On the existence of Hamiltonians for non-holonomic systems

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ABSTRACT

We consider the existence of Hamiltonians for autonomous non-holonomic mechanical systems. The approach is elementary in that the existence of a Hamiltonian for a non-holonomic system is equivalent to the existence of an appropriate Lagrangian for the system in question. The existence of such a Lagrangian is related to the inverse problem of constructing a Lagrangian from the equations of motion. A simple example in three dimensions with one non-holonomic constraint is analyzed in detail. In this case there is no Lagrangian reproducing the equations of motion in three dimensions. Thus the system does not admit a variational formulation in three dimensions. However, the system in question is equivalent to a two-dimensional system which does admit a variational formulation. Two distinct Lagrangians and their corresponding Hamiltonians are constructed explicitly for this two-dimensional system.

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1 Introduction

Hamilton’s principle for mechanical systems with non-holonomic constraints has recently been discussed by Flannery [1]. In particular a variational formulation of the equations of motion of a mechanical system was discussed both for holonomic and non-holonomic constraints. It was shown that while the equations of motion for a system with holonomic constraints can be obtained as variational equations, with the constraints being taken into account by the multiplication rule in the calculus of variations [2], the corresponding procedure with non-holonomic constraints leads to equations which differ from the correct equations of motion.

The problems discussed by Flannery are not new; they have been discussed in the literature at least since Hertz’s textbook [3], in which the use of variational principles in mechanics was questioned. Two papers published by Jeffreys [4] and Pars [5] consider Hamilton’s principle for non-holonomic systems, and propose rectification of previous papers in which the variational procedures discussed by Flannery had been used also for non-holonomic systems.

Several papers have advocated the use of a variational principle involving the multiplication rule in the calculus of variations for non-holonomic systems. In addition to the papers of this kind quoted by Flannery and by Pars and Jeffreys, respectively, we mention a paper by Berezin [6], in which no distinction is made between holonomic and non-holonomic systems.

It is also appropriate to mention that, in contradistinction to the original ”Classical Mechanics” by Goldstein [7], the 3rd edition of this classical mechanics textbook advocates the use of a variational principle involving the the multiplication rule for non-holonomic systems [8]. However, the use of this principle for non-holonomic systems was later retracted [9]. This fact was pointed out already by Flannery.

It appears that if a system with non-holonomic constraints does not admit a variational formulation, then the dynamics of the system is not governed by a Hamiltonian $H$. This is the question we address in this paper: Can a non-holonomic system be described in terms of Hamiltonian equations of motion? We confine the detailed discussion to a simple example in three dimensions introduced by Pars [5]. We show that in this case the equations of motion are reducible to a set of equations for a two-dimensional autonomous system, which can be formulated as Hamiltonian equations. However, the original equations of motion in three dimensions do not admit a Hamiltonian formulation.

Our analysis is elementary in that the existence of a Hamiltonian for a given non-holonomic system is considered to be equivalent to the existence of an appropriate Lagrangian $L(q, \dot{q})$ for the system in question. By appropriate is meant that the Lagrangian is non-degenerate, i.e. that the equations defining the canonical momenta $p^j$,

$$ p^j := \frac{\partial L}{\partial \dot{q}_j}, \quad (1) $$

are solvable for the generalized velocities $\dot{q}_j$. It should be noted that we discuss only autonomous systems. Hence the Lagrangian is allowed to depend on time only through the coordinates $q$ and velocities $\dot{q}$. 

1
The existence of an appropriate Lagrangian is related to the inverse problem of constructing a Lagrangian from the appropriate equations of motion. To the best of our knowledge, a complete solution to the inverse problem is not known in the general n-dimensional case for \( n \geq 3 \).

In the next section we consider the Lagrange equations of motion for an autonomous mechanical system with both holonomic and non-holonomic constraints. This problem was considered in some detail by Flannery [1]. For the sake of completeness we consider the equations obtained from the generalized form of d’Alembert’s principle and the equations which follow from a variational procedure with constraints implemented by the multiplication rule. In the non-holonomic case these equations are not identical.

