FIRST EIGENVALUE ESTIMATES OF 
DIRICHLET-TO-NEUMANN OPERATORS ON GRAPHS

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Abstract. Following Escobar [Esc97] and Jammes [Jam15], we introduce two types of isoperimetric constants and give lower bound estimates for the first nontrivial eigenvalues of Dirichlet-to-Neumann operators on finite graphs with boundary respectively.

1. Introduction

Let \((M, g)\) be a compact manifold with boundary \(\partial M\). The Dirichlet-to-Neumann operator \(\Lambda : H^{\frac{1}{2}}(\partial M) \to H^{-\frac{1}{2}}(\partial M)\) is defined as
\[
\Lambda(f) = \frac{\partial u_f}{\partial n},
\]
where \(u_f\) is the harmonic extension of \(f \in H^{\frac{1}{2}}(\partial M)\). The Dirichlet-to-Neumann operator is a first order elliptic pseudo-differential operator [Tay96, page 37] and its associated eigenvalue problem is also known as the Steklov problem, see [KKK+14] for a historical discussion. Since \(\partial M\) is compact, the spectrum of \(\Lambda\) is nonnegative, discrete and unbounded [Ban80, page 95].

The Dirichlet-to-Neumann operator is closely related to the Calderón problem [Cal80] of determining the anisotropic conductivity of a body from current and voltage measurements at its boundary. This makes it useful for applications to electrical impedance tomography, which is used in medical and geophysical imaging, see [Uhl14] for a recent survey.

Eigenvalue estimates are of interest in spectral geometry. In [Che70], Cheeger discovered a close relation between the geometric quantity, the isoperimetric constant (also called the Cheeger constant), and the analytic quantity, the first nontrivial eigenvalue of the Laplace-Beltrami operator on a closed manifold. Estimate of this type is called the Cheeger estimate.

For the first nontrivial eigenvalue of the Dirichlet-to-Neumann operator, two different types of lower bound estimates, which are similar to the classical Cheeger estimate, have been obtained by Escobar and Jammes respectively in [Esc97] and [Jam15]. For convenience, we call them the Escobar-type Cheeger estimate and the Jammes-type Cheeger estimate.

The Cheeger constant introduced by Escobar [Esc97], which we call the Escobar-type Cheeger constant, is defined as

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\[
h_E(M) := \inf_{\text{Area}(\Omega \cap \partial M) \leq \frac{1}{2}\text{Area}(\partial M)} \frac{\text{Area}(\Omega \cap \text{int}(M))}{\text{Area}(\Omega \cap \partial M)},
\]

where Area(\cdot) denotes the codimensional one Hausdorff measure, i.e. the Riemannian area, and int(M) denotes the interior of the manifold M. Let \(\sigma_1\) be the first nontrivial eigenvalue of the Dirichlet-to-Neumann operator \(\Lambda\), then the Escobar-type Cheeger estimate [Esc97, Theorem 10] reads as

\[
\sigma_1 \geq \frac{(h_E(M)\mu_1(k) - ak)a}{a^2 + \mu_1(k)},
\]

where \(a > 0, k > 0\) are arbitrary positive constants, and \(\mu_1(k)\) is the first eigenvalue of the following Robin problem

\[
\begin{align*}
\Delta u + \mu_1(k)u &= 0, & \text{on } M, \\
\frac{\partial u}{\partial n} + ku &= 0, & \text{on } \partial M.
\end{align*}
\]

The Jammes-type Cheeger constant, which was introduced in [Jam15], is defined as

\[
h_J(M) = \inf_{\text{Vol}(\Omega) \leq \frac{1}{2}\text{Vol}(M)} \frac{\text{Area}(\partial \Omega \cap \text{int}(M))}{\text{Vol}(\Omega)},
\]

where Vol(\cdot) denotes the Riemannian volume. The Jammes-type Cheeger estimate [Jam15, Theorem 1] is given as follows

\[
\sigma_1 \geq \frac{1}{4}h(M)h_J(M),
\]

where \(h(M)\) is the classical Cheeger constant associated to the Laplacian operator with Neumann boundary condition on M, which is defined as

\[
h(M) = \inf_{\text{Vol}(\Omega) \leq \frac{1}{2}\text{Vol}(M)} \frac{\text{Area}(\partial \Omega \cap \text{int}(M))}{\text{Vol}(\Omega)}.
\]

The Dirichlet-to-Neumann operator is naturally defined in the discrete setting. We recall some basic definitions on graphs. Let \(V\) be a countable set which serves as the set of vertices of a graph and \(\mu\) be a symmetric weight function given by

\[
\mu : V \times V \rightarrow [0, \infty),
\]

\[(x, y) \mapsto \mu_{xy} = \mu_{yx}.
\]

This induces a graph structure \(G = (V, E)\) with the set of vertices \(V\) and the set of edges \(E\) which is defined as \(\{x, y\} \in E\) if and only if \(\mu_{xy} > 0\), in symbols \(x \sim y\). Note that we do allow self-loops in the graph, i.e. \(x \sim x\) if \(\mu_{xx} > 0\). Given \(\Omega_1, \Omega_2 \subset V\), the set of edges between \(\Omega_1\) and \(\Omega_2\) is denoted by

\[
E(\Omega_1, \Omega_2) := \{e = \{x, y\} \in E| x \in \Omega_1, y \in \Omega_2\}.
\]

For any subset \(\Omega \subset V\), there are two notions of boundary, i.e. the edge boundary and the vertex boundary. The edge boundary of \(\Omega\), denoted by \(\partial \Omega\), is defined as

\[
\partial \Omega := E(\Omega, \Omega^c),
\]
where $\Omega^c := V \setminus \Omega$. The vertex boundary of $\Omega$, denoted by $\delta \Omega$, is defined as

$$\delta \Omega := \{ x \in V \setminus \Omega | \ x \sim y \text{ for some } y \in \Omega \}.$$ 

Set $\overline{\Omega} := \Omega \cup \delta \Omega$. We introduce a measure on $\overline{\Omega}$, $m : \overline{\Omega} \to (0, \infty)$, as follows

$$m_x = \begin{cases} \sum_{y \in V, y \sim x} \mu_{xy}, & x \in \Omega, \\ \sum_{y \in \delta \Omega, y \sim x} \mu_{xy}, & x \in \delta \Omega. \end{cases}$$

Accordingly, $m(A) := \sum_{x \in A} m_x$ denotes the measure of any subset $A \subset \overline{\Omega}$. Given any set $F$, we denote by $\mathbb{R}^F$ the collection of all real functions defined on $F$.

For any finite subset $\Omega \subset V$, in analogy to the Riemannian case, one can define the Dirichlet-to-Neumann operator in the discrete setting to be

$$\Lambda : \mathbb{R}^\delta \Omega \to \mathbb{R}^\delta \Omega,$$

$$\varphi \mapsto \Lambda \varphi := \frac{\partial u_\varphi}{\partial n},$$

where $u_\varphi$ is the harmonic extension of $\varphi$ to $\Omega$, and $\frac{\partial}{\partial n}$ is the outward normal derivative in the discrete setting defined as in (2.1) in section 2. We call $\Lambda$ the DtN operator for short. Let $\sigma(\Lambda)$ denote the spectrum of $\Lambda$. By the definition of $\Lambda$ and Green’s formula, see Lemma 2.1, $\Lambda$ is a nonnegative self-adjoint operator, i.e. $\sigma(\Lambda)$ is a set of nonnegative real numbers. In fact, $\sigma(\Lambda)$ is contained in $[0, 1]$, see Proposition 3.1.

