INDUCTIVELY FREE MULTIDERIVATIONS
OF BRAID ARRANGEMENTS

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Abstract. The reflection arrangement of a Coxeter group is a well known instance of a free hyperplane arrangement. In 2002, Terao showed that equipped with a constant multiplicity each such reflection arrangement gives rise to a free multiarrangement. In this note we show that this multiarrangement satisfies the stronger property of inductive freeness in case the Coxeter group is of type $A$.

1. Introduction

Arnold and independently Saito proved that the reflection arrangements of Coxeter groups are free, [OT92, §6]. They play a special role in the class of free hyperplane arrangements.

In his seminal work [Z89], Ziegler introduced the notion of multiarrangements and initiated the study of their freeness. In general, for a free hyperplane arrangement, an arbitrary multiplicity need not afford a free multiarrangement, e.g. see [Z89, Ex. 14].

By constructing an explicit basis of the module of derivations, Terao showed in [Ter02] that each Coxeter arrangement gives rise to a free multiarrangement when endowed with a constant multiplicity.

In their ground breaking work [ATW08, Thm. 0.8], Abe, Terao and Wakefield proved the Addition-Deletion Theorem for multiarrangements. This naturally leads to the class of inductively free multiarrangements, see Definition 2.7 below.

Let $\mathcal{B}_\ell$ be the braid arrangement in $\mathbb{C}^\ell$. It is the direct product of the empty 1-arrangement $\Phi_1$ and the irreducible Coxeter arrangement $\mathcal{A}_{\ell-1}$ of type $A_{\ell-1}$, [OT92, §6.4]. It follows from Definition 2.7 and Theorem 2.11 that a multiplicity on $\mathcal{B}_\ell$ is inductively free if and only if the corresponding multiplicity on the factor $\mathcal{A}_{\ell-1}$ is inductively free.

Our main result shows that the irreducible Coxeter arrangement $\mathcal{A}_{\ell-1}$ of type $A_{\ell-1}$ when equipped with a constant multiplicity is an inductively free multiarrangement.

Theorem 1.1. Let $\mathcal{B}_\ell$ be the braid arrangement. Then, for $m \in \mathbb{Z}_{\geq 1}$, the multiarrangement $(\mathcal{B}_\ell, m)$ with defining polynomial

$$Q(\mathcal{B}_\ell, m) = \prod_{1 \leq i < j \leq \ell} (x_i - x_j)^m$$

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is inductively free. In particular, for the irreducible Coxeter arrangement \( \mathcal{A}_{\ell-1} \) of type \( A_{\ell-1} \), the multiarrangement \((\mathcal{A}_{\ell-1}, m)\) is inductively free with exponents given by

\[
\exp(\mathcal{A}_{\ell-1}, m) = \{ \frac{m\ell}{2}, \ldots, \frac{m\ell}{2} \}
\]

for \( m \) even, respectively

\[
\exp(\mathcal{A}_{\ell-1}, m) = \left\{ \frac{(m-1)\ell}{2} + 1, \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \right\}
\]

for \( m \) odd.

The exponents in Theorem 1.1 have been determined by Terao [Ter02, Thm. 1.1]. Note that \( \ell \) is the Coxeter number of the irreducible Coxeter group of type \( A_{\ell-1} \) and \( \{1, 2, \ldots, \ell - 1\} \) is its set of exponents, see [Bou68, V 6.2].

As a consequence of our proof of Theorem 1.1, we also obtain that certain non-constant multiplicities give rise to inductively free multiarrangements of the braid arrangement. These occur as restrictions in our induction tables.

**Corollary 1.2.** Let \( \mathcal{A} \) be the irreducible Coxeter arrangement of type \( A_{\ell-1} \). Then, for \( m, q \in \mathbb{Z}_{\geq 1} \), the multiarrangement \((\mathcal{A}; m, q)\) with defining polynomial

\[
Q(\mathcal{A}; m, q) = \prod_{2 \leq j \leq \ell} (x_1 - x_j)^{m+q} \prod_{2 \leq i < j \leq \ell} (x_i - x_j)^m
\]

is inductively free with

\[
\exp(\mathcal{A}; m, q) = \{ \frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q \}
\]

when \( m \) is even or

\[
\exp(\mathcal{A}; m, q) = \left\{ \frac{(m-1)\ell}{2} + 1 + q, \frac{(m-1)\ell}{2} + 2 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 + q \right\}
\]

when \( m \) is odd.

It can be rather challenging to prove or disprove that a given arrangement is inductively free, e.g. see [AHR14, Lem. 4.2], [BC12, §5.2], and [HR15, Lem. 3.5]. In principle, one might have to search through all possible chains of free subarrangements. We prove Theorem 1.1 by exhibiting an induction table of inductively free submultiarrangements, see Remark 2.10.

If \( \mathcal{A} \) is an inductively free simple arrangement, then for \( m \in \mathbb{Z}_{\geq 1} \) the multiarrangement \((\mathcal{A}, m)\) need not be inductively free in general (indeed it need not even be free), e.g. see [Z89, Ex. 14]. So the situation for Coxeter groups as suggested by Theorem 1.1 is very special.

In [ANN09], Abe, Nuda and Numata determine a large class of non-constant free multiplicities of the braid arrangement. It is natural to ask whether these are inductively free as well. More generally, it is likely that the reflection arrangement of any Coxeter group with constant multiplicity is inductively free. It is also natural to investigate inductive freeness for non-constant free multiplicities of Coxeter arrangements, cf. [AY09], [ATW12].

We refer to [Ter02, Rem. 1.6] and [Y04] for the connection of the question of freeness of a Coxeter arrangement \( \mathcal{A} \) endowed with a constant multiplicity and the question of freeness of extended Shi and extended Catalan arrangements. The latter was a conjecture of Edelman and Reiner and is proved by Yoshinaga in [Y04].
2. Recollection and Preliminaries

2.1. Hyperplane Arrangements. Let $V = \mathbb{K}^\ell$ be an $\ell$-dimensional $\mathbb{K}$-vector space. A hyperplane arrangement is a pair $(\mathcal{A}, V)$, where $\mathcal{A}$ is a finite collection of hyperplanes in $V$. Usually, we simply write $\mathcal{A}$ in place of $(\mathcal{A}, V)$. We write $|\mathcal{A}|$ for the number of hyperplanes in $\mathcal{A}$. The empty arrangement in $V$ is denoted by $\emptyset$. The lattice $L(\mathcal{A})$ of $\mathcal{A}$ is the set of subspaces of $V$ of the form $H_1 \cap \cdots \cap H_i$ where $\{H_1, \ldots, H_i\}$ is a subset of $\mathcal{A}$. For $X \in L(\mathcal{A})$, we have two associated arrangements, firstly $\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \subseteq \mathcal{A}$, the localization of $\mathcal{A}$ at $X$, and secondly, the restriction of $\mathcal{A}$ to $X$, $(\mathcal{A}^X, X)$, where $\mathcal{A}^X := \{X \cap H \mid H \in \mathcal{A} \setminus \mathcal{A}_X\}$. Note that $V$ belongs to $L(\mathcal{A})$ as the intersection of the empty collection of hyperplanes and $\mathcal{A}^V = \mathcal{A}$.

