Calculation of Rayleigh type sums for zeros of the equation arising in spectral problem

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Abstract. For zeros of the equation (arising in the oblique derivative problem)
\[ \mu J_n'(\mu) \cos \alpha + i n J_n(\mu) \sin \alpha = 0, \quad \mu \in \mathbb{C}, \]
with parameters \( n \in \mathbb{Z}, \alpha \in [-\pi/2, \pi/2] \) and the Bessel function \( J_n(\mu) \) special summation relationships are proved. The obtained results are consistent with the theory of well-known Rayleigh sums calculating by zeros of the Bessel function.

1. Introduction
In [1] the authors was studied the question on the complex location of eigenvalues \( \lambda = \mu^2 \) of following spectral problem for Laplace operator in disk \( D = \{(x, y) \mid x^2 + y^2 < 1\} \):
\[ \Delta w + \mu^2 w = 0 \quad \text{in} \quad D, \quad \frac{\partial w}{\partial l} = 0 \quad \text{on} \quad \partial D. \quad (1) \]
Here, \( \Delta \) is the Laplace operator, \( \mu^2 \in \mathbb{C} \) is a spectral parameter, \( l \) is a direction forming a fixed angle \( \alpha \in (-\pi/2, \pi/2) \) with outer normal \( n \) to \( \partial D \). Our interest in this subjects have been stimulated by a well-known work [2] in which the absence of the basis property in \( L_2(D) \) system of root functions of the problem (1) was proved. In [2] (see, e.g. [1]) have been shown that all eigenvalues of the spectral problem (1) are described by zeros of the equation
\[ \mu J_n'(\mu) \cos \alpha + i n J_n(\mu) \sin \alpha = 0, \quad n \in \mathbb{Z}, \quad (2) \]
with given \( \alpha \in (-\pi/2, \pi/2) \) and the Bessel function \( J_n(\mu) \) of variable \( \mu \in \mathbb{C} \). For the aims of the work we expand change area of parameters \( n, \alpha \). We assume that pairs \((n, \alpha)\) belong to the set of permissible values
\[ \mathcal{P} \equiv (\mathbb{Z} \times [-\pi/2, \pi/2]) \setminus (\{0\} \times [-\pi/2, \pi/2]). \quad (3) \]
For \((n, \alpha) \in \{0\} \times [-\pi/2, \pi/2]\) the equation (2) is degenerate and its zeros fill the whole complex plane. Introducing the set (3) we except such combination of parameters. It is also obviously

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that \( \mu = 0 \) is a zero (2) for any values \( n \) and \( \alpha \) in (3). The others zeros of equation (2) which we call non-trivial are of interest. For the permissible values of parameters when \( \alpha = 0 \) (or \( n = 0 \)) and \( \alpha = \pm \pi/2 \) zeros of corresponding equations \( \mu J'_\nu(\mu) = 0 \) and \( J'_\nu(\mu) = 0 \) are real and well studied (see [3, Ch. 15]). For the main case \( 0 < |\alpha| < \pi/2 \) from the general theory of entire functions and from the results of [1] follows that for a fixed \( n \neq 0 \) the equation (2) has an infinite countable set of non-trivial zeros at that all of them are simple, don’t lie on real and imaginary axes and are located on the complex plane symmetrically with respect to the point \( \mu = 0 \).

The report will present further results on the properties of zeros of the equation of form (2). To describe the the results we introduce the even entire function of exponential type

\[
L(\mu; n, \alpha) \equiv \frac{\mu J'_n(\mu) \cos \alpha + i \, n \, J_n(\mu) \sin \alpha}{(\mu/2)^n}, \quad \mu \in \mathbb{C}. \tag{4}
\]

