On the absence of global weak solutions for a nonlinear time-fractional Schrödinger equation

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ABSTRACT
In this paper, an initial value problem for a nonlinear time-fractional Schrödinger equation with a singular logarithmic potential term is investigated. The considered problem involves the left/forward Hadamard-Caputo fractional derivative with respect to the time variable. Using the test function method with a judicious choice of the test function, we obtain sufficient criteria for the absence of global weak solutions.

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1. Introduction
This paper is concerned with the nonexistence of global weak solutions to the initial value problem for the time-fractional Schrödinger equation

\[
\begin{aligned}
\begin{cases}
\Delta^\alpha u + \Delta u = \lambda \left( \ln \frac{|x|}{a} \right)^\gamma |u|^p, & t > a, \ x \in \mathbb{R}^N, \\
u(a, x) = f(x), & x \in \mathbb{R}^N,
\end{cases}
\end{aligned}
\]

(1)

where \( u = u(t, x) \) is the complex-valued unknown function, \( N \geq 1, a > 0, i \in \mathbb{C} \) is the imaginary unit \((i^2 = -1), 0 < \alpha < 1, \Delta^\alpha = e^{i\alpha \pi/2}, \Delta^\alpha \) is the left/forward Hadamard-Caputo fractional derivative of order \( \alpha \) with respect to the time variable \( t \) (see Section 2), \( \Delta \) is the Laplacian operator with respect to the space variable \( x, \lambda \in \mathbb{C} \setminus \{0\}, \gamma \in \mathbb{R}, p > 1, \) and \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \). Namely, we are interested in obtaining sufficient conditions under which (1) admits no global weak solution, in the sense that will be specified later.

The theory of fractional calculus has received a great attention from several researchers working in various disciplines. Namely, it was shown that many real-world phenomena can be better modeled using fractional operators, see e.g. [1–7], and the references therein. Due to this fact, the study of fractional evolution equations has become increasingly popular. In particular, the study of time-fractional
schrödinger equations in both, theoretical and numerical aspects, has been attracted a great deal of attention, see e.g. [8–16], and the references therein.

For any complex number \( z \in \mathbb{C} \), we denote by \( \Re z \) and \( \Im z \) the real and the imaginary parts of \( z \), respectively.

In [17], Kirane and Nabti considered the initial value problem for the nonlocal in time nonlinear Schrödinger equation

\[
\begin{cases}
    i \frac{\partial u}{\partial t} + \Delta u = \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s,x)|^p \, ds, & t > 0, \, x \in \mathbb{R}^N, \\
    u(0,x) = f(x), & x \in \mathbb{R}^N,
\end{cases}
\]

(2)

where \( 0 < \alpha < 1 \) and \( \Gamma(\cdot) \) is the Gamma function. It was shown that, if

\[
\Re f \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \exists \lambda \int_{\mathbb{R}^N} \Re f(x) \, dx > 0 \quad \text{or} \quad \Im f \in L^1(\mathbb{R}^N, \mathbb{R}), \quad \Re \lambda \int_{\mathbb{R}^N} \Im f(x) \, dx < 0,
\]

and

\[
1 < p \leq 1 + \frac{2(\alpha + 1)}{N - 2\alpha}, \quad N > 2\alpha,
\]

then (2) has no global weak solution.

In [18], Zhang et al. considered the nonlinear time-fractional Schrödinger equation

\[
\begin{cases}
    i^{\alpha} \mathcal{D}_{0,t}^{\alpha} u + \Delta u = \lambda |u|^p, & t > 0, \, x \in \mathbb{R}^N, \\
    u(0,x) = f(x), & x \in \mathbb{R}^N,
\end{cases}
\]

(3)

where \( 0 < \alpha < 1 \) and \( \mathcal{D}_{0,t}^{\alpha} \) is the Caputo fractional derivative of order \( \alpha \) with respect to the time variable \( t \) (see [19] for the definition of Caputo fractional derivative). It was shown that, if \( 1 < p < 1 + \frac{2}{N}, f \in L^1(\mathbb{R}^N, \mathbb{C}) \), and

\[
\Re \lambda \int_{\mathbb{R}^N} F_1(x) \, dx > 0 \quad \text{or} \quad \Im \lambda \int_{\mathbb{R}^N} F_2(x) \, dx > 0,
\]

where

\[
F_1(x) = \cos \left( \frac{\pi \alpha}{2} \right) \Re f(x) - \sin \left( \frac{\pi \alpha}{2} \right) \Im f(x)
\]

and

\[
F_2(x) = \cos \left( \frac{\pi \alpha}{2} \right) \Im f(x) + \sin \left( \frac{\pi \alpha}{2} \right) \Re f(x),
\]

then (3) admits no global weak solution. Let us mention that \( 1 + \frac{2}{N} \) is the Fujita critical exponent for the semilinear heat equation \( \frac{\partial u}{\partial t} - \Delta u = |u|^p, t > 0, x \in \mathbb{R}^N \) (see Fujita [20]).

Let us mention that in the limit case \( \alpha \to 1^- \), problem (3) reduces to the nonlinear time Schrödinger equation (see e.g. [19])

\[
\begin{cases}
    i \frac{\partial u}{\partial t} + \Delta u = \lambda |u|^p, & t > 0, \, x \in \mathbb{R}^N, \\
    u(0,x) = f(x), & x \in \mathbb{R}^N.
\end{cases}
\]

(4)

In [21], Ikeda and Wakasugi considered the global behavior of solutions to problem (4), then they established a finite-time blow-up result of an \( L^2 \)-solution, whenever \( p \in (1, 1 + 2/N) \). Later, the same problem was discussed by Ikeda and Inui [22], where they established a small data blow-up result of \( H^1 \)-solution, whenever \( p \in (1, 1 + 4/N) \).
In the above mentioned papers, the time fractional derivative was considered in the Caputo sense. In this paper, we investigate the nonlinear time-fractional Schrödinger Equation (1), which involves the left/forward Hadamard-Caputo fractional derivative introduced in [23]. This fractional differential operator differs from the preceding ones in the sense that the kernel of the integral in its definition contains a logarithmic function.