If $0 \in H$ for each $H$ in $\mathcal{A}$, then $\mathcal{A}$ is called central. If $\mathcal{A}$ is central, then $T_\mathcal{A} := \cap_{H \in \mathcal{A}} H$ is the center of $\mathcal{A}$. We have a rank function on $L(\mathcal{A})$: $r(X) := \text{codim}_V(X)$. The rank $r := r(\mathcal{A})$ of $\mathcal{A}$ is the rank of $T_\mathcal{A}$.

2.2. Free Hyperplane Arrangements. Let $S = S(V^*)$ be the symmetric algebra of the dual space $V^*$ of $V$. If $x_1, \ldots, x_\ell$ is a basis of $V^*$, then we identify $S$ with the polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell]$. Letting $S_p$ denote the $\mathbb{K}$-subspace of $S$ consisting of the homogeneous polynomials of degree $p$ (along with 0), $S$ is naturally $\mathbb{Z}$-graded: $S = \oplus_{p \in \mathbb{Z}} S_p$, where $S_p = 0$ in case $p < 0$.

Let $\text{Der}(S)$ be the $S$-module of algebraic $\mathbb{K}$-derivations of $S$. Using the $\mathbb{Z}$-grading on $S$, $\text{Der}(S)$ becomes a graded $S$-module. For $i = 1, \ldots, \ell$, let $D_i := \partial/\partial x_i$ be the usual derivation of $S$. Then $D_1, \ldots, D_\ell$ is an $S$-basis of $\text{Der}(S)$. We say that $\theta \in \text{Der}(S)$ is homogeneous of polynomial degree $p$ provided $\theta = \sum_{i=1}^\ell f_i D_i$, where $f_i$ is either 0 or homogeneous of degree $p$ for each $1 \leq i \leq \ell$. In this case we write $\text{pdeg} \theta = p$.

Let $\mathcal{A}$ be an arrangement in $V$. Then for $H \in \mathcal{A}$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. The defining polynomial $Q(\mathcal{A})$ of $\mathcal{A}$ is given by $Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S$.

The module of $\mathcal{A}$-derivations of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) := \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for each } H \in \mathcal{A}\}.$$ 

We say that $\mathcal{A}$ is free if the module of $\mathcal{A}$-derivations $D(\mathcal{A})$ is a free $S$-module.

With the $\mathbb{Z}$-grading of $\text{Der}(S)$, also $D(\mathcal{A})$ becomes a graded $S$-module, [OT92, Prop. 4.10]. If $\mathcal{A}$ is a free arrangement, then the $S$-module $D(\mathcal{A})$ admits a basis of $\ell$ homogeneous derivations, say $\theta_1, \ldots, \theta_\ell$, [OT92, Prop. 4.18]. While the $\theta_i$'s are not unique, their polynomial degrees $\text{pdeg} \theta_i$ are unique (up to ordering). This multiset is the set of exponents of the free arrangement $\mathcal{A}$ and is denoted by $\exp \mathcal{A}$.

2.3. Multiarrangements. A multiarrangement is a pair $(\mathcal{A}, \nu)$ consisting of a hyperplane arrangement $\mathcal{A}$ and a multiplicity function $\nu : \mathcal{A} \to \mathbb{Z}_{\geq 0}$ associating to each hyperplane $H$ in $\mathcal{A}$ a non-negative integer $\nu(H)$. Alternately, the multiarrangement $(\mathcal{A}, \nu)$ can also be thought of as the multiset of hyperplanes

$$(\mathcal{A}, \nu) = \{H^{\nu(H)} \mid H \in \mathcal{A}\}.$$
We say that $\nu$ is a \textit{constant} multiplicity provided there is some fixed $m \in \mathbb{Z}_{\geq 0}$ so that $\nu(H) = m$ for every $H \in \mathcal{A}$. In that case we also say that $\nu$ is constant of \textit{weight} $m$ and frequently write $(\mathcal{A}, m)$ in place of $(\mathcal{A}, \nu)$.

The \textit{order} of the multiarrangement $(\mathcal{A}, \nu)$ is the cardinality of the multiset $(\mathcal{A}, \nu)$; we write $|\nu| := |(\mathcal{A}, \nu)| = \sum_{H \in \mathcal{A}} \nu(H)$. For a multiarrangement $(\mathcal{A}, \nu)$, the underlying arrangement $\mathcal{A}$ is sometimes called the associated \textit{simple} arrangement, and so $(\mathcal{A}, \nu)$ itself is simple if and only if $\nu(H) = 1$ for each $H \in \mathcal{A}$.

Let $\mathcal{A} = \{H_1, H_2, \ldots\}$ be a simple arrangement. Then sometimes it is convenient to denote a multiplicity function $\nu$ on $\mathcal{A}$ simply by the ordered tuple of its values $[\nu(H_1), \nu(H_2), \ldots]$.

\textbf{Definition 2.1.} Let $\nu_i$ be a multiplicity of $\mathcal{A}_i$ for $i = 1, 2$. When viewed as multisets, suppose that $(\mathcal{A}_1, \nu_1)$ is a subset of $(\mathcal{A}_2, \nu_2)$. Then we say that $(\mathcal{A}_1, \nu_1)$ is a \textit{submultiarrangement} of $(\mathcal{A}_2, \nu_2)$ and write $(\mathcal{A}_1, \nu_1) \subseteq (\mathcal{A}_2, \nu_2)$, i.e. we have $\nu_1(H) \leq \nu_2(H)$ for each $H \in \mathcal{A}_1$.

\textbf{Definition 2.2.} Let $(\mathcal{A}, \nu)$ be a multiarrangement in $\mathcal{V}$ and let $X$ be in the lattice of $\mathcal{A}$. The \textit{localization} of $(\mathcal{A}, \nu)$ at $X$ is $(\mathcal{A}_X, \nu_X)$, where $\nu_X = \nu|_{\mathcal{A}_X}$.

\textbf{2.4. Freeness of multiarrangements.} Following Ziegler [Z89], we extend the notion of freeness to multiarrangements as follows. The \textit{defining polynomial} $Q(\mathcal{A}, \nu)$ of the multiarrangement $(\mathcal{A}, \nu)$ is given by
\[
Q(\mathcal{A}, \nu) := \prod_{H \in \mathcal{A}} \alpha_H^{\nu(H)},
\]
a polynomial of degree $|\nu|$ in $S$.

