Torsional rigidity, isospectrality and quantum graphs

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Abstract
We study torsional rigidity for graph and quantum graph analogs of well-known pairs of isospectral non-isometric planar domains. We prove that such isospectral pairs are distinguished by torsional rigidity.

Keywords: quantum graphs, torsional rigidity, heat content, isospectral

(Some figures may appear in colour only in the online journal)

1. Introduction

In 1966, Marc Kac offered a poetic formulation of what came to be a much studied problem involving the geometry of planar domains: Does Dirichlet spectrum determine a planar domain up to isometry? Kac’s problem inspired a great deal of work much of which centered on developments involving heat invariants. The appearance of Sunada’s work involving the construction of isospectral metrics for Riemannian manifolds provided powerful new methods for investigating diverse spectral phenomena; in particular, Sunada’s method led to the discovery of a number of negative results for inverse spectral problems. In 1992 Gordon, Webb and Wolpert used an extension of Sunada’s method to construct a pair of isospectral, non-isometric planar domains [GWW]. Soon after, Buser, Conway Doyle and Semmler produced a collection of families of pairs of isospectral non-isometric planar polygonal domains (we refer to these examples as BCDS pairs). Their construction, again going back to Sunada, involves reflecting a ‘seed’ triangle across edges (see [BCDS]). An example of their construction can be found in section 3 below and plays an important role in our work.

Given these negative results in reference to Kac’s question$^3$, one might ask: What are good invariants for planar domains and/or quantum graphs?

One approach to finding new invariants is to use data which, while not spectral, arise from the normalized Dirichlet eigenfunctions in a natural fashion. An example is afforded by the

$^3$ For smoothly bounded domains Kac’s problem is still open.
sequence generated by counting nodal domains. This approach has been investigated in a recent series of papers in which quantum graph analogs of BCDS pairs are produced and their corresponding nodal domain counts are compared (see [BPB, BSS]).

Another approach to producing non-spectral invariants is given by integration against the heat kernel. More precisely, suppose $p_D(t, x, y)$ is the Dirichlet heat kernel for the domain $D$ and $u(t, x) = \int_D p_D(t, x, y)dy$ is the solution of the initial value problem

$$\frac{\partial u}{\partial t} = \Delta u \text{ on } (0, \infty) \times D$$

$$\lim_{t \to 0} u(t, x) = 1.$$ 

Then one can define the heat content of $D$

$$q_D(t) = \int_D u(t, x)dx$$

as well as a sequence of invariants

$$A_k = k \int_0^\infty t^{k-1} q_D(t)dt.$$ 

(1.1)

The invariant $A_1$, sometimes referred to as the torsional rigidity of the domain $D$, can be computed via the solution of a Poisson problem. More precisely, if $u_1$ solves

$$\Delta u_1 = -1 \text{ on } D$$

$$u_1 = 0 \text{ on } \partial D$$

then

$$A_1 = \int_D u_1(x)dx.$$ 

Torsional rigidity arises in the study of elasticity and has a long history (a source for early work is [Po1]). In addition to its role in the mechanics of solid bodies, torsional rigidity is closely related to the expected exit time of Brownian motion from the domain $D$ and higher values of $k$ in (1.1) correspond to higher moments of the exit time (see [Mc1, KMM] and references within). For this reason the sequence $\{A_k\}$ defined by (1.1) has been labeled the $L^1$-moment spectrum associated to $D$.

It is easy to see that both heat content and the moment spectrum are invariants of the metric. Recent work suggest they play a valuable role in understanding fundamental geometric properties of a given ambient space (for example, isoperimetry [HMP1, HMP2, Mc2]), and one might reasonably ask how such objects compare to Dirichlet spectrum as tools to classify domain behavior up to isometry. Such a study has been initiated in [MM1]. For the purpose of this note, the main results of [MM1] can be summarized as follows:

(1) For smoothly bounded domains, the moment spectrum determines the heat content.
(2) For generic smoothly bounded domains, the moment spectrum determines the Dirichlet spectrum.

