Supersymmetry of the planar Dirac – Deser-Jackiw-Templeton system, and of its non-relativistic limit

Peter A. Horváthy\textsuperscript{a}, Mikhail S. Plyushchay\textsuperscript{b,c}, Mauricio Valenzuela\textsuperscript{a} \(*

\textsuperscript{a}Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, Parc de Grandmont, F-37200 Tours, France
\textsuperscript{b}Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile
\textsuperscript{c}Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47071, Valladolid, Spain

October 11, 2010

Abstract

The planar Dirac and the topologically massive vector gauge fields are unified into a supermultiplet involving no auxiliary fields. The superPoincaré symmetry emerges from the \( \mathfrak{osp}(1|2) \) supersymmetry realized in terms of the deformed Heisenberg algebra underlying the construction. The non-relativistic limit yields spin 1/2 as well as new, spin 1 “Lévy-Leblond-type” equations which, together, carry an \( N = 2 \) superSchrödinger symmetry. Part of the latter has its origin in the universal enveloping algebra of the superPoincaré algebra.

1 Introduction

The two main descriptions of massive relativistic spinning particles in the plane, namely those with spins 1/2 and with spin 1, are given by the Dirac, and by the topologically massive gauge theory of Deser, Jackiw and Templeton (DJT)\cite{1,2,3,4}. 

\[
D^b_a \psi_b \equiv (P^b_a \gamma^\mu - m) \epsilon^b_a \psi_b = 0, \tag{1.1}
\]

\[
\mathcal{D}^\mu_\nu F_\nu \equiv \left( -i \epsilon^{\mu\lambda}_\nu P^\lambda + m \delta^\nu_\nu \right) F_\nu = 0, \tag{1.2}
\]

respectively, where, in (1.1), \( \psi = \psi_a \) is a two-component Dirac spinor and the \(- (1/2) \gamma_\mu \), where \( \gamma_0 = \sigma^3 \), \( \gamma_1 = i \sigma^2 \), \( \gamma_2 = i \sigma^1 \), generate the spin 1/2 representation of the planar Lorentz group. Similarly in (1.2), the \(-i \epsilon^{\mu\lambda}_\nu \) generate the 3-dimensional vector representation, to which the \( F_\nu \) belongs. The equivalence of the dual formulation we use here with the Chern-Simons approach in [1] is discussed in [5].

The supersymmetric unification of these two theories is usually realized in the superfield formulation, which involves auxiliary fields, and relates the Dirac and the gauge vector fields. In such an approach, the supersymmetry between the “Dirac” and “topological” masses in (1.1) and

\*e-mails: horvathy-at-univ-tours.fr; mplyushc-at-lauca.usach.cl; valenzuela-at-lmpt.univ-tours.fr
respectively, was noticed in [1, 4], and was extended to topologically massive supergravity in [6]. Below we unify the two, Dirac and DJT, systems into a single supermultiplet with no gauge or auxiliary fields. Our results differ, hence, from those in [3].

The non-relativistic limit provides us then with another minimally realized supersymmetric system composed of the planar version of Lévy-Leblond’s “non-relativistic Dirac equation” [7], whose superpartner is a new, non-relativistic version of the DJT equation we construct here below.

Recently [8], a supersymmetric extension of Galilei symmetry was obtained by contraction, namely as the non-relativistic limit of superPoincaré symmetry. Our results here further extend those in [8]: the non-relativistic system we obtain is shown to carry an $N = 2$ superSchrödinger symmetry [9, 10].

Schrödinger symmetry is, in fact, “more”, and not “less” than Poincaré symmetry. It is well-known that the Schrödinger symmetry can not be derived from the relativistic counterpart by Inonü-Wigner contraction. Extending the usual contraction to the conformal group is, on the one hand, unjustified, since the latter is not a symmetry of the (massive) relativistic system one starts with. On the other hand, the standard contraction procedure yields, instead of Schrödinger’s, the conformal Galilei algebra, whose physical interest is limited [11].

Here we show, however, that the full superSchrödinger symmetry of the non-relativistic system emerges by contracting higher symmetry generators of the relativistic system, namely, certain elements of the universal enveloping superPoincaré algebra.

While the relativistic SUSY has been known before [4], the super-Schrödinger symmetry of its non-relativistic counterpart is a new result which, to our knowledge, has not been discussed so far.

2 The Dirac/Deser-Jackiw-Templeton supermultiplet

We start with considering the direct sum, $D^{1/2} \oplus D^{1}$, of the two, spin 1/2 and spin 1, representations, with Lorentz generators,

$$
J_{\mu} = \begin{pmatrix}
J_{\mu}^- & 0 \\
0 & J_{\mu}^+
\end{pmatrix}, \quad (J_{\mu}^-)_{a}^{b} = -\frac{1}{2}(\gamma_{\mu})_{a}^{b}, \quad (J_{\mu}^+)_{\nu \lambda}^{\lambda} = i \epsilon_{\mu \nu \lambda}.
$$

(2.1)

A unified wave function can be represented by the 5-tuplet

$$
\Psi(x) = \begin{pmatrix}
\psi_{a} \\
F_{\mu}
\end{pmatrix}, \quad (\psi_{a}) = \begin{pmatrix}
\psi_{1}(x) \\
\psi_{2}(x)
\end{pmatrix}, \quad (F_{\mu}) = \begin{pmatrix}
F_{0}(x) \\
F_{1}(x) \\
F_{2}(x)
\end{pmatrix}.
$$

(2.2)

The 3-dimensional Lorentz algebra generated by $J_{\mu}$ can be completed to the superalgebra $osp(1|2)$ by adding the two off-diagonal matrices

$$
L_{\underline{\Delta}} = \sqrt{2} \begin{pmatrix}
0 & Q_{\underline{\Delta}} \xi_{\mu}^{a} \\
-Q_{\underline{\Delta}} \xi_{\mu}^{a} & 0
\end{pmatrix},
$$

(2.3)

which satisfy $Q_{\underline{\Delta}} \xi_{\mu}^{a} = \eta_{\mu \nu} (Q_{\underline{\Delta}} \xi_{\nu}^{b})^{T} \epsilon^{ba}$, where $T$ means transposition, $\epsilon^{12} = -\epsilon^{21} = 1$, and underlined capitals label $osp(1|2)$ spinors ($\underline{\Delta} = 1, 2$). Here the space-time metric is $\eta_{\mu \nu} = diag(-1, 1, 1)$.
and \( \epsilon^{ab} = \epsilon_{ab} \) is used to rise and lower the spinor indices, \( \chi^a = \chi \epsilon^{ba} \), \( \chi_a = \epsilon_{ab} \chi^b \). Explicitly,

\[
Q_{\pm a}^\mu = \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Q_{\mp a}^\mu = \begin{pmatrix} 0 & 0 & -i \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},
\]

(2.4)

related as \( (Q_{\pm a}^\mu)^\dagger = -Q_{\mp a}^{\dagger \mu} \), \( (Q_{\mp a}^{\dagger \mu})^\dagger = Q_{\pm a}^\mu \). The operators \( L_\pm, L_\mp \) are hermitian conjugate with respect to the scalar product, \( \Phi^\dagger \tilde{\eta} \Psi \). Hence \( L_+ = \tilde{\eta} L_+^{\dagger \mp} \tilde{\eta} \), where \( \tilde{\eta} = \text{diag}(\gamma^0, \eta) \), and \( \eta = \eta_{\mu\nu} \).

