On multidimensional solutions in the Einstein-Gauss-Bonnet model with a cosmological term

A.A. Kobtsev¹, V.D. Ivashchuk² and K.K. Ernazarov³

Moscow Mesons Factory of INR RAS, Moscow, Troitsk, 142190, Russia;
Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya ul.,
Moscow, 119361, Russia;
Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, 6
Miklukho-Maklaya ul., Moscow, 117198, Russia.

Abstract

A $D$-dimensional gravitational model with Gauss-Bonnet and cosmological term $\Lambda$ is considered. When ansatz with diagonal cosmological metrics is adopted, we overview recent solutions for $\Lambda = 0$ and find new examples of solutions for $\Lambda \neq 0$ and $D = 8$ with exponential dependence of scale factors which describe an expansion of “our” 3-dimensional factor-space and contraction of 4-dimensional internal space.

¹e-mail: aak@inr.ru
²e-mail: ivashchuk@mail.ru
³e-mail: kubantai80@mail.ru
1 Introduction

Here we consider $D$-dimensional gravitational model with the Gauss-Bonnet term. The action reads

$$S = \int_M d^Dz \sqrt{|g|} \{ \alpha_1 (R[g] - 2\Lambda) + \alpha_2 \mathcal{L}_2[g] \},$$  \hspace{1cm} (1.1)

where $g = g_{MN} dz^M \otimes dz^N$ is the metric defined on the manifold $M$, $\dim M = D$, $|g| = |\det(g_{MN})|$ and

$$\mathcal{L}_2 = R_{MNPQ} R^{MNPQ} - 4R_{MN}R^{MN} + R^2$$  \hspace{1cm} (1.2)

is the standard Gauss-Bonnet term. Here $\alpha_1$ and $\alpha_2$ are non-zero constants.

Earlier the appearance of the Gauss-Bonnet term was motivated by string theory \[1\] \[2\] \[3\] \[4\] \[5\].

At present, the (so-called) Einstein-Gauss-Bonnet (EGB) gravitational model and its modifications are intensively used in cosmology, see \[6\] (for $D = 4$), \[7\] \[8\] \[9\] \[10\] \[11\] \[12\] \[13\] \[14\] \[15\] \[16\] \[17\] \[18\] and references therein, e.g. for explanation of accelerating expansion of the Universe following from supernovae (type Ia) observational data \[16\] \[17\] \[18\]. Certain exact solutions in multidimesional EGB cosmology were obtained in \[7\]-\[15\] and some other papers.

Here we are dealing with the cosmological type solutions with diagonal metrics (of Bianchi-I-like type) governed by $n$ scale factors depending upon one variable, where $n > 3$. Moreover, we restrict ourselves by the solutions with exponential dependence of scale factors. We present new examples of exact solutions in dimension $D = 8$ which describe an exponential expansion of 3-dimensional factor-space and contraction of 4-dimensional internal space.

2 The Cosmological Model

Here we consider the manifold

$$M = \mathbb{R} \times \mathbb{R}^n$$  \hspace{1cm} (2.1)

with the metric

$$g = -dt \otimes dt + \sum_{i=1}^{n} e^{2\beta_i(t)} dy^i \otimes dy^i,$$  \hspace{1cm} (2.2)

where $\beta_i(t)$ are smooth functions, $i = 1, \ldots, n$.

We introduce “Hubble-like” variables $h^i = d\beta^i/dt$. The equations of motion for the action (1.1) read as follows

$$\alpha_1 (G_{ij} h^j + 2\Lambda) - \alpha_2 G_{ijkl} h^i h^j h^k h^l = 0,$$  \hspace{1cm} (2.3)

$$\left[ 2\alpha_1 G_{ij} h^j - \frac{4}{3} \alpha_2 G_{ijkl} h^i h^j h^k h^l \right] \sum_{i=1}^{n} h^i$$

$$+ \frac{d}{dt} \left[ 2\alpha_1 G_{ij} h^j - \frac{4}{3} \alpha_2 G_{ijkl} h^i h^j h^k h^l \right] - L = 0,$$  \hspace{1cm} (2.4)
\[ L = \alpha_1 (G_{ij} h^i h^j + 2\Lambda) - \frac{1}{3} \alpha_2 G_{ijkl} h^i h^j h^k h^l. \] (2.5)

Here

\[ G_{ij} = \delta_{ij} - 1, \] (2.6)
\[ G_{ijkl} = G_{ij} G_{ik} G_{il} G_{jk} G_{jl} G_{kl}. \] (2.7)

are respectively the components of two metrics on \( \mathbb{R}^n \) [19, 20]. The first one is the well-known “minisupermetric” - 2-metric of pseudo-Euclidean signature and the second one is the Finslerian 4-metric.