It is well known that the eigenvalues of (normalized) Laplace operators on finite graphs without any boundary condition are contained in $[0, 2]$. The multiplicity of eigenvalue 0 is equal to the number of connected components of the graph, see [BH12, Prop.1.3.7]. The largest possible eigenvalue 2 is achieved if and only if the graph is bipartite, see [Gri09, Theorem 2.3]. Similar results can be generalized to the case of DtN operators. For convenience, let $\overline{\Omega}$ denote the graph with vertices in $\overline{\Omega}$ and edges in $E(\Omega, \overline{\Omega})$, i.e.

\begin{equation}
(1.1) \quad \overline{\Omega} := (\overline{\Omega}, E(\Omega, \overline{\Omega})).
\end{equation}

From Proposition 3.2 we know that the multiplicity of eigenvalue 0 of $\Lambda$ is equal to the number of connected components of $\overline{\Omega}$. Moreover, we show that the eigenspace associated to the eigenvalue 1 which is the largest possible eigenvalue is the kernel of a linear operator $Q$ whose definition is given in (3.2) below.

Given $A \subset \overline{\Omega}$, we denote by

$$\partial \Omega A := \partial A \cap E(\Omega, \overline{\Omega})$$

the relative boundary and $A^\vee := \overline{\Omega} \setminus A$ the relative complement of $A$ in $\overline{\Omega}$. Without loss of generality, we always assume that $\overline{\Omega}$ is connected.

Now we consider the first nontrivial eigenvalue estimates of DtN operators on finite graphs. We introduce two isoperimetric constants following Escobar and Jammes, which we call Escobar-type Cheeger constant and Jammes-type Cheeger constant respectively.
Definition 1.1. The Escobar-type Cheeger constant for \( \tilde{Ω} \) is defined as

\[
h_E(\tilde{Ω}) := \inf_{m(A \cap \delta\Omega) \leq \frac{1}{2} m(\Omega)} \frac{\mu(\partial\Omega \setminus A)}{m(A \cap \delta\Omega)}.
\]

The Jammes-type Cheeger constant for \( \tilde{Ω} \) is defined as

\[
h_J(\tilde{Ω}) := \inf_{m(A) \leq \frac{1}{2} m(\Omega)} \frac{\mu(\partial\Omega \setminus A)}{m(A \cap \delta\Omega)}.
\]

By definitions, one has \( h_J(\tilde{Ω}) \leq h_E(\tilde{Ω}) \), see Proposition 4.1. This estimate is optimal, see Example 4.1 in the paper.

To derive the Cheeger estimates, we first show that \( h_E(\tilde{Ω}) \) is equal to a type of Sobolev constant. Similar results can be found in both Riemannian and discrete case, see e.g. [Li12, Cha16, Chu97].

Theorem 1.1. Let \( G \) be a finite graph and \( \Omega \subset V \), then

\[
h_E(\tilde{Ω}) = \inf_{f \in \mathbb{R}^Ω} \frac{\sum_{e=\{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}|f(x) - f(y)|}{\inf_{a \in \mathbb{R}} \sum_{x \in \delta\Omega} m_x|f(x) - a|}.
\]

Without loss of generality, we assume that the number of vertices in \( \delta\Omega \) is at least two. From now on, we denote by \( \lambda_1(\Omega) \) the first nontrivial eigenvalue of the DtN operator \( \Lambda \) on \( \Omega \) in a finite graph. In the discrete setting, it’s easy to obtain an upper bound estimate as

\[
\lambda_1(\Omega) \leq 2 h_E(\tilde{Ω}),
\]

see Proposition 4.2. The sharpness of this upper bound can be seen from Example 4.2. For the lower bound estimate, we obtain the Escobar-type Cheeger estimate as follows.

Theorem 1.2. Let \( G \) be a finite graph and \( \Omega \subset V \), then

\[
\lambda_1(\Omega) \geq \frac{\left(2h_E(\tilde{Ω})\mu_1(k) - a(k + \mu_1(k))\right)a}{a^2 + 2\mu_1(k)},
\]

where \( a > 0, k > 0 \) are arbitrary positive constants, and \( \mu_1(k) \) is the first eigenvalue of the Robin problem

\[
\begin{align*}
\Delta u + \mu_1(k)u &= 0, & \text{on } \Omega, \\
\frac{\partial u}{\partial n} + ku &= 0, & \text{on } \partial\Omega.
\end{align*}
\]

Analogous to the Riemannian case, we obtain Jammes-type Cheeger estimate following [Jam15].

Theorem 1.3. Let \( G \) be a finite graph and \( \Omega \subset V \), then

\[
\lambda_1(\Omega) \geq \frac{1}{2} h(\tilde{Ω}) h_J(\tilde{Ω}),
\]

where \( h(\tilde{Ω}) \) is the classical Cheeger constant of the graph \( \tilde{Ω} \) viewed as a graph without any boundary condition.
Remark 1.1. The Jammes-type Cheeger estimate is asymptotically sharp, see Example 5.1.

For the completeness, we recall the definition of $h(\Omega)$,
\[ h(\Omega) := \inf_{m(A) \leq \frac{1}{2} m(\Omega)} \frac{\mu(\partial \Omega A)}{m(A)}, \]
where $A$ is any nonempty subset of $\Omega$, see e.g. [Chu97, p.24]. By the definitions of $h(\Omega)$ and $h_J(\Omega)$, one is ready to see that $h(\Omega) \leq h_J(\Omega)$. Notably, the classical Cheeger estimate uses only one Cheeger constant while the Jammes-type Cheeger estimate involves two. One may ask whether $\lambda_1(\Omega)$ can be bounded from below using only $h_J(\Omega)$. The answer is negative and we give a counterexample in Example 5.1.

As a corollary of Theorem 1.3, we have the following interesting eigenvalue estimate, which has no counterpart in the Riemannian setting.

Corollary 1.1.
\[ \lambda_1(\Omega) \geq \frac{1}{8} \left( \zeta_1(\Omega) \right)^2, \]
where $\zeta_1(\Omega)$ is the first nontrivial eigenvalue of the Laplace operator with no boundary condition on $\Omega$.

The organization of the paper is as follows: In Section 2, we collect some basic facts about the DtN operators on graphs. In Section 3, we study the spectrum of the DtN operators. In Section 4 and Section 5, we give the proof of the main theorems: Theorem 1.2 and 1.3.

2. Preliminaries

Let $(X, \nu)$ be a discrete measure space, i.e. $X$ is a discrete space equipped with a Borel measure $\nu$. The spaces of $\ell^p$, $p \in [1, \infty]$, summable functions on $(X, \nu)$, are defined routinely: Given a function $f \in \mathbb{R}^X$, for $p \in [1, \infty)$, we denote by
\[ \|f\|_{\ell^p} = \left( \sum_{x \in X} |f(x)|^p \nu(x) \right)^{1/p}, \]
the $\ell^p$ norm of $f$. For $p = \infty$,
\[ \|f\|_{\ell^\infty} = \sup_{x \in X} |f(x)|. \]
Let $\ell^p(X, \nu) := \{ f \in \mathbb{R}^X \|f\|_{\ell^p} < \infty \}$ be the space of $\ell^p$ summable functions on $(X, \nu)$. In our setting, these definitions apply to $(X, \nu) = (\Omega, m)$ or $(\delta\Omega, m)$. The case where $p = 2$ is of particular interest, as we have the Hilbert spaces $\ell^2(\Omega, m) := \{ f \in \mathbb{R}^\Omega \|f\|_{\ell^2} < \infty \}$ and $\ell^2(\delta\Omega, m) := \{ \varphi \in$
\( \mathbb{R}^{\delta\Omega} \) equipped with the standard inner products
\[
\langle f, g \rangle_{\Omega} = \sum_{x \in \Omega} f(x)g(x)m_x, \quad f, g : \Omega \to \mathbb{R},
\]
\[
\langle \varphi, \psi \rangle_{\delta\Omega} = \sum_{x \in \delta\Omega} \varphi(x)\psi(x)m_x, \quad \varphi, \psi : \delta\Omega \to \mathbb{R}.
\]

Given \( \Omega \subset V \), an associated quadratic form is defined as
\[
D_{\Omega}(f, g) = \sum_{e=\{x:y\} \in E(\Omega,\overline{\Omega})} \mu_{xy}(f(x) - f(y))(g(x) - g(y)), \quad f, g \in \mathbb{R}^{\Omega}.
\]
The Dirichlet energy of \( f \in \mathbb{R}^{\Omega} \) can be written as
\[
D_{\Omega}(f) := D_{\Omega}(f, f).
\]
For any \( f \in \mathbb{R}^{\Omega} \), the Laplacian operator is defined as
\[
\Delta f(x) := \frac{1}{m_x} \sum_{y \in V : y \sim x} \mu_{xy}(f(y) - f(x)), \quad x \in \Omega
\]
and the outward normal derivative of \( f \) is defined as
\[
\frac{\partial f}{\partial n}(z) := \frac{1}{m_z} \sum_{x \in \Omega : x \sim z} \mu_{zx}(f(z) - f(x)), \quad z \in \delta\Omega.
\]

We recall the following two well-known results on Laplace operators.

