TIME–DEPENDENT HAMILTON–JACOBI EQUATIONS ON NETWORKS

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ABSTRACT. We study well posedness of time–dependent Hamilton–Jacobi equations on a network, coupled with a continuous initial datum and a flux limiter. We show existence and uniqueness of solutions as well as stability properties. The novelty of our approach is that comparison results are proved linking the equation to a suitable semidiscrete problem, bypassing doubling variable method. Further, we do not need special test functions, and perform tests relative to the equations on different arcs separately.

1. Introduction

True to the title, the purpose of this paper is to study the well posedness of a time–dependent Hamilton–Jacobi equation, coupled with suitable additional conditions, posed on a network.

We consider a connected network Γ embedded in \( \mathbb{R}^N \) with a finite number of arcs \( \gamma \), which are regular simple curves parametrized in \([0, 1]\), linking points of \( \mathbb{R}^N \) called vertices, which make up a set we denote by \( V \). We define a Hamiltonian on Γ as a collection of Hamiltonians \( H_\gamma : [0, 1] \times \mathbb{R} \to \mathbb{R} \), indexed by arcs, with the crucial feature that Hamiltonians associated to arcs possessing different support, are totally unrelated.

The equations we deal with are accordingly of the form

\[
    u_t + H_\gamma(s, u') = 0 \quad \text{in } (0, 1) \times (0, +\infty)
\]

on each arc \( \gamma \), the aim being to uniquely select distinguished viscosity type solutions of each equation which can be assembled together continuously, making up a continuous function \( u : \Gamma \times (0, +\infty) \to \mathbb{R} \) with \( u(\gamma(s), t) \) solution of \((1)\) for each \( \gamma \). To accomplish it, one has to appropriately exploit the network geometry, via the adjacency condition between arcs and vertices, and the decisive issue for that is the right definition of supersolution. The subtle point in fact is that the conditions for

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supersolutions are not the same at all vertices, but are given taking into account the network structure, as made precise in Definition 3.1 (ii).

The problem becomes discontinuous across all the one–dimensional interfaces of the form
\[ \{(x, t), t \in [0, +\infty)\} \] with \( x \in V \),
in contrast to what happens for the stationary version of this kind of equations, where the discontinuities are located at the vertices, that is to say: they are finite and of zero dimension. This dimensional change explains why the analysis of evolutive equations on networks is by far more challenging than the stationary ones.

There are consequently few results available in the literature. The basic reference paper is [4] by Imbert and Monneau, where the topic is treated through PDE techniques, adapting tools from viscosity solutions theory, under the assumptions that the Hamiltonians in play are continuous, semiconvex and coercive. See also [1], [3] for applications of this theory. A previous contribution of the same authors, with an additional coauthor, see [5], requires instead the Hamiltonians to be convex, and attacks the problem using control theoretic representation formulae.

Here we prove existence, uniqueness and stability of solutions on the network assuming convexity of the Hamiltonians, but without the growth conditions which allow applying Fenchel transform, so that an action functional cannot be defined. In addition, the Hamiltonians we consider cannot be put in relation to any control model. In conclusion, though the Hamiltonians are convex, we do not have representation formulae for solutions at hand, and our techniques employ purely PDE methods.

One of the main discoveries in [4] is that to get well posedness of the evolutive problem, the assignment of an initial datum at \( t = 0 \) is not enough. It must actually be coupled with a condition regarding the time derivative of solutions on the discontinuity interfaces. They qualify as flux–limited the corresponding solutions. We adopt here the same point of view, and the terminology of flux limiter as well.

We make in Remark 3.4 a comparison between our definition of solution and the one of [4]. They are clearly the same outside the discontinuity interfaces, namely classical viscosity solutions. On the interfaces, the definition of subsolution coincides as well, while regarding supersolution, which is the most delicate point, the formulation is different, and our definition is stronger. We believe that our pattern is more related to the geometrical sense of the definition, and is more simple to write down, in particular because we take into account, for any arc, also the arc with the opposite orientation.

In [4], the method is first developed in the context of junctions, namely networks with a single vertex, for Hamiltonians only depending on the momentum variable. It is then generalized to Hamiltonians also depending on state variable and time, and
defined on general networks. In our opinion this last part, which contains interesting ideas, would deserve to be developed more.

We do not need the preliminary step of junctions. We directly work, with Hamiltonian depending on state variable and momentum, on a compact network, namely such that any arc has bounded length, with a general geometry and the unique limitation that no loops are admitted, namely arcs for which initial and final point coincide. We believe that our approach can also include the presence of loops, but this should require nontrivial adjustments. In [4], unbounded arcs are admitted, but no loops.

The approach of [4] is based on the construction of special test functions at the vertices, and a clever adaptation of Crandall–Lions doubling variable method to get the comparison result. Perron–Ishii method is used to prove existence of solutions.

Our method is different. First of all, we do not use doubling variable techniques, but instead we prove a comparison principle by associating the Hamilton–Jacobi equation to a semidiscrete problem posed on the discontinuity interfaces. This is the same road walked in [6], [8] for the stationary case, even if the evolutive setting brings in some complications. The proof of the comparison result for the semidiscrete problem turns out to be quite simple, and it is then transferred to the initial equation exploiting the fundamental property that a continuous function $u: \Gamma \times [0, +\infty) \to \mathbb{R}$ is solution of the main problem if and only $u(\gamma(s), t)$ solves (1) in the viscosity sense for any $\gamma$, and its trace on the discontinuity interfaces is solution of the semidiscrete problem.

A further relevant peculiarity of our techniques with respect to those in [4], is that we do not use special test functions at the vertices, more generally, we do not need functions testing at the same time solutions of equations with different Hamiltonians. For our definition, it is enough to consider viscosity test functions for the equations (1), separately considered, plus test functions on the discontinuity interfaces. Finally, we do not use Perron–Ishii method to prove existence of solutions, but rely on a more constructive technique, showing first existence for small time interval and then gluing together the local solutions to get a solution global in time.

All in all, the main outputs of the paper are:

- comparison principle for uniformly continuous sub and supersolutions, see Theorem 7.1
- existence results for Lipschitz continuous initial data, see Theorem 7.4 and continuous initial data, see Proposition 7.6
- existence of unique continuous solution, which is the maximal among the continuous subsolutions, it is in addition uniformly continuous if the initial datum is continuous, and Lipschitz continuous for Lipschitz continuous initial data, see Theorem 7.7 and Proposition 7.8
stability results, see Corollary 7.2 and Theorem 7.9.

The paper is organized as follows: In Section 2 we provide some preliminaries, we give the assumptions on the Hamiltonians, the definition of flux limiter and of solution in Section 3. The Section 4 is devoted to the definition and the study of the main properties of an operator, denoted by $G$, which enters, together with the operator $F_x$ defined in Section 6, in the definition of the semidiscrete equation. Roughly speaking, the operator $G$ allows taking into account the constraint due to the presence of the flux limiter. The proofs in this section are elementary.

In Section 5 we gather material regarding one–dimensional evolutive Hamilton–Jacobi equation on an interval. The focus is on (sub)solutions less than or equal to a given datum on part of the parabolic boundary. Since this point of view is quite unusual, we did not find the needed statements in the literature. Therefore we have chosen to prove everything in full details to make the presentation self-contained. This part is quite lengthy, but the arguments are traditional and plain. The corresponding proofs are mainly in Appendix B.

The core and the most innovative part of the paper is assembled in Sections 6 and 7. In Section 6 we write down the semidiscrete problem, prove a comparison principle for it, see Theorem 6.3 and study the connection with the time–dependent Hamilton–Jacobi equation. The part regarding supersolutions is the most demanding one, see in particular Proposition 6.5. In Section 7 we set down our main results, all the proofs are rather simple except that of the existence Theorem 7.4.

Finally in Appendix A we record some well–known results on $t$–partial sup convolutions we need in the paper.

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2. Preliminaries

2.1. Notations and basic definitions. We denote by $C(\cdot)$, $UC(\cdot)$, the spaces of real valued continuous and uniformly continuous functions, respectively. We further denote by $|\cdot|_\infty$ the uniform norm. Given $a, b$ in $\mathbb{R}$, we set

$$a \vee b = \max\{a, b\} \quad a \wedge b = \min\{a, b\}.$$ 

We set

$$\mathbb{R}^+ = [0, +\infty), \quad Q = (0, 1) \times (0, +\infty)$$
Given an open rectangle $R = (a, b) \times (T_0, T_1) \subset Q$, with $T_1 \in \mathbb{R} \cup \{+\infty\}$, we further set
\[
\partial^+_p R = [a, b] \times \{T_0\} \cup \{b\} \times (T_0, T_1) \\
\partial^-_p R = [a, b] \times \{T_0\} \cup \{a\} \times (T_0, T_1) \\
\partial_p R = \partial^-_p R \cup \partial^+_p R
\]
For any $C^1$ function $\psi : Q \to \mathbb{R}$ and $(s_0, t_0) \in Q$, we denote by $\psi'(s_0, t_0)$ the space derivative, with respect to $s$, at $(s_0, t_0)$, and by $\psi_t(s_0, t_0)$ or $\frac{d}{dt} \psi(s_0, t_0)$ the time derivative. Given a Lipschitz continuous function, $w : Q \to \mathbb{R}$ we define
\[
\text{Lip } w = \sup_{(s_1, t_1) \neq (s_2, t_2)} \frac{|w(s_1, t_1) - w(s_2, t_2)|}{|t_1 - t_2| + |s_1 - s_2|}.
\]
The (Clarke) generalized gradient of a Lipschitz continuous function $u : Q \to \mathbb{R}$ at $(s_0, t_0)$ is given by
\[
\overline{\text{co}} \{ (p, r) = \lim_{\substack{(s, t) \to (s_0, t_0) \in \mathbb{R}^+ \times \mathbb{R} \setminus \{(0, +\infty)\}\cap Q}} (u'(s_i, t_i), u_t(s_i, t_i)) \mid (s_i, t_i) \text{ diffe. pts of } u, (s_i, t_i) \to (s_0, t_0) \},
\]
where co stands for convex hull, and is indicated by $\partial u(s_0, t_0)$.

Given a continuous function $u : \Gamma \times \mathbb{R}^+ \to \mathbb{R}$ and an arc $\gamma$ of $\Gamma$, we define $u \circ \gamma : [0, 1] \times [0, +\infty) \to \mathbb{R}$ as
\[
u \circ \gamma(s, t) = u(\gamma(s), t) \quad \text{for any } (s, t) \in Q.
\]

Given a continuous function $u : Q \to \mathbb{R}$, we call supertangents (resp. subtangents) to $u$ at $(s_0, t_0) \in Q$ the viscosity test functions from above (resp. below). If needed, we take, without explicitly mentioning, $u$ and test function coinciding at $(s_0, t_0)$ and test function strictly greater (resp. less) than $u$ in a punctured neighborhood of $(s_0, t_0)$.

Given a closed subset $C \subset \overline{Q}$, where $\overline{Q}$ stands for the closure of $Q$, we say that a supertangent (resp. subtangent) $\varphi$ to $u$ at $(s_0, t_0) \in C$ is constrained to $C$ if $(s_0, t_0)$ is maximizer (resp. a minimizer) of $u - \varphi$ in a neighborhood of $(s_0, t_0)$ intersected with $C$.

The same notions apply, with obvious adaptations, to continuous function from $\mathbb{R}^+$ to $\mathbb{R}$.

2.2. **Networks.** An embedded network, is a subset $\Gamma \subset \mathbb{R}^N$ of the form
\[
\Gamma = \bigcup_{\gamma \in \mathcal{E}} \gamma([0, 1]) \subset \mathbb{R}^N,
\]
where \( E \) is a finite collection of regular (\( i.e., C^1 \) with non-vanishing derivative) simple oriented curves, called \( \text{arcs} \) of the network, that we assume, without any loss of generality, parameterized on \([0, 1]\).

Observe that on the support of any arc \( \gamma \), we also consider the inverse parametrization defined as
\[
-\gamma(s) = \gamma(1 - s) \quad \text{for } s \in [0, 1].
\]
We call \(-\gamma\) the \textit{inverse arc} of \( \gamma \). We assume
\[
\gamma((0, 1)) \cap \gamma'([0, 1]) = \emptyset \quad \text{whenever } \gamma \neq \gamma', \gamma \neq -\gamma'.
\]

We call \textit{vertices} the initial and terminal points of the arcs, and denote by \( V \) the sets of all such vertices. Note that (2) implies that
\[
\gamma((0, 1)) \cap V = \emptyset \quad \text{for any } \gamma \in E.
\]
We assume that the network is \textit{connected}, namely given two vertices there is a finite concatenation of arcs linking them. The unique restriction we assume on the geometry of the network is the nonexistence of loops, namely arcs with initial and final point coinciding. See [9] for a comprehensive treatment on graphs and networks.

Given \( x \in V \), we define
\[
\Gamma_x = \{ \gamma \mid \gamma(1) = x \}.
\]

A Hamiltonian on a network \( \Gamma \) is a collection of Hamiltonians \( H_\gamma : [0, 1] \times \mathbb{R} \to \mathbb{R} \), indexed by the arcs satisfying
\[
H_{-\gamma}(s, p) = H_\gamma(1 - s, -p) \quad \text{for any } \gamma \in E
\]
Apart the above compatibility condition, the Hamiltonians \( H_\gamma \) are unrelated.

3. Setting of the problem and definition of solution

We consider a Hamiltonian \( \{H_\gamma\} \) on the network \( \Gamma \). We require any \( H_\gamma \) to be:

\textbf{(H1)} continuous in both arguments;

\textbf{(H2)} convex in the momentum variable;

\textbf{(H3)} coercive in the momentum variable, uniformly in \( s \);

\textbf{(H4)} uniformly local Lipschitz continuous in \( p \). Namely, given \( M > 0 \), there exists \( C_M \) such that
\[
H_\gamma(s, p) - H_\gamma(s, q) \leq C_M |p - q| \quad \text{for any } s \in [0, 1], q, p \in (-M, M)
\]
We set
\[ c_\gamma = - \max_s \min_p H_\gamma(s, p) \]
for any arc \( \gamma \).

Note that the stationary equation
\[ H_\gamma(s, u') = a \]
admits (viscosity) subsolutions in \((0, 1)\) if and only if \( a \geq -c_\gamma \).

Following [4], we call flux limiter any function \( x \mapsto c_x \) from \( V \) to \( \mathbb{R} \) satisfying
\[ c_x \leq \min_{\gamma \in \Gamma_x} c_\gamma \]
for any \( x \in V \).

Let \((T_0, T_1)\) be an interval, possibly unbounded, contained in \( \mathbb{R}^+ \). For any given arc \( \gamma \), we consider the time–dependent equation
\[ (HJ_\gamma) \quad u_t + H_\gamma(s, u') = 0 \]
in \((0, 1) \times (T_0, T_1)\).

