A rigid elliptical cross-sectional projectile with different geometrical characteristics penetration into concrete target

X H Dai, K H Wang, J Duan, M R Li, B W Qian and G Zhou
Laboratory of Intense Dynamic Loading and Effect, Northwest Institute of Nuclear Technology, Xi’an, Shaanxi 710024, P R China

Abstract. Elliptical cross-sectional projectiles have recently attracted attention because they fit the flattened shape of earth-penetrating weapons. This study aims to investigate a rigid elliptical cross-sectional projectile with different geometrical characteristics penetration into the semi-infinite concrete targets by implementing a theoretical method. The general geometric models of four types of elliptical cross-sectional projectiles are introduced; closed-form penetration equations are then derived based on the dynamic cavity-expansion theory. Furthermore, the present models are validated by comparing the predicted penetration depths with the predictions obtained using the semi-empirical formulae and test data; the maximum deviation from the test data is 15.8%. In addition, the deceleration, velocity, and displacement of the projectiles during the penetration process are obtained based on the present models, and the penetration performance of the four types of elliptical cross-sectional projectiles is discussed by comparing the penetration depths. The conical-nose elliptical cross-sectional projectile exhibits the best penetration performance than the other three types if the nose length is sufficiently large, and the ogive-nose elliptical cross-sectional projectile gradually exhibits its penetration performance advantage with the increase in the nose length.

Nomenclature

| Symbol | Description |
|--------|-------------|
| a, a₁ | Semi-major axis of the initial/terminal cross-sectional ellipse shown in figure 1/3 |
| b, b₁ | Semi-minor axis of the initial/terminal cross-sectional ellipse shown in figure 1/3 |
| A | Static resistance |
| B | Dynamic resistance coefficient |
| c | A constant given by equation (44a) |
| d | Diameter of the projectile |
| f_c | Unconfined compressive strength of the concrete target |
| F_z | Axis resistance of the projectile nose |
| s | Curvature radius of the arbitrary ogival curve shown in figure 1 |
| S | Empirical constant of a concrete target and related to f_c |
| S_A | Area of the ogive-nose surface |
| t₁ | Time at z = kd given by equation (44d) |
| t₂ | Total penetration time given by equation (45) |
| V | Instantaneous velocity |
| V₀ | Initial striking velocity |
| V₁ | Velocity at z = kd given by equation (44c) |
| V_n | Normal expansion velocity |
| x | X coordinate shown in figure 1 |
1. **Introduction**

Earth-penetrating weapons (EPWs) are effective means of attacking underground buildings, and a flattened shape has been gradually used to improve their stealth performance. The projectile mounted on an EPW adopts an elliptical instead of a circular cross section, and it can more effectively utilize the installation space. The projectile mass can be increased by increasing the size of its outline and eventually increasing the initial kinetic energy for penetration into underground buildings. However, the penetration performance of elliptical cross-sectional projectiles into concrete targets is still not fully understood because of its unique shape.

The analytical methods implemented for penetration began with the work of Bishop et al. [1]. They developed equations for the quasi-static expansion of cylindrical and spherical cavities, and used these equations to estimate the force applied on the conical nose when it punches slowly into a metal target. Hill [2] and Hopkins [3] developed the dynamic, spherically symmetric, cavity-expansion equations for an incompressible target material, Goodier [4] applied these equations and developed a model to predict the penetration depth of rigid spheres launched into metal targets. Furthermore, Forrestal and Frew [5-10] developed a spherical cavity-expansion penetration model for concrete targets and proposed penetration formulae to predict the penetration depth of rigid circular cross-sectional ogive-nose projectiles into concrete targets. Moreover, by introducing two dimensionless numbers, namely, geometry function N and impact factor I, and considering the projectile–target interfacial frictions, Li and Chen [11-14] extended the Forrestal formula to a dimensionless form that can be applied for projectiles with arbitrary nose profiles penetrating into diversified targets. In addition, by assuming that the resistance acting on the projectile remains constant during the penetration process, Peng et al. [15] proposed a new mean resistance approach based on the dynamic cavity-expansion theory, and a simple unified model was further presented to predict the penetration depth. By introducing a hyperbolic yield criterion and the Murnaghan equation of state to describe the plastic behavior of concrete under projectile penetration, Kong et al. [16] proposed an extended dynamic cavity-
expansion theory and formulated a unified one-dimensional resistance of the concrete target to rigid and eroding projectile penetrations.

In general, the penetration mechanism of rigid circular cross-sectional projectiles into concrete targets based on the cavity-expansion theory has been fully understood. Relevant theories have been fully established, and excellent agreement has been obtained between the high-precision experimental data and the predictions obtained using the cavity-expansion theory, which serves as a theoretical foundation to study the penetration of rigid elliptical cross-sectional projectiles.

Recently, several studies have been conducted to evaluate the penetration of elliptical cross-sectional projectiles into concrete targets. Guo et al. [17] investigated the penetration performance of elliptical cross-sectional ogive-nose projectiles using experimental and numerical methods and concluded that the penetration provided by elliptical cross-sectional ogive-nose projectiles is deeper than that provided by circular cross-sectional ogive-nose projectiles. Wang et al. [18] presented an analytical model to calculate the penetration depth of an elliptical cross-sectional ogive-nose projectile based on the dynamic spherical cavity-expansion theory. However, a few simplifications were considered in the solution process, and the analytical model contained room for further improvement. Dong et al. [19] and Liu et al. [20] conducted a series of concrete target penetration tests using one type of circular cross-sectional projectile and two elliptical cross-sectional projectiles with shape ratios of 1.5 and 2.0. They also studied the stress characteristics of an elliptical cross-sectional ogive-nose projectile during the penetration process using numerical simulation and established a formula for the calculation of the penetration depth based on the assumption of the relationship between the normal stress and normal velocity of the projectile nose. However, the penetration depth formula and conclusions based on the numerical simulation results need further verification. In general, although various analytical models for concrete target penetration provided by elliptical cross-sectional ogive-nose projectiles have been established in previous studies [18-20] based on some simplifications and assumptions, a convincing, effective, and reliable analytical model that describes the penetration process is yet to be developed. In addition, the analytical model for an elliptical cross-sectional projectile with different nose shapes also needs to be improved.

