Second Eigenvalue of Paneitz Operators and Mean Curvature

Daguang Chen, Haizhong Li

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China.
E-mail: dgchen@math.tsinghua.edu.cn; hli@math.tsinghua.edu.cn

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Abstract: For $n \geq 7$, we give the optimal estimate for the second eigenvalue of Paneitz operators for compact $n$-dimensional submanifolds in an $(n+p)$-dimensional space form in terms of the mean curvature and the Q-curvature.

1. Introduction

Assume that $M^n$ is a compact Riemannian manifold immersed into Euclidean space $\mathbb{R}^{n+p}$. In [9], Reilly obtained the estimate for the first eigenvalue $\lambda_1$ of Laplacian

$$\lambda_1 \leq \frac{n}{V(M^n)} \int_{M^n} |H|^2,$$

where $H$ is the mean curvature vector of the immersion $M^n$ in $\mathbb{R}^{n+p}$, $V(M^n)$ is the volume of $M^n$. In [11], El Soufi and Ilias obtained the corresponding estimates for submanifolds in the unit sphere $S^{n+p}(1)$, hyperbolic space $\mathbb{H}^{n+p}(-1)$ and some other ambient spaces. Motivated by certain variational stability issues, El Soufi and Ilias [12] obtained the sharp estimates for the second eigenvalue of the Schrödinger operator for compact submanifolds $M^n$ in space form $\mathbb{R}^{n+p}$, $S^{n+p}(1)$ and hyperbolic space $\mathbb{H}^{n+p}(-1)$.

Given a smooth 4-dimensional Riemannian manifold $(M^4, g)$, the Paneitz operator, discovered in [8], is the fourth-order operator defined by

$$P^4 f = \Delta^2 f - \text{div} \left( \frac{2}{3} R \text{Id} - 2 Ric \right) df,$$

for $f \in C^\infty(M^4)$,

where $\Delta$ is the scalar Laplacian defined by $\Delta = \text{div} d$, div is the divergence with respect to $g$, $R$, $Ric$ are the scalar curvature and Ricci curvature respectively. The Paneitz operator was generalized to higher dimensions by Branson [1]. Given a smooth compact

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Riemannian n-manifold \((M^n, g), n \geq 5\), let \(P\) be the operator defined by (see also [2])

\[
Pf = \Delta^2 f - \text{div}(a_n R \text{Id} + b_n \text{Ric}) df + \frac{n-4}{2} Qf,
\]

(1.2)

where

\[
Q = c_n |\text{Ric}|^2 + d_n R^2 - \frac{1}{2(n-1)} \Delta R = \frac{n^2 - 4}{8n(n-1)^2} R^2 - \frac{2}{(n-2)^2} |E|^2 - \frac{1}{2(n-1)} \Delta R,
\]

(1.3)

and

\[
a_n = \frac{(n-2)^2 + 4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2}, \quad c_n = -\frac{2}{(n-2)^2}, \quad d_n = \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2}.
\]

(1.4)

The operator \(P\) is also called Paneitz operator (or Branson-Paneitz operator).

In [4,5,13], the authors investigated positivity of the Paneitz operator. In analogy with the conformal volume in [7], Xu and Yang [14] defined N-conformal energy for compact 4-dimensional Riemannian manifold immersed in the N-dimensional sphere \(S^N(1)\). In the same paper [14], the upper bound for the first eigenvalue of the Paneitz operator was bounded using n-conformal energy. In [3], we obtained the sharp estimates for the first eigenvalue of the Paneitz operator for compact 4-dimensional submanifolds in Euclidean space and unit sphere.

The aim of this paper is to obtain the optimal estimates for the second eigenvalue of the Paneitz operator in terms of the extrinsic geometry of the compact submanifold \(M^n\) in space form \(R^{n+p}(c)\) (the Euclidean space \(\mathbb{R}^{n+p}\) for \(c = 0\), the Euclidean unit sphere \(S^{n+p}(1)\) for \(c = 1\) and the hyperbolic space \(\mathbb{H}^{n+p}(-1)\) for \(c = -1\)). Considering the first eigenvalue \(\Lambda_1\) of \(P\), it is easy to see that it is bounded by the mean value of the Q-curvature on \(M^n\),

\[
\Lambda_1 \leq \frac{n-4}{2V(M^n)} \int_{M^n} Q dv_g.
\]

(1.5)

Moreover, the inequality (1.5) is strict unless \(Q\) is constant.

