Dynamical localization of gap-solitons by time periodic forces

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Abstract – The phenomenon of dynamical localization of matter wave solitons in optical lattices is first demonstrated and the conditions for its existence are discussed. In addition to the trapping linear periodic potential we use a periodic modulation of the nonlinearity in space to eliminate nonexistence regions of gap-solitons in reciprocal space. We show that when this condition is achieved, the observation of dynamical localization in true nonlinear regime becomes possible. The results apply to all systems described by the periodic nonlinear Schrödinger equation, including Bose-Einstein condensates of ultracold atoms trapped in optical lattices and arrays of waveguides or photonic crystals in nonlinear optics.

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An interesting phenomenon occurring in periodic nonlinear systems is the possibility to sustain different types of localized waves for arbitrarily long times due to the interplay between periodicity, dispersion and nonlinearity. In particular, optical lattices (OL) can support the so-called gap-solitons, i.e. localized solutions of the periodic nonlinear system with energies located inside forbidden gaps of the spectrum. These solutions exist both for focusing (self-attractive) and defocusing (self-repulsing) interactions due to the coupling (induced by the nonlinearity) between forward and backward propagating waves undergoing Bragg scattering [1]. Gap solitons can be found in many physical systems including photonic crystals and layered structures [2], arrays of optical fibers [3] and Bose-Einstein condensates (BECs) in OL. In this last case, gap-solitons have been experimentally observed in [4] for BEC with positive scattering lengths (repulsive interactions) and for continuous BEC [7,8] (see also reviews [9–11]). As stationary states with energies inside band gaps they are intrinsically localized and remain stable for long time. By applying external forces, such as the one generated by the gravity (for BEC in periodic vertical traps) or by the acceleration of the OL, solitons can be forced to move. A moving soliton however has a frequency belonging to an allowed band. It is no longer an exact solutions of the one-dimensional Gross-Pitaevskii equation with a periodic potential, but can be viewed as a persisting wave packet.

On the other hand, it is known from the linear theory, that a time periodic force applied to a quantum particle in a lattice can result in the phenomenon of dynamical localization [12]. In the simplest terms this effect can be viewed as a resonance which occurs when the strength of the force and its frequency take certain discrete values. Then a particle undergoes oscillatory motion in a finite spatial domain (namely in this sense the localization is understood). Outside of these resonances the particle acquires a drift velocity which enables transport. For sinusoidal time-dependent forces resonances are found to be roots of the zeroth-order Bessel function. On a pure quantum-mechanical level [13] dynamical localization can also be characterized as a complete suppression of

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the inter-well tunneling\(^1\) of the particle in a periodic potential [14,15].

The linear dynamical localization has been experimentally observed in several systems including arrays of curved optical waveguides [16] and BEC trapped [15,17]. Properties of the ac forces (electrical fields) that result in the dynamical localization of an electron in a periodic potential were theoretically investigated in [18] where it was shown that for generic (i.e. beyond tight-binding) band structures exact dynamical localization is possible only for electric field displaying discontinuities at all changes of the sign of the field (the dynamical localization phenomenon, however, was shown to be surprisingly tolerant to the smoothing of these discontinuities [19]).

Thus, while the concept of a gap-soliton is intrinsically nonlinear the concept of dynamical localization is independent of nonlinearity. At the same time, the arguments applicable to a single particle do not mean of course that a wave packet composed of oscillating atoms will oscillate without dispersion, i.e. preserving its width. Then the natural question arises: Is it possible to observe dynamical localization of a nonspreading wave packets, i.e. of gap-solitons? Possibility of positive answer to this question stems from the results reported in ref. [20], where it was shown that properly designed nonlinearity can guarantee long-lived Bloch oscillations of gap-solitons of small amplitude originated by constant forces.

The aim of the present letter is to show for the first time how the dynamical localization of gap-solitons of the continuous periodic nonlinear Schrödinger (NLS) models can be achieved by means of suitable spatial modulations of the nonlinearity (nonlinear OL). For BEC in OL this modulation can be obtained by changing the scattering length along the condensate using the optically induced Feshbach resonance technique. We shall use the spatial modulation to suppress (or strongly reduce) nonexistence regions for gap-solitons in the Brillouin zone (BZ). As a result we have that the resonances for the dynamical localization of gap-solitons are the same as the one obtained in the seemingly very different linear system.

