Stable equivalence of bridge positions of a handlebody-knot

Makoto Ozawa

Abstract. We show that any two bridge positions of a handlebody-knot are stably equivalent.

1. Introduction

Reidemeister ([11]) and Singer ([15]) independently proved the stable equivalence theorem, that is, for any two Heegaard splittings $H_1$ and $H_2$ of a 3-manifold $M$, there exists a third Heegaard splitting $H$ which is a stabilization of both. There are another proofs in [2], [14]. A modern, simple proof of the Reidemeister–Singer Theorem is given by Lei ([9]). Those proofs are worked in the piecewise linear category. On the other hand, in the smooth category, Johnson ([8]) gives a proof of Reidemeister–Singer Theorem using Rubinstein and Scharlemann’s graphic, and Laudenbach ([10]) gives a proof of Reidemeister–Singer Theorem by Cerf’s methods.

In the context of knot theory, bridge positions (or bridge decompositions) of a knot correspond to Heegaard splittings of a 3-manifold. Birman ([1]) proved that the stable equivalence of plat representations, namely, bridge positions. Hayashi ([5]) proved that the stable equivalence theorem for a pair of (3-manifold, 1-submanifold), which generalizes both of Reidemeister–Singer Theorem and Birman’s theorem. Along the proof by Lei ([9]), Zupan ([16]) gives a short proof of Hayashi’s theorem.

Ishii ([7]) opened a road to handlebody-knots, that is, handlebodies embedded in the 3-sphere. He classified the moves on diagrams of trivalent graphs embedded in the 3-sphere, which define the same handlebody-knot ([7, Corollary 2]). See Theorem 3.9 for the details. In this paper, we will prove the stable equivalence of bridge positions of a handlebody-knot (Theorem 3.2), and equivalently, the stable equivalence of bridge positions of a trivalent graph (Theorem 3.3). To prove Theorem 3.2 and 3.2 we introduce a regular bridge position of a knotted trivalent graph. We will show that any bridge position can be isotoped by a horizontal isotopy so that it is regular (Lemma 3.6). Then we can handle regular diagrams and bridge positions simultaneously, and by virtue of Reidemeister moves for trivalent graphs

2010 Mathematics Subject Classification. Primary 57M25.

The author is partially supported by Grant-in-Aid for Scientific Research (C) (No. 17K05262) and (B) (No. 16H03928), The Ministry of Education, Culture, Sports, Science and Technology, Japan.
(7 Corollary 2), we will show that stable equivalence of regular bridge positions (Theorem 3.10).

In the proof of the stable equivalence of bridge positions of a trivalent graph, the second stabilization move $S_2$ (Figure 3) is the key to a solution. As far as the author knows, the stabilization $S_2$ was first appeared in [3] as moves 3, 4, 5, 6. Dancso probably has known all moves which relate all Morse positions of a trivalent graph embedded in the 3-sphere, but the proof was not included in the paper [3]. Ishihara–Ishii (6) defined the sliced diagram of trivalent tangles, that is, a diagram in Morse position, and showed that two sliced diagrams are related by an isotopy of the plane if and only if they are related by a finite sequence of moves $P_i$, $P_{ii}$, $P_{iii}$, $P_{iv}$ ($= S_2$) (6 Theorem 5.1 (1)). This refines Dancso’s unproved theorem in the context of diagrams in Morse position. Dancso’s unproved theorem or Ishihara–Ishii’s theorem overlaps our main result Theorem 3.3, however it does not imply Theorem 3.3 since for given two bridge positions of two trivalent graphs, we will construct two sequences of stabilizations and moves from two bridge positions to a third bridge position. Thus it is irreversible. By Theorem 3.3 we also give a proof of Dancso’s unproved theorem in Proposition 3.15.

Contents

1. Introduction 1
2. Definitions 2
2.1. Height function and projection 2
2.2. Essential saddles, $\lambda$-vertices and $Y$-vertices 3
2.3. Morse positions for handlebody-knots 3
2.4. Morse positions for knotted trivalent graphs 3
2.5. Several moves for Morse positions 4
3. Stable equivalence of bridge positions 5
3.1. Regular bridge position 6
3.2. Stable equivalence of regular bridge positions 7
3.3. Relation among Morse positions 13
References 14

2. Definitions

2.1. Height function and projection. Recall that $S^3 = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1\}$ is decomposed by $S^2 = \{(x, y, z, 0) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 = 1\}$ into two 3-balls $B_+ = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1, w \geq 0\}$ and $B_- = \{(x, y, z, w) \in \mathbb{R}^4 \mid x^2 + y^2 + z^2 + w^2 = 1, w \leq 0\}$. We call two points $(0, 0, 0, 1)$ and $(0, 0, 0, -1)$ the north pole and south pole of $S^3$, and denote by $+\infty$ and $-\infty$ respectively.