### 2 Lagrange equations with constraints

Consider an autonomous mechanical system with independent generalized coordinates \( q_1, \ldots, q_n \), and velocities \( \dot{q}_1, \ldots, \dot{q}_n \). Let the kinetic energy be \( T \), and the generalized applied forces on the system be \( Q^j, j = 1, \ldots, n \). The generalized principle of d’Alembert (see e.g. the classical texts by Goldstein [7] or Whittaker [10]) then gives the following equation,

\[
\sum_{j=1}^{n} \left\{ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q^j \right\} \delta q_j = 0,
\]

where the quantities \( \delta q_j \) are virtual displacements of the system. If the virtual displacements \( \delta q_j, j = 1, \ldots, n \) are independent, then the equation (2) results in the Lagrange equations of motion,

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q^j, \ j = 1, \ldots, n,
\]

We generalize to systems with \( 1 \leq m < n \) independent non-holonomic constraints, which are taken to be linear and homogeneous in the velocities. The constraint equations are of the following form,

\[
\sum_{j=1}^{n} a_i^j (q_1, \ldots, q_n) \dot{q}_j = 0, \ i = 1, \ldots, m < n,
\]

where the quantities \( a_i^j, i = 1, \ldots, m, j = 1, \ldots, n \), are given functions of the variables \( q_1, \ldots, q_n \).

The derivation given below of the equations of motion for this non-holonomic system can be found e.g. in the textbook by Whittaker [10].

Implement the constraints (4) by regarding the system to be acted on by external applied forces \( Q^j \) and by certain additional forces of constraint \( Q'^j, j = 1, \ldots, n \), which force the system to satisfy the non-holonomic conditions (4). The equation (2) is then replaced by the following equation,

\[
\sum_{j=1}^{n} \left\{ \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q^j - Q'^j \right\} \delta q_j = 0,
\]
In Eq. (5) the virtual displacements \( \delta q_j, j = 1, \ldots, n \), can now be regarded as independent. Thus one obtains the equations of motion,

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q^j + Q'^j, \ j = 1, \ldots, n.
\]  

(6)

The forces of constraint, \( Q'^j, j = 1, \ldots, n \), are \textit{a priori} unknown, but they are such that, in any instantaneous displacement \( \delta q^j, j = 1, \ldots, n \), consistent with the constraints (4), they do no work. The non-holonomic constraints (4) imply the following conditions on the possible instantaneous displacements \( \delta q^j, j = 1, \ldots, n \) of the system,

\[
\sum_{j=1}^{n} a^j_i(q_1, \ldots, q_n) \delta q_j = 0, \ i = 1, \ldots, m < n.
\]  

(7)

For any instantaneous displacements \( \delta q^j, j = 1, \ldots, n \), which satisfy the conditions (7), the work \( \delta W' \) done by the constraint forces \( Q'^j, j = 1, \ldots, n \) equals zero,

\[
\delta W' \equiv \sum_{j=1}^{n} Q'^j \delta q^j = 0.
\]  

(8)

The conditions (7) and (8) together imply that

\[
Q'^j = \sum_{i=1}^{m} \lambda^i a^j_i, \ j = 1, \ldots, n,
\]  

(9)

where the quantities \( \lambda^i, i = 1, \ldots, m \), are time-dependent parameters. The equations (6) have been reduced to

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} = Q^j + \sum_{i=1}^{m} \lambda^i a^j_i, \ j = 1, \ldots, n.
\]  

(10)

To these \( n \) equations of motion one should add the \( m \) equations of constraint (4). We have \( n + m \) equations for the determination of \( n + m \) quantities \( q_j(t), j = 1, \ldots, n \), and \( \lambda^i(t), i = 1, \ldots, m \).

It should be observed that in the argument above, one has not required the constraint equations (4) to be in force under general variations \( q_j \rightarrow q_j + \delta q_j \); the constraints (4) are only imposed on the actual motion of the system.

Now assume that the external applied forces \( Q^j, j = 1, \ldots, n \), can be expressed in terms of a potential \( V \) such that,

\[
Q^j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_j} \right), \ j = 1, \ldots, n.
\]  

(11)

Using the notation

\[
L_0 := T - V,
\]  

(12)

the equations (10) can be written as,

\[
\frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{q}_j} \right) - \frac{\partial L_0}{\partial q_j} = \sum_{i=1}^{m} \lambda^i a^j_i, \ j = 1, \ldots, n.
\]  

(13)
It should be observed that the \( m \) one-forms (7) are non-integrable by assumption, for otherwise the system would be holonomic. In the integrable case (after multiplying the conditions (4) with integrating factors if necessary) one would have

\[ a_i^j = \frac{\partial G_i}{\partial q_j}, i = 1, \ldots, m, \]  

