to-nearest neighbor coupling which skips over the nearest neighbors: then $ω(k) = \sqrt{ω₀² + 4 \sin²(2πk)}$ and thus for $k₀ = 1/2$ we have $ω(k + k₀/2) = ω(k - k₀/2)$ for all $k$, hence the integral in (4.8) evaluates to zero at $k₀ = 1/2$. Instead of including a formal proof of these statements, we have depicted the values of the above integrals for one choice of parameters in Fig. 1.

We prove in this section that these assumptions suffice to have the following pointwise behavior of the temperature profile.
Theorem 4.4 Suppose $\Phi$ and $\gamma$ satisfy the conditions in Assumption 4.3. Then there is $L_0 \in \mathbb{N}$ such that equation (4.2) has a unique continuous solution $T_{t,x}$ for every $L \geq L_0$ whenever all second moments of the initial field $X(0)$ exist. Let $E := |\Lambda_L|^{-1} \langle H_L(X) \rangle < \infty$ denote the total energy density. Then there are constants $C, d > 0$, independent of the initial data and of $L$, such that for this solution

$$|T_{t,x} - E| \leq C E e^{\tau_0} e^{-C t L^{-3/2}},$$

for all $t \geq 0$ and $x \in \Lambda_L$. In addition, we can choose $C$ and define $\tau_x \in \mathbb{R}$ and $\hat{p}_x \geq 0$, $x \in \Lambda_L$, so that for all $t > 0$ and $x \in \Lambda_L$

$$|T_{t,x} - (e^{-tD} \tau)_x| \leq C E L^{t - 3/2},$$

where the operator $D$ is defined by

$$(D \tau)_x := \sum_{y \in \Lambda_L} \hat{p}_y (2 \tau_x - \tau_{x+y} - \tau_{x-y}), \quad x \in \Lambda_L. \tag{4.11}$$

Both $\tau_x$ and $\hat{p}_x$ in the statement have explicit definitions which can be found in the beginning of the proof of the Theorem. To summarize in words, the first of the bounds implies that the temperature profile equilibrates, and the relative error is exponentially decaying on the diffusive time scale, i.e., as $t L^{-2}$ becomes large. The second statement says that solving the “lattice diffusion equation” $\partial T_{t,x} = -(D T_{t,x})_x$ with initial data $T_0 = \tau$ provides an approximation to the temperature profile which is accurate even before the diffusive time scale, for $t \gtrsim L^{2/3}$.

A closer inspection of the proof of the Theorem reveals that the main contribution to the error bound given in (4.10) comes from “memory effects” of the original time evolution. These corrections can estimated using a bound which for large $t$ and $L$ behaves as $\int_0^\infty dk k^2 e^{-t k^2} = O(t^{-3/2})$. The rest of the factors can be uniformly bounded using the total energy, resulting in the bound in (4.10). We do not know if the bound is optimal, although this could well be true for generic initial data. The worst case scenario for thermalization should be given by initial data in which all energy is localized to one site. It would thus be of interest to study the solution of (4.2) with initial data $q(0)_x = 0$ and $p(0)_x = \sqrt{2L} 1(x=0)$ in more detail to settle the issue.

The following corollary makes the connection to diffusion more explicit. Its physical motivation is to show that the Fourier’s law can here be used to predict results from temperature measurements, as soon as these are not sensitive to the lattice structure. Explicitly, we assume that the measurement device detects only the cumulative effect of thermal movement of the particles, say via thermal radiation, and thus can only measure a smeared temperature profile. The smearing is assumed to be linear and given by a convolution with some fixed function $\varphi$ which for convenience we assume to be smooth and rapidly decaying, i.e., that it should belong to the Schwartz space. The Corollary implies that then for large systems it is possible to obtain excellent predictions for future measurements of the temperature profile by first waiting a time $t_0 \gg L^{2/3}$, measuring
the temperature profile, and then using the profile as initial data for the time-dependent Fourier’s law. The diffusion constant $\kappa_L$ of the Fourier’s law depends on the harmonic dynamics and is given by (4.14) below. We prove later, in Corollary 4.7 and Proposition 4.9 that the constant remains uniformly bounded away from 0 and infinity. Hence, the present assumptions are sufficient to guarantee normal heat conduction. We have also computed the values of $\kappa_L$ numerically for a nearest neighborhood interaction and plotted these in Fig. 2. The results indicate that the limit $L \to \infty$ exists and agrees with $\kappa_\infty = \gamma^{-1}/(2 + \omega_0^2 + \omega_0\sqrt{\omega_0^2 + 4})$ which is the value obtained in previous works on this model [8, 9, 12].

**Corollary 4.5** Suppose the assumptions of Theorem 4.4 hold, $L \geq L_0$, and all second moments of the initial field exist. Let $T_{t,x}$ denote the corresponding solution to (4.2) and $E := \|\Lambda_L^{-1}(H_L)\|$ the energy density. For any kernel function $\varphi \in \mathcal{S}(\mathbb{R})$ and initialization time $t_0 > 0$, define the corresponding observed temperature profile by

$$T^{(obs)}(t, \xi) := \sum_{y \in \mathbb{Z}} \varphi(\xi - y)T_{t_0 + t, y \mod \Lambda_L}, \quad t \geq 0, \quad \xi \in LT.$$  (4.12)

Let the predicted temperature profile be defined as the solution of the diffusion equation on the circle $LT$ with initial data $T^{(obs)}(0, \cdot)$, i.e., let $T^{(pred)} \in C^2([0, \infty) \times LT)$ be the unique solution to the Cauchy problem

$$\partial_t T^{(pred)}(t, \xi) = \kappa_L \partial_\xi^2 T^{(pred)}(t, \xi), \quad T^{(pred)}(0, \xi) = T^{(obs)}(0, \xi)$$  (4.13)
where $t \geq 0$ and $\xi \in L_T$, and the diffusion constant is defined by

$$
\kappa_L := \sum_{y \in \Lambda_L} y^2 \tilde{p}_y > 0.
$$

(4.14)

Then there is a constant $C > 0$, independent of the initial data and of $L$, $\varphi$ and $t_0$, such that

$$
|T^{(pred)}(t, \xi) - T^{(obs)}(t, \xi)| \leq C E L t_0^{-3/2} \left( \|\varphi\|_1 + \sup_{\xi} \sum_{y \in \mathbb{Z}} |\varphi(\xi - y)| \right) + C E \sum_{|n| \geq L/2} |\hat{\varphi}(n/L)|,
$$

(4.15)

for all $t \geq 0$ and $\xi \in L_T$. In particular, if $\varphi$ is a “macroscopic averaging kernel” and satisfies additionally $\varphi \geq 0$, $\int_{\mathbb{R}} dx \varphi(x) = 1$ and $\hat{\varphi}(p) = 0$ for all $|p| \geq \frac{1}{2}$, then for the same $C$ as above

$$
|T^{(pred)}(t, \xi) - T^{(obs)}(t, \xi)| \leq 2C E L t_0^{-3/2}.
$$

(4.16)

The rest of this section is used for proving the above statements. However, as the arguments get somewhat technical and will not be used in the remaining sections, it is possible to skip over the details in the first reading. We begin with a Lemma collecting the main consequences of our assumptions.

**Lemma 4.6** Suppose $\Phi$ and $\gamma$ satisfy the conditions in Assumption 4.3. Use (4.6) to define $\hat{A}_i : [0, \infty) \times T \to \mathbb{R}$ for $i = 1, 2$, and set $\delta_0 = \omega_0^2/\gamma$. Then we can find constants $c_0, c_1, c_2, \gamma_2 > 0$, $t_1 \geq 0$, such that

1. The functions $\hat{A}_i, i = 1, 2$, belong to $C^{(1)}([0, \infty) \times T)$.

2. $|\hat{A}_i(k)|, |\partial_k \hat{A}_i(k)| \leq c_0 e^{-\delta_0 t}$ and $|\partial_k \hat{A}_i(k)| \leq c_0 e^{-\delta_0 t/2}$ for every $i = 1, 2$, $t \geq 0$ and $k \in T$.

3. $-\hat{A}_i(k) \geq c_1 e^{-\gamma t/2}$ for all $t \geq t_1$ and $k \in T$.

4. The functions $\hat{A}_{i,x} := \int_T dk e^{i2\pi x k} \hat{A}_i(k), x \in \mathbb{Z}$, satisfy $|\hat{A}_{i,x}| \leq c_2 e^{-\delta_0 t/2-\gamma_2 |x|}$ for all $i = 1, 2$, $t \geq 0$, $x \in \mathbb{Z}$.

**Proof:** The first item follows straightforwardly from the definitions. As in the statement, set $\delta_0 := \omega_0^2/\gamma > 0$ and recall the functions $u$ and $\mu_\pm$ defined in (4.7). Since $\omega(k)^2/\mu_+(k) = \mu_-(k) < \mu_+(k) \leq \gamma$, we have $\mu_+(k) > \mu_-(k) \geq \delta_0$ for all $k$ and thus a direct computation shows that $c_0$ for the first two upper bounds in item 2 can be found. The bound for $|\partial_k \hat{A}_i(k)|$ follows similarly, using the estimate $te^{-\delta_0 t/2} \leq e^{-1/2}/\delta_0$ and possibly increasing $c_0$ to accommodate the extra factors resulting from taking the derivative, such as $\max_k |\omega(k)| < \infty$. The lower bound in item 3 is a direct consequence of the identity $-\hat{A}_i^2(k) = (2u)^{-1} \mu_- e^{-\mu_-}(1-e^{-t_2 u} \mu_+/\mu_-)$ where $u = u(k) \geq
\(u_0 := \sqrt{(\gamma/2)^2 - \max_k \Phi(k)} > 0\) and \(\mu_- \leq \gamma/2\). (We may define, for instance, \(t_1 := (2u_0)^{-1} \ln(2(\gamma^2/\omega_0^2))\) and \(c_1 := \omega_0^2/(2\gamma^2)\).)

All of the maps \(k \mapsto \hat{A}_t^i(k)\) can be represented as a composition of a function analytic on \(\mathbb{C} \setminus \{0\}\) and the function \(u\) (note that \(\omega(k)^2 = (\gamma/2)^2 - u(k)^2\)). Then, by assumption, \(0 < u_0 \leq u(k) \leq \sqrt{(\gamma/2)^2 - \omega_0^2} \leq \frac{1}{2}\gamma - \delta_0\) for real \(k\), and there is a strip on which \((\gamma/2)^2 - \omega(z)^2\) is an analytic, 1-periodic function. Therefore, we can find \(\varepsilon_0 > 0\) such that \(u(z)\) is an analytic, 1-periodic continuation of \(u\) to a neighborhood of \(U := \mathbb{R} + i[-\varepsilon_0,\varepsilon_0]\) with \(u_0/2 \leq \Re u(z) \leq \frac{1}{2}\gamma - \delta_0/2\). Hence, \(\hat{A}_t^i|_{u-u(z)}\) is also analytic and 1-periodic on \(U\) and it is bounded there by \(4u_0^{-1}(1 + (\gamma/2)^2 + \max_{z \in U} |u(z)|^2)e^{-\delta_0 t/2}\) for \(i = 1, 2, t \geq 0\).

