Observer for non-linear systems with sampled measurements: Application to the friction factor estimation of a pipeline

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Abstract
This paper presents a new approach of the continuous-discrete observer design for a class of uncertain state-affine non-linear systems. A high-gain observer redesign is developed and analysed under insightful conditions. The proposed observer estimates the state vector by using system output measurements with long sampling times. This result is achieved by considering a persistent excitation condition that can be validated online. The performance of the proposed algorithm is evaluated under time-varying sampled measurements to estimate the friction factor of a pipeline where the case of noisy sampled output measurements is also considered.

1 | INTRODUCTION

A good fluid administration in pipes is a task that requires a major effort, therefore fluid distribution systems, its monitoring, and automatic fault diagnosis are areas that need continuous improvement. The main objective of automatic pipeline monitoring systems is to estimate pressure losses, distortions, or sensor failures, hence optimising instrumentation and economy [1–3]. Leaks in pipelines have different causes, resulting in financial losses, destruction of the biosphere, and serious risks to human health. The percentage of drinking water volume loss has been calculated at about 21%, although in countries such as Mexico it reaches an average value of 40% [4]. Also in Mexico, the state oil company PEMEX lacks adequate technology to deal with this, leading to losses of around two billion dollars a year due to leaks in oil pipelines provoked by clandestine taps which even cause human losses [5]. Therefore, the use of modern techniques of automatic control theory, such as the so-called observers, helps and improves the processes of monitoring states and failures to mitigate the problem mentioned above [6].

The observer design problem has been investigated by numerous researchers, but it is still an open problem. Many different classes of systems have been considered, and mainly for continuously available measurements [7, 8]. A strong property of linear systems is the fact that the observability does not depend on the input. This idea has been taken to non-linear systems giving birth to the general high-gain observer design in the seminal paper [9]. Contrarily to the linear case, the observability property of a non-linear system can fail depending on the input. This idea has been taken to non-linear systems giving birth to the general high-gain observer design in the seminal paper [9]. Contrarily to the linear case, the observability property of a non-linear system can fail depending on the input. Several different general forms have been proposed for the observer design of non-uniformly observable systems. Such design usually requires additional assumptions on the input, called persistent excitation condition [10–12].

A common situation encountered for observer design is that typically the output of the systems is assumed to be continuous time. However, in many practical cases, the output is only available at discrete-time intervals. Several results are proposed for
this challenging problem, for example, [12]. The first observer considering this constraint has been presented in [9] and is based on a Kalman filter, whereas the first observer applied to a real process has been presented in [13] to estimate the evolution of the copolymer composition in a free radical emulsion copolymerization reactor. In the continuous case, a continuous estimation of the output is used in [14] and reconsidered in [15]. Following the same idea, an observer for a class of multi-output systems is proposed in [16] and the effect of uncertainty on the model is further studied in [17]. A similar design based on the continuous case is proposed in [18] but the correction term is kept constant between two measures. The case of systems whose observability depends on the input is considered in [19]. It is based on a state estimator which is re-initialised when a new measurement is available.

Design efforts have been made in the past to overcome the observer design problem for non-linear systems with sampled outputs. Some of the first approaches are exhibited in [20, 21]. Here, dynamical models are used to provide a state prediction over the sampling intervals. The state prediction is updated by the output measurements, which are sampled at each instant $t_k$. Another work related to this technique is presented in [14]. An impulsive continuous-discrete time observer for uniformly observable systems with sampled outputs is presented in [18], where the gain of the correction term is computed through the resolution of Linear Matrix Inequalities (LMIs). In the same context, an impulse observer design based on a hybrid model is proposed in [22], which considers a restriction of a Lyapunov function. In the study of observer design to estimate time-varying parameters, there exist several approaches, among which stand out the adaptive observer presented in [23]. In this work, the estimation problem is tackled through the use of cooperative linear systems, which are based on lower and upper bounds of the system state. Moreover, the LMI approach presents disadvantages, for example, the resolution turns in a large number of inequalities to be solved. When different problems such as noise, sampled time, disturbances are considered in the observer design, the complexity and constraints of the LMIs increase and different approaches are necessary to obtain a solution [24, 25]. Although LMI approaches represent a powerful tool to consider control problems that appear hard to be solved in an analytic way, we are interested in proposing an estimation method with an easy and online adjustment. Furthermore, the adaptive observer design with sampled output measurements is provided in [26], where a predictor is used to estimate the output during the sampling intervals. Moreover, some engineering applications are reported in the following works: in [27], the state vector of an electro-hydraulic actuator system is estimated via a continuous-discrete observer, in [28] a continuous-discrete observer is applied to perform a predictive control design for a coupled four-tank system, and finally, in [29] an observer with sampled output measurements is designed to provide estimations of a quadrotor.

This paper focuses on a class of state-affine uncertain non-linear system with sampled output measurements, that is, measurements are available at sampling instants $t_k$ (Section 2). The non-linear system is observable for any persistent input. An observer for these systems is developed in [10, 30], where no uncertainties are considered. In this work, the design of the continuous observer with sampled measurements is achieved from a redesigned version of the continuous-time observer proposed in [30] (Section 3). The two main contributions of this observer are: (i) the observer is able to provide continuous-time estimation despite uncertainties and the sampled output. A convergence analysis is presented in Section 4 in order to demonstrate that the observation error lies in a region centred at the origin whose radius depends on the bounds of the uncertainties and the maximum sampling; (ii) the easiness to compute the correction term of the observer which is updated at sampling instants. These important features contribute to providing an adequate continuous-time estimation in the presence of reasonably long sampling periods affecting the output measurements. These features are highlighted in Section 5 through numerical simulations conducted to estimate the friction factor of a pipeline where the output measurements are sampled by time-varying random sampling periods, whose outputs are also affected by measurement noise; finally, this behaviour is compared to an existing approach from the literature.

2 \hfill PROBLEM STATEMENT

Consider the following class of multi-variable state-affine non-linear system with sampled measurements

\[
\begin{align*}
\dot{x}(t) &= A(u(t), x(t))x(t) + \varphi(x(t), u(t)) + B\varepsilon(t), \\
y(t_k) &= Cx(t_k) = x^1(t_k),
\end{align*}
\]

where

\[
A(u(t), x(t)) = \begin{bmatrix}
0 & A_1(u(t), x^1(t)) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & A_{p-1}(u(t), x^1(t), \ldots, x^{p-1}(t)) & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
\varphi(u(t), x(t)) = \begin{bmatrix}
\varphi_1(u(t), x^1(t)) \\
\varphi_2(u(t), x^1(t), x^2(t)) \\
\vdots \\
\varphi_{p-1}(u(t), x^1(t), \ldots, x^{p-1}(t)) \\
\varphi_p(u(t), x(t))
\end{bmatrix},
\]

and

\[
C = \begin{bmatrix}
I_{s_1 \times s_1} & 0_{s_1 \times s_2} & \cdots & 0_{s_1 \times s_q}
\end{bmatrix}
\]
\( B = \begin{bmatrix} 0_{n_1} & 0_{n_1} & \cdots & 1_{n_1} \end{bmatrix}^T. \)

