Fast Estimation of Orbital Parameters in Milky Way-like Potentials

J. Ted Mackereth¹ and Jo Bovy²,³

¹ Astrophysics Research Institute, Liverpool John Moores University, 146 Brownlow Hill, Liverpool, L3 5RF, UK; J.E.Mackereth@2011.ljmu.ac.uk
² Department of Astronomy & Astrophysics, University of Toronto, 50 St. George Street, Toronto, ON M5S 3H4, Canada

Received 2018 March 28; accepted 2018 August 24; published 2018 October 5

Abstract

Orbital parameters, such as eccentricity and maximum vertical excursion, of stars in the Milky Way are an important tool for understanding its dynamics and evolution, but calculation of such parameters usually relies on computationally expensive numerical orbit integration. We present and test a fast method for estimating these parameters using an application of the Stäckel fudge, used previously for the estimation of action-angle variables. We show that the method is highly accurate, to a level of <1% in eccentricity, over a large range of relevant orbits and in different Milky Way-like potentials, and demonstrate its validity by estimating the eccentricity distribution of the RAVE-TGAS data set and comparing it with that from orbit integration. Using this method, the orbital characteristics of the ∼7 million Gaia DR2 stars with radial velocity measurements are computed with Monte Carlo sampled errors in ∼116 hours of parallelized cpu time, at a speed that we estimate to be ∼3 to 4 orders of magnitude faster than using numerical orbit integration. We demonstrate using this catalog that Gaia DR2 samples a large range of orbits in the solar vicinity, down to those with \( r_{ap} \lesssim 2.5 \) kpc, and out to \( r_{peri} \gtrsim 13 \) kpc. We also show that many of the features present in orbital parameter space have a low mean \( z_{\text{max}} \), suggesting that they likely result from disk dynamical effects.

Key words: galaxies: kinematics and dynamics – stars: kinematics and dynamics – methods: data analysis – methods: numerical

Online material: color figures

1. Introduction

The orbit integration of stars from their observed 6D phase-space coordinates is an important method for better understanding the kinematic and dynamical properties of galaxies and their stellar populations, but can be time consuming computationally, especially when large sample sizes are involved. While various methods have been devised for estimating the angle-action variables for given sets of phase-space coordinates (see, e.g., Sanders & Binney 2016 and references therein), more conventional orbital parameters such as eccentricity for increasingly large samples in the era of Gaia may still offer important insights into the nature of the Milky Way.

As an example, orbital eccentricities have already been effective in trying to understand the origins of the thicker disk component in the Galaxy. It was suggested by Sales et al. (2009) that the eccentricity distribution as a function of height above the midplane could be constraining to thick disk formation models. Calculations of the orbital eccentricity from orbit integration of 31,535 stars from SDSS DR7 were later employed to test this idea, and place constraints on the origin of the thicker disk components (Dierickx et al. 2010). The apocenter and pericenter radii of orbits (as well as their angular momenta) have also been employed as a means of finding substructure in the Milky Way disk (the APL space, Helmi et al. 2006).

While actions are more fundamental in labeling orbits and perhaps more useful for dynamical modeling (being distinguished from other labels by their adiabatic invariance), we argue that the regular orbit parameters: maximum vertical excursion \( z_{\text{max}} \), pericenter, and apocenter radius \( r_{\text{peri}}, r_{ap} \), and their transformation into orbital eccentricity, \( e \), can still be useful for kinematics studies and are more naturally related to the orbital configuration of the Galaxy and its relation to its formation history. Used in tandem with computed orbital actions, these orbit labels can aid in the understanding and disentangling of, for example, substructures that are discovered in action space, as the parameters are expressed as physical distances. In this paper, we present a method for the rapid estimation of these orbital parameters analytically, without recourse to numerical orbit integration. This method is a simplified application of the Stäckel fudge presented by Binney (2012) to compute actions and angles for axisymmetric potentials.

