On the Tradeoff Between Computation and Communication Costs for Distributed Linearly Separable Computation

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Abstract

This paper studies the distributed linearly separable computation problem, which is a generalization of many existing distributed computing problems such as distributed gradient descent and distributed linear transform. In this problem, a master asks N distributed workers to compute a linearly separable function of K datasets, which is a set of Kc linear combinations of K messages (each message is a function of one dataset). We assign some datasets to each worker, which then computes the corresponding messages and returns some function of these messages, such that from the answers of any Nr out of N workers the master can recover the task function. In the literature, the specific case where Kc = 1 or where the computation cost is minimum has been considered. In this paper, we focus on the general case (i.e., general Kc and general computation cost) and aim to find the minimum communication cost.

We first propose a novel converse bound on the communication cost under the constraint of the popular cyclic assignment (widely considered in the literature), which assigns the datasets to the workers in a cyclic way. Motivated by the observation that existing strategies for distributed computing fall short of achieving the converse bound, we propose a novel distributed computing scheme for some system parameters. The proposed computing scheme is optimal for any assignment when Kc is large and is optimal under cyclic assignment when the numbers of workers and datasets are equal or Kc is small. In addition, it is order optimal within a factor of 2 under cyclic assignment for the remaining cases.
Index Terms

Distributed computation, linearly separable function, communication and computation costs tradeoff

I. INTRODUCTION

Nowadays to cope with the emergence of big data and the complexity of data mining algorithm, using cloud computing infrastructures such as Amazon Web Services (AWS) [1], Google Cloud Platform [2], and Microsoft Azure [3] becomes an efficient and popular solution. While large scale distributed computing algorithms and simulations have the potential for achieving unprecedented levels of accuracy and providing dramatic insights into complex phenomena, they are also presenting new challenges. This paper mainly refers to two important challenges of cloud distributed computing. The first is the relation between the computation and communication costs. It is critically important to understand the fundamental tradeoff between computation and communication costs for large scale distributed computing algorithms. The second is to tackle the existence of straggler workers (i.e., machines) in applications, such that it is not necessary to wait for the computation of slow workers. Coding techniques have been introduced into the cloud distributed computing scenarios [4] and have attracted significant attention recently. The strategy of this paper is to use coding techniques to characterize the tradeoff between computation and communication costs, while mitigating the straggler effect.

This paper specially considers a distributed linearly separable computation problem recently formulated in [5]. A master aims to compute a linearly separable function $f$ on $K$ datasets $(D_1, \ldots, D_K)$, where

$$ f(D_1, \ldots, D_K) = g(f_1(D_1), \ldots, f_K(D_K)) = g(W_1, \ldots, W_K). $$

$W_k = f_k(D_k)$ for all $k \in \{1, \ldots, K\}$ is the outcome of the partial function $f_k(\cdot)$ applied to dataset $D_k$. $g(W_1, \ldots, W_K)$ can be seen as a set of $K_c$ linear combinations of the messages $W_1, \ldots, W_K$ with uniformly i.i.d. coefficients. The task function is computed by $N$ workers in the following three phases. During the data assignment phase, we assign each dataset to a subset of workers, and the number of datasets assigned to each worker is defined as the computation cost. During the computing phase, each worker should compute and send data packets as functions of the

\footnote{One of the major differences between this problem and the existing distributed matrix-matrix multiplication problems [6]–[12] is that in the considered problem we can only assign the datasets in an uncoded manner to the workers.}
datasets assigned to it, such that from the answers of any \( N_r \) workers, the master can recover the task function. During the decoding phase, the master should recover the task function by receiving the answers of the \( N_r \) fastest workers. The communication cost is defined as the total number of transmissions which should be received by the master in order to recover the task function. The objective is to characterize the tradeoff between the computation-communication costs.

In the literature, some sub-cases of the considered problem have been considered. When \( K_c = 1 \), the considered problem becomes the distributed gradient descent problem considered in \([13]–[17]\). The optimal computation-communication costs tradeoff was characterized in \([16]\) under the constraint of linear coding in the computing phase and symmetric transmission (i.e., the number of packets transmitted by each worker is the same). When each worker is limited to send one linear combination of messages, the considered problem becomes the distributed linear transform problem treated in \([18]\). The “Short-Dot” distributed computing scheme was proposed in \([18]\), which offers significant speed-up compared to uncoded computing techniques. When the computation cost is minimum (equal to \( \frac{K}{N}(N - N_r + 1) \)), a distributed computing scheme based on linear space intersection was proposed in \([5]\), which is exactly optimal when \( N = K \); and is optimal under the constraint of cyclic assignment.\(^2\)

**Contributions**

In this paper, as in \([16]\), we assume that the computation cost of each worker is \( \frac{K}{N}(N - N_r + m) \) where \( m \in [1 : N_r] \). Our main contributions are as follows.

- For any \( m \in [1 : N_r] \), under the constraint of cyclic assignment, we propose an information theoretic converse bound on the minimum communication cost \( R_{\text{cyc}}^* \).
- On the observation that the existing distributed computing schemes \([5], [16], [17]\) for the case \( K_c = 1 \) or \( m = 1 \) cannot be used to achieve the converse bound when \( K_c > 1 \) and \( m > 1 \), we propose a novel distributed computing scheme under the constraint that 
  \[ N \geq \frac{m+u-1}{u} + u(N_r - m - u + 1) \]
  where \( u := \left \lceil \frac{K_c N}{K} \right \rceil \).

\(^2\) The cyclic assignment was widely used in the existing works on the sub-problems or related problems of the considered problem such as \([5], [13], [14], [16], [17], [19]\). The main advantages of the cyclic assignment are that it can be used for any case where \( N \) divides \( K \) regardless of other system parameters, and its simplicity. According to our knowledge, the other existing assignments, such as the repetition assignments in \([13], [20]\) and the caching-like assignment in \([5]\), can only be used for very limited number of cases. In addition, the cyclic assignment is independent of the task function; thus if the master has multiple tasks in different times, we need not assign the datasets in each time.
Compared to the proposed converse bound, for the considered problem satisfying $N \geq \frac{m+u-1}{u} + u(N_r - m - u + 1)$, the proposed computing scheme is exactly optimal when $K_c \in [N_r - m + 1 : K]$ and is optimal under the constraint of cyclic assignment when $K = N$ or $K_c \in \left[ 1 : \frac{K}{N} \right]$. In addition, it is order optimal within a factor of 2 under the constraint of cyclic assignment for the remaining cases.

**Paper Organization**

The rest of this paper is organized as follows. Section II introduces the distributed linearly separable computation problem and reviews the existing schemes for the case $K_c = 1$ or $m = 1$. Section III provides the main results in this paper and provide some numerical evaluations. Section IV proves the proposed converse bound. Section V describes the proposed distributed computing scheme. Section VI concludes the paper and some of the proofs are given in the Appendices.

**Notation Convention**

Calligraphic symbols denote sets, bold symbols denote vectors and matrices, and sans-serif symbols denote system parameters. We use $| \cdot |$ to represent the cardinality of a set or the length of a vector; $[a : b] := \{a, a + 1, \ldots, b\}$ and $[n] := \{1 : n\}$; $a! = a \times (a - 1) \times \ldots \times 1$ represents the factorial of $a$; $\mathbb{F}_q$ represents a finite field with order $q$; $M^T$ and $M^{-1}$ represent the transpose and the inverse of matrix $M$, respectively; the matrix $[a; b]$ is written in a Matlab form, representing $[a, b]^T$; $\text{rank}(M)$ represents the rank of matrix $M$; $0_{m \times n}$ represents the zero matrix with dimension $m \times n$; $(M)_{m \times n}$ represents the dimension of matrix $M$ is $m \times n$; $M^{(S)_r}$ represents the sub-matrix of $M$ which is composed of the rows of $M$ with indices in $S$ (here $r$ represents ‘rows’); $M^{(S)_c}$ represents the sub-matrix of $M$ which is composed of the columns of $M$ with indices in $S$ (here $c$ represents ‘columns’); $\text{det}(M)$ represents the determinant matrix $M$; $\text{Mod}(b, a)$ represents the modulo operation with integer divisor $a$ and in this paper we let $\text{Mod}(b, a) \in \{1, \ldots, a\}$ (i.e., we let $\text{Mod}(b, a) = a$ if $a$ divides $b$); we let $\binom{x}{y} = 0$ if $x < 0$ or $y < 0$ or $x < y$. In this paper, for each set of integers $S$, we sort the elements in $S$ in an increasing order and denote the $i^{th}$ smallest element by $S(i)$, i.e., $S(1) < \ldots < S(|S|)$. 
II. SYSTEM MODEL

A. Problem formulation

We consider a \((K, N, N_r, K_c, m)\) distributed linearly separable computation problem over the canonical master-worker distributed system, formulated in [5]. The master wants to compute a linearly separable function on \(K\) statistically independent datasets \(D_1, \ldots, D_K\),

\[
f(D_1, \ldots, D_K) = g(f_1(D_1), \ldots, f_K(D_K))
\]

where we model \(f_k(D_k), k \in [K]\) as the \(k\)-th message \(W_k\) and \(f_k(\cdot)\) is an arbitrary function. We assume that the \(K\) messages are independent and that each message is composed of \(L\) uniformly i.i.d. symbols over a finite field \(\mathbb{F}_q\) for some large enough prime-power \(q\). As in [5], we assume that the function \(g(\cdot)\) is a linear mapping as follows,

\[
g(W_1, \ldots, W_K) = \mathbf{F} \begin{bmatrix} W_1 \\ \vdots \\ W_K \end{bmatrix} = \begin{bmatrix} F_1 \\ \vdots \\ F_{K_c} \end{bmatrix},
\]

where \(\mathbf{F}\) is a matrix known by the master and the workers. The dimension of \(\mathbf{F}\) is \(K_c \times K\), with elements uniformly i.i.d. over \(\mathbb{F}_q\). The \(i\)th row of \(\mathbf{F}\), denoted by \(f_i\), is referred to as the \(i\)th demand vector. The \(j\)th element of \(f_i\) is denoted by \(f_{i,j}\). It can be seen that \(g(W_1, \ldots, W_K)\) contains \(K_c \leq K\) linear combinations of the \(K\) messages, whose coefficients are uniformly i.i.d. over \(\mathbb{F}_q\). In this paper, we assume that \(\frac{K}{N}\) is an integer\(^3\).

A distributed computing scheme for our problem contains three phases, data assignment, computing, and decoding.

**Data assignment phase:** Each dataset \(D_k\) where \(k \in [K]\) is assigned to a subset of \(N\) workers in an uncoded manner. Define \(Z_n \subseteq [K]\) as the set of datasets assigned to worker \(n \in [N]\). The assignment constraint is that

\[
|Z_n| \leq M := \frac{K}{N} \left( N - N_r + m \right), \quad \forall n \in [N],
\]

where \(M := \frac{K}{N} \left( N - N_r + m \right)\) represents the computation cost, and \(m\) represents the computation cost factor\(^4\).

\(^3\)When \(N\) does not divide \(K\), as shown in [5, Section V-A], we can simply add \(\left\lceil \frac{K}{N} \right\rceil N - K\) virtual datasets.