Let $h : \mathbb{R}^4 \to \mathbb{R}$ be a height function defined by $(x, y, z, w) \mapsto (0, 0, 0, w)$. We denote the restriction $h_{|S^3}$ by the same symbol $h$.

Let $r : \mathbb{R}^4 - (w\text{-axis}) \to S^2 \times \mathbb{R}$ be a retraction defined by

$$(x, y, z, w) \mapsto \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}, w\right).$$
Then the restriction \( r|_{S^3 - \{\pm\infty\}} \) defines a map \( S^3 - \{\pm\infty\} \rightarrow S^2 \times (-1,1) \) and denote it by the same symbol \( r \). Let \( \pi : S^2 \times (-1,1) \rightarrow S^2 \) be a projection defined by \( (x, y, z, w) \mapsto (x, y, z, 0) \). Then the composition \( \pi \circ r \) defines a projection \( S^3 - \{\pm\infty\} \rightarrow S^2 \) and denote it by \( p \).

### 2.2. Essential saddles, \( \lambda \)-vertices and \( Y \)-vertices

Let \( F \) be a closed surface embedded in \( S^3 \) and suppose that \( h|_F \) is a Morse function and all critical points have distinct critical values. Let \( x \) be a saddle point of \( F \) which corresponds to the critical value \( t_x \in \mathbb{R} \). Let \( P_x \) be a pair of pants component of \( F \cap h^{-1}(\{t_x - \epsilon, t_x + \epsilon\}) \) containing \( x \) for a sufficiently small positive real number \( \epsilon \). Let \( C^1_x, C^2_x \) and \( C^3_x \) be the boundary components of \( P_x \), where we assume that \( C^1_x \) and \( C^2_x \) are contained in the same level \( h^{-1}(t_x - \epsilon) \), and \( C^3_x \) is contained in the another level \( h^{-1}(t_x + \epsilon) \). A saddle point \( x \) of \( F \) is upper (resp. lower) if \( C^1_x \) and \( C^2_x \) are contained in \( h^{-1}(t_x - \epsilon) \) (resp. \( h^{-1}(t_x + \epsilon) \)). A saddle point \( x \) of \( F \) is essential if both of the two loops \( C^1_x \) and \( C^2_x \) are essential in \( F \), and it is inessential if it is not essential.

Let \( \Gamma \) be a trivalent graph in \( S^3 \). A vertex \( v \) of \( \Gamma \) is called a \( \lambda \)-vertex (resp. \( Y \)-vertex) with respect to the height function \( h \) if two ends of incident edges lie below \( v \) (resp. above \( v \)).

### 2.3. Morse positions for handlebody-knots

Let \( v, v' \subset S^3 \) be two handlebodies embedded in the 3-sphere. We say that \( v, v' \) are equivalent if there exists an ambient isotopy \( \{f_t\}_{t \in [0,1]} \) of \( S^3 \) taking \( v \) to \( v' \).

Let \( V \) be a handlebody-knot type (i.e. an equivalence class of handlebody-knots). We always assume that \( v \in V \) does not contain the north and south poles \( \pm\infty = (0,0,0,\pm1) \) of \( S^3 \).

**Definition 2.1.** We say that \( v \in V \) is a Morse position if

1. \( h|_{\partial v} \) is a Morse function.
2. All critical points of \( h|_{\partial v} \) have distinct values.
3. For any regular value \( t \) of \( h|_{\partial v} \), \( v \) intersects \( h^{-1}(t) \) only in disks.
4. Each saddle point of \( h|_{\partial v} \) is essential.

**Definition 2.2.** Two Morse positions \( v, v' \in V \) are equivalent if there exists an ambient isotopy \( \{f_t\}_{t \in [0,1]} \) of \( S^3 \) taking \( v \) to \( v' \) such that

1. For any \( t \in [0,1] \), \( h|_{f_t(\partial v)} \) is a Morse function.
2. For any \( t \in [0,1] \), there is no pair of minimum/lower saddle and maximum/upper saddle of \( h|_{f_t(\partial v)} \) with the same critical value.

