Admissible subsets and Littelmann paths in affine Kazhdan-Lusztig theory

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Abstract

The center of an extended affine Hecke algebra is known to be isomorphic to the ring of symmetric functions associated to the underlying finite Weyl group $W_0$. The set of Weyl characters $s_\lambda$ forms a basis of the center and this character acts as translation on the Kazhdan-Lusztig basis element $C_{w_0}$ where $w_0$ is the longest element of $W_0$, that is we have $s_\lambda C_{w_0} = C_{p_\lambda w_0}$. As a consequence, we see that the coefficients that appears when decomposing $s_r s_\lambda C_{w_0} = s_r C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis are tensor multiplicities of the Lie algebra with Weyl group $W_0$. The aim of this paper is to explain how admissible subsets and Littelmann paths, which are models to compute such multiplicities, naturally appear when working out the decomposition of $s_r C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis.

1 Introduction

Let $W_e$ be an extended affine Weyl group with underlying finite Weyl group $W_0$. Then $W_e = W_0 \ltimes P$ where $P$ denotes the set of weights associated to $W_0$. Let $\mathcal{H}_e$ be the associated generic affine Hecke algebra defined over the ring $\mathcal{A}$ of Laurent polynomials with one indeterminate and let $\{C_w \mid w \in W_e\}$ be the Kazhdan-Lusztig basis of $\mathcal{H}_e$. The center of the affine Hecke algebra $\mathcal{H}_e$ associated to $W_e$ is known to be isomorphic to the ring of symmetric functions $\mathcal{A}[P]^{W_0}$. The set of Weyl characters $\{s_\lambda \mid \lambda \in P^+\}$ forms a basis of $\mathcal{A}[P]^{W_0}$ and we have $s_\lambda C_{w_0} = C_{p_\lambda w_0}$ where $p_\lambda$ denotes the translation by $\lambda \in P^+$ in $W_e$; see [7].

Denote by $V(\tau)$ the irreducible highest weight module of weight $\tau \in P^+$ for the simple Lie algebra over $\mathbb{C}$ with Weyl group $W_0$. Then the character of $V(\lambda)$ is $s_\lambda$ and for all $\tau, \lambda \in P^+$ we have $s_r s_\lambda = \sum m^{\mu}_{r,\lambda} s_\mu$ where $m^{\mu}_{r,\lambda}$ is the multiplicity of $V(\mu)$ in the tensor product $V(\tau) \otimes V(\lambda)$. Computing the multiplicities $m^{\mu}_{r,\lambda}$ is one of the most basic question in representation theory of simple Lie algebras over $\mathbb{C}$. Littelmann showed [10] that such multiplicities can be determined by counting certain kind of paths in the weight lattice $P$ constrained to stay in the fundamental chamber. Later on, Lenart and Postnikov [8, 9] showed that these multiplicities can be determined using admissible subsets associated to a fix reduced expression of $p_\tau \in W_e$. Their model can be viewed as a discrete counterpart of Littelmann paths model and they explicitly constructed a bijection between admissible subsets and Littelmann paths.

In the extended affine algebra, we must have

$$s_r C_{p_\lambda w_0} = s_r s_\lambda C_{w_0} = \sum m^{\mu}_{r,\lambda} C_{p_\mu w_0}$$

and the aim of this paper is to explain why these multiplicities appear when decomposing $s_r C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis. This will be done in two steps:

1) we will show that to determine the decomposition of the product $s_\lambda C_{p_\mu w_0}$ it is sufficient to study products of the form $T_{p_\lambda} T_{p_\mu w_0}$ in the standard basis and to find the terms of maximal degrees;

2) we will show that the terms of maximal degree in products of the form $T_{p_\lambda} T_{p_\mu w_0}$ are indexed by the set of admissible subsets $J$ associated to a fix reduced expression of $p_\lambda$.

Then, according to the results of Lenart and Postnikov [9], we will obtain the desired equality.

The paper is organised as follows. In Section 2, we introduce all the needed material on (extended) affine Weyl groups. In Section 3, we present Kazhdan-Lusztig theory for affine Hecke algebras with unequal parameters and we describe the center of this algebra. In Section 4 and Section 5, we prove (1) and (2) above. Finally, we prove the desired equality using the theory of admissible subsets in Section 6. In Section 7, following [9], we describe the connection between admissible subsets and Lakshmibai-Seshadri paths.
2 Affine Weyl groups

Let $V$ be an Euclidean space with scalar product $(\cdot, \cdot)$. We denote by $V^*$ the dual of $V$ and by $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$ the canonical pairing. Let $\Phi$ be a root system and let $\Phi^+$ be the dual root system. If $\alpha \in \Phi$ then $\alpha^\vee \in \Phi^+$ is defined by $\langle x, \alpha^\vee \rangle = 2(x, \alpha)/(\alpha, \alpha)$. We fix a set of positive roots $\Phi^+$ and a simple system $\Delta$ such that $\Delta \subset \Phi^+$.

2.1 Geometric presentation of an affine Weyl group

We denote by $H_{\alpha,n}$ the hyperplane defined by the equation $\langle x, \alpha^\vee \rangle = n$ and by $\mathcal{F}$ the collection of all such hyperplanes. We will say that an hyperplane $H$ is of direction $\alpha \in \Phi^+$ if there exists a pair $(\alpha, n) \in \Phi^+ \times \mathbb{Z}$ such that $H = H_{\alpha,n}$. We then write $\mathcal{F} = \phi$.

Let $\Omega$ be the group generated by the set of orthogonal reflections $\sigma_{\alpha,n}$ with respect to $H_{\alpha,n}$ where $\alpha \in \Phi$ and $n \in \mathbb{Z}$. The group $\Omega$ is an affine Weyl group of type $\Phi^\vee$. It is well known that $\Omega$ is generated by the set $\{\sigma_{\alpha,0} \mid \alpha \in \Delta\} \cup \{\sigma_{\alpha,1}\}$ where $\alpha$ is such that $\alpha^\vee$ is the highest root of $\Phi^\vee$. To simplify notation, we will simply write $\sigma_\alpha$ for $\sigma_{\alpha,0}$. Let $\Omega_0$ be the stabilizer of 0 in $\Omega$: clearly $\Omega_0 = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$. Let $\sigma_{\Omega_0}$ be the longest element of $\Omega_0$.

The set of alcoves, denoted $\text{Alc}(\mathcal{F})$, is the set of connected components of $V \setminus \mathcal{F}$. The fundamental alcove $A_0$ is defined by

$$A_0 = \{ \lambda \in V \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \forall \alpha \in \Phi^+ \}$$

$$= \{ \lambda \in V \mid 0 < \langle \lambda, \alpha^\vee \rangle < 1, \forall \alpha \in \Delta \}.$$ 

The group $\Omega$ acts simply transitively on the set of alcoves. We set $A_0^- = A_0 \sigma_{\Omega_0}$ and we have $A_0^- = A_0 \sigma_0$.

The group $\Omega$ acts on the set of faces (codimension 1 facet) of alcoves and we denote by $S$ the set of orbits. Then to each $s \in S$, one can associate an involution on the set of $\text{Alc}(\mathcal{F})$: for all $A \in \text{Alc}(\mathcal{F})$, $sA$ is the unique alcove which shares with $A$ a face of type $s$. The set of all such involutions generate a Coxeter group $W_A$ which is isomorphic to $\Omega$. Both groups $W_A$ and $\Omega$ acts transitively on the set of alcoves. We will write the action of $W_A$ on the left and the action of $\Omega$ on the right. It is a well known fact that these two actions commute. For all $s \in S$ and all $A \in \text{Alc}(\mathcal{F})$, there exists a unique pair $(\alpha, n) \in \Phi^+ \times \mathbb{Z}$ such that the hyperplane $H_{\alpha,n}$ separate the alcoves $A$ and $sA$ and we have $sA = A s_{\alpha,n}$.

Given two alcoves $A, B \in \text{Alc}(\mathcal{F})$, we set

$$H(A, B) = \{ H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B \} \quad \text{and} \quad \overline{H(A, B)} = \{ \overline{H} \mid H \in H(A, B) \}.$$ 

Let $A \in \text{Alc}(\mathcal{F})$, $\beta \in \Phi$ and $n \in \mathbb{Z}$. We write $n < A[\beta]$ if and only if for all $x \in A$ we have $n < \langle x, \beta^\vee \rangle$. Similarly, we write $n > A[\beta]$ if and only if for all $x \in A$ we have $n > \langle x, \beta^\vee \rangle$. For all alcoves $A$ and all roots $\beta \in \Phi^+$, there exists a unique $n \in \mathbb{Z}$ such that $n < A[\beta] < n + 1$.

The following proposition gather some well known facts about the two actions introduced above.

Proposition 2.1. Let $w \in W_A$. We have

1) $\ell(w) = |H(A_0, w A_0)|$

2) Let $s \in S$ and let $H_{\alpha,n}$ be the unique hyperplane separating $w A_0$ and $sw A_0$. We have $sw < w$ if and only if we are in one of the following situation:

(a) $w A_0 \in H_{\alpha,n}^+$, $sw A_0 \in H_{\alpha,n}^-$ and $n > 0$,

(b) $w A_0 \in H_{\alpha,n}^-$, $sw A_0 \in H_{\alpha,n}^+$ and $n \leq 0$.

