SIMPLE FOLD MAPS WITH REGULAR FIBERS CONSISTING OF PRODUCTS OF SPHERES

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Abstract. In this paper, we study simple fold maps or fold maps such that any connected component of the fiber of any singular value includes at most 1 singular point. More precisely, we study simple fold maps whose regular fibers are disjoint unions of products of spheres. We consider Lemma 1 of [6], which states that a closed manifold admitting a simple fold map whose regular fibers are always disjoint unions of spheres bound a nice manifold and Corollary 3 of [6], which states that the quotient map from the closed (and connected) manifold into a quotient space (the Reeb space) induced from the map gives some isomorphisms of homotopy groups: these are extensions of the results of [17] under some constraints. We extend these facts to simple fold maps whose regular fibers, or fibers of regular values, may have connected components homeomorphic to products of two spheres under technical conditions.

Fold maps are important in generalizing the theory of Morse functions. Studies of such maps were started by Whitney ([23]) and Thom ([20]) in the 1950’s. A fold map is a $C^\infty$ map whose singular points are of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)$$

for two positive integers $m \geq n$ and an integer $0 \leq i \leq m - n + 1$. A Morse function is naturally regarded as a fold map ($n = 1$). For a fold map from a closed $C^\infty$ manifold of dimension $m$ into a $C^\infty$ manifold of dimension $n$ (without boundary) the followings hold where $m \geq n \geq 1$.

1. The singular set, or the set of all the singular points, is a closed $C^\infty$ submanifold of dimension $n - 1$ of the source manifold.
2. The restriction map to the singular set is a $C^\infty$ immersion of codimension 1.

We also note that if the restriction map to the singular set is an immersion with normal crossings, it is stable (stable maps are important in the theory of global singularity; see [3] for example).

Since around the 1990’s, fold maps with additional conditions have been actively studied. For example, in [1], [2], [15], [16] and [19], special generic maps, which are defined as fold maps whose singular points are of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m} x_k^2)$$

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for two positive integers \( m \geq n \), were studied. In \([19]\), Sakuma studied simple fold maps, which are defined as fold maps such that fibers of singular values do not have any connected component with more than one singular points (see also \([14]\)). For example, special generic maps are simple. In \([12]\), Kobayashi and Saeki investigated topology of stable maps including fold maps which are stable into the plane. In \([17]\), Saeki and Suzuoka found good properties of manifolds admitting stable maps whose regular fibers, or fibers of regular values, are disjoint unions of spheres.

The main theorem of \([17]\) or Theorem 4.1 of \([17]\) states that a closed \( C^\infty \) manifold of dimension 4 admitting such a stable map into a surface without boundary bounds a nice compact \( C^\infty \) manifold of dimension 5. In \([5]\) and \([6]\), the theorem is generalized as a proposition for a simple fold map whose regular fibers are disjoint unions of spheres from a closed \( C^\infty \) manifold of dimension \( m \) into a \( C^\infty \) manifold of dimension \( n \) without boundary (see Lemma 1 of \([6]\) and see also Proposition 3.12 and its proof of \([14]\)).

In this paper, we study simple fold maps whose regular fibers are disjoint unions of spheres and products of two spheres. A stable fold map from a closed \( C^\infty \) manifold of dimension 4 into the plane whose regular fibers are connected and homeomorphic to the sphere \( S^2 \) or the torus \( T^2 \) is a good example. It is called a stable fibration of torus type or a simple mapping with \( g \leq 1 \) (see \([7]\), \([8]\), \([11]\) and \([12]\) for example).

In this paper, we generalize Theorem 4.1 of \([17]\) and Lemma 1 of \([6]\) for simple fold maps whose regular fibers may have products of two spheres as connected components under technical conditions.

This paper is organized as follows.

Section 1 is for preliminaries. We recall fold maps including stable fold maps. We also recall special generic maps and simple fold maps. Finally we review the Reeb space of a smooth map, which is the space consisting of all the connected components of all the fibers of the map.

In section 2, we recall results on simple fold maps whose regular fibers are disjoint unions of spheres first: we review Theorem 4.1 of \([17]\), which states that a closed \( C^\infty \) manifold of dimension 4 admitting a stable map into a surface without boundary whose regular fibers are disjoint unions of standard spheres bounds a compact \( C^\infty \) manifold of dimension 5 simple homotopy equivalent to the Reeb space (Proposition 2), and Lemma 1 of \([6]\), a generalization of the theorem for a closed \( C^\infty \) manifold admitting a simple fold map with regular fibers consisting of spheres into a \( C^\infty \) manifold of lower dimension without boundary (Proposition 3).

After that, as a main theorem, we prove a similar theorem for a closed \( C^\infty \) manifold admitting a simple fold map whose regular fibers are disjoint unions of spheres or products of two spheres with conditions defined as Definitions 4, 5 and 6 by analogy of the proofs of the original propositions (Theorem 1). More precisely, we first decompose the Reeb space \( W_f \) of the given simple fold map \( f : M \rightarrow N \) into some pieces by using its simplicial structure, construct manifolds piece by piece and then we glue them together to obtain the desired manifold \( W \) such that \( \partial W = M \).

For example, as statements of Theorem 1, we show that the source manifold bounds a compact PL manifold simple homotopy equivalent to a polyhedron whose dimension is not smaller than that of the Reeb space and not larger than that of the source manifold and that the polyhedron admits a nice surjective simplicial map
onto the Reeb space.

We also introduce some examples (Example 3). As a corollary to Theorem 1, we also show that the composition of the inclusion map from the connected source manifold \(M\) into the connected compact manifold \(W\) bounded by \(M\) and a map \(r\) constructed in the proof of Theorem 1 from the compact manifold \(W\) into the polyhedron \(V\) simple homotopy equivalent to \(W\) gives isomorphisms of some homotopy groups (Corollary 3). In [17], as a corollary to the main theorem of the paper, it has been shown that the fact similar to this holds for a stable map from a closed and connected \(C^\infty\) manifold of dimension 4 into a surface without boundary whose regular fibers are disjoint unions of standard spheres (Corollary 1) and in [6], it has been shown that the fact similar to this holds for a simple fold map from a closed and connected \(C^\infty\) manifold of dimension \(m\) into a \(C^\infty\) manifold of dimension \(n\) \((m > n \geq 1)\) without boundary whose regular fibers are spheres with some additional conditions (see Corollary 3 of [6] or Corollary 2 of the present paper).

In section 3, we introduce an application of Theorem 1.

In [5] and [6], the author introduced and studied round fold maps; a round fold map is defined as a fold map such that the singular set is a disjoint union of standard spheres, that the restriction map to the singular set is a \(C^\infty\) embedding and that the singular value set is a disjoint union of spheres embedded concentrically. We first review the definition of a round fold map and a method of construction. Then we give a round fold map satisfying the assumption of Theorem 1 whose singular set consists of two connected components. We apply the theorem to show that the source manifold of this round fold map bounds a manifold simple homotopy equivalent to a standard sphere (Theorem 2). By applying this theorem and the theory of topology of manifolds (results of [21] and [22] on closed \((k−1)\)-connected \(C^\infty\) manifolds of dimensions \(2k\) where \(k \geq 3\)), we study the diffeomorphism types of manifolds admitting such maps (Corollary 5 and Example 5).

In this paper, we mainly work in the \(C^\infty\) category. Throughout this paper, we assume that \(M\) is a closed \(C^\infty\) manifold of dimension \(m\) \((m \geq 1)\) and that \(N\) is a \(C^\infty\) manifold of dimension \(n\) without boundary \((m \geq n \geq 1)\). We denote a \(C^\infty\) map from \(M\) into \(N\) by \(f : M \to N\) and the singular set of the map, or the set of all the singular points of the map, by \(S(f)\).

We note about homeomorphisms. In this paper, we often consider PL homeomorphisms between two polyhedra including ones between two PL manifolds. In this paper, for two polyhedra \(X_1\) and \(X_2\), a homeomorphism \(\phi : X_1 \to X_2\) which is isotopic (in the topology category) to a homeomorphism from \(X_1\) onto \(X_2\) which gives an isomorphism between underlying simplicial complexes is said to be a PL homeomorphism.

For example, any diffeomorphism between two diffeomorphic manifolds is regarded as a PL homeomorphism, where we consider canonical PL structures.

We also note on the mapping cylinder of a continuous map \(c : X \to Y\). We define the mapping cylinder as the quotient space of \((X \times [0, 1]) \cup Y\) obtained by identifying \((x, 1)\) and \(c(x)\) \((x \in X)\).

1. Preliminaries

For preliminaries, see also section 1 of [6].

1.1. Fold maps. First, we recall fold maps, which are simplest generalizations of Morse functions. See also [3] and [13], for example.
**Definition 1.** For a $C^\infty$ map $f : M \to N$, a point $p \in M$ is a *fold point* of $f$ if at $p$, $f$ has the normal form

$$f(x_1, \ldots, x_m) := (x_1, \ldots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)$$

and if all the singular points of $f$ are fold, we say $f$ is a *fold map*.

