Refined decay bounds on the entries of spectral projectors associated with sparse Hermitian matrices

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Due Giorni di Algebra Lineare Numerica e Applicazioni
Napoli, 14-15 Febbraio 2022

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Overview

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Introduction
Decaying matrices

Exponential off-diagonal decay: \( \{A_n\}_n, A_n \in \mathbb{C}^{n \times n}, \)

\[
|[A_n]_{ij}| \leq C \rho^{|i-j|}, \quad \text{for all } i, j,
\]

\( C > 0 \) and \( 0 < \rho < 1 \) are independent of \( n \).
Decaying matrices

**Exponential off-diagonal decay:** \( \{A_n\}_n, A_n \in \mathbb{C}^{n \times n}, \)

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\]

\( C > 0 \) and \( 0 < \rho < 1 \) are independent of \( n \).

For all \( m \) define the \( m \)-banded truncation of \( A_n \) as

\[
[A_n^{(m)}]_{ij} = \begin{cases} [A_n]_{ij} & \text{if } |i - j| \leq m, \\ 0 & \text{otherwise.} \end{cases}
\]

For all \( \varepsilon > 0 \) there is \( m \) independent of \( n \) s.t. \( \|A_n - A_n^{(m)}\|_p \leq \varepsilon \) for all \( n \),
where \( p = 1, 2, \infty \) [Benzi-Razouk, 2007].

\( A_n^{(m)} \) is \( m \)-banded \( \implies \) has \( \mathcal{O}(n) \) non-zero entries.
Matrix functions: $A \in \mathbb{C}^{n \times n}$ Hermitian, $A = U\Lambda U^*$ spectral decomposition, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then $f(A)$ is defined by

$$f(A) = Uf(\Lambda)U^*, \quad f(\Lambda) = \text{diag}(f(\lambda_1), \ldots, f(\lambda_n)).$$
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Given a set $S$, define $E_k(f, S) = \inf_{p_k \in \mathcal{P}_k} \sup_{x \in S} |f(x) - p_k(x)|$, where $\mathcal{P}_k$ is the set of polynomials with degree at most $k$. 

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The bound depends only on $S$ and on $m$, not on $n$.
Decay for matrix functions

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- $A \in \mathbb{C}^{n \times n}$ Hermitian and $m$-banded, $\sigma(A) \subset S$, $E_k(f, S) \leq C \rho^k$ for all $k \geq 0$. Then

$$|[f(A)]_{ij}| \leq C \rho^{\frac{|i-j|}{m} - 1} \quad \text{for all } i \neq j.$$  

The bound depends only on $S$ and on $m$, not on $n$.

If $\{A_n\}_n$ is s.t. $\sigma(A_n) \subset S$ and $A_n$ is $m$-banded for all $n$, then $\{f(A_n)\}_n$ has an exponential off-diagonal decay.
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Given a set \( S \), define \( E_k(f, S) = \inf_{p_k \in \mathcal{P}_k} \sup_{x \in S} |f(x) - p_k(x)| \), where \( \mathcal{P}_k \) is the set of polynomials with degree at most \( k \).

- \( A \in \mathbb{C}^{n \times n} \) Hermitian and \( m \)-banded, \( \sigma(A) \subset S \), \( E_k(f, S) \leq C\rho^k \) for all \( k \geq 0 \). Then
  \[
  |[f(A)]_{ij}| \leq C\rho \frac{|i-j|}{m}^{-1} \quad \text{for all } i \neq j.
  \]
  The bound depends only on \( S \) and on \( m \), not on \( n \).
  If \( \{A_n\}_n \) is s.t. \( \sigma(A_n) \subset S \) and \( A_n \) is \( m \)-banded for all \( n \), then \( \{f(A_n)\}_n \) has an exponential off-diagonal decay.

- If \( A \) is not \( m \)-banded, it still holds that \( |[f(A)]_{ij}| \leq C\rho^{d(i, j)} \) for all \( i \neq j \), where \( d(i, j) \) is the geodesic distance on \( G(A) \).

Under suitable hypotheses on \( G(A) \) [Frommer-Schimmel-Schweitzer, 2021], \( f(A) \) is close to a sparse matrix.
Spectral projector

$H \in \mathbb{C}^{n \times n}$ Hermitian, $\sigma(H) \subset [b_1, a_1] \cup [a_2, b_2]$, $b_1 < a_1 < a_2 < b_2$.