(14)

where the functions \( G_i, i = 1, \ldots, m \), are \( m \) independent functions of the variables \( q_j, j = 1, \ldots, n \),

\[ G_i = G_i(q_1, \ldots, q_n), i = 1, \ldots, m. \]  

(15)

The \( m \) constraint equations (4) would then be equivalent to the following \( m \) holonomic constraints,

\[ G_i(q_1, \ldots, q_n) = C_i, i = 1, \ldots, m, \]  

(16)

where the quantities \( C_i, i = 1, \ldots, m \) are constants. In this case the equations (13) are the Euler-Lagrange equations of the variational problem

\[ \delta \int dt L_0 = 0. \]  

(17)

under the constraints (16). These constraints can be implemented with the multiplication rule in the calculus of variations. This leads to the following free variational problem with Lagrange multipliers \( \lambda^i, i = 1, \ldots, m \),

\[ \delta \int dt \left[ L_0 + m \sum_{i=1}^m \lambda^i (G_i - C_i) \right] = 0. \]  

(18)

The variational problem yields

\[ \frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{q}_j} \right) - \frac{\partial L_0}{\partial q_j} - m \sum_{i=1}^m \lambda^i \frac{\partial G_i}{\partial q_j} = 0, j = 1, \ldots, n. \]  

(19)

The system of equations (19), together with the constraints (16), are the correct equations of motion for the system under consideration in the integrable (holonomic) case. These equations are a set of Euler-Lagrange equations with the integrand in Eq. (18) as a Lagrangian \( L \),

\[ L := L_0 + m \sum_{i=1}^m \lambda^i (G_i - C_i), \]  

(20)

provided one adjoins the time-dependent parameters \( \lambda^i, i = 1, \ldots, m \), as new coordinates to the system. It should be noted that the usual method of transition to a Hamiltonian from the Lagrangian (20) does not apply, since the momenta conjugate to the new coordinates \( \lambda^j \) vanish identically.

Contrary to the assertions in some of the papers referred to in the references [1], [4], [5], as well as in reference [6], a similar procedure in the non-holonomic case does not lead to the correct equations of motion. Specifically, if one considers the variational problem (17) under
the constraints (1) using the multiplication rule, one is led to the following free variational problem,

$$
\delta \int dt \left[ L_0 - \sum_{i=1}^{m} \mu^i \sum_{j=1}^{n} a_i^j(q_1, ..., q_n) \dot{q}_j \right] = 0,
$$

(21)

where the Lagrange multipliers are now denoted by $\mu^i, i = 1, \ldots, m$. The variational equations following from Eq. (21) are,

$$
\frac{d}{dt} \left( \frac{\partial L_0}{\partial \dot{q}_j} \right) - \frac{\partial L_0}{\partial q_j} - \sum_{i=1}^{m} \dot{\mu}^i a_i^j\dot{q}_j - \sum_{i=1}^{m} \mu^i \sum_{k=1}^{n} \left( \frac{\partial a_i^j}{\partial q_k} - \frac{\partial a_i^k}{\partial q_j} \right) \dot{q}_k = 0.
$$

(22)

The equations (22) are not identical to the correct equations of motion (13) for the non-holonomic system under consideration. However, if the integrability conditions

$$
\frac{\partial a_i^j}{\partial q_k} - \frac{\partial a_i^k}{\partial q_j} = 0, i = 1, \ldots, m, j, k = 1, \ldots, n,
$$

(23)

are valid, in which case the system becomes holonomic, the equations of motion (22) coincide with the corresponding correct equations of motion (13) [equivalently Eqns, (19)] upon a change of notation $\dot{\mu}^i \rightarrow \lambda^i, i = 1, \ldots, m$.

The generalized principle of d’Alembert differs from the variational principle involving the multiplication rule in the case of non-holonomic constraints. One consequence of this difference is the fact that the equations of motion following from the principle of d’Alembert differ in form from the equations of motion which follow from the variational principle. It is not excluded that these equations may have the same solutions, however. It thus remains to consider the question of whether the equations of motion (13) and the variational equations (22) can have coinciding solutions in general. In his discussion of this problem Pars [5] used a simple yet non-trivial example in three-dimensional configuration space to show that the equations (13) and (22) in that case can not have coincident general solutions. We will give a detailed discussion of Pars’ example below, adding a few details related to the relevance of the initial values. For clarity, we also pay attention to the dimensions of the quantities in the example, by including appropriate dimensional constants.

