Combinatorial Reduction of Set Functions and Matroid Permutations through Minor Product Assignment

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Abstract

We introduce an algebraic model, based on the determinantal expansion of the product of two matrices, to test combinatorial reductions of set functions. Each term of the determinantal expansion is deformed through a monomial factor in $d$ indeterminates, whose exponents define a $\mathbb{Z}^d$-valued set function. By combining the Grassmann-Plücker relations for the two matrices, we derive a family of sparse polynomials, whose factorisation properties in a Laurent polynomial ring are studied and related to information-theoretic notions.

Under a given genericity condition, we prove the equivalence between combinatorial reductions and determinantal expansions with invertible minor products; specifically, a deformation returns a determinantal expansion if and only if it is induced by a diagonal matrix of units in $\mathbb{C}(t)$ acting as a kernel in the original determinant expression. This characterisation supports the definition of a new method for checking and recovering combinatorial reductions for matroid permutations.

1 Introduction

A natural question about set functions regards conditions that imply a form of complexity reduction: given sets $X$ and $M$, a family $\mathcal{X}$ of subsets of $X$, and a map $\Psi : \mathcal{X} \rightarrow M$, we ask whether there is an underlying function $\psi : X \rightarrow M$ such that, for all $I \in \mathcal{X}$, we can recover $\Psi(I)$ from $\{\psi(\alpha) : \alpha \in I\}$. The additivity axiom for countable probability distributions is a fundamental example of such a condition; but we can extend this investigation to more general cases where $M$ is a $R$-module for a ring $R$. In this way, we can reconstruct $\Psi$ from $\psi$ by taking advantage of algebraic relations.

This work focuses on determinant expansions as a means to encode information about set functions, relying on maximal minors of two $(k \times n)$-dimensional full-rank complex matrices to specify the type of combinatorial reduction. In fact, Grassmann-Plücker relations among these minors are algebraic conditions entailing a reduction for $k$-forms on $\mathbb{C}^n$ [5, Ch.3.1]: formally, they characterise totally decomposable $k$-forms $v_{i_1} \wedge \ldots \wedge v_{i_k} \in \wedge^k \mathbb{C}^n$, which are labelled by $k$-subsets of $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and are expressed as the wedge product of $1$-forms $v_{i_1}, \ldots, v_{i_k}$.

The motivation behind the study of such determinantal relations is the wide range of applications of combinatorial, algebraic, and geometric properties of determinantal varieties and Grassmann-Plücker relations. To elucidate the main notions underlying this research, we start by recalling the explicit connection between determinantal varieties and one of these application areas.
1.1 Relations with previous work: determinantal expansions of Wronskian soliton solutions

The bilinear form of the Kadomtsev-Petviashvili (KP) II hierarchy, which is a family of PDEs expressing Grassmann-Plücker relations for an infinite-dimensional space [7], includes a special class of solutions (τ-functions) with a rich combinatorial structure [10], namely, Wronskian solutions of the type

\[ \tau(x) = \det(A \cdot \Theta(x) \cdot K) \]  

(1)

where \( A \) is a matrix of constant coefficients,

\[ \Theta := \text{diag}\left( \exp\left( \sum_{r=1}^{d} \kappa_r x_r \right), \ldots, \exp\left( \sum_{r=1}^{n} \kappa_r x_r \right) \right) \]  

(2)

and \( K \) is the Vandermonde matrix associated with the \( n \)-tuple \((\kappa_1, \ldots, \kappa_n)\). For these solutions of the KP II hierarchy, the determinant (1) is converted into an exponential sum through the well-known Cauchy-Binet expansion for two \((k \times n)\)-dimensional matrices \( A, K^T \) over a ring \( (k \leq n) \)

\[ \det(A \cdot K) = \sum_{\mathcal{I} \in \wp_k[n]} \Delta_A(\mathcal{I}) \cdot \Delta_K(\mathcal{I}). \]  

(3)

where \( \wp_k[n] := \{ \mathcal{I} \subseteq \{1, \ldots, n\} : \#\mathcal{I} = k \} \) and \( \Delta_A(\mathcal{I}) \) (respectively, \( \Delta_K(\mathcal{I}) \)) is the maximal minor of \( A \) extracted from columns (respectively, rows) indexed by \( \mathcal{I} \subseteq \{1, \ldots, n\} \). From (3), the solution (1) is expressed as

\[ \det(A \cdot \Theta(x) \cdot K) = \sum_{\mathcal{I} \in \wp_k[n]} \Delta_A(\mathcal{I}) \cdot \Delta_K(\mathcal{I}) \cdot e^{\sum_{\alpha \in \mathcal{I}} x_{\alpha} \cdot K\{\alpha\}} \]  

(4)

so the \( \tau \)-function is a superposition of elementary waves (exponential functions) whose phases are set functions defined by the soliton parameters.

The combinatorial properties derived from (4) (see e.g. [10]) suggest broadening the exploration of the information provided by deformations of the individual terms of such expansions that are compatible with the determinantal constraints, i.e. return another determinant expansion. The introduction of deformations is a way to investigate the rigidity of a given structure in terms of possible configurations that are consistent with algebraic constraints. In [1], we focused on the information-theoretic aspects of expressions (4) preserving the KP II equation under discrete deformations

\[ \Delta_A(\mathcal{I}) \mapsto \sigma_{\mathcal{I}} \cdot \Delta_A(\mathcal{I}), \quad \sigma_{\mathcal{I}} \in \{1, -1\} \quad \mathcal{I} \in \wp_k[n]. \]  

(5)

It should be noted that other determinantal expansions and their variations have been addressed in the literature, highlighting complexity aspects and computational advantages arising from the symmetries of the determinant, compared to other immanants such as the permanent [16].

1.2 Scope of this work: from algebraic to combinatorial conditions on set functions

In this paper, we extend the previous investigation by focusing on algebraic and combinatorial properties entailed by deformations

\[ \Delta_A(\mathcal{I}) \cdot \Delta_K(\mathcal{I}) \mapsto \Delta_A(\mathcal{I}) \cdot \Delta_K(\mathcal{I}) \cdot c_{\mathcal{I}} \cdot t^{e(\mathcal{I})}, \quad \mathcal{I} \in \wp_k[n], \Delta_A(\mathcal{I}) \neq 0, c_{\mathcal{I}} \in \mathbb{C} \setminus \{0\} \]  

(6)
of each minor product in (3) through a monomial in \(d\) indeterminates, where we have adopted the notation

\[
t^a(I) := \prod_{u=1}^d t_u^{e_u(I)}.
\]  

Note that the deformations (5) studied in [1] can be seen as a reduction of (6), choosing \(d = 1\) and specifying \(t_1 = -1\) and \(c_I = 1\) for all the sets \(I \in \wp_k[n]\) that contribute to (6). Moreover, (6) generalises the functional form of the phases in (4) for rational soliton parameters \(\kappa_1, \ldots, \kappa_n\), as follows from the change of variables \(e^\kappa/M := r_\alpha\) for all \(\alpha \in \{1, \ldots, n\}\) and an appropriate choice of \(M \in \mathbb{N}\).

We recall that, given a matrix \(A \in C^{k \times n}\), the set

\[
\Theta(A) := \{I \in \wp_k[n] : \Delta_A(I) \neq 0\}.
\]

is a matroid, i.e. a non-empty set characterised by the following exchange relation [13]

\[
\forall A, B \in \Theta(A), \alpha \in A \setminus B : \exists \beta \in B \setminus A. A_{\beta}^\alpha \in \Theta(A).
\]

Then, a deformation (6) can be described by a mapping that associates each subset \(I \in \Theta(A)\) with an element \(\Psi(I) := (e_1(I), \ldots, e_d(I)) \in \mathbb{Z}^d\). From this, we can explore the existence of a “potential” \(\psi : \{1, \ldots, n\} \rightarrow \mathbb{Z}^d\) such that \(\Psi(I)\) can be uniquely recovered from values \(\psi(\alpha)\) with \(\alpha \in I\).

To relate \(\Psi\) and \(\psi\), we use the \(Z\)-module structure of \(\mathbb{Z}^d\) looking for additive and affine set functions. Additivity generalises (4) and, in turn, extends to affine set functions since the set of configurations \(e\) in (6) that return the terms of a determinantal expansion is closed under translations \(e \mapsto e + m_0\) for any constant \(m_0 \in \mathbb{Z}^d\). More importantly, affine set functions let us introduce consistency (another notion often related to integrability, see e.g. [2]) in our combinatorial context: for all \(\mathcal{H} \subseteq \{1, \ldots, n\}\) with \(\#\mathcal{H} = k - 2\) and \(\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{1, \ldots, n\} \setminus \mathcal{H}\), the object of interest is the quantity

\[
\Psi(\mathcal{H} \cup \{\alpha_1, \alpha_2\}) + \Psi(\mathcal{H} \cup \{\beta_1, \beta_2\}) - \Psi(\mathcal{H} \cup \{\alpha_1, \beta_2\}) - \Psi(\mathcal{H} \cup \{\beta_1, \alpha_2\})
\]

whenever the arguments lie in the domain of \(\Psi\), that is, \(\Theta(A)\). Setting (10) equal to zero entails the consistency of the differences \(\Psi(I \setminus \{\alpha_1\} \cup \{\beta_1\}) - \Psi(I)\) for two different choices of the basis, indexed by \(I := \mathcal{H} \cup \{\alpha_1, \beta_2\}\) or \(I := \mathcal{H} \cup \{\alpha_1, \beta_2\}\), respectively. When, in addition, we have \(\mathcal{H} \cup \{\alpha_u, \beta_u\} \in \Theta(A)\) for both \(u \in \{1, 2\}\), we also get consistency between two different sequences \((\alpha_1, \alpha_2) \rightarrow (\alpha_1, \beta_u) \rightarrow (\beta_1, \beta_2)\), defined by \(u \in \{1, 2\}\), of single-index exchanges connecting the bases labelled by \(\mathcal{H} \cup \{\alpha_1, \alpha_2\}\) and \(\mathcal{H} \cup \{\beta_1, \beta_2\}\), respectively.

1.3 Overview of main results and implications for permutation testing

The first contribution of this work is to characterise the set functions that are compatible with the determinantal expansion.

**Theorem 1.** Let \(L(t), R(t)^T\) be two \((k \times n)\)-dimensional matrices depending on \(d\) indeterminates \(t\) such that \(\max\{n - k, k\} \geq 5\), \(R(t)\) is generic (i.e. no maximal minor vanishes for a generic choice of \(t\)), and \(L(1) \in C^{k \times n}\) has two generic columns, that is, there exist \(I \in \wp_k[n]\) and \(\alpha_1, \alpha_2 \in \{1, \ldots, n\} \setminus I\) such that

\[
J \setminus I \subseteq \{\alpha_1, \alpha_2\} \Rightarrow \Delta_{L(1)}(J) \neq 0, \quad J \in \wp_k[n].
\]

If the terms in the Cauchy-Binet expansion of \(\det(L(t) \cdot R(t))\) satisfy the monomial condition

\[
\Delta_{L(I)}(I) \cdot \Delta_{R(I)}(I) = g_I \cdot t^\Psi(I), \quad I \in \Theta(I), \quad g_I \in \mathbb{C}, \quad \Psi(I) \in \mathbb{Z}^d
\]

then there exist \(a(\mathcal{H}) : \mathcal{H} \in \wp_k[n]\) such that

\[
\sum_{\mathcal{H} \in \wp_k[n]} a(\mathcal{H}) \Delta_{L(I)}(I) \Delta_{R(I)}(I) = g(I) \cdot t^\Psi(I),
\]

for all \(I \in \wp_k[n]\), where

\[
g(I) := \sum_{\mathcal{H} \in \wp_k[n]} a(\mathcal{H})
\]

satisfies the Cauchy-Binet expansion of \(\det(L(t) \cdot R(t))\) for all \(I \in \wp_k[n]\).
then \( \Psi \) is an affine set function, that is, there exist an element \( m_0 \in \mathbb{Z}^d \) and a map \( \psi : \{1, \ldots, n\} \rightarrow \mathbb{Z}^d \) such that

\[
\Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) = t^{m_0} \cdot \Delta_{L(1)}(I) \cdot \Delta_{R(1)}(I) \prod_{\alpha \in \mathcal{I}} t^{\psi(\alpha)}, \quad I \in \mathcal{A}_n[n].
\] (13)

Equivalently, the genericity condition on \( G(L(1)) \) in (11) guarantees that the compatibility with determinantal expansions for a generic \( R(t) \) is achieved if and only if we can reduce to the expansion of

\[
\det \left( L(1) \cdot \text{diag}(t^{\psi(\alpha)})_{\alpha \in \{1, \ldots, n\}} \cdot R(1) \right)
\] (14)

apart from a common unit \( t^{m_0} \) that is irrelevant in terms of Plücker coordinates. This factor can be absorbed into the determinant by translation \( \psi \mapsto \psi + m_0/n \) and change of variables \( t \mapsto s^n \) to deal with units of \( C(t, s)/I \), where \( I \) is the ideal generated by binomials \( t_u - s^n_u, \ u \in \{1, \ldots, d\} \).

Theorem 1 may fail when there are not enough non-vanishing minors of \( L(t) \): under the genericity conditions for \( R \) and for two columns of \( L \), the reduced number of minors may only come from the low dimensionality of the two matrices. When the genericity condition for \( L(1) \) is violated, we can find a counterexample to Theorem 1.

**Example 1.** Let us take \( k = 6, n = 12 \) (so \( n - k \geq \max\{5, k\} \)), and matrices

\[
L_c := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}, \quad R_c := \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & t & t+1 & t+1 & t+1 & t+1 & t+1 & t+1 \\
0 & 1 & 0 & 0 & 0 & t-1 & t & t-2 & t-1 & t-6 & t-6 & t-6 \\
0 & 0 & 1 & 0 & 0 & 0 & t+2 & t & t+5 & t+3 & t+10 & t+10 \\
0 & 0 & 0 & 1 & 0 & 0 & t-1 & t-1 & t-5 & t & t-14 & t-7 \\
0 & 0 & 0 & 0 & 1 & t-1 & t-2 & t-10 & t+7 & t & t-13 & t
\end{pmatrix}
\] (15)

It is easily checked that \( \Delta_{L_c}(I) \neq 0 \) if and only if \( I \setminus \{1, 2, 3, 4, 5, 6\} \) is obtained from \( \{1, 2, 3, 4, 5, 6\} \) by adding 6 to all its elements. It follows that condition (11) is not verified. On the other hand, we can also check that all non-vanishing minors \( \Delta_{L_c}(I) \cdot \Delta_{R_c}(I) \neq 0 \) are monomials in \( t \) of degree 0 or 1. Evaluating \( R_c \) at \( t = 9 \), all the maximal minors of \( R_c(9) \) are non-vanishing, where \( \min_{I \in \mathcal{P}(12)} | \Delta_{R_c(9)}(I) | = 1 \) is achieved, for example, at \( I = \{1, 2, 3, 4, 5, 6\} \); by the continuity of the determinant as a function of its arguments, there is a neighbourhood of \( t = 9 \) where \( R_c(t) \) remains generic, so the genericity condition for \( R_c(t) \) is also verified. We see that

\[
\Psi (\{1, 2, 3, 4, 5, 6\}) + \Psi (\{3, 4, 5, 6, 7, 8\}) - \Psi (\{1, 3, 4, 5, 6, 8\}) - \Psi (\{2, 3, 4, 5, 6, 7\}) = -2
\] (16)

in contradiction to (10), which is a necessary condition for (13) to hold. Then, the combinatorial reduction does not take place in this case.

More general independence structures associated with \( L(1) \) require an additional study of minimal conditions that guarantee the combinatorial reduction (13), and we dedicate a separate work to this investigation.

The algebraic structure resulting from this model combines two quadratic contributions, namely the Grassmann-Plücker relations for two matrices in (3). This results in a set of quartic equations that constrain the form of the minors, and hence the deformations preserving the determinantal form. Our second contribution delves into the relations between the set of allowed deformations and the factorisation properties of polynomials derived from this combination of Grassmann-Plücker relations, in particular their squarefree decomposition. In turn, we will find that this squarefree decomposition leads to a characterisation of Shannon’s entropy for a Bernoulli random variable (see Remark 6). These observations highlight an information-theoretic interpretation of the model: \( L(t) \) is a source of structural information represented by the matroid \( G(L(1)) \), and a generic matrix \( R(t) \) is selected to explore this structure using accessible information (12).
The assumptions of Theorem 1 specify neither the form of $g_{\mathcal{I}}$ nor the exponents $\Psi(\mathcal{I})$, which makes the Cauchy-Binet expansion a sparse polynomial [9] with a given upper bound for its sparsity. This argument does not rely upon the association $\mathcal{I} \mapsto (g_{\mathcal{I}}, \Psi(\mathcal{I}))$ or potential reductions arising from the cancellation of terms in the expansion (3): if we are able to find an assignment of non-vanishing terms $g_{\mathcal{I}}$ that is consistent with (3) and the assumptions of Theorem 1, then the reduction (13) follows.

The third contribution of this work exploits Theorem 1 and the previous observation, providing a verification protocol to check the existence and, if so, recover a given permutation of $\{1, \ldots, n\}$ when available information is only provided by the unlabelled collection of terms (12). When the set function $\Psi$ corresponds to a permutation $\hat{\Psi} : \mathcal{G}(L(1)) \rightarrow \mathcal{G}(L(1))$, Theorem 1 can be used to check if it is induced by a permutation $\hat{\psi} : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$ even when we do not have explicit information about $\Psi$, but only through the terms of a candidate determinantal expansion.

**Theorem 2.** Let $k, n \in \mathbb{N}$ with $k \leq n$ and $\max\{n - k, k\} \geq 5$, and $\hat{\Psi} : \mathcal{G} \rightarrow \mathcal{G}$ be a permutation of a matroid $\mathcal{G}$ on $\{1, \ldots, n\}$ with rank $k$. Assume that each $n$-tuple $t \in \mathbb{C}^n$ returns a list, i.e. a tuple standing for an unlabelled multiset of $\# \mathcal{G}$ complex numbers, which represent the candidate components of a determinantal expansion. Then, we can choose two $n$-tuples $t_u, u \in \{1, 2\}$, to verify the fulfilment of the assumptions of Theorem 1, the existence of a permutation $\psi$ of $\{1, \ldots, n\}$ that induces $\Psi$ and, in the affirmative case, recover $\psi$ and matrices $a$ and $q$ that generate the terms of the expansion.

Additional a priori knowledge of the domain of the entries of $a$, e.g. $\mathbb{Z}$ or an algebraic number field, can be used to extend the previous argument, enabling specific methods to perform the verification based on less information. This is shown in Corollary 4, where only two numbers suffice to verify complexity reduction and retrieve two candidate matrices that define the determinantal expansion, if they exist.

This application provides a complementary view to distance-based permutation testing (see e.g. [4]) and relates to the geometric interpretation of maximal minors, which serve as homogeneous coordinates, in the projectivization $\mathbb{P}(\Lambda^k \mathbb{C}^n)$, for $k$-subspaces of $\mathbb{C}^n$ associated with $(k \times n)$-dimensional full-rank matrices. Indeed, the expansion (3) arises in the study of principal angles between subspaces of a vector space [11] and consequent applications [6, 3]. Theorem 2 allows us to deal with the lack of explicit information on coordinate labelling, which is a phenomenon that arises in unlabelled sensing or shuffled data [15, 14]; furthermore, it extracts sufficient information to decompose candidate minor products, identifying in this way two subspaces involved in the determinantal expansion.

