The Feigin-Fuks construction of irreducible lowest-weight Virasoro representations is reviewed using physics terminology. The procedure consists of two steps: constructing invariants and applying them to the Fock vacuum. We attempt to generalize this construction to the diffeomorphism algebra in higher dimensions. The first step is straightforward, but the second is difficult, due to the appearance of infinite Schwinger terms. This might be avoided by imposing constraints on the fields, which should be of the recently discovered conformal type. The resulting representations are reminiscent of quantum gravity.

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1. Introduction

Conformal field theory applied to two-dimensional phase transitions is one of the most powerful theories in physics.\textsuperscript{1,2} A striking success is the explanation of universality, i.e. the phenomenon that similar but different models have exactly the same critical exponents. Namely, because the simplest consistent continuum theories fall into a discrete series, a small change in the model must either result in a large change in the universal behavior, or in no change at all.

Universality is seen experimentally also in three dimensions, and it is built into the renormalization group approach to critical phenomena.\textsuperscript{3} It would therefore be desirable to have a theory with the same predictive power as conformal field theory also in higher dimensions. At first sight, this appears impossible, because the conformal group in two dimensions is intimately tied to the existence of complex numbers, which are inherently two-dimensional. However, if the relevant group is enlarged already in two dimensions, this larger group may have a generalization to higher dimensions. There is only one viable candidate, namely the diffeomorphism group in \(N\) dimensions.

It can also be argued on purely mathematical grounds that any sensible object transforms as a representation of the diffeomorphism group, i.e. that it is generally covariant. From a passive point of view, a diffeomorphism is simply a coordinate transformation, so representations of the diffeomorphism group have a meaning irrespective of the choice of coordinate system. In the language of differential geometry, they are intrinsic objects. In fact, differential geometry\textsuperscript{4,5} deals with the properties of such objects, and consequently that entire subject can be described as diffeomorphism group representation theory.

The diffeomorphism group is also essential to quantum gravity, in approximately the same way as \(SU(2)\) is important to the quantum theory of spin. Its representation theory thus seems to be an important subject to study, and it has recently attracted some interest by physicists.\textsuperscript{6–12} For simplicity, we limit ourselves to the infinitesimal form of the diffeomorphism group, i.e. the Lie algebra of vector fields in \(N\) dimensions, \(\text{Vect}(N)\). This notation makes sense because we are only interested in local aspects; the only important parameter is the dimension of space.

The basic question underlying this work is the following. How do quantum fields transform under arbitrary coordinate transformations? In one complex dimension, the answer is well known: quantum fields transform as unitary, irreducible, lowest-weight representations of the Virasoro algebra, and it therefore seems to be a reasonable assumption that quantum fields in higher dimensions likewise transform as unitary, irreducible, lowest-weight representations of \(\text{Vect}(N)\), extended by some Schwinger terms (anomalies). In principle, this statement would amount to a classification of inequivalent quantum fields. Since critical exponents arise as dilatation eigenvalues, and a dilatation is certainly a diffeomorphism, we would also obtain a classification of higher-dimensional critical exponents, which would be very important in statistical physics.

Unfortunately, there is a slight complication. The classification of the above-mentioned class of representations has not been achieved except in one dimension, and we are in fact not aware of any non-trivial higher-dimensional example. In one real dimension, the algebra has no non-trivial unitary representations, but if the dimension is complex the situation
is more interesting. We can use the monomial basis

\[ L_m = z^{m+1} \frac{d}{dz}, \]  

which leads to the Witt algebra

\[ [L_m, L_n] = (n - m)L_{m+n}. \]  

(1.1)

However, we must recall that (1.2) arose as the infinitesimal form of the diffeomorphism group close to the origin. If the generators should act as infinitesimal transformations, they must in particular not be infinite in some neighborhood of the origin, including the origin itself. Hence any generator with a pole at the origin should be discarded. From (1.1) it is then clear that relevant algebra must be restricted to the following “amputated” Witt algebra (algebra of holomorphic vector fields),

\[ [L_m, L_n] = (n - m)L_{m+n}, \quad m, n \geq -1. \]  

(1.3)

The amputated algebra has many interesting representations, because it is a subalgebra of the Virasoro algebra for any value of \( c \). In the Virasoro algebra,

\[ [L_m, L_n] = (n - m)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n}, \]  

(1.4)

the central extension only appears in brackets where either \( m \) or \( n \) is less than \(-2\), and hence any Virasoro representation yields a representation of (1.3) by restriction.

An important technical point is that the Verma module must be inverted, compared to the usual convention in the physics literature. Let \( |h\rangle \) be the state characterized by

\[ L_0 |h\rangle = h |h\rangle, \quad L_{-1} |h\rangle = 0. \]  

(1.5)

The Verma module then has the basis

\[ \ldots L_3^{p_3}L_2^{p_2}L_1^{p_1} |h\rangle, \]  

with only finitely many \( p_k \) non-zero. This module can be extended to a Virasoro representation by

\[ L_{-m} |h\rangle = 0, \quad \text{for all } -m < 0. \]  

(1.6)

Although mathematically isomorphic, the inversion of the Verma module has important physical consequences. Because \( L_{-1} \) is the translation operator, the ground state \( |h\rangle \) is translationally invariant. This is natural if we want to identify it with a state of uniform incipient magnetization. On the other hand, \( |h\rangle \) is not conformally invariant, but it is conformally covariant by (1.6). This should be contrasted to the opposite convention, which is prevailing in physics, with a ground state with is not translationally invariant. It is very difficult for us to picture such a ground state.
So where did the central extension go? The answer is that it resurfaces when we consider the singular transformations $L_m$, $m \leq -2$. Classically, (1.3) can obviously extended to negative $m$ just by lifting the restriction, but in the quantum case a surprise turns up in the form of a non-zero central extension. Moreover, we are really interested in unitary representations, which need an involution to be defined. The natural definition $L_m^\dagger = L_{-m}$ is consistent with the grading, but contrary to the situation for finite-dimensional Lie algebras, there are many inequivalent ways to define the bracket between the original and involuted generators. These inequivalent involutions are parametrized by the value of $c$. Similarly, we expect that Schwinger terms appear when we consider singular transformations in higher dimensions.

The present work makes some progress towards irreducible representations of $Vect(N)$, at least by generalizing half of Feigin’s and Fuks’ construction of irreducible Virasoro representations. As explained in Sec. 2, their work proceeds in two steps: constructing invariants in an exterior algebra, and relating them to lowest-weight modules in a Fock space. The first step can readily be generalized to higher dimensions, which is done in Sec. 4. The second step is more difficult, because the anomalies tend to diverge, but we think that this problem can be resolved in a multi-field Fock space by imposing constraints on the fields. Indeed, all possible covariant equations have to be satisfied, or else the representation is decomposable. Some possible constraints on tensor fields are listed in Sec. 5. In Sec. 6 a similar list is given for the recently discovered conformal fields, which in a sense are more natural than tensor fields, and an algebraic structure not completely dissimilar to canonical quantum gravity arises. Because this structure appears in a mathematically very natural context, namely pure representation theory, we suggest that we might have found a viable approach to consistent quantum gravity. At least, we do know that $Vect(N)$ plays an important role in quantum gravity, namely as the symmetry algebra.

2. Discrete Virasoro series

The Witt algebra $Vect(1)$,  

$$[L_m, L_n] = (n - m)L_{m+n},$$  

(2.1)

can be realized as vector fields on the circle or on the line,

$$L_m = -ie^{imx} \frac{d}{dx} = -ie^{imx} \partial,$$

(2.2)

where $m \in \mathbb{Z}$ on the circle and $m \in \mathbb{R}$ on the line. The Witt algebra also has the realization (1.1), with $m \in \mathbb{Z}$. Since we are working on the level of linear representations in this paper, the imaginary unit can be absorbed into the definition of $m$,  

$$L_m = e^{mx} \partial,$$

(2.3)

which saves many explicit references to $i$. All cases are summarized by $m \in \Lambda$, where $\Lambda$ is an abelian group, possibly continuous or imaginary.
An important $\text{Vect}(1)$ module is the primary field $\mathbf{T}(\lambda, w)$. The action is given by

$$[L_m, \psi(x)] = -e^{mx}(\partial + \lambda m + w)\psi(x)$$

(2.4)

In the Fourier transformed basis, defined by

$$\psi(x) = \sum_{n \in \Lambda} \psi_n e^{-nx},$$

(2.5)

(2.4) takes the form

$$[L_m, \psi_n] = -(\partial + \lambda m + w)\psi_{m+n} \equiv (n + (1 - \lambda)m - w)\psi_{m+n}$$

(2.6)

where the derivative in momentum space simply is multiplication by a constant: $\partial \psi_n = -n\psi_n$, $\partial \psi_{m+n} = -(m + n)\psi_{m+n}$. The adjoint representation is clearly $\mathbf{T}(2, 0)$. The parameter $w$ can be shifted to $w + s$ by relabelling the Fourier components as $\psi_n \rightarrow \psi_{n+s}$. It is thus only defined modulo $\Lambda$, and particularly for the plane-wave basis on the line it can be eliminated altogether. However, on the circle and for the basis (1.1), it is an important parameter.

Another natural module is the connection (one-dimensional Christoffel symbol),

$$[L_m, \Gamma(x)] = -e^{mx}((\partial + m)\Gamma(x) - m^2),$$

(2.7)

transforming as the primary field $\mathbf{T}(1, 0)$ apart from an inhomogeneous term. The covariant derivative is a map $\mathbf{T}(\lambda, w) \rightarrow \mathbf{T}(\lambda + 1, w)$, which depends on the connection. It is explicitly given by

$$\nabla \psi = (\partial + \lambda \Gamma + w)\psi$$

(2.8)

It is not difficult to prove by direct computation that (2.7) is consistent and that (2.8) defines a module map.

The above modules extend by means of Leibniz’ rule to tensor products of $\psi$ and $\Gamma$. The field $\psi(x)$ can be either bosonic or fermionic;

$$[\psi(x), \psi(y)] = 0, \quad \text{or} \quad \{\psi(x), \psi(y)\} = 0,$$

(2.9)

respectively. The connection $\Gamma(x)$ must be bosonic, because the inhomogeneous term in its transformation law is so. A basis for these modules is given by all states of the form

$$\psi(x_1) \ldots \psi(x_p)\Gamma(x_{p+1}) \ldots \Gamma(x_{p+q}),$$

(2.10)

modulo the relations above. The number operators $N_\psi$ and $N_\Gamma$ satisfy

$$[N_\psi, \psi(x)] = \psi(x), \quad [N_\Gamma, \Gamma(x)] = \Gamma(x).$$

(2.11)

$N_\psi$ commutes with every $L_m$ and thus the module (2.10) decomposes into sectors with a fixed number of $\psi$’s. However, $N_\Gamma$ does not commute with $\text{Vect}(1)$ due to the inhomogeneous term in (2.7).
We will use a terminology inspired by physics and refer to (2.4) as the transformation law of a particle and (2.10) as a state with \( p \) fundamental particles and \( q \) gauge bosons. It is natural to try to construct a molecule, i.e. a composite particle, in this multiparticle state, and to ask what the corresponding values of \( \lambda \) and \( w \) are.

Because of locality, a molecule must be constructed out of \( \psi(x) \), \( \Gamma(x) \), and finitely many of their derivatives at the same point \( x \). Using the fact that the covariant derivative maps primary fields to primary fields, the most general local expression reads

\[
\Phi^P(x) \equiv \Phi^{(p_0,p_1,p_2,\ldots)}(x) = \psi(x)^{p_0}(\nabla\psi(x))^{p_1}(\nabla^2\psi(x))^{p_2}\ldots
\]  

(2.12)

where only finitely many \( p_i \) are non-zero. Each application of the covariant derivative increases the conformal weight \( \lambda \) by one, and thus if \( \psi(x) \in T(\lambda, w) \), \( \nabla^k\psi(x) \in T(\lambda+k, w) \). Hence

\[
-[L_m, \Phi^P(x)] = \sum_k \psi(x)^{p_0}\ldots p_k e^{m\lambda}(\partial + (\lambda + k)m + w)\nabla^k\psi(x)(\nabla^k\psi)^{p_{k-1}}(x)\ldots
\]

\[
= e^{m\lambda}\left(\partial\Phi^P(x) + (\lambda m + w) \sum_k p_k \Phi^P(x) + \sum_k kmp_k \Phi^P(x)\right)
\]

\[
= e^{m\lambda}(\partial + A(P)(\lambda m + w) + B(P)m)\Phi^P(x),
\]

(2.13)

where \( A(P) = \sum_k p_k \) and \( B(P) = \sum_k kp_k \). We conclude that the molecule \( \Phi^P(x) \in T(A(P)\lambda + B(P), A(P)w) \).

