Floppy Curves
With Applications to Real Algebraic Curves

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Abstract. We show how one may sometimes perform singular ambient surgery on the complex locus of a real algebraic curve and obtain what we call a floppy curve. A floppy curve is a certain kind of singular surface in \( \mathbb{C}P(2) \), more general than the complex locus of a real nodal curve. We derive analogs for floppy curves of known restrictions on real nodal curves. In particular we derive analogs of Fiedler’s congruence for certain nonsingular curves and Viro’s inequalities for nodal curves which generalize those of Arnold and Petrovskii for nonsingular curves. We also obtain a determinant condition for certain curves which are extremal with respect to some of these equalities. One may prohibit certain schemes for real algebraic curves by prohibiting floppy curves which result from singular ambient surgery. In this way, we give a new proof of Shustin’s prohibition of the scheme \( 1 < 2 \bigcup 1 < 18 \rangle \) for a real algebraic curve of degree eight.

Introduction

A real algebraic curve \( \mathbb{R}F \) in \( \mathbb{R}P(2) \) of degree \( m \) is the set of zeros of a real homogeneous polynomial \( F \) of degree \( m \) in three variables. We will refer to \( \mathbb{R}F \) as a real curve. It is nonsingular if the gradient of \( F \) is nonzero on \( \mathbb{R}F \). In this case \( \mathbb{R}F \) is a collection of smoothly embedded simple closed curves in \( \mathbb{R}P(2) \). We let \( \mathbb{C}F \) be the set of zeros of \( F \) in \( \mathbb{C}P(2) \), and refer to \( \mathbb{C}F \) as a complex curve. If \( \mathbb{C}F \) has only nodal (ordinary double point) singularities and no multiple components, then we call \( \mathbb{R}F \) a nodal curve, and \( \mathbb{C}F \) a complex nodal curve. \( \mathbb{R}A \) will consist of a collection of immersed curves in the plane with only double point singularities which intersect transversely away from these singularities together with a finite set of points. The possible ambient isotopy types of such subsets of \( \mathbb{R}P(2) \) is interesting. In 1978, Viro [V3] found the analogs for even degree nodal real algebraic curves of the strengthened Petrovskii and the strengthened Arnold inequalities [W, (7.4)], [V2, (3.4), (3.5), (3.6), (3.7)]. The original inequalities applied to nonsingular curves of even degree. (At the same conference Kharlamov and Viro independently stated a version of the original Petrovskii inequality for more general singular curves [Kh1] [V3]). Kharlamov and Viro have also studied analogs of the Gudkov-Rokhlin congruence for real nodal curves [KV](they also consider more general singularities). Fiedler has also given some generalization of his congruences for nonsingular curves [F1] which also applies to nodal curves [F2].

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Viro introduced the notion of a flexible curve [V2,p.59]. This is a surface in \( \mathbb{C}P(2) \) which is homologous to \( m[\mathbb{C}P(1)] \) satisfying three further conditions which are automatically satisfied by a complex curve. Viro observed that many known restrictions apply equally well to flexible curves. We will define a similar weakening of the notion of a nodal curve. As our notion allows considerably more freedom than one might expect, we will refer to this new kind of curve as a floppy curve. In this paper we show that the existence of a real nonsingular curve frequently implies the existence of related floppy curves. If these floppy curves can be prohibited, so are the original real nonsingular curves.

We further generalize Viro’s inequalities for nodal curves so that they apply to floppy curves as well. Our results [G1] relating nonsingular curves to link cobordisms also apply to floppy curves. Thus the restrictions on cobordisms given using signatures and Arf invariants given in [G2] also apply. Thus many theorems are available to prohibit floppy curves. In particular we derive Theorem (3.3) which gives restrictions on floppy curves based on the Arf invariants of associated links in the tangent circle bundle of \( RP(2) \). Theorem (3.3) when applied to nonsingular curves yields Fiedler’s congruence [F1]. Fiedler observes that his result [F2] also applies to flexible nodal curves. Theorem (3.3) also probably implies [F2], but we have not checked this yet. However Theorem (3.3) applies to some curves to which [F2] does not apply. To illustrate our methods, we use Theorem (3.3) to give a new proof of Shustin’s prohibition of the scheme \( 1 < 2 \coprod 1 < 18 \coprod > > \) for a real algebraic curve of degree eight.

Our method of constructing floppy curves makes use of Fiedler’s concept of a chain of ovals [F3], and Viro’s idea [V2,p.67],[K-S,§5] of using chains of ovals to find membranes for \( \mathbb{C}F \). Our method requires that one deduce from the isotopy class of a hypothetical real non-singular curve some set of chains of ovals associated to a given pencil of lines (or a list of possibilities for such sets of chains of ovals, one of which must occur). There are many arguments in the literature [V4] [K-S], [K1], [K2] for deducing chains of ovals.

There are other reasons to study nodal curves. The union of a line and say a nonsingular curve of degree \( m’ \) is a nodal curve of degree \( m’ + 1 \). As the complement of a line is affine space, the study of such arrangements is the study of affine curves of degree \( m’ \). This question has received a lot of attention [F3,Theorem 2],[P],[K-S]. Besides its intrinsic interest, one reason to study this case is that it is related to smoothings of non-degenerate singular points [S1],[K-S]. This in turn is important for the construction of nonsingular curves by Viro’s method [V5]. Another way the above theorems may prove useful is as follows. Sometimes one can use Bézout’s theorem to show an isotopy class of a nonsingular curve implies an isotopy class for the arrangement (or one of several possible arrangements) made up of a given curve and some auxiliary lines (or conics). Then one can rule out the original curve by showing the resulting arrangements are impossible [V4], [K1], [K2]. We provide new methods which may be added to the arsenal of methods which can be used to rule out the resulting arrangements.

Section one contains preliminaries on nodal curves and the definition of a floppy curve. Section two describes how floppy curves can be constructed by ambient singular surgery on real algebraic curves. In section three, we discuss link cobordisms related to flexible nodal curves. We also give our generalization of Fiedler’s congruence. In section four, we state the generalized Viro inequalities for floppy curves. Section five contains a further restriction on floppy curves involving the determi-
nant of some matrices used in the statement of the generalized Viro inequalities for floppy curves. Sections six through nine provides a complete proof the generalized Viro inequalities for floppy curves, as well as the further determinant conditions. This should prove useful even for those only interested in Viro’s original theorem as [V3] is an announcement and contains no proofs. As we were completing an earlier version of this paper in April 1996, we learned of a recent paper by Finashin [Fi] which derives the Viro-inequalities by somewhat different approach. We thank Viro for useful conversations. The basic method of proof which we use is the same as Viro’s which in turn is based on Arnold’s original method [A]. We were greatly influenced by Wilson’s exposition and extensions [W].

We let \( \beta_k(X, \mathbb{Z}_p) \) denotes \( \dim H_k(X, \mathbb{Z}_p) \). Let \( T_p(M) \) denote the real tangent space of a manifold \( M \) at a point \( p \in M \). Let \( T \) denote the involution on \( \mathbb{C}P(2) \) given by complex conjugation, i.e. \( T[z_0 : z_1 : z_2] = [\bar{z}_0 : \bar{z}_1 : \bar{z}_2] \).

§1 NODAL CURVES AND FLOPPY CURVES

Let \( G \) be a complex polynomial factor of \( F \), and let \( \mathbb{C}G \) and \( \mathbb{R}G \) denote the associated zero sets in \( \mathbb{C}P(2) \) and \( \mathbb{R}P(2) \). If \( G \) is irreducible over \( \mathbb{C} \), we call \( \mathbb{C}G \) an \( \mathbb{C} \)-irreducible component of \( \mathbb{C}F \). If \( G \) is real, and is irreducible over \( \mathbb{R} \), \( \mathbb{C}G \) is called a \( \mathbb{R} \)-irreducible component. If \( \mathbb{C}G \) is \( \mathbb{R} \)-irreducible but not \( \mathbb{C} \)-irreducible, then \( G = g\bar{g} \) where \( g \) is a complex polynomial irreducible over \( \mathbb{C} \) and \( \mathbb{R}G \) must consist of isolated points which are nodal singularities for \( \mathbb{C}G \). Otherwise \( \mathbb{C}g \) and \( \mathbb{C}\bar{g} \) would be distinct irreducible curves with an infinite number of intersections.

Let \( m \) be a non-negative integer. By a floppy curve of degree \( m \), we mean the image \( \mathcal{F} \) of an immersion \( j \) of a closed oriented surface \( \hat{\mathcal{F}} \) in \( \mathbb{C}P(2) \) with only ordinary double point singularities which is invariant under \( T \). Note that \( T_{|\mathcal{F}} \) lifts to an involution on \( \hat{\mathcal{F}} \), which we also denote by \( T \). Let \( \mathbb{R}\mathcal{F} \) denote \( \mathcal{F} \cap \mathbb{R}P(2) \) and \( \mathbb{R}\hat{\mathcal{F}} \) the inverse image of \( \mathbb{R}\mathcal{F} \) in \( \hat{\mathcal{F}} \). \( \mathbb{R}\hat{\mathcal{F}} \) is a disjoint collection of simple closed curves and some isolated points. \( \mathbb{R}\mathcal{F} \) consists of a collection of immersed circles in \( \mathbb{R}P(2) \) together with a finite number of isolated double points. Each of these immersed circles is called a constituent of \( \mathbb{R}\mathcal{F} \). Let \( \mathbb{R}\mathcal{F}^* \) denote \( \mathbb{R}\mathcal{F} \) with any isolated points deleted, and \( \mathbb{R}\hat{\mathcal{F}}^* \) its inverse image in \( \hat{\mathcal{F}} \). We insist that the following conditions hold.

1. \( \mathcal{F} \) represents the class \( m[\mathbb{C}P(1)] \) in \( H_2(\mathbb{C}P(2)) \).
2. For each \( p \in \mathbb{R}\hat{\mathcal{F}}^* \), there is a line \( L_p \) in \( T_{j(p)}(\mathbb{R}P(2)) \subset T_{j(p)}(\mathbb{C}P(2)) \) such that \( j_*(T_p(\mathbb{R}\hat{\mathcal{F}}^*)) \oplus i(L_p) = j_*(T_p(\hat{\mathcal{F}})) \).
3. If \( p \in \mathbb{R}\hat{\mathcal{F}} - \mathbb{R}\mathcal{F}^* \), we insist that \( j_*(T(\hat{\mathcal{F}})|_p) \) is the realization of a complex line in the complex tangent bundle of \( \mathbb{C}P(2) \) and the orientation on \( \mathcal{F} \) agrees with the orientation arising from the complex structure.

Note that for each \( p \in \mathbb{R}\hat{\mathcal{F}}^* \), there can be at most one line \( L_p \) satisfying the above condition. We denote the function \( p \to L_p \) by \( L_{\mathbb{R}\mathcal{F}} \). If \( j(p_1) = j(p_2) \) is a double point of \( \mathbb{R}\mathcal{F}^* \), then \( L_{p_1} \neq L_{p_2} \). The orientation on \( \mathcal{F} \) induces an orientation \( \mathfrak{o}_{\mathbb{R}\mathcal{F}} \) on \( j_*(T_{\mathcal{F}}(\mathbb{R}\mathcal{F})^*) \oplus i(L_{\mathbb{R}\mathcal{F}}(p)) \).

We write \( \mathbb{R}\mathcal{F} \) when we mean the triple \( (\mathbb{R}\mathcal{F}, L_{\mathbb{R}\mathcal{F}}, \mathfrak{o}_{\mathbb{R}\mathcal{F}}) \) and call \( \mathbb{R}\mathcal{F} \) the real part of \( \mathcal{F} \). We refer to \( \mathbb{R}\mathcal{F} \) as a real floppy curve. Thus our real floppy curves come equipped with a line field \( L_{\mathbb{R}\mathcal{F}} \), and an orientation \( \mathfrak{o}_{\mathbb{R}\mathcal{F}} \) as above. Note that condition (3) above forces all isolated real double points to be positive double points of \( \mathcal{F} \).
Sketching a line field can be quite tedious. However there is an easy way to encode in a drawing all the data of a triple \((RF, L_{RF}, o_{RF})\), up to isotopy. Note that we can isotope the line field so that it is tangent to \(RF^*\) in a neighborhood of every double point. Then we may further isotope the line field so that it is tangent everywhere except along some short segments where it looks like the field illustrated in Figure 1a. We call these segments flops. It is not hard to see that one may isotope the floppy curve \(F\) so that \(RF\) has this new field. In this situation we say that \(LRF\) is mostly tangent to \(RF\). We can indicate in a drawing of \(RF\) that the line field has a flop by replacing such a segment by a segment with a “cusp” as illustrated in Figure 1b.

![Figure 1a](image)

![Figure 1b](image)

So as not to encourage any further associations with cusp singularities beyond the visual associations, we will refer to these cusp-like segments of the encoded drawing as flops. Notice that the direction of the flop is determined by the line field. At a point \(p\) where the line field is tangent to the appropriate branch of \(RF\), the orientation given by a tangent vector \(v\) followed by \(iv\) will either agree or disagree with \(o\). This is independent of our choice of \(v\). We say \(o(p)\) is \(\pm 1\), depending on whether these orientations agree or disagree. This function changes exactly when one passes through a flop. It follows that each constituent of \(RF\) contains an even number of flops. We can encode this by drawing the segments where \(o\) is positive thicker. Thus we can encode the isotopy type of a real floppy curve with a collection closed curves in \(RP(2)\) which meet transversely and each of which has an even number of flops. Moreover the thickness of the curves should change at each flop as we travel along each constituent of \(RF\).

A nonisolated real double point of \(RF\) will be a positive double point of \(F\) if and only if the value of \(o\) agrees on the two branches which cross at the double point. In our encoded pictures this mean both branches are thick or both branches are thin. If a floppy curve has a nonisolated real double point, we may obtain another floppy curve by resolving the double point in either of the two possible ways as in Figure 2. Figure 2a shows the case where both curves are thick, there is a similar case where both curves are thin.
The topology of \( \mathcal{F} \) is modified under these moves by deleting a cone on the disjoint union of two circles and gluing in an annulus. To describe this precisely recall there is a orientation reversing diffeomorphism \( \mathcal{D} \) from the tangent disk bundle of \( \mathbb{R}P(2) \) to its tubular neighborhood in \( \mathbb{C}P(2) \) given by multiplication by \( i \). One defines on the shaded regions in Figures 2a and 2b, unit line fields. In Figure 2a it is \( \pm < x, -y > \), after we have chosen coordinates which make the lines the \( x \) and \( y \) axes. In Figure 2b it is \( \pm < x, y > \). One completes these by an arc of lines located at the double points. Under \( \mathcal{D} \), these describe a pair bands \( I \times I \) in the sphere bundle of \( \mathbb{R}P(2) \) which meet \( \mathcal{F} \) intersected with the before pictures of along \( \partial I \times I \) and meets the local picture of the floppy curve given by the first “after” picture of the “after” picture along \( I \times \partial I \). Each band is given by a unit vector field parallel to the line field together with an arc of unit vectors located at the double point. On may then replace \( \mathcal{F} \) intersected the disk bundle of the before picture by the two bands union the two disks which are given by the tangent unit disk bundles of the first after picture under \( \mathcal{D} \). To obtain the second “after” pictures one just considers the bands given by the above line fields defined on the unshaded regions. These are local moves and do not change \( \mathcal{F} \) away from a small neighborhood of the point in question. The next section will show that under certain circumstances the reverse process of Figure 2a can be applied to a nonsingular real algebraic curves.

From a floppy curve with some number of pairs of complex conjugate double points, one may obtain a floppy curve with one fewer pair of complex conjugate double points by equivariantly resolving the pair of double points.

Let \( n(\mathcal{F}) \) denote the number of connected components of \( \hat{\mathcal{F}} \). We will call the image of a component of \( \hat{\mathcal{F}} \) an \( \mathbb{C} \)-irreducible component of \( \mathcal{F} \). The degree of a \( \mathbb{C} \)-
irreducible component of $F$ is the nonzero multiple of the homology class of $\mathbb{C}(P(1))$ that it represents in $H_2(\mathbb{C}P(2))$. Note that a $C$-irreducible component of $F$ is not necessarily a floppy curve as it need not be invariant under $T$. A floppy curve $F$ is said to be $R$-irreducible if either it is $C$-irreducible or $F$ consists of pair of $C$-irreducible components which are interchanged by $T$. If $F$ is $R$-irreducible but not $C$-irreducible, then $RF$ consists of a collection of isolated points. $F$ is the union of a collection $R$-irreducible curves, called the $R$-irreducible components. Let $c(F)$ denote the number of $R$-irreducible components of $\hat{F}$. This is also the number of components of the orbit space $\hat{F}$ under $T$. If a $R$-irreducible component of $F$ is $C$-irreducible, we will call it a strongly-irreducible component. If a $R$-irreducible component of $F$ is not irreducible, we will call it a weakly-irreducible component.