If $p \in M$ is a fold point of $f$, then we can define $j := \min\{i, m - n + 1 - i\}$ uniquely in the previous definition. We call $p$ a *fold point of index* $j$ of $f$. We also call a fold point of index 0 a *definite* fold point of $f$ and $f$ a *special generic* map if all the singular points are definite fold (for special generic maps, see [1], [2], [15] and [16] for example). Let $f$ be a fold map. Then the singular set $S(f)$ and the set of all the fold points whose indices are $i$ of $f$, which we denote by $F_i(f)$, are $C^\infty$ $(n - 1)$-submanifolds of $M$. The restriction map $f|_{S(f)}$ is a $C^\infty$ immersion.

**Example 1.** A Morse function on a closed manifold is naturally regarded as a fold map ($n = 1$). A Morse function which has just two singular points is naturally regarded as a special generic map.

In this paper, we sometimes need Morse functions on compact $C^\infty$ manifolds possibly with non-empty boundaries. We call a Morse function on such a manifold *good* if it is constant and minimal on the boundary, singular points of it are not on the boundary and at two distinct singular points, the values are always distinct. We call a Morse function on a closed $C^\infty$ manifold *good* if at two distinct singular points, the values are always distinct.

We introduce *simple* fibers of fold maps and *simple* fold maps.

**Definition 2** (see e.g. [14] and [19] for example). For a fold map $f$ and $p \in f(S(f))$, the fiber $f^{-1}(p)$ is *simple* if each connected component of $f^{-1}(p)$ includes at most one singular point of $f$. $f$ is a *simple* fold map if $f$ is a fold map and for each $p \in f(S(f))$, $f^{-1}(p)$ is simple.

**Example 2.**

1. A good Morse function on a closed manifold is also simple.
2. A fold map $f : M \to \mathbb{R}^n$ is simple if $f|_{S(f)}$ is a $C^\infty$ embedding.

1.2. **Reeb spaces.** We introduce the *Reeb space* of a continuous map.

**Definition 3.** Let $X$, $Y$ be topological spaces. For $p_1, p_2 \in X$ and for a map $e : X \to Y$, we define as $p_1 \sim_p p_2$ if and only if $p_1$ and $p_2$ are in the same connected component of $e^{-1}(p)$ for some $p \in Y$. $\sim_c$ is an equivalence relation.

We set the quotient space $W_c := X/\sim_c$. we call $W_c$ the *Reeb space* of $c$.

We denote the induced quotient map from $X$ into $W_c$ by $q_c$. We define $\bar{e} : W_c \to Y$ so that $c = \bar{e} \circ q_c$. For example, for a Morse function, the Reeb space is a graph and for a simple fold map, the Reeb space is homeomorphic to a polyhedron which is not so complex (see Proposition 1 later). For a special generic map, the Reeb space is homeomorphic to a $C^\infty$ manifold (see section 2 of [15]).

Proposition 1 is well-known and we omit the proof. See also [12], [14] and [17] for example. This proposition is a basic tool in the proofs of Theorem 4.1 of [17] (section 5 of the paper), its generalization for simple fold maps whose regular fibers are disjoint unions of spheres (Lemma 1 of [6]) and their extension in this paper (Theorem 1).
We note that in this paper, an almost-sphere means a $C^\infty$ homotopy sphere given by gluing same dimensional two standard discs together by a diffeomorphism between the boundaries. We note that the underlying PL manifold of an almost-sphere is a standard sphere.

We often use terminologies on fiber bundles in this paper (see also [18] for example). For a topological space $X$, an $X$-bundle is a bundle whose fiber is $X$. A bundle whose structure group is $G$ is a trivial bundle if it is equivalent to the product bundle as a bundle whose structure group is $G$. In this paper, a $C^\infty$ (PL) bundle means a bundle whose fiber is a $C^\infty$ (resp. PL) manifold and whose structure group is a group consisting of some diffeomorphisms (resp. PL homeomorphisms) of the fiber. A linear bundle is a bundle whose fiber is a standard sphere or a standard disc and whose structure group is an orthogonal group.

In this paper, we also use terminologies on polyhedra. In this paper, for a polyhedron of dimension $k \geq 1$, a branched point means a point such that every open neighborhood of the point is not homeomorphic to any open set of $\mathbb{R}^k$ or $\mathbb{R}^k_+ := \{(x_1, \ldots, x_k) \in \mathbb{R}^k \mid x_k \geq 0\}$. If a polyhedron $X$ of dimension $k$ does not have branched points, then it is a manifold with triangulation and we can define the interior $\text{Int} X$ and the boundary $\partial X$.

**Proposition 1.** Let $f : M \to N$ be a special generic map, or a simple fold map, or a stable fold map. Then $W_f$ has the structure of a polyhedron and the followings hold.

1. $W_f - q_f(S(f))$ is uniquely given the structure of a $C^\infty$ manifold of dimension $n$ such that $q_f : M - S(f) \to W_f - q_f(S(f))$ is a $C^\infty$ submersion. For any compact subset $P$ of any connected component of $W_f - q_f(S(f))$, $q_f|_{q_f^{-1}(P)} : q_f^{-1}(P) \to P$ gives the structure of a $C^\infty$ bundle whose fiber is a connected $C^\infty$ $(m - n)$-manifold.

2. The restriction of $q_f$ to the set $F_0(f)$ of all the definite fold points is injective.

3. $f$ is simple if and only if $q_f|_{S(f)} : S(f) \to W_f$ is injective.

4. Suppose that $f$ is simple. Then for any connected component $C$ of $S(f)$, there exists a small regular neighborhood $N(q_f(C))$ of $q_f(C)$ in $W_f$ such that $N(q_f(C))$ has the structure of a PL bundle over $q_f(C)$ and that $q_f^{-1}(N(q_f(C)))$ has the structure of a $C^\infty$ bundle over $q_f(C)$.

Furthermore, assume that $N(q_f(C))$ does not contain branched points and that $N(q_f(C)) - q_f(C)$ is not connected, then $N(q_f(C))$ has the structure of a trivial PL $[-1, 1]$-bundle over $q_f(C)$ and $q_f(C)$ corresponds to the 0-section. $q_f^{-1}(N(q_f(C)))$ has the structure of a $C^\infty$ bundle over $q_f(C)$ and the bundle structure is given by the composition of $q_f|_{q_f^{-1}(N(q_f(C)))} : q_f^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$.

5. For any connected component $C$ of $F_0(f)$, any small regular neighborhood of $q_f(C)$ has the structure of a trivial PL $[0, 1]$-bundle over $q_f(C)$ such that $q_f(C)$ corresponds to the 0-section. We can take a small regular neighborhood $N(q_f(C))$ of $q_f(C)$ so that $q_f^{-1}(N(q_f(C)))$ has the structure of a linear $D^{m-n+1}$-bundle over $q_f(C)$. More precisely, the bundle structure is given by the composition of $q_f|_{q_f^{-1}(N(q_f(C)))} : q_f^{-1}(N(q_f(C))) \to N(q_f(C))$ and the projection to $q_f(C)$. 
(6) Let $f$ be simple and $m - n \geq 1$. If $m - n = 1$, then we also assume that $M$ is orientable.

Then for any connected component $C$ of the set $F_1(f)$ of all the fold points of index 1 such that for any point $p \in q_f(C)$, any small neighborhood $N(p)$ of $p$ and any point $q$ in $N(p) - q_f(C)$, the inverse image $q_f^{-1}(q)$ is an almost-sphere, any small regular neighborhood $N(q_f(C))$ of $q_f(C)$ has the structure of a $K$-bundle over $q_f(C)$, where $K := \{r \exp(2\pi i \theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, \frac{1}{3}, \frac{2}{3}\}$ with the structure group consisting of just two elements, where one element is defined as the identity transformation and the other element is defined as the transformation defined by $z \rightarrow \bar{z}$ ($\bar{z} \in K$ is the complex conjugation of $z \in K$), such that $q_f(C)$ corresponds to the 0-section. We can take the neighborhood $N(q_f(C))$ of $q_f(C)$ so that $q_f^{-1}(N(q_f(C)))$ has the structure of a $C^\infty$ bundle over $q_f(C)$ with a fiber PL homeomorphic to $S^{m-n+1}$ with the interior of a union of disjoint three $(m-n+1)$-dimensional standard closed discs removed. More precisely, the bundle structure is given by the composition of $q_f|_{q_f^{-1}(N(q_f(C)))} : q_f^{-1}(N(q_f(C))) \rightarrow N(q_f(C))$ and the projection to $q_f(C)$.

2. Simple fold maps with regular fibers homeomorphic to products of spheres

In this section, we study simple fold maps with regular fibers homeomorphic to products of spheres.

2.1. Simple fold maps whose regular fibers are disjoint unions of spheres.

We recall some results on simple fold maps or stable maps whose regular fibers are disjoint unions of spheres.

The following theorem is a part of Theorem 4.1 of [17].

**Proposition 2 ([17])**. Let $f : M \rightarrow N$ be a simple fold map from a closed $C^\infty$ manifold of dimension 4 into a $C^\infty$ manifold of dimension 2 without boundary. For each regular value $p$, let $f^{-1}(p)$ be a disjoint union of finite copies of $S^2$. Then there exist a compact $C^\infty$ manifold $W$ of dimension 5 such that $\partial W = M$ and a continuous map $r : W \rightarrow W_f$ such that $r|_{\partial W}$ coincides with $q_f : M \rightarrow W_f$ and the followings hold.

