Characteristic Dynkin Diagrams
and $W$ algebras.

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Abstract

We present a classification of characteristic Dynkin diagrams for the $A_N$, $B_N$, $C_N$ and $D_N$ algebras. This classification is related to the classification of $W(G,K)$ algebras arising from non-Abelian Toda models, and we argue that it can give new insight on the structure of $W$ algebras.
1 Introduction

These recent years, a lot of efforts have been made in order to study the extended conformal algebras $[1]$, also called $W$ algebras. Two of the main problems concerning these algebras are that, first, they have non linear commutation relations, so that there are technical difficulties to deal with, and, second, that there exists no classification scheme, so that a global point of view for their study is hard to find. In order to classify these algebras, one needs to determine which features are relevant to fully distinguish their structure. Of course, the conformal spin contents will be one of them, but it is known that it is not sufficient to define a $W$ algebra. Some results have been obtained in that direction $[2]$, for a wide class of $W$ algebras, namely the so-called $W(G,K)$ algebras that arise in the framework of non-Abelian Toda theories $[3]$.

In this paper, we propose what could be thought of as the Dynkin diagrams (DDs) for $W(G,K)$ algebras: the Characteristic Dynkin Diagrams (CDDs). The CDDs (introduced by Dynkin $[4]$) are DDs where each node carry a grade 0, $\frac{1}{2}$, or 1, and are in one to one correspondence with the $Sl(2)$ subalgebra of an algebra $G$ (themselves being known as classifying the $W(G,K)$ algebras). Until now, only the CDDs of exceptional algebras were classified $[4]$, and we present in the first sections of this paper a classification of CDDs for the other algebras. Then, we will connect the CDDs to the $W(G,K)$ algebras and show how they can be of some help in the study of these algebras. Finally, we will discuss some open problems in the conclusion.

2 Some basic notations and definitions

A Characteristic Dynkin Diagram (CDD) is built from a Dynkin Diagram (DD) of a simple Lie algebra $G$ in such a way that it characterizes an $Sl(2)$ subalgebra of $G$. Practically, to any node in the DD of $G$, one attaches the grade of the corresponding simple root generator under the $Sl(2)$-Cartan generator. It can be shown that these grades can only be 0, $\frac{1}{2}$, or 1. However, to any choice of grades (in $\{0, \frac{1}{2}, 1\}$) will not always correspond an $Sl(2)$ algebra, since there exist some grading operators in $G$ that are not Cartan generator of an $Sl(2)$ $G$-subalgebra. In other words, among the $3^n$ possibilities of constructing a CDD from a rank $n$ algebra, only a certain number of them will really be CDDs. In sections 3,4,5 we present a complete classification of the CDDs for the algebras $A_N$, $B_N$, $C_N$, and $D_N$. The CDDs of the exceptional algebras have already being classified by Dynkin $[4]$, and we will not study them here.

Let us also recall that the $Sl(2)$ subalgebras of $A_N$, $B_N$ and $C_N$ are classified by the regular subalgebras of these algebras. In the case of $D_N$, one has to add the irregular embeddings of the type $B_M \oplus B_{N-M}$ with $2M \neq N$. Still for $D_N$ algebras, some $Sl(2)$ subalgebras have a multiplicity 2 (3 for $D_4$), which means that there are two (three) non-conjugate $Sl(2)$ related by an outer automorphism of $D_N$ ($D_4$). Using CDDs, this multiplicity is given by the quotient of the number of symmetries of its CDD by the number of symmetries of the DD: thus, the multiplicity of any $Sl(2)$ will come naturally with its CDD, and we do not have to pay attention to it.

We will use intensively the connection that exists between CDDs and defining vectors. A defining vector $f = (f_1, f_2, ..., f_N)$ is formed by the components $f_i$ of the $Sl(2)$-Cartan generator in a given basis. For a good choice of basis, the grades of the simple roots (i.e. the nodes of the CDD) can be expressed in a simple way as function of the $f_i$’s: for any algebra $G=A,B,C$ or $D$, and denoting by $e_i$ a basis of $\mathbb{R}^N$ where $N=\text{rank}G$ (rank$G+1$ for $A_M$), the grade of simple roots
of type $e_{i+1} \pm e_i$ (when they exist in $G$) will be $f_{i+1} \pm f_i$ and the grade of $2e_i$ and $e_i$ will be $2f_i$ and $f_i$ (respectively). With this normalisation, the defining vector of any $Sl(2)$ can be directly read from its CDD, and vice versa.

Finally, let us introduce some notations:

We will call integral CDD, a CDD where all the coefficients are 0 or 1. The corresponding gradation of the algebra will be integral. Among the integral CDDs, the principal CDD is the one where all the nodes have grade 1: it correspond to the principal $Sl(2)$ in $G$.

A half-integral CDD will be a CDD where the coefficients are only 0 or $\frac{1}{2}$, whereas a mixed CDD will contain 0, $\frac{1}{2}$ and 1. These two types of CDD correspond to half-integral gradations.

We will call a piece of length $p$ any part of a CDD formed by $p$ nodes. Thus, a node is also a piece of length 1.

We will also draw some CDDs as

$$ (s_1 \bigcirc \cdots \bigcirc s_n) $$

where the notation $(s_1 \bigcirc)$ means that both CDDs with and without the node $s_1 \bigcirc$ have to be considered. For this kind of CDDs, we will use a plus (+) and a minus (−) sign to distinguish the CDD with the node from the other (without the node). In the above example, these CDDs will be denoted as $(2.1^+)$ and $(2.1^-)$.

### 3 Characteristic Dynkin Diagrams for $A_N$

Let us first remark that the CDDs of $A_N$ are symmetric. This can be proven in two ways. First, because the $Sl(2)$-Cartan generator can be chosen diagonal in $A_N$: as it is traceless, it can be chosen symmetric, so that the corresponding CDD is also symmetric. Second, because any $Sl(2)$ has a multiplicity one: since the DD of $A_N$ possesses a $\mathbb{Z}_2$ symmetry, so must any CDD. Notice that the symmetry centre of $A_N$ is either a node (for $A_{2N+1}$), or a link between two nodes (for $A_{2N}$). We will indicate the middle (i.e. the symmetry centre) by a double arrow $\bigcirc$, and draw only half of each CDD of $A_N$.

#### 3.1 Integral CDDs of $A_N$

Let us look first at an integral CDD. It is of the form

$$ (1 \bigcirc 0 \bigcirc \cdots \bigcirc 1) $$

where we can suppose $p_i \neq 0 (\forall i)$ without any loss of generality. The gradation $H$ associated to this CDD will be of the form

$$ (x, x-1, x-2, \ldots, x-n_1, x-n_1, \ldots, x-n_1-1, x-n_1-n_2, x-n_1-n_2, \ldots, x-n_1-n_2, \ldots) $$

with $x$ determined by $trH = 0$. $H$ is an $Sl(2)$-Cartan generator, so that its eigenvalues can be gathered in $D_j$ representations. This implies that for any eigenvalue $y$ in $(3.2)$, $y - 1$ must also
appear with a multiplicity which is greater than or equal to the multiplicity of \( y \). For \( y = x - n_1 \), this means \( x - n_1 - 1 \) must appear at least \((p_1 + 1)\) times, so that \( n_2 = 1 \) and \( p_2 \geq p_1 \). The same argument for all the eigenvalues \( y_j = x - \sum_{i=1}^{j} n_i \) leads us to the constraints:

\[
  n_i = 1, \quad \text{and} \quad p_i \geq p_{i-1} \quad \forall i \geq 2
\]  

(3.3)

