The quantum erasure channel (QEC) is considered. Codes for the QEC have to correct for erasures, i.e., arbitrary errors at known positions. We show that four qubits are necessary and sufficient to encode one qubit and correct one erasure, in contrast to five qubits for unknown positions. Moreover, a family of quantum codes for the QEC, the quantum BCH codes, that can be efficiently decoded is introduced.

I. INTRODUCTION

The prospect of speeding up certain classes of computations by utilizing the quantum mechanical superposition principle and the physics of entanglement has received a great deal of attention lately. The potentially useful quantum algorithms so far include factorization of large numbers, database search and simulation of quantum mechanical systems. Recent theoretical and experimental progress in atomic physics and quantum optics has shown that small-scale quantum computing is feasible.

However, building a quantum computer is an extremely difficult task. The major obstacle is the coupling of the quantum computer to the environment which destroys quantum mechanical superpositions very rapidly. This effect is usually referred to as decoherence. It is thus of crucial importance to find schemes to actively suppress and undo the effects of decoherence.

Schemes to protect static quantum states against decoherence were first found independently by Peter Shor and Andrew Steane. Their proposals gave rise to a large number of subsequent publications (see for example and references therein). Thus the theory of quantum error-correcting codes is increasingly well understood.

In most publications the focus is on finding quantum codes for the most general error model. These quantum codes can correct for arbitrary errors at unknown positions in the codeword. However, in many realistic situations additional information on possible errors is available. For example, the physical system may permit dephasing errors only or bit-flip errors only. Of course more efficient codes are possible for restricted error models. For example, the smallest quantum code to correct for errors due to dephasing (or due to bit-flips) has length three. On the other hand, Knill and Laflamme have shown that the length of the smallest quantum code for arbitrary errors is five qubits.

In this paper we consider an error model where the position of the erroneous qubits is known. In accordance with classical coding theory we shall call this model the quantum erasure channel (QEC). Below a few physical systems are discussed where this model is applicable. The main results of the present paper are: (i) an explicit example of a code for the quantum erasure channel (QEC code) with four qubits which can correct one erasure is presented; (ii) a proof is presented that four qubits are minimal; (iii) a construction for a family of QEC codes based on classical BCH codes is given. For these codes efficient algorithms for correcting erasures exist.

The paper is organized as follows. In Section II we introduce the quantum erasure channel. The error model is discussed and a physical motivation is given. In Section III a four–qubit code for the QEC is given and the proof is presented that four qubits are minimal. A construction for quantum BCH codes is given in Section IV.

II. THE QUANTUM ERASURE CHANNEL (QEC)

Whenever the position of an error can be determined by an appropriate measurement the QEC error model applies. In the following we give a few examples for physical scenarios where this is the case.

(i) If errors are accompanied by the emission of quanta they can in principle be detected. For example, if the qubits are represented by atoms an important source of errors is spontaneous emission. Spontaneous photons can be observed by photodetection techniques. There is, however, the difficulty that spontaneous photons from free atoms are emitted in a solid angle of $4\pi$ and will very likely elude observation. One may circumvent this problem by modifying the modal structure of the surrounding electromagnetic field by placing the atoms within a cavity and thereby channeling spontaneous decay. Under appropriate conditions photons escape primarily via cavity decay through the cavity mirrors in a well defined spatial direction. There may also be the possibility to detect the emission of photons by other means, for example via the photon recoil. Similarly, if quantum bits are stored in quantized cavity modes a detected cavity photon indicates an error.

