The Proximal Distance Algorithm

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Abstract. The MM principle is a device for creating optimization algorithms satisfying the ascent or descent property. The current survey emphasizes the role of the MM principle in nonlinear programming. For smooth functions, one can construct an adaptive interior point method based on scaled Bregman barriers. This algorithm does not follow the central path. For convex programming subject to nonsmooth constraints, one can combine an exact penalty method with distance majorization to create versatile algorithms that are effective even in discrete optimization. These proximal distance algorithms are highly modular and reduce to set projections and proximal mappings, both very well-understood techniques in optimization. We illustrate the possibilities in linear programming, binary piecewise-linear programming, nonnegative quadratic programming, ℓ₀ regression, matrix completion, and inverse sparse covariance estimation.

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1. Introduction

The MM principle is a device for constructing optimization algorithms [4, 25, 28, 29, 30]. In essence, it replaces the objective function f(x) by a simpler surrogate function g(x | x_n) anchored at the current iterate x_n and majorizing or minorizing f(x). As a byproduct of optimizing g(x | x_n) with respect to x, the objective function f(x) is sent downhill or uphill, depending on whether the purpose is minimization or maximization. The next iterate x_{n+1} is chosen to optimize the surrogate g(x | x_n) subject to any relevant constraints. Majorization combines two conditions: the tangency condition g(x_n | x_n) = f(x_n) and the domination condition g(x | x_n) ≥ f(x) for all x. In minimization these conditions and the definition of x_{n+1} lead to the descent property

\[ f(x_{n+1}) \leq g(x_{n+1} \mid x_n) \leq g(x_n \mid x_n) = f(x_n). \]

Minorization reverses the domination inequality and produces an ascent algorithm. Under appropriate regularity conditions, an MM algorithm is guaranteed to converge to a stationary point of the objective function [30]. From the perspective of dynamical systems, the objective function serves as a Liapunov function for the algorithm map.

The MM principle simplifies optimization by: (a) separating the variables of a problem, (b) avoiding large matrix inversions, (c) linearizing a problem, (d)
restoring symmetry, (e) dealing with equality and inequality constraints gracefully, and (f) turning a nondifferentiable problem into a smooth problem. Choosing a tractable surrogate function \( g(x | x_n) \) that hugs the objective function \( f(x) \) as tightly as possible requires experience and skill with inequalities. The majorization relation between functions is closed under the formation of sums, nonnegative products, limits, and composition with an increasing function. Hence, it is possible to work piecemeal in majorizing complicated objective functions.

It is impossible to do justice to the complex history of the MM principle in a paragraph. The celebrated EM (expectation-maximization) principle of computational statistics is a special case of the MM principle \[33\]. Specific MM and EM algorithms appeared years before the principle was well understood \[22, 32, 38, 40, 41\]. The widely applied projected gradient and proximal gradient algorithms can be motivated from the MM perspective, but the early emphasis on operators and fixed points obscured this distinction. Although Dempster, Laird, and Rubin \[15\] formally named the EM algorithm, many of their contributions were anticipated by Baum \[1\] and Sundberg \[39\]. The MM principle was clearly stated by Ortega and Rheinboldt \[36\]. de Leeuw \[13\] is generally credited with recognizing the importance of the principle in practice. The EM algorithm had an immediate and large impact in computational statistics. The more general MM principle was much slower to take hold. The papers \[14, 23, 26\] by the Dutch school of psychometricians solidified its position. (In this early literature the MM principle is called iterative majorization.) The related Dinklebach \[17\] maneuver in fractional linear programming also highlighted the importance of the descent property in algorithm construction.

Before moving on, let us record some notational conventions. All vectors and matrices appear in boldface. The * superscript indicates a vector or matrix transpose. The Euclidean norm of a vector \( x \) is denoted by \( \|x\| \) and the Frobenius norm of a matrix \( M \) by \( \|M\|_F \). For a smooth real-valued function \( f(x) \), we write its gradient (column vector of partial derivatives) as \( \nabla f(x) \), its first differential (row vector of derivatives) as \( df(x) = \nabla f(x)^* \), and its second differential (Hessian matrix) as \( d^2f(x) \).

2. An Adaptive Barrier Method

In convex programming it simplifies matters notationally to replace a convex inequality constraint \( h_j(x) \leq 0 \) by the concave constraint \( v_j(x) = -h_j(x) \geq 0 \). Barrier methods operate on the relative interior of the feasible region where all \( v_j(x) > 0 \). Adding an appropriate barrier term to the objective function \( f(x) \) keeps an initially inactive constraint \( v_j(x) \) inactive throughout an optimization search. If the barrier function is well designed, it should adapt and permit convergence to a feasible point \( y \) with one or more inequality constraints active.

We now briefly summarize an adaptive barrier method that does not follow the central path \[27\]. Because the logarithm of a concave function is concave, the
The Proximal Distance Algorithm

Bregman majorization

\[-\ln v_j(x) + \ln v_j(x_n) + \frac{1}{v_j(x_n)} dv_j(x_n)(x - x_n) \geq 0\]

acts as a convex barrier for a smooth constraint \(v_j(x) \geq 0\). To make the barrier adaptive, we scale it by the current value \(v_j(x_n)\) of the constraint. These considerations suggest an MM algorithm based on the surrogate function

\[g(x | x_n) = f(x) - \rho \sum_{j=1}^{s} v_j(x_n) \ln v_j(x) + \rho \sum_{j=1}^{s} dv_j(x_n)(x - x_n)\]

for \(s\) inequality constraints. Minimizing the surrogate subject to relevant linear equality constraints \(Ax = b\) produces the next iterate \(x_{n+1}\). The constant \(\rho\) determines the tradeoff between keeping the constraints inactive and minimizing \(f(x)\). One can show that the MM algorithm with exact minimization converges to the constrained minimum of \(f(x)\) [30].

In practice one step of Newton’s method is usually adequate to decrease \(f(x)\). The first step of Newton’s method minimizes the second-order Taylor expansion of \(g(x | x_n)\) around \(x_n\) subject to the equality constraints. Given smooth functions, the two differentials

\[dg(x_n | x_n) = df(x_n)\]
\[d^2g(x_n | x_n) = d^2f(x_n) - \rho \sum_{j=1}^{s} d^2v_j(x_n) + \rho \sum_{j=1}^{s} \frac{1}{v_j(x_n)} \nabla v_j(x_n) dv_j(x_n)\]

are the core ingredients in the quadratic approximation of \(g(x | x_n)\). Unfortunately, one step of Newton’s method is neither guaranteed to decrease \(f(x)\) nor to respect the nonnegativity constraints.

**Example 1. Adaptive Barrier Method for Linear Programming**

For instance, the standard form of linear programming requires minimizing a linear function \(f(x) = c^*x\) subject to \(Ax = b\) and \(x \geq 0\). The quadratic approximation to the surrogate \(g(x | x_n)\) amounts to

\[c^*x_n + c^*(x - x_n) + \frac{\rho}{2} \sum_{j=1}^{p} \frac{1}{x_{nj}} (x_j - x_{nj})^2.\]

The minimum of this quadratic subject to the linear equality constraints occurs at the point

\[x_{n+1} = x_n - D_n^{-1}c + D_n^{-1}A^*(AD_n^{-1}A^*)^{-1}(b - Ax_n + AD_n^{-1}c).\]
Here \( D_n \) is the diagonal matrix with \( i \)th diagonal entry \( \rho x_{ni}^{-1} \), and the increment \( x_{n+1} - x_n \) satisfies the linear equality constraint \( A(x_{n+1} - x_n) = b - Ax_n \).