The spectral projector associated with $[b_1, a_1]$ is given by:

$$P = h_\mu(H), \quad h_\mu(x) = \begin{cases} 1 & \text{if } x < \mu, \\ 1/2 & \text{if } x = \mu, \\ 0 & \text{if } x > \mu, \end{cases}$$

where $\mu$ is arbitrary between $a_1$ and $a_2$. 
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where $\mu$ is arbitrary between $a_1$ and $a_2$.

- The function $h_\mu(x)$ is not continuous over $[b_1, b_2]$.
- **Invariance for linear transformations**: if $\tilde{H} = cH + d$, then $P = \tilde{P} = h_{\tilde{\mu}}(\tilde{H})$, where $\tilde{\mu} = c\mu + d$. Note that $\sigma(\tilde{H}) = c\sigma(H) + d$.
- We can assume $\sigma(H) \subset [-b, -a] \cup [a, b]$, $0 < a < b$, so $\mu = 0$ and $h_0(x) =: h(x)$.
- $h(x) = (1 - \text{sign}(x))/2$, so $|[P]_{ij}| = |[\text{sign}(H)]_{ij}|/2$ for $i \neq j$. 
Idea:
\[ \text{sign}(x) = x(x^2 - 1/2). \]

\[ q_k(y) \approx y - 1/2, \quad y \in [a_2, b_2] \Rightarrow xq_k(x^2) \approx \text{sign}(x), \quad x \in [-b, -a] \cup [a, b]. \]

Theorem [Benzi-Boito-Razouk, 2013]
Let \( H \) be Hermitian and \( m \)-banded with \( \sigma(H) \subset [-b, -a] \cup [a, b] \).

Then, for \( 1 < \chi < \bar{\chi} := b + a \chi - 1 \): \[
|P_{ij}| = |\text{sign}(H)_{ij}| \leq 2bM(\chi) \chi - 1 (1 \chi) \]
for all \( i \neq j \),
where \( M(\chi) = \frac{1}{\sqrt{z_0}}, \quad z_0 = \begin{cases} b_2 + a_2 b_2 - a_2 - \chi^2 + 1/2 \chi \quad & \text{if } \chi^2 < b_2 - a_2^2, \\ b_2 - a_2^2/2. & \end{cases} \]

For \( i, j \) fixed, we can minimize in \( \chi \).

The optimized bound numerically behaves as \( (b - a) b + a b \chi \). \( |i - j|^2 \) \( m - 1/2 \).
\textbf{Idea:} \( \text{sign}(x) = x(x^2)^{-1/2} \).

\( q_k(y) \approx y^{-1/2}, \ y \in [a^2, b^2] \implies xq_k(x^2) \approx \text{sign}(x), \ x \in [-b, -a] \cup [a, b] \).
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**Theorem** [Benzi-Boito-Razouk, 2013]

Let \( H \) be Hermitian and \( m \)-banded with \( \sigma(H) \subset [-b, -a] \cup [a, b] \).

Then, for \( 1 < \chi < \bar{\chi} := \frac{b+a}{b-a} \),

\[
2|P_{ij}| = |[\text{sign}(H)]_{ij}| \leq \frac{2bM(\chi)}{\chi - 1} \left( \frac{1}{\chi} \right)^{\frac{|i-j|}{2m}} - \frac{1}{2}
\]

for all \( i \neq j \),

where \( M(\chi) = \frac{1}{\sqrt{z_0}} \), \( z_0 = \left[ \frac{b^2+a^2}{b^2-a^2} - \frac{\chi^2+1}{2\chi} \right] \frac{b^2-a^2}{2} \).
**Idea:** \( \text{sign}(x) = x(x^2)^{-1/2} \).

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For \( i, j \) fixed, we can minimize in \( \chi \).

