G-FRAME REPRESENTATIONS WITH BOUNDED OPERATORS

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Abstract. Dynamical sampling, as introduced by Aldroubi et al., deals with frame properties of sequences of the form \( \{T_i f_1\}_{i \in \mathbb{N}} \), where \( f_1 \) belongs to Hilbert space \( \mathcal{H} \) and \( T : \mathcal{H} \to \mathcal{H} \) belongs to certain classes of the bounded operators. Christensen et al., study frames for \( \mathcal{H} \) with index set \( \mathbb{N} \) (or \( \mathbb{Z} \)), that have representations in the form \( \{T^{-1} f_1\}_{i \in \mathbb{N}} \) (or \( \{T^i f_0\}_{i \in \mathbb{Z}} \)). As frames of subspaces, fusion frames and generalized translation invariant systems are the spacial cases of \( g\)-frames, the purpose of this paper is to study \( g\)-frames \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in I\} \) (\( I = \mathbb{N} \) or \( \mathbb{Z} \)) having the form \( \Lambda_{i+1} = \Lambda_1 T^i \), for \( T \in B(\mathcal{H}) \).

1. Introduction

In 1952, the concept of frames for Hilbert spaces was defined by Duffin and Schaeffer [11]. Frames are important tools in the signal/image processing [3, 4, 12], data compression [10, 16], dynamical sampling [1, 2] and etc.

Throughout this paper, \( I \) is a countable index set, \( \mathcal{H} \) and \( \mathcal{K} \) are seperable Hilbert spaces, \( \{\mathcal{K}_i : i \in I\} \) is a family of seperable Hilbert spaces, \( Id_{\mathcal{H}} \) denotes the identity operator on \( \mathcal{H} \), \( B(\mathcal{H}) \) and \( GL(\mathcal{H}) \) denote the set of all bounded linear operators and the set of all invertible bounded linear operators on \( \mathcal{H} \), respectively, \( l^2(\mathcal{H}, I) = \{\{g_i\}_{i \in I} : g_i \in \mathcal{K}, \sum_{i \in \mathbb{Z}} \|g_i\|^2 < \infty\} \) as well. Also, we will apply \( B(\mathcal{H}, \mathcal{K}) \) for the set of all bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \). We use \( \ker T \) and \( \text{ran} T \) for the null space and range \( T \in B(\mathcal{H}) \), respectively. We denote the natural, integer and complex numbers by \( \mathbb{N}, \mathbb{Z} \) and \( \mathbb{C} \), respectively.

A sequence \( F = \{f_i\}_{i \in I} \) in \( \mathcal{H} \) is called a frame for \( \mathcal{H} \) if there exist two constants \( A_F, B_F > 0 \) such that

\[
A_F \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B_F \|f\|^2, \quad f \in \mathcal{H}.
\]

2000 Mathematics Subject Classification. Primary 41A58, 42C15, 47A05.
Key words and phrases. representation, \( g\)-frame, duality.
Let $F = \{f_i\}_{i \in I}$ be a frame for $\mathcal{H}$, then the operator

$$T_F : l^2(\mathcal{H}, I) \to \mathcal{H}, \quad T_F(\{c_i\}_{i \in I}) = \sum_{i \in I} c_i f_i,$$

is well defined and onto, also its adjoint is

$$T_F^* : \mathcal{H} \to l^2(\mathcal{H}, I), \quad T_F^* f = \{\langle f, f_i \rangle\}_{i \in I}.$$

The operators $T_F$ and $T_F^*$ are called the synthesis and analysis operators of frame $F$, respectively.

Frames for $\mathcal{H}$ allow each $f \in \mathcal{H}$ to be expanded as an (infinite) linear combination of the frame elements. For more on frame we refer to the book [5]. Aldroubi et al., introduced the concept of dynamical sampling deals with frame properties of sequences of the form $\{T^i f_1\}_{i \in \mathbb{N}}$, for $f_1 \in \mathcal{H}$ and $T : \mathcal{H} \to \mathcal{H}$ belongs to certain classes of the bounded operators. Christensen and Hassannasab analyze frames $F = \{f_i\}_{i \in \mathbb{Z}}$ having the form $F = \{T^i f_0\}_{i \in \mathbb{Z}}$, where $T$ is a bijective linear operator (not necessarily bounded) on $\text{span}\{f_i\}_{i \in \mathbb{Z}}$. They show, $(T^*)^{-1}$ is the only possibility of the representing operator for the duals of the frame $F = \{f_i\}_{i \in \mathbb{Z}} = \{T^i f_0\}_{i \in \mathbb{Z}}, T \in GL(\mathcal{H})$ [6]. They even clarify stability of the representation of frames. Christensen et al., determine the frames that have a representation with a bounded operator, and survey the properties of this operators [8].

**Proposition 1.1.** [6] Consider a frame sequence $F = \{f_i\}_{i \in \mathbb{Z}}$ in $\mathcal{H}$ which spans an infinite dimentional subspace. The following are equivalent:

(i) $F$ is linearly independent.

(ii) The map $T f_i := f_{i+1}$ is well-defined and extends to a linear and invertible operator $T : \text{span}\{f_i\}_{i \in \mathbb{Z}} \to \text{span}\{f_i\}_{i \in \mathbb{Z}}$.

