Coulomb interactions and renormalization of semi-Dirac fermions near a topological Lifshitz transition

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We aim to understand how the spectrum of semi-Dirac fermions is renormalized due to long-range Coulomb electron-electron interactions at a topological Lifshitz transition, where two Dirac cones merge. At the transition, the electronic spectrum is characterized by massive quadratic dispersion in one direction, while it remains linear in the other. We have found that, to lowest order, the unconventional log squared correction to the quasiparticle mass in bare perturbation theory leads to resummation into strong mass renormalization in the exact full solution of the perturbative renormalization group equations. This behavior effectively wipes out the curvature of the dispersion and leads to Dirac cone restoration at low energy: the system flows towards Dirac dispersion which is anisotropic but linear in momentum, with interaction-dependent logarithmic modulation. This effect contrasts with the behavior that has been found within the large-$N$ approach.

I. INTRODUCTION

Semi-Dirac fermions are chiral quasiparticles in two dimensions (2D) that propagate as Galilean invariant particles as they move in one direction and as relativistic ones in the other direction. Such quasiparticles emerge at a topological Lifshitz transition, where two Dirac cones merge. Strongly anisotropic Dirac fermions, eventually transforming into semi-Dirac particles at a topological quantum critical point, appear in a variety of physical situations, from strained graphene-based structures, black phosphorus under pressure and doping, BEDT-TTF$_2$I$_3$ salt under pressure, VO$_2$/TO$_2$ heterostructures, photonic crystals and atomic (cold atom) physics. In solid state context the prototypical example is strained graphene. It is known that by applying uniaxial strain in the the zig-zag direction in graphene one can induce a transition into a gapped state. In the gapless regime (before the transition), the electronic spectrum consists of separated anisotropic (elliptic) Dirac cones, while at the transition the spectrum becomes quadratic in one direction, remaining linear in the other.

The universal effective Hamiltonian describing the physics outlined above is

\[ H(p) = \left( \frac{p_x^2}{2m} + \Delta \right) \hat{\sigma}_x + v p_y \hat{\sigma}_y, \]

which \( \Delta \) depends on the (anisotropic) hopping parameters, in the case of strained graphene. We will keep in mind this example, while the results will be of course applicable to all systems falling within the same universality class. The case \( \Delta < 0 \) corresponds to separated anisotropic (elliptic) Dirac cones (gapless phase, weak strain), the value \( \Delta = 0 \) is the critical point, and \( \Delta > 0 \) corresponds to the gapped phase (strong strain), as shown in Fig. 1. The chemical potential is set to zero.

At the critical point the spectrum is

\[ \epsilon(p) = \pm \sqrt{\left( \frac{p_x^2}{2m} \right)^2 + v^2 p_y^2}, \quad \Delta = 0. \]

From now on we set \( \hbar = 1 \) and all lengths will be measured in units of the lattice spacing (which we set to one), with \( \pm \) indexing the two particle-hole branches. In particular, at the critical point induced by zig-zag strain, by taking into account the strain dependence of the tight-binding Hamiltonian parameters, one can deduce the following relationship, \( mv = 2 \), in units of the inverse lattice spacing. This is the only remnant of non-universal (system specific) physics at the critical point and we use it for illustration purposes in our plots describing interaction effects (whose structure itself is universal.)

An important issue is how interactions (both short and long-range) affect the fermion spectrum at and around the critical point, and the various phenomena associated with it. For example short range interactions can in-

![Fig. 1. Topological Lifshitz phase transition across a quantum critical point (QCP) at \( \Delta = 0 \). For \( \Delta < 0 \), approaching the QCP from the left, two Dirac cones merge, producing a single touching point with semi-Dirac fermion excitations. For \( \Delta > 0 \), a trivial insulating phase forms.](image-url)
fluence the Dirac cone merger and shift the critical point itself (i.e., affect the gap). Such interactions can also affect the appearance of various instabilities (such as charge and spin density waves, etc) at criticality.