\( L_\pm \) and \( L_\mp \) span the deformed Heisenberg algebra [12], [\( L_\pm, L_\mp = -\epsilon_{\pm a} \) (1 + \( \nu R \)), with deformation parameter \( \nu = -5 \), where

\[
R = \text{diag}( -1, 2, 1, 1) ,
\]

(2.5)

\( R^2 = 1 \), \{\( L_\pm, R \} = 0 \), is the reflection operator. The operators \( L_\pm \) extend the Lorentz algebra generated by (2.1) to \( \text{osp}(1|2) \),

\[
[\mathcal{J}_\mu, \mathcal{J}_\nu] = -i \epsilon_{\mu\nu\lambda} \mathcal{J}_\lambda , \quad \{L_\pm, L_\mp\} = 4(\mathcal{J}_\gamma)_{\pm \mp}, \quad [\mathcal{J}_\mu, L_\pm] = \frac{1}{2} (\gamma_\mu)_{\pm \mp} L_\mp,
\]

(2.6)

where \( (\gamma_\mu)_{\pm \mp} = \epsilon_{\mu\nu\lambda}(\gamma_\mu)_{\pm \mp} \). The role of the grading operator is played by the reflection operator \( R \). Then the super-Casimir operator is

\[
\mathcal{C} = \mathcal{J}_\mu \mathcal{J}^\mu + \frac{1}{8} [L_+, L_-] = \mathcal{J}_\mu \mathcal{J}^\mu + \frac{1}{8} L_\mp L_\pm = -\frac{3}{2}.
\]

(2.7)

The representation of \( \text{osp}(1|2) \) is therefore irreducible. The original ingredients, \( \psi_a \) and \( F_{\mu} \), can plainly be recovered by projecting onto the \( \mp 1 \) eigenspaces of the reflection operator, \( R \). On these subspaces the Casimir of the Lorentz subalgebra is

\[
\mathcal{J}_\mu \mathcal{J}^\mu = -\hat{\alpha} (\hat{\alpha} - 1) \quad \text{with} \quad \hat{\alpha} = -\frac{1}{4} (3 + R) .
\]

(2.8)

The operator \( \hat{\alpha} \) has, hence, eigenvalues \( \alpha_- = -\frac{1}{2} \) and \( \alpha_+ = -1 \), proving that the \( \mp 1 \) eigenspaces carry indeed the irreducible spin-1/2 (Dirac) and resp. spin-1 DJT representations. Moreover, using \( \hat{\alpha} \) our two, Dirac and DJT, systems can be written in the same unified form,

\[
(P_\mu \mathcal{J}^\mu - \hat{\alpha} m) \Psi = 0.
\]

(2.9)

The operators \( L_\pm \) interchange \( \psi \) and \( F \), but they do not preserve the physical states (defined as solutions of the Dirac and DJT equations, respectively). This can be achieved, however, by considering instead the two supercharges,

\[
Q_\pm = \frac{1}{2 \sqrt{m}} \big((P_\mu (\gamma_\mu)_{\pm \mp} - R m \delta_{\pm \mp}) L_\mp \big),
\]

(2.10)

\( A, B = 1, 2 \), whose components in explicit form are,

\[
Q_+ = \frac{1}{2 \sqrt{m}} (L_2 P_+ + L_1 (m R - P_0)) , \quad Q_- = \frac{1}{2 \sqrt{m}} (-L_2 P_- + L_1 (m R + P_0)) ,
\]

(2.10)

where \( P_\pm = P_1 \pm i P_2 \). The action of (2.10) on the spin-1 (\( F_{\mu} \)) and spin-1/2 (\( \psi_a \)) components is found, respectively, as

\[
\Psi' = \begin{pmatrix} \psi'_a \\ F_{\mu} \end{pmatrix} = \zeta^+ Q_+ \Psi = \zeta^+ \begin{pmatrix} Q_{\pm a}^\mu F_{\mu} \\ Q_{\pm a}^\mu \psi_a \end{pmatrix},
\]

(2.11)
where $\zeta^A$ are the parameters of the supersymmetry transformation. Hence, a two-component Dirac field is transformed into a three-component DJT field $F'$ and conversely. Furthermore,

$$D_a^b \psi_b' = \zeta^A \left( Q_{a\mu} \partial_{\mu} F'_\nu + \frac{1}{2\sqrt{m}} Q_{a\mu} (P^2 + m^2) F_\mu \right), \quad (2.12)$$

$$D_{\mu}^\nu F'_\nu = \zeta^A \left( -\frac{1}{2} Q_{a\mu} D_{ab}^\nu \psi_b - \frac{1}{2\sqrt{m}} Q_{a\mu} (P^2 + m^2) \psi_a \right). \quad (2.13)$$

Both the Dirac and DJT equations imply the Klein-Gordon equation, allowing us to conclude that the transformed fields satisfy the Dirac and DJT equations, respectively, if the original ones satisfy them (in the reversed order).

Adding the two supercharges (2.10) to the Poincaré generators of the space-time translations, $P_\mu$, and of the Lorentz transformations, $M_\mu = -\epsilon_{\mu\nu\lambda} x^\nu P^\lambda + J_\mu$, yields the off-shell relations,

$$[P_\mu, P_\nu] = 0, \quad [M_\mu, P_\nu] = -i\epsilon_{\mu\nu\lambda} P^\lambda, \quad [M_\mu, M_\nu] = -i\epsilon_{\mu\nu\lambda} M^\lambda, \quad (2.14)$$

$$[P_\mu, Q_\lambda] = 0, \quad [M_\mu, Q_\lambda] = \frac{1}{2} (\gamma_\mu)_{\lambda\beta} Q_\beta. \quad (2.15)$$

$$\{Q_\lambda, Q_\mu\} = 2(P\gamma)_\lambda\mu + \frac{1}{2m} \left[ (J\gamma)_{\lambda\mu} (P^2 + m^2) - 2(P\gamma)_{\lambda\mu} (P J - \hat{\alpha}m) \right]. \quad (2.16)$$

The second term on the r.h.s. of (2.16) vanishes on-shell, leaving us with the usual $N = 1$ planar super-Poincaré algebra, $\mathfrak{iso}(1|2,1)$.

