What Becomes of Vortices in Theories with Flat Directions

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Abstract

In many theories with flat directions of scalar potential, static vortex solutions do not exist for a generic choice of vacuum. In two Euclidean dimensions, we find their substitutes — constrained instantons consisting of compact core formed by Abrikosov–Nielsen–Olesen vortex and long-ranged cloud of modulus field. In (3+1) dimensions, an initial compact configuration of string topology evolves in such a way that at every point in space the modulus relaxes, in a universal manner, to one and the same value characteristic to the theory.
1. Flat directions of scalar potential determining moduli spaces of physically inequivalent vacua are inherent in many supersymmetric gauge theories. Some of these theories exhibit topological properties allowing for Abrikosov–Nielsen–Olesen vortices. However, at least in several examples, static vortex solutions actually do not exist for generic choice of vacuum \[1\]. The purpose of this paper is to analyse this situation in some detail.

This analysis may be of interest from two points of view. First, vortices are instantons in two dimensions. In the absence of exact instanton solutions, the instanton transitions are dominated by constrained instantons \[2\]. Hence, our first purpose is to describe constrained instantons in two-dimensional theories with flat directions. We shall see that, unlike the situation in Yang–Mills–Higgs theories in four dimensions, the constraint is required to stabilize the instanton against extending to infinite size. This and other peculiarities of the constrained instantons in two dimensions, which we describe below, are obviously due to the logarithmic behavior of classical massless moduli fields at large distances.

The second aspect is that vortices are cosmic strings in four dimensions. Since the static vortex solutions do not exist in models under discussion, an initial configuration with vortex topology (created, say, by the Kibble mechanism) will evolve in time. We shall see that this evolution is rather peculiar: i) at given point in space and at sufficiently large times the evolution is slow (logarithmic in time) and independent of the details of the initial configuration; ii) the evolution (slowly) drives the moduli field to a certain value everywhere in space; in other words, the presence of the string results in slow transition from the original vacuum to a particular one which is fixed in a given theory.

2. To be specific, let us consider $U(1)$ gauge theory with two complex scalars of opposite charges, described by the following Lagrangian,

$$L = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi_1|^2 + |D_\mu \phi_2|^2 - V(\phi_1, \phi_2),$$

(1)

where

$$V(\phi_1, \phi_2) = \frac{\lambda}{2} (|\phi_1|^2 - |\phi_2|^2 - \eta^2)^2.$$

This theory at $\lambda = e^2$ ($e$ is the gauge coupling) may be viewed as the bosonic part of $N = 1$, $d = 4$ or $N = 2$, $d = 2$ supersymmetric theory with zero superpotential and with Fayet–Illiopoulos $D$-term proportional to $\eta^2$. The moduli space, up to gauge and global transformations, is parameterized by one parameter $u$,

$$\phi_1 = \text{real} = v_1 = \eta ch u,$$

$$\phi_2 = \text{real} = v_2 = \eta sh u.$$  

(2)
The gauge symmetry is broken for any choice of vacuum, the mass of the gauge boson being

\[ m_V = e \eta \sqrt{2 \text{ch}^2 u}. \]

There exists one massless Goldstone field corresponding to broken global symmetry \( \phi_1, \phi_2 \to e^{i \alpha} \phi_1, e^{i \alpha} \phi_2 \), and massless modulus. The latter will be of primary interest for our purposes. The modulus field with canonical kinetic term is related to the variable \( u \) as follows,

\[ z = \eta \int_0^u du \sqrt{2 \text{ch}^2 u}, \quad (3) \]

so that \( dz/\sqrt{2} \) is the length element along the moduli curve \((2)\).