Continuing the physics terminology, \( \nabla^k\psi \) can be thought of as the \( k:th \) shell of the molecule and \( p_k \) is the corresponding occupation number. If we think of \( \lambda \) as an energy, we see that every particle in the \( k:th \) shell has the same energy \( (k + \lambda) \), and hence the shells play the role of energy levels. The ground state of a \( p \)-particle molecule is the state with the lowest possible energy, i.e. the shells are filled from the bottom up with totally \( p \) particles. Other combinations are referred to as excited states.

If \( \psi \) is bosonic the occupation numbers can be arbitrary non-negative integers. The ground state is thus \( \psi(x)^{p} \in T(p\lambda, pw) \), and it is special because it does not involve the gauge boson. We have thus constructed a map

\[
S^pT(\lambda, w) \longrightarrow T(p\lambda, pw),
\]

(2.14)

where \( S^p \) denotes the \( p:th \) symmetric power.

The fermionic case is more interesting, because the occupation numbers can only be zero or one. The \( p \)-fermion ground state \( \Psi^{(p)}(x) \) is thus defined by \( p_k = 1 \) if \( k < p \) and \( p_k = 0 \) if \( k \geq p \). Clearly,

\[
A(P) = \sum_{k=0}^{p-1} 1 = p \quad \text{and} \quad B(P) = \sum_{k=0}^{p-1} k = \binom{p}{2} = \frac{p(p-1)}{2}.
\]

(2.15)

The ground state is again independent of the gauge bosons. This is not as manifest as in the bosonic case, but it follows from the anti-commutation relations. For example, the two-fermion molecule

\[
\psi\nabla\psi = \psi(\partial + \lambda\Gamma + w)\psi = \psi\partial\psi + (\lambda\Gamma + w)\psi^2,
\]

(2.16)
and the second term vanishes because $\psi^2 = 0$. An analogous argument shows that

$$\Psi^{(p)}(x) \equiv \psi(x) \nabla \psi(x) \nabla^2 \psi(x) \ldots \nabla^{p-1} \psi(x) = \psi(x) \partial \psi(x) \partial^2 \psi(x) \ldots \partial^{p-1} \psi(x), \quad (2.17)$$

which thus defines a map

$$\Lambda^p T(\lambda, w) \longrightarrow T(p\lambda + \binom{p}{2}, pw), \quad (2.18)$$

where $\Lambda^p$ denotes the $p$:th exterior power.

Feigin and Fuks\textsuperscript{13} call this map an operator in general position. To see that our expression is the same as theirs requires some extra work. Up to a constant factor, (2.17) can be rewritten as

$$\Delta(\partial_1, \ldots, \partial_p)(\psi(x_1) \ldots \psi(x_p)) \bigg|_{x_1 = \ldots = x_p = x}, \quad (2.19)$$

where $\partial_k$ is the derivative with respect to $x_k$ and the Vandermonde determinant is

$$\Delta(\partial_1, \ldots, \partial_p) = \prod_{1 \leq i < j \leq p} (\partial_i - \partial_j). \quad (2.20)$$

We have e.g.

$$(\partial_1 - \partial_2)\psi(x_1)\psi(x_2) \bigg|_{x_1 = x_2 = x} = \partial(\psi(x)\psi(x)) - 2\psi(x)\partial\psi(x), \quad (2.21)$$

and the first term vanishes because $\psi$ is a fermion.

For certain values of $\lambda$ and $w$ the size of the module can be decreased by factoring out an invariant, which commutes with the entire Witt algebra. This is most easily explained in the Fourier basis (2.6). $\Theta_n$ is invariant if $\Theta \in T(\lambda = 1, w = n)$, because then

$$[L_m, \Theta_n] = (n - w)\Theta_{m+n} \equiv 0. \quad (2.22)$$

Comparing with (2.18), we see that $\Psi^{(p)}_n$ is invariant provided that

$$p\lambda + \frac{p(p-1)}{2} = 1, \quad pw = n. \quad (2.23)$$

This equation selects infinitely many points on a curve in $(\lambda, w)$-space, which is defined by regarding $p$ as a continuous parameter. For a given value of $\lambda$, (2.23) is a quadratic equation for $p$ which generically has two complex solutions,

$$p_\pm = -(\lambda - \frac{1}{2}) \pm \sqrt{(\lambda - \frac{1}{2})^2 + 2}, \quad (2.24)$$
We can ask if \((\lambda, w)\) lies on the intersection of two curves of the type (2.23); the answer is positive provided that there are two integers \(n_+\) and \(n_-\) such that

\[ w = \frac{n_+}{p_+} = \frac{n_-}{p_-}. \tag{2.25} \]

This Diophantine equation has a solution if

\[ p_+ = \sqrt{\frac{-2n_-}{n_+}}, \quad p_- = -\sqrt{\frac{-2n_+}{n_-}}, \tag{2.26} \]

\((n_+n_- < 0)\), or

\[ (\lambda - \frac{1}{2})^2 = -\frac{(n_+ + n_-)^2}{2n_+n_-}. \tag{2.27} \]

Let us now explain how this is connected to the discrete series of Virasoro representations. It terms of the fermion \(\psi\) and its conjugate \(\psi^\dagger\), which satisfy canonical anticommutation relations (CAR),

\[
\{\psi^\dagger(x), \psi(y)\} = \delta(x - y), \quad \{\psi^\dagger(x), \psi^\dagger(y)\} = \{\psi(x), \psi(y)\} = 0.
\]

\[
\{\psi_m^\dagger, \psi_n\} = \delta_{m+n}, \quad \{\psi_m^\dagger, \psi_n^\dagger\} = \{\psi_m, \psi_n\} = 0.
\tag{2.28}
\]

We can think of \(\psi(x)\) as the anti-commuting coordinates of some vector space \(V_\psi\) and of \(\psi^\dagger(x)\) as the corresponding derivative. The enveloping algebra of (2.28) is then the algebra of all differential operators on this space, i.e. a fermionic Weyl algebra. \(Vect(1)\) acts on this Weyl algebra as the vector field

\[ L_m = \int dx \, e^{mx} \, \psi^\dagger(x)(\partial + \lambda m + w)\psi(x) = \sum_s (-s + \lambda m + w)\psi^\dagger_{m-s}\psi_s. \tag{2.29} \]

This means that we have a representation of the Witt algebra on the fermionic Weyl algebra. If the Witt algebra were finite-dimensional, it would inherit a representation for any representation of (2.28), but this is not quite true in the infinite-dimensional setting.

We have on purpose not specified the domain of the summation variable \(s\). Since \(m\) is an integer it is clear that \(s\) must run over numbers which differ by integers, i.e. \(s \in \mathbb{Z} + \alpha\) for some real number \(\alpha\). It is natural to demand that \(\alpha = 1/2\), because this choice makes \(\psi\) into a kind of “one-dimensional spinor”. Moreover, for \(\alpha\) integer or half-integer \(\psi\) and \(\psi^\dagger\) appear symmetrically. However, the ultimate reason for this choice is that it yields the correct Kac table below.

The irreducible representation of the CAR is the fermionic Fock module, which is obtained from the enveloping algebra of (2.28) by deleting all strings containing elements of negative degree. Equivalently, we introduce the vacuum \(|0\rangle\), satisfying

\[ \psi_{-s}|0\rangle = \psi_{-s}^\dagger|0\rangle = 0, \quad \text{for all } s > 0. \tag{2.30} \]
We also introduce the shifted vacuum $|q\rangle$, defined by
\[
|q\rangle = \psi_{q-1/2} \cdots \psi_{1/2} |0\rangle,
\] (2.31)
which satisfies
\[
\psi_{q-s} |0\rangle = \psi_{q-s}^\dagger |0\rangle = 0, \quad \text{for all } s > 0.
\] (2.32)
$q$ can also be negative; $\psi$ is then replaced by $\psi^\dagger$ in (2.31).

The inherited $\text{Vect}(1)$ module is defined by
\[
L_m |q\rangle = \sum_{s=q+1/2}^{m+q-1/2} (-s + \lambda m + w) \psi_{m-s}^\dagger \psi_s |q\rangle
\] (2.33)
\[
L_0 |q\rangle = h |q\rangle
\]
\[
L_{-m} |q\rangle = 0,
\]
together with
\[
[L_m, \psi_n] = (n + (1 - \lambda)m - w) \psi_{m+n},
\]
\[
[L_m, \psi_n^\dagger] = (n + \lambda m + w) \psi_{m+n}^\dagger.
\] (2.34)

Because of the infinite dimensionality, (2.33) and (2.34) are actually not representations of the Witt algebra but of the Virasoro algebra (1.4), because our procedure is nothing but the standard normal ordering recipe. By demanding that $L_0$ should have a finite eigenvalue, we have subtracted off an infinite constant. After a straightforward calculation it is found that
\[
c = -2(6\lambda^2 - 6\lambda + 1) = 1 - 12\left(\lambda - \frac{1}{2}\right)^2
\]
\[
2h = (w - q)^2 - (\lambda - \frac{1}{2})^2
\] (2.35)

The Fock module decomposes into modules where the difference between the numbers of $\psi$'s and $\psi^\dagger$'s is fixed, because the fermionic number operator commutes with $L_m$. In a obvious notation we denote by $\psi^q |0\rangle$ the homogeneous component where this difference equals $q$; $|q\rangle$ is the vacuum in this submodule.

A singular vector is an element which satisfies the same conditions as the vacuum. Given the invariant $\Psi_t^{(p)}$ above, we obtain the singular vector by applying it to the vacuum. $\Psi_t^{(p)} |0\rangle$ is hence a singular vector in the module $\psi^p |0\rangle$, and $(\Psi_t^{(p)})^j |0\rangle$ is singular in $\psi^{jp} |0\rangle$. Explicitly
\[
L_{-m}(\Psi_t^{(p)})^j |0\rangle = [L_{-m}, (\Psi_t^{(p)})^j] |0\rangle + (\Psi_t^{(p)})^j L_{-m} |0\rangle = 0 + 0.
\] (2.36)

If $p$ is odd, $\Psi_t^{(p)}$ is fermionic and the singular vector vanishes identically. The interesting case is thus $p$ even, and hence the procedure can be continued to $j$ half-integer. With $j = s/2, s \in \mathbb{N}$, there is a singular vector in $\psi^{sp/2} |0\rangle$, which is characterized by (2.23) and (2.35), where $q = sp/2$. Hence, the module is reducible if
\[
c = 1 - 12\left(\frac{1}{p} - \frac{p}{2}\right)^2
\]
\[
2h = \left(\frac{t - sp}{2}\right)^2 - \left(\frac{1}{p} - \frac{p}{2}\right)^2
\] (2.37)
By eliminating $p$ we obtain Kac’ formula for reducibility of the Verma module. In particular, for the the special values of $\lambda$ given by (2.27),

$$
\begin{align*}
c &= 1 - \frac{6(m-n)^2}{nm}, \\
h &= \frac{(tm - sn)^2 - (m-n)^2}{4mn}
\end{align*}
$$

where $m = n_+, n = -n_-, m, n, s, t \in \mathbb{N}$. This is the discrete series of irreducible Virasoro representations.\(^2\) Eq. (2.38) was obtained from (2.37) by substituting $p = p_+$ from (2.26). If we instead use $p = p_-$, we find the same formula for $h$ but with $t$ and $s$ interchanged. This is a consistency check on the formula.

Our construction can not really be considered as a proof of Kac’ formula, because there is no guarantee that the singular vectors are non-trivial. There are two possibilities to find singular vectors which are not covered by the list (2.38). First we can use fermions whose Fourier components are not half-integers. Second, we could consider the singular vectors $(\Psi_t^{(p)})^{s/k} |0\rangle$ for every $p$ divisible by $k$. Since these generalizations would lead outside Kac’ formula, we conclude that these new singular vectors are trivial.

We can consider more complicated Fock modules by including the gauge boson $\Gamma(x)$ and its canonical conjugate $\Gamma^\dagger(x)$. Their Fourier components must be labelled by integers, and they satisfy

$$
[\Gamma_m, \Gamma^\dagger_n] = \delta_{m+n}, \quad [\Gamma_m, \Gamma_n] = [\Gamma^\dagger_m, \Gamma^\dagger_n] = 0.
$$

The contribution to the $Vect(1)$ generators is

$$
L_m = \sum_{n \in \mathbb{Z}} (-n + m) \Gamma^\dagger_{m-n} \Gamma_n - m^2 \Gamma^\dagger_m,
$$

and the vacuum satisfies

$$
\begin{align*}
\Gamma_{-n} |0\rangle &= \Gamma^\dagger_{-n} |0\rangle = 0, \quad \text{for all } n > 0. \\
\Gamma^\dagger_0 |0\rangle &= 0.
\end{align*}
$$

When (2.40) is applied to the vacuum we find the contribution $c = 2, h = 0$, which should be added to (2.35). In the combined Fock space containing both fermions and gauge bosons, new singular vectors can be constructed by demanding that some of the excited states (2.12) be invariant and applying them to the vacuum. The relevance of these more complicated modules is not clear to us.