\(^4\)It was proved in [5] that in order to tolerate \(N - N_r\) stragglers, the minimum computation cost is \(\frac{K}{N} (N - N_r + 1)\).
The assignment function of worker $n$ is denoted by $\varphi_n$, where

$$Z_n = \varphi_n(F),$$  
(4)

$$\varphi_n : [\mathbb{F}_q]^{K_cK} \to \Omega_M(K),$$  
(5)

and $\Omega_M(K)$ represents the set of all subsets of $[K]$ of size not larger than $M$. In addition, for each dataset $D_k$ where $k \in [K]$, we define $\mathcal{H}_k$ as the set of workers to whom dataset $D_k$ is assigned. For each set of datasets $\mathcal{K}$ where $\mathcal{K} \subseteq [K]$, we define $\mathcal{H}_\mathcal{K} := \bigcup_{k \in [K]} \mathcal{H}_k$ as the set of workers to whom there exists some dataset in $\mathcal{K}$ assigned.

**Computing phase:** Each worker $n \in [N]$ first computes the message $W_k = f_k(D_k)$ for each $k \in Z_n$. Worker $n$ then computes

$$X_n = \psi_n(\{W_k : k \in Z_n\}, F)$$  
(6)

where the encoding function $\psi_n$ is such that

$$\psi_n : [\mathbb{F}_q]^{Z_nL} \times [\mathbb{F}_q]^{K_cK} \to [\mathbb{F}_q]^{T_n},$$  
(7)

and $T_n$ represents the length of $X_n$. Finally, worker $n$ sends $X_n$ to the master.

**Decoding phase:** The master only waits for the $N_r$ fastest workers’ answers to compute $g(W_1, \ldots, W_K)$. In other words, the computation scheme can tolerate $N - N_r$ stragglers. Since the master does not know a priori which workers are stragglers, the computation scheme should be designed so that from the answers of any $N_r$ workers, the master should recover $g(W_1, \ldots, W_K)$. More precisely, for any subset of workers $\mathcal{A} \subseteq [N]$ where $|\mathcal{A}| = N_r$, with the definition

$$X_\mathcal{A} := \{X_n : n \in \mathcal{A}\},$$  
(8)

there exists a decoding function $\phi_\mathcal{A}$ such that

$$\hat{g}_\mathcal{A} = \phi_\mathcal{A}(X_\mathcal{A}, F),$$  
(9)

where the decoding function $\phi_\mathcal{A}$ is such that

$$\phi_\mathcal{A} : [\mathbb{F}_q]^{\sum_{n \in \mathcal{A}} T_n} \times [\mathbb{F}_q]^{K_cK} \to [\mathbb{F}_q]^{K_cL}.$$  
(10)

The worst-case probability of error is defined as

$$\varepsilon := \max_{\mathcal{A} \subseteq [N] ; |\mathcal{A}| = N_r} \Pr\{\hat{g}_\mathcal{A} \neq g(W_1, \ldots, W_K)\}.$$  
(11)
In addition, we denote the communication cost by,

$$R := \max_{\mathcal{A} \subseteq [N]: |\mathcal{A}| = N_r} \frac{\sum_{n \in \mathcal{A}} T_n}{L},$$

representing the maximum normalized number of symbols downloaded by the master from any $N_r$ responding workers. The communication cost $R$ is achievable if there exists a computation scheme with assignment, encoding, and decoding functions such that

$$\lim_{q \to \infty} \lim_{L \to \infty} \varepsilon = 0.$$  \hfill (13)

The objective is to characterize the optimal tradeoff between the computation and communication costs $(m, R^*)$, i.e., for each $m \in [N_r]$, we aim to find the minimum communication cost $R^*$. The cyclic assignment was widely used in the existing works on the distributed computing problems [5], [13]–[17]. For each dataset $D_k$ where $k \in [K]$, we assign $D_k$ to the workers in $\mathcal{H}_k$ where (recall that by convention, we let $\text{Mod}(b, a) = a$ if $a$ divides $b$)

$$\mathcal{H}_k = \{\text{Mod}(k, N), \text{Mod}(k - 1, N), \ldots, \text{Mod}(k - N + N_r - m + 1, N)\}.$$ \hfill (14)

Thus the set of datasets assigned to worker $n \in [N]$ is

$$Z_n = \bigcup_{p \in [0, \frac{K}{N} - 1]} \{\text{Mod}(n, N) + pN, \text{Mod}(n + 1, N) + pN, \ldots, \text{Mod}(n + N - N_r + m - 1, N) + pN\}$$ \hfill (15)

with cardinality $\frac{N_r}{N}(N - N_r + m)$. For each $m \in [N_r]$, the minimum communication cost under the cyclic assignment in (15) is denoted by $R^*_{cyc}$.

**Remark 1.** In the considered problem, the assumption that the desired function’s coefficients (i.e., the coefficients in demand matrix $F$) are uniformly i.i.d., is needed to get information theoretic converses and achievability with vanishing probability of error. As shown in [5, Remark 3], to satisfy some specific demand matrices, the optimal communication costs can be strictly higher than $R^*$. It is one of our on-going works to study the arbitrary demand matrices.

In contrast, the assumption that the symbols in each message are uniformly i.i.d., is only needed for the information theoretic converses, while the proposed computing scheme in this paper works for any arbitrary component functions $f_k(D_k)$ where $k \in [K]$. \hfill \Box
B. Review of the existing results for \( K_c = 1 \) or \( m = 1 \)

The sub-case of the considered problem for \( K_c = 1 \) and any \( m \) was studied in \([16], [17]\) and the sub-case for \( m = 1 \) and any \( K_c \) was studied in \([5]\). In the following, we review the computing schemes in the literature for the above two sub-cases.

1) \( K_c = 1 \): We first review the computing scheme in \([16], [17]\) for the case \( K_c = 1 \). The cyclic assignment described above is used for the data assignment phase. In the computing phase, we divide each message \( W_k, k \in [K] \), into \( m \) non-overlapping and equal-length sub-messages \( W_k = \{W_{k,i} : i \in [m]\} \) where each sub-message contains \( \frac{1}{m} \) symbols. Thus the desired linear combination by the master can be seen as \( m \) linear combinations of sub-messages with the same coefficients. The main idea is to let each worker send one linear combination of sub-messages, such that the master receives \( N_r \) linear combinations of sub-messages, among which it then recovers the \( m \) desired ones. We generate \( v = N_r - m \) virtually demanded linear combinations of sub-messages, such that the effective demand matrix (containing original and virtual demands) is with dimension \( N_r \times mK \) and with the form

\[
F' = \begin{bmatrix}
    f_{1,1} & \cdots & f_{1,K} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
    0 & \cdots & 0 & f_{1,1} & \cdots & f_{1,K} & \cdots & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & f_{1,1} & \cdots & f_{1,K} \\
    a_{1,1} & \cdots & a_{1,K} & a_{1,K+1} & \cdots & a_{1,2K} & \cdots & a_{1,(m-1)K+1} & \cdots & a_{1,mK} \\
    \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    a_{v,1} & \cdots & a_{v,K} & a_{v,K+1} & \cdots & a_{v,2K} & \cdots & a_{v,(m-1)K+1} & \cdots & a_{v,mK}
\end{bmatrix} \quad (16)
\]

The transmission of worker \( n \in [N] \) can be expressed as

\[
s^{n,1} F'[W_{1,1}; W_{2,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m}],
\]

where \( s^{n,1} = (s_1^{n,1}, \ldots, s_{N_t}^{n,1}) \) is the transmission vector for worker \( n \). The next step is to determine the values for each \( s^{n,1} \) where \( n \in [N] \). The authors in \([16]\) choose these values from a specific matrix while the authors in \([17]\) choose the value of each element in these vectors uniformly i.i.d over \( \mathbb{F}_q \). Here we use the random generation in \([17]\). Let us then focus on each column in \( F' \), which corresponds to a sub-message. For example, the first column of \( F' \) corresponds to \( W_{1,1} \), which cannot be computed by \( N_r - m = v \) workers, i.e., the workers in \( [K] \setminus H_1 \). Hence,
for each worker $n \in ([k] \setminus {h_1})$, it should satisfy

$$
0 = s_{n,1} f_{1,1} + s_{n,1}^1 0 + \ldots + s_{m,1}^1 a_{1,1} + s_{m+2,1}^1 a_{2,1} + \ldots + s_{n,1}^1 a_{v,1},
$$

such that in the transmitted linear combination of worker $n$ the coefficient of $W_{1,1}$ is 0. Since there are totally $v$ variables (i.e., $a_{1,1}, \ldots, a_{v,1}$) and $v$ linear constraints over these variables whose coefficients are uniformly i.i.d. over $\mathbb{F}_q$, we can solve these $v$ variables with high probability. By considering all the columns in $F'$, we can guarantee that in the transmitted linear combination of each worker, the coefficients of the sub-messages which it cannot compute are 0. Moreover, for each set $A \subseteq [n]$ where $|A| = n_r$, the $n_r$ vectors, $s^{A(1),1}, \ldots, s^{A(n_r),1}$, are linearly independent with high probability. Hence, the master can recover $F'[W_{1,1}; W_{2,1}; \ldots; W_{k,1}; W_{1,2}; \ldots; W_{k,m}]$ from the answer of workers in $A$.

It was proved in [16] that when $K_c = 1$, the communication cost $\frac{n_r}{m}$ is optimal under the constraint of linear coding in the computing phase and symmetric transmission (i.e., the number of symbols transmitted by each worker is the same).

2) $m = 1$: We then review the computing scheme in [5] for the case where $m = 1$. Here we focus on the regime where $\frac{K}{N} < K_c \leq \frac{K}{N} N_r$, because the remaining regimes of $K_c$ can be solved by an extension of the computing scheme in [5] for the above considered regime. The cyclic assignment is also used for the data assignment phase. In the computing phase, the main idea is to let each worker send $\frac{K}{N}$ linear combinations of messages, such that the master receives $N_r \frac{K}{N}$ linear combinations of messages, among which it then recovers the $K_c$ desired ones. We generate $\nu = \frac{K}{N} N_r - K_c$ virtually demanded linear combinations of messages, such that the effective demand matrix is

$$
F' = \begin{bmatrix}
  f_{1,1} & \cdots & f_{1,K} \\
  \vdots & \ddots & \vdots \\
  f_{K_c,1} & \cdots & f_{K_c,K} \\
  a_{1,1} & \cdots & a_{1,K} \\
  \vdots & \ddots & \vdots \\
  a_{\nu,1} & \cdots & a_{\nu,K}
\end{bmatrix}.
$$

Different from the computing scheme in [16], [17] for the case $K_c = 1$ where the transmission vectors of workers are first randomly picked, the computing scheme in [5] first choose the value
of each $a_{i,k}$ where $i \in [v]$ and $k \in [K]$ uniformly i.i.d over $\mathbb{F}_q$. The next step is to determine the transmission vectors of each worker $n \in [N]$, denoted by $s^{n,j}$ for $j \in [\frac{K}{N}]$, where the $j^{th}$ transmitted linear combination by worker $n$ is
\[
 s^{n,j} \mathbf{F}' [W_1; W_2; \ldots; W_K].
\] (19)

Notice that the number of messages which worker $n$ cannot compute is $|\{K\} \setminus Z_n| = \frac{K}{N}(N_r - 1)$. The sub-matrix of $\mathbf{F}'$ including the columns with the indices in $\{K\} \setminus Z_n$ has the dimension $\frac{K}{N} N_r \times \frac{K}{N}(N_r - 1)$. Since the elements in this sub-matrix are uniformly i.i.d. over $\mathbb{F}_q$, a vector basis for the left-side null space of this sub-matrix is the set of $\frac{K}{N}$ linearly independent vectors with high probability. Hence, we let $s^{n,j}$ where $j \in [\frac{K}{N}]$ be each of this left-side null space vector, such that in the linear combination $s^{n,j} \mathbf{F}' [W_1; W_2; \ldots; W_K]$ the coefficients of the messages which worker $n$ cannot compute are 0. It was also proved in [5] that for each set $A \subseteq [N]$ where $|A| = N_r$, the set of vectors $s^{n,j}$ where $n \in A$ and $j \in [\frac{K}{N}]$ are linearly independent with high probability, such that the master can recover $\mathbf{F}' [W_1; W_2; \ldots; W_K]$ from the answer of workers in $A$.