(ii) It is usually assumed that the system space $H_{\text{sys}}$ is a tensor product of two–dimensional spaces $H_2$ (qubits), i.e.,

$$H_{\text{sys}} = H_2 \otimes \ldots \otimes H_2.$$
However, this is an approximation. For example, atoms usually have many levels which may be populated due to an unwanted dynamical evolution of the system. Thus the Hilbert space of the system $\mathcal{H}_{\text{sys}}$ is a tensor product of multi-dimensional spaces with two-dimensional subspaces used for computing:

\[
\mathcal{H}_{\text{sys}} = \mathcal{H}_k \otimes \ldots \otimes \mathcal{H}_k \quad \text{and}
\]
\[
\mathcal{H}_{\text{comp}} = \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_2,
\]

where $\mathcal{H}_{\text{comp}}$ is the subspace of allowed computational states. Each two-dimensional space $\mathcal{H}_2$ is a subspace of $\mathcal{H}_k$, but not necessarily a tensor factor of $\mathcal{H}_k$. (For simplicity we assume that the dimension of all tensor factors is equal.) Therefore, the system space $\mathcal{H}_{\text{sys}}$ can only be decomposed as a direct sum of subspaces

\[
\mathcal{H}_{\text{sys}} = \mathcal{H}_{\text{comp}} \oplus \mathcal{H}_{\text{comp}}^\perp,
\]

and generally not as a tensor product. During error–free computations the system remains in $\mathcal{H}_{\text{comp}}$. Any population found in $\mathcal{H}_{\text{comp}}^\perp$ is the signature of an error. Besides, we can learn about the position of the error by determining which subsystem has left the allowed Hilbert space $\mathcal{H}_2$. The erroneous subsystem can then be reset by hand to an arbitrary state in $\mathcal{H}_2$, $|0\rangle$ say. As an example we may think of an atom in which unwanted levels are coupled to the “allowed” two–level system by non–resonant laser interaction. We can measure the population in these levels for example by applying the quantum jump technique \[18\].

(iii) QEC codes may be useful in fault tolerant quantum computing. This scheme was recently proposed by Peter Shor and permits to perform quantum computations and error correction with a network of erroneous quantum gates \[19\]. We may assume that only quantum gates introduce errors and that errors can be detected by appropriate measurements. In this case it is not necessary to use a quantum code for the most general error model because it is known to which qubits the quantum gate was applied when an error is detected. For example, in the cavity QED quantum computer model system proposed by Pellizzari et al. \[20\] the quantum information is safely stored in stable Zeeman ground state levels while no computations are performed. However, during gate operation a single mode of a quantized cavity is excited, which is much more fragile a quantum system. A photodetector which records photons leaking out of the cavity indicates errors in those atoms that are involved in the current quantum gate.

(iv) It is worthwhile noting that there is a strong connection of codes for the QEC to the error correction scheme for quantum gates recently proposed by Cirac et al. \[21\]. This scheme is designed to correct for a specific but important error in the ion trap quantum computer during quantum gates. In this error model errors are caused by decays in the center–of–mass phonon mode which is temporarily excited during quantum gate operation. If a residual population in the phonon mode is found an error is detected. As above in (iii) the position of the error is known and thus the QEC error model applies. In this scheme each logical qubit is encoded in two physical qubits. One might expect that a four qubit code is required for this scheme since the smallest code conforming to the QEC has length four. However, two qubits are sufficient because specific assumptions about the type of errors are made.

### III. CODES FOR THE QEC

#### A. Conditions on Codes for the QEC

For the general case, Knill and Laflamme \[13\] derived necessary and sufficient conditions on quantum error–correcting codes $\mathcal{QC}$. Given a set of error operators $\{A_i\}$ the conditions on states $|c_k\rangle \in \mathcal{QC}$ are

\[
\langle c_k | A_i^\dagger A_j | c_k \rangle = \langle c_l | A_i^\dagger A_j | c_l \rangle \quad (1)
\]
\[
\langle c_k | A_i^\dagger A_j | c_l \rangle = 0 \quad \text{for} \quad \langle c_k | c_l \rangle = 0. \quad (2)
\]

For a code of length $N$ that can correct $t$ errors the error operators $\{A_i\}$ are of a special form. They are all $t$–error operators, i.e., operators that differ on at most $t$ of the tensor factors of $\mathcal{H} = \mathcal{H}_2^\otimes N$ from identity. In $\mathcal{H}$ and $\mathcal{B}$ it is sufficient to consider algebra bases for $t$–error operators. The bases might be tensor products of local bases, e.g., the identity $\mathbb{1}$ and the Pauli spin matrices $\{\sigma_x, \sigma_y, \sigma_z\}$, or the operators $|0\rangle\langle 0|$, $|1\rangle\langle 0|$, $|0\rangle\langle 1|$, and $|1\rangle\langle 1|$. In this paper, we consider the one–error operators $P_{ij}^l$ that are the operators $|i\rangle\langle j|$ applied to the $k$–th qubit.