The \textit{module of $\mathcal{A}$-derivations} of $(\mathcal{A}, \nu)$ is defined by
\[
D(\mathcal{A}, \nu) := \{\theta \in \text{Der}(S) \mid \theta (\alpha_H) \in \alpha_H^{\nu(H)} S \text{ for each } H \in \mathcal{A}\}.
\]

We say that $(\mathcal{A}, \nu)$ is \textit{free} if $D(\mathcal{A}, \nu)$ is a free $S$-module, [Z89, Def. 6].

As in the case of simple arrangements, $D(\mathcal{A}, \nu)$ is a $\mathbb{Z}$-graded $S$-module and thus, if $(\mathcal{A}, \nu)$ is free, there is a homogeneous basis $\theta_1, \ldots, \theta_\ell$ of $D(\mathcal{A}, \nu)$. The multiset of the unique polynomial degrees $\text{pdeg} \theta_i$ forms the set of \textit{exponents} of the free multiarrangement $(\mathcal{A}, \nu)$ and is denoted by $\text{exp}(\mathcal{A}, \nu)$. It follows from Ziegler’s analogue of Saito’s criterion [Z89, Thm. 8] that $\sum \text{pdeg} \theta_i = \deg Q(\mathcal{A}, \nu) = |\nu|$.

\textbf{Remark 2.3.} A product of multiarrangements is free if and only if each factor is free: using [ATW08, Lem. 1.3], the proof of [OT92, Thm. 4.28] readily extends to multiarrangements, thanks to Ziegler’s analogue of Saito’s criterion [Z89, Thm. 8]. Moreover, in that case the set of exponents of the product is the union of the sets of exponents of the factors.

\textbf{2.5. The Addition-Deletion Theorem for Multiarrangements.} We recall the construction from [ATW08].

\textbf{Definition 2.4.} Let $(\mathcal{A}, \nu) \neq \Phi$ be a multiarrangement. Fix $H_0$ in $\mathcal{A}$. We define the \textit{deletion} $(\mathcal{A}', \nu')$ and \textit{restriction} $(\mathcal{A}'', \nu'')$ of $(\mathcal{A}, \nu)$ with respect to $H_0$ as follows. If $\nu(H_0) = 1$, then set $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and define $\nu'(H) = \nu(H)$ for all $H \in \mathcal{A}'$. If $\nu(H_0) > 1$, then set $\mathcal{A}' = \mathcal{A}$ and define $\nu'(H_0) = \nu(H_0) - 1$ and $\nu'(H) = \nu(H)$ for all $H \neq H_0$. 

\[\]
Let $\mathcal{A}'' = \{H \cap H_0 \mid H \in \mathcal{A} \setminus \{H_0\}\}$. The Euler multiplicity $\nu^*$ of $\mathcal{A}''$ is defined as follows. Let $Y \in \mathcal{A}''$. Since the localization $\mathcal{A}_Y$ is of rank 2, the multiarrangement $(\mathcal{A}_Y, \nu_Y)$ is free, [Z89, Cor. 7]. According to [ATW08, Prop. 2.1], the module of derivations $D(\mathcal{A}_Y, \nu_Y)$ admits a particular homogeneous basis $\{\theta_Y, \psi_Y, D_3, \ldots, D_\ell\}$, where $\theta_Y$ is identified by the property that $\theta_Y \notin \alpha_0\text{Der}(S)$ and $\psi_Y$ by the property that $\psi_Y \in \alpha_0\text{Der}(S)$, where $H_0 = \ker \alpha_0$. Then the Euler multiplicity $\nu^*$ is defined on $Y$ as $\nu^*(Y) = \text{pdeg} \theta_Y$.

We refer to $(\mathcal{A}, \nu), (\mathcal{A}', \nu')$ and $(\mathcal{A}'', \nu^*)$ as the triple of $(\mathcal{A}, \nu)$ with respect to $H_0$.

**Theorem 2.5** ([ATW08, Thm. 0.8] Addition-Deletion-Theorem for Multiarrangements). Suppose that $(\mathcal{A}, \nu) \neq \Phi_\ell$. Fix $H_0$ in $\mathcal{A}$ and let $(\mathcal{A}, \nu), (\mathcal{A}', \nu')$ and $(\mathcal{A}'', \nu^*)$ be the triple with respect to $H_0$. Then any two of the following statements imply the third:

1. $(\mathcal{A}, \nu)$ is free with $\exp(\mathcal{A}, \nu) = \{b_1, \ldots, b_\ell-1, b_\ell\}$;
2. $(\mathcal{A}', \nu')$ is free with $\exp(\mathcal{A}', \nu') = \{b_1, \ldots, b_\ell-1, b_\ell - 1\}$;
3. $(\mathcal{A}'', \nu^*)$ is free with $\exp(\mathcal{A}'', \nu^*) = \{b_1, \ldots, b_\ell-1\}$.

Let $H_0 \in \mathcal{A}$, $\mathcal{A}'' = \mathcal{A}' \cap H_0$ and let $X \in \mathcal{A}''$. Let $\nu$ be a multiplicity on $\mathcal{A}$. Let $\nu_0 = \nu(H_0)$. Further let $k = |\mathcal{A}_X|$ and $\nu_1 = \max\{\nu(H) \mid H \in \mathcal{A}_X \setminus \{H_0\}\}$.

1. If $k = 3$, $2\nu_0 \leq |\nu_X|$, and $2\nu_1 \leq |\nu_X|$, then $\nu^*(X) = \lceil|\nu_X|/2\rceil$.
2. If $k = 2$, then $\nu^*(X) = \nu_1$.
3. If $2\nu_1 \geq |\nu_X| - 1$, then $\nu^*(X) = \nu_1$.

### 2.6. Inductive Freeness for Multiarrangements

As in the simple case, Theorem 2.5 motivates the notion of inductive freeness.

**Definition 2.7** ([ATW08, Def. 0.9]). The class $\mathcal{IFM}$ of inductively free multiarrangements is the smallest class of arrangements subject to

1. $\Phi_\ell \in \mathcal{IFM}$ for each $\ell \geq 0$;
2. for a multiarrangement $(\mathcal{A}, \nu)$, if there exists a hyperplane $H_0 \in \mathcal{A}$ such that both $(\mathcal{A}', \nu')$ and $(\mathcal{A}'', \nu^*)$ belong to $\mathcal{IFM}$, and $\exp(\mathcal{A}'', \nu^*) \subseteq \exp(\mathcal{A}', \nu')$, then $(\mathcal{A}, \nu)$ also belongs to $\mathcal{IFM}$.

**Remark 2.8** ([ATW08, Rem. 0.10]). The intersection of $\mathcal{IFM}$ with the class of simple arrangements is the class $\mathcal{IF}$ of inductively free arrangements.

**Remark 2.9.** As in the simple case, if $r(\mathcal{A}) \leq 2$, then $(\mathcal{A}, \nu)$ is inductively free, [Z89, Cor. 7].

