Problem of stabilizing the nonlinear "seesaw" model object

G A Frantsuzova 1*, O A Votrina 1, and K N Meleshkin 1

1Novosibirsk State Technical University, Novosibirsk, 630073, Russia
E-mail: *olga_votrina@mail.ru

Abstract. The objective of this paper is to discuss the problem of stabilizing the nonlinear type of oscillatory objects. The object under research is “Seesaw” model object which has many interpretations with different physical properties and mathematical models. Such a construction is used in biomechanics for implementing the “natural” gait of a human simulating the knee joint, for instance. This object can be stabilized by well-known techniques such as relay control, transitions controller, PI-regulator and others. Such a model has a lot of nonlinearities that makes the process of constructing the mathematical model and designing the stabilizing system more complicated. One of “Seesaw” models that consists of a mass suspended by an inelastic fiber was stabilized by methods of maximizing and minimizing the domain of attraction. The results of calculating and simulating the object within its stabilizing system show that the object can be successfully stabilized by relay control in case of minimizing the domain of attraction. However, the case of maximizing the domain of attraction leads to opposite transient process.

1. Introduction
Nowadays the problem of stabilizing objects in oscillatory mode is urgent. A lot of models simulating different processes and constructions become unstable under the influence of external forces or random disturbances [1-6]. For instance, the model object “Seesaw” which is used in biomechanics for implementing the “natural” knee motion is needed to be stabilized during all motion simulation including the one support leg phase [7]. Such a model object consists of a mass suspended by inelastic fiber (see figure 1).

Figure 1. A “Seesaw” object [7].
However, the model object “Seesaw” has a lot of interpretations within its physical construction, different parameters, or nonlinearities. One of them is used in civil engineering as a part of the versatile seismic force resisting systems for steel structures. It consists of a pin-supported seesaw, two spiral strand ropes (cables) with turnbuckles that intersect from the edges of the seesaw and are anchored using gusset plates to the bottom end of beam-to-column interfaces, as well as a couple of dampers installed vertically on the seesaw [8]. Regardless of the axis of the lateral seismic motion in relation to the plane of the steel frame, the two clamps of the seesaw system, for instance, the spiral cables, possess only tension (see figure 2).

![Figure 2. A steel frame equipped with a seesaw system, [8].](image)

Such a seesaw system is used in earthquake resistant constructions due to its abilities of significant seismic response and damping capacity.

The most interesting application of seesaw systems is the postural control of human body. Thus, many of well-known companies are interested in studies on the robots, especially biped robots. Such robots have two legs and great number of degrees of freedom which are used to make robot moves naturally as the human body does. In order to make the torso of the robot in correlation with legs and arms asymptotically stable, the stabilizing system is constructed. Usually, the calculations are based on a control law for the exact nonlinear or linear mathematical model extracted from the physical object under the influence of external forces or disturbances.

The seesaw model plays a great role in robot movement. A knee joint can be simulated by using such a model as the basis of the swinging leg motion. The legs movement and the gait itself of 7-degree-of-freedom biped robot, which consists of two legs with knee joints, hips and torso, can be stabilized by different methods. This robot within its stabilizing system successfully walks for two support legs phase, phase transitions and one support leg phase (see figure 3).

The first of known approaches for such a model is the use of feedback controller with high-gain value. However, the difficulty appears if analyzing mathematically the existence and stability of periodic orbits induced by the controller.

The Poincaré’s method is the appropriate tool when working with periodic orbits [9].

![Figure 3. Schematic representation of a biped robot torso and leg, [9].](image)
According to this method, in a row with the high-gain controller the induced discrete-time dynamics from a hyper-surface transversal to the orbit back to the hyper-surface should be computed [10-11]. The induced discrete-time dynamics, in other words, the Poincaré map, which can be calculated numerically, leads to conclusive existence and stability properties for periodic orbits.