The role of long-range Coulomb interactions is expected to be even more dramatic. It has been argued that a non-Fermi liquid (NFL) state emerges in the large $N$ limit, where the quasiparticle residue $(Z)$ approaches zero as a power law at low energy. This behavior is governed by the $N\alpha \gg 1$ limit. At the lowest energies, this state crosses over to a marginal Fermi liquid (MFL) where $Z$ exhibits a weaker logarithmic renormalization, governed by a weak coupling (in a sense that $N\alpha \ll 1$) fixed point. Here $N$ is the number of fermion flavors (equal to four) and $\alpha$ is the effective Coulomb coupling constant. This overall behavior can be compared with previous results for simple, isotropic Dirac cones in graphene within the same approximation, where $Z$ does not vanish and the interacting isotropic Dirac liquid remains coherent. The peculiar incoherent behavior of the semi-Dirac fermions can be traced back to the appearance of higher powers of logarithms in perturbation theory (log squared contributions even at first order of perturbation theory, compared to simple logs for isotropic graphene). It should be emphasized that this result is based on the large $N$ scheme, i.e., assuming the dominance of polarization bubbles. The alternative to large $N$ is the “conventional” perturbative renormalization group (RG) in powers of the Coulomb coupling $\alpha$. While in isotropic graphene the two approaches connect smoothly and describe the same state (interacting Dirac liquid), for semi-Dirac fermions the results are drastically different, as we will show below.

The purpose of the present paper is to point out that for semi-Dirac fermions the “NFL–MFL” fixed point obtained in the large $N$ limit is not the only possible scenario. The presence of log squared terms in first order of perturbation theory does not by itself justify non-perturbative RG when $\alpha$ is small and $N \sim 1$. We show that after taking into account the unconventional log squared contributions that appear in the self-energy for semi-Dirac fermions, and performing perturbative RG to lowest order in $\alpha$, the resulting fixed point is characterized by restoration of linear quasiparticle dispersion in the direction where it was originally quadratic. The resulting Dirac cone is not necessarily isotropic but the “semi-Diracness” has disappeared. This is in contrast to the MFL state where the dispersion retains its semi-Dirac features. While we have not addressed the issue how the quasiparticle residue behaves, since it appears at the next order in $\alpha$, we do not expect our main conclusion about Dirac cone restoration to be altered due to the fact that the residue affects the terms in the different momentum directions in the same manner. Thus our results indicate that the perturbative RG and the large $N$ version lead to different fixed points, and this can have far-reaching consequences for properties of interacting semi-Dirac fermions.

For instance it has been claimed that, at large $N$, the ratio between the shear viscosity and the entropy of semi-Dirac fermions violates the conjectured lower bound $\eta/s \geq \hbar/(4\pi k_B)$ derived in an infinitely strongly coupled conformal field theory. This ratio is usually taken as a universal measure of the strength of interactions in the hydrodynamic regime of quantum fluids. The violation was attributed to the strongly anisotropic nature of semi-Dirac fermions. In contrast, conventional Dirac fermions are known to satisfy the lower bound. In the present work we find that, at least in the perturbative regime, Coulomb interactions lead to restoration of the linearity of the spectrum. This effect may have relevant implications for the solution of the quantum kinetic equation in the collision dominated regime.

The rest of the paper is organized as follows. In Section II we present a detailed formulation and results of the perturbative RG for semi-Dirac fermions at criticality. In Section III we also extend our treatment away from the critical point. Section IV contains our conclusions.