2. Preliminaries

Let \((a, T) \in \mathbb{R}^2\) be such that \(0 < a < T\). We denote by \(AC([a, T], \mathbb{R})\) the space of real-valued absolutely continuous functions on \([a, T]\).

Let \(f \in L^1([a, T], \mathbb{R})\). The left-sided and right-sided Riemann-Liouville fractional integrals of order \(\sigma > 0\) of \(f\), are defined respectively by (see [19])

\[
(I_a^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_a^t (t - s)^{\sigma - 1} f(s) \, \text{d}s
\]

and

\[
(I_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (s - t)^{\sigma - 1} f(s) \, \text{d}s,
\]

for almost everywhere \(t \in [a, T]\).

We have the following integration by parts rule.

**Lemma 2.1 (see [19]):** Let \(\sigma > 0, q, r \geq 1, \text{ and } \frac{1}{q} + \frac{1}{r} = 1 + \sigma\) (\(q = 1, r = 1\), in the case \(\frac{1}{q} + \frac{1}{r} = 1 + \sigma\)). If \(f \in L^q([a, T], \mathbb{R})\) and \(g \in L^r([a, T], \mathbb{R})\), then

\[
\int_a^T (I_a^\sigma f)(t) g(t) \, \text{d}t = \int_a^T f(t) (I_T^\sigma g)(t) \, \text{d}t.
\]

For \(r \geq 1\), we denote by \(L^r([a, T], \mathbb{R}, \frac{1}{t} \, \text{d}t)\) the weighted Lebesgue space of real-valued measurable functions \(f : [a, T] \rightarrow \mathbb{R}\) satisfying

\[
\int_a^T |f(t)|^r \frac{1}{t} \, \text{d}t < \infty.
\]

Let \(f \in L^1([a, T], \mathbb{R}, \frac{1}{t} \, \text{d}t)\). The left-sided and right-sided Hadamard fractional integrals of order \(\sigma > 0\) of \(f\), are defined respectively by (see [19])

\[
(J_a^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_a^t \left(\ln \frac{t}{s}\right)^{\sigma - 1} f(s) \frac{1}{s} \, \text{d}s
\]

and

\[
(J_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T \left(\ln \frac{s}{t}\right)^{\sigma - 1} f(s) \frac{1}{s} \, \text{d}s,
\]

for almost everywhere \(t \in [a, T]\).

**Remark 2.1:** It can be easily seen that, if \(f \in C([a, T], \mathbb{R})\) and \(\sigma > 0\), then

\[
\lim_{t \to a^+} (J_a^\sigma f)(t) = \lim_{t \to T^-} (J_T^\sigma f)(t) = 0.
\]

The following integration by parts rule holds.
Lemma 2.2: Let \( \sigma > 0, q, r \geq 1 \), and \( \frac{1}{q} + \frac{1}{r} \leq 1 + \sigma \) \((q = 1, r = 1, \text{in the case } \frac{1}{q} + \frac{1}{r} = 1 + \sigma)\). If \( f \in L^q([a, T], \mathbb{R}, \frac{1}{t} \, dt) \) and \( g \in L^r([a, T], \frac{1}{t} \, dt) \), then
\[
\int_a^T (J_a^\sigma f)(t)g(t)\frac{1}{t} \, dt = \int_a^T f(t)(J_T^\sigma g)(t)\frac{1}{t} \, dt.
\]

Proof: Using the change of variable \( \tau = \ln t \), we obtain
\[
(J_a^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_{\ln a}^{\ln t} (\ln t - \tau)^{\sigma-1} (f \circ \exp)(\tau) \, d\tau,
\]
that is,
\[
(J_a^\sigma f)(t) = (I_{\ln a}^\sigma f \circ \exp)(\ln t).
\] (5)

Using the same change of variable, we obtain
\[
(J_T^\sigma g)(t) = (I_{\ln T}^\sigma g \circ \exp)(\ln t).
\] (6)

By (5), there holds
\[
\int_a^T (J_a^\sigma f)(t)g(t)\frac{1}{t} \, dt = \int_a^T (I_{\ln a}^\sigma f \circ \exp)(\ln t)g(t)\frac{1}{t} \, dt.
\]

Using the change of variable \( \tau = \ln t \), we obtain
\[
\int_a^T (J_a^\sigma f)(t)g(t)\frac{1}{t} \, dt = \int_{\ln a}^{\ln T} (I_{\ln a}^\sigma f \circ \exp)(\tau)(g \circ \exp)(\tau) \, d\tau.
\]
Notice that, since \( f \in L^q([a, T], \mathbb{R}, \frac{1}{t} \, dt) \) and \( g \in L^r([a, T], \frac{1}{t} \, dt) \), then \( f \circ \exp \in L^q([\ln a, \ln T], \mathbb{R}) \) and \( g \circ \exp \in L^r([\ln a, \ln T], \mathbb{R}) \). Then, using Lemma 2.1, we obtain
\[
\int_a^T (J_a^\sigma f)(t)g(t)\frac{1}{t} \, dt = \int_{\ln a}^{\ln T} (f \circ \exp)(\tau)(I_{\ln T}^\sigma g \circ \exp)(\tau) \, d\tau.
\]

Using the above change of variable, there holds
\[
\int_a^T (J_a^\sigma f)(t)g(t)\frac{1}{t} \, dt = \int_a^T f(t)(I_{\ln T}^\sigma g \circ \exp)(\ln t)\frac{1}{t} \, dt.
\]