In the affirmative case, $F = \{T^i f_0\}_{i \in \mathbb{Z}}$.

**Theorem 1.2.** [8] Consider a frame $F = \{f_i\}_{i \in \mathbb{N}}$ in $\mathcal{H}$. Then the following are equivalent:

(i) $F$ has a representation $F = \{T^{i-1} f_1\}_{i \in \mathbb{N}}$ for some $T \in B(\mathcal{H})$.

(ii) For some dual frame $G = \{g_i\}_{i \in \mathbb{N}}$ (and hence all)

$$f_{j+1} = \sum_{i \in \mathbb{N}} \langle f_j, g_i \rangle f_{i+1}, \forall j \in \mathbb{N}.$$

(iii) The ker $T_F$ is invariant under the right-shift operator.
In the affirmative case, letting $G = \{g_i\}_{i \in \mathbb{N}}$ denote an arbitrary dual frame of $F$, the operator $T$ has the form
\[ Tf = \sum_{i \in \mathbb{N}} \langle f, g_i \rangle f_{i+1}, \forall f \in \mathcal{H}, \]
and $1 \leq \|T\| \leq \sqrt{B_F A_F^{-1}}$.

We will need the right-shift operator on $l^2(\mathcal{H}, \mathbb{N})$ and $l^2(\mathcal{H}, \mathbb{Z})$, defined by $T(\{c_i\}_{i \in \mathbb{N}}) = (0, c_1, c_2, \ldots)$ and $T(\{c_i\}_{i \in \mathbb{Z}}) = \{c_{i-1}\}_{i \in \mathbb{Z}}$. Clearly, the right-shift operator on $l^2(\mathcal{H}, \mathbb{Z})$ is unitary and $T^*$ is the left-shift operator, i.e. $T^*(\{c_i\}_{i \in \mathbb{Z}}) = \{c_{i+1}\}_{i \in \mathbb{Z}}$. A subspace $V \subseteq l^2(\mathcal{H}, \mathbb{N})$ is invariant under right-shift (left-shift) if $T(V) \subseteq V$ ($T^*(V) \subseteq V$).

In 2006, generalized frames (or simply $g$-frames) and $g$-Riesz bases were introduced by Sun [17]. "$G$-frames are natural generalizations of frames which cover many other recent generalizations of frames, e.g., bounded quasi-projectors, frames of subspaces, outer frames, oblique frames, pseudo-frames and a class of time-frequency localization operators [18]. The interest in $g$-frames arises from the fact that they provide more choices on analyzing functions than frame expansion coefficients [17], and also, every fusion frame is a $g$-frame [5]." Generalized translation invariant (GTI) frames can be realized as $g$-frames [13], so for motivating to answer the similar problems relevant to shift invariant systames and GTI systems in [7], we generalize some results of the frame representations with bounded operators in [6] to $g$-frames. Now, we summarize some facts about $g$-frames from [14, 17].

**Definition 1.3.** We say that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\}$ is a generalized frame, or simply $g$-frame, for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ if there are two constants $0 < A_\Lambda \leq B_\Lambda < \infty$ such that
\[
(1.2) \quad A_\Lambda \|f\|^2 \leq \sum_{i \in I} \|\Lambda_i f\|^2 \leq B_\Lambda \|f\|^2, \quad f \in \mathcal{H}.
\]

We call $A_\Lambda, B_\Lambda$ the lower and upper $g$-frame bounds, respectively. $\Lambda$ is called a tight $g$-frame if $A_\Lambda = B_\Lambda$, and a Parseval $g$-frame if $A_\Lambda = B_\Lambda = 1$. If for each $i \in I$, $\mathcal{K}_i = \mathcal{K}$, then, $\Lambda$ is called a $g$-frame for $\mathcal{H}$ with respect to $\mathcal{K}$. Note that for a family $\{\mathcal{K}_i\}_{i \in I}$ of Hilbert spaces, there exists a Hilbert space $\mathcal{K} = \bigoplus_{i \in I} \mathcal{K}_i$ such that for all $i \in I$, $\mathcal{K}_i \subseteq \mathcal{K}$, where $\bigoplus_{i \in I} \mathcal{K}_i$ is the direct sum of $\{\mathcal{K}_i\}_{i \in I}$. A family $\Lambda$ is called a $g$-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ if the right hand inequality in (1.2) holds for all $f \in \mathcal{H}$, in this case, $B_\Lambda$ is called the $g$-Bessel bound.

If there is no confusion, we use $g$-frame ($g$-Bessel family) instead of $g$-frame for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$ ($g$-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{K}_i : i \in I\}$).
Example 1.4. [17] Let \( \{f_i\}_{i \in I} \) be a frame for \( \mathcal{H} \). Suppose that \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in I\} \), where
\[
\Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H}.
\]
It is easy to see that \( \Lambda \) is a \( g \)-frame.