## II. Renormalization Group at Criticality: Restoration of Dirac Spectrum at Low Energy

In this section we consider the critical point $\Delta = 0$. Let us introduce interactions via the non-retarded Coulomb potential

$$V(p) = \frac{2\pi e^2}{|p|}. \quad (3)$$

We will take into account the interaction at first order in perturbation theory. The self-energy shown in Fig. 2 is

$$\Sigma(p) = i \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int \frac{d^2 k}{(2\pi)^2} \hat{G}(k, \nu) V(k - p), \quad (4)$$

where

$$\hat{G}^{-1}(p, \nu) = \nu - \hat{H}(p) + i 0^+ \text{sign}(\nu) \quad (5)$$

is the fermionic Green’s function. The frequency integral can be easily evaluated,

$$\Sigma(p) = \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \frac{2\pi e^2}{|k - p|} \frac{1}{|\epsilon(k)|} \left( \frac{k^2}{2m} \hat{\sigma}_x + v k_y \hat{\sigma}_y \right). \quad (6)$$

In this order, the self-energy is frequency independent. When evaluating logarithmic corrections it is useful to look at the behavior at small external momenta $p \to 0$ and expand

$$\frac{1}{|k - p|} = \frac{1}{k} \left( 1 + \frac{k p}{k^2} - \frac{p^2}{2k^2} + \frac{3(k p)^2}{2k^4} \right) + O(p^3). \quad (7)$$

Here $k = |k|$. As usual, we introduce the dimensionless coupling

$$\alpha = e^2/v. \quad (8)$$
mass gap evaluated from Eq. (6) is implying that a gap is generated by the interactions. The phase (∆ is the lattice spacing). Thus we can conclude that the interaction effects drive the system away from criticality, towards the gapped regime. The phase ∆ off Λ which follows the structure of the dispersion when we can study the spectrum renormalization at criticality. We will return to the issue of gap renormalization in Section III.

B. Mass and Velocity Renormalization

We now proceed to calculate the first order corrections to the velocity and mass parameters. These will exhibit logarithmic divergences and we will adopt an “on-shell” renormalization procedure with an ultraviolet energy cutoff Λ which follows the structure of the dispersion ε(p), and therefore depends on direction in momentum space. To extract the log divergence with an energy cutoff we introduce a change of variables,

\[ \frac{k^2}{2m} = \epsilon \sin \varphi, \quad \nu k_y = \epsilon \cos \varphi, \]

where \( \epsilon \in [0, \Lambda] \) and \( \varphi \in [0, \pi] \). Then we have

\[ \int d^2k = (1/v) \int_0^\Lambda \sqrt{2m\epsilon} d\epsilon \int_0^\pi \frac{d\varphi}{\sqrt{\sin \varphi}}. \]

Integration over the variables \( \epsilon \) and \( \varphi \) in the self-energy [6] gives

\[ \hat{\Sigma}(p) = \left( \frac{p_x^2}{2m} \Sigma_x + \frac{p_y^2}{2m} \Sigma_{y,m} \right) \sigma_x + vp_y \Sigma_{y,e} \sigma_y. \] (13)

The term

\[ \Sigma_{y,v} = \frac{\alpha}{4} \int_{E_x}^{E_a} \frac{dE}{E} L_1(E) = \frac{\alpha}{\pi} \ln(\Lambda/\omega), \] (14)

gives the self-energy correction to the velocity \( v \), where

\[ L_1(E) = \int_0^\pi \frac{d\varphi}{\pi} \frac{E \cos^2 \varphi}{\sin \varphi} \left( E \cos^2 \varphi + \sin \varphi \right)^{3/2} \frac{E^{\epsilon \ll 1}}{\pi}. \]

is an angular integral, and

\[ E = \epsilon/\epsilon_0, \quad \epsilon_0 \equiv 2mv^2, \]

is the dimensionless energy integrated in the interval \( E \in [E_\omega, E_\Lambda] \), with \( E_\omega = \omega/\epsilon_0 \) and \( E_\Lambda = \Lambda/\epsilon_0 \). The renormalization is done “on-shell” in the low-energy limit,

\[ \omega \equiv |\epsilon(p)| = \sqrt{\left( \frac{p_x^2}{2m} \right)^2 + v^2 p_y^2} \ll \Lambda. \] (16)

The first term in [13] gives correction to the mass \( m \) for quasiparticles moving along \( p_x \).

