Single-Photon Detection Using Quantum Phase Transitions

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The temporal dynamics of large quantum systems perturbed weakly by a single excitation can give rise to unique phenomena at the quantum phase boundaries. Here, we develop a time-dependent model to study the temporal dynamics of a single photon interacting with a defect within a large system of interacting spins. Our model can lead to a single-photon detector if the system of spins is engineered to simulate a first-order quantum phase transition (QPT). We show that the absorption of a single photon pulse by an engineered defect (i.e., a nitrogen-vacancy center) can nucleate a single shot quantum measurement. This concept of a single-shot detection event ("click") is different from parameter estimation which requires repeated measurements. The crucial step of amplifying the weak quantum signal occurs by coupling the defect to a system of interacting spins biased close to a QPT point. The macroscopic spin-order change during the QPT generates an amplified signal, which can be read out by a classical device. Our work paves the way for studying the temporal dynamics of large quantum systems interacting with a single photon.

Utilizing the high sensitivity of thermodynamic (classical) phase transitions (TPTs) has a long and successful history for weak signal detection, such as the Wilson cloud chamber \cite{1} and the bubble chamber \cite{2}. One recent prominent example is the superconducting nanowire single-photon detector (SNSPD) \cite{3}, which has achieved the state-of-art of quantum efficiency \cite{4}, timing jitter \cite{5} and dark counting rate in single-photon detection \cite{6}. The defects in the superconductor absorb the incident single-photon pulse (SPP) nucleating a hotspot, which diffuses and finally breaks the superconductivity of the nanowire to generate an amplified signal. The essential amplification mechanism relies on the concept of a optimum critical bias to ensure that even an SPP can trigger a phase transition from the superconducting to the normal domain.

The success of thermodynamic phase transitions for single photon detection leads to the important question whether their quantum counterparts could be systematically exploited for the same. We emphasize that recent developments in engineering quantum phase transitions in cold atoms systems\cite{7}, ion traps \cite{8}, and superconducting qubit systems\cite{9} can be leveraged to answer this question. We note that multiple recent proposals on parameter estimation utilize continuous quantum phase transitions \cite{10}. However, the single photon detection problem is fundamentally different (see table I) and requires discontinuous QPTs. We exploit recent developments in quantum pulse scattering theory \cite{11–13} to analyze the dynamics of a single photon interacting with a defect state coupled to a large system of collective spins (N>100). We believe this opens the route to studying the time dynamics of large quantum systems excited by a single photon. While we capture the essential physics through a minimalistic model, it points to single photon nucleated space-time theory of quantum phase transitions where even excited states along with ground states play an important role. This can lead to an exciting frontier at the interface of condensed matter physics and quantum optics.

The proposed detector functions by a weak signal within a defect triggering a first-order QPT \cite{14, 15}. We note that it is a fundamental open question whether a single photon perturbation can trigger the QPT which is the focus of this work (see table I for differences from a quantum sensor). In contrast with TPTs, QPTs occur at zero temperature and can be triggered even by varying a physical parameter \cite{16}, such as magnetic field or interaction strength. Consequently, the quantum fluctuations, arising from Heisenberg’s uncertainty principle, dominate and drive the transition to realize the amplification of the quantum signal. Our amplification scheme is based on the giant sensitivity of the first-order QPT and originates from a mechanism overlooked previously: diverging higher-order spin-spin correlation at the phase transition point.

Another important goal of the current paper is to advance the quantum theory of photodetection. Historically, the pioneering work of Glauber \cite{17}, Mandel \cite{18}, and Mollow \cite{19} explained the concept of detector counts and connection to the single photon absorption process \cite{11}. However, these early models do not take into account amplification which is the key step for converting a quantum time-dependent signal to a classical read-out pulse. In our work, this amplification of a single photon signal occurs due to a phase transition in a system of spins with long-range interactions.

The proposed detector is fundamentally different from quantum interferometers used for parameter estimation or quantum sensing/metrology \cite{20–22} (see Table I for a summary of these differences). The enhanced sensitivity in quantum interferometers benefits from the accumulated phase from a large number of synchronized non-interacting particles in repeated measurements \cite{23–25}. In stark contrast, the giant sensitivity in our scheme originates from the singular behavior of strongly correlated systems at the phase transition point \cite{14, 15}. The detection events in the presented scheme are single-shot measurements. Our approach is also distinct from the well-established quantum linear amplifiers \cite{26, 27}. The gain of linear quantum amplifiers arises from the coher-
Quantum Interferometers for Metrology/Sensing | Quantum Counters for Single-Photon Detection
---|---
Phase interference based | Not an interference measurement; Non-adiabatic transitions dominate dynamics
Repetitive measurements | Single-shot measurement
Slope of the continuous transition determines the sensitivity | Discontinuity leads to clicks
No concept of timing jitter and dark counting rate | High counting rate and low dark counting rate is required
NOON (GHZ) state to reach the Heisenberg limit in a Mach–Zehnder (Ramsey) Interferometer [20] | Single-photon avalanche diode; superconducting nanowire single-photon detectors

TABLE I. Contrast between single-photon detection (quantum counters) and quantum metrology (sensing).

![FIG. 1. Single photon detection via quantum phase transition (QPT).](image)

The interacting spins at the bottom, which function as the amplifier of the detector, are critically biased close to the first-order QPT point. The three states in the absorber on the top form a Λ-structure. After absorption of a single-photon pulse, the absorber is excited from the ground state |g⟩ and finally relaxes to the meta-stable state |e⟩. After the \( g \rightarrow e \) transition, the absorber exerts an effective magnetic field on the amplifier spins. This magnetic field triggers a QPT in the spins underneath. Initially, the spins are polarized in the yz-plane (see panel a). After the phase transition, the spins rotate to the xz-plane (see panel b). This macroscopic spin-order change functions as the output signal of the single-photon detector.