We call such an ambient isotopy \( \{f_t\}_{t \in [0,1]} \) a Morse isotopy.

### 2.4. Morse positions for knotted trivalent graphs

Let \( \gamma, \gamma' \subset S^3 \) be two trivalent graphs embedded in the 3-sphere. We say that \( \gamma, \gamma' \) are equivalent if there exists an ambient isotopy \( \{f_t\}_{t \in [0,1]} \) of \( S^3 \) taking \( \gamma \) to \( \gamma' \).

Let \( \Gamma \) be a knotted trivalent graph type (i.e. an equivalence class of knotted trivalent graphs). We always assume that \( \gamma \in \Gamma \) does not intersect the north and south poles \( \pm\infty = (0,0,0,\pm1) \) of \( S^3 \).

**Definition 2.3** (Normal form of [4], [12], [13]). We say that \( \gamma \in \Gamma \) is a Morse position if

1. For each edge \( e \), the critical points of \( h|_{e} \) are nondegenerate and each lies in the interior of \( e \).
2. All critical points of \( h|_{\gamma} \) and vertices have distinct values.
(3) Each vertex is either $Y$-vertex or $\lambda$-vertex.

**Definition 2.4.** Two Morse positions $\gamma, \gamma' \in \Gamma$ are equivalent if there exists an ambient isotopy $\{f_t\}_{t \in [0,1]}$ of $S^3$ taking $\gamma$ to $\gamma'$ such that

1. For any $t \in [0,1]$ and for each edge $e$, the critical points of $h|_{f_t(e)}$ are nondegenerate and each lies in the interior of $e$.
2. For any $t \in [0,1]$, each vertex $f_t(v)$ is either $Y$-vertex or $\lambda$-vertex.
3. For any $t \in [0,1]$, there is no pair of minimum/$Y$-vertex and maximum/$\lambda$-vertex of $h|_{f_t(\gamma)}$ with the same critical value.

We call such an ambient isotopy $\{f_t\}_{t \in [0,1]}$ a Morse isotopy.

**2.5. Several moves for Morse positions.** We prepare several moves for Morse positions of knotted trivalent graphs. By taking the neighborhood, these moves can be considered as moves for Morse positions of handlebody-knots.

First we define $B_1, B_2, B_3$ moves as shown in Figure 1. These moves are reversible and obtained by Morse isotopies for both knotted trivalent graphs and handlebody-knots.

![Figure 1. B1, B2, B3 moves](image1)

Next we define $B_4, B_5$ moves as shown in Figure 2. These moves are reversible and obtained by Morse isotopies for only handlebody-knots. Those are not Morse isotopies for knotted trivalent graphs. The $B_4$ move is appeared as 7, 8 moves in [3]. The $B_5$ move is appeared as $Rvi$ in [6].

![Figure 2. B4, B5 moves](image2)

Next we define stabilizations $S_1, S_2, S_3$ as shown in Figure 3. These stabilizations are irreversible and not obtained by Morse isotopies for both knotted trivalent graphs and handlebody-knots. The reverse move of a stabilization is called a destabilization. The stabilization $S_1$ and its destabilization are appeared as 1, 2 moves in [3] and $Pii$ in [6]. The stabilization $S_2$ and its destabilization are appeared as 3, 4, 5, 6 moves in [3] and $Piv$ in [6].
Finally we define $M_1, M_2, M_3$ moves as shown in Figure 4. These moves are irreversible. Those are not Morse isotopies for both knotted trivalent graphs and handlebody-knots.

**2.6. Correspondence between two Morse positions.** For a given Morse position $\gamma$ of a knotted trivalent graph type $\Gamma$, by taking a regular neighborhood $N(\gamma)$ of $\gamma$, we obtain a Morse position $v$ of a handlebody-knot type $V$ such that $v = N(\gamma)$. Conversely, for a given Morse position $v$ of a handlebody-knot type $V$, by taking a spine $\sigma$ of $v$, we obtain a Morse position $\gamma$ of a knotted trivalent graph type $\Gamma$ such that $\gamma = \sigma$. In this correspondence, the definitions of equivalence for Morse positions coincides except for $B_4, B_5$ moves. Hence the following holds.