In the present study, general geometry functions are introduced to define the geometrical characteristics of four types of elliptical cross-sectional projectiles. According to the dynamic cavity-expansion theory, theoretical models for deep penetration of these projectiles into concrete targets are developed, and the present models are validated using existing semi-empirical formulae and test data. The deceleration, velocity, and displacement of the projectiles during the penetration process are then obtained based on the present models, and comparative analysis of the penetration performance of these projectiles is conducted.

2. Theoretical model

For a rigid elliptical cross-sectional projectile impacting a uniform concrete target at normal incidence with initial striking velocity $V_0$ and proceeding to penetrate at an instantaneous velocity of $V$, the penetration process parameters can be calculated when the resistance of the projectile is known. First, we establish the nose surface geometry function. Then, we model the target resistance. Finally, we calculate the penetration process parameters.

2.1. Nose surface geometry functions of elliptical cross-sectional projectiles

2.1.1. Ogive-nose projectile. Figure 1 shows the geometry of an ogive-nose projectile. The ogival curve represents the arc of a circle and is tangent to the shank of the projectile. $a$ and $b$ are the semi-major and semi-minor axis of the initial cross-sectional ellipse, respectively. $l$ is the length of the projectile nose, $s$ is the radius of curvature of the arbitrary ogival curve, $\theta$ is the azimuth between the plane of the arbitrary ogival curve and the XOZ plane, $\varphi$ is the angle between the outward normal vector of the ogive-nose surface and projectile axis, $r$ is the radius of the intersection of the arbitrary ogival curve and initial cross-sectional ellipse, $r_1$ is the radius of the intersection of the arbitrary ogival
curve and arbitrary cross-sectional ellipse, and $S_a$ is the area of the ogive-nose surface. The arrow shows the direction where the projectile is moving.

![Figure 1. Ogive-nose projectile geometry.](image)

The equation for the initial cross-sectional ellipse shown in figure 1 is expressed as

$$
\left(\frac{r \cos \theta}{a}\right)^2 + \left(\frac{r \sin \theta}{b}\right)^2 = 1
$$

(1)

where

$$
\cos \theta = \frac{x}{\sqrt{x^2 + y^2}}
$$

(2a)

$$
\sin \theta = \frac{y}{\sqrt{x^2 + y^2}}
$$

(2b)

Substituting equation (2) into equation (1) yields

$$
r = \frac{ab\sqrt{x^2 + y^2}}{\sqrt{(bx)^2 + (ay)^2}}
$$

(3)

From the geometry shown in figure 1,

$$
l^2 + (s - r)^2 = s^2
$$

(4a)

$$
(r_1 + s - r)^2 + z^2 = s^2
$$

(4b)

where $r_1 = (x^2 + y^2)^{1/2}$.

Substituting equation (3) into equation (4a) yields

$$
s = \frac{ab\sqrt{x^2 + y^2}}{2\sqrt{(bx)^2 + (ay)^2}} + \frac{l^2\sqrt{(bx)^2 + (ay)^2}}{2ab\sqrt{x^2 + y^2}}
$$

(5)
From equations (3), (4b), and (5) we obtain

\[ z = \sqrt{r^2 - x^2 - y^2} - \frac{l^2 \sqrt{b^2 x^2 + a^2 y^2}}{ab} + \frac{ab(x^2 + y^2)}{\sqrt{b^2 x^2 + a^2 y^2}} \]

(6)

The ogive-nose surface geometry function is given by

\[ f(x, y, z) = x^2 + y^2 + (y^2 + \frac{l^2 \sqrt{b^2 x^2 + a^2 y^2}}{ab} - \frac{ab(x^2 + y^2)}{\sqrt{b^2 x^2 + a^2 y^2}}) = l^2 = 0 \]

(7)

2.1.2. Conical-nose projectile. Figure 2 shows the geometry of the conical-nose projectile. The geometric parameters of the conical-nose projectile are defined to be identical to those of the ogive-nose projectile shown in figure 1. The only difference is that the generatrix of the projectile nose is a straight line.

![Conical-nose projectile geometry](image)

**Figure 2.** Conical-nose projectile geometry.

From the geometry shown in figure 2, we obtain

\[ r = \frac{l \sqrt{x^2 + y^2}}{l - z} \]

(8)

Substituting equation (8) into equation (3) yields

\[ z = \left( 1 - \frac{\sqrt{b^2 x^2 + a^2 y^2}}{ab} \right) l \]

(9)

The conical-nose surface geometry function is given by

\[ f(x, y, z) = \left( \frac{\sqrt{b^2 x^2 + a^2 y^2}}{ab} - 1 \right) l + z = 0 \]

(10)

2.1.3. Truncated-ogive-nose projectile. Figure 3 shows the geometry of the truncated-ogive-nose projectile. The geometric parameters of the truncated-ogive-nose projectile are defined to be identical
to those of the ogive-nose projectile shown in figure 1; however, $l$ represents the actual nose length and $l_N$ represents the total nose length. The truncated-ogive-nose projectile has the same nose surface geometry function, i.e., equation (7), as the ogive-nose projectile, but $l$ needs to be substituted by $l_N$. $a_1$ and $b_1$ are respectively the semi-major and semi-minor axis of the terminal cross-sectional quasi-ellipse [which can be proven using equation (7) that the $z = l$ cross section is not an ellipse but is very close to an ellipse].

The geometry shown in figure 3 yields

$$\left( s - a \right)^2 + l_N^2 = s^2$$  \hspace{1cm} (11a)

$$\left( s - a + a_1 \right)^2 + l^2 = s^2$$  \hspace{1cm} (11b)

Equation (11) yields

$$l_N = \sqrt{\frac{a \left( l^2 - aa_1 + a_1^2 \right)}{a - a_1}}$$  \hspace{1cm} (12)

Substituting $l = l_N$, $z = l$, and $x = 0$ into equation (7) gives

$$b_1 = \frac{b^2 - l_N^2 + \sqrt{l_N^2 - b^2} + 4b^2 \left( l_N^2 - l^2 \right)}{2b}$$  \hspace{1cm} (13)

Equations (12) and (13) indicate that $l_N$ is a function of $a$, $a_1$, and $l$, whereas $b_1$ is a function of $b$, $l$, and $l_N$. Hence, the nose of the truncated-ogive-nose projectile can be characterized by $a$, $b$, $a_1$, and $l$.