Shells can also be defined by using bosons rather than fermions, but this is less interesting. In one dimension bosons and fermions are equivalent,\(^14\) wherefore the Feigin-Fuks construction can be rewritten in terms of bosons; they are essentially the vertex operators.\(^15\) The fermion approach seems nicer because it generalizes to higher dimensions. Moreover, if we seriously believe that $Vect(N)$ is relevant to physics, it is natural to expect that the interesting representations deal with fundamental fermions and gauge bosons, because that is the kind of fundamental particles which are seen experimentally.
3. Kac-Moody algebras

The construction of the previous section immediately suggests a generalization to Kac-Moody algebras. Recall that the Kac-Moody algebra is a central extension of the loop algebra based on a finite-dimensional Lie group $\mathfrak{g}$. The brackets are

$$[J^a_m, J^b_n] = f^{abc} J^c_{m+n} + km\delta^{ab}\delta_{m+n} \tag{3.1}$$

where $f^{abc}$ are the totally skew-symmetric structure constants of $\mathfrak{g}$ and $\delta^{ab}$ is the Killing metric. Because of this metric, there is no need to distinguish between upper and lower indices.

Actually, we are only interested in representations which admit an intertwining action of $\text{Vect}(1)$, because the generators must transform covariantly under arbitrary reparametrizations of the loop. The interesting algebra is thus the semi-direct product $\text{Vect}(1) \ltimes \text{Map}(1, \mathfrak{g})$, where

$$[L_m, J^b_n] = n J^b_{m+n}, \tag{3.2}$$

and its central extensions. The classical representations of (3.1–2) are

$$[J^a_m, \psi^\alpha_n] = -M^a_{\beta\alpha} \psi^\beta_{m+n} \tag{3.3}$$

where $M^a = (M^a_{\beta\alpha})$ are matrices of some finite-dimensional $\mathfrak{g}$ representation, and $\psi^\alpha_n$ transforms as (2.6) under $\text{Vect}(1)$. Alternatively, the transformation law (3.3) can be recast in real space

$$[J^a_m, \psi(x)] = -e^{m\alpha} M^a \psi(x), \tag{3.4}$$

with representation indices suppressed. We denote this representation by $T(\lambda, w) \times M$.

To construct a first-order differential operator we must covariantize the map (2.8) with respect to $\mathfrak{g}$. It is straightforward to check that the gauge connection $\omega^a(x)$, which transforms as

$$[J^a_m, \omega^b(x)] = e^{m\alpha} (f^{abc} \omega^c(x) + m\delta^{ab}) \tag{3.5}$$

and as the $\text{Vect}(1)$ primary field $T(1, 0)$, is defined consistently. The covariant derivative is now the map

$$T(\lambda, w) \times M \longrightarrow T(\lambda + 1, w) \times M$$

$$D\psi(x) = \nabla \psi(x) + \omega^a(x) M^a \psi(x) \tag{3.6}$$

where $\nabla$ is given by (2.8).

A molecule can be constructed from fundamental particles transforming in the $r$-dimensional representation $M$ in the following fashion.

$$\Phi^P(x) = \psi^1(x)^{p_1^0} \cdots \psi^r(x)^{p_r^0} (D\psi^1(x))^{p_1^1} \cdots (D\psi^r(x))^{p_r^1} (D\psi^1(x))^{p_1^2} \cdots (D\psi^r(x))^{p_r^2} \cdots, \tag{3.7}$$

where $P = \{p^\alpha_k : 1 \leq \alpha \leq r, k \geq 0\}$, and only finitely many $p^\alpha_k$ are non-zero. Because there are now $r$ different states with the same energy, we say that all of them belong to the same shell, and $p_k = \sum_{\alpha=1}^r p_k^\alpha$ is the total occupation number of the $k$:th shell. The ground
states are again independent of both the connection $\Gamma(x)$ and the gauge connection $\omega^a(x)$. In the bosonic case we thus have a map

$$S^p(T(\lambda, w) \times M) \longrightarrow T(p\lambda, pw) \times S^p M,$$

and by appropriate symmetrization of the $g$ indices, the image of this map can be restricted to any representation $N \subset S^p M$.

The case of fermions is again more interesting. The occupation numbers can now range between zero (empty shell) and $r$ (full shell). A ground state, which we denote by $\Psi(p, b)(x)$, consists of $pr + b$ fermions, $0 \leq b < r$, where $p$ is the number of full shells and $b$ is the number of fermions in the valence shell. It is clear that a full shell, e.g.

$$\psi^1(x)\psi^2(x)\ldots\psi^r(x)$$

is a gauge singlet, so full shells do not contribute to the gauge representation. Because the gauge bosons factor out in the ground state, we have constructed a map

$$\Lambda^{pr+b}(T(\lambda, w) \times M) \longrightarrow T((pr + b)\lambda + r\left(\frac{p}{2}\right) + bp, (pr + b)w) \times \Lambda^b M.$$ 

By appropriate symmetrization of the $g$ indices, the image of this map can be restricted to any $g$ representation $N \subset \Lambda^b M$.

An invariant must be a $g$ singlet, which is possible only if $b = 0$. $\Psi_n^{(p,0)}$ is an invariant in $\Lambda^{pr}(T(\lambda, w) \times M)$ provided that

$$pr\lambda + r\frac{p(p-1)}{2} = 1, \quad prw = n,$$

for some integer $n$. The discrete series of $\lambda$’s which admit two invariants are again characterized by $n_p = n_{-p}$, i.e.

$$p_+ = \sqrt{-\frac{2n_-}{rn_+}}, \quad p_- = -\sqrt{-\frac{2n_+}{rn_-}},$$

and

$$\left(\lambda - \frac{1}{2}\right)^2 = -\frac{(n_+ + n_-)^2}{2rn_+n_-},$$

where $n_+n_- < 0$. The only difference compared to (2.27) is the factor $r$ in the denominator.

This result can immediately be used to find singular vectors in fermionic Fock modules. Introduce the canonical conjugate of the fermions,

$$\{\psi_{\beta m}^+, \psi_m^\alpha\} = \delta_\beta^\alpha \delta_{m+n}, \quad \{\psi_{am}^\dagger, \psi_{bm}^\dagger\} = \{\psi_m^\alpha, \psi_n^\beta\} = 0,$$

and let $|0\rangle$ be the vacuum as in Sec. 2. Other vacua are constructed from this vector by filling the states immediately above the Fermi level. Since there is room for $r$ fermions with the same momentum, the following expression is a gauge singlet

$$|q\rangle = \psi_{q-1/2}^1 \ldots \psi_{q-1/2}^r \ldots \psi_{1/2}^1 \ldots \psi_{1/2}^r |0\rangle.$$
In the notation of Sec. 2, this is the vacuum of the module $\psi^{qr}|0\rangle$. Other vacua carrying non-trivial representations of $\mathfrak{g}$ can of course also be constructed, but we limit ourselves to the class (3.15).

It can be checked that the following generators satisfy Virasoro and Kac-Moody algebras, respectively.

$$L_m |q\rangle = \sum_{s=q+1/2}^{m+q-1/2} (-s + \lambda m + w) \psi^\dagger_{\alpha,m-s} \psi^\alpha_s |q\rangle$$

$$L_0 |q\rangle = h |q\rangle$$

$$L_{-m} |q\rangle = 0$$

$$J^a_m |q\rangle = \sum_{s=q+1/2}^{m+q-1/2} M^\alpha_\beta \psi^\dagger_{\alpha,m-s} \psi^\beta_s |q\rangle$$

$$J^a_0 |q\rangle = J^a_{-m} |q\rangle = 0.$$  

(3.16)

The parameters are given by

$$c = r \left( 1 - 12 \left( \lambda - \frac{1}{2} \right)^2 \right)$$

$$k = Q_M$$

$$2h = r \left( (w - q)^2 - \left( \lambda - \frac{1}{2} \right)^2 \right)$$

(3.17)

where $Q_M$ is the value of the quadratic Casimir in the representation $M$, i.e. $\text{tr} \ M^a M^b = Q_M \delta^{ab}$.

The singular vector in $\psi^{jpr}|0\rangle$, $p$ even, is obtained by applying the invariant molecule with $p$ full shells $j$ times to the vacuum, i.e. $\left( \Psi^{(p,0)} \right)^j |0\rangle$. This singular vector lies in the module with vacuum $|jp\rangle$. We can formally continue this to the case that $2j \in \mathbb{Z}$. If we set $j = s/2$, $q = sp/2$, the reducible modules are

$$c = r \left( 1 - 12 \left( \frac{1}{rp} - \frac{p}{2} \right)^2 \right)$$

$$k = Q_M$$

$$2h = r \left( \left( \frac{t}{pr} - \frac{sp}{2} \right)^2 - \left( \frac{1}{rp} - \frac{p}{2} \right)^2 \right)$$

(3.18)

In particular in the case that there are two different singular vectors for the same value of $c$,

$$c = r - \frac{6(m-n)^2}{mn}$$

$$k = Q_M$$

$$h = \frac{(tm - sn)^2 - (m-n)^2}{4mn}$$

(3.19)
where \( m = n_+ \) and \( n = -n_- \). The only difference compared to the previous section, apart from the appearance of the Kac-Moody algebra, is that the central charge has been increased by \( r - 1 \).

Just as in the previous section, we can not be sure that the singular vectors constructed above are non-trivial. Moreover, other possibilities arise in a module whose vacuum is not a gauge singlet. The purpose of this section is thus not to present an exhaustive list of all analogs of Kac’ table for every Kac-Moody algebra, but rather to illustrate that molecules yield a simple method for concretely constructing singular vectors in quite general situations. This will be further emphasized in the next section, where we turn to higher dimensions.

4. Invariants in several dimensions

We now generalize the concept of a molecule to more than one dimension. In a plane-wave basis, the generators of \( \text{Vect}(N) \) are

\[
L^\mu(m) = e^{m \cdot x} \partial^\mu, \tag{4.1}
\]

where \( x = (x_1, \ldots, x_N) \), \([\partial^\mu, x^\nu] = \delta^\mu_\nu\), and \( m = (m^1, \ldots, m^N) \) belongs to an \( N \)-dimensional imaginary lattice \( \Lambda \). The algebra thus reads

\[
[L^\mu(m), L^\nu(n)] = n^\mu L^\nu(m + n) - m^\nu L^\mu(m + n). \tag{4.2}
\]

Globally, this is the algebra of vector fields on an \( N \)-dimensional torus, but since any two manifolds of the same dimension are locally diffeomorphic, (4.2) applies locally to any \( N \)-dimensional manifold.

An important class of \( \text{Vect}(N) \) representations are tensor fields (or densities), which are constructed from \( gl(N) \) representations as follows. Assume that \( \{T^\mu_{\nu}\}_{\mu, \nu=1}^N \) satisfies \( gl(N) \), i.e.

\[
[T^\mu_{\sigma}, T^\nu_{\tau}] = \delta^\mu_\tau T^\nu_{\sigma} - \delta^\nu_{\tau} T^\mu_{\sigma}. \tag{4.3}
\]

Then it is easy to check that

\[
L^\mu(m) = e^{m \cdot x} \left( \partial^\mu + w^\mu + m^\sigma T^\mu_{\sigma} \right) \tag{4.4}
\]

satisfies \( \text{Vect}(N) \), where \( w^\mu \) is a constant vector which is defined modulo \( \Lambda \). Hence we have a \( \text{Vect}(N) \) representation for each \( gl(N) \) representation. From a \( gl(N) \) tensor with \( p \) upper and \( q \) lower indices and conformal weight \( \lambda \), we obtain the \( \text{Vect}(N) \) module \( T^p_q(\lambda, w) \) with action

\[
[L^\mu(m), \psi^{\nu_1 \cdots \nu_p}_{\tau_1 \cdots \tau_q}(x)] = -e^{m \cdot x} \left( (\partial^\mu + w^\mu + \lambda m^\mu)\psi^{\nu_1 \cdots \nu_p}_{\tau_1 \cdots \tau_q}(x) \right. \\
+ \sum_{i=1}^p m^\nu_i \psi^{\nu_1 \cdots \nu_p}_{\tau_1 \cdots \tau_q}(x) - \sum_{j=1}^q \delta^\mu_{\tau_j} m^\sigma \psi^{\nu_1 \cdots \nu_p}_{\tau_1 \cdots \sigma \cdots \tau_q}(x) \Big). \tag{4.5}
\]
We write $T^p_q(\lambda) = T^p_q(\lambda, 0)$. This equation clearly reduces to (2.4) in one dimension, with $\lambda$ replaced by $\lambda + p - q$. In the Fourier basis,

$$[L^\mu(m), \psi^{\nu_1\cdots\nu_p}(n)] = (n^\mu - w^\mu + (1 - \lambda)m^\mu)\psi^{\nu_1\cdots\nu_p}(m + n)$$

$$- \sum_{i=1}^p \mu g^{\mu\nu_i} \psi^{\nu_1\cdots\nu_{i-1}\nu_{i+1}\cdots\nu_p}(m + n) + \sum_{j=1}^q \delta^\mu_{\tau_j} m^\sigma \psi^{\nu_1\cdots\nu_{\tau_j}\cdots\nu_p}(m + n).$$

(4.6)

In particular, we have for a scalar field

$$[L^\mu(m), \psi(n)] = (n^\mu - w^\mu + (1 - \lambda)m^\mu)\psi(m + n).$$

(4.7)

The adjoint is $T^p_0(1)$.