The communication cost by the computing scheme in [5] is $N_r K_c$ when $K_r \leq \frac{K}{N}$; is $\frac{KN_r}{N}$ when $\frac{K}{N} \leq K_c \leq \frac{K}{N} N_r$; is $K_c$ when $K_c \geq \frac{KN_r}{N}$. The communication cost is exactly optimal when $K = N$, or when $K_c \in \left[ \frac{K}{N} \frac{N_r + 1}{N_r - 1} \right]$, or when $K_c \in \left[ \frac{KN_r}{N} : K \right]$. In addition, it is optimal under the constraint of cyclic assignment when $N$ divides $K$.

## III. MAIN RESULTS

In this section, we present our novel results in this paper. We first provide a converse bound under the constraint of cyclic assignment, which will be proved in Section [IV]

**Theorem 1.** For the $(K, N, N_r, K_c, m)$ distributed linearly separable computation problem,

- when $K_c \in \left[ \frac{K}{N}(N_r - m + 1) \right]$, by defining $u := \left\lceil \frac{KN_r}{K} \right\rceil$, we have
  \[
  R^*_{\text{cyc}} \geq \frac{N_rK_c}{m + u - 1}.
  \] (20a)

- when $K_c \in \left[ \frac{K}{N}(N_r - m + 1) : K \right]$, we have
  \[
  R^*_{\text{cyc}} \geq R^* \geq K_c.
  \] (20b)
We then introduce the computation-communication costs tradeoff by the novel computing scheme in the following theorem.

**Theorem 2.** For the \((K, N, N_r, K_c, m)\) distributed linearly separable computation problem where

\[
40 \geq N \geq \frac{m + u - 1}{u} + u(N_r - m - u + 1),
\]

the computation-communication costs tradeoff \((m, R_{ach})\) is achievable, where

- when \(K_c \in \left[\frac{K}{N}\right]\),
  \[
  R_{ach} = \frac{KcN_r}{m}, \tag{22a}
  \]
- when \(K_c \in \left[\frac{K}{N} : \frac{K}{N}(N_r - m + 1)\right]\),
  \[
  R_{ach} = \frac{N_rKu}{N(m + u - 1)}; \tag{22b}
  \]
- when \(K_c \in \left[\frac{K}{N}(N_r - m + 1) : K\right]\),
  \[
  R_{ach} = K_c. \tag{22c}
  \]

□

Notice that the RHS of the constraint (21)

\[
N \geq \frac{m + u - 1}{u} + u(N_r - m - u + 1),
\]

will be explained in Remark 2 from a viewpoint of linear space dimension. It can be seen that in the first case of the proposed computing scheme (i.e., \(K_c \in \left[\frac{K}{N}\right]\)), we have \(u = 1\) and thus the constraint (23) always holds. In the third case of the proposed computing scheme (i.e., \(K_c \in \left[\frac{K}{N}(N_r - m + 1) : K\right]\)), we have \(u \geq N_r - m + 1\) and thus the constraint in (23) always holds.

While proving the decodability of the proposed computing scheme in Theorem 2, we use the Schwartz-Zippel lemma [21]–[23] in Appendix A. For the non-zero polynomial condition for the Schwartz-Zippel lemma, we numerically verify all cases that \(40 \geq N \geq \frac{m + u - 1}{u} + u(N_r - m - u + 1)\), and conjecture in the rest of the paper that the condition holds for any case where \(N \geq \frac{m + u - 1}{u} + u(N_r - m - u + 1)\), i.e., in Theorem 2 we replace the constraint (21) by (23).

In Section V for the sake of space limitation, we will only provide our novel computing scheme for the second case (22b) (i.e., \(K_c \in \left[\frac{K}{N} : \frac{K}{N}(N_r - m + 1)\right]\)). By the exactly same method
as described in [5] Sections IV-B and IV-C, the computing schemes for the first and third cases can be obtained by the direct extensions of the computing scheme for the second case. More precisely,

- \( K_c \in \left[ \frac{K}{N} \right] \). When \( K_c = 1 \), it can be easily shown (see [5] Section IV-B) that the \((K, N, N_r, 1, m)\) distributed linearly separable computation problem is equivalent to the \((N, N, N_r, 1, m)\) distributed linearly separable computation problem, which needs the communication cost \( \frac{N_r}{m} \) from (22b). For \( K_c \in \left[ 2 : \frac{K}{N} \right] \), we can treat the \((K, N, N_r, K_c, m)\) distributed linearly separable computation problem as \( K_c \) independent \((K, N, N_r, 1, m)\) distributed linearly separable computation problems; thus the communication cost is \( \frac{K N_r}{m} \), coinciding with (22a).

- \( K_c \in \left[ \frac{K}{N} (N_r - m + 1) : K \right] \). When \( K_c = \frac{K}{N} (N_r - m + 1) \), from (22b) it can be seen that the communication cost is \( \frac{N_r K u}{N (m + u - 1)} = \frac{K u}{N} = K_c \), coinciding with (22c). When \( K_c > \frac{K}{N} (N_r - m + 1) \), as in [5] Section IV-C, we can divide each demanded linear combination into \( \left( \frac{K}{N_r} (N_r - m + 1) - 1 \right) \) equal-length sub-combinations, each of which has \( \frac{L}{\left( \frac{K}{N_r} (N_r - m + 1) - 1 \right)} \) symbols.

We then treat the \((K, N, N_r, K_c, m)\) distributed linearly separable computation problem as \( \left( \frac{K}{N_r} (N_r - m + 1) \right) \) independent \((K, N, N_r, \frac{K}{N_r} (N_r - m + 1), m)\) distributed linearly separable computation sub-problems, where in each sub-problem we let the master recover \( \frac{K}{N_r} (N_r - m + 1) \) sub-combinations, with the communication cost \( \frac{N_r (N_r - m + 1)}{K_c} \); thus the total communication cost is

\[
\left( \frac{K}{N_r} (N_r - m + 1) \right) \frac{K}{N_r} (N_r - m + 1) \frac{K_c - 1}{K_c} = K_c
\]

coinciding with (22c).

By comparing the proposed converse bound in Theorem 1 and the proposed scheme in Theorem 2, we can directly have the following (order) optimality results.

**Theorem 3.** For the \((K, N, N_r, K_c, m)\) distributed linearly separable computation problem where \( N \geq \frac{m + u - 1}{u} + u(N_r - m - u + 1) \),

- when \( K = N \), we have

\[
R_{cyc}^* = R_{ach} = \begin{cases} 
\frac{N_r K c}{m + K_c - 1}, & \text{if } K_c \in [N_r - m + 1]; \\
K_c, & \text{if } K_c \in [N_r - m + 1 : K];
\end{cases}
\]  

(24)

- when \( K_c \in \left[ \frac{K}{N_r} \right] \), we have

\[
R_{cyc}^* = R_{ach} = \frac{N_r K_c}{m}.
\]  

(25)
• when $K_c \in \left[ \frac{K}{N} + 1 : \frac{K}{N}(N_r - m + 1) - 1 \right]$, we have

$$R^*_\text{cyc} \geq \frac{K_c}{K_u} R_{\text{ach}} \geq \frac{R_{\text{ach}}}{2};$$

(26)

• when $K_c \in \left[ \frac{K}{N}(N_r - m + 1) : K \right]$, we have

$$R^* = R^*_\text{cyc} = R_{\text{ach}} = K_c;$$

(27)

In words, for the considered problem satisfying the constraint in (23), when $K_c \in \left[ N_r - m + 1 : K \right]$, the proposed computing scheme is exactly optimal; when $K = N$ or $K_c \in \left[ \frac{K}{N} \right]$, the proposed computing scheme is optimal under the constraint of cyclic assignment; when $N$ divides $K$ and $K_c \in \left[ \frac{K}{N} + 1 : \frac{K}{N}(N_r - m + 1) - 1 \right]$, the proposed scheme is order optimal within a factor of $\frac{K_u}{K_c} \leq 2$ under the constraint of cyclic assignment.

Notice that when $K_c = 1$, the proposed computing scheme achieves the same communication load as in [16], [17], which was proved to be optimal under the constraint of linear coding in the computing phase and symmetric transmission. Instead, in this paper we prove that it is optimal only under the constraint of cyclic assignment.

In Fig. 1, we provide some numerical evaluations on the proposed converse and achievable bounds. For the sake of comparison, we introduce a baseline scheme. For the case where $K_c = 1$, the computing scheme in [16], [17] (reviewed in Section II-B) needs the communication cost $\frac{N_c}{m}$ for each $m \in [N]$. Hence, a simple baseline scheme can be obtained by treating the considered problem as $K_c$ independent sub-problems, where in each sub-problem the master recover one of its desired linear combination. Thus the communication cost for the baseline scheme is

$$R_{\text{base}} = \frac{K_c N_r}{m}, \ \forall m \in [N_r].$$

(28)

In Fig. 1a we consider the distributed linearly separable computation problem where $K = 20$, $N = 10$, $N_r = 8$, and $K_c = 8$. In this example, the constraint in (23) always holds. It can be seen from Fig. 1a that the proposed computing scheme outperforms the baseline scheme and coincides with the proposed converse bound.

In Fig. 1b we consider the distributed linearly separable computation problem where $K = 20$, $N = 10$, $N_r = 7$, $m = 2$. For each $K_c \in [20]$, we plot the communication costs. In this example, the constraint in (23) also always holds. It can be seen from Fig. 1b that the proposed computing scheme outperforms the baseline scheme. The propose scheme coincides with the proposed
IV. PROOF OF THEOREM 1

As shown in [5, Section II], since the elements of the demand matrix $F$ are uniformly i.i.d. over larger enough field $\mathbb{F}_q$, a simple cut-set bound argument yields

$$R^*_cyc \geq R^* \geq K_c,$$

which coincides with the converse bound in (20b) for the case $K_c \in \left[\frac{K}{N} (N_r - m + 1) : K\right]$. Hence, in the following we focus on the case $K_c \in \left[\frac{K}{N} (N_r - m + 1) \right]$.

We will use an example to illustrate the main idea.