An strongly-irreducible component is called dividing if it becomes disconnected when its real part and its non-real double points are deleted. For such a curve, a choice of one half of $F - RF$ induces a semi-orientation on $RF^*$ called the complex orientation. We can indicate a complex orientation on a real floppy curve by drawing arrows on each constituent of $RF^*$ in a consistent way. $F$ is called dividing, if each strongly-irreducible component is dividing.

If a floppy curve has an isolated real point then one may at will either delete it or replace it with a small empty oval with no flops and $\sigma$ identically one on the added oval. Again the topology of $F$ is changed in this process by replacing a cone on two circles with an annulus. This would not be possible if we did not insist on the orientation part of condition (4). Gudkov showed that modifications similar to this one and Figure 2a could be made to (non-flexible) nodal curves. See [Gu, p12.]

If the original curve is dividing and we resolve it by creating a small empty oval then so is the resulting curve with an induced complex conjugation which remains constant on the unaffected portions of $RF$. Suppose we have chosen a component of $F_i - RF_i$ for each strongly-irreducible component $F_i$ and also chosen a component of $F_j$ for each weakly-irreducible component $F_j$. Then the resolved curve we obtain by replacing an isolated double point of a dividing floppy curve with an empty oval comes with such choices made in a natural way. Thus we can speak of a complex orientation for a dividing floppy curve which includes this information. This can be indicated in a picture of $RF$ by an arrow curved around the isolated point showing the way an added oval would be oriented. In the moves described by Figures (2) and (2b), if the original floppy curve is dividing, then the resulting curve will be if we make the above resolution compatible with the complex orientation. Otherwise the resulting curve will not be dividing.

**Lemma (1.1).** $F$ is a floppy curve of even degree if and only if $RF^*$ is null-homologous modulo two in $\mathbb{R}P(2)$.

**Proof.** We may isotope $F$ slightly through floppy curves so that it meets $\mathbb{C}P(1)$ transversely in points. The number of intersections is even and all the nonreal intersections arise in pairs. The intersection of $RF^*$ with $\mathbb{R}P(1)$ is even. $\square$

We emphasize that $F$ may have both positive and negative double points. We let $d_+(F)$ denote the number of positive (respectively negative) double points and let $d(F)$ denote the total number of double points. Define

\begin{equation}
\Delta(F) = \chi(F) - d_+(F) + d_-(F) + \frac{m^2}{2} - 3m.
\end{equation}
Note that a nodal curve $\mathbb{C}F$ of degree $m$ is a floppy curve of degree $m$. Although a nodal curve will automatically satisfy the following additional conditions which can be phrased topologically, we do not require that floppy curves satisfy them:

1. All the double points are positive, i.e. $d_-(\mathbb{C}F) = 0$.
2. The oriented tangent planes to $\mathbb{C}F$ at points in $\mathbb{C}F \cap \mathbb{R}P(2)$ are complex lines in the tangent space of $\mathbb{C}P(2)$ and the orientation structure induced by the complex structure agrees with the given orientation.
3. $\Delta(\mathbb{C}F) = 0$.

To see that condition seven is satisfied. Perturb $F$ slightly obtaining $\tilde{F}$ with $\mathbb{C}\tilde{F}$ is nonsingular. One has $\chi(\mathbb{C}\tilde{F}) = 3m - m^2$, and $\chi(\mathbb{C}\tilde{F}) = \chi(\mathbb{C}F) - d_+(F)$. A floppy curve $F$ may be isotoped so that Condition (5) holds if and only if $\mathbb{R}F$ has no flops and $\varphi$ is identically $+1$. It is because we do not require floppy curves to satisfy conditions (4), (5), and (6) that we did not use the name flexible to describe them. We define a flexible nodal curve to be a flexible curve which satisfies these last three conditions. The flexible curves of Viro are then flexible nodal curves without any double points.

We also have the following generalization of Harnack’s theorem, and a related congruence for dividing curves [V2,(3.12)]. Let $r(F)$ denote the number double points which lie in $\mathbb{R}P(2)$ and are not isolated in $\mathbb{R}F$. So $d(F) = 2\nu(F) + i(F) + r(F)$, where $\nu(F)$ denotes the number of pairs of complex conjugate double points of $F$ in $\mathbb{C}P(2) - \mathbb{R}P(2)$. We will also let $\nu_+(F)$ ($\nu_-(F)$) denotes the number such pairs of positive (negative) double points. Let $\ell(F)$ denote the number of constituents of $\mathbb{R}\tilde{F}^*$. Let

$$h(F) = 4c(F) - \chi(F) - 2\ell(F) - d(F).$$

**Proposition (1.2).** $h(F) \geq 0$. This is an equality modulo two. For dividing curves it is an equality modulo 4. If $h(F) = 0$, then the curve is dividing.

**Proof.** Let $\tilde{G}$ denote $F$ with $n(\mathbb{R}F)$, the interior a small closed neighborhood of $\mathbb{R}F$ (invariant under complex conjugation), deleted. $\tilde{G} \cap \overline{n(\mathbb{R}F)}$ consists of $2\ell(F) + 2i(F)$ circles and so has zero Euler characteristic. Thus $\chi(\tilde{G}) = \chi(F) - \chi(\mathbb{R}F)$. Let $G$ denote the orbit space of $\tilde{G}$ under complex conjugation. We have $\chi(G) = 2\chi(\tilde{G})$. $G$ is a surface with $\ell(F) + i(F)$ boundary components with $\nu(F)$ interior points identified. Let $\tilde{G}$ denote the closed surface with $c(F)$ components that obtained by capping off each boundary component with a disk and unidentifying any interior points.

$$\chi(\tilde{G}) = \chi(G) + \ell(F) + i(F) + \nu(F).$$

The above inequalities simply say that $\chi(\tilde{G})$ must be an integer less than or equal to $2c(F)$ and must be even if $G$ is orientable. If $h(F) = 0$, then $\tilde{G}$ is a collection of 2-spheres, and the curve must be dividing. \qed

**§2 Pencils of Lines, and Floppy Curves**

Fiedler introduced pencils of lines to the study of real algebraic curves [F3]. Let $\mathbb{R}F$ be a nodal real algebraic curve. Let $P \in \mathbb{R}P(2)$ be point which does not lie on $\mathbb{R}F$ or on any inflectional tangent lines to $\mathbb{C}F$ or on any real line which is tangent to $\mathbb{C}F - \mathbb{R}F$ or on any real line which passes through a node on $\mathbb{C}F - \mathbb{R}F$. Here a real line is simply a degree one curve with real equation. The set of such $P$ is open and dense as the set of illegal points is a proper real algebraic set. Let
\( \mathcal{L}(P) = \{ \mathbb{R}L_t \}_{t \in \mathbb{R}P(1)} \) be the pencil of all real lines through a point \( P \). Here 
\( L_{[u_0; u_1]} = u_0L_0 + u_1L_1 \) where \( \mathbb{R}L_0 \) and \( \mathbb{R}L_1 \) are distinct real lines through \( P \). For \( t \in \mathbb{R} \), we let \( L_t \) denote \( L_{[1-t; t]} \). This is consistent with the above notations for \( L_0 \) and \( L_1 \). We call \( \{L_t | t \in [0, 1] \} \) a segment of the pencil of lines joining \( L_0 \) to \( L_1 \). There are only two such segments, one of which is picked out by a choice of sign for
\( L \) and \( P \in \mathcal{L}(P) \) induces the complex orientation. The index of a point of tangency of \( L \) in \( \mathcal{L}(P) \) is a consistent way “so that the orientations pass into one another under the natural isotopy” and denote by \( \mathbb{C}L_t^+ \) the closure of the component of \( \mathbb{C}L_t - \mathbb{R}L_t \) which induces the complex orientation. The index of a point of tangency of \( \mathbb{R}F \) with a line in \( L \in \mathcal{P} \) is the Morse index of the function which sends \( x \) to \( t \) whenever \( x \in L_t \).
Consider a pair of tangencies of lines in \( \mathcal{P} \) with a component \( C \) of \( \mathbb{R}F \) where one has index one and the other has index zero. We say that such a pair of tangencies is inessential if and only if the complex orientation on the tangent lines induce the same orientation on \( C \). Otherwise such a pair of tangencies is called essential. See [V4,p.413]. Following Viro [V2] we denote the closure of \( \mathbb{C}F \cap \bigcup_{L \in \mathcal{L}(P)} \mathbb{C}L \) by \( S_P(F) \). \( S_P(F) \) is a smooth closed one dimensional submanifold of \( \mathbb{C}P(2) \) lying on \( \mathbb{C}F \). \( S_P(F) \cap \mathbb{R}F \) consists of the points of tangency of \( \mathcal{L}(P) \) with \( \mathbb{R}F \).

In [V2,p.67], Viro announced some new prohibitions for curves of degree eight obtained by a new method of using embedded membranes for \( \mathbb{C}F \) with boundary the union of some of the components of \( S_P(F) \). By an embedded pseudo-membrane (e-p-membrane for short) for \( \mathbb{C}F \) we mean an embedded surface \( \mathcal{M} \) such that
\( \partial \mathcal{M} \subset \mathcal{M} \cap \mathbb{C}F \). By an embedded membrane (e-membrane for short), we mean an e-p-membrane whose interior is transverse to \( \mathbb{C}F \). Viro’s method was first described in detail in [KS,§5] where it is applied to the study of affine curves of degree six. An e-membrane can be found in the following situation which is more general than that described in [KS]. Still we follow the exposition in [KS] quite closely. There is a misprint in the English translation [KS,p509]: \( \mathbf{C}L \setminus \mathbf{R}L \) should read \( \mathbf{C}L \cap \mathbf{R}L \). Note in [KS], the membrane is used to construct homology classes in a branched double cover along \( \mathbb{C}F \). We plan to use it to perform surgery on \( \mathbb{C}F \). This accounts for some differences in our treatment. Let \( \mathcal{P} \) be a segment of the pencil of lines joining two lines through \( P: L_0 \) and \( L_1 \). Let \( \mathcal{P}^+ = \cup_t \mathbf{C}L_t^+ \). and \( \mathbb{R}P = \mathcal{P}^+ \cap \mathbb{R}P(2) \). Note that \( \mathcal{P}^+ \) is homeomorphic to \( D^2 \times I/\{p \} \times I \) where \( p \) is a point on the boundary of \( D^2 \). We assume that the following conditions hold:

1. \( \mathcal{P} \) contains an even number of simple tangents to the curve \( \mathbb{C}F \). These tangents correspond to tangents of \( \mathbb{R}L_t \) and \( \mathbb{R}F \) in \( \mathbb{R}P(2) \). The remaining lines in \( \mathcal{P} \) intersect \( \mathbb{C}F \) transversely in \( 2k \) points, of which at least \( 2k - 4 \) are real.
2. \( L_0 \) and \( L_1 \) are tangent to \( \mathbb{C}F \) and intersect it only in real points.
3. The points of tangencies of the lines in \( \mathcal{P} \) with \( \mathbb{R}F \) can be grouped into pairs with each pair joined by an arc, such that the arcs are disjoint and every line \( L_t \) in \( \mathcal{P} \) which is not a tangent to \( \mathbb{R}F \) transversely intersects exactly \( 2 - s/2 \) arcs, where \( s \) is the number of imaginary points in \( \mathbb{C}L_t \cap \mathbb{C}F \). Note \( s \) is either zero, two or four. The arcs must be disjoint and must be one of following two types. The first type, called inessential, join an inessential pair of tangencies on the same oval, meets \( \mathbb{R}F \) only at its endpoints, is tangent to \( \mathbb{R}F \) at its endpoints and is a small perturbation of an arc on this oval joining these points. The second kind, called essential, is transverse to \( \mathbb{R}F \), misses the double points of \( \mathbb{R}F \) and either joins points on distinct ovals.
or a pair of essential tangencies.

Let \( \{ \Lambda_i \} \) denote the set of inessential arcs and let \( \{ J_j \} \) denote the set of essential arcs. Let \( S = (\cup \Lambda_i) \cup (\cup J_j) \cup (P^+ \cap (CF - RF)) \). Each disk \( CL_t^+ \) for \( t \neq 0, 1 \) contains exactly two points of \( S \) and \( CL_t^+ \) and \( CL_t^+ \) contains exactly one point of \( S \). Thus \( S \) is a circle, which may be spanned by a disk \( \Delta \) whose interior consists of one arc in each \( CL_t^+ \) for \( t \neq 0, 1 \). Note \( M = \Delta \cup T(\Delta) \) is a planar surface with Euler characteristic two minus the number of arcs, and is an ep-membrane for \( CF \) with boundary some of the components of \( SP(F) \).

In deducing the occurrence of the above setup, from the topology of a hypothetical isotopy type for \( RF \), one usually has no control over how many inessential arcs are needed. However one can find a nearby e-membrane \( M' \) with Euler characteristic two minus the number of essential arcs. First we consider the e-p-membrane \( M'' \) constructed as above only replacing the inessential arcs by the unperturbed arcs \( \Gamma_i \) on the components joining inessential pairs. Let \( \Gamma = \cup \Gamma_i \), and let \( S'_{P}(F) \) be \( S_{P}(F) \) surgered in \( CF \) along the \( \Gamma \). We will construct \( M' \) to have boundary the union of some of the components of \( S'_{P}(F) \). We may isotope (see below) \( M \) to say \( M'' \) in a neighborhood of \( \Gamma \) so that \( M'' \cap CF \) is a tubular neighborhood of \( \Gamma \) in \( CF \). Let \( M' \) denote \( M'' - \text{Int}(M'' \cap CF) \) smoothed.

We must describe the isotopy of \( M \) to \( M'' \) precisely. Recall that multiplication by \( i \) defines an orientation reversing diffeomorphism from a tubular neighborhood \( \mathcal{N} \) of \( \mathbb{R}P^2 \subset \mathbb{CP}(2) \) to the tangent disk bundle of \( \mathbb{R}P^2 \) which sends \( CF \cap \mathcal{N} \) to the total space of the line bundle given by \( \mathcal{L}_{RF} \). Moreover it sends \( \mathcal{M} \cap \mathcal{N} \) to the total space of the line bundle over \( \cup (\Lambda_i) \cup (\cup (J_j)) \) given by the line field which sends each point \( p \) to the line joining \( p \) to \( P \) which we denote by \( \mathcal{L}_P \). Our isotopy takes place completely inside of \( \mathcal{N} \). Under our diffeomorphism it completely lies over the disks \( D_i \) which span the union of inessential \( \Lambda_i \) and the associated \( \Gamma_i \). We pick a line segment field of fixed length over each \( D_i \) which restricted to \( \Gamma_i \) is tangent to \( RF \) i.e. given by \( \mathcal{L}_{RF} \) and which restricted to \( \Lambda_i \) is parallel to \( \mathcal{L}_P \). See Figure 3. Then the total space of this line field is a 4-disk we push \( M \) across to get \( M'' \). It is important to note that \( M \) lies completely in \( P^+ \cup T(P^+) \), and that \( M' \) lies completely in \( (P^+) \cup T(P^+) \) except for a portion which lies over \( \cup D_i \) under the diffeomorphism.