## 3 Solutions of the supersymmetric equation

The equations are solved following the method outlined in [8]. It requires expanding the fields in the lowest weight representation basis of the Lorentz generators (2.1).

Since the representation of the Lorentz algebra (2.1) is reducible, there are two lowest non-trivial vectors such that, $J_-|0\rangle_D = 0, J_-|0\rangle_{DJT} = 0$. These are just the lowest spin states in the spin $1/2$ (Dirac) and spin 1 (DJT) sectors, respectively.

The irreducible spaces of spin $1/2$ and 1 representations are generated by the ladder operator $J_+ = J_1 + iJ_2$, which acts as

$$|1\rangle_D = J_+|0\rangle_D, \quad |1\rangle_{DJT} = \frac{1}{\sqrt{2}} J_+|0\rangle_{DJT}, \quad |2\rangle_{DJT} = \frac{1}{\sqrt{2}} J_+|1\rangle_{DJT}. \quad (3.1)$$

Both subspaces have highest spin states, $J_+|1\rangle_D = 0, J_+|2\rangle_{DJT} = 0$ and the ladder operator $J_- = J_1 - iJ_2$ acts as,

$$J_-|1\rangle_D = -|0\rangle_D, \quad \frac{1}{\sqrt{2}} J_-|1\rangle_{DJT} = -|0\rangle_{DJT}, \quad \frac{1}{\sqrt{2}} J_-|2\rangle_{DJT} = -|1\rangle_{DJT}. \quad (3.2)$$

$J_0$ acts diagonally,

$$J_0|0\rangle_D = -\frac{1}{2}|0\rangle_D, \quad J_0|1\rangle_D = \frac{1}{2}|0\rangle_D, \quad (3.3)$$

$$J_0|0\rangle_{DJT} = -|0\rangle_{DJT}, \quad J_0|1\rangle_{DJT} = 0, \quad J_0|2\rangle_{DJT} = |2\rangle_{DJT}. \quad (3.4)$$
The Dirac and DJT fields are written in this basis as,

$$\psi = \psi_0|0\rangle_D + \psi_1|1\rangle_D, \quad F = \frac{1}{\sqrt{2}} F_+|0\rangle_{DJT} + F_0|1\rangle_{DJT} + \frac{1}{\sqrt{2}} F_-|2\rangle_{DJT}. \quad (3.5)$$

Here, $F_\pm = F_1 \pm iF_2$.

For the Dirac and DJT fields, equation (2.9) yields,

$$\frac{1}{\sqrt{2}} [(m - P^0)\psi_0 + P_+ \psi_1]|0\rangle_D + \frac{1}{\sqrt{2}} [(m + P^0)\psi_1 + P_- \psi_0]|1\rangle_D = 0, \quad (3.6)$$

$$\frac{1}{\sqrt{2}} [(m + P_0)F_+ - P_+F_0]|0\rangle_{DJT} + \left[ mF_0 + \frac{P_- F_+ - P_+ F_-}{2}\right]|1\rangle_{DJT} + \frac{1}{\sqrt{2}} [(m - P_0)F_0 + P_- F_0]|2\rangle_{DJT} = 0. \quad (3.7)$$

For positive energy solutions ($P^0 = -P_0 > 0$), the operator $P^0 + m$ can be inverted. Hence,

$$\frac{(P \cdot P + m^2)}{(P^0 + m)^2} F_0 + F_0 = -\frac{P_-}{P^0 + m} F_+; \quad F_- = \left(\frac{P_-}{P^0 + m}\right)^2 F_+; \quad \psi_1 = -\frac{P_-}{P^0 + m} \psi_0. \quad (3.8)$$

The first term in (3.8) vanishes by the Klein-Gordon equation, so that all components of the DJT (Dirac) field are obtained from the lowest spin state $-1$ (and $-1/2$),

$$F_+ = A \Phi(x), \quad \psi_0 = B \Phi(x), \quad \Phi(x) = \exp \left\{ -ix_0 \sqrt{p_i^2 + m^2 + ix_i p_i} \right\}. \quad (3.9)$$

where $A, B$, are arbitrary constants, and $p_i$ are the eigenvalues of $P_i$.

Negative energy solutions can be obtained by an analogous procedure, considering the highest spin components, $\psi_2$ and $F_-$, of the spin $1/2$ and resp 1 sectors.

4 Nonrelativistic counterpart of the Dirac-DJT supermultiplet

Taking the nonrelativistic limit is subtle. For example, central extensions correspond to cohomology [13, 14, 15, 16]: that of the Poincaré group is trivial, while the one of the Galilei group is not. How can nontrivial cohomology arise in the nonrelativistic limit? As explained in Ref. [15], one should start with the trivial $U(1)$ extension (i.e. with trivial two-cocycle) of the universal covering of the Poincaré group, and then Inönü-Wigner contraction yields the universal covering of the Galilei group, namely an $U(1)$ extension of the Galilei group (with nontrivial two-cocycle). The latter is necessary to support the mass-central-charge extension. It is in fact the rest frame energy $mc^2$ that generates the nontrivial two-cocycle in the nonrelativistic limit.

