Noncompact, Gauge-Invariant Simulations of $U(1)$, $SU(2)$, and $SU(3)$

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Abstract

We have applied a new noncompact, gauge-invariant, Monte Carlo method to simulate the $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. For $U(1)$ the Creutz ratios of the Wilson loops agree with the exact results for $\beta \geq 0.5$ after a renormalization of the charge. The $SU(2)$ and $SU(3)$ Creutz ratios robustly display quark confinement at $\beta = 0.5$ and $\beta = 2$, respectively. At much weaker coupling, the $SU(2)$ and $SU(3)$ Creutz ratios agree with perturbation theory after a renormalization of the coupling constant. For $SU(3)$ without quarks, our lattice QCD parameter is $\Lambda_L = 130 \pm 18$ MeV.
1 Introduction

In compact lattice gauge theory, gauge fields are represented by group elements rather than by fields, and the action is a periodic function of a gauge-invariant lattice field strength. The periodicity of the action entails spurious vacua. The principal advantage of noncompact actions, in which gauge fields are represented by fields, is that they avoid multiple vacua.

Palumbo, Polikarpov, and Veselov [1] carried out the first gauge-invariant noncompact simulations. They saw a confinement signal. Their action contains five terms, constructed from two invariants, and involves (noncompact) auxiliary fields and an adjustable parameter.

The present paper describes a test of a new way [2] of performing gauge-invariant noncompact simulations. Our action, which is similar to one term of Palumbo’s action, is exactly invariant under compact gauge transformations, is a natural discretization of the classical Yang-Mills action, and reduces to Wilson’s action when the gauge fields are compactified. In our version of Palumbo’s method, there are fewer auxiliary fields, and they are compact group elements related to gauge transformations.

We have used this method to simulate the $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. For $U(1)$ our Creutz ratios of Wilson loops agree with the exact ratios of the free continuum theory after a renormalization of the charge when the inverse coupling $\beta$ exceeds 0.5. Our $SU(2)$ and $SU(3)$ Creutz ratios clearly show quark confinement at $\beta = 0.5$ and $\beta = 2$, respectively. At much weaker coupling, our $SU(2)$ and $SU(3)$ Creutz ratios approximate those of continuum perturbation theory when the coupling constant is suitably renormalized. For $SU(3)$ there is a scaling in the transition region $2.2 \leq \beta \leq 3$ with a lattice QCD parameter $\Lambda_L \approx 130\pm18$ MeV, which is to be compared with the continuum value $\Lambda_{\text{MS}}^{(0)} \approx 210$ MeV.

2 The Method

For massless fermions, the continuum action density is $\bar{\psi}i\gamma_{\mu}\partial_{\mu}\psi$. A suitable discretization of this quantity is $i\bar{\psi}(n)\gamma_{\mu}[\psi(n + e_{\mu}) - \psi(n)]/a$ in which $n$ is
a four-vector of integers representing an arbitrary vertex of the lattice, \( e_\mu \) is a unit vector in the \( \mu \)th direction, and \( a \) is the lattice spacing. The product of Fermi fields at the same point is gauge invariant as it stands. The other product of Fermi fields becomes gauge invariant if we insert a matrix \( A_\mu(n) \) of gauge fields

\[
\bar{\psi}(n)\gamma_\mu [1 + i g a A_\mu(n)] \psi(n + e_\mu)
\]

that transforms appropriately. Under a gauge transformation represented by the group elements \( U(n) \) and \( U(n + e_\mu) \), the required response is

\[
1 + i g a A'_\mu(n) = U(n)[1 + i g a A_\mu(n)]U^{-1}(n + e_\mu)
\]

or equivalently

\[
A'_\mu(n) = U(n)A_\mu(n)U^{-1}(n + e_\mu) + \frac{i}{a g} U(n) \left[ U^{-1}(n) - U^{-1}(n + e_\mu) \right].
\]