**Lemma 2.1.** (Green’s formula) Let \( f, g \in \mathbb{R}^{\overline{\Omega}} \). Then
\[
\langle \Delta f, g \rangle_{\Omega} = -D_{\Omega}(f, g) + \langle \frac{\partial f}{\partial n}, g \rangle_{\delta\Omega}.
\]

**Lemma 2.2.** Given any \( \varphi \in \mathbb{R}^{\delta\Omega} \), there is a unique function \( u_{\varphi} \in \mathbb{R}^{\overline{\Omega}} \) satisfying the Laplace equation
\[
\Delta u_{\varphi}(x) = 0, \quad x \in \Omega,
\]
and the boundary condition
\[
u_{\varphi}(x) = \varphi(x), \quad x \in \delta\Omega.
\]

**Remark 2.1.** We will denote by \( u_{\varphi} \) the harmonic extension of \( \varphi \in \mathbb{R}^{\delta\Omega} \) to \( \overline{\Omega} \) in this paper. For the proof of Lemma 2.1 and 2.2 one can see e.g. [Gri09].

**Proposition 2.1.** \( \Lambda \) is a nonnegative self-adjoint linear operator on \( \ell^2(\delta\Omega, m) \).

**Proof.** For any \( f, g \in \ell^2(\delta\Omega) \), by Green’s formula, we have
\[
0 = \langle \Delta f, u_g \rangle_{\Omega} = -D_{\Omega}(u_f, u_g) + \langle \frac{\partial f}{\partial n}, g \rangle_{\delta\Omega},
\]
\[
0 = \langle u_f, \Delta u_g \rangle_{\Omega} = -D_{\Omega}(u_f, u_g) + \langle f, \frac{\partial g}{\partial n} \rangle_{\delta\Omega}.
\]
Hence
\[
\langle \Lambda f, g \rangle = \langle f, \Lambda g \rangle.
\]
In particular,

\[ 0 = \langle \Delta u_f, u_f \rangle_{\Omega} = -D_{\Omega}(u_f, u_f) + \langle \frac{\partial f}{\partial n}, f \rangle_{\delta \Omega}, \]

then

\[ \langle \Lambda f, f \rangle_{\delta \Omega} = D_{\Omega}(u_f, u_f) \geq 0. \]

So we complete the proof. \(\square\)

For any \(y \in \delta \Omega\), let \(\delta_y(z)\) denote the delta function at \(y\), i.e. \(\delta_y(y) = 1\) and \(\delta_y(z) = 0\) for any \(z \in \delta \Omega, z \neq y\). We denote by \(P(\cdot, y)\) the solution of equation \([2.3]\) with Dirichlet boundary condition \(\frac{1}{m_y} \delta_y\). The family \(\{P(\cdot, y)\}_{y \in \delta \Omega}\) are called Poission kernels associated to the Dirchlet boundary conditions on \(\Omega\). They have interesting probabilistic explanations using simple random walks, see e.g. [Law10, p.25] and [LL10, chapter 8]. Using Poission kernels, one can regard the DtN operator on \(\Omega\) as a Laplace operator defined on a graph with the set of vertices \(\delta \Omega\) and modified edges.

**Proposition 2.2.** The DtN operator can be written as

\[ \Lambda \varphi(x) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \tilde{\mu}_{xy}(\varphi(x) - \varphi(y)), \]

where \(\tilde{\mu}_{xy} = \sum_{z \in \Omega} \mu_{xz} P(z, y) m_y\) and \(\sum_{y \in \delta \Omega} \tilde{\mu}_{xy} = m_x\).

**Proof.** For any given \(\varphi \in \mathbb{R}^{\delta \Omega}\), we have

\[ \varphi(x) = \sum_{y \in \delta \Omega} \varphi(y) m_y \cdot \frac{\delta_y(x)}{m_y}. \]

By the linearity of equation \([2.3]\), we have

\[ u_{\varphi}(x) = \sum_{y \in \delta \Omega} \varphi(y) m_y P(x, y). \]

Hence by the definition of \(\Lambda\),

\[
\Lambda \varphi(x) = \frac{1}{m_x} \sum_{z \in \Omega} \mu_{xz} \left( \varphi(x) - \sum_{y \in \delta \Omega} \varphi(y) m_y P(z, y) \right) \\
= \varphi(x) - \frac{1}{m_x} \sum_{y \in \delta \Omega} \left( \sum_{z \in \Omega} \mu_{xz} P(z, y) m_y \right) \varphi(y) \\
= \varphi(x) - \frac{1}{m_x} \sum_{y \in \delta \Omega} \tilde{\mu}_{xy} \varphi(y).
\]
Notice that \( u_\varphi(x) \equiv 1 \), if we choose \( \varphi(x) = 1, \ x \in \Omega \). Hence combining with (2.4), we have

\[
\sum_{y \in \delta \Omega} \tilde{\mu}_{xy} = \sum_{z \in \Omega} \mu_{xz} \left( \sum_{y \in \delta \Omega} P(z, y) m_y \right) = \sum_{z \in \Omega} \mu_{xz} \cdot 1 = m_x.
\]

Then the proposition follows. \( \square \)

From Proposition 2.2, the DtN operator \( \Lambda \) can be written in a matrix form. We define matrices \( D_{\delta \Omega}, A_{\delta \Omega \times \Omega}, P_{\Omega \times \delta \Omega} \) as

\[
(D_{\delta \Omega})_{xy} = m_x \delta_z(y), \quad x, y \in \delta \Omega,
\]

\[
(A_{\delta \Omega \times \Omega})_{xz} = \mu_{xz}, \quad x \in \delta \Omega, z \in \Omega,
\]

\[
(P_{\Omega \times \delta \Omega})_{zx} = P(z, x), \quad x \in \delta \Omega, z \in \Omega.
\]

Then we have

**Corollary 2.1.** The DtN operator \( \Lambda \) can be written as

\[
\Lambda = I - D^{-1} APD,
\]

where \( I \) is the identity map.

### 3. Spectrum of the DtN Operator

Given \( \varphi \in \mathbb{R}^{\delta \Omega} \), for simplicity we denote by \( \bar{\varphi} \) the null-extension of \( \varphi \) to \( \Omega \), which is defined as

\[
\bar{\varphi}(x) = \begin{cases} 
0, & x \in \Omega, \\
\varphi(x), & x \in \delta \Omega.
\end{cases}
\]

For any \( p \in [1, \infty] \), the \( \ell^p-\ell^p \) norm of the operator \( \Lambda \) is defined as

\[
\|\Lambda\|_{p,p} := \sup_{\varphi \in \mathbb{R}^{\delta \Omega}, \|\varphi\|_{\ell^p} = 1} \|\Lambda \varphi\|_{\ell^p}.
\]

**Proposition 3.1.** The \( \ell^2-\ell^2 \) and \( \ell^\infty-\ell^\infty \) norm of the operator \( \Lambda \) are bounded, in particular

\[
\|\Lambda\|_{2,2} \leq 1, \quad \|\Lambda\|_{\infty,\infty} \leq 2.
\]
Proof. Let \( \varphi \in \mathbb{R}^{\delta \Omega} \). According to Hölder’s inequality,

\[
\| \Lambda \varphi \|_{\ell^2(\delta \Omega, m)}^2 = \sum_{x \in \delta \Omega} m_x \left| \sum_{y \in \Omega} \frac{\mu_{xy}}{m_x} (\varphi(x) - u_\varphi(y)) \right|^2 \\
\leq \sum_{x \in \delta \Omega} \sum_{y \in \Omega} \mu_{xy} |\varphi(x) - u_\varphi(y)|^2 \leq D_\Omega(u_\varphi) \\
\leq D_\Omega(\varphi) = \sum_{x \in \delta \Omega, y \in \Omega} \mu_{xy} \varphi^2(x) \\
= \| \varphi \|_{\ell^2(\delta \Omega, m)}^2,
\]

where (3.1) follows from the fact that harmonic functions minimize the Dirichlet energy in the class of functions with the same boundary condition. This proves the \( \ell^2-\ell^2 \) norm bound.

