Pfaff $\tau$-functions

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0 Introduction

Throughout, let $A := \mathbb{Z}$ ("bi-infinite" case) or $A := \mathbb{Z}_{\geq 0} = \{0, 1, \ldots\}$ ("semi-infinite" case). Consider the set of equations

$$\frac{\partial m_{\infty}}{\partial t_n} = \Lambda^n m_{\infty}, \quad \frac{\partial m_{\infty}}{\partial s_n} = -m_{\infty}(\Lambda^\top)^n, \quad n = 1, 2, \ldots,$$

(0.1)
on bi- or semi-infinite (i.e., $A \times A$) matrices $m_{\infty} = m_{\infty}(t, s)$, where the matrix $\Lambda = (\delta_{i,j-1})_{i,j \in A}$ is the shift matrix, and $\Lambda^\top$ its transpose. In [2, 4], it was shown that Borel decomposing

$$m_{\infty}(t, s) = (\mu_{ij})_{i,j \in A} = S_1^{-1} S_2,$$

(0.2)
into lower- and upper-triangular matrices $S_1(t, s)$ and $S_2(t, s)$, leads to a two-Toda (two-dimensional Toda) system for $L_1 := S_1 \Lambda S_1^{-1}$ and $L_2 = S_2 \Lambda^\top S_2^{-1}$, with $\tau$-functions given by

$$\tau_n(t, s) = \det m_{\infty}(t, s), \quad n \in A.$$  

This paper deals with skew-symmetric initial data $m_{\infty}(0, 0)$. As readily seen from formula (0.1), the 2-Toda flow then maintains the relation $m_{\infty}(t, s) = -m_{\infty}(-s, -t)^\top$, and hence, by formula (0.3) and the interpretation of its right hand side in footnote 2,

$$\tau_n(t, s) = (-1)^n \tau_n(-s, -t).$$  

(0.4)

The main point of this paper is to study the reduction $s = -t$, as used in the theory of random matrices in H. Peng’s doctoral dissertation [16]. When $s \to -t$, formula (0.4) shows that in the limit the odd $\tau$-functions vanish, whereas the even $\tau$-functions are determinants of skew-symmetric matrices. In particular, the factorization (0.2) fails; in fact in the limit $s \to -t$, the

1 Here "$t, s \in \mathbb{C}^\infty$" is an informal way of saying that $t$ and $s$ are two sequences of independent scalar variables; a function of those variables may be defined only in an open subset of $\mathbb{C}^\infty \times \mathbb{C}^\infty$, or may even be a formal power series in $t$ and $s$.

2 This formula will be used mainly in the semi-finite case, with $m_n = (\mu_{ij})_{0 \leq i,j < n}$. In the bi-infinite case, $m_n = (\mu_{ij})_{-\infty < i,j < n}$ and the determinant is interpreted as

$$\lim_{k \to \infty} \det(\mu_{ij})_{-2k \leq i,j < q}.$$  

(*)

assuming the limit makes sense.
system leaves the main stratum to penetrate a deeper stratum in the Borel decomposition. In this paper we show this stratum leads to its own system, whereas in a forthcoming paper with Horozov [7], we show this system is integrable by producing its Lax pair.

Thus, we are led to considering Pfaffians:

\[ \tilde{\tau}_n(t) := \text{Pfaff } m_n(t, -t) = (\det m_n(t, -t))^{1/2} = \tau_n(t, -t)^{1/2}, \]  

(0.5)

for every even \( n \in A \) (the same remark as in footnote 2 applies here). The “Pfaffian \( \tilde{\tau} \)-function” is itself not a 2-Toda \( \tau \)-function, but it ties up remarkably with the 2-Toda \( \tau \)-function \( \tau \) as follows:

\[ \tau_{2n}(t, -t - [\alpha] + [\beta]) = \tilde{\tau}_{2n}(t) \tilde{\tau}_{2n}(t + [\alpha] - [\beta]), \]
\[ \tau_{2n+1}(t, -t - [\alpha] + [\beta]) = (\beta - \alpha) \tilde{\tau}_{2n}(t - [\beta]) \tilde{\tau}_{2n+2}(t + [\alpha]). \]

(0.6)

When \( \beta \to \alpha \), we approach the deeper stratum in the Borel decomposition of \( m_\infty \) in a very specific way. It also shows that the odd \( \tau \)-functions \( \tau_{2n+1}(t, -t - [\alpha] + [\beta]) \) approach zero linearly as \( \beta \to \alpha \), at the rate depending on \( \alpha \):

\[ \lim_{\beta \to \alpha} \frac{\tau_{2n+1}(t, -t - [\alpha] + [\beta])}{(\beta - \alpha)} = \tilde{\tau}_{2n}(t - [\alpha]) \tilde{\tau}_{2n+2}(t + [\alpha]). \]

Equations (0.6) are crucial in establishing bilinear relations\(^3\) for Pfaffian \( \tilde{\tau} \)-functions, where \( n, m \in A \):

\[ \sum_{j, k \geq 0, j - k = -2n + 2m + 1} p_j(-2y) e^{\sum_i -y_i D_i} p_k(-\tilde{D}) \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2} \]
\[ + \sum_{j, k \geq 0, k - j = -2n + 2m - 1} p_j(2y) e^{\sum_i -y_i D_i} p_k(\tilde{D}) \tilde{\tau}_{2n+2} \cdot \tilde{\tau}_{2m} = 0. \]

(0.7)

This is a generating function for the Hirota equations satisfied by \( \tilde{\tau}(t) \); for each \( n \) and \( m \), after expanding (0.7) into a power series in \( y = (y_1, y_2, \ldots) \), the coefficient of each monomial in \( y \) gives a Hirota equation. For example,

\[^3\; [\alpha] := (\alpha, \alpha^2, \alpha^3, \ldots)\]
\[^4\; \tilde{D} = (\frac{\partial}{\partial \alpha}, \frac{1}{2} \frac{\partial}{\partial \alpha^2}, \frac{1}{3} \frac{\partial}{\partial \alpha^3}, \ldots), \text{ and } \tilde{D} = (D_1, \frac{1}{2} D_2, \frac{1}{3} D_3, \ldots) \text{ is the corresponding Hirota symbol, i.e., } P(\tilde{D})f \cdot g := P(\partial/\partial y_1, \frac{1}{2} \partial/\partial y_2, \frac{1}{3} \partial/\partial y_3, \ldots) f(t + y) g(t - y) \big|_{y = 0} \text{ for any polynomial } P, \text{ and } p_k \text{ are the elementary Schur functions: } \sum_0^\infty p_k(t) z^k := \exp(\sum_1^\infty t_i z^i).\]

3 \([\alpha] := (\alpha, \alpha^2, \alpha^3, \ldots)\)

4 \(\tilde{D} = (\frac{\partial}{\partial \alpha}, \frac{1}{2} \frac{\partial}{\partial \alpha^2}, \frac{1}{3} \frac{\partial}{\partial \alpha^3}, \ldots), \text{ and } \tilde{D} = (D_1, \frac{1}{2} D_2, \frac{1}{3} D_3, \ldots) \text{ is the corresponding Hirota symbol, i.e., } P(\tilde{D})f \cdot g := P(\partial/\partial y_1, \frac{1}{2} \partial/\partial y_2, \frac{1}{3} \partial/\partial y_3, \ldots) f(t + y) g(t - y) \big|_{y = 0} \text{ for any polynomial } P, \text{ and } p_k \text{ are the elementary Schur functions: } \sum_0^\infty p_k(t) z^k := \exp(\sum_1^\infty t_i z^i).\)
for $m = n - 1$, the coefficient of the linear terms ($y_{k+3}$, to be more specific) gives

$$
\left( p_{k+4}(-D) - \frac{1}{2}D_1D_{k+3} \right) \tau_{2n} \cdot \tau_{2n} = p_k(D)\tau_{2n+2} \cdot \tau_{2n-2},
$$

(0.8)

where $k \in \mathbb{Z}_{\geq 0}$ and $2n - 2 \in A$. For $k = 0$, this equation can be viewed as an inductive expression of $\tau_{2n+2}$ in terms of $\tau_{2n-2}$ and derivatives of $\tau_{2n}$. These equations already appear in the work of Kac and van de Leur [12], in the context of the DKP hierarchy. On the exact connection, see forthcoming work by J. van de Leur [18].

In analogy with the 2-Toda or KP theory, one establishes Fay identities for the Pfaff $\tau$-functions. In this instance, they involve Pfaffians rather than determinants:

$$
Pfaff(\frac{(z_j - z_i)\tau_{2n-2}(t - [z_j] - [z_i])}{\tau_{2n}(t)})_{1 \leq i,j \leq 2k} = \Delta(z)\frac{\tau_{2n-2k}(t - \sum_{k-1}^{2k}[z_i])}{\tau_{2n}(t)}.
$$

(0.9)

In the semi-infinite case, the latter has a useful interpretation in terms of Pfaffians of “Christoffel-Darboux” kernels of the form

$$
K_n(\mu, \lambda) = e^{\sum_{i=1}^{\infty} t_i(\mu^i + \lambda^i)} \sum_0^{n-1} \left( q_{2k}(t, \mu)q_{2k+1}(t, \mu) - q_{2k}(t, \mu)q_{2k+1}(t, \lambda) \right).
$$

(0.10)

where the $q_m(t, \lambda)$ form a system of skew-orthogonal polynomials [4]. This is the analogue of the Christoffel-Darboux kernel for orthogonal polynomials. So, formula (0.9) can be rewritten as

$$
Pfaff(K_n(z_i, z_j))_{1 \leq i,j \leq 2k} = \left( \frac{1}{2} \prod_{i=1}^{2k} X(t; z_i)\tau \right)^{2n},
$$

(0.11)

where $X(t; z)$ is a vertex operator for the corresponding Pfaff lattice (see [7, 6]):

$$
X(t; z) := \Lambda^{-1} e^{\sum_{i=1}^{\infty} t_i z^i} e^{-\sum_{i=1}^{\infty} \frac{z^i}{i} \frac{\partial}{\partial t}} \chi(z).
$$
This vertex operator also has the remarkable property that for a Pfaffian \( \tilde{\tau} \)-function,
\[
\tilde{\tau} + aX(\lambda)X(\mu)\tilde{\tau} = \tilde{\tau}_{2n}(t) + a \left( 1 - \frac{\mu}{\lambda} \right) \lambda^{2n-2} \mu^{2n-1} e^{\sum t_{i}(\lambda^{i} + \mu^{i})} \tilde{\tau}_{2n-2}(t - [\lambda^{-1}] - [\mu^{-1}]).
\]
is again a Pfaffian \( \tilde{\tau} \)-function.

As was shown in [2, 3], the 2-Toda lattice has four distinct vertex operators. Upon using the reduction \( s = -t \), the 2-Toda vertex operators reduce to vertex operators for the Pfaff lattice. This enables us to give the action of Virasoro generators on Pfaff \( \tilde{\tau} \)-functions, in terms of the restriction (to \( s = -t \)) of actions on 2-Toda \( \tau \)-functions:
\[
\left( J_{i}^{(k)}(t) + (-1)^{k} J_{i}^{(k)}(s) \right) \tau_{2n}(t, s)|_{s=-t} = 2\tau_{2n}(t) J_{i}^{(k)}(t) \tilde{\tau}_{2n}(t).
\]

Finally, we discuss two examples, a first sketchy one, involving a semi-infinite Pfaff lattice and matrix integrals; this is extensively discussed in [6]. A second example, genuinely bi-infinite, will be given in the context of curves with fixed point free involutions \( \iota \), equipped with a line bundle \( L \) having a suitable antisymmetry condition with respect to \( \iota \).

1 Borel decomposition and the 2-Toda lattice

In [4, 2], we considered the following differential equations for the bi-infinite or semi-infinite moment matrix \( m_{\infty} \)
\[
\frac{\partial m_{\infty}}{\partial t_{n}} = \Lambda^{n} m_{\infty}, \quad \frac{\partial m_{\infty}}{\partial s_{n}} = -m_{\infty}(\Lambda^{T})^{n}, \quad n = 1, 2, \ldots, \tag{1.1}
\]
where the matrix \( \Lambda = (\delta_{i,j-1})_{i,j \in A} \) is the shift matrix; then (1.1) has the following solution
\[
m_{\infty}(t, s) = e^{\sum t_{n}\Lambda^{n}} m_{\infty}(0, 0) e^{-\sum s_{n}(\Lambda^{T})^{n}} \tag{1.2}
\]
in terms of the initial data \( m_{\infty}(0, 0) \).

Assume \( m_{\infty} \) allows, for “generic” \( (t, s) \), the Borel decomposition \( m_{\infty} = S_{1}^{-1} S_{2} \), for
\[
S_{1} \in G_{-} := \left\{ \begin{array}{l}
\text{lower-triangular matrices} \\
\text{with 1’s on the diagonal}
\end{array} \right\},
\]
\[
S_{2} \in G_{+} := \left\{ \begin{array}{l}
\text{upper-triangular matrices} \\
\text{with non-zero diagonal entries}
\end{array} \right\}.
\]
with corresponding Lie algebras $g_-, g_+$. Assume moreover that $m_\infty, S_1$ and $S_2$ are nice in the sense of Remark at the end of this section. Then setting $L_1 := S_1 \Lambda S_1^{-1}$,

$$S_1 \frac{\partial m_\infty}{\partial t} S_2^{-1} = \begin{cases} S_1 (\partial / \partial t_1) (S_1^{-1} S_2) S_2^{-1} = -\dot{S}_1 S_1 + \dot{S}_2 S_2^{-1} \in g_- + g_+, \\
S_1 \Lambda^n m_\infty S_2^{-1} = S_1 \Lambda^n S_1^{-1} = L_1^n = (L_1^n)^- + (L_1^n)^+ \in g_- + g_+; \end{cases}$$

the uniqueness of the decomposition $g_- + g_+$ leads to

$$- \frac{\partial S_1}{\partial t} S_1^{-1} = (L_1^n)^-, \quad \frac{\partial S_2}{\partial t} S_2^{-1} = (L_1^n)^+.$$ 

Similarly, setting $L_2 = S_2 \Lambda^T S_2^{-1}$, we find

$$- \frac{\partial S_1}{\partial s} S_1^{-1} = -(L_2^n)^-, \quad \frac{\partial S_2}{\partial s} S_2^{-1} = -(L_2^n)^+.$$ 

This leads to the 2-Toda equations \[17\] for $S_1, S_2$ and $L_1, L_2$:

$$\frac{\partial}{\partial t_n} S_{\{1\}}^{\{2\}} = \pm (L_1^n)^+ S_{\{1\}}^{\{2\}}, \quad \frac{\partial}{\partial s_n} S_{\{1\}}^{\{2\}} = \pm (L_2^n)^+ S_{\{1\}}^{\{2\}};$$

$$\frac{\partial L_i}{\partial t_n} = [(L_1^n)^+, L_i], \quad \frac{\partial L_i}{\partial s_n} = [(L_2^n)^-, L_i], \quad i = 1, 2, \ldots,$$ \[1.4\]

and conversely, reading this argument backwards, we observe that the 2-Toda equations \[1.3\] imply the time evolutions \[1.4\] for $m_\infty$.

