On the optimality of gluing over scales

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Abstract

We show that for every $\alpha > 0$, there exist $n$-point metric spaces $(X, d)$ where every “scale” admits a Euclidean embedding with distortion at most $\alpha$, but the whole space requires distortion at least $\Omega(\sqrt{\alpha \log n})$. This shows that the scale-gluing lemma [Lee, SODA 2005] is tight, and disproves a conjecture stated there. This matching upper bound was known to be tight at both endpoints, i.e. when $\alpha = \Theta(1)$ and $\alpha = \Theta(\log n)$, but nowhere in between.

More specifically, we exhibit $n$-point spaces with doubling constant $\lambda$ requiring Euclidean distortion $\Omega(\sqrt{\log \lambda \log n})$, which also shows that the technique of “measured descent” [Krauthgamer, et. al., Geometric and Functional Analysis] is optimal. We extend this to $L_p$ spaces with $p > 1$, where one requires distortion at least $\Omega((\log n)^{1/q}(\log \lambda)^{1-1/q})$ when $q = \max\{p, 2\}$, a result which is tight for every $p > 1$.

1 Introduction

Suppose one is given a collection of mappings from some finite metric space $(X, d)$ into a Euclidean space, each of which reflects the geometry at some “scale” of $X$. Is there a non-trivial way of gluing these mappings together to form a global mapping which reflects the entire geometry of $X$? The answers to such questions have played a fundamental role in the best-known approximation algorithms for Sparsest Cut [7, 10, 4, 1] and Graph Bandwidth [17, 7, 11], and have found applications in approximate multi-commodity max-flow/min-cut theorems in graphs [17, 7]. In the present paper, we show that the approaches of [7] and [10] are optimal, disproving a conjecture stated in [10].

Let $(X, d)$ be an $n$-point metric space, and suppose that for every $k \in \mathbb{Z}$, we are given a non-expansive mapping $\phi_k : X \to L_2$ which satisfies the following. For every $x, y \in X$ with $d(x, y) \geq 2^k$, we have

$$\|\phi_k(x) - \phi_k(y)\| \geq \frac{2^k}{\alpha}.$$ 

The Gluing Lemma of [10] (generalizing the approach of [7]) shows that the existence of such a collection $\{\phi_k\}$ yields a Euclidean embedding of $(X, d)$ with distortion $O(\sqrt{\alpha \log n})$. (See Section 1.1 for the relevant definitions on embeddings and distortion.) This is known to be tight when $\alpha = \Theta(1)$ [10] and also when $\alpha = \Theta(\log n)$ [13, 2], but nowhere in between. In fact, in [10], the second named author conjectured that one could achieve $O(\alpha + \sqrt{\log n})$ (this is indeed stronger, since one can always construct $\{\phi_k\}$ with $\alpha = O(\log n)$).

In the present paper, we give a family of examples which shows that the $\sqrt{\alpha \log n}$ bound is tight for any dependence $\alpha(n) = O(\log n)$. In fact, we show more. Let $\lambda(X)$ denote the doubling

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constant of \( X \), i.e. the smallest number \( \lambda \) so that every open ball in \( X \) can be covered by \( \lambda \) balls of half the radius. In [7], using the method of “measure descent,” the authors show that \((X,d)\) admits a Euclidean embedding with distortion \( O(\sqrt{\log \lambda(X) \log n}) \). (This is a special case of the Gluing Lemma since one can always find \( \{\phi_k\} \) with \( \alpha = O(\log \lambda(X)) \) [[5]].) Again, this bound was known to be tight for \( \lambda(X) = \Theta(1) \) [[8]], and \( \lambda(X) = n^{\Theta(1)} \) [[13],[2]], but nowhere in between. We provide the matching lower bound for any dependence of \( \lambda(X) \) on \( n \). We also generalize our method to give tight lower bounds on \( L_p \) distortion for every fixed \( p > 1 \).

Construction and analysis. In some sense, our lower bound examples are an interpolation between the multi-scale method of [16] and [8], and the expander Poincaré inequalities of [13],[2],[14].

We start with a vertex-transitive expander graph \( G \). Sometimes we will equip \( G \) with a non-negative length function \( \text{len} : E(G) \to \mathbb{R}_+ \), and we let \( d_{\text{len}} \) denote the shortest-path (semi-)metric on \( G \). We refer to the pair \((G,\text{len})\) as a metric graph, and often \( \text{len} \) will be implicit, in which case we use \( d_G \) to denote the path metric. We use \( \text{Aut}(G) \) to denote the group of automorphisms of \( G \).