3 Pars’ example

The example considered by Pars [5] is the case of an otherwise free particle of mass $m$ in three-dimensional Euclidean space with coordinates designated by $(x, y, z)$, except that the motion of the particle is subjected to the following non-holonomic constraint,

$$
z\ddot{x} - \ell \ddot{y} = 0,
$$

(24)

where $\ell$ is a constant with the dimension length. The dimensional parameters $m$ and $\ell$, which were absent in the formulation given by Pars, are introduced here for clarity.
The Lagrangian $L_0$ in this case is the following,
\[ L_0 = \frac{m}{2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right). \] (25)

The Lagrange equations of motion (13) reduce now to the following equations,
\[ m\ddot{x} = \lambda z, m\ddot{y} = -\lambda \ell, m\ddot{z} = 0, \] (26)
where $\lambda$ is a parameter, which is to be determined by solving the system of equations (26) and (24). The initial conditions are as follows,
\[ (x, y, z)|_{t=0} = 0, \] (27)
and
\[ (\dot{x}, \dot{y}, \dot{z})|_{t=0} = (u, 0, w), \] (28)
where $u$ and $w$ are parameters at our disposal, except for the conditions $u \neq 0$ and $w \neq 0$. If $u = 0$ or $w = 0$ then the solutions to the equations (26) with the initial conditions (27) and (28) are trivial and uninteresting. It should also be noted that the condition $\dot{y}(0) = 0$ in (28) above is not a free choice, but a consequence of the constraint equation (24) and the initial values (27).

We now consider an alternative form of the Lagrange equations (26) and the constraints (24). Differentiating the constraint equation (24) one obtains
\[ z\ddot{x} + \dot{x}\dot{z} - \ell\ddot{y} = 0. \] (29)
Eliminating the quantities $\ddot{x}$ and $\ddot{y}$ from the equation (29) above with the aid of the equations (26), one obtains the following expression for the quantity $\lambda$:
\[ \lambda = -\frac{m\dot{x}\dot{z}}{\ell^2 + z^2}. \] (30)
Inserting the expression (30) for the parameter $\lambda$ into the original equations (26) one obtains the following three equations,
\[ \ddot{x} = -\frac{\dot{x}\dot{z}}{\ell^2 + z^2}, \] (31)
\[ \ddot{y} = \frac{\ell\dot{x}\dot{z}}{\ell^2 + z^2}, \] (32)
and
\[ \ddot{z} = 0. \] (33)
It should be noted that the mass $m$ does not appear in the equations (31) - (33).

Before proceeding further, we demonstrate that the set of equations (31) - (33) above are essentially equivalent to the original equations (26) and the constraints (24). Multiplying the equation (31) with $z$, and subtracting the equation (32) from the result, one obtains
\[ z\ddot{x} - \ell\ddot{y} = -\dot{x}\dot{z} \iff \frac{d}{dt}(z\dot{x} - \ell\dot{y}) = 0 \iff (z\dot{x} - \ell\dot{y}) = C, \] (34)
where $C$ is a constant. Using finally the initial conditions (27) and (28) to evaluate this constant one obtains

$$C = 0.$$  \hfill (35)

The equations (31) - (33) thus imply the constraints (24) when one also uses the information encoded in the initial conditions (27) and (28). The equations (31) - (33) are indeed of the form (26), where the parameter $\lambda$ is identified with the expression (30). We have now demonstrated that the equations (31) - (33) together with the initial conditions (27) and (28) are equivalent to the original equations (26) and the constraints (24). The wording "essentially equivalent" used above was meant to reflect the fact that one had to invoke the initial conditions (27) and (28) in order to show that the constraints (24) are a consequence of the alternative equations (31) - (33) and not a separate condition, as in the formulation (26) which involves the parameter $\lambda$.

We note that the equation (31) can be integrated, yielding

$$\dot{x}\sqrt{\ell^2 + z^2} = \ell u,$$  \hfill (36)

were $u$ is the initial value at $t = 0$ for the quantity $\dot{x}$. The result (36) will be used shortly.

