Magnetic moments of mesons

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Abstract
Magnetic moments of charged and neutral mesons are calculated with the use of the relativistic Hamiltonian derived from the path integral form of the $q_1\bar{q}_2$ Green’s function. The magnetic moments are shown to be expressed via the average quark energies which are defined by the fundamental quantities: the string tension $\sigma$, the current quark masses, and the strong coupling constant $\alpha_s$. Resulting values for vector, axial, and tensor light and $K$ mesons agree well with all available lattice data.

1 Introduction
Magnetic moments of hadrons are important dynamical characteristics, which can be useful in many areas, e.g for nucleons, and can give new information on the hadron dynamics, being also a serious test of the dynamics put in a model. In this paper we shall exploit the QCD dynamics in the form of the relativistic Hamiltonian (RH) of a meson, derived from the QCD path integral [1], which was already extensively used in the studies of hadron spectra without external fields [2]. The extension of the RH to the case of external magnetic field (MF) has been done recently in Ref. [3], where the meson spectrum as a function of MF was calculated.

Within this method the magnetic moments of baryons have also been studied analytically in Ref. [4] and for the lowest hyperons their values, calculated in the first approximation, agree with experiment within 10% accuracy.
(The same accuracy was achieved for the baryon magnetic moments before, in Ref. [5], where the QCD string dynamics was exploited with approximated wave function).

At the same time for last decades a thorough analysis of the hadron magnetic moments is also being done in the lattice QCD (see [6] for the review).

In this paper we are mainly interested in the magnetic moments of vector, axial, and tensor mesons and to this end, develop further the method of Ref. [4], suggested for baryons. For the sake of generality, here we shall use the RH in MF, both for charged and neutral mesons, from Refs. [3, 7] and in this way define all terms in the expansion of the hadron mass in powers of MF. (This is done in Section 2). We show that the meson magnetic moments are easily calculated through the average quark energies $\omega_i$, which here are defined by the minimal set of the QCD fundamental parameters: the string tension $\sigma$, $\alpha_s$, and the current quark masses, not introducing any fitting parameters.

Having done our analytical calculations, in Section 3 we compare our results for light and $K$ mesons with the lattice calculations of the meson magnetic moments [8, 9] and find a good agreement within the accuracy of the lattice and our calculations. Discussion of the results and prospectives are given in Section 4.

2 The Hamiltonian for a meson in magnetic field

The path-integral Hamiltonian for the $q_1\bar{q}_2$ system in MF was derived in Ref. [3, 7] and has the form,

$$ H = H_0 + H_\sigma + W, $$

where

$$ H_0 = \sum_{i=1}^{2} \frac{\left(\mathbf{p}^{(i)} - \frac{e_i}{2}(\mathbf{B} \times \mathbf{z}^{(i)})\right)^2 + m_i^2 + \omega_i^2}{2\omega_i}, $$

$$ H_\sigma = \frac{e_1 \sigma_1 B}{2\omega_1} - \frac{e_2 \sigma_2 B}{2\omega_2}, $$

$$ W = \sum_{i=1}^{2} \frac{1}{2} \kappa_i (\mathbf{z}^{(i)} - \mathbf{z}_0)^2, $$
In (1) the term \( W \) contains the confinement potential \( V_{\text{conf}} \), the perturbative gluon-exchange (GE) potential \( V_{\text{GE}} \), and spin-dependent interaction \( V_{\text{SD}} \), as well as the nonperturbative self-energy term \( \Delta M_{\text{SE}} \). 

\[
W = V_{\text{conf}} + V_{\text{GE}} + V_{\text{SD}} + \Delta M_{\text{SE}}.
\] (4)

All these terms have been introduced and extensively studied in case without MF in Refs. [2], [10]-[12].

In (2),(3) the following basic elements of the path-integral approach enter: the average energies \( \omega_i \), which play actually the role of the einbein parameters [13], being defined from the eigenvalues \( M_n(\omega_1, \omega_2) \) of the Hamiltonian \( H \).

\[
H \Psi = M_n \Psi,
\] (5)

using the stationary point equations:

\[
\frac{\partial M_n(\omega_1 \omega_2)}{\partial \omega_1} \bigg|_{\omega_i = \omega_i^{(0)}} = 0, \quad i = 1, 2.
\] (6)

As a result \( M_n(\omega_1^{(0)}, \omega_2^{(0)}) \) is our prediction for the mass of a given meson.

In the case without MF the eigenvalues \( M_n \) were already calculated for all kinds of mesons: light-light [2], heavy-light [14], heavy quarkonia [15], and in all cases good agreement with experiment was obtained. Note, that \( M(\omega_1^{(0)}, \omega_2^{(0)}) \) are the functions of the current quark masses, the string tension \( \sigma \), and the strong coupling \( \alpha_s(r) \), i.e. do not contain any fitting parameters, in contrast to other relativistic model approaches.