For \( \mu \neq 0 \) the equation (2) is equivalent to the equation \( L(\mu; n, \alpha) = 0 \) with coincidence of the multiplicities of zeros. It was noted (see also section 2) that the function (4) has an infinite countable set of zeros. We denote the set by \( \{\pm \mu_{n,k}(\alpha)\}_{k \in \mathbb{N}} \), where \( \mu_{n,k}(\alpha) \) are located in half-plane \( \Re \mu > 0 \), rearranged on increasing of modules and \( \mu_{n,k}(\alpha) \to \infty \) at \( k \to \infty \). In view of the obvious chain of relationships

\[
L(\mu; n, \alpha) = 0 \iff L(\overline{\mu}; n, -\alpha) = 0 \iff L(\overline{\mu}; -n, \alpha) = 0 \iff L(\mu; -n, -\alpha) = 0 \tag{5}
\]

it is enough to study zeros \( \mu \in \mathbb{C} \) of the equation

\[
L(\mu; n, \alpha) = 0, \quad n \in \mathbb{N}, \quad \alpha \in [0, \pi/2]. \tag{6}
\]

Systematically investigating the problem on the location of zeros of the equation (6) on half-plane \( \mathbb{C} \) of variable \( \mu \), we have got explicit formulas for sums of special structure containing zeros \( \mu_{n,k}(\alpha) \). For example we established the following relationship

\[
\sum_{k=1}^{\infty} \frac{1}{\mu_{n,k}^2(\alpha)} = \frac{n + 2 \cos^2 \alpha}{4n(n+1)} - \frac{i \sin 2\alpha}{4n(n+1)} = \frac{n + 1 + \exp(-2\alpha \, i)}{4n(n+1)}, \quad n \in \mathbb{N}, \quad \alpha \in [0, \pi/2]. \tag{7}
\]

If \( \alpha = \pi/2 \) then the formula (7) coincides with the particular expression

\[
\sigma^{(2)}(n) \equiv \sum_{k=1}^{\infty} \frac{1}{j_{n,k}^2} = \frac{1}{4(n+1)}, \tag{8}
\]

for the known Rayleigh sums (see [4], [3], [5], [6])

\[
\sigma^{(2p)}(n) \equiv \sum_{k=1}^{\infty} \frac{1}{j_{n,k}^{2p}}, \quad p \in \mathbb{N},
\]

where \( j_{n,k} \) are positive zeros of the Bessel function \( J_n(\mu) \). If \( \alpha = 0 \) then the formula (7) again gives well-known expression (see [7], [8], [9])

\[
\sum_{k=1}^{\infty} \frac{1}{j_{n,k}^2} = \frac{n + 2}{4n(n+1)},
\]

where \( j'_{n,k} \) are positive zeros of the derivative \( J'_n(\mu) \) of the Bessel function.

The authors know a number of others studies (see, e.g. the bibliography in reviews [6], [10]), dedicated to close questions for the equation of form

\[
A \mu J'_\nu(\mu) + B \, J_\nu(\mu) = 0,
\]

but as a rule either with real parameters \( A, B, \nu \) or without explicit indication of the range of parameters. In this connection, even the formula (7) appears to be a new.

In the main part of the work we will found the basic properties of zeros of the entire function (4) and then we will obtain some summation relationships like the formula (7).

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2. The properties of zeros of the function \( L(\mu; n, \alpha) \)

Let us study in more detail the properties of zeros of the entire function \( L(\mu; n, \alpha) \) denoting it for simplifying through \( L(\mu) \) for fixed values \( n \in \mathbb{N}, \alpha \in [0, \pi/2] \). We briefly explain why the function \( L(\mu) \) has an infinite set of zeros. Using the expansion in the power series of the Bessel function by the coefficients of the Taylor series (see [11, p. 6]) we obtain that \( L(\mu) \) can be represented in the form

\[
L(\mu) = \sum_{k=0}^{\infty} \left( \frac{-1}{k!} \right)^k \frac{(2k + n) \cos \alpha + i n \sin \alpha}{(k + n)!} \left( \frac{\mu}{2} \right)^{2k}, \quad \mu \in \mathbb{C}. \tag{9}
\]