Results

Single-photon detection—We now discuss the working mechanism and implementation of our model for SPP detection. The first step of the detection event is the transduction (absorption) of the incident single photon in an engineered defect. This process is similar to the generation of the first electron–hole pair in single-photon avalanche diode or the first photo-emission event in the photo-multiplier tube. The highly efficient transduction is realized via a Λ-structure transition as shown in Fig 1. In contrast to a two-level absorber, this Λ-transition defect has three main benefits: (1) higher absorption probability [12, 28]; (2) longer lifetime of the destination state |e⟩ [11] conducive for efficient read-out; (3) connection of the optical transition in the absorber and the RF-frequency dynamics in the amplifier. One promising example of such kind of absorber is a nitrogen-vacancy (NV) center. The states |g⟩ and |e⟩ correspond to the two ground spin states |0⟩ and |+1⟩ of the NV. The \( T_1 \) time (lifetime of the state |e⟩ = |+1⟩) of NV centers is few milliseconds at room temperature and even much longer at lower temperatures [29]. The Λ transition can be realized with the spin non-conserving transition [30](see the supplementary material [31]). After the transduction, the information of the SPP is written in the |e⟩ state of the absorber.

The second principle is effective engineering of the absorber-amplifier interaction to guarantee that the absorbed energy is transferred to the readout channel to trigger the QPT. In our detector, the coupling between the absorber and the amplifier is engineered in \( x \)-direction

\[
\hat{H}_{\text{int}} = B_x \langle e | \sum_j \hat{\sigma}_j^x \rangle.
\]

(1)

This dispersive coupling with strength \( B_x \) acts an effective magnetic field for the amplifier spins. As shown in the following, the defect functions as a control of the QPT in the amplifier. More importantly, the dispersive coupling avoids additional decoherence of the amplifier induced by the SPP. The NV center couples to its surrounding spins dispersively as in equation (1) when the strength \( B_x \) is much smaller the ground-state zero-field splitting \( \Delta_{g0} \approx 2.87 \text{ GHz} \) [32].

The essential step of an SPP detection is the amplification, since the excitation in the defect after the transduction is still an extremely weak quantum signal. In our detector, the amplification is realized by exploiting the giant sensitivity of the
first-order QPT. With the mean-field theory, we predicted a universal first-order QPT in interacting spin systems [15]

\[ \hat{H}_{\text{AM}} = \frac{1}{2} \varepsilon \sum_{j=1}^{N} \sigma_j^z - \frac{1}{n} \sum_{\langle i,j \rangle} (J_x \sigma_i^x \sigma_j^x + J_y \sigma_i^y \sigma_j^y), \]  

(2)

where \( \varepsilon \) is the energy splitting of the spins along \( z \) direction, \( J_x \) and \( J_y \) are the strengths of the ferromagnetic spin-spin couplings in \( x \)- and \( y \)-direction respectively, and \( \sigma_j^a \) \((a = x, y, z)\) are the Pauli matrices of the \( j \)th spin. The summation \((i < j)\) runs over \( n \) coupled neighbours. For the 1-dimensional Ising chain with \( n = 1 \), the short-range coupling only exists between the nearest neighbours [16]. For the Lipkin-Meshkov-Glick (LMG) model with \( n = N - 1 \) \((N\) the total spin number\) [33], all the spins are coupled with each other. The interacting spins function as the amplifier of our proposed SPP counter.

The amplifier has two ferromagnetic (FM) phases: FM-X and FM-Y with long-range spin order in \( x \)- and \( y \)-direction. The competition between these two FM phases results in the first-order QPT, which exhibits giant sensitivity for weak signal detection [15]. In Fig. 2 (a), we present the schematic of the first-order QPT boundary (the red line) in the phase diagram. The quantum phases and the corresponding QPTs can be characterized by two magnetic order parameters

\[ \xi_x = \langle \hat{S}_x^2 \rangle_0/N^2 \text{ and } \xi_y = \langle \hat{S}_y^2 \rangle_0/N^2, \]  

(3)

which describe the magnetic fluctuations in the \( xy \)-plane. Here, \( \hat{S}_a = \sum_{j} \sigma_j^a/2 \) are the collective spin operators and \( \langle \cdots \rangle_0 \) means average on the ground-state of the amplifier. The second-order QPTs in interacting spin systems have been extensively demonstrated in recent experiments [7-9]. Specifically, the second-order QPT in the LMG model (with long-range spin-spin coupling only in \( x \)-axis) has also been demonstrated in a recent experiment with 16 Dysprosium atoms [34].

We suggest that by adding an additional laser to induce the long-range coupling in \( y \)-direction, the first-order QPT due to the competition between the two FM phases can also be observed. This provides a promising platform to build a single-photon detector utilizing first-order QPT in the LMG model. In the following, we numerically demonstrate the detection of SPPs with such a QCD. The first-order QPT in the LMG amplifier occurs at \( J_x = J_y > \varepsilon/2 \) [15].

The amplification and single-shot readout of the quantum information stored in the state \( |e\rangle \) is realized by exploiting the first-order QPT in the amplifier. Initially, the spin-spin coupling \( J_x \) is pre-biased slightly below the phase transition point \( J_{x,c} \equiv J_x \) [see the red star in Fig. 2 (a)] and the amplifier is initialized in its ground state of the FM-Y phase. After absorption of a SPP pulse, the absorber is flipped to the state \( |e\rangle \) with probability \( P_e(t) \) [31]. Thus, the additional effective magnetic field experienced by the amplifier spins is \( \hat{B}_x \times P_e(t) \). The initial critical bias guarantees that the small magnetic field perturbation \( \hat{B}_x \times P_e(t) \) from the absorber can trigger a QPT and leads to efficient amplification.

There are two ways to read out the amplified signal in practice. One is to directly measure the spontaneous magnetization \( \sqrt{\sum_{i} \sigma_i^x} \) of the amplifier in \( x \)-direction, which increases from an extremely small value to a finite value after the collective rotation of the spins. Another option is to couple the amplifier spin with a cavity as proposed in our previous works [14, 15]. The energy prestored in the spins is transferred to the cavity mode generating macroscopic excitations after the QPT. The photons leak out from cavity can be directly measured with classical photodetectors.