The assumption of monomial deformation of terms in the Cauchy-Binet expansion is equivalent to the invertibility in the ring $\mathbb{C}(t)$ of the non-vanishing principal minors of the composed matrix

\[
\begin{pmatrix}
0_k & L(t) \\
R(t) & 0_n
\end{pmatrix}
\]

indexed by subsets $\{1, \ldots, k\} \cup \mathcal{I}$ with $\mathcal{I} \subseteq \{k + 1, \ldots, k + n\}$ and $\# \mathcal{I} = k$ (here $0_k$ denotes the $(k \times k)$-dimensional null matrix). This connection with non-constant, invertible minor assignment in a given ring is the basis for further studies.

### 1.4 Organisation of the paper

After Section 2, where we fix the notation that will be used in the rest of the paper, we derive the polynomial equations that are the basis of our study (Section 3) and the constraints on algebraic extensions (Section 4) resulting from these equations. In Section 5, we explore the constraints for non-vanishing minors in the three-term Grassmann-Plücker relations and describe allowed configurations (Subsections 5.1-5.2). From these results, in Section 6 we prove Theorem 1, also investigating the
occurrence of multiple distinct configurations to assess the minimal dimensions that guarantee combinatorial reduction. We provide an application to permutations on matroids in Section 7.

2 Preliminaries

2.1 Notation

Let $\varphi[n]$ be the power set of $[n] := \{1, \ldots, n\}$ and $\varphi_k[n] := \{I \subseteq [n] : \#I = k\} \subseteq \varphi[n]$. We adopt the shortening expressions

$$
\ell_\beta := \ell \{\alpha\} \cup \{\beta\}, \quad I \in \varphi[n], \alpha \in I, \beta \notin I
$$

and similarly, $\ell_\alpha := \ell \{\beta\}, \ell^\alpha := \ell \{\alpha\}, \ell^{\alpha_1, \alpha_2} := \ell \{\alpha_1, \alpha_2\}$, etc. The notation $\ell_\beta$ implicitly assumes that $\alpha \in I$ and $\beta \notin I$, unless $\alpha = \beta$ where $\ell^{\alpha_1}_1 := I$. We denote by $\ell^C := [n] \setminus I$ the complement of $I$ in $[n]$. We adopt the notation $A_{\alpha_1, \beta}$ to denote the submatrix of $A$ with rows extracted from $\alpha_1 \subseteq [k]$ and columns extracted from $\beta \subseteq [n]$.

Rather than working with exponential sums as in (4), we adopt a different parameterisation, moving to a polynomial ring with indeterminates $t := (t_u : u \in [d])$. For each index $w \in [d]$, the symbol $t_{\hat{w}}$ denotes the $(d - 1)$-tuple obtained from $t$ by omission of the $w$-th component. Given the ring $C(t) := C[t, t^{-1}]$ of the Laurent polynomials in $t$, let $F$ be the field of fractions of $C(t)$. Given two polynomials $P, Q$, the symbol $P \mid Q$ means that $P$ is a factor of $Q$.

For a non-vanishing polynomial $P \in C(t)$, we use the standard inner product $\langle \cdot, \cdot \rangle$ between polynomials to formulate the support and the exponent set of $P$, i.e.

$$
P \in C(t) \quad \Rightarrow \quad \text{Supp}(P) := \{m \in C(t) : m^{-1} \in C(t), \langle P, m \rangle = \langle m, m \rangle\},
$$

$$
P \in C(t) \quad \Rightarrow \quad \Psi(P) := \{e \in \mathbb{Z}^d : \exists c_e \in C \setminus \{0\}, c_e \cdot t^e \in \text{Supp}(P)\}.
$$

When $P := c \cdot t^e$ is invertible in $C(t)$ and no ambiguity arises, we use the symbol $\Psi(P)$ to refer to the vector of exponents $e \in \mathbb{Z}^d$, in line with the function $\Psi$ in (12). When $\Psi(P) \subseteq \{0, 1\}^d$, it reduces to the characteristic function of a finite subset of $\varphi[d]$.

**Remark 1.** Any unimodular matrix $V \in \mathbb{Z}^{d \times d}$ defines an invertible transformation $s_u := t^V u$, where $V_u$ is the $u$-th column of $V$ and $u \in [d]$, so $t_u = s \hat{V}^{-1}$. Each $P \in C(t)$ corresponds to a polynomial $P_s := P(\hat{V}^{-1}) \in C(s)$, which induces a bijection between $\text{Supp}(P)$ and $\text{Supp}(P_s)$: distinct monomials $a_1 m_1$ and $a_2 m_2$ in $\text{Supp}(P)$ are mapped into distinct monomials $a_1 s^{-1} \Psi(m_1)$ and $a_2 s^{-1} \Psi(m_2)$ in $\text{Supp}(P_s)$ due to $\ker V^{-1} = \{0\}$, and vice versa. This bijection extends to a ring isomorphism between $C(t)$ and $C(s)$.

We will make use of monomial orders: let us recall that a monomial order $\preceq$ is a total order on the class of monic monomials that is compatible with multiplication:

$$
\forall x, y, z \text{ monic monomials} : x \preceq y \Rightarrow x \cdot z \preceq y \cdot z.
$$

We express the left-hand side of (12) as

$$
h(I) := \Delta_{L(t)}(I) \cdot \Delta_{R(t)}(I), \quad I \in \varphi_k[n].
$$

**Definition 1.** The multiset

$$
\chi(I \mid _{\alpha, \beta}) := \{h(I) \cdot h(T_\alpha^\beta), h(T_\alpha^\beta) \cdot h(I_\alpha^\beta), h(I_\alpha^\beta) \cdot h(T_\alpha^\beta)\}
$$

(22)
is called observable when $\chi(\mathcal{I}^{|I|\alpha^3}) \neq \{0\}$ and integrable when

$$
\#\Psi \left( \chi(\mathcal{I}^{|I|\alpha^3}) \setminus \{0\} \right) = 1,
$$

(23)

which implies that $\chi(\mathcal{I}^{|I|\alpha^3})$ is also observable. We say that $\mathcal{I} \in \mathfrak{S}(\mathbf{L}(1))$ is an integrable basis when (23) holds for all observable sets $\chi(\mathcal{I}^{|I|\alpha^3})$ with basis $\mathcal{I}$. We call $h(\mathcal{I}) \cdot h(\mathcal{I}^{|I|\alpha^3})$ the central term of $\chi(\mathcal{I}^{|I|\alpha^3})$, while we refer to a term in $\chi(\mathcal{I}^{|I|\alpha^3})$ that is algebraically independent of the others as unique term. We will refer to (23) as a set to ease the notation, bearing in mind that repeated elements are allowed.

Subsets $\mathcal{I}$ are considered unordered: signed minors are introduced as

$$
\Delta_{\mathcal{A}}(\alpha, \beta | \mathcal{H}) := \Delta_{\mathcal{A}}(\mathcal{H}_{|\alpha,\beta}) \cdot (-1)^{1 + \#\mathcal{A} + S(\alpha, \mathcal{H})} \cdot \text{sign} (\alpha - \beta), \quad \mathcal{H} \in \wp_{k-2}[n]
$$

(24)

where

$$
S(\alpha, \mathcal{H}) := \#\{\beta \in \mathcal{H} : \beta < \alpha\}, \quad \alpha \in \mathcal{H}^C
$$

(25)

takes into account permutations of $[n]$ and makes us express the three-terms Plücker relations [5] as

$$
\Delta_{\mathcal{A}}(\delta_1, \delta_2 | \mathcal{H}) \cdot \Delta_{\mathcal{A}}(\delta_3, \delta_4 | \mathcal{H}) = \Delta_{\mathcal{A}}(\delta_1, \delta_3 | \mathcal{H}) \cdot \Delta_{\mathcal{A}}(\delta_2, \delta_4 | \mathcal{H}) - \Delta_{\mathcal{A}}(\delta_1, \delta_4 | \mathcal{H}) \cdot \Delta_{\mathcal{A}}(\delta_2, \delta_3 | \mathcal{H})
$$

(26)

for any $\mathcal{H} \in \wp_{k-2}[n]$ and pairwise distinct elements $\delta_a \in \mathcal{H}^C, a \in [4]$.

### 3 Coupled Plücker relations

Let us consider the Grassmann-Plücker relations in the form (26) for $\mathbf{L}(t)$, that is,

$$
\Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta}) \cdot \Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta}) = c_1 \Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta}) \cdot \Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta}) + c_2 \Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta}) \cdot \Delta_{\mathbf{L}(t)}(\mathcal{H}_{|\alpha,\beta})
$$

(27)

where $\mathcal{H} \in \wp_{k-2}[n], i, j, \alpha, \beta \in \mathcal{H}^C$, and signs

$$
c_1 := \text{sign} [(i - j) \cdot (\alpha - \beta) \cdot (i - \beta) \cdot (\alpha - j)],
$$
$$
c_2 := \text{sign} [(i - j) \cdot (\alpha - \beta) \cdot (i - \alpha) \cdot (j - \beta)]
$$

(28)

take into account the permutation of indices $i, j, \alpha, \beta$ with respect to a given order. The same relations hold for $\mathbf{R}(t)$. Multiplying term by term the resulting equations (27) for $\mathbf{L}(t)$ and $\mathbf{R}(t)$, we obtain

$$
h(\mathcal{I}) \cdot h(\mathcal{I}^{|I|\alpha^3}) = h(\mathcal{I}^{|I|\alpha^3}) \cdot h(\mathcal{I}^{|I|\alpha^3}) + h(\mathcal{I}^{|I|\alpha^3}) \cdot h(\mathcal{I}^{|I|\alpha^3})
$$

(29)

since $c_1, c_2$ depend only on the indices $i, j, \alpha, \beta$. This expression can be written as

$$
h(\mathcal{I}) \cdot h(\mathcal{I}^{|I|\alpha^3}) = h(\mathcal{I}^{|I|\alpha^3}) \cdot h(\mathcal{I}^{|I|\alpha^3}) + h(\mathcal{I}^{|I|\alpha^3}) \cdot h(\mathcal{I}^{|I|\alpha^3})
$$

(30)
where (28) gives

Furthermore, we introduce

\[ d_{\text{Plücker coordinates; this additional factor can be absorbed through a left multiplication of} \]

preserves the determinantal expansion up to a common factor

where the

\[ k \]

In addition, we consider the right multiplication of

Definition 2. We define the \( Y \)-terms, or cross-ratios, as

\[
Y(I)_{ij}^{αβ} := c_1c_2 \cdot \frac{\Delta_{R(t)}(I_α)}{\Delta_{R(t)}(I_β)} \cdot \frac{\Delta_{R(t)}(I_j)}{\Delta_{R(t)}(I_j)}
\]

where (28) gives

\[
c_1c_2 = -\text{sign}[(i-\alpha) \cdot (i-\beta) \cdot (j-\alpha) \cdot (j-\beta)].
\]

Furthermore, we introduce

\[
Y(I) := \left\{ Y(I)_{ij}^{αβ} : \chi(I)_{ij}^{αβ} \text{ is observable} \right\}.
\]

We extend the notation introduced in Definition 1 stating that \( Y(I)_{ij}^{αβ} \) is observable if \( \chi(I)_{ij}^{αβ} \) is an observable set.

The dependence of \( Y_{ij}^{αβ} := Y(I)_{ij}^{αβ} \) on the basis \( I \) will be implicit when no ambiguity arises. Being \( Y(I)_{ij}^{αβ} = Y(I)_{ij}^{αβ} \)\(^{-1}\), Assumption 1 about \( R(t) \) is equivalent to the existence of \( Y(I)_{ij}^{αβ} \) for all choices of bases and indices. For each generalised permutation matrix \( D(t) \) dependent on \( t \), the \( Y \)-terms (31) are invariant with respect to the action

\[
L(t) \mapsto L(t) \cdot D(t)^{-1}, \quad R(t) \mapsto D(t) \cdot R(t).
\]

Using this invariance, we can fix a form for the matrices \( R(t) \) and \( L(t) \) while preserving \( Y \)-terms. In particular, we choose

\[
D(t) := \text{diag}\left( R_{k+1,1}(t) \quad \ldots \quad R_{k+1,k}(t) \quad \frac{R_{k+1,1}(t)}{R_{k+2,1}(t)} \quad \ldots \quad \frac{R_{k+1,k}(t)}{R_{k+1,1}(t)} \right).
\]

In addition, we consider the right multiplication of \( R(t) \) by

\[
d(t) := \text{rcf}(R(t)) \cdot \text{diag}\left( R_{k+1,1}(t)^{-1} \quad \ldots \quad R_{k+1,k}(t)^{-1} \right)
\]

where the \( k \)-dimensional matrix \( \text{rcf}(R(t)) \) transforms \( R(t) \) into its reduced column echelon form. This preserves the determinantal expansion up to a common factor \( \det d(t) \) that is irrelevant in terms of Plücker coordinates; this additional factor can be absorbed through a left multiplication of \( L(t) \), e.g. by \( d(t)^{-1} \). In this way, \( R(t) \) takes the form

\[
R(t) \mapsto D(t) \cdot R(t) \cdot d(t) = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 1 & \ldots & 1 \\
\frac{R_{k+1,1}R_{k+2,2}}{R_{k+2,1}R_{k+1,1}} & \frac{R_{k+1,1}R_{k+2,2}}{R_{k+2,1}R_{k+1,1}} & \ldots & \frac{R_{k+1,1}R_{k+2,2}}{R_{k+2,1}R_{k+1,1}} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \frac{R_{k+1,1}R_{k+2,2}}{R_{k+2,1}R_{k+1,1}} & \ldots & \frac{R_{k+1,1}R_{k+2,2}}{R_{k+2,1}R_{k+1,1}} \\
\end{pmatrix}
\]
so that, for each \( j \in \{2, \ldots, k\} \) and \( \alpha \in \{k + 2, \ldots, n\} \), we get

\[
R_{ja}(t) \mapsto \frac{\Delta_{R(t)}(V_{k+1}^j)}{\Delta_{R(t)}(V_{k+1}^0)}, \quad \frac{\Delta_{R(t)}(V_{r}^j)}{\Delta_{R(t)}(V_{r}^0)} = c_1 c_2 \cdot Y_{(k+1)\alpha}^{ij}.
\]  

(38)

**Remark 2.** The previous representation allows us to discuss the duality defined by a pair of matrices \((L(t), R(t))\) representing the orthogonal complements of the subspaces associated with \((L(t), R(t))\). Given any \( I \subset \mathcal{P}(L(1)) \), an appropriate choice of a permutation matrix \( P \) lets us map \( I \) into \([k]\) through \((L(t), R(t)) \mapsto (L(t) \cdot P, P^{-1} \cdot R(t)) \) while preserving the expansion \((3)\). Then we consider the reduced row echelon form \( L(t) =: (1_k | I) \) of \( L(t) \cdot P \) and the reduced column echelon form \( R(t) =: (1_k | r)^T \) of \( P^{-1} \cdot R(t) \), where \( 1_k \) denotes the \( k \)-dimensional identity matrix. A well-known result (see e.g. [13, 8] and references therein for more details) asserts that \( L(t) \) (respectively, \( R(t) \)) spans the same subspaces as \( L(t) \) (respectively, \( R(t) \)) and, moreover, there exist two non-vanishing coefficients \( C_L, C_R \) such that

\[
\Delta_{alt(L(t))}(J^T) = C_L \cdot \Delta_L(t)(J), \quad \Delta_{alt(R(t))}(J^T) = C_R \cdot \Delta_R(t)(J)
\]

for all \( J \in \wp_k[n] \), where

\[
alt(L(t)) := (I^T | -I_{n-k}) \cdot \mathrm{diag}((-1)^{i+1})_{i \in [n]},
\]

\[
alt(R(t)) := \mathrm{diag}((-1)^{i+1})_{i \in [n]} \cdot (r^T | -I_{n-k})^T.
\]

In particular, if \( n < 2 \cdot k \) we can work with the matrices \( alt(L(t)) \) and \( alt(R(t)) \) obtaining the same minor products as in \((3)\) up to a common multiplicative factor. Minors extracted from these matrices have dimensions \((n-k, n)\) instead of \((k, n)\). Therefore, the condition \( \max\{n-k, k\} \geq 5 \) in the statement of Theorem 1 becomes \( n-k \geq 5 \) hereinafter, without loss of generality.

It is also easy to check that

\[
Y_{\alpha\beta}^{ij} Y_{\beta\gamma}^{ij} = -Y_{\alpha\gamma}^{ij}, \quad (39)
\]

\[
Y_{\alpha\beta}^{im} Y_{\alpha\beta}^{mj} = -Y_{\alpha\beta}^{ij}. \quad (40)
\]

Iterating \((39)\), we obtain an identity that will be used several times in the rest of this work, which we refer to as **quadrilateral decomposition**:

\[
Y_{\alpha\beta}^{ij} = -Y_{\alpha\beta}^{ij} \cdot Y_{\beta\gamma}^{ij} = -Y_{\alpha\beta}^{im} \cdot Y_{\alpha\beta}^{mj}, \quad i, j, m \in I, \alpha, \beta, \delta \in \mathcal{C}. \quad (41)
\]

Then, \((29)\) is equivalent to

\[
h(I) \cdot h(T_{\alpha\beta}^{ij}) = h(T_{\alpha\beta}^i) \cdot h(T_{\beta\gamma}^j) + Y_{\alpha\beta}^{ij} \cdot h(T_{\delta}^i) \cdot h(T_{\delta}^j) + \frac{1}{Y_{\alpha\beta}^{ij}} \cdot h(T_{\alpha\beta}^i) \cdot h(T_{\beta\gamma}^j) + h(T_{\alpha\beta}^{ij}) \cdot h(T_{\beta\gamma}^j) \quad (42)
\]

where the dependence on \( t \) is implicit.

**Remark 3.** Note that \( Y_{\alpha\beta}^{ij} \neq -1 \) for all \( I \in \wp_k[n] \), \( i, j \in I \), and \( \alpha, \beta \in \mathcal{C} \), unless \( i = j \) or \( \alpha = \beta \). Indeed, \( Y_{\alpha\beta}^{ij} = -1 \) is equivalent to

\[
\Delta_{R(t)}(I_{\alpha\beta}^{ij}) \cdot \Delta_{R(t)}(I_{\alpha\beta}^{ij}) = -c_1 c_2 \cdot \Delta_{R(t)}(I_{\alpha\beta}^{ij}) \cdot \Delta_{R(t)}(I_{\alpha\beta}^{ij})
\]

which implies \( \Delta_{R(t)}(I) \cdot \Delta_{R(t)}(I_{\alpha\beta}^{ij}) = 0 \) by the Plücker relations \((27)\), and this contradicts the assumption \( \Delta_{R(t)}(I_{\alpha\beta}^{ij}) \neq 0 \).

We also note that \((39)\) and \((40)\) are special instances of \((41)\); these correspond to the degenerate
cases where the lower, respectively, upper indices in some $Y$-terms coincide, leading to the only allowed case of $Y$-terms equal to $-1$, according to the definition (31) and Remark 3.