It is a result of Gilkey that heat content cannot distinguish isospectral planar domains constructed via the Sunada method [Gi1]. On the other hand, in [BDK], the authors establish that heat content distinguishes the isospectral domains constructed by Chapman and in [MM2] the authors construct combinatorial analogs of BCDS pairs and explicit check that their heat contents differ (at the fifth coefficient). The main result of this paper is that with respect to heat
content, quantum graph analogs of BCDS pairs behave more like combinatorial graphs than
planar domains, and that torsional rigidity is sufficient to distinguish well-known examples of
isospectral pairs. More precisely, we have

**Theorem 1.1.** Let $D_1$ and $D_2$ be the 7-pair of isospectral non-isometric planar polygonal
domains as constructed by Buser, Conway, Doyle and Semmler and let $G_1$ and $G_2$ be their
quantum graph analogs (see section 3 below). Then $G_1$ and $G_2$ are isospectral, non-isometric
and distinguished by their torsional rigidity.

In addition to establishing theorem 1.1, we prove

**Theorem 1.2.** The moment spectrum of a quantum graph with Dirichlet standard boundary
conditions (see section 2 below) determines the heat content of the quantum graph.

As an immediate corollary, we conclude that our isospectral non-isometric pairs are distin-
guished by their heat content. Our technique applies to the other families of isospectral pairs
constructed in [BCDS].

The remainder of this note is organized as follows. In the second section we establish
the required background information concerning quantum graphs, establish our notational
conventions and prove theorem 1.2. In the third section we review the Sunada construction
in the context of planar domains following Buser, Conway, Doyle and Semmler and include
the construction of isospectral non-isometric quantum graph analogs of BCDS pairs. In the
fourth section of the paper we explicitly compute the torsional rigidity for a pair of isospec-
tal non-isometric quantum graphs and check they differ. We end the paper with a previously
studied example ([MM2]) of a pair of isospectral non-isomorphic weighted combinatorial
graphs which arise as analogs of BCDS pairs and check that they too are distinguished by
their torsional rigidity.

We thank the referees for suggestions that greatly improved the paper, including corollary 2.3.

2. Quantum graphs

Let $G$ be a graph with finite vertex set $V$ and edge set $E$. For $v \in V$, denote by $d(v)$ the cardinal-
ity of the set $E_v = \{ e \in E : v \in e \}$. We will often find it convenient to assign an orientation to
each edge $e = \{ u, v \}$, such an orientation can be represented as either $u$ or $v$.

By a path in $G$ we will mean a sequence of vertices, $v_{i_0}, v_{i_1}, \ldots , v_{i_n}$ with $\{ v_{i_1}, v_{i_2} \} \in E$. We say $G$ is connected if every pair of vertices can be connected by a path in $G$. We restrict our
attention to graphs which are connected and contain no edges of the form $\{ v, v \}$.

We endow $G$ with a metric structure as follows. For each edge $e \in E$, choose a positive real
number $l_e$. Identify each edge $e \in E$ with the interval $[0, l_e]$ where $0$ is identified with
$u$ and $l_e$ is identified with the remaining vertex of $e$. This identification allows us to introduce a
natural coordinate on each edge; the coordinate $x_e$ along the interval $[0, l_e]$. These coordinates give
rise to a natural metric structure for $G$. The pair $(G, \{ l_e \}_{e \in E})$ is called a metric graph. Because
$G$ and $l_e$ are finite, the resulting metric graph is compact.