Then, the symmetry of the CDD will lead to the two types of integral CDDs for \( A_N \):

\[
\begin{array}{c}
  \includegraphics[width=0.8\textwidth]{integral_cdds.png}
\
p_1 \leq p_2 \leq \ldots \leq p_{\ell}
\end{array}
\]

(3.4)

\[
\begin{array}{c}
  \includegraphics[width=0.8\textwidth]{half_integral_cdds.png}
\
p_1 \leq p_2 \leq \ldots \leq p_{\ell}
\end{array}
\]

(3.5)

### 3.2 Half-integral CDDs of \( A_N \)

For half-integral CDDs, the above reasoning is still valid, with the difference that between the eigenvalues \( y \) and \( y - 1 \), there will occur the eigenvalue \( y - \frac{1}{2} \), which is in a different \( Sl(2) \) representation. Instead of one series of \( p_i \)'s (as in the integral case), we obtain two independent series \((p_i \text{ and } q_i)\) which are intertwined. Thus, the half-integral CDDs take the form:

\[
\begin{array}{c}
  \includegraphics[width=0.8\textwidth]{half_integral_cdds.png}
\
p_1 \leq p_2 \leq \ldots \leq p_{\ell} \quad \text{and} \quad q_1 \leq q_2 \leq \ldots \leq q_{\ell'} \quad ; \quad u = p_\ell \text{ or } q_\ell \text{ depending on } \ell' = \ell - 1 \text{ or } \ell
\end{array}
\]

(3.6)

Notice that the CDD (3.6) can be split in two different CDDs, depending on the two different values \((\ell - 1)\) and \(\ell\) that \(\ell'\) can take.

### 3.3 Mixed CDDs for \( A_N \)

The computation seems very much more complicated when looking at the mixed CDDs. However, it can be greatly simplified if one remarks that the pieces of CDDs that contains nodes of grade 1 must be outside of the pieces containing nodes with grade \( \frac{1}{2} \). In fact, looking at a piece of CDD which possesses a grade 1 node inside two grade \( \frac{1}{2} \) nodes, one computes:

\[
\begin{array}{c}
  \includegraphics[width=0.8\textwidth]{mixed_cdds.png}
\
  H = (\ldots, x, x - \frac{1}{2}, \ldots, x - \frac{1}{2}, x - \frac{3}{2}, \ldots, x - \frac{3}{2}, x - 2, \ldots)
\end{array}
\]

so that the eigenvalue \( x - 1 \) does not appear, as it should do. Thus, the mixed CDDs are formed "outside" by an integral CDD, and "inside" by a half-integral CDD. To study the "transition" between these two pieces of the mixed CDD is as easy as for the preceding cases. Altogether, the mixed CDDs of \( A_N \) take the form:
3.4 $SL(2)$ decomposition associated to a CDD

In the previous section, we have used the Cartan generator of the $SL(2)$ subalgebra to determine all the possible CDDs. This generator also allows us to know the $SL(2)$ representations that enter into the fundamental representation of $A_N$. A $D_j$ representation will be characterized by the eigenvalues $(j, j - 1, ..., -j)$ of the Cartan generator. We give in the following table the $SL(2)$ decomposition for each CDD described in the previous section. To simplify the formulae, we define $p_0 = 0$ and $q_0 = 0$.

| CDD | Decomposition of the fundamental of $A_{N-1}$ |
|-----|-----------------------------------------------|
| $(3.4)$ | $N = D_{n+\ell-1} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i)D_{\ell-1-i}$ |
| $(3.5)$ | $N = D_{n+\ell-2} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i)D_{\ell-2-i}$ |
| $(3.6)$ | $N = D_{(\ell+\ell'+n-1)/2} \oplus D_{(\ell+\ell'+n-2)/2} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i)D_{(\ell+\ell'-1)/2} \oplus_{i=0}^{\ell'-1} (q_{i+1} - q_i)D_{(\ell+\ell'-2)/2-i}$ |
| $(3.7)$ | $N = D_{n+(m+\ell+\ell'-1)/2} \oplus D_{(m+\ell+\ell'-2)/2} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i)D_{(\ell+\ell'-1)/2-i} \oplus_{i=0}^{\ell'-1} (q_{i+1} - q_i)D_{(\ell+\ell'-2)/2-i}$ |
| $(3.8)$ | $N = D_{n+\ell+(\ell+\ell'-1)/2} \oplus_{i=0}^{\ell+\ell'-1} (p_{i+1} - p_i)D_{\ell+(\ell+\ell'-1)/2-i} \oplus_{i=0}^{\ell'-1} (q_{i+1} - q_i)D_{(\ell+\ell'-2)/2-i}$ |

Let us remark that these five decompositions represent all the types of decomposition one can obtain from the reduction of $A_N$ with respect to any of its $SL(2)$ subalgebras.

4 CDDs of $B_N$ and $C_N$

The algebras $B_N$ and $C_N$ are irregular subalgebras of $A_M$ ones. A nice way to see how they are
embedded in \(A_M\), is to use the \(\mathbb{Z}_2\) symmetry of \(A_M\)-DD. This symmetry consists in exchanging the simple root \(\alpha_i\) with the simple root \(\alpha_{M+1-i}\). The invariant subalgebra is built on the generators \(E_{\alpha_i} + E_{\alpha_{M+1-i}}\), so that the DD of this subalgebra is obtained by folding the \(A_M\)-DD. By such a folding, we directly get the \(B_N\) and \(C_N\) DDs. Here, we will use these foldings \(A_{2N} \rightarrow B_N\) and \(A_{2N-1} \rightarrow C_N\) to construct not only the DDs, but the CDDs of \(B_N\) and \(C_N\) from those of \(A_M\). The folding can be extended from the DD to the CDDs of \(A_M\), because the CDDs are themselves symmetric, so that the roots that are summed have the same grade. Of course, only some of the CDDs of \(A_{2N}\) (\(A_{2N-1}\)) will lead to CDDs of \(B_N\) (\(C_N\)). The criteria to select them will be the \(\text{Sl}(2)\) decomposition of the fundamental of \(A_M\), and we will use intensively the table [3.9]. As for the \(A_M\) case, the integers \(p_i, q_i\) and also \(k_i\) will satisfy the conditions:

\[\begin{align*}
p_1 &\leq p_2 \leq p_3 \leq \ldots \leq p_\ell \leq p_{\ell+1} \leq \ldots \leq p_{\ell+q} \quad p_0 = 0 \\
q_1 &\leq q_2 \leq q_3 \leq \ldots \leq q_{\ell} \quad q_0 = 0 \\
k_1 &\leq k_2 \leq k_3 \leq \ldots \leq k_{\ell+q} \quad k_0 = -\frac{1}{2} \quad (\text{so that } 2k_0 + 1 = 0)
\end{align*}\]

### 4.1 CDD for \(B_N\)

A CDD of \(A_{2N}\) will lead to a CDD of \(B_N\) iff all the \(\mathcal{D}_j\) with \(j \in \frac{1}{2} + \mathbb{Z}\) have an even multiplicity (in the fundamental of \(A_{2N}\)). For each decomposition given in table [3.9], we determine which are the cases where the folding can be effectively done, and then give the folded diagram and the corresponding decomposition of the fundamental. Of course, as we are folding \(A_{2N}\), we will consider only the CDDs with an even number of nodes, so that the center of the CDD is always on a link. Notice that under the folding, a CDD of the type \((3.6)\) will behave in two different ways, depending on whether \(u\) is equal to or different from 0. If \(u = 0\), then the short root of \(B_N\) will have a grade \(\frac{1}{2}\), while for \(u \neq 0\) it will possess a 0-grade.