Then we introduce the c.m. and relative coordinates of the \( q_1 \bar{q}_2 \) system,

\[
R = \frac{\omega_1 z_1^{(1)} + \omega_2 z_2^{(2)}}{\omega_1 + \omega_2}, \quad \eta = z_1^{(1)} - z_2^{(2)},
\] (7)

and also make an ansatz for the wave function,

\[
\Psi(\eta, R) = \exp(i\Gamma)\varphi(\eta, R),
\] (8)

where \( \Gamma = PR - \frac{e}{2}(B \times \eta)R \). Then one can get the Hamiltonian \( H'_0 \), acting on \( \varphi(\eta, R) \). If the c.m. motion is chosen to associate with the total charge of the meson, equal \( e_1 + e_2 \), then one has to put \( \bar{e} = \frac{e_1 - e_2}{2} \) and the new Hamiltonian \( H'_0 \) is obtained in the form,

\[
H'_0 = \frac{P^2}{2(\omega_1 + \omega_2)} + \frac{(\omega_1 + \omega_2)\Omega_R^2 R_1^2}{2} + \frac{\pi^2}{2\bar{\omega}} + \frac{\bar{\omega}\Omega_\eta^2 \eta_1^2}{2} + X_{LP}B'L_P + \ldots
\]
+X_{L_{\eta}} B L_{\eta} + X_{1} P(B \times \eta) + X_{2}(B \times R)(B \times \eta) + X_{3}\pi(B \times R) + \frac{m^2_1 + \omega^2_1}{2\omega_1} + \frac{m^2_2 + \omega^2_2}{2\omega_2}. \tag{9}

Here \( L_{\eta} = (\eta \times \frac{\partial}{\partial \eta}) \), \( L_{P} = (R \times \frac{\partial}{\partial R}) \), \( \tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2} \). All coefficients \( X_i \) are given in Appendix 1, while \( \Omega_R, \Omega_\eta \) are following,

\[ \Omega^2_R = B^2 \frac{(e_1 + e_2)^2}{16 \omega_1 \omega_2}, \tag{10} \]
\[ \Omega^2_\eta = \frac{B^2}{2\tilde{\omega}(\omega_1 + \omega_2)^2} \left[ \frac{(\epsilon_1 \omega_2 + \epsilon \omega_1)^2}{2\omega_1} + \frac{(\epsilon_2 \omega_1 - \epsilon \omega_2)^2}{2\omega_2} \right]. \tag{11} \]

Our purpose here is to study the first order corrections \( O(eB) \), proportional to spin and the angular momentum \( L_\eta \), to the meson mass \( M(0) \) in the total expansion of the meson mass \( M(B) \) in powers of the MF \( B \) (here the relation \( L_P \equiv 0 \) is assumed):

\[ M(B) = M(0) - \mu_s B + X_{L_{\eta}} L_{\eta} B + \sum_{n=1}^{\infty} \kappa_n B^n. \tag{12} \]

Taking into account (9), (10), (11), one arrives at the mass formula for a meson in the form,

\[ M(B) = \frac{p^2}{2(\omega_1 + \omega_2)} + \Omega_R(2n_R + 1) + \frac{\tilde{\omega} \Omega^2_\eta \Omega^2_{\eta\perp}}{2} + \frac{m^2_1 + \omega^2_1}{2\omega_1} + \frac{m^2_2 + \omega^2_2}{2\omega_2} - \frac{e_1 \sigma_1 B}{2\omega_1} - \frac{e_2 \sigma_2 B}{2\omega_2} + X_{L_{\eta}} L_{\eta} B + \langle \Delta M_X \rangle. \tag{13} \]

In (13) the term \( \Delta M_X \) implies the sum of all terms with coefficients \( X_i \) \((i = 1, 2, 3)\).

Now, expanding (13) in powers of \( B \), one can write \( (L_\eta \equiv 1) \)

\[ M(B) = M(0) - \mu B + B \kappa_1 + B^2 \kappa_2 + X_1 B + O(B^3), \tag{14} \]

where

\[ \mu = \frac{e_1 \sigma_1}{2\omega_1^{(0)}(B = 0)} + \frac{e_2 \sigma_2}{2\omega_2^{(0)}(B = 0)} - X_1. \tag{15} \]

Here the term \( B \kappa_1 \) can be obtained, expanding \( \omega_i^{(0)}(B) \) in \( B \) and keeping the first order term, but all these terms do not contribute to the magnetic
moments. Also in (13) \( \langle \Delta M_X \rangle \) contributes only to terms \( O(B^2) \), if the vector state considered has no internal angular momenta \( L_P \). In this case all terms in \( \Delta M_X \) vanish in the first order, \( (\Psi \Delta M_X \Psi) = 0 \). As a result magnetic moments of mesons in (15) are calculated through only \( B = 0 \) characteristics of the \( q \bar{q} \) system, namely only \( \omega_i^{(0)}(B = 0) \) are needed, while magnetic polarizabilities require higher terms in \( (eB) \) from (13).

## 3 Results for vector, axial, and tensor mesons

The expression (15) allows to calculate the magnetic moments of mesons with different quantum numbers, both for the angular momentum \( l = 0 \) and \( l \neq 0 \). Below we perform calculations for the light and \( K \) mesons, while in similar way the magnetic moments of heavy-light and heavy-heavy mesons can be also defined.

a) The case of zero internal angular momentum, \( l = 0 \)

The absolute value \( \mu_S \) of the magnetic moment \( \mu = \mu_S \mathbf{S} \) for the \( S \)-wave mesons with \( l = 0 \) can be obtained, taking \( S_z = \frac{1}{2}(\sigma_{1z} + \sigma_{2z}) = +1 \), so that one can write

\[
\mu_S = \frac{e_1}{2\omega_1^{(0)}(B = 0)} + \frac{e_2}{2\omega_2^{(0)}(B = 0)},
\]

where \( e_1(e_2) \) refers to the charge of quark \( q_1 \) (antiquark \( \bar{q}_2 \)). Both \( \omega_i^{(0)}(B = 0) \) are supposed to be found from the same Hamiltonian (11) with \( B = 0 \), using as the only input \( \sigma, \alpha_s(r) \), and the current quark masses \( m_i \) ; their values are discussed in details in Appendix 2.