Differential equations describing the system within its phase of leg transferring with three degrees of freedom obtained via the Lagrangian method are following:

\[ D(\theta)\ddot{\theta}+C(\theta)\dot{\theta}+G(\theta)=B(\theta)u \]  

where \( u=(u_1,u_2)' \), \( \theta=(\theta_1,\theta_2,\theta_3)' \) are parametrizing the support leg, \( \Theta_2 \) – leg in the air, \( \alpha \) \( \Theta_3 \) – a torso.

The system (1) of a second order can be written in state-space form by defining [3]:

\[ \dot{x}=\begin{bmatrix} 0 \\ D^{-1}(\theta)(-C(\theta,\omega)\dot{\theta}-G(\theta)+Bu) \end{bmatrix}=-f(x)+g(x)u \]  

The second existing approach for seesaw stabilizing is the PI control method with transitions controllers. An event-based controller, which changes the parameters on the output of the model object, can help to obtain movement at a continuum of rates for biped model of robot (see figure 4). In other words, walking at more than one discrete rate can be achieved by switching law and a one-step transition controller which computation should be based on a hybrid zero dynamics. In addition to the transition controller an event-based PI controller can be embedded to the stabilizing system. Calculated with the use of restricted Poincaré map of the hybrid zero dynamics, it provides an ability to adjust the motion rate to a continuum of values. Such a controller can change the parameters due to the integral action. For constant values it induces a periodic orbit that is exponentially stable. Parameters are changing after some impact, e.g. when a leg touch the ground in the one support leg phase of movement [12].

The third one is applicable for a bit different seesaw model. Such a system consists of the body standing on the seesaw platform (see figure 5). The stabilizing algorithm is discussed within the upper unstable equilibrium position.

Figure 4. Schematic of a biped robot [9].
Figure 5. The inverted pendulum object on a seesaw [13].

The control within the model system is momentum appeared in engine of pendulum (which simulates the body) rotation axis which is limited by strict value. This system is the simplest example of a human body standing on the seesaw neglected the human simulation was conducted by rod.

Equations of kinetic, potential energy of gravitational force and work of a possible movement of forces actuated in joint bearing are presented in following form:

\[ T = \frac{1}{2} \Phi(\phi) \dot{\phi}^2 + \frac{1}{2} M r^2 \dot{\alpha}^2 + \Psi(\alpha, \phi) \dot{\phi} \]  

(3)

\[ \Pi = mg(R - r \cos \phi) + Mg(R - h \cos \phi + l \cos \alpha) \]  

(4)

\[ \delta W = Q(\delta \alpha - \delta \phi) \]  

(5)

\[ \Phi(\phi) = m (R^2 + r^2 + \rho^2 - 2Rr \cos \phi) + M (R^2 + h^2 - 2Rh \cos \phi) \]  

(6)

\[ \Psi(\alpha, \phi) = M |R \cos \alpha - h \cos (\phi - \alpha)| \]  

(7)

Then the equation of movement in Lagrange form of II type was obtained in matrix-like system:

\[ A(\alpha) \ddot{q} + F(q) \dot{q}^2 + C \sin q = DQ, \]  

(8)

where \( q = \begin{bmatrix} \phi \end{bmatrix}, \dot{q} = \begin{bmatrix} \dot{\phi} \end{bmatrix}, \sin q = \begin{bmatrix} \sin \phi \end{bmatrix}, \]

\[ A(\alpha) = \begin{bmatrix} M r^2 & \Psi(\alpha, \phi) \\ \Psi(\alpha, \phi) & \Phi(\phi) \end{bmatrix}, \]

\[ F(q) = \begin{bmatrix} 0 \\ -M [R \sin \alpha + h \sin (\phi - \alpha)] \end{bmatrix}, \]

\[ C = \begin{bmatrix} -M g l & 0 \\ 0 & (M h + m r) g \end{bmatrix}, \quad D = \begin{bmatrix} -1 & \end{bmatrix} \]

The linearized model of the discussed object was obtained neglecting the tilting angles \( \phi \) and \( \alpha \) during all period of motion.