Therefore, Cauchy’s theorem can be used to change the integration contour in the definition of \(\hat{A}_{t,x}^i\) from \([-1/2,1/2]\) to \([-1/2,1/2] + i\text{sign}(x)\varepsilon_0\) without altering the value of the integral. Thus we can define \(\gamma_2 := 2\pi \varepsilon_0\) and find \(c_2 > 0\) independent of \(x, t, i\) such that \(|\hat{A}_{t,x}^i| \leq c_2e^{-\delta_0 t/2 - \gamma_2 |x|}\) for all \(i = 1, 2, t \geq 0, x \in \mathbb{Z}\). This concludes the proof of the Lemma. \(\square\)

**Corollary 4.7** Suppose \(\Phi\) and \(\gamma\) satisfy the conditions in Assumption [4.3] and let \(\delta_0, \gamma_2, t_1\) be constants for which Lemma 4.6 holds. For each \(L \geq 1\) define

\[
p_{t,x} := 2\gamma \left( \int_{\Lambda_L} dk \ e^{-i2\pi k \cdot x} \hat{A}_t^i(k) \right)^2, \quad \rho_t := \sum_{y \in \Lambda_L} p_{t,y},
\]

for \(t \geq 0\) and \(x \in \Lambda_L\). Then there are constants \(C_0, C_1, C_2 > 0,\) all independent of \(L,\) such that

1. \(t \mapsto p_{t,x}\) belongs to \(C^1([0, \infty))\) for all \(x \in \Lambda_L.\)
2. \(p_{t,-x} = p_{t,x}\) for all \(x, t.\)
3. \(p_{t,x} \geq 0\) and \(|\partial tp_{t,x}|, p_{t,x} \leq C_0 e^{-\gamma_2 |x| - \delta_0 t}\) for all \(x \in \Lambda_L\) and \(t \geq 0.\)
4. \(\int_0^\infty dt \sum_{x \in \Lambda_L} p_{t,x} = 1.\)
5. \(\int_0^\infty dt \sum_{x \in \Lambda_L} tp_{t,x} = \gamma^{-1}.\)
6. \(0 \leq \rho_t \leq C_1 e^{-\delta_0 t},\) for all \(t,\) and \(\rho_t \geq C_2 e^{-\gamma t}\) for \(t \geq t_1.\)

**Proof:** The first item follows directly from the corresponding item in Lemma 4.6. Consider then a fixed \(L \geq 1\) and denote \(A_{t,x}^2 := \int_{\Lambda_L} dk \ e^{i2\pi k \cdot x} \hat{A}_t^2(k)\) for \(x \in \Lambda_L, t \geq 0.\) Since \(\hat{A}_t^2(k) \in \mathbb{R}\) with \(\hat{A}_t^2(-k) = \hat{A}_t^2(k),\) we have \(A_{t,x}^2 \in \mathbb{R}\) with \(A_{t,-x}^2 = A_{t,x}^2.\) Hence, \(p_{t,-x} = p_{t,x}\) and \(p_{t,x} \geq 0.\) In particular, item 2 holds.

By Lemma 4.6, the Fourier-transform of \(\hat{A}_t^2(k),\) denoted \(\hat{A}_{t,x}^2,\) is absolutely summable, and thus \(\hat{A}_t^2(k) = \sum_{y \in \mathbb{Z}} e^{-2\pi y \cdot k} \hat{A}_{t,y}^2\) for every \(k \in \mathbb{T}.\) Inserting the formula in the definition of \(A_{t,x}^2\) yields

\[
A_{t,x}^2 = \sum_{n \in \mathbb{Z}} \hat{A}_{t,x+nL}^2, \quad (4.18)
\]
for all $x, t$. In the above sum, the definition of $\Lambda_L$ implies that $|x+nL| \geq L(|n| - 1) + L/2 \geq L(|n| - 1) + |x|$ if $n \neq 0$. The exponential bound in item 1 of Lemma 4.6 thus shows that $|A^2_{t,x}| \leq c_2 e^{-\delta t/2 - \gamma_2|x|}(1 + 2/(1 - e^{-\gamma_2}))$. As mentioned above, $A^2_{t,x} \in \mathbb{R}$ and thus $0 \leq p_{t,x} \leq 2\gamma c_0 c_2 (1 + 2/(1 - e^{-\gamma_2})) e^{-\delta t/2 - \gamma_2|x|}$. Since $\partial_t p_{t,x} = 4\gamma A^2_{t,x} \int_{\Lambda^*_L} dk e^{\delta x k \cdot x} \partial_t \hat{A}^2_{t}(k)$ we find using item 2 in Lemma 4.6 that $|\partial_t p_{t,x}| \leq 4\gamma c_0 c_2 (1 + 2/(1 - e^{-\gamma_2})) e^{-\delta t/2 - \gamma_2|x|}$. Choosing $C_0 := 4\gamma c_0 c_2 (1 + 2/(1 - e^{-\gamma_2}))$ thus implies that item 3 holds.

Using the discrete Parseval’s theorem in the definition of $\rho_t$ yields an alternative representation $\rho_t = 2\gamma \int_{\Lambda^*_L} dk (\hat{A}^2_{t}(k))^2$. Hence, the bounds in Lemma 4.6 imply that the bounds in item 6 hold for the choices $C_1 := 2\gamma c_0^2$ and $C_2 := 2\gamma c_1^2$. A direct computation using the definition of $\hat{A}^2_{t}(k)$ shows that $\int_0^\infty dt (\hat{A}^2_{t}(k))^2 = (2\gamma)^{-1}$ independently of $k$. Therefore, $\int_0^\infty dt \sum_x p_t(x) = \int_0^\infty dt \rho_t = 1$ and item 4 holds. The equality $\int_0^\infty dt \sum_{x \in \Lambda_L} t p_{t,x} = \gamma^{-1}$ can be checked analogously. This concludes the proof of the Corollary.

\[\]

The normalization in item 4 is a crucial identity which makes the structure of (1.2) to be that of a renewal equation. Instead of the explicit computation referred to in the above proof, the identity can also be inferred by noting that the integral is equal to a $(2, 2)$-diagonal component of $\int_0^\infty dt e^{-t M^T} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} e^{-t M}$. By the results proven in [6] the integral yields $G_{L}^{-1}$, and thus its $(2, 2)$-component has only ones on the diagonal.

The main properties of the “source term”, $g_{t,x}$, are summarized in the following Proposition.

\[\]

**Proposition 4.8** Suppose $\Phi$ and $\gamma$ satisfy the conditions in Assumption 4.3 and let $\delta_0 = \omega_0^2/\gamma$ as in Lemma 4.6. If $L \geq 1$ and all second moments of the initial field $X(0)$ exist, we define for $t \geq 0$ and $x \in \Lambda_L$

\[
g_{t,x} := \sum_{i',i=1,2} \sum_{y,z} A^i_{t,y} A^{i'}_{t,z} \langle X(0)_{x+y}, X(0)_{x+y} \rangle,
\]

where $A^2_{t,x} := \int_{\Lambda^*_L} dk e^{\delta x k \cdot x} \hat{A}^2_{t}(k)$. Then there is a constant $C'$, independent of $L$ and the choice of initial state, such that all of the following statements hold with $E_L := \langle H_L(X(0)) \rangle < \infty$:

1. $g_{t,x} \geq 0$ for all $t, x$.
2. $\sum_x g_{t,x} \leq C' e^{-\delta t} E_L$ for all $t$.
3. $\int_0^\infty dt \sum_{x \in \Lambda_L} g_{t,x} = \gamma^{-1} E_L$.

**Proof**: The assumptions imply that Lemma 4.6 holds. In the following, the constants $c_{0, c_1, c_2, \gamma_2, t_1}$ refer to those appearing in the Lemma.

Since it follows from the definition that $g_{t,x} = \langle \sum_{i,y} A^i_{t,y} X(0)_{x+y}^2 \rangle$, obviously $g_{t,x} \geq 0$.

As in the proof of Corollary 4.7 Lemma 4.6 implies that now $|A^i_{t,y}| \leq c_2 e^{-\delta t/2 - \gamma_2|y|}(1 + $
By the Schwarz inequality $\sum_{x,y} |\langle X(0)_{x+y},X(0)_{x+y} \rangle| \leq 2\|X(0)\|_2^2$. Since $\|X\|_2^2 \leq H_L(X) 2/\min(1,\omega_0^2)$, we find that item $\boxed{2}$ holds for $C'' = 4c_2^2 \max(1,\omega_0^2)(1 + 2/(1-e^{-\gamma}))^4$.

Therefore, $\int_0^\infty dt \sum_{x \in \Lambda_L} g_{t,x} = \langle X(0)^T Q X(0) \rangle$ where $Q := \int_0^\infty dt e^{-tM_\gamma} P^{(2)} e^{-tM_{\gamma}^T}$. Using the definition of $M_\gamma$ in (3.6), we can then verify that $Q = \frac{1}{2\gamma} \begin{pmatrix} \Phi_L & 0 \\ 0 & 1 \end{pmatrix}$. Thus $X^T Q X = H_L(X)/\gamma$, and we can conclude that item $\boxed{3}$ holds. 

Renewal equations can conveniently be studied via Laplace transforms. We have included a proof in Appendix B how the above bounds allow for an explicit representation of the solution of (4.2) for any continuous “initial data” $g_{t,x}$. The solution is also unique, at least in the class of continuous functions. We denote the solution by $T_{t,x}$ and conclude that

$$T_{t,x} = g_{t,x} + \int_0^t ds \sum_{y \in \Lambda_L} G(t-s,x-y) g_{s,y},$$

where for any $\varepsilon > 0$

$$G(t,x) := p_{t,x} + \int_{\Lambda_L} dk e^{i2\pi k \cdot x} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d\lambda}{2\pi i} e^\lambda \frac{\tilde{p}(\lambda,k)^2}{1 - \tilde{p}(\lambda,k)},$$

and $\tilde{p}(\lambda,k) := \int_0^\infty ds \sum_{y \in \Lambda_L} P_{s,y} e^{-s}\lambda e^{-i2\pi k \cdot y}$ is analytic for $\Re \lambda > -\delta_0$. The estimates proven in Proposition B.1 also imply that $|\tilde{p}(\lambda,k)| \leq C'/(1 + |\lambda|)$ for all $k$ if $\Re \lambda \geq -\delta_0/2$, where $C'$ can be chosen independently of $L$. In particular, the above integral is absolutely convergent for any choice of $\varepsilon > 0$.

Many of the properties below could be derived more easily by relying on standard results, such as the implicit function theorem. However, for such bounds to be useful here, it is crucial to obtain them with $L$-independent constants. To convince the reader that no $L$-dependence is sneaking in, we provide here detailed estimates with examples of such $L$-independent constants albeit at the cost of some repetition of standard computations. No claim is made that the given choices for the constants would be optimal.

**Proposition 4.9** Suppose $\Phi$ and $\gamma$ satisfy the conditions in Assumption 4.3 and set $\delta_0 = \omega_0^2/\gamma$. Then we can find constants $c'_0, \delta, \epsilon_0, \beta > 0$ and $L_0 \geq 1$, such that $\beta \leq \delta_0/2$ and for all $L \geq L_0$, $x \in \Lambda_L$, $t \geq 0$,

$$G(t,x) = \int_{\Lambda_L} dk e^{i2\pi k \cdot x} a(k)e^{-lR(k)} + \Delta(t,x),$$

where $\Delta(t,x)$ is analytic for $\Re \lambda > -\delta_0$ and $|\Delta(t,x)| \leq C''/(1 + |\lambda|)$ for all $k$ if $\Re \lambda \geq -\delta_0/2$, where $C''$ can be chosen independently of $L$. In particular, the above integral is absolutely convergent for any choice of $\varepsilon > 0$.
where
\[ |\Delta(t, x)| \leq c_0 e^{-\delta t}. \] (4.24)

Here \( a(k) \) and \( R(k) \) are defined for \( |k| > \varepsilon_0 \) by \( a(k) = 0 \) and \( R(k) = \beta \), and for \( |k| \leq \varepsilon_0 \) there is a unique \( R(k) \in [0, \beta] \), such that
\[ 1 = \int_0^\infty ds \ e^{s R(k)} \sum_{y \in \Lambda_L} p_{s,y} \cos(2\pi k \cdot y), \] (4.25)

and then
\[ \frac{1}{a(k)} = \int_0^\infty ds \ e^{s R(k)} \sum_{y \in \Lambda_L} p_{s,y} \cos(2\pi k \cdot y). \] (4.26)

In addition, we can choose the constants so that there are \( c'_1, c'_2, c'_3, c'_4, \kappa' > 0 \), all independent of \( L \), such that for all \( L \geq L_0 \) and \( k \in \Lambda_L^* \) with \( |k| \leq \varepsilon_0 \) all of the following estimates hold:

1. \( 0 < a(k) \leq c'_0 \).
2. \( \int_0^\infty ds \ \sum_{y \in \Lambda_L} y^2 p_{s,y} \geq \kappa' \).
3. \( c'_1 k^2 \leq R(k) \leq c'_2 k^2 \) and \( |\hat{D}(k; L) - R(k)| \leq c'_4 k^4 \) with
   \[ \hat{D}(k'; L) := \gamma \sum_{y \in \Lambda_L} (1 - \cos(2\pi k' \cdot y)) \int_0^\infty ds \ p_{s,y}, \quad k' \in \Lambda_L^*. \] (4.27)

In addition, we may assume \( \hat{D}(k'; L) \geq c'_3 \min(|k'|, \varepsilon_0)^2 \) for all \( k' \in \Lambda_L^* \).

**Proof:** The goal is to use Cauchy’s theorem to move the integration contour in (4.22) to the left half-plane, in which case the factor \( e^{M} \) produces exponential decay in time. To do this, it is crucial to study the zeroes of \( 1 - \hat{p} \) since these will correspond to poles of the integrand determining the dominant modes of decay. For notational simplicity, let us for the moment consider some fixed \( k_0 \in \Lambda_L^* \) and set \( F(\lambda) := 1 - \hat{p}(\lambda, k_0) \). As proven in Appendix B, \( |\hat{p}(\lambda, k_0)| < 1 \) if Re \( \lambda > 0 \) and thus \( F \) is an analytic function for Re \( \lambda > -\delta_0 \) which has no zeroes in the right half plane. It turns out that under the present assumptions, in particular, when the nondegeneracy condition in Assumption 1.3 holds, only the case with small \( k_0 \) and \( \lambda \) will be relevant, and we begin by considering that case.