\( x(t) = (x^1 \quad \cdots \quad x^q)^T \in \mathbb{R}^q \) is the state vector where \( x^k \in \mathbb{R}^q, \ k = 1, \ldots, q \) with \( n_1 = p \) and \( \sum_{k=1}^{q} n_k = n \). Each \( A_k(u,x) \) is an \( n_k \times n_{k+1} \) matrix which is triangular with respect to \( x \), that is, \( A_k(u,x) = A(u,x^1,\ldots,x^k), \ k = 1, \ldots, q - 1; \ \varphi(x(t),u(t)) \) is a non-linear vector function having a triangular structure with respect to \( x; u \in \mathbb{R}^q \) is the system input; \( \varepsilon(t) \) is an unknown function describing the system uncertainties and may depend on the state, \( \varepsilon : \mathbb{R}^+ \mapsto \mathbb{R}^q \) and \( y(t_k) \in \mathbb{R}^p \) is the discrete-time output. Furthermore, \( 0 \leq t_0 < t_1 < \cdots < t_k < \cdots, \Delta_k = t_{k+1} - t_k \) and \( \lim_{k \to +\infty} t_k = +\infty \).

It is assumed that there exists a maximum value of the sampling rate denoted as \( \Delta_{\text{max}} > 0 \) such that \( 0 < \Delta_m < \Delta_{\text{max}}, \forall k \geq 0, \Delta_{\text{max}}. \)

Now, some assumptions are provided (see [31, 32]):

**Assumption 1.** The state vector \( x(t) \) and the control signal \( u(t) \) are bounded, that is, \( x(t) \in X \) and \( u(t) \in U \), where \( X \subset \mathbb{R}^q \) and \( U \subset \mathbb{R}^m \) are compacts sets.

**Assumption 2.** The functions \( A(u,x), \varphi(x,u) \) are Lipschitz with respect to \( x \), uniformly with respect to \( u \) where \( (u,x) \in U \times X \). Their Lipschitz constants are denoted by \( L_A \) and \( L_{\varphi} \), respectively.

**Assumption 3.** The unknown function \( \varepsilon(t) \) is essentially bounded, that is, \( \exists \delta_\varepsilon > 0 \quad \text{Ess.sup.}_{t \geq 0} \| \varepsilon(t) \| \leq \delta_\varepsilon. \)

Since the state is confined to the bounded set \( X \), one can assume the Lipschitz prolongations of the non-linearities, using smooth saturation functions. In what follows, it is assumed that the prolongations have been carried out and that the functions \( A(u,x), \varphi(x,u) \) are provided from these prolongations. This allows to conclude that for any bounded input \( u \in U \), the functions \( A(u,x), \varphi(x,u) \) are globally Lipschitz with respect to \( x \) and they are bounded for all \( x \in \mathbb{R}^q \).

Furthermore, the estimation process under sampled output measurements is handled by the following lemma [33]:

**Lemma 1.** A differentiable function \( \mu : t \in \mathbb{R}^+ \mapsto \mu(t) \in \mathbb{R}^p \) that satisfies the following inequality:

\[
\mu(t) \leq -a_{\mu} \mu(t) + b_{\mu} \int_{t_k}^{t} \mu(s) ds + p(t), \quad \forall t \in [t_k,t_{k+1}) \quad (2)
\]

with \( k \in \mathbb{N}, \ t_0 \geq 0, \) where \( 0 < \Delta_m \leq \Delta_k = t_{k+1} - t_k \leq \Delta_{\text{max}} < +\infty \), and function \( p(t) : \mathbb{R}^+ \to \mathbb{R} \) is an essentially bounded function with \( a_{\mu} = \sup_{t \geq 0} p(t) ; \) the positive reals \( a_{\mu} \) and \( b_{\mu} \) satisfy the inequality \( \frac{b_{\mu}\Delta_{\text{max}}}{a_{\mu}} < 1 \). Then, the function \( \mu(t) \) satisfies

\[
\mu(t) \leq e^{-\chi_{\mu}\Delta_{\text{max}}}(\mu(t_0) + a_{\mu}\Delta_{\text{max}}) + e^{-\chi_{\mu}\Delta_{\text{max}}^2} - e^{-\chi_{\mu}\Delta_{\text{max}}^2} \quad (3)
\]

with \( 0 < \chi_{\mu}(\Delta_{\text{max}}) = (a_{\mu} - b_{\mu}\Delta_{\text{max}})e^{-a_{\mu}\Delta_{\text{max}}} \).

### 3 | OBSERVER WITH CONTINUOUS-TIME OUTPUT MEASUREMENTS

Consider the system (1) with a continuous-time output:

\[
\begin{aligned}
\dot{x}(t) &= A(u(t),x(t))x(t) + \varphi(x(t),u(t)), \\
y(t) &= Cx(t) = x^1(t). 
\end{aligned}
\]

According to Assumption 1, an upper bound for the state and the function \( A(x(t),u(t)) \) are

\[
x_{\text{max}} = \sup_{t \geq 0} \| x(t) \|, \quad \bar{a} = \sup_{t \geq 0} \| A(u(t),x(t)) \|. \quad (5)
\]

A candidate high-gain observer is

\[
\dot{\hat{x}}(t) = A(u(t),\hat{x}(t))\hat{x}(t) + \varphi(u(t),\hat{x}(t)) - \theta \Delta_k^{-1}S(t)^{T}(C\hat{x}(t) - y(t)), \quad (6)
\]

where \( \hat{x} = (\hat{x}^1 \quad \cdots \quad \hat{x}^q)^T \in \mathbb{R}^q \); \( \hat{x}^k \in \mathbb{R}^{q_k} \) represents the estimated state; \( u \) and \( y \) are, respectively, the input signal and the measured output of the system (4); \( S(t) \) is a semi-positive-definite (SPD) matrix governed by the following Lyapunov differential equation:

\[
\dot{S}(t) = \theta \left( -S(t) - A(u(t),\hat{x}(t))^T S(t) - S(t)A(u(t),\hat{x}(t)) + C^T C \right), \quad (7)
\]

where \( S(0) = S^T(0) > 0 \) and \( \theta > 0 \) is a scalar tuning parameter. According to the high-gain approach, the diagonal matrix \( \Delta_\theta \) is

\[
\Delta_\theta = \text{diag}[I_{n_1} I_{n_2}/\theta \cdots I_{n_q}/\theta^{q-1}], \quad (8)
\]

It is worth to note that the non-linear system (4) is not necessarily uniformly observable, that is, its observability depends on the input and the state. Therefore, observability needs to be guaranteed at arbitrarily short times for observer design. Thus, a specific excitation is usually required for this kind of systems, it is called local regularity. Regularity qualifies the behaviour of the input for small times [32]. To adopt an additional hypothesis required for the observer design, the following definitions are given.