2. Method

Many galactic mass distributions, and in particular that of the Milky Way, are well approximated by a Stäckel potential (e.g.,
These potentials are defined in terms of prolate confocal coordinates (see Binney & Tremaine 2008), with
\[ R = \Delta \sinh u \sin v, \]  
\[ z = \Delta \cosh u \cos v, \]  
where \( \Delta \) is a parameter that specifies the focal point of the coordinate system, placed at \( R = 0, z = \pm \Delta \). The momenta in these coordinates are then given by
\[ p_u = \Delta (p_k \cosh u \sin v + p_r \sinh u \cos v) \]  
\[ p_v = \Delta (p_k \sinh u \cos v - p_r \cosh u \sin v). \]  
In these coordinates, an oblate, axisymmetric Stäckel potential is a potential that can be written in terms of two functions \( U(u) \) and \( V(v) \) of one variable as
\[ \Phi_S(u, v) = \frac{U(u) - V(v)}{\sinh^2 u + \sin^2 v}. \]  
For a potential of this form, the Hamilton-Jacobi equation can be solved using the separation-of-variables method, the motions in \( u \) and \( v \) decouple, and we have that
\[ \frac{p_u^2}{2\Delta^2} = E \sinh^2 u - I_3 - U(u) - \frac{L_z^2}{2\Delta^2 \sinh^2 u}, \]  
\[ \frac{p_v^2}{2\Delta^2} = E \sin^2 v - I_3 - V(v) - \frac{L_z^2}{2\Delta^2 \sin^2 v}, \]  
where \( E \) and \( I_3 \) denote the energy and vertical component of the angular momentum of the orbit, respectively, and \( I_3 \) is a constant of separation—the third integral. By numerically solving the equations for the turning points, \( p_u(u) = 0 \) and \( p_v(v) = 0 \), one can determine the spatial boundary of the orbit, which is rectangular in \( (u, v) \): \( u_{\text{min}}/\text{max} \) in \( u \) and \( v_{\text{min}}/\text{max} \) in \( v \). For a Stäckel potential that is symmetric around the midplane, \( v_{\text{min}} = \pi - v_{\text{max}} \) and we will assume that this holds hereafter.

For galactic potentials, \( \Phi \), that are close to a Stäckel potential but not exactly equal to one, Equation (4) only approximately holds. Following Binney (2012), for such potentials we can define functions \( U(u) \) and \( V(v) \) as
\[ U(u) \equiv \cosh^2 u \Phi(u, \pi/2), \]  
\[ V(v) \equiv \cosh^2 u_0 \Phi(u_0, \pi/2) - (\sinh^2 u_0 + \sin^2 v)\Phi(u_0, v), \]  
using a reference point \((u, v) = (u_0, \pi/2)\), to create an approximate Stäckel potential \( \Phi_S \) for \( \Phi \) using Equation (4). Using these functions, we can solve for the spatial boundary of the orbit in \( \Phi_S \), which approximates the boundary in the desired potential \( \Phi \). When computing the boundary for a single phase-space point we simply set \( u_0 \) to the \( u \) coordinate of the phase-space point; when we generate an interpolation grid as described below, we use the method described by Binney (2012) to determine a good \( u_0 \) as a function of \( \dot{E} \) and \( L_z \).

By transforming the rectangular boundary back to \((R, z)\) using Equations (1) and (2), we can determine the orbit’s spatial boundary in regular cylindrical coordinates. This boundary is typically summarized using the peri- and apogalacticon distances \( r_{\text{peri/ap}} \)—which we take to be the closest and furthest 3D distance to the Galactic center along the orbit—and the maximum height \( z_{\text{max}} \) above the plane. From the geometry of the prolate spheroidal coordinate system, it is straightforward to see that the perigalacticon is attained at \( z = 0 (v = \pi/2) \) and \( u = u_{\text{min}} \)
\[ r_{\text{peri}} = \Delta \sinh u_{\text{min}}, \]  
while \( z_{\text{max}} \) and the apogalacticon are both attained at \((u, v) = (u_{\text{max}}, v_{\text{max}})\)
\[ z_{\text{max}} = \Delta \cosh u_{\text{max}} \cos v_{\text{min}}, \]
\[ r_{\text{ap}} = \sqrt{(\Delta \sinh u_{\text{max}} \sin v_{\text{min}})^2 + z_{\text{max}}^2}. \]  
We can then also compute the orbital eccentricity using its usual definition for Galactic orbits
\[ e = \frac{(r_{\text{ap}} - r_{\text{peri}})}{(r_{\text{ap}} + r_{\text{peri}})}. \]  

This computation can be performed relatively quickly and inexpensively and is straightforward to parallelize for large numbers of orbits. A good value for the parameter \( \Delta \), which defines the prolate coordinate system, can be computed for a given \((R, z)\) position using Equation (9) of Sanders (2012), which exploits the relation between \( \Delta \) and the first and second derivatives of the potential that holds for a Stäckel potential to determine a good \( \Delta \) using the derivatives of any axisymmetric potential.

We can further speed up the computation of the orbital parameters \((e, z_{\text{max}}, r_{\text{peri}}, r_{\text{ap}})\) by building an interpolation grid using the implementation described in Section 5.4 of Bovy (2015) of the interpolation method first discussed in Binney (2012).