The pairs of wave and adjoint wave functions $\Psi = (\Psi_1, \Psi_2)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)$, defined by

$$\Psi_{\{1\}}^{\{2\}}(t, s, z) = e^{\sum_{i,k} \{t_i, s_i\} z_i} S_{\{1\}}^{\{2\}} \chi(z),$$

$$\Psi_{\{1\}}^{\{2\}}(t, s, z) = e^{-\sum_{i,k} \{t_i, s_i\} z_i} (S_{\{1\}}^{\{2\}})^{-1} \chi(z^{-1}),$$ \[1.5\]

where $\chi(z)$ is the column vector $(z^n)_{n \in A}$, satisfy

$$L \Psi = (z, z^{-1}) \Psi, \quad L^* \Psi^* = (z, z^{-1}) \Psi^*,$$
and

\[
\frac{\partial}{\partial t_n} \Psi = ((L^n_1)_+, (L^n_2)_+) \Psi,
\]

\[
\frac{\partial}{\partial s_n} \Psi = ((L^n_1)_-, (L^n_2)_-) \Psi,
\]

\[
\frac{\partial}{\partial t_n} \Psi^* = -(((L^n_1)_+)^\top, ((L^n_2)_+)\top) \Psi^*,
\]

\[
\frac{\partial}{\partial s_n} \Psi^* = -(((L^n_1)_-)^\top, ((L^n_2)_-)\top) \Psi^*,
\]

which are equivalent to (1.3), and are further equivalent to the following bilinear identities, for all \(m, n \in A\) and \(t, s, t', s' \in \mathbb{C}^\infty\):

\[
\oint_{z=\infty} \Psi_1^n(t, s, z) \Psi_1^m(t', s', z') \frac{dz}{2\pi i z} = \oint_{z=0} \Psi_2^n(t, s, z) \Psi_2^m(t', s', z') \frac{dz}{2\pi i z}.
\]

(1.7)

By 2-Toda theory [17, 4], the problem is solved in terms of a sequence of tau-functions

\[
\tau_n(t, s) = \det m_n(t, s),
\]

with \(m_n(t, s)\) defined in (and \(\det m_n\) interpreted as in) footnote 2:

\[
m_n(t, s) := \begin{cases} (\mu_{ij}(t, s))_{-\infty < i, j < n} & \text{(bi-infinite case),} \\ (\mu_{ij}(t, s))_{0 \leq i, j < n} & \text{(semi-infinite case, with } \tau_0 = 1\text{),} \end{cases}
\]

as

\[
\Psi_1(t, s; z) = \left( \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t, s)} e^{\sum_{i=1}^{\infty} t_i z^i} z^{-n} \right)_{n \in A},
\]

\[
\Psi_2(t, s; z) = \left( \frac{\tau_{n+1}(t, s) - [z]}{\tau_n(t, s)} e^{\sum_{i=1}^{\infty} s_i z^{-i}} z^{-n} \right)_{n \in A},
\]

\[
\Psi_1^*(t, s, z) = \left( \frac{\tau_{n+1}(t + [z^{-1}], s)}{\tau_{n+1}(t, s)} e^{-\sum_{i=1}^{\infty} t_i z^i} z^{-n} \right)_{n \in A},
\]

\[
\Psi_2^*(t, s, z) = \left( \frac{\tau_n(t, s + [z])}{\tau_{n+1}(t, s)} e^{-\sum_{i=1}^{\infty} s_i z^{-i}} z^{-n} \right)_{n \in A}.
\]

Footnote 5: The contour integral around \(z = \infty\) is taken clockwise about a small circle around \(z = \infty\), while the one around \(z = 0\) is taken counter-clockwise about \(z = 0\).
Note (1.5) and (1.10) yield
\[ h(t, s) := \text{(diagonal part of } S_2) = \text{diag} \left( \frac{\tau_{n+1}(t, s)}{\tau_n(t, s)} \right)_{n \in \mathbb{A}}. \] (1.11)

Formulas (1.7) and (1.10) imply the following bilinear identities
\[
\oint_{z=\infty} \tau_n(t - [z^{-1}], s) \tau_{m+1}(t', s') e^{\sum_{i=1}^{\infty}(t_i-t'_i)z^{i-1}z^n-1} dz \\
= \oint_{z=0} \tau_{n+1}(t, s - [z]) \tau_m(t', s' + [z]) e^{\sum_{i=1}^{\infty}(s_i-s'_i)z^{i-1}z^n-1} dz,
\] (1.12)

where \( m, n \in \mathbb{A} \), satisfied by and characterizing the 2-Toda \( \tau \)-functions.

**Remark:**
In the bi-infinite case the factorization \( m_\infty = S_1^{-1}S_2 \) in (0.2) or the determinant formula (0.3) may fail to make sense. Nevertheless, we can take (0.4) as a starting point, use (0.6) to define \( \tilde{\tau} \) up to the sign, and make sense of the \( \tau \)-side of the whole story.

In the bi-infinite case, factorization as in (0.2) is not unique in general. This is responsible for the Backlund transform of a finite band matrix having a continuous family of solutions. However, it means the matrix multiplication may not be associative in the bi-infinite case: if
\[ S_1^{-1}S_2 = S'_1^{-1}S'_2, \]
with \( S_1, S'_1 \in G_- := \{ \text{lower triangular matrices with 1's on the diagonal} \} \) and \( S_2, S'_2 \in G_+ := \{ \text{upper triangular matrices with non-zero diagonal} \} \), and if the matrix multiplication was always associative, we should have
\[ S'_1S_1^{-1} = S'_2S_2^{-1} \in G_+ \cap G_- = \{ 1 \}, \]
so that \( S_1 = S'_1 \) and \( S_2 = S'_2 \).

The associativity is important in establishing the relation between equation (0.1) and the 2-Toda flows on \((S_1, S_2)\). Moreover, we are mainly interested in the semi-infinite case, in which the associativity clearly holds. So we assume that, for generic \((t, s)\), \( S_1, S_2 \) and \( m_\infty \) actually belong to suitable subgroups \( G'_\pm \) of \( G_\pm \) and a suitable subspace Mat’ of the space Mat of all infinite matrices, respectively, in which the multiplications
\[
G'_- \times \text{Mat’} \rightarrow \text{Mat’} \quad \text{and} \quad \text{Mat’} \times G'_+ \rightarrow \text{Mat’}
\]
\[
(S, m) \rightarrow Sm \quad \text{and} \quad (m, S) \rightarrow mS
\]
are associative:

\[(SS')m = S(S'm), \quad m(S''S''') = (mS'')S''' \quad \text{and} \quad (Sm)S'' = S(mS'')\]

hold for any \(S, S' \in G'_-, S'', S''' \in G'_+\) and \(m \in \text{Mat}'\).

For instance these conditions are satisfied if the \((i,j)\) entries of every matrix in \(G'_\pm\) and \(\text{Mat}'\) tend to 0 quickly enough as \(i \to -\infty\) uniformly in \(j > i + a\), and as \(j \to -\infty\) uniformly in \(i > j + a\), for some constant \(a\).

2 Two-Toda \(\tau\)-functions versus Pfaffian \(\tilde{\tau}\)-functions

In this section, we assume either the matrix \(m_{\infty}\) is semi-infinite, or \(\det m_n\) can be interpreted as in formula (\(*\)) in footnote 4, and we exhibit the properties of the 2-Toda lattice, associated with a skew-symmetric initial matrix \(m_{\infty}(0, 0)\). The \(\tau\)-functions \(\tau_n(t, s)\) then have the property

\[\tau_n(t, s) = (-1)^n\tau_n(-s, -t).\]

**Theorem 2.1** If the initial matrix \(m_{\infty}(0, 0)\) is skew-symmetric, then under the 2-Toda flow, \(m_{\infty}(t, s)\) maintains the relation

\[m_{\infty}(t, s) = -m_{\infty}(-s, -t)^\top. \quad (2.1)\]

Moreover,

\[h^{-1}S_1(t, s) = -(S_2^\top)^{-1}(-s, -t), \quad h^{-1}S_2(t, s) = (S_1^\top)^{-1}(-s, -t), \quad (2.2)\]

\[h^{-1}\Psi_1(t, s, z) = -\Psi_2^*(-s, -t, z^{-1}), \quad h^{-1}\Psi_2(t, s, z) = \Psi_1^*(-s, -t, z^{-1}), \quad (2.3)\]

\[L_1(t, s) = hL_2^\top h^{-1}(-s, -t) \quad \text{and} \quad L_2(t, s) = hL_1^\top h^{-1}(-s, -t), \quad (2.4)\]

with \(h\), defined by (1.11), satisfying

\[h(-s, -t) = -h(t, s). \quad (2.5)\]

Finally,

\[\tau_n(-s, -t) = (-1)^n\tau_n(t, s). \quad (2.6)\]
Proof: Formula (2.1) is an immediate consequence of (1.2) and the skew-symmetry of $m_\infty(0,0)$. Formula (2.2) follows from (2.1) and the Borel decomposition of $m_\infty(t,s)$ and $-m_\infty(-s,-t)^\top$:

$$m_\infty(t,s) = S_1^{-1}(t,s)S_2(t,s),$$

$$-m_\infty(-s,-t)^\top = - S_2^\top(-s,-t)S_1^{-1\top}(-s,-t) = (S_2^\top(-s,-t)h^{-1}(-s,-t))(-h(-s,-t)S_1^{-1\top}(-s,-t)).$$

by the uniqueness of the Borel decomposition of $m_\infty(t,s) = -m_\infty(-s,-t)^\top$, we have

$$S_1^{-1}(t,s) = S_2^\top(-s,-t)h^{-1}(-s,-t) \in G_-$$

$$S_2(t,s) = -h(-s,-t)S_1^{-1\top}(-s,-t) \in G_+.$$

Substituting $(t,s) \rightarrow (-s,-t)$ in the second equation and comparing it to the first one, yields $h(t,s) = -h(-s,-t)$, which is (2.3). Substituting this relation into the first and second equations yields (2.2), which by (1.3) and the definition of $L_1$ and $L_2$, amounts to (2.3) and (2.4). Relation (2.6) follows from (1.8), (2.1), footnote 2 and the multilinearity of determinant; or, in the semi-infinite case, from (2.5), using $\tau_0(t,s) = 1$:

$$\frac{\tau_n(t,s)}{\tau_n(-s,-t)} = -\frac{\tau_{n-1}(t,s)}{\tau_{n-1}(-s,-t)} = \cdots = (-1)^n \frac{\tau_0(t,s)}{\tau_0(-t,-s)} = (-1)^n.$$

For a skew-symmetric initial matrix $m_\infty(0,0)$, relation (2.1) implies the skew-symmetry of $m_\infty(t,-t)$. Therefore the odd $\tau$-functions vanish and the even ones have a natural square root, the Pfaffian $\tilde{\tau}_{2n}(t)$:

$$\tilde{\tau}_{2n+1}(t,-t) = 0, \quad \tilde{\tau}_{2n}(t,-t) =: \tilde{\tau}_{2n}^2(t), \quad (2.7)$$

where the Pfaffian, together with its sign specification, is also determined by the formula:

$$\tilde{\tau}_{2n}(t)dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{2n-1} := \frac{1}{n!} \left( \sum_{0 \leq i < j \leq 2n-1} \mu_{ij}(t,-t)dx_i \wedge dx_j \right)^n. \quad (2.8)$$
Theorem 2.2 For $\tau$ satisfying (2.4), and hence in particular for a skew-symmetric initial condition $m_\infty(0,0)$, the 2-Toda $\tau$-function $\tau(t,s)$ and the Pfaffians $\tilde{\tau}(t)$ are related by

\begin{alignat*}{2}
\tau_{2n}(t + [\alpha] - [\beta],-t) &= \tilde{\tau}_{2n}(t)\tilde{\tau}_{2n}(t + [\alpha] - [\beta]), \\
\tau_{2n+1}(t + [\alpha] - [\beta],-t) &= (\alpha - \beta)\tilde{\tau}_{2n}(t - [\beta])\tilde{\tau}_{2n+2}(t + [\alpha]),
\end{alignat*}

or alternatively

\begin{alignat*}{2}
\tau_{2n}(t - [\beta],-t + [\alpha]) &= \tilde{\tau}_{2n}(t - [\alpha])\tilde{\tau}_{2n}(t - [\beta]), \\
\tau_{2n+1}(t + [\alpha],-t - [\beta]) &= (\alpha - \beta)\tilde{\tau}_{2n}(t - [\alpha] - [\beta])\tilde{\tau}_{2n+2}(t), \\
\tau_{2n+1}(t - [\beta],-t + [\alpha]) &= (\alpha - \beta)\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t + [\alpha] + [\beta]).
\end{alignat*}