For a \( g \)-frame \( \Lambda \), there exists a unique positive and invertible operator \( S_\Lambda : \mathcal{H} \to \mathcal{H} \) such that
\[
S_\Lambda f = \sum_{i \in I} \Lambda_i^* \Lambda_i f, \quad f \in \mathcal{H},
\]
and \( A_\Lambda \text{Id}_\mathcal{H} \leq S_\Lambda \leq B_\Lambda \text{Id}_\mathcal{H} \). Consider the space
\[
\left( \sum_{i \in I} \bigoplus \mathcal{K}_i \right)_{l^2} = \left\{ \{g_i\}_{i \in I} : g_i \in \mathcal{K}_i, \ i \in I \text{ and } \sum_{i \in I} \|g_i\|^2 < \infty \right\}.
\]
It is clear that, \( \left( \sum_{i \in I} \bigoplus \mathcal{K}_i \right)_{l^2} \) is a Hilbert space with pointwise operations and with the inner product given by
\[
\langle \{f_i\}_{i \in I}, \{g_i\}_{i \in I} \rangle = \sum_{i \in I} \langle f_i, g_i \rangle.
\]
For a \( g \)-Bessel \( \Lambda \), the synthesis operator \( T_\Lambda : \left( \sum_{i \in I} \bigoplus \mathcal{K}_i \right)_{l^2} \to \mathcal{H} \) is defined by
\[
T_\Lambda (\{g_i\}_{i \in I}) = \sum_{i \in I} \Lambda_i^* g_i.
\]
The adjoint of \( T_\Lambda \), \( T_\Lambda^* : \mathcal{H} \to \left( \sum_{i \in I} \bigoplus \mathcal{K}_i \right)_{l^2} \) is called the analysis operator of \( \Lambda \) and is as follows
\[
T_\Lambda^* f = \{\Lambda_i f\}_{i \in I}, \quad f \in \mathcal{H}.
\]
It is obvious that \( S_\Lambda = T_\Lambda T_\Lambda^* \).

Definition 1.5. Two \( g \)-frames \( \Lambda \) and \( \Theta \) are called dual if
\[
\sum_{i \in I} \Lambda_i^* \Theta_i f = f, \quad f \in \mathcal{H}.
\]
For a \( g \)-frame \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\} \), the \( g \)-frame \( \tilde{\Lambda} = \{\Lambda_i S_\Lambda^{-1} \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\} \) is a dual of \( \Lambda \), that is called canonical dual.

Definition 1.6. Consider a family \( \Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I\} \).

(i) We say that \( \Lambda \) is \( g \)-complete if \( \{f : \Lambda_i f = 0, i \in I\} = \{0\} \).
(ii) We say that \( \Lambda \) is a \( \text{g-Riesz} \) basis if \( \Lambda \) is \( \text{g-complete} \) and there are two constants \( 0 < A_\Lambda \leq B_\Lambda < \infty \) such that for any finite set \( I_n \subset I \)
\[
A_\Lambda \sum_{i \in I_n} \| g_i \|^2 \leq \| \sum_{i \in I_n} \Lambda_i^* g_i \|^2 \leq B_\Lambda \sum_{i \in I_n} \| g_i \|^2, \quad g_i \in \mathcal{K}_i.
\]

(iii) We say that \( \Lambda \) is a \( \text{g-orthonormal} \) basis if it satisfies the following:
\[
\langle \Lambda_i^* g_i, \Lambda_j^* g_j \rangle = \delta_{i,j} \langle g_i, g_j \rangle, \quad i, j \in I, g_i \in \mathcal{K}_i, g_j \in \mathcal{K}_j,
\]
\[
\sum_{i \in I} \| \Lambda_i f \|^2 = \| f \|^2, \quad f \in \mathcal{H}.
\]

**Theorem 1.7.** A family \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in I \} \) is a \( \text{g-Riesz} \) basis if and only if there exist a \( \text{g-orthonormal} \) basis \( \Theta \) and \( U \in GL(\mathcal{H}) \) such that \( \Lambda_i = \Theta_i U, \ i \in I \).

**Theorem 1.8.** [17] Let for \( i \in I \), \( \{ e_{i,j} \}_{j \in J_i} \) be an orthonormal basis for \( \mathcal{K}_i \),

(i) \( \Lambda \) is a \( \text{g-frame} \) (respectively, \( \text{g-Bessel family, g-Riesz basis, g-orthonormal basis} \) if and only if \( \{ \Lambda_i^* e_{i,j} \}_{i \in I, j \in J_i} \) is a frame (respectively, Bessel sequence, Riesz basis, orthonormal basis).

(ii) \( \Lambda \) and \( \Theta \) are dual if and only if \( \{ \Lambda_i^* e_{i,j} \}_{i \in I, j \in J_i} \) and \( \{ \Theta_i^* e_{i,j} \}_{i \in I, j \in J_i} \) are dual.

In this paper, we generalize some recent results of Christensen et al., [6, 8] to investigate representations for \( \text{g-frames} \) with bounded operators.

### 2. Representations of \( \text{g-frames} \)

In this section, by generalizing some results of [6, 8], we introduce representation for \( \text{g-frames} \) with bounded operators and give some examples of \( \text{g-frames} \) with a representation and without any representations. In the Theorem 2.5, we get sufficient conditions for \( \text{g-frames} \) to have a representation with bounded operator. Also, the Theorem 2.5 and the Proposition 2.10 show that for \( \text{g-frames} \) \( \Lambda = \{ \Lambda_i T^{i-1} : i \in \mathbb{N} \} \), the boundedness of \( T \) is equivalent to the invariant of the \( \ker T_\Lambda \) under right-shift.