For the \( \ell^\infty-\ell^\infty \) norm bound, we have

\[
\| \Lambda \varphi \|_\infty = \sup_{x \in \delta \Omega} \left| \frac{1}{m_x} \sum_{y \in \Omega} \mu_{xy} (\varphi(x) - u_\varphi(y)) \right| \\
\leq \sup_{x \in \delta \Omega} \frac{1}{m_x} \sum_{y \in \Omega} \mu_{xy} |(\varphi(x) - u_\varphi(y))| \\
\leq 2 \| \varphi \|_\infty.
\]

Remark 3.1. From Proposition 3.1, we have \( \sigma(\Lambda) \subset [0, 1] \).

By interpolation, we have

**Corollary 3.1.** \( \| \Lambda \|_{p, p} \leq 2^{1-\theta} \), where \( \theta \in [0, 1) \) and \( p = \frac{2}{\theta} \in [2, \infty] \).

**Proposition 3.2.** The multiplicity of the eigenvalue 0 of \( \Lambda \), i.e. \( \dim \ker \Lambda \), is equal to the number of connected components of the graph \( \tilde{\Omega} \).

Proof. Let \( \varphi \in \mathbb{R}^{\delta \Omega} \) be an eigenfunction associated to eigenvalue 0 of \( \Lambda \), i.e. \( \Lambda \varphi = 0 \). By Green’s formula we have

\[
0 = \langle \Delta u_\varphi, u_\varphi \rangle \Omega = -D_\Omega(u_\varphi) + \langle \Lambda \varphi, \varphi \rangle \delta \Omega = -D_\Omega(u_\varphi).
\]

Hence \( u_\varphi \) is constant on each connected component of \( \tilde{\Omega} \). \( \square \)

From now on, we always assume that the graph \( \tilde{\Omega} \) is connected and \( \Lambda \) is an operator from \( \ell^2(\delta \Omega, m) \) to \( \ell^2(\delta \Omega, m) \). Let

\[
E_1(\Lambda) := \{ \varphi \in \mathbb{R}^{\delta \Omega} \mid \Lambda \varphi = \varphi \}
\]

be the space of eigenvectors associated to the eigenvalue 1 which might be empty. Set \( \delta_1 \Omega := \{ x \in \Omega \mid x \sim y \text{ for some } y \in \delta \Omega \} \). For any \( \varphi \in \mathbb{R}^{\delta \Omega} \), we
introduce a linear operator \( Q : \mathbb{R}^{\delta \Omega} \rightarrow \mathbb{R}^{\delta I \Omega} \), which is defined as
\[
Q \varphi(x) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \mu_{xy} \varphi(y), \quad x \in \delta I \Omega.
\]
(3.2)

Let \( \#(\delta \Omega) \) (\( \#(\delta I \Omega) \) resp.) denote the number of vertices in \( \delta \Omega \) (\( \delta I \Omega \) resp.). We order the eigenvalues of the DtN operator \( \Lambda \) in the nondecreasing way:
\[
0 = \lambda_0(\Omega) < \lambda_1(\Omega) \leq \cdots \leq \lambda_{N-1}(\Omega) \leq 1,
\]
where \( N = \#(\delta \Omega) \).
We obtain some characterisations of \( E_1(\Lambda) \) in the following proposition.

**Proposition 3.3.** (1) For \( \varphi \in \mathbb{R}^{\delta \Omega} \), \( \varphi \in E_1(\Lambda) \) if and only if
\[
u \varphi = \varphi.
\]
(2) \( E_1(\Lambda) = \text{Ker} Q \).

**Proof.** (1) For the ”if ” part,
\[
\Delta \varphi(x) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \mu_{xy} (\varphi(y) - \varphi(x)) = \varphi(x).
\]

For the ”only if ” part, if \( \varphi \in E_1(\Lambda) \), then all the inequalities in (3.1) are equalities. Hence \( D_{\Omega}(u_{\varphi}) = D_{\Omega}(\varphi) \). This implies that \( \varphi \) attains the minimal Dirichlet energy with fixed boundary condition, i.e. \( \varphi \) is harmonic. By the uniqueness of harmonic functions with fixed boundary condition, we have \( \varphi = u_{\varphi} \).

(2) If \( \varphi \in E_1(\Lambda) \), then \( u_{\varphi} = \varphi \). For any \( x \in \delta I \Omega \),
\[
Q \varphi(x) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \mu_{xy} \varphi(y) \\
= \frac{1}{m_x} \left( \sum_{y \in \delta \Omega} \mu_{xy} (\varphi(y) - \varphi(x)) + \sum_{y \in \Omega} \mu_{xy} (\varphi(y) - \varphi(x)) \right) \\
= \Delta u_{\varphi}(x) = 0.
\]

Hence \( E_1(\Lambda) \subset \text{Ker} Q \). On the other hand, If \( \varphi \in \text{Ker} Q \), then for any \( x \in \delta I \Omega \),
\[
\Delta \varphi(x) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \mu_{xy} (\varphi(y) - \varphi(x)) = \frac{1}{m_x} \sum_{y \in \delta \Omega} \mu_{xy} \varphi(y) = 0.
\]
Hence \( u_{\varphi} = \varphi \), i.e. \( \text{Ker} Q \subset E_1(\Lambda) \), and the proof is completed.

**Remark 3.2.** By Proposition 3.3, the problem of determining \( E_1(\Lambda) \) can be reduced to the properties of the combinatorial structure of \( \delta I \Omega \cup \delta \Omega \), which is independent of the inner structure \( \Omega \setminus \delta I \Omega \).
From Proposition 3.3, we obtain a sufficient condition for \( E_1(\Lambda) \) to be nonempty as follows.

**Corollary 3.2.**

\[
\dim E_1(\Lambda) \geq \#\delta\Omega - \#\delta_I\Omega.
\]

In particular, \( E_1(\Lambda) \neq \emptyset \) if \( \#\delta\Omega > \#\delta_I\Omega \).

**Proof.** It directly follows from the fact that \( \dim \ker Q + \dim \text{Im} Q = \#\delta\Omega \). \( \square \)

### 4. Escobar-type Cheeger estimate

**Proposition 4.1.** Let \( G \) be a finite graph and \( \Omega \subset V \), then we have

\[
h_J(\tilde{\Omega}) \leq h_E(\tilde{\Omega}).
\]

**Proof.** Choose \( A \subset \overline{\Omega} \) that achieves \( h_E(\tilde{\Omega}) \), i.e.

\[
m(A \cap \delta \Omega) \leq \frac{m(\delta \Omega)}{2} \quad \text{and} \quad h_E(\tilde{\Omega}) = \frac{\mu(\partial \Omega A)}{m(A \cap \delta \Omega)}.
\]

If \( m(A) \leq \frac{m(\overline{\Omega})}{2} \), then \( h_J(\overline{\Omega}) \leq h_E(\overline{\Omega}) \). If \( m(A) \geq \frac{m(\overline{\Omega})}{2} \), i.e. \( m(A^c) \leq \frac{m(\overline{\Omega})}{2} \), then

\[
\frac{\mu(\partial \Omega A^c)}{m(A^c \cap \delta \Omega)} = \frac{\mu(\partial \Omega A)}{m(A \cap \delta \Omega)} \leq \frac{\mu(\partial \Omega A)}{m(A \cap \delta \Omega)}.
\]

Hence in both cases we have \( h_J(\overline{\Omega}) \leq h_E(\overline{\Omega}) \). \( \square \)

From the following example, we know that the equality in the above proposition can be achieved.