Suppose first that \( \lambda, \lambda_0 \in \mathbb{C} \) with Re \( \lambda \), Re \( \lambda_0 > -\delta_0 \), and \( n \in \mathbb{N} \), with \( n = 0 \) also allowed. The derivatives of \( \hat{p} \) can be computed by differentiating the defining integrand. Therefore, the \( n \)th derivative of \( F \) is equal to \( 1(n = 0) + (-1)^{n+1} \int_0^\infty ds \ \sum_{y \in \Lambda_L} p_{s,y} s^n e^{-\delta s} e^{-i2\pi k_0 y} \). Thus by item 6 in Corollary 4.7, for any \( n \geq 0 \), we have
\[
|F^{(n)}(\lambda) - F^{(n)}(\lambda_0)| \leq \int_0^\infty ds \ p_{s} s^n |e^{-s\lambda_0} - e^{-s\lambda}| \leq |\lambda - \lambda_0| C_1 \int_0^\infty ds \ s^{n+1} e^{-\delta s} = |\lambda - \lambda_0| n! C_1 \delta_0^{-(n+2)}. \] (4.28)
Here, the second bound can be derived for instance from the representation \( e^{-s\lambda_0} - e^{-s\lambda} = \int_0^1 dr \, s(\lambda - \lambda_0) e^{-s(\lambda+r(\lambda_0-\lambda))} \) where in the exponent for any \( r \) the real part is bounded by \( \delta_0 s \). Since \( p_{t-y} = p_{t,y} \), we also have

\[
F'(0) = \int_0^\infty ds \sum_{y \in \Lambda_L} sp_{s,y} \cos(2\pi k_0 \cdot y) = \int_0^\infty ds \, sp_s - \int_0^\infty ds \sum_{y \in \Lambda_L} sp_{s,y} 2\sin^2(\pi k_0 \cdot y)
\]

\[
\geq C_2 \int_{t_1}^\infty ds \, s e^{-\gamma s} - 2\pi^2 |k_0|^2 \int_0^\infty ds \sum_{y \in \Lambda_L} y^2 s p_{s,y}
\]

\[
\geq C_2 (1 + \gamma t_1) \gamma^{-2} e^{-\gamma t_1} - 4\pi^2 |k_0|^2 C_0 \delta_0^{-2} \sum_{n=1}^\infty n^2 e^{-\gamma^2 n}, \tag{4.29}
\]

where we have used that \(| \sin x | \leq |x|\), for any \( x \in \mathbb{R} \), and applied the bounds in Corollary 4.7. Here the constants \( b_0 := C_2 (1 + \gamma t_1) \gamma^{-2} e^{-\gamma t_1} \) and \( C_4 := 4\pi^2 C_0 \delta_0^{-2} \sum_{n=1}^\infty n^2 e^{-\gamma^2 n} \) are strictly positive and independent of \( L \). Therefore, so is \( \varepsilon_1 := \sqrt{b_0/(2C_4)} \) and we can conclude that whenever \(|k_0| \leq \varepsilon_1\), we have \( F'(0) \geq b_0/2 \).

Consider then the case \(|k_0| \leq \varepsilon_1\), with \( \varepsilon_1 > 0 \) defined above. Let \( r_0 > 0 \) be given such that \( r_0 < \delta_0 \) and suppose that \( \lambda \) satisfies \(|\lambda| \leq r_0\). Then \( F'(0) \geq b_0/2 \) and by \( \tag{4.28} \) we have \(|F'(\lambda) - F'(0)| \leq r_0 C_1 \delta_0^{-3} \). Hence, \( \text{Re} F'(\lambda) \geq b_0/2 - r_0 C_1 \delta_0^{-3} \). We set \( r_0 := \min(\delta_0/2, b_0 \delta_0^3/(4C_1)) \) which is \( L \)-independent and strictly positive, and conclude that then we have a lower bound \( \text{Re} F'(\lambda) \geq b_0/4 > 0 \) for all \(|\lambda| \leq r_0\). On the other hand, if \(|\lambda|, |\lambda_0| \leq r_0 \) with \( \lambda \neq \lambda_0 \), then the identity \( F(\lambda) = F(\lambda_0) + (\lambda - \lambda_0) \int_0^1 dr \, F'(\lambda_0 + r(\lambda - \lambda_0)) \) implies a bound

\[
\text{Re} \left( \frac{F(\lambda) - F(\lambda_0)}{\lambda - \lambda_0} \right) \geq \frac{b_0}{4} > 0. \tag{4.30}
\]

Suppose that \( \lambda_0 \) is a zero of \( F \) in the closed ball of radius \( r_0 \). Since \( \text{Re} F'(\lambda_0) > 0 \), then \( \lambda_0 \) has multiplicity one. Also, by \( \tag{4.30} \), we have \(|F(\lambda)| \geq |\lambda - \lambda_0| b_0/4 \) for all \(|\lambda| \leq r_0\), and thus there can then be no other zeros of \( F \) in the ball. Since \( F(\lambda^*) = F(\lambda)^* \) and \( F(r) > 0 \) for all \( r > 0 \), we can also conclude that then necessarily \( \lambda_0 = -R_0 \) with \( 0 \leq R_0 \leq r_0 \). Therefore, if \( \lambda = -\beta + i\alpha \), with \( \beta \neq R_0 \), \( 0 \leq \beta \leq r_0/2 \) and \( \alpha \) is real and satisfies \(|\alpha| \leq r_0/2 \), we may always use the estimate \(|1/F(\lambda)| \leq 4b_0^{-1} / |R_0 - \beta| \).

Consider then the case in which there are no zeros of \( F \) in the closed ball of radius \( r_0 \). The map \( r \mapsto F(r) \) is continuous, it maps real values to real values, and \( F(r_0) > 0 \). Hence now \( F(-r) > 0 \) for all \( 0 \leq r \leq r_0 \). We apply \( \tag{4.30} \) with \( \lambda_0 = -r_0 \) to conclude that for all \(|\lambda| \leq r_0 \) with \( \lambda \neq -r_0 \)

\[
\text{Re} \left( \frac{F(\lambda)}{\lambda + r_0} \right) \geq \frac{b_0}{4} + \text{Re} \left( \frac{F(-r_0)}{\lambda + r_0} \right) \geq \frac{b_0}{4} > 0.
\]

Therefore,

\[
|F(\lambda)| \geq |\lambda + r_0| \left| \text{Re} \left( \frac{F(\lambda)}{\lambda + r_0} \right) \right| \geq |\lambda + r_0| \frac{b_0}{4}. \tag{4.32}
\]
We can then conclude that \(|1/F(\lambda)| \leq 8/(r_0b_0)| \) whenever \(\lambda = -\beta + i\alpha\) with \(0 \leq \beta \leq r_0/2\) and \(\alpha\) is real and satisfies \(|\alpha| \leq r_0/2\).

The above estimates are sufficient to control the \(r_0\)-neighborhood of zero for small \(k_0\). Coming back to general \(k_0\) we next study the properties of \(F\) on the imaginary axis, for \(\lambda = i\alpha\) with \(\alpha \in \mathbb{R}\). Using item 4 in Corollary 4.7 and the notations introduced in the proof of the Corollary shows that

\[
\mathrm{Re} \ F(i\alpha) = \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} (1 - \cos(s\alpha + 2\pi k_0 \cdot y))
\]

\[
= 4\gamma \int_0^\infty ds \sum_{y \in \Lambda_L} A_{s,y}^2 \sin\left(\frac{1}{2} s\alpha + \pi k_0 \cdot y\right)^2
\]

\[
= 4\gamma \int_0^\infty ds \sum_{y \in \Lambda_L} \left| e^{-i2\pi k_0 y} A_{s,y}^2 \sin\left(\frac{1}{2} s\alpha + \pi k_0 \cdot y\right)\right|^2. \quad (4.33)
\]

Since \(k_0 \in \Lambda_L^r\), there is \(n_0 \in \Lambda_L\) such that \(k_0 = n_0/L\). To derive lower bounds for (4.33), it suffices to consider the case in which \(n_0 \geq 0\), since then for \(n_0 < 0\) we can use the symmetry of cosine and apply the bounds derived for the case where the signs of \(\alpha\) and \(k_0\) are reversed.

Consider first the case in which \(n_0\) is even. Then there is \(n_1 \in \Lambda_L\) such that \(0 \leq n_1 \leq L/4\) and \(n_0 = 2n_1\). In this case, \(k_0/2 \in \Lambda_L^r\), the Fourier-transform of \(y \mapsto A_{s,y}^2(k)\) equals \(\hat{A}_s^2(k)\) and thus by using \(\sin x = (e^{ix} - e^{-ix})/(2i)\) in (4.33) yields

\[
\mathrm{Re} \ F(i\alpha) = \gamma \int_0^\infty ds \int_{\Lambda_L^r} dk \left| e^{isa/2} \hat{A}_s^2(k - k_0/2) - e^{-isa/2} \hat{A}_s^2(k + k_0/2) \right|^2
\]

\[
\geq \gamma \int_{t_1}^\infty ds \int_{\Lambda_L^r} dk \left( \hat{A}_s^2(k - k_0/2) - \hat{A}_s^2(k + k_0/2) \right)^2
\]

\[
+ 2\gamma \int_{t_1}^\infty ds (1 - \cos(s\alpha)) \int_{\Lambda_L^r} dk \hat{A}_s^2(k - k_0/2) \hat{A}_s^2(k + k_0/2) \quad (4.34)
\]

where in the last step we used the fact that \(\hat{A}_s^2\) are real. Applying the lower bound in item 3 of Lemma 4.6 thus proves that

\[
\mathrm{Re} \ F(i\alpha) \geq 2\gamma \int_{t_1}^\infty ds (1 - \cos(s\alpha)) c_1^2 e^{-\gamma s}. \quad (4.35)
\]

For instance by representing the cosine as a sum of two exponential terms, we find that \(\int \mathrm{d}s (1 - \cos(s\alpha)) e^{-\gamma s} = \gamma^{-1} e^{-\gamma \alpha^2/(\alpha^2 + \gamma^2)}\) if \(|\alpha| t \in 2\pi\mathbb{Z}\). Therefore, now

\[
\mathrm{Re} \ F(i\alpha) \geq 2c_1^2 \gamma \int_{t_1}^\infty e^{-\gamma t} e^{-2\pi|\gamma/\alpha|} \frac{1}{1 + |\gamma/\alpha|^2}. \quad (4.36)
\]

For any \(r > 0\), set \(\hat{C}(r)\) to be equal to the right hand side at \(\alpha = r\). Then \(\hat{C}(r) > 0\), it is independent of \(L\), and we can conclude that \(\mathrm{Re} \ F(i\alpha) \geq \hat{C}(r)\) whenever \(|\alpha| \geq r\) and \(Lk_0\) is even.
In the remaining cases \( n_0 \) is odd and positive. Then there is \( n_1 \in \Lambda_L \) such that \( 0 \leq n_1 \leq L/4 \) and \( n_0 = 2n_1 + 1 \). Thus we can apply the above estimate for \( \text{Re} \, F(\alpha) \) at \( k_0 - 1/L = 2n_1/L \). On the other hand,

\[
|\cos(s\alpha + 2\pi k_0 \cdot y) - \cos(s\alpha + 2\pi(k_0 - 1/L) \cdot y)| \leq \frac{2\pi|y|}{L},
\]

and we can conclude that for odd \( n_0 \) and every \( |\alpha| \geq r > 0 \)

\[
\text{Re} \, F(\alpha) \geq \hat{C}(r) - \int_0^\infty \text{d}s \sum_{y \in \Lambda_L} p_{s,y} \frac{2\pi|y|}{L} \geq \hat{C}(r) - L^{-1} 4\pi C_0 \frac{\delta_0}{\delta_0} \sum_{n=1}^\infty ne^{-\gamma_n},
\]

where in the second inequality we have applied the bounds in item \( 3 \) of Corollary 4.7. Therefore, to every \( r > 0 \) there is \( \hat{L}(r) \in N_+ \) such that the final bound is greater than \( \hat{C}(r)/2 \) for every \( L \geq \hat{L}(r) \). Thus we can conclude that, if \( r > 0 \) and \( L \geq \hat{L}(r) \), then \( \text{Re} \, F(\alpha; k_0, L) \geq \hat{C}(r)/2 \) for all \( |\alpha| \geq r \) and \( k_0 \in \Lambda_L^* \).