#### 4.1.1 Integral CDDs

The CDD \((3.4)\) can always be folded in \(A_{2N}\), however, for \(p_\ell = 2u + 2 \neq 0\) we get the CDD \((4.2)\) of \(B_N\), while for \(p_\ell = 0\) (which implies \(p_i = 0 \forall i\)), we obtain the principal CDD \((4.3)\). The CDD \((3.3)\) corresponds to \(A_N\) with \(N\) odd, so that we do not have to consider it here.

\[(4.2)\] \[
\begin{align*}
2N + 1 & = \mathcal{D}_{n+\ell-1} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i) \mathcal{D}_0 \quad p_\ell = 2u + 2 \neq 0 \\
2N + 1 & = \mathcal{D}_n
\end{align*}
\]

#### 4.1.2 Half-integral CDDs

The CDD \((3.6)\) provides three types of \(B_N\)-CDDs, depending on \((n + \ell + \ell' - 1)\) even or odd, \(n = 0\) or 1 (only allowed values here) and also \(q_\ell = 0\) or \(q_\ell > 0\).

\[(4.4)\] \[\text{is obtained from (3.6) with the constraints } n = 0, \ell' = \ell, p_i = 2k_i + 1, \text{ and } q_\ell = 2u + 2 \neq 0 \text{ for (4.4)}; \text{ and } n = 1, \ell' = \ell, p_i = 2k_i + 1, \text{ and } q_\ell = 2u + 2 \neq 0 \text{ for (4.4').}
\]

Note that \(q_\ell = 0\) implies \(q_i = 0 \forall i\).
The conditions \((n = 0, \ell' = \ell, p_i = 2k_i + 1, \text{ and } q_i = 0)\) and \((n = 1, \ell' = \ell - 1, q_i = 2k_i + 1, \text{ and } p_i = 0)\) on (3.6) give the CDDs (4.3) and (4.3+) respectively.

\((4.4)\) is obtained from (3.6) with the conditions \(n = 0, \ell' = \ell - 1, q_i = 2k_i + 1, \text{ and } p_i = 2u + 2 \neq 0\). We recover the CDD (4.3+) with the conditions \(n = 0, \ell' = \ell - 1, q_i = 2k_i + 1 \text{ and } p_i = 0\) on (3.4).

\[
2N + 1 = \begin{cases} 
2D_{\ell-1/2} \oplus D_{\ell-1} \oplus_{i=0}^{\ell-1} 2(k_i+1 - k_i)D_{\ell-i-1/2} \oplus (q_{i+1} - q_i)D_{\ell-i-1} & \text{without } \frac{i}{2} \\
D_\ell \oplus D_{\ell-1/2} \oplus_{i=0}^{\ell-1} 2(k_i+1 - k_i)D_{\ell-i-1/2} \oplus (q_{i+1} - q_i)D_{\ell-i-1} & \text{with } \frac{i}{2} 
\end{cases}
q_i = 2u + 2
\]

\[
2N + 1 = \begin{cases} 
D_{\ell-1/2} \oplus D_{\ell-1} \oplus_{i=0}^{\ell-1} 2(k_i+1 - k_i)D_{\ell-i-1/2} & \text{without } \frac{i}{2} \\
D_\ell \oplus D_{\ell-1/2} \oplus_{i=0}^{\ell-1} 2(k_i+1 - k_i)D_{\ell-i-1/2} & \text{with } \frac{i}{2} 
\end{cases}
p_i = 2u + 2
\]

\[
2N + 1 = D_{\ell-1} \oplus D_{\ell-3/2} \oplus_{i=0}^{\ell-1} (p_{i+1} - p_i)D_{\ell-i-1} \oplus_{i=0}^{\ell-2} (k_i+1 - k_i)D_{\ell-i-3/2} 
q_i = 2u + 2
\]

### 4.1.3 Mixed CDDs

Studying (3.7) and (3.8) leads to the following CDDs:

\((4.7)\) is obtained from (3.7) by the restrictions \(m = 1, p_i = 2k_i + 1, \ell' = \ell \text{ and } q_i = 2u + 2 \neq 0\).

\((4.8)\) with \(\ell = 0\) comes from (3.7) by the conditions \(m = 0, \ell' = \ell - 1, q_i = 2k_i + 1\) and \(p_i = 2u + 2 \neq 0\), while the general case (for \((4.8)\)) is given by (3.8) with \(\ell'' = \ell' - 1, q_i = 2k_i + 1 \text{ and } p_i = 2u + 2 \neq 0\).

\((4.9)\) is the folding of (3.8) when applying \(\ell'' = \ell'\), \(n = 0, p_i = 2k_i + 1 \text{ and } q_{\ell''} = 2u + 2 \neq 0\).

\((4.10)\) is obtained from (3.8) by the restrictions \(\ell'' = \ell', n = 0, p_i = 2k_i + 1 \text{ and } q_{\ell'} = 0\).

\((4.11)\) can be obtained in several ways: from (3.7) with the conditions \(\ell' = \ell - 1, m = 0, q_i = 2k_i + 1 \text{ and } p_i = 0\) or \((\ell' = \ell, m = 1, p_i = 2k_i + 1 \text{ and } q_{\ell'} = 0)\), and also from (3.8) when imposing \(\ell'' = \ell' - 1, q_i = 2k_i + 1 \text{ and } p_{\ell''} = 0\).

\[
2N + 1 = D_{n+\ell} \oplus D_{\ell-1/2} \oplus_{i=0}^{\ell-1} 2(k_i+1 - k_i)D_{\ell-i-1/2} \oplus (q_{i+1} - q_i)D_{\ell-i-1} 
q_i = 2u + 2
\]
\[2N + 1 = D_{n+\ell} \oplus \ell+1 \quad (p_{i+1} - p_i) D_{\ell-i-1} \oplus D_{\ell-3/2} \oplus 2(k_{i+1} - k_i) D_{\ell-i-3/2}\]

\[q_{\ell+\ell'} = 2u + 2\]

\[2N + 1 = D_{\ell+\ell'}/2 \oplus \ell+1 \quad 2(k_{i+1} - k_i) D_{\ell+i-1/2} \oplus D_{\ell-1} \oplus 2(1) \quad (q_{i+1} - q_i) D_{\ell-i-1}\]

\[q_{\ell'} = 2u + 2\]

\[2N + 1 = D_{n+\ell} \oplus D_{\ell-1}/2 \oplus \ell-1 \quad 2(k_{i+1} - k_i) D_{\ell-i-1/2}\]

\[2N + 1 = D_{n+\ell-1} \oplus \ell-1 \quad 2(k_{i+1} - k_i) D_{\ell-i-1}\]

\[2N = D_{\ell-1} \oplus \ell-1 \quad 2(k_{i+1} - k_i) D_{\ell-i-1}\]

\[2N = D_{n+\ell-1/2} \oplus \ell-1 \quad (p_{i+1} - p_i) D_{\ell-i-1/2}\]

### 4.2 CDD for \(C_N\)

We obtain the CDD for \(C_N\) by folding \(A_{2N-1}\). A \(A_N\)-CDD will give a CDD of \(C_N\) iff all the \(D_j\) with \(j \in \mathbb{Z}\) have an even multiplicity (in the fundamental of \(A_{2N-1}\)).