One can overcome the objections to Newton updates by taking a controlled step along the Newton direction \( u_n = x_{n+1} - x_n \). The key is to exploit the theory of self-concordant functions \([5, 35]\). A thrice differentiable convex function \( h(t) \) is said to be self-concordant if it satisfies the inequality

\[
|h'''(t)| \leq 2ch''(t)^{3/2}
\]

for some constant \( c \geq 0 \) and all \( t \) in the essential domain of \( h(t) \). All convex quadratic functions qualify as self-concordant with \( c = 0 \). The function \( h(t) = -\ln(at + b) \) is self-concordant with constant 1. The class of self-concordant functions is closed under sums and composition with linear functions. A convex function \( k(x) \) with domain \( \mathbb{R}^p \) is said to be self-concordant if every slice \( h(t) = k(x + tu) \) is self-concordant.

Rather than conduct an expensive one-dimensional search along the Newton direction \( x_{n+1} + tu_n \), one can majorize the surrogate function \( h(t) = g(x_n + tu_n | x_n) \) along the half-line \( t \geq 0 \). The clever majorization

\[
h(t) \leq h(0) + h'(0)t - \frac{1}{c}h''(0)^{1/2}t - \frac{1}{c^2} \ln[1 - ch''(0)^{1/2}]
\]

serves the dual purpose of guaranteeing a decrease in \( f(x) \) and preventing a violation of the inequality constraints \([35]\). Here \( c \) is the self-concordance constant associated with the surrogate. The optimal choice of \( t \) reduces to the damped Newton update

\[
t = \frac{h'(0)}{h''(0) - ch''(0)^{1/2}}.
\]

The first two derivatives of \( h(t) \) are clearly

\[
h'(0) = df(x_n)u_n
\]

\[
h''(0) = u_n^* d^2 f(x_n)u_n - \rho \sum_{j=1}^s u_n^* d^2 v_j(x_n)u_n + \rho \sum_{j=1}^s \frac{1}{v_j(x_n)} |d v_j(x_n)u_n|^2.
\]

The first of these derivatives is nonpositive because \( u_n \) is a descent direction for \( f(x) \). The second is generally positive because all of the contributing terms are nonnegative.

When \( f(x) \) is quadratic and the inequality constraints are affine, detailed calculations show that the surrogate function \( g(x | x_n) \) is self-concordant with constant

\[
c = \frac{1}{\sqrt{\rho \min\{v_1(x_n), \ldots, v_s(x_n)\}}}.
\]
The Proximal Distance Algorithm

Table 1. Performance of the adaptive barrier method in linear programming.

| Iteration $n$ | No Safeguard | Self-concordant Safeguard |
|--------------|--------------|----------------------------|
|              | $c^\ast x_n$ | $\|\Delta_n\|$ | $c^\ast x_n$ | $\|\Delta_n\|$ | $t_n$ |
| 1            | -1.20000     | 0.25820            | -1.11270     | 0.14550     | 0.56351 |
| 2            | -1.33333     | 0.17213            | -1.20437     | 0.11835     | 0.55578 |
| 3            | -1.41176     | 0.10125            | -1.37561     | 0.05517     | 0.54345 |
| 4            | -1.45455     | 0.05523            | -1.47289     | 0.01264     | 0.53746 |
| 5            | -1.47692     | 0.02889            | -1.49426     | 0.00271     | 0.53622 |
| 10           | -1.49927     | 0.00003            | -1.49995     | 0.00003     | 0.53597 |
| 15           | -1.50000     | 0.00000            | -1.50000     | 0.00000     | 0.53590 |
| 20           | -1.50000     | 0.00000            | -1.50000     | 0.00000     | 0.53590 |
| 25           | -1.50000     | 0.00000            | -1.50000     | 0.00000     | 0.53590 |
| 30           | -1.50000     | 0.00000            | -1.50000     | 0.00000     | 0.53590 |
| 35           | -1.50000     | 0.00000            | -1.50000     | 0.00000     | 0.53590 |

Taking the damped Newton’s step with step length (3) keeps $x_n + t_n u_n$ in the relative interior of the feasible region while decreasing the surrogate and hence the objective function $f(x)$. When $f(x)$ is not quadratic but can be majorized by a quadratic $q(x \mid x_n)$, one can replace $f(x)$ by $q(x \mid x_n)$ in calculating the adaptive-barrier update. The next iterate $x_{n+1}$ retains the descent property.

As a toy example consider the linear programming problem of minimizing $c^\ast x$ subject to $Ax = b$ and $x \geq 0$. Applying the adaptive barrier method to the choices

$$A = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

and to the feasible initial point $x_0 = \frac{1}{3}1$ produces the results displayed in Table 1. Not shown is the minimum point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0)^\ast$. Columns two and three of the table record the progress of the unadorned adaptive barrier method. The quantity $\|\Delta_n\|$ equals the Euclidean norm of the difference vector $\Delta_n = x_n - x_{n-1}$. Columns four and five repeat this information for the algorithm modified by the self-concordant majorization (2). The quantity $t_n$ in column six represents the optimal step length (3) in going from $x_{n-1}$ to $x_n$ along the Newton direction $u_{n-1}$. Clearly, there is a price to be paid in implementing a safeguarded Newton step. In practice, this price is well worth paying.
3. Distance Majorization

On a Euclidean space, the distance to a closed set $S$ is a Lipschitz function $\text{dist}(x, S)$ with Lipschitz constant 1. If $S$ is also convex, then $\text{dist}(x, S)$ is a convex function. Projection onto $S$ is intimately tied to $\text{dist}(x, S)$. Unless $S$ is convex, the projection operator $P_S(x)$ is multi-valued for at least one argument $x$. Fortunately, it is possible to majorize $\text{dist}(x, S)$ at $x_n$ by $\|x - P_S(x_n)\|$. This simple observation is the key to the proximal distance algorithm to be discussed later. In the meantime, let us show how to derive two feasibility algorithms by distance majorization [9]. Let $S_1, \ldots, S_m$ be closed sets. The method of averaged projections attempts to find a point in their intersection $S = \cap_{j=1}^m S_j$. To derive the algorithm, consider the convex combination 

$$f(x) = \sum_{j=1}^m \alpha_j \text{dist}(x, S_j)^2$$

of squared distance functions. Obviously, $f(x)$ vanishes precisely on $S$ when all $\alpha_j > 0$. The majorization 

$$g(x \mid x_n) = \sum_{j=1}^m \alpha_j \|x - P_{S_j}(x_n)\|^2$$

of $f(x)$ is easily minimized. The minimum point of $g(x \mid x_n)$, 

$$x_{n+1} = \sum_{j=1}^m \alpha_j P_{S_j}(x_n),$$

defines the averaged operator. The MM principle guarantees that $x_{n+1}$ decreases the objective function.