For the QEC there are similar conditions. Since the positions of the errors are known by definition there is no need to separate the spaces corresponding to errors at different positions. Therefore, in $\mathcal{H}$ and $\mathcal{B}$ only $t$–error operators $A_i$ and $A_j$ that differ from identity at the same positions have to be considered. But the product of such $t$–error operators is also a $t$–error operator and can be written as linear combination of the $A_i$ since they are an algebra basis. Hence, (1) and (2) reduce to

\[
\langle c_k | A_i | c_k \rangle = \langle c_l | A_i | c_l \rangle \quad (3)
\]
\[
\langle c_k | A_i | c_l \rangle = 0 \quad \text{for} \quad \langle c_k | c_l \rangle = 0. \quad (4)
\]

Equations (1) and (2) for $t$–error operators $A_i$ imply equation (3) and (4) for $2t$–error operators since the operators $A_i^\dagger A_j$ are bases for $2t$–error operators. Hence, a quantum error–correcting code correcting $t$ errors is a $2t$ erasure–correcting code.

#### B. QEC Code with 4 Qubits

For the general situation it was shown that the shortest code to encode one qubit and to correct one error...
has length five \[ |0\rangle = (0000) + |1111\rangle \]
\[ |1\rangle = |0101\rangle + |0110\rangle. \]

(To simplify the notation, normalization factors are omitted here and in the remainder of the paper.) In \[11,12\] it is shown that it is sufficient to correct bit–flips in two bases that are Hadamard transforms of each other. The Hadamard transform of the code \( QC \) corresponds to the “dual” code \( QC^\perp \) given by

\[
|0^+\rangle = H|0\rangle = (0000) + |0011\rangle + |0101\rangle + |0110\rangle + |1001\rangle + |1100\rangle + |1111\rangle
\]
\[
|1^+\rangle = H|1\rangle = (0000) - |0011\rangle - |0101\rangle - |0110\rangle + |1001\rangle - |1100\rangle + |1111\rangle.
\]

By definition of an erasure the position of the error is known, but it is not known what the error is. Since all states of both the code \( QC \) and its “dual” \( QC^\perp \) have even weight, for both bases a single bit–flip error can be detected by computing the overall parity. Odd parity indicates an error. Thus, any one–bit error can be corrected since correcting single bit–flips in both bases is sufficient.

The code \( QC \) can be extended by the following two states \[2\] and \[3\]
\[
|2\rangle = |1100\rangle + |0011\rangle
\]
\[
|3\rangle = |1010\rangle + |0101\rangle
\]

with

\[
|2^+\rangle = H|2\rangle = (0000) + |0011\rangle - |0101\rangle - |0110\rangle - |1001\rangle + |1100\rangle + |1111\rangle
\]
\[
|3^+\rangle = H|3\rangle = (0000) - |0011\rangle + |0101\rangle - |0110\rangle + |1001\rangle - |1100\rangle + |1111\rangle.
\]

Thus, the extended code encodes not only one, but two qubits and corrects for one erasure. Note that this code is equivalent to the code used for error detection in \[2\].

The existence of a code with these parameters was shown e.g. in \[14\].

Theorem 1 Let \( QC \) be a quantum error–correcting code that can correct at least one erasure. If a one–qubit state \( |\theta_0\rangle \) is a factor of a state \( |\phi_0\rangle \in QC \) it is a factor of all states \( |\phi\rangle \in QC \).

Proof: Assume w.l.o.g. that the first qubit is a factor, i.e., \(|\phi_0\rangle = |\theta_0\rangle|\psi_0\rangle\). Inserting the local operator \( P_{|\theta_0\rangle} = |\theta_0\rangle\langle\theta_0| \) in \[3\] yields for any state \( |\phi\rangle \in QC \)

\[ \langle\phi|P_{|\theta_0\rangle}|\phi\rangle = \langle\phi|P_{|\theta_0\rangle}|\phi\rangle = 1. \]

Hence, \( |\theta_0\rangle \) is a factor of every code state.