Let \( \Phi(t,\hat{x}) \) be the state transition matrix of the state-affine system:

\[
\dot{\xi}(t) = A(u(t),\hat{x}(t))\xi(t), \quad (9)
\]
where $\xi \in \mathbb{R}^{n}$ is the state vector, $u$ is the input of system (4) and $\hat{x}$ is the estimated state of system (6). Recall that the matrix $\Phi_{\hat{x},x}(t, s)$ is defined as

$$\left\{ \begin{array}{l}
\frac{d\Phi_{\hat{x},x}(t, s)}{dt} = A(u(t), \hat{x}(t))\Phi_{\hat{x},x}(t, s), \quad \forall t \geq s \geq 0, \\
\Phi_{\hat{x},x}(t, t) = I_n, \quad \forall t \geq 0,
\end{array} \right.$$ (10)

where $I_n$ denotes the identity matrix in $\mathbb{R}^{n \times n}$.

The following additional assumption is given:

**Assumption 4.** The input $u$ is such that for any trajectory $\hat{x}$ of system (6) starting from $\hat{x}(0) \in X$, $\exists \delta_0 > 0, \forall \theta \geq 0$ and $\forall t \geq 1/\theta$, the following persistent excitation condition is satisfied

$$\int_{t-1/\theta}^{t} \Phi_{\hat{x},x}(s, t)^T C^T C\Phi_{\hat{x},x}(s, t) ds \geq \frac{\delta_0}{\theta \alpha(\theta)} \Delta^2_\delta,$$ (11)

where $\alpha(\theta) \geq 1$ is a function satisfying

$$\lim_{\theta \to \infty} \frac{\alpha(\theta)}{\theta^2} = 0.$$ (12)

Therefore, the following theorem can be established:

**Theorem 1.** Consider system (4), satisfying Assumptions 1–3. Then, for every bounded input satisfying Assumption 4, there exists a constant $\Theta^*$ such that for every $\theta > \Theta^*$, system (6) is a state observer for system (4) with an exponential error convergence to the origin for sufficiently high values of $\theta$, that is, for any initial conditions $(\hat{x}(0), \hat{\xi}(0)) \in X$, the observation error $\hat{x}(t) - x(t)$ exponentially tends to a finite value when $t \to \infty$, in presence of the bounded uncertainties $\delta_\varepsilon$.

**Proof of Theorem 1.** It is worth mentioning that the matrix $S(t)$ is SPD and the lower bound for its smallest eigenvalue should be calculated. The transition matrix $\Phi_{\hat{x},x}(t, s)$ of the state affine system

$$\dot{\xi}(t) = \theta A(u(t), \hat{x}(t))\xi(t)$$ (13)

is $\Phi_{\hat{x},x} = \Delta_\theta \Phi_{\hat{x},x}(t, s)\Delta_\theta^{-1}$, where $\Phi_{\hat{x},x}$ is defined by (10).

As a result, the matrix $S(t)$ (the solution of the ODE (7)), can be expressed as

$$S(t) \geq e^{-\frac{1}{\alpha(\theta)} \frac{\delta_0}{\theta} I_n},$$ (14)

where $\delta_0$ and $\alpha(\theta)$ are given by Assumption 4. Clearly, according to inequality (14), the lower bound of $S(t)$ is given by

$$\lambda_{\text{min}}(S) \geq e^{-\frac{1}{\alpha(\theta)} \frac{\delta_0}{\theta}}.$$ (15)

To show that $\lambda_{\text{max}}(S)$ is bounded with an upper bound independent of $\theta$, it can be shown that this property is satisfied for each entry of the matrix $S(t)$. Denote by $S_{i,j}$ the block entry of matrix $S$ located at the row $i$ and the column $j$. Then, according to Equation (7):

$$\dot{\hat{S}}_{i,j} = -\theta (S_{i,j}(t) - I_p),$$ (16)

$$\dot{S}_{i,j} = -\theta (S_{i,j}(t) + S_{i,j-1}(t)A_{j-1}(u(t), \hat{x}(t)))$$ (17)

$$+ A_{j-1}^T(u(t), \hat{x}(t))S_{i,j-1}(t))$$ (18)

According to (16),

$$\|S_{i1}(t)\| \leq e^{-\theta t} \|S_{i1}(0)\| + \theta \int_{0}^{t} e^{\theta (r-t)} \|I_p\| dr$$ (19)

For $j \geq 2$, by induction on $j$ it can be proved that $S_{i,j}$ is bounded with a bound that does not depend on $\theta$. Assume that $S_{i,j-1}$ is bounded and denote

$$S_M = \sup_{t \geq 0} \|S_{i,j-1}(t)\|.$$ (20)

According to Assumptions 1 and 2, the matrices $A_k(u, \hat{x})$, $k = 0, \ldots, q - 1$ are bounded. Setting

$$A_M = \sup_{t \geq 0} \|A_k(u(t), \hat{x}(t))\|,$$ (21)

thus it can be established that all the entries of the matrix $S(t)$ are bounded with an upper bound independent of $\theta$. As a result, the largest eigenvalue of $S(t)$, $\lambda_{\text{max}}(S)$, is also independent of $\theta$.

The solution of each one of the block entry of matrix $S(t)$ ((17) and (18)) is further detailed in [10].

Now the exponential convergence to zero of the observation error $\hat{x}$ is proved. Set $\hat{x} = \Delta_\theta \hat{\xi}$, where $\hat{x} = \hat{x} - x$, while taking into consideration: $\Delta_\theta A_{j-1}(u, \hat{x})\Delta_\theta^{-1} = \theta A(U, \hat{x})$ and $C\Delta_\theta^{-1} = C$. Therefore, the error dynamics is given by

$$\dot{\hat{\xi}}(t) = \theta [A(u(t), \hat{x}(t)) - S^{-1}(t)C^T C] \hat{x}(t)$$
$$+ \Delta_\theta [\hat{A}(u(t), \hat{x}(t))\times(t) - \Phi(u(t), \hat{x}(t), x(t)) - B\xi(t)],$$ (22)
where
\[
\dot{A}(u(t), \hat{x}(t), x(t)) = A(u(t), \hat{x}(t)) - A(u(t), x(t)),
\]
\[
\dot{\phi}(u(t), \hat{x}(t), x(t)) = \phi(u(t), \hat{x}(t)) - \phi(u(t), x(t)).
\]

Consider the Lyapunov candidate function
\[
V(\hat{x}) = \hat{x}^T(t)S(t)\dot{x}(t).
\]

By using (7), the derivative of \( V(\hat{x}) \) is
\[
\dot{V}(\hat{x}) = -\theta \hat{x}^T(t)S(t)\dot{x}(t) - \theta \hat{x}^T(t)C^T(t)\Sigma(t)
+ 2\theta \hat{x}^T(t)S(t)\Delta_\theta \left[ \dot{A}(u(t), \hat{x}(t), x(t)) \right]
+ \phi(u(t), \hat{x}(t), x(t)) \right] - 2\theta \hat{x}^T(t)S(t)\Delta_\theta \delta e(t).
\]

Proceeding as in [34], it can be shown that for \( \theta > 0 \):
\[
\|2\dot{\Sigma}(t)\Delta_\theta \dot{A}(u(t), \hat{x}(t), x(t))\| \leq 2\sqrt{\frac{\lambda_{\text{max}}(S)}{\lambda_{\text{min}}(S)}} V(\hat{x}) L_\lambda \lambda_{\text{max}}
\]
\[
\leq 2\sqrt{\frac{\lambda_{\text{max}}(S)}{\lambda_{\text{min}}(S)}} V(\hat{x}) L_\lambda L_{\lambda_{\text{max}}}
\]
\[
\leq 2\alpha(\theta) V(\hat{x}) L_\lambda L_{\lambda_{\text{max}}} \tag{26}
\]

and
\[
\|2\dot{\Sigma}(t)\Delta_\theta \dot{\phi}(u(t), \hat{x}(t), x(t))\| \leq 2\sqrt{\frac{\lambda_{\text{max}}(S)}{\lambda_{\text{min}}(S)}} V(\hat{x}) L_{\phi}
\]
\[
\leq 2\alpha(\theta) V(\hat{x}) L_{\phi} \tag{27}
\]

where \( L_\lambda \) and \( L_{\phi} \) are the Lipschitz constants defined in Assumption 2 and \( \lambda_{\text{min}}(S) \) is given in (15).