We identify functions on $(G, \{ l_e \}_{e \in E})$ with functions along the open edges together with
values at each vertex: For $\phi : G \to \mathbb{C}$, we write $\phi = \bigoplus_{e \in E} \phi_e$. The metric structure on each edge
gives rise to a natural Hilbert space associated to $(G, \{ l_e \}_{e \in E})$. Write

$$\mathcal{H}_e = L^2([0, l_e])$$

where the inner product associated to edge $e$ is defined by

$$\langle f, g \rangle_e = \int_0^{l_e} f(x_e)\overline{g(x_e)}dx_e.$$
Let \[ \mathcal{H} = \bigoplus_{e \in E} \mathcal{H}_e \]
and denote the inner product on \( \mathcal{H} \) by
\[ \langle f, g \rangle = \int_G f(x) \overline{g(x)} \, dx \]
\[ \equiv \sum_{e \in E} \langle f, g \rangle_e. \]
In particular,
\[ \langle 1, 1 \rangle = L(G) \]
where \( L(G) \) is the total length of the graph \( G \).

There is a natural differential operator acting on functions on the interior of each edge: \( \frac{d^2}{dx_e^2} \).

We wish to extend this operator to a self-adjoint operator on \( L^2 \)-functions on the metric graph.

We sketch the approach developed in [KPS1] (see [KPS1] for details).

For each \( e \in E \), let \( D_e \) denote \( \{ \phi_e \in \mathcal{H}_e : \phi_e, \phi'_e \text{ are absolutely continuous}, \phi'_e \in \mathcal{H}_e \} \).

Let \( D_e^0 = \{ \phi_e \in D_e : \phi_e(0) = \phi_e(l_e) = \phi'_e(0) = \phi'_e(l_e) = 0 \} \)
and set
\[ D = \bigoplus_{e \in E} D_e \]
\[ D^0 = \bigoplus_{e \in E} D_e^0. \]

Then the operator \( \Delta^0 \) which acts on \( \phi \in D^0 \) according to
\[ (\Delta^0 \phi)_e(x) = \frac{d^2 \phi_e}{dx_e^2}(x_e) \]
is closed, symmetric and densely defined. We seek to impose boundary conditions which give self-adjoint extensions of \( \Delta^0 \) to \( \mathcal{H} \).

We begin by noting that boundary values of functions along edges lie in the space \( B = \mathbb{C}^{|E|} \times \mathbb{C}^{|E|} \) where the order of the components is fixed by the orientation. More precisely, given \( \phi \in D \), let \( \phi \in B \) be given by
\[ \phi = (\{ \phi_e(0) \}_{e \in E}, \{ \phi_e(l_e) \}_{e \in E}). \]
Boundary conditions can then be described by a pair of linear operators \( A, B : B \to B \) satisfying
\[ A \phi + B \phi' = 0. \]

Those pairs of linear operators \( (A,B) \) leading to self-adjoint extensions of \( \Delta^0 \) have been classified by Kostrykin et al [KPS1] using a symplectic formalism going back to at least to Novikov.

We are interested in a single special case, Dirichlet standard boundary conditions, which we now describe.
Definition 2.1. Denote by DSBC the collection of continuous functions defined by

\[
\text{DSBC} = \left\{ \phi \in \mathcal{D} : \phi(v) = 0 \quad \text{if} \quad d(v) = 1, \quad \sum_{e \in E_v, d(v) = 1} \phi(v) = 0 \quad \text{if} \quad d(v) \neq 1 \right\}
\]  
(2.1)

where the derivative occurring in (2.1) is always directed into the vertex \( v \) and, as before, \( E_v = \{ e \in E : v \in e \} \) is the collection of edges on which \( v \) is incident.

A few remarks concerning our choice of boundary conditions are in order. The literature often refers to ‘standard boundary conditions’ as those involving Kirchoff conditions at internal vertices (ie vertices of degree at least 2) and Neumann conditions at boundary vertices (ie vertices of degree 1). Dirichlet standard boundary conditions are local in the sense of [KPS3] and do not mix derivatives with function values. Finally, Dirichlet standard boundary conditions force continuity across all internal vertices for all elements in the corresponding domain of the self-adjoint extension of \( \Delta^0 \).