An intuitive understanding of why the oscillatory motion of single particles is accompanied by the spreading out of the wave packets and why this situation can be inverted in presence of space-dependent nonlinearities can be obtained from the analysis of nonlinear discrete models. It is known that the usual (e.g., physical) discretization of the NLS equation (DNLS) [21],

\[
 i\partial_n c_n = \omega_0 c_n + \omega_1 (c_{n-1} + c_{n+1}) + \chi |c_n|^2 c_n , \tag{1}
\]

has the onsite nonlinearity and does not display any dynamical localization phenomenon in presence of ac linear forces (any initially localized wave packet quickly spread out under the action of the force). In contrast with its continuous limit, eq. (1) is nonintegrable, having only two conserved quantities: the energy and the norm. In BEC contexts this equation arises as a tight binding model of the NLS equation and is used to describe arrays of coupled BECs [5,6]. The integrable discrete version of the NLS equation, known as the Ablowitz-Ladik (AL) model, has the nonlinearity nonlocal i.e. splitted on two adjacent sites: \( \chi |c_n|^{a}|c_{n+1}| \), (which must be substituted in (1) instead of \( \chi |c_n|^{a}_{c} \)) and it is known that it does display the dynamical localization phenomenon. The first observation of dynamical localization for discrete solitons was indeed achieved for this model [22,23]. The generalization of the theory of dynamical localization by including the one-site nonlinearity, as well the understanding of the physical picture, was suggested in [23] using the discrete nonlinear Schrödinger (DNLS) equation as a governing model, where again the same frequencies leading to the dynamical localization were obtained.

To understand why the dynamical localization phenomenon of undistorted spatially localized wave packets occurs in the AL model and does not occur in the DNLS chain it is of interest to consider how the stability of linear modes of the underlying discrete linear chains (they are discrete analogies of the Bloch states of the continuum models) is affected by the nonlinearity in the two cases. One finds that the stationary solutions of eq. (1) at different band edges are connected by a staggering transformation. It is indeed easy to prove that if \( c_n = \exp(-i\omega t) f_n \) is a stationary solution of eq. (1) for frequency \( \omega \) and nonlinearity \( \chi \), then \( \tilde{c}_n = \exp(-i(2\omega_0 - \omega)t - i\pi n) f_n \) is a solution for frequency \( 2\omega_0 - \omega \) and nonlinearity \( \tilde{\chi} = -\chi \). This means that if at one edge of the band a mode is modulationally unstable (and, hence the systems admits the existence of small-amplitude solitons), at the other edge of the band for the same nonlinearity the same mode is modulationally stable and small-amplitude solitons cannot exist. For the AL model, however, one finds that to a stationary solution \( c_n \) for frequency \( \omega \) and nonlinearity \( \chi \) corresponds a solution \( \tilde{c}_n \) for frequency \( 2\omega_0 - \omega \) and the same nonlinearity \( \chi \). This means that for the AL model it is possible to have stable small-amplitude solitons at both edges of the band for a fixed nonlinearity. Actually one can prove that in this case solitons can exist for all values of the crystal momentum since the corresponding plane waves are modulationally unstable in the whole BZ. This means that AL solitons can slide along the whole band under the action of a time-dependent force without undergoing any passage through the nonexistence region.