Whether $\{0\} \subsetneq \chi(I \mid i_j^\alpha)$, (42) is equivalent to $P(Y^{ij}_\alpha) = 0$ where $P$ is a monic polynomial over $C(t)$, since $h(T^2_\beta) \cdot h(T^1_\alpha) \neq 0$ is invertible in $C(t)$ by assumption; therefore, $Y^{ij}_\alpha$ is integral over $C(t)$, and we introduce the notation

\[
A^{ij}_{\alpha \beta} := \frac{h(I) \cdot h(T^2_{\alpha \beta}) - h(T^1_\beta) \cdot h(T^1_\alpha) - h(T^2_\alpha) \cdot h(T^2_\beta)}{2 \cdot h(T^2_\beta) \cdot h(T^2_\alpha)},
\]

\[
B^{ij}_{\alpha \beta} := \left( A^{ij}_{\alpha \beta} \right)^2 - 4 \cdot h(T^1_\alpha) \cdot h(T^1_\beta) \cdot h(T^2_\beta) \cdot h(T^2_\alpha)
\]

to express

\[
Y^{ij}_{\alpha \beta} = \frac{\varepsilon^{ij}_{\alpha \beta} \cdot \sqrt{B^{ij}_{\alpha \beta} + A^{ij}_{\alpha \beta}}}{2 \cdot h(T^2_\beta) \cdot h(T^2_\alpha)}
\]

as a root of

\[
F^{ij}_{\alpha \beta}(X) := (h(T^1_\beta) \cdot h(T^1_\alpha)) \cdot X^2 - A^{ij}_{\alpha \beta} \cdot X + h(T^1_\alpha) \cdot h(T^1_\beta)
\]

where $\varepsilon^{ij}_{\alpha \beta} \in \{+1, -1\}$ and $\sqrt{B^{ij}_{\alpha \beta}}$ is a fixed square root of $B^{ij}_{\alpha \beta}$. Even for the $A$-terms (43) and the $B$-terms (44) the dependence on $I$ will be implicit where no ambiguity arises. For future convenience, we also introduce the following function based on (44)

\[
B(x, y, z) := (x - y - z)^2 - 4yz.
\]

**Definition 3.** We call $Y^{ij}_{\alpha \beta}$ a radical term if $0 \notin \chi(I \mid i_j^\alpha)$ and $Y^{ij}_{\alpha \beta} \notin \mathbb{F}$. By extension, we say that the associated set $\chi(I \mid i_j^\alpha)$ is a radical set. We refer to $(Y^{ij}_{\alpha \beta}, Y^{ij}_{\beta \gamma}, Y^{ij}_{\gamma \alpha})$ a radical triple if all its components are radical $Y$-terms.

The exchange $\alpha \leftrightarrow \beta$ affects $Y_{\alpha \beta}^{ij} \mapsto Y_{\beta \alpha}^{ij} = (Y_{\alpha \beta}^{ij})^{-1}$ through the change of sign $\pm \mapsto \mp$ in (45), while $c_1 \cdot c_2$, $A^{ij}_{\alpha \beta}$, and $B^{ij}_{\alpha \beta}$ are preserved. From the Plücker relations (27), we also find

\[
Y(I_{\alpha \beta}^{i_j}) = -c_2 \left. \frac{\Delta_{R(t)}(T)}{\Delta_{R(t)}(T^{ij}_{\alpha \beta})} \right|_{T_{\alpha \beta}^{i_j}} = -Y(I^{ij}_{\alpha \beta}) - 1,
\]

\[
Y(I^{ij}_{\beta \alpha}) = -c_1 \left. \frac{\Delta_{R(t)}(T^{ij}_{\beta \alpha})}{\Delta_{R(t)}(T)} \right|_{T^{ij}_{\beta \alpha}} = \frac{1}{1 + \left( Y(I^{ij}_{\alpha \beta}) \right)^{-1}}.
\]

Together with the exchange $\alpha \leftrightarrow \beta$, (48) and (49) define a set of local transformations (with respect to the basis $I$ and the indices $i, j, \alpha, \beta$) that will be useful in the following.

**Lemma 1.** Any radical term $Y^{ij}_{\alpha \beta}$ forces $0 \notin \chi(I \mid i_j^\alpha)$ for all $\gamma \in I^\circ$, $\gamma \neq \alpha$.

**Proof.** All observable sets $\chi(I \mid i_j^\alpha)$ with $0 \notin \chi(I \mid i_j^\alpha)$ lie in $\mathbb{F}$, as can easily be seen from (42). It follows that any radical $Y$-term $Y^{ij}_{\alpha \beta}$ satisfies

\[
\frac{h(I) \cdot h(T^{ij}_{\alpha \beta}) \cdot h(T^1_\alpha) \cdot h(T^1_\beta) \cdot h(T^2_\beta) \cdot h(T^2_\alpha)}{h(T^2_\beta)} \neq 0
\]

so $T^{i_j}_m \in \mathfrak{G}(L(1))$ for all $\gamma \in \{\alpha, \beta\}$ and $m \in \{i, j\}$. The term $Y^{ij}_{\alpha \beta}$ remains radical under each change of basis $I \mapsto I^{i_j}_\gamma$, so whether $0 \notin \chi(I \mid i_j^\alpha)$ we can assume $h(T^2_\beta) = 0$, since the other cases are obtained from changes of bases generated by local transformations. Due to the lack of null columns, we can find $p \in I$ such that $h(T^2_\beta) \neq 0$, and we choose $p = j$ if $h(T^2_\alpha) \neq 0$. So $\chi(I \mid i_j^\alpha)$ is observable and contains 0 for all $\gamma \in \{\alpha, \beta\}$ and $m \in \{i, j\}$, then $Y^{ij}_{\gamma \omega} \in \mathbb{F}$ and, from (41), $Y^{ij}_{\alpha \beta} \in \mathbb{F}$, i.e. a contradiction.
Remark 4. There are three basic, non-equivalent actions that can be carried out on the indices of \( \chi(\mathcal{I}|_{\alpha,\beta}) \): the identity, the exchange of upper indices \( i \rightleftharpoons j \) (equivalently, lower indices \( \alpha \rightleftharpoons \beta \)), and the change of basis \( \mathcal{I} \rightarrow \mathcal{T}_\alpha \) (equivalently, \( \mathcal{I} \rightarrow \mathcal{T}_\beta^j \)). Other operations can be obtained from the composition of the previous ones: for instance, \( \mathcal{I} \rightarrow \mathcal{T}_\alpha^j \) is obtained from the composition of \( i \rightleftharpoons j \), the above-mentioned exchange of basis, and a second exchange \( i \rightleftharpoons \alpha \).

Looking at the action of these transformations on the \( Y \)-terms, they are obtained from the composition of the inversion \( f_h(Y) := Y \rightarrow Y^{-1} \) associated with \( i \rightleftharpoons j \), and the transformation \( f_c(Y) := -1 - Y \) associated with \( i \rightleftharpoons \alpha \) as in (48). These two functions are involutions and can be matched with transpositions in the permutation group \( S_3 \): the identity is assigned to the trivial permutation, \( f_h \) is assigned to the transposition \( (12) \), and \( f_c \) is assigned to \( (23) \). In this way, a combination of these two functions corresponds to the product of the associated transpositions, and the different combinations are recovered by decomposing the elements in \( S_3 \) in terms of \( (12) \) and \( (23) \). In particular, (49) comes from the composition \( f_h \circ f_c \circ f_h \), which is mapped to the product of transpositions \( (12)(23)(12) = (13) \).

4 Constraints on algebraic extensions

For a given unit \( G^{ij}_{\alpha,\beta} \in \mathbb{C}(t) \), we introduce

\[
\hat{\chi}(\mathcal{I}|_{\alpha,\beta}) := \left\{ (G^{ij}_{\alpha,\beta})^{-1} \cdot X, X \in \chi(\mathcal{I}|_{\alpha,\beta}) \right\}.
\]

We anticipate that the scaling (51) will be used later in this work with different definitions of the monomial \( G^{ij}_{\alpha,\beta} \). In this subsection, we consider the “ground” monic monomial in \( \mathbb{C}(t) \)

\[
G^{ij}_{\alpha,\beta} := \prod_{c=1}^d \min \left\{ \Psi(h(\mathcal{I}_c), h(\mathcal{T}_\alpha^j)), \Psi(h(\mathcal{I}_c)), \Psi(h(\mathcal{T}_\beta^j)) \right\}.
\]

Note that \( B^{ij}_{\alpha,\beta} \) and \( G^{ij}_{\alpha,\beta} \) are symmetric under permutations of elements of \( \chi(\mathcal{I}|_{\alpha,\beta}) \), so these quantities are preserved by the simultaneous application of a bijection \( \pi : \{i,j,\alpha,\beta\} \rightarrow \{i,j,\alpha,\beta\} \) and the change of basis induced by \( \pi \).

We start from the following lemma, which contains a very basic result of practical relevance for the discussion, as it will be repeatedly recalled in the rest of the paper.

Lemma 2. If (50) is verified and \( B^{ij}_{\alpha,\beta} \) is a perfect square in \( \mathbb{C}(t) \), then \( Y^{ij}_{\alpha,\beta} \in \mathbb{C} \).

Proof. Let \( B^{ij}_{\alpha,\beta} = P^2 \) for some \( P \in \mathbb{C}(t) \). From the definitions (43)-(44), we get

\[
\left( A^{ij}_{\alpha,\beta} - P \right) \cdot \left( A^{ij}_{\alpha,\beta} + P \right) = 4 \cdot h(\mathcal{T}_\alpha) \cdot h(\mathcal{T}_\beta^j) \cdot h(\mathcal{T}_\alpha^j).
\]

Therefore, both \( A^{ij}_{\alpha,\beta} - P \) and \( A^{ij}_{\alpha,\beta} + P \) are invertible in \( \mathbb{C}(t) \), and their sum \( 2A^{ij}_{\alpha,\beta} \) has sparsity at most 2 (as well as their difference \( 2P \)). From (43), this means that at least two of the elements in \( \chi(\mathcal{I}|_{\alpha,\beta}) \) are proportional over \( \mathbb{C}(t) \). Changing the basis \( \mathcal{I} \rightarrow \mathcal{J} \) through a local transformation \( \mathcal{J} \in \{ \mathcal{I}, \mathcal{T}_\alpha, \mathcal{T}_\beta^j \} \), we can get \( h(\mathcal{J}_c) \cdot h(\mathcal{J}_c^\alpha) = c \cdot h(\mathcal{J}_c^\alpha) \cdot h(\mathcal{J}_c^\beta^j), c \in \mathbb{C} \setminus \{0\} \), while preserving the factors of \( B^{ij}_{\alpha,\beta} \) in \( \mathbb{C}(t) \). We label the unit \( \tau := h(\mathcal{J}_c^\alpha) \cdot h(\mathcal{J}_c^\beta^j) \cdot h(\mathcal{J}_c^\beta^j)^{-1} \cdot h(\mathcal{J}_c^\beta^j)^{-1} \) to express

\[
\frac{B^{ij}_{\alpha,\beta}}{h(\mathcal{J}_c^\beta^j)^2 \cdot h(\mathcal{J}_c^\beta)^2} = (1 - c)^2 \cdot \tau^2 - 2(c + 1) \cdot \tau + 1.
\]

Being \( c \neq 0 \), \( B^{ij}_{\alpha,\beta} \) is a square in \( \mathbb{C}(t) \) only if \( \tau \in \mathbb{C} \); by definition, this means that \( \chi(\mathcal{I}|_{\alpha,\beta}) \) is integrable and hence \( Y^{ij}_{\alpha,\beta} \in \mathbb{C} \).
Lemma 3. All observable Y-terms with basis $I$ lie in the same quadratic extension of $\mathbb{F}$.

Proof. We can assume the existence of a radical term $Y^{ij}_{\alpha\beta}$, otherwise all the observable Y-terms lie in $\mathbb{F}$ and the thesis follows. By Lemma 1, we can focus on observable sets that satisfy (50) for the rest of the proof: being $\mathbb{C}(t)$ a unique factorisation domain, we can express

$$B^{ij}_{\alpha\beta} := \left(Q^{ij}_{\alpha\beta}\right)^2 \cdot D$$

where $Q^{ij}_{\alpha\beta}, D \in \mathbb{C}(t)$ and $D$ is a squarefree polynomial, that is, there exists no $P \in \mathbb{C}(t)$ such that $P^2 \mid D$. Then, we prove

$$Y^{m1}_{\gamma\delta} \in \mathbb{F}\left(\sqrt{Y^{ij}_{\alpha\beta}}\right) = \mathbb{F}(\sqrt{D})$$

for observable terms $Y^{m1}_{\gamma\delta}$, where $\mathbb{F}(\sqrt{Y^{ij}_{\alpha\beta}})$ denotes the algebraic extension of $\mathbb{F}$ by a square root of $Y^{ij}_{\alpha\beta}$.

We start from the instance $(m, l) = \{i, j\}$ and $\delta \in \{\alpha, \beta\}$, noting that we only have to consider radical triples $(Y^{ij}_{\alpha\beta}, Y^{ij}_{\gamma\delta}, Y^{ij}_{\alpha\beta})$: being (50) verified by $\chi(I^{ij}_{\alpha\beta})$ and $\chi(I^{ij}_{\gamma\delta})$, all the components of this triple are observable, and from (39) we get the thesis (56) when $Y^{ij}_{\gamma\delta} \in \mathbb{F}$ or $Y^{ij}_{\gamma\delta} \in \mathbb{F}$. Therefore, taking into account (45) for the Y-terms in the expression $Y^{ij}_{\alpha\beta} = -Y^{ij}_{\alpha\gamma} \cdot Y^{ij}_{\beta\gamma}$, we get

$$Y^{ij}_{\alpha\gamma} \in \mathbb{F}\left(\sqrt{B^{ij}_{\alpha\beta}}\right) \Leftrightarrow Y^{ij}_{\beta\gamma} \in \mathbb{F}\left(\sqrt{B^{ij}_{\alpha\beta}}\right).$$

Using again the assumption $Y^{ij}_{\alpha\beta}, Y^{ij}_{\alpha\gamma}, Y^{ij}_{\beta\gamma} \notin \mathbb{F}$, the product $Y^{ij}_{\alpha\gamma} \cdot Y^{ij}_{\beta\gamma}$ lies in a quadratic extension of $\mathbb{F}$ only if (56) holds. This argument can be adapted to triples $(Y^{ij}_{\alpha\beta}, Y^{ij}_{\alpha\gamma}, Y^{ij}_{\beta\gamma})$ by transposition of the upper and lower indices. Consequently, for all $\gamma_1, \gamma_2 \in \mathbb{C}$ and $m_1, m_2 \in I$, such that $Y^{ij}_{\gamma_1\gamma_2}$ and $Y^{m1}_{\alpha\beta}$ are observable, $Y^{ij}_{\alpha\gamma}$ and $Y^{m1}_{\alpha\beta}$ are also observable for all $u, w \in [2]$, and we have

$$Y^{ij}_{\gamma_1\gamma_2} = -Y^{ij}_{\gamma_1\gamma_2} \cdot Y^{ij}_{\alpha\gamma} \in \mathbb{F}(\sqrt{D}), \quad Y^{m1}_{\alpha\beta} = -Y^{m1}_{\alpha\beta} \cdot Y^{m1}_{\alpha\beta} \in \mathbb{F}(\sqrt{D}).$$

Then, we move to radical terms $Y^{m1}_{\alpha\beta}$ with $\{\alpha, \beta\} \neq \{\gamma_1, \gamma_2\}$ and $\{i, j\} \neq \{m_1, m_2\}$: for all $a, b, u, w \in [2], i_a \in \{i, j\}$, and $a\in \{\alpha, \beta\}$, the set $\chi(I^{m1}_{\alpha\gamma})$ is observable since $h(I^{m1}_{\alpha\gamma}) \cdot h(I^{m1}_{\beta\gamma}) \neq 0$. For any $u, w \in [2]$, we use such sets and (41) to express

$$Y^{ij}_{\alpha\beta} = -Y^{m1}_{\alpha\gamma} Y^{m1}_{\alpha\gamma} Y^{m1}_{\alpha\gamma} \in \mathbb{F}(\sqrt{D}) \setminus \mathbb{F}$$

which implies that there exists at least one pair in $\{i, j\} \times \{\alpha, \beta\}$, say $(i, \alpha)$ with proper labelling, such that $Y^{m1}_{\alpha\gamma}$ is radical. This condition forces $h(I^{m1}_{\alpha\gamma}) \cdot h(I^{m1}_{\beta\gamma}) \neq 0$ too: indeed, at $h(I^{m1}_{\alpha\gamma}) = 0$ we would have $Y^{m1}_{\alpha\gamma} \notin \mathbb{F}$, since they derive from observable sets containing 0, so their product would return $Y^{m1}_{\alpha\gamma} \notin \mathbb{F}$ by (40). A similar argument gives $h(I^{m1}_{\alpha\gamma}) \neq 0$. In this way, we get

$$h(I^{m1}_{\alpha\gamma}) \cdot h(I^{m1}_{\beta\gamma}) \neq 0, \quad a, b, u, w \in [2], i_a \in \{i, j\}, a_b \in \{\alpha, \beta\}.$$

When $\chi(I^{m1}_{\alpha\beta})$ is radical, we apply twice (58) to obtain

$$Y^{ij}_{\alpha\beta} \in \mathbb{F}(\sqrt{D}) \setminus \mathbb{F} \Rightarrow Y^{m1}_{\alpha\beta} \in \mathbb{F}(\sqrt{D}) \setminus \mathbb{F} \Rightarrow Y^{m1}_{\gamma_1\gamma_2} \in \mathbb{F}(\sqrt{D}) \setminus \mathbb{F}$$

and a similar implication holds for radical $\chi(I^{ij}_{\gamma_1\gamma_2})$. When both $\chi(I^{m1}_{\alpha\beta})$ and $\chi(I^{ij}_{\gamma_1\gamma_2})$ contain 0, (60) entails $h(I^{m1}_{\alpha\beta}) = h(I^{ij}_{\gamma_1\gamma_2}) = 0$; being $Y^{ij}_{\alpha\beta}, Y^{m1}_{\gamma_1\gamma_2} \notin \mathbb{F}$, this means that we can find $\tilde{f} \in \{i, j\}$ and $\tilde{m} \in \{m_1, m_2\}$ such that $Y^{m1}_{\alpha\beta}$ and $Y^{m1}_{\alpha\gamma}$ are radical for all $u \in [2]$ and $x \in \{i, j\}$.
uniquely characterised by any radical. Assumption 1.

5 Reduction of set functions: quadratic case

Proposition 1. The quantity \( D \) in (55) is the same for each choice of basis \( J \in \mathfrak{S}(L(1)) \).

Proof. Here, we make explicit the dependence of \( Y(J)^{ij}_{\alpha\beta} \) on the basis \( I \). For any \( J \in \mathfrak{S}(L(1)) \), we denote by \( D(J) \) the squarefree part of any \( B \)-term associated with a radical \( Y \)-term. In particular, \( D(J) \subseteq \mathbb{C} \) indicates that there is no radical \( Y \)-term with basis \( J \). Let us label \( I \setminus J =: \{m_1, \ldots, m_r\} \), \( r \leq k \). As remarked in [1, Lem.6], the exchange property (9) for the matroid \( \mathfrak{S}(L(1)) \) implies that there exists a labelling \( J \setminus I =: \{\delta_1, \ldots, \delta_u\} \) such that

\[
\mathcal{L}_u := I \setminus \{m_1, \ldots, m_u\} \cup \{\delta_1, \ldots, \delta_u\} \in \mathfrak{S}(L(1)), \quad u \in [r].
\]

Note that \( \delta_u \neq \delta_t \) for all \( t < u \), since \( \delta_t \in \mathcal{L}_{u-1} \) and \( \delta_u \notin \mathcal{L}_{u-1} \). We set \( \mathcal{L}_0 := I \) to unify the notation.