1. For each $p \in W_f - q_f(S(f))$, $r^{-1}(p)$ is diffeomorphic to $D^3$.
2. $f \circ r$ is a $C^\infty$ submersion.
3. There exist a $C^\infty$ triangulation of $W$ and a triangulation of $W_f$ such that $r$ is a simplicial map.
4. For each $p \in W_f$, $r^{-1}(p)$ collapses to a point and $r$ is a homotopy equivalence.
5. $W$ collapses to a subpolyhedron $W'_f$ such that $r|_{W'_f} : W'_f \rightarrow W_f$ is a PL homeomorphism.

If $M$ is orientable, then we can construct $W$ as an orientable manifold.

As a corollary to the proposition, we can prove the following.

**Corollary 1 ([17])**. In the situation of Proposition 3, let $M$ be connected and $i : M \rightarrow W$ be the natural inclusion. Then

$$q_{f*} = r_* \circ i_* : \pi_k(M) \rightarrow \pi_k(W_f)$$
gives an isomorphism for \( k = 0, 1 \).

In [6], the following was shown as Lemma 1.

**Proposition 3 ([6])**. Let \( m > n \geq 1 \) and \( f : M \to N \) be a simple fold map such that the followings hold. If \( m - n = 1 \), then we also assume \( M \) is orientable.

1. For each regular value \( p \), \( f^{-1}(p) \) is a disjoint union of almost-spheres.
2. The indices of all the fold points of \( f \) are 0 or 1.

Then there exist a compact PL manifold \( W \) of dimension \( m + 1 \) such that \( \partial W = M \) and a continuous map \( r : W \to W_f \) such that \( r |_{\partial W} \) coincides with \( q_f : M \to W_f \) and the followings hold.

1. For each \( p \in W_f - q_f(S(f)) \), \( r^{-1}(p) \) is PL homeomorphic to \( D^{m-n+1} \).
2. There exist a triangulation of \( W \) and a triangulation of \( W_f \) such that \( r \) is a simplicial map.
3. For each \( p \in W_f \), \( r^{-1}(p) \) collapses to a point and \( r \) is a homotopy equivalence.
4. \( W \) collapses to a subpolyhedron \( W_{f'} \) such that \( r |_{W_{f'}} : W_{f'} \to W_f \) is a PL homeomorphism.

If \( M \) is orientable, then we can construct \( W \) as an orientable manifold.

We have the following.

**Corollary 2 ([6])**. In the situation of Proposition 2, let \( M \) be connected and \( i : M \to W \) be the natural inclusion. Then

\[
q_{f*} = r_* \circ i_* : \pi_k(M) \to \pi_k(W_f)
\]

gives an isomorphism for \( 0 \leq k \leq m - n - 1 \).

Remark 5.2 of [17] says that we can prove a theorem similar to Proposition 2 of the present paper, or Theorem 4.1 of [17] in the case where the dimension of the source manifold is even and greater than 4. The remark also says that \( W \) is not always smooth. Remark 5.3 of [17] says that we don’t know whether we can prove the theorem in the case where the dimension of the target manifold is greater than 2. In Proposition 3 of the present paper, or Lemma 1 of [6], we concretely generalize the original theorem under some constraints; see also Proposition 3.12 and its proof of [14]. See also Remark 6 of [6].

2.2. Simple fold maps whose regular fibers are not always disjoint unions of spheres. In the previous subsection, we review some results on simple fold maps whose regular fibers are disjoint unions of spheres. If we have a good Morse function on a compact and connected manifold with non-empty boundary with exactly two singular points, where one is a definite fold point and the other singular point is not a definite fold point, then in many cases the manifold is a disc bundle over a sphere and the boundary of the manifold is a sphere bundle over a sphere. From this viewpoint of Morse theory and the theory of handle decompositions, it is natural to consider simple fold maps whose regular fibers are disjoint unions of spheres or products of two spheres.

In this subsection, we prove a similar result for simple fold maps whose regular fibers are not always disjoint unions of spheres and may have connected components homeomorphic to products of two spheres, under some constraints. We introduce technical conditions on maps and show the result.
**Definition 4.** Let \( f : M \to N \) be a simple fold map and \( m - n \geq 2 \).

(1) Let \( C \) be a connected component of \( q_f(S(f)) - q_f(F_0(f)) \) consisting of non-branched points. Assume that there exists a small regular neighborhood \( N(C) \) of \( C \) in \( W_f \) having the structure of a trivial PL \([-1, 1]\)-bundle over \( q_f(C) \) \((q_f(C) \text{ corresponds to the } 0\text{-section})\) and that the composition of \( q_f|_{q_f^{-1}(N(C))} : q_f^{-1}(N(C)) \to N(C) \) and the projection to \( q_f(C) \) gives \( q_f^{-1}(N(C)) \) the structure of a \( C^\infty \) bundle over \( C \) whose fiber is PL homeomorphic to \( D^{k+1} \times S^{m-n-k} - \text{Int}D^{m-n+1} \) for an integer \( 1 \leq k < m - n \).

Then \( C \) is said to be a \( k \) S-locus.

(2) Let \( R \) be a connected component of \( W_f - q_f(S(f)) \).

If for any compact subset \( P \) in \( R \), \( q_f|_{q_f^{-1}(P)} : q_f^{-1}(P) \to P \) gives the structure of a bundle over \( P \) whose fiber is PL homeomorphic to \( S^k \times S^{m-n-k} \) for an integer \( 1 \leq k \leq \lfloor \frac{m-n}{2} \rfloor \), then \( R \) is said to be a \( k \) S-region or an \( m-n-k \) S-loci.(note that \( k = m-n-k \) \( \text{if} \ m-n \) is even and \( k = \frac{m-n}{2} \)).

For example, on \( S^{k_1} \times S^{k_2} \) \((k_1, k_2 \in \mathbb{N}, k_1, k_2 \geq 2)\), there exists a good Morse function \( f : S^{k_1} \times S^{k_2} \to \mathbb{R} \) with four singular points such that \( W_f \) contains two \( k_1-1 \) \((k_2-1)\) S-loci.

Let \( f : M \to N \) be a simple fold map and \( p \in S(f) \). Let \( q_f(p) \) be in a \( k \) S-locus. If \( k+1 \leq \lfloor \frac{m-n+1}{2} \rfloor \), then \( p \in F_{k+1}(f) \) and if \( k+1 > \lfloor \frac{m-n+1}{2} \rfloor \), then \( p \in F_{m-n-k}(f) \).

**Definition 5.** If a simple fold map \( f : M \to N \) \((m-n \geq 2)\) satisfies the followings, then \( f \) is said to be a normal simple fold map with regular fibers of two spheres.

(1) Any connected component of \( q_f(S(f)) \) in \( W_f - q_f(F_0(f)) \) consisting of non-branched points is an S-locus.

(2) For any connected component \( R \) of \( W_f - q_f(S(f)) \) such that the intersection of the closure \( \overline{R} \) and the set of all the branched points of \( W_f \) is non-empty, the fiber of each point in \( R \) is an almost-sphere.

For example, on \( S^{k_1} \times S^{k_2} \) \((k_1, k_2 \in \mathbb{N})\), there exists a good Morse function \( f : S^{k_1} \times S^{k_2} \to \mathbb{R} \) with four singular points which is also a normal simple fold map with regular fibers of two spheres.

**Definition 6.** Let \( f \) be a normal simple fold map with regular fibers of two spheres.

(1) If for any \( k \) S-loci \( C \), the connected component \( R \) of \( W_f - q_f(S(f)) \) which is not an AS-region such that \( C \) is in the boundary of the closure \( \overline{R} \), is a \( k \) S-region and the boundary of the closure is a disjoint union of \( k \) S-loci, then we say that \( f \) decomposes into S-systems.

(2) Assume that \( f \) decomposes into S-systems.

We say that \( f \) has a family of S-identifications if there exists a family of small regular neighborhoods \( \{N(C_\lambda)\}_{\lambda \in \Lambda} \) of all the S-loci \( \{C_\lambda\}_{\lambda \in \Lambda} \) such that the followings hold.

(a) (The bundle structure of \( q_f^{-1}(N(C_\lambda)) \) over \( C_\lambda \))

Let \( C_\lambda \) be a \( k \) S-locus. \( q_f^{-1}(N(C_\lambda)) \) has the structure of a \( C^\infty \) bundle over \( C_\lambda \) with the fiber PL homeomorphic to \( D^{k+1} \times S^{m-n-k} - \text{Int}D^{m-n+1} \subset D^{k+1} \times S^{m-n-k} \) for an integer \( 1 \leq k < m - n \) such that for the connected component \( C \) of \( \partial N(C_\lambda) \) in a \( k \) S-region, \( q_f^{-1}(C) \) has
the structure of a subbundle of the bundle $q_f^{-1}(N(C_\lambda))$ over $C_\lambda$ with the fiber PL homeomorphic to $\partial D^{k+1} \times S^{m-n-k}$ and that the structure group of $q_f^{-1}(C)$ is a subgroup of the diffeomorphism group of the fiber, consisting of some diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $\partial D^{k+1} \times S^{m-n-k}$ over $\partial D^{k+1}$ inducing PL homeomorphisms on the base space $\partial D^{k+1} = S^k$ ($1 \leq k < m - n$).

