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Abstract. Branched flow is a universal phenomenon of wave propagation in random media. From a classical point of view, branched flow is the overall pattern of classical electron trajectories moving in a potential with randomly placed weak scatterers. Individually, each electron trajectory is exponentially unstable to small perturbations due to the chaotic nature of the classical dynamics of electrons moving in a random potential. However, the overall pattern, branched flow, displays strong stability against large perturbations. In this paper, we discuss both the classical and quantum stability of branched flow against perturbations.

1. Discovery of Branched Flow

Coherent electron transport in disordered media is a widely studied topic in condensed matter physics. Different physics phenomena arise in different regimes. When the disorder is strong enough or the sample is long enough such that the sample size is larger than the localization length (the mean free path of the electrons adjusted by the number of open conduction modes), it leads to the Anderson Localization\cite{1}. When the size of the sample is smaller than the localization length, but much larger than the mean free path, it leads to the Universal Conductance Fluctuations\cite{2}. Branched flow arises in a yet different regime with much weaker disorders. In this case, the mean free path of the electrons is even larger than the sample size. Therefore, it falls under the category of coherent ballistic transport.

Branched flow was first observed in an experiment by Prof. Westervelt\cite{3, 4} where he measured the two dimensional electron transport pattern through a random potential in a GaAs heterostructure. The system is cooled down to liquid Helium temperature (4.2K) to ensure coherent transport. In the experiment, the electrons are confined to move only in two dimensions due to a lateral confinement in the third dimension. The randomly placed donor atoms and impurities generate a weakly correlated random potential that can scatter the electrons. The potential due to the donor atoms is usually weak compared to the energy of the electrons, leading only to deflection of the electron trajectories. The impurities, on the other hand, can contribute to a strong potential that results in backscattering. Overall, the random potential is still weak and has a standard deviation of only 8\% the energy of the electrons. Thus, the electron can in theory fly through the whole sample without being deflected much.

In the experiment, these two dimensional electrons are launched into the random potential through a narrow injector called the Quantum Point Contact(QPC). The QPC can be thought as an electron waveguide with only one or two open modes. The electron wavelength in the
Figure 1. Experimentally observed branched flow. This figure is taken from Ref. 2 with copyright permissions from Nature Publishing Group. In this figure, we see the experimentally observed branched flow pattern. The two patterns in (b) correspond to two different random potential. This figure is obtained with the copyright permission from Nature Publishing Group.

experiment is 37nm, and the narrowest opening in the QPC is on the same order of magnitude. Semiclassically, one can imagine the effect of this QPC as aligning the initial momenta of the electrons in one direction while confining their positions down to nanoscales in the other direction[5]. This effectively creates an initial condition resembling that from a “point” injector. These highly aligned electrons are then launched into the random potential where they are being scattered. Magnified by the chaotic classical dynamics, the tiny differences in initial conditions lead to different classical electron trajectories, the overall pattern of which is called the branched flow.

As the name suggests, the electrons move along a small number of narrow branches in the random potential. Fig. 1 shows the experimentally measured electron flow pattern that displays the branching behavior and this figure is taken from Ref.[4] with copyright permissions from Nature Publishing Group. This branched flow pattern is measured by a technique called the Scanning Gate Microscopy(SGM). In the SGM, a negatively charged metallic tip is held on top of the sample, generating a potential barrier that can backscatter the electrons impinging on it. These backscattered electron flux leads to a reduction in the measured conductance, which is measured as a function of the position of the metallic tip. If the tip is placed in a region with high electron flux, a larger reduction in conductance will be recorded, while a smaller change will be recorded if the tip is placed in a region with small electron flux. By moving the tip across the surface of the sample and recording the changes in conductance, it produces a conductance map that reveals information of the electron flux pattern in the random potential.
What makes branched flow particularly interesting is the contrast between the relatively weak random potential and the significant effect on the overall electron flow pattern. As mentioned before, the random potential has a mean energy of zero and standard deviation of 8% the energy of the electrons. Moreover, the sample size of the system is smaller than the classical mean free path of the electrons. If one thinks in terms of perturbation theory, it would not be expected that the random potential will be able to cause branching. Moreover, the classical dynamics of electrons moving a random potential is chaotic. Naively, chaos should separate electron trajectories with small differences in initial conditions exponentially fast, leading to a more uniform flow pattern rather than a branched flow pattern.

