Einstein Homogeneous Riemannian Fibrations

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To my godparents Amadeu and Conceição
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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(Fátima Araújo)
Abstract

This thesis is dedicated to the study of the existence of homogeneous Einstein metrics on the total space of homogeneous fibrations such that the fibers are totally geodesic manifolds. We obtain the Ricci curvature of an invariant metric with totally geodesic fibers and some necessary conditions for the existence of Einstein metrics with totally geodesic fibers in terms of Casimir operators. Some particular cases are studied, for instance, for normal base or fiber, symmetric fiber, Einstein base or fiber, for which the Einstein equations are manageable. We investigate the existence of such Einstein metrics for invariant bisymmetric fibrations of maximal rank, i.e., when both the base and the fiber are symmetric spaces and the base is an isotropy irreducible space of maximal rank. We find this way new Einstein metrics. For such spaces we describe explicitly the isotropy representation in terms subsets of roots and compute the eigenvalues of the Casimir operators of the fiber along the horizontal direction. Results for compact simply connected 4-symmetric spaces of maximal rank follow from this. Also, new invariant Einstein metrics are found on Kowalski $n$-symmetric spaces.
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Introduction

The principal topic of study in this thesis is the existence of Einstein invariant metrics on the total space of homogeneous Riemannian fibrations such that the fibers are totally geodesic submanifolds. In chapter 1 we introduce some main definitions and notation and deduce some essential formulas for the Ricci curvature of an invariant metric. Then we consider a fibration $G/L \to G/K$ such that $G$ is a compact connected semisimple Lie group, and $L \subset K \subset G$ are connected closed non-trivial subgroups of $G$. We assume that $G/L$ has simple spectrum. On the total space $G/L$, we consider a $G$-invariant Riemannian metric such that the natural projection $G/L \to G/K$ is a Riemannian submersion and the fibers are totally geodesic submanifolds. We shall briefly call such a metric an Einstein adapted metric. We describe the Ricci curvature of any adapted metric in terms of Casimir operators and obtain two necessary conditions for existence of an Einstein adapted metric expressed only in terms of the Casimir operators of the horizontal and vertical directions. These will provide a tool to show that in many cases such a metric cannot exist without further study, i.e., only by studying eigenvalues of certain Casimir operators.

In chapter 2 we restrict the object of study to some special cases, where the Einstein equations are simpler. We consider the cases where the metric on the fiber or on the base is a multiple of the Killing form of $G$, in which cases are included those with isotropy irreducible fiber or base. The case when both these two conditions are satisfied gives rise to the study of the, throughout called, Einstein binormal metrics. The existence of Einstein binormal metrics translates into very simple algebraic conditions which shall allow us to find out new Einstein metrics. Also we obtain necessary conditions for existence of an Einstein adapted metric such that the metric on the base space or the metric on the fiber are also Einstein. Finally, we apply the results obtained so far to the case when the fiber is a symmetric space and $N$ is isotropy irreducible.

Chapter 3 is devoted to bisymmetric fibrations of maximal rank, i.e., we consider a fibration $G/L \to G/K$, as in chapter 1, such that $L$ is a subgroup of maximal rank, $K/L$ is a symmetric space and $G/L$ is an isotropy irreducible symmetric space. We introduce the notion of a bisymmetric triple $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ associated to a bisymmetric fibration. We obtain all the bisymmetric triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ in the case
when $g$ is a simple Lie algebra and classify them into two different types, I and II. We classify all the Einstein adapted metrics when $g$ is an exceptional Lie algebra, for both Type I and II. When $g$ is a classical Lie algebra, we classify all the Einstein adapted metrics for Type I. For Type II in the classical case, we classify all Einstein binormal metrics and all Einstein adapted metrics whose restriction to the fiber is also Einstein; moreover, if one of these metrics exists we obtain all the other Einstein adapted metrics. Finally, we apply the results obtained to compact simply-connected irreducible 4-symmetric spaces. In appendix A we obtain all the necessary eigenvalues for the Einstein equations for each bisymmetric triple considered in this chapter.

In chapter 4 we study the existence of Einstein adapted metrics on the Kowalski $n$-symmetric spaces, i.e., we consider a fibration

$$
\frac{\Delta^pG_0 \times \Delta^qG_0}{\Delta^nG_0} \rightarrow \frac{G^n_0}{\Delta^nG_0} \rightarrow \frac{G^p_0}{\Delta^pG_0} \times \frac{G^q_0}{\Delta^qG_0},
$$

where $G_0$ is compact connected simple Lie group and $\Delta^mG_0$ is the diagonal subgroup in $G^m_0$, for $m = p, q, n$. It is known that $\frac{G^n_0}{\Delta^nG_0}$ is a standard Einstein manifold and we prove that, for $n > 4$, there exists another Einstein adapted metric, whereas, for $n = 4$, the standard metric is the only Einstein adapted metric.
CHAPTER 1

In Section 1 we introduce some essential definitions and notation. We deduce a formula for the Ricci curvature of an invariant metric on a reductive homogeneous space. In Section 2 we obtain the Ricci curvature of an invariant metric with totally geodesic fibers on the total space of a homogeneous fibration and some necessary conditions for that metric to be Einstein.

1.1 The Ricci Curvature of a Riemannian Homogeneous Space

A Riemannian metric $g$ is said to be Einstein if its Ricci curvature satisfies an equation of the form $\text{Ric} = Eg$, for some constant $E$, the Einstein constant of $g$ ([10]). This equation, commonly called the Einstein equation, is in general a very complicated system of partial differential equations of second order. Although so far no fully general results are known for existence of Einstein metrics, many results of existence and classification are known for many classes of spaces. Two examples of this are the Kähler-Einstein metrics ([49], [5], [10], [43]) and the Sasakian-Einstein metrics ([17]). Many results are known on homogeneous Einstein metrics. For Riemannian homogeneous spaces the Einstein equation translates into a system of algebraic equations, which is an easier problem than its general version. However, even for this class of spaces we are far from knowing full answers. Einstein normal homogeneous manifolds were classified by Wang and Ziller ([44]). Nowadays, it is known that every compact simply connected homogeneous manifold with dimension less or equal to 11 admits a homogeneous Einstein metric: any such manifold with dimension 2 or 3 has constant sectional curvature [10]; in dimension 4, the result was shown by Jensen ([18]), and by Alekseevsky, Dotti and Ferraris in dimension 5 ([4]); in dimension 6, the result is due to Nikonorov and Rodionov ([31]), and in dimension 7 it is due to Castellani, Romans and Warner ([14]). All the 7-dimensional homogeneous Einstein manifolds ([29]) were obtained by Nikonorov. These results were extended to dimension up to 11 by Böhm and Kerr ([12]). Many examples of homogeneous Einstein manifolds with dimension arbitrary big are known. Spheres and projective spaces
are examples of this, where all homogeneous Einstein metrics were classified by Ziller ([51]). Also isotropy irreducible spaces ([47]), symmetric spaces ([16],[22]), flag manifolds, among many others, provide examples of Einstein manifolds with arbitrary big dimension. Einstein homogeneous fibrations have also been the object of study. We recall the work of Jensen on principal fibers bundles ([19]), where new invariant Einstein metrics are found on the total space of certain homogeneous fibrations, and the work of Wang and Ziller on principal torus bundles ([46]). Einstein homogeneous fibrations are the main focus of this thesis.

Let $G$ be a Lie group and $L$ a closed subgroup. We denote by $\mathfrak{g}$ and $\mathfrak{l}$ the Lie algebras of $G$ and $L$, respectively. The homogeneous space $M = G/L$ is the space of all cosets $\{aL : a \in G\}$ endowed with the unique differentiable structure such that the canonical projection

$$
\pi : G \to M \\
a \mapsto aL
$$

is a submersion, i.e., $d\pi_a$ is onto for all $a \in G$, and with the natural transitive left action of $G$,

$$
\tau : G \times M \to M \\
(b,aL) \mapsto (ba)L
$$

Let $X \in \mathfrak{g}$ and let $exp tX$ be the one-parameter subgroup generated by $X$. For every $a \in G$,

$$
d\pi_a(X) = \frac{d}{dt} (exp tX)aL \mid_{t=0}
$$

and, in particular, for $o = \pi(e) = L$, this map yields an isomorphism

$$
d\pi_e : \mathfrak{g}/\mathfrak{l} \cong T_o M.
$$

For every $X \in \mathfrak{g}$ we define a $G$-invariant vector field on $M$ by

$$
X_{aL}^* = d\pi_a(X) = \frac{d}{dt} (exp tX)gL \mid_{t=0}.
$$

The homogeneous space $M$ is called reductive if there exists a direct complement $\mathfrak{m}$ of $\mathfrak{l}$ in $\mathfrak{g}$ which is $Ad L$-invariant, i.e.,

$$
\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \text{ and } Ad L(\mathfrak{m}) \subset \mathfrak{m}.
$$

The inclusion $Ad L(\mathfrak{m}) \subset \mathfrak{m}$ implies

$$
[\mathfrak{l}, \mathfrak{m}] \subset \mathfrak{m}
$$
and the converse holds if \( L \) is connected. The homogeneous space \( M \) is reductive if \( L \) is compact. Throughout, we suppose that \( L \) is compact and we denote by \( \mathfrak{m} \) a reductive complement of \( \mathfrak{l} \) on \( \mathfrak{g} \). If \( M \) is reductive we have an isomorphism

\[ \mathfrak{m} \cong T_0M \]

and the tangent space \( T_0M \) is identified with \( \mathfrak{m} \) and consequently the vector field \( X^* \) on \( M \) is identified with \( X \in \mathfrak{m} \). Under this identification we shall simply write \( X \) for \( X^*_o \). Furthermore, the isotropy representation of \( M \),

\[ Ad^M : L \to GL(T_0M), \]

is equivalent to the adjoint representation of \( L \) on \( \mathfrak{m} \). Consequently, there is a one-to-one correspondence between \( G \)-invariant objects on \( M \) and \( Ad L \)-invariant objects on \( \mathfrak{m} \). In particular, \( G \)-invariant metrics on \( M \) correspond to \( Ad L \)-invariant scalar products on \( \mathfrak{m} \). More precisely, a metric \( g \) on \( M \) is said to be \( G \)-invariant if, for every \( a \in G \),

\[ \tau^*_a g = g \]

and the correspondence between \( G \)-invariant metrics on \( M \) and \( Ad L \)-scalar products on \( \mathfrak{m} \) is given

\[ g_a(X_a^*, Y_a^*) = \langle X, Y \rangle, \text{ for all } a \in G. \] (1.8)

Let \( \text{Kill} \) be the Killing form of \( \mathfrak{g} \). We recall that \( \text{Kill} \) is the bilinear form on \( \mathfrak{g} \) defined by

\[ \text{Kill}(X, Y) = \text{tr}(ad_X ad_Y), \ X, Y \in \mathfrak{g} \] (1.9)

where, for each \( X \in \mathfrak{g} \), \( ad_X \) denotes the adjoint map

\[ \mathfrak{g} \to \mathfrak{g}, \ Y \mapsto [X, Y]. \] (1.10)

The Killing form of \( \mathfrak{g} \) is an \( Ad G \)-invariant bilinear form and it is non-degenerate if \( \mathfrak{g} \) is semisimple. Moreover, if \( G \) is a compact connected semisimple Lie group, \( \text{Kill} \) is negative definite. In this case, by (1.8), the negative of the Killing form induces a \( G \)-invariant metric on \( M \), the \textit{standard Riemannian metric} on \( M \).

With respect to the decomposition \( \mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m} \), we write

\[ ad_X = \begin{pmatrix} 0 & C_X \\ B_X & P_X \end{pmatrix}, \text{ for every } X \in \mathfrak{m}. \] (1.11)
Hence, for $Y \in \mathfrak{m}$,

$$(ad_X Y)_m = P_X Y \text{ and } (ad_X^2 Y)_m = (B_X C_X + P_X^2) Y,$$  

(1.12)

where the subscript $\mathfrak{m}$ denotes the projection onto $\mathfrak{m}$.

Let $g_M$ be a $G$-invariant metric on $M$. As was explained above, there is a one-to-one correspondence between $G$-invariant metrics on $M$ and $Ad L$-invariant scalar products on $\mathfrak{m}$. So let $<,>$ be the $Ad L$-invariant scalar product on $\mathfrak{m}$ corresponding to $g_M$.

For every $X \in \mathfrak{m}$, let $T_X$ be the endomorphism of $\mathfrak{m}$ defined by

$$< T_X Y, Z >= < X, P_Y Z >, \text{ for every } Y, Z \in \mathfrak{m}.$$  

(1.13)

The Nomizu operator of the scalar product $<,>$ (cf. [28], [25]) is

$$L_X \in \text{End}(\mathfrak{m}), X \in \mathfrak{g},$$

defined by

$$L_X Y = -\nabla_Y X^*, Y \in \mathfrak{m}$$  

(1.14)

where $\nabla$ is the Riemannian connection of $g_M$. We have

$$L_X Y = \frac{1}{2} P_X Y + U(X, Y),$$  

(1.15)

where $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is the operator

$$U(X, Y) = -\frac{1}{2} (T_X Y + T_Y X), \text{ } X, Y \in \mathfrak{m}.$$  

(1.16)

The metric $g_M$ is called naturally reductive if $U = 0$. The curvature tensor, the sectional curvature and the Ricci curvature of $g_M$ are $G$-invariant tensors and thus they are determined by the following identities ([28]), which represent their values at the point $o$. By using $L$ the curvature tensor of $g_M$ at $o$ can be written as

$$R(X, Y) = [L_X, L_Y]_m - L_{[X,Y]}_m - ad_{[X,Y]},$$  

(1.17)

for every $X, Y \in \mathfrak{m}$. The sectional curvature $K$ of $g_M$ is defined by

$$K(Z, X) = < R(Z, X) X, Z >,$$  

(1.18)

for every $X, Z \in \mathfrak{m}$ orthonormal with respect to $<,>$. The Ricci curvature of $g_M$ is determined by

---
\[ \text{Ric}(X, X) = \sum_i K(Z_i, X), \ X \in \mathfrak{m} \]  

(1.19)

where \((Z_i)_i\) is an orthonormal basis of \(\mathfrak{m}\) with respect to \(<,>\). The metric \(g_M\) is said to be an \textit{Einstein metric} if

\[ \text{Ric} = Eg_M, \]  

(1.20)

for some constant \(E\) called the \textit{Einstein constant} of \(g_M\). Equipped with a \(G\)-invariant Einstein metric, \(M\) is called an Einstein homogeneous manifold.

Below we show some elementary properties of the Nomizu operator:

**Lemma 1.1.** (i) \(L_X\) is skew-symmetric with respect to \(<,>\), i.e.,

\[ < L_X Y, Z > + < Y, L_X Z > = 0, \ X, Y, \in \mathfrak{m}; \]  

(1.21)

(ii) for every \(X, Y \in \mathfrak{m},\)

\[ L_X Y - L_Y X = [X, Y]_\mathfrak{m} = P_{XY}. \]  

(1.22)

**Proof:** From identities (1.13), (1.15) and (1.16) we deduce

\[ 2 < L_X Y, Z > = < P_X Y, Z > - < T_X Y, Z > - < T_Y X, Z > = < T_Z X, Y > + < T_X Z, Y > - < P_Z Y > = -2 < Y, L_X Z > \]

and this shows the skew-symmetry of \(L_X\). To show (ii) we just observe that \(U(X, Y) = U(Y, X)\) and \(P_X Y = -P_Y X\). Equivalently, this assertion just means that the Levi-Civita connection is torsion free since \(\nabla_X Y^* = L_X Y\).

\(\square\)

For every \(X, Y \in \mathfrak{m},\) we define the operator

\[ R_X Y = L_Y X \]  

(1.23)

and the vector

\[ V_X = L_X X. \]  

(1.24)

**Lemma 1.2.** For every \(X \in \mathfrak{m},\)

\[ R_X = -\frac{1}{2}(P_X + P_X^* + T_X) \text{ and } R_X^* = -\frac{1}{2}(P_X + P_X^* - T_X). \]
Proof: Let $X, Y, Z \in \mathfrak{m}$.

$$2 < R_X Y, Z > = 2 < L_Y X, Z >$$

$$= < [Y, X]_m, Z > + < Y, [Z, X]_m, > + < X, [Y, Z]_m >$$

$$= - < P_X Y, Z > - < P^*_X Y, Z > - < T_X Y, Z > .$$

Thus we obtain the required expression for $R_X$. The formula for $R^*_X$ follows from the fact that $P_X + P^*_X$ is symmetric and $T_X$ is skew symmetric with respect to $<, >$.

□

Lemma 1.3. The sectional curvature of $g_M$ is

$$K(Z, X) = < (R^*_X R_X - P_X^2 - P_X^* P_X - B_X C_X + P_{V_X}) Z, Z >,$$

for every $Z, X \in \mathfrak{m}$ orthonormal with respect to $<, >$.