(b) (The bundle structure of $q_f^{-1}(R)$ over $R$ for a connected component $R$ of $W_f - \text{Int} \bigcup_{\lambda \in A} N(C_\lambda)$ in an $S$-region)
For any connected component $R$ of $W_f - \text{Int} \bigcup_{\lambda \in A} N(C_\lambda)$ in a $k$-S-region such that the closure of the $k$-S-region is bounded by a disjoint union of $k$ $S$-loci, $q_f^{-1}(R)$ has the structure of a $C^\infty$ bundle over $R$ with the fiber PL homeomorphic to $S^k \times S^{m-n-k}$ such that the structure group is a subgroup of the diffeomorphism group of the fiber, consisting of some diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $S^k \times S^{m-n-k}$ over $S^k$ inducing PL homeomorphisms on the base space $S^k$.

(c) (Identifications of pieces of the source manifold on their boundaries)
For any connected component $R$ of $W_f - \text{Int} \bigcup_{\lambda \in A} N(C_\lambda)$ in a $k$-S-region such that the closure of the $k$-S-region is bounded by a disjoint union of $k$ $S$-loci, the identification map of the restriction of a bundle $q_f^{-1}(R)$ over $R$ satisfying (b) of this definition to any connected component $C$ of the boundary $\partial R$ and the subbundle $q_f^{-1}(C)$ of a bundle $q_f^{-1}(N(C_\lambda))$ ($C \subset \partial N(C_\lambda)$) over $C$ satisfying (a) of this definition is a bundle map; the structure groups are a subgroup of the diffeomorphism group of the fiber, consisting of some diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $\partial D^{k+1} \times S^{m-n-k}$ over $\partial D^{k+1}$ inducing PL homeomorphisms on the base space $\partial D^{k+1} = S^k$. (Here the base space of the fiber of the bundle satisfying (a) means $\partial D^{k+1}$ and the base space of the fiber of the bundle satisfying (b) means $S^k$.

We should be careful in the case where $k = m - n - k$.)

For example, on $S^{k_1} \times S^{k_2}$ ($k_1, k_2 \in \mathbb{N}$), there exists a good Morse function $f : S^{k_1} \times S^{k_2} \to \mathbb{R}$ with four singular points which has a family of $S$-identifications. After following the theorem and its proof, we introduce other examples.

**Theorem 1.** Let $M$ be a closed $C^\infty$ manifold of dimension $m$, $N$ be a $C^\infty$ manifold of dimension $n$ without boundary and $f : M \to N$ be a simple fold map. Let $m - n \geq 2$.
We assume that $f$ has a family of $S$-identifications.

Then there exist a compact PL manifold $W$ of dimension $m + 1$ such that $\partial W = M$, a polyhedron $V$ and continuous maps $r : W \to V$ and $s : V \to W_f$ and the followings hold.

1. There exist a triangulation of $W$, a triangulation of $V$ and a triangulation of $W_f$ such that $r$ is a simplicial map and that $s$ is a simplicial map and the followings hold.
   a. For each $p \in V$, $r^{-1}(p)$ collapses to a point and $r$ is a homotopy equivalence.
(b) If \( p \) is in the closure of an \( AS \)-region in \( W_f \), then \( s^{-1}(p) \) is a point. If \( p \) is in an \( AS \)-region in \( W_f \) and \( q \in s^{-1}(p) \), then \( r^{-1}(q) \) is PL homeomorphic to \( D^{m-n+1} \).

(c) If \( p \) is in a \( k \) \( S \)-region in \( W_f \) whose closure is bounded by a disjoint union of \( k \) \( S \)-loci, then \( s^{-1}(p) \) is PL homeomorphic to \( S^k \). If \( p \) is in a \( k \) \( S \)-region in \( W_f \) whose closure is bounded by a disjoint union of \( k \) \( S \)-loci and \( q \in s^{-1}(p) \), then \( r^{-1}(q) \) is PL homeomorphic to \( D^{m-n-k+1} \).

(2) \( W \) collapses to a subpolyhedron \( V' \) such that \( r|_{V'} : V' \to V \) is a PL homeomorphism.

If \( M \) is orientable, then we can construct \( W \) as an orientable manifold.

We prove the theorem by analogy of the proof of Theorem 4.1 of [17] and the proof of Lemma 1 of [6]. In the proof, we often apply Proposition 1, which is on important properties of the Reeb spaces of simple fold maps, implicitly.

**Proof.** We construct a compact manifold of dimension \( m+1 \) bounded by \( M \).

**Step 1** Around a regular neighborhood of \( q_f(F_0(f)) \)

\( q_f(F_0(f)) \) is the image of the set \( F_0(f) \) of all the definite fold points of \( f \). Let \( N(q_f(F_0(f))) \) be a small regular neighborhood of \( q_f(F_0(f)) \). We note that \( N(q_f(F_0(f))) \) has the structure of a trivial PL bundle over \( q_f(F_0(f)) \) with the fiber \([0, 1] \). We may assume that \( q_f(F_0(f)) \) corresponds to the 0-section \((0 \in [0, 1]) \).

For each \( p \in q_f(F_0(f)) \), set \( K_p := q_f^{-1}([p] \times [0, 1]) \) for a fiber \([p] \times [0, 1] \) of the bundle \( N(q_f(F_0(f))) \) over \( q_f(F_0(f)) \). It is diffeomorphic to \( D^{m-n+1} \). We may assume that \( q_f^{-1}(N(q_f(F_0(f)))) \) has the structure of a \( C^\infty \) bundle over \( q_f(F_0(f)) \) with the fiber \( D^{m-n+1} \) and \( K_p \) is the fiber over \( p \in q_f(F_0(f)) \).

We obtain the following objects (see also \textsc{Figure} 1).

1. A compact \( C^\infty \) \( (m+1) \)-manifold \( V_0 \) having the structure of a \( C^\infty \) \( D^{m-n+2} \)-bundle over \( q_f(F_0(f)) \) (we denote by \( \tilde{K}_p \) the fiber over \( p \in q_f(F_0(f)) \)) such that \( q_f^{-1}(N(q_f(F_0(f)))) \) has the structure of a subbundle of the bundle \( V_0 \) with the fiber \( D^{m-n+1} \) and that \( q_f^{-1}(N(q_f(F_0(f)))) \) is in the boundary of \( V_0 \).

2. A simplicial map \( r_0 : V_0 \to N(q_f(F_0(f))) \) such that \( r_0^{-1}(p, t) \) is PL homeomorphic to \( D^{m-n+1} \) for \( p \in q_f(F_0(f)) \) and \( t \in (0, 1] \) and that \( r_0^{-1}(p, t) \) is a point for \( p \in q_f(F_0(f)) \) and \( t = 0 \).

3. A PL manifold \( N(q_f(F_0(f))) \subset V_0 \) of dimension \( n \) such that the following hold.
   (a) \( N(q_f(F_0(f))) \) has the structure of a subbundle of the bundle \( V_0 \).
   (b) \( r_0|_{N(q_f(F_0(f)))} : N(q_f(F_0(f))) \to N(q_f(F_0(f))) \) is a PL homeomorphism (a bundle isomorphism between the two PL bundles).
   (c) \( N(q_f(F_0(f))) \cap \partial \tilde{K}_p \) consists of two points \((p, 0), (p, 1) \in [p] \times [0, 1] \).
      One of the two points is in \( q_f^{-1}(N(q_f(F_0(f)))) \) and the other point is not.
   (d) \( V_0 \) collapses to \( N(q_f(F_0(f))) \).

Finally we set \( s_0 := \text{id}_{N(q_f(F_0(f)))} \) for latter discussions.

**Step 2** Around a regular neighborhood of \( q_f(F_k(f)) \) \((k \neq 0)\)
We obtain the following objects. Let $K_p$ denote the fiber over $p$ of the bundle $q_f^{-1}(N(q_f(F_0(f))))$ which is the spherical part of the boundary of $K_p$.

$q_f(F_k(f))$ is the image of the set $F_k(f)$ of all the fold points of $f$ whose indices are $k (1 \leq k \leq \lceil \frac{m-2}{2} \rceil)$. Let $Q(f)$ be the intersection of the image of the set of all the fold points of $f$ whose indices are 1 and the set of all the branched points of $W_f$.

Step 2-1 A regular neighborhood of $Q(f)$.

Let $N(Q(f))$ be a small regular neighborhood of $Q(f)$. We note that $N(Q(f))$ has the structure of a PL $K$-bundle over $Q(f)$, where $K := \{r \exp(2\pi i \theta) \in \mathbb{C} \mid 0 \leq r \leq 1, \theta = 0, \frac{1}{3}, \frac{2}{3}\}$. We may assume that $Q(f)$ corresponds to the 0-section ($0 \in K$). For each $p \in Q(f)$, set $K_p := q_f^{-1}(\{p\} \times K)$ for a fiber $\{p\} \times K$ of the bundle $N(Q(f))$ over $Q(f)$. It is PL homeomorphic to $S^{m-n+1}$ with the interior of a union of disjoint three standard closed $(m-n+1)$-discs removed. We may assume that $q_f^{-1}(N(Q(f)))$ has the structure of a $C^\infty$ bundle over $Q(f)$ with a fiber PL homeomorphic to $S^{m-n+1}$ with the interior of a union of disjoint three standard closed $(m-n+1)$-discs removed and $K_p$ is the fiber over $p \in Q(f)$.