**Proposition 2.5.** There is a correspondence between two Morse positions $v \in V$ and $\gamma \in \Gamma$. Two Morse positions $v, v' \in V$ are equivalent if and only if corresponding $\gamma \in \Gamma$, $\gamma' \in \Gamma'$ are equivalent up to $B_4, B_5$ moves.

**3. Stable equivalence of bridge positions**

**Definition 3.1.** Let $\gamma \in \Gamma$ be a Morse position. We say that $\gamma$ is a bridge position if all maximum and $\lambda$-vertices have the positive critical values, and all minimum and $Y$-vertices have the negative critical values.

**Theorem 3.2.** Let $V$ be a handlebody-knot type and $v, v' \in V$ be two bridge positions. Then there exists a bridge position $v'' \in V$ which is obtained from both $v$ and $v'$ by finite sequences of stabilizations $S_1, S_2$ and Morse isotopies.
THEOREM 3.3. Let \( \Gamma, \Gamma' \) be two knotted trivalent graph types and \( \gamma \in \Gamma, \gamma' \in \Gamma' \) be two bridge positions. If \( N(\gamma) \) and \( N(\gamma') \) are equivalent as handlebody-knots, then there exist a knotted trivalent graph type \( \Gamma'' \) and a bridge position \( \gamma'' \in \Gamma'' \) which is obtained from both \( v \) and \( v' \) by finite sequences of stabilizations \( S1, S2, B4, B5 \) moves and Morse isotopies.

By Proposition 2.5 Theorem 3.2 and 3.3 are equivalent.

3.1. Regular bridge position. An ambient isotopy \( \{f_t\}_{t \in [0,1]} \) of \( S^3 \) is called a horizontal isotopy (resp. vertical isotopy) if for any \( t \in [0,1], h \circ f_t = h \circ id_{S^3} \) (resp. \( p \circ f_t = p \circ id_{S^3} \)).

DEFINITION 3.4. Let \( \Gamma \) be a knotted trivalent graph type and \( \gamma \in \Gamma \) be a bridge position. We say that \( \gamma \) is regular if

1. \( p|_{\gamma \cap B_\pm} \) is an injection.
2. \( p|_{\gamma} \) is regular (i.e. whose multiple points are only finitely many transversal double points of the edges.).

DEFINITION 3.5. Let \( \gamma \in \Gamma \) be a bridge position. Put \( \gamma_\pm = \gamma \cap B_\pm \). Then \( \gamma_\pm \) is isotopic in \( B_\pm \) fixing \( \partial\gamma_\pm = \gamma_\pm \cap \partial B_\pm \) so that \( \gamma_\pm \) is contained in \( \partial B_\pm \). Therefore, there exists a complex \( \Sigma_\pm = \gamma_\pm \times [0,1]/(\partial\gamma_\pm \times \{t\} \sim \partial\gamma_\pm \times \{t'\}) \) \( (t,t' \in [0,1]) \) embedded in \( B_\pm \) such that \( \gamma_\pm \times \{1\} = \gamma_{\pm}, \Sigma_\pm \cap \partial B_\pm = \gamma_\pm \times \{0\} \). We call this complex \( \Sigma_\pm \) a canceling complex for \( \gamma_\pm \). We say that a canceling complex \( \Sigma_\pm \) is monotone if \( h|_{\text{int}\Sigma_\pm} \) has no critical point, where \( \text{int}\Sigma_\pm = \Sigma_\pm - (\gamma_\pm \times \{0,1\}) \). Moreover, we say that a canceling complex \( \Sigma_\pm \) is vertical if \( p(\Sigma_\pm) = \gamma_\pm \times \{0\} \).

LEMMA 3.6. Let \( \Gamma \) be a knotted trivalent graph type and \( \gamma \in \Gamma \) be a bridge position. Then there exists a horizontal isotopy \( \{f_t\}_{t \in [0,1]} \) of \( S^3 \) such that \( f_1(\gamma) \) is regular.

PROOF.

CLAIM 3.7. There exists a monotone canceling complex \( \Sigma_\pm \) for \( \gamma_\pm \).