Proof: Let $Z, X \in \mathfrak{m}$. The proof is a straightforward calculation by using the identities (1.17), (1.21) and (1.22):

$$K(Z, X) = < R(Z, X) X, Z >$$

$$= < [L_Z, L_X]_m X, Z > - < L_{[Z, X]_m} X, Z > - < ad_{[Z, X]_m}, X, Z >$$

$$= < L_Z L_X X, Z > - < L_X L_Z X, Z > - < L_X [Z, X]_m, Z > -$$

$$- < [[Z, X]_m, X]_m, Z > - < [[Z, X]_m, X]_m, Z >$$

$$= - < L_X X, L_Z Z, > + < L_Z X, L_X Z > + < [Z, X]_m, L_X Z > -$$

$$- < [[Z, X]_m, X]_m, Z >$$

$$= - < L_X X, L_Z Z, > + < L_Z X, L_Z X > + < L_Z X, [X, Z]_m > +$$

$$+ < [Z, X]_m, L_Z X > + < [Z, X]_m, [X, Z]_m > -$$

$$- < [X, [X, Z]]_m, Z >$$

$$= - < L_X X, L_Z Z, > + < L_Z X, L_Z X > - < [Z, X]_m, [Z, X]_m > -$$

$$- < (P^*_X Z)_m, Z >$$
\[ \begin{align*}
&= <LZX, LZX>-<[X, Z]_m, [X, Z]_m>-<(P^2_X Z)_m, Z > \\
&\quad -<LZV_X, Z>
\end{align*} \]
\[\begin{align*}
&= <RXZ, RXZ>-<PXZ, PXZ>-<(P^2_X + BXC_X)Z, Z > \\
&\quad -<LV_X Z + [Z, V_X]_m, Z>
\end{align*} \]
\[\begin{align*}
&= <(R^*_X RX - P^*_X PX - BXC_X - P^2_X)Z, Z > + <(PV_X - LV_X)Z, Z >.
\end{align*} \]

Since \(L_X\) is skew-symmetric with respect to \(<,>\), we have \(<LV_X Z, Z> = 0, \) for every \(Z \in m, \) and we obtain

\[K(Z, X) = <(R^*_X RX - P^*_X PX - BXC_X - P^2_X + PV_X)Z, Z >,\]

as required. \(\square\)

**Theorem 1.1.** Let \(X, Y \in m.\) Then

\[ Ric(X, Y) = -\frac{1}{2} tr(2RXRY + BXC_Y + BYC_X - 2P_{U(X,Y)}). \]

In particular,

\[ Ric(X, X) = -tr(R^2_X + BXC_X - PV_X). \]

**Proof:** We first compute \(Ric(X, X)\) and then obtain \(Ric(X, Y)\) by polarization. Since \(T_X\) is a skew-symmetric operator and \(P_X + P^*_X\) is symmetric,

\[ tr((P_X + P^*_X)T_X) = -tr(T_X(P_X + P^*_X)) = tr((P_X + P^*_X)T_X) \]

and thus all the terms vanish. Therefore, by using Lemma 1.2 we obtain

\[ tr(R^*_X RX) = \frac{1}{4}((P_X + P^*_X)^2 - T^2_X) = \frac{1}{4} tr(2P^2_X + 2P^*_X PX - T^2_X) \]

and

\[ tr(R^2_X) = \frac{1}{4}((P_X + P^*_X)^2 + T^2_X) = \frac{1}{4} tr(2P^2_X + 2P^*_X PX + T^2_X). \]

Hence, \(tr(R^*_X RX - P^2_X - P^*_X PX) = -tr(R^2_X).\)

Let us suppose that \(<X, X> = 1\) and let \(\{e_i\},\) be an orthonormal basis for \(m\) with respect to \(<,>\) such that \(X = e_1.\) Hence, we can apply Lemma 1.3 to obtain the following:
\[Ric(X, X) = \sum_i K(e_i, X)\]
\[= \langle (R^*_XR_X - P^*_XP_X - B_XC_X - P^2_X + P_{VX})e_i, e_i \rangle\]
\[= tr(R^*_XR_X - P^2_X - P^*_XP_X - B_XC_X + P_{VX})\]
\[= -tr(R^2_X + B_XC_X - P_{VX}).\]

Since both \(Ric\) and the map \(X \mapsto tr(R^2_X + B_XC_X - P_{VX})\) are bilinear maps, the identity above holds even if \(X\) is not unit. Hence, for every \(X \in \mathfrak{m}\),

\[Ric(X, X) = -tr(R^2_X + B_XC_X - P_{VX}).\]

Now we compute \(Ric(X, Y)\). Since \(Ric\) is a symmetric bilinear operator we have

\[2Ric(X, Y) = Ric(X + Y, X + Y) - Ric(X, X) - Ric(Y, Y).\]

By using the expression above for \(Ric(X, X)\) we get

\[2Ric(X, Y) = tr(-R^2_{X+Y} + R^2_X + R^2_Y) +\]
\[+ tr(-B_{X+Y}C_{X+Y} + B_XC_X + B_YC_Y) +\]
\[+ tr(P_{VX+Y} - P_{VX} - P_{VY}).\]

By bilinearity of \(L\) and property [1.22] we obtain

\[V_{X+Y} = L_{X+Y}(X + Y)\]
\[= L_XX + L_YY + L_XY + L_YX\]
\[= V_X + V_Y + 2L_XY - [X, Y]_m\]
\[= V_X + V_Y + 2L_XY + P_XY.\]

Therefore, \(P_{VX+Y} - P_{VX} - P_{VY} = P_{2L_XY - P_XY}.\)

Also the identity \(B_{X+Y}C_{X+Y} = B_XC_X + B_YC_Y + B_XC_Y + B_YC_X\) implies that

\[-B_{X+Y}C_{X+Y} + B_XC_X + B_YC_Y = -B_XC_Y - B_YC_X.\]

Moreover, \(R^2_{X+Y} = R^2_X + R^2_Y + R_XR_Y + R_YR_X\) and thus

\[tr(-R^2_{X+Y} + R^2_X + R^2_Y) = -tr(R_XR_Y + R_YR_X) = -tr(2R_XR_Y).\]
Therefore, we obtain
\[
2Ric(X, Y) = -tr(2R_X R_Y + B_X C_Y + B_Y C_X) + tr(P_{2L_X Y - P_X Y})
= -tr(2R_X R_Y + B_X C_Y + B_Y C_X) + tr(P_{2U(X, Y)}).
\]
□

**Corollary 1.1.** Let \( X, Y \in \mathfrak{m} \).

\[
Ric(X, Y) = -\frac{1}{4} tr(2P^*_X P_Y + T_X T_Y) - \frac{1}{2} Kill(X, Y) + tr(P_{U(X, Y)}).
\]

**Proof:** By using Theorem 1.1 and Lemma 1.2, we write \( Ric \) as follows:

\[
Ric(X, Y) = -\frac{1}{2} \left( \frac{1}{2}(P_X + P^*_X + T_X)(P_Y + P^*_Y + T_Y) + B_X C_Y + B_Y C_X - 2P_{U(X, Y)} \right).
\]

(1.25)

Since \( P_X + P^*_X \) and \( P_Y + P^*_Y \) are symmetric linear maps and \( T_X \) and \( T_Y \) are skew-symmetric, we have

\[
tr((P_X + P^*_X)T_Y) = tr(T_X(P_Y + P^*_Y)) = 0.
\]

(1.26)

Moreover,

\[
tr(P_X P_Y) = tr(P^*_X P^*_Y) \text{ and } tr(P^*_X P_Y) = tr(P_X P^*_Y).
\]

(1.27)

We can use (1.11) to write \( Kill \) as follows:

\[
Kill(X, Y) = tr(ad_X ad_Y) = tr(C_X B_Y + B_X C_Y + P_X P_Y).
\]

(1.28)

Finally, by using (1.26), (1.27) and (1.28) we simplify (1.25) to obtain the expression stated for \( Ric \).

□

**Definition 1.1.** A symmetric bilinear map \( \beta \) on \( \mathfrak{m} \) is said to be associative if \( \beta([u, v]_m, w) = \beta(u, [v, w]_m) \), for every \( u, v, w \in \mathfrak{m} \).

**Remark 1.1.** If there exists on \( \mathfrak{m} \) an associative symmetric bilinear form \( \beta \) such that \( \beta \) is non-degenerate, then \( tr(P_{U(X, Y)}) = 0 \), for all \( X, Y \in \mathfrak{m} \). Indeed, if such a bilinear form exists, \( tr P_a = 0 \), for every \( a \in \mathfrak{m} \). Let \( \{w_i\}_i \) and \( \{w'_i\}_i \) be bases of \( \mathfrak{m} \) dual with respect to \( \beta \), i.e., \( \beta(w_i, w'_j) = \delta_{ij} \). Then, for every \( a \in \mathfrak{m} \),

\[
\beta(P_a w_i, w'_j) = \beta([a, w_i]_m, w'_j)
= -\beta(w_i, [a, w'_j]_m)
= -\beta(P_a w'_j, w_i).
\]
Hence, $tr(P_a) = 0$. Also, if the metric $g_M$ on $M$ is naturally reductive, then 
$P_{U(X,Y)} = 0$, for all $X, Y \in m$, since, in this case, $U$ is identically zero.

\diamond

**Definition 1.2.** Let $U, V$ be $Ad L$-invariant vector subspaces of $g$. We define a bilinear map $Q^{UV} : m \times m \rightarrow \mathbb{R}$ by

$$Q^{UV}_{XY} = tr([X, [Y, \cdot]_V]_U), \ X, Y \in m,$$
where the subscripts $U$ and $V$ denote the projections onto $U$ and $V$, respectively.

**Definition 1.3.** Let $U$ be an $Ad L$-invariant vector subspace of $g$ such that the restriction of Kill to $U$ is non-degenerate. The Casimir operator of $U$ with respect to the Killing form of $g$ is the operator

$$C_U = \sum_i ad_{u_i} ad_{u'_i},$$
where $\{u_i\}_i$ and $\{u'_i\}_i$ are bases of $U$ which are dual with respect to Kill, i.e.,

$$Kill(u_i, u'_j) = \delta_{ij}.$$

The Casimir operator $C_U$ is an $Ad L$-invariant linear map and thus it is scalar on any irreducible $Ad L$-module. In particular, if $g$ is simple, then $C_g = Id_g$.

**Lemma 1.4.** Suppose that $g$ is semisimple. Let $U, V$ be $Ad L$-invariant vector subspaces of $g$ such that the restrictions of the Killing form to $U$ and $V$ are both non-degenerate.

(i) $Q^{UV}$ is an $Ad L$-invariant symmetric bilinear map. Hence, if $W \subset g$ is any irreducible $Ad L$-submodule, then $Q^{UV} |_{W \times W}$ is a multiple of $Kill |_{W \times W}$.

(ii) $Q^{UV} = Q^{VU}$.

Proof: Since $g$ is a semisimple Lie algebra its Killing form is non-degenerate. As in addition $Kill |_{U \times U}$ and $Kill |_{V \times V}$ are non-degenerate, we may consider the orthogonal complements $U^\perp$ and $V^\perp$ of $U$ and $V$, respectively, in $g$ with respect to $Kill$. It follows that

$$Kill |_{U \times g} = Kill(\cdot, p_U \cdot) |_{U \times g}$$

and

$$Kill |_{V \times g} = Kill(\cdot, p_V \cdot) |_{V \times g},$$
where $p_U$ and $p_V$ are the projections onto $U$ and $V$, respectively.
Also, we may consider bases $\{w_i\}_i$ and $\{w'_i\}_i$ of $U$ which are dual with respect to $\text{Kill}$. By using these facts we have the following:

(i) Let $X, Y \in \mathfrak{m}$ and $g \in L$.

\[
\text{Kill}([X, [Y, w_i]_V]_U, w'_i) = \text{Kill}([X, [Y, w_i]_V], w'_i) \\
= -\text{Kill}([Y, w_i]_V, [X, w'_i]) \\
= -\text{Kill}([Y, w_i], [X, w'_i]_V) \\
= \text{Kill}(w_i, [Y, [X, w'_i]_V]) \\
= \text{Kill}(w_i, [Y, [X, w'_i]_V]_U).
\]

Therefore, $\text{tr}([X, [Y, \cdot]_V]_U) = \text{tr}([Y, [X, \cdot]_V]_U)$ and thus $Q^{UV}_{XY} = Q^{UV}_{YX}$. So $Q^{UV}$ is symmetric.

To show the $\text{Ad } L$-invariance of $Q^{UV}$ we note that since $V$ and $V^\perp$ are $\text{Ad } L$-invariant subspaces and $g = V \oplus V^\perp$, the projections on $V$ and $V^\perp$ are also $\text{Ad } L$-invariant linear maps.

\[
\text{Kill}([\text{Ad}_g X, [\text{Ad}_g Y, w_i]_V]_U, w'_i) = \text{Kill}([\text{Ad}_g X, [\text{Ad}_g Y, w_i]_V], w'_i) \\
= \text{Kill}(\text{Ad}_{g^{-1}} [\text{Ad}_g X, [\text{Ad}_g Y, w_i]_V], \text{Ad}_{g^{-1}} w'_i) \\
= \text{Kill}([X, \text{Ad}_{g^{-1}} [\text{Ad}_g Y, w_i]_V], \text{Ad}_{g^{-1}} w'_i) \\
= \text{Kill}([X, [Y, \text{Ad}_{g^{-1}} w_i]_V], \text{Ad}_{g^{-1}} w'_i) \\
= \text{Kill}([X, [Y, \text{Ad}_{g^{-1}} w_i]_V]_U, \text{Ad}_{g^{-1}} w'_i).
\]

If $\{w_i\}_i$ and $\{w'_i\}_i$ are dual bases of $U$ with respect to $\text{Kill}$, then $\{\text{Ad}_{g^{-1}} w_i\}_i$ and $\{\text{Ad}_{g^{-1}} w'_i\}_i$ are still dual bases as well since the Killing form is invariant under inner automorphisms. So by the above we conclude that

\[
\text{tr}([\text{Ad}_g X, [\text{Ad}_g Y, \cdot]_V]_U) = \text{tr}([X, [Y, \cdot]_V]_U)
\]

and thus $Q^{UV}$ is $\text{Ad } L$-invariant.

(ii) For $Z \in \mathfrak{m}$, let $A_Z = (ad_Z |_U)_V$ and $B_Z = (ad_Z |_V)_U$. We have

\[
Q^{UV}_{XY} = \text{tr}(A_X B_Y) = \text{tr}(B_Y A_X) = Q^{UV}_{YX}.
\]

Hence, by symmetry of $Q^{UV}$, we conclude that $Q^{UV}_{XY} = Q^{UV}_{YX} = Q^{UV}_{XY}$, for every $X, Y \in \mathfrak{m}$. Therefore, $Q^{UV} = Q^{VU}$. 

\[\square\]
Lemma 1.5. Suppose that $g$ is semisimple. Let $U, V$ be $\text{Ad} L$-invariant vector subspaces of $g$ such that the restrictions of the Killing form to $U$ and $V$ are both non-degenerate. For every $X, Y \in \mathfrak{m}$,

(i) if $\text{ad} X U \subset V$ or $\text{ad} Y U \subset V$, then $Q_{X Y}^{U V} = \text{Kill}(C_U X, Y) = \text{Kill}(X, C_U Y)$;

(ii) if $\text{ad} X V \bot U$ or $\text{ad} Y V \bot U$, then $Q_{X Y}^{U V} = 0$;

(iv) if $\text{ad} X \text{ad} Y U \bot U$ or $\text{ad} Y \text{ad} X U \bot U$, then $Q_{X Y}^{U V} = 0$.

Proof: (i) Let $C_U = \sum_i \text{ad} w_i \text{ad} w_i'$ be the Casimir operator of $U$. Since $Q_{X Y}^{U V} = Q_{V X}^{U V}$ it suffices to suppose that $\text{ad} Y U \subset V$. If $\text{ad} Y U \subset V$, then

$$Q_{X Y}^{U V} = \text{tr}([X, [Y, \cdot]] U) = \text{tr}(\text{ad} X \text{ad} Y \mid_U).$$

Since

$$\text{Kill}([X, [Y, w_i]] U, w_i') = \text{Kill}([X, [Y, w_i]], w_i')$$

$$= -\text{Kill}([Y, w_i], [X, w_i'])$$

$$= \text{Kill}(Y, [w_i, [w_i', X]]),$$

we have $Q_{X Y}^{U V} = \sum_i \text{Kill}(Y, [w_i, [w_i', X]]) = \text{Kill}(Y, C_U X)$. By symmetry of $Q_{X Y}^{U V}$ we also get $Q_{X Y}^{U V} = \text{Kill}(X, C_U Y)$.

(ii) If $\text{ad} X V \bot U$, then, for every $w, w' \in U$, $\text{Kill}([X, [Y, w]] U, w') = 0$ and thus $Q_{X Y}^{U V} = 0$, for every $Y \in \mathfrak{m}$. By symmetry, the same conclusion holds if $\text{ad} Y V \bot U$.

(iii) If $\text{ad} X \text{ad} Y U \bot U$, then, for every $w, w' \in U$, $\text{Kill}([X, [Y, w]] U, w') = 0$ and thus $\text{Kill}([X, [Y, w]] U, w') = 0$. Hence $Q_{X Y}^{U V} = 0$. If $\text{ad} Y \text{ad} X U \bot U$, then $Q_{X Y}^{U V} = 0$ by symmetry.

$\square$

Remark 1.2. In Lemmas 1.4 and 1.5 the condition that $g$ is semisimple may be replaced by requiring that there is on $g$ a non-degenerate $\text{Ad} L$-invariant symmetric bilinear form $\beta$, since in the proofs above the Killing form may be replaced by any such form $\beta$. In this case, the orthogonality conditions in Lemma 1.5 should be understood as conditions with respect to $\beta$.

Theorem 1.2. Let $\beta$ be an associative $\text{Ad} L$-invariant non-degenerate symmetric bilinear form on $\mathfrak{m}$. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_m$ be a decomposition of $\mathfrak{m}$ into $\text{Ad} L$-invariant subspaces such that $\beta \mid_{\mathfrak{m}_i \times \mathfrak{m}_j} = 0$, if $i \neq j$. Let $g_M$ be the $G$-invariant pseudo-Riemannian metric on $G/L$ induced by the scalar product of the form

$$\langle \cdot, \cdot \rangle = \bigoplus_{j=1}^m \nu_j \beta \mid_{\mathfrak{m}_j \times \mathfrak{m}_j}, \quad \text{(1.29)}$$

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for $\nu_j > 0$, for every $j = 1, \ldots, m$. For every $X \in m_a$ and $Y \in m_b$, the Ricci curvature of $g_M$ is given as follows:

$$Ric(X, Y) = \frac{1}{2} \sum_{j,k=1}^m \left( \frac{\nu_k}{\nu_j} - \frac{\nu_a \nu_b}{2 \nu_k \nu_j} \right) Q_X^{m_j m_k} \frac{1}{2} \text{Kill}(X, Y).$$

Proof: First we observe that the non-degeneracy of $\beta$ and the condition of pairwise orthogonality of the $m_j$'s imply that $\beta_{m_j \times m_j}$ is in fact non-degenerate. Let $X \in m_a$ and $Y \in m_b$. By Corollary 1.1 we have

$$Ric(X, Y) = -\frac{1}{4} tr(2P_X P_Y + T_X T_Y) - \frac{1}{2} \text{Kill}(X, Y) + tr(P_{U(Y,Y)}).$$

According to Remark 1.1 we have $tr(P_{U(Y,Y)}) = 0$.