Under this gauge transformation, the lattice field strength

\[
F_{\mu\nu}(n) = \frac{1}{a} [A_\mu(n + e_\nu) - A_\mu(n)] - \frac{1}{a} [A_\nu(n + e_\mu) - A_\nu(n)]
\]

\[
+ i g [A_\nu(n)A_\mu(n + e_\nu) - A_\mu(n)A_\nu(n + e_\mu)]
\]

which reduces to the continuum Yang-Mills field strength in the limit \( a \to 0 \), transforms as

\[
F'_{\mu\nu}(n) = U(n)F_{\mu\nu}(n)U^{-1}(n + e_\mu + e_\nu).
\]

The field strength \( F_{\mu\nu}(n) \) is antisymmetric in the indices \( \mu \) and \( \nu \), but it is not hermitian. To make a positive plaquette action density, we use the Hilbert-Schmidt norm of \( F_{\mu\nu}(n) \)

\[
S = \frac{1}{4k} \text{Tr}[F_{\mu\nu}(n)F_{\mu\nu}(n)],
\]

in which the generators \( T_a \) of the gauge group are normalized as \( \text{Tr}(T_a T_b) = k \delta_{ab} \). Because \( F_{\mu\nu}(n) \) transforms covariantly (3), this action density is exactly invariant under the noncompact gauge transformation (3).

The gauge transformation (3) with group element \( U(n) = \exp(-i g a \omega^a T_a) \) typically maps the matrix of gauge fields \( A_\mu(n) = T_a A_\mu^a(n) \) outside the Lie algebra, apart from terms of lowest (zeroth) order in the lattice spacing \( a \). We accept this larger space of matrices. We use the action (6) in which the
field strength \( (4) \) is defined in terms of gauge-field matrices \( A_\mu(n) \) that are the images under arbitrary gauge transformations of matrices \( A^0_\mu(n) \) of gauge fields defined in the usual way, \( A^0_\mu(n) \equiv T_\alpha A^{\alpha,0}_\mu(n) \) where the fields \( A^{\alpha,0}_\mu(n) \) are real:

\[
1 + iag_{\alpha}(n) = V_\mu(n)[1 + iag_{\alpha}(n)]W^{-1}_\mu(n + e_\mu).
\]

These gauge transformations are applied separately and link-wise to the gauge-field matrices \( A^0_\mu(n) \). In general the group elements \( V_\mu(n) \) and \( W_\mu(n + e_\mu) \) associated with the gauge field \( A_\mu(n) \) are unrelated to those associated with the neighboring gauge fields \( A_\mu(n + e_\nu), A_\nu(n), \) and \( A_\nu(n + e_\mu) \). But when the matrices \( V_\mu \) and \( W_\mu \) are equal and independent of direction

\[
V_\mu(n) = W_\mu(n) = U(n)
\]

for every vertex \( n \), then they constitute a gauge transformation.

Actually there is only one independent group element associated with each link. For by writing the equation (7) in the form

\[
1 + iag_{\alpha}(n) = V_\mu(n)[1 + iag_{\alpha}(n)]V^{-1}_\mu(n)
\]

we see that the departure of the gauge-field matrix \( A_\mu(n) \) from the Lie algebra is entirely due to the last matrix product \( V_\mu(n)W^{-1}_\mu(n + e_\mu) \). Thus there are as many auxiliary fields in this method as there are generators of the gauge group. For \( SU(N) \), the product \( VW^{-1} \) has \( N^2 - 1 \) generators, so our method involves \( N^2 - 1 \) auxiliary fields in this case. We have not tried to parameterize these products and have instead accepted whatever matrices the link-wise gauge transformation (7) generated.

Palumbo, in his version of this method [1], proposed a parameterization for the gauge-field matrices \( A_\mu \) that is slightly more general than is necessary and that leads to two more auxiliary fields than in our version. For instance, in the case of \( SU(N) \) his procedure involves \( N^2 + 1 \) auxiliary fields which support a \( U(N) \) gauge invariance. His action also has a special term that may be needed to suppress these extra auxiliary fields.