We are interested in finding a function \( v: \Gamma \times [T_0, T_1) \to \mathbb{R} \) such that \( v \circ \gamma \) solves \((HJ_\gamma)\) in \( R \), for any \( \gamma \), taking into account, in the sense we are going to specify, a flux limiter \( c_x \) at any vertex. We denote by \((HJ\Gamma)\) the problem as a whole.

The definition of (sub / super) solution to \((HJ\Gamma)\) is as follows:

**Definition 3.1.** We say that a continuous function \( v(x, t), v: \Gamma \times [T_0, T_1) \to \mathbb{R} \), is a supersolution in \((T_0, T_1)\) if
(i) \( v \circ \gamma \) is a viscosity supersolution of \((HJ_\gamma)\) in \((0, 1) \times (T_0, T_1)\) for any arc \( \gamma \);
(ii) for any vertex \( x \) and time \( t_0 \in (T_0, T_1) \), if
\[ \frac{d}{dt} \phi(t_0) < c_x \]
for some \( C^1 \) subtangent \( \phi \) to \( v(x, \cdot) \) at \( t_0 \), then there is an arc \( \gamma \in \Gamma_x \) such that all the \( C^1 \) subtangents \( \varphi \), constrained to \([0, 1] \times (T_0, T_1)\), to \( v \circ \gamma \) at \((1, t_0)\) satisfy
\[ \varphi_t(1, t_0) + H_\gamma(1, \varphi'(1, t_0)) \geq 0. \]

Note that the arc \( \gamma \), with \( \gamma(1) = x \), where condition (ii) holds true changes in function of the time.

**Definition 3.2.** We say that a continuous function \( v(x, t), v: \Gamma \times [T_0, T_1) \to \mathbb{R} \), is a subsolution to \((HJ\Gamma)\) in \((T_0, T_1)\) if
(i) \( v \circ \gamma \) is a viscosity subsolution of \((HJ_\gamma)\) in \((0, 1) \times (T_0, T_1)\) for any arc \( \gamma \);
(ii) for any vertex $x$ and time $t_0 \in (T_0, T_1)$, all supertangents $\psi(t)$ to $v(x, \cdot)$ at $t_0$ satisfy

$$\frac{d}{dt}\psi(t_0) \leq c_x.$$ 

We finally say that a continuous function $v$ is solution to (HJ$\Gamma$) in $(T_0, T_1)$ if it subsolution and supersolution at the same time.

**Remark 3.3.** Given a constant $a \in \mathbb{R}$ and the family of Hamiltonians

$$H'\gamma(s, p) = H_\gamma(s, p) + a \quad \text{for any } \gamma,$$

it is apparent that $c_x - a$ is a flux limiter for the $H'\gamma$. It is also apparent that $u$ is solution to (HJ$\Gamma$) with $H'\gamma$ in place of $H_\gamma$, initial datum $g$ and flux limiter $c_x - a$, if and only if $u + at$ is solution of the original problem (HJ$\Gamma$). This means that the analysis of the problem is not affected if we add to all Hamiltonians the same constant. We can therefore assume, without any loss of generality, all the Hamiltonians $H_\gamma$ to be strictly positive. This implies that flux limiter is negative at any vertex, and consequently all subsolutions to (HJ$\gamma$) are decreasing in time.

**Remark 3.4.** We make some comparisons between our definition of solution and the one in [4]. Clearly, the point is to look at the conditions required on the interfaces

$$\{(x, t) \mid t \in [T_0, T_1]\} \quad \text{with } x \in V.$$ 

According to [4, Theorem 2.10], the notion of subsolution is the same. As first pointed out in [7], the definition of supersolution, for equations posed in networks, is more delicate.

Our definition reads roughly like: if at a vertex $x$, for some instant of time, the constraint given by the flux limiter is non active, then at least for the equation on one arc $\gamma$ ending at $x$, some state constraint conditions must be satisfied. This follows along the same line of the definition given in [6], [8] for stationary equations. In this case, due to the absence of the time variable, there is no flux limiter, so that just the state constraint condition survives at the vertices.

As far as we can see, our definition is stronger in two respects. First, we have more test functions from below at points on the interfaces because we perform tests separately on any branch joining at the vertex under exam. On the contrary, in [4] $C^1$ functions testing from below at the same time all the equations on the branches are used. Secondly, we require the supersolution property to be satisfied for all the test functions relative to a distinguished equation. On the contrary, in [4], given a joint test function, the validity of the supersolution property is just assumed for some equation, which should change together with the test function.
4. The operator $G$

Loosely speaking, the operator $G$ allows taking into account the constraint given, in problem (HJΓ), by the flux limiter. Given a time interval $[T_0, T_1) \subset \mathbb{R}^+$, with $T_1 \leq +\infty$, we define $G : C([T_0, T_1) \times (-\infty, 0)) \to C([T_0, T_1))$ via

$$G[\psi, a](t) = \min\{\psi(r) + a(t - r), r \in [T_0, t]\}$$

We recall two basic results that we will exploit in this section.

**Lemma 4.1.** Let $u$ be a continuous function in an interval $[\alpha, \beta]$ satisfying

$$\frac{d}{dt} \varphi(t) \leq 0 \quad (\text{resp. } \geq 0)$$

for any $t \in (\alpha, \beta)$, any $C^1$ supertangent to $u$ at $t$. Then $u$ is nonincreasing (resp. nondecreasing) in $[\alpha, \beta]$.

**Proof:** We treat the case $\leq 0$. Given $\varepsilon > 0$, the function $u_\varepsilon := u(t) - \varepsilon t$ satisfies the assumptions with strict inequality. This implies that it cannot have local maximizers in $(\alpha, \beta)$ and consequently it has at most one local minimizer. If it is not strictly decreasing in $[\alpha, \beta]$, there is a nontrivial subinterval, say $(\gamma, \delta)$, where it is nondecreasing. Since the set of points where $u_\varepsilon$ admits $C^1$ supertangent is dense in $[\alpha, \gamma]$, we find $t_0 \in (\gamma, \delta)$ and $\varphi$ supertangent to $u_\varepsilon$ at $t_0$. Summing up: there is a neighborhood of $t_0$ where $\varphi$ is strictly decreasing, $u_\varepsilon$ nondecreasing and $\varphi$ supertangent to $u_\varepsilon$ at $t_0$. This is clearly impossible.

We derive that $u_\varepsilon$ is decreasing in $[\alpha, \beta]$ and, passing at the limit as $\varepsilon \to 0$, we find that $u$ is nonincreasing, as was claimed. The case $\geq 0$ can be proved arguing similarly.

The same statement of above also holds by replacing supertangents by subtangents.

**Lemma 4.2.** Let $u$ be a continuous function in an interval $[\alpha, \beta]$ satisfying

$$\frac{d}{dt} \varphi(t) \leq 0 \quad (\text{resp. } \geq 0)$$

for any $t \in (\alpha, \beta)$ any $C^1$ subtangent to $u$ at $t$. Then $u$ is nonincreasing (resp. nondecreasing) in $[\alpha, \beta]$.

**Lemma 4.3.** We have

$$G[\psi, a](t) = \min\{G[\psi, a](r) + a(t - r), r \in [T_0, t]\} \quad \text{for any } t \in [T_0, T_1).$$
Proof: We set \( w(t) = G[\psi, a](t) \). It is apparent that \( w(t) \) is greater than or equal to the function in the right hand–side of (5). Given \( r \leq t \), we find for a suitable \( r' \leq r \)

\[
    w(r) + a(t - r) = \psi(r') + a(r - r') + a(t - r) = \psi(r') + a(t - r) \geq w(t),
\]

which proves

\[
    w(t) \leq \min\{w(r) + a(t - r), r \in [T_0, t]\} \quad \text{for any } t \in [T_0, T_1]
\]

and concludes the proof.

Lemma 4.4. \( G[\psi, a] \) is the maximal continuous function in \([T_0, T_1]\) less than or equal to \( \psi \) satisfying

\[
(6) \quad \frac{d}{dt} \varphi(t) \leq a \quad \text{for any } t \in (T_0, T_1), \text{ any } C^1 \text{ supertangent } \varphi \text{ to } G[\psi, a] \text{ at } t.
\]

Proof: We set \( w(t) = G[\psi, a](t) \). We deduce from (5) that \( w \) is decreasing. Given \( T > T_0 \); we denote by \( \omega \) a continuity modulus of \( w \) in \([T_0, T]\). We consider \( r, t \) in \([T_0, T]\) with \( r < t \), and denote by \( r_0 \in [T_0, t] \) a time realizing the equality in (4). If \( r_0 \leq r \) then

\[
    |w(r) - w(t)| = w(r) - w(t) \leq \psi(r_0) + a(r - r_0) - \psi(r_0) - a(t - r_0)
    \]
\[
    = -a|t - r|.
\]

If instead \( r_0 > r \) we obtain

\[
    |w(r) - w(t)| \leq \psi(r) - \psi(r_0) - a(t - r_0) \leq \omega(r_0 - r) - a(t - r_0)
    \]
\[
    \leq \omega(|t - r|) - a|t - r|.
\]

The above formulae show that \( w \) is continuous. If \( \varphi \) is a \( C^1 \) differentiable supertangent to \( w \) at \( t \), we have by (5)

\[
    -\frac{d}{dt} \varphi(t) = \lim_{h \to 0^+} \frac{\varphi(t - h) - \varphi(t)}{h} \geq \lim_{h \to 0^+} \frac{w(t - h) - w(t) - a h}{h} = -a.
\]

Finally, If \( v \leq \psi \) is a continuous function in \([T_0, T_1]\) satisfying (4) with \( v \) in place of \( G[\psi, a] \), then, given \( t \in [T_0, T_1] \), we have by Lemma 4.1

\[
    v(t) \leq v(r) + a(t - r) \leq \psi(r) + a(t - r) \quad \text{for any } r \in [T_0, t].
\]

This implies

\[
    v(t) \leq w(t),
\]

and concludes the proof.

□
Remark 4.5. We deduce from the proof of the above lemma that if \( \psi \) is uniformly continuous with continuity modulus \( \omega \) in \((T_0, T_1)\), then \( G[\psi, a] \) is uniformly continuous as well with continuity modulus \( r \mapsto \omega(r) - a r \).

In addition, if \( \psi \) is Lipschitz continuous, then \( G[\psi, a] \) is Lipschitz continuous as well, it is maximal in the family of Lipschitz continuous functions \( w : [T_0, T_1) \to \mathbb{R} \) satisfying

\[
v(t) \leq \psi(t) \quad \text{and} \quad \frac{d}{dt} v(t) \leq a \quad \text{for a.e. } t,
\]

and has Lipschitz constant \(-a \vee \text{Lip } \psi\).

We record for later use:

Lemma 4.6. Assume that \( \psi_n \) is a sequence of continuous functions uniformly converging to a function \( \psi \) in \([T_0, T_1)\), then

\[
G[\psi_n, a] \to G[\psi, a] \quad \text{uniformly, in } [T_0, T_1).
\]

Proof: Given \( \varepsilon > 0 \), we have for \( n \) large

\[
G[\psi_n, a] + \varepsilon = G[\psi_n + \varepsilon, a] \geq G[\psi, a] \geq G[\psi_n - \varepsilon, a] = G[\psi_n, a] - \varepsilon.
\]

\qed

Lemma 4.7. Assume \( \psi_1, \psi_2 \) to be continuous functions from \([T_0, T_1)\) to \( \mathbb{R} \) satisfying

\[
\psi_1 > \psi_2 \quad \text{in } [T_0, T], \text{ for some } T > T_0
\]

then

\[
G[\psi_1, a] > G[\psi_2, a] \quad \text{in } [T_0, T], \text{ for any } a < 0.
\]

Proof: Because of the continuity of \( \psi_1, \psi_2 \) there exists \( b > 0 \) with

\[
\psi_1 > \psi_2 + b \quad \text{in } [T_0, T]
\]

therefore

\[
G[\psi_1, a] \geq G[\psi_2 + b, a] = G[\psi_2, a] + b > G[\psi_2, a]
\]

\qed

The next result will be used in Proposition 6.6.

Proposition 4.8. Let \((\psi, a) \in C([T_0, T_1]) \times (-\infty, 0)\). If \( G[\psi, a] \) admits a \( C^1 \) sub-tangent \( \varphi \) at \( t_0 \in (T_0, T_1) \) with \( \frac{d}{dt} \varphi(t_0) < a \) then \( G[\psi, a](t_0) = \psi(t_0) \).
Proof: We set \( w = G[\psi, a] \). We take \( r_0 \) realizing the equality in (5) for \( t_0 \) and assume \( r_0 \in [T_0, t_0); \) for \( r \in [r_0, t_0] \), we have

\[
   w(t_0) \leq w(r) + a(t_0 - r) \leq w(r_0) + a(r - r_0) + a(t_0 - r) = w(r_0) + a(t - r_0)
\]

and since the first and last term in the above formula are equal, we conclude

\[
   w(r) = w(r_0) + a(r - r_0) \quad \text{for} \quad r \in [r_0, t_0].
\]

This is in contrast with the existence of a subtangent \( \varphi \) to \( w \) at \( t_0 \) with \( \frac{d}{dt}\varphi(t_0) < a \). We conclude that \( r_0 = t_0 \), which proves the assertion by the very definition of \( w \), see (4).

We finally have:

**Proposition 4.9.** Let \( w \) be a continuous function in \([T_0, T_1] \) and \((\psi, a) \in C([T_0, T_1]) \times (-\infty, 0) \). Assume that \( w(T_0) \geq \psi(T_0) \), and \( w(t) \geq \psi(t) \) whenever there is a \( C^1 \) subtangent \( \varphi \) to \( w \) at \( t \) with \( \frac{d}{dt}\varphi(t) < a \). Then

\[
   w(t) \geq G[\psi, a](t) \quad \text{for any} \quad t \in [T_0, T_1]
\]

Proof: Given \( t \in (T_0, T_1) \), we define \( E \) as the set of points \( r \in (T_0, t) \) where there is a subtangent \( \varphi \) to \( w \) at \( r \) with \( \frac{d}{dt}\varphi(r) < a \). We set

\[
   r_0 = \begin{cases} 
   \sup E & \text{if} \ E \neq \emptyset \\
   T_0 & \text{if} \ E = \emptyset 
   \end{cases}
\]

By the assumption and \( w(T_0) \geq \psi(T_0) \), we have \( w(r_0) \geq \psi(r_0) \) and by Lemma 4.1

\[
   w(t) \geq w(r_0) + a(t - r_0) \geq \psi(r_0) + a(t - r_0).
\]

Therefore \( w(t) \geq G[\psi, a](t) \). This concludes the proof. \( \square \)

5. Hamilton Jacobi equations in an interval.

In this section we consider a single Hamiltonian \( H : [0, 1] \times \mathbb{R} \to \mathbb{R} \) satisfying (H1), (H2), (H3), (H4) plus

\[
   (7) \quad \max_{s \in [0, 1]} \min_{p \in \mathbb{R}} H(s, p) > 0,
\]

see Remark 3.3 we further consider the equation

\[
   (HJ) \quad u_t + H(s, u') = 0.
\]
We fix an open rectangle $R = (a, b) \times (T_0, T_1) \subset Q$, possibly unbounded. We call admissible an uniformly continuous function $w_0$ defined on $\partial_p^{-} R$ (resp. $\partial_p^{+} R$, $\partial_p R$) if there is an uniformly continuous subsolution of $\text{(HJ)}$ in $R$ agreeing with $w_0$ on $\partial_p^{-} R$ (resp. $\partial_p^{+} R$, $\partial_p R$).