Let $j = 1, \ldots, m$ and let $\{w_i\}_1$ and $\{w'_i\}_1$ be dual bases for $m_j$ with respect to $\beta$. We note that such bases exist as $\beta_{m_j \times m_j}$ is non-degenerate.

$$< T_X T_Y w_i, w'_i > = < X, [T_Y w_i, w'_i]_m >$$

$$= \nu_a \beta(X, [T_Y w_i, w'_i])$$

$$= -\nu_a \beta(T_Y w_i, [X, w'_i])$$

$$= -\nu_a \sum_{k=1}^m \beta(T_Y w_i, [X, w'_i]_{m_k})$$

$$= -\nu_a \sum_{k=1}^m \frac{1}{\nu_k} < T_Y w_i, [X, w'_i]_{m_k} >$$

$$= -\nu_a \sum_{k=1}^m \frac{1}{\nu_k} < Y, [w_i, [X, w'_i]_{m_k}]_m >$$

$$= -\nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k} \beta(Y, [w_i, [X, w'_i]_{m_k}]_m)$$

$$= -\nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k} \beta([Y, w_i], [X, w'_i]_{m_k})$$

$$= -\nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k} \beta([Y, w_i]_{m_k}, [X, w'_i])$$

$$= \nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k} \beta(w'_i, [X, [Y, w_i]_{m_k}]_m)$$

$$= \nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k} \beta(w'_i, [X, [Y, w_i]_{m_k}]_{m_j})$$

$$= \nu_a \nu_b \sum_{k=1}^m \frac{1}{\nu_k \nu_j} < w'_i, [X, [Y, w_i]_{m_k}]_m >.$$
Hence,

\[ tr(T_X T_Y \mid m_j) = \nu_a \nu_b \sum_{k=1}^{m} \frac{1}{\nu_k \nu_j} tr([X, [Y, \cdot]_{m_k}]_{m_j}) = \nu_a \nu_b \sum_{k=1}^{m} \frac{1}{\nu_k \nu_j} Q_{XY}^{m_j m_k} \]

and thus

\[ tr(T_X T_Y) = \nu_a \nu_b \sum_{j,k=1}^{m} \frac{1}{\nu_k \nu_j} Q_{XY}^{m_j m_k}. \]

and thus

\[ < P_X^* P_Y w_i, w_i' > = < P_Y w_i, P_X w_i' > \]

\[ = \sum_{k=1}^{m} < [Y, w_i]_{m_k}, [X, w_i']_{m_k} > \]

\[ = \sum_{k=1}^{m} \nu_k \beta([Y, w_i]_{m_k}, [X, w_i']) \]

\[ = - \sum_{k=1}^{m} \nu_k \beta(w_i', [X, [Y, w_i]_{m_k}]) \]

\[ = - \sum_{k=1}^{m} \nu_k \beta(w_i', [X, [Y, w_i]_{m_k}]_{m_j}) \]

\[ = - \sum_{k=1}^{m} \frac{\nu_k}{\nu_j} < w_i', [X, [Y, w_i]_{m_k}]_{m_j} > . \]

Then

\[ tr(P_X^* P_Y \mid m_j) = - \sum_{k=1}^{m} \frac{\nu_k}{\nu_j} tr([X, [Y, \cdot]_{m_k}]_{m_j}) = - \sum_{k=1}^{m} \frac{\nu_k}{\nu_j} Q_{XY}^{m_j m_k} \]

and thus we get

\[ tr(P_X^* P_Y) = - \sum_{j,k=1}^{m} \frac{\nu_k}{\nu_j} Q_{XY}^{m_j m_k}. \]

By using Corollary 1.1 we finally obtain the required expression for \( Ric(X, Y) \).

\( \square \)

We recall that a metric \( g_M \) is said to be normal if it is a multiple of an associative \( Ad L \)-invariant non-degenerate symmetric bilinear form on \( m \).

**Corollary 1.2.** If \( g_M \) is a normal metric, then \( Ric(m_i, m_j) = 0 \), for all \( i \neq j \).

For every \( X \in m_j \),

\[ Ric(X, X) = - \frac{1}{4} Kill(X, X) - \frac{1}{2} Kill(C_i X, X), \]

where \( C_i \) is the Casimir operator of \( l \) with respect to the Killing form. Furthermore, if \( m_j \) is irreducible, then

\[ Ric \mid_{m_j \times m_j} = - \frac{1}{2} \left( \frac{1}{2} + c_{i,j} \right) Kill \mid_{m_j \times m_j}, \]

where \( c_{i,j} \) is the eigenvalue of \( C_i \) on \( m_j \).
Proof: Let $X \in \mathfrak{m}_a$ and $Y \in \mathfrak{m}_b$. If $g_M$ is a normal metric, then there exists an $Ad L$-invariant non-degenerate symmetric bilinear form $\beta$ on $\mathfrak{m}$ which induces $g_M$. Hence, in Theorem 1.2 we can take $\nu_1 = \ldots = \nu_m = 1$ and obtain the following:

$$Ric(X, Y) = \frac{1}{4} \sum_{j,k=1}^m Q_{XY}^{mjmk} - \frac{1}{2} Kill(X, Y)$$

$$= \frac{1}{4} Q_{XY}^{mm} - \frac{1}{2} Kill(X, Y)$$

$$= \frac{1}{4} Q_{XY}^{mg} - \frac{1}{4} Q_{XY}^{ml} - \frac{1}{2} Kill(X, Y)$$

$$= \frac{1}{4} Q_{XY}^{mg} - \frac{1}{2} Q_{XY}^{im} - \frac{1}{2} Kill(X, Y)$$

$$= \frac{1}{4} Kill(C_mX, Y) - \frac{1}{4} Kill(C_mX, Y) - \frac{1}{2} Kill(X, Y)$$

$$= -\frac{1}{4} Kill(X, Y) - \frac{1}{2} Kill(C_mX, Y).$$

Since $C_l(\mathfrak{m}_a) \subset \mathfrak{m}_a$, it is clear that $Ric(X, Y) = 0$ if $a \neq b$ and $Ric$ is well determined by elements $Ric(X, X)$ with $X \in \mathfrak{m}_a$. If $\mathfrak{m}_j$ is irreducible, then $C_l$ is scalar on $\mathfrak{m}_j$ and we obtain the identity given for $Ric$. □

The formula above for the Ricci curvature of a normal metric was first found by M.Y. Wang and W. Ziller in [44]. From Corollary 1.2 it is clear that a necessary and sufficient condition for a normal metric to be Einstein is that the Casimir operator of $l$ is scalar on the isotropy space $\mathfrak{m}$. For instance, this condition holds if $\mathfrak{m}$ is irreducible. Simply connected non-strongly isotropy irreducible homogeneous spaces which admit a normal Einstein metric were classified by M.Y. Wang and W. Ziller in [44], when $G$ is a compact connected simple group. Also, more generally, simply connected compact standard homogeneous manifolds were studied by E.D. Rodionov in [39].

We obtain a similar formula to that of Corollary 1.2 in the case when the submodules $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ pairwise commute.

**Corollary 1.3.** If $\mathfrak{m}_1, \ldots, \mathfrak{m}_m$ pairwise commute, i.e., $[\mathfrak{m}_a, \mathfrak{m}_b] = 0$, for all $a \neq b$, then $Ric(\mathfrak{m}_a, \mathfrak{m}_b) = 0$, for all $a \neq b$. For every $X \in \mathfrak{m}_a$,

$$Ric(X, X) = -\frac{1}{4} Kill(X, X) - \frac{1}{2} Kill(C_lX, X),$$

where $C_l$ is the Casimir operator of $l$ with respect to the Killing form. Furthermore, if $\mathfrak{m}_j$ is irreducible, then

$$Ric \mid_{\mathfrak{m}_j \times \mathfrak{m}_j} = -\frac{1}{2} \left( \frac{1}{2} + c_{l,j} \right) Kill \mid_{\mathfrak{m}_j \times \mathfrak{m}_j},$$

where $c_{l,j}$ is the eigenvalue of $C_l$ on $\mathfrak{m}_j$. 17
Proof: Let $X \in m_a$ and $Y \in m_b$. If $m_1, \ldots, m_m$ pairwise commute, then, by Lemma 1.5, we have $Q_{XY}^{m_j m_k} = 0$, for every $j, k \neq a, b$. In particular, all these bilinear maps vanish if $a \neq b$. Hence, if $a \neq b$, $Ric(X, Y) = -\frac{1}{2}Kill(X, Y) = 0$. Therefore, the Ricci curvature is well determined by elements of the form $Ric(X, X)$, with $X \in m_a$, and

$$Ric(X, X) = \frac{1}{4}Q_{XX}^{ma} - \frac{1}{2}Kill(X, X).$$

Since $Q_{XX}^{ma} = Q_{XX}^{mm}$, the rest of the proof is similar to the proof of Corollary 1.2.

$\square$
1.2 Homogeneous Riemannian Fibrations

1.2.1 Notation and Hypothesis

In this Section we obtain the Ricci curvature of an invariant metric with totally geodesic fibers on the total space of a homogeneous fibration. We start by settling once and for all the notation used throughout.

Let $G$ be a compact connected semisimple Lie group and $L \subset K \subset G$ connected closed non-trivial subgroups of $G$. We denote $M = G/L$, $N = G/K$ and $F = K/L$. We consider the natural fibration

$$\pi : M \rightarrow N$$

$$aL \mapsto aK$$

(1.30)

with fiber $F$ and structural group $K$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$ the Lie algebras of $G$, $K$ and $L$, respectively. By $\text{Kill}$ we denote the Killing form of $G$ and we set $B = -\text{Kill}$. Also, we denote the Killing forms of $K$ and $L$ by $\text{Kill}_K$ and $\text{Kill}_L$, respectively. As $G$ is compact and semisimple, the Killing form of $G$ is negative definite and thus $B$ is positive definite. We consider an orthogonal decomposition of $\mathfrak{g}$ with respect to $B$ given by

$$\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} \oplus \mathfrak{n},$$

(1.31)

where $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$. Clearly, $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{m}$, $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}$ are reductive decompositions for $M$, $N$ and $F$, respectively. Hence, we have the following inclusions

$$[\mathfrak{k}, \mathfrak{n}] \subset \mathfrak{n}, \ [\mathfrak{l}, \mathfrak{n}] \subset \mathfrak{n}, \ [\mathfrak{l}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{n}] \subset \mathfrak{n}.$$  

(1.32)

An $AdK$-invariant scalar product on $\mathfrak{n}$ induces a $G$-invariant Riemannian metric $g_N$ on $N$ and an $AdL$-invariant scalar product on $\mathfrak{p}$ induces a $G$-invariant Riemannian metric $g_F$ on $F$. The orthogonal direct sum of these scalar products on $\mathfrak{m}$ defines a $G$-invariant Riemannian metric $g_M$ on $M$ which projects onto a $G$-invariant metric on $N$. Moreover, if $\mathfrak{p}$ and $\mathfrak{n}$ do not contain any equivalent $AdL$-submodules, then any $G$-invariant metric which projects onto a $G$-invariant metric on $N$ is constructed in this fashion. We recall the following result due to L.Bérard-Bergery ([9], [10] §H):

**Theorem 1.3.** The natural projection $\pi : M \rightarrow N$ is a Riemannian submersion from $(M, g_M)$ to $(N, g_N)$ with totally geodesic fibers.

Throughout this thesis we shall refer to such a metric $g_M$ as an adapted metric:
Definition 1.4. An adapted metric on $M$ is a $G$-invariant Riemannian metric $g_M$ such that the natural projection $\pi : M \to N$ is a Riemannian submersion and consequently the fibers are totally geodesic submanifolds. The fibration $M \to N$ equipped with an adapted metric $g$ is then called a Riemannian fibration.

An adapted metric on $M$ shall be denoted by $g_M$ and, as already introduced above, $g_F$ and $g_N$ shall denote the projection of $g_M$ onto the base space $N$ and $g_F$ its restriction to the fiber $F$.

We consider a decomposition $p = p_1 \oplus \ldots \oplus p_s$ into irreducible $Ad L$-submodules pairwise orthogonal with respect to $B$. Also let $n = n_1 \oplus \ldots \oplus n_n$ be an orthogonal decomposition into irreducible $Ad K$-submodules. Throughout we assume the following hypothesis:

(i) $p_1, \ldots, p_s$ are pairwise inequivalent irreducible $Ad L$-submodules;
(ii) $n_1, \ldots, n_n$ are pairwise inequivalent irreducible $Ad K$-submodules;
(iii) $p$ and $n$ do not contain equivalent $Ad L$-submodules.

We shall refer to this hypothesis by saying that $M$ has simple spectrum. Under this hypothesis, according to Schur’s Lemma, any $Ad L$-invariant scalar product on $m = p \oplus n$ which restricts to an $Ad K$-invariant scalar product on $n$ is of the form

$$<, > = (\oplus_{a=1}^s \lambda_a B_{|p_a \times p_a}) \oplus (\oplus_{k=1}^n \mu_k B_{|n_k \times n_k}),$$

(1.33)

for some $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_n > 0$. Since an adapted metric $g_M$ on $M$ projects onto a $G$-invariant Riemannian metric on $N$, $g_M$ is necessarily induced by a scalar product of the form (1.33). To denote that $g_M$ is induced by a scalar product as in (1.33) we shall write

$$g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n).$$

(1.34)

Similarly, we write

$$g_F = g_F(\lambda_1, \ldots, \lambda_s) \text{ and } g_N = g_N(\mu_1, \ldots, \mu_n).$$

(1.35)

By $Ric$ we mean the Ricci curvature of $g_M$ and by $Ric^F$ and $Ric^N$ the Ricci curvature of $g_N$ and $g_F$, respectively.

In the following Sections we compute the Ricci curvature for $g_M$ and find some necessary conditions so that $g_M$ is an Einstein metric. We recall that in Theorem 1.2 we have shown that the Ricci curvature of any metric on $M$ can be described using the bilinear maps $Q_{XY}$ defined in 1.2.
1.2.2 The Ricci Curvature in the Direction of the Fiber

In this Section we obtain the Ricci curvature of the adapted metric

\[ g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \]

in the vertical direction \( p \). We recall that \( p \) decomposes into the direct sum of the pairwise inequivalent irreducible \( Ad L \)-submodules \( p_1, \ldots, p_s \) and, as explained above, \( g_M \) is induced by the scalar product (1.33)

\[
(\oplus_{a=1}^{s} \lambda_a B |_{p_a \times p_a}) \oplus (\oplus_{k=1}^{n} \mu_k B |_{n_k \times n_k}),
\]

while \( g_F \) is the restriction of \( g_M \) to the fiber, i.e.,

\[ g_F = g_F(\lambda_1, \ldots, \lambda_s). \]

Lemma 1.6. Let \( X \in p \) and \( Y \in m \).

(i) \( Q^{n_j p_a}_{XY} = Q^{p_a n_j}_{XY} = 0, a = 1, \ldots, s, j = 1, \ldots, n; \)

(ii) \( Q^{p_a n_j}_{XY} = 0, \text{ if } i \neq j, i, j = 1, \ldots, n; \)

(iii) \( Q^{n_j n_j}_{XY} = \text{Kill}(C_{n_j} X, Y), j = 1, \ldots, n. \)

Proof: Since \( \text{ad}_X p \subset n \perp n \) we have \( Q^{n_j p_a}_{XY} = 0 \), from Lemma 1.5. From Lemma 1.4 (ii), \( Q^{p_a n_j}_{XY} = Q^{p_a p_a}_{XY} = 0. \)

As \( \text{ad}_X n_j \subset n_j \), we have \( Q^{n_j n_j}_{XY} = \text{Kill}(C_{n_j} X, Y). \) Moreover, since \( n_j \perp n_i \), for every \( i \neq j \), we also conclude that \( Q^{n_j n_j}_{XY} = 0, \text{ if } i \neq j. \)

\[ \Box \]

Lemma 1.7. The Ricci curvature of \( g_f = g_F(\lambda_1, \ldots, \lambda_s) \) is of the form \( \text{Ric}^F = \oplus_{a=1}^{s} q_a B |_{p_a \times p_a} \), with

\[ q_a = \frac{1}{2} \sum_{b,c=1}^{s} \left( \frac{\lambda_a^2}{2 \lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_c^{b} + \frac{\gamma_a}{2} \]

where the constants \( q_a^{b} \) and \( \gamma_a \) are such that

\[ \text{Kill}_f |_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a} \]

and

\[ Q^{p_b p_c} |_{p_a \times p_a} = q_a^{b} \text{Kill} |_{p_a \times p_a}. \]

In particular, \( \text{Ric}^F(p_a, p_b) = 0, \text{ if } a \neq b. \)

Proof: Since \( p_1, \ldots, p_s \) are pairwise inequivalent irreducible \( Ad L \)-submodules and \( \text{Ric}^F \) is an \( Ad L \)-invariant symmetric bilinear form, we may write \( \text{Ric}^F = \)}
\( \oplus_{a=1}^{s} g_a B \mid_{{p_a} \times {p_a}} \), for some constants \( q_1, \ldots, q_s \). In particular, we have \( \text{Ric}^F(p_a, p_b) = 0 \), for every \( a, b = 1, \ldots, s \) such that \( a \neq b \). By Theorem 1.2, the Ricci curvature of \( g_F \) is

\[
\text{Ric}^F(X, X) = \frac{1}{2} \sum_{b, c=1}^{s} \left( \frac{\lambda_b}{\lambda_c} - \frac{\lambda_c^2}{2 \lambda_c \lambda_b} \right) Q_{X}^{p_c p_b} - \frac{1}{2} \text{Kill}_\mathfrak{g}(X, X).
\]

By Lemma 1.4 the maps \( Q^{p_c p_b} \) are \( \text{Ad} L \)-invariant symmetric bilinear maps. Since \( p_a \) is \( \text{Ad} L \)-irreducible, there are constants \( q_a^{cb} \) such that

\[
Q^{p_c p_b} \mid_{{p_a} \times {p_a}} = q_a^{cb} \text{Kill} \mid_{{p_a} \times {p_a}}.
\]

Similarly, the Killing form of \( \mathfrak{t} \), \( \text{Kill}_\mathfrak{t} \), is an \( \text{Ad} L \)-invariant symmetric bilinear form on \( p_a \). So, by irreducibility of \( p_a \), there is a constant \( \gamma_a \) such that

\[
\text{Kill}(C_{\mathfrak{n}_j} \cdot, \cdot) \mid_{{p_a} \times {p_a}} = \gamma_a \text{Kill} \mid_{{p_a} \times {p_a}}.
\]

By the expression above for \( \text{Ric}^F \), we must have

\[
q_a = \frac{1}{2} \sum_{b, c=1}^{s} \left( \frac{\lambda_a^2}{2 \lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2},
\]

and the result follows from this.