Now we wish to perform “ambient singular surgery” to \( CF \) along \( M \) so as to
obtain a floppy curve $\mathcal{F} \subset \mathbb{C}P(2)$. What we actually do is perform a similar construction using a segment of the pencil of lines through $P^*$ where $P^*$ is near $P$ but not in $\mathbb{R}P$. Note that for each inessential path $\Lambda_i$, we then obtain two paths $\Gamma_i$ and $\Gamma^*_i$ on $\mathbb{R}F$ and one of these will be nested within the other. For notational convenience assume that inner segment is $\Gamma^*_i$. If we choose our inessential $\Lambda^*_i$ so that $D^*_i$ is also nested within $D_i$, and we choose a shorter length for the line segment field over the $D^*_i$, then our resulting membranes $M'$ and $M''$ will only intersect in $\mathbb{R}P(2)$. Moreover $(M' \cup M'') \cap \mathbb{C}F$ will be the boundary of a union of annuli. Thus we may delete the interior of these annuli and adjoin $M' \cup M''$ to obtain an immersed surface $\mathfrak{F}$. This surface may not be orientable. However if it is orientable then it will be a floppy curve of degree $m$. $\mathbb{R}\mathfrak{F} = \mathfrak{F} \cap \mathbb{R}P(2)$ may be obtained by performing “ambient singular surgery” to $\mathbb{R}F$ along $\cup_j \mathcal{J}_j$. In other words, we delete neighborhoods of the end points of the essential arcs and attach $(\cup_j \mathcal{J}_j) \cup (\cup_j \mathcal{J}^*_j)$. The line field $\mathcal{L}_{\mathfrak{F}}$ along $(\cup_j \mathcal{J}_j)$ is given by $\mathcal{L}_P$ and the line field along $(\cup_j \mathcal{J}^*_j)$ is given by $\mathcal{L}_{P^*}$.

Note that $\mathfrak{F} = \mathfrak{F} \cap \mathbb{R}P(2)$ is oriented. Thus $\mathfrak{F}$ will be orientable if and only if $\mathfrak{o}_F$ restricted to $\mathfrak{F}$ (neighborhoods of essential arcs) extends to an orientation $\mathfrak{o}_\mathfrak{F}$. Below will given an example where the orientation does not extend and the resulting $\mathfrak{F}$ is nonorientable. One could study “non-orientable” floppy curves with the extra data of a normal Euler number. We do not pursue this now.

If $\mathfrak{F}$ is orientable we denote it by $\mathfrak{F}$. $\mathfrak{F}$ will be dividing if and only if $\mathbb{C}F$ is dividing and the complex orientation on $\mathbb{R}F$ (neighborhoods of essential arcs) extends to an orientation on $\mathbb{R}\mathfrak{F}$.

We may also perform a sequence of these singular surgeries to $\mathbb{C}F$ along a collection of membranes which arise from a disjoint sequence of pencil segments around the same point $P$. There are probably situations of interest where one can do a sequence of such modifications using membranes arising from pencils around different points as well. But one would need to know that the portions of the membranes lying over disks near the inessential segments do not intersect. For now, we will only consider disjoint segments of pencils around a single point.

We will present in Figures 4, 5, and 6 three typical examples of how to obtain a encoded picture of the $\mathbb{R}F$ from $\mathbb{R}F$. We only draw relevant portions of $\mathbb{R}F$ in a region around $P^+$. We draw in gray some of the lines in the pencil of curves, as this helps us see the line fields $\mathcal{L}_P$ and $\mathcal{L}_P^*$. We also draw the essential arcs with dashed lines in the “before” pictures. A membrane like that used in Figure 4 will be called a simple membrane, and the resulting surgery be called a simple surgery.

Figure 5 has a thin part between the two flops. The “figure eight” in Figure 6 also has a thin part and a thickened part so the double point is a negative double point. Figures 4 and 6 use membranes of the type used by Viro and Shustin. However the membrane used in Figure 5 is of a different type as an essential arc crosses $\mathbb{R}F$. It is useful to know how these moves effect $\Delta(\mathcal{F})$. A simple surgery leaves $\Delta(\mathcal{F})$ unchanged. The moves of Figures 5 and 6 lower $\Delta(\mathcal{F})$ by two. The move of Figure 6 will create a nondiving curve from a dividing curve. In Figure 7 we show an ambient singular surgery which leads to a nonorientable curve.

The example in Figure 4 could be a hyperbola that was constructed by perturbing a pair of straight lines intersecting in a single double point as in Gudkov. Then our singular ambient surgery simply undoes the perturbation giving us a pair of 2-spheres meeting transversely in point. We may also topologically undo the move.
that replaces an isolated double point with a small oval around that point as follows. Suppose that we have a floppy curve with an empty oval $C$, and suppose that $L_{RF}$ is tangent to $C$. Then we may extend this to a vector field $f$ on the interior of $C$, and replace a tubular neighborhood of $C$ in $F$ with the pair of disks traced out by the endpoints of $\pm if$. If the original floppy curve is dividing, the new one will be dividing as well. This is also a singular ambient surgery.

§3 Floppy curves and link cobordism

This section generalizes [G1] so that it applies to floppy curves. The later sections do not depend on this section. The reader will find consulting [G1] helpful. Let $Q$ be the projective tangent bundle of $\mathbb{R}P(2)$. A point in $Q$ is a pair consisting of a point $p \in \mathbb{R}P(2)$ and a line in $T_p(\mathbb{R}P(2))$. We let $f \in H_1(Q)$ be the class generated by the fiber. Since this is a 2-torsion class, we do not need to specify an orientation.

Suppose that $C$ is the image of an immersion $j$ of a 1-manifold $\hat{C}$ into $\mathbb{R}P(2)$,
and and $C$ comes equipped with a line field $L_C$ which associates to $p \in \hat{C}$ a line in $T_{j(p)}(\mathbb{R}P(2))$. We assume that $L_C$ has the property that if $p_1 \neq p_2$, $L_C(p_1) \neq L_C(p_2)$. The image of a single circle is called a constituent of $C$. Then we may define a link $L(C)$ in $Q$ which lies over $C$, it is the image of the map $\hat{j}$ from $\hat{C}$ to $Q$ which sends $p$ to the pair $(j(p), L_C(p))$. If $C$ has an orientation we may lift the orientation to $L(C)$. We may encode the isotopy type of the pair $(C, L_C)$ by a curve with flops in the same way we encoded the isotopy type of the real part of a floppy curve. The isotopy type of $L(C)$ only depends on the isotopy type of $(C, L_C)$. We will also wish to consider $(C, L_C)$ enriched with an orientation $\sigma_C$ which assigns continuously for each $p \in C$ an orientation on $j_*(T_p(C) \oplus (L_C(p)))$ thought of as a plane in the tangent bundle to the the tangent disk bundle of $\mathbb{R}P(2)$ at $(p,0) \in T(\mathbb{R}P(2))$. The isotopy type of a triple $(C, L_C, \sigma_C)$ may be encoded by a curve with flops and thickenings. The existence of an orientation $\sigma$ implies that each constituent must have an even number of flops.

We wish to specify a framing for $L(D)$. For each orientation preserving constituent $\delta$ of $\delta'$, we draw a parallel curve $\hat{\delta}$ whose only intersections with $\delta$ are due to double points of $\delta$. For each orientation reversing constituent $\delta$ of $\delta'$, we draw a parallel curve $\delta$ with only one intersection with $\delta$ not due to double points of $\delta$. 
Let $\tilde{D}$ be the union of these pushoffs with line field $L_D$ obtained by translating $L_D$. The $L(\tilde{D})$ is a push-off of $L(D)$ and so gives a framing.

We let $\tilde{Q}$ denote the tangent circle bundle of $\mathbb{R}P(2)$. Let $\tilde{L}(C)$ be the inverse image of $L(C)$ under this two fold covering map. We can use $\sigma$ to define an orientation on $\tilde{L}(C)$ whether or not $C$ has one. This is called the natural orientation. For this purpose it is best to assume that our triple $(C, \mathcal{L}_C, \sigma_C)$ has been isotoped so it is given by a curve with flops and thickenings. Then for almost all points in $\tilde{p} \in \tilde{L}(C)$ are given by a point $p \in C$ together with a direction tangent to $C$ at $p$. This tangent direction to $C$ at $p$ can be lifted to an orientation to $\tilde{L}(C)$ at $\tilde{p}$. If $C$ is thick at $p$ we choose this orientation at $\tilde{p}$. Otherwise we choose the opposite orientation. If $C$ is oriented we may pick out a sublink $\tilde{L}_+(C)$ of $\tilde{L}$ with one component covering each constituent of $C$. One simply chooses the component of $\tilde{L}$ which induces the given orientation on $C$ under projection. The isotopy classes of all these links only depend on the isotopy of $C$ with its extra data (line field and orientation $\sigma$). We let $g$ denote the generator of $H_1(Q) \approx \mathbb{Z}_4$ represented by $L_+$ of a straight oriented line.

A small isotopy will make a floppy curve transverse to the Fermat quadric $\Sigma$ in $\mathbb{C}P(2)$ with equation $z_0^2 + z_1^2 + z_2^2 = 0$. From now on we assume $\mathcal{F}$ is transverse to $\Sigma$. Let $\epsilon(\mathcal{F})$ be half the number of negative intersections of $\mathcal{F}$ with $\Sigma$. So the number of intersections is $2m + 4\epsilon(\mathcal{F})$. Let $w(\mathbb{R}\mathcal{F})$ denote the number of orientation reversing constituents of $\mathbb{R}\mathcal{F}$. Let $r(\mathbb{R}\mathcal{F})$ denote the number of real nonisolated double points of $\mathbb{R}\mathcal{F}^*$ and let $r_{\pm}(\mathbb{R}\mathcal{F})$ denote the number these points which are positive/negative double points of $\mathbb{R}\mathcal{F}^*$. Let $\Sigma_k$ denote some collection of $k$ thick lines in general position in $\mathbb{R}P(2)$. Let $\Sigma_{m,\epsilon}$ denote some collection of $m + 2\epsilon$ lines with $m + \epsilon$ thick and $\epsilon$ thin. Let $L_k$, $L_{m,\epsilon}$, $L_{k}$, $L_{m,\epsilon}$, $L_{+,k}$, $L_{+,m,\epsilon}$ denote respectively $L(\Sigma_k)$, $L(\Sigma_{m,\epsilon})$, $L_{k}$, $L_{m,\epsilon}$, $L_{+,k}$, $L_{+,m,\epsilon}$, where for the last two links we give the lines arbitrary orientations. These links are well-defined up to isotopy and do not depend on the specific choice of lines or orientations. The following theorem is derived by exactly the same method as [G1, Theorem (6.1)]. Let $\tilde{\mathcal{F}}$ denote the curve obtained from $\mathbb{R}\mathcal{F}$ by replacing every isolated point by a small oval. Recall that if $\mathbb{R}\mathcal{F}$ is dividing, $\tilde{\mathcal{F}}$ has a complex orientation. A generalized cobordism is a cobordism which allows ordinary double points [G2]. It is the union of generalized cobordisms which are the images connected surfaces with boundary. These are called components of the cobordism.

**Theorem (3.1).** Given a floppy curve $\mathcal{F}$ transverse to $\Sigma$ of degree $m$ with real part $\mathbb{R}\mathcal{F}$ then there is an associated generalized cobordism $G$ in $I \times Q$ from $L(\mathbb{R}\mathcal{F})$ to $L_{m,\epsilon}(\mathcal{F})$ with $\nu_{+}(\mathcal{F})$ positive double points and $\nu_{-}(\mathcal{F})$ negative double points such that

1. $2\chi(G) = \chi(\mathcal{F}) + r(\mathcal{F}) - \nu(\mathcal{F})$
2. One may orient $G$, the inverse image of $G$ in $I \times \tilde{Q}$, so that it becomes an oriented generalized cobordism from $\tilde{L}(\mathbb{R}\mathcal{F})$ to $\tilde{L}_{m,\epsilon}(\mathcal{F})$ with their natural orientations. This orientation is reversed by the covering transformation of the cover $G \to G$.
3. $G \circ G = (1/2)(m^2 - w(\mathbb{R}\mathcal{F})) - r_{-}(\mathbb{R}\mathcal{F}) + r_{+}(\mathbb{R}\mathcal{F})$. Here $G$ is pushed off itself in $I \times Q$ so that along $L(A)$ and $L_m$, it has been pushed off with their given framings.
4. If $|L_+(\mathbb{R}\mathcal{F})| = mg$, then for any orientation of $L(\mathbb{R}\mathcal{F})$ and some orientation
Moreover we have $\gamma(G) = 0$. If $[L_+ (\mathbb{R} F)] = (m+2)g$, then for any orientation of $L(\mathbb{R} F)$ and some orientation of $L_{m,e}(\mathcal{F})$, $\gamma(G) = 1$.

(5) $\mathcal{F}$ is dividing if and only if $G$ is orientable. If $G$ is orientable, an orientation of $G$ extends the orientation on $L(\mathbb{R} F)$ induced by some complex orientation of $\mathbb{R} F$.

(6) There is a 1-1 correspondence between the $\mathbb{R}$-irreducible components of $\mathcal{F}$ and the components of $G$.

Moreover if $G$ is a generalized cobordism in $I \times Q$ from $L(\mathbb{R} F)$ to $L_{m,e}(\mathcal{F})$ satisfying (1),(2), and (3), then there is a floppy curve $\mathcal{F}$ transverse to $\Sigma$ of degree $m$ with real part $\mathbb{R} F$ with $G$ as its associated generalized cobordism.

The invariant $\gamma(G) \in H_1(Q)$ of condition (4) is defined in [G2,§2]. Recall $\gamma(G)$ one can calculate $[L_+(\mathbb{R} F)]$ as follows. First one resolves each double point of $\mathbb{R} F$ as in Figure 2a or 2b. Let $C$ be the resulting curve. Then $[L_+(\mathbb{R} F)] = ag$ where $a$ is twice the number of flops in $C$ plus twice the number of 2-sided components of $C$ plus the number of proper components minus the number of thick one-sided components. Note there can be at most one one-sided component in $C$.

If one applied the generalized Tristram-Murasugi inequalities [G2,8.3] to the generalized cobordism $G$ given above one would probably obtain the results which we describe in the next section. At least this is what happens in the case $F$ is a nonsingular real algebraic curve [G3]. Similarly applying [G1,8.5] to $G$ leads to a generalization of the Viro-Zvonilov inequalities. We are working on deriving a further generalization of the Viro-Zvonilov inequalities by generalizing the method of sections six and later of this paper. In the case where $\nu = 0$, and $G$ is nearly planar (this means Proposition (1.2) is nearly sharp), we may apply [G2,Theorem (6.7)] on the Arf invariant of links. For nonsingular curves, we obtain [G3], the Gudkov-Rokhlin congruence and related congruences. Presumably one should also be able to obtain some of the results of [KV] in this way. In general one may apply the results of [G2] to the link cobordisms arising from components of the inverse image of $G$ in $I \times \tilde{Q}$ where $\tilde{Q}$ is any one of the five nontrivial covering spaces of $Q$.

In particular if $\mathcal{F}$ is dividing, we have a generalized cobordism $G_+ in I \times \tilde{Q}$ between $L_+(\mathbb{R} F)$ to $L_{+,m,e}(\mathcal{F})$. By the proof of (1.2), if $h(\mathcal{F}) = 0$, $n(\mathcal{F}) = c(\mathcal{F})$, and $\nu(\mathcal{F}) = 0$, then $G_+ is planar. We may then apply [G2,Theorem (6.8)] which asserts that the Arf invariant of proper links (with extra data) is an invariant of planar cobordism. Recall that $\tilde{Q}$ is oriented diffeomorphic to $L(4,3)$. Its first homology is generated by a class $g$ which is represented by $L_1$. If $L$ is a link in $\tilde{Q}$, and $\gamma \in H_1(\tilde{Q})$ such that $[L] = 2\gamma$, and $q$ is a quadratic refinement of the linking form $\ell$ on $H_1(\tilde{Q})$, then $(L, \gamma, q)$ is proper if for each component $K$ of $L$, $q([K]) + \frac{\ell_k(K,L-K)}{2} \equiv \ell([K], \gamma) \pmod{1}$. In [G2,§6], we defined an Arf invariant for proper triples $(L, \gamma, q)$. It takes values in $\mathbb{Q}/8\mathbb{Z}$. Let $q_\pm$, respectively $q_\mp$ denote the quadratic refinement of the linking form on $H_1(\tilde{Q})$ which sends $g$ to $-1/8$, respectively $3/8$.