\[ \Sigma_x = -\frac{\alpha}{8} \int_{E_\omega}^{E_a} \frac{dE}{E} L_2(E) + 3\alpha \int_{E_\omega}^{E_a} \frac{dE}{E} L_3(E) \]

\[ = \frac{\alpha}{4\pi} \ln^2(\Lambda/\omega) + \frac{\alpha}{4\pi} F \ln(\Lambda/\omega), \]

where

\[ L_2(E) = \int_0^\pi \frac{d\varphi}{\pi} \frac{\sqrt{\sin \varphi}}{E \cos^2 \varphi + \sin \varphi} \left( E \cos^2 \varphi + \sin \varphi \right)^{3/2} \frac{E^{\epsilon \ll 1}}{\pi}, \]

\[ L_3(E) = \int_0^\pi \frac{d\varphi}{\pi} \frac{\sin \varphi^{3/2}}{E \cos^2 \varphi + \sin \varphi} \left( E \cos^2 \varphi + \sin \varphi \right)^{5/2} \frac{E^{\epsilon \ll 1}}{\pi}, \]

with the numerical constants \( c = 1.1, d = 0.56 \), and \( F \equiv \ln \left( [d/\pi c]/(\Lambda^2 c) \right) \).

Finally, the second term in Eq. [13] gives an induced mass in the \( p_y \) direction, which is generated by interactions,

\[ \Sigma_{y,m} = -\frac{\alpha}{8} \int_{E_\omega}^{E_a} \frac{dE}{E} L_2(E) + 3\alpha \int_{E_\omega}^{E_a} \frac{dE}{E} L_4(E) \]

\[ = -\frac{\alpha}{8\pi} \ln^2(\Lambda/\omega) - \frac{\alpha}{4\pi} G \ln(\Lambda/\omega), \]

where

\[ L_4(E) = \int_0^\pi \frac{d\varphi}{\pi} \frac{E \sqrt{\sin \varphi} \cos^2 \varphi}{E \cos^2 \varphi + \sin \varphi} \left( E \cos^2 \varphi + \sin \varphi \right)^{3/2} \frac{E^{\epsilon \ll 1}}{3\pi}, \] (21)
and $G \equiv \ln [\epsilon_0/\Lambda] - 2$.

On the basis of the above results, the renormalized Hamiltonian to leading order in the interaction has the form:

$$\mathcal{H}(\mathbf{p}) = \left(g_1(\omega)\frac{p_x^2}{2} - g_2(\omega)\frac{p_y^2}{2}\right)\hat{\sigma}_x + \nu(\omega)p_y\hat{\sigma}_y.$$  (22)

We define the inverse masses as

$$g_1(\omega) = m_x^{-1}(\omega), \quad g_2(\omega) = m_y^{-1}(\omega).$$  (23)

The functions $g_1(\omega), g_2(\omega), \nu(\omega)$ will be found below from the solution of the RG equations. The bare values of all parameters, i.e. the values at the lattice (ultraviolet) energy scale, are determined by the parameters of the Hamiltonian without interactions: $g_1(\omega = \Lambda) \equiv g_{10} = m^{-1}$, $g_2(\omega = \Lambda) \equiv g_{20} = 0$ and $\nu(\omega = \Lambda) \equiv \nu_0 = \nu$. Similarly, $\alpha_0 = e^2/\nu = \alpha$ (from Eq. (5)) is defined as the bare value of the Coulomb coupling, corresponding to the bare value of $\nu_0 = \nu$.