**Remark 2.1.** Consider a frame \( F = \{ f_i \}_{i \in \mathbb{N}} = \{ T^{i-1} f_1 \}_{i \in \mathbb{N}} \) for \( \mathcal{H} \) with \( T \in B(\mathcal{H}) \). For the \( \text{g-frame} \) \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N} \} \) where
\[
\Lambda_i f = \langle f, f_i \rangle, \quad f \in \mathcal{H},
\]
we have

\[ \Lambda_{i+1}f = \langle f, f_{i+1} \rangle = \langle f, Tf_i \rangle = \langle T^*f, f_i \rangle = \Lambda_i T^*f, \quad f \in \mathcal{H}. \]

Therefore, there exists \( S \in B(\mathcal{H}) \) such that \( \Lambda_i = \Lambda_1 S^{i-1}, i \in \mathbb{N} \). Conversely, if \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{N} \} = \{ \Lambda_1 T^{i-1} : i \in \mathbb{N} \} \) for \( T \in B(\mathcal{H}) \), then \( F = \{ f_i \}_{i \in \mathbb{N}} = \{(T^*)^{-1} f_i \}_{i \in \mathbb{N}} \), where \( \Lambda_i f = \langle f, f_i \rangle, i \in \mathbb{N}, f \in \mathcal{H} \).

We are motivated to study \( g \)-frames \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \), where \( \Lambda_i = \Lambda_1 T^{i-1} \) with \( T \in B(\mathcal{H}) \).

**Definition 2.2.** We say that a \( g \)-frame \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) has a representation, if there is a \( T \in B(\mathcal{H}) \) such that \( \Lambda_i = \Lambda_1 T^{i-1}, i \in \mathbb{N} \). In the affirmative case, we say that \( \Lambda \) is represented by \( T \).

In the following, we give some \( g \)-frames that have a representation.

**Example 2.3.**

(i) The \( g \)-frame \( \Lambda = \{ \Lambda_i \in GL(\mathcal{H}) : i = 1, 2 \} \) is represented by \( \Lambda_1^{-1} \Lambda_2 \).

(ii) The tight \( g \)-frame \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}) : i \in \mathbb{N} \} \) with \( \Lambda_i = \frac{2^{i-1}}{\sqrt{n}} \text{Id}_\mathcal{H} \) is represented by \( \frac{2}{3} \text{Id}_\mathcal{H} \).

(iii) Let \( F = \{ f_i \}_{i \in \mathbb{N}} = \{ T^{i-1} f_1 \}_{i \in \mathbb{N}} \) be a frame for \( \mathcal{H} \). Then the \( g \)-frame \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}^2) : i \in \mathbb{N} \} \) with \( \Lambda_i f = \langle f, f_i \rangle, \langle f, f_{i+1} \rangle \), \( f \in \mathcal{H} \), is represented by \( T^* \).

Now, we give a \( g \)-frame without any representations.

**Example 2.4.** Consider the tight \( g \)-frame \( \Lambda = \{ \Lambda_n \in B(\mathcal{H}) : n \in \mathbb{N} \} \) with \( \Lambda_n = \frac{1}{n+1} \text{Id}_\mathcal{H} \). Since \( \Lambda_1 = \frac{1}{2} \text{Id}_\mathcal{H} \) and \( \Lambda_2 = \frac{1}{4} \text{Id}_\mathcal{H} \), the \( g \)-frame \( \Lambda \) has not any representation.

By generalizing a result of the [6], the following theorem give sufficient conditions for \( g \)-frame \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) to have a representation.

**Theorem 2.5.** Let \( \Lambda = \{ \Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N} \} \) be a \( g \)-frame that for any finite set \( I_n \subset \mathbb{N} \), and \( \{ g_i \}_{i \in I_n} \subset \mathcal{K} \), \( \sum_{i \in I_n} \Lambda_i^* g_i = 0 \) implies \( g_i = 0 \) for any \( i \in I_n \). Suppose that \( \ker T_\Lambda \) is invariant under the right-shift operator \( T \). Then \( \Lambda \) is represented by \( T \in B(\mathcal{H}) \), where \( \|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}} \).

**Proof.** Let \( \{ e_i \}_{i \in I} \) be an orthonormal basis for \( \mathcal{K} \). We define the linear map \( S : \text{span}\{ \Lambda_i^* (\mathcal{K}) \}_{i \in \mathbb{N}} \to \text{span}\{ \Lambda_i^* (\mathcal{K}) \}_{i \in \mathbb{N}} \) with