PROOF. Since \( \gamma_+ \) has only maxima, there exists a horizontal isotopy \( \{f_t\}_{t \in [0,1]} \) of \( B_+ \) such that \( p|_{f_t(\gamma_+)} \) is a homeomorphism. Then there exists a vertical canceling complex \( \Sigma_+ \) for \( f_1(\gamma_+) \). Now we pull back \( \Sigma_+ \) by the inverse function \( \{f_t^{-1}\}_{t \in [0,1]} \) to get a monotone canceling complex \( f_1^{-1}(\Sigma_+) \) for \( \gamma_+ \). □

CLAIM 3.8. There exists a horizontal isotopy \( \{f_t\}_{t \in [0,1]} \) taking a monotone canceling complex \( \Sigma_\pm \) for \( \gamma_\pm \) to a vertical canceling complex \( f_1(\Sigma_\pm) \) for \( f_1(\gamma_+) \).

PROOF. There exists a sufficiently small positive real number \( \epsilon \) such that \( \Sigma_+ \cap h^{-1}([0,\epsilon]) \) is contained in \( N(\Sigma_+ \cap h^{-1}(0)) \times [0,\epsilon] \). Since \( \Sigma_+ \) is monotone, there exists a horizontal isotopy \( \{g_t\}_{t \in [0,1]} \) of \( h^{-1}([0,\epsilon]) \) such that \( g_1(\Sigma_+ \cap h^{-1}(0)) \times [0,\epsilon] \) is vertical. To define a horizontal isotopy of the rest \( h^{-1}([\epsilon,\infty)) \), let \( g_t|_{h^{-1}(x)} = g_t|_{h^{-1}(x)} \), for any \( x \in (\epsilon,\infty) \) and \( t \in [0,1] \). Repeating this procedure, we will obtain a horizontal isotopy \( \{f_t\}_{t \in [0,1]} \) as desired. □

The above two claims show Lemma 3.6 □

Let \( \gamma \in \Gamma \) be a regular bridge position. Then, we have a projection \( p(\gamma) \). For each double point \( x \) of \( p(\gamma) \), there are two points \( p^{-1}(x) \) one of which lies \( \gamma_+ \) and
another lies $\gamma_-$. We obtain a diagram $\hat{p}(\gamma)$ by adding an over/under information on each double point $x$ depending on whether each point of $p^{-1}(x)$ lies $\gamma_+$ or $\gamma_-$. To prove Theorem 3.3 we use the following theorem.

**Theorem 3.9 ([7, Corollary 2]).** Let $D_1$ and $D_2$ be diagrams of spatial trivalent graphs $L_1$ and $L_2$, respectively. Then $L_1$ and $L_2$ are neighborhood equivalent if and only if $D_1$ and $D_2$ are related by a finite sequence of the moves $R_1 - 6$ depicted in Figure 5.

![Figure 5. Moves for diagrams of knotted trivalent graphs](image)

**3.2. Stable equivalence of regular bridge positions.** By Lemma 3.6 we may assume that $\gamma$ and $\gamma'$ are regular by horizontal isotopies. To prove Theorem 3.2 or equivalently Theorem 3.3 it suffices to show the following theorem.

**Theorem 3.10.** Let $\gamma$, $\gamma'$ be two regular bridge positions. Suppose that two diagrams $\hat{p}(\gamma)$ and $\hat{p}(\gamma')$ are related by a finite sequence of the moves $R_1 - 6$. Then there exist two sequences of regular bridge positions $\gamma = \gamma_1, \gamma_2, \ldots, \gamma_n = \gamma''$ and $\gamma' = \gamma'_1, \gamma'_2, \ldots, \gamma'_m = \gamma''$, where $\gamma_{i+1}$ (resp. $\gamma'_{j+1}$) is obtained from $\gamma_i$ (resp. $\gamma'_j$) by one from among stabilizations $S_1$, $S_2$, $B_4$, $B_5$ moves and a Morse isotopy ($i = 1, \ldots, n - 1; j = 1, \ldots, m - 1$).

**Definition 3.11.** For a regular bridge position $\gamma$, both stabilizations $S_1$ and $S_2$ can be performed by vertical isotopies. We call such stabilizations *vertical stabilizations*.

![Figure 6. A vertical stabilization $S_2$ with a vertical canceling complex](image)
Lemma 3.12. Let $\gamma$ be a regular bridge position. By a finite sequence of vertical stabilizations $S_1, S_2$ and vertical isotopies, we may assume that for any subarc $\alpha$ of $\gamma_+$ (resp. $\gamma_-$) except for all crossings of $p(\gamma)$, $\alpha$ is a component of $\gamma_-$ (resp. $\gamma_+$).