Thus, we have the following corollary.

Corollary 2 If a quantum code \( QC \) of length \( N \) has a one–qubit factor deleting this position yields a quantum code \( QC' \) of length \( N - 1 \) and equal dimension with same error–correcting capabilities.

Next, we show that every two–dimensional subspace of \( H_2 \otimes H_2 \) contains at least one product state.

Lemma 3 For every two–dimensional subspace of \( H_2 \otimes H_2 \) there is a basis that contains at least one product state, i.e., a state \( |\pi\rangle = |\pi_1\rangle|\pi_2\rangle \).

Proof: Let the subspace be generated by \( \{|b_1\rangle, |b_2\rangle\} \). A product state \( |\pi\rangle \in H_2 \otimes H_2 \) is characterized by

\[ \langle 00| |11\rangle |\pi\rangle = \langle 01| |10\rangle |\pi\rangle. \]

Inserting \( |\pi\rangle = \eta_1 |b_1\rangle + \eta_2 |b_2\rangle \) in \[4\] yields a quadratic equation for the complex coefficients \( \eta_1 \) and \( \eta_2 \):

\[ 0 = c_1 \eta_1^2 + c_2 \eta_1 \eta_2 + c_2 \eta_2^2 \]

with

\[ c_1 = \langle 00| |11\rangle |b_1\rangle - \langle 01| |10\rangle |b_1\rangle \]
\[ c_12 = \langle 00| |11\rangle |b_2\rangle + \langle 11| |00\rangle |b_2\rangle - \langle 01| |10\rangle |b_2\rangle - \langle 10| |01\rangle |b_2\rangle \]
\[ c_2 = \langle 00| |11\rangle |b_2\rangle - \langle 01| |10\rangle |b_2\rangle - \langle 10| |01\rangle |b_2\rangle. \]

If \( c_1 \) vanishes \( |b_1\rangle \) is a product state and the lemma holds. Similarly, \( |b_2\rangle \) is a product state if \( c_2 = 0 \). Now consider the case \( c_1 \neq 0 \) and \( c_2 \neq 0 \). The solutions of \[4\] are given by

\[ \eta_1 = \frac{-c_12 \pm \sqrt{c_1^2 - 4c_1c_2}}{2c_1} \eta_2. \]

For \( c_1 \neq 0 \) and \( c_2 \neq 0 \) there is at least one non–trivial solution with \( \eta_1 \neq 0 \) and \( \eta_2 \neq 0 \) and thus a product state exists.

Using Lemma 3 we are able to prove the following theorem.

Theorem 4 There is no quantum error–correcting code of length two that can correct one erasure and encodes one qubit.

Proof: Assume that such a code exists. The states \(|0\rangle\) and \(|1\rangle\) span a two–dimensional subspace \( QC \) of \( H_2 \otimes H_2 \). According to Lemma 3, \( QC \) contains a product state \( |\pi_1\rangle|\pi_2\rangle \). From Theorem 1 it follows that both \( |\pi_1\rangle \) and \( |\pi_2\rangle \) are factors of all code states and thus the code cannot be two–dimensional.

Theorem 5 There is no quantum error–correcting code of length three that can correct one erasure and encodes one qubit.
Proof: Assume that such a code exists. Since there is no code of length two the states in the code cannot be factored. With reference to the first qubit the encoding can be written as
\[
\begin{align*}
|0\rangle &= |0\rangle|\Phi_0\rangle + |1\rangle|\Phi_1\rangle \\
|1\rangle &= |0\rangle|\Theta_0\rangle + |1\rangle|\Theta_1\rangle,
\end{align*}
\]
where $|\Phi_1\rangle$, $|\Theta_1\rangle$ are, in general, unnormalized and non-orthogonal states. The states $|\Phi_0\rangle$ and $|\Phi_1\rangle$ have to be linearly independent since otherwise $|0\rangle$ is a product state and a code of length one exists (cf. Corollary 2). Similarly, $|\Theta_0\rangle$ and $|\Theta_1\rangle$ have to be linearly independent.

