The radiation instability in modified gravity

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Abstract

We study the problem of the instability of inhomogeneous radiation universes in quadratic lagrangian theories of gravity written as a system of evolution equations with constraints. We construct formal series expansions and show that the resulting solutions have a smaller number of arbitrary functions than that required in a general solution. These results continue to hold for more general polynomial extensions of general relativity.
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1 Introduction

The Friedman cosmologies are a central theme in modern physical cosmology because they provide an excellent comparison with cosmological observations. The initial singularity in these models is, however, a feature not only generic in general relativity, but also one that persists when these models are analysed in geometric extensions of general relativity such as modified gravity. An important feature of these generalized models that was known very early is that if some special FRW solution avoided the initial singularity in the past, it failed to approach the Friedman solution of general relativity and experienced a curvature singularity in the future (cf. [1] and earlier references therein).

A basic such model is the flat, isotropic, radiation universe, which in the context of extensions of general relativity possesses interesting further properties such as the non-existence of horizons near the singularity [2], [3], and, for certain non-analytic choices of lagrangian, the avoidance of the initial singularity. The existence and (in)stability properties of the flat, radiation solution in analytic $f(R)$ gravity, on the other hand, is a well-studied problem (cf. [1], [4], [5], and references therein). It is known that the singular behaviour of this solution towards the past singularity persists and is stable in the sense that all non-flat, radiation solutions of the same theory asymptotically approach the flat one at early times [6]. The radiation stage is also very important in recent studies of dark energy, in particular, of the radiation-matter transition in modified gravity [7], and a very prolonged radiation era may result in incompatibilities of some modified gravity models with current CMB observations [8].

However, the past-instability conjecture states that all homogeneous and isotropic solutions of general relativity as in general past-unstable when viewed as solutions of higher order gravity [2]. Such an instability is a prerequisite in any attempt to examine the cosmological viability of geometric extensions and generalizations of general relativity through arguments relying on a transition between different isotropic solutions. These transitions are needed in dark energy models based on relativistic extensions. In this paper, we prove this conjecture for the case of radiation solutions of general relativity
in the framework of the quadratic lagrangian theory $R + \epsilon R^2$. The method of proof consists in constructing formal series expansions and counting the free functions in the resulting solutions. Then we find that the final solution so constructed cannot have the full number of such functions to qualify as a general solution of the evolution and constraint equations of the problem.

The plan of this paper is as follows. In the next Section, we write down the basic system of evolution and constraint equations for the quadratic $f(R)$ theory as a dynamical system for the two tensors $\gamma, K$, and high-order ones $D, W$ od the space slices, and give a count of the total number of arbitrary data for the radiation solution to be regarded as a general solution of the theory. In Section 3, we give a new proof of a well known result about the existence of radiation isotropic solutions in quadratic gravity, and provide formal series expansions for the metric and the extrinsic and Ricci curvatures. In Section 4, we calculate the asymptotic structure of the various terms in the basic system of evolution and constraint equations, and in Section 5 we count the free data present in the general perturbation of the radiation solution. Alongside, we prove several technical results that are important for the function counting arguments used in this work, but more importantly because they show that our conclusions have a wider significance. A discussion of these implications appears in the last Section.

2 Splittings

In this Section, we write down the basic dynamical system which consists of evolution and constraint equations for the quadratic lagrangian theory $f(R) = R + \epsilon R^2$ written in a splitted form in suitable coordinates as in a Cauchy problem formulation of the theory. We then count the free functions necessary for the radiation solution to be a general solution of the theory, and this number turns out to be 20 such functions.
2.1 Metric

We consider a spacetime \((V, g_{ij})\) where \(V = \mathbb{R} \times M\), with \(M\) being an orientable 3-manifold, and \(g_{ij}\) is a Lorentzian metric, analytic and with signature \((+,-,-,-)\). We also consider a diffeomorphism \(\phi : V \rightarrow M \times I, I \subseteq \mathbb{R}\), such that, the submanifolds \(\phi^{-1}(\{t\} \times M) = M_t, t \in I\) are spacelike and the curves \(\phi^{-1}(\{x\} \times I), x \in M\) are timelike. Such a frame is called a Cauchy adapted frame [9]. A vector field \(\partial/\partial t\) on \(V\) is defined by these curves and it can be decomposed into normal and parallel components relative to the slicing as follows:

\[
\partial_t = Nn + \underline{N}.
\]  

(2.1)

Here \(N\) is a positive function on \(V\), \(n\) is a future-directed, unit, normal field, and \(\underline{N}\) is a vector field tangent to the slices \(M_t\). As usual we call \(N\) the lapse function and \(\underline{N}\) the shift vector field. Then we set,

\[
\underline{N} = N^\alpha \partial_\alpha,
\]

(2.2)

and due to the fact that \(\partial_t\) is a timelike vector field on \(V\) we obtain,

\[
g_{00} = g(\partial_t, \partial_t) = N^2 + N^\alpha N_\alpha > 0,
\]

\[
g_{0\alpha} = g(\partial_t, \partial_\alpha) = N_\alpha, \quad g^{0\alpha} = N^\alpha/N^2,
\]

(2.3)  (2.4)

so that,

\[
ds^2 = N^2 dt^2 - (\gamma_{\alpha\beta} - \frac{N_\alpha N_\beta}{N^2 + N_\gamma N_\gamma})(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt).
\]

(2.5)

Using the choice (geodesic slicing) \(N = 1, N^\alpha = 0\), that is \(g_{00} = 1, g_{0\alpha} = 0\), we arrive at the synchronous system of local coordinates (the spacetime \(V\) is a sliced spacetime in this system) where the 4-metric \(g_{ij}\) is

\[
ds^2 = dt^2 - \gamma_{\alpha\beta} dx^\alpha dx^\beta,
\]

(2.6)

with the time \(t\) measuring proper time.
2.2 Radiation fluid

We assume a radiation content for the model treated below. In particular, we define a cosmological perfect fluid is defined to be a continuous distribution of matter with energy-momentum tensor of the form,

\[ T^i_j = (\rho + p)u^i u_j - p\delta^i_j, \]  

where \( \rho \) is the energy density and \( p \) is the pressure satisfying the equation of state,

\[ p = w\rho, \]  

where \( w \in \mathbb{R} \) is the equation of state parameter. In this paper, we take \( w = 1/3 \), although many of the results hold obviously true for \( w \) in an interval. We are not going to repeat this statement every time is true. We assume that the velocity vector,

\[ u^i = \left( \frac{dt}{ds}, \frac{dx^\alpha}{ds} \right) = (u^0, u^\alpha), \]  

associated with any timelike curve parametrized by the proper time \( s \) has unit length,

\[ 1 = u_i u^i = u_0 u^0 + u_\alpha u^\alpha. \]  

The 4-velocity \( u^i \) is used for a covariant description of continuous matter distributions given by the energy-momentum tensor \( T_{ij} \).

2.3 Evolution and constraint equations

We start from the field equations of the higher-order gravity theory derived from an analytic lagrangian \( f(R) \),

\[ L_{ij} = f'(R)R_{ij} - \frac{1}{2}f(R)g_{ij} - \nabla_i \nabla_j f'(R) + g_{ij} \Box g f'(R) = 8\pi G T_{ij}, \]  

where \( G \) is Newton’s gravitational constant, and \( T_{ij} \) is the energy-momentum tensor satisfying the conservations laws of energy and momentum, given by the equations of motion,

\[ \nabla_i T^i_j = 0. \]
In this paper, we specialize in the quadratic theory \( f(R) = R + \epsilon R^2 \), where the field equations (2.11) in a Cauchy adapted frame split as follows:

\[
L_{00} = (1 + 2\epsilon R) R_{00} - \frac{1}{2} (1 + \epsilon R) R + 2\epsilon g^{\alpha\beta} \nabla_\alpha \nabla_\beta R = 8\pi G T_{00},
\]

(2.13)

\[
L_{0\alpha} = (1 + 2\epsilon R) R_{0\alpha} - 2\epsilon \nabla_\alpha R = 8\pi G T_{0\alpha},
\]

(2.14)

\[
L_{\alpha\beta} = (1 + 2\epsilon R) R_{\alpha\beta} - \frac{1}{2} (1 + \epsilon R) R g_{\alpha\beta} - 2\epsilon \nabla_\alpha \nabla_\beta R + 2\epsilon g_{\alpha\beta} \Box g R = 8\pi G T_{\alpha\beta}.
\]

(2.15)

Here, using standard methods for splitting the Riemann tensor and the connection \([9]\), the various components of the Ricci 4-curvature are given by the formulae,

\[
R_{00} = -\frac{1}{2} \partial_t K - \frac{1}{4} K^\beta_\alpha K^\alpha_\beta,
\]

(2.16)

\[
R_{0\alpha} = \frac{1}{2} (\nabla_\beta K^\beta_\alpha - \nabla_\alpha K),
\]

(2.17)

and also

\[
R_{\alpha\beta} = P_{\alpha\beta} + \frac{1}{2} \partial_t K_{\alpha\beta} + \frac{1}{4} K K_{\alpha\beta} - \frac{1}{2} K^\gamma_\alpha K^\gamma_\beta.
\]

(2.18)

Here, \( P_{\alpha\beta} \) denotes the three-dimensional Ricci tensor associated with \( \gamma_{\alpha\beta} \), and \( K = \text{tr} \gamma K_{\alpha\beta} = \gamma^{\alpha\beta} K_{\alpha\beta} \) is the mean curvature of the slices \( \mathcal{M}_t \), with \( K_{\alpha\beta} \) being the extrinsic curvature (or, second fundamental form) of the spatial slices \( \mathcal{M}_t \), defined by the first variational equation

\[
\partial_t \gamma_{\alpha\beta} = K_{\alpha\beta}.
\]

(2.19)

Because of the higher-than-second-order derivatives present in the field equations, we need to introduce here further important symbols, namely, the symmetric acceleration tensor \( D_{\alpha\beta} \) through the corresponding second variational equation

\[
\partial_t K_{\alpha\beta} = D_{\alpha\beta},
\]