Proof: In formula (1.12), set $n = m - 1$, $s = -t + [\beta]$, $t' = t + [\alpha] - [\beta]$ and $s' = s - [\alpha] - [\beta] = -t - [\alpha]$; then using

\begin{align*}
\frac{1}{2\pi i} \oint_{z=\infty} \tau_n(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')e^{\sum_{i=1}^{n}(t'_i-t_i)z^i}z^{n-m-1}dz \\
&= \frac{1}{2\pi i} \oint_{z=\infty} \tau_{m-1}(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')\frac{1-\alpha z}{1-\beta z}dz \\
&= -\text{Res}_{z=\beta^{-1}} \tau_{m-1}(t-[z^{-1}],s)\tau_{m+1}(t'+[z^{-1}],s')\frac{1-\alpha z}{1-\beta z}dz \\
&= (\beta - \alpha)\tau_{m-1}(t-[\beta],s)\tau_{m+1}(t'+[\beta],s') \\
&= (\beta - \alpha)\tau_{m-1}(t-[\beta],-t+[\beta])\tau_{m+1}(t+[\alpha],-t-[\alpha]),
\end{align*}

\begin{align*}
\frac{1}{2\pi i} \oint_{z=0} \tau_m(t,s-[z])\tau_m(t',s'+[z])e^{\sum_{i=1}^{n}(s_i-s'_i)z^i}z^{n-m-1}dz \\
&= \frac{1}{2\pi i} \oint_{z=0} \tau_m(t,s-[z])\tau_m(t',s'+[z])\frac{1}{1-\alpha/z}dz \\
&= (\text{Res}_{z=\alpha} + \text{Res}_{z=\beta})\tau_m(t,s-[z])\tau_m(t',s'+[z])dz \\
&= \frac{1}{\alpha - \beta} \left( \tau_m(t,s-[\alpha])\tau_m(t',s'+[\alpha]) - \tau_m(t,s-[\beta])\tau_m(t',s'+[\beta]) \right) \\
&= \frac{1}{\alpha - \beta} \left( \tau_m(t,-t+[\beta]-[\alpha])\tau_m(t+[\alpha]-[\beta],-t) \\
&\quad - \tau_m(t,-t)\tau_m(t+[\alpha]-[\beta],-t-[\alpha]+[\beta]) \right),
\end{align*}
and (2.6), we have

\[-(\beta - \alpha)^2 \tau_{m-1}(t - [\beta], -t + [\beta]) \tau_{m+1}(t + [\alpha], -t - [\alpha])\]

\[= (-1)^m \tau_m(t + [\alpha] - [\beta], -t)^2 - \tau_m(t, -t) \tau_m(t + [\alpha] - [\beta], -t - [\alpha] + [\beta]).\]

Setting first \(m = 2l\) and then \(m = 2l + 1\), one finds respectively, since odd \(\tau\)-functions vanish on \(\{s = -t\}\) in view of (2.6):

\[0 = \tau_{2l}(t + [\alpha] - [\beta], -t)^2 - \tau_{2l}(t, -t) \tau_{2l}(t + [\alpha] - [\beta], -t - [\alpha] + [\beta]),\]

(2.11)

and

\[-(\beta - \alpha)^2 \tau_{2l}(t - [\beta], -t + [\beta]) \tau_{2l+2}(t + [\alpha], -t - [\alpha])\]

\[= -\tau_{2l+1}(t + [\alpha] - [\beta], -t)^2.\]

(2.12)

Taking the square root, with the consistent choice of sign,\(^6\) (2.8) yields (2.9), and then (2.10) upon setting \(t \rightarrow t - [\alpha]\) or \(t \rightarrow t + [\beta]\).

**Corollary 2.3** Under the assumption of theorem 2.2, the wave and adjoint

\(^6\) It suffices to check that (2.8) yields the correct sign in the second equation of (2.9) at \(\beta = 0, t = 0\) and modulo \(O(\alpha^2)\), i.e.,

\[\frac{\partial}{\partial t_1} \tau_{2n+1}(0, 0) \tau_{2n+2}(0),\]

for some \(m_\infty(0, 0)\) for which the right hand side does not vanish. This can be checked easily, e.g., for \(m_\infty(0, 0)\) made of \(2 \times 2\) blocks \(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\) on the diagonal.
wave functions $\Psi, \Psi^*$ along the locus $\{s = -t\}$ satisfy the relations

$$
\Psi_{1,2n}(t, -t, z) = -\left.\left(\frac{\tau_{2n+1}}{\sqrt{\tau_{2n}\tau_{2n+2}}} \Psi_{1,2n+1}(z)\right)\right|_{s=-t}
$$

$$
= \left(\frac{\tau_{2n+1}}{\tau_{2n}} \Psi_{2,2n}^*(z^{-1})\right)_{s=-t} = \left(\sqrt{\frac{\tau_{2n+2}}{\tau_{2n}}} \Psi_{2,2n+1}^*(z^{-1})\right)_{s=-t}
$$

$$
= \tilde{\tau}_{2n}(t - [z^{-1}])z^{2n}e^\sum \iota z^i,
$$

$$
\Psi_{1,2n-1}^*(t, -t, z) = \left(\frac{\tau_{2n-1}}{\sqrt{\tau_{2n-2}\tau_{2n}}} \Psi_{1,2n-2}^*(z)\right)_{s=-t}
$$

$$
= \left(\frac{\tau_{2n-1}}{\tau_{2n}} \Psi_{2,2n-1}(z^{-1})\right)_{s=-t} = -\left(\sqrt{\frac{\tau_{2n-2}}{\tau_{2n}}} \Psi_{2,2n-2}(z^{-1})\right)_{s=-t}
$$

$$
= \tilde{\tau}_{2n}(t + [z^{-1}])z^{-(2n-1)}e^\sum \iota z^i.
$$

Proof: These follow from (1.10), (2.9) and (2.10) by straightforward calculations.

Corollary 2.4 Under the assumption of theorem 2.2, we have

(i) for $k \geq 1$:

$$
\frac{\partial \tau_{2n}}{\partial t_k}_{s=-t} = \tilde{\tau}_{2n}(t)\frac{\partial \tilde{\tau}_{2n}}{\partial t_k} (t),
$$

$$
\frac{\partial \tau_{2n+1}}{\partial t_k}_{s=-t} = p_{k-1}(-\tilde{D}_t)\tilde{\tau}_{2n} \cdot \tilde{\tau}_{2n+2} (t)
$$

$$
:= \sum_{i+j=k-1} (p_i(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t))(p_j(\tilde{\partial}_t)\tilde{\tau}_{2n+2} (t)),
$$

(ii) for $m \geq 2$:

$$
\sum_{k+l=m} \frac{\partial^2 \tau_{2n}}{\partial t_k \partial t_l}_{s=-t} = \tilde{\tau}_{2n}(t) \sum_{k+l=m} \frac{\partial^2 \tilde{\tau}_{2n}}{\partial t_k \partial t_l} (t),
$$

$$
\sum_{k+l=m} \frac{\partial^2 \tau_{2n+1}}{\partial t_k \partial t_l}_{s=-t} = \sum_{k+l=m-1} (k-l)(p_k(-\tilde{\partial}_t)\tilde{\tau}_{2n}(t))(p_l(\tilde{\partial}_t)\tilde{\tau}_{2n+2} (t)),
$$

$$
- \sum_{k+l=m} \frac{\partial^2 \tau_{2n}}{\partial t_k \partial s_l}_{s=-t} = \sum_{k+l=m} \frac{\partial \tilde{\tau}_{2n}}{\partial t_k} (t) \frac{\partial \tilde{\tau}_{2n}}{\partial s_l} (t),
$$
(iii) for \( k, l \geq 0 \):
\[
\begin{align*}
p_k(\partial_t) & p_s(-\partial_t) \tau_{2n}(t, s) |_{s=-t} = \tilde{\tau}_{2n}(t)p_k(\partial_t)p_t(-\partial_t) \tilde{\tau}_{2n}(t), \\
p_k(\partial_t) & p_s(-\partial_t) \tau_{2n+1}(t, s) |_{s=-t} = p_t(-\partial_t) \tilde{\tau}_{2n}(t) \cdot p_{k-1}(\partial_t) \tilde{\tau}_{2n+2}(t) \\
& - p_{t-1}(-\partial_t) \tilde{\tau}_{2n}(t) \cdot p_k(\partial_t) \tilde{\tau}_{2n+2}(t),
\end{align*}
\]
where \( p_k(\cdot) \) are the elementary Schur functions, with \( p_{-1}(\cdot) = 0 \), and \( D_t = (D_t, (1/2)D_{t_2}, \ldots) \) are Hirota’s symbols.

Proof: Relations (i) are obtained by differentiating formulas (2.9) in \( \alpha \), setting \( \beta = \alpha \) and identifying the coefficients of \( \alpha^{k-1} \). The first two relations in (ii) are obtained by differentiating formulas (2.9) in \( \alpha \) and \( \beta \) (i.e., applying \( \partial^2/\partial\alpha\partial\beta \)), setting \( \beta = \alpha \) and identifying the coefficients of \( \alpha^{m-2} \). The last relation in (ii) is obtained by differentiating the first formula in (2.10) in \( \alpha \) and \( \beta \), setting \( \beta = \alpha \), substituting \( t + [\alpha] \) for \( t \), and then identifying the coefficients of \( \alpha^{m-2} \). Finally, expanding both identities (2.9) in \( \alpha \) and \( \beta \), e.g.,

\[
\tau_{2n}(t + [\alpha] - [\beta], s) = \sum_{k, l=0}^\infty \alpha^k\beta^l p_k(\partial_t)p_l(-\partial_t)\tau_{2n}(t, s)
\]

and identifying the powers of \( \alpha \) and \( \beta \) yields relations (iii). ■

Variants of formulas (2.9) and the formulas in the corollary can be obtained by using (2.3) and the following consequence of it:

\[
\left. \frac{\partial^{|I|+|J|}}{\partial t^I\partial s^J} \tau_n \right|_{s=-t} = (-1)^{|I|+|J|+n} \left. \frac{\partial^{|J|+|I|}}{\partial t^J\partial s^I} \tau_n \right|_{s=-t},
\]

where \( I = (i_1, i_2, \ldots) \) and \( J = (j_1, j_2, \ldots) \) are multiindices, \( |I| = i_1 + i_2 + \cdots \), \( \partial t^I = \partial t_1^1\partial t_2^2 \cdots \) etc. In particular, \( (\partial^2/\partial t_k\partial s_l + \partial^2/\partial t_l\partial s_k)\tau_{2n+1} = 0 \), so we get the (rather trivial) counterpart of the last formula in part (ii) of the corollary:

\[
\sum_{k+l=m} \frac{\partial^2 \tau_{2n+1}}{\partial t_k\partial s_l} \bigg|_{s=-t} = 0.
\]

3 Equations satisfied by Pfaffian \( \tilde{\tau} \)-functions

In this section, we exhibit the properties of the Pfaffian \( \tilde{\tau} \)-function introduced above, for the 2-Toda \( \tau \)-function satisfying (2.9), or the skew-symmetric initial data \( m_\infty(0, 0) \). As in the last section, whenever we make a connection
with the matrix \(m_\infty\), we assume either \(m_\infty\) is semi-infinite, or \(\det m_n\) can be interpreted as in formula (\(*\)) in footnote \([2]\).

**Theorem 3.1** The \(\tilde{\tau}\)-functions satisfy the bilinear relations

\[
\oint_{z=\infty} \tilde{\tau}_{2n}(t - [z^{-1}]) \tilde{\tau}_{2m+2}(t' + [z^{-1}]) e^{\sum_{i=0}^\infty (t_i-t'_i)z^i} z^{2n-2m-2} dz

+ \oint_{z=0} \tilde{\tau}_{2n+2}(t + [z]) \tilde{\tau}_{2m}(t' - [z]) e^{\sum_{i=0}^\infty (t'_i-t_i)z^{-i}} z^{2n-2m} dz = 0, \quad (3.1)
\]

or equivalently\([\text{footnote 2}]\)

\[
\sum_{j,k \geq 0} p_j(-2y)e^{\sum_{i=0}^\infty -y_i D_i} p_k(-\tilde{\mathcal{D}}) \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2}

+ \sum_{j,k \geq 0 \atop k-j=-2n+2m-1} p_j(2y)e^{\sum_{i=0}^\infty -y_i D_i} p_k(\tilde{\mathcal{D}}) \tilde{\tau}_{2n+2} \cdot \tilde{\tau}_{2m} = 0. \quad (3.2)
\]

**Proof:** Formula (3.1) follows from (1.12) upon replacing \(n\) by \(2n\) and \(m\) by \(2m\)\([\text{footnote 3}]\) using (2.9) and (2.10), with \(\beta = 0\), to eliminate \(\tau_{2n}(t - [z^{-1}], -t)\), \(\tau_{2m+1}(t' - [z])\), \(\tau_{2n+1}(t, -t - [z])\) and \(\tau_{2m}(t' - [z], -t')\) and, upon dividing both sides by \(\tilde{\tau}_{2n}(t)\tilde{\tau}_{2m}(t)\).

---

7 The \(\tilde{\mathcal{D}} = (\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \frac{\partial}{\partial t_3}, \ldots)\), and \(\tilde{\mathcal{D}} = (D_1, \frac{1}{2}D_2, \frac{1}{4}D_3, \ldots)\) is the corresponding Hirota symbol, i.e., \(P(\tilde{\mathcal{D}})f \cdot g := P(\partial/\partial y_1, \frac{1}{2}\partial/\partial y_2, \frac{1}{4}\partial/\partial y_3, \ldots) f(t + y)g(t - y)\big|_{y=0}\) for any polynomial \(P\); and \(p_k\) are the elementary Schur functions: \(\sum_{i} p_k(t)z^i := \exp(\sum_{i} t_i z^i)\).

8 One can check that all the other choices of parities of \(n\) and \(m\) yield the same formula.
Substituting $t - y$ and $t + y$ for $t$ and $t'$, respectively, into the left hand side of (3.1) and Taylor expanding it in $y$, we obtain

$$\oint_{z=\infty} e^{-\sum_{i=1}^{\infty} 2y_i z^i} \tilde{\tau}_{2n}(t - y - [z^{-1}]) \tilde{\tau}_{2m+2}(t + y + [z^{-1}]) z^{2n-2m-2} \, dz$$

$$+ \oint_{z=0} e^{-\sum_{i=1}^{\infty} 2y_i z^i} \sum_{-y_i D_i} e^{-\sum_{i=1}^{\infty} z^{-1} D_i / i} \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2} z^{2n-2m-2} \, dz$$

$$= \oint_{z=\infty} e^{-\sum_{i=1}^{\infty} 2y_i z^i} \sum_{-y_i D_i} e^{-\sum_{i=1}^{\infty} z^{-1} D_i / i} \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2} z^{2n-2m-2} \, dz$$

$$+ \oint_{z=0} e^{-\sum_{i=1}^{\infty} 2y_i z^i} \sum_{-y_i D_i} e^{-\sum_{i=1}^{\infty} z^{-1} D_i / i} \tilde{\tau}_{2n+2} \cdot \tilde{\tau}_{2m} z^{2n-2m} \, dz$$

$$= \oint_{z=\infty} \sum_{j=0}^{\infty} p_j (-2y) z^j e^{-\sum_{i=1}^{\infty} y_i D_i} \sum_{k=0}^{\infty} p_k (-\tilde{D}) z^{-k} \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2} z^{2n-2m-2} \, dz$$

$$+ \oint_{z=0} \sum_{j=0}^{\infty} p_j (2y) z^j e^{-\sum_{i=1}^{\infty} y_i D_i} \sum_{k=0}^{\infty} p_k (\tilde{D}) z^k \tilde{\tau}_{2n+2} \cdot \tilde{\tau}_{2m} z^{2n-2m} \, dz$$

$$= 2\pi i \left( \sum_{j-k=-2n+2m+1} p_j (-2y) e^{-\sum_{i=1}^{\infty} y_i D_i} p_k (-\tilde{D}) \tilde{\tau}_{2n} \cdot \tilde{\tau}_{2m+2} \right)$$

$$+ \sum_{j-k=-2n+2m-1} p_j (2y) e^{-\sum_{i=1}^{\infty} y_i D_i} p_k (\tilde{D}) \tilde{\tau}_{2n+2} \cdot \tilde{\tau}_{2m} \right),$$

showing the equivalence of (3.1) and (3.2).