\( \square \)

**Proposition 1.1.** Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \).

For every \( a, b = 1, \ldots, s \) such that \( a \neq b \), \( \text{Ric}(p_a, p_b) = 0 \). For every \( X \in p_a, a = 1, \ldots, s \),

\[
\text{Ric}(X, X) = \left( q_a + \frac{\lambda_a^2}{4} \sum_{j=1}^{n} \frac{c_{n_j, a}}{\mu_j^2} \right) B(X, X),
\]

where, for \( j = 1, \ldots, n \), the constants \( c_{n_j, a} \) are such that

\[
\text{Kill}(C_{n_j} \cdot, \cdot) \mid_{{p_a} \times {p_a}} = c_{n_j, a} \text{Kill} \mid_{{p_a} \times {p_a}}
\]

and \( C_{n_j} \) is the Casimir operator of \( n_j \) with respect to \( \text{Kill} \). The constant \( q_a \) is such that

\[
q_a = \frac{1}{2} \sum_{b, c=1}^{s} \left( \frac{\lambda_a^2}{2 \lambda_c \lambda_b} - \frac{\lambda_c}{\lambda_b} \right) q_a^{cb} + \frac{\gamma_a}{2},
\]

with \( q_a^{cb} \) and \( \gamma_a \) defined by
\[ \text{Kill}_t |_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a} \]

\[ Q^{p_b \cdot p_c} |_{p_a \times p_a} = \alpha_a \text{Kill} |_{p_a \times p_a} . \]

**Proof:** Since \( p_1, \ldots, p_s \) are pairwise inequivalent irreducible \( Ad L \)-submodules and \( \text{Ric} |_{p \times p} \) is an \( Ad L \)-invariant symmetric bilinear form, we have that \( \text{Ric} |_{p \times p} \) is diagonal, i.e.,

\[ \text{Ric} |_{p \times p} = a_1 B |_{p_1 \times p_1} + \ldots + a_s B |_{p_s \times p_s} , \]

for some constants \( a_1, \ldots, a_s \). In particular, we have \( \text{Ric}(p_a, p_b) = 0 \), for every \( a, b = 1, \ldots, s \) such that \( a \neq b \). Hence, \( \text{Ric} |_{p \times p} \) is determined by elements \( \text{Ric}(X, X) \) with \( X \in p_a, a = 1, \ldots, s \).

By Lemma 1.6 we obtain that only \( Q_{X \cdot X}^{n_j} = \text{Kill}(C_{n_j} X, X) \) and \( Q_{X \cdot X}^{p_b \cdot p_c} \) are non-zero. Therefore, by Theorem 1.2 we obtain that

\[ \text{Ric}(X, X) = \frac{1}{2} \sum_{j,k=1}^{s} \left( \frac{\lambda_k}{\lambda_j} - \frac{\lambda_j^2}{2\lambda_j \lambda_k} \right) Q_{X \cdot X}^{p_j \cdot p_k} + \frac{1}{2} \sum_{j=1}^{n} \left( 1 - \frac{\lambda_j^2}{2\mu_j^2} \right) Q_{X \cdot X}^{n_j} - \frac{1}{2} \text{Kill}(X, X) . \]

We have

\[ \sum_{j=1}^{m} Q_{X \cdot X}^{n_j} = \sum_{j=1}^{m} \text{Kill}(C_{n_j} X, Y) = \text{Kill}(C_n X, X) = \text{Kill}(X, X) - \text{Kill}(C_p X, X) = \text{Kill}(X, X) - \text{Kill}_l(X, X) . \]

Hence we can rewrite \( \text{Ric}(X, X) \) as follows:

\[ \frac{1}{2} \sum_{a,b=1}^{s} \left( \frac{\lambda_b}{\lambda_c} - \frac{\lambda_c^2}{2\lambda_c \lambda_b} \right) Q_{X \cdot X}^{p_p \cdot p_b} - \frac{1}{2} \text{Kill}_t(X, X) - \frac{1}{2} \sum_{j=1}^{n} \frac{\lambda_j^2}{2\mu_j^2} \text{Kill}(C_{n_j} X, X). \]

As we saw in the proof of Lemma 1.7 the summand (1) is just \( \text{Ric}^F(X, X) = q_a B(X, X) \).

Furthermore, since \( \text{Kill}(C_{n_j} \cdot, \cdot) = Q_{X \cdot X}^{n_j} \) are \( Ad L \)-invariant symmetric bilinear maps (Lemma 1.4) and \( p_a \) is \( Ad L \)-irreducible, there are constants \( c_{n_j, a} \) such that

\[ \text{Kill}(C_{n_j} \cdot, \cdot) |_{p_a \times p_a} = c_{n_j, a} \text{Kill} |_{p_a \times p_a} . \]
Therefore,

\[ \text{Ric}(X, X) = \left(q_a + \frac{1}{4} \sum_{j=1}^{n} \frac{\lambda_j^2}{\mu_j^2} c_{n_j,a} \right) B(X, X). \]

\[ \square \]

1.2.3 The Ricci Curvature in the Horizontal Direction

We obtain the Ricci curvature of an adapted metric \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) in the horizontal direction \( n \). We recall that \( n \) decomposes into the direct sum of the pairwise inequivalent irreducible \( Ad K \)-submodules \( n_1, \ldots, n_n \) and, as explained above, \( g_M \) is induced by the scalar product (1.33)

\[ (\bigoplus_{a=1}^{s} \lambda_a B |_{p_a \times p_a}) \oplus (\bigoplus_{k=1}^{n} \mu_k B |_{n_k \times n_k}) \]

and \( g_N \) is the projection of \( g_M \) onto the base space, i.e.,

\[ g_N = g_N(\mu_1, \ldots, \mu_n). \]

**Lemma 1.8.** Let \( X \in n_k \) and \( Y \in m \).

(i) \( Q^{n_j,p_a}_{X,Y} = Q^{n_a,p_j}_{X,Y} = 0 \), for \( j \neq k \);

(ii) \( Q^{n_j,p_k}_{X,Y} = Q^{n_k,p_a}_{X,Y} = \text{Kill}(C_{p_a} X, Y) \);

(iii) \( Q^{n_k,p_a}_{X,Y} = 0 \), for \( i, j = 1, \ldots, s \).

**Proof:** We have \( ad_X p_a \subset n_k \perp p_j, \) for \( j \neq k \). Thus, \( Q^{n_j,p_a}_{X,Y} = 0 \), for \( j \neq k \) and \( Q^{n_j,p_k}_{X,Y} = 0 \), for \( i, j = 1, \ldots, s \), from Lemma 1.5. Also from \( ad_X p_a \subset n_k \) we deduce that \( Q^{n_k,p_a}_{X,Y} = \text{Kill}(C_{p_a} X, Y) \). From Lemma 1.4 (ii), we also obtain \( Q^{n_j,p_a}_{X,Y} = 0 \), for \( j \neq k \) and \( Q^{n_k,p_a}_{X,Y} = Q^{n_k,p_a}_{X,Y} = \text{Kill}(C_{p_a} X, Y) \), for \( j = k \).

\[ \square \]

**Lemma 1.9.** The Ricci curvature of \( g_N = g_N(\mu_1, \ldots, \mu_n) \) is of the form \( \text{Ric}^N = \bigoplus_{k=1}^{n} r_k B |_{n_k \times n_k} \), where

\[ r_k = \frac{1}{2} \sum_{j=1}^{n} \left( \frac{\mu_j^2}{2 \mu_i \mu_j} - \frac{\mu_i}{\mu_j} \right) r_{j}^{ji} + \frac{1}{2} \]

and the constants \( r_{j}^{ji} \) are such that

\[ Q^{n_j,n_i}_{n_k \times n_k} = r_k^{ji} \text{Kill} |_{n_k \times n_k}. \]

In particular, \( \text{Ric}^N(n_k, n_j) = 0 \), if \( k \neq j \).
Proof: The metric \( g_N \) is induced by an \( Ad K \)-invariant scalar product on \( \mathfrak{n} \). Hence, \( \text{Ric}^N \) is an \( Ad K \)-invariant symmetric bilinear form on \( \mathfrak{n} \). Since the subspaces \( \mathfrak{n}_j, j = 1, \ldots, n \), are irreducible pairwise inequivalent \( Ad K \)-submodules, we may write

\[
\text{Ric}^N = \oplus_{k=1}^m r_k B \mid_{\mathfrak{n}_k \times \mathfrak{n}_k},
\]

for some constants \( r_1, \ldots, r_n \). It is clear that \( \text{Ric}^N(\mathfrak{n}_k, \mathfrak{n}_j) = 0 \), if \( k \neq j \).

It follows from Theorem 1.2 that, for every \( X \in \mathfrak{n}_k \),

\[
\text{Ric}^N(X, X) = \frac{1}{2} \sum_{j, i=1}^n \left( \frac{\mu_i^2}{\mu_j} - \frac{\mu_j^2}{2\mu_i\mu_j} \right) Q_{XX}^{\mathfrak{n}_n} - \frac{1}{2} \text{Kill}(X, X).
\]

Since each subspace \( \mathfrak{n}_j \) is \( Ad K \)-invariant, the bilinear maps \( Q_{XX}^{\mathfrak{n}_n} \) are \( Ad K \)-invariant symmetric bilinear maps (Lemma 1.4). Hence, by irreducibility of the \( \mathfrak{n}_k \)'s it follows that \( Q_{XX}^{\mathfrak{n}_n} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_{ji}^k \text{Kill} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} \), for some constants \( r_{ji}^k \), \( j, i, k = 1, \ldots, n \).

By definition of the \( r_k \)'s it must be

\[
r_k = \frac{1}{2} \sum_{j, i=1}^n \left( \frac{\mu_i^2}{\mu_j} - \frac{\mu_j^2}{2\mu_i\mu_j} \right) r_{ji}^k + \frac{1}{2}.
\]

□

Proposition 1.2. Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \). For every \( k, j = 1, \ldots, n \), such that \( j \neq k \), \( \text{Ric}(\mathfrak{n}_k, \mathfrak{n}_j) = 0 \). For every \( X \in \mathfrak{n}_k \),

\[
\text{Ric}(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^s \lambda_a B(C_{pa} X, X) + r_k B(X, X),
\]

where, for every \( a = 1, \ldots, s \), \( C_{pa} \) is the Casimir operator of \( p_a \) with respect to \( \text{Kill} \),

\[
r_k = \frac{1}{2} \sum_{j, i=1}^n \left( \frac{\mu_i^2}{\mu_j} - \frac{\mu_j^2}{2\mu_i\mu_j} \right) r_{ji}^k + \frac{1}{2}
\]

and the constants \( r_{ji}^k \) are such that

\[
Q_{XX}^{\mathfrak{n}_n} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k} = r_{ji}^k \text{Kill} \mid_{\mathfrak{n}_k \times \mathfrak{n}_k}.
\]

Proof: Let \( X \in \mathfrak{n}_k \) and \( Y \in \mathfrak{n}_{k'} \). By Lemma 1.8 we have \( Q_{XY}^{pa} = 0 \), for every \( a, b = 1, \ldots, s \). Also, \( Q_{XY}^{pa} = Q_{XY}^{pa} = 0 \), if \( j \neq k, k' \). Therefore, it follows from Theorem 1.2 and Lemma 1.9 that, if \( k \neq k' \), then
Ric\((X, Y) = \frac{1}{2} \sum_{j,i=1}^{n} \left( \frac{\mu_i}{\mu_j} - \frac{\mu_k \mu_k'}{2 \mu_i \mu_j} \right) Q_{n,n}^{j,i} - \frac{1}{2} \text{Kill}(X, Y) = Ric^N(X, Y) = 0.\)

Hence, \(Ric\)|_{n×n} is determined by elements \(Ric(X, X)\) with \(X \in n_k, k = 1, \ldots, n\). For \(X \in n_k\), by Theorem 1.2 we get

\[
Ric(X, X) = \frac{1}{2} \sum_{k=1}^{s} \left( \frac{\mu_k}{\lambda_a} - \frac{\mu_k^2}{2 \mu_k \lambda_a} \right) Q^{p_a}_{n,n} + \frac{1}{2} \sum_{a=1}^{s} \left( \frac{\lambda_a}{\mu_k} - \frac{\mu_k^2}{2 \mu_k \lambda_a} \right) Q^{n_{p_a}}_{X,X} + Ric^N(X, X).
\]

From Lemma 1.8 we know that \(Q^{n_{p_a}}_{X,X} = Q^{n_{p_a}}_{X,X} = \text{Kill}(C_{p_a} X, X)\). Hence, we simplify the expression above obtaining

\[
Ric(X, X) = \frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu_k} \text{Kill}(C_{p_a} X, X) + Ric^N(X, X).
\]

Finally, since \(Ric^N(X, X) = r_k B(X, X)\), using the notation of Lemma 1.9 we conclude that

\[
Ric(X, X) = -\frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu_k} B(C_{p_a} X, X) + r_k B(X, X).
\]

\(\square\)

### 1.2.4 The Ricci Curvature in the Mixed Direction

In the previous two sections we determined the Ricci curvature of

\(g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)\)

on the directions of \(p\) and \(n\). Here we obtain the Ricci curvature in the direction of \(p \times n\).

**Proposition 1.3.** Let \(g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)\) be an adapted metric on \(M\). For every \(X \in p_a\) and \(Y \in n_k\),

\[
Ric(X, Y) = \frac{\lambda_a \mu_k}{4} \sum_{j=1}^{n} \frac{B(C_{n_j} X, Y)}{\mu_j^2},
\]

where, for every \(j = 1, \ldots, n\), \(C_{n_j}\) is the Casimir operator of \(n_j\) with respect to \(\text{Kill}\).
Proof: For \( X \in \mathfrak{p} \) we know from Lemma 1.6 that \( Q^{n_p}_{XY} = Q^{p_n}_{XY} = 0 \), for every \( a = 1, \ldots, s, \) \( j = 1, \ldots, n \), and \( Q^{n_n}_{XY} = 0 \), if \( i \neq j, \) \( i, j = 1, \ldots, n \), whereas \( Q^{n_j}_{XY} = \text{Kill}(C_n, X, Y), j = 1, \ldots, n \). Moreover, for \( Y \in \mathfrak{n}_k \), since \( ad_X ad_Y \mathfrak{p} \subset \mathfrak{n}_k \perp \mathfrak{p} \), from Lemma 1.6 we also obtain that \( Q^{p_p}_{XY} = 0 \), for every \( b, c = 1, \ldots, s \). Therefore, only \( Q^{n_j}_{XY} = \text{Kill}(C_n, X, Y), j = 1, \ldots, n \), may not be zero. Furthermore, \( \text{Kill}(X, Y) = 0 \). Hence, from Theorem 1.2 we get

\[
\text{Ric}(X, Y) = \frac{1}{2} \sum_{j=1}^{n} \left( 1 - \frac{\lambda_a \mu_k}{\mu_j^2} \right) Q^{n_j}_{XY} = \frac{1}{2} \sum_{j=1}^{n} \left( 1 - \frac{\lambda_a \mu_k}{\mu_j^2} \right) \text{Kill}(C_n, X, Y).
\]

On the other hand,

\[
\sum_{j=1}^{n} \text{Kill}(C_n, X, Y) = \text{Kill}(C_n X, Y) = \text{Kill}(X, Y) - \text{Kill}(C t X, Y) = 0,
\]

since \( C t \mathfrak{p} \subset \mathfrak{k} \perp \mathfrak{n} \). Therefore,

\[
\text{Ric}(X, Y) = -\frac{\lambda_a \mu_k}{4} \sum_{j=1}^{n} \frac{\text{Kill}(C_n, X, Y)}{\mu_j^2}.
\]

\[\square\]

1.2.5 Necessary Conditions for the Existence of an Adapted Einstein Metric

From the expressions obtained previously for the Ricci curvature in the horizontal direction and in the direction of \( \mathfrak{p} \times \mathfrak{n} \) we obtain two necessary conditions for the existence of an adapted Einstein metric on \( M \). These are restrictions on Casimir operators and shall be extremely useful in the chapters ahead.

Corollary 1.4. Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \). If \( g_M \) is Einstein, then the operator \( \sum_{a=1}^{s} \lambda_a C_{p_a} |_{\mathfrak{n}_k} \) is scalar.

Proof: Let \( g_M \) be an adapted metric as defined in (1.33). If \( g_M \) is Einstein with Einstein constant \( E \), then, \( \text{Ric} |_{\mathfrak{n} \times \mathfrak{n}} = E <, > |_{\mathfrak{n} \times \mathfrak{n}} \) and thus \( \text{Ric} |_{\mathfrak{n} \times \mathfrak{n}} \) is \( \text{Ad} K \)-invariant. Therefore, by Proposition 1.2, we conclude that \( \sum_{a=1}^{s} \lambda_a C_{p_a} |_{\mathfrak{n}} \) has to be \( \text{Ad} K \)-invariant. Hence, \( \sum_{a=1}^{s} \lambda_a C_{p_a} |_{\mathfrak{n}_k} \) is scalar, by irreducibility of \( \mathfrak{n}_k \).

\[\square\]

Corollary 1.5. Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \). The orthogonality condition \( \text{Ric}(\mathfrak{p}, \mathfrak{n}) = 0 \) holds if and only if

\[
\sum_{j=1}^{n} \frac{1}{\mu_j^2} C_{n_j}(\mathfrak{p}) \subset \mathfrak{k}.
\]

Moreover, if \( g_M \) is Einstein, then (1.36) holds.
Proof: Let $g_M$ be an adapted metric of the form $g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$. From Proposition 1.3 we obtain that $Ric(p, n) = 0$ if and only if, for every $X \in p_a$ and $Y \in n_b$,

$$K\text{ill} \left( \sum_{j=1}^{n} \frac{C_{n_j}}{\mu_j^2} X, Y \right) = 0.$$  

This holds if only if $\sum_{j=1}^{n} \frac{C_{n_j}}{\mu_j^2} X \subset \mathfrak{k}$, for every $X \in p$.