3) We have $\overline{H(A_0, A_0 \sigma)} = \overline{H(A_0^+, A_0^+ \sigma)} = \{ \gamma \in \Phi^+ \mid \gamma \sigma^{-1} \in \Phi^- \}$.

2.2 Weight functions and special points

Let $L$ be a positive weight function on $W_A$, that is a function $L : W_A \to \mathbb{N}$ such that $L(w w') = L(w) + L(w')$ whenever $\ell(w w') = \ell(w) + \ell(w')$. To determine a weight function, it is enough to give its values on the conjugacy classes of generators of $S$. From now on, we fix such a positive weight function...
Let $L$ on $W_a$. Note that this also defines a weight function on $\Omega$ simply by setting $L(\sigma) = L(w)$ whenever $wA_0 = A_0\sigma$.

Let $H$ be an hyperplane in $\mathcal{F}$. We say that $H$ is of weight $L(r)$ if it contains a face of type $r \in S$. This is well-defined since if $H$ contains a face of type $r$ and $r'$ then $r$ and $r'$ are conjugate and $L(r) = L(r')$ [2 Lemma 2.1]. We denote the weight of an hyperplane $H$ by $L(H)$. If $\beta \in \Phi$, we set $L(\beta) = \max_{\alpha \in \Phi} L(H)$.

For any $\lambda \in V$ we set

$$L(\lambda) = \sum_{H, \lambda \in H} L(H).$$

Let $\nu = \max_{\lambda \in V} L(\lambda)$. We call $\lambda$ an $L$-weight if $L(\lambda) = \nu$ and we denote by $P$ the set $L$-weights. Further we denote by $P^+$ the set of dominant $L$-weights that is $P^+ := \{ \lambda \in P \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Phi^+ \}$ and by $P^-$ the set of antidominant weights, that is $P^- = -P^+$. For $\lambda \in P$ and $\beta \in \Phi^+$ we set $\lambda_\beta := (\lambda, \beta^\vee)$. For each root $\alpha$, we set

$$b_\alpha = \begin{cases} 1 & \text{if } L_{H_{\alpha_0,0}} = L_{H_{\alpha_1,1}}, \\ 2 & \text{otherwise.} \end{cases}$$

Remark 2.2. The only case where there can be parallel hyperplanes with different weights and therefore some $b_\alpha$ equal to 2 is when $W$ is of type $C$ and when the extremal generator in the Dynkin diagram have same weights. We refer to [2 Lemma 2.2] for details on this case.

The action of the longest element $\sigma_{\Omega_\lambda}$ on the set of weights is an involution, we set $\lambda^* = \lambda \sigma_{\Omega_\lambda}$. Note that we do not have in general $\lambda \sigma_{\Omega_\lambda} = -\lambda$ but if $\lambda \in P^+$ then $\lambda^* \in P^-$. If $\sigma = \sigma_{\beta_1} \ldots \sigma_{\beta_n}$ where $\beta_i \in \Phi$ we set $\sigma^* = \sigma_{\beta_n} \sigma_{\beta_{n-1}} \ldots \sigma_{\beta_1}$. Let $v \in W_0$ and $\sigma_v \in \Omega_0$ be such that $vA_0 = A_0\sigma_v$. With our notation we have $vA_0 = A_0^\sigma_v$.

For $\lambda \in P$, we denote by $W_\lambda$ the stabiliser in $W_a$ of the set of alcoves that contain $\lambda$ in their closure and by $\Omega_\lambda$ the stabiliser of $\lambda$ in $\Omega$. This notation is coherent since for all $\lambda \in P$ we have $W_\lambda \simeq \Omega_\lambda \simeq \Omega_0$ where $\Omega_0 = \langle \sigma_\alpha \mid \alpha \in \Delta \rangle$.

2.3 Extended affine Weyl groups

The extended affine Weyl group is defined by $\Omega_e = \Omega_0 \ltimes P$; it acts naturally on (the right) of $V$ and on $\text{Alc}(\mathcal{F})$ but the action is no longer faithful. If we denote by $\Pi$ the stabiliser of $A_0$ in $\Omega_e$ then we have $\Omega_e = \Pi \ltimes \Omega$. Note that $\Pi$ permutes the weights that belongs to the closure of $A_0$. Further the group $\Pi$ is isomorphic to $P/Q$, hence abelian, and its action on $\Omega$ is given by an automorphism of the Dynkin diagram; see Planches I-IX in [1].

An extended alcove is a pair $(A, \mu)$ where $A \in \text{Alc}(\mathcal{F})$ and $\mu$ is a vertex of $A$ which lies in $P$. We denote by $\text{Alc}_{e}(\mathcal{F})$ the set extended alcoves, then, the group $\Omega_e$ acts naturally on $\text{Alc}_{e}(\mathcal{F})$ and the action is faithfull and transitive.

We set $W_e \simeq \Pi \ltimes W_a$. We define a left action of $W_e$ on $\text{Alc}_{e}(\mathcal{F})$ as follows. Let $(A, \mu) \in \text{Alc}_{e}(\mathcal{F})$ and let $(a, s) \in \Pi \times S$. Let $f_s$ be the face of type $s$ of $A \in \text{Alc}(\mathcal{F})$, by which we mean that $sA_0 = A_0\sigma_{f_s}$ where $\sigma_{f_s}$ is the reflection with respect to the hyperplane that contains $f_s$. Let $\sigma_{A,\mu} \in \Omega_e$ be such that $(A_0, 0)\sigma_{A,\mu} = (A, \mu)$. We set

$$s(A, \mu) = (A, \sigma_{\Omega_e} \mu)$$

where $\sigma$ is the reflection that contains the face $f_s\sigma_{A,\mu}$.

$$a(A, \mu) = (A, \mu_0)$$

where $\mu_0 = \sigma_{A,\mu}$.

It can be shown that the action of $\Omega_e$ and the action of $W_e$ commute.

To simplify notation we will simply write $A_0 \in \text{Alc}_{e}(\mathcal{F})$ for the alcove $(A_0, 0)$. Similarly, we write $A_{\lambda}$ and $A_{\lambda}^\ast$ for the alcoves $(A_\lambda, \lambda)$ and $(A_{\lambda}^\ast, \lambda)$. Let $p_w \in W_e$ and $p_w^\lambda$ be such that $p_wA_0 = A_\lambda$ and $p_w^\lambda A_0 = A_{\lambda}^\ast$. Finally for all $w \in W_e$ we will denote by $\sigma_w$ the unique element of $\Omega_e$ such that $w(A_0, 0) = (A_0, 0)\sigma_w$.

All the notion and notation for alcoves can be extended to $\text{Alc}_{e}(\mathcal{F})$. We just omit the part with the weight when needed. For instance if $A' = (A, \lambda) \in \text{Alc}_{e}(\mathcal{F})$, we write $A' [\gamma] < 0$ to mean $A[\gamma] < 0$. The length function, the weight function, the Bruhat order all naturally extend to $W_e$ and $\Omega_e$ by setting $\ell(wa) = \ell(w)$, $L(wa) = L(w)$ and $aw < a'w'$ if and only if $a = a'$ and $w < w'$ where $a, a' \in \Pi$ and $w, w' \in W_e$. 

2.4 Quarter of vertex $\lambda$

The quarters of vertex $\lambda$ are the connected components of

$$V \setminus \bigcup_{H \in \mathcal{P}, \lambda \in H} H.$$  

Given $\lambda \in P$ and $v \in W_0$, we let $\mathcal{C}^+_\lambda,v$ and $\mathcal{C}^-_\lambda,v$ the quarter of vertex $\lambda$ which contains $vA_\lambda$ and $vA_\lambda^-$ respectively. When $v = 1$, we will omit the 1 in the notation $\mathcal{C}^+_\lambda,1$. The fundamental Weyl chamber is

$$\mathcal{C}^+_0 := \{ x \in V \mid \langle x, \alpha^\vee \rangle > 0 \text{ for all } \alpha \in \Phi^+ \}.$$  

Let $X_0$ be the set of distinguished left coset representatives of $W_0$ in $W_e$. Any element $w$ of $W_e$ can be uniquely written under the form $w = xv$ where $v \in W_0$ and $x \in X_0$. In view of Proposition 2.1 we have

$$x \in X_0 \iff xA_0 \in \mathcal{C}^+_0 \iff xA_0[\alpha] > 0 \text{ for all } \alpha \in \Delta.$$  

For all $x \in X_0$, $\lambda \in P$ and $v \in W_0$ we have $xvA_\lambda \in \mathcal{C}^+_\lambda,v$.

Let $\mathcal{C}$ be a quarter of vertex $\lambda \in P$ and fix $\gamma \in \Phi^+$. We have either

$$\{ (x, \gamma^\vee) \mid x \in \mathcal{C} \} = [\lambda_\gamma, +\infty[ \text{ or } \{ (x, \gamma^\vee) \mid x \in \mathcal{C} \} = ]-\infty, \lambda_\gamma[.$$  

In the first case we say that $\mathcal{C}$ is oriented toward $+\infty$ in the direction $\gamma$. In the second case we say that $\mathcal{C}$ is oriented toward $-\infty$ in the direction $\gamma$.