If \( \beta \) satisfies \( 0 \leq \beta < \delta_0 \), then by \( (4.28) \) we have \( |F(-\beta + i\alpha) - F(\alpha)| \leq C_1 \delta_0^{-2} \) for all real \( \alpha \). Therefore, if we set \( \tilde{\beta}(r) := \frac{1}{\delta_0} \min(1, \hat{C}(r)\delta_0/(2C_1)) \) for \( r > 0 \), then we have found strictly positive, \( L \)-independent constants such that for any \( r > 0 \) and \( L \geq \hat{L}(r) \)

\[
\text{Re} \, F(-\beta + i\alpha; k_0, L) \geq \frac{1}{4} \hat{C}(r) > 0,
\]

for all \( k_0 \in \Lambda_L^* \), \( |\alpha| \geq r \), and \( 0 \leq \beta \leq \tilde{\beta}(r) \leq \delta_0/2 \).

It is now possible to conclude the estimates for the case when \( |k_0| \leq \varepsilon_1 \). Recall the earlier definition of \( r_0 \) and set \( C_5 := \hat{C}(r_0/2) \), \( L_5 := \hat{L}(r_0/2) \) and \( \beta_1 := \tilde{\beta}(r_0/2) \). Assume that \( L \geq L_5 \). We use Cauchy’s theorem and, when necessary, the residue theorem to change the integration contour from \( \varepsilon' + i\mathbb{R} \) to \( -\beta + i\mathbb{R} \) with some \( \beta > 0 \). If there are no zeroes of \( F \) in the closed ball of radius \( r_0 \), we choose \( \beta = \beta_0 \) with \( \beta_0 := \min(\beta_1, r_0/2) \) and the above results imply that the integrand in \( (4.22) \) is analytic for \( \text{Re} \, \lambda \geq -\beta_0 \) and have \( |1/F| \leq \max(4/C_5, 8/(r_0b_0)) \) on the integration contour. If there are zeroes in the ball, then the unique zero and lies at \( -R_0 \) with \( 0 \leq R_0 \leq r_0 \) and \( 1/F \) has a first order pole at \( -R_0 \). If \( R_0 > \beta_0/2 \), we choose \( \beta = \beta_0/4 < \beta_1 \) when the integrand is analytic to the right of the final contour and hence the pole does not contribute. Then also \( |1/F| \leq \max(4/C_5, 16/(\beta_0b_0)) \) on the integration contour. If \( R_0 \leq \beta_0/2 \), we choose \( \beta = \beta_0 \). Then the residue theorem can be used to evaluate the contribution from the pole, and the remaining integral over \( -\beta + i\mathbb{R} \) can be bounded using \( |1/F| \leq \max(4/C_5, 8/(\beta_0b_0)) \).

Assume then that \( L \geq L_5 \) and \( |k_0| \leq \varepsilon_1 \). Following the above steps, we find that, if there is \( 0 \leq R_0 \leq \beta_0/2 \) such that

\[
1 = \int_0^\infty \text{d}s \, e^{sR_0} \sum_{y \in \Lambda_L} p_{s,y} \cos(2\pi k_0 \cdot y),
\]

then \( \widehat{\rho}(-R_0, k_0) = 1 \) and

\[
\int_{\varepsilon'-i\infty}^{\varepsilon'+i\infty} \frac{\text{d}\lambda}{2\pi i} e^{\lambda t} \frac{\widehat{\rho}(\lambda, k_0)^2}{1 - \widehat{\rho}(\lambda, k_0)} = \frac{1}{m(k_0)} e^{-tR_0} + \Delta.
\]
Here \( m(k_0) = F'(-R_0) \), implying

\[
m(k_0) := \int_0^\infty ds \, se^{sR_0} \sum_{y \in \Lambda_L} p_{s,y} \cos(2\pi k_0 \cdot y) \ge \frac{b_0}{4} > 0,
\]

(4.42)

and there is a constant \( C_6 > 0 \), independent of \( L \), such that

\[
|\Delta| \le C_6 e^{-\beta_0 t}.
\]

(4.43)

(Recall that \( |\hat{p}(\lambda, k_0)| \le C'/(1 + |\lambda|) \) for \( \text{Re} \lambda \ge -\delta_0/2 \).) If no such \( R_0 \) can be found, then \( \hat{p}(-r, k_0) < 1 \) for all \( r \le \beta_0/2 \) and one of the remaining cases is realized. Hence, then

\[
\int_{\varepsilon' - \infty}^{\varepsilon' + \infty} \frac{d\lambda}{2\pi i} \frac{e^{\lambda t} \hat{p}(\lambda, k_0)^2}{1 - \hat{p}(\lambda, k_0)} \le C_6 e^{-\beta_0 t/4},
\]

(4.44)

with some \( L \)-independent \( C_6 > 0 \).

The following Lemma will be used to study the remaining values of \( k_0 \).

**Lemma 4.10** Suppose Assumption \ref{assumption:4.3} holds. Then for every \( \varepsilon > 0 \) we can find \( C(\varepsilon) > 0 \), \( L(\varepsilon) \in \mathbb{N}_+ \) and \( \beta(\varepsilon) \in (0, \delta_0/2] \) such that, if \( L \ge L(\varepsilon) \) and \( k_0 \in \Lambda_L^* \) with \( |k_0| \ge \varepsilon \), then \( F(0; k_0, L) \ge C(\varepsilon) \) and \( \text{Re} F(\lambda; k_0, L) \ge C(\varepsilon)/2 \) for all \( \lambda \) with \( 0 \ge \text{Re} \lambda \ge -\beta(\varepsilon) \).

**Proof:** Fix \( \varepsilon > 0 \) and let \( C_\varepsilon > 0 \) denote the corresponding constant in Assumption \ref{assumption:4.3}. As proven above, if \( L \ge 1 \) and \( k_0 \in \Lambda_L^* \) is such that \( Lk_0 \) is a even and nonnegative, then

\[
F(0; k_0, L) = \gamma \int_0^\infty ds \int_{\Lambda_L^*} dk \, f_s(k, k_0), \quad f_s(k, k_0) := \left( \hat{A}_L^2(k - k_0/2) - \hat{A}_L^2(k + k_0/2) \right)^2.
\]

(4.45)

If \( h \in C^{(1)}(\mathbb{T}) \), then \( |\int_{\mathbb{T}} \! dk \, h(k) - \int_{\Lambda_L^*} \! dk \, h(k)| \le \sum_{n \in \Lambda_L} ||h'||_\infty \int_{|k-n/L| \le (2L-1)} \! dk \, |k-n/L| \le ||h'||_\infty / (2L). \) By Lemma \ref{lemma:4.6}, we can apply this in the above with \( ||h'||_\infty \le 8\varepsilon_0^2 e^{-\delta_0 s}. \) Therefore,

\[
|F(0; k_0, L) - \gamma \int_0^\infty ds \int_{\mathbb{T}} dk \, f_s(k, k_0)| \le \frac{4\varepsilon_0^2}{L \delta_0}.
\]

(4.46)

If \( Lk_0 \) is odd and nonnegative, we have by (4.37)

\[
|F(0; k_0, L) - F(0; k_0 - 1/L, L)| \le \frac{4\pi C_0}{L \delta_0} \sum_{n=1}^\infty ne^{-\gamma_2 n}
\]

(4.47)

and also \( |f_s(k, k_0) - f_s(k, k_0 - 1/L)| \le 4\varepsilon_0^2 e^{-\delta_0 s} L^{-1} \), by Lemma \ref{lemma:4.6}. Therefore, there is an \( L \)-independent constant \( C > 0 \) such that \( |F(0; k_0, L) - \gamma \int_0^\infty ds \int_{\mathbb{T}} dk \, f_s(k, k_0)| \le C/L \) for all \( k_0 \in \Lambda_L^* \). If \( |k_0| \ge \varepsilon \), then by assumption \( \gamma \int_0^\infty ds \int_{\mathbb{T}} dk \, f_s(k, k_0) \ge \gamma C_\varepsilon \) and hence
\[ F(0; k_0, L) \geq \gamma C_\varepsilon - C/L. \] Thus by choosing \( L'(\varepsilon) \) such that \( L'(\varepsilon) \geq 2C/(\gamma C_\varepsilon) \) we have \( F(0; k_0, L) \geq \gamma C_\varepsilon/2 \) whenever \( L \geq L'(\varepsilon) \) and \( |k_0| \geq \varepsilon \).

Consider then some fixed \( L \geq L'(\varepsilon) \) and \( |k_0| \geq \varepsilon \). By the earlier results, \( |F(\lambda) - F(0)| \leq rC_1 \delta_0^{-2} \) if \( |\lambda| \leq r < \delta_0 \). (The constant \( C_1 \) here should not be confused with \( C_\varepsilon \) at \( \varepsilon = 1 \).) Thus if \( r_1 := \min(\delta_0/2, \gamma C_\varepsilon \delta_0^2/(4C_1)) > 0 \), then \( \Re F(\lambda) \geq \gamma C_\varepsilon/4 \) for all \( |\lambda| \leq r_1 \). On the other hand, if also \( L \geq \tilde{L}(r_1/2) \), then we have \( \Re F(-\beta + i\alpha) \geq \tilde{C}(r_1/2)/4 \) for all \( 0 \leq \beta \leq \tilde{\beta}(r_1/2) \) and \( |\alpha| \geq r_1/2 \). Combining the above estimates yields constants such that the Lemma holds for all \( L \geq L(\varepsilon) := \max(L'(\varepsilon), \tilde{L}(r_1/2)) \).

We next apply Lemma 4.10 with \( \varepsilon = \varepsilon_1/2 > 0 \). Set thus \( L_7 := L(\varepsilon_1/2), C_7 := C(\varepsilon_1/2), \) and \( \beta_2 := \beta(\varepsilon_1/2) \). Assume \( L \geq L_7 \) and \( k_0 \in \Lambda^*_L \) with \( |k_0| \geq \varepsilon_1/2 \). Then by the Lemma, for any \( \lambda = -\beta + i\alpha \), with \( 0 \leq \beta \leq \beta_2 \) and \( \alpha \in \mathbb{R} \) we have \( |1/F(\lambda)| \leq 2/C_7 \). Therefore, we can change the contour to \(-\beta_2 + i\mathbb{R} \) without encountering any singularities. This proves that there is an \( L \)-independent constant \( C_8 \) such that for \( |k_0| \geq \varepsilon_1 \) and \( L \geq L_7 \)

\[
\left| \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{d\lambda}{2\pi i} e^{\lambda t} \frac{\tilde{p}(\lambda, k_0)^2}{1 - \tilde{p}(\lambda, k_0)} \right| \leq C_8 e^{-\beta_2 t}. \tag{4.48}
\]

Collecting the above estimates together proves that there are constants \( c'_0 > 0, \beta' > 0, L' \in \mathbb{N}_+ \), such that if \( L \geq L' \), then for all \( k_0 \in \Lambda^*_L \) either

\[
\left| \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{d\lambda}{2\pi i} e^{\lambda t} \frac{\tilde{p}(\lambda, k_0)^2}{1 - \tilde{p}(\lambda, k_0)} \right| \leq c'_0 e^{-\beta' t}, \tag{4.49}
\]

or \( |k_0| \leq \varepsilon_1 \) and there are \( R(k_0) \) and \( a(k_0) := 1/m_0(k_0) \) satisfying (4.25) and (4.26) such that

\[
\left| \int_{\varepsilon' - i\infty}^{\varepsilon' + i\infty} \frac{d\lambda}{2\pi i} e^{\lambda t} \frac{\tilde{p}(\lambda, k_0)^2}{1 - \tilde{p}(\lambda, k_0)} - a(k_0)e^{-tR(k_0)} \right| \leq c'_0 e^{-\beta' t}. \tag{4.50}
\]