Based on the assumption that the unknown function \( \varepsilon(t) \) is essentially bounded, it can be shown that \( \theta > 0 \), as in [33]:
\[
\|2\dot{\Sigma}(t)\Delta_\theta \delta e(t)\| \leq \frac{2}{\theta} \sqrt{\frac{\lambda_{\text{max}}(S)}{\lambda_{\text{min}}(S)}} \sqrt{V(\hat{x})} \delta_e, \tag{28}
\]

where \( \delta_e \) is defined in Assumption 3.

By substituting (26)–(28) in (25):
\[
\dot{V}(\hat{x}) \leq -\theta V(\hat{x})
+ 2\sqrt{\frac{\alpha(\theta)}{\theta}} \sqrt{\frac{\lambda_{\text{max}}(S)e}{\delta_0}} \left( L_{\lambda_{\text{max}}} \lambda_{\text{max}} + L_\phi \right) V(\hat{x})
+ 2\left( \frac{\delta_e}{\theta} \right) \sqrt{\frac{\lambda_{\text{max}}(S)}{\theta}} \sqrt{V(\hat{x})}. \tag{29}
\]

Equation (29) can be rewritten as
\[
\frac{d}{dt} \sqrt{V(\hat{x})} \leq -\theta \nu_0 \sqrt{V(\hat{x})} + 2 \left( \frac{\delta_e}{\theta} \right) \sqrt{\frac{\lambda_{\text{max}}(S)}{\theta}} \sqrt{V(\hat{x})}, \tag{30}
\]

where
\[
\nu_0 = 1 - 2\sqrt{\frac{\alpha(\theta)}{\theta}} \sqrt{\frac{\lambda_{\text{max}}(S)e}{\delta_0}} \left( L_{\lambda_{\text{max}}} \lambda_{\text{max}} + L_\phi \right). \tag{31}
\]

According to (12), \( V(\hat{x}) \) converges exponentially to sufficiently large values of \( \theta \).

Finally, by using of the comparison Lemma:
\[
\sqrt{V(\hat{x})} \leq e^{-\theta \nu_0 t} \sqrt{V(\hat{x}(0))} + \frac{2}{\theta \nu_0} \sqrt{\frac{\alpha(\theta)}{\theta}} \sqrt{\frac{\lambda_{\text{max}}(S)}{\theta}} \delta_e. \tag{32}
\]

In order to get the estimation error, firstly, one should come back to \( \tilde{x} \):
\[
\|\tilde{x}(t)\| \leq \sqrt{\frac{\lambda_{\text{max}}(S)}{\lambda_{\text{min}}(S)}} e^{-\theta \nu_0 t} \|\tilde{x}(0)\|
+ \frac{2}{\theta \nu_0} \sqrt{\frac{\alpha(\theta)}{\theta}} \sqrt{\frac{\lambda_{\text{max}}(S)}{\theta}} \delta_e, \tag{33}
\]

where \( \zeta = \sqrt{\frac{\lambda_{\text{max}}(S)e}{\delta_0}} \). Thus, the estimation error \( \tilde{x}(t) = \Delta_\theta \hat{x} \), since \( \|\tilde{x}(t)\| \leq \|\tilde{x}(t)\| \leq \theta \nu_0^{-1} \|\tilde{x}(t)\| \), one has
\[
\|\tilde{x}(t)\| \leq \theta \nu_0^{-1} \sqrt{\frac{\alpha(\theta)}{\theta}} e^{-\theta \nu_0 t} \|\tilde{x}(0)\| + \frac{2}{\theta \nu_0} \sqrt{\frac{\alpha(\theta)}{\theta}} \sqrt{\frac{\lambda_{\text{max}}(S)}{\theta}} \delta_e. \tag{35}
\]

It can be concluded that the observation error \( \tilde{x}(t) \) converges exponentially to a finite value which is subject to the upper bound of the uncertainties \( \delta_e \). This completes the proof of Theorem 1. \( \square \)

Remark 1. It is worth to note that when there are no uncertainties, that is, \( \delta_e = 0 \), the observation error converges exponentially to zero. This is not the case with \( \delta_e \neq 0 \). In this case, the observation error is finite, since, the observation error converges into a ball centred at the origin with a radius \( \delta_e \) which can be made smaller by choosing values of tuning parameter \( \theta \) sufficiently high. However, high values of \( \theta \) should be avoided in practice, since higher values of \( \theta \) implies amplification of the measurement noise.
4 | OBSERVER UNDER SAMPLED OUTPUT MEASUREMENTS

The observer candidate for system (1) with sampled output measurements is defined by the following equations:

\[
\begin{align*}
\hat{x}(t) &= A(u(t), \hat{x}(t))\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_{\theta}^{-1} S^{-1}(t)C^T \eta(t), \quad (36) \\
\dot{\eta}(t) &= -\theta CS^{-1}(t)C^T \eta(t) \quad t \in [t_k, t_{k+1}), k \in N, \quad (37) \\
\eta(t_k) &= C\hat{x}(t_k) - y(t_k), \quad t = t_k, \quad (38)
\end{align*}
\]

where \( \hat{x} = [\hat{x}_1, \ldots, \hat{x}_n]^T \) is the estimated state and \( \Delta_{\theta} \) is a block-diagonal matrix defined in (8) with \( \theta > 0 \).

The next theorem is the main result of this work:

**Theorem 2.** Consider the non-linear system (1) with uncertainties, satisfying Assumptions 1–4, with bounded input \( u(t) \) making \( A(u(t), x(t)) \) a bounded function. There exist \( \theta_0 > 0 \) such that for all \( \theta \geq \theta_0 \), if the maximum value of the sampling rate \( \Delta_{\theta \max} \) satisfies

\[
\Delta_{\theta \max} < \frac{\alpha_{\theta}}{\beta_{\theta}}, \quad (40)
\]

then the estimation error of the continuous-time observer (36)–(39) converges towards a neighborhood of the origin. This neighborhood depends on the ultimate bound of the uncertainties \( \Delta_{\theta} \) and the functions \( \Pi(\Delta_{\theta \max}) \) and \( \Psi(\Delta_{\theta \max}) \), since the ultimate bound of the error is an increasing function of \( \Delta_{\theta \max} \).