Taking into account Eqs. (13)–(21), we have the one-loop perturbation theory results:

$$v(\omega) = v \left(1 + \frac{\alpha}{\pi} \ln(\Lambda/\omega)\right),$$  (24)

$$g_1(\omega) = g_{10} \left(1 + \frac{\alpha}{4\pi} \ln^2(\Lambda/\omega) + \frac{\alpha}{4\pi} F \ln(\Lambda/\omega)\right),$$  (25)

and

$$g_2(\omega) = g_{10} \left(\frac{\alpha}{8\pi} \ln^2(\Lambda/\omega) + \frac{\alpha}{4\pi} G \ln(\Lambda/\omega)\right).$$  (26)

Here $g_{10} = m^{-1}$, as explained previously. We find, in particular, that a mass term is generated in the $p_y$ direction, where the dispersion was originally linear. The most important feature of the mass renormalization formulas above is that both masses contain a log squared contribution at leading order in the coupling $\alpha$. In addition, the two mass terms have different signs upon renormalization (with the sign in front of $m_y$ being negative). In Eqs. (25, 26) we have also kept sub-leading (first power) log contributions which strictly speaking is not necessary; however we retain them in our calculations for completeness.

C. Renormalization Group Equations and their Solutions

Given that the Coulomb interaction in 2D is a non-analytic function, the electron charge does not renormalize in the RG flow. Next, define the RG scale

$$\ell \equiv \ln(\Lambda/\omega).$$  (27)

From Eqs. (24)–(26), we obtain the RG equations

$$\frac{dv(\ell)}{d\ell} = v(\ell) \alpha(\ell)/\pi = e^2/\pi,$$  (28)

$$\frac{dg_1(\ell)}{d\ell} = g_1(\ell)\left(\frac{\alpha(\ell)}{2\pi} \ell + \frac{\alpha(\ell)}{4\pi} F\right),$$  (29)

and

$$\frac{dg_2(\ell)}{d\ell} = g_1(\ell)\left(\frac{\alpha(\ell)}{4\pi} \ell + \frac{\alpha(\ell)}{4\pi} G\right).$$  (30)

Integrating the velocity, Eq. (28), we obtain

$$v(\ell) = v \left(1 + \frac{\alpha}{\pi} \ell\right) \Rightarrow \alpha(\ell) = \frac{\alpha}{1 + \frac{2}{\pi} \ell},$$  (31)

which in turn determines the running of the interaction coupling constant. Similarly to isotropic graphene, the velocity increases logarithmically as energy decreases, leading to a logarithmic decrease of the interaction, which flows to weak coupling,

$$\alpha(\ell = \ln(\Lambda/\omega)) = \frac{\alpha}{1 + \frac{2}{\pi} \ln(\Lambda/\omega)}. $$  (32)

Eq. (29) can be integrated with the result

$$g_1(\ell)/g_{10} = \left(1 + \frac{\alpha}{\pi} \ell\right)^{F/4} e^{\ell/2 - \frac{\pi}{\alpha} \ln(1 + \frac{2}{\pi} \ell)}. $$  (33)

Rewriting this result as a function of energy $\omega$, by taking into account Eq. (27), we obtain:

$$g_1(\omega)/g_{10} = \frac{\sqrt{\Lambda/\omega}}{(1 + \frac{2}{\pi} \ln(\Lambda/\omega))^{(\pi/2\alpha) - F/4}}.$$  (34)

It is instructive to expand Eq. (34) for small values of the bare coupling (we set $F = 0$ in this formula for clarity),

$$g_1(\omega)/g_{10} \approx 1 + \frac{\alpha}{4\pi} \ln^2(\Lambda/\omega) + \frac{\alpha^2}{\pi^2} \left(\frac{1}{32} \ln^4(\Lambda/\omega) - \frac{1}{6} \ln^3(\Lambda/\omega)\right) + O(\alpha^3),$$  (35)

which gives an idea of the structure of higher orders of perturbation theory, re-summed by the RG. We note that the expansion is well controlled at all orders when $\alpha/\pi \ll 1$. This inequality defines the validity of the perturbative regime.