In Figure 1, we demonstrate the appearance of an exemplar orbit in the MWPotential2014 Milky Way-like potential described in Bovy (2015), in both cylindrical \( R, z \) coordinates, and under the transformation into the \( u, v \) plane described above. The orbit shown is at a random energy equivalent to \( \log_{10}(E_{\text{u}}/E_{\text{u}(L)}) = -0.8 \), and an angular momentum \( \log_{10}(L) = -0.1 \) in units of the angular momentum of the circular orbit at the Sun. The orbit is squashed from a cone-like shape in \( R - z \) to an approximate box-like geometry in the \( u - v \) plane. In this geometry the vertical and radial oscillation is readily separable, allowing the simple calculation of the parameters.
2.1. Implementation in galpy

We have implemented the method described above in the galpy galactic dynamics Python package\(^2\) (Bovy 2015) and included it in its recent v1.3 release, both in its direct form and in the grid-based form. In this way, the method can be used for any orbit in any axisymmetric potential that is implemented in galpy. We briefly describe the details of this implementation here.

The novel method for determining orbital parameters described here is naturally a part of the Stäckel approximation for computing orbital actions and angles, which is implemented in galpy as a class actionAngleStaeckel with methods that return actions, frequencies, and angles. We therefore implemented a new method EccZmacRperiRap of these objects that uses the formalism above to compute the orbital eccentricity, \(e\), maximum vertical excursion, \(z_{\text{max}}\), and peri- and apogalacticon radii, \(r_{\text{peri}}\) and \(r_{\text{ap}}\). The EccZmacRperiRap method uses a C implementation of the method if the provided gravitational potential has a C implementation—which is the case for almost all built-in potentials—and falls back onto a pure Python implementation otherwise. The EccZmacRperiRap method can be applied to arrays of phase-space positions and can use a different \(\Delta\) parameter for each phase-space position. The C implementation can furthermore make use of OpenMP to parallelize the calculation for different phase-space positions. The grid-based method is implemented by adding a method EccZmacRperiRap to the actionAngleStaeckelGrid class in galpy—which implements the grid-based version of the algorithm of Binney (2012)—that uses a grid of \(e\), \(z_{\text{max}}\), \(r_{\text{peri}}\), and \(r_{\text{ap}}\) pre-computed during the instantiation of a actionAngleStaeckelGrid object. The interpolation is performed in the same way as that of the actions (see Bovy 2015 for details on this).

The interface through the actionAngleStaeckel class allows large numbers of phase-space points to be processed quickly, but requires the phase-space points to be input in Galactocentric cylindrical coordinates and a \(\Delta\) parameter (or array of such parameters) to be given. A simpler interface to the same method is provided through galpy’s Orbit class, which represents Galactic orbits and forms the basis of orbit integration in galpy. Orbit instances can be initialized in a variety of ways, including from observed positions and velocities (sky coordinates, distances, proper motions, and line-of-sight velocities). The fast method of this paper is implemented as part of the existing \(e, z_{\text{max}}, r_{\text{peri}},\) and \(r_{\text{ap}}\) methods of Orbit instances. This interface performs an automatic determination of a good \(\Delta\) parameter (using Equation (9) of Sanders 2012 applied to the current position). Moreover, for spherical potentials \(\Delta = 0\), and the Orbit methods automatically detect this and use the simpler version of the method above that is appropriate for spherical potentials.

We provide some explicit code examples in Section 3.6 below.

3. Tests and Applications

In the following, we demonstrate the accuracy of the estimation of orbit parameters via the methodology described in Section 2. We use the parameters calculated using an orbit

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\(^2\) https://github.com/jobovy/galpy
integration technique as “truth” values in each case, but note that these calculations are subject to some uncertainty, arising from the (small) error in the integration, and in the subsequent calculation of parameters such as the eccentricity, which can be underestimated if the orbit torus is not fully filled by the integration.

3.1. Estimating Parameters of Disk Orbits

First, we demonstrate the accuracy of estimation of $z_{\text{max}}$ and $e$ in a grid of orbits in MWPotential2014 spanning the range of angular momentum $-0.3 < \log_{10}(L) < 0.3$ (angular momentum here and everywhere below is expressed in units of the angular momentum of the circular orbit at the Sun) and covering at each $L$ the range of random energy $-2 < \log_{10}(E - E_{\infty}/E(L)) < -0.5$. This region of orbital space roughly corresponds to that of stars on disk orbits, with angular momenta corresponding to guiding radii $3 \lesssim R_g \lesssim 30$ kpc. At each energy and angular momentum point, we integrate an orbit for 20 azimuthal periods (at fixed timesteps), initialized at $R = R_g$ and $z = 0$ kpc, with a tangential velocity $v_T = L/R_g$ and radial and vertical velocity $v_R$ and $v_z$, such that