In a given vacuum, the static \( O(2) \)-symmetric configurations with vortex topology have the following structure,

\[ \begin{align*}
\phi_1 &= f_1(r) e^{i \theta}, \\
\phi_2 &= f_2(r) e^{-i \theta}, \\
A_\alpha &= -\frac{1}{e} \epsilon_{\alpha \beta} a(r) \frac{x_\beta}{r}, \quad \alpha, \beta = 1, 2
\end{align*} \]

with \( f_1(0) = f_2(0) = a(0) = 0 \) and

\[ \begin{align*}
f_1(r \to \infty) &= v_1, \\
f_2(r \to \infty) &= v_2, \\
a(r \to \infty) &\to 1/r.
\end{align*} \quad (4) \]

The reason for the absence of the static vortex solution at \( v_2 \neq 0 \), i.e. at \( z_0 \neq 0 \), where \( z_0 \) is the value of the modulus field in the prescribed vacuum, is as follows. The modulus field extends to large distances. At these distances other fields of vortex-like solutions die away, and the modulus field becomes free. The solution to free massless equation logarithmically increases with \( r \), so the modulus field grows to infinity instead of approaching its vacuum value \( z_0 \). Hence, the boundary conditions \((4)\) cannot be satisfied. This argument does not work for a special choice of vacuum,

\[ \begin{align*}
\phi_1^{(0)} &= \eta, \\
\phi_2^{(0)} &= 0.
\end{align*} \quad (5) \]

In this special case, the field \( \phi_2 \), and hence the modulus field, can be set to zero everywhere in space, and the vortex solution exists and coincides with the standard one appearing in the theory with one Higgs field.
3. Let us consider first the theory (1) in two Euclidean dimensions. To obtain the constrained instanton in a general vacuum, let us impose the constraint that the fields \( \phi_1 \) and \( \phi_2 \) are equal, up to gauge transformation, to their vacuum values (2) at large but finite distance \( r = R \), i.e.

\[
\begin{align*}
    f_1(R) &= v_1 \\
    f_2(R) &= v_2, \quad R \gg \frac{1}{e\eta}
\end{align*}
\]

Outside this circle the fields remain constant. It is straightforward to solve the field equations numerically; the results for \( f_1(r) \) and \( f_2(r) \) are presented in fig.1. At large enough \( R \), the solution has the core of the size

\[ r_0 \sim 1/e\eta \]

and the cloud extending to \( r = R \). For very large \( R \), the function \( f_1 \) reaches the value \( f_1 = \eta \) just outside the core, while the function \( f_2 \) remains small in this region. The gauge field configuration \( a(r) \) coincides with the gauge field of the vortex in the model with one Higgs field with expectation value \( \eta \). In other words, the field configuration inside and just outside the core is the same for all vacua and coincides with the vortex solution existing in the special vacuum (5) (up to corrections of order \( 1/\ln R \)). Outside the core, the fields \( f_1 \) and \( f_2 \) logarithmically tend to their values (6) in such a way that \( f_1^2 - f_2^2 = \eta^2 \), i.e., only the modulus field varies.

This behavior is straightforward to understand analytically. Let us assume for a moment that the structure of the core is the same for all choices of vacua and coincides with the core structure of the vortex solution existing in the special vacuum (5). In particular, up to corrections of order \( 1/\ln R \),

\[
\begin{align*}
    f_1 &= \eta, \\
    f_2 &= 0
\end{align*}
\]

just outside the core. The relations \( f_1^2 - f_2^2 = \eta^2 \) and \( a(r) = 1/r \) are required well outside the core for the action not to increase with \( R \). The remaining modulus field \( z(r) \) obeys the free massless equation

\[ z'' + \frac{1}{r} z' = 0 \]

whose solution satisfying the boundary conditions (6), (7) is, again up to corrections suppressed by \( 1/\ln R \),

\[ z(r) = z_0 \frac{\ln r/r_0}{\ln R/r_0}, \]

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1One can show that these relations are indeed consistent with the field equations at \( r \gg 1/e\eta \).
where \( z_0 \) is the value of the modulus in the prescribed vacuum. The fields \( f_1(r) \) and \( f_2(r) \) outside the core are then equal to \( f_1(r) = \eta ch u(r) \), \( f_2(r) = \eta sh u(r) \), where \( u(r) \) and \( z(r) \) are related by eq.(3). Therefore, the fields \( f_1 \) and \( f_2 \) approach their boundary values logarithmically.