**Example 1.** In this example, we have $N = K = 5$, $N_r = 4$, $m = 2$, and $K_c = 2$. Hence, the number of datasets assigned to each worker is $M = \frac{K}{N} (N - N_r + m) = 3$. Each dataset is assigned to 3 workers. With the cyclic assignment, we assign

| Worker 1 | Worker 2 | Worker 3 | Worker 4 | Worker 5 |
|----------|----------|----------|----------|----------|
| $D_1$    | $D_2$    | $D_3$    | $D_4$    | $D_5$    |
| $D_2$    | $D_3$    | $D_4$    | $D_5$    | $D_1$    |
| $D_3$    | $D_4$    | $D_5$    | $D_1$    | $D_2$    |
We consider the demand matrix $F$ whose dimension is $2 \times 5$ with elements uniformly i.i.d. over large field $\mathbb{F}_q$. Hence, the sub-matrix including each $K_c = 2$ columns is full-rank with high probability.

Notice that in this example the number of stragglers is $N - N_r = 1$. We first consider that worker 5 is the straggler; thus the master should recover $F[W_1; \ldots; W_5]$ from the answers of workers in $A = [4]$. In addition, each dataset is assigned to $N - N_r + m = 3$ workers. Hence, there must exist one dataset assigned to all the straggler(s) which is also assigned to $m$ responding workers. In this example, all of $D_1$, $D_2$, and $D_5$ belong to such datasets. Now we select one of them, e.g., $D_2$. Note that $D_2$ is assigned to workers $\mathcal{H}_2 = \{1, 2, 5\}$. We then consider the next dataset $D_{\text{Mod}(2+1, K)} = D_3$. The workers storing dataset $D_3$ (denoted by $\mathcal{H}_3$) is obtained by right-shifting $\mathcal{H}_2$ by one position, i.e., $\mathcal{H}_3 = \{1, 2, 3\}$. Hence, there is exactly one new worker in $\mathcal{H}_3$ who is not in $\mathcal{H}_2 \cap A$, which is worker 3. So we have

$$|\mathcal{H}_2 \cup \mathcal{H}_3 \cap A| = m + (2 - 1) = 3 = m + K_c - 1;$$

in other words, in the set of responding workers $A$, the number of workers who can compute $W_2$ or $W_3$ is equal to 3. In addition, the sub-matrix of $F$ including the columns in $\{2, 3\}$ is full-rank (with rank $K_c = 2$). Recall that each message has $L$ uniformly i.i.d. symbols. Hence, the number of transmitted symbols by workers in $(\mathcal{H}_2 \cup \mathcal{H}_3) \cap A$ should be no less than $2L$; thus

$$\sum_{n \in (\mathcal{H}_2 \cup \mathcal{H}_3) \cap A} T_n = T_1 + T_2 + T_3 \geq K_c L = 2L. \quad (30)$$

Similarly, considering that worker 4 is the straggler, we have

$$T_5 + T_1 + T_2 \geq K_c L = 2L. \quad (31)$$

Considering that worker 3 is the straggler, we have

$$T_4 + T_5 + T_1 \geq K_c L = 2L. \quad (32)$$

Considering that worker 2 is the straggler, we have

$$T_3 + T_4 + T_5 \geq K_c L = 2L. \quad (33)$$

Considering that worker 1 is the straggler, we have

$$T_2 + T_3 + T_4 \geq K_c L = 2L. \quad (34)$$
By summing (30)-(34), we have
\[
T_1 + T_2 + T_3 + T_4 + T_5 \geq \frac{10}{3} L,
\]
which leads that
\[
R^*_\text{cyc} \geq \max_{A \subseteq [9]: |A| = N - 4} \frac{\sum_{j \in A} T_j}{L} \geq \frac{8}{3},
\]
as the converse bound in (20a).

We are now ready to generalize the proposed converse bound under the constraint of cyclic assignment in Example 1. Recall that we consider the case where \( K_c \in \left[ \frac{K}{N} (N_r - m + 1) \right] \) and that \( u = \left\lfloor \frac{K N}{K} \right\rfloor \). The demand matrix \( F \) has dimension \( K_c \times K \) with elements uniformly i.i.d. over large field. Hence, the sub-matrix including each \( K \) columns is full-rank with high probability. By the cyclic assignment, as shown in (14), each dataset \( D_k \) is assigned to workers \( H_k = \{ \text{Mod}(k, N), \text{Mod}(k - 1, N), \ldots, \text{Mod}(k - N + N_r - m + 1, N) \} \).

We consider the set of stragglers whose are adjacent. Thus each time we choose one integer \( n \in [N] \), let \( S_n := \{ \text{Mod}(n, N), \text{Mod}(n - 1, N), \ldots, \text{Mod}(n - N + N_r + 1, N) \} \) where \( |S_n| = N - N_r \), be the set of stragglers. The master should recover \( F[W_1; \ldots; W_K] \) from the answers of workers in \([N] \setminus S_n\). From the cyclic assignment, there are exactly \( \frac{K}{N} \) datasets, denoted by \( U_0 = \{ \text{Mod}(n + m, N) + pN : p \in \left[ 0 : \frac{K}{N} - 1 \right] \} \), which are exclusively assigned to the workers in
\[
H_{U_0} = S_n \cup \{ \text{Mod}(n + 1, N), \text{Mod}(n + 2, N), \ldots, \text{Mod}(n + m, N) \},
\]
are exclusively assigned to the workers in
\[
H_{U_t} = \{ \text{Mod}(n - N + N_r + i + 1, N), \text{Mod}(n - N + N_r + i + 2, N), \ldots, \text{Mod}(n + m + i, N) \}.
\]

It can be seen that there are totally \( \frac{K}{N} u \) datasets in \( \cup_{i \in [0:u-1]} U_t \), which are exclusively assigned to the workers in
\[
\cup_{i \in [0:u-1]} H_{U_t} = \{ \text{Mod}(n - N + N_r + 1, N), \text{Mod}(n - N + N_r + 2, N), \ldots, \text{Mod}(n + m + u - 1, N) \} = S_n \cup \{ \text{Mod}(n + 1, N), \ldots, \text{Mod}(n + m + u - 1, N) \}.
\]
Note that since \( u \leq N_r - m + 1 \), we have \( S_n \cap \{ \text{Mod}(n + 1, N), \ldots, \text{Mod}(n + m + u - 1, N) \} = \emptyset \).
In other words, the number of responding workers in $\bigcup_{i \in [0:u-1]} \mathcal{H}_{U_i}$ is
$$\left| \left( \bigcup_{i \in [0:u-1]} \mathcal{H}_{U_i} \right) \cap ([N] \setminus S_n) \right| = |\{\text{Mod}(n + 1, N), \ldots, \text{Mod}(n + m + u - 1, N)\}| = m + u - 1.$$ Since $\frac{K}{N}u \geq K_c$, the sub-matrix of the demand matrix including the columns in $\bigcup_{i \in [0:u-1]} U_i$ has a rank equal to $K_c$. Hence, the number of transmitted symbols by workers in $\{\text{Mod}(n + 1, N), \ldots, \text{Mod}(n + m + u - 1, N)\}$ should be no less than $K_c L$; thus
$$\sum_{j \in \{\text{Mod}(n+1,N),\ldots,\text{Mod}(n+m+u-1,N)\}} T_j \geq K_c L. \quad (37)$$
By considering all $n \in [N]$ and summing all the inequalities as in (37), we have
$$\sum_{j \in [N]} T_j \geq \frac{NK_c}{m + u - 1} L, \quad (38)$$
which leads that
$$R^*_{\text{cyc}} \geq \frac{\max_{A \subseteq [N] : |A| = N_r} \sum_{j \in A} T_j}{L} \geq \frac{N_r K_c}{m + u - 1}, \quad (39)$$
as the converse bound in (20a).

V. PROOF OF (22b)

We focus on the case where $K_c \in \left[\frac{K}{N} : \frac{K}{N}(N_r - m + 1)\right]$. We first illustrate the main idea in the following example.

Example 2. In this example, we have $N = K = 6$, $N_r = 5$, $m = 2$, and $K_c = 2$. Since $N = K$ in this example, we have $u = K_c = 2$. We assume the demand matrix is
$$F = \begin{bmatrix} f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & f_{1,5} & f_{1,6} \\ f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & f_{2,5} & f_{2,6} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}. \quad (40)$$

Data assignment phase: The number of datasets assigned to each worker is $M = \frac{K}{N}(N - N_r + m) = 3$. We use the cyclic assignment, to assign

| Worker 1 | Worker 2 | Worker 3 | Worker 4 | Worker 5 | Worker 6 |
|----------|----------|----------|----------|----------|----------|
| $D_1$    | $D_2$    | $D_3$    | $D_4$    | $D_5$    | $D_6$    |
| $D_2$    | $D_3$    | $D_4$    | $D_5$    | $D_6$    | $D_1$    |
| $D_3$    | $D_1$    | $D_5$    | $D_6$    | $D_1$    | $D_2$    |

Computing phase: Since the communication cost is no less than $N_r \frac{K_c}{m + K_c - 1} = \frac{10}{3}$ from the converse bound (20a), we divide each message $W_k$ where $k \in [6]$ into $m + K_c - 1 = 3$ non-
overlapping and equal-length sub-messages, \( W_k = \{ W_{k,j} : j \in [3] \} \). Each worker should send \( K_c = 2 \) linear combinations of sub-messages. From the answers of \( N_c = 5 \) workers, the master totally receives \( N_c K_c = 10 \) linear combinations of sub-messages, which contain the desired \((m+K_c-1)K_c = 6 \) linear combinations. Hence, we generate \( v = 10 - 6 = 4 \) virtually demanded linear combinations of sub-messages; thus the effective demand matrix (i.e., containing original and virtual demands) is

\[
F'[W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}]
\]

where \( F' \) has dimension \( N_c K_c \times K(m+K_c-1) = 10 \times 18 \), with the form

\[
F' = \begin{bmatrix}
1 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 5 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 1 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 5 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \ldots & 0 & \ldots & 0 \\
\end{bmatrix}
\]

The transmissions of the 6 workers can be expressed as

\[
S \ F' [W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}] = [s^{1,1}; s^{1,2}; s^{2,1}; \ldots; s^{6,2}] \ F' [W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}],
\]

where the row vector \( s^{n,j} \) represents the \( j \)th transmission vector of worker \( n \); in other words, \( s^{n,j} F' [W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}] \) represents the \( j \)th transmitted linear combination by worker \( n \). We can further expand \( S \) as follows,

\[
S = \begin{bmatrix}
s^{1,1} \\ s^{1,2} \\ s^{2,1} \\ s^{2,2} \\ \vdots \\ s^{6,2}
\end{bmatrix} = \begin{bmatrix}
s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b_1 & b_2 & b_3 & b_4 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b_1 & b_2 & b_3 & b_4 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b_1 & b_2 & b_3 & b_4 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b_1 & b_2 & b_3 & b_4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & b_1 & b_2 & b_3 & b_4
\end{bmatrix}
\]
Now the $j^{th}$ transmitted linear combination by worker $n$ can be expressed as

$$s^{n,j}_d_1 W_{1,1} + s^{n,j}_d_2 W_{2,1} + \cdots + s^{n,j}_d_6 W_{6,1} + s^{n,j}_d_7 W_{1,2} + \cdots + s^{n,j}_d_18 W_{6,3}, \quad (45)$$

where $d_i$ represents the $i^{th}$ column of $F'$. Recall that $Z_n \subseteq [K]$ represents the set of messages which are not assigned to worker $n$. Hence, to guarantee that the linear combination in (45) can be transmitted by worker $n$, we should have

$$s^{n,j}_d_k + (t - 1)K = 0, \quad \forall n \in [6], j \in [2], t \in [3], k \in Z_n. \quad (46)$$

In addition, for each set $A \subseteq [6]$ where $|A| = 5$, by receiving the linear combinations transmitted by the workers in $A$, the master should recover the desired linear combinations. Hence, we should have (recalling that $A(i)$ represents the $i^{th}$ smallest element of $A$)

$$[s^{A(1),1}; s^{A(1),2}; s^{A(2),1}; \ldots; s^{A(5),2}] \text{ is full rank, } \forall A \subseteq [6] : |A| = 5. \quad (47)$$

Our objective is to determine the variables in $S$ and in $F'$ such that the constraints in (46) and (47) are satisfied.