Consider then the variational problem (21) for the case at hand, \textit{i.e.}

$$\delta \int dt \left[ L_0 - \mu (z\dot{x} - \ell\dot{y}) \right] = 0,$$  \hfill (37)

where the function $L_0$ is given in Eq. (25). The differential equations which follow from Eq. (37) are the following

$$\frac{d}{dt} (m\dot{x} - \mu z) = 0,$$  \hfill (38)

$$\frac{d}{dt} (m\dot{y} + \mu \ell) = 0,$$  \hfill (39)

and

$$m\ddot{z} + \mu \dot{x} = 0.$$  \hfill (40)

To the equations (38) - (40) one should still add the constraint equation (24).

It will be shown that the solutions to the equations (31) - (33) with the initial conditions (27) and (28) can not satisfy the variational equations (38) - (40) and the constraint equation (24), except in certain trivial cases, as shown below.

Assume now that there are appropriate solutions $x(t)$, $y(t)$ and $z(t)$, which satisfy both sets of equations (31) - (33) and (38) - (40) together with the constraint (24), respectively, under the initial conditions (27) and (28). From Eq. (33) and Eq. (40) then follows that

$$\mu \dot{x} = 0.$$  \hfill (41)

From the equations (41) and (36) then follows that

$$\mu u = 0.$$  \hfill (42)
The condition (41), or equivalently (42), is thus necessary for the existence of functions \( x(t), \ y(t) \) and \( z(t) \), which satisfy both the set of equations (31) - (33) and the set of equations (38) - (40) together with the constraint (24) under the initial conditions (27) and (28). There are three possible cases to be considered:

\[
\mu = 0, \ u = 0, \tag{43}
\]

\[
\mu = 0, \ u \neq 0, \tag{44}
\]

and

\[
\mu \neq 0, \ u = 0. \tag{45}
\]

If the conditions (43) are valid, then one one finds readily that the only common solutions of the equations (31) - (33) and the equations (38) - (40) as well as the constraint equation (24) which satisfy the initial conditions are the following,

\[
x = 0, \ y = 0, \ z = wt. \tag{46}
\]

Likewise, if the conditions (44) are in force, then the only possible solutions are

\[
x = ut, \ y = 0, \ z = 0. \tag{47}
\]

Finally, if the conditions (45) are valid, one finds the following solution,

\[
x = 0, \ y = 0, \ z = 0, \ \mu = c, \tag{48}
\]

where \( c \) is a constant.

The solutions (46), (47), and (48), respectively, are the only functions which satisfy both the Lagrange equations of motion in the form (31) - (33) and the variational equations (38) - (40) together with the constraint (24), under the initial conditions (27) and (28). These solutions are clearly exceptional in that the non-holonomic constraint (24) is no constraint at all for these solutions.

4 Existence of Lagrangians and Hamiltonians in Pars’ example

It should be observed that the fact that the variational procedure involving the multiplication rule does not lead to equations of motion identical to those which follow from the generalized principle of d’Alembert in the case of non-holonomic systems, does not prove that there is no variational principle at all for non-holonomic systems. One may still wonder whether non-holonomic systems may nevertheless admit some kind of variational formulation. A straightforward answer to this question is obtained if one can show that the correct equations of motion (13) together with the constraints (4) constitute a set of Lagrangian equations with some appropriate Lagrangian. This is an inverse problem, which is trivial in the case of one-dimensional
systems. Complete results on the inverse problem in question exist for two-dimensional systems, but not for systems of dimension three or higher. We analyze the problem posed here only in the non-holonomic three-dimensional special case considered by Pars, which was analyzed in some detail above.

The question is now whether the equations (31) - (33) are the Euler-Lagrange equations with some appropriately chosen Lagrangian, or linearly equivalent to such Euler-Lagrange equations in three space dimensions. For this problem we refer to a paper by Douglas [11] on the inverse problem in the calculus of variations as well as to a paper by Crampin et al. [12], which gives a geometric formulation of the inverse problem, with due reference to the paper of Douglas.

Using results given in the papers by Douglas and Crampin et al., referred to above, one finds that the equations (31) - (33) can not be recast into linearly equivalent equations involving three variables, such that these equivalent equations are the Euler-Lagrange equations of some appropriate functional. We know that the space of dynamically accessible paths in the problem under consideration is in fact two-dimensional, so it is then natural to look for a variational formulation in a two-dimensional space. It will be shown that the system of equations (31) - (33) can be reduced to an equivalent two-dimensional autonomous system, for which there exist Lagrangians.