The identity (3.1) gives various bilinear relations satisfied by $\tilde{\tau}$. We show that the Pfaffian $\tilde{\tau}$-functions satisfy identities reminiscent of the Fay and differential Fay identities for the KP or 2-Toda $\tau$-functions (e.g., see [1]). From this we deduce a sequence of Hirota bilinear equations for $\tilde{\tau}$, which can be interpreted as a recursion relation for $\tilde{\tau}_{2n}(t)$. 

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Theorem 3.2  The functions $\bar{\tau}_{2n}(t)$ satisfy the following “Fay identity”:

$$
\sum_{i=1}^{r} \bar{\tau}_{2n}\left(t - \sum_{j=1}^{l} [z_j] - [\zeta_i]\right) \bar{\tau}_{2m+2}\left(t - \sum_{j=1}^{r} [\zeta_j]\right) \prod_{k=1}^{l} (\zeta_j - \zeta_k) \prod_{1 \leq k \leq r (\zeta_i - \zeta_k)} = 0,
$$

(3.3)

the “differential Fay identity”:

$$
\{\bar{\tau}_{2n}(t - [u]), \bar{\tau}_{2n}(t - [v])\} + (u^{-1} - v^{-1})(\bar{\tau}_{2n}(t - [u])\bar{\tau}_{2n}(t - [v]) - \bar{\tau}_{2n}(t)\bar{\tau}_{2n}(t - [u] - [v])) = uv(u - v)\bar{\tau}_{2n-2}(t - [u] - [v])\bar{\tau}_{2n+2}(t),
$$

(3.4)

and Hirota bilinear equations, involving nearest neighbors:

$$
\left(p_{k+4}(\bar{D}) - \frac{1}{2}D_1D_{k+3}\right) \bar{\tau}_{2n} \cdot \bar{\tau}_{2n} = p_k(\bar{D})\bar{\tau}_{2n+2} \cdot \bar{\tau}_{2n-2}.
$$

(3.5)

Here, in (3.3) 2n, 2m ∈ A, l, r ≥ 0 such that r - l = 2n - 2m, z_i (1 ≤ i ≤ l) and ζ_i (1 ≤ i ≤ r) are scalar parameters near 0; in (3.4) 2n - 2 ∈ A (hence 2n, 2n + 2 ∈ A), and u and v are scalar parameters near 0; and in (3.3) 2n - 2 ∈ A, k = 0, 1, 2, …, and {f, g} := f'g - fg' = D_1f · g is the Wronskian of f and g, where’ = ∂/∂t_1.

Proof:  The Fay identity (3.3) follows from the bilinear identity (3.4) by substitutions

$$
t \mapsto t - [z_1] - \cdots - [z_l] \quad \text{and} \quad t' \mapsto t - [\zeta_1] - \cdots - [\zeta_r].
$$

Indeed, since r - l = 2n - 2m, we have

$$
\exp(\sum (t_i - t'_i)z^i)z^{2n-2m-2}dz = \prod_{k=1}^{l} (1 - zz_k)z^{r-l-2}dz = -\prod_{k=1}^{l} (1/z - z_k) d(1/z)
$$

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and
\[
\exp\left(\sum (t'_i - t_i)z^{-i}\right)z^{2n-2m}dz = \frac{\prod_{k=1}^r (1 - \zeta_k/z)}{\prod_{k=1}^l (1 - z_k/z)}z^{r-i}dz
\]
\[
= \frac{\prod_{k=1}^r (z - \zeta_k)}{\prod_{k=1}^l (z - z_k)}dz,
\]

so the first and second terms on the left hand side of (3.1), divided by $2\pi i$, become, respectively,

\[
\frac{1}{2\pi i} \oint_{z=\infty} \ldots \quad = - \sum_{i=1}^r \text{Res}_{z=\zeta_i^{-1}} \tau_{2n}\left(t - \sum_{j=1}^l [z_j] - [z^{-1}]\right)
\]
\[
\quad \cdot \tau_{2m+2}\left(t - \sum_{j=1}^r [\zeta_j] + [\zeta^{-1}]\right) \frac{\prod_{k=1}^l (1 - z\zeta_k)}{\prod_{k=1}^r (1 - z_k)}dz
\]
\[
= \sum_{i=1}^r \text{Res}_{z=\zeta_i} \tau_{2n}\left(t - \sum_{j=1}^l [z_j] - [\zeta]\right)
\]
\[
\quad \cdot \tau_{2m+2}\left(t - \sum_{j=1}^r [\zeta_j] + [\zeta]\right) \frac{\prod_{k=1}^l (\zeta - z_k)}{\prod_{k=1}^r (\zeta - \zeta_k)}d\zeta \quad (\zeta := z^{-1})
\]
\[
= \sum_{i=1}^r \tau_{2n}\left(t - \sum_{j=1}^l [z_j] - [\zeta_i]\right) \tau_{2m+2}\left(t - \sum_{j=1}^r [\zeta_j] + [\zeta_i]\right) \frac{\prod_{k=1}^l (\zeta_i - z_k)}{\prod_{1 \leq k \leq r \atop k \neq i} (\zeta_i - \zeta_k)},
\]

and

\[
\frac{1}{2\pi i} \oint_{z=0} \ldots \quad = \sum_{i=1}^l \text{Res}_{z=z_i} \tau_{2n+2}\left(t - \sum_{j=1}^l [z_j] + [z]\right)
\]
\[
\quad \cdot \tau_{2m}\left(t - \sum_{j=1}^r [\zeta_j] - [z]\right) \frac{\prod_{k=1}^r (z - \zeta_k)}{\prod_{k=1}^l (z - z_k)}dz
\]
\[
= \sum_{i=1}^l \tau_{2n+2}\left(t - \sum_{j=1}^l [z_j] + [z_i]\right) \tau_{2m}\left(t - \sum_{j=1}^r [\zeta_j] - [z_i]\right) \frac{\prod_{k=1}^r (z_i - \zeta_k)}{\prod_{1 \leq k \leq r \atop k \neq i} (z_i - z_k)},
\]
(z_2 - z_1)(z_3 - z_4)\tilde{\tau}_{2n}(t - [z_1] - [z_2])\tilde{\tau}_{2n}(t - [z_3] - [z_4])
- (z_3 - z_1)(z_2 - z_4)\tilde{\tau}_{2n}(t - [z_1] - [z_3])\tilde{\tau}_{2n}(t - [z_2] - [z_4])
+ (z_4 - z_1)(z_2 - z_3)\tilde{\tau}_{2n}(t - [z_1] - [z_4])\tilde{\tau}_{2n}(t - [z_2] - [z_3])
+ \left(\prod_{1 \leq i < j \leq 4} (z_i - z_j)\right)\tilde{\tau}_{2n+2}(t)\tilde{\tau}_{2n-2}(t - [z_1] - [z_2] - [z_3] - [z_4]) = 0. \quad (3.6)

The differential Fay identity (3.4) follows from (3.6) by taking a limit (set \(z_4 = 0\), divide by \(z_3\) and let \(z_3 \to 0\)). Alternatively, we can prove (3.4) directly from (3.1): Set \(t - t' = [u] - [v], 2m = 2n - 2\) in (3.1) and in the clockwise integral about \(z = \infty\), set \(z \mapsto 1/z\), yielding

\[
\oint_0 \tilde{\tau}_{2n}(t - [z])\tilde{\tau}_{2n}(t' + [z])\frac{1 - v/z}{1 - u/z} \frac{dz}{z^2} = -\oint_0 \tilde{\tau}_{2n+2}(t + [z])\tilde{\tau}_{2n-2}(t' - [z])\frac{1 - u/z}{1 - v/z} z^2 dz.
\]

The first integral has a simple pole at \(z = u\) and a double pole at \(z = 0\), while the second integral has a simple pole at \(z = v\) only, yielding, after substitution \(t' = t - [u] + [v]\),

\[
\tilde{\tau}_{2n}(t - [u])\tilde{\tau}_{2n}(t + [v])(u - v)\frac{1}{u^2}
+ \left.\frac{d}{dz}\left(\tilde{\tau}_{2n}(t - [z])\tilde{\tau}_{2n}(t - [u] + [v] + [z])\frac{z - v}{z - u}\right)\right|_{z=0}
= -\tilde{\tau}_{2n+2}(t + [v])\tilde{\tau}_{2n-2}(t - [u])(v - u)v^2,
\]

or, after carrying out \(d/dz|_{z=0}\) on the left hand side,

\[
\tilde{\tau}_{2n}(t - [u])\tilde{\tau}_{2n}(t + [v])(u - v)\frac{1}{u^2}
+ \tilde{\tau}_{2n}(t - [u] + [v])\frac{v - u}{u^2} - D_1 \tilde{\tau}_{2n}(t) \cdot \tilde{\tau}_{2n}(t - [u] + [v])\frac{v}{u}
= -\tilde{\tau}_{2n+2}(t + [v])\tilde{\tau}_{2n-2}(t - [u])(v - u)v^2. \quad (3.7)
\]
Shifting \( t \mapsto t - [v] \) and multiplying both sides by \( \frac{u}{v} \) yield (3.4).

Since \( P(-D)f \cdot f = P(D)f \cdot f \) by the definition of Hirota operator, (3.5) is the same as (??), which, as we have pointed out, are nothing but the coefficients of \( y_{k+3} \) in (0.7), or (3.2). It also follows from (3.4), since, for any power series \( F(t, t') \) which satisfies \( F(t, t') \equiv 0 \),

\[
\text{coefficient of } y_{k+3} \text{ in } F(t - y, t + y) = \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t_n} \right) F(t, t)
\]

\[
= 2 \frac{\partial}{\partial t_n} F(t, t') = 2 \times \text{coefficient of } u^{k+2} \text{ in } \left. \frac{d}{dv} F(t, t - [u] + [v]) \right|_{v = u}.
\]

Indeed, differentiating (3.7), which is equivalent to (3.4), in \( v \), setting \( v = u \) and using \( D_1 f \cdot f = 0 \),

\[
\frac{\partial}{\partial v} (D_1 f(t) \cdot g(t + [v])) = - \frac{1}{2} D_1 D_2 f(t) \cdot g(t + [v])
\]

\[
= - \frac{1}{2} \sum_{j=1}^{\infty} u^{j-1} D_1 D_j f(t) \cdot g(t + [v]),
\]

etc., we have

\[
- \tau_{2n}(t - [u]) \tau_{2n}(t + [u]) \frac{1}{u^2} + \tau_{2n}(t)^2 \frac{1}{u^2} + \frac{1}{2} \sum_{j=1}^{\infty} u^{j-1} D_1 D_j \tau_{2n}(t)
\]

\[
= - \tau_{2n+2}(t + [u]) \tau_{2n-2}(t - [u]) u^2,
\]

which, noting \( f(t + [u]) g(t - [u]) = \sum u^k p_k (D) f \cdot g \), is a generating function for (3.3).

As in the case of KP or 2-Toda \( \tau \)-functions, Pfaffian \( \tau \)-functions satisfy higher degree identities:

**Theorem 3.3**

\[
\text{Pfaff} \left( \frac{(z_j - z_i) \tau_{2n-2}(t - [z_i] - [z_j])}{\tau_{2n}(t)} \right)_{1 \leq i, j \leq 2k}
\]

\[
= \Delta(z) \frac{\tau_{2n-2k}(t - \sum_{i=1}^{2k} [z_i])}{\tau_{2n}(t)} , \quad (3.8)
\]

where \( k \geq 1, 2n - 2k \in A, z_i \) are scalar parameters near 0, and \( \Delta(u_1, \ldots, u_n) := \prod_{i<j} (u_j - u_i) \).
Proof: This may be obtained, up to the sign, from the second identity in Theorem 4.2 of [3]:

\[
\det \left( \frac{\tau_{N-1}(t-[z_i], s+[y_j])}{\tau_N(t, s)} \right)_{1 \leq i, j \leq k} = \Delta(y) \Delta(z) \left( \frac{\tau_{N-k} \left( t - \sum_{1 \leq i \leq 1}^k [z_i], s + \sum_{i=1}^k [y_i] \right)}{\tau_N(t, s)} \right),
\]

by setting \( N \mapsto 2n, k \mapsto 2k, y_i = z_i \), taking the square roots of both sides and using (2.10). Rather than taking this route, here we prove (3.8) by induction on \( k \), using the bilinear Fay identity (3.3). First, (3.8) is trivial when \( k = 1 \). (Note also that it gives (3.6) when \( k = 2 \).) Suppose (3.8) holds for \( k - 1 \). Then we have, for every \( p \in \{2, \ldots, 2k\} \),

\[
Pfaff \left( \frac{(z_j - z_i)\bar{\tau}_{2n-2} \left( t - [z_i] - [z_j] \right)}{\bar{\tau}_{2n}(t)} \right)_{2 \leq i, j \leq 2k \atop i \neq j} = \Delta(z_2, \ldots, \hat{z}_p, \ldots, z_{2k}) \left( \frac{\bar{\tau}_{2n-2k+2} \left( t - \sum_{2 \leq i \leq 2k, i \neq p} [z_i] \right)}{\bar{\tau}_{2n}(t)} \right).
\]