**Remark 2.10.** In analogy to the simple case, cf. [OT92, §4.3, p. 119], [HR15, Rem. 2.9], it is possible to describe an inductively free multiarrangement $(\mathcal{A}, \nu)$ by means of a so-called induction table. In this process we start with an inductively free multiarrangement (frequently $\Phi_\ell$) and add hyperplanes successively ensuring that part (ii) of Definition 2.7 is satisfied. We refer to this process as induction of hyperplanes. This procedure amounts to choosing a total order on the multiset $(\mathcal{A}, \nu)$, say $\mathcal{A} = \{H_1, \ldots, H_n\}$, where $n = |\nu|$, so
that each of the submultiarrangements $A_0 := \Phi_\ell$, $(A_i, \nu_i) := \{H_1, \ldots, H_i\}$ (viewed again as multiset) and each of the restrictions $(A_i^H, \nu_i^*)$ is inductively free for $i = 1, \ldots, n$. As in the simple case, in the associated induction table we record in the $i$-th row the information of the $i$-th step of this process, by listing $\exp(A_i', \nu_i') = \exp(A_{i-1}, \nu_{i-1})$, the defining form $\alpha_{H_i}$ of $H_i$, as well as $\exp(A_i'', \nu_i^*) = \exp(A_i^H, \nu_i^*)$, for $i = 1, \ldots, n$. Frequently, we refer to a triple $(A_i, \nu_i), (A_{i-1}, \nu_{i-1}), (A_i^H, \nu_i^*)$ in such an induction table as an inductive triple. In addition we also record the Euler multiplicity and in part the relevant data from Proposition 2.6. For instance, see Tables 1 up to 7 below.

We also require the following result from [HRS15, Thm. 1.4]; this extends the compatibility of freeness with products from Remark 2.3 to inductive freeness.

**Theorem 2.11.** A product of multiarrangements belongs to $\mathcal{IFM}$ if and only if each factor belongs to $\mathcal{IFM}$.

**Remark 2.12.** Since localization is compatible with the product construction, it follows from the definition of the Euler multiplicity that it is also compatible with this product construction. In particular, the Euler multiplicity of the restriction of a product to a hyperplane only depends on the relevant factor. We use this fact throughout without further comment.

### 3. Proof of Theorem 1.1

In order to prove Theorem 1.1, we perform an induction of hyperplanes, see Remark 2.10. By [OT92, Prop. 6.73], every restricted arrangement $A''$ is of Coxeter type $A$ again. However, calculating the corresponding Euler multiplicities of these restrictions, we see that we do not always get a constant multiplicity. If $(A, \nu) = (A, m)$ has a constant multiplicity of weight $m$, then during the induction of hyperplanes, $(A'', \nu^*)$ has multiplicity given by the following defining polynomial

$$Q(A; m, q) := \prod_{1 < j \leq \ell - 1} (x_1 - x_j)^{m+q} \prod_{2 \leq i < j \leq \ell - 1} (x_i - x_j)^m$$

for some non-negative integer $q$.

If $q = 0$ (i.e. when $\nu^*$ is a constant multiplicity), then the exponents are given by Theorem 1.1. In any case, irrespective of being able to determine the exponents in our induction, we do not know a priori whether or not the restricted multiarrangements that occur are inductively free. In this context, the next result is very useful. It states that such arrangements with described multiplicities are indeed inductively free assuming Theorem 1.1 holds.

**Lemma 3.1.** Let $A$ be the Coxeter arrangement of type $A_{\ell-1}$ and let $\nu: A \to \mathbb{Z}_{\geq 0}$ be a constant multiplicity of weight $m$. Suppose that the multiarrangement $(A, \nu)$ is inductively free. Then, for any $q \in \mathbb{Z}_{\geq 0}$, the multiarrangement $(A; m, q)$ with defining polynomial

$$Q(A; m, q) := \prod_{2 \leq j \leq \ell} (x_1 - x_j)^{m+q} \prod_{2 \leq i < j \leq \ell} (x_i - x_j)^m$$

is inductively free with

$$\exp(A; m, q) = \{\frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q\}$$
when \( m \) is even, respectively

\[
\exp(A; m, q) = \left\{ \frac{(m-1)\ell}{2} + 1 + q, \frac{(m-1)\ell}{2} + 2 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 + q \right\}
\]

when \( m \) is odd.

**Proof.** Let \( A_{\ell-1} \) be the Coxeter arrangement of type \( A_{\ell-1} \). We argue by induction on \( \ell \). For \( \ell = 2 \), it follows from Remark 2.9 that \((A_1; m, q)\) is inductively free. We have

\[
Q(A_1; m, q) = (x_1 - x_2)^{m+q}.
\]

This is a Coxeter arrangement of type \( A_1 \) with a constant multiplicity \( m + q \) and so, its set of exponents is \( \{m + q\} \), as given by Theorem 1.1, thanks to [Ter02, Thm. 1.1]. Note that this does not depend on the parity of \( m + q \). So the result follows for \( \ell = 2 \).

Strictly speaking, the case \( \ell = 3 \) is not necessary in our induction. It is however very instructive to see the arguments in this case, as this is an instance of a non-constant multiplicity.

For \( \ell = 3 \) we see again by Remark 2.9 that \((A_2; m, q)\) is inductively free. But this time we do not have a constant multiplicity. Here the multiplicity is given by \( [m + q, m + q, m] \). Therefore, we perform an induction of hyperplanes, starting with the case in which \( m \) is even (including \( m = 0 \)). By assumption of the lemma, the multiarrangement \((A_2, \nu) = (A_2; m, q)\) is inductively free. Therefore, we may initialize the induction table with the multiarrangement \((A_2; m, q)\) with

\[
\exp(A_2; m, q) = \left\{ \frac{3m-1}{2}, \frac{3m}{2}, \frac{3m+1}{2} \right\},
\]

where the exponents are again given by [Ter02, Thm. 1.1]. Our aim is to add the hyperplanes of type \( \ker(x_1 - x_j) \) (\( j = 2, 3 \)) \( q \) times successively. In this 3-dimensional case, determining the restriction in each step is very simple. We have \( A'' = \{x_1 = x_2 = x_3\} \) (except for the first step of the case \( m = 0 \) where \( A'' = \Phi_2 \)). The Euler multiplicity can easily be calculated using Proposition 2.6(1), because we have \( k = 3 \) in every step. The resulting multiarrangement \((A'', \nu^*)\) is always inductively free because it has rank 1. Since \( A'' \cong A_1 \) is again a Coxeter arrangement of type \( A_1 \) with a constant multiplicity \( |\nu^*| \), its exponents are as given by Theorem 1.1. In the first step we obtain \( \nu^*(X) = \left\lfloor \frac{3m-1}{2} \right\rfloor \), so the multiplicity of the single hyperplane is \( 3m/2 \) and so \( \exp(A'', \nu^*) = \{3m/2\} \). Applying Theorem 2.5, we can easily determine the exponents of the new multiarrangement in every step, see Table 1.