Now consider any \( p \in [r] \): the thesis holds for \( \mathcal{L}_{p-1} \) and \( \mathcal{L}_p \) when \( D(\mathcal{L}_{p-1}), D(\mathcal{L}_p) \in \mathbb{C} \), so let us assume the contrary, say \( D(\mathcal{L}_{p-1}) \notin \mathbb{C} \) with an appropriate labelling of \( \{I, J\} \). This means that there exists a radical term \( Y(\mathcal{L}_{p-1})^{ij}_{\alpha\beta} \) with \( i, j \in \mathcal{L}_{p-1} \) and \( \alpha, \beta \in \mathcal{L}_{p-1} \). We invoke the decompositions (41)

\[
Y(\mathcal{L}_{p-1})^{ij}_{\alpha\beta} = -Y(\mathcal{L}_{p-1})^{imp}_{\alpha \delta_p} \cdot Y(\mathcal{L}_{p-1})^{mpj}_{\alpha \delta_p} \cdot Y(\mathcal{L}_{p-1})^{m \beta}_{\delta_p \beta} \cdot Y(\mathcal{L}_{p-1})^{m \beta}_{\delta_p \beta}.
\]

Since \( \chi(\mathcal{L}_{p-1})^{ij}_{\alpha\beta} \) satisfies (50) by assumption and \( h((\mathcal{L}_{p-1})^{m \beta}_{\delta_p \beta}) \neq 0 \) by construction (63), all factors on the right-hand side of (64) derive from observable sets. By Lemma 3, they lie in the same quadratic extension of \( \mathbb{F} \); furthermore, by \( Y(\mathcal{L}_{p-1})^{ij}_{\alpha\beta} \notin \mathbb{F} \), at least one of these \( Y \)-terms does not lie in \( \mathbb{F} \). These two properties are preserved under the transformation rules (48)-(49), which let us move from \( L_{p-1} \) to \( L_p \), finding \( D(\mathcal{L}_{p-1}) = D(\mathcal{L}_p) \). Concatenating these equalities for all \( p \in [r] \), we get \( D(\mathcal{I}) = D(\mathcal{J}) \). \( \square \)

A consequence of the previous proposition is that the set \( \Psi(D) \) of monomials appearing in \( D \) is uniquely characterised by any radical \( Y \)-term, modulo multiplication by invertible elements in \( \mathbb{C}(t) \), independently of the choice of \( I \in \mathfrak{S}(L(1)) \).

5 Reduction of set functions: quadratic case

As mentioned in the Introduction, in this paper we concentrate on the following:

Assumption 1. All the maximal minors of \( R(1) \) are non-vanishing, and there is \( I \in \mathfrak{S}(L(1)) \), \( \alpha_1, \alpha_2 \in \mathcal{T} \) such that (11) is verified, equivalently, \( \chi(I, m_1 m_2) \) satisfies (50) for all distinct \( m_1, m_2 \in I \).

Before deepening the investigation of algebraic conditions started in the previous section, we provide another example (in addition to Example 1) to show that a constraint on \( G(L(1)) \) is required to guarantee combinatorial reductions.

Example 2. Consider \( a \in \mathcal{L}^k \setminus \{0_k\} \) and a \((k \times s)\)-dimensional matrix \( e \) of units of \( \mathbb{C}(t) \) such that each minor \( \det(e, A, B) \), where \( A \in \varphi_2[k] \) and \( B \in \varphi_2[s] \) for some \( z \leq \min\{k, s\} \), does not vanish for a generic
choice of \( t \). Then, we introduce

\[
\mathbf{L}_a(t) := (1_k | a \cdot 1^r_k) \in \mathbb{C}^{k \times (k + s)}, \quad \mathbf{R}_a(t) := (1_k | e(t))^T \in \mathbb{C}((t))^{(k + s) \times k}.
\]

In other words, \( \mathbf{L}_a \) is constructed by appending \( s \) replicas of the column \( a \) to \( 1_k \). Each non-vanishing minor of \( \mathbf{L}_a(t) \) has the form \([1]^r_k\) for some \((i, \alpha) \in [k] \times [k + 1; k + s] \), and the associated term \( \Delta_{\mathbf{L}_a(t)}([1]^r_k) \) is proportional to \( e_\alpha \) and hence invertible in \( \mathbb{C}(t) \). However, the condition (10) is not verified in general, in which case the reduction (13) does not occur.

### 5.1 Allowed configurations and squarefree decomposition

**Proposition 2.** If \( Q^{ij}_{\alpha \beta} \) in (55) is not invertible in \( \mathbb{C}(t) \), then it is a binomial.

**Proof.** Taking into account the definition (52), we introduce

\[
\hat{A}^{ij}_{\alpha \beta} := \frac{A^{ij}_{\alpha \beta}}{G^{ij}_{\alpha \beta}}, \quad \hat{B}^{ij}_{\alpha \beta} := \frac{B^{ij}_{\alpha \beta}}{(G^{ij}_{\alpha \beta})^2}, \quad \hat{Q}^{ij}_{\alpha \beta} := \frac{Q^{ij}_{\alpha \beta}}{G^{ij}_{\alpha \beta}}.
\]

Whenever \( D \in \mathbb{C} \), we find \((\hat{Q}^{ij}_{\alpha \beta})^2 = \hat{B}^{ij}_{\alpha \beta} \), which means \( Q^{ij}_{\alpha \beta}, B^{ij}_{\alpha \beta} \in \mathbb{C} \) by Lemma 2, while \( Q^{ij}_{\alpha \beta} \) is invertible in \( \mathbb{C}(t) \) at \( D = \hat{B}^{ij}_{\alpha \beta} \). Focusing on the remaining cases, we take \( p \in [d] \) satisfying \( \partial_t^p \hat{Q}^{ij}_{\alpha \beta} \neq 0 \), where \( \partial_t^p \) denotes the partial derivative with respect to \( t_p \), so \( \hat{Q}^{ij}_{\alpha \beta} \) is a common factor for \( B^{ij}_{\alpha \beta} \) and \( \partial_t^p \hat{B}^{ij}_{\alpha \beta} \). By (65) there is a term in \( \bar{\chi}(\mathcal{I}^{ij}_{\alpha \beta}) := \{ E_1, E_2, E_3 \} \) that is constant with respect to \( t_p \), therefore we can write

\[
\hat{B}^{ij}_{\alpha \beta} = E_1^2 + E_2^2 + E_3^2 - 2E_1E_2 - 2E_1E_3 - 2E_2E_3
\]

\[
= c_1t_p^{2d_1} + c_2t_p^{2d_2} + c_3 - 2c_1c_2t_p^{d_1+d_2} - 2c_1c_3t_p^{d_1} - 2c_2c_3t_p^{d_2}
\]

with \( c_a \in \mathbb{C}[t_p], E_a = c_at_p^{d_a}, u \in [3], \) and \( d_1 \geq d_2 \geq d_3 = 0 \). Each common factor of \( \hat{B}^{ij}_{\alpha \beta} \) and \( \partial_t^p \hat{B}^{ij}_{\alpha \beta} \) also divides the combination

\[
t_p \frac{\partial_t^p \hat{B}^{ij}_{\alpha \beta} - \hat{B}^{ij}_{\alpha \beta}}{\partial_t^p \hat{A}^{ij}_{\alpha \beta}} = (c_1t_p^{d_1} - c_2t_p^{d_2} + c_3) \cdot (c_1t_p^{d_1} + c_2(1 - 2d_1^{-1}d_2)t_p^{d_2} - c_3).
\]

However, we also find

\[
\hat{B}^{ij}_{\alpha \beta} - (c_1t_p^{d_1} - c_2t_p^{d_2} + c_3)^2 = -4c_1c_3t_p^{d_1},
\]

\[
\hat{B}^{ij}_{\alpha \beta} - (c_1t_p^{d_1} - c_2t_p^{d_2} - c_3)^2 = -4c_2c_3t_p^{d_2}.
\]

From (69) and \( c_3 \neq 0 \) we see that \( \hat{B}^{ij}_{\alpha \beta} \) and \( c_1t_p^{d_1} - c_2t_p^{d_2} + c_3 \) are coprime in \( \mathbb{C}[t] \), so from (68) any common factor between \( \hat{B}^{ij}_{\alpha \beta} \) and \( \partial_t^p \hat{B}^{ij}_{\alpha \beta} \) also divides

\[
w_1 := c_1t_p^{d_1} + c_2(1 - 2d_1^{-1}d_2)t_p^{d_2} - c_3.
\]

If \( d_1 = d_2 \) we find the reductions

\[
w_{1,d_1=d_2} := (c_1 - c_2)t_p^{d_1} - c_3,
\]

\[
(B^{ij}_{\alpha \beta})_{d_1=d_2} := (c_1 - c_2)^2t_p^{2d_1} + c_3^2 - 2(c_1 + c_2)c_3t_p^{d_1}.
\]

Taking into account their combination

\[
-(B^{ij}_{\alpha \beta})_{d_1=d_2} + w_{1,d_1=d_2}^2 = 4c_2c_3 \cdot t_p^{d_1}
\]
we find that \( w_1, d_1 = d_2 \) and \( (B^{ij}_{\alpha\beta})_{d_1 = d_2} \) are coprime. Thus, we get \( d_1 \neq d_2 \) and look at the combination

\[
-t_p \cdot \partial_{t_p} B^{ij}_{\alpha\beta} = \frac{2c_3 d_2 - 2(d_1 - d_2) \cdot (c_1 t^d_p - c_2 t^d_p - c_3)}{d_1 - d_2} \cdot d_1 \cdot w_1
\]

\[
= 2(d_1 - d_2) \left( c_1 t^d_p - c_2 t^d_p - c_3 \right) \left( c_2 \cdot t^d_p - c_3 d_2^2 (d_1 - d_2)^{-2} \right). \tag{74}
\]

Noting that \( c_1 t^d_p - c_2 t^d_p - c_3 \) and \( B^{ij}_{\alpha\beta} \) are coprime by \((70)\), any common factor has to divide

\[
w_2 := c_2 \cdot t^d_p - c_3 d_2^2 (d_1 - d_2)^{-2} \tag{75}
\]

which also entails \( d_2 > 0 \) so as to have \( \partial_{t_p} \hat{Q}^{ij}_{\alpha\beta} \neq 0 \). Finally, each common divisor of \( B^{ij}_{\alpha\beta} \) and \( \partial_{t_p} \hat{B}^{ij}_{\alpha\beta} \) also divides

\[
w_3 := w_1 - (1 - 2d_1^{-1}d_2) \cdot w_2
\]

\[
= c_1 t^d_p - (d_1 - d_2)^{-2} c_3 d_2^2. \tag{76}
\]

The two factors \((75)\) and \((76)\) have a non-trivial common factor only if the condition

\[
c_1^{-d_2}(d_1 - d_2)^{-2d_2} c_3 d_2^2 d_2^d_1 d_1 - d_2)^{-2d_1} \tag{77}
\]

is satisfied, equivalently, only if

\[
\frac{E^{d_1}}{d_2} = \frac{E^{d_1}}{d_2} \cdot \frac{E^{d_2}}{d_2}, \quad d_1 > d_2 > 0. \tag{78}
\]

If this relation holds, then we can introduce

\[
\rho := d_2/d_1, \quad \rho_0 := \gcd(d_1, d_2)/d_1 \tag{79}
\]

and identify a unit \( r \in C(t) \) such that

\[
r^{1/\rho_0} := \left( \rho^{-1} - 1 \right)^2 E^{2-1}_3 E.
\tag{80}
\]

In more detail, from \((78)\) we can write \( \Psi(E_u \cdot E_{\cdot}^{-1}) = \gcd(\Psi(E_u \cdot E_{\cdot}^{-1})) \cdot \mathbf{f} \) where \( \mathbf{f} \) is a primitive vector independent of \( u \), which can be completed to a unimodular matrix \( \mathbf{V} \in \mathbb{Z}^{d \times d} \) \([12]\). Following Remark 1, we get an associated ring isomorphism that maps \( w_2 \) and \( w_3 \) into univariate binomials in \( C(s_1) \). Then, any non-trivial common divisor of \( w_2 \) and \( w_3 \) corresponds to a common factor for the images of these two binomials in \( C(s_1) \), so it divides \( r - 1 \) where \( r \) satisfies \((80)\). Using \((80)\) and \((78)\) to write \( \hat{B}^{ij}_{\alpha\beta} \) in terms of \( r \) up to units of \( C(t) \), it is easily checked that it has a double root at \( r = 1 \). Thus, we get

\[
\hat{Q}^{ij}_{\alpha\beta} = E^{\rho_0}_1 - \left( \rho^{-1} - 1 \right)^{-2} E^{\rho_0}_1 \tag{81}
\]

where the complex phases of the two summands are determined by \( r \) up to a common invertible factor. \( \square \)

**Remark 5.** Situations where \( \hat{Q}^{ij}_{\alpha\beta} \) is not invertible in \( C(t) \) are associated with two disjoint tuples of variables \( t_1, t_3 \) extracted from \( t \), which satisfy \( E_u = c_u t_1^{\rho} / \rho_0 \in \zeta(I^{(ij)}_{\alpha\beta}) \) with \( c_u \in C \setminus \{0\} \) and \( e_u \in \mathbb{N}^{\mathbb{R}^+} \) for both \( u \in \{1, 3\} \). The label switch \( t_1 \leftrightarrow t_3 \) induces \( \rho = 1 - \rho_0 \) consistently with \((78)\) and preserves the factors of \( \hat{Q}^{ij}_{\alpha\beta} \), as it only contributes as a constant factor \((\rho^{-1} - 1)^{-2} \rho_0 \) multiplying \( \hat{Q}^{ij}_{\alpha\beta} \) in \((81)\). In terms of the univariate notation \((80)\), this exchange leads to the change of variable \( r \mapsto r^{-1} \).

The previous proposition leads us to introduce the following notation.
Definition 4. Given \( f_1, f_3 \in \mathbb{Z}^d \), we say that \( M \) is \((f_1, f_3)\)-homogeneous if, for all \( x \in M \), \( \Psi(x^{-1} \cdot M) \) lies in a unidimensional submodule of the span of \( f_1 \) and \( f_3 \) in \( \mathbb{Z}^d \) (therefore, \( \Psi(M) \) lies in a unidimensional affine submodule of \( \mathbb{Z}^d \)).

The following corollary guarantees that even \( D \) can be expressed as a polynomial in the variable \( r \) satisfying (80) when \( \hat{Q}_{ij}^{(r)} \) is not invertible.

**Corollary 1.** When \( \hat{Q}_{ij}^{(r)} \) is not invertible in \( \mathbb{C}(t) \), \( \Psi(D) \) is \((e_1, e_3)\)-homogeneous where \( \{e_1, e_3\} = \Psi(\hat{Q}_{ij}^{(r)}) \).

**Proof.** Consider the two sub-tuples \( t_1, t_3 \) of \( t \) introduced in Remark 5. Both \( \text{Supp}(\hat{B}_{ij}^{(r)}) \) and \( \text{Supp}(\hat{Q}_{ij}^{(r)}) \) are \((e_1, e_3)\)-homogeneous, as they can be expressed in terms of \( r \) through (80) and (78). For any monomial order \( \preceq \) on \( \text{Supp}(D) \cup \text{Supp}(\hat{Q}_{ij}^{(r)}) \), if there exists a set \( F \subseteq \text{Supp}(D) \) that makes \( \text{Supp}(D) \) \((e_1, e_3)\)-inhomogeneous, then we can look at the minimum of such monomials with respect to \( \preceq \). Specifically, the product \((\min \text{Supp}(\hat{Q}_{ij}^{(r)})) \cdot (\min F)\) appears in \( \text{Supp}(\hat{B}_{ij}^{(r)}) \) and makes this set \((e_1, e_3)\)-inhomogeneous too, i.e. a contradiction. Therefore, \( \text{Supp}(D) \) is \((e_1, e_3)\)-homogeneous, which implies that \( \Psi(D) \) lies in the \( \mathbb{Z} \)-submodule generated by \( \{e_1, e_3\} = \Psi(\hat{Q}_{ij}^{(r)}) \). \( \square \)

**Remark 6** (Characterization of the binary entropy function). We can write (78) as

\[
\frac{E_{d_1}^2}{E_{d_3}^{d_1 - d_2} E_{d_2}^{d_2}} = \left( \frac{d_1 - d_2}{d_1} \right)^{-2(d_1 - d_2)} \cdot \left( \frac{d_2}{d_1} \right)^{-2d_2} \in \mathbb{Q}_+ \tag{82}
\]

which leads to

\[
\frac{1}{2d_1} \log \left( \frac{E_{d_1}^2}{E_{d_3}^{d_1 - d_2} E_{d_2}^{d_2}} \right) = H \left( \frac{d_2}{d_1} \right) \tag{83}
\]

where \( H(p) \) is the entropy associated with a Bernoulli random variable with parameter \( p \in (0, 1) \).

The Bernoulli distribution is the basic law for dealing with random binary outcomes, whose functional form here is characterised by an algebraic condition: the discriminant \( B_{ij}^{(r)} \) is not squarefree if and only if (83) holds, which entails a normalisation condition for exponents \( \Psi(\chi(I_{ij}^{(r)})) \) (equivalently, the homogeneity of \( \hat{B}_{ij}^{(r)}, \hat{Q}_{ij}^{(r)}, \) and \( D \) discussed in Corollary 1) and determines the form of \( H \) as a function of \( d_2/d_1 = \varrho \). As a basis for future research, this suggests a deeper study of the connections between the factorisation properties of sparse polynomials obtained from the deformation of different determinantal expansions and entropy functions.

**Example 3.** The first case with \( \hat{Q}_{ij}^{(r)} \notin \mathbb{C} \) is found at \( d_1 = 2d_2 \) in (78), leading to the ratios \( \gcd(d_1, d_2)/d_1 = \frac{1}{2} \) and \( d_2/(d_1 - d_2) = 1 \). Thus, we get

\[
D = \varepsilon_1^2 - 6\varepsilon_1 \varepsilon_3 + \varepsilon_3^2 \tag{84}
\]

where \( \varepsilon_u = E_u, \ u \in \{1, 3\} \), and the factorisation (55) is given by

\[
B_{ij}^{(r)} = (\varepsilon_1 - \varepsilon_3)^2 \cdot (\varepsilon_1^2 - 6\varepsilon_1 \varepsilon_3 + \varepsilon_3^2). \tag{85}
\]

This example also shows that ambiguity can arise when reconstructing \( B_{ij}^{(r)} \) from \( D \): indeed, the case \( D = \varepsilon_1^2 - 6\varepsilon_1 \varepsilon_3 + \varepsilon_3^2 \) can be associated with both the configurations \( \chi_{1,a}(I_{ij}^{(r)}) = \{\varepsilon_1, 2\varepsilon_1, \varepsilon_3\} \) and \( \chi_{1,b}(I_{ij}^{(r)}) = \{\varepsilon_1, 2\varepsilon_3, \varepsilon_3\} \) with \( \hat{Q}_{ij}^{(r)} = 1 \), as well as \( \chi_{1,1}(I_{ij}^{(r)}) = \{\varepsilon_1, 4\varepsilon_1 \varepsilon_3, \varepsilon_3^2\} \) with \( \hat{Q}_{ij}^{(r)} = \varepsilon_1 - \varepsilon_3 \). We highlight that the sets \( \chi_{1,a}(I_{ij}^{(r)}) \) and \( \chi_{1,b}(I_{ij}^{(r)}) \) are \((\Psi(\varepsilon_1), \Psi(\varepsilon_3))\)-homogeneous as well, in line with the proof of Corollary 1.
5.2 Recovering local from global information

Now we address the reverse problem of recovering local data that generate $Y$-terms, i.e. the sets $\chi(I_{i\alpha\beta}^i)$, starting from the global information provided by $D$. In the present context, the term global means independent of the choice of the basis and the indices of the $Y$-term.