The values of \( \omega_i^{(0)}(B = 0) \) for \( \rho \) and \( K^* \) were taken from [2], [16] and listed in Tables I, II. Notice that the values of \( \omega_i^{(0)} \) given there refer to the case when GE interaction with the standard form of \( \alpha_s(r)(n_f = 3) \) is taken into account, while in the absence of GE potential their values for the ground states of light and \( K \) mesons would be by \( \sim 15\% \) smaller (see (A2.14)).

The most simple case refers to \( \rho^\pm \), when neglecting quark masses \( (m_u = m_d = 0) \), the following values were calculated: \( \omega_1^{(0)}(B = 0) = \omega_2^{(0)}(B = 0) \equiv \omega_0 = 0.397 \text{ GeV} \). Then in nuclear magnetons (n.m.)

\[
\mu_S(\rho) \equiv \mu_\rho = \frac{e}{2\omega_0} = \frac{M_P}{\omega_0} \text{ (n.m.)}.\]

(17)
In this case the magnetic moment, \( \mu_{\rho}(1S) = 2.37 \text{ n.m.} \), appears to be large (see Table I) and close to that calculated on the lattice in Refs. [8], [9], where \( \mu_{\rho}(1S, \text{lat}) \approx 2.4 \text{ n.m.} \). In the same way for higher radial excitations the magnetic moments \( \mu_{\rho}(2S) \) and \( \mu_{\rho}(3S) \), given in last column of Table I, are obtained. Note, that in our calculations we have neglected the S-D mixing of excited \( \rho \) states, as well as the influence of the spin-spin interaction \( V_{ss} \) on the value of \( \omega_0 \), i.e. \( V_{ss} \) is considered as the first order correction to the \( \rho \) mass, which is defined via \( \omega_0 \).

The same procedure is used for the \( K^*\pm \) mesons, for which important values are given in Table II. For \( K^* \) a close agreement with lattice data from Refs. [8], [9] also takes place (see Table III).

Of special interest are the magnetic moment of neutral vector meson \( K^{*0} \). As it follows from (15), if \( e_1 = -e_2 \), then the magnetic moment is proportional to the difference \( (\omega_1^{(0)} - \omega_2^{(0)}) \), which in its turn is proportional to \( (m_1^2 - m_2^2) \) and vanishes, when the current masses of the strange quark and d-quark are taken to be equal, \( \tilde{m}_s = m_d \). For \( K^{*0}(d\bar{s}) \), using (16) with \( e_1 = -\frac{\xi}{3}, e_2 = +\frac{\xi}{3} \) and taking \( \omega_1 = \omega_n, \omega_2 = \omega_s \) for \( m_d = 0, m_s = 0.2 \text{ GeV} \) from Table II, one obtains the magnetic moment of \( K^{*0}(d\bar{s}) \):

\[
\mu(K^{*0}) = -0.0972 \text{e (GeV)}^{-1} = -0.183 \text{ n.m.,}
\]

which is much smaller than the magnetic moment of \( K^{*+} \). The same result is clearly seen in lattice data [8], where for a neutral meson the linear dependence of its magnetic moment on the squared mass \( m_q^2 \), which is proportional to \( m_q^2 \), is observed, thus corresponding to small magnetic moment.

b) The case of nonzero \( l \)

Firstly one can consider the simple situation, when \( S = 0, \ l \neq 0 \) and in this case clearly \( J = l \) and \( \mu = \mu_lJ \). Hence, one has a simple correspondence:

\[
\tilde{\mu} = \mu_l \cdot \frac{2M_p}{e} \text{ (n.m.)}, \ \mu_l \equiv -X_{2\eta} = \frac{e_1\omega_2^2 + e_2\omega_1^2}{2\omega_1\omega_2(\omega_1 + \omega_2)}.
\]

The values of \( \tilde{\mu} \) for \( l = 1, 2 \) are given in Tables II, III.

In another case, if \( S = 1 \), one should define the average value of \( \mu \) in the situation, when \( J \) is the only vector of the system, as
\[ \bar{\mu} = \langle J, J_z | \mu_z | J, J_z \rangle; \quad J_z = J; \quad \bar{\mu} = \sum_{m_s, m_l} |C_{Sm \bar{s}lml}^J|^2 (\mu_S m_S + \mu_l m_l), \] (20)

where \( \mu \) is given by

\[ \mu = \mu_S S + \mu_l l, \quad \mu_S = \frac{e_1}{2\omega_1} + \frac{e_2}{2\omega_2}; \] (21)

and \( J = l + S \). As a result, one obtains the averaged magnetic moment in units \( e(\text{GeV}^{-1}) \),

\[ \bar{\mu} = \frac{\mu_S + \mu_l}{2} (J = l = S = 1), \quad \bar{\mu} = \frac{3\mu_l - \mu_S}{2} (J = 1, l = 2, S = 1), \] (22)

where \( \mu_S, \mu_l \) are given in (21) and (19), respectively.