The auxiliary fields that describe the group element \( VW^{-1} \) are the principal defect of the present method. Their presence may be detected by measuring the average value of the path-ordered products of the factors...
$V_\mu(n)W_\mu^{-1}(n+e_\mu)$ around the plaquettes of the lattice. For when the gauge-transformation condition (3) is satisfied, the product around each plaquette

$$P_{\mu\nu}(n) = V_\mu(n)W_\mu^{-1}(n+e_\mu)V_\nu(n+e_\mu)W_\nu^{-1}(n+e_\mu+e_\nu)$$

$$\times [V_\mu(n+e_\nu)W_\mu^{-1}(n+e_\mu+e_\nu)]^\dagger [V_\nu(n)W_\nu^{-1}(n+e_\nu)]^\dagger$$

is unity.

To estimate the effects of the auxiliary fields, we have measured the mean values of these products

$$\langle P \rangle = \frac{1}{6L^4} \sum_{n,\mu\nu} \langle P_{\mu\nu}(n) \rangle$$

(11)

at various values of $\beta$ in our $SU(3)$ simulations. We found that the auxiliary fields do reduce to gauge transformations in the continuum limit, $\beta \to \infty$, but slowly: at $\beta = 2$, $\langle P \rangle = 0.18$; at $\beta = 3$, $\langle P \rangle = 0.30$; at $\beta = 100$, $\langle P \rangle = 0.71$; and at $\beta = 1000$, $\langle P \rangle = 0.87$.

The quantity $1 + i ga A_\mu(n)$ is not an element $L_\mu(n)$ of the gauge group, except when the real gauge fields $A_\mu^{a,0}(n)$ all vanish. But if one compactified the fields by requiring $1 + i ga A_\mu(n)$ to be an element of the gauge group, then the matrix $A_\mu(n)$ of gauge fields would be related to the link $L_\mu(n)$ by $A_\mu(n) = (L_\mu(n) - 1)/(iga)$, and the action (3) defined in terms of the field strength (4) would be, *mirabile dictu*, Wilson’s action:

$$S = \frac{k - \Re \text{Tr} L_\mu(n) L_\nu(n + e_\mu) L_\mu^\dagger(n + e_\nu) L_\nu^\dagger(n)}{2a^4 g^2 k}.$$  

(12)

### 3 Results

We have tested this method by applying it to the $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. We used a multi-hit Metropolis algorithm with a flat measure for the real gauge fields $A_\mu^{a,0}(n)$ and the relevant Haar measure for the group elements $V_\mu(n)$ and $W_\mu(n)$. In most of our initial configurations, the unitary matrices $V$ and $W$ and the hermitian gauge fields $A_\mu^0$ were randomized. For thermalization we allowed 50,000 sweeps for $U(1)$, 10,000 for $SU(2)$, and 100,000 for $SU(3)$. We allowed at least 20
sweeps between measurements. Our Wilson loops are ensemble averages of ordered products of the binomials \(1 + iagA_\mu(n)\) rather than of the exponentials \(\exp[iagA_\mu(n)]\) around the loop.

For \(U(1)\) and for \(\beta = 1/g^2 \geq 0.5\), our measured Creutz ratios \(^3\) of Wilson loops agree with the exact ones of the free continuum theory apart from finite-size effects and a renormalization of the charge. For instance at \(\beta = 0.5\), on the \(12^4\) lattice, we found \(\chi(2,2) = 0.147(1), \chi(2,3) = 0.103(1), \chi(2,4) = 0.090(1), \chi(3,3) = 0.049(1), \chi(3,4) = 0.034(1),\) and \(\chi(4,4) = 0.020(2)\). At the value \(\beta_r = 0.45\) of the renormalized charge, the exact Creutz ratios of the continuum theory are: \(\chi(2,2) = 0.147, \chi(2,3) = 0.102, \chi(2,4) = 0.093, \chi(3,3) = 0.048, \chi(3,4) = 0.037,\) and \(\chi(4,4) = 0.024\). The agreement is excellent apart from the two largest loops which are too small by 8\% and by 17\%, respectively, largely due to finite-size effects.