5.1. Basic facts. We start recalling some well known results on $\text{(HJ)}$. the first one is a comparison result when the boundary datum is assigned on the whole of parabolic boundary.

**Theorem 5.1.** Let $u, v$ be continuous sub and supersolution, respectively, to $\text{(HJ)}$, in $R$ with $u$ Lipschitz continuous. If $u \leq v$ on $\partial_p R$ then $u \leq v$ in $R$.

This result can be generalized using $t$– partial sup–convolutions, see Appendix A.

**Theorem 5.2.** Let $u, v$ be continuous sub and supersolution, respectively, to $\text{(HJ)}$, in $R$ with $u$ uniformly continuous. If $u \leq v$ on $\partial_p R$ then $u \leq v$ in $R$.

The proof is in Appendix B.

The next proposition says that the vertical (in time) gluing of two (sub/super) solutions is still a (sub/super) solution.

**Proposition 5.3.** Let $t^* \in (T_0, T_1)$. Let $u$ be a continuous function from $R$ to $\mathbb{R}$. Assume $u$ to be (sub/ super)solution of $\text{(HJ)}$ in $(a, b) \times (T_0, t^*)$ and in $(a, b) \times (t^*, T_1)$. Then $u$ is (sub/super)solution in $R$.

We proceed stating a result on maximal subsolutions.

**Proposition 5.4.** Let $w_0$ be a continuous admissible datum on $\partial_p^{+} R$ (resp. $\partial_p^{-} R$, $[a, b] \times \{T_0\}$). The maximal subsolution of $\text{(HJ)}$ attaining the datum $w_0$ on $\partial_p^{+} R$ (resp. $\partial_p^{-} R$, $[a, b] \times \{T_0\}$), denoted by $w$, is characterized by the properties of being solution in $R$, and to satisfy for any $(s^*, t^*) \in \partial_p R \setminus \partial_p^{+} R$ (resp. $\partial_p^{-} R$, $\partial_p^{-} R \setminus \partial_p^{+} R$, $\partial_p R \setminus \partial_p^{-} R$, $\partial_p^{-} R \setminus ([a, b] \times \{T_0\})$ any subtangent $\varphi$, constrained to $R$, to $w$ at $(s^*, t^*)$

$$\varphi_t(s^*, t^*) + H(1, \varphi'(s^*, t^*)) \geq 0.$$ 

We derive the following stability property:

**Corollary 5.5.** Let $H_n$ be a sequence of Hamiltonians in $\overline{R}$ satisfying the same assumptions of $H$ and locally uniformly convergent to $H$, and $w_0^n$ a sequence of continuous initial data in $\partial_p^{-} R$ locally uniformly convergent to $w_0$. Then the maximal
subsolution \( w_n \) of \((HJ)\), with \( H_n \) in place of \( H \), attaining the datum \( w_0^n \) on \( \partial^- R \) locally uniformly converges in \( \overline{R} \) to the maximal subsolution \( w \) of \((HJ)\) agreeing with \( w_0 \) on \( \partial^- R \).

We finally record for later use:

**Proposition 5.6.** Assume that \( u \) is a Lipschitz continuous subsolution to \((HJ)\) in \( R \). Assume further that

\[
\text{Lip } u(s, \cdot) \leq M \quad \text{for some } M > 0, \text{ any } s \in [a, b].
\]

Then \( u(\cdot, t) \) is subsolution to

\[
H(s, v') \leq M \quad \text{in } (a, b), \text{ for any } t \in [T_0, T_1).
\]

The proof is in Appendix B.

5.2. Maximal subsolutions.

**Proposition 5.7.** Let \( w_0 \) be a Lipschitz continuous boundary datum assigned on \( \partial^- R \). Then the function

\[
v(s, t) := \sup \{ u(s, t) \mid u \text{ un. cont. subsoln of } (HJ) \text{ in } R, u \leq w_0 \text{ on } \partial^- R \}
\]

is a Lipschitz continuous solution to \((HJ)\) in \( R \) with

\[
|v(s_1, t) - v(s_2, t)| \leq M_0 |t_1 - t_2|,
\]

\[
|v(s_1, t) - v(s_2, t)| \leq L_0 |s_1 - s_2|,
\]

where

\[
M_0 = \min \{ m \mid H(s, w_0'(\cdot, T_0)) \leq m \text{ a.e. } s \} \vee \text{Lip } w_0(a, \cdot)
\]

\[
L_0 = \max \{|p| \mid H(s, p) \leq M_0 \forall s \}.
\]

In addition, it coincides with \( w_0 \) in \( [a, b] \times \{T_0\} \).

In other terms, \( M_0 \) is the minimal constant such that the initial datum is subsolution of the corresponding stationary equation, and \( L_0 \) is a constant estimating from above the Lipschitz constants of all subsolutions to such stationary equation. The proof is in Appendix B.

**Remark 5.8.** If the boundary datum \( w_0 \) is assigned on \([a, b] \times \{T_0\}\) and we define \( v \) as in \((HJ)\), we get, slightly adapting the argument of Proposition 5.7, that \( v \) is
Lipschitz continuous solution to (HJ) agreeing with \(w_0\) in \([a, b] \times \{T_0\}\) and satisfying the estimates (13), (14) with \(M_0, L_0\) replaced by

\[
M = \min \{m \mid H(s, w_0'(\cdot, T_0)) \leq m \text{ a.e. } s\}
\]

\[
L = \max \{|p| \mid H(s, p) \leq M \forall s\}.
\]

If the datum \(w_0\) is assigned on \(\partial^+ R\) the constants to put in (13), (14) are

\[
M_1 = \min \{m \mid H(s, w_0'(\cdot, T_0)) \leq m \text{ a.e. } s\} \vee \text{Lip } w_0(b, \cdot)
\]

\[
L_1 = \max \{|p| \mid H(s, p) \leq M \forall s\}.
\]

Finally, if the datum \(w_0\) is given on the whole of parabolic boundary \(\partial_p R\), we have the constants

\[
M_2 = \min \{m \mid H(s, w_0'(\cdot, T_0)) \leq m \text{ a.e. } s\} \vee \text{Lip } w_0(a, \cdot) \vee \text{Lip } w_0(b, \cdot)
\]

\[
L_2 = \max \{|p| \mid H(s, p) \leq M \forall s\}.
\]

We generalize Proposition 5.7 to absolutely continuous boundary data.

**Proposition 5.9.** Let \(w_0\) be an uniformly continuous boundary datum assigned on \(\partial_p^+ R\). Then the function

\[
v(s, t) := \sup \{u(s, t) \mid u \text{ un. cont. subsoln of (HJ) in } R, u \leq w_0 \text{ on } \partial_p^- R\}
\]

is an uniformly continuous solution to (HJ) in \(R\), and coincides with \(w_0\) in \([a, b] \times \{T_0\}\).

We preliminarily need introducing a regularization device. Given an uniformly continuous function \(u\) defined in a closed set \(C \subset \overline{Q}\), we define, for \(n \in \mathbb{N}\), the following approximations from above and below

\[
u^{[n]}(s, t) = \sup \{u(z, r) - n(|z - s| + |t - r|) \mid (z, r) \in C\}
\]

\[
u_{[n]}(s, t) = \inf \{u(z, r) + n(|z - s| + |t - r|) \mid (z, r) \in C\}
\]

The following properties hold:

**Lemma 5.10.**

(i) For \(n\) sufficiently large, the functions \(u^{[n]}\) and \(u_{[n]}\) are Lipschitz continuous in \(C\) with Lipschitz constant \(n\), and \(u^{[n]} \geq u \geq u_{[n]}\);

(ii) \(u^{[n]}\) and \(u_{[n]}\) uniformly converge to \(u\) in \(C\) as \(n\) goes to infinity, with \(|u^{[n]} - u|_\infty, |u_{[n]} - u|_\infty\) only depending on the continuity modulus of \(u\).

The proof is in Appendix B.
Proof of Proposition 5.9: According to Lemma 5.10, we find a sequence of Lipschitz continuous functions \( w_n \) uniformly converging to \( w_0 \) in \( \partial_p R \). We denote by \( v_n \) the maximal subsolutions of \((HJ)\) not exceeding \( w_n \) in \( \partial_p R \). According to Proposition 5.7, the \( v_n \)'s are Lipschitz continuous subsolutions of \((HJ)\), and any \( v_n \) agrees with \( w_n \) on \( \partial_p R \). Given \( \varepsilon > 0 \), we have
\[
w_n - \varepsilon \leq w_0 \leq w_n + \varepsilon \quad \text{for } n \text{ large, in } \partial_p R
\]
which implies by the very definition of maximal subsolution
\[
v_n - \varepsilon \leq v \leq v_n + \varepsilon \quad \text{for } n \text{ large, in } R
\]
so that \( v_n \) uniformly converges to \( v \) in \( R \). This gives the assertion.

We also derive from the argument of the above proposition:

Corollary 5.11. Let \( w_0 \) be an uniformly continuous function in \( \partial_p R \), and \( w_n \) a sequence of uniformly continuous functions uniformly approximating it in \( \partial_p R \). Then the maximal subsolutions of \((HJ)\) among those less than or equal to \( w_n \) in \( \partial_p R \) uniformly converge to the maximal subsolutions less than or equal to \( w_0 \) in \( \partial_p R \).

5.3. Admissible traces. In this section we investigate the possibility of modifying an admissible boundary datum on part of the parabolic boundary, still getting an admissible datum. The first statement in this respect is:

Proposition 5.12. Let \( u_0 \) be a Lipschitz continuous admissible trace on \( \partial_p R \), and assume \( w_0 \) to be a Lipschitz continuous function defined in \( \partial_p R \) with
\[
u_0 \leq w_0 \text{ in } [a,b] \times \{T_0\} \quad \text{and} \quad u_0 = w_0 \text{ in } \{a\} \times [T_0,T_1]
\]
then \( w_0 \) is admissible on \( \partial_p R \) as well.

Proof: We take \( M \) with
\[
\text{Lip } w_0, \text{Lip } u_0 < M \quad \text{in } \partial_p R
\]
\[
H(s,w'_0(\cdot,0)) < M \quad \text{a.e. in } (a,b)
\]
Therefore \( \overline{w}(s,t) := w_0(s,T_0) - M (t-T_0) \) is a subsolution to \((HJ)\) in \( R \) such that
\[
\overline{w} \leq u_0 = w_0 \quad \{a\} \times (T_0,T_1)
\]
\[
\overline{w} \geq w_0 \quad \text{in } [a,b] \times \{T_0\}
\]
We denote by \( \overline{v} \) a subsolution with trace \( u_0 \) on \( \partial_p R \), then the function
\[
(s,t) \mapsto \max\{\overline{w}(s,t), \overline{v}(s,t)\}
\]
is a subsolution agreeing with $w_0$ on $\partial^-_p R$, as was claimed.

□

The following result somehow complements Proposition [5.12] We show that we can fix an admissible boundary datum on $\partial^-_p R$ (resp. $\partial^+_p R$) and make it decrease on $\{a\} \times [T_0, T_1]$ (resp. $\{b\} \times [T_0, T_1]$) still obtaining an admissible datum. We need for that an additional assumption on the time derivative of the new datum on $\{a\} \times [T_0, T_1]$ (resp. $\{b\} \times [T_0, T_1]$).

**Proposition 5.13.** Let $u_0$ be an admissible Lipschitz continuous boundary datum for (HJ) in $\partial^-_p R$ (resp. $\partial^+_p R$), and $v_0$ a Lipschitz continuous function defined in $\partial^-_p R$ (resp. $\partial^+_p R$) with $u_0 = v_0$ on $[a, b] \times \{T_0\}$ and $v_0 \leq u_0$ on $\{a\} \times [T_0, T_1]$ (resp. $\{b\} \times [T_0, T_1]$). We further assume that

\[ \frac{d}{dt}v_0(a, t) \leq -\max_{s \in [0,1]} \min_{p \in \mathbb{R}} H(s, p) \quad \text{a.e. in } (T_0, T_1). \]

(24) \[ \left( \frac{d}{dt}v_0(b, t) \leq -\max_{s \in [0,1]} \min_{p \in \mathbb{R}} H(s, p) \quad \text{a.e. in } (T_0, T_1). \right) \]

Then $v_0$ is admissible for (HJ) on $\partial^-_p R$ (resp. $\partial^+_p R$).

We need a preliminary elementary lemma, we provide the proof in the Appendix [3] for reader’s convenience.

**Lemma 5.14.** The minimum of two Lipschitz continuous subsolutions to (HJ) is a subsolution.

**Proof of Proposition 5.13** We give the proof for functions defined on $\partial^-_p R$. The argument for $\partial^+_p R$ is the same. We set

\[ c = -\max_{s \in [0,1]} \min_{p \in \mathbb{R}} H(s, p). \]

We denote by $w_0(s)$ a function satisfying

\[ H(s, w_0') = -c \quad \text{in the viscosity sense in } (a, b) \]

and $u_0(a) = 0$. Due to (24), the function

\[ w(s, t) := w_0(s) + v_0(a, t) \]
is a Lipschitz continuous subsolution to \((HJ)\) in \(R\). Applying Proposition 5.12 with admissible trace \(w\), we find that the trace
\[
\max\{w, u_0\} = \max\{w, v_0\} \quad \text{on } [a, b] \times \{T_0\},
\]
\[
v_0(a, t) \quad \text{on } \{a\} \times [T_0, T_1)
\]
is admissible for \((HJ)\) on \(\partial^- p R\). Finally, by exploiting Lemma 5.14 we see that \(v_0\), being the minimum of \(u_0\) and the function in (25), (26), is admissible. \(\square\)

We say that a continuous function \(u\) is strict subsolution in \(R\) of \((HJ)\) if
\[
(27) \quad u_t + H(s, u') \leq -\delta \quad \text{in } R, \text{ in the viscosity sense}
\]
for some \(\delta > 0\).