The optimized bound numerically behaves as \( \left( \frac{b-a}{b+a} \right)^{\frac{|i-j|}{2m}} \).
Refined decay bounds
Exploiting an integral representation of $\text{sign}(x)$

\[
\text{sign}(H) = \frac{2}{\pi} \int_{0}^{\infty} H(H^2 + t^2 I)^{-1} \, dt,
\]

\[
||[\text{sign}(H)]_{ij}| \leq \frac{2}{\pi} \int_{0}^{\infty} ||[H(H^2 + t^2 I)^{-1}]_{ij}|| \, dt.
\]

**Idea:** Bound $||[H(H^2 + t^2 I)^{-1}]_{ij}||$ and integrate [Benzi-Simoncini, 2015].

$q_k(y) \approx (y + t^2)^{-1} \Rightarrow xq_k(x^2) \approx x(x^2 + t^2)^{-1}$, and the best polynomial approximation of the inverse gives a single bound.
Exploiting an integral representation of $\text{sign}(x)$

$$\text{sign}(H) = \frac{2}{\pi} \int_{0}^{\infty} H(H^2 + t^2 I)^{-1} \, dt,$$

$$|[\text{sign}(H)]_{ij}| \leq \frac{2}{\pi} \int_{0}^{\infty} |[H(H^2 + t^2 I)^{-1}]_{ij}| \, dt.$$

**Idea:** Bound $|[H(H^2 + t^2 I)^{-1}]_{ij}|$ and integrate [Benzi-Simoncini, 2015].

$q_k(y) \approx (y + t^2)^{-1} \Rightarrow xq_k(x^2) \approx x(x^2 + t^2)^{-1}$, and the best polynomial approximation of the inverse gives a single bound.

**Theorem** [Benzi-R., 2021]

Let $H$ be Hermitian, $m$-banded, $\sigma(H) \subset [-b, -a] \cup [a, b]$. Then

$$|[P]_{ij}| \leq \left( \frac{1 + \sqrt{b/a}}{4} \right)^2 \left( \frac{b - a}{b + a} \right)^{\frac{|i-j|}{2m} - \frac{1}{2}} \quad \text{for all } i, j. \quad (1)$$

We get the previous behaviour without optimization.
The last bound was not the best from an asymptotic point of view.
Asymptotically optimal bound

The last bound was not the best from an asymptotic point of view. It is shown (see [Eremenko-Yuditskii, 2007]) that

\[ E_k(\text{sign}(x), [-b, -a] \cup [a, b]) = O \left( \frac{1}{\sqrt{k}} \left( \frac{b - a}{b + a} \right)^{k/2} \right) \quad \text{as } k \to \infty. \]

If \( H = H^* \) is Hermitian and \( m \)-banded, \( \sigma(H) \subset [-b, -a] \cup [a, b] \), then

\[ |[\text{sign}(H)]_{ij}| \leq \frac{C}{\sqrt{\frac{|i-j|}{m} - 1}} \left( \frac{b - a}{b + a} \right)^{|i-j|/2m - 1/2}, \]

for some \( C > 0 \).

However, such \( C \) is still unknown.
Asymptotically optimal bound

**Theorem** [Benzi-R., 2021]

Let $H$ be Hermitian, $m$-banded, $\sigma(H) \subset [-b, -a] \cup [a, b]$. Let $C_1 = \frac{1}{2ab}$, $C_2 = \frac{a^2 + ab + b^2}{8a^3b^3}$, and $0 < \tau < \bar{\tau} := \sqrt{\frac{C_1}{C_2}}$. Then

\[
|[P]_{ij}| \leq \frac{K_1(\tau)}{\sqrt{\frac{|i-j|m}{m} - 1}} \left( \frac{b - a}{b + a} \right)^{\frac{|i-j|}{2m} - \frac{1}{2}} + K_2 \, q(\tau)^{\frac{|i-j|}{2m} - \frac{1}{2}}
\]

for $|i - j| \geq m$, where $q(\tau) = \frac{\sqrt{b^2 + \tau^2} - \sqrt{a^2 + \tau^2}}{\sqrt{b^2 + \tau^2} + \sqrt{a^2 + \tau^2}}$ and

\[
K_1(\tau) = \frac{(1 + b/a)^2}{2\sqrt{2\pi}(C_1 - \tau^2 C_2)}, \quad K_2 = \frac{1}{4} \left( 1 + \sqrt{b/a} \right)^2.
\]

$q(\tau) < q(0) = \frac{b - a}{b + a} \implies$ optimal asymptotic behaviour for all $\tau > 0$.

We can also optimize to get the best possible bound.
Comparison between the bounds

\[ H \in \mathbb{C}^{150 \times 150} \text{ Hermitian and tridiagonal.} \]

\[ \sigma(H) \subset [-1, -0.2] \cup [0.2, 1] \text{ uniformly distributed.} \]

**Solid line:** \( d_P(k) = \max_{|i-j|=k} |P_{ij}|. \) **Other lines:** bounds for \( |i-j| = k. \)

**Remark:** The bounds are independent of the size.
Comparison between the bounds

\( H \in \mathbb{C}^{2000 \times 2000} \) Hermitian, 20-banded.