$$v_R = \sqrt{x(E - E_c(L))}v_z = \sqrt{(1-x)[E - E_c(L)]},$$

(11)

where here, we let $x = 4/5$. We find that integration for 20 azimuthal periods is sufficient to estimate the orbital parameters with a precision better than a hundredth of a percent for the orbits shown. The “true” parameters of the orbit are then calculated based on this integration. We then estimate the $\Delta$ parameter required for the application of the method using the method from Sanders (2012) as described above, taking the median estimated value for a range of phase-space points along a small part of the orbit. Using this, we estimate the parameters again using the Stäckel approximation method, then compare these with the integrated value. We show the results in Figure 2 by plotting the difference between the integrated parameter and estimated parameter, $\Delta P = P_{\text{integrated}} - P_{\text{estimation}}$, where $P$ represents the parameter in question and where, in the case of $z_{\text{max}}$, we normalize this value by the integrated parameter. The values of energy and angular momenta where $r_{\text{peri}} = 4$ kpc, $r_{\text{ap}} = 20$ kpc, and $z_{\text{max}} = 4$ kpc are indicated by dashed, dashed-dotted, and dotted lines, respectively.

For both of the parameters, $z_{\text{max}}$ and $e$, the median difference between the estimated and integrated parameter is much less than 1%. For the majority of orbits in the region bounded by the lines of constant $r_{\text{peri}}$, $r_{\text{ap}}$, and $z_{\text{max}}$ and centered on $\log_{10}(L) = 0$ (hereafter referred to as the disk region), the parameters are extremely well-estimated. The left panel, showing $\Delta z_{\text{max}}/z_{\text{max}},$ demonstrates that there is a small, roughly systematic offset between integration and estimation, at a level of $\sim 10^{-3}$. A number of orbits are more strongly overestimated relative to the integration value, with significant, sharp substructure in energy-angular momentum space. However, the median $|\Delta z_{\text{max}}/z_{\text{max}}|$ across all the orbits shown is still small, at $\sim 4 \times 10^{-3}$. We demonstrate in the right panel, showing $\Delta e$, that the estimation is accurate to a level less than

![Figure 2](image-url)
0.1% in the disk region, with no obvious systematic offset relative to the integration. The median $|\Delta e|$ over the full range of energy and angular momentum shown is small, at $\sim 6 \times 10^{-5}$. Although there are regions of the space where the parameters are over- and underestimated, these are localized to the sharp regions, which appear to correspond to areas in the energy-angular momentum space where the orbital tori are not well filled by the orbit integration.

We also repeat this test across a grid in the $x$ parameter in Equation (11), performing the estimation in the same random energy and angular momentum grid, but varying $x$ between 0 and 1. We find that this initial ratio between the radial and vertical velocities makes little difference to the performance of the estimation, and that $\Delta P$ correlates more strongly with the energy and angular momentum. We note, however, that the performance of the estimation is slightly worsened, to the order of a few tenths of a percent in all the parameters, at the edges of the $x$ grid, where the random velocity is concentrated almost entirely radially or vertically.

### 3.2. Applying the Method to a Wider Range of Orbits

We now extend the range of energy and angular momenta to look at orbits in with $-2 < \log_{10}(L) < 1$ (with guiding radii ranging from $\sim 0.2$ to 117 kpc) and $-2 < \log_{10}\left(\frac{E-L(R)}{E(L)}\right) < 0$. We study the accuracy of estimation in $r_{\text{peri}}$ and $r_{\text{ap}}$, as well as the parameters shown in the previous section. We follow the same procedure as described in Section 3.1, showing the results in Figure 3, retaining the definition of $\Delta P$, and choosing to also normalize this value by the integrated parameter in the cases of $r_{\text{peri}}$ and $r_{\text{ap}}$.

In this wider range of orbits, there are clearly cases where the estimation has difficulty matching the calculation returned by the orbit integration, with some parameters being returned at values greater than $\sim 60\%$ different in isolated regions of energy and angular momentum space. The method particularly breaks down at lower angular momenta ($\log_{10}(L) \lesssim -1$ or $R_e \lesssim 1.17$ kpc). Direct inspection of the orbits in these regions shows that they are generally occupied by resonant orbits, which do not fill the orbital tori which are described by the Stäckel approximation. It is also worth noting that the Stäckel approximation breaks down in the regions of parameter space where the potential deviates strongly from a Stäckel potential. Regardless, across the whole range of orbits, the method still achieves median variations of order $10^{-3}$ over all the parameters, demonstrating the isolation of the regions where it is not so accurate.