Multiplying both sides by \((-1)^p(z_p - z_1)\bar{\tau}_{2n-2} \left( t - [z_1] - [z_p] \right) / \bar{\tau}_{2n}(t)\), summing it up for \( p = 2, \ldots, 2k \), and using

\[
(-1)^p(z_p - z_1)\Delta(z_2, \ldots, \hat{z}_p, \ldots, z_{2k}) = \frac{\Delta(z)}{\prod_{2 \leq i \leq 2k}(z_i - z_1) \prod_{i \neq p}^2(\bar{z}_p - z_i)}
\]

and the identity

\[
Pfaff(a_{ij})_{1 \leq i, j \leq 2k} = \sum_{p=2}^{2k} (-1)^p a_{1p} Pfaff(a_{ij})_{2 \leq i, j \leq 2k \atop i, j \neq p} \quad \forall (a_{ij}) \text{ skew symmetric}
\]

\[21\]
which follows from definition (2.8) of the Pfaffian, we have
\[
\text{Pfaff}\left( \frac{(z_j - z_i)\tilde{\tau}_{2n-2}(t - [z_i] - [z_j])}{\tilde{\tau}_{2n}(t)} \right)_{1\leq i,j\leq 2k} = \frac{\Delta(z)}{\prod_{2\leq i\leq 2k}(z_i - z_1)} \cdot \frac{1}{\tilde{\tau}_{2n}(t)^2} \sum_{p=2}^{2k} \frac{(z_p - z_1)}{\prod_{2\leq i\leq 2k}(z_p - z_i)} \cdot \tilde{\tau}_{2n-2}(t - [z_1] - [z_p]) \tilde{\tau}_{2n-2k+2}\left( t - \sum_{2\leq i\leq 2k} [z_i] \right)
\]
using the bilinear Fay identity (3.3) with \( r = 2k - 1, \ l = 1, \ \zeta_i := z_{i+1} \ (1 \leq i \leq 2k - 1) \) and (\(2n, 2m\)) replaced by (\(2n - 2, 2n - 2k\)) this becomes
\[
= \frac{\Delta(z)}{\prod_{2\leq i\leq 2k}(z_i - z_1)} \cdot \frac{1}{\tilde{\tau}_{2n}(t)^2} \\
\cdot (-1) \left( \prod_{i=2}^{2k} (z_1 - z_i) \right) \tilde{\tau}_{2n}(t) \tilde{\tau}_{2n-2k}\left( t - \sum_{i=1}^{2k} [z_i] \right)
= \Delta(z) \frac{\tilde{\tau}_{2n-2k}(t - \sum_{i=1}^{2k} [z_i])}{\tilde{\tau}_{2n}(t)},
\]
completing the proof of (3.8) by induction. \( \blacksquare \)

4 Vertex operators for Pfaffian \( \tilde{\tau} \)-functions

In terms of the operators
\[
X(t, \lambda) := e^{\sum_{i=1}^{\infty} \lambda^k \frac{\partial}{\partial t_i}} e^{-\sum_{i=1}^{\infty} \frac{\lambda^{-k}}{k} \frac{\partial}{\partial s_i}} \quad \text{and} \quad X^*(t, \lambda) := e^{-\sum_{i=1}^{\infty} \lambda^k \frac{\partial}{\partial t_i}} e^{\sum_{i=1}^{\infty} \frac{\lambda^{-k}}{k} \frac{\partial}{\partial s_i}},
\]
acting on functions \( f(t) \) of \( t = (t_1, t_2, \ldots) \in \mathbb{C}^\infty \), define the following four operators\(^9\) acting on column vectors \( g = (g_n(t))_{n \in A} \),
\[
\begin{align*}
X_1(\mu) := X(t, \mu)\chi(\mu), & \quad X_1^*(\lambda) := -\chi^*(\lambda)X^*(t, \lambda), \\
X_2(\mu) := -X(s, \mu)\chi^*(\mu)\Lambda, & \quad X_2^*(\lambda) := \Lambda^\top \chi(\lambda)X^*(s, \lambda),
\end{align*}
\]
\(^9\) Here \( X(s, \lambda) \) has \( s_i \) in place of \( t_i \), as well as \( \partial/\partial s_i \) in place of \( \partial/\partial t_i \), in the definition of \( X(t, \lambda) \), etc.; \( \chi(\mu) := (\mu^n)_{n \in A} \), and \( \chi^*(\mu) = \chi(\mu^{-1}) \).
and their compositions

\[ X_{ij}(\mu, \lambda) := X_j^*(\lambda)X_i(\mu), \quad i, j = 1, 2. \]

They form a set of four vertex operators associated with the 2-Toda lattice. Among those, \( X_{12} \) is important in the semi-infinite case, related to the study of orthogonal polynomials. In [3], we showed that

\[
\sum_{m \leq j < n} \Psi_{1,j}(\mu) \Psi_{2,j}^*(\lambda^{-1}) = \frac{(X_{12}(\mu, \lambda)\tau_n - (X_{12}(\mu, \lambda)\tau_m)(n\tau_n - (X_{12}(\mu, \lambda)\tau_m)(m\tau_m)}{n\tau_n - (m\tau_m)(4.1)}
\]

for any \( n, m \in A, n \geq m \). Note on the right hand side the limit exists as \( s \to -t \) if \( n \) and \( m \) are even, so in particular, taking \( n = m + 1 \), we see the poles along \( s = -t \) cancel out in \( \Psi_{1,2m}(\mu)\Psi_{2,2m}^*(\lambda^{-1}) + \Psi_{1,2m+1}(\mu)\Psi_{2,2m+1}^*(\lambda^{-1}) \).

We shall come back to this point after proving the following theorem and its corollary.

Suppose \( \tau \) satisfies (2.6), and let \( \tilde{\tau} \) be the vector of corresponding Pfaffian \( \tilde{\tau} \)-functions. Let \( X_{1}, X_{1}^* \) and \( X_{11} \) act on \( \tilde{\tau} \) as if they are acting on the extended vector \( (\tilde{\tau}_n)_{n \in A} \), where \( \tilde{\tau}_n \equiv 0 \) if \( n \) is odd, so that \( \chi(\mu) \) (resp. \( \chi^*(\lambda) \)) acts on \( \tilde{\tau}_{2n} \) by multiplication of \( \mu^{2n} \) (resp. \( \lambda^{-2n} \)). Then we have

\[
\text{10 When } i = j, \quad X_i^* \text{ interacts with } X_i \text{ nontrivially, yielding the factor } \exp(\sum(\mu/\lambda)^k/k) = 1/(1 - \mu/\lambda) \text{ if we bring the multiplication operators to the left and the shift operators in } t \text{ or } s \text{ to the right. So if we denote by } : : \text{ the usual normal ordering of operators in } t, s \text{ (but not in the discrete index } n), \text{ we have}
\]

\[
\text{11 The product } X_1(\lambda)X_1(\mu) \text{ in (L.4) is computed the same way as in footnote 10.}
Theorem 4.1

$$\begin{align*}
(X_{11}(\mu, \lambda) \tau)_N|_{s=-t} &= \begin{cases} 
\bar{\tau}_{2n}(t)X_{11}(\mu, \lambda)\bar{\tau}_{2n}(t) & (N = 2n) \\
-\lambda(X_1(\mu)\bar{\tau}_{2n}(t))X_1^*(\lambda)\bar{\tau}_{2n+2} & (N = 2n + 1)
\end{cases} \\
(X_{22}(\mu, \lambda) \tau)_N|_{s=-t} &= \begin{cases} 
-\tau_{2n}(t)X_{11}(\lambda, \mu)\bar{\tau}_{2n}(t) & (N = 2n) \\
-\mu(X_1(\lambda)\bar{\tau}_{2n})X_1^*(\mu)\bar{\tau}_{2n+2} & (N = 2n + 1)
\end{cases} \\
(X_{12}(\mu, \lambda) \tau)_N|_{s=-t} &= \begin{cases} 
-\lambda\tau_{2n}(t)X_1(\lambda)X_1(\mu)\bar{\tau}_{2n-2}(t) & (N = 2n) \\
(X_1(\mu)\bar{\tau}_{2n})X_1^*(\lambda)\bar{\tau}_{2n} & (N = 2n + 1)
\end{cases}
\end{align*}$$

Corollary 4.2 For $k = 1, 2$, the following holds:

$$J_i^{(k)}(t)\tau_{2n}(t, s)|_{s=-t} = \bar{\tau}_{2n}(t)J_i^{(k)}(t)\bar{\tau}_{2n}(t),$$
$$J_i^{(k)}(s)\tau_{2n}(t, s)|_{s=-t} = (-1)^k\bar{\tau}_{2n}(t)J_i^{(k)}(t)\bar{\tau}_{2n}(t),$$
and so

$$(J_i^{(k)}(t) + (-1)^kJ_i^{(k)}(s))\tau_{2n}(t, s)|_{s=-t} = 2\bar{\tau}_{2n}(t)J_i^{(k)}(t)\bar{\tau}_{2n}(t).$$

Proof: The theorem follows from formulas (2.9) and (2.10) by straightforward calculations:

$$\begin{align*}
(X_{11}(\mu, \lambda) \tau)_N|_{s=-t} &= -\left(\frac{\mu}{\lambda}\right)^Ne^{\sum t_i(\mu^{-1}-\lambda)}\tau_N(t - \lceil \mu^{-1}\rceil - \lceil \lambda^{-1}\rceil, -t) \\
&= -\left(\frac{\mu}{\lambda}\right)^Ne^{\sum t_i(\mu^{-1}-\lambda)}\bar{\tau}_{2n}(t)\bar{\tau}_{2n}(t - \lceil \mu^{-1}\rceil + \lceil \lambda^{-1}\rceil) \\
&= \bar{\tau}_{2n}(t)X_{11}(\mu, \lambda)\bar{\tau}_{2n}(t),
\end{align*}$$

for $N = 2n$:

$$\begin{align*}
&= -\left(\frac{\mu}{\lambda}\right)^Ne^{\sum t_i(\mu^{-1}-\lambda)}\bar{\tau}_{2n}(t)\bar{\tau}_{2n}(t - \lceil \mu^{-1}\rceil + \lceil \lambda^{-1}\rceil) \\
&= \bar{\tau}_{2n}(t)X_{11}(\mu, \lambda)\bar{\tau}_{2n}(t),
\end{align*}$$

for $N = 2n + 1$:

$$\begin{align*}
&= \left(\frac{\mu}{\lambda}\right)^Ne^{\sum t_i(\mu^{-1}-\lambda)}\bar{\tau}_{2n}(t - \lceil \mu^{-1}\rceil + \lceil \lambda^{-1}\rceil)\bar{\tau}_{2n+2}(t + \lceil \lambda^{-1}\rceil) \\
&= \lambda(X_1(\mu)\bar{\tau}_{2n})(X_1^*(\lambda)\bar{\tau}_{2n+2});
\end{align*}$$
$$\forall 2n+1$$
The corollary is shown by expanding $X_{11}$ in $\lambda$ and $\mu - \lambda$. Recall that
\[
X_{11}(\mu, \lambda) = -\frac{\lambda}{\lambda - \mu} \left( \left( \frac{\mu}{\lambda} \right)^n X(\mu, \lambda) \right)_{n \in A}
= -\frac{\lambda}{\lambda - \mu} \left( \sum_{k=0}^{\infty} \frac{(\mu - \lambda)^k}{k!} \sum_{l=-\infty}^{\infty} \lambda^{-l-k} W^{(k)}_{n,l}(t) \right)_{n \in A},
\]
where $X(\mu, \lambda)$ is the vertex operator in the KP theory \cite{KP}, and
\[
W^{(k)}_{n,l}(t) = \sum_{j=0}^{k} \binom{n}{j} (k)_j W^{(k-j)}_{l},
\]
with $W^{(k)}_{l}$ the coefficients of similar expansion of $X(\mu, \lambda)$.

Expanding $X_{11}$ in (4.2) as above leads to
\[
W^{(k)}_{2n,l}(t) \tau_{2n}(t, s)|_{s=-t} = \tilde{\tau}_{2n}(t) W^{(k)}_{2n,l}(t) \tilde{\tau}_{2n}(t).
\]
In particular, since $J_i^{(k)}$ ($k \leq 2$) and $W_{(n,i)}^{(k)}$ ($k \leq 2$) are linear combinations of each other \cite{KP}: 
\[
W^{(0)}_{n,i} = J_i^{(0)} = \delta_{i,0}, \\
W^{(1)}_{n,i} = J_i^{(1)} + nJ_i^{(0)}, \\
W^{(2)}_{n,i} = J_i^{(2)} + (2n - i - 1)J_i^{(1)} + n(n - 1)J_i^{(0)},
\]
on one sees for $k = 1, 2$ that
\[
J_i^{(k)}(t) \tau_{2n}(t, s)|_{s=-t} = \tilde{\tau}_{2n}(t) J_i^{(k)}(t) \tilde{\tau}_{2n}(t).
\]

Consider the following vertex operator\footnote{As before, $X$ treats $\tilde{\tau}$ as an extended vector $(\tilde{\tau}_n)_{n \in A}$, where $\tilde{\tau}_n \equiv 0$ for $n$ odd. So $\chi(z)$ appearing in $X(z)$ acts on $\tilde{\tau}_n$ as multiplication by $z^n$, and $\Lambda^*$ acts on $\tilde{\tau}$ as $(\Lambda^* \tilde{\tau})_n = \tilde{\tau}_{n-1}$. In practice, we always have even number of $X$'s acting on $\tilde{\tau}$, so there is always an even power of $\Lambda^*$, and $\tilde{\tau}_n$ for odd $n$ will never appear in our formulas.} 
\[
X(z) := \Lambda^* X_1(z) = \Lambda^* e^{\sum t_i z^i} e^{-\sum \frac{z^{-i}}{\lambda^i} \partial_i} \chi(z),
\]
and define the kernel

\[ K_n(y, z) := \left( \frac{1}{\tau} X(y) X(z) \tilde{\tau} \right)_{2n} \]

It is easy to see that \( (X(y) X(z) \tilde{\tau})_{2n} = y X_1(y) X_1(z) \tilde{\tau}_{2n-2} \), so by (4.3)

\[ K_n(y, z) = -\left( \frac{X_{12}(y, z) \tilde{\tau}}{\tau_{2n}} \right)_{2n} \]

and by (4.1)

\[ \left( \sum_{2m \leq j < 2n} \Psi_{1,j} (\mu) \Psi_{2,j}^* (\lambda^{-1}) \right) \bigg|_{s=\tau} = K_n(\mu, \lambda) - K_m(\mu, \lambda). \]

Here each term \( \Psi_{1,j} (\mu) \Psi_{2,j}^* (\lambda^{-1}) \) on the left hand side blows up along \( s = -t \), but the poles from two successive terms (for \( j = 2k \) and \( j = 2k + 1 \)) cancel, as we saw earlier.