The case where \( m \) is odd is treated in a similar way. Starting with

\[
\exp(A_2; m, q) = \left\{ \frac{3(m-1)}{2} + 1, \frac{3(m-1)}{2} + 2 \right\},
\]

we add the hyperplanes of type \( \ker(x_1 - x_j) \) (\( j = 2, 3 \)) \( q \) times successively until we get

\[
\exp(A_2; m, q) = \left\{ \frac{3(m-1)}{2} + q + 1, \frac{3(m-1)}{2} + q + 2 \right\}.
\]

Now suppose that \( m \) is even, \( \ell > 3 \), and that the statement of the lemma holds for all values of \( q \) for smaller ranks. By hypothesis of the lemma, the multiarrangement \((A_{\ell-1}; m)\) is inductively free with exponents

\[
\exp(A_{\ell-1}; m, q) = \left\{ \frac{m\ell}{2}, \ldots, \frac{m\ell}{2} \right\}.
\]
Consequently, by Proposition 3.2.4, we may assume that $\nu_1 = m + 1$ and $\nu_0 = m + 1$, hence, by Proposition 3.2.5, we have

$$\nu_1 = m + 1, \quad \nu_0 = m + 1.$$ 

We initialize our induction table with this inductively free multiarrangement. Then we consider the restriction to $\ker(x_1 - x_j)$ without loss, so that

$$A'' := \{x_1 = x_j = x_a; x_1 = x_j, x_b = x_c\} \cong A_{\ell-2},$$

where $1 < j \leq \ell$ and $a, b, c \neq 1, j$ and $b \neq c$. We denote the members of $A''$ by $Y_a^j := \{x_1 = x_j = x_a\}$ and $Y_{b,c}^j := \{x_1 = x_j, x_b = x_c\}$. In the first step we restrict to $\ker(x_1 - x_2)$ and calculate the Euler multiplicities using Proposition 2.6.1 and (2) as follows: for $Y_a^2$, we have

$$Q(A_{Y_a^2}, \nu_{Y_a^2}) = (x_1 - x_2)^{m+1}(x_1 - x_a)^m(x_2 - x_a)^m.$$ 

Hence, by Proposition 2.6.1,

$$\nu^*(Y_a^2) = \left\lfloor \frac{3m+1}{2} \right\rfloor = \frac{3m}{2}.$$ 

For $Y_{b,c}^2$ we have

$$Q(A_{Y_{b,c}^2}, \nu_{Y_{b,c}^2}) = (x_1 - x_2)^{m+1}(x_b - x_c)^m.$$ 

Consequently, by Proposition 2.6.2, we get $\nu^*(Y_{b,c}^2) = m$. Therefore, the Euler multiplicity $\nu^*$ on $A''$ is given by

$$\left[ \frac{3m}{2}, \ldots, \frac{3m}{2}, m, \ldots, m \right] = \left[ m + \frac{m}{2}, \ldots, m + \frac{m}{2}, m, \ldots, m \right].$$ 

Due to our induction hypothesis and the fact that $A'' \cong A_{\ell-2}$, the resulting multiarrangement $(A'', \nu^*)$ is inductively free with

$$\exp(A'', \nu^*) = \left\{ \frac{m(\ell-1)}{2} + \frac{m}{2}, \ldots, \frac{m(\ell-1)}{2} + \frac{m}{2} \right\} = \{ m \ell \frac{1}{2}, \ldots, m \ell \frac{1}{2} \}$$
(note here \( q = m/2 \)). By the addition part of Theorem \( 2.5 \) and Definition \( 2.7 \), we see that \((A', \nu')\) in the next step is inductively free with

\[
\exp(A', \nu') = \left\{ \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1 \right\}.
\]

Restricting to \( \ker(x_1 - x_3) \) leads to a similar arrangement as in the previous step, but this time the multiplicity \( \nu^* \) of \( A'' \) is given by

\[
\left[ \frac{3m}{2} + 1, \frac{3m}{2}, \ldots, \frac{3m}{2}, m, \ldots, m \right].
\]

| \( \exp(A', \nu') \) | \( \alpha_H \) | \( \exp(A'', \nu^*) \) and Euler multiplicity |
|----------------------|--------------|------------------------------------------|
| \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2} \) | \( x_1 - x_2 \) | \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2} \) [\( \frac{3m}{2}, \ldots, \frac{3m}{2}, m, \ldots, m \) (\( \ell - 1 \) times) |
| \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1 \) | \( x_1 - x_3 \) | \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1 \) [\( \frac{3m}{2} + 1, \frac{3m}{2}, \ldots, \frac{3m}{2}, m, \ldots, m \) (\( \ell-2 \) times) |
| \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1, \frac{m\ell}{2} + 1 \) | \( x_1 - x_4 \) | \( \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1, \frac{m\ell}{2} + 1 \) [\( \frac{3m}{2} + 1, \frac{3m}{2} + 1, \frac{3m}{2}, \ldots, \frac{3m}{2}, m, \ldots, m \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \frac{m\ell}{2}, \frac{m\ell}{2} + 1, \ldots, \frac{m\ell}{2} + 1 \) | \( x_1 - x_\ell \) | \( \frac{m\ell}{2} + 1, \ldots, \frac{m\ell}{2} + 1 \) [\( \frac{3m}{2} + 1, \frac{3m}{2} + 1, m, \ldots, m \) |
| \( \frac{m\ell}{2} + 1, \ldots, \frac{m\ell}{2} + 1 \) | \( x_1 - x_2 \) | \( \vdots \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \frac{m\ell}{2} + q - 1, \ldots, \frac{m\ell}{2} + q - 1 \) | \( x_1 - x_2 \) | \( \frac{m\ell}{2} + q - 1, \ldots, \frac{m\ell}{2} + q - 1 \) [\( \frac{3m}{2} + q - 1, \ldots, \frac{3m}{2} + q - 1, m, \ldots, m \) |
| \( \frac{m\ell}{2} + q - 1, \ldots, \frac{m\ell}{2} + q - 1, \frac{m\ell}{2} + q \) | \( x_1 - x_3 \) | \( \frac{m\ell}{2} + q - 1, \ldots, \frac{m\ell}{2} + q - 1, \frac{m\ell}{2} + q \) [\( \frac{3m}{2} + q, \frac{3m}{2} + q - 1, \ldots, \frac{3m}{2} + q - 1, m, \ldots, m \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( \frac{m\ell}{2} + q - 1, \frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q \) | \( x_1 - x_\ell \) | \( \frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q \) [\( \frac{3m}{2} + q, q, m, \ldots, m \) |
| \( \frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q \) | \( \frac{m\ell}{2} + q, \ldots, \frac{m\ell}{2} + q \) [\( \frac{3m}{2} + q + q, q, m, \ldots, m \) |

**Table 2.** Lemma 3.1: Induction of hyperplanes for \( \ell > 3 \) and \( m \) even

Obviously, such multiplicities occur while adding hyperplanes of the type \( Y^m_\alpha \) to the underlying multiarrangement with constant multiplicity \( m \). In this case, the hyperplanes of type \( Y^m_\alpha \) have been added \( \frac{m}{2} \) times except for \( Y^3_2 = \{ x_1 = x_2 = x_3 \} \) which already has multiplicity \( \frac{m}{2} + 1 \) and we have

\[
\exp(A'', \nu^*) = \left\{ \frac{m\ell}{2}, \ldots, \frac{m\ell}{2}, \frac{m\ell}{2} + 1 \right\}.
\]
Continuing in the same way, we can easily complete our induction of hyperplanes when \( m \) is even, see Table 2.