Von Neumann’s method of alternating projections can also be derived from this perspective. For two sets $S_1$ and $S_2$, consider the problem of minimizing the objective function $f(x) = \text{dist}(x, S_2)^2$ subject to the constraint $x \in S_1$. The function 

$$g(x \mid x_n) = \|x - P_{S_2}(x_n)\|^2$$

majorizes $f(x)$. Indeed, the domination condition $g(x \mid x_n) \geq f(x)$ holds because $P_{S_2}(x_n)$ belongs to $S_2$; the tangency condition $g(x_n \mid x_n) = f(x_n)$ holds because $P_{S_2}(x_n)$ is the closest point in $S_2$ to $x_n$. The surrogate function $g(x \mid x_n)$ is minimized subject to the constraint by taking $x_{n+1} = P_{S_1} \circ P_{S_2}(x_n)$. The MM principle again ensures that $x_{n+1}$ decreases the objective function. When the two sets intersect, the least distance of 0 is achieved at any point in the intersection. One can extend this derivation to three sets by minimizing the objective function $f(x) = \text{dist}(x, S_2)^2 + \text{dist}(x, S_3)^2$ subject to $x \in S_1$. The surrogate 

$$g(x \mid x_n) = \|x - P_{S_2}(x_n)\|^2 + \|x - P_{S_3}(x_n)\|^2$$

$$= 2\|x - \frac{1}{2}[P_{S_2}(x_n) + P_{S_3}(x_n)]\|^2 + c_n$$
relying on an irrelevant constant $c_n$. The closest point in $S_1$ is

$$x_{n+1} = P_{S_1} \left\{ \frac{1}{2} [P_{S_2}(x_n) + P_{S_3}(x_n)] \right\}.$$ 

This construction clearly generalizes to more than three sets.

**4. The Proximal Distance Method**

We now turn to an exact penalty method that applies to nonsmooth functions. Clarke’s exact penalty method [10] turns the constrained problem of minimizing a function $f(y)$ over a closed set $S$ into the unconstrained problem of minimizing the function $f(y) + \rho \text{dist}(y, S)$ for $\rho$ sufficiently large. Here is a precise statement of a generalization of Clarke’s result [6, 10, 11].

**Proposition 1.** Suppose $f(y)$ achieves a local minimum on $S$ at the point $x$. Let $\phi_S(y)$ denote a function that vanishes on $S$ and satisfies $\phi_S(y) \geq c \text{dist}(y, S)$ for all $x$ and some positive constant $c$. If $f(y)$ is locally Lipschitz around $x$ with constant $L$, then for every $\rho \geq c^{-1}L$, $F_\rho(y) = f(y) + \rho \phi_S(y)$ achieves a local unconstrained minimum at $x$.

Classically the choice $\phi_S(x) = \text{dist}(x, S)$ was preferred. For affine equality constraints $g_i(x) = 0$ and affine inequality constraints $h_j(x) \leq 0$, Hoffman’s bound

$$\text{dist}(y, S) \leq \tau \left\| \begin{array}{c} G(y) \\ H(y) \end{array} \right\|$$

applies, where $\tau$ is some positive constant, $S$ is the feasible set where $G(y) = 0$, and $H(y)_+ \leq 0$ [24]. The vector $H(y)_+$ has components $h_j(x)_+ = \max\{h_j(y), 0\}$. When $S$ is the intersection of several closed sets $S_1, \ldots, S_m$, the alternative

$$\phi_S(y) = \sqrt{\sum_{i=1}^m \text{dist}(y, S_i)^2}$$

(4)

is attractive. The next proposition gives sufficient conditions under which the crucial bound $\phi_S(y) \geq c \text{dist}(y, S)$ is valid for the function [11].

**Proposition 2.** Suppose $S_1, \ldots, S_m$ are closed convex sets in $\mathbb{R}^p$ with the first $j$ sets polyhedral. Assume further that the intersection

$$S = (\cap_{i=1}^j S_i) \cap (\cap_{i=j+1}^m \text{ri } S_i)$$

is nonempty and bounded. Then there exists a constant $\tau > 0$ such that

$$\text{dist}(x, S) \leq \tau \sum_{i=1}^m \text{dist}(x, S_i) \leq \tau \sqrt{m} \sqrt{\sum_{i=1}^m \text{dist}(x, S_i)^2}$$

for all $x$. The sets $S_1, \ldots, S_m$ are said to be linearly regular.
Proof. See the references [2, 10] for all details. A polyhedral set is the nonempty intersection of a finite number of half-spaces. The operator $\text{ri } K$ forms the relative interior of the convex set $K$, namely, the interior of $K$ relative to the affine hull of $K$. When $K$ is nonempty, its relative interior is nonempty and generates the same affine hull as $K$ itself.

In general, we will require $f(x)$ and $\phi_S(x)$ to be continuous functions and the sum $F_\rho(y) = f(y) + \rho \phi_S(y)$ to be coercive for some value $\rho = \rho_0$. It then follows that $F_\rho(y)$ is coercive and attains its minimum for all $\rho \geq \rho_0$. One can prove a partial converse to Clarke’s theorem [11, 12]. This requires the enlarged set $S_\epsilon = \{x : \phi_S(x) < \epsilon\}$ of points lying close to $S$ as measured by $\phi_S(x)$.

**Proposition 3.** Suppose that $f(y)$ is Lipschitz on $S$, for some $\epsilon > 0$. Then under the stated assumptions on $f(x)$ and $\phi_S(x)$, a global minimizer of $F_\rho(y)$ is a constrained minimizer of $f(y)$ for all sufficiently large $\rho$.

When the constraint set $S$ is compact and $f(y)$ has a continuously varying local Lipschitz constant, the hypotheses of Proposition 3 are fulfilled. This is the case, for instance, when $f(y)$ is continuously differentiable. With this background on the exact penalty method in mind, we now sketch an approximate MM algorithm for convex programming that is motivated by distance majorization. This algorithm is designed to exploit set projections and proximal maps. The proximal map $\text{prox}_h(y)$ associated with a convex function $h(x)$ satisfies

$$\text{prox}_h(y) = \arg\min_x \left[ h(x) + \frac{1}{2} \| y - x \|^2 \right].$$

A huge literature and software base exist for computing projections and proximal maps [3].