**Proposition (3.2).** The triples $(L_+(\mathcal{L}_{2k,e}), kg, q_\pm)$, $(L_+(\mathcal{L}_{2k,e}), (k+2)g, q_\pm)$ are proper. The triples $(L_+(\mathcal{L}_{2k,e}), kg, q_\pm)$, $(L_+(\mathcal{L}_{2k,e}), (k+2)g, q_\pm)$ are not proper. Moreover we have

$$Arf(L_+(\mathcal{L}_{2k,e}), kg, q_\pm) = \frac{k^2}{2} = Arf(L_+(\mathcal{L}_{2k,e}), (k+2)g, q_\pm) + 2$$
Proof. The results on properness follow easily from the definition. By [G2,(6.5)] we need only work out \( \text{Arf}(L_+(\mathcal{L}_{2k},\epsilon),kg,q_{-\frac{1}{8}}) \). There is a spanning surface \( F_{2k,\epsilon} \) consisting of \( k + \epsilon \) annuli with \( \gamma = kg \). Each annuli is given by a pencil of lines joining two distinct lines. The Brown invariant \( \varphi \) \( F_{2k,\epsilon} \) must vanish [G2,5.8]. Thus we are left with
\[
\text{Arf}(L_+(\mathcal{L}_{2k},\epsilon),kg,q_{-\frac{1}{8}}) = \frac{1}{2}e(F_{2k,\epsilon}) - \lambda(L_{2k,\epsilon}).
\]
One calculates that \( e(F_{2k,\epsilon}) = k + \epsilon, \lambda(L_{2k,\epsilon}) = \frac{e-k(2k-1)}{4} \).
□

Theorem (3.3). Let \( F \) be a floppy curve of degree \( 2k \). Suppose that \( h(F) = 0, n(F) = c(F), \) and \( \nu(F) = 0 \). If \( \mathbb{R}F \) is equipped with one of its complex orientations and \( (L_+(\mathbb{R}F),kg,q_{-\frac{1}{8}}) \) is proper then
\[
\text{Arf}(L_+(\mathbb{R}F),kg,q_{-\frac{1}{8}}) = \frac{k^2}{2} = \text{Arf}(L_+(\mathbb{R}F),(k + 2)g,q_{\frac{1}{8}}) + 2.
\]

Proof. Note that by (1.2), \( F \) is dividing. Also by the proof of (1.2), \( \tilde{G} \) is planar. Therefore the cobordism \( G \) of (3.1) is planar as well. By (6.8) of [G2],
\[
\text{Arf}(L_+(\mathbb{R}F),kg,q_{-\frac{1}{8}}) = \text{Arf}(L_+(\mathbb{L}_{2k,\epsilon},kg,q_{-\frac{1}{8}})).
\]
□

When Theorem (3.3) is applied to nonsingular curves, one obtains Fiedler’s congruences [F1]. This will be discussed in [G4]. One would expect that Fiedler’s congruences [F2] for certain flexible nodal curves could also be obtained but we have not verified this. Nevertheless the example given below shows that Theorem (3.3) applies in some situations where [F2] does not.

Theorem (3.4)(Shustin). There is no nonsingular real algebraic curve of degree eight with isotopy type \( 1 < 2 \prod 1 < 18 > > \).

Proof. We let \( \mathbb{R}F \) denote such a curve and seek a contradiction. Pick a point \( P \) interior to one of the empty odd ovals. Consider the pencil of lines through \( P \) as it sweeps across the nonempty odd ovals. As the lines in the pencil pass from one empty even oval to the next either the other empty odd oval is not encountered or it is. In the first case one can find simple membranes joining consecutive ovals. In the second case one can find two membranes of the type pictured in Figure 5 joining the empty even ovals to the empty odd oval. Depending on where the empty odd oval appears either all the even empty ovals may be partitioned into pairs of ovals joined by a simple membrane (Case 1) or all but two of the can paired in this way (Case 2).

In Case (1) by Fiedler’s theorem [F3] [V2,(4.9)], half of the empty even ovals are oriented one way and half the other. By Rokhlin’s formula [V2,3.13], the only possible complex orientation for the remaining ovals is as pictured in Figure 8a. This orientation may be excluded by Paris’s extremal property of Rokhlin’s inequality [Pa1,Pa2]. However to illustrate our methods we now prohibit this orientation using Theorem (3.3), just as we did in a preliminary version of this paper written before
Perform singular ambient surgery along these membranes and obtain a floppy curve $\mathcal{F}$ with $\mathbb{R}\mathcal{F}$ given by Figure 8a with nine additional figure eights floating inside one of the odd ovals. Then $L_+ (\mathbb{R}\mathcal{F})$ is given by this same picture. But note that each component of $\mathbb{R}\mathcal{F}$ which is given by a figure eight is actually a small unknot unlinked from the rest of the link. This is because when traveling around a figure eight ones tangent vector does not point in every direction. Thus the link given by a figure eight can be described by the same diagram in a 3-ball and so bounds a disk disjoint from the rest of the link. Let $L_1$ be the link in $\hat{Q}$ which is defined by Figure 8a. $L_1$ is proper for all $(\gamma, q)$. $L_+ (\mathbb{R}\mathcal{F})$ and $L_1$ will have the same Arf invariants for all $(\gamma, q)$. We can find a vector field on the shaded region of Figure 8b tangent to boundary along the boundary and pointing in the direction of the orientation of the boundary along the boundary which has with a single zero of index minus 2 in the interior. Let $F_1$ denote this region blown up at this zero. In other words a neighborhood of this zero is replaced by a Mobius band with core say $\alpha$. As in [G3], [G4], the vector field then defines and embedding of $F$ into $\hat{Q}$ with the boundary of $\mathcal{F}$ given by $L_1$. $\alpha$ is a fiber of the bundle $\hat{Q} \rightarrow \mathbb{RP}(2)$ and so represents $2g \in H_1 (\hat{Q})$. We have that $\gamma (F) = 2g$, $e(F) = -4$, $\lambda (L) = -3$, $F$ capped of is just an $\mathbb{RP}(2)$. One calculates that $\varphi_{\hat{Q}}^{F_1} (\alpha) = +1$. See [G2,§6], [G4]. So $\beta (\varphi_{\hat{Q}}^{F_1}) = 1$. Thus $Arf (L_+ (\mathbb{R}\mathcal{F}), 2g, q_{\hat{Q}}) = Arf (L_+ (L_1), 2g, q_{\hat{Q}}) = 2$. This contradicts Theorem (3.3).

We now consider Case (2). We first observe that the arrangement in Figure 9a is ruled out by Bezout’s Theorem and a conic. By Fiedler’s theorem [F3] [V2,(4.9)], the orientations of all but two empty even ovals alternate and these two ovals must be oriented opposite to the empty odd oval which does not include $P$. By Rokhlin’s formula [V2,3.13], there are only two only possible complex orientation for the remaining ovals as pictured in Figures 9b and 9c. However the one shown in 9b is shown to be impossible if we consider a pencil of curves through a point inside the other empty odd oval instead. Thus we have the complex orientation as in Figure 9c.
We perform eight simple surgeries and two of the type illustrated in Figure 5. Let \( \mathcal{F} \) be the result. \( \mathbb{R}\mathcal{F} \) with eight figure eights deleted is pictured in Figure 10a. \( L_+ (\mathbb{R}\mathcal{F}) \) is then the oriented link \( L_2 \) in \( \tilde{Q} \) given by Figure 10a together with eight unknots.

Figure 10 illustrates a sequence of isotopic links in \( \tilde{Q} \). From Figure 10d we see that \( L_2 \) is proper for all \( (\gamma, q) \). \( L_+ (\mathbb{R}\mathcal{F}) \) and \( L_2 \) will have the same Arf invariants for all \( (\gamma, q) \). Shade Figure 10d according to a “checkerboard” pattern, and then pick a vector field on the shaded region tangent to boundary along the boundary and pointing in the direction of the orientation of the boundary along the boundary which has a single zero of index minus 2 in the interior. As above this determines a spanning surface \( F_2 \) for \( L_2 \). We have that \( \gamma (F_2) = 0, e(F) = 0, \lambda (L_2) = -3, \beta (\phi_{q-1}^F) = 1 \). Thus \( \text{Arf}(L_+ (\mathbb{R}\mathcal{F}), 0, q_{\downarrow}) = \text{Arf}(L_+ (L_2), 0, q_{\downarrow}) = 4 \). This contradicts Theorem (3.3). \( \square \)

Remark \( L_+ (1 < 2 \big| 1 < 18 >> \) (with any orientation)) is not proper for any \( (\gamma, q) \) because the components covering the empty even ovals are improper. A simple surgery and also the surgery of the type of Figure 5 change the links in \( Q \) and \( \tilde{Q} \) corresponding to the real parts of the floppy curves by band moves. The band moves associated to a simple surgery are the band moves which undo the band moves described in Figure 2a. The band moves associated to the surgery of the type of Figure 5 are described by the line field with integral curves pictured in Figure 11. In each case the surgery to \( \mathcal{F} \) leads to a single band move to \( L(\mathbb{R}\mathcal{F}) \).
and $L_+(\mathbb{R}F)$ and a double band move to $\tilde{L}(\mathbb{R}F)$. Thus by [G2, 6.10] the links $L_1$ and $L_2$ are guaranteed to be proper. Thus ambient surgery sometimes allows us to move from a situation where our results on Arf invariants do not apply directly to one where they do apply.

![Figure 11](image)

§4 Generalized Viro-inequalities for floppy curves of even degree

For the rest of this paper we assume that the degree of $F$ is $2k$, and that $F$ is connected. In this section we generalize the inequalities that Viro originally found for nodal curves so that they apply to floppy curves. By a region $R$ of $\mathbb{R}F$ we will mean a connected region of $\mathbb{R}P(2) - \mathbb{R}F^*$. A point $c$ on the boundary of $R$ which is a double point of $\mathbb{R}F$ is called a corner of $R$. A corner of $R$ is called simple if it has an open neighborhood which intersects $R$ in a connected set. If $\mathbb{R}P(2) - \mathbb{R}F$ is nonorientable, we let $w$ be a fixed orientation reversing simple closed curve in $\mathbb{R}P(2) - \mathbb{R}F$. If $\mathbb{R}P(2) - \mathbb{R}F$ is orientable, we pick a fixed orientation reversing simple closed curve $w$ on $\mathbb{R}F$. A non-simple corner of $R$ is said to cross $w$ if a neighborhood of the corner intersected with the closure of $R$ is disconnected by its intersection with $w$.

Let $s(R)$ be the number of simple corners of $R$. If $k$ is even, we let $p(R)$ the number of non-simple corners of $R$ which do not cross $w$. If $k$ is odd we set $p(R)$ to be the number of non-simple corners of $R$. Let $\iota(R)$ denote the number of isolated singularities of $\mathbb{R}F$ in the interior of $R$. Let $O(R)$ denote the number of flops on the boundary of $R$ which point out of $R$ minus the number of flops which point into $R$. Let

$$\langle R, R \rangle = -2\chi(R) + \frac{s(R)}{2} + 2p(R) + 2\iota(R) + O(R)$$

Note our regions $R$ are open. If $R$ and $R'$ are two distinct regions of $B^\pm$ and $k$ is odd, let $\langle R, R' \rangle$ denote one half the number of points in the intersection of the closures of $R$ and $R'$. If $k$ is even, $\langle R, R' \rangle$ is one half the difference of the number of intersection points which do not cross $w$ and the number of intersection points which do cross $w$. We call a region $R$ relevant if it is orientable or if $k$ is odd. Let $M^\pm$ denote the matrix whose rows and columns are indexed by the relevant regions of $B^\pm$ with $R, R'$ entry $\langle R, R' \rangle$. 
Given a matrix $M$, let $\sigma_\pm(M)$ denote the number of positive, respectively negative eigenvalues of $M$. Let $\eta(M)$ denote the nullity of $M$. A vector $\vec{k}$ with odd integer entries such that $M^\pm \cdot \vec{k} = 0$ is called a kernel vector for $M^\pm$. Let $\epsilon(F)$ be one if $F$ has a $\mathbb{C}$-irreducible component of odd degree and be zero otherwise.

For a nodal curve $\mathbb{R}F$ one lets $B^+$ and $B^-$ denote respectively the subsets of $\mathbb{R}P(2)$ where $F$ is greater than or equal to zero and less than or equal to zero. We need to define analogous regions for a floppy curve. Consider the new curve, say $\mathcal{R}f$ consisting of simple closed curves obtained if we “resolve” each double point of $\mathbb{R}F^*$ as in Figure 2. We make an arbitrary choice at each crossing. $\mathcal{R}f$ will be null-homologous as well and thus it can contain no 1-sided curve. Now we can color the components of the complement of $\mathcal{R}f$ in $\mathbb{R}P(2)$ either black or white such that each component of $\mathcal{R}f$ is on the boundary of exactly one black region. There are exactly two such colorings. Then it is possible to pick a related coloring of the complement of $\mathcal{R}f$ in a more or less obvious way. Let $B^{++}$ denote the closure of the black region and $B^{+-}$ denote the closure of the white regions. Let $I^\pm$ denote the isolated points of $\mathbb{R}F$ in $B^{\pm}$, and let $B^\pm = B^{\pm^{+}} \cup I^\pm$. Let $O^\pm(\mathcal{F})$ denote the number of flops in the line field which point out of $B^\pm$ minus the number which point into $B^\pm$. Note $O^+(\mathcal{F}) = -O^-(\mathcal{F})$. Let $\nu^\pm(\mathcal{F})$ denote the cardinality of $I^\pm(\mathcal{F})$. Let $\nu(\mathcal{F}) = \nu^+(\mathcal{F}) + \nu^-(\mathcal{F})$. Of course if we had made the opposite choice of original coloring, the roles of $B^\pm$ would be reversed. Note that in (4.1$^\pm$), (4.2$^\pm$), and (5.1$^\pm$), the roles of these regions is completely symmetric.

From now on we will abbreviate $d_+(\mathcal{F})$ by $d_+$, $\nu^\pm(\mathcal{F})$ by $\nu^\pm$, $\Delta(\mathcal{F})$ by $\Delta$ etc. Let

$$\mathcal{E}^\pm = \text{Max}\{0, \eta(M^\pm) - n + \epsilon\}.$$

**Theorem (4.1$^\pm$).** We have:

$$\begin{align*}
(1^\pm) & \quad \sigma_+(M^\pm) + \mathcal{E}^\pm \leq (k-1)(k-2)/2 - \frac{\Delta - O^\pm}{4} \\
(2^\pm) & \quad \sigma_-(M^\pm) + \mathcal{E}^\pm \leq 3k(k-1)/2 + \chi(B^\pm) - \frac{\Delta + O^\pm}{4} + \frac{r - \nu - d_+ + d_-}{2}
\end{align*}$$

For nonsingular curves Equations (1$^\pm$) are the strengthened Arnold inequalities and Equations (2$^\pm$) are the strengthened Petrovskii inequalities. One obtains from the Addenda given below exactly the same extremal properties for these inequalities as are given in [W].

For nodal real algebraic curves, Theorem (4.1$^\pm$) is almost the same as a theorem Viro announced in his abstract [V3], which he only states in complete detail for $k \equiv 1 \pmod{2}$. There is a misprint in the formula which corresponds to Equation (2$^\pm$): $-2\chi(\Gamma)$ should read $-2\chi(\Gamma')$. Our $\mathcal{E}^\pm$ is sometimes a little bigger than the corresponding term in [V3]. Viro was aware that this could be improved. The left hand side of Equation (2$^\pm$) is smaller by $\nu$ than the equation in [V3]. We improved our earlier version of Lemma (9.6) below after reading [Fi]. This lead to an improvement in (4.1) so that our version of the Viro-inequalities for nodal curves became the same as Finashin’s.

We will say $\mathcal{F}$ is 2-irreducible, if $\mathcal{F}$ is either strongly-irreducible, or if $n(\mathcal{F}) = 2$ and each $\mathbb{C}$-irreducible component of $\mathcal{F}$ has odd degree. If $\mathcal{F}$ is 2-irreducible these inequalities can sometimes be improved. The proof we give uses an adaptation of
a lemma of Viro and Zvonilov [ZV] that we give below as Lemma (7.3), as well as arguments from [W]. Consider the equations:

\[
3^\pm \quad \sigma_+(M^\pm) + \eta(M^\pm) \leq (k-1)(k-2)/2 - \frac{\Delta - O^\pm}{4}
\]

\[
4^\pm \quad \sigma_-(M^\pm) + \eta(M^\pm) \leq 3k(k-1)/2 + \chi(B^\pm) - \frac{\Delta + O^\pm}{4} + \frac{r - \ell - d_+ + d_-}{2}
\]

These are just Equations (1\pm) and (2\pm) with \(E^\pm\) replaced by \(\eta(M^\pm)\). We have:

**Addendum (4.2\pm)**. Suppose that \(\mathcal{F}\) is 2-irreducible, then \((3^\pm)\) and \((4^\pm)\) can fail by at most one. If either of \((3^\pm)\) or \((4^\pm)\) fail then all of the following conditions hold:

(a) \(\mathbb{R}\mathcal{F}\) is dividing

(b\pm) All of the regions of \(B^\pm\) are relevant

(c\pm) There is a kernel vector for \(M^\pm\)

Moreover, if \(\mathcal{F}\) is strongly-irreducible, and either of \((3^\pm)\) or \((4^\pm)\) fail to hold then both of \((3^\pm)\) and \((4^\pm)\) will hold.