\[ S(\Lambda_i^* e_j) = \Lambda_i^* e_{j+1}. \]
For any finite index sets $I_n \subset \mathbb{N}$ and $J_m \subset I$, $\sum_{i \in I_n, j \in J_m} c_{ij} \Lambda^*_i e_j = \sum_{i \in I_n} \Lambda^*_i (\sum_{j \in J_m} c_{ij} e_j) = 0$ implies $c_{ij} = 0$ for $i \in I_n$, $j \in J_m$. Therefore, $S$ is well-defined. Now, we show that $S$ is bounded. Let $f = \sum_{i \in I_n, j \in J_m} c_{ij} \Lambda^*_i e_j$ for $c_{ij} \in \ell^2(\mathbb{C}, \mathbb{N})$ with $c_{ij} = 0$, $i \notin I_n$ or $j \notin J_m$. By the Theorem [13], $F = \{\Lambda^*_i e_j\}_{i \in I_n, j \in J}$ is a frame for $\mathcal{H}$ with lower and upper frame bounds $A_\Lambda$ and $B_\Lambda$, respectively. We can write $c_{ij} = d_{ij} + r_{ij}$ with $d_{ij} \in \ker T_F$ and $r_{ij} \in \text{ran} T_F^*$. From $\sum_{i \in I_n} \Lambda^*_i \left( \sum_{j \in J_m} d_{ij} e_j \right) = \sum_{i \in I_n, j \in J_m} d_{ij} \Lambda^*_i e_j = 0$, we conclude that $\sum_{i \in I_n, j \in J_m} d_{ij} \Lambda^*_i e_j = 0$. The same as the proof of [8], we have

$$\|Sf\|^2 = \left\| \sum_{i \in I_n, j \in J_m} c_{ij} \Lambda^*_i e_j \right\|^2 = \left\| \sum_{i \in I_n, j \in J_m} r_{ij} \Lambda^*_i e_j \right\|^2 \leq B_\Lambda \sum_{i \in I_n, j \in J_m} |r_{ij}|^2.$$

Since $\{r_{ij}\}_{i \in I_n, j \in J_m} \in (\ker T_\Lambda)^\perp$, by the [5, Lemma 5.5.5] we have

$$A_\Lambda \sum_{i \in I_n, j \in J_m} |r_{ij}|^2 \leq \left\| \sum_{i \in I_n, j \in J_m} r_{ij} \Lambda^*_i e_j \right\|^2.$$

Therefore

$$\|Sf\|^2 \leq B_\Lambda A_\Lambda^{-1} \left\| \sum_{i \in I_n, j \in J_m} r_{ij} \Lambda^*_i e_j \right\|^2 = B_\Lambda A_\Lambda^{-1} \left\| \sum_{i \in I_n, j \in J_m} (d_{ij} + r_{ij}) \Lambda^*_i e_j \right\|^2 \leq B_\Lambda A_\Lambda^{-1} \left\| \sum_{i \in I_n, j \in J_m} c_{ij} \Lambda^*_i e_j \right\|^2 = B_\Lambda A_\Lambda^{-1} \|f\|^2.$$

So, $S$ is bounded and can be extended to $\bar{S} \in B(\mathcal{H})$. It is obvious that $\Lambda$ is represented by $T = \bar{S}^*$ and $\|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}$. \hfill $\Box$

**Corollary 2.6.** Every $g$-orthonormal basis has a representation.

**Proof.** For every finite sequence $\{g_i\}_{i \in I_n} \subset \mathcal{K}$, we have

$$\left\| \sum_{i \in I_n} \Lambda^*_i g_i \right\|^2 = \left\langle \sum_{i \in I_n} \Lambda^*_i g_i, \sum_{j \in I_n} \Lambda^*_j g_j \right\rangle = \sum_{i \in I_n, j \in I_n} \left\langle \Lambda^*_i g_i, \Lambda^*_j g_j \right\rangle = \sum_{i \in I_n} \left\langle g_i, g_i \right\rangle = \sum_{i \in I_n} \|g_i\|^2.$$

So $\sum_{i \in I_n} \Lambda^*_i g_i = 0$ implies $g_i = 0$ for any $i \in I_n$. Similarly, we have $\ker T_\Lambda = \{0\}$, that is invariant under right-shift operator. Then, by the Theorem [25] the proof is completed. \hfill $\Box$
Remark 2.7. Consider a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ that is represented by $T$. For $S \in GL(\mathcal{H})$, the family $\Lambda S = \{\Lambda_i S \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is a $g$-frame [15, Corollary 2.26], that is represented by $T^{-1}TS$.

Corollary 2.8. Every $g$-Riesz basis has a representation.

Proof. By the Theorem 1.7, Proposition 2.6 and Remark 2.7, the proof is completed. □

Now, we give an example to show that the converse of the Theorem 2.5 is not satisfied.

Example 2.9. Consider the tight $g$-frame $\Lambda = \{\Lambda_i \in B(l^2(\mathcal{H}, \mathbb{N})) : i \in \mathbb{N}\}$ with $\Lambda_i = (\frac{1}{2})^{i-1}I\!d_{l^2(\mathcal{H}, \mathbb{N})}$. It is obvious that $\Lambda$ is represented by $\frac{1}{2}I\!d_{l^2(\mathcal{H}, \mathbb{N})}$, but $\Lambda_1^{*} (\frac{1}{2}e_1) + \Lambda_2^{*} (-e_1) = 0$ for $e_1 = (1, 0, 0, ...)$.

Proposition 2.10. Let the $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be represented by $T$. Then $\ker T_\Lambda$ is invariant under right-shift $\mathcal{T}$.

Proof. For any $\{g_i\}_{i \in \mathbb{N}} \in \ker T_\Lambda$, we have

$$T_\Lambda \mathcal{T}\{g_i\}_{i \in \mathbb{N}} = \sum_{i \in \mathbb{N}} \Lambda_i^{*} g_i = \sum_{i \in \mathbb{N}} T^{*} \Lambda_i^{*} g_i = T^{*} \left( \sum_{i \in \mathbb{N}} \Lambda_i^{*} g_i \right) = 0.$$ □

The following Proposition shows that the converse of the Theorem 2.5 is satisfied for one-dimentional Hilbert space $\mathcal{K}$.