The RG solution, Eq. (34), is one of our main results. Examining the $p_x$ direction part of the dispersion, we clearly see that in the low energy limit $\Lambda/\omega \gg 1$, we have the dominant behavior $g_1(\omega)/g_{10} \sim \sqrt{\Lambda/\omega}$, up to logarithmic corrections. This in turn implies that the mass term in the renormalized Hamiltonian (22), which has the structure $g_1(\omega)p_x^2$, effectively becomes linear in momentum when the energy is on-shell, as defined in Eq. (16). More precisely, for low momenta, $p_x/\sqrt{2m \Lambda} \ll 1$, provided also $\frac{2}{\pi} \ln(2m \Lambda/p_x^2) \gg 1$, we have

$$g_1(\omega)\frac{p_x^2}{2} = \sqrt{\frac{\Lambda}{2m\left[\frac{2}{\pi} \ln(2m \Lambda/p_x^2)\right]^{\pi/2\alpha}} - F/4}.$$  (36)
We see from here that the dispersion becomes linear, with log correction whose power depends on the value of $\alpha$ (the value of the subleading piece $F$ is conceptually and numerically not important; for our parameter values we have $|F|/4 \approx 0.1$). Therefore for small values of $\alpha$ when the power $\frac{2}{2\pi}$ is large, the log term presence will provide some bending to the dispersion; as $\alpha$ increases the linearity becomes gradually more pronounced. We will see shortly that the numerical plot of the RG dispersion confirms this behavior.

Finally, integration of Eq. (30)

$$g_2(\ell) = \frac{1}{4\pi} \int_0^\ell [\xi g_1(\xi)\alpha(\xi) + Gg_1(\xi)\alpha(\xi)]d\xi$$

leads to a cumbersome expression which is not particularly illuminating and will be taken into account numerically. We can deduce however both analytically and numerically that in the extreme low energy limit ($\ell \rightarrow \infty$)

$$g_2(\omega)/g_1(\omega) \rightarrow 1/2, \quad \omega \rightarrow 0.$$  \hspace{1cm} (38)

This is related to the factor of two difference which appears in the RG Equations (29,30). Therefore the induced $g_2$ term plays a marginal role and modifies somewhat the dispersion in the $p_y$ direction at intermediate energies, while at low energies it does not change the preexistent linear behavior.

Our numerical results for the renormalized dispersion,

$$\tilde{\varepsilon}(p) = \pm \sqrt{\left(\frac{g_1(\omega)p_x^2}{2} - \frac{g_2(\omega)p_y^2}{2}\right)^2 + v(\omega)^2p_y^2},$$

evaluated simultaneously with Eq. (40), are presented in Fig. 3. We use the following values of parameters for these plots, setting $v = 1$: $m = 2$, $\Lambda = 2$, $F = -0.4$, $G = -1.2$. In the full units, $mv = 2\hbar/a$, $\Lambda = 2\hbar v/a$, where $a$ is the lattice spacing. The overall behavior is quite robust and not sensitive to these particular values (in particular the subleading pieces $F,G$ follow from the previously derived formulas and are non-universal, although the results are very weakly dependent on their exact values, as expected). We see that the spectrum undergoes a profound transformation from parabolic towards linear, thus recovering a more conventional Dirac cone shape. In the $p_y$ direction the spectrum remains linear even though it undergoes renormalization due to the increase of the velocity at low energy.

A different way to detect the transition towards Dirac cone behavior is to monitor the density of states which can be expressed in the following way for the renormalized spectrum:

$$D(E) = \frac{\sqrt{2m}}{v(2\pi)^2} \int d\xi \int_0^\pi d\varphi \frac{\sqrt{\varepsilon}}{\sin \varphi} \delta(E - \tilde{\varepsilon}(\varepsilon, \varphi)).$$  \hspace{1cm} (40)

Here the notation $\tilde{\varepsilon}(\varepsilon, \varphi)$ means that the momenta are ex-
can be determined similarly to the procedure from the previous section. We will keep only the leading log contributions. Our final result is

\[ \Delta_1(\omega) = \Delta_1 \left( 1 + \frac{\alpha}{4\pi} \ln^2 \left( \frac{\Lambda}{\omega} \right) + \cdots \right) , \quad (44) \]

\[ \Delta_2(\omega) = \Delta_2 \left( 1 + \frac{\alpha}{4\pi} \ln^2 \left( \frac{\Lambda}{\omega} \right) + \cdots \right) . \quad (45) \]