At stronger coupling, the extra terms
\[
ig \left[ A_\nu(n)A_\mu(n + e_\nu) - A_\mu(n)A_\nu(n + e_\mu) \right]
\] (13)
in the lattice field strength \(F_{\mu\nu}(n)\) eventually do produce a confinement signal. For example, at \(\beta = 0.375\), our measured Creutz ratios on the \(12^4\) lattice are: \(\chi(2,2) = 0.906(5), \chi(2,3) = 0.909(21), \chi(2,4) = 0.85(10), \chi(3,3) = 0.62(24),\) and \(\chi(3,4) = 0.6(16)\).

For \(SU(2)\) on the \(8^4\) lattice at \(\beta = 4/g^2 = 0.5\), we found \(\chi(2,2) = 0.835(3), \chi(2,3) = 0.852(12), \chi(2,4) = 0.865(60),\) and \(\chi(3,3) = 0.94(23)\) which within the limited statistics clearly exhibit confinement.

To test this method at weaker coupling, we compared our Creutz ratios with those of continuum perturbation theory. The tree-level perturbative formula for \(SU(n)\) may be expressed \(^4\) in terms of the function \(u(i,j)\)
\[
u(i,j) = \frac{i}{j} \arctan \frac{i}{j} + \frac{j}{i} \arctan \frac{j}{i} - \log \left( \frac{1}{i^2} + \frac{1}{j^2} \right)
\] (14)
as
\[
\chi(i,j) = \frac{n^2 - 1}{2\pi^2 \beta} \left[ -u(i,j) - u(i-1,j-1) + u(i,j-1) + u(i-1,j) \right].
\] (15)

For \(SU(2)\) at \(\beta = 1\), our six Creutz ratios \(\chi(2,2) = 0.1087(5), \chi(2,3) = 0.0783(5), \chi(2,4) = 0.0696(5), \chi(3,3) = 0.0414(7), \chi(3,4) = 0.0299(8),\) and
\( \chi(4, 4) = 0.0189(10) \) are close to the \( \chi \)'s of the perturbative formula (15) at the renormalized value of \( \beta_r = 1.75 \), to wit: \( \chi(2, 2) = 0.1133, \chi(2, 3) = 0.0786, \chi(2, 4) = 0.0721, \chi(3, 3) = 0.0374, \chi(3, 4) = 0.0286, \) and \( \chi(4, 4) = 0.0187. \)

For \( SU(3) \) at \( \beta = 6/g^2 = 2 \) on the \( 12^4 \) lattice, we found in ten independent runs \( \chi(2, 2) = 0.838(1), \chi(2, 3) = 0.826(3), \chi(2, 4) = 0.828(13), \chi(3, 3) = 0.793(42), \chi(3, 4) = 0.47(25), \) and \( \chi(4, 4) = 1.2(86) \). Within the statistics, these results robustly exhibit confinement.

At much weaker coupling, our ratios approximate those of the tree-level formula (15) of continuum perturbation theory apart from finite-size effects and after a renormalization of the coupling constant. At \( \beta = 4 \) on the \( 8^4 \) lattice, for instance, we found in one run \( \chi(2, 2) = 0.0878(1), \chi(2, 3) = 0.0622(1), \chi(2, 4) = 0.0554(2), \chi(3, 3) = 0.0319(2), \chi(3, 4) = 0.0228(6), \) and \( \chi(4, 4) = 0.0132(1) \); whereas the tree-level formula (15) at the renormalized value of \( \beta_r = 6.03 \) gives \( \chi(2, 2) = 0.0878, \chi(2, 3) = 0.0609, \chi(2, 4) = 0.0559, \chi(3, 3) = 0.0290, \chi(3, 4) = 0.0222, \) and \( \chi(4, 4) = 0.0145. \) Similarly at \( \beta = 100 \) we found in one run \( \chi(2, 2) = 0.00477(1), \chi(2, 3) = 0.00325(1), \chi(2, 4) = 0.00289(1), \chi(3, 3) = 0.00150(1), \chi(3, 4) = 0.00104(2), \) and \( \chi(4, 4) = 0.00053(3) \); whereas the perturbative formula (15) at \( \beta_r = 111 \) gives \( \chi(2, 2) = 0.00476, \chi(2, 3) = 0.00328, \chi(2, 4) = 0.00303, \chi(3, 3) = 0.00157, \chi(3, 4) = 0.00120, \) and \( \chi(4, 4) = 0.00078. \) The better agreement for the smaller loops is a finite-size effect.