For the second eigenvalue we have the following:

**Theorem 1.1.** Let \(\phi : M^n \rightarrow R^{n+p}(c)\) be an n-dimensional \((n \geq 7)\) compact submanifold. Then the second eigenvalue \(\Lambda_2\) of Paneitz operator satisfies

\[
\Lambda_2 V(M^n) \leq \frac{1}{2} n(n^2 - 4) \int_{M^n} \left(|H|^2 + c\right)^2 dv_g + \frac{n-4}{2} \int_{M^n} Q dv_g.
\]

(1.6)

Moreover, the equality holds if and only if \(\phi(M^n)\) is an n-dimensional geodesic sphere \(S^n(r_c)\) in \(R^{n+p}(c)\), where

\[
r_0 = \frac{1}{2} \left(\frac{n(n+4)(n^2 - 4)}{\Lambda_2}\right)^{1/4}, \quad r_1 = \arcsin r_0, \quad r_1 = \sinh^{-1} r_0.
\]

(1.7)
Remark 1.1. For the $n$-dimensional geodesic sphere $S^n(r_c)$ in $R^{n+p}(c)$, we have
\[
\Lambda_1 = \frac{1}{16}n(n-4)(n^2-4)(|H|^2 + c)^2,
\]
\[
\Lambda_2 = \frac{1}{16}n(n+4)(n^2-4)(|H|^2 + c)^2,
\]
\[
Q = \frac{1}{8}n(n^2-4)\left(|H|^2 + c\right)^2.
\]

From (1.3) and (1.6), we can reach

Corollary 1.2. Under the same assumptions as in Theorem 1.1, then
\[
\Lambda_2 V(M^n) \leq \frac{1}{2}n(n^2-4) \int_{M^n} |H|^2 d\nu_g + \frac{(n-4)(n^2-4)}{16n(n-1)^2} \int_{M^n} R^2 d\nu_g.
\] (1.8)

Moreover, the equality holds if and only if $M^n$ is an $n$-dimensional geodesic sphere.

Remark 1.2. We note that our technique in the proof of Theorem 1.1 does not work for $3 \leq n \leq 6$, so it is interesting to know whether Theorem 1.1 is true or not for $3 \leq n \leq 6$.

Remark 1.3. The authors in [11] and [12] applied the estimate for the second eigenvalue of the Schrödinger operator to the stability of constant mean curvature hypersurfaces and the stability of the interfaces in the Allen-Cahn reaction diffusion model. We expect that the estimate of the second eigenvalue of the Paneitz operator here may be related to some variational stability problems. We also mention that the proof of Theorem 1.1 has taken the similar strategy as in [11] and [12].

2. Some Lemmas

Assume that $\phi : M^n \to R^{n+p}(c)$ is an $n$-dimensional compact submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. From [7] (see also [12]), it is known that

Lemma 2.1. Let $w$ be the first eigenfunction of the Paneitz operator $P$ on $M^n$. Then there exists a regular conformal map
\[
\Gamma : R^{n+p}(c) \to S^{n+p}(1) \subset R^{n+p+1}
\]
(2.1)
such that for all $1 \leq \alpha \leq n+p+1$, the immersion $X = \Gamma \circ \phi = (X^1, \ldots, X^{n+p+1})$ satisfies
\[
\int_{M^n} X^\alpha w d\nu_g = 0,
\]
where $g$ is the induced metric of $\phi : M^n \to R^{n+p}(c)$.

Assume that $\tilde{g} = e^{2u} g$ is a conformal transformation for $u \in C^\infty(M^n)$, then the scalar curvature obeys [10]
\[
e^{2u} \tilde{R} = R - 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2,
\] (2.2)
the gradient operator and the Laplacian follows
\[
\tilde{\Delta} f = e^{-2u} [\Delta f + (n - 2) \nabla u \cdot \nabla f],
\]
where \( \nabla \) and \( \Delta \) (resp. \( \tilde{\nabla} \) and \( \tilde{\Delta} \)) are the Levi-Civita connection and Laplacian with respect to \( g \) (resp. \( \tilde{g} \)).