Proof. Without loss of generality, let $\alpha$ be a subarc of $\gamma_+$ except for all crossings of $\tilde{p}(\gamma)$. Let $e$ be an edge of $\gamma$ containing $\alpha$.

First suppose that $e$ intersects the bridge sphere $S_2$. As shown in Figure 7, we perform a vertical stabilization $S_1$ at the point of $e \cap S_2$, and then two small upper bridge and lower bridge are created. This lower bridge can be moved into the below of $\alpha$ by a vertical isotopy. Hence $\alpha$ is a component of $\gamma_-$. See Figure 8. The same applies to $e'_3$.

![Figure 7. A deformation of $\alpha$ by a stabilization $S_1$ and a vertical isotopy](image)

![Figure 8. A deformation of $e'_2$ by a stabilization $S_1$ and a vertical isotopy](image)

By repeating these operations, the edge $e$ eventually intersects the bridge sphere $S^2$. Then the argument follows the previous case. $\square$

By Theorem 3.9, two diagrams $\tilde{p}(\gamma)$ and $\tilde{p}(\gamma')$ are related by a finite sequence of the moves $R1-6$. 
Lemma 3.13. A finite sequence of stabilizations $S_1$, $S_2$, $B_4$, $B_5$ moves and Morse isotopies can be substituted for the finite sequence of the moves $R_1 - 6$ and planar isotopies.

Proof. $(R_1)$ Figure 9 shows that Morse isotopies can be substituted for one direction of $R_1$.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig9.jpg}
\caption{Morse isotopies can be substituted for $R_1$.}
\end{figure}

For another direction of $R_1$, by Lemma 3.12 we may assume that any subarc in a given edge intersects the bridge sphere $S^2$. Then, by the converse deformation of Figure 9, Morse isotopies are substituted for another direction of $R_1$.

$(R_2)$ Following a deformation as shown in Figure 7, we may assume that any subarc does not intersect the bridge sphere $S^2$. Then Figure 10 shows that Morse isotopies can be substituted for one direction of $R_2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig10.jpg}
\caption{Morse isotopies can be substituted for $R_2$.}
\end{figure}

For another direction of $R_2$, by Lemma 3.12 we may assume that any two subarcs lie in $B_+$ and $B_-$ respectively. Then, by the converse deformation of Figure 10, Morse isotopies are substituted for another direction of $R_2$.

$(R_3)$ Figure 11 shows that a stabilization $S_1$ and Morse isotopies can be substituted for one direction of $R_3$. The same applies to another direction of $R_3$.

$(R_4)$ We consider one direction of $R_4$ as shown in Figure 12, where around the vertex $x$, two edges $e_1$ and $e_2$ contain over and under crossings respectively, $e_3$ contains no crossing. Following a deformation as shown in Figure 7, we may assume that around the vertex $x$, only $e_2$ intersects the bridge sphere $S^2$.

Since $x$ is a λ-vertex, there are three cases for consideration.

Case 1: One end of $e_1$ lies above $x$ and two ends of $e_2, e_3$ lie below $x$.
Case 2: One end of $e_2$ lies above $x$ and two ends of $e_1, e_3$ lie below $x$.
Case 3: One end of $e_3$ lies above $x$ and two ends of $e_1, e_2$ lie below $x$. 
In Case 1, around the vertex $x$, $e_1$ lies entirely above $e_2$. Hence one direction of $R4$ can be realized by a Morse isotopy.

In Case 3, around the vertex $x$, we may assume that $e_1$ lies entirely above $e_2$ by a Morse isotopy. Then one direction of $R4$ can be realized by a Morse isotopy.

In Case 2, $e_2$ must have a maximal point, say $y$. By a B4 move, we slide $e_3$ beyond $y$. Then the situation is similar to Case 1, and one direction of $R4$ can be realized by a Morse isotopy. See Figure 13.

Next we consider the reverse direction of $R4$ as shown in Figure 12. Since $x$ is a $\lambda$-vertex, there are three cases for consideration.