It is sometimes convenient to describe $\text{Vect}(N)$ more invariantly. Define

$$L(f_\mu) = \sum_{m \in \Lambda} f_\mu(m)L^\mu(m)$$

(4.8)

for each function $f_\mu(x)$ with Fourier coefficients $f_\mu(m)$. The algebra then takes the form

$$[L(f_\mu), L(g_\mu)] = L(f_\nu \partial^\nu g_\mu - g_\nu \partial^\nu f_\mu) = L(f_\nu \partial^\nu g_\mu) - L(g_\nu \partial^\nu f_\mu).$$

(4.9)

Tensor fields are given by

$$L(f_\mu) = f_\mu(x)\partial^\mu + f_\mu(x)w^\mu(x) + \partial^\sigma f_\mu(x)\Gamma^\mu_{\sigma\tau},$$

(4.10)

where the function $w^\mu(x)$ satisfies

$$\partial^\mu w^\nu(x) - \partial^\nu w^\mu(x) + [w^\mu(x), w^\nu(x)] = 0.$$  

(4.11)

This zero-curvature condition has the solution $w^\mu(x) = u^{-1}(x)\partial^\mu u(x)$. Substituting this solution into (4.10), we obtain

$$L(f_\mu) = u^{-1}(x)(f_\mu(x)\partial^\mu + \partial^\sigma f_\mu(x)\Gamma^\mu_{\sigma\tau})u(x)$$

(4.12)

On the torus we can make the non-trivial choice $u(x) = \exp(w \cdot x)$, which gives back (4.4). An advantage of the formulation (4.12) is that it suggests a generalization of $w$ when $f_\mu(x)$ is expanded in another set of basis functions. This is important because Kac’ formula (2.37) is essentially a relation between the parameters $\lambda$ and $w$. E.g., in three dimensions, $f_\mu(x) = f_\mu(r, \theta, \varphi)$ can be expanded in the basis $r^n Y_{lm}(\theta, \varphi), n \in \mathbb{Z}$, and a non-trivial function is $u(x) = r^w, 0 < w < 1$. For the rest of this section we limit ourselves to the torus.

The connection is a central extension of the representation $T^2_1(0)$, with transformation law

$$[L^\mu(m), \Gamma^\sigma_{\tau}(x)] = e^{m \cdot x} \left( - \partial^\mu \Gamma^\sigma_{\tau}(x) - m^\sigma \Gamma^\mu_{\tau}(x) - m^\nu \Gamma^\tau_{\sigma\nu}(x) + \delta^\mu_\sigma m^\tau \Gamma^\nu_{\sigma\tau} + \delta^\mu_\tau m^\sigma m^\nu \right),$$

(4.13)
and the covariant derivative is the map
\[ T_q^p(\lambda, w) \rightarrow T_q^{p+1}(\lambda, w) \]
\[ \nabla^\nu \psi(x) = (\partial^\nu + w^\nu)\psi(x) + \Gamma^\nu_{\sigma}(x)[T^\tau_{\sigma}, \psi(x)] \]

(4.14)

For a scalar field,
\[ \nabla^\nu \psi(x) = (\partial^\nu + w^\nu + \lambda \Gamma^\nu_{\sigma}(x))\psi(x). \]

(4.15)

Just as in Sec. 2 we can now build a molecule out of fundamental particles and their covariant derivatives at the same point \( x \). For simplicity we limit ourselves to scalar fields and fermions. A fermion in the \( k \)-th shell enters through the expression
\[ \nabla^\nu_1 \ldots \nabla^\nu_k \psi \in T_0^k(k\lambda, kw) \]

(4.16)

which depends on the connection. The ground state with the \( p \) lowest shells being filled is
\[ \Psi(p)(x) = \psi(x) \nabla^\nu_1 \psi(x) \ldots \nabla^\nu_N \psi(x) \nabla^\sigma_{11} \nabla^\tau_{11} \psi(x) \nabla^\sigma_{12} \nabla^\tau_{12} \psi(x) \ldots \]
\[ = \psi(x) \prod_{i=1}^N \nabla^\nu_i \psi(x) \prod_{j=1}^N \prod_{k=1}^N \nabla^\sigma_{jk} \nabla^\tau_{jk} \psi(x) \ldots \]

(4.17)

In this ground state all references to the connection vanishes due to anti-symmetry, and therefore we can replace covariant derivatives with usual ones. The expression (4.16) is replaced by
\[ \partial^\nu_1 \ldots \partial^\nu_k \psi, \]

which is manifestly symmetric in \( \nu_1 \ldots \nu_k \), and the ground state becomes
\[ \Psi(p)(x) = \psi(x) \prod_{i=1}^N \partial^\nu_i \psi(x) \prod_{j=1}^N \prod_{k=1}^N \partial^\sigma_{jk} \partial^\tau_{jk} \psi(x) \ldots \]

(4.19)

The number of different states in the \( k \)-th shell is thus equal to the number of symmetric combinations of \( k \) indices which can take \( N \) different values, i.e.
\[ n_k = \binom{N - 1 + k}{k} = \binom{N - 1 + k}{N - 1}. \]

(4.20)

In the molecule with \( p \) full shells, (4.20) equals the occupation number for the \( p \) lowest shells. Each fermion in the \( k \)-th shell contributes with \( k \) upper indices, wherefore
\[ \Psi(p)(x) \in T_0^{B_N(p)}(A_N(p)\lambda, A_N(p)w), \]

(4.21)

where
\[ A_N(p) = \sum_{k=0}^{p-1} n_k = \sum_{k=0}^{p-1} \binom{N - 1 + k}{N - 1} = \binom{N - 1 + p}{N}, \]

\[ B_N(p) = \sum_{k=0}^{p-1} kn_k = \sum_{k=0}^{p-1} k \binom{N - 1 + k}{N - 1} = N \sum_{j=0}^{p-2} \binom{N + j}{N} = N \binom{N - 1 + p}{N + 1}. \]
Since the total number of fermions in the \( p \)-shell molecule is \( A_N(p) \), we have constructed a map
\[
\Lambda^{A_N(p)} T^0_0(\lambda, w) \longrightarrow T^B_N(p)(A_N(p)\lambda, A_N(p)w). \quad (4.23)
\]

The range of this map is actually a submodule, characterized by certain symmetries. E.g., the three-shell ground state in two dimensions,
\[
\Psi^{\nu_1 \nu_2 \sigma_{11} \tau_{11} \sigma_{12} \tau_{12} \sigma_{22} \tau_{22}} = \psi \partial^{\nu_1} \psi \partial^{\nu_2} \psi \partial^{\sigma_{11}} \psi \partial^{\sigma_{12}} \psi \partial^{\sigma_{22}} \psi \partial^{\tau_{11}} \psi \partial^{\tau_{12}} \psi \partial^{\tau_{22}} \psi.
\]
(4.24)
is skew in \( \nu_1 \) and \( \nu_2 \), symmetric in \( \sigma_{ij} \) and \( \tau_{ij} \), and skew under interchange of any pairs \( \sigma_{ij} \tau_{ij} \leftrightarrow \sigma_{kl} \tau_{kl} \).

Some more work is needed before (4.19) can be used to construct an invariant. Consider the submodule \( \Omega_p \subset T^0_0(1) \) consisting of totally skew tensors with \( p \) lower indices. There is a sequence of maps
\[
\Omega_N \xrightarrow{\tilde{d}_N} \Omega_{N-1} \xrightarrow{\tilde{d}_{N-1}} \ldots \xrightarrow{\tilde{d}_1} \Omega_0 \xrightarrow{\tilde{d}_0} C,
\]
given by
\[
(\tilde{d}_p \phi(x))_{\nu_1 \ldots \nu_{p-1}} = \partial^\sigma \phi(x)_{\nu_1 \ldots \nu_{p-1} \sigma} \quad (p \geq 1)
\]
and
\[
\tilde{d}_0 \phi = \phi(n = 0).
\]
Moreover, \( \tilde{d}_{p-1} \tilde{d}_p = 0 \). The elements in \( \Omega_p \) can be thought of as \( p \)-dimensional integral operators,
\[
\phi_{\nu_1 \ldots \nu_p}(n) = \int \ldots \int_{\nu_p} dx_{\nu_1} \ldots dx_{\nu_p} e^{inx},
\]
(4.27)
and \( \tilde{d}_p \) is the boundary map. There is also a dual sequence of differential forms.

The important map for our purposes is the last one. In the Fourier basis, an element in \( \Omega_0 = T^0_0(1) \) satisfies
\[
[L^\mu(m), \Theta(n)] = n^\mu \Theta(m + n),
\]
(4.28)
and it is clear that \( \Theta(0) \) is an invariant; it transforms trivially and therefore we can consistently set it equal to zero. This can be slightly generalized by a shift in \( n \).

\[
[L^\mu(m), \Theta(n)] = (n^\mu - w^\mu) \Theta(m + n),
\]
(4.29)
which shows that \( \Theta(n) \) is an invariant in \( T^0_0(1, n) \).

The object (4.19), which only has upper indices with certain symmetries, can be transformed into a scalar provided that we introduce a field with lower indices. The resulting invariant depends on this new field, which is undesirable unless it can be introduced in a canonical manner. However, there is a unique object with \( N \) skew lower indices, namely the permutation symbol \( \epsilon_{\nu_1 \ldots \nu_N} \), which is defined by \( \epsilon_{12 \ldots N} = 1 \) and total skewness. The permutation symbol can be considered as a constant tensor field \( \epsilon_{\nu_1 \ldots \nu_N}(x) \in \Omega_N \subset T^0_N(1) \).

To see this, consider
\[
[L^\mu(m), \epsilon_{\nu_1 \ldots \nu_N}(x)] = -e^{inx}((\partial^\mu + m^\mu)\epsilon_{\nu_1 \ldots \nu_N}(x) - \sum_{i=1}^N \delta^\mu_{\nu_i} m^\sigma \epsilon_{\nu_1 \ldots \sigma \ldots \nu_N}(x)).
\]
(4.30)
If $\mu = \nu_1$, say, the only term that survives in the last sum is $m^\sigma \epsilon_{\sigma \nu_2 \ldots \nu_N}$, which cancels the first term because the permutation symbol is non-zero only when $\sigma = \nu_1$. In one dimension the permutation symbol becomes the one-component vector $\epsilon_\nu = \epsilon_1 = 1$, transforming as

$$[L_m, \epsilon] = -e^{mx}((\partial + m)\epsilon - m\epsilon) = 0. \quad (4.31)$$

Dually, we may regard the permutation symbol as a tensor field with upper indices and weight $-1$; $\epsilon_{\nu_1 \ldots \nu_N}(x) \in T^N_0(-1)$.