2.1.4. Truncated-conical-nose projectile. Figure 4 shows the geometry of the truncated-conical-nose projectile. The geometric parameters of the truncated-conical-nose projectile are defined to be identical to those of the projectiles shown in figures 2 and 3. The truncated-conical-nose projectile has the same nose surface geometry function, i.e., equation (10), as the conical-nose projectile, but $l$ also needs to be substituted by $l_N$.
From the geometry shown in figure 4, we obtain

\[ \frac{a_1}{a} = \frac{b_1}{b} = \frac{l_N - l}{l_N} \]  

Equation (14) yields

\[ l_N = \frac{a l}{a - a_1} \]  

Equations (15) and (16) indicate that the nose of the truncated-conical-nose projectile can also be characterized by \( a, b, a_1, \) and \( l \).

2.2. Resistance of the elliptical cross-sectional projectiles

2.2.1. Ogive-nose projectile. Normal expansion velocity \( V_n \) of the projectile–target interface caused by the projectile penetrating with rigid-body velocity \( V \) is expressed as

\[ V_n = V \cos \phi \]  

where

\[ \cos \phi = \frac{f_x(x, y, z)}{\sqrt{f_x^2(x, y, z) + f_y^2(x, y, z) + f_z^2(x, y, z)}} \]  

Equations (15) and (16) indicate that the nose of the truncated-conical-nose projectile can also be characterized by \( a, b, a_1, \) and \( l \).

Figure 4. Truncated-conical-nose projectile geometry.

From the geometry shown in figure 4, we obtain

\[ \frac{a_1}{a} = \frac{b_1}{b} = \frac{l_N - l}{l_N} \]  

Equation (14) yields

\[ l_N = \frac{a l}{a - a_1} \]  

Equations (15) and (16) indicate that the nose of the truncated-conical-nose projectile can also be characterized by \( a, b, a_1, \) and \( l \).

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Equations (15) and (16) indicate that the nose of the truncated-conical-nose projectile can also be characterized by \( a, b, a_1, \) and \( l \).
\[ f_i(x, y, z) = 2y + \frac{a'l^2y}{b\sqrt{(bx)^2 + (ay)^2}} - \frac{2ab}{\sqrt{(bx)^2 + (ay)^2}} + a'by(x^2 + y^2)\left[\frac{(bx)^2 + (ay)^2}{(bx)^2 + (ay)^2}\right]^{3/2} \] (19b)
\[ f_i(x, y, z) = 2z \] (19c)

According to the spherical dynamic cavity-expansion theory, normal stress \( \sigma_n \) acting on the projectile nose and cavity-expansion velocity \( V_n \) satisfy the following relationship in [6]:

\[ \sigma_n = R + \rho_1 V_n^2 \] (20)

where \( \rho_1 \) and \( R \) are the density and resistance parameter of the concrete target, respectively. According to Frew et al. [9], \( R = S\ell_c \), where \( S \) and \( \ell_c \) are the empirical constant and unconfined compressive strength of the concrete target, respectively. \( S = 82.6\ell_c^{-0.544} \) and \( S = 72.0\ell_c^{-0.5} \), as presented by Frew et al. [9] and Li and Chen [12], respectively. We choose \( S = 82.6\ell_c^{-0.544} \) in the present study.

Because the sliding frictional resistance of the projectile–target interface was considered by Frew et al. [9] when fitting \( S \), it is neglected in the present study. Substituting equation (17) into equation (20) and integrating yield the axial resistance of the projectile nose, which is given by

\[ F_z = \iiint_{\Omega} \sigma_z dS_n = \iiint_{\Omega} \cos \phi \left[ R + \rho_1 (V \cos \phi)^2 \right] dS_n \] (21)

where \( \Omega \) represents the entire ogive-nose surface and \( \sigma_z \) is the axial stress acting on the projectile nose.

By converting the surface integral into a double integral, equation (21) is transformed into

\[ F_z = \iint_{x^2 + y^2 \leq l^2} \cos \phi \left[ R + \rho_1 (V \cos \phi)^2 \right] \sqrt{1 + z_1^2(x, y) + z_2^2(x, y)} dxdy \] (22)

where \( z_1(x, y) \) and \( z_2(x, y) \) are the derivatives of \( z(x, y) \) with respect to \( x \) and \( y \), respectively.

Differentiating both sides of equation (6) with respect to \( x \) and \( y \), respectively, yields the following:

\[ z_1(x, y) = \frac{1}{2} \left[ l^2 - x^2 - y^2 - \frac{l^2\sqrt{(bx)^2 + (ay)^2}}{ab} + \frac{ab(x^2 + y^2)}{\sqrt{(bx)^2 + (ay)^2}} \right]^{-1/2} \]
\[ \times \left\{ -2x - \frac{bl^2x}{a\sqrt{(bx)^2 + (ay)^2}} + \frac{2abx}{\sqrt{(bx)^2 + (ay)^2}} - \frac{ab^3(x^2 + y^2)((bx)^2 + (ay)^2)^{3/2}}{ab^3(x^2 + y^2)((bx)^2 + (ay)^2)^{3/2}} \right\} \] (23a)
\[ z_2(x, y) = \frac{1}{2} \left[ l^2 - x^2 - y^2 - \frac{l^2\sqrt{(bx)^2 + (ay)^2}}{ab} + \frac{ab(x^2 + y^2)}{\sqrt{(bx)^2 + (ay)^2}} \right]^{-1/2} \]
\[ \times \left\{ -2y - \frac{al^2y}{b\sqrt{(bx)^2 + (ay)^2}} + \frac{2aby}{\sqrt{(bx)^2 + (ay)^2}} - \frac{ab^3a^3(x^2 + y^2)((bx)^2 + (ay)^2)^{3/2}}{ab^3a^3(x^2 + y^2)((bx)^2 + (ay)^2)^{3/2}} \right\} \] (23b)

To more conveniently solve equation (22), the generalized polar-coordinate transformation is performed as follows:

\[ \begin{cases} x = a\rho \cos \alpha \\ y = b\rho \sin \alpha \end{cases} \] (24)
where $0 \leq \rho \leq 1, \ 0 \leq \alpha \leq 2\pi$.