The hypothesis that \(\mathcal{F}\) is strongly-irreducible in the last line of (4.2) is necessary as the example of two transverse real lines shows.

For the next addendum, we need to specify precisely which region is \(B^+\), in the case where every constituent on \(\mathbb{R}\mathcal{F}\) is null-homologous in \(\mathbb{R}P(2)\). Suppose one assigns orientations arbitrarily to the constituents of \(\mathbb{R}\mathcal{F}\). Choose \(B^+\) so that \(\text{Int}(B^+)|_{\mathbb{R}\mathcal{F}^*}\) is homologous to twice a generator for \(H_1(\mathbb{R}P(2) - x)\). \(B^+\) does not depend on the arbitrary choice of orientations for the constituents. Regions of the complement of \(\mathbb{R}\mathcal{F}^*\), which can be joined by an arc which meets \(\mathbb{R}\mathcal{F}^*\) exactly once transversely are assigned opposite values by this procedure. Note that in the case of a nonsingular curve \(\mathbb{R}\mathcal{F}\), this agrees with the usual convention than \(B^-\) should be non-orientable. Moreover if we smooth the intersections compatibly with the orientation, then we will obtain a disjoint collection of ovals. Then the \(B^+\) for this collection of ovals determines a choice of \(B^+\) for \(\mathbb{R}\mathcal{F}\). This is the same \(B^+\) defined above.

**Addendum (4.3)**. Suppose that \(\mathcal{F}\) is strongly-irreducible, every immersed closed curve on \(\mathbb{R}\mathcal{F}\) is null-homologous in \(\mathbb{R}P(2)\), and we have chosen \(B^+\) by the above scheme. If either \(3^+\) or \(4^+\) does not hold then \(k\) is even. If either \(3^-\) or \(4^-\) does not hold then \(k\) is odd.

We also have the following proposition which follows from Equation (21) in §6, and the fact that each constituent of \(\mathbb{R}\mathcal{F}^*\) has an even number of flops.

**Proposition (4.4)**. If \(\mathcal{F}\) is a floppy curve of degree \(2k\), then \(\Delta(\mathcal{F}) \equiv O^\pm (\text{mod } 4)\). Also \(O^\pm\) is even. So \(\Delta(\mathcal{F})\) is even, and \(\chi(\mathcal{F}) \equiv d(\mathcal{F}) (\text{mod } 2)\).

Suppose we have found some simple membranes for a scheme for a hypothetical complex curve \(\mathbb{C}\mathcal{A}\). We can either follow Viro’s method of looking at the homology classes in the double branched cover of \(\mathbb{C}\mathcal{A}\) given by the inverse image of the membrane, and then follow the proofs of Arnold and Petrovskii inequalities. Alternatively we could perform simple surgery to obtain \(\mathcal{F}\) and apply (4.1), (4.2), (4.3) or (5.1) to \(\mathcal{F}\). Preliminary investigations indicate that one will get equivalent results either way. Note this is only for simple membranes as in Figure 4.
§5 The Determinant of $2M^\pm$

The idea for the further conditions in this section came from the determinant or discriminant conditions mentioned in [S1,§2] and [KS,5.1]. Let $\rho^\pm(\mathcal{F})$ denote the number of relevant regions in $B^\pm$. Let $\pi^\pm(\mathcal{F})$ be the number of non-relevant regions in $B^\pm$. There can be at most one such region. Let

$$h^\pm(\mathcal{F}) = 3 + r + \chi(B^\pm) + \frac{d - \chi(F)}{2} - 2\rho^\pm - 2\iota^\pm - 2\pi^\pm,$$

(5)

$$P^\pm(\mathcal{F}) = 1 + \chi(B^\pm) + \frac{r - \iota - \chi(F)}{2}.$$

(6)

**Theorem (5.1±).** Let $\mathcal{F}$ be 2-irreducible. We have that $h^\pm(\mathcal{F})$ is a nonnegative integer. If $\rho^\pm(\mathcal{F}) = P^\pm(\mathcal{F})$ then, $|\det(2M^\pm)| = b^2h^\pm + \rho^\pm - d$ for some integer $b$.

The proof of this theorem will be given in §10. The following observation follows easily from the material in §10: in the situation where $\eta(M^\pm) = 0$, one has that $\rho^\pm(\mathcal{F}) \leq P^\pm(\mathcal{F})$ and moreover one has that: $\rho^\pm(\mathcal{F}) = P^\pm(\mathcal{F})$ if and only if Equations (3±) and (4±) are both sharp. The inequality $h^\pm \geq 0$ when applied to a nonsingular real algebraic curve, where we choose $B^-$ to be the nonorientable region, becomes Harnack’s inequality. The only nonsingular real algebraic curve where either of $\rho^\pm(\mathcal{F}) = P^\pm(\mathcal{F})$ holds is the degree two curve consisting of a single oval. The nodal real algebraic curve of degree four constructed by perturbing some of the singularities in a collection of four lines illustrated in Figure 12 is a simple example of a curve where $\rho^+(\mathcal{F}) = P^+(\mathcal{F})$ holds. As yet we have no interesting prohibitions arising from Theorem (5.1).

![Figure 12](image-url)

§6 The double branched cover of $CP(2)$ along $\mathcal{F}$

Let $\mathcal{F}$ be the $C$-irreducible components of $\mathcal{F}$, and $[\mathcal{F}_i] = m_i[CP(1)] \in H_2(CP(2))$. Let $N$ denote a regular neighborhood of $\mathcal{F}$ in $CP(2)$ which is invariant under $T$. By studying the long exact sequence of the pair, $(CP(2), CP(2) - N)$, one can show that there is a isomorphism from $H_1(CP(2) - \mathcal{F})$ to $\mathbb{Z}^n/a\mathbb{Z}$ which sends the meridians of $\mathcal{F}_i$ to $e_i$ for each $i$, where $a = \sum m_i e_i$. Since, $\sum m_i$ is even, there is a unique homomorphism $\phi : H_1(CP(2) - \mathcal{F}) \rightarrow \mathbb{Z}_2$ sending the meridians of each $\mathcal{F}_i$ to the residue class of one. Note $T|_{CP(2) - N} \circ \phi = \phi$. We now let $Y$ denote the associated branched cover of $CP(2)$. $Y$ is a manifold except above the double points of $\mathcal{F}$. As the double branched cover of the Hopf link is $RP(3)$ [R,p302], the points above the double points have neighborhoods which are cones on $RP(3)$. So if $Y$ is a rational homology manifold and satisfies Poincare duality with rational coefficients. Let $\theta$ denote the covering involution on $Y$ and $\pi$ denote the covering map from $Y$ to $CP(2)$. There are two the involutions on $Y$ which cover $T$. One will
have \( \pi^{-1}(B^+) \) as fixedpoint set. The other will have have \( \pi^{-1}(B^-) \) as fixedpoint set. We name the first \( T^+ \) and the other \( T^- \). Thus \( T^- \) is the composition of \( T^+ \) and \( \theta \). Thus \( Y \) is a space with a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) action. Let \( Y^\pm = \pi^{-1}(B^\pm) \), the fixedpoint set of \( T^\pm \). The orbit space of this action, \( \mathbb{C}P(2) \) modulo complex conjugation is a smooth manifold (actually the 4-sphere). We can give a Whitney stratification of this orbit space so that the images of \( F \) and \( \mathbb{R}P(2) \) are unions of strata. Thus we can triangulate the 4-sphere so that the images of \( F \) and \( \mathbb{R}P(2) \) are sub-complexes [GM, p37 and references therein]. If we then lift this triangulation to \( Y \), we have a simplicial \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) action on \( Y \). By counting cells we see that

\[
\chi(Y^+) + \chi(Y^-) = 2\chi(\mathbb{R}P(2)) = 2.
\]

Because \( F \) is connected, the long exact sequence for the pair \((\mathbb{C}P(2), F)\) shows that \( \beta_1(\mathbb{C}P(2), F, \mathbb{Z}_2) \) is zero. Thus by the sequence of Lee and Weintraub [LW, Thm 1], \( \beta_1(Y, F, \mathbb{Z}_2) \) is zero. Here we could as well use the Smith theory exact sequence given in [W, Appendix]. The universal coefficient theorem then gives that \( \beta_1(Y, F) \) is zero. So by the long exact sequence of the pair \((Y, F)\), \( \beta_1(Y) \) is zero. By duality with rational coefficients \( \beta_3(Y) \) also vanishes.

Let \( aE_\beta \) denote the subspace of \( H_2(Y, \mathbb{Q}) \) consisting of those elements in both the \( \alpha \) eigenspace of \( T^- \) and the \( \beta \) eigenspace of \( T^+ \), where \( \alpha, \beta \) belong to the set \( \{-1, 1\} \). These subspaces are orthogonal with respect to the intersection form on \( H_2(Y, \mathbb{Q}) \). Following Wilson [W], we let \( aE_\beta^+ \) denote maximal subspaces of \( aE_\beta \) on which the intersection form is respectively positive and negative definite. The +1 eigenspace for \( \theta \) is \( -1E_{-1} \oplus +1E_{+1} \) can be identified by the transfer with \( H_2(\mathbb{C}P(2), \mathbb{Q}) \). Since \( T \) acts by multiplication by minus one on \( H_2(\mathbb{C}P(2), \mathbb{Q}) \), the intersection form is positive definite on \( H_2(\mathbb{C}P(2), \mathbb{Q}) \), and \( \beta_2(\mathbb{C}P(2) = 1 \), we conclude that \( \dim(-1E_{-1}^+) = 1 \), and \( \dim(-1E_{-1}^-) = +1E_{+1}^- = 0 \). Thus \( H_2(Y, \mathbb{Q}) \) is written as a direct sum of five spaces:

\[
-1E_{+1}^+ \oplus -1E_{+1}^- \oplus +1E_{-1}^+ \oplus +1E_{-1}^- \oplus -1E_{+1}^-.
\]

Let \( a, b, c, \) and \( d \) denote the dimensions of the first four of these in order. Recall that the signature, \( \text{Sign}(J) \) of an involution \( J \) on a rational homology 4-manifold \( X \) is the difference of the signature of the intersection form restricted to the +1 eigenspace of \( H_2(X, \mathbb{Q}) \) and the signature of the intersection form restricted to the −1 eigenspace of \( H_2(X, \mathbb{Q}) \). We also let \( L(J) \) denote the Lefschetz number of \( J \).

We have:

\[
-(a + b) - (c + d) + 1 + 2 = L(\theta)
\]

(8)

\[
-(a - b) - (c - d) + 1 = \text{Sign}(\theta)
\]

(9)

\[
(a + b) - (c + d) - 1 + 2 = L(T^+)
\]

(10)

\[
(a - b) - (c - d) - 1 = \text{Sign}(T^+).
\]

(11)
We now solve these equations obtaining:

\[(12) \quad a = (1/4)[L(T^+) - L(\theta) + \text{Sign}(T^+) - \text{Sign}(\theta)] + 1\]

\[(13) \quad b = (1/4)[L(T^+) - L(\theta) - \text{Sign}(T^+) + \text{Sign}(\theta)]\]

\[(14) \quad c = (1/4)[-L(T^+) - L(\theta) - \text{Sign}(T^+) - \text{Sign}(\theta)] + 1\]

\[(15) \quad d = (1/4)[-L(T^+) - L(\theta) + \text{Sign}(T^+) + \text{Sign}(\theta)] + 1.\]

If \(x\) and \(y\) are 2-dimensional homology classes in a closed rational homology 4-manifold, we denote their intersection number by \(x \circ y\). According to Theorem (7.1) and Lemma (7.2) in the next section, we have that

\[(16) \quad \text{Sign}(\theta) = (1/2)[\mathcal{F}] \circ [\mathcal{F}] - d_+ + d_- = m^2/2 - d_+ + d_- .\]

By Lemma (9.5), we will show:

\[(17) \quad \text{Sign}(T^+) = -\chi(Y^+) + O^+.\]

Using the oriented simplicial homology of \(Y\) [Sp.p.159], and the Hopf trace formula [Sp.p.195], we see that:

\[(18) \quad L(\theta) = \chi(\mathcal{F})\]

\[(19) \quad L(T^+) = \chi(Y^+).\]

By Equation (*) of §1, we have

\[(20) \quad \chi(\mathcal{F}) = 3m - m^2 + d_+ - d_- + \Delta.\]

Substituting (16)–(19) in (11)–(15), and then substituting (20),(7) and \(m = 2k\) yields:

\[(21) \quad a = (k - 1)(k - 2)/2 - (\Delta - O^+)/4\]

\[(21') \quad c = (k - 1)(k - 2)/2 - (\Delta + O^+)/4\]

\[(22) \quad b = 3k(k - 1)/2 + (\chi(Y^+) - d_+ + d_-)/2 - (\Delta + O^+)/4\]

\[(22') \quad d = 3k(k - 1)/2 + (\chi(Y^-) - d_+ + d_-)/2 - (\Delta - O^+)/4\]

Counting cells we see that:

\[(23) \quad \chi(Y^\pm) = 2\chi(B^\pm) + r(\mathbb{R}\mathcal{F}) - \iota(\mathbb{R}\mathcal{F})\]

We will estimate the ranks of the \(\alpha E^\pm_\beta\) from below using the homology classes residing on \(Y^\pm\) to obtain Theorems (4.1) and (4.2).
§7 Nice Involutions

In this section, we describe a class of nice involutions, and prove a version of the G–Signature Theorem for them. We will see that the involutions $\theta$ and $T^\pm$ on $Y$ are nice. Let $M'$ be a compact smooth oriented 4–manifold with boundary a disjoint union of $RP(3)$’s. Let $M$ be the compact oriented rational homology 4–manifold obtained by attaching the cone on $RP(3)$ to each boundary component. Let $T'$ be an involution on $M'$ whose fixed points set is a properly embedded (not necessarily orientable or connected) surface $F'$. We assume that $T'$ acts on the boundary in the following specified way. $T'$ may permute some pairs of boundary components. Each of the boundary components invariant under $T'$ is oriented diffeomorphic as a manifold with action to either the double branched cover of the Hopf link in $S^3$, the double covering of $L(4,1)$ or the double covering of $L(4,3)$. Then we may extend $T'$ to an action $T$ on $M$ by coning off the action on each boundary component which is not permuted and permuting the cones on the boundary components which are permuted. We will call any involution which can be constructed in this way nice.

Each boundary component of $M'$ with a free involution will contribute one isolated point to the fix point set of the involution on $M$. Let $f_\pm$ denote respectively the number of boundary components of $M'$ oriented as boundary of $M'$ with action the double covering of $L(4,1)$, respectively $L(4,3)$. Note the oriented boundary of a neighborhood of a fixed point contributing to $f_+$ is then the double covering of $L(4,3)$. Let $F$ denote the fixedpoint set of $T$ less the isolated fixed points set. We let $\bar{M}$ denote the orbit space of the involution on $M$, and $\bar{F}$ denote the image of $F$ in $\bar{M}$. $\bar{M}$ is also a rational homology manifold. Its only singularities are cones on $RP(3)$ or $\pm L(4,1)$. $\bar{F}$ is the image of a immersed surfaces with only ordinary double point singularities. Note $\bar{F}$ lies completely in the nonsingular part of $\bar{M}$. We can view the pair $(M,T)$ as the double branched cover of $\bar{M}$ along the collection of immersed surfaces $\bar{F}_i$. We let $e(\bar{F})$ denote the Euler number of the normal bundle of the immersion of $\bar{F}$ in $\bar{M}$.

**Theorem (7.1).** With $T$, $M$, and $\bar{F}_i$ as above:

\[(24) \quad \text{Sign}(T, M) = f_+ - f_- + (1/2) e(\bar{F}).\]

**Proof.** If there are no singularities in $M$, this is just a special case of the G–Signature Theorem [AS]. See [JO] for a topological proof of the G–Signature Theorem for involutions or [Go] for a topological proof of this theorem for 4-manifolds.