According to (9) and standard formulas for calculation of the order and the type of an entire function by the coefficients of the Taylor series (see [11, p. 6]) we obtain that \( L(\mu) \) is a function of the order \( \rho_L = 1 \) and of the type \( \sigma_L = 1 \). We suppose that the entire function \( L(\mu) \) has the finite number of zeros or doesn’t have them at all. It is well known that then \( L(\mu) \) can be represented in the form

\[
L(\mu) = e^{a\mu} P(\mu), \quad \mu \in \mathbb{C},
\]

where \( a \in \mathbb{C}, P(\mu) \) is a polynomial. As \( L(-\mu) = L(\mu) \) then \( a = 0 \) that is \( L(\mu) = P(\mu) \). But it contradicts to the expression (9). Since the zeros of an entire function are isolated, it follows that the set of zeros \( L(\mu) \) is countable and doesn’t have finite limit points (the only limit point is \( \mu = \infty \)). The parity \( L(\mu) \) leads to the symmetry of the zeros with respect to the point \( \mu = 0 \).

The simplicity of zeros of the function \( L(\mu) \) is an obvious consequence of the general fact about the simplicity of non-trivial zeros of the equation (2) that was proved in [16].

Now we show that all zeros of the equation (6) lie in some specific domain (depending on \( n, \alpha \), contained in the horizontal strip. We use the expansion of the Bessel function in the infinite product

\[
J_n(\mu) = \frac{1}{n!} \left( \frac{\mu}{2} \right)^n \prod_{k=1}^{\infty} \left( 1 - \frac{\mu^2}{j_{n,k}^2} \right), \quad \mu \in \mathbb{C},
\]

and we calculate its logarithmic derivative. We have

\[
\frac{J_n'(\mu)}{J_n(\mu)} = \sum_{k=1}^{\infty} \frac{2\mu}{\mu^2 - \nu_{n,k}} + \frac{n}{\mu}, \tag{10}
\]

where \( \nu_{n,k} \equiv j_{n,k}^2 \). Differentiating (10) we obtain the equality

\[
\left( \frac{J_n'(\mu)}{J_n(\mu)} \right)' = 2 \sum_{k=1}^{\infty} \frac{1}{\mu^2 - \nu_{n,k}} - 4\mu^2 \sum_{k=1}^{\infty} \frac{1}{(\mu^2 - \nu_{n,k})^2} - \frac{n}{\mu^2}. \tag{11}
\]

On the other hand the Bessel equation

\[
\mu^2 J_n''(\mu) + \mu J_n'(\mu) + (\mu^2 - n^2) J_n(\mu) = 0
\]

gives the relationship

\[
\left( \frac{J_n'(\mu)}{J_n(\mu)} \right)' = \frac{n^2 - \mu^2}{\mu^2} - \frac{J_n'(\mu)}{\mu J_n(\mu)} - \left( \frac{J_n'(\mu)}{J_n(\mu)} \right)^2. \tag{12}
\]

From (10)–(12) we conclude

\[
4\mu^2 \sum_{k=1}^{\infty} \frac{1}{(\mu^2 - \nu_{n,k})^2} = 4\mu^2 \left( \sum_{k=1}^{\infty} \frac{1}{\mu^2 - \nu_{n,k}} \right)^2 + 4(n + 1) \sum_{k=1}^{\infty} \frac{1}{\mu^2 - \nu_{n,k}} + 1.
\]
Replacing $\lambda = \mu^2$ and denoting

$$S^{(1)}(\lambda; n) \equiv \sum_{k=1}^{\infty} \frac{1}{\lambda - \nu_{n,k}}$$

and

$$S^{(2)}(\lambda; n) \equiv \sum_{k=1}^{\infty} \frac{1}{(\lambda - \nu_{n,k})^2},$$

we obtain the identity

$$4\lambda S^{(2)}(\lambda; n) = 4\lambda \left( S^{(1)}(\lambda; n) \right)^2 + 4(n + 1) S^{(1)}(\lambda; n) + 1. \quad (13)$$

If the parameters satisfy the conditions $n \in \mathbb{N}, \alpha \in [0, \pi/2)$, we rewrite the equation (2) in the equivalent form at $\mu \neq 0$:

$$\frac{J_n'(\mu)}{2\mu J_n(\mu)} + \frac{i n \tan \alpha}{2\mu^2} = 0. \quad (14)$$