The case where \( m \) is odd is again treated in an analogous way. By hypothesis of the lemma, the multiarrangement \((\mathcal{A}_{\ell - 1}, m)\) is inductively free with exponents

\[
\exp(\mathcal{A}_{\ell - 1}, m) = \left\{ \frac{(m-1)\ell}{2} + 1, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \right\}.
\]

We initialize our induction table with this inductively free multiarrangement. The Euler multiplicities can then be calculated again using Proposition 2.6(1) and (2), see Table 3 for details.

| \( \exp(\mathcal{A}', \nu') \) | \( \alpha_H \) | \( \exp(\mathcal{A}'', \nu^*) \) and Euler multiplicities |
|--------------------------------|-------------|---------------------------------------------------|
| \( \frac{(m-1)\ell}{2} + 1, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \) | \( x_1 - x_2 \) | \( \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \) |
| \( \frac{(m-1)\ell}{2} + 2, \frac{(m-1)\ell}{2} + 2, \frac{(m-1)\ell}{2} + \ell - 1 \) | \( x_1 - x_3 \) | \( \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell - 1, \frac{(m-1)\ell}{2} + \ell - 1 \) | \( x_1 - x_\ell \) | \( \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 \) |
| \( \frac{(m-1)\ell}{2} + 2, \ldots, \frac{(m-1)\ell}{2} + \ell \) | \( x_1 - x_2 \) | \( \ldots \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( \frac{(m-1)\ell}{2} + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) | \( x_1 - x_2 \) | \( \frac{(m-1)\ell}{2} + 1 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) |
| \( \frac{(m-1)\ell}{2} + 1 + q, \frac{(m-1)\ell}{2} + 1 + q, \frac{m-1}2 + 1 \) | \( x_1 - x_3 \) | \( \frac{(m-1)\ell}{2} + 1 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) |
| \( \frac{(m-1)\ell}{2} + 2 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) | \( \ldots \) | \( \ldots \) |
| \( \ldots \) | \( \ldots \) | \( \ldots \) |
| \( \frac{(m-1)\ell}{2} + 1 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) | \( x_1 - x_\ell \) | \( \frac{(m-1)\ell}{2} + 1 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 2 + q \) |
| \( \frac{(m-1)\ell}{2} + \ell - 2 + q, \frac{(m-1)\ell}{2} + \ell - 2 + q \) | \( \ldots \) | \( \ldots \) |
| \( \frac{(m-1)\ell}{2} + 1 + q, \ldots, \frac{(m-1)\ell}{2} + \ell - 1 + q \) | \( \ldots \) | \( \ldots \) |

Table 3. Lemma 3.1: Induction of hyperplanes for \( \ell > 3 \) and \( m \) odd

This completes the proof of the lemma.  

\[ \square \]
We prove Theorem 1.1 by induction on the rank \( \ell \). For \( \ell = 2 \), \( \mathcal{A} \) is a Coxeter arrangement of type \( A_1 \) and the multiarrangement \((\mathcal{A}, m)\) is inductively free thanks to Remark 2.9, with \( \exp(\mathcal{A}, m) = \{m\} \), thanks to [Ter02, Thm. 1.1].

Now let \( \ell = 3 \). The underlying simple arrangement \( \mathcal{A}_2 \) is inductively free due to Remark 2.9 with \( \exp(\mathcal{A}_2) = \{1, 2\} \). Thus we initialize our induction table with the simple inductively free arrangement \( \mathcal{A}_2 = (\mathcal{A}_2, 1) \). In our induction of hyperplanes, each of the three hyperplanes is added in turn until each has multiplicity \( m \). Since in every step \( \mathcal{A}' \cong \mathcal{A}_1 \) is a Coxeter arrangement of type \( A_1 \) necessarily with a constant multiplicity, we readily obtain \( \exp(\mathcal{A}', \nu^*) = \{|\nu^*|\} \), thanks to [Ter02, Thm. 1.1]. It is again very easy to determine the multiplicity \( \nu^* \) at each step, using Proposition 2.6(1); see Table 4.

| \( \exp(\mathcal{A}', \nu') \) | \( \alpha_H \) | \( \exp(\mathcal{A}', \nu^*) \) |
|-----------------|-----------------|-----------------|
| 1, 2            | \( x_1 - x_2 \) | 2               |
| 2, 2            | \( x_1 - x_3 \) | 2               |
| 2, 3            | \( x_2 - x_3 \) | 3               |
| 3, 3            | \( x_1 - x_2 \) | 3               |
| 3, 4            | \( x_1 - x_3 \) | 4               |
| 4, 4            | \( x_2 - x_3 \) | 4               |
| :              | :              | :               |

when \( m \) is even:

| \( \frac{3m}{2} - 2, \frac{3m}{2} - 1 \) | \( x_1 - x_2 \) | \( \frac{3m}{2} - 1 \) |
| \( \frac{3m}{2} - 1, \frac{3m}{2} - 1 \) | \( x_1 - x_3 \) | \( \frac{3m}{2} - 1 \) |
| \( \frac{3m}{2} - 1, \frac{3m}{2} \) | \( x_2 - x_3 \) | \( \frac{3m}{2} \) |

when \( m \) is odd:

| \( \frac{3(m-1)}{2} \) | \( \frac{3(m-1)}{2} \) | \( x_1 - x_2 \) | \( \frac{3(m-1)}{2} \) |
| \( \frac{3(m-1)}{2} \) | \( \frac{3(m-1)}{2} + 1 \) | \( x_1 - x_3 \) | \( \frac{3(m-1)}{2} + 1 \) |
| \( \frac{3(m-1)}{2} + 1, \frac{3(m-1)}{2} + 1 \) | \( x_2 - x_3 \) | \( \frac{3(m-1)}{2} + 1 \) |
| \( \frac{3(m-1)}{2} + 1 \) | \( \frac{3(m-1)}{2} + 2 \) | \( x_2 - x_3 \) | \( \frac{3(m-1)}{2} + 2 \) |

Table 4. Theorem 1.1; induction of hyperplanes for \( \ell = 3 \)