**Lemma 2.3.** Let $\lambda \in P$ and $v \in W_0$. We have for all $\gamma \in \Phi^+$ :

$$\mathcal{C}^+_{\lambda,v}[\gamma] = \begin{cases} +\infty & \text{if } \gamma \cdot \sigma_v^{-1} \in \Phi^+ \\ -\infty & \text{if } \gamma \cdot \sigma_v^{-1} \in \Phi^- \end{cases}$$

**Proof.** Let $x \in \mathcal{C}^+_{\lambda,v}$. There exists $x_0 \in \mathcal{C}^+_{0,v}$ and $y_0 \in \mathcal{C}^+_0$ such that $x = x_0 + \lambda$ and $x_0 = y_0 \sigma_v$. We have

$$\langle x, \gamma^\vee \rangle = \langle x_0 + \lambda, \gamma^\vee \rangle = \langle y_0 \sigma_v, \gamma^\vee \rangle + \lambda_\gamma = \lambda_\gamma + \langle y_0, (\gamma \sigma_v^{-1})^\vee \rangle$$

and the result follows by definition of $\mathcal{C}^+_0$. \hfill $\Box$

Similarly, we have

$$\mathcal{C}^-_{\lambda,v}[\gamma] = \begin{cases} -\infty & \text{if } \gamma \cdot \sigma_v^{+1} \in \Phi^+ \\ +\infty & \text{if } \gamma \cdot \sigma_v^{+1} \in \Phi^- \end{cases}.$$

3 Affine Hecke algebras with unequal parameters

3.1 Affine Hecke algebra

Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$ where $q$ is an indeterminate. The Iwahori-Hecke algebra $\mathcal{H}$ associated to $W_e$ is the free $\mathcal{A}$-module with basis $\{ T_w \mid w \in W_e \}$ and relation given by

$$T_uT_v = T_{uv} \text{ whenever } \ell(uv) = \ell(u) + \ell(v) \quad \text{and} \quad (T_s - q^L(s))(T_s + q^{-L(s)}) = 0 \text{ if } s \in S.$$  

From this relation, we easily find that for all $s \in S$ and all $w \in W$, we have

$$T_sT_w = \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ T_{sw} + \xi_s T_w & \text{if } \ell(sw) < \ell(w) \end{cases} \quad \text{where } \xi_s = q^{L(s)} - q^{-L(s)}.$$  

The basis $(T_w)_{w \in W_e}$ is called the standard basis. We write $f_{x,y,z}$ for the structure constants associated to the standard basis:

$$T_xT_y = \sum_{z \in W_e} f_{x,y,z} T_z.$$  

The elements $f_{x,y,z}$ are polynomials in $\{ \xi_s \mid s \in S \}$ with positive coefficients. The degree of $f_{x,y,z}$ will be denoted $\text{deg}(f_{x,y,z})$ and is the highest power of $q$ that appear in $f_{x,y,z}$. 
3.2 Multiplication of the standard basis

In this section, we present a result of [4] on a bound on the degree of the polynomials \(f_{x,y,z}\). Recall that for \(A, B \in \text{Alg}_e(\mathcal{F})\), we have set

\[ H(A, B) = \{ H \in \mathcal{F} \mid H \text{ separates } A \text{ and } B \}. \]

Then for \(x, y \in W_e\) we set

\[ H_{x,y} = H(A_0, yA_0) \cap H(xyA_0, yA_0). \]

For \(\alpha \in \Phi^+\) we set

\[ c_{x,y}(\alpha) := \max_{H \in H(x,y)} L_H. \]

Then according to [4, Theorem 2.4] we have:

**Theorem 3.1.** The degrees of the polynomials \(f_{x,y,z}\) are bounded by \(\sum_{\alpha \in \Phi^+} c_{x,y}(\alpha)\).

We obtain the following corollary [5, Proposition 5.3] which will be crucial in the following section.

**Corollary 3.2.** Let \(y \in W_e\) and \(x \in X_0\). Let \((y_0, y_r) \in W_0 \times X_0^{-1}\) be such that \(y = y_0y_r\). Then

\[ \deg(f_{x,y,z}) \leq L(y_0) + L(y_0). \]

3.3 Kazhdan-Lusztig basis

Let \(T\) the ring involution of \(\mathcal{H}\) which takes \(q\) to \(q^{-1}\). This involution can be extended to a ring involution of \(\mathcal{H}\) via the formula

\[ \sum_{w \in W_e} a_w T_w = \sum_{w \in W_e} T^{-1}_{w^{-1}} a_w \quad (a_w \in \mathcal{A}). \]

We set

\[ \mathcal{A}_{\leq 0} = q^{-1}\mathbb{Z}[q^{-1}] \quad \text{and} \quad \mathcal{H}_{\leq 0} = \bigoplus_{w \in W_e} \mathcal{A}_{\leq 0} T_w. \]

We define \(\mathcal{A}_{<0}\) and \(\mathcal{H}_{<0}\) in a similar manner. For each \(w \in W_e\) there exists a unique element \(C_w \in \mathcal{H}\) (see [II, Theorem 5.2]) such that (1) \(C_w \equiv C_w\) and (2) \(C_w \equiv T_w \mod \mathcal{H}_{<0}\). For any \(w \in W_e\) we set

\[ C_w = T_w + \sum_{y \in W_e} P_{y,w} T_y \quad \text{where } P_{y,w} \in \mathcal{A}_{<0}. \]

The coefficients \(P_{y,w}\) are called as the Kazhdan-Lusztig polynomials. It is well known ([II, §5.3]) that \(P_{y,w} = 0\) whenever \(y \not\leq w\). It follows that \((C_w)_{w \in W_e}\) forms an \(\mathcal{A}\)-basis of \(\mathcal{H}\) known as the Kazhdan-Lusztig basis.

**Remark 3.3.** Using Corollary 3.2 and the definition of the Kazhdan-Lusztig basis, one can show that \(T_{x_{\lambda}}C_{P_{\lambda}}w_0 \in \mathcal{H}_{\leq 0}\) for all \(\lambda \in P^+\).

3.4 The center of the affine Hecke algebra

In this section we follow the presentation of Nelsen and Ram [7] and we refer to it and the references therein for details and proofs.

We start by introducing another presentation of the affine Hecke algebra which is more convenient to describe its center \(Z(\mathcal{H})\). For each \(\lambda \in P\), we set \(e^\lambda := T_{p_\mu} T_{p_\nu}^{-1}\) where \(\mu, \nu \in P^+\) are such that \(\mu - \nu = \lambda\). This can be shown to be independent of the choice of \(\mu\) and \(\nu\). Then \(\mathcal{H}\) is generated by the sets \(\{T_s \mid s \in S_0\} \text{ and } \{e^\lambda \mid \lambda \in P\}\) and we have the relations

\[ e^\lambda e^\mu = e^{\lambda+\mu} = e^\mu e^\lambda \quad \text{and} \quad e^\lambda T_\alpha = T_\alpha e^{\lambda-\alpha} + \xi_\alpha e^{\lambda-\alpha} \quad \text{for all } \alpha \in \Phi. \]

where \(s\) is such that \(sA_0 = A_0\sigma_\mu\). Let \(\mathcal{A}[P]\) be the subalgebra of \(\mathcal{H}\) generated by \(\{e^\lambda \mid \lambda \in P\}\). The group \(\Omega_0\) acts naturally on \(\mathcal{A}[P]\) via \(e^\lambda \sigma = e^{\lambda \sigma}\).

**Theorem 3.4.** The sets \(\{e^\lambda T_w \mid \lambda \in P, w \in W_0\}\) and \(\{T_w e^\lambda \mid w \in W_0, \lambda \in P\}\) are \(\mathcal{A}\)-basis of \(\mathcal{H}\). The center \(Z(\mathcal{H})\) of \(\mathcal{H}\) is

\[ \mathcal{A}[P]^{\Omega_0} = \{ f \in \mathcal{A}[P] \mid f \cdot \sigma = f \text{ for all } \sigma \in \Omega_0 \}. \]
Let \( \mathcal{A}[P]^{\Omega_0} = \{ f \in \mathcal{A}[P] \mid f \cdot \sigma = (-1)^{\ell(\sigma)} f \text{ for all } \sigma \in \Omega_0 \} \) be the sets of antisymmetric functions and let

\[ a_\mu := \sum_{\sigma \in \Omega_0} (-1)^{\ell(\sigma)} \nu_{\mu \sigma} \quad \text{and} \quad s_\mu = \frac{a_{\mu + \rho}}{a_\rho} \quad \text{where } \rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha. \]

The elements \( s_\mu \) which are called the Weyl characters are well-defined and the set \( \{ s_\mu \mid \mu \in P^+ \} \) forms a basis of \( \mathcal{A}[P]^{\Omega_0} \). Recall the definition of the coefficients \( m_{r,\lambda}^{\mu} \) in the introduction.

**Theorem 3.5.** We have \( s_\tau C_{w_0} = C_{w_0} s_\tau = C_{p_\tau w_0} \) for all \( \tau \in P^+ \) and

\[ s_\tau C_{p_\lambda w_0} = s_\tau s_\lambda C_{w_0} = \sum_{\mu} m_{r,\lambda}^{\mu} C_{p_\mu w_0}. \]

### 4 Decomposition into the Kazhdan-Lusztig basis

The aim of this section is to prove Statement (1) in the introduction, that is, in order to determine the decomposition of \( s_\tau C_{p_\lambda w_0} \) in the Kazhdan-Lusztig basis, it is enough to determine the terms of maximal degree in product of the form \( T_{p_\tau} T_{p_\lambda w_0} \).