Lemma 5.15. Let \(u\) be an uniformly continuous strict subsolution to \((HJ)\) in \(R\). The maximal subsolution agreeing with \(u\) on \(\partial^- p R\) is strictly greater than \(u\) in \(R \cup \{\{b\} \times (T_0, T_1)\}\).

Proof: We can assume that \(u\) is the maximal uniformly continuous subsolution to (27) among those with the same trace on \(\partial^- p R\). We denote by \(u_n\) a sequence of Lipschitz continuous functions uniformly approximating \(u\) on \(\partial^- p R\) and by \(\overline{u}_n\) the maximal subsolution of (27) less that or equal to \(u_n\) on \(\partial^- p R\). We know by Corollary 5.11 that the \(\overline{u}_n\)'s uniformly converge to \(u\) on \(R\). We finally denote by \(v\) the maximal subsolution of \((HJ)\) agreeing with \(u\) on \(\partial^- p R\).

It is clear that \(u \leq v\) on \(R\). if \(u = v\) at a point \((s_0, T_0) \in R \setminus \partial^- p R\), then, taking into account that \(\overline{u}_n\) uniformly converges to \(u\), we find that \(\overline{u}_n\) is subtangent to \(v\) at some point \((s_n, t_n) \in R \setminus \partial^- p R\), for \(n\) large. Exploiting that \(\overline{u}_n\) is subsolution to (27), we can construct, using Perron–Ishii method, a subsolution to \((HJ)\) strictly greater than \(v\) in a neighborhood of \((x_n, t_n)\) and equal to \(u\) in \(\partial^- p R\). This is impossible by the very definition of \(v\). \(\square\)

We record for later use in the proof of Proposition 6.5

Proposition 5.16. Given \(s_0 \in (a, b), c \leq -\max_s \min_p H(s, p)\), we set
\[
A = (a, s_0) \times (T_0, T_1) \quad \text{and} \quad B = (s_0, b) \times (T_0, T_1).
\]
Let \(u\) be an uniformly continuous supersolution of \((HJ)\) in \(R\), we denote by \(v\) the maximal subsolution in \(A\) with trace less than or equal to \(u\) on \(\partial^- p A\), and by \(w\) the maximal subsolution in \(B\) with trace less than or equal to \(u\) on \(\partial^- p B\). Then
\[
(28) \quad G[u(s_0, \cdot), c](t) \geq \min\{G[v(s_0, \cdot), c], G[w(s_0, \cdot), c]\}(t) \quad \text{for any } t \in (T_0, T_1).
\]
We recall that the operator $G$ is defined in Section 4.

**Proof:** We denote by $u_n$, with $u_n \leq u$, the Lipschitz continuous approximation from below of $u$ in $\partial_p R$ introduced in Lemma 5.10. We further denote $\overline{v}_n$, $\overline{w}_n$ the maximal subsolutions on $A$, $B$, respectively, with trace less than or equal to $u_n$ on $\partial^- A$, $\partial^+ B$, respectively. We know by Corollary 5.11 that

$$v_n \to v \text{ uniformly in } A \quad \text{and} \quad w_n \to v \text{ uniformly in } B$$

Owing to Proposition 5.13 the function equal to $u_n$ in $[a,s_0] \times \{T_0\}$ and

$$\min\{G[\overline{v}_n(s_0, \cdot), c], G[\overline{w}_n(s_0, \cdot), c]\}$$

on $\{s_0\} \times (T_0, T_1)$ is admissible on $\partial^+ A$, and the same holds true on $\partial^- B$ for the function equal to $u_n$ on $[s_0, b] \times \{T_0\}$ and to the function in (30) on $\{s_0\} \times (T_0, T_1)$.

We denote by $\tilde{v}_n, \tilde{w}_n$ the corresponding maximal subsolutions on $A, B$, respectively. We further set

$$v^*_n = \min\{\overline{v}_n, \tilde{v}_n\} \quad \text{and} \quad w^*_n = \min\{\overline{w}_n, \tilde{w}_n\},$$

by Lemma 5.14 $v^*_n$ and $w^*_n$ are subsolutions to (HJ) in $A$ and $B$, respectively. We have

$${v}^*_n \leq u_n \leq u \quad \text{on } \partial^- R$$

$${w}^*_n \leq u_n \leq u \quad \text{on } \partial^+ R$$

$$v^*_n = u_n \leq u \quad \text{on } [a, s_0] \times \{T_0\}$$

$$w^*_n = u_n \leq u \quad \text{on } [s_0, b] \times \{T_0\}$$

$$v^*_n = w^*_n \quad \text{on } \{s_0\} \times (T_0, T_1)$$

We consider the function $\varphi_n$ defined in the whole of $R$ by merging together $v^*_n, w^*_n$.

Note that $\varphi_n$ is Lipschitz continuous. In addition, since $\varphi_n$ is subsolution in $A$ and in $B$, it is subsolution in the whole of $R$ for the Hamiltonian is convex in $p$ so that the notions of viscosity and a.e. subsolution coincide. We also have

$$\varphi_n \leq u \quad \text{for any } n, \text{ on } \partial R.$$ 

We therefore get by the comparison principle given in Theorem 5.2

$$\varphi_n \leq u \quad \text{in } R.$$ 

and consequently

$$\varphi_n(s_0, \cdot) = \min\{G[\overline{v}_n(s_0, \cdot), c], G[\overline{w}_n(s_0, \cdot), c]\}(t) \leq G[u(s_0, \cdot), c](t) \quad \text{for any } t.$$ 

We get (28) passing at the limit as $n$ goes to infinity in the above formula and taking into account (29) plus Lemma 4.6.

□
5.4. Finite speed of propagation. In this section we assume the rectangle $R = (a, b) \times (T_0, T_1)$ to be bounded with $a = 0$, $b = 1$.

**Lemma 5.17.** Let $u_0$ be a Lipschitz continuous initial datum on $[0, 1] \times \{T_0\}$. We consider two Lipschitz continuous solutions $u$, $v$ to $(HJ)$ in $R$ agreeing with $u_0$ on $[0, 1] \times \{T_0\}$. Then there is $\delta > 0$ depending on $H$ and the Lipschitz constants of $u$, $v$, such that

$$u = v \quad \text{in } [1/2 - \delta, 1/2 + \delta] \times [T_0, T_0 + \delta].$$

**Proof:** We denote by $L$ a Lipschitz constant of both $u$, $v$ in $R$, we further denote by $M$ a Lipschitz constant of $H(s, \cdot)$ in $[-L, L]$, for any $s \in [0, 1]$, see assumption (H4), that we can assume greater than 3, so that

$$|H(s, u'(s, t)) - H(s, v'(s, t))| \leq M |u'(s, t) - v'(s, t)|$$

for a.e. $(s, t) \in R$. We then have

$$0 = u_t(s, t) + H(s, u'(s, t)) - v_t(s, t) - H(s, v'(s, t))$$

$$\geq (u_t(s, t) - v_t(s, t)) - M |u'(s, t) - v'(s, t)|$$

a.e. $(s, t) \in R$. Consequently $(u - v)$ and similarly $(v - u)$ are a.e. and viscosity subsolutions to the equation

$$(31) \quad w_t - M |w'| = 0 \quad \text{in } R$$

attaining the value 0 on $[0, 1] \times \{T_0\}$ and $u - v$ (resp. $v - u$) on the rest of the parabolic boundary of $R$. We know that the solution of (31) with these boundary conditions is given by

$$(s, t) \mapsto \max\{(u-v)(s^*, t^*) \mid (s^*, t^*) \in \partial_p((0, 1) \times (T_0, t)), |(s, t) - (s^*, t^*)| \leq M (t-t^*)\}.$$ 

We take

$$(s, t) \in (1/2 - 1/M, 1/2 + 1/M) \times (T_0, T_0 + \delta) =: U,$$

where $\delta$ is a positive constant to be determined, we recall that $M$ has been taken greater that 3. If $(s^*, t^*)$ is in the lateral part of the parabolic boundary of $R$ then

$$|(s, t) - (s^*, t^*)| \geq \min\{1 - s, s\} \geq \frac{1}{2} - \frac{1}{M} = M \left(\frac{1}{2M} - \frac{1}{M^2}\right).$$

It is then enough to take

$$\delta < \frac{1}{2M} - \frac{1}{M^2} < \frac{1}{M}$$

to see that the lateral boundary does not have influence in the above formula of the solution at $(s, t) \in U$. This shows that the solution is vanishing in $U$, since $u - v$...
and \( v - u \) are both less than or equal the solution by the comparison principle, we deduce

\[
u = v \quad \text{in } U.
\]

Since \( U \supset [1/2 - \delta, 1/2 + \delta] \times [T_0, T_0 + \delta] \), we get the assertion. \( \square \)

We derive:

**Corollary 5.18.** Let \( u_0, v_0 \) be admissible Lipschitz continuous boundary data for (HJ) in \( \partial^- R \) and \( \partial^+ R \), respectively, with \( u_0 = v_0 \) on \([0, 1] \times \{T_0\}\). Then the merge of the two functions is admissible on \( \partial_p ([0, 1] \times [T_0, T_0 + \delta]) \), for a suitable constant \( \delta > 0 \) solely depending on the Lipschitz constants of \( u_0, v_0 \) and \( H \).

**Proof:** We denote by \( u, v \) the maximal (sub)solutions agreeing with \( u_0 \) on \( \partial^- R \) and \( v_0 \) on \( \partial^+ R \), respectively. We derive from Proposition 5.7 and Remark 5.8 that their Lipschitz constants depend on those of \( u_0, v_0 \) and clearly on \( H \). By Lemma 5.17 \( u \) and \( v \) coincide in

\[
[1/2 - \delta, 1/2 + \delta] \times [T_0, T_0 + \delta],
\]

for a suitable \( \delta \), we define a new solution \( w \) setting

\[
w = \begin{cases}
u & \text{on } [T_0, 1/2 - \delta] \times [T_0, T_0 + \delta] \\
u & \text{on } (1/2 + \delta, 1] \times [T_0, T_0 + \delta] \\
 = v & \text{on } [1/2 - \delta, 1/2 + \delta] \times [T_0, T_0 + \delta]
\end{cases}
\]

The function \( w \) coincides with \( u_0 \) on \( \partial^- ([0, 1] \times [0, \delta]) \) and with \( v_0 \) in \( \partial^+ ([0, 1] \times [0, \delta]) \). This proves the assertion \( \square \)

6. A semidiscrete equation

We set

\[
\mathcal{F} = \text{UC}((\Gamma \times \{T_0\}) \cup (V \times [T_0, T_1]))
\]

\[
R = (0, 1) \times (T_0, T_1) \quad \text{with } T_1 \leq +\infty.
\]

We define, for any \( x \in V \), the operator \( F_x : \mathcal{F} \to \text{UC}([T_0, T_1]) \) in the interval \([T_0, T_1]\) through two steps:

- Given \( u \in \mathcal{F} \) and \( \gamma \in \Gamma_x \) we indicate by \( (s, t) \mapsto F_{\gamma}(u)(s, t) \), the maximal among the uniformly continuous subsolutions to (HJ) in \( R \) with trace less than or equal to the merge of \( u \circ \gamma \) and \( u(\gamma(0), t) \) on \( \partial^- R \). Note that \( F_{\gamma}(u) \) is uniformly continuous by Proposition 5.9 and in addition Lipschitz continuous if \( u \) is Lipschitz continuous by Proposition 5.7.
we set
\[ F_x[u] = \min_{\gamma \in \Gamma_x} F_\gamma[u](1, \cdot). \]

Note that
\[ F_x[u](T_0) = u(x, T_0) \quad \text{for any } x \in V, \ u \in F \]

We directly derive from the definition of \( F_x \):

**Lemma 6.1.** For any \( u \in F, \ x \in V \)

(i) \( F_x[u + a] = F_x[u] + a \) for any \( a \in \mathbb{R} \);

(ii) if \( v \in F \) with \( v \geq u \) then \( F_x[v] \geq F_x[u] \) in \([T_0, T_1)\).

---

6.1. **Definition of the problem and comparison result.** Given a flux limiter \( x \mapsto c_x \) on \( V \), we consider the semidiscrete equation

\[(\text{Discr}) \quad u(x, t) = G[F_x[u], c_x](t).\]

See Section [4] for the definition of the operator \( G \). By uniformly continuous \((\text{sub} / \text{super})\) solution of it in the interval \((T_0, T_1)\), we mean a function

\[ v \in \text{UC}((\Gamma \times \{0\}) \cup (V \times [T_0, T_1])) \]

which satisfies pointwise the (in)equalities in \((\text{Discr})\) for any \((x, t) \in V \times (T_0, T_1)\).

If

\[ v(x, t) < G[F_x[v], c_x](t) \quad \text{for any } x \in V, \ t \in (T_0, T_1) \]

then we say that \( v \) is a strict subsolution.

**Lemma 6.2.** Let \( u \) be an uniformly continuous subsolution to \((\text{Discr})\) in \((T_0, T_1)\), then \( u - \varepsilon(t - T_0) \) is a strict subsolution for any \( \varepsilon > 0 \).