We still need to make sure that all the claimed bounds will hold. From now on we assume that \( L \geq L' \) so that all of the earlier derived bounds can be used. Suppose then that \( |k_0| \leq \varepsilon_1 \). Since \( F(-R) \in \mathbb{R} \), for \( 0 \leq R \leq r_0 \), (4.30) implies that \( F(-R) \leq F(0) - Rb_0/4 \) for these \( R \). Therefore, if \( F(-R(k_0)) = 0 \), we have \( 0 \leq R(k_0) \leq 4F(0)/b_0 \). Since

\[
F(0) = 2 \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} \sin^2(\pi k_0 \cdot y) \leq 2\pi^2 k_0^2 \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} y^2 \leq k_0^2 \frac{4\pi^2 C_0}{\delta_0} \sum_{n=1}^{\infty} n^2 e^{-\gamma_2 n}, \tag{4.51}
\]

we can conclude that with \( \xi'_2 := (4\pi)^2 C_0/(b_0 \delta_0) \sum_{n=1}^{\infty} n^2 e^{-\gamma_2 n} \) we have \( R(k_0) \leq \xi'_2 k_0^2 \). Also, whenever \( L \geq L_8 := \max(L', L_7, 1/\varepsilon_1) \), there exists \( k_\varepsilon \in [\varepsilon_1/2, \varepsilon_1] \cap \Lambda^*_L \), and by Lemma 4.10 then \( F(0; k_\varepsilon, L) \geq C_T > 0 \). Therefore, for \( L \geq L_8 \) also \( \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} y^2 \geq C_T/(2\pi^2 \varepsilon_1^2) \). Thus if we set \( \kappa' := C_T/(2\pi^2 \varepsilon_1^2) \), then item 2 holds.
To get a lower bound, we assume $L \geq L_8$, and use the fact that $|\sin x| \geq |x|/2$ for all $|x| \leq \pi/2$. This shows that if $0 < \varepsilon_0 \leq \varepsilon_1$, then for all $|k_0| \leq \varepsilon_0$

$$F(0) \geq k_0^2 \delta \int_0^\infty ds \sum_{y \in \Lambda_L} y^2 p_{s,y} \mathbb{1}(\|y\| \leq \frac{1}{2 \varepsilon_0}) \geq k_0^2 \delta \sum_{n \geq 1} n^2 e^{-n^2 \varepsilon_0}. \tag{4.52}$$

Therefore, by choosing any $L$-independent $\varepsilon_0 \in \left(0, \sqrt{r_0/\varepsilon_0^2}\right)$ such that $\varepsilon_0 \leq \varepsilon_1$ and for which $\sum_{n \geq 1} n^2 e^{-n^2 \varepsilon_0} \leq \kappa' \delta_0/(4C_0), \varepsilon_0$, we have $F(0) \geq k_0^2 \delta_0/4 \kappa'$ for all $|k_0| \leq \varepsilon_0$. It follows from (4.28) that $F(0) - F(-R) = |F(0) - F(-R)| \leq RC_1 \delta_0^2/2$ for $0 \leq R \leq r_0$. Therefore, if $F(-R(k_0)) = 0$, we have $R(k_0) \geq \delta_0 F(0)/C_1 \geq c'_1 k_0^2$ with $c'_1 := 4 \kappa' \delta_0^2/C_1 > 0$ for all $|k_0| \leq \varepsilon_0$.

Set then $\beta := c'_2 \varepsilon_0^2 \in (0, r_0]$. Collecting the above estimates together, we can now conclude that if $|k_0| \leq \varepsilon_0$, then $F(-\beta) \leq (c'_2 k_0^2 - \beta) b_0/4 \leq 0$, and thus there is a unique $R(k_0) \in [0, \beta]$ such that $F(-R(k_0)) = 0$, and then also $c'_1 k_0^2 \leq R(k_0) \leq c'_1 k_0^2$. As $|k_0| \leq \varepsilon_1$, then $0 < a(k_0) \leq b_0/2$ for $a(k_0) := 1/m_0(k_0)$. If $0 < |k_0| \leq \varepsilon_1$, then either $R(k_0) \geq \delta_0 F(0)/C_1$, or there is no zero of $F$ in $[0, \beta_0/2]$. In the first case, we have $F(0; k_0, L) \geq C(\varepsilon_1) > 0$ for all $L \geq L(\varepsilon_0)$ and thus then $|e^{-R(k_0)}|/m_0(k_0)| \leq b_0/2 e^{-C(\varepsilon_0) \delta_0^2/C_1}$. In the second case, the previous estimates apply. Thus by setting $\delta := \min(C(\varepsilon_0) \delta_0^2/C_1, \beta_2, \beta_0/4) > 0$ we have also proven the exponential upper bound for the correction. (Note that $|p_{k,x}| \leq C_0 e^{-\delta t}$ decays always faster than $e^{-\delta t}$.)

Finally, define $\hat{D}(k')$ by (4.27) for all $k' \in \Lambda^*_L$. Comparing the definition to (4.51) shows that then in fact $\hat{D}(k') = \gamma F(0; k')$. Therefore, if $L$ is large enough, then by the above estimates, we have $\hat{D}(k') \geq \gamma C(\varepsilon_0) |k'| > \varepsilon_0,$ and $\hat{D}(k') \geq 4\gamma \kappa'(k')^2$ for $|k'| \leq \varepsilon_0$. Hence, we can arrange that $\hat{D}(k') \geq c'_2 \min(|k'|, \varepsilon_0)^2$ for some $c'_2 > 0$, independent of $L$, as claimed in the Proposition. Using item 5 in Corollary 4.7, and the above estimates then shows that whenever $|k| \leq \varepsilon_0$

$$\gamma^{-1} \left( R(k) - \hat{D}(k) \right) = -1 + \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} (s R(k) + \cos(2\pi k \cdot y))$$

$$= \int_0^\infty ds \sum_{y \in \Lambda_L} p_{s,y} s R(k) (1 - \cos(2\pi k \cdot y))$$

$$+ \int_0^\infty ds \left( 1 + s R(k) - e^{s R(k)} \right) \sum_{y \in \Lambda_L} p_{s,y} \cos(2\pi k \cdot y), \tag{4.53}$$

where in the second step we used the defining relation of $R(k)$, equation (4.25). The first term in the sum is bounded by $k^4 c'_2 C_0(2\pi/\delta_0)^2 \sum_{n=1}^\infty n^2 e^{-n^2 \varepsilon_0}$, and the second one is bounded by $R(k)^2 \int_0^\infty ds \rho s^2 e^{s R(k)} \leq k^4 C_1(c'_2)^2 \delta_0^{-3}$. Choosing the sum of the two factors multiplying $k^4$ as $c'_4/\gamma$ then implies $|R(k) - \hat{D}(k)| \leq c'_4 k^4$. This concludes the proof of the Proposition.

The following observation will provide a convenient estimate for the proof of the main theorem.
Lemma 4.11 \( \sum_{n=1}^{\infty} n^2 e^{-en^2} \leq 2e^{-\frac{1}{2}} \) for all \( \varepsilon > 0 \).

Proof: Fix \( \varepsilon > 0 \) and consider the function \( f(x) = x^2 e^{-\varepsilon x^2} \) for \( x \geq 0 \). It is strictly increasing on \( [0, x_\varepsilon] \) and strictly decreasing for \( x > x_\varepsilon \), with \( x_\varepsilon := \varepsilon^{-\frac{1}{2}} \). If \( \varepsilon < 1 \), we have \( x_\varepsilon > 1 \), and we set \( n_\varepsilon \geq 1 \) as the integer part of \( x_\varepsilon \). We estimate the sum as an integral over a step function containing values of \( f \). This shows that \( \sum_{n=1}^{n_\varepsilon} n^2 e^{-en^2} \leq \int_0^{x_\varepsilon} dx f(x) \) and \( \sum_{n=n_\varepsilon+1}^{\infty} n^2 e^{-en^2} \leq \int_{x_\varepsilon}^{\infty} dx f(x) \). Hence, \( \sum_{n=1}^{\infty} n^2 e^{-en^2} \leq \int_0^{\infty} dx f(x) + 2f(x_\varepsilon) \leq \varepsilon^{-\frac{1}{2}} \left( \int_0^{\infty} dy y^2 e^{-y^2} + 1 \right) \). The constant is equal to \( 1 + \sqrt{\pi}/4 < 2 \).

If \( \varepsilon \geq 1 \), we have \( x_\varepsilon \leq 1 \) and thus \( f \) is strictly decreasing for \( x \geq 1 \). As above, this implies \( \sum_{n=1}^{\infty} n^2 e^{-en^2} \leq \int_1^{\infty} dx f(x) + e^{-\varepsilon} \leq \varepsilon^{-\frac{1}{2}} \left( \int_0^{\infty} dy y^2 e^{-y^2} + (3/(2\varepsilon))^{3/2} \right) < 2e^{-\frac{1}{2}} . \)

Proof of Theorem 4.4 Suppose now that \( L \geq L_0 \) which together with the assumptions of the Theorem allows using the formulae and constants given in Proposition 4.9 with \( \delta_0 := \omega_0^2/\gamma \). In particular, let \( \hat{D}(k) \) be defined by the formula (4.27), \( a(k) \) for \( |k| \leq \varepsilon_0 \) by (4.26), and set

\[ \tau_x := \sum_{y \in \Lambda_L} \int_{\Lambda_L^*} dk e^{i2\pi k \cdot (x-y)} a(k) \int_0^\infty ds g_{s,y}, \]

and

\[ (D\tau)_x := \sum_{y \in \Lambda_L} \hat{p}_y (2\tau_x - \tau_{x+y} - \tau_{x-y}), \quad \hat{p}_x := \frac{\gamma}{2} \int_0^\infty ds p_{s,x}. \]

Clearly, \( \hat{p}_x \geq 0 \), and since \( a(-k) = a(k) \), \( \tau_x \in \mathbb{R} \). By the discussion before Proposition 4.9, we can now conclude that there is a unique continuous solution \( T_{i,t} \) to (4.2). It satisfies

\[ T_{i,t} = g_{i,t} + \int_0^t ds \sum_{y \in \Lambda_L} \Delta(t-s, x-y) g_{s,y} + \sum_{y \in \Lambda_L} \int_{\Lambda_L^*} dk e^{i2\pi k \cdot (x-y)} a(k) \int_0^t ds e^{-(t-s)R(k)} g_{s,y}. \]

By Proposition 4.8, the first term is bounded by \( C' e^{-\delta_0 t} E_L \), where \( E_L = L \mathcal{E} \) denotes the average total energy. Applying also Proposition 4.9 to the second term shows that it is bounded by \( \int_0^t ds \sum_{y \in \Lambda_L} g_{s,y} \rho_0 e^{-\delta(t-s)} \leq 2C' \rho_0 \varepsilon e^{-\delta t} E_L \), since \( \delta \leq \delta_0/2 \). In the third term, we separate the term with \( k = 0 \), for which \( R(0) = 0 \). By item 5 in Corollary 4.7 then \( 1/a(0) = \int_0^\infty ds s \rho_s = \gamma^{-1} \), and thus the \( k = 0 \) term is equal to \( L_L^{-1} \int_0^t ds \sum y g_{s,y} \).

By Proposition 4.8, this differs from \( \mathcal{E} \) maximally by \( \mathcal{E} C' \gamma / \delta_0 e^{-\delta_0 t} \). Therefore, with \( c := C' \max(1 + 2C_0 \gamma_0, \gamma \delta_0^{-1}) \),

\[ |T_{i,t} - \mathcal{E}| \leq c \mathcal{E}(1 + L) e^{-\delta t} + c_0 C' \mathcal{E} \sum_{1 \leq |n| \leq L_0} e^{-t \varepsilon_0 n^2 L^{-2}} \int_0^t ds e^{-\delta_0 s/2} \]

\[ \leq c \mathcal{E}(1 + L) e^{-\delta t} + 4c_0 C' \delta_0^{-1} \mathcal{E} \sum_{n=1}^{\infty} e^{-t \varepsilon_0 n^2 L^{-2}}. \]
Now for any \( \varepsilon > 0 \), \( \sum_{n=1}^{\infty} e^{-\varepsilon n^2} \leq e^{-\varepsilon} \sum_{n'=0}^{\infty} e^{-2\varepsilon n'} = e^{-\varepsilon}/(1 - e^{-2 \varepsilon}) \). Then for \( \varepsilon = tc_1 L^{-2} \) we also have \( L^2 \leq 2tc_1/(1 - e^{-2 \varepsilon}) \). Therefore, we can find a constant \( C \) such that (4.9) holds for \( d := \min(c'_1, \delta L^2_0/2) > 0 \).