Due to (2.3)
\[ L = \frac{2}{3} \alpha_1 (G_{ij} h^i h^j - 4\Lambda). \] (2.8)

In this paper we deal with the following solutions to equations (2.3) and (2.4)
\[ h^i(t) = v^i, \] (2.9)
with constant \( v^i \), which corresponding to the solutions
\[ \beta^i = v^i t + \beta^i_0, \] (2.10)
where \( \beta^i_0 \) are constants, \( i = 1, \ldots , n \).

In this case we obtain the metric (2.2) with the exponential dependence of scale factors
\[ g = -dt \otimes dt + \sum_{i=1}^{n} B_i e^{2v^i t} dy^i \otimes dy^i, \] (2.11)
where \( B_i > 0 \) are arbitrary constants.

For the fixed point \( v = (v^i) \) we have the set of polynomial equations
\[ G_{ij} v^i v^j + 2\Lambda - \alpha G_{ijkl} v^i v^j v^k v^l = 0, \] (2.12)
\[ \left[ 2G_{ij} v^j - \frac{4}{3} \alpha G_{ijkl} v^i v^j v^k v^l \right] \sum_{i=1}^{n} v^i - \frac{2}{3} G_{ij} v^i v^j + \frac{8}{3} \Lambda = 0, \] (2.13)
i = 1, \ldots , n, where \( \alpha = \alpha_2 / \alpha_1 \). For \( n > 3 \) this is a set of forth-order polynomial equations.

For \( \Lambda = 0 \) and \( n > 3 \) the set of equations (2.12) and (2.13) has an isotropic solution \( v^1 = \ldots = v^n = H \), only if \( \alpha < 0 \) [19, 20] \( H = \pm 1/\sqrt{|\alpha|} (n - 2)(n - 3) \). This solution was generalized in [15] to the case \( \Lambda \neq 0 \).

It was shown in [19, 20] that there are no more than three different numbers among \( v^1, \ldots , v^n \) when \( \Lambda = 0 \). This is valid also for \( \Lambda \neq 0 \).
3 Examples of Cosmological Solutions

In this section we consider some solutions to the set of equations (2.12), (2.13) of the following form

\[ v = (H, \ldots, H, h, \ldots, h) \]  

(3.1)

where \( H \) the “Hubble-like” parameter corresponding to \( m \)-dimensional isotropic subspace with \( m > 3 \) and \( h \) is the “Hubble-like” parameter corresponding to \( l \)-dimensional isotropic subspace, \( l > 2 \).

These solutions should satisfy the following conditions: \( H > 0, \ h < 0 \). The first inequality \( H > 0 \) is necessary for a description of accelerated expansion of 3-dimensional subspace, which may describe our Universe, while the second inequality \( h < 0 \) is necessary for contraction of internal space volume.

3.1 Polynomial equations

According to our ansatz (3.1), we have \( m \) dimensions expanding with the Hubble parameter \( H > 0 \) and \( l \) dimensions contracting with the “Hubble-like” parameter \( h < 0 \). The set of polynomial equations (2.12), (2.13) reads

\[
H^2(m - m^2) + h^2(l - l^2) - 2mlHh \\
- \alpha(H^4m(m - 1)(m - 2)(m - 3) + h^4l(l - 1)(l - 2)(l - 3) \\
+ 4H^3hm(m - 1)(m - 2)l + 4h^3Hl(l - 1)(l - 2)m \\
+ 6H^2h^2m(m - 1)l(l - 1)) + 2\Lambda = 0,
\]  

(3.2)

\[
m(1 - m)H^2 - (1/2)lh^2l(1 + 2l) + 2Hh((3/4 - m) \\
- \alpha(H^4m(m - 1)(m - 2)(m - 3) + H^3l(m - 1)(m - 2)(4m - 3) \\
+ 3H^2h^2l(m - 1)(2m - 2l - m) \\
+ Hh^3l(l - 1)(4lm - 3l - 2m) + h^4l^2(l - 1)(l - 2)) + 2\Lambda = 0,
\]  