We deal first with the singularities of $M$ that are permuted in pairs by $T$. Let $P$ be some oriented smooth 4-manifold with boundary $RP(3)$. Replace neighborhoods of each singular point in $M$ with copies of $P$ to obtain a new action. It is easy to see the signature of the new involution is the same as the old. The left hand side of (7.1) is also unchanged. So it is enough to prove the theorem in the case there are no singularities permuted in pairs.

Next we deal with singularities of $\bar{M}$ which are cones on $L(4,3)$. We can replace a neighborhood of each such singularity with a disk bundle over a 2-sphere with Euler number four, and simultaneously replace a neighborhood of the point above it with a disk bundle over a 2-sphere with Euler number two. The involution extends over this new manifold. The signature of the new involution is one more than the old. The right hand side of (7.1) changes precisely the same way. We deal with singularities of $\bar{M}$ which are cones on $L(4,1)$ in a similar way: using a disk bundle
with Euler number minus four etc. Thus without loss of generality we may assume that their are no isolated fixed points or singularities permuted in pairs.

Form $M^-$ by deleting open neighborhoods of each singularities. Let $\tilde{M}^-$ be the orbit space. The boundary of $M^-$ is a collection of 3-spheres. $\tilde{F}$ meets each one in a Hopf link. Form $M^+$ by adding a pair of 2-handles to each 3-sphere along these Hopf links, the first two handle one with framing one and the other with framing minus one. Then the branched cover of $M^-$ extends uniquely to a branched cover of $\tilde{M}^+$ along $\tilde{F}^+$ which denotes $\tilde{F} \cap \tilde{M}^-$ union the cores of the 2-handles with covering transformation $T^+$. Above the branched cover of each pair of 2-handles, we have a pair of 2-handles which have been added to $M^-$ to form say $M^+$. Each time we add a pair of 2-handles we do not change the signature of the intersection pairing restricted to the $-1$ eigenspace. Nor do we change the signature of the pairing restricted to the $+1$ eigenspace or equivalently the signature of the orbit space by 1. This is because the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ has zero signature. We also have $e(\tilde{F}^+) = e(\tilde{F})$. Thus we only need to show:

$$\text{Sign}(T^+, M^+) = (1/2)e(\tilde{F}^+).$$

The boundary of $\tilde{M}^+$ consists of copies of $RP(3)$ and the boundary of $M^+$ consists a copy of the 3-sphere covering each of the $RP(3)$'s. The cover of $RP(3)$ by $S^3$ has an orientation reversing diffeomorphism, as $RP(3)$ does and the non-trivial cover is unique. Thus we may take two copies of $M^+$ and glue them together by an equivariant orientation reversing diffeomorphism on the boundary. In this way, we form $\bar{M}$ with involution $\bar{T}$. By Novikov additivity,

$$\text{Sign}(\bar{T}, \bar{M}) = 2 \text{Sign}(T^+, M^+).$$

$\bar{M}$ is a smooth closed manifold and $\bar{T}$ is smooth. We know the theorem holds in this case. Thus twice Equation (25) holds. \qed

If we take the double branched cover of the 4-disk over the cone of the Hopf link, we obtain a neighborhood above a double point in the above construction. The branch set in this case is two 2-disks which we can orient so that the double point is positive. Downstairs the oriented boundary of these two 2-disks is the Hopf link with linking number one. Upstairs the oriented boundary of the lifts of these two 2-disks is a link in $RP(3)$ with linking number one half. To see this push of parallel copies of both disks, they lift upstairs to two 2-disks each lies in a neighborhood of a disk of the branch set which double covers this disk. These two lifts have four positive transverse intersections. However the boundary of each disk is two parallel copies of a component of the boundary of the branch set.

This gives us a way to distinguish points lying above a positive double point of $\bar{G}$ and points lying above negative double point of $\bar{G}$, when $\bar{G}$ is oriented. If $G$ intersects the $RP(3)$ about the singularity in a link with linking number $1/2 (-1/2)$, the double point below the singularity is a positive (negative) double point of $\bar{G}$. In this situation, we will refer the double point of $G$ as a positive (negative) singular double point.

The $G$-Signature Theorem for involutions is normally stated in terms of the self intersection of the fixed point set. We will need to apply it in this form as well. If we assume that $G \subset F$ is the image of an immersion of a closed orientable surface
with ordinary double points, then we may distinguish between positive and negative
double points. Let $G \subset F$ be the corresponding subset of $F$. We let both $d(G)$ and
denote the number of double points of $G$ counted with sign.

**Lemma (7.2).** For $G$ orientable as above, we have:

\[
(1/2)e(G) = [G] \circ [G] - d(G) = (1/2)[\bar{G}] \circ [\bar{G}] - d(\bar{G}).
\]

**Proof.** $e(G)$ is a signed count of the number of zeros of a transverse section of
the normal bundle of the immersion. We push off a parallel copy $G'$ of $G$ using
such a section. For each positive (negative) index zero of the section, we have
one positive (negative) point of intersection between $G$ and $G'$. Near each positive
(negative) double point of the immersion we get an extra pair of positive (negative)
intersections. Thus $[G'] \circ [G] = 2d(\bar{G}) + e(\bar{G})$. Now if we consider $G'$ the lift of
$G'$ in $M$. We have $[G'] = 2[\bar{G}]$. On the other hand, we see that $[G'] \circ [G]$ is also
$2d(\bar{G}) + e(\bar{G})$. So

\[
([G] \circ [G] = (1/2)([\bar{G}] \circ [\bar{G}])
\]

The Lemma follows. \hfill \Box

If $\mathcal{M}$ is surface (possibly non-orientable) with boundary (we allow the boundary
to have corners) mapped into $\bar{M}$ such that $\partial \mathcal{M} \subset \bar{F}$, $\text{Int} \mathcal{M}$ is immersed in $M$
minus the singular set and $\text{Int} \mathcal{M}$ is transverse to $F$, we call $\mathcal{M}$ a membrane for $\bar{F}$.
Let $\mathcal{M}$ denote the inverse image of $\mathcal{M}$ in $M$. Let $H_2(M, \mathbb{Z}_2)^T$ denote the subgroup
of $H_2(M, \mathbb{Z}_2)$ which is fixed by $T$. We have that $H_2(\mathcal{M}, \mathbb{Z}_2)$ is one dimensional and
is fixed by $T$, so $\mathcal{M}$ defines a class $[\mathcal{M}] \in H_2(M, \mathbb{Z}_2)^T$. We will need the following
lemma to prove Addenda (4.2). Here we adapt an argument of Viro and Zvonilov's
[VZ §2.2] to our situation. We are working with a possibly nonorientable and
possibly singular branch set. Also $M$ may have singularities and is not necessarily
a $\mathbb{Z}_2$-homology manifold. However the first geometric argument in [VZ] does go
through.

**Lemma (7.3).** There is a well-defined map $\mathcal{I} : H_5(M, \mathbb{Z}_2) \to H_1(F, \mathbb{Z}_2)$, given by
intersection. There is a homomorphism $\mathcal{B}$ from $H_2(M, \mathbb{Z}_2)^T$ to
$H_1(F, \mathbb{Z}_2)/\text{Image} \mathcal{I}$. If $\mathcal{M}$ is a membrane for $\bar{F}$, $\mathcal{B}[\mathcal{M}] = [\partial \mathcal{M}]/\text{Image} \mathcal{I}.$

**Proof.** We will say a singular $n$-chain in $C_n(M, \mathbb{Z}_2)$ is good if it is a sum of singular
$n$-simplices of one of two kinds. The first kind has image in $M'$, is smooth, and
is transverse to $F$. The second kind is the join of a singular point with a singular
$(n-1)$-simplex of the first kind using the cone structure of neighborhoods. We can
pick a good 3-cycle $\alpha$ representing a class $\mathfrak{a} \in H_3(M, \mathbb{Z}_2)$. The intersections of the
images of its simplices with $F$ can be pieced together to form a 1-cycle in $F$. We
define $\mathcal{I}(\mathfrak{a})$ to be the homology class represented by this 1-cycle. If $\alpha'$ is another
good 3-cycle representing $\mathfrak{a}$, then $\alpha + \alpha'$ is a good cycle which must be the boundary
of a good 4-chain $\beta$. Note that $\beta$ intersected with $F$ provides a null-homology of
$\alpha + \alpha'$ intersect $F$. This shows that $\mathcal{I}$ is well-defined.

If $\mathfrak{g} \in H_2(M, \mathbb{Z}_2)^T$, we can pick $\gamma$, a good cycle representing $\mathfrak{g}$. Then $\gamma + T(\gamma)$ is
the boundary of a good 3-chain $\delta$. Then $\mathcal{B}(\mathfrak{g})$ is represented by the cycle obtained
by piecing together the intersection of the images of the simplices in $\delta$ with $F$.
If $\delta'$ is another 3-chain with boundary $\gamma + T(\gamma)$, then $\delta + \delta'$ is a 3-cycle whose
intersection with $F$ defines an element in the image of $I$. Thus the value of $B(g)$ does not depend on the choice of $\delta$.

To see that the choice of $\gamma$ is not important, let $\gamma'$ denote a second choice. Then $\gamma + \gamma'$ is the boundary of a good 3-chain $\zeta$. Then $\delta + \zeta + T(\zeta)$ is a 3-cycle with boundary $\gamma' + T\gamma'$. As $\zeta$ and $T(\zeta)$ have the same intersection with $F$, $B$ is well-defined. It is not hard to see $B[M] = [\partial M]/\text{image } I$. □

§8 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action above the real double points of $F$

First we need to understand the involutions $T^{\pm}$ around the singularities of $Y$ above the real double points of $\mathbb{R}F$. For this purpose we perform stereographic projection from a 3-sphere $S$ in $\mathbb{C}P(2)$ surrounding such a point to see how the graph $(\mathbb{R}F \cup \mathbb{R}P(2)) \cap S$ sits in $S$. This is an easy exercise. We may take local coordinates so the point is the origin in $\mathbb{C}^2$. $F$ is locally given by the graph of $z_1z_2 = 0$, and $T(z_1, z_2) = (\overline{z}_1, \overline{z}_2)$. The result is shown in Figure 13a. The inverse image $R$ of $S$ in $Y$ is thus the double branched cover of $S$ along a Hopf link $H$, and so is diffeomorphic to $RP(3)$. We first study the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $R$.

![Figure 13a](image1)

![Figure 13b](image2)

The orbit of the restriction of the involution $T$ to $S$ is a 3-sphere $\overline{S}$. Figure 13b shows $\Gamma$ the image of $(F \cup \mathbb{R}P(2)) \cap S$ in the orbit space of $T$. The image of $\mathbb{R}P(2) \cap S$ consists of the edges labeled $a$ and $b$. The image of $F \cap S$ consists of the edges labeled $c$. The result is a completely symmetrical embedding of the complete graph on four vertices. In fact it is isotopic to the standard embedding of a tetrahedron. There are both orientation preserving diffeomorphisms and orientation reversing diffeomorphisms of the 3-sphere which send the graph to itself and performs any given permutation of the vertices. To see this we make the following two observations. An orientation reversing diffeomorphism which performs any transposition is given by a reflection of the tetrahedron. The effect on the pictured graph of reflection in the plane of the circle composed of the edges labeled $a$ and $b$ can also be achieved by an isotopy.
The inverse image of $S - (\mathcal{F} \cup \mathbb{R}P(2))$ in $Y$ is a regular $\mathbb{Z}_2 \times \mathbb{Z}_2$ cover of the complement of $\Gamma$ in $\mathbb{S}^3$. This cover is classified by a surjection $\phi : H_1(\mathbb{S}^3 - \Gamma) \to \mathbb{Z}_2 \times \mathbb{Z}_2$. $R$ is the associated branched cover. By studying the relations coming from the 3–punctured spheres around each vertex of $\Gamma$, we conclude that the $\phi$ must send the meridian of an edge to its label where $a, b, c$ now the distinct nonzero elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$. The complete symmetry of embedding of $\Gamma$ implies that the involutions $T^\pm$ and $\Theta$ restricted to $R$ are all conjugate.

We also need to know certain linking numbers. $\mathbb{R}F \cap S$ consists of four points. Let $x$ and $y$ be two points of $(\mathbb{R}F \cap S)$ adjacent on $\mathbb{R}P(2) \cap S$ and $\gamma$ be the arc of $\mathbb{R}P(2) \cap S$ joining them. Consider a nonzero vector field on a neighborhood of $\gamma$ in $\mathbb{R}P(2)$ which is tangent to $\mathbb{R}F$ at $x$ and $y$ but points toward the central double point at $x$ and away from it at $y$ and is only tangent to $\gamma$ at exactly one point. If we multiply this vector field by $i$ it will be a normal vector field to $\mathbb{R}F$ and therefore tangent to $S$ along $\gamma$. Let $\gamma'$ be an arc in $S$ with boundary in $\mathbb{F}$ obtained by pushing $\gamma$ off itself with this vector field. Let $\alpha$ and $\alpha'$ denote the inverse images of $\gamma$ and $\gamma'$ in the double branched cover of $S$ along $\mathbb{R}F \cap S$. $\alpha$ is a simple closed curve in $R$ and $\alpha'$ is a push off of $\alpha$. Here the one parameter family of curves upstairs traced out during the push-off upstairs covers the one parameter family of arcs traced out by the push-off downstairs. If we assign an arbitrary orientation to $\alpha$ and the parallel orientation of $\alpha'$, then the resulting linking number of $\alpha$ and $\alpha'$ is well defined.

**Lemma (8.1).** $Lk(\alpha, \alpha') = \frac{1}{2}$.

**Proof.** To see this we pick local coordinates so that $\mathbb{R}F$ is the union of the coordinate axes, and take the vector field to be tangent to the family of hyperbola given by $xy = c$ is a constant. We parameterize $\gamma$ and $\gamma'$ and view the images of the resulting curves under stereographic projection. See Figure 14a. In Figure 14b, we have redrawn the Hopf link given by $z_1 = 0$, and $z_2 = 0$. We have also shaded the band between $\alpha$ and $\alpha'$ traced out by the one parameter family of arcs. We use the method in [R,p302] to see $\alpha$ and $\alpha'$ in $R$. If we describe $R$ as $-2$ framed surgery to the unknot, then $\alpha$ is a meridian to the unknot and $\alpha'$ is a parallel meridian. It is not hard to see the linking number is $1/2$. \[\square\]

We will also need to know another linking number. Consider a 4-ball $D$ neighborhood of our double point in $\mathbb{C}P(2)$ with boundary $S$. In $D$ there is an annulus $A$ invariant under complex conjugation with boundary $\mathcal{F} \cap S$ which one obtains by resolving the double point. $A \cap \mathbb{R}P(2)$ is a “hyperbola”. Let $\bar{N}$ be the double branched cover of $D$ along $A$. Consider the region $W \in N \cap \mathbb{R}P(2)$ pictured in Figure 15.

Let $\hat{W}$ be the inverse image of $W$ in the branched cover. $\hat{W}$ is an annulus. Give $\hat{W}$ an orientation then the boundary of $\hat{W}$ is an oriented link with two components. One of these components is $\alpha$ lying over $\gamma$. The other is say $\alpha^*$ lying over $\gamma^*$.