Since the function $\text{dist}(x, S)$ is merely continuous, we advocate approximating it by the differentiable function

$$\text{dist}_\epsilon(x, S) = \sqrt{\text{dist}(x, S)^2 + \epsilon}$$

for $\epsilon > 0$ small. The composite function $\text{dist}_\epsilon(x, S)$ is convex when $S$ is convex because the function $\sqrt{t^2 + \epsilon}$ is increasing and convex on $[0, \infty)$. Instead of minimizing $f(x) + \rho \text{dist}(x, S)$, we suggest minimizing the differentiable convex function $f(x) + \rho \text{dist}_\epsilon(x, S)$ by an MM algorithm. Regardless of whether $S$ is convex, the majorization

$$\text{dist}_\epsilon(x, S) \leq \sqrt{\| x - P_S(x_n) \|^2 + \epsilon} \quad (5)$$

holds. If $S$ is nonconvex, there may be a multiplicity of closest points, and one must choose a representative of the set $P_S(x_n)$. In any event one can invoke the univariate majorization

$$\sqrt{t} \geq \sqrt{t_n} + \frac{t - t_n}{2\sqrt{t_n}} \quad (6)$$
of the concave function $\sqrt{t}$ on the interval $t > 0$ and majorize the majorization \[ (5) \]
by
\[
\sqrt{\|x - PS(x_n)\|^2 + \epsilon} \leq \frac{1}{2\sqrt{\|x_n - PS(x_n)\|^2 + \epsilon}} \|x - PS(x_n)\|^2 + c_n
\]
for some irrelevant constant $c_n$. The second step of our proposed MM algorithm consists of minimizing the surrogate function
\[
g(x | x_n) = f(x) + \frac{w_n}{2} \|x - PS(x_n)\|^2
\]
\[
w_n = \frac{\rho}{\sqrt{\|x_n - PS(x_n)\|^2 + \epsilon}}
\]
The corresponding proximal map drives $f(x) + \rho \text{dist}_c(x, S)$ downhill. Under the more general exact penalty \[ (1) \], the surrogate function depends on a sum of spherical quadratics rather than a single spherical quadratic.

It is possible to project onto a variety of closed nonconvex sets. For example, if $S$ is the set of integers, then projection amounts to rounding. An ambiguous point $n + \frac{1}{2}$ can be projected to either $n$ or $n + 1$. Projection onto a finite set simply tests each point separately. Projection onto a Cartesian product is achieved via the Cartesian product of the projections. One can also project onto many continuous sets of interest. For example, to project onto the closed set of points having at most $k$ nonzero coordinates, one zeros out all but the $k$ largest coordinates in absolute value. Projection onto the sphere of center $z$ and radius $r$ takes $y \neq z$ into the point $z + \frac{r}{\|y - z\|} (y - z)$. All points of the sphere are equidistant from its center.

By definition the update $x_{n+1} = \text{prox}_{w_n^{-1} f/PS(x_n)}$ minimizes $g(x | x_n)$. We will refer to this MM algorithm as the \textbf{proximal distance algorithm}. It enjoys several virtues. First, it allows one to exploit the extensive body of results on proximal maps and projections. Second, it does not demand that the constraint set $S$ be convex. Third, it does not require the objective function $f(x)$ to be convex or smooth. Finally, the minimum values and minimum points of the functions $f(x) + \rho \text{dist}_c(x, S)$ and $f(x) + \rho \text{dist}_c(x, S)$ are close when $\epsilon > 0$ is small.

In implementing the proximal distance algorithm, the constants $L$ and $\epsilon$ must be specified. For many norms the Lipschitz constant $L$ is known. For a differentiable function $f(x)$, the mean value inequality suggests taking $L$ equal to the maximal value of $\|\nabla f(x)\|$ in a neighborhood of the optimal point. In specific problems a priori bounds can be derived. If no such prior bound is known, then one has to guess an appropriate $\rho$ and see if it leads to a constrained minimum. If not, $\rho$ should be systematically increased until a constrained minimum is reached. Even with a justifiable bound, it is prudent to start $\rho$ well below its intended upper bound to emphasize minimization of the loss function in early iterations. Experience shows that gradually decreasing $\epsilon$ is also a good tactic; otherwise, one again runs the risk of putting too much early stress on satisfying the constraints. In practice the sequences $\rho_n = \min\{\alpha^n \rho_0, \rho_{\text{max}}\}$ and $\epsilon_n = \max\{\beta^{-n} \epsilon_0, \epsilon_{\text{min}}\}$ work well for $\alpha$ and $\beta$ slightly larger than 1, say 1.2, and $\rho_0 = \epsilon_0 = 1$. On many problems more aggressive choices of $\alpha$ and $\beta$ are possible. The values of $\rho_{\text{max}}$ and $\epsilon_{\text{min}}$ are
problem specific, but taking $\rho_{\text{max}}$ substantially greater than a known Lipschitz constant slows convergence. Taking $\epsilon_{\text{min}}$ too large leads to a poor approximate solution.

5. Sample Problems

We now explore some typical applications of the proximal distance algorithm. In all cases we are able to establish local Lipschitz constants. Comparisons with standard optimization software serve as performance benchmarks.

**Example 2. Projection onto an Intersection of Closed Convex Sets**

Let $S_1, \ldots, S_k$ be closed convex sets, and assume that projection onto each $S_j$ is straightforward. Dykstra’s algorithm \[16, 18\] is designed to find the projection of an external point $y$ onto $S = \cap_{j=1}^k S_j$. The proximal distance algorithm provides an alternative based on the convex function

$$f(x) = \sqrt{\|x - y\|^2 + \delta}$$

for $\delta$ positive, say $\delta = 1$. The choice $f(x)$ is preferable to the obvious choice $\|x - y\|^2$ because $f(x)$ is Lipschitz with Lipschitz constant 1. In the proximal distance algorithm, we take

$$\phi_S(x) = \sqrt{\sum_{j=1}^k \text{dist}(x, S_j)^2}$$

and minimize the surrogate function

$$g(x | x_n) = f(x) + \frac{w_n}{2} \sum_{j=1}^k \|x - p_{nj}\|^2 = f(x) + \frac{kw_n}{2} \|x - \bar{p}_n\|^2 + c_n,$$

where $p_{nj}$ is the projection of $x_n$ onto $S_j$, $\bar{p}_n$ is the average of the projections $p_{nj}$, $c_n$ is an irrelevant constant, and

$$w_n = \frac{\rho}{\sqrt{\sum_{j=1}^k \|x_n - p_{nj}\|^2 + \epsilon}}.$$

After rearrangement, the stationarity condition for optimality reads

$$x = (1 - \alpha)y + \alpha \bar{p}_n, \quad \alpha = \frac{kw_n}{\sqrt{\|x - y\|^2 + \delta} + kw_n}.$$

In other words, $x_{n+1}$ is a convex combination of $y$ and $\bar{p}_n$.

To calculate the optimal coefficient $\alpha$, we minimize the convex surrogate

$$h(\alpha) = g(1 - \alpha)y + \alpha \bar{p}_n | x_n = \sqrt{\alpha^2d^2 + \delta} + \frac{kw_n}{2}(1 - \alpha)^2d^2 + c_n.$$
Dykstra Proximal Distance Iteration

| Iteration n | Dykstra            | Proximal Distance |
|-------------|--------------------|-------------------|
|             | $x_{n1}$           | $x_{n2}$          |
| 0           | -1.00000           | -1.00000          |
| 1           | -0.44721           | 0.89443           |
| 2           | 0.00000            | 0.89443           |
| 3           | -0.26640           | 0.96386           |
| 4           | 0.00000            | 0.96386           |
| 5           | -0.14175           | 0.98990           |
| 10          | 0.00000            | 0.99934           |
| 15          | -0.00454           | 0.99999           |
| 20          | 0.00000            | 1.00000           |
| 25          | -0.00014           | 1.00000           |
| 30          | 0.00000            | 1.00000           |
| 35          | 0.00000            | 1.00000           |

Table 2. Dykstra’s algorithm versus the proximal distance algorithm.

for $d = \|y - \bar{p}_n\|$. Its derivative

$$h'(\alpha) = \frac{\alpha d^2}{\sqrt{\alpha^2 d^2 + \delta}} - kw_n(1 - \alpha)d^2$$

satisfies $h'(0) < 0$ and $h'(1) > 0$ and possesses a unique root on the open interval $(0, 1)$. This root can be easily computed by bisection or Newton’s method.