#### 4.2.1 Integral CDDs

We start again with the CDD \((3.4)\). To get an even multiplicity for \(D_{n+\ell}\) one must impose \(n = 0\) and \(p_1\) odd. Then, looking at the other \(D\)-representations leads to the constraints \((n = 0, p_i = 2k_i + 1)\), and the folded CDD \((4.12)\). The CDD \((3.3)\) can always be folded: we obtain then the CDD \((4.13)\).
4.2.2 Half-integral CDDs

Starting with (4.6), we have to know whether \( n + \ell + \ell' - 1 \) is even or odd. In the first case, one must impose \( n = 0 \), so that we have the constraints \( (n = 0, \ell' = \ell - 1, p_i = 2k_i + 1) \) and obtain (4.15). In the second case, one is left with two possibilities: \( (n = 0, \ell' = \ell, q_i = 2k_i + 1) \) that leads to (4.14), or \( (n = 1, \ell' = \ell - 1, p_i = 2k_i + 1) \) that leads to (4.13).

\[
2N = D_{\ell-1/2} \oplus D_{\ell-1} \oplus \bigoplus_{i=0}^{\ell-1} (p_{i+1} - p_i) D_{\ell-i-1/2} \oplus 2(k_{i+1} - k_i)D_{\ell-i-1}
\]

\[
2N = \begin{cases} 
D_{\ell-1/2} \oplus D_{\ell-1} \oplus \bigoplus_{i=0}^{\ell-2} (q_{i+1} - q_i)D_{\ell-i-3/2} & \text{with } \frac{1}{2} \\
D_{\ell-1} \oplus D_{\ell-3/2} \oplus \bigoplus_{i=0}^{\ell-2} (q_{i+1} - q_i)D_{\ell-i-3/2} & \text{without} 
\end{cases}
\]

4.2.3 Mixed CDDs

For the CDD (4.4), and recalling that \( n \neq 0 \), we have the constraints \( (m = 0, \ell' = \ell, q_i = 2k_i + 1) \) to get (4.16), and \( (m = 1, \ell' = \ell - 1, p_i = 2k_i + 1) \) to find (4.17).

For the CDD (3.8), \( (n = 0, \ell' = \ell - 1, p_i = 2k_i + 1) \) leads to (4.18), while \( (\ell'' = \ell', q_i = 2k_i + 1) \) produces (4.19).

\[
2N = D_{n+\ell-1/2} \oplus D_{\ell-1} \oplus \bigoplus_{i=0}^{\ell-1} (p_{i+1} - p_i) D_{\ell-i-1/2} \oplus 2(k_{i+1} - k_i)D_{\ell-i-1}
\]

\[
2N = D_{n+\ell-1/2} \oplus D_{\ell-1} \oplus \bigoplus_{i=0}^{\ell-2} (q_{i+1} - q_i)D_{\ell-i-3/2}
\]

\[
2N = D_{\ell+\ell'-1} \oplus \bigoplus_{i=0}^{\ell+\ell'-1} 2(k_{i+1} - k_i)D_{\ell+\ell'-i-1/2} \oplus 2(k_{i+1} - k_i)D_{\ell+\ell'-i-3/2}
\]

\[
2N = D_{n+\ell+\ell'-1/2} \oplus \bigoplus_{i=0}^{n+\ell+\ell'-1} (p_{i+1} - p_i)D_{\ell+\ell'-i-1/2} \oplus 2(k_{i+1} - k_i)D_{\ell+\ell'-i-1}
\]
5 CDDs of \( D_N \)

The CDDs of \( D_N \) are obtained thanks to the inclusion \( D_N \subset B_N \). As the dimension of the fundamental representation of \( D_N \) is the one of \( B_N \) minus 1, we must select in the CDDs of \( B_N \) those which correspond to fundamental representation decomposition with at least one \( D_0 \). Then, from the relation between the defining vectors and the CDDs:

\[
\begin{align*}
\bigcirc \cdots \bigcirc & \quad \text{with } s_i = f_i - f_{i+1}, \; i = 1, \ldots, n-1 \quad \text{and} \quad s_n = f_n \\
\bigcirc \cdots \bigcirc & \quad \text{with } s_i = f_i - f_{i+1}, \; i = 1, \ldots, n-1 \quad \text{and} \quad s'_n = f_{n-1} + f_n \\
\bigcirc \cdots \bigcirc & \quad \text{with } s_i = f_i - f_{i+1}, \; i = 1, \ldots, n-1 \quad \text{and} \quad s'_n = 2s_n + s_{n-1}
\end{align*}
\]

it is easy to deduce the CDDs of \( D_N \) from the CDDs of \( B_N \). We recall that the condition (4.1) is always understood.

As far as the decomposition of the fundamental is concerned, the different cases related to the same CDD of \( B_N \) will provide the same sort of decomposition. In fact these decompositions are just the ones of \( B_N \) (minus one \( D_0 \)) so that we do not re-write them in this section.

5.1 Integral CDDs

Let us start with the CDD (4.2). To get a \( D_0 \) representation, we must impose \( p_\ell = 2u + 2 > p_{\ell-1} \). Then, the computation of \( s'_n \) and of the grades entering in the "tail" of the CDD will lead to four different cases: \( u \geq 2, \; u = 1, \; u = 0 \) and \( \ell > 1, \) or \( u = 0 \) and \( \ell = 1 \). We can suppose in (4.2) that \( p_i \neq 0 \) since \( p_i = 0 \) can be replaced by \( n \rightarrow (n+1) \) and \( p_i \rightarrow p_{i-1} \) (\( i > 1 \)). If \( u \geq 2 \), then \( s'_n = s_{n-1} = s_{n-2} = 0 \) (CDD (5.1)); for \( u = 1 \), we get \( s'_n = s_{n-1} = 0 \) but \( s_{n-2} = 1 \) (CDD (5.2)); finally, \( u = 0 \) provides \( s'_n = s_{n-1} = 1 \) and \( s_{n-2} = 1 \) or 0, depending on \( \ell = 1 \) or \( \ell > 1 \) respectively (CDDs (5.3) and (5.4) respectively). Note that because of the constraints (4.1) and \( 2u + 2 > p_{\ell-1}, \; u = 0 \) leads to \( p_i = 1 \), while \( u = 1 \) imposes \( p_i \leq 3 \).

The CDD (4.3) is of course not allowed, since it has no \( D_0 \) representation, but two other integral CDDs (5.5 and 5.6) are provided by mixed CDDs: we write these CDDs here, but explain them in the section dealing with mixed CDDs.

\[
\begin{align*}
\text{5} \leq p_{\ell-1} \leq 2u + 1
\end{align*}
\]

\[
\begin{align*}
0 \leq p_i \leq 3
\end{align*}
\]

\[
\begin{align*}
(\ell - 1) \text{ nodes}
\end{align*}
\]

\[
\begin{align*}
\text{(5.1)}
\end{align*}
\]

\[
\begin{align*}
\text{(5.2)}
\end{align*}
\]

\[
\begin{align*}
\text{(5.3)}
\end{align*}
\]

\[
\begin{align*}
\text{(5.4)}
\end{align*}
\]

\[
\begin{align*}
\text{(5.5)}
\end{align*}
\]
5.2 Half-integral CDDs

As for the integral case, one has to distinguish \( u = 0 \), \( u = 1 \) and \( u \geq 2 \). However, as there are two intertwined series (the odd one being never absent), \( \ell = 1 \) (which should be replaced here by \( p_{\ell-1} = 0 \)) is no more particular with respect to \( \ell > 1 \) (\( p_{\ell-1} > 0 \)).

Starting from (4.4), one obtains the CDDs (5.7), (5.8), and (5.9) for \( (5.6) \) can be considered only for the case (4.5) with \( \ell = 1 \); it produces the integral CDD (5.5) with \( \ell = 0 \).