In our particular case, the nonrelativistic limit is carried out first by reinstating the velocity of light, $c$, and putting $m \to mc, x^0 = ct$. $P^0$ diverges in the nonrelativistic limit $c \to \infty$ as $mc$ and must be renormalized therefore. Similar considerations indicate that, when compared to $F_+$, the components $F_0$ and $F_-$ are suppressed by factors of order $c^{-1}$ and $c^{-2}$, respectively, on account of equation (3.8). Analogously for the Dirac field, $\psi_1$ is suppressed by $c^{-1}$ compared $\psi_0$. Rescaling the field components by suitable powers of $c$ yields nontrivial components with nonrelativistic spin. Consider, in fact, $\phi_0 = e^{-imc^2t}\psi_0, \phi_1 = cc^{-imc^2t}\psi_1$ for the Dirac field, and
\[ f_+ = e^{-imc^2t}F_+ , \quad f_0 = ce^{-imc^2t}F_0 , \quad f_- = c^2e^{-imc^2t}F_- \] for the DJT field. These transformations can be written in a compact form in terms of the supermultiplet (2.2), namely as \( \Phi = M \Psi, \quad M = \text{diag}(M^-, M^+) \), where \( M \) is a block-diagonal matrix, composed of

\[
M^- = e^{-imc^2t} \text{diag}(1, c), \quad M^+ = e^{-imc^2t} \begin{pmatrix} c & 0 & 0 \\ 0 & 1 + c^2 & i(1 - c^2) \\ 0 & -i(1 - c^2) & 1 + c^2 \end{pmatrix}.
\]  

(4.1)

The relativistic operators transform according to

\[
O \rightarrow O' = M\!O\!M^{-1}.
\]

(4.2)

In terms of

\[
P_0' = -c^{-1}(i\partial_t + mc^2), \quad P_i' = P_i, \quad J'_0 = J_0, \quad J'_+ = cJ_+, \quad J'_- = \frac{1}{c}J_-
\]

(4.3)

Eqn (2.9) can therefore be rewritten as \( (P'_\mu J'^\mu - \dot{\alpha}mc)\Phi = 0 \). Switching to primed variables is, in fact, an automorphism of the Poincaré algebra (2.14), so that the value of the \( \mathfrak{so}(2, 1) \) Casimir operator of \( J' \) is left invariant. We have, moreover,

\[
P'_\mu J'^\mu - \dot{\alpha}mc = \frac{1}{c} \left( iJ_0 \partial_t + \frac{1}{2}P_+J_- \right) + c \left( m(J_0 - \dot{\alpha}) + \frac{1}{2}P_-J_+ \right).
\]

(4.4)

This operator diverges in the nonrelativistic limit. Consistency in the nonrelativistic limit requires, therefore,

\[
\left( m(J_0 - \dot{\alpha}) + \frac{1}{2}P_-J_+ \right) \Phi = 0.
\]

(4.5)

In the rest frame, this equation is equivalent to \( J_0 - \dot{\alpha} = 0 \), which fixes the spin of the nonrelativistic particle. Note that the nonrelativistic Hamiltonian, \( \mathcal{H} = i\partial_t = cP^0 - mc^2 \), is not obtained here, since the first term in (4.4) drops out when \( c \rightarrow \infty \). The Schrödinger equation is obtained, however, from the transformed Klein-Gordon equation [which is, as said before, a consequence of the first-order equations (1.1) and (1.2)],

\[
(i\partial_t - \frac{1}{2m}P_+P_-)\Phi = \lim_{c \rightarrow \infty} \left( -\frac{1}{2m}(P^2 + m^2c^2)\Phi \right) = 0.
\]

(4.6)

Furthermore, (4.5) and (4.6) allow us to infer,

\[
((J_0 - \dot{\alpha})P_+ + iJ_+ \partial_t)\Phi = 0.
\]

(4.7)

Eqn (4.7), together with (4.5) allows us to recover, once again, the Schrödinger equation \( (i\partial_t - (2m)^{-1}P_i^2)\Phi = 0 \) as consistency condition, namely by commuting the operators in front the field \( \Phi \). Hence, (4.5), (4.6) and (4.7) form a self consistent system. In fact, Eqns (4.5) and (4.7) alone are enough to describe our massive nonrelativistic supermultiplet. Projecting these equations to the spin 1/2 and 1 subspaces yields indeed the independent equations (written in component form),

\[
\begin{align*}
\text{spin } \frac{1}{2} : & \quad \begin{cases} i\partial_t \phi_0 + P_+ \phi_1 = 0, \\
2m\phi_1 + P_- \phi_0 = 0, \end{cases} \\
\text{spin } 1 : & \quad \begin{cases} i\partial_t f_+ - iP_+ f_0 = 0, \\
2mf_0 + iP_- f_+ = 0, \\
2mf_- + iP_- f_0 = 0. \end{cases}
\end{align*}
\]

(4.8) \quad (4.9)
Equations (4.8) are the (2 + 1)D Lévy-Leblond equations [7]. (4.9) is in turn the non-relativistic limit of the spin-1 Deser-Jackiw-Templeton system, cf. [8]. \( \psi_1, f_0 \) and \( f_- \) are auxiliary fields and may be expressed in terms of the lowest spin states, \( \psi_0 \) and \( f_+ \), respectively.

These equations imply the Schrödinger equation for each component.

With the nonrelativistic limit is associated a contraction of the superPoincaré algebra (2.14), that produces a symmetry of the nonrelativistic system (4.6) and (4.7). Defining

\[
K_i = -\lim_{c \to \infty} \epsilon_{ij} M'_j / c, \quad \mathcal{H} = c P^0 - mc^2, \quad J = M'_0 \quad \text{where} \quad \epsilon_{ij} = -\epsilon_{ji}, \quad \epsilon_{12} = 1,
\]

we get

\[
K_i = -t P_i + mx_i + \hat{s}_i, \quad \mathcal{H} = i \frac{D}{Dt}, \quad J = \epsilon_{ij} x_i P_j + J_0, \quad (4.10)
\]

where \( \hat{s}_1 = i \hat{s}_2 = \frac{i}{2} J_+ \). \( J_+ \) has diagonal blocks \( J_+ \Pi_- = \text{diag}(J_+, \Pi_-) \), \( J_+ \Pi_+ = \text{diag}(J_+, \Pi_+) \) which act independently in the spin \( 1/2 \) and spin 1 subspaces,

\[
J_+ \Pi_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_+ \Pi_+ = \begin{pmatrix} 0 & 1 & i \\ 1 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad (4.11)
\]

where \( \Pi_\pm = \frac{1}{2}(1 \pm R) \). The boost operator we find is consistent with the known result for spin 1/2 [7], recently generalized to spin 1 [8]. Note the spin contribution, \( J_0 \), to the angular momentum. Together with the \( P_i \), the operators (4.10) generate the seven-dimensional, one-parameter centrally extended Galilei (also called “Bargmann”) algebra \( \text{gal} = \{ \text{translations}, P_i, \text{time translations}, \mathcal{H}, \text{Galilei boosts}, K_i, \text{rotations}, J, \text{mass-central-charge}, m \} \),

\[
[K_i, P_j] = im \delta_{ij}, \quad [P_i, P_j] = 0, \quad [\mathcal{H}, P_i] = 0, \quad [K_i, K_j] = 0, \quad [K_i, \mathcal{H}] = i P_i, \quad [J, P_i] = i \epsilon_{ij} P_j, \quad [J, K_i] = i \epsilon_{ij} K_j. \quad (4.12)
\]

The Casimir operators of the algebra (4.12) are

\[
C_1 = P_i^2 - 2m \mathcal{H}, \quad C_2 = mJ - \epsilon_{ij} K_i P_j = mJ_0 + \frac{1}{2} P_- J_+. \quad (4.13)
\]

The internal energy represented by \( C_1 \) vanishes, owing to the Schrödinger equation. In virtue of equation (4.5), the second Casimir is, however, operator valued,

\[
C_2 = m \hat{\alpha}.
\]

The Galilei algebra we have found is therefore reducible. This has been expected, owing to the supersymmetry of the Dirac-DJT multiplet, whose nonrelativistic limit we have taken. The algebra becomes, however, irreducible if we restrict ourselves to the \( \mp 1 \) subspaces of the operator \( R \), i.e., to spin \( 1/2 \) and spin 1, respectively.