If $g_M$ is Einstein with Einstein constant $E$, then $Ric(p, n) = E < p, n > = 0$. This shows the last assertion of the Corollary.

□

The two previous Corollaries may be restated in the following way, which emphasizes the fact that the two necessary conditions obtained for existence of an Einstein adapted metric are just algebraic conditions on the Casimir operators of the submodules $p_a$ and $n_k$.

**Corollary 1.6.** If there exists on $M$ an Einstein adapted metric, then there are positive constants $\lambda_1, \ldots, \lambda_s$ such that the operator $\sum_{a=1}^{s} \lambda_a C_{p_a} |_{n_k}$ is scalar. Furthermore, if $g_N$ is not the standard metric, then there are positive constants $\nu_1, \ldots, \nu_n$, not all equal, such that

$$\sum_{j=1}^{n} \nu_j C_{n_j}(p) \subset \mathfrak{k}.$$  

Proof: The assertions follow from Corollaries 1.4 and 1.5. In 1.5 we set $\nu_k = 1/\mu_k^2$. Hence, $\nu_1 = \ldots = \nu_n$ occurs when $g_N$ is the standard metric. Moreover, if $\nu_1 = \ldots = \nu_n$, the inclusion in 1.5 is equivalent to $C_n(p) \subset \mathfrak{k}$, which always holds since $C_n = Id - C_{\mathfrak{k}}$ and $C_{\mathfrak{k}}$ maps $p$ into $\mathfrak{k}$. So we obtain a condition on the $C_{n_j}$’s only when $g_N$ is not standard.

□
CHAPTER 2

As in Chapter 1, we consider a homogeneous fibration $F \to M \to N$, for $M = G/L$, $N = G/K$ and $F = K/L$, where $G$ is a compact connected semisimple Lie group and $L \subset K \subset G$ are connected closed non-trivial subgroups, and an adapted metric $g_M$ on $M$. We consider some particular cases by imposing restrictions on the metric $g_M$, which shall lead to simpler expressions of the Ricci curvature and thus allow us to determine further conditions for the existence of an Einstein adapted metric. Unless otherwise stated we follow the notation used in Chapter 1. Thus, as before, $p_1, \ldots, p_s$ are the irreducible pairwise inequivalent $Ad L$-submodules of $p$, the tangent space to the fiber, and $n_1, \ldots, n_n$ are the irreducible pairwise inequivalent $Ad K$-submodules of $n$, the tangent space to the base. An adapted metric $g_M$ on $M$ is written as

$$g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$$

meaning that $g_M$ is induced by the scalar product

$$(\oplus_{a=1}^s \lambda_a B |_{p_a \times p_a}) \oplus (\oplus_{k=1}^n \mu_k B |_{n_k \times n_k}),$$

on the tangent space $m = p \oplus n$ of $M$. We assume that $M$ has simple spectrum as in Section 1.2.1.

2.1 Riemannian Fibrations with Normal Fiber

In this section we consider an adapted metric $g_M$ whose restriction to the fiber, $g_F$, is a multiple of the Killing form of $g$. Hence, we have

$$g_M = g_M(\underbrace{\lambda, \ldots, \lambda}_s; \mu_1, \ldots, \mu_n) \quad (2.1)$$

and

$$g_F = g_F(\underbrace{\lambda, \ldots, \lambda}_s) \quad (2.2)$$

by setting $\lambda_1 = \ldots = \lambda_s = \lambda$ in (1.33) and (1.34). Clearly, when equipped with $g_F$, $F$ becomes a normal Riemannian manifold. In particular, if the Killing form
of \( \mathfrak{t} \) is a multiple of the Killing form of \( \mathfrak{g} \), then \( F \) is a standard Riemannian manifold. This shall be the case when, for instance, \( \mathfrak{p} \) is irreducible, but it will not be the case in general.

**Proposition 2.1.** Let \( g_M \) be an adapted metric on \( M \) of the form

\[
g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n).
\]

The Ricci curvature of \( g_M \) is as follows:

(i) For every \( X \in \mathfrak{p}_a \),

\[
Ric(X, X) = \left( q_a + \frac{\lambda^2}{4} \sum_{j=1}^{n} \frac{c_{n,a}}{\mu_j^2} \right) B(X, X),
\]

with

\[
q_a = \frac{1}{2} \left( c_{l,a} + \gamma_a \right),
\]

where \( c_{l,a} \) is the eigenvalue of the Casimir operator of \( l \) on \( \mathfrak{p}_a \), \( \gamma_a \) is defined by

\[
Kill_{\mathfrak{t}} |_{\mathfrak{p}_a \times \mathfrak{p}_a} = \gamma_a Kill |_{\mathfrak{p}_a \times \mathfrak{p}_a}
\]

and \( c_{n,j,a} \) is given by

\[
Kill(\mathbf{C}_{n,j} \cdot, \cdot) |_{\mathfrak{p}_a \times \mathfrak{p}_a} = c_{n,j,a} Kill |_{\mathfrak{p}_a \times \mathfrak{p}_a},
\]

where \( \mathbf{C}_{n,j} \) is the Casimir operator of \( \mathfrak{n}_j \) with respect to \( \text{Kill} \).

(ii) For every \( X \in \mathfrak{n}_k \),

\[
Ric(X, X) = -\frac{\lambda}{2\mu_k} B(C_{p}X, X) + r_k B(X, X),
\]

where \( r_k \) is as defined in Lemma 1.9.

(iii) For every \( X \in \mathfrak{p}_a \) and \( Y \in \mathfrak{n}_k \),

\[
Ric(X, Y) = -\frac{\lambda \mu_k}{4} \sum_{j=1}^{n} \frac{B(C_{n,j}X, Y)}{\mu_j^2};
\]

(iv) \( Ric(\mathfrak{p}_a, \mathfrak{p}_b) = 0 \), for every \( a, b = 1, \ldots, s \) such that \( a \neq b \), and \( Ric(\mathfrak{n}_i, \mathfrak{n}_j) = 0 \), for every \( i, j = 1, \ldots, n \) such that \( i \neq j \).

**Proof:** (i) By Corollary 1.2 if \( g_F \) is a multiple of \( B \), then we obtain that

\[
Ric^F(X, X) = -\frac{1}{2} \left( \frac{1}{2} + c_{l,a} \right) Kill_{\mathfrak{t}}(X, X),
\]

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for $X \in p_a$, where $c'_{la}$ is the eigenvalue of the Casimir operator of $l$ with respect to the Killing form of $t$ on $p_a$. Clearly,
\[
c'_{la} \text{Kill}_t(X, X) = \text{Kill}_t(C'_l X, X) \\
= \text{tr}(ad^2_{X} |_{p_a}) \\
= \text{Kill}(C_l X, X) \\
= c_{la} \text{Kill}(X, X).
\]

By recalling that $\text{Kill}_t |_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a}$, we write $\text{Ric}^F = \frac{1}{2} \left( \frac{\gamma_a}{2} + c_{la} \right) B$ and by following the notation in Lemma 1.7 we have
\[
q_a = \frac{1}{2} \left( \frac{\gamma_a}{2} + c_{la} \right).
\]
The result then follows from this and Proposition 1.1.

(ii) follows directly from Proposition 1.2 by observing that $\sum_{a=1}^s C_{p_a} = C_p$.

(iii) The Ricci curvature in the direction $p \times n$ essentially remains unchanged; the expression given is just that of Proposition 1.3 after replacing $\lambda_1, \ldots, \lambda_s$ by $\lambda$.

(iv) These orthogonality conditions are satisfied by any adapted metric on $M$ and were shown to hold in Propositions 1.1 and 1.2. □

**Proposition 2.2.** Let $g_M(\lambda, \ldots, \lambda; \mu_1, \ldots, \mu_n)$ be any adapted metric on $M$ and suppose that $p_1, \ldots, p_s$ pairwise commute, i.e., $[p_a, p_b] = 0$ if $a \neq b$. For every $X \in p_a$,
\[
\text{Ric}(X, X) = \left( q_a + \frac{\lambda_a^2}{4} \sum_{j=1}^n \frac{c_{n_j, a}}{\mu_j^2} \right) B(X, X),
\]
where all the constants are as in Proposition 2.1.

**Proof:** This follows from the fact that, in this case, Corollary 1.3 gives also
\[
\text{Ric}^F(X, X) = -\frac{1}{2} \left( \frac{1}{2} + c'_{la} \right) \text{Kill}_t(X, X),
\]
as in the previous proof. Hence, the expression for $\text{Ric}(X, X)$ is exactly the same obtained above for (i) in Proposition 2.1. □

**Corollary 2.1.** If there exists on $M$ an Einstein adapted metric of the form $g_M(\lambda, \ldots, \lambda; \mu_1, \ldots, \mu_n)$, then $C_p$ and $C_l$ are scalar on $n_k$, $k = 1, \ldots, n$. 

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Proof: Since $\sum_{a=1}^{s} C_{p_a} = C_p$, the necessary condition for $g_M$ to be Einstein given in Corollary 1.4 translates into the condition that $C_p$ is scalar on $n_j$, if $\lambda_1 = \ldots = \lambda_s = \lambda$. We have that $C_t = C_p + C_l$ is scalar on $n_j$, since $n_j$ is irreducible as a $K$-module. Then $C_p$ is scalar on $n_j$ if and only if $C_l$ is.

$\square$

2.2 Riemannian Fibrations with Standard Base

In this section we consider an adapted metric $g_M$ whose projection onto the base space, $g_N$, is a multiple of the Killing form of $g$. Hence, we have

$$g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu, \ldots, \mu)$$

and

$$g_N = g_N(\mu, \ldots, \mu)$$

by setting $\mu_1 = \ldots = \mu_n = \mu$ in (1.33) and (1.34). In this particular case, when equipped with $g_N$, $N$ is a standard Riemannian manifold.

**Proposition 2.3.** Let $g_M$ be an adapted metric on $M$ of the form

$$g_M(\lambda_1, \ldots, \lambda_s; \mu, \ldots, \mu).$$

The Ricci curvature of $g_M$ is as follows:

(i) For every $X \in p_a$,

$$Ric(X, X) = \left(q_a + \frac{\lambda_a^2}{4\mu^2}(1 - \gamma_a)\right) B(X, X),$$

where $q_a$ and $\gamma_a$ are as defined in Lemma 1.7, i.e., they are defined by the identities

$$\text{Kill}_t|_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a} \quad \text{and} \quad \text{Ric}^F |_{p_a \times p_a} = q_a B |_{p_a \times p_a};$$

(ii) For every $X \in n_k$,

$$Ric(X, X) = -\frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu} B(C_{p_a}X, X) + r_k B(X, X),$$

with

$$r_k = \frac{1}{2} \left(\frac{1}{2} + c_{t,k}\right),$$

where $c_{t,k}$ is the eigenvalue of the Casimir operator $C_t$ on $n_k$;

(iii) $Ric(p, n) = 0$;

(iv) $Ric(p_a, p_b) = 0$, for every $a, b = 1, \ldots, s$ such that $a \neq b$, and $Ric(n_i, n_j) = 0$, for every $i, j = 1, \ldots, n$ such that $i \neq j$.

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Proof: (i) From the fact that \( \gamma + \sum_{j=1}^{n} c_{nj,a} = 1 \), we obtain

\[
\sum_{j=1}^{n} \frac{\lambda_a^2}{\mu^2} c_{nj,a} = \frac{\lambda_a^2}{\mu^2} (1 - \gamma).
\]

The required expression follows immediately from Proposition 1.1.

(ii) From Corollary 1.2 we obtain that \( r_k = \frac{1}{2} \left( \frac{1}{2} + c_{tk,k} \right) \), where \( r_k \) is as defined in Lemma 1.9. The expression then follows from Proposition 1.2.

(iii) By using the fact that \( C_n = \sum_{j=1}^{n} C_{nj} \), from Proposition 1.3 it follows that

\[
\text{Ric}(X,Y) = \frac{\lambda_a}{4\mu} B(C_n X,Y),
\]

for every \( X \in p_a \) and \( Y \in n_k \). Moreover, since \( C_n = C_g - C_t = Id - C_t \) and \( C_t(p) \subset \mathfrak{t} \), we have that \( C_n(X) \in \mathfrak{t} \) is orthogonal to \( Y \in n \) with respect to \( B \). Hence, \( \text{Ric}(X,Y) = 0 \).

(iv) these orthogonality conditions are simply those in Propositions 1.1 and 1.2.

\square

**Corollary 2.2.** Let \( g_M \) be any adapted metric on \( M \) and suppose that \( n_1, \ldots, n_n \) pairwise commute, i.e., \( [n_j, n_k] = 0 \), for \( k \neq j \). Then, for every \( X \in n_k \),

\[
\text{Ric}(X,X) = -\frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu_k} B(C_p a X, X) + r_k B(X,X),
\]

where \( r_k = \frac{1}{2} \left( \frac{1}{2} + c_{tk,k} \right) \) and \( c_{tk,k} \) is the eigenvalue of the Casimir operator \( C_t \) on \( n_k \).

Proof: The proof is immediate by using Corollary 1.3 and Proposition 1.2

\square

### 2.3 Binormal Riemannian Fibrations

A \( G \)-invariant metric \( g_M \) on \( M \) of the form

\[
g_M = (\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)
\]

is called **binormal**. That is, a binormal metric is induced by the scalar product

\[
\lambda B \mid_{p \times p} \oplus \mu B \mid_{n \times n}
\]

on \( m \). The fibration \( F \to M \to N \) is then called a **binormal Riemannian fibration**. Clearly, a binormal metric projects onto an invariant metric on the
base space $N$ and thus it is an adapted metric. For a binormal metric $g_M$, both $g_N$ and $g_F$ are multiples of the Killing form of $g$

$$g_F = (\lambda, \ldots, \lambda) \quad \text{and} \quad g_N = (\mu, \ldots, \mu)$$

and thus $F$ is a normal Riemannian manifold, which is standard if $\text{Kill}_k$ is a multiple of $\text{Kill}$, and $N$ is a standard Riemannian manifold.

In this Section we obtain the Ricci curvature of a binormal metric $g_M$ on $M$ and conditions for such a metric to be Einstein. As we shall see, the conditions for the existence of an Einstein binormal metric translate in very simple conditions on the Casimir operators of $k$, $l$ and $p_a$, $a = 1, \ldots, s$. The results that we found in Sections 2.1 and 2.2 yield the following description of the Ricci curvature:

**Corollary 2.3.** Let $g_M = g_M(\lambda, \ldots, \lambda; \mu, \ldots, \mu)$ be a binormal metric on $M$.

(i) For every $X \in p_a$,

$$\text{Ric}(X, X) = \left( q_a + \lambda^2 \frac{1}{4\mu^2} (1 - \gamma_a) \right) B(X, X),$$

where $q_a = \frac{1}{2} \left( \frac{c_{k,a}}{2} + c_{l,a} \right)$, $c_{l,a}$ is the eigenvalue of $C_l$ on $p_a$ and $\gamma_a$ is determined by

$$\text{Kill}_k |_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a};$$

(ii) For every $Y \in n_j$,

$$\text{Ric}(Y, Y) = -\lambda \frac{1}{2\mu} B(C_p Y, Y) + r_j B(Y, Y),$$

where $r_j = \frac{1}{2} \left( \frac{1}{2} + c_{t,j} \right)$ and $c_{t,j}$ is the eigenvalue of $C_t$ on $n_j$;

(iii) Moreover, $\text{Ric}(p, n) = 0$;

(iv) $\text{Ric}(p_a, p_b) = 0$, for every $a, b = 1, \ldots, s$ such that $a \neq b$, and $\text{Ric}(n_i, n_j) = 0$, for every $i, j = 1, \ldots, n$ such that $i \neq j$.