We have the following relation under conformal transformation \( \tilde{g} = e^{2u} g \) (see p. 766 in [12]):

**Lemma 2.2.** Let \( \phi : M^n \to \mathbb{R}^{n+p}(c) \) be an \( n \)-dimensional submanifold and \( X = \Gamma \circ \phi \) as before. Then we have
\[
e^{2u} (|\tilde{h}|^2 - n|\tilde{H}|^2) = |h|^2 - n|H|^2,
\]
where \( h, \tilde{h} \) are the second fundamental form of the immersion \( \phi \) and \( X \) respectively, \( H = \frac{1}{n} tr h \) and \( \tilde{H} = \frac{1}{n} tr \tilde{h} \) are the mean curvature vectors, \( u \) is defined by
\[
e^{2u} = \frac{1}{n} |\nabla (\Gamma \circ \phi)|^2.
\]

We need also the following result (see [11, 12]):

**Lemma 2.3.** Let \( \phi : M^n \to \mathbb{R}^{n+p}(c) \) be an \( n \)-dimensional submanifold and \( X = \Gamma \circ \phi \) as before. Then we have
\[
e^{2u} (|\tilde{H}|^2 + 1) = |H|^2 + c - \frac{2}{n} \Delta u - \frac{n - 2}{n} |\nabla u|^2.
\]

The following lemma is crucial in the proof of our Theorem 1.1.

**Lemma 2.4.** Let \( \phi : M^n \to \mathbb{R}^{n+p}(c) \) be an \( n \)-dimensional compact submanifold, \( X = \Gamma \circ \phi \) as before and \( u \) be defined by (2.6), then
\[
\int_{M^n} e^{2u} (|H|^2 + c) \leq \int_{M^n} (|H|^2 + c)^2 - \frac{n - 6}{n} \int_{M^n} e^{2u} |\nabla u|^2.
\]

**Proof.** Multiplying \( e^{2u} \) on both sides of (2.7), we have
\[
e^{4u} (|\tilde{H}|^2 + 1) = e^{2u} (|H|^2 + c) - \frac{2}{n} e^{2u} \Delta u - \frac{n - 2}{n} e^{2u} |\nabla u|^2.
\]

Integrating (2.9) over \( M^n \) and noting
\[
\int_{M^n} e^{2u} \Delta u = -2 \int_{M^n} e^{2u} |\nabla u|^2,
\]
we can get
\[
\int_{M^n} e^{4u} \leq \int_{M^n} e^{2u} (|H|^2 + c) - \frac{n - 6}{n} \int_{M^n} e^{2u} |\nabla u|^2.
\]

From the Cauchy-Schwartz inequality and (2.10), we have
\[
2 \int_{M^n} e^{2u} (|H|^2 + c) \leq \int_{M^n} e^{4u} + \int_{M^n} (|H|^2 + c)^2 \leq \int_{M^n} e^{2u} (|H|^2 + c) - \frac{n - 6}{n} \int_{M^n} e^{2u} |\nabla u|^2 + \int_{M^n} (|H|^2 + c)^2.
\]
This inequality implies (2.8). \( \square \)
3. Proof of Theorem 1.1

Assume that \( \phi : M^n \to R^{n+p} (c) \) is an \( n \)-dimensional compact submanifold in an \( (n+p) \)-dimensional space form \( R^{n+p} (c) \). From Lemma 2.1, there exists a regular conformal map

\[
\Gamma : R^{n+p} (c) \to S^{n+p}(1) \subset \mathbb{R}^{n+p+1}
\]

such that the immersion \( X = \Gamma \circ \phi = (X^1, \ldots , X^{n+p+1}) \) satisfies

\[
\int_{M^n} X^\alpha w \, dv_g = 0, \quad \text{for all } 1 \leq \alpha \leq n + p + 1,
\]

where \( w \) is the first eigenfunction of the Paneitz operator on \( M^n \).

Let \( \Lambda_2 \) be the second eigenvalue of the Paneitz operator \( P \). From the max-min principle for the Paneitz operator, we have

\[
\Lambda_2 \int_{M^n} (X^\alpha)^2 \, dv_g \leq \int_{M^n} P(X^\alpha) \cdot X^\alpha \, dv_g, \quad 1 \leq \alpha \leq n + p + 1. \tag{3.1}
\]

Making summation over \( \alpha \) from 1 to \( n+p \) in (3.1), using the fact \( \sum_{\alpha=1}^{n+p+1} (X^\alpha)^2 = 1 \) and (1.2), we can obtain

\[
\Lambda_2 V(M) \leq \sum_{\alpha=1}^{n+p+1} \int_{M^n} P(X^\alpha) \cdot X^\alpha \, dv_g
\]
\[
= \int_{M^n} \left[ \sum_{\alpha=1}^{n+p+1} \Delta^2 X^\alpha \cdot X^\alpha - \sum_{j,k=1}^n (a_n R \delta_{jk} + b_n R_{jk}) X_j X_k \right] \, dv_g
\]
\[
+ \frac{n-4}{2} \int_{M^n} Q |X|^2 \, dv_g
\]
\[
= \int_{M^n} < \Delta X, \Delta X > \, dv_g + \int_{M^n} \sum_{j,k=1}^n (a_n R \delta_{jk} + b_n R_{jk}) X_j X_k \, dv_g
\]
\[
+ \frac{n-4}{2} \int_{M^n} Q |X|^2 \, dv_g, \tag{3.2}
\]

where we use Stokes’ formula in the second equality.