Let \( x \in X_0 \) (respectively \( y \in X_0^{-1} \)). There exists a unique family of polynomials \( (p_{x',x})_{x',x} \in X_0 \) in \( \mathcal{A}[x_0] \) (respectively \( (p_{x',x})_{x',x} \in X_0 \) in \( \mathcal{A}[x_0^{-1}] \)) such that \( p_{x',x} = 0 \) whenever \( x' \not\leq x \) (respectively \( p_{y',y} = 0 \)) and

\[ C_{x w_0} = T_x C_{w_0} + \sum_{x' \in X_0} p_{x',x} T_{x'} C_{w_0} \quad \text{and} \quad C_{w_0 y} = C_{w_0} T_y + \sum_{y' \in X_0^{-1}} p_{y',y} C_{w_0} T_{y'}. \]

The polynomials \( p \) are called relative Kazhdan-Lusztig polynomials [3]. In our case, it can be shown [12] that we have \( p_{x',x} = P_{x w_0, x' w_0} \) although it is sufficient for us to know that they belong to \( \mathcal{A}[x_0] \). Finally, we set

\[ P(x) = \sum_{x' \in X_0} p_{x',x} T_{x'} \quad \text{and} \quad P_R(y) = \sum_{y' \in X_0^{-1}} p_{y',y} T_{y'}. \]

so that \( P(x) C_{w_0} = C_{x w_0} \) and \( C_{w_0} P_R(y) = C_{w_0 y} \). When \( \tau \in P^+ \), we know that \( p_\tau \in X_0 \). To lighten notation, we will write \( P(\tau) \) instead of \( P(p_\tau) \). For all \( \tau, \lambda \in P^+ \) we have

\[ s_\tau C_{w_0} = P(\tau) C_{w_0} \quad \text{and} \quad s_\tau C_{p_\lambda w_0} = s_\tau s_\lambda C_{w_0} = P(\tau) C_{w_0} s_\lambda = P(\tau) C_{p_\lambda w_0}. \]

Let \( \tau, \lambda \in P^+ \). Using Remark [3,3] we get

\[ P(\tau) C_{p_\lambda w_0} = T_{p_\tau} C_{p_\lambda w_0} + \sum_{x' \in X_0} p_{x',x} T_{x'} C_{p_\lambda w_0} \]

\[ \equiv T_{p_\tau} C_{p_\lambda w_0} \mod \mathcal{A}[x_0]. \]

Next

\[ T_{p_\tau} C_{p_\lambda w_0} = T_{p_\tau} T_{p_\lambda w_0} + \sum_{y < p_\lambda w_0} P_{y, p_\lambda w_0} T_{p_\tau} T_y. \]

Let \( y \in W_2 \) and \( (y_0, y) \in W_0 \times X_0^{-1} \) be such \( y = y_0 y \). On the one hand, we have

\[ P_{y, p_\lambda w_0} = P_{y_0 y_0, p_\lambda w_0} = q^{L(y_0) - L(w_0)} P_{y_0 y_0, p_\lambda w_0}. \]

On the other hand the maximal degree that can appear in \( T_{p_\tau} T_y \) is \( L(w_0) - L(y_0) \). Therefore if \( w_0 y_0 < w_0 p_\lambda \), we get that \( P_{y, p_\lambda w_0} T_{p_\tau} T_y \in \mathcal{A}[x_0] \) and

\[ T_{p_\tau} C_{p_\lambda w_0} \equiv T_{p_\tau} T_{p_\lambda w_0} + \sum_{y \in W_0} q^{L(y) - L(w_0)} T_{p_\tau} T_{y_0 p_\lambda} \mod \mathcal{A}[x_0] \]

\[ \equiv T_{p_\tau} T_{w_0 p_\lambda} + \sum_{v \in W_0} q^{-L(v)} T_{p_\tau} T_{v w_0 p_\lambda} \mod \mathcal{A}[x_0]. \]

Finally

\[ P(\tau) C_{p_\lambda w_0} \equiv T_{p_\tau + \lambda} w_0 + \sum_{v \in W_0} q^{-L(v)} T_{p_\tau} T_{v w_0 p_\lambda} \mod \mathcal{A}[x_0]. \]
The element $P(\tau)C_{w_0p_\lambda}$ are stable under the $\tau$-involution, thus to determine the decomposition of $P(\tau)C_{p_\lambda w_0}$ in the Kazhdan-Lusztig basis, we need to determine which products $T_{\tau p_\lambda}T_{w_0p_\lambda}$ can actually give rise to a coefficient of degree $L(\nu)$. To simplify notation we now take $\lambda \in P^-$. Let $x,y \in W_\nu$ and fix a reduced expression $as_n\ldots s_1$ of $x$ where $a \in \Pi$ and $s_i \in S$ for all $i$. Let $J = \{i_1, \ldots, i_p\}$ be a subset of $\{1, \ldots, n\}$. For all $1 \leq \ell \leq k < n$, we set

$$x^J[k,\ell] = \prod_{r=\ell}^k s_r \quad \text{and} \quad x^J[n,\ell] = \prod_{r=\ell}^n s_r.$$ 

We denote by $J_{x,y}$ the set of all subsets $\{i_1, \ldots, i_p\}$ of $\{1, \ldots, n\}$ such that $1 \leq i_1 < \ldots < i_p \leq k$ and $s_{i_1}x^J[i_1-1,i] < x^J[i_1-1,i]$ for all $t \in \{1, \ldots, p\}$.

It should be noted that the set $J_{x,y}$ depends on the reduced expression of $x$. For $J = \{i_1, \ldots, i_p\}$ in $J_{x,y}$, we set $\xi_J = \prod_{k=1}^n \xi_{s_k}$. Then we have \[2\] Proof of Proposition 5.1

$$T_xT_y = \sum_{J \in J_{x,y}} \xi_J T_{x^J[n,1]y^J[n,1]}.$$ 

We fix a reduced expression $as_n\ldots s_1$ of $p_\tau$. Then

$$T_{p_\tau}T_{w_0p_\lambda} = T_{p_\tau}T_{v\lambda} = \sum_{J \in J_{p_\tau,v\lambda}} \xi_J T_{p_\tau^J[n,1]v\lambda}.$$ 

If we set (keeping in mind that a reduced expression for $p_\tau$ is fixed)

$$J_{\tau,v\lambda} := J_{p_\tau,v\lambda}^- \quad \text{and} \quad J_{\tau,v\lambda}^\max := \{J \in J_{p_\tau,v\lambda} \mid \xi_J = L(\nu)\}$$

then, we have

$$P(\tau)C_{w_0p_\lambda} = \sum_{\nu \in W_0} \sum_{J \in J_{\tau,v\lambda}^\max} C_{p_\lambda^J[n,1]v\lambda}.$$ 

we now need to determine the sets $J_{\tau,v\lambda}^\max$ for all $\nu \in W_0$ and all $\lambda \in P^-$. 

5 Description of $J_{\tau,v_\lambda}^\max$ in terms of admissible subsets 

Once and for all in this section, we fix a reduced expression $p_\tau = as_n\ldots s_1$ where $a \in \Pi$ and $s_i \in S$. Let $(\beta_1, \ldots, \beta_n) \in (\Phi^+)^n$ be such that

$$s_i \ldots s_1A_0 = A_0\sigma_{\beta_i}N_1 \ldots \sigma_{\beta_i}N_i \quad \text{for all} \quad i \in \{1, \ldots, n\}.$$ 

Recall that for $\beta \in \Phi^+$, we have set $\beta^* := \beta\sigma_0 \in \Phi^-$. We have

$$p_\tau A_0 = p_\tau A_0 \sigma_0 \sigma_0 = A_0\sigma_{\beta_1}N_1 \ldots \sigma_{\beta_n}N_n \sigma_0 = A_0^\tau \sigma_{\beta_1}N_1 \ldots \sigma_{\beta_n}N_n.$$ 

Following \cite{9}, we now introduce the concept of admissible subsets. Recall that a chain $(\sigma_0, \sigma_1, \ldots, \sigma_p) \in \Omega_0^p$ is an increasing saturated chain in the Bruhat order if and only if

$$\sigma_{i-1} < \sigma_i \quad \text{and} \quad \ell(\sigma_i) = \ell(\sigma_{i-1}) + 1 \quad \text{for all} \quad i \in \{1, \ldots, p\}.$$ 

Definition 5.1. A subset $J = \{i_1, \ldots, i_p\}$ of $\{1, \ldots, n\}$ will be called an admissible subset if

$$e < \sigma_{\beta_{i_p}} < \sigma_{\beta_{i_{p-1}}} \sigma_{\beta_{i_p}} < \ldots < \sigma_{\beta_1} \ldots \sigma_{\beta_{i_{p-1}}} \sigma_{\beta_{i_p}}$$ 

is a saturated chain in the Bruhat order on $\Omega_0$. 