**Proof:** For a binormal metric on $M$, $g_F$ and $g_N$ are multiples of $\text{Kill}$, so we use Propositions 2.1 and 2.3

□

**Definition 2.1.** For $i, j = 1, \ldots, n$ and $a, b = 1, \ldots, s$, we set

(i) $\delta_{ij}^k = c_{t,i} - c_{t,j}$ and $\delta_{ij}^l = c_{l,i} - c_{l,j}$;

(ii) $\delta_{ab}^k = \gamma_a - \gamma_b$ and $\delta_{ab}^l = c_{l,a} - c_{l,b}$.
Theorem 2.1. (i) If $C_p$ is not scalar on each $n_j$, then there are no binormal Einstein metrics on $M$;
(ii) Suppose that $C_p$ is scalar on each $n_j$ and write $C_p|_{n_j} = b^j Id_{n_j}$, for $j = 1, \ldots, n$. Then there is a one-to-one correspondence, up to homothety, between binormal Einstein metrics on $M$ and positive solutions of the following set of equations on the unknown $X$:

$$\delta^t_{ij}(1 - X) = \delta^t_{ij}, \text{ if } n > 1,$$

$$\delta^t_{ab} + \delta^t_{ab}X^2 = \delta^t_{ab}, \text{ if } s > 1,$$

$$(\gamma_a + 2c_{t,a})X^2 - (1 + 2c_{t,j})X + (1 - \gamma_a + 2b^j) = 0.$$  \hfill (2.9)

for every $a, b = 1, \ldots, s$ and $i, j = 1, \ldots, n$, where $c_{t,a}$ is the eigenvalue of $C_t$ on $p_a$, $\gamma_a$ is determined by $\text{Kill}_t|_{p_a \times p_a} = \gamma_a \text{Kill}_t|_{p_a \times p_a}$, $c_{t,j}$ is the eigenvalue of $C_t$ on $n_j$ and the $\delta$’s are as in Definition 2.1. If such a positive solution $X$ exists, then binormal Einstein metrics are, up to homothety, given by

$$<,> = B|_{p \times p} \oplus B|_{n \times n}.$$  \hfill (2.12)

Proof: Let $g_M(\lambda, \ldots , \lambda; \mu, \ldots , \mu)$ be a binormal metric on $M$ and

$$X = \frac{\mu}{\lambda}.$$  \hfill (2.6)

By Lemma 2.1 we have that, if $g_M$ is Einstein, then $C_p$ and $C_t$ are scalar on $n_j$, for every $j = 1, \ldots, n$. Say

$$C_p|_{n_j} = b^j Id \text{ and } C_t|_{n_j} = c_{t,j} Id.$$  \hfill (2.7)

Suppose that $g$ is Einstein with constant $E$. From Corollary 2.3 we obtain the Einstein equations

$$-\frac{\lambda}{2\mu} b^j + r_j = \mu E, \text{ for } j = 1, \ldots, n \hfill (2.10)$$

$$\frac{1}{2} \left( \frac{\gamma_a}{2} + c_{t,a} + \frac{\lambda^2}{2\mu^2}(1 - \gamma_a) \right) = \lambda E, \text{ for } a = 1, \ldots, s.$$  \hfill (2.11)
If $n > 1$, from Equation (2.10) we obtain the following:

$$\frac{\lambda}{2\mu}(b^i - b^j) = r_i - r_j, \ i, j = 1, \ldots, n.$$  \hfill (2.12)

By using Lemma 1.9 we have

$$r_i - r_j = \frac{1}{2} \left( \frac{1}{2} + c_{t,i} \right) - \frac{1}{2} \left( \frac{1}{2} + c_{t,j} \right) = \frac{1}{2}(c_{t,i} - c_{t,j}),$$

whereas

$$b^i - b^j = (c_{t,i} - c_{t,j}) - (c_{l,i} - c_{l,j}).$$

Therefore, Equation (2.12) becomes

$$-\lambda \mu \frac{c_{t,i} - c_{t,j}}{\delta_{ij}} = \left(1 - \frac{\lambda}{\mu}\right) (c_{t,i} - c_{t,j}).$$

By using the variable $X$, we rewrite the equation above as $-\frac{1}{X} \delta_{ij}^t = \left(1 - \frac{1}{X}\right) \delta_{ij}^s$, and this yields $\delta_{ij}^t = (1 - X) \delta_{ij}^s$.

Equation (2.11) may be rewritten as

$$\frac{1}{2} \left( \frac{\gamma_a}{2} + c_{l,a} \right) X + (1 - \gamma_a) \frac{1}{4X} = \mu E.$$  \hfill (2.13)

Hence, if $s > 1$, for $a, b = 1, \ldots, s$, we get

$$\frac{1}{2} \left( \frac{\gamma_a}{2} + c_{l,a} \right) X + (1 - \gamma_a) \frac{1}{4X} = \frac{1}{2} \left( \frac{\gamma_b}{2} + c_{l,b} \right) X + (1 - \gamma_b) \frac{1}{4X},$$

which yields

$$c_{l,a} - c_{l,b} = \frac{1}{2} \left( \frac{1}{X^2} - 1 \right) (\gamma_a - \gamma_b).$$

By solving this equation we obtain

$$(2\delta_{ab}^t + \delta_{ab}^s)X^2 = \delta_{ab}^s.$$  \hfill □

Finally, by using Equations (2.10) and (2.13) we obtain the equality

$$\frac{1}{2} \left( \frac{\gamma_a}{2} + c_{l,a} \right) X + (1 - \gamma_a) \frac{1}{4X} = -\frac{b^j}{2X} + \frac{1}{2} \left( \frac{1}{2} + c_{t,j} \right),$$

which rearranged gives

$$\left( \frac{\gamma_a}{2} + c_{l,a} \right) X^2 - \left( \frac{1}{2} + c_{t,j} \right) X + \frac{1}{2}(1 - \gamma_a + 2b^j) = 0.$$  \hfill □

An immediate Corollary is the following:
Corollary 2.4. Suppose that $F$ and $N$ are isotropy irreducible spaces such that $\text{dim} \ F > 1$. There exists on $M$ an Einstein adapted metric if and only if $C_p$ is scalar on $n$ and $\Delta \geq 0$, where

$$\Delta = (1 + 2c_{t,n})^2 - 4(\gamma + 2c_{l,p})(1 - \gamma + 2b),$$

$C_{t,n}$ is the eigenvalue of $C_t$ on $n$, $C_{l,p}$ is the eigenvalue of $C_l$ on $p$, $b$ is the eigenvalue of $C_p$ on $n$ and $\gamma$ is such that $\text{Kill}_t |_{p \times p} = \gamma \text{Kill} |_{p \times p}$.

If all these conditions are satisfied, then Einstein adapted metrics are, up to homothety, given by

$$g_M = B |_{p \times p} \oplus X B |_{n \times n}, \text{ where } X = \frac{1 + 2c_{t,n} \pm \sqrt{\Delta}}{2(\gamma + 2c_{l,p})}.$$  

Proof: Since $p$ is an irreducible $Ad \ L$-module and $n$ is an irreducible $Ad \ K$-module, then any adapted metric on $M$ is binormal. Hence, we use Theorem 2.1. By the irreducibility of $p$ and $n$, we have $s = 1$ and $n = 1$ and thus Einstein binormal metrics are given by positive solutions of (2.9), if $C_p$ is scalar on $n$. Hence, from Theorem 2.1 we conclude that there exists on $M$ an Einstein binormal metric if and only if $C_p$ is scalar on $n$ and $\Delta \geq 0$, where

$$\Delta = (1 + 2c_{t,n})^2 - 4(\gamma + 2c_{l,p})(1 - \gamma + 2b).$$

Since $F$ is isotropy irreducible and $\text{dim} \ F > 1$, we have $\gamma + 2c_{l,p} \neq 0$ and the polynomial in (2.9) has exactly degree two. In fact, if $\gamma + 2c_{l,p} = 0$, then $\gamma = c_{l,p} = 0$ and thus, in particular, $p$ lies in the center of $\mathfrak{k}$. But the hypothesis that $p$ is irreducible and abelian implies that $p$ is 1-dimensional which contradicts the hypothesis that $\text{dim} \ F > 1$. Therefore, $\gamma + 2c_{l,p} \neq 0$. In this case, the solutions of (2.9) are

$$X = \frac{1 + 2c_{t,n} \pm \sqrt{\Delta}}{2(\gamma + 2c_{l,p})}.$$  

□

In the case when $F$ is 1-dimensional, the fibration $M \to N$ is a principal circle bundle, since $F$ is an abelian compact connected 1-dimensional group. We recall that Einstein metrics on principal fiber bundles have been widely studied ([19], [46]) and, in particular, homogeneous Einstein metrics on circle bundles were classified McKenzie Y. Wang and Wolfgang Ziller in [46]. We revisit metrics on circle bundles by stating the following:

Corollary 2.5. Suppose that $N$ is isotropy irreducible and $F$ is isomorphic to the circle group. There exists on $M$ exactly one $G$-invariant Einstein metric, up to homothety, given by

\[ \]
\[ g_M = B \mid_{p \times p} \oplus X B \mid_{n \times n}, \text{ with } X = \frac{2 + m}{m(1 + 2c_{t,n})}, \]

where \( c_{t,n} \) is the eigenvalue of \( C_t \) on \( n \) and \( m = \dim G/K \).

**Proof:** The fact that \( p \) is 1-dimensional implies that \( p \) lies in the center of \( \mathfrak{t} \). Hence, in the notation of Corollary 2.1, \( \gamma = c_{t,p} = 0 \). On the other hand, if \( n \) is \( \text{Ad} \, K \)-irreducible then, the semisimple part of \( K \) acts transitively on \( n \). Moreover, since \( p \) lies in the center of \( \mathfrak{t} \), then the semisimple part of \( l \) coincides with the semisimple part of \( \mathfrak{t} \). Hence, \( L \) also acts transitively on \( n \) and \( n \) is an irreducible \( \text{Ad} \, L \)-module as well. Consequently, any \( G \)-invariant metric on \( M \) is adapted and moreover is binormal, by the irreducibility of \( p \) and \( n \). Furthermore, \( C_p \) must be scalar on \( n \), since \( C_t \) and \( C_l \) are scalar on \( n \). Therefore, \( G \)-invariant Einstein metrics are given by positive solutions of (2.9) in Theorem 2.1. Since \( \gamma = c_{t,p} = 0 \), (2.9) is just a degree-one equation whose solution is

\[ X = \frac{1 + 2b}{1 + 2c_{t,n}}, \]  

(2.14)

where \( b \) is the eigenvalue of \( C_p \) on \( n \). Now we compute \( b \), which is the eigenvalue of \( C_p \) on \( n \). Since \( \mathfrak{g} \) is simple we have \( \text{tr}(C_p) = \dim p = 1 \). Since \( p \) lies in the center of \( \mathfrak{t} \), \( C_p \) vanishes on \( \mathfrak{t} \) and thus \( \text{tr}(C_p) = \text{tr}(C_p \mid_n) = b \dim n = bm \). Hence,

\[ b = \frac{1}{m}. \]

By replacing \( b \) on (2.14) we obtain the desired expression for \( X \).

\[ \square \]

**Example 2.1. Circle Bundles over Compact Irreducible Hermitian Symmetric Spaces.** An application of Corollary 2.5 occurs when the base space is an irreducible symmetric space. So let us consider a fibration \( F \to M \to N \) where \( F \) is isomorphic to the circle group and \( N \) is an isotropy irreducible symmetric space. Since \( F \) is the circle group, \( p \) lies in the center of \( \mathfrak{t} \). Hence, \( K \) has one-dimensional center, since for a compact irreducible symmetric space the center of \( K \) has at most dimension 1. Moreover, in this case \( N \) is a compact irreducible Hermitian symmetric space. In particular, \( L \) must coincide with the semisimple part of \( K \). Compact irreducible Hermitian symmetric spaces \( G/K \) are classified (see e.g. [10]). All the possible \( G, K \) and \( L \) are listed in Table 2.1, together with the coefficient \( X \) of the, unique, Einstein adapted metric on \( G/L \), as in Corollary 2.5.

Finally, if \( F \) is not isotropy irreducible, under some hypothesis we can show the following:
Table 2.1: Circle Bundles Over irreducible Hermitian Symmetric Spaces.

| $G$     | $K$           | $L$                      | $X$                      |
|---------|---------------|--------------------------|--------------------------|
| $SU(n)$ | $SU(p) \times U(n-p)$ | $SU(p) \times SU(n-p)$   | $\frac{p(n-p)+1}{2p(n-p)}$ |
| $SO(2n)$ | $U(n)$       | $SU(n)$                  | $\frac{n(n-1)+2}{2n(n-1)}$ |
| $SO(n)$  | $SO(2) \times SO(n-2)$ | $SO(n-2)$                | $\frac{n-1}{n}$          |
| $Sp(n)$  | $U(n)$       | $SU(n)$                  | $\frac{n(n+1)+2}{2n(n+1)}$ |
| $E_6$    | $SO(10) \times U(1)$ | $SO(10)$                 | $\frac{17}{32}$          |
| $E_7$    | $E_6 \times U(1)$ | $E_6$                    | $\frac{14}{27}$          |

**Corollary 2.6.** Suppose $F$ is not isotropy irreducible and that there exists a constant $\alpha$ such that

$$\text{Kill}_l |_{p \times p} = \alpha \text{Kill}_t |_{p \times p}.$$  

For $a = 1, \ldots, s$, let $\gamma_a$ be the constant determined by

$$\text{Kill}_t |_{p_a \times p_a} = \gamma_a \text{Kill}_l |_{p_a \times p_a}.$$  

If for some $a, b = 1, \ldots, s$, $\gamma_a \neq \gamma_b$, then there exists a binormal Einstein metric on $M$ if and only if, for every $j = 1, \ldots, n$,

$$c_{l,j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) \left(c_{t,j} + \frac{1}{2}\right) \quad (2.15)$$

and $C_p$ is scalar on each $n_j$, where $c_{l,j}$ and $c_{t,j}$ are the eigenvalues of $C_l$ and $C_t$, respectively, on $n_j$. In this case, there is a unique binormal Einstein metric, up to homothety, given by

$$B |_{p \times p} \oplus \frac{1}{\sqrt{2\alpha + 1}} B |_{n \times n}.$$  

**Proof:** If $\text{Kill}_l |_{p \times p} = \alpha \text{Kill}_t |_{p \times p}$, then

$$c_{l,a} = \alpha \gamma_a, \text{ for every } a = 1, \ldots, s. \quad (2.16)$$

Therefore, for any $a, b = 1, \ldots, s$, if $s > 1$, $2\delta_{ab}^l + \delta_{ab}^t = (2\alpha + 1)\delta_{ab}^t$ and, thus, Equation (2.8) in Theorem 2.1 becomes

$$(2\alpha + 1)\delta_{ab}^t X^2 = \delta_{ab}^t.$$  

In particular, (2.16) implies that $c_{l,a} = 0$ if and only if $\gamma_a = 0$ (thus if $p$ has submodules where $L$ acts trivially, then $\text{Kill}_t$ vanish on those submodules and then they lie in the center of $\mathfrak{t}$. If $K$ is semisimple, then the isotropy representation
of $K/L$ is faithful). The fact that the isotropy representation of $K/L$ is not irreducible implies that $p$ decomposes as a direct sum $p_1 \oplus \ldots \oplus p_s$ with $s > 1$. For the indices for which $\gamma_a \neq \gamma_b$, we have $\delta_{ab}^s \neq 0$ and (2.17) implies that

$$X = \frac{1}{\sqrt{2\alpha + 1}}.$$  

Hence, $X = \frac{1}{\sqrt{2\alpha + 1}}$ must be a root of the polynomial in (2.9). By using the fact that $c_{t,a} = \alpha \gamma_a$ and $b_j = c_{t,j} - c_{l,j}$, simple calculations show that

$$c_{t,j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) \left(c_{t,j} + \frac{1}{2}\right).$$  

(2.18)

We observe that this condition implies (2.7) in Theorem 2.1 as we can see by the equalities below:

$$\delta_{ij}^l = c_{t,i} - c_{l,j} = \left(1 - \frac{1}{\sqrt{2\alpha + 1}}\right) \left(c_{t,i} - c_{t,j}\right) = (1 - X)\delta_{ij}^s.$$  

Hence, there is a binormal Einstein metric if and only if (2.18) is satisfied and the operator $C_k$ is scalar on $n_j$, for every $j = 1, \ldots, n$. In this case, according also to Theorem 2.1 such metric is, up to homothety, given by $B |_{p \times p} \oplus \frac{1}{\sqrt{2\alpha + 1}}B |_{n \times n}$.

$\square$

**Corollary 2.7.** Suppose $F$ is not isotropy irreducible and that there exists a constant $\alpha$ such that

$$\text{Kill}_l |_{p \times p} = \alpha \text{Kill}_t |_{p \times p}.$$  

For $a = 1, \ldots, s$, let $\gamma_a$ be the constant determined by

$$\text{Kill}_t |_{p_a \times p_a} = \gamma_a \text{Kill}_l |_{p_a \times p_a}.$$  

If for some $a, b = 1, \ldots, s$, $\gamma_a \neq \gamma_b$ and there exists on $M$ an Einstein binormal metric, then the number $\sqrt{2\alpha + 1}$ is a rational.

**Proof:** This follows from the fact that the eigenvalues of $C_t$ and $C_l$ on $n_j$ are rational numbers. Since $\mathfrak{f}$ is a compact algebra, the eigenvalue of its Casimir operator on the complex representation on $n_j^C$ is given by

$$\frac{<\lambda_j, \lambda_j + 2\delta>}{2h^*(g)} \in \mathbb{Q},$$

where $\lambda_j$ is the highest weight for $n_j^C$, $2\delta$ is the sum of all positive roots of $\mathfrak{f}$ and $h^*(g)$ is the dual Coxeter number of $g$ (17, 35). A similar formula holds for

1This condition implies that the representation of $L$ on at least one of the $p_a$’s is faithful.
and we conclude that $C_{t,j}$ and $C_{t,j}$ are rational numbers. If there exists a binormal Einstein metric on $M$, then $C_{t,j}$ and $C_{t,j}$ are related by formula (2.15) in Corollary 2.6. This implies that $\sqrt{2\alpha + 1}$ is a rational number.

2.4 Riemannian Fibrations with Einstein Fiber and Einstein Base

In this section we investigate conditions for the existence of an Einstein adapted metric $g_M = (\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ on $M$ such that $g_F$ or $g_N$ are also Einstein.

**Theorem 2.2.** Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on $M$. If $g_M$ and $g_N$ are both Einstein and $n > 1$, then

$$\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k} = \left(\frac{b^j}{b^k}\right)^{\frac{1}{2}}, \text{ for } j, k = 1, \ldots, n,$$

where $b^j$ is the eigenvalue of the operator $\sum_{a=1}^s \lambda_a C_{p_a}$ on $n_j$, for $j = 1, \ldots, n$, and the $r_j$'s are determined by $\text{Ric}^N = \oplus_{k=1}^n r_k B|_{n_k \times n_k}$ as in Lemma 1.9. Up to homothety, there exists at most one Einstein metric $g_N$ on $N$ such that the corresponding $g_M$ on $M$ is Einstein.