By (2.3) and (2.4), we have the following calculations:

\[
< \Delta X, \Delta X >
\]
\[
= e^{4u} < \tilde{\Delta} X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X , \tilde{\Delta} X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X >
\]
\[
= e^{4u} < n \tilde{H} - n X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X , n \tilde{H} - n X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X >
\]
\[
= e^{4u} [n^2 |\tilde{H}|^2 + n^2 + (n-2)^2 |\nabla u|^2]
\]
\[
= e^{2u} [n^2 c^2 |\tilde{H}|^2 + n^2 c^2 + (n-2)^2 |\nabla u|^2], \tag{3.3}
\]

where \( \tilde{H} \) is the mean curvature vector of \( X = \Gamma \circ \phi : M^n \to S^{n+p}(1) \); here we used in the second equality the well-known formula \( \tilde{\Delta} X = n \tilde{H} - n X \).

Noting

\[
< X_j, X_k > = e^{2u} \delta_{jk}, \tag{3.4}
\]
and putting (3.3) into (3.2), we have
\[
\Lambda_2 V(M) \leq \int_{M^n} e^{2u} \left[ n^2 e^{2u} \left( |\tilde{H}|^2 + 1 \right) + (n - 2)^2 |\nabla u|^2 \right] dv_g
\]
\[+ (n a_n + b_n) \int_{M^n} R e^{2u} dv_g + \frac{n - 4}{2} \int_{M^n} Q dv_g. \tag{3.5}\]

Putting (2.7) into (3.5) and by use of the definitions of \(a_n, b_n\) in (1.4), we obtain
\[
\Lambda_2 V(M) \leq \int_{M^n} e^{2u} \left[ n^2 \left( |\tilde{H}|^2 + c - \frac{2}{n} \Delta u - \frac{n - 2}{n} |\nabla u|^2 \right) \right]
\[+ (n - 2)^2 |\nabla u|^2 \right] dv_g + (n a_n + b_n) \int_{M^n} R e^{2u} dv_g
\[+ \frac{n - 4}{2} \int_{M^n} Q dv_g + \int_{M^n} e^{2u} \left( (n - 2)^2 - (n - 2)n + 4n \right) |\nabla u|^2 dv_g
\]
\[= n^2 \int_{M^n} e^{2u} \left( |\tilde{H}|^2 + c \right) dv_g + (n a_n + b_n) \int_{M^n} R e^{2u} dv_g
\[+ \frac{n - 4}{2} \int_{M^n} Q dv_g + \int_{M^n} e^{2u} \left( (n - 2)^2 - (n - 2)n + 4n \right) |\nabla u|^2 dv_g
\]
\[= n^2 \int_{M^n} e^{2u} \left( |\tilde{H}|^2 + c \right) dv_g + 2(n + 2) \int_{M^n} e^{2u} |\nabla u|^2 dv_g
\[+ \frac{n^2 - 2n - 4}{2(n - 1)} \int_{M^n} R e^{2u} dv_g + \frac{n - 4}{2} \int_{M^n} Q dv_g. \tag{3.6}\]

From Gauss equation of \(\phi : M^n \rightarrow R^{n+p}(c)\),
\[
R = n(n - 1)c + n^2 |\tilde{H}|^2 - |\mathbf{h}|^2
\]
and \(|\mathbf{h}|^2 \geq n|\tilde{H}|^2\), we have
\[
R \leq n(n - 1)(|\tilde{H}|^2 + c). \tag{3.7}\]

The equality holds in (3.7) if and only if \(\phi : M^n \rightarrow R^{n+p}(c)\) is a total umbilical submanifold (see [6]).