We divide matrix $F'$ into 3 sub-matrices, $F'_1, F'_2, F'_3$ each of which has the dimension $10 \times 6$, as illustrated in (42). We also divide matrix $S$ into 4 sub-matrices, $S_1, S_2, S_3$ each of which has the dimension $12 \times 2$ and $S_4$ with dimension $12 \times 4$, as illustrated in (44).

The proposed computing scheme in the computing phase contains three main steps:

1) we first choose the values for the variables in $S_4$;

2) after determining $S_4$, the constraints in (45) become linear in terms of the remaining variables (i.e., the variables in $F'_1, F'_2, F'_3, S_1, S_2, S_3$). Hence, we can obtain the values for these remaining variables by solving linear equations;

3) after determining all the variables, we check that the constraints in (47) such that the proposed scheme is decodable.

---

5 Notice that the computing schemes in [16], [17] for the case $K_e = 1$ and in [5] for the case where $m = 1$ cannot be used in this example to achieve the converse bound. The idea of the computing schemes in [16], [17] is first to randomly determine the variables in $S$, and then to determine the coefficients of the virtually demanded linear combinations in $F'$ in order to satisfy the constraints in (46). One can check that if we randomly choose all the variables in $S$, there does not exist any solution on $F'$ which satisfies the constraints in (46), because there will be more linearly independent constraints than the variables. The idea of the computing scheme in [5] is first to randomly determine the coefficients of the virtually demanded linear combinations in $F'$, and then to determine the variables in $S$ in order to satisfy the constraints in (46). However, one can check that if we randomly determine the coefficients of the virtually demanded linear combinations in $F'$, we cannot find any solution of $S$ satisfying the constraints in (46), where the two transmission vectors of each worker in $S$ are linearly independent.
Step 1: We choose the values for $S_4$ with the following form,

$$
S_4 = \begin{bmatrix}
  b_{1,1}^{1,1} & b_{2,1}^{1,1} & b_{3,1}^{1,1} & b_{4,1}^{1,1} \\
  b_{1,1}^{1,2} & b_{2,2}^{1,2} & b_{3,1}^{1,2} & b_{4,2}^{1,2} \\
  b_{1,1}^{2,1} & b_{2,1}^{2,1} & b_{3,1}^{2,1} & b_{4,1}^{2,1} \\
  b_{1,1}^{2,2} & b_{2,2}^{2,2} & b_{3,1}^{2,2} & b_{4,2}^{2,2} \\
  b_{1,1}^{3,1} & b_{2,1}^{3,1} & b_{3,1}^{3,1} & b_{4,1}^{3,1} \\
  b_{1,1}^{3,2} & b_{2,1}^{3,2} & b_{3,1}^{3,2} & b_{4,1}^{3,2} \\
  b_{1,1}^{4,1} & b_{2,1}^{4,1} & b_{3,1}^{4,1} & b_{4,1}^{4,1} \\
  b_{1,1}^{4,2} & b_{2,1}^{4,2} & b_{3,1}^{4,2} & b_{4,1}^{4,2} \\
  b_{1,1}^{5,1} & b_{2,1}^{5,1} & b_{3,1}^{5,1} & b_{4,1}^{5,1} \\
  b_{1,1}^{5,2} & b_{2,1}^{5,2} & b_{3,1}^{5,2} & b_{4,1}^{5,2} \\
  b_{1,1}^{6,1} & b_{2,1}^{6,1} & b_{3,1}^{6,1} & b_{4,1}^{6,1} \\
  b_{1,1}^{6,2} & b_{2,1}^{6,2} & b_{3,1}^{6,2} & b_{4,1}^{6,2} \\
\end{bmatrix}
\begin{bmatrix}
  * & * & 0 & 0 \\
  0 & 0 & * & * \\
  * & * & 0 & 0 \\
  0 & 0 & * & * \\
  * & * & 0 & 0 \\
  0 & 0 & * & * \\
  * & * & 0 & 0 \\
  0 & 0 & * & * \\
  * & * & 0 & 0 \\
  0 & 0 & * & * \\
\end{bmatrix}
= \begin{bmatrix}
  0 & 2 & 0 & 0 \\
  0 & 0 & 2 & 0 \\
  2 & 0 & 0 & 0 \\
  0 & 0 & 0 & 2 \\
  1 & 2 & 0 & 0 \\
  0 & 0 & 2 & 1 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  1 & 0 & 0 & 0 \\
  0 & 0 & 2 & 1 \\
  2 & 2 & 0 & 0 \\
  0 & 0 & 1 & 1 \\
\end{bmatrix},
$$

where each ‘*’ represents an uniform i.i.d. symbol on $\mathbb{F}_q$. More precisely, for the first linear combination transmitted by each worker $n \in [6]$, we choose $b_1^{n,1}$ and $b_2^{n,1}$ uniformly i.i.d. over $\mathbb{F}_q$, while letting $b_3^{n,1}$ and $b_4^{n,1}$ be zero. For the second linear combination transmitted by each worker $n$, we choose $b_3^{n,2}$ and $b_4^{n,2}$ uniformly i.i.d. over $\mathbb{F}_q$, while letting $b_1^{n,2}$ and $b_2^{n,2}$ be zero. The above choice on $S_4$ will guarantee that the constraints in (45) become linearly independent in terms of the remaining variables to be decided in the next step.

Step 2: Let us focus on the constraints in (46) for $t = 1$, which corresponds to the variables in $S_1$ and $S'_1$.

When $(t, j) = (1, 1)$, the constraints in (46) become

$$
 s_1^{n,1} f_{1,k} + s_2^{n,1} f_{2,k} + b_1^{n,1} a_{1,k} + b_2^{n,1} a_{2,k} + b_3^{n,1} a_{3,k} + b_4^{n,1} a_{4,k} = 0, \ \forall n \in [6], k \in \mathbb{F}_q, \tag{49}
$$

where $f_{1,k}$ represents the $k^{th}$ element in the first demand vector, $f_{2,k}$ represents the $k^{th}$ element in the second demand vector, and the values of $b_i^{n,1}$ where $i \in [4]$ have been chosen in (48). For example, if $n = 1$, we have the set of datasets which are not assigned to worker 1 is $\mathbb{F}_1 = \{4, 5, 6\}$. Hence, we have the following three constraints

$$
s_1^{1,1} f_{1,4} + s_2^{1,1} f_{2,4} + b_1^{1,1} a_{1,4} + b_2^{1,1} a_{2,4} + b_3^{1,1} a_{3,4} + b_4^{1,1} a_{4,4} = 1s_1^{1,1} + 3s_2^{1,1} + 0a_{1,4} + 2a_{2,4} = 0, \\
s_1^{1,1} f_{1,5} + s_2^{1,1} f_{2,5} + b_1^{1,1} a_{1,5} + b_2^{1,1} a_{2,5} + b_3^{1,1} a_{3,5} + b_4^{1,1} a_{4,5} = 1s_1^{1,1} + 4s_2^{1,1} + 0a_{1,5} + 2a_{2,5} = 0, \\
s_1^{1,1} f_{1,6} + s_2^{1,1} f_{2,6} + b_1^{1,1} a_{1,6} + b_2^{1,1} a_{2,6} + b_3^{1,1} a_{3,6} + b_4^{1,1} a_{4,6} = 1s_1^{1,1} + 5s_2^{1,1} + 0a_{1,6} + 2a_{2,6} = 0.
$$
Similarly, if \( n = 2 \), with \( \overline{Z}_2 = \{1, 5, 6\} \) we have the following three constraints

\[
\begin{align*}
& s_{1}^{2,1} f_{1,1} + s_{2}^{2,1} f_{2,1} + b_{1}^{2,1} a_{1,1} + b_{2}^{2,1} a_{2,1} + b_{3}^{2,1} a_{3,1} + b_{4}^{2,1} a_{4,1} = 1 s_{1}^{2,1} + 0 s_{2}^{2,1} + 2 a_{1,1} + 2 a_{2,1} = 0, \\
& s_{1}^{2,1} f_{1,5} + s_{2}^{2,1} f_{2,5} + b_{1}^{2,1} a_{1,5} + b_{2}^{2,1} a_{2,5} + b_{3}^{2,1} a_{3,5} + b_{4}^{2,1} a_{4,5} = 1 s_{1}^{2,1} + 4 s_{2}^{2,1} + 2 a_{1,5} + 2 a_{2,5} = 0, \\
& s_{1}^{2,1} f_{1,6} + s_{2}^{2,1} f_{2,6} + b_{1}^{2,1} a_{1,6} + b_{2}^{2,1} a_{2,6} + b_{3}^{2,1} a_{3,6} + b_{4}^{2,1} a_{4,6} = 1 s_{1}^{2,1} + 5 s_{2}^{2,1} + 2 a_{1,6} + 2 a_{2,6} = 0.
\end{align*}
\]

If \( n = 3 \), with \( \overline{Z}_3 = \{1, 2, 6\} \) we have the following three constraints

\[
\begin{align*}
& s_{1}^{3,1} f_{1,1} + s_{2}^{3,1} f_{2,1} + b_{1}^{3,1} a_{1,1} + b_{2}^{3,1} a_{2,1} + b_{3}^{3,1} a_{3,1} + b_{4}^{3,1} a_{4,1} = 1 s_{1}^{3,1} + 0 s_{2}^{3,1} + 1 a_{1,1} + 2 a_{2,1} = 0, \\
& s_{1}^{3,1} f_{1,2} + s_{2}^{3,1} f_{2,2} + b_{1}^{3,1} a_{1,2} + b_{2}^{3,1} a_{2,2} + b_{3}^{3,1} a_{3,2} + b_{4}^{3,1} a_{4,2} = 1 s_{1}^{3,1} + 1 s_{2}^{3,1} + 1 a_{1,2} + 2 a_{2,2} = 0, \\
& s_{1}^{3,1} f_{1,6} + s_{2}^{3,1} f_{2,6} + b_{1}^{3,1} a_{1,6} + b_{2}^{3,1} a_{2,6} + b_{3}^{3,1} a_{3,6} + b_{4}^{3,1} a_{4,6} = 1 s_{1}^{3,1} + 5 s_{2}^{3,1} + 1 a_{1,6} + 2 a_{2,6} = 0.
\end{align*}
\]