While the AL model itself has restricted physical applications, it shows the route for achieving the stability of gap-solitons in the whole band with the help of nonlocal nonlinear terms. At the same time, such terms naturally appear in systems with inhomogeneous nonlinearities (for a number of nonlocal Hamiltonian lattices and their properties see e.g., [24]). Thus one can expect that the dynamical localization of continuous gap-solitons could be achieved in presence of a \( x \)-dependent nonlinearity. This conclusion can be reached also from another perspective

\(^1\) The complete suppression of inter-well tunneling is a valid statement only in the high-frequency limit.
The conditions of the soliton existence in the whole band was depicted with respect to the spectral width of the soliton. True when the nonexistence regions are sufficiently wide, leading to its spreading and destruction (this is especially the soliton will not exist at both edges of the band, this effectivenonlinearity can be changed only if the nonlinear- and therefore on the crystal momentum, the sign of the effective mass depend on the band. While the sign of the effective mass depends on the band and therefore on the crystal momentum, the sign of the effective nonlinearity can be changed only if the nonlinearity is space dependent. For a constant nonlinearity, indeed, the soliton will not exist at both edges of the band, this leading to its spreading and destruction (this is especially true when the nonexistence regions are sufficiently wide with respect to the spectral width of the soliton)\(^2\).

The main idea for dynamical localization in the nonlinear case then is to suppress (or strongly reduce) nonexistence regions to allow the gap-soliton to safely slide along the whole band under the action of the external force. To show how this can be done, we consider the specific case of the NLS equation with periodic potential and ac linear force:

\[
\dot{\psi}_0 = -\psi_0 + i(U(x)\psi + \gamma(t)x\psi + \mathcal{G}(x)|\psi|^2\psi. \tag{2}
\]

In the BEC context the spatial and temporal variables in eq. (2) are measured in the units of \(d\) and \(\hbar E_r\), respectively, while the energy is measured in units of the recoil energy \(E_r = \hbar^2 \pi^2 / (2md^2)\), where \(d\) is the lattice period and \(m\) is the atomic mass. In a typical situations, the external force \(\gamma(t)\) arises from the acceleration of the linear OL. Then eq. (2) describes the condensate in the moving frame. This implies that the nonlinear OL should be also accelerated in the same manner as the linear one. Notice that eq. (2) appears both in BEC and in nonlinear optics contexts, although the physical contexts are completely different.

\(\gamma(t)\) is the zeroth-order Bessel function of the first kind and \(\mathcal{G}(Q)\) is the Bloch velocity. Thus, the mean velocity of the soliton is given by

\[
\langle V_S \rangle = \mathcal{V}_B(Q_0) J_0 \left( \frac{\Gamma \pi}{\Omega} \right), \tag{6}
\]

i.e., it coincides with the Bloch velocity at the time \(t_0\) of switching on the time-dependent force but scaled by the factor \(J_0\left( \frac{\Gamma \pi}{\Omega} \right)\).

As can be seen from eq. (6), if \(\Omega = \Omega_n = \Gamma \pi / z_n\) (\(z_n\) is the \(n\)-th zero of the Bessel function \(J_0(z_n) = 0\), then \(\langle V_S \rangle = 0\) so that the motion becomes perfectly periodic \(X(t_0 + T) = X(t_0)\) and the soliton wave packet is dynamically localized. In fig. 1b we depict the soliton mean velocity as a function of \(Q(t_0)\) for fixed parameters of the OLs and fixed frequencies \(\Omega\). From this figure it is evident that the soliton becomes dynamically localized for any \(\Omega\) at the points \(Q(t_0) = m, m = -1, 0, 1\) in the BZ. The absence of transport at these points, however, is not due to force resonances but simply to the Bloch velocity becoming zero (from this point of view the dynamics at these points should not be considered as dynamical localization).

The external force will be assumed to oscillate periodically in time. While this is not important for the theory, to simplify the formulas, and to bring the model closer to the experimentally feasible situations in the theory of matter waves, we consider the cos-like law \(\gamma(t) = \Gamma \cos[\Omega(t - t_0)]\) with amplitude \(\Gamma\) and frequency \(\Omega\) (time \(t_0\) characterizes the initial phase).

Let us now briefly recall the semiclassical arguments which allow one to account for the linear force (see, e.g., [13]). Identifying the center of a Bloch wave packet, \(X\), with the coordinate of a quasiparticle (which in the nonlinear case is nothing but a matter wave soliton) one finds that it is described by the equations

\[
\dot{X} = v(Q) = \frac{dE(q)}{dq}|_{q=Q}, \quad \dot{Q} = -\gamma(t), \tag{3}
\]

where we also introduce the wave-packet center \(Q\) in the reciprocal space, use \(v(Q)\) for the Bloch velocity and \(E(Q)\) for the linear OL energy band structure (see fig. 1a).