Let $\mu = \mu_0$ be an any fixed non-trivial zero of the equation (2). Comparing (10) and (14) we see that

$$S^{(1)}(\lambda_0; n) = -\frac{n(1 + i \tan \alpha)}{2\lambda_0}, \quad (15)$$

where $\lambda_0 \equiv \mu_0^2 = \xi_0 + i\eta_0$. Substituting $\lambda_0$ in (13) and taking into account (15) after groups of transformations we obtain

$$4|\lambda_0|^4 \sum_{k=1}^{\infty} \frac{(\xi_0 - \nu_{n,k})^2 - \eta_0^2}{|\lambda_0 - \nu_{n,k}|^4} = \left( \xi_0 - \frac{n^2}{\cos^2 \alpha} - 2n \right) \left( \xi_0^2 - \eta_0^2 \right) + 2\xi_0 \eta_0 (\eta_0 - 2n \tan \alpha),$$

and

$$8\eta_0 |\lambda_0|^4 \sum_{k=1}^{\infty} \frac{\xi_0 - \nu_{n,k}}{|\lambda_0 - \nu_{n,k}|^4} = 2\xi_0 \eta_0 \left( \xi_0 - \frac{n^2}{\cos^2 \alpha} - 2n \right) - (\eta_0 - 2n \tan \alpha)(\xi_0^2 - \eta_0^2).$$

Combining the two last formulas with the formula (15) we have the equality

$$8\eta_0^3 |\lambda_0|^4 \sum_{k=1}^{\infty} \frac{\nu_{n,k}}{|\lambda_0 - \nu_{n,k}|^4} = (2n \tan \alpha - \eta_0)|\lambda_0|^2 + \xi_0 \eta_0 \frac{n^2}{\cos^2 \alpha}. \quad (16)$$

By the results [1] the condition $\eta_0 \geq 0$ is satisfied. From here we have

$$(2n \tan \alpha - \eta_0)|\lambda_0|^2 + \xi_0 \eta_0 \frac{n^2}{\cos^2 \alpha} \geq 0. \quad (16)$$

The inequality (16) means that for fixed $n \in \mathbb{N}, \alpha \in [0, \pi/2)$ the squares of zeros of the equation (6) on the plane $\mathbb{C}_\lambda$, where $\lambda = \mu^2 = \xi + i\eta = r(\cos \varphi + i \sin \varphi)$, are located in the domain $\Omega$ specified by the conditions

$$r \sin \varphi \leq 2n \tan \alpha + \frac{n^2}{2 \cos^2 \alpha} \sin 2 \varphi, \quad 0 \leq \varphi \leq \pi/2. \quad (17)$$

From [1] it follows that in the condition (17) one can take $0 \leq \varphi \leq \alpha$. For $\lambda = \xi + i\eta \in \Omega$ we obviously have

$$0 \leq \eta \leq 2n \tan \alpha + \frac{n^2}{2 \cos^2 \alpha}. \quad (17)$$

In other words all points are located in the half-strip

$$0 \leq \text{Im} \lambda \leq 2n \tan \alpha + \frac{n^2}{2 \cos^2 \alpha}, \quad \text{Re} \lambda \geq 0.$$
of the plane $\mathbb{C}_\lambda$. Further, from (17) at the same $n, \alpha$ we extract that all zeros $\mu_{n,k}(\alpha)$ on the plane $\mathbb{C}_\mu$ where $\mu = \rho(\cos \psi + i \sin \psi)$ lie in the domain

$$0 \leq \text{Im} \mu \leq \sqrt{\left(n \tan \alpha + \frac{n^2}{4\cos^2 \alpha} \sin 4\psi\right) \tan \psi}, \quad 0 \leq \psi \leq \alpha/2, \quad \text{Re} \mu \geq 0. \quad (18)$$

Taking into account the relationship (5) from (18) now we easily obtain the following properties of zeros of the equations (2) and (6).

(i) For given $n \in \mathbb{Z}, \alpha \in (-\pi/2, \pi/2)$ all zeros of the equation (2) lie in the strip

$$|\text{Im} \mu| \leq \sqrt{|n|(|n| + 2\sin 2|\alpha|)\tan \psi}, \quad 0 \leq \psi \leq \alpha/2,$$

and for $n \in \mathbb{Z} \setminus \{0\}, \alpha = \pm \pi/2$ the equation (2) has only real zeros.

(ii) For fixed $n \in \mathbb{N}, \alpha \in (0, \pi/2)$ and $k \to \infty$ it’s satisfied $\text{Im} \mu_{n,k}(\alpha) \to 0$.

(iii) For fixed $(n,\alpha) \in \mathcal{P}$ and for any $h > 0$ outside the strip $|\text{Im} \mu| \leq h$ it lies not more than the finite number of zeros of the equation (2).

3. The Rayleigh type sums

As in the previous section for convenience of exposition we choose and fix parameters in the equation (2) so that $n \in \mathbb{N}, \alpha \in [0, \pi/2]$. In fact, this allow us to work with the equation (6). At the end of the section we give the general result.

By analogy with the Rayleigh sums [4] we define more general sums of the Rayleigh type

$$\eta^{(r)}(n,\alpha) \equiv \frac{1}{2} \sum \frac{1}{(\pm \mu_{n,k}(\alpha))^r}, \quad r \in \mathbb{N}, \quad (19)$$

where the summation is performed for all non-trivial zeros $\pm \mu_{n,k}(\alpha)$ of the equation (2). From the theory of entire functions it follows that the convergence exponent of the sequence of zeros of the function $L(\mu) = L(\mu; n, \alpha)$ doesn’t exceed a value of the order $\rho_L = 1$ (in our case these characteristics coincide). Thus, the series in (19) converges absolutely for $r > 1$. For all odd exponents $r = 2m - 1 \in \mathbb{N}$ the sum $\eta^{(2m-1)}(n,\alpha)$ is equal to zero. For even exponents $r = 2m \in \mathbb{N}$ we have

$$\eta^{(2m)}(n,\alpha) = \sum_{k=1}^{\infty} \frac{1}{\mu_{2m,k}(\alpha)}, \quad m \in \mathbb{N}. \quad (20)$$

We remind that through $\mu_{n,k}(\alpha)$ are denoted zeros of the equation (6) (non-trivial zeros of the equation (2)) located in the half-plane $\text{Re} \mu > 0$.

We set a problem of calculating of the sums (20). Above it was noted that $L(\mu)$ is the even entire function of exponential type with simple zeros $\{\pm \mu_{n,k}(\alpha)\}_{k \in \mathbb{N}}$. From the expansion $L(\mu)$ in the power series (9) we have

$$L(0) = \frac{\cos \alpha + i \sin \alpha}{(n-1)!} = \frac{\exp (\alpha i)}{(n-1)!}.$$

Representing $L(\mu)$ as the canonical product

$$L(\mu) = \frac{\exp (\alpha i)}{(n-1)!} \prod_{k=1}^{\infty} \left(1 - \frac{\mu^2}{\mu_{n,k}(\alpha)}\right), \quad (21)$$
and selecting its logarithmic derivative we write the identity

$$\frac{L'(\mu)}{2\mu L(\mu)} = \sum_{k=1}^{\infty} \frac{1}{\mu^2 - \mu^2_{n,k}(\alpha)}.$$  \hspace{1cm} (22)

We calculate of the value of the left hand side (22) at \( \mu = 0 \). Using the expansion (9) after standard transformations we find

$$\left. \frac{L'(\mu)}{2\mu L(\mu)} \right|_{\mu=0} = -\frac{(n + 2) \cos \alpha + i \, n \, \sin \alpha}{4n(n + 1) (\cos \alpha + i \, \sin \alpha)}.$$  

We substitute \( \mu = 0 \) in (22) and one obtain the relationship

$$\eta^{(2)}(n, \alpha) = \sum_{k=1}^{\infty} \frac{1}{\mu^2_{n,k}(\alpha)} = \frac{(n + 2) \cos \alpha + i \, n \, \sin \alpha}{4n(n + 1) (\cos \alpha + i \, \sin \alpha)} \equiv \frac{n + 1 + \exp(-2\alpha i)}{4n(n + 1)},$$

coinciding with (7).

