ALMOST SURE GLOBAL WELL-POSEDNESS FOR THE ENERGY SUPERCRITICAL
SCHRÖDINGER EQUATIONS

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ABSTRACT. We consider the Schrödinger equations with arbitrary (large) power non-linearity on the three-dimensional torus. We construct non-trivial probability measures supported on Sobolev spaces and show that the equations are globally well-posed on the supports of these measures, respectively. Moreover, these measures are invariant under the flows that are constructed. Therefore, the constructed solutions are recurrent in time.

Also, we show slow growth control on the time evolution of the solutions. A generalization to any dimension is given. Our proof relies on a new approach combining the fluctuation-dissipation method and some features of the Gibbs measures theory for Hamiltonian PDEs. The strategy of the paper applies to other contexts.

Résumé: Nous considérons des équations de Schrödinger avec des puissance arbitrairement grandes de la nonlinéarité sur le tore tri-dimensionel. Nous construisons des mesures de probabilité non triviales supportées sur des espaces de Sobolev et montrons que les équations sont globalement bien posées sur les supports de ces mesures, respectivement. De plus, ces mesures sont invariantes sous les flots qui sont construits. Par conséquent, les solutions construites sont récurrentes en temps.

Nous établissons également des contrôles de faible croissance sur l’évolution en temps des solutions. Une généralisation à toutes dimensions est fournie. Notre preuve se base sur une nouvelle approche combinant la méthode de fluctuation-dissipation et certaines caractéristiques de la théorie des mesures de Gibbs pour les EDPs hamiltoniennes.

La stratégie de cet article s’applique à d’autres contextes.

Keywords: Supercritical Schrödinger equation, global solutions, invariant measure, long time behavior, statistical ensemble.

MSC Classification: 28D05, 60H30, 35R60, 60H15, 37L50.

1. INTRODUCTION

1.1. Context. We consider the following Schrödinger equations

(1.1) \[ \partial_t u = i(\Delta u - |u|^{p-1}u), \]

where \( i \) is the complex number that satisfies \( i^2 = -1 \). Here we consider that the unknown function \( u = u(t,x) \) is defined on \( \mathbb{R} \times \mathbb{T}^3 \) and takes values in \( \mathbb{C} \), where \( \mathbb{T}^3 = \left( \frac{\mathbb{R}}{2\pi\mathbb{Z}} \right)^3 \) is a three-dimensional torus. Throughout the paper, the parameter \( p \) can be taken to be any real number bigger or equal to 3, although we highlight the case \( p > 5 \) where the equation is much less understood.

The present work is devoted to proving a probabilistic global well-posedness for (1.1) and to establishing strong controls on the growth in time of the solutions by using an invariant measure technique. Before we present our method and results, let us recall the structure of the equation and some known results. The equation (1.1) is Hamiltonian and is derived from the energy

\[ H(u) = \int_{\mathbb{T}^3} \left( \frac{1}{2} |\nabla u(t,x)|^2 + \frac{1}{p+1} |u(t,x)|^{p+1} \right) dx. \]
It also preserves the following quantity (its mass)

\begin{equation}
M(u) = \frac{1}{2} \int_{\mathbb{T}^3} |u(t,x)|^2 \, dx,
\end{equation}

and obeys the scaling invariance

\begin{equation}
u_{\lambda}(t,x) = \lambda^d u(\lambda^{d-1} t, \lambda^{\frac{d-4}{2}} x).
\end{equation}

A direct computation shows that, for the homogeneous Sobolev norm of order \( s \),
\[
\|D^s u_{\lambda}\| = \lambda^{1+(p-1)(\frac{d}{2}-\frac{s}{2})} \|D^s u\|.
\]

It follows that the critical exponent of (1.1) is \( s_c = \frac{3}{2} - \frac{2}{p-1} \). For \( p > 5 \), the space \( H^{s_c} \) is smoother than the energy space \( H^1 \); we have that (1.1) is energy supercritical. This results in that the global regularity problem of (1.1) is an extremely challenging open problem. We do not know any global well-posedness result on a Sobolev space for (1.1) in dimension 3. In the best of our knowledge, the Cauchy problem for (1.1) considered in \( \mathbb{R}^3 \) is solved only locally in time or globally for small data on some Sobolev spaces [CW90]. On the torus \( \mathbb{T}^3 \), we can establish a local theory as well, at least for smooth enough data, for example beyond the \( H^2 \)-regularity. But, at this level of regularity, the energy does not control the relevant norm, and then cannot be an argument of globalization. In the present work, we globalize the local theory of (1.1) posed on \( H^s(\mathbb{T}^3) \), \( s \geq 2 \), for data belonging to subsets of these spaces, where we have proved a control on the relevant norm. These subsets are constructed using a probabilistic method and they satisfy some qualitative and quantitative properties. For example, we prove that they contain data of any given size.

The critical and subcritical cases of (1.4) \( (p \leq 5) \) were widely studied by many authors (see [Bou93, Bou99a, Gri00, BGT04, BGT05, BGT07, CKS’08, HTT11, IP12, PTW14] and references therein), in particular global well-posedness of the Cauchy theory was proved, as well as long time dynamics properties.

For the supercritical setting \( (p > 5) \), let us mention the work of Wang [Wan16] who used bifurcation analysis to construct global quasi-periodic solutions to energy supercritical NLS and Beceanu-Deng-Soffer-Wu [BDSW19] who constructed global scattering solutions for the equation as well. A difference between the present setting and that of these two papers is that, in one hand, the solutions we investigate are not small compared to those in [Wan16]; on the other hand, our physical space is the torus compared to \( \mathbb{R}^3 \) in [BDSW19]. In particular the solutions constructed in the present paper do not scatter. However, we do not know if the small solutions contained in the constructed statistical ensemble have a quasi-periodicity property as for those in [Wan16].

1.2. Invariant measures methods in dispersive PDEs. The Gibbs measures techniques for dispersive PDEs go back to Lebowitz, Rose and Speer [LRS88], where one-dimensional nonlinear Schrödinger equations (1D NLS) are considered and Gibbs measures are constructed, the question of existence of global flow matching the regularity of the support of the measure was left open, and then so does the question of invariance. Zhidkov [Zhi91] redefined these measures via finite-dimensional approximations and a passage to the limit, he showed the invariance in the case of the wave equation [Zhi94]. Bourgain [Bou94, Bou96] constructed global flow and proved the invariance of the measures in question for 1D (subquintic) NLS. He then showed similar result for the 2D defocusing cubic Schrödinger equation posed on \( \mathbb{T}^2 \), under a suitable renormalization. Tzvetkov [Tzv06, Tzv08] considered the subquintic NLS equations (including the focusing nonlinearity for the subcubic case) on the disc of \( \mathbb{R}^2 \) and constructed invariant Gibbs measures, he proved a probabilistic global well-posedness on relevant spaces. Three-dimensional results have followed in [BT07, BT08b, dS11, BB14a, BB14b]. Several other contexts were analyzed on the line of these techniques. An important feature in this approach is that the Gibbs measures are concentrated on relatively rough spaces. Namely, their supports are \( \frac{d}{2} + \) degrees of regularity weaker than that of the energies on which they are based. Here \( d \) is the (effective) dimension of the
physical space, and is sensitive to some symmetry assumptions. That is why these objects are often used to deal with the global well-posedness on spaces of low regularity.

A second approach to construct invariant measures is the so-called fluctuation-dissipation method, based on ‘compact approximations’ of the equation and a use of stochastic tools. An inviscid limit is then considered. We go back to the work of Kuksin [Kuk04] and Kuksin and Shirikyan [KS04, KS12] for the 2D Euler equation and the cubic defocusing Schrödinger equation (in dimension $\leq 4$). For both of these equations, an invariant measure on the Sobolev space $H^2$ is obtained. Let us also mention some results of the author [Sy18, Sy19] using this approach.

It is worth mentioning some ‘non-invariance’ probabilistic methods in the Cauchy problem of PDEs, see for instance [BT08a, Tho09, CO12, NPS13, BT14, Poc14, OP16].

Let us also notice the works [Tzv15, OT17, OST18] that showed quasi-invariance properties of Gaussian measures under Hamiltonian flows.

1.3. The methodology of the paper. In the present work, we introduce a new approach which is somehow hybrid to deal with the supercriticality present in (1.1). Namely, we combine the fluctuation-dissipation method with some features of the approach based on Gibbs measures. Indeed, the nonlinearity is such that usual energy methods do not provide continuity of the flow in the context the standard fluctuation-dissipation for Hamiltonian PDEs, essentially because of a lack of time integrability. Also, the Gibbs measure approach does not work either due to a lack of space regularity obstructing the analysis of (the large power of) the nonlinearity in dimension 3 and higher. Our solution to overcome these two serious issues is to combine the two approaches in a single new one. We plug fluctuation-dissipation tools into the setting of the Gibbs measures as developed in [Bou94]. More precisely, we consider damped/driven Galerkin approximations of the NLS (1.1), construct invariant measures that enjoy bounds that are uniform both in the viscosity parameter and in the dimension of the approximating equation. We pass first to the limit when the viscosity goes to 0 and obtain a sequence of invariant measures associated to the (deterministic) Galerkin approximations for (1.1). We study the infinite-dimensional limit in spirit of Bourgain [Bou94]. In order to obtain large deviation bounds that are exploitable in the Bourgain argument, we introduce a new and carefully prepared dissipation operator as discussed below.

Once these bounds are obtained, we perform an extension of the Bourgain framework to the measures that are not necessarily of Gibbs type. Despite the lack of information on our measures, occasioned by the compactness method, we were able to make the infinite-dimensional data (living on the support of the limiting measure) inheriting the good properties of their finite-dimensional approximations. To achieve this, a suitable family of restriction measures (that are conditional probabilities) is introduced and the Skorokhod representation theorem is used.

Let us discuss a first new ingredient of our proof: in a step of the approximation argument (see Section 3), we considered fluctuation-dissipation on the Galerkin equation having this form:

$$du = i[(\Delta - 1)u - P_0(|u|^{p-1}u)]dt - \alpha[(1 - \Delta)^{s-1} + \epsilon^2\rho(||u||_{s-})]u dt + \sqrt{\alpha}d\eta_N,$$

where $\eta_N$ is a noise and $s^- = s - \epsilon$, for some $\epsilon > 0$ close enough to 0 (we use $s^+$ in a similar way). Let us focus on the factor $\rho(||u||_{s-})$ in the dissipation operator. Remark that an application of the Itô formula to the mass, given by (1.2), provides a statistical control (under an eventual invariant measure) of the quantity

$$E\rho(||u||_{s-})||u||^2,$$

and therefore, to some extent, of the quantity

$$E\rho(||u||_{s-}).$$

Such a control is a crucial step in the use of the argument of Bourgain [Bou94] (see also [Tzv06, Tzv08]). The local existence time for (1.4) depends on the size of the data (that is in our situation $T \sim ||u||_{1-2}^{-1+}$. Without
the factor $e^{\rho([u])}$, we should control only a quadratic power, which seems to be not enough (see the proof of Proposition 5.2). The ‘miracle’ of this factor is that, beside its exponential strength, it does not participate to usual estimation computations (such as integration in $x$, projection, etc.) And it is of an assigned regularity. That means its regularity is chosen and does not directly depend on the structure of the equation; the function $\rho$ is also our choice modulo some weak constraints. Furthermore, the stronger this factor the slower the growth in time of the constructed solutions. This new ingredient should be a trick that can be used in many other situations and its flexibility can be exploited further. More generally, all this new approach might provide a new way to construct global solutions and invariant measures and to establish slow growth properties, specially for PDEs presenting strong supercriticality.

The overall message of this strategy is that, by employing the performed generalization of the Bourgain framework, one can extract individual (pointwise) bounds from the statistical (integral) ones provided by the fluctuation-dissipation setting. The regularity of the controlled quantities allows, in particular, to obtain uniqueness and continuity properties (that are missing if we use only the classical fluctuation-dissipation). The other bounds arise as a byproduct of the method.

1.4. Main results. Set $v = e^{-k}u$, we see that if $u$ solves (1.1), then $v$ solves the following equation

$$
\partial_t v = i[(\Delta - 1)v - |v|^{p-1}v].
$$

This formulation is more appropriate for dealing with the zero frequency. Its energy is the following

$$
E(v) = \int_{\mathbb{T}^d} \frac{1}{2}|v|^2 + \frac{1}{2}|v|^2 + \frac{1}{p+1}|v|^{p+1}dx.
$$

Let us state the main result of the paper.

**Theorem 1.1.** For any $s \geq 2$ and any increasing concave function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, there is a measure $\mu = \mu_{s, \xi}$ concentrated on $H^s$ such that

1. For $\mu$-almost any $u_0 \in H^s$, there is a unique solution $u \in C(\mathbb{R}, H^s)$ to (1.4) such that $u(0) = u_0$;
2. the distributions of $M(u)$ and $E(u)$ via $\mu$ admit densities with respect to the Lebesgue measure on $\mathbb{R}$.
3. For any $n > 0$, there is a set $S_n$ such that $\mu(S_n) > 0$, and for any $u_0 \in S_n$, $\|u_0\|_s \geq n$.
4. The flow $\phi^t$ implied by the statement 1 satisfies the following properties:
   a. For any $T_0 > 0$, there is $C(T_0) > 0$ such that for $\mu$-almost any $u$ and $v$ in $H^s$ we have
      \[
      \sup_{t \in [-T_0, T_0]} \|\phi^t u - \phi^t v\|_s \leq C(T_0) \|u - v\|_s.
      \]
   b. The measure $\mu$ is invariant under $\phi^t$.
   c. For $\mu$-almost all $u_0 \in H^s$ we have the slow growth property
      \[
      \|\phi^t u_0\|_s \leq C_\xi(\|u_0\|_s) \xi(\ln(1 + |t|)) \quad \text{for all } t \in \mathbb{R}.
      \]

Consequently, using the Poincaré recurrence theorem, we have that for $\mu$-almost any $u_0 \in H^s$, there is a sequence $t_k \uparrow \infty$ such that

$$
\lim_{t_k \to \infty} \|\phi^{\pm t_k} u_0 - u_0\|_s = 0.
$$

That gives a long time property of the flow $\phi^t$.

Also, the point 3 of the statement expresses the fact that our result is not of small data type.

Now, in the control (1.5) the function $\xi$ being concave gives us the desired slow growth: concretely we can choose $\xi$ to be log or log$\circ$log or even ‘better’ (as long as it is increasing concave), and our result ensures then the existence of a measure with the mentioned qualitative/quantitative properties which provides the chosen control on the growth of the solutions.
This may be put in contrast with the Gibbs measure techniques, where the Fernique theorem provides an estimate of the quantity $\mathbb{E}e^{\|u\|^2}$, giving rise to a control of type $\sqrt{\ln(1+|t|)}$.