Substituting equation (24) into equations (18) and (19) yields

$$\cos \varphi = \frac{2(1-\rho)^{1/2} l_i}{\left\{ \frac{\cos^2 \alpha \left[ 2a^2 (\rho - 1) + l_i^2 \right]}{a^2} + \frac{\sin^2 \alpha \left[ 2b^2 (\rho - 1) + l_i^2 \right]}{b^2} + 4(1-\rho) l_i^2 \right\}^{1/2}}$$

(25)

where $l_i$ is a length variable and is defined to simplify the equations in this study as follows:

$$l_i = \sqrt{l_i^2 + a^2 \rho \cos^2 \alpha + b^2 \rho \sin^2 \alpha}$$

(26)

Substituting equation (24) into equation (23) gives

$$z_x (\rho, \alpha) = \frac{\cos \alpha \left[ 2a^2 (1-\rho) - l_i^2 \right]}{2a \left[ (1-\rho) (l_i^2 + a^2 \rho \cos^2 \alpha + b^2 \rho \sin^2 \alpha) \right]^{1/2}}$$

(27a)

$$z_y (\rho, \alpha) = \frac{\sin \alpha \left[ 2b^2 (1-\rho) - l_i^2 \right]}{2b \left[ (1-\rho) (l_i^2 + a^2 \rho \cos^2 \alpha + b^2 \rho \sin^2 \alpha) \right]^{1/2}}$$

(27b)

According to the polar-coordinate transformation rule of double integration, equation (22) becomes

$$F_z = \int_0^{2\pi} \int_0^1 \cos \varphi \left[ R + \rho \left( V \cos \varphi \right)^2 \right] \sqrt{1 + z_x^2 (\rho, \alpha) + z_y^2 (\rho, \alpha)} \rho d\rho d\alpha$$

(28)

Equation (28) takes the following form:

$$F_z = A + BV^2$$

(29)

where $A$ and $B$ are the static resistance and dynamic resistance coefficients, respectively.

Substituting equations (25) and (27) into equation (28) and combining them with equation (29) yields

$$A = Rab \int_0^{2\pi} \int_0^1 \frac{2(1-\rho)^{1/2} l_i}{\left\{ \frac{\cos^2 \alpha \left[ 2a^2 (\rho - 1) + l_i^2 \right]}{a^2} + \frac{\sin^2 \alpha \left[ 2b^2 (\rho - 1) + l_i^2 \right]}{b^2} + 4(1-\rho) l_i^2 \right\}^{1/2}} \times \left\{ 1 + \frac{1}{4(1-\rho) l_i^2} \left[ \frac{\cos^2 \alpha \left[ 2a^2 (\rho - 1) + l_i^2 \right]}{a^2} + \frac{\sin^2 \alpha \left[ 2b^2 (\rho - 1) + l_i^2 \right]}{b^2} \right]^{1/2} \right\} \rho d\rho d\alpha$$

(30a)
2.2.2. Conical-nose projectile. For the conical-nose projectile, $f_x(x, y, z), f_y(x, y, z),$ and $f_z(x, y, z)$ are different from those of the ogive-nose projectile, i.e.,

$$f_x(x, y, z) = \frac{bx}{a\sqrt{(bx)^2 + (ay)^2}}$$  
(31a)

$$f_y(x, y, z) = \frac{ay}{b\sqrt{(bx)^2 + (ay)^2}}$$  
(31b)

$$f_z(x, y, z) = 1$$  
(31c)

Correspondingly, $z_x(x, y)$ and $z_y(x, y)$ of the conical-nose projectile are different from those of the ogive-nose projectile, i.e.,

$$z_x(x, y) = \frac{bx}{a\sqrt{(bx)^2 + (ay)^2}}$$  
(32a)

$$z_y(x, y) = \frac{ay}{b\sqrt{(bx)^2 + (ay)^2}}$$  
(32b)

The generalized polar-coordinate transformation is performed similar to equation (24), which yields

$$\cos \phi = \frac{1}{\sqrt{\left(\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}\right) + 1}}$$  
(33)

$$z_x(\rho, \alpha) = \frac{\rho \cos \alpha}{a}$$  
(34a)

$$z_y(\rho, \alpha) = \frac{\rho \sin \alpha}{b}$$  
(34b)

Substituting equations (33) and (34) into equation (28) and combining them with equation (29) yields

$$A = R \pi ab$$  
(35a)

$$B = \frac{\rho^2 ab}{2} \int_0^{2\pi} \int_0^1 \frac{1}{l^2} \left(\frac{\cos^2 \alpha}{a^2} + \frac{\sin^2 \alpha}{b^2}\right) + 1 \, d\alpha$$  
(35b)
2.2.3. **Truncated-ogive-nose projectile.** For the truncated-ogive-nose projectile, the solution process of the resistance is similar to that of the ogive-nose projectile. The only difference is that the range of \( \rho \) is from \( a_1/a \) to one for the double integration. The analytical solutions for \( A \) and \( B \) are as follows:

\[
A = R ab \int_0^{2\pi} \int_0^1 \frac{2(1-\rho)^{1/2} l_z}{a^2} \left[ \cos^2 \alpha \left( 2a^2 (\rho - 1) + l_z^2 \right)^2 + \sin^2 \alpha \left( 2b^2 (\rho - 1) + l_z^2 \right)^2 + 4(1-\rho)l_z^2 \right]^{1/2} \rho d\rho d\alpha \\
+ R \pi a_i b^2 - l_n^2 + \sqrt{l_n^2 - b^2} + 4b^2 (l_n^2 - l^2) \\
\frac{2b}{8(1-\rho)^{1/2} l_z^2}
\]

\[
B = \rho ab \int_0^{2\pi} \int_0^1 \left[ \cos^2 \alpha \left( 2a^2 (\rho - 1) + l_z^2 \right)^2 + \sin^2 \alpha \left( 2b^2 (\rho - 1) + l_z^2 \right)^2 + 4(1-\rho)l_z^2 \right]^{1/2} \rho d\rho d\alpha \\
+ \rho \pi a_i b^2 - l_n^2 + \sqrt{l_n^2 - b^2} + 4b^2 (l_n^2 - l^2) \\
\frac{2b}{8(1-\rho)^{1/2} l_z^2}
\]

