A COMMENT ON THE LOW-DIMENSIONAL BUSEMANN-PETTY PROBLEM

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Abstract. The generalized Busemann-Petty problem asks whether centrally-symmetric convex bodies having larger volume of all m-dimensional sections necessarily have larger volume. When m > 3 this is known to be false, but the cases m = 2, 3 are still open. In those cases, it is shown that when the smaller body’s radial function is a n − m-th root of the radial function of a convex body, the answer to the generalized Busemann-Petty problem is positive (for any larger star-body). Several immediate corollaries of this observation are also discussed.

1. Introduction

Let Vol(L) denote the Lebesgue measure of a set L ⊂ ℜ^n in its affine hull, and let G(n, k) denote the Grassmann manifold of k dimensional subspaces of ℜ^n. Let D_n denote the Euclidean unit ball, and S^{n−1} the Euclidean sphere. All of the bodies considered in this note will be assumed to be centrally symmetric star-bodies, defined by a continuous radial function ρ_K(θ) = max{r ≥ 0 | rθ ∈ K} for θ ∈ S^{n−1} and a star-body K.

The Busemann-Petty problem, first posed in [BP56], asks whether two centrally-symmetric convex bodies K and L in ℜ^n satisfying:

(1.1) Vol(K ∩ H) ≤ Vol(L ∩ H) ∀H ∈ G(n, n − 1)

necessarily satisfy Vol(K) ≤ Vol(L). For a long time this was believed to be true (this is certainly true for n = 2), until a first counterexample was given in [LR75] for a large value of n. In the same year, the notion of an intersection-body was first introduced by Lutwak in [Lut75] (see also [Lut88] and Section 2 for definitions) in connection to the Busemann-Petty problem. It was shown in [Lut88] (and refined in

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that the answer to the Busemann-Petty problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are intersection bodies. Subsequently, it was shown in a series of results ([LR75], [Bal88], [Bon91], [Gia90], [Pap92], [Gar94a], [Gar94c], [Kol98b], [Zha99], [GKS99]), that this is true for $n \leq 4$, but false for $n \geq 5$.

In [Zha96], Zhang considered a natural generalization of the Busemann-Petty problem, which asks whether two centrally-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfying:

\begin{equation}
\text{Vol}(K \cap H) \leq \text{Vol}(L \cap H) \quad \forall H \in G(n, n - k)
\end{equation}

necessarily satisfy $\text{Vol}(K) \leq \text{Vol}(L)$, where $k$ is some integer between 1 and $n - 1$. Zhang showed that the generalized $k$-codimensional Busemann-Petty problem is also naturally associated to another class of bodies, which will be referred to as $k$-Busemann-Petty bodies (note that these bodies are referred to as $n - k$-intersection bodies in [Zha96] and generalized $k$-intersection bodies in [Kol00]), and that the generalized $k$-codimensional problem is equivalent to whether all convex bodies in $\mathbb{R}^n$ are $k$-Busemann-Petty bodies. Analogously to the original problem, it was shown in [Zha96] that if $K$ and $L$ are two centrally-symmetric star-bodies (not necessarily convex) satisfying (1.2), and if $K$ is a $k$-Busemann-Petty body, then $\text{Vol}(K) \leq \text{Vol}(L)$.

It was shown in [BZ98], and later in [Kol00], that the answer to the generalized $k$-codimensional problem is negative for $k < n - 3$, but the cases $k = n - 3$ and $k = n - 2$ still remain open (the case $k = n - 1$ is obviously true). A partial answer to the case $k = n - 2$ was given in [BZ98], where it was shown that when $L$ is a Euclidean ball and $K$ is convex and sufficiently close to $L$, the answer is positive. Our main observation in this note concerns the cases $k = n - 2, n - 3$ and reads as follows:

**Theorem 1.1.** Let $K$ denote a centrally-symmetric convex body in $\mathbb{R}^n$. For $a = 2, 3$, let $K_a$ be the star-body defined by $\rho_{K_a} = \rho_K^{1/(n-a)}$. Then $K_a$ is a $(n - a)$-Busemann-Petty body, implying a positive answer to the $(n - a)$-codimensional Busemann-Petty problem (1.2) for the pair $K_a, L$ for any star-body $L$.