Then from Eq. (22) one can easily find magnetic moments of all meson states with \( l = 1, 2 \) and \( S = 1 \); they are given in Table 1 for light mesons and in Table 2 for strange mesons.

These results refer to the positive (negative) charge mesons, while for neutral light mesons with \( \omega_1 = \omega_2 \) their magnetic moments are identically zero. For the strange neutral mesons, as mentioned above, their magnetic moments are proportional to \( (\omega_1 - \omega_2) \sim m_1^2 - m_2^2 \) and therefore they are also very small. While using corresponding values of \( \mu_S, \mu_l \) with \( e_1 = -e_2 \) from (19), (21), one can find the values of \( \bar{\mu} \) for any neutral meson with \( S = 1 \), using (22). It is important that the Eq. (20) is applicable for the states with all possible values of \( S, l, \) and \( J \).

4 Discussion of results and conclusions

The main result of our study is the expression (20) for the magnetic moment of an arbitrary meson, expressed through the factors \( \mu_S, \mu_l \) and hence, through the averaged quark energies \( \omega_1, \omega_2 \). The latter are calculated here within the same relativistic Hamiltonian (see Appendix 2), which contains only the first principle input of QCD: current quark masses, string tension, and \( \alpha_s \) in coordinate space. In this way magnetic moments of mesons in the \( LS \) scheme, neglecting the \( L, L \pm 2 \) mixing, were calculated. This mixing can be easily included within our method, provided the mixing amplitudes are
known from experiment ($e^+e^-$ cross sections of mesons), or from theoretical models.

Our main results for the light and $K$ mesons are presented in Tables 1,2 and in Tables 3-5 they are compared to other calculations. The most significant our result is for the mesons with the spin $S = 1$ (see Table 3), where the magnetic moments of different mesons are compared to all existing lattice calculations, since both approaches are of the first principle QCD calculations.

From Table 3 one can see an encouraging agreement within the accuracy of the lattice calculations for all mesons considered. This agreement can be further detailed, using dependence of the averaged energies $\omega_i$ on varying current quark masses: it can be translated in the lattice study of the magnetic moment dependence on the pion mass squared – see a quantitative analytic analysis of this dependence in a recent publication [17].

In Tables 4,5 calculated magnetic moments are compared to the existing model calculations: the sum rule approach [18]-[20], the constituent quark model [21], the Dyson-Schwinger approach [22]. One can see a reasonable agreement only for the $S$-wave mesons, like $\rho^+, K^{*+}$; however, in other cases a strong disagreement is obtained and this implies that the sum rule method is less reliable for higher excitations of mesons. Another line of possible development is the calculation of magnetic moments for growing MF, in which case for some mesons, the energies $\omega_i$ are growing like $\sqrt{eB}$ and therefore their magnetic moment decrease. For other mesons, having different spin projection on the MF, their $\omega_i$ decrease [3], [7] and in this case their magnetic moments are growing with the MF.

As it is, our results give an additional support for the relativistic Hamiltonian approach, generalized here to the case of arbitrary strong magnetic and electric fields as in Ref. [3]. We have shown that the varying quark energies $\omega_i$ and their final stationary point values $\omega_i^{(0)}$, used in our approach, yield the relevant physical information, which can be applied in different directions: to calculate meson masses with and without magnetic field, the meson magnetic or electric moments of different mesons in strong magnetic field. Our results on magnetic moments can be easily generalized to the case of heavy-light or heavy-heavy mesons.

One should stress that our approach, which is quite successful also in case of magnetic moments, considers only valence part of the wave functional, while sea quark part is ignores. One might expect, that the additional $q\bar{q}$
Table 1: The light meson magnetic moments $\bar{\mu}$ ($\sigma = 0.18$ GeV$^2$, $m_u = m_d = 0$, $\omega(nl) = \omega_1 = \omega_2$)$^a$.

| meson state | $\omega(nl)$ (GeV) | $\bar{\mu}$ (n. m.) |
|-------------|---------------------|---------------------|
| 1S          | 0.397               | 2.37                |
| 2S          | 0.549               | 1.71                |
| 3S          | 0.667               | 1.41                |
| 1P          | 0.489               | $a_1^\pm (^3P_1)$ 1.44; $a_2^\pm (^3P_2)$ 2.88 $b(^1P_1)$ 0.96 |
| 2P          | 0.616               | $a_1^\pm$ 1.14; $a_2^\pm$ 2.28 $b_1(^1P_1)$ 0.76 |
| 1D          | 0.571               | $a_3^\pm$ 3.29; $\rho(^3D_1)$ 0.411 |
| 2D          | 0.681               | $a_3^\pm$ 2.76; $\rho(^3D_1)$ 0.345 |

$^a$ The parameters of $\alpha_{GE}(r)$ are given in Appendix 2, Eqs. (A.2.8)-(A.2.12). Components are important for high excited mesons near decay thresholds, and they can be accounted for by the multichannel formalism.

The same method can be used to study other relativistic systems of two or more constituents: atoms or positronium, different hadrons, and also nuclei in strong MF. The relevant physical situation may exist in astrophysics (magnetars) and in colliding ions, as discussed in [3].