If \( n = 4 \), with \( \overline{Z}_4 = \{1, 2, 3\} \) we have the following three constraints

\[
\begin{align*}
& s_{1}^{4,1} f_{1,1} + s_{2}^{4,1} f_{2,1} + b_{1}^{4,1} a_{1,1} + b_{2}^{4,1} a_{2,1} + b_{3}^{4,1} a_{3,1} + b_{4}^{4,1} a_{4,1} = 1 s_{1}^{4,1} + 0 s_{2}^{4,1} + 0 a_{1,1} + 1 a_{2,1} = 0, \\
& s_{1}^{4,1} f_{1,2} + s_{2}^{4,1} f_{2,2} + b_{1}^{4,1} a_{1,2} + b_{2}^{4,1} a_{2,2} + b_{3}^{4,1} a_{3,2} + b_{4}^{4,1} a_{4,2} = 1 s_{1}^{4,1} + 1 s_{2}^{4,1} + 0 a_{1,2} + 1 a_{2,2} = 0, \\
& s_{1}^{4,1} f_{1,3} + s_{2}^{4,1} f_{2,3} + b_{1}^{4,1} a_{1,3} + b_{2}^{4,1} a_{2,3} + b_{3}^{4,1} a_{3,3} + b_{4}^{4,1} a_{4,3} = 1 s_{1}^{4,1} + 2 s_{2}^{4,1} + 0 a_{1,3} + 1 a_{2,3} = 0.
\end{align*}
\]

If \( n = 5 \), with \( \overline{Z}_5 = \{2, 3, 4\} \) we have the following three constraints

\[
\begin{align*}
& s_{1}^{5,1} f_{1,2} + s_{2}^{5,1} f_{2,2} + b_{1}^{5,1} a_{1,2} + b_{2}^{5,1} a_{2,2} + b_{3}^{5,1} a_{3,2} + b_{4}^{5,1} a_{4,2} = 1 s_{1}^{5,1} + 1 s_{2}^{5,1} + 1 a_{1,2} + 2 a_{2,2} = 0, \\
& s_{1}^{5,1} f_{1,3} + s_{2}^{5,1} f_{2,3} + b_{1}^{5,1} a_{1,3} + b_{2}^{5,1} a_{2,3} + b_{3}^{5,1} a_{3,3} + b_{4}^{5,1} a_{4,3} = 1 s_{1}^{5,1} + 2 s_{2}^{5,1} + 1 a_{1,3} + 0 a_{2,3} = 0, \\
& s_{1}^{5,1} f_{1,4} + s_{2}^{5,1} f_{2,4} + b_{1}^{5,1} a_{1,4} + b_{2}^{5,1} a_{2,4} + b_{3}^{5,1} a_{3,4} + b_{4}^{5,1} a_{4,4} = 1 s_{1}^{5,1} + 3 s_{2}^{5,1} + 1 a_{1,4} + 0 a_{2,4} = 0.
\end{align*}
\]

If \( n = 6 \), with \( \overline{Z}_6 = \{3, 4, 5\} \) we have the following three constraints

\[
\begin{align*}
& s_{1}^{6,1} f_{1,3} + s_{2}^{6,1} f_{2,3} + b_{1}^{6,1} a_{1,3} + b_{2}^{6,1} a_{2,3} + b_{3}^{6,1} a_{3,3} + b_{4}^{6,1} a_{4,3} = 1 s_{1}^{6,1} + 2 s_{2}^{6,1} + 2 a_{1,3} + 2 a_{2,3} = 0, \\
& s_{1}^{6,1} f_{1,4} + s_{2}^{6,1} f_{2,4} + b_{1}^{6,1} a_{1,4} + b_{2}^{6,1} a_{2,4} + b_{3}^{6,1} a_{3,4} + b_{4}^{6,1} a_{4,4} = 1 s_{1}^{6,1} + 3 s_{2}^{6,1} + 2 a_{1,4} + 2 a_{2,4} = 0, \\
& s_{1}^{6,1} f_{1,5} + s_{2}^{6,1} f_{2,5} + b_{1}^{6,1} a_{1,5} + b_{2}^{6,1} a_{2,5} + b_{3}^{6,1} a_{3,5} + b_{4}^{6,1} a_{4,5} = 1 s_{1}^{6,1} + 4 s_{2}^{6,1} + 2 a_{1,5} + 2 a_{2,5} = 0.
\end{align*}
\]

Hence, there are totally \( 6 \times 3 = 18 \) constraints on 24 variables, which are

\[
a_{1,1}, \ldots, a_{1,6}, a_{2,1}, \ldots, a_{2,6}, b_{1}^{1,1}, b_{2}^{1,1}, b_{1}^{2,1}, b_{2}^{2,1}, \ldots, b_{1}^{6,1}.
\]

(56)

We then give a random value to each of \( s_{1}^{1,1}, s_{2}^{2,1}, s_{3}^{3,1}, s_{4}^{4,1}, s_{5}^{5,1}, s_{6}^{6,1} \), totally 6 variables among
variables, such that we can solve
\[ \begin{bmatrix} 1 & 5 & 5 & 15 & 21 & 27 \\ 4 & 8 & 8 & 8 & 8 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -33 & -57 & -49 & 139 & 161 & -191 \end{bmatrix}; \] (59a)

\[ [a_{1,1}, \ldots, a_{1,18}] = \begin{bmatrix} 1 & 5 & 5 & 15 & 21 & 27 \\ 4 & 8 & 8 & 8 & 8 & 8 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -33 & -57 & -49 & 139 & 161 & -191 \end{bmatrix}; \] (59b)

\[ [a_{2,1}, \ldots, a_{2,18}] = \begin{bmatrix} -5 & -13 & -21 & -15 & -5 & -25 \\ 8 & 8 & 8 & 4 & 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 25 & 25 & 25 & 33 & 11 & -55 \end{bmatrix}; \] (59c)

\[ [a_{3,1}, \ldots, a_{3,18}] = \begin{bmatrix} 19 & 19 & 19 & 41 & 55 & 69 \\ 2 & 8 & 2 & 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} -41 & -43 & -45 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -109 & -68 & 0 & 0 & 0 & 0 \end{bmatrix}; \] (59d)

\[ [a_{4,1}, \ldots, a_{4,18}] = \begin{bmatrix} -20 & -10, 0 & -12 & -20 & -20, 41 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} -25 & -13 & -21 \end{bmatrix} \begin{bmatrix} 51 & 39 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -109 & -68 & 0 & 0 & 0 & 0 \end{bmatrix}; \] (59e)

the 24 variables in (56), as follows,
\[ s_{1,1}^{1} = 0, \ s_{2,1}^{1} = 1, \ s_{3,1}^{1} = 1, \ s_{4,1}^{1} = 1, \ s_{5,1}^{1} = 0, \ s_{6,1}^{1} = 1. \] (57)

After determining the 6 variables in (57), it can be checked that the above 18 constraints are
linearly independent on the remaining 18 variables, such that we can solve

\[ a_{1,1} = 1/4, \ a_{1,2} = 5/8, \ a_{1,3} = 5/4, \ a_{1,4} = 15/8, \ a_{1,5} = 21/8, \ a_{1,6} = 27/8, \] (58a)

\[ a_{2,1} = -5/8, \ a_{2,2} = -13/8, \ a_{2,3} = -21/8, \ a_{2,4} = -15/4, \ a_{2,5} = -5, \ a_{2,6} = -25/4, \] (58b)

\[ s_{1,1}^{2} = 5/2, \ s_{2,1}^{2} = 3/4, \ s_{3,1}^{2} = 13/8, \ s_{4,1}^{2} = 5/8, \ s_{5,1}^{2} = -5/8, \ s_{6,1}^{2} = 3/4. \] (58c)

Similarly, by considering all pairs \((t, j)\) where \(t \in [3]\) and \(j \in [2]\), we can determine (59).
Step 3: For each subset of workers $\mathcal{A} \subseteq [6]$ where $|\mathcal{A}| = 5$, it can be seen that the constraints in (47) holds. For example, if $\mathcal{A} = [5]$, the sub-matrix $S^{(10)}$, including the first 10 rows of $S$ is full-rank. Hence, we let each worker $n$ compute and send two linear combinations of sub-messages, $s^{n,1}F'[W_{1,1}; \ldots; W_{6,3}]$ and $s^{n,2}F'[W_{1,1}; \ldots; W_{6,3}]$.

Decoding phase: Assume that the set of responding workers is $\mathcal{A}$ where $\mathcal{A} \subseteq [6]$ and $|\mathcal{A}| = 5$. The master receives
\[
X_\mathcal{A} = [s^{A(1),1}; s^{A(1),2}; s^{A(2),1}; \ldots; s^{A(5),2}] F' [W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}].
\] Since $[s^{A(1),1}; s^{A(1),2}; s^{A(2),1}; \ldots; s^{A(5),2}]$ is full-rank, the master then computes
\[
[s^{A(1),1}; s^{A(1),2}; s^{A(2),1}; \ldots; s^{A(5),2}]^{-1} X_\mathcal{A}
\] to obtain $F' [W_{1,1}; \ldots; W_{6,1}; W_{1,2}; \ldots; W_{6,3}]$, which contains its demanded linear combinations.

Performance: Since each worker sends $\frac{2K_u}{3}$ symbols, the communication cost is $\frac{10L}{3K_u} = \frac{10}{3}$, coinciding with the converse bound in (20b).

We are ready to generalize the proposed distributed computing scheme in Example 2. First we focus on $K_c = \frac{K_u}{u}$, where $u \in [N, m + 1]$ and $N \geq \frac{m+u-1}{u} + u(N_t - m - u + 1)$. During the data assignment phase, we use the cyclic assignment.

Computing phase: Since the communication cost is no less than $N_t \frac{K_u}{m+u-1}$, from the converse bound (20b), we divide each message $W_k$ where $k \in [K]$ into $m+u-1$ non-overlapping and equal-length sub-messages, $W_k = \{W_{k,j} : j \in [m+u-1]\}$. Each worker should send $K_c$ linear combinations of sub-messages. From the answers of $N_t$ workers, the master totally receives $N_t K_c$ linear combinations of sub-messages. Hence, we generate
\[
v = N_t K_c - (m+u-1)K_c = K_c(N_t - m - u + 1)
\] virtually requested linear combinations; thus the effective demand matrix $F'$ has dimension $N_t K_c \times K(m+u-1)$, with the form in (61).