The case $a = 1$ is also true, but follows trivially since it is easy to see (e.g. [Mil05]) that any star-body is an $n - 1$-Busemann-Petty body. The case $a = 2$ follows from $a = 3$ by a general result from [Mil05], stating that if $K$ is a $k$-Busemann-Petty body and $L$ is given by $\rho_L = \rho_K^{k/l}$ for $1 \leq k < l \leq n - 1$, then $L$ is a $l$-Busemann-Petty body.

Theorem 1.1 has several interesting consequences. The first one is the following complementary result to the one aforementioned from
Roughly speaking, it states that any small enough perturbation $K$ of the Euclidean ball, for which we have control over the second derivatives of $\rho_K$, satisfies the low-dimensional generalized Busemann-Petty problem \textcolor{red}{(1.2)} with any star-body $L$.

**Corollary 1.2.** For any $n$, there exists a function $\gamma : [0, \infty) \to (0, 1)$, such that the following holds: let $\varphi$ denote a twice continuously differentiable function on $S^{n-1}$ such that:

$$\max_{\theta \in S^{n-1}} |\varphi(\theta)| \leq 1, \max_{\theta \in S^{n-1}} |\varphi_i(\theta)| \leq M, \max_{\theta \in S^{n-1}} |\varphi_{i,j}(\theta)| \leq M,$$

for every $i, j = 1, \ldots, n - 1$, where $\varphi_i$ and $\varphi_{i,j}$ denote the first and second partial derivatives of $\varphi$ (w.r.t. any local coordinate system of $S^{n-1}$), respectively. Then the star-body $K^\varepsilon$ defined by $\rho_{K^\varepsilon} = 1 + \varepsilon \varphi$ for any $|\varepsilon| < \gamma(M)$ is a $(n-a)$-Busemann-Petty body for $a = 2, 3$, implying a positive answer to the $(n-a)$-codimensional Busemann-Petty problem \textcolor{red}{(1.2)} for $K^\varepsilon$ and any star-body $L$.

Note that the definition of $K_a$ in Theorem \textcolor{red}{1.1} is highly non-linear with respect to $K$. Since the class of $k$-Busemann-Petty bodies is closed under certain natural operations (see \textcolor{red}{Mil05} for the latest known results), we can take advantage of this fact to strengthen the result of Theorem \textcolor{red}{1.1}. For instance, it is well known (e.g., \textcolor{red}{GZ99}, \textcolor{red}{Mil05}) that the class of $k$-Busemann-Petty bodies is closed under taking $k$-radial sums. The $k$-radial sum of two star-bodies $L_1, L_2$ is defined as the star-body $L$ satisfying $\rho_k^L = \rho_k^L_1 + \rho_k^L_2$. When $k = 1$ this operation will simply be referred to as radial sum. The space of star-bodies in $\mathbb{R}^n$ is endowed with the natural radial metric $d_r$, defined as $d_r(L_1, L_2) = \max_{\theta \in S^{n-1}} |\rho_{L_1}(\theta) - \rho_{L_2}(\theta)|$. We will denote by $\mathcal{RC}^n$ the closure in the radial metric of the class of all star-bodies in $\mathbb{R}^n$ which are finite radial sums of centrally-symmetric convex bodies. It should then be clear that:

**Corollary 1.3.** Theorem \textcolor{red}{1.1} holds for any $K \in \mathcal{RC}^n$.

Our last remark in this note is again an immediate consequence of Theorem \textcolor{red}{1.1} and the following characterization of $k$-Busemann-Petty bodies due to Grinberg and Zhang (\textcolor{red}{GZ99}), which generalizes the characterization of intersection-bodies (the case $k = 1$) given by Goodey and Weil (\textcolor{red}{GW95}):