### 3.3. Using Different Potentials

All tests thus far have used MWPotential2014. The components of this potential, and the dynamical constraints used to fit their parameters are described in Bovy (2015). Here, we test an alternate and more complex Milky Way-like potential to understand how the assumed potential might affect the estimation. We choose to implement the best-fitting Milky Way potential of McMillan (2017), consisting of a flattened axisymmetric bulge, an NFW halo, pure exponential thin and thick disks, and two gas disks, representing the H I and molecular gas, with sech² vertical profiles and exponential radial profile (including a hole in the center with an exponential scale length). We approximate the bulge and the four disks using galpy’s SCFPotential and DiskSCFPotential.
respectively. These are implementations of the Self-Consistent Field (SCF; Hernquist & Ostriker 1992) method for generating potentials from general density functions: DiskSCFPotential uses the approach of Kuijken & Dubinski (1995) to approximate the disk contribution to a potential and solves for the (approximately spherical) difference between the approximate and true potential using the SCF method. Our implementation approximates the McMillan (2017) density to better than 1% everywhere. We numerically compute the second derivatives of the potential for the estimation of the Stäckel \( \Delta \) parameter.

We repeat the comparison between orbit integration and the estimation method as in Section 3.1 using the alternative potential. We find a median \( |\Delta z_{\text{max}}/z_{\text{max}}| = 0.034 \), in a region of energy and angular momentum equivalent to that shown in Figure 2, which also contains a region bounded by orbit parameters consistent a “disk” orbit in this potential. We find a median \( |\Delta e| = 3 \times 10^{-3} \) in the same region. The estimation of \( z_{\text{max}} \) in this potential is significantly worse than that in MWPotential2014, but \( e \) is still estimated to a high level of accuracy. The worse performance for \( z_{\text{max}} \) is due to the fact that the vertical structure of this potential is more complex than that of MWPotential2014 and the potential is therefore less well approximated as a Stäckel potential. The structure in the energy-angular momentum plane as shown in Figures 2 and 3 is still present, with most of the differences between the potentials being systematic in nature.

We also compare the estimation of parameters in this potential with the same in MWPotential2014. We find that there are significant systematic offsets between the two sets of results. The disk region, which is similar between the two potentials has a median difference in \( z_{\text{max}} \) of \( \sim 15\% \), and in \( e \) of \( \sim 5\% \). The offset between potentials is roughly systematic in \( z_{\text{max}} \), whereas those in \( e \) vary as a function of angular momentum.

3.4. Validating the Method with a Real Data Set

To demonstrate the validity of the method in comparison to orbit integration, we apply it to the cross section of the RAVE and Gaia-TGAS data sets; a sample of 216,201 stars with position, distance, proper motion, and heliocentric line-of-sight velocity measurements. This data set is useful for validating the method, as it is possible to (relatively) quickly measure the parameters via orbit integration as a cross-check. We cross match the two data sets by sky position, then convert the observed coordinates into Galactocentric cylindrical coordinates, assuming the solar radius and height above the midplane \( R_0 = 8 \) kpc, \( Z_0 = 0.025 \) kpc and a circular velocity \( V_c = 220 \) km s\(^{-1}\) (e.g., Bovy et al. 2012). We assume the solar motion of Schönrich et al. (2010): \( [U, V, W]_\odot = [-11.1, 12.24, 7.25] \) km s\(^{-1}\). We then estimate the orbital eccentricities in MWPotential2014 using the direct method, estimating \( \Delta \) at each phase-space point, and also applying the grid-based method with a fixed \( \Delta = 0.4 R_0 \). We find that the regular method returns values for the entire sample in \( 20 \) s, whereas the grid-based method performs the same estimation in \( 2 \) s. The speed of the estimation depends on the complexity of the potential. Under a simpler, flattened logarithmic halo potential results are returned in \( 400 \) ms.

We show the estimated eccentricity distribution across three bins in vertical distance from the midplane \([Z]\), in the upper panel of Figure 4. The lower panel shows a comparison between the regular and grid method estimation values. We find that the median eccentricity increases with \([Z]\), and that the distribution becomes broader, such that the relative fraction of stars on eccentric orbits becomes larger as \([Z]\) increases, in good agreement with existing measurements (e.g., Dierickx et al. 2010; Adibekyan et al. 2013; Kordopatis et al. 2015). The grid-based method generally agrees very well with the regular
method, with a standard deviation in \( e - e_{\text{grid}} \) less than \( \sim 10^{-2} \). It is noteworthy that the grid estimation returns values marginally closer to the regular method (\( \leq 0.5\% \) more accurate) for stars at intermediate \([Z]\), with the smallest and largest \([Z]\) bins returning values with similar accuracy. It should be noted that stars which are in regions of angular momentum-energy space which are not well approximated by the Stäckel fudge can be subject to systematic offsets in \( e \) up to \( \sim 10^{-2} \). These offsets may act to artificially broaden the eccentricity distributions, but the observed broadening as a function of \([Z]\) exceeds these uncertainties.