This shows that the two gaps are renormalized exactly the same way and again the unconventional log squared behavior is the dominant one even at first order in the interaction. The next steps are identical to the ones performed in the previous section for the mass terms. The corresponding RG equations are

\[ \frac{d\Delta_i(\ell)}{d\ell} = \Delta_i(\ell) \frac{\alpha(\ell)}{2\pi} \ell , \quad (i = 1, 2) . \quad (46) \]

Their solution leads to the following results:

\[ \Delta_i(\omega) = \Delta_i \frac{\sqrt{\Lambda/\omega}}{\left( 1 + \frac{\alpha}{\pi} \ln \left( \frac{\Lambda}{\omega} \right) \right)^{(\pi/2\alpha)^1}} , \quad (i = 1, 2) . \quad (47) \]

These demonstrate that if the initial “bare” gaps (\(\Delta_{1,2}\)) are present, the gap values will increase quite strongly \(\sim \sqrt{\Lambda/\omega}\) under renormalization at low energy (with additional, interaction-dependent log variation). In particular if \(\Delta_2 = 0\) (no excitonic pairing), the sign and value of \(\Delta_1 = \Delta\) controls the distance from criticality (\(\Delta < 0\), gapless phase; \(\Delta > 0\), gapped phase) and therefore if the system is initially on either side of criticality, it will keep flowing away from it. Similarly, if excitonic pairing is present, it will increase under renormalization. Such tendency (for excitonic pairing) is similar to the case of graphene, except that in our case the renormalization is much stronger (related to the log squared behavior in perturbation theory).

### III. GAP RENORMALIZATION AWAY FROM CRITICALITY

For completeness, we also consider behavior away from the critical point in order to assess how the gap changes under interaction-induced renormalization. In fact we consider modification of the Hamiltonian to include two gap-producing pieces: \(1\) \(\Delta_1\), already mentioned previously, and \(2\) \(\Delta_2\), which could be generated by excitonic pairing,

\[ \mathcal{H}(p) = \left( \frac{p_x^2}{2m} + \Delta_1 \right) \hat{\sigma}_x + vp_y \hat{\sigma}_y + \Delta_2 \hat{\sigma}_z . \quad (41) \]

The spectrum now obviously becomes:

\[ \varepsilon(p) = \sqrt{\left( \frac{p_x^2}{2m} + \Delta_1 \right)^2 + v^2 p_y^2 + \Delta_2^2} . \quad (42) \]

The renormalization of the two gaps in the frequency regime of interest

\[ \sqrt{\Delta_1^2 + \Delta_2^2} \leq \omega \ll \Lambda , \quad (43) \]
point in either direction we find that gap renormalization is also very strong and the system flows further away from criticality.

Overall, we have shown that the full weak coupling RG implementation gives results that are very different from the large \( N \) approach, which favors a fixed point with renormalized semi-Dirac dispersion and also exhibits incoherent ("NFL-MFL") behavior. Our results therefore can have profound consequences for understanding systems with interacting semi-Dirac fermions.

