Environment Viewed from the Particle and Slowdown Estimates for Ballistic RWRE on $\mathbb{Z}^2$ and $\mathbb{Z}^3$

TAL PERETZ*

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Abstract

We consider a random walk in a random environment on $\mathbb{Z}^d$ under Sznitman’s ballisticity condition ($T$). Using techniques from [3], we show the existence of the invariant measure $Q$ with respect to the environment viewed from the particle for $d = 2$ and $d = 3$. This disproves a conjecture made in [3] that the invariant measure does not exist in dimension 2. As a corollary, we prove sharp tail bounds for regeneration times for $d = 3$. Finally, we also provide tail estimates for the Radon-Nikodym derivative $dQ/dP$, where $P$ is the original distribution on the environment.

Keywords and phrases. Random walks in random environment; ballisticity; equivalence of static and dynamic points of view; slowdown of random walk.

MSC 2020 subject classifications. 60K37, 82D30.

1 Introduction

1.1 Background

Random Walk in a Random Environment (RWRE) is one the central models for random motion in non-homogeneous media. The model is defined by creating a random environment, and then defining a Markov chain that depends on the environment. More formally, denote by $M_1$ the space of all probability measures on $\mathcal{E}_d = \{ \pm e_j \}_{j=1}^d$, where $e_j$ are the canonical coordinate vectors for $\mathbb{Z}^d$, and define $\Omega = M_1^{\mathbb{Z}^d}$. An environment is an element $\omega \in \Omega$, and for $z \in \mathbb{Z}^d$ we denote $\omega(z, e)$ the probability assigned to a coordinate vector $e$ by $\omega(z)$. Given $\omega \in \Omega$ and $z \in \mathbb{Z}^d$, we can define a time-homogeneous Markov chain $\{X_n\}_{n=0}^\infty$ on $\mathbb{Z}^d$ starting at $z$ with transition probabilities

$$P^z_n(X_{n+1} = y + e | X_n = y) = \omega(y, e) \quad \forall y \in \mathbb{Z}^d, e \in \mathcal{E}_d.$$ 

$P^z_n(\cdot)$ is called the quenched measure of the random walk. Let $P$ be a probability measure defined on $\Omega$, and let $E$ be its expectation. We can average the quenched measure over all of the environments and define a new distribution for the random walk

$$\mathbb{P}^z(\cdot) = E[P^z_n(\cdot)],$$

*Technion - Israel Institute of Technology. E-mail: tal.peretz@campus.technion.ac.il
which we call the annealed measure. We write $E^z_\omega[\cdot]$ and $E^z[\cdot]$ for the expectation with respect to the quenched and annealed measure, respectively. In this paper, we will consider RWRE that satisfy

- $P$ is an i.i.d. measure: $P = \nu^{Z^d}$ for some measure $\nu$ on $M_1$.
- $P$ is uniformly elliptic: there exists $\kappa > 0$ such that for all $x \in Z^d$,
  \[ P\left(\min_{e \in E_d} \omega(x,e) > \kappa\right) = 1. \]

In [21, 22, 23], the authors proved a law of large numbers when $P$ satisfies the above conditions:

\[ \lim_{n \to \infty} \frac{X_n}{n} = v, \quad \mathbb{P}^0 - \text{a.s.} \quad (1.1) \]

However, from the statement it is not clear whether $v$ is deterministic and non-vanishing. When $v$ is a $\mathbb{P}$-almost sure non-zero constant, we say the random walk is ballistic. In the last two decades, there has been a research focus in studying ballistic RWRE and answering questions surrounding (1.1). In [19, 20], Sznitman introduced a new criterion for ballisticity. To define this, let $L > 0$ and $\ell \in S^{d-1}$, where $S^{d-1} = \{z \in \mathbb{R}^d : |z| = 1\}$. Define the hitting times

\[ T_L = T^{(\ell)}_L = \inf\{n \geq 0 : \langle X_n, \ell \rangle \geq L\}, \]

and for $A \subset Z^d$ set

\[ T_A = \inf\{n \geq 0 : X_n \in A\}. \]

**Definition 1.1.** (Sznitman [20]) We say that $P$ satisfies condition ($T$) in direction $\ell_0 \in S^{d-1}$ if for every $\ell \in S^{d-1}$ in some open neighborhood of $\ell_0$ there exist positive constants $C$ and $c$ such that

\[ \mathbb{P}^0(T^{(-\ell)}_L < T^{(\ell)}_L) \leq Ce^{-cL}, \]

for every $L > 0$.

Condition ($T$) has two other equivalent formulations, one being an effective criterion, see [20], and the other a moment assumption on regeneration times, see [19]. This condition was introduced in order to guarantee ballisticity, which the following theorem justifies.

**Theorem 1.2** (Sznitman [20]). If condition ($T$) holds for some direction $\ell_0$, then

\[ \lim_{n \to \infty} \frac{X_n}{n} = v \]

for some deterministic constant $v \in \mathbb{R}^d$ that satisfies $\langle v, \ell_0 \rangle > 0$.

At the moment, condition ($T$) is the most general assumption that implies ballisticity, and it is even conjectured to be equivalent to ballisticity. There has been a research focus in generalizing this condition and relaxing the exponential decay assumption in Definition 1.1, see [4, 9, 10, 12] for examples.
1.2 Environment Viewed from the Particle

One of the major tools in studying RWRE is an auxiliary Markov chain on the space of environments \( \Omega \) called the environment viewed from the particle

\[
\overline{\omega}_n = \sigma_{X_n} \omega = \omega(X_n + \cdot) \text{ for } n \in \mathbb{N}_0,
\]

where \( \sigma_x \) for \( x \in \mathbb{Z}^d \) denotes the canonical shift on \( \Omega \). One of the main technical difficulties with RWRE is that the random walk is not a Markov chain under the annealed distribution. An advantage of \( \overline{\omega}_n \) is that under \( \mathbb{P} \) it is a Markov chain with compact state space \( \Omega \), initial distribution \( \mathbb{P} \) and transition kernel

\[
\mathcal{R}f(\omega) = \sum_{e \in \mathcal{E}_d} \omega(x, e) f(\sigma_e \omega)
\]

for every bounded measurable \( f : \Omega \to \mathbb{R} \). To prove limit theorems for RWRE, it will be advantageous for \( \overline{\omega}_n \) to converge to some invariant distribution. For this reason, we say that a probability measure \( Q \) on \( \Omega \) is invariant with respect to the point of view of the particle if

\[
\int_{\Omega} \mathcal{R}g(\omega)dQ(\omega) = \int_{\Omega} g(\omega)dQ(\omega),
\]

for every bounded continuous \( g : \Omega \to \mathbb{R} \). While invariant measure are not difficult to find, they are extremely useful if they are also equivalent to \( \mathbb{P} \). When such a measure exists, we say the static and dynamic points of view of the environment are equivalent. Such invariant measures have been used to prove central limit theorems and law of large numbers, see \([1, 6, 7, 11, 13, 22]\) for such examples. Consequently, there is a research focus to find invariant measures which are equivalent to \( \mathbb{P} \).

In \([6]\), Bolthausen and Sznitman showed that \( Q \) exists for certain high-dimensional ballistic random walks. Their methods relied on bounding the number of intersections of independent random walks. In \([3]\), Berger, Cohen and Rosenthal extended their result by proving that for \( d \geq 4 \) and under condition \((T)\), the static and dynamic points of view of the environment are equivalent. In addition, they also proved a local limit theorem for the quenched heat kernel. The main technical step in their proof, which goes back to \([2]\), is bounding certain martingale differences in terms of the number of intersection of two independent random walks in the same environment. Due to the small number of intersection in high dimensions, \([3]\) and \([6]\) were able to prove the existence of \( Q \) for \( d \geq 4 \). The authors also conjectured that for \( \mathbb{Z}^2 \), where there are many intersections, there is no invariant measure equivalent to \( \mathbb{P} \). Our first result is that for \( d = 2 \) and \( d = 3 \) such an invariant measure exists, which disproves their conjecture, as well as a local limit theorem.

**Theorem 1.3.** Let \( d \geq 2 \), and assume \( \mathbb{P} \) is uniformly elliptic, i.i.d. and satisfies condition \((T)\). Then there exists a unique invariant probability measure \( Q \) on \( \Omega \) which is invariant with respect to \( \mathcal{R} \) and is equivalent to \( \mathbb{P} \). Furthermore, we have for \( \mathbb{P} \)-almost every \( \omega \in \Omega \)

\[
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \left| P_0^\omega(X_n = x) - \mathbb{P}^0(X_n = x) \frac{dQ}{dP}(\sigma_x \omega) \right| = 0.
\]

**Remark 1.4.** This theorem was proved in \([3]\) for \( d \geq 4 \). For the one-dimensional, Alili proved in \([1]\) the existence of \( Q \) whenever the random walk is ballistic.
1.3 Slowdown

Another major tool in studying RWRE is regeneration times, which were introduced in [21] for the purpose of proving (1.1).

**Definition 1.5.** Fix $\ell \in S^{d-1}$. We define regeneration times for $\{X_n\}$ in direction $\ell$ as

$$\tau_1 = \inf \{n \in \mathbb{N} : \langle X_k, \ell \rangle < \langle X_n, \ell \rangle \text{ for all } k < n \text{ and } \langle X_k, \ell \rangle \geq \langle X_n, \ell \rangle \text{ for all } k > n\},$$

and for $j \in \mathbb{N}$

$$\tau_{j+1} = \inf \{n > \tau_j : \langle X_k, \ell \rangle < \langle X_n, \ell \rangle \text{ for all } k < n \text{ and } \langle X_k, \ell \rangle \geq \langle X_n, \ell \rangle \text{ for all } k > n\}.$$ 

To guarantee regeneration times in direction $\ell$, we will need to assume the random walk is transient in this direction:

$$\lim_{n \to \infty} \langle X_n, \ell \rangle = +\infty \quad \mathbb{P}\text{-a.s.}$$

**Theorem 1.6** (Sznitman, Zerner [21]). Assume $P$ is uniformly elliptic, i.i.d. and transient in direction $\ell_0$. Then $\mathbb{P}$-a.s. there exist infinitely many regeneration times $\tau_1 < \tau_2 < \ldots$. Furthermore, $\{(\tau_{k+1} - \tau_k, X_{\tau_{k+1}} - X_{\tau_k})\}_{k \in \mathbb{N}}$ is an i.i.d ensemble and $\mathbb{P}(\tau_2 - \tau_1 > u) \leq C \mathbb{P}(\tau_1 > u)$ for some constant $C > 0$ and all $u > 0$.

The theorem implies that $X_n$ can roughly be considered as a sum of i.i.d. random variables. Hence questions regarding limit theorems comes down to understanding the moments of the regeneration times. For example, Sznitman and Zerner in [21] showed the random walk is ballistic when the first moment of $\tau_1$ is finite, and Sznitman proved in [17] an annealed invariance principle when the second moment of $\tau_1$ is finite. In [5], Berger and Zeitouni proved for ballistic RWRE a quenched invariance principle assuming a fixed number of moments of the regeneration times exist and an annealed invariance principle. For this reason, there has been much effort in understanding the tail behavior of regeneration times. In [17, 18], Sznitman showed that they are dominated by traps, atypical pockets in the environment where the random walk is likely to spend a long time.

Define the local drift at $x$ for the environment $\omega$ by

$$d(x, \omega) = E^x_\omega[X_1 - X_0] = \sum_{|e| = 1} \omega(x, e)e.$$ 

**Definition 1.7** (Sznitman [18]). Let $K_0$ be the convex hull of the support of the law of $d(0, \omega)$. We say $P$ is nestling if $0$ is in the interior of $K_0$.

Sznitman showed that for $d \geq 1$ and when $P$ is nestling

$$\mathbb{P}^0(\tau_1 > u) > Ce^{-c(\log u)^d} \tag{1.2}$$

for some fixed constants $C, c > 0$ by creating the na"ive trap, see Lemma 4.6 for a precise definition. Deriving a semi-local limit theorem akin to Proposition 2.10 and then applying it to a renormalization scheme, Berger was able to prove a matching upper bound in [2]: for $d \geq 4$, and for every $\alpha < d$, there exists $C > 0$ such that

$$\forall u \geq 0, \mathbb{P}^0(\tau_1 > u) < Ce^{-(\log u)^\alpha}.$$ 

The following result extends this upper bound to $d = 3$. 

4
Theorem 1.8. Let $d = 3$ and assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). Then for every $\alpha < 3$ there exists $C > 0$ such that

$$\forall u \geq 0, \ P^0(\tau_1 > u) \leq Ce^{-(\log u)\alpha}.$$ 

Remark 1.9. An upper bound for $P^0(\tau_1 > u)$ in the one-dimensional case was proven in [3], and so the only remaining dimension left is $d = 2$. We were not able to prove this since the proof requires the variance of $E_\omega X_n$ to be sufficiently small, which is not the case in dimension 2.

Our last result draws a connection between the slowdown effect and the environment viewed from the particle. Intuitively, for an invariant measure $Q$ which is equivalent to $P$ to exist, the random walk should be well mixing in the environment. However, if the random walk spends atypical amount of time in a trap, this will not be the case. For example, Sabot showed in [16] that for random walks on Dirichlet environments, the invariant measure exists if and only if the expected occupation time of a ball is finite. This is related to the following open question (see [11] for a discussion): assuming the random walk is transient in a certain direction, is the existence of $Q$ equivalent to the random walk being ballistic? All of this suggests that in our setting, the tail behavior of $dQ/dP$ is dominated by traps in the environment, just like regeneration times. The following theorem justifies this intuition.