(2.20)

as well as the jerk tensor \( W_{\alpha\beta} \) satisfying the equation,

\[
\partial_t D_{\alpha\beta} = W_{\alpha\beta}.
\]

(2.21)
Then the final evolution equation of the quadratic theory, called here the *snap* equation, becomes,

\[
\partial_t W = \frac{1}{6\epsilon}(8\pi GT^\alpha_\alpha + \frac{1}{2} P + \frac{1}{8} K^2 - \frac{5}{8} K^{\alpha\beta} K_{\alpha\beta} + D) + \\
\frac{1}{6}\left[\frac{1}{2} P^2 + \frac{1}{4} PK^2 - \frac{1}{4} PK^{\alpha\beta} K_{\alpha\beta} + \frac{1}{32} K^4 - \frac{1}{16} K^2 K^{\alpha\beta} K_{\alpha\beta} - \\
6K^{\alpha\beta} K_{\gamma} K_{\gamma} - \frac{99}{32} (K^{\alpha\beta} K_{\alpha\beta})^2 + 27K^{\alpha\beta} K_{\gamma} K_{\delta} K_{\delta} + 9K K^{\alpha\beta} D_{\alpha\beta} - \\
57K^{\alpha\beta} K_{\gamma} D_{\alpha} + \frac{13}{2} DK^{\alpha\beta} K_{\alpha\beta} - \frac{7}{2} D^2 + 15D^{\alpha\beta} D_{\alpha\beta} - 3KW + \\
15K^{\alpha\beta} W_{\alpha\beta} - 6\partial_t (\partial_t P) - \\
4\gamma^{\alpha\beta} \nabla_\alpha \nabla_\beta (-P - D + \frac{3}{4} K^{\alpha\beta} K_{\alpha\beta} - \frac{1}{4} K^2)\right].
\tag{2.22}
\]

In addition, we find that the theory contains *constraint equations*, which read as follows,

**Hamiltonian Constraint**

\[
C_0 : \quad \frac{1}{2} P + \frac{1}{8} K^2 - \frac{1}{8} K^{\alpha\beta} K_{\alpha\beta} + \\
\epsilon\left[-\frac{1}{2} P^2 + \frac{1}{4} PK^2 + \frac{1}{4} PK^{\alpha\beta} K_{\alpha\beta} - \frac{1}{32} K^4 + \frac{1}{16} K^2 K^{\alpha\beta} K_{\alpha\beta} + \\
\frac{3}{32} (K^{\alpha\beta} K_{\alpha\beta})^2 - \frac{1}{2} DK^{\alpha\beta} K_{\alpha\beta} + \frac{1}{2} D^2 - \\
2\gamma^{\alpha\beta} \nabla_\alpha \nabla_\beta (-P - \frac{1}{4} K^2 + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - D)\right] = 8\pi GT_{00},
\tag{2.23}
\]

**Momentum Constraint**

\[
C_\alpha : \quad \frac{1}{2}(\nabla_\beta K^\beta_\alpha - \nabla_\alpha K) + \\
\epsilon\left[(-P + \frac{1}{4} K^2 + \frac{3}{4} K^{\gamma\delta} K_{\gamma\delta} - D)(\nabla_\beta K^\beta_\alpha - \nabla_\alpha K) - \\
\nabla_\alpha (-2\partial_t P + KK^{\gamma\delta} K_{\gamma\delta} - 3K^{\gamma\delta} K_{\gamma} K_{\delta} - KD + 5K^{\gamma\delta} D_{\gamma\delta} - 2W)\right] \\
= 8\pi GT_{0\alpha}.
\tag{2.24}
\]

We note that in the two constraints above the brackets multiplied by \(\epsilon\) contain the extra higher-order terms not present in the Einstein equations.
2.4 Counting

The four evolution equations (2.19), (2.20), (2.21) and (2.22) together with the constraints (2.23) and (2.24), the equation of state (2.8), and the identity (2.10), constitute the basic system studied in this paper. They describe the time development \( (V, g_{ij}) \) of any initial data set \( (\mathcal{M}_t, \gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W_{\alpha\beta}) \) together with the quantities \( p, \rho, u^i \) in the present theory. The vacuum parts of the equations are identical to those found in [10], but we give them here for easy reference. This system satisfies the Cauchy-Kovalewski property, but since the proof is identical to that in Ref. [10], we refer the reader to that reference.

Based on the previous relations, we end up with a dynamical system consisting of 30 arbitrary functions, the 24 initial data \( (\gamma_{\alpha\beta}, K_{\alpha\beta}, D_{\alpha\beta}, W_{\alpha\beta}) \) together with the 6 functions \( p, \rho, u^i \) satisfying evolution equations (2.19), (2.20), (2.21) and (2.22) as well as the four constraint equations (2.23) and (2.24) on each slice \( \mathcal{M}_t \). The quantities \( p, \rho \) satisfy the equation of state (2.8), and the velocities the identity (2.10).

Hence, the number of arbitrary functions that have to be specified initially is equal to \( 30 - 4 - 1 - 1 - 4 = 20 \), that is from the initial 30 functions we have to subtract in turn 4 from the constraint equations, 1 from the equation of state, 1 from the identity for the velocities, and the 4 diffeomorphisms. This number 20 is the correct number for any kind of perfect fluid matter including radiation \( (w = 1/3) \), also consistent with that calculated in [11] (corresponding to the notation \( F = 1, S = 0 \) of that paper).

3 Curvatures

In this Section, after giving a new proof of the existence of exact radiation solutions in the quadratic theory, we calculate the necessary formal expansions for the mean and extrinsic curvatures, acceleration and jerk tensors, the spatial Ricci curvature, and so for the Ricci 4-curvature components and the scalar curvature. These expansions will be used in the next Section when we expand the constraint and evolution equations.
3.1 The exact radiation solution

For the FRW metric,\n
\[ ds^2 = dt^2 - a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (3.1) \]

the off-diagonal components of the Ricci tensor are equal to zero, and the diagonal components are given by,\n
\[ R_{00} = -\frac{3\ddot{a}}{a}, \quad (3.2) \]
\[ R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2}, \quad (3.3) \]
\[ R_{22} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2}, \quad (3.4) \]
\[ R_{33} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{r^2\sin^2\theta}, \quad (3.5) \]

while for the scalar curvature we obtain,\n
\[ R = -6 \left( \frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right), \quad (3.6) \]

where dot denotes differentiation with respect to \( t \). Working with the lagrangian \( f(R) = R + \epsilon R^2 \) and the field equations (2.11), in addition to the Eqns. (3.3)-(3.6), we find that,\n
\[ \partial_t R = -6 \left( \frac{\dddot{a}}{a} + \frac{\dot{a}^2}{a^2} - 2\frac{\dot{a}^3}{a^3} - 2k\frac{\dot{a}}{a^2} \right), \quad (3.7) \]
\[ \partial_t^2 R = -6 \left( \frac{a^{(4)}}{a} + \frac{\dot{a}^2}{a^2} - 8\frac{a^2\ddot{a}}{a^3} + 6\frac{\dot{a}^4}{a^4} - 2k\frac{\dot{a}}{a^3} + 6k\frac{\dot{a}^2}{a^4} \right), \quad (3.8) \]

as well as,\n
\[ \nabla_1 \nabla_1 R = \frac{6}{1 - kr^2} \left( \dddot{a} + \frac{\ddot{a}^2\ddot{a}}{a} - 2\frac{\dot{a}^4}{a^2} - 2k\frac{\dot{a}^2}{a^2} \right), \quad (3.9) \]
\[ \nabla_2 \nabla_2 R = 6r^2 \left( \dddot{a} + \frac{\ddot{a}^2\ddot{a}}{a} - 2\frac{\dot{a}^4}{a^2} - 2k\frac{\dot{a}^2}{a^2} \right), \quad (3.10) \]
\[ \nabla_3 \nabla_3 R = 6r^2 \sin \theta \left( \dddot{a} + \frac{\ddot{a}^2\ddot{a}}{a} - 2\frac{\dot{a}^4}{a^2} - 2k\frac{\dot{a}^2}{a^2} \right), \quad (3.11) \]
\[ \nabla_\alpha \nabla_\beta R = 0, \; \text{if} \; \alpha \neq \beta. \quad (3.12) \]
Then, the 00-component and $\alpha\alpha$-components of the generalized Friedmann equations for the energy density $\rho$ and the pressure $p$, are given by \[6\],

$$\frac{8\pi G\rho}{3} = \frac{k + \dot{a}^2}{a^2} + 6\epsilon \left( 2\frac{\ddot{a}\dot{a}}{a^2} + 2\frac{\dot{a}^2\ddot{a}}{a^3} - \frac{\ddot{a}^2}{a^2} - 3\frac{\dot{a}^4}{a^4} - 2k\frac{\dot{a}^2}{a^4} + \frac{k^2}{a^4} \right), \quad (3.13)$$

and,

$$8\pi Gp = -2\frac{\dddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} + 6\epsilon \left( -2\frac{a^{(4)}}{a} - 4\frac{\ddot{a}^2}{a^2} - 3\frac{\dot{a}^2\dddot{a}}{a^3} - 3\frac{\dot{a}^4}{a^4} + 4k\frac{\dddot{a}}{a^3} - 2k\frac{\dot{a}^2}{a^4} + \frac{k^2}{a^4} \right), \quad (3.14)$$

where the general relativity case is recovered when $\epsilon = 0$. The parts not multiplied by the coefficient $6\epsilon$ in the right-hand-sides are identical to the standard general relativistic expressions $\frac{8\pi G\rho_{GR}}{3}$ and $8\pi Gp_{GR}$ respectively. The extra parts multiplied by $6\epsilon$ become zero if we substitute the standard forms for radiation, namely, for $k = -1, 0, +1$ are respectively (cf. \[12\], chap. 12):

$$a = \sqrt{c\left[1 + \left(\frac{t}{\sqrt{c}}\right)^2 - 1\right]}^{1/2} = (2\sqrt{ct} + t^2)^{1/2}, \quad (3.15)$$

$$a = (4c)^{1/4}t^{1/2}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
3.2 Radiation metric expansions

Since in the quadratic theory the asymptotic form of the scale factor for the exact radiation solution of the previous Subsection is given by,
\[ a \propto t^{1/2}, \]
we assume a formal series representation of the spatial metric for a generic solution in this case of the form,
\[ \gamma_{\alpha\beta} = \gamma^{(1)}_{\alpha\beta} + \gamma^{(2)}_{\alpha\beta} t^2 + \gamma^{(3)}_{\alpha\beta} t^3 + \gamma^{(4)}_{\alpha\beta} t^4 + \cdots \]
\[ = a_{\alpha\beta} t + b_{\alpha\beta} t^2 + c_{\alpha\beta} t^3 + d_{\alpha\beta} t^4 + \cdots, \tag{3.21} \]
where the \( \gamma^{(1)}_{\alpha\beta}, \gamma^{(2)}_{\alpha\beta}, \gamma^{(3)}_{\alpha\beta}, \gamma^{(4)}_{\alpha\beta}, \cdots \) are arbitrary, nontrivial analytic functions of the space coordinates.

It is not difficult to count the number of arbitrary functions present in this expansion. In Section 5.2, we shall show that keeping all terms of order higher than four leads to qualitatively the same results, and therefore we can keep only terms of order up to four in this series and remove the dots at the end from now on. Hence, before substitution to the evolution and constraint higher order equations, the basic metric expansion (3.21) contains 24 degrees of freedom (6 arbitrary functions in each spatial matrix \( \gamma^{(n)}_{\alpha\beta}, n = 1, \cdots, 4 \)).

To calculate the formal expansion of the inverse metric \( \gamma^{\alpha\beta} = \sum_{n=-1}^{\infty} (\gamma^{\alpha\beta})^{(n)} t^n \), we use the identity \( \gamma_{\alpha\beta} \gamma^{\beta\gamma} = \delta^\gamma_\alpha \). Then for the components \( \gamma^{(\mu)}_{\alpha\beta}, \mu = 1, \cdots, 4 \), we obtain,
\[ \gamma^{\alpha\beta} = \frac{1}{t} a^{\alpha\beta} - b^{\alpha\beta} + t(-c^{\alpha\beta} + b^{\alpha\gamma} b^\gamma_\beta) + t^2(-d^{\alpha\beta} + b^{\alpha\gamma} c^\gamma_\beta - b^{\alpha\gamma} b^\gamma_\delta b^\delta_\beta + c^{\alpha\gamma} b^\gamma_\beta). \tag{3.22} \]
Here \( a_{\alpha\beta} a^{\beta\gamma} = \delta^\gamma_\alpha \) and the indices in \( b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta} \) are raised by \( a^{\alpha\beta} \).

Using the metric expansion (3.21), we may calculate the coefficients of any tensor \( X \) in the expansion
\[ X_{\alpha\beta} = \sum X^{(n)}_{\alpha\beta} t^n, \tag{3.23} \]
where the values of \( n \) depend on the tensor. In the remaining of this Section, we do this for the various curvatures.
3.3 Extrinsic curvature, acceleration and jerk tensors

For the extrinsic curvature, in terms of the data $a_{\alpha \beta}, b_{\alpha \beta}, c_{\alpha \beta}, d_{\alpha \beta}$, we have explicitly,

$$K_{\alpha \beta} = \partial_t \gamma_{\alpha \beta} = a_{\alpha \beta} + 2tb_{\alpha \beta} + 3t^2c_{\alpha \beta} + 4t^3d_{\alpha \beta},$$

(3.24)

and so for the mixed components we find,

$$K^\alpha_\beta = \frac{1}{t}\delta^\alpha_\beta + b^\beta_\alpha + t(2c^\beta_\alpha - b^\beta_\gamma b^\gamma_\alpha) + t^2(3d^\beta_\alpha - 2b^\beta_\gamma c^\gamma_\alpha - c^\beta_\gamma b^\gamma_\alpha + b^\beta_\gamma b^\gamma_\alpha b^\gamma_\delta).$$

(3.25)

Additionally, since the spatial metric determinant satisfies $\gamma > 0$ [13], for the mean curvature,

$$K = K^\alpha_\alpha = \gamma^{\alpha \beta} \partial_t \gamma_{\alpha \beta} = \partial_t \ln(\gamma),$$

(3.26)

we have the expansion,

$$K = \frac{3}{t} + b + t(2c - b^\beta_\alpha b^\alpha_\beta) + t^2(3d - 3b^\beta_\gamma c^\gamma_\alpha + b^\beta_\gamma b^\gamma_\alpha b^\alpha_\beta).$$

(3.27)

The corresponding expressions for the coefficients $(K^{\alpha \beta})^{(n)}$ of the fully contravariant symbols are,

$$K^{\alpha \beta} = \frac{1}{t^2}a^{\alpha \beta} + (c^{\alpha \beta} - b^{\alpha \gamma} b^\gamma_\beta) + t(2d^{\alpha \beta} - 2b^\alpha_\gamma c^{\gamma \beta} - 2b^\alpha_\gamma b^\gamma_\beta + 2b^\alpha_\gamma b^\gamma_\beta b^\gamma_\delta).$$

(3.28)

We can now find the various components of the acceleration and jerk tensors to the required order. For the acceleration we have,

$$D_{\alpha \beta} = \partial_t K_{\alpha \beta} = 2b_{\alpha \beta} + 6c_{\alpha \beta}t + 12d_{\alpha \beta}t^2,$$

(3.29)

and for the mixed components we find,

$$D^{\beta}_\alpha = \frac{2}{t}b^\beta_\alpha + 2(3c^\beta_\alpha - b^\beta_\gamma b^\gamma_\alpha) + 2t(6d^\beta_\alpha - 3b^\beta_\gamma c^\gamma_\alpha - c^\beta_\gamma b^\gamma_\alpha + b^\beta_\gamma b^\gamma_\alpha b^\gamma_\delta),$$

(3.30)

so that,

$$D = \frac{2}{t}b + 2(3c - b^\beta_\alpha b^\alpha_\beta) + 2t(6d - 4b^\beta_\alpha c^\alpha_\beta + b^\beta_\alpha b^\alpha_\beta b^\alpha_\beta),$$

(3.31)

where the trace is given by,

$$D = D^\alpha_\alpha = \gamma^{\alpha \beta} \partial_t K_{\alpha \beta}. $$

(3.32)
The fully contravariant components are given by,
\[ D^{\alpha\beta} = \frac{2}{t^2} b^{\alpha\beta} + \frac{2}{t} (3c^{\alpha\beta} - 2b^{\beta}_\gamma b^{\gamma\alpha}) + 2(6d^{\alpha\beta} - 4b^{\beta}_\gamma c^{\gamma\alpha} - 4b^{\alpha}_\gamma c^{\gamma\beta} + 3b^{\gamma\alpha} b^{\delta}_\gamma b^{\delta\beta}). \] (3.33)
For the jerk, we have
\[ W_{\alpha\beta} = \partial_t D_{\alpha\beta} = 6c_{\alpha\beta} + 24d_{\alpha\beta}t, \] (3.34)
so that,
\[ W = 6t c + 6(4d - b^\beta_\gamma c^\alpha_\beta), \] (3.35)
with
\[ W = W^\alpha = \gamma^{\alpha\beta} \partial_t D_{\alpha\beta}. \] (3.36)
This completes the extrinsic curvature calculations.

3.4 Ricci and scalar curvatures

For the three-dimensional Ricci tensor \( P_{\alpha\beta} \) and its trace \( P = \text{tr}_\gamma P_{\alpha\beta} \) we may also apply the above method to get the corresponding expansions. The tensor \( P_{\alpha\beta} \) satisfies,
\[ P_{\alpha\beta} = \partial_\mu \Gamma^\mu_{\alpha\beta} - \partial_\beta \Gamma^\mu_{\alpha\mu} + \Gamma^\mu_{\alpha\beta} \Gamma^\epsilon_{\mu\epsilon} - \Gamma^\mu_{\alpha\epsilon} \Gamma^\epsilon_{\beta\mu}, \] (3.37)
and so finding the expansion of \( P_{\alpha\beta} \) depends on the corresponding ones for the Christoffel symbols,
\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} \gamma^{\mu\epsilon} (\partial_\gamma \gamma_{\alpha\epsilon} + \partial_\alpha \gamma_{\beta\epsilon} - \partial_\epsilon \gamma_{\alpha\beta}), \] (3.38)
The final result which we now prove is as follows,
\[ P_{\alpha\beta} = \tilde{P}_{\alpha\beta} + tH_{\alpha\beta} + t^2 I_{\alpha\beta} + t^3 J_{\alpha\beta}, \] (3.39)
where the coefficients \( \tilde{P}, H, I, J \) are purely spatial functions of the metric coefficients \( a, b, c, d \). To show this result, we use the metric series (3.21) in the Christoffel symbols (3.38) to obtain,
\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} \left[ \frac{1}{t} a^{\mu\epsilon} - b^{\mu\epsilon} + t(-c^{\mu\epsilon} + b^{\beta\gamma} b^{\gamma\epsilon}_\gamma) + t^2 (-d^{\mu\epsilon} + b^{\mu\gamma} c^{\epsilon}_\gamma + c^{\mu\gamma} b^{\epsilon}_\gamma - b^{\gamma\epsilon} b^{\delta}_\gamma b^{\delta\epsilon}_\delta) \right] \times \left[ t(\partial_\beta a_{\alpha\epsilon} + \partial_\alpha a_{\beta\epsilon} - \partial_{\epsilon} a_{\alpha\beta}) + t^2 (\partial_\beta b_{\alpha\epsilon} + \partial_\alpha b_{\beta\epsilon} - \partial_{\epsilon} b_{\alpha\beta}) + t^3 (\partial_\beta c_{\alpha\epsilon} + \partial_\alpha c_{\beta\epsilon} - \partial_{\epsilon} c_{\alpha\beta}) + t^4 (\partial_\beta d_{\alpha\epsilon} + \partial_\alpha d_{\beta\epsilon} - \partial_{\epsilon} d_{\alpha\beta}) \right]. \] (3.40)
In order to simplify the ensuing calculations we set,

\[ A_{\alpha\beta\epsilon} = \partial_\beta a_{\alpha\epsilon} + \partial_\alpha a_{\beta\epsilon} - \partial_\epsilon a_{\alpha\beta}, \]  
(3.41)

\[ B_{\alpha\beta\epsilon} = \partial_\beta b_{\alpha\epsilon} + \partial_\alpha b_{\beta\epsilon} - \partial_\epsilon b_{\alpha\beta}, \]  
(3.42)

\[ C_{\alpha\beta\epsilon} = \partial_\beta c_{\alpha\epsilon} + \partial_\alpha c_{\beta\epsilon} - \partial_\epsilon c_{\alpha\beta}, \]  
(3.43)

\[ D_{\alpha\beta\epsilon} = \partial_\beta d_{\alpha\epsilon} + \partial_\alpha d_{\beta\epsilon} - \partial_\epsilon d_{\alpha\beta}. \]  
(3.44)