**Lemma (8.2).** $Lk(\alpha, \alpha^*) = 1/2$.

**Proof.** Let $L$ be the link in $R$ consisting of $\alpha$ and $\alpha^*$. The curve labeled $\gamma'$ in Figure 14a can be isotoped in $S - \gamma$ ($S$ is the boundary of $N$) keeping its boundary on our branching set $S \cap \mathcal{F}$ to the curve, $\gamma^*$, given by reflection of $\gamma$ through the origin in Figure 14a. Therefore $L$ is isotopic to the link consisting of $\alpha$ and $\alpha'$ with some orientation. However we have already seen that the linking number of the lift of $\alpha$ and $\alpha'$ with (a push-off) orientation is $1/2$. So $Lk(\alpha, \alpha^*) = \pm Lk(\alpha, \alpha^*) = \pm 1/2$. \[\square\]
We define a vector field $\vec{w}$ on $W$ which is tangent to $A \cap \mathbb{R}P(2)$ agrees with the vector field defined above along $\gamma$. Along $\gamma^*$, it is obtained by reflection through the central point. We extend the vector field to the interior of $W$, so that it has a single zero of order minus one. We push $W$ off itself with $i\vec{w}$ to obtain $W'$. Let $W'$ denote inverse image of $W'$ in the branched cover. We orient $\hat{W}'$ by pushing off the orientation on $W$. $L$ is the boundary of $\hat{W}$, and we let $L'$ denote the boundary of $\hat{W}'$. $\hat{W}$ and $\hat{W}'$ will intersect in two points above the zero of $\vec{w}$. Their intersection number is 2. See [A, Lemma 6], or [W, proof of (2.4)]. The intersection form on $H_2(\hat{N})$ is infinite cyclic, and self intersection of this generator is $-2$ [V6], [Ka]. So $H_2(\hat{N}, \partial\hat{N})$ is infinite cyclic and $[\hat{W}, \partial\hat{W}]$ will represent some multiple of the generator say, $\lambda$. It follows that the total linking number between $L$ and $L'$ is $2 + \lambda^2/2$. The formula we use here may be derived similarly to way one derives the well-known formula which describes the $\mathbb{Q}/\mathbb{Z}$-linking form of a 3-manifold given by surgery on a framed link with the inverse of the linking matrix. As this number is also $1 + 2Lk(\alpha, \alpha^*)$, and $Lk(\alpha, \alpha^*) = \pm 1/2$, we conclude $\lambda = 0$.
and $Lk(\alpha, \alpha^*) = 1/2$. □

![Figure 16a](image1)

![Figure 16b](image2)

We also must consider the involutions $T^\pm$ around the singularities of $Y$ above the isolated points of $\mathbb{R}F$. In [G1, Figure 4a], we have already described how $\mathcal{F} \cup \mathbb{R}P(2)$ intersects a small 3-sphere $\tilde{S}$ around such a point. See Figure 16a. $\mathcal{F} \cap S$ consists of the components labeled $L_{\pm i}$. $\mathcal{R}$ is $\mathbb{R}P(2) \cap S$. We also described the image of $(\mathcal{F} \cup \mathbb{R}P(2)) \cap S$ in the orbit space $\overline{S}$ of $T$ restricted to $S$ [G1, Figure 4b]. See Figure 16b where we have labeled the image of $L_{\pm i}$ simply $L$. This link of two components is symmetric in the sense that there is an isotopy which switches the components. The link $L_{+i} \cup L_{-i}$ is a Hopf link and so the inverse image $\tilde{S}$ of $S$ in $Y$ is a copy of $\mathbb{R}P(3)$. We have a $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on $\tilde{S}$. The three non-trivial elements $\mathbb{Z}_2 \times \mathbb{Z}_2$ each induce involutions on $\tilde{S}$. The orbit spaces of these involutions must be the different 2-fold branched covers of $\overline{S}$ along one or both the components of the link in Figure 16b. If we form the branched cover along just one component we get $S^3$. Thus two of the orbit spaces are 3-spheres. The link in Figure 16b is isotopic to a $(2, -4)$ torus link and so using the method of Kirby and Akbulut [AK] the double cover is surgery on the unknot with framing -4 and so is the lens space $L(4, 1)$. Suppose that $p$ is an isolated double point in $B^-$ and $\tilde{p} = \pi^{-1}(p)$. Then $T^+$ acts freely on the boundary of a neighborhood of $\tilde{p}$, and the orbit space of $T^+$ on this $\mathbb{R}P(3)$ is $L(4, 1)$. The orbit space of $T^-$ will be a 3-sphere. If $p$ lies in $B^+$ instead the roles of $T^+$ and $T^-$ are reversed.

Consider the component labeled $\mathcal{R}$ in Figure 16a, we wish to know the linking number of the two lifts of this component in $\tilde{S}$. Viewing $L_{+i} \cup L_{-i}$ as a $(2, -2)$ torus link and using the method of constructing double branched covers in [R,p.302] or [AK], we see that the lift of this component consists of two oppositely oriented meridians in the description of $\tilde{S}$ as surgery on an unknot with framing $-2$. Thus their linking number is $-1/2$.

§9 PROOF OF THEOREM (4.1) AND ADDENDA

Let $\phi : H_1(\mathbb{C}P(2) - F) \to \mathbb{Z}_2$ be as chosen in §6.

**Lemma (9.1).** Let $D$ be a oriented simple closed curve in $\mathbb{R}P(2) - \mathbb{R}F$. If $D$ is one sided then $\phi([D]) = \frac{m}{2} \quad (\text{mod } 2)$. If $D$ is 2-sided, $\phi([D]) = 0 \quad (\text{mod } 2)$.

**Proof.** In the case $\mathcal{F}$ is a nonsingular real algebraic curve, Lemma 1 of [Pa1] provides an oriented 2-sphere $\Sigma \subset \mathbb{C}P(2)$ such that $\Sigma \cap \mathbb{R}P(2) = D$, $\Sigma \circ \mathcal{F} = 2k$, and
\(\Sigma = D_1 \cup D_2\), where \(D_1\) and \(D_2\) are disks, \(D_1 \cap D_2 = D\), and \(T\) restricted to \(D_1\) is an orientation reversing homeomorphism to \(D_2\). The proof goes through in this context. So \(\phi([D]) = D_1 \circ \mathcal{F} = k\).

If \(D\) is 2-sided, a similar proof gives a \(\Sigma\) as above except \(\Sigma \circ \mathcal{F} = 4k\). Thus \(\phi([D]) = D_1 \circ \mathcal{F} = 2k = 0\). \(\square\)

Recall that \(Y^+\), the fix point set of \(T^+\), is \(\pi^{-1}(B^+)\). If \(R\) is a connected region of \(B^+\), let \(\hat{R} = \pi^{-1}\text{closure}(R)\). Each \(\hat{R}\) is the continuous image of a 2-manifold under a map which is one to one except at a finite set where the map is two to one. By an orientation on \(\hat{R}\), we mean an orientation on its preimage. If \(R\) is orientable, then so is \(\hat{R}\) by Lemma (9.1), but an orientation on \(\hat{R}\) does not induce one on \(\hat{R}\). We remark that even for nonsingular curves it not obvious that \(\hat{R}\) will be orientable whenever \(R\) is orientable. It is easy to construct an involution on a Klein bottle with fix point set two disjoint circles with orbit space an annulus. See [M, Remark (3.2)] for a discussion of this issue for curves in \(RP(1) \times RP(1)\). Also by Lemma (9.1), if \(k\) is odd, \(\hat{R}\) is orientable whether or not \(R\) is. For nonsingular curves these facts are known [W,(5.1)]. To assign orientations to the orientable \(\hat{R}\), we consider the methods of removing double points discussed in §1. Let \(\mathcal{F}_t\) denote the new floppy curve obtained in this manner if we remove nonisolated double points by opening channels which join regions of \(B^+\), and delete isolated double points. We can actually construct a 1-parameter family of floppy curves \(\mathcal{F}_t\) for \(t \in [0,1]\) such that \(\mathcal{F}_t\) for \(t \in (0,1]\) is an isotopy, and \(\mathcal{F}_0 = \mathcal{F}\). We may then form \(\hat{\gamma}\) the double branched cover of \(CP(2) \times [0,1]\) along \(\cup_{t \in [0,1]}(\mathcal{F}_t \times t)\). Let \(S_t\) denote a region of \(B^+_t\), which includes a region \(R\) of \(B^+\). Then we can view \(\hat{R}\) as a part of the limit of \(\hat{S}_t \times t\) as \(t\) approaches zero.

We choose an orientation for each orientable \(\hat{S}_t\). If \(\hat{S}_t\) is orientable and \(R\) is included in \(S_t\), then the orientation on \(\hat{S}_t\), induces one on \(\hat{R}\). Recall the fixed orientation reversing simple closed curve \(w\) chosen in the introduction. If \(\hat{S}_t\) is non-orientable then \(S_t\) will include \(w\), and \(\hat{S}_t = \pi^{-1}(w)\) is orientable. In this case choose an orientation on \(\hat{S}_t = \pi^{-1}(w)\) and this will induce an orientation on \(\hat{R}\) if \(R\) is included in \(S_t = \{w\}\). We have now chosen orientations on \(\hat{R}\) for each relevant region \(R\) in \(B^+\).

**Lemma (9.2).** Each \(\hat{R}\) will have a positive singular double point over each isolated point in \(R\). \(\hat{R}\) will also have singular double points above the nonsimple corners of \(R\). Such a singular double point \(c\) will be positive if \(k\) is odd or \(c\) does not cross \(w\). Such a singular double point \(c\) will be negative, if \(k\) is even and \(c\) does cross \(w\). Thus \(\hat{R}\) will have \(p(R) + c(R)\) positive singular double points.

**Proof.** The singular double points over each isolated points in \(R\) will all be positive by the last remark in §8. To see this we must note that an orientation on \(\hat{R}\) will induce orientations on the two sheets passing through \(\hat{p}\) lying over an isolated point \(p\) which map down to opposite local orientations at \(p\).

Note a neighborhood of a non-simple corner \(p\), will intersect \(R\) in a region which looks locally like a neighborhood of the origin intersected with the first and third quadrants in the plane. It is clear that \(\hat{R}\) will have a singular double point above such a corner of \(R\). It follows from our method of orienting \(\hat{R}\) and Lemma (8.2) that the sign of the double points will be as stated. \(\square\)

\(Y^+\) is the union of all the \(\hat{R}\) for \(R\) in \(B^+\) and \(c^-\) isolated points lying above \(I^-\).
$H_2(Y^+)$ is a free Abelian group on the relevant regions of $B^+$. We wish to figure out the intersection form on this group. Our choice of orientations, picks out a set of generators for this group: $[\hat{R}]$ as $R$ varies over the relevant regions of $B^+$.  

**Lemma (9.3).** For $R$ a relevant region of $B^+$, and any orientation on $\hat{R}$, $[\hat{R}] \circ [\hat{R}] = \langle R, R \rangle$.

**Proof.** For simplicity we consider first the case when $O(R)$ is even. We assume that we have already isotoped the floppy curve so that so $L_{RF}$ is mostly tangent. We pick a vector field $\vec{v}$ on $R$ with isolated zeros that is parallel to $L_{RF}$. We also model the vector field at the corners by the vector field $yi - xj$ near the origin and in the first quadrant of the $xy$ plane. We also insist that the zeros of $\vec{v}$ be simple in the interior of $R$ and that there be a simple zero of order one at each of the $\iota(R)$ isolated double points of $RF$ in $R$. The sum of the $\pm 1$ indices of these zeros away from the isolated double points is exactly $\chi(R) - \iota(R) - O(R)/2$. Then we can attempt to push $R$ off itself in $CP(2)$ with the vector field $i\vec{v}$. Let $R^+$ denote the result and $\hat{R}^+ = \pi^{-1}R^+$. We will succeed except at the isolated zeros of $\vec{v}$, which include the corners. $R^+$ and $R$ will represent the same homology class in $Y^+$. They will intersect each other transversely at nonsingular points of $Y^+$ except above the corners and the isolated singular points. The sum of the signs of these intersection numbers will be $-\chi(R) + \iota(R) + O(R)/2$. See [A, Lemma 6], or [W, proof of (2.4)]. Thus the sum of the intersection indices of $\hat{R}$ and $\hat{R}^+$ except above the corners and isolated singularities will be $-2\chi(R) + 2\iota(R) + O(R)$. Above each simple corner, the contribution of the intersection of $\hat{R}$ and $\hat{R}^+$ to $[\hat{R}] \circ [\hat{R}]$ is be computed as the linking number in $RP(3)$ of the curves $\alpha$ and $\alpha'$ which we considered in Lemma (8.1). This is $1/2$.

The contribution above a non-simple corner $c$ is given by a linking number in the copy of $RP(3)$ which is the boundary of a small neighborhood of $c$. $\hat{R}$ and $\hat{R}'$ will each intersect this $RP(3)$ in a link of two components. The contribution at a negative singular double point of $[\hat{R}]$ is zero and the contribution at a positive singular double point of $[\hat{R}]$ is two. See the proof of (8.2).

The contribution at an isolated singularity is zero. This is calculated as follows. Push off with framing $-1$ a parallel copy of the component labeled $\mathcal{R}$ in Figure 16a. Call it $\mathcal{R}^+$. $\mathcal{R}$ and $\mathcal{R}^+$ describe the intersection of $R$ and $R^+$ with a small 3-sphere $S$ centered at the singular point. $L_{\pm}$ describes the intersection of $S$ with $\mathcal{F}$. $\hat{S}$, the branched cover along the link consisting of both $L_{\pm}$, is the link of the singularity upstairs. Our contribution is the linking number of the inverse image of $\mathcal{R}$ with inverse image of $\mathcal{R}^+$. However we must orient the two components of each inverse image so that they induce opposite orientations downstairs. See the very end of §8. One checks that the linking number is zero.

For the case $O(R)$ odd, we need to replace $\vec{v}$ with $i$, a line field with isolated singularities. We choose $i$ to be parallel to $L_{RF}$ and at the corners to be parallel to the vector field modeled on $yi - xj$ near the origin and in the first quadrant of the $xy$ plane. We also insist that there be a simple singularity of order two at each of the $\iota(R)$ isolated double points of $RF$ in $R$. Then we let $R^+$ denote the double branched cover of $R$ along the singularities of $i$ and corners of $R$, given by the endpoint of a small I-bundle parallel to $i$ completed by cones at the singularities of $i$. Let $\hat{R}^+$ be the inverse image of $R^+$ in $Y$. Then $\hat{R}^+$ represents twice the class of $\hat{R}$ and so the sum of the intersection indices of $\hat{R}$ and $\hat{R}^+$ is twice $[\hat{R}] \circ [\hat{R}]$. The
rest of the argument is similar. \(\square\)

**Lemma (9.4).** For \(R\) and \(R'\) distinct relevant regions of \(B^+\), with the given orientations on \(R\) and \(R'\), \([\hat{R}] \circ [\hat{R}'] = (R, R')\).

**Proof.** \(\hat{R}\) and \(\hat{R}'\) intersects only above the points of intersection of closures of \(R\), and \(R'\). The contribution there is given by the linking number we considered at the end of the proof of Lemma (9.2). \(\square\)

Thus the matrix \(M^+\) is the matrix which gives the intersection form on \(H_2(Y^+)\) induced from the intersection form on \(H_2(Y)\) by the inclusion, with respect to the basis \([\hat{R}]\) as \(R\) varies over the relevant regions of \(B^+\). Replacing \(F\) by \(-F\), and yields the analogous result for \(M^-\).

**Lemma (9.5)**. \(\text{Sign}(T^\pm) = -\chi(Y^\pm) + O^\pm\).

**Proof.** By §8, we see that the action of \(T^\pm\) on \(Y\) is nice in the sense of §7. Temporarily switching the names of \(B^\pm\) if necessary we may assume that every region of \(B^+\) is relevant. Recall that \(Y^+\), the fix point set of \(T^+\), is the union of all the \(\hat{R}\) for \(R \in B^+\) and \(\iota^-\) isolated points. By Theorem (7.1) and the discussion in §8, each isolated point contributes \(-1\) to \(\text{Sign}(T^+)\). By Lemma (7.2), the contribution of each \(\hat{R}\) is \([\hat{R}] \circ [\hat{R}] - d(\hat{R})\). Let \(n(R)\) be such that \(p(R) + n(R)\) is the number of non-simple corners of \(R\). Let \(c(R) = s(R) + 2p(R) + 2n(R)\). So \(c(R)\) is the number of corners of \(R\) where we count a corner twice if it is not simple (i.e., if it occurs twice while traveling around the perimeter of \(R\)). By Lemma (9.2), \(\hat{R}\) will have \(p(R) + \iota(R)\) positive singular double points and \(n(R)\) negative singular double points. So \(d(\hat{R}) = p(R) - n(R) + \iota(R)\). On the other hand we can rewrite

\[
[\hat{R}] \circ [\hat{R}] = (R, R') = -2\chi(R) + c(R)/2 + p(R) - n(R) + 2\iota(R) + O(R).
\]

Thus the contribution of \(\hat{R}\) to \(\text{Sign}(T^+)\) is \(-2\chi(R) + c(R)/2 + \iota(R) + O(R)\). If we sum this contribution over all \(R\) in \(B^+\), we will get \(-2\chi(\text{Int}(B^+)) + r + \iota^+ + O^+\). So

\[
\text{Sign}(T^+) = -2\chi(\text{Int}(B^+)) + r + \iota^+ + \iota^- + O^+ = -\chi(Y^+) + O^+.
\]

In the notation of §6, \(\text{Sign}(T^-)\) is \((c - d) - (a - b) - 1\). If we plug in Equations (21), (21')(22), (22') and (7), we see that \(\text{Sign}(T^-) = -\chi(Y^-) + O^-\). \(\square\)

The logic is slightly tricky here. We derive the equations of §6 for the above choice of \(B^+\) first, using Lemma (9.5)^+. Then using the equations of §6 for this choice, we derive Lemma (9.5)^-. This may then be used to derive the equations of §6 for the other choice of \(B^+\).