The CDD (4.6) produces (5.10), (5.11), and (5.12) in the same way we have expressed for (4.4).
5.3 Mixed CDDs

For the three first mixed CDDs of $B_N$, the calculation is the same as for the half-integral case. The CDD (5.7) give rise to the CDDs (5.13), (5.14), and (5.15). The CDDs (5.8) and (5.9) produce the CDDs (5.16), (5.17), and (5.18) for the former, and (5.19), (5.20), and (5.21) for the latter.

The CDD (4.10) can contribute only when $\ell' = 1$, which leads to two integral CDDs for $D_N$. For these CDDs, one has to distinguish if $k_{\ell+1} > 0$ or $k_{\ell+1} = 0$. In the first case, one obtains (5.11), and in the second (5.6).

(4.11) do not contribute at all.
5 \leq q_{v-1} \leq 2u + 1

(5.19)

0 \leq q_{v-1} \leq 3

(5.20)

0 \leq q_{v-1} \leq 1

(5.21)

5.4 Folding of the CDDs of $D_N$

In the same way that we have folded $A_M$ to get $B_N$ and $C_N$, it is possible, using the $\mathbb{Z}_2$ symmetry of the ”tail” of $D_N$, to fold it and get $B_{N-1}$. Note however that the CDDs of $D_N$ do not possess a priori the symmetry that allows the folding, so that we have to select the subset of symmetric CDDs, that is to discard the CDDs (5.5) and (5.6). Moreover, the fundamental representation $2N - 1$ of $B_{N-1}$ is obtained from the $2N$ representation of $D_N$ by $2N - 1 = 2N - D_0$: we must also select the CDDs that correspond to a decomposition with at least one representation $D_0$. This prescription will suppress all the CDDs associated with $u = 0$. Then, one can fold the remaining CDDs of $D_N$, and re-find the results given in section 4.1, the cases $u = 1$ becoming a ”real” case of $u \geq 2$.

As an exercise, let us also compute the CDDs of $G_2$ thanks to the $\mathbb{Z}_3$ symmetry of $D_4$. $G_2$-CDDs are obtained from $D_4$-CDDs by identifying its three external nodes. We have also to impose that the decomposition of the 8 of $D_4$ possesses a $D_0$ representation, since the fundamental of $G_2$ has dimension 7. The CDDs of $D_4$ are:

(5.22)

The first line corresponds to non $\mathbb{Z}_3$-symmetric CDDs. Note however that this line shows how the ”new” outer isomorphism (induced by the enlarged symmetry of $D_4$ w.r.t. $D_N$, $N \neq 4$) relates
some of the \( Sl(2) \) of \( D_4 \). On the second line lies the only CDD of \( D_4 \) which is \( \mathbb{Z}_3 \) symmetric but which does not possess a \( \mathcal{D}_0 \) representation in its fundamental representation. Thus, we fold the CDDs of the last line and get the CDDs of \( G_2 \):

\[
\begin{array}{cccc}
1 & 1 & 0 & 1 \\
\bullet & \equiv & \bullet & \equiv \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{array}
\quad (5.23)
\]

6 Application to \( W \) algebras

Most (and perhaps all) the \( W \) algebras can be realized in the context of non Abelian Toda theories. Such models are deduced from a WZW action \( S_0(g) \), based on a group \( G \) of Lie algebra \( \mathcal{G} \), by imposing some constraints on the (chiral) Kac-Moody currents of \( S_0(g) \). These constraints are themselves induced by a gradation \( H \) of \( \mathcal{G} \). Then a Lagrange multiplier treatment and a Gauss decomposition \( g = g_+ g_0 g_- \) lead to a new action \( S(g_0) \) that depends only on the 0-grade (w.r.t. \( H \)) component \( g_0 \). This provides an easy-to-handle framework for the realization of \( W \) algebras (which are symmetry of non Abelian Toda theories) in term of the Kac-Moody generators on the one hand, and in terms of free fields on the other hand. The \( W \) algebra obtained is naturally associated to an \( Sl(2) \) \( \mathcal{G} \)-subalgebra, \( H \) being the Cartan generator of this \( Sl(2) \) subalgebra. In fact, the \( Sl(2) \) embeddings in \( \mathcal{G} \) classify the different \( W \) algebras which can be obtained by such a procedure. These algebras are denoted \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \), where \( \mathcal{K} \) is the \( \mathcal{G} \)-subalgebra whose principal \( Sl(2) \) is the \( Sl(2) \) algebra under consideration. Since these \( W \) algebras are classified by the different \( Sl(2) \) in \( \mathcal{G} \), it is natural to relate also these algebras to CDDs. Let us immediately remark that such an approach will be particularly useful for the folding of \( W \) algebras \( [5] \), since the CDDs of \( B_N \) and \( C_N \) already have been construct using this method. We give below other examples of the use of CDDs in the study of \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) algebras.

6.1 Characterization of \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \)

As explain above, since the \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) are classified by \( Sl(2) \) subalgebras in \( \mathcal{G} \), we can think of the \( \mathcal{G} \)-CDDs as the Dynkin Diagrams for the \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) algebras. We want here to show how the use of the CDDs, combined with the results obtained in \( [2] \) give a lot of informations on the structure of \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) with a minimal of calculation. Using only the decomposition of the fundamental \( \mathcal{G} \)-representation with respect to this \( Sl(2) \), \( [4] \) were able to compute the spin contents of all the \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) algebras. Moreover, extending the \( Sl(2) \) algebra to an \( Sl(2) \oplus U(1) \) algebra, they also give the eigenvalues of all the \( W \) generators with respect to some special \( U(1) \). From this work, and the knowledge of the fundamental representation decomposition, it is possible to make contact between CDDs and \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \) algebras. Moreover, the knowledge of the grades of the simple roots allows us to compute the Kac-Moody subalgebra which is in \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \), up to \( U(1) \) factors. In fact, this Kac-Moody algebra is just the ”affinization” of the algebra one obtains by removing in the CDD all the nodes that have not a 0-grade. Note that the \( U(1) \) factors that the CDD cannot determine are just the ones whose eigenvalues are computed in \( [4] \), so that combining the two approaches gives almost all the informations on \( \mathcal{W}(\mathcal{G}, \mathcal{K}) \). The results are given in the tables below. We will use the notations

\[
\begin{align*}
 r_i &= p_{i+1} - p_i \quad i \geq 1 \quad \text{and} \quad r_0 = p_1 \\
 s_i &= q_{i+1} - q_i \quad i \geq 1 \quad \text{and} \quad s_0 = q_1 \\
 l_i &= k_{i+1} - k_i \quad i \geq 1
\end{align*}
\]
6.1.1 $\mathcal{W}(A_N,K)$ algebras

The only regular subalgebras of $A_N$ are (sum of) $A_M$ subalgebras. When reducing the fundamental representation with respect to the principal $Sl(2)$ of $A_M$, a $D_{M/2}$ representation will appear. Thus, we identify the form of $K$ thanks to the rule

$$\mathcal{D}_{p/2} \text{ in fundamental} \rightarrow A_p \text{ in } K$$

(6.2)