To characterize the detection sensitivity, we define the quantum gain of the amplifier as

\[ G(t) = \langle \hat{S}_x^2(t) \rangle/\langle \hat{S}_x^2(t_0) \rangle. \]  

(4)

We contrast the time-dependent quantum gain for the cases...
of critical bias (the red-solid curve) and non-critical bias (the blue dotted curve) in Fig. 2 (b). It is clearly seen that the efficient amplification can only be obtained if the system is optimally biased close to the phase transition point [14]. We also note that for the critical bias case, the amplifier spins finally evolve to an excited state in the FM-X phase with macroscopic spins polarized in $xz$-plane as shown in following.

To reveal the intrinsic change within the amplifier, we contrast the time-dependent spin $Q$-function of the amplifier for different biases in Fig. 3. The first row (a-d) and the second row (e-h) correspond to critical and non-critical bias cases, respectively. In both cases, the amplifier starts from the FM-Y phase with spins polarized in the $yz$-plane. The two arms of the $Q$-function in the $yz$-plane at time $t_0 = -5/\epsilon$ (the time before the absorption of the pulse) correspond to the two degenerate ground states of the FM-Y phase [15]. For the first row, the incident SPP triggers a phase transition to the FM-X phase. The spins rotate $90^\circ$ to the $xz$-plane at time $t_0 = 18/\epsilon$ in Fig. 3 (d). This reveals the dynamic change in the long-range spin order within the amplifier and clearly shows the signature of the detection event. In contrast, no macroscopic spin order change occurs when the amplifier is biased far from the phase transition point. The polarization of the spins marginally varies with time in Fig. 3 (e-h).

Our simulation of the amplifier dynamics has ignored the decoherence of the interacting spins that may degrade the performance of a realistic device. However, the amplification has completed within the time $\epsilon T_{\text{Am}} \approx 15$ [31], which is usually much shorter than the decoherence time of the spins. If the amplifier is composed of electron spins with typical energy splitting $\epsilon \sim 1$ GHz and coherence time $T_2^* \sim 1\mu s$ [35], we have $\epsilon T_2^* \approx 1000 > \epsilon T_{\text{Am}}$. For nuclear spins with typical energy splitting 1 MHz and coherence time $T_2^* \sim 1$ ms [36], the decoherence time is still much longer than the amplification time. With dynamical decoupling techniques [37, 38], the coherence time of the spins can be further prolonged $2 \sim 3$ orders of magnitude [39–41], which is far more than the required time for amplification. The dipole-dipole interaction between the NV center and nuclear spins at the typical distance 1 nm is around 20 kHz. This effective magnetic field ($B_s/\epsilon \approx 0.02$) is large enough to trigger the QPT.

**Discussion**

**The singular scaling of the detector**—The giant sensitivity of the detector fundamentally originates from the singular behaviors of the system at the phase transition point. We now show the singular scalings of the amplifier. We also notice that in most cases, it is difficult for weak input signals to change the coupling strength within the amplifier [14]. Here, we show that a weak magnetic field perturbation can also break the balance of the two FM phases at the phase boundary $J_s = J_y$ to trigger the first-order QPT. This also lays the foundation of the amplification mechanism in our proposed single-photon detector. As shown in the subgraph of Fig. 4 (a), the order parameter $\zeta_x$ increases swiftly with the perturbation magnetic field in $x$-direction and the other order parameter $\zeta_y$ drops. The sensitivity to the magnetic field is characterized by the susceptibility of the spontaneous magnetization

$$\chi = \frac{d \sqrt{\zeta_x}}{dB_s} \bigg|_{B_s \rightarrow 0} \propto \left| B_s \right|^\gamma,$$

which is symmetric on the two sides of the transition with singular exponent $\gamma \approx 1.525$. The same susceptibility for the spontaneous magnetization $\sqrt{\zeta_x}$ with respect to a magnetic field in $y$-axis can also be obtained (data not shown). The susceptibility diverges linearly with the spin number $\chi \sim N$ as shown in Fig. 4 (b).

We emphasize that in first-order QPTs, a singularity occurs on the higher-order magnetic correlation. This is fundamentally different from the traditional TPTs, in which the diverging spatial correlation length $\xi$ in the microscopic correlator $\langle (\hat{\sigma}_i^x - \langle \hat{\sigma}_i^x \rangle)(\hat{\sigma}_{i+\xi}^z - \langle \hat{\sigma}_{i+\xi}^z \rangle) \rangle$ leads to the divergence of the magnetic susceptibility [42]. However, in the LMG model, the spins are all coupled with each other with homogeneous strength and the spins are indistinguishable. Thus, we can-
FIG. 4. Singular behavior in the susceptibility. (a) The susceptibility $\chi$ diverges at the phase transition point $J_x = J_y = 0.7 \epsilon$. The subgraph shows the abrupt changes in the order parameters in the first-order QPT transition with spin number $N = 1000$. (b) The susceptibility $\chi$ near the phase transition point increases linearly with the spin number $N$. Here, the perturbation magnetic field is set as $B_z = 10^{-5} \epsilon$.

FIG. 5. Singular behavior of the higher-order correlation and energy gap. (a) The higher-order correlation $C_{\text{xyy}}$ diverges at the phase transition point $J_x = J_y = 0.7 \epsilon$. (b) The energy gap vanishes at the phase transition point. The subgraphs show the same curves in the linear coordinates. Both curves are symmetric on the two sides of the phase transition. The solid black lines are the algebraic fittings. The spin number is set as $N = 1000$.

not define a simple correlation length $\xi$ for the LMG model. Alternatively, we define a higher-order correlation function

$$C_{\text{xyy}} = \frac{1}{2} \langle \hat{S}_x^2 \hat{S}_y^2 + \hat{S}_y^2 \hat{S}_x^2 \rangle_{0} - \langle \hat{S}_x^2 \rangle_{0} \langle \hat{S}_y^2 \rangle_{0} \propto |B_z|^y, \tag{6}$$

to characterize the macroscopic correlation between the magnetic fluctuations in x- and y- axis.