For a graph \( G \), we will use \( V(G), E(G) \) to denote the sets of vertices and edges of \( G \), respectively. Sometimes we will equip \( G \) with a non-negative length function \( \text{len} : E(G) \to \mathbb{R}_+ \), and we let \( d_{\text{len}} \) denote the shortest-path (semi-)metric on \( G \). We refer to the pair \((G,\text{len})\) as a metric graph, and often \( \text{len} \) will be implicit, in which case we use \( d_G \) to denote the path metric. We use \( \text{Aut}(G) \) to denote the group of automorphisms of \( G \).

Given two expressions \( E \) and \( E' \) (possibly depending on a number of parameters), we write \( E = O(E') \) to mean that \( E \leq CE' \) for some constant \( C > 0 \) which is independent of the parameters. Similarly, \( E = \Omega(E') \) implies that \( E \geq CE' \) for some \( C > 0 \). We also write \( E \lesssim E' \) as a synonym for \( E = O(E') \). Finally, we write \( E \approx E' \) to denote the conjunction of \( E \lesssim E' \) and \( E \gtrsim E' \).
Embeddings and distortion. If \((X, d_X), (Y, d_Y)\) are metric spaces, and \(f : X \to Y\), then we write
\[
\|f\|_{\text{Lip}} = \sup_{x \neq y \in X} \frac{d_Y(f(x), f(y))}{d_X(x, y)}.
\]
If \(f\) is injective, then the distortion of \(f\) is defined by \(\text{dist}(f) = \|f\|_{\text{Lip}} \cdot \|f^{-1}\|_{\text{Lip}}\). A map with distortion \(D\) will sometimes be referred to as \(D\)-bi-lipschitz. If \(d_Y(f(x), f(y)) \leq d_X(x, y)\) for every \(x, y \in X\), we say that \(f\) is non-expansive. If \(d_Y(f(x), f(y)) \geq d_X(x, y)\) for every \(x, y \in X\), we say that \(f\) is non-contracting. For a metric space \(X\), we use \(c_p(X)\) to denote the least distortion required to embed \(X\) into some \(L_p\) space.

Finally, for \(x \in X\), \(r \in \mathbb{R}_+\), we define the open ball \(B(x, r) = \{y \in X : d(x, y) < r\}\). Recall that the doubling constant of a metric space \((X, d)\) is the infimum over all values \(\lambda\) such that every ball in \(X\) can be covered by \(\lambda\) balls of half the radius. We use \(\lambda(X, d)\) to denote this value.

We now state the main theorem of the paper.

**Theorem 1.1.** For any positive nondecreasing function \(\lambda(n)\), there exists a family of \(n\)-vertex metric graphs \(\tilde{G}^{\otimes k}\) such that \(\lambda(\tilde{G}^{\otimes k}) \lesssim \lambda(n)\), and for every fixed \(p > 1\),
\[
c_p(\tilde{G}^{\otimes k}) \gtrsim (\log n)^{1/q}(\log \lambda(n))^{1-1/q},
\]
where \(q = \max\{p, 2\}\).

## 2 Metric construction

### 2.1 \(\otimes\)-products

An \(s\)-\(t\) graph \(G\) is a graph which has two distinguished vertices \(s, t \in V(G)\). For an \(s\)-\(t\) graph, we use \(s(G)\) and \(t(G)\) to denote the vertices labeled \(s\) and \(t\), respectively. We define the length of an \(s\)-\(t\) graph \(G\) as \(\text{len}(G) = d_{\text{len}}(s, t)\).

**Definition 2.1** (Composition of \(s\)-\(t\) graphs). Given two \(s\)-\(t\) graphs \(H\) and \(G\), define \(H \otimes G\) to be the \(s\)-\(t\) graph obtained by replacing each edge \((u, v) \in E(H)\) by a copy of \(G\) (see Figure 1). Formally,
- \(V(H \otimes G) = V(H) \cup (E(H) \times (V(G) \setminus \{s(G), t(G)\}))\).
- For every edge \(e = (u, v) \in E(H)\), there are \(|E(G)|\) edges,
  \[
  \left\{ \left( (e, v_1), (e, v_2) \right) \ | \ (v_1, v_2) \in E(G) \text{ and } v_1, v_2 \notin \{s(G), t(G)\} \right\} \cup \\
  \left\{ (u, (e, w)) \ | \ (s(G), w) \in E(G) \right\} \cup \left\{ ((e, w), v) \ | \ (w, t(G)) \in E(G) \right\}.
  \]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure1.png}
\caption{A single edge \(H, H \otimes K_{2,3}\), and \(H \otimes K_{2,3} \otimes K_{2,2}\).}
\end{figure}
\[ s(H \odot G) = s(H) \] and \[ t(H \odot G) = t(H). \]

If \( H \) and \( G \) are equipped with length functions \( \text{len}_H, \text{len}_G \), respectively, we define \( \text{len} = \text{len}_{H \odot G} \) as follows. Using the preceding notation, for every edge \( e = (u, v) \in E(H) \),

\[
\text{len} \left( (e, v_1), (e, v_2) \right) = \frac{\text{len}_H(e)}{\text{len}_G(\text{s}(G), \text{t}(G))} \text{len}_G(v_1, v_2)
\]

\[
\text{len} \left( (u, e, w) \right) = \frac{\text{len}_H(e)}{\text{len}_G(\text{s}(G), \text{t}(G))} \text{len}_G(s(G), w)
\]

\[
\text{len} \left( (e, w), v \right) = \frac{\text{len}_H(e)}{\text{len}_G(\text{s}(G), \text{t}(G))} \text{len}_G(w, t(G)).
\]

This choice implies that \( H \odot G \) contains an isometric copy of \((V(H), d_{\text{len}_H})\).

Observe that there is some ambiguity in the definition above, as there are two ways to substitute an edge of \( H \) with a copy of \( G \), thus we assume that there exists some arbitrary orientation of the edges of \( H \). However, for our purposes the graph \( G \) will be symmetric, and thus the orientations are irrelevant.

**Definition 2.2** (Recursive composition). For an \( s \)-\( t \) graph \( G \) and a number \( k \in \mathbb{N} \), we define \( G^{\odot k} \) inductively by letting \( G^{\odot 0} \) be a single edge of unit length, and setting \( G^{\odot k} = G^{\odot k-1} \odot G \).

The following result is straightforward.

**Lemma 2.3** (Associativity of \( \odot \)). For any three graphs \( A, B, C \), we have \((A \odot B) \odot C = A \odot (B \odot C)\), both graph-theoretically and as metric spaces.

**Definition 2.4.** For two graphs \( G, H \), a subset of vertices \( X \subseteq V(H) \) is said to be a copy of \( G \) if there exists a bijection \( f : V(G) \to X \) with distortion 1, i.e. \( d_H(f(u), f(v)) = C \cdot d_G(u, v) \) for some constant \( C > 0 \).

Now we make the following two simple observations about copies of \( H \) and \( G \) in \( H \odot G \).

**Observation 2.5.** The graph \( H \odot G \) contains \(|E(H)|\) distinguished copies of the graph \( G \), one copy corresponding to each edge in \( H \).

**Observation 2.6.** The subset of vertices \( V(H) \subseteq V(H \odot G) \) form an isometric copy of \( H \).

### 2.2 A stretched \( \tilde{G} \)

Let \( G = (V, E) \) be an unweighted graph, and put \( D = \text{diam}(G) \). We define a metric \( s \)-\( t \) graph \( \tilde{G} \) which has \( D + 1 \) layers isomorphic to \( G \), with edges between the layers, and a pair of endpoints \( s, t \). Formally,

\[
V(\tilde{G}) = \{s, t\} \cup \{v^{(i)} : v \in V, i \in [D + 1]\}
\]

\[
E(\tilde{G}) = \{(s, v^{(1)}), (v^{(D+1)}, t) : v \in V\}
\]

\[
\cup \left\{ \left( u^{(i)}, v^{(i+1)} \right), \left( u^{(j)}, v^{(j)} \right) : (u, v) \in E, i \in [D], j \in [D + 1]\right\}
\]

\[
\cup \left\{ \left( v^{(i)}, v^{(i+1)} \right) : v \in V, i \in [D]\right\}.
\]
We put \( \text{len}(s, v^{(1)}) = \text{len}(v^{(D+1)}, t) = D \) for \( v \in V \), \( \text{len}(u^{(i)}, v^{(i+1)}) = \text{len}(u^{(j)}, v^{(j)}) = 1 \) for \( (u, v) \in E \), \( i \in [D], j \in [D+1] \) and \( \text{len}(u^{(i)}, v^{(i+1)}) = 1 \) for \( v \in V, i \in [D] \). We refer to edges of the form \((u^{(i)}, v^{(i)})\) as vertical edges. All other edges are called horizontal edges. In particular, there are \( D + 1 \) copies \( G^{(1)}, \ldots, G^{(D+1)} \) of \( G \) in \( \tilde{G} \) which are isometric to \( G \) itself, and their edges are all vertical.

A doubling version, following Laakso. Let \( \tilde{G} \) be a stretched graph as in Section 2.2 with \( D = \text{diam}(G) \), and let \( s' = s(\tilde{G}), t' = t(\tilde{G}) \). Consider a new metric \( s-t \) graph \( \tilde{G} \), which has two new vertices \( s, t \) and two new edges \((s, s'), (t', t)\) with \( \text{len}(s, s') = \text{len}(t', t) = 3D \).

Claim 2.7. For any graph \( G \) with \( |V(G)| = m \), and any \( k \in \mathbb{N} \), we have \( \log \lambda(\tilde{G}^{\otimes k}) \lesssim \log m \).

The proof of the claim is similar to [8, 9], and follows from the following three results.

We define \( \text{tri}(G) = \max_{v \in V(G)}(d_{\text{len}}(s, v) + d_{\text{len}}(v, t)) \). For any graph \( G \), we have \( \text{len}(\tilde{G}) = d(s, t) = 9D \), and it is not hard to verify that \( \text{tri}(\tilde{G}^{\otimes k}) \leq \text{len}(\tilde{G}^{\otimes k})(1 + \frac{1}{9D-1}) \). For convenience, let \( G_0 \) be the top-level copy of \( \tilde{G} \) in \( \tilde{G}^{\otimes k} \), and \( H \) be the graph \( \tilde{G}^{\otimes k-1} \). Then for any \( e \in E(G_0) \), we refer to the copy of \( H \) along edge \( e \) as \( H_e \).

Observation 2.8. If \( r > \frac{\text{tri}(\tilde{G}^{\otimes k})}{3} \), then the ball \( B(x, r) \) in \( \tilde{G}^{\otimes k} \) may be covered by at most \( |V(\tilde{G})| \) balls of radius \( r/2 \).

Proof. For any \( e \in E(G_0) \), we have \( r > \frac{\text{len}(e)}{\text{len}(H)} \cdot \text{tri}(H) \), so every point in \( H_e \) is less than \( r/2 \) from an endpoint of \( e \). Thus all of \( \tilde{G}^{\otimes k} \) is covered by placing balls of radius \( \frac{\text{tri}(\tilde{G}^{\otimes k})}{6} \) around each vertex of \( G_0 \).

Lemma 2.9. If \( s \in B(x, r) \), then one can cover the ball \( B(x, r) \) in \( \tilde{G}^{\otimes k} \) with at most \( |E(\tilde{G})||V(\tilde{G})| \) balls of radius \( r/2 \).

Proof. First consider the case in which \( r > \frac{\text{len}(\tilde{G}^{\otimes k})}{3} \). Then for any edge \( e \) in \( \tilde{G}^{\otimes k} \), we have \( r > \frac{\text{len}(e)}{\text{len}(H)} \cdot \frac{\text{tri}(H)}{3} \). Thus by Observation 2.8, we may cover \( H_e \) by \( |V(\tilde{G})| \) balls of radius \( r/2 \). This gives a covering of all of \( \tilde{G}^{\otimes k} \) by at most \( |E(\tilde{G})||V(\tilde{G})| \) balls of radius \( r/2 \).

Otherwise, assume \( \frac{\text{len}(\tilde{G}^{\otimes k})}{6} \geq r \). Since \( s \in B(x, r) \), but \( 2r < \frac{\text{len}(\tilde{G}^{\otimes k})}{3} \), the ball must be completely contained inside \( H_{\{s, s'\}} \). By induction, we can find a sufficient cover of this smaller graph.

Lemma 2.10. We can cover any ball \( B(x, r) \) in \( \tilde{G}^{\otimes k} \) with at most \( 2|V(\tilde{G})||E(\tilde{G})|^2 \) balls of radius \( r/2 \).

Proof. We prove this lemma using induction. For \( \tilde{G}^{\otimes 0} \), the claim holds trivially. Next, if any \( H_e \) contains all of \( B(x, r) \), then by induction we are done. Otherwise, for each \( H_e \) containing \( x \), \( B(x, r) \) contains an endpoint of \( e \). Then by Lemma 2.9, we may cover \( H_e \) by at most \( |E(\tilde{G})||V(\tilde{G})| \) balls of radius \( r/2 \). For all other edges \( e' = (u, v), x \notin H_{e'} \), we have:

\[
V(H_{e'}) \cap B(x, r) \subseteq B(v, \max(0, r - d(x, v))) \cup B(u, \max(0, r - d(x, u))).
\]

Thus, using Lemma 2.9 on both of the above balls, we may cover \( V(H_{e'}) \cap B(x, r) \) by at most \( 2|E(\tilde{G})||V(\tilde{G})| \) balls of radius \( r/2 \). Hence, in total, we need at most \( 2|V(\tilde{G})||E(\tilde{G})|^2 \) balls of radius \( r/2 \) to cover all of \( B(x, r) \).
Proof of Claim 2.7. First note that \(|V(\tilde{G})| = m(D + 1) + 2 \lesssim m^2\). By Lemma 2.10 we have
\[
\lambda(\tilde{G}^{\otimes k}) \leq 2|V(\tilde{G})||E(\tilde{G})|^2 \leq 2|V(\tilde{G})|^5 \lesssim m^{10}.
\]
Hence \(\log \lambda(\tilde{G}^{\otimes k}) \lesssim \log m\). \(\square\)

3 Lower bound

For any \(\pi \in \text{Aut}(G)\), we define a corresponding automorphism \(\tilde{\pi}\) of \(\tilde{G}\) by \(\tilde{\pi}(s) = s, \tilde{\pi}(t) = t,\) \(\tilde{\pi}(s') = s', \tilde{\pi}(t') = t'\), and \(\tilde{\pi}(v(i)) = \pi(v(i))\) for \(v \in V, i \in [D + 1]\).

Lemma 3.1. Let \(G\) be a vertex transitive graph. Let \(f : V(\tilde{G}) \to L_2\) be an injective mapping and define \(\tilde{f} : V(\tilde{G}) \to L_2\) by
\[
\tilde{f}(x) = \frac{1}{\sqrt{\text{|Aut}(G)|}} \left(f(\bar{x})\right)_{\pi \in \text{Aut}(G)}.
\]

Let \(\beta\) be such that for every \(i \in [D+1]\) there exists a vertical edge \((u(i), v(i))\) with \(\|\tilde{f}(u(i)) - \tilde{f}(v(i))\| \geq \beta\). Then there exists a horizontal edge \((x, y) \in E(\tilde{G})\) such that
\[
\frac{\|\tilde{f}(x) - \tilde{f}(y)\|^2}{d_G(x, y)^2} \geq \frac{\|f(s) - \tilde{f}(t)\|^2}{d_G(s, t)^2} + \frac{\beta^2}{36} (1)
\]

Proof. Let \(D = \text{diam}(G)\). We first observe three facts about \(\tilde{f}\), which rely on the fact that when \(\text{Aut}(G)\) is transitive, for every \(x \in V\), the orbits \(\{\pi(x)\}_{\pi \in \text{Aut}(G)}\) all have the same cardinality.

(F1) \(\|\tilde{f}(s) - \tilde{f}(t)\| = \|f(s) - f(t)\|\)

(F2) For all \(u, v \in V\),
\[
\|\tilde{f}(s) - \tilde{f}(v^{(i)})\| = \|\tilde{f}(s) - \tilde{f}(u^{(i)})\|, \quad \|\tilde{f}(t) - \tilde{f}(v^{(D+1)})\| = \|\tilde{f}(t) - \tilde{f}(u^{(D+1)})\|.
\]

(F3) For every \(u, v \in V, i \in [D]\),
\[
\|\tilde{f}(v^{(i)}) - \tilde{f}(v^{(i+1)})\| = \|\tilde{f}(u^{(i)}) - \tilde{f}(u^{(i+1)})\|.
\]

(F4) For every pair of vertices \(u, v \in V\) and \(i \in [D + 1]\),
\[
\langle \tilde{f}(s) - \tilde{f}(t), \tilde{f}(u^{(i)}) - \tilde{f}(v^{(i)}) \rangle = 0.
\]

Let \(z = \frac{\tilde{f}(s) - \tilde{f}(t)}{\|\tilde{f}(s) - \tilde{f}(t)\|}\). Fix some \(r \in V\) and let \(\rho_0 = |\langle z, \tilde{f}(s) - \tilde{f}(r^{(1)})\rangle|, \rho_i = |\langle z, \tilde{f}(r^{(i)}) - \tilde{f}(r^{(i+1)})\rangle|\) for \(i = 1, 2, \ldots, D\) and \(\rho_{D+1} = |\langle z, \tilde{f}(t) - \tilde{f}(r^{(D+1)})\rangle|\). Note that, by (F2) and (F3) above, the values \(\{\rho_i\}\) do not depend on the representative \(r \in V\). In this case, we have
\[
\sum_{i=0}^{D+1} \rho_i \geq \|\tilde{f}(s) - \tilde{f}(t)\| = 9\gamma D, \quad (2)
\]
Lemma 3.3. Let $f : G \rightarrow L_2$ be a non-contractive mapping, then there exists a horizontal edge $(x, y) \in E(G)$ with

$$\frac{\|f(x) - f(y)\|^2}{d_G(x, y)^2} \geq \frac{\|f(s) - f(t)\|^2}{d_G(s, t)^2} + \Omega \left( \frac{\mu_2}{d} \left( \log_d m \right)^2 \right).$$
Proof. We need only prove the existence of an $(x, y) \in E(\tilde{G})$ such that (4) is satisfied for $\tilde{f}$ (as defined in Lemma 3.1), as this implies it is also satisfied for $f$ (possibly for some other edge $(x, y)$).

Consider any layer $G^{(i)}$ in $\tilde{G}$, for $i \in [D + 1]$. Applying (3) and using the fact that $f$ is non-contracting, we have

$$
\mathbb{E}_{(u, v) \in E} \|\tilde{f}(u^{(i)}) - \tilde{f}(v^{(i)})\|^2 = \mathbb{E}_{(u, v) \in E} \|f(u^{(i)}) - f(v^{(i)})\|^2
\geq \frac{\mu_2}{d} \mathbb{E}_{u, v \in V} \|f(u^{(i)}) - f(v^{(i)})\|^2
\geq \frac{\mu_2}{d} \mathbb{E}_{u, v \in V} d_G(u, v)^2
\geq \frac{\mu_2}{d} (\log d m)^2.
$$

In particular, in every layer $i \in [D + 1]$, at least one vertical edge $(u^{(i)}, v^{(i)})$ has $\|\tilde{f}(u^{(i)}) - \tilde{f}(v^{(i)})\| \geq \sqrt{\frac{\mu_2}{d} \log d m}$. Therefore the desired result follows from Lemma 3.1.

We now come our main theorem.

**Theorem 3.4.** If $G = (V, E)$ is a $d$-regular, $m$-vertex, vertex-transitive graph with $\mu_2 = \mu_2(G)$, then

$$
c_2(\tilde{G}^{\otimes k}) \geq \sqrt{\frac{\mu_2}{d} \log d m}.
$$

Proof. Let $f : V(\tilde{G}^{\otimes k}) \to L_2$ be any non-contracting embedding. The theorem follows almost immediately by induction: Consider the top level copy of $\tilde{G}$ in $G^{\otimes k}$, and call it $G_0$. Let $(x, y) \in E(G_0)$ be the horizontal edge for which $\|f(x) - f(y)\|$ is longest. Clearly this edge spans a copy of $G^{\otimes k-1}$, which we call $G_1$. By induction and an application of Lemma 3.3, there exists a (universal) constant $c > 0$ and an edge $(u, v) \in E(G_1)$ such that

$$
\|f(u) - f(v)\|^2 \geq \frac{c\mu_2(k - 1)}{d} (\log d m)^2 + \frac{\|f(x) - f(y)\|^2}{d_G(u, v)^2}
\geq \frac{c\mu_2(k - 1)}{d} (\log d m)^2 + \frac{c\mu_2}{d} (\log d m)^2 + \frac{\|f(s) - f(t)\|^2}{d_G(s, t)},
$$

completing the proof.

**Corollary 3.5.** If $G = (V, E)$ is an $O(1)$-regular $m$-vertex, vertex-transitive graph with $\mu_2 = \Omega(1)$, then

$$
c_2(\tilde{G}^{\otimes k}) \geq \sqrt{k} \log m \approx \sqrt{\log m} \log N,
$$

where $N = |V(\tilde{G}^{\otimes k})| = 2^{\Theta(k \log m)}$.

We remark that infinite families of $O(1)$-regular vertex-transitive graphs with $\mu_2 \geq \Omega(1)$ are well-known. In particular, one can take any construction coming from the Cayley graphs of finitely generated groups. We refer to the survey [6]; see, in particular, Margulis’ construction in Section 8.
3.1 Extension to other $L_p$ spaces

Our previous lower bound dealt only with $L_2$. We now prove the following.

**Theorem 3.6.** If $G = (V, E)$ is an $O(1)$-regular $m$-vertex, vertex-transitive graph with $\mu_2 = \Omega(1)$, for any $p > 1$, there exists a constant $C(p)$ such that

$$c_p(\tilde{G}^{\otimes k}) \gtrsim C(p)k^{1/q} \log m \approx C(p)(\log m)^{1-1/q}(\log N)^{1/q}$$

were $N = |V(\tilde{G}^{\otimes k})|$ and $q = \max\{p, 2\}$.

The only changes required are to Lemma 3.2 and Lemma 3.1 (which uses orthogonality). The first can be replaced by Matoušek’s Poincaré inequality: If $G = (V, E)$ is an $O(1)$-regular expander graph with $\mu_2 = \Omega(1)$, then for any $p \in [1, \infty)$ and $f : V \to L_p$,

$$\mathbb{E}_{x,y \in V} \|f(x) - f(y)\|_p^p \leq O(2p)^p \mathbb{E}_{(x,y) \in E} \|f(x) - f(y)\|_p^p.$$  

Generalizing Lemma 3.7 is more involved. We need the following well-known 4-point inequalities for $L_p$ spaces.

**Lemma 3.7.** Consider any $p \geq 1$ and $u, v, w, x \in L_p$. If $1 \leq p \leq 2$, then

$$\|u - w\|_p^2 + (p - 1)\|x - v\|_p^2 \leq \|u - v\|_p^2 + \|v - w\|_p^2 + \|x - w\|_p^2 + \|u - x\|_p^2. \quad (5)$$

If $p \geq 2$, then

$$\|u - w\|_p^p + \|x - v\|_p^p \leq 2^{p-2}\left(\|u - v\|_p^p + \|v - w\|_p^p + \|x - w\|_p^p + \|u - x\|_p^p\right). \quad (6)$$

**Proof.** The following inequalities are known for $a, b \in L_p$ (see, e.g. [3]). If $1 \leq p \leq 2$, then

$$\left\| \frac{a + b}{2} \right\|_p^2 + (p - 1)\left\| \frac{a - b}{2} \right\|_p^2 \leq \frac{\|a\|_p^2 + \|b\|_p^2}{2}.$$  

On the other hand, if $p \geq 2$, then

$$\left\| \frac{a + b}{2} \right\|_p^p + \left\| \frac{a - b}{2} \right\|_p^p \leq \frac{\|a\|_p^p + \|b\|_p^p}{2}.$$  

In both cases, the desired 4-point inequalities are obtained by averaging two incarnations of one of the above inequalities with $a = u - v, b = v - w$ and then $a = u - x, b = x - w$ and using convexity of the $L_p$ norm (see, e.g. [12, Lem. 2.1]).

**Lemma 3.8.** Let $G$ be a vertex transitive graph, and suppose $p > 1$. If $q = \max\{p, 2\}$, then there exists a constant $K(p) > 0$ such that the following holds. Let $f : V(G) \to L_p$ be an injective mapping and define $\tilde{f} : V(\tilde{G}) \to L_p$ by

$$\tilde{f}(x) = \frac{1}{|\text{Aut}(G)|^{1/p}} \left(f(\pi x)\right)_{\pi \in \text{Aut}(G)}.$$  

Suppose that $\beta$ is such that for every $i \in [D+1]$, there exists a vertical edge $(u^{(i)}, v^{(i)})$ which satisfies $\|\tilde{f}(u^{(i)}) - \tilde{f}(v^{(i)})\|_p \geq \beta$. Then there exists a horizontal edge $(x, y) \in E(\tilde{G})$ such that

$$\frac{\|\tilde{f}(x) - \tilde{f}(y)\|_p^p}{d_{\tilde{G}}(x, y)^q} \geq \frac{\|f(s) - f(t)\|_p^q}{d_G(s, t)^q} + K(p)\beta^q. \quad (7)$$

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Proof. Let \( D = \text{diam}(G) \). For simplicity, we assume that \( D \) is even in what follows.

(F1) \( \|\tilde{f}(s) - \tilde{f}(t)\|_p = \|f(s) - f(t)\|_p \)

(F2) For all \( u, v \in V \),

\[
\|\tilde{f}(s) - \tilde{f}(v^{(1)})\|_p = \|\tilde{f}(s) - \tilde{f}(u^{(1)})\|_p,
\|\tilde{f}(t) - \tilde{f}(v^{(D+1)})\|_p = \|\tilde{f}(t) - \tilde{f}(u^{(D+1)})\|_p.
\]

(F3) For every \( u, v \in V, i \in [D] \),

\[
\|\tilde{f}(v^{(i)}) - \tilde{f}(v^{(i+1)})\|_p = \|f(v^{(i)}) - f(v^{(i+1)})\|_p.
\]

Fix some \( r \in V \) and let \( \rho_0 = \|\tilde{f}(s) - \tilde{f}(r^{(1)})\|_p, \rho_i = \|\tilde{f}(r^{(2i-1)}) - \tilde{f}(r^{(2i+1)})\|_p \) for \( i = 1, \ldots, D/2 \), \( \rho_{D/2+1} = \|f(t) - f(r^{(D+1)})\|_p \). Also let \( \rho_{i,1} = \|f(r^{(2i-1)}) - f(r^{(2i)})\|_p \) and \( \rho_{i,2} = \|f(r^{(2i)}) - f(r^{(2i+1)})\|_p \) for \( i = 1, \ldots, D/2 \).

Note that, by (F2) and (F3) above, the values \( \{\rho_i\} \) do not depend on the representative \( r \in V \).

In this case, we have

\[
\sum_{i=0}^{D/2+1} \rho_i \geq \|\tilde{f}(s) - \tilde{f}(t)\|_p = 9\gamma D,
\]

where we put \( \gamma = \frac{\|f(s)-f(t)\|_p}{d_G(s,t)} \). Note that \( \gamma > 0 \) since \( f \) is injective.

Let \( \delta = \delta(p) \) be a constant to be chosen shortly. Recalling that \( d_G(s,t) = 9D \) and \( d_G(s,r^{(1)}) = 4D \), observe that if \( \rho_0^q \geq \left(1 + \delta \frac{\beta \gamma}{\gamma^q}\right) (4\gamma D)^q \), then

\[
\max\left(\frac{\|\tilde{f}(s) - \tilde{f}(s')\|_p^q}{d_G(s,s')^q}, \frac{\|\tilde{f}(s') - \tilde{f}(r^{(1)})\|_p^q}{d_G(s',r^{(1)})^q}\right) \geq \gamma^q + \delta \beta q,
\]

verifying (7). The symmetric argument holds for \( \rho_{D/2+1} \), thus we may assume that

\[
\rho_0, \rho_{D/2+1} \leq 4\gamma D \left(1 + \delta \frac{\beta \gamma}{\gamma^q}\right)^{1/q} \leq 4\gamma D \left(1 + \delta \frac{\beta \gamma}{\gamma^q}\right).
\]

Similarly, we may assume that \( \rho_{i,1}, \rho_{i,2} \leq \gamma \left(1 + \delta \frac{\beta \gamma}{\gamma^q}\right)^{1/q} \) for every \( i \in [D/2] \).

In this case, by (8), there must exist an index \( j \in \{1, 2, \ldots, D/2\} \) such that

\[
\rho_j \geq \left(1 - 8\delta \frac{\beta \gamma}{\gamma^q}\right) 2\gamma.
\]

Now, consider a vertical edge \((u^{(2j)}, v^{(2j)})\) with \( \|f(u^{(2j)}) - f(v^{(2j)})\|_p \geq \beta \). Also consider the vertices \( v^{(2j-1)} \) and \( v^{(2j+1)} \). We now replace the use of orthogonality ((F4) in Lemma 3.1) with Lemma 3.7

We apply one of (5) or (6) of these two inequalities with \( x = f(u^{(2j)}), v = f(v^{(2j)}), u = f(v^{(2j-1)}), w = f(v^{(2j+1)}) \). In the case \( p \geq 2 \), we use (5) to conclude that

\[
\|f(u^{(2j)}) - f(v^{(2j-1)})\|_p + \|f(u^{(2j)}) - f(v^{(2j+1)})\|_p \geq 2^{-p+2} \rho_j^p + 2^{-q+2} \beta^p - \rho_j^2 - \rho_j^p \geq 2\gamma^p + 2^{-p+2} \beta^p - 34\delta \beta \rho^p.
\]
Thus choosing $\delta = \frac{2^{1-p}}{34p}$ yields the desired result for one of $(u^{(2j)}, v^{(2j-1)})$ or $(u^{(2j)}, v^{(2j+1)})$.

In the case $1 \leq p \leq 2$, we use (6) to conclude that

$$
\|f(u^{(2j)}) - f(v^{(2j-1)})\|_p^2 + \|f(u^{(2j)}) - f(v^{(2j+1)})\|_p^2 \geq \rho_j^2 + (p-1)\beta^2 - \rho_{j,1}^2 - \rho_{j,2}^2.
$$

A similar choice of $\delta$ again yields the desired result. \qed

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