Looking at all the CDDs of $A_N$ we obtain the following table

| CDD  | $\mathcal{K}$ in $\mathcal{W}(A_N,K)$                                                                                                                                                                                                 | KM subalg. in $\mathcal{W}$ (up to $U(1)$ factors)                                                                 |
|------|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|
| (3.4)| $A_{2n+2\ell-2} \oplus \ell^{-2} r_i A_{2\ell-2i-2}$                                                                                       | $A_{p_i} \oplus \ell^{-1} 2A_{p_i}$                                                                                         |
| (3.5)| $A_{2n+2\ell-1} \oplus \ell^{-2} r_i A_{2\ell-2i-1}$                                                                                       | $2 \oplus \ell^{-1} A_{p_i}$                                                                                                                  |
| (3.6) | $A_{n+\ell+\ell-1} \oplus A_{n+\ell+\ell-2} \oplus \ell^{-1} r_i A_{\ell+\ell-2i-1} \oplus \ell^{-1} \sum s_i A_{\ell+\ell-2i-2}$                                                                                                         | $2 \oplus \ell^{-1} [A_{p_i} \oplus A_{q_i}] \oplus A_{p_i} \oplus A_{p_i} \oplus A_{q_i}$ (if $\ell' = \ell - 1$) |
| (3.7) | $A_{2n+m+\ell+\ell-1} \oplus A_{m+\ell+\ell-2} \oplus \ell^{-1} r_i A_{\ell+\ell-2i-1} \oplus \ell^{-1} \sum s_i A_{\ell+\ell-2i-2}$                                                                                                         | $2 \oplus \ell^{-1} [A_{p_i} \oplus A_{q_i}] \oplus A_{p_i} \oplus A_{p_i} \oplus A_{q_i}$ (if $\ell' = \ell$)  |
| (3.8) | $A_{2n+2\ell+\ell'+\ell''-1} \oplus \ell^{+1} r_i A_{2\ell+\ell'+\ell''-2i-1}$                                                                  | $2 \oplus \ell^{+1} [A_{p_i} \oplus A_{q_i}] \oplus A_{p_i} \oplus A_{p_i} \oplus A_{q_i}$ (if $\ell'' = \ell' - 1$) |
|      | $\oplus A_{\ell+\ell-2} \oplus \ell^{\prime+1} \sum s_i A_{\ell+\ell-2i-2}$                                                                                                         | $2 \oplus \ell^{\prime+1} [A_{p_i} \oplus A_{q_i}] \oplus A_{p_i} \oplus A_{p_i} \oplus A_{q_i}$ (if $\ell'' = \ell'$) |

(6.3)

The spin contents and the hypercharges are then easily computed from the decompositions given in table 3.9, following the scheme developed in [2].

6.1.2 $\mathcal{W}(B_N,K)$ algebras

$B_N$ contains as regular subalgebras $A_M$, $B_M$ and $D_M$. The relation between these subalgebras and the decomposition of the fundamental with respect to their principal $Sl(2)$ have been given in [4]: each $A_M$ contributes to the fundamental decomposition by a term $2D_{M/2}$, and $B_M$ or $D_M$ provide a $D_M$ representation (we use the notation $B_1$ for the $A_1$ algebra of index 2, constructed on a short root). Moreover, in $B_N$, the principal $Sl(2)$ constructed on a $D_M$ algebra is the same as the one constructed on a $B_M$ algebra, so that we can neglect these inclusions. Thus, the rules are in this case

$$\begin{align*}
\mathcal{D}_p & \rightarrow B_p \\
2\mathcal{D}_{p/2} & \rightarrow A_{p-1}
\end{align*}$$

(6.4)
| CDD | $K$ in $\mathcal{W}(B_N, K)$ | KM subalg. in $\mathcal{W}$ (up to $U(1)$ factors) |
|-----|-----------------|-----------------------------------------------|
| (4.2) | $B_{n+\ell-1} \oplus p_1 B_{\ell-1} \oplus \bigoplus_{i=1}^{\ell-2} r_i B_{\ell-i-1}$ | $B_{u+1} \oplus \bigoplus_{i=1}^{\ell-1} A_{p_i}$ |
| (4.3) | $B_N$ | $\emptyset$ |
| (4.4) | $(k_1 + 1) A_{2\ell-1} \oplus (q_1 + 1) B_{\ell-1} \oplus \bigoplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \bigoplus_{i=1}^{\ell-2} s_i B_{\ell-i-1}$ | $B_{u+1} \oplus A_{2k_{\ell+1}} \oplus \bigoplus_{i=1}^{\ell-1} [A_{2k_{\ell+1}} \oplus A_{q_{}\ell-1}]$ |
| (4.5) | $(k_1 + 1) A_{2\ell-1} \oplus (q_1 + 1) B_{\ell-1} \oplus \bigoplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1}$ | $B_{u+1} \oplus A_{2k_{\ell+1}} \oplus \bigoplus_{i=1}^{\ell-1} A_{2k_{\ell+1}}$ |
| (4.6) | $(p_1 + 1) B_{\ell-1} \oplus (k_1 + 1) A_{2\ell-3} \oplus \bigoplus_{i=1}^{\ell-2} r_i B_{\ell-i-1} \oplus l_i A_{2\ell-2i-3}$ | $B_{u+1} \oplus \bigoplus_{i=1}^{\ell-1} [A_{p_1} \oplus A_{2k_{\ell+1}}]$ |
| (4.7) | $B_{n+\ell} \oplus (k_1 + 1) A_{2\ell-1} \oplus q_1 B_\ell \oplus \bigoplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \bigoplus_{i=1}^{\ell-2} s_i B_{\ell-i-1}$ | $A_{2k_{\ell+1}} \oplus B_{u+1} \oplus \bigoplus_{i=1}^{\ell-1} [A_{2k_{\ell+1}} \oplus A_{q_{}\ell-1}]$ |
| (4.8) | $B_{n+\ell+\ell'-1} \oplus q_1 B_{\ell+\ell'-1} \oplus \bigoplus_{i=1}^{\ell+\ell'-2} r_i B_{\ell+\ell'-i-1} \oplus (k_1 + 1) A_{2\ell'-3} \oplus \bigoplus_{i=0}^{\ell'-2} l_i A_{2\ell'-2i-3}$ | $B_{u+1} \oplus \bigoplus_{i=1}^{\ell+\ell'-1} A_{p_1} \oplus \bigoplus_{i=1}^{\ell-1} A_{2k_{\ell+1}}$ |
| (4.9) | $(k_1 + 1) A_{2\ell+2\ell'-1} \oplus \bigoplus_{i=1}^{\ell+\ell'-1} r_i A_{2\ell+2\ell'-2i-1} \oplus (q_1 + 1) B_{\ell'-1} \oplus \bigoplus_{i=1}^{\ell'-2} s_i B_{\ell'-i-1}$ | $B_{u+1} \oplus \bigoplus_{i=1}^{\ell+\ell'} A_{2k_{\ell+1}} \oplus A_{q_{}\ell-1}$ |
| (4.10) | $(k_1 + 1) A_{2\ell+2\ell'-1} \oplus B_{\ell'-1} \oplus \bigoplus_{i=1}^{\ell+\ell'-1} l_i A_{2\ell+2\ell'-2i-1}$ | $\bigoplus_{i=1}^{\ell+\ell'} A_{2k_{\ell+1}}$ |
| (4.11) | $B_{n+\ell} \oplus (k_1 + 1) A_{2\ell-1} \oplus \bigoplus_{i=1}^{\ell+\ell'-1} l_i A_{2\ell-2i-1}$ | $\bigoplus_{i=1}^{\ell+\ell'} A_{2k_{\ell+1}}$ |

### 6.1.3 $\mathcal{W}(C_N, K)$ algebras

As in the previous case, we can write $C_1$ to distinguish the index 1 $A_1$ algebra constructed on a long root, from the index 2 $A_M$ subalgebras. Then, the correspondence between $D_M$ representations and (regular) subalgebras will be:

$$\begin{align*}
D_{p-1/2} & \longrightarrow C_p \\
2D_{p/2} & \longrightarrow A_p
\end{align*}$$