To understand why the structure of the core is universal (i.e., does not depend on the choice of the vacuum), let us calculate the part of the action which comes from the region outside the core,

\[
S_{\text{outside}} = \int_{r_0 < r < R} d^2x \left( \frac{1}{2} (\partial_\mu z)^2 \right) = \frac{\pi z_0^2}{\ln R/r_0}.
\]

This part is small at large \( R \). In other words, it costs essentially no action to develop the modulus field far outside the core. Hence the core is free to settle down in such a way that its own action is minimized, with no condition imposed on the modulus field just outside the core. This minimum is clearly universal and coincides with the solution in the vacuum (5).

This argument is valid up to corrections of order \((\ln R/r_0)^{-1}\). Hence, the corrections to the action of this order, coming from the core region, are not yet ruled out. To see that there are no such corrections, let us make use of the following scaling argument. The constrained instanton solutions corresponding to two sets of parameters \((z_0, R)\) and \((z'_0, R')\) coincide at \( r < R, r < R' \) provided that (see eq. (8))

\[
\frac{z_0}{\ln R/r_0} = \frac{z'_0}{\ln R'/r_0}.
\]

So, the contribution of the core region to the action, \( S_{\text{core}} \), depends only on the combination \( z_0 (\ln R/r_0)^{-1} \). At small values of this parameter, the field \( z(r) \), and therefore, the function \( f_2(r) \), is small inside the core, and its contribution to \( S_{\text{core}} \) is quadratic in its amplitude. Hence

\[
S_{\text{core}} = S_0(\eta) + O \left( \frac{z_0^2}{\ln^2 R/r_0} \right),
\]

where \( S_0(\eta) \) is the action for the vortex solution in the special vacuum (5) (say, \( S_0 = \pi \eta^2 \) at \( \lambda = e^2 \)). We conclude that the action of the constrained instanton at large \( R \) is

\[
S = S_0(\eta) + \frac{\pi z_0^2}{\ln R/r_0} + O \left( \frac{1}{\ln^2 R/r_0} \right).
\]

A somewhat surprising feature of this expression is that at sufficiently large \( R \), the action is proportional to \( \eta^2 \) and not to the vacuum expectation values of the fields \( \phi_1 \) or \( \phi_2 \). This means that the instanton contribution to the path integral is sizeable even at large \( v_1 \) and \( v_2 \) provided that \( \eta \) is not too large.
To summarize, the constrained instanton consists of the universal (independent of the vacuum at $r = \infty$) core of size $r_0 \sim 1/e\eta$ that dominates the instanton action, and the long–ranged cloud of the modulus field determining the correction to the action of order $(\ln R/r_0)^{-1}$.

4. Let us now turn to (3 + 1) dimensions. Suppose that initially a compact configuration of string topology exists in a generic vacuum of the model (1). As is clear from the above discussion, the configuration will evolve in such a way that the size of its massless cloud, $R$, will increase and the vacuum (5) will finally settle down everywhere in space, with the usual Abrikosov–Nielsen–Olesen string remaining near $r = 0$. Indeed, the latter evolution reduces the static energy per unit length, eq.(11).

Our purpose is to evaluate the behavior of the fields during this process. Clearly, the fields near the light cone depend on the details of the initial configuration. On the other hand, the behavior of the fields at large times and deep inside the light cone is universal. To show this, let us first note that outside the core, the only varying field is the modulus $z$. It obeys the free massless equation,

\[-\partial_t^2 z + z'' + \frac{1}{r} z' = 0\]  
(12)

(we still consider $O(2)$-symmetric configurations; this restriction in fact does not lead to loss of generality, as only $s$-waves survive at large $r$). Let us consider an initial configuration $z = z_i(r)$ that vanishes at $r = r_0$ where $r_0$ is of order of the core radius, and approaches the vacuum value at some $r = r_1 > r_0$ (but still $r_1$ is of order $r_0$),

\[z_i(r) = z_0, \quad r > r_1.\]  
(13)

Let us assume for simplicity that initially

\[\partial_z z(t = 0) = 0\]  
(14)

(this assumption is unimportant and may be safely dropped). Let us note also that the modulus field is zero in the core at all times (otherwise the energy of the core would become larger than $S_0$), so that we have a condition

\[z(r_0) = 0\]  
(15)

at any $t$. These conditions determine the solution to eq.(12) uniquely.