Thus, by (6), the desired result follows. \(\square\)

For \( \kappa \gg 1 \) (\( \kappa \) is sufficiently large), let
\[
\mu(t) = \left( \frac{T}{a} \right)^{-\kappa} \left( \frac{T}{t} \right)^{\kappa}, \quad a \leq t \leq T.
\]

Lemma 2.3: Let \( \sigma > 0 \). Then
\[
(J_T^\sigma \mu)(t) = \frac{\Gamma(\kappa + 1)}{\Gamma(\sigma + \kappa + 1)} \left( \frac{T}{a} \right)^{-\kappa} \left( \frac{T}{t} \right)^{\sigma + \kappa},
\] (7)
\[
t(J_T^\sigma \mu)'(t) = -\frac{\Gamma(\kappa + 1)}{\Gamma(\sigma + \kappa)} \left( \frac{T}{a} \right)^{-\kappa} \left( \frac{T}{t} \right)^{\sigma + \kappa - 1}.
\] (8)
Proof: We have

\[
(J^\sigma \mu)_a(T)(t) = \left(\ln \frac{T}{a}\right)^{-\kappa} \frac{1}{\Gamma(\sigma)} \int_t^T (\ln s - \ln t)^{\sigma - 1} \left(\ln T - \ln s\right)^{\frac{1}{s}} \, ds
\]

\[
= \left(\ln \frac{T}{a}\right)^{-\kappa} \frac{1}{\Gamma(\sigma)} \int_t^T (\ln s - \ln t)^{\sigma - 1} \left(\ln T - \ln t - (\ln s - \ln t)\right)^{\kappa} \frac{1}{s} \, ds
\]

\[
= \left(\ln \frac{T}{a}\right)^{-\kappa} \left(\ln \frac{T}{t}\right)^\kappa \frac{1}{\Gamma(\sigma)} \int_t^T (\ln s - \ln t)^{\sigma - 1} \left(1 - \frac{\ln s - \ln t}{\ln T - \ln t}\right)^{\kappa} \frac{1}{s} \, ds.
\]

Using the change of variable \( \tau = \frac{\ln s - \ln t}{\ln T - \ln t} \), we obtain

\[
(J^\sigma \mu)_a(T)(t) = \left(\ln \frac{T}{a}\right)^{-\kappa} \left(\ln \frac{T}{t}\right)^{\kappa + \sigma} \frac{1}{\Gamma(\sigma)} \int_0^1 \tau^{\sigma - 1} (1 - \tau)^{\kappa} \, d\tau
\]

\[
= \left(\ln \frac{T}{a}\right)^{-\kappa} \left(\ln \frac{T}{t}\right)^{\kappa + \sigma} \frac{1}{\Gamma(\sigma)} B(\sigma, \kappa + 1),
\]

where \( B(\cdot, \cdot) \) is the Beta function. Using the property (see e.g. [19])

\[
B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x + y)}, \quad x > 0, \quad y > 0,
\]

we get

\[
(J^\sigma \mu)_a(T)(t) = \left(\ln \frac{T}{a}\right)^{-\kappa} \left(\ln \frac{T}{t}\right)^{\kappa + \sigma} \frac{1}{\Gamma(\sigma)} \frac{\Gamma(\sigma) \Gamma(\kappa + 1)}{\Gamma(\sigma + \kappa + 1)}
\]

\[
= \frac{\Gamma(\kappa + 1)}{\Gamma(\sigma + \kappa + 1)} \left(\ln \frac{T}{a}\right)^{-\kappa} \left(\ln \frac{T}{t}\right)^{\kappa + \sigma},
\]

which proves (7). Differentiating (7) and using the property (see e.g. [19])

\[
x \Gamma(x) = \Gamma(x + 1), \quad x > 0,
\]

(8) follows.

\[
\square
\]

Let \( f \in AC([a, T], \mathbb{R}) \). The left/forward Hadamard-Caputo fractional derivative of order \( \alpha \in (0, 1) \) of \( f \), is defined by (see Agrawal [23])

\[
(D^\alpha_a f)(t) = f^1_{a}^{1-\alpha} \left( tf' \right) (t)
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{s} \right)^{-\alpha} f'(s) \, ds,
\]

for almost everywhere \( t \in [a, T] \).

Let \( u : [a, T] \times \mathbb{R}^N \rightarrow \mathbb{C} \) be a given complex-valued function. For a fixed \( x \in \mathbb{R}^N \), we denote by \( u(\cdot, x) : [a, T] \rightarrow \mathbb{C} \) the function defined by

\[
u(\cdot, x)(t) = u(t, x), \quad t \in [a, T].
\]

The left-sided and right-sided Hadamard fractional integrals of order \( \sigma \geq 0 \) of \( u \) with respect to the time variable \( t \), are defined respectively by

\[
(J^\sigma_{a[t]} u)(t, x) = (J^\sigma_{a} u(\cdot, x))(t)
\]
Definition 3.1: We say that weak solutions to (1) as follows.