**Proof:** By the very definition of \( F_\gamma \), the function \( F_\gamma[u](s, r) - \varepsilon(r - T_0) \) is a strict subsolution of \((\text{HJ}_\gamma)\) in \( R \) for any arc \( \gamma \), then we apply Lemma 5.15 to \( F_\gamma[u](s, r) - \varepsilon r \) to get

\[ F_\gamma[u](s, r) - \varepsilon(r - T_0) < F_\gamma[u - \varepsilon(t - T_0)](s, r) \quad \text{in } \overline{R} \setminus \partial^-R. \]

and in particular

\[ F_\gamma[u](1, r) - \varepsilon(r - T_0) < F_\gamma[u - \varepsilon(t - T_0)](1, r) \quad \text{for any } \gamma, \ r \in (T_0, T_1). \]

This implies

\[ F_x[u](r) - \varepsilon(r - T_0) < F_x[u - \varepsilon(t - T_0)](r) \quad \text{for any } r \in (T_0, T_1). \]
Since \(u\) is subsolution to \([\text{Discr}]\), we derive from (33) and the definition of \(G\), see (1), that for any \((x,t_0) \in V \times (T_0,T_1), r \in (T_0,t_0)\)

\[
\begin{align*}
u(x,t_0) - \varepsilon (t_0 - T_0) \leq G[F_x[u],c_x](t_0) - \varepsilon (t_0 - T_0) \\
\leq F_x[u](r) + c_x(t_0 - r) - \varepsilon (t_0 - T_0) \\
\leq F_x[u](r) + c_x(t_0 - r) - \varepsilon r \\
< F_x[u - \varepsilon (t - T_0)](r) + c_x(t_0 - r).
\end{align*}
\]

This shows the assertion provided that
\[
G[F_x[u - \varepsilon (t - T_0)],c_x](t_0) = F_x[u - \varepsilon (t - T_0)](r) + c_x(t_0 - r)
\]

for some \(r \in (T_0,t_0)\), otherwise we have
\[
G[F_x[u - \varepsilon (t - T_0)],c_x](t_0) = u(x,T_0) + c_x(t_0 - T_0).
\]

In this case we exploit that \(u\) is subsolution of \([\text{Discr}]\) in \((T_0,T_1)\) and (32) to obtain
\[
u(x,t_0) - \varepsilon (t_0 - T_0) < G[F_x(u),c_x](t_0) \leq u(x,T_0) + c_x(t_0 - T_0)
\]
\[
= G[F_x[u - \varepsilon (t - T_0)],c_x](t_0).
\]

This concludes the proof.

\[\square\]

**Theorem 6.3.** Let \(u, v\) be uniformly continuous sub and supersolution to \([\text{Discr}]\), respectively, in \((T_0,T_1)\) with

\[
u(\cdot, T_0) \leq v(\cdot, T_0) \quad \text{in } V
\]

then

\[
u \leq v \quad \text{on } V \times [T_0,T_1).
\]

**Proof:** We assume for purposes of contradiction that \(u > v\) for some \((y,t_0) \in V \times (T_0,T_1)\). By taking \(\varepsilon\) small, we get

\[
u(y,t_0) - \varepsilon (t_0 - T_0) - v(y,t_0) > 0.
\]

By replacing \(u\) by \(u - \varepsilon (t - T_0)\), for such a small \(\varepsilon\), and bearing in mind Lemma 6.2, we can therefore assume without loosing generality that \(u\) is a strict subsolution.

We denote by \((y,t^*)\), \(t^* \in (T_0,t_0]\), a maximizer of \(u - v\) in \(V \times [T_0,t_0]\); it does exist because \(V \times [T_0,t_0]\) is a compact set and \(u(y,\cdot) - v(y,\cdot)\) is continuous. We have

\[
a := u(y,t^*) - v(y,t^*) > 0.
\]

Let \(r\) be a time with \(r \in [T_0,t^*]\) such that

\[
v(y,t^*) \geq G[F_y[v],c_y](t^*) = F_y[v](r) + c_y(t^* - r),
\]

where \(F_y[v] \in (c,c\cdot)\) and \(c \in (0,1)\), and

\[
\sum_{i,j=1}^p a_{ij} \int_{T_0}^{T_1} (u,v) < \int_{T_0}^{T_1} a(x,t) \cdot \langle Du, Dv \rangle.
\]

where \(a \in C^1\) and \(a(x,t) \geq 0\) for all \((x,t) \in V\).
then taking into account that
\[ v(x, t) + a \geq u(x, t) \quad \text{in} \ V \times [T_0, t_0] \]
\[ u(y, t^*) < F_y[u](r) + c_y(t^* - r) \]
we can use Lemma 6.1 (i), (ii) to get
\[ v(y, t^*) + a \geq F_y[v](r) + c_y(t^* - r) + a = F_y[v + a](r) + c_y(t^* - r) \]
\[ \geq F_y[u](r) + c_y(t^* - r) > u(y, t^*) , \]
in contrast with (34). □

6.2. Links between semidiscrete and Hamilton–Jacobi equation. We proceed linking (Discr) to (HJΓ). We consider a time interval \([T_0, T_1] \subset \mathbb{R}^+\) with \(T_1 \leq +\infty\).

**Proposition 6.4.** Let \( u \) be an uniformly continuous subsolution to (HJΓ) in \([T_0, T_1]\) then the trace of \( u \) on \((\Gamma \times \{T_0\}) \cup (V \times [T_0, T_1])\) is a subsolution of (Discr) in \([T_0, T_1]\).

**Proof:** We apply the definition of subsolution to (HJΓ) and of \( F_x[u] \) to get for every \( x \in V, t \in [T_0, T_1], \) every \( C^1 \) supertangent \( \varphi \) to \( u(x, \cdot) \) at \( t \)
\[ u(x, t) \leq F_x[u](t) \]
\[ \frac{d}{dt}\varphi(t) \leq c_x. \]
We deduce from Lemma 4.4
\[ u(x, t) \leq G[F_x[u], c_x](t) \quad \text{for any} \ (x, t) \in V \times [T_0, T_1]. \]
□

**Proposition 6.5.** Let \( u \) be an uniformly continuous supersolution to (HJΓ) in \([T_0, T_1]\) then the trace of \( u \) on \((\Gamma \times \{T_0\}) \cup (V \times [T_0, T_1])\) is a supersolution of (Discr) in \([T_0, T_1]\).

**Proof:** We fix a vertex \( x \) and an arc \( \gamma \in \Gamma_x \). According to Proposition 4.9 to prove that
\[ u(x, t) \geq G[F_x[u], c_x](t) \quad \text{for any} \ t \geq 0 \]
it is enough showing that for any \( t_0 \in [T_0, T_1] \) where there is a \( C^1 \) subtangent \( \varphi \) to \( u(x, \cdot) \) with \( \frac{d}{dt}\varphi(t_0) < c_x \), one has
\[ u(x, t_0) \geq F_x[u](t_0). \]
**Step 1.** We fix $a$ with

$$0 > c_x > a > \frac{d}{dt} \varphi(t_0)$$

and $p_0 \in \mathbb{R}$ such that

$$a + H_\gamma(1, p_0) < 0,$$

which implies

$$\frac{d}{dt} \varphi(t) + H_\gamma(s, p_0) < 0 \quad \text{for} \quad (s, t) \in Q \text{ close to } (1, t_0)$$

We can assume without loosing generality that $\varphi$ is a strict subtangent such that

$$u(x, t_0) = \varphi(t_0) \quad \text{and} \quad (39) \quad u(x, t) > \varphi(t) \quad \text{for} \quad t \neq t_0, \ t \ close \ to \ t_0.$$  

We can therefore determine $\delta > 0$ such that $t_0 + \delta < T_1$ and both $(38), (39)$ hold true for $(s, t) \in [1 - \delta, 1] \times [t_0 - \delta, t_0 + \delta]$; in addition, since

$$s \mapsto u(\gamma(s), t_0 - \delta) - \varphi(t_0 - \delta) - p_0 (s - 1)$$

is continuous and positive at $s = 1$ by $(39)$, we can also get

$$\quad u(\gamma(s), t_0 - \delta) > \varphi(t_0 - \delta) + p_0 (s - 1) \quad \text{for} \quad s \in [1 - \delta, 1].$$

If we assume by contradiction that

$$F_x[u](t_0) > u(x, t_0)$$

then we further have, up to shrinking $\delta$

$$F_\gamma[u](s, t) > u(\gamma(s), t) \quad \text{for} \quad (s, t) \in [1 - \delta, 1] \times [t_0 - \delta, t_0 + \delta].$$

**Step 2** We claim that the function

$$\phi(s, t) = \varphi(t) + p_0 (s - 1).$$

is a subtangent, constrained to $Q$, to $u \circ \gamma$ at $(1, t_0)$. By applying Proposition 5.16 to $s_0 = 1 - \delta$, $c = c_x$, the supersolution $u \circ \gamma$, the sets

$$A = (0, 1 - \delta) \times (t_0 - \delta, t_0 + \delta) \quad \text{and} \quad B = (1 - \delta, 1) \times (t_0 - \delta, t_0 + \delta),$$

the maximal subsolutions, denoted by $v, w$ to $[HJ]$, with trace less than or equal to $u \circ \gamma$ on $\partial_p^- A, \partial_p^+ B$, respectively, we derive

$$G[u(\gamma(1 - \delta), \cdot), c_x](t) \geq \min\{G[v(1 - \delta, \cdot), c_x], G[w(1 - \delta, \cdot), c_x]\}(t).$$
for any \( t \in [t_0 - \delta, t_0 + \delta] \). We further deduce from (42) and the relation \( v \geq F_\gamma[u] \) in \( A \)
\[
v(s, t) > u(\gamma(s), t) \quad \text{for} \quad (s, t) \in [1 - \delta] \times [t_0 - \delta, t_0 + \delta]
\]
and consequently by Lemma 4.7
\[
G[v(1 - \delta, \cdot), c_x](t) > G[u(\gamma(1 - \delta), \cdot), c_x](t) \quad \text{for} \quad t \in [t_0 - \delta, t_0 + \delta].
\]
We then have by (44)
\[
G[u(\gamma(1 - \delta), \cdot), c_x](t) \geq G[w(1 - \delta, \cdot), c_x](t) \quad \text{for} \quad t \in [t_0 - \delta, t_0 + \delta].
\]
We know by (39), (40) that \( \phi \leq u \circ \gamma \) on \( \partial_+ p \), and consequently by (38) that it is a subsolution of \((HJ_\gamma)\) in \( B \). We derive taking into account the maximality property of \( w \)
\[
\phi(1 - \delta, t) = G[\phi(1 - \delta, \cdot), c_x] \leq w(1 - \delta, t) \quad \text{for} \quad t \in [t_0 - \delta, t_0 + \delta]
\]
which implies
\[
\phi(1 - \delta, t) = G[\phi(1 - \delta, \cdot), c_x] \leq G[w(1 - \delta, \cdot), c_x](t) \leq u(\gamma(1 - \delta), t)
\]
for \( t \in [t_0 - \delta, t_0 + \delta] \). Summing up, \( u \circ \gamma \geq \phi \) on the whole of the parabolic boundary of \( B \). We derive from the comparison principle given in Theorem 5.2 that
\[
\phi(1 - \delta, t) = G[\phi(1 - \delta, \cdot), c_x] \leq w(1 - \delta, t) \quad \text{for} \quad t \in [t_0 - \delta, t_0 + \delta]
\]
which implies
\[
\phi(1 - \delta, t) = G[\phi(1 - \delta, \cdot), c_x] \leq G[w(1 - \delta, \cdot), c_x](t) \leq u(\gamma(1 - \delta), t)
\]
for \( t \in [t_0 - \delta, t_0 + \delta] \). Summing up, \( u \circ \gamma \geq \phi \) on the whole of the parabolic boundary of \( B \). We derive from the comparison principle given in Theorem 5.2 that
\[
u(\gamma(s), t) \geq \phi(s, t) \quad \text{for} \quad (s, t) \in B
\]
Taking into account that \( u = \phi \) at \((1, t_0)\), we conclude that the function \( \phi \) is a subtangent, constrained to \( Q \), to \( u \circ \gamma \) at \((1, t_0)\), as was claimed.

**Step 3** We therefore reach a contradiction with \( u \) being supersolution and \( \frac{d}{dt} \phi(t_0) < c_x \), since \( \gamma \) is an arbitrary arc in \( \Gamma_x \) and by (37)
\[
\phi_t(1, t_0) + H_\gamma(1, \phi'(1, t_0)) < 0.
\]
This shows (35) and ends the proof. \( \square \)

We recall that
\[
R = (0, 1) \times (T_0, T_1).
\]

**Proposition 6.6.** Let \( u : \Gamma \times [T_0, T_1] \to \mathbb{R} \) be an uniformly continuous function such that \( u \circ \gamma \) is solution to \((HJ_\gamma)\) in \((0, 1) \times (T_0, T_1)\) for any arc \( \gamma \), and the trace of \( u \) on \((\Gamma \times \{0\}) \cup (V \times [T_0, T_1])\) solves \((\text{Discr})\) in \((T_0, T_1)\), then \( u \) is solution of \((HJ_\Gamma)\) in \((T_0, T_1)\).
**Proof:** It is clear from the assumptions that \( (x, t) \mapsto u(x, t) \) is subsolution to \((HJ\Gamma)\) in \([T_0, T_1]\). We fix \( x_0 \in V \). It is left checking condition (ii) in the definition of supersolution. Let \( t_0 \in (T_0, T_1) \), we select \( \gamma \in \Gamma_{x_0} \) with
\[
F_{\gamma}[u](1, t_0) = F_{x_0}[u](t_0).
\]
Let \( \psi \) be a \( C^1 \) subtangent to \( u(x_0, \cdot) \) at \( t_0 \) with \( \partial_t \psi(t_0) < c_{x_0} \), therefore by Proposition 4.8
\[
u(x_0, t_0) = F_{x_0}[u](t_0) = F_{\gamma}[u](1, t_0).
\]
In addition
\[
u(\gamma(s), t) \leq F_{\gamma}[u](s, t) \quad \text{for} \quad (s, t) \in \overline{R}
\]
so that any subtangent \( \phi \), constrained to \( \overline{R} \), to \( u \) at \((1, t_0)\) is also subtangent to \( F_{\gamma}[u] \). If we now assume by contradiction that
\[
\phi_t(1, t_0) + H_{\gamma}(1, \phi'(1, t_0)) < 0
\]
we can construct via Perron–Ishii method a subsolution to \((HJ_{\gamma})\) strictly greater than \( F_{\gamma}[u](s, t) \) in a neighborhood of \((1, t_0)\) intersected with \( \overline{R} \). This contradicts the maximality of \( F_{\gamma}[u] \), and concludes the proof. \( \square \)

The above argument also allows pointing out a property of solutions we will use to prove some stability results.

**Corollary 6.7.** Assume \( u \) to be an uniformly continuous solution of \((HJ\Gamma)\) in \((T_0, T_1)\), and consider \( x_0 \in V \). Assume that there is a \( C^1 \) subtangent \( \varphi \) to \( u(x_0, \cdot) \) at \( t_0 \), for some \( t_0 \in (T_0, T_1) \) with \( \frac{d}{dt}\varphi(t_0) < c_{x_0} \). Then there exists \( \gamma \in \Gamma_{x_0} \) such that \( u(x_0, t_0) \) coincides with the maximal subsolution of \((HJ_{\gamma})\) in \( R \), with trace \( u \circ \gamma \) on \([0, 1] \times \{T_0\}\) and \( u(\gamma(0), t) \) on \( \{0\} \times [T_0, T_1) \), computed at \((1, t_0)\).

**Proof:** We derive from the assumptions, arguing as in Proposition 6.6 that
\[
u(x_0, t_0) = F_{x_0}[u](t_0).
\]
To conclude the proof it is then enough to take \( \gamma \in \Gamma_{x_0} \) with
\[
F_{x_0}[u](t_0) = F_{\gamma}[u](1, t_0)
\]
and recall the definition of \( F_{\gamma}[u] \). \( \square \)

We further show that Proposition 5.3 can be extended to (sub)solutions of \((HJ\Gamma)\).
Corollary 6.8. Let $t^* \in (T_0, T_1)$. Let $u$ be an uniformly continuous function from $\Gamma \times [T_0, T_1]$ to $\mathbb{R}$. Assume $u$ to be (sub)solution of $(HJ\Gamma)$ in $(T_0, t^*)$ and in $(t^*, T_1)$. Then $u$ is (sub)solution in $(T_0, T_1)$ as well.