In order to prove the lattice diffusion equation, we come back to (4.50). For the third term we now apply the estimates

\[
\left| \int_0^t ds g_{s,y} e^{-(t-s)R(k)} - e^{-t\hat{D}(k)} \int_0^\infty ds g_{s,y} \right| 
\leq \left| e^{-tR(k)} - e^{-t\hat{D}(k)} \right| \int_0^\infty ds g_{s,y} + \int_t^\infty ds g_{s,y} + \int_0^t ds g_{s,y} e^{-(t-s)R(k)} \left| 1 - e^{-sR(k)} \right| .
\]

(4.58)

Splitting the final integral into two parts at \( s = t/2 \) then yields

\[
\sum_{y \in \Lambda_L} \left| \int_0^t ds g_{s,y} e^{-(t-s)R(k)} - e^{-t\hat{D}(k)} \int_0^\infty ds g_{s,y} \right| 
\leq C' \delta_0^{-1} E_L \left[ t \left| \hat{D}(k) - R(k) \right| e^{-t \min(\hat{R}(k), \hat{D}(k))} + e^{-\delta_0 t} + \delta_0^{-1} R(k) e^{-\frac{t}{2} R(k)} + \delta_0 t e^{-\frac{t}{2} \delta_0} \right] .
\]

(4.59)

Using the known properties of \( R \) and \( \hat{D} \), we can now conclude that there are constants \( c, m > 0 \), independent of \( L \) and the initial state, such that

\[
\left| T_{t,x} - \int_{\Lambda_L} dk e^{-t\hat{D}(k)} \sum_{y \in \Lambda_L} e^{i2\pi k \cdot (x-y)} a(k) \int_0^\infty ds g_{s,y} \right| 
\leq cE_L \left[ e^{-\frac{t}{4}} + \int_{\Lambda_L} dk \, 1(|k| \leq \varepsilon_0) k^2 e^{-tm^2 k^2} \right] .
\]

(4.60)

To arrive at the above bound, we choose \( m := \min(c'_1, c')/2 \) and estimate \( tk^2 e^{-2mtk^2} \leq m^{-1} e^{-mtk^2} \). Here, by Lemma 4.11, \( \sum_{y \in \Lambda_L} \sum_{k \in \Lambda_L} dk \, 1(|k| \leq \varepsilon_0) k^2 e^{-tk^2} \leq L^{-3/2} \sum_{n=1}^{\infty} n^2 e^{-tm^2} n^2 \leq 4m^{-3/2} t^{-3/2} \). Thus for \( C := 4(\delta_0^{-1/2} + m^{-3/2})C' \), the right hand side of (4.60) is bounded by \( C E_L t^{-3/2} \). On the other hand, since the Fourier-transform of the operator \( D \) is equal to multiplication by \( \hat{D}(k) \), we can now conclude that (4.10) holds. □

**Proof of Corollary 4.11**

Fix an allowed \( L \geq L_0 \), and some \( t_0 > 0 \) and the function \( \varphi \). For any initial data \( f_0 \in C(\mathbb{R}^2) \) the solution of the heat equation (4.13) on the circle is standard and can be done using Fourier series. Explicitly, then

\[
f(t, \xi) = \sum_{n \in \mathbb{Z}} e^{i2\pi n \xi/L} e^{-t\kappa_L(2\pi n/L)^2} \hat{f}_0(n)
\]

(4.61)

where

\[
\hat{f}_0(n) := \frac{1}{L} \int_{LT} d\xi \, e^{-i2\pi n \xi/L} f_0(\xi) .
\]

(4.62)
On the other hand, a solution to the periodic heat equation coincides with the solution to the heat equation on \( \mathbb{R} \) with periodic initial data, and thus also \( |f(t, x)| \leq \|f\|_{\infty} \) for all \( t, x \).

As intermediate approximations, set \( \tau_{t,x} := (e^{-tD}t)x \), for \( t \geq 0 \), \( x \in \Lambda_L \), and, for \( t \geq 0 \), \( \xi \in \Lambda_T \), set \( \tilde{T}(t, \xi) := \sum_{y \in \mathbb{Z}} \varphi(\xi - y)\tau_{t+0,y \mod \Lambda_T} \) and define \( f(t, \xi) \) as the solution to the heat equation (4.13) with initial data \( f(0, \xi) := \tilde{T}(0, \xi) \). Theorem 4.4 implies the following bound for the error: \( |T^{\text{obs}}(t, \xi) - \tilde{T}(t, \xi)| \leq C\varepsilon L_t^{-3/2} \sum_{y \in \mathbb{Z}} |\varphi(\xi - y)| \). Since the difference \( T^{\text{pred}} - f \) is a solution to the heat equation, we also obtain

\[
|T^{\text{pred}}(t, \xi) - f(t, \xi)| \leq \sup_{\xi'} |T^{\text{pred}}(0, \xi') - f(0, \xi')| \leq C\varepsilon L_t^{-3/2} \sup_{\xi' \in \mathbb{Z}} |\varphi(\xi' - y)|. \tag{4.63}
\]

Hence, it suffices to study the difference \( f(t, \xi) - \tilde{T}(t, \xi) \). For any vector \( \psi \in \mathbb{C}^{\Lambda_T} \), the Fourier transform of \( h(\xi) := \sum_{y \in \mathbb{Z}} \varphi(\xi - y)\psi_{y \mod \Lambda_T} \) satisfies

\[
\hat{h}(n) = \frac{1}{L} \int_{L\mathbb{T}} d\xi e^{-i2\pi n L \xi} \sum_{x \in \Lambda_L} \sum_{m \in \mathbb{Z}} \varphi(\xi - x + mL) \\
= \sum_{x \in \Lambda_L} \sum_{m \in \mathbb{Z}} \int_{\mathbb{R}} dq e^{-i2\pi n q L} \varphi(L(q + m - x)/L) \\
= \sum_{x \in \Lambda_L} \int_{\mathbb{R}} dq e^{-i2\pi n q L} \varphi(L(q - x)/L) \\
= \frac{1}{L} \hat{\varphi}(n/L) \sum_{x \in \Lambda_L} \psi_x e^{-i2\pi n x/L}. \tag{4.64}
\]

Since \( h \in C^{(2)} \), its Fourier transform is pointwise invertible, and thus at every \( \xi \) we then have

\[
h(\xi) = \sum_{n \in \mathbb{Z}} e^{i2\pi n L \xi} \frac{1}{L} \hat{\varphi}(\frac{n}{L}) \hat{\psi}(\frac{n \mod \Lambda_L}{L}). \tag{4.65}
\]

Therefore, using the definition of \( \tau_0 \) and (4.61) to represent \( f \), we have

\[
f(t, \xi) - \tilde{T}(t, \xi) = \frac{1}{L} \sum_{n \in \mathbb{Z}} e^{i2\pi n L \xi} \hat{\varphi}(\frac{n}{L}) \left( e^{-t\varphi(2\pi n L)} - e^{-t\tilde{D}(k)} \right) e^{-t_0 \tilde{D}(k)} a(k) \sum_{y \in \Lambda_L} e^{-i2\pi k y} \int_0^\infty ds g_{s,y}, \tag{4.66}
\]

where \( k \) is a shorthand for \((n \mod \Lambda_L)/L\). Here \( \tilde{D}(k) = 4 \sum_{y \in \Lambda_L} \tilde{p}_y \sin^2(\pi k \cdot y) \) and thus

\[
\kappa_L(2\pi k)^2 - \tilde{D}(k) = 4 \sum_{y \in \Lambda_L} \tilde{p}_y ((\pi k \cdot y)^2 - \sin^2(\pi k \cdot y)) \leq k^4 \frac{4\pi^4}{3} \sum_{y \in \Lambda_L} \tilde{p}_y y^4, \tag{4.67}
\]

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since $0 \leq x^2 - \sin^2 x \leq x^4/3$ for any $x \in \mathbb{R}$. Here $\sum_{y \in \Lambda_L} \hat{n}_y y^4$ is bounded by the $L$-independent constant $\gamma C_0 k^{-1} \sum_{m=1}^{\infty} m^4 e^{-\gamma^2 m} < \infty$ and thus we can now conclude that there is a constant $c' > 0$, independent of $L$ and the initial state, such that $0 \leq \kappa_L (2\pi k)^2 - \hat{D}(k) \leq c' k^4$. On the other hand, by Proposition 4.9 for any $k \in \Lambda_L$ either $a(k) = 0$ or $\hat{D}(k) \geq c'_2 k^2$ and $0 < a(k) \leq c'_0$. Together with Proposition 4.8 it follows that

$$|f(t, \xi) - \tilde{T}(t, \xi)| \leq \frac{c'}{\gamma} LE \left[ c' \|\varphi\|_1 \int_{\Lambda_L} dk t k^4 e^{-(t+t_0)c'_1 k^2} + \frac{1}{L} \sum_{|n| \geq L/2} |\hat{\varphi}(n/L)| \right],$$

where we have first treated separately the sum over $n \in \Lambda_L$ for which $n/L = k$. Here $\int_{\Lambda_L} dk t k^4 e^{-(t+t_0)c'_1 k^2}$ can be estimated as in the proof of Theorem 4.4, which implies that it is bounded by a constant times $t_0^{-3/2}$. Therefore, collecting the above three estimates together, and readjusting the constant $C$, we find that (4.15) holds.

To prove the final statement, assume additionally that $\varphi \geq 0$, $\int_{\mathbb{R}} dx \varphi(x) = 1$, and $\hat{\varphi}(p) = 0$ for all $|p| \geq \frac{1}{2}$. Then for any $\xi \in LT$ we can apply (4.65) with $\psi_x = 1$ and conclude

$$\sum_{y \in \mathbb{Z}} |\varphi(\xi - y)| = \sum_{y \in \mathbb{Z}} \varphi(\xi - y) = \sum_{n \in \mathbb{Z}} e^{i2\pi n \cdot \xi/L} \frac{1}{L} \hat{\varphi} \left( \frac{n}{L} \right) \sum_{x \in \Lambda_L} e^{-i2\pi n \cdot x/L} \hat{\varphi}(m) = \hat{\varphi}(0) = 1.$$

As now $\|\varphi\|_1 = 1$ and $\sum_{|n| \geq L/2} |\hat{\varphi}(n/L)| = 0$, the second bound (4.16) follows.

5 Proving complete thermalization via local dynamic replicas?

As a conclusion, let us present a local version of the dynamic replica method introduced in Sec. 3 and comment on how this might be used to extend the results derived in Sec. 4 into a proof of complete thermalization. For the present homogeneous system with periodic boundary conditions, the benefits of using a local version of the replicas are perhaps not immediately apparent. However, for any system which is not totally translation invariant, either due to boundary effects or to inhomogeneities, the use of local replicas should be helpful since it allows using different local dynamics for different lattice sites. For instance, near the boundary it will be necessary to incorporate the correct boundary conditions to properly account for reflection and absorption at the boundary, whereas in the “bulk” one could use the simpler periodic dynamics. At present, this is mere speculation, and it remains to be seen how well such a division can be implemented in practice.

For simplicity, let us again assume that the harmonic interactions have a finite range $r_\phi$. The lattice size is still assumed to be $L \gg 1$, but the local dynamics will live on a smaller lattice $\Lambda_R$. For convenience, we assume that $R$ is odd and satisfies $r_\phi \leq R \leq L/2$.
which implies that $R + r_\phi \leq L$. The replicated generating functional is then defined as

$$h_t(\zeta; R) := \left\langle \exp \left[ i \sum_{x \in \Lambda_R, y \in \Lambda_R, i = 1, 2} \zeta^i_{x,y} X(t)_{[x+y],L} \right] \right\rangle, \quad t \geq 0, \quad \zeta \in \mathbb{R}^{2\times \Lambda_L \times \Lambda_R},$$

(5.1)

where we recall the notation $[x]_L$ used for projection onto $\Lambda_L$. Let us also drop “$R$” from the notation from now on. If $y \in \Lambda_R$ and $z \in \Lambda_L$ are such that $[y - z]_L \neq y - z$, then $L/2 \leq |y - z| \leq L$ and thus $|[y - z]_L| \geq L - |y - z| \geq (L - R)/2 \geq r_\phi/2$, implying $\Phi([y - z]_L) = 0$. Hence, $\sum_{x \in \Lambda_L} \Phi([y - z]_L) X^1_{[x+z],L} = \sum_{x \in \Lambda_L} \Phi(y - z) X^1_{[x+z],L} = \sum_{y' \in \Lambda_R} \sum_{\tau \in \{-1,0,1\}} \Phi(y - y' - \tau R) X^1_{[x+y'+\tau R],L}$ for any $y \in \Lambda_R$. Since for any $y, y' \in \Lambda_R$ also $\sum_{\tau \in \{-1,0,1\}} \Phi(y - y' - \tau R) = \Phi([y - y']_R)$, we then obtain as in Sec. 3 the following evolution equation

$$\partial_t h_t(\zeta) = \frac{\gamma}{2} \sum_{x \in \Lambda_L} \left( h_t(\sigma_{x,0} \zeta) - h_t(\zeta) - (\sigma_{x,0} - 1) \zeta \cdot \nabla \zeta h_t(\zeta) - (\mathcal{M}_\gamma \zeta) \cdot \nabla \zeta h_t(\zeta) \right)$$

$$- \sum_{x \in \Lambda_L} \sum_{y' \in \Lambda_R} \sum_{\tau = \pm 1} \Phi(y' - y + \tau R) \zeta^2_{x,y} \left( i X^1_{[x+y+\tau R],L} - X^1_{[x+y],L} \right) e^{i \sum_{x' \in \Lambda_L} \zeta^i_{x',y} X(t)_{x'+y_0}},$$

(5.2)

where the last term collects the terms left over from the periodic harmonic evolution in the replicated direction. The matrix operations in the equation are defined analogously to those appearing in Sec. 3 as

$$(\sigma_{x,0} \zeta)^i_{x,y} := \begin{cases} -\zeta^i_{x,y}, & \text{if } i = 2 \text{ and } [x+y]_L = x_0, \\ \zeta^i_{x,y}, & \text{otherwise,} \end{cases}$$

(5.3)

and, similarly to (3.6),

$$\mathcal{M}_\gamma := \bigoplus_{x_0 \in \Lambda_L} M^i_{\gamma(x_0)}, \quad (M^i_{\gamma(x_0)})^j_{x,y} := \mathbb{1}(x = x_0) (M_{\gamma(x_0)})^j_y, \quad M_{\gamma} := \begin{pmatrix} 0 & \Phi_R \\ -1 & \gamma^1 \end{pmatrix}.$$

(5.4)

The correction term can be expressed via derivatives of $h_t$ using the matrices

$$(D_R \zeta^2)^{x,y} := \sum_{y' \in \Lambda_R} \sum_{\tau = \pm 1} \Phi(y' - y + \tau R) (\zeta^2_{[x-\tau R],y'} - \zeta^2_{x,y'}), \quad D := \begin{pmatrix} 0 & D_R \\ 0 & 0 \end{pmatrix}.$$\ (5.5)

As in Sec. 3 we denote $\zeta_s := e^{-s\mathcal{M}_\gamma} \zeta$ and obtain the equality

$$h_t(\zeta) = h_0(\zeta_t) - \int_0^t ds \left( (D_s \zeta) \cdot \nabla \zeta h_{t-s}(\zeta_s) \\ + \int_0^s \frac{\gamma}{2} \sum_{x_0 \in \Lambda_L} \left( h_{t-s}(\sigma_{x_0} \zeta) - h_{t-s}(\zeta) - (\sigma_{x_0} - 1) \zeta_s \cdot \nabla \zeta h_{t-s}(\zeta_s) \right) \right).$$

(5.6)
We recall the definition of the local statistics generating function and choose \( R_0 = R \) here. Explicitly, \( f_{t,x_0}(ξ) := h_t(ξ[ξ, x_0]) \) with \( ξ[ξ, x_0]_y := Π(x = x_0)ξ_y^0 \) for \( ξ \in \mathbb{R}^{2×Λ_R} \), \( x_0 \in Λ_L \). We denote \( ξ_s := e^{-sM_\varepsilon}ξ \), for which clearly \( ξ[ξ, x_0]_s = ξ[ξ_s, x_0] \). Similarly, setting \( (S_{y_0}ξ)_y := (-1)^{(i+2y=y_0)}ξ_y \) implies that \( (1-σ(x))ξ[ξ, x_0] = \sum_{y_0∈Λ_R} Π(y_0 = [x'-x_0]_L) \times ξ[(1-S_{y_0})ξ, x_0] \). Then it is straightforward to check that \( f_{t,x_0} \) satisfies

\[
(1 - C)f_t(ξ) = f_0(ξ_t) - \int_0^t ds \left( Dξ_s \cdot ∇ξh_{t-s}(ξ_s)\right)_{ξ_s=ξ[ξ_s, x_0]}, \quad \text{with}
\]

\[
(Cf)_t(ξ) := \int_0^t ds \left. \frac{γ}{2} \sum_{y_0∈Λ_R} (f_{t-s}(S_{y_0}ξ_0) - f_{t-s}(ξ_0) - (S_{y_0} - 1)ξ_0 \cdot ∇ξf_{t-s}(ξ_0)) \right). \tag{5.7}
\]

The operator \( C \) is essentially the same as the one appearing in Sec. 3. For the sake of argument, suppose that we could extend the strong estimate in (4.9) and show that, if \( g \) is exponentially decaying in time, then \( (1 - C)^{-1}g \) is always a sum of an equilibrium generating function and a term which is exponentially small on the \( R \)-diffusive time scale, in \( tR^{-2} \). Then the above formula would imply that also \( f \) has this property, i.e., that strong local equilibrium holds. The main additional hurdle for such analysis is to find a separate control for the current term, \( \int_0^t ds (Dξ_s) \cdot ∇ξh_{t-s}(ξ_s) \), since the term needs to be sufficiently small and slowly varying for the local equilibrium approximation to hold. Proving this might be possible with an iterative argument. However, further developments are needed to put any such scheme on solid ground.

### 6 On applications to other dynamical systems

Although the result in Theorem 4.4 requires sufficiently large lattice sizes and times, it does not involve taking any direct scaling limits. This is somewhat surprising considering that up to know nearly all mathematically rigorous work on energy diffusion in particle systems of the present kind has relied on control of either a hydrodynamic scaling limit, or as a middle step, a kinetic scaling limit leading to a Boltzmann equation, see for instance 23 [24, 25].

It is thus fair to ask if the present results are just particular properties of the harmonic particle chain with velocity flips. Indeed, it is unlikely that the above computations for the velocity flip model can directly be carried over to other models, such as purely Hamiltonian dynamical systems. In the present case the evolution equation of second moments is closed, in the sense that it does not depend on any higher order moments, and hence we do not need to invert the full evolution equation of the characteristic function, (3.10), but instead we can use an equation derived for its second derivatives evaluated at zero, (4.2). In addition, for the present system it is straightforward to separate a part of the generator related to the “perturbation” (the flips), and use this to produce a spectral gap for the exponentiated linear term: this results in the change of \( e^{-tM_0} \) to \( e^{-tM_\varepsilon} \), and is responsible for the exponential decay in time of the memory kernel \( p_{t,x} \) in (4.2).

To give an indication of the structure for other dynamical systems, we give below an outline of an application of the dynamic replica method for an anharmonic particle chain.
The application mimics the standard perturbation theory, and should be taken with a grain of salt: there could well exist another way of organizing the interactions so that the replica evolution semigroup has better decay properties. For this reason, we skip all technical details of the computations below.

We consider the Hamiltonian system obtained by replacing the earlier velocity flips by an anharmonic pinning potential $\gamma q_x^4$. Explicitly, fix $L$ and let $\Phi_L$ and $H_L$ be defined as before, by (2.3) and (2.9). Define then for $X \in \mathbb{R}^{\Lambda_L} \times \mathbb{R}^{\Lambda_L}$

$$V_L(X) := \gamma \sum_{x \in \Lambda_L} \frac{1}{4}(X_1^x)^4,$$  

(6.1)

choose some coupling $\gamma > 0$, and set

$$H_L^{\text{anh}}(X) := H_L(X) + \gamma V_L(X) = \sum_{x \in \Lambda_L} \frac{1}{2} p_x^2 + \sum_{x',x \in \Lambda_L} \frac{1}{2} (\Phi_L)_{x',x} q_{x'} q_x + \gamma \sum_{x \in \Lambda_L} \frac{1}{4} q_x^4.$$  

(6.2)

The corresponding Hamiltonian evolution equations are

$$\partial_t q(t)_x = p(t)_x, \quad \partial_t p(t)_x = -(\Phi_L q(t))_x - \gamma q(t)_x^3.$$  

(6.3)

For any initial data $q(0), p(0)$ there is a unique differentiable solution to these equations, and this defines $X(t; X(0))$. We choose some random distribution $\mu_0$ for the initial data $X(0)$ and use this to define a random vector $X(t)$. The corresponding replicated generating function is

$$h_t(\zeta) := \langle e^{i Y_t} \rangle, \quad Y_t := \sum_{x,y,i} \zeta_{x,y}^i X(t)^i_{x+y},$$  

(6.4)

and it satisfies an evolution equation

$$\partial_t h_t(\zeta) = \sum_{x,y \in \Lambda_L} \zeta_{x,y}^1 \langle i X_{x+y}^1 e^{i Y_t} \rangle - \sum_{x,y,z \in \Lambda_L} \zeta_{x,y}^2 (\Phi_L)_{x+y,x+z} \langle i X_{x+z}^1 e^{i Y_t} \rangle$$

$$- \gamma \sum_{x,y \in \Lambda_L} \zeta_{x,y}^2 \langle i (X_{x+y}^1)^3 e^{i Y_t} \rangle.$$  

(6.5)

We close the equation by using the previous computations for the first two terms and set the anharmonic term to act at the origin of the replicated direction. This yields

$$\partial_t h_t(\zeta) = -(\mathcal{M}_0 \zeta) \cdot \nabla h_t(\zeta) + \gamma \sum_{x_0 \in \Lambda_L} \left( \sum_{y \in \Lambda_L} \zeta_{x_0-y,y}^2 \right) \partial_{\zeta_{x_0,0}}^1 h_t(\zeta).$$  

(6.6)

Proceeding as in Section 3 we obtain the following Duhamel formula for $h_t$: with $\zeta_t := e^{-t \mathcal{M}_0} \zeta$ we have

$$h_t(\zeta) = h_0(\zeta_t) + \gamma \int_0^t ds \sum_{x \in \Lambda_L} \left( \sum_{y \in \Lambda_L} (\zeta_s)_{x-y,y}^2 \right) \partial_{\zeta_{x,0}}^1 h_{t-s} \bigg|_{\zeta_s}.$$  

(6.7)
The formula (6.7) could then be used as a starting point for further analysis. For instance, if the goal is the study of kinetic scaling limits, which involve only times $0 \leq t \leq \tau \gamma^{-2}$, $\tau > 0$ fixed and $\gamma \ll 1$, it could be iterated once to obtain

$$h_t(\zeta) = h_0(\zeta)$$

$$+ \gamma \sum_{x \in \Lambda_{L}^*} \sum_{i \in \{1,2\}} \sum_y \left( \sum_{x' \in \gamma_{x-i} \Lambda_{L}^*} \right) \partial_{\zeta_{x-i}} \partial_{\zeta_{x+i}} \partial_{\zeta_{x+2i}} h_0 |_{\zeta} \int_0^t ds \left( e^{sM_0} \right)_{i,0}^{2,i} \prod_j (e^{-sM_0})_{z_j,0}^{3,i}$$

$$+ \gamma^2 \int_0^t dt' \sum_{x \in \Lambda_{L}^*} \sum_{i \in \{1,2\}} \sum_{x' \in \gamma_x \Lambda_{L}^*} \left( \sum_{y \in \gamma_y \Lambda_{L}^*} \right) \partial_{\zeta_{x-i}} \partial_{\zeta_{x+i}} \partial_{\zeta_{x+2i}} \partial_{\zeta_{x+3i}} h_{t-t'} |_{\zeta'} \prod_j (e^{-sM_0})_{z_j,0}^{3,i}$$

$$+ 3 \gamma^2 \int_0^t dt' \sum_{x \in \Lambda_{L}^*} \sum_{i \in \{1,2\}} \sum_{x' \in \gamma_x \Lambda_{L}^*} \left( \sum_{y \in \gamma_y \Lambda_{L}^*} \right) \partial_{\zeta_{x-i}} \partial_{\zeta_{x+i}} \partial_{\zeta_{x+2i}} \partial_{\zeta_{x+3i}} h_{t-t'} |_{\zeta'} \prod_j (e^{-sM_0})_{z_j,0}^{3,i}.$$  (6.8)

The new terms all have additional decay arising from the integration over $s$ which involves four oscillating factors. This should facilitate rigorous analysis of the kinetic scaling limits for a suitable class of initial states $\mu_0$.

However, let us stress once more that this approach might not be optimal since it uses the unmodified free evolution in the replicated directions. Settling the question will likely require careful consideration of which initial data to allow and of what function space to use for the solutions $h_t$.

## A Local dynamics

In this section we collect some basic properties of the semigroup $e^{-tM}$, for

$$M_\gamma := \begin{pmatrix} 0 & \Phi_L \\ -1 & \gamma I \end{pmatrix}. \quad (A.1)$$

Since $\Phi_L$ is invariant under periodic translations, it can be diagonalized by Fourier transform. A discrete Fourier transform then yields, for $k \in \Lambda_{L}^*$,

$$\widehat{M}_\gamma(k) := \begin{pmatrix} 0 & \omega(k)^2 \\ -1 & \gamma \end{pmatrix}. \quad (A.2)$$

We make a Jordan decomposition of this matrix. The two eigenvalues, labeled by $\sigma = \pm 1$, are

$$\mu_\sigma(k) := \frac{\gamma}{2} + \sigma \sqrt{(\gamma/2)^2 - \omega(k)^2} = \frac{\gamma}{2} + i\sigma \sqrt{\omega(k)^2 - (\gamma/2)^2}, \quad (A.3)$$

\[33\]
where the first formula is used if \( \gamma > 2\omega(k) \), and the second formula otherwise. The case \( \gamma = 2\omega(k) \) has a nontrivial Jordan block and needs to be treated separately. Using the decomposition, it is straightforward to conclude that

\[
(e^{-tM_i})^{y_i} = \int_{\Lambda_L} dk e^{i2\pi k \cdot (y'-y)} (e^{-t\tilde{M}_i(k)})^{y_i}
\]  

(A.4)

where

\[
e^{-t\tilde{M}_i(k)} = \sum_{\sigma = \pm 1} e^{-t\mu_\sigma(k)} P_\sigma(k),
\]  

(A.5)

\[
P_\sigma(k) := \frac{1}{\mu_\sigma - \mu_{-\sigma}} \left( \begin{array}{c} -\mu_{-\sigma} & \omega^2 \\ -1 & \mu_\sigma \end{array} \right) \bigg|_{\mu_{\pm} = \mu_\pm(k), \omega := \omega(k)}, \quad \text{if } |\omega(k)| \neq \frac{\gamma}{2},
\]  

(A.6)

and taking the formal limit \( \omega(k) \to \gamma/2 \) yields the correct expressions also for the degenerate case. The eigenvalues satisfy, for \( \omega(k) \geq \gamma/2 \) and dropping the variable \( k \) from the notations,

\[
\mu_\sigma = \frac{\gamma}{2} + i\sigma \sqrt{\omega^2 - (\gamma/2)^2}, \quad |\mu_\sigma| = \omega, \quad \mu_\sigma^* = \mu_{-\sigma}, \quad \mu_\sigma - \mu_{-\sigma} = 2i\sigma \sqrt{\omega^2 - (\gamma/2)^2}.
\]  

(A.7)

If \( \omega(k) < \gamma/2 \), then

\[
\mu_- < \omega < \mu_+, \quad \mu_+ \in \left[\frac{1}{2}, 1\right], \quad \mu_- = \frac{\omega^2}{\mu_+} \in \left[\frac{\omega^2}{\gamma} [1, 2]\right], \quad \mu_\sigma - \mu_{-\sigma} = 2\sigma \sqrt{(\gamma/2)^2 - \omega^2}.
\]  

(A.8)

B Solution of the lattice renewal equation

In this appendix, we recall some basic mathematical properties of the Green’s function solution of the renewal equations encountered in the main text. By Corollary 4.7 the corresponding functions \( p_{t,x} \) satisfy all of the assumptions below.

**Proposition B.1** Consider some \( L \geq 1 \) and \( p_{t,x} \), given for \( t \geq 0, x \in \Lambda_L \). Suppose that there are constants \( C_0, \delta_0, \gamma_2 > 0 \) such that all of the following statements hold:

1. \( t \mapsto p_{t,x} \) belongs to \( C^{(1)}([0, \infty)) \) for all \( x \in \Lambda_L \).
2. \( p_{t,x} \geq 0 \) and \( |\partial_t p_{t,x}|, p_{t,x} \leq C_0 e^{-\gamma_2 |x| - \delta_0 t} \) for all \( x \in \Lambda_L \) and \( t > 0 \).
3. \( \int_0^\infty dt \sum_{x \in \Lambda_L} p_{t,x} = 1 \) and \( \int_0^t \sum_{x \in \Lambda_L} p_{t,x} < 1 \) for all \( t_0 \geq 0 \).

Then there is a unique continuous function \( G \in C([0, \infty) \times \Lambda_L) \) for which

\[
G(t, x) = p_{t,x} + \int_0^t ds \sum_{y \in \Lambda_L} G(t - s, x - y) p_{s,y}, \quad t \geq 0, \ x \in \Lambda_L.
\]  

(B.1)
In addition, this $G$ is non-negative, bounded, continuously differentiable, and has the following pointwise integral representation, valid for any $\varepsilon > 0$,

$$G(t, x) = p_{t,x} + \int_{\Lambda_k^*} \text{d}k \, e^{i2\pi k \cdot x} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{\lambda}{2\pi i} e^{\lambda t} \frac{\hat{p}(\lambda, k)^2}{1 - \hat{p}(\lambda, k)}. \quad (B.2)$$

In the integrand, the function $\hat{p}$ denotes the Laplace-Fourier transform of $p$, defined for all $k \in \Lambda_k^*$ and $\lambda \in \mathbb{C}$ with Re $\lambda > -\delta_0$ by

$$\hat{p}(\lambda, k) := \int_0^\infty \text{d}s \sum_{y \in \Lambda_L} p_{s,y} e^{-s\lambda} e^{-i2\pi k \cdot y}. \quad (B.3)$$

It satisfies the following properties:

1. The map $\lambda \mapsto \hat{p}(\lambda, k)$ is an analytic function on the half plane Re $\lambda > -\delta_0$ for any $k$, and there all of its $\lambda$-derivatives can be computed by differentiating the integrand.
2. For any $\varepsilon > 0$, there is $c_\varepsilon > 0$ such that $|\hat{p}(\varepsilon + i\alpha, k)| \leq 1 - c_\varepsilon$ for all $\alpha \in \mathbb{R}$ and $k \in \Lambda_k^*$.
3. There is $C' > 0$, which depends only on the input constants $C_0, \delta_0, \gamma_2 > 0$, such that $|\hat{p}(\lambda, k)| \leq C'(1 + |\lambda|)^{-1}$ whenever Re $\lambda \geq -\delta_0/2$.

In particular, the integral in (B.3) is always absolutely convergent.

Proof: Assume that the constants $C_0, \delta_0, \gamma_2 > 0$, which will be called the input parameters, and $p_{t,x}$ are given as above. Then the integrand in the definition (B.3) is bounded by $C_0 e^{-\gamma_2 |y| - s(\delta_0 + \text{Re} \lambda)}$, hence it is bounded by an $L^1$-function for any compact subset of the half-plane Re $\lambda > -\delta_0$. Since the integrand is an entire function of $\lambda$, we can now conclude (for instance via Morera’s theorem) that for any $k$ formula (B.3) defines an analytic function $\hat{p}(\lambda, k)$ on the half plane. In addition, we can compute the derivatives by differentiation inside the integral for these values of $\lambda$; this can be concluded for instance by relying on Cauchy’s integral formula for derivatives of an analytic function, and then using Fubini’s theorem. In particular, $\hat{p}$ satisfies the properties in item 1

If Re $\lambda \geq \varepsilon > 0$, then $|\hat{p}(\lambda, k)| \leq \int_0^\infty \text{d}s \sum_{y \in \Lambda_L} p_{s,y} e^{-s\varepsilon} =: 1 - c_\varepsilon$. Applying the assumptions, then $c_\varepsilon = \int_0^\infty \text{d}s \sum_{y} p_{s,y} (1 - e^{-s\varepsilon}) = \varepsilon \int_0^\infty \text{d}s' e^{-s' \varepsilon} \int_s^\infty \text{d}s \sum_{y} p_{s,y} \geq 0$. If $c_\varepsilon = 0$, then $\int_s^\infty \text{d}s' \sum_{y} p_{s,y} = 0$ for almost every $s' > 0$, which contradicts the assumed normalization condition. Therefore, $\hat{p}$ satisfies also item 2. In particular, $1 - \hat{p}(\lambda, k)$ has no zeroes on the positive right half-plane.

The assumed differentiability of $p$ allows to use partial integration to conclude that for any $\lambda \neq 0$ with Re $\lambda > -\delta_0$ we have

$$\hat{p}(\lambda, k) = \frac{1}{\lambda} \sum_{y \in \Lambda_L} e^{-i2\pi k \cdot y} \left[ p_{0,y} + \int_0^\infty \text{d}s e^{-s\lambda} \partial_s p_{s,y} \right], \quad (B.4)$$
Therefore, item 3 holds with $C$ as mentioned whose steps can be justified by using Fubini’s theorem and the above observation. Then $G$ is continuous. Also, the above bounds imply that the value of the integral in (B.2) is zero in (B.2) for all $x \in \Lambda_L$. Relying on dominated convergence, we conclude that then $G$ is continuous. Also, the above bounds imply that the value of the integral in (B.2) is zero for any $t \geq 0$ since in that case Cauchy’s theorem allows taking $\varepsilon \to \infty$. In particular, then $G(0,x) = p_{0,x}$ and thus (B.1) holds at $t = 0$.

In order to check that $G$ satisfies (B.1) also for $t > 0$, we rely on the following computation, whose steps can be justified by using Fubini’s theorem and the above observation about the vanishing of the integral for negative $t$:

$$\int_0^t ds \sum_y p_{s,y} G(t-s, x-y) - \int_0^t ds \sum_y p_{s,y} p_{t-s,x-y}$$

$$= \int_0^t ds \sum_y p_{s,y} \int_{\Lambda_L} dk e^{i2\pi k(x-y)} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d\lambda}{2\pi i} e^{\lambda(t-s)} \frac{\hat{p}(\lambda, k)^2}{1 - \hat{p}(\lambda, k)}$$

$$= \int_0^t ds \sum_y p_{s,y} \int_{\Lambda_L} dk e^{i2\pi k(x-y)} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d\lambda}{2\pi i} e^{\lambda(t-s)} \frac{\hat{p}(\lambda, k)^2}{1 - \hat{p}(\lambda, k)}$$

$$= G(t,x) - p_{t,x} - \int_{\Lambda_L} dk e^{i2\pi k x} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{d\lambda}{2\pi i} e^{\lambda t} \hat{p}(\lambda, k)^2.$$  \hspace{1cm} (B.5)

Since the final integral is equal to $\int_0^t ds \sum_y p_{s,y} p_{t-s,x-y}$, we have proven that (B.1) holds for all $t \geq 0$ and $x \in \Lambda_L$.

Hence, the above function $G$ provides a continuous solution to (B.1). Let us next prove that this solution is unique. Consider an arbitrary $t_0 > 0$ and the Banach space $X = C([0,t_0] \times \Lambda_L, \mathbb{C})$ endowed with the sup-norm. For $f \in X$ and $0 \leq t \leq t_0$, $x \in \Lambda_L$, set $(Bf)(t,x) := \int_0^t ds \sum_{y \in \Lambda_L} f(t-s,x-y)p_{s,y}$. Then $Bf$ is continuous and bounded by $r_0\|f\|$, where $r_0 := \int_0^t ds \sum_{y \in \Lambda_L} p_{s,y}$. Hence $\|B\| \leq r_0$ and it follows from the assumptions that $r_0 < 1$. Therefore, the inverse of $1 - B$ exists and is a bounded operator on $X$ defined by the convergent Dyson series formula, $(1 - B)^{-1} = \sum_{n=0}^{\infty} B^n$. Suppose $f$ is a continuous function which solves (B.1) and let $g$ denote the restriction of $f$ to $[0,t_0]$. Then we have $g \in X$ and $(1 - B)g$ is equal to $h$, the restriction of $p$ to $[0,t_0]$. Thus necessarily $g = (1 - B)^{-1}h$. Hence for all $0 \leq t \leq t_0$ we have $f(t,x) = \sum_{n=0}^{\infty} (B^n h)(t,x)$, and as $t_0$ can be taken arbitrarily large, this proves the uniqueness of the solution. Since $B$ is obviously positivity preserving and $h$ is nonnegative, this also implies that the unique
solution, coinciding with $G$ defined in (B.2), is pointwise nonnegative. As $p$ is assumed to be continuously differentiable, the continuous solution $G$ to (B.1) is that, as well. This concludes the proof of the Proposition.

**Corollary B.2** Suppose that $p$ and $G$ are given as in Proposition B.1. Then for any $h \in C([0, \infty) \times \Lambda_L, \mathbb{C})$ the formula

$$f(t, x) := h(t, x) + \int_0^t ds \sum_{y \in \Lambda_L} G(t - s, x - y) h(s, y), \quad t \geq 0, \ x \in \Lambda_L,$$

defines the unique continuous solution to the equation

$$f(t, x) = h(t, x) + \int_0^t ds \sum_{y \in \Lambda_L} p_{s,y} f(t - s, x - y).$$

**Proof:** Since $h$ is continuous, dominated convergence theorem immediately implies that (B.6) defines a continuous function. Then a straightforward application of Fubini’s theorem and (B.1) proves that (B.7) holds for all $t, x$. The uniqueness can be proven via the same argument which was used in the proof of Proposition B.1.

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