Now suppose that \( \ell > 3 \) and that the statement of the theorem holds for smaller ranks. In particular, the multiarrangement \((\mathcal{A}_{\ell-2}, \nu) = (\mathcal{A}_{\ell-2}, m)\) with constant multiplicity \( m \) is inductively free. By Theorem 2.11, the multiarrangement \((\mathcal{A}_{\ell-2}, m) \times \Phi_1\) is inductively free as well. It has exponents \( \{0, \exp(\mathcal{A}_{\ell-2}, m)\} \). In our induction of hyperplanes we now add the hyperplanes of type \( \ker(x_i - x_\ell) \) (for \( 1 \leq i < \ell \)) \( m \) times. The first \( \frac{m}{2} \), respectively \( \frac{m+1}{2} \) rounds adding those hyperplanes, the parity of \( m \) does not matter. In order to describe the
restrictions we use the following notation. We denote the members of $\mathcal{A}''$ by $Y^j_a := \{ x_j = x_\ell = x_a \}$ and $Y^j_{b,c} := \{ x_j = x_\ell, x_b = x_c \}$, where $1 \leq j < \ell$ and $a, b, c \neq j, \ell$ and $b \neq c$. Starting with $\ker(x_1 - x_\ell)$ we have

$$\mathcal{A}'' = \{ x_1 = x_\ell = x_a; x_1 = x_\ell, x_b = x_c \}$$

and of course $\mathcal{A}'' \cong \mathcal{A}_{\ell-2}$. Using Proposition 2.6(2), we get $\nu^*(Y^1_a) = \nu^*(Y^1_{b,c}) = m$ which implies that $(\mathcal{A}''', \nu'H)$ is a multiarrangement with a constant multiplicity $m$ and it is of Coxeter type $A_{\ell-2}$, hence it is inductively free due to our induction hypothesis. Continuing on, restricting to $\ker(x_i - x_\ell)$ for $2 \leq i \leq \ell - 1$, we need to make use of Proposition 2.6(2) and (3), to derive that again $(\mathcal{A}''', \nu'H)$ is a multiarrangement with a constant multiplicity $m$ and at each step $\mathcal{A}''$ it is still of Coxeter type $A_{\ell-2}$, hence it is inductively free due to our induction hypothesis. After the first round of adding hyperplanes the set of exponents of the new multiarrangement is $\{ \ell - 1, \exp(\mathcal{A}_{\ell-2}, m) \}$ and the multiplicity is

$$[m, \ldots, m, 1, \ldots, 1],$$

$(\ell-1)$ times

because the hyperplanes $\ker(x_i - x_\ell)$ now have multiplicity 1.

| $\exp(\mathcal{A}', \nu')$ | $\alpha_H$ | $\exp(\mathcal{A}'', \nu'')$ |
|---------------------------|-------------|-----------------------------|
| 0, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_1 - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| 1, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_2 - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| 2, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_3 - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| 3, $\exp(\mathcal{A}_{\ell-2}, m)$ | $\ldots$ | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $\ell - 1$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_1 - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| $\ell$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $\ldots$ | $\ldots$ |
| $\vdots$ | $\vdots$ | $\vdots$ |

when $m$ is even:

| $\frac{m}{2}(\ell - 1) - 2$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_{\ell-2} - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| $\frac{m}{2}(\ell - 1) - 1$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_{\ell-1} - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| $\frac{m}{2}(\ell - 1)$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $\ldots$ | $\ldots$ |

when $m$ is odd:

| $\frac{m+1}{2}(\ell - 1) - 2$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_{\ell-2} - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| $\frac{m+1}{2}(\ell - 1) - 1$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $x_{\ell-1} - x_\ell$ | $\exp(\mathcal{A}_{\ell-2}, m)$ |
| $\frac{m+1}{2}(\ell - 1)$, $\exp(\mathcal{A}_{\ell-2}, m)$ | $\ldots$ | $\ldots$ |

Table 5. Theorem 1.1; the first $\frac{m}{2}$ resp. $\frac{m+1}{2}$ rounds of the induction of hyperplanes when $m$ is even resp. when $m$ is odd and $\ell > 3$
In the subsequent rounds we always get the same restrictions and the Euler multiplicities are calculated using Proposition 2.6(3). Let $r$ be the number of rounds of adding the hyperplanes $\ker(x_i - x_\ell)$ for $1 \leq i \leq \ell - 1$. We consider the next $m \over 2$ resp. $m+1 \over 2$ rounds of the induction of hyperplanes when $m$ is even resp. when $m$ is odd. As long as $1 \leq r \leq m \over 2$ in the case where $m$ is even and $1 \leq r \leq m+1 \over 2$ when $m$ is odd, Proposition 2.6(3) applies and gives the Euler multiplicities, as shown in Table 5.

Table 6 shows the final $m \over 2$ rounds in the case where $m$ is even. The initial inductively free arrangement $(\mathcal{A}', \nu')$ here is the final arrangement from Table 5, where $m \over 2$ rounds of adding the hyperplanes $\ker(x_i - x_\ell)$ have already been performed. Its defining polynomial is

$$Q(\mathcal{A}', \nu') := \prod_{1 \leq i < j \leq \ell-1} (x_i - x_j)^m \prod_{1 \leq j \leq \ell-1} (x_j - x_\ell)^{m/2}$$

with set of exponents

$$\exp(\mathcal{A}', \nu') = \left\{ {m \over 2}(\ell - 1), \exp(\mathcal{A}_{\ell-2}, m) \right\}.$$ 

Now we add $\ker(x_1 - x_\ell)$ again and get

$$Q(\mathcal{A}_{Y_1^2}, \nu_{Y_1^2}) = (x_1 - x_\ell)^{m+1}(x_1 - x_a)^m(x_a - x_\ell)^{m \over 2}$$

and hence $\nu^*(Y_1^1) = m$, by Proposition 2.6(1). Moreover, since

$$Q(\mathcal{A}_{Y_2^1}, \nu_{Y_2^1}) = (x_1 - x_\ell)^{m+1}(x_b - x_c)^m$$

and so $\nu^*(Y_2^1) = m$, by Proposition 2.6(2). So once again we have $(\mathcal{A}''', \nu^*) = (\mathcal{A}_{\ell-2}, m)$ with constant multiplicity $m$ again. This is not any different from all the steps before. However, this round’s second step, restricting to $\ker(x_2 - x_\ell)$, leads to a different multiplicity. For, here we have

$$Q(\mathcal{A}_{Y_2^1}, \nu_{Y_2^1}) = (x_1 - x_\ell)^m(x_1 - x_\ell)^{m \over 2+1}(x_2 - x_\ell)^{m \over 2+1}.$$ 

It follows from Proposition 2.6(1) that $\nu^*(Y_2^1) = m + 1$. One checks that the multiplicity in this case is given by $[m+1, m, \ldots, m]$. It follows from the proof of Lemma 3.1 that this restriction $(\mathcal{A}''', \nu^*)$ is also inductively free, because its multiplicity occurs in the induction of hyperplanes in the proof of the lemma, see Table 2. Consequently, it has exponents

$$\left\{ m \over 2(\ell-1), \ldots, m \over 2(\ell-1), m \over 2(\ell-1) + 1 \right\}.$$ 