Lemma 2.2, the initial conditions in (9), and taking in consideration Remark 2.1, we define global ϕ

Multiplying the first two equations in (9) by t and

and

\[ (J^0_T, u)(t, x) = (J^0_T, u(\cdot, x))(t) \]
\[ = (J^0_T, u(\cdot, x))(t) + i(J^0_T, \Im u(\cdot, x))(t) \]
\[ = \frac{1}{\Gamma(\sigma)} \int_t^T \left( \ln \frac{t}{s} \right)^{\sigma - 1} \Re u(s, x) \frac{1}{s} \, ds + \frac{i}{\Gamma(\sigma)} \int_t^T \left( \ln \frac{t}{s} \right)^{\sigma - 1} \Im u(s, x) \frac{1}{s} \, ds. \]

The left-forward Hadamard-Caputo fractional derivative of order \( \alpha \in (0, 1) \) of \( u \) with respect to the time variable \( t \), is defined by

\[
(D^{\alpha}_{at}u)(t, x) = \frac{1}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{s} \right)^{\alpha - 1} \frac{\partial \Re u}{\partial s}(s, x) \, ds + \frac{i}{\Gamma(1 - \alpha)} \int_a^t \left( \ln \frac{t}{s} \right)^{\alpha - 1} \frac{\partial \Im u}{\partial s}(s, x) \, ds.
\]

3. Main results

Before stating our main results, let us give the definition of global weak solutions to (1). Observe that (1) is equivalent to the system

\[
\begin{align*}
    r_a D^{\alpha}_{at}u_1 - s_a D^{\alpha}_{at}u_2 + \Delta u_1 &= \lambda_1 \left( \ln \frac{t}{a} \right)^{\gamma} |u|^p, \quad t > a, \ x \in \mathbb{R}^N, \\
    s_a D^{\alpha}_{at}u_1 + r_a D^{\alpha}_{at}u_2 + \Delta u_2 &= \lambda_2 \left( \ln \frac{t}{a} \right)^{\gamma} |u|^p, \quad t > a, \ x \in \mathbb{R}^N, \\
    (u_1(a, x), u_2(a, x)) &= (f_1(x), f_2(x)), \quad x \in \mathbb{R}^N, \\
\end{align*}
\]

where

\[
(r_a, s_a) = \left( \cos \left( \frac{\pi \alpha}{2} \right), \sin \left( \frac{\pi \alpha}{2} \right) \right), \quad (u_1, u_2) = (\Re u, \Im u), \\
(f_1, f_2) = (\Re f, \Im f), \quad (\lambda_1, \lambda_2) = (\Re \lambda, \Im \lambda).
\]

For \( T > 0 \), let

\[ Q_T = [a, T] \times \mathbb{R}^N \]

and \( \Phi_T \) be the set of functions \( \varphi \) satisfying:

\[ \varphi \in C^{1,2}_{t,x}(Q_T, \mathbb{R}), \quad \text{supp}_x \varphi \subset \subset \mathbb{R}^N. \]

Multiplying the first two equations in (9) by \( \varphi \in \Phi_T \), using the integration by parts rule provided by Lemma 2.2, the initial conditions in (9), and taking into consideration Remark 2.1, we define global weak solutions to (1) as follows.

Definition 3.1: We say that \( u \) is a global weak solution to (1), if
(i) \( u \in L^1_{loc}([a, \infty) \times \mathbb{R}^N, \mathbb{C}), \quad \left( \ln \frac{t}{a} \right)^\gamma |u|^p \in L^1_{loc}([a, \infty) \times \mathbb{R}^N, \mathbb{R}), \)

(ii) for all \( T > 0 \) and \( \varphi \in \Phi_T \), there holds

\[
\lambda_1 \int_{Q_T} \left( \ln \frac{t}{a} \right)^\gamma |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} \left( r_\alpha f_1(x) - s_\alpha f_2(x) \right) \left( (J_t^{1-\alpha} t\varphi)(a, x) \right) \, dx
\]

\[
= \int_{Q_T} u_1 \Delta \varphi \, dx \, dt - \int_{Q_T} \left( r_\alpha u_1 - s_\alpha u_2 \right) \frac{\partial \left( J_t^{1-\alpha} t\varphi \right)}{\partial t} \, dx \, dt \tag{10}
\]

and

\[
\lambda_2 \int_{Q_T} \left( \ln \frac{t}{a} \right)^\gamma |u|^p \varphi \, dx \, dt + \int_{\mathbb{R}^N} \left( s_\alpha f_1(x) + r_\alpha f_2(x) \right) \left( (J_t^{1-\alpha} t\varphi)(a, x) \right) \, dx
\]

\[
= \int_{Q_T} u_2 \Delta \varphi \, dx \, dt - \int_{Q_T} \left( s_\alpha u_1 + r_\alpha u_2 \right) \frac{\partial \left( J_t^{1-\alpha} t\varphi \right)}{\partial t} \, dx \, dt. \tag{11}
\]

Our main results are the following.

**Theorem 3.1:** Let \( \gamma > -\alpha, \gamma(N\alpha - 2) < 2\alpha, \) and

\[
\max\{1, 1 + \gamma\} < p < 1 + \frac{2(\alpha + \gamma)}{N\alpha}. \tag{12}
\]

If the initial value \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \) satisfies

\[
\lambda_1 \int_{\mathbb{R}^N} \left( r_\alpha f_1(x) - s_\alpha f_2(x) \right) \, dx > 0 \quad \text{or} \quad \lambda_2 \int_{\mathbb{R}^N} \left( s_\alpha f_1(x) + r_\alpha f_2(x) \right) \, dx > 0, \tag{13}
\]

then (1) admits no global weak solution.

**Remark 3.1:** Notice that under the conditions \( \gamma > -\alpha \) and \( \gamma(N\alpha - 2) < 2\alpha, \) the set of \( p \) satisfying (12) is nonempty.

**Theorem 3.2:** Let \( \gamma > 0 \) and

\[
1 + \gamma < p < 1 + \frac{\gamma}{\alpha}. \tag{14}
\]

If the initial value \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \) satisfies (13), then (1) admits no global weak solution.

In the case \( \gamma > 0, \) we deduce from Theorems 3.1 and 3.2 the following result.