Then Eq. (3.40) becomes,

\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} (a^\mu \epsilon - b^\mu \epsilon + t(\gamma^\mu)_{(1)} + t^2(\gamma^\mu)_{(2)}) \times (tA_{\alpha\beta\epsilon} + t^2B_{\alpha\beta\epsilon} + t^3C_{\alpha\beta\epsilon} + t^4D_{\alpha\beta\epsilon}), \]  
(3.45)

and so,

\[ \Gamma^\mu_{\alpha\beta} = \frac{1}{2} a^\mu A_{\alpha\beta\epsilon} + \frac{1}{2} (a^\mu B_{\alpha\beta\epsilon} - b^\mu A_{\alpha\beta\epsilon})t + \frac{1}{2} (a^\mu C_{\alpha\beta\epsilon} - b^\mu B_{\alpha\beta\epsilon} + \gamma^{(1)\mu}_{\epsilon\alpha\beta})t^2 \]
\[ + \frac{1}{2} (a^\mu D_{\alpha\beta\epsilon} - b^\mu C_{\alpha\beta\epsilon} + (\gamma^\mu)_{(1)} B_{\alpha\beta\epsilon} + (\gamma^\mu)_{(2)} A_{\alpha\beta\epsilon})t^3. \]  
(3.46)

We further set,

\[ \tilde{\Gamma}^\mu_{\alpha\beta} = \frac{1}{2} a^\mu A_{\alpha\beta\epsilon}, \]  
(3.47)

\[ E^\mu_{\alpha\beta} = \frac{1}{2} (a^\mu B_{\alpha\beta\epsilon} - b^\mu A_{\alpha\beta\epsilon}), \]  
(3.48)

\[ F^\mu_{\alpha\beta} = \frac{1}{2} (a^\mu C_{\alpha\beta\epsilon} - b^\mu B_{\alpha\beta\epsilon} + \epsilon^\mu \epsilon A_{\alpha\beta\epsilon}), \]  
(3.49)

\[ G^\mu_{\alpha\beta} = \frac{1}{2} (a^\mu D_{\alpha\beta\epsilon} - b^\mu C_{\alpha\beta\epsilon} + (\gamma^\mu)_{(1)} B_{\alpha\beta\epsilon} + (\gamma^\mu)_{(2)} A_{\alpha\beta\epsilon}), \]  
(3.50)

so that for the Christoffel symbols we find,

\[ \Gamma^\mu_{\alpha\beta} = \tilde{\Gamma}^\mu_{\alpha\beta} + tE^\mu_{\alpha\beta} + t^2F^\mu_{\alpha\beta} + t^3G^\mu_{\alpha\beta}. \]  
(3.51)

Moreover, from (3.37) using (3.51) we find,

\[ P_{\alpha\beta} = (\partial_\mu \tilde{\Gamma}^\mu_{\alpha\beta} + t\partial_\mu E^\mu_{\alpha\beta} + t^2\partial_\mu F^\mu_{\alpha\beta} + t^3 \partial_\mu G^\mu_{\alpha\beta}) \]
\[ - (\partial_\mu \tilde{\Gamma}^\mu_{\alpha\beta} + t\partial_\beta E^\mu_{\alpha\epsilon} + t^2\partial_\beta F^\mu_{\alpha\epsilon} + t^3 \partial_\beta G^\mu_{\alpha\epsilon}) \]
\[ + (\tilde{\Gamma}^\mu_{\alpha\beta} + tE^\mu_{\alpha\beta} + t^2 F^\mu_{\alpha\beta} + t^3 G^\mu_{\alpha\beta}) \times (\tilde{\Gamma}^\epsilon_{\mu\alpha\beta} + tE^\epsilon_{\mu\alpha\beta} + t^2 F^\epsilon_{\mu\alpha\beta} + t^3 G^\epsilon_{\mu\alpha\beta}) \]
\[ - (\tilde{\Gamma}^\mu_{\alpha\beta} + tE^\mu_{\alpha\beta} + t^2 F^\mu_{\alpha\beta} + t^3 G^\mu_{\alpha\beta}) \times (\tilde{\Gamma}^\epsilon_{\beta\mu\alpha} + tE^\epsilon_{\beta\mu\alpha} + t^2 F^\epsilon_{\beta\mu\alpha} + t^3 G^\epsilon_{\beta\mu\alpha}), \]  
(3.52)
and setting further,

\[
\tilde{P}_{\alpha\beta} = \partial_\mu \tilde{\Gamma}^\mu_{\alpha\beta} - \partial_\beta \tilde{\Gamma}^\mu_{\alpha\mu} + \tilde{\Gamma}^\mu_{\alpha\beta} \tilde{\Gamma}_{\mu\epsilon} - \tilde{\Gamma}^\mu_{\alpha\epsilon} \tilde{\Gamma}_{\beta\mu},
\]

\[
H_{\alpha\beta} = \partial_\mu E^\mu_{\alpha\beta} - \partial_\beta E^\mu_{\alpha\mu} + \tilde{\Gamma}^\mu_{\alpha\beta} E^\epsilon_{\mu\epsilon} + \tilde{\Gamma}^\mu_{\alpha\mu} E^\epsilon_{\beta\mu} - \tilde{\Gamma}^\mu_{\alpha\epsilon} E^\epsilon_{\beta\mu},
\]

\[
I_{\alpha\beta} = \partial_\mu F^\mu_{\alpha\beta} - \partial_\beta F^\mu_{\alpha\mu} + \tilde{\Gamma}^\mu_{\alpha\beta} F^\epsilon_{\mu\epsilon} + \tilde{\Gamma}^\mu_{\alpha\mu} F^\epsilon_{\beta\mu} - \tilde{\Gamma}^\mu_{\alpha\epsilon} F^\epsilon_{\beta\mu} + E^\mu_{\alpha\beta} E^\epsilon_{\mu\epsilon} - E^\mu_{\alpha\epsilon} E^\epsilon_{\beta\mu},
\]

\[
J_{\alpha\beta} = \partial_\mu G^\mu_{\alpha\beta} - \partial_\beta G^\mu_{\alpha\mu} + \tilde{\Gamma}^\mu_{\alpha\beta} G^\epsilon_{\mu\epsilon} + \tilde{\Gamma}^\mu_{\alpha\mu} G^\epsilon_{\beta\mu} - \tilde{\Gamma}^\mu_{\alpha\epsilon} G^\epsilon_{\beta\mu} + E^\mu_{\alpha\beta} F^\epsilon_{\mu\epsilon} - E^\mu_{\alpha\epsilon} F^\epsilon_{\beta\mu},
\]

we obtain the announced result (3.39).

For the mixed three-dimensional Ricci tensor we have,

\[
P^\beta_{\alpha} = \gamma^\beta_{\mu} P^\mu_{\alpha} = \left(\frac{1}{t} a^\beta_{\mu} - b^\beta_{\mu} + t \gamma^{(1)}_{\beta\mu} + t^2 \gamma^{(2)}_{\beta\mu}\right) \times (\tilde{P}^\mu_{\alpha} + t H^\mu_{\alpha} + t^2 I^\mu_{\alpha} + t^3 J^\mu_{\alpha}),
\]

and setting,

\[
\kappa^\beta_{\alpha} = a^\beta_{\mu} H^\mu_{\alpha} - b^\beta_{\mu} \tilde{P}^\mu_{\alpha},
\]

\[
\lambda^\beta_{\alpha} = a^\beta_{\mu} I^\mu_{\alpha} - b^\beta_{\mu} H^\mu_{\alpha} + (\gamma^\beta_{\mu})^{(1)} \tilde{P}^\mu_{\alpha},
\]

\[
\mu^\beta_{\alpha} = a^\beta_{\mu} J^\mu_{\alpha} - b^\beta_{\mu} I^\mu_{\alpha} + (\gamma^\beta_{\mu})^{(1)} H^\mu_{\alpha} + (\gamma^\beta_{\mu})^{(2)} \tilde{P}^\mu_{\alpha},
\]

we find that the form of the mixed three-dimensional Ricci tensor is,

\[
P^\beta_{\alpha} = \frac{1}{t} \tilde{P}^\beta_{\alpha} + \kappa^\beta_{\alpha} + t \lambda^\beta_{\alpha} + t^2 \mu^\beta_{\alpha}.
\]