4 Scaling

We used an \( 8^4 \) lattice to study the scaling of the lattice spacing \( a \) with the coupling constant \( g \) for \( SU(3) \). The two-loop result for the dependence of the string tension \( \sigma a^2 \) upon the inverse coupling \( \beta \) is

\[
\sigma a^2 \approx \frac{\sigma}{\Lambda_L^2} \exp \left[ -\frac{8\pi^2 \beta}{33} + \frac{102}{121} \log \left( \frac{8\pi^2 \beta}{33} \right) \right].
\]

(16)

We expect this scaling formula to hold in a transition region where perturbation theory is still valid and where the quark-antiquark static potential
$V(r)$ is a linear combination of a confining linear potential and a Coulomb potential. At one end of this region, $V(r)$ is mostly linear; at the other end it, is mostly Coulomb. We found a transition region between $\beta = 2.2$ and $\beta = 3$. In this region, we used the interpolation

$$\chi(i,j) = \frac{1}{3} \left[ (4 - \beta)\sigma a^2 + (\beta - 1)\chi_0(i,j) \right]$$

in which the string tension $\sigma a^2$ is given by the scaling formula (16) and the perturbative Creutz ratio $\chi_0(i,j)$ is given by the tree-level formula (15).

In the figure we have plotted the Creutz ratios $\chi(i,j)$ that we measured on an $8^4$ lattice for $1.5 \leq \beta \leq 3$. We also have plotted the interpolative formula (17) for $2.2 \leq \beta \leq 3$ for various values of $\sigma/\Lambda^2$ from 8 to 14 as solid curves. Our $2 \times 2$ Creutz ratios fit the interpolation (17) in the transition region $2.2 \leq \beta \leq 3$ with $\sigma/\Lambda^2_L = 14$. Our ratios $\chi(2,3)$ and $\chi(2,4)$ fit it with $\sigma/\Lambda^2_L = 12$ and 11, respectively. Our ratios $\chi(3,3)$ fit it with $\sigma/\Lambda^2_L = 8$. Altogether our $\chi$’s fit the interpolation (17) for $2.2 \leq \beta \leq 3$ with $\sigma/\Lambda^2_L = 11 \pm 3$. A string tension $\sqrt{\sigma} \approx 420$ MeV implies that $\Lambda_L \approx 130 \pm 18$ MeV, which is to be compared with the continuum value of $\Lambda^{(0)}_{\text{MS}} \approx 210$ MeV and with the lattice parameter $\Lambda_{LW} \approx 7.9$ MeV of Wilson’s method.

At stronger coupling where the transition to a linear potential is complete, our $\chi(i,j)$’s fit the scaling formula (16) without any Coulomb term for $1.9 < \beta < 2.1$ if we set $\sigma/\Lambda^2_L \approx 25.0 \pm 4$. The corresponding value of $\Lambda_L$ is $\Lambda_L \approx 85 \pm 7$ MeV.

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Figure Caption

The $SU(3)$ Creutz ratios $\chi(i,j)$ and the scaling predictions for the string tension $\sigma a^2$ are plotted against $\beta$. The solid lines for $1.9 < \beta < 2.1$ represent formula (16) for $\sigma a^2$ with $\sigma/\Lambda^2_L = 25 \pm 4$. The solid curves for $2.2 < \beta < 3.0$ are the interpolation (17) between the tree-level formula (15) for the Creutz ratios $\chi(i,j)$ and the scaling prediction (14) for $\sigma a^2$ with $\sigma/\Lambda^2_L = 11 \pm 3$. 

8
\( \beta \)

\( \chi(2,2) \)

\( \chi(2,3) \)

\( \chi(2,4) \)

\( \chi(3,3) \)

\( \chi(3,4) \)

\( \chi(4,4) \)

\( SU(3) \ 8^4 \)