Conjugating by $\sigma_\Omega_0$, we see that $J = \{i_1, \ldots, i_p\}$ is admissible if and only if

$$e < \sigma_{\beta_{i_p}}^* < \sigma_{\beta_{i_{p-1}}}^* \sigma_{\beta_{i_p}}^* < \ldots < \sigma_{\beta_1}^* \ldots \sigma_{\beta_{i_{p-1}}}^* \sigma_{\beta_{i_p}}^*$$

is a saturated chain in the Bruhat order on $\Omega_0$. 

Remark 5.2. Our definition of admissible subset is slightly different than the one in \cite{9} where they multiply on the right in the definition. This is because, in our work, reduced expressions of $p_{-\tau}$ naturally appear whereas the authors in \cite{9} work with reduced expressions of $p_{-\tau}$. The relationship between those two definitions will be made explicit in the next section.
Definition 5.3. Let $J = \{i_1, \ldots, i_p\}$ be an admissible subset. Let $\sigma_J = \sigma_{i_1} \ldots \sigma_{i_{p-1}} \sigma_{i_p}$ and $v_J \in W_0$ be such that $v_J A_0 = A_0 \sigma_J$. We say that $J$ is

1) $\mu$-dominant for $\mu \in P^+$ if the alcoses $p_r^{\mu} [k, 1] v_J A_\mu \in C_0^+$ for all $k \in \{1, \ldots, n\}$,

2) maximal if $L(\ell_j) = L(\ell_j)$ for all $1 \leq \ell \leq p$,

If $\mu \in P^-$, we will say that $J$ is $\lambda$-antidominant if $p_r^{\mu} [k, 1] v_J A_\mu \in C_0^-$ for all $k \in \{1, \ldots, n\}$. We can easily see that $J$ is $\mu$-antidominant if and only if $J$ is $\mu^*$-dominant.

We have $L(\ell_j) = L(\ell_j)$ if and only if $\tau_j = N_{i_j} A_{i_j}$. In particular, we see that if $J \in P_{\tau, \varepsilon, \lambda}$, then $J$ must be maximal in order to have deg($\xi_j$).

Let $\beta \in \Phi^+$ and $\sigma \in \Omega_0$. We have [6, §5.7] $\sigma < \sigma_\beta$ if and only if $\beta \sigma \in \Phi^+$. If $J = \{i_1, \ldots, i_p\}$ is an admissible subset, then, since $\sigma_{i_1} \ldots \sigma_{i_p} < \sigma_{i_1} \ldots \sigma_{i_p}$, we have $\beta_{i_1} \sigma_{i_1} \ldots \sigma_{i_p} \in \Phi^+$. Similarly, since $\beta_{i_1}^\tau$ is a negative root, we have

$$\sigma_{i_{p+1}} \ldots \sigma_{i_p} < \sigma_{i_{p+1}} \ldots \sigma_{i_p} \Rightarrow (\beta_{i_{p+1}}^\tau) \sigma_{i_{p+1}} \ldots \sigma_{i_p} \in \Phi^+ \Rightarrow (\beta_{i_p}^\tau) \sigma_{i_p} \ldots \sigma_{i_p} \in \Phi^+.$$

Note that if $\{i_1, \ldots, i_p\}$ is an admissible subset then so is $\{i_1, \ldots, i_p\}$ for all $\ell \leq p$; we’ll denote this subset by $J_{\ell - 1}$ so that $J_0 = J$ and $J_p = \emptyset$. Then according to Definition 5.3 we have

$$\sigma_J = \sigma_{i_{p+1}} \ldots \sigma_{i_p} \quad \text{and} \quad v_J A_0 = A_0 \sigma_J.$$

We are now ready to state the main result of this section. Recall the notation introduced at the end of the previous section.

Theorem 5.4. Let $\lambda \in P^-$ and $v \in W_0$.

1) If $J$ is admissible, $\lambda$-antidominant and maximal then $J \in P_{\tau, \varepsilon, \lambda}$.

2) If $J \in P_{\tau, \varepsilon, \lambda}$ then $v = v_J$, $J$ is admissible, $\lambda$-antidominant and maximal.

The rest of this section is devoted to the proof of this theorem.

Proposition 5.5. Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \ldots, n\}$.

1) The unique hyperplane separating the alcoses $p_r [k-1, 1] v A_\lambda$ and $p_r [k, 1] v A_\lambda$ is $H_k := H_{\beta_{i_1} \sigma_{i_1} (\lambda, \beta_{i_1}^\tau)}$.

2) We have $p_r [k-1, 1] v A_\lambda = p_r [k, 1] \psi' A_{\psi'}$ where $\sigma_{i'_{p+1}} \lambda = \lambda + N_{i'} \beta_{i'} \sigma_{i'}$.

Proof. The (unique) hyperplane separating $p_r [k-1, 1] A_0$ and $p_r [k, 1] A_0$ is $H_{\beta_{i_1} \sigma_{i_1}}$. It follows that the hyperplane separating $p_r [k-1, 1] v A_0$ and $p_r [k, 1] v A_0$ is $H_{\beta_{i_1} \sigma_{i_1}}$. Finally, the hyperplane separating $p_r [k-1, 1] v A_\lambda$ and $p_r [k, 1] v A_\lambda$ is $H_{\beta_{i_1} \sigma_{i_1} (\lambda, \beta_{i_1}^\tau)}$. Next we have

$$p_r [k, 1] v A_\lambda = p_r [k, 1] v A_\lambda \sigma H_k = p_r [k, 1] v A_\lambda \sigma_{i_1} \sigma H_k = p_r [k, 1] v A_0 \sigma_{i_1} \lambda.$$

as required.

Given any triplet $(\lambda, v, J)$ such that $(\lambda, v) \in X \times W_0$ and $J = \{i_1, \ldots, i_p\} \in P_{\tau, \varepsilon, \lambda}$, we associate sequences $\nu = (\nu_1, \ldots, \nu_p)$, $\gamma = (\gamma_1, \ldots, \gamma_p)$ and $\lambda = (\lambda_0, \ldots, \lambda_p)$ defined by the following relations for $\ell \in \{1, \ldots, p\}$:

- $v_0 = v$, $\sigma_0 = \sigma v$ and $v_\ell A_0 = A_0 \sigma_{i_\ell} \ldots \sigma_{i_1} \sigma v$;
- $\gamma_\ell = \beta_{i_\ell} \gamma_{i_\ell - 1}$;
- $\lambda_0 = \lambda$ and $\lambda_\ell = \lambda_{\ell - 1} + N_{i_\ell} \gamma_{i_\ell} = \lambda_{\ell - 1} - \sigma_{i_\ell} N_{i_\ell} (\lambda_{\ell - 1} + \gamma_{i_\ell})$.

With these notation, we have for all $0 \leq \ell \leq p$:

$$p_r [\nu_{\ell - 1}, 1] v A_\lambda = p_r [\nu_{\ell - 1}, 1] v_{\ell - 1} A_\lambda = p_r [\nu_{\ell}, 1] v A_\lambda.$$

If we choose $v = v_J$ then $v_\ell = v_{J_\ell}$, $\gamma_\ell = \beta_{i_\ell} \gamma_{i_\ell - 1}$ and

$$\sigma_{i_\ell} \ldots \sigma_{i_p} \sigma_{i_p} = \gamma_{i_p} \ldots \gamma_{i_\ell}.$$

We are now ready to prove the first part of Theorem 5.4.
Proposition 5.6. Let $\lambda \in P^-$. If $J$ is admissible, $\lambda$-antidominant and maximal then $J \in \mathcal{J}^\max_{\tau,v,\lambda}$.

Proof. In order to prove that $J = \{i_1, \ldots, i_p\} \in \mathcal{J}^\max_{\tau,v,\lambda}$, since $J$ is maximal, we only need to show that

$$s_{i_\ell}p^H_{\ell}[i_{\ell-1},1]v_Jp^\lambda_\tau < p^H_{\ell}[i_{\ell-1},1]v_Jp^\lambda_\tau$$

for all $1 \leq \ell \leq p$.

We have

1) $p^H_{\ell}[i_{\ell-1},1]v_Jp^\lambda_\tau = p^H_{\ell}[i_{\ell-1},1]v_Jp^\lambda_{\tau-1}$;

2) $(-\beta^*_p)\sigma_{i_{\ell+1}} \cdots \sigma_{i_p} = \beta^*_p \sigma_{i_{\ell+1}} = \gamma_{i_{\ell}} \in \Phi^+$ (since $J$ is admissible);

3) the hyperplane separating $p^H_{\ell}[i_{\ell-1},1]v_JA^\lambda_\tau$ and $s_{i_\ell}p^H_{\ell}[i_{\ell-1},1]v_JA^\lambda_\tau$ is equal to $H_{\gamma_{i_{\ell}}}$, where $m < 0$ (since $J$ is $\lambda$-antidominant).

It follows from the second statement that $\gamma_{i_{\ell}}\sigma_{j_{\ell-1}}^\lambda = \beta^*_p \in \Phi^-$. The quarter $\mathcal{C}^-_{\lambda,-1,\tau}$ is therefore oriented towards $+\infty$ in the direction $\gamma_{i_{\ell}}$. The result follows by Proposition 2.4 and statement 3.

We now focus on the second part. We start by proving some technical lemmas.