**Corollary 3.1:** Let \( \gamma > 0, \gamma(N\alpha - 2) < 2\alpha, \) and

\[
1 + \gamma < p < \max \left\{ 1 + \frac{2(\alpha + \gamma)}{N\alpha}, 1 + \frac{\gamma}{\alpha} \right\}.
\]

If the initial value \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \) satisfies (13), then (1) admits no global weak solution.
Remark 3.2: (i) Notice that for $\gamma > 0$, we have
\[
\max \left\{ 1 + \frac{2(\alpha + \gamma)}{N\alpha}, 1 + \frac{\gamma}{\alpha} \right\} = \begin{cases} 
1 + \frac{2(\alpha + \gamma)}{N\alpha} & \text{if } (N-2)\gamma < 2\alpha, \\
1 + \frac{\gamma}{\alpha} & \text{if } \gamma(N\alpha - 2) < 2\alpha \leq (N-2)\gamma.
\end{cases}
\]

(ii) Observe that, if $N \in \{1, 2\}$ and $\gamma > 0$, then $(N-2)\gamma < 2\alpha$. Hence, by (i), we deduce that
\[
\max \left\{ 1 + \frac{2(\alpha + \gamma)}{N\alpha}, 1 + \frac{\gamma}{\alpha} \right\} = 1 + \frac{2(\alpha + \gamma)}{N\alpha}.
\]

We provide below some examples to illustrate our obtained results.

Example 3.1: Consider the initial value problem for the nonlinear time-fractional Schrödinger equation
\[
\begin{cases}
\sqrt{iD_{t}^{1/2}} u + \Delta u = \left( \ln \frac{t}{a} \right)^{-1/4} |u|^p, & t > a, \ x \in \mathbb{R}^N, \\
u(a, x) = \frac{1}{|x|^{N-1}(1 + |x|^2)}, & x \in \mathbb{R}^N,
\end{cases}
\tag{15}
\]
where $a > 0$ and $N \geq 5$. Then (15) is a special case of (1) with
\[
\alpha = \frac{1}{2}, \quad \gamma = -\frac{1}{4}, \quad \lambda = \lambda_1 = 1, \quad f(x) = f_1(x) = \frac{1}{|x|^{N-1}(1 + |x|^2)}.
\]
Observe that $f \in L^1(\mathbb{R}^N, \mathbb{R})$ and
\[
\lambda_1 \int_{\mathbb{R}^N} (r_0 f_1(x) - s_{0} f_2(x)) \ dx = \frac{\sqrt{2}}{2} \int_{\mathbb{R}^N} \frac{1}{|x|^{N-1}(1 + |x|^2)} \ dx > 0.
\]
Moreover, we have
\[
\gamma > -\alpha, \quad \gamma(N\alpha - 2) = -\frac{1}{8}(N-4) < 0 < 2\alpha, \quad \max\{1, 1 + \gamma\} = 1, \quad 1 + \frac{2(\alpha + \gamma)}{N\alpha} = 1 + \frac{1}{N}.
\]
Hence, by Theorem 3.1, we deduce that for all
\[
1 < p < 1 + \frac{1}{N},
\]
(15) admits no global weak solution.

Example 3.2: Consider the initial value problem for the nonlinear time-fractional Schrödinger equation
\[
\begin{cases}
\sqrt{iD_{t}^{1/2}} u + \Delta u = -\left( \ln \frac{t}{a} \right)^{1/N} |u|^p, & t > a, \ x \in \mathbb{R}^N, \\
u(a, x) = i|x|^{2-N} \exp(-|x|^2), & x \in \mathbb{R}^N,
\end{cases}
\tag{16}
\]
where $a > 0$ and $N \geq 1$. Then (16) is a special case of (1) with
\[
\alpha = \frac{1}{2}, \quad \gamma = \frac{1}{N}, \quad \lambda = \lambda_1 = 1, \quad f(x) = f_2(x) = i|x|^{2-N} \exp(-|x|^2).
\]
Observe that \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \) and
\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \, dx = \frac{\sqrt{2}}{2} \int_{\mathbb{R}^N} |x|^{2-N} \exp(-|x|^2) \, dx > 0.
\]
Moreover, we have
\[
\gamma > 0, \quad (N - 2)\gamma = 1 - \frac{2}{N} < 1 = 2\alpha, \quad \max\{1, 1 + \gamma\} = 1 + \frac{1}{N},
\]
\[
1 + \frac{2(\alpha + \gamma)}{N\alpha} = 1 + \frac{2}{N} + \frac{4}{N^2}.
\]
Hence by Corollary 3.1, and taking in consideration Remark 3.2(i), we deduce that for all
\[
1 + \frac{1}{N} < p < 1 + \frac{2}{N} + \frac{4}{N^2},
\]
(16) admits no global weak solution.

**Example 3.3:** Consider the initial value problem for the nonlinear time-fractional Schrödinger equation
\[
\left\{
\begin{array}{l}
\sqrt{iD_{a_t}^{1/2}} u + \Delta u = \left(\ln \frac{t}{a}\right)^{1/(N-2)} |u|^p, \quad t > a, \; x \in \mathbb{R}^N, \quad \\
u(a, x) = \exp(-|x|), \quad x \in \mathbb{R}^N,
\end{array}
\right.
\]
where \( a > 0 \) and \( N \geq 3 \). Then (17) is a special case of (1) with
\[
\alpha = \frac{1}{2}, \quad \gamma = \frac{1}{N - 2}, \quad \lambda = \lambda_1 = 1, \quad f(x) = f_1(x) = \exp(-|x|).
\]
Observe that \( f \in L^1(\mathbb{R}^N, \mathbb{R}) \) and
\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \, dx = \frac{\sqrt{2}}{2} \int_{\mathbb{R}^N} \exp(-|x|) \, dx > 0.
\]
Moreover, we have
\[
\gamma > 0, \quad \gamma (N\alpha - 2) = \frac{N - 4}{2(N - 2)} < 2\alpha = (N - 2)\gamma, \quad 1 + \frac{\gamma}{\alpha} = 1 + \frac{2}{N - 2}.
\]
Hence by Corollary 3.1, and taking in consideration Remark 3.2(i), we deduce that for all
\[
1 + \frac{1}{N - 2} < p < 1 + \frac{2}{N - 2},
\]
(17) admits no global weak solution.