Finally, we obtain,

\[
P = P^\alpha_{\alpha} = \frac{1}{t} \tilde{P} + \kappa + t \lambda + t^2 \mu,
\]

where obviously,

\[
\kappa = \delta^\alpha_{\beta} \kappa^\beta_{\alpha},
\]

\[
\lambda = \delta^\alpha_{\beta} \lambda^\beta_{\alpha},
\]

\[
\mu = \delta^\alpha_{\beta} \mu^\beta_{\alpha}.
\]
Using the above results, we can express the components of the Ricci curvature as well as its trace in terms of the asymptotic data $a_{\alpha \beta}, b_{\alpha \beta}, c_{\alpha \beta}, d_{\alpha \beta}$. Based on Eqns. (2.16), (2.17) and (2.18) and our previous results concerning the quantities $P_{\alpha \beta}$, the components of the Ricci curvature become,

$$R_0^0 = \frac{3}{4t^2} - \frac{1}{2t} b + (-2c + \frac{3}{4} b^3 b_\alpha^\alpha) + (-\frac{9}{2} d + \frac{7}{2} b_\alpha^\alpha c_\beta^\beta) - b_\gamma^\gamma b^\alpha_\alpha b_\beta^\beta) t,$$

$$R_0^\alpha = \frac{1}{2} \left( \nabla_\beta b_\alpha^\beta - \nabla_\alpha b \right) + \left[ \nabla_\alpha c_\beta^\beta - \nabla_\beta c_\alpha^\alpha \right] - \frac{1}{2} \nabla_\beta (b_\gamma^\gamma b_\alpha^\alpha) + \frac{1}{2} \nabla_\alpha (b_\gamma^\gamma b_\beta^\beta) t$$

$$+ \frac{3}{2} \left( \nabla_\beta a_\beta^\beta - \nabla_\alpha d \right) - \nabla_\alpha (b_\gamma^\gamma c_\alpha^\alpha) - \frac{1}{2} \nabla_\beta (c_\gamma^\gamma b_\alpha^\alpha) + \frac{3}{2} \nabla_\alpha (b_\gamma^\gamma c_\beta^\beta) + \frac{1}{2} \nabla_\beta (b_\gamma^\gamma b_\alpha^\alpha b_\beta^\beta) - \frac{1}{2} \nabla_\alpha (b_\gamma^\gamma b_\beta^\beta b_\gamma^\gamma) t, \quad (3.66)$$

$$R_\beta^\alpha = -\frac{1}{4t^2} \delta_\alpha^\beta - \frac{1}{4t} (4 \tilde{P}_\alpha^\alpha + 3b_\alpha^\alpha + b_\delta^\delta) + (-\frac{5}{2} c_\alpha^\alpha + \frac{5}{4} b_\gamma^\gamma b_\alpha^\alpha - \frac{1}{2} b_\beta^\beta - \frac{1}{2} c_\alpha^\alpha$$

$$+ \frac{1}{4} b_\gamma^\gamma b_\delta^\delta - \kappa_\alpha^\alpha) + (-\frac{21}{4} d_\alpha^\alpha + \frac{7}{2} b_\gamma^\gamma c_\alpha^\alpha + \frac{7}{4} c_\gamma^\gamma b_\alpha^\alpha - \frac{7}{4} b_\gamma^\gamma b_\alpha^\alpha b_\alpha^\alpha - \frac{1}{2} b_\beta^\beta + \frac{1}{4} b_\gamma^\gamma b_\alpha^\alpha - \frac{1}{2} c_\beta^\beta$$

$$+ \frac{1}{4} b_\gamma^\gamma b_\alpha^\alpha b_\beta^\beta - \frac{3}{4} d_\delta^\delta + \frac{3}{4} b_\gamma^\gamma c_\delta^\delta \delta_\alpha^\alpha - \frac{1}{4} b_\gamma^\gamma b_\delta^\delta \delta_\alpha^\alpha - \lambda_\beta^\beta) t, \quad (3.67)$$

For the scalar curvature we find,

$$R = \frac{1}{t} R^{(-1)} + R^{(0)} + R^{(1)} t,$$

explicitly,

$$R = -\frac{1}{t} (\tilde{P} + 2b) + (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\alpha b_\beta^\beta - \kappa)$$

$$+ (-12d - bc + 11 b_\alpha^\alpha c_\beta^\beta - \frac{7}{2} b_\gamma^\gamma b_\alpha^\alpha b_\beta^\beta + \frac{1}{2} b_\alpha^\alpha b_\beta^\beta - \lambda) t. \quad (3.70)$$

### 4 Constraints and evolution

This Section, which is the heart of this paper, we work out the asymptotic structure of the various terms appearing in the system (2.13), (2.14), (2.15) to prepare them for the final counting and balancing done later. In the next Subsection, we express them in terms of the metric coefficients, while later in this Section we arrive at the final asymptotic forms of the splitted equations.
4.1 Formal expansions

It is now straightforward to use the previous results to write down analogously the evolution and constraints in a splitted form in terms of $a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}$. We have: The hamiltonian constraint (2.23) is,

$$C_0 = \frac{1}{E^3} \left( \frac{3}{2} \epsilon (\tilde{P} + 2b) \right)$$

+ $\frac{1}{t^2} \left\{ \frac{3}{4} + \epsilon \left[ -\frac{1}{2} (\tilde{P}^2 - 4b^2) + \frac{3}{2} (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_{\alpha\beta} - \kappa) \right] 
+ 2a^{\alpha\beta} \partial_{\alpha} (\tilde{P} + 2b) \right\}$

+ $\frac{1}{t} \left\{ \frac{1}{2} (\tilde{P} + b) + \epsilon \left[ (\tilde{P} + 2b) \left( \frac{1}{4} b_{\alpha} b_{\gamma} - \frac{1}{4} b_{\gamma} - \kappa \right) - b (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_{\alpha\beta} - \kappa) \right] 
+ \frac{9}{2} (-12d - bc + 11 b_{\alpha} c_{\beta} - \frac{7}{2} b_{\alpha} b_{\beta} + \frac{1}{2} b_{\gamma} b_{\delta} - \lambda) - \frac{2}{a} \partial_{\alpha} (\tilde{P} + 2b) \right\}$

+ $2a^{\alpha\beta} \tilde{\Gamma}_{\alpha\beta} \partial_{\mu} (\tilde{P} + 2b) - 2a^{\alpha\beta} \tilde{E}_{\alpha\beta} \partial_{\mu} (\tilde{P} + 2b) \right\}$

= $8\pi G T^0_\alpha$. (4.1)

From Eq. (2.24), the momentum constraints become,

$$C_\alpha = \frac{1}{t^2} \left( -3 \epsilon \partial_{\alpha} (\tilde{P} + 2b) \right)$$

+ $\frac{1}{t} \left\{ \epsilon \left[ -(\tilde{P} + 2b) (\nabla_{\beta} b_{\alpha} - \nabla_{\alpha} b) + \partial_{\alpha} (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_{\beta} - \kappa) \right] 
+ b_{\alpha} \partial_{\beta} (\tilde{P} + 2b) \right\}$

+ $t \left\{ \frac{1}{2} (\nabla_{\beta} b_{\alpha} - \nabla_{\alpha} b) + \epsilon \left[ -2 (\tilde{P} + 2b) \left( \nabla_{\beta} c_{\alpha} - \nabla_{\alpha} c \right) - \frac{1}{2} \nabla_{\beta} (b_{\gamma} b_{\alpha}) + \frac{1}{2} \nabla_{\alpha} (b_{\gamma} b_{\beta}) \right] \right.$

+ $(\nabla_{\beta} b_{\alpha} - \nabla_{\alpha} b)$

- $\partial_{\alpha} (-12d - bc + 11 b_{\gamma} c_{\delta} - \frac{7}{2} b_{\gamma} b_{\delta} + \frac{1}{2} b_{\gamma} b_{\delta} - \lambda)$

+ $b_{\alpha} \partial_{\beta} (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_{\gamma} - \kappa) - (2 c_{\alpha} - b_{\gamma} b_{\alpha}) \partial_{\beta} (\tilde{P} + 2b) \right\}$

= $8\pi G T^0_\alpha$. (4.2)
Finally, the snap equation (2.22) gives,

\[
\frac{1}{t^3} \left( -\frac{9}{2} \epsilon (\tilde{P} + 2b) \right) + \frac{1}{t^2} \left\{ -\frac{3}{4} + \epsilon \left[ (\tilde{P} + 2b) \left( \frac{1}{2} \tilde{P} + 2b \right) - \frac{3}{2} \left( -6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\alpha b_\beta^\beta - \kappa \right) \right] + 2a^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) + 2a^\alpha\beta \tilde{T}_\alpha^\mu \partial_\mu (\tilde{P} + 2b) + 6a^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) - 6a^\alpha\beta \tilde{T}_\alpha^\mu \partial_\mu (\tilde{P} + 2b) \right\} + \frac{1}{t} \left\{ -1 \frac{1}{2} \tilde{P} + 3b + \epsilon \left[ \frac{9}{2} (-12d - bc + 11b_\alpha^\alpha c_\beta - \frac{7}{2} b_\gamma^\gamma b_\alpha^\alpha + \frac{1}{2} b b_\beta^\alpha - \lambda) + (\tilde{P} + 2b) (\tilde{P} + 2b) + 2b^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) - 2a^\alpha\beta \tilde{T}_\alpha^\mu \partial_\mu \left( -6c - \frac{1}{4} b^2 + \frac{11}{4} b_\gamma^\gamma b_\delta^\delta - \kappa \right) - 6b^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) + 6a^\alpha\beta \tilde{T}_\alpha^\mu \partial_\mu (\tilde{P} + 2b) + 2a^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) - 6b^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) + 6a^\alpha\beta \tilde{T}_\alpha^\mu \partial_\mu (\tilde{P} + 2b) + 2a^\alpha\beta \partial_\alpha \left( \partial_\beta (\tilde{P} + 2b) \right) - \frac{1}{3} \left[ \tilde{P} (\tilde{P} + 2b) \right] \right\} \right\}
\]

= \frac{8}{3} \pi G T_\alpha^\alpha, \quad (4.3)

where the components of the energy-momentum tensor \( T^i_j \) and its trace are given by the relations \( (2.7) \), namely,

\[
T^0_0 = \frac{1}{3} \rho (4u_0^2 - 1), \quad (4.4)
\]

\[
T^0_\alpha = 4 \frac{3}{3} \rho u_\alpha u_0, \quad (4.5)
\]

\[
T^\beta_\alpha = \frac{1}{3} \rho (4u^\beta u_\alpha - \delta^\beta_\alpha), \quad (4.6)
\]

and,

\[
T = \text{tr} T_{ij} = 0. \quad (4.7)
\]
In view of the identity (2.10) and the form (3.21), we further get,

\[ 1 = u_i u^i \approx u_0^2 - \frac{1}{t} a^{\alpha \beta} u_\alpha u_\beta. \] (4.8)

Hence, the constraint and evolution equations are now in terms of the initial data \( a_{\alpha \beta}, b_{\alpha \beta}, c_{\alpha \beta}, d_{\alpha \beta} \). This will allow us later to give a count of how many of the initial data are free.