To end our discussion let us state the generalization of the results to all dimensions, specially to dimensions $d > 3$:

**Remark 1.2.** The results of this paper remain true if we consider the equation (1.4) to be posed on a $d$-dimensional torus $\mathbb{T}^d$, $d \geq 3$. The computations are the same modulo some minor adaptations in the conditions of some statements. However, we need to assume that $s > \frac{d}{2}$ so that Proposition 2.1 survives.

1.5. **Organization of the paper.** In section 2 we present a local well-posedness result of (1.4) and its Galerkin approximations on smooth spaces. We emphasis on the fact that the time of existence can be taken independently of the dimension. We show a convergence result.

In section 3, we study the fluctuation-dissipation equations based on the Galerkin approximations of (1.4), we establish stochastic global well-posedness, existence of stationary measures and we derive uniform estimates for them. Then, the section 4 is devoted to the study of inviscid limits, it is shown that the inviscid measures are invariant under the flows of the approximating problems for (1.4), uniform in $N$ bounds are proved. In section 5, we construct the infinite-dimensional statistical ensemble, derive bounds for the approximating dynamics, and use them to construct global flows for (1.4) on $H^r$ for $r \leq s$—and for data living on the statistical ensemble. Section 6 is concerned with the invariance of the infinite-dimensional limiting measure. In section 7, we use an argument based on the propagation of regularity principle to state the almost sure global wellposedness, existence of stationary measures and we derive uniform estimates of some statements. However, we need to assume that $s > \frac{d}{2}$.

We see then the $H^r$-regularity. We also deal with the size of the data by constructing a cumulative measure. And finally, we derive qualitative properties for the constructed measure in Section 8.

**General notations.** Consider the sequence $\left( \frac{2\pi}{2^k} e^{ikx} \right)_{k \in \mathbb{N}}$ whose elements are normalized eigenfunctions of the Laplace operator $-\Delta$ on $\mathbb{T}^3 = \left( \mathbb{R} / \mathbb{Z} \right)^3$. The associated eigenvalues are $|k|^2 = k_1^2 + k_2^2 + k_3^2$. We shall arrange the eigenfunctions in the increasing order of eigenvalues. Namely, denoting the latters as $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$, we obtain the corresponding sequence of eigenfunctions $(\varphi_m)_{m \in \mathbb{N}}$. The Weyl asymptotic states that $\lambda_m \sim m^2$. Let us denote by $e_{-m}$ the eigenfunction $i\varphi_m$, we have that the sequence $(e_m)_{m \in \mathbb{Z}}$ forms a basis of $L^2 = L^2(\mathbb{T}^3, \mathbb{C})$.

Therefore for $u \in L^2$, we have the representation

$$u(x) = \sum_{m \in \mathbb{Z}} u_m e_m(x).$$

We have the Parseval identity

$$\|u\|^2_{L^2} = \|u\|^2 = \int_{\mathbb{T}^3} |u(x)|^2 dx = \sum_{m \in \mathbb{Z}} |u_m|^2.$$

Let $s > 0$. The Sobolev space $H^s := H^s(\mathbb{T}^3; \mathbb{C})$ is defined by the norm

$$\|u\|_s := \sqrt{\|\Delta^{\frac{s}{2}} u\|^2} = \sqrt{\sum_{m \in \mathbb{Z}} (1 + \lambda_m)^s |u_m|^2}.$$

Since $\lambda_m$ are all non-negative integers, we have that $(1 + \lambda_m)^s \leq (1 + \lambda_m)^r$ for any $m$, if $s \leq r$. We see then the embedding inequality

$$\|u\|_s \leq \|u\|_r \quad \text{if} \quad s \leq r. \quad (1.6)$$

Let us define a real inner product on $L^2$ by

$$(u, v) = \Re \int_{\mathbb{T}^3} u(x) \bar{v}(x) dx, \quad (1.7)$$
where $\Re z$ stands for the real part of the complex number $z$. Hence, we have the property

\[(u, iu) = 0.\]  

(1.8)

We denote by $E_N$ the subspace of $L^2$ generated by the finite family $\{e_m, |m| \in [0, N]\}$, the operator $P_N$ is the projector onto $E_N$.

For a functional $F : L^2 \to \mathbb{C}$, we denote by $F'(u; v)$ and $F''(u; v, w)$ its first derivative at $u$, evaluated at $v$, and its second derivative at $u$ evaluated at $(v, w) \in L^2 \times L^2$, respectively.

On the space $E_N$, we define a Brownian motion by

\[\zeta_N(t, x) = \sum_{|m| \leq N} a_m \beta_m(t) e_m(x)\]  

(1.9)

where $(a_m)$ is a family of complex numbers, and $(\beta_m)$ is a sequence of independent standard real Brownian motions with respect to a filtration $(\mathcal{F}_t)$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

The noise $\eta$ is defined as

\[\eta_N(t, x) = \frac{d}{dt} \zeta_N(t, x).\]

Set the numbers

\[A_{x,N} = \sum_{|m| \leq N} |a_m|^2 \lambda_m^x.\]

For a Banach space $H$, we denote by $C_b(H)$ the space of bounded continuous functions on $H$ with range in $\mathbb{R}$, and $p(H)$ the set of all the probability measures on $H$.

For a Banach space $X$ and an interval $I \subset \mathbb{R}$, we denote by $C(I, X) = C_t X$ the space of continuous functions $f : I \to X$. The corresponding norm is $\|f\|_{C_t X} = \sup_{t \in I} \|f(t)\|_X$.

For $q \in [1, \infty)$, we also denote by $L^q(I, X) = L^q_t X$, the Lebesgue’s spaces given by the norm

\[\|f\|_{L^q_t X} = \left( \int_I \|f(t)\|_X^q dt \right)^{\frac{1}{q}}.\]

The inequality $A \lesssim B$ between two positive quantities $A$ and $B$ means $A \leq CB$ for some $C > 0$.

For a measure $\mu$, we denote by Supp($\mu$) the support of $\mu$.

2. Uniform Local Well-Posedness and (Deterministic) Convergence

We have the following expansion of an element $u$ in $E_N = P_N L^2$:

\[u = \sum_{|m| \leq N} u_m e_m(x).\]

Notice that $E_\infty$ refers to $L^2$, and the projector $P_\infty$ to the identity operator.

In the sequel, the notation $B_R(X)$ refers to the closed ball with center 0 and radius $R > 0$ of the Banach space $X$.

Let us consider the problem

\[\partial_t u = i(\Delta - 1)P_N u - P_N(|P_N u|^{p-1}P_N u),\]  

(2.1)

\[u(t_0) = P_N u_0.\]  

(2.2)
2.1. Uniform LWP.

**Proposition 2.1.** Let $s > \frac{3}{2}$. For any $R > 0$, there is a constant $T := T(R,s)$ such that for every $N \in \mathbb{N}^* \cup \{\infty\}$, any $u_0 \in B_R(H^s)$, there is a unique $u_N \in X^s_T := C((-T,T),H^s)$ satisfying (2.1) and (2.2). Moreover, we have

\[ \|u\|_{X^s_T} := \sup_{t \in (-T,T)} \|u(t)\|_s \leq 2\|P_N u_0\|_s. \]  

**Proof.** Fix $u_0$ in $B_R(H^s)$ and set the map

\[ F(u) = S(t)P_N u_0 - i \int_0^T S(t - \tau)P_N(|P_N u|^{p-2}P_N u)d\tau \quad C_1H^s \rightarrow C_1H^s, \]

where $S(t)$ stands for the group $e^{it\Delta}$. We see that an eventual fixed point of $F$ must belong to $E_N$ and be a solution to (2.1), (2.2). Using the algebra structure of $X^s_T$, we have

\[ \|F(u)\|_{X^s_T} \leq \|P_N u_0\|_s + \int_0^T \|u\|^p\|d\tau \leq \|u_0\|_s + T\|u\|^p_{X^s_T}. \]

For $T \leq \frac{1}{2^{p-1}}$ for some constant $c \geq 1$, we have for all $u \in B(R, X^s_T)$

\[ \|F(u)\|_{X^s_T} \leq 2R, \]

hence $F(u) \in B(R, X^s_T)$. Now let $u_1$ and $u_2$ be two element of $B(R, X^s_T)$, we have that

\[ \|F(u_1) - F(u_2)\|_{X^s_T} \leq CTR(\|u_1\|^p_{X^s_T} + \|u_2\|^p_{X^s_T})\|u_1 - u_2\|_{X^s_T}. \]

Taking $c = C \vee 1$ in the choice of $T$, we obtain

\[ \|F(u_1) - F(u_2)\|_{X^s_T} \leq \frac{1}{2}\|u_1 - u_2\|_{X^s_T}. \]

Therefore $F$ is a contraction on $B(R, X^s_T)$ and we obtain the claimed existence and uniqueness. Now, to see the last claim, let us observe that the constructed solution stay in $B(R, X)$ for $|t| < T$. Therefore, using the Duhamel formula, we have

\[ \|u\|_{X^s_T} \leq \|P_N u_0\|_s + \frac{1}{2}\|u\|_{X^s_T}, \quad |t| < T. \]

This is (2.3). $\square$

**Remark 2.2.** An important property of the local time existence $T$ in the Proposition 2.1 above is that it does not depend on $N$.

2.2. Local uniform convergence.

**Lemma 2.3.** Let $s > \frac{3}{2}$, $R > 0$ and $B_R := B_R(H^s)$. Let $T := T(s,R)$ be the associated (uniform) existence time for the problem (2.1), (2.2), we have that for every $r < s$,

\[ \sup_{u_0 \in B_R} \|\phi^{(r)}(u_0) - \phi^{(s)}(P_N u_0)\|_{X^s_T} \rightarrow 0, \quad \text{as } N \rightarrow \infty. \]

**Proof.** Let us write the Duhamel formulas of $\phi^{(r)}(u_0)$ and $\phi^{(s)}(u_0)$:

\[ \phi^{(r)}(u_0) = S(t)u_0 - i \int_0^T S(t - \tau)|\phi^{(r)}(u_0)|^{p-1}\phi^{(r)}(u_0)d\tau, \]

\[ \phi^{(s)}(P_N u_0) = S(t)P_N u_0 - i \int_0^T S(t - \tau)P_N(|\phi^{(s)}(P_N u_0)|^{p-1}\phi^{(s)}(P_N u_0)). \]
Taking the difference between the two equations above and using the decomposition \( f = P_N f + f - P_N f \), we obtain for any \( t \in [0,T_R] \) that
\[
\phi'(u_0) - \phi'_N(P_Nu_0) = S(t)(u_0 - P_Nu_0) - \int_0^t S(t - \tau) \left( P_N(|\phi^\tau(u_0)|^{p-1}\phi^\tau(u_0) - |\phi'_N(P_Nu_0)|^{p-1}\phi'_N(P_Nu_0)) \right) d\tau
\]
\[
- \int_0^t S(t - \tau) \left( |\phi^\tau(u_0)|^{p-1}\phi^\tau(u_0) - P_N(|\phi^\tau(u_0)|^{p-1}\phi^\tau(u_0)) \right) d\tau.
\]
Now we use the fact that \( \|P_N f\|_r \leq \|f\|_r \) and \( \|S(t)\|_{H^r \rightarrow H^s} \leq 1 \), to obtain
\[
\|\phi'(u_0) - \phi'_N(P_Nu_0)\|_s \leq \|(1-P_N)u_0\|_r + \int_0^t \|\phi^\tau(u_0)|^{p-1}\phi^\tau(u_0) - |\phi'_N(P_Nu_0)|^{p-1}\phi'_N(P_Nu_0)\|_r d\tau.
\]
Now using the algebra structure of \( H^s \) and the fact that, on \([0,T_R]\) we have \( \|\phi^\tau u_0\|_{L^r}, \|\phi'_N u_0\|_{L^r} \leq \text{const}(R) \), we obtain
\[
\|\phi^\tau(u_0)|^{p-1}\phi^\tau(u_0) - |\phi'_N(P_Nu_0)|^{p-1}\phi'_N(P_Nu_0)\|_r \leq C\|\phi^\tau(u_0)\|_{L^p-1}^{p-1} + \|\phi'_N(P_Nu_0)\|_{L^p-1}^{-1}\|\phi^\tau(u_0) - \phi'_N(P_Nu_0)\|_r,
\]
Remark that for \( r < s \) and \( f \in H^s \), we have
\[
\|(1-P_N)f\|_r \leq (1 + \lambda_N)^\frac{1}{2} \|(1-P_N)f\|_s \leq (1 + \lambda_N)^\frac{1}{2} \|f\|_s.
\]
We use the Gronwall lemma to get
\[
\sup_{u_0 \in B_R} \|\phi'(u_0) - \phi'_N(P_Nu_0)\|_{X^r} \leq (1 + \lambda_N)^\frac{1}{2} e^{C_1(s,R)} (\|u_0\|_r + C_2(s,R)).
\]
Whence follows
\[
\sup_{u_0 \in B_R} \|\phi'(u_0) - \phi'_N(P_Nu_0)\|_{X^r} \leq (1 + \lambda_N)^\frac{1}{2} C_3(s,R).
\]
We finish the proof by recalling that \( r < s \) and letting \( N \) go to \( \infty \).

A sufficient condition of globalization. Now let us remark the following a priori bound
\[
\|\phi^\tau u_0\|_s \leq e^{C_0' \|\phi^\tau u_0\|_{L^p-1}^{p-1}} \|\phi^\tau u_0\|_s.
\]

Then, if for some initial datum \( u_0 \in H^s \) we have that
\[
\int_0^t \|\phi^\tau u_0\|_{L^p-1}^{p-1} d\tau < \infty \text{ for any } t > 0,
\]
then the solution \( \phi^\tau u_0 \) is global in time on \( H^s \).

3. Fluctuation-Dissipation for the Approximating Equations

In this section we consider fluctuation-dissipation based on the Galerkin approximations of (1.4). We will prove that they are globally well-posed on the approximating spaces \( E_N \), then we construct a sequence of stationary measures and derive uniform bounds. Also, using the projector \( P_N \), we recall the notation \( E_N = P_NL^2 \).

The finite-dimensional property of \( E_N \) makes all norms well-defined on it to be equivalent. Therefore, unless we need uniformity for an estimate, we may work only with the \( L^2 \)-norm and the result will be automatically valid for the other.
Now, for
\[ \partial_t u = i(\Delta - 1)u - P_N(|u|^{p-1}u) - \alpha[(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u + \sqrt{\epsilon} \eta_N \]
\[ u|_{t=0} = w \in E_N. \]

Here \( \rho : \mathbb{R}_+ \to \mathbb{R}_+ \) satisfies
\[ C(\rho, r)\epsilon^0(x) \geq x' \quad \text{for any} \ r > 1, \]
for some constant \( C(\rho, r) \) depending only on \( \rho \) and \( r \) (we can think \( \rho \) as a suitable convex function as it will be taken later).