One essential problem of the human standing on seesaw is returning to equilibrium position when deviations from such a state are significant. In other words, it is obvious that a human tends to maximize the area of initial disturbances which can be overcome.
The control stabilizing the system equilibrium can be calculated for this case [13]. The control law was constructed for providing the maximized area of attraction. By the area of attraction, the variety of initial conditions from which the system asymptotically tends to origin of coordinates is meant.

The existing approach was used for such a type of system [7]. The method is to compute the regulator stabilizing an unstable upper position of the inverted pendulum and lateral position of the platform with the maximized area of attraction. Unstable mode of the system is overcome due to the feedback control. The linear feedback with saturation should be used instead of common linear feedback indicating the limitation of control described above. Final control law maximizes the system area of attraction what underlines the optimal way of proposed control in this context.

As well as the abovementioned facts, the other fact is proved: the higher the gain in the system of a pendulum with a movable suspension point, the faster the output value tends to zero [7]. A delay within the feedback chain leads to the limitation of the amplification gain. By neglecting this condition, the system can lose its stability. A delay is caused by delays within the control circuit or also through the data of sensors.

The problem of acceptable delay in feedback chain estimation is significant within the task of stabilizing of any unstable object as far as at least the lowest delay can make the stabilizing itself an impossible task. In order to stabilize the system with the control correlated with pure delay in feedback it is needed to distinguish the equations of movement with linear feedback.

Such an approach for synthesis of feedback is appropriate for using within systems with the degree of instability equal to one. In addition, the full nonlinear pendulum model with fixed point of suspension is discussed. There is also a provided evidence that equilibrium state can be achieved under any initial state with a help of control limited by absolute value.

The fourth known approach provides the solution for seesaw system within the five-link biped robot moving by stairs [14-15]. The length and height of the step is fixed and is discussed as initial parameter.

The task of obtaining the periodical movement by stairs of biped five-link robot with knee joints is discussed in [14]. The vector-mode output which is the basis of the constructed normal form of the system is provided. System normal form describes the robot behavior in a one support leg phase.

The controls providing a robot movement by stairs in accordance with predefined parameters are obtained. The only solution was chosen from a variety of system solutions describing the movement model of robot with knee joints. This is because in random moment \( t_0 \), which is initial, the conditions for torso coordinates and angular velocity were defined.

Resulting solution \( x(t, t_0, x_0) \) fulfils all requirements indicated during the computation of control (see figure 6). The tilting angle of torso tends to the trajectory of hips and the end of support leg movement for a finite time. The trajectories become parabolic; the hips position is fixed in the middle between the ends of legs.

The system of equations describing the behavior of robot within the one support leg phase was accumulated by Lagrange equations of II type:

\[
D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = Bu
\]

\[
q = (q_1, q_{31}, q_{32}, q_{41}, q_{42})^T
\]

where \( D(q) \) – symmetric positive-defined matrix of the fifth order, \( C(q, \dot{q}) \) – square matrix of the fifth order, \( G(q) \) – vector of the averaged gravitational forces, \( B \) – matrix of coefficients within control:

\[
B = \begin{pmatrix}
-1 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
The disadvantage of the assumed model is that it is impossible to anticipate the robot behavior in
the following periods of time when only the initial condition \( x(t_0) = x_0 \) is defined. This is the chance
that the condition of changing the roles of legs would be done earlier than the moment \( t = T_s \) came. It
means that resulting control \( u \) would not perform the stabilizing of the system movement.

\[ \frac{\ddot{\alpha}}{I+Mu^2} = \frac{K}{I+Mu^2} \]

\[ \dot{\alpha} = -\frac{ck}{I+Mu^2} \cdot (M+mb) \sin \alpha \]

where \( \alpha \) – angle of tilting from vertical position, \( \dot{\alpha} \) - angular velocity, \( K \) – kinetic momentum of the
system regarding the suspension point \( O \), \( M \) – single mass point, \( I \) - object moment of inertia regarding
the point of suspension \( O \), \( u \) – distance between \( M \) and \( O \), \( c \) – mass center, \( b \) – distance between
the point \( O \) and mass center \( c \), \( m \) – pendulum mass.