**Theorem (Grinberg and Zhang).** A star-body $K$ is a $k$-Busemann-Petty body if and only if it is the limit of $\{K_i\}$ in the radial metric $d_\star$, where each $K_i$ is a finite $k$-radial sums of ellipsoids $\{E_i\}$:

$$\rho_{K_i}^k = \rho_{E_i}^k + \ldots + \rho_{E_i}^k,$$
Applying Grinberg and Zhang’s Theorem to the bodies \(K_a\) from Theorem 1.1, we immediately have:

**Corollary 1.4.** Let \(K\) denote a centrally-symmetric convex body in \(\mathbb{R}^n\). Then for \(a = 2, 3\), \(K\) is the limit in the radial metric \(d_r\) of star-bodies \(K_i\) having the form:

\[
\rho_{K_i} = \rho_{E_1}^{n-a} + \ldots + \rho_{E_{m_i}}^{n-a},
\]

where \(\{E_i\}\) are ellipsoids.

### 2. Definitions and Notations

A star body \(K\) is said to be an intersection body of a star body \(L\), if \(\rho_K(\theta) = \text{Vol}(L \cap \theta^\perp)\) for every \(\theta \in S^{n-1}\). \(K\) is said to be an intersection body, if it is the limit in the radial metric \(d_r\) of intersection bodies \(\{K_i\}\) of star bodies \(\{L_i\}\), where \(d_r(K_1, K_2) = \sup_{\theta \in S^{n-1}} |\rho_{K_1}(\theta) - \rho_{K_2}(\theta)|\).

This is equivalent (e.g. [Lut88], [Gar94a]) to \(\rho_K = R^*(d\mu)\), where \(\mu\) is a non-negative Borel measure on \(S^{n-1}\), \(R^*\) is the dual transform (as in (2.1)) to the Spherical Radon Transform \(R : C(S^{n-1}) \to C(S^{n-1})\), which is defined for \(f \in C(S^{n-1})\) as:

\[
R(f)(\theta) = \int_{S^{n-1} \cap \theta^\perp} f(\xi) d\sigma_{n-1}(\xi),
\]

where \(\sigma_{n-1}\) the Haar probability measure on \(S^{n-2}\) (and we have identified \(S^{n-2}\) with \(S^{n-1} \cap \theta^\perp\)).

Before defining the class of \(k\)-Busemann-Petty bodies we shall need to introduce the \(m\)-dimensional Spherical Radon Transform, acting on spaces of continuous functions as follows:

\[
R_m : C(S^{n-1}) \to C(G(n,m))
\]

\[
R_m(f)(E) = \int_{S^{n-1} \cap E} f(\theta) d\sigma_m(\theta),
\]

where \(\sigma_m\) is the Haar probability measure on \(S^{m-1}\) (and we have identified \(S^{m-1}\) with \(S^{n-1} \cap E\)). The dual transform is defined on spaces of signed Borel measures \(\mathcal{M}\) by:

\[
R^*_m : \mathcal{M}(G(n,m)) \to \mathcal{M}(S^{n-1})
\]

\[
\int_{S^{n-1}} fR^*_m(d\mu) = \int_{G(n,m)} R_m(f)d\mu \quad \forall f \in C(S^{n-1}),
\]

and for a measure \(\mu\) with continuous density \(g\), the transform may be explicitly written in terms of \(g\) (see [Zha96]):

\[
R^*_mg(\theta) = \int_{\theta \in E \in G(n,m)} g(E)d\nu_m(E),
\]
where $\nu_m$ is the Haar probability measure on $G(n-1, m-1)$.

We shall say that a body $K$ is a $k$-Busemann-Petty body if $\rho_k^K = R_{n-k}^*(d\mu)$ as measures in $\mathcal{M}(S^{n-1})$, where $\mu$ is a non-negative Borel measure on $G(n, n-k)$. We shall denote the class of such bodies by $\mathcal{BP}_k^n$. Choosing $k = 1$, for which $G(n, n-1)$ is isometric to $S^{n-1}/\mathbb{Z}_2$ by mapping $H$ to $S^{n-1} \cap H^\perp$, and noticing that $R$ is equivalent to $R_{n-1}$ under this map, we see that $\mathcal{BP}_1^n$ is exactly the class of intersection bodies.