**4. Proof of the main results**

In the sequel, we use \( C \) to denote a positive constant which may vary from line to line, but its value is not essential to the analysis of the problem. The proof of our main results is based on the test function method developed by Mitidieri and Pohozaev [24], and a judicious choice of the test function.
Proof of Theorem 3.1.: We argue by contradiction. Namely, we suppose that \( u \) is a global weak solution to (1). Let
\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \, dx > 0.
\] (18)

Then, by (10), for all \( T > 0 \) and \( \varphi \in \Phi_T \), we have (after a multiplication by \( \lambda_1 \))
\[
\lambda_1 \int_t^{T_t} \left( \int \frac{t}{a} \right)^\gamma |u|^{p} \varphi \, dx \, dt + \lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) (J_{1,t}^{1-\alpha} t\varphi)(a, x) \, dx
\]
\[
= \lambda_1 \int_{Q_T} u_1 \Delta \varphi \, dx \, dt - \lambda_1 \int_{Q_T} (r_\alpha u_1 - s_\alpha u_2) \frac{\partial J_{1,t}^{1-\alpha} t\varphi}{\partial t} \, dx \, dt,
\]
which yields
\[
\lambda_1 \int_{Q_T} \left( \int \frac{t}{a} \right)^\gamma |u|^{p} \varphi \, dx \, dt + \lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) (J_{1,t}^{1-\alpha} t\varphi)(a, x) \, dx
\]
\[
\leq |\lambda_1| \int_{Q_T} |u| |\Delta \varphi| \, dx \, dt + 2|\lambda_1| \int_{Q_T} |u| \left| \frac{\partial J_{1,t}^{1-\alpha} t\varphi}{\partial t} \right| \, dx \, dt.
\] (19)

On the other hand, by \( \varepsilon \)-Young inequality with \( \varepsilon = \frac{\lambda_1}{4} > 0 \), we have
\[
\int_{Q_T} |u| |\Delta \varphi| \, dx \, dt \leq \frac{|\lambda_1|}{2} \int_{Q_T} \left( \int \frac{t}{a} \right)^\gamma |u|^{p} \varphi \, dx \, dt + C \int_{Q_T} \left( \int \frac{t}{a} \right)^{-\frac{\gamma}{p-1}} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} \, dx \, dt.
\] (20)

Similarly, using \( \varepsilon \)-Young inequality with \( \varepsilon = \frac{\lambda_1}{4} \), we get
\[
\int_{Q_T} \left| u \right| \left| \frac{\partial J_{1,t}^{1-\alpha} t\varphi}{\partial t} \right| \, dx \, dt
\]
\[
\leq \frac{|\lambda_1|}{4} \int_{Q_T} \left( \int \frac{t}{a} \right)^\gamma |u|^{p} \varphi \, dx \, dt + C \int_{Q_T} \left( \int \frac{t}{a} \right)^{-\frac{\gamma}{p-1}} \varphi^{-\frac{1}{p-1}} \left| \frac{\partial J_{1,t}^{1-\alpha} t\varphi}{\partial t} \right|^{\frac{p}{p-1}} \, dx \, dt.
\] (21)

Hence, combining (19)–(21), we obtain
\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) (J_{1,t}^{1-\alpha} t\varphi)(a, x) \, dx \leq C(K_1(\varphi) + K_2(\varphi)),
\] (22)

where
\[
K_1(\varphi) = \int_{Q_T} \left( \int \frac{t}{a} \right)^{-\frac{\gamma}{p-1}} \varphi^{-\frac{1}{p-1}} \left| \frac{\partial J_{1,t}^{1-\alpha} t\varphi}{\partial t} \right|^{\frac{p}{p-1}} \, dx \, dt
\]
and
\[
K_2(\varphi) = \int_{Q_T} \left( \int \frac{t}{a} \right)^{-\frac{\gamma}{p-1}} \varphi^{-\frac{1}{p-1}} |\Delta \varphi|^{\frac{p}{p-1}} \, dx \, dt.
\]

Now, consider a family of cut-off functions \( \{\xi_R\}_{R \geq 1} \subset C_\infty(\mathbb{R}^N, \mathbb{R}) \) \( (\xi_R \in C_\infty(\mathbb{R}^N, \mathbb{R}) \) and \( \text{supp} \xi_R \subset \subset \mathbb{R}^N \) satisfying the following properties:

(a) \( 0 \leq \xi_R \leq 1, \xi_R |_{BR} \equiv 1 \),
(b) \( \text{supp}(\xi_R) \subseteq B_{2R} \),
(c) \( |\nabla \xi_R| \leq \frac{C}{R} \),
(d) \( |\Delta \xi_R| \leq \frac{C}{R^2} \),

where for \( \rho > 0 \),

\[
B_\rho = \{ x \in \mathbb{R}^N : |x| < \rho \}.
\]

For \( \kappa \gg 1 \) and \( \ell \gg 1 \), let us introduce the test function

\[
\varphi(t, x) = \eta(t) \xi_\ell^R(x), \quad a \leq t \leq T,
\]

where

\[
\eta(t) = \frac{1}{t} \left( \ln \frac{T}{a} \right)^{-\gamma} \left( \frac{\ln T}{t} \right)^{\kappa} = \frac{1}{t} \mu(t).
\]

It can be easily seen that for all \( T > 0 \), the function \( \varphi \) defined by (23) belongs to \( \Phi_T \), and thus, it satisfies the estimate (22).