(36a, 36b)

where

\[
l_z = \sqrt{l_n^2 + a^2 \rho \cos^2 \alpha + b^2 \rho \sin^2 \alpha} 
\]

(37)

2.2.4. **Truncated-conical-nose projectile.** For the truncated-conical-nose projectile, the solution process of the resistance is similar to that of the conical-nose projectile. The only difference is that the range of \( \rho \) is also from \( a_1/a \) to one for the double integration. The analytical solutions for \( A \) and \( B \) are as follows:

\[
A = R \pi ab \int_0^{2\pi} \int_0^1 \frac{1}{a} \left( a^2 - a_1^2 \right) b^2 \left( \cos^2 \alpha + \sin^2 \alpha \right) + 1 d\alpha + \rho \pi a_i^2 b^2 
\]

(38a, 38b)

2.3. **Penetration process**

Consider a projectile impacting a concrete target normally with initial velocity \( V_0 \). If the thickness of the concrete target is sufficiently large, the projectile stops and is embedded in the target medium, and the two-stage (cratering + tunneling) penetration model of the circular cross-sectional ogive-nose projectile is always utilized for thick concrete targets \([6,12]\). A conical crater with depth \( kd \) and a tunnel with diameter \( d \), \( k = 2.0 \) and \( k = 0.707 + l/d \) are suggested by Forrestal et al. \([6]\) and Li et al.
respectively. In the present study, the two-stage penetration model is also utilized for the elliptical cross-sectional projectile, as shown in figure 5, where $H$ is the height of the concrete target, $H^*$ is the final penetration depth, and $d = 2(ab)^{1/2}$ is the equivalent diameter.

![Figure 5. Schematic of normal penetration of concrete target by an elliptical cross-sectional projectile.](image)

The axial resistance of the projectile nose during the cratering and tunneling processes are given as follows [6]:

\begin{align}
F_z &= cz_1, \quad 0 \leq z_1 \leq kd \quad (39a) \\
F_z &= A + BV^2, \quad kd < z_1 \leq H^* \quad (39b)
\end{align}

where $c$ is a constant and $z_1$ is the instantaneous penetration depth.

The deceleration versus time relationship is expressed as

\begin{align}
\frac{dV(t)}{dt} &= -\omega V_0 \sin \omega t, \quad 0 \leq z_1 \leq kd \quad (40a) \\
\frac{dV(t)}{dt} &= -\frac{m \cos^2 \theta}{A} \left[ \tan^{-1} \left( \frac{B V}{A} \right) - \frac{\sqrt{AB}}{m} \left( t - t_1 \right) \right], \quad kd < z_1 \leq H^* \quad (40b)
\end{align}

where $m$ is the projectile mass.

The rigid-body velocity versus time relationship is given by

\begin{align}
V(t) &= V_0 \cos \omega t, \quad 0 \leq z_1 \leq kd \quad (41a) \\
V(t) &= \sqrt{A B} \tan \left[ \tan^{-1} \left( \frac{B V}{A} \right) - \frac{\sqrt{AB}}{m} \left( t - t_1 \right) \right], \quad kd < z_1 \leq H^* \quad (41b)
\end{align}

The displacement versus time relationship is expressed as

\begin{align}
z_1(t) &= \frac{V_0}{\omega} \sin \omega t, \quad 0 \leq z_1 \leq kd \quad (42a)
\end{align}
The final penetration depth is given by

$$H^* = \frac{m}{2B} \ln \left( 1 + \frac{BV_0^2}{A} \right) + kd$$

(43)

In these equations, we have

$$c = \frac{A + BV_0^2}{kd + Bk^2d^2/m}$$

(44a)

$$\omega = (c/m)^{1/2}$$

(44b)

$$V_1 = \left( V_0^2 - Akd/m \right) \left( 1 + Bkd/m \right)^{1/2}$$

(44c)

$$t_1 = \frac{\cos^{-1} (V_1/V_0)}{(c/m)^{1/2}}$$

(44d)

The total penetration time is given by

$$t_2 = \frac{m}{\sqrt{AB}} \tan^{-1} \left( \frac{B}{\sqrt{A}V_1} \right) + t_1$$

(45)

3. Validation

This section presents the use of the nose surface geometry functions, penetration depth predictions of the circular cross-sectional projectile, and experimental analyses of the elliptical cross-sectional ogive-nose projectiles to validate the present models.

3.1. Nose surface geometry functions

According to the nose surface geometry functions presented in Section 2.1, the nose shapes of the elliptical cross-sectional projectiles with $a = 34$ mm, $b = 17$ mm, $a_1 = 6$ mm, and $l = 79.7$ mm are accurately drawn, as shown in figure 6, which indicate that the nose surface geometry functions and geometric modeling process are correct.
3.2. Penetration depth predictions of the circular cross-sectional projectiles

When the semi-major axis is equal to the semi-minor axis, the elliptical cross-sectional projectile degenerates into a circular cross-sectional projectile. The penetration depths of the circular cross-sectional projectiles can be calculated using the semi-empirical formulae given in [6] and [11]. Then, they are employed to validate the present models.

The calculation parameters of the projectiles and concrete targets are as follows: \(a = b = 24\) mm, \(a_1 = 6\) mm, \(m = 1.8\) kg, \(l = 63.6, 79.7, 93.1,\) and 104.8 mm for CRH = 2, 3, 4, and 5, respectively, of the circular cross-sectional ogive-nose projectile. \(\rho_t = 2450\) kg/m\(^3\) and \(f_c = 40\) MPa.