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Table 2: $K$ meson magnetic moments ($m_n = 0, m_s = 0.20$ GeV, $\sigma = 0.18$ GeV$^2$, $\alpha_{GE}^a$)

| state | $\omega_n$ (GeV) | $\omega_s$ (GeV) | $\bar{\mu}$ (n.m.) |
|-------|-----------------|-----------------|-------------------|
| 1S    | 0.411           | 0.467           | 219               |
| 2S    | 0.559           | 0.612           | 1.73              |
| 3S    | 0.676           | 0.722           | 1.36              |
| 1P    | 0.500           | 0.544           | $K^{*0}(3P_2)$, $-0.10$ $K^*(3P_1)$, 1.38 $K_1^*(4P_1)$, 0.93 $K_2^*(3P_2)$, 2.76 |
| 2P    | 0.625           | 0.667           | $K^*(3D_1)$, 0.405 |
| 1D    | 0.580           | 0.617           | $K^*(3D_1)$, 0.183 |
| 2D    | 0.689           | 0.725           |                   |

$^a$) See the footnote to Table 1.

Table 3: Magnetic moments (in n.m.) of the lowest vector and axial mesons in comparison with lattice calculations

| meson       | $\rho^+(1S)$ | $K^{*+}$ | $a_1^+(1^3P_1)$ | $K^{*+}(3P_1)$ | $\rho_1(3D_1)$ | $K^{*+}(3D_1)$ |
|-------------|---------------|---------|-----------------|----------------|---------------|----------------|
| m.m. (this paper) | 2.37          | 2.194   | 1.44            | 1.38           | 0.411         | 0.405          |
| m.m. $^\ast$) lattice [8] | 2.4         | 2.4     | 1.5             | 1.5            | 0.5           | 0.5            |
| m.m. $^\ast$) lattice [9] | 2.3         | 2.1     | -               | -              | -             | -              |

$^\ast$) The values of lattice magnetic moments are taken from the Figs. [8],[9] and their accuracy is $\gtrsim 10\%$.

Table 4: Magnetic moments of vector and axial mesons (in n.m.) in comparison with other model calculations

| $\rho^+$ | $K^{*+}$ | $K^{*[0]}$ | $a_1^+$ | source                                      |
|----------|----------|------------|--------|---------------------------------------------|
| 2.37     | 2.19     | -0.183     | 1.44   | this paper                                  |
| 2.4 $\pm$ 0.4 | 2.0 $\pm$ 0.4 | -0.28 $\pm$ 0.04 | 3.8 $\pm$ 0.6 | [18] QCD sum rules                          |
| 1.92     | -        | -          | -      | [20] light-front                            |
| 2.14     | -        | -          | -      | [21] light-front                            |
| 2.2      | 2.08     | -0.08      | -      | [22] Schwinger-Dyson eq.                     |
Table 5: Magnetic moments (in n.m.) of the lowest tensor mesons

| meson          | $f_2^l$ | $f_2^l$ | $a_2^l$ | $\alpha_2^l$ | $K_2^+(1430)$ | $K_2^{*0}(1430)$ |
|----------------|---------|---------|---------|---------------|---------------|-----------------|
| m.m. (this paper) | 2.88    | 0       | 2.88    | 0             | 2.76          | -0.10           |
| m.m. [19]      | 2.1±0.5 | 0       | 1.88±0.4| 0             | 0.75±0.08     | 0.076±0.008     |

Appendix 1

Coefficients $X_i, i = L\eta, LP, 1, 2, 3$

$X_{L\eta} = -\frac{e_1\omega_2^2 + e_2\omega_1^2}{2\omega_1\omega_2(\omega_1 + \omega_2)}$ (A1.1)

$X_{LP} = -\frac{e_1 + e_2}{2(\omega_1 + \omega_2)}$ (A1.2)

$X_1 = \frac{e_2\omega_1 - e_1\omega_2 - \bar{e}(\omega_1 + \omega_2)}{2(\omega_1 + \omega_2)^2}$ (A1.3)

$X_2 = \frac{-\omega_2(\bar{e} - e_1)(e_1\omega_2 + \omega_1\bar{e}) - (\bar{e} + e_2)(e_2\omega_1 - \bar{e}\omega_2)\omega_1}{4\omega_1\omega_2(\omega_1 + \omega_2)}$ (A1.4)

$X_3 = \frac{\bar{e}(\omega_1 + \omega_2) - e_1\omega_2 + e_2\omega_1}{2\omega_1\omega_2}, \quad \bar{e} = \frac{e_1 - e_2}{2}$ (A1.5)

Appendix 2

The Hamiltonian without magnetic fields

If there are no MF $B$, the relativistic Hamiltonian $H_0$ [1] can be presented as

$$\tilde{H}_0 = \frac{\omega_1}{2} + \frac{m_1^2}{2\omega_1} + \frac{\omega_2}{2} + \frac{m_2^2}{2\omega_2} + \frac{p^2}{2\omega} + V_{\text{static}}(r) \equiv T_R + V_{\text{stat}}(r),$$ (A2.1)
where by derivation the quark mass cannot be chosen arbitrarily and must be equal to the current mass $\bar{m}_q$ for the $u, d$, and $s$ quarks. In our calculations $\bar{m}_q = 0$ for the $u, d$ quarks, $m_s \simeq \bar{m}_s(1 \text{ GeV}) = 200 \text{ MeV}$ for the $s$ quark \cite{23}. Here the mass $m_s$ is larger than the conventional $\bar{m}_s(2 \text{ GeV}) = 95 \pm 20 \text{ MeV}$, taken at the scale $\mu = 2 \text{ GeV}$, and the reason for that difference originates from the fact that in the static interaction the $s$-quark current mass $m_s(\mu)$ enters at a smaller scale, $\mu \sim 1 \text{ GeV}$. It is important that the current quark masses, used in our relativistic Hamiltonian, allow to avoid such fitting parameters as the constituent quark masses, usually present in other models.