For \( n \in A \), let

\[ q_n(t, \lambda) := \begin{cases} \lambda^{2m} \frac{\tilde{\tau}_{2m} (t - [\lambda^{-1}])}{\sqrt{\tilde{\tau}_{2m} (t) \tilde{\tau}_{2m+2} (t)}} & \text{if } n = 2m, \\ \lambda^{2m} \frac{(\partial / \partial t_1 + \lambda) \tilde{\tau}_{2m} (t - [\lambda^{-1}])}{\sqrt{\tilde{\tau}_{2m} (t) \tilde{\tau}_{2m+2} (t)}} & \text{if } n = 2m + 1. \end{cases} \quad (4.5) \]

In the semi-infinite case, the \( q_n \)’s form a system of skew-orthogonal polynomials [7].

**Theorem 4.3** The following holds:

\[ \text{Pfaff} (K_n(z_i, z_j))_{1 \leq i, j \leq 2k} = \left( \frac{1}{\tau} \prod_{i=1}^{2k} X(z_i) \tilde{\tau} \right)_{2n}, \quad (4.6) \]

\[ K_{n+1}(\mu, \lambda) - K_n(\mu, \lambda) = e^{\sum t_i (\mu^i + \lambda^i)} \left( q_{2n}(t, \lambda) q_{2n+1}(t, \mu) - q_{2n}(t, \mu) q_{2n+1}(t, \lambda) \right), \quad (4.7) \]

so in the semi-infinite case

\[ K_N(\mu, \lambda) = e^{\sum t_i (\mu^i + \lambda^i)} \sum_{0}^{N-1} \left( q_{2n}(t, \lambda) q_{2n+1}(t, \mu) - q_{2n}(t, \mu) q_{2n+1}(t, \lambda) \right). \quad (4.8) \]
Proof: Using (3.8) and
\[ K_n(\mu, \lambda) = \left( \frac{\bar{X}(\mu)\bar{X}(\lambda)\bar{t}}{\bar{t}} \right)_{2n} \]
\[ = (\mu - \lambda)(\mu\lambda)^{2n-2} e^{\sum_{i=1}^{\infty} t_i(\mu^i + \lambda^i)} \frac{\tilde{\tau}_{2n-2}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n}(t)} \]
\[ = (\lambda^{-1} - \mu^{-1})(\mu\lambda)^{2n-1} e^{\sum_{i=1}^{\infty} t_i(\mu^i + \lambda^i)} \frac{\tilde{\tau}_{2n-2}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n}(t)}, \quad (4.9) \]
the left hand side of (4.3) becomes
\[ (z_1 \cdots z_{2k})^{2n-1} e^{\sum_{j=1}^{2k} \sum_{i=1}^{\infty} t_i} \Delta(z^{-1}) \frac{\tilde{\tau}_{2n-2k}(t - \sum_{j=1}^{2k} [z_j^{-1}])}{\tilde{\tau}_{2n}(t)}. \]
This equals the right hand side of (4.6), because
\[ (\bar{X}(z_{2k})\bar{X}(z_{2k-1}) \cdots \bar{X}(z_2)\bar{X}(z_1)\bar{t})_{2n} \]
\[ = (z_{2k}^{2n-2} \cdot z_{2k-2}^{2n-2} \cdots z_1^{2n-2k}) e^{\sum_{i=1}^{2k} t_i(z_1^i + \cdots + z_{2k}^i)} \]
\[ \times \left[ \prod_{1<j \leq 2k} \prod_{1 \leq i < j} \left( 1 - \frac{z_i}{z_j} \right) \right] \frac{\tilde{\tau}_{2n-2k}(t - \sum_{j=1}^{2k} [z_j^{-1}])}{\tilde{\tau}_{2n}(t)} \]
\[ = (z_1 \cdots z_{2k})^{2n-1} \Delta(z^{-1}) e^{\sum_{i=1}^{2k} t_i(z_1^i + \cdots + z_{2k}^i)} \frac{\tilde{\tau}_{2n-2k}(t - \sum_{j=1}^{2k} [z_j^{-1}])}{\tilde{\tau}_{2n}(t)}. \]
To prove (4.7), we have
\[ (\mu - \lambda) \left( (\mu\lambda)^{2n} \frac{\tilde{\tau}_{2n}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n+2}(t)} - (\mu\lambda)^{2n-2} \frac{\tilde{\tau}_{2n-2}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n}(t)} \right) \]
\[ = (\mu - \lambda)(\mu\lambda)^{2n} \frac{\tilde{\tau}_{2n}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n+2}(t)} \frac{\tilde{\tau}_{2n-2}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2n}(t)} \]
\[ = (\mu\lambda)^{2n} \left( \frac{\tilde{\tau}_{2n}(t - [\mu^{-1}])}{\tilde{\tau}_{2n+2}(t)} \right) + (\mu - \lambda) \frac{\tilde{\tau}_{2n}(t - [\mu^{-1}])}{\tilde{\tau}_{2n+2}(t)} \frac{\tilde{\tau}_{2n}(t - [\lambda^{-1}])}{\tilde{\tau}_{2n}(t)} \]
using (3.4),
\[ = \left( \lambda^{2n} \frac{\tilde{\tau}_{2n}(t - [\lambda^{-1}])}{\sqrt{\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t)}} \frac{\mu^{2n} \partial \tilde{\tau}_{2n}(t - [\mu^{-1}])}{\sqrt{\tilde{\tau}_{2n}(t)\tilde{\tau}_{2n+2}(t)}} - (\lambda \leftrightarrow \mu) \right) \]
\[ = (q_{2n}(t, \lambda)q_{2n+1}(t, \mu) - q_{2n}(t, \mu)q_{2n+1}(t, \lambda)) \]
in terms of the skew-orthogonal polynomials \((4.3)\). Multiplying this with an exponential and noting \((4.9)\), one obtains \((4.7)\). Summing up this telescoping sum yields \((4.8)\):

\[
K_N(\mu, \lambda) = \left( \frac{\mathcal{X}(\mu)\mathcal{X}(\lambda)\tilde{\tau}}{\tilde{\tau}} \right)_{2N} \\
= (\mu - \lambda)(\mu \lambda)^{2N-2}e^{\sum_{i=1}^{\infty} t_i(\mu + \lambda)} \frac{\tilde{\tau}_{2N-2}(t - [\mu^{-1}] - [\lambda^{-1}])}{\tilde{\tau}_{2N}(t)} \\
= \sum_{\nu=0}^{N-1} e^{\sum_{i=1}^{\infty} t_i(\mu + \lambda)}(q_{2\nu}(t, \lambda)q_{2\nu+1}(t, \mu) - q_{2\nu}(t, \mu)q_{2\nu+1}(t, \lambda)).
\]

\section{The exponential of the vertex operator maintains \tilde{\tau}-functions}

The purpose of this section is to show the following theorem:

\textbf{Theorem 5.1} \textit{For a Pfaffian \tilde{\tau}-function,}

\[
\tilde{\tau} + a\mathcal{X}(\lambda)\mathcal{X}(\mu)\tilde{\tau}
\]

\textit{is again a Pfaffian \tilde{\tau}-function.}

Remember that \(\mathcal{X}(\lambda)\mathcal{X}(\mu)\) acts on \(\tilde{\tau}_{2\nu}(t)\), as follows:

\[
\mathcal{X}(\lambda)\mathcal{X}(\mu)\tilde{\tau}_{2\nu}(t) = \left(1 - \frac{\mu}{\lambda}\right)\lambda^{2\nu-2}\mu^{2\nu-1}e^{\sum_{i=1}^{\infty} t_i(\lambda + \mu)}\tilde{\tau}_{2\nu-2}(t - [\lambda^{-1}] - [\mu^{-1}]).
\]

(5.2)

It is convenient to relabel \(\tilde{\tau}_{2\nu} \rightarrow \tilde{\tau}_n\)

\[
\mathcal{X}(\lambda)\mathcal{X}(\mu)\tilde{\tau}_n(t) = \left(1 - \frac{\mu}{\lambda}\right)\lambda^{2n-2}\mu^{2n-1}e^{\sum_{i=1}^{\infty} t_i(\lambda + \mu)}\tilde{\tau}_{n-2}(t - [\lambda^{-1}] - [\mu^{-1}])
\]

\[
= \frac{\mu}{\lambda}(\lambda - \mu)^{\Lambda^{-1}e^{\sum_{i=1}^{\infty} t_i(\lambda + \mu)}}e^{-\sum_{i=1}^{\infty} \left(\frac{\lambda^{-1} + \mu^{-1}}{\mu}\right)i} \mathcal{X}(\lambda^2\mu^2)\tilde{\tau}^n_n
\]

\[
= \frac{\mu}{\lambda}(\lambda - \mu)\mathcal{Y}(\lambda, \mu).
\]

(5.3)
With this relabeling, the bilinear identity takes on the form
\[
\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum_{i=1}^{\infty} (t_i - t'_i)z^i} z^{2n-2m-2} dz \\
+ \oint_{z=0} \tau_{n+1}(t + [z])\tau_m(t' - [z])e^{\sum_{i=1}^{\infty} (t'_i - t_i)z^{-i}} z^{2n-2m} dz = 0. \tag{5.4}
\]

**Lemma 5.2** We have:
\[
(1 - \lambda z)^{-1}(1 - \mu z)^{-1} - \frac{1}{\lambda \mu z^2} \left( \frac{1}{1 - \lambda z} \right)^{-1} \left( \frac{1}{1 - \mu z} \right)^{-1} = \frac{1}{\mu - \lambda} \left( z - \frac{1}{\lambda} \right) + \frac{1}{\lambda - \mu} \left( z - \frac{1}{\mu} \right).
\]

**Proof:** See for instance [8].

**Lemma 5.3**
\[
\begin{align*}
\oint_{z=\infty} \mathbb{Y} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum_{i=1}^{\infty} (t_i - t'_i)z^i} z^{2n-2m-2} dz \\
+ \oint_{z=0} \mathbb{Y} \tau_{n+1}(t + [z])\tau_m(t' - [z])e^{\sum_{i=1}^{\infty} (t'_i - t_i)z^{-i}} z^{2n-2m} dz \\
= \frac{1}{\mu - \lambda} \left( \mu^{2n}\lambda^{2m}\tau_n(t - [\mu^{-1}])\tau_m(t' - [\lambda^{-1}])e^{\sum_{i=1}^{\infty} (t'_i \lambda^i + t_i \mu^i)} \\
- \lambda^{2n}\mu^{2m}\tau_n(t - [\lambda^{-1}])\tau_m(t' - [\mu^{-1}])e^{\sum_{i=1}^{\infty} (t_i \lambda^i + t'_i \mu^i)} \right)
\end{align*}
\]

**Proof:** Upon performing the following operations
\[
\begin{align*}
&\{ n \mapsto n - 1 \\
&\{ t \mapsto t - [\mu^{-1}] - [\lambda^{-1}] \\
&\{ \text{multiplication by } (\lambda \mu)^{2n-1} e^{\sum_{i=1}^{\infty} t_i (\mu^i + \lambda^i)} \}
\end{align*}
\]
the bilinear identity (5.3) yields

\[
0 = \oint_{z=\infty} \tau_{n-1}(t - [z^{-1}] - [\lambda^{-1}] - [\mu^{-1}]) \tau_{m+1}(t' + [z^{-1}]) (\lambda \mu)^{2n-2} \\
\left(1 - \frac{z}{\lambda} \right) \left(1 - \frac{z}{\mu} \right) \frac{\lambda \mu}{z^2} e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} z^{2n-2m-2} dz \\
+ \oint_{z=0} \tau_n(t + [z] - [\lambda^{-1}] - [\mu^{-1}]) \tau_m(t' + [z]) (\lambda \mu)^{2n} \\
\left(1 - \frac{1}{\lambda z} \right)^{-1} \left(1 - \frac{1}{\mu z} \right)^{-1} \frac{1}{\lambda \mu z^2} e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} z^{2n-2m} dz.
\]

Subtracting this expression (which is \(= 0\)), the left hand side of (5.5) equals

\[
\oint_{z=\infty} \tau_{n-1}(t - [z^{-1}] - [\lambda^{-1}] - [\mu^{-1}]) \tau_{m+1}(t' + [z^{-1}]) \\
e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} (\lambda \mu)^{2(n-1)} z^{2n-2m-2} dz \\
+ \oint_{z=0} \tau_n(t + [z] - [\lambda^{-1}] - [\mu^{-1}]) \tau_m(t' + [z]) \\
e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} (\lambda \mu)^{2n} z^{2n-2m} dz \\
= \oint_{z=\infty} \tau_{n-1}(t - [z^{-1}] - [\lambda^{-1}] - [\mu^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} \\
(\lambda \mu)^{2n-2} \left(1 - \frac{\lambda}{z} \right) \left(1 - \frac{\mu}{z} \right) \frac{\lambda \mu}{z^2} \left(1 - \frac{z}{\lambda} \right) \left(1 - \frac{z}{\mu} \right) z^{2n-2m-2} dz \\
+ \oint_{z=0} \tau_n(t + [z] - [\lambda^{-1}] - [\mu^{-1}]) \tau_m(t' + [z]) e^{\sum_i^\infty ((t_i - t_i')z^i + t_i(\mu^i + \lambda^i))} (\lambda \mu)^{2n} \\
\left(1 - \frac{1}{\lambda z} \right)^{-1} \left(1 - \frac{1}{\mu z} \right)^{-1} \left(1 - \frac{1}{\lambda z} \right) \left(1 - \frac{1}{\mu z} \right) z^{2n-2m} dz \\
= \frac{1}{\mu - \lambda} \left(\mu^{2n} \lambda^{2m} \tau_n(t - [\mu^{-1}]) \tau_m(t' - [\lambda^{-1}]) e^{\sum_i^\infty ((t_i + t_i')(\mu^i + \lambda^i))} \\
\right.
\left.\left. - \lambda^{2n} \mu^{2m} \tau_n(t - [\lambda^{-1}]) \tau_m(t' - [\mu^{-1}]) e^{\sum_i^\infty ((t_i + t_i')(\mu^i + \lambda^i))} \right) \right) ,
\]

ending the proof of the lemma.
Proof of theorem 5.1: It suffices to prove

\[ 0 = \oint_{z=\infty} (a + b Y) \tau_n(t - [z^{-1}]) (a + b Y) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_{i=1}^{\infty} (t_i - t'_i) z^i} z^{2n-2m-2} dz \]

\[ + \oint_{z=0} (a + b Y) \tau_{n+1}(t + [z]) (a + b Y) \tau_m(t' - [z]) e^{\sum_{i=1}^{\infty} (t'_i - t_i) z^{-i}} z^{2n-2m} dz. \]