We obtain the following objects.

1. A compact PL $(m+1)$-manifold $V_1$ having the structure of a PL $D^{m-n+2}$-bundle over $Q(f)$ (we denote by $\widetilde{K}_p$ the fiber over $p \in Q(f)$) such that the bundle $q_f^{-1}(N(Q(f)))$ is a subbundle of the bundle $V_1$ and that $q_f^{-1}(N(Q(f)))$ is in the boundary of $V_1$.

2. A simplicial map $r_1 : V_1 \rightarrow N(Q(f))$ such that $r_1^{-1}(p, t)$ is PL homeomorphic to $D^{m-n+1}$ for $p \in Q(f)$ and $t \in K - \{0\}$ and that $r_1^{-1}(p, t)$ collapses to a point for $p \in Q(f)$ and $t = 0$.

3. A subpolyhedron $\widetilde{N(Q(f))} \subset V_1$ of dimension $n$ such that the followings hold.

   a. $\widetilde{N(q_f(F_0(f)))}$ has the structure of a subbundle of the bundle $V_1$.

   b. $r_1 \mid_{\widetilde{N(Q(f))}} : \widetilde{N(Q(f))} \rightarrow N(Q(f))$ is a PL homeomorphism (a bundle isomorphism between the two PL bundles).

   c. $\widetilde{N(Q(f))} \cap \partial \widetilde{K}_p$ consists of three points $(p, \exp(\frac{2}{3}k\pi)) \in \{p\} \times K (k = 0, 1, 2)$ and is not in $q_f^{-1}(N(Q(f)))$. Furthermore, each connected component of $\partial \widetilde{K}_p - q_f^{-1}(N(Q(f)))$ includes one of these points and $V_1$ collapses to $\widetilde{N(Q(f))}$.

   d. $V_1$ collapses to $\widetilde{N(q_f(F_1(f)))}$.
Finally we set $s_1 := \text{id}_{\mathcal{N}(Q(f))}$ for latter discussions.

Step 2-2 Around a regular neighborhood of the set of all the non-branched points in $q_f(F_k(f))$ ($1 \leq k \leq \lfloor \frac{m-n+1}{2} \rfloor$) for $2 \leq k \leq \lfloor \frac{m-n+1}{2} \rfloor$, let $G_k(f) \subset q_f(F_k(f))$ be the disjoint union of all the $k - 1$ S-loci. Set $H_k(f) := q_f(F_k(f)) - G_k(f)$. We also note that $H_k(f)$ is the disjoint union of all the $m - n - k$ S-loci unless $m - n$ is odd and $k = \frac{m-n+1}{2}$; if $m - n$ is odd and $k = \frac{m-n+1}{2}$, then $H_k(f)$ is empty. Furthermore, we can define $H_1(f)$ as the disjoint union of all the $m - n - 1$ S-loci.

By the assumption, we can take a regular neighborhood $N(G_k(f))$ so that $q_f^{-1}(N(G_k(f)))$ has the structure of a $C^\infty$ bundle over $G_k(f)$ with a fiber PL homeomorphic to $D^k \times S^{m-n-k+1} - \text{Int}D^m$. We note that $N(G_k(f))$ has the structure of a trivial PL $[1,1]$-bundle over $G_k(f)$ and we may assume that $G_k(f)$ corresponds to the 0-section ($0 \in [1,1]$). We may assume that there exists a PL bundle over $G_k(f)$ with a fiber PL homeomorphic to $(S^{m-n-k+1} \times D^k) / \phi_1 \times \phi_2 (D^m \times S^{k+2} \times S^{k-1})$ and $(S^{m-n-k+1})$ for a pair of PL homeomorphisms $(\phi_1 : \partial D^m \times S^{k+2} \to S^{m-n-k+1}, \phi_2 : S^{k-1} \to \partial D^k)$ and that the bundle $q_f^{-1}(N(G_k(f)))$ is a subbundle of the $(S^{m-n-k+1} \times D^k)$-bundle. For this fact, remember that by the definition of S-identifications, for any connected component $C$ of $\partial N(G_k(f))$ in a $k - 1$ S-region, $q_f^{-1}(C)$ has the structure of a subbundle of the bundle $q_f^{-1}(N(G_k(f)))$ over $C$ with a fiber PL homeomorphic to $\partial D^k \times S^{m-n-k+1}$ and that the structure group of $q_f^{-1}(C)$ is a subgroup of the diffeomorphism group of the fiber, consisting of some diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle $\partial D^k \times S^{m-n-k+1}$ over $\partial D^k$ inducing PL homeomorphisms on the base space $\partial D^k = S^{k-1}$.

We obtain the following objects.

(1) A compact PL manifold $V_{k-1}$ of dimension $m + 1$ having the structure of a PL $D^m \times D^{k-1}$ bundle over $G_k(f)$ (let $V_{k-1}^p$ denote the fiber over $p$ of the base space) such that the subbundle corresponding to $\partial D^m \times D^{k-2} = (S^{m-n-k+1} \times D^k) / \phi_1 \times \phi_2 (D^m \times S^{k+2} \times S^{k-1})$ is the PL $S^{m-n}$-bundle in the previous
paragraph. The bundle $q_{f}^{-1}(N(G_{k}(f)))$ (over $G_{k}(f)$) is a subbundle of the bundle $V_{k}^{1}$ and is in the boundary of $V_{k}^{1}$ (we regard it is also a subbundle of the $(S^{m-n-k+1} \times D^{k})$-bundle).

(2) A PL bundle $P(G_{k}(f))$ whose fiber is the mapping cylinder of the constant map $c : S^{k} \to [0,1]$ satisfying $c(S^{k-1}) = \{0\}$ and whose base space is $G_{k}(f)$.

(3) Simplicial maps $r_{k}^{1} : V_{k}^{1} \to P(G_{k}(f))$ and $s_{k}^{1} : P(G_{k}(f)) \to N(G_{k}(f))$ such that $s_{k}^{1-1}(p)$ is a point for $p$ in the closure of an AS-region, that $s_{k}^{1-1}(p)$ is PL homeomorphic to $S^{k-1}$ for $p$ in a $k-1$ S-region, that $r_{k}^{1-1}(q)$ is PL homeomorphic to $D^{m-n+1}$ for $q \in s_{k}^{1-1}(p)$ for $p$ in an AS-region, that $r_{k}^{1-1}(q)$ is PL homeomorphic to $D^{m-n-k+2}$ for $q \in s_{k}^{1-1}(p)$ for $p$ in a $k-1$ S-region and that $(s_{k}^{1} \circ r_{k}^{1})^{-1}([p] \times [-1,1]) = V_{k}^{1} \ p$ for all $p \in G_{k}(f)$ ($[p] \times [-1,1]$ is a fiber of the bundle $N(G_{k}(f))$ over $G_{k}(f)$; $N(G_{k}(f))$ has the structure of a trivial PL $[-1,1]$-bundle over $G_{k}(f)$ and $G_{k}(f)$ corresponds to the 0-section). Furthermore, $r_{k}^{1-1}(q)$ collapses to a point for $q \in P(G_{k}(f))$.

(4) A subpolyhedron $P(G_{k}(f)) \subset V_{k}^{1}$ of dimension $n + k - 1$ such that the followings hold.

(a) $P(G_{k}(f))$ has the structure of a subbundle of the bundle $V_{k}^{1}$.

(b) $r_{k}^{1} \mid P(G_{k}(f)) : P(G_{k}(f)) \to P(G_{k}(f))$ is a PL homeomorphism and a bundle isomorphism between the two PL bundles.

(c) $P(G_{k}(f)) \cap \partial V_{k}^{1}$ does not contain any points in $q_{f}^{-1}(N(G_{k}(f)))$ and is the disjoint union of a point in $S^{m-n-k+1} \times D^{k}$ and a $(k-1)$-sphere $\{0\} \times S^{k-1} \subset D^{m-n-k+2} \times S^{k-1}$, where $S^{m-n-k+1} \times D^{k}$ and $D^{m-n-k+2} \times S^{k-1}$ are subspaces of the fiber $V_{k}^{1}$ over $p$ of the bundle $V_{k}^{1}$, which is PL homeomorphic to $(S^{m-n-k+1} \times D^{k}) \cup \{p \times \phi_{1} \times \phi_{2} (D^{m-n-k+2} \times S^{k-1}) \}

(d) V_{k}^{1}$ collapses to $P(G_{k}(f))$.