**Lemma 4.** If $\#\text{Supp}(D) = 2$, then $B_{ij}^{ij}$ is squarefree in $\mathbb{C}(t)$ and there exist two distinct terms $E_u, E_w \in \chi(I_{i\alpha\beta}^i)$ and $cs \in \{1, -1\}$ such that $E_u = cs \cdot E_w$.

**Proof.** From (81), a non-invertible square factor $\hat{Q}_{ij}^{ij} \in \mathbb{C}(t)$ of $\hat{B}_{ij}^{ij}$, when $\#\text{Supp}(D) = 2$ entails the existence of $e_1, e_3 \in \mathbb{Z}$ such that

$$\Psi(D) = \{2 \cdot (q_0^{-1} - 1) \cdot e_1, 2 \cdot (q_0^{-1} - 1) \cdot e_3\}$$  \hspace{1cm} (86)

We can order the terms in $\hat{B}_{ij}^{ij}$, $\hat{Q}_{ij}^{ij}$, and $D$ based on their fractional degree $\nu_1(\cdot)$ with respect to a non-empty tuple between those involved in Remark 5 and Definition 4, say $t_1$; we normalise $\nu_1(E_1) = 1$, while $\nu_1(E_3) = 0$ and, from (78), $\nu_1(E_2) = 2$. The induced order is total since $\hat{B}_{ij}^{ij}$, $\hat{Q}_{ij}^{ij}$, and $D$ are $(e_1, e_3)$-homogeneous.

This allows comparing the exponents in $\nu_1(\text{Supp}(\hat{B}_{ij}^{ij}))$ with the corresponding ones in the expansion $\nu_1(\text{Supp}(D \cdot (\hat{Q}_{ij}^{ij})^2))$: the minimum is 0 for all sets, while the least non-vanishing weights are

$$\min_{(0,1)} \nu_1(\text{Supp}((\hat{Q}_{ij}^{ij})^2)) = q_0, \quad \min_{(0,1)} \nu_1(\text{Supp}(D)) = 2 \cdot (1 - q_0), \quad \min_{(0,1)} \nu_1(\text{Supp}(\hat{B}_{ij}^{ij})) = 0$$

where the last equality follows from $0 < d_2 < d_1$, as stated in (78). The same condition, combined with (79), entails $0 \leq q_0 \leq \frac{1}{2}$ and $0 \leq \varrho < 1$, which can be summarised by

$$0 < q_0 \leq \varrho < 1 \leq 2 \cdot (1 - q_0).$$  \hspace{1cm} (87)

To satisfy (55), two elements of $\{2 \cdot (1 - q_0), q_0, \varrho\}$ must coincide, and (87) forces $q_0 = \varrho$, that is, $d_2 \mid d_1$.

We can repeat this argument considering the dual of the previous order, which coincides with the order induced by the valuation $\nu_3(\cdot)$ with respect to $t_1$ when it is non-empty, or equivalently considering maxima instead of minima. This corresponds to the transformations $e_1 \leftarrow e_3$ and $\varrho \leftarrow 1 - \varrho$, according to (87). Thus, we also get $q_0 = 1 - \varrho$, which means $\varrho = q_0 = \frac{1}{2}$. This leads to (84) and does not satisfy $\#\text{Supp}(D) = 2$. Therefore, $D = \hat{B}_{ij}^{ij}$ and to obtain $\#\text{Supp}(D) = 2$, we find two linearly dependent elements of $\hat{\chi}(I_{i\alpha\beta}^i)$, say $E_2 = cs \cdot E_3$, where $cs \in \mathbb{C}$ satisfies $D = (cs - 1)^2E_3^2 - 2(cs + 1)E_3E_1 + E_1^2$. This is a binomial if and only if $cs \in \{1, -1\}$. \hfill $\Box$

A possible issue arising from this process has been highlighted in Example 3. Before analysing it in more detail, the results of Proposition 2 and Lemma 4 suggest the following definition.

**Definition 5. [Type and class of radical terms]** We say that a configuration $\chi(I_{i\alpha\beta}^i)$ (or the associated polynomials $B_{ij}^{ij}$ and $Y_{ij}^{ij}$) is of $G$-type (generic type) if $\#\Psi(D) > 2$ and is of $S$-type (singular type) if $\#\Psi(D) = 2$. A $G$-type configuration $\chi(I_{i\alpha\beta}^i)$ is class-I if $Q_{ij}^{ij}$ is a monomial, and it is class-II if $\hat{Q}_{ij}^{ij}$ is a binomial (81).

We stress that the type of configuration is independent of the set $\chi(I_{i\alpha\beta}^i)$, since it depends only on $D$, while this may not hold for the class, as shown in Example 3.

**Proposition 3.** For $G$-type configurations, we can reconstruct $\hat{B}_{ij}^{ij}$ from $D$ and the knowledge of the class of $\hat{B}_{ij}^{ij}$.
Proof. We focus on class-II configurations; otherwise, $\tilde{B}^i_{\alpha\beta} = D$. Let $\Omega := \#\text{Supp}(D) > 2$ and proceed as follows: choose any variable $t_p$ such that $\partial t_p D \neq 0$ and order the elements in $\text{Supp}(D)$ according to their degree with respect to $t_p$. This order is total since all the elements in $\Psi(D)$ belong to a $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ generated by two vectors $f_1, f_2$, and $\text{Supp}(D)$ is $(f_1, f_2)$-homogeneous by Corollary 1. We can identify the maximum $d_1$ and minimum $d_0$ in $\text{Supp}(D)$, so the same order is induced by the valuation $\nu_{d_i}(d_u) := \text{deg}_{t_p}(d_u)/\text{deg}_{t_1}(d_1), u \in [\Omega]$. This function is related to the valuation $\nu_{E_i}$ with respect to the monomial $E_1$ introduced in (66) via

$$\nu_{d_i}(E_1)^{-1} = \nu_{E_i}(d_1) = \max(\nu_{E_i}(\tilde{B}^i_{\alpha\beta})) - \max(\nu_{E_i}((\tilde{Q}^i_{\alpha\beta})^2)) = 2 \cdot (1 - \varrho_0).$$

(88)

Denoting $q_u := \nu_{d_i}(d_u), u \in [\Omega]$, we recover $\varrho_0 := \gcd(d_1, d_2)/d_1$ by looking at the minimal gap

$$\mu := \min_u \{q_u - q_{u+1}\}.$$  

(89)

Indeed, we can evaluate $\nu_{d_i}$ at monomials in (66), using the expansion of (55) to relate them to monomials in $\text{Supp}(D)$: we have

$$q_1 - q_2 > \nu_{d_i}(E_1)\varrho_0 \Rightarrow \varrho_0 = \frac{\nu_{d_i}(E_1) - \nu_{d_i}(E_2)}{\nu_{d_i}(E_1)} = 1 - \varrho$$

(90)

$$q_{\Omega-1} - q_0 > \nu_{d_i}(E_1)\varrho_0 \Rightarrow \varrho_0 = \frac{\nu_{d_i}(E_2) - \nu_{d_i}(E_3)}{\nu_{d_i}(E_1)} = \varrho$$

(91)

so the combination of the two previous assumptions entails $\varrho = \varrho_0 = \frac{1}{2}$ too; from (88), in this case we can conclude $\nu_{d_i}(\varrho_0) = \frac{1}{2}$, which is an upper bound for $\mu$. Even when (90) and (91) do not hold simultaneously, by contraposition, we infer

$$\mu \leq \min\{q_1 - q_2, q_{\Omega-1} - q_0\} \leq \nu_{d_i}(E_1) \cdot \varrho_0.$$  

On the other hand, from Corollary 1 we know that $\Psi(D)$ lies in the $\mathbb{Z}$-submodule generated by $\Psi(\tilde{Q}^i_{\alpha\beta})$, so $\mu$ is bounded from below by $\max\nu_{d_i}(\text{Supp}(\tilde{Q}^i_{\alpha\beta})) = \nu_{d_i}(E_1)\varrho_0$. Therefore, $\mu = \nu_{d_i}(E_1) \cdot \varrho_0$ and we can recover $\varrho_0$ from $\mu$ using (88). Furthermore, from the knowledge of $\varrho_0$ we can consider an appropriate scaling for elements $q_u$, that is, $s_u := \nu_{E_i}(d_u) = 2 \cdot (1 - \varrho_0) \cdot q_u, u \in [\Omega]$.

When $\varrho_0 < \frac{1}{2}$ we infer $\varrho = \varrho_0$ and recover the decomposition shown in Example 3, so we focus on $\varrho_0 < \frac{1}{2}$. Looking at the expansion of $\tilde{B}^i_{\alpha\beta} = (\tilde{Q}^i_{\alpha\beta})^2 \cdot D$, a necessary and sufficient condition for $\varrho_0 < \frac{1}{2}$ is that a cancelation occurs and one between

$$0 = E_1^{2\varrho_0}d_2 - (\varrho^{-1} - 1)^{-2\varrho_0} E_1^{\varrho_0} E_3^{\varrho_0} d_1,$$

(92)

$$0 = (\varrho^{-1} - 1)^{-4\varrho_0} E_3^{2\varrho_0} d_{\Omega-1} - 2 (\varrho^{-1} - 1)^{-2\varrho_0} E_1^{\varrho_0} E_3^{\varrho_0} d_{\Omega}$$

(93)

holds. We can discriminate the compatibility of the previous two relations by looking at $\Xi := 4d_2d_1d_2^{-1}d_{\Omega-1}^{-1}$. At $\Xi = 1$, (92) and (93) are consistent and define a unique value $r = E_1^{\varrho_0} (\varrho^{-1} - 1)^{-2\varrho_0} E_3^{-\varrho_0} = 2d_2^{-1} d_1$. On the other hand, $\Xi \neq 1$ means that exactly one of these relations holds, i.e. (92) at $\varrho = \varrho_0$ and (93) at $\varrho = 1 - \varrho_0$. The relation associated with a choice of $\varrho$ determines the corresponding unit $r_\varrho$ introduced in (80), and hence the B-term, which we denote by $B_\varrho(r_\varrho)$. Given $B_\varrho(r_\varrho)$, the polynomial $r^{2\varrho_0} \cdot B_1 - \varrho_0 (r^{-1}_\varrho)$ has the same factors in $\mathbb{C}(t)$ as noted in Remark 5, thus they are the unique B-terms compatible with $D$ at $\varrho_0$ and $1 - \varrho_0$, respectively.

In this way, we obtain information that uniquely defines $(Q^i_{\alpha\beta})^2$ and, from (55), $B^i_{\alpha\beta}$, up to units. □

Remark 7. The last step of the previous proof can also be checked by expanding $(r - k_\varrho)^2 \cdot D$ with $k_\varrho \in \mathbb{C}$,
verifying the compatibility of $D$ with the two choices $\varrho \in \{\varrho_0, 1 - \varrho_0\}$. Fix the unit $r$ satisfying (80) at $\varrho = \varrho_0$, so that $k_{\varrho_0} = 1$. Looking at the $(2/\varrho_0)$-degree term in the expansion, we get $r^2d_1 = E_1^2$; as for the $(2/\varrho_0 - 1)$-degree term, we find $r^2d_2 - 2k_{\varrho_0}rd_1 = 0$ and $r^2d_2 - 2k_{1-\varrho_0}r \cdot d_1 = 2E_1E_2$, where $E_1, E_2$ refer to the expression (66) at $\varrho = 1 - \varrho_0$. Solving for $r$ and $E_2$, we obtain

$$r = 2d_1d_2^{-1}, \quad E_2 = 2^{-1}d_1^{-1}d_2 \cdot E_1 \cdot (1 - k_{1-\varrho_0}).$$  \hfill (94)

If $\varrho_0 < \frac{1}{2}$, looking at the $(2/\varrho_0 - 2)$-degree term, we also get $r^2d_3 - 2k_{\varrho_0}rd_2 + d_1 = 0$ and $r^2d_3 - 2k_{1-\varrho_0}rd_2 + k_{1-\varrho_0}^2d_1 = E_2^2$, which can be combined with (94) to infer $d_3 = \frac{1}{4}(4k_{\varrho_0} - 1) \cdot d_1^{-1}d_2^2$ and $d_3 = \frac{1}{4}(2k_{1-\varrho_0} + 1) \cdot d_1^{-1}d_2^2$. Together with $k_{\varrho_0} = 1$ and $d_1^{-1}d_2^2 \neq 0$, the previous equations are compatible only if $k_{1-\varrho_0} = 1$ too, so the two choices of $r$ return the same terms $Q_{ij}^{\alpha \beta}$ and $B_{ij}^{\alpha \beta}$ up to units in $\mathbb{C}(t)$. The same result is easily checked by direct verification at $\varrho_0 = \frac{1}{8}$.

**Theorem 3.** From any radical $Y_{ij}^{\alpha \beta}$ we can recover a finite number $N_D$ of monomials configurations $(E_1, E_2, E_3) \in \mathbb{C}(t)^3$ defining radical $Y$-terms. In particular, there are no radical terms if $n - k > 2N_D + 2$.

**Proof.** All $B$-terms are monomial expressions of the elements of $(\mathcal{I}_{ij}^{\alpha \beta})$, and the only choices are $\varrho$, $\hat{\varrho}$, $Q_{ij}^{\alpha \beta}$, and the set $\{\varrho, 1 - \varrho\}$ are uniquely defined by $D$; for each individual choice of the value $\varrho$ or $1 - \varrho$, we can reconstruct $\hat{\chi}(\mathcal{I}_{ij}^{\alpha \beta})$ from $\hat{\chi}(\hat{\mathcal{I}}_{ij}^{\alpha \beta})$ with this information. According to Remark 5, the two choices are related by $\varrho \mapsto 1 - \varrho$ and induce a permutation of the roots of $B_{ij}^{\alpha \beta}$.

Now we move to class-I configurations and look at $d_1 = \max \text{Supp}(D)$ and $d_3 = \min \text{Supp}(D)$, which coincide if and only if $Y_{ij}^{\alpha \beta} \in \mathbb{C}$. We fully recover $\hat{\chi}(\mathcal{I}_{ij}^{\alpha \beta}) = \{E_1, E_2, E_3\}$ up to common factors when $\Omega \geq 4$, since we can fix a square root of $E_1^2 := d_1$ and obtain $E_2 = \frac{1}{2}E_1 \cdot d_2d_2^{-1}$ and $E_3 := 2E_2 \cdot d_2d_2^{-1}$. The occurrence $\Omega = 3$ means that two terms in $\hat{\chi}(\mathcal{I}_{ij}^{\alpha \beta})$ are linearly proportional. To unify the notation, we still denote by $\varrho = 0$, respectively $\varrho = 1$, the choice $\Psi(E_2) = \Psi(E_3)$, respectively $\Psi(E_2) = \Psi(E_1)$. Each of the two choices $u \in \{1, 3\}$ allows us to uniquely recover the constant $c = B(E_1, c \cdot E_u, E_3) = D$.

Possible ambiguity between the two classes may arise, as shown in Example 3. Thus, when $\#\text{Supp}(D) > 2$, the possible configurations follow from: i. the knowledge of the class; ii. the choice in $\{\varrho, 1 - \varrho\}$ defining $E_2$ ($\varrho \in [0, 1]$); iii. the choice of the square root of $B_{ij}^{\alpha \beta}$; iv. the ordering of $\hat{\chi}(\mathcal{I}_{ij}^{\alpha \beta})$ defining $(E_1, E_2, E_3)$.

When $\#\text{Supp}(D) = 2$, the possible configurations for $Y_{ij}^{\alpha \beta}$ generated by $D$ come from the following choices: i. the choice of the square root of the B-term ii. the constant $c \in \{1, -1\}$ mentioned in Lemma 4; iii. with reference to the same comment, the choice of $e \in \text{Supp}(D)$ such that $e = E_u = c \cdot E_w$, which extends the definition of $\varrho$ already specified at $\#\text{Supp}(D) > 0$; iv. the ordering $(E_1, E_2, E_3)$. There are two choices for both the root of the B-term and the identification of the independent term in $\text{Supp}(D)$; at $c = +1$ there are only three distinct configurations for $(E_1, E_2, E_3)$ associated with the cyclic subgroup of $S_3$, while there are six configurations at $c = -1$, namely the full $S_3$ group.

Finally, let us denote by $N_D$ the number of distinct forms for radical terms. We suppose that there exists a radical term $Y_{ij}^{\alpha \beta}$. From Lemma 1, all sets $\chi(\mathcal{I}_{ij}^{\alpha \beta})$ and $\hat{\chi}(\hat{\mathcal{I}}_{ij}^{\alpha \beta})$ satisfy (50). Therefore, for each $\gamma \in \mathcal{T}^2$, at least one between $Y(\mathcal{I}_{ij}^{\alpha \beta}) \not\in \mathbb{F}$ and $Y(\hat{\mathcal{I}}_{ij}^{\alpha \beta}) \not\in \mathbb{F}$ holds. If $n - k > 2N_D + 2$, Dirichlet’s box principle would imply that one between $\alpha$ and $\beta$, say $\alpha$ with appropriate labelling, satisfies $Y_{ij}^{\alpha \gamma} \not\in \mathbb{F}$ for $N_D + 1$ distinct indices $\gamma_u \in \mathcal{T}^2$. From the previous argument, a second application of Dirichlet’s box principle lets us infer that there exist two indices, say $\gamma_p$ and $\gamma_q$, such that $Y_{ij}^{\alpha \gamma_p} = Y_{ij}^{\alpha \gamma_q}$. The identity (39) implies $Y_{ij}^{\alpha \gamma_p \gamma_q} = -1$, which contradicts Remark 3, so $n - k \leq 2N_D + 2$. \hfill $\blacksquare$

The proof of Theorem 3 highlights the multiplicity sources for sets $\hat{\chi}(\mathcal{I}_{ij}^{\alpha \beta})$ that are compatible with
same set is a special case of the former, where (80). In this way, we can unify the choices shown in Table 1. In conclusion, we can consider the variables that can write $S \in \mathbb{C}$.

We briefly discuss the situation where $\chi$ does not span a 3-dimensional sublattice of $\mathbb{Z}^4$ and $\chi \in \mathbb{C}$ (80), which also applies when two terms in $\chi$ are proportional over $\mathbb{C}$.