**Example 4.1.** Consider the path graph \( P_6 \) as shown in Figure 1 with unit edge weights, \( \Omega = \{v_2, v_3, v_4, v_5\} \) and \( \delta \Omega := \{v_1, v_6\} \). By computation, we have \( h_E(\overline{\Omega}) = h_J(\overline{\Omega}) = 1 \).

![Figure 1](image_url)

For convenience, we need the following notion.
Definition 4.1. For any $f \in \mathbb{R}^{\delta \Omega}$, we call $k \in \mathbb{R}$ the $L^1$-mean value of $f$ over $\delta \Omega$ if $k$ satisfies
\[
m(\{x \in \delta \Omega | f(x) \geq k\}) \geq \frac{1}{2} m(\delta \Omega)
\]
and
\[
m(\{x \in \delta \Omega | f(x) \leq k\}) \geq \frac{1}{2} m(\delta \Omega).
\]

Remark 4.1. The $L^1$-mean value may not be unique in general. For simplicity, we denote by $\overline{f}$ the $L^1$-mean value of $f \in \mathbb{R}^{\delta \Omega}$.

Lemma 4.1.
\[
\arg \min_{k \in \mathbb{R}} \sum_{x \in \delta \Omega} m_x |f(x) - k| = \overline{f},
\]
where $\arg \min$ denotes the value $k$ at which $\sum_{x \in \delta \Omega} m_x |f(x) - k|$ attains the minimum.

Remark 4.2. (a) For the proof of Lemma 4.1 one refers to e.g. [CSZ15].
(b) From Lemma 4.1 Theorem 1.1 is equivalent to
\[
h_E(\overline{\Omega}) = \inf_{f \in \mathbb{R}^{\Omega}} \frac{\sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy} |f(x) - f(y)|}{\sum_{x \in \delta \Omega} m_x |f(x) - \overline{f}|}.
\]

We will need the following discrete version of Co-area formula. For discrete Co-area formula, see e.g. [Gri09, Lemma 3.3].

Lemma 4.2. For any $f \in \mathbb{R}^{\Omega}$, we have
\[
\sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy} |f(x) - f(y)| = \int_{-\infty}^{\infty} \sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega}), f(x) < \sigma \leq f(y)} \mu_{xy} d\sigma.
\]

Proof. For any interval $(a, b)$, we denote by $\chi_{(a,b]}$ the characteristic function on $(a, b]$, i.e.
\[
\chi_{(a,b]}(x) = \begin{cases} 
0, & x \notin (a, b], \\
1, & x \in (a, b].
\end{cases}
\]

\[
\int_{-\infty}^{\infty} \sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega}), f(x) < \sigma \leq f(y)} \mu_{xy} d\sigma = \int_{-\infty}^{\infty} \sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy} \chi_{(f(x), f(y)]}(\sigma) d\sigma
\]
\[
= \sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy} \int_{-\infty}^{\infty} \chi_{(f(x), f(y)]}(\sigma) d\sigma
\]
\[
= \sum_{e \in \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy} |f(x) - f(y)|.
\]

\hfill \Box

Now we are ready to prove Theorem 1.1.
Proof of Theorem 4.1. From Remark 4.2 it suffices to prove (4.1). Choose $A \subset \Omega$ that achieves $h_E(\Omega)$, i.e.
\[ m(A \cap \delta \Omega) \leq m(A^\vee \cap \delta \Omega), \quad h_E(\tilde{\Omega}) = \frac{\mu(\partial A)}{m(A \cap \delta \Omega)}. \]

Set 
\[ u(x) = \begin{cases} 1, & x \in A, \\ 0, & x \in \Omega \setminus A. \end{cases} \]

We observe that $\bar{u}$, the $L^1$-mean value of $u$, is contained in $[0,1]$. To see this, one just need to notice that if $m(A \cap \delta \Omega) = m(A^\vee \cap \delta \Omega)$, then $\bar{u} \in [0,1]$; if $m(A \cap \delta \Omega) < m(A^\vee \cap \delta \Omega)$, then $\bar{u} = 0$.

For any $t \in [0,1]$, 
\[
\sum_{x \in \delta \Omega} m_x |u(x) - t| = \sum_{x \in A \cap \delta \Omega} m_x (1 - t) + \sum_{x \in A^\vee \cap \delta \Omega} m_x t \\
= m(A \cap \delta \Omega) + t \cdot (m(A^\vee \cap \delta \Omega) - m(A \cap \delta \Omega)) \\
\geq m(A \cap \delta \Omega).
\]

Hence 
\[
\frac{\sum_{e=(x,y) \in E(\Omega, \Omega)} \mu_{xy} |u(x) - u(y)|}{\sum_{x \in \delta \Omega} m_x |u(x) - \bar{u}|} \leq \frac{\mu(\partial A \cap E(\Omega, \Omega))}{m(A \cap \delta \Omega)} = h_E(\tilde{\Omega}).
\]

Then it follows that 
\[ h_E(\tilde{\Omega}) \geq \inf_{f \in \mathbb{R}^\Omega} \frac{\sum_{e=(x,y) \in E(\Omega, \Omega)} \mu_{xy} |f(x) - f(y)|}{\sum_{x \in \delta \Omega} m_x |f(x) - f(y)|}. \]

Now we prove the opposite direction. For any nonconstant function $f \in \mathbb{R}^\Omega$, choose a constant $c$ such that 
\[ m(\{f < c\} \cap \delta \Omega) \leq m(\{f \geq c\} \cap \delta \Omega), \]
\[ m(\{f \leq c\} \cap \delta \Omega) \geq m(\{f > c\} \cap \delta \Omega). \]

Set $g := f - c$, then we have 
\[ m(\{g < \sigma\} \cap \delta \Omega) \leq m(\{g \geq \sigma\} \cap \delta \Omega), \quad \text{for } \sigma \leq 0 \]
and 
\[ m(\{g < \sigma\} \cap \delta \Omega) \geq m(\{g \geq \sigma\} \cap \delta \Omega), \quad \text{for } \sigma > 0. \]

For any $\sigma \in \mathbb{R}$, set 
\[ G(\sigma) := \sum_{e=(x,y) \in E(\Omega, \Omega), g(x) < \sigma \leq g(y)} \mu_{xy}. \]

Then by Lemma 4.2 we have 
\[
\sum_{e=(x,y) \in E(\Omega, \Omega)} \mu_{xy} |f(x) - f(y)| = \int_{-\infty}^{\infty} G(\sigma) d\sigma.
\]
Set $A := \{g < \sigma\}$ for $\sigma \leq 0$ and $A := \{g \geq \sigma\}$ for $\sigma > 0$. In either case, we have $m(A \cap \delta \Omega) \leq m(A^\vee \cap \delta \Omega)$ and

$$G(\sigma) \geq h_E(\tilde{\Omega}) \cdot m(A \cap \delta \Omega) = h_E(\tilde{\Omega}) \cdot \begin{cases} m(\{g < \sigma\} \cap \Omega), & \text{for } \sigma \leq 0, \\ m(\{g \geq \sigma\} \cap \Omega), & \text{for } \sigma > 0 \end{cases}$$

by the definition of $h_E(\tilde{\Omega})$. Hence

$$\sum_{e=(x,y) \in E(\Omega, \Omega)} \mu_{xy}|f(x) - f(y)| = \int_{-\infty}^{0} G(\sigma) d\sigma + \int_{0}^{\infty} G(\sigma) d\sigma \geq h_E(\tilde{\Omega}) \left( \int_{-\infty}^{0} m(\{g < \sigma\} \cap \Omega) d\sigma + \int_{0}^{\infty} m(\{g \geq \sigma\} \cap \Omega) d\sigma \right)$$

$$= h_E(\tilde{\Omega}) \left( \int_{-\infty}^{0} m(\{g < \sigma\} \cap \Omega) d\sigma + \int_{0}^{\infty} m(\{g \geq \sigma\} \cap \Omega) d\sigma \right)$$

$$= h_E(\tilde{\Omega}) \left( \int_{-\infty}^{0} \sum_{x \in \delta \Omega} \chi_{g(x),0} m_x d\sigma + \int_{0}^{\infty} \sum_{x \in \delta \Omega} \chi_{0, g(x)} m_x d\sigma \right)$$

$$= h_E(\tilde{\Omega}) \sum_{x \in \delta \Omega} |f(x) - c| m_x.$$

Then the other direction follows and we complete the proof of the theorem.