**Proof of Theorem 2.** In this proof, the exponential convergence to zero of the observation error is demonstrated. Set \( \hat{x} = \Delta_{\theta} \hat{x} \), where \( \hat{x} = \hat{x} - x \), again, taking into account : \( \Delta_{\theta} \hat{A}(u, \hat{x}) \Delta_{\theta}^{-1} = \theta A(u, x) \) and \( C \Delta_{\theta}^{-1} = C \). The dynamical equation of the error is

\[
\begin{align*}
\dot{x}(t) &= \theta A(u(t), \hat{x}(t))\hat{x}(t) - \theta S^{-1}(t)C^T \eta(t) - \Delta_{\theta} B \xi(t) \\
+ \Delta_{\theta} \tilde{A}(u(t), \hat{x}(t), x(t)) \hat{x}(t) + \Delta_{\theta} \tilde{\varphi}(u(t), \hat{x}(t), x(t)) \\
= & \theta \left[ A(u(t), \hat{x}(t)) - S^{-1}(t)C^T C \right] \hat{x}(t) + \theta S^{-1}(t)C^T \eta(t) \\
+ \Delta_{\theta} \left[ \tilde{A}(u(t), \hat{x}(t), x(t)) \hat{x}(t) + \tilde{\varphi}(u(t), \hat{x}(t), x(t)) - B \xi(t) \right], \quad (41)
\end{align*}
\]

where \( \tilde{A}(u(t), \hat{x}(t), x(t)) \) and \( \tilde{\varphi}(u(t), \hat{x}(t), x(t)) \) are defined in (23) and (24), and \( \eta(t) = C \hat{x}(t) - \eta(t) \).

By considering the fact that \( \eta(t) \) is governed by the ODE (38), it can be shown that

\[
\begin{align*}
\hat{\eta}(t) &= C \theta A(u(t), \hat{x}(t))\hat{x}(t) + \Delta_{\theta} \tilde{A}(u(t), \hat{x}(t), x(t)) \hat{x}(t) \\
+ \Delta_{\theta} \tilde{\varphi}(u(t), \hat{x}(t), x(t)) \hat{x}(t) - \Delta_{\theta} B \xi(t), \quad (42)
\end{align*}
\]

where \( \Delta_{\theta} B = 0 \). Equation (42) can be rewritten as

\[
\begin{align*}
\dot{\xi}(t) &= C \left[ \theta A(u(t), \hat{x}(t))\hat{x}(t) + \Delta_{\theta} \tilde{A}(u(t), \hat{x}(t), x(t)) \hat{x}(t) \\
+ \Delta_{\theta} \tilde{\varphi}(u(t), \hat{x}(t), x(t)) \right] - C \Delta_{\theta} B \xi(t), \quad (43)
\end{align*}
\]

A candidate Lyapunov function is given by \( V(\xi) = C^T \xi(t) \xi(t) \). By using (37), the time-derivative of \( V \) is

\[
\begin{align*}
\dot{V}(\xi(t)) &= -\theta C^T \xi(t) S^{-1}(t)S^{-1}(t) C^T \xi(t) \\
+ 2 \theta \Delta_{\theta}^{-1} \xi(t) C^T \xi(t) - 2 \xi(t) S^{-1}(t) \Delta_{\theta} B \xi(t) \\
+ 2 \xi(t) S^{-1}(t) \Delta_{\theta} \left[ A(u(t), \hat{x}(t), x(t)) \right] x(t) \\
+ \phi(u(t), \hat{x}(t), x(t)).
\end{align*}
\]

According to Equation (42), an over-valuation of \( |\xi(t)| \) can be obtained as

\[
\begin{align*}
|\xi(t)| &= C \int_{t_k}^{t} \left[ \theta A(u(s), \hat{x}(s))\hat{x}(s) + \Delta_{\theta} \tilde{A}(u(s), \hat{x}(s), x(s)) \hat{x}(s) \\
+ \Delta_{\theta} \tilde{\varphi}(u(s), \hat{x}(s), x(s)) \right] ds.
\end{align*}
\]

Then

\[
\begin{align*}
|\xi(t)| &\leq \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} + \sqrt{n L \lambda_{\theta \max}} \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} + \sqrt{n L \lambda_{\theta \max}} \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} \\
\times \int_{t_k}^{t} \sqrt{V(\xi(s))} ds, \quad (46)
\end{align*}
\]

where \( \lambda_{\theta \max} \) and \( \alpha \) were defined by (5). It follows that

\[
\begin{align*}
\|2\xi(t) C^T \xi(t)\| &\leq 2 \theta \sqrt{V(\xi)} \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} + \sqrt{n L \lambda_{\theta \max}} \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} + \sqrt{n L \lambda_{\theta \max}} \frac{\theta \alpha}{\lambda_{\theta \max} (\delta)} \\
\times \int_{t_k}^{t} \sqrt{V(\xi(s))} ds, \quad (47)
\end{align*}
\]

By combining Equations (26), (27), (28), (44) and (47), the time derivative of \( V \) becomes

\[
\begin{align*}
\dot{V}(\xi) &\leq -\theta \sqrt{V(\xi)} + 2 \theta \sqrt{\frac{\theta \alpha}{\lambda_{\theta \max} (\delta)}} \left[ L \lambda_{\theta \max} + L \lambda_{\theta \max} \right] \sqrt{V(\xi)} \\
+ 2 \theta \sqrt{\frac{\theta \alpha}{\lambda_{\theta \max} (\delta)}} \left[ \theta \alpha + \theta \alpha \sqrt{n L \lambda_{\theta \max}} + \theta \alpha \sqrt{n L \lambda_{\theta \max}} \right].
\end{align*}
\]
This means that the function $V'(\tilde{x})$ is a continuously decreasing function, which implies that the observation error will converge to a small region.

Now, as described in the previous section, one should come back to the original coordinates of the observation error $\tilde{x}$, it yields

$$\|\tilde{x}(t)\| \leq \Theta^{-1} \sqrt{\Lambda_{\text{max}}(\Theta) e^{-\Theta'(\Lambda_{\text{max}})\delta_\varepsilon}}$$

where $\Theta = \Theta\nu_\Theta$ and

$$h_\Theta = \frac{2\Theta\left(\Theta_0 + \sqrt{nL_{\gamma_0}}\gamma_{\text{max}} + \sqrt{nL_{\phi}}\right)}{\Lambda_{\text{max}}(\Theta)}$$

$$q_\Theta = \frac{2}{\Theta^{-1}} \sqrt{\Lambda_{\text{max}}(\Theta)\delta_\varepsilon}.$$

According to Theorem 2, the maximum value of the sampling rate $\Delta_{\text{max}}$ satisfies the following condition

$$\Delta_{\text{max}} < \frac{\nu_\Theta\Lambda_{\text{max}}(\Theta)}{2(\Theta_0 + \sqrt{nL_{\gamma_0}}\gamma_{\text{max}} + \sqrt{nL_{\phi}})}.$$  \hspace{1cm} (51)

By applying Lemma 1 with $a_\mu = a_\Theta$, $b_\mu = b_\Theta$, and $c_\mu = q_\Theta$, it results

$$\chi_\Theta(\Delta_{\text{max}}) = (a_\Theta - b_\Theta\Delta_{\text{max}}) e^{-a_\Theta\Delta_{\text{max}}}.$$  \hspace{1cm} (52)

Hence, the function $V'(\tilde{x})$ satisfies Lemma 1:

$$\sqrt{V'(\tilde{x})} \leq e^{-\chi_\Theta(\Delta_{\text{max}})} \sqrt{V'(\tilde{x}(0))}$$

$$+ \phi\Delta_{\text{max}} \frac{2 - e^{-\chi_\Theta(\Delta_{\text{max}})}\lambda_w}{1 - e^{-\chi_\Theta(\Delta_{\text{max}})}\lambda_w}.$$  \hspace{1cm} (53)

From this convergence analysis, it is easy to appreciate that the observation error $\tilde{x}(t)$ exponentially tends to a finite value, which depends on the uncertainties $\delta_\varepsilon$ and the upper bound of the sampling rate $\Delta_{\text{max}}$. This concludes the proof of Theorem 2.