We will denote the self-adjoint extension of \( \Delta^0 \) with Dirichlet standard boundary conditions as \( \Delta \). The pair consisting of the metric graph \( G \) and the operator \( \Delta \) is the quantum graph we study.

The spectrum of \( \Delta \) is discrete and of finite multiplicity. We will denote the spectrum of \( \Delta \) by spec(\( \Delta \)). We will assume there is at least one boundary vertex and we will list elements of spec(\( \Delta \)) in increasing order with multiplicity:

\[ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \]

There is a great deal known about the spectrum for arbitrary self-adjoint boundary conditions. For our purposes it is important that there is a heat kernel which can be written as

\[
p_G(t, x, y) = \sum_{\lambda \in \text{spec}(\Delta)} e^{-\lambda t} \phi_\lambda(x)\phi_\lambda(y)
\]  
(2.2)

where the \( \phi_\lambda \) form an \( L^2 \)-orthonormal family of eigenfunctions. The heat content of \( G \) can then be defined via integration as for domains:

\[
q_G(t) = \int_G \int_G p_G(t, x, y) \, dx \, dy = \sum_{\lambda \in \text{spec}(\Delta)} e^{-\lambda t} \left( \int_G \phi_\lambda(x) \, dx \right)^2.
\]

Note that the integrals occurring in (2.3) involve information which is, in general, not spectral. We let spec'(\( \Delta \)) denote the set of values defined by the Dirichlet standard boundary conditions (ie we disregard multiplicity) for which the corresponding eigenspace is not orthogonal to constant functions. Then we can write the heat content as

\[
q_G(t) = \sum_{\lambda \in \text{spec}'(\Delta)} a^2 e^{-\lambda t}
\]  
(2.4)

where \( a^2 \) is the square of the \( L^2 \)-norm of the projection of the constant function 1 on the eigenspace defined by \( \lambda \).

The moments of \( q_G(t) \) define a sequence associated to the quantum graph \( G \). More precisely, as in the introduction, for \( k \) a natural number, set

\[
A_k = k \int_0^t \tau^{k-1} q_G(\tau) \, d\tau.
\]
The moment spectrum of the graph $G$ is the sequence $\{A_k\}_{k=1}^{\infty}$.

From the definition it is clear that the heat content determines the moment spectrum. The converse (theorem 1.2 above) is also true:

**Theorem 2.2.** The moment spectrum of the quantum graph $G$ determines the heat content of the quantum graph $G$.

**Proof.** The proof follows the case of domains, where the same result holds (see [MM1, Mc2]). For completeness we provide the argument given in [MM1].

It suffices to show that the moment spectrum determines both $\text{spec}^*(\Delta)$ and the collection of coefficients $\{\lambda^a\}$. The important observation is that the moment spectrum describes special values of the Mellin transform of the heat content. More precisely, using (2.4), the Mellin transform of the heat content is given by the Dirichlet series

$$\zeta_G(s) = \sum_{\lambda \in \text{spec}^*(\Delta)} a_\lambda^2 \left(\frac{1}{\lambda}\right)^s$$

which converges for real part of $s$ nonnegative and admits a meromorphic extension to the plane, with poles at the negative half-integers (see [MM1]). The moment spectrum then satisfies

$$\zeta_G(n) = \frac{A_n}{\Gamma(n+1)}$$

Using (2.5) and (2.6), there is a recursion: Write $\text{spec}^*(\Delta) = \{\mu_n\}_{n=1}^{\infty}$ with $\mu_n$ strictly increasing. Then

$$\mu_1 = \sup \left\{ \mu \geq 0 : \lim_{n \to \infty} \sup (\mu)^n \frac{A_n}{\Gamma(n+1)} < \infty \right\}$$