The complete set of solutions to eq.(12) with the conditions (14), (15) is

\[\cos(\omega t) Z_\omega(r),\]
where \( \omega \) is a continuous parameter,

\[
Z_{\omega}(r) = N_{\omega} \left[ J_0(\omega r_0) N_0(\omega r) - N_0(\omega r_0) J_0(\omega r) \right],
\]

\( J_0 \) and \( N_0 \) are the Bessel functions and

\[
N_{\omega} = \left[ \frac{\omega}{J_0^2(\omega r_0) + N_0^2(\omega r_0)} \right]^{1/2}.
\]

The functions \( Z_{\omega}(r) \) are normalized (with measure \( r \, dr \)) to \( \delta(\omega - \omega') \). Therefore, the solution to the problem (12), (13), (14), (15) is

\[
z(r, t) = \int d\omega \, z_{\omega} \cos(\omega t) Z_{\omega}(r)
\]

with

\[
z_{\omega} = \int r \, dr \, z_i(r) Z_{\omega}(r).
\]

At large \( t \) and \( r \), and at \( (t - r) \sim t \) (deep inside the light cone and far from the core) the integral (16) is dominated by small \( \omega \), namely, \( \omega \sim t^{-1} \). It is straightforward to see that at small \( \omega \), the function \( z_{\omega} \) is independent of the details of the initial configuration and is equal to

\[
z_{\omega} = \frac{z_0}{\omega^{3/2} \ln(\omega r_0)}.
\]

So, in this regime

\[
z(r, t) = \frac{\pi}{2} z_0 \int \frac{d\omega}{\omega \ln^2(\omega r_0)} \cos(\omega t) \left[ N_0(\omega r) - \frac{2}{\pi} \ln(\omega r_0) J_0(\omega r) \right].
\]

Clearly, \( z(r, t) \) depends on \( r \) and \( t \) logarithmically, and at \( r_0 \ll r \ll t \) the solution has a particularly simple form

\[
z(r, t) = z_0 \frac{\ln r/r_0}{\ln t/r_0}
\]

up to corrections suppressed by \( (\ln r/r_0)^{-1} \), \( (\ln t/r_0)^{-1} \). Note that eq.(18) coincides with the constrained instanton solution (8) whose radius \( R \) linearly increases with time.

The solution deep inside the light cone, eq.(17), does not depend on the details of the initial configuration, i.e., it is universal. At a given point \( r \) in space, the modulus field relaxes to zero logarithmically in time, eq.(15), i.e., the vacuum (7) logarithmically settles down. A typical evolution of the field \( f_2 \) obtained by solving numerically the exact non-linear field equations coming from the Lagrangian (1), is presented in fig.2. Fig.3 demonstrates that the numerical solutions are in agreement with eq.(15).
The peculiar properties of string-like objects in the model (1) are entirely due to the presence of the modulus field in the theory. So, these properties should be generic in many models with flat directions. On the other hand, these features occur because of the specific behavior of massless classical fields in two dimensions, so they do not show up in the case of point-like objects such as monopoles.

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Figure captions

Figure 1. The functions $f_1(r)$ and $f_2(r)$ of constrained instanton at $\lambda = e^2 = 1$, $\eta = 1$, $v_1 = 1.34$, $v_2 = 0.89$, for two different values of the constraint parameter $R_1 = 25$ (dashed curves) and $R_2 = 100$ (solid curves).

Figure 2. The component $f_2(r)$ of a numerical solution to the exact field equations at different times, $t_1 = 50$, $t_2 = 150$, $t_3 = 250$, $t_4 = 350$. The function $f_1(r,t)$ for this solution obeys $f_1^2 - f_2^2 = \eta^2$ to better than 0.5 per cent outside the core. The parameters are the same as in fig.1. The initial configuration had the size of order one.

Figure 3. The value of $f_2$ at $r = 200$ as function of $1/\ln t$ for the exact solution shown in fig.2. Solid line is a linear fit $f_2 = v_2 \ln(r/r_0)/\ln(t/r_0)$ with fitted value of $r_0$ equal to 2.39. Note that at small $z(r,t)$ one has $f_2(r,t) = \text{const} \cdot z(r,t)$. 

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Fig. 1
Fig. 2
Fig. 3