Theorem 1.10. Assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). For $d \geq 3$, for every $\alpha < d$ there exist constants $C, c > 0$ such that

$$\forall u \geq 0, \ P\left(\frac{dQ}{dP} > u\right) \leq Ce^{c(\log u)^{\alpha}}.$$ 

For $d \geq 2$ and assuming $P$ is nestling, there exist constants $C, c > 0$ such that

$$\forall u \geq 0, \ P\left(\frac{dQ}{dP} > u\right) \geq Ce^{c(\log u)^{d}}.$$ 

Remark 1.11. The lower bound in Theorem 1.10 is closely related to the event yielding the lower bound for regeneration times derived by Sznitman. This theorem shows that the static and dynamic points of view of the environment differ due to traps in the environment.

The rest of the paper is organized as follows. The main ingredient for this paper is an estimate for the quenched heat kernel, see Proposition 2.10, which we prove in Section 2 by using the same martingale argument used in [3]. For our proofs, we will need new intersection estimates for $d = 2$ and $d = 3$. In Section 3 we prove Theorem 1.8 which is an easy corollary of the semi-local limit theorem. In Section 4 we prove Theorem 1.10. This requires extending the semi-local limit theorem to finite boxes, see Lemma 4.4, and proving the lower bound, see Lemma 4.6. A few remarks about the proofs in this paper:

- The constants $c$ and $C$ may change from line to line, while numbered constants $c_1, c_2, \ldots$ are fixed at their first appearance.
- When assuming $P$ satisfies condition (T) in direction $\ell$, without any loss we will assume $\ell = e_1$. In this case, by Theorem 1.2 the limiting velocity $v$ satisfies $\langle v, e_1 \rangle > 0$. 

5


2 Quenched Heat Kernel Estimates

2.1 Preliminaries

We begin by introducing notation and results from [2, 3].

Definition 2.1. For positive sequences $a_n$ and $b_n$, we write

\[ a_n = o(b_n) \quad \text{or} \quad a_n \ll b_n \quad \text{if} \quad \lim_{n \to \infty} a_n/b_n = 0 \]

\[ a_n = \xi(b_n) \quad \text{or} \quad a_n \gg b_n \quad \text{if} \quad \lim_{n \to \infty} b_n/a_n = 0 \]

\[ a_n \asymp b_n \quad \text{if} \quad 0 < \liminf_{n \to \infty} a_n/b_n \leq \limsup_{n \to \infty} a_n/b_n < \infty \]

\[ a_n = O(b_n) \quad \text{if} \quad \limsup_{n \to \infty} a_n/b_n < \infty. \]

Note that in our notation, $n^{-\xi(1)}$ is a sequence going to zero faster than any polynomial of $n$.

Definition 2.2. Define the asymptotic direction of the random walk

\[ \vartheta = \lim_{n \to \infty} \frac{X_n}{|X_n|} \in S^{d-1}, \]

where $|\cdot|$ is the Euclidean norm. Note that by our earlier assumption, $\langle \vartheta, e_1 \rangle > 0$.

Definition 2.3. Define the sequence $R_j(N) = \exp((\log N)^{\frac{j+1}{j+2}})$. We will often use the fact that for any $\epsilon > 0$ and any $\kappa > 0$, we have

\[ \log N \ll R_\kappa^\epsilon(N) \ll R_{j+1}(N) \ll N^\epsilon \]

as $N \to \infty$.

Definition 2.4. For $z \in \mathbb{Z}^d$ and $N \in \mathbb{N}$, define

1. The parallelogram of side-length $N$ and center $z$ to be

\[ \mathcal{P}(z, N) = \left\{ x \in \mathbb{Z}^d : |\langle x - z, e_1 \rangle| < N^2, \left\| x - z - \vartheta \cdot \frac{\langle x - z, e_1 \rangle}{\langle \vartheta, e_1 \rangle} \right\|_\infty < NR_5(N) \right\}. \]

2. The middle third of $\mathcal{P}(z, N)$

\[ \tilde{\mathcal{P}}(z, N) = \left\{ x \in \mathbb{Z}^d : |\langle x - z, e_1 \rangle| < N^2/3, \left\| x - z - \vartheta \cdot \frac{\langle x - z, e_1 \rangle}{\langle \vartheta, e_1 \rangle} \right\|_\infty < NR_5(N)/3 \right\}. \]

3. The right boundary of $\mathcal{P}(z, n)$

\[ \partial^+ \mathcal{P}(z, N) = \left\{ x \in \mathcal{P}(z, n) : \langle x - z, e_1 \rangle = N^2 \right\}. \]

We will often need estimates for the probability that the random walk does not exit $\mathcal{P}(z, N)$ through the right, or spends a long time in a parallelogram.
2 Quenched Heat Kernel Estimates

2. The middle third of $P = \mathbb{P}$ and $P(\cdot, N)$.

We begin by introducing notation and results from [2, 3].

\[ b \]

2. The middle third of $P$.

Note that in our notation, $b$.

\[ \log : \mathbb{P}(\cdot, N) \to \mathbb{P}(\cdot, N) \]

The fact that for any $x \in \mathcal{P}(z, N)$, the random walk starting at $x$ will exit the parallelogram through $\partial^+ \mathcal{P}(z, N)$ with very high probability.

Lemma 2.5 (Lemma 3.4 of [3]). Let $d \geq 2$, and assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). For every $z \in \mathcal{P}(0, N)$, every $\alpha < d$, and every $j \in \mathbb{N}$, there exist constants $C, c > 0$ such that

\[ \mathbb{P}^z(T_{\partial^+ \mathcal{P}(0, N)} \neq T_{\partial \mathcal{P}(0, N)}) \leq C \exp(-cR_5(N)) \]

and

\[ \mathbb{P}^z(\|T_{\partial \mathcal{P}(0, N)} - \mathbb{E}^zT_{\partial \mathcal{P}(0, N)}\| > R_{j+1}(N)N) \leq C \exp(-c \log(R_j(N))^\alpha) \]

for every $n \in \mathbb{N}$.

The next lemma estimates the probability the random walk deviates from its mean.

Lemma 2.6 (Lemma 2.16 of [3]). Assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). Then there exist constants $C, c > 0$ such that

\[ \mathbb{P}^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > R_5(n)n^{1/2}) \leq C \exp(-cR_5(n)) \]

for every $n \in \mathbb{N}$.

The following slowdown result was proven by Sznitman in [15]. While it is not sharp due to [1.2], it has the advantage of holding for $d \geq 2$ and suffices for our purposes.
\textbf{Theorem 2.7} (Theorem 3.5 of [18]). Let \( d \geq 2 \). Assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). Then for any \( \alpha < 1 + \frac{d-1}{3d} \), there exist constants \( C, \epsilon > 0 \) such that

\[
P(\tau_1 > u) \leq Ce^{-c(\log u)^{\alpha}}
\]

for every \( u > 0 \).

The next two lemmas provide annealed heat kernel bounds.

\textbf{Lemma 2.8} (Lemma 2.14 of [3]). Assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). There exists a constant \( C > 0 \) such that for all \( n \in \mathbb{N} \), and \( x, y, z, w \in \mathbb{Z}^d \) such that \( \|x - y\|_1 = 1 \) and \( \|z - w\|_1 = 1 \)

\[
\begin{align*}
\mathbb{P}(X_n = x) &\leq Cn^{-d/2} \\
|\mathbb{P}(X_n = x) - \mathbb{P}(X_{n+1} = y)| &\leq Cn^{-(d+1)/2} \\
|\mathbb{P}(X_n = x) - \mathbb{P}(X_{n+1} = x)| &\leq Cn^{-(d+1)/2}. 
\end{align*}
\]

\textbf{Lemma 2.9} (Lemma 3.3 of [3]). Assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). There exists a constant \( C > 0 \), such that for all \( z_1 \in \mathbb{Z}^d \), \( N \in \mathbb{N} \) and \( z \in \mathbb{P}(z_1, N) \), we have

1. For every \( m \in \mathbb{N} \) and \( x \in \partial^+ \mathbb{P}(z_1, N) \)

\[
\mathbb{P}^{\omega}(T_{\partial \mathbb{P}(z_1, N)} = m, X_{T_{\partial \mathbb{P}(z_1, N)}} = x) < CN^{-d}. 
\]

2. For every \( m \in \mathbb{N} \) and \( x, y \in \partial^+ \mathbb{P}(z_1, N) \) such that \( \|x - y\|_1 = 1 \)

\[
|\mathbb{P}^{\omega}(T_{\partial \mathbb{P}(z_1, N)} = m, X_{T_{\partial \mathbb{P}(z_1, N)}} = x) - \mathbb{P}^{\omega}(T_{\partial \mathbb{P}(z_1, N)} = m, X_{T_{\partial \mathbb{P}(z_1, N)}} = y)| < CN^{-d-1}. 
\]

3. For every \( m \in \mathbb{N} \), every \( x \in \partial^+ \mathbb{P}(z_1, N) \) and every \( \epsilon \in \mathcal{E}_d \)

\[
|\mathbb{P}^{\omega}(T_{\partial \mathbb{P}(z_1, N)} = m, X_{T_{\partial \mathbb{P}(z_1, N)}} = x) - \mathbb{P}^{\omega}(T_{\partial \mathbb{P}(z_1, N)} = m + 1, X_{T_{\partial \mathbb{P}(z_1, N)}} = x)| < CN^{-d-1}. 
\]

The following proposition is a semi-local limit theorem which is the key estimate in all of our proofs. A version of this result was proven in Proposition 3.1 in [3].

\textbf{Proposition 2.10.} Let \( d \geq 2 \). Assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). For every \( \theta \in (0, 1] \), let \( F(N) = F(N, \theta) \) be the event that for every \( z \in \mathbb{P}(0, N) \), every \( (d-1) \)-dimensional cube \( \Delta \subset \partial^+ \mathbb{P}(0, N) \) of side-length \( N^{\theta} \) and every interval \( I \) of length \( N^\theta \)

\[
|\mathbb{P}_{\omega}(X_{T_{\partial \mathbb{P}(0, N)}} \in \Delta, T_{\partial \mathbb{P}(0, N)} \in I) - \mathbb{P}^{\omega}(X_{T_{\partial \mathbb{P}(0, N)}} \in \Delta, T_{\partial \mathbb{P}(0, N)} \in I)| \leq C N^{\theta d - \frac{d}{d+2}} R^4(N) N^d \]

Then for any \( \alpha < 1 + \frac{d-1}{3d} \), there exist constants \( C, \epsilon > 0 \) such that \( P(F(N)) > 1 - Ce^{-c(\log N)^{\alpha}} \) for all \( N \in \mathbb{N} \).
2.2 Intersections of Random Walks

To prove Proposition 2.10 we will need bounds on the expected number of intersections of two independent random walks in the same environment. Let $X^{(1)}$ and $X^{(2)}$ be two independent random walks in the same environment starting at the origin, i.e. their joint law is given by

$$P^0_\omega \otimes P^0_\omega (X^{(1)} \in \cdot, X^{(2)} \in \cdot) = P^0_\omega (X^{(1)} \in \cdot) P^0_\omega (X^{(2)} \in \cdot).$$

Lemma 2.11. Assume $d \geq 2$, and assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). For $N \in \mathbb{N}$, define

$$I_n = \sum_{z \in \mathbb{Z}^d : \|z\|_1 \leq n} 1 \{X^{(1)} \text{ visits } z\} 1 \{X^{(2)} \text{ visits } z\}$$

to be the number of intersection points of $X^{(1)}$ and $X^{(2)}$ in $\{z \in \mathbb{Z}^d : \|z\|_1 \leq n\}$. Then for any $\alpha < 1 + \frac{d-1}{3d}$ and any $\epsilon > 0$, there exist constants $C, c > 0$ such that

$$P \left( E^0_\omega \otimes E^0_\omega [I_n] > n^{1/2+\epsilon} \right) < C \exp(-c(\log n)^\alpha)$$

for every $n \in \mathbb{N}$.

Proof. Our argument follows the one in Lemma 4.2 in [5], which roughly says that while the two random walks might intersect often, the time between each intersection is long. Recall that $v$ is the limiting velocity of the random walk, and define

$$G(t) = \sum_{i,j=0}^{\infty} 1 \{X^{(1)}_i = X^{(2)}_j\} 1 \{\langle X^{(1)}_i, v \rangle \in [t-0.5, t+0.5]\}.$$ 

We have

$$E^0_\omega \otimes E^0_\omega [I_n] \leq \sum_{z \in \mathbb{Z}^d : \|z\|_1 \leq n} E^0_\omega \otimes E^0_\omega \left[ \sum_{i,j=0}^{\infty} 1 \{X^{(1)}_i = X^{(2)}_j = z\} \right]$$

$$\leq \sum_{t=-n}^{n} \sum_{z \in \mathbb{Z}^d : \|z\|_1 \leq n} E^0_\omega \otimes E^0_\omega \left[ \sum_{i,j=0}^{\infty} 1 \{X^{(1)}_i = X^{(2)}_j = z\} \right]$$

$$= \sum_{t=-n}^{n} E^0_\omega \otimes E^0_\omega \left[ G(t) \right] 1 \{G(t) \neq 0\}. $$
Let $\tau_k^{(i)}$ denote the respective regeneration times of $X^{(i)}$, and define $Y_n^{(i)} = \max\{\tau_k^{(i)} - \tau_k^{(i)} : 1 \leq k \leq n\}$. Define the event

$$A_n(j) = A_n = \{\omega \in \Omega : \max\{Y_n^{(1)}, Y_n^{(2)}\} < R_j(n)\}.$$ 

From Theorem 2.7, for every $\alpha < 1 + \frac{d - 1}{d}$, there exist constants $C, c > 0$ such that

$$P(A_n(j)) > 1 - Ce^{-c(\log n)^{\alpha(j+2)/(j+3)}} \quad (2.2)$$

for all $n \in \mathbb{N}$. Since $G(t)$ is the number of intersection points in $\{z \in \mathbb{Z}^d : \langle z, v \rangle \in [t - 0.5, t + 0.5]\}$, it is bounded by the length of the $X^{(1)}$ regeneration interval containing $t$, multiplied by the length of the $X^{(2)}$ regeneration interval containing $t$. Hence for $\omega \in A_n$ and for any $-n \leq t \leq n$, we have

$$G(t) < Y_n^{(1)} \cdot Y_n^{(2)} < R_j^2(n),$$

and so

$$E_\omega^0 \otimes E_\omega^0[I_n] \leq R_j^2(n)E_\omega^0 \otimes E_\omega^0 \left[\sum_{t=-n}^{n} \mathbf{1}_{\{G(t) \neq 0\}}\right]$$

for $\omega \in A_n$.