Table 2 compares Dykstra’s algorithm and the proximal distance algorithm on a simple planar example. Here $S_1$ is the closed unit ball in $\mathbb{R}^2$, and $S_2$ is the closed halfspace with $x_1 \geq 0$. The intersection $S$ reduces to the right half ball centered at the origin. The table records the iterates of the two algorithms from the starting point $x_0 = (-1, 2)^\ast$ until their eventual convergence to the geometrically obvious solution $(0, 1)^\ast$. In the proximal distance method we set $\rho_n = 2$ and aggressively $\epsilon_n = 4^{-n}$. The two algorithms exhibit similar performance but take rather different trajectories.

Example 3. Binary Piecewise-Linear Functions

The problem of minimizing the binary piecewise-linear function

$$f(x) = \sum_{i<j} w_{ij} |x_i - x_j| + b^* x$$

subject to $x \in \{0, 1\}^d$ and nonnegative weights $w_{ij}$ is a typical discrete optimization problem with applications in graph cuts. If we invoke the majorization

$$|x_i - x_j| \leq \left| x_i - \frac{x_{ni} + x_{nj}}{2} \right| + \left| x_j - \frac{x_{ni} + x_{nj}}{2} \right|$$
prior to applying the proximal operator, then the proximal distance algorithm separates the parameters. Parameter separation promotes parallelization and benefits from a fast algorithm for computing proximal maps in one dimension. The one-dimensional algorithm is similar to but faster than bisection \cite{37}. Finally, the objective function is Lipschitz with the explicit constant

\[
L = \sum_i \sqrt{\sum_{j \neq i} w_{ij}^2} + \|b\|.
\]  

This assertion follows from the simple bound

\[
|f(x) - f(y)| \leq \sum_i \sum_{j \neq i} w_{ij} |x_j - y_j| + |b^*(x - y)|
\]

\[
\leq \sum_i \sqrt{\sum_{j \neq i} w_{ij}^2} \cdot \|x - y\| + \|b\| \cdot \|x - y\|
\]

under the symmetry convention \(w_{ij} = w_{ji}\).

| CPU times |
|-----------|
| Dimension MM CVX Iterations |
| 2 0.038 0.080 9 |
| 4 0.052 0.060 18 |
| 8 2.007 0.050 200 |
| 16 2.416 0.100 200 |
| 32 2.251 0.130 200 |
| 64 4.134 0.400 200 |
| 128 0.212 2.980 32 |
| 256 0.868 62.63 200 |
| 512 68.27 1534 200 |
| 1024 526.6 * 200 |
| 2048 127.2 * 200 |
| 4096 547.4 * 200 |

Table 3. CPU times in seconds and MM iterations until convergence for binary piecewise linear functions. Asterisks denote computer runs exceeding computer memory limits. Iterations were capped at 200.

Table 3 displays the numerical results for a few typical examples. For each dimension \(d\) we filled \(b\) with standard normal deviates and the upper triangle of the weight matrix \(W\) with the absolute values of such deviates. The lower triangle of \(W\) was determined by symmetry. Small values of \(b\) often lead to degenerate solutions \(x\) with all entries 0 or 1. To avoid this possibility, we multiplied each entry of \(b\) by \(d\). In the graph cut context, a degenerate solution corresponds to no cuts at all or a completely cut graph. These examples depend on the schedules
\[ \rho_n = \min\{1.2^n, L\} \text{ and } \epsilon_n = \max\{1.2^{-n}, 10^{-15}\} \text{ for the two tuning constants and the local Lipschitz constant } (\theta) \]

Although the MM proximal distance algorithm makes good progress towards the minimum in the first 100 iterations, it sometimes hovers around its limit without fully converging. This translates into fickle compute times, and for this reason we capped the number of MM iterations at 200. For small dimensions MM can be much slower than CVX. Fortunately, the performance of the MM algorithm improves markedly as \( d \) increases. In all runs the two algorithms reach the same solution after rounding components to the nearest integer. MM also requires much less storage than CVX. Asterisks appear in the table where CVX demanded more memory than our laptop computer could deliver.

**Example 4. Nonnegative Quadratic Programming**

The proximal distance algorithm is applicable in minimizing a convex quadratic \( f(x) = \frac{1}{2}x^T Ax + b^T x \) subject to the constraint \( x \geq 0 \). In this nonnegative quadratic programming program, let \( y_n \) be the projection of the current iterate \( x_n \) onto \( S = \mathbb{R}_d^+ \). If we define the weight

\[ w_n = \frac{\rho}{\sqrt{\|x_n - y_n\|^2 + \epsilon}}, \]

then the next iterate can be expressed as

\[ x_{n+1} = (A + w_n I)^{-1}(w_n y_n - b). \]

The multiple matrix inversions implied by the update can be avoided by extracting and caching the spectral decomposition \( U^* DU \) of \( A \) at the start of the algorithm. The inverse \( (A + w_n I)^{-1} \) then reduces to \( U^* (D + w_n I)^{-1} U \). The diagonal matrix \( D + w_n I \) is obviously trivial to invert. The remaining operations in computing \( x_{n+1} \) collapse to matrix times vector multiplications. Nonnegative least squares is a special case of nonnegative quadratic programming.

One can estimate an approximate Lipschitz constant for this problem. Note that \( f(0) = 0 \) and that

\[ f(x) \geq \frac{1}{2} \lambda_{\min} \|x\|^2 - \|b\| \cdot \|x\|, \]

where \( \lambda_{\min} \) is the smallest eigenvalue of \( A \). It follows that any point \( x \) with \( \|x\| > \frac{2}{\lambda_{\min}} \|b\| \) cannot minimize \( f(x) \) subject to the nonnegativity constraint. On the other hand, the gradient of \( f(x) \) satisfies

\[ \|\nabla f(x)\| \leq \|A\| \|x\| + \|b\| \leq \lambda_{\max} \|x\| + \|b\|. \]

In view of the mean-value inequality, these bounds suggest that

\[ L = \left( \frac{2\lambda_{\max}}{\lambda_{\min}} + 1 \right) \|b\| = [2 \text{ cond}_2(A) + 1]\|b\|. \]
provides an approximate Lipschitz constant for $f(x)$ on the region harboring the minimum point. This bound on $\rho$ is usually too large. One remedy is to multiply the bound by a deflation factor such as 0.1. Another remedy is to replace the covariance $A$ by the corresponding correlation matrix. Thus, one solves the problem for the preconditioned matrix $D^{-1}AD^{-1}$, where $D$ is the diagonal matrix whose entries are the square roots of the corresponding diagonal entries of $A$. The transformed parameters $y = Dx$ obey the same nonnegativity constraints as $x$.