For the projections $P_{ij}^{(1)} = |i\rangle \langle j| \otimes 1 \otimes 1$, $i, j \in \{0, 1\}$ equation (1) implies
\[
\begin{align*}
(1) P_{00}^{(1)} |0\rangle &= (\Theta_0|\Phi_0\rangle = 0 \\
(1) P_{01}^{(1)} |0\rangle &= (\Theta_1|\Phi_0\rangle = 0 \\
(1) P_{10}^{(1)} |0\rangle &= (\Theta_0|\Phi_1\rangle = 0 \\
(1) P_{11}^{(1)} |0\rangle &= (\Theta_1|\Phi_1\rangle = 0.
\end{align*}
\]

Thus, the subspaces $\mathcal{H}_0$, spanned by $\{|\Phi_0\rangle, |\Phi_1\rangle\}$ and $\mathcal{H}_1$ spanned by $\{|\Theta_0\rangle, |\Theta_1\rangle\}$ are two-dimensional and orthogonal. This yields a decomposition of the joint Hilbert space of the second and third qubit:
\[
\mathcal{H}_2 \otimes \mathcal{H}_2 = \mathcal{H}_0 + \mathcal{H}_1.
\]

We now choose an orthonormal basis $B = \{ |b_1\rangle, |b_2\rangle, |b_3\rangle, |b_4\rangle \}$ for the Hilbert space of the second and third qubit such that \{ $|b_1\rangle, |b_2\rangle$} and \{ $|b_3\rangle, |b_4\rangle$} span $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively. In the orthonormal basis $B$, the codewords can be written as
\[
\begin{align*}
|0\rangle &= |\alpha\rangle|b_1\rangle + |\beta\rangle|b_2\rangle \\
|1\rangle &= |\gamma\rangle|b_3\rangle + |\delta\rangle|b_4\rangle,
\end{align*}
\]
where $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$, and $|\delta\rangle$ are, in general, unnormalized and non-orthogonal states in the Hilbert space of the first qubit.

According to Lemma 3, w.l.o.g. $|b_1\rangle$ and $|b_3\rangle$ can be assumed to be product states. Since a local unitary transformation of a QC yields another QC with same parameters, w.l.o.g. $|b_1\rangle = |00\rangle$ can be chosen. At least one factor of the product state $|b_3\rangle$, w.l.o.g. the first one, has to be $|1\rangle$ since $|b_1|b_3\rangle = 0$.

Therefore, the orthonormal basis $B$ has the form
\[
\begin{align*}
|b_1\rangle &= |00\rangle \\
|b_2\rangle &= k_1|01\rangle - k_2b_{13}^*|10\rangle + k_3b_{10}^*|11\rangle \\
|b_3\rangle &= b_{30}|10\rangle + b_{31}|11\rangle \\
|b_4\rangle &= -k_2^*|01\rangle - k_1^*b_{31}^*|10\rangle + k_1^*b_{30}^*|11\rangle
\end{align*}
\]
with $|k_1|^2 + |k_2|^2 = 1$ and $|b_{30}|^2 + |b_{31}|^2 = 1$. The codewords are of the form
\[
\begin{align*}
|0\rangle &= |\alpha\rangle|00\rangle + k_1|\beta\rangle|01\rangle - k_2b_{13}^*|\beta\rangle|10\rangle + k_3b_{10}^*|\beta\rangle|11\rangle \\
|1\rangle &= -k_2^*|\beta\rangle|01\rangle + b_{30}|\gamma\rangle|10\rangle - k_1^*b_{31}^*|\gamma\rangle|10\rangle + b_{31}|\gamma\rangle|11\rangle + k_1^*b_{30}^*|\delta\rangle|11\rangle.
\end{align*}
\]
If $k_2 = 0$ the state $|0\rangle$ would have the factor $|0\rangle$ at the second position and a code of length two would exist. Therefore we have $k_2 \neq 0$.