Define the variables $\{\psi_n\}_{n \in \mathbb{N}}$ and $\{\theta_n\}_{n \in \mathbb{N}}$ inductively: $\theta_0 = 0$, $\psi_1 = \max\{\tau_1^{(1)}, \tau_1^{(2)}\}$, and for $n \geq 1$

$$\theta_n = \min\{k > \psi_n : G(k) \neq 0\}, \quad \psi_{n+1} = \max\{\tau^{(1)}(\theta_n), \tau^{(2)}(\theta_n)\},$$

where

$$\tau^{(1)}(k) = \min\{\langle X_{\tau_m^{(1)}}, v \rangle : \langle X_{\tau_m^{(1)}}, v \rangle > k + 1\}.$$ 

Define $j_n = \theta_n - \psi_n$ and set

$$K = \min\left\{m \in \mathbb{N} : \sum_{i=1}^{m} j_i > n\right\}.$$

In [5], the authors showed that

$$\mathbb{P}^0 \otimes \mathbb{P}^0(K > t) \leq \exp\left(-c \frac{t}{n^{1/2 + \delta_1}}\right),$$

where $\delta_1 > 0$ is arbitrary, and $\mathbb{P}^0 \otimes \mathbb{P}^0$ is the law of two independent annealed random walks, i.e.

$$\mathbb{P}^0 \otimes \mathbb{P}^0(X^{(1)} \in \cdot, X^{(2)} \in \cdot) = \mathbb{P}^0(X^{(1)} \in \cdot)\mathbb{P}^0(X^{(2)} \in \cdot).$$

Let $\omega_1, \omega_2$ be independent environments. We observe that the annealed distribution of two random walks in the same environment that do not intersect have the annealed distribution of two random walks in independent environments:

$$E[P_\omega \otimes P_\omega(A)] = E[P_{\omega_1} \otimes P_{\omega_2}(A)] = \mathbb{P} \otimes \mathbb{P}(A)$$
where $A$ is a non-intersection event. Since \( \{ K > t \} \) is such an event, the last inequality and uniform ellipticity implies

\[
E[P^0_\omega \otimes P^0_\omega (K > t)] \leq C \exp \left( -c \frac{t}{n^{1/2+\delta_1}} \right).
\]

For $\delta_2 > \delta_1 > 0$, define the event

\[
B_n = \left\{ \omega \in \Omega : P^0_\omega \otimes P^0_\omega (K > n^{1/2+\delta_2}) < \exp \left( -\frac{c}{2} n^{1/2+\delta_1} \right) \right\}.
\]

From Markov’s inequality we have

\[
P(B^c_n) < \frac{E[P^0_\omega \otimes P^0_\omega (K > n^{1/2+\delta_2})]}{\exp \left( -\frac{c}{2} n^{1/2+\delta_1} \right)} \leq e^{-\frac{c}{2} n^{\delta_2-\delta_1}}.
\]

Note that

\[
\sum_{t=-n}^{n} 1_{\{G(t) \neq 0\}} \leq K \cdot \max\{Y_n^{(1)}, Y_n^{(2)}\},
\]

for $\omega \in A_n$, and so $E^0_\omega \otimes E^0_\omega [I_n] < R^3_j(n) E^0_\omega \otimes E^0_\omega [K]$. This implies that for $\omega \in A_n \cap B_n$, we have for large enough $n$

\[
E^0_\omega \otimes E^0_\omega [K] \leq R^3_j(n) (n^{1/2+\delta_2} + n \cdot P^0_\omega \otimes P^0_\omega (K > n^{1/2+\delta_2})) \leq n^{1/2+\delta_2+\epsilon},
\]

for arbitrary $\epsilon > 0$, where the second inequality follows from $K \leq n$ by definition. From (2.2) and (2.3), we see that for all $\alpha < 1 + \frac{d-1}{3d}$

\[
P((A_n \cup B_n)^c) \leq C e^{-c(\log n)^\alpha(j+2)/(j+3)} + e^{-cn^{\delta_2-\delta_1}} = O \left( e^{-c(\log n)^\alpha(j+2)/(j+3)} \right).
\]

If $\alpha'$ satisfy $\alpha' < \alpha(j+2)/(j+3) < 1 + \frac{d-1}{3d}$, then

\[
P((A_n \cup B_n)^c) \leq C e^{-c(\log n)^{\alpha'}}.
\]

Since $j$ is arbitrary we conclude that the bound holds for any $\alpha' < 1 + \frac{d-1}{3d}$, which finishes the proof.

2.3 Proof of Proposition 2.10

The proof of Proposition 2.10 will be nearly identical to the proof of Proposition 3.1 in [3], except we will use Lemma 2.11 rather than Lemma 2.12 in [3]. The proof follows in three steps. First, in Lemma 2.12 we use our intersection bounds to estimate the quenched heat kernel for large boxes. Second, in Lemma 2.13 we use an induction argument to get an
estimate for boxes of all sizes. Lastly, in Lemma 2.14 we get sharper quenched bounds by bootstrapping the previous heat kernel estimates.

Fix $j \in \mathbb{N}$ and define the event

$$B_N = B_N(j) = \{ \forall 1 \leq k \leq N^2, \tau_k - \tau_{k-1} \leq R_j(N) \}.$$  

For $L \in \mathbb{Z}$, denote the hyperplane

$$H_L = \{ z \in \mathbb{Z}^d : \langle z, e_1 \rangle = L \}.$$

**Lemma 2.12.** Let $d \geq 2$, and assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). For every $\theta \in (0, 1]$, let $L(N) = L(N, \theta)$ be the event that for every $\frac{2}{5} N^2 \leq M \leq N^2$, every $z \in \mathcal{P}(0, N)$, every $(d-1)$-dimensional cube $\Delta \subset H_M$ of side-length $N^\theta$ and every interval $I$ of length $N^\theta$ 

$$|P_z^\omega(X_{T_m} \in \Delta, T_m \in I, B_N) - \mathbb{P}(X_{T_m} \in \Delta, T_m \in I, B_N)| \leq CN^{\alpha d}/N^d.$$  

For every $\theta > (2d+1)/(2d+2)$ and $\alpha < 1 + \frac{d-1}{3d}$, there exist constants $C, c > 0$ such that

$$P(L(\theta, N)) > 1 - C \exp(-c(\log N)^\alpha)$$  

for any $n \in \mathbb{N}$.

**Proof.** Fix $j \in \mathbb{N}$. Let $\theta' \in (0, \theta)$ and define $V = \lfloor N^{2\theta'} \rfloor$. Let $\mathcal{F}$ be the $\sigma$-algebra generated by

$$\{ \omega(x) : x \in \mathcal{P}^M(0, N) \},$$

where $\mathcal{P}^M(0, N) = \mathcal{P}(0, N) \cap \{ x \in \mathbb{Z}^d : \langle x, e_1 \rangle \leq M \}$. Let $x_1, x_2, \ldots$ be any lexicographic ordering of the vertices in $\mathcal{P}^M(0, N)$ with the first coordinate being the most significant, and let $\mathcal{F}_k$ be the $\sigma$-algebra generated by $\omega(x_1), \ldots, \omega(x_k)$. Define the martingale

$$M_k = E[P_z^\omega(X_{T_{m+v}} = v, T_{m+v} = m | B_N) | \mathcal{F}_k],$$

and the martingale difference $U_k = \text{esssup}(|M_k - M_{k-1}| | \mathcal{F}_k)$. We will first bound the martingale differences, and then finish the proof by applying McDiarmid’s inequality. We first show

$$U_k \leq CR_j(N)V^{-(d+1)/2}E[P_z^\omega(x_k \text{ is visited} | B_N) | \mathcal{F}_k]. \quad (2.4)$$

Let $P_k(\cdot) = E[P_z^\omega(\cdot | B_N) | \mathcal{F}_k]$. The random walk under $P_{k-1}$ has quenched transition probabilities on $\{x_1, \ldots, x_{k-1}\}$ and annealed transition probabilities on $\mathbb{Z}^d \setminus \{x_1, \ldots, x_{k-1}\}$, while $P_k$ has the same distribution except that on $x_k$ it has quenched transition probability. Hence both distributions are identical on the event the random walk never hits $x_k$, and so

$$|M_k - M_{k-1}| = |P_k(X_{T_{m+v}} = v, T_{m+v} = m) - P_{k-1}(X_{T_{m+v}} = v, T_{m+v} = m)|$$

$$= |P_k(X_{T_{m+v}} = v, T_{m+v} = m, x_k \text{ is visited}) - P_{k-1}(X_{T_{m+v}} = v, T_{m+v} = m, x_k \text{ is visited})|. $$
We observe that until the first time \( x_k \) is hit, \( P_k \) and \( P_{k-1} \) have the same distribution, and so we can couple them until the first hitting time of \( x_k \). Using annealed derivative estimates from Lemma 2.8 and that under \( B_N \) we have bounded regeneration times, we get

\[
U_k \leq P_k(x_k \text{ is visited}) CR_j(N)V^{-(d+1)/2},
\]

which yields (2.4). Let \( B(x) = \{ y \in H_{(x,\epsilon_1)}^{-1} : \|y - x\|_{\infty} < R_j(N) \} \). Under the event \( B_N \), we have

\[
P_k(x_k \text{ is visited}) = E[P_\omega^z(x_k \text{ is visited}|B_N)|F_k]
\]

\[
\leq \sum_{y \in B(x_k)} E[P_\omega^z(T_{(x_k,\epsilon_1)}^{-1} = y|B_N)|F_k]
\]

\[
= \sum_{y \in B(x_k)} P_\omega^z(T_{(x_k,\epsilon_1)}^{-1} = y|B_N)
\]

\[
\leq \sum_{y \in B(x_k)} P_\omega^z(y \text{ is visited}|B_N).
\]

Since \( |B(x)| \leq C 2^d R_j^d(N) \), and every \( y \in \mathbb{Z}^d \) is in \( B(x) \) for at most \( 2^d R_j^d(N) \) points \( x \in \mathbb{Z}^d \), we have

\[
U := \sum_k U_k^2 \leq CR_j^{2d+2}(N) \sum_k P_\omega^z(x_k \text{ is visited})^2 V^{-(d+1)}.
\]

Fix some \( \epsilon > 0 \), and define the event

\[
A_N = \{ \omega \in \Omega : E_\omega^z \otimes E_\omega^z[I_{N^2}] \leq N^{1+\epsilon} \},
\]

which by Lemma 2.11 satisfies \( P(A_N^c) \leq Ce^{-c(log N)^{\alpha}} \). For \( \omega \in A_N \), we have

\[
U \leq CR_j^{2d+2}(N) \sum_k P_\omega^z(x_k \text{ is visited})^2 V^{-(d+1)}
\]

\[
\leq CR_j^{2d+2}(N)V^{-(d+1)} E_\omega[I_{N^2}]
\]

\[
\leq R_j^{2d+1}(N)V^{-(d+1)} N^{1+\epsilon}.
\]

Applying McDiarmid’s inequality, see Theorem 3.14 in [15], we have

\[
P(|E[P_\omega^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N)|F] - \mathbb{E}(X_{T_{M+V}} = v, T_{M+V} = m, B_N)| > N^{-d})
\]

\[
\leq P(|M_k - M_0| > N^{-d}, U \leq R_j^{2d+1}(N)V^{-(d+1)} N^{1+\epsilon}) + P(A_N^c)
\]

\[
\leq \exp \left( -\frac{N^{-2d}}{R_j^{2d+1}(N)V^{-(d+1)} N^{1+\epsilon}} \right) + C \exp \left( -c(log N)^{\alpha} \right).
\]

For \( \theta' > \theta \), we have

\[
\exp \left( -\frac{N^{-2d}}{R_j^{2d+1}(N)V^{-(d+1)} N^{1+\epsilon}} \right) \leq \exp \left( -\frac{N^{2\theta(d+1)+\delta}}{N^{2d+1}} \right),
\]

13
where $\delta > 0$ can be taken arbitrarily small since $\theta'$ can be taken arbitrarily close to $\theta$. Note that this exponent is negligible as long as $\theta > (2d+1)/(2d+2)$. Hence if we let $W(N) \subseteq \Omega$ be the event

$$|E[P^z( X_{T_{M+V}} = v, T_{M+V} = m, B_N) | \mathcal{F}] - P^z( X_{T_{M+V}} = v, T_{M+V} = m, B_N) | < N^{-d}$$

for every $z \in \mathcal{P}(0, N)$, every $v \in H_{M+V} \cap \mathcal{P}(0, 2N)$ and every $m \in \mathbb{N}$, we conclude that $P(W(N)) < e^{-c(\log N)\alpha}$. We can now continue exactly as in the proof of Lemma 3.5. The only estimates left to show are equations (3.6), (3.7) in [3], but these hold for any $\theta' < \theta$ with probability $C \exp(-c(\log n)\alpha)$. This finishes the proof. \qed

The next result extends the previous lemma to boxes of all sizes, but with a sub-optimal bound.

**Lemma 2.13.** Let $d \geq 2$ and assume $P$ is uniformly elliptic, i.i.d. and satisfies (T). For every $\theta \in (0, 1]$ and $h \in \mathbb{N}$, let $D^{(\theta, h)}(N)$ be the event that for every $z \in \mathcal{P}(0, N)$, every $1/2 N^2 \leq M \leq N^2$, every $(d-1)$-dimensional cube $\Delta \subseteq H_M$ of side-length $N^\theta$ and every interval $I \subseteq \mathbb{N}$ of length $N^\theta$

$$P^z( X_{T_M} \in \Delta, T_M \in I ) \leq R_h(N) N^{\alpha h / d} / N^d. \quad (2.5)$$

Then for every $\theta \in (0, 1]$, there exists $h = h(\theta)$ such that for every $\alpha < 1 + (d-1)/3d$, there exist constants $C, c > 0$ such that $P(D^{(\theta, h)}(N)) > 1 - C \exp(-c(\log n)\alpha)$ for every $n \in \mathbb{N}$.

**Proof.** We will prove the lemma by descending induction on $\theta$. First note that if $(2d+1)/(2d+2) < \theta' < 1$, then by Lemma 2.12 we have $P(D^{(\theta', h)}(N)) > 1 - C \exp(-c(\log n)\alpha)$. By induction, assume the statement of the lemma holds for some $\theta'$, and fix any $\theta$ satisfying $\theta'_{2d+1}^{2d+2} < \theta < \theta'$. Let $h' = h(\theta')$, and let $\rho = \theta/\theta'$. Define the event

$$S(N) = D^{(\rho, 1)}(N) \cap \bigcap_{z \in \mathcal{P}(0, 2N)} \sigma_z \zeta_s( D^{(\theta', h')}([N^{\theta'}]) \cap T(N, \rho) ,$$

where $\zeta$ is the time shift of the random walk defined as $\zeta_s( X_1, X_2, X_3, \ldots ) = ( X_{s+1}, X_{s+2}, \ldots )$, and

$$T(N, \rho) = \left\{ \omega \in \Omega : \begin{array}{c} \forall v \in \mathcal{P}(0, N) P^\omega( X_{T_{\mathcal{P}(v, [N^\rho])}} \notin \partial^+ \mathcal{P}(v, [N^\rho]) ) < C e^{-\rho R(N)} , \\
P^\omega( [T_{\partial \mathcal{P}(v, [N^\rho])} - \mathbb{E}[T_{\partial \mathcal{P}(v, [N^\rho])}]] > N^\rho R_j(N^\rho) ) \leq C e^{-c(\log R(N))^{\alpha'/2}} \end{array} \right\} .$$

In [3], the authors showed that $S(N) \subseteq D^{(\theta, h)}(N)$, and so we just have to bound $P(S^c(N))$. By the induction assumption, we only need to bound $P(T(N, \rho))$. By Lemma 2.5 and Markov’s inequality, we have for every $\alpha < 1 + (d-1)/3d$ and for every $j \in \mathbb{N}$

$$P(T(N, \rho)) \leq \exp(-c(\log N)^{\alpha(j+2)/(j+3)/2}).$$

Letting $\alpha' < \alpha(j+2)/(j+3) < 1 + (d-1)/3d$, we have

$$P(T(N, \rho)) \leq \exp(-c(\log N)^{\alpha'/2}).$$

Since $j$ is arbitrary the result holds for any $\alpha' < 1 + (d-1)/3d$, which finishes the proof. \qed

14
The next lemma bootstraps the estimates from Lemma 2.13 to obtain sharper bounds. This is done by using the new quenched heat kernel estimates to bound the number of intersection points $I_n$, replacing Lemma 2.11.

**Lemma 2.14.** Let $d \geq 2$, and assume $P$ is uniformly elliptic, i.i.d. and satisfies (T). Let $\mathcal{F}$ be the $\sigma$-algebra generated by $\{\omega(z) : (z, e_1) \leq N^2\}$. Let $\eta > 0$ such that $\eta < 2/(d - 1)$, $V = [N^n]$ and define $R(N, \eta)$ to be the event that for every $z \in \hat{\mathcal{P}}(0, N)$, every $v \in H_{N^2+V}$ and every $m \in \mathbb{N}$

$$|E[P^z_\omega(X_{T_{N^2+V}} = v, T_{N^2+V} = m)|\mathcal{F}] - \mathbb{P}^x(X_{T_{N^2+V}} = v, T_{N^2+V} = m)| \leq CN^{-d}V^{-(d-1)/5}. $$

Then for every $\alpha < 1 + (d - 1)/3d$, there exist constants $C, c > 0$ such that $P(R(N, \eta)) > 1 - C \exp(-c(\log N)^\alpha)$ for every $n \in \mathbb{N}$.

**Proof.** Fix some $\theta \in (0, 1)$. Let $K$ be an integer such that $2^{-K-1}N^2 \leq V < 2^{-K}N^2$, and for $0 \leq k \leq K$, define

$$\mathcal{P}^{(k)} = \{x \in \mathcal{P}(0, N) : 2^{-k-1}N^2 \leq N^2 - \langle x, e_1 \rangle < 2^{-k}N^2\},$$

$$\mathcal{P}^{(0)} = \{x \in \mathcal{P}(0, N) : N^2/2 \leq N^2 - \langle x, e_1 \rangle\},$$

$$F(v) = \{x \in \mathcal{P}(0, N) : \|x - v - \vartheta \langle x - v, e_1 \rangle\|_\infty \leq R_j(N)|\langle v - x, e_1 \rangle|^{1/2}\},$$

$$\mathcal{P}^{(k)}(v) = \mathcal{P}^{(k)} \cap F(v).$$

We repeat the argument from Lemma 2.12. Let $x_1, x_2, \ldots$ be a lexicographic ordering of the vertices of $\mathcal{P}(0, N)$ emphasizing the first coordinate, and let $\mathcal{F}_i$ be the $\sigma$-algebra generated by $\{\omega(x_j) : j = 1, \ldots, i\}$. Define the martingale

$$M_i = E[P^z_\omega(X_{T_{N^2+V}} = v, T_{N^2+V} = m|B_i)|\mathcal{F}_i],$$

as well as the martingale difference $U_i = \text{esssup}(|M_i - M_{i-1}||\mathcal{F}_{i-1})$. As in the proof of Lemma 2.12, we have

$$U_i \leq |P_i(X_{T_{M+V}} = v, T_{M+V} = m, x_i \text{ is visited}) - P_{i-1}(X_{T_{M+V}} = v, T_{M+V} = m, x_i \text{ is visited})|.$$

For $x \in \mathcal{P}(0, N)$, let $L_x = N^2 + V - \langle x, e_1 \rangle$ and define the interval

$$I(x, m) = [m - \mathbb{E}^x[T_{L_x}] - R_j(L_x)L_x^{1/2}, m - \mathbb{E}^x[T_{L_x}] + R_j(L_x)L_x^{1/2}].$$

By Lemma 2.5

$$\mathbb{P}^x(X \text{ visits } x \text{ in } I(x, m)^c, T_{N^2+V} = m) \leq C \exp(-c(\log R_j(L_x))^\alpha).$$

Let $G_N$ be the event

$$\left\{ \begin{array}{c}
\forall x \in \mathcal{P}(0, N), \forall z \in \hat{\mathcal{P}}(0, N), \forall m \in [0, CN^2], \\
\omega \in \Omega : \\
P^z_\omega(X \text{ visits } x \text{ in } I(x, m)^c, X_{T_{N^2+V}} = v, T_{N^2+V} = m) < C \exp(-c(\log R_j(N))^\alpha/2) \\
\end{array} \right\}. $$

15
Then by Markov’s inequality and the union bound, $P(G_N^c) < C \exp(-c(\log R_j)^\alpha/2)$. For $\omega \in G_N$, we have

$$U_i \leq |P_i(X_{T_{M+V}} = v, T_{M+V} = m, x_i \text{ is visited in } I(x_i, m)) - P_{i-1}(X_{T_{M+V}} = v, T_{M+V} = m, x_i \text{ is visited in } I(x_i, m))| + N^{-\xi(1)}. $$

By applying annealed difference estimates from Lemma 2.8 and noting that we are under the event $B_N$, we get for $x_i \in F(v)$

$$U_i \leq P_i(x_i \text{ is visited in } I(x_i, m)) \cdot CR_j(N)L_{x_i}^{-(d+1)/2},$$

while for $x_i \notin F(v)$ we have $U_i = N^{-\xi(1)}$. Applying these bounds, as well as similar arguments from Lemma 2.12, we have

$$U := \sum_i U_i^2 \leq \sum_{k=0}^K \sum_{x_i \in P^{(k)}(v)} P^x_\omega(x_i \text{ is visited in } I(x_i, m))^2 \frac{CR_j^{2d+2}}{L_{x_i}^{d+1}} + N^{-\xi(1)}.$$ 

To bound $U$, define the event $E_N = D_{\theta,h} \cap \{ \omega \in \Omega : E_\omega^x \otimes E_\omega^x[I_{N^2}] < N^{1+\epsilon} \}$, which for appropriate $h$ satisfies $P(E_N^c) \leq C \exp(-c(\log N)^\alpha)$ by Lemmas 2.11 and 2.13. We first bound the $k = 0$ summand. For $x_i \in P^{(0)}(v)$, $L_{x_i} \geq N^2/2$ and so for $\omega \in E_N$ we have

$$\sum_{x_i \in P^{(0)}(v)} P^x_\omega(x_i \text{ is visited in } I(x_i, m))^2 \frac{CR_j^{2d+2}}{L_{x_i}^{d+1}} \leq CR_j^{2d+2}(N)N^{-2(d+1)} \sum_{x_i \in P^{(0)}(v)} P^x_\omega(x_i \text{ is visited})^2 \leq CR_j^{2d+2}(N)N^{-2(d+1)} E_\omega^x \otimes E_\omega^x[I_{N^2}] \leq CR_{j+1}(N)N^{-2d-1+\epsilon}.$$ 

Note that when $x_i \in P^{(k)}(v)$ for $k > 0$, we have $V + 2^{-k-1}N^2 \leq L_{x_i} < V + 2^{-k}N^2$. Hence when $k > 0$, we have for $\omega \in E_N$

$$\sum_{x_i \in P^{(k)}(v)} P^x_\omega(x_i \text{ is visited in } I(x_i, m))^2 \frac{CR_j^{2d+2}}{L_{x_i}^{d+1}} \leq \frac{CR_j^{2d+2}}{(V + N2^2-2-k)^{d+1}} \sum_{x_i \in P^{(k)}(v)} P^x_\omega(x_i \text{ is visited in } I(x_i, m))^2 \leq \frac{CR_j^{2d+2}}{(V + N2^2-2-k)^{d+1}} \sum_{x_i \in P^{(k)}(v)} R_{h}(N)N^{2d}L_{x_i} \frac{N^{2d}L_{x_i}}{N^{2d}} \leq \frac{CR_j^{2d+2} R_{h}(N) N^{2d} (V + N2^2-2-k)}{(V + N2^2-2-k)^{d+1} N^{2d}} |P^{(k)}(v)| \leq \frac{CR_j^{2d+2} R_{h}(N) N^{2d}}{(V + N2^2-2-k)^{d} N^{2d}} |P^{(k)}(v)| \leq \frac{CR_j^{2d+2} R_{h}(N) N^{2d} N^{d+1}2^{-k(d+1)/2} N^{2d}}{(V + N2^2-2-k)^{d} N^{2d}} \leq 16.$$
number of intersections, which is the content of the following lemma.

Theorem 2.7 implies

\[ \eta < \theta \frac{C R_{j+1}}{V (d-1)/2} N^{2d} \]

Putting both estimates together, we conclude that for \( \omega \in E_N \cap G_N \)

\[ U \leq C R_{j+1} N^{-2d-1+\epsilon} + \sum_{k=1}^{K} \frac{C R_{j+1}^{2d+2}}{(V + N^{22-k})^{(d-1)/2}} N^{2d} \]

where the last inequality follows from assuming \( \eta < 2/(d-1) \), and that \( \theta \) can be chosen arbitrarily close to 0. We can now apply McDiarmid’s inequality

\[ P\left( \left| E[P^\omega (X_{T_{N^2+v}} = v, T_{N^2+v} = m, B_N)] \right| \geq C N^{-d} V^{-(d-1)/5} \right) \]

\[ \leq P \left( \{ |M_K - M_0| > C N^{-d} V^{-(d-1)/5} \} \right) \]

\[ \leq P \left( \{ |M_K - M_0| > C N^{-d} V^{-(d-1)/5} \} \cap \{ U < CR_{max(j+1,h+1)} N^{-2d} V^{-(d-1)/2} \} \right) \]

\[ \leq \exp \left( -C \frac{N^{-2d} V^{-(d-1)/2}}{N^{-2d} V^{-(d-1)/2}} \right) + C \exp (-c (\log (R_j(N)))^\alpha) \]

\[ \leq 2C \exp (-c (\log N)^{\alpha(j+2)/(j+3)}). \]

Theorem 2.7 implies \( \mathbb{P}(B_N^\omega) \leq C \exp (-c (\log R_j(N))^\alpha) \) for any \( \alpha < 1 + (d-1)/3d \), and so by Markov’s inequality

\[ P (P^\omega (B_N^\omega) > C \exp (-c (\log R_j(N))^\alpha)/2) \leq C \exp (-c (\log R_j(N))^\alpha/2). \]

With this estimate, we conclude that \( P(R(N, \eta)) > 1 - \exp(-c (\log N)^\alpha) \) for any \( \alpha < 1 + (d-1)/3d \).

\[ \square \]

**Proof of Proposition 2.10** Continue as in the proof of Proposition 3.1 in [3], replacing Lemma 3.7 in [3] with Lemma 2.14.

**Proof of Theorems 1.3** In [3], the only parts of the proof of Theorems 1.10 and 1.11 that depend on dimension are the quenched heat kernel bounds in Proposition 3.1. Since Proposition 2.10 provides the equivalent heat kernel bounds for \( d = 2 \) and \( d = 3 \), we are done.

\[ \square \]

# 3 Slowdown Estimates

The purpose of this section is to prove Theorem 1.8. To prove this slowdown result for \( d = 3 \), we need a control on the quenched mean of the position of the random walk when exiting \( P(0, N) \), see Lemma 3.2.