By the assumption, we can also take a regular neighborhood $N(H_{k}(f))$ so that $q_{f}^{-1}(N(H_{k}(f)))$ has the structure of a $C^\infty$ bundle over $H_{k}(f)$ with a fiber PL homeomorphic to $S^{k} \times D^{m-n-k+1} - \text{Int}D^{m-n+1}$ ($2 \leq k \leq \left\lfloor \frac{m+1}{2} \right\rfloor$). We can construct a compact PL manifold $V_{k}^{2}$ of dimension $m+1$, a PL bundle $P(H_{k}(f))$ whose fiber is the mapping cylinder of the constant map $c : S^{m-n-k} \to [0,1]$ satisfying $c(S^{m-n-k}) = \{0\}$ and whose base space is $H_{k}(f)$, simplicial maps $r_{k}^{2} : V_{k}^{2} \to P(H_{k}(f))$ and $s_{k}^{2} : P(H_{k}(f)) \to N(H_{k}(f))$ and a subpolyhedron $P(H_{k}(f)) \subset V_{k}^{2}$ of dimension $m - k$ with properties similar to those of $V_{k}^{1}$, $P(G_{k}(f))$, simplicial maps $r_{k}^{1}$ and $s_{k}^{1}$ and $P(G_{k}(f))$, respectively. If $k = 1$, then we can construct similar objects $V_{1}^{2}$, $P(H_{1}(f))$, $r_{1}^{2}$, $s_{1}^{2}$ and $P(H_{1}(f))$.

Now we define disjoint unions as the followings.

$$V_{S} := V_{0} \cup V_{1} \cup V_{1}^{2} \sqcup (\bigcup_{k=2}^{\left\lfloor \frac{m-n+1}{2} \right\rfloor} (V_{k}^{1} \cup V_{k}^{2}))$$

$$P_{S} := N(q_{f}(F_{0}(f))) \cup N(Q(f)) \cup P(H_{1}(f)) \sqcup (\bigcup_{k=2}^{\left\lfloor \frac{m-n+1}{2} \right\rfloor} (P(G_{k}(f)) \cup P(H_{k}(f))))$$
Now we set disjoint unions

\[ N_S := N(q_f(F_0(f))) \sqcup N(Q(f)) \sqcup N(H_1(f)) \sqcup (\sqcup_{k=2}^{m-n+1} N(G_k(f)) \cup N(H_k(f))) \]

\[ r_S := r_0 \sqcup r_1 \sqcup r_k^2 \sqcup (\sqcup_{k=2}^{m-n+1} (r_k^1 \sqcup r_k^2)) : V_S \to P_S \]

\[ s_S := s_0 \sqcup s_1 \sqcup s_k^2 \sqcup (\sqcup_{k=2}^{m-n+1} (s_k^1 \sqcup s_k^2)) : P_S \to N_S \]

\[ \tilde{P}_S := N(q_f(F_0(f))) \sqcup N(Q(f)) \sqcup P(H_1(f)) \sqcup (\sqcup_{k=2}^{m-n+1} (P(G_k(f)) \cup P(H_k(f)))) \]

Step 3 Around \( R := W_f \setminus \text{Int} N_S \)

\( R \) is a compact manifold of dimension \( n \) and let \( \{R_\lambda\}_{\lambda \in \Lambda} \) be the family of all the connected components of \( R \). Since each connected component of each regular fiber of \( f \) is an almost-sphere or PL homeomorphic to a product of two standard spheres, \( q_f^{-1}(R_\lambda) \) has the structure of a bundle over \( R_\lambda \) whose fiber is an almost-sphere or PL homeomorphic to \( S^k \times S^{m-n-k} \) for an integer \( 1 \leq k \leq m-n-1 \).

If \( R_\lambda \) is in a \( k \) S-region whose closure is bounded by a disjoint union of \( k \) S-loci, then we may assume that the fiber of the bundle \( q_f^{-1}(R_\lambda) \) is PL homeomorphic to \( S^k \times S^{m-n-k} \) and that the structure group of the bundle \( q_f^{-1}(R_\lambda) \) is a subgroup of the diffeomorphism group of the fiber, consisting of some diffeomorphisms regarded as bundle isomorphisms on the trivial PL bundle \( S^k \times S^{m-n-k} \) over \( S^k \) inducing PL homeomorphisms on the base space \( S^k \).

Let the fiber of a point in \( R_\lambda \) be an almost-sphere and \( q_f^{-1}(R_\lambda) \) have the structure of a \( C^\infty \) bundle over \( R_\lambda \) with a fiber PL homeomorphic to \( S^m-n \). Let \( r_{R_\lambda} : V_{R_\lambda} \to R_\lambda \) give the structure of a PL \( D^{m-n+1} \)-bundle which is an associated bundle of the bundle \( q_f^{-1}(R_\lambda) \) over \( R_\lambda \) (we regard \( S^m-n = \partial D^{m-n+1} \)). We take the associated bundle so that the structure group is a group consisting of PL homeomorphisms \( r \) on \( D^{m-n+1} \) such that \( r(0) = 0 \) and that for a PL homeomorphism \( r' \) on \( S^m-n \),

\[ r(x) = r'(\frac{x}{|x|}) (x \neq 0) \] Let \( P(R_\lambda) \subset V_{R_\lambda} \) be the 0-section of the associated bundle.

Finally we set \( P(R_\lambda) := R_\lambda \) and let \( s_{R_\lambda} : P(R_\lambda) \to R_\lambda \) be the identity map.

Let \( R_\lambda \) be in a \( k \) S-region whose closure is bounded by a disjoint union of \( k \) S-loci and \( q_f^{-1}(R_\lambda) \) have the structure of a \( C^\infty \) bundle over \( R_\lambda \) with a fiber PL homeomorphic to \( S^k \times S^{m-n-k} \). We can take a PL manifold \( V_{R_\lambda} \) having the structure of a PL \( (S^k \times D^{m-n-k+1}) \)-bundle over \( R_\lambda \) which is an associated bundle of the bundle \( q_f^{-1}(R_\lambda) \) over \( R_\lambda \) (we regard \( S^k \times S^{m-n-k} = S^k \times \partial D^{m-n-k+1} \)). We take the associated bundle so that the structure group is a group consisting of some PL homeomorphisms \( r \) on \( S^k \times D^{m-n-k+1} \) such that \( r \) is a bundle isomorphism on a PL bundle \( S^k \times D^{m-n-k+1} \) over \( S^k \) whose structure group consists of PL homeomorphisms \( r_1 \) on \( D^{m-n-k+1} \), where \( r_1(0) = 0 \) and for a PL homeomorphism \( r_2 \) on \( S^m-n-k+1 \), \( r_2(x) = r_2(\frac{x}{|x|}) (x \neq 0) \) and that \( r \) induces a PL homeomorphism on the base space \( S^k \). Let \( P(R_\lambda) \subset V_{R_\lambda} \) be the subbundle whose fiber is \( S^k \times \{0\} \subset S^k \times D^{m-n-k+1} \). The bundle \( V_{R_\lambda} \) over \( R_\lambda \) is also given by the composition of two PL maps \( r_{R_\lambda} : V_{R_\lambda} \to P(R_\lambda) \) and \( s_{R_\lambda} : P(R_\lambda) \to R_\lambda \), where \( P(R_\lambda) \) is a PL manifold and \( s_{R_\lambda} \) gives \( P(R_\lambda) \) the structure of a bundle over \( R_\lambda \) equivalent to the bundle \( P(R_\lambda) \) over \( R_\lambda \).

Now we set disjoint unions

\[ V_R := \sqcup_{\lambda \in \Lambda} V_{R_\lambda}, \quad P_R := \sqcup_{\lambda \in \Lambda} P(R_\lambda), \quad r_R := \sqcup_{\lambda \in \Lambda} r_{R_\lambda}, \quad s_R := \sqcup_{\lambda \in \Lambda} s_{R_\lambda} \quad \text{and} \quad \tilde{P}_R := \sqcup_{\lambda \in \Lambda} \tilde{P}(R_\lambda). \]

\( f \) has a family of S-identifications and we can take the regular neighborhoods and the bundles as in the definition of a family of S-identifications. So we may assume that we can glue \( V_S \) and \( V_R \) together to give a compact PL \((m+1)\)-manifold \( W, P_S \) and \( P_R \) together to give a polyhedron \( V \), the maps \( r_S : V_S \to P_S \) and \( r_R : V_R \to P_R \).
together to give a map \( r : W \to V \), the maps \( s_S : P_S \to N_S \) and \( s_R : P_R \to R \) together to give a map \( s : V \to W_f \) and polyhedra \( \tilde{P}_S \) and \( \tilde{P}_R \) together to give a polyhedron \( V' \). From the construction, it is now easy to verify all the required conditions. This completes the proof. \( \square \)

For example, maps satisfying the assumption of Proposition 3 or Lemma 1 of [6] satisfies the assumption of Theorem 1. Special generic maps are examples of such maps. Let \( f : M \to N \) be a normal simple fold map with regular fibers of two spheres satisfying the followings.