We will also require, although indirectly, several notions regarding Fourier transforms of homogeneous distributions. We denote by $\mathcal{S}(\mathbb{R}^n)$ the space of rapidly decreasing infinitely differentiable test functions in $\mathbb{R}^n$, and by $\mathcal{S}'(\mathbb{R}^n)$ the space of distributions over $\mathcal{S}(\mathbb{R}^n)$. The Fourier Transform $\hat{f}$ of a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function $\phi$, where $\hat{\phi}(y) = \int \phi(x) \exp(-i(x, y))dx$. A distribution $f$ is called homogeneous of degree $p \in \mathbb{R}$ if $\langle f, \phi(\cdot/t) \rangle = |t|^n p \langle f, \phi \rangle$ for every $t > 0$, and it is called even if the same is true for $t = -1$. An even distribution $f$ always satisfies $(\hat{f})^\wedge = (2\pi)^n \hat{f}$. The Fourier Transform of an even homogeneous distribution of degree $p$ is an even homogeneous distribution of degree $-n-p$.

We will denote the space of continuous functions on the sphere by $C(S^{n-1})$. The spaces of even continuous and infinitely smooth functions will be denoted $C_e(S^{n-1})$ and $C^\infty(S^{n-1})$, respectively.

For a star-body $K$ (not necessarily convex), we define its Minkowski functional as $\|x\|_K = \min \{ t \geq 0 \mid x/t \in K \}$. When $K$ is a centrally-symmetric convex body, this of course coincides with the natural norm associated with it. Obviously $\rho_K(\theta) = \|\theta\|_K^{-1}$ for $\theta \in S^{n-1}$.

3. PROOFS OF THE STATEMENTS

Before we begin, we shall need to recall several known facts about the Spherical Radon Transform $R$, and its connection to the Fourier transform of homogeneous distributions. It is well known (e.g. [Gro96 Chapter 3]) that $R : C_e(S^{n-1}) \to C_e(S^{n-1})$ is an injective operator, and that it is onto a dense set in $C_e(S^{n-1})$ which contains $C^\infty_e(S^{n-1})$. The connection with Fourier transforms of homogeneous distributions was demonstrated by Koldobsky, who showed (e.g. [Kol98a]) the following:

**Lemma 3.1.** Let $L$ denote a star-body in $\mathbb{R}^n$. Then for all $\theta \in S^{n-1}$:

$$\langle \|\cdot\|_{L}^{-n+1} \rangle^\wedge (\theta) = \pi (n-1) \text{Vol}(D_{n-1}) R(\|\cdot\|_{L}^{-n+1})(\theta).$$

In particular $(\|\cdot\|_{L}^{-n+1})^\wedge$ is continuous, and of course homogeneous of degree $-1$. Hence, if we denote $\rho_K(\theta) = \|\theta\|_K^{-1} = (\|\cdot\|_{L}^{-n+1})^\wedge (\theta)$ for
\( \theta \in S^{n-1} \) and use \((\| \cdot \|_K^{-1})^\wedge(\theta) = (2\pi)^n \| \theta \|_L^{-n+1}\), we immediately get the following inversion formula for the Spherical Radon transform:

**Lemma 3.2.** Let \( K \) denote a star-body in \( \mathbb{R}^n \) such that \( \rho_K \) is in the range of the Spherical Radon Transform. Then for all \( \theta \in S^{n-1} \):

\[
R^{-1}(\rho_K)(\theta) = \frac{\pi(n-1) Vol(D_{n-1})}{(2\pi)^n} (\| \cdot \|_K^{-1})^\wedge(\theta).
\]

Koldobsky also discovered the following property of the Fourier transform of a norm of a convex body ([Kol00, Corollary 2]):

**Lemma 3.3.** Let \( K \) be an infinitely smooth centrally-symmetric convex body in \( \mathbb{R}^n \). Then for every \( E \in G(n,k) \):

\[
\int_{S^{n-1} \cap E} (\| \cdot \|_K^{-n+k+2})^\wedge(\theta)d\theta \geq 0.
\]