Taking into account cos-like dependence of \(\gamma(t)\), the expression for the coordinate can be represented as

\[
X(t) = X_0 - 2\omega_1 \pi \int_{t_0}^{t} \sin \left\{ \pi Q_0 - \Gamma \pi \sin[\Omega(t - t_0)] \right\} d\tau,
\]

where we have left only the two leading terms in the Fourier expansion of the energy \(E(q) \approx \omega_0 + 2\omega_1 \cos(\pi q)\), and use the notations \(X(t_0) = X_0\) and \(Q(t_0) = Q_0\). In this case after one period \(T = 2\pi / \Omega\) of the external force oscillation, the coordinate of the wave packet will be

\[
X(t_0 + T) = X_0 + v_B(Q_0) J_0 \left( \frac{\Gamma \pi}{\Omega} \right) T \tag{5}
\]

The drift velocity as a function of the initial crystal period and \(\gamma = 0.001\) (thin line). The linear and nonlinear OLs are fixed as \(U(x) = -3 \cos(2x)\) and \(\mathcal{G}(x) = -0.777 + \cos(2x)\).

\(\gamma\) is the effective mass of the soliton. The lowest linear band energy \(E\) vs. crystal momentum \(Q\) (thick line) and the evolution of energy of the soliton with norm \(N = 5.42\) during its movement under the constant external force \(\gamma(t) = 0.001\) (thin line). The evolution of the soliton crystal momentum is \(Q(t) = -1 + \Gamma t\). The drift velocity as a function of the initial crystal momentum of the gap-soliton for the nonresonant frequency \(\Omega = 0.001437\). The linear and nonlinear OLs are fixed as \(U(x) = -3 \cos(2x)\) and \(\mathcal{G}(x) = -0.777 + \cos(2x)\).
Fig. 2: (Colour on-line) Soliton center position $X$ vs. time $t$ (upper row, panels a–c) and particle density $|\psi|^2$ vs. time $t$ (lower row, panels a–c) and coordinate $x$ in linear OL $U(x) = -3 \cos(2x)$ and $G(x) = -0.777 + \cos(2x)$, and external force parameters $\Gamma = -0.001$, $t_0 = 500.0$ (upper and middle rows, panels a–f), $t_0 = 1000.0$ (lower row, panels g–i), $\Omega \approx 0.001306$ (left column, panels a, d, g), $\Omega \approx 0.001176$ (middle column, panels b, e, h) and $\Omega \approx 0.001437$ (right column, panels c, f, i). The initial condition at $t = 0$ is a stationary gap-soliton located at $X(0) = 0$ with energy $E = -0.7324$ near the upper edge of the band ($Q(0) = 1.0$). In panels a–c thick (red) lines represent the calculation according to approximation (4), and thin (blue) lines the direct numerical integration of eq. (2).

To check these predictions, we performed direct numerical integrations of the NLS equation (2) to obtain the evolution of the center of the soliton $X(t)$ when an external force starts to oscillate at time $t = t_0$

$$\gamma(t) = \begin{cases} \Gamma, & t < t_0, \\ \Gamma \cos[\Omega(t-t_0)], & t \geq t_0, \end{cases}$$

(7)

The results are depicted in fig. 2. As can be seen from panel (a), when $\Omega = \Omega_1$ (corresponds to the first zero of Bessel function), the dependence, calculated from approximation (4), exhibits the periodic motion. Nevertheless, the period of this motion is two times less than that of external force oscillations $2\pi/\Omega_1 \approx 4810$, because the period is imposed mainly by the Bloch oscillations, whose period is $2/\Gamma = 2000.0$. The dependencies in the upper panel, calculated from direct simulation, exhibit movement, slightly different from periodic due to the fact, that we restricted expansion of $E(q)$ into Fourier series by zeroth and first harmonics only. In cases, when $\Omega$ differs from $\Omega_1$ in panels (b) $\Omega = 0.9\Omega_1$ and in panel (c) $\Omega = 1.1\Omega_1$, the soliton movement looses its periodicity.