1. Let \( \{C_\lambda\}_{\lambda \in \Lambda} \) be the family of all the connected components of \( q_f(S(f)) \). For any connected component \( C_\lambda \subset q_f(S(f)) \), there exists a small regular neighborhood \( N(C_\lambda) \) in \( W_f \) such that \( q_f^{-1}(N(C_\lambda)) \) has the structure of a trivial \( C^\infty \) bundle over \( C_\lambda \) (the bundle structure is given by the composition of \( q_f|_{q_f^{-1}(N(C_\lambda))} \) and the natural projection to \( C_\lambda \); note that \( N(C_\lambda) \) has the structure of a PL bundle over \( C_\lambda \) mentioned in Proposition 1(4)) and that \( W_f - \bigcup_{\lambda \in \Lambda} \text{Int} N(C_\lambda) \) is a \( C^\infty \) submanifold of \( W_f - q_f(S(f)) \).
2. For any connected component \( R \) of \( W_f - \bigcup_{\lambda \in \Lambda} \text{Int} N(C_\lambda) \), \( q_f|_{q_f^{-1}(R)} : q_f^{-1}(R) \to R \) gives the structure of a trivial \( C^\infty \) bundle.

Then, we say that \( f \) is locally product. For \( f \), easily we have a map satisfying the assumption of Theorem 1 by changing the identification of \( q_f^{-1}(\partial \cup_{\lambda \in \Lambda} N(C_\lambda)) \) and \( \partial q_f^{-1}(W_f - \bigcup_{\lambda \in \Lambda} \text{Int} N(C_\lambda)) \).

**Example 3.**

1. Special generic maps on spheres into Euclidean spaces are locally product and satisfy the assumption of Theorem 1 (see section 5 of [15] or Example 3 (1) of [6]). Discussions of section 5 of [15] imply that special generic maps from closed and orientable \( C^\infty \) manifolds into the plane whose Reeb spaces are homeomorphic to the standard 2-dimensional sphere \( S^2 \) with the interior of a disjoint union of a finite number of closed standard 2-dimensional discs removed are also locally product and satisfy the assumption of Theorem 1.

2. In [7], [8], [11] and [12], Kobayashi and Saeki studied stable fibrations of torus type or simple mappings with \( g \leq 1 \). A stable fibration of torus type or a simple mapping with \( g \leq 1 \) is defined as a stable map from a closed, connected and orientable \( C^\infty \) manifold of dimension 4 into the plane whose regular fibers are \( S^2 \) or \( T^2 \) (note that the fibers are connected). The singular set of such a map is a disjoint union of finite number of circles in the source manifold and the restriction map to the singular set is an embedding (a topological embedding). If such a map is a fold map, then the restriction map is a \( C^\infty \) embedding.

\( S^4 \) admits a map satisfying the definition of a stable fibration of torus type. In fact, a special generic map from \( S^4 \) into \( \mathbb{R}^2 \) such that the restriction map to the singular set is a \( C^\infty \) embedding is an example of such maps. It is locally product and satisfies the assumption of Theorem 1.

In [8] and [11], the diffeomorphism types of manifolds admitting stable fibrations of torus type which are locally product were studied under various conditions. Some of such maps in these papers satisfy the assumption of Theorem 1 and if such a map does not satisfy the assumption of Theorem 1, then we obtain a new map satisfying the assumption by the method
mentioned just before this example. On the other hand, for example, the connected sum \(\mathbb{CP}^2 \sharp \mathbb{CP}^2\) of two copies of \(\mathbb{CP}^2\) does not admit a map satisfying the assumption, although it admits a stable fibration of torus type which is also a fold map ([7]). In fact this does not bound any compact orientable topological 5-dimensional manifold since the signature is not 0. Furthermore, results of [8] and [11] implicitly state that \(\mathbb{CP}^2 \sharp \mathbb{CP}^2\) does not admit stable fibrations of torus type into the plane which are locally product.

We have the following corollary.

**Corollary 3.** In the situation of Theorem 1, let \(M\) be connected and \(i : M \to W\) be the natural inclusion. Then

\[ r_* \circ i_* : \pi_k(M) \to \pi_k(V) \]

gives an isomorphism for \(0 \leq k \leq m - \dim V - 1\).

**Proof.** Since \(r\) is a homotopy equivalence, we have only to show that \(i_* : \pi_k(M) \to \pi_k(W)\) \((0 \leq k \leq m - \dim V - 1)\) is an isomorphism. Since \(W\) collapses to a polyhedron of dimension \(\dim V\), it admits a PL-handlebody decomposition consisting of handles whose indices are not larger than \(\dim V\). Dualizing the handles, we see that \(W\) is obtained from \(M \times [0,1]\) by attaching handles whose indices are not smaller than \(m - \dim V + 1\) along \(M \times \{1\}\). Hence, \(i_* : \pi_k(M) \to \pi_k(W)\) \((0 \leq k \leq m - \dim V - 1)\) is an isomorphism. This completes the proof. \(\square\)

3. **Applications**

In this section, we recall *round fold maps* and then, we apply Theorem 1 to a round fold map which satisfies the assumption of Theorem 1.

3.1. **Round fold maps.** We review the definition of a *round fold map*, a method of construction and examples of round fold maps (see also [5] and [6]). Before that, we recall \(C^\infty\) equivalence (see also [3] for example). For two \(C^\infty\) maps \(f_1 : X_1 \to Y_1\) and \(f_2 : X_2 \to Y_2\), they are said to be \(C^\infty\) equivalent if there exist diffeomorphisms \(\phi_X : X_1 \to X_2\) and \(\phi_Y : Y_1 \to Y_2\) such that the following diagram commutes.

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\phi_X} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{\phi_Y} & Y_2
\end{array}
\]

**Definition 7** (round fold map). \(f : M \to \mathbb{R}^n\) \((n \geq 2)\) is said to be a *round fold map* if \(f\) is \(C^\infty\) equivalent to a fold map \(f_0 : M_0 \to \mathbb{R}^n\) on a closed \(C^\infty\) manifold \(M_0\) such that the followings hold.

1. The singular set \(S(f_0)\) is a disjoint union of standard spheres and consists of \(l \in \mathbb{N}\) connected components.
2. The restriction map \(f_0|_{S(f_0)}\) is a \(C^\infty\) embedding.
3. Let \(D^n_r := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^n x_k^2 \leq r\}\). \(f_0(S(f_0)) = \sqcup_{k=1}^l \partial D^n_{k}\).
We call $f_0$ a normal form of $f$.

We call a ray $L_0$ from $0 \in \mathbb{R}^n$ an axis of $f_0$ and $D_n^\frac{1}{2}$ the proper core of $f_0$.

Suppose that for a round fold map $f$, its normal form $f_0$ and diffeomorphisms $\Phi : M \to M_0$ and $\phi : \mathbb{R}^n \to \mathbb{R}^n$, the following diagram commutes.

\[
\begin{array}{ccc}
M & \xrightarrow{\Phi} & M_0 \\
\downarrow f & & \downarrow f_0 \\
\mathbb{R}^n & \xrightarrow{\phi} & \mathbb{R}^n
\end{array}
\]

Then for an axis $L$ of $f_0$, we also call $\phi^{-1}(L)$ an axis of $f$ and for the proper core $D_n^\frac{1}{2}$ of $f_0$, we also call $\phi^{-1}(D_n^\frac{1}{2})$ a proper core of $f$.

See also FIGURE 3.

We give a method of construction of a round fold map.

Let $\bar{M}$ be a compact $C^\infty$ manifold with non-empty boundary. There exists a good Morse function $\tilde{f} : \bar{M} \to [a, +\infty)$, where $a$ is the minimal value.

Let $\Phi : \partial(\bar{M} \times \partial(\mathbb{R}^n \setminus \text{Int}D^n)) \to \partial(\bar{M} \times D^n)$ be a diffeomorphism, $\phi : \partial(\mathbb{R}^n \setminus \text{Int}D^n) \to \partial D^n$ be a $C^\infty$ diffeomorphism and $p_1$ and $p_2$ be the canonical projections. Suppose that the following holds.

\[
\begin{array}{ccc}
\partial M \times \partial(\mathbb{R}^n \setminus \text{Int}D^n) & \xrightarrow{\Phi} & \partial M \times \partial D^n \\
\downarrow p_1 & & \downarrow p_2 \\
\partial(\mathbb{R}^n \setminus \text{Int}D^n) & \xrightarrow{\phi} & \partial D^n
\end{array}
\]

By glueing two manifolds by $\Phi$, we construct $M := (\partial \bar{M} \times D^n) \cup_{\Phi} (\bar{M} \times \partial(\mathbb{R}^n \setminus \text{Int}D^n))$. Let $p : \partial \bar{M} \times D^n \to D^n$ be the canonical projection. Then by glueing $p$ and $f \times \text{id}_{S^{n-1}}$ together by a pair of diffeomorphisms $(\Phi, \phi)$, $f = p \cup_{\Phi, \phi} (f \times \text{id}_{S^{n-1}})$ is constructed and $f : M \to \mathbb{R}^n$ is a round fold map.
3.2. An application of Theorem 1 to a round fold map. In this subsection, we consider a round fold map whose singular set consists of two connected components and whose regular fibers of points in a proper core are PL homeomorphic to \( S^k \times S^{m-n-k} \) for an integer \( 1 \leq k < m - n \) \( (m - n \geq 2) \).

For example, let \( M = S^{m-n-k} \times D^{k+1} \) in the situation of the construction mentioned in the previous subsection. Then there exists a good Morse function with two singular points \( f : M \rightarrow [0, +\infty) \), where \( a \) is the minimal value of \( f \). By the method, we obtain a round fold map \( f : M \rightarrow \mathbb{R}^n \) such that the singular set consists of two connected components and that regular fibers of points in a proper core are diffeomorphic to \( S^k \times S^{m-n-k} \). \( f \) is also a normal simple fold map with regular fibers of two spheres.

By applying Theorem 1, we have the following theorem.

**Theorem 2.** Let \( M \) be a closed and connected \( C^\infty \) manifold of dimension \( m \) and \( f : M \rightarrow \mathbb{R}^n \) \( (n \geq 2) \) be a round fold map. Let \( m - n \geq 2 \). Suppose that \( q_f(S(f)) \) consists of two connected components and that one of the connected components is a \( k \) \( S \)-locus.

We also assume that \( f \) has a family of \( S \)-identifications.

Then there exist a compact PL manifold \( W \) of dimension \( m + 1 \), a polyhedron \( V \) and continuous maps \( r : W \rightarrow V \) and \( s : V \rightarrow W_f \) such that the followings hold.

1. \( \partial W = M \) holds.
2. \( V \) is a polyhedron of dimension \( n + k \).
3. There exist triangulations of \( W, V \) and \( W_f \) such that the followings holds.
   a. \( r \) and \( s \) are simplicial maps.
   b. For each \( p \in V \), \( r^{-1}(p) \) collapses to a point and \( r \) is a homotopy equivalence.
   c. If \( p \) is in the closure of the AS-region in \( W_f \), then \( s^{-1}(p) \) is a point. If \( p \) is in the \( AS \)-region in \( W_f \) and \( q \in s^{-1}(p) \), then \( r^{-1}(q) \) is PL homeomorphic to \( D^{m-n+1} \).
   d. If \( p \) is in the \( k \) \( S \)-region in \( W_f \), then \( s^{-1}(p) \) is PL homeomorphic to \( S^k \). If \( p \) is in the \( k \) \( S \)-region in \( W_f \) and \( q \in s^{-1}(p) \), then \( r^{-1}(q) \) is PL homeomorphic to \( D^{m-n+k+1} \).
4. \( W \) collapses to a subpolyhedron \( V' \) such that \( r|_{V'} : V' \rightarrow V \) is a PL homeomorphism.
5. \( r_* \circ i_* : \pi_j(M) \rightarrow \pi_j(V) \) gives an isomorphism for \( 0 \leq j \leq m - \dim V - 1 = m - n - k - 1 \) where \( i : M \rightarrow W \) is the inclusion.
6. \( V \) is PL homeomorphic to \( S^{n+k} \cup \psi(S^{n-1} \times [0, 1]) \) for a PL homeomorphism \( \psi : B \rightarrow A \) (a polyhedron obtained by gluing \( S^{n+k} \) and \( S^{n-1} \times [0, 1] \) by \( \psi \)) where \( A \) is PL homeomorphic to \( S^{n-1} \) and trivially embedded in \( S^{n+k} \) in the PL category and where \( B := S^{n-1} \times \{0\} \subset S^{n-1} \times [0, 1] \).

**Example 4.**

1. A round fold map above satisfies the assumption of Theorem 2 for some of pairs of diffeomorphisms \( (\Phi, \phi) \) in the situation of the construction in subsection 3.1.

2. Let \( m \geq 4 \) and \( n = 2 \). A round fold map \( f : M^m \rightarrow \mathbb{R}^n \) the inverse image of whose axis is PL homeomorphic to \( D^{k+1} \times S^{m-2-k} \) \( (1 \leq k \leq m - 3) \) and whose singular set consists of two connected components satisfies the assumption for some of identifications of the boundary of \( f^{-1}(\mathbb{R}^n - \text{Int}P) \) and the boundary of \( f^{-1}(P) \) for a proper core \( P \) of \( f \) if the subgroup of
the PL homeomorphism group of \( D^{k+1} \times S^{m-2-k} \) consisting of all the PL homeomorphisms fixing all the points of the boundary is connected.

**Proof of Theorem 2.** From Theorem 1 and Corollary 3, the existence of \( W, V, r : W \to V \) and \( s : V \to W_f \) such that the conditions except the last one hold is clear.

We prove that the last condition holds. We may assume that \( V \) is PL homeomorphic to \((D^n \times S^k) \cup_{\psi} (S^{n-1} \times P)\), where \( P \) is the mapping cylinder of the constant map \( c : S^k \to [0,1] \) satisfying \( c(S^k) = \{0\} \) and \( \psi : S^{n-1} \times P' \to \partial D^n \times S^k \) is a product of two PL homeomorphisms \( (a \text{ PL homeomorphism from } S^{n-1} \text{ onto } \partial D^n \text{ and one from } P' \text{ onto } S^k) \) for \( P' := S^k \times \{0\} \subset P \). Note that \((D^n \times S^k) \cup_{\psi} (S^{n-1} \times P)\) is PL homeomorphic to \( S^{n+k} \cup_{\psi} (S^{n-1} \times [0,1]) \) where \( \psi : S^{n-1} \times \{0\} \to S^{n+k} \) is a trivial PL embedding.

This completes the proof. \( \square \)

In the situation of Theorem 2, if \( m = 2(n+k) \), then \( M \) is a closed and \((n+k-1)\)-connected manifold of dimension \( m = 2(n+k) \) by Corollary 3. We recall the following fact. We denote the \( h \)-cobordism group of \( C^{\infty} \) oriented homotopy spheres whose dimensions are \( k \) by \( \Theta_k \).

**Proposition 4** (Wall, [21]). Let \( X \) be a closed \( C^{\infty} \) oriented manifold of dimension \( 2k \) having the homotopy type same as that of a connected sum of finite copies of \( S^k \times S^k \). If \( k \equiv 3, 5, 6, 7 \mod 8 \), then \( X \) is diffeomorphic to a connected sum of finite copies of \( S^k \times S^k \) and an oriented almost-sphere of dimension \( 2k \). If \( k \equiv 3, 5, 6, 7 \mod 8 \) and \( \Theta_{2k} \equiv \{0\} \) (Ex. \( k = 3, 6 \)), then \( X \) is diffeomorphic to a connected sum of finite copies of \( S^k \times S^k \).

We denote by \( b_k(X) \) the \( k \)-th Betti number of a topological space \( X \). We prove the proposition by the method similar to that of the proof of Corollary 4.10 of [17].

**Proposition 5.** In the situation of Corollary 3, if \( m = 2 \dim V \) and \( V \) is \((\dim V - 1)\)-connected, then \( b_{\dim V}(M) = 2b_{\dim V}(V) \).

**Proof.** \( W \) is simple homotopy equivalent to \( V \). \( M \) and \( W \) are \((\dim V - 1)\)-connected. We glue two copies of \( W \) on the boundaries, which are PL homeomorphic to \( M \), and we consider the Euler number of the resulting closed and connected manifold. Then we have \( 2b_0(W) + 2 \times (-1)^{\dim V} b_{\dim V}(W) - b_0(M) - (-1)^{\dim V} b_{\dim V}(M) - b_{2 \dim V}(M) = 2 + 2 \times (-1)^{\dim V} b_{\dim V}(V) - 1 - (-1)^{\dim V} b_{\dim V}(M) = 0 \). This completes the proof. \( \square \)

From Theorem 2 and Proposition 5, we have the following corollaries to Theorem 2, immediately.

**Corollary 4.** In the situation of Theorem 2, if \( m = 2(n+k) \), then \( H_{n+k}(M; \mathbb{Z}) \cong \mathbb{Z}^2 \) holds.

**Corollary 5.** In the situation of Theorem 2, let \( m = 2(n+k) \) and \( n+k \equiv 3, 5, 6, 7 \mod 8 \). Then \( M \) is \((n+k-1)\)-connected. If \( M \) has the same homotopy type as that of \( S^{n+k} \times S^{n+k} \), then it is diffeomorphic to a connected sum of \( S^{n+k} \times S^{n+k} \) and an oriented almost-sphere of dimension \( 2(n+k) \). If \( \Theta_{2(n+k)} \equiv \{0\} \) (Ex. \( n + k = 3, 6 \)) and \( M \) has the homotopy type same as that of \( S^{n+k} \times S^{n+k} \), then \( M \) is diffeomorphic to \( S^{n+k} \times S^{n+k} \).
**Example 5.** Any closed and 2-connected manifold of dimension 6 is known to have a $C^\infty$ differentiable structure and the resulting 6-dimensional manifold is always diffeomorphic to a connected sum of finite copies of $S^3 \times S^3$ ([22]). So in the situation of Corollary 5, if $m = 6$, then $M$ is diffeomorphic to $S^3 \times S^3$ without the assumption about the homotopy type.

**Remark 1.** In [9], [10] and [11], Kobayashi studied maps satisfying the definition of a round fold map independently. In these papers, he introduced and studied surgery operations (bubbling surgeries). For example, he constructed a new round fold map whose singular set consists of two connected components by such a surgery operation on a round fold map whose singular set is connected, which is a special generic map on a sphere. A lot of new maps and source manifolds are obtained by such operations and some maps seem to satisfy the assumption of Theorem 2.

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