Remark 2. The estimation error of the sampled output observer is a bounded function since inequation (55) is equivalent to Equation (35) achieved with the continuous output observer. This statement can be verified by means of the following relations $\Delta_{\text{max}} = \Delta_{\mu} = \Delta_{\gamma}$, where $\Delta_{\gamma}$, implies a constant sampling time. One then proceeds to prove that the function $\chi_\Theta(\Delta_{\gamma})$ is decreasing with respect to $\Delta_{\gamma}$, one gets

$$\lim_{\Delta_{\gamma} \to 0} \chi_\Theta(\Delta_{\gamma}) = a_\Theta = \Theta\nu_\Theta.$$  \hspace{1cm} (57)

Instead, the function $\Pi(\Delta_{\gamma})$ is not decreasing with respect to $\Delta_{\gamma}$, as follows:

$$\lim_{\Delta_{\gamma} \to 0} \Pi(\Delta_{\gamma}) = \lim_{\Delta_{\gamma} \to 0} \frac{2\Theta\Delta_{\gamma} \frac{2 - e^{-\chi_\Theta(\Delta_{\gamma})\lambda_w}}{1 - e^{-\chi_\Theta(\Delta_{\gamma})\lambda_w}}}{\Delta_{\gamma}}.$$

$$= \lim_{\Delta_{\gamma} \to 0} 2\Theta\Delta_{\gamma} \left[ 1 + \frac{1}{1 - e^{-\chi_\Theta(\Delta_{\gamma})\lambda_w}} \right].$$

This means that the function $V'(\tilde{x})$ is a continuously decreasing function, which implies that the observation error will converge to a small region.
NUMERICAL SIMULATIONS

The smaller the sampling time period \( \Delta t \) is, the stronger the convergence of the observation error is strongly correlated with the observer \( (35) \), therefore, we conclude that the rate of convergence of the pipeline leak detection is an effective measure \( (35) \). An alternative to solve the above problem is to use a mathematical model describing the complex dynamics found in the pipeline, together with the state estimation technique shown in the previous section.

Figure 1 Pump with a pipeline and a tank system

One of the great challenges in this field is to know with precision the friction factor of a pipeline, which is involved in the dynamics of a fluid in a horizontal pipeline. In the beginning, the Moody's Diagram proposed in \( (36) \) was used to compute graphically the friction factor in a pipeline, which is accomplished through the knowledge of the Reynolds number and the absolute roughness. However, this is a rough estimation and it is not meant to be implemented in real-time through computer algorithms. A numeric alternative is the Colebrook–White equation \( (37, 38) \), which considers the relative roughness and the diameter of the pipeline. This approach provides an implicit solution in which a numerical method is required and therefore it represents a considerable computational cost. In this sense, other innovative explicit approaches based on the Colebrook–White equation are presented in \( (39, 40) \). It is clear that the aim is to provide explicit methods to further approximate the friction coefficient. Another explicit approximation has been proposed in \( (41) \), which introduces the Swamee–Jain equation. Nevertheless, in practice, the roughness can be altered by the internal corrosion or the surface wear, and the diameter is affected by tuberculation or mineral deposits, therefore, it is clear that the friction factor varies over time. In order to deal with this problem, a power-law function is proposed in \( (42) \). This function can be used when the friction factor is needed but the physical parameters to calculate it are unknown and it can be used during the pipeline operation. However, the power-law has to be obtained in a two-step process making use of an auxiliary differentiator to estimate the derivative output. Hence, this method can lack precision.

In order to demonstrate the observer functionality, a pump feeding a pipeline-tank system, as depicted in Figure 1, is considered. By taking into consideration the assumptions described in \( (42) \), the momentum and the continuity equations that describe the dynamics of the fluid in a horizontal pipeline are \( (43) \):

\[
\begin{align*}
\frac{\partial Q(z,t)}{\partial t} + gAr\frac{\partial H(z,t)}{\partial z} + f(Q(z,t)) &= 0, \\
\frac{\partial H(z,t)}{\partial t} + \frac{b^2}{gAr} \frac{\partial Q(z,t)}{\partial z} &= 0,
\end{align*}
\]

where \((z,t) \in [0,L] \times [0,\infty)\) involve the space \((m)\) and time \((s)\); \(L\) is the length of the pipeline; \(H(z,t)\) is the pressure head; \(Q(z,t)\) is the flow rate; \(b\) is the wave speed in the fluid; \(g\) is the gravitational acceleration; \(Ar\) is the cross-sectional area of the pipe; \(\sigma\) is the inside diameter of the pipe; and \(f\) is the quasi-steady friction term, which may be expressed by the Darcy–Weisbach relation as follows:

\[
\begin{align*}
f(Q(z,t)) &= \frac{f(Q(z,t))}{2\sigma A_r} Q(z,t)(Q(z,t)), \quad f(Q(z,t)) = \alpha Q'(z,t)Q(z,t),
\end{align*}
\]

where \(f\) is the Darcy–Weisbach friction factor.

Indeed, the friction factor can be accurately calculated by using the Colebrook–White equation, which depends on several parameters: the Reynolds number, the kinematic viscosity, and the pipe’s effective roughness height. This factor is implicit and it has to be solved by using iterative methods which demand a high computational cost. An example of this estimation is shown in \( (44) \). The authors in \( (42) \) propose a novel way of calculating the friction factor, as follows:

\[
\begin{align*}
J_i(Q(z,t)) &= \alpha Q'(z,t)Q(z,t),
\end{align*}
\]
where two empirical parameters are introduced, \( \alpha > 0 \) and \( \gamma \in (0, 1) \). Therefore, the friction factor is based on a power law, this is approximated to two unknown parameters, which are obtained in an empirical way.

Based on the novel power law, the dynamics of the fluid becomes:

\[
\dot{Q}(t) = -\alpha \frac{Q'(t)}{Q(t)} Q(t) + \beta (H_{in}(t) - H_{out}(t)),
\]

(64)

where \( \beta = \frac{\alpha L}{T} \).