and

$$a_{\mu_1}^2 = \lim_{n \to \infty} \sup (\mu)^n \frac{A_n}{\Gamma(n+1)}.$$

Having determined $\mu_j$ and $a_{\mu_j}^2$ for $j < k$, we have

$$\mu_k = \sup \left\{ \mu \geq 0 : \lim_{n \to \infty} \sup (\mu)^n \left( \frac{A_n}{\Gamma(n+1)} - \sum_{j=1}^{k-1} a_{\mu_j}^2 \left(\frac{1}{\mu_j}\right)^n \right) < \infty \right\}$$

and

$$a_{\mu_k}^2 = \lim_{n \to \infty} \sup (\mu)^n \left( \frac{A_n}{\Gamma(n+1)} - \sum_{j=1}^{k-1} a_{\mu_j}^2 \left(\frac{1}{\mu_j}\right)^n \right).$$

This shows that both $\text{spec}^*(\Delta)$ and the coefficients $a_{\mu_k}^2$ are determined by the $L^1$-moment spectrum. Using the spectral representation of the heat content (2.3), it is clear that $\text{spec}^*(\Delta)$ and the coefficients $a_{\mu_k}^2$ determine the heat content. This proves that the $L^1$-moment spectrum determines heat content and completes the proof of theorem 1.2. □
As an immediate corollary, we have

**Corollary 2.3.** For generic quantum graphs with Dirichlet standard boundary conditions, the moment spectrum determines the Dirichlet standard spectrum.

**Proof.** It is a result of Friedlander [Fr1] (see also [Be1]) that for metric graphs with standard boundary conditions, the spectrum of the associated Laplace operator has multiplicity one. Because the $L^1$-moment spectrum determines $\text{spec}^c(\Delta)$, the result follows.

For the remainder of the paper our primary interest is in $A_1$, which, as in the case of smooth domains, can be described via a solution of a Poisson problem on the graph $G$. More precisely, let $u_1$ solve

$$\begin{align*}
\Delta u_1 &= -1 \text{ on interior}(G) \\
u_1 &= 0 \text{ on } \partial G.
\end{align*}$$

(2.7)

We can apply Fubini’s theorem to interchange the order of integration in the definition of the heat content and $u_1(x)$ to obtain

$$A_1 = \int_G u_1(x) \, dx.$$  

(2.8)

Given any quantum graph with Dirichlet standard boundary conditions, the computation of the right-hand-side of (2.8) is an exercise in linear algebra. We compute torsional rigidity for certain pairs of quantum graphs constructed in the next section.

### 3. The Sunada construction, BCDS pairs and isospectral quantum graphs

As mentioned in the introduction, Buser, Conway, Doyle and Semmler have constructed families of pairs of isospectral planar domains that are not isometric. Their construction and investigation of the resulting pairs is spectacularly simple. To construct families of pairs of isospectral planar domains,

- Fix a seed triangle with edges labeled;
- Reflect the seed about its edges to produce a second generation of labeled triangles;
- For each of the progeny, reflect and label across edges determined by two rules derived from subgroups which are ‘nearly conjugate’ in Sunada’s sense
- Iterate
- Interpret the edges as distinct lengths of an acute triangle.

An example of the construction is given in figure 1: the so-called $7_1$ pairs (for complete details see [BCDS]).

Having constructed families of pairs as above, Buser et al prove that the associated pairs are isospectral using the technique of eigenfunction transplantation. Given an eigenfunction $f$ on one domain, say $D_1$, they use the process by which the domain is constructed to naturally partition the domain into a collection of isometric subdomains, say $T_{1,i}$ where $i$ indexes the number of triangles occurring in the construction, say $1 \leq i \leq n$. The same procedure carried out on the domain pair, say $D_2$, results in a collection $T_{2,i}$, $1 \leq i \leq n$, of isometric subdomains of $D_2$. Having partitioned both domains into collections of subdomains, they restrict the given eigenfunction to each subdomain to produce an associated collection of functions, $f_i$, $1 \leq i \leq n$, on isometric copies of the same subdomain. They then use the geometry of the underlying domains $D_1$ and $D_2$ (ie, how the domains are constructed from the collections of subdomains) to construct a linear map which prescribes how to build an associated...
eigenfunction on the domain $D_2$ by combining the function elements $f_i$ on each subdomain of $D_2$. The process by which an explicit transplantation map is constructed is described neatly in the references [BCDS] and [Ch]. For our purposes it is important to note that the process is essentially combinatorial in the sense that it depends only on the following:

- The eigenfunction $f$ vanishes at the boundary of $D_1$.
- To smoothly continue the eigenfunction $f$ across the boundary of the domain $D_1$, the reflection principle requires that the value of $f(x^*)$ be prescribed to be $-f(x)$ where $x$ is the reflection image of $x^*$.
- If $T_{1,1}$ and $T_{1,2}$ are subdomains which share a boundary internal to $D_1$, the associated eigenfunction elements $f_{11}$ and $f_{12}$ and their respective normal derivatives must match along the common boundary.

There are also techniques to produce quantum graph analogs [BPB, SS, BSS]. To proceed, we recall the required definitions.

Following earlier work (see [SS, BSS, BPB] and references therein) Band et al have described an extension of eigenfunction transplantation in the context of quantum graphs [BPB]. They provide a method which associates to every family of pairs constructed in [BCDS], a family of pairs of quantum graphs which are isospectral but not isometric. The construction technique is straightforward, as is the check that the objects satisfy the required conditions. We sketch the process below and produce the pairs associated to the 71 pairs in figure 2.

For each pair of isospectral nonisometric domains found in [BCDS],

- For each triangle appearing in the BCDS construction, introduce a 3-star graph consisting of three edges, with edges labeled with a label from the edges of the corresponding triangle, joined at a central vertex.
- Join edges in 3-stars at a common degree two vertex if corresponding edges in the triangles to which they are associated overlap.

We carry out the process on quantum graphs corresponding to 71 pairs in figure 3. Note that for the graphs to be isometric, it would be necessary to map the 3-stars corresponding to generating triangles onto each other.

For Neumann standard boundary conditions, the quantum graph analogs are shown to be isospectral and non-isometric in [BSS] (see also [BPB]). This is accomplished by showing...
that the graphs share the same secular equation. The same argument can be made to work in the current context, but instead we provide an explicit transplantation map. In fact, it’s easy to check that the transplantation map used to establish that the BCDS pairs are isometric works to show that the associated quantum graphs pairs are isospectral. We do this as follows:

- Label the edges of the quantum graph $G_1$ and the corresponding partner $G_2$.
- Given an eigenfunction on $G_1$, restrict to 3-star subgraphs to obtain a collection of functions defined on isometric 3-stars.
- Use the linear map constructed following [BCDS] to construct the required transplantation map taking eigenfunctions of $G_1$ to eigenfunctions of $G_2$.

We can formalize these observations with

**Proposition 3.1.** The quantum graph (with Dirichlet standard boundary conditions) analogs of the isospectral non-isometric 71 pairs constructed in [BCDS] are isospectral and non-isometric.

**Proof.** By construction, the domains constructed in [BCDS] correspond to acute seed triangles with $a$, $b$ and $c$ distinct edge lengths. The corresponding quantum graph is obtained by replacing subdomains with 3-stars, gluing 3-stars along boundaries which correspond to boundaries which are glued in subdomains. The transplantation map for domains may be

![Figure 2. Quantum graph analogs of pairs of isospectral non-isometric planar domains.](image)
represented by a matrix acting on function elements defined on subdomains, all of which can be identified. This defines a linear map from eigenfunctions on $G_1$ to functions on $G_2$, which we will write as $L$. Given an eigenfunction $\phi$ on $G_1$ corresponding to eigenvalue $\lambda$, to see that the image $\phi L$ is an eigenfunction on $G_2$ first note that on the interior of every edge, $\Delta \phi = \lambda \phi$ by linearity. Thus, it suffices to check that $\phi L$ has value zero at the boundary of $G_2$ and that $L\phi$ behaves as it should across glued edges. But for this to be the case requires only the three properties listed in our description of the transplantation process given above. Since the gluing relations for quantum graphs are precisely those that are prescribed by the domains and the boundary conditions are Dirichlet in both cases, these three properties must hold for the image of $\phi$ under $L$. In particular, $L\phi$ must be an eigenfunction corresponding to eigenvalue $\lambda$.