**Case 1:** One end of $e_1$ lies above $x$ and two ends of $e_2, e_3$ lie below $x$.

**Case 2:** One end of $e_2$ lies above $x$ and two ends of $e_1, e_3$ lie below $x$.

**Case 3:** One end of $e_3$ lies above $x$ and two ends of $e_1, e_2$ lie below $x$. 

Figure 11. A stabilization $S1$ and Morse isotopies can be substituted for $R3$.

Figure 12. One direction of $R4$

Figure 13. One direction of $R4$
In Case 1, around the vertex $x$, $e_1$ lies entirely above $e_2$. Hence the reverse direction of $R4$ can be realized by a Morse isotopy.

In Case 3, around the vertex $x$, we may assume that $e_1$ lies entirely above $e_2$ except for $x$ by a Morse isotopy. Then the reverse direction of $R4$ can be realized by a Morse isotopy.

In Case 2, by Lemma 3.12 we may assume that around the vertex $x$, $e_1$ and $e_3$ intersect the bridge sphere $S^2$. By a vertical stabilization $S^2$, $x$ becomes a $Y$-vertex, $e_1$ intersects $S^2$ in two points, $e_2$ intersects $S^2$ in one point and $e_3$ does not intersect $S^2$. Then the reverse direction of $R4$ can be realized by a Morse isotopy. See Figure 14.

(Figure 14) The reverse direction of $R4$

(R5) Following a deformation as shown in Figure 7, we may assume that $e_4$ does not intersect the bridge sphere $S^2$. Then by a vertical isotopy, we may assume that $e_4$ lies above $e_1, e_2, e_3$, and one direction of $R5$ can be realized by a Morse isotopy as shown in Figure 15.

(Figure 15) One direction of $R5$.

For another direction of $R5$, by Lemma 3.12 we may assume that $e_4$ lies in $B_+$ and lies above $e_1, e_2, e_3$ by a vertical isotopy. Then, the reverse direction of $R5$ can be realized by a Morse isotopy.

(R6) We label each edges and vertices as shown in Figure 16. We note that $p(e_1) p(\gamma)$ does not contain any crossing of $p(\gamma)$. First we consider the case that $e_1$ does not intersect the bridge sphere $S^2$, and $x_1, x_2$ lies in $B_+$. If $e_1$ has a maximal point, then by a $B4$ move, we may assume that $e_1$ does not have a maximal point. Then one direction of $R6$ can be realized by a $B5$ move.

Next we consider the case that $e_1$ intersects the bridge sphere $S^2$. By repeating a stabilization $S^2$ and a $B4$ move alternatively, $e_1$ eventually does not intersect $S^2$ as follows. Then as the previous case, one direction of $R6$ can be realized by a $B5$ move.
To show that the intersection of $e_1$ and $S^2$ can be reduced by repeating a stabilization $S_2$ and a $B_4$ move alternatively, we may assume without loss of generality that $x_2$ lies in $B_+$. First we consider the case that the portion of $e_1$ which incidents to $x_2$ contains a maximal point $y$. Then by a $B_4$ move, we slide $e_4$ so that the portion of $e_1$ does not contain $y$, but instead, $e_5$ contains a maximal point. See Figure 17.

Next we consider the case that the portion of $e_1$ which incidents to $x_2$ does not contain a maximal point. We may assume without loss of generality that $e_5$ contains a maximal point $y$ and following a deformation as shown in Figure 7, $e_4$ intersects the bridge sphere $S^2$ around $x_2$. By a vertical stabilization $S_2$, the number of the intersection of $e_1$ and $S^2$ can be reduced by one. See Figure 18.

This completes the proof of Lemma 3.13. □

By Lemma 3.13, $\gamma$ and $\gamma'$ are same as diagrams. However, $\gamma$ and $\gamma'$ are not necessarily same as regular bridge positions.

**Lemma 3.14.** Let $\gamma$ and $\gamma'$ be regular bridge positions such that $p(\gamma) = p(\gamma')$. Then by a finite sequence of vertical stabilizations $S_1$, $S_2$ and $B_4$ moves, we have $\gamma = \gamma'$.