We obtain the following upper bound estimate for $\lambda_1(\Omega)$.

**Proposition 4.2.** Let $G$ be a finite graph and $\Omega \subset V$, then we have

$$\lambda_1(\Omega) \leq 2h_E(\tilde{\Omega}).$$

**Proof.** Choose $A \subset \Omega$ that achieves $h_E(\tilde{\Omega})$, i.e. $m(A \cap \delta \Omega) \leq \frac{1}{2} m(\delta \Omega)$ and $\frac{\mu(\partial A, A)}{m(A \cap \delta \Omega)} = h_E(\tilde{\Omega})$. Set $\varphi(x) \in \mathbb{R}^{\delta \Omega}$ as

$$\varphi(x) = \begin{cases} \frac{1}{m(A \cap \delta \Omega)}, & x \in A \cap \delta \Omega, \\ -\frac{1}{m(A^\vee \cap \delta \Omega)}, & x \in A^\vee \cap \delta \Omega. \end{cases}$$

Similarly set $\tilde{\varphi}(x)$ to be

$$\tilde{\varphi}(x) = \begin{cases} \frac{1}{m(A \cap \delta \Omega)}, & x \in A, \\ -\frac{1}{m(A^\vee \cap \delta \Omega)}, & x \in A^\vee. \end{cases}$$

Then we have

$$\lambda_1(\Omega) \leq \frac{D_\Omega(u_x)}{\sum_{x \in \delta \Omega} \varphi^2(x)m_x} \leq \frac{D_\Omega(\tilde{\varphi})}{\sum_{x \in \delta \Omega} \tilde{\varphi}^2(x)m_x}$$

$$= \left( \frac{1}{m(A \cap \delta \Omega)} + \frac{1}{m(A^\vee \cap \delta \Omega)} \right) \mu(\partial A \cap E(\Omega, \tilde{\Omega}))$$

$$\leq 2 \frac{\mu(\partial A, A)}{m(A \cap \delta \Omega)} = 2h_E(\tilde{\Omega}).$$
The second inequality follows from the fact that harmonic functions minimize the Dirichlet energy among functions with the same boundary condition. □

Figure 2.

Remark 4.3. The above upper bound estimate for \( \lambda_1(\Omega) \) is sharp. From the following example, we can see that the factor 2 in the upper bound estimate can’t be reduced.

Example 4.2. Consider a sequence of graphs \( \{G_n\}_{n=1}^{\infty} \) as shown in Figure 2 with \( \Omega_n = \{w_1, w_2\} \), \( \delta \Omega_n = \{v_1, v_2, \ldots, v_{4n}\} \) and unit edge weights. By calculation, \( \lambda_1(G_n) = \frac{1}{n+1} \) and \( h_E(G_n) = \frac{1}{2n} \). Hence \( \lambda_1(G_n) = \frac{1}{n+1} \leq \frac{1}{n} = 2h_E(G_n) \).

At the end of this section, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let \( u \) be the first eigenvector associated to \( \lambda_1(\Omega) \). For simplicity, we still denote by \( u \) the harmonic extension of \( u \). Set \( v = (u - \overline{u})_+ \) and choose \( \overline{v} = 0 \). Then

\[
m(\{v = 0\} \cap \delta \Omega) \geq \frac{1}{2} m(\delta \Omega).
\]

Applying Theorem 1.1 by choosing \( f = v^2 \), we have

\[
h_E(\Omega_1) \cdot \sum_{x \in \delta \Omega} v^2(x) m_x \leq \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} |v^2(x) - v^2(y)|.
\]

For the right hand side of the above inequality,

\[
\left( \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} |v^2(x) - v^2(y)| \right)^2 \\
\leq \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} (v(x) + v(y))^2 \cdot \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} (v(x) - v(y))^2 \\
\leq 2 \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} (v^2(x) + v^2(y)) \cdot \sum_{e = \{x, y\} \in E(\Omega_1, \overline{\Omega_1})} \mu_{xy} (v(x) - v(y))^2.
\]
Notice that it suffices to consider the graph \( \tilde{\Omega} \), then
\[
\sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(v^2(x) + v^2(y))
\]
\[
= \frac{1}{2} \sum_{x,y \in \Omega} \mu_{xy}(v^2(x) + v^2(y)) + \sum_{x \in \Omega} \sum_{y \in \delta \Omega} \mu_{xy}(v^2(x) + v^2(y))
\]
\[
= \sum_{x \in \Omega} v^2(x) \left( \sum_{y \in \Omega} + \sum_{y \in \delta \Omega} \right) \mu_{xy} + \sum_{y \in \delta \Omega} v^2(y) \sum_{x \in \Omega} \mu_{xy}
\]
\[
= \sum_{x \in \Omega} v^2(x) m_x.
\]
Hence
\[
h_E(\tilde{\Omega}) \cdot \sum_{x \in \delta \Omega} v^2(x) m_x \leq \left( \sum_{x \in \Omega} v^2(x) m_x \cdot \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(v(x) - v(y))^2 \right)^{1/2},
\]
i.e.
\[
h_E(\tilde{\Omega}) \cdot \sum_{x \in \delta \Omega} (u - \overline{u})^2 m_x \leq \left( \sum_{x \in \Omega} (u - \overline{u})^2 m_x \cdot \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(u(x) - u(y))^2 \right)^{1/2}.
\]
Similarly, we have
\[
h_E(\tilde{\Omega}) \cdot \sum_{x \in \delta \Omega} (u - \overline{u})^2 m_x \leq \left( \sum_{x \in \Omega} (u - \overline{u})^2 m_x \cdot \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(u(x) - u(y))^2 \right)^{1/2}.
\]
Hence
\[
h_E(\tilde{\Omega}) \cdot \sum_{x \in \delta \Omega} (u - \overline{u})^2 m_x 
\]
\[
\leq \left( \sum_{x \in \Omega} (u - \overline{u})^2 m_x \cdot \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(u(x) - u(y))^2 \right)^{1/2}
\]
\[
\leq \frac{a}{2} \sum_{x \in \Omega} (u - \overline{u})^2 m_x + \frac{1}{a} \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(u(x) - u(y))^2
\]
\[
= \frac{a}{2} \left( \sum_{x \in \Omega} + \sum_{x \in \delta \Omega} \right) (u - \overline{u})^2 m_x + \frac{1}{a} \sum_{e = \{x,y\} \in E(\Omega, \overline{\Omega})} \mu_{xy}(u(x) - u(y))^2.
\]
(4.2)
For any \( \varphi \in \mathbb{R}^\Pi \) we have
\[
\mu_1(k) \leq \frac{D_\Omega(\varphi) + k \sum_{x \in \delta \Omega} \varphi^2(x)m_x}{\sum_{x \in \Omega} \varphi^2(x)m_x}.
\]
Hence
\[
\sum_{x \in \Omega} (u - \bar{u})^2 m_x \leq \frac{D_{\Omega}(u)}{\mu_1(k)} + \frac{k}{\mu_1(k)} \sum_{x \in \partial \Omega} (u - \bar{u})^2 m_x.
\]
Combining with the above inequality and (4.2), we have
\[
\left( h_E(\Omega) - \frac{a(k + \mu_1(k))}{2\mu_1(k)} \right) \frac{2a\mu_1(k)}{a^2 + 2\mu_1(k)} \leq \frac{D_{\Omega}(u)}{\sum_{x \in \partial \Omega} (u - \bar{u})^2 m_x}.
\]
Using the fact that \( u \) is the first eigenfunction associated to \( \lambda_1 \) we find that
\[
\left( h_E(\Omega) - \frac{a(k + \mu_1(k))}{2\mu_1(k)} \right) \frac{2a\mu_1(k)}{a^2 + 2\mu_1(k)} \leq \frac{\lambda_1 \sum_{x \in \partial \Omega} u^2(x) m_x}{\sum_{x \in \partial \Omega} (u - \bar{u})^2 m_x}.
\]
Since \( \langle u, 1 \rangle_{\partial \Omega} = 0 \), we have
\[
\frac{\sum_{x \in \partial \Omega} u^2(x) m_x}{\sum_{x \in \partial \Omega} (u - \bar{u})^2 m_x} \leq 1
\]
and therefore
\[
\lambda_1(\Omega) \geq \left( h_E(\Omega) - \frac{a(k + \mu_1(k))}{2\mu_1(k)} \right) \frac{2a\mu_1(k)}{a^2 + 2\mu_1(k)} = \frac{2h_E(\Omega)\mu_1(k) - a(k + \mu_1(k))}{a^2 + 2\mu_1(k)}.
\]
\[\square\]