We give the required transplantation map in figure 3.

4. Computing torsional rigidity

We are now in a position to prove theorem 1.1 for the $7_1$ pairs of figure 2. We begin by noting that we can eliminate vertices of degree 2 and replace the corresponding two edges by a single edge of length equal to the sum of the corresponding edges (following Friedlander [Fr1], we call such quantum graphs clean). This results in the pair of quantum graphs given in figure 4. Note that there is a symmetry: interchanging $b$ and $c$ in the two graphs maps one graph onto the other. We use this symmetry in the computation of the torsional rigidity of each graph.

Figure 3. Eigenfunction transplantation for quantum graphs.
Suppose we denote by $x_{ij}$ for the coordinate on the $j$th edge of the $i$th graph. Then we must solve

$$\frac{d^2 u_{ij}}{dx_{ij}^2} = -1$$

with boundary conditions determined by our choice of Dirichlet standard boundary conditions. The solution on each edge is of the form

$$u_{ij}(x_{ij}) = -\frac{1}{2}v_{ij}^2 + \alpha_{ij}x_{ij} + \beta_{ij}$$

(4.1)

where the constants $\alpha_{ij}$ and $\beta_{ij}$ are to be determined from the Dirichlet standard boundary conditions. Thus, for each graph there are 30 constants to be determined using nine Dirichlet boundary conditions at vertices of degree 1, seven Kirchoff boundary conditions at vertices of degree 3, and fourteen continuity conditions at vertices of degree 3. This determines a linear system $La = v$ where the coefficient matrix $L$ is a sparse $21 \times 21$ matrix and the constant vector $v$ is defined in terms of continuity constraints and conservation at internal nodes. The structure of the matrix will be determined by our choice of edge labels and orientation. We
choose the first nine edges to be those incident to boundary vertices and we orient the edges to be inward facing relative to the boundary. With this choice the coefficients $\beta_{ij} = 0$ for $i = 1, 2$ and $1 \leq j \leq 9$. With the remaining edges oriented to point toward the ‘central node’, the representation of $L$ for $G_1$ as pictured in figure 4, is given by the coefficient matrix

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
$$

The continuity and conservation constraints define a vector, $v_1$, given by

$$
v_1 = (c^2/2, a^2/2, b^2/2, a^2/2, a^2/2, c^2/2, b^2/2, c^2/2, 2b^2, 2c^2 - 2a^2, 2a^2 - 2b^2, 2c^2, 2a^2,
\begin{array}{c}
a + c, a + b, b + c, a + 2b, b + 2c, 2a + c, 2a + 2b + 2c
\end{array}^T
$$

where ‘$T$’ denotes transpose. To obtain the data for the second graph, simply interchange $b$ and $c$.

We can use these systems to carry out the required calculation

$$
\int_{G_1} u_1(x) dx - \int_{G_2} u_2(x) dx = \frac{1}{2} \sum_{j=1}^{15} \left( \int_{0}^{l_{1j}} u_{1j}(x_{1j}) dx_{1j} - \int_{0}^{l_{2j}} u_{2j}(x_{2j}) dx_{2j} \right)
$$

$$
= \frac{1}{2} \sum_{j=1}^{15} \left( (\alpha_{1j} l_{1j}^2 - \alpha_{2j} l_{2j}^2) + \sum_{j=1}^{15} (\beta_{1j} l_{1j} - \beta_{2j} l_{2j}) \right)
$$

$$
= \langle L_1^T v_1, l_1 \rangle - \langle L_2^T v_2, l_2 \rangle \tag{4.2}
$$