Proposition 2.11. Let $\mathcal{K}$ be a one-dimentional Hilber space and $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ be represented by $T$. If for finite index set $I_n \subset \mathbb{N}$ and $\{g_i\}_{i \in I_n} \subset \mathcal{K}$, we have $\sum_{i \in I_n} \Lambda_i^{*} g_i = 0$, then $g_i = 0$ for any $i \in I_n$.

Proof. Let $\{e_1\}$ be a basis for $\mathcal{K}$. By the Theorem 1.8, the sequence $F = \{\Lambda_i^{*} e_1\}_{i \in \mathbb{N}} = \{(T^{*})^{i-1} \Lambda_i^{*} e_1\}_{i \in \mathbb{N}}$ is a frame for $\mathcal{H}$, and so by the Proposition 1.1, $F$ is linearly independent. We have

$$0 = \sum_{i \in I_n} \Lambda_i^{*} g_i = \sum_{i \in I_n} \Lambda_i^{*} (\alpha_i e_1) = \sum_{i \in I_n} \alpha_i \Lambda_i^{*} e_1,$$

therefore, for any $i \in I_n$, $\alpha_i = 0$ and so $g_i = 0$. □

Remark 2.12. The above Proposition shows that for one dimentional Hilbert space $\mathcal{K}$ with basis $\{e_1\}$, when a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ has a representation, then the frame $\{\Lambda_i^{*} e_1\}_{i \in \mathbb{N}}$ has a representation. For finite dimentional Hilbert space $\mathcal{K}$ with orthonormal basis $\{e_j\}_{j=1}^{n}$, when a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is represented by $T$, then the frame $F = \{\Lambda_i^{*} e_j, j = 1, ..., n\}_{i \in \mathbb{N}}$ can be represented
by $T^*$ and finite vectors $\{\Lambda^*_i e_1, ..., \Lambda^*_i e_n\}$, i.e., $F = \{(T^*)^{i-1} \Lambda^*_i e_j, j = 1, ..., n\}_{i \in \mathbb{N}}$, then, it can be worked on $g$-frames that be represented with finite family of operators. But, the Example 2.9 shows that for infinite dimension Hilbert space $K = l^2(\mathcal{H}, \mathbb{N})$ with orthonormal basis $\{e_j\}_{j \in I}$, it does not happen, i.e., a $g$-frame $\Lambda$ has a representation and the frame $\{\Lambda^*_i e_j\}_{i, j \in \mathbb{N}}$ does not have. Note that for a $g$-Reisz basis $\Lambda = \{\Lambda^*_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$, the sequence $F = \{\Lambda^*_i e_j\}_{i \in \mathbb{N}, j \in I}$ is a Riesz basis [17]. By the Corollary 2.8 and [8 Example 2.2], both of the $\Lambda$ and $F$ have representations. What is the relation between these two representations (open problem)?

Now, we want to discuss the concept of representation for $g$-frames with index set $\mathbb{Z}$.

**Definition 2.13.** We say that a $g$-frame $\Lambda = \{\Lambda^*_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\}$ has a representation, if there is a $T \in GL(\mathcal{H})$ such that $\Lambda^*_i = \Lambda^*_0 T^i$, $i \in \mathbb{Z}$. In the affirmative case, we say that $\Lambda$ is represented by $T$.

**Example 2.14.** Consider the tight $g$-frame $\Lambda = \{\Lambda_n \in B(\mathcal{C}) : n \in \mathbb{Z}\}$ with $\Lambda_n = \frac{1}{n^2 + 1}Id_{\mathcal{C}}$. Since $\Lambda_1 = \frac{1}{3}Id_{\mathcal{C}}$ and $\Lambda_3 = \frac{1}{7}Id_{\mathcal{C}}$, the $g$-frame $\Lambda$ has not any representation.

**Remark 2.15.** Note that, all results that we have investigated for $g$-frames with index set $\mathbb{N}$, are satisfied for $g$-frames with index set $\mathbb{Z}$, as well.

**Theorem 2.16.** Let a $g$-frame $\Lambda = \{\Lambda^*_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\}$ is represented by $T$, then $\ker T_\Lambda$ is invariant under right-shift and left-shift and

$$1 \leq \|T\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}, \quad 1 \leq \|T^{-1}\| \leq \sqrt{B_\Lambda A_\Lambda^{-1}}.$$