Non-relativistic supersymmetry is inherited from the relativistic one of the Dirac-DJT supermultiplet. In fact, the nonrelativistic supercharges, are obtained by taking the limit, \( \Omega_\Delta = \lim_{c \to \infty} \frac{1}{c} Q'_\Delta \), of transformed expressions \( Q'_\Delta = M Q_\Delta M^{-1} \). Explicitly, for \( \Omega_1 = \frac{1}{\sqrt{2}}(\Omega_\perp + \Omega_\parallel) \)
and \( \Omega_2 = \frac{1}{\sqrt{2}i}(\Omega_1 - \Omega_2) \) we find

\[
\Omega_1 = \frac{1}{\sqrt{2}} \begin{pmatrix}
0 & \sqrt{m} Q_1 a^\mu \\
-\frac{1}{2\sqrt{m}}(P_\mu Q_1 a^\mu + 2mP_\mu Q_2 a^\mu) & 0
\end{pmatrix},
\]

\( (4.14) \)

\[
\Omega_2 = \frac{1}{\sqrt{2}i} \begin{pmatrix}
0 & \sqrt{m} Q_1 a^\mu \\
\frac{1}{2\sqrt{m}}(P_\mu Q_1 a^\mu + 2mP_\mu Q_2 a^\mu) & 0
\end{pmatrix}.
\]

\( (4.15) \)

Note that these supercharges are related through the grading operator \( [2,5] \), \( \Omega_2 = iR \Omega_1 \). The action of the supercharge \( \Omega_1 \) on the nonrelativistic field \( \Phi \) reads

\[
\Phi' = \begin{pmatrix}
\phi'_a \\
f'_\mu
\end{pmatrix} = \begin{pmatrix}
\sqrt{m} Q_1 a^\mu f_\mu \\
-\frac{1}{2\sqrt{m}}(P_\mu Q_1 a^\mu + 2mP_\mu Q_2 a^\mu) \phi_a
\end{pmatrix},
\]

\( (4.16) \)

and the action of \( \Omega_2 \) follows analogously. Here, \( f_1 = (f_+ + f_-)/2 \) and, \( f_2 = -i(f_+ - f_-)/2 \). This formula shows how the supercharges indeed interchange the non-relativistic “Dirac” \( (\phi_a) \) and “DJT” \( (f_\mu) \) components, cf. \( (2.11) \). Our supercharges \( \Omega_i \), extend \( (4.12) \) to a superalgebra \( [9] \) with (anti)commutation relations,

\[
\{J, \Omega_i\} = i\epsilon_{ij} \Omega_j, \quad \{\Omega_i, \Omega_j\} = \delta_{ij} \left( 4m + (m(J_0 - \hat{\alpha}) + \frac{1}{2}P_-J_+ \right) 
\]

\( (4.17) \)

\[
\{K_i, \Omega_i\} = [P_i, \Omega_i] = [H, \Omega_i] = 0.
\]

Note that the supercharge is a vector w.r.t. a rotation. Observe also that the supercharge-anticommutator involves the operator in Eqn. \( (4.5) \), which vanishes when acting on a wave function on account of Eqn. \( (4.5) \). On-shell we have therefore

\[
\{\Omega_i, \Omega_j\} = 4\delta_{ij} m.
\]

\( (4.18) \)

The non-relativistic theory actually has more symmetries, which do not derive directly from the Lie (super)algebraic structure of the parent relativistic theory \( [9, 10] \) (see the discussion below). The two “helicities”

\[
Q = \frac{1}{2m} \vec{\Omega} \cdot \vec{P}, \quad Q^* = \frac{1}{2m} \vec{\Omega} \times \vec{P},
\]

\( (4.19) \)

related by \( Q^* = iRQ \), are both new supercharges for the non-relativistic system which yield on-shell, with the previous operators, a closed superalgebra,

\[
\{Q, K_i\} = -\frac{i}{2} \Omega_i, \quad \{Q, P_i\} = 0, \quad \{Q, H\} = 0,
\]

\[
\{Q^*, K_i\} = -\frac{i}{2} \epsilon_{ij} \Omega_j, \quad \{Q^*, P_i\} = 0, \quad \{Q^*, H\} = 0,
\]

\[
\{Q, Q\} = 2H, \quad \{Q^*, Q^*\} = 2H, \quad \{Q, Q^*\} = 0,
\]

\[
\{\Omega_i, Q\} = 2P_i, \quad \{\Omega_i, Q^*\} = 2\epsilon_{ij} P_j,
\]

\( (4.20) \)

called the [centrally extended] \( N = 2 \) superGalilei algebra \( [9, 10] \), \( sgal \), which extends the Galilei algebra, \( gal \), by the odd supercharges \( \Omega_i \), \( Q \), and \( Q^* \). In particular, both \( Q \) and \( Q^* \) are “square
roots” of the Hamiltonian, $\mathcal{H}$ – just like the $Q_i$ in (4.18) are “square roots” of the mass. The super-Casimir operator reads

\[ C_{\text{susy}} = C_2 + \frac{i}{16} [Q_1, Q_2] = -\frac{3}{4} m, \quad (4.21) \]

where (4.5) was taken into account. Note that the representation of the superalgebra is irreducible, since $C_{\text{susy}}$ is a constant.

By (4.19), and remembering that $m$ plays the role of the central charge, our $N = 2$ superGalilei algebra also has two odd Casimir operators, namely

\[ C_3 = mQ - \frac{1}{2} \vec{Q} \cdot \vec{P}, \quad C_4 = mQ^* - \frac{1}{2} \vec{Q} \times \vec{P}. \quad (4.22) \]

They take here zero values.