The diverging $C_{\text{xyy}}$ in the subgraph of Fig. 5 (a) shows the strong negative correlation between $\hat{S}_x^2$ and $\hat{S}_y^2$ at the phase transition point. The negative correlation reveals the fact that the order parameter $\zeta_x$ decreases as the other one $\zeta_y$ increases. The corresponding singular exponent is $\tilde{v} \approx 0.919$ as shown by the black fitting curve. This exponent is universal for the LMG model, as it is independent on the spin number $N$ as well as the position on the first-order QPT boundary in Fig. 2 [31]. We note that $\tilde{v}$ is similar to the traditional correlation length critical exponent [42, 43]. We also find that the lower-order correlation $(1/2) \langle \hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x \rangle_{0} - \langle \hat{S}_x \rangle_{0} \langle \hat{S}_y \rangle_{0}$ shows no singularity [31].

Another typical character of QPTs is that the energy gap $\Delta$ vanishes at the phase transition point as shown in Fig. 5b. The corresponding exponent is given by $\Delta \sim |B_z|^{1/2}$, which is same as the second-order QPT in LMG model [44–46]. Previous study on the size scaling for the LMG model shows that $\Delta \sim 1/N$ at the phase transition point [47, 48].

Our work provides a practical framework for single photon pulse detection using QPTs. This defect-controlled-QPT device is based on the fact that the first-order QPT in interaction spin systems can be induced by a weak in-plane magnetic field. Our theoretical proposal can be directly implemented in current QPT simulators [7–9]. We note that for microscopic systems, zero temperature generally implies preparing a system in a pure quantum (ground) state. Future work will explore the role of dark counts, dead time, timing jitter and quantum efficiency of our proposed detector in a practical system to identify the operational figures of merit.

This work is supported by the DARPA DETECT ARO award (W911NF-18-1-0074).

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[1] N. N. D. Gupta and S. K. Ghosh, Rev. Mod. Phys. 18, 225 (1946).
[2] D. A. Glaser, Phys. Rev. 87, 665 (1952).
[3] G. Goltseman, O. Okunev, G. Challita, A. Lipatov, A. Semenov, K. Smirnov, B. Voronov, A. Dzardanov, C. Williams, and S. Sobolewski, Applied physics letters 79, 705 (2001).
[4] F. Marsili, V. B. Verma, J. A. Stern, S. Harrington, A. E. Lita, T. Gerrits, I. Vayshenker, B. Baek, M. D. Shaw, R. P. Mirin, et al., Nature Photonics 7, 210 (2013).
[5] J. Wu, L. You, S. Chen, H. Li, Y. He, C. Lv, Z. Wang, and X. Xie, Applied optics 56, 2195 (2017).
[6] Q.-Y. Zhao, D. Zhu, N. Calandri, A. E. Dane, A. N. McCaughan, F. Bellei, H.-Z. Wang, D. F. Santavicca, and K. K. Berggren, Nature Photonics 11, 247 (2017).
[7] H. Bernien, S. Schwartz, A. Keesling, H. Levine, A. Omran, H. Pichler, S. Choi, A. S. Zibrov, M. Endres, M. Greiner, et al., Nature 551, 579 (2017).
[8] J. Zhang, G. Pagano, P. W. Hess, A. Kyprianidis, P. Becker, H. Kaplan, A. V. Gorshkov, Z.-X. Gong, and C. Monroe, Nature 551, 601 (2017).
[9] Harris et al., Science 361, 162 (2018).
[10] I. Frérot and T. Roscilde, Phys. Rev. Lett. 121, 020402 (2018).
[11] L.-P. Yang, H. X. Tang, and Z. Jacob, Phys. Rev. A 97, 013833 (2018).
[12] Y. Wang, J. c. v. Minář, L. Sheridan, and V. Scarani, Phys. Rev. A 83, 063842 (2011).
[13] B. Q. Baragiola, R. L. Cook, A. M. Brafczuk, and J. Combes, Phys. Rev. A 86, 013811 (2012).
[14] L.-P. Yang and Z. Jacob, Optics express 27, 10482 (2019).
[15] L.-P. Yang and Z. Jacob, arXiv preprint arXiv:1905.07420 (2019).
[16] S. Sachdev, Quantum phase transitions (Wiley Online Library, 2007).
[17] R. J. Glauber, Physical Review 130, 2529 (1963).
[18] L. Mandel, E. G. Sudarshan, and E. Wolf, Proceedings of the Physical Society 84, 435 (1964).
[19] B. Mollow, Physical Review 168, 1896 (1968).
[20] V. Giovannetti, S. Lloyd, and M. Maccone, Science 306, 1330 (2004).
[21] C. L. Degen, F. Reinhard, and P. Cappellaro, Rev. Mod. Phys. 89, 035002 (2017).
[22] R. Demkowicz-Dobrzański, J. Kolodyński, and M. Guţă, Nature communications 3, 1063 (2012).
[23] J. Zhang, M. Um, D. Lv, J.-N. Zhang, L.-M. Duan, and K. Kim, Phys. Rev. Lett. 121, 160502 (2018).
[24] E. Davis, G. Bentsen, and M. Schleier-Smith, Phys. Rev. Lett. 116, 053601 (2016).
[25] K. S. Thorne, Rev. Mod. Phys. 90, 040503 (2018).
[26] C. M. Caves, J. Combes, Z. Jiang, and S. Pandey, Phys. Rev. A 86, 063802 (2012).
[27] N. Bergeal, R. Vijay, V. Manucharyan, I. Siddiqi, R. Schoelkopf, S. Girvin, and M. Devoret, Nature Physics 6, 296 (2010).
[28] S. M. Young, M. Sarovar, and F. m. c. Léonard, Phys. Rev. A 97, 033836 (2018).
[29] A. Jarmola, V. M. Acosta, K. Jensen, S. Chemerisov, and D. Budker, Phys. Rev. Lett. 108, 197601 (2012).
[30] Y. Chu, M. Markham, D. J. Twitchen, and M. D. Lukin, Phys. Rev. A 91, 021801 (2015).
[31] See Supplemental Material for information about the details of single-photon transduction and higher-order correlations of amplifier spins.
[32] N. Zhao, J.-L. Hu, S.-W. Ho, J. T. Wan, and R. Liu, Nature nanotechnology 6, 242 (2011).
[33] H. J. Lipkin, N. Meshkov, and A. Glick, Nuclear Physics 62, 188 (1965).
[34] V. Makhalov, T. Satoor, A. Evrard, T. Chalopin, R. Lopes, and S. Nascimbene, Phys. Rev. Lett. 123, 120601 (2019).
[35] C. A. Ryan, J. S. Hodges, and D. G. Cory, Phys. Rev. Lett. 105, 200402 (2010).
[36] B. Smeltzer, J. McIntyre, and L. Childress, Phys. Rev. A 80, 050302 (2009).
[37] N. Zhao, S.-W. Ho, and R.-B. Liu, Phys. Rev. B 85, 115303 (2012).
[38] W. Yang, W.-L. Ma, and R.-B. Liu, Reports on Progress in Physics 80, 016001 (2016).
[39] G. Balasubramanian, P. Neumann, D. Twitchen, M. Markham, R. Kolesov, N. Mizuochi, J. Isoya, J. Achard, J. Beck, J. Tissler, et al., Nature materials 8, 383 (2009).
[40] T. D. Ladd, D. Maryenko, Y. Yamamoto, E. Abe, and K. M. Itoh, Phys. Rev. B 71, 014401 (2005).
[41] P. C. Maurer, G. Kucsko, C. Latta, L. Jiang, N. Y. Yao, S. D. Bennett, F. Pastawski, D. Hunger, N. Chisholm, M. Markham, et al., Science 336, 1283 (2012).
[42] M. Kardar, Statistical physics of fields (Cambridge University Press, 2007), Chap. 1.
[43] J. Dziarmaga, Advances in Physics 59, 1063 (2010).
[44] N. Defenu, T. Enss, M. Kastner, and G. Morigi, Phys. Rev. Lett. 121, 240403 (2018).
[45] M. Xue, S. Yin, and L. You, Phys. Rev. A 98, 013619 (2018).
[46] P. Ribeiro, J. Vidal, and R. Mosseri, Phys. Rev. E 78, 021106 (2008).
[47] R. Botet and R. Jullien, Phys. Rev. B 28, 3955 (1983).
[48] S. Dusuel and J. Vidal, Phys. Rev. B 71, 224420 (2005).
Supplementary Material for: “Single photon detection using quantum phase transitions”