Proof: We set $S = (0, 1) \times (T_0, t^*)$. Taking into account Proposition 5.3, the assertion is immediate in the case of subsolutions. In the case of solutions, we need just checking property (ii) in the definition of supersolution at $t^*$. Given $x \in V$, we therefore assume that there is a $C^1$ subtangent $\psi$ to $u(x, \cdot)$ at $t^*$ with

$$d\frac{dt}{t^*} \psi(t^*) < c_x. \tag{46}$$

We claim that there exists a sequence $t_n < t^*$, $t_n \to t$ and $C^1$ subtangents $\psi_n$ to $u(x, \cdot)$ at $t_n$ with

$$d\frac{dt}{t_n} \psi_n(t_n) < c_x. \tag{47}$$

If on the contrary, one can find an interval $(t^* - \delta, t^*)$, for some $\delta > 0$, such that

$$d\frac{dt}{\varphi(t)} \geq c_x$$

for any $t \in (t^* - \delta, t^*)$, any $C^1$ sub tangent $\psi$ to $u(x, \cdot)$ at $t$, then, according to Lemma 4.2

$$u(x, t) - c_x (t - t^*)$$

is nondecreasing in $[t^* - \delta, t^*]$. In contrast with (46). By applying Corollary 6.7 to the $t_n$'s satisfying (47), we find a sequence $\gamma_n$ of arcs in $\Gamma_x$ such that $u(x, t_n)$ coincides with the maximal subsolution of $(HJ\gamma)$, with $\gamma$ replaced by $\gamma_n$, in $S$ with trace equal to $u \circ \gamma_n$ on $[0, 1] \times \{T_0\}$ and $u(\gamma_n(0), t)$ on $\{0\} \times [T_0, t^*)$, computed at $t_n$. The sequence $\gamma_n$ be must definitively constant, equal, say, to $\gamma \in \Gamma_x$. Accordingly, we have that

$$u(x, t^*) = \lim_{n \to \infty} w(1, t_n),$$

where we have denoted by $w(s, t)$ the maximal subsolution to $(HJ\gamma)$ in $S$ with trace $u \circ \gamma$ on $\partial_p^- S$. Now assume for purposes of contradiction that there is a $C^1$ sub tangent $\phi$, constrained to $[0, 1] \times [T_0, T_1)$, to $u \circ \gamma$ at $(1, t^*)$, with

$$\phi_t(1, t^*) + H_\gamma(1, \phi'(1, t^*)) < 0. \tag{48}$$

We can further assume that

$$\phi(1, t^*) = u(x, t^*) \quad \text{and} \quad \phi < u \circ \gamma \quad \text{in} \quad U \cap ([0, 1] \times [T_0, T_1)) \setminus \{(1, t^*)\},$$
where $U$ is a suitable neighborhood of $(1, t^*)$ in $\mathbb{R}^2$. Exploiting the maximality of $w$, we deduce that

$$w > \phi \quad \text{in } (U \cap S) \setminus \{(1, t^*)\},$$

so that using (48) we can construct via Perron–Ishtii method a subsolution to $\mathbf{HJ}_\gamma$ agreeing with $w$ in $\partial_p^- S$ and strictly greater to $w$ at points of $S$ suitably close to $(1, t^*)$. We have therefore reached a contradiction, which shows that (48) is not possible. This ends the proof.

7. Well posedness of $\mathbf{HJ}_\Gamma$

As already clarified in the Introduction and Section 3, the well posedness is relative to the time dependent Hamilton–Jacobi equation coupled with a continuous initial datum plus a flux limiter at the vertices.

7.1. Comparison result. We start with a comparison result based on Theorem 6.3 and the links established between Hamilton–Jacobi equation and semidiscrete problem.

**Theorem 7.1.** Let $u$, $v$ be uniformly continuous sub and supersolution of $(\mathbf{HJ}_\Gamma)$ respectively, in $(T_0, T_1)$ with $u(\cdot, T_0) \leq v(\cdot, T_0)$ in $[0, 1]$, then $u \leq v$ in $\Gamma \times [T_0, T_1]$.

**Proof:** The trace of $u$ and $v$ on $(\Gamma \times \{T_0\}) \cup (V \times [T_0, T_1])$ are subsolution and supersolution to $(\mathbf{Discr})$ by Propositions 6.4, 6.5 respectively. We then invoke Theorem 6.3 to get

$$u \leq v \quad \text{in } (\Gamma \times \{T_0\}) \cup (V \times \mathbb{R}^+).$$

We apply the comparison result in Theorem 5.2 to finally obtain

$$u \circ \gamma \leq v \circ \gamma \quad \text{in } [0, 1] \times [T_0, T_1], \text{ for any } \gamma.$$

This proves the assertion. □

We derive:

**Corollary 7.2.** Let $g$ be a continuous datum on $\Gamma$, there exists at most one uniformly continuous solution to $(\mathbf{HJ}_\Gamma)$ in $(T_0, T_1)$ agreeing with $g$ at $t = T_0$.

We further derive a first stability result.
Corollary 7.3. Let $g_n$ a sequence of continuous functions in $\Gamma$ uniformly converging to a function $g$. Assume that there exist (unique) uniformly continuous solutions $u_n$ of $(HJ_\Gamma)$ in $(T_0, T_1)$ with initial data $g_n$ at $t = T_0$ and flux limiter $c_x$. Then the $u_n$’s are uniformly convergent in $\Gamma \times [T_0, T_1]$ to a solution $u$ of $(HJ_\Gamma)$ in $(T_0, T_1)$ agreeing with $g$ at $t = T_0$, and

$$|u_n - u|_\infty \leq |g_n - g|_\infty \quad \text{for every } n.$$

Proof: We have

$$g_m - |g_n - g_m|_\infty \leq g_n \leq g_m + |g_n - g_m|_\infty \quad \text{in } \Gamma, \text{ for } n, m \in \mathbb{N}.$$

We derive from the comparison principle in Theorem 7.1 that

$$u_m - |g_n - g_m|_\infty \leq u_n \leq u_m + |g_n - g_m|_\infty \quad \text{in } \Gamma \times [T_0, T_1],$$

which implies

$$|u_n - u_m|_\infty \leq |g_n - g_m|_\infty$$

then $u_n$ uniformly converges in $\Gamma \times [T_0, T_1]$ to a function $u$ and

$$|u_n - u|_\infty = \lim_m |u_n - u_m|_\infty \leq \lim_m |g_n - g_m|_\infty = |g_n - g|_\infty.$$

Taking into account the usual stability results in viscosity solutions theory, to prove that $u$ is solution of $(HJ_\Gamma)$ in $(T_0, T_1)$, it is enough to check that it satisfies the condition (ii) in the definition of supersolution.

We assume that for a given $(x_0, t_0) \in V \times (T_0, T_1)$ there exists a $C^1$ subtangent $\varphi$ to $u(x_0, \cdot)$ at $t_0$ with $\frac{d}{dt} \varphi(t_0) < c_{x_0}$. We derive that $\varphi$ is subtangent to $u_n(x_0, \cdot)$ at points $t_n$ with $t_n \to t_0$, and

$$\frac{d}{dt} \varphi(t_n) < c_{x_0} \quad \text{for } n \text{ large enough.}$$

We further derive from Corollary 6.7, applied to $u_n$ and $(x_0, t_n)$, that there is $\gamma_n \in \Gamma_{x_0}$ such that $u_n(x_0, t_n)$ coincides with the maximal subsolution of $(HJ_{\gamma_n})$, with $\gamma_n$ in place of $\gamma$, with $g_n \circ \gamma_n$ on $[0, 1] \times \{0\}$ and $u_n(\gamma_n(0), t)$ on $\{0\} \times \mathbb{R}^+$, computed at $(1, t_0)$.

Since $\Gamma_{x_0}$ contains finite elements, we can assume, up to further extracting a subsequence, that the sequence $\gamma_n$ is constant, equal, say, to $\gamma$. We then invoke Corollary 5.5 to conclude that $u(x_0, t_0)$ coincides with the maximal solution of $(HJ_{\gamma})$ with $g \circ \gamma$ on $[0, 1] \times \{0\}$ and $u(\gamma(0), t)$ on $\{0\} \times \mathbb{R}^+$, computed at $(1, t_0)$. Arguing as in Proposition 6.6, we deduce that

$$\psi_t(1, t_0) + H_{\gamma}(1, \psi(1, t_0)) \geq 0$$
for any subtangent $\psi$, constrained to $[0, 1] \times (T_0, T_1)$, to $u \circ \gamma$ at $(1, t_0)$. This concludes the proof.

7.2. Existence of solutions and stability properties. In this section we show the existence of a solution to $(HJ_\Gamma)$ in $(T_0, T_1)$, with $T_1 \leq +\infty$, and consequently to $(\text{Discr})$, coupled with any continuous initial datum $g$ and any flux limiter $c_x$. We first assume the initial datum $g$ to be Lipschitz continuous in $\Gamma$, and set

$$(49) \quad M_0 = \left( \max_{\gamma} \min_m \{ H_\gamma(s, (g \circ \gamma)'(1)) \leq m \} \right) \vee \left( \max_{x \in V} |c_x| \right)$$

**Theorem 7.4.** For any given Lipschitz continuous initial datum $g$ at $t = T_0$, any flux limiter $c_x$, there exists a Lipschitz continuous solution $u$ of $(HJ_\Gamma)$ in $(T_0, T_1)$ with

$$(50) \quad \text{Lip} \left( u \circ \gamma(s, \cdot) \right) \leq M_0 \quad \text{for any } \gamma, \text{ any } s \in [0, 1].$$

**Proof:** We define

$${\cal T} = \{ T > 0 \mid \exists \text{ Lip. sol. } u \text{ to } (HJ_\Gamma) \text{ with } g, c_x \text{ in } (T_0, T) \text{ sat. } (50) \}.$$ 

We first prove that $\cal T \neq \emptyset$. We set $R = (0, 1) \times (T_0, T_1)$.

**Step 1.** Given any arc $\gamma$, let $v_\gamma$ be the maximal subsolution of $(HJ_{\gamma})$ in $R$ with initial datum $g \circ \gamma$ at $t = T_0$, we set

$$v(x, t) = \min_{\gamma \in \Gamma_x} v_\gamma(1, t) \quad \text{for any } x \in V, t \in (T_0, T_1)$$

$$v(y, T_0) = g(y) \quad \text{for any } y \in \Gamma.$$

**Step 2.** We have

$$F_\gamma[v](1, t) \leq v_\gamma(1, t) \quad \text{in } [T_0, T_1), \text{ for any } \gamma.$$ 

Since, given $x \in V$, $t \in [T_0, T_1)$

$$v(x, t) = v_{\gamma_x}(1, t) \quad \text{for a suitable } \gamma_x \in \Gamma_x$$

we deduce that

$$(51) \quad v(x, t) \geq F_{\gamma_x}[v](1, t) \geq F_x[v](t) \quad \text{for any } x \in V, t \in [T_0, T_1)$$

**Step 3.** We define

$$(52) \quad \overline{u}(x, t) = G[v(x, \cdot), c_x](t) \leq v(x, t) \quad x \in V, t \in [T_0, T_1)$$

$$(53) \quad \overline{u}(y, T_0) = g(y) \quad y \in \Gamma.$$
To sum up: \( \overline{u}(x,t) \) from \( V \times [T_0, T_1) \to \mathbb{R} \) has been defined through three steps:

- first, we have set \( \nu_\gamma(1,t) \), for \( \gamma \in \Gamma_x \), as the maximal subsolution of (HJ)\( _\gamma \)
in \( R \) among those agreeing with \( g \circ \gamma \) at \( t = T_0 \);
- second, we have defined \( v(x,t) = \min_{\gamma \in \Gamma_x} \nu_\gamma(1,t) \);
- and, finally, \( \overline{u}(x,t) = G[v(x,\cdot), c_x](t) \).

We see through estimate (53) that \( \text{Lip} v_\gamma(1,\cdot) \leq M_0 \), see (52), for any \( \gamma \in \Gamma_x \), and consequently \( \text{Lip} v(x,\cdot) \leq M_0 \), so that we have, according to Remark 4.5

\[
(54) \quad \text{Lip} \overline{u}(x,\cdot) \leq M_0 \quad \text{for any } x \in V.
\]

**Step 4.** We apply, for any \( \gamma \), Proposition 5.13 to the admissible boundary datum \( \nu_\gamma \) in \( \partial^- R \) and derive, via (52) and the very definition of \( v \), that \( (g \circ \gamma, \overline{u}(\gamma(0),\cdot)) \) is an admissible datum for (HJ)\( _\gamma \) on \( \partial^- R \), for any arc \( \gamma \), and the same holds true for \( (g \circ \gamma, \overline{u}(\gamma(1),t)) \) in \( \partial^+ R \).

We denote by \( u_1 \circ \gamma, u_2 \circ \gamma \) the corresponding maximal subsolutions of (HJ)\( _\gamma \) with the above data taken on \( \partial^- R, \partial^+ R \), respectively. According to the estimates given in (13), (14), (17), (18) and (54), we get for any \( \gamma \)

\[
(55) \quad \text{Lip} (u_1 \circ \gamma), \text{Lip} (u_2 \circ \gamma) \leq M_0 \lor \big( \max \{|p| \mid H_\gamma(s,p) \leq M_0 \} \big).
\]

and

\[
(56) \quad \text{Lip} (u_1 \circ \gamma(s,\cdot)), \text{Lip} (u_2 \circ \gamma(s,\cdot)) \leq M_0 \quad \text{for any } s \in [0, 1]
\]

By monotonicity of \( G \), (51) and (52), we further have

\[
(57) \quad G[F_x[u], c_x](t) \leq G[F_x[v], c_x](t) \leq G[v(x,\cdot), c_x](t) = \overline{u}(x,t)
\]

for any \( x \in V, t \in [T_0, T_1) \).