After its initial discovery in two dimensional electron gases (2DEGs), branched flow has later been observed and studied in other systems with much different length scales. For 2DEGs, the wavelength of the electrons is on the scale of nanometers[3, 4]. On the micrometer scales, branched flow has been experimentally observed in quasi-two-dimensional microwave resonators[6]. Branched flow is also used to study sound propagation in oceans with megameter length scales[7] and is found to have implications for the formation of freak waves in oceans[8].

2. Classical Stability of Branched Flow: ”Unexpected Features”

The formation of branched flow has long been explained using classical theory. In Ref.[4], the authors reported a numerical simulation based on classical electron trajectories that reproduces many of the key features of branched flow. Moreover, with a clever choice of the initial conditions, the classical simulation closely resembles the result of a quantum simulation on the same random potential.

According to the classical theory, branched flow can form due to two mechanisms. One mechanism is by forming caustics in phase space. Caustics forms after electrons pass through a potential dip, which focuses parallel electron trajectories. Immediately after the focal point, electrons travel along caustics with high flux density. One theory based on this caustics idea is called the ”kick and drift” model[9]. In this model, the random potential is modeled as a series of potential dips that focus electron trajectories. As the electrons move across the random potential, a series of random caustics form and they are where branches appear. Detailed statistical analysis of the distribution of these caustics has also been reported[10, 11].

Another mechanism that could lead to branched flow is called stable regions in the phase space[9, 12]. Similarly to caustics, stable regions in phase space form by chance. Qualitatively, stable regions in phase space refer to regions in phase space that is less sensitive to changes in initial conditions. One way to understand those stable regions is like this: one fundamental property of any phase space manifold is that the total electron count can not change over time. In the case of branched flow, the total area being occupied by the electrons in the phase space can not expand exponentially over time since the sample size is smaller than the mean free path. As a result, when certain region’s electron density is diminished, this decrease in electron density has to be compensated with a higher electron density somewhere else and these more densely populated regions are stable regions. Quantitatively, a rarefaction exponent[9, 13], which is a variant of the Lyapunov exponent, is developed to characterize those stable regions. More detailed account of these two mechanisms can be found in Ref.[12, 13, 14].

One consequence of this classical theory is that the dynamics becomes chaotic, meaning that each individual classical trajectory is exponentially unstable to perturbations. This chaotic nature of the classical dynamics motivated a new experiment to test the stability of branched flow[15]. The setup of this new experiment is almost identical to the setup used to observe branched flow in the first set of experiments[3, 4]. The only difference is that the branched flow patterns are observed for two sets of initial conditions. This difference in initial conditions is achieved by shifting the position of the QPC, the electron injector. In the experiment, the authors first measure the branched flow for a fixed QPC position. After the branched flow
pattern has been mapped out, the QPC is shifted away from the previous position by about 60 nanometers in the direction of confinement and the same procedure of measurement of the branched flow pattern is repeated with the new shifted QPC position over the same random potential. This shifting of the QPC position corresponds to a change in the initial conditions of the electrons. Moreover, this 60nm change in initial conditions is actually very significant in terms of both the classical dynamics and the quantum dynamics. Classically, this 60nm change in initial conditions is comparable to the correlation length of the random potential, which measures how fast the self-correlation of the random potential decays over distance. Quantum mechanically, this 60nm change is also comparable to the narrowest opening in the QPC. Thus, if one imagines channeling two wavepackets through two different QPCs separated by 60nm, the coherent overlap between the two wavepackets is estimated to be less than five percent[15].