5 Schrödinger (super)symmetry

For a spinless particle, the (free) Schrödinger equation is known to be symmetric under the “conformal” extension of the Galilei group, obtained by adding dilations and expansions

\[ D = 2t\mathcal{H} - \vec{x} \cdot \vec{P} + i, \quad C = t^2 \mathcal{H} - t\vec{x} \cdot \vec{P} + it + \frac{m}{2} \vec{x}^2. \quad (5.1) \]

Since the nonrelativistic spin 1/2 and spin 1 equations, (4.8) and (4.9), also describe free particles, their Schrödinger symmetry is expected (and has actually been proved for the spin 1/2 model of Lévy-Leblond [18].) Now we prove that the operators

\[ \mathcal{D} = -\frac{1}{2m} (\mathcal{K} \cdot P + P \cdot \mathcal{K}) = D - \frac{i}{2m} P_- \mathcal{J}_+, \]

\[ \mathcal{C} = \frac{1}{2m} \mathcal{K} \cdot \mathcal{K} = C + \frac{i}{2m} (tP_- + mx_-) \mathcal{J}_+, \quad (5.2) \]

extend the Galilei algebra (4.12) into the Schrödinger algebra, $\mathfrak{sch}$, with non-trivial additional commutation relations,

\[ [\mathcal{D}, C] = 2i\mathcal{C}, \quad [\mathcal{D}, \mathcal{H}] = -2i\mathcal{H}, \quad [\mathcal{H}, C] = i\mathcal{D}, \quad (5.3) \]

This representation is reducible: the operators in (5.2) act diagonally on the spin 1/2 and spin 1 subsystems. Projected to the spin-1/2 subspace we obtain, using $P_- \mathcal{J}_+ = -2m(\mathcal{J}_0 - \hat{a})$ (cf. Eqn. (4.5)),

\[ \mathcal{D}^- = \mathcal{D} \Pi_- \approx \begin{pmatrix} D & 0 \\ 0 & D + i \end{pmatrix}, \quad \mathcal{C}^- = \mathcal{C} \Pi_- \approx \begin{pmatrix} C & 0 \\ \frac{i}{2} x_- & C + it \end{pmatrix}, \quad (5.4) \]

which are equivalent, on-shell, with those in Ref. [18].

Projecting instead onto the spin-1 sector we obtain the new result,

\[ \mathcal{D}^+ = \mathcal{D} \Pi_+ \approx \begin{pmatrix} D + i & 0 & 0 \\ 0 & D + i & 1 \\ 0 & -1 & D + i \end{pmatrix}, \quad \mathcal{C}^+ = \mathcal{C} \Pi_+ \approx \begin{pmatrix} C & -x_- & -ix_- \\ -x_- & C & 0 \\ -ix_- & 0 & C \end{pmatrix}. \quad (5.5) \]
Here \( \approx \) means after using Eq. \( \text{[4.5]} \), and we put \( x_- = x_1 - ix_2 \).

To prove that this operators are symmetries, notice first that \( \text{[5.2]} \) are elements of the universal enveloping algebra of the Galilei algebra, namely polynomials in the Galilei algebra generators (boosts and translations). Now, we write Eqns \( \text{[4.5]} \), \( \text{[4.6]} \) and \( \text{[4.7]} \) symbolically as \( \mathfrak{D}\Phi = 0 \), where \( \mathfrak{D} \) is the respective differential operator.

Consider now two operators \( \mathcal{A} \) and \( \mathcal{B} \) such that \( [\mathfrak{D}, \mathcal{A}] = [\mathfrak{D}, \mathcal{B}] = 0 \). They both preserve the space of solutions of \( \mathfrak{D}\Phi = 0 \), and can be treated therefore as symmetry generators. Then it is straightforward to show that the product of two such symmetry generators \( \mathcal{A}\mathcal{B} \), is also a symmetry \( [\mathfrak{D}, \mathcal{A}\mathcal{B}] = 0 \). Choosing, in particular, \( \mathcal{A} \) and \( \mathcal{B} \) to be Galilei boost or momentum generators, it follows that \( \mathcal{C} \) and \( \mathcal{D} \), constructed of them according to \( \text{[5.2]} \), are also (explicitly \( t \)-dependent) symmetries.

The system of equation \( \text{[4.5]}, \text{[4.6]} \) and \( \text{[4.9]} \) is therefore Schrödinger symmetric. The same arguments explain the origin of the helicity supercharges \( \text{[4.19]} \) introduced above.

Now the superGalilei symmetry combines with the conformal extension,

\[
S = \frac{1}{2m} \vec{\mathcal{\Omega}} \cdot \vec{\mathcal{\Omega}}, \quad S^* = \frac{1}{2m} \vec{\mathcal{\Omega}} \times \vec{\mathcal{\Omega}}^2,
\]

(5.6)

related by \( S^* = iRS \), are both supercharges for the non-relativistic system. They are both “square roots” of expansions,

\[
\{S, S\} = \{S^*, S^*\} = 2C, \quad \{S, S^*\} = 0.
\]

(5.7)

Moreover,

\[
\{Q, S\} = \{Q^*, S^*\} = -D, \quad \{Q^*, S\} = -\{Q, S^*\} = \mathcal{Y},
\]

(5.8)

where \( \mathcal{Y} = \frac{1}{m} \vec{\mathcal{K}} \times \vec{P} \). On shell, \( \mathcal{Y} \) is just \( J - \hat{\alpha} \), cf. \( \text{[10]} \). Note for further reference that the conserved quantities \( \text{[5.6]} \) are obtained by the commuting the generator of the special conformal transformations, \( \mathcal{C} \), with the supercharges \( Q, Q^* \), see below.