This estimate will require an improved bound on the number of intersections, which is the content of the following lemma.
Lemma 3.1. Let \( d \geq 3 \) and assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). Then for each \( \epsilon > 0 \),

\[
P\left( E^0_\omega \otimes E^0_\omega[I_n] > n^\epsilon \right) = n^{-\epsilon(1)}.
\]

Proof. Recall that \( X^{(1)} \) and \( X^{(2)} \) are independent random walks in the same environment. Before we bound the number of their intersections, we will condition the environment to have good properties. Fix \( \delta \in (0, 1) \), and define the set

\[
A = \bigcup_{m=\lfloor n^\delta \rfloor}^n \left\{ z \in \mathbb{Z}^d : \| z - \mathbb{E}^0[X_m] \|_\infty < R_5(m) \sqrt{m} \right\}.
\]

For \( z \in \mathbb{Z}^d \) and \( r \in \mathbb{R} \), define the ball \( B(z, r) = \{ x \in \mathbb{Z}^d : \| z - x \|_\infty \leq r \} \). From the union bound and Lemma 2.6, we have

\[
\mathbb{P}^0( X \text{ visits } A^c ) \leq \sum_{m=\lfloor n^\delta \rfloor}^n \mathbb{P}(X_m \notin B(\mathbb{E}^0[X_m], R_5(m)m^{1/2})) \leq \sum_{m=\lfloor n^\delta \rfloor}^n C e^{-cR_5(m)} \leq (n - \lfloor n^\delta \rfloor) C e^{-cR_5(\lfloor n^\delta \rfloor)} \leq \tilde{C} e^{-\tilde{c}(\delta) R_5(n)/2},
\]

where \( \tilde{C}, \tilde{c} \) are some constants. From Corollary 5.4 in [18], we have

\[
\mathbb{P}^0\left( T_{\lfloor n^\delta \rfloor} > 1.5 \cdot \mathbb{E}^0[T_{\lfloor n^\delta \rfloor}^1] \right) \leq C e^{-c(\log n)^{10/9}}.
\]

Define the event

\[
E_\omega = \{ \omega \in \Omega : P^0_\omega(X \text{ visits } A)^c \leq \tilde{C} e^{-\tilde{c}(\delta) R_5(n)/4} \} \cap \{ \omega \in \Omega : P^0_\omega(T_{\lfloor n^\delta \rfloor} > 1.5 \cdot \mathbb{E}^0[T_{\lfloor n^\delta \rfloor}]) < e^{-c(\log n)^{10/9}/2} \} \cap \{ \omega \in \Omega : k = 1, \ldots, n, P^0_\omega(\tau_{k+1} - \tau_k > R(n)) < e^{-c(\log n)^{10/9}} \}.
\]

By Markov’s inequality, (3.1), (3.2) and Theorem 2.7, we have \( P(E^c_\omega) = n^{-\epsilon(1)} \).

Note that for \( \omega \in E_\omega \), the number of intersection points of \( X^{(1)} \) and \( X^{(2)} \) in \( \{ z \in \mathbb{Z}^d : \| z \|_1 \leq n^\delta \} \) is bounded by \( Cn^{\delta d} \). Hence for \( \omega \in E_\omega \), we have

\[
E^0_\omega \otimes E^0_\omega[I_n] \leq \sum_{x \in \mathbb{Z}^d} P^0_\omega( X^{(1)} \text{ visits } x ) P^0_\omega( X^{(2)} \text{ visits } x ) = \sum_{x \in \mathbb{Z}^d} P^0_\omega(X \text{ visits } x)^2 \leq \sum_{x \in \mathbb{Z}^d} P^0_\omega(X \text{ visits } x)^2 + \sum_{x \in A} P^0_\omega(X \text{ visits } x)^2 + \sum_{x \in A^c} P^0_\omega(X \text{ visits } x)^2 \leq Cn^{\delta d} + \sum_{x \in A} P^0_\omega(X \text{ visits } x)^2 + n^{-\epsilon(1)}.
\]
To bound the remaining number of intersection points, we will use Proposition 2.10. Observe the inclusion
\[
A \cap \{x \in \mathbb{Z}^d : \|x\|_1 \geq n^\delta\} \subset \bigcup_{m=[n^\delta]}^{n} B(\mathbb{E}^0[X_{T_m}], R_6(m) m^{1/2}) \cap H_m =: \bigcup_{m=[n^\delta]}^{n} D_m.
\]

Fix \(\epsilon > 0\), and let \(F_n\) be the event that for all \(m > \lfloor n^\delta \rfloor\), for all \((d-1)\)-dimensional boxes \(\Delta\) of side-length \(\lfloor m^\epsilon \rfloor\), we have \(P^0_\omega(X_{T_m} \in \Delta) \leq C m^{2(d-1)} (m^d)^{1/2}\). By Proposition 2.10 and the union bound, \(P(F_n^c) = n^{-\xi(1)}\). For \(\omega \in F_n\), we have
\[
\sum_{x \in A} P^0_\omega(\text{X visits } x)^2 \leq \sum_{m=[n^\delta]}^{n} \sum_{x \in D_m} P^0_\omega(\text{X visits } x)^2
\]
\[
\leq C \sum_{m=[n^\delta]}^{n} |D_m| \cdot \frac{m^{2d-1}}{m^{d-1}}
\]
\[
\leq C \sum_{m=[n^\delta]}^{n} R^{d-1}(m) m^{(d-1)/2} \frac{m^{2d-1}}{m^{d-1}}
\]
\[
\leq C \sum_{m=[n^\delta]}^{n} R_2(m) \frac{m^{2d-1}}{m^{d-1}/2}
\]
\[
\leq C n^{d\epsilon}
\]
where the last inequality used the fact that \(d \geq 3\) and \(\epsilon\) is chosen small enough. We have shown that under the events \(E_n \cap F_n\), we have \(E_n^c \otimes E_n[I_n] < C n^\delta + C n^{c\delta} + n^{-\xi(1)}\). Since \(\delta\) is arbitrary, we are done.

We use this intersection result to provide estimates for the mean of the quenched exit distribution.

**Lemma 3.2.** Let \(d \geq 3\) and assume \(P\) is uniformly elliptic, i.i.d. and satisfies condition \((T)\). Fix \(\epsilon > 0\), and let \(K(N, \epsilon)\) be the event that for every \(z \in \hat{P}(0, N)\),
\[
\left\| E^\epsilon[z_{T_{\beta P(0, N)}}] - E^\epsilon[z_{T_{\beta P(0, N)}}] \right\|_{\infty} \leq N^{\epsilon}.
\]
Then \(P(K(N, \epsilon)) = 1 - N^{-\xi(1)}\).

**Proof.** Rerun the same proof of Lemma 4.11 in [2], replacing the intersection bounds with Lemma 3.1.

**Proof of Theorem 1.8.** The only place in [2] which uses intersection estimates is Proposition 4.5. In that proposition, the estimate
\[
\left\| E^\epsilon[z_{T_{\beta P(0, N)}}] - E^\epsilon[z_{T_{\beta P(0, N)}}] \right\|_{\infty} \leq R_3(N)
\]
is used, which is better than our estimate. However, Proposition 4.5 is only used in order to show \(\mathcal{D}\) and \(\mathbb{D}\) are \((N^{-d-1/(d+1)} \times (d+1)N^\theta)\)-close for any \(\theta \in (0, 1/2)\). Here, \(\mathcal{D}\) is the
quenched exit distribution from $\mathcal{P}(0,N)$ conditioned on exiting through $\partial^+\mathcal{P}(0,N)$, and similarly $\mathbb{W}$ is the annealed exit distribution conditioned on exiting through $\partial^+\mathcal{P}(0,N)$. Reviewing the proof, to prove that these two distribution are $(N^{-\theta}\frac{d+1}{d+2},(d+1)N^\theta)$-close, it is enough to show
\[
\left\|\mathbb{E}[Z_{T_{\mathcal{P}(0,N)}}] - \mathbb{E}[Z_{T_{\mathbb{W}(0,N)}}]\right\|_\infty \leq N^\epsilon.
\]
for arbitrarily small $\epsilon$. The proof of Theorem 1.8 follows from this observation and Lemma 3.2.

**Lemma 3.3.** Let $d \geq 3$, and recall the event $F = F(N,\theta)$ in Proposition 2.10. For any $\alpha < d$, there exist constants $C, c > 0$ such that $\mathbb{P}(F^c) \leq Ce^{-c(log n)^{\alpha}}$ for all $n \in \mathbb{N}$.

**Proof.** Reviewing the proof of Proposition 2.10 we see that $\mathbb{P}(F^c)$ is dominated by tail estimates for regeneration times. Plugging in the new estimates from Theorem 1.8 yields the result. 

**4 Tail estimates for $dQ/dP$**

The purpose of this section is to prove Theorem 1.10. Let $Q_n$ be the law of $\mathbb{X}_n$ and define
\[
f_n(\omega) = \sum_{z \in \mathbb{Z}^d} P^\omega_{z}(X_n = 0).
\]
We observe that $dQ_n = f_n dP$, since for measurable $A \subset \Omega$
\[
Q_n(A) = E \left[ \sum_{z \in \mathbb{Z}^d} P^0_{\omega}(X_n = z) 1_{z \in A} \right] = E \left[ \sum_{z \in \mathbb{Z}^d} P^\omega_{z}(X_n = 0) 1_{\omega \in A} \right] = E[f_n(\omega)1_{\omega \in A}],
\]
where the first equality is by definition of $\mathbb{X}_n$, and the second equality is by translation invariance of $P$. Hence to prove the theorem, we will prove tail estimates for $f_n$ and show $Q_n \overset{d}{\to} Q$.

**Proposition 4.1.** Assume $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). For every $d \geq 3$ and $\alpha < d$, there exist constants $C, c > 0$ such that
\[
\forall u > 0, \limsup_{n \to \infty} \mathbb{P}(f_n > u) \leq Ce^{-c(log n)^\alpha}.
\]
Furthermore, if $d \geq 2$ and $P$ is nestling, then there exist constants $C, c > 0$ such that
\[
\forall u > 0, \liminf_{n \to \infty} \mathbb{P}(f_n > u) \geq Ce^{-c(log n)^d}.
\]
To prove $Q_n$ converges to $Q$ we will need the following theorem, which is stated in a weaker version.

**Theorem 4.2** (Kozlov [13]). Assume $P$ is uniformly elliptic and i.i.d. Assume there exists an invariant probability measure $Q$ with respect to $\mathbb{X}_n$ such that $Q \ll P$. Then the following hold:

20
1. $Q$ is equivalent to $P$.

2. $\{\omega_n\}_{n \in \mathbb{N}_0}$ with initial law $Q$ is ergodic.

3. $Q$ is the unique invariant probability measure for $\{\omega_n\}_{n \in \mathbb{N}_0}$ which is absolutely continuous with respect to $P$.

Proof of Theorem 1.10 assuming Proposition 4.1. Let $Q_n$ denote the law of $\omega_N$. In Theorem 3.1 in [21], Sznitman and Zerner proved that under Kalikow’s condition, $Q_n$ converges weakly to some invariant distribution. Their proof relies on Kalikow’s condition in order to assume finite moments for the regeneration times. Since this is implied by condition $(T)$ (for example Theorem 2.7), we can rerun their same proof to conclude that $Q_n \to Q_\infty$ weakly, where $Q_\infty$ is an invariant distribution. By Theorem 4.2, to show that $Q_\infty$ is the unique invariant probability measure $Q$ which is equivalent to $P$, it is enough to show $Q_\infty \ll P$. Let $A \subset \Omega$ be a measurable set such that $P(A) = 0$. Applying the Cauchy-Schwartz inequality, we have

$$Q_n(A) = E[f_n(\omega)\mathbb{1}_{\omega \in A}] \leq (Ef_n^2)^{1/2}P(A)^{1/2}.$$ 

From the tail estimates of Proposition 4.1, we conclude that $\limsup_{n \to \infty} E[f_n^2] < \infty$. Taking limits on both sides, we see that there exists a constant $C > 0$ such that

$$Q_\infty(A) < CP(A)^{1/2},$$

which implies that $Q_\infty \ll P$. Hence $Q_n \overset{d}{\to} Q_\infty = Q$ and so for any measurable $A$, we have

$$\lim_{n \to \infty} E[f_n(\omega)\mathbb{1}_{\omega \in A}] = E[dQ/dP(\omega)\mathbb{1}_{\omega \in A}].$$

We claim that this implies $f_n$ converges to $dQ/dP$ in probability, which will conclude the proof of the theorem. By contradiction, if $f_n$ does not converge to $dQ/dP$ in probability, then there exist $\epsilon, \delta > 0$ and a subsequence $\{n_k\}_{k \in \mathbb{N}}$ such that

$$\forall k \in \mathbb{N}, P(E_{n_k}) := P(\omega \in \Omega : f_{n_k}(\omega) - dQ/dP(\omega) > \epsilon) > \delta.$$

Define the event $E = \limsup_{k \to \infty} E_{n_k}$, and observe that

$$P(E) = \lim_{k \to \infty} P \left( \bigcup_{j=k}^{\infty} E_{n_j} \right) > \delta.$$

For $\omega \in E$, there exists a sub-subsequence $n_{km}$ such that $f_{n_{km}}(\omega) - dQ/dP(\omega) > \epsilon$. By Fatou’s lemma, we have

$$\liminf_{m \to \infty} E[(f_{n_{km}}(\omega) - dQ/dP(\omega))\mathbb{1}_{\omega \in E}] \geq \epsilon \cdot P(E) > \epsilon \cdot \delta > 0,$$

which is a contradiction.

The rest of this section is dedicated to proving Proposition 4.1. We will need to prove a version of Proposition 2.10 for boxes whose size do not depend on $n$. 21
Definition 4.3. For sets $A, B \subset \mathbb{Z}^d$, define $\text{dist}(A, B) = \min \{ \| x - y \|_1 : x \in A, y \in B \}$. For $x \in \mathbb{Z}^d$, denote $x + A = \{ x + z : z \in A \}$.