### 4.2 The intermediate asymptotic system

We now rearrange the terms in the equations given in the previous subsection so as to appear in a form amenable to further simplification. We have:

\[ 8\pi G \rho = C_0 = (L_0^0)^{(-3)} \frac{1}{t^3} + (L_0^0)^{(-2)} \frac{1}{t^2} + (L_0^0)^{(-1)} \frac{1}{t}, \] (4.9)

\[ \frac{32\pi G}{3} \rho u_\alpha = C_\alpha = (L_\alpha^0)^{(-3)} \frac{1}{t^3} + (L_\alpha^0)^{(-2)} \frac{1}{t^2} + (L_\alpha^0)^{(0)}, \] (4.10)

and,

\[-8\pi G \rho \delta^\beta_\alpha = L_\alpha^\beta = (L_\alpha^\beta)^{(-3)} \frac{1}{t^3} + (L_\alpha^\beta)^{(-2)} \frac{1}{t^2} + (L_\alpha^\beta)^{(-1)} \frac{1}{t}. \] (4.11)

In terms of \( a_{\alpha \beta}, b_{\alpha \beta}, c_{\alpha \beta}, d_{\alpha \beta} \), the quantities appearing in Eq. (4.9) become,

\[(L_0^0)^{(-3)} = \epsilon \left[ 2 R^{(-1)} (R_0^0)^{(-2)} - K^{(-1)} R^{(-1)} \right] = \frac{3}{2} \epsilon (\tilde{P} + 2b), \] (4.12)

\[(L_0^0)^{(-2)} = (R_0^0)^{(-2)} + \epsilon \left[ 2 (R^{(-1)} (R_0^0)^{(-1)} + R^{(0)} (R_0^0)^{(-2)}) - \frac{1}{2} (R^{(-1)})^2 
- \left. 2(\gamma^{\alpha \beta})^{(-1)} \partial_\alpha (\partial_\beta R^{(-1)}) \right] + 2(\gamma^{\alpha \beta})^{(-1)} (\Gamma^\mu_{\alpha \beta})^{(0)} \partial_\mu R^{(-1)} - K^{(0)} R^{(-1)} \right]
= \frac{3}{4} + \epsilon \left[ -\frac{1}{2} (\tilde{P}^2 - 4b^2) + \frac{3}{2} (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\beta b_\beta^\alpha - \kappa) 
+ 2a^{\alpha \beta} \partial_\alpha (\tilde{P} + 2b) \right] - 2a^{\alpha \beta} \Gamma^\mu_{\alpha \beta} \partial_\mu (\tilde{P} + 2b), \] (4.13)
\((L_0^0)^{(-1)}\) = \((R_0^0)^{(-1)}\) - \frac{1}{2} R^{(-1)} + \epsilon \left\{ 2 \left[ R^{(-1)} R_0^0 R_0^0 + R_0^0 R_0^0 R_0^0 \right] \right\} \\
\text{Further, for Eq. (4.10), we find } \(L_0^0)^{(-1)}\) = \epsilon \left\{ \begin{array}{l}
(2 \partial_\alpha R^{(-1)} + (K_\alpha^0)^{(-1)} \partial_\beta R^{(-1)}) \\
-3 \epsilon \partial_\alpha (\bar{P} + 2b),
\end{array} \right\} \\
\text{Further, for Eq. (4.11), we find}
\text{Further, for Eq. (4.12), we find}
\text{Further, for Eq. (4.13), we find}
\text{Further, for Eq. (4.14), we find}
\text{Further, for Eq. (4.15), we find}
\text{Further, for Eq. (4.16), we find}
\text{Further, for Eq. (4.17), we find}
and finally for Eq. (4.11) we have,

\[
(L^\beta_\alpha)^{(-3)} = \epsilon \left[ 2R^{(-1)}(R^\beta_\alpha)^{(-2)} + (K^\beta_\alpha)^{(-1)}R^{(-1)} + 4\delta^\beta_\alpha R^{(-1)} - \delta^\beta_\alpha K^{(-1)}R^{(-1)} \right] \\
= -\frac{3}{2} \epsilon \delta^\beta_\alpha (\bar{P} + 2b),
\]  

(4.18)

\[
(L^\beta_\alpha)^{(-2)} = (R^\beta_\alpha)^{(-2)} + \epsilon \left\{ 2 \left[ R^{(-1)}(R^\beta_\alpha)^{(-1)} + R^{(0)}(R^\beta_\alpha)^{(-2)} \right] - \frac{1}{2} \delta^\beta_\alpha (R^{(-1)})^2 \\
+ 2(\gamma^\beta\gamma)^{(-1)} \partial_\gamma (\partial_\alpha R^{(-1)}) - 2(\gamma^\beta\gamma)^{(-1)}(\Gamma^\mu_{\gamma\alpha})^{(0)} \partial_\mu R^{(-1)} + (K^\beta_\alpha)^{(0)}R^{(-1)} \\
- 2\delta^\beta_\alpha (\gamma^\gamma)^{(-1)} \partial_\gamma (\partial_\alpha R^{(-1)}) + 2\delta^\beta_\alpha (\gamma^\gamma)^{(-1)}(\Gamma^\mu_{\gamma\delta})^{(0)}(\partial_\mu R^{(-1)}) - \delta^\beta_\alpha K^{(0)}R^{(-1)} \right\} \\
= -\frac{1}{4} \delta^\beta_\alpha + \epsilon \left[ (\bar{P} + 2b)(2\bar{P}^\beta_\alpha + \frac{1}{2}b^\beta_\alpha + \frac{1}{2}b\delta^\beta_\alpha - \frac{1}{2}\bar{P}\delta^\beta_\alpha) \\
- \frac{1}{2} \delta^\beta_\alpha (-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\gamma b_\delta - \kappa) \\
- 2a^\beta\gamma \partial_\gamma (\partial_\alpha (\bar{P} + 2b)) + 2a^\beta\gamma \tilde{\Gamma}^\mu_{\gamma\alpha} \partial_\mu (\bar{P} + 2b) + 2\delta^\beta_\alpha a^\gamma\delta \partial_\gamma (\partial_\delta (\bar{P} + 2b)) \\
- 2\delta^\beta_\alpha a^\gamma\delta \tilde{\Gamma}^\mu_{\gamma\delta} \partial_\mu (\bar{P} + 2b) \right],
\]  

(4.19)
\[(L_α^β)^{(-1)} = (R_α^β)^{(-1)} - \frac{1}{2} \delta_α^β R^{(-1)} + \epsilon \left\{ 2 \left[ R^{(-1)} (R_α^β)^{(0)} + R^{(0)} (R_α^β)^{(-1)} + R^{(1)} (R_α^β)^{(-2)} \right] \right. \\
- \left. R^{(-1)} R^{(0)} \delta_α^β + 2 \left[ (\gamma_β^γ)^{(-1)} \partial_γ (\partial_α R^{(0)}) + (\gamma_β^γ)^{(0)} \partial_γ (\partial_α R^{(-1)}) \right] \right. \\
+ \left. 2 \left[ (\gamma_β^γ)^{(1)} (\Gamma_γ^μ)^{(0)} \partial_μ R^{(0)} + (\gamma_β^γ)^{(0)} (\Gamma_γ^μ)^{(0)} \partial_μ R^{(-1)} + (K_α^β)^{(1)} R^{(1)} + (K_α^β)^{(1)} R^{(-1)} \right] \right. \\
- \left. 2 \delta_α^β \left[ (\gamma_δ^β)^{(1)} \partial_δ (\partial_β R^{(0)}) + (\gamma_δ^β)^{(0)} \partial_δ (\partial_β R^{(-1)}) \right] \right. \\
+ \left. 2 \delta_α^β \left[ (\gamma_δ^β)^{(1)} (\Gamma_δ^μ)^{(0)} \partial_μ R^{(0)} + (\gamma_δ^β)^{(0)} (\Gamma_δ^μ)^{(0)} \partial_μ R^{(-1)} \right] \right. \\
+ \left. (\gamma_δ^β)^{(1)} (\Gamma_δ^μ)^{(1)} \partial_μ R^{(-1)} \right) + \delta_α^β \left( K^{(-1)} R^{(1)} - K^{(-1)} R^{(-1)} \right) \right\} \\
- \frac{3}{2} \delta_α^β \left( -12d - bc + 11b_γ^δ c_δ - \frac{7}{2} b_γ^δ b_δ^γ b_δ + \frac{1}{2} b b_γ^δ b_δ^γ - \lambda \right) \\
+ \left( \tilde{P} + 2b \right) \left( 3c_α - 3c_δ^α + \frac{5}{4} b_γ^δ b_δ^γ \delta^α - \frac{1}{4} b^2 \delta^α - \frac{3}{2} b_γ^δ b_δ^γ + \frac{1}{2} b b_γ^δ - \delta_α^β \kappa + 2\kappa_α \right) \\
- \frac{1}{2} \left( 4\tilde{P}_α + 3b_β^α + b_δ^β \right) \left( -6c - \frac{1}{4} b^2 + \frac{11}{4} b_δ^γ - \kappa \right) - 2\delta_α^β a_δ^γ E_δ^μ \partial_μ (\tilde{P} + 2b) \\
+ \left( \partial_α (\tilde{P} + 2b) \right) - 2a_δ^γ \Gamma_γ^μ \partial_μ (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_δ^γ - \kappa) \\
- \left( 2a_δ^γ \Gamma_γ^μ \partial_μ (\tilde{P} + 2b) \right) + 2a_δ^γ E_γ^μ \partial_μ (\tilde{P} + 2b) \\
- \left( 2\delta_α^β a_δ^γ \partial_γ \left( \partial_β (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_δ^γ - \kappa) \right) \right) \\
+ \left( 2\delta_α^β b_δ^γ \partial_γ \left( \partial_β (\tilde{P} + 2b) \right) + 2\delta_α^β a_δ^γ \Gamma_γ^μ \partial_μ (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_δ^γ - \kappa) \right) \\
+ \left( 2\delta_α^β b_δ^γ \Gamma_γ^μ \partial_μ (\tilde{P} + 2b) + 2a_δ^γ \partial_γ \left( \partial_α (-6c - \frac{1}{4} b^2 + \frac{11}{4} b_δ^γ - \kappa) \right) \right) \]. \quad (4.20)

This intermediate form of the constraint and evolution equations is now suitable for further simplification.