In the static interaction, $V_{\text{stat}} = V_{\text{conf}} + V_{\text{GE}}$, the linear confining potential $V_{\text{conf}} = \sigma \cdot r$ is taken here with the string tension $\sigma = 0.18 \text{ GeV}^2$, which cannot be considered as a fitting parameter, since its value follows from the slope of the Regge trajectories for light mesons \cite{2}.

The choice of GE potential is important for low-lying light and $K$ mesons, while its influence is much smaller for high excitations. Here we use the vector strong coupling, denoted as $\alpha_B(r)$, which possesses the asymptotic freedom property and freezes at large distances at the value $\alpha_{\text{crit}}$, and its parameters are not arbitrary, as we show below.

The variables $\omega_i$, entering RH $\tilde{H}_0$, have to be determined from the extremum conditions: $\frac{\partial H}{\partial \omega_i} = 0 \,(i = 1, 2)$, that gives

$$\omega_i(nl) = \langle \sqrt{\mathbf{p}^2 + m_i^2} \rangle_{nl} \quad (i = 1, 2). \quad (A2.2)$$

These average energies, $\omega_1(nl)$ and $\omega_2(nl)$, refer to the quark $q_1$ and the antiquark $\bar{q}_2$, while $\tilde{\omega}$ is the reduced mass, $\tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}$, and $l = l_1 + l_2$. Then putting $\omega_i$ into Eq. (A2.1), one arrives at a different form of the kinetic energy term, denoted as $T_R$:

$$T_R = \sqrt{\mathbf{P}^2 + m_q^2} + \sqrt{\mathbf{P}^2 + m_c^2}. \quad (A2.3)$$

Rigorously, the expression (A2.3) for $T_R$ is valid only for $l = 0$, while in general case, for $l \neq 0$, $T = T_R + T_{\text{str}}$ contains additional kinetic energy term, $T_{\text{str}}$, which appears because, besides a standard rotation of a quark and an antiquark, the string rotates itself. It was shown in Refs. \cite{2}, \cite{24} that for $l \leq 4$ the contribution from $T_{\text{str}}$ is relatively small compared to the e.v. $M_0(nl)$ and therefore $T_{\text{str}}$ can be considered as a perturbation. Still its matrix element (m.e.) $\Delta_{\text{str}}(nL) = \langle T_{\text{str}} \rangle_{nl}$ has to be included in the mass formula of a meson.
Then the e.v. \( M_0(nl) \) and the meson w.f. are defined by the spinless Salpeter equation (SSE):

\[
[T_R + V_B(r)] \varphi_{nl} = M_0(nl) \varphi_{nl}.
\]

However, the spin-averaged meson mass \( M(nl) \equiv M_{\text{cog}}(nl) \) includes not only the e.v. \( M_0(nl) \) \((A2.4)\), but also two additional negative contributions: the string correction \( \Delta_{\text{str}}(nl) = \langle H_{\text{str}}\rangle_{nl} \) \([2]\), if \( l \neq 0 \), and the nonperturbative self-energy (SE) term \( \Delta_{\text{SE}}(nl) \) \([10]\):

\[
M(nl) = M_0(nl) + \Delta_{\text{str}}(nl) + \Delta_{\text{SE}}(nl).
\]

For given quantum numbers \( n, l \) the string correction increases for larger \( l \), while for a fixed \( l \) it decreases for higher radial excitations. For the \( 1P \) and \( 1D \) light mesons their values are typically equal to \( \sim -40 \text{ MeV}, -70 \text{ MeV} \), respectively (they were calculated using analytical expressions for \( \Delta_{\text{str}} \) from \([1],[2],[24]\)).

The nonperturbative SE correction to the quark (antiquark) mass is of great importance to provide linear behavior of the Regge trajectories \([2]\). As shown in Ref. \([10]\), this correction is flavor-dependent, depends on the averaged energy of a quark, being very small for a heavy quark and large for a light (strange) quark:

\[
\Delta_{\text{SE}} = -\frac{3\sigma}{2\pi} \left( \frac{\eta_1}{\omega_1(nl)} - \frac{\eta_2}{\omega_2(nl)} \right).
\]

The factor \( \eta_f \) \((f = 1, 2)\) depends on the quark flavor and the vacuum correlation length: \( \eta_1 = 1.0 \) for a light quark, \( \eta_s = 0.70 \) for the \( s \) quark. Notice that the number \( 3/2 \) enters the SE term \((A2.6)\), instead of the number \( 2 \) derived before in \([10]\); this change comes from more exact definition of the vacuum correlation length \([25]\).