Since \( C^\infty_c(S^{n-1}) \) is in the range of the Spherical Radon Transform, applying Lemma 3.3 with \( k = n - 3 \) and using Lemma 3.2, we have:

**Proposition 3.4.** Let \( K \) be an infinitely smooth centrally-symmetric convex body in \( \mathbb{R}^n \). Then for every \( E \in G(n,n-3) \):

\[
\int_{S^{n-1} \cap E} R^{-1}(\rho_K)(\theta)d\theta \geq 0.
\]

We are now ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** First, assume that \( K \) is infinitely smooth and fix \( \theta \in S^{n-1} \). Denote by \( H_\theta \in G(n,n-1) \) the hyperplane \( \theta^\perp \), and let \( \sigma_{H_\theta} \) denote the Haar probability measure on \( S^{n-1} \cap H_\theta \). Let \( \eta_{H_\theta} \) denote the Haar probability measure on the homogeneous space \( G^{H_\theta}(n,n-3) := \{ E \in G(n,n-3) | E \subset H_\theta \} \), and let \( \sigma_E \) denote the Haar probability measure on \( S^{n-1} \cap E \) for \( E \in G(n,n-3) \). Then:

\[
\rho_K(\theta) = R(R^{-1}(\rho_K))(\theta) = \int_{S^{n-1} \cap H_\theta} R^{-1}(\rho_K)(\xi)d\sigma_{H_\theta}(\xi)
\]

\[
= \int_{E \in G^{H_\theta}(n,n-3)} \int_{S^{n-1} \cap E} R^{-1}(\rho_K)(\xi)d\sigma_E(\xi)d\eta_{H_\theta}(E).
\]

The last transition is explained by the fact that the measure \( d\sigma_E(\xi)d\eta_{H_\theta}(E) \) is invariant under orthogonal transformations preserving \( H_\theta \), so by the uniqueness of the Haar probability measure, it must coincide with \( d\sigma_{H_\theta}(\xi) \). Denoting:

\[
g(F) = \int_{S^{n-1} \cap F^\perp} R^{-1}(\rho_K)(\xi)d\sigma_E(\xi)
\]
for $F \in G(n, 3)$, we see by Proposition 3.1 that $g \geq 0$. Plugging the definition of $g$ in (3.1), we have:

$$\rho_K(\theta) = \int_{E \in G^{H_\theta(n,n-3)}} g(E^\perp) d\eta_{H_\theta}(E) = \int_{F \in G_\theta(n,3)} g(F) d\nu_\theta(F),$$

where $\nu_\theta$ is the Haar probability measure on the homogeneous space $G_\theta(n, 3) := \{ F \in G(n, 3) | \theta \in F \}$ and the transition is justified as above. By (2.2), we conclude that $\rho_K = R_3^*(g)$ with $g \geq 0$, implying that the body $K_3$ satisfying $\rho_{K_3}^{n-3} = \rho_K$ is in $\mathcal{BP}_{n-3}$.

As mentioned in the Introduction, the case $a = 2$ follows from $a = 3$ by a general result from [Mil05], but for completeness we reproduce the easy argument. Using double-integration as before:

$$\rho_K(\theta) = \int_{F \in G_\theta(n,3)} g(F) d\nu_\theta(F) = \int_{J \in G_\theta(n,2)} \int_{F \in G_J(n,3)} g(F) d\nu_J(F) d\mu_\theta(J),$$

where $\mu_\theta$ and $\nu_J$ are the Haar probability measures on the homogeneous spaces $G_\theta(n, 2) := \{ J \in G(n, 2) | \theta \in J \}$ and $G_J(n, 3) := \{ F \in G(n, 3) | J \subset F \}$, respectively. Denoting:

$$h(J) = \int_{F \in G_J(n,3)} g(F) d\nu_J(F),$$

we see that $h \geq 0$ and $\rho_K = R_2^*(h)$, implying that the body $K_2$ satisfying $\rho_{K_2}^{n-2} = \rho_K$ is in $\mathcal{BP}_{n-2}$.