\( \sigma(H) \subset [-1, -0.3] \cup [0.3, 1] \) uniformly distributed.

Solid line: \( d_P(k) = \max_{|i-j|=k} |P_{ij}|. \) Other lines: bounds for \( |i-j| = k. \)

Remark: The bounds are independent of the size.
The decay of the entries of $A^{-1}$ benefits from certain eigenvalue distributions [Frommer-Schimmel-Schweitzer, 2018].

Does a similar property hold for spectral projectors?
Bounds related with the eigenvalue distribution

**Theorem** [Benzi-R., 2021]

Let $H = H^*$ be $m$-banded with $\sigma(H) \subset [-b, -a] \cup [a, b]$. Let $b = b_0 > b_1 > \ldots, > b_\nu = a$, with $\nu \leq n$, be the distinct values of $|\lambda|$ for $\lambda \in \sigma(H)$. Then

$$|[P_{ij}]| \leq C_\ell q_\ell \left(\frac{|i-j|}{2m}\right)^{\frac{1}{2} - \ell}$$

for $\ell = 0, 1, \ldots, \left\lceil \frac{|i-j|}{2m} - \frac{1}{2} \right\rceil$, \hspace{1cm} (3)

where $C_\ell = \frac{1}{4} \left(1 + \sqrt{\frac{b_\ell}{a}}\right)^2$, $q_\ell = \frac{b_\ell - a}{b_\ell + a}$. 
Bounds related with the eigenvalue distribution

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Let $H = H^*$ be $m$-banded with $\sigma(H) \subset [-b, -a] \cup [a, b]$. Let $b = b_0 > b_1 > \ldots, > b_\nu = a$, with $\nu \leq n$, be the distinct values of $|\lambda|$ for $\lambda \in \sigma(H)$. Then

$$|[P_{ij}]| \leq C_\ell q_\ell \frac{|i-j|}{2m} - \frac{1}{2} - \ell \quad \text{for } \ell = 0, 1, \ldots, \left\lfloor \frac{|i-j|}{2m} - \frac{1}{2} \right\rfloor,$$

where $C_\ell = \frac{1}{4} \left( 1 + \sqrt{\frac{b_\ell}{a}} \right)^2$, $q_\ell = \frac{b_\ell - a}{b_\ell + a}$.

For fixed $i, j$, when $\ell$ increases:

- $q_\ell << q_{\ell-1}$ when $b_\ell << b_{\ell-1}$.
- Smaller exponent: trade-off with the geometric rate.
- One or few isolated eigenvalue of maximum modulus do not contribute to the decay.
- Certain eigenvalue distributions lead to superexponential decay.
One isolated eigenvalue

\[ H \in \mathbb{C}^{3000\times 3000}, \text{ Hermitian, 20-banded,} \]
\[ \sigma(H) \subset \{-1\} \cup [-0.5, -0.1] \cup [0.1, 0.5], \text{ and } -1 \text{ has multiplicity 10.} \]

Bound (3) with \( \ell = 1 \) catches the behaviour.
$H \in \mathbb{C}^{300 \times 300}$, Hermitian, tridiagonal, $\sigma(H) \subset [-1, -0.1] \cup [0.1, 1]$.

**Left:** $\sigma(H)$, symmetric with respect to the origin. There are several isolated eigenvalues at the extremes. They cluster near the spectral gap.

**Right:** Exact decay compared with the bounds.
Superexponential decay

$H \in \mathbb{C}^{300 \times 300}$, Hermitian, tridiagonal, $\sigma(H) \subset [-1, -0.1] \cup [0.1, 1]$.

**Left:** $\sigma(H)$, no symmetry is present. There are several isolated eigenvalues at the extremes. They cluster near the spectral gap.

**Right:** Exact decay compared with the bounds.
Conclusions

We developed three new decay bounds for the entries of spectral projectors.

- The first is a single bound that describes well the decay.
- The second is optimal in the sense of polynomial approximation.
- The third catches the behaviour in presence of extremal isolated eigenvalues.

Some open problems are:

- Find an appropriate bound for the case of nonsymmetric intervals.
- Try new strategies to obtain smaller constant factors.
- Establish connections with more complicated eigenvalue distributions.
Thank you for the attention!