(3.3)

\[
l(1 - l)h^2 - (1/2)mH^2(1 + 2m) + 2mHh((3/4 - l) \\
- \alpha(h^4l(l - 1)(l - 2)(l - 3) + h^3Hl(l - 1)(l - 2)(4l - 3) \\
+ 3h^2H^2m(l - 1)(2lm - 2m - l) \\
+ hH^3m(m - 1)(4lm - 3m - 2l) + H^4m^2(m - 1)(m - 2)) + 2\Lambda = 0.
\]  

(3.4)

We put \( \alpha = \pm 1 \) and denote \( \Lambda = \lambda \), keeping in mind the general \( \alpha \)-dependent form of solution

\[ H(\alpha) = H|\alpha|^{-1/2}, \quad h(\alpha) = h|\alpha|^{-1/2} \quad \Lambda = \lambda|\alpha|^{-1}. \]  

(3.5)
3.2 Solutions with \( \Lambda = 0 \)

Let \( \Lambda = 0 \) and \( \alpha = 1 \). It was shown in [21] that, for \( m = 9 \) there exists an infinite series of cosmological solutions with \( l = 3000, 3001, \ldots \), any of which describes an accelerated expansion of the 3-dimensional factor space with sufficiently small value of the variation of the effective gravitational constant \( G \) obeying the observational restrictions [22], see also [23]. This variation may be arbitrary small for a big enough value of \( l \). We remind that the effective gravitational constant \( G \) is proportional to the inverse volume scale factor of the internal space, see [24, 25, 26, 27, 28] and references therein.

For \( m = 11 \) and \( l = 16 \) it was found in [21] a solution with

\[
H = \frac{1}{\sqrt{15}}, \quad h = -\frac{1}{2\sqrt{15}}, \tag{3.6}
\]

which describes a zero variation of effective cosmological constant \( G \).

Another solution of such type which was found in [21] appears for \( m = 15 \) and \( l = 6 \) with

\[
H = \frac{1}{6}, \quad h = -\frac{1}{3}. \tag{3.7}
\]

3.3 Solutions with \( \Lambda \neq 0 \)

Here we present several new cosmological solutions for \( \Lambda \neq 0 \), \( \alpha = 1 \), \( m = 3 \) and \( l = 4 \). The first solution takes place for \( \lambda = 3/16 \):

\[
H = \frac{1}{4}\sqrt{2}, \quad h = -\frac{1}{4}\sqrt{2}. \tag{3.8}
\]

The second one is valid for \( \lambda = 13/48 \):

\[
H = \frac{1}{4}\sqrt{6}, \quad h = -\frac{1}{12}\sqrt{6}. \tag{3.9}
\]

The third one

\[
H = \frac{2}{29}\sqrt{29}, \quad h = -\frac{3}{58}\sqrt{29}. \tag{3.10}
\]

corresponds to \( \lambda = 21/116 \).

Any of these solutions describe accelerated expansion of 3-dimensional factor space and contraction of internal space. All of these solutions take place for fixed positive values of \( \lambda \) (see (3.5)). There exist also examples of solutions with negative cosmological constant \( \lambda = -21/80 \):

\[
H_\pm = \frac{1}{60320}(248 \pm 32\sqrt{30})\sqrt{68150 \mp 9280\sqrt{30}}, \tag{3.11}
\]

\[
h_\pm = -\frac{1}{580}\sqrt{68150 \mp 9280\sqrt{30}}. \tag{3.12}
\]

These solutions obey inequalities \( H_\pm > 0 \) and \( h_\pm < 0 \).
4 Conclusions

We have considered the $D$-dimensional Einstein-Gauss-Bonnet (EGB) model with with $\Lambda$-term. By using the ansatz with diagonal cosmological type metrics, we have found new solutions with exponential dependence of scale factors with respect to synchronous time variable $t$ in dimension $D = 1 + 3 + 4$. Any of these solutions describes an exponential expansion of “our” 3-dimensional factor-space with the Hubble parameter $H > 0$ and exponential contraction of 4$d$ internal space. An open question arising here is to find solutions with $\Lambda \neq 0$ which obey the observational constraints on the temporal variation of the effective gravitational constant $G$. This question will be addressed in a separate publication.

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