The transmissions of the $K$ workers can be expressed as
\[
S F'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}]
\]
\[
= [s^{1,1}; \ldots; s^{1,K_c}; s^{2,1}; \ldots; s^{N,K_c}] F'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}],
\] where $s^{n,j}F'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}]$ represents the $j$th transmitted linear combina-
To guarantee that the linear combination in (45) can be transmitted by worker $n$, we can further expand $S$ as follows,

$$S = \begin{bmatrix}
    S_{1,1} & \cdots & S_{1,K_c} \\
    \vdots & \ddots & \vdots \\
    S_{N,K_c} & \cdots & S_{N,K_c}
\end{bmatrix}
= \begin{bmatrix}
    S_{1,1} & \cdots & S_{1,K_c} & \cdots & S_{(m+u-2)K_c+1} & \cdots & S_{(m+u-1)K_c} \\
    \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
    S_{N,K_c} & \cdots & S_{N,K_c} & \cdots & S_{N,K_c} & \cdots & S_{N,K_c}
\end{bmatrix} = \begin{bmatrix}
    b_{1,1} & \cdots & b_{1,K_c} \\
    \vdots & \ddots & \vdots \\
    b_{N,K_c} & \cdots & b_{N,K_c}
\end{bmatrix}.
$$

By defining $d_i$ as the $i$th column of $F'$, the $j$th transmitted linear combination by worker $n$ can be expressed as

$$s^{n,j}d_1W_{1,1} + \cdots + s^{n,j}d_KW_{1,K} + s^{n,j}d_{K+1}W_{1,2} + \cdots + s^{n,j}d_{(m+u-1)K}W_{1,m+u-1}.$$  

To guarantee that the linear combination in (45) can be transmitted by worker $n$, the coefficients of the sub-messages which worker $n$ cannot compute should be 0; that is

$$s^{n,j}d_{k+(t-1)K} = 0, \quad \forall n \in [N], j \in [K_c], t \in [m + u - 1], k \in \mathbb{Z}_n.$$  

In addition, for each set $A \subseteq [N]$ where $|A| = N_t$, by receiving the linear combinations
transmitted by the workers in $A$, the master should recover the desired linear combinations. Hence, we should have

$$ [s^{A(1)}, \ldots; s^{A(K_c)}; s^{A(N_r)\cdot K_c}] \text{ is full rank, } \forall A \subseteq [N] : |A| = N_r. \quad (66) $$

Our objective is to determine the variables in $S$ (i.e., $s^{n\cdot j}_i$ where $n \in [N]$, $j \in [K_c]$, $i \in [(m + u - 1)K_c]$; $b^{n\cdot j}_i$ where $n \in [N]$, $j \in [K_c]$, $i \in [v]$) and in $F'$ (i.e., $a_{i,k}$ where $i \in [v]$ and $k \in [(m + u - 1)K]$) such that the constraints in (65) and (66) are satisfied.

We divide matrix $F'$ into $m + u - 1$ sub-matrices, $F'_1, \ldots, F'_{m+u-1}$ each of which has the dimension $N_rK_c \times K$, as illustrated in (61). We also divide matrix $S$ into $m + u$ sub-matrices, $S_1, \ldots, S_{m+u-1}$ each of which has the dimension $NK_c \times K_c$ and $S_{m+u}$ with dimension $NK_c \times v$, as illustrated in (63). As in Example 2, the proposed computing scheme contains three main steps:

1) we first choose the values for the variables in $S_{m+u}$;
2) after determining the variables in $S_{m+u}$, the constraints in (65) become linear in terms of the remaining variables, which are then determined by solving linear equations;
3) after determining all the variables, we check that the constraints in (66) such that the proposed scheme is decodable.

**Step 1:** We choose the values for $S_{m+u}$ with the following form,

$$ S_{m+u} = \begin{bmatrix}
  b^{1,1}_{1,1} & \ldots & b^{1,1}_{1,K_c} & b^{1,1}_{2,1} & \ldots & b^{1,1}_{2,K_c} & \ldots & b^{1,1}_{1,v} & b^{1,2}_{1,v} & \ldots & b^{1,2}_{v-1,v} & b^{1,v}_{v-1,v} & b^{1,v}_{v,v} \\
  b^{1,2}_{1,1} & \ldots & b^{1,2}_{1,K_c} & b^{1,2}_{2,1} & \ldots & b^{1,2}_{2,K_c} & \ldots & b^{1,2}_{1,v} & b^{1,2}_{2,v} & \ldots & b^{1,2}_{v-1,v} & b^{1,v}_{v-1,v} & b^{1,v}_{v,v} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  b^{1,K_c}_{1,1} & \ldots & b^{1,K_c}_{1,K_c} & b^{1,K_c}_{2,1} & \ldots & b^{1,K_c}_{2,K_c} & \ldots & b^{1,K_c}_{1,v} & b^{1,K_c}_{2,v} & \ldots & b^{1,K_c}_{v-1,v} & b^{1,K_c}_{v,v} \\
  b^{2,1}_{1,1} & \ldots & b^{2,1}_{1,K_c} & b^{2,1}_{2,1} & \ldots & b^{2,1}_{2,K_c} & \ldots & b^{2,1}_{1,v} & b^{2,1}_{2,v} & \ldots & b^{2,1}_{v-1,v} & b^{2,1}_{v,v} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  b^{N,K_c}_{1,1} & \ldots & b^{N,K_c}_{1,K_c} & \ldots & b^{N,K_c}_{2,v} & \ldots & b^{N,K_c}_{v-1,v} & \ldots & b^{N,K_c}_{v,v} \\
  \end{bmatrix} $$
\[
\begin{bmatrix}
* & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
* & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{bmatrix},
\]
(67)

where each ‘*’ represents an uniformly i.i.d. symbol on \( \mathbb{F}_q \). More precisely, for the \( j \)th linear combination transmitted by worker \( n \) where \( n \in [6] \), we choose each of \( b_{n,j}^{i_1} \) uniformly i.i.d. over \( \mathbb{F}_q \), while setting the other variables in this linear combination be 0. The above choice on \( S_{m+u} \) will guarantee that the constraints in (65) become linearly independent in terms of the remaining variables to be determined in the next step.

**Step 2:** We then fix one \( t \in [m+u-1] \) and one \( j \in [K_c] \); thus the constraints in (65) become

\[
0 = s_{n,j}^{n,j} d_{k+(t-1)K_c} = \sum_{i_1 \in [K_c]} f_{i_1,k} s_{(t-1)K_c+i_1}^{n,j} + \sum_{i_2 \in [v]} b_{i_2}^{n,j} a_{i_2,(t-1)K_c+k}
\]
(68a)

\[
= \sum_{i_1 \in [K_c]} f_{i_1,k} s_{(t-1)K_c+i_1}^{n,j} + \sum_{i_3 \in \left[ \left( \frac{(j-1)v}{K_c} + 1 : \frac{jv}{K_c} \right) \right]} b_{i_3}^{n,j} a_{i_3,(t-1)K_c+k}, \forall n \in [N], k \in \mathbb{Z}_n.
\]
(68b)

Notice that in (68b) the coefficients \( f_{i_1,k} \) are the elements in the demand matrix \( \mathbf{F} \) and \( b_{i_3}^{n,j} \) have been already determined in Step 1. Hence, the constraints (68b) are linear in terms of the variables

\[
s_{(t-1)K_c+i_1}^{n,j} \text{ and } a_{i_3,k_1}, \forall n \in [N], i_1 \in [K_c], i_3 \in \left[ \left( \frac{(j-1)v}{K_c} + 1 : \frac{jv}{K_c} \right) \right], k_1 \in [(t-1)K_c+1 : tK_c].
\]
(69)

Next, we determine the values of the variables in (69) by solving linear equations. In (69), there are totally

\[
NK_c + \frac{v}{K_c} K = N \frac{K}{N} u + (N_r - m - u + 1) K = K (N_r - m + 1)
\]

variables while in (68b) there are totally

\[
N \frac{K}{N} (N_r - m) = K (N_r - m)
\]
constraints. In order to determine all the variables in (69) while satisfying the constraints in (68b), for each \( n \in [N] \), we first choose each of

\[
s_{(i-1)K_u+(i-1)u+\text{Mod}(n,u)}^i \quad \forall i \in [K/N];
\]  

(70)uniformly i.i.d. over \( \mathbb{F}_q \). Hence, among all the \( K(N_i-m+1) \) variables in (69), we have determined \( N^K_N = K \) variables. Thus there are \( K(N_i-m) \) variables to be solved by \( K(N_i-m) \) linear equations in (68b). It will be proved in Appendix A that with high probability, these \( K(N_i-m) \) linear equations are linearly independent over these remaining \( K(N_i-m) \) variables. As a result, we have determined all the variables in (69).

By considering all the pairs \((t, j)\) where \( t \in [m+u-1] \) and \( j \in [K_c] \), we can determine all the elements in \( S \) and \( F' \).

**Step 3:** It will be proved in Appendix A that the constraints in (66) hold with high probability. Hence, we let each worker \( n \) compute and send \( K_c \) linear combinations, i.e., \( s_{n,j}^nF'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}] \) where \( j \in [K_c] \).

**Decoding phase:** Assume that the set of responding workers is \( A \) where \( A \subseteq [K] \) where \( |A| = N_i \). The master receives

\[
X_A = [s^{A(1),1}; \ldots; s^{A(1),K_c}; s^{A(2),1}; \ldots; s^{A(N_i),K_c}] F'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}].
\]  

(71)

Since \([s^{A(1),1}; \ldots; s^{A(1),K_c}; s^{A(2),1}; \ldots; s^{A(N_i),K_c}]\) is full-rank, the master then computes

\[
[s^{A(1),1}; \ldots; s^{A(1),K_c}; s^{A(2),1}; \ldots; s^{A(N_i),K_c}]^{-1} X_A
\]
to obtain \( F'[W_{1,1}; \ldots; W_{K,1}; W_{1,2}; \ldots; W_{K,m+u-1}] \), which contains its demanded linear combinations.

**Performance:** Since each worker sends \( \frac{K_cL}{m+u-1} \) symbols, the communication cost is \( \frac{N_iK_cL}{(m+u-1)L} = \frac{N_iK_c}{m+u-1} \), coinciding with (22a).

**Remark 2.** The proposed scheme works for the case where

\[
N \geq \frac{m+u-1}{u} + u(N_i - m - u + 1),
\]

(72)which can be explained intuitively in the following way. It will be proved in Appendix A that if the proposed scheme works for the \((N, N, N_i, u, m)\) distributed linearly separable computation problem (i.e., the number of messages is equal to \( N \)) with high probability, then with high prob-
ability the proposed scheme also works for the \((K, N, N_t, \frac{K}{N} u, m)\) distributed linearly separable computation problem where \(N\) divides \(K\). Hence, let us then analyse the case \(K = N\).

We fix one \(t \in [m + u − 1]\) in the constraints (65). In Step 2 of the computing phase, we should solve the following problem:

**Problem t**: Determine the values of the variables

\[
s_{(t-1)u+i_1}^{n,j} \text{ and } a_{i_3,k}, \quad \forall n \in [N], j \in [u], i_1 \in [u], i_3 \in [v], k \in [(t-1)K : tK]
\]  

(73)

satisfying the constraints

\[
\sum_{i_3 \in [u]} f_{i_1,k} s_{(t-1)u+i_1}^{n,j} + \sum_{i_3 \in [(\frac{t-1}{u})r+1; \frac{t-1}{u}]} b_{i_3}^{n,j} a_{i_3,(t-1)K+k} = 0, \quad \forall j \in [u], n \in [N], k \in \bar{Z}_n.
\]  

(74)

Notice that by solving Problem \(t\), for each \(i \in [v]\), we can determine

\[
[s_{(t-1)u+i_1}^{1,1}; s_{(t-1)u+i_1}^{1,u}; \ldots; s_{(t-1)u+i_1}^{u,1}; \ldots; s_{(t-1)u+i_1}^{N,u}]
\]

which is the \(((t-1)u + i)\text{th}\) column of \(S\). Another important observation is that, Problem \(t_1\) is totally equivalent to Problem \(t_2\) for any \(t_1 \neq t_2\). Thus, we can introduce the following unified problem.