According to the classical theory, branched flow is the overall pattern of individually chaotic classical trajectories moving through a random potential. Since each classical trajectory is chaotic and exponentially unstable to perturbation, it seems reasonable to assume that this large change in initial conditions due to the shifting of the QPC will result in a significant change in the measured branched flow pattern. However, the experimental results display an unexpected long-range stability of the branched flow. It was observed that about seventy correlation lengths away from the injection points, the electrons move along almost identical branches for the two sets of initial conditions. This recovery of branches was named "unexpected features of branched flow" and was suspected to be of pure quantum origin without any classical analogy. However, as we explained in Ref.[5], this long range stability of branched flow is actually of classical origin and we reported in that paper classical simulations based on chaotic classical trajectories that reproduce the long range stability. Our theory shows that this long range stability of branched flow derives its origin from the QPC's ability to correlate classical trajectories. We showed in Ref.[5] that even if one starts with two initial conditions that have zero overlap to begin with in the phase space, the QPCs can boost their overlap to almost eighty percent for the experimental conditions. The key point to notice is that shifting the QPC corresponds to a change in the Hamiltonian of the physical system. Even though the overlap between any two manifolds in the phase space is guaranteed to be preserved over time if they are propagated according to the same Hamiltonian, there is no mathematical constraint on how the overlap will evolve if the Hamiltonians are different. In this particular example, we have shown that the overlap in phase space is monotonically increased as the electrons move through the QPCs. Thus, this unexpected feature of branched flow has nothing to do with the chaotic nature of the classical dynamics of electrons moving in a random potential and is completely generated by the ability of the QPC to produce overlap in the phase space.

3. Quantum Stability of Branched Flow

In Ref.[5], we reported additional stability of branched flow against the change in the open modes of QPC. We showed that the branched flow pattern for the first open mode in the QPC bears close similarity to the pattern for the second mode. In terms of quantum mechanics, the two modes are orthogonal to each other and thus the overlap is zero to start with. Moreover, the two modes are propagated under the same Hamiltonian, so even the QPC could not boost their coherent overlap. We gave a qualitative explanation of this additional stability in terms of the Wigner quasiprobability distribution[16] and attributed its formation to overlap in the classical phase space.

We have also studied the stability of branched flow against energy changes in Ref.[12] and have shown how this additional stability is also compatible with the classical interpretation of branched flow. In Ref.[12], we presented numerical simulations showing that the classical branched flow pattern is stable against a 22% change in the electron's energy and argued how this stability is not due to quantum interference, which is destroyed by thermal averaging. In
this section, we report one more stability of branched flow that we believe is of quantum origin. More detailed quantitative analysis of its origin will be left for a later publication[17] and we will present only some preliminary results here.

In this case, we are interested in the stability of branched flow against strong perturbations without moving the QPC. We numerically simulate both the quantum flow patterns and the classical branched flow patterns after an area of strong perturbations is introduced into the systems. Our numerical methods are primarily based on the thermal wavepacket approach developed in Ref.[18] and more detailed information about its implementation can be found in Ref.[5].

The results are shown in Fig.2. All parameters(Fermi energy $E_F$, wavelength $\lambda_F$, donor to two-dimensional electron gases(2DEGs) distance, sample mobility etc.) are chosen to match that in a previous experiment[15]. The random potential has a standard deviation of 8%$E_F$ and correlation length $0.9\lambda_F$, as estimated in that experiment. The perturbation in this case is introduced by imposing a region of random potential on top of the original random potential. This region of strong perturbations is introduced at $25\lambda_F$ from the injection point and lasts for $10\lambda_F$ as shown in Fig.2. In this region, a random potential of twice the original standard deviation(16%$E_F$) and the same correlation length is superimposed on the original random potential. This perturbation is classically strong enough such that it destroys all the classical branches in the long range(long in terms of the correlation length of the random potential), as shown in Fig.2c and Fig.2d. However, for the quantum simulations in Fig.2a and Fig.2b, the branches are only distorted inside and near the region of perturbations and recover themselves in the long range. It is our belief that this recovery of branches in the long range for the quantum dynamics is closely tied to the piecewise nature of interference in chaotic systems, which is a key difference between the quantum and classical dynamics in chaotic systems[17].

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