Eliminating the quantity $\dot{x}$ from equation in the system (32) with the aid of the relation (36) above, one obtains a two-dimensional autonomous system from the equations (31) - (33), which involves the variables $y$ and $z$ only,

\begin{align*}
\ddot{y} &= \ell^2 u \dot{z} \left( \ell^2 + z^2 \right)^{-\frac{3}{2}}, \\
\ddot{z} &= 0.
\end{align*}

(49)

The simple system of equations (49) is indeed obtainable from a principle of stationary action in a space of two dimensions. There is in fact more than one Lagrangian for which the equations (49) are the Euler-Lagrange equations. It is known that Lagrangians which are derived from the equations of motion are not necessarily unique. [13] We display below two such Lagrangians $L_I$ and $L_{II}$, whose difference is not a time derivative of some appropriate function:

\begin{align*}
L_I := m \dot{y} \dot{z} - \frac{mu \dot{z}}{\sqrt{\ell^2 + z^2}} \log \left( \frac{\dot{z}}{c_0} \right),
\end{align*}

(50)

where $c_0$ is a constant with the dimension of velocity. It should be noted that the second equation in (49) implies that $\dot{z}(t)$ is of constant sign for $t > 0$. The sign of the constant $c_0$ in Eq. (50) should be chosen to be the same as the sign of the initial value $w$, so that $\dot{z}(t)/c_0 > 0$ for $t > 0$. It should also be noted that the absolute value of the dimensional constant $c_0$ is of no consequence for the equations of motion. The difference of two Lagrangians corresponding to two different constants $c_0$ and $c'_0$, respectively, is

\begin{align*}
mu \left( \log \frac{c_0}{c_0'} \right) \frac{d}{dt} \sqrt{\ell^2 + z^2}.
\end{align*}

(51)

Since the difference (51) is a time derivative, the Lagrangians corresponding to the different constants $c_0$ and $c'_0$ are equivalent.
The second Lagrangian is,
\[ L_{II} := \frac{m\dot{y}\dot{z}^2}{2c_0} - \frac{muz\dot{z}^2}{c_0\sqrt{\ell^2 + z^2}}, \] (52)
where \(c_0\) is again a constant with the dimension of velocity.

In the Lagrangians above, the mass \(m\) occurs only as a multiplicative factor. It has been included for convenience in order to keep track of the dimensions in the Lagrangians in question.

The construction of the Lagrangians (50) and (52) using the methods developed in Douglas paper [11] is a task which involves lengthy calculations, which we will not present here. However one can easily verify that the curious Lagrangians \(L_I\) and \(L_{II}\) above do give rise to the system of equations (49). We will consider the Lagrangian \(L_I\) above only, leaving the detailed consideration of the second Lagrangian \(L_{II}\) to the interested reader.

From Eq. (50) follows readily that
\[ \frac{\partial L_I}{\partial y} = 0, \] (53)
and
\[ \frac{\partial L_I}{\partial \dot{y}} = m\dot{z}, \] (54)
whence the Euler equation
\[ \frac{\partial L_I}{\partial y} - \frac{d}{dt} \left( \frac{\partial L_I}{\partial \dot{y}} \right) = 0, \] (55)
implies that
\[ \ddot{z} = 0. \] (56)
Likewise,
\[ \frac{\partial L_I}{\partial z} = -\frac{m\ell^2u\dot{z}}{\ell^2 + z^2} \left( \frac{\log \dot{z}}{c_0} \right), \] (57)
and
\[ \frac{\partial L_I}{\partial \dot{z}} = m\dot{y} - \frac{muz}{\sqrt{\ell^2 + z^2}} \left[ 1 + \log \left( \frac{\dot{z}}{c_0} \right) \right]. \] (58)
Inserting the expressions (57) and (58) in the appropriate Euler equation
\[ \frac{\partial L_I}{\partial z} - \frac{d}{dt} \left( \frac{\partial L_I}{\partial \dot{z}} \right) = 0, \] (59)
one obtains the expression
\[ \ddot{y} = \frac{\ell^2 u \dot{z}}{(\ell^2 + z^2)^{3/2}} + \frac{uz}{(\ell^2 + z^2)^{1/2}} \dot{z}. \] (60)
Using the equation (56) in the expression (60) above, one ends up with the equation
\[ \ddot{y} = \ell^2 u \dot{z} (\ell^2 + z^2)^{-3/2}. \] (61)
This demonstrates that the equations of motion (49) are equivalent to the Euler equations with the Lagrangian $L_I$ given in Eq. (50).