(6.6)
\[
\begin{array}{|c|c|c|}
\hline
\text{CDD} & \mathcal{K} \text{ in } \mathcal{W}(C_N, \mathcal{K}) & \text{KM subalg. in } \mathcal{W} \text{ (up to } U(1) \text{ factors)} \\
\hline
(4.12) & (k_1 + 1)A_{2\ell-2} \oplus \ell-2 \sum_{i=1}^{\ell-2} l_i A_{2\ell-2i-2} & C_{k_{i+1}} \oplus \ell-1 \sum_{i=1}^{\ell-1} A_{2k_{i+1}} \\
(4.13) & (p_1 + 1)C_{n+\ell} \oplus \ell-2 \sum_{i=1}^{\ell-2} r_i C_{\ell-i} & \oplus \ell-1 \sum_{i=1}^{\ell} A_{p_{i+1}} \\
(4.14) & (p_1 + 1)C_{\ell} \oplus (k_1 + 1)A_{2\ell-2} \oplus \ell-2 \sum_{i=1}^{\ell-2} [r_i C_{\ell-i} \oplus l_i A_{2\ell-2i-2}] & \oplus \ell-1 [A_{p_{i+1}} \oplus A_{2k_{i+1}}] \oplus A_{p_{i+1}} \oplus C_{k_{i+1}} \\
(4.15) & (k_1 + 1)A_{2\ell-2} \oplus (q_1 + 1)C_{\ell-1} \oplus \ell-2 \sum_{i=1}^{\ell-2} [l_i A_{2\ell-2i-2} \oplus s_i C_{\ell-i-1}] & \oplus \ell-1 [A_{2k_{i+1}} \oplus A_{q_{i+1}}] \oplus C_{k_{i+1}} \\
(4.16) & C_{n+\ell} \oplus (k_1 + 1)A_{2\ell-2} \oplus \ell-2 \sum_{i=1}^{\ell-2} [r_i C_{\ell-i-1} \oplus l_i A_{2\ell-2i-2}] & \oplus \ell-1 A_{p_{i+1}} \oplus \ell-1 A_{2k_{i+1}} \oplus C_{k_{i+1}} \\
(4.17) & C_{n+\ell} \oplus (k_1 + 1)A_{2\ell-2} \oplus \ell-2 \sum_{i=1}^{\ell-2} [l_i A_{2\ell-2i-2} \oplus s_i C_{\ell-i-1}] & \oplus \ell-1 [A_{2k_{i+1}} \oplus A_{q_{i+1}}] \oplus C_{k_{i+1}} \\
(4.18) & (k_1 + 1)A_{2\ell+2\ell'}-2 \oplus \ell+\ell' \sum_{i=1}^{\ell+\ell'-2} [l_i A_{2\ell+2\ell'-2i-2} \oplus (q_1 + 1)C_{\ell'-1} \oplus \ell-2 \sum_{i=1}^{\ell' \ell'-2} s_i C_{\ell'-i-1}] & \oplus \ell+\ell'-1 A_{2k_{i+1}} \oplus \ell-1 A_{q_{i+1}} \oplus C_{k_{i+\ell'+1}} \\
(4.19) & C_{n+\ell+\ell'} \oplus p_1 C_{\ell+\ell'} \oplus \ell+\ell' \sum_{i=1}^{\ell+\ell'-2} [r_i C_{\ell+\ell'-i} \oplus (k_1 + 1)A_{2\ell+2\ell'-2} \oplus \ell-2 \sum_{i=1}^{\ell'} l_i A_{2\ell-2i-2}] & \oplus \ell+\ell' \sum_{i=1}^{\ell} A_{p_{i+1}} \oplus \ell-1 A_{2k_{i+1}} \oplus C_{k_{i+\ell'+1}} \\
\hline
\end{array}
\]

### 6.1.4 \( W(D_N, \mathcal{K}) \) algebras

As far as \( D_N \) algebra are concerned, one can proceed in the same way as for \( B_N \) (replacing \( B_M \) by \( D_M \) ) except for the irregular embeddings \( B_k \oplus B_{N-k} \subset D_N \), that cannot be replaced by \( D_M \) embeddings. It is natural to think of them as related to the particular cases \( u = 0 \) or \( u = 1 \) of the integral CDDs. As these irregular embeddings should not survive to the folding \( D_N \to B_{N-1} \), we focus on \( u = 0 \) and \( \ell > 1 \) \((\ell = 1 \text{ is the principal CDD})\). Then, looking at the \( Sl(2) \) decomposition of the CDD \((5.4)\), we obtain \( 2N = D_{n+\ell-1} \oplus D_{\ell-1} \) which indeed corresponds to the reduction with respect to the principal \( Sl(2) \) of \( B_{\ell-1} \oplus B_{N-\ell+1} \). Note that the case \( \ell = 1 \) could also be included in the exceptions (with the convention \( B_0 = 0 \)), since the \( Sl(2) \)-principal embedding of \( B_{N-1} \) in \( D_N \) is also the principal \( Sl(2) \) of \( D_N \). Thus the rules are

\[
\text{CDDs \((3.3)\) and \((3.4)\)} \to B_{n+\ell-1} \oplus B_{\ell-1} \tag{6.8}
\]

\(^2\text{We have used the equality } 2N = 2n + 2\ell - 1 + 2\ell - 1, \text{ resulting from the decomposition of the fundamental.}\)
other CDDs \[
\begin{align*}
D_p & \rightarrow D_p \\
D_{p/2} & \rightarrow A_p
\end{align*}
\] (6.9)

For the three \( D_N \)-CDDs \((u = 0, \ u = 1, \ u \geq 2)\) associated to one \( B_N \)-CDD, the decomposition of the fundamental representation has the same form, so that we will always (up to the above exceptions) get the same type of subalgebra \( \mathcal{K} \). However, because the tails have different grades the \( \mathcal{G}_0 \) subalgebras are different. In the case where \( u = 1 \), we get two \( A_1 \) subalgebras from the tail, instead of the \( D_{u+1} \) subalgebra arising when \( u \geq 2 \). But for \( u = 1 \), the algebra \( D_{u+1} = D_2 \) is isomorphic to \( 2A_1 \), so that we include \( u = 1 \) in the general treatment \( u \geq 2 \). Note that once more, it is the case \( u = 0 \) that appears much different from the other ones.

When \( u = 0 \), we call \( i_0 \) the only index such that \( p_{i_0} = 0 \) and \( p_{i_0+1} = 1 \) (or \( q_{i_0} = 0 \) and \( q_{i_0+1} = 1 \)): it is the only index such that \( r_{i_0} = 1 \neq 0 \) (or \( s_{i_0} = 1 \neq 0 \)). In the case of integral grading we have \( i_0 = \ell - 1 \).