All these generators close, at last, into an \( N = 2 \) superSchrödinger algebra \( \mathfrak{gsch} \), that includes the supercharges \( \Omega_i, Q, Q^*, S \) and \( S^* \), with additional commutation relations

\[
[\mathfrak{D}, Q] = -iQ, \quad [\mathfrak{C}, Q] = iS, \quad [\mathcal{Y}, Q] = iQ^*,
\]

\[
[\mathfrak{D}, Q^*] = -iQ^*, \quad [\mathfrak{C}, Q^*] = iS^*, \quad [\mathcal{Y}, Q^*] = -iQ,
\]

\[
[\mathfrak{H}, S] = -iQ, \quad [\mathfrak{D}, S] = iS, \quad [\mathfrak{C}, S] = 0, \quad [\mathcal{Y}, S] = iS^*,
\]

\[
[\mathfrak{H}, S^*] = -iQ^*, \quad [\mathfrak{D}, S^*] = iS^*, \quad [\mathfrak{C}, S^*] = 0, \quad [\mathcal{Y}, S^*] = -iS,
\]

\[
[S, \mathcal{K}_i] = 0, \quad [S, P_i] = i\frac{1}{2} \Omega_i, \quad [S, \mathcal{H}] = iQ,
\]

\[
[S^*, \mathcal{K}_i] = 0, \quad [S^*, P_i] = -i\frac{1}{2} \epsilon_{i j k} \Omega_j, \quad [S^*, \mathcal{H}] = iQ^*.
\]

(5.9)

6 The relativistic origin of (super)Schrödinger symmetry

It is well-known that while the Galilei symmetry is obtained from the Poincaré symmetry by contraction, Schrödinger symmetry, its conformal extension, can not be derived in such a way \( \text{[11]} \). Below we show, however, that the latter, and in fact superGalilei symmetry, can be obtained from a relativistic theory, — but one has to start with a larger structure.

Consider all operators which are quadratic in the generators of the superPoincaré algebra. Their commutators with the superPoincaré generators are again quadratic. The commutators of the quadratic operators between themselves give rise, however, to the operators which are cubic
in the superPoincaré generators. Continuing this procedure, we end with the universal enveloping algebra of the superPoincaré algebra.

Restricting ourselves to a certain subset of the quadratic operators, apply the similarity transformation (4.2) to the commutators of these operators between themselves, and with the generators of the superPoincaré algebra; then divide both sides of these commutation relations by appropriate powers of velocity of light, \( c \), and take, finally, the limit \( c \to \infty \). This procedure can yield a closed Lie superalgebra structure.

To identify an appropriate quadratic subset of the universal enveloping algebra, we note that the non-relativistic symmetry generators (5.2), (4.19) and (5.6) can be identified as the non-relativistic limits of
\[
\tilde{D} = \frac{1}{2m} \epsilon_{ij} (P_i \mathcal{M}_j + \mathcal{M}_j P_i), \quad \tilde{C} = \mathcal{M}_i \mathcal{M}_i, \quad \tilde{Q} = \frac{1}{2 \sqrt{2m}} Q_2 P_+ + Q_2 P_+ + \frac{i}{2 \sqrt{2m}} (Q_2 \mathcal{M}_+ - Q_2 \mathcal{M}_-) ,
\]
and of \( \tilde{Q}^* = iR \tilde{Q} \) and \( \tilde{S}^* = iR \tilde{S} \),
\[
\mathcal{D} = \lim_{c \to \infty} \frac{\tilde{D}'}{c}, \quad \mathcal{C} = \lim_{c \to \infty} \frac{\tilde{C}'}{c^2}, \quad \mathcal{Q} = \lim_{c \to \infty} \frac{\tilde{Q}'}{c}, \quad \mathcal{S} = \lim_{c \to \infty} \frac{\tilde{S}'}{c^2} .
\]

If we take now, for instance, the commutator of the \( \tilde{D} \) with \( P_i \), we get a new element of the universal enveloping algebra of the superPoincaré algebra, \([\tilde{D}, P_i] = -\frac{i}{m} P_i P_0 \). On account of the definition (6.3) and of the first relation from (4.3), this reduces, after applying the similarity transformation and taking the non-relativistic limit, to one of the Lie algebraic relations from (5.3), \([\mathcal{D}, P_i] = -iP_i \). One can check then that in a similar way all the rest of the (anti)commutation relations of the superSchrödinger algebra can be reproduced proceeding from (6.1), (6.2) and (6.3).

In conclusion, Schrödinger supersymmetry is inherited from its relativistic predecessor, but this requires the extension of the superPoincaré algebra by certain elements of its universal enveloping algebra, which, in the nonrelativistic limit, become genuine space-time transformations.

7 Discussion

Contraction from the Poincaré algebra yields the Galilei algebra. Extending the contraction to the whole superPoincaré algebra only yields some, but not all, of the nonrelativistic symmetries.

Those which are not obtained emerge from higher order tensor products of the Galilei generators and the supercharges \( \Omega_i \) in (4.14)-(4.15). These products form indeed a finite subset of the universal enveloping algebra of the Galilei algebra (4.12) extended with the supercharges \( \Omega_i \). The latter is endowed with the supercommutator product, cf. [20, 21]. It follows that the new generators descend from some of the generators of the universal enveloping of the superPoincaré algebra (2.14)-(2.15) when the nonrelativistic limit is taken. In contrast, the relativistic counterpart of the new nonrelativistic symmetries would not close into a finite dimensional superextension of Poincaré, but generates instead its whole universal enveloping algebra.

We mention that this approach allowed to show that a free scalar nonrelativistic particle in \( d \)-spatial dimensions exhibits an \( Sp(2d) \) symmetry, which extends its well known conformal \( Sl(2, \mathbb{R}) \approx Sp(2) \) symmetry [21].

Note that the contraction endows the higher-order operators (6.1) and (6.2) of the universal enveloping superPoincaré algebra, with a clear-cut geometrical meaning: dilations and expansion.
generators are genuine space-time transformations, with the supercharges becoming the square roots of the time translations and expansions, respectively.

Our considerations here are based on a particular representation of the deformed Heisenberg algebra [12], that carries a suitable irreducible representation of $\mathfrak{osp}(1|2)$. It is this representation that is promoted to an irreducible representation of the superPoincaré symmetry of the Dirac–DJT system. The advantage of this approach is that it allows us to work with physical fields only and to identify the supersymmetries of the corresponding nonrelativistic limit. It has also a universal character, since by taking other representations of the deformed algebra, one can describe any (including $N$-extended and anyonic) representations of the superPoincaré algebra [8]. However, unlike in the superfield formulation [2–4], the supersymmetry algebra is closed here only on-shell. This is a price we pay for a minimality of the supermultiplet involving no auxiliary fields.

Let us note that the supersymmetry studied here has been, since, extended to anyons [22].

Acknowledgements. MSP is indebted to the Laboratoire de Mathématiques et de Physique Théorique of Tours University, and PAH is indebted to the Departamento de Física, Universidad de Santiago de Chile, respectively, for hospitality. Partial support by the FONDECYT (Chile) under the grant 1095027 and by DICYT (USACH), and by Spanish Ministerio de Educación under Project SAB2009-0181 (sabbatical grant of MSP), is acknowledged. MV has been supported by CNRS postdoctoral grant (contract number 87366). We are grateful to S. Deser, R. Jackiw and J. Lukierski for correspondence. We thank to P.M. Zhang for careful reading of our draft.

References

[1] S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” Annals Phys. 140 (1982) 372; “Three-dimensional massive gauge theories,” Phys. Rev. Lett. 48 (1982) 975.

[2] W. Siegel, “Unextended Superfields In Extended Supersymmetry,” Nucl. Phys. B 156 (1979) 135.

[3] R. Jackiw and S. Templeton, “How Superrenormalizable Interactions Cure Their Infrared Divergences,” Phys. Rev. D 23 (1981) 2291.

[4] J. F. Schonfeld, “A Mass Term For Three-Dimensional Gauge Fields,” Nucl. Phys. B 185 (1981) 157.

[5] P. K. Townsend, K. Pilch, P. van Nieuwenhuizen, “Selfduality In Odd Dimensions,” Phys. Lett. 136B (1984) 38, Addendum-ibid.137B(1984) 443; S. Deser and R. Jackiw, ““Self-duality” of topologically massive gauge theories,” Phys. Lett. 139B (1984) 371.

[6] S. Deser and J. H. Kay, “Topologically Massive Supergravity,” Phys. Lett. B 120 (1983) 97.

[7] J-M. Lévy-Leblond, “Non-relativistic particles and wave equations,” Comm. Math. Phys. 6(1967) 286.

[8] P. A. Horvathy, M. S. Plyushchay and M. Valenzuela, “Bosons, fermions and anyons in the plane, & supersymmetry,” Annals Phys. 325 (2010) 1931-1975, [arXiv:1001.0274]. See also M. S. Plyushchay, “R deformed Heisenberg algebra, anyons and $d = (2+1)$ supersymmetry,” Mod. Phys. Lett. A12 (1997) 1153 [arXiv:hep-th/9705034].
[9] J. P. Gauntlett, J. Gomis and P. K. Townsend, “Supersymmetry and the physical-phase-space formulation of spinning particles,” *Phys. Lett. B* **248** (1990) 288; J. A. de Azcarraga and D. Ginestar, “Nonrelativistic limit of supersymmetric theories,” *Journ. Math. Phys.* **32** (1991) 3500.

[10] M. Leblanc, G. Lozano and H. Min, “Extended superconformal Galilean symmetry in Chern-Simons matter systems,” *Annals Phys.* **219** (1992) 328 [arXiv:hep-th/9206039]; C. Duval and P. A. Horvathy, “On Schrödinger superalgebras,” *Journ. Math. Phys.* **35** (1994) 2516 [hep-th/0508079].

[11] A. O. Barut, “Conformal group → Schrödinger group → dynamical group – the maximal kinematical group of the massive Schrödinger particle,” *Helv. Phys. Acta* **46** (1973) 496; P. Havas, and J. Plebański, “Conformal extensions of the Galilei group and their relation to the Schrödinger group,” *J. Math. Phys.* **19** (1978) 482; M. Henkel and J. Unterberger, “Schrödinger invariance and space-time symmetries,” *Nucl. Phys. B* **660** (2003) 407 [arXiv:hep-th/0302187].

[12] E. P. Wigner, “Do the equations of motion determine the quantum mechanical commutation relations?” *Phys. Rev.* **77** (1950) 711; L. M. Yang, “A Note on the quantum rule of the harmonic oscillator,” *Phys. Rev.* **84** (1951) 788. For further developments see, e.g., M. A. Vasiliev, “Higher spin algebras and quantization on the sphere and hyperboloid,” *Int. J. Mod. Phys. A* **6** (1991) 1115; M. S. Plyushchay, “Deformed Heisenberg algebra, fractional spin fields and supersymmetry without fermions,” *Annals Phys.* **245** (1996) 339 [arXiv:hep-th/9601116]; “Deformed Heisenberg algebra with reflection,” *Nucl. Phys. B* **491** (1997) 619 [arXiv:hep-th/9701091]; “Hidden nonlinear supersymmetries in pure parabosonic systems,” *Int. J. Mod. Phys. A* **15** (2000) 3679 [arXiv:hep-th/9903130].
[16] G. Tuynman, W. Wiegerinck, “Central extensions in Physics,” *J. Geom. Phys.* 4 (1987) 2007; G. Marmo, G. Morandi, A. Simoni and E. C. G. Sudarshan, “Quasi-invariance and central extensions,” *Phys. Rev.* D 37 (1988) 2196.

[17] R. Jackiw, “Introducing scaling symmetry,” *Phys. Today* 25 (1) (1972) 23; C. R. Hagen, “Scale and conformal transformations in Galilean-covariant field theory,” *Phys. Rev.* D 5 (1972) 377; U. Niederer, “The maximal kinematical symmetry group of the free Schrödinger equation,” *Helv. Phys. Acta* 45 (1972) 802. The two extra conserved quantities $\mathcal{D}$ and $\mathcal{K}$ were already known to Jacobi in 1842, see, e.g., C. Duval and P. A. Horvathy, “Non-relativistic conformal symmetries and Newton-Cartan structures,” *J. Phys.* A 42 (2009) 465206 [arXiv:0904.0531], which also provides a rather exhaustive list of references.

[18] C. Duval, P. A. Horvathy and L. Palla, “Spinors in non-relativistic Chern-Simons electromagnetism,” *Ann. Phys.* (N. Y.) 249 (1996) 265 [hep-th/9510114]; “Spinor vortices in non-relativistic Chern-Simons theory,” *Phys. Rev.* D 52 (1995) 4700 [hep-th/9503061].

[19] P. A. Horvathy, “Dynamical (super)symmetries of monopoles and vortices,” *Rev. Math. Phys.* 18 (2006) 329 [arXiv:hep-th/0512233]; A. Anabalon and M. S. Plyushchay, “Interaction via reduction and nonlinear superconformal symmetry,” *Phys. Lett.* B 572 (2003) 202 [arXiv:hep-th/0306210].

[20] F. Correa, M. A. del Olmo and M. S. Plyushchay, “On hidden broken nonlinear superconformal symmetry of conformal mechanics and nature of double nonlinear superconformal symmetry,” *Phys. Lett.* B 628 (2005) 157 [arXiv:hep-th/0508223].

[21] M. Valenzuela, “Hidden (super)symmetry of the Schrödinger equation,” arXiv:0912.0789.

[22] P. A. Horvathy, M. S. Plyushchay and M. Valenzuela, “Supersymmetry between Jackiw-Nair and Dirac-Majorana anyons,” *Phys. Rev.* D 81 (2010) 127701 [arXiv:1004.2676].