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Non-symmetric intervals

\[ E_k(\text{sign}(x), [-b_1, -a] \cup [a, b_2]) = \mathcal{O}(k^{-\frac{1}{2}} e^{-\eta k}), \]
\[ \eta = \int_{-1}^{K} \frac{K-x}{\sqrt{(1-x^2)(x+b_1/a)(x-b_2/a)}} \, dx \left( = \log \left( \frac{b+a}{b-a} \right) \text{ if } b_1 = b_2 = b \right), \text{ where} \]
\[ K = \frac{\int_{-1}^{1} x((1-x^2)(x+b_1/a)(x-b_2/a))^{-1/2} \, dx}{\int_{-1}^{1} ((1-x^2)(x+b_1/a)(x-b_2/a))^{-1/2} \, dx}. \]

\[ H \in \mathbb{C}^{300 \times 300}, \text{ tridiagonal, } \sigma(H) \subset [-0.5, -0.1] \cup [0.1, 1]. \text{ Rate with } C = 1. \]
Dashed line: \( b_1 = 0.5, b_2 = 1. \) Dotted line: \( b_1 = b_2 = 1. \)
Inverse function and general analytic functions

• \( f(A) = A^{-1} \), \( \sigma(A) \subset [a, b] \), \( 0 < a < b \). In this case
  \( E_k(x^{-1}, [a, b]) = Cq^{k+1} \) [Meinardus, 1967], where

  \[
  C = \frac{(1 + \sqrt{b/a})^2}{2b}, \quad q = \frac{(\sqrt{b} - \sqrt{a})}{(\sqrt{b} + \sqrt{a})}.
  \]

  \( A \) is \( m \)-banded \( \implies |[A^{-1}]_{ij}| \leq Cq^{\frac{|i-j|}{m}}, \ i \neq j \) [Demko-Moss-Smith, 1984].

  **Remark:** \( q = (\sqrt{b/a - 1})/(\sqrt{b/a + 1}) \implies \) connection with CG.

• \( \sigma(A) \subset [-1, 1] \), \( f \) analytic over the ellipse \( \mathcal{E}_\chi \) with foci in \( \pm 1 \) and
  sum of semiaxes \( \chi > 1 \). From Bernstein’s Theorem [Meinardus, 1967]

  \[
  E_k(f, [-1, 1]) \leq \frac{2M(\chi)}{\chi - 1} \left( \frac{1}{\chi} \right)^k, \quad M(\chi) = \max_{z \in \mathcal{E}_\chi} |f(z)|.
  \]

  \( A \) is \( m \)-banded \( \implies |[f(A)]_{ij}| \leq \frac{2M(\chi)}{\chi - 1} \left( \frac{1}{\chi} \right)^{\frac{|i-j|}{m}-1} \) [Benzi-Golub, 1999].

  **Remark:** We can shift and scale any \( A \) to have \( \sigma(A) \subset [-1, 1] \).
Insights on bound (1)

- \[ |[\text{sign}(H)]_{ij}| \leq \frac{2}{\pi} \int_0^\infty |[H(H^2 + t^2 I)^{-1}]_{ij}| \, dt. \]
- \[ q_k(y) \approx (y + t^2)^{-1} \text{ of best approximation, } y \in [a^2, b^2]. \text{ Then} \]
  \[
  E_{2k+1}(x(x^2 + t^2)^{-1}, [-b, -a] \cup [a, b]) \leq \|x(x^2 + t^2)^{-1} - xq_k(x^2)\|_\infty \\
  \leq b \|(y + t^2)^{-1} - q_k(y)\| = b E_k((y + t^2)^{-1}, [a^2, b^2]). \\
  = b C(t)q(t)^{k+1},
  \]
  where \( C(t) = (1 + \sqrt{\frac{b^2 + t^2}{a^2 + t^2}})^2/2(b^2 + t^2) \), \( q(t) = \frac{\sqrt{b^2 + t^2} - \sqrt{a^2 + t^2}}{\sqrt{b^2 + t^2} + \sqrt{a^2 + t^2}} \).
- \[ |[H(H^2 + t^2)^{-1}]_{ij}| \leq b C(t)q(t)^{\frac{|i-j|}{2m} - \frac{1}{2}}. \]
- \[ \int_0^\infty |[H(H^2 + t^2 I)^{-1}]_{ij}| \, dt \leq \int_0^\infty C(t)q(t) \, dt \leq \int_0^\infty b C(t) \, dt \cdot q(0). \]
- \[ \int_0^\infty C(t) \, dt = \int_0^\infty \frac{1}{2(b^2 + t^2)} \, dt + \int_0^\infty \frac{1}{2(a^2 + t^2)} \, dt + \int_0^\infty \frac{1}{\sqrt{b^2 + t^2} \sqrt{a^2 + t^2}} \, dt. \]
  First = \( \pi/4b \); Second = \( \pi/4a \); Third \( \leq \pi/2\sqrt{ab} \).
- \[ |[\text{sign}(H)]_{ij}| \leq \frac{1}{2} \left(1 + 2\sqrt{\frac{b}{a} + b/a}\right) q(0) = \frac{1}{2} \left(1 + \sqrt{\frac{b}{a}}\right)^2 \left(\frac{b-a}{b+a}\right) \]
**Insights on bound (2)**