3.5. A Catalog of Estimated Orbital Characteristics for the Gaia DR2 RV Sample

We now demonstrate the usefulness of this method and the previously examined Stäckel approximation for action-angle coordinates of Binney (2012) in estimating orbital characteristics for a sample of a size that would be computationally intractable using orbit integration. We release with this paper a catalog of estimated orbit parameters: \( e, r_{\text{ap}}, r_{\text{peri}}, z_{\text{max}}, \) action-angle coordinates: \( J_R, L_z, J_z, \theta_R, \theta_\phi, \theta_z, \) orbital frequencies: \( \Omega_R, \Omega_\phi, \Omega_z, \) orbit energies and guiding radii for the Gaia DR2 stars with a 6D phase-space solution (parallax, celestial position, proper motion, and radial velocity). The catalog assumes a left-handed coordinate frame (positive solar angular momentum). The angles are approximated such that the radial angle \( \theta_R \) is zero at pericenter, increasing towards the apocenter, the vertical angle \( \theta_z \) is zero at \( z = 0 \), increasing toward positive \( z_{\text{max}} \).

We take the full gaia_source_with_rvs table from the Gaia Archive, and for each object with a measurement of all the necessary parameters (i.e., no NULL fields, a total of 6643147 stars), we sample 100 realizations of the observation by reconstruction of the covariance matrix of the astrometric parameters, which are given in the table. The orbital characteristics listed above are then estimated for each realization, given the simple MWPotential2014, and the mean and standard deviation reported for the parameter value and its associated error. We also compute and report the correlation coefficients between the orbital parameters and the actions. This process is obviously more time consuming than a simple operation of the method over a list of 6D coordinates, as it requires the sampling of the covariance matrix, and the computation of the correlation coefficients for 6643147 stars. Estimation of the uncertainties by this method also means that we effectively perform 100 times more estimations. In total, the estimations that form the catalog take \( \sim 116 \) hours, parallelized across 16 cores. Performing a similar computation using orbit integration would obviously take considerably longer.

The table is available at http://cdsarc.u-strasbg.fr/. We use the data model provided in Table 1. The provided table is ordered by the Gaia integer source ID, and contains rows populated with NaN values for objects that did not have the necessary astrometric parameters, so that it can be directly joined row-for-row to a flattened gaia_source_with_rv file. The table also includes Gaia source ID’s, and so is readily joinable with the tables available on the Gaia archive.

As a simple demonstration of the scientific value of the catalog as we provide it, in Figure 5 we show the density of stars in \( r_{\text{ap}} - r_{\text{peri}} \) space, for stars with parallax signal-to-noise ratio \( >20 \). While the appearance of this plane is clearly affected by the selection function of the higher quality part of the Gaia RVs sample, which is restricted mainly to regions close to the Sun, it shows that the orbits sampled by Gaia extend from the very inner galaxy, well into the halo. This is a clear testament to the scientific value of this data set. There are also substructures visible in this plane, the understanding of which

| Column ID | Quantity | Description | Units |
|-----------|----------|-------------|-------|
| source_id | Gaia DR2 source ID | Identifier provided in DR2 |
| ra        | R.A.    | The object right ascension | Deg. |
| dec       | Decl.   | The object declination | Deg. |
| e         | e       | Orbital eccentricity as defined in Equation (10) |
| z_max     | z_max   | Maximum vertical excursion from the midplane | kpc |
| r_peri    | r_peri  | 3D pericenter radius | kpc |
| r_ap      | r_ap    | 3D apocenter radius | kpc |
| jr        | J_R     | Radial action | km s^{-1} kpc |
| Lz        | L_z(J_z) | Azimuthal action (equivalent to vertical component of angular momentum) | km s^{-1} kpc |
| jz        | J_z     | Vertical action | km s^{-1} kpc |
| omega_r   | \Omega_R| Radial frequency | Gyr^{-1} |
| omega_phi | \Omega_\phi| Azimuthal frequency | Gyr^{-1} |
| omega_z   | \Omega_z | Vertical frequency | Gyr^{-1} |
| theta_r   | \theta_R| Radial angle | Rad. |
| theta_phi | \theta_\phi| Azimuthal angle | Rad. |
| theta_z   | \theta_z | Vertical angle | Rad. |
| rl        | R_{guide} | Radius of a circular orbit at the same L_z | kpc |
| E         | E       | Orbital energy | km^2 s^{-2} |
| EminusEc  | E - E_z | Difference between orbit energy and energy of a circular orbit of the same L_z | km^2 s^{-2} |
| *_err     | Error on each quantity |
| *_x_corr  | Correlation between the estimation of quantities |