**Proof:** Let $g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on $M$. From Corollary 1.4 we know that if $g_M$ is Einstein, then there are constants $b^j$ such that

$$\sum_{a=1}^s \lambda_a C_{p_a}|_{n_j} = b^j \text{Id}_{n_j}.$$ We recall from Lemma 1.9 that $\text{Ric}^N = \oplus_{k=1}^n r_k B|_{n_k \times n_k}$. Hence, if $g_N$ is Einstein, then

$$\frac{r_1}{\mu_1} = \ldots = \frac{r_n}{\mu_n}.$$ (2.19)

From this equalities we obtain that

$$\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k}, \text{ for } j, k = 1, \ldots, n.$$ (2.20)

From Proposition 1.2, for $X \in n_k$, the Ricci curvature of $g_M$ is

$$\text{Ric}(X, X) = -\frac{1}{2\mu_k} \sum_{a=1}^s \lambda_a B(C_{p_a} X, X) + r_k B(X, X) = \left( -\frac{b^k}{2\mu_k} + r_k \right) B(X, X).$$
If $g_M$ is Einstein, then from the expression above we obtain the following equations

$$-\frac{b_k}{2\mu_k^2} + \frac{r_k}{\mu_k} = -\frac{b_j}{2\mu_j^2} + \frac{r_j}{\mu_j}. \quad (2.21)$$

The identities (2.19) and (2.21) imply that

$$\frac{b_k}{\mu_k^2} = \frac{b_j}{\mu_j^2}$$

and consequently, by using (2.20),

$$\left(\frac{r_j}{r_k}\right)^2 = \left(\frac{\mu_j}{\mu_k}\right)^2 = \frac{b_j}{b_k}.$$ 

Finally, we observe that although the fact that $g_N$ is Einstein implies the equalities $\frac{\mu_j}{\mu_k} = \frac{r_j}{r_k}$, there might be more than one solution for the $n$-tuples $(\mu_1, \ldots, \mu_n)$, up to scalar multiplication, since the $r_j$’s in general depend on the $\mu_i$’s. This is obvious since clearly there might be many distinct Einstein metrics on $N$, up to homothety. This is explicit in the formula given in Lemma 1.9. However, as the eigenvalues $b_j$ are independent of the constants $\mu_1, \ldots, \mu_n$, the ratios $\frac{\mu_j}{\mu_k} = \left(\frac{b_j}{b_k}\right)^{\frac{1}{2}}$ imply that there is at most one possible choice for $g_N$, up to scalar multiplication.

□

**Theorem 2.3.** Let $g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on $M$. If $g_M$ and $g_F$ are both Einstein and $s > 1$, then

$$\frac{\lambda_a}{\lambda_b} = \frac{q_a}{q_b} = \sum_{j=1}^{n} \frac{C_{nj,b}}{\mu_j^2} / \sum_{j=1}^{n} \frac{C_{nj,a}}{\mu_j^2}, \text{ for } a, b = 1, \ldots, s,$$

where $c_{nj,a}$ is such that $\text{Kill}(C_{nj}, \cdot) |_{pa \times pa} = c_{nj,a} \text{Kill} |_{pa \times pa}$, for $a = 1, \ldots, s$ and the $q_a$’s are determined by $\text{Ric}^F = \oplus_{a=1}^{s} q_a B |_{pa \times pa}$ as in Lemma 1.7. Up to scalar multiplication, there exists at most one Einstein metric $g_F$ on $F$ such that the corresponding metric $g_M$ on $M$ is Einstein.

**Proof:** The proof is similar to that of Theorem 2.2 by using Lemma 1.7 and Proposition 1.1. □

**Corollary 2.8.** Let $g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n)$ be an adapted metric on $M$. If $g_M$, $g_N$ and $g_F$ are Einstein, then

$$\frac{r_j}{r_k} = \left(\frac{b_j}{b_k}\right)^{\frac{1}{2}}, \text{ for } j, k = 1, \ldots, n,$$
and

\[ \frac{q_a}{q_b} = \sum_{j=1}^{n} \frac{C_{nj,b}}{b_j} / \sum_{j=1}^{n} \frac{C_{nj,a}}{b_j}, \quad \text{for } a, b = 1, \ldots, s, \]

where all the constants are as in Theorems 2.2 and 2.3.

Proof: Using Theorem 2.2, we write \( \mu_j^2 = \frac{b_j}{\sigma^2} \mu_1^2 \). The second formula follows immediately from this and Theorem 2.3.

\[ \square \]

Theorem 2.4. Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \). Suppose that \( g_M, g_N \) and \( g_F \) are Einstein and let \( E, E_F \) and \( E_N \) be the corresponding Einstein constants. If \( E \neq E_N \), then

\[ \mu_j = \left( \frac{b_j}{2(E_N - E)} \right)^{\frac{1}{2}}, \]

\[ \lambda_a = \frac{2(E - E_F)}{E - E_N} \left( \frac{C_{nj,a}}{b_j} \right)^{-1}, \]

where \( b^i \) is the eigenvalue of the operator \( \sum_{a=1}^{s} \lambda_a C_{pa} \) on \( n_j \) and \( c_{nj,a} \) is such that \( \text{Kill}(C_{nj}, \cdot) |_{p_a \times p_a} = c_{nj,a} \text{Kill} |_{p_a \times p_a} \).

Proof: Let \( g_M = (\lambda_1, \ldots, \lambda_s; \mu_1, \ldots, \mu_n) \) be an adapted metric on \( M \). If \( g_M, g_N \) and \( g_F \) are all Einstein, from Propositions 1.1 and 1.2 we get

\[ -\frac{1}{2} b^i + \mu_j E_N = \mu_j E, \]

from which, if \( E_N \neq E \), we deduce

\[ \mu_j^2 = \frac{b_j}{2(E_N - E)} \quad (2.22) \]

and

\[ \lambda_a E_F + \frac{\lambda_a^2 C_{nj,a}}{4 \mu_j^2} = \lambda_a E. \]

From this we get

\[ \lambda_a \frac{C_{nj,a}}{\mu_j^2} = 4(E - E_F). \quad (2.23) \]

We obtain the required formula by replacing (2.22) in the equation above.

\[ \square \]




2.5 Riemannian Fibrations with Symmetric Fiber

In this section we consider a fibration $F \to M \to N$ such that $F$ is a symmetric space and $N$ is isotropy irreducible. We specify the Ricci curvature of an adapted metric on $M$ and obtain the Einstein equations in some particular cases.

If $F = K/L$ is a symmetric space, then we consider its DeRham decomposition

$$K/L = K_0/L_0 \times K_1/L_1 \times \ldots \times K_s/L_s,$$

where $K_0$ is the center of $K$ and, for $a = 1, \ldots, s$, $K_a$ is simple. By $\mathfrak{k}_a$ and $\mathfrak{l}_a$ we denote the Lie algebras of $K_a$ and $L_a$, respectively. In particular, for $a = 1, \ldots, s$, $K_a/L_a$ is an irreducible symmetric space. Thus $p_a$ may be chosen as a symmetric reductive complement of $l_a$ in $k_a$. Since $k_a$ is simple, the Casimir operator of $k$ is scalar on $k_a$. Hence, in the equality $\text{Kill}_\mathfrak{k} |_{p_a \times p_a} = \gamma_a \text{Kill} |_{p_a \times p_a}$, the constant $\gamma_a$ is simply the eigenvalue of the Casimir operator of $\mathfrak{k}$ on $\mathfrak{k}_a$, because $\text{Kill}_\mathfrak{k} = \text{Kill}(C_\mathfrak{k}, \cdot)$. For $a = 0$ this is still true with $\gamma_0 = 0$.

**Proposition 2.4.** Suppose that $F$ is a symmetric space and $N$ is isotropy irreducible and let $g_M = (\lambda_0, \ldots, \lambda_s; \mu)$ be an adapted metric on $M$. The Ricci curvature of $g_M$ is as follows:

(i) For every $X \in p_a$, $a = 0, \ldots, s$,

$$\text{Ric}(X, X) = \left(\frac{\gamma_a}{2} + \frac{\lambda_a^2}{4\mu^2}(1 - \gamma_a)\right) B(X, X),$$

where $\gamma_a$ is the eigenvalue of $C_\mathfrak{k}$ on $\mathfrak{k}_a$;

(ii) For every $X \in n$,

$$\text{Ric}(X, X) = -\frac{1}{2} \sum_{a=1}^s \lambda_a B(C_{\mathfrak{p}_a}X, X) + r B(X, X),$$

where $r = \frac{1}{2}(\frac{1}{2} + c_{\mathfrak{k},\mathfrak{n}})$ and $c_{\mathfrak{k},\mathfrak{n}}$ is the eigenvalue of $C_\mathfrak{k}$ on $\mathfrak{n}$;

(iii) $\text{Ric}(\mathfrak{p}, \mathfrak{n}) = 0$;

(iv) For every $a, b = 0, \ldots, s$ such that $a \neq b$, $\text{Ric}(\mathfrak{p}_a, \mathfrak{p}_b) = 0$ and for every $i, j = 1, \ldots, n$ such that $i \neq j$, $\text{Ric}(\mathfrak{n}_i, \mathfrak{n}_j) = 0$.

**Proof:** Since $N$ is isotropy irreducible the expressions for the Ricci curvature of $g_M$ are given by Proposition 2.3. In particular, for $X \in \mathfrak{p}_a$,

$$\text{Ric}(X, X) = \left(q_a + \frac{\lambda_a^2}{4\mu^2}(1 - \gamma_a)\right) B(X, X).$$

If we consider the DeRham decomposition of $F$ as in (2.24), $[\mathfrak{p}_a, \mathfrak{p}_b] = 0$, for every $a \neq b$. Hence, from Proposition 2.2 we have $q_a = \frac{1}{2} \left(\frac{x_a}{2} + c_{\mathfrak{k},\mathfrak{n}}\right)$. Since $F$ is a...
symmetric space, then $\text{Kill}_1 |_{p\times p} = \frac{1}{2} \text{Kill}_1 |_{p\times p}$ and thus $C_1 |_{p\times p} = \frac{1}{2} C_1 |_{p\times p}$. Hence, $c_{t,a} = \frac{a}{2}$. Therefore $q_a = \frac{\gamma_a}{2}$.

Since $N$ is irreducible, by using Proposition 2.3, for $X \in \mathfrak{n}$, we write

$$Ric(X, X) = -\frac{1}{2} \sum_{a=1}^{s} \frac{\lambda_a}{\mu} B(C_{p_a} X, X) + r B(X, X),$$

where

$$r = \frac{1}{2} \left( \frac{1}{2} + c_{t,n} \right)$$

and $c_{t,n}$ is the eigenvalue of the Casimir operator $C_t$ on $\mathfrak{n}$.

(iii) and (iv) follow directly from Proposition 2.3 as well.

□

**Theorem 2.5.** Suppose that $F$ is a symmetric space and $N$ is an isotropy irreducible space. Moreover, suppose that $C_{p_a} |_{\mathfrak{n}} = b_a \text{Id}_{\mathfrak{n}}$, for some constants $b_a$, for every $a = 1, \ldots, s$. There exists on $M$ an Einstein adapted metric if and only if there are positive solutions of the following system of $s$ algebraic equations in the unknowns $X_1, \ldots, X_s$.

$$2\gamma_1 X_1^2 + (1 - \gamma_1) X_a - 2\gamma_a X_1 X_a^2 - (1 - \gamma_a) X_1 = 0, a = 2, \ldots, s$$

$$2 \sum_{a=1}^{s} b_a X_1 \ldots \widehat{X}_a \ldots X_s = 4r X_1 \ldots X_s + 2\gamma_1 X_1^2 X_2 \ldots X_s + (1 - \gamma_1) X_2 \ldots X_s = 0,$$

where $\gamma_a$ is the eigenvalue of $C_t$ on $p_a$, $r = \frac{1}{2} \left( \frac{1}{2} + c_{t,n} \right)$ and $c_{t,n}$ is the eigenvalue of $C_t$ on $\mathfrak{n}$. To each $s$-tuple $(X_1, \ldots, X_s)$ corresponds a family of Einstein adapted metrics on $M$ given, up to homothety, by

$$g_M = \bigoplus_{a=1}^{s} \frac{1}{X_a} B |_{p_a \times p_a} \oplus B |_{\mathfrak{n} \times \mathfrak{n}}.$$

**Proof:** Let $g_M = (\lambda_1, \ldots, \lambda_s; \mu)$ be an adapted metric on $M$. First we observe that the hypothesis $C_{p_a} |_{\mathfrak{n}} = b_a \text{Id}_{\mathfrak{n}}$, for every $a = 1, \ldots, s$, implies that $\sum_{a=1}^{s} \lambda_a C_{p_a}$ is scalar for any choice of $\lambda_a$’s. Hence, the necessary condition for the existence of an Einstein adapted metric on $M$ given by Corollary 1.4 is satisfied. Moreover, (iii) and (iv) of Proposition 2.3 imply that for $g_M$ to be Einstein, it suffices to analyze the equations

$$\text{Ric} |_{p_a \times p_a} = \lambda_a E B |_{p_a \times p_a}, a = 1, \ldots, s \quad (2.25)$$

$$\text{Ric} |_{\mathfrak{n} \times \mathfrak{n}} = \mu E B |_{\mathfrak{n} \times \mathfrak{n}} \quad (2.26)$$

$\widehat{X}_a$ means that $X_a$ does not occur in the product.
where $E$ is the Einstein constant of $g_M$.

We introduce the unknowns

$$X_a = \frac{\mu}{\lambda_a}, \ a = 1, \ldots, s.$$ 

By using $C_p \mid_n = b_a Id_n$ and the $X_a$’s, Equation (2.26) may be rewritten as

$$- \sum_{a=1}^{s} \frac{b_a}{2X_a} + r = \mu E. \quad (2.27)$$

Also, by using Proposition 2.4 and the $X_a$’s, Equation (2.25) may be rewritten as

$$\frac{\gamma_a}{2} + \frac{1 - \gamma_a}{4X_a^2} = \lambda_a E. \quad (2.28)$$

By multiplying (2.28) by $X_a$ we get

$$\frac{2\gamma_aX_a^2 + 1 - \gamma_a}{4X_a} = \mu E. \quad (2.29)$$

Therefore, the Einstein Equations are just

$$\frac{2\gamma_aX_a^2 + 1 - \gamma_a}{4X_a} = \frac{2\gamma_1X_1^2 + 1 - \gamma_1}{4X_1}, \ a = 1, \ldots, s \quad (2.30)$$

$$- \sum_{a=1}^{s} \frac{b_a}{2X_a} + r = \frac{2\gamma_1X_1^2 + 1 - \gamma_1}{4X_1}. \quad (2.31)$$

We obtain the equations stated in the theorem simply by rearranging (2.30) and (2.31). We recall that since $N$ is irreducible, we have $r = \frac{1}{2} (\frac{1}{2} + c_{t,n})$, as in Proposition 2.4 and thus $r$ does not depend on $\mu$. So $X_1, \ldots, X_s$ are actually the only unknowns of the system above.

□

**Corollary 2.9.** Suppose that $F$ and $N$ are irreducible symmetric spaces and $\dim F > 1$. There exists on $M$ an Einstein adapted metric if and only if $C_p$ is scalar on $n$ and $\Delta' \geq 0$, where

$$\Delta' = 1 - 2\gamma(1 - \gamma + 2b),$$

$\gamma$ is the eigenvalue of $C_p$ on $p$ and $b$ is the eigenvalue of $C_p$ on $n$. If these two conditions are satisfied, then Einstein adapted metrics are homothetic to $g_M = B \mid_p \otimes \mathbb{R} \mid_n$, where

$$X = \frac{1 \pm \sqrt{\Delta'}}{2\gamma}.$$
Proof: It follows from Corollary 2.4 and from the fact that, since $F$ and $N$ are irreducible symmetric spaces, $c_{t,n} = \frac{1}{2}$ and $c_{l,p} = \frac{\gamma}{2}$.

□

Corollary 2.10. Suppose that $F$ is a symmetric space and $N$ is isotropy irreducible.

(i) If $C_p$ is not scalar on $n$ or $C_t$ is not scalar on $p$, then there is no binormal Einstein metric on $M$.

(ii) Suppose that $C_p$ is scalar on $n$ and $C_t$ is scalar on $p$, and write $C_p|_n = bId_n$ and $C_t|_p = \gamma Id_p$. There is an one-to-one correspondence between binormal Einstein metrics on $M$ and positive roots of the polynomial

$$2\gamma X^2 - (1 + 2c_{t,n})X + (1 - \gamma + 2b) = 0.$$  \hspace{1cm} (2.32)

for every $a, b = 1, \ldots, s$ and $i, j = 1, \ldots, n$, where $c_{t,n}$ is the eigenvalue of $C_t$ on $n$. If such a positive solution $X$ exists, then binormal Einstein metrics are, up to homothety, given by

$$<,> = B|_{p \times p} \oplus X B|_{n \times n}.$$ 

Proof: If $F$ is a symmetric space, then $\text{Kill}_t|_{p \times p} = \alpha \text{Kill}_t|_{p \times p}$, with $\alpha = \frac{1}{2}$. The number $\sqrt{2\alpha + 1} = \sqrt{2}$ is not a rational. Hence, Corollary 2.7 implies that if there exists a binormal Einstein metric on $M$, then $\gamma_1 = \ldots = \gamma_s = \gamma$ for some constant $\gamma$, i.e., the Casimir operator of $t$ is scalar on $p$.

Moreover, we know from Theorem 2.1 that the condition that $C_p$ is scalar on $n$ is also a necessary condition for the existence of a binormal Einstein metric. The polynomial (2.32) is just (2.9) from Theorem 2.1, for $c_{l,p} = \frac{\gamma}{2}$ and $n = 1$. Also, the condition (2.8) from Theorem 2.1 is satisfied since for $\gamma_1 = \ldots = \gamma_s$, we have $\delta_{ab}^t = \delta_{ab}^t = 0$.

□

Corollary 2.11. Suppose that $F$ is a symmetric space. If there exists on $M$ a binormal Einstein metric $g_M$, then $g_F$ is Einstein. The converse holds if $C_t$ is scalar on $p$.

Proof: If $F$ is irreducible, then any metric on $F$ is Einstein. So let us suppose that $F$ is a reducible symmetric space. Then, by Corollary 2.10, the existence of a binormal Einstein metric $g_M$ on $M$ implies that $\gamma_1 = \ldots = \gamma_s = \gamma$ for some $\gamma$. From Proposition 2.1, we have $\text{Ric}^F|_{p_a \times p_a} = q_a B(X, X)$, where $q_a = \frac{1}{2} \left( \frac{\gamma_a}{2} + c_{l,a} \right) = \frac{2\gamma_a}{2} = \frac{\gamma}{2}$, for every $a = 1, \ldots, s$. Therefore, $g_F$ is Einstein with Einstein constant $E_F = \frac{\gamma}{2\gamma}$. 