Remark 8. For a $G$-type configuration $(E_1, E_2, E_3) = (r, c_1 \cdot r, q_1)$ with $c_1 \in \mathbb{C}$, this constant can only assume values from a set $\{\kappa, \kappa^{-1}\}$ that is uniquely defined knowing $B_{\alpha \beta}$. The two choices correspond to different labels for the two proportional terms $E_1, E_2$. Different configurations for $\chi$ are described by permutations of two triples $(r, c_1 \cdot r, q_1)$ and $(r, c_2, q_2)$ such that $B(r, c_1 \cdot r, q_1)$ is equal to $B(r, c_2, q_2)$, up to units. In turn, both $(r, c_2, q_2)$ and $q_1 r^{-1} (r, c_2, q_2) = (q_1, c_2 q_1 r^{-1}, q_2 q_1 r^{-1})$ return the same $B$-term, up to units. Therefore, the ratio $c_2 / q_2 = (c_2 q_1 r^{-1}) / (q_2 q_1 r^{-1})$ in the triple $(r, c_2, q_2)$ belong to $\{\kappa, \kappa^{-1}\}$ as well. The explicit sets returning the same $B$-term are

$$C_1 := \{r, \kappa \cdot g, g\}, \quad C_2 := \left\{1 - \kappa \right\} g, \quad \frac{\kappa r}{1 - \kappa} \right\}.$$

We also note that the set $C_2$ can be obtained from $C_1$ starting from (39): two $Y$-terms generated by the same set $C_1$ but different roots of $B$ in (45) produce a term derived from $C_2$, that is,

$$\frac{\kappa g - g - r + \sqrt{B}}{2g}, \quad \frac{g - \kappa g - r - \sqrt{B}}{2r} = \frac{\kappa - 1}{2r} \left(1 - \kappa\right) g - \frac{r}{1 - \kappa} - \frac{\kappa r}{1 - \kappa} \right\}.$$

For $S$-type configurations, Proposition 2 and Corollary 1 imply that $B_{\alpha \beta}$ is squarefree in $\mathbb{C}(t)$. We can write $D = e_1^2 + e_2^2$ with linearly independent monomials $e_1, e_2$: as in the proof of Theorem 3, the components of $\chi$ are defined by the constant $C_1 \in \{1, -1\}$ in Lemma 4 and the choice of $p \in \{2\}$ in the relation $\Psi(e_p) = \Psi(E_u) = \Psi(E_w)$ for two distinct $u, w \in \{3\}$. This corresponds to the choices shown in Table 1. In conclusion, we can consider the variables $r := e_2^{-1} e_1$ or $r^{-1}$ in analogy to (80). In this way, we can unify $G$-type and $S$-type factors at $c_0 = -1$ noting that the latter configuration is a special case of the former, where $g = 1/2$ and $\kappa = -1$ in (96).

### 6 Proof of the main result and dimensional bounds

This section explores the constrained form of radical terms derived in Theorem 3 together with the relation (39) and the condition in Remark 3. The details of this section improve the qualitative results of Theorem 3 with respect to the dimensional constraints in the presence of radical terms. A deeper analysis of the implications of (39) provides quantitative bounds and leads to a better understanding of the possible monomial combinations.

Lemma 5. Let $Y_{\alpha \alpha \alpha}^{ij}$ and $Y_{\alpha \alpha \alpha}^{ij}$ be radical, while $Y_{\alpha \alpha \alpha}^{ij}$ and $Y_{\alpha \alpha \alpha}^{ij}$ are defined by the constant $C_1 \in \{1, -1\}$ in Lemma 4 and the choice of $p \in \{2\}$ in the relation $\Psi(e_p) = \Psi(E_u) = \Psi(E_w)$ for two distinct $u, w \in \{3\}$. $\chi$ contains at least two proportional, but not equal, terms.

Proof. Say $\kappa := -Y_{\alpha \alpha \alpha}^{ij}$, $\hat{Y}_{\alpha \alpha \alpha}^{ij}$, and $B_{\alpha \beta}$, $C_1 \in \{1, -1\}$ in Lemma 4 and the choice of $p \in \{2\}$ in the relation $\Psi(e_p) = \Psi(E_u) = \Psi(E_w)$ for two distinct $u, w \in \{3\}$. This corresponds to the choices shown in Table 1. In conclusion, we can consider the variables $r := e_2^{-1} e_1$ or $r^{-1}$ in analogy to (80). In this way, we can unify $G$-type and $S$-type factors at $c_0 = -1$ noting that the latter configuration is a special case of the former, where $g = 1/2$ and $\kappa = -1$ in (96).
proportionality of \( Y_{ij}^{(i)} \) and \( Y_{ij}^{(i)} \) means that the corresponding \( B \)-terms have the same factors in \( C(t) \), and an appropriate choice of the scaling (51) for the elements in \( \chi(I|_{\alpha_1 \alpha_2}) \) lets us fix \( B_{ij}^{(i)} = B_{ij}^{(i)} =: B \) without affecting the associated \( Y \)-terms. When \( \chi(I|_{\alpha_1 \alpha_2}) \) contains three pairwise independent elements, they can be recovered uniquely from \( B \) as discussed in Remark 8. This forces \( Y_{ij}^{(i)} = Y_{ij}^{(i)} \), in contradiction to Remark 3. Therefore, each of the sets \( \chi(I|_{\alpha_1 \alpha_2}) \) and \( \chi(I|_{\alpha_1 \alpha_3}) \) contains two proportional terms. A similar argument holds for \( S \)-type configurations: \( A_{ij}^{(i)} \) and \( A_{ij}^{(i)} \) must have the same factors even under the changes of basis allowed values for \( B \).

**Remark 9.** From Remark 4, we get six transformations acting on \( \chi(I|_{\alpha_2 \alpha_3}) \), which form a group under composition that is isomorphic to \( S_3 \). Their action on \( \kappa \) is defined by (48) and (49), which act as \( -\kappa \rightarrow \kappa - 1 \) and \( -\kappa \rightarrow (\kappa - 1)^{-1} \), respectively, together with their reciprocals. So we get a finite set of allowed values for \( Y_{ij}^{(i)} \) that is closed under the transformation rules (48) and (49), that is, under the action of functions \( f_b \) and \( f_t \) introduced in Remark 4. When \( D \) is of \( S \)-type, this reduces the possible values for \( Y_{ij}^{(i)} \) to the set \( \{-1/2, 1, -2\} \), in line with allowed expressions in Table 1.

**Lemma 6.** The only form for \( D \) that is compatible with both classes I and II is (84). In particular, all radical terms have the same class when \( D \) does not have this form.

**Proof.** Let \( D \) be a squarefree polynomial in \( C(t) \) such that there exist three units \( c_u \in C(t) \), \( u \in [3] \), satisfying \( B(e_1, e_2, e_3) = D \). This is a class-I configuration, and we look for a non-invertible polynomial \( Q_{ij}^{(i)} \in C(t) \) such that \( D \cdot (Q_{ij}^{(i)})^2 \) is an allowed class-II \( B \)-term. We exploit the homogeneity property discussed in Corollary 1 and the proof of Proposition 3 to infer the existence of a unit \( r \in C(t) \) such that \( Q_{ij}^{(i)} = r - 1 \) and \( D \) is a univariate polynomial in \( r \), up to units. With appropriate labelling, we can set \( \Psi(e_3) = 0, \Psi(e_2) = q, \Psi(r), \) and \( \Psi(e_1) = p \cdot \Psi(r) \) where \( p, q \in \mathbb{N} \) and \( q < p \).

We focus on the cases where (90) and (91) do not simultaneously hold, otherwise we recover Example 3 as in the proof of Proposition 3. So, \( q \in \{1, p - 1\} \), say \( q = 1 \) without loss of generality, then we prove that \( (r - 1)^2 \cdot D \), where \( D = B(r^\omega, c_{11}, c_2) \) for some \( \omega \in \mathbb{N} \), has a sparsity strictly greater than 6 at \( \omega \geq 3 \), so it cannot be represented as a \( B \)-term. This claim is easily verified at \( \omega \in \{3, 4\} \), where (90) or (91) are violated only if \( c_1 = c_2 \), hence we get
\[
(r - 1)^2 \cdot B(r^3, c_1 r, -c_1) = c_1^2 - 2c_1^2 r^2 + 2c_1 r^3 + c_1 (c_1 - 6) r^4 + 6c_1 r^5 + (1 - 2c_1) r^6 - 2r^7 + r^8
\]
and
\[
(r - 1)^2 \cdot B(r^4, c_1 r, -c_1) = c_1^2 - 2c_1^2 r^2 + c_1 (c_1 + 2) r^4 - 6c_1 r^5 + 6c_1 r^6 - 2c_1 r^7 + r^8 - 2r^9 + r^{10}.
\]

At most one coefficient can vanish in these expressions, so the sparsity is at least 7. At \( \omega > 4 \), we get
\[
(r - 1)^2 \cdot B(r^\omega, c_1 r, c_2) = c_1^2 - 2c_1 (c_1 + c_2) r + (c_1^2 + 4c_1 c_2 + c_2^2) r^2 - 2c_1 (c_1 + c_2) r^3 + c_1^2 r^4 - 2c_1 r^5
+ 2(2c_2 - c_1) \cdot r^{\omega + 1} + 2(2c_1 - c_2) \cdot r^{\omega + 2} - 2c_1 r^{\omega + 3} + r^{2\omega} - 2r^{2\omega + 1} + r^{2\omega + 2}
\]
which has a sparsity at least 7. The remaining exponent \( \omega = 2 \) returns \( (r - 1)^2 \cdot B(r^2, c_1 r, c_2) \), which does not reproduce a \( B \)-term as is easily checked. Therefore, when \( D \) is not of the form (84) we can recover the class of any radical term, and all radical terms have the same class.

We can now analyse the additional constraints on \( Y_{ij}^{(i)} \in C \setminus \{0\} \) generated by the existence of radical terms \( Y_{ij}^{(i)}, \gamma \in T^i \). Prior to that, we introduce the notation that will be used in the rest of this section.
Definition 6. When two components of $\chi (\mathcal{I} | i_{\alpha \beta})$ are proportional, we denote by $\Lambda \alpha \beta \in [3]$ the position of the unique independent component in the triple $\chi (\mathcal{I} | i_{\alpha \beta})$ when $Y\alpha \beta \notin \mathcal{F}$; we extend this definition setting $\Lambda \alpha \beta = 0$ to indicate the condition $Y\alpha \beta \in \mathcal{C}$. Furthermore, we introduce the symbols

$$h(T_{\delta i})h(T_{\delta j}) := c_{\delta i} c_{\delta j}, \quad h(T_{\delta i})h(T_{\delta j}) = T_{\delta w} \in \mathfrak{G}(L(1)),$$  \(w \in \{0,1\}, \ m \in \{i,j\} \). \hfill (98)

The consistency with this definition implies $c_{\delta i} c_{\delta j} = c_{\delta i} c_{\delta j}$ for all indices $\delta, \delta, \delta, \delta$. 

Proposition 4. When $n - k \geq 5$, for all $\mathcal{I} \in \mathfrak{G}(L(t))$, $i, j \in \mathcal{I}$, and $\alpha, \beta \in \mathcal{F}$, the triple $\chi (\mathcal{I} | i_{\alpha \beta})$ returning a radical term $Y(\mathcal{I})_{\alpha \beta}$ contains two proportional but not equal components.

Proof. Let $\chi (\mathcal{I} | i_{\alpha \beta}) = (E_1, E_2, E_3)$ violate the theorem, that is, it contains pairwise independent monomials, or two components coincide. Being $Y\alpha \beta \notin \mathcal{F}$, this is summarised in the following proposition:

$$\#\Psi (\chi (\mathcal{I} | i_{\alpha \beta})) = \#\text{Supp}(\chi (\mathcal{I} | i_{\alpha \beta})) > 1 \quad \text{(99)}$$

where, in this case, $\#\text{Supp}(\chi (\mathcal{I} | i_{\alpha \beta}))$ denotes the support, i.e. the underlying set of the multiset $\chi (\mathcal{I} | i_{\alpha \beta})$. By contraposition of Lemma 5, (99) implies $Y\alpha \beta \notin \mathcal{C}$ for all $\alpha \in \mathcal{F}$. Together with the assumption $Y\alpha \beta \notin \mathcal{F}$, we infer $Y\alpha \beta \notin \mathcal{F}$, by Lemma 1.

First, we look at configurations where $\chi (\mathcal{I} | i_{\alpha \beta})$ is the same for all $u < w$, which implies $B_{\alpha \beta} \neq \mathcal{B}$. The only configuration that returns $Y\alpha \beta \cdot Y\alpha \beta \cdot Y\alpha \beta \in \mathcal{F}$, as required by (39), is

$$E_1 - E_2 - E_3 \sqrt{B} = E_1 - E_3 \sqrt{B} = E_3 - E_1 - E_2 \sqrt{B} = \frac{E_1 E_2 E_3}{E_1 E_2 E_3}.$$  \hfill (100)

Indeed, at least two Y-terms of the form $Y\alpha \beta \cdot Y\alpha \beta$, say $Y\alpha \beta$ and $Y\alpha \beta$, involve the same square root of B, by Dirichlet’s box principle; on the other hand, $A\alpha \beta$ and $A\alpha \beta$ cannot have the same factors in C(t), unless they are invertible, otherwise $B_{\alpha \beta}$ would not coincide with B up to units. Under the hypothesis (99), this forces the form (100), possibly with $E_u = E_u$ for some $u, w \in [3]$. The conditions $E_x \in \{E_1, E_3\}$, $E_y \in \{E_1, E_3\}$, and $E_z \in \{E_1, E_2\}$ imply $E_1 E_2 E_3 = E_1 E_2 E_1$ for some $s, t \in [3]$, so this ratio never equals $-1$ when (99) holds.

From the previous argument, we infer that multiple configurations for $\chi$-sets are involved, which allows us to move to univariate polynomials in $C(r)$, according to Remarks 5 and 8, with an appropriate choice of the unit r. For G-type configurations satisfying (99), multiple $\chi$-sets can arise when both classes I and II appear: from Lemma 6, this can only be achieved by (84), so we choose the form

$$\chi (\mathcal{I} | i_{\alpha \beta}) = (r^2, 4r, 1)$$

get $c_{\alpha \beta} \cdot c_{\alpha \beta} = c_{\alpha \beta} = 4 \cdot r$. We infer $\{c_{\alpha \beta}, c_{\alpha \beta}\} = \{2 \cdot r, 2\}$, since the alternative $\{2 \cdot r, 2 \cdot r\}$ returns (100), and this set can only be generated by

$$\{\chi (\mathcal{I} | i_{\alpha \beta}), \chi (\mathcal{I} | i_{\alpha \beta})\} = \{(r, 2 \cdot r, 1), (r, 2, 1)\}. \hfill (101)$$

since $\{(r, 2 \cdot r, 1), (2 \cdot r, r)\}$ returns (100) as well. For S-type configurations, we choose $\chi (\mathcal{I} | i_{\alpha \beta}) \neq (r^2, 4r, 1)$, so we have $c_{\alpha \beta} = -4 \cdot r^2$: excluding $\{c_{\alpha \beta}, c_{\alpha \beta}\} = \{-4 \cdot r^2, 1\}$, which returns (100), we infer $\{c_{\alpha \beta}, c_{\alpha \beta}\} = \{2 \cdot r, 2 \cdot r\}$. These values are generated by

$$\{\chi (\mathcal{I} | i_{\alpha \beta}), \chi (\mathcal{I} | i_{\alpha \beta})\} = \left\{\left(\frac{1}{2} \cdot r, -\frac{1}{2} \right), \left(-\frac{1}{2} \cdot r, \frac{1}{2} \right)\right\}. \hfill (102)$$

In both cases, at $n - k \geq 5$ we can invoke Dirichlet’s box principle to identify two indices $\gamma_1, \gamma_2 \in (\mathcal{I} | i_{\alpha \beta})$ such that $\chi (\mathcal{I} | i_{\gamma_1}) = \chi (\mathcal{I} | i_{\gamma_2})$: the only possibility to make them different, according to Remark 3,
is that \( h(T_{ij})Y_{\alpha_1\gamma_1}^{ij} \) and \( h(T_{ij})Y_{\alpha_1\gamma_2}^{ij} \) are conjugate over \( \mathbb{F} \), hence

\[
Y_{\gamma_1\gamma_2}^{ij} = -Y_{\gamma_1\alpha_2}^{ij} = \frac{h(T_{ij})h(T_{ij})}{h(T_{ij})h(T_{ij})} (Y_{\alpha_1\gamma_2}^{ij})^2.
\]

This expression for \( Y_{\gamma_1\gamma_2}^{ij} \) is not compatible with \( Y_{\alpha_1\gamma_2}^{ij} \) as derived from one of the triples in (101) or (102), since \( B_{\gamma_1\gamma_2}^{ij} \) has neither the same factors as \( D \), nor it is consistent with (85). Thus, \( n - k \leq 4 \).

Configurations allowing both class-I and class-II terms exist at \( n - k \leq 4 \), e.g.

\[
\frac{-r + 1 + \sqrt{D}}{2} \cdot \frac{r - 3 - \sqrt{D}}{2} = \frac{-r^2 - 4r + 1 + (r - 1) \cdot \sqrt{D}}{2}.
\]

**Corollary 2.** For each radical \( \gamma \), there is at most one index \( \gamma \) with \( \gamma \) \in \( \mathbb{C} \).

**Proof.** Let \( Y_{\alpha_1\alpha_2}^{ij} \) be radical and take \( \alpha_3 \) such that \( \alpha := -Y_{\alpha_2\alpha_3}^{ij} \) is not radical. All sets \( \chi((I |_{i,j}^Y \alpha_4 \alpha_3), 1 \leq u < w \leq 3 \), satisfy (50) by Lemma 1, then \( \kappa \in \mathbb{C} \) by Lemma 2, and we can find two proportional terms in \( \chi((I |_{i,j}^Y \alpha_2 \alpha_2) \) by Lemma 5. Recalling the definitions of central and unique terms provided in Section 2.1, we can assume, moving to a basis \( I \in \{I, T_{ij}, T_{ij}^2, T_{ij}^3\} \) if necessary, that the central term in \( \chi((I |_{i,j}^Y \alpha_1 \alpha_1) \) is unique while preserving the relation \( Y_{\alpha_1\alpha_2}^{ij} \notin I \) and \( Y_{\alpha_2\alpha_4}^{ij} \notin \mathbb{C} \). With the notation (98), this means \( c_{\alpha_1\alpha_2} \in \mathbb{C} \), while \( c_{\alpha_2\alpha_4} \in \mathbb{C} \) by Lemma 2 applied to \( \chi((I |_{i,j}^Y \alpha_2 \alpha_4) \); therefore, \( c_{\alpha_1\alpha_3} = c_{\alpha_1\alpha_2} \cdot c_{\alpha_2\alpha_4} \in \mathbb{C} \) and, being \( Y_{\alpha_1\alpha_3}^{ij} \notin I \), we find that the central term in \( \chi((I |_{i,j}^Y \alpha_1 \alpha_1) \) is also unique, letting us state

\[
\Lambda_{\alpha_1\alpha_2} = 1, \ Y_{\alpha_2\alpha_3} \in \mathbb{C} \quad \Rightarrow \quad \Lambda_{\alpha_1\alpha_3}^{ij} = 1.
\]

This generates two possible \( G \)-type configurations, namely the two sets \( C_1 \) and \( C_2 \) in (96). The A-terms (43) corresponding to these two sets, according to the position of the central terms discussed above, are \( r - (\kappa + 1)g \) and \( (1 + \kappa) \cdot r - (1 - \kappa)^2 \cdot g \): being \( \kappa \neq -1 \) for \( G \)-type configurations, these polynomials do not have the same factors in \( \mathbb{C} \). Therefore, \( \chi((I |_{i,j}^Y \alpha_1 \alpha_2) = \chi((I |_{i,j}^Y \alpha_2 \alpha_4) \) and there cannot be another index \( \alpha_4 \neq \alpha_3 \) such that \( Y_{\alpha_2\alpha_4}^{ij} \in \mathbb{C} \), as it would entail \( Y_{\alpha_2\alpha_4}^{ij} = Y_{\alpha_2\alpha_4}^{ij} \) contradicting Remark 3. For \( S \)-type configurations, the possible values for \( Y_{\alpha_2\alpha_4}^{ij} \) are in \( \{-\frac{1}{2}, 1, -2\} \) by Remark 9. The existence of these two different indices \( \alpha_3, \alpha_4 \) such that \( Y_{\alpha_2\alpha_4}^{ij}, Y_{\alpha_2\alpha_4}^{ij} \notin \mathbb{C} \) implies \( Y_{\alpha_2\alpha_4}^{ij} \notin \mathbb{C} \), and hence \( Y_{\alpha_2\alpha_4}^{ij}, Y_{\alpha_2\alpha_4}^{ij} \notin \mathbb{C} \), \( Y_{\alpha_2\alpha_4}^{ij} \notin \mathbb{C} \) in \( \{-\frac{1}{2}, 1, -2\} \). But \( Y_{\alpha_2\alpha_4}^{ij} = -Y_{\alpha_2\alpha_4}^{ij} \cdot Y_{\alpha_2\alpha_4}^{ij} \) and there exist no elements \( a, b, c \in \{-\frac{1}{2}, 1, -2\} \) such that \( ab = c \).
for some $\sigma_{u,1}, \sigma_{u,2} \in \{1, -1\}$, $u \in \{+1, -1\}$, that depend on the set configuration (C1 or C2) and $\Lambda_{ij}^{\alpha\beta}$ in (2, 3).