Let us estimate the terms \( K_j(\varphi), \ j = 1, 2 \), where \( \varphi \) is defined by (23). The term \( K_1(\varphi) \) can be written as

\[
K_1(\varphi) = K_{11}(\varphi)K_{12}(\varphi),
\]

where

\[
K_{11}(\varphi) = \int_a^T \left( \ln \frac{t}{a} \right)^{-\gamma} t^{-\frac{\gamma}{p-1}} \frac{1}{t^{\frac{\gamma}{p-1}}} \mu^{-\frac{1}{p-1}} \left( \int_a^T \left( \ln \frac{t}{a} \right)^{-\gamma} t^{-\frac{\gamma}{p-1}} \right) t^{-\frac{p}{p-1}} \ dt
\]

and

\[
K_{12}(\varphi) = \int_{\mathbb{R}^N} \xi_\ell^R(x) \ dx.
\]

On the other hand, using (8), for \( a < t < T \), we get

\[
\mu^{-\frac{1}{p-1}} (t) \left( \int_a^T (\ln \frac{t}{a})^{-\gamma} t^{-\frac{\gamma}{p-1}} \right) \left( \int_a^T (\ln \frac{t}{a})^{-\gamma} t^{-\frac{\gamma}{p-1}} \right) \leq \frac{\Gamma(\kappa + 1)}{\Gamma(1 - \alpha + \kappa)},
\]

which yields

\[
K_{11}(\varphi) = \left[ \frac{\Gamma(\kappa + 1)}{\Gamma(1 - \alpha + \kappa)} \right]^{\frac{p}{p-1}} \left( \ln \frac{T}{a} \right)^{-\gamma} \int_a^T \left( \ln \frac{t}{a} \right)^{-\gamma} t^{-\frac{\gamma}{p-1}} \left( \int_a^T \left( \ln \frac{t}{a} \right)^{-\gamma} t^{-\frac{\gamma}{p-1}} \right) \frac{1}{t} \ dt
\]

Then, since \( \gamma < p - 1 \) by (12), there holds

\[
K_{11}(\varphi) \leq \left[ \frac{\Gamma(\kappa + 1)}{\Gamma(1 - \alpha + \kappa)} \right]^{\frac{p}{p-1}} \left( 1 - \frac{\gamma}{p - 1} \right)^{-1} \left( \ln \frac{T}{a} \right)^{1 - \frac{\gamma + ap}{p-1}}.
\]

Next, by the properties (a) and (b) of \( \xi_R \), we get

\[
K_{12}(\varphi) = \int_{B_{2R}} \xi_\ell^R(x) \ dx
\]

\[
\leq \text{Vol}(B_{2R})
\]
Hence, it follows from (24)–(26) that

\[ K_1(\varphi) \leq C \left( \ln \frac{T}{\alpha} \right) \frac{1}{\frac{1}{\frac{1}{p}-1}} R^N. \]  

(27)

The term \( K_2(\varphi) \) can be written as

\[ K_2(\varphi) = K_{21}(\varphi)K_{22}(\varphi), \]  

(28)

where

\[ K_{21}(\varphi) = \int_{\alpha}^{T} \left( \ln \frac{t}{\alpha} \right) \frac{-\frac{\gamma}{p-1}}{t} \eta(t) \, dt \]

and

\[ K_{22}(\varphi) = \int_{R^N} \xi_{R}^{-\frac{\ell}{p-1}}(x) \left| \Delta[\xi_{R}^\ell(x)] \right| \frac{p}{\frac{1}{p}-1} \, dx. \]

On the other hand, we have

\[ K_{21}(\varphi) = \left( \ln \frac{T}{\alpha} \right) -\kappa \int_{\alpha}^{T} \left( \ln \frac{t}{\alpha} \right) \frac{-\frac{\gamma}{p-1}}{t} \frac{1}{t} \left( \ln \frac{T}{t} \right)^{\kappa} \]

\[ \leq \int_{\alpha}^{T} \left( \ln \frac{t}{\alpha} \right) \frac{-\frac{\gamma}{p-1}}{t} \frac{1}{t} \, dt \]

\[ = \left( 1 - \frac{\gamma}{p-1} \right)^{-1} \left( \ln \frac{T}{\alpha} \right) \frac{1}{\frac{1}{\frac{1}{p}-1}} \]  

(29)

Next, using the property

\[ \Delta(\xi_{R}^\ell) = \ell^{2-\ell} \left( (\ell - 1)|\nabla \xi_{R}|^2 + \xi_{R} \Delta \xi_{R} \right) \]

as well as the properties (a)–(d) of \( \xi_{R} \), we obtain

\[ \left| \Delta[\xi_{R}^\ell(x)] \right| \leq CR^{-2}\xi_{R}^{\ell-2}, \quad R < |x| < 2R, \]

which yields

\[ K_{22}(\varphi) \leq CR^{-\frac{2p}{p-1}} \int_{R<|x|<2R} \xi_{R}^{\ell-\frac{2p}{p-1}}(x) \, dx \]

\[ \leq CR^{-\frac{2p}{p-1}} \text{Vol}(B_{2R}) \]

\[ = CR^{N-\frac{2p}{p-1}}. \]  

(30)

Hence, by (28)–(30), we deduce that

\[ K_2(\varphi) \leq C \left( \ln \frac{T}{\alpha} \right) \frac{1}{\frac{1}{\frac{1}{p}-1}} R^N \frac{1}{\frac{1}{p}-1}. \]  

(31)