3.1. Dissipation rates of the mass and the energy. In the equation (3.1), the mass and the energy given by
\[ M(u) = \frac{1}{2} \int_{\mathbb{T}^3} |u(x)|^2 \, dx, \]
\[ E(u) = \int_{\mathbb{T}^3} \left( \frac{1}{2} |\nabla u(x)|^2 + \frac{1}{2} |u(x)|^2 + \frac{1}{p+1} |u(x)|^{p+1} \right) \, dx. \]

interact with the damping term \( \alpha[(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u. \)

Let us remark that, in this interaction, the quantity \( \epsilon^0(|u|^{s-1}) \) is a constant in \( x \) and does not "participate" to any integration w.r.t. the \( x \) variable. Namely, for a functional \( G(u) = \int_{\mathbb{T}^3} g(u) \, dx \), we have, formally, that
\[ G'(u, \epsilon^0(|u|^{s-1})h(u)) = \int_{\mathbb{T}^3} g'(u)\epsilon^0(|u|^{s-1})h(u) \, dx = \epsilon^0(|u|^{s-1})G'(u, h(u)). \]

The resulting dissipation rates are formally given by
\[ \alpha M(u) := M'(u, \alpha[(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u) \]
\[ = \alpha[M'(u, (1 - \Delta)^{s-1}u) + \epsilon^0(|u|^{s-1})M'(u, u)] \]

and
\[ \alpha E(u) := E'(u, \alpha[(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u) \]
\[ = \alpha[E'(u, (1 - \Delta)^{s-1}u) + \epsilon^0(|u|^{s-1})E'(u, u)], \]

respectively. These quantity are well-defined for regular enough solutions. Here, we give some useful properties concerning them. Let us, first, observe that
\[ M'(u; v) = (u, v), \]
\[ E'(u; v) = (-\Delta u + u + |u|^{p-1}u, v). \]

We have, using that \( P_N u = u \), that
\[ \alpha M'(u) = M'(u, [(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u) = ||u||^2_{L^1} + \epsilon^0(|u|^{s-1})||u||^2. \]

Also,
\[ E'(u, [(1 - \Delta)^{s-1} + \epsilon^0(|u|^{s-1})]u) = ((1 - \Delta)u + |u|^{p-1}u, (1 - \Delta)^{s-1}u) + \epsilon^0(|u|^{s-1})((1 - \Delta)u + |u|^{p-1}u, u) \]
\[ = ||u||^2 + (|u|^{p-1}u, (1 - \Delta)^{s-1}u) + \epsilon^0(|u|^{s-1})(||u||^2 + ||u||^2_{L^1} + ||u||^{p+1}_{L^1}). \]

For \( s = 2 \), we use an integration by parts to find that
\[ (|u|^{p-1}u, (1 - \Delta)^{s-1}u) \geq 0. \]

Now, for \( s > 2 \), using the Agmon’s inequality,
\[ ||((1 - \Delta)^{s-1}u, |u|^{p-1}u)|| \leq C_0 ||u||^2_{L^1}||u||^{p-1}_{L^1} \leq C_0 ||u||^2 ||u||^2_{L^1}^2 ||u||_{L^1}^{3(p-1)/2}. \]
Since $s > 2$, we can always find, by performing a Young inequality and the properties of $\rho$ (see (3.3)), a constant $K = K(s, p, \rho) > 0$ such that
\[ C_s \|u\|_{L^2}^2 \|u\|_{L^p}^{\frac{s}{2}} \|\nabla u\|_{L^{p+1}}^2 \leq K + \frac{1}{2} \|u\|_2^2 e^{\rho(\|u\|_{L^2})}. \]
Overall, one obtains, for all $s \geq 2$, that
\[ e^s(u) = E^s(u) \left[ (1 - \Delta)^{s-1} + e^{\rho(\|u\|_{L^{p+1}})} \right] \geq \|u\|_{L^p}^2 + \left( \frac{\|u\|_{L^{p+1}}}{2} + \|u\|_{L^{p+1}} \right) e^{\rho(\|u\|_{L^2})} - K. \]

## 3.2. Global well-posedness for the fluctuation-dissipation problems on $E_N$.

Let us introduce the following definition.

**Definition 3.1.** Let $N \geq 2$. The equation (3.1) is said to be stochastically globally well-posed on $E_N$ if for all the following properties hold

1. For any random variable $u_0$ in $E_N$ which is independent of $\mathcal{F}_t$, we have, for almost all $\omega \in \Omega$,
   a. (Existence) there exists $u \in C(\mathbb{R}_+, E_N)$ satisfying (3.1) and (3.2) in which $u_0$ is replaced by $u_0(\omega)$. We denote the solution by $u^{\omega}(t, u_0)$.
   b. (Uniqueness) if $u_1, u_2 \in C(\mathbb{R}_+, E_N)$ are two solutions starting at $u_0$ then $u_1 \equiv u_2$.
2. (Continuity w.r.t. initial data) for almost all $\omega$, we have
   \[ \lim_{u_0 \to u_0'} u^{\omega}(\cdot, u_0) = u^{\omega}(\cdot, u_0') \text{ in } C_t E_N, \]
   where $u_0$ and $u_0'$ are deterministic data in $H^s$;
3. the process $(\omega, t) \mapsto u^{\omega}(t)$ is adapted to the filtration $\sigma(u_0, \mathcal{F}_t)$.

We claim that the problem (3.1), (3.2) is stochastically globally well-posed on $E_N$ in the sense of Definition 3.1. The proof of this fact is rather classical and is going to be presented here following the few steps below.

1. **Existence of a global solution.** Consider the stochastic convolution
   \[ z(t) := z_{\alpha} = \sqrt{\alpha} \int_0^t e^{i(t-s)(\Delta-1) - \alpha(1-\Delta)^{s-1}} d\xi_N(s), \]
   this is a well-defined for $\mathbb{P}$-almost all $\omega \in \Omega$; we see that $z$ is the unique solution of the equation
   \[ dz = [i(\Delta - 1)v - \alpha(1 - \Delta)^{s-1}]z dt + \sqrt{\alpha} d\xi_N, \quad z|_{t=0} = 0. \]

   We see without difficulties that, for $\mathbb{P}$-almost all $\omega \in \Omega$, $z$ belongs in $C_t(\mathbb{R}_+, C^\infty(E_N))$ (one can apply the Itô formula to derivatives of (3.10)).

   Now any $\omega \in \Omega$ such that $z^{\omega}$ belongs to $C_t(\mathbb{R}_+, C^\infty(E_N))$, we set the problem
   \[ \dot{\alpha}_v = i[\Delta - 1)v - \rho N(v + z)^{p-1}(v + z)] - \alpha[(\Delta - 1)^{s-1}v + e^{\rho(\|v+z\|_{L^2})}(v + z)], \]
   \[ v|_{t=0} = u_0 \in E_N. \]
   Since the map $E_N \to E_N : v \mapsto i[(\Delta - 1)v - N(v + z)^{p-1}(v + z)] - \rho[(\Delta - 1)^{s-1}v + e^{\rho(\|v+z\|_{L^2})}(v + z)]$ is smooth, thanks to the classical Cauchy-Lipschitz theorem, the problem (3.11) has a local in time smooth solution. We see that this solution is in fact global in time by using the Proposition 3.2 below.

2. Now, observe that the sum $v + z$ is a solution to (3.1), (3.2). Also, $u \in C_t(\mathbb{R}_+, C^\infty)$ since $z \in C_t(\mathbb{R}_+, C^\infty)$ and $F \in C^\infty(E_N, E_N)$.

**Proposition 3.2.** The local solution $v$ constructed above exists globally in time, $\mathbb{P}$—almost surely.
Proof. Let us compute the derivative of \( \|w\|^2 \) and use the equation (3.11) and (3.3), we obtain

\[
\frac{d}{dt} \left( \frac{\|v\|^2}{2} \right) = -(z, i|v + z|^{p-1}(v + z)) - \alpha \|v\|^{2p} - \alpha \|v\|^2 e^\rho(\|v + z\|^{\alpha}) + \alpha \|v, z\| e^\rho(\|v + z\|^{\alpha})
\]

\[
\leq \frac{\|z\|^2}{2\alpha} + \alpha \|v + z\|^{2p} - \alpha \|v\|^2 e^\rho(\|v + z\|^{\alpha}) + \alpha \left[ \frac{\|v\|^2}{2} + \frac{\|z\|^2}{2} \right] e^\rho(\|v + z\|^{\alpha})
\]

\[
\leq \frac{\|z\|^2}{2\alpha} + \frac{\alpha}{2} |C(p, p) + \|z\|^2 - \|v\|^2| e^\rho(\|v + z\|^{\alpha}).
\]

Now we have that for \( P \) almost all \( \omega \in \Omega \) for all \( T \), there is a constant \( C_{\alpha}(\omega, T) \) such that

\[
(3.12) \quad \sup_{t \in [0, T]} \|z(t, \omega)\| \leq C_{\alpha}(\omega, T).
\]

Now, for a fixed \( \omega \) such that (3.12) holds, fix any \( T > 0 \). Let \( t \in [0, T] \), we have the following two complementary scenarios:

(a) either \( \|v(\omega, t)\| \leq C(p, p) + C_{\alpha}(\omega, T) \)

(b) or \( \|v(\omega, t)\| > C(p, p) + C_{\alpha}(\omega, T) \). In this case \( [C(p, p) + \|z\|^2 - \|v\|^2] e^\rho(\|v + z\|^{\alpha}) < 0 \) and

\[
\frac{d}{dt} \left( \frac{\|v\|^2}{2} \right) \leq \frac{\|z\|^2}{2\alpha} \leq C_{\alpha}(\omega, T).
\]

Therefore

\[
\frac{d}{dt} \left( \frac{\|v\|^2}{2} \right) \leq C_{\alpha}^2(\omega, T) \quad \text{for any } t \in [0, T].
\]

Overall

\[
\sup_{t \in [0, T]} \|v(t)\|^2 \leq \|u_0\|^2 + C_{\alpha}^2(\omega, T) \quad \text{for any } T > 0.
\]

In conclusion, we obtain the claim by using an iteration argument. \( \square \)

\[ \text{(2) Uniqueness and continuity.} \]

For a fixed \( \omega \in \Omega \), let \( u_i, i = 1, 2 \) two solutions to (3.1) starting at \( u_{0i}, i = 1, 2 \), respectively. Let \( w := u_1 - u_2 \), and \( F(u) = i[(\Delta - 1)u - P_N(|u|^2)u] - \alpha[(1 - \Delta) + e^\rho(\|u\|^{\alpha})]u \). \( F \) is clearly in \( C^\omega(E^N \rightarrow E^N) \). Using the difference of the corresponding two equations, we see readily that \( w \) satisfies the equation

\[
\partial_t w = F(u_1) - F(u_2) = w \int_0^1 F'(su_1 + (1 - s)u_2) ds.
\]

Taking the inner product with \( w \), we have

\[
\partial_t \|w\|^2 \leq \|w\|^2 \sup_{x \in \mathbb{T}^3} s \int_0^1 |F'(su_1 + (1 - s)u_2)| ds.
\]

Using the Gronwall lemma, we obtain

\[
\|u_1(t) - u_2(t)\| \leq \|u_{0,1} - u_{0,2}\| \exp \left( \frac{1}{2} \sup_{x \in \mathbb{T}^3, t \in [0, T]} \int_0^1 |F'(su_1 + (1 - s)u_2)| ds \right).
\]

This estimate implies uniqueness in \( C_{\gamma}L^2 \), as well as continuity with respect to the initial datum in any Sobolev type norm of \( E_N \), because of the finite-dimensionality.

\[ \text{(3) Adaptation.} \]

It is clear that \( z \) is adapted to \( \mathcal{F}_t \), since \( v \) is constructed by a fixed point argument, then it is adapted to \( \sigma(u_0, \mathcal{F}_t) \). We obtain the claim.
Let us denote by \( u_α(t, u_0) \) the unique solution to (3.1), (3.2).

3.3. **Stationary solutions and uniform estimates.**

3.3.1. **A Markov framework.** Let us define the transition probability

\[
T^t_{α,N}(w, Γ) = \mathbb{P}(u_α(t, P_N w) \in Γ) \quad w \in L^2, \quad Γ \in \text{Bor}(L^2), \quad t ≥ 0,
\]

and define the Markov semi-groups

\[
\Psi_{α,N}^t f(v) = \int_{L^2} f(w) T^t_{α,N}(v, dw) \quad L^∞(L^2; \mathbb{R}) \rightarrow L^∞(L^2; \mathbb{R}),
\]

\[
\Psi_{α,N}^t \lambda(Γ) = \int_{L^2} \lambda(dw) T^t_{α,N}(P_N w, Γ) \quad p(L^2) \rightarrow p(L^2).
\]

Since the solution \( u(t, u_0) \) is continuous in \( u_0 \), the Markov semi-group \( \Psi_{α,N}^t \) is Feller: for any \( t ≥ 0 \), \( \Psi_{α,N}^t C_b(L^2) \subset C_b(L^2) \). Hence we can consider it as acting on this space.

3.3.2. **Statistical estimates of the flow.** Set the truncated constants

\[
A_{s,N} = \sum_{|m| ≤ N} λ^s_m |a_m|^2.
\]

Of course, these constants are bounded respectively by

\[
A_s = \sum_{m \in \mathbb{N}} λ^s_m |a_m|^2,
\]

that we assume to be finite for \( s = 1 \). Also we have then the obvious convergence \( A_{s,N} \rightarrow A_s \), as \( N \rightarrow ∞ \) for \( s ≤ 1 \).

**Proposition 3.3.** Let \( u_0 \) be a random variable in \( E_N \) independent of \( \mathcal{F}_i \) such that \( \mathbb{E}M(u_0) < ∞ \). Let \( u \) be the solution to (3.1) starting at \( u_0 \). Then we have

\[
\mathbb{E}M(u) + α \int_0^t \mathbb{E}M(u) dτ = \mathbb{E}M(u_0) + \frac{αA_{0,N}}{2} t.
\]

**Proof.** We apply the finite-dimensional Itô formula to the functional \( M(u) : \)

\[
dM(u) = M'(u, du) + \frac{α}{2} \sum_{|m| ≤ N} a^2_m M''(u; e_m, e_m) dt.
\]

Now, using the fact that \( M'(u, 1(|Δu - |u|^{p-1}u)) = 0 \) and (3.6), we have

\[
M(u, du) = -αM(u) dt + √α \sum_{|m| ≤ N} a_m(u, e_m) dβ_m.
\]

On the other hand,

\[
M''(u; e_m, e_m) = \|e_m\|^2 = 1.
\]

Then, after integration in \( t \) and taking the expectation, we arrive at the (3.15).