\[ u_0 \leq u \leq u_1 \]

where \( u_0, u_1 = \text{const} \), \( u_0 < u_1 \).
The second step is the transformation of the mathematical model into the form of state variables. It is needed there to set the state variable. It could be the $\alpha$ which is an angle of tilting of object from vertical position. In such case the state vector should have the following form:

$$x_1 = \alpha, \quad x_2 = \dot{\alpha}$$

(14)

The output variable is the whole output of the system:

$$y_1 = x_1$$

(15)

The synthesis task formulation has the following form:

$$\begin{align*}
\lim_{t \to \infty} \begin{cases}
\alpha(t) = 0 \\
t_n < t^*_n
\end{cases}
\end{align*}$$

(16)

However, the stabilizing system with the angular velocity as a state variable is not an optimal way of overcoming the instability. In order to eliminate the movement velocity of single mass point from equations of motion and design the full figure of synthesis the kinetic momentum can be used as a second state variable instead of the angular velocity [7]. So, in such case the state vector will have the following form:

$$x_1 = \alpha, \quad x_2 = K,$$

(17)

where kinetic momentum of system is described in following way:

$$K = (I + Mu^2) \ddot{u}$$

(18)

The output variable has the same form as in not optimal case.

Thus, we can transform the mathematical model into the form of state variables for seesaw system with state variables of tilting angle and kinetic momentum:

$$\begin{align*}
\dot{x}_1 &= x_2 - \frac{1}{I + Mu^2} \\
\dot{x}_2 &= -\sin x_1 (Mu + mb)x_2 - \frac{c}{I + Mu^2} \\
y &= x_1 
\end{align*}$$

(19)

However, this form of state variable equations is inconvenient to process it within verification of stability, controllability, observability and it is hard to get control $u$ from the equation, thus, it was decided to move to the other approach.

Such a technique is relay control [7]. This technique is appropriate for situations when control impact of some value should be applied to object in some exact state. An impact of another value should be applied when the state of the object is switched to opposite condition.

The main type of seesaw motion is swinging from one side to the other. It is obvious that external disturbance or force can make the object unstable involving it into such a state. In this case it is necessary to minimize the oscillations to achieve the equilibrium position or be close to it. On the other hand, maximizing of the oscillations can be useful in some cases. For instance, the force applied to robot knee could be multiplied, so the amplitude of a step and leg motion would be increased.

So, the task is to maximize the amplitude of the seesaw oscillations under certain conditions and to minimize it under other conditions.

Firstly, the maximizing task will be discussed. Note that if the conditions are following:

$$\alpha(0) = 0, \quad K(0) = 0, \quad \dot{\alpha}(0) = 0$$

(20)

the parameters will be following within any of control impact $u(t)$:
In other words, if the seesaw object is in its equilibrium state, it is impossible to change this state by any control \( u(t) \). It is also important that the model object cannot be set to the equilibrium state by any control \( u(t) \) if it was not in this state before.

The initial state of the system is set in following form:

\[
\alpha(0) = 0, \quad K(0) = 0
\]

(21)

The problem is to obtain the law of changing the distance \( u \) in interval (12) when the maximum of the tilting angle \( \alpha \) is achieved in the moment \( \theta \) along with the kinetic momentum \( K \) and the velocity \( \dot{\alpha} \) which becomes equal to zero \( (K(\theta) = 0) \) first time after the beginning of motion [7]. So, it is needed to maximize the deviation of seesaw from the upper position at the end of oscillations half-period.