Lemma 4.4. Let $d \geq 3$. Fix $n \in \mathbb{N}$ and let $L \in (0, n^{1/2}] \cap \mathbb{N}$. Let $x \in \mathbb{Z}^d$, $z \in \mathcal{P}(x, n^{1/2})$, and let $\Delta \subset \partial^+ \mathcal{P}(x, n^{1/2})$ be a $(d - 1)$-dimensional box with side-length $\lambda$, and let $I \subset [\mathbb{E}[\tau_{T_{\partial^+ \mathcal{P}(x, n^{1/2})}}] - R_5(n)n^{1/2}, \mathbb{E}[\tau_{T_{\partial^+ \mathcal{P}(x, n^{1/2})}}] + R_5(n)n^{1/2}]$ an interval of length $L$. Let $G = G(\Delta, I)$ be the event that for every $z \in \mathcal{P}(x, n^{1/2})$ we have

$$|P_\omega^z(X_{\tau_{T_{\partial^+ \mathcal{P}(x, n^{1/2})}}} \in \Delta, T_{\partial^+ \mathcal{P}(x, n^{1/2})} \in I) - \mathbb{P}(X_{\tau_{T_{\partial^+ \mathcal{P}(x, n^{1/2})}}} \in \Delta, T_{\partial^+ \mathcal{P}(x, n^{1/2})} \in I)| \leq \frac{C}{n^{d/2}} \frac{R_6(L)}{L^{(d-1)/2(d+2)}}.$$

Then for any $\alpha < d$, there exist constants $C, c > 0$ such that $P(G^c) \leq Ce^{-(\log L)^\alpha}$ for any $n \in \mathbb{N}$ and $L \in (0, n^{1/2}] \cap \mathbb{N}$.

Proof. By translation invariance of $P$, we will only consider the case when $x$ is the origin. The proof will use descending induction on the size of the boxes. Fix $\theta \in (0, 1/5)$. For $j \in \mathbb{N}$, let $N_j = [n^{1/2}]$ and let $r(n) = [\frac{\log n}{\log L}]$. Note that $r(n)$ is the minimal natural number satisfying $N_\theta^r(n) \leq L$. Denote $n_0 = n - \sum_{j=1}^{r(n)} N_j$ and $n_k = \sum_{j=1}^{k} N_j$. For $0 \leq k \leq r(n)$, let $\Pi_k$ be a partition of $\partial^+ \mathcal{P}(0, n_{k+1}^{1/2})$ into $(d - 1)$-dimensional boxes of side-length $N_{k+1}^\theta$. Let $I_k$ be a partition of $[\mathbb{E}[\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}] - R_5(n_k)n_k^{1/2}, \mathbb{E}[\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}] + R_5(n_k)n_k^{1/2}]$ into intervals of side-length $N_k^\theta$. Define

$$\lambda_k = \max_{\Delta \in \Pi_k} \max_{I \in I_k} |P_\omega^z(X_{T_{\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}} \in \Delta, T_{\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}} \in I) - \mathbb{P}(X_{T_{\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}} \in \Delta, T_{\tau_{T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}}} \in I)|.$$

To simplify notation, denote $T_{\partial^+ \mathcal{P}(0, n_k^{1/2})}$ and $T_{\partial^+ \mathcal{P}(0, n_{k-1}^{1/2})}$ by $T$ and $T'$, respectively. For fixed $\Delta \in \Pi_k$ and $I \in I_k$, we have

$$|P_\omega^z(X_T \in \Delta, T \in I) - \mathbb{P}(X_T \in \Delta, T \in I)| \leq \sum_{\Delta \in \Pi_{k-1}} \sum_{I' \in I_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} P_\omega^z(X_{T'} = y, T' = m, X_T \in \Delta, T \in I) - \mathbb{P}(X_{T'} = y, T' = m, X_T \in \Delta, T \in I) \right| \leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in I_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} P_\omega^z(X_{T'} = y, T' = m) \times [P_\omega^z(X_T \in \Delta, T \in I - m) - \mathbb{P}(X_T \in \Delta, T \in I - m)] \right| + \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in I_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} \mathbb{P}(X_T \in \Delta, T \in I - m) \times [P_\omega^z(X_{T'} = y, T' = m) - \mathbb{P}(X_{T'} = y, T' = m)] \right| + \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in I_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} \mathbb{P}(X_{T'} = y, T' = m) \right|.$$
\[ \times \left| P^y(X_T \in \Delta, T \in I - m) - P^z(X_T \in \Delta, T \in I | X_{T'} = y, T' = m) \right| \]

\[ =: S_1 + S_2 + S_3. \]

We first bound \( S_1 \):

\[ S_1 = \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} P^z_\omega(X_{T'} = y, T' = m) \right| \times \left| P^y_\omega(X_T \in \Delta, T \in I - m) - P^y(X_T \in \Delta, T \in I - m) \right| \]

\[ \leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} P^z(X_{T'} \in \Delta', T' \in I') \]

\[ \times \max_{y \in \Delta'} \max_{m \in I'} \left| P^y_\omega(X_T \in \Delta, T \in I - m) - P^y(X_T \in \Delta, T \in I - m) \right|. \]

By our induction hypothesis and Lemma 2.9, we have

\[ P^z_\omega(X_{T'} \in \Delta', T' \in I') \]

\[ \leq P^z(X_{T'} \in \Delta', T' \in I') + |P^z_\omega(X_{T'} \in \Delta', T' \in I') - P^z(X_{T'} \in \Delta', T' \in I')| \]

\[ \leq P^z(X_{T'} \in \Delta', T' \in I') \]

\[ \leq C N_{k-1}^{\theta d} + \frac{C N_{k-1}^{\theta d}}{\frac{R_3(N_k)}{\sum_{m \in I'} \max \sum_{y \in \Delta'}} \left| P^y_\omega(X_T \in \Delta, T \in I - m) - P^y(X_T \in \Delta, T \in I - m) \right| \]}

which implies

\[ S_1 \leq \frac{C N_{k-1}^{\theta d}}{n^{d/2}} \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \max_{y \in \Delta'} \max_{m \in I'} \left| P^y_\omega(X_T \in \Delta, T \in I - m) - P^y(X_T \in \Delta, T \in I - m) \right|. \]

We say the pair \((\Delta', I') \in \Pi_{k-1} \times \mathcal{I}_{k-1}\) is good if for every \(y \in \Delta'\) and \(m \in I'\), we have for every \(\Delta \in \Pi_k\) and \(I \in \mathcal{I}_k\)

\[ |P^y_\omega(X_T \in \Delta, T \in I - m) - P^y(X_T \in \Delta, T \in I - m)| \leq \frac{C N_{k-1}^{\theta d}}{n^{d/2}} \frac{R_3(N_k)}{N_k^{\theta(d-1)/2(d+2)}}, \]

(4.1)

and

\[ P^y_\omega(T_{\theta P(y,N_k)} \neq T_{\theta + P(y,N_k)}) \leq \exp(-cR_5(N_k)), \]

(4.2)

\[ P^y_\omega(|T_{\theta P(y,N_k)} - \mathbb{E}^zT_{\theta P(y,N_k)}| > R_{j+1}(N_k)N_k) \leq C \exp(-c \log(R_j(N_k)\alpha)). \]

By Lemma 2.5, Proposition 3.3 and Markov’s inequality, we have for any \((\Delta', I')\)

\[ P((\Delta', I') \text{ is bad}) \leq C e^{-c(\log N_k)\alpha}. \]

Let \(A_k\) be the event that for all boxes \(\Delta' \in \Pi_{k-1}\) satisfying \(\text{dist}(\Delta, \Delta') < N_{k-1}\) and all intervals \(I' \in \mathcal{I}_{k-1}\) satisfying \(\text{dist}(I', I) < N_{k-1}\), \((\Delta', I')\) is good. From the union
bound, \(P(A'_k) \leq C \exp(-c(\log N_k)^\alpha)\). For \(\omega \in A_k\), we have by (4.2) that if \(\Delta'\) satisfies \(\text{dist}(\Delta, \Delta') > R_5(N_k)\), or \(I'\) satisfies \(\text{dist}(I, I') > R_5(N_k)\), then

\[
\max_{y \in \Delta'} \max_{m \in I'} |P^\omega_\Delta(X_T \in \Delta, T \in I - m) - P^\omega(X_T \in \Delta, T \in I - m)| = N_k^{-\xi(1)}.
\]

Hence we only need to consider at most

\[
\frac{CR_5(N_k)N_k^{d/2}}{|\Delta'|} \times \frac{CR_5(N_k)N_k^{d/2}}{|I'|} \leq \frac{CR_5(N_k)N_k^{d/2}}{N_{k-1}^{d/2}N_k^{\theta(d-1)/(2(d+2))}}
\]

such pairs \((\Delta', I')\). Thus for \(\omega \in A_k\), we can apply (4.1) and get

\[
S_1 \leq \frac{CN_{k-1}^{d/2} \left( CR_5(N_k)N_k^{d/2} \frac{R_3(N_k)}{N_{k-1}^{d/2}N_k^{\theta(d-1)/(2(d+2))}} \right)}{n^{d/2}}
\]

Next, we bound \(S_2:\)

\[
S_2 = \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} \sum \mathbb{P}^y(X_T \in \Delta, T \in I - m) \times \left[ P^\omega_\Delta(X_T = y, T' = m) - \mathbb{P}^\omega(X_T = y, T' = m) \right] \right|
\]

\[
\leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \sum_{\Delta'} \sum_{y \in \Delta'} \sum_{m \in I'} \max \mathbb{P}^y(X_T \in \Delta, T \in I - m) \times \left[ P^\omega_\Delta(X_T' \in \Delta', T' \in I') - \mathbb{P}^\omega(X_T' \in \Delta', T' \in I') \right]
\]

\[
\leq \lambda_{k-1} \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \sum_{y \in \Delta'} \max \mathbb{P}^y(X_T \in \Delta, T \in I - m).
\]

Since we are dealing with the annealed measure, we can apply the same analysis for bounding \(S_1\) and get

\[
\sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \max \mathbb{P}^y(X_T \in \Delta, T \in I - m) \leq \frac{CR_5(N_k)N_k^{d/2}}{N_{k-1}^{d/2}N_k^{\theta(d-1)/(2(d+2))}} \leq \frac{CR_5^2(N_k)}{N_k^{d/2}N_{k-1}^{d/2}}.
\]

On the other hand, by the induction hypothesis, we have

\[
S_2 \leq \lambda_{k-1} \frac{CR_5^2(N_k)}{N_k^{\theta d}} \leq \frac{CN_{k-1}^{d/2} \left( CR_5^2(N_k) \frac{R_5(N_k)}{N_{k-1}^{d/2}N_k^{\theta(d-1)/(2(d+2))}} \right)}{n^{d/2}} \leq \frac{CN_{k-1}^{d/2} R_5(N_k) R_5^2(N_k)}{n^{d/2} N_{k-1}^{\theta(d-1)/(2(d+2))}}.
\]
Finally, we bound $S_3$:

$$S_3 = \sum_{\Delta \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \left| \sum_{y \in \Delta'} \sum_{m \in I'} \mathbb{P}^z(X_{T'} = y, T' = m) \right|$$

$$\times \left| \mathbb{P}^z(X_T \in \Delta, T \in I - m) - \mathbb{P}^z(X_T \in \Delta, T \in I | X_{T'} = y, T' = m) \right|$$

$$\leq \frac{N_{k-1}^{d \theta}}{n^{d/2}} \sum_{\Delta' \in \Pi_{k-1}} \sum_{I' \in \mathcal{I}_{k-1}} \left| \max_{y \in \Delta'} \mathbb{P}^y(X_T \in \Delta, T \in I - m) - \min_{y \in \Delta'} \mathbb{P}^y(X_T \in \Delta, T \in I | X_{T'} = y, T' = m) \right| .$$

For any $y, x \in \Delta'$ and $k \in I'$, we have from Lemma 2.9

$$\left| \mathbb{P}^y(X_T \in \Delta, T \in I - m) - \mathbb{P}^z(X_T \in \Delta, T \in I | X_{T'} = x, T' = k) \right|$$

$$\leq \left| \mathbb{P}^y(X_T \in \Delta, T \in I - m) - \mathbb{P}^z(X_T \in \Delta, T \in I | X_{T'} = x, T' = m) \right|$$

$$+ \left| \mathbb{P}(X_T \in \Delta, T \in I | X_{T'} = x, T' = m) - \mathbb{P}(X_T \in \Delta, T \in I | X_{T'} = x, T' = k) \right|$$

$$\leq \frac{|x - y| \cdot CN_k^{\theta d}}{N_k^{(d+1)/2}} + \frac{|m - k| \cdot CN_k^{\theta d}}{N_k^{(d+1)/2}}$$

$$\leq \frac{CN_k^{\theta d} N_{k-1}^{\theta d}}{N_k^{(d+1)/2}} .$$

We thus get

$$S_3 \leq \frac{N_{k-1}^{d \theta}}{n^{d/2}} \frac{R_3(N_k) N_k^{d/2}}{N_k^{(d+1)/2}} \frac{CN_k^{\theta d} N_{k-1}^{\theta d}}{N_k^{(d+1)/2}} \leq \frac{CN_k^{\theta d} N_{k-1}^{\theta d}}{n^{d/2} N_k^{(d+1)/2}} .$$