**Proposition 3.4.** Let \( u_0 \) be a random variable in \( E_N \) independent of \( \mathcal{F}_i \). Suppose that \( \mathbb{E}E(u_0) < ∞ \), then we have

\[
\mathbb{E}E(u) + α \int_0^t \mathbb{E}E(u) dτ \leq \mathbb{E}E(u_0) + \frac{α}{2} \left( A_{1,N} t + A_{0,N} (2π)^{-3} \int_0^t \mathbb{E}\|u\|^{p-1}_{L^{p-1}} dτ \right),
\]

where \( u \) is the solution to (3.1) starting at \( u_0 \).
Proof. We apply the Itô’s formula to \( E(u) \), and use the fact that \( E'(u, i(\Delta u - |u|^{p-1}u)) = 0 \) and (3.8), we obtain

\[
E(u) + \alpha \int_0^t \mathcal{E}(u) \, d\tau \leq E(u_0) + \frac{\alpha}{2} \left( A_{1,N} t + \sum_{|m| \leq N} a_m^2 \int_0^t (|u|^{p-1}, e_m) \, d\tau \right)
+ \sqrt{\alpha} \sum_{|m| \leq N} a_m \int_0^t \left( \lambda_m(u, e_m) + (|u|^{p-1}u, e_m) \right) \, d\beta_m(\tau).
\]

Taking the expectation, we obtain

\[
\mathbb{E}E(u) + \alpha \mathbb{E} \int_0^t \mathcal{E}(u) \, d\tau \leq \mathbb{E}E(u_0) + \frac{\alpha}{2} \left( A_{1,N} t + \mathbb{E} \sum_{|m| \leq N} a_m^2 \int_0^t (|u|^{p-1}, e_m) \, d\tau \right)
\]

\[
\leq \mathbb{E}E(u_0) + \frac{\alpha}{2} \left( A_{1,N} t + A_{0,N} (2\pi)^{-3} \mathbb{E} \int_0^t \|u\|_{L^p}^{p-1} \, d\tau \right).
\]

The proof is finished. \( \square \)

We can see without difficulties the following statement:

**Proposition 3.5.** The solution \( z_{\alpha} \) to (3.10) satisfies the estimate

\[
(3.17) \quad \mathbb{E}\|z_{\alpha}(t)\|^{2p} \leq \alpha C(p, A_0, t),
\]

where, \( C(p, A_0, t) \) does not depend on \( \alpha \), and \( p \geq 1 \).

**Corollary 3.6.** The solution \( z_{\alpha} \) to (3.10) satisfies the estimate

\[
(3.18) \quad \mathbb{E} \sup_{t \in [0,T]} \|z_{\alpha}(t)\|^{2p} \leq C(p, A_0, T) \alpha,
\]

where, \( C(p, A_0, T) \) does not depend on \( \alpha \), and \( p \geq 1 \).

**Proof.** We have that \( z_{\alpha} \) is a martingale adapted to \( \mathcal{F}_s \), thanks to the well known properties of the Itô integral. Since the function \( u \mapsto \|u\|^{2p}, p \geq 1 \) is convex, then \( \|z_{\alpha}\|^{2p} \) is a submartingale. Then by the Doob inequality,

\[
(3.19) \quad \mathbb{E} \sup_{t \in [0,T]} \|z_{\alpha}(t)\|^{2p} \leq C_p \mathbb{E}\|z_{\alpha}(T)\|^{2p}.
\]

We finish the proof after a use of the estimate (3.17). \( \square \)

3.3.3. Existence of stationary measures and uniform bounds.

**Theorem 3.7.** For any \( N \geq 2 \) and any \( \alpha \in (0, 1) \), there is an stationary measure \( \mu_{\alpha,N} \) to (3.1) concentrated on \( H^3 \). Moreover, we have the following estimates

\[
(3.20) \quad \int_{L^2} \mathcal{M}(u) \mu_{\alpha,N}(du) = \frac{A_0}{2} \leq \frac{A_0}{2},
\]

\[
(3.21) \quad \int_{L^2} \mathcal{E}(u) \mu_{\alpha,N}(du) \leq C.
\]

where \( C \) does not depend on \( \alpha \) and \( N \).
Proof. **Existence of stationary measures.** Let \( B_R \) be the ball of \( H^1 \) with center 0 and radius \( R \). We have, with the use of the Chebyshev inequality, that
\[
\frac{1}{t} \int_0^t \int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) = \int_{L^2} \lambda_t(dw) \int_0^t \int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) \leq \int_{L^2} \lambda_t(dw) \int_0^t \int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) \leq \frac{1}{R^2} \left( \frac{\|v\|^2_1}{\alpha t} + \frac{A_{0,N}}{2} \right).
\]
Choose \( \lambda \) to be the Dirac measure concentrated at 0, \( \delta_0 \). Then we obtain
\[
\frac{1}{t} \int_0^t \int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) \leq \frac{A_{0,N}}{2R^2}.
\]
Whence follows the compactness of the family \( \{ \frac{1}{t} \int_0^t \int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) : t \geq 0 \} \) on \( E_N \).

Let \( \mu_{\alpha,N} \in p(E_N) \) be an accumulation point at \( \infty \), that is,
\[
\mu_{\alpha,N} = \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \mathcal{M}(v) \chi_R(\|v\|) \delta_0(dv).
\]
The well known Bogoliubov-Krylov argument states that \( \mu_{\alpha,N} \) is stationary for (3.1).

**Estimates of the stationary measures.** Denote by \( \lambda_t \) the measure \( \frac{1}{t} \int_0^t \mathcal{M}(v) \chi_R(\|v\|) \delta_0(dv) \). We have that, using (3.15), that
\[
\int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \lambda_t(dv) \leq \int_{L^2} \mathcal{M}(v) \lambda_t(dv) \leq \frac{A_{0,N}}{2}.
\]
Now, since \( v \mapsto \mathcal{M}(v) \chi_R(\|v\|) \) is continuous bounded on \( L^2 \), we have by passing to the limit on a subsequence \( t_n \to \infty \)
\[
\int_{L^2} \mathcal{M}(v) \chi_R(\|v\|) \mu(dv) \leq \frac{A_{0,N}}{2}.
\]
We know use the Fatou’s lemma (under the limit \( R \to \infty \)) to obtain
\[
\int_{L^2} \mathcal{M}(v) \mu(dv) \leq \frac{A_{0,N}}{2}.
\]
In particular
\[
(3.22) \quad \int_{L^2} \|v\|^q \mu(dv) \leq \frac{A_{0,N}}{2} \quad \text{for any } q \geq 1,
\]
and using finite-dimensionality (with (3.22)), we remark that
\[
(3.23) \quad \int_{L^2} E(v) \mu(dv) < \infty.
\]
Therefore, (3.22) provides the requirement of Proposition 3.3, then we obtain the identity (3.15). But since the measure \( \mu_{\alpha} \) is stationary, we arrive at the identity (3.20).

To establish the estimate (3.21), let us observe that (3.23) implies the condition of Proposition 3.4. Therefore we have the estimate (3.16). Again, thanks to the stationarity of \( \mu_{\alpha} \) and the fact that \( A_{\alpha,N} \leq A_{\alpha} \), we obtain (3.21) with the required constant \( C \). □
Let us give an additional estimate for the measures \( \mu_{\alpha,N} \). Let \( \chi \) be a \( C^\infty \) function having value 1 on \([0,1]\) and 0 on \([2,\infty)\). Clearly the function \( \chi \) and its derivative are bounded, we can take a universal constant that bounds \( \chi \) and its first two derivative, so without loss of generality we can consider this constant to be 1. Set \( \chi_R(x) = \chi(\frac{x}{R}) \). We then have the following estimates on the derivatives of \( \chi_R \):

\[
|\chi_R^{(m)}(x)| \leq R^{-m}.
\]

We have that

**Proposition 3.8.** For any \( R > 0 \), the following estimate holds

\[
\int_{\mathbb{R}^3} \mathcal{M}(u)(1 - \chi_R(||u||^2)) \mu_{\alpha,N}(du) \leq C_2 R^{-1}.
\]

where \( C_2 \) is independent of \((\alpha,N)\).

**Proof.** Let \( F_R(u) = ||u||^2(1 - \chi_R(||u||^2)) \). Applying Ito’s formula we see the following

\[
dF_R + 2\alpha \mathcal{M}(u)(1 - \chi_R(||u||^2)) = -2\alpha \mathcal{M}(u)\chi_R(||u||^2) + \frac{\alpha}{2} \left( (1 - \chi_R(||u||^2)) A_{0}^N + 2 \chi_R(||u||^2) \sum_{|m|=0}^{N} a_m(u,e_m)^2 + ||u||^2 \left[ A_{0}^N \chi_R(||u||^2) + \chi_R''(||u||^2) \sum_{|m|=0}^{N} a_m(u,e_m)^2 \right] \right).
\]

Let us use the invariance and (3.20) to get

\[
\mathbb{E} \mathcal{M}(u)(1 - \chi_R(||u||^2)) \leq A_0^N \mathbb{E}(1 - \chi_R(||u||^2)) + \frac{C(A_0)}{R}.
\]

Now using the Markov inequality and (3.20), we have

\[
\int_{\mathbb{R}^3} (1 - \chi_R(||u||^2)) \mu_{\alpha,N} \leq C R^{-1}
\]

where \( C \) is independent of \((\alpha,N)\). Overall,

\[
\int_{\mathbb{R}^3} \mathcal{M}(u)(1 - \chi_R(||u||^2)) \mu_{\alpha,N} \leq C R^{-1},
\]

which is the claim. \( \square \)

4. **Inviscid limit towards the approximating NLS-7 equations**

We consider now the truncated NLS equations

\[
\partial_t u = i(\Delta - 1) u + P_N(||u||^{p-1} u),
\]
\[
u|_{t=0} = u_0 \in E_N.
\]

Using the preservation of the \( L^2 \)–norm, we see that the local solutions constructed in Proposition 2.1 are in fact global. Uniqueness and continuity follow, through usual methods, from the regularity of the non-linearity, we then obtain global well-posedness. Define the associated global flow \( \phi_N(t) : E_N \to E_N \). \( u_0 \mapsto \phi_N^{t_0} u_0 \), where \( \phi_N(t_0) =: u(t,u_0) \) represents the solution to (4.1) starting at \( u_0 \). Let us set the corresponding Markov groups

\[
\Phi_N^t f(y) = f(\phi_N^t(y)); \quad \mathcal{C}_{\text{c}}(L^2) \to \mathcal{C}_{\text{c}}(L^2),
\]
\[
\Phi_N^t \lambda(\Gamma) = \lambda(\phi_N^t(\Gamma)); \quad \text{p}(L^2) \to \text{p}(L^2).
\]

From the estimate (3.20), we have the weak compactness of any sequence \((\mu_{\alpha,N})\) with respect to the topology of \( H^{1-\varepsilon} \), therefore there exists a subsequence \((\mu_{\alpha,N}) \Rightarrow (\mu_{\alpha,N})\), converging to a measure \( \mu_N \) on \( L^2 \). We have the following
Proposition 4.1. Let $N \geq 2$, the measure $\mu_N$ is invariant under $\Phi_N(t)$ and satisfies the estimates

\begin{align}
\int_{L^2} \mathcal{H}(u) \mu_N(du) &= \frac{A_{0,N}}{2} \leq \frac{A_0}{2}, \\
\int_{L^2} \mathcal{E}(u) \mu_N(du) &\leq C_1, \\
\int_{L^2} \mathcal{H}(u)(1 - \chi_R(\|u\|^2)) \mu_N(du) &\leq C_2 R^{-1},
\end{align}

where $C_1$ and $C_2$ are independent of $N$.

Below, the subscript $k$ stands for $\alpha_k$. We do this abuse of notation to simplify the formulas.

Proof. (1) Estimates. The estimates (4.4) and (4.5) follow respectively from (3.21) and (3.25) and the lower semicontinuity of $\mathcal{E}(u)$ and $\mathcal{H}(u)$. Now let us prove (4.3): let $\chi_R$ be a bump function on $\mathbb{R}$ having the value 1 on $[0, 1]$ and the value 0 on $[2, \infty)$, we write

$$\frac{A_{0,N}}{2} - \int_{L^2} (1 - \chi_R(\|u\|^2)) \mathcal{H}(u) \mu_{k,N}(du) \leq \int_{L^2} \chi_R(\|u\|^2) \mathcal{H}(u) \mu_{k,N}(du) \leq \frac{A_{0,N}}{2}.$$ 

Now, using (3.25),

$$\frac{A_{0,N}}{2} - C_2 R^{-1} \leq \int_{L^2} \chi_R(\|u\|^2) \mathcal{H}(u) \mu_{k,N}(du) \leq \frac{A_{0,N}}{2}.$$ 

It remains to pass to the limits $k \to \infty$, then $R \to \infty$ to arrive at the claim.

(2) Invariance. It suffices to show the invariance under $\phi_{\alpha, \epsilon}$, $\epsilon > 0$. Indeed For $t < 0$, we have, using the invariance for positive times, that

$$\mu_N(\Gamma) = \mu_N(\phi_{\alpha, \epsilon}\Gamma) = \mu_N(\phi_{\alpha, \epsilon}^{-1}\Gamma) = \mu_N(\phi_{\alpha, \epsilon}^{-1}\Gamma),$$

which is the needed property. Now the proof of the invariance for positive times is summarized in the following diagram

\begin{align*}
\Phi_{\alpha, \epsilon}^N \mu_{k,N} \xrightarrow[]{(I)} \mu_{k,N} \\
\downarrow \quad (II) \\
\Phi_{\alpha, \epsilon}^N \mu_N \xrightarrow[]{(IV)} \mu_N
\end{align*}

The equality $(I)$ represents the stationarity of $\mu_{k,N}$ under $\Phi_{\alpha, \epsilon}^N$. $(II)$ is the weak convergence of $\mu_{k,N}$ towards $\mu_N$. The equality $(IV)$ represents the (claimed) invariance of $\mu_N$ under $\phi_{\alpha, \epsilon}$, that will follow once we prove the convergence $(III)$ in the weak topology of $L^2$. To this end, let $f : L^2 \to \mathbb{R}$ be a Lipschitz function that is also bounded by 1. We have

$$\langle \Phi_{\alpha, \epsilon}^N \mu_{k,N}, f \rangle - \langle \Phi_{\alpha, \epsilon}^N \mu_N, f \rangle = \langle \mu_{k,N}, \Phi_{\alpha, \epsilon}^N f \rangle - \langle \mu_N, \Phi_{\alpha, \epsilon}^N f \rangle = \langle \mu_{k,N}, \Phi_{\alpha, \epsilon}^N f - \Phi_{\alpha, \epsilon}^N f \rangle - \langle \mu_N - \mu_{k,N}, \Phi_{\alpha, \epsilon}^N f \rangle = A - B.$$ 

Since $\Phi_{\alpha, \epsilon}^N$ is Feller, we have that $B \to 0$ as $k \to \infty$. Now, using the boundedness property of $f$, we have

$$|A| \leq \int_{B_R(L^2)} |\Phi_{\alpha, \epsilon}^N f(u) - \Phi_{\alpha, \epsilon}^N f(u)| \mu_{k,N}(du) + 2 \mu_{k,N}(L^2 \setminus B_R(L^2)) =: A_1 + A_2.$$
Here $C_f$ is the Lipschitz constant of $f$ and $u_k(t, P_N u)$ is the solution to (3.1) at time $t$ and starting from $P_N u$. Now from (3.20), we have

$$A_2 \leq \frac{C}{R^2}$$

To treat the term $A_1$, let us consider the set

$$S_r = \left\{ \omega \in \Omega \mid \max \left( |\sqrt{\alpha_k} \sum_{|m| \leq N} \lambda_m \int_0^t (u, e_m) d\beta_m|, \|z_k\| \right) \leq r \sqrt{\alpha_k} \right\}, \quad r > 0,$$

we have the following statement.