From Eq. \((A2.6)\) one can see that the averaged quark energies \( \omega_i \) play a special role: they determine both the string and the SE contributions, and also enter all spin-dependent potentials. In some potential models a negative overall constant is often introduced in the mass term (which plays the role of a self-energy correction), however, such a constant violates the linear behavior of the Regge trajectories.

The “linear+GE” potential \( V_B(r) \), was already tested in a large number of previous studies of heavy-light mesons \([14]\) and heavy-quarkonia \([15]\):

\[
V_B(r) = \sigma r - \frac{4\alpha_B(r)}{3r},
\]

\[13\]
where the vector coupling $\alpha_B(r)$ is taken as in background perturbation theory \[26\] with the freezing value $\alpha_{\text{crit}} = 0.495 \ (n_f = 3)$.

The vector coupling in coordinate space is defined through the vector coupling $\alpha_B(q^2)$ in the momentum space:

$$\alpha_B(r) = \frac{2}{\pi} \int_0^\infty dq \frac{\sin(qr)}{q} \alpha_B(q), \quad (A2.8)$$

which is taken in two-loop approximation,

$$\alpha_B(q) = \frac{4\pi}{\beta_0 t_B} \left( 1 - \frac{\beta_1 \ln t_B}{\beta_0^2 t_B} \right). \quad (A2.9)$$

Here the logarithm,

$$t_B = \frac{q^2 + M_B^2}{\Lambda_B^2}, \quad (A2.10)$$

contains the QCD constant $\Lambda_B(n_f)$, which is defined via the QCD constant $\Lambda_{\overline{MS}}(n_f)$ in the $\overline{MS}$ scheme. The relation between them has been established in \[27\]:

$$\Lambda_B(n_f) = \Lambda_{\overline{MS}} \exp \left( -\frac{a_1}{2\beta_0} \right), \quad (A2.11)$$

with $\beta_0 = 11 - \frac{2}{3} n_f$ and $a_1 = \frac{31}{3} - \frac{10}{3} n_f$. From the relation \[A2.11\] one can see that for a given $n_f$ the constant $\Lambda_B(n_f)$ is significantly larger than $\Lambda_{\overline{MS}}$:

$$\Lambda^{(5)}_B = 1.3656 \Lambda^{(5)}_{\overline{MS}} \ (n_f = 5); \quad \Lambda^{(3)}_B = 1.4753 \Lambda^{(3)}_{\overline{MS}} \ (n_f = 3). \quad (A2.12)$$

At present only the QCD constant $\Lambda^{(5)}_{\overline{MS}}$ (for $n_f = 5$) is well known from experimental value of $\alpha_s(M_Z) = 0.1182 \pm 0.0012$; then in two-loop approximation it gives $\Lambda^{(5)}_{\overline{MS}}$ (two-loop) = 232(12) MeV. For $n_f = 3$ the QCD constant $\Lambda^{(3)}_{\overline{MS}}$ is not known with a good accuracy and to define it we fix here the freezing value $\alpha_{\text{crit}}$: $\alpha_{\text{crit}}(n_f = 3) \simeq 0.495$. In \[A2.10\] the background mass $M_B$ also enters; its value is proportional to $\sqrt{\sigma}$ and for $\sigma = 0.18$ GeV$^2$ the number $M_B = 1.0 \pm 0.05$ GeV was extracted from a detailed comparison of the static force in the field correlator method used and in the lattice QCD \[28\] (here we take $M_B = 1.0$ GeV).

The important feature of the critical couplings is that in the momentum and coordinate space they coincide, $\alpha_B(\text{crit}) = \alpha_B(q^2 = 0) = \alpha_B(r \to \infty)$:

$$\alpha_B(\text{crit}) = \alpha_B(r \to \infty) = \alpha_B(q = 0) = \frac{4\pi}{\beta_0 t_0} \left( 1 - \frac{\beta_1 \ln t_0}{\beta_0^2 t_0} \right), \quad (A2.13)$$

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with \( t_0 = t_B(q^2 = 0) = \ln\left( \frac{M_B^2}{\Lambda_B^2} \right) \). Thus in our calculations \( \Lambda_B(n_f = 3) = 360 \) MeV, \( M_B = 1.0 \) GeV, \( \alpha_{\text{crit}} = 0.4945 \), and \( \sigma = 0.18 \) GeV².

In [2, 16] it was shown that the GE interaction remains important for the ground states of light, K, and \( \phi \) mesons, e.g. if GE interaction is neglected, then

\[
\begin{align*}
\omega_1^{(0)}(1S) &= \omega_2^{(0)}(1S) = 335 \text{ MeV for light mesons} \\
\omega_1^{(0)}(1S) &= 347 \text{ Mev} \\
\omega_2^{(0)}(1S) &= 411 \text{ MeV}
\end{align*}
\]

for K mesons, while their values \( \omega_1(nl), \omega_2(nl) \) increase if GE interaction is taken into account (see Tables I,II). Thus for a light meson \( \omega(1S) \) appears to be \( \sim 18\% \) larger and such the growth of \( \omega_i(nl) \) is important for more precise definition of the magnetic moments of the \( \rho \) and \( K^* \) mesons. Notice, that for higher excitations the GE interaction provides an increase of \( \omega_i(nl) \) by only \( \sim 5\% \).