We conclude that for $\omega \in A_k$ and $\lambda_{k-1}$ satisfying the induction hypothesis, then

$$\lambda_k \leq \frac{CN_k^{\theta d}}{n^{d/2}} \left( \frac{R_3(N_k)}{N_k^{\theta (d-1)/2(d+2)}} + \frac{R_5(N_{k-1}) N_k^{\theta d}}{N_{k-1}^{d (d+1)/2(d+2)}} + \frac{N_{k-1}^{\theta d}}{N_k^{1/2}} \right)$$

$$\leq \frac{CN_k^{\theta d}}{n^{d/2}} \frac{R_5(N_k)}{N_k^{\theta (d-1)/2(d+2)}} ,$$

where the last inequality is due to $N_k < N_{k-1}$ and $\theta < 1/5$. Hence $\cap_{k=1}^{r(n)} A_k \subset G(\Delta, I)$, and we finish the proof by observing

$$P \left( \bigcup_{k=1}^{r(n)} A_k^c \right) \leq \sum_{k=1}^{r(n)} 2e^{-c(\log N_k)\alpha} \leq C e^{-c(\log L)\alpha}$$

for any $\alpha < d$. \qed

To find a lower bound for $f_n$, we will need a lower bound for the annealed heat kernel.
Lemma 4.5. Let $d \geq 2$. Suppose $P$ is uniformly elliptic, i.i.d. and satisfies condition (T). There exist constants $C, c > 0$ such that for $z \in \mathbb{Z}^d$ and $x \in H_n$ satisfying $\|x - \mathbb{E}^z[T_n]\|_\infty < Cn^{1/2}$ and $m \in [\mathbb{E}[T_n] - Cn^{1/2}, \mathbb{E}[T_n] + Cn^{1/2}]$, we have
\[
P^z(X_{T_n} = x, T_n = m) > c/n^{d/2}.
\]

Proof. The argument closely follows the proof of Lemma 4.4 in [2]. Denote $B(n, k) = \{(X_{n_k}, e_1) = n\}$ and $B(n) = \bigcup_k B(n, k)$. Suppose that $x \in H_n$ and $m \in \mathbb{N}$ satisfy
\[
|x - k\mathbb{E}^z[X_{\tau_k} - \tau_1]| < C k^{1/2} \quad \text{and} \quad |m - k\mathbb{E}^z[\tau_2 - \tau_1]| < C k^{1/2}
\]
for $k \in [M - M^{1/2}, M + M^{1/2}]$ for some $M = O(n)$. By Theorem 1.6 and the local central limit theorem for sum of i.i.d. lattice-valued random variables, see Theorem 2.1.1 in [14], under (4.3) we have
\[
P^z(X_{T_n} = x, T_n = m, B(n, k)) = P^z(X_{\tau_k} = x, \tau_k = m, B(n, k)) \geq c/k^{(d+1)/2}.
\]
This yields us
\[
P^z(X_{T_n} = x, T_n = m) \geq P^z(X_{T_n} = x, T_n = m, B(n)) \\
\geq \sum_{k=M-M^{1/2}}^{M+M^{1/2}} P^z(X_{\tau_k} = x, \tau_k = m, B(n, k)) \\
\geq 2cM^{1/2}/n^{(d+1)/2} \\
= O(n^{-d/2}).
\]
Hence it is enough to show that the assumptions of the lemma imply (4.3). Let $\beta_n = \inf\{j \in \mathbb{N} : \langle X_{\tau_{j+1}} - \tau_1, \ell \rangle \geq n\}$, and note that
\[
\{\beta_n = k\} = \left\{\sum_{j=1}^{k-1} \langle X_{\tau_{j+1}} - \tau_1, \ell \rangle < n \leq \sum_{j=1}^{k} \langle X_{\tau_{j+1}} - \tau_1, \ell \rangle \right\}.
\]
Thus $\beta_n$ is a stopping time for the filtration for the process $\{X_{\tau_{j+1}} - X_{\tau_1} : j \in \mathbb{N}\}$, and by Wald’s identity we have
\[
\mathbb{E}^z[\beta_n] \mathbb{E}^z[X_{\tau_2} - X_{\tau_1}] = \mathbb{E}^z \left[\sum_{j=1}^{\beta_n} (X_{\tau_{j+1}} - X_{\tau_j}) \right],
\]
and we can now write
\[
\mathbb{E}^z[X_{T_n}] = \mathbb{E}^z[X_{T_n} - X_{\beta_n+1}] + \mathbb{E}^z[\beta_n] \mathbb{E}^z[X_{\tau_2} - X_{\tau_1}].
\]
Let $M = \mathbb{E}^z[\beta_n]$ and note that $M \asymp n$, see Lemma 5.1 in [18]. For $k \in [M - M^{1/2}, M + M^{1/2}]$, we have $\|x - \mathbb{E}^z[T_n]\|_\infty < Cn^{1/2} = O(k^{1/2})$, which implies
\[
\|x - k\mathbb{E}^z[X_{\tau_2} - X_{\tau_1}]\|_\infty \leq \|x - \mathbb{E}^z[T_n]\|_\infty + \|\mathbb{E}^z[X_{T_n}] - k\mathbb{E}^z[X_{\tau_2} - X_{\tau_1}]\|_\infty \\
\leq Ck^{1/2} + \|M\mathbb{E}^z[X_{\tau_2} - X_{\tau_1}] + \mathbb{E}^z[X_{T_n} - X_{\beta_n+1}] - k\mathbb{E}^z[X_{\tau_2} - X_{\tau_1}]\|_\infty \\
\leq Ck^{1/2} + CM^{1/2}\|\mathbb{E}^z[X_{\tau_2} - X_{\tau_1}]\|_\infty \\
= O(k^{1/2}).
\]
Figure 2: The naïve trap is the environment where all the drifts in $\Delta$ are roughly pointing towards the origin. The random walk (in red) will typically spend a long time in this trap.

Applying a similar analysis, we have that if $m \in [\mathbb{E}[T_n] - Cn^{1/2}, \mathbb{E}[T_n] + Cn^{1/2}]$, then $m \in [k\mathbb{E}^z[\tau_2 - \tau_1] - Ck^{1/2}, k\mathbb{E}^z[\tau_2 - \tau_1] + Ck^{1/2}]$ for $k \in [M - M^{1/2}, M + M^{1/2}]$. This finishes the proof.

**Lemma 4.6.** Let $d \geq 2$. There exist constants $c, C > 0$ such that for all $u > 0$ and for large enough $n \in \mathbb{N}$, we have

$$P(\forall z \in B(-nv, cn^{1/2}), P^x_\omega(X_n = 0) > u/n^{d/2}) \geq Ce^{-c(\log u)^d}.$$

**Proof.** To prove the lower bound, we will need to consider two events. The first is the atypical event that there is a trap at the origin. The second is the typical event that quenched random walk behaves like the annealed random walk up until the first hitting of a box around the origin. Define the parallelogram centered at the origin

$$\Delta = \left\{ x \in \mathbb{Z}^d : |\langle x, e_1 \rangle| < L, \left\| x - \vartheta \cdot \frac{\langle x, e_1 \rangle}{\langle \vartheta, e_1 \rangle} \right\|_\infty < L \right\}.$$

We first recall the naïve trap event from [18]

$$E_L = \{ \omega \in \Omega : \forall y \in \Delta \setminus \{0\}, \langle d(y, \omega), y/|y| \rangle \leq -c_1 \},$$

where $c_1 > 0$ is some constant. For $z \in \mathbb{Z}^d$ and $r > 0$, denote $B_2(z, r) = \{ x \in \mathbb{Z}^d : |x - z| \leq r \}$. We will need the following estimate.

**Lemma 4.7.** Suppose $\omega \in E_L$. There exist positive constants $c_2, c_4, p_0$ such that for $y \in \Delta \setminus B(0, c_2)$

$$P^y_\omega(T_{\{0\}} > c_4L) < 1 - p_0.$$

**Proof.** We first recall a few facts about the naïve trap. From Lemma 2.8 in [18], we see that there exist positive constants $c_2, c_3$, such that for $y \in \Delta \setminus B_2(0, c_2)$ and $\omega \in E_L$, $\exp(c_3|X_n \wedge T_{B_2(0, c_2)} \wedge T_\Delta|)$ is a $P^y_\omega$-supermartingale. By uniform ellipticity, it is enough to prove the statement of the lemma with $T_{\{0\}}$ replaced by $T_{B_2(0, c_2)}$. We decompose this probability

$$P^y_\omega(T_{B_2(0, c_2)} > c_4L) \leq P^y_\omega(T_{\Delta^c} < T_{B_2(0, c_2)}) + P^y_\omega(T_{\Delta^c} > T_{B_2(0, c_2)} > c_4L).$$
By the optional stopping theorem, we obtain for any $y \in \Delta \setminus B_2(0, c_2)$ and $u > 0$
\[ P^y_w(\{X_n \cap T_{B_2(0, c_2)} \cap T_{\Delta}^u \} > u) \leq e^{-c_3(|y| - u)}. \quad (4.4) \]
Applying this inequality and uniform ellipticity, we obtain
\[ P^y_w(T_{\Delta} < T_{B_2(0, c_2)}) \leq (1 - p_0)/2 \]
for some $p_0 \in (0, 1)$. To bound the second summand, we note that from the proof of Lemma 2.8 in [18], for $x \in \Delta \setminus B_2(0, c_2)$ we have
\[ E^x_w[e^{\lambda|x|}] \leq e^{\lambda|x|} \left( 1 + \lambda(d(x, \omega), \frac{x}{|x|}) + O(\lambda/|x|) + O(\lambda^2) \right). \]
Since $|x| \geq c_2$, and $\omega \in E_L$, we can choose $\lambda$ small enough and $c_2$ large enough such that
\[ E^x_w[e^{\lambda|x|}] \leq e^{\lambda|x|} (1 - c_1 \lambda/2) \leq e^{\lambda|x| - c_1 \lambda^2/2}. \]
From this estimate and the Markov property,
\[ E^x_w[e^{\lambda|x_n|1}_n\{T_{\Delta} \cap T_{B_2(0, c_2)} \geq n\}] = E^x_w[E^x_{n-1}[e^{\lambda|x_n|1}_n\{T_{\Delta} \cap T_{B_2(0, c_2)} \geq n\}]] \leq e^{-c_1 \lambda/2}E^x_w[e^{\lambda|x_n-1|1}_n\{T_{\Delta} \cap T_{B_2(0, c_2)} \geq n\}], \]
and continuing iteratively, we get
\[ E^x_w[e^{\lambda|x_n|1}_n\{T_{\Delta} \cap T_{B_2(0, c_2)} \geq n\}] \leq e^{-c_1 \lambda n/2 + \lambda|x|}. \]
Applying Chebyshev’s inequality and setting $n = c_4 L$, we obtain
\[ P^y_w(T_{\Delta} > T_{B_2(0, c_2)} > c_4 L) \leq P^y_w(|X_{c_4 L}| > c_2, T_{\Delta} \cap T_{B_2(0, c_2)} \geq c_4 L) \leq \exp(-c_1 c_4 \lambda L/2 + \lambda L). \]
Taking $c_4$ large enough finishes the proof. \( \square \)

Next, we define the typical event. Fix $\epsilon > 0$. Define $\partial^\Delta = \Delta \cap H_{-L}$, and partition it into $(d - 1)$-dimensional boxes $\delta$ of side-length $[L']$. Let $c$ be as in Lemma 4.5 and let $I = \left[ n - L/\langle v, e_1 \rangle - cn^{1/2}, n - L/\langle v, e_1 \rangle + cn^{1/2} \right]$ and partition it into intervals $\eta$ of side-length $[L']$. Define
\[ g(\delta, \eta) = |P^z_w(X_{T_{-L}} \in \delta, T_{-L} \in \eta) - \mathbb{P}^z(X_{T_{-L}} \in \delta, T_{-L} \in \eta)|, \]
and note that from Lemma 4.4 $P(g(\delta, \eta) > CL^d/n^{d/2}) = L^{-\xi(1)}$. Define the event
\[ F = F(n, L, \epsilon) = \left\{ \omega \in \Omega : \forall \delta \subset \partial^\Delta, \forall \eta \subset I, g(\delta, \eta) \leq \frac{CL^d}{n^{d/2}} \frac{R_3(L')}{L^{(d+1)/2(d+1)}} \right\}. \]
From the union bound, we have $P(F) = 1 - L^{-\xi(1)}$. Choose $L$ large enough such that $P(F) > 1/2$. For $\omega \in F$, and by Lemma 4.5 we have
\[ P^z_w(X_{T_{-L}} \in \delta, T_{-L} \in \eta) \geq \mathbb{P}^z(X_{T_{-L}} \in \delta, T_{-L} \in \eta) - \frac{CL^d}{n^{d/2}} \frac{R_3(L')}{L^{(d+1)/2(d+1)}} \geq \frac{CL^d}{n^{d/2}} \frac{R_3(L')}{L^{(d+1)/2(d+1)}} \geq \frac{CL^d}{n^{d/2}}. \]
\[ \frac{CL^d}{n^{d/2}}. \]
We will now show $P^x_\omega(X_n = 0)$ is large when $\omega \in F \cap E_L$. We have

$$P^x_\omega(X_n = 0) = \sum_{y \in H} \sum_{m < n} P^x_\omega(X_{T(y)} = y, T(y) = m) P^y_\omega(X_{n-m} = 0)$$

$$\geq \sum_{y \in \partial \Delta} \sum_{m \in I} P^x_\omega(X_{T-L} = y, T-L = m) P^y_\omega(X_{n-m} = 0)$$

$$= \sum_{\delta} \sum_{\eta} \sum_{y \in \delta} \sum_{m \in \eta} P^x_\omega(X_{T-L} = y, T-L = m) P^y_\omega(X_{n-m} = 0).$$

We force the random walk to hit the origin

$$P^y_\omega(X_{n-m} = 0) > P^y_\omega(X_{n-m} = 0, T\{0\} < cL)$$

$$\geq \sum_{k = L} \frac{P^y_\omega(T\{0\} = k) P^0_\omega(X_{n-m-k} = 0)}{p_0 \min_{k \in [L,cL]} P^0_\omega(X_{n-m-k} = 0)},$$

where the last inequality is from Lemma 4.7. Applying this, and that $\omega \in F \cap E_L$, we have

$$P^\omega_\omega(X_n = 0) \geq C \sum_{\delta} \sum_{\eta} \sum_{m \in \eta} P^\omega_\omega(X_{T-L} = y, T-L = m) \min_{k \in [L,cL]} P^\omega_\omega(X_{n-m-k} = 0)$$

$$\geq C \sum_{\delta} \sum_{\eta} P^\omega_\omega(X_{T-L} \in \delta, T-L \in I) \min_{m \in \eta} \min_{k \in [L,cL]} P^\omega_\omega(X_{n-m-k} = 0)$$

$$\geq C \frac{L^{d+1} \min_{\eta} \min_{k \in [L,cL]} P^\omega_\omega(X_{n-m-k} = 0)}{n^{d/2}}.$$