**Lemma 4.2.** We have that, for any $R > 0$, any $r > 0$,

$$\sup_{u \in B_R(L^2)} \mathbb{E}(\|\phi_k^* P_N u - u_k(t, P_N u)\|_{S_r}) \to 0, \quad \text{as } k \to \infty.$$  \hspace{1cm} (4.6)

Now let us split $A_1$, and use the Lipschitz and boundedness properties of $f$

$$A_1 \leq C_f \int_{B_R} \mathbb{E}(\|\phi_k^* P_N u - u_k(t, P_N u)\|_{S_r}) \mu_{k+N}(du) + 2 \int_{B_R} \mathbb{E}(\|\phi_k\|) \mu_k N(du) =: A_{1,1} + A_{1,2}.$$  

It follows from the Lemma 4.2 above that, for any fixed $R > 0$ and $r > 0$, $\lim_{k \to \infty} A_{1,1} = 0$. Now, it follows from the classical Itô isometry and (3.20) that

$$\mathbb{E}(\|z_k\|^2) \leq C \alpha_k,$$

where $C$ does not depend on $k$. Also, from (3.17),

$$\mathbb{E}(\|z_k\|^2) \leq C \alpha_k,$$

where $C$ is independent of $k$. Therefore, using the Chebyshev inequality, we have

$$\mathbb{E}(\|z_k\|^2) = \mathbb{P} \left\{ \omega \mid \max \left( |\sqrt{\alpha_k} \sum_{|m| \leq N} \lambda_m \int_0^t (u, e_m) d\beta_m|, \|z_k\| \right) \geq r \sqrt{\alpha_k} \right\} \leq \frac{C \alpha_k}{r^2 \alpha_k} = \frac{C}{r^2}.$$

Passing to the limits $k \to \infty$, $R \to \infty$, $r \to \infty$ (respecting this order), we obtain (III), and hence (IV). \hfill \Box

**Proof of Lemma 4.2.** Set $w_k = u - v_k := \phi_k^* P_N u_0 - v_k(t, P_N u_0)$, where $v_k(t, P_N u_0)$ is the solution of (3.11), with $\alpha = \alpha_k$ and that starts from $P_N u_0$. We recall that $u_k = v_k + z_k$, where $z_k$ solves the problem (3.10) with $\alpha = \alpha_k$. Now, thanks to (3.18), we have that $\mathbb{E}(\|z_k\|^2) \to 0$ as $k \to \infty$. Therefore, it suffices to show that

$$\sup_{u_0 \in B_R(L^2)} \mathbb{E}(\|w_k\|_{S_r}) \to 0, \quad \text{as } k \to \infty.$$  \hspace{1cm} (4.7)

to complete the proof of the Lemma 4.2.

Let us take the solution to the equation (3.1) and (3.11):}

$$\partial_t w_k = i[(\Delta - 1)w_k - P_N (w_k f_{p-1}(u, v_k))] + i P_N (g_{p-1}(u, v_k, z_k) z_k)$$

$$- \alpha_k [(1 - \Delta)^{p-1} v_k + e^{p(\|v_k + z_k\|_{-1})} (v_k + z_k)],$$

where $f_{p-1}$ and $g_{p-1}$ are polynomial of degree $p - 1$ in the given variables. We observe that $|v_k + z_k|^{p-1}(v_k + z_k) - |u|^{p-1}u = |v_k|^{p-1}v_k - |u|^{p-1}u + z_k g_{p-1}(v_k, z_k) = w f_{p-1}(u, v_k) + z_k g_{p-1}(v_k, z_k)$.\hfill \Box
Taking the inner product with $w$, we obtain
\[
\partial_t\|w_k\|^2 \leq 2\|w_k\|^2(1 + \lambda_2^2 + \|f_{p-1}(u, v_k)\|_{L_t^\infty}) + 2\|z_k\|^2\|g_{p-1}(v_k, z_k)\|_{L_t^\infty}
\]
\[
+ \alpha_k C_0(N)\|w_k\|\|z_k\|e^{\rho_1(c(N)(\|v_k\| + \|z_k\|))}
\]
\[
\leq C_1(N)\|w_k\|^2(1 + \lambda_2^2 + \|u\|_{L_t^\infty}^{p-1} + \|v_k\|_{L_t^\infty}^{p-1}) + C_2(N)\|z_k\|^2\left(\|v_k\|_{L_t^\infty}^{2p-2} + \|z_k\|_{L_t^\infty}^{2p-2}\right)
\]
\[
+ \alpha_k C_3(N)\|v_k\|^2 + \alpha_k \|z_k\|^2)e^{\rho_1(c(N)(\|v_k\| + \|z_k\|))}.
\]

Using the Gronwall lemma, the fact that $w_k(0) = 0$ and using (3.3), we arrive at
\[
\|w_k(t)\|^2 \leq C_4(N)e^{C_2(N)t(1 + \lambda_2^2 + \|u\|_{L_t^\infty}^{p-1} + \|v_k\|_{L_t^\infty}^{p-1})\|z_k\|d\tau} \left(\int_0^t \|z_k\|^2d\tau + \alpha_k t\right) \left[1 + e^{\rho_1(c(N)(\|v_k\|_{L_t^\infty}^{2p-2} + \|z_k\|_{L_t^\infty}^{2p-2}))}\right]
\]
and the estimate (3.18), we have that, up to a subsequence,
\[
\lim_{k \to \infty} \sup_{t \in [0, T]} \|w_k\| = 0, \quad \mathbb{P} - \text{almost surely}.
\]

Now, writing the Itô formula for $\|u\|^2$, we have
\[
\|u_k\|^2 + 2\alpha_k \int_0^t \mathcal{H}(u_k)d\tau = \|P_N u_0\|^2 + \alpha_k \frac{A_0 N}{2} t + 2\sqrt{\alpha_k} \sum_{|m| \leq N} \lambda_m \int_0^t (u_k, e_m)d\beta_m.
\]
Therefore, recalling that $\alpha_k \leq 1$, we have that, on the set $S_r$,
\[
\|u_k\|^2 \leq \|P_N u_0\|^2 + C(r, N)t,
\]
where $C(r, N)$ does not depend on $k$. Hence we see that, on $S_r$,
\[
(4.9) \quad \|w_k\| \leq \|v_k\| + \|z_k\| \leq \|u_k\| + 2\|z_k\| \leq \|u_0\| + 3C(r, N)t.
\]
In particular, we have the following two estimates:
\[
(4.10) \quad \sup_{u_0 \in B_R} \|w_k\|_{L_t^\infty} \leq R + 3C(r, N)T
\]
\[
(4.11) \quad \sup_{k \geq 1} \sup_{u_0 \in B_R} \|w_k\|_{L_t^\infty} \leq R + 3C(r, N)T.
\]
Hence coming back to (4.8) and using the (deterministic) conservation $\|u(t)\| = \|P_N u_0\|$ and the estimate (4.10), we obtain
\[
(4.12) \quad \sup_{u_0 \in B_R} \|w_k\|^2_{L_t^\infty} \leq A(R, N, r, T)\|z_k\|_{L_t^\infty}.
\]
Therefore, using again the bound (3.18), we obtain the almost sure convergence $\|z_k\| \to 0$ (as $k \to \infty$, up to a subsequence), we obtain then the almost sure convergence
\[
(4.13) \quad \lim_{k \to \infty} \sup_{u_0 \in B_R} \|w_k\|^2_{L_t^\infty} \|z_k\|_{S_t} = 0.
\]
Now, taking into account the bound (4.11), we can then use the Lebesgue dominated convergence theorem to obtain
\[
\mathbb{E} \sup_{u_0 \in B_R} \|w_k\|_{L_t^\infty} \to 0, \quad \text{as } k \to \infty.
\]
Now, for $u_0 \in B_R$, we have
\[
\|w_k(t, P_N u_0)\|_{S_r} \leq \sup_{u_0 \in B_R} \|w_k(t, P_N u_0)\|_{S_r},
\]
then
\[
\mathbb{E} \|w_k(t, P_N u_0)\|_{S_r} \leq \mathbb{E} \sup_{u_0 \in B_R} \|w_k(t, P_N u_0)\|_{S_r},
\]
and finally,
\[
\sup_{u_0 \in B_R} \mathbb{E} \|w_k(t, P_N u_0)\|_{S_r} \leq \mathbb{E} \sup_{u_0 \in B_R} \|w_k(t, P_N u_0)\|_{S_r}.
\]
The proof is finished. \qed

5. Statistical ensemble for NLS-7 and almost sure GWP

In this section, we consider the Schrödinger equations
\begin{align}
\partial_t u &= i[(\Delta - 1)u - |u|^{p-1}u] \\
|u|_{t=0} &= u_0.
\end{align}

We follow closely the arguments of [Bou94] (see also [Tzv06, Tzv08]) in the construction of an statistical ensemble for the NLS equations (5.1). We show that on this set, the equation is globally well-posed, and the probability measure used in the construction is left invariant under the flow that has been established. In contrast with the ‘Gaussianity’ of the measures in [Bou94], here we do not have many information about the relations between the approximating measures and the limiting measure. Therefore in establishing the statistical ensemble, we need additional tools; that is why we introduce the restricted measures that, combined with the Skorokhod representation theorem, allow to defined an ‘almost sure limiting’ set whose elements can be compared with finite-dimensional data for which the associated solutions are controlled. These controls are inherited by the infinite-dimensional solutions living on the limiting set by the use of the Bourgain iteration procedure [Bou94]; we then obtain global wellposedness on the constructed set.

In the sequel we consider any function increasing concave function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ and take the function $\rho$ to be $\rho(x) = \frac{\alpha^2}{2} x^{-1}(x)$, where $\xi^{-1}$ is the inverse of $\xi$. Remark that $\xi^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing convex function. Then we remark that $e^{3\xi^{-1}}$ satisfies (3.3). Also,
\[
\xi(x) \leq C_\xi x,
\]
where $C_\xi$ depends only on $\xi$.

To proof Theorem 1.1, we can consider exterior of $B_a$, where $B_a = \{ u \in L^2 \mid \|u\| \leq a \}$, the number $a > 0$ being arbitrary small, invoking the conservation of the $L^2$-norm under the NLS equation (5.1). Let us set $L_a^2 = L^2 \setminus B_a$, and $E_a^N = \{ u \in E_a^N \mid \|u\| \geq a \}$. Notice that
\[
\int_{E_a^N} e^{3\xi^{-1}(|u|_{L^2})} \mu_N(du) = \int_{L_a^2} e^{3\xi^{-1}(|u|_{L^2})} \mu_N(du) \leq \int_{L_a^2} \frac{|u|^2}{a^2} e^{3\xi^{-1}(|u|_{L^2})} \mu_N(du) \leq \frac{A_0}{2a^2}.
\]

Now using (4.4), (3.8), we obtain
\[
\int_{L_a^2} \left[ |u|^2 + (|u|^2 + \|u\|_{L^{p+1}}^{p+1}) e^{3\xi^{-1}(|u|_{L^2})} \right] \mu_N(du) \leq C_1 + K := C,
\]
where $C$ does not depend on $N$. 
Proposition 5.1. There are a subsequence \((\mu_{\mathcal{W}(N)}) \subset (\mu_N)\) and a measure \(\mu\) on \(L^2\) such that
\[
\lim_{N \to \infty} \mu_{\mathcal{W}(N)} = \mu, \quad \text{weakly on } H^r, \quad \forall r < s.
\]
Moreover, we have the estimates
\[
\int_{L^2} \mathcal{M}(u) \mu(du) = \frac{A_0}{2},
\]
\[
\int_{L^2} \left[ \|u\|^2 + \left( \|u\|^2 + \|u\|_{L^{p+1}}^{p+1} \right) e^{3\|u\|^2} \right] \mu(du) \leq C,
\]
where \(C\) does not depend on \(t\).

**Proof.** The independence in \(N\) of the constance \(C\) in (4.4) ensures the tightness of the sequence \((\mu_N)\) on \(H^r, r < s\), thanks to the Prokhorov theorem. We obtain the first statement of the proposition.

The estimate (5.8) follows from (5.5) and the lower semicontinuity of \(\mathcal{G}(u)\). Now let us prove (5.7): let \(\chi_R\) be a cut-off function on \(\mathbb{R}\) having the value 1 on \([0, 1]\) and the value 0 on \([2, \infty)\), we write
\[
\frac{A_{0,N}}{2} - \int_{L^2} (1 - \chi_R(\|u\|^2)) \mathcal{M}(u) \mu_N(du) \leq \int_{L^2} \chi_R(\|u\|^2) \mathcal{M}(u) \mu_N(du) \leq \frac{A_{0,N}}{2}.
\]
Now, we obtain, with the use of (4.5),
\[
\frac{A_{0,N}}{2} - C_2 R^{-1} \leq \int_{L^2} \chi_R(\|u\|^2) \mathcal{M}(u) \mu_N(du) \leq \frac{A_{0,N}}{2}.
\]
It remains to pass to the limits \(N \to \infty\), then \(R \to \infty\) to arrive at the claim. \(\Box\)

Recall that \(E^N_a = \{ u \in E^N_0 \mid \|u\| \geq a \}\), for fixed small enough \(a > 0\).