The penetration depths of the circular cross-sectional projectiles can be calculated using equation (43) and the semi-empirical formulae given in [6] and [11]. Figure 7 shows the comparison of the predictions between the present models and semi-empirical formulae. We can observe that the penetration depth predictions of the present models are consistent with the results of the semi-empirical formulae. Thus, the present models are confirmed to incorporate the circular cross-sectional projectile penetration depth models of Forrestal et al. [6] and Chen et al. [11].
3.3. Experimental analyses of the elliptical cross-sectional ogive-nose projectiles

In the current literature, only the test data of the penetration depth of the elliptical cross-sectional ogive-nose projectiles can be obtained, and these test data are presented in this section to validate the present model.

A collection of test data from [17]–[19] is compared with the penetration depth predictions of the present model, as listed in table 1, where \( L \) is the length of the projectile and \( \varepsilon \) is the deviation of the model prediction from the test data. The penetration depth predictions agree well with the test data, and the maximum deviation is 15.8%. In general, the deviations are within the allowable range.

| Type no. | Test no. | \( m \) (g) | \( a \) (mm) | \( b \) (mm) | \( \lambda \) | \( l \) (mm) | \( L \) (mm) | \( \rho_s \) (kg\( \cdot \)m\(^{-3} \)) | \( f_c \) (MPa) | \( V_0 \) (m\( \cdot \)s\(^{-1} \)) | \( H^* \) (m) | \( \varepsilon \) (%) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| T1-1    | 448.00  | 15.00   | 9.60    | 1.56    | 58.1    | 180     | 2150    | 40.00   | 755.0   | 0.573   | 0.617   | 7.7     |
| T1-2    | 450.00  | 15.00   | 9.60    | 1.56    | 58.1    | 180     | 2150    | 40.00   | 747.0   | 0.553   | 0.608   | 9.9     |
| T1-3    | 447.00  | 15.00   | 9.60    | 1.56    | 58.1    | 180     | 2150    | 40.00   | 748.0   | 0.472   | 0.451   | 2.4     |
| T2-1    | 451.00  | 15.00   | 12.00   | 1.25    | 58.1    | 180     | 2150    | 40.00   | 740.0   | 0.471   | 0.462   | 0.0     |
| T2-2    | 443.00  | 15.00   | 12.00   | 1.25    | 58.1    | 180     | 2150    | 40.00   | 740.0   | 0.471   | 0.462   | 0.0     |
| T2-3    | 451.00  | 15.00   | 12.00   | 1.25    | 58.1    | 180     | 2150    | 40.00   | 740.0   | 0.471   | 0.462   | 0.0     |
| T3-1    | 93.90   | 14.00   | 5.00    | 2.80    | 26.0    | 70      | 2400    | 40.00   | 805.0   | 0.101   | 0.093   | 7.9     |
| T3-2    | 93.90   | 14.00   | 5.00    | 2.80    | 26.0    | 70      | 2400    | 40.00   | 805.0   | 0.101   | 0.093   | 7.9     |
| T3-3    | 93.90   | 14.00   | 5.00    | 2.80    | 26.0    | 70      | 2400    | 40.00   | 824.0   | 0.347   | 0.294   | 15.3    |
| E1-1    | 222.43  | 12.12   | 8.08    | 1.50    | 65.7    | 120     | 2400    | 42.43   | 879.0   | 0.530   | 0.599   | 13.1    |
| E1-2    | 222.81  | 12.12   | 8.08    | 1.50    | 65.7    | 120     | 2400    | 42.43   | 879.0   | 0.530   | 0.599   | 13.1    |

Table 1. Comparison of the penetration depths of the elliptical cross-sectional ogive-nose projectiles.
Figure 8 shows the test data and model predictions from equation (43), using an average value of \( m = 0.448 \) kg for projectile type nos. T1 and T2, \( m = 0.22257 \) kg for projectile type no. E1, and \( m = 0.22228 \) kg for projectile type no. E2 in table 1. Although significant differences exist between the projectiles and striking velocities, the model predictions are in good agreement with the test data, especially for projectile type nos. T1 and T2, which have a larger size. Thus, the efficacy of the present model for elliptical cross-sectional ogive-nose projectile is verified by its overall agreement with the test data.

Figure 8. Test data and model predictions of the elliptical cross-sectional ogive-nose projectiles.

In this section, we present the validation of the nose surface geometry functions by accurately drawing the nose shapes. Moreover, according to the semi-empirical formulae and existing test data, the present models are validated by comparing the penetration depths.

4. Projectile motion prediction
According to equations (40)–(42), we can calculate the deceleration, velocity, and displacement of the four types of elliptical cross-sectional projectiles during the penetration process. The calculation parameters of the projectiles and concrete targets are identical to those presented in Section 3.2 except for \( a = 34 \) mm and \( b = 17 \) mm. As shown in figure 9, the nose length of the four types of elliptical cross-sectional projectiles is exactly the same. Although the truncated-ogive-nose and truncated-conical-nose projectiles are flat-headed, the nose are sharper than the ogive-nose and conical-nose projectiles. The striking velocity of the projectiles is 1200 m/s.

Figure 9. Side outline of the four types of elliptical cross-sectional projectiles.
Figure 10 shows the comparison between the deceleration and time of various elliptical cross-sectional projectiles. During the cratering phase, the deceleration continuously increases prior to reaching a plateau corresponding to the tunneling phase. In the tunneling phase, the deceleration exhibits a gradual decrease with time. As the projectile stops inside the target, the deceleration plateaus followed by a sudden drop to zero. Furthermore, the conical-nose projectile demonstrates the smallest deceleration, followed by the truncated-conical-nose projectile, the ogive-nose projectile, and finally the truncated-ogive-nose projectile. However, when $l = 63.6$ mm, the deceleration of the truncated-conical-nose projectile is slightly smaller than that of the conical-nose projectile. It shows that when the projectile nose is short, the sharper nose can make up for the disadvantage of the flat head. Moreover, the peak deceleration decreases with the increase in the nose length. In addition, the maximum peak deceleration is close to 100,000 g ($g = 9.81 \text{ m/s}^2$) for the truncated-ogive-nose projectile shown in figure 10(a), but the minimum peak deceleration is only 62,500 g for the conical-nose projectile shown in figure 10(d).