By (3.6) and (3.7), we have
\[
\Lambda_2 V(M) \leq \frac{1}{2} n(n^2 - 4) \int_{M^n} e^{2u} \left( |\tilde{H}|^2 + c \right) dv_g + \frac{n - 4}{2} \int_{M^n} Q dv_g
\[+ 2(n + 2) \int_{M^n} e^{2u} |\nabla u|^2 dv_g. \tag{3.8}\]

From (2.8), we have
\[
\Lambda_2 V(M^n) \leq \frac{1}{2} n(n^2 - 4) \int_{M^n} \left( |\tilde{H}|^2 + c \right)^2 dv_g + \frac{n - 4}{2} \int_{M^n} Q dv_g
\[+ \left( \frac{1}{2} (n - 6)(n^2 - 4) - 2(n + 2) \right) \int_{M^n} e^{2u} |\nabla u|^2 dv_g
\]
\[= \frac{1}{2} n(n^2 - 4) \int_{M^n} \left( |\tilde{H}|^2 + c \right)^2 dv_g + \frac{n - 4}{2} \int_{M^n} Q dv_g
\[+ \frac{1}{2} (n + 2)(n^2 - 8n + 8) \int_{M^n} e^{2u} |\nabla u|^2 dv_g. \tag{3.9}\]

Therefore, the inequality (1.6) follows immediately from inequality (3.9) if \(n \geq 7\).
If the equality holds in (1.6), all the inequalities become equalities from (3.1) to (3.9). From (3.9), we can get $\nabla u = 0$, i.e. $u = constant$. In this case, (2.10) becomes equality, and then we can infer $\tilde{H} = 0$. Equation (2.7) implies

$$|H|^2 + c = e^{2u} = constant.$$

The equality case in (3.7) gives us $|h|^2 = n|H|^2$, that is,

$$h^\alpha_{ij} = H^\alpha \delta_{ij},$$

i.e., $\phi(M^n)$ is a totally umbilical submanifold in $R^{n+p}(c)$ (in [12], also called a geodesic sphere).

From (3.11) and Gauss equation of $\phi$, we have

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk} = (|H|^2 + c)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}),$$

$$R_{ij} = (n - 1)(|H|^2 + c)\delta_{ij},$$

$$R = n(n - 1)(|H|^2 + c),$$

where $R_{ijkl}$, $R_{ij}$ and $R$ are the components of Riemannian curvature tensor, the Ricci tensor and scalar curvature of $M^n$, respectively.

By the definition of the $Q$-curvature in (1.3), we have by (1.4),

$$Q = \frac{-2}{(n - 2)^2} |Ric|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n - 1)^2(n - 2)^2} R^2$$

$$= \frac{-2}{(n - 2)^2} (n - 1)^2 (|H|^2 + c)^2 \delta_{ik} \delta_{jk}$$

$$+ \frac{n^3 - 4n^2 + 16n - 16}{8(n - 1)^2(n - 2)^2} n^2(n - 1)^2 (|H|^2 + c)^2$$

$$= \frac{1}{8} n(n^2 - 4) \left(|H|^2 + c\right)^2.$$  (3.13)

Therefore, for the equality case in (1.6), we have

$$\Lambda_2 = \frac{1}{16} n(n + 4)(n^2 - 4)(|H|^2 + c)^2.$$  (3.14)

From (3.14), we have

$$|H|^2 + c = 4 \sqrt{\frac{\Lambda_2}{n(n + 4)(n^2 - 4)}}.$$  (3.15)

Therefore, from (3.12) and (3.15), we deduce that $\phi(M^n)$ is a geodesic sphere $S^n(r_c)$ with radius $r_c$ defined by (1.7).

Conversely, suppose that $\phi(M^n)$ is a geodesic sphere $S^n(r_c)$ with radius $r_c$ defined by (1.7) in space form $R^{n+p}(c)$. It is easily deduced that the section curvature

$$R_{ijij} = 4 \sqrt{\frac{\Lambda_2}{n(n + 4)(n^2 - 4)}}, \quad i \neq j.$$  (3.16)
From (3.12), we obtain (3.15). Therefore the equality holds in (1.6). We complete the proof of Theorem 1.1.

**Remark 3.1.** If we assume that the scalar curvature $R$ is nonnegative, from (3.7) we have

$$R^2 \leq n^2 (n-1)^2 (|H|^2 + c)^2. \quad (3.17)$$

Inserting (3.17) into (1.8), we have under $R \geq 0$ and the same assumptions as in Theorem 1.1.

$$\Lambda_2 V(M) \leq \frac{1}{16} n(n+4)(n^2 - 4) \int_M (|H|^2 + c)^2. \quad (3.18)$$

Moreover, the equality holds if and only if $M^n$ is an n-dimensional geodesic sphere.

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