To get a lower bound for $P^\omega_\omega(X_{n-m-k} = 0)$, we can force the random walk to hit $\partial B_2(0,c_2)$, then successively make sojourns to $\Delta \setminus B_2(0,c_2)$ for at most $n - m - k$ times, and then return to the origin. This, with (4.4) and uniform ellipticity, yields

$$P^\omega_\omega(X_{n-m-k} = 0) \geq \kappa^{2\sqrt{d}c_2} \left( \min_{x \in \partial B_2(0,c_2)} P^x_\omega(T_{B_2(0,c_2)} < T_{\Delta^c}) \right)^{n - m - k} \geq \kappa^{2\sqrt{d}c_2} \left( 1 - e^{-c_3L} \right)^{n - m - k}.$$

Applying this inequality, we get

$$\sum_{\eta} \min_{k \in [L,cL]} \min_{m \in \eta} P^\omega_\omega(X_{n-m-k} = 0) \geq \sum_{\eta} \min_{m \in \eta} \min_{k \in [L,cL]} \kappa^{2\sqrt{d}c_2} \left( 1 - e^{-c_3L} \right)^{n - m - k}$$

$$\geq \sum_{\eta} \min_{m \in \eta} \kappa^{2\sqrt{d}c_2} \left( 1 - e^{-c_3L} \right)^{n - m - L}.$$

We choose the following partition for $I$: let

$$\eta_j = \{ m \in \mathbb{N} : m \in [n - L/(v,e_1) + j[L^c], n - L/(v,e_1) + (j+1)[L^c]) \}$$
for \( j \in J \) where \( J \) is the corresponding index set. We thus have
\[
\sum_{\eta} \min_{m \in \eta} \kappa^{2\sqrt{d}c_2} (1 - e^{-c_3 L})^{n-m-L} = \sum_{j \in J} \min_{m \in \eta} \kappa^{2\sqrt{d}c_2} (1 - e^{-c_3 L})^{n-m-L}
\geq \sum_{j=1}^{\lceil \eta \sqrt{d}/L \rceil} \kappa^{2\sqrt{d}c_2} (1 - e^{-c_3 L})^{\frac{\eta}{(\eta+1)^2 L} - 2jL^2}
\geq C L^{-e^{cL}}.
\]
Putting everything together, we get that for \( \omega \in F \cap E_L \),
\[
P^\omega(X_n = x) \geq C \frac{L^{d-1}}{n^{d/2}} e^{cL} = C \exp(cL - (d-1) \log L)/n^{d/2} > e^{cL}/n^{d/2},
\]
for some \( \hat{c} > 0 \) for large \( L \). Letting \( L = \lceil \log u/\hat{c} \rceil \), we conclude that
\[
P(F \cap E_{\log u/\hat{c}}) \geq P(F \cap E_{\log u/\hat{c}}).
\]
To finish the proof, we note that for any \( L > 0 \), \( F \) and \( E_L \) are independent. Indeed \( F \) is measurable with respect to \( \{ \omega(x) : \langle x, e_1 \rangle < -L \} \) and \( E_L \) is measurable with respect to \( \{ \omega(x) : x \in \Delta \} \). Since both sets do not intersect, and \( P \) is an i.i.d. measure, we get \( P(F \cap E_{\log u/\hat{c}}) = P(F)P(E_{\log u/\hat{c}}) \). Finally, we recall that \( P(F) > 1/2 \) and \( P(E_{\log u/\hat{c}}) > Ce^{-c(\log u)^d} \), which finishes the proof.

Before we prove the proposition, we need one last auxiliary result.

**Definition 4.8.** For \( x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \), we denote \( x \leftrightarrow n \) if \( x \) and \( n \) have the same parity, i.e. \( \sum_{i=1}^d x_i + n \) is even. Note that since our random walk is nearest-neighbor, \( \mathbb{P}_0(X_n = x) > 0 \) if and only if \( x \leftrightarrow n \).

**Lemma 4.9** (Proposition A.1 in [3]). Let \( d \geq 2 \) and assume \( P \) is uniformly elliptic, i.i.d. and satisfies condition (T). Then
\[
\lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d \atop x \leftrightarrow n} \mathbb{P}_0(X_n = x) - \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} (x - \mathbb{E}^0 X_n)^T \Sigma^{-1} (x - \mathbb{E}^0 X_n) \right) = 0,
\]
where \( \Sigma \) is some non-degenerate covariance matrix.

**Proof of Proposition 4.1.** We first prove the lower bound. For any \( \epsilon > 0 \) there exists a \( C > 0 \) such that
\[
\left\{ \omega \in \Omega : \sum_{z \in \mathbb{Z}^d} P^\omega_z(X_n = 0) > u \right\} \supset \left\{ \omega \in \Omega : \forall z \in B(-nv, cn^{1/2}), \ P^\omega_z(X_n = 0) > Cu/n^{d/2} \right\}.
\]
Choosing \( \epsilon \) from Lemma 4.6, we conclude that the probability of the right-hand event has the lower bound \( C e^{-c(\log u)^d} \). We are left to prove the upper bound. For any \( \epsilon > 0 \), we
have
\[ P \left( \sum_{z \in \mathbb{Z}^d} P^z_{\omega}(X_n = 0) > u \right) \leq P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} P^z_{\omega}(X_n = 0) > u/2 \right) \]
\[ + P \left( \sum_{z \not\in B(-nv, L \cdot n^{1/2})} P^z_{\omega}(X_n = 0) > u/2 \right). \]

Let \( \Delta \) be a box with side-length \( L \), where \( L \) is the largest natural number satisfying \( |\Delta| \leq u/4 \). We bound the first term:

\[ P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} P^z_{\omega}(X_n = 0) > u/2 \right) \]
\[ \leq P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} P^z_{\omega}(X_n \in \Delta) > u/2 \right) \]
\[ \leq P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} (|\mathbb{P}^z(X_n \in \Delta) + |\mathbb{P}^z(X_n \in \Delta) - P^z_{\omega}(X_n \in \Delta)| > u/2 \right) \).

Using translation invariance of \( \mathbb{P} \),
\[ \sum_{z \in \mathbb{Z}^d} \mathbb{P}^z(X_n \in \Delta) = \sum_{z \in \mathbb{Z}^d} \mathbb{P}^0(X_n \in \Delta - z) \]
\[ = \mathbb{E}^0 \left[ \sum_{x \in \mathbb{Z}^d} 1_{\{X_n = x\}} \sum_{z \in \mathbb{Z}^d} 1_{\{x + z \in \Delta\}} \right] \]
\[ = |\Delta| \cdot \mathbb{E}^0 \left[ \sum_{x \in \mathbb{Z}^d} 1_{\{X_n = x\}} \right] \]
\[ = |\Delta|. \]

We thus get
\[ P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} P^z_{\omega}(X_n = 0) > u/2 \right) \]
\[ \leq P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} |\mathbb{P}^z(X_n \in \Delta) - P^z_{\omega}(X_n \in \Delta)| > u/2 - |\Delta| \right) \]
\[ \leq P \left( \sum_{z \in B(-nv, L \cdot n^{1/2})} |\mathbb{P}^z(X_n \in \Delta) - P^z_{\omega}(X_n \in \Delta)| > u/4 \right) \]
\[ \leq P \left( \exists z \in B(-nv, L \cdot n^{1/2}), |\mathbb{P}^z(X_n \in \Delta) - P^z_{\omega}(X_n \in \Delta)| > C\frac{u}{n^{d/2}L^{\ell d}} \right) \]

31
\[ \leq P \left( \exists z \in B(-nv, L^* \cdot n^{1/2}), \ |\mathbb{P}^z(X_n \in \Delta) - P^z_\omega(X_n \in \Delta)| > C \frac{u \cdot R_6(L)}{n^{d/2} \cdot L(d-1)/2(d+2)} \right) \]
\[ \leq C \exp(-c(\log u)^\alpha), \]

where the last inequality holds from Lemma 4.4 and the fact that \( L \asymp u^{1/d} \). Note that the fourth inequality holds by choosing \( \epsilon > 0 \) small enough. We now bound the second term by Markov’s inequality:

\[ P \left( \sum_{z \notin B(-nv, L^* \cdot n^{1/2})} P^z_\omega(X_n = 0) > u/2 \right) \]
\[ \leq \frac{2}{u} \sum_{z \notin B(-nv, L^* \cdot n^{1/2})} \mathbb{P}^z(X_n = 0) \]
\[ = \frac{2}{u} \sum_{z \notin B(nv, L^* \cdot n^{1/2})} \mathbb{P}^0(X_n = z) \]
\[ \leq \frac{2}{u} \sum_{z \notin B(nv, L^* \cdot n^{1/2})} \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp \left( -\frac{1}{2n} (z - \mathbb{E}^0 X_n)^T \Sigma^{-1} (z - \mathbb{E}^0 X_n) \right) + o(1) \]
\[ \leq Ce^{-cL^2u} + o(1), \]

where the first inequality is Markov’s inequality, the equality is by translation invariance of \( \mathbb{P} \), and the second inequality is from Lemma 4.9. Here we write \( o(1) \) with respect to \( n \). We conclude that

\[ \limsup_{n \to \infty} P \left( \sum_{z \in \mathbb{Z}^d} P^z_\omega(X_n = 0) > u \right) \leq Ce^{-c(\log u)^\alpha} + Ce^{-cu^2/d}. \]

Since \( e^{-c(\log u)^\alpha} \gg e^{-cu^2/d} \), we are finished with the proof.

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References

[1] Alili, S. Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.* 36, 2 (1999), 334–349.

[2] Berger, N. Slowdown estimates for ballistic random walk in random environment. *J. Eur. Math. Soc. (JEMS)* 14, 1 (2012), 127–174.

[3] Berger, N., Cohen, M., and Rosenthal, R. Local limit theorem and equivalence of dynamic and static points of view for certain ballistic random walks in i.i.d. environments. *Ann. Probab.* 44, 4 (2016), 2889–2979.
[4] Berger, N., Drewitz, A., and Ramírez, A. F. Effective polynomial ballisticity conditions for random walk in random environment. *Comm. Pure Appl. Math.* 67, 12 (2014), 1947–1973.

[5] Berger, N., and Zeitouni, O. A quenched invariance principle for certain ballistic random walks in i.i.d. environments. In *In and out of equilibrium. 2*, vol. 60 of *Progr. Probab.* Birkhäuser, Basel, 2008, pp. 137–160.

[6] Bolthausen, E., and Sznitman, A.-S. On the static and dynamic points of view for certain random walks in random environment. *Methods Appl. Anal.* 9, 3 (2002), 345–375.

[7] Bolthausen, E., and Sznitman, A.-S. *Ten lectures on random media*, vol. 32 of *DMV Seminar*. Birkhäuser Verlag, Basel, 2002.

[8] Dembo, A., Peres, Y., and Zeitouni, O. Tail estimates for one-dimensional random walk in random environment. *Comm. Math. Phys.* 181, 3 (1996), 667–683.

[9] Drewitz, A., and Ramírez, A. F. Ballisticity conditions for random walk in random environment. *Probab. Theory Related Fields* 150, 1-2 (2011), 61–75.

[10] Drewitz, A., and Ramírez, A. F. Quenched exit estimates and ballisticity conditions for higher-dimensional random walk in random environment. *Ann. Probab.* 40, 2 (2012), 459–534.

[11] Drewitz, A., and Ramírez, A. F. Selected topics in random walks in random environment. In *Topics in percolative and disordered systems*, vol. 69 of *Springer Proc. Math. Stat.* Springer, New York, 2014, pp. 23–83.

[12] Guerra, E., and Ramírez, A. F. A proof of Sznitman’s conjecture about ballistic RWRE. *Comm. Pure Appl. Math.* 73, 10 (2020), 2087–2103.

[13] Kozlov, S. M. The averaging method and walks in inhomogeneous environments. *Uspekhi Mat. Nauk* 40, 2(242) (1985), 61–120, 238.

[14] Lawler, G. F., and Limic, V. *Random walk: a modern introduction*, vol. 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[15] McDiarmid, C. Concentration. In *Probabilistic methods for algorithmic discrete mathematics*, vol. 16 of *Algorithms Combin.* Springer, Berlin, 1998, pp. 195–248.

[16] Sabot, C. Random Dirichlet environment viewed from the particle in dimension $d \geq 3$. *Ann. Probab.* 41, 2 (2013), 722–743.

[17] Sznitman, A.-S. Slowdown and neutral pockets for a random walk in random environment. *Probab. Theory Related Fields* 115, 3 (1999), 287–323.

[18] Sznitman, A.-S. Slowdown estimates and central limit theorem for random walks in random environment. *J. Eur. Math. Soc. (JEMS)* 2, 2 (2000), 93–143.
[19] Sznitman, A.-S. On a class of transient random walks in random environment. 
*Ann. Probab.* 29, 2 (2001), 724–765.

[20] Sznitman, A.-S. An effective criterion for ballistic behavior of random walks in 
random environment. *Probab. Theory Related Fields* 122, 4 (2002), 509–544.

[21] Sznitman, A.-S., and Zerner, M. A law of large numbers for random walks in 
random environment. *Ann. Probab.* 27, 4 (1999), 1851–1869.

[22] Zeitouni, O. Random walks in random environment. In *Lectures on probability 
theory and statistics*, vol. 1837 of *Lecture Notes in Math.* Springer, Berlin, 2004, 
pp. 189–312.

[23] Zerner, M. P. W. A non-ballistic law of large numbers for random walks in i.i.d. 
random environment. *Electron. Comm. Probab.* 7 (2002), 191–197.