For testing purposes we filled a $d \times d$ matrix $M$ with independent standard normal deviates and set $A = M^*M + I$. Addition of the identity matrix avoids ill conditioning. We also filled the vector $b$ with independent standard normal deviates. Our gentle tuning constant schedule $\epsilon_n = \max\{1.005^{-n}, 10^{-15}\}$ and $\rho_n = \min\{1.005^n, 0.1 \times L\}$ adjusts $\rho$ and $\epsilon$ so slowly that their limits are not actually met in practice. In any event $L$ is the a priori bound for the correlation matrix derived from $A$. Table 4 compares the performance of the MM proximal distance algorithm to MATLAB’s `quadprog`, CVX with the SDPT3 solver, and YALMIP with the MOSEK solver. MATLAB’s `quadprog` is clearly the fastest of the four tested methods on these problems. The relative speed of the MM algorithm improves as the problem dimension $d$ increases.

Table 4. CPU times in seconds and optima for the nonnegative quadratic program. Abbreviations: $d$ stands for problem dimension, MM for the proximal distance algorithm, CV for CVX, MA for MATLAB’s `quadprog`, and YA for YALMIP.

| $d$ | MM | CV | MA | YA | MM | CV | MA | YA |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 8   | 0.97 | 0.23 | 0.01 | 0.13 | -0.0172 | -0.0172 | -0.0172 | -0.0172 |
| 16  | 0.50 | 0.24 | 0.01 | 0.11 | -1.1295 | -1.1295 | -1.1295 | -1.1295 |
| 32  | 0.50 | 0.24 | 0.01 | 0.14 | -1.3811 | -1.3811 | -1.3811 | -1.3811 |
| 64  | 0.57 | 0.28 | 0.01 | 0.13 | -0.5641 | -0.5641 | -0.5641 | -0.5641 |
| 128 | 0.79 | 0.36 | 0.02 | 0.14 | -0.7018 | -0.7018 | -0.7018 | -0.7018 |
| 256 | 1.66 | 0.65 | 0.06 | 0.22 | -0.6890 | -0.6890 | -0.6890 | -0.6890 |
| 512 | 5.61 | 2.95 | 0.26 | 0.73 | -0.5971 | -0.5968 | -0.5970 | -0.5970 |
| 1024| 32.69| 21.90| 1.32 | 2.91 | -0.4944 | -0.4940 | -0.4944 | -0.4944 |
| 2048| 156.7| 178.8| 8.96 | 15.89| -0.4514 | -0.4505 | -0.4512 | -0.4512 |
| 4096| 695.1| 1551| 57.73| 91.54| -0.4690 | -0.4678 | -0.4686 | -0.4686 |
Table 5. Numerical experiments comparing MM to MATLAB’s lasso. Each row presents averages over 100 independent simulations. Abbreviations: \( m \) the number of cases, \( n \) the number of predictors, \( df \) the number of actual predictors in the generating model, \( tp_1 \) the number of true predictors selected by MM, \( tp_2 \) the number of true predictors selected by the lasso, \( \lambda \) the regularization parameter at the lasso optimal loss, \( L_1 \) the optimal loss from MM, \( L_1/L_2 \) the ratio of \( L_1 \) to the optimal lasso loss, \( T_1 \) the total computation time in seconds for MM, and \( T_1/T_2 \) the ratio of \( T_1 \) to the total computation time of the lasso.

|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
|    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |    |
| 256 | 128 | 10 | 5.97 | 3.32 | 0.143 | 248.763 | 0.808 | 0.603 | 8.098 |
| 128 | 256 | 10 | 3.83 | 1.91 | 0.214 | 106.234 | 0.744 | 0.999 | 10.254 |
| 512 | 256 | 10 | 6.51 | 2.88 | 0.119 | 506.570 | 0.900 | 0.907 | 6.262 |
| 256 | 512 | 10 | 4.50 | 1.82 | 0.172 | 241.678 | 0.835 | 1.743 | 8.687 |
| 1024 | 512 | 10 | 7.80 | 5.25 | 0.101 | 1029.333 | 0.921 | 2.597 | 5.057 |
| 512 | 1024 | 10 | 5.54 | 2.58 | 0.138 | 507.451 | 0.881 | 8.235 | 13.532 |
| 2048 | 1024 | 10 | 8.98 | 4.89 | 0.080 | 2047.098 | 0.945 | 15.460 | 8.858 |
| 1024 | 2048 | 10 | 6.80 | 2.93 | 0.110 | 1044.640 | 0.916 | 34.997 | 18.433 |
| 4096 | 2048 | 10 | 9.75 | 9.90 | 0.060 | 4086.886 | 0.966 | 89.684 | 10.956 |
| 2048 | 4096 | 10 | 8.36 | 6.60 | 0.086 | 2045.645 | 0.942 | 166.386 | 25.821 |

Example 5. Linear Regression under an \( \ell_0 \) Constraint

In this example the objective function is the sum of squares \( \frac{1}{2} \| y - X\beta \|^2 \), where \( y \) is the response vector, \( X \) is the design matrix, and \( \beta \) is the vector of regression coefficients. The constraint set \( S_d^k \) consists of those \( \beta \) with at most \( k \) nonzero entries. Projection onto the closed but nonconvex set \( S_d^k \) is achieved by zeroing out all but the \( k \) largest coordinates in absolute value. These coordinates will be unique except in the rare circumstances of ties. The proximal distance algorithm for this problem coincides with that of the previous problem if we substitute \( X^*X \) for \( A \), \(-X^*y\) for \( b \), \( \beta \) for \( x \), and the projection operator \( P_{S_d^k} \) for \( P_{R^k} \). Better accuracy can be maintained if the MM update exploits the singular value decomposition of \( X \) in forming the spectral decomposition of \( X^*X \). Although the proximal distance algorithm carries no absolute guarantee of finding the optimal set of \( k \) regression coefficients, it is far more efficient than sifting through all \( \binom{d}{k} \) sets of size \( k \). The alternative of lasso-guided model selection must contend with strong shrinkage and a surplus of false positives.

Table 5 compares the MM proximal distance algorithm to MATLAB’s lasso function. In simulating data, we filled \( X \) with standard normal deviates, set all components of \( \beta \) to 0 except for \( \beta_i = 1/i \) for \( 1 \leq i \leq 10 \), and added a vector of standard normal deviates to \( X\beta \) to determine \( y \). For a given choice of \( m \) and \( n \) we ran each experiment 100 times and averaged the results. The table demonstrates the superior speed of the lasso and the superior accuracy of the MM algorithm as measured by optimal loss and model selection.
Example 6. Matrix Completion

Let $Y = (y_{ij})$ denote a partially observed $p \times q$ matrix and $\Delta$ the set of index pairs $(i, j)$ with $y_{ij}$ observed. Matrix completion [8] imputes the missing entries by approximating $Y$ with a low rank matrix $X$. Imputation relies on the singular value decomposition

$$X = \sum_{i=1}^{r} \sigma_i u_i v_i^T,$$

where $r$ is the rank of $X$, the nonnegative singular values $\sigma_i$ are presented in decreasing order, the left singular vectors $u_i$ are orthonormal, and the right singular vectors $v_i$ are also orthonormal [20]. The set $R_k$ of $p \times q$ matrices of rank $k$ or less is closed. Projection onto $R_k$ is accomplished by truncating the sum (8) to

$$P_{R_k}(X) = \sum_{i=1}^{\min(r,k)} \sigma_i u_i v_i^T.$$ When $r > k$ and $\sigma_{k+1} = \sigma_k$, the projection operator is multi-valued.