To calculate the sums of the Rayleigh type of higher orders it is convenient to make a substitute \( t = \mu^2 \) in (22) and to briefly write

$$\sum_{k=1}^{\infty} \frac{1}{t - \mu^2_{n,k}(\alpha)} = -\frac{1}{4} \left. \frac{u(t)}{v(t)} \right|_{t=0},$$  \hspace{1cm} (23)

where

$$u(t) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} \frac{(n + 2k + 2) \cos \alpha + i \, n \, \sin \alpha}{(n + k + 1)!} \left( \frac{t}{4} \right)^k,$$

$$v(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(n + 2k) \cos \alpha + i \, n \, \sin \alpha}{(n + k)!} \left( \frac{t}{4} \right)^k.$$

Differentiating the necessary number of times for the equality (23) we obtain the following general formula for sums of the Rayleigh type

$$\eta^{(2m)}(n, \alpha) = \sum_{k=1}^{\infty} \frac{1}{\mu^2_{n,k}(\alpha)} = \frac{1}{4(m - 1)!} \left. \left( \frac{u(t)}{v(t)} \right)^{(m-1)} \right|_{t=0}, \quad m \in \mathbb{N}. \hspace{1cm} (24)$$

We exemplify to obtain the sums \( \eta^{(2m)}(n, \alpha) \) at \( m = 2, 3 \). We give only the results of calculations.

For \( m = 2 \) we have

$$\eta^{(4)}(n, \alpha) = \frac{1}{4} \left\{ \frac{u'(0)}{v(0)} - \frac{v'(0)u(0)}{v^2(0)} \right\},$$

whence

$$\eta^{(4)}(n, \alpha) \equiv \sum_{k=1}^{\infty} \frac{1}{\mu^4_{n,k}(\alpha)} = \frac{(n + 1)(n + 2) + 4(n + 1) \exp(-2\alpha i) + (n + 2) \exp(-4\alpha i)}{2^4 n^2 (n + 1)^2 (n + 2)}.$$  

In particular, for \( \alpha = 0 \) we have the formula

$$\sum_{k=1}^{\infty} \frac{1}{\mu^4_{n,k}} = \frac{n^2 + 8n + 8}{2^4 n^2 (n + 1)^2 (n + 2)}.$$
coinciding with the formula P. L. Kapitsa (see [8]). In the other extreme case \( \alpha = \pi/2 \) the obtained relationship is transformed in the well-known Rayleigh sum

\[
\sum_{k=1}^{\infty} \frac{1}{J_{n,k}^4} = \frac{1}{2^4 (n+1)^2 (n+2)} \equiv \sigma^{(4)}(n).
\]

For \( m = 3 \) we have

\[
\eta^{(6)}(n, \alpha) = \frac{1}{8} \left\{ \frac{u''(0)}{v(0)} - 2 \frac{v'(0)u'(0)}{v^2(0)} - \frac{v''(0)u(0)}{v^3(0)} + 2 \frac{v^2(0)u(0)}{v^3(0)} \right\},
\]

whence

\[
\eta^{(6)}(n, \alpha) = \frac{1}{\mu^6_{n,k}(\alpha)} = \frac{2(n+1)^2(n+3) + 3(n+1)(5n+6) e^{-2\alpha i} + 6(n+1)(n+3) e^{-4\alpha i} + (n+2)(n+3) e^{-6\alpha i}}{2^6 n^3 (n+1)^3 (n+2)(n+3)}.
\]

For \( \alpha = 0 \) it leads to the sum

\[
\sum_{k=1}^{\infty} \frac{1}{j_{n,k}^6} = \frac{n^3 + 16n^2 + 38n + 24}{2^5 n^3 (n+1)^3 (n+2)(n+3)},
\]

and for \( \alpha = \pi/2 \) it leads to the corresponding Rayleigh sum

\[
\sum_{k=1}^{\infty} \frac{1}{\mu^6_{n,k}(\alpha)} = \frac{1}{2^5 (n+1)^3 (n+2)(n+3)} \equiv \sigma^{(6)}(n).
\]