The motion is performed in time interval \( 0 < t < \theta \) under the condition of \( K > 0 \), the system could be formulated as:

\[
\frac{dK}{d\alpha} = -c - \frac{(Mu + mb)(1 + Mu^2) \sin \alpha}{K}
\]

(23)

The right part of the equation can be maximized by \( u \) value in interval (12). Thus, the optimal way of control law for exact case of seesaw object is obtained for a half-period of oscillations, where \( K > 0 \) [7]. Following conditions are valid:

\[
u = u_1 \text{ if } \alpha < 0, \quad u = u_0 \text{ if } \alpha > 0
\]

(24)

Similarly to the first half-period of oscillations, the second half-period with \( K < 0 \) has an optimal control low of such form:

\[
u = u_1 \text{ if } \alpha > 0, \quad u = u_0 \text{ if } \alpha < 0
\]

(25)

Finally, the full control law will be described in such way:

\[
u = u(\alpha, \dot{\alpha}) = \begin{cases} 
  u_1 \text{ when } \alpha \dot{\alpha} > 0 \\
  u_0 \text{ when } \alpha \dot{\alpha} < 0
\end{cases}
\]

(26)

The distribution between values of control impact has following interpretation on phase plane (see figure 7):

![Figure 7. Synthesis of the optimal control for seesaw object swinging [7].](image)

The single mass point M should move up to the point of suspension by the influence of the optimal control when seesaw object passes through the lowest position. The single mass point M should move
down till the end by the influence of the optimal control when seesaw object tilts to the maximum from vertical position, in other words, when the angular velocity becomes equal to zero.

In addition, the control law for seesaw object could be designed in the form of dependency on the phase coordinates what allows to solve the problem for any initial states.

3. Results

The results were obtained for the task of maximizing and minimizing the amplitude of seesaw swinging under the relay control.

Experiments were conducted for following parameters of the object:

\[
m = 5 \text{ kg},
\]
\[
b = 4 \text{ m},
\]
\[
I = 26.67 \text{ kg} \cdot \text{m}^2,
\]
\[
M = 70 \text{ kg},
\]
\[
u_0 = 3 \text{ m},
\]
\[
u_1 = 3.75 \text{ m},
\]
\[
c = 2 \text{ N} \cdot \text{m} \cdot \text{s},
\]
\[
e = 1, \quad v = 3 \text{ N} \cdot \text{m}.
\]

The following control law was chosen for the case of maximizing the amplitude of the seesaw:

\[
u = u(\alpha, \dot{\alpha}) = \begin{cases} u_1 & \text{when } \alpha \dot{\alpha} < 0 \\ u_0 & \text{when } \alpha \dot{\alpha} > 0 \end{cases}
\]  (27)

Initial conditions were following:

\[
\alpha(0) = -0.1, \quad K(0) = \dot{\alpha}(0) = 0
\]  (28)

The simulation was performed within MATLAB Simulink integrated environment. The block diagram has the following structure:

\[\text{Figure 8. A block diagram for the case of maximizing the swinging amplitude.}\]

The transient processes of the system with maximized amplitude of swinging were computed for the parameters of tilting angle and angular velocities (see figures 9, 10).
Figure 9. A transient process within the tilting angle.

The control input is shown in figure 11.

Figure 10. A transient process within the angular velocity.

The following control law was chosen for the case of minimizing the amplitude of the seesaw:

\[ u = u(\alpha, \dot{\alpha}) = \begin{cases} 
    u_1 & \text{when } \alpha \dot{\alpha} > 0 \\
    u_0 & \text{when } \alpha \dot{\alpha} < 0 
\end{cases} \]  

(29)
Initial conditions were following:

\[
\alpha(0) = -1.75, \ K(0) = \dot{\alpha}(0) = 0
\]  

MATLAB Simulink block diagram for the case of minimizing is shown in figure 12:

**Figure 12.** A block diagram for the case of minimizing the swinging amplitude.

The transient processes of the system with minimized amplitude of swinging were computed for the parameters of tilting angle and angular velocities (see figures 13, 14).

**Figure 13.** A transient process within the tilting angle.
4. Discussion
The sections of results show that the minimizing of the seesaw swinging amplitude could be performed successfully. As can be seen from the transient process, the high oscillatory mode was transformed to the mode without oscillations, in other words, to the equilibrium state.