Now, we fix any radical term and, choosing a basis $\mathcal{J} \in \{\mathcal{I}, \mathcal{I}_{\alpha_1}^{\gamma_1}, \mathcal{I}_{\alpha_2}^{\gamma_2}\}$, we assume $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$. When $Y_{\alpha_1\gamma_1}^{ij}$ and $Y_{\alpha_2\gamma_2}^{ij}$ are both radical, $\gamma \in \mathcal{T}^\mathcal{C}$, the conditions $c_{\alpha_1\alpha_2} \in \{\kappa, \kappa^{-1}\}$ (by Remark 8), $\kappa \neq 1$ (by Proposition 4), and $c_{\alpha_1\alpha_2} = c_{\alpha_1\alpha_2} \cdot c_{\alpha_1\alpha_2}$ require at least one $u \in [2]$ such that $c_{\alpha_1\gamma_1} \in \{\kappa, \kappa^{-1}\}$, hence $\Lambda_{ij}^{\alpha_1\alpha_2} \neq 1$. Being $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$ and $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$ equivalent when $\Lambda_{ij}^{\alpha_1\alpha_2} = 1$, we conclude $\Lambda_{ij}^{\alpha_1\alpha_2}, \Lambda_{ij}^{\alpha_2\alpha_2} \in \{2, 3\}$.

On the other hand, $\Lambda_{ij}^{\alpha_1\alpha_2}$ and $\Lambda_{ij}^{\alpha_2\alpha_2}$ do not share the same factors, otherwise the relation $Y_{\alpha_1\alpha_2}^{ij} = -Y_{\alpha_1\alpha_2}^{ij} \cdot Y_{\gamma_2}^{ij}$ would generate different factors in $B_{\alpha_1\alpha_2}^{ij}$ and $B_{\gamma_2}^{ij}$. This argument forces

$$\{\hat{\Lambda}_{ij}^{\alpha_1\alpha_2}, \hat{\Lambda}_{ij}^{\alpha_2\alpha_2}\} = \{F_{(+1)}, F_{(-1)}\}. \quad (106)$$

Considering (105), this condition allows us to express $c_{\alpha_1\alpha_2} = c_{\alpha_1\gamma_1} \cdot c_{\alpha_2\gamma_2} = c_{(+1)} \cdot c_{(-1)}$, where $c_{\alpha_1\alpha_2} \in \mathbb{C}$ entails $\sigma_{+1,1} = -\sigma_{-1,1}$, so the previous relation is equivalent to

$$(1 - \kappa)^{\sigma_{+1,2}-\sigma_{-1,2}} = \kappa^{\sigma_{-1,1}-\sigma_{-1,2}}. \quad (107)$$

This relation is trivially satisfied only if $\hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\alpha_2}) \neq \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_1})$ for both $u \in [2]$, i.e. they do not share the same set configuration (96) and an identity analogous to (97) holds. Otherwise, (107) restricts the allowed values of $\kappa$, that is, $\kappa \in \{\frac{1}{2}, -1, 2\}$, noting that $\kappa = 1$ is excluded by Proposition 4. In particular, this constraint is required by the following implications

$$\gamma_1, \gamma_2 \in \mathcal{T}^\mathcal{C}, \hat{\Lambda}_{ij}^{\alpha_1\alpha_2} = \hat{\Lambda}_{ij}^{\alpha_1\gamma_1} \in \{F_{(+1)}, F_{(-1)}\} \Rightarrow Y_{\gamma_1\gamma_2}^{ij} \in \mathbb{F} \Rightarrow Y_{\gamma_1\gamma_2}^{ij} \in \mathbb{C} \quad (108)$$

by Lemmas 1 and 2; the last condition is achievable only if $\hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\gamma_1}) \neq \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_2})$ at $\Lambda_{ij}^{\alpha_1\gamma_1} \neq 1$, forcing $\hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\alpha_2}) = \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_2})$ for some $u \in [2]$, and thus requiring $\kappa \in \{\frac{1}{2}, -1, 2\}$ as above. On the other hand, the proportionality constant $Y_{\gamma_1\gamma_2}^{ij}$ must belong to $\{\kappa^{u} - 1, (\kappa^{u} - 1)^{-1}\}$ for both $u \in [2]$, as it is accessible from both terms in $\{\hat{\Lambda}_{ij}^{\alpha_1\gamma_1}, \hat{\Lambda}_{ij}^{\alpha_3\gamma_1}\} = \{F_{(+1)}, F_{(-1)}\}$; therefore, $\kappa = (1 - \kappa)^{\sigma_{+1,2}}$ for some $\sigma \in \{1, -1\}$, which is not verified at $\kappa \in \{\frac{1}{2}, -1, 2\}$.

The previous argument returns at most two indices $\gamma_1, \gamma_2$ such that $Y_{\alpha_1\gamma_1}^{ij}$ and $Y_{\alpha_2\gamma_2}^{ij}$ are radical, specifically $\{\hat{\Lambda}_{ij}^{\alpha_1\gamma_1}, \hat{\Lambda}_{ij}^{\alpha_2\gamma_2}\} = \{F_{(+1)}, F_{(-1)}\}$ for both $u \in [2]$, and at most two indices $\omega_1, \omega_2$ exist such that $Y_{\alpha_1\omega_1}^{ij}, Y_{\alpha_2\omega_2}^{ij} \in \mathbb{C}$, according to Corollary 2. However, by the same argument, any three such indices are mutually exclusive: for any $\gamma_1$ with $\hat{\Lambda}_{ij}^{\alpha_1\gamma_1} = \hat{\Lambda}_{ij}^{u_1\gamma_1}, u \in \{+1, -1\}$, and $\omega_1, \omega_2$ as above, we infer the existence of $\sigma_1, \sigma_2 \in \{1, -1\}$ such that

$$Y_{\alpha_1\omega_1}^{ij} \in \{-\kappa, -\kappa^{-1}\} \cap \{\kappa^{u} - 1, (\kappa^{u} - 1)^{-1}\} \Rightarrow \kappa = (1 - \kappa)^{\sigma_1},$$

$$Y_{\alpha_2\omega_2}^{ij} \in \{-\kappa, -\kappa^{-1}\} \cap \{\kappa^{u} - 1, (\kappa^{u} - 1)^{-1}\} \Rightarrow \kappa = (1 - \kappa)^{\sigma_2}. \quad (109)$$

These relations are not compatible with the condition $\kappa \in \{\frac{1}{2}, -1, 2\}$ obtained from $\hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\alpha_2}) = \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_1})$ or $\hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\alpha_2}) = \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_1})$, one of which follows from the conjunction of

$$\Lambda_{ij}^{\alpha_1\alpha_2} = \Lambda_{ij}^{\alpha_1\gamma_1} = 2, Y_{\alpha_1\omega_1}^{ij} \in \mathbb{C} \Rightarrow \hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\alpha_2}) = \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_1}),$$

$$\hat{\Lambda}_{ij}^{\alpha_1\gamma_1} = \hat{\Lambda}_{ij}^{\alpha_1\gamma_1} \in \{F_{(+1)}, F_{(-1)}\}, Y_{\alpha_1\omega_1}^{ij} \in \mathbb{C} \Rightarrow \hat{\chi}(\mathcal{I}^{ij}_{\alpha_1\gamma_1}) \neq \chi(\mathcal{I}^{ij}_{\alpha_1\gamma_1}). \quad (110)$$

The same reasoning applies by considering a single $Y_{\alpha_1\omega_1}^{ij} \in \mathbb{C}$ but two terms in $\{\hat{\Lambda}_{ij}^{\alpha_1\gamma_1}, \hat{\Lambda}_{ij}^{\alpha_1\gamma_2}\} = \{F_{(+1)}, F_{(-1)}\}$, since we recover (109) by connecting all terms $\hat{\Lambda}_{ij}^{\alpha_1\alpha_2}, \hat{\Lambda}_{ij}^{\alpha_1\gamma_1}$, and $\hat{\Lambda}_{ij}^{\alpha_1\gamma_2}$ to the same constant $Y_{\alpha_1\omega_1}^{ij}$, and (110) with $\gamma = \gamma_1$. Thus, we find at most two indices in $\mathcal{T}^\mathcal{C}$ besides $\alpha_1$ and $\alpha_2$. □

This bound represents minimal information required to recover the consistency conditions forcing
combinatorial reduction, under Assumption (50), as shown in the following example.

**Example 4.** When \( n - k = 4 \) we can obtain radical terms through the configuration \( \mathbf{L}_{ex} = \mathbf{E}_{-1} \) and \( \mathbf{R}_{ex} = \mathbf{E}_{1} \) with

\[
\mathbf{E}_{c} := \begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
r + \varepsilon \cdot \sqrt{r^2 + 1} & r - \varepsilon \cdot \sqrt{r^2 + 1} & -1 & 1
\end{pmatrix}.
\]

It is easily checked by direct computation that all the minors of \( \mathbf{R}_{ex} \) are non-vanishing, the products \( \Delta_{\mathbf{L}_{u}}(I) \cdot \Delta_{\mathbf{R}_{u}}(I), I \in \varphi_{2}[6], \) are monomials in \( C(r) \), but (10) does not vanish for every choice of indices.

Finally, we can summarise the previous results to prove Theorem 1.

**Proof.** Call \( \alpha_{1}, \alpha_{2} \) two generic columns of \( \mathbf{L}(1) \), whose existence is stated in Assumption 1, and fix \( i \in \mathcal{I} \).

Take any \( (m, \omega) \in \mathcal{I} \times \mathcal{T}^{c} \): we assume that \( \mathbf{L}(1) \) does not have null columns, since they do not enter the deformation (12) of the determinantal expansion, so we can find \( p \in \mathcal{I} \) satisfying \( h(\mathcal{I}_{p}^{c}) \neq 0 \). We choose a basis \( \mathcal{J} \in \{ \mathcal{I}, \mathcal{J}_{p}^{c} \} \) such that \( h(\mathcal{J}_{p}^{m}) \neq 0 \), setting \( \mathcal{J} = \mathcal{I} \) at \( m = p \); then, define the singletons \( \{ \alpha \} := \{ p, \alpha_{2} \} \cap \mathcal{J} \) and \( \{ \pi_{1} \} := \{ p, \alpha_{2} \} \cap \mathcal{J}^{c} \), so that \( \alpha_{1} := \alpha_{2} \) and \( \pi_{2} \) are generic columns for \( \mathcal{J} \), in particular \( \alpha \notin \chi(\mathcal{J} | \mathcal{I}_{\alpha_{2}}^{m}, \pi_{1}, \pi_{2}) \) at \( m \neq p \). By Proposition 5 and Lemma 2, this means \( Y(\mathcal{J} | \mathcal{I}_{\alpha_{2}}^{m}, \pi_{1}, \pi_{2}) \subset \mathbb{C} \). We can also find \( u \in [2] \) such that \( h(\mathcal{I}_{\alpha_{1}}^{m}) \neq 0 \) when \( m \neq p \), which makes \( \alpha \notin \chi(\mathcal{J} | \mathcal{I}_{\alpha_{2}}^{m}, \pi_{1}, \pi_{2}) \); as before, from Proposition 5 and Lemma 2 we can infer \( Y(\mathcal{J} | \mathcal{I}_{\alpha_{2}}^{m}, \pi_{1}, \pi_{2}) \subset \mathbb{C} \). We can repeat this procedure with \( i \) instead of \( m \), getting \( \alpha_{1} \cdot Y_{\alpha_{1} \omega}^{m} \subset \mathbb{C} \), so we conclude \( Y_{\alpha_{1} \omega}^{m} = -Y_{\alpha_{1} \omega}^{m} \cdot Y_{\alpha_{1} \omega}^{m} \subset \mathbb{C} \).

In this configuration, we exploit the invariance (34) to obtain the form (37) with the entries \( Y_{\alpha_{1} \omega}^{m} \) in (38), finding \( \mathbf{R}(t) := \mathbf{D}(t) \cdot \mathbf{R}(t) \cdot \mathbf{d}(t) \in \mathbb{C}^{n \times k} \). Condition (12) involving \( \mathbf{L}(t) := \mathbf{d}(t)^{-1} \cdot \mathbf{L}(t) \cdot \mathbf{D}(t)^{-1} \) and \( \mathbf{R}(t) \) forces all minors \( \Delta_{\mathbf{L}(t)}(I) \), with \( I \in \mathcal{G}(\mathbf{L}(1)) \), to be monomials. Setting

\[
\chi_{\mathcal{J}}(j; \beta) := \Psi \left( \Delta_{\mathbf{L}(t)}(I)^{-1} \cdot \Delta_{\mathbf{L}(t)}(J) \right), \quad j \in \mathcal{J}, \beta \in \mathcal{J}^{c}
\]

we easily prove that \( \chi_{\mathcal{J}}(j; \beta) \) does not depend on \( \mathcal{J} \): taking two bases \( \mathcal{J}_{0}, \mathcal{J}_{1} \) satisfying \( j \in \mathcal{J}_{0}, \beta \in \mathcal{J}_{1}^{c} \) for both \( u \in \{ 0, r \} \) with \( r := \#(\mathcal{J}_{1} \Delta \mathcal{J}_{0}) \), we introduce a finite sequence of bases \( \mathcal{J}_{u} \) as in (63), \( u \in [r - 1] \), noting that \( j, \beta \notin \{ m_{1}, \ldots, m_{r} \} \cup \{ \delta_{1}, \ldots, \delta_{r} \} \). Then, for each \( u \in [r - 1] \), the relation \( \chi_{\mathcal{J}_{u}}(j; \beta) = \chi_{\mathcal{J}_{u}}(j; \beta) \) is entailed by the three-term Grassmann-Plücker relation (27), since a non-trivial linear combination over \( \mathbb{C} \) of at most three units in \( \mathbb{C}(t) \) can vanish only if they are pairwise proportional. Iterating this argument for each \( u \in [r - 1] \), we find that \( \chi_{\mathcal{J}}(j; \beta) \) does not depend on \( \mathcal{J} \).

Finally, we fix the basis \( \mathcal{I} \) and construct the function \( \psi \) in (13) setting \( \psi(m) := \chi_{\mathcal{J}}(m, \alpha_{1}) \) for all \( m \in \mathcal{I} \) and \( \psi(\omega) := \chi(\omega, m_{\omega}) + \psi(m_{\omega}) \) for any \( \omega \in \mathcal{T}^{c} \), where \( m_{\omega} \in \mathcal{I} \) satisfies \( h(\mathcal{I}_{\omega}^{m}) \neq 0 \). Proceeding as in the proof in [1, Th.15], we get the thesis. \( \square \)

7 Combinatorial reduction from set permutations to element permutations

An implication of the previous results concerns the extraction of information from a given combinatorial structure: specifically, we can embed the permutations of \( [n] \) into the permutations of \( \mathcal{G}(\mathbf{L}(1)) \subseteq \varphi_{2}[n] \), which induce a relabelling of the collection \( \{ \Delta_{\mathbf{L}}(I), I \in \mathcal{G}(\mathbf{L}(1)) \} \). This lets us take advantage of the apparent complexity arising from the embedding, while still allowing us to recover the original permutation and matrices that provide the decomposition.
We adapt each permutation $\hat{\Psi} : \mathfrak{S}(L(1)) \rightarrow \mathfrak{S}(L(1))$ to our formalism through the composition
\[
\Psi := \text{Ind} \circ \hat{\Psi} : \wp_k[n] \rightarrow \{0,1\}^n
\] (112)
where Ind is the map that associates each set with its indicator function. This defines a set function with values in $\mathbb{Z}^n$. With these premises, we have the following consequence of Theorem 1:

Corollary 3. Let $\hat{\Psi} : \mathfrak{S} \rightarrow \mathfrak{S}$ be a permutation of a matroid $\mathfrak{S}$ with two generic columns, that is, satisfying Assumption 1. Using the correspondence (112), for any mapping $g : \mathfrak{S} \rightarrow \mathbb{C}$ there exist two $(k \times n)$-dimensional matrices $A(t), Q^I(t)$ satisfying $\max\{k, n-k\} \geq 5$, $\mathfrak{S} = \mathfrak{S}(A(1))$, and
\[
\Delta_{A(t)}(I) \cdot \Delta_{Q^I(t)}(I) = g(I) \cdot t^{\Psi(I)}, \quad I \in \mathfrak{S}
\] (113)
if and only if there exist matrices $a, q \in \mathbb{C}^{k \times n}$ and a permutation $\psi \in S_n$ such that the pairs $(A(t), Q(t))$ and $(a, \text{diag}(t_{\psi(1)}, \ldots, t_{\psi(n)}) \cdot q)$ have the same Cauchy-Binet expansion, i.e.
\[
\Delta_{A(t)}(I) \cdot \Delta_{Q(t)}(I) = \Delta_a(I) \cdot \Delta_q(I) \cdot \prod_{\alpha \in I} t_{\psi(\alpha)}, \quad I \in \wp_k[n].
\] (114)

Thereafter we will use the same symbol for $\Psi$ and $\hat{\Psi}$ in (112) with a slight abuse of notation.

Analysing the statement of Corollary 3 from an information-theoretic perspective, if the labelling $I \mapsto g(I)$ was known, we could check the existence of a from $g(I) = \Delta_a(I) \cdot \Delta_q(I)$ and the knowledge about $q := Q(1)$, obtaining $\Delta_a(I)$ and checking the Grassmann-Plücker relations. On the other hand, Corollary 3 suggests an approach to check if a (possibly implicitly defined) permutation $\Psi$ of $\mathfrak{S}$ is induced by a permutation $\psi \in S_n$: products $g(I) \cdot t^{\Psi(I)}$ in (113) encode accessible information about the permutation $\Psi$ and, based on the previous observation, we have the freedom to choose different evaluation points $t_0$ to infer information about the candidate $a$ and $\Psi$ from these products.