The desired scalar field is now formed by contracting (4.19) with the permutation symbol in a way which respects the symmetries. From the two-dimensional three-shell ground state (4.24), we obtain

$$\epsilon_{\nu_1 \nu_2} \epsilon_{\sigma_1 \sigma_2 \sigma_3} \epsilon_{\tau_1 \tau_2 \tau_3} \psi \partial^{\nu_1} \psi \partial^{\nu_2} \psi \partial^{\sigma_1} \partial^{\tau_1} \psi \partial^{\sigma_2} \partial^{\tau_2} \psi \partial^{\sigma_3} \partial^{\tau_3} \psi. \quad (4.32)$$

A little thought reveals that this procedure is well-defined for arbitrary molecules with full shells. There are $B_N(p)$ upper indices to contract. Since each $\epsilon_{\nu_1 \ldots \nu_N}$ has $N$ lower indices, we need a total of $B_N(p)/N$ permutation symbols, each contributing unity to the parameter $\lambda$ and nothing to $w$. Hence we have a map

$$\Lambda^{A_N(p)} T^0_0(\lambda, w) \longrightarrow T^0_0(A_N(p)\lambda + \frac{1}{N} B_N(p), A_N(p)w). \quad (4.33)$$

The map (4.33) defines an invariant provided that

$$A_N(p)\lambda + \frac{1}{N} B_N(p) = 1, \quad A_N(p)w^\mu = n^\mu, \quad (4.34)$$

for some $n \in \Lambda$. Equivalently,

$$\lambda - \frac{1}{N+1} = \frac{1}{A_N(p)} - \frac{p}{N+1}, \quad w^\mu = \frac{n^\mu}{A_N(p)}, \quad (4.35)$$

because $B_N(p) = N(p-1)A_N(p)/(N+1)$. Note that every solution to the second equation is parallel to $w^\mu$. If we introduce $\kappa = (N+1)\lambda - 1$, the first equation takes the form

$$(p + \kappa) \left(\frac{N + p - 1}{N}\right) = N + 1, \quad (4.36)$$

i.e.

$$p(p + 1) \ldots (p + N - 1)(p + \kappa) = (N + 1)!. \quad (4.37)$$

This is a polynomial equation of degree $N + 1$, which generically has $N + 1$ complex solutions $p_i$. The maximal number of invariants in this module is thus $N + 1$, which is reached if all $p_i$ are real and different and satisfy the simultaneous Diophantine equations

$$w^\mu \propto \frac{n_i}{A_N(p_i)} = \frac{n_j}{A_N(p_j)} \quad i, j = 1, \ldots N + 1. \quad (4.38)$$
Instead of solving these solutions, we will at once generalize to the case that we have an action of the the gauge algebra \( \text{Map}(N, \mathfrak{g}) \). As in Sec. 3, there are additional brackets

\[
\begin{align*}
[J^a(m), J^b(n)] &= f^{abc} J^c(m + n), \\
[L^\mu(m), J^b(n)] &= n^\mu J^b(m + n).
\end{align*}
\] (4.39)

If the field \( \psi(x) \) takes values in an \( r \)-dimensional representation of \( \mathfrak{g} \), the entire analysis above is unchanged except that the RHS of (4.37) is replaced by \((N + 1)!/r\), because each shell can be filled with \( r \) times as many fermions as before.

For \( N = 1 \), \( p_i = n_i x \) and (4.37) reads

\[
n_1 n_2 x^2 = -\frac{2}{r}, \quad (n_1 + n_2)x = -\kappa
\] (4.40)

with the solution

\[
n_1 n_2 < 0, \quad x = \sqrt{-\frac{2}{r n_1 n_2}}, \quad \kappa^2 = -\frac{2(n_1 + n_2)^2}{r n_1 n_2}.
\] (4.41)

This is the discrete series found in (3.13).

For \( N = 2 \),

\[
\begin{align*}
1 + \kappa &= -(p_1 + p_2 + p_3), \\
\kappa &= p_1 p_2 + p_1 p_3 + p_2 p_3, \\
\frac{6}{r} &= p_1 p_2 p_3,
\end{align*}
\] (4.42)

where the \( p_i \) are related by

\[
\frac{p_i(p_i + 1)}{2} = n_i x, \quad n_i \in \mathbb{Z}.
\] (4.43)

As a last example, the specialization of (4.37) to \( N = 3 \) reads

\[
\begin{align*}
3 + \kappa &= -\sum_{i=1}^{4} p_i, \\
2 + 3\kappa &= \sum_{1 \leq i < j \leq 4} p_i p_j, \\
2\kappa &= -\sum_{1 \leq i < j < k \leq 4} p_i p_j p_k, \\
\frac{24}{r} &= -p_1 p_2 p_3 p_4,
\end{align*}
\] (4.44)

where the \( p_i \) are related by

\[
\frac{p_i(p_i + 1)(p_i + 2)}{6} = n_i x, \quad n_i \in \mathbb{Z}.
\] (4.45)
It is not clear to us that any solution exists when $N \geq 2$, but if it does, it generalizes the discrete series of $\lambda$’s. Even if it is impossible to find the maximal number of invariants, we can give up a few of these equations and still obtain a discrete set of exceptionally small modules, by factoring out the remaining invariants.

The shell construction can be generalized in several directions. Instead of starting with a scalar field, we could use a vector field $\psi^\nu(x)$. Because $\nu$ can take $N$ different values, each shell can be occupied by $N$ times as many fermions as in the scalar case. Explicitly, the $p$-shell molecule has the form

$$\prod_{i=1}^{N} \psi^{\nu_i} \prod_{j=1}^{N} \prod_{k=1}^{N} \nabla^{\sigma_j} \psi^{\tau_k} \ldots = \prod_{i=1}^{N} \psi^{\nu_i} \prod_{j=1}^{N} \prod_{k=1}^{N} \partial^{\sigma_j} \psi^{\tau_k} \ldots.$$  \hspace{1cm} (4.46)

The molecule can be contracted with the permutation symbol to obtain an invariant. Since it now takes $NA_N(p)$ fermions to fill $p$ shells, we have the maps

$$\Lambda^{NA_N(p)} T^1_{0}(\lambda, w) \rightarrow T^0_{NB_N(p)+NA_N(p)}(NA_N(p)\lambda, NA_N(p)w) \rightarrow T^0_{0}(A_N(p)(N\lambda + 1) + B_N(p), NA_N(p)w)$$ \hspace{1cm} (4.47)

Similarly, if we start from a contravariant vector field,

$$\Lambda^{NA_N(p)} T^0_{2}(\lambda, w) \rightarrow T^{NB_N(p)}_{NA_N(p)}(NA_N(p)\lambda, NA_N(p)w) \rightarrow T^0_{0}(A_N(p)(N\lambda - 1) + B_N(p), NA_N(p)w)$$ \hspace{1cm} (4.48)

Another generalization deals with the recently discovered $Vect(N)$ representations which we called conformal fields, because they transform naturally under the “conformal” subalgebra $sl(N+1) \subset Vect(N)$. The generators, in a form analogous to (4.12), are

$$L(\mu) = u^{-1}(x) \left( f_\mu(x) \partial^\mu + (\partial^B + k^B)f_\mu(x)T^A_B + cx A \partial^B \partial^\mu f_\mu(x)T^A_B \right) u(x),$$ \hspace{1cm} (4.49)

where $A, B = 0, 1, \ldots, N$ are “conformal” indices, which take on $N + 1$ different values. $T^A_B$ satisfies the algebra $gl(N+1)$,

$$[T^A_B, T^C_D] = \delta^A_C T^B_D - \delta^B_C T^A_D,$$ \hspace{1cm} (50.50)

and

$$\partial^A \equiv (\partial^0, \partial^\mu) = (-x \cdot \partial, \partial^\mu),$$

$$x_B \equiv (x_0, x_\nu) = (1, x_\nu),$$

$$k^A \equiv (k^0, k^\nu) = (1, 0).$$ \hspace{1cm} (50.51)

Moreover, $c$ is a c-number parameter and $u(x)$ is the same arbitrary function as in (4.12); on the torus, $u(x) = \exp(w \cdot x)$.

Denote the module with $p$ upper and $q$ lower conformal indices and conformal weight $\lambda$ by $C^p_q(\lambda, c, w)$; when $w = 0$ we write $C^p_q(\lambda, c)$. E.g., a conformal vector transforms as

$$[-L(f_\mu), \psi^B(x)] = f_\mu(x)(\partial^\mu + w^\mu(x))\psi^B(x) + \lambda \partial^\mu f_\mu(x)\psi^B(x) + (\partial^B + k^B)f_B(x)\psi^\mu(x) + c(\partial^B \partial^\mu f_\mu(x))x_C^\nu \psi^C(x).$$ \hspace{1cm} (4.52)
Scalar fields are a special case of both tensor fields and conformal fields \((T^A_B = \lambda \delta^A_B)\). It can be shown that there is a map

\[
\mathbf{C}_0^0(\lambda, c, w) \to \mathbf{C}_0^1(\lambda, c, w)
\]

\[
(d\psi)^A = (\partial^A + w^A + \frac{\lambda}{c} k^A)\psi
\]  

(4.53)

However, the ground state molecules constructed with help of this map are essentially the same as above. To see this, consider the molecule with one full shell.

\[
\psi \partial^0 \psi \partial^1 \psi \ldots \partial^N \psi = -x^{\nu} \psi \partial^{\nu} \psi \partial^1 \psi \ldots \partial^N \psi \equiv 0,
\]  

(4.54)

because \(\partial^{\nu} \psi\) appears twice for every value of \(\nu\). If we start with non-scalar conformal fields, new molecules can be found. Invariants are then obtained by contracting with the \((N + 1)\)-dimensional permutation symbol, considered as the totally skew, constant conformal field

\[
\varepsilon_{A_0 A_1 \ldots A_N}(x) \in \mathbf{C}_N^{0}(1, c),
\]  

(4.55)

or, dually,

\[
\varepsilon^{A_0 A_1 \ldots A_N}(x) \in \mathbf{C}_N^{N+1}(-1, c).
\]  

(4.56)

That this definition is consistent is established analogously to (4.30). It should be noted that the value of \(c\) is arbitrary, because it will only enter the transformation law multiplied by the factor

\[
x_B \partial^B \partial^{\mu} f_\mu(x) \varepsilon_{A_0 A_1 \ldots A_N}(x) \equiv 0.
\]  

(4.57)

For example, there are maps

\[
\Lambda^{(N+1)A_N(p)} C_0^1(\lambda, c, w)
\]

\[
\to \mathbf{C}_0^{(N+1)B_N(p)+(N+1)A_N(p)}((N + 1)A_N(p)\lambda, c, (N + 1)A_N(p)w)
\]

\[
\to \mathbf{C}_0^{0}(A_N(p)((N + 1)\lambda + 1) + \frac{(N + 1)B_N(p)}{N}, c, (N + 1)A_N(p)w).
\]  

(4.58)

Using the fact that \(\mathbf{C}_0^0(\lambda, c, w) = T_0^0(\lambda, w)\), it is now straightforward to check when these maps give invariants.

5. **Fock modules**

The first part of the Feigin-Fuks procedure, the construction of invariants, was generalized to higher dimensions in the previous section. The second step consists of applying these invariants to the Fock vacuum in order to obtain singular vectors. Unfortunately, the construction of Fock modules is technically and conceptually difficult for \(N > 1\), because infinities arise that can not be removed by normal ordering. The problem is essentially that subspaces of fixed energy are infinite-dimensional. In this section we discuss this issue.
and indicate a way out of the mathematical difficulty; however, it is not the physically
relevant solution.

We first explain why it is not possible to find representations on the unconstrained
Fock space. Let the fermion \( \psi(x) \) and its canonical conjugate \( \psi^\dagger(x) \) satisfy CAR,
\[
\{ \psi^\dagger(x), \psi(y) \} = \mathbf{1} \delta(x-y) \quad \{ \psi^\dagger(x), \psi^\dagger(y) \} = \{ \psi(x), \psi(y) \} = 0,
\]
\[
\{ \psi^\dagger(m), \psi(n) \} = \mathbf{1} \delta(m+n) \quad \{ \psi^\dagger(m), \psi^\dagger(n) \} = \{ \psi(m), \psi(n) \} = 0.
\]
(5.1)

Any tensor, conformal or internal indices are suppressed and \( \mathbf{1} \) is the unit matrix.

As in Sec. 2, we consider \( \psi(x) \) as coordinates and \( \psi^\dagger(x) \) as derivatives on a vector
space \( V_\psi \), and the enveloping algebra of (5.1) is the fermionic Weyl algebra on this space.
\( \text{Vect}(N) \) acts (by commutation) as the first-order differential operators
\[
L^\mu(m) = \int d^N x \, e^{m \cdot x} \psi^\dagger(x) \left( (\partial^\mu + w^\mu) \psi(x) + m^\sigma [T^\mu_\sigma, \psi(x)] \right)
= \sum_s \left( (-s^\mu + w^\mu) \psi^\dagger(m-s) \psi(s) + m^\sigma \psi^\dagger(m-s) [T^\mu_\sigma, \psi(s)] \right),
\]
(5.2)
where the sum runs over \( s \in \Lambda_\psi \), where \( \Lambda_\psi \) is obtained from the lattice \( \Lambda \) by a constant
translation (analogous to \( \frac{1}{2} \) in one dimension). If \( \psi \in \text{T}_p^q(\lambda, w) \), we read off from (5.2)
that its conjugate transforms as \( \psi^\dagger \in \text{T}_q^p(1-\lambda, -w) \). The problem of representing \( \text{Vect}(N) \)
has thus essentially been reduced to the apparently simpler problem of representing
the Weyl algebra. This intuition is false, and even more so than in one dimension, due to the
infinite dimensionality of \( \text{Vect}(N) \).