**Proposition 5.2.** Let \(s - \frac{1}{4} < r < s\), \(N \geq 0\) and \(a > 0\). There \(C = C(a) > 0\), such that for any \(i \in \mathbb{N}^+\), there is a set \(\Sigma^i_{N,r}\) verifying
\[
\mu_N(E^N_0 \setminus \Sigma^i_{N,r}) \leq Ce^{-2i},
\]
and having the property: For all \(u_0 \in \Sigma^i_{N,r}\), we have
\[
\|\phi^i_N u_0\|_r \leq 2 \xi \left( 1 + i + \ln(1 + |t|) \right), \quad \forall t \in \mathbb{R}.
\]

**Proof.** Without loss of generality, let us work with non-negative times. Define, for \(j \geq 1\), the set
\[
B^{i,j}_{N,r} = \{ u \in E^N_0 \mid \|u\|_r \leq \xi (i + j) \},
\]
Let \(T \sim (\xi (i + j))^{1-p}\), this is smaller than the time existence defined in Proposition 2.1. Then, according to the same proposition, we know that for \(t \in [0, T]\),
\[
\phi^i_N B^{i,j}_{N,r} \subset \{ u \in E^N_0 \mid \|u\|_r \leq 2 \xi (i + j) \}.
\]

Define the set
\[
\Sigma^i_{N,r} = \bigcup_{k=0}^{[T]} \phi^{-1}_N(B^{i,j}_{N,r} T^k).
\]
Using the invariance of \(\mu_N\) under \(\phi^i_N\), we have
\[
\mu_N(E^N_0 \setminus \Sigma^i_{N,r}) = \mu_N \left( \bigcup_{k=0}^{[T]} E^N_0 \setminus (B^{i,j}_{N,r} T^k) \right) \leq \left( \frac{e^i}{T} + 1 \right) \mu_N(E^N_0 \setminus B^{i,j}_{N,r}).
\]
Now since \( r \leq s - \), we have from (5.4) that \( \mathbb{E} e^{\frac{\mu_2}{2} - 1(\|u\|)} \leq C \), where \( C_1 = C_1(a) > 0 \) is a constant independent of \( r \) and \( N \). One has, with the use of the Chebyshev inequality,

\[
\mu_N (E_N \setminus \Sigma_{N,j}^{i}) \leq C_1 \left( \frac{\epsilon^j}{T} \right) e^{-\frac{\mu_2}{2} - 1(\xi (i + j))} \lesssim e^{\xi (i + j)} e^{-3(i + j)}
\]

(5.14)

Now let us define the needed set as

\[
\Sigma_{N,j}^{i} = \bigcap_{j \geq 1} \Sigma_{N,j}^{i}
\]

(5.15)

We verify easily (5.9) using the fact that the series \( \sum_{j \geq 1} e^{-j} \) converges.

Next, let us observe that for \( u_0 \in \Sigma_{N,j}^{i} \), we have

\[
\| \phi_N u_0 \|_r \leq 2 \xi (i + j) \quad \forall t \leq e^j.
\]

(5.16)

Indeed, for \( t \leq e^j \), we can write \( t = kT + \tau \), where \( k \) is an integer in \([0, \frac{e^j}{T}]\) and \( \tau \in [0, T] \). Also, by definition of \( \Sigma_{N,j}^{i} \), we have that \( u_0 \) can be written as \( \phi^{-kT} w \) for any fixed integer \( k \in [0, \frac{e^j}{T}] \) and a corresponding \( w \in B_{N,j}^{i} \).

We then have

\[
\phi^t u_0 = \phi^T \phi^{kT} u_0 = \phi^T w.
\]

Now, using (5.12), we obtain (5.16).

Let \( t \geq 0 \), there is \( j \geq 1 \) and such that \( e^{j-1} \leq 1 + t \leq e^j \), therefore

\[
j - 1 \leq \ln (1 + t),
\]

then

\[
j \leq 1 + \ln (1 + t).
\]

And then,

\[
\| \phi_N^{i} u_0 \|_r \leq 2 \xi (i + j) \leq 2 \xi (1 + i + \ln (1 + t)),
\]

then we arrive at the estimate (5.10).

\[ \square \]

**Proposition 5.3.** For any \( s - \frac{1}{2} < r \leq s - \), any \( s - \frac{1}{2} < r_1 < r \), for every \( t \in \mathbb{R} \), there is \( i_1 \in \mathbb{N}_+ \) such that for any \( i \in \mathbb{N}_+ \), if \( u_0 \in \Sigma_{N,j}^{i} \), then we have \( \phi_{N,j}^{i} (u_0) \in \Sigma_{N,j}^{i+i_1} \).

**Proof.** Fix \( r \in (s - \frac{1}{2}, s - \). Without loss of generality, assume \( t > 0 \). Let \( u_0 \in \Sigma_{N,j}^{i} \), then for any \( j \geq 1 \), we have

\[
\| \phi_N^{i} u_0 \|_r \leq 2 \xi (i + j), \quad t_1 \leq e^j.
\]

Let \( i_1 := i_1 (t) \) be such that for every \( j \geq 1 \), \( e^j + t \leq e^j + i_1 \). We then have

\[
\| \phi_N^{i_1} u_0 \|_r \leq 2 \xi (i + j + i_1), \quad t_1 \leq e^j.
\]

Now, thanks to (5.10), we have, for every \( u_0 \in \Sigma_{N,j}^{i} \),

\[
\| u_0 \| \leq \| u_0 \|_r \leq 2 \xi (1 + i),
\]

therefore, since the \( L^2 \)-norm is preserved, we have, for every \( u_0 \in \Sigma_{N,j}^{i} \), that

\[
\| \phi_N^{i_1} u_0 \| \leq 2 \xi (1 + i).
\]
Hence for every \( r_1 \in (s - \frac{1}{2}, s - ) \), we use an interpolation to see that there is \( \theta \in (0, 1) \) such that
\[
\|\phi_N^{i_1 + i_1} u_0\|_{r_1} \leq \|\phi_N^{i_1 + i_1} u_0\|^{1 - \theta} \|\phi_N^{i_1 + i_1} u_0\|^\theta \leq 2^{1 - \theta} 2^\theta (\xi (1 + i))^1 - \theta (\xi (i + j + i_1))^\theta
\]
the last inequality above follows the fact that \( j \geq 1 \) and \( \xi \) is an increasing function, therefore the inequality holds for \( i_1 (i) \) large enough. Thus we obtain that \( \phi_N^{i_1 + i_1} (u_0) \) belongs to \( B_{N,r_1}^{i_1} \) for the constructed \( i_1 (i) \) and \( r_1 \), for all \( i_1 \leq i \), for all \( j \geq 1 \). The proof is finished.

Let us introduce the restriction measures (or conditional probabilities)
\[
\mu_{N,i,r}(\Gamma) = \frac{\mu_N(\Gamma \cap \Sigma_{N,r})}{\mu_N(\Sigma_{N,r})}, \quad \Gamma \in \text{Bor}(L^2).
\]
We do not claim any invariance of these measures under the corresponding dynamics.

**Proposition 5.4.** For any \( i \in \mathbb{N}^* \) any \( r < s \), the sequence \( (\mu_{N,i,r})_{N \geq 1} \) is tight on \( H' \), \( r < s \). In particular, there is a subsequence that we denote by \( (\mu_{i,r}) \) and that converges weakly to a measure \( \mu_{i,r} \) on \( H' \), \( r < s \).

**Proof.** We see, using (5.5), that
\[
E_{\mu_{N,i,r}}\|u\|^2 \leq \frac{E_{\mu_N}\|u\|^2}{\mu_N(\Sigma_{N,r})} \leq \frac{C}{1 - Ce^{-2t}}.
\]
This gives the claimed tightness by using the Chebyshev theorem. The compactness follows from the Prokhorov theorem.

Now, by invoking the Skorokhod representation theorem (see Theorem 11.7.2 in [Dud02]), we obtain a probability space still denoted \((\Omega, \mathcal{F}, \mathbb{P})\) on which are defined random variables \( u_{N,i,r} \) and \( u_{i,r} \) satisfying the following

1. \( u_{i,r} \) is distributed by \( \mu_{i,r} \), and for every \( N \), \( u_{N,i,r} \) is distributed by \( \mu_{N,i,r} \);
2. \( u_{N,i,r} \) converges to \( u \) almost surely in \( H' \).

Let us introduce the sets
\[
\Sigma_i^l = \{u \in H' | \exists k \rightarrow \infty as k \rightarrow \infty, \exists (u_{N_k}), u_{N_k} \rightarrow u as k \rightarrow \infty and u_{N_k} \in \Sigma_{i,r}^l\}.
\]

**Remark 5.5.**

1. For any fixed \( N \), \( \Sigma_{N,i,r}^l \) is obviously included in \( \Sigma_i^l \), for instance we can take constant sequences of \( \Sigma_{N,i,r}^l \).
2. We have that \( \Sigma_i^l \) is non-decreasing. Indeed \( \Sigma_i^l \) is non-decreasing (see (5.11)). Then, by definition, so does \( \Sigma_i^l \) (see (5.13)) and then \( \Sigma_{N,i,r}^l \) (see (5.15)). And it is clear that this property is preserved by the definition of \( \Sigma_i^l \).

Let us set \( \Sigma_r = \bigcup_{i \geq 1} \Sigma_i^l \).

**Proposition 5.6.** The following holds

1. The support of \( \mu_{i,r} \) is contained in \( \Sigma_i^l \), up to a set of \( \mu_{i,r} \)-measure 0. Hence \( \mu_{i,r}(\Sigma_i^l) = 1 \).
2. We have that
\[
\mu(\Sigma_r) = 1.
\]
(3) For any \( f \in C_b(H^r) \) bounded by 1, we have the inequalities
\[
\mu(f) \leq \mu_{i,r}(f) + Ce^{-2i},
\]
and
\[
\mu_{i,r}(f) \leq \frac{1}{1 - Ce^{-2i}} \mu(f),
\]
where \( v(f) := \int_{L^2} f(u)v(du) \). In particular
\[
\lim_{i \to \infty} \mu_{i,r} = \mu \text{ weakly on } H^r, \ r < s.
\]

**Proof.** Using the Skorokhod representation theorem, we have that the support of \( \mu_{i,r} \) contains essentially the almost sure limits of a sequence of random variables whose elements are distributed by the measures \( \mu_{N,i,r} \), respectively. Now, by definition of \( \mu_{N,i,r} \), these Skorokhod’s random variables distribute in \( \Sigma_{N,i}^i \) respectively. Hence, we get the inclusion in 1.

Next, using the Portmanteau theorem, the inclusion \( \Sigma_{N,i}^i \subset \Sigma_i^i \) and then the definition of \( \Sigma_{N,i}^i \), we have
\[
\mu(\Sigma_i^i) \geq \lim\mu_N(\Sigma_i^i) \geq \lim\mu_N(\Sigma_{N,i}^i) \geq \lim\mu_N(\Sigma_{N,i}^i) \geq 1 - Ce^{-2i}.
\]

Since \( (\Sigma_i^i)_{i \geq 1} \) is non-decreasing, then so does \( (\Sigma_{N,i}^i)_{i \geq 1} \), therefore we obtain
\[
\mu(\Sigma_i^i) = \mu\left(\bigcup_{i \geq 1} \Sigma_i^i\right) = \lim_{i \to \infty} \mu(\Sigma_i^i) \geq 1.
\]
Since \( \mu \) is a probability measure, we get
\[
\mu(\Sigma_i^i) = 1.
\]

Next, let us prove the inequalities in point 3:
\[
\int_{L^2} f(u)\mu_{i,r}(du) = \lim_{N \to \infty} \int_{L^2} f(u)\mu_{N,i,r}(du) \leq \frac{1}{1 - Ce^{-2i}} \lim_{N \to \infty} \int_{L^2} f(u)\mu_N(du) = \frac{1}{1 - Ce^{-2i}} \int_{L^2} f(u)\mu(du),
\]
that is (5.19). Also, using the fact that \( \mu_N(\Sigma_{N,i}^i) \leq 1 \), we have
\[
\int_{L^2} f(u)\mu_N(du) = \int_{\Sigma_{N,i}^i} f(u)\mu_N(du) + \int_{L^2 \setminus \Sigma_{N,i}^i} f(u)\mu_N(du)
\leq \int_{L^2} f(u)\mu_{N,i,r}(du) + \mu_N(\Sigma_{N,i}^i \setminus \Sigma_{N,i}^i) \leq \int_{L^2} f(u)\mu_{N,i,r}(du) + Ce^{-2i}.
\]
After passing to the limit \( N \to \infty \), we obtain the inequality (5.18). \( \square \)

Now, we state the well-posedness result.

**Proposition 5.7.** Let \( r \leq s^- \). For any \( u_0 \in \Sigma_r \cap \text{Supp}(\mu) \), there is a unique global in time solution to (5.1).
Therefore we obtain a global flow \( \phi_t \) defined on \( \Sigma_r \).
For any \( T_0 > 0 \), there is \( C(T_0) > 0 \) such that for any \( u, v \in \Sigma_r \)
\[
\sup_{t \in [-T_0,T_0]} \| \phi_t(u) \|_r \leq C(T_0),
\]
\[
\sup_{t \in [-T_0,T_0]} \| \phi_t(u) - \phi_t(v) \|_r \leq C(T_0)\| u - v \|_r,
\]
\[
\| \phi_t(u) \|_{s^-} \leq 2^\frac{r}{2} (1 + i + \ln(1 + |t|)).
\]
Proof. Let us fix an arbitrary $T_0 > 0$. Recall that $\mu (\Sigma_r \cap \text{Supp}(\mu)) = 1$ (Proposition 5.6). We wish to show that for $u_0 \in \Sigma_r \cap \text{Supp}(\mu)$, the solution $\phi'^u_0$ constructed in Proposition 2.1 exists in fact on $[-T_0, T_0]$. Then assume that $T_0$ is greater than the time of Proposition 2.1. Remark also that, from the bound (5.8), $\text{Supp}(\mu) \subset H'$. Then in particular, $u_0 \in H'$. Now, by the construction of $\Sigma_r$, any $u_0$ in $\Sigma_r$ belongs to $\Sigma'_i$ for some $i$. Let us consider the two cases (not necessary disjoint):

- $u_0 \in \Sigma'_i$
- $u_0 \in \partial \Sigma'_i$

In the first case, there is a sequence $(u_{0,N})_N$ such that $u_{0,N} \in \Sigma'_{N,r}$. Using the estimate (5.10), we have that

$$||\phi'_1 u_{0,N}||_r \leq 2\xi (1 + i + \ln(1 + |r|)), \quad t \in \mathbb{R}.$$ 

Therefore we have the bound

$$||\phi'_1 u_{0,N}||_r \leq 2\xi (1 + i + \ln(1 + |T_0|)), \quad |r| \leq T_0.$$ 

And, at $t = 0$, we see that

$$||u_{0,N}||_r \leq 2\xi (1 + i), \quad (5.23)$$

hence, by passing to the limit $N \to \infty$,

$$||u_0||_r \leq 2\xi (1 + i).$$

Let us remark for $u_0 \in \partial \Sigma'_i$, there is a sequence $(u^k_0)_k \in \Sigma'_i$ that converges to $u_0$ in $H'$. We see easily that (5.23) holds also on $\partial \Sigma'_i$ and then on $\Sigma'_i$.

Set $\Lambda = 2\xi (1 + i + \ln(1 + |T_0|))$, and $R = \Lambda + 1$. From Proposition 2.1, we have a uniform existence time associated to the ball $B_R$ is greater than $T \sim R^{1-p}$. Let $u_0 \in \Sigma'_i$ and $|r| \leq T$, we have that

$$\phi'(u_0) - \phi'((u_{0,N})) = S(t)(u_0 - u_{0,N}) - i\int_0^t S(t - \tau) \left(P_N(|\phi^T(u_0)|^{p-1} \phi^T(u_0) - |\phi^T_N(u_{0,N})|^{p-1} \phi^T_N(u_{0,N}))\right) d\tau$$

$$- i\int_0^t S(t - \tau) \left((1 - P_N)|\phi^T(u_0)|^{p-1} \phi^T(u_0)\right) d\tau.$$ 

Therefore, using in particular the fact that $u_0$ belongs to $H'$ (implying that $\phi'^u_0 \in H'$ for $|r| \leq T$) to treat the last term in the RHS, we have

$$||\phi'(u_0) - \phi'((u_{0,N}))||_r \leq ||u_0 - u_{0,N}||_r + C(r) \int_0^t ||\phi^T(u_0) - \phi^T_N(u_{0,N})||_r d\tau + C\Lambda N^{-\frac{\xi}{p}}.$$ 

Using the Gronwall lemma, and letting $N \to \infty$, we that

$$||\phi'((u_0) - \phi'((u_{0,N}))||_r \to 0, \quad |r| \leq T.$$ 

Now, by the triangle inequality

$$||\phi'((u_0))||_r \leq ||\phi'((u_0) - \phi'((u_{0,N})))||_r + ||\phi'((u_{0,N}))||_r \leq ||\phi'((u_0) - \phi'((u_{0,N}))||_r + \Lambda,$n

passing to the limit on $N$, we obtain

$$||\phi'((u_0))||_r \leq \Lambda \quad |r| \leq T. \quad (5.24)$$

Then $\phi^T(u_0)$ still belongs to the ball $B_R$, and we can iterate the procedure. Repeating the argument above, we have

$$||\phi^T((u_0) - \phi^T((u_{0,N}))||_r \leq ||\phi^T((u_0) - \phi^T_N((u_{0,N})))||_r + C \int_T^T ||\phi^T((u_0) - \phi^T_N((u_{0,N})))||_r d\tau + C\Lambda N^{-\frac{\xi}{p}} \quad T \leq |r| \leq 2T.$$
Again, we obtain that for $T \leq |t| \leq 2T$, $\| \phi'(u_0) - \phi'_N(u_{0,N}) \|_r \to 0$ as $N \to 0$, leading to the estimate $\| \phi'(u_0) \|_r \leq \Lambda$. $T \leq |t| \leq 2T$, as above. We see that after the $n$th step, $\phi^{nt}(u_0)$ remains in the ball $B_\Lambda$, allowing the next iteration. Then we arrive at the claim after iterating a sufficient number of times (recall that $\| \phi'_N(u_{0,N}) \|_r$ remains bounded by $\Lambda$ on $[-T_0, T_0]$).

The bound (5.20) for $u_0 \in \Sigma_i^j$ follows from the iteration of (5.24). Now let $u_0 \in \partial \Sigma_i^j$, take a $(u^k_0)_k \subset \Sigma_i^j$ converging to $u_0$. Recall that the bound (5.23) holds for both $u^k_0$, for all $k$, and $u_0$. In particular these elements belong to the ball $BR$ where $R = \Lambda + 1$, the same as above. Denote again the time existence of this ball by $T$. We have by continuity that

$$\lim_{k \to \infty} \| \phi^t(u_0) - \phi^t(u^k_0) \|_r = 0.$$ 

Combining this convergence with the triangle inequality, we have

$$\| \phi^t(u_0) \|_r \leq \Lambda \quad \forall t \leq T.$$ 

This allows to iterate the procedure as above. We arrive at global existence for data in $\partial \Sigma_i^j$ and completed the globalization on $\Sigma_i^j$. Also (5.20) is established on $\Sigma_i^j$.

Now, using the Duhamel formula, it is not difficult to see that

$$\| \phi^t u - \phi^t v \|_r = \| u - v \|_r + C \int_0^t \left( \| \phi^\tau v \|_{L_r}^{p-1} + \| \phi^\tau u \|_{L_r}^{p-1} \right) \| \phi^\tau u - \phi^\tau v \|_r d \tau \leq \| u - v \|_r + 2C \Lambda^{p-1} \int_0^t \| \phi^\tau u - \phi^\tau v \|_r d \tau.$$ 

We use the Gronwall lemma and take the sup over $[-T_0, T_0]$ to obtain (5.21). The inequality (5.22) follows from (5.10).

**Remark 5.8.** From the proof above, we have that for any $i \geq 1$, any $u_0 \in \Sigma_i^j$, any $t \in \mathbb{R}$,

$$(5.25) \quad \lim_{N \to \infty} \| \phi^t u_0 - \phi^t_N u_{0,N} \|_r = 0,$$

where $(u_{0,N})$ is a sequence in $\Sigma_{N,r}^i$ that converges to $u_0$ in $H^s$.

Consider an increasing sequence $l = (l_n)_{n \in \mathbb{N}}$ such that $l_0 = s - \frac{1}{2}$ and $\lim_{n \to \infty} l_n = s -$. Set

$$\Sigma = \bigcap_{r \in l} \Sigma_r.$$ 

We have the following result.

**Proposition 5.9.** The set $\Sigma$ is of full $\mu$—measure. Moreover, the flow $\phi^t$ constructed in Proposition 5.7 satisfies $\phi^t \Sigma = \Sigma$, for any $t \in \mathbb{R}$.

**Proof.** Since any $\Sigma_r$ is of full $\mu$—measure and the intersection is countable, we obtain the first statement.

To prove the second statement, let us take $u_0 \in \Sigma$, then $u_0$ belong to each $\Sigma_r$, $r \in l$.

First, consider $u_0 \in \Sigma_i^j$. Therefore $u_0$ is the limit of a sequence $(u_{0,N})$ such that $u_{0,N} \in \Sigma_{N,r}^i$ for every $N$.

Now from the Proposition 5.3, there is $i_1 := i_1(t)$ such that $\phi^{n_i}_{N}(u_{0,N}) \in \Sigma_{i_1}^{i+1}$. Using the convergence (5.25), we see that $\phi^t(u_0) \in \Sigma_{r_1}^{i+1}$. Now if $u_0 \in \partial \Sigma_i^j$, there is $(u^k_0)_k \subset \Sigma_i^j$ that converges to $u_0$ in $H^s$. Since we showed that $\phi^t \Sigma_i^j \subset \Sigma_{i_1}^{i_1+1}$ and $\phi^t(\cdot)$ is continuous, we see that $\phi^t(u_0) = \lim_k \phi^t(u^k_0) \in \Sigma_{r_1}^{i+1}$. We conclude that $\phi^t \Sigma_i^j \subset \Sigma_{r_1}^{i_1+1} \subset \Sigma_{i_1}$. It follows that $\phi^t \Sigma_i^j \subset \Sigma$. 

Now, let $u$ be in $\Sigma$, since $\phi^t$ is well-defined on $\Sigma$ we can set $u_0 = \phi^{-t} u$, then we have $u = \phi^t u_0$ and hence $\Sigma \subset \phi^t \Sigma$. That finishes the proof. \qed
6. INVARIANCE OF THE MEASURE

**Theorem 6.1.** The measure $\mu$ is invariant under $\phi^t$.

**Proof:** The measure $\mu$ is a Borel probability defined on a Polish space. The Ulam’s theorem (see Theorem 7.1.4 in [Dud02]) states that such a measure is regular: for any $S \in \text{Bor}(H^\gamma)$

$$\mu(S) = \sup\{\mu(K), \ K \subset S \text{ compact}\}.$$

Therefore it suffices to prove invariance for compact sets. Indeed, we then obtain, for any $t$,

\begin{align}
(6.1) \quad \mu(\phi^{-t}S) &= \sup\{\mu(K), \ K \subset \phi^{-t}S \text{ compact}\} = \sup\{\mu(\phi^tK), \ K \subset \phi^{-t}S \text{ compact}\} \\
(6.2) \quad &= \sup\{\mu(\phi^tK), \ \phi^tK \subset S, \ K \text{ compact}\} \leq \sup\{\mu(C), \ C \subset S \text{ compact}\} = \mu(S),
\end{align}

where we used the fact that $\phi^t$ is continuous in space, therefore it transforms compact sets into compact sets. Using the inequality above, we also have for any $t$ that

$$\mu(S) = \mu(\phi^{-t}\phi^tS) \leq \mu(\phi^tS).$$

since $t$ is arbitrary, we then obtain the invariance.

Now we claim that it also suffices to show the invariance only on a fixed interval $[-\tau, \tau]$, where $\tau > 0$ can be as small as we want. Indeed for $\tau \leq t \leq 2\tau$, one has $\mu(\phi^{-t}K) = \mu(\phi^{-2\tau}K) = \mu(\phi^{1-2\tau}K) = \mu(K)$ (using that $0 \leq t - \tau \leq \tau$), and for greater values of $t$ we can iterate. A same argument works for negative values of $t$.

Our proof is then reduced to showing invariance for compact sets on a small time interval. Therefore, it suffices to show it on the balls of $H^\gamma$. Here is the idea of the proof:

$$\Phi^t_\nu \mu_k \xrightarrow{(I)} \mu_k\xrightarrow{(III)} \mu_k \xrightarrow{(IV)} \mu\xrightarrow{(II)} \mu.$$

The equality (I) is the invariance of $\mu_N$ under $\Phi^t_\nu$, and (II) is the weak convergence $\mu_N \rightharpoonup \mu$. Then (IV) is proved once (III) is verified.

Let $f \in C_b(H^\gamma)$, supported on a ball $B_R(H^\gamma)$. Assume that $f$ is Lipschitz in the topology of $H^r$, $r < s$. Let $\tau$ be the associated time existence provided by Proposition 2.1. Then for $t < \tau$, we have

\begin{align*}
(\Phi^t_\nu \mu_N, f) - (\Phi^t_\nu \mu, f) &= (\mu_N, \Phi^t_\nu f) - (\mu, \Phi^t_\nu f) \\
&= (\mu_N, \Phi^t_\nu f - \Phi^t_\nu f) - (\mu - \mu_N, \Phi^t_\nu f) \\
&= A - B.
\end{align*}

By the continuity property of $\phi^t$, we have that $\Phi^t_\nu f \in C_b(H^\gamma)$. Then by weak convergence of $\mu_N$ to $\mu$ on $H^r$, we have that $B \rightarrow 0$ as $N \rightarrow \infty$.

Now using the Lipschitz property of $f$, we have, with the use of Lemma 2.3,

$$|A| \leq C_f \sup_{u \in B_R(H^\gamma)} \|\phi^t_\nu(u) - \phi^t(u)\| \mu_N(B_R(H^\gamma)) \leq C_f \sup_{u \in B_R(H^\gamma)} \|\phi^t_\nu(u) - \phi^t(u)\| \rightarrow 0, \text{ as } N \rightarrow \infty.$$

We obtain the claim. \qed
7. Almost sure GWP on $H^s(\mathbb{T}^3)$ and Remark on the Size of the Data

We have shown the global well-posedness on the support of $\mu$ viewed as a subset of $\cap_{\sigma<s} H^\sigma$ (Proposition 5.7). But the estimate (5.8) (in particular the control on $\|u\|_s$) shows us that $\mu$ is in fact concentrated on $H^s$. As a consequence, we give here the argument that the global well-posedness holds with respect to the topology of $H^s$. This fact relies on the propagation of regularity principle, very well known in the context of dispersive equations. Afterwards, we give an argument showing that large data are concerned by our result.

From Subsection 2.2, we have the statement that if the quantity $\int_0^t \|\phi^\prime u_0\|_{L^r}^{p-1} d\tau$ remains finite for all times, then the solution issued from $u_0 \in H^s$ is global in $H^r$. Now let $u_0$ belong to the support of $\mu$, thanks to Proposition 5.7, the solution of (1.4) issued to $u_0$ is global and belongs to $C_t H^s$ for any $r \in (s - \frac{1}{2}, s)$. In particular, the quantity $\int_0^t \|\phi^\prime u_0\|_{L^r}^{p-1} d\tau$ remains finite for all $t$. By this way, we see that the local solutions on $H^s$ stated in Proposition 2.1 are global on the support of $\mu$ viewed as a subset of $H^s$. The invoked control allows also uniqueness and continuity with respect to the initial datum by following usual estimation procedures.

Now let us turn our attention to the size of the data. We remark that the ensemble constructed in this work does not concern only small data. In fact, by an scaling of the measure, we have that for any $\Lambda > 0$, there is a non-degenerate measure $\mu^\Lambda$ concentrated on $H^s$ such that

$$E_{\mu^\Lambda} (\mathcal{M}(u)) = \Lambda,$$

and we have global wellposedness on the support of $\mu^\Lambda$. To see the construction of such a measure, it suffices to change the numbers $(a_m)$ entering the definition of the noise in (1.9) into $(\frac{a_m}{\sqrt{\Lambda_0}})$. Therefore, the number $A_0$ is changed into $\Lambda$, the numbers $A_{0,N}$ into $\Lambda_N := \frac{A_{0,N}}{\Lambda_0} \Lambda$ that converge clearly to $\Lambda$. Also, all the analysis done here remains unchanged (because the scaling in $\alpha$ between the fluctuation and the dissipation in (3.1) is not affected: we still keep $\alpha$ as the size of the dissipation for a fluctuation of intensity $\sqrt{\alpha}$). Therefore the following statement is a consequence of the results that have been establish so far:

**Theorem 7.1.** Let $\Lambda > 0$, there is a measure $\mu^\Lambda$ concentrated on $H^s$ and having the following properties

1. The NLS equation (1.4) is globally well-posed on the support of $\mu^\Lambda$;
2. The identity (7.1) holds true;
3. The measure $\mu^\Lambda$ is invariant under the flow $\phi^\prime$ of (1.4) defined on its support $S^\Lambda$.

Recall that $\mathcal{M}(u) = \|u\|^2_{L^l} + e^{\frac{s}{2}} \|\mathcal{I} u\|_{L^r} \|u\|_s^2 \leq \left(1 + e^{\frac{s}{2}} |\mathcal{I} u|_{L^r}\right) \|u\|^2_{L^l}$. Therefore, the estimate (7.1) provides data on the support $S^\Lambda$ of $\mu^\Lambda$ whose $H^r$-sizes are larger than $C(\Lambda)$, where $C(\Lambda) \to \infty$ as $\Lambda \to \infty$. We see from (7.1) that the set of such data is of positive $\mu^\Lambda$-measure.

Furthermore, we can define a cumulative probability measure

$$\mu^* = \sum_{n=1}^{\infty} \frac{\mu^n}{2^n},$$

where we have taken $\Lambda = n, n \in \mathbb{N}^*$. The support of $\mu^*$ is the set

$$S^* = \bigcup_{n \in \mathbb{N}^*} S^n.$$

It follows from Theorem 7.1 that a global flow for (1.4) that we write again $\phi^\prime$ is defined on $S^*$.

Since for any $n$, $E_{\mu^n} \mathcal{M}(u) = n$, we have that for any $n$, there is a set of positive $\mu_n$—measure containing initial data whose sizes are bigger than $C(n)$, where $C(n)$ goes to infinity with $n$. Hence, we obtain the following statement:

"forall $n > 0$, there is a set $W_n$ such that $\mu^*(W_n) > 0$, and any $u_0 \in W_n$ satisfies $\|u_0\|_s \geq n."
Moreover since $\phi_t^\mu = \mu^n$ for any $n$, we see that $\mu^*$ is invariant under the flow $\phi_t$. This finishes the discussion of this section.

8. Density for the distributions of the conservation laws.

Let $\mu_{a,N}$ be an stationary measure of (3.1) and $\mu$ the invariant measure for (1.4) that has been constructed in the previous sections. The quantity $E_\nu \mu = \mu(E \in \cdot)$ is the law of $E(u)$, where $u$ is distributed as $\mu$. The similar notation is used for $M$ and for the measures $\mu_{a,N}$.

**Theorem 8.1.** Suppose $a_m$ is non-zero for any $m \geq 0$. Then, the measures $E_\nu \mu$ and $M_\nu \mu$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Before presenting the proof of the theorem, we establish some results concerning a quite general context. Consider a general equation

$$du = f(u)dt + d\zeta,$$

where $\zeta$ is a Brownian motion in some separable Hilbert space $X$, given by

$$\zeta(t, x) = \sum_{|m| \geq 0} a_m e_m(x) \beta_m(t),$$

where the parameters entering the sum are similar to (1.9). Suppose that the equation admits an stationary measure $\nu$ concentrated on $X$, the corresponding solution is denoted by $u$. For a functional $F : X \rightarrow \mathbb{R}$, we denote by $F, \nu$ the distribution of $F(u)$, that is $F, \nu(\cdot) = \nu(F(\cdot))$.

**Theorem 8.2.** Let $F$ be in $C^2(X, \mathbb{R})$ satisfying the Itô change of variable

$$dF(u) = \left( F'(u; f(u)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(u; e_m, e_m) \right) dt + \sum_{|m| \geq 0} a_m F'(u; e_m) d\beta_m.$$ 

Let $O \subset X$ be an open set and $c$ and $C$ be two positive constants such that

$$Q(\nu) := \sum_{|m| \geq 0} |a_m|^2 |F'(\nu, e_m)|^2 \geq c \quad \text{for } \nu \text{ almost all } \nu \text{ in } O,$$

(8.1)

$$\int_X |F'(v; f(v)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(v; e_m, e_m)| \nu(dv) \leq C.$$ 

(8.2)

Then for any non-negative function $g \in C^0_0(\mathbb{R})$ we have

$$\int_O g(F(\nu)) \nu(dv) \leq \frac{C}{c} \int_{\mathbb{R}} g(x) dx.$$ 

(8.3)

**Proof.** Let $g$ be a positive $C^0_0$-function on $\mathbb{R}$, set the function

$$\Phi_\lambda(x) = \frac{1}{\sqrt{2\lambda}} \int_{\mathbb{R}} g(y) e^{-|x-y|\sqrt{2\lambda}} dy = \frac{1}{\sqrt{2\lambda}} \left( \int_{-\infty}^{x} g(y) e^{-(x-y)\sqrt{2\lambda}} dy + \int_{x}^{\infty} g(y) e^{-(x-y)\sqrt{2\lambda}} dy \right).$$

Thanks to the properties of $g$, we can differentiate this function and obtain

$$\Phi'_\lambda(x) = \int_{x}^{\infty} g(y) e^{(x-y)\sqrt{2\lambda}} dy - \int_{-\infty}^{x} g(y) e^{-(x-y)\sqrt{2\lambda}} dy.$$ 

Computing the second derivative of $\Phi_\lambda$, we obtain that

$$\frac{1}{2} \Phi''_\lambda + g = \lambda \Phi_\lambda.$$ 

(8.4)
We apply the Itô formula to $\Phi^\lambda \circ F(u)$:

$$\Phi^\lambda(F(u)) = \Phi^\lambda(F(u_0)) + \int_0^t \left( \Phi^\lambda(F(u)) \left\{ F'(u,f(u)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(u;e_m,e_m) \right\} + \Phi''^\lambda(F(u))Q(u) \right) ds + \sum_{|m| \geq 0} \int_0^t \Phi^\lambda(F(u))F'(u,e_m)d\beta_m(s).$$

Integrating with respect to $v$ and using its stationarity, we get

$$\int_X \left( \Phi^\lambda(F(v)) \left\{ F'(v,f(v)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(v;e_m,e_m) \right\} + \Phi''^\lambda(F(v))Q(v) \right) v(dv) = 0. \tag{8.5}$$

Now, evaluate the equation (8.4) at the point $F(v)$, $v \in O$, multiply by $Q(v)$, then integrate over $O$ against $v$. Using (8.5), we find

$$\int_O Q(v)g(F(v))v(dv) = \int_O \lambda \Phi^\lambda(F(v))Q(v)v(dv) - \frac{1}{2} \int_O \Phi''^\lambda(F(v))Q(v)v(dv)$$

$$= \int_O \lambda \Phi^\lambda(F(v))Q(v)v(dv) - \frac{1}{2} \int_X \Phi''^\lambda(F(v))Q(v)v(dv)$$

$$+ \frac{1}{2} \int_{X \setminus O} \Phi''^\lambda(F(v))Q(v)v(dv)$$

$$= \int_O \lambda \Phi^\lambda(F(v))Q(v)v(dv) + \frac{1}{2} \int_X \Phi''^\lambda(F(v)) \left\{ F'(v,f(v)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(v;e_m,e_m) \right\} v(dv)$$

$$+ \frac{1}{2} \int_{X \setminus O} \Phi''^\lambda(F(v))Q(v)v(dv).$$

Now, in view of the definition of $\Phi^\lambda$, we see clearly that

$$\lambda \Phi^\lambda(x) \rightarrow 0 \quad \text{as} \quad \lambda \rightarrow 0, \quad \forall x \in \mathbb{R}.$$

Also, as $\lambda \rightarrow 0$, we have, using the non-negativity of $g$, that

$$\Phi'(\lambda) (x) \rightarrow \int_x^\infty g(y)dy - \int_{-\infty}^x g(y)dy \leq \int_{-\infty}^\infty g(y)dy \leq \int_{\mathbb{R}} g(y)dy \quad \forall x \in \mathbb{R},$$

and, using again the sign of $g$, we obtain as $\lambda \rightarrow 0$

$$\Phi''(\lambda) (x) \rightarrow -2g(x) \leq 0 \quad \forall x \in \mathbb{R}.$$

Finally, with the use of the Lebesgue’s dominated convergence theorem, we arrive at

$$\int_O g(F(v))v(dv) \leq \frac{1}{2} \int_X \left| F'(v,f(v)) + \frac{1}{2} \sum_{|m| \geq 0} |a_m|^2 F''(v;e_m,e_m) \right| v(dv) \int_{\mathbb{R}} g(x)dx.$$

It remains to use the hypothesis (8.1) and (8.2) to obtain the claim. \hfill \Box

Let us consider now the NLS equation for which we have constructed an invariant measure $\mu$. Let $M(u)$ and $E(u)$ be the mass and energy of the equation.
Corollary 8.3. Suppose the numbers $a_m$ are nonzero for all indices. The laws under $\mu$ of the quantities $M(u)$ and $E(u)$ are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}\setminus(-a,a)$ for any $a > 0$. More precisely, there is a positive constant $C(a)$ such that for any Borel set $\Gamma \subset \mathbb{R}\setminus(-a,a)$,

$$M_{\alpha}\mu(\Gamma), E_{\alpha}\mu(\Gamma) \leq C(a)\ell(\Gamma).$$

Proof. It suffices to prove (8.6) for the measures $\mu_{\alpha,N}$ where $C$ is independent of $\alpha$ and $N$. Indeed, once we have such a bound, we can finish the argument by invoking the Portmanteau theorem.

Since the measure $\mu_{\alpha,N}$ is concentrated on $E_N$, let us set $X = E_N$ and $B_\varepsilon$ be the closed ball in $E_N$, with center 0 and radius $0 < \varepsilon < 1$.

Set the quadratic variations for $M(u)$ and $E(u)$:

$$\frac{\alpha}{2} Q_M(u) = \frac{\alpha}{2} \sum_{|m| \leq N} |a_m|^2 (u, e_m)^2,$$

$$\frac{\alpha}{2} Q_E(u) = \frac{\alpha}{2} \sum_{|m| \leq N} |a_m|^2 (-\Delta u + |u|^{p-1} u, e_m)^2.$$

Since $a_m \neq 0$ for any $m$, we see that $Q_M$ and $Q_E$ vanish only at 0. In what follows the symbol $Q$ denote both $Q_M$ and $Q_E$. We claim that (8.1) holds on the set $O = X \setminus B_\varepsilon$ for $Q$ with a constant $c = c(\varepsilon)$. Indeed, since $Q(v) = 0$ only for $v = 0$ and $B_\varepsilon$ is compact in $X$, we have, from the continuity of $Q$ on $B_\varepsilon$, that $Q(B_\varepsilon)$ is a compact interval (non reduced to $\{0\}$ because of the non-vanishing property of $Q$ outside 0) $I_\varepsilon \subset [0,\infty)$ containing 0. Therefore, if we denote by $c(\varepsilon)$ the upper point of $I_\varepsilon$, we have that

$$Q(v) \geq c(\varepsilon) \quad \text{for any } v \in X \setminus B_\varepsilon.$$

Therefore

$$\frac{\alpha}{2} Q(v) \geq \frac{\alpha}{2} c(\varepsilon) \quad \text{for any } v \in X \setminus B_\varepsilon.$$

Now, using Theorem 8.2, we claim that for a constant $C$ independent of $\alpha$ and $N$, we have

$$\frac{\alpha}{2} Q_M(u) = \frac{\alpha}{2} \sum_{|m| \leq N} |a_m|^2 (u, e_m)^2,$$

$$\frac{\alpha}{2} Q_E(u) = \frac{\alpha}{2} \sum_{|m| \leq N} |a_m|^2 (-\Delta u + |u|^{p-1} u, e_m)^2.$$

Since $a_m \neq 0$ for any $m$, we see that $Q_M$ and $Q_E$ vanish only at 0.

In what follows the symbol $Q$ denote both $Q_M$ and $Q_E$. We claim that (8.1) holds on the set $O = X \setminus B_\varepsilon$ for $Q$ with a constant $c = c(\varepsilon)$. Indeed, since $Q(v) = 0$ only for $v = 0$ and $B_\varepsilon$ is compact in $X$, we have, from the continuity of $Q$ on $B_\varepsilon$, that $Q(B_\varepsilon)$ is a compact interval (non reduced to $\{0\}$ because of the non-vanishing property of $Q$ outside 0) $I_\varepsilon \subset [0,\infty)$ containing 0. Therefore, if we denote by $c(\varepsilon)$ the upper point of $I_\varepsilon$, we have that

$$Q(v) \geq c(\varepsilon) \quad \text{for any } v \in X \setminus B_\varepsilon.$$

Therefore

$$\frac{\alpha}{2} Q(v) \geq \frac{\alpha}{2} c(\varepsilon) \quad \text{for any } v \in X \setminus B_\varepsilon.$$

Now, using Theorem 8.2, we claim that for a constant $C$ independent of $\alpha$ and $N$, we have

$$\int_X g(F(v)) \mu_{\alpha,N}(dv) = \int_{X \setminus B_\varepsilon} g(F(v)) \mu_{\alpha,N}(dv) + \int_{B_\varepsilon} g(F(v)) \mu_{\alpha,N}(dv)$$

$$\leq \frac{C}{c(\varepsilon)} \int_X g(x) dx + \int_{B_\varepsilon} g(F(v)) \mu_{\alpha,N}(dv).$$

Indeed, according to (8.2), $C$ must be a bound for the following quantities (drifts of $M(u)$ and $E(u)$):

$$\mathbb{E} |M'(u, i[(\Delta - 1)u - P_N(|u|^{p-1} u)] - \alpha[(1 - \Delta)^{s-1} + \rho(|u|^{p-1})] u)| = \alpha \mathbb{E} M(u),$$

or (using (3.8))

$$\mathbb{E} |E'(u, i[(\Delta - 1)u - P_N(|u|^{p-1} u)] - \alpha[(1 - \Delta)^{s-1} + \rho(|u|^{p-1})] u)| \leq \alpha \mathbb{E} E(u).$$

depending on the functional we consider. But in both cases, the estimates (3.20) and (3.21) provide bounds for $\mathbb{E} M(u)$, $\mathbb{E} E(u)$ that are independent of both $\alpha$ and $N$. Then we consider such bounds $C$.

By an standard approximation argument, we pass from $C^\infty$-functions to indicator functions in the above inequality. We arrive at, for every $a > 0$, for every Borel set $\Gamma$ in $\mathbb{R}$ contained in $\mathbb{R}\setminus(-a,a)$, and for any $\varepsilon > 0$,

$$F_\varepsilon \mu_{\alpha,N}(\Gamma) \leq \frac{C}{c(\varepsilon)} \ell(\Gamma) + F_\varepsilon \nu(\Gamma \cap [-\varepsilon, \varepsilon]).$$

Choosing $\varepsilon = a/2$, we obtain

$$F_\varepsilon \mu_{\alpha,N}(\Gamma) \leq C(a) \ell(\Gamma) \quad \text{for any Borel set } \Gamma \subset \mathbb{R}\setminus(-a,a).$$
Let us present here a result of estimation of the measure $\mu$ around 0. The strategy of its proof is due to Shirikyan [Shi11] and uses the properties of the local time of a functional based on the $L^2$-norm of the fluctuation-dissipation stationary solutions. The preservation of this norm by the limiting flow is crucial to obtain uniform bounds that allow to pass to the limit, we refer to [Shi11] for a complete proof.

**Proposition 8.4.** Let $\lambda_m \neq 0$, at least for two indices. There is a constant $C > 0$ such that

$$\mu(B_\delta(L^2)) \leq C\delta, \quad \text{for any } \delta > 0.$$  

**Proof of Theorem 8.1.** For $\Gamma \in \text{Bor}(\mathbb{R})$, let $\delta > 0$, we write

$$M_*\mu(\Gamma) = M_*\mu(\Gamma \cap [-\delta, \delta]) + M_*\mu(\Gamma \cap (\mathbb{R} \setminus [-\delta, \delta])).$$

It remains to apply the Corollary 8.3, and the Proposition 8.4 to obtain the claimed absolute continuity for $M_*\mu$. We do the same for the measure $E_*\mu$. That finishes the proof. □

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