The MM principle allows one to restore the symmetry lost in the missing entries [34]. Suppose $X_n$ is the current approximation to $X$. One simply replaces a missing entry $y_{ij}$ of $Y$ for $(i, j) \notin \Delta$ by the corresponding entry $x_{nij}$ of $X_n$ and adds the term $\frac{1}{2} (x_{nij} - x_{ij})^2$ to the least squares criterion

$$f(X) = \frac{1}{2} \sum_{(i,j) \in \Delta} (y_{ij} - x_{ij})^2.$$ Since the added terms majorize 0, they create a legitimate surrogate function. One can rephrase the surrogate by defining the orthogonal complement operator $P_{\Delta}^\perp(Y)$ via the equation $P_{\Delta}^\perp(Y) + P_{\Delta}(Y) = Y$. The matrix $Z_n = P_{\Delta}(Y) + P_{\Delta}^\perp(X_n)$ temporarily completes $Y$ and yields the surrogate function $\frac{1}{2} \|Z_n - X\|^2_F$. In implementing a slightly modified version of the proximal distance algorithm, one must solve for the minimum of the Moreau function

$$\frac{1}{2} \|Z_n - X\|^2_F + \frac{w_n}{2} \|X - P_{R_k}(X_n)\|^2_F.$$ The stationarity condition

$$0 = X - Z_n + w_n [X - P_{R_k}(X_n)]$$ yields the trivial solution

$$X_{n+1} = \frac{1}{1 + w_n} Z_n + \frac{w_n}{1 + w_n} P_{R_k}(X_n).$$ Again this is guaranteed to decrease the objective function

$$F_p(X) = \frac{1}{2} \sum_{(i,j) \in \Delta} (y_{ij} - x_{ij})^2 + \frac{\rho}{2} \text{dist}_\varepsilon(X, R_k).$$
Table 6. Comparison of the MM proximal distance algorithm to SoftImpute. Abbreviations: $p$ is the number of rows, $q$ is the number of columns, $\alpha$ is the ratio of observed entries to total entries, $L_1$ is the optimal loss under MM, $L_2$ is the optimal loss under SoftImpute, $T_1$ is the total computation time (in seconds) for MM, and $T_2$ is the total computation time for SoftImpute.

| $p$ | $q$ | $\alpha$ | rank | $L_1$ | $L_1/L_2$ | $T_1$ | $T_1/T_2$ |
|-----|-----|----------|------|-------|-----------|-------|-----------|
| 200 | 250 | 0.05     | 20   | 1598  | 0.251     | 4.66  | 7         |
| 800 | 1000| 0.20     | 80   | 571949| 0.253     | 131.02| 18.1      |
| 1000| 1250| 0.25     | 100  | 1112604| 0.24     | 222.2 | 15.1      |
| 1200| 1500| 0.15     | 40   | 793126| 0.361     | 161.51| 3.6       |
| 1200| 1500| 0.30     | 120  | 1569105| 0.235    | 367.78| 12.3      |
| 1400| 1750| 0.35     | 140  | 1642661| 0.236    | 561.76| 9         |
| 1800| 2250| 0.45     | 180  | 2955533| 0.171    | 1176.22| 10.1     |
| 2000| 2500| 0.10     | 20   | 822673 | 0.50     | 307.89| 1.9       |
| 2000| 2500| 0.50     | 200  | 1087404| 0.192    | 2342.32| 2         |
| 5000| 5000| 0.05     | 30   | 7647707| 0.664    | 1827.16| 2         |

\[ f(0) = \frac{1}{2} \sum_{(i,j) \in \Delta} y_{ij}^2. \]

In the spirit of Example 4, let us derive a local Lipschitz constant based on the value $f(0) = \frac{1}{2} \sum_{(i,j) \in \Delta} y_{ij}^2$. The inequality

\[ \frac{1}{2} \sum_{(i,j) \in \Delta} y_{ij}^2 < \frac{1}{2} \sum_{(i,j) \in \Delta} (y_{ij} - x_{ij})^2 = \frac{1}{2} \sum_{(i,j) \in \Delta} (y_{ij}^2 - 2y_{ij}x_{ij} + x_{ij}^2) \]

is equivalent to the inequality

\[ 2 \sum_{(i,j) \in \Delta} y_{ij}x_{ij} < \sum_{(i,j) \in \Delta} x_{ij}^2. \]

In view of the Cauchy-Schwarz inequality

\[ \sum_{(i,j) \in \Delta} y_{ij}x_{ij} \leq \sqrt{\sum_{(i,j) \in \Delta} y_{ij}^2} \sqrt{\sum_{(i,j) \in \Delta} x_{ij}^2}, \]

no solution $x$ of the constrained problem can satisfy

\[ \sqrt{\sum_{(i,j) \in \Delta} x_{ij}^2} > 2 \sqrt{\sum_{(i,j) \in \Delta} y_{ij}^2}. \]

When the opposite inequality holds,

\[ \| \nabla f(x) \|_F = \sqrt{\sum_{(i,j) \in \Delta} (x_{ij} - y_{ij})^2} \leq \sqrt{\sum_{(i,j) \in \Delta} x_{ij}^2} + \sqrt{\sum_{(i,j) \in \Delta} y_{ij}^2} \leq 3 \sqrt{\sum_{(i,j) \in \Delta} y_{ij}^2}. \]
Table 6 compares the performance of the MM proximal distance algorithm and a MATLAB implementation of SoftImpute [34]. Although the proximal distance algorithm is noticeably slower, it substantially lowers the optimal loss and improves in relative speed as problem dimensions grow.

Example 7. Sparse Inverse Covariance Estimation

The graphical lasso has applications in estimating sparse inverse covariance matrices [19]. In this context, one minimizes the convex criterion

$$-\ln \det \Theta + \text{tr}(S\Theta) + \rho \|\Theta\|_1,$$

where $\Theta^{-1}$ is a $p \times p$ theoretical covariance matrix, $S$ is a corresponding sample covariance matrix, and the graphical lasso penalty $\|\Theta\|_1$ equals the sum of the absolute values of the off-diagonal entries of $\Theta$. The solution exhibits both sparsity and shrinkage. One can avoid shrinkage by minimizing

$$f(\Theta) = -\ln \det \Theta + \text{tr}(S\Theta)$$

subject to $\Theta$ having at most $2k$ nonzero off-diagonal entries. Let $T^p_k$ be the closed set of $p \times p$ symmetric matrices possessing this property. Projection of a symmetric matrix $M$ onto $T^p_k$ can be achieved by arranging the above-diagonal entries of $M$ in decreasing absolute value and replacing all but the first $k$ of these entries by 0. The below-diagonal entries are treated similarly.