### References

[1] Yu. A. Simonov, Nucl.Phys. B307,512 (1988); Phys. Lett. B226, 151 (1989); Nucl. Phys. B324, 67 (1989); A. Yu. Dubin, A. B. Kaidalov, and Yu. A. Simonov, Phys. At. Nucl. 56, 1745 (1993); E. Gubankova and A. Yu. Dubin Phys. Lett. B 334,180 (1994).

[2] A. M. Badalian and B. L. G. Bakker, Phys. Rev. D 66, 034025 (2002); A. M. Badalian, B. L. G. Bakker, and Yu. A. Simonov, Phys. Rev. D 66, 034026 (2002).

[3] M. A. Andreichikov, B. O. Kerbikov, and Yu. A. Simonov, [arXiv:1210.0227][hep-ph].

[4] B. O. Kerbikov and Yu. A. Simonov, Phys. Rev. D 62, 093016 (2000).

[5] Yu. A. Simonov, J. A. Tjon, and J. Weda, Phys. Rev. D 65, 094013 (2002); [hep-ph/0111344]

[6] P. Haegler, Phys. Rept. 490, 49 (2010); arXiv:0912.5483

[7] M. A. Andreichikov, B. O. Kerbikov, and Yu. A. Simonov (in preparation).
[8] F. X. Lee, S. Moerschbacher, and W. Wilcox, Phys. Rev. D 78, 094502 (2008).

[9] J. N. Hedditch, W. Kamlech, B. G. Lasscock, D. B. Leinweb, G. Williams, and J. M. Zanotti, Phys. Rev. D 75, 094504 (2007); [hep-lat/0703014]

[10] Yu. A. Simonov, Phys. Lett. B 515, 137 (2001).

[11] Yu. S. Kalashnikova, A. V. Nefediev and Yu. A. Simonov, Phys. Rev. D64, 014037 (2001); A. B. Kaidalov, and Yu. A. Simonov, Phys. Lett. B 477, 163 (2001).

[12] A. Di Giacomo, H. G. Dosch, V. I. Shevchenko, and Yu. A. Simonov, Phys. Rept. 372, 319 (2002); Yu. A. Simonov, Phys. Usp. 39, 313 (1996); [hep-ph/9709344]; Yu. A. Simonov, QCD and Theory of Hadrons in “QCD: Perturbative or Nonperturbative”, Interscience, Singapore, 2000; [hep-ph/9911237].

[13] L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. B 118, 76 (1977); Yu. S. Kalashnikova, A. V. Nefediev, and Yu. A. Simonov, Phys. Rev. D 64, 014037 (2001).

[14] A. M. Badalian and B. L. G. Bakker, Phys. Rev. D 84, 034006 (2011); A. M. Badalian, B. L. G. Bakker, and I. V. Danilkin, Phys. Rev. D 81, 071502 (2010); A. M. Badalian, Yu. A. Simonov, and M. A. Trusov, Phys. Rev. D 77, 074017 (2008); A. M. Badalian, B. L. G. Bakker, and Yu. A. Simonov, Phys. Rev. D 75, 116001 (2007).

[15] A. M. Badalian, B. L. G. Bakker, and I. V. Danilkin, Phys. Rev. D 79, 037505 (2009); Phys. At. Nucl. 72, 638 (2009); ibid. 73, 138 (2010).

[16] A. M. Badalian and B. L. G. Bakker, (in preparation). In this paper the averaged energies $\omega_i(nl)$ of light, $K$, and $\phi$ mesons are calculated for the gluon-exchanged potential $V_{GE} = 0$, as well as when asymptotic freedom behavior of $V_{GE}$ is taken into account.

[17] Yu. A. Simonov, Phys. At. Nucl. (to be published), [arXiv:1205.0692](http://arxiv.org/abs/1205.0692) [hep-ph].
[18] T. M. Aliev, A. Ozpineci, and M. Savci, Phys. Lett. B 678, 470 (2009); arXiv: 0902.4627 [hep-ph].

[19] T. M. Aliev, K. Azizi, and M. Savci, J. Phys. G 37, 075008 (2010).

[20] H. M. Choi and C. R. Ji, Phys. Rev. D 70, 053015 (2004).

[21] J. P. B. de Melo and T. Frederico, Phys.Rev. C 55, 2043 (1997).

[22] M. S. Bhagwat and P. Maris, Phys.Rev. C 77, 025203 (2008).

[23] A. M. Badalian and B. L. G. Bakker, JETP Lett. 86, 631 (2008); hep-ph/0702229.

[24] V. L. Morgunov, A. V. Nefediev, and Yu. A. Simonov, Phys. Lett. B 459, 653 (1999).

[25] A. Di Giacomo and Yu. A. Simonov, Phys. Lett. B 595, 368 (2004).

[26] Yu. A. Simonov, Phys. At. Nucl. 74, 1223 (2011) and references therein; arXiv:1011.5386 [hep-ph]; A. M. Badalian and D. S. Kuzmenko, Phys. Rev. D 65, 016004 (2002).

[27] M. Peter, Phys. Rev. Lett. 78, 602 (1997); Y. Schröder, Phys. Lett. B 447, 321 (1999).

[28] A. M. Badalan and A. I. Veselov, Phys. At. Nucl. 68, 582 (2005).