The coefficient of \( a^2 \) and \( b^2 \) vanishes, on view of the fact that \( \tau_n \) and \( Y \tau_n \) are Pfaffian \( \tau \)-functions. So it suffices to show the vanishing of the \( ab \)-term.

\[
ab\text{-coefficient} = \oint_{z=\infty} \left( Y \tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) + \tau_n(t - [z^{-1}]) Y \tau_{m+1}(t' + [z^{-1}]) \right) e^{\sum_{i=1}^{\infty} (t_i - t'_i) z^i} z^{2n-2m-2} dz \\
+ \oint_{z=0} \left( Y \tau_{n+1}(t + [z]) \tau_m(t' - [z]) + \tau_{n+1}(t + [z]) Y \tau_m(t' - [z]) \right) e^{\sum_{i=1}^{\infty} (t'_i - t_i) z^{-i}} z^{2n-2m} dz.
\]

The first terms in each of the integrals can be evaluated by means of lemma. The sum of the two terms equals

\[
\frac{1}{\mu - \lambda} \left( \mu^{2n} \lambda^{2m} \tau_n(t - [\mu^{-1}]) \tau_m(t' - [\lambda^{-1}]) e^{\sum_{i=1}^{\infty} (t_i \lambda^{-1} + t'_i \mu^{-1})} \right) \\
- \left( \lambda^{2n} \mu^{2m} \tau_n(t - [\lambda^{-1}]) \tau_m(t' - [\mu^{-1}]) e^{\sum_{i=1}^{\infty} (t_i \lambda^{-1} + t'_i \mu^{-1})} \right). \tag{5.6}
\]

Performing the exchange

\[ n \leftrightarrow m, \quad t \leftrightarrow t', \quad z \leftrightarrow z^{-1} \]

gives an expression for the sum of the second terms in the integrals; the sum of expression (5.6) and the same expression with the exchange above is obviously zero.

\[ \square \]

6 Examples
6.1 Symmetric matrix integrals

Consider the matrix $m_{n}(t, s)$ of $(t, s)$-dependent moments,

$$
\mu_{k\ell}(t, s) := \int \int_{\mathbb{R}^2} x^k y^\ell e^{\sum_{i=1}^{\infty} (t_i x_i - s_i y_i)} F(x, y) \, dx \, dy, \quad t, s \in \mathbb{C}^\infty,
$$

with regard to a skew-symmetric weight $F(x, y)$, satisfying $F(x, y) = -F(y, x)$.

Then

$$
\tau_n(t, s) := \det m_{n}(t, s)
$$

and

$$
\tau_2(t) := \text{Pfaff } m_2(t, -t) = \sqrt{\tau_2(t, -t)}
$$

satisfies equations (0.6) up to (0.8).

Moreover, the moments $\mu_{ij}$ in (6.1) satisfy the equations

$$
\frac{\partial \mu_{ij}}{\partial t_k} = \mu_{i+k,j} \quad \text{and} \quad \frac{\partial \mu_{ij}}{\partial s_k} = -\mu_{i,j+k},
$$

and so $m := m_{\infty}$ satisfies (1.1).

The skew-symmetry of $F$ above implies the skewness of $m_{\infty}(0, 0)$; so, by theorem 2.1, we have

$$
\mu_{ij}(t, s) = -\mu_{ji}(-s, -t).
$$

The skew-symmetric weights $F(x, y)$ of the special form

$$
F(x, y) := e^{V(x)+V(y)} I_E(x) I_E(y) \text{sign}(x - y), \quad \text{for an interval } E \subset \mathbb{R},
$$
and for a union of intervals $E \subset \mathbb{R}$, the expression $\tilde{\tau}_{2n} = \tau_{2n}(t,-t)^{1/2}$ equals the integral over symmetric matrices, given in

$$\tilde{\tau}_{2n}(t) = \sqrt{\tau_{2n}(t,-t)} = \int_{S_{2n}(E)} e^{\text{Tr}(V(X)+\sum_{i=1}^{\infty} t_i X_i)} dX,$$

for the Haar measure $dX$ on symmetric matrices and $S_{2n}(E) := \{2n \times 2n \text{ symmetric matrices } X \text{ with spectrum } \subset E\}$.

In [5], we worked out the Virasoro constraints satisfied by $(??)$, which then leads to inductive expressions for those integrals, involving Painlevé-like expressions.

### 6.2 Quasiperiodic solutions

In this subsection, we shall combine the construction of quasi-periodic solutions of 2-Toda lattice [14, 17] and the theory of Prym varieties [15] to obtain quasiperiodic solutions of the Pfaff lattice. While we put stress on the semi-infinite case in the present paper, this gives a non-trivial example in the bi-infinite case.

A 2-Toda quasiperiodic solution is given by some deformation of a line bundle $L$ on a complex curve (Riemann surface) $C$, with the time variables playing the role of deformation parameters, so the orbit under the 2-Toda flows is parametrized by the Jacobian of $C$. If $C$ is equipped with an involution $\iota: C \to C$, and if $L$ satisfies a suitable antisymmetry condition with respect to $\iota$, then the 2-Toda flows can be restricted to preserve the antisymmetry of $L$, giving a solution of Pfaff lattice. The Prym variety $P$ of $(C, \iota)$ naturally appears as the restricted parameter space. The vanishing of every other $\tau_n(t,-t)$ (see (1.4) or (2.6)) indicates that the space of $L$’s which satisfy the antisymmetry condition must consist of two connected components, $P$ and $P^-$. This means the involution $\iota$ has no fixed points. So, in general a quasiperiodic solution of the Pfaff lattice does not satisfy the BKP equation and vice versa, since the orbit of a quasiperiodic solution of the BKP equation is isomorphic to the Prym variety of a curve with involution having at least two fixed points.

**Preliminary on the geometry of curves**

A line bundle on a complex curve $C$ is defined by a divisor $D = \sum m_i p_i$, $m_i \in \mathbb{Z}$, $p_i \in C$, i.e., a set of points $p_i$ on $C$ with (positive or negative)
multiplicities $m_i$, as $\mathcal{L} = \mathcal{O}(D)$. Its local sections (on an open set $U \subset C$, say) are meromorphic functions on $U$ which have poles of order at most $m_i$ (zeros of order at least $-m_i$) at $p_i$. The number $d := \sum m_i$ is called the degree of $\mathcal{L}$. For $\mathcal{L} = \mathcal{O}(D)$ and $m, n \in \mathbb{Z}$, $p, q \in C$, we denote $\mathcal{L}(mp + nq) = \mathcal{O}(D + mp + nq)$ etc. A deformation of $\mathcal{L}$ can be described as a deformation of $D$, like $D_{t,s} = \sum m_i p_i(t,s)$, but in the 2-Toda theory it is more convenient to describe it by requiring its local sections to have some exponential behaviors at prescribed points, as we shall see later.

Two line bundles $\mathcal{O}(D_1)$ and $\mathcal{O}(D_2)$ are isomorphic if the divisors $D_1$ and $D_2$ are “linearly equivalent,” i.e., if they differ by the divisor of a global meromorphic function on $C$. Jacobian $J$ of $C$ is the space (Lie group) of isomorphism classes of degree 0 line bundles on $C$. It becomes a principally polarized abelian variety of dimension $g := \text{genus of } C$, i.e., $J$ is a complex torus $\mathbb{C}^g/\Gamma$, $\mathbb{C}^g \supset \Gamma \cong \mathbb{Z}^{2g}$, for which there is a divisor (codimension 1 subvariety) $\Theta \subset J$, such that some positive integer multiple of $\Theta$ defines an embedding of $J$ into a complex projective space, and $\Theta$ is “rigid” in the sense that it has no deformation in $J$ except parallel translations. A complex torus $\mathbb{C}^g/\Gamma$ is a principally polarized abelian variety if and only if, after some change of coordinates by $GL(g, \mathbb{C})$, the lattice $\Gamma$ becomes $\mathbb{Z}^g + \Omega \mathbb{Z}^g$ for some complex symmetric $g \times g$ matrix $\Omega$ with positive definite imaginary part.

On a principally polarized abelian variety $\mathbb{C}^g/\Gamma$, there is a special quasiperiodic function (i.e., holomorphic function on $\mathbb{C}^g$ that satisfies some quasiperiodicity condition with respect to $\mathbb{Z}^g + \Omega \mathbb{Z}^g$) called Riemann’s theta function $\vartheta$, defined by

$$\vartheta(z) = \sum_{m \in \mathbb{Z}^g} \exp(2\pi i m^t z + \pi i m^t \Omega m).$$

The theta divisor $\Theta$ becomes the zero divisor of $\vartheta$.

If $C$ has a (holomorphic) involution $\iota: C \to C$ (i.e., $\iota^2 = \text{id}$), $J$ gets an involution $\iota^*$ induced by $\iota$. The Jacobian $J'$ of the quotient curve $C' = C/\iota$, and the Prym variety $P$ of the pair $(C, \iota)$ (or $(C, C')$) appear in $J$ roughly as the $\pm 1$ eigenspaces of $\iota$: $J' = J'/(\text{some subgroup of order 2}) \subset J$ and $P \subset J$ are subabelian varieties of $J$, such that $\iota|_{J'} = +1$, $\iota|_P = -1$, and $J \cong (J' \times P)/(\text{finite subgroup})$. When $\iota$ has at most two fixed points, the restriction of $\Theta$ on $P$ gives twice some principal polarization on $P$ (the restriction $\vartheta|_P$ becomes the square of the Riemann theta function on $P$ defined by this polarization).
Quasiperiodic solutions of 2-Toda lattice

Let $C$ be a nonsingular complete curve on $\mathbb{C}$ (compact Riemann surface) of genus $g$, let $\mathcal{L}$ be a line bundle of degree $g-1$ on $C$, let $p, q \in C$ be distinct points. Let us choose local coordinates $z^{-1}$ at $p$ and $z$ at $q$, and trivializations of $\mathcal{L}(p)$ at $p$ and $q$,

$$\sigma_p : \mathcal{L}_p(p) \simeq \mathcal{O}_p \quad \text{and} \quad \sigma_q : \mathcal{L}_q \simeq \mathcal{O}_q.$$ 

For $t, s \in \mathbb{C}^\infty$, let $\mathcal{L}_{t,s}$ be the line bundle whose (local holomorphic) sections are (local holomorphic) sections of $\mathcal{L}$ away from $p$ and $q$, and at $p$ (resp. $q$) have singularities of the form $e^{\sum t_i z^i}$ (holomorphic) (resp. $e^{\sum s_i z^{-i}}$ (holomorphic)). For “generic” $(n, t, s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty$, the wave functions $\Psi_{1,n}, \Psi_{2,n}$ are obtained from a (unique) section $\varphi_n(t, s)$ of $\mathcal{L}_{t,s}((n+1)p - nq)$, which has the form $z^n e^{\sum t_i z^i} (1 + O(z^{-1}))$ at $p$ via $\sigma_p$, i.e.,

$$\Psi_{1,n}(t, s, z) := \sigma_p(\varphi_n(t, s)) = z^n e^{\sum t_i z^i} (1 + O(z^{-1})), \quad \Psi_{2,n}(t, s, z) := \sigma_q(\varphi_n(t, s)) = z^n e^{\sum s_i z^{-i}} (h_n(t, s) + O(z)).$$  \hspace{1cm} (6.4)

The adjoint wave functions

$$\Psi_{1,n}^* = z^{-n} e^{-\sum t_i z^i} (1 + O(z^{-1})), \quad \Psi_{2,n}^* = z^{-n} e^{-\sum s_i z^{-i}} (h_n(t, s)^{-1} + O(z))$$

are defined similarly, by using

$$(\mathcal{L}_{t,s})^*(-np + (n+1)q) = (\mathcal{L}^*)_{-t,-s}(-np + (n+1)q),$$

in place of $\mathcal{L}_{t,s}((n+1)p - nq)$, where we denote

$$\mathcal{L}^* := \mathcal{H}om(\mathcal{L}, \omega) = \mathcal{L}^{-1} \otimes \omega,$$

with $\omega$ being the dualizing sheaf (the canonical bundle, i.e., the line bundle of holomorphic 1-forms), and, in place of $\sigma_p$ and $\sigma_q$, trivializations

$$\sigma_p^* : \mathcal{L}_p^* \simeq \mathcal{O}_p \quad \text{and} \quad \sigma_q^* : \mathcal{L}_q^* \simeq \mathcal{O}_q,$$

\hspace{1cm}13 Here generic means that $\Gamma(\mathcal{L}_{t,s}(-np + nq)) = \{0\}$ holds. For a degree $g-1$ line bundle $\mathcal{L}$, this condition holds for almost all $(n, t, s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty$, and implies that $\dim \Gamma(\mathcal{L}_{t,s}((n+1)p - nq)) = 1$. 

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for which the maps
\[
\begin{align*}
\mathcal{L}_p(p) \otimes \mathcal{L}_p^* & \ni (\phi, \psi) \mapsto \sigma_p(\phi) \sigma_p^*(\psi) dz/z \in \omega(p)_p, \\
\mathcal{L}_q \otimes \mathcal{L}_q^* & \ni (\phi, \psi) \mapsto \sigma_q(\phi) \sigma_q^*(\psi) dz/z \in \omega(q)_q
\end{align*}
\]
(6.5)
extend to the canonical map
\[
\mathcal{L}(p) \otimes \mathcal{L}^*(q) \Rightarrow \omega(p + q).
\]
Hence for general \((n, t, s), (m, t', s') \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty,\)
\[
\Psi_{i,n}(t, s, z) \Psi_{i,m}^*(t', s', z) dz/z, \quad i = 1, 2
\]
become expansions at \(p\) and \(q\), respectively, of a holomorphic 1-form on \(C \setminus \{p, q\}\), so by the residue calculus the pair \(\Psi, \Psi^*\) satisfies the bilinear identities (1.7).