4.3 The asymptotic system

From Eq. \((4.9)\), the degree of the energy density \(\rho\) seems to be equal to \(-3\), which is not in accordance with the Friedmann solution \((3.20)\) and Eq. \((3.19)\). However, in this subsection we prove that the term \((L_0^β)^{(-3)}\) is equal to zero, and the term \((L_0^β)^{(-2)}\)
does not vanish. Therefore the degree of the energy density becomes equal to $-2$. The method of proof below is based on the use of the trace of the higher-order equations to simplify the nine terms $(L^i_j)^{(-n)}$.

In case of radiation $(p = \rho/3)$, the trace of the field equations (2.11) gives the identity,

$$R - 6\epsilon\Box_g R = 0.$$  \hfill (4.21)

Then, for the $t^{-3}$ order terms we get,

$$\tilde{P} + 2b = 0,$$ \hfill (4.22)

for the $t^{-2}$ order terms we obtain,

$$-12d - bc + 11b^\beta\alpha c^\alpha - \frac{7}{2}b^\beta\gamma b^\delta\alpha b^\alpha + \frac{1}{2}bb^\beta b^\alpha - \lambda = 0,$$ \hfill (4.23)

and finally for the $t^{-1}$ order terms we have,

$$\frac{1}{2}a^\alpha\beta a^{\mu\nu}A_{\alpha\beta\epsilon}\partial_{\mu} \left( -6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa \right) = a^\alpha\beta\partial_{\alpha} \left( \partial_{\beta}(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa) \right).$$ \hfill (4.24)

Using the identities (4.22), (4.23) and (4.24), the terms $(L_0^0)^{(-3)}$, $(L_0^0)^{(-2)}$, $(L_0^0)^{(-3)}$ vanish identically, while the six further quantities appearing in (4.19)-(4.11) become,

$$(L_0^0)^{(-2)} = \frac{3}{4} + \frac{3}{2}\epsilon(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa),$$ \hfill (4.25)

$$(L_0^0)^{(-1)} = \frac{1}{2}b - \epsilon b(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa),$$ \hfill (4.26)

and,

$$(L_0^0)^{(-1)} = \epsilon \partial_{\alpha}(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa),$$ \hfill (4.27)

$$(L_0^0)^{(0)} = \frac{1}{2}(\nabla_\beta b^\alpha - \nabla_\alpha b) + \epsilon \left[ (-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa)(\nabla_\beta b^\alpha - \nabla_\alpha b) + b^\beta\gamma \partial_{\beta}(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa) \right],$$ \hfill (4.28)

and also,

$$(L_0^0)^{(-2)} = -\frac{1}{4}\delta^\beta\gamma - \frac{1}{2}\epsilon\delta^\beta\gamma(-6c - \frac{1}{4}b^2 + \frac{11}{4}b^\beta\gamma b^\alpha - \kappa),$$ \hfill (4.29)
\[(L^\beta_\alpha)^{(-1)} = -\vec{P}_\alpha^\beta - \frac{3}{4}b_\alpha^\beta - \frac{1}{4}b_\alpha^\delta \]
\[+ \epsilon \left[ \frac{1}{2}(4\vec{P}_\alpha^\beta + 3b_\alpha^\beta + b_\alpha^\delta)(-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\gamma^\delta b_\delta^\gamma - \kappa) \right. \]
\[\left. - \epsilon a^\beta\gamma a^\mu A_{\gamma\alpha\epsilon} \partial_\mu (-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\epsilon^\delta b_\delta^\epsilon - \kappa) \right] + 2\epsilon a^\beta\gamma \partial_\gamma \left( \partial_\alpha (-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\epsilon^\delta b_\delta^\epsilon - \kappa) \right). \tag{4.30} \]

These can all be further simplified. Using (4.21) and the identity,
\[\nabla_0 L_0^\beta + \nabla_\beta L_0^\beta = 0, \tag{4.31} \]
we find that for \(j = \alpha\),
\[\partial_\alpha L_0^\alpha - \frac{1}{2}K_\alpha^\beta L_0^\beta + \partial_\beta L_0^\beta + KL_0^\beta - \Gamma_\alpha^\gamma L_0^\gamma + \Gamma_\beta^\beta L_0^\gamma = 0. \tag{4.32} \]
Substituting their series for each of the various terms in (4.32) we find that balancing at the \(t^{-2}\) order, the coefficient \((L_0^\alpha)^{(-1)}\) vanishes identically, namely,
\[\partial_\alpha (-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\gamma^\delta b_\delta^\gamma - \kappa) = 0. \tag{4.33} \]
Consequently, the terms of (4.28) and (4.30) become,
\[(L^\alpha_0)^{(0)} = \frac{1}{2}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) + \epsilon(-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\gamma^\delta b_\delta^\gamma - \kappa)(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b), \tag{4.34} \]
and,
\[(L^\beta_\alpha)^{(-1)} = -\vec{P}_\alpha^\beta - \frac{3}{4}b_\alpha^\beta - \frac{1}{4}b_\alpha^\delta \]
\[\left. - \frac{1}{2}\epsilon(2\vec{P}_\alpha^\beta + 3b_\alpha^\beta + b_\alpha^\delta)(-6c - \frac{1}{4}b^2 + \frac{11}{4}b_\gamma^\delta b_\delta^\gamma - \kappa). \tag{4.35} \right] \]
Therefore the system of equations (4.9)-(4.11) which describes the dependence between the initial data becomes,
\[8\pi G\rho = (L_0^0)^{(-2)} - \frac{1}{t^2} + (L_0^\beta)^{(-1)} \frac{1}{t}, \tag{4.36} \]
\[\frac{32\pi G}{3} \rho u_\alpha = (L^\alpha_0)^{(0)}. \tag{4.37} \]
\[-\frac{8\pi G}{3} \rho \delta_\alpha = (L_\alpha^\beta)(-2) \frac{1}{t^2} + (L_\alpha^\beta)(-1) \frac{1}{t}. \tag{4.38}\]

Then, from Eq. (4.36) we find one relation for the energy density \(\rho\), namely,

\[8\pi G \rho = \left[ \frac{3}{4} + \frac{3}{2} \epsilon(-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\beta b_\beta^\alpha - \kappa) \right] \frac{1}{t^2} - \left[ \frac{1}{2} b + \epsilon b(-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\beta b_\beta^\alpha - \kappa) \right] \frac{1}{t}. \tag{4.39}\]

Substituting \(\rho\) from Eq. (4.39) to Eq. (4.37), we find three more relations for the velocities \(u_\alpha\) and the spatial tensors \(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}\), namely,

\[u_\alpha = 3 \left( \frac{L_0^0}{t^2} \right)^{(0)} \tag{4.40}\]

These last relations concerning the velocities \(u_\alpha\) and the spatial tensors \(a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}\) read,

\[u_\alpha = \frac{3t^2}{4} \left[ \frac{1}{2}(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) + \epsilon(-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\gamma^\beta b_\delta^\gamma - \kappa)(\nabla_\beta b_\alpha^\beta - \nabla_\alpha b) \right] \times \left[ \frac{3}{4} + \frac{3}{2} \epsilon(-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\alpha^\beta b_\beta^\alpha - \kappa) \right]^{-1}. \tag{4.41}\]

Continuing, we substitute \(\rho\) from Eq. (4.39) to Eq. (4.38) and we obtain,

\[\left( L_\alpha^\beta \right)(-2) \frac{1}{t^2} + \left( L_\alpha^\beta \right)(-1) \frac{1}{t} = - \frac{1}{3} \delta_\alpha^\beta \left[ \left( L_0^0 \right)(-2) \frac{1}{t^2} + \left( L_0^0 \right)(-1) \frac{1}{t} \right]. \tag{4.42}\]

Using the last equation and the results for the required terms \((L_\alpha^\beta)^{-n}\), we find that the terms of order \(t^{-2}\) cancel, while the terms of order \(t^{-1}\) give,

\[-3(\tilde{P}_\alpha^\beta + \frac{3}{4} b_\alpha^\beta + \frac{5}{4} b_\alpha^\beta) - \frac{1}{2} \epsilon(12 \tilde{P}_\alpha^\beta + 9 b_\alpha^\beta + 5 b_\alpha^\beta)(-6c - \frac{1}{4} b^2 + \frac{11}{4} b_\gamma^\beta b_\delta^\gamma - \kappa) = 0. \tag{4.43}\]

Notice that the trace of Eq. (4.43) gives the identity (4.22).

5 The final balance

In this Section, we arrive at the final number of free functions possible in the radiation solution of the theory without assuming any symmetry. This number (equal to 15 arbitrary functions) is less than that required (20) for the radiation solution to be a general
one in the quadratic theory. Then we show that this number would remain the same had we considered higher-than-fourth-order terms in the original metric expansion. This completes the proof of the general structure of the radiation solution in the quadratic theory.

5.1 Counting

From equations (4.39), (4.40) and (4.43), we already found 1 relation for the energy density \( \rho \), 3 for the velocities \( u^\alpha \) and additional 6 concerning the initial data \( a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta} \). Also, we should not make the mistake to count as extra conditions between the initial data the identities (4.22)-(4.24) as well as the identity (4.33), because the trace of the gravitational field equations and the conservation laws yield directly to the same relations.

Nevertheless, we have to take into consideration the fact that the choice of the time \( t \) in the metric (3.21) is completely determined by the condition \( t = 0 \) at the singularity, while the space coordinates still permit arbitrary transformations that do not involve the time. These arbitrary transformations can be used, for example, to bring tensor \( a_{\alpha\beta} \) to diagonal form.

Thus, from the initial 28 functions \( a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}, \rho, u^\alpha \), we have to subtract 13 that are not arbitrary, and we finally find that the solution contains all together 15 physically different arbitrary functions (below we also give the possible choice of the initial data that we can use for the particular solution that we find after this analysis). In comparison with the number 20 which corresponds to a general solution of the problem, we see that after the imposition of the higher-order evolution and constraint equations, the tensor \( \gamma_{\alpha\beta} \) of the form (3.21) cannot correspond to a general solution.