From equations (4) and (7) we obtain the following conditions:
\[
\begin{align*}
0 &= (1) F_{00}^{(2)} |0\rangle = -k_1k_2|\delta\rangle |\beta\rangle \\
0 &= (1) F_{01}^{(3)} |0\rangle = k_2(b_{30})^*|\gamma\rangle |\beta\rangle - k_1k_2b_{30}^*b_{31}|\gamma\rangle |\beta\rangle \\
0 &= (1) F_{10}^{(3)} |0\rangle = -k_2^*b_{30}^*|\gamma\rangle |\delta\rangle + k_1b_{31}|\gamma\rangle |\beta\rangle + k_2^*b_{30}^*|\delta\rangle |\beta\rangle.
\end{align*}
\]
In the sequel we distinguish whether $k_1$ and $b_{30}$ vanish or not:

1. $k_1 \neq 0$, $b_{30} \neq 0$:
   From (8) follows $\langle \delta | \beta \rangle = 0$ and thus (4) reduces to $\langle \gamma | \beta \rangle = 0$.

2. $k_1 = 0$, $b_{30} = 0$:
   From (8) follows $\langle \delta | \beta \rangle = 0$. Equations (10) and (11) reduce to
   \[
   -k_2\langle \delta | \alpha \rangle - k_2(b_{31}^*)^2\langle \gamma | \beta \rangle = 0 \\
   -k_1b_{31}\langle \delta | \alpha \rangle + k_1b_{31}^*\langle \gamma | \beta \rangle = 0.
   \]
   This implies $\langle \delta | \alpha \rangle = 0$ and $\langle \gamma | \beta \rangle = 0$.

3. $k_1 = 0$, $b_{30} \neq 0$:
   From (11) and (12) follows $\langle \delta | \beta \rangle = 0$ and $\langle \gamma | \beta \rangle = 0$.

4. $k_1 = 0$, $b_{30} = 0$:
   The basis states $|b_3\rangle$ and $|b_4\rangle$ are $|11\rangle$ and $|01\rangle$, i.e., $|1\rangle$ has the factor $|1\rangle$ in the third position.

For the first three cases $\langle \delta | \beta \rangle = 0$ and $\langle \gamma | \beta \rangle = 0$ implies that $|\gamma\rangle$ and $|\delta\rangle$ are linearly dependent or $|\beta\rangle = 0$. Both results in a factorization of the code. Thus, for all cases the code can be factored and thus reduced to a code of length two which contradicts Theorem 4.

IV. QUANTUM BCH (QBCH) CODES

In principle every quantum error-correcting code applies for the quantum erasure channel since a $t$ error-correcting code is a $2t$ erasure-correcting code. But even
for classical codes, error correction is a hard task \cite{23}. The same is true for the correction of erasures.

But for some codes there are efficient algorithms to correct erasures and errors. Using the algorithm of Berlekamp and Massey \cite{24} for decoding binary BCH codes with designed distance $d_{\text{BCH}}$, \nu erasures and \kt errors can be corrected provided that $\nu + 2t < d_{\text{BCH}}$.

In this section we present a construction of quantum error–correcting codes based on certain binary BCH codes that can be decoded efficiently using the algorithm of Berlekamp and Massey.

In a recent preprint \cite{14} the term quantum BCH code is used for codes derived from BCH codes over $GF(4)$. This definition is more general than ours since every cyclic code over $GF(2)$ is a subcode of a cyclic code over $GF(4)$. But a BCH code over $GF(4)$ need not be a binary BCH code and thus correction of erasures for the codes defined in \cite{14} is not straightforward.

The construction of quantum codes from classical codes is based on the following theorem \cite{13}:

**Theorem 6** Given two classical binary error–correcting codes $C_1 = [N, K_1, d_1]$ and $C_2 = [N, K_2, d_2]$ such that $C_1$ contains the dual of $C_2$, i.e., $C^\perp_2 \subseteq C_1$, a quantum error–correcting code $QC = [[N, K_1 - (N - K_2), \min(d_1, d_2)]]$ exists.