It is important to keep in mind that the estimation approach provided in [42] relies on a two-step estimation process: First, an observer estimates the unknown parameter \( \alpha \), and then, a second observer estimates \( \gamma \). In addition, it makes use of an auxiliary differentiator to estimate the derivative output \( \dot{Q}(t) \). All of the above conditions can mean important precision losses in the procedure. In order to achieve a less restrictive design, the following state vector is considered in order to synthesize an observer for the estimation of the unknown parameters \( \gamma \) and \( \alpha \):

\[
\psi(t) = [Q(t) \hspace{1cm} \alpha Q'(t) \hspace{1cm} \gamma]^T.
\]

The derivative of the unknown parameters are zero, that is, \( \dot{\gamma} = 0 \) and \( \dot{\alpha} = 0 \). This is because these parameters are positive constants. By considering the above extended state vector, the system given in Equation (64) can be written as system (1) as follows:

\[
\begin{align*}
\dot{\psi}(t) &= A(\psi(t), u(t))\psi(t) + f(\psi(t), u(t)) \\
\gamma(t_k) &= C\psi(t_k) = \psi^1(t_k) = Q(t_k)
\end{align*}
\]

(66)

where

\[
A(\psi(t), u(t)) = \begin{bmatrix} 0 & -[Q(t)] & 0 \\ 0 & 0 & \frac{\alpha Q'(t)}{Q(t)} g(\psi(t), u(t)) \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
f(\psi(t), u(t)) = \begin{bmatrix} \beta u(t) \\ 0 \end{bmatrix}^T \text{ and } C = [1 \hspace{1cm} 0 \hspace{1cm} 0].
\]

The pressure difference represents the input of the system \( u(t) = H_{in}(t) - H_{out}(t) \), and the sampled output measurements is \( \gamma(t_k) = C\psi(t_k) = \psi^1(t_k) \), that is, the flow rate of the pipeline. Moreover, the non-linear function \( g(\psi(t), u(t)) \) is

\[
g(\psi(t), u(t)) = -[Q(t)]\alpha Q'(t) + \beta u(t).
\]

The original states are derived from the new coordinates (65) as follows:

\[
Q(t) = \psi_1(t) \quad \gamma = \psi_3(t) \quad \text{and} \quad \alpha = \psi_2(t)/\psi_1(t)^\gamma.
\]

It is worth to note that the system (66) is non-uniformly observable with sampled output in presence of the uncertainties, since the matrix \( A(\psi(t), u(t)) \) depends on the state vector and the input of the system.

For this example, it is easy to show that Assumption 1 is verified. The flow \( Q(t) \) is a physical variable whose value varies from a minimum to a maximum value, that is, \( Q(t) \in (Q_{min}, Q_{max}) \). On the other hand, if \( \gamma \in (0, 1) \), then \( \alpha Q'(t) \in (\alpha, \alpha Q_{max}) \), where \( \alpha > 0 \) is an unknown positive constant parameter.

Indeed, Assumption 2 is also verified, since \( A(\psi, u) \) and \( f(\psi, u) \) are Lipschitz. By taking into account the infinity norm:

\[
||A(\psi, u) - A(\psi', u)||_{\infty} = \max \{ |\psi_1 - \psi'_1|, |\psi_3 - \psi'_3|, |\bar{a}_{1,3}(\psi, \psi', u)| \}
\]

\[
= \max \{ |\psi_1 - \psi'_1|, [\bar{a}_{1,3}(\psi, \psi', u)] \},
\]

where

\[
\bar{a}_{1,3}(\psi, \psi', u) = \frac{\psi'_2}{\psi_1} g(\psi, u) - \frac{\psi'_2}{\psi_1} g(\psi', u).
\]

Because of boundedness of state variables, it is evident that there exists a big-enough Lipschitz constant \( L_1 > 0 \) such that

\[
\max \{ |\psi_1 - \psi'_1|, [\bar{a}_{1,3}(\psi, \psi', u)] \} \leq L_1 \max \{ |\psi_1 - \psi'_1|, [\psi_2 - \psi'_2] \},
\]

where

\[
L_1 \cdot \max \{ |\psi_1 - \psi'_1|, [\psi_2 - \psi'_2] \} = L_1 \|\psi - \psi'\|_{\infty}.
\]

On the other hand, by considering boundedness of the input \( u(t) \) it is evident that \( f(\psi, u) \) is Lipschitz with respect to \( \psi \) for a big-enough Lipschitz constant \( I_{\psi} \), that is,

\[
\|f(\psi, u) - f(\psi', u)\|_{\infty} = \beta u \leq I_{\psi} \|\psi - \psi'\|_{\infty}.
\]

This means, for the example, that Assumption 3 is satisfied since the unknown parameters are physically bounded.

In this regard, Assumption 4 is the well-known definition of a regularly persistent input. In this case, this implies that the estimated state \( \hat{\psi}(t) \) and the input \( u(t) \) of system (66) must be such that inequality (11) must not be singular in order to accomplish the new state \( \xi(t) \bar{\psi}_{\xi}(s) \). In other words, the inputs \( s \) and \( \bar{\psi}_{\xi}(s) \) of the system \( \hat{\xi}(t) \bar{\psi}_{\xi}(s) \) must vary in order to have a non-zero transition matrix \( \Phi_{\bar{\psi}_{\xi}}(s) \). In the application example, the input \( u(t) = H_{in}(t) - H_{out}(t) \), with \( H_{in}(t) = 30 + \sin(0.08t) \) mwc and \( H_{out}(t) = 10 \) mwc, the effect of this persistent input can be appreciated in Figure 5. It would be worthwhile indicating that the items \( a_{1,2}(\psi', u) \) and \( a_{2,3}(\psi', u) \) of the matrix \( A(\psi(t), u(t)) \) are not equal to 0, since the state vector \( \hat{\psi}(t) > 0 \), this is further reinforced in the state estimate \( \hat{\psi}(t) \), see Figures 3–5. It is also important to mention that Assumption 4 permits to relax the conditions proposed in [42], which makes use of the derivative output, which entails serious estimation costs. In accordance to
As mentioned above, Assumption 4 has been further verified on system (66), which also implies the fulfillment of Assumptions 1 and 2.

The simulations are performed with MATLAB/Simulink® using the ODE 4 solver (Runge–Kutta) with a step time of $\Delta t = 0.002$ s. The known parameters are the following: $L = 2000$[m], $g = 9.81$[m/s²], $D = 0.3$[m], $b = 1400$[m/s] and $A_r = 0.0707$[m²]. The input is $u(t) = H_{in}(t) - H_{out}(t)$, with $H_{in}(t) = 30 + \sin(0.08t)$ mwc and $H_{out}(t) = 10$ mwc.

The parameters $\gamma$ and $\alpha$ are not available for the proposed observer. The initial conditions are $\psi(0) = [0.12, 0.06, 0.09]$, $\varphi(0) = [0.12, 0.01, 0.08]$, and $S(0) = I_{3 \times 3}$. According to Theorem 1, the tuning parameter is $\theta = 1.25$.

In order to verify the observer performance, two cases are considered. The first case considers a continuous observer design that feeds with sampled output measurements under different sampling time conditions. The second case addresses a continuous-discrete observer (36)–(39) under sampled output measurements. Both simulations are carried out with different sampling periods: $\Delta_{max_1} = 0.002$ s = 500 Hz, $\Delta_{max_2} = 0.01$ s = 100 Hz, $\Delta_{max_3} = 0.1$ s = 10 Hz and $\Delta_{max_4} = 0.3$ s = 3.3 Hz.