**Note.** Correlation coefficients are only computed for the orbital parameters and actions, and are computed separately for each set of quantities.
Figure 5. Distribution of stars in the Gaia DR2 RVS sample, with \( \omega/\xi > 20 \), in rap and \( r_{\text{peri}} \). Bins with <10 stars are not colored. There are many substructures in this plane, the understanding of which will likely be of great importance to the understanding of the formation and evolution of the Milky Way. However, for the purpose of this work, it demonstrates that while spatially limited, Gaia DR2 samples a wide range of Galactic orbits, that range from the center of the Galaxy to far out into the Galactic Halo.

(A color version of this figure is available in the online journal.)

will provide detailed insight into the orbital structure of the Galaxy.

A simple exploration of the orbit space structures in Figure 5, which may provide insight into their origin, is shown in Figure 6. The left hand figure shows the density of stars in the space of \((r_{\text{ap}} + r_{\text{peri}})/2 = r_{\text{mean}}\) versus \(\epsilon\), and the mean \(z_{\text{max}}\) in the same bins on the right. This plane is roughly analogous in the space of \(J_R-L_z\) plane in action space (explored in Gaia DR2 by Trick et al. 2018), but offers a more direct and intuitive link to the spatial and kinematic structure of the disk. As in action space, and in our Figure 5, there are many clumpy features in this space, which suggest that the orbital distribution function is not smooth, but rather stars are “trapped” in certain regions of the space.

In the right-hand panel, we show that the clumpy features in \(r_{\text{mean}}-\epsilon\) space correspond to regions where the mean \(z_{\text{max}}\) is low. This suggests that these features are more likely to be induced by disk dynamical effects which act in-plane, rather than halo dynamics, such as satellite flybys and bombardment by small subhalos. This is consistent with the results of Trick et al. (2018), who showed that the clumps in action space had a low vertical action, \(J_z\) (which is analogous to \(z_{\text{max}}\)). Through proper modeling of this plane in orbital parameter and action space, it will be possible to begin linking these features to the complex dynamics of the Milky Way, which will give way to new insights into its present day structure and history of its formation and evolution.

In terms of uncertainties on the estimated parameters, and their implications as to the conclusions drawn from the catalog, there are three main sources of uncertainty to consider: observational errors, choice of potential, and systematics from the application of the estimation method. In general, uncertainties arising from observational errors are small, and of the order of a few percent. We have shown in the earlier sections of this paper that systematic uncertainties arising from the use of the estimation method are also small, at a level less than a percent over wide ranges of energy-angular momentum space. Most importantly, we have discussed that the choice of potential imposes significant systematic uncertainties on these estimates. Adopting a more complex potential such as that of McMillan (2017) can change these estimates systematically by as much as 20%, so this should be taken into account when interpreting the tabulated parameters.

### 3.6. A Basic Python Example

We now briefly demonstrate the use of the method in galpy. First we show the simplest implementation, which performs estimation within a galpy.Orbit instance in a given potential (in this case, MWPotential2014)

```python
from galpy.orbit import Orbit
from galpy.potential import MWPotential2014
o = Orbit(vxvv, pot=MWPotential2014)
ec = o.ecentricity(analytic=True,type='staeckel')
rmap = o.rap(analytic=True,type='staeckel')
rperi = o.rperi(analytic=True,type='staeckel')
zmax = o.zmax(analytic=True,type='staeckel')
```

where `vxvv` gives the phase-space coordinate of the object in question (galpy accepts various different forms of this input). Notice that it was not necessary to integrate the orbit before producing the estimates. It is also possible to run the estimation for large sets of phase-space points using an instance of the `actionAngle.actionAngleStaeckel` class:

```python
from galpy.actionAngle import actionAngleStaeckel
aAS = actionAngleStaeckel(pot=MWPotential2014, delta=0.4)
e, zmax, rperi, rap = 
    aAS.EccZmaxRperiRap(R, vR, vT, vz, phi)
```

where \(R, vR, vT, z, vz,\) and \(\phi\) are each an array of the corresponding coordinates, and each estimation is performed assuming \(\Delta = 0.4\). It is possible to calculate a separate \(\Delta\) parameter for each of a set of objects by using the `estimateDeltaStaeckel` function.
from galpy.actionAngle import estimateDeltaStaeckel

delta=estimateDeltaStaeckel(mp, R, z,  
no_median=True)

these estimated Δ parameters can then be passed to aAS.  
Ecc2maxRperiRap using the delta keyword argument.  
The process can be sped up further by using the grid-based  
estimation method, which is implemented in galpy as  
actionAngleStaeckelGrid.  
A full tutorial which shows this implementation and the  
others listed here, as well as an example using real data, is available online.5