**Proof.** Similar to the Proposition 2.10, $\ker T_\Lambda$ is invariant under right-shift. For $\{g_i\}_{i \in \mathbb{Z}} \in \ker T_\Lambda$,

$$T_\Lambda T^* \{g_i\}_{i \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} \Lambda^*_i g_i = \sum_{i \in \mathbb{Z}} (T^{i-1})^* \Lambda^*_0 g_i = (T^{-1})^* (\sum_{i \in \mathbb{Z}} (T^i)^* \Lambda^*_0 g_i) = (T^{-1})^* (\sum_{i \in \mathbb{Z}} \Lambda^*_i g_i) = (T^{-1})^* T_\Lambda \{g_i\}_{i \in \mathbb{Z}} = 0.$$
So, ker $T_\Lambda$ is also invariant under left-shift. Now for some fixed $n \in \mathbb{N}$ and $0 \neq f \in \mathcal{H}$ we have

$$A_\Lambda \|f\|^2 \leq \sum_{i \in \mathbb{Z}} \|\Lambda_i f\|^2 = \sum_{i \in \mathbb{Z}} \|\Lambda_0 T^{-n} T^n f\|^2 = \sum_{i \in \mathbb{Z}} \|\Lambda_0 T^n f\|^2 \leq B_\Lambda \|T^n f\|^2 \leq B_\Lambda \|T\|^{2n} \|f\|^2,$$

that implies $\|T\| \geq 1$. Since for any $i \in \mathbb{Z}$, $\Lambda_i T = \Lambda_{i+1}$, we have $T^* \Lambda_i e_j = \Lambda_{i+1} e_j$. So, $T^*$ is the operator $\tilde{S}$ that is defined in the proof of the Theorem 2.5 just on span$\{\Lambda_i(K)\}_{i \in \mathbb{Z}}$, and therefore we have $\|T\| \leq \sqrt{B_\Lambda A^{-1}_\Lambda}$, alike. Since $\Lambda = \{\Lambda_{-i} \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\} = \{\Lambda_0(T^{-1})^i : i \in \mathbb{Z}\}$, by replacing $T^{-1}$ instead of $T$, we get $1 \leq \|T^{-1}\| \leq \sqrt{B_\Lambda A^{-1}_\Lambda}$. \hfill \Box

Part (ii) of the Example 2.3 shows that for the index set $\mathbb{N}$, $1 \leq \|T\|$ does not happen, in general.

**Corollary 2.17.** Let a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\}$ is represented by $T \in GL(\mathcal{H})$. Then the following hold:

(i) If $\Lambda$ is a tight $g$-frame, then $\|T\| = \|T^{-1}\| = 1$, and so $T$ is isometry.

(ii) $\|S^{1/2}_\Lambda T S^{-1/2}_\Lambda\| = \|S^{1/2}_\Lambda T^{-1} S^{-1/2}_\Lambda\| = 1$.

Authors of the paper [9] have considered sequences in $\mathcal{H}$ of the form $F = \{T^i f_0\}_{i \in \mathbb{I}}$, with a linear operator $T$ to study for which bounded operator $T$ and vector $f_0 \in \mathcal{H}$, $F$ is a frame for $\mathcal{H}$. In [8, Proposition 3.5], it was proved that if the operator $T \in B(\mathcal{H})$ is compact, then the sequence $\{T^i f_0\}_{i \in \mathbb{I}}$ can not be a frame for infinite dimensional $\mathcal{H}$. Someone can study these results for family of operators $\{\Lambda_0 T^i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\}$ for $T \in B(\mathcal{H})$ and $\Lambda_0 \in B(\mathcal{H}, \mathcal{K})$.

### 3. Duality

The purpose of this section is to get a necessary and sufficient condition for a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}_i) : i \in \mathbb{N}\}$ to have a representation, by applying the concept of duality. Also, for some $g$-frames with representation we get a dual with representation and in one case
without representation. At the end, we get the relation between representations of dual $g$-frames by index set $\mathbb{Z}$. The proofs of the results are similar to [6] [8].

**Theorem 3.1.** A $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is represented by $T$ if and only if for a dual $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ of $\Lambda$ (and hence all),

$$\Lambda_{k+1} = \sum_{i \in \mathbb{N}} \Lambda_k \Theta_i^* \Lambda_{i+1}.$$ 

**Proof.** First, assume that $\Lambda$ is represented by $T$. For any $g \in \mathcal{K}$ we have

$$T^*\Lambda_k^* g = T^* \left( \sum_{i \in \mathbb{N}} \Lambda_i^* \Theta_i \Lambda_k^* g \right) = \sum_{i \in \mathbb{N}} T^* \Lambda_i^* \Theta_i \Lambda_k^* g$$

$$= \sum_{i \in \mathbb{N}} \Lambda_i^* \Theta_i \Lambda_k^* g = \sum_{i \in \mathbb{N}} (\Lambda_k \Theta_i^* \Lambda_{i+1})^* g,$$

then, $\Lambda_{k+1} = \sum_{i \in \mathbb{N}} \Lambda_k \Theta_i^* \Lambda_{i+1}$.

Conversely, it is obvious that $\Lambda_i T = \Lambda_{i+1}$ for $T f = \sum_{i \in \mathbb{N}} \Theta_i^* \Lambda_{i+1} f$. $\square$

**Remark 3.2.** By [17] Corollary 3.3], for a $g$-Riesz basis $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$, we have

$$\left( \sum_{i \in \mathbb{N}} \Lambda_k \tilde{\Lambda}_i^* \Lambda_{i+1} f, g \right) = \sum_{i \in \mathbb{N}} \langle \tilde{\Lambda}_i^* \Lambda_{i+1} f, \Lambda_k^* g \rangle$$

$$= \sum_{i \in \mathbb{N}} \delta_{i,k} \langle \Lambda_{i+1} f, g \rangle = \langle \Lambda_{k+1} f, g \rangle, \quad f \in \mathcal{H}, g \in \mathcal{K},$$

therefore, by the Theorem 3.1 $\Lambda$ has a representation.