where the vectors $l_i$ are constructed from the lengths of the edges of the graphs $G_i$:

$$
l_1 = 1/2(c^2, a^3, b^3, a^2, b^2, c^2, a^2, b^2, c^2, 4b^2, 4c^2, 4b^2, 4c^2, 4a^2, 4b, 4c, 4b, 4b, 4a, 4c, 4a)^T
$$

$$
l_2 = 1/2(a^2, b^2, c^2, a^2, b^2, c^2, a^2, b^2, c^2, 4b^2, 4c^2, 4b^2, 4c^2, 4a^2, 4b, 4c, 4b, 4b, 4a, 4c, 4a)^T
$$
and $l_2$ is obtained from $l_1$ by interchanging $b$ and $c$. The right hand side of (4.2) is a rational function in the variables $a$, $b$, and $c$:

$$
(L_1^{-1}v_1, l_1) - (L_2^{-1}v_2, l_2) = (a - b)(a - c)(b - c)R(a, b, c)
$$

(4.3)

where the rational function $R(a, b, c) = \frac{N(a, b, c)}{D(a, b, c)}$ has a sign:

$$
N(a, b, c) = -4(bc + a(b + c))(11a^3(b + c) + bc(11b^2 + 23bc + 11c^2) + a(b + c)(11b^2 + 48bc + 11c^2) + a^2(23b^2 + 59bc + 23c^2))
$$

$$
D(a, b, c) = 32a^4(b + c)^2 + 8b^2c^2(4b^2 + 9bc + 4c^2) + 8abc(b + c)(8b^2 + 23bc + 8c^2) + 8a^3(b + c)(9b^2 + 22bc + 9c^2) + a^2(32b^4 + 248b^3c + 439b^2c^2 + 248bc^3 + 32c^4).
$$

From (4.3) and the explicit expression of $R(a, b, c)$ it is easy to see that the difference in torsional rigidities is nonzero whenever $a$, $b$ and $c$ are distinct. This proves theorem 1.1.

As a corollary of theorems 1.1 and 1.2, we obtain:

**Corollary 4.1.** The quantum graphs $G_1$ and $G_2$ are distinguished by their heat content.

5. Combinatorial graphs

In [MM2] the authors construct combinatorial weighted graph analogs of BCDS pairs. The result of these constructions for the $7_1$-pairs given above are the two isospectral, non-isomorphic weighted combinatorial graphs given in figure 5:

There is a natural weighted Laplacian associated to such weighted graphs. For functions on the vertices we write

$$
Df(x) = \sum_{y \neq x} W_{E}(x, y)f(y) - f(x)
$$

where the sum is over vertices $y$ adjacent to $x$ and $W_{E}(x, y)$ denotes the weight along the edge defined by the vertices $x$ and $y$. For the above two graphs this results in Laplace operators given by
It is easy to check that these matrices have the same characteristic equation.

To compute the difference in torsional rigidity for the corresponding combinatorial graphs we let $v = (1, 1, 1, 1, 1, 1)^T$ and we compute:

$$
(D_1^{-1}v, v) - (D_2^{-1}v, v) = (a - b)(a - c)(b - c)r(a, b, c)
$$

where $r(a, b, c) = \frac{n(a, b, c)}{d(a, b, c)}$ is the rational function defined by

$$
n(a, b, c) = 56(bc + a(b + c))
$$

$$
d(a, b, c) = (32a^4(b + c)^2 + 8b^2c^2(4b^2 + 9bc + 4c^2) + 8abc(b + c)(8b^2 + 23bc + 8c^2) + 8a^2(b + c)(9b^2 + 22bc + 9c^2) + a^4(32b^4 + 248b^2c + 439bc^2 + 248bc^3 + 32c^4))
$$

In particular, for distinct values of $a$, $b$, and $c$ the isospectral non-isomorphic combinatorial graphs constructed in [MM2] are distinguished by their torsional rigidity.

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