4. Conclusions

We have demonstrated a new application of the Binney  
(2012) Stäckel fudge for the rapid calculation of the orbit  
parameters rperi, rap, zmax, and e, which does not depend on  
orbit integration.  
We have shown that for disk orbits, each  
parameter can generally be estimated to within less than a  
percent of the orbit integration value.  
We have demonstrated that this estimation is also valid outside the disk,  
but should be used cautiously for such orbits, where resonances can cause  
problems.  
We applied the method to the RAVE-TGAS data set  
of 216, 201 stars, demonstrating the utility and speed of the  
method, which can return results as fast as 9 μs per object when  
used in its grid-based application.  
Thus, this technique can compute point estimates for orbital parameters for  
e.g., all 150 million Gaia stars with RVs at end of mission,  
in about 2.5 hr on a single CPU, which can be brought down to a few  
minutes of wall time because the computation can be trivially  
parallelized (It should be noted that this time is inflated by a  
factor of ~100 when uncertainties are propagated).  
We calculate the orbital parameters, as well as action-angle  
coordinates, frequencies, and energies for the sample of stars  
with full 6D phase-space coordinates from Gaia DR2,  
performing a full error propagation and estimation of the  
correlation between the parameters and the actions.  
This more robust estimation, which includes the estimation of the  
parameters for 100 realizations of the observed coordinates  
per star, takes ~116 hours of parallelized CPU time.  

Using the catalog, which is available at http://cdsarc.u-
strasbg.fr/, we show that the rperi-rap distribution for the Gaia  
RV sample demonstrates the extensive “dynamical sampling”  
of the data, even when constrained to a relatively small sphere  
around the Sun.  
We demonstrated that the clumpy features which are apparent in that space and the rmean-e plane are  
coincident with regions of low mean zmax, indicating that these

Figure 6. Distribution of stars in a local (1/ϖ < 1.5 kpc) Gaia DR2 RVS sample in rmean and e space, where  
rmean = (rap + rperi)/2.  
In the left panel, we show the density of stars in the space, where many substructures are visible, which are at least qualitatively similar to those seen in action space.  
In the right-hand panel, we demonstrate that these overdensities in rmean-e correspond to regions with lower mean zmax, suggesting that they may have arisen due to resonances from “disk” dynamics related to, for example, bars and spiral features, rather than e.g., “halo” dynamics such as satellite flybys or subhalo bombardment.  
(A color version of this figure is available in the online journal.)

5 http://galpy.readthedocs.io/en/latest/orbit.html
features are likely the result of disk dynamical effects rather than halo dynamics. While angle-action variables may be a more fundamental label of stellar orbits, this tool may still be able to provide insight into the orbital structure of the Galaxy throughout the coming era of extremely large sets of stellar phase-space coordinates.

The authors thank the anonymous reviewer for helpful reports, which improved the clarity and content of this paper. JTM acknowledges an STFC doctoral studentship, as well as the Dunlap Visitorship program and an RAS Grant, which funded an extended visit to the Dunlap Institute at the University of Toronto, to work on this project. JTM is also grateful for the hospitality of the Center for Computational Astrophysics at the Simons Foundation in New York City for a short period while working on this project. J.B. acknowledges the support of the Natural Sciences and Engineering Research Council of Canada (NSERC), funding reference number RGPIN-2015-05235, and from an Alfred P. Sloan Fellowship.

This work has made use of data from the European Space Agency (ESA) mission Gaia (http://www.cosmos.esa.int/gaia), processed by the Gaia Data Processing and Analysis Consortium (DPAC; http://www.cosmos.esa.int/web/gaia/dpac/consortium). Funding for the DPAC has been provided by national institutions, in particular the institutions participating in the Gaia Multilateral Agreement. This project was developed in part at the 2018 NYC Gaia Sprint, hosted by the Center for Computational Astrophysics of the Flatiron Institute in New York City.

ORCID iDs

J. Ted Mackereth @ https://orcid.org/0000-0001-8108-0935

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