**Proof.** By Lemma 3.12 we may assume that each edge of $\gamma$ and $\gamma'$ intersects the bridge sphere $S^2$. Moreover, for simplicity, we may assume that any subarc between two crossings or between a crossing and a vertex of $\gamma$ and $\gamma'$ intersects $S^2$. On each crossing, by vertical isotopies, $\gamma$ and $\gamma'$ coincide. On each corresponding vertices $x$ and $x'$, we may assume that both of $x$ and $x'$ lie in $B_+$ or $B_-$ by
a stabilization $S2$. Then by a $B4$ move, $\gamma$ and $\gamma'$ coincide around the vertices. Finally, by stabilizations $S1$, we may assume that the intersection number of $S^2$ and each subarc between two crossings or between a crossing and a vertex of $\gamma$ and $\gamma'$ coincide. Hence by a vertical isotopy, we have $\gamma = \gamma'$. □

3.3. Relation among Morse positions.

Proposition 3.15 (cf. [3, 6] Theorem 5.1 (1)). Let $V$ be a handlebody-knot type and $v, v' \in V$ be two Morse positions. Then two equivalence classes $[v]$ and $[v']$ are related by a finite sequence of stabilizations $S1$, $S2$ and their destabilizations and $M1$, $M2$, $M3$ moves.

Proof. First by applying $S3$, $M1$, $M2$, $M3$ to $v, v'$, we obtain two bridge positions $v, v'$. We remark that a stabilization $S3$ is obtained by a sequence of $S2$, $M2$, $B5$ (Morse isotopy) and a destabilization of $S2$. See Figure 19.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure19.png}
\caption{A stabilization $S3$ is obtained by $S2$, $M2$, $B5$, $S2^{-1}$}
\end{figure}

Then by Theorem 3.2, two bridge positions $v, v'$ are related by $S1$, $S2$ and their destabilizations. □

Acknowledgements. The author would like to thank Kazuto Takao and Atsushi Ishii for useful comments.
References

1. J. S. Birman, On the stable equivalence of plat representations of knots and links, Canad. J. Math. 28 (1976) 264–290.
2. R. Craggs, A new proof of the Reidemeister–Singer theorem on stable equivalence of Heegaard splittings, Proc. Amer. Math. Soc. 57 (1976), 143–147.
3. Z. Dancso, On the Kontsevich integral for knotted trivalent graphs, Algebr. Geom. Topol. 10 (2010), 1317–1365.
4. H. Goda, M. Scharlemann, A. Thompson, Leveling an unknotting tunnel, Geom. Topol. 4 (2000), 243–275.
5. C. Hayashi, Stable equivalence of Heegaard splittings of 1-submanifolds in 3-manifolds, Kobe J. Math. 15 (1998), 147–156.
6. K. Ishihara, A. Ishii, An operator invariant for handlebody-knots, Fund. Math. 217 (2012), 233–247.
7. A. Ishii, Moves and invariants for knotted handlebodies, Algebr. Geom. Topol. 8 (2008), 1403–1418.
8. J. Johnson, Stable functions and common stabilizations of Heegaard splittings, Trans. Amer. Math. Soc. 361 (2009), 3747–3765.
9. F. Lei, On stability of Heegaard splittings, Math. Proc. Camb. Phil. Soc. 129 (2000), 55–57.
10. F. Laudenbach, A proof of Reidemeister–Singer’s theorem by Cerf’s methods, Ann. Fac. Sci. Toulouse Math. (6) 23 (2014), 197–221.
11. K. Reidemeister, Zur dreidimensionalen Topologie, Abh. Math. Sem. Univ. Hamburg 11 (1933), 189–194.
12. M. Scharlemann, Thin Position in the Theory of Classical Knots, in Handbook of Knot Theory, Elsevier Science (2005), 429–459.
13. M. Scharlemann, A. Thompson, Thin position and Heegaard splittings of the 3-sphere, J. Diff. Geom. 39 (1994), 343–357.
14. L. Siebenmann, Les bissections expliquent le théorème de Reidemeister–Singer : Un retour aux sources, Ann. Fac. Sci. Toulouse Math. (6) 24 (2015), 1025–1056.
15. J. Singer, Three-dimensional manifolds and their Heegaard diagrams, Trans. Amer. Math. Soc. 35 (1933), 88–111.
16. A. Zupan, Bridge and pants complexities of knots, J. London Math. Soc. (2) 87 (2013), 43–68.

Department of Natural Sciences, Faculty of Arts and Sciences, Komazawa University, 1-23-1 Komazawa, Setagaya-ku, Tokyo, 154-8525, Japan
Email address: w3c@komazawa-u.ac.jp