We will refer to a given choice of the evaluation point $t_0$ as a query, which returns the unlabeled multiset of values, that is, a tuple with components $g(I) \cdot t_0^{\Psi(I)}$, $I \in \wp_k[n]$, ordered without explicit dependence on $I$. For a given query $t_0$ and a permutation $\Psi \in S_\mathfrak{S}$, we set
\[
G(\Psi ; t_0) := \left\{ \left| g(I) \cdot t_0^{\Psi(I)} \right| \right\} \setminus \{0\}, \quad \Delta(\Psi ; t_0) := \sum_{I \in \mathfrak{S}} g(I) \cdot t_0^{\Psi(I)}.
\] (115)
In this framework, an approach to check the hypothesis $\exists \Psi : \Psi \text{ is induced by a permutation } \psi_{\mathfrak{S}} \in S_n$ of the elements of $[n]$" proceeds through the algorithm that we describe after the following proposition, which also applies to the constant matrices $L = L(1)$ and $R = R(1)$.

Proposition 6. Let there exist an observable set $\chi(\mathcal{I}_{ij})$ whose corresponding polynomial $F_{ij,k}$ in (46) has distinct roots. Then, starting from the family of minor products $\{h(I) : I \in \mathfrak{S}(L)\}$, each choice of a root of $F_{ij,k}$ for $Y_{ij}$ determines all the $Y$-terms associated with observable sets.

Proof. Let us assume that there exists an observable set $\chi(\mathcal{I}_{ij})$ such that both the roots of $F_{ij,k}$, call them $\tilde{Y}_{ij}$ and $\tilde{Y}_{ij}$ with $\tilde{Y}_{ij} \neq \tilde{Y}_{ij}$, are compatible with the family $\{h(I) : I \in \mathfrak{S}(L)\}$. Then, consider any $\gamma \in \mathcal{E}$ such that $\chi(\mathcal{I}_{ij})$ and $\chi(\mathcal{I}_{ij})$ are observable too: supposing that the same root $Y_{ij} \gamma$ of $F_{ij}$ is associated with both the choices $\tilde{Y}_{ij}$ and $\tilde{Y}_{ij}$ for $Y_{ij}$, the aforementioned compatibility imposes that $F_{ij}(\gamma) (\tilde{Y}_{ij} \gamma, \tilde{Y}_{ij} \gamma) = F_{ij}(\gamma) (\tilde{Y}_{ij} \gamma, \tilde{Y}_{ij} \gamma) = 0$ can hold only if the arguments $\tilde{Y}_{ij} \gamma$, $Y_{ij} \gamma$ and $\tilde{Y}_{ij} \gamma$, $Y_{ij} \gamma$ are the distinct roots of the quadratic polynomial $F_{ij}$. Vietà’s formula gives
\[
\frac{h(\mathcal{I}_a) \cdot h(\mathcal{I}_b)}{h(\mathcal{I}_c) \cdot h(\mathcal{I}_d)} \cdot (Y_{ij})^2 = \frac{\tilde{Y}_{ij} \gamma \cdot Y_{ij} \gamma \tilde{Y}_{ij} \gamma \cdot Y_{ij} \gamma}{h(\mathcal{I}_a) \cdot h(\mathcal{I}_b)}
\]
so \( \hat{Y}^{ij}_{\beta \gamma} = \hat{Y}^{ij}_{\beta \gamma} \) since \( Y^{ij}_{\beta \gamma} \in \{ \hat{Y}^{ij}_{\beta \gamma}, \check{Y}^{ij}_{\beta \gamma} \} \). Therefore, when \( \hat{Y}^{ij}_{\beta \gamma} \) is the \( Y \)-term associated with the pair \((\beta, \gamma)\) in the configuration that includes \( \check{Y}^{ij}_{\beta \gamma} \), the second root \( \hat{Y}^{ij}_{\beta \gamma} \) of \( F^{ij}_{\gamma \delta} \) appears in the configuration that includes \( \check{Y}^{ij}_{\alpha \beta} \), possibly entailing \( \hat{Y}^{ij}_{\beta \gamma} = \check{Y}^{ij}_{\beta \gamma} \) as in (116). On the other hand, for any observable set \( \chi(\mathcal{I}^{\mu}_{\alpha \beta}) \) that does not fulfil (50) the corresponding \( Y \)-term is uniquely defined by (42), since the root 0 is not acceptable for \( Y \)-terms.

Now we introduce some notation that will be used in the rest of this proof: we say that \( Y^{ij}_{\alpha \beta} \) is ambiguous if (50) holds for \( \chi(\mathcal{I}^{\mu}_{\alpha \beta}) \) and the two roots of \( F^{ij}_{\gamma \delta} \) are distinct. On the other hand, a non-ambiguous \( Y \)-term \( Y^{ij}_{\alpha \beta} \) still derives from an observable set \( \chi(\mathcal{I}^{\mu}_{\alpha \beta}) \), but (50) is violated or \( F^{ij}_{\gamma \delta} \) has not two distinct roots. The latter condition includes the degenerate cases \( \gamma = \delta \) or \( l = m \) that return polynomials \( P^{lm}_{\gamma \gamma}(X) \) or \( F^{ij}_{\gamma \delta}(X) \), respectively, that are proportional to \( (X + 1)^2 \).

Let \( \mathcal{Y} \) be the set of ambiguous \( Y \)-terms \( Y^{ij}_{\alpha \beta} \) such that, for any \( c \in \mathbb{N} \), \( (-1)^{c+1} \cdot Y^{ij}_{\alpha \beta} \) cannot be represented as the product of \( c \) non-ambiguous \( Y \)-terms by iterations of (39) and (40). For any \( Y^{ij}_{\alpha \beta} \in \mathcal{Y} \) the co-implication \( Y^{ij}_{\alpha \beta} \in \mathcal{Y} \Rightarrow Y^{ij}_{\alpha \beta} \in \mathcal{Y} \) holds and, given \( \gamma \in \mathcal{I}^{\mu}_{\alpha \beta} \) such that \( \chi(\mathcal{I}^{\mu}_{\alpha \beta}) \) and \( \chi(\mathcal{I}^{\mu}_{\gamma \delta}) \) are observable, we find \( \{ Y^{ij}_{\alpha \beta}, Y^{ij}_{\gamma \delta} \} \cap \mathcal{Y} \neq \emptyset \), say \( Y^{ij}_{\alpha \beta} \in \mathcal{Y} \). From the derivation of (116), each choice of the root of \( F^{ij}_{\gamma \delta} \) for \( Y^{ij}_{\alpha \beta} \) determines the choice of the root of \( F^{ij}_{\gamma \delta} \) for \( Y^{ij}_{\alpha \beta} \); we denote this relation as \( Y^{ij}_{\alpha \beta} \rightarrow Y^{ij}_{\alpha \beta} \) and, similarly, the notation \( Y^{ij}_{\alpha \beta} \rightarrow Y^{ij}_{\alpha \beta} \) will be used for analogous variations of upper indices.

The thesis holds if \( \mathcal{Y} = \emptyset \), so fix \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \) and take any other \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \). From \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} = -Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \cdot Y^{m_{1}m_{2}}_{\alpha\gamma_{2}m_{2}} \), by definition there is at least one \( w \in [2] \) such that \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \), say \( w = 1 \). Analogously, for at least one \( u \in [2] \) we get \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \), say \( u = 1 \). Whether \( \{ Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}}, Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \} \subset \mathcal{Y} \neq \emptyset \), say \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \), we get \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \), otherwise we infer \( Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}}, Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \in \mathcal{Y} \) and

\[
Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}}.
\]

Finally, when \( Y^{ij}_{\alpha \beta} \in \mathcal{Y} \), (117) gives

\[
Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}} \rightarrow Y^{m_{1}m_{2}}_{\gamma_{1}\gamma_{2}}
\]

otherwise we conclude \( Y^{m_{1}m_{2}}_{\alpha \beta} \in \mathcal{Y} \) too: therefore, \( Y^{m_{1}m_{2}}_{\alpha \beta} \rightarrow Y^{m_{1}m_{2}}_{\alpha \beta} \) and, together with (117), \( Y^{m_{1}m_{2}}_{\alpha \beta} \) determines each term \( Y^{m_{1}m_{2}}_{\alpha \beta} \) for all \( A \in \{i, j\} \) and \( \Omega \in \{\alpha, \beta\} \), as well as the product \(-Y^{m_{1}m_{2}}_{\alpha \beta} \cdot Y^{m_{1}m_{2}}_{\alpha \beta} \cdot Y^{m_{1}m_{2}}_{\alpha \beta} \rightarrow Y^{m_{1}m_{2}}_{\alpha \beta} \) and

In particular, when \( \Theta(L) = \emptyset \cdot \mathbb{N} \) and for any \( Y^{ij}_{\alpha \beta} \), both the choices of a root of \( F^{ij}_{\alpha \beta} \) are acceptable, as they correspond to the exchange of the roles of \( L \) and \( R \), i.e. the transposition \( (L, R) \rightarrow (R, L) \), which preserves the terms of the determinantal expansion.

**Proof. (of Theorem 2)**

1. Using a query \( t_{*} \), we gain information on the candidates \( a \) and \( q \), specifically, we can choose \( t_{*} := 1 \) to obtain two bounds

\[
\lambda < \min \mathcal{G}(\Psi; 1), \quad \mu > \max \mathcal{G}(\Psi; 1).
\]

As will be shown below, this allows one to choose the evaluation points \( t_{0} \) "generically", that is, in such a way that different permutations \( \Psi_{1} \neq \Psi_{2} \) produce different sums \( \Delta(\Psi_{1}; t_{0}) \neq \Delta(\Psi_{2}; t_{0}) \).

2. We choose \( t_{0} \) so that

\[
|t_{0,s+1}| \cdot |t_{0,1}^{k-1}| \cdot |t_{0,s}^{k-1}| > (2 \cdot \# \Theta - 1) \cdot \mu \cdot \lambda^{-1}, \quad s \in \mathbb{N} - 1.
\]

\[
19
\]
This provides a strict order $> \mid \mathcal{G}(\Psi; t_0)$ that is compatible with the lexicographic order induced by $|t_{0,n}| > \cdots > |t_{0,1}|$. Whether $\Psi_1(\mathcal{I}) > \Psi_2(\mathcal{J})$ according to this lexicographic order, we can define $z := \max(\Psi_1(\mathcal{I}) \setminus \Psi_2(\mathcal{J}))$ and find
\[
\left| t_0^{\Psi_1(\mathcal{I})} - t_0^{\Psi_2(\mathcal{J})} \right| > \left| t_{0,z}^{k_{0,\min} - 1} \Psi_1(\mathcal{I}) \right| - \left| t_{0,k_{0,\max} - 1}^{\Psi_2(\mathcal{J})} \setminus \Psi_1(\mathcal{I}) \right|.
\]
In this way, the sums $\sum_{g \in Z} |g|$ with $Z \subseteq \mathcal{G}(\Psi; t_0)$ are pairwise different: given $\Psi_1 \neq \Psi_2$, label the two permutations so that $\Psi_1(\mathcal{Z}) > \Psi_2(\mathcal{Z})$, consider $\mathcal{Z} := \arg\max\{\Psi_1(\mathcal{I}) : \mathcal{I} \in \mathcal{G}, \Psi_1(\mathcal{I}) \neq \Psi_2(\mathcal{I})\}$, and recall the notation $z := \max\left(\Psi_1(\mathcal{Z}) \setminus \Psi_2(\mathcal{Z})\right)$; then, we have
\[
\frac{|\Delta(\Psi_1; t_0) - \Delta(\Psi_2; t_0)|}{|g(\mathcal{Z})|} \cdot \left(\left| t_0^{\Psi_1(\mathcal{Z})} \right| - \left| t_0^{\Psi_2(\mathcal{Z})} \right| \right)
\]
\[
\geq - \sum_{\mathcal{I} \in \mathcal{G}, \Psi_1(\mathcal{I}) > \Psi_2(\mathcal{I})} \frac{|g(\mathcal{I})|}{|g(\mathcal{Z})|} \cdot \left(\left| t_0^{\Psi_1(\mathcal{I})} \right| - \left| t_0^{\Psi_2(\mathcal{I})} \right| \right) \quad \text{(by (122))}
\]
\[
\geq 1 - (2 \cdot |\mathcal{G}| - 1) \cdot \frac{\mu}{\lambda} \cdot |t_{0,z}^{k} \cdot t_{0,1}^{k-1} \cdot t_{0,z-1}^{k} | > 0.
\]
In particular, the order obtained from this query produces the mapping $\Gamma(A) := |g(\Psi^{-1}(A))| t_0^{\text{Ind}(A)}$ with $A \in \mathcal{G}$: being the terms in $\mathcal{G}(\Psi; t_0)$ distinct, $\Gamma$ is injective.

3. The ordering of $\mathcal{G}(\Psi; t_0)$ is total since $\Gamma$ is injective, so we consider two consecutive terms $g_u, g_{u+1}$. By (121), we can identify the sets $\mathcal{I}(\Psi) \cap \mathcal{J}(\Psi)$ such that $g_u = \Gamma(\mathcal{I}(\Psi))$ and $g_{u+1} = \Gamma(\mathcal{J}(\Psi))$. Due to the exchange relation (9) with $A := \mathcal{I}(\Psi), B := \mathcal{J}(\Psi), \alpha := \max(\Psi(\mathcal{I}) \setminus \Psi(\mathcal{J}))$, there is $\beta \in \Psi(\mathcal{I}) \setminus \Psi(\mathcal{J})$ such that $\Psi(\mathcal{J})^\beta = \Psi(\mathcal{I})$. From $\Gamma(\mathcal{J}(\Psi)) > \Gamma(\mathcal{I}(\Psi))$ we infer $\alpha > \beta$; this means $\Gamma(\mathcal{J}(\Psi)) > \Gamma(\mathcal{J}(\Psi)^\beta)$, forcing $\Psi(\mathcal{J})(\beta) = \Psi(\mathcal{I})$. We can specify $\mathcal{I}(k-1) := \Gamma^{-1}(\text{argmin}\mathcal{G}(\Psi; t_0))$ and, for all $k \leq u \leq n$,
\[
T_u := \mu \prod_{\beta = u-k+1}^{k} |t_{0,\beta}|, \quad \mathcal{I}^{(u)} := \Gamma^{-1}(\text{argmin}\mathcal{G}(\Psi; t_0) \cap \mathcal{I}^{(u)}). \quad \text{(124)}
\]
so that $T_u$ is an upper bound for $\Psi(\mathcal{I})$ whether $\max(\Psi(\mathcal{I}) \leq u$; under the hypothesis $\mathcal{H}_p$, we get
\[
\forall u \in [k + 1; n] : \quad \mathcal{I}^{(u)} \setminus \mathcal{I}^{(u-1)} = \{\psi_p(u)\}. \quad \text{(125)}
\]
Similarly, we can iteratively construct $\psi_p(u)$ for all $u \in [k]$ from the sets $\mathcal{I}^{(u-1)} \setminus \mathcal{I}^{(u)}$.

4. The previous steps hold independently on the determinantal structure we are investigating: now we can check if the query has returned the terms of a determinantal expansion, making use of the previous results. If we attest the condition (125), then we can verify with a single check whether
The method described above lets us distinguish a permutation \( \psi \) of \([n]\), if we know \( \mathbf{q} \), we can use explicit information on \( \mathbf{G} \) provided by the previous steps: first, we check assumption (11); if it is verified, then we construct the terms \( \mathbf{Y}_{\omega}^{m} \) for each \((m, \omega) \in \mathcal{I} \times \mathcal{I}^{2} \) by using Proposition 6 for observable sets and getting the remaining ones through the procedure defined in the proof of Theorem 1. In this way, we recover \( \mathbf{q} \) in the form (37) and, from this information, we retrieve \( \mathbf{a} \) using \( \Delta_{\mathbf{a}}(\mathcal{I}) = \mathbf{g}(\mathcal{I}) \cdot \Delta_{\mathbf{q}}(\mathcal{I})^{-1} \) as before.

The method described above lets us distinguish a permutation \( \psi \) of \([n]\), if it exists, even when the explicit form of the permutation \( \Psi \) of \( \mathcal{G} \) is unknown, that is, when we do not have information on \( \Psi \) beyond \( \mathcal{G}(\Psi; t_{0}) \) in (115). In fact, the input, the intermediate steps, and the output produced by a query involve only unlabelled lists of minor products.

The method uses two queries, namely, \( \Psi \) (introduced in Step 1) and \( t_{0} \) (introduced in Step 2): even if the two permutations \( \Psi_{*} \) and \( \Psi \) involved in these steps differ, the method still provides an answer on the existence and form of a combinatorial reduction for the second query \( \Psi \), as 1 returns information on \( \mathbf{a} \) that is independent of \( \Psi \). Even Assumption 1, which entails Theorem 1, holds independently of the exponents \( \Psi(\mathcal{I}) \) and the matrix \( \mathbf{Q}(\mathbf{t}) \), since the existence of two generic columns only refers to the matroid of \( \mathbf{A}(1) \). This observation further stresses the robustness of the method against the uncertainty about \( \Psi \), which fits into unlabelled sensing related to homogeneous coordinates of \( k \)-dimensional subspaces of \( \mathbb{C}^{n} \), as noted in the Introduction.

Finally, we specify Theorem 2 for rational determinantal expansions, i.e. \( \mathbf{g}_{\mathcal{I}} \in \mathbb{Q} \) for all \( \mathcal{I} \in \mathfrak{G} \).

**Corollary 4.** Under the hypotheses of Theorem 2 and, in addition, the assumption \( \mathbf{g}(\mathcal{I}) \in \mathbb{Q}, \mathcal{I} \in \mathfrak{G} \), we can check the occurrence of a combinatorial reduction and retrieve two matrices \( \mathbf{a} \) and \( \mathbf{q} \) as in (126) only based on two scalar outputs, namely, the maximum \( \max_{\mathcal{G}} \mathcal{G}(\Psi; 1) \) for a first query \( 1 \) and the candidate determinant \( \Delta(\Psi; t_{0}) \) for a second query \( t_{0} \).

**Proof.** We can reduce to \( \mathbf{g}(\mathcal{I}) \in \mathbb{Z}, \mathcal{I} \in \mathfrak{G} \), by scaling each term through a common integer factor, which corresponds to the left action of \( \mathbf{GL}_{k}(\mathbb{Z}) \) on \( \mathbb{C}^{k \times n} \). Then, we can set \( \lambda = 1 \) in (119) and use
\( L := \log(2 + \max_{\Psi} \mathcal{G}(\Psi; 1)) > 1 \) to choose \( \mu := \exp(L) > \max_{\Psi} \mathcal{G}(\Psi; 1) \). Based on the same argument as in Step 2 in the previous proof, we further specify \( t_0 \) setting

\[
t_{0,s} := \exp \left( \frac{L}{k-1} \left[ \log(2 \cdot \# \mathcal{G}) + L \right] \cdot (k^{s-1} - 1) \right), \quad s \in [n]
\]

(127)

where \( [\cdot] \) denotes the ceiling function. Indeed, we obtain \( t_0 \in \mathbb{N}^n \) and

\[
\log t_{0,s+1} - k \cdot \log t_{0,s} = L \cdot \left[ \log(2 \cdot \# \mathcal{G}) + L \right] > \log(2 \cdot \# \mathcal{G} - 1) + L
\]

(128)

which is equivalent to (120). We split the base-\( e^L \) representation of \( \Delta(\Psi; t_0) \) into blocks of length \( \left\lceil \log(2 \cdot \# \mathcal{G}) + L \right\rceil \), numbering their position starting from the least significant block: each non-vanishing block encodes a term \( g(\Psi^{-1}(I)) \), and the label \( \Psi^{-1}(I) \) of the block at position \( P \) is obtained by the expansion of \( P \) in base \( k \). Adapting Steps 4 and 6 of the previous proof, we get the thesis.

\[
\square
\]

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