Lemma 5.7. Let $\lambda \in P^-$ and $v \in W_0$. We have

$$\overline{H^\lambda_{p,v,\lambda}} \subset \{\delta \in \Phi^+ | \delta \sigma^\lambda_{v} \in \Phi^-\}.$$

Proof. Let $\delta \in \Phi^+$ be such that $H_{\delta,N} \in H_{p,v,\lambda} = H(A_0, vA^\lambda_\tau) \cap H(p,vA^\lambda_\tau, vA^\lambda_\tau)$. First of all, since $\lambda \in P^-$, we have $vA^\lambda_\tau[\delta] < 1$. It follows that $vA^\lambda_\tau[\delta] = N \leq 0$. If the quarter $\mathcal{C}^-_{\tau,v,\lambda}$ is oriented towards $-\infty$ in the direction $\delta$ then $vA^\lambda_\tau[\delta] \geq p_v vA^\lambda_\tau[\delta]$ so that $vA^\lambda_\tau[\delta] > N$ which is impossible. This shows that the quarter $\mathcal{C}^-_{\lambda,v}$ is oriented towards $+\infty$ in the direction $\delta$ and thus $\delta \sigma^\lambda_{v} \in \Phi^-$ as required.

We remark that if $vA^\lambda_\tau \notin \mathcal{C}^-_{0}$ then there exists a root $\delta \in \Phi^+$ such that $0 < vA^\lambda_\tau[\delta] < 1$ and $\delta \sigma^\lambda_{v} \in \Phi^-$ which in turn implies that the inclusion $\mathcal{C}^-_{p,v,\lambda} \subset \{\delta \in \Phi^+ | \delta \sigma^\lambda_{v} \in \Phi^-\}$ is strict. Then $\mathcal{J}^\max_{\tau,v,\lambda}$ has to be empty by Theorem 3.4.

Lemma 5.8. Let $\lambda \in P^-$ and $v \in W_0$. If $J = \{i_1, \ldots, i_p\} \in \mathcal{J}_{\tau,v,\lambda}$ is such that $p_v[i_0,1]vA^\lambda_\tau \notin \mathcal{C}^-_{0}$ for some $k_0 \in \{0, \ldots, i_1-1\}$ then

$$\overline{H_{p_v[i_1,1],p_v[i_1,1],v,\lambda}} \subset \{\delta \in \Phi^+ | \delta \sigma^\lambda_{v} \in \Phi^-\}.$$

Proof. Let $\delta \in \Phi^+$ be such that $H_{\delta,N} \in H_{p_v[i_1,1],p_v[i_1,1],v,\lambda}$ where we recall that

$$H_{p_v[i_1,1],p_v[i_1,1],v,\lambda} = H(A_0, p_v[i_1-1,1]vA^\lambda_\tau) \cap H(p_v vA^\lambda_\tau, p_v[i_1-1,1]vA^\lambda_\tau).$$

Assume first that the $\mathcal{C}^-_{\lambda,v}$ is oriented towards $-\infty$. Then we must have

$$vA^\lambda_\tau[\delta] \geq p_v[i_0,1]vA^\lambda_\tau[\delta] \geq p_v[i_1-1,1]vA^\lambda_\tau[\delta] \geq p_v vA^\lambda_\tau[\delta].$$

But $H_{\delta,N} \in H(A_0, p_v[i_1-1,1]vA^\lambda_\tau)$ implies that $0 \geq N > p_v[i_1-1,1]vA^\lambda_\tau[\delta]$. Therefore in this case we cannot have $H_{\delta,N} \in H(p_v vA^\lambda_\tau, p_v[i_1-1,1]vA^\lambda_\tau)$. This shows that the quarter $\mathcal{C}^-_{\tau,v}$ has to be oriented towards $+\infty$ in the direction $\delta$ and therefore we have $\delta \sigma^\lambda_{v} \in \Phi^-$. 

Lemma 5.9. Let $v \in W_0$, $\lambda \in P$ and fix $k \in \{1, \ldots, n\}$ such that

$$s_k p_v[k-1,1]v, \lambda < p_v[k-1,1]v, \lambda \quad \text{and} \quad p_v[k-1,1]v, \lambda \in \mathcal{C}^-_{0}.$$

Then $\beta^*_k \sigma^\lambda_{v} \in \Phi^+.$

Proof. The unique hyperplane $H$ that separates $p_v[k-1,1]v, \lambda$ and $s_k p_v[k-1,1]v, \lambda$ is

$$H := H_{\beta^*_k \sigma^\lambda_{v},N_k,\lambda}.$$

Since $p_v[k-1,1]v, \lambda \in \mathcal{C}^-_{0}$, we must have $p_v[k-1,1]v, \lambda \in H^-$ and $s_k p_v[k-1,1]v, \lambda \in H^+$, which means that the quarter $\mathcal{C}^-_{\lambda,v}$ has to be oriented toward $+\infty$ in the direction $|\beta^*_k \sigma^\lambda_{v}| \in \Phi^+$. According to Lemma 2.3 since $\beta^*_k \sigma^\lambda_{v} \sigma^\lambda_{v} \in \Phi^-$, we must have $\beta^*_k \sigma^\lambda_{v} \in \Phi^+$.

Proposition 5.10. Let $\lambda \in P^-$, $v \in W_0$. If $J \in \mathcal{J}^\max_{\tau,v,\lambda}$ then $J$ is $\lambda$-antidominant.
Proof. Let \( J = \{i_1, \ldots, i_p\} \in \mathcal{J}^\text{max}_m \). We have already seen that if \( vA^-_\lambda \notin C_0^- \) then \( \mathcal{J}^\text{max}_m = \emptyset \). Thus we must have \( vA^-_\lambda \in C_0^- \). Assume that there exists an index \( k_0 \) such that \( p_{r^+}[k_0+1,1]vA^-_\lambda \notin C_0^- \) and choose \( k_0 \) to be minimal with this property. Let \( 0 < \ell \leq p \) be such that \( i_{\ell+1} > k_0 + 1 \) and \( i_{\ell+1} \equiv +\infty \). By minimality of \( k_0 \) we get \( p_r^+[\ell,v^+]vA^-_\lambda \in C_0^- \) for all \( k' \leq k_0 \).

We construct the sequences \( \underline{w} \) and \( \underline{\lambda} \) associated to the triplet \( (\lambda, v, J) \). By construction and by hypothesis we have
\[
s_i[p_r[v_{\ell+1},1]v_{\ell+1}p_{r^-}_\lambda,1] < p_r[v_{\ell+1},1]v_{\ell+1}p_{r^-}_\lambda,1 \quad \text{and} \quad p_r[v_{\ell+1},1]v_{\ell+1}p_{r^-}_\lambda,1 \notin C_0^-.
\]
Therefore, by Lemma 5.3 we have \( \beta^+_{\ell+1} \sigma_{\ell+1}^+, v_{\ell+1} \in \Phi^+ \), in other words \( -\beta^+_{\ell+1} \sigma_{\ell+1}^+ \in \Phi^+ \). This implies in particular that \( \sigma_{\ell+1} = \sigma_{\ell+1}^++ \sigma_{\ell+1}^- < \sigma_{\ell+1}^- \). Thus we have \( \sigma_{\ell+1} > \sigma_{\ell+1} \). We have \( L(\sigma_{\ell+1}) = L(\sigma_{\ell+1}) - \sum_{k=1}^\ell L(\beta^+_k) = \sum_{k=1}^\ell L(\beta^+_k) = \sum_{k=1}^\ell L(s_{ik}) \).

In particular, if \( \ell = p \), then we must have \( v_p = 1 \). We know that
\[
p_r^+[k_0,1]vA^- = p_r[k_0,1]vA^-_\lambda \quad \text{and} \quad s_{k_0+1}p_r^+[k_0,1]vA^- = s_{k_0+1}p_r[k_0,1]vA^-_\lambda.
\]
We denote by \( H := H_{k_0} \) the unique hyperplane separating \( s_{k_0+1}p_r^+[k_0,1]vA^-_\lambda \) and \( p_r^+[k_0,1]vA^-_\lambda \). Since \( p_r^+[k_0,1]vA^-_\lambda \in C_0^- \) and \( s_{k_0+1}p_r^+[k_0,1]vA^- \notin C_0^- \), the quarter \( C_{\lambda, v} \) has to be oriented toward \(+\infty \) in the direction \( \delta_0 \). This implies \( \delta_0 \in \{ \gamma \in \Phi^+ \mid |\gamma^+_{\ell+1}^+ \in \Phi^- \} \). Further, we have \( \ell < p \) since otherwise \( v_{\ell+1} = 1 \) and there is no such \( \delta_0 \).

The set \( J_\ell = \{i_{\ell+1}, \ldots, i_p\} \in \mathcal{J}_{v,A_\lambda} \) is non-empty and \( p_r^+[k_0,1]A^-_\lambda \in C_0^- \) (where \( k_0 < i_{\ell+1} \)) thus by Lemma 5.8 we know that
\[
H_{p_r^+[k_0,1], p_r^+[i_{\ell+1},1]v, p_r^+} \subseteq \{ \gamma \in \Phi^+ \mid |\gamma^+_{\ell+1}^+ \in \Phi^- \}.
\]
By definition of \( \delta_0 \) and since \( i_{\ell+1} > k_0 + 1 \) we have
\[
H_{p_r^+[k_0,1], p_r^+[i_{\ell+1},1]v, p_r^+} \subseteq \{ \gamma \in \Phi^+ \mid |\gamma^+_{\ell+1}^+ \in \Phi^- \} \backslash \{\delta_0\}
\]
and by Theorem 5.4 we obtain
\[
\sum_{k=1}^p L(s_{ik}) < \sum_{\gamma \in \Phi^+, |\gamma^+_{\ell+1}^+ \in \Phi^-} L(\gamma) = L(v_\ell).
\]
This contradicts \((*)\) and therefore, there cannot be such an index \( k_0 \).

Now that we know that if \( J \in \mathcal{J}^\text{max}_m \) then \( J \) is \( \lambda \)-antidominant, we can re-do the first part of the proof and show that we must have \( v = v_J \) (indeed in the case where \( J \) is \( \lambda \)-antidominant there is no index \( k_0 \) and therefore \( \ell = p \)).

The proof of the second part of Theorem 5.4 follows from Proposition 5.10.

**Corollary 5.11.** Let \( \lambda \in P^- \) and \( \tau \in P^+ \). We have
\[
P(\tau)C_{\lambda P^+} = \sum_J C_{p_r^+[v_J, w_0 P^+]}
\]
where the sum is taken over all \( \lambda \)-antidominant maximal admissible subset \( J \).

**Remark 5.12.** If we are working in the case of equal parameters, we can remove the condition of maximality since all parallel hyperplanes have same weights and therefore all admissible subsets are maximal.
6 Proof of the main result

Fix a reduced expression \( as_n \ldots s_1 \) of \( p_\tau \) and denote by \( H_{\beta_i,N_i} \) the corresponding hyperplanes as in the beginning of the previous section. As mentioned in Remark 3.2, the authors in [9] work with reduced expression of \( p_\tau \). Let \( a^{-1}s'_n \ldots s'_1 \) be the reduced expression of \( p_{-\tau} \) obtained by inverting the reduced expression above and moving \( a^{-1} \) to the left. Then, the hyperplane separating \( s'_n \ldots s'_1A_0 \) and \( s'_{n-1} \ldots s'_1A_0 \) is \( H_{\beta'_n,N'_n} \) where \( \beta'_n = \beta_{n-i+1} - (\tau, \beta'^{\tau}_{n-i+1}) \). The map \( i \mapsto n - i + 1 \) induces a bijection \( J \mapsto J' \) between admissible subsets in the sense of Definition 5.1 and admissible subsets in the sense of Lenart and Postnikov [9, Definition 6.1]. To distinguish between those two definitions, we will say that a subset is LP-admissible if it is admissible in the sense of [9].

We start with the definition of a gallery.

**Definition 6.1.** A gallery is a sequence \( \gamma = (\mu_0, B_0, F_1, B_1, \ldots, B_\ell, \mu_{\ell+1}) \) such that

1) \( B_0, \ldots, B_\ell \) are alcoves in \( \text{Alc}(\mathcal{F}) \);
2) for all \( k \in \{1, \ldots, \ell\} \), \( F_k \) is the common facet of \( B_k \) and \( B_{k-1} \);
3) \( \mu_0 \) and \( \mu_{\ell+1} \) are vertices of \( B_0 \) and \( B_\ell \) respectively which are elements of \( P \).

The weight of the gallery is \( F_{\ell+1} - F_0 \) and we denote it by \( \mathbf{w}(\gamma) \).

Let \( \gamma = (\mu_0, B_0, F_1, B_1, \ldots, B_\ell, \mu_{\ell+1}) \) be a gallery and let \( \sigma_k \) be the affine reflection with respect to the hyperplane which contains the facet \( F_k \). The \( k \)-th tail flip operator \( f_k \) is defined by

\[
 f_k(\gamma) = (F_0, B_0, F_0, \ldots, B_{k-1}, F_k = F_k, B_k', F_k', B_{k+1}, \ldots, B_{\ell}', F_{\ell+1}).
\]

The operators \( f_k \) commutes. For any subset \( J = \{i_1, \ldots, i_p\} \) of \( \{1, \ldots, \ell\} \), the weight of \( f_{i_1} \ldots f_{i_p} \gamma \) is \( \mu_{\ell+1}^{\sigma_{i_p}} \ldots \sigma_{i_1}^{\mu_0} \).

Let \( x = x_0p_\mu \in W_e \) where \( (x_0, \mu) \in W_0 \times P \) and let \( bt_n \ldots t_1 \) be an expression of \( x \) such that \( b \in \Pi \) and \( t_i \in S \) for all \( i \). Let \( (A, \mu_0) \in \text{Alc}(\mathcal{F}) \). To the reduced expression of \( x \) and the alcove \( (A, \mu_0) \) we associate a gallery \( \gamma = (\mu_0, A, F_1, A_1, \ldots, F_n, A_n, \mu_{n+1}) \) such that \( A_i = t_i \ldots t_1 A \) for all \( 1 \leq i \leq n \) and \( \mu_{n+1} \) is such that \( x(A, \mu_0) = (A_0, \mu_{n+1}) \). We denote by \( \gamma_{-\tau} \) the gallery associated to our reduced expression of \( p_{-\tau} \) and the alcove \( (A_0, 0) \). If \( J \) is an admissible subset we denote by \( \gamma_{J} \) the gallery associated to our reduced expression of \( p_{-\tau} \) and the alcove \( (v_J A_0, 0) \). Finally, let \( \sharp \) be the involution on the set of galleries defined by reading \( \gamma \) backward and by translating all of its components by \( -\mathbf{w}(\gamma) \).

**Proposition 6.2.** Let \( J = \{i_1, \ldots, i_p\} \) be a admissible subset and let \( J^\perp = \{j_1, \ldots, j_p\} \) be the corresponding LP-admissible subset. We have

\[
 (f_{j_1} \ldots f_{j_p}, \gamma_{-\tau})^2 = f_{i_1} \ldots f_{i_p} \gamma_{J}.
\]

**Proof.** The gallery \( f_{j_1} \ldots f_{j_p} \gamma_{-\tau} \) is completely determined by the action of \( p_{j_1}^J \) on \( (A_0, 0) \). If we denote by \( \mu \) its weight, we have

\[
 p_{-\tau}^J A_0 = A_{-\tau} \sigma_{\beta_{j_1}'}, \ldots \sigma_{\beta_{j_p}'}, N_{j_1}', \ldots, N_{j_p}'.
\]

Since \( p_{j_1}^J p_{-\tau}^J A_0 = A_0 = p_{j_1}^J v_J A_\mu \), we get the result.

We will denote by \( \mu(J) \) the weight of the gallery \( f_{i_1} \ldots f_{i_p} \gamma_{J} \). According to the proposition above, it is equal to \( -\mathbf{w}(f_{j_1} \ldots f_{j_p} \gamma_{-\tau}) \).

Lenart and Postnikov [9, Corollary 8.3] showed that in a simple Lie algebra with Weyl group \( W_0 \), we have for \( \lambda, \tau \in P^+ \)

\[
 s_{\tau}s_{\lambda} = \sum_J s_{\lambda + \mu(J)}.
\]
where the sum is over all LP-admissible subsets $J = \{j_1, \ldots, j_p\}$ that are $\lambda$-dominant and where $-\mu(J)$ is the weight of the gallery $f_{j_1} \cdots f_{j_p} \gamma_{-\tau}$. Using the previous proposition together with expression of $P(\tau)C_{p_xw_0}$ obtained at the end of the last section in the case of equal parameters, we get

$$P(\tau)C_{p_xw_0} = \sum_j C_{p_j \tau v_j, p_xw_0} + \sum_j C_{p_x + \mu(\tau), w_0}.$$

Remark 6.3. The condition at the end of Corollary 8.3 in [9] is not formulate in the same fashion but can be seen to be equivalent.

7 Lakshmibai-Seshadri paths

In this last section, we explain following [9 §9], how to construct an Lakshmibai-Seshadri path (LS paths for short) from an admissible subset $J$. There is no new result in this section but this construction can be helpful in order to understand the proof of Theorem 5.4.

A rational $\Omega$-path of shape $\mu \in P$ is a pair of sequences $(\underline{q}, \underline{a})$ such that $\underline{q} = (\sigma_0, \sigma_1, \ldots, \sigma_r)$ is a strictly decreasing chain (in the Bruhat order) of distinguished left coset representatives of $\text{Stab}_\mu(\Omega_0)$ and $\underline{a} = (a_0, a_1, \ldots, a_r)$ is an increasing sequence of rational numbers such that $a_0 = 0$ and $a_{r+1} = 1$.

We identify $\pi$ with the path $\pi : [0, 1] \to V$ defined by

$$\pi(t) := \sum_{i=1}^{j-1} (a_i - a_{i-1})/\mu a_i + (t - a_{j-1})/\mu a_j \text{ for } a_{j-1} \leq t \leq a_j.$$

The weight of $\pi$ is equal to $\pi(1)$. We set $\pi^* := (\underline{a}^*, \underline{q})$ where $\underline{a}^* := (a_0^*, a_1^*, \ldots, a_r^*)$. The path $\pi^*$ is then of weight $\pi(1)\sigma_{\Omega_0}$.