**Quasiperiodic solutions of Pfaff lattice**

In the above construction, suppose \(C\) has an involution \(\iota: C \to C\) with no fixed point. In this case \(g\) is odd, \(g = 2g' - 1\), with \(g'\) being the genus of the quotient curve \(C' = C/\iota\). Suppose \(q = \iota(p)\), and \(\mathcal{L}\) satisfies
\[
\iota^*(\mathcal{L}) \simeq \mathcal{L}^*, \quad \text{so that} \quad \mathcal{L} \otimes \iota^* \mathcal{L} \simeq \omega.
\]
Choose the local coordinates \(z^{\mp 1}\) and the trivializations \(\sigma_p, \sigma_q, \sigma_p^*, \sigma_q^*\) at \(p\) and \(q = \iota(p)\), such that \(z \cdot \iota^* z \equiv 1\) and \(\sigma_q = \iota^* \sigma_p^* \circ \iota^*\) hold. (We then have \(\sigma_q^* = -\iota^* \circ \sigma_p \circ \iota^*\), with the minus sign due to the fact that \(dz/z\), which appear in (6.3), satisfy \(\iota^*(dz/z) = -dz/z\).) Then the wave and adjoint wave functions constructed above satisfy (2.3), so they lead to a quasiperiodic solution of the Pfaff lattice when \(s = -t\) (and skipping every other \(n\)).

The orbit of the 2-Toda flows is parametrized by the Jacobian \(J\) of \(C\), and the \(\tau\)-functions are written in terms of Riemann’s theta function of \(J\). The orbit of the Pfaff flows will become the Prym variety \(P\) of \((C, \iota)\), with \(\tilde{\tau}\) given by the Prym theta function. To be more precise, let \(J_{g-1}\) be the moduli space of the isomorphism classes of line bundles of degree \(g - 1\) on \(C\). This is a principal homogeneous space \(14\) over \(J\), on which the theta divisor
\[
\Theta := \{ \mathcal{L} \in J_{g-1} \mid \Gamma(\mathcal{L}) \neq (0) \}
\]

\(^{14}\) Hence \(J_{g-1}\) is (non-canonically) isomorphic to \(J\). We choose this isomorphism in such a way that \(\Theta \subset J_{g-1}\) is identified with the zero locus of Riemann’s theta function for \(J\).
is canonically defined. The set of $L \in J_{g-1}$ satisfying (6.6) becomes the disjoint union $P_{g-1} \cup P_{g-1}^-$, where

\[
\begin{align*}
P_{g-1} &:= \{ L \in J_{g-1} \mid L \text{ satisfies (6.6) and } \dim \Gamma(L) \text{ is even} \}, \\
P_{g-1}^- &:= \{ L \in J_{g-1} \mid L \text{ satisfies (6.6) and } \dim \Gamma(L) \text{ is odd} \}.
\end{align*}
\]

are principal homogeneous spaces over the Prym $P$. We have

\[
P_{g-1}^- \subset \Theta \quad \text{and} \quad P_{g-1}^- \cdot \Theta = 2\Xi,
\]

for some divisor $\Xi \subset P_{g-1}$ which gives a principal polarization on $P_{g-1}$. Since $\Theta$ is the zero locus of Riemann’s theta function $\vartheta$ of the Jacobian $J$, this means $\vartheta$ vanishes identically on $P_{g-1}^-$, and the restriction $\vartheta|_{P_{g-1}^-}$ becomes the square of Riemann’s theta function $\vartheta_P$ of $(P, \Xi)$, which is called the Prym theta function.

For a 2-Toda quasiperiodic solution, the discrete time flow (shift of $n$ by 1) is given by the shift $L \mapsto L(p - q)$. In the present case, since $q = \iota(p)$, this flow preserves condition (6.6). Moreover, we have

\[
\forall p \in C, \forall L \in J_{g-1} : \begin{cases} L \in P_{g-1} & \Rightarrow L(p - \iota(p)) \in P_{g-1}^-; \\
L \in P_{g-1}^- & \Rightarrow L(p - \iota(p)) \in P_{g-1}, \end{cases}
\]

so that $L(np - n\iota(p))$’s alternate between $P_{g-1}$ and $P_{g-1}^-$, and every other $\tau$ function vanishes identically when $s = -t$. Shifting the discrete index $n$ by 1 if necessary, we may assume that $\tau_n(t, s)$ satisfies (0.4) or (2.6).

**Explicit formulas**

Explicit formulas for $\Psi$, $\Psi^*$ and $\tau$ can be given in terms of Riemann’s theta function for $J$, and hence explicit formulas for $\tilde{\tau}$ can be given in terms of the Prym theta function for $P$.

Taking a basis $A_i, B_i$ ($i = 1, \ldots, g$) of $H_1(C, \mathbb{Z})$ such that $A_i \cdot B_j = \delta_{i,j}$ and $A_i \cdot A_j = B_i \cdot B_j = 0$, let $\omega_i$ ($i = 1, \ldots, g$) be a basis of the space of holomorphic 1-forms such that

\[
\int_{A_i} \omega_j = \delta_{i,j}.
\]

Then

\[
\int_{B_i} \omega_j = \Omega_{i,j}
\]
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gives a complex symmetric matrix \( \Omega \) with positive definite imaginary part, and \( J = \mathbb{C}^g/(\mathbb{Z}^g + \Omega \mathbb{Z}^g) \) becomes the Jacobian of \( C \). Choosing a point \( p \in C \), the map

\[
\alpha : C \ni x \mapsto \left( \int_p^x \omega_1, \ldots, \int_p^x \omega_g \right) \in J
\]

is well-defined and gives an embedding of \( C \) into \( J \). Composing \( \alpha \) with a translate of Riemann’s theta function:

\[
q(x) := \vartheta(\alpha(x) + a), \quad a \in \mathbb{C}^g, \quad (6.8)
\]

one obtains a multi-valued function on \( C \) which is single-valued around the \( A \)-cycles.

Next, let \( \zeta_n^{(\ell)} \), \( \ell = 1, 2, \ldots \), be the differentials of the second kind (meromorphic 1-forms with no residues) with poles only at \( p \) of the form \( d(z^n + O(1)) \) and no \( A \)-periods (\( \int_A \zeta_n^{(p)} = 0 \)), and let \( \zeta_n^{(q)} \), \( \ell = 1, 2, \ldots \), be defined similarly, with \( p \) replaced by \( q \) and \( z \) by \( z^{-1} \) (recall that \( z^{-1} \) (resp. \( z \)) is the local coordinate at \( p \) (resp. \( q \))). Let \( \zeta_0 \) be the differential of the third kind (meromorphic 1-form with simple poles) with no \( A \)-periods and poles only at \( p \) and \( q \) of the form \( dz/z + O(1) \). Then, given \( (n,t,s) \in \mathbb{Z} \times \mathbb{C}^\infty \times \mathbb{C}^\infty \), the multi-valued holomorphic function

\[
C \ni x \mapsto \varepsilon(x) := \exp\left( \int_x^p \left( n \zeta_0 + \sum_{n=1}^\infty t_n \zeta_n^{(p)} + \sum_{n=1}^\infty s_n \zeta_n^{(q)} \right) \right) \quad (6.9)
\]

has singularities at \( p \) and \( q \) of the form \( z^n e^{\sum t_n z^n} \) and \( z^n e^{\sum s_n z^{-n}} \), respectively, and is single-valued around \( A \)-cycles. The product of the form \( \varepsilon(x) q(x)/q(p) \), where \( \varepsilon(x) \) and \( q(x) \) are as in (6.8) and (6.9), with

\[
a = a(n,t,s) = n a(q) + \sum_{i=1}^\infty t_i U_i + \sum_{i=1}^\infty s_i V_i + a_0, \quad \forall a_0 \in \mathbb{C}^g, \quad (6.10)
\]

and \( U_i = -(d/d(z^{-1}))^i \alpha(p)/(i-1)! \), \( V_i = -(d/dz)^i \alpha(q)/(i-1)! \), gives function on \( (n,t,s) \), and hence the wave functions \( \Psi \), with the desired properties:

- \( \varphi_n(t,s;x) \) is single-valued around the \( A \)-cycles, and when \( x \) goes around \( B_i \), it is multiplied by a factor independent of \( n,t,s \),

- \( \varphi_n(t,s;x) \simeq z^n e^{\sum t_i z^i} (1 + O(z^{-1})) \) at \( x \simeq p \), and

- \( \varphi_n(t,s;x) \simeq z^n e^{\sum s_i z^{-i}} (h_n(t,s) + O(z)) \) at \( x \simeq q \).
The adjoint wave functions \( \Psi^* \) are obtained similarly, from \( \varepsilon(x)^{-1} \vartheta(\alpha(x) - a)/\vartheta(-a) \) with the same \( a \) as above.

The 2-Toda \( \tau \)-function can be computed from those formulas as

\[
\tau_n(t, s) = \exp(Q(n, t, s)) \vartheta(a(n, t, s))
\]

for some quadratic form \( Q(n, t, s) \), i.e.,

\[
Q(n, t, s) = \sum_{i,j=1}^{\infty} Q_{i,j} t^i t_j + \sum_{i,j=1}^{\infty} Q'_{i,j} s_i s_j + \sum_{i=1}^{\infty} n(q_i t_i + q'_i s_i),
\]

with \( Q_{i,j} = Q_{j,i} \) appearing in the Laurent expansion of the integral of \( \zeta_i^{(p)} \) or \( \zeta_j^{(q)} \) as

\[
\int x \zeta_i^{(p)} = z^i - 2 \sum_{j=1}^{\infty} Q_{i,j} z^{-j} / j \quad \text{for} \quad x \approx p,
\]

\( Q'_{i,j} = Q'_{j,i} \) appearing similarly in the Laurent expansion of the integral of \( \zeta_i^{(q)} \) or \( \zeta_j^{(g)} \) as

\[
\int x \zeta_i^{(q)} = z^{-i} - 2 \sum_{j=1}^{\infty} Q'_{i,j} z^j / j \quad \text{for} \quad x \approx q,
\]

and \( q_i \) and \( q'_i \) appearing similarly in the expansions

\[
\int x \zeta_0 = \log z - \sum_{j=1}^{\infty} q_j z^{-j} / j \quad \text{for} \quad x \approx p
\]

and

\[
\int x \zeta_0 = \log z - \sum_{j=1}^{\infty} q'_j z^j / j \quad \text{for} \quad x \approx q.
\]

Suppose \( C \) has an involution \( \iota \) with no fixed points, so that \( g = 2g' - 1 \) with \( g' \) being the genus of the quotient curve \( C' = C/\iota \). Suppose \( q = \iota(p) \).

Take the cycles \( A_i, B_i \) \( (i = 1, \ldots, g) \) in such a way that \( \iota(A_i) \simeq A_{g+1-i} \), \( \iota(B_i) \simeq B_{g+1-i} \). Then \( \iota^*(\omega_i) = \omega_{g+1-i} \), and \( \Omega \) satisfies \( \Omega_{i,j} = \Omega_{g+1-i,g+1-j} \).

The map \( \iota: \mathbb{C}^g \ni (z_1, \ldots, z_g) \mapsto (z_g, \ldots, z_1) \in \mathbb{C}^g \) maps the lattice \( \Gamma := \mathbb{Z}^g + \Omega \mathbb{Z}^g \) onto itself, and the embeddings

\[
\bar{J}' = J'/(\mathbb{Z}/2\mathbb{Z}) \subset J \quad \text{and} \quad P \subset J
\]
are given by the images under $\pi_L: \mathbb{C}^g \to \mathbb{C}^g/\Gamma$ of the $\pm 1$-eigenspaces of $\bar{i}$: Setting

$$R' := (\delta_{i,j} + \delta_{i,g+1-j})_{1 \leq i \leq g, 1 \leq j \leq g'} \quad \text{and} \quad R'' := (\delta_{i,j} - \delta_{i,g+1-j})_{1 \leq i \leq g, 1 \leq j \leq g'-1},$$

so that $\mathbb{C}^g_+ := R'\mathbb{C}^g$ and $\mathbb{C}^g_- := R''\mathbb{C}^{g'-1}$ are the $\pm 1$-eigenspaces of $\bar{i}$, and for any $z \in \mathbb{C}^g$, $z' := (1/2)(R')^t z$ and $z'' := (1/2)(R'')^t z$ give the decomposition $z = R'z' + R''z'' \in \mathbb{C}^g_+ \oplus \mathbb{C}^g_-$, we have

$$\bar{J}' = \mathbb{C}^g / (\varepsilon \mathbb{Z}^g + \Omega' \mathbb{Z}^g) \simeq \pi_L(R'\mathbb{C}^g) \subset \mathbb{C}^g / (\mathbb{Z}^g + \Omega' \mathbb{Z}^g) \quad z' \mapsto R'z'$$

and

$$P = \mathbb{C}^g / (\mathbb{Z}^{g'-1} + \Omega'' \mathbb{Z}^{g'-1}) \simeq \pi_L(R''\mathbb{C}^{g'-1}) \subset \mathbb{C}^g / (\mathbb{Z}^g + \Omega' \mathbb{Z}^g) \quad z'' \mapsto R''z'', \quad (6.12)$$

where $\varepsilon = \text{diag}(1,1,\ldots,1,1/2)$,

$$\Omega' = \left( \frac{\Omega_{i,j} + \Omega_{i,g+1-j}}{(1 + \delta_{i,g'})(1 + \delta_{j,g'})} \right)_{1 \leq i,j \leq g'} \quad \text{and} \quad \Omega'' = (\Omega_{i,j} - \Omega_{i,g+1-j})_{1 \leq i,j \leq g'-1}. $$

In (5.10), suppose $a_0 = R''a''_0 \in R''\mathbb{C}^{g'-1}$. Since, by definition, $\alpha(q) = \alpha(q) - \alpha(p) \in \pi_L(\mathbb{C}^g)$ and $\bar{i}(U_i) = V_i$, we then have $a(n,t,-t) = R''a''(n,t)$, where

$$a''(n,t) = \frac{1}{2}(R'')^t a(n,t,-t) = (R'')^t \left( \frac{1}{2}n\alpha(q) + \sum_{i=1}^{\infty} t_i U_i \right) + a''_0. $$

Hence by using (5.11) and (5.12), and noting that $Q'_{i,j} = Q_{i,j}$ and $q'_i = -q_i$, we have

$$\tilde{\tau}(t) = \exp(\tilde{Q}(n,t)) \vartheta_P(a''(n,t)), $$

where

$$\tilde{Q}(n,t) = \sum_{i,j=1}^{\infty} Q_{i,j}t_i t_j + \sum_{i=1}^{\infty} q_i n t_i, $$

and

$$\vartheta_P(z) = \sum_{m \in \mathbb{Z}^{g'-1}} \exp(2\pi im^t z + \pi im^t \Omega'' m), \quad \text{for} \quad z \in \mathbb{C}^{g'-1}. $$
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