Now, consider the term from the left-hand side of (22). By (23) and using (7), we have

\[ (J_{1/\alpha}^{1-\alpha} t\varphi)(a,x) = (J_{1/\alpha}^{1-\alpha} t\eta \xi_{R}^\ell)(a,x) \]
On the other hand, observe that for \( \theta = \frac{2}{\alpha} \), we have

\[
N + \theta \left( \alpha - \frac{\gamma + \alpha p}{p - 1} \right) = N - \frac{2p}{p - 1} + \theta \left( \alpha - \frac{\gamma}{p - 1} \right) = \frac{N\alpha(p - 1) - 2(\alpha + \gamma)}{\alpha(p - 1)}.
\]

Hence, for this value of \( \theta \), we have

\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \xi_R^\ell(x) \, dx \leq C R^{N+\theta \left( \alpha - \frac{\gamma + \alpha p}{p - 1} \right)} + R^{N-\frac{2p}{p - 1} + \theta \left( \alpha - \frac{\gamma}{p - 1} \right)}.
\]  

(32)

On the other hand, observe that for \( \theta = \frac{2}{\alpha} \), we have

\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \xi_R^\ell(x) \, dx \leq C \left( R^{N+\theta \left( \alpha - \frac{\gamma + \alpha p}{p - 1} \right)} + R^{N-\frac{2p}{p - 1} + \theta \left( \alpha - \frac{\gamma}{p - 1} \right)} \right).
\]

On the other hand, observe that for \( \theta = \frac{2}{\alpha} \), we have

\[
N + \theta \left( \alpha - \frac{\gamma + \alpha p}{p - 1} \right) = N - \frac{2p}{p - 1} + \theta \left( \alpha - \frac{\gamma}{p - 1} \right) = \frac{N\alpha(p - 1) - 2(\alpha + \gamma)}{\alpha(p - 1)}.
\]

Hence, for this value of \( \theta \), we have

\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \xi_R^\ell(x) \, dx \leq CR \frac{N\alpha(p - 1) - 2(\alpha + \gamma)}{\alpha(p - 1)}.
\]  

(34)

Notice that by (12), we have

\[
\frac{N\alpha(p - 1) - 2(\alpha + \gamma)}{\alpha(p - 1)} < 0.
\]

Hence, since \( f \in L^1(\mathbb{R}^N, \mathbb{C}) \), using the property (a) of the cut-off function \( \xi_R \), the dominated convergence theorem, and passing to the limit as \( R \to \infty \) in (34), we obtain

\[
\lambda_1 \int_{\mathbb{R}^N} (r_\alpha f_1(x) - s_\alpha f_2(x)) \, dx \leq 0,
\]

which contradicts (18).

Suppose now that

\[
\lambda_2 \int_{\mathbb{R}^N} (s_\alpha f_1(x) + r_\alpha f_2(x)) \, dx > 0.
\]  

(35)

Then, by (11), we have (after a multiplication by \( \lambda_2 \))

\[
\lambda_2^2 \int_{Q_T} \left( \ln \frac{t}{a} \right)^{\gamma} |u|^p \varphi \, dx \, dt + \lambda_2 \int_{\mathbb{R}^N} (s_\alpha f_1(x) + r_\alpha f_2(x)) \left( J_{1/\alpha}^{1-\alpha} \varphi \right)(a, x) \, dx
\]

\[
= \lambda_2 \int_{Q_T} u_2 \Delta \varphi \, dx \, dt - \lambda_2 \int_{Q_T} (s_\alpha u_1 + r_\alpha u_2) \frac{\partial J_{1/\alpha}^{1-\alpha} \varphi}{\partial t} \, dx \, dt,
\]

(33)
which yields
\[
\lambda_2^2 \int_{Q_T} \left( \ln \frac{t}{a} \right)^\gamma |u|^p \varphi \, dx \, dt + \lambda_2 \int_{\mathbb{R}^N} \left( s_{\alpha f_1}(x) + r_{\alpha f_2}(x) \right) (f_1^{1-\alpha} t \varphi)(a, x) \, dx \\
\leq |\lambda_2| \int_{Q_T} |u| |\nabla \varphi| \, dx \, dt + 2|\lambda_2| \int_{Q_T} |u| \left| \frac{\partial f_1^{1-\alpha} t \varphi}{\partial t} \right| \, dx \, dt.
\]

Then, repeating the same procedures as in the previous case, we arrive at
\[
\lambda_2 \int_{\mathbb{R}^N} \left( s_{\alpha f_1}(x) + r_{\alpha f_2}(x) \right) \xi_{\ell R}(x) \, dx \leq C \left( (\ln T)^{\alpha - \frac{\nu + \alpha p}{p - 1}} R^N + (\ln T)^{\alpha - \frac{\nu}{p - 1}} R^N - \frac{2^p}{p - 1} \right).
\]

Following exactly the same steps as above, we reach a contradiction with (35). The proof of Theorem 3.1 is completed.

\textbf{Proof of Theorem 3.2.:} Suppose that \( u \) is a global weak solution to (1). We only consider the case (18), since the case (35) can be treated in the same way. From the proof of Theorem 3.1, (33) holds for \( T \gg 1 \). Observe that for \( \gamma > 0 \) and under condition (14), we have
\[
\alpha - \gamma + \alpha p \frac{p}{p - 1} = - \frac{\alpha + \gamma}{p - 1} < 0, \quad \alpha - \gamma \frac{p}{p - 1} < 0.
\]

Hence, fixing \( R \), and passing to the limit as \( T \to \infty \) in (33), we obtain
\[
\lambda_1 \int_{\mathbb{R}^N} \left( r_{\alpha f_1}(x) - s_{\alpha f_2}(x) \right) \xi_{\ell R}(x) \, dx \leq 0.
\]

Next, passing to the limit as \( R \to \infty \) in the above inequality, we reach a contradiction with (18). The proof of Theorem 3.2 is completed.

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