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Conversely, let $g_M = (\lambda_1, \ldots, \lambda_s, \mu)$ be any Einstein adapted metric on $M$. If $C_t$ is scalar on $p$, then $\gamma_1 = \ldots = \gamma_s$. Hence, if $g_F$ is Einstein, we have from Theorem 2.3 that

$$\frac{\lambda_a}{\lambda_b} = \frac{q_a}{q_b} = \frac{\gamma_a/2}{\gamma_b/2} = 1$$

and $g_M$ is binormal.

□

Hence, binormal Einstein metrics are such that the restriction to the fiber is Einstein. As the next two results show, there might exist other Einstein adapted metrics satisfying this property.

**Corollary 2.12.** Suppose that $F$ is a symmetric space and $N$ is isotropy irreducible. Let $\gamma_a$ be the eigenvalue of $C_t$ on $p_a$, $a = 1, \ldots, s$, and $g_M$ an adapted metric on $M$. If $g_M$ and $g_F$ are both Einstein, then

$$\gamma_a = \gamma_b \text{ or } \gamma_a = 1 - \gamma_b, \ a, b = 1, \ldots, s.$$ 

*Proof:* If $F$ is a symmetric space, we have $q_a = \frac{b_a}{2}$, for every $a = 1, \ldots, s$. On the other hand, if $N$ is isotropy irreducible, then $n$ is an irreducible $Ad K$-module, and since $C_{n,a} = 1 - \gamma_a$, the identity in Theorem 2.3 becomes

$$\frac{\gamma_a/2}{\gamma_b/2} = \frac{1 - \gamma_b}{\mu^2} = \frac{1 - \gamma_a}{\mu^2}.$$ 

Hence, we obtain the equation

$$\gamma_a(1 - \gamma_a) = \gamma_b(1 - \gamma_b), \text{ for } a, b = 1, \ldots, s,$$

whose solutions are $\gamma_a = \gamma_b$ or $\gamma_a = 1 - \gamma_b$.

□

**Corollary 2.13.** Suppose that $F$ is a symmetric space such that $p = p_1 \oplus p_2$, where $p_1$ and $p_2$ are non-abelian, and $N$ is isotropy irreducible. Suppose that $C_{p_a \mid n} = b_a Id_n$, for some constants $b_a$, $a = 1, 2$. Let $\gamma_a$ be the eigenvalue of $C_t$ on $p_a$, $a = 1, 2$, and $c_{t,n}$ be the eigenvalue of $C_t$ on $n$.

If there exists on $M$ an Einstein adapted metric $g_M$ such that $g_F$ is also Einstein, then one of the following cases holds:

(i) $\gamma_2 = \gamma_1$ and $\Delta \geq 0$, where

$$\Delta = (1 + c_{t,n})^2 - 8\gamma_1(1 - \gamma_1 + 2b).$$

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If these two conditions are satisfied the metric, then \( g_M \) is the binormal metric given, up to homothety, by

\[
g_M = B |_{p \times p} \oplus X B |_{n \times n}, \text{ where } X = \frac{1 + c_{t,n} \pm \sqrt{\Delta}}{2\gamma_1}.
\]

(ii) \( \gamma_2 = 1 - \gamma_1 \) and \( D(\gamma_1) \geq 0 \), where

\[
D(\gamma_1) = 4r^2 - 4b_1\gamma_1 - 4b_2(1 - \gamma_1) - 2\gamma_1(1 - \gamma_1)
\]

and \( r = \frac{1}{2} \left( \frac{1}{2} + c_{t,n} \right) \). If these two conditions are satisfied the metric \( g_M \) is given, up to homothety, by

\[
g_M = \frac{1}{X_1} B |_{p_1 \times p_1} \oplus \frac{1}{X_2} B |_{p_2 \times p_2} \oplus B |_{n \times n},
\]

where

\[
X_2 = \frac{\gamma_1 X_1}{1 - \gamma_1} \text{ and } X_1 = \frac{2r \pm \sqrt{D(\gamma_1)}}{2\gamma_1}.
\]

Proof: First we observe that the hypothesis that \( p_1 \) and \( p_2 \) are non-abelian implies that \( \gamma_1, \gamma_2 \neq 0 \). Let \( g_M = g_M(\lambda_1, \ldots, \lambda_s, \mu) \) be an Einstein adapted metric on \( M \) such that \( g_F \) is also Einstein. Corollary 2.12 implies that either \( \gamma_2 = \gamma_1 \) or \( \gamma_2 = 1 - \gamma_1 \).

In the case \( \gamma_2 = \gamma_1 \), the statement follows from Corollaries 2.10 and 2.11. In the case \( \gamma_2 = 1 - \gamma_1 \), we obtain from Theorem 2.3, that

\[
\frac{\lambda_1}{\lambda_2} = \frac{\gamma_1/2}{\gamma_2/2} = \frac{\gamma_1}{1 - \gamma_1}.
\]

(2.33)

On the other hand, according to Theorem 2.5, an adapted Einstein metric on \( M \) corresponds to positive solutions of the equations

\[
2\gamma_1 X_1^2 X_2 + (1 - \gamma_1)X_2 - 2\gamma_2 X_1 X_2^2 - (1 - \gamma_2)X_1 = 0 \quad (2.34)
\]

\[
2b_1 X_2 + 2b_2 X_1 - 4r X_1 X_2 + 2\gamma_1 X_1^2 X_2 + (1 - \gamma_1)X_2 = 0, \quad (2.35)
\]

where \( X_a = \frac{\mu_a}{\lambda_a} \). By using the identity (2.33), we solve the system of equations (2.34) and (2.35) for \( X_2 = \frac{\gamma_1}{1 - \gamma_1} X_1 \) and \( \gamma_2 = 1 - \gamma_1 \), in order to obtain the solutions stated in (ii).

□

The following two Corollaries classify all the Einstein adapted metrics in the cases when \( \gamma_2 = \gamma_1 \) or \( \gamma_2 = 1 - \gamma_1 \). These results follow immediately from Corollary 2.13 and from solving the equations (2.34) and (2.35) for \( \gamma_2 = \gamma_1 \) or \( \gamma_2 = 1 - \gamma_1 \), respectively.

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Corollary 2.14. Suppose that $F$ is a symmetric space such that $p = p_1 \oplus p_2$, where $p_1$ and $p_2$ are non-abelian, and $N$ is isotropy irreducible. Suppose that $C_{p_a} |_{n} = b_a \text{Id}_n$, for some constants $b_a$, $a = 1, 2$, and let $\gamma_a$ be the eigenvalue of $C_t$ on $p_a$, $a = 1, 2$, and $c_{t,n}$ the eigenvalue of $C_t$ on $n$.

Suppose that $\gamma_2 = \gamma_1$, i.e., $C_t$ is scalar on $p$. If there exists on $M$ an Einstein adapted metric $g_M$, then one of the following two cases holds:

(i) $g_F$ is also Einstein and $g_M$ is a binormal metric given by Corollary 2.13 (i).

(ii) $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2(1 - \gamma_1) - 2\gamma_1(2b_2 + 1 - \gamma_1)(2b_1 + 1 - \gamma_1)$$

and $r = \frac{1}{2} \left( \frac{1}{2} + c_{t,n} \right)$. The metric $g_M$ is given, up to homothety, by

$$g_M = \frac{1}{X_1} B |_{p_1 \times p_1} \oplus \frac{1}{X_2} B |_{p_2 \times p_2} \oplus B |_{n \times n},$$

where

$$X_2 = \frac{1 - \gamma_1}{2\gamma_1 X_1} \quad \text{and} \quad X_1 = \frac{2r(1 - \gamma_1) \pm \sqrt{(1 - \gamma_1)D(\gamma_1)}}{2\gamma_1(2b_2 + 1 - \gamma_1)}.$$

In this second case, $g_F$ is not Einstein and $g_M$ is not binormal.

Corollary 2.15. Suppose that $F$ is a symmetric space such that $p = p_1 \oplus p_2$, where $p_1$ and $p_2$ are non-abelian, and $N$ is isotropy irreducible. Suppose that $C_{p_a} |_{n} = b_a \text{Id}_n$, for some constants $b_a$, $a = 1, 2$, and let $\gamma_a$ be the eigenvalue of $C_t$ on $p_a$, $a = 1, 2$, and $c_{t,n}$ the eigenvalue of $C_t$ on $n$.

Suppose that $\gamma_2 = 1 - \gamma_1$. If there exists on $M$ an Einstein adapted metric $g_M$, then one of the following two cases holds:

(i) $g_F$ is also Einstein and $g_M$ is the metric given by Corollary 2.13 (ii).

(ii) $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2 - 2(2b_2 + \gamma_1)(2b_1 + 1 - \gamma_1)$$

and $r = \frac{1}{2} \left( \frac{1}{2} + c_{t,n} \right)$. The metric $g_M$ is given, up to homothety, by

$$g_M = \frac{1}{X_1} B |_{p_1 \times p_1} \oplus \frac{1}{X_2} B |_{p_2 \times p_2} \oplus B |_{n \times n},$$

where

$$X_2 = \frac{1}{2X_1} \quad \text{and} \quad X_1 = \frac{2r \pm \sqrt{D(\gamma_1)}}{2(2b_2 + \gamma_1)}.$$

$g_M$ is never binormal and in the second case $g_F$ is not Einstein.
CHAPTER 3

As in the previous chapters, we consider a homogeneous fibration \( F \to M \to N \), for \( M = G/L \), \( N = G/K \) and \( F = K/L \), where \( G \) is a compact connected semisimple Lie group and \( L \subset K \subset G \) connected closed non-trivial subgroups.

In this chapter we suppose that both the fiber \( F \) and the base space \( N \) are symmetric spaces of maximal rank and, moreover, \( N \) is isotropy irreducible. The triple formed by the Lie algebras of \( G, K, L \), denoted by \( (\mathfrak{g}, \mathfrak{k}, \mathfrak{l}) \), shall be called a bisymmetric triple of maximal rank. We classify all the bisymmetric triple of maximal rank when \( \mathfrak{g} \) is simple and obtain formulas to compute the eigenvalues which are necessary to decide about the existence of Einstein adapted metrics. For each triple, we present the eigenvalues of the Casimir operators of the irreducible \( L \)-invariant subspaces of the fiber on the horizontal direction and the eigenvalues of the Casimir operator of \( \mathfrak{k} \) on the vertical direction. The computations for these eigenvalues are in Appendix A as well as a description of the isotropy representation in terms of subset of roots for each triple. Finally, we study the existence of adapted Einstein metrics by using the results in previous chapters. Tables are presented in the end of this chapter. We use the notation used in previous chapters unless stated otherwise.

3.1 Introduction

For all the definitions and properties concerning the roots system of a Lie algebra please see [16] or [34]. Let \( G \) be a compact connected semisimple Lie group and \( L \subset K \subset G \) connected closed non-trivial subgroups such that \( N = G/K \) is isotropy irreducible. As in section 1.2 of Chapter 1, \( \mathfrak{n} \) and \( \mathfrak{p} \) denote the reductive complements of \( \mathfrak{k} \) in \( \mathfrak{g} \) and of \( \mathfrak{l} \) in \( \mathfrak{k} \), respectively. The subspace \( \mathfrak{n} \) is irreducible as an \( Ad\mathfrak{K} \)-module and \( \mathfrak{p} \) may decompose into the direct sum \( \mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_s \) of irreducible \( Ad\mathfrak{L} \)-modules. We suppose that \( M \) has simple spectrum, i.e., \( \mathfrak{n} \) do not contain any \( Ad\mathfrak{L} \)-submodule equivalent to any of the \( \mathfrak{p}_1, \ldots, \mathfrak{p}_s \) and \( \mathfrak{p}_1, \ldots, \mathfrak{p}_s \) are pairwise inequivalent.

Initially, we only suppose that \( L \) is a subgroup of maximal rank in \( G \). We choose a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g}^\mathbb{C} \) such that \( \mathfrak{h} \subset \mathfrak{k}^\mathbb{C} \). Let \( \mathcal{R} \) be a system of nonzero
roots for $\mathfrak{g}^C$ with respect to $\mathfrak{h}$. As usual we have a decomposition of $\mathfrak{g}^C$ into root subspaces

$$\mathfrak{g}^C = \mathfrak{h} \oplus (\oplus_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha),$$

where $\mathfrak{g}^\alpha = \{ u \in \mathfrak{g}^C : \text{ad } h(u) = \alpha u, \forall h \in \mathfrak{h} \}$. We have $\text{Kill}(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) \neq 0$ if and only if $\alpha + \beta = 0$ and thus, for every $\alpha \in \mathcal{R}$, we can take $E_\alpha \in \mathfrak{g}^\alpha$ such that $\text{Kill}(E_\alpha, E_{-\alpha}) = 1$. Since $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \subset \mathfrak{h}$, each pair $E_\alpha, E_{-\alpha}$ determines an element $H_\alpha$ in the Cartan subalgebra $\mathfrak{h}$ given by $H_\alpha = [E_\alpha, E_{-\alpha}]$.

The vectors $H_\alpha$ are such that $\text{Kill}(H_\alpha, h) = \alpha(h)$, for every $h \in \mathfrak{h}$. In particular, the length $|\alpha|$ of a root $\alpha \in \mathcal{R}$ in $\mathfrak{g}$ is defined by

$$|\alpha|^2 = \alpha(H_\alpha) = \text{Kill}(H_\alpha, H_\alpha). \quad (3.1)$$

For every $\alpha, \beta \in \mathcal{R}$ such that $\alpha + \beta \in \mathcal{R}$, since $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subset \mathfrak{g}^{\alpha+\beta}$, we define numbers $N_{\alpha, \beta} \in \mathbb{C}$ by

$$[E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha+\beta}, \quad (3.2)$$

called the structure constants. The $N_{\alpha, \beta}$’s satisfy the following properties:

$$N_{\alpha, \beta} = -N_{\beta, \alpha} \quad (3.3)$$

$$N_{-\alpha, \beta+\alpha} = N_{-\beta, -\alpha} = N_{\alpha, \beta}, \quad (3.4)$$

for every $\alpha, \beta \in \mathcal{R}$ such that $\alpha + \beta \in \mathcal{R}$.

A basis $\{E_\alpha\}_{\alpha \in \mathcal{R}}$ of $\oplus_{\alpha \in \mathcal{R}} \mathfrak{g}^\alpha$ formed by elements chosen as above is called a standard normalized basis and that is what we will use throughout. By using such a basis we construct the elements

$$X_\alpha = \frac{E_\alpha - E_{-\alpha}}{\sqrt{2}} \quad \text{and} \quad Y_\alpha = \frac{i(E_\alpha + E_{-\alpha})}{\sqrt{2}}. \quad (3.5)$$

The vectors $X_\alpha$ and $Y_\alpha$ are unit vectors with respect to $B$. Together with the maximal toral subalgebra $i\mathfrak{h}_\mathbb{R}$, $X_\alpha$ and $Y_\alpha$ generate a compact real form for $\mathfrak{g}^C$ which we identify with $\mathfrak{g}$ (see e.g. [10], ch.III).

Since $L \subset K$, $\mathfrak{h}$ is also a Cartan subalgebra for $\mathfrak{k}^C$. We define the following subsets of roots
\[ R_l = \{ \alpha \in \mathbb{R} : E_\alpha \in l^C \} \quad (3.6) \]

\[ R_t = \{ \alpha \in \mathbb{R} : E_\alpha \in t^C \} \quad (3.7) \]

\[ R_n = R - R_t = \{ \alpha \in \mathbb{R} : E_\alpha \in n^C \} \quad (3.8) \]

\[ R_p = R_t - R_l = \{ \alpha \in \mathbb{R} : E_\alpha \in p^C \} \quad (3.9) \]

We may also consider the subsets of roots

\[ R_{p_a} = \{ \alpha \in \mathbb{R} : E_\alpha \in p_a^C \}, \quad a = 1, \ldots, s, \quad (3.10) \]

and, since \( I \) has maximal rank and \( k = l \oplus p \), \( p_a = \langle X_\alpha, Y_\alpha : \alpha \in R_{p_a}^+ \rangle \). Since \( \text{Kill}(E_\alpha, E_{-\alpha}) = 1 \), the bases \( \{ E_\alpha \}_{\alpha \in R_{p_a}} \) and \( \{ E_{-\alpha} \}_{\alpha \in R_{p_a}} \) of \( p_a^C \) are dual with respect to \( \text{Kill} \). Moreover, \( \{ X_\alpha, Y_\alpha \}_{\alpha \in R_{p_a}^+} \) is an orthonormal basis for \( p_a \) with respect to \( B \). Consequently, the Casimir operators of \( p_a^C \) and of \( p_a \) are

\[ C_{p_a^c} = \sum_{\alpha \in R_{p_a}} \text{ad}_{E_\alpha} \text{ad}_{E_{-\alpha}} \quad (3.11) \]

\[ C_{p_a} = -\sum_{\alpha \in R_{p_a}^+} \left( \text{ad}_{X_\alpha}^2 + \text{ad}_{Y_\alpha}^2 \right). \quad (3.12) \]

Since \( \mathfrak{k} \) has maximal rank \( g = \mathfrak{k} \oplus \mathfrak{n} \), we have \( n^C = \langle E_\alpha : \alpha \in R_n^+ \rangle \) and \( n = \langle X_\alpha, Y_\alpha : \alpha \in R_n^+ \rangle \). The subspace \( n \) is by hypothesis irreducible as an \( \text{Ad} K \)-submodule. If \( n = \bigoplus_j n^j \) is a decomposition of \( n \) into irreducible \( \text{Ad} L \)-modules, we write

\[ R_{n^j} = \{ \phi \in \mathbb{R} : E_\phi \in (n^j)^C \} \quad \text{and} \quad n^j = \{ X_\phi, Y_\phi : \phi \in R_{n^j}^+ \}. \quad (3.13) \]

We recall that one of the conditions for existence of an Einstein adapted metric on \( M \), given in Corollary 1.4, is that there are \( \lambda_1, \ldots, \lambda_s > 0 \) such that the operator \( \sum_{a=1}^s \lambda_a C_{p_a} \) is scalar on \( n \). The Casimir operator \( C_{p_a} \) