Every linear group acts by isometries on some properly convex domain in real projective space. This follows from the fact that action of SL(n, R) on the space of quadratic form in n variables preserves the properly convex cone consisting of positive definite forms. If Γ is the holonomy of a properly convex orbifold of finite volume then every virtually nilpotent group is virtually abelian, moreover every unipotent element is conjugate into PO(n, 1). A reference for all this is [1]. This paper gives the first example of a unipotent group that is not virtually abelian and preserves a strictly convex domain. It answers a question asked by Misha Kapovich.

The Heisenberg group is the subgroup $H \subset$ SL(3, R) of unipotent upper-triangular matrices. Define $\theta : H \rightarrow$ SL(10, R) and $G = \theta(H)$ where

$$
\theta \left( \begin{array}{ccc}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{array} \right) = \left( \begin{array}{cccccccccc}
1 & 2a & 2c & a & a^2/2 & a^3/6 & b & 2a^2 + b^2/2 & b^3/6 + 2ac & (a^4 + b^4)/24 + c^2 \\
0 & 1 & b & 0 & 0 & 0 & 2a & ab + c & bc & 0 \\
0 & 0 & 1 & a & a^2/2 & 0 & 0 & a & c & 0 \\
0 & 0 & 0 & 1 & a & 0 & 0 & 0 & a^2/2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & a & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array} \right)
$$

It is clear that $\theta$ is injective and easy to check that it is a homomorphism. Since the center of $H$ is $Z \cong R$ and $H/Z \cong R^2$ it is also easy to check that every non-trivial element of $G$ has a unique largest Jordan block, and that this block has odd size. It easily follows that each element of $G$ preserves some properly convex domain depending on that element, cf the discussion of parabolics in (2.9) of [1].

**Theorem 0.1.** There is a strictly convex domain $\Omega \subset \mathbb{RP}^9$ that is preserved by $G$. This is an effective action of the Heisenberg group on $\Omega$ by parabolic isometries that are unipotent.

**Proof.** The group $G$ acts affinely on the affine patch $[x_1 : x_2 : x_3 : x_4 : x_5 : x_6 : x_7 : x_8 : x_9 : 1]$ that we identify with $\mathbb{R}^9$. Let $p \in \mathbb{R}^9$ be the origin. Then $G \cdot p$ is

$$(a^4 + b^4)/24 + c^2, bc, c, a^3/6, a^2/2, a, b^3/6, b^2/2, b)$$

This orbit is an algebraic embedding $\mathbb{R}^3 \hookrightarrow \mathbb{R}^9$ which limits on the single point $q = [1 : 0 : 0 : 0 : 0 : 0 : 0 : 0 : 0] \in \mathbb{RP}^9$ in the hyperplane at infinity, $P\infty$. This follows from the fact that $(a^4 + b^4)/24 + c^2$ dominates all the other entries whenever at least one of $|a|, |b|, |c|$ is large.

*Date: May 20, 2018.*

The author acknowledges support from U.S. National Science Foundation grants DMS 1107452, 1107263, 1107367 “RNMS: GEometric structures And Representation varieties” (the GEAR Network).

Cooper was partially supported by NSF grants DMS 1065939, 1207068 and 1045292

MSC 57N16, 57M50.
Let $S \subset \mathbb{R}^9$ be this orbit. Choose 10 random points on $S \subset \mathbb{R}^9$ and compute the determinant, $d$, of the corresponding 10 vectors in $\mathbb{R}^{10}$. Then $d \neq 0$ therefore the interior $\Omega^+ \subset \mathbb{R}^9$ of the convex hull of $S$ has dimension 9.

Moreover the closure $\Omega'$ of $\Omega^+$ in $\mathbb{R}^9$ is disjoint from the closure of the affine hyperplane $x_1 = -1$, hence $\Omega'$ is properly convex. Since $\Omega' \cap P_{\infty} = q$ and $G$ preserves $q$ and $P_{\infty}$ and $G$ is unipotent, it follows from (5.8) in [1] that $G$ preserves some strictly convex domain $\Omega \subset \Omega'$.

**Corollary 0.2.** There is a strictly convex real projective manifold $\Omega/\Gamma$ of dimension 9 with nilpotent fundamental group $\Gamma \cong \langle \alpha, \beta : [\alpha, [\alpha, \beta]], [\beta, [\alpha, \beta]] \rangle$ that is not virtually abelian. Moreover $\Gamma$ is unipotent.

**Proof.** If $\Gamma$ is a lattice in $G$ then $\Omega/\Gamma$ is a strictly convex manifold with unipotent holonomy and $\Gamma$ is nilpotent but not virtually abelian. □

The genesis of this example is as follows. The image of $H$ in $\text{SL}(6, \mathbb{R})$ under the irreducible representation $\text{SL}(3, \mathbb{R}) \rightarrow \text{SL}(6, \mathbb{R})$ is

\[
\begin{pmatrix}
1 & 2a & a^2 & 2c & 2ac & c^2 \\
0 & 1 & a & b & ab + c & bc \\
0 & 0 & 1 & 0 & 2b & b^2 \\
0 & 0 & 0 & 1 & a & c \\
0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

and preserves the properly convex domain $Q \subset \mathbb{RP}^9$ that is the projectivization of the space of positive definite quadratic forms on $\mathbb{R}^3$.

The boundary of the closure of $Q$ consists of semi-definite forms and contains flats, so $Q$ is *not* strictly convex. Let $A, B, C \in \text{SL}(6, \mathbb{R})$ be the elements corresponding to one of $a, b, c$ being 1 and the others 0. Each of $A, B, C$ has a parabolic fixed point in $\partial Q$ corresponding to a rank 1 quadratic form. Every point in $Q$ converges to this parabolic fixed point under iteration by the given group element. The fixed point for $A$ and $B$ are distinct and lie in a flat in $\partial Q$.

The idea is to increase the dimension of the representation and use the extra dimensions to add parabolic blocks of size 5 onto $A$ (row 1 and rows 7-10) and onto $B$ (row 1 and rows 11-14) that commute and the parabolic fixed point of each block is the rank-1 form that is a fixed point of $C$. This gives a 14-dimensional representation of $H$:

\[
\begin{pmatrix}
1 & 2a & a^2 & 2c & 2ac & c^2 & a & a^2/2 & a^3/6 & a^4/24 & b & b^2/2 & b^3/6 & b^4/24 \\
0 & 1 & a & b & ab + c & bc & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2b & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & a^3/6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & a^2/2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 & b^3/6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b & b^2/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & b \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The top-left $6 \times 6$ block is the image of $H$ in $\text{SL}(6, \mathbb{R})$. The entries in $A^n$ and $B^n$ grow like $n^2$. This is beaten by the growth of some entries in the added blocks of size 5 which grow like $n^4$. This
gives rise to a representation of $H$ of dimension $6 + 4 + 4 = 14$. The orbit of
\[ [0 : 0 : 0 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1 : 0 : 0 : 0 : 1] \]
is
\[ [(a^3 + b^4)/24 + c^2 : bc : b^2 : c : b : 1 : a^3/6 : a^2/2 : a : 1 : b^3/6 : b^2/2 : b : 1] \]
so there is a codimension-4 projective hyperplane that is preserved, and which is defined by
\[ x_6 = x_{10} = x_{14} \quad x_5 = x_{13} \quad x_3 = 2x_{12} \]
The restriction to this hyperplane gives $\theta$.

References

[1] D. Cooper, D. D. Long, and S. Tillmann. On convex projective manifolds and cusps. *Adv. Math.*, 277:181–251, 2015.

Department of Mathematics, University of California, Santa Barbara, CA 93106, USA

E-mail address: cooper@math.ucsb.edu