**Remark 4.4.** The maximum of the right hand side of (1.2) with respect to \( a \) can be achieved at
\[
a_0 = \frac{2h\mu}{\sqrt{(k + \mu)^2 + 2h^2\mu} + (k + \mu)}
\]
and the maximum is
\[
\frac{h^2\mu\sqrt{(k + \mu)^2 + 2h^2\mu}}{2h^2\mu + (k + \mu)^2 + (k + \mu)\sqrt{(k + \mu)^2 + 2h^2\mu}}.
\]

5. **Jammes-type Cheeger estimate**

**Proposition 5.1.** Let \( G \) be a finite graph and \( \Omega \subset V \), then we have
\[
h_{J}(\tilde{\Omega}) \leq 1.
\]

**Proof.** Choose \( A = \{v\} \), where \( v \in \partial \Omega \). Then by the definition of \( h_{J}(\tilde{\Omega}) \), we have \( h_{J}(\tilde{\Omega}) \leq \frac{\mu(\partial_{G} A)}{m_{(\partial_{G} A)}} = \frac{m_v}{m_v} = 1. \)

The eigenvalues of \( \Lambda \) can be characterised by Rayleigh quotient as follows
\[
\lambda_1(\Omega) = \inf_{\varphi \in \mathbb{R}^{\partial \Omega}, \|\varphi\|_2 = 1, \varphi \perp 1} \langle \Lambda \varphi, \varphi \rangle = \inf_{\varphi \in \mathbb{R}^{\partial \Omega}, \|\varphi\|_2 = 1, \varphi \perp 1} D_{\Omega}(u_{\varphi}),
\]
\[
\lambda_{N-1}(\Omega) = \sup_{\varphi \in \mathbb{R}^{\partial \Omega}, \|\varphi\|_2 = 1} \langle \Lambda \varphi, \varphi \rangle = \sup_{\varphi \in \mathbb{R}^{\partial \Omega}, \|\varphi\|_2 = 1} D_{\Omega}(u_{\varphi}).
\]
We denote by \( \sigma_1 \) the first nontrivial eigenvalue of the Dirichlet-to-Neumann operator on a compact manifold with boundary. For convenience, we recall the idea of the proof of Jammes-type Cheeger estimate, which can be divided into four steps.

Step 1: Choosing \( f \) as the eigenfunction associated to \( \sigma_1 \), we still denote by \( f \) the harmonic extension of \( f \) to \( M \) with \( \text{vol}(M^+) \leq \frac{\text{vol}(M)}{2} \), where \( M^+ := \{ x \in M \mid f(x) > 0 \} \).

Step 2: Show that \( \sigma_1 = \frac{\int_{M^+} |\partial f|^2}{\int_{\partial M^+} f^2} \), where \( \partial M^+ = M^+ \cap \partial M \).

Step 3: By Hölder’s inequality, \( \sigma_1 = \frac{(\int_{M^+} f^2)(\int_{\partial M^+} |\partial f|^2)}{(\int_{M^+} |\partial f|^2)(\int_{\partial M^+} f^2)} \geq 1 \frac{(\int_{M^+} |\partial f|^2)^2}{(\int_{\partial M^+} f^2)^2} \).

Step 4: Set \( D_t := f^{-1}([\sqrt{t}, \infty)) \), \( \partial_t D_t := \partial D_t \cap \text{int}(M) \) and \( \partial E D_t := \partial D_t \setminus \partial_t D_t \). Use Co-area formula to show that \( \int_{M^+} |\partial f|^2| = \int_{t \geq 0} \text{Area}(\partial_t D_t) \), \( \int_{M^+} f^2 = \int_{t \geq 0} \text{vol}(D_t) \) and \( \int_{\partial M^+} f^2 = \int_{t \geq 0} \text{Area}(\partial E D_t) \).

Then by the definitions of \( h_M \) and \( h_J(M) \) the lower bound estimate of \( \sigma_1 \) follows.

Inspired by the Riemannian case, we can prove the Jammes-type Cheeger constant for \( \lambda_1(\Omega) \) in the discrete setting. Let \( 0 \neq f \in \mathbb{R}^{\delta \Omega} \) be an eigenfunction associated to \( \lambda_1(\Omega) \). For convenience, we still denote \( u_f \) by \( f(x) \). Set \( \overline{\Omega}^+ = \{ x \in \overline{\Omega} \mid f(x) > 0 \} \) with \( m(\overline{\Omega}^+) \leq \frac{m(\overline{\Omega})}{2} \) (otherwise we can change the sign of \( f \)) and set

\[
(5.1) \quad g(x) = \begin{cases} 
  f(x), & x \in \overline{\Omega}^+, \\
  0, & \text{otherwise}.
\end{cases}
\]

For simplicity, we set \( \overline{\Omega}^- := \overline{\Omega} \setminus \overline{\Omega}^+ \), \( \Omega^+ := \overline{\Omega}^+ \cap \Omega \), \( \Omega^- := \Omega \setminus \Omega^+ \), \( \delta^+ \Omega := \overline{\Omega}^+ \cap \delta \Omega \) and \( \delta^- \Omega := \delta \Omega \setminus \delta^+ \Omega \). In order to prove Jammes-type Cheeger estimate, we need the following Lemmas.