The result of the first simulation is illustrated in Figures 2–4. Figure 2 shows that the estimation errors of the flow rate $|$ are gradually becoming degraded, increasing the sampling time instants. This increment of the sampling time compromises further the performance of estimations of the unknown parameters $\alpha$ and $\gamma$, as can be seen in Figures 3 and 4.

The performance of the observer described as our main contribution is presented in the second simulation. The estimations of the flow rate and the unknown parameters are provided by the proposed continuous-discrete observer, as is shown in Figures 5–8. From these figures, it is clear the satisfactory performance of the proposed observer with sampled output.
measurements. It is worth to highlight that the good estimation has been obtained even in the presence of the relative long sampling time, both on the estimates of the flow rate (see Figures 5 and 6) and the unknown parameters (see Figures 7 and 8). One can also note that the proposed continuous-discrete observer presents a response closely matched to the continuous observer in presence of the short sampling times.

From the obtained results, it is clearly evident than the unknown parameters $\alpha$ and $\gamma$ converge to 0.4638 and 0.9917, respectively. It can be concluded that the friction factor function can be obtained, as follows:

$$\dot{j}_f(Q(z, t)) = 0.4638 Q^{0.9917}(z, t) \frac{\partial}{\partial Q(z, t)}.$$  \hspace{1cm} (67)

5.2 | Sampled output measurements under long time-varying sampling instants

In order to evaluate the effectiveness when the continuous-discrete observer deals with long time-varying sampling instants, the observer is tested in two time-varying sampling intervals, which vary randomly in the intervals: $\Delta_1 \in [0.6, 1] \text{ s}$ and $\Delta_2 \in [1.08, 1.8] \text{ s}$ = $[0.55, 0.92] \text{ Hz}$. Figures 11–13 illustrate the performance of the proposed observer. Firstly, Figure 11 shows the original sampled output in both intervals $\Delta_1$ and $\Delta_2$, and their respective estimation of the flow rate. In both cases, convergence to the correct value has been accomplished without any trouble. The unknown parameters $\alpha$ and $\gamma$ are estimated successfully, as can be seen in Figures 12 and 13. Obviously, intervals take an important role in the convergence time. It is clear that the proposed observer with the maximum sample time reaches its convergence in less time than...
the other. Despite this drawback, in both cases it is achieved an asymptotic convergence within a reasonable period of time.

5.3 | Sampled output with measurement noise

In order to highlight the behaviour of the proposed observers (continuous and continuous-discrete) in a practical case, it is assumed the presence of a noisy output measurement. This evaluation assumes that \( Q(t) \) is corrupted by additive Gaussian noise with zero mean value and a variance equal to \( 1 \times 10^{-10} \), such as in [42]. This noisy output is carried out under different sampling time conditions: constant time and time-varying time. Considering that the output is noisy, the tuning parameter should be readjusted. Accordingly \( \sigma = 0.3 \), to reduce the effect of noise. In spite of the convergence time slightly greater than the aforementioned, the observer still guarantees its convergence.

**Case: Constant sampling time**

The noisy output is sampled in two sampling periods: \( \Delta_{\max} = 0.5 \) s = 2 Hz and \( \Delta_{\max} = 1 \) s = 1 Hz. For the first period (\( \Delta_{\max} \)), both observers are tested, and for the second one (\( \Delta_{\max} \)) only the continuous-discrete observer is tested. Figure 14 illustrates the error norm of the proposed observers under two sampled time. It is clear that the continuous observer shows a greater error than the continuous-discrete case, even in the face of the longer sampling rates.

**Case: Time-varying sampling instant**

A series of tests to determine the influence of time-varying sampling noisy measurements are carried out. The noisy output is measured in two random sampling interval varying: \( \Delta_1 \in [0.3, 0.5] \) s = [2, 3.33] Hz and \( \Delta_2 \in [0.6, 1] \) s = [1, 1.67] Hz. Figure 15 plots the observation error norm of the observers. The observers are tested under the first random time-varying sampling interval (\( \Delta_1 \)). It is evident that continuous-discrete observer presents lower values and less oscillations than the continuous one. The continuous-discrete observer also allows to handle the relative long sampling time, for example, (\( \Delta_2 \)), where its performance keeps values close to zero.

To summarise, both observers achieve to converge within a bounded ratio, which is strongly linked to the ultimate bound of the noise. To minimise the noise effect, the rate of convergence is compromised by the adjusting of the tuning parameter \( \sigma > 0 \). It is evident that the observation error of the continuous-discrete observer shows lower values than the continuous one. Moreover, it is important to highlight that even in long time-varying sampling interval the observer achieved to converge, which is a challenging practical case.

5.4 | High-gain proposed versus high-gain Kalman-like observer

To compare the proposed methods to other existing ones, a set of simulations has been carried out. Therefore, a comparison between both proposed high-gain observers (\( (\cdot)_C \), \( (\cdot)_{CD} \)) versus the high-gain Kalman-like (\( (\cdot)_{KL} \)) proposed in [42] is presented. The observers have the same vector state proposed in Equation (65), and the noisy output is sampled in the random sampling interval \( \Delta \in [0.6, 1] \) s = [1, 1.67] Hz, and assuming the same measurement noise of the section above. For the high-gain Kalman-like observer, the tuning parameters are \( \lambda = 1 \) and \( \sigma_{KL} = 0.5 \), the same as the original proposed in [42]. For both proposed observers in this work (\( (\cdot)_C \), \( (\cdot)_{CD} \)), the tuning parameter is \( \sigma = 0.3 \). Figure 16 illustrates the observation error norm of the three observers, by considering only the continuous case (\( (\cdot)_{KL} \), \( (\cdot)_C \)). It is clear that the high-gain Kalman-like observer presents greater values than the high-gain observer proposed here. It is important to emphasise that the observation error of the continuous-discrete observer converges to a much lower value than the two continuous observers due to its structure. Moreover, both proposed observers present lower values than the high-gain Kalman-like observer proposed in [42].
6 | CONCLUSION

In this work, the problem of observer design for a class of non-uniformly observable state-affine non-linear systems under-sampled output measurements, in the presence of uncertainties of the systems, is considered. A simple observer design for the continuous case based on a high-gain structure is provided. The proposed observer is redesigned to tackle the sampled output problem. The originality of the proposed observer is based on the fact that the observability of the uncertain non-linear systems depends on the state vector and the input. This drawback may be accomplished due to the proposed persistent excitation condition. This condition allows to address a more general class of non-linear systems than those studied in previous works. Besides, this approach allows to overcome the sample output measurements in the presence of relative long sampling time.

To highlight the characteristics of the proposed algorithm, numerical simulations of a system consisting of a pump with a pipeline and a tank were presented. The objective of the numerical experiments was to perform the estimation of unknown parameters of the system in order to outline a simple methodology for obtaining the friction factor of the pipeline. Moreover, the numerical experiments considered different sampling instants and reasonably long varying-time sampling periods, whose outputs were also affected with measurement noise; finally, the provided design was compared with an existing approach from the literature. The proposed design was able to overcome all these inconveniences.

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