In panels d–i of fig. 2 we depict the time evolution of the matter density as obtained from direct integration of the NLS equation. Panels d, e, f correspond to panels a, b, c, respectively. Notice that in all the cases the soliton is long lived as a consequence of a proper design of the OLs. In the nonresonant case $J_0 \left( \frac{\Gamma \pi}{\Omega_1} \right) \neq 0$ (panels e, f) the mean velocity of the soliton is $\langle V_S \rangle \approx 0.03722$ (panel e) and $\langle V_S \rangle \approx -0.04192$, this agreeing very well with the value predicted by eq. (6) (see fig. 1).

In panels g–i of fig. 2 we have shown that localization can also be induced in nonresonant case if the initial Bloch velocity at time $t_0$ is zero, in agreement with what predicted by eq. (6) (notice that $t_0 = 1000$ and the external force oscillation starts when the soliton is at the bottom of the band $Q(0) = 0$). In this case the dynamical localization can occur for any frequency $\Omega$ of the external force, as expected from eq. (5).

To demonstrate the importance of the nonlinear OL for the stability of dynamical localization of solitons, we display in fig. 3a, b, the dynamics of soliton under the periodic external force with the same parameters, as those of fig. 2d, e, except for the case of coordinate-independent nonlinearity $\mathcal{G}(x) = 1$. As evident from fig. 3, in the case of constant nonlinearity solitonic oscillations in real space are not stable. This happens due the above-described mechanism of soliton distortion in the modulationally stable regions of BZ. This situation differs
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considerably from the case, depicted in fig. 2, where special shape of nonlinear OL results in narrow width of the modulationally stable regions in the BZ (see ref. [20]) so that they do not affect considerably on the soliton stability during its propagation in the real and reciprocal spaces.

The phenomenon of dynamical localization can occur also, when external force oscillation frequency corresponds to the higher roots of Bessel function. In other words, within the framework of the meanfield theory, there exist no lower limit on the frequency leading to the dynamical localization. Using the well-known asymptotic of the Bessel functions one obtains the behavior of the low-frequency limit (recall that they are measured in the units \(E_r/h\)): \(\Omega \sim \Gamma/(n - \frac{1}{4})\). Reducing the frequency however, generally speaking leads to larger amplitudes of soliton oscillations and more sophisticated trajectories. An example of this is reported in fig. 4 where the dynamical localization is shown for the case \(\Omega = \Omega_2\). In this case, however, we see a small discrepancy between a tight-binding approximation and direct numerical simulation of the NLS system in eq. (2) (fig. 4a).

In closing this letter we wish to remark that periodically driven OLs have also been recently used experimentally to achieve coherent control of matter waves [25] and theoretically to investigate phase transition from superfluid to a Mott insulator by means of Bose-Hubbard model [14]. In both cases an effective renormalization of the tunneling matrix element in the Bose-Hubbard Hamiltonian was assumed. In the present letter we have dealt with a quite different situation in which each lattice site is populated by many atoms so that the mean field approach is perfectly valid. The presence of an inhomogeneous nonlinearity in our case permits the existence of gap soliton for all points in the BZ. The gap-soliton dynamics is then well described by the semiclassical equation of motion and is very stable, without any visible increase of the local density, this ruling out any possibility for occurrence of phase transitions or failure of the 1D approximation (the latter occurring when the two-body interaction energy become comparable to the transverse kinetic energy).

In conclusion, we have shown for the first time the possibility of nonlinear dynamical localization of matter wave solitons by means of a spatial modulation of the nonlinearity. In a BEC this can be achieved by changing the scattering length by means of optically (or magnetically) induced Feshbach resonances while the ac force can be generated by a time periodically accelerated OL. For other possible applications of the theory, which are arrays of coupled nonlinear waveguides, the spatial change of the nonlinearity can be made by changing the refractive index in the fibers along the array with dopants, while the periodic linear force could be applied by curving the fibers along the propagation direction.

The phenomenon of dynamical localization of solitons in OL is expected then to be observed both in BEC and in nonlinear optical systems.

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