**Lemma (9.6).** The \((-1)\)-eigenspace for the action of \(\theta\) on \(H_3(Y, Y^+, \mathbb{Q})\) has dimension less than or equal to \(n - \epsilon\).

**Proof.** Let \(\alpha \varepsilon_\beta\) denote the dimension of the subspace of \(H_3(Y, Y^+\mathbb{Q})\) consisting of those elements in both the \(\alpha\) eigenspace of \(T^-\) and the \(\beta\) eigenspace of \(T^+\), where \(\alpha, \beta\) belong to the set \([-1, 1]\). Let \(X_\pm\) denote the orbit space of the involution \(T^\pm\) on \(Y\), then as in [Fi4.3], one may use the Smith theory exact sequence for an involution and the universal coefficient theorem to calculate that \(\beta_3(X_+, Y^+) \leq n - \epsilon\). Using the transfer \(B[2.4], (\varepsilon_+ + (-\varepsilon) = \beta_3(X_+, Y^+)\). By the same argument, \((\varepsilon_+ + (-\varepsilon) = \beta_3(X_-, B^+). Using [LW], we have that \(\beta_1(X_-) = 0\), as there is
an involution on $X_-$ with $S^4$ as an orbit space with a connected fixed point set. By duality $\beta_3(X_-) = 0$, and so $\beta_3(X_-, B^+) = 0$. Thus both $\pm e_+$ and $\pm e_-$ are zero. So $\mp e_+ - \beta_3(X_+, Y^+) \leq n - \epsilon$. The eigenspace in question has dimension: $(-e_+) + (+e_-) \leq n - \epsilon$. □

**Proof of Theorem (4.1).**

Let $j^+$ denote the inclusion of $Y^+$ into $Y$. Then $j^+_s$ maps $H_2(Y^+, \mathbb{Q})$ into $-1E_{+1}$. By Lemma (9.6), the kernel of $j^+_s$ on $H_2(Y^+, \mathbb{Q})$ has dimension less than $n - \epsilon$. It also has dimension less than $\eta(M^+)$, as a class which represents zero is in the radical of the intersection form. Pick a basis of eigenvectors for the matrix $M^+$. Let $S^+$ denote the space spanned by the $\sigma_+(M^+)$ basis elements with positive eigenvalue. Let $S^0$ denote the space spanned by the $\eta(M^+)$ basis elements with eigenvalue zero. Then the image of $S^+ \oplus S^0$ in $H_2(Y, \mathbb{Q})$ will not intersect any of the summands $-1E_{+1} \oplus +1E_{+1} \oplus +1E_{-1} \oplus -1E_{-1}$.

So the inclusion of $S^+ \oplus S^0$ in $H_2(Y, \mathbb{Q})$ followed by the projection to $-1E_{+1}$ will have kernel of dimension less than or equal to $\min\{\eta(M^+), n - \epsilon\}$. So we have:

$$\sigma_+(M^+) + \eta(M^+) \leq \dim(-1E_{+1}) + \min\{\eta(M^+), n - \epsilon\}.$$ 

Equation (1$^+$) follows easily from this and Equation (21).

Let $S^-$ denote the space spanned by the $\sigma_+(M^+)$ basis elements with positive eigenvalue. Considering the image of $S^- \oplus S^0$ and Equation (22) as above yields Equation (2$^+$). Equations (1$^-$) and (2$^-$) are obtained in exactly the same way, or we may obtain them by reversing the roles of $B^+$ and $B^-$ by changing the sign of the defining polynomial $F$. □

The following lemma follows easily from the Smith theory exact sequence or the exact sequence due to Lee and Weintraub.

**Lemma (9.7).** $\beta_3(Y, \mathbb{Z}_2) = n(\mathcal{F}) - \epsilon(\mathcal{F}) - 1$. Thus $\mathcal{F}$ is 2-irreducible if and only if $\beta_3(Y, \mathbb{Z}_2) = 0$.

**Proof of Addendum (4.2).**

By (9.7), $H_3(Y, \mathbb{Z}_2) = 0$. By Lemma (7.3), the only possible nonzero element of the kernel of $j^+_s: H_2(Y^+, \mathbb{Z}_2) \to H_2(Y, \mathbb{Z}_2)$ is $[Y^\pm]$, and this element can not be in the kernel unless condition (a) holds. Here $[Y^\pm] \in H_2(Y^\pm, \mathbb{Z}_2)$ is the class of a cycle given the sum of all the simplices in some triangulation of $Y^\pm$.

Thus $\beta_3(Y, Y^\pm, \mathbb{Z}_2) \leq 1$. By the proof of (4$^+$), Equations (3$^+$) and (4$^+$) can fail by at most $\beta_3(Y, Y^\pm, \mathbb{Q})$. By the universal coefficient theorem, $\beta_3(Y, Y^\pm, \mathbb{Q}) \leq \beta_3(Y, Y^\pm, \mathbb{Z}_2) \leq 1$. Thus Equations (3$^+$) and (4$^+$) can fail by at most one, and then (a) holds.

We now show that the failure of condition (b$^\pm$) implies that $j^+_s: H_2(Y^\pm, \mathbb{Q}) \to H_2(Y, \mathbb{Q})$ is injective. Let $\hat{Y}$ denote the inverse image in $\hat{Y}$ of the union of the relevant regions in $B^\pm$. Then the inclusion induces an isomorphism from $H_2(\hat{Y}, \mathbb{Q})$ to $H_2(Y^\pm, \mathbb{Q})$. Thus it suffices to show that $H_3(Y, \hat{Y}, \mathbb{Q})$ is zero. This will hold if $H_3(Y, \hat{Y}, \mathbb{Z}_2)$ is zero. The boundary of the closure of the union of any collection of relevant regions in $B^+$ can not be null-homologous in $\mathcal{F}$. So by Lemma (7.3), $j^+_s: H_2(\hat{Y}, \mathbb{Z}_2) \to H_2(Y, \mathbb{Z}_2)$ is injective. It follows that $H_3(Y, \hat{Y}, \mathbb{Z}_2)$ is zero.
We now assume that \((3^\pm)\) or \((4^\pm)\) fails, and wish to establish condition \((c^\pm)\), having already established conditions \((a)\) and \((b^\pm)\). \(H_2(Y^\pm)\) is free Abelian generated by \(\{\hat{R}_i\}\), where \(R_i\) ranges over all the regions of \(B^+\), each of which must be relevant. It follows that \(j_i^+: H_2(Y^+, \mathbb{Z}) \to H_2(Y, \mathbb{Z})/\text{torsion}\) is not injective. So there must be an element in the kernel of the form \(\sum_i k_i[\hat{R}_i]\) where the integral vector \(\vec{k} = \{k_i\}\) is primitive. In particular some \(k_i\) is odd. Also we have that \(M^\pm \vec{k} = 0\).

\(H_2(Y^\pm, \mathbb{Z}_2)\) also has as a basis \(\{[\hat{R}_i]\}\) (reduced modulo two). As \(H_3(Y, \mathbb{Z}_2)\) is zero, \(H^3(Y, \mathbb{Z}_2) = 0\), and so \(H_2(Y, \mathbb{Z})\) has no 2-torsion. Thus the map given by reduction modulo two from \(H_2(Y, \mathbb{Z})\) to \(H_2(Y, \mathbb{Z}_2)\) factors through \(H_2(Y, \mathbb{Z})/\text{Torsion}\). It follows that \(\sum_i k_i[\hat{R}_i]\) reduced modulo two is in the kernel of \(j_i^+: H_2(Y^+, \mathbb{Z}_2) \to H_2(Y, \mathbb{Z}_2)\). It is nonzero, since one \(k_i\) is odd. In the proof of \((a)\), we saw the only possible such element is \([Y^\pm] = \sum_i k_i[\hat{R}_i]\). It follows that all \(k_i\) are odd, and \(\vec{k}\) is a kernel vector for \(M^\pm\). So condition \((c^\pm)\) holds.

To see the second statement, we note that \(\mathbb{R}F\) has at least one constituent as \(F\) is strongly irreducible and must be dividing. Then an argument in the proof of [W, 7.4] shows that not both of \([Y^+]\) and \([Y^-]\) can be zero in \(H_2(Y, \mathbb{Z}_2)\). \(\square\)

**Proof of Addendum (4.3).**

With our choice of \(B^+\) the proofs of Lemmas (6.5), (6.6) and (6.7) of [W] all go through. Then the proof [W, 7.4 (iii)] completes the proof. \(\square\)

**\S 10 Derivation of the Determinant Condition**

**Proof of Theorem (5.1\pm).**

We only need to prove the plus version. We desire to work with an actual manifold rather than a rational homology manifold so that we have a unimodular intersection pairing over \(\mathbb{Z}\). So we equivariantly resolve \(d\) singularities in the space with involution \((Y, T^+)^\). Each singularity is a cone on a copy of \(\mathbb{R}P(3)\). Let \(\mathbb{D}\) denote the disk bundle over \(S^2\) with Euler class \(-2\).

For each of the \(\nu\) pairs of complex conjugate double points, we have a pair of cones on \(\mathbb{R}P(3)\) which are interchanged by \(T^+\). We replace each of these pairs with two copies of \(\mathbb{D}\) which are interchanged by \(T^+\). The sum of their fundamental classes is fixed by \(T^+\).

Above each of the \(i^-\) isolated double points in \(B^-\), we have an isolated fixed point of \(T^+\) whose neighborhood is a cone on a copy of \(\mathbb{R}P(3)\) with a free action. We can equivariantly replace this neighborhood with a copy of \(\mathbb{D}\) with the involution which is the antipodal map on the disk fibers. The core 2-sphere is fixed by the involution, so its fundamental class is also fixed.

Above each of the \(i^+\) isolated double points in \(B^+\) and above each of the \(r\) nonisolated real double points, we have a fixed point of \(T^+\) whose neighborhood with involution is a copy of the double branched cover of \(B^4\) along the cone on the Hopf link. We can replace this with the double branched cover of the 2-disk bundle over the 2-sphere with Euler class \(-1\) along two disjoint disk fibers. This is just another copy of \(\mathbb{D}\) with a different involution. This changes the fixed point set by locally replacing the cone on two circles with the disjoint union of two disks. The homology class of the core 2-sphere is fixed by the involution.

Let \((Y, T)\) denote the manifold with involution we obtain if we resolve the singularities of \((Y, T^+)\) as above. As \(F\) is 2-irreducible, \(H_3(Y, \mathbb{Z}_2) = 0\). Comparing the
Mayer-Vietoris sequence for \( Y \) as the union on the complement of singularity set and a neighborhood of the singularity set and a similar Mayer-Vietoris sequence for \( \mathcal{Y} \), shows that \( H_3(\mathcal{Y}, \mathbb{Z}_2) = 0 \) and thus, using Poincare duality in an actual manifold, \( H_1(\mathcal{Y}, \mathbb{Z}_2) = 0 \). Thus \( H_2(\mathcal{Y}) \) can only have odd torsion. We have

\[
\beta_2(\mathcal{Y}) = \beta_2(Y) + d = \chi(Y) - 2 + d \\
= 2\chi(\mathbb{C}P(2)) - \chi(\mathcal{F}) - 2 + d = 4 - \chi(\mathcal{F}) + d.
\]

The fix point set \( \mathcal{F} \) of \( T \) can be obtained from \( Y^+ \) by replacing \( i^- \) isolated points with 2-spheres and by replacing \( r + i^+ \) cones on circles with pairs of disks. Thus we can find a basis \( \mathbb{B} \) for \( H_2(\mathcal{F}) \) indexed by the relevant regions of \( B^+ \) and the isolated points in \( B^- \). Thus \( \beta_2(\mathcal{F}) = \rho^+ + i^- \) and \( \beta_2(\mathcal{F}, \mathbb{Z}_2) = \beta_0(\mathcal{F}) = \rho^+ + i^- + \pi^+ \). We have that \( \chi(\mathcal{F}) = \chi(Y^+) + i^- + r + i^+ \). Then by Equation (23) we have:

\[
\beta_1(\mathcal{F}, \mathbb{Z}_2) = 2\beta_0(\mathcal{F}) - \chi(\mathcal{F}) = 2(\rho^+ + \pi^+ + i^- - r - \chi(B^+)).
\]

By a theorem of Floyd [Fl,4.4], [B,p.126], [W,A2], \( \sum\beta_k(\mathcal{F}, \mathbb{Z}_2) \leq \sum\beta_k(\mathcal{Y}, \mathbb{Z}_2) \). Equivalently their difference: \( 2 - 2\beta_0(\mathcal{F}) + \beta_2(\mathcal{Y}) - \beta_1(\mathcal{F}, \mathbb{Z}_2) \) is positive. By the above equations, this last quantity is \( 2h^+ \). Thus \( h^+ \) is nonnegative. The integrality of \( h^+ \) follows from Proposition (4.4).

Let \( H_+ \) denote the kernel of the endomorphism of \( H_2(\mathcal{Y})/\text{Torsion} \) defined by \( 1 - T_+ \). Let \( L \) be the subspace of \( H_+ \) generated by the images of elements of \( \mathbb{B} \) in \( H_2(\mathcal{Y}), \) together with the \( r + i^+ \) classes coming from the 2-spheres with involution replacing the singularities above the nonisolated real points and the isolated points in \( B^+ \) and the \( \nu/2 \) classes given by the sums of paired of 2-spheres arising from the pairs of complex conjugate double points. \( \text{dim}(L) \leq \text{L} \) where we let \( \text{L} \) denote \( \rho^+ + r + i + \nu/2 \). By §6, subspace of on \( H_2(Y, \mathbb{Q}) \) fixed by the action of \( T^+ \) on \( H_2(Y, \mathbb{Q}) \) has dimension \( a + b = P^+ \). By our hypothesis, this is also \( \rho^+ \). So \( \text{dim}(H) \) equals \( \text{L} \). If \( \eta(M^+) \) is nonzero, \( \text{det}(M^+) = 0 \) and the conclusion holds. Thus we may assume that \( \eta(M^+) = 0 \). It follows that the inclusion of \( Y^+ \) into \( Y \) induces an injection of \( H_2(Y^+, \mathbb{Q}) \) into \( H_2(Y, \mathbb{Q}) \). Let \( \mathcal{M} \) denote the matrix for the intersection pairing on \( H_2(Y) \) with respect to the generators for \( L \). It is not hard to see that \( \mathcal{M} \) can be obtained from \( M^+ \) direct sum a diagonal matrix with \( r + i \) minus two’s and \( \nu \) minus four’s on the diagonal by a sequence of row and column operations of the form where one modifies one row (or column) by a multiple of another. It follows that \( |\text{det}(\mathcal{M})| = 2^d|\text{det}(M^+)| \). Thus \( |\text{det}(2M^+)| = 2^{p^+ - d}|\text{det}(\mathcal{M})| \).

Since \( |\text{det}(\mathcal{M})| \neq 0 \), the inclusion of \( \mathcal{F} \) into \( Y \) induces an injection of \( H_2(\mathcal{F}, \mathbb{Q}) \) into \( H_2(Y, \mathbb{Q}) \). It follows that \( \text{dim} L = \text{L} \). Thus \( L \) is a lattice of maximal rank in \( H_+ \). Let \( \text{det} H_+ \) denote the absolute value of the determinant of the matrix for the restriction of the nonsingular intersection pairing on \( H_2(\mathcal{Y}) \) to \( H_+ \) with respect to some basis for \( H_+ \). We have that \( |\text{det}(\mathcal{M})| = b^2 \text{det}(H_+) \) where \( b \) is an integer.

The proof of [Kh2,Lemma 3.7] then shows that \( |\text{det}(H_+)| = 2^{h^+} \). The proof of [Kh2,Lemma 3.7] refers to [Kh3,Lemma 2.4] which is not precisely stated in the English translation and is given without proof. This same lemma is stated precisely and given with proof in [W, Lemma 3.14]. Thus \( |\text{det}(2M^+)| = 2^{h^+ + p^+ - d + 2} \). 

\[ \square \]

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