**Unified Problem**: Determine the values of the variables

\[
p_{i_3,j}^{n,k} \text{ and } q_{i_3,k}, \quad \forall n \in [N], j \in [u], i_1 \in [u], i_3 \in [v], k \in [K]
\]  

(75)

satisfying the constraints

\[
\sum_{i_3 \in [u]} f_{i_1,k} p_{i_1}^{n,j} + \sum_{i_3 \in [(\frac{t-1}{u})r+1; \frac{t-1}{u}]} b_{i_3}^{n,j} q_{i_3,k} = 0, \quad \forall j \in [u], n \in [N], k \in \bar{Z}_n.
\]  

(76)

In the unified problem, there are

\[N_{uu} + vK = Nu(u + N_t - m - u + 1) = Nu(N_t - m + 1)\]

variables and \(Nu(N_t - m)\) constraints. Hence, the number of linearly independent solutions of the unified problem is no less than \(Nu(N_t - m + 1) - Nu(N_t - m) = Nu\), where the equality holds when the constraints in the unified problem is linearly independent. To guarantee that all the columns in \(S\) are linearly independent, we should assign \(m + u - 1\) linearly independent solutions to Problems 1, 2, \ldots, \(m + u - 1\).

In addition, among all the linearly independent solutions of the unified problem, there are \(uv\) trivial solutions which we cannot pick. More precisely, for each \(i \in [v]\) and \(d \in [u]\), one possible
solution is to set (recall that $f_d$ represents the $d$th demand vector)

$$(q_{i,1}, q_{i,2}, \ldots, q_{i,3}, k) = f_d,$$

while setting $q_{i,3,k} = 0$ if $i_3 \neq i$. In addition, we set

$$p_{i}^{n,j} = -b_{i}^{n,j}, \forall n \in [N], j \in [u],$$

while setting $p_{i_1}^{n,j} = 0$ if $i_1 \neq i$. It can be easily checked that by the above choice of variables, the constraints in (76) holds. Hence, the above choice is one possible solution of the unified problem. There are totally $uv$ such possible solutions. However, any combination of such $uv$ solutions cannot be chosen as a solution of Problem $t$. This is because in each of the above solutions, there is a column of $S$ (i.e., $[p_{1}^{1,1}; \ldots; p_{1}^{1,u}; p_{1}^{1,2}; \ldots; p_{1}^{N,u}]$), which can be expressed by a fixed column of $S$ (i.e., $[b_{1}^{1,1}; \ldots; b_{1}^{1,u}; b_{1}^{1,2}; \ldots; b_{1}^{N,u}]$). Hence, the full-rank constraints in (66) cannot hold.

As a result, if we have

$$Nu \geq m + u - 1 + uv = m + u - 1 + u^2(N_r - m - u + 1) \tag{77}$$

which is equivalent to (72), it can be guaranteed that we can assign one linearly independent non-trivial solution to each Problem $t$.

\[ \square \]

For each $\frac{K_N^c}{N}(u - 1) < K_c < \frac{K_N^c}{N} u$ where $u \in [N_r - m + 1]$, we first generate $\frac{K_N^c}{N} u - K_c$ demand vectors whose elements are uniformly i.i.d. over $F_q$, and add these vectors into the demand matrix $F$. Next, we use the above distributed computing scheme with $K_c' = \frac{K_N^c}{N} u$. Hence, the communication cost is

$$\frac{N_rK_c'}{m+u-1} = \frac{N_rK_N^c}{N(m+u-1)}$$

coinciding with (22a).

VI. CONCLUSIONS AND FUTURE RESEARCH DIRECTIONS

In this paper, we studied the computation-communication costs tradeoff for the distributed linearly separable computation problem. A converse bound under the constraint of cyclic assignment was proposed, and we also proposed a novel distributed computing scheme under some parameter regimes. Some exact optimality results were derived with or without the constraint of cyclic assignment. The proposed computing scheme was also proved to be generally order optimal within a factor of 2 under the constraint of cyclic assignment.
The simplest open which the proposed scheme cannot work is the case where $K = N = N_t = 5$, $K_c = 2$, and $m = 2$. Further works include the design of the distributed computing scheme for the open cases and the derivation of the converse bound for any dataset assignment.

**APPENDIX A**

**Feasibility Proof of the Proposed Computing Scheme in Section V**

In the following, we first show that for the $(K, N, N_t, K_c, m) = (N, N, N_t, u, m)$ distributed linearly separable computation problem, where $N \geq \frac{m+u-1}{u} + u(N_t - m - u + 1)$, the proposed computing scheme works with high probability. Next we show that if the proposed scheme works for the $(N, N, N_t, u, m)$ distributed linearly separable computation problem with high probability, then with high probability the proposed scheme also works for the $(K, N, N_t, \frac{K}{N}u, m)$ distributed linearly separable computation problem, where $\frac{K}{N}$ is a positive integer.

A. $K = N$

The feasibility of the proposed computing scheme is proved by the Schwartz-Zippel Lemma [21]–[23] as we used in [5, Appendix C] for the computing scheme where $m = 1$. For the sake of simplicity, in the following we provide the sketch of the feasibility proof.

Recall that in Step 2 of the proposed computing scheme, for each pair $(t, j)$ where $t \in [m+u-1]$ and $j \in [u]$, we need to determine the values of the variables in (69) while satisfying the linear constraints in (68b). In addition, among all the variables in (69), we choose the values of the variables in (70) uniformly i.i.d. over $\mathbb{F}_q$. Then there are remaining $K(N_t - m)$ variables (the vector of these $K(N_t - m)$ variables is assumed to be $b$) and $K(N_t - m)$ linear equations over these variables, and thus we can express these linear equations as (recall that $(M)_{m \times n}$ indicates that the dimension of matrix $M$ is $m \times n$)

\[
(A)_{K(N_t - m) \times K(N_t - m)} \ (b)_{K(N_t - m) \times 1} = (c)_{K(N_t - m) \times 1},
\]

where the coefficients in $A$ and $c$ are composed of the elements in $F$, $S_{m+u}$, and the variables in (70) which are all generated uniformly i.i.d. over $\mathbb{F}_q$. Hence, the determinant of $A$ can be seen as a multivariate polynomial of the elements in $F$, $S_{m+u}$ and the variables in (70). Since we assume $q \to \infty$, by the Schwartz-Zippel Lemma [21]–[23], if this polynomial is a non-zero multivariate polynomial (i.e., a multivariate polynomial whose coefficients are not all 0), the probability that the polynomial is equal to 0 over all possible realization of $F$, $S_{m+u}$, and the
variables in (70), goes to 0. In other words, the determinant is non-zero with high probability.

So the next step is to show this polynomial is non-zero. This means that we need to find one realization of F, S_{m+u}, and the variables in (70), such that this polynomial is not equal to zero. By random generation of F, S_{m+u}, and the variables in (70), we have tested all cases where N = K ≤ 40 satisfying the constraint N ≥ \( \frac{m+u-1}{u} + u(N_r - m - u + 1) \). Hence, for each pair \((t, j)\), the probability that Step 2 of the proposed computing scheme is feasible goes to 1. By the probability union bound, the probability that Step 2 of the proposed computing scheme is feasible for all pairs of \((t, j)\), also goes to 1. Moreover, by using the the Cramer’s rule, each element in b can be seen as a ratio of two polynomials of the elements in F, S_{m+u} and the variables in (70), where the polynomial in the denominator is non-zero with high probability. As a result, each element in S can be seen as ratio of two polynomials of the elements in F, S_{m+u} and the variables in (70) for all pairs \((t, j)\). So for each \(A \subseteq [N]\) where \(|A| = N_r\), the determinant of the matrix \([s^{A(1)}, \ldots; s^{A(Kc)}; s^{A(2)}, \ldots; s^{A(N_r)}, Kc]\) can be expressed as

\[
Y_A = \sum_{i \in [(N_r)!!]} \frac{P_i}{Q_i},
\]

where \(P_i\) and \(Q_i\) are polynomial of the elements in F, S_{m+u} and the variables in (70) for all pairs \((t, j)\). We want to prove that \(Y_A \prod_{i \in [(N_r)!!]} Q_i\) is a non-zero polynomial such that we can use the Schwartz-Zippel Lemma [21]–[23] to show that the determinant \(Y_A\) is not equal to zero with high probability. Again, by random generation of F, S_{m+u}, and the variables in (70) for all pairs \((t, j)\), we have tested all cases where N = K ≤ 40 satisfying the constraint N ≥ \( \frac{m+u-1}{u} + u(N_r - m - u + 1) \). In these cases, with the random choices, both \(\prod_{i \in [(N_r)!!]} Q_i\) and \(Y_A\) are not equal to zero, and thus \(Y_A \prod_{i \in [(N_r)!!]} Q_i\) is not equal to 0.

In conclusion, we prove the feasibility of the proposed computing scheme in Steps 2 and 3 with high probability, for the case where \(\frac{m+u-1}{u} + u(N_r - m - u + 1) \leq K = N \leq 40\).

\textbf{B. N divides K}

We then consider the \((K, N, N_r, K_c, m) = (K, N, N_r, K_u, m)\) distributed linearly separable computation problem, where N ≥ \( \frac{m+u-1}{u} + u(N_r - m - u + 1) \) and \(K_u\) is a positive integer. Similar to the proof for the case where K = N, we also aim to find a specific realization of F, S_{m+u} and the variables in (70) for all pairs \((t, j)\), such that Steps 2 and 3 of the proposed scheme are feasible (i.e., the determinant polynomials are non-zero).
We construct the demand matrix (i.e., $\mathbf{F}$ with dimension $\frac{K}{N}u \times K$) as follows,

$$
\mathbf{F} = \begin{bmatrix}
(F_1)_{u \times N} & 0_{u \times N} & \cdots & 0_{u \times N} \\
0_{u \times N} & (F_2)_{u \times N} & \cdots & 0_{u \times N} \\
\vdots & \vdots & \ddots & \vdots \\
0_{u \times N} & 0_{u \times N} & \cdots & (F_{K/N})_{u \times N}
\end{bmatrix},
$$

(79)

where each element in $F_i$, $i \in \left[ \frac{K}{N} \right]$ is generated uniformly i.i.d. over $\mathbb{F}_q$. In the above construction, the $(K, N, N_r, \frac{K}{N}u, m)$ distributed linearly separable computation problem is divided into $\frac{K}{N}$ independent/disjoint $(N, N, N_r, u, m)$ distributed linearly separable computation sub-problems.

Since the determinant polynomials are non-zero with high probability for each sub-problem as we proved in Appendix A-A, it can be seen that the determinant polynomials for the $(K, N, N_r, \frac{K}{N}u, m)$ distributed linearly separable computation problem are also non-zero with high probability.

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