In the following, we want to investigate that if a $g$-frame $\Lambda$ has a representation, its duals have representations or not. If so, what is the relation between their representations?

**Example 3.3.**

(i) Assume that a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ is represented by $T$. Then, by the Remark 2.7, the canonical dual $\tilde{\Lambda}$ is represented by $S_{\Lambda} T S_{\Lambda}^{-1}$.

(ii) Consider the $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}) : i \in \mathbb{N}\}$ with $\Lambda_i = (\frac{3}{2})^i Id_\mathcal{H}$, that is represented by $\frac{3}{2}^i Id_\mathcal{H}$. The $g$-frame $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{N}\}$ with $\Theta_i = (\frac{3}{4})^i Id_\mathcal{H}$ is a dual of $\Lambda$ that is represented by $\frac{3}{4}^i Id_\mathcal{H}$.

(iii) The $g$-frame $\Lambda = \{\Lambda_i \in B(\mathbb{C}) : i = 1, 2, 3\}$ with $\Lambda_i = 2^{i-1} Id_\mathbb{C}$ is represented by $2 Id_\mathbb{C}$, but the dual $\Theta = \{\Theta_i \in B(\mathbb{C}) : i = 1, 2, 3\}$ of $\Lambda$ with $\Theta_1 = -2 Id_\mathbb{C}, \Theta_2 = Id_\mathbb{C}$ and $\Theta_3 = \frac{1}{4} Id_\mathbb{C}$ does not have
any representation. Note that the dual $\Gamma = \{\Gamma_i \in B(\mathbb{C}) : i = 1, 2, 3\}$ of $\Lambda$ with $\Gamma_i = \frac{1}{2}(\frac{1}{2})^{i-1}Id_{\mathbb{C}}$ is represented by $\frac{1}{2}Id_{\mathbb{C}}$.

**Proposition 3.4.** Let a $g$-frame $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\}$ be represented by $T \in GL(\mathcal{H})$. Then, the canonical dual $\tilde{\Lambda}$ is represented by $S_\Lambda TS_\Lambda^{-1} = (T^*)^{-1}$.

**Proof.** It is obvious that $\tilde{\Lambda}$ is represented by $S_\Lambda TS_\Lambda^{-1}$. For any $\{g_i\}_{i \in \mathbb{Z}} \in \ell^2(\mathbb{K}, \mathbb{Z})$,

$$T^*T_\Lambda\{g_i\}_{i \in \mathbb{Z}} = \sum_{i \in \mathbb{Z}} T^*\Lambda_i^* g_i = \sum_{i \in \mathbb{Z}} (\Lambda_i T)^* g_i$$

$$= \sum_{i \in \mathbb{Z}} \Lambda_{i+1}^* g_i = T_\Lambda T\{g_i\}_{i \in \mathbb{Z}},$$

So, we have

$$T^*S_\Lambda T = T^*T_\Lambda T^*T_\Lambda = T^*T_\Lambda T^* = T_\Lambda T^* T_\Lambda = S_\Lambda.$$

Therefore, $S_\Lambda TS_\Lambda^{-1} = (T^*)^{-1}$. \qed

**Remark 3.5.** Let $F = \{f_i\}_{i \in \mathbb{Z}}$ and $G = \{g_i\}_{i \in \mathbb{Z}}$ be dual frames that is represented by $T, S \in GL(\mathcal{H})$, respectively. Then, by the Remark 2.1 the dual $g$-frames $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{Z}\}$ with $\Lambda_i f = \langle f, f_i \rangle$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathbb{C}) : i \in \mathbb{Z}\}$ with $\Theta_i f = \langle f, g_i \rangle$ are represented by $T^*, S^* \in GL(\mathcal{H})$, respectively. By the [6, Lemma 3.3], $S = (T^*)^{-1}$.

The relation between representations of dual $g$-frames by index set $\mathbb{Z}$ is given in below.

**Theorem 3.6.** Assume that $\Lambda = \{\Lambda_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\} = \{\Lambda_0 T^i : i \in \mathbb{Z}\}$ and $\Theta = \{\Theta_i \in B(\mathcal{H}, \mathcal{K}) : i \in \mathbb{Z}\} = \{\Theta_0 S^i : i \in \mathbb{Z}\}$ are dual $g$-frames, where $T, S \in GL(\mathcal{H})$. Then, $S = (T^*)^{-1}$.

**Proof.** For any $f \in \mathcal{H}$, we have

$$f = \sum_{i \in \mathbb{Z}} \Lambda_i^* \Theta_i f = \sum_{i \in \mathbb{Z}} (T^*)^i \Lambda_0^* \Theta_0 S^i f = T^* \left( \sum_{i \in \mathbb{Z}} (T^*)^{i-1} \Lambda_0^* \Theta_0 S^i \right) S f$$

$$= T^* \left( \sum_{i \in \mathbb{Z}} \Lambda_i^* \Theta_i \right) S f = T^* S f.$$

Since $T \in GL(\mathcal{H})$, the proof is completed. \qed

In general, the Theorem 3.6 is not satisfied for index set $\mathbb{N}$ (see the Example 3.1 (ii)).

**Acknowledgment:**
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