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
CDD & \( \mathcal{K} \) in \( \mathcal{W}(D_N, \mathcal{K}) \) & KM subalg. in \( \mathcal{W} \) (up to \( U(1) \) factors) \\
\hline
\( (5.1) \) & \( D_{n+\ell-1} \oplus p_1 D_{\ell-1} \oplus \ell A_\ell \sum_{i=1}^{\ell-2} r_i D_{\ell-i-1} \) & \( D_{u+1} \oplus \ell A_{p_1} \) \\
\( (5.2) \) & \( B_{N-\ell+1} \oplus B_{\ell-1} \) & \((\ell-1) A_{p_{\ell-1}} \) with \( p_{\ell-1} = \begin{cases} 1 \\
0 \end{cases} \) \\
\( (5.3) \) & \( (k_1 + 1) A_{2\ell+1} \oplus \ell A_{2\ell-2} \sum_{i=1}^{\ell+1} A_{k_1} \) & \( \oplus \ell A_{2k_1+1} \) \\
\( (5.4) \) & \( (k_1 + 1) A_1 \) & \( \ell A_1 \) \\
\hline
\end{tabular}
\end{table}
| CDD | $\mathcal{K}$ in $\mathcal{W}(D_N, \mathcal{K})$ | KM subalg. in $\mathcal{W}$ (up to $U(1)$ factors) |
|-----|---------------------------------|---------------------------------|
| (5.7$^+$) | $(k_1 + 1)A_{2\ell-1} \oplus (q_1 + 1)D_{\ell} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $D_{u+1} \oplus A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.8$^+$) | $(k_1 + 1)A_{2\ell-1} \oplus (q_1 + 1)D_{\ell-1} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $D_{u+1} \oplus A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.9) | $(k_1 + 1)A_{2\ell-1} \oplus D_{\ell-1} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.10) | $(p_1 + 1)D_{\ell} \oplus (k_1 + 1)A_{2\ell-3} \oplus \oplus_{i=1}^{\ell-2} r_i D_{\ell-i-1} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1}$ | $D_{u+1} \oplus \oplus_{i=1}^{\ell-1} [A_{p_1} \oplus A_{2k_{r+1}}]$ |
| (5.12) | $D_{\ell} \oplus (k_1 + 1)A_{2\ell-3} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.13) | $(k_1 + 1)A_{2\ell-1} \oplus D_{n+\ell} \oplus q_1 D_{\ell} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $D_{u+1} \oplus A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.14) | $(k_1 + 1)A_{2\ell-1} \oplus D_{n+\ell} \oplus (k_1 + 1)A_{2\ell-3} \oplus \oplus_{i=1}^{\ell-1} l_i A_{2\ell-2i-1} \oplus \oplus_{i=1}^{\ell-2} s_i D_{\ell-i-1}$ | $A_{2k_{r+1}} \oplus \oplus_{i=1}^{\ell-1} [A_{2k_{r+1}} \oplus A_{q_i}]$ |
| (5.16) | $D_{n+\ell'+e'-1} \oplus \oplus_{i=1}^{\ell'+e'-1} r_i D_{\ell+e'-i-1} \oplus p_1 D_{\ell+e'-1} \oplus \oplus_{i=1}^{\ell'} l_i A_{2\ell'+2e'-2i-3}$ | $D_{u+1} \oplus \oplus_{i=1}^{\ell'+e'-1} [A_{p_1} \oplus \oplus_{i=1}^{\ell'} A_{2k_{r+1}}]$ |
| (5.18) | $D_{n+\ell'+e'-1} \oplus D_{\ell+e'-i_0-1} \oplus \oplus_{i=1}^{\ell'} l_i A_{2\ell'+2e'-2i-3}$ | $\oplus_{i=1}^{\ell'+e'-1} [A_{p_1} \oplus \oplus_{i=1}^{\ell'} A_{2k_{r+1}}]$ |
| (5.19) | $(k_1 + 1)A_{2\ell+2e'-1} \oplus \oplus_{i=1}^{\ell} l_i A_{2\ell+2e'-2i-1}$ | $D_{u+1} \oplus \oplus_{i=1}^{\ell} [A_{q_i} \oplus \oplus_{i=1}^{\ell'} A_{2k_{r+1}}]$ |
| (5.20) | $(k_1 + 1)A_{2\ell+2e'-1} \oplus D_{\ell-1} \oplus D_{\ell-i_0-1} \oplus \oplus_{i=1}^{\ell'+e'-1} l_i A_{2\ell+2e'-2i-1}$ | $\oplus_{i=1}^{\ell'+e'-1} [A_{q_i} \oplus \oplus_{i=1}^{\ell'} A_{2k_{r+1}}]$ |
| (5.21) | $(k_1 + 1)A_{2\ell+2e'-1} \oplus D_{\ell-1} \oplus D_{\ell-i_0-1} \oplus \oplus_{i=1}^{\ell'+e'-1} l_i A_{2\ell+2e'-2i-1}$ | $\oplus_{i=1}^{\ell'+e'-1} [A_{q_i} \oplus \oplus_{i=1}^{\ell'} A_{2k_{r+1}}]$ |
6.2 Duality between $B_N$ and $C_N$

It has been proven in [3] that the $W(B_N, B_N)$ and $W(C_N, C_N)$ exhibit a duality, in the sense that the Abelian Toda model constructed on $C_N$ is obtained from the Abelian Toda model built on $B_N$ by replacing the roots system by its dual, and the coupling constant $\beta$ by $\frac{1}{\beta}$.

Using the CDDs, it is possible to see which of the (other) $W(B_N, K)$ and $W(C_N, \bar{K})$ algebras can be dual. Indeed, using the duality between $B_N$ and $C_N$ DDs that consists in exchanging the colors black and white in the nodes, one obtains the following candidates for the "B-C duality"

\[ (6.12) \]
\[ (6.13) \]
\[ (6.14) \]
\[ (6.15) \]
\[ (6.16) \]
\[ (6.17) \]
\[ (6.18) \]

Notice that one has to pay attention, when comparing $B_N$ and $C_N$ CDDs, to the difference between $\ell' = \ell$ and $\ell' = \ell - 1$. For instance, (4.7) seems very similar to (4.13), but has one piece of length $q$ more than the other: to get (6.17), one has to take (4.7) with $q_i$ odd $(i \neq \ell)$ on the one hand, but (4.19) with $\ell = 0$, $p_i$ odd $(i \geq 2)$ and $p_1 = 0$ on the other hand.

Looking at the fundamental representation, it is easy to see that in most of the cases we have the same decomposition for the $B_N$ and $C_N$ algebras (except one $D_0$ that allows to go from $2N$ to $2N+1$). This means that the spin contents of the corresponding $W$ algebras will be different, since it is obtained by a symmetric (antisymmetric) product of the fundamental by its contragredient for $C_N$ ($B_N$). The exceptions that have different fundamental decomposition for $B_N$ and $C_N$ are (5.13) and (5.14). The principal CDD (5.13) lead to the well-known algebras $W(B_N, B_N)$ and $W(C_N, C_N)$, and a direct calculation of the adjoint decomposition for (5.17) shows that the spin content is still different.

Thus, the only $W$ algebras related by a "B-C duality" are $W(B_N, B_N)$ and $W(C_N, C_N)$. It is nevertheless possible that the $W$ algebras pointed out in this paragraph are related through another relation weaker than this duality.
7 Conclusion

In this paper, we have presented a classification of CDDs for A,B,C,D algebras, and connected them to the classification of $\mathcal{W}(\mathcal{G},\mathcal{K})$ algebras. The CDD may be viewed as a kind of Dynkin diagram for these algebras, so that one can wonder whether it is still true for a more general class of $W$ algebras. For such a purpose, one can think of enlarging the definition of CDDs by allowing the grades to take any values and try to connect $W$ algebras with ”generalized” CDDs in a different way than the $Sl(2)$ classification used here. If it is possible, this will imply that a ”generalised” CDD encode all the relevant informations concerning the $W$ algebras (as it is the case for CDDs with $\mathcal{W}(\mathcal{G},\mathcal{K})$ algebras). Note however that a generalised CDD will always be related to a Cartan generator of $\mathcal{G}$ (but not of a $Sl(2)\mathcal{G}$-subalgebra), through the grades of the simple roots.

Finally, let us remark that a similar treatment for super-$W$ algebras, with some ”super-CDDs” classifying $OSp(1|2)$ sub-superalgebras, may perhaps be done. One has first to know if the $OSp(1|2)$ embeddings can be classify by such diagrams: to our knowledge, this has not yet been studied, but the notion of super-defining vectors has already being introduced in [2].

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