To construct a Fock module, we must choose a polarization which separates the elements
into two sets: raising and lowering operators. One possibility is to divide the lattice
into two parts: \( \Lambda_\psi = \Lambda_\psi^{(+)} \oplus \Lambda_\psi^{(-)} \). We write \( m > 0 \) (\( m < 0 \)) if \( m \in \Lambda_\psi^{(+)} \) (\( m \in \Lambda_\psi^{(-)} \)). The decomposition defines an order of the lattice points
provided that \( m, n > 0 \) implies that \( m+n > 0 \) and \( -m < 0 \). For example, introduce a constant vector \( t_\mu \), say \( t_\mu = (0, \ldots, 0, 1) \),
and proclaim that \( m > 0 \) (\( m < 0 \)) if \( t_\mu m^\mu > 0 \) (\( t_\mu m^\mu < 0 \)). This procedure can be ex-
pressed physically as follows. Introduce a Hamiltonian
\[
H = t_\mu L^\mu(0) = L^N(0),
\]
(5.3)
that generates rigid translations in the “time” direction \( N \). Every Fourier component of
the field has a definite energy, because
\[
[ H, \psi(n) ] = t_\mu n^\mu \psi(n).
\]
(5.4)
The vector space associated to the field thus has a decomposition into energy eigenspaces,
\[
V_\psi = \bigoplus_{j=-\infty}^{\infty} V_\psi^{(j)} = V_\psi^{(+)} \oplus V_\psi^{(-)}.
\]
(5.5)
\[\text{Page 22}\]
A Fock module is now defined by a vacuum state \( |0\rangle \), defined by the relations
\[
\psi(-n) |0\rangle = \psi^\dagger(-n) |0\rangle = 0, \quad \text{for all } n > 0 \tag{5.6}
\]
This is a representation of the Weyl algebra, but the inherited \( Vect(N) \) representation has an infinite central extension. For a scalar field, the relevant calculation is
\[
[L^\mu(m), L^\nu(-m)] |0\rangle = \sum_{0<s<m} (-s^\mu + \lambda m^\mu + w^\mu)(s^\nu + (\lambda - 1)m^\nu - w^\nu) |0\rangle \tag{5.7}
\]
and this sum is infinite except in one dimension, because there are infinitely many points perpendicular to the “time” direction. An infinite extension is definitely not acceptable, so some additional idea has to be invoked.

The Hamiltonian (5.3) was introduced on purely mathematical grounds to generate a \( \mathbb{Z} \)-gradation by energy. A more natural Hamiltonian is the scaling operator
\[
H = x^\mu \partial^\mu,
\]
which corresponds to a polarization by the radial component. Express an arbitrary vector field in spherical coordinates.
\[
L(f_\mu) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{\mu, nlm} L^{\mu}_{nlm} r^n Y_{lm}(\theta, \phi). \tag{5.8}
\]
We have specialized to three dimensions, but (5.8) is readily generalized to arbitrary dimension by considering hyper-spherical harmonics. The energy is now given by the radial power \( n \), and the algebra \( Vect(N) \equiv \mathcal{L} \) has a decomposition in homogeneous components
\[
\mathcal{L} = \bigoplus_{n=-\infty}^{\infty} \mathcal{L}^{(n)} = \mathcal{L}^{(+)} + \mathcal{L}^{(-)}, \tag{5.9}
\]
where \( \{L^{\mu}_{nlm} : l \in [0, \infty], m \in [-l, l], \mu \in [1, N]\} \) is a basis for \( \mathcal{L}^{(n)} \). The vacuum obeys \( \mathcal{L}^{(-)} |0\rangle = 0 \), i.e.
\[
L^{\mu}_{nlm} |0\rangle = 0, \tag{5.10}
\]
for every \( n < 0 \), and for every \( \mu, l \) and \( m \).

\( \mathcal{L}^{(+)} \) is the algebra of vector fields which are non-singular in some neighborhood of the origin; because of general covariance, any sensible object must be a representation of it. Schematically,
\[
[\mathcal{L}^{(+)}, \mathcal{L}^{(+)}] \subset \mathcal{L}^{(+)}. \tag{5.11}
\]
Similarly, the elements of \( \mathcal{L}^{(-)} \) are non-singular close to infinity.
\[
[\mathcal{L}^{(-)}, \mathcal{L}^{(-)}] \subset \mathcal{L}^{(-)}. \tag{5.12}
\]
However, \( \mathcal{L}^{(+)} \) and \( \mathcal{L}^{(-)} \) are nowhere simultaneously non-singular. Commutators involving both subalgebras can not be trusted to have their classical form. Rather, we expect that
\[
[\mathcal{L}^{(+)}, \mathcal{L}^{(-)}] \subset \mathcal{L}^{(+)} \oplus \mathcal{L}^{(-)} \oplus \text{anomaly}. \tag{5.13}
\]
In one dimension, the anomaly (Schwinger term) is simply the central extension of the Virasoro algebra. The presence of this anomaly does not really violate general covariance, because a singular vector field is not truly an infinitesimal coordinate transformation.

We naively expect to obtain a module of the type above by expanding the fermion field in spherical coordinates,

$$\psi(x) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \psi_{nlm} r^n Y_{lm}(\theta, \varphi), \quad (5.14)$$

and similar for $\psi^\dagger(x)$, and inserting these expressions into (5.2). However, it is easy to see that we pick up an infinite anomaly for precisely the same reason as with the Fourier polarization (5.3): every subspace of fixed energy is infinite-dimensional,

$$\dim V^{(n)}_\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} 1 = \infty. \quad (5.15)$$

The same kind of infinities arise also in other bases, and in other algebras that act on $N$-dimensional space, e.g. $Map(N, g)$. The problem is thus generic. If we regard the presence of an infinite anomaly as a fundamental inconsistency, as we do, the conclusion must be that these quantized fields do not make sense except in one dimension. In the remainder of this paper we discuss various means to deal with this problem.

Physically, we might expect to remove the infinities by renormalization. Normal ordering, which has a natural role in representation theory, is equivalent to mass renormalization, but in physics there is also wave-function renormalization. However, it is not clear to us how renormalization ideas can be implemented within our formalism. Moreover, we are dealing with the diffeomorphism group, which is intimately linked to unrenormalizable gravity.

Related ideas been investigated by Mickelsson and collaborators in the context of current algebras, extending the Pressley-Segal approach to loop groups to higher dimensions.\cite{17-20} Eq. (5.2) defines an embedding of $Vect(N)$ in $GL(V_\psi)$, the group of linear transformations of $V_\psi$. One can define various restrictions of $GL(V_\psi)$, and attempt to represent the algebra on these restrictions.

In Ref. 11 we proposed to introduce fundamental bosons, because they yield infinities of opposite sign. By matching fermionic and bosonic degrees of freedom, the infinities would cancel. This is not a satisfactory solution, because the bosonic and fermionic sectors decouple, and hence such a module would decompose into a direct sum of two unacceptable modules. If one could introduce interaction terms in the $Vect(N)$ generators, e.g. a term which destroys a boson and creates two fermions, the different sectors would not decouple, and the matching argument might deserve further attention. However, we have found no way to achieve this.

As an alternative to the approaches above, we propose to circumvent the problem in a mathematically very simple manner. Eq. (5.2) defines an action of $Vect(N)$ on the vector space $V_\psi$, but this vector space is too large. By imposing some constraints on the fields, we can restrict the action to some submanifold (the constraint manifold).
The constraints must evidently transform covariantly under arbitrary diffeomorphisms, to make the submanifold stable under the action of $\text{Vect}(N)$. If such a constraint is found, we automatically get an action on the function and Weyl algebras on the constraint manifold, and these algebras might be small enough to admit a $\mathbb{Z}$-gradations by finite-dimensional subspaces. If so, we can reduce the representations further by introducing a vacuum that annihilates all negative components. The basic idea is hence that a free field has too many degrees of freedom, but the constraints cut down the size of the field enough to make it manageable.

It should be noted that the idea of imposing constraints is mathematically quite inevitable, because our goal is to construct irreducible representations. Consider some set of fields as a $\text{Vect}(N)$ module. If it is possible to write down a non-trivial covariant equation, the module decomposes into the solution space and its complement, both of which are submodules. Hence, unless one of these submodules is trivial (and the other is the original module), we have managed to reduce our module. For example, a metric can be considered as the $\text{Vect}(N)$ module $T^2_0(0)$, which can be decomposed into one submodule with and one without curvature. From this point of view, every conceivable equation must be satisfied.

To illustrate that constraints may lead to finite anomalies, we consider a fermion satisfying the massless Dirac equation. There is an obstacle here because $\text{Vect}(N)$ has no natural spinor representation. To deal with this we introduce the frame algebra $\text{Map}(N,\text{so}(N))$ as in (4.39).

\[ [J^{ij}(m), J^{kl}(n)] = \eta^{ik} J^{jl}(m+n) - \eta^{il} J^{jk}(m+n) - \eta^{jl} J^{ik}(m+n) + \eta^{ll} J^{lk}(m+n). \] (5.16)

There is no distinction between upper and lower frame indices, due to the constant metric $\eta_{ij}$, and we will therefore mix them freely. A spinor is defined to transform as the scalar $T^0_0(0)$ under $\text{Vect}(N)$ and as a spinor under the frame algebra. The total algebra has thus been enlarged to $\text{Vect}(N) \times \text{Map}(N,\text{so}(N))$, and we want to construct representations of this larger algebra.

The massless Dirac equation reads

\[ \gamma^i \gamma^i(x) (\partial^\mu + \omega^\mu(x)) \psi(x) = 0 \] (5.17)

Here $\gamma^i$ are the constant gamma matrices, $\sigma_{ij} = [\gamma_i, \gamma_j]/4i$, and $e^i_\mu(x)$ is a vielbein field, which is a frame vector transforming as $T^0_1(0)$ under $\text{Vect}(N)$. Further, the vielbein has an inverse

\[ e^i_\mu(x) e^\mu_j(x) = \delta^i_j, \quad e^i_\nu(x) e^\mu_i(x) = \delta^\mu_\nu, \] (5.18)

and $\omega^\mu(x) = \omega^i_\mu(x) \sigma^{ij}$ is the spin connection, which is defined in terms of the vielbein by

\[ \partial^{[\mu} e_{i]}^\nu + \omega^i_{\mu} e^\nu_j = 0. \] (5.19)

To solve (5.17) for $\psi(x)$ we adopt a special coordinate system in which

\[ e^i_\mu(x) = \delta^i_\mu, \quad \omega^i_\mu(x) = 0 \] (5.20)
It must be emphasized that this choice is not covariant and thus not really acceptable, but it serves to illustrate our point. Using (5.20), we have $\gamma_i \partial^i \psi(x) = 0$, which as usual implies that the components satisfy the Laplace equation. Specializing to spherical coordinates in three dimensions, the general solution reads

$$\psi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \psi_{lm}^{(+)} r^l + \psi_{lm}^{(-)} r^{-l-1} \right) Y_{lm}(\theta, \varphi). \tag{5.21}$$

The gradation is given by

$$\deg \psi_{lm}^{(+)} = l, \quad \deg \psi_{lm}^{(-)} = -l - 1. \tag{5.22}$$

and the dimension of the homogeneous subspace

$$\dim V_{\psi}^{(l)} = \dim V_{\psi}^{(-l-1)} = \sum_{m=-l}^{l} 1 = 2l + 1 \tag{5.23}$$

is finite. Similarly, $\dim V_{\psi}^{(l)} \propto |l|^{N-2}$ for large $|l|$ in $N$ dimensions. The coefficients $\psi_{lm}^{(\pm)}$ are coordinates on the constraint manifold and the conjugate variables $\psi_{lm}^{(\pm)\dagger}$,

$$\{ \psi_{lm}^{(\pm)\dagger}, \psi_{l'm'}^{(\mp)} \} = \delta_{ll'} \delta_{mm'}, \tag{5.24}$$

are the corresponding tangential derivatives. Since the constraint manifold is stable, we can express the $\text{Vect}(N)$ generators in terms of these conjugate variables, essentially as in (5.2). We can now define the vacuum by

$$\psi_{lm}^{(-)} |0\rangle = \psi_{lm}^{(-)\dagger} |0\rangle = 0. \tag{5.25}$$

A serious flaw in the above argument is that the equation (5.20) is not covariant, wherefore the fermions can not invariantly be expanded in solutions to the flat Laplace equation. However, for continuity reasons we expect the expansion (5.21) to be almost correct in coordinate systems close to (5.20). A smooth deformation of (5.21) is

$$\psi(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \psi_{lm}^{(+)} F_{lm}(r, \theta, \varphi; e^i) + \psi_{lm}^{(-)} F_{l-1,m}(r, \theta, \varphi; e^i) \right), \tag{5.26}$$

where the Laplace eigenfunctions $F_{lm}(r, \theta, \varphi; e)$ depend on the vielbein, as does the vacuum $|0; e^i(\mu)(x)\rangle$. In the case of a diagonal vielbein,

$$F_{lm}(r, \theta, \varphi; \delta^i_\mu) = r^i Y_{lm}(\theta, \varphi) \tag{5.27}$$

Both the eigenfunctions and the vacuum must transform non-trivially under $\text{Vect}(N)$ because of this dependence, but the number of eigenfunctions, and thus the dimensions of
the spaces \( V^{(l)}_\psi \), can not change continuously. This makes it plausible that these dimensions remain finite for all vielbeins, at least in some suitable class, and consequently the anomaly does not diverge. We have thus defined, modulo these technical assumptions about continuity, a \( Vect(N) \) module which is partially of lowest-weight type.