**Step 5.** We apply Corollary 5.18 to all arcs \( \gamma \) and deduce for a suitable \( \delta > 0 \), with \( T_0 + \delta < T_1 \), depending on the \( H_\gamma \) and the Lipschitz constants of \( u_1 \circ \gamma, u_2 \circ \gamma \), see (53), that, for any \( \gamma \), the merge of all three functions \( g \circ \gamma, \overline{u}(\gamma(0),t), \overline{u}(\gamma(1),t) \) is admissible in \( \partial_p ((0,1) \times (T_0, T_0 + \delta)) \). We denote by \( u \circ \gamma \) the corresponding solutions, with \( u \circ \gamma : [0, 1] \times [T_0, T_0 + \delta] \to \mathbb{R} \). We have for any \( x \in V \)

\[
F_x[u](t) = F_x[\overline{u}](t) \geq \overline{u}(x,t) = u(x,t) \quad \text{in } [T_0, T_0 + \delta]
\]

and accordingly

\[
(58) \quad G[F_x[u], c_x](t) = G[F_x[\overline{u}], c_x](t) \geq \overline{u}(x,t) = u(x,t) \quad \text{in } [T_0, T_0 + \delta].
\]
By combining (57), (58), we obtain
\[ G[F_x[u], c_x](t) = G[F_x[w], c_x](t) = w(x, t) = u(x, t). \]
This implies by Proposition 6.6 that \( u \) is solution to (HJΓ) in \( \Gamma \times (T_0, T_0 + \delta) \). We therefore see, taking into account (56), that \( T_0 + \delta \in \mathcal{T} \).

**Step 6.** We set \( T^* = \sup \mathcal{T} \), and proceed proving that there is a solution of (HJΓ) in \( [T_0, T^*) \) satisfying (50). We select an increasing sequence \( T_n \in \mathcal{T} \) with \( T_n \to T^* \), and denote by \( u_n \) the corresponding solutions satisfying (50) in \( (T_0, T_n) \). By the uniqueness result in Corollary 7.2, we have
\[ u_n(x, t) = u_{n+1}(x, t) \quad \text{for } (x, t) \in \Gamma \times (T_0, T_n), \text{ for any } n \in \mathbb{N}, \]
so that a solution \( u \) in \( [T_0, T^*) \) can be unambiguously defined via
\[ u(x, t) = u_n(x, t) \quad \text{for } n \text{ with } T_n > t. \]
Since all the \( u_n \) satisfy (50), the same holds true for \( u \).

**Step 7.** To conclude the proof, it is then enough proving that \( T^* = T_1 \). We assume for purposes of contradiction that \( T^* < T_1 \), and iterate the construction performed in the first part of the proof in the interval \( (T_n, T_1) \) starting from \( u(\cdot, T_n) \) as initial datum at \( t = T_n \), where \( u \) and \( T_n \) are defined as in Step 6.

Since \( u \) satisfies (50), we get, according to Proposition 5.6
\[ \max_{\gamma} \min\{m \mid H_\gamma(s, u \circ \gamma')(s, T_n) \leq m\} \leq M_0. \]

Arguing as in the first part of the proof, we show that we can define a solution of (HJΓ) with initial datum \( u(\cdot, T_n) \) in an interval \( (T_n, T_n + \delta_n) \) with \( T_n + \delta_n < T_1 \). By the gluing result given in Corollary 6.8, we get altogether a solution in \( (T_0, T_n + \delta_n) \).

The crucial point is that, apart the restriction \( T_n + \delta_n < T_1 \), \( \delta_n \) does not depend further on \( n \). It is in fact the positive constant for which Corollary 5.18 holds true, and it depends on \( H_\gamma \) and the Lipschitz constants of the solutions of (HJΓ), for any \( \gamma \), in \( (T_n, T_1) \) to be merged. Due to (59), they satisfy estimates as in (55), independent, to repeat, of \( n \).

To get a contradiction, it is then enough to take, for a suitable \( n, \delta_n \) with
\[ T_1 > T_n + \delta_n > T^*. \]
This ends the proof. \( \square \)

**Remark 7.5.** We denote by \( u \) the unique Lipschitz continuous solutions of (HJΓ). According to (19), (20), we see that
\[ \text{Lip} (u \circ \gamma) \leq M_0 \lor \left( \max\{|p| \mid H_\gamma(s, p) \leq M_0\} \right) \quad \text{for any } \gamma. \]
The Lipschitz constant of the solution therefore only depends on the Hamiltonians \( H_\gamma \) and the Lipschitz constant of the initial datum.

We proceed proving, through Corollary 7.3, existence of solutions of (HJ\( \Gamma \)) for any continuous initial datum.

**Proposition 7.6.** For any continuous initial datum \( g \), any flux limiter \( c_x \), there exists an uniformly continuous solution \( u \) of (HJ\( \Gamma \)) in \((T_0, T_1)\).

**Proof:** Let \( g_n \) be a sequence of Lipschitz continuous function in \( \Gamma \) uniformly converging to \( g \), see Lemma 5.10 for the existence of such approximations. By applying Corollary 7.3, we see that the unique Lipschitz continuous solutions \( u_n \) to (HJ\( \Gamma \)) in \((T_0, T_1)\) with initial datum \( g_n \), which do exist by Theorem 7.4, uniformly converge to an uniformly continuous solution of (HJ\( \Gamma \)) in \((T_0, T_1)\) with initial datum \( g \).

We proceed giving a new version of Theorem 7.1.

**Theorem 7.7.** Let \( u, v \) be continuous subsolution and solution of (HJ\( \Gamma \)) in \((T_0, T_1)\), respectively, with \( u(\cdot, T_0) \leq v(\cdot, T_0) \), then \( u \leq v \) in \( \Gamma \times [T_0, T_1] \).

**Proof:** Assume by contradiction that there exists \((x_0, t_0) \in \Gamma \times (T_0, T_1)\) with

\[
u(x_0, t_0) > u(x_0, t_0).
\]

According to Proposition 7.6 there exist uniformly continuous solutions \( \overline{u}, \overline{v} \) to (HJ\( \Gamma \)) in \((t_0, T_0)\) with initial data \( u(\cdot, t_0), v(\cdot, t_0) \), respectively, at \( t = t_0 \). We set

\[
\overline{u}(x,t) = \begin{cases} 
  u(x,t) & \text{for } (x,t) \in \Gamma \times [T_0, t_0) \\
  \overline{u}(x,t) & \text{for } (x,t) \in \Gamma \times (t_0, T_1) 
\end{cases}
\]

and

\[
\overline{v}(x,t) = \begin{cases} 
  v(x,t) & \text{for } (x,t) \in \Gamma \times [T_0, t_0) \\
  \overline{v}(x,t) & \text{for } (x,t) \in \Gamma \times (t_0, T_1) 
\end{cases}
\]

According to Corollary 6.8 \( \overline{u}, \overline{v} \) are uniformly continuous subsolution and solution to (HJ\( \Gamma \)) in \((T_0, T_1)\), respectively, but then the inequality \( \overline{u} > \overline{v} \) at \((x_0, t_0)\) is in contrast with Theorem 7.1.

By summarizing the information gathered in Theorem 7.4, Proposition 7.6 and Theorem 7.7, we can state:
Proposition 7.8. For any continuous initial datum \( g \) and flux limiter \( c_x \), there exists one and only one continuous solution to \((HJ_\Gamma)\) in \((T_0, T_1)\). This solution is in addition uniformly continuous in \( \Gamma \times [T_0, T_1) \). If \( g \) is Lipschitz continuous, the solution is Lipschitz continuous as well.

We now consider a sequence of Hamiltonians \( H^n_\gamma \) for any arc \( \gamma \), a sequence of continuous functions \( g^n \) from \( \Gamma \) to \( \mathbb{R} \), and a sequence of real numbers \( c^n_x \), for any vertex \( x \), with

\[
c^n_x \leq \min_{\gamma \in \Gamma_x} c^n_\gamma,
\]

where

\[
c^n_\gamma = -\max_s \min_p H^n_\gamma(s, p) \quad \text{for any} \ \gamma.
\]

By adapting the argument of Corollary 7.3, we can finally prove:

**Theorem 7.9.** Assume that \( H^n_\gamma(s, p) \longrightarrow H_\gamma(s, p) \) uniformly in \([0, 1] \times [T_0, T_1)\), for any \( \gamma \), that \( g^n \) is uniformly convergent to a function \( g \) in \( \Gamma \), and that \( c^n_x \longrightarrow c_x \) for any \( x \). Let us denote by \( u^n \) the sequence of continuous solutions to \((HJ_\Gamma)\) with \( H^n_\gamma, c^n_x \) in place of \( H_\gamma, c_x \), respectively, and initial datum \( g^n \). Then the \( u^n \) locally uniformly converge in \( \Gamma \times [T_0, T_1) \) to the continuous solution \( u \) of \((HJ_\Gamma)\) with initial datum \( g \) and flux limiter \( c_x \).

**Proof:** We claim that the \( u^n \)'s are equibounded and equicontinuous. We denote by \( g^n_k \) sequences of Lipschitz continuous functions on \( \Gamma \) with

\[
\lim_k g^n_k = g_n \quad \text{for any} \ n, \ \text{uniformly in} \ \Gamma \quad \text{and} \ \text{Lip} g^n_k = k,
\]

see Lemma 5.10 for the existence of such approximations. We denote by \( u^n_k \) the solutions corresponding to \( g^n_k \) and the flux limiter \( c^n_x \). Since, in view of Remark 7.3, the Lipschitz constants of \( u^n_k \) only depend on the Hamiltonians \( H^n_\gamma \), which uniformly converge to \( H_\gamma \), and the Lipschitz constants of \( g^n_k \), we derive that the family of \( u^n_k \), with \( k \) fixed and \( n \) varying in \( \mathbb{N} \), is equiLipschitz continuous. We set

\[
\ell_n = \sup_k \text{Lip} u^n_k.
\]

Since the rate of convergence of \( g^n_k \) to \( g_n \) depends on the continuity modulus of \( g_n \), see Lemma 5.10 \((ii)\), and the \( g^n \)'s are equicontinuous, there exists an infinitesimal sequence \( a_k > 0 \) with

\[
a_k \geq |g^n_k - g_n|_\infty \quad \text{for any} \ n.
\]
This implies by Corollary 7.3
\[ a_k \geq |u^k_n - u_n|_\infty \text{ for any } n. \]

We derive that
\[ |u_n(s_1, t_1) - u_n(s_2, t_2)| \leq \inf_k \{a_k + \ell_k (|s_1 - t_1| + |s_2 - t_2|)\} \text{ for any } n. \]

Therefore the \( u_n \)'s are equicontinuous and, since they start from the initial data \( g_n \) which are equibounded, they are locally equibounded as well, which proves the claim. This in turn implies by Ascoli Theorem that the \( u_n \) are locally uniformly convergent, up to subsequences, to a continuous function denoted by \( u \).

Taking into account the usual stability results in viscosity solutions theory, to prove that \( u \) is solution of (HJΓ), it is enough to check that it satisfies the condition (ii) in the definition of supersolution. This can be done arguing as in the last part of Corollary 7.3 with obvious modification. \( \square \)

Appendix A. \( t \)-partial sup convolutions

We set, as usual, \( R = (a,b) \times (T_0, T_1) \), with \( T_1 \leq +\infty \). Given an uniformly continuous function \( u \) in \( \overline{R} \) with continuity modulus \( \omega \), we define its \( t \)-partial sup–convolutions via:

\[
(60) \quad u^\delta(s,t) = \max \left\{ u(s,r) - \frac{1}{2\delta} (r-t)^2 \mid r \in [T_0, T_1] \right\},
\]

for \( \delta > 0 \), note that the maximum in (60) does exist, even if \( T_1 = +\infty \), because \( u \) has sublinear growth for \( t \) going to +\( \infty \). The next proposition summarizes some properties of interest of this regularization.

**Proposition A.1.** We have

1. \( u^\delta \) uniformly converges to \( u \) in \( R \) as \( \delta \) goes to 0;
2. if \( u \) is a subsolution of (HJ) in \( R \), then for any \( \delta \), there exists \( T_\delta = O(\sqrt{\delta}) \) such that \( u^\delta \) is a Lipschitz continuous subsolution of (HJ) in \( (a,b) \times (T_0 + T_\delta, T_1) \).

**Proof:** We denote by \( \omega \) a continuity modulus of \( u \) in \( R \), and consider two positive constants \( a, \ell \) with

\[ |u(s_i, t_1) - u(s_2, t_2)| \leq a + \ell (|s_1 - s_2| + |t_1 - t_2|) \quad (s_i, t_i), \ i = 1, 2 \text{ in } R. \]
Given \((s,t) \in \mathcal{R}\), we say that \(r\) is \(u^\delta\)-optimal for \((s,t)\) if it realizes the maximum in \((60)\). To estimate \(|r - t|\), we start from

\[
0 \leq u^\delta(s,t) - u(s,t) = u(s,r) - u(s,t) - \frac{1}{2\delta}(t-r)^2
\]

which implies

\[
\frac{1}{2\delta} (r-t)^2 \leq a + \ell |t-r|.
\]

We deduce that there exists \(T_\delta = O(\sqrt{\delta})\) such that

\[
|t-r| \leq T_\delta \quad \text{for any } (s,t), \text{ } r \text{ } u^\delta\text{-optimal for } (s,t).
\]

We know that \(u^\delta(s,\cdot)\) is semiconvex and consequently locally Lipschitz continuous, for any \(s\). We derive from \((61)\) that it is globally Lipschitz continuous in \(\mathbb{R}^+\) and the Lipschitz constant is independent of \(s\). This property depends on the fact that if \(u(s,\cdot)\) is differentiable at \(t\) then the \(u^\delta\)-optimal point for \((s,t)\) is unique and the derivative is given by \(\frac{r-t}{\delta}\). We therefore have that the Lipschitz constant of \(u^\delta(s,\cdot)\) in \([T_0, T_1]\) is estimated from above by \(\frac{T_\delta}{\delta}\). We further derive

\[
u^\delta(s_1, t_1) - u^\delta(s_2, t_2) = \left(\nu^\delta(s_1, t_1) - u^\delta(s_1, t_2)\right) + \left(u^\delta(s_1, t_2) - u^\delta(s_2, t_2)\right)
\]

\[
\leq \left(\nu^\delta(s_1, t_1) - u^\delta(s_1, t_2)\right) + \left(u(s_1, r_1) - u(s_2, r_1)\right)
\]

\[
\leq \frac{T_\delta}{\delta} |t_1 - t_2| + \omega(|s_1 - s_2|),
\]

where \(r_1\) is \(u^\delta\)-optimal for \((s_1, t_1)\), and

\[
u^\delta(s, t) - u(s, t) \leq u(s,r) - u(s,t) \leq \omega(T_\delta) \quad \text{for any } (s,t),
\]

which shows that \(u^\delta\) is uniformly continuous in \((s,t)\), and item (i). Taking into account that by \((61)\) \(T_0\) cannot be \(u^\delta\)-optimal for \((s,t)\) whenever \(t > T_0 + T_\delta\), we have that if \(\varphi\) is supertangent to \(u^\delta\) at a point \((s_0, t_0)\) with \(t_0 \in (T_0 + T_\delta, T_1)\) and \(r_0\) is \(u^\delta\)-optimal for \((s_0, t_0)\) then

\[
s(t) \mapsto \varphi(s, t + (t_0 - r_0))
\]

is supertangent to \(u\) at \((s_0, r_0)\), constrained to \(\overline{R}\) if \(r = T_1\). Therefore, if \(u\) is subsolution of \((HJ)\) in \(R\), then

\[
\varphi_t(s_0, t_0) + H(s_0, \varphi'(s_0, t_0)) \leq 0
\]
which shows that \(u^\delta\) is subsolution of the same equation in \((a, b) \times (T_0 + T_\delta, T_1)\). Consequently, taking into account that the Hamiltonian is coercive and \(u\) is continuous in \((s, t)\) plus Lipschitz continuous in \(t\) with Lipschitz constant independent of \(s\), \(u^\delta\) is Lipschitz continuous in \([a, b] \times [T + T_\delta, T_1]\). This shows item (ii), and concludes the proof.

\[\square\]

Appendix B. Some proofs of results in Section 5

We recall that \(R = (a, b) \times (T_0, T_1)\).