![Figure 10](image1.png)

**Figure 10.** Deceleration versus time of various elliptical cross-sectional projectiles.

Figure 11 shows the comparison between the velocity and time of various elliptical cross-sectional projectiles. The projectile velocity appears to almost linearly decrease with the penetration time, and it finally drops to zero. In addition, the velocity more slowly decreases, and the penetration process lasts longer for the conical-nose projectile, followed by the truncated-conical-nose projectile, the ogive-nose projectile, and finally the truncated-ogive-nose projectile. However, when $l = 63.6$ mm, the velocity of the truncated-conical-nose projectile slightly decreases and becomes slower than that of the conical-nose projectile, and the penetration time of the truncated-conical-nose projectile is slightly longer than that of the conical-nose projectile.

![Figure 11](image2.png)

**Figure 11.** Velocity versus time of various elliptical cross-sectional projectiles.
Figure 11. Velocity versus time of various elliptical cross-sectional projectiles.

Figure 12 shows the comparison between the displacement and time of various elliptical cross-sectional projectiles. The displacement appears to logarithmically increase with the penetration time. Furthermore, the conical-nose projectile demonstrates the deepest penetration depth, followed by the truncated-conical-nose projectile, the ogive-nose projectile, and finally the truncated-ogive-nose projectile. However, when \( l = 63.6 \) mm, the penetration depth of the truncated-conical-nose projectile is slightly deeper than that of the conical-nose projectile. Moreover, the penetration depth increases with the nose length, and the maximum penetration depth is approximately 1.4 m for the conical-nose projectile shown in figure 12(d).
5. Penetration performance

In this section, according to the present models, the comparative analysis of the penetration performance of the four types of elliptical cross-sectional projectiles with various striking velocities is presented.

Figure 13 shows the comparison between the penetration depth and striking velocity of various elliptical cross-sectional projectiles from equation (43). The calculation parameters of the projectiles and concrete targets are identical to those presented in Section 3.2 except for $a = 34$ mm and $b = 17$ mm.

The results in figure 13 show that the penetration depth exponentially increases with the striking velocity and increases with the increase in the nose length. Furthermore, the conical-nose projectile achieves the deepest penetration depth, followed by the truncated-conical-nose projectile, the ogive-nose projectile, and finally the truncated-ogive-nose projectile. However, when $l = 63.6$ mm, the penetration depth of the truncated-conical-nose projectile is slightly larger than that of the conical-nose projectile. Moreover, as the nose length increases, the penetration depth of the ogive-nose projectile gradually approaches or even exceeds that of the truncated-conical-nose projectile.
6. Discussion

In the present models, the erosion and deformation of the projectile are not considered. However, the erosion and deformation of projectiles has already been observed in penetration experiments. Local erosion and melting of the projectile surface have been investigated in experiments conducted by Forrestal et al. [7] and Frew et al. [9]. The ogive nose becomes blunt, and the mass loss is approximately 7% when the striking velocity reaches up to 1200 m/s. According to [21], in penetration tests using 45# steel projectiles and mortar targets with a compressive strength of 50 MPa, apparent erosions and deformations may occur when the impacting velocity exceeds 710 m/s. According to [22], the 30CrMnSiNi2A high-strength steel alloy projectiles can be treated as rigid bodies when the striking velocity is slower than 1402 m/s, and they are eroded but not deformed when the striking velocity reaches 1596 m/s. In addition, the effects of erosion and deformation of the circular cross-sectional ogive-nose projectiles were investigated in [23]–[32]. Considering the effects of the varying nose factor and mass of the projectile, a modified model was proposed by Zhao et al. [25] to calculate the penetration depth. To improve the present models, the effects of erosion and deformation of the elliptical cross-sectional projectiles on the penetration process with high striking velocity will be investigated in the future.

In addition, a series of experiments for 1.8-kg elliptical cross-sectional projectiles that penetrate into concrete targets will be conducted in the future. With regard to the predictions made by the present models, the peak deceleration is approximately 100,000 g at a striking velocity of \( V_0 = 1200 \) m/s into an \( f_c = 40 \) MPa concrete target, as shown in figure (9). Thus, the measurement range of the accelerometers would be slightly larger than the peak deceleration predicted by the present models. Meanwhile, the onboard recorder and accelerometers must be structurally mounted within the projectiles to measure the projectile motion and transient structural response. Finally, we can obtain the projectile deceleration by filtering the deceleration–time data and further verify the present models using the test data.

The present models contain dimensionless empirical parameter \( S \) that was proposed by Frew et al. [9], and \( S \) is determined from the comparison between the penetration depth and striking velocity data of circular cross-sectional ogive-nose projectiles. Moreover, \( S \) suffers from certain application limitations as described by Frew et al. [9]. Exceeding these limitations may result in inaccurate predictions. Although the predictions made by the present model for elliptical cross-sectional ogive-nose projectiles are in good agreement with the test data in [17]–[19], if we want to obtain more accurate predictions under other conditions, especially beyond the application limitations of \( S \) in [9], more test data must be obtained to refit \( S \).
7. Conclusions

Theoretical models of a rigid elliptical cross-sectional projectile with different geometrical characteristics penetration into a semi-infinite concrete target are presented based on the dynamic cavity-expansion theory. Moreover, comparative analysis of the motion and penetration performance on four types of elliptical cross-sectional projectiles is conducted based on the present models. The following main conclusions are drawn:

(1) The penetration equations of the four types of elliptical cross-sectional projectiles are in closed form and depend on the striking velocity, geometry, projectile mass, and material properties of the concrete target.

(2) The nose surface geometry functions are validated by accurately drawing the nose shapes. Furthermore, the penetration depth predictions of the present models are consistent with the results of the semi-empirical formulae for circular cross-sectional projectiles. Moreover, the present model is validated by comparing the predicted penetration depths with the test data for elliptical cross-sectional ogive-nose projectiles, and the maximum deviation is 15.8%.

(3) The tendency of the deceleration–time, velocity–time, and displacement–time curves of the four types of elliptical cross-sectional projectiles are similar, and the differences lie in the deceleration amplitude, penetration time, and penetration depth. When the nose length is sufficiently large, the conical-nose elliptical cross-sectional projectile tends to be subjected to lower deceleration, exhibits a slower drop in velocity during the penetration process, and achieves the deepest penetration depth.