Therefore, at the end of this round the restriction has multiplicity

$$[m+1, \ldots, m+1, m, \ldots, m].$$

Therefore, using our induction hypothesis that Theorem 1.1 holds for lower ranks and by Lemma 3.1 applied in case $\ell - 1$, we have

$$\exp(\mathcal{A}', \nu') = \left\{ {m(\ell-1) \over 2} + 1, \ldots, {m(\ell-1) \over 2} + 1 \right\}.$$ 

In the next round, the restriction’s multiplicity builds up from $[m+1, \ldots, m+1, m, \ldots, m]$ over $[m+2, m+1, \ldots, m+1, m, \ldots, m]$ to $[m+2, \ldots, m+2, m, \ldots, m]$ and we can argue in the same way as in the preceding round, using again the induction hypothesis on $\ell$, Lemma 3.1
for the restriction and the addition part of Theorem 2.5. The same applies for the remaining rounds as the multiplicity of the restriction increases to

\[ [3m/2, \ldots, 3m/2, m, \ldots, m] = [m + m/2, \ldots, m + m/2, m, \ldots, m] \]

when we restrict to \( \ker(x_{\ell-1} - x_\ell) \). As before we use induction on \( \ell \), Lemma 3.1 for the restriction and then the addition part of Theorem 2.5 again and obtain the expected exponents \( \{m^{\ell}/2, \ldots, m^\ell/2\} \), see Table 6.

| \( \exp(A', \nu') \) | \( \alpha_H \) | \( \exp(A'', \nu^*) \) |
|----------------|----------------|----------------|
| \( m/2(\ell-1), \exp(A_{\ell-2}, m) \) | \( x_1 - x_\ell \) | \( \exp(A_{\ell-2}, m) \) |
| \( m(\ell-1)/2, \ldots, m(\ell-1)/2, m(\ell-1)/2 + 1 \) | \( x_2 - x_\ell \) | \( m(\ell-1)/2, \ldots, m(\ell-1)/2 + 1 \) |
| \( (\ell-1) \) elements | | \( (\ell-2) \) elements |
| \( m(\ell-1)/2, \ldots, m(\ell-1)/2 + 1, m(\ell-1)/2 + 1 \) | | |
| \( : \) | | \( : \) |
| \( m(\ell-1)/2 + 1, \ldots, m(\ell-1)/2 + 1 \) | \( x_1 - x_\ell \) | \( m(\ell-1)/2 + 1, \ldots, m(\ell-1)/2 + 1 \) |
| \( m(\ell-1)/2 + 1, \ldots, m(\ell-1)/2 + 1, m(\ell-1)/2 + 2 \) | | \( : \) |
| \( : \) | | \( : \) |
| \( m^\ell/2 - 1, m^\ell/2, \ldots, m^\ell/2 \) | \( x_{\ell-1} - x_\ell \) | \( m^\ell/2, \ldots, m^\ell/2 \) |
| \( m^\ell/2, \ldots, m^\ell/2 \) | | |

Table 6. Theorem 1.1; induction of hyperplanes for \( \ell > 3 \) and \( m \) even

Table 7 shows the remaining \( m-1/2 \) rounds in the case where \( m \) is odd. Comparing to the case where \( m \) is even we see that the restriction’s multiplicity does not change in the course of one round of adding hyperplanes. In the \( m+3/2 \)-th round we have

\[ [m + 1, \ldots, m + 1, m, \ldots, m] \]

and in the last round we have

\[ [3m-1/2, \ldots, 3m-1/2, m, \ldots, m] = [m + m-1/2, \ldots, m + m-1/2, m, \ldots, m] \]

where these multiplicities can be calculated again using Proposition 2.6(1). Consequently, the restriction’s exponents do not change during any of these rounds either. So there is only one element in the set \( \exp(A', \nu') \) that increases by 1 in every step. As before, arguing by induction on \( \ell \), employing Lemma 3.1 and the addition part of Theorem 2.5 in each round, we obtain the expected exponents

\[ \left\{ \left( (m-1)\ell/2 \right) + 1, \ldots, \left( (m-1)\ell/2 \right) + \ell - 1 \right\} \]

This completes the proof of Theorem 1.1.

Corollary 1.2 follows from Theorem 1.1 and Lemma 3.1.
| \(\exp(A', \nu')\) | \(\alpha_H\) | \(\exp(A'', \nu'')\) |
|---|---|---|
| \(m-1/2(\ell-1), \exp(A_{\ell-2}, m)\) | \(x_1 - x_\ell\) | \(\exp(A_{\ell-2}, m)\) |
| \(m-1/2(\ell-1) + 1, \exp(A_{\ell-2}, m)\) | \(\ldots\) | \(\ldots\) |
| \(\exp(A_{\ell-2}, m), m-1/2(\ell-1) + \ell - 1\) | \(x_1 - x_\ell\) | \(\frac{(m-1)(\ell-1)}{2} + 2, \frac{(m-1)(\ell-1)}{2} + 3, \ldots, \frac{(m-1)(\ell-1)}{2} + \ell - 1\) |
| \(\frac{(m-1)(\ell-1)}{2} + 2, \frac{(m-1)(\ell-1)}{2} + 2, \frac{(m-1)(\ell-1)}{2} + 3, \ldots, \frac{(m-1)(\ell-1)}{2} + \ell - 1\) | \(x_2 - x_\ell\) | \(\frac{(m-1)(\ell-1)}{2} + 2, \frac{(m-1)(\ell-1)}{2} + 3, \ldots, \frac{(m-1)(\ell-1)}{2} + \ell - 1\) |
| \(\frac{(m-1)(\ell-1)}{2} + 3, \frac{(m-1)(\ell-1)}{2} + 3, \ldots, \frac{(m-1)(\ell-1)}{2} + \ell - 1\) | \(\ldots\) | \(\ldots\) |
| \(\frac{(m-1)(\ell-1)}{2} + 4, \frac{(m-1)(\ell-1)}{2} + 4, \ldots, \frac{(m-1)(\ell-1)}{2} + \ell\) | \(\ldots\) | \(\ldots\) |
| \(\ldots\) | \(\ldots\) | \(\ldots\) |
| \(\frac{(m-1)(\ell-2)}{2} + 1, \frac{(m-1)(\ell-2)}{2} + \ell - 2\) | \(x_1 - x_\ell\) | \(\frac{(m-1)(\ell-2)}{2} + 1, \ldots, \frac{(m-1)(\ell-2)}{2} + \ell - 2\) |
| \(\frac{(m-1)(\ell-2)}{2} + 1, \frac{(m-1)(\ell-2)}{2} + \ell - 2\) | \(x_\ell - x_{\ell-1}\) | \(\frac{(m-1)(\ell-2)}{2} + 1, \ldots, \frac{(m-1)(\ell-2)}{2} + \ell - 2\) |

| Table 7. Theorem 1.1: induction of hyperplanes for \(\ell > 3\) and \(m\) odd |

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