For completeness we also record the Hamiltonian $H_I$ which corresponds to the Lagrangian (50). The canonical momenta are defined in the standard manner,

$$p_y := \frac{\partial L_I}{\partial \dot{y}} = m \dot{z}, \quad (62)$$

and

$$p_z := \frac{\partial L_I}{\partial \dot{z}} = m \dot{y} - \frac{m u z}{\sqrt{\ell^2 + z^2}} \left[1 + \log \left(\frac{\dot{z}}{c_0}\right)\right]. \quad (63)$$

The equations (62) and (63) can be solved for the velocities $(\dot{y}, \dot{z})$ in terms of the canonical momenta $(p_y, p_z),

$$\dot{z} = \frac{p_y}{m}, \quad (64)$$

and

$$\dot{y} = \frac{p_z}{m} + \frac{u z}{\sqrt{\ell^2 + z^2}} \left[1 + \log \left(\frac{p_y}{m c_0}\right)\right]. \quad (65)$$

This leads to the following Hamiltonian $H_I$,

$$H_I = \frac{1}{m} p_y p_z + \frac{u z}{\sqrt{\ell^2 + z^2}} p_y \log \left(\frac{p_y}{m c_0}\right). \quad (66)$$

The Hamiltonian equations involving the Hamiltonian $H_I$ in Eq. (66) are as follows,

$$\dot{y} := \frac{\partial H_I}{\partial p_y} = \frac{1}{m} p_z + \frac{u z}{\sqrt{\ell^2 + z^2}} \left[1 + \log \left(\frac{p_y}{m c_0}\right)\right], \quad (67)$$

$$\dot{z} := \frac{\partial H_I}{\partial p_z} = \frac{1}{m} p_y, \quad (68)$$

$$\dot{p}_y := -\frac{\partial H_I}{\partial y} = 0, \quad (69)$$

and

$$\dot{p}_z := -\frac{\partial H_I}{\partial z} = -\frac{\ell^2 u}{(\ell^2 + z^2)^{3/2}} p_y \log \left(\frac{p_y}{m c_0}\right). \quad (70)$$

Combining Eqns. (67) and (70) one obtains the first equation in the two-dimensional system (49). Likewise, combining Eqns. (68) and (69) one obtains the second equation in (49).

Using the Lagrangian $L_{II}$ given above as a starting point, one obtains an alternative canonical Hamiltonian formulation of the two-dimensional system of equations (49). The canonical momenta $p_y$ and $p_z$ are now related to the velocities $\dot{y}$ and $\dot{z}$ as follows,

$$p_y = \frac{m z^2}{2 c_0}, \quad (71)$$
and
\[ p_z = \sqrt{\frac{2mp_y}{c_0}} \left( \dot{y} - \frac{2uz}{\sqrt{\ell^2 + z^2}} \right). \]  
(72)

The Hamiltonian \( H_{II} \) is
\[ H_{II} = \sqrt{\frac{2c_0p_y}{m}} p_z + \frac{2uz}{\sqrt{\ell^2 + z^2}} p_y. \]  
(73)

It is readily verified that the canonical equations with the Hamiltonian (73) likewise reproduce the equations (49).

5 Concluding remarks

We consider the principle of stationary action for autonomous mechanical systems. The principle of stationary action, Hamilton’s principle, can be generalized to systems with holonomic constraints by including the constraints in the variational procedure by means of the multiplication rule in the calculus of variations. The corresponding procedure in the case of non-holonomic constraints leads to equations which are not, in general, identical to the correct equations of motion. This fact has been known for some time.

It has been shown, in the case of a particular three-dimensional autonomous non-holonomic system, that a variational principle exists; the equations of motion for the system can be reduced to a two-dimensional system of autonomous equations which admit a variational formulation and also a canonical Hamiltonian formulation. Explicit expressions for two nonequivalent Lagrangians together with the corresponding Hamiltonians are given. The example shows that one can not a priori rule out the existence of variational principles and Hamiltonians for non-holonomic systems, however artificial.

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