The proximal distance algorithm for minimizing $f(\Theta)$ subject to the set constraints operates through the convex surrogate

$$g(\Theta | \Theta_n) = f(\Theta) + \frac{w_n}{2} \|\Theta - P_{T^p_k}(\Theta_n)\|^2_F.$$

$$w_n = \frac{\rho}{\sqrt{\|\Theta_n - P_{T^p_k}(\Theta_n)\|^2_F + \epsilon}}.$$

A stationary point minimizes the surrogate and satisfies

$$0 = -\Theta^{-1} + w_n \Theta + S - w_n P_{T^p_k}(\Theta_n).$$

If the constant matrix $S - w_n P_{T^p_k}(\Theta_n)$ has spectral decomposition $U_nD_nU_n^*$, then multiplying equation (9) on the left by $U_n^*$ and on the right by $U_n$ gives

$$0 = -U_n^* \Theta^{-1}U_n + w_nU_n^* \Theta U_n + D_n.$$

This suggests that we take $E = U_n^* \Theta U_n$ to be diagonal and require its diagonal entries $e_i$ to satisfy

$$0 = -\frac{1}{e_i} + w_n e_i + d_{ni}.$$
Multiplying this identity by $e_i$ and solving for the positive root of the resulting quadratic yields

$$e_i = -d_{ni} + \sqrt{d_{ni}^2 + 4w_n}$$

Given the solution matrix $E_{n+1}$, we reconstruct $\Theta_{n+1}$ as $U_nE_{n+1}U_n^*$. Finding a local Lipschitz constant is more challenging in this example. Because the identity matrix is feasible, the minimum cannot exceed

$$-\ln\det I + \text{tr}(SI) = \text{tr}(S) = \sum_{i=1}^p \omega_i,$$

where $S$ is assumed positive definite with eigenvalues $\omega_i$ ordered from largest to smallest. If the candidate matrix $\Theta$ is positive definite with ordered eigenvalues $\lambda_i$, then the von Neumann-Fan inequality [6] implies

$$f(\Theta) \geq -\sum_{i=1}^p \ln \lambda_i + \sum_{i=1}^p \lambda_i \omega_{p-i+1} - \sum_{j \neq i} \ln \omega_{p-j+1} + 1.$$  

To show that $f(\Theta) > f(I)$ whenever any $\lambda_i$ falls outside a designated interval, note that the contribution $-\ln \lambda_i + \lambda_i \omega_{p-i+1}$ to the right side of inequality (10) is bounded below by $\ln \omega_{p-i+1} + 1$ when $\lambda_i = \omega_{p-i+1}^{-1}$. Hence, $f(\Theta) > f(I)$ whenever

$$-\ln \lambda_i + \lambda_i \omega_{p-i+1} > \sum_{i=1}^p \omega_i - \sum_{j \neq i} (\ln \omega_{p-j+1} + 1).$$  

Given the strict convexity of the function $-\ln \lambda_i + \lambda_i \omega_{p-i+1}$, equality holds in inequality (11) at exactly two points $\lambda_{\text{min}} > 0$ and $\lambda_{\text{max}} > \lambda_{\text{min}}$. These roots can be readily extracted by bisection or Newton’s method. The strict inequality $f(\Theta) > f(I)$ holds when any $\lambda_i$ falls to the left of $\lambda_{\text{min}}$ or to the right of $\lambda_{\text{max}}$. Within the intersection of the intervals $[\lambda_{\text{max}}, \lambda_{\text{min}}]$, the gradient of $f(\Theta)$ satisfies

$$\|\nabla f(\Theta)\|_F \leq \|\Theta^{-1}\|_F + \|S\|_F \leq \sqrt{\sum_{i=1}^p \lambda_i^{-2} + \|S\|_F} \leq \sqrt{\sum_{i=1}^p \lambda_{\text{min}}^{-2} + \|S\|_F}.$$  

This bound serves as a local Lipschitz constant near the optimal point.

Table 7 compares the performance of the MM algorithm to that of the R glasso package [19]. The sample precision matrix $S^{-1} = LL^* + \delta MM^*$ was generated by filling the diagonal and first three subdiagonals of the banded lower triangular matrix $L$ with standard normal deviates. Filling $M$ with standard normal deviates and choosing $\delta = 0.01$ imposed a small amount of noise obscuring the band nature of $LL^*$. All table statistics represent averages over 10 runs started at $\Theta = S^{-1}$ with $k$ equal to the true number of nonzero entries in $LL^*$. The MM algorithm performs better in minimizing average loss and recovering nonzero entries.
Kenneth Lange and Kevin L. Keys

Table 7. Numerical results for precision matrix estimation. Abbreviations: \( p \) for matrix dimension, \( k_t \) for the number of nonzero entries in the true model, \( k_1 \) for the number of true nonzero entries recovered by the MM algorithm, \( k_2 \) for the number of true nonzero entries recovered by \texttt{glasso}, \( \rho \) the average tuning constant for \texttt{glasso} for a given \( k_t \), \( L_1 \) the average loss from the MM algorithm, \( L_1 - L_2 \) the difference between \( L_1 \) and the average loss from \texttt{glasso}, \( T_1 \) the average compute time in seconds for the MM algorithm, and \( T_1/T_2 \) the ratio of \( T_1 \) to the average compute time for \texttt{glasso}.

| \( p \) | \( k_t \) | \( k_1 \) | \( k_2 \) | \( \rho \) | \( L_1 \) | \( L_1 - L_2 \) | \( T_1 \) | \( T_1/T_2 \) |
|---|---|---|---|---|---|---|---|---|
| 8  | 18 | 14.0 | 14.0 | 0.00186 | −12.35 | 0.01 | 0.022 | 43.458 |
| 16 | 42 | 30.5 | 28.7 | 0.00305 | −25.17 | 0.08 | 0.026 | 43.732 |
| 32 | 90 | 53.5 | 49.9 | 0.00330 | −50.75 | 0.17 | 0.054 | 31.639 |
| 64 | 186 | 97.8 | 89.3 | 0.00445 | −98.72 | 0.53 | 0.234 | 28.542 |
| 128| 378 | 191.6 | 169.9 | 0.00507 | −196.09 | 1.14 | 1.060 | 18.693 |
| 256| 762 | 345.0 | 304.2 | 0.00662 | −369.62 | 2.55 | 4.253 | 9.559 |
| 512| 1530 | 636.4 | 566.8 | 0.00983 | −641.89 | 6.72 | 19.324 | 5.679 |

6. Discussion

The MM principle offers a unique and potent perspective on high-dimensional optimization. The current survey emphasizes proximal distance algorithms and their applications in nonlinear programming. Our construction of this new class of algorithms relies on the exact penalty method of Clarke [10] and majorization of a smooth approximation to the Euclidean distance to the constraint set. Well-studied proximal maps and Euclidean projections constitute the key ingredients of seven realistic examples. These examples illustrate the versatility of the method in handling nonconvex constraints, its improvement as problem dimension increases, and the pitfalls in sending the tuning constants \( \rho \) and \( \epsilon \) too quickly to their limits. Certainly, the proximal distance algorithm is not a panacea for optimization problems. For example, the proximal distance algorithm as formulated exhibits remarkably fickle behavior on linear programming problems. For linear programming, we ensure numerical stability and guard against premature convergence only by great care in parameter tuning and updating. Nonetheless, we are sufficiently encouraged to pursue this research further, particularly in statistical applications where model fitting and selection are compromised by aggressive penalization.

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