**Idea:** \( \alpha = \frac{|i-j|}{2m} - \frac{1}{2}, \ C_1 = \frac{1}{2ab}, \ C_2 = \frac{a^2+ab+b^2}{8a^3b^3}, \ 0 < \tau < \bar{\tau} := \sqrt{\frac{C_1}{C_2}}, \)

\[
\int_0^\infty C(t)q(t)^\alpha dt = \int_0^\tau C(t)q(t)^\alpha dt + \int_\tau^\infty C(t)q(t)^\alpha dt
\]

- \( q(t)^\alpha \leq q(0)^\alpha e^{-(C_1-\tau^2 C_2)\alpha t^2} \) for \( 0 \leq t \leq \tau \). Then
  \[
  \int_0^\tau C(t)q(t)^\alpha dt \leq C(0) \int_0^\tau q(t) dt \leq C(0)q(0) \int_0^\infty e^{-(C_1-\tau^2 C_2)\alpha t^2} dt
  \]
  \[
  \approx C(0)q(0)^\alpha / \sqrt{\frac{|i-j|}{m} - 1}.
  \]
- \( \int_\tau^\infty C(t)q(t)^\alpha dt \leq \int_0^\infty C(t) dt \cdot q(\tau)^\alpha. \)
Insights on bound (3)

Bound for $| [H(H^2 + t^2 I)^{-1}]_{ij} |$.

$b = b_0 > b_1 > \cdots > b_\nu = a$ moduli of eigenvalues of $H$.

$$R_\ell(x) = \prod_{i=0}^{\ell-1} \left( 1 - \frac{x}{b_i^2 + t^2} \right)$$

$R_\ell(b_i^2 + t^2) = 0$ for $i = 0, \ldots, \ell - 1$, $|R_\ell(b_i^2 + t^2)| < 1$ for $i = \ell, \ldots, \nu$ and $R_\ell(0) = 1$.

$$p_k(x) = \frac{1 - R_\ell(x)}{x} - q_{k-\ell}(x), \quad q_{k-\ell}(x) \approx 1/x \text{ best, } x \in [a^2 + t^2, b_\ell^2 + t^2],$$

so

$$\frac{x}{x^2 + t^2} - xp_k(x^2 + t^2) = xR_\ell(x^2 + t^2) \left( \frac{1}{x^2 + t^2} - q_{k-\ell}(x^2 + t^2) \right)$$

$$\max_{x = b_0, \ldots, b_\nu} \left| \frac{x}{x^2 + t^2} - xp_k(x^2 + t^2) \right| \leq b_\ell \max_{x = b_\ell, \ldots, b_\nu} \left| \frac{1}{x^2 + t^2} - q_{k-\ell}(x^2 + t^2) \right| \leq b_\ell C_\ell(t) q_\ell(t)^{k+1}.$$
Spectra used for the experiments

Symmetric:

\[ \lambda_i^{(j)} = (-1)^j \left[ 1 + 0.9 \left( 1 - \frac{i - 1}{149} - 2\sqrt{1 - \frac{i - 1}{149}} \right) \right] \in [-1, -0.1] \cup [0.1, 1], \]

for \( i = 1, \ldots, 150 \) and \( j = 0, 1 \).

Non symmetric:

\[ \lambda_i = (-1)^i \left[ 1 + 0.9 \left( 1 - \frac{i - 1}{299} - 2\sqrt{1 - \frac{i - 1}{299}} \right) \right] \in [-1, -0.1] \cup [0.1, 1], \]

for \( i = 1, \ldots, 300 \).