However, provided in [7] method for maximizing the swinging amplitude leads to the minimizing of it. The system became stable but in the opposite way. Probably, such state was achieved due to some exact value of seesaw model parameter or the relay component needs some adjustment.

5. Conclusion
The problem of stabilizing the nonlinear type of oscillatory objects, especially the “Seesaw” model object, was discussed in this paper. Different interpretations of such a model was presented within its physical construction. Several techniques for stabilizing the object were defined, such as relay control, transitions controller, PI-regulator, and others.

The main applicable seesaw model which consists of the mass suspended by an inelastic fiber was discussed. The mathematical model for the object was described. The model was stabilized by methods of maximizing and minimizing the domain of attraction.

The results of calculating and simulating the object within its stabilizing system show that object can be successfully stabilized by relay control in case of minimizing the domain of attraction. However, the case of maximizing the domain of attraction leads to opposite transient process.

References
[1] Sablina G V, Sazhin A I 2016 Designing and research of the real sliding mode in the inverted pendulum modelling, 11 International forum on strategic technology (IFOST 2016): proc., Novosibirsk, 1-3 June, 2016, Novosibirsk, NSTU, Pt. 1, pp 571-575. ISBN 978-1-5090-0853-7. DOI: 10.1109/IFOST.2016.7884182
[2] Lin J, Guo S Y, Julian Chang, Fuzzy coordinator compensation for balancing control of cart-seesaw system, Journal of Sound and Vibration, Accepted 5 August,
[3] Grizzle J W, Abba G, Plestan F 2001 Asymptotically Stable Walking for Biped Robots: Analysis via Systems with Impulse Effects, IEEE Trans., On Automatic Control, 46, 8, pp 51-64
[4] Westervelt E R, Grizzle J W, Koditschek D E 2003 Hybrid Zero Dynamics of Planar Biped Walkers, IEEE Trans. on Automatic Control, 48, 1, pp 42-56
[5] Chevallereau C, Abba G, Auoust Y, Plestan E, Westervelt F 2003 Rabbit: A testbed for advanced control theory, IEEE Control Systems Magazine, 23, 5, pp 57-79
[6] Morris B, Grizzle J W 2005 *A restricted Poincar’e map for determining exponentially stable periodic orbits in systems with impulse effects: Application to bipedal robots*, Proceedings of the IEEE International Conference on Decision and Control, Seville, Spain, pp 4199–4206

[7] Formalskii A M 2012 *Control of nonlinear objects motion*, M.: Fizmatlit, p 232. ISBN 978-5-9221-1460-8

[8] Panagiota S K, George A P, Paraskevi K A 2020 *Seismic response of low-rise 3-D steel structures equipped with the seesaw system*, Soil Dynamics and Earthquake Engineering, 128, 105877

[9] Plestan F, Grizzle J W, Westervelt E R, G. Abba G 2003, *Stable walking of a 7-DOF biped robot*, IEEE Trans. Robotics and Automation, 19, 4, pp 653-668

[10] Guckenheimer J, and Holmes P 1996 *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, New York: Springer-Verlag, Applied Mathematical Sciences, 42

[11] Parkerand T S, Chua L O 1989 *Practical Numerical Algorithms for Chaotic Systems*, New York: Springer-Verlag

[12] Westervelt E R, Grizzle J W 2003 *Switching and PI Control of Walking Motions of Planar Biped Walkers*, IEEE Transactions on automatic control, 48, 2

[13] Gugaev K V, Kruchinin P A, Formalskii A M 2016 *A model of maintaining balance by a person on the seesaw*, Journal of Applied Mathematics and Mechanics, Received 5 February

[14] Kolesnikova G P, Formalskii A M 2014 *About one way of the human gait simulation*, Engineering journal: Science and Innovations, 1

[15] Krishhenko A P, Tkachev S B, Fetisov D A 2006 Control of the biped 5-link robot flat movement on stairs, ISSN 1812-3368, Vestnik MSTU im. N.Je. Bauman, Ser. “Natural Sciences”, 1