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I. \(\Lambda\)-TRANSDUCTION IN NV CENTER

Usually, the optical transition in nitrogen-vacancy (NV) center does not change the state of the spin degree of freedom. Thus, a single-photon pulse (SPP) cannot induce a \(\Lambda\)-type transition to realize the flip of the two ground spin states \(|0\rangle \rightarrow |1\rangle\) in an NV center. To solve this problem, we need to construct a spin non-conservation transition in NV. One approach is combining a linearly polarized laser with an additional circularly polarized laser, which has been demonstrated to realized all-optical control of the NV ground-state spin [1]. Here, we use another method by utilizing the energy crossing in the excited states of the NV as shown in Fig. 1. We add strain to the NV at the energy crossing point of the two excited states \(|E_y\rangle = |a_1 e_y - e_y a_1\rangle \otimes |0\rangle\) and \(|E_1\rangle = |E_-\rangle \otimes |1\rangle - |E_+\rangle \otimes |+1\rangle\) (with \(E_\pm = a_1 e_\pm - e_\pm a_1\) and \(e_\pm = \mp(e_x \pm i e_y)\)) [2]. Here, \(|a_1\rangle, |e_x\rangle, |e_y\rangle\) and \(|-1\rangle, |0\rangle, |+1\rangle\) are the orbital basis of the excited states and the triplet spin states (the two-hole representation). The coupling \(\Delta''\) between \(|E_y\rangle\) and \(|E_1\rangle\) realizes the spin non-conservation transition. Now, the effective Hamiltonian for the NV reads

\[
\hat{H}_{NV} = \Delta_{gs} |1\rangle \langle 1| + \omega_y |E_y\rangle \langle E_y| + \omega_1 |E_1\rangle \langle E_1| + \Delta'' (|E_y\rangle \langle E_1| + |E_1\rangle \langle E_y|),
\]

where the energy of the ground spin state \(|0\rangle\) is set as zero, \(\Delta_{gs} \approx 2.87 \text{ GHz}\) the zero-field splitting between ground spin states, and \(\omega_y = \omega_1\) (the strain has been taken into account) are the energy difference between the two excited states and the ground state \(|0\rangle\).

The interaction between the NV and the incident SPP is given by

\[
\hat{H}_{\text{pump}} = i\hbar \int_0^\infty d\omega [g(\omega) e^{i\tau_{\text{NV}} \hat{a}(\omega)} |E_y\rangle \langle 0| - \text{h.c.}],
\]

Figure 1. Single-photon transduction (absorption) process. The incident single-photon pulse excited the NV center to the excited state \(|f\rangle = |E_y\rangle\). Via the spin non-conservation coupling \(\Delta''\) between states \(|E_y\rangle\) and \(|h\rangle = |E_1\rangle\), this excitation can be transferred to the ground spin state \(|e\rangle = |1\rangle\) after the spontaneous decay. To realize an efficient transduction, the bandwidth matching between pulse length \(\tau_f\), the spontaneous decay rates \(\gamma_{fg}\) and \(\gamma_{he}\), and the coupling strength \(\Delta''\) must be carefully considered (see follow).

