PROOF OF SWISS CHEESE CONJECTURE

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To the beautiful country of Confoederatio Helvetica.

ABSTRACT: For an associative algebra $A$ we consider the pair “the Hochschild cochain complex $C^\bullet(A, A)$ and the algebra $A$”. There is a natural 2-colored operad which acts on this pair. We show that this operad is quasi-isomorphic to the singular chain operad of Voronov’s Swiss Cheese operad. This statement is the 2-dimensional case of the conjecture formulated by M. Kontsevich in [19].

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1. Introduction

The interest to various versions [4], [6], [15], [17], [20], [21], [22], [24], [29], [31] of the Deligne conjecture on Hochschild complex is motivated by generalizations of the famous Kontsevich’s formality theorem [18]. Thus, in recent preprint [21] M. Kontsevich and Y. Soibelman proposed a proof of the chain version of Deligne’s conjecture for Hochschild complexes of an $A_\infty$-algebra. This is an important step in proving the formality for the homotopy calculus algebra of Hochschild (co)chains [10].

Let $A$ be an associative algebra and $C^\bullet(A, A)$ be the Hochschild cochain complex of $A$. The original version of Deligne’s conjecture says that the operad of natural operations on $C^\bullet(A, A)$ is quasi-isomorphic to the singular chain operad of the operad $E_2$ of little discs [8], [23]. This statement is not very precise because there are different choices of what one may call “the operad of natural operations on $C^\bullet(A, A)$.” One may use the so-called minimal operad of M. Kontsevich and Y. Soibelman [20] or the operad of braces [13], [16] as in [24] and [32] or the “big operad” of M. Batanin and M. Markl [3]. Due to works of various people [3], [6], [20], [24], [27], and [32] it is now known that all these operads are quasi-isomorphic to the singular chain operad of the operad $E_2$.

The topological operad $E_2$ of little discs admits a natural extension to a 2-colored topological operad which is called the Swiss Cheese operad $SC_2$. This operad was proposed by A. Voronov in [33].

In [33] A. Voronov also described the homology operad $H_{-\bullet}(SC_2)$. More precisely, he showed that an algebra over the operad $H_{-\bullet}(SC_2)$ is a pair of graded vector spaces $(V_1, V_2)$, where $V_1$ is a Gerstenhaber algebra, and $V_2$ is an associative algebra equipped with a module structure over the commutative algebra $V_1$

\begin{equation}
V_1 \otimes V_2 \rightarrow V_2,
\end{equation}

satisfying the following condition

\begin{equation}
(u_1 \cdot v_1) \ldots (u_n \cdot v_n) = (u_1 \ldots u_n) \cdot (v_1 \ldots v_n),
\end{equation}

where $u_i \in V_1$, $v_i \in V_2$, and for the multiplication of the corresponding elements we use either the associative algebra structure in $V_2$ or the commutative algebra structure in $V_1$.

It is not hard to prove the following proposition:

**Proposition 1.1.** If $A$ is an associative algebra and $HH^\bullet(A, A)$ is its Hochschild cohomology then the pair $(HH^\bullet(A, A), A)$ forms an algebra over the operad $H_{-\bullet}(SC_2)$.

**Proof.** Indeed the associative algebra structure on $A$ is already given. $HH^\bullet(A, A)$ is a Gerstenhaber algebra due to [12]. Finally, to define the module structure on $A$ over the commutative algebra $HH^\bullet(A, A)$ we use the fact that the zeroth Hochschild cohomology $HH^0(A, A)$ is the center $Z(A)$ of $A$. Namely, we declare

\[ z \cdot a = \begin{cases} z \cdot a & \text{if } z \in HH^0(A, A) = Z(A), \\ 0 & \text{otherwise}. \end{cases} \]

Equation (1.2) is nontrivial only when $u_i \in HH^0(A, A)$. In this case the required condition is automatically satisfied since $u_i$’s are elements of the center $Z(A)$ of $A$. \[ \square \]

In this paper we prove the Swiss Cheese version of Deligne’s conjecture which extends Proposition 1.1 to the level of cochains.

\footnote{In particular, it means that $V_1$ is a commutative algebra.}
To formulate this version of Deligne’s conjecture we, first, construct a 2-colored DG operad $\Lambda$ of natural operations on the pair $(C^\bullet(A, A); A)$. Roughly speaking, this operad is generated by the insertions of a cochain into a cochain, the cup-product of cochains and the insertions of elements of the algebra $A$ into a cochain. The precise description of $\Lambda$ is given in Section 2.

The main result of this paper is the following theorem

**Theorem 1.2.** The 2-colored DG operad $\Lambda$ of natural operations on the pair $(C^\bullet(A, A), A)$ is quasi-isomorphic to the singular chain operad of Voronov’s Swiss Cheese operad $SC_2$. The induced action of the homology operad $H_{-\bullet}(SC_2)$ on the pair $(HH^\bullet(A), A)$ recovers the one from Proposition 1.1.

We prove this theorem using ideas from [27] and Batanin’s theorem [2] which identifies the homotopy type of Voronov’s Swiss Cheese operad with that of the symmetrization of a contractible cofibrant Swiss Cheese type 2-operad. The required facts about 2-operads are reviewed in Section 4.

**1.1. Remarks on higher dimensional versions.** Voronov’s Swiss Cheese operad admits the obvious higher dimensional analogue $SC_d$ ($d \geq 2$). This operad extends the operad of $d$-cubes in the same way as the operad $SC_2$ extends the operad of little disks. From this point of view, Theorem 1.2 is a 2-dimensional case of the generalized Deligne conjecture proposed by M. Kontsevich in paper [19]. In order to formulate this conjecture we need to recall from [14] that a $d$-algebra is an algebra over the homology operad $H_{-\bullet}(E_d)$ of the operad of little $d$-cubes $E_d$.

The generalized Deligne conjecture of M. Kontsevich [19] says that the DG operad of natural operations on the pair “a $d$-algebra and its Hochschild complex” is quasi-isomorphic to the singular chain operad of $SC_{d+1}$. This statement would imply that the Hochschild complex of $d$-algebra is equipped with an action of a DG operad which is quasi-isomorphic to the singular chain operad of the operad $E_{d+1}$ of little $d + 1$-cubes.

This corollary of Kontsevich’s conjecture was proved in the full generality by P. Hu, I. Kriz and A. Voronov [15]. In paper [26] this statement was proved under the assumption that the ground field has characteristic zero.

In [11] J.N.K. Francis showed that an appropriate deformation complex for a $d$-algebra $A$ is an extension of its Hochschild complex by $A$. In the spirit of this result Kontsevich’s conjecture [19] can be reformulated as follows: the DG operad of natural operations on the deformation complex of a $d$-algebra is quasi-isomorphic to the singular chain operad of $SC_{d+1}$.

**Notation and conventions.** We denote by $k$ the ground field and by “(co)chain complexes” we mean (co)chain complexes of vector spaces over $k$. $A$ is a unital associative algebra over $k$ and $C^\bullet(A, A)$ is the normalized Hochschild cochain complex of $A$ with coefficients in $A$

$$C^\bullet(A, A) = \text{hom}(A/k)^{\otimes \bullet}, A).$$

The abbreviation SMC stands for “symmetric monoidal category” and the notation $1$ is reserved for the unit of a symmetric monoidal category. We also use the abbreviation SC for “Swiss Cheese type” when we discuss the Swiss Cheese type symmetric operads, 2-operads, sets, ordinals, and 2-trees.

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\footnote{See the paragraph after Definition 8 in Section 2.5 in [19].}
2. The DG operad $\Lambda$ of natural operations on the pair $(C^\bullet(A,A); A)$

We construct the DG operad $\Lambda$ using an auxiliary operad of sets $\mathcal{O}$. In order to introduce this operad we slightly extend our consideration and let $A$ be a unital monoid in some tensor (not necessarily symmetric) category. Then we consider the full nonsymmetric endomorphism operad of $A$

$$C_A(n) := \text{hom}(A^\otimes n; A), \quad n \geq 0.$$  

It is clear that $A$ is naturally a $C_A$-algebra. The associative unital structure on $A$ gives rise to a map of nonsymmetric operads $\text{assoc} \to C_A$, where $\text{assoc}$ is the nonsymmetric operad of sets controlling unital monoids; each space $\text{assoc}(n), n \geq 0,$ is a point.

We fix a set of colors $X_\rho := \mathbb{N} \sqcup \{a\}$ and define a $X_\rho$-colored symmetric operad $\mathcal{O}$ in the category of sets as an operad whose algebra structure on an $X_\rho$-family of objects $(C(n), n \in \mathbb{N}; A)$ is:

- a nonsymmetric operad structure on the collection of objects $C(n)$;
- a map of nonsymmetric operads $\text{assoc} \to C$;
- a $C$-algebra structure on $A$.

The operad $\mathcal{O}$ has the following sets of operations:

- $\mathcal{O}(k)_{n_1,n_2,\ldots,n_k}^n := \mathcal{O}((n_1,n_2,\ldots,n_k) \mapsto n))$ where all the entries are in $\mathbb{N}$;

- $\mathcal{O}(k, N)_{n_1,n_2,\ldots,n_k} := \mathcal{O}((n_1,n_2,\ldots,n_k, a, a, \ldots a) \mapsto a), \quad N \geq 0.$

The operadic sets for other colorings are empty.

These sets can be described explicitly in terms of planar trees (see 3.0.5)

2.0.1. The unary operations in a colored operad endow the set of colors with a category structure. In our case, this category structure is a disjoint union of the simplicial category $\Delta$ and a point.

More precisely:

- $\text{hom}(n, a) = \text{hom}(a, n) = \emptyset$ for all $n \in \mathbb{N}$;
- $\text{hom}(n, m) = \text{hom}_\Delta([n], [m])$ for all $n, m \in \mathbb{N}$;
- $\text{hom}(a, a) = \{\text{Id}\}$.

This implies that the operadic sets of our colored operad have a natural polysimplicial/cosimplicial structure, namely:

the collection of sets

$$\mathcal{O}(k)_{n_1,n_2,\ldots,n_k}^n,$$

as $n_1, n$ run through $\mathbb{N}$, is a functor

$$\mathcal{O}(k) : (\Delta^{op})^k \times \Delta \to \text{Sets},$$

(the functor is simplicial in each of the arguments $n_1, n_2, \ldots, n_k$ and cosimplicial in $n$);

likewise, for each $N$, the collection of sets

$$\mathcal{O}(k, N)_{n_1,n_2,\ldots,n_k}$$
forms a functor $O(k, N) : (\Delta^{op})^k \to \text{Sets}$.  

2.0.2. Let $S$ be a cosimplicial complex given by 

$$ S([n])^k := C_n^k(\Delta^n, k), $$

where the complex on the right hand side is the normalized chain complex of the simplex $\Delta^n$ put in the non-positive degrees.  

Using this complex, we can convert polysimplicial/cosimplicial sets into complexes. Namely, let $F : (\Delta^{op})^k \to \text{Sets}$ be a functor. Set 

$$ |F| := k[F] \otimes_{(\Delta^{op})^k} S^{\otimes k}, $$

where $S^{\otimes k} : \Delta^k \to \text{complexes}$:

$$ S^{\otimes k}([n_1], [n_2], \ldots, [n_k]) := \bigotimes_{i=1}^k S([n_i]). $$

Given a functor 

$$ G : (\Delta^{op})^k \times \Delta \to \text{Sets}, $$

denote by $G^n$ the evaluation at $[n] \in \Delta$ so that 

$$ G^n : (\Delta^{op})^k \to \text{Sets} $$

and 

$$ n \mapsto G^n $$

is a functor from $\Delta$ to the category of $k$-simplicial sets. Set 

$$ |G| := \text{hom}_\Delta(S^*, |G^*|). $$

2.0.3. Set 

$$ |O|(k) := |O(k)|; $$

$$ |O|(k, N) := |O(k, N)|. $$

We see that these spaces form a 2-colored DG operad. Denote this two-colored operad by $|O|$.  

Now let $A$ be a unital associative algebra over the field $k$. It is easy to see that the normalized Hochschild cochain complex $C^\bullet(A, A)$ (1.3) can be written as 

$$ C^\bullet(A, A) := \text{hom}_\Delta(S^*, C_A(*)). $$

Therefore the DG operad $|O|$ acts on the pair $(C^\bullet(A, A), A)$. This two-colored DG operad $|O|$ is the desired operad $\Lambda$ of natural operations on the pair $(C^\bullet(A, A), A)$ and our Theorem 1.2 can be reformulated as 

**Theorem 2.1.** The operad $|O|$ is weakly equivalent to the singular chain operad of Voronov’s Swiss Cheese operad $SC_2$. The induced action of the homology operad $H_\text{-}^\bullet(SC_2)$ on the pair $(HH^\bullet(A), A)$ recovers the one from Proposition 1.1. 

We prove this theorem in Section 8.  

**Remark.** Our method also works in the topological setting: one can apply the topological realization functors to the polysimplicial/cosimplicial sets from 2.0.1 so as to get a topological colored operad $|O|_{\text{top}}$. This operad can be proven to be weakly equivalent to Voronov’s Swiss Cheese operad.
3. Studying the operad $\mathcal{O}$

3.0.4. Let us pass to a slightly more invariant language. Recall that our set of colors is $\mathbb{N} \cup \{a\}$, and that an $\mathcal{O}$-algebra structure on the collection of spaces $(C(n), n \in \mathbb{N}; A)$ is the same as a nonsymmetric operad structure on the collection of spaces $C(n)$, $n \in \mathbb{N}$, a map of operads $\text{assoc} \to C$, and a $C$-algebra structure on $A$. The definition of a nonsymmetric operad implies that we have a total order on the set of arguments so that it is better to replace the natural numbers with isomorphism classes of finite ordinals: the number $n$ gets replaced with the ordinal $< n >\{1 < 2 < \cdots < n\}$.

Given finite sets $S$, $S_c$, an $S$-family $\{I_s\}_{s \in S}$ and an $S_c$-family $\{I_s\}_{s \in S}$, of finite (possibly empty) ordinals, an ordinal $J$, and a set $S_a$, we then have the following operadic sets:

\begin{equation}
\mathcal{O}(S)_{\{I_s\}_{s \in S}}
\end{equation}

\begin{equation}
\mathcal{O}(S_c, S_a)_{\{I_s\}_{s \in S}}
\end{equation}

where in (3.1) the set of arguments is $S$ and the coloring of $s \in S$ is $I_s$, the result has the color $J$. In (3.2), the set of arguments is $S_c \sqcup S_a$ the argument $s \in S_c$ has color $I_s$ and all arguments from $S_a$ have color $a$. The result also has color $a$.

3.0.5. Planar trees. For a finite set $S$ and ordinals $I_s, s \in S; J$, we describe

\begin{equation}
\mathcal{O}(S)_{\{I_s\}_{s \in S}}
\end{equation}

as the set of equivalence classes of planar trees $T$ with the following structure:

— a subset of the set of vertices of a tree $T$ is identified with $S \sqcup J$ in such a way that with elements of $J$ we may only identify the terminal vertices of $T$. We call the vertices identified with elements of $S \sqcup J$ marked.

— the ordered set of edges originating at the vertex marked by $s \in S$ is identified with $I_s$.

Notice that, the subset of vertices identified with $J$ acquires from $J$ a natural linear order. We require that this linear order coincides with the order which is obtained by going around the tree in the clockwise direction starting from the root vertex.

The equivalence relation is the finest one in which two such trees are equivalent if one of them can be obtained from the other by either:

the contraction of an edge with unmarked ends

or: removing an unmarked vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

Example. The planar tree $T$ in figure 1 represents an element in $\mathcal{O}(S)_{\{I_s\}_{s \in S}}$ with $S = \{s_1, s_2\}$, $J = \{j_1, j_2, j_3\}$, $I_{s_1} = \emptyset$, and $I_{s_2} = \{3\}$. In all the figures we use circles to denote the vertices marked by elements of $S$ and arrows to denote vertices marked by elements of $J$. Thus, in figure 1 the vertices $a_1$, $a_2$, $a_3$, and $a_4$ are unmarked. The vertices $a_1$ and $a_2$ correspond to the product in $\text{assoc}(2)$, $a_3$ corresponds to the identity operation in $\text{assoc}(1)$, and $a_4$ corresponds to the unit in $\text{assoc}(0)$.

In figures 2 and 3 we depict the trees $T_1$ and $T_2$ which are equivalent to the original tree $T$.

The tree $T_1$ is obtained from $T$ by removing the unmarked vertex $a_3$ and joining the two edges adjacent to this vertex into one edge. The tree $T_2$ is obtained from $T$ by contracting the edge with the unmarked ends $a_1$ and $a_2$. The unmarked vertex $a$ of the tree $T_2$ (figure 3) corresponds to the unique element of $\text{assoc}(3)$.

Applying both of the equivalence operations to the tree $T$ in figure 1 we obtain the tree $T_3$ depicted in figure 4. Although the tree $T_3$ has unmarked vertices $a$ and $a_4$, it is no longer possible
to apply any equivalence operation to $T_3$. We call such trees minimal. It is obvious that every equivalence class of $\mathcal{O}(S)_{(I_s)_{s \in S}}$ contains at least one minimal tree.

The equivalence class containing all these planar trees $T$, $T_1$, $T_2$, and $T_3$ corresponds to the operation which sends a Hochschild cochain $P_1 \in C_A(0)$ and a Hochschild cochain $P_2 \in C_A(3)$ to the Hochschild cochain $Q \in C_A(3)$ defined by the formula

$$Q(b_1, b_2, b_3) = P_2(b_1, 1, b_2) b_3 P_1,$$

$$b_1, b_2, b_3 \in A.$$
3.0.6. Let us now describe the set

$$\mathcal{O}(S_c, S_a) \{ I_s \}_{s \in S_c},$$

where we use the same notation as above.

Each element of this set can be represented by a planar tree $T$ with the following additional structure:

— a subset of the set of vertices of $T$ is identified with $S_c \sqcup S_a$ in such a way that with elements of $S_a$ we may only identify the terminal vertices of $T$. We call the vertices identified with elements of $S_c \sqcup S_a$ marked;

— the ordered set of edges originating at the vertex marked by $s \in S_c$ is identified with $I_s$.

The equivalence relation on the set of isomorphism classes of such trees is defined in the same way as in the previous section.

This description implies the following identification:

$$\mathcal{O}(S_c, S_a) \{ I_s \}_{s \in S_c} = \bigcup_{\succ \in \text{ord}(S_a)} \mathcal{O}(S_c^{S_a, \succ}) \{ I_s \}_{s \in S_c},$$

where $\text{ord}(S_a)$ is the set of all total orders on $S_a$.

Let us also describe the degenerate cases. In the case $S$ is the empty set $\emptyset$ we have

$$\mathcal{O}(\emptyset)^J = \text{assoc}(J).$$

If $S_a = \emptyset$ then

$$\mathcal{O}(S_c, \emptyset) \{ I_s \}_{s \in S_c} = \mathcal{O}(S_c, \emptyset)^\emptyset \{ I_s \}_{s \in S_c}.$$ 

Finally, if $S_c$ is empty then

$$\mathcal{O}(\emptyset, S_a) = \bigcup_{\succ \in \text{ord}(S_a)} \text{assoc}(S_a, \succ).$$

3.0.7. Replacing trees with sequences. There is another way to label the elements of $\mathcal{O}$.

We need the following notation. Given a vertex $v$ of a planar tree marked by an element $s \in S$, let us draw a little circle centered at this vertex. This circle gets split into sectors, the set of these sectors is totally ordered in the clockwise order. Denote this ordered set by $I'_s$. The set of edges originating at $v$ is naturally identified with $I'_s$, where $I'_s$ is the set of pairs $i_1 i_2$, where $i_2$ is an immediate successor of $i_1$ and $i_1, i_2 \in I'_s$. We see that $I'_s$ is the next ordinal after $I_s$. Below, given an ordinal $K$, we denote by $K'$ its next ordinal.

Given a planar tree $T$ which defines an element $\mathcal{T} \in \mathcal{O}(S)^\{ I_s \}_{s \in S}$, let us consider its small tubular neighborhood and let us walk along its boundary starting from the root vertex of our tree in the clockwise direction. On our way, we will meet the vertices marked by elements of $S$ and vertices.
marked by elements of $J$. (The latter ones are terminal according to our requirement.) Every time we approach a vertex $v$ marked by $s \in S$, we are at a certain sector from $I_s'$. Thus, given a planar tree $T$ representing an element $\mathcal{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}$ we obtain a total order $>_T$ on the set $\bigsqcup_{s \in S} I_s' \sqcup J$.

**Example.** Let us show how we obtain the order for the tree $T_3$ given in figure 4. This tree represents an element in $\mathcal{O}(\{s_1, s_2\}_{I_{s_1}, I_{s_2}})$ where $I_{s_1}$ is empty and $I_{s_2} = \{<3\}$. This means that the vertex labeled by $s_1$ (see figure 5) is surrounded by a single sector $s_1^1$, while the vertex labeled by $s_2$ is surrounded by four sectors $s_2^1, s_2^2, s_2^3, s_2^4$ which we number in the clockwise direction. Walking along the boundary of a small tubular neighborhood of $T_3$, as it is shown on figure 5, we get the following order on the set $\{s_1^1, s_2^1, s_2^2, s_2^3, s_2^4, j_1, j_2, j_3\}$:

$$s_2^1 < j_1 < s_2^2 < s_2^3 < j_2 < s_2^4 < j_3 < s_1^1.$$ 

For every planar tree $T$ representing an element $\mathcal{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}$ the corresponding total order $>_T$ satisfies:

1) let $T_1, T_2$ be planar trees representing the same element $\mathcal{T} = \mathcal{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}$

Then $>_T \Rightarrow T_1 = T_2$. Hence, for each $\mathcal{T} \in \mathcal{O}(S)_{\{I_s\}_{s \in S}}$ we have a well defined order, to be denoted by $>_T$:

2) The total order $>_T$ agrees with the existing orders on $I_s', J$;

3) given distinct $s_1, s_2 \in S$ it is impossible to find $i_1, j_1 \in I_{s_1}'; i_2, j_2 \in I_{s_2}'$ such that

$$i_1 <_T i_2 <_T j_1 <_T j_2.$$

Let us denote the set of all total orders satisfying conditions 2) and 3) by $\text{Ord}(S)_{\{I_s\}_{s \in S}}$.

**Proposition 3.1.** The set $\text{Ord}(S)_{\{I_s\}_{s \in S}}$ is in 1-to-1 correspondence with $\mathcal{O}(S)_{\{I_s\}_{s \in S}}$.

**Proof.** Let us describe an inductive construction which assigns to each total order $\pi$ on $\bigsqcup_{s \in S} I_s' \sqcup J$
satisfying conditions 2) and 3) a minimal tree \( T \) which recovers the order \( \pi \) by walking along a small tubular neighborhood of \( T \).

The induction goes by the order \(|S|\) of the set \( S \).

For \( S = \emptyset \) the set \( \text{Ord}(S)_{\{I_s\}_{s \in S}} \) consists of a single element. That is the given order on \( J \). In this case it is very easy to find a minimal tree which recovers this order. It is also easy to see that such a tree is unique.

Let us suppose that we can construct a desired minimal tree for all elements of \( \text{Ord}(S_0)_{\{I_s\}_{s \in S_0}} \) if \( |S_0| < |S| \). We need to present a construction for every \( \pi \in \text{Ord}(S)_{\{I_s\}_{s \in S}} \).

Condition 3) implies that for an arbitrary pair \( s, \tilde{s} \in S \) exactly one of the following options realizes:

1. all elements of \( I'_s \) are smaller than elements of \( I'_\tilde{s} \),
2. all elements of \( I'_s \) are greater than elements of \( I'_\tilde{s} \),
3. \( I'_s \) splits into two non-empty subsets such that all elements of the first subset are smaller than all elements of \( I'_\tilde{s} \) while all the elements of the second subset are greater than elements of \( I'_\tilde{s} \),
4. same as (3) with \( s \) and \( \tilde{s} \) interchanged.

If the third (resp. fourth) option realizes we say that \( s < \tilde{s} \) (resp. \( \tilde{s} < s \)). Thus we get a partial order on the set \( S \).

Since \( S \) is finite, it has at least one minimal element. Let us denote this element by \( s_{\min} \) and introduce the interval \( \bar{I}_{s_{\min}} \) of the ordinal (3.3) between the minimal element of \( I'_{s_{\min}} \) and the maximal element of \( I'_{s_{\min}} \). It is obvious that \( \bar{I}_{s_{\min}} \) consists of elements of \( I'_{s_{\min}} \) and some elements of \( J \).

Let us consider the set

\[(3.4) \bigcup_{s \in S^{(1)}} I'_s \sqcup J^{(1)},\]

where \( S^{(1)} = S \setminus \{s_{\min}\} \) and \( J^{(1)} \) is obtained from \( J \) by attaching the element \( s_{\min} \) and removing those elements of \( J \) which belong to the interval \( \bar{I}_{s_{\min}} \). In other words,

\[(3.5) J^{(1)} = J \sqcup \{s_{\min}\} \setminus (J \cap \bar{I}_{s_{\min}}).\]

Notice that, the set (3.4) is obtained from (3.3) by replacing the interval \( \bar{I}_{s_{\min}} \) by a single element \( s_{\min} \). Hence, (3.4) acquires a natural total order. Let us denote this order by \( \pi^{(1)} \).

It is not hard to see that \( \pi^{(1)} \) satisfies conditions 2) and 3) and hence is an element of the set

\[\text{Ord}(S^{(1)}_{\{I_s\}_{s \in S^{(1)}}})^{(1)} \]

Since \(|S^{(1)}| < |S|\) we can assign to \( \pi^{(1)} \) a minimal tree \( T^{(1)} \) which recovers the order \( \pi^{(1)} \) on (3.4).

To construct the desired tree \( T \) we observe that the element \( s_{\min} \) is identified with an external vertex \( v \) of \( T^{(1)} \). So, we draw from this vertex \( v \) edges labeled by elements \( i_1i_2 \) of \( \bar{T}_{s_{\min}} \). Recall that \( \bar{T}_{s_{\min}} \) consists of pairs \( i_1i_2 \), where \( i_2 \) is an immediate successor of \( i_1 \) and \( i_1, i_2 \in I'_{s_{\min}} \).

Let us denote by \( t^{i_1i_2} \) the terminal vertex of the edge corresponding to \( i_1i_2 \).

If there are no elements of \( J \) between \( i_1 \) and \( i_2 \) then we leave \( t^{i_1i_2} \) as an unmarked terminal vertex of the tree \( T \).

If there is only one element \( j \) of \( J \) between \( i_1 \) and \( i_2 \) we leave \( t^{i_1i_2} \) as a terminal vertex of \( T \) and mark it by \( j \).
Finally, if we have elements \( j_1, \ldots, j_m \in J \) \((m > 1)\) between \( i_1 \) and \( i_2 \), then we draw from the vertex \( t^{i_1i_2} \) exactly \( m \) terminal edges. We leave \( t^{i_1i_2} \) unmarked and mark the corresponding terminal vertices by \( j_1, \ldots, j_m \) in the clockwise direction.

Let us denote the resulting tree by \( T \). It is not hard to see that, since \( T^{(1)} \) recovers the order \( \pi^{(1)} \) on (3.4) the tree \( T \) recovers the order \( \pi \) on (3.3). It is also obvious that, since the tree \( T^{(1)} \) is minimal, so is \( T \).

We already have a map from the set \( \mathcal{O}(S)^J_{\{I_s\}_{s \in S}} \) to the set \( \text{Ord}(S)^J_{\{I_s\}_{s \in S}} \) which is defined by assigning the total order to a tree. Let us denote this map by \( \nu_{\text{ord}} \)

\[
\nu_{\text{ord}} : \mathcal{O}(S)^J_{\{I_s\}_{s \in S}} \to \text{Ord}(S)^J_{\{I_s\}_{s \in S}}.
\]

The above construction provides us with the map in the opposite direction:

\[
\nu_{\text{tree}} : \text{Ord}(S)^J_{\{I_s\}_{s \in S}} \to \mathcal{O}(S)^J_{\{I_s\}_{s \in S}}.
\]

It is clear from the construction that the composition \( \nu_{\text{ord}} \circ \nu_{\text{tree}} \) is the identity on \( \text{Ord}(S)^J_{\{I_s\}_{s \in S}} \).

It is not hard to verify that if we start with a minimal tree \( T \) representing an element \( T \in \mathcal{O}(S)^J_{\{I_s\}_{s \in S}} \), and assign to \( T \) the total order \( \pi \) from \( \text{Ord}(S)^J_{\{I_s\}_{s \in S}} \), then the above construction gives us back exactly the same minimal tree \( T \). This implies that the composition \( \nu_{\text{tree}} \circ \nu_{\text{ord}} \) is the identity on the set \( \mathcal{O}(S)^J_{\{I_s\}_{s \in S}} \) and the proposition follows\(^3\). \( \square \)

**Remark.** The construction presented in the proof is reminiscent of Kontsevich-Soibelman pairs of complementary orders [20].

3.0.8. Modification. Given a total order as above, we can construct a map

\[
Q : \bigsqcup_{s \in S} I'_s \to J'
\]

as follows. We identify \( J' = J \cup \{M\} \), where \( M > J \). Set \( Q(x) = j \) if \( j \) is the minimal element from \( J \) such that \( j > x \); if there is no such \( j \), set \( Q(x) = M \).

Thus, given a total order as in the previous subsection, we obtain the following data:
— a total order on the set

\[
\mathcal{I} := \bigsqcup_{s \in S} I'_s;
\]

a non-decreasing map

\[
\mathcal{I} \to J'.
\]

These data should satisfy:

i) the order on \( \mathcal{I} \) agrees with those on each \( I'_s \);

ii) same as condition 3) from Sec 3.0.7.

Denote the set of such objects by

\[
s\mathcal{O}(S)^{J'}_{\{I'_s\}_{s \in S}}.
\]

This set is in 1-to-1 correspondence with the set of total orders from the previous subsection, hence we have a bijection with the set \( \mathcal{O}(S)^J_{\{I_s\}_{s \in S}} \):

\[
(3.6) \quad s\mathcal{O}(S)^{J'}_{\{I'_s\}_{s \in S}} \to \mathcal{O}(S)^J_{\{I_s\}_{s \in S}}.
\]

\(^3\)In particular, it implies that in each equivalence class of trees there is exactly one minimal tree.
3.0.9. Likewise, one identifies the set $\mathcal{O}(S_{c}, S_{a})_{\{I_s\}_{s \in S_t}}$ with the set of total orders on

$$\bigsqcup_{s \in S_c} I'_s \sqcup S_a$$

satisfying:

— the total order agrees with those on each $I'_s$;
— same as condition 3) from Sec. 3.0.7.

Denote the set of such total orders by $s\mathcal{O}(S_{c}, S_{a})_{\{I_s\}_{s \in S_t}}$.

The construction of the 1-to-1 correspondence

$$s\mathcal{O}(S_{c}, S_{a})_{\{I_s\}_{s \in S_t}} \to \mathcal{O}(S_{c}, S_{a})_{\{I_s\}_{s \in S_t}}$$

is the same as in the previous subsection.

3.1. Operadic structure on $s\mathcal{O}$. Let $\mathbb{N}'$ be the set of isomorphism classes of non-empty finite ordinals. The identifications (3.6), (3.7) imply that the colored operad structure on $\mathcal{O}$ induces a colored operad structure on the collection of spaces $s\mathcal{O}$. It turns out that this operadic structure can be naturally formulated in terms of $s\mathcal{O}$.

**Warning.** We will not use the symbol ‘’ anymore when talking about ordinals from $\mathbb{N}'$. The reason is that in the sequel, instead of the operad $\mathcal{O}$, the isomorphic operad $s\mathcal{O}$ will be used.

3.1.1. Let $T$ be a finite set and let $S_t$ be a $T$-family of finite sets. Let $S := \sqcup_t S_t$ and $p : S \to T$ be the map which sends $S_t$ to $t$.

Suppose we are given ordinals $I_s$, $s \in S$; $J_t$, $t \in T$, and $J$.

Describe the operadic composition

$$s\mathcal{O}(T)_{\{J_t\}_{t \in T}} \times \prod_{t \in T} s\mathcal{O}(S_t)_{\{I_s\}_{s \in S_t}} \to s\mathcal{O}(S)_{\{I_s\}_{s \in S}}.$$

Let $u \in s\mathcal{O}(T)_{\{J_t\}_{t \in T}}$ and $u_t \in s\mathcal{O}(S_t)_{\{I_s\}_{s \in S_t}}$. Let us describe the composition $v$ of these elements.

1) the total order $>_v$ is defined as a unique one
— which agrees with the orders $>_u$, on

$$\sqcup_{s \in S_t} I_s \subset \sqcup_{s \in S} I_s$$

for each $t \in T$;
— for which the map

$$\sqcup_{t \in T} F_{u_t} : \sqcup_{s \in S} I_s \to (\sqcup_{t \in T} J_t, >_u)$$

is non-decreasing.

2) the map $F_v$ is just the composition of (3.8) with the map $F_u$.

3.1.2. To describe the remaining composition maps we consider sets $S_{c}, S_{a}, T_{c}, T_{a}$ and let $P : S_{c} \sqcup S_{a} \to T_{c} \sqcup T_{a}$ be a map such that $P^{-1} T_{c} \subset S_{c}$. For $t \in T_a$ we set $(P^{-1} t)_{a} := P^{-1} t \cap S_{a}$; $(P^{-1} t)_{c} := P^{-1} t \cap S_{c}$.

Let $\{I_s\}_{s \in S_t}$; $\{J_t\}_{t \in T_t}$; be non-empty ordinals. We need to define the following composition map:

$$s\mathcal{O}(T_{c}, T_{a})_{\{J_t\}_{t \in T_t}} \times \prod_{t \in T_t} s\mathcal{O}(P^{-1} t)_{a} \times \prod_{t \in T_a} s\mathcal{O}((P^{-1} t)_{a}, (P^{-1} t)_{a})_{\{I_s\}_{s \in (P^{-1} t)_{a}}}$$

$$\to s\mathcal{O}(S_{c}, S_{a})_{\{I_s\}_{s \in S_t}}.$$
Choose elements
\[ v \in sO(T_c, T_a)_{\{J_t\}_{t \in T_c}}; \]
\[ u_t \in sO(P^{-1}t)_{\{I_s\}_{s \in (P^{-1}t)}}; \quad t \in T_c; \]
\[ u_t \in sO((P^{-1}t)_c, (P^{-1}t)_a)_{\{I_s\}_{s \in (P^{-1}t)_c}}; \quad t \in T_a \]
and denote their composition by \( w \).

Let us set
\[ \mathcal{I}_w := \bigsqcup_{s \in S_t} I_s \sqcup S_a. \]
and define a map
\[ F : \mathcal{I}_w \to \mathcal{I}_v, \]
where
\[ \mathcal{I}_v = \bigsqcup_{t \in T_c} J_t \sqcup T_a, \]
as follows:
— If \( t \in T_c \) then the restriction of \( F \) to the subset
\[ \mathcal{I}_u_t := \bigsqcup_{s \in P^{-1}t} I_s, \]
should coincide with the map \( F_{u_t} : \mathcal{I}_{u_t} \to J_t; \)
— if \( t \in T_a \) then the restriction of \( F \) to the subset
\[ \mathcal{I}_u_t := \bigsqcup_{s \in (P^{-1}t)_c} I_s \sqcup (P^{-1}t)_a \]
should send every element to \( t \).

We define the order \( >_w \) as the unique one for which the map \( F \) is non-decreasing and which agrees with the orders \( >_{u_t} \) on \( \mathcal{I}_{u_t}, t \in T_c \sqcup T_a \).

4. Review of 2-operads

We are going to remind the basic definitions from Batanin’s theory of 2-operads which will be used below.

An ordinal is a finite totally ordered set. Another name for ordinals is a 1-tree.
A 2-tree \( t \) is a pair of ordinals \( S, T \) along with an order preserving map \( t : S \to T \).
A 2-tree is called pruned if the map \( t \) is surjective.

A map of 2-trees
\[ P : (t : S \to T) \to (t_1 : S_1 \to T_1) \]
is a pair of maps \( P_S : S \to S_1; \quad P_T : T \to T_1 \) such that \( t_1 P_S = P_T t; \) \( P_T \) is order preserving; \( P_S \) preserves the order on each set \( t^{-1} t, t \in T \).

This way, 2-trees form a category 2-trees.

Given \( s_1 \in S_1 \), we define a 2-tree \( P^{-1}s_1 \) as follows:
\[ t \bigg|_{(P_S)^{-1} s_1} : (P_S)^{-1} s_1 \to (P_T)^{-1} t_1(s_1). \]

A 2-operad in a symmetric monoidal category (SMC) \( C \) is defined as:
— a functor \( O : 2\text{-trees}^\times \to C \), where \( 2\text{-trees}^\times \) is the groupoid of isomorphisms of 2-trees (note that every object in this groupoid has the trivial automorphism group);
— for every map of 2-trees \( P : t \to t_1 \), where \( t : S \to T ; t_1 : S_1 \to T_1 \), there should be given a map
\[
\mathcal{O}(t_1) \otimes \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1} s_1) \to \mathcal{O}(t)
\]
called the operadic composition map.

These maps should satisfy a certain associativity property. In order to formulate it let us define the objects \( \mathcal{O}(P) \), where \( P : t \to t_1 \) is a map of 2-trees as follows:
\[
\mathcal{O}(P) := \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1} s_1).
\]
The operadic insertion maps can be rewritten as
\[
\mathcal{O}(t_1) \otimes \mathcal{O}(P) \to \mathcal{O}(t).
\]
Given a chain of maps of 2-trees
\[
t \xrightarrow{P} t_1 \xrightarrow{Q} t_2,
\]
the operadic insertion maps naturally give rise to a map
\begin{equation}
\mathcal{O}(Q) \otimes \mathcal{O}(P) \to \mathcal{O}(QP).
\end{equation}
Indeed, for every \( s_2 \in S_2 \), where \( t_2 : S_2 \to T_2 \), the map \( P \) naturally restricts to a map of 2-trees
\[
P_{s_2} : (QP)^{-1} s_2 \to Q^{-1} s_2
\]
and we have
\[
\mathcal{O}(P) \cong \bigotimes_{s_2 \in S_2} \mathcal{O}(P_{s_2}).
\]

We then define the map (4.1) as follows:
\[
\mathcal{O}(Q) \otimes \mathcal{O}(P) \cong \bigotimes_{s_2 \in S_2} \mathcal{O}(Q^{-1} s_2) \otimes \mathcal{O}(P_{s_2}) \to \bigotimes_{s_2 \in S_2} \mathcal{O}((QP)^{-1} s_2) = \mathcal{O}(QP).
\]
The associativity axiom requires that the map (4.1) be associative: the following maps should coincide:
\[
\mathcal{O}(R) \otimes \mathcal{O}(Q) \otimes \mathcal{O}(P) \to \mathcal{O}(RQ) \otimes \mathcal{O}(P) \to \mathcal{O}(RQP)
\]
and
\[
\mathcal{O}(R) \otimes \mathcal{O}(Q) \otimes \mathcal{O}(P) \to \mathcal{O}(R) \otimes \mathcal{O}(QP) \to \mathcal{O}(RQP).
\]

**Remark.** 1-trees are simply ordinals and the definition of a 1-operad based on 1-trees coincides with the definition of a nonsymmetric operad.

### 4.1. Colored 2-operads.

#### 4.1.1. Colored 2-trees. Fix a set of colors \( X_\rho \). Define a colored 2-tree \( \tau \) as:

— a 2-tree \( t_\tau : S_\tau \to T_\tau \);
— a map \( \chi_\tau : S_\tau \to X_\rho \);
— an element \( c_\tau \in X_\rho \).
4.1.2. Given colored 2-trees $\tau_1, \tau_2$ we define their map $P : \tau_1 \to \tau_2$ as follows:
- if $c_{\tau_1} = c_{\tau_2}$, then it is just a map $P : t_{\tau_1} \to t_{\tau_2}$ of the underlying 2-trees;
- if $c_{\tau_1} \neq c_{\tau_2}$, then we declare that there are no maps $\tau_1 \to \tau_2$.
This way colored 2-trees form a category.

Given such a map and $s_2 \in S_{\tau_2}$ the 2-tree $P^{-1}s_2$ naturally receives a coloring as follows.
Recall that the 2-tree $P^{-1}s_2$ is defined as
\[
\tau_1 \bigg|_{(P_S)^{-1}s_2} : (P_S)^{-1}s_2 \to (P_T)^{-1}t_{\tau_2}s_2.
\]
We then define
\[
\chi_{P^{-1}s_2} : P_S^{-1}s_2 \to X\rho
\]
as the restriction of $\chi_{\tau_1}$ and set
\[
c_{P^{-1}s_2} := \chi_{\tau_2}(s_2).
\]

4.1.3. We then define a colored 2-operad in a SMC $\mathcal{C}$ as:
- a functor $\mathcal{O}$ from the isomorphism groupoid of the category of colored 2-trees to the category $\mathcal{C}$;
- for every map $P : \tau_1 \to \tau_2$ of colored 2-trees there should be given the operadic composition map
\[
\mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2) \to \mathcal{O}(\tau_1).
\]
Next, given a map $P : \tau_1 \to \tau_2$ we define
\[
\mathcal{O}(P) := \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2)
\]
and observe that the operadic composition maps naturally produce maps
\[
\mathcal{O}(Q) \otimes \mathcal{O}(P) \to \mathcal{O}(QP),
\]
where $P : \tau_1 \to \tau_2$, $Q : \tau_2 \to \tau_3$.
Lastly we require the associativity of this map in the same way as for the non-colored 2-operads.

4.1.4. **Desymmetrization.** Given a colored symmetric operad $\mathcal{O}$, one can define a colored 2-operad $\text{des} \mathcal{O}$ by setting
\[
\text{des} \mathcal{O}(\tau) = \mathcal{O}(S_{\tau}),
\]
where the coloring on the right hand side is determined by that of $\tau$, and the operadic composition maps are inherited from those of $\mathcal{O}$.

4.2. **Unital colored 2-operads.**

4.2.1. Given a color $c \in X\rho$, consider a special 2-tree $u_c : pt \to pt$ such that $\chi_{u_c}$ sends $pt$ to $c$ and $c_{u_c} := c$.
For every isomorphism $P : \tau_1 \to \tau_2$ of colored 2-trees every pre-image $P^{-1}s_2$, $s_2 \in S_{\tau_2}$, is isomorphic (canonically) to $u_c$, where $c = \chi_{\tau_2}(s_2)$. Furthermore, for every colored 2-tree $\tau$ the pre-image $Q^{-1}pt$ of the point for a unique map $Q : \tau \to u_{c_\tau}$ is equal to $\tau$.
Let $1$ be the unit of the underlying symmetric monoidal category. Define a *unital 2-operad* as a colored 2-operad $\mathcal{O}$ along with maps $1 \to \mathcal{O}(u_c)$ for each $c \in X\rho$ satisfying:
- for every isomorphism $P : \tau_1 \to \tau_2$, the map
\[
\mathcal{O}(\tau_2) \cong \mathcal{O}(\tau_2) \otimes 1^\otimes_{S_{\tau_2}} \to \mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} \mathcal{O}(P^{-1}s_2) \to \mathcal{O}(\tau_1)
\]
coincides with the map $\mathcal{O}(\tau_2) \to \mathcal{O}(\tau_1)$ induced by $P^{-1}$ from the definition of $\mathcal{O}$ as a functor from the isomorphism groupoid of the category of colored 2-trees.

— for every colored 2-tree $\tau$ the composition

$$\mathcal{O}(\tau) \cong 1 \otimes \mathcal{O}(\tau) \to \mathcal{O}(u_{c_\tau}) \otimes \mathcal{O}(\tau) \to \mathcal{O}(\tau)$$

is the identity on $\mathcal{O}(\tau)$.

4.2.2. **Pruned colored operads.** Let

$$P : \tau_1 \to \tau_2$$

be a map of colored 2-trees. According to M. Batanin [2] $P$ is called a full injection if $P_S : S_{\tau_1} \to S_{\tau_2}$ is a color-preserving isomorphism and $P_T : T_{\tau_1} \to T_{\tau_2}$ is an injection.

Let $\mathcal{O}$ be a unital colored 2-operad. Consider the composition map associated with $P$:

$$\mathcal{O}(\tau_2) \otimes \bigotimes_{s_2 \in S_2} \mathcal{O}(P^{-1}s_2) \to \mathcal{O}(\tau_1).$$

It is clear that each $P^{-1}s_2$ is a 2-tree of the form $u_{c_\tau}, c \in X_{\rho}$. Hence we have unital maps $1 \to \mathcal{O}(P^{-1}s_2)$. Pre-composition with these maps gives rise to a map

$$\mathcal{O}(\tau_2) \to \mathcal{O}(\tau_1).$$

**Definition 4.1.** We call $\mathcal{O}$ a pruned 2-operad if for every full injection $P$ the map (4.2) is an isomorphism.

For every colored 2-tree $\tau$ there exists a unique (up-to an isomorphism) pruned colored 2-tree $\tau'$ together with a full injection $\tau' \to \tau$. Thus, a pruned 2-operad is completely determined by prescribing its spaces for each pruned 2-tree.

4.2.3. **Algebras over colored 2-operads.** Given an $X_{\rho}$-colored 2-operad $\mathcal{O}$ and an $X_{\rho}$-family of objects $X_c \in \mathcal{C}, c \in X_{\rho}$, we define an $\mathcal{O}$-algebra structure on $\{X_c\}_{c \in X_{\rho}}$ as a map

$$f : \mathcal{O} \to \text{des full}(\{X_c\}_{c \in X_{\rho}}),$$

where $\text{full}(X)$ is the full colored symmetric endomorphism operad of $X$. If $\mathcal{O}$ is unital, then we additionally require that $f$ matches the units.

4.3. **Symmetric Swiss Cheese type operads.** In this subsection we recall from [2] the notion of the symmetric Swiss Cheese type operads. Here, we call them symmetric SC operads for short.

Let $X_{\rho} := \{a, c\}$ be the set of colors. An SC-set is a $X_{\rho}$-colored set $S$ satisfying:

— if $a \in \chi(S)$, then $c_S = a$.

A map of SC-sets is a usual map $P : S_1 \to S_2$ satisfying: if $s_1 \in S_1$ is such that $\chi_{S_1}(s_1) = a$, then $\chi_{S_2}(P(s_1)) = a$. Given such $P$ and $s_2 \in S_2$, $P^{-1}s_2$ is naturally an SC-set: $\chi_{P^{-1}s_2}$ is the restriction of $\chi_{S_1}$; $c_{P^{-1}s_2} = \chi_{S_2}(s_2)$.

An SC-operad in a symmetric monoidal category $\mathcal{C}$ is a functor $\mathcal{O}$ from the groupoid of SC-sets and their color preserving bijections to $\mathcal{C}$.

For every map $P : S_1 \to S_2$ of SC-sets, there should be given a composition map

$$\mathcal{O}(S_2) \otimes \bigotimes_{s_2 \in S_2} \mathcal{O}(P^{-1}s_2) \to \mathcal{O}(S_1).$$

These compositions should satisfy the associativity law which is similar to that for usual operads. It is clear how to define unital symmetric SC operads. In this paper all our symmetric SC operads are unital.
4.3.1. **Reduced symmetric SC operads.** We say that a unital symmetric SC operad $O$ is reduced if for every SC set $S$ with at most one element

$$O(S) \cong 1.$$  

For the one element SC sets these isomorphisms should coincide with the unit maps. Furthermore, the operadic compositions of zero-ary and unary operations send products of $1$ to $1$ via the corresponding isomorphism of the symmetric monoidal category.

4.3.2. **Colored symmetric SC-operads.** Fix two sets of colors $X_{\rho_c}$ and $X_{\rho_a}$. A colored SC-set $S$ is a map $\chi_S : S \to X_{\rho_c} \sqcup X_{\rho_a}$ and an element $c_S \in X_{\rho_c} \sqcup X_{\rho_a}$ satisfying: if $\chi_S^{-1}X_{\rho_a}$ is non-empty, then $c_S \in X_{\rho_a}$.

We declare that there are no maps between colored SC-sets $S_1$ and $S_2$ if $c_{S_1} \neq c_{S_2}$. On the other hand if $c_{S_1} = c_{S_2}$ then a map from $S_1$ to $S_2$ is a map of sets $P : S_1 \to S_2$ satisfying the property: for any $s_2 \in S_2$, the set $P^{-1}s_2$ along with the map $\chi_{{S}_1}|_{P^{-1}s_2} : P^{-1}s_2 \to X_{\rho_c} \sqcup X_{\rho_a}$ and the element $c_{P^{-1}s_2} := \chi_{{S}_1}(s_2)$ is a colored SC-set. Thus, given a map of colored SC-sets $P : S_1 \to S_2$ and $s_2 \in S_2$, we have a colored SC-set $P^{-1}s_2$.

A colored SC-operad $O$ in an SMC $C$ is a functor $O$ from the isomorphism groupoid of colored SC-sets to $C$ along with the composition maps: given a map $P : S_1 \to S_2$ of colored sets, one should have a map

$$O(S_2) \otimes \bigotimes_{s_2 \in S_2} O(P^{-1}s_2) \to O(S_1)$$

satisfying the associativity property as above.

The operad $sO$ is an example of colored SC operad. Indeed, let $X_{\rho_c} := \mathbb{N}$ and $X_{\rho_a} := \{a\}$ and let $S$ be a colored SC-set. Let $S_c := \chi^{-1}X_{\rho_c}$ and $S_a := \chi^{-1}X_{\rho_a}$.

In the case $c_S \in X_{\rho_c}$, set

$$sO(S) := sO(S)^{c_S}_{\{\chi(s)\}_{s \in S}};$$

if $c_S = a$, we set

$$sO(S) := sO(S_c, S_a)^{\{\chi(s)\}_{s \in S_t}}.$$  

4.4. **SC 2-operads.** Let us review Batanin’s definition of a Swiss Cheese type (or simply SC) 2-operad from [2]. This notion is obtained via modifying the definition of a usual 2-operad as follows:

1) An SC-ordinal is any non-empty ordinal; its minimum is considered to be marked.

2) A map of SC-ordinals is a monotonous map preserving the minima.

3) An SC 2-tree $t$ is a monotonous map $t : S \to T$ where $S$ is a usual ordinal and $T$ is an SC-ordinal. A map of SC 2-trees

$$(t : S \to T) \to (t_1 : S_1 \to T_1)$$

is a map of sets $P_S : S \to S_1$ as well as a map of SC-ordinals $P_T : T \to T_1$ such that $t_1P_S = P_Tt_2$; the map $P_S$ must preserve the order on each set $t^{-1}t$, $t \in T$. Given $s_1 \in S_1$ such that $t_1(s_1)$ is not the minimum of $T_1$, we define a usual 2-tree $P^{-1}s_1$ in the same way as for the usual 2-trees (see the beginning of Sec. 4); in the case $t_1(s_1)$ is the minimum of $T_1$, we naturally get an SC 2-tree $P^{-1}s_1$.

4) We define an SC 2-operad in a symmetric monoidal category $C$ as:

— a functor

$$O : \text{2-trees}^x \sqcup \text{SC 2-trees}^x \to C;$$
— for every map of 2-trees or SC 2-trees \( P : t \to t_1 \), there should be given a map
\[
\mathcal{O}(t_1) \otimes \bigotimes_{s_1 \in S_1} \mathcal{O}(P^{-1}s_1) \to \mathcal{O}(t).
\]

These maps should satisfy the associativity property which is similar to that for usual 2-operads.

4.4.1. **Unital SC 2-operads.** In this paper all SC 2-operads are assumed to be unital.

To introduce the notion of unital SC 2-operads we define \( u_c \) to be the ordinary 2-tree \( pt \to pt \).
We also define \( u_a \) to be an SC 2-tree in which a 1-element ordinal is mapped into a one-element SC ordinal.

For every isomorphism \( P : t_1 \to t_2 \) of 2-trees or SC 2-trees every pre-image \( P^{-1}s_2, s_2 \in S_{t_2} \), is either \( u_c \) or \( u_a \). For every 2-tree \( t \) the pre-image \( Q_c^{-1}pt \) of the point for a unique map \( Q_c : t \to u_c \) is equal to \( t \). Furthermore, for every SC 2-tree \( t \) the pre-image \( Q_a^{-1}pt \) of the point for a unique map \( Q_a : t \to u_a \) is also equal to \( t \).

Define a **unital SC 2-operad** as an SC 2-operad \( \mathcal{O} \) along with maps \( 1 \to \mathcal{O}(u_c) \) and \( 1 \to \mathcal{O}(u_a) \), satisfying:

— for every isomorphism \( P : t_1 \to t_2 \) of 2-trees or SC 2-trees the map
\[
\mathcal{O}(t_2) \cong \mathcal{O}(t_2) \otimes 1^{\otimes S_{t_2}} \to \mathcal{O}(t_2) \otimes \bigotimes_{s_2 \in S_{t_2}} \mathcal{O}(P^{-1}s_2) \to \mathcal{O}(t_1)
\]

coincides with the map \( \mathcal{O}(t_2) \to \mathcal{O}(t_1) \) induced by \( P^{-1} \) from the definition of \( \mathcal{O} \) as a functor from the corresponding groupoid.

— for every 2-tree \( t \) the composition
\[
\mathcal{O}(t) \cong 1 \otimes \mathcal{O}(t) \to \mathcal{O}(u_c) \otimes \mathcal{O}(t) \to \mathcal{O}(t)
\]
is the identity on \( \mathcal{O}(t) \).

— for every SC 2-tree \( t \) the composition
\[
\mathcal{O}(t) \cong 1 \otimes \mathcal{O}(t) \to \mathcal{O}(u_a) \otimes \mathcal{O}(t) \to \mathcal{O}(t)
\]
is the identity on \( \mathcal{O}(t) \).

4.4.2. We define the trivial SC 2-operad \( \text{triv} \) by setting
\[
\text{triv}(t) = 1,
\]
for all 2-trees and SC 2-trees \( t \). Here \( 1 \) is the unit of the SMC and the operadic multiplications are the canonical maps sending tensor products of \( 1 \) to \( 1 \).

4.4.3. **Colored SC 2-operads.** We define a colored SC 2-operad as follows. Fix 2 sets of colors: \( X_{\rho_c} \) and \( X_{\rho_a} \). Define a coloring of an SC 2-tree \( t_\tau : S_\tau \to T_\tau \) as follows.

First decompose \( S_\tau = S_{\tau,a} \sqcup S_{\tau,c} \), where \( S_{\tau,a} \) is the \( t_\tau \)-preimage of the minimum of \( T_\tau \), and \( S_{\tau,c} \) is the complement.

A \((X_{\rho_c}, X_{\rho_a})\)-coloring of \( \tau \) is a prescription of maps \( \chi_{\tau,c} : S_{\tau,c} \to X_{\rho_c} \); \( \chi_{\tau,a} : S_{\tau,a} \to X_{\rho_a} \) and an element \( c_\tau \in X_{\rho_a} \).

As well as for ordinary colored 2-trees we declare that there are no maps between colored SC 2-trees \( \tau \) and \( \tau_1 \) if \( c_\tau \neq c_{\tau_1} \). On the other hand, if \( c_\tau = c_{\tau_1} \) then a map \( P : \tau \to \tau_1 \) is just the map of the underlying SC 2-trees. Then it is clear that for every \( s_1 \in S_{\tau_1} \) such that \( t_{\tau_1,s_1} \) is the minimum, the SC 2-tree \( P^{-1}s_1 \) is naturally \((X_{\rho_c}, X_{\rho_a})\)-colored. Furthermore, for every \( s_1 \in S_{\tau_1} \) such that \( t_{\tau_1,s_1} \) is not the minimum, the 2-tree \( P^{-1}s_1 \) is naturally \( X_{\rho_c} \)-colored.

We define a \((X_{\rho_c}, X_{\rho_a})\)-colored SC 2-operad as a functor \( \mathcal{O} \) from the disjoint union of the groupoid of \( X_{\rho_c} \)-colored 2-trees and the groupoid of \((X_{\rho_c}, X_{\rho_a})\)-colored SC 2-trees to \( \mathcal{C} \). Given a
map $P : \tau \rightarrow \tau_1$ of $X\rho_c$-colored 2-trees or $(X\rho_c,X\rho_a)$-colored SC 2-trees there should be given a map

$$O(\tau_1) \otimes \bigotimes_{s_1 \in S_1} O(P^{-1}s_1) \rightarrow O(\tau).$$

The associativity axiom should be satisfied.

4.4.4. Unital colored SC 2-operads. As well as SC 2-operads all colored SC 2-operads are assumed to be unital.

To introduce the notion of unital SC 2-operads we define $u_c$, $c \in X\rho_c$, be the colored 2-tree $pt \rightarrow pt$ for which the point $pt$ has the color $c$ and $c_{uc} = c$. Similarly, we define $u_a$, $a \in X\rho_a$ to be the colored SC 2-tree in which the one-element ordinal is mapped into the one-element SC ordinal and all colorings are $a$.

For every isomorphism $P : \tau_1 \rightarrow \tau_2$ of $X\rho_c$-colored 2-trees or SC 2-trees every pre-image $P^{-1}s_2$, $s_2 \in S_{\tau_2}$, is either $u_c$ or $u_a$. For every colored 2-tree or colored SC 2-tree $\tau$ the pre-image $Q^{-1}_\tau pt$ of the point for a unique map $Q_\tau : \tau \rightarrow u_{c\tau}$ is equal to $\tau$.

Define a unital colored SC 2-operad as a colored SC 2-operad $O$ along with maps $1 \rightarrow O(u_c)$ and $1 \rightarrow O(u_a)$ for all $c \in X\rho_c$ and $a \in X\rho_a$ satisfying:

— for every isomorphism $P : \tau_1 \rightarrow \tau_2$, of colored 2-trees or colored SC 2-trees the map

$$O(\tau_2) \cong O(\tau_2) \otimes 1 \otimes_{S_{\tau_2}} \rightarrow O(\tau_2) \otimes \bigotimes_{s_2 \in S_{\tau_2}} O(P^{-1}s_2) \rightarrow O(\tau_1)$$

coincides with the map $O(\tau_2) \rightarrow O(\tau_1)$ induced by $P^{-1}$ from the definition of $O$ as a functor from the corresponding groupoid.

— for every colored 2-tree or colored SC 2-tree $\tau$ the composition

$$O(\tau) \cong 1 \otimes O(\tau) \rightarrow O(u_{c\tau}) \otimes O(\tau) \rightarrow O(\tau)$$

is the identity on $O(\tau)$.

4.5. Reduced SC 2-operads. Technically, it turns out to be convenient to consider SC-operads satisfying certain additional properties. We are going to define these properties.

4.5.1. Pruned SC 2-operads. A (colored) SC 2-tree $\tau$ is called pruned if $\text{Im}(t_\tau) \supset T_\tau \setminus m_{T_\tau}$, where $m_{T_\tau}$ is the marked minimum of $T_\tau$.

For every colored SC 2-tree $\tau$ there exists a unique up to isomorphism pruned colored SC 2-tree $\tau'$ and a map $P : \tau' \rightarrow \tau$ such that $P_S : S_{\tau'} \rightarrow S_{\tau}$ is a bijection; $P_T$ is injective, and $P$ induces an isomorphism of colorings. For every such $P$ the pre-images $P^{-1}s$ are of the form $u_c$ or $u_a$, therefore, given a unital operad $O$, we have a map

$$(4.3) \quad O(\tau') \rightarrow O(\tau).$$

By analogy with ordinary 2-operads (see Subsection 4.2.2) $O$ is called pruned if all such maps (4.3) are isomorphisms.

4.5.2. Definitions. We say that a pruned (colored) SC 2-operad $O$ is reduced if

— all the unit maps $1 \rightarrow O(u_c)$, $1 \rightarrow O(u_a)$ are isomorphisms;

— $O(\tau) = 1$ whenever $|S_\tau| \leq 1$ so that we have an identification $O(\tau) = \text{triv}(\tau)$ for all such $\tau$;

— for every map $P : \tau_1 \rightarrow \tau_2$ where $|S_{\tau_1}|, |S_{\tau_2}| \leq 1$ the corresponding operadic composition law coincides with that of $\text{triv}$.

Note that, equivalently, one can only require that the conditions are the case for pruned 2-trees $\tau$. 
4.6. Desymmetrization. Given a symmetric SC-operad $Q$, Batanin defines its desymmetrization $\text{des} Q$ by setting $\text{des} Q(t) := Q(S_t)$ for all 2-trees and SC 2-trees $t$. Here $S_t$ is treated as an SC-set as follows:

- if $t$ is a usual 2-tree then we define all the colorings to be $c$;
- if $t$ is an SC 2-tree, we set $c_{S_t} := a$ and we give the preimage of marked element of $T_t$ the color $a$, the remaining elements of $S_t$ receive color $c$.

Given a reduced symmetric SC-operad $Q$, its desymmetrization $\text{des} Q$ is a reduced SC 2-operad so that $\text{des}$ is a functor from the category of reduced symmetric SC operads to that of reduced SC 2-operads.

4.6.1. In the same spirit, one defines the desymmetrization of a colored symmetric SC-operad $O$. Let $\tau$ be a colored SC 2-tree; we then see that $S_\tau$ is a colored SC-set in the natural way: the map $\chi_{S_\tau} = \chi_\tau$ and $\chi_{S_\tau} := \chi_{\tau,a}$. Finally, $c_{S_\tau} := c_\tau$. We then set

$$\text{des}(O)(\tau) := O(S_\tau)$$

with the composition law determined by that in $O$.

4.7. Symmetrization. If the SMC $C$ has small colimits then the functor $\text{des}$ has a left adjoint $\text{sym}$. For many categories of higher operads the functor $\text{sym}$ can be elegantly expressed using colimits [2]. Here we recall from [2] a description of the functor $\text{sym}$ for the category of reduced SC 2-operads.

For every SC set $S$ we define a category $J(S)$.

If $c_S = c$ then objects of $J(S)$ are pruned 2-trees of the form

$$t : S \rightarrow T.$$  

Morphisms are the maps between 2-trees which induce the identity map on $S$.

If $c_S = a$ then objects of $J(S)$ are pruned SC 2-trees $t : S \rightarrow T$ such that the preimage of the minimal element of $T$ coincides with $S_a = \chi^{-1}(a)$. Morphisms are the maps between SC 2-trees which induce the identity map on $S$.

Notice that, although elements of an SC set $S$ are not ordered, choosing an object of the category $J(S)$ we equip $S$ with a total order.

Remark. It is not hard to show that for every SC set $S$ the category $J(S)$ is a poset. In fact if $c_S = c$ then $J(S)$ is the opposite of the Milgram poset from [1].

Let $O$ be a reduced SC 2-operad.

For every SC set $S$ the SC 2-operad $O$ gives us an obvious (contravariant) functor from the category $J(S)$ to the underlying SMC $C$. We denote this functor by $O_S$.

According to Theorem 9.1 from [2] we have

$$\text{sym} O(S) = \operatorname{colim}_{J(S)} O_S.$$  

The operadic multiplications of $\text{sym} O$ can be easily obtained from those of $O$ using the properties of colimits.

4.8. Batanin’s theorem. Let $C$ be either the category of topological spaces or the category of chain complexes of vector spaces over the field $k$. Then, according to Theorem 5.3 from [2], the category of reduced SC 2-operads has a closed model structure transferred from $C$.

We observe that the operad $\text{triv}$ is reduced and set $R\text{triv} \rightarrow \text{triv}$ to be its cofibrant resolution in the category of reduced SC 2-operads.
Theorem 4.2 (Theorem 9.2, 9.4, [2]). The symmetric SC operad \( \text{sym} \mathcal{R}_{\text{triv}} \) is weakly equivalent to Voronov’s Swiss Cheese operad if \( \mathcal{C} \) is the category of topological spaces, and to the singular chain operad of Voronov’s Swiss Cheese operad if \( \mathcal{C} \) is the category of chain complexes of \( k \)-vector spaces.

Let us sketch its proof for \( \mathcal{C} \) being the category of topological spaces.

First, we observe that the SC 2-operad \( \mathcal{R}_{\text{triv}} \) can be replaced with any weakly equivalent one. Batanin uses the Swiss Cheese version \( \mathcal{SCGJ} \) of the Getzler-Jones 2-operad \( \mathcal{GJ} \).

This SC 2-operad is constructed in [2] as a sub SC 2-operad of the desymmetrization \( \text{des}(\mathcal{SCFM}) \) of the Fulton-MacPherson version \( \mathcal{SCFM} \) of Voronov’s Swiss Cheese operad.

Then, since the desymmetrization functor \( \text{des} \) admits the left adjoint \( \text{sym} \), the inclusion

\[
\mathcal{SCGJ} \hookrightarrow \text{des}(\mathcal{SCFM})
\]

produces the following map

\[
\text{sym}(\mathcal{SCGJ}) \to \mathcal{SCFM}
\]

which can be shown to be an isomorphism, hence a weak equivalence. This completes the proof.

The case when \( \mathcal{C} \) is the category of chain complexes is treated by applying the singular chain functor.

5. Linking the operad \( \text{sO} \) with 2 operads: a 2-operad \( \text{seq} \)

We have a \( \mathbb{N} \)-colored 2-operad \( \text{seq} \) of sets defined in Section 6.1 in [27].

Let us recall the definition. Given a colored 2-tree \( \tau \) with the underlying 2-tree \( t : S \to T \) and a \( \mathbb{N} \)-coloring such that an \( s \in S \) has a color \( \chi_{\tau}(s) = I_s \), where \( I_s \) is a non-empty finite ordinal, and the color of the result is \( c_{\tau} = J \) we define a set

\[
\text{seq}(\tau) = \text{seq}(t)_{\{I_s\}_{s \in S}}^J
\]

whose each element \( u \) is a collection of the following data:

- a total order \( >_u \) on \( I := \bigsqcup_{s \in S} I_s \);
- a non-decreasing map \( I \to J \).

The following conditions should be satisfied:
- the order \( >_u \) agrees with those on each \( I_s \);
- if \( i, k \in I_{s_1} \), \( j \in I_{s_2} \), \( s_1 \neq s_2 \) and \( i < u \) \( j < u \) \( k \), then \( t(s_2) < t(s_1) \);
- if \( s_1, s_2 \in S \), \( s_1 < s_2 \), and \( t(s_1) = t(s_2) \), then \( I_{s_1} < u I_{s_2} \).

The 2-operadic composition law on the collection of sets \( \text{seq} \) is defined by the same rules as on \( \text{sO} \).

Let us define a colored SC 2-operad \( \text{seq} \) by modifying the definition of \( \text{seq} \) as follows.

First of all we fix the sets of colors:
- the set \( X_{\rho_a} \) is the same as the set of colors of \( \text{seq} \), i.e. \( \mathbb{N} \);
- the set \( X_{\rho_a} \) is the one element set \{\( a \)\}; we identify a unique element of \( X_{\rho_a} \) with the ordinal consisting of 1 element.
- Given a usual colored 2-tree \( \tau \) we set \( \text{seq}(\tau) := \text{seq}(\tau') \);
- given a colored SC 2-tree \( \tau \), let us construct a usual colored 2-tree \( \tau' \) with the underlying 2-tree \( t' = t : S \to T \). Define a map \( \chi_{\tau'} : S \to X_{\rho_a} = \mathbb{N} \) by setting
  a) if \( s \in S \) and \( t(s) \) is the minimum of \( T \), then we set \( \chi_{\tau'}(s) \) to be the one-element ordinal;
  b) if \( s \in S \) and \( t(s) \) is not the minimum of \( T \), then we set \( \chi_{\tau'}(s) = \chi_{\tau,c}(s) \), where \( \chi_{\tau,c} \) is a defining map of the coloring for \( \tau \) (see Sec. 4.4.3).

Lastly, we set \( c_{\tau'} \) to be the one-element ordinal.

We then define \( \text{seq}(\tau) := \text{seq}(\tau') \).
Note that we have natural inclusions

\[ \text{seq}^{\text{SC}}(\tau) \subset \text{sO}(S) = (\text{des sO})(\tau), \]

where \( S_\tau \) is the colored SC-set corresponding to the colored 2-tree or the colored SC 2-tree \( \tau \) as defined in Sec 4.6. Thus \( \text{seq}^{\text{SC}} \) is a colored SC 2-suboperad of \( \text{des sO} \).

5.1. The SC 2-operad \( \text{seq}^{\text{SC}} \) and the SC operad \( \text{sO} \). The spaces of unary operations in both \( \text{seq}^{\text{SC}} \) and \( \text{sO} \) give a category structure on \( \text{N} \) hom\((I_1, I_2) = \text{seq}^{\text{SC}}(t)_{I_1, I_2} = \text{sO}(t)_{I_1, I_2} \),

where \( t_0 : \text{pt} \to \text{pt} \). This category is isomorphic to the simplicial category \( \Delta \).

The action of these unary operations defines a polysimplicial/cosimplicial structure on the collection of operadic sets. In order to define this structures we need some notation.

Given a 2-tree \( t : S \to S_1 \), a collection of non-empty ordinals \( I_s, s \in S \) and a non-empty ordinal \( J \) we get a \( \text{N} \)-colored 2-tree \( \tau := t(I_s : s \in S; J) \) in the obvious way. Write

\[ \text{seq}^{\text{SC}}(t)_{I_s : s \in S} = \text{seq}(\tau). \]

Next, given an SC 2-tree \( t : S \to S_1 \), let us decompose \( S = S_1 \sqcup S_a \), where \( S_a \) is the pre-image of the minimum of \( S_1 \). Suppose we are given ordinals \( I_s, s \in S_c \). Using these data, we naturally get a colored SC 2-tree \( \tau := \tau(t, I_s : s \in S_c) \), where the coloring sets are \( X \rho_c = \text{N} \) and \( X \rho_a = \{a\} \). Each element of \( s \in S_c \) receives color \( I_s \); each element of \( S_a \) gets colored in \( a \); we set \( c_\tau = a \).

We set

\[ \text{seq}^{\text{SC}}(t)_{I_s : s \in S_1} = \text{seq}(\tau). \]

Thus, given a 2-tree \( t : S \to S_1 \), we have a polysimplicial/cosimplicial set

\[ \text{seq}^{\text{SC}}(t) : \Delta \times (\Delta^{\text{op}})^S \to \text{Sets}. \]

Given an SC 2-tree \( t : S \to S_1 \), we get a polysimplicial set

\[ \text{seq}^{\text{SC}}(t) : (\Delta^{\text{op}})^S \to \text{Sets}. \]

Likewise, we can get polysimplicial sets out of the symmetric \( \text{N} \sqcup \{a\} \)-colored operad \( \text{sO} \):

\[ \text{sO}(S) : \Delta \times (\Delta^{\text{op}})^S \to \text{Sets}; \]
\[ \text{sO}(S_c, S_a) : (\Delta^{\text{op}})^S \to \text{Sets}. \]

Using the functor \( S : \Delta \to \text{complexes} \) we can take the total complexes of these polysimplicial (cosimplicial) sets, in the same way as in Subsection 2.0.2.

Let \( t \) be a 2-tree. Set

\[ |\text{seq}^{\text{SC}}(t)| := |\text{seq}^{\text{SC}}(t)|; \]

Given an SC 2-tree \( t \), we set

\[ |\text{seq}^{\text{SC}}(t)| := |\text{seq}^{\text{SC}}(t)|. \]

Next, we set

\[ |\text{sO}(S)| := |\text{sO}(S)|; \]
\[ |\text{sO}(S_c, S_a)| := |\text{sO}(S_c, S_a)|. \]

Since the operad \( \text{sO} \) is isomorphic to \( \emptyset \) the DG operad \( |\text{sO}| \) is isomorphic to the DG operad \( \Lambda = |\emptyset| \) of natural operations on the pair \( \text{C}^\bullet(A, A); A \).
The complexes \( |\text{SC} \text{seq}|(t) \), where \( t \) is either a 2-tree or an SC 2-tree, form a dg SC 2-operad in which the composition maps come from those in \( |\text{SC} \text{seq}| \). Likewise, the complexes \( |s\text{O}|(S) \) form a symmetric dg SC operad.

Since \( |\text{SC} \text{seq}| \) is a colored SC 2-suboperad of \( |\text{des} |s\text{O}| \), the dg SC 2-operad \( |\text{SC} \text{seq}| \) is a dg SC 2-suboperad of \( |\text{des} |s\text{O}| \). In other words, we have the inclusion of dg SC 2-operads

\[
|\text{SC} \text{seq}| \hookrightarrow |\text{des} |s\text{O}| .
\]

6. The SC 2-operad \( \text{br} \) and the SC operad \( \text{braces} \)

It is not hard to see that the SC 2-operad \( |\text{SC} \text{seq}| \) is pruned. However, neither \( |\text{SC} \text{seq}| \) nor \( |s\text{O}| \) is reduced. In this section we construct a reduced SC 2-operad \( \text{br} \) which is quasi-isomorphic to the SC 2-operad \( |\text{SC} \text{seq}| \). Similarly, we construct a reduced SC operad \( \text{braces} \) which is quasi-isomorphic to the SC operad \( |s\text{O}| \). Both \( \text{br} \) and \( \text{braces} \) are obtained as suboperads of \( |\text{SC} \text{seq}| \) and \( |s\text{O}| \), respectively.

6.1. An increasing filtration on \( |\text{SC} \text{seq}| \).

Let \( t : S \to T \) and

\[
v \in |\text{SC} \text{seq}(t)|_{\{I_s\}_{s \in S}} .
\]

Consider the order \( >_v \) on

\[
\mathcal{I} := \mathcal{I}_S := \bigsqcup_{s \in S} I_s .
\]

Call two elements \( i_1, i_2 \in \mathcal{I}_S \) elementary equivalent if \( i_1, i_2 \in I_s \) for some \( s \in S \) and for every \( i \in \mathcal{I}_S \) between \( i_1 \) and \( i_2 \) with respect to the order \( <_v \) the element \( i \) belongs to \( I_s \). In this way we get an equivalence relation on \( \mathcal{I}_S \). Denote by \( |v| \) the number of equivalence classes with respect to this relation.

Let \( F_N^{|\text{SC} \text{seq}(t)|}_{\{I_s\}_{s \in S}} \) be the subset consisting of all elements \( v \) with \( |v| \leq N + |S| \). Roughly speaking, the difference \( |v| - |S| \) counts how many times the order \( <_v \) cuts the ordinals \( I_s, s \in S \) into subordinals.

6.1.1. Let \( t : S \to T \) be an SC 2-tree. As above, we set \( S_a \) to be the pre-image of the minimum of \( T \) and \( S_c := S \setminus S_a \).

Recall that an element

\[
v \in |\text{SC} \text{seq}(t)|_{\{I_s\}_{s \in S_c}}
\]

is nothing else but a total order \( >_v \) on

\[
\bigsqcup_{s \in S_c} I_s \sqcup S_a
\]

subject to certain conditions.

In order to define the elementary equivalence relation on (6.2) we replace (6.2) by the isomorphic set

\[
\mathcal{I}_{S_c \sqcup S_a} = \bigsqcup_{s \in S} I_s ,
\]

where \( I_s \) is the one element ordinal for every \( s \in S_a \).

Using the total order \( >_v \) on \( \mathcal{I}_{S_c \sqcup S_a} \) and the construction from the previous subsection we get the elementary equivalence relation on the set \( \mathcal{I}_{S_c \sqcup S_a} \) and hence on (6.2).
On the set (6.2) the elementary equivalence relation can be described as follows. The restriction of this relation onto \( S_a \) coincides with the identity relation, there is no element of \( S_a \) which is equivalent to an element

\[
i \in \bigcup_{s \in S_c} I_s.
\]

Finally we call two elements

\[
i_1, i_2 \in \bigcup_{s \in S_c} I_s
\]
elementary equivalent iff

— \( i_1, i_2 \in I_s \) for some \( s \in S_c \),
— for every element \( i \) of the set (6.2) between \( i_1 \) and \( i_2 \) with respect to the order \( <_v \) we have \( i \in I_s \).

We denote the number of equivalence classes in (6.3) \( I_{S_c \sqcup S_a} \) by \(|v|\) and define the subset

\[F_{N_{seq}(t)}\{I_{s}\} \subset seq(t)\{I_{s}\} \subset seq(t)\{I_{s}\} \subset SC\]
to consist of all elements \( v \) with \(|v| \leq N + |S|\).

**Lemma 6.1.** The filtration \( F \) is compatible with operadic compositions on \( SC \).

**Proof.** Consider operadic compositions of the following type:

Let \( P : t_1 \to t_2 \) be a map of 2-trees, where \( t_1 : S_1 \to T_1 \) and \( t_2 : S_2 \to T_2 \). Let \( P_S : S_1 \to S_2 \) be the induced map. For every \( s_2 \in S_2 \), we have a pre-image \( P_S^{-1}(s_2) \subset S_1 \).

Let \( I_{s_1}, s_1 \in S_1 ; J_{s_2}, s_2 \in S_2 ; J \) be non-empty ordinals.
Let

\[
w \in \text{seq}(t_2)\{J_{s_2}\} \subset \text{seq}(t_2)\{J_{s_2}\},
\]

\[
v_{s_2} \in \text{seq}(P^{-1}s_2)\{I_{s_1}\} \subset \text{seq}(P^{-1}s_2)\{I_{s_1}\}.
\]

Let us denote by \( z \) the composition of these elements and estimate \(|z|\). Suppose that

\[
J_{\sigma} \subset (\bigcup_{s_2 \in S_2} J_{s_2}, >_w)
\]

for \( \sigma \in S_2 \) is split into \(|\sigma|\) equivalence classes.

Consider the map

\[
I_{\sigma} := \bigcup_{s_1 \in P^{-1}_S \sigma} I_{s_1} \rightarrow J_{\sigma}.
\]

It is clear that the number of equivalence classes of

\[
I_{\sigma} \subset (\bigcup_{s_1 \in S_1} I_{s_1}, >_z)
\]
does not exceed \(|v_\sigma| + |\sigma| - 1\). Therefore

\[
|z| \leq \sum_{\sigma \in S_2} (|v_\sigma| + |\sigma| - 1) = |w| - |S_2| + \sum_{\sigma} |v_\sigma|.
\]

Hence,

\[
|z| - |S_1| \leq |w| - |S_2| + \sum_{\sigma} (|v_\sigma| - |P^{-1}_S \sigma|)
\]

which means that this composition is compatible with the filtration \( F \). The compositions of other types can be considered in a similar way. \( \square \)
This Lemma, in particular implies that the polysimplicial/cosimplicial structure on \( \text{SC}_{\text{seq}} \) is compatible with the filtration \( F \). Therefore, the filtration \( F \) descends onto the level of total complexes so that we have an increasing filtration on each operadic complex \( \text{SC}_{\text{seq}}(t) \): 

\( F_N \text{SC}_{\text{seq}}(t) \subset \text{SC}_{\text{seq}}(t) \).

**Lemma 6.2.** The filtration \( F \) on \( \text{SC}_{\text{seq}} \) satisfies the following properties:

1. The operadic compositions in \( \text{SC}_{\text{seq}} \) are compatible with the filtration.
2. The complex \( F_N \text{SC}_{\text{seq}}(t) \) is concentrated in the degrees \( \geq -N \).
3. The quotient \( F_N \text{SC}_{\text{seq}}(t) / F_{N-1} \text{SC}_{\text{seq}}(t) \) only has cohomology concentrated in degree \( -N \).

**Proof.**

(1) Follows from the previous lemma.

(2) We start with the case when \( t : S \to T \) is a usual 2-tree.

Let us consider the simplicial realization with respect to the lower indices for

\( \text{SC}_{\text{seq}}(t)_{\{I_s\}_{s \in S}} \).

Let

\( v \in \text{SC}_{\text{seq}}(t)_{\{I_s\}_{s \in S}} \).

According to Subsection 6.1 the order \( >_v \) on

\( I_S := \bigcup_{s \in S} I_s \)

defines on \( I_S \) an equivalence relation.

If

\( |v| + |J| - 1 < \sum_{s \in S} |I_s| \).

then there exist two different but equivalent elements of \( I_S \) which go to the same element in \( J \). In this case the element \( v \) is obtained from another element by applying a degeneracy.

Thus if inequality (6.6) holds for \( v \) then \( v \) does not contribute to the realization of (6.4).

Therefore if \( v \) contributes to the realization then

\[ |J| - 1 - \sum_{s \in S}(|I_s| - 1) \geq -|v| + |S| \]

and hence the complex

\( F_N \text{SC}_{\text{seq}}(t) \)

is concentrated in degrees

\[ \geq -N. \]

The case when \( t : S \to T \) is an SC 2-tree is very similar.

Let \( S_a \) be the pre-image of the minimal element of \( T \) and \( S_c = S \setminus S_a \). An element \( v \) of

\( \text{SC}_{\text{seq}}(t)_{\{I_s\}_{s \in S_c}} \)

is a total order \( >_v \) on

\( I_s \sqcup S_a \)

subject to certain conditions.

According to Subsection 6.1.1 the order \( >_v \) gives us the elementary equivalence relation on the set (6.8).
If at least one equivalence class in (6.8) contains more than 1 element then the corresponding element \( v \) in (6.7) is obtained from another element by applying a degeneracy. Indeed, only the equivalence classes in 

\[
\bigcup_{s \in S_c} I_s
\]

may contain more than one element. And if at least one class contains more than 1 element then there are distinct elements \( i_1, i_2 \in I_s \) for some \( s \in S_c \) such that one of them goes right after another in the ordinal (6.8).

Therefore, if \( v \) contributes to the realization of (6.7) then

\[
\sum_{s \in S_c} |I_s| + |S_a| = |v|.
\]

Hence

\[
\sum_{s \in S_c} (|I_s| - 1) = |v| - |S_a| - |S_c|
\]

or equivalently

\[
- \sum_{s \in S_c} (|I_s| - 1) = |S| - |v|.
\]

If \( v \in F_N^{SC_{seq}}(I_s)_{s \in S_t} \) then the right hand side of the latter equation is \( \geq -N \). Thus the complex

\[
F_N^{SC_{seq}}(t)
\]

is concentrated in degrees

\[
\geq -N.
\]

(3) Let us first consider the cochain complex

(6.9)

\[
F_N^{SC_{seq}}(t)/F_{N-1}^{SC_{seq}}(t)
\]

in the case when \( t : S \to T \) is a usual 2-tree.

If an element \( v \) in (6.4) represents a non-zero vector in (6.9) then the set \( I_S \) (6.5) has exactly \( N + |S| \) equivalence classes. The total order on \( I_S \) gives a total order on the set of these equivalence classes. Hence the set of equivalence classes in \( I_S \) can be identified with the ordinal \( \{1, 2, \ldots, N + |S|\} \). Furthermore each equivalence class is a subset of \( I_s \) for some \( s \in S \).

Thus to every such element \( v \) in (6.4) we assign a surjection

(6.10)

\[
\sigma : \{1, 2, \ldots, N + |S|\} \to S
\]

from the ordinal \( \{1, 2, \ldots, N + |S|\} \) to the set\(^4\) \( S \).

Not all such surjections can be gotten from the elements of (6.4) representing non-zero vectors in (6.9). The 2-tree \( t : S \to T \), the definition of \( \text{SC}_{seq} \), and the definition of the elementary equivalence relation impose the following conditions on the possible surjections (6.10):

A \( \sigma(i) \neq \sigma(i + 1) \) \( \forall i = 1, 2, \ldots, N + |S| - 1 \),

B if \( s \neq \tilde{s} \) and \( j_1 < i < j_2 \) for \( i \in \sigma^{-1}(s) \) and \( j_1, j_2 \in \sigma^{-1}(\tilde{s}) \) then \( t(s) < t(\tilde{s}) \) in \( T \),

C if \( t(s) = t(\tilde{s}) \) and \( s < \tilde{s} \) then all elements of \( \sigma^{-1}(s) \) are smaller than all elements of \( \sigma^{-1}(\tilde{s}) \).

\(^4\)Recall that \( S \) is also equipped with a total order but in general \( \sigma \) is not a map of ordinals.
Let us denote by \( D(t, N) \) the set of all surjections (6.10) satisfying above conditions \( A, B, \) and \( C. \)

It is not hard to see that the elements of (6.4) representing non-zero vectors in (6.9) and corresponding to the same surjection (6.10) span a subcomplex of (6.9). Furthermore for every map (6.10) this subcomplex is isomorphic to the cochain complex \( |\Xi_{N+|S|}|^{*+N} \), where \( |\Xi_k|^* \) are the complexes described in the Appendix.

Thus (6.9) is isomorphic to the direct sum of identical cochain complexes

\[
F_N^{\text{sp}}(t)/F_{N-1}^{\text{sp}}(t) \cong \bigoplus_{\sigma \in D(t, N)} |\Xi_{N+|S|}|^{*+N}.
\]

Therefore, due to Proposition 8.2 from the Appendix we have,

\[
H^\bullet\left(F_N^{\text{sp}}(t)/F_{N-1}^{\text{sp}}(t)\right) = \begin{cases} \bigoplus_{\sigma \in D(t, N)} k, & \text{if } \bullet = -N, \\ 0, & \text{otherwise}. \end{cases}
\]

Let us now consider the cochain complex

\[
F_N^{\text{sp}}(t)/F_{N-1}^{\text{sp}}(t)
\]

in the case when \( t : S \to T \) is an SC 2-tree.

As above \( S_a \) is the pre-image of the minimal element of \( T \) and \( S_c = S \setminus S_a \).

If an element \( v \) of (6.7) represents a non-zero vector in (6.13) then the set

\[
I_{S_c \sqcup S_a} = \bigcup_{s \in S_c} I_s \sqcup S_a
\]

has exactly \( N + |S| \) equivalence classes. The total order on \( I_{S_c \sqcup S_a} \) gives us a total order on the set of its equivalence classes. Hence the set of the equivalence classes can be identified with the standard ordinal \( \{1, 2, \ldots, N + |S|\} \). Furthermore, each equivalence class is either a subset of \( I_s \) for some \( s \in S_c \) or a one element subset of \( S_a \). Thus we get a surjection

\[
\sigma : \{1, 2, \ldots, N + |S|\} \to S
\]

from the ordinal \( \{1, 2, \ldots, N + |S|\} \) to the set \( S \).

As well as in the case of the usual 2-tree this surjection satisfies above conditions \( A, B, \) and \( C. \)

Let us remark that, since \( S_a \) is the pre-image of the minimal element of \( T \), conditions \( A, B, \) and \( C \) imposed on the surjection (6.15) imply that for every \( s \in S_a \) the pre-image \( \sigma^{-1}(s) \) is a one element set.

As above we denote by \( D(t, N) \) the set of all surjections (6.15) satisfying above conditions \( A, B, \) and \( C. \)

Similarly to the case of a usual 2-tree the set of elements of (6.7) representing non-zero vectors in (6.13) splits into the disjoint union of subsets, corresponding surjections \( \sigma \in D(t, N) \). And similarly the elements of (6.7) representing non-zero vectors in (6.13) and corresponding to the same map (6.15) span a subcomplex of (6.13). These subcomplexes are all isomorphic to the cochain complex

\[
|\Xi_{N+|S_c|}|^{*+N,0},
\]

where the bicomplexes \( |\Xi_k|^* \) are described in the Appendix.

It is not hard to see that the complex \( |\Xi_{N+|S_c|}|^{*+0} \) consists of the field \( k \) placed in degree 0.

Thus for an SC 2-tree \( t \) we have

\[
\left(F_N^{\text{sp}}(t)/F_{N-1}^{\text{sp}}(t)\right)^* \cong \begin{cases} \bigoplus_{\sigma \in D(t, N)} k, & \text{if } \bullet = -N, \\ 0, & \text{otherwise} \end{cases}
\]
and statement (3) holds in this case too.

Using this filtration we give the following definition.

**Definition 6.3.** We define the dg SC 2-operad \( \mathbf{br} \) as a suboperad of \( \mathbf{SC}_{\text{seq}} \) with

\[
\mathbf{br}(t) = \bigoplus_{N \geq 0} G^N \mathbf{SC}_{\text{seq}}(t),
\]

where

\[
G^N \mathbf{SC}_{\text{seq}}(t) = \{ v \in F^N \mathbf{SC}_{\text{seq}}(t) \mid dv \in F^{N-1} \mathbf{SC}_{\text{seq}}(t) \},
\]

and \( t \) is either a 2-tree or an SC 2-tree.

Lemma 6.2 implies that the inclusion

\( \mathbf{br} \hookrightarrow \mathbf{SC}_{\text{seq}} \)

is a quasi-isomorphism. Furthermore,

**Proposition 6.4.** The SC 2-operad \( \mathbf{br} \) is reduced.

*Proof.* Let \( t : S \to T \) be a 2-tree or an SC 2-tree with \( |S| \leq 1 \). The condition \( |S| \leq 1 \) implies that the filtration \( F \) on \( \mathbf{SC}_{\text{seq}}(t) \) is trivial: \( F^{-1} \mathbf{SC}_{\text{seq}}(t) = 0 \) and \( F^N \mathbf{SC}_{\text{seq}}(t) = \mathbf{SC}_{\text{seq}}(t) \) for all \( N \geq 0 \).

Therefore, \( \mathbf{br}(t) \) is simply the vector space of degree 0 cocycles in \( \mathbf{SC}_{\text{seq}}(t) \)

\[
\mathbf{br}(t) = \mathbf{SC}_{\text{seq}}(t)^0 \cap \ker d.
\]

Due to Lemma 6.2 the complex \( \mathbf{SC}_{\text{seq}}(t) \) is concentrated in nonnegative degrees. Hence

\[
H^0(\mathbf{SC}_{\text{seq}}(t)) = \mathbf{SC}_{\text{seq}}(t)^0 \cap \ker d.
\]

On the other hand, equations (6.12) and (6.16) imply that

\[
H^0(\mathbf{seq}(t)) = k[D(t, 0)]
\]

and it is easy to see that if \( |S| \leq 1 \) then \( D(t, 0) \) is a one element set.

Thus \( \mathbf{br}(t) \) is indeed isomorphic to \( k \).

It is not hard to check that the isomorphisms \( k \cong \mathbf{br}(u_c) \) and \( k \cong \mathbf{br}(u_a) \) are given by the unit maps. \( \square \)

### 6.2. An increasing filtration on \( s\mathcal{O} \)

Recall that if \( S \) is an SC set with \( c_S = \mathfrak{c} \) then an element

\[
u \in s\mathcal{O}_{\{I_s\}_{s \in S}}^J\]

gives us a total order \( >_u \) on the set

\[
\mathcal{I}_S := \bigsqcup_{s \in S} I_s.
\]

Following Subsection 6.1 this order gives us the elementary equivalence relation on \( \mathcal{I}_S \). We denote the number of equivalence classes in \( \mathcal{I}_S \) by \( |u| \) and define \( F_N s\mathcal{O}_{\{I_s\}_{s \in S}}^J \) as the subset consisting of all elements \( u \in s\mathcal{O}_{\{I_s\}_{s \in S}}^J \) with \( |u| \leq N + |S| \).

For an SC 2-set \( S \) with \( c_S = \mathfrak{a} \) we split \( S \) as \( S = S_\mathfrak{c} \sqcup S_\mathfrak{a} \) where \( S_\mathfrak{c} = \chi^{-1}(\mathfrak{c}) \) and \( S_\mathfrak{a} = \chi^{-1}(\mathfrak{a}) \).

By definition an element

\[
u \in s\mathcal{O}(S_\mathfrak{c}, S_\mathfrak{a})_{\{I_s\}_{s \in S_\mathfrak{c}}}\]
is a total order on the set
\[ I_{S_t \sqcup S_a} = \bigsqcup_{s \in S_t} I_s \sqcup S_a \]
subject to certain conditions.

Following Subsection 6.1.1 this order gives us the elementary equivalence relation on \( I_{S_t \sqcup S_a} \).
Let us denote the number of the equivalence classes in \( I_{S_t \sqcup S_a} \) by \(|u|\) and define
\[ F_N sO(I_{I_s}) \]
as the set of all elements \( u \in sO(I_{I_s}) \) with \(|u| \leq N + |S|\).

We claim that

**Lemma 6.5.** The filtration \( F \) is compatible with operadic compositions on \( sO \).

**Proof.** Similar to proof of Lemma 6.1. \( \square \)

This lemma implies that the filtration \( F \) on \( sO \) is compatible with the polysimplicial/cosimplicial structure. Therefore, the formula
\[ F_N |sO|(S) = |F_N(sO)(S)| \]
defines an increasing filtration on the dg SC operad \(|sO|\).

**Lemma 6.6.** The filtration \( F \) on \(|sO|\) satisfies the following properties:

1. The operadic compositions in \(|sO|\) are compatible with the filtration \( F \).
2. The complexes \( F_N |sO|(S) \) are concentrated in the degrees \( \geq -N \).
3. The cohomology of the quotient \( F_N |sO|(S)/F_{N-1} |sO|(S) \) is concentrated in the degree \(-N\).

**Proof.** Since the proof is very similar to that of Lemma 6.2 we will only briefly outline the proof of (3).

Let \( S \) be an SC set with \( c_S = \mathfrak{c} \).

As well as for the SC 2-operad \(|\text{seq}^{\text{SC}}|\) the cochain complex \( F_N |sO|(S)/F_{N-1} |sO|(S) \) is isomorphic to a direct sum of identical complexes
\[ (6.19) \quad F_N |sO|(S)/F_{N-1} |sO|(S) \cong \bigoplus_{\sigma \in D(S,N)} |\Xi_{N+|S|}|^{*+N}, \]
where the complexes \(|\Xi_k|\) are described in the Appendix and \( D(S,N) \) is the set of surjections
\[ (6.20) \quad \sigma : \{1,2,\ldots,N + |S|\} \to S \]
satisfying the following conditions

- **I** \( \sigma(i) \neq \sigma(i+1) \) \( \forall \ i = 1,2,\ldots,N + |S| - 1 \),
- **II** if \( s \neq \tilde{s} \in S \) then it is impossible to have \( i_1,i_2 \in \sigma^{-1}(s) \), and \( j_1,j_2 \in \sigma^{-1}(\tilde{s}) \) such that \( i_1 < j_1 < i_2 < j_2 \).

Thus Proposition 8.2 implies statement (3) in the case \( c_S = \mathfrak{c} \).

Similarly, if \( S \) is an SC set with \( c_S = \mathfrak{a}, S_t = \chi^{-1}(\mathfrak{c}) \) and \( S_a = \chi^{-1}(\mathfrak{a}) \) then the complex \( F_N |sO|(S)/F_{N-1} |sO|(S) \) is isomorphic to a direct sum of identical complexes
\[ (6.21) \quad F_N |sO|(S)/F_{N-1} |sO|(S) \cong \bigoplus_{\sigma \in D(S,N)} |\Xi_{N+|S_t|}|^{*+N,0}, \]
where the bicomplexes \(|\Xi_k|^{*,*}\) are described in the Appendix and \( D(S,N) \) is the set of surjections (6.20) satisfying above conditions **I**, **II** and the additional condition:

- **III** if \( s \in S_a \) then \( \sigma^{-1}(s) \) consists of exactly one element.
Since the complex $|\Xi_{N+|S|}|_0$ consists of the field $k$ placed in degree $0$, statement (3) follows in this case too.

We would like to remark that condition $B$ in the proof of Lemma 6.2 implies condition $II$ in the proof of Lemma 6.6. Therefore, for every 2-tree $t : S \to T$ we have the inclusion

$$D(t, N) \subset D(S, N).$$

Similarly, if $t$ is an SC 2-tree then conditions $A$, $B$, and $C$ imply conditions $I$, $II$, and $III$. Therefore, we have the inclusion (6.22) for SC 2-trees $t$ as well. We will use this inclusion later.

We now define a useful suboperad of $|s\emptyset|$

**Definition 6.7.** We define the dg SC operad braces as a suboperad of $|s\emptyset|$ with

$$\text{braces}(S) = \bigoplus_{N \geq 0} G^N|s\emptyset|(S),$$

where

$$G^N|s\emptyset|(S) = \{ v \in F_N|s\emptyset|(S)^{-N} \mid dv \in F_{N-1}|s\emptyset|(S) \},$$

and $S$ is an SC set.

Lemma 6.6 implies that the inclusion

$$\text{braces} \hookrightarrow |s\emptyset|$$

is a quasi-isomorphism.

**Proposition 6.8.** The dg SC operad braces is reduced.

**Proof.** Let $S$ be an SC set with $|S| \leq 1$. It is not hard to construct a pruned 2-tree or a pruned SC 2-tree $t : S \to T$ with $S$ being the source ordinal.

It is easy to see that if $|S| < 1$ then

$$\text{br}(t) = \text{braces}(S)$$

as cochain complexes.

Thus the desired statement follows immediately from Proposition 6.4.

Let us now consider a cofibrant resolution $R\text{br} \to \text{br}$ of $\text{br}$ in the closed model category of reduced dg SC 2-operads.

It is clear from the definitions of $\text{br}$ and braces that we have the embedding of dg SC 2-operads

$$\text{br} \hookrightarrow \text{des braces}.$$ 

Since $\text{sym}$ is the left adjoint functor for $\text{des}$ this embedding produces the map

$$\text{sym } \text{br} \to \text{braces}.$$ 

Composing (6.24) with the map

$$\text{sym } R\text{br} \to \text{sym } \text{br}$$

we get the map

$$\text{sym } R\text{br} \to \text{braces}.$$ 

We claim that

**Theorem 6.9.** The map (6.25) is a quasi-isomorphism of dg SC 2-operads.

This theorem plays a crucial role in proving our main result (Theorem 2.1). We devote the next section to the proof of this theorem.
7. Proof of Theorem 6.9

We need to show that for every SC set $S$ the map

\[(\text{sym } R \text{br})(S) \to \text{braces}(S)\]

is a quasi-isomorphism of cochain complexes.

Due to the symmetrization formula (see equation (4.4))

\[\text{sym } R \text{br}(S) = \text{colim}_{J(S)} R \text{br}_S.\]

Therefore, since $R \text{br}$ is a cofibrant resolution of $\text{br}$, we conclude that

\[\text{sym } R \text{br}(S) = \text{hocolim}_{J(S)} \text{br}_S.\]

Thus we need to show that the map

\[\text{hocolim}_{J(S)} \text{br}_S \to \text{braces}(S)\]

is a quasi-isomorphism of cochain complexes.

For this, it suffices to show that so is the map

\[\text{hocolim}_{J(S)}(F_N \text{br}/F_{N-1} \text{br})_S \to F_N \text{braces}(S)/F_{N-1} \text{braces}(S)\]

for every $N$.

Equations (6.12), (6.16) and statement (2) of Lemma 6.2 imply that for every 2-tree or SC 2-tree

\[F_N \text{br}(t)/F_{N-1} \text{br}(t) = k[D(t, N)][N],\]

where $k[D(t, N)][N]$ is considered as a cochain complex with the zero differential.

Similarly, equations (6.19), (6.21) and statement (2) of Lemma 6.6 imply that for every SC set $S$

\[F_N \text{braces}(S)/F_{N-1} \text{braces}(S) = k[D(S, N)][N],\]

where $k[D(S, N)][N]$ is considered as a cochain complex with the zero differential.

Let us recall that for every SC set $S$ and for every $t \in J(S)$ we have the inclusion

\[D(t, N) \subset D(S, N).\]

If $S$ is an SC set with $c_S = c$ (resp. $c_S = a$) then for $\sigma \in D(S, N)$ we set $J(\sigma) \subset J(S)$ to be the full subcategory of all 2-trees (resp. SC 2-trees) $t$ such that

\[\sigma \in D(t, N).\]

Recall that for every SC set $S$ the category $J(S)$ is a poset. It is not hard to see that for every morphism

\[P : t \to \tilde{t}\]

in the category $J(S)$ we have the inclusion

\[D(\tilde{t}, N) \subset D(t, N).\]

Furthermore, the morphism

\[F_N \text{br}(\tilde{t})/F_{N-1} \text{br}(\tilde{t}) \to F_N \text{br}(t)/F_{N-1} \text{br}(t)\]

corresponding to $P : t \to \tilde{t}$ is given by this inclusion.

Combining this observation with equations (7.3) and (7.4) we conclude that

\[\text{hocolim}_{J(S)}(F_N \text{br}/F_{N-1} \text{br})_S = \bigoplus_{\sigma \in D(S, N)} \text{hocolim}_{J(\sigma)} k,\]
\[(F_N/F_{N-1}) \text{braces}(S)(N) = \bigoplus_{\sigma \in D(S,N)} k,\]

and (7.2) is induced by the natural maps
\[
(7.7) \quad \text{hocolim}_{\mathcal{J}(\sigma)} k \to k,
\]
where, by abuse of notation, \(k\) denotes both the underlying field and the functor which assigns \(k\) to every object of \(\mathcal{J}(\sigma)\).

Thus it suffices to show that the map (7.7) is a quasi-isomorphism for every \(\sigma \in D(S,N)\).

The obvious topological counterpart of this statement can be formulated as

**Proposition 7.1.** For every SC set \(S\) and every element \(\sigma \in D(S,N)\) the natural map
\[
(7.8) \quad \text{hocolim}_{\mathcal{J}(\sigma)} \text{pt} \to \text{pt}
\]

is a weak equivalence.

In what follows, by abuse of notation, we denote a constant functor from \(\mathcal{J}(\sigma)\) to another category by the underlying object. For example, in (7.8) \(\text{pt}\) denotes both the one-point space and the functor from \(\mathcal{J}(\sigma)\) to the category of topological spaces which assigns \(\text{pt}\) to every object of \(\mathcal{J}(\sigma)\).

Let us postpone the proof of Proposition 7.1 to the end of the section and show that this proposition indeed implies that (7.7) is a quasi-isomorphism.

We, first, use the adjunction
\[
(7.9) \quad | |_{\text{top}} : \text{sSets} \leftrightarrow \text{Top} : C^*_{\text{sing}}
\]

between the category \(\text{Top}\) of topological spaces and the category \(\text{sSets}\) of simplicial sets. Here \(| |_{\text{top}}\) denotes the realization functor and \(C^*_{\text{sing}}\) is the singular chain functor.

Using the fact that the adjunction (7.9) gives a Quillen equivalence between \(\text{Top}\) and \(\text{sSets}\) it is not hard to deduce from Proposition 7.1 its counterpart for simplicial sets. Namely, Proposition 7.1 implies that for every \(\sigma \in D(S,N)\) the natural map
\[
(7.10) \quad \text{hocolim}_{\mathcal{J}(\sigma)} \Delta^0 \to \Delta^0
\]
is a weak equivalence of simplicial sets, where

\[
\Delta^0 = \text{hom}_\Delta (\ , [0])
\]
is the terminal object of the category \(\text{sSets}\).

Therefore, for the simplicial Abelian group \(\mathbb{Z}\Delta^0\), the natural map
\[
(7.11) \quad \text{hocolim}_{\mathcal{J}(\sigma)} \mathbb{Z}\Delta^0 \to \mathbb{Z}\Delta^0
\]
is a weak equivalence.

Notice that, via the Dold-Kan correspondence, (7.11) can be viewed as a map of cochain\(^5\) complexes of Abelian groups. Furthermore, to say that (7.11) is a weak equivalence of simplicial Abelian groups is to say that (7.11) is a quasi-isomorphism of the corresponding cochain complexes.

Recall that the forgetful functor
\[
\Psi : k - \text{Vect} \to \text{Ab}
\]
from the category \(k - \text{Vect}\) of \(k\)-vector spaces to the category \(\text{Ab}\) of Abelian groups admits the left adjoint functor
\[
k \otimes \mathbb{Z} : \text{Ab} \to k - \text{Vect}.
\]

\(^5\)Here we reverse the standard grading of the Dold-Kan correspondence.
Using this adjunction and the quasi-isomorphism (7.11) we deduce that the natural map
\[ \text{hocolim}_{\mathcal{J}} \mathbf{k}\Delta^0 \to \mathbf{k}\Delta^0 \]
is a quasi-isomorphism of cochain complexes of \(\mathbf{k}\)-vector spaces. Here \(\mathbf{k}\Delta^0\) is the cochain complex
\[ \cdots \to \mathbf{k} \to \mathbf{k} \to \mathbf{k} \to \cdots \]
with the right most term placed in degree 0. This complex is obviously quasi-isomorphic to \(\mathbf{k}\) placed in degree 0. And hence the map (7.7) is indeed a quasi-isomorphism of cochain complexes.

In order to complete the proof of Theorem 6.9 it remains to prove Proposition 7.1.

7.1. Proof of Proposition 7.1. We need a cofibrant resolution of the trivial functor from the poset \(\mathcal{J}(\sigma)\) to the category of topological spaces. The closed model structure on the category of functors from \(\mathcal{J}(\sigma)\) to \(\text{Top}\) is obtained from that on topological spaces using the transfer principle of C. Berger and I. Moerdijk [7]. In other words, fibrations (resp. weak equivalences) between functors from \(\mathcal{J}(\sigma)\) are object-wise fibrations (resp. object-wise weak equivalences).

To construct this resolution we use the configuration space \(\text{Conf}(S)\) of distinct points on \(\mathbb{R}^2\) labeled by elements of a finite set \(S\).

It is known that the space \(\text{Conf}(S)\) admits a cellular subdivision into the Fox-Neuwirth cells [5], [14], [32]. Each Fox-Neuwirth cell \(\text{FN}_t\) corresponds to a pruned 2-tree \(t : S \to T\) and it can be defined as the space of all injective maps from the 2-tree \(t\) to the generalized 2-tree:
\[(x, y) \to x : \mathbb{R}^2 \to \mathbb{R},\]
where on \(\mathbb{R}^2\) we use the lexicographic order.

In other words, a configuration \(\{(x_s, y_s)\}_{s \in S}\) belongs to \(\text{FN}_t\) iff the following conditions are satisfied:

— if \(t(s) = t(\tilde{s})\) and \(s < \tilde{s}\) then \(x_s = x_{\tilde{s}}\) and \(y_s < y_{\tilde{s}}\),
— if \(t(s) < t(\tilde{s})\) then \(x_s < x_{\tilde{s}}\).

An example of a configuration from \(\text{FN}_{t_1}\) for the 2-tree
\[ t_1 : \{1, 2, 3, 4, 5\} \to \{1, 2, 3\} \]
\[ t_1(1) = t_1(2) = 1, \quad t_1(3) = t_1(4) = 2, \quad t_1(5) = 3 \]
is depicted in figure 6

\[ \text{Figure 6. A typical point of } \text{FN}_{t_1} \]

\[ \text{Figure 6. A typical point of } \text{FN}_{t_1} \]

\[ \text{Figure 6. A typical point of } \text{FN}_{t_1} \]
This construction can be easily generalized to pruned SC 2-trees. Namely, if \( t : S \to T \) is a pruned SC 2-tree with \( S_a \) being the preimage of the minimal element of \( T \) and \( S_\epsilon = S \setminus S_a \) then \( \text{FN}_t \) consists of configurations \( \{(x_s, y_s)\}_{s \in S} \) satisfying the following conditions:

— if \( s \in S_a \) then \( x_s = 0 \); if \( s \in S_\epsilon \) then \( x_s > 0 \),

— if \( t(s) = t(\tilde{s}) \) and \( s < \tilde{s} \) then \( x_s = x_\tilde{s} \) and \( y_s < y_\tilde{s} \),

— if \( t(s) < t(\tilde{s}) \) then \( x_s < x_\tilde{s} \).

Recall that for pruned SC 2-trees the range \( t(S) \) does not in general include the minimal element. In other words, the subset \( S_a \) may be empty. In this case we still require that \( x_s > 0 \) for \( s \in S_\epsilon \).

If \( S \) is an SC set then for every map \( P : t \to \tilde{t} \) of pruned (SC) 2-trees in the category \( \mathcal{J}(S) \) we have the obvious inclusion

\[
\text{FN}_{\tilde{t}} \hookrightarrow \partial \text{FN}_t,
\]

where \( \partial \text{FN}_t \) denotes the boundary of the Fox-Neuwirth cell \( \text{FN}_t \).

For example, we may consider the 2-tree

\[
t_2 : \{1, 2, 3, 4, 5\} \to \{1, 2\}
\]

\[
t_2(1) = t_2(2) = 1, \quad t_2(3) = t_2(4) = t_2(5) = 2
\]

with a (unique) map in \( \mathcal{J}(\{1, 2, 3, 4, 5\}) \)

\[
P : t_1 \to t_2,
\]

\[
P_S = id, \quad P_T(1) = 1, \quad P_T(2) = P_T(3) = 2.
\]

A configurations from \( \text{FN}_{t_2} \) consists of a pair of distinct vertical lines; the left line carries points 1 and 2 such that 1 is below 2; the right line carries points 3, 4, 5 which are put in the order from the bottom to the top. (See figure 7.) It is clear that \( \text{FN}_{t_2} \) belongs to the boundary of \( \text{FN}_{t_1} \).

\[\text{Figure 7. A typical point of } \text{FN}_{t_2}\]

Let \( \sigma \in D(S, N) \) and \( \mathcal{J}(\sigma) \) be the sub-poset of \( \mathcal{J}(S) \) defined above. Using the inclusion (7.13) we upgrade the correspondence

\[
t \to \Phi_\sigma(t) = \bigcup_{\tilde{t} \in \mathcal{J}(\sigma); t \to \tilde{t}} \text{FN}_{\tilde{t}}
\]

to the functor

\[
\Phi_\sigma : \mathcal{J}(\sigma) \to \text{Top}.
\]

The union in (7.14) is taken over all the pruned (SC) 2-trees \( \tilde{t} \in \mathcal{J}(\sigma) \) for which we have a map from \( t \) to \( \tilde{t} \).
Example 7.2. We consider $S = \{\alpha, \beta, \gamma, \delta\}$ with $c_S = a$, $\chi(\alpha) = \chi(\gamma) = \chi(\delta) = c$, $\chi(\beta) = a$, and $\sigma$ being the following map
\[
\sigma : \{1, 2, 3, 4, 5, 6\} \rightarrow S
\]
\[
\sigma(1) = \alpha, \quad \sigma(2) = \delta, \quad \sigma(3) = \gamma, \quad \sigma(4) = \beta, \quad \sigma(5) = \gamma, \quad \sigma(6) = \delta.
\]
The map $\sigma$ is an element of $D(S, 2)$ and the SC 2-tree $t : \{\beta < \gamma < \alpha < \delta\} \rightarrow \{1, 2, 3, 4\}$
\[
t(\beta) = 1, \quad t(\gamma) = 2, \quad t(\alpha) = 3, \quad t(\delta) = 4
\]
is an object of $J(\sigma)$.

There are exactly three pruned SC 2-trees $\tilde{t} \in J(\sigma)$ for which there is a map $t \rightarrow \tilde{t}$. The first one is $\tilde{t}_1 = t$ and the second one is $\tilde{t}_2 : \{\beta < \gamma < \alpha < \delta\} \rightarrow \{1, 2, 3\}$
\[
\tilde{t}_2(\beta) = 1, \quad \tilde{t}_2(\gamma) = 2, \quad \tilde{t}_2(\alpha) = \tilde{t}_2(\delta) = 3.
\]
The third SC 2-tree $\tilde{t}_3 : \{\beta < \alpha < \gamma < \delta\} \rightarrow \{1, 2, 3\}$
\[
\tilde{t}_3(\beta) = 1, \quad \tilde{t}_3(\alpha) = \tilde{t}_3(\gamma) = 2, \quad \tilde{t}_3(\delta) = 3.
\]
So the space $\Phi_\sigma(t)$ consists of configurations $\{(x_s, y_s)\}_{s \in \{\alpha, \beta, \gamma, \delta\}}$ satisfying the following conditions:

- if $x_\beta = 0 < x_\gamma \leq x_\alpha \leq x_\delta$, and $x_\gamma < x_\delta$,
- if $x_\alpha = x_\gamma$ then $y_\alpha < y_\gamma$,
- if $x_\alpha = x_\delta$ then $y_\alpha < y_\gamma$.

Proposition 7.3. Let $S$ be an SC set and $\sigma \in D(S, N)$. Then the functor $\Phi_\sigma$ (7.14) is a cofibrant resolution of the trivial functor from $J(\sigma)$ to the category of topological spaces.

Proof. Let $S$ be an SC set and $\sigma \in D(S, N)$. Let us show that $\Phi_\sigma(t)$ is contractible for every pruned (SC) 2-tree $t : S \rightarrow T$ for which $\sigma \in D(t, N)$.

We give a detailed proof of contractibility of $\Phi_\sigma(t)$ in the case when $c_S = c$ and hence $t$ is a pruned non-SC 2-tree. The case $c_S = a$ is very similar.

The 2-tree $t : S \rightarrow T$ gives us a total order on the set $S$. So we identify $S$ with the ordinal $\{1, 2, 3, \ldots, |S|\}$ and denote by $(x_i, y_i)$ the coordinates of the point labeled by $i \in \{1, 2, 3, \ldots, |S|\}$.

Next, we consider the following sequence of subspaces
\[
\Phi_\sigma(t) = F_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots \supseteq F_{|S|}
\]
where $F_k$ consists of configurations $\{(x_i, y_i)\} \in \Phi_\sigma(t)$ with
\[
y_i = i, \quad \forall i \leq k.
\]
Let us show that $F_{k+1}$ is a deformation retract of $F_k$ for all $k = 0, 1, 2, \ldots, |S| - 1$.

A deformation retraction $f : F_k \times [0, 1] \rightarrow F_k$ of $F_k$ onto $F_{k+1}$ is given by the formula:
\[
f(\{(x_i, y_i)\}, t) = \{(x_i, y_i(t))\},
\]
where
\[
y_i(t) = \begin{cases} i, & \text{if } i \leq k, \\ (1 - t)y_i + t(k + 1 + y_i - y_{i+1}), & \text{if } i > k. \end{cases}
\]
We need to show that for all $t \in [0, 1]$ and for all configurations $\{(x_i, y_i)\} \in F_k$ the point $f(\{(x_i, y_i)\}, t)$ belongs to $\Phi_\sigma(t)$. More precisely, we need to check that if $x_i = x_j$ and $i < j$ then $y_i(t) < y_j(t)$ for all $t \in [0, 1]$.

First, it is obvious that if $i < j \leq k$ the $y_i(t) < y_j(t)$ regardless of whether $x_i$ equals $x_j$ or not.

Second, it is not hard to see that if $k < i < j$ and $y_i < y_j$ then $y_i(t) < y_j(t)$ for all $t \in [0, 1]$. 
Finally, if $i \leq k < j$ and $x_i = x_j$ then the configuration $\{(x_i, y_i)\}$ belongs to the Fox-Neuwirth cell $\text{FN}_t$ corresponding to a pruned 2-tree $\tilde{t}$ for which

$$\tilde{t}(i) = \tilde{t}(j).$$

The latter implies that $\tilde{t}(i) = \tilde{t}(i + 1) = \cdots = \tilde{t}(j - 1) = \tilde{t}(j)$ and hence

$$x_i = x_{i+1} = \cdots = x_{j-1} = x_j.$$

Therefore $y_i < y_{i+1} < \cdots < y_{j-1} < y_j$ and, in particular, $y_j \geq y_{k+1} > y_k = k$.

Using these inequalities we conclude that for all $t \in [0, 1]$

$$y_j(t) = (1 - t)y_j + t(k + 1) + t(y_j - y_{k+1}) > k + t(y_j - y_{k+1}) \geq k.$$

On the other hand $y_i(t) \leq k$. Thus, if $x_i = x_j$ and $i < j$ then

$$y_j(t) > y_i(t)$$

for all $t \in [0, 1]$.

Furthermore, if $y_i = i$ for all $i \leq k + 1$ then $y_i(t) \equiv i$ for all $i \leq k + 1$. Thus $f$ is indeed a deformation retraction of $F_k$ onto $F_{k+1}$.

Let us now identify $T$ with the standard ordinal $\{1, 2, 3, \ldots, |T|\}$. Next we note that if $\tilde{t} \in J(\sigma)$ admits a map $t \to \tilde{t}$ then equality $t(i) = t(j)$ implies the equality $\tilde{t}(i) = \tilde{t}(j)$. Hence, if $t(i) = t(j)$ then $x_i = x_j$ for every configuration $\{(x_i, y_i)\} \in \Phi_\sigma(t)$.

Therefore the function $i \to x_i$ factors through

$$t : \{1, 2, 3, \ldots, |S|\} \to \{1, 2, 3, \ldots, |T|\}$$

and hence, we may describe configurations from $\Phi_\sigma(t)$ using the collections of coordinates $\{z_l, y_i\}$, $z_l, y_i \in \mathbb{R}$ where $l \in \{1, 2, 3, \ldots, |T|\}$ and $i \in \{1, 2, 3, \ldots, |S|\}$.

For every configuration $\{z_l, y_i\}$ from $\Phi_\sigma(t)$ we have

$$z_1 \leq z_2 \leq z_3 \leq \cdots \leq z_{|T|}$$

and if $z_l = z_m$ for $l \neq m$ then the corresponding configuration belongs to the Fox-Neuwirth cell $\text{FN}_t$ of a 2-tree $\tilde{t} \neq t$.

To show the contractibility of $F_{|S|}$ we consider the following sequence of subspaces:

$$F_{|S|} = G_0 \supset G_1 \supset G_2 \supset \cdots \supset G_{|T|} \cong pt.$$

where $G_k$ consists of configurations $\{z_l, y_i\} \in F_{|S|}$ satisfying the following condition

$$z_l = l, \quad \forall \ l \leq k.$$

In terms of the original coordinates $(x_i, y_i)$ the latter condition reads

$$x_i = t(i), \quad \text{if} \quad t(i) \leq k.$$

We show that for all $k \leq |T| - 1$ the space $G_{k+1}$ is a deformation retract of $G_k$.

The desired deformation retraction is defined by the formula

$$g(\{z_l, y_i\}, t) = \{z_l(t), y_i\},$$

where

$$z_l(t) = \begin{cases} l, & \text{if} \quad l \leq k, \\ (1 - t)z_l + t(k + 1 + z_l - z_{k+1}), & \text{if} \quad l > k. \end{cases}$$

\footnote{In this case $y_j = y_{k+1}$ only if $j = k + 1$}
To prove that the configuration \( \{ z(t), y(t) \} \) belongs to \( F_{|S|} \) for all \( t \in [0,1] \) we need to check that inequalities
\[
(7.18) \quad z_1(t) \leq z_2(t) \leq z_3(t) \leq \cdots \leq z_{|T|}(t)
\]
hold for all \( t \in [0,1] \). Furthermore we need to check that if \( z_l < z_m \) then \( z_l(t) < z_m(t) \) for all \( t \in [0,1] \).

In the case \( l < m \leq k \) we simply have the inequality \( z_l(t) < z_m(t) \). Also it is not hard to see that in the case \( k < l < m \) the inequality \( z_l(t) \leq z_m(t) \) (resp. \( z_l(t) < z_m(t) \)) follows from \( z_l \leq z_m \) (resp. \( z_l < z_m \)).

Thus it remains to consider the case \( l = k \) and \( m = k + 1 \).

In this case we have \( z_k(t) \equiv z_k = k \). Furthermore, due to (7.16) we have \( z_{k+1} \geq k \) and hence
\[
z_{k+1}(t) = (1-t)z_{k+1} + t(k+1) \geq (1-t)k + tk = k.
\]
It is also obvious that if \( z_{k+1} > k \) then
\[
z_{k+1}(t) = (1-t)z_{k+1} + t(k+1) > (1-t)k + tk = k = z_k(t)
\]
for all \( t \in [0,1] \).

Finally, it is clear that if \( z_{k+1} = k + 1 \) then
\[
z_{k+1}(t) \equiv k + 1.
\]

Thus \( g (7.17) \) is indeed a deformation retraction of \( G_k \) onto \( G_{k+1} \).

Since \( G_{|T|} \) is a one-point space we conclude that \( F_{|S|} \), and hence, the space \( \Phi_\sigma(t) \) is contractible.

The proof of the fact that \( \Phi_\sigma \) is a cofibrant object in the category of functors from \( J(\sigma) \) to \( \text{Top} \) is very similar to the proof of Theorem 7.2 from [2].

Following the arguments of [2] we define the following sequence of functors
\[
\Phi^m_\sigma, \quad m \in \mathbb{Z}, \quad m \geq 0.
\]
On the level of objects the functor \( \Phi^m_\sigma \) operates as
\[
(7.19) \quad \Phi^m_\sigma(t) = \begin{cases} 
\Phi_\sigma(t), & \text{if } |S| + |T| < m, \text{ and } t \text{ is a 2-tree,} \\
\Phi_\sigma(t), & \text{if } |S| + |T| - 1 < m, \text{ and } t \text{ is an SC 2-tree,} \\
\emptyset, & \text{otherwise.}
\end{cases}
\]
We would like to remark that the number \( |S| + |T| \) (resp. \( |S| + |T| - 1 \)) for a 2-tree \( t : S \to T \) (resp. for an SC 2-tree \( t : S \to T \)) is the dimension of the Fox-Neuwirth cell \( \text{FN}_t \). Thus the collection \( \Phi^m_\sigma \) may be considered as a filtration of \( \Phi_\sigma \) by dimension.

We have the obvious sequence of natural transformations
\[
\Phi^0_\sigma \to \Phi^1_\sigma \to \Phi^2_\sigma \to \ldots
\]
and the functor \( \Phi_\sigma \) is the sequential colimit
\[
\Phi_\sigma = \text{colim}_m \Phi^m_\sigma.
\]

Similarly to the proof of Theorem 7.2 from [2] we show that for every \( m \) the natural transformation
\[
\Phi^m_\sigma \to \Phi^{m+1}_\sigma
\]
is a cellular extension generated by a cofibration.

Thus \( \Phi_\sigma \) is indeed a cofibrant object in the category of functors from \( J(\sigma) \) to the category of topological spaces.

This completes the proof of Proposition 7.3.
Now that we have a cofibrant resolution $\Phi_\sigma$ of the trivial functor from $\mathcal{J}(\sigma)$ to $\text{Top}$ we prove Proposition 7.1 by showing that the space

\begin{equation}
X_\sigma = \colim_{\mathcal{J}(\sigma)} \Phi_\sigma
\end{equation}

is contractible for every surjection $\sigma \in D(S,N)$, where $S$ is an SC set.

It is easy to see that

\begin{equation}
X_\sigma = \bigcup_{t \in \mathcal{J}(\sigma)} FN_t.
\end{equation}

To get a more explicit description of the space $X_\sigma$ (7.21) we recall that the set $D(S,N)$ consists of surjections

$\sigma : \{1,2,3,\ldots,|S|+N\} \to S$

from the standard ordinal $\{1,2,3,\ldots,|S|+N\}$ to the set $S$; the surjections $\sigma$ should satisfy two conditions I and II from the proof of Lemma 6.6; if, in addition $c_S = a$, then we should also impose on $\sigma$ condition III from the proof of the same lemma.

Let us also recall that a 2-tree (an SC 2-tree) $t : S \to T$ belongs to $\mathcal{J}(\sigma)$ iff the following conditions are met:

- if for $s \neq \tilde{s}$ there exist $i_1,i_2 \in \sigma^{-1}(\tilde{s})$ and $i \in \sigma^{-1}(s)$ such that $i_1 < i < i_2$ then $t(s) < t(\tilde{s})$ in the (SC) ordinal $T$,
- if $t(s) = t(\tilde{s})$ and $s < t \tilde{s}$ then all elements of $\sigma^{-1}(s)$ are smaller than all elements of $\sigma^{-1}(\tilde{s})$.

Here $<_t$ is the total order on $S$ coming from the structure of the (SC) 2-tree $t$.

Thus the space $X_\sigma$ (7.21) consists of the configurations $\{(x_s,y_s)\}$ from $\text{Conf}(S)$ satisfying the following conditions:

**C1** if $\exists i_1,i_2 \in \sigma^{-1}(\tilde{s})$ and $i \in \sigma^{-1}(s)$ such that $i_1 < i < i_2$ then $x_s < x_{\tilde{s}}$

**C2** if $x_s = x_{\tilde{s}}$ and all elements of $\sigma^{-1}(s)$ are smaller than all elements of $\sigma^{-1}(\tilde{s})$ then $y_s < y_{\tilde{s}}$.

If $c_S = a$ then we have to impose on the configuration $\{(x_s,y_s)\}$ the additional condition

**C3** if $\chi(s) = a$ then $x_s = 0$ and if $\chi(s) = c$ then $x_s > 0$.

**Remark.** Let $S$ be an SC set with $c_S = c$. It can be shown that every surjection $\sigma \in D(S,N)$ gives us a pair of complementary orders on the set $S$ in the sense of M. Kontsevich and Y. Soibelman [20]. (See also Section 2 in [2] about complementary orders and higher trees.) To a pair of complementary orders $>_0$ and $>_1$ M. Kontsevich and Y. Soibelman assign a subspace $X_{>0,>_1}$ [20] of the compactified configuration space of points on $\mathbb{R}^2$ labeled by elements of $S$. Our space $X_\sigma$ is an uncompactified version of the subspace considered by M. Kontsevich and Y. Soibelman in [20].

7.1.1. **Contractibility of $X_\sigma$.** We give a detailed proof of the contractibility of $X_\sigma$ (7.21) in the case when the color $c_S$ of the SC set $S$ is $a$. The case $c_S = c$ is very similar.

Although every SC 2-tree $t \in \mathcal{J}(\sigma)$ gives us a total order $>_t$ on $S$, we equip the set $S$ with yet another total order which we denote by $<_\sigma$. Namely, we set $s <_\sigma \tilde{s}$ iff

- either $\exists i_1,i_2 \in \sigma^{-1}(\tilde{s})$ and $i \in \sigma^{-1}(s)$ such that $i_1 < i < i_2$ or
- all elements of $\sigma^{-1}(s)$ are smaller than all elements of $\sigma^{-1}(\tilde{s})$.

**Warning.** In general, the order total $>_t$ on $S$ coming from the structure of an SC 2-tree $t \in \mathcal{J}(\sigma)$ does not coincide with the order $>_\sigma$. Thus, in Example 7.2, the map $\sigma$ induces on the SC set $S$ the order

$\alpha < \beta < \gamma < \delta$. 

On the other hand we have a pruned SC 2-tree $t : \{\beta < \gamma < \alpha < \delta\} \to \{1, 2, 3, 4\}$ which belongs to $J(\sigma)$. A similar example can be found for an SC set $S$ with $c_S = c$.

Using the total order $>_\sigma$ we identify $S$ with the standard ordinal $\{1 < 2 < 3 < \cdots < |S|\}$.

Next we define the following functions on $\text{Conf}(S)$
\begin{equation}
\mu_k(\{(x_s, y_s)\}) = \min(y_k, y_{k+1}, \ldots, y_{|S|})
\end{equation}
which are obviously continuous.

Then we introduce the sequence of subspaces
\begin{equation}
X_\sigma = Y_0 \supset Y_1 \supset \cdots \supset Y_{|S|},
\end{equation}
where $Y_k$ consists of configurations $\{(x_s, y_s)\} \in X_\sigma$ satisfying the properties
\begin{equation}
y_s = y_1 + s - 1, \quad \forall \ s \leq k,
\end{equation}
\begin{equation}
\mu_{k+1}(\{(x_s, y_s)\}) = y_k + 1.
\end{equation}

Let us show that $Y_k$ is homotopy equivalent to $Y_k$ for all $k < |S|$.

For this purpose we introduce an intermediate subspace $Z_k$
\begin{equation}
Y_k \supset Z_k \supset Y_{k+1}.
\end{equation}
This subspace consists of configurations $\{(x_s, y_s)\} \in Y_k$ satisfying the property
\begin{equation}
y_{k+1} = \mu_{k+1}(\{(x_s, y_s)\}).
\end{equation}

Let us consider the map $h : Y_k \times [0, 1] \to Y_k$
\begin{equation}
h(\{(x_s, y_s)\}, t) = \{(x_s, y_s(t))\},
\end{equation}
where
\begin{equation}
y_s(t) = \begin{cases}
y_s, & \text{if } s \neq k + 1, \\
(1-t)y_{k+1} + ty_{k+1}(\{(x_s, y_s)\}), & \text{if } s = k + 1.
\end{cases}
\end{equation}

In order to show that $h(\{(x_s, y_s)\}, t) \in Y_k$ we only need to check condition C2 for all $t \in [0, 1]$.

It is clear that
\begin{equation}
y_{k+1}(t) \geq y_{k+1}(t) \geq \mu_{k+1}(\{(x_s, y_s)\}), \quad \forall \ t \in [0, 1].
\end{equation}

Since $\{(x_s, y_s)\} \in Y_k$ we have
\begin{equation}
\mu_{k+1}(\{(x_s, y_s)\}) > y_s, \quad \forall \ s \leq k
\end{equation}
and hence
\begin{equation}
y_{k+1}(t) > y_s, \quad \forall \ s \leq k, \quad t \in [0, 1].
\end{equation}

Furthermore, since condition C2 is satisfied for $\{(x_s, y_s)\}$ we conclude that all points $(x_s, y_s)$ with $s > k + 1$ and $x_s = x_{k+1}$ lie above the point $(x_{k+1}, y_{k+1})$. Combining this observation with inequality (7.27) we conclude that if $s > k + 1$ and $x_s = x_{k+1}$ then $y_{k+1}(t) < y_s$ for all $t \in [0, 1]$.

It is clear that $h(\{(x_s, y_s)\}, 1) \in Z_k$ and for all $\{(x_s, y_s)\} \in Z_k$ we have
\begin{equation}
h(\{(x_s, y_s)\}, t) = \{(x_s, y_s)\}, \quad \forall \ t \in [0, 1].
\end{equation}

Thus $h$ is a deformation retraction of $Y_k$ onto $Z_k$.

It is clear that the subspace $Y_{k+1}$ consists of configurations $\{(x_s, y_s)\} \in Z_k$ satisfying the additional property
\begin{equation}
\mu_{k+2}(\{(x_s, y_s)\}) = y_{k+1} + 1.
\end{equation}
Thus we conclude that condition (7.25) is satisfied for every configuration \( \{ (x_s, y_s) \} \in X_\sigma \) satisfying the property
\[
y_s = y_1 + s - 1, \quad \forall \ s \in S.
\]

To show that \( Y_{|S|} \) is contractible we set, as above, \( S_\alpha = \chi^{-1}(a) \) and \( S_\epsilon = \chi^{-1}(\epsilon) \).

Due to Condition C3 \( x_s = 0 \) for all \( s \in S_\alpha \) and \( x_s > 0 \) for all \( s \in S_\epsilon \).

Restricting the total order \( >_\sigma \) from \( S \) to \( S_\epsilon \) we get an isomorphism
\[
\beta : S_\epsilon \to \{ 1 < 2 < 3 < \cdots < |S_\epsilon| \}
\]
from \( S_\epsilon \) to the standard ordinal \( \{ 1 < 2 < 3 < \cdots < |S_\epsilon| \} \).

Using this isomorphism we define the following map \( H : Y_{|S|} \times [0,1] \to Y_{|S|} \)
\[
H(\{ (x_s, y_s) \}, t) = \{ (x_s(t), y_s) \},
\]
where
\[
x_s(t) = \begin{cases} 
0, & \text{if } s \in S_\alpha, \\
(1 - t)x_s + t \beta(s), & \text{if } s \in S_\epsilon.
\end{cases}
\]

Let us show that \( H \) indeed lands in \( X_\sigma \).
Since \(x_s(t) > 0\) for all \(s \in S_\varepsilon\) and \(t \in [0, 1]\) we need to check Condition C1 only for \(s, \tilde{s} \in S_\varepsilon\).

If \(s, \tilde{s} \in S_\varepsilon\), \(s \neq \tilde{s}\) and there exists \(i_1, i_2 \in \sigma^{-1}(\tilde{s})\) and \(i \in \sigma^{-1}(s)\) such that \(i_1 < i < i_2\) then \(x_s < x_{\tilde{s}}\) and \(\beta(s) < \beta(\tilde{s})\) according to the definition of the total order \(<_\sigma\) on \(S\). Hence

\[
(1-t)x_s + t\beta(s) < (1-t)x_{\tilde{s}} + t\beta(\tilde{s}), \quad \forall \ t \in [0,1].
\]

Condition C2 is satisfied automatically because for every configuration in \(Y_{[S]}\) we have (7.29).

Condition C3 is also obviously satisfied.

It also follows from the construction that

\[
H(\{(x_s, y_s)\}, t) \in Y_{[S]}
\]

for all \(\{(x_s, y_s)\} \in Y_{[S]}\) and \(t \in [0, 1]\).

Furthermore, it is clear that \(H\) is a deformation retraction of \(Y_{[S]}\) onto the subspace \(L\) of configurations \(\{(x_s, y_s)\} \in X_\sigma\) with

\[
y_s = y_1 + s - 1, \quad \forall \ s \in S,
\]

\[
x_s = 0, \quad \forall \ s \in S_\alpha,
\]

and

\[
x_s = \beta(s), \quad \forall \ s \in S_\varepsilon.
\]

The subspace \(L\) is obviously homeomorphic to the real line \(\mathbb{R}\).

Thus we conclude that \(Y_{[S]}\) and hence \(X_\sigma\) is contractible.

This completes the proof of Proposition 7.1 and hence the proof of Theorem 6.9.

**Example 7.4.** Let us illustrate the proof of contractibility for \(X_\sigma\) with the map

\[
\sigma : \{1, 2, 3, 4, 5, 6\} \rightarrow \{\alpha, \beta, \gamma, \delta\}
\]

from Example 7.2. Recall that \(c_S = \mathcal{A}\). \(\chi(\alpha) = \chi(\gamma) = \chi(\delta) = c\), and \(\chi(\beta) = a\).

The space \(X_\sigma\) consists of configurations from \(\text{Conf}(\{\alpha, \beta, \gamma, \delta\})\) satisfying the following conditions:

i) \(x_\beta = 0 < x_\gamma < x_\delta\),

ii) \(x_\alpha > 0\),

iii) if \(x_\alpha = x_\gamma\) then \(y_\alpha < y_\gamma\),

iv) if \(x_\alpha = x_\delta\) then \(y_\alpha < y_\delta\).

In the first step of the above proof we retract \(X_\sigma\) onto the subspace \(Z_0\) of configurations satisfying the property

\[
y_\alpha = \min(y_\alpha, y_\beta, y_\gamma, y_\delta).
\]

Second, we retract \(Z_0\) to the subspace \(Y_1\) of configurations satisfying in addition the property

\[
\min(y_\beta, y_\gamma, y_\delta) = y_\alpha + 1.
\]

Next, we retract \(Y_1\) to the subspace \(Z_1\) which consists of configurations \(\{(x_s, y_s)\} \in Y_1\) with

\[
y_\beta = \min(y_\beta, y_\gamma, y_\delta).
\]

We keep doing so until we get the subspace \(Y_4\) of configurations \(\{(x_s, y_s)\} \in X_\sigma\) with

\[
y_\delta = y_\gamma + 1 = y_\beta + 2 = y_\alpha + 3.
\]

Then we retract the resulting space \(Y_4\) to the subspace \(L\) of configurations \(\{(x_s, y_s)\} \in X_\sigma\) satisfying (7.31) and

\[
x_\alpha = 1, \quad x_\beta = 0, \quad x_\gamma = 2, \quad x_\delta = 3.
\]
Performing the latter retraction we may need to move horizontally the point labeled by $\alpha$ through the vertical lines containing the points labeled by $\gamma$ and $\delta$. In doing so we will not violate conditions $iii$ and $iv$ because the inequalities $y_{\alpha} < y_{\gamma}$ and $y_{\alpha} < y_{\delta}$ are already achieved at the previous steps.

The subspace $L$ is obviously homeomorphic to the real line. Thus contractibility of $X_\sigma$ follows.

8. Proof of Theorem 2.1

Let us return to the dg SC 2-operad $\text{br}$ introduced in Definition 6.3 and show that

**Proposition 8.1.** For every pruned 2-tree (pruned SC 2-tree) $t$

1) the cochain complex $\text{br}(t)$ is contractible;

2) there exist natural identifications

$$H^0(\text{br}(t)) = k$$

under which all operadic composition maps of the operad $H^*(\text{br})$ evaluated on $1 \in k$ produce $1 \in k$.

**Proof.** Due to Lemma 6.2 the inclusion

$$\text{br} \hookrightarrow \text{SC \seq}$$

is a quasi-isomorphism of dg SC 2-operads.

Thus we need show that for every pruned 2-tree $t : S \to T$ and for every pruned SC 2-tree $t : S \to T$ the cochain complex

$$\text{SC \seq}(t)$$

is contractible.

For a pruned 2-tree $t$

$$\text{SC \seq}(t) = |\text{seq}(t)|$$

and contractibility of $|\text{seq}(t)|$ was proved in Proposition 6.4 in [27]. For the convenience of the reader we briefly recall the proof of contractibility for $|\text{seq}(t)|$.

By definition $|\text{seq}(t)|$ is the realization of the cosimplicial/polysimplicial set (see Section 5)

$$\{\{I_s\}_{s \in S}; J\} \to \text{seq}(t)_{\{I_s\}_{s \in S}}$$

in the category of cochain complexes.

Thus we need to show that realizing (8.1) in the category of topological spaces we get a contractible space.

For this purpose we fix the ordinal $J$ and consider the corresponding polysimplicial set

$$\{\{I_s\}_{s \in S}\} \to \text{seq}(t)_{\{I_s\}_{s \in S}}.$$

It is shown in [27] that for every (non-empty) ordinal $J$

$$|\text{seq}(t)_{\bullet, \ldots, \bullet\text{top}}| \cong |\text{seq}(t)_{0, \ldots, 0\text{top}}| \times \Delta^J$$

and moreover the collection of homeomorphisms (8.3) gives an isomorphism of the corresponding cosimplicial topological spaces. Here $[0]$ is the one element ordinal.

Thus, in order to prove contractibility of the realization of (8.1) we need to prove contractibility of the topological space

$$|\text{seq}(t)_{0, \ldots, 0\text{top}}|.$$

This space admits the following explicit description. A point of $|\text{seq}(t)_{0, \ldots, 0\text{top}}|$ is given by an equivalence class of decompositions of the segment $[0, |S|]$ into a number of subsegments labeled by elements of $S$. The labeling should satisfy the following conditions:
\textbf{Theorem 1} if \( a, b \in S \) and a segment labeled by \( b \) lies between segments labeled by \( a \) then \( t(a) > t(b) \) in \( T \).

\textbf{Theorem 2} if for \( a, b \in S \) we have \( t(a) = t(b) \) and \( a < b \) then all segments labeled by \( a \) are on the left-hand side of all segments labeled by \( b \).

\textbf{Theorem 3} for every \( b \in S \) the total length of all segments labeled by \( b \) is 1.

Two such decompositions are equivalent if one is obtained from the other by a number of operations of the following two types:

\begin{itemize}
  \item[a)] adding into or deleting from our decomposition a number of labeled segments of length 0,
  \item[b)] joining two neighboring segments of our decomposition labeled by an element \( a \in S \) and its length is 1; if for \( a, b \in S \) we have \( a < b \) then the segment labeled by \( b \) is on the left-hand side of the segment labeled by \( a \).
\end{itemize}

In \cite{27} it was proved, by induction on \( |T| \), that the space (8.4) is a product of simplices and hence (8.4) is contractible. Thus we deduce that so is the cochain complex \(|\text{seq}(t)|_{\text{top}}\).

\textbf{Theorem 4} let now \( t : S \to T \) be a pruned SC 2-tree with \( S = S_a \sqcup S_c \), where \( S_a \) is the preimage of the minimal element of \( T \) and \( S_c = S \setminus S_a \). The subset \( S_a \) may, in principle, be empty.

Recall that \(|\text{seq}(t)|_{\text{top}}\) is the realization of the polysimplicial set

\begin{equation}
\{ (I_s)_{s \in S_c} \} \to \text{seq}(t)(I_s)_{s \in S_c}
\end{equation}

in the category of cochain complexes.

Each element \( u \) of \( \text{seq}(t)(I_s)_{s \in S_c} \) is a total order \( >_u \) on

\[ T = \bigsqcup_{s \in S_c} I_s \sqcup S_a \]

satisfying the following conditions:

\begin{itemize}
  \item[a)] it agrees with the total order on each \( I_s \) and with the total order on \( S_a \),
  \item[b)] if \( i, k \in I_{s_1} \), \( j \in I_{s_2} \), \( s_1 \neq s_2 \) and \( i <_u j <_u k \), then \( t(s_2) < t(s_1) \),
  \item[c)] if \( s_1, s_2 \in S_c \), \( s_1 < s_2 \), and \( t(s_1) = t(s_2) \), then all elements of \( I_{s_1} \) are strictly smaller than all elements of \( I_{s_2} \).
\end{itemize}

As well as the space (8.4) the realization \(|\text{seq}(t)|_{\text{top}}\) of (8.5) has the following explicit description.

A point of \(|\text{seq}(t)|_{\text{top}}\) is given by an equivalence class of decompositions of the segment \([0, |S|]\) into a number of subsegments labeled by elements of \( S \). The labeling should satisfy the following conditions:

\begin{itemize}
  \item[a)] for each \( s \in S_a \) there is exactly one segment labeled by \( s \) and its length is 1; if for \( s_1, s_2 \in S_a \) we have \( s_1 < s_2 \) then the segment labeled by \( s_1 \) is on the left-hand side of the segment labeled by \( s_2 \).
  \item[b)] if \( s_1, s_2 \in S_c \) and a segment labeled by \( s_2 \) lies between segments labeled by \( s_1 \) then \( t(s_1) > t(s_2) \),
  \item[c)] if for \( s_1, s_2 \in S_c \) we have \( t(s_1) = t(s_2) \) and \( s_1 < s_2 \) then all segments labeled by \( s_1 \) are on the left-hand side of all segments labeled by \( s_2 \),
  \item[d)] for every \( s \in S_c \) the total length of all segments labeled by \( s \) is 1.
\end{itemize}

Two such decompositions are equivalent if one is obtained from the other by a number of operations of the following two types:

\begin{itemize}
  \item[a)] adding into or deleting from our decomposition a number of labeled segments of length 0,
  \item[b)] joining two neighboring segments of our decomposition labeled by an element \( s \in S_c \) into one segment labeled by \( s \), or the inverse operation.
If we remove all elements of $S_a$ from $S$ and the minimal element $t_{\text{min}}$ from $T$ then we get a usual pruned (non-SC) 2-tree
\begin{equation}
\tilde{t} = t \big|_{S_c} : S_c \to T \setminus \{t_{\text{min}}\}.
\end{equation}
To this 2-tree we assign the following polysimplicial set
\begin{equation}
\{\{I_s\}_{s \in S_c}\} \to \text{seq}(\tilde{t})^{[0]}_{\{I_s\}_{s \in S_c}}
\end{equation}
and the corresponding topological space
\begin{equation}
|\text{seq}(\tilde{t})^{[0]}\ldots\bullet|_{\text{top}}
\end{equation}
which was explicitly described above. (The space (8.8) is obtained from the space (8.4) via replacing $t$ by $\tilde{t}$.)
We have the obvious projection
\[ P : |\text{seq}(t)|_{\text{top}} \to |\text{seq}(\tilde{t})^{[0]}_{\bullet\ldots\bullet}|_{\text{top}} \]
which sends a point of $|\text{seq}(t)|_{\text{top}}$ to a point of $|\text{seq}(\tilde{t})^{[0]}_{\bullet\ldots\bullet}|_{\text{top}}$ by collapsing each segment labeled by an element of $S_a$ to a point.

Conversely, given:
1) a point $x \in |\text{seq}(\tilde{t})^{[0]}_{\bullet\ldots\bullet}|_{\text{top}}$, and
2) a monotonous map $U : S_a \to [0, |S_c|]$
one can reconstruct a point in $|\text{seq}(t)|_{\text{top}}$ by inserting unit segments labeled by $s \in S_a$ in the place of the point $U(s)$.
Thus we conclude that
\[ |\text{seq}(t)|_{\text{top}} \cong |\text{seq}(\tilde{t})^{[0]}_{\bullet\ldots\bullet}|_{\text{top}} \times \Delta^{|S_a|}. \]
Due to Proposition 6.4 from [27] the first component $|\text{seq}(\tilde{t})^{[0]}_{\bullet\ldots\bullet}|_{\text{top}}$ is contractible. Hence so is $|\text{seq}(t)|_{\text{top}}$.
Thus we proved that $|\text{seq}|_{\text{seq}}(t)$ is contractible for every pruned SC 2-tree $t$.
The identifications from Part 2) of this proposition come from the fact that the topological spaces
\[ |\text{seq}(t)^{[0]}_{\bullet\ldots\bullet}|_{\text{top}} \]
for pruned 2-trees $t$ and
\[ |\text{seq}(t)^{[0]}_{\bullet\ldots\bullet}|_{\text{top}} \]
for pruned SC 2-trees $t$ are contractible. These topological realizations inherit the operadic compositions, whence Part 2) of this proposition.

Proposition 8.1 implies that the cofibrant resolution $\mathcal{R}_{\text{br}}$ of $\text{br}$ is also a cofibrant resolution of the trivial SC 2-operad $\text{triv}$ in the category of reduced SC 2-operads over cochain complexes.
Therefore, due to Batanin’s theorem (Theorem 4.2) the symmetrization $\text{sym} \mathcal{R}_{\text{br}}$ of $\mathcal{R}_{\text{br}}$ is quasi-isomorphic to the singular chain operad of Voronov’s Swiss Cheese operad $\text{SC}_2$.
Due to Theorem 6.9 the SC operad $\text{sym} \mathcal{R}_{\text{br}}$ is quasi-isomorphic to $\text{braces}$ which is, in turn, quasi-isomorphic to the SC operad $|\text{SC}_2|$ by Lemma 6.6.
Finally, by construction the SC operad $|\text{SC}_2|$ is isomorphic to the operad $|\text{SC}_2|$.
Thus we conclude that the two-colored operad $|\text{SC}_2|$ is quasi-isomorphic to the singular chain operad of Voronov’s Swiss Cheese operad $\text{SC}_2$. 

\[ \square \]
It remains to show that the induced action of $H_{-•}(SC_2)$ on the pair $(HH^•(A, A), A)$ coincides with the one given in Proposition 1.1. For this purpose we present operations on the pair
\[(C^•(A, A), A)\]
which come from the action of $|Ø|$ and which induce on $(HH^•(A, A), A)$ the $H_{-•}(SC_2)$-algebra structure from Proposition 1.1.

These operations are the cup-product and the Gerstenhaber bracket \[12\] on $C^•(A, A)$, the associative product on $A$, and the following contraction of a cochain $P$ with elements of the algebra $A$:
\[
i(P, a) = a P(1, 1, \ldots, 1): C^•(A, A) \otimes A \to A.
\]
We would like to remark that since $C^•(A, A)$ is the normalized Hochschild complex only degree zero cochains contribute to the contraction.

These operations induce the desired $H_{-•}(SC_2)$-algebra structure on $(HH^•(A, A), A)$ and they obviously come from the action of the SC operad $|Ø|$ on the pair (8.9).

Since the cohomology operad $H^•(|Ø|)$ of $|Ø|$ is isomorphic to $H_{-•}(SC_2)$ we conclude that the action of $|Ø|$ on (8.9) induce the desired $H_{-•}(SC_2)$-algebra structure on $(HH^•(A, A), A)$.

Theorem 2.1 is proved. □

### Appendix

Let $[n]$ be the standard ordinal $\{0, 1, 2, \ldots, n\}$.

Given a collection of $k$ ordinals $[n_1], [n_2], \ldots, [n_k]$ we consider the following ordinal
\[
I_{n_1, \ldots, n_k} = [n_1] \sqcup [n_2] \sqcup \cdots \sqcup [n_k],
\]
where the order is defined by the following rule: for $i_1 \in [n_{l_1}]$ and $i_2 \in [n_{l_2}]$ $i_1 < i_2$ if
- $l_1 < l_2$ or
- $l_1 = l_2$ and $i_1 < i_2$ in $[n_{l_1}]$.

Given ordinals $J, [n_1], [n_2], \ldots, [n_k]$ the collection
\[
(Ξ_k)^J_{n_1, \ldots, n_k} = \text{hom}_\Delta(I_{n_1, \ldots, n_k}, J)
\]
form a polysimplicial/cosimplicial set. Indeed $(Ξ_k)^J_{n_1, \ldots, n_k}$ is simplicial in $[n_1], [n_2], \ldots, [n_k]$ and cosimplicial in $J$.

In this appendix we show that

**Proposition 8.2.** The cochain complex $|Ξ_k|$ is concentrated in nonnegative degrees. Furthermore,
\[
H^•(|Ξ_k|) = \begin{cases} k, & \text{if } \bullet = 0, \\ 0, & \text{otherwise}. \end{cases}
\]

**Proof.** The first statement is very easy. Indeed, an element $v \in \text{hom}_\Delta(I_{n_1, \ldots, n_k}, J)$ will not contribute to the realization if it is degenerate. It is clear that if
\[
|J| < \sum_{i=1}^k (n_i + 1) - k + 1
\]
then $v$ is degenerate. Therefore, elements $v \in \text{hom}_\Delta(I_{n_1, \ldots, n_k}, J)$ with
\[
|J| - 1 - \sum_{i=1}^k n_i < 0
\]
will not contribute to the realization. Hence the cochain complex $|\Xi_k|$ is indeed concentrated in nonnegative degrees.

The cochain complex $|\Xi_k|$ can be considered as bicomplex

$$(8.14) \quad |\Xi_k| = |\Xi_k|^{\bullet,\bullet}.$$ 

The first degree is the total degree of the simplicial indices. According to our conventions this degree is nonpositive. The second degree is the degree in the cosimplicial index and this degree is nonnegative. Let us denote by $\partial^s$ the part of the differential in $|\Xi_k|$ which comes from the simplicial indices and by $\partial^c$ the part of the differential in $|\Xi_k|$ coming from the cosimplicial structure.

Fixing the second degree we get the cochain complex

$$(8.15) \quad |\Xi_k|^{\bullet,m}$$

which is the realization of the polysimplicial set

$$(8.16) \quad ([n_1], [n_2], \ldots, [n_k]) \rightarrow \text{hom}_\Delta(\mathcal{I}_{n_1,n_2,\ldots,n_k}, [m]).$$

It is not hard to see that the realization of (8.16) in the category of topological spaces is the following stretched $m$-simplex:

$$\{ (x_0, x_1, \ldots, x_m) \mid x_i \geq 0, \quad x_0 + x_1 + x_2 + \cdots + x_m = k \}.$$ 

Therefore for each $m$ the complex $|\Xi_k|^{\bullet,m}$ has non-trivial cohomology only in degree 0 and

$$(8.17) \quad H^0(|\Xi_k|^{\bullet,m}) = k.$$ 

The class which generates $H^0(|\Xi_k|^{\bullet,m})$ is represented by the map

$$(8.18) \quad c \in \text{hom}_\Delta(\mathcal{I}_{0,\ldots,0}, [m]),$$

which sends all elements of $\mathcal{I}_{0,\ldots,0}$ to the same element $0 \in [m]$. All other maps in $\text{hom}_\Delta(\mathcal{I}_{0,\ldots,0}, [m])$ are cohomologous to the cocycle (8.18).

It is not hard to see that

$$(8.19) \quad \Theta = \bigoplus_{q<0} |\Xi_k|^q,^\bullet \oplus \partial^s(|\Xi_k|^{-1,\bullet})$$

is a subcomplex of the bicomplex $|\Xi_k|$.

Equation (8.17) implies that each term of the quotient complex $|\Xi_k|/\Theta$ is $k$. Using the explicit cocycle (8.18) it is not hard to see that the quotient complex $|\Xi_k|/\Theta$ is

$$k \overset{0}{\rightarrow} k \overset{id}{\rightarrow} k \overset{0}{\rightarrow} k \overset{id}{\rightarrow} k \overset{0}{\rightarrow} \ldots$$

and hence

$$(8.20) \quad H^\bullet(|\Xi_k|/\Theta) = \begin{cases} k, & \text{if } \bullet = 0, \\ 0, & \text{otherwise}. \end{cases}$$

We see from the construction that the bicomplex $\Theta$ (8.19) is acyclic in the first degree. Therefore $\Theta$ is acyclic as the total complex.

Thus $H^\bullet(|\Xi_k|) = H^\bullet(|\Xi_k|/\Theta)$ and the proposition follows. $\square$
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