**Proof of Theorem 5.2** We consider the \(t\)-partial sup convolutions of \(u\) denoted by \(u^\delta\). We exploit that \(u^\delta\) uniformly converges to \(u\) in \(R\) as \(\delta \to 0\), see Proposition A.1 and \(T_\delta \to 0\) as \(\delta \to 0\), see the statement of Proposition A.1 for the definition of \(T_\delta\). Given an arbitrary \(\varepsilon > 0\), we then have

\[(62)\quad v + \varepsilon > u^\delta \quad \text{in} \quad \partial_p R \cup [a, b] \times [T_0, T_0 + T_\delta], \quad \text{for} \ \delta \ \text{small.}\]

Let \((s_0, t_0) \in R\), then \((s_0, t_0) \in (a, b) \times (T_0 + T_\delta, T_1)\) for \(\delta\) sufficiently small. Since by Proposition A.1 \(u^\delta\) is a Lipschitz continuous subsolution to \([HJ]\) in \((a, b) \times (T_0 + T_\delta, T_1)\), and by \(62\)

\[v + \varepsilon > u^\delta \quad \text{in} \quad \partial_p ((a, b) \times (T_0 + T_\delta, T_1)), \quad \text{for} \ \delta \ \text{small,}\]

we derive from Theorem 5.1 that

\[v(s_0, t_0) + \varepsilon \geq u^\delta(s_0, t_0) \quad \text{for} \ \delta \ \text{small.}\]

Passing at the limit for \(\delta \to 0\) we then have

\[v(s_0, t_0) + \varepsilon \geq u(s_0, t_0).\]

This proves the assertion for \(\varepsilon\), \((s_0, t_0)\) have been arbitrarily chosen. \(\square\)

**Proof of Proposition 5.6** Let \((s_0, t_0)\) a differentiability point of \(u\) in \(R\), then

\[u_t(s_0, t_0) + H(s_0, u'(s_0, t_0)) \leq 0,\]

and \(|u_t(s_0, t_0)| \leq M\) by \(8\), consequently

\[H(s_0, u'(s_0, t_0)) \leq M,\]

since by \(7\) \(u_t(s_0, t_0) < 0\). We deduce from the convex character of the Hamiltonian and from [2, Proposition 2.3.16]

\[H(s, p) \leq M \quad \text{for any} \ (s, t) \in R, \ p \in \partial u(t, \cdot) \ \text{at} \ s.\]
We therefore get (9) letting \( t \) go to the boundary of \([T_0, T_1]\) and exploiting the stability properties of viscosity subsolutions.

\[ \square \]

We need introducing some preliminary material before attacking the proof of Proposition 5.7.

**Lemma B.1.** Due to condition (7), every uniformly continuous subsolution of (HJ) in \( R \) is nonincreasing in \( t \).

**Proof:** Let \( u \) be a function as in the statement. We fix \( t_1 > t_2 \in (T_0, T_1) \). We know from Proposition A.1 (ii) that for \( \delta \) sufficiently small, the \( u^\delta \)'s are subsolutions of (HJ) in \((a, b) \times (T, T_1)\) for some \( T < t_2 \). Since the Hamiltonian is convex and \( u^\delta \) is Lipschitz continuous, we have

\[ r + H(s, p) \leq 0 \quad \text{for any } (s, t) \in (0, 1) \times (T, +\infty), \text{ any } (p, r) \in \partial u^\delta(s, t). \]

We deduce from condition (7) and \cite[Proposition 2.3.16]{2}

\[ (63) \quad u^\delta_t(s, t) \leq 0 \quad \text{at any point } (s, t) \text{ where } u \text{ is } t\text{-differentiable}. \]

We now fix \( s \), we derive from (63) and the fact that \( u^\delta(s, \cdot) \) is Lipschitz continuous that

\[ u^\delta(s, t_1) \geq u^\delta(s, t_2) \]

and consequently taking into account that \( u^\delta \) uniformly converges to \( u \) in \( R \)

\[ u(s, t_1) \geq u(s, t_2). \]

This concludes the proof since \( s, t_1, t_2 \) have been arbitrarily chosen \( \square \)

**Lemma B.2.** Let \( \mathcal{A} \) be a family of uniformly continuous functions from \( R \) to \( \mathbb{R} \) locally equibounded, with a common continuity modulus \( \omega \), and closed in the local uniform topology, then

\[ u(s, t) = \sup \{v(s, t) \mid v \in \mathcal{A}\} \]

is uniformly continuous with continuity modulus \( \omega \).

**Proof:** Given \((s, t) \in R\), we consider a sequence \( v_n \) contained in \( \mathcal{A} \) with

\[ v_n(s, t) \to u(s, t). \]

By applying Ascoli theorem, we see that the \( v_n \) locally uniformly converges, up to subsequences, to some \( v \in \mathcal{A} \). We conclude that, given \((s, t) \in R\)

\[ u(s, t) = v(s, t) \quad \text{for some } v \in \mathcal{A}. \]
Given \((s_i, t_i) \in R, i = 1, 2\), we denote by \(v_i\) the functions of \(A\) satisfying the above property for \((s_i, t_i)\). We have

\[
u(s_1, t_1) - u(s_2, t_2) \leq u_1(s_1, t_1) - u_1(s_2, t_2) \leq \omega(|s_1 - s_2| + |t_1 - t_2|)
\]

and

\[
u(s_1, t_1) - u(s_2, t_2) \geq u_2(s_1, t_1) - u_2(s_2, t_2) \geq -\omega(|s_1 - s_2| + |t_1 - t_2|).
\]

This shows the assertion. \(\Box\)

**Proof of Proposition 5.7** Let \(\tilde{u}\) be a function in the set appearing in (10), we denote by \(\omega\) its uniform continuity modulus. We define

\[
\tilde{S} = \{ u \text{ un. cont. subsol of (HJ) in } R \text{ with cont. modulus } \omega + M_0 r, u \leq w_0 \text{ on } \partial_p^- R \},
\]

and

\[
\tilde{v}(s,t) = \sup\{ u(s,t) \mid u \in \tilde{S} \}.
\]

Since all uniformly continuous subsolutions of (HJ) are nonincreasing by Lemma B.1 and the function \(w_0(s, T_0) - M_0(t - T_0)\) belongs to \(\tilde{S}\), \(\tilde{v}\) is not affected if we assume the \(u\)’s of \(\tilde{S}\) to satisfy in addition

\[
w_0(s, T_0) - M_0(t - T_0) \leq u(s,t) \leq w_0(s, T_0) \quad \text{for } (s,t) \in R
\]

Therefore the functions of \(\tilde{S}\) have a common uniformity modulus and are locally equibounded. We derive that \(\tilde{v}\) is uniformly continuous with continuity modulus \(\omega + M_0 r\) by Lemma B.2 and subsolution to (HJ) by basic properties of viscosity solutions theory. We further have \(\tilde{v}(\cdot, T_0) = w_0(\cdot, T_0)\).

We fix a time \(h > 0\) with \(T_0 + h < T_1\), and consider the family of functions \(\overline{S}\) defined as

\[
\{ u(s,t) \text{ un. cont. subsol of (HJ) with cont. modulus } \omega + M_0 r, u \leq w_0 - M_0 h \text{ on } \partial_p^- R \},
\]

it is clear that

\[
(64) \quad \overline{v}(s,t) - M_0 h = \sup\{ u(s,t) \mid u \in \overline{S} \}.
\]

We consider an \(u \in \overline{S}\) coinciding with \(w_0 - M_0 h\) in \([a,b] \times \{ T_0 \}\), and define

\[
\overline{u}(s,t) = \begin{cases} 
  w_0(s, T_0) - M_0(t - T_0) & s \in [a,b], t \in [T_0, T_0 + h) \\
  u(s, t - h) & s \in [a,b], t \in [T_0 + h, T_1) 
\end{cases}
\]
We have
\[
\begin{align*}
\overline{u}(s,t_0) &= w_0(s,t_0) \quad \text{for } s \in [a, b] \\
\overline{u}(a,t) &= w_0(a,t) - M_0(t - T_0) \leq w_0(a,t) \quad \text{for } t \in [T_0, T_0 + h] \\
\overline{u}(a,t) &= u(a, t - h) \leq w_0(a, t - h) - M_0 h \leq w_0(a, t) \quad \text{for } t \in [T_0 + h, T_1),
\end{align*}
\]
in addition \(\overline{u}\) is subsolution of \((HJ)\) by Proposition 5.3, and has continuity modulus \(\omega + M_0 r\). Note that for this last property we have used that it is Lipschitz continuous with Lipschitz constant \(M_0\) in \([a, b] \times [t_0, t_0 + h]\). We conclude that \(\overline{u}\) belongs to \(\tilde{S}\), so that
\[
(65) \quad \tilde{v}(s, t + h) \geq \overline{u}(s, t + h) = u(s, t) \quad \text{for any } s \in [a, b], t \in [T_0, T_1 - h).
\]
Since \(u\) has been arbitrarily taken in \(\overline{S}\), the prescription of the value on \([a, b] \times \{t_0\}\) being not a real restriction, we derive from (64), (65) that
\[
\tilde{v}(s, t + h) \geq \tilde{v}(s, t) - M_0 h \quad \text{in } [a, b] \times [T_0, T_1 - h).
\]
Taking into account that \(\tilde{v}\) is nonincreasing in \(t\), we finally get
\[
|\tilde{v}(s, t + h) - \tilde{v}(s, t)| \leq M_0 h \quad \text{for any } s, t,
\]
which implies, \(h\) being arbitrary, that \(\tilde{v}(s, \cdot)\) is Lipschitz continuous with Lipschitz constant \(M_0\) for any \(s\). Taking into account the coercivity of \(H\) and that \(\tilde{v}\) is subsolution to \((HJ)\), we derive that \((\tilde{v}', \tilde{v}_t)\) is bounded in the viscosity sense in \(R\) so that \(\tilde{v}\) is Lipschitz continuous and the estimates (11), (12) holds true with \(\tilde{v}\) in place of \(v\). We define
\[
S = \{u(s, t) \mid u \text{ Lip. subsol. of } (HJ) \text{ in } R \text{ satisfying (11), (12), } u \leq w_0 \text{ in } \partial^-_p R\}
\]
Since \(\tilde{u}\) has been arbitrarily taken in the family of functions defining \(v\), \(\tilde{u} \leq \tilde{v}\) and \(\tilde{v} \in S\), we have that
\[
v(s, t) = \sup\{u(s, t) \mid u \in S\}.
\]
Arguing as in the first part of the proof, we see that \(v\) is subsolution of \((HJ)\), that it is Lipschitz continuous satisfying (11), (12) and coincides with \(w_0\) on \([a, b] \times \{T_0\}\). Finally, by exploiting its maximality, we show, via Perron–Ishii method, that \(v\) is solution to \((HJ)\). \(\square\)

**Proof of Lemma 5.10.** We denote by \(\omega\) a continuity modulus for \(u\) in \(C\) that can be taken in the form
\[
\omega(r) = \inf_k \{a_k + \ell_k r\} \quad \text{with } a_k, \ell_k \text{ positive, } a_k \to 0, \text{ for } r \geq 0.
\]
We fix $k_0 \in \mathbb{N}$ and set $a = a_{k_0}$, $\ell = \ell_{k_0}$, so that
\[ |u(s_1, t_1) - u(s_2, t_2)| \leq a + \ell (|s_1 - s_2| + |t_1 - t_2|). \]
This implies that there are maximizers/minimizers in the formulae yielding $u^{[n]}$, $u^{[n]}$ for $n > \ell$. Let $(s_1, t_1)$, $(s_2, t_2)$ with $u^{[n]}(s_1, t_1) \geq u^{[n]}(s_2, t_2)$, $n > \ell$, we denote by $(z_1, t_1)$ a maximizer for $u^{[n]}(s_1, t_1)$. We have
\[ u^{[n]}(s_1, t_1) - u^{[n]}(s_2, t_2) \leq n (|s_1 - z_1| + |t_1 - r_1| - |s_2 - z_1| - |t_2 - r_1|) \]
which shows item (i) for $u^{[n]}$, the same argument applies to $u^{[n]}$. Given $(s, t)$, if $(z_n, r_n)$ is a point realizing the maximum for $u^{[n]}(s, t)$, $n > \ell$, then
\[ |s - z_n| + |t - r_n| \leq \frac{1}{n} (u(z_n, r_n) - u(s, t)) \leq \frac{a}{n} + \frac{\ell}{n} (|s - z_n| + |t - r_n|) \]
which implies
\[ |s - z_n| + |t - r_n| = O(1/n). \]
We then have
\[ u^{[n]}(s, t) - u(s, t) \leq u(z_n, r_n) - u(s, t) \leq \omega(O(1/n)) \]
and consequently $u^{[n]}$ uniformly converge to $u$ in $\overline{Q}$. The same property holds for $u^{[n]}$. We see, in addition, that $|u^{[n]}-u|_\infty (|u^{[n]}-u|_\infty)$ solely depends on the continuity modulus of $u$. \qed

**Proof of Lemma 5.14.** We denote by $u$, $v$ two Lipschitz continuous subsolutions and by $w$ their minimum. Let $(s_0, t_0)$ be a differentiability point of $w$ and assume that $w(s_0, t_0) = u(s_0, t_0)$, then
\[ (w_t(s_0, t_0), Dw(s_0, t_0)) \in D^- u(s_0, t_0) \subset \partial u(s_0, t_0), \]
where $D^-$ denotes the viscosity subdifferential, and by the convexity of $H$
\[ w_t(s_0, t_0) + H(s_0, Dw(s_0, t_0)) \leq 0. \]
This shows that $w$ is a.e. subsolution which is equivalent, thanks again to the convexity of $H$, of being a viscosity subsolution. \qed
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