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Figure 2. **Time-dependent absorption probability.** The length of the Gaussian pulse is set as $\epsilon \tau_f = 1$. The coupling between the two excited state $|f\rangle$ and $|h\rangle$ is $\Lambda'' = 5\epsilon$ and the spontaneous decay rates of these two excited states are $\gamma_{fg} = \gamma_{he} = \Gamma = 10\epsilon$.

where $\hat{a}(\omega)$ is the bosonic operator of the pulse mode with frequency $\omega = c|\vec{k}|$, $\vec{r}_{NV}$ is the position of the NV center, and the rotating-wave approximation has been taken. The amplitude of the NV-SPP interaction spectrum is given by

$$g(\omega) = \sqrt{\frac{\omega}{4\pi\varepsilon_0\hbar c A}} (\hat{\epsilon} \cdot \vec{d}_{0y}),$$

where $A$ is the effective transverse cross section of the pulse [3], $\vec{d}_{0y}$ the electric dipole vector of the $|0\rangle \rightarrow |E_y\rangle$ transition, and the unit vector $\hat{\epsilon}$ denotes the polarization of the pulse. The wave-packet amplitude of a Gaussian SPP is given by

$$\xi(t) = \left(\frac{1}{2\pi\tau_f^2}\right)^{1/4} \exp \left[ -\frac{t^2}{4\tau_f^2} - i\omega_0 t \right], \quad (3)$$

with center frequency $\omega_0$ and pulse length $\tau_f$ [3]. The incident SPP pulse is resonant with $|0\rangle \rightarrow |E_y\rangle$ transition, i.e., $\omega_0 = \omega_y = \omega_1$.

We note that the excited state $|E_1\rangle$ is a superposition of states with spin $|-1\rangle$ and $|+1\rangle$. After the spontaneous decay, a quantum entanglement state $|\psi\rangle = (|\sigma_-\rangle - 1) - |\sigma_+\rangle + 1)$ between a outgoing circularly polarized single photon and the ground spin states of the NV is obtained [4]. The amplifier performs a projection measurement on the spin state of the NV. Each time, only one of the spin states can be detected. Actually, both of the ground spin states $|\pm\rangle$ can provide an effective magnetic field (with inverse direction) for amplifier spins to trigger the quantum phase transition. If the coherence of the NV center has been preserved during the amplification process, the detector will finally go to a NV-amplifier entangled state, i.e., $(|+\rangle \otimes |-M_x\rangle + |1\rangle \otimes |+M_x\rangle)/\sqrt{2}$ ($|-M_x\rangle$ are the excited states of the amplifier with positive and negative spontaneous magnetization in $x$-axis, respectively). Here, without loss of generality, we only take the case that the NV decays to the ground spin state $|1\rangle$ as an example. For simplicity, we use the following denotation hereafter

$\begin{align*}
|0\rangle &= |g\rangle \quad (4) \\
|1\rangle &= |e\rangle \quad (5) \\
|E_y\rangle &= |f\rangle \quad (6) \\
|E_1\rangle &= |h\rangle \quad (7)
\end{align*}$
II. DYNAMICS OF SINGLE-PHOTON DETECTIONS

The full dynamics of the detector under the pumping of the center spin by a single-photon pulse is given by a time-dependent master equation [5, 6]

\[
\frac{d}{dt}\rho_{\text{tot}}(t) = -i[\hat{H}, \rho_{\text{tot}}(t)] + \mathcal{L}_P(t)\rho_{\text{tot}}(t) + \mathcal{L}_{\text{SD}}\rho_{\text{tot}}(t).
\] (8)

Here, \( \hat{H} = \hat{H}_{\text{NV}} + \hat{H}_{\text{Am}} + \hat{H}_{\text{int}} \) is the total Hamiltonian of the detector. The Hamiltonian of the NV is given in equation (1). The the amplifier is described by the Lipkin-Meshkov-Glick (LMG) model [7–9],

\[
\hat{H}_{\text{Am}} = \frac{1}{2} \sum_{j=1}^{N} \hat{\sigma}_j^x - \frac{1}{N} \sum_{i<j} (J_x \hat{\sigma}_i^x \hat{\sigma}_j^x + J_y \hat{\sigma}_i^y \hat{\sigma}_j^y)
\] (9)

with energy splitting \( \epsilon \) in z-direction and the homogeneous long-range couplings \( J_x \) and \( J_y \) in \( xy \)-plane. The interaction between the NV absorber and the amplifier spins is described by

\[
\hat{H}_{\text{int}} = B_x |e\rangle \langle e| \sum_j \hat{\sigma}_j^x.
\] (10)

The initial density matrix \( \rho_{\text{tot}}(t_0) = I_p \otimes \rho_{\text{NV}}(t_0) \otimes \rho_{\text{Am}}(t_0) \) of the whole system is composed of three parts: (1) \( I_p \) is the \( 2 \times 2 \) identity matrix for a \( n \)-photon Fock-state pulse; (2) \( \rho_{\text{NV}}(t_0) = |g\rangle \langle g| \) for the ground-state NV center; (3) \( \rho_{\text{Am}}(t_0) \) the ground-state of \( H_{\text{Am}} \) with specifically engineered bias couplings \( J_x \) and \( J_y \).

The pumping from a quantum pulse is given by

\[
\mathcal{L}_P\rho_{\text{tot}} = \sqrt{\gamma_{fg}} \{ \xi(t-t_0)e^{i\phi}[\hat{\sigma}_+\rho_{\text{tot}}, \hat{\sigma}_f] + \xi^*(t-t_0)e^{-i\phi}[\hat{\sigma}_g, \rho_{\text{tot}}\hat{\sigma}_-] \},
\] (11)

with the spontaneous decay rate \( \gamma_{fg} \) from the excited state \( |f\rangle \) back to the ground state \( |g\rangle \) and ladder operators \( \hat{\sigma}_g = |f\rangle \langle g| \) and \( \hat{\sigma}_f = |g\rangle \langle f| \). Here, \( \eta_j \) characterizes the scattering efficiency of the NV, \( \phi = \vec{k}_0 \cdot \vec{r}_{\text{NV}} \) \( (\omega_0 = c|\vec{k}_0|) \), \( t_0 \) is the propagating time for pulse to arrive at the NV center, and the time-dependent wave-packet amplitude \( \xi(t) \) of the Fock-state pulse is given in equation (3). The raising operator \( (\hat{\sigma}_-)^\dagger = \hat{\sigma}_+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) couples the different photon-number subspace for Fock-state pulse.