An analogous module has been studied by Mickelsson and Rajeev in the gauge case, where the role of the vielbein is played by a gauge potential.\(^{17-18}\) It appears that they have encountered serious problems (non-unitarizability) with this approach. It is quite clear that this module is not what we want: the vielbein field is still classical, i.e. not of lowest-weight type. To remedy this, consider the Weyl algebra built out of both vielbeins and fermions and their conjugates. To (5.1) we add the canonical commutation relations,

\[
[e^\mu_j(x), e^{i\nu}(y)] = \delta^\mu_i \delta^j_\nu (x-y), \quad [e^\mu_j(x), e^{i\nu}(y)] = [e^\mu_i(x), e^{i\nu}(y)] = 0. \tag{5.28}
\]

The vielbein contribution to \( Vect(N) \),

\[
L^\mu_{\psi}(m) = \int d^N x \ e^{m-x} e^{i\psi}(x) \left( \partial^\mu e^{i\nu}(x) + m^\nu e^\mu_i(x) \right), \tag{5.29}
\]

is added to (5.2) to get the total generator.

We now introduce the Dirac equation (5.17) and consider the Weyl algebra on the constraint manifold. It is a \( Vect(N) \) module, with action given in principle by (5.2) and (5.29). Unfortunately, the description of this module is severely complicated by the fact that the constraint is non-linear in the vielbein. Previously, the constraint was linear in \( \psi(x) \), although it was parametrized by the classical field \( e^\mu_i(x) \), and the submanifold was a family of vector spaces. We could therefore write down the general solution as a sum of eigenfunctions. This is no longer the case.

Another complication is that the quantized vielbein will give rise to the same kind of infinite anomalies as the unconstrained fermion did above. To see this, note that the constraint is preserved by the fermionic number operator,

\[
[N_\psi, \text{constraint}] = \text{constraint}. \tag{5.30}
\]

We can thus limit our attention to modules with a fixed number of fermions. The zero-fermion sector is completely unconstrained because the Dirac equation is an identity there. The analysis that led to (5.7) remains unchanged except that the fermion is replaced by a boson, and an infinite anomaly arises.

In view of the previous development it is now clear how to proceed: introduce another constraint that cuts down the number of degrees of freedom, to obtain an appropriate \( \mathbb{Z} \)-gradation. The constraint must transform covariantly and it can only depend on the vielbein since we are in the zero-fermion sector. Moreover, it should be as restrictive as possible without making the vielbein trivial. The simplest conceivable equations thus involve the curvature, or some of its derivates, such as the Ricci or Einstein tensors.

The strategy for constructing more complicated modules is now clear: find a suitable set of constraints to impose on some Weyl algebra. A constraint should preferably be as simple as possible, for maximal reduction of the degrees of freedom. On the other hand,
it must not be so restrictive as to trivialize any field. We presumably strike a sensible balance by considering first- and second-order differential equations.

For a scalar field $\psi(x)$ with non-zero conformal weight $\lambda$, the Dirac equation has to be modified, because the gradient of $\psi(x)$ is no longer a tensor field. The modified Dirac equation reads

$$\gamma_k e^k_\mu(x)(\partial^\mu + \lambda \Gamma^\sigma_\mu(x) + \omega_i^{j \mu}(x)\sigma_{ij})\psi(x) = iM\psi(x),$$  \hspace{1cm} (5.31)

where

$$\Gamma^\sigma_\mu(x) = e^\sigma_i(x)(\partial^\mu \delta_i^j + \omega_i^{j \mu}(x))e^j_\tau(x)$$ \hspace{1cm} (5.32)

is the Christoffel symbol and we have introduced a non-zero constant mass $M$. Eq. (5.31) is manifestly covariant both under $\text{Vect}(N)$ and $\text{Map}(N, so(N))$ and is hence a meaningful equation.

The number of fundamental fields can be increased further by the introduction of a Yang-Mills field. Every field must then transform consistently under some additional gauge algebra (4.39), in addition to the frame algebra $so(N)$ (or $so(N+1)$ for conformal fields). The equations must be covariantized with respect to this extra gauge algebra, which amounts to the introduction of the gauge potential in the Dirac equation and in the energy-momentum tensor. If the Yang-Mills field were free it would generate an infinite Schwinger term by the same mechanism as before. This is avoided by demanding that the field strength satisfy the Yang-Mills equation with some current constructed from the fermions.

6. Conformal fields

The maximally constrained modules of the previous section were mathematically appealing: they were of lowest-weight type, presumably had a finite anomaly, and could not be further reduced by imposing equations. However, they are not the modules of interest in physics. Classical physics fields obey certain equations of motion in space-time, but the fields and momenta on a space-like surface are freely specifiable. On the other hand, a mathematically sensible action of $\text{Vect}(N)$ on the quantized phase space is only possible if the fields obey some equations already at fixed time. The fields must thus obey two fundamentally different types of equations.

Or must they? With the recent discovery of conformal fields, a new possibility opens up, which roughly can be described as projecting the time derivative onto space derivatives. We can then take this “time” derivative, substitute it into conformal analogs of the classical equations of motion, and obtain kinematical constraints on the quantum fields at fixed time. This idea of trading dynamics for kinematics may admittedly sound absurd, but it does make mathematical sense, as we now proceed to show.

In the plane-wave basis, the conformal fields (4.49) are defined by

$$L^\mu(m) = e^{mx}\left(\partial^\mu + (m^A + k^A)T^\mu_A + cm^\mu m^A T^B_A x_B\right),$$ \hspace{1cm} (6.1)
where the \((N + 1)\)-dimensional coordinate is

\[ x_B = (t, x_\nu), \quad (6.2) \]

with \(t\) being an arbitrary constant. Further, \(T^A_B\) generate \(gl(N + 1)\), i.e.

\[ [T^A_B, T^C_D] = \delta^A_D T^C_B - \delta^C_D T^A_B, \quad (6.3) \]

and

\[ k^A = (t^{-1}, 0), \quad \partial^A = (-t^{-1}x \cdot \partial, \partial^\mu), \quad m^A = (-t^{-1}x \cdot m, m^\mu), \quad (6.4) \]

The introduction of the parameter \(t\) makes (6.4) slightly more general than (4.51) and the expression in Ref. 11. However, it is readily verified that (6.2) and (6.4) still obey the algebraic relations

\[ m^Ax_A = x_A \partial^A = 0, \quad k^Ax_A = 1, \quad (6.5) \]

and

\[ [\partial^A, \partial^B] = k^A \partial^B - k^B \partial^A \quad [\partial^A, m^B] = -k^B m^A \]

\[ [\partial^A, x_B] = \delta^A_B - k^A x_B \quad [\partial^A, e^{m \cdot x}] = m^A e^{m \cdot x}, \quad (6.6) \]

and all commutators between \(m^A, x_B\) and \(k^C\) vanish. The proof that (6.1) satisfies \(\text{Vect}(N)\) is repeated in the appendix.

From the expression for \(x_B\) we see that conformal fields are naturally equipped with a length scale \(t\), so in that sense their name might be poorly chosen. We will think of this parameter as “time”, which is fixed on the space-like surface where \(\text{Vect}(N)\) acts. The units are such that the velocity of light is one. Of course, the algebraic structure is independent of this physical interpretation, but it is useful at least as a naming convention. Any conformal field can be divided into time and space components. E.g., if we split a conformal vector as

\[ \psi^A(x) = (\phi(x), \psi^\nu(x)), \quad (6.7) \]

the transformation law

\[ [L^\mu(m), \psi^A(x)] = -e^{m \cdot x} \left( (\partial^\mu + \lambda m^\mu)\psi^A(x) + (m^A + k^A)\psi^\mu(x) + cm^\mu m^A x_B \psi^B(x) \right) \quad (6.8) \]

takes the form

\[ [L^\mu(m), \phi(x)] = -e^{m \cdot x} \left( (\partial^\mu + \lambda m^\mu)\phi(x) + t^{-1}(-m \cdot x + 1)\psi^\mu(x) \right. \]

\[ \quad - ct^{-1}m^\mu m \cdot x(t\phi(x) + x_\nu \psi^\nu(x)) \right) \]

\[ [L^\mu(m), \psi^\nu(x)] = -e^{m \cdot x} \left( (\partial^\mu + \lambda m^\mu)\psi^\nu(x) + m^\nu \psi^\mu(x) + cm^\mu m^\nu(t\phi(x) + x_\nu \psi^\nu(x)) \right). \quad (6.9) \]

Similarly,

\[ [L^\mu(m), \psi_B(x)] = -e^{m \cdot x} \left( (\partial^\mu + \lambda m^\mu)\psi_B(x) - \delta^\mu_B (m^A + k^A)\psi_A(x) - cm^\mu m^A x_B \psi_A(x) \right) \quad (6.10) \]

29
becomes

\[ [L^\mu(m), \phi(x)] = -e^{m\cdot x}\left((\partial^\mu + \lambda m^\mu)\phi(x) - c t m^\mu (-t^{-1}m \cdot x \phi(x) + m^\nu \psi_\nu(x))\right) \]

\[ [L^\mu(m), \psi_\nu(x)] = -e^{m\cdot x}\left((\partial^\mu + \lambda m^\mu)\psi_\nu(x) - \delta^\mu_\nu (t^{-1}(-m \cdot x + 1)\phi(x) + m^\sigma \psi_\sigma(x))
- cm^\mu (-t^{-1}m \cdot x \phi(x) + m^\nu \psi_\nu(x))\right), \] (6.11)

when divided into time and space components:

\[ \psi_B(x) = (\phi(x), \psi_\nu(x)). \] (6.12)

The time parameter can be eliminated by replacing \( \phi(x) \) by \( t\phi(x) \) in both (6.7) and (6.12). However, it is useful for showing that a tensor field in a sense is a special case of a conformal field. Set \( c = 0 \) and let \( t \to \infty \) in (6.1). In this limit, \( k^A \to 0 \) and \( m^A \to m^\mu \), so

\[ L^\mu(m) \to e^{m\cdot x}(\partial^\mu + m^\sigma T^\mu_\sigma), \] (6.13)

which defines a tensor field. In this limit, a conformal vector decomposes into a direct sum of its time and space components,

\[ C^1_0(\lambda, 0) \to T^0_0(\lambda) \oplus T^1_0(\lambda). \] (6.14)

A generic conformal field with \( c \neq 0 \) does not correspond to a tensor field.

We have recently studied intertwining operators that connect different conformal fields.\(^{12}\) First, conformal fields with the same value of \( c \) can be multiplied, because Leibniz’ rule holds both for the derivative \( \partial^A \) and for the \( gl(N + 1) \) generators \( T^A_B \). Second, there are two types of first-order differential operators. The first involves the totally skew positive conformal forms \( \Omega^p(\lambda, c) \subset C^p_0(\lambda, c) \).

\[
d_p(\lambda, c) : \quad \Omega^p(\lambda, c) \longrightarrow \Omega^{p+1}(\lambda, c)
(\phi_p)^{A_1 \ldots A_p} \mapsto (d_p(\lambda, c)\phi_p)^{A_1 \ldots A_{p+1}} = \frac{1}{(p+1)!} (\partial^{[A_1} + \gamma_p(\lambda, c) k^{[A_1]})(\phi_p)^{A_2 \ldots A_{p+1}]} \] (6.15)

where \( \gamma_p(\lambda, c) = \lambda/c - p \). The second map involves the totally skew negative conformal forms \( \Omega_p(\lambda, c) \subset C^p_0(\lambda, c) \).

\[
d_{\tilde{p}}(\lambda, c) : \quad \Omega_p(\lambda, c) \longrightarrow \Omega_{p-1}(\lambda, c)
(\phi_{\tilde{p}})^{A_1 \ldots A_p} \mapsto (d_{\tilde{p}}(\lambda, c)\phi_{\tilde{p}})^{A_1 \ldots A_{p-1}} = (\partial^{B} + \tilde{\gamma}_p(\lambda, c) k^{B})(\phi_{\tilde{p}})^{A_1 \ldots A_{p-1} B} \] (6.16)

where \( \tilde{\gamma}_p(\lambda, c) = (\lambda - 1)/c + p - N - 1 \).