When $K$ is a general convex body, the result follows by approximation. It is well known (e.g. [Sch93 Theorem 3.3.1]) that any centrally-symmetric convex body $K$ may be approximated (for instance in the radial metric) by a series of infinitely smooth centrally-symmetric convex bodies $\{ K^i \}$. Denoting by $K_a^i$ the star-bodies satisfying $\rho_{K_a^i} = \rho_{K_a}^{1/(n-a)}$ for $a = 2, 3$, we have seen that $K_a^i \in \mathcal{BP}^{n-2}_{n-a}$. Obviously the series $\{ K_a^i \}$ tends to $K_a$ in the radial metric, and since $\mathcal{BP}^{n-2}_{n-a}$ is closed under taking radial limit (see [Mil05]), the result follows.

We now turn to close a few loose ends in the proof of Corollary 1.3. Since $\mathcal{BP}_k^n$ is closed under $k$-radial sums, it is immediate that if $K^1$ and $K^2$ are two convex bodies, $L$ is their radial sum, and $\rho_{T_a} = \rho_T^{1/(n-a)}$ for $T = K_1, K_2, L$, then:

$$\rho_L^{n-a} = \rho_T = \rho_{K_1} + \rho_{K_2} = \rho_{K_a}^{n-a} + \rho_{K_a}^{n-a},$$

and therefore $L_a \in \mathcal{BP}^{n-a}_{n-a}$. This argument of course extends to any finite radial sum of convex bodies, and since $\mathcal{BP}_k^n$ is closed under taking limit in the radial metric, the argument extends to the entire class $\mathcal{RC}^n$ defined in the Introduction.
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It remains to prove Corollary 1.2.

Proof of Corollary 1.2. By Theorem 1.1, it is enough to show that for a small enough $|\varepsilon|$ (which depends on $n$ and $M$), the star-bodies $L_a^\varepsilon$ defined by $\rho_{L_a^\varepsilon} = \rho_{K_a^\varepsilon}^{n-a}$ are in fact convex. Since $\rho_{L_a^\varepsilon} = (1 + \varepsilon \varphi)^{n-a}$, it is clear that for every $\theta \in S^{n-1}$:

$$|\rho_{L_a^\varepsilon}(\theta)| \leq f_0(\varepsilon, n), \quad |(\rho_{L_a^\varepsilon})_i(\theta)| \leq f_1(\varepsilon, n, M), \quad |(\rho_{L_a^\varepsilon})_{ij}(\theta)| \leq f_2(\varepsilon, n, M),$$

for every $i, j = 1, \ldots, n-1$, where $f_0$ tends to 1 and $f_1, f_2$ tend to 0, as $\varepsilon \to 0$. It should be intuitively clear that the convexity of $L_a^\varepsilon$ depends only on the behaviour of the derivatives of order 0,1 and 2 of $\rho_{L_a^\varepsilon}$, and since we have uniform convergence of these derivatives to those of the Euclidean ball as $\varepsilon$ tends to 0, $L_a^\varepsilon$ is convex for small enough $\varepsilon$. To make this argument formal, we follow [Gar94b], and use a formula for the Gaussian curvature of a star-body $L$ whose radial function $\rho_L$ is twice continuously differentiable, which was explicitly calculated in [Oli84, 2.5]. In particular, it follows that $M_L(\theta)$, the Gaussian curvature of $\partial L$ (the hypersurface given by the boundary of $L$) at $\rho_L(\theta)\theta$, is a continuous function of the derivatives of order 0,1 and 2 of $\rho_L$ at the point $\theta$. Since the Gaussian curvature of the boundary of the Euclidean ball is a constant 1, it follows that for small enough $\varepsilon$, the boundary of $L_a^\varepsilon$ has everywhere positive Gaussian curvature. By a standard result in differential geometry (e.g. [KN69, p. 41]), this implies that $L_a^\varepsilon$ is convex. This concludes the proof.

\[\square\]

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