As usually the coherence time of the amplifier is much longer than amplification time, we only consider the decay of the center spin from the electronic excited state

\[
\mathcal{L}_{\text{SD}}\rho_{\text{tot}} = \gamma_{fg}[\hat{\sigma}_g\rho_{\text{tot}}\hat{\sigma}_f - \frac{1}{2}\hat{\sigma}_f\hat{\sigma}_g\rho_{\text{tot}} - \frac{1}{2}\rho_{\text{tot}}\hat{\sigma}_f\hat{\sigma}_g] + \gamma_{he}[\hat{\sigma}_h\rho_{\text{tot}}\hat{\sigma}_e - \frac{1}{2}\hat{\sigma}_e\hat{\sigma}_h\rho_{\text{tot}} - \frac{1}{2}\rho_{\text{tot}}\hat{\sigma}_e\hat{\sigma}_h].
\] (12)
Quantum Gain \( +,-(\frac{1}{\gamma^2}) \) a

0 20 40 60 80 ... spin number \( N \). However, the quantum gain increases with \( N \) linearly, which is consistent with our previous result \([15]\) where

\[ |\alpha\rangle = 0.2 \times 10^6 \text{ spins}. \]

Only if the amplifier is optically biased around the critical point \( J_{x,c} \equiv J_y \), the time varying field can trigger a first-order quantum phase transition. Here, the spin-spin coupling in \( y \)-direction is fixed at \( J_y = 0.7\epsilon \) and the absorber-amplifier coupling \( B_x = 0.01\epsilon \). The time-dependent quantum gain \( G(t) \) for different spin-number is shown. Here, the spin-spin couplings are \( J_x = 0.675\epsilon \) and \( J_y = 0.7\epsilon \).

We note that due to the dispersive coupling between the NV center and the amplifier, the dynamics of the absorber and the amplifier are almost "decoupled". After trace off the NV degree of freedom, the dynamics of the amplifier is given by

\[
\frac{d}{dt}\rho_{Am}(t) = -i[H'_{Am},\rho],
\]

where

\[
H'_{Am} = H_{Am} + P_e(t)B_x \sum_j \hat{\sigma}_j^z,
\]

and \( P_e(t) \) is the population of the NV in the state \( |e\\rangle \). Here, we see that after transduction, the amplifier spins experience an effective magnetic field \( P_e(t) \times B_x \) from the absorber. The dynamics of the absorber and the amplifier can be evaluated separately.

In Fig. 2, we give the time-dependent population (the net absorption probability) \( P_e(t) \) of the state \( |e\\rangle \). The absorption probability \( P_e \) increases after the pulse arrives and finally reaches a steady-state value \( P_{e,s} \). Here, the dissipation of the state back to \( |g\\rangle \) has been neglected due to the long life time of the metastate \( |e\\rangle \).

To realize an efficient single-photon transduction, we need to optimize the pulse length \( \tau_f \), the coupling strength \( \Delta'' \) between the two excited states, and the two spontaneous decay rates \( \gamma_{fg} \) and \( \gamma_{he} \). In experiment, we can use filters to tailor the pulse spectrum and change the pulse length \([10]\). The typical value of the coupling strength \( \Delta'' \) is around 1 GHz. Usually, it is hard to tune this coupling strength. However, we can engineer the density of state of the electromagnetic fields to tune the decay rates \( \gamma_{fg} \) and \( \gamma_{he} \) \([11–13]\) to enhance the transduction efficiency. In Fig. 3a, we show the optimization conditions for larger transduction probability. It has been shown that nearly unit transduction probability \( P_{e,s} \) can be realized when \( \gamma_{fg} = \gamma_{he} = \Gamma \gg 1/\tau_f \) for three-level atom system \([14]\). In Fig. 3b, we show that for four-level systems, unit probability \( P_{e,s} \) can also be obtained when \( \Gamma = 2\Delta'' \gg 1/\tau_f \).

In Fig. 4a, we show that only if the spin-spin coupling \( J_x \) is biased close to the phase transition point \( J_{x,c} = J_y \), a large quantum gain can be obtained. The red curve and the yellow curve correspond to \( J_x = 0.675\epsilon \) and \( J_x = 0.5\epsilon \), respectively. In Fig. 4b, we show that time \( T_{Am} \) to reach the maximum of the quantum gain is almost independent on the spin number \( N \). However, the quantum gain increases with \( N \) linearly, which is consistent with our previous result \([15]\).
III. HIGHER-ORDER CORRELATION

In thermodynamic phase transitions, the divergences of the magnetic susceptibility and spatial correlation length are directly related. The Gibbs partition function in a magnetic field \( h \) is given by [16]

\[
Z = \text{Tr} \exp(-\beta \hat{H}_0 + \beta h \hat{M}_a),
\]

where \( \hat{H}_0 \) describes the internal energy of the magnet including spin-spin interactions and \( -h \hat{M}_a \) is the work done against the magnetic field to produce a magnetization \( \langle \hat{M}_a \rangle \) in the direction \( \alpha = x, y, \text{or } z \). The equilibrium magnetization is computed from

\[
\langle \hat{M}_a \rangle = \frac{\partial \ln Z}{\partial h_a} = \frac{1}{Z} \text{Tr}[\hat{M} \exp(-\beta \hat{H}_0 + \beta h_a \hat{M}_a)],
\]

and the susceptibility is then related to the variance of the magnetization by

\[
\chi_\alpha = \frac{\partial \langle \hat{M}_a \rangle}{\partial h_\alpha} = \beta \left\{ \frac{1}{Z} \text{Tr}[\hat{M}_a^2 \exp(-\beta \hat{H}_0 + \beta h_a \hat{M}_a)] - \frac{1}{Z^2} \text{Tr}[\hat{M}_a \exp(-\beta \hat{H}_0 + \beta h_a \hat{M}_a)]^2 \right\},
\]

\[
= \frac{1}{k_B T} \left( \langle \hat{M}_a^2 \rangle - \langle \hat{M}_a \rangle^2 \right).
\]