**Lemma 5.1.** For \( g \) as in \( (5.1) \), we have

\[
(5.2) \quad \lambda_1(\Omega) \geq \frac{\sum_{e \in \{x,y\} \in E(\overline{\Omega}^+, \overline{\Omega}^-)} \mu_{xy}(g(y) - g(x))^2}{\sum_{x \in \delta^+ \Omega} g^2(x) m_x}.
\]

**Proof.** Notice that it suffices to consider graph \( \overline{\Omega} \). Hence for any \( x \in \delta \Omega \), we have

\[
\Delta f(x) = \frac{1}{m_x} \sum_{y \in \Omega} \mu_{xy}(f(y) - f(x)) = -\frac{\partial f(x)}{\partial n}.
\]

Then

\[
\langle \Delta f(x), g(x) \rangle_{\overline{\Omega}^+} = \sum_{x \in \Omega^+} \Delta f(x) g(x) m_x + \sum_{x \in \delta^+ \Omega} \Delta f(x) g(x) m_x
\]

\[
= -\lambda_1(\Omega) \sum_{x \in \delta^+ \Omega} g^2(x) m_x.
\]

\[
(5.3)
\]
Notice that

\[
\langle \Delta f(x), g(x) \rangle_{\Omega^+} = \left( \sum_{x \in \Omega^+} \sum_{y \in \Pi^+} + \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} \right) \mu_{xy} (f(x) - f(y))g(x)
\]

\[
= -\frac{1}{2} \sum_{x,y \in \Pi^+} \mu_{xy} (f(y) - f(x))(g(y) - g(x)) - \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} \mu_{xy} (f(x) - f(y))(g(x) - g(y))
\]

\[
\leq - \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^+)} \mu_{xy} (g(y) - g(x))^2 - \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} \mu_{xy} (g(x) - g(y))^2
\]

\[
= - \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^-)} \mu_{xy} (g(y) - g(x))^2.
\]

Then the lemma follows in view of \[\text{(5.3)}\]. \hfill \square

Multiplying both the numerator and denominator of the fraction in the right hand side of \[\text{(5.2)}\] by \[\sum_{x \in \Pi^+} g^2(x) m_x\] and setting

\[
P \geq \frac{\sum_{x \in \Pi^+} g^2(x) m_x \cdot \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^+) \mu_{xy} (g(y) - g(x))^2}{\sum_{x \in \Pi^+} g^2(x) m_x \cdot \sum_{x \in \Pi^+} g^2(x) m_x},
\]

we have

\[\text{(5.4)}\]

\[
\lambda_1(\Omega) \geq \frac{P}{Q}.
\]

**Lemma 5.2.**

\[
P \geq \frac{1}{2} \left( \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^-)} \frac{|g^2(x) - g^2(y)| \mu_{xy}}{g^2(x) m_x} \right)^2.
\]

**Proof.** Note that

\[
\sum_{x \in \Pi^+} g^2(x) m_x = \left( \sum_{x \in \Pi^+} \sum_{y \in \Pi^+} + \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} \right) g^2(x) \mu_{xy}
\]

\[
= \frac{1}{2} \sum_{x,y \in \Pi^+} (g^2(x) + g^2(y)) \mu_{xy} + \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} g^2(x) \mu_{xy}
\]

\[
= \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^+) \mu_{xy} + \sum_{x \in \Pi^+} \sum_{y \in \Pi^-} g^2(x) \mu_{xy}
\]

\[
= \sum_{e = \{x,y\} \in E(\Pi^+,\Pi^-)} (g^2(x) + g^2(y)) \mu_{xy}.
\]
Hence
\[ P = \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} (g^2(x) + g^2(y))\mu_{xy} \cdot \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} \mu_{xy}(g(y) - g(x))^2 \]
\[ \geq \frac{1}{2} \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} (g(x) + g(y))^2\mu_{xy} \cdot \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} \mu_{xy}(g(y) - g(x))^2 \]
\[ \geq \frac{1}{2} \left( \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} |g^2(x) - g^2(y)|\mu_{xy} \right)^2. \]
The last inequality follows from Hölder’s inequality. \( \square \)

For any \( t > 0 \), set \( D_t := g^{-1}([\sqrt{t}, +\infty)) = \{ x \in \overline{\Omega} | g^2(x) \geq t \} \). Then we have \( m(D_t) \leq m(\overline{\Omega}^+) \leq \frac{m(\overline{\Omega})}{2} \).

**Lemma 5.3.**
\[ \int_0^\infty \mu(\partial D_t \cap E(\Omega, \overline{\Omega}))dt = \sum_{e=(x,y) \in E(\overline{\Omega}^+,\Omega)} \mu_{xy}|g^2(x) - g^2(y)|. \]

**Proof.** It follows from Lemma 4.2 by setting \( f = g^2 \) and considering the edge set \( E(\overline{\Omega}^+,\overline{\Omega}) \). \( \square \)

**Lemma 5.4.**
\[ \int_0^\infty m(D_t)dt = \sum_{x \in \overline{\Omega}^+} g^2(x)m_x. \]
\[ \int_0^\infty m(D_t \cap \delta \Omega)dt = \sum_{x \in \delta^+\Omega} g^2(x)m_x. \]

**Proof.** Similar to Lemma 4.2 we have
\[ \int_0^\infty m(D_t)dt = \int_0^\infty \sum_{x \in D_t} m_x dt = \int_0^\infty \sum_{x \in \overline{\Omega}^+} m_x \chi_{(0,g^2(x))}(t)dt = \sum_{x \in \overline{\Omega}^+} g^2(x)m_x \]
and
\[ \int_0^\infty m(D_t \cap \delta \Omega)dt = \int_0^\infty \sum_{x \in \delta^+\Omega} m_x \chi_{(0,g^2(x))}(t)dt = \sum_{x \in \delta^+\Omega} g^2(x)m_x. \]
\( \square \)

Now we are ready to prove Theorem 1.3.
Proof of Theorem 1.3. Combining with the above Lemmas (Lemma 5.1 to 5.4), we have

\[
\lambda_1(\Omega) \geq \frac{1}{2} \int_0^\infty \mu(\partial D_t \cap E(\Omega, \overline{\Omega})) dt \cdot \int_0^\infty \mu(\partial D_t \cap E(\Omega, \overline{\Omega})) dt
\]

\[
= \frac{1}{2} \int_0^\infty h(\overline{\Omega}) m(D_t) dt \cdot \int_0^\infty h_J(\overline{\Omega}) m(D_t \cap \delta \Omega) dt
\]

\[
= \frac{h(\overline{\Omega}) h_J(\overline{\Omega})}{2} \int_0^\infty m(D_t) dt \cdot \int_0^\infty m(D_t \cap \delta \Omega) dt
\]

\[
= \frac{h(\overline{\Omega}) h_J(\overline{\Omega})}{2}.
\]

\[
□
\]

From Theorem 1.3, we know that \( h(\overline{\Omega}) \) plays an important role in Jammu-type Cheeger estimate. Recall that \( \zeta_1(\overline{\Omega}) \) is the first nontrivial eigenvalue of the Laplace operator with no boundary condition on \( \overline{\Omega} \) and the classical Cheeger estimates reads as

\[
2h(\overline{\Omega}) \geq \zeta_1(\overline{\Omega}) \geq \frac{h(\overline{\Omega})^2}{2},
\]

see [Chu97, p.26]. Then we are ready to prove Corollary 1.1

Proof of Corollary 1.1. Recall that \( h(\overline{\Omega}) \leq h_J(\overline{\Omega}) \). Combining with Theorem 1.3 and (5.5), we have

\[
\lambda_1(\Omega) \geq \frac{h(\overline{\Omega})^2}{2} \geq \frac{(\zeta_1(\overline{\Omega}))^2}{8}.
\]

\[
□
\]

Finally we give an example to show that \( \lambda_1 \) can’t be bounded from below using only \( h_J(\overline{\Omega}) \).

Example 5.1. Consider a path graph with even vertices \( n (n \geq 6) \) and unit edge weights as shown in Figure 1. Set \( \Omega = \{v_2, v_3, \cdots, v_{n-1}\} \), \( \delta E \Omega = \{v_1, v_n\} \). Then we have

\[
\Lambda = \begin{pmatrix}
\frac{1}{n-1} & -\frac{1}{n-1} \\
-\frac{1}{n-1} & \frac{1}{n-1}
\end{pmatrix}.
\]

Hence \( \lambda_0(\Omega) = 0 \) and \( \lambda_1(\Omega) = \frac{2}{n-1} \). Choosing \( A = \{v_1, v_2, \cdots, v_{n-1}\} \), we obtain that \( h(\overline{\Omega}) = \frac{1}{n-1} \) and \( h_J(\overline{\Omega}) = 1 \). Hence the Jammes-type Cheeger estimate we obtained is asymptotically sharp of the same order \( \frac{1}{n-1} \) on both sides as \( n \to \infty \). Moreover, one can not obtain that \( \lambda_1(\Omega) \geq F(h_J(\overline{\Omega})) \) for any positive function \( F \).
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