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1. G. Montambaux, F. Piéchon, J.-N. Fuchs, and O. M. Goerbig, *The European Physical Journal B* **72**, 509 (2009)
2. G. Montambaux, F. Piéchon, J.-N. Fuchs, and O. M. Goerbig, *Phys. Rev. B* **80**, 153412 (2009)
3. P. Adroguer, D. Carpentier, G. Montambaux, and E. Origanc, *Phys. Rev. B* **93**, 125113 (2016)
4. M. Bellec, U. Kuhl, G. Montambaux, and F. Mortessagne, *Phys. Rev. Lett.* **110**, 033902 (2013)
5. L.-K. Lim, J.-N. Fuchs, and G. Montambaux, *Phys. Rev. Lett.* **108**, 175303 (2012)
6. S. Banerjee, R. R. P. Singh, V. Pardo, and W. E. Pickett, *Phys. Rev. Lett.* **103**, 016402 (2009)
7. S. Banerjee and W. E. Pickett, *Phys. Rev. B* **86**, 075124 (2012)
8. B. Amorim, A. Cortijo, F. de Juan, A. Grushin, F. Guinea, A. Gutiérrez-Rubio, H. Ochoa, V. Parente, R. Roldán, P. San-Jose, J. Schiefele, M. Sturla, and M. Vozmediano, *Physics Reports* **617**, 1 (2016)
9. A. S. Rodin, A. Carvalho, and A. H. Castro Neto, *Phys. Rev. Lett.* **112**, 176801 (2014)
10. J. Kim, S. S. Balk, S. H. Ryu, Y. Sohn, S. Park, B.-G. Park, J. Denlinger, Y. Yi, H. J. Choi, and K. S. Kim, *Science* **349**, 723 (2015)
11. S. Katayama, A. Kobayashi, and Y. Suzumura, *Journal of the Physical Society of Japan* **75**, 054705 (2006)
12. V. Pardo and W. E. Pickett, *Phys. Rev. Lett.* **102**, 166803 (2009)
13. H. Huang, Z. Liu, H. Zhang, W. Duan, and D. Vanderbilt, *Phys. Rev. B* **92**, 161115 (2015)
14. M. C. Rechtsman, Y. Plotnik, J. M. Zeuner, D. Song, Z. Chen, A. Szameit, and M. Segev, *Phys. Rev. Lett.* **111**, 103901 (2013)
15. M. Polini, F. Guinea, M. Lewenstein, H. C. Manoharan, and V. Pellegrini, *Nature Nanotechnology* **8**, 625 (2013)
16. V. M. Pereira, A. H. Castro Neto, and N. M. R. Peres, *Phys. Rev. B* **80**, 045401 (2009)
17. S.-M. Choi, S.-H. Jhi, and Y.-W. Son, *Phys. Rev. B* **81**, 081407 (2010)
18. B. Dóra, I. F. Herbut, and R. Moessner, *Phys. Rev. B* **88**, 075126 (2013)
19. M. D. Uryszek, E. Christou, A. Jafarzadeh, K. Kruger, and B. Uchoa, *Phys. Rev. B* **100**, 155101 (2019)
20. B. Uchoa, and K. S. Kim, *Phys. Rev. B* **96**, 115103 (2017)
21. B. Roy and M. S. Foster, *Phys. Rev. X* **8**, 011049 (2018)
22. H. Isobe, B.-J. Yang, A. Chubukov, J. Schmalian, and N. Nagaosa, *Phys. Rev. Lett.* **116**, 076803 (2016)
23. G. Y. Cho and E.-G. Moon, *Sci. Rep.* **6**, 19198 (2016)
24. J. M. Link, B. N. Narozhny, E. I. Kiselev, and J. Schmalian, *Phys. Rev. Lett.* **120**, 196801 (2018)
25. D. T. Son, *Phys. Rev. B* **75**, 235423 (2007)
26. V. N. Kotov, B. Uchoa, and A. H. Castro Neto, *Phys. Rev. B* **80**, 165424 (2009)
27. V. N. Kotov, B. Uchoa, V. M. Pereira, F. Guinea, and A. H. Castro Neto, *Rev. Mod. Phys.* **84**, 1067 (2012)
28. P. K. Kortun, D. T. Son, and A. O. Starinets, *Phys. Rev. Lett.* **94**, 116401 (2005)
29. M. Müller, J. Schmalian, and L. Fritz, *Phys. Rev. Lett.* **103**, 025301 (2009)
30. I. F. Herbut, *Phys. Rev. Lett.* **97**, 146401 (2006)
31. J. Ye and S. Sachdev, *Phys. Rev. Lett.* **80**, 5409 (1998)