(4) The conical-nose elliptical cross-sectional projectile achieves the best penetration performance than the other three types of elliptical cross-sectional projectiles if the nose length is sufficiently large, and the penetration performance of the ogive-nose elliptical cross-sectional projectile gradually approaches or even exceeds that of the truncated-conical-nose elliptical cross-sectional projectile with the increase in the nose length.

To further verify the present models, more systematic penetration experiments for a 1.8-kg elliptical cross-sectional projectile with different geometrical characteristics are urgently needed, and this will be the focus of our future study.

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References

[1] Bishop R, Hill R and Mott N 1945 The theory of indentation and hardness tests Proc Phys Soc 147-59
[2] Hill R 1948 A theory of earth movement near a deep underground explosion Memo No 21-48 (Armament Research Establishment, Front Halstead, Kent, UK)
[3] Hopkins H 1960 Dynamic expansion of spherical cavities in metals Progress in solid mechanics vol 1 (New York: North Holland) p 85-164
[4] Goodier J 1965 On the mechanics of indentation and cratering in solid targets of strain-hardening metals by impact of hard and soft sphere Proc. of the 7th Symp Hypervel Impact,III p 215-59
[5] Forrestal M and Tzou D 1997 A spherical cavity-expansion penetration model for concrete targets Int J Solids Struct 4127-46
[6] Forrestal M, Altman B and Cargile J 1994 An empirical equation for penetration depth of ogive-nose projectiles into concrete targets Int J Impact Eng 4 395-405
[7] Forrestal M, Frew D, Hanchak S and Brar N 1996 Penetration of grout and concrete targets with ogive-nose steel projectiles Int J Impact Eng 5 465-76
[8] Forrestal M, Frew D, Hickerson J and Rohwer T 2003 Penetration of concrete targets with deceleration-time measurements Int J Impact Eng 5 479-97
[9] Frew D, Hanchak S, Green M and Forrestal M 1998 Penetration of concrete targets with ogive-nose steel rods Int J Impact Eng 6 489-97
[10] Frew D, Forrestal M and Cargile J 2006 The effect of concrete target diameter on projectile deceleration and penetration depth Int J Impact Eng 10 1584-94
[11] Chen X and Li Q 2002 Deep penetration of a rigid projectile with different geometrical characteristics Int J Impact Eng 6 619-37
[12] Li Q and Chen X 2003 Dimensionless formulae for penetration depth of concrete target impacted by a rigid projectile Int J Impact Eng 1 93-116
[13] Chen X, Fan S and Li Q 2004 Oblique and normal perforation of concrete targets by a rigid projectile Int J Impact Eng 6 617-37
[14] Li Q, Reid S, Wen H and Telford A 2005 Local impact effects of hard missiles on concrete targets Int J Impact Eng 1-4 224-84
[15] Peng Y, Wu H, Fang Q, Gong Z and Kong X 2015 A note on the deep penetration and perforation of hard projectiles into thick targets Int J Impact Eng 37-43
[16] Kong X, Wu H, Fang Q and Peng Y 2017 Rigid and eroding projectile penetration into concrete targets based on an extended dynamic cavity-expansion theory Int J Impact Eng 13-22
[17] Guo L, He Y, Pan X, He X, Tu J and Qiao L 2019 Experimental and numerical investigation on the penetration mechanism of non-circular projectile into concrete Ordnance Material Science and Engineering 3 62-7
[18] Wang W, Zhang X, Deng J, Zheng Y and Liu C 2018 Analysis of projectile penetrating into mortar target with elliptical cross-section Explosive Shock Waves 1 164-73
[19] Dong H, Liu Z, Wu H, Gao X, Pi A and Huang F 2019 Study on penetration characteristics of high-speed elliptical cross-sectional projectiles into concrete Int J Impact Eng 1-12
[20] Liu Z, Wu H, Gao X, Pi A and Huang F 2019 Study on the resistance characteristics of elliptical cross-section ogive-nose projectile penetrating concrete Transactions of Beijing Institute of Technology 2 135-46
[21] Kong X, Wu H, Fang Q, Zhang W and Xiao Y 2017 Projectile penetration into mortar targets with a broad range of striking velocities: Test and analyses Int J Impact Eng 18-29
[22] Feng J, Song M, Sun W, Wang L, Li W and Li W 2018 Thick plain concrete targets subjected to high speed penetration of 30CrMnSiNi2A steel projectiles: Tests and analyses Int J Impact Eng 305-17
[23] Silling S and Forrestal M 2007 Mass loss from abrasion on ogive-nose steel projectiles that penetrate concrete targets Int J Impact Eng 11 1814-20
[24] Chen X, He L and Yang S 2010 Modeling on mass abrasion of kinetic energy penetrator European Journal of Mechanics A/Solids 1 7-17
[25] Zhao J, Chen X, Jin F and Xu Y 2010 Depth of penetration of high-speed penetrator with including the effect of mass abrasion Int J Impact Eng 9 971-9
[26] Zhao J, Chen X, Jin F and Xu Y 2012 Analysis on the bending of a projectile induced by asymmetrical mass abrasion Int J Impact Eng 1 16-27
[27] He L and Chen X 2011 Analyses of the penetration process considering mass loss European Journal of Mechanics A/Solids 2 145-57
[28] He L, Chen X and He X 2010 Parametric study on mass loss of penetrators Acta Mech Sin 4 585-97
[29] He L and Chen X 2012 Simulation of shape variation of projectile nose during high-speed penetration into concrete Theor. Appl. Mech. Lett. 2 021006
[30] Beissel S and Johnson G 2000 An abrasion algorithm for projectile mass loss during penetration Int J Impact Eng 2 103-16
[31] Beissel S and Johnson G 2002 A three-dimensional abrasion algorithm for projectile mass loss during penetration Int J Impact Eng 7 771-89
[32] Qian F, Wu H and Huang F 2014 Projectile Mass Loss Model Using the Coefficient of Friction for High-Speed Penetration into Concrete Applied Mechanics and Materials 365-70