The magnetization operator for discrete lattice systems is given by

\[
\hat{M}_a = \frac{1}{2} \sum_j \hat{\sigma}_j^a,
\]

for lattice system. Then, the relation between the susceptibility and the spatial correlation function is given by

\[
\chi = \frac{1}{4k_B T} \sum_{ij} \langle (\hat{\sigma}_i^a \hat{\sigma}_j^a) - \langle \hat{\sigma}_i^a \rangle \langle \hat{\sigma}_j^a \rangle \rangle \equiv \frac{1}{k_B T} C_{aa}.
\]

Utilizing the transnational symmetry of a homogeneous system, we can connect the bulk response function with the microscopic two point correlation functions,

\[
C_{aa} = \frac{N}{4} \sum_j (\langle \hat{\sigma}_j^a \rangle - \langle \hat{\sigma}_1^a \rangle)(\hat{\sigma}_j^a - \langle \hat{\sigma}_1^a \rangle) \equiv \frac{N}{4} \sum_j C_{1j}^a.
\]
Figure 6. **Re-scaled correlation function.** Different curves denote different spin number. Here, the spin-spin coupling is fixed at \( J_x = J_y = 0.7/\epsilon \).

In many cases, the correlation function decays as \( G_{\alpha\beta} \propto \exp(-|j - 1|/\xi) \) at separations \( |j - 1| > \xi \). Here, \( \xi \) called the correlation length is the only relevant length at the phase transition point.

However, at the first-order quantum phase transition (QPT) points, the singular behaviors occur on the correlation between the magnetic fluctuations in \( x \) and \( y \) directions. It can be easily verified that the lowest-order symmetrized macroscopic correlation function

\[
C_{xy} = \frac{1}{2} \langle \hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x \rangle - \langle \hat{S}_x \rangle \langle \hat{S}_y \rangle,
\]

(22)
due to the symmetry of the spontaneous magnetization in \( xy \)-plane. Here, \( \hat{S}_\alpha = \sum_i \hat{\sigma}_\alpha^i / 2 \) is the collective angular momentum operator. Thus, we need to consider the higher-order correlation

\[
C_{xxyy} = \frac{1}{2} \langle \hat{S}_x^2 \hat{S}_y^2 + \hat{S}_y^2 \hat{S}_x^2 \rangle - \langle \hat{S}_x^2 \rangle \langle \hat{S}_y^2 \rangle.
\]

(23)

The singular scaling of \( C_{xxyy} \) has been shown in Fig. 4 (a) in the main text. Here, we show that this scaling is independent on the spin number and the position on the phase transition boundary. In Fig. 5a, we contrast the correlation \( C_{xxyy} \) with different spin number. In Fig. 5b, we contrast the correlation \( C_{xxyy} \) with different spin-spin coupling \( J_x = J_y = J \). We see that the value of \( C_{xxyy} \) changes, but the scaling exponent \( \gamma \) of \( C_{xxyy} \propto |B_x|^{-\gamma} \) at the phase transition point remains the same.

As explained in the main text, we can not define a simple correlation length \( \xi \) for the LMG model with indistinguishable spins. However, we may used the rescaled correlation function

\[
\eta = \frac{2}{N} |C_{xxyy}|^{1/4},
\]

(24)
to characterize the proportion of correlated spins. As shown in Fig. 6, the size of correlated spin clusters decreases away from the phase transition point.

[1] Y. Chu, M. Markham, D. J. Twitchen, and M. D. Lukin, Phys. Rev. A 91, 021801 (2015).
[2] J. R. Maze, A. Gali, E. Togan, Y. Chu, A. Trifonov, E. Kaxiras, and M. D. Lukin, New Journal of Physics 13, 025025 (2011).
[3] L.-P. Yang, H. X. Tang, and Z. Jacob, Phys. Rev. A 97, 013833 (2018).
[4] E. Togan, Y. Chu, A. Trifonov, L. Jiang, J. Maze, L. Childress, M. G. Dutt, A. S. Sørensen, P. Hemmer, A. S. Zibrov, et al., *Nature* **466**, 730 (2010).

[5] L.-P. Yang, C. Khandekar, T. Li, and Z. Jacob, arXiv preprint arXiv:1904.02796 (2019).

[6] B. Q. Baragiola, R. L. Cook, A. M. Brańczyk, and J. Combes, *Phys. Rev. A* **86**, 013811 (2012).

[7] H. J. Lipkin, N. Meshkov, and A. Glick, *Nuclear Physics* **62**, 188 (1965).

[8] N. Meshkov, A. Glick, and H. Lipkin, *Nuclear Physics* **62**, 199 (1965).

[9] A. Glick, H. Lipkin, and N. Meshkov, *Nuclear Physics* **62**, 211 (1965).

[10] A. M. Weiner, *Progress in Quantum Electronics* **19**, 161 (1995).

[11] E. M. Purcell, H. C. Torrey, and R. V. Pound, *Phys. Rev.* **69**, 37 (1946).

[12] D. Kleppner, *Phys. Rev. Lett.* **47**, 233 (1981).

[13] E. Yablonovitch, *Phys. Rev. Lett.* **58**, 2059 (1987).

[14] S. M. Young, M. Sarovar, and F. m. c. Léonard, *Phys. Rev. A* **97**, 033836 (2018).

[15] L.-P. Yang and Z. Jacob, *Optics express* **27**, 10482 (2019).

[16] M. Kardar, *Statistical physics of fields* (Cambridge University Press, 2007) , Chap. 1.