It is also worth noting that setting \( \epsilon = 0 \), the basic equations (4.39), (4.40) and (4.43) lead us to conclude that we have the exact same results as those found in general relativity [13].

The last question is: Out of the 28 different functions \( a_{\alpha\beta}, b_{\alpha\beta}, c_{\alpha\beta}, d_{\alpha\beta}, \rho, u^\alpha \), which
15 of those should be chosen as our initial data? We have shown that in the case of radiation the higher-order gravity equations (2.19), (2.20), (2.21) and (2.22) together with the constraint equations (2.23) and (2.24) admit a singular formal series expansion of the form (3.21) leading to a solution which is not general and requires 15 smooth initial data.

If we prescribe the 28 data

\[ a_{\alpha\beta}, \ b_{\alpha\beta}, \ c_{\alpha\beta}, \ d_{\alpha\beta}, \ \rho, \ u^\alpha \]  \hspace{1cm} (5.1)\n
initially, we still have the freedom to fix 13 of them. We choose to leave the twelve components of the metrics \( c_{\alpha\beta} \) and \( d_{\alpha\beta} \) free, and we choose the symmetric space tensor \( a_{\alpha\beta} \) to be diagonal. Then we proceed to count the number of free functions in three steps, starting from these \( 3 + 0 + 2 \times 6 + 1 + 3 = 19 \) functions. First, (4.36) fixes the function \( \rho \) and secondly Eq. (4.40) fixes the 3 more components of \( u^\alpha \). Lastly, we use the 6 relations in (4.43) to completely fix the 6 components of \( b_{\alpha\beta} \). Summing up the free functions we have found, we end up with,

\[ 3 \quad \text{from} \ a_{\alpha\beta} + 0 \quad \text{from} \ b_{\alpha\beta} + 12 \quad \text{from} \ c_{\alpha\beta} \text{ and } d_{\alpha\beta} + 0 \quad \text{from} \ \rho + 0 \quad \text{from} \ u^\alpha = 15, \]  \hspace{1cm} (5.2)\n
which are suitable free data as required for the specific particular solution. Obviously this is not the only way to choose among the initial data. For instance, we can choose the space tensor \( a_{\alpha\beta} \) not to be diagonal and fix three components of \( c_{\alpha\beta} \). Then, we end up with,

\[ 6 \quad \text{from} \ a_{\alpha\beta} + 0 \quad \text{from} \ b_{\alpha\beta} + 3 \quad \text{from} \ c_{\alpha\beta} + 6 \quad \text{from} \ d_{\alpha\beta} + 0 \quad \text{from} \ \rho + 0 \quad \text{from} \ u^\alpha = 15, \]  \hspace{1cm} (5.3)\n
suitable free data. Other choices are also possible.

### 5.2 The minimal-order expansion

In this subsection, we show that the number of degrees of freedom of the gravitational field equations in \( R + \epsilon R^2 \) theory plus radiation would remain the same if instead of the
singular form \( (3.21) \) we take a singular expression of the form,

\[
\gamma_{\alpha\beta} = \sum \gamma^{(n)}_{\alpha\beta} t^n, \quad n > 4.
\] \( (5.4) \)

We assume a formal series representation of the spatial metric of the form:

\[
\gamma_{\alpha\beta} = (\gamma_{\alpha\beta})^{(1)} t + (\gamma_{\alpha\beta})^{(2)} t^2 + (\gamma_{\alpha\beta})^{(3)} t^3 + (\gamma_{\alpha\beta})^{(4)} t^4 + \cdots + (\gamma_{\alpha\beta})^{(n)} t^n,
\] \( (5.5) \)

where \( n \) is a natural number, with \( n > 4 \). Obviously, we have,

\[
(\gamma_{\alpha\beta})^{(1)} = a_{\alpha\beta}, \quad (\gamma_{\alpha\beta})^{(2)} = b_{\alpha\beta}, \quad (\gamma_{\alpha\beta})^{(3)} = c_{\alpha\beta}, \quad (\gamma_{\alpha\beta})^{(4)} = d_{\alpha\beta}.
\] \( (5.6) \)

We note that the expression \( (5.5) \) contains \( 6n \) degrees of freedom (6 of each one of the \( n \) spatial matrices). Adding 4 additional degrees of freedom of the energy density \( \rho \) and the velocities \( u^\alpha \), the main question now becomes: how many of these \( 6n + 4 \) data are independent? (That is when we assume the validity of our dynamical system of equations, and the metric \( (5.5) \) is taken to be a solution of it, that is of the evolution equations \( (2.19), (2.20), (2.21) \) and \( (2.22) \), together with the constraint equations \( (2.23), (2.24) \), and also the radiation equation of state, and relation \( (2.10) \) on each slice \( \mathcal{M}_t \).)

To answer this question, we consider again our basic system of equations \( (4.36), (4.37) \) and \( (4.38) \) which now due to \( (5.5) \) takes the form,

\[
8\pi G \rho = (L_0^0)^{(-2)} \frac{1}{t^2} + (L_0^0)^{(-1)} \frac{1}{t} + \cdots + (L_0^0)^{(n-5)} t^{n-5},
\] \( (5.7) \)

\[
\frac{32\pi G}{3} \rho u_\alpha = (L_0^\alpha)^{(0)} + \cdots + (L_0^\alpha)^{(n-4)} t^{n-4},
\] \( (5.8) \)

\[
- \frac{8\pi G}{3} \rho \delta_\alpha^\beta = (L_0^\beta)^{(-2)} \frac{1}{t^2} + (L_0^\beta)^{(-1)} \frac{1}{t} + \cdots + (L_0^\beta)^{(n-5)} t^{n-5}.
\] \( (5.9) \)

Then \( (5.7) \) gives us 1 relation for the energy density \( \rho \), which when substituted in \( (5.8) \) gives 3 relations for the velocities,

\[
u_\alpha = \frac{3(L_0^\alpha)^{(0)}}{4(L_0^0)^{(-2)}} t^2.
\] \( (5.10) \)
This is exactly the same relation between the velocities $u_\alpha$ and the spatial matrices as that we found in the case of only using the first four terms of $\gamma_{\alpha\beta}$. Finally, substituting $\rho$ from (5.7) in Eq. (5.9), we obtain,

$$
(L_\beta^\alpha)^{(-2)} \frac{1}{t^2} + (L_\beta^\alpha)^{(-1)} \frac{1}{t} + \ldots + (L_\beta^\alpha)^{(n-5)} t^{n-5} = -\frac{1}{3} \delta_\beta^\alpha (L_0^0)^{(-2)} \frac{1}{t^2} - \frac{1}{3} \delta_\beta^\alpha (L_0^0)^{(-1)} \frac{1}{t} - \ldots$$

$$- \frac{1}{3} \delta_\beta^\alpha (L_0^0)^{(n-5)} t^{n-5}.
$$

(5.11)

The terms of order $t^{-2}$ in (5.11) cancel as before, and the $t^{-1}$ order terms give 6 relations between the initial data, which is the equation (4.43). For the $t^k$-order terms of (5.11), where $k = 0, 1, \ldots, n - 5$, we obtain the following equations,

$$
(L_\beta^\alpha)^{(k)} = -\frac{1}{3} \delta_\beta^\alpha (L_0^0)^{(k)}.
$$

(5.12)

Apparently, each one of the previous $n - 4$ equations gives 6 relations between the initial data. Therefore, from (5.12) we find $6 \times (n - 4) = 6n - 24$ relations between the data $(\gamma_{\alpha\beta})^{(0)}, \ldots, (\gamma_{\alpha\beta})^{(n)}$ and so, counting the $t^{-2}$ and $t^{-1}$ order terms together with the next $n - 4$ terms from (5.11) we get $0 + 6 + (6n - 24) = 6n - 18$ relations in total. Consequently, taking further into account one relation from (5.7), three relations from (5.10) and $(6n - 18)$ relations from (5.11), the counting give us $1 + 3 + (6n - 18) = 6n - 14$ relations between the initial data. Subtracting this from the total $6n + 4$ data that we started from, we conclude that only 18 functions can be arbitrary. Hence, subtracting 3 diffeomorphisms gives the final number 15 of free functions for the problem. Accordingly, we have shown that without loss of generality we can stop the series (5.5) at the order 4.

6 Discussion

In this paper we have analysed the genericity aspect of radiation cosmologies in quadratic gravity. We have shown that perturbations of the general relativistic radiation isotropic solution in the framework of the quadratic theory cannot sustain a generic radiation solution because the final solution has fewer than necessary arbitrary functions than
those which must be present in a general radiation solution in quadratic gravity. The unperturbed GR radiation isotropic solution \( a \sim \sqrt{t} \) as a solution in quadratic gravity is consequently non-generic and unstable, and so we expect it to be replaced by another solution. Starting with a radiation stage at early times as in the standard cosmological model, our results indicate that there must be a future replacement, and the radiation-matter transition in the standard model in the quadratic theory might be justified in this light. Similarly, a transition from a radiation regime to an earlier one such as an inflationary stage, is also dependent on the instability of the former as in the present work.

There are two aspects of these results worth noting. The first is that although not general, our solution contains 15 arbitrary functions and so we expect slightly inhomogeneous radiation solutions to have a similar instability and decay into something else (either in the future or past direction). The second feature is that the non-genericity of the radiation solution will hold true for any polynomial lagrangian theory, not just the quadratic, for instance a theory like \( R + \epsilon R^n, n > 2 \). This follows from an examination of the various terms and we may conclude that \( n \)-th order terms will not change the qualitative nature of the final result.

There is finally the question of what is the nature of the general radiation solution with 20 arbitrary functions (instead of the one with 15 data constructed here) in the framework of the quadratic theory? Such a generic radiation solution is predicted by the analysis of this paper, but its construction requires techniques beyond those advanced here. A radiation solution with 20 arbitrary functions would necessarily be global and may not be constructed perturbatively, as the present work implied. It is unknown whether such generic radiation models will necessarily be singular or not.
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