Here $C = [N, K, d]$ denotes a classical binary linear error–correcting code of length $N$, dimension $K$, and minimum distance $d$; $QC = [[N, K, d]]$ denotes a quantum error–correcting code with $N$ qubits that encodes $K$ qubits and allows correction of arbitrary errors of at least $t < d/2$ qubits. Decoding of $QC$ is based on (classical) decoding algorithms for $C_1$ and $C_2$.

We consider the special case where $C_1 = C_2 = C$. Then $C^\perp \subseteq C$ is required, i.e., $C^\perp$ has to be a weakly self dual code and an efficient decoding algorithm for $C$ is needed.

For the construction of quantum BCH codes, $C$ is chosen to be a binary BCH code with $C^\perp$ weakly self dual.

**Definition 1 (Quantum BCH codes)**

Let $C$ be a binary BCH code with $C^\perp$ weakly self dual. The states of the quantum BCH code $QBCH$ code are given (up to normalization) by

$$|\psi_\nu\rangle = \sum_{c \in C^\perp} |c + \nu\rangle \quad \text{for } \nu \in C/C^\perp.$$  

In the remainder of this section we show how to construct the BCH codes needed for QBCH codes. First we recall some properties of BCH codes (for proofs and details see for example \cite{23}).

A cyclic code of length $N$ is defined by the set of roots of its generator polynomial. The roots are distinct powers of a primitive $N$–th root $\alpha$. Equivalently, the code corresponds to the set of exponents of the roots of its generator polynomial, the **defining set** $I_C$. For binary cyclic codes the defining set is a union of cyclotomic cosets $C_i = \{i 2^k \mod N : k = 0, 1, 2, \ldots\}$. For the construction of a binary BCH code with designed distance $d_{\text{BCH}}$, $I_C$ is chosen as $I_C = C_b \cup C_{b+1} \cup \ldots \cup C_{b+d_{\text{BCH}}-2}$, i.e., the union of cyclotomic cosets of $d_{\text{BCH}} - 1$ consecutive numbers.

The defining set $I_{C^\perp}$ of the dual code $C^\perp$ can be computed from that of the code in the following manner:

$$I_{C^\perp} = \bigcup_{i \in I_C} C_{-i},$$

where $\bar{I}_C = \{0, \ldots, N-1\} \setminus I_C$. A cyclic code $C$ is weakly self dual if and only if the defining set $I_C$ contains that of its dual, i.e., $I_{C^\perp} \subseteq I_C$ or, equivalently, $I_{C^\perp} \cap \bar{I}_C = \emptyset$.

For the QBCH codes, a binary BCH code $C$ with $C^\perp$ weakly self dual is needed. Therefore, the condition for $C$ is

$$\bigcup_{i \in I_C} C_i = I_C \subseteq I_{C^\perp} = \bigcup_{j \in \bar{I}_C} C_{-j} = \bigcup_{j \notin I_C} C_{-j}.$$  

Thus, $I_C$ must not contain both $C_i$ and $C_{-i}$. Especially, $I_C$ must not contain cyclotomic cosets with $C_i = C_{-i}$.

The following lemma summarizes the preceding.

**Lemma 7 (BCH codes for QBCH codes)**

Let $C$ be a binary BCH code of length $N$ and defining set $I_C$ such that

$$\forall i : (i \in I_C \implies (-i \mod N) \notin I_C).$$

Then the dual code $C^\perp$ is weakly self dual and a QBCH code can be constructed.

**V. CONCLUSIONS**

We conclude by noting that finding efficient codes for restricted error models is relevant for proof–of–principle demonstrations of quantum error correction in the near future. The first prototype quantum computers will presumably have only a few qubits and will not be powerful enough to implement the most general error correction schemes. For example, a simplified demonstration of quantum error correction could consist in deliberately inducing an error in a known qubit. In this case the QEC error model applies.

**VI. ACKNOWLEDGEMENTS**

The authors acknowledge fruitful discussions with Peter Zoller and helpful comments from David DiVincenzo and the referee. T. P. is supported by the Austrian Science Foundation under grant S06514PHY. This research was supported in part by the National Science Foundation under grant no. PHY94-07194.
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