A rational $\Omega$-path of shape $\mu$ is called a Lakshmibai-Seshadri path if there exists a sequence of positive roots $\{\delta_1, \ldots, \delta_r\}$ such that

$$\sigma_0 > \sigma_1 = \sigma_0 \sigma_{\delta_1} > \sigma_2 = \sigma_0 \sigma_{\delta_1} \sigma_{\delta_2} > \ldots > \sigma_r = \sigma_0 \sigma_{\delta_1} \cdots \sigma_{\delta_r},$$

$$\ell(\sigma_i) = \ell(\sigma_{i-1}) - 1 \text{ and } a_i(\mu a_i, \delta_0^*) \in \mathbb{Z}.$$ 

Remark 7.1. Our definition of LS paths is slightly different from the one of Littelmann in [10] but nearly obviously equivalent. One only needs to notice that if $\alpha = a_0 = a_{q+1} = \ldots = a_{q+r}$ then the chain $(\sigma_q, \ldots, \sigma_{q+r})$ is an $a$-chain as defined by Littelmann.

For the rest of this section we will need to work with a specific reduced expression of $p_\tau$. The reason for this choice will become clear later. Fix a total order on the set of simple roots $\{\alpha_1, \ldots, \alpha_d\}$ and write $\{\omega_1, \ldots, \omega_d\}$ for the corresponding fundamental weights. We define the map

$$h : H(A_0, A_\tau) \to \mathbb{R}^{d+1} \to (k, (\omega_1, \delta^\nu), \ldots, (\omega_d, \delta^\nu))$$

The lexicographic order on $\mathbb{R}^{d+1}$ induces a total order on the set $H(A_0, A_\tau)$. Let

$$H(A_0, A_\tau) = \{H_{\beta_1, N_1}, \ldots, H_{\beta_n, N_n}\}$$

be such that $H_{\beta_i, N_i} < H_{\beta_{i+1}, N_{i+1}}$ for all $i$. Then there exists a reduced expression of $p_\tau$ of the form $a_{s_n} \cdots s_1$ where $a \in \Pi$, $s_i \in S$ and such that $H_{\beta_i, N_i}$ is the unique hyperplane separating $s_1 \ldots s_1 A_0$ and $s_{i-1} \cdots s_1 A_0$.

Let $J = \{i_1, \ldots, i_p\}$ be an admissible subset. For all $k \in \{0, \ldots, p\}$, we set (compare to the construction following Proposition 5.3)

* $J_k = \{i_{k+1}, \ldots, i_p\}$ where by convention $J_p = \emptyset$.
* $\sigma_{j_k} = \sigma_{j_{k+1}} \cdots \sigma_{j_p}$ and $v_{j_k} \in W_0$ is such that $v_{j_k} A_0 = A_0 \sigma_{j_k}$.
* $\gamma_{i_k} = \beta_{i_k} \sigma_{j_k}$ for $k \geq 1$ so that $\sigma_{j_k} = \sigma_{i_{1p}} \cdots \sigma_{i_{k+1}}$.
* $\tau_k = \tau \sigma_{j_k}$.
* $\mu_0 = 0$ and $\mu_k = \mu_{k-1} \gamma_{i_k} \cdots \gamma_{i_k}$. 
* $\pi_k$ is the straight path defined by

$$
\pi_k: [0,1] \rightarrow V
\quad t \mapsto \mu_k + t \cdot \tau_k
$$

* $a_k = \frac{N_{ik}}{\langle \gamma_k - 1, \gamma^\vee_{ik} \rangle} = \frac{N_{ik}}{\langle \gamma, \beta^\vee_{ik} \rangle}$.

Finally, we set $\varrho = (\sigma_{j_0}, \ldots, \sigma_{j_p})$ and $\varrho = (a_0, \ldots, a_{p+1})$ where $a_0 = 0$ and $a_{p+1} = 1$. By construction, the path $\pi = (\varrho, \varrho)$ coincide with the path $\pi_k$ for all $a_k \leq t \leq a_{k+1}$. Then according to [11] one can show the following result where once again, we assume that we are in the equal parameter case. The choice of our specific reduced expression plays a crucial role in the proof of Statement 3). Recall the definition of $\mu(J)$ in the previous section.

**Theorem 7.2.** Let $J$ be an admissible subset. Let $\lambda \in P^+$ and $\pi = (\sigma, \varrho)$ be the $\Omega$-rational path defined above. We have

1) $\pi$ is a LS-path of shape $\tau$ and weight $\mu(J)$.

2) The set of LS paths of shape $\tau$ is in bijection with the set of admissible subsets.

3) $\pi$ is $\lambda$-dominant (i.e. $\lambda + \pi(t)$ lies in the closure of $C_\varrho$ for all $t \in [0,1]$) if and only if the set of alcoves $p^\bullet_\varrho(k,1)v_J A_\lambda \in C_\varrho$ for all $k \in \{1, \ldots, n\}$.

As a direct consequence of this theorem and of Theorem 5.4 we obtain for $\lambda, \tau \in P^+$

$$
P(\tau)C_{P_\lambda \rightarrow 0} = \sum_{\pi} C_{P_\lambda + \pi(1) \rightarrow 0}
$$

where the sum is over all $\lambda$-dominant LS-paths of shape $\tau$.

**Example 7.3.** Let $\Phi^+ = \{\delta_1, \ldots, \delta_6\}$ be a root system of type $G_2$:

The associated simple system is $\{\delta_1, \delta_6\}$. The Weyl group of type $G_2$ is generated by $\sigma_{\delta_1}$ and $\sigma_{\delta_6}$ and the affine Weyl group of type $\tilde{G}_2$ is generated by $\{\sigma_{\delta_1}, \sigma_{\delta_6}, \sigma_{\delta_6,1}\}$. We'll denote the corresponding element of $S$ by $\{t_1, t_2, t_3\}$. Let $\tau$ be the fundamental weight associated to the root $\delta_6$. In this case, we have $\tau = \delta_4$. It is shown in [11] §18] that

$$
l_\tau = \sigma_{\delta_6,1} \sigma_{\delta_6,1} \sigma_{\delta_6,1} \sigma_{\delta_6,1} \sigma_{\delta_6,2} \sigma_{\delta_6,2} \sigma_{\delta_6,3} \sigma_{\delta_6,3} \sigma_{\delta_6,1} \quad \text{and} \quad p_\tau = t_2t_1t_2t_1t_3t_1t_2t_1t_3.
$$

Following [11] Example 10.2], we know that there are 14 admissible subsets. We describe these sets in the table below. In the column saturated chain, we only put the extremal element and one can recover the full chain by adding to the chain all the elements in the columns above: for instance, the saturated chain associated to the admissible subset $\{3,9,10\}$ is $\sigma_{\delta_6} \sigma_{\delta_6} > \sigma_{\delta_5} \sigma_{\delta_6} > \sigma_{\delta_6} > e$.

| Saturated chains | reduced expression | admissible subset |
|------------------|--------------------|-------------------|
| 1                | 1                  | $\emptyset$       |
| $\sigma_{\delta_6}$ | $t_2$               | 10                |
| $\sigma_{\delta_6} \sigma_{\delta_6}$ | $t_2t_1$           | 9,10, {5,10}, {2,10} |
| $\sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6}$ | $t_2t_1t_2$ | 8,9,10, {3,9,10}, {3,5,10} |
| $\sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6}$ | $t_2t_1t_2t_1$ | 7,8,9,10, {4,8,9,10}, {1,8,9,10}, {1,3,9,10}, {1,3,5,10} |
| $\sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6}$ | $t_2t_1t_2t_1t_2$ | 6,7,8,9,10 |
| $\sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6} \sigma_{\delta_6}$ | $t_2t_1t_2t_1t_2$ | 6,7,8,9,10 |
We now compute the different elements needed in the construction of the path $\pi_J$ where $J = \{3, 5, 10\}$.

| $k$ | $J_k$ | $\beta_{i_k}$ | $\gamma_{i_k}$ | $\sigma_{J_k}$ | $\tau_k$ | $N_{i_k}$ | $a_k$ |
|-----|-------|---------------|----------------|----------------|---------|----------|------|
| 0   | $\{3, 5, 10\}$ |               | $\delta_4\sigma_5\sigma_6\delta_6$ | $-\delta_6$ | 0       |          |      |
| 1   | $\{5, 10\}$   | $\delta_4$   | $\delta_6$    | $\sigma_5\sigma_6\delta_6$ | $\delta_6$ | 1        | 1/2  |
| 2   | $\{10\}$      | $\delta_5$   | $\delta_1$    | $\sigma_6\delta_6$       | $\delta_2$ | 2        | 2/3  |
| 3   | $\emptyset$    | $\delta_6$   | $\delta_6$    | $\sigma_6\delta_6$       | $\delta_4$ | 1        | 1    |

The LS path associated to $J$ is represented in red in Figure 1: we see that

- it follows the direction $\delta_4\sigma_5\sigma_6\delta_6 = -\delta_6$ for a time $1/2$;
- it follows the direction $\delta_4\sigma_5\delta_6 = \delta_6$ for a time $1/6$;
- it follows the direction $\delta_4\sigma_6 = \delta_2$ for a time $1/3$.

![Diagram](image1.png)

Fig. 1: LS path associated to the set $J = \{3, 5, 10\}$.

When doing it for all $J$ we obtain 14 paths corresponding to the 14 admissible subsets as shown in Figure 2.

![Diagram](image2.png)

Fig. 2: LS paths in type $G_2$.

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