Let us give some expressions for special double sums. With fixed \( n \in \mathbb{N}, \alpha \in [0, \pi/2) \) for any zero \( \mu_{n,k}(\alpha) \) from (10), (14) we have

\[
\sum_{m=1}^{\infty} \frac{1}{\mu^2_{n,k}(\alpha)} - \frac{1}{j_{n,m}^2} = - \frac{n(1 + i \tan \alpha)}{2 \mu^2_{n,k}(\alpha)}.
\]

Now summing over all \( k \), we obtain the relationship

\[
\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{j_{n,m}^2 - \mu^2_{n,k}(\alpha)} = \frac{n + 2 + i n \tan \alpha}{8(n+1)}, \quad (25)
\]

containing both zeros of the equation (2) and zeros of the Bessel function \( J_n(\mu) \). In particular, for \( \alpha = 0 \) the equality (25) takes the form

\[
\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{j_{n,m}^2 - j_{n,k}^2} = \frac{n + 2}{8(n+1)}.
\]

The authors didn’t find the two last formulas in the literature.

In conclusion we give a generalization of the formula (24) for the sums \( \eta^{(2m)}(n, \alpha) \) on the permissible values of parameters \( (n, \alpha) \in P \). For \( n \in \mathbb{N}, \alpha \in [0, \pi/2] \) and \( m = 1, 2, 3 \) the necessary formulas are given above and the method of calculating such sums for subsequent \( m \in \mathbb{N} \) is also indicated. Let us analyze other cases.
(i) Let $n = 0, \alpha \in (-\pi/2, \pi/2)$. The equation (2) at $\mu \neq 0$ is equivalent to each from the equations

$$J_0'(\mu) = 0 \Leftrightarrow J_1(\mu) = 0,$$

and the required sums are $\eta^{(2m)}(0, \alpha) \equiv \sigma^{(2m)}(1)$.

(ii) Let $n \in \mathbb{N}, \alpha = -\pi/2$. The equation (2) is equivalent to the equation

$$J_n(\mu) = 0,$$

and the required sums are $\eta^{(2m)}(n, -\pi/2) \equiv \sigma^{(2m)}(n)$.

(iii) Let $n \in \{-1, -2, \ldots\}, \alpha = \pm \pi/2$. The equation (2) is equivalent to the equation

$$J_p(\mu) = 0, \quad p = -n \in \mathbb{N},$$

and the required sums are $\eta^{(2m)}(n, \pm \pi/2) \equiv \sigma^{(2m)}(|n|)$.

(iv) Let $n \in \{-1, -2, \ldots\}, \alpha \in [0, \pi/2]$. The equation (2) is equivalent to the equation

$$\mu J_p'(\mu) \cos \alpha - i p J_p(\mu) \sin \alpha = 0, \quad p = -n \in \mathbb{N}.$$

Zeros of the equation are the numbers $\overline{\mu}_{p,k}(\alpha)$ thus $\eta^{(2m)}(n, \alpha) = \overline{\eta}^{(2m)}(|n|, \alpha)$.

(v) Let $n \in \mathbb{N}, \alpha \in [-\pi/2, 0]$. After denoting $\beta = -\alpha \in [0, \pi/2]$ the equation (2) takes the form

$$\mu J_n'(\mu) \cos \beta - i n J_n(\mu) \sin \beta = 0.$$

Zeros of the equation are the numbers $\overline{\mu}_{n,k}(\beta)$ and therefore $\eta^{(2m)}(n, \alpha) = \overline{\eta}^{(2m)}(n, |\alpha|)$.

(vi) Let $n \in \{-1, -2, \ldots\}, \alpha \in [-\pi/2, 0]$. We denote $p = -n \in \mathbb{N}, \beta = -\alpha \in [0, \pi/2]$. Then the equation (2) takes the form

$$\mu J_p'(\mu) \cos \beta + i p J_p(\mu) \sin \beta = 0,$$

zeros of the equation are the numbers $\mu_{p,k}(\beta)$ and therefore $\eta^{(2m)}(n, \alpha) = \eta^{(2m)}(|n|, |\alpha|)$.

Thereby, all possible cases are sorted and the method of calculating of the Rayleigh type sums for all values $(n, \alpha) \in \mathcal{P}$ is specified.

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