These maps satisfy \( d_{p+1}d_p = 0 \) and \( d_{p-1}\tilde{d}_p = 0 \), with the exception \( d_0\tilde{d}_1 \neq 0 \). This is the natural generalization of the exterior derivative to conformal fields, but it should be noted that these conformal exterior derivatives are much more abundant than the usual ones; they can be defined for any value of the conformal weight \( \lambda \).
With these tools we can now immediately write down conformal generalizations of most equations in physics. The recipe is simply to substitute

$$\frac{\partial}{\partial t} \rightarrow \partial^0 + \frac{\lambda}{c} k^0 = -x \cdot \partial + \frac{\lambda}{ct}$$

in derivatives acting on scalar fields. When acting on non-scalar conformal fields, this recipe has to be modified in accordance with (6.15-16). This is what we mean by “projecting the time derivative onto space”. We stress that this procedure does make sense, because the conformal exterior derivatives are invariantly defined.

The simplest conceivable conformal equation involves a single scalar field $\phi(x)$.

$$(d\phi)^A(x) \equiv (\partial^A + \frac{\lambda}{c} k^A)\phi(x) = 0.$$  (6.18)

Multiplying with $x_A$ we find that $\lambda \phi(x) = 0$, i.e. either $\lambda = 0$ or $\phi(x) \equiv 0$. In the former case, the space components of (6.18) read $\partial^\nu \phi(x) = 0$. The only non-zero solution is thus that $\phi(x)$ is constant, which is too restrictive.

To write down covariant equations with non-trivial solutions, we must introduce more conformal fields. A conformal vielbein $e^A_i(x)$, which is a $C^1_{0}(0,c)$ field transforming as a vector under the frame algebra $\text{Map}(N,\text{so}(N + 1))$. This is of course the same algebra as (5.16), apart from the dimension of the frame vectors. The vielbein is defined by the non-trivial property of having a two-sided inverse $e^i_A(x)$ everywhere.

$$e^A_i(x)e^B_j(x) = \delta^A_B, \quad e^A_i(x)e^A_i(x) = \delta^i_j.$$  (6.19)

It is this property that selects the gauge group $\text{so}(N + 1)$ ($\text{sl}(N + 1)$ would also be possible, but it has no spinor representations). Clearly, the inverse vielbein transforms as a $C^0_1(0,c)$ and as an $\text{so}(N + 1)$ vector.

A conformal Laplace equation reads

$$e^i_A(x)\left(\partial^A \eta_{ij} + \frac{\lambda}{c} k^A \eta_{ij} + \omega^A_{ij}(x)\right)e^j_B(x)\left(\partial^B + \frac{\lambda}{c} k^B\right)\phi(x) = 0$$  (6.20)

where the conformal spin connection $\omega^A_{ij} \in C^A_0(0,c)$ transforms as a connection under the frame algebra, i.e.

$$[J_{ij}, \omega^B_{kl}(x)] = e^{mx}\left(\eta_{ik}\omega^B_{jl}(x) - \eta_{il}\omega^B_{jk}(x) - \eta_{jk}\omega^B_{il}(x) + \eta_{jl}\omega^B_{ik}(x) + m^B \eta_{ij}\right).$$  (6.21)

The spin connection can as usual be defined in terms of the vielbein by the condition of vanishing torsion,

$$\partial[A e^B_i](x) + \left(\frac{\lambda}{c} - 1\right) k[A e^B_i](x) + \omega^B_{ij}(x) e^B_j(x) = 0.$$  (6.22)

That (6.20) and (6.22) are covariant both under $\text{Vect}(N)$ and $\text{Map}(N, \text{so}(N + 1))$ follows immediately from the existence of the conformal exterior derivatives. The conformal
Laplace equation is actually somewhat simpler than its standard counterpart, which would require the use of Christoffel symbols for non-zero $\lambda$.

Similarly, we can write down a conformal Dirac equation if $\psi(x)$ is a frame spinor.

\[
\gamma_i e^i_A(x) \left( \partial^A + \frac{\lambda}{c} k^A + \omega^A(x) \right) \psi(x) = iM \psi(x),
\]  
(6.23)

where $\gamma_i$ and $\sigma_{ij}$ are the $\text{so}(N+1)$ gamma and spin matrices and

\[
\omega^A(x) = \omega^i_A(x) \sigma_{ij}.
\]  
(6.24)

The number of fermionic degrees of freedom can be halved when $M = 0$ by considering chiral spinors. The conformal Dirac equation (6.23) is nevertheless reminiscent of a massive equation, because in flat space it has the structure

\[
\left( \gamma_i \partial^i + \frac{\lambda}{c} \gamma^0 \right) \psi(x) = 0,
\]  
(6.25)

and the role of mass is played by the parameter $\lambda/c$.

Skew conformal fields may be dualized in the same fashion as tensor fields. A volume element can be defined as

\[
v = e^{A_0}_{i_0} e^{A_1}_{i_1} \cdots e^{A_N}_{i_N} \epsilon_{A_0A_1\ldots A_N} e^{i_0i_1\ldots i_N} \in C^0_{0}(1,0),
\]  
(6.26)

where the transformation law of the permutation symbol (4.55) was used. A dual conformal field is defined by contraction with one of

\[
E_{A_0A_1\ldots A_N} = v^{-1} \epsilon_{A_0A_1\ldots A_N} \in C^0_{N+1}(0,c),
\]
\[
E^{A_0A_1\ldots A_N} = v \epsilon^{A_0A_1\ldots A_N} \in C^0_{N+1}(0,c),
\]  
(6.27)

This notion of duality allows us to write down conformal analogs of the Einstein equation. The conformal spin connection has the curvature

\[
R_{ij}^{AB} = \left( \partial^A + \left( \frac{\lambda}{c} - 1 \right) k^A \right) \omega^B_{ij} + \omega^A_{ik} \omega^B_{kj},
\]  
(6.28)

i.e.

\[
R_{CD}^{AB} = e^i_C e^j_D R_{ij}^{AB},
\]  
(6.29)

and the Einstein tensor is the contraction of the double dual of (6.29).

The preceding paragraphs illustrate that it is possible to construct conformal analogs of most equations in differential geometry (see also Ref. 12). We will not pursue this topic further, but hope to return to it in a forthcoming publication. In the remainder of this section we discuss the choice of Hamiltonian. It is tempting to extend the definition (6.1) by equipping $L^\mu(m)$ with a time component,

\[
L^A(m) = e^{m^x} \left( \partial^A + (m^B + k^B) T_B^A + cm^A m^B T_B^C x_C \right).
\]  
(6.30)
A slightly more general expression is also possible.\textsuperscript{12} Since the space components $L^\mu(m)$ generate diffeomorphisms in space, it is natural to think of the new time components as the generators of diffeomorphisms in the time direction, and the Hamiltonian can be identified as the generator of rigid time translations, $H = L^0(0)$. From

$$
H = \partial^0 + k^B T^0_B = -x_\mu \partial^\mu - T^\mu_\mu = -\frac{\partial L^\mu(m)}{\partial m^\mu} \bigg|_{m=0},
$$

we see that our Hamiltonian in fact is the dilatation operator. This presumably indicates that the natural $\mathbb{Z}$-gradation is by the radial exponent, at least for modules built from conformal fields.

Equivalently, the Hamiltonian acts by commutation on the fermionic Weyl algebra as the operator

$$
H_\psi = \int d^N x \, \psi^\dagger(x) \left( x_\mu \partial^\mu \psi(x) + [T^\mu_\mu, \psi(x)] \right)
$$

$$
= \int d^N x \, \psi^\dagger(x) (x_\mu \partial^\mu - \lambda) \psi(x),
$$

where the last expression holds for $\psi(x) \in C^0_0(\lambda, c)$. Eq. (6.32) is bilinear, but it can be recast in a non-linear form if the constraints are taken into account. In this sense, the Hamiltonian contains interactions.

\section{Discussion}

As stated in the introduction, our primary goal is to construct irreps of $Vect(N)$ (with anomalies). To achieve this goal, we start from a set of modules which we understand well, (tensor and conformal fields), and construct the corresponding Weyl algebras. These are of course huge modules, but we may hope that they contain irreducible components which can be isolated by factoring out certain relations. In this paper we have employed several methods of reduction, namely

- Invariants. A single Fourier component of a molecule is set to zero. This mechanism is responsible for quantization of the parameters $\lambda$ and $\omega$.
- Constraints. The fields are restricted to the solutions of certain equations, involving all Fourier components.
- Vacuum. A Fock module is much smaller than its parent Weyl algebra, because all strings containing oscillators of negative degree are eliminated. Na\"ively, a vacuum leads to infinities, which is taken care of by normal ordering and imposing constraints.
- Central charges. Any operator which commutes with all of $Vect(N)$, e.g. the fermionic number operator, has a single value in an irrep. Hence any module decomposes into submodules of fixed value.
- Gauge orbits. If some field admits a nontrivial action of a gauge algebra, the $Vect(N)$ module decomposes into an infinite direct sum of isomorphic copies. Each copy can be considered as a representation on a gauge orbit. Equivalently, we can enlarge the
algebra to $\text{Vect}(N) \times \text{Map}(N, g)$, which is indeed done in rational conformal field theory. Representations on gauge orbits are also closely related to the theory of Hamiltonian systems with first-class constraints.\textsuperscript{21}

We believe that this programme to a large extent has been carried out in this paper, at least conceptually, but many problems of technical nature remain. The major difficulty lies perhaps in the non-linearity of the constraints, which prevents us from writing down the general solution as a sum of eigenmodes. Another problem is to decide which fields to start from. Mathematically, the ideal choice would be the minimal set of fields from which all irreps can be built, but it is not obvious what this is. It would also be desirable to have a more concrete description of a single nontrivial irrep. Finally, we should of course check whether the modules are unitary, but there seems to be little point in doing so before we know if they are irreducible.

Although the problem in this paper is purely mathematical, we expect that it has consequences for physics. Our original reason for studying $\text{Vect}(N)$ was to extend the classification of critical exponents to $N$ dimensions. This goal now seems closer, because exceptionally small modules can only be constructed for a discrete set of the parameters, as was explained in Sec. 4.

However, the most striking result is the appearance of algebraic structures similar to quantum gravity. If we by the word “quantum” mean that there is a lowest-weight state and by the word “gravity” mean that a conformal analog of the Einstein equation is satisfied, then the modules in the previous section qualify as “quantum gravity”. Moreover, they are presumably consistent in the sense that there are only finite anomalies. Of course, this is not what is usually meant by quantum gravity. On the other hand, conformal equations could not have been written down before conformal fields were discovered, so this option has never been tested before. In any event, we know of no other route to quantized gravity which is not manifestly inconsistent, with the possible exception of string theory.

We can evidently build some kind of classical gravity theory using conformal fields, as we indicated in Sec. 6. It is not clear to us if such a theory will differ in any significant way from the standard Einstein theory for large time scales, compared to the Planck time. It should in this connection be noted that $N$-dimensional conformal fields in many respects are similar to $(N + 1)$-dimensional tensor fields, because vectors have an equal number of components.

Finally, we want to emphasize that all physical objects, including those involved in quantum gravity, must transform consistently under arbitrary coordinate transformations and hence they must be $\text{Vect}(N)$ representations. We find it plausible that Nature would use irreducible representations as her fundamental building blocks.

Appendix
The proof that (6.1) satisfies $\text{Vect}(N)$ is given here for convenience.

\[
[L^\mu(m), L^\nu(n)] = [e^{m\cdot x} (\partial^\mu + (m^A + k^A)T_A^\mu + cm^\mu m^A T_B^B x_B),
\]
\[
e^{n\cdot x} (\partial^\nu + (n^C + k^C)T_C^\nu + cn^\nu n^C T_D^D x_D)]
\]
\[
= e^{(m+n)\cdot x} \left( n^\mu (\partial^\nu + (n^C + k^C)T_C^\nu + cn^\nu n^C T_D^D x_D) - n^\mu k^C T_C^\nu 
\right.
\]
\[
+ cn^\nu (-n^\mu k^C T_C^D x_D + n^C T_C^\nu) + (n^\mu + k^\mu)(m^A + k^A)T_A^\nu 
\]
\[
+ cn^\nu (m^A + k^A)(n^\nu T_A^D x_D - n^C T_C^\nu x_A) 
\]
\[
+ c^2 m^\mu n^\nu m^B n^B T_A^D x_D) - m \leftrightarrow n 
\]
\[
= n^\mu e^{(m+n)\cdot x} \left( \partial^\nu + (m^C + n^C + k^C)T_C^\nu + cn^\nu (m^C + n^C)T_C^D x_D \right) - m \leftrightarrow n 
\]
\[
= n^\mu L^\nu(m + n) - m \leftrightarrow n. \quad (A.1)
\]

We used that $m^A x_A = n^A x_A = k^\mu = 0$ and that

\[
n^\nu n^\nu f(m + n) - m \leftrightarrow n = n^\nu (m^\nu + n^\nu) f(m + n) - m \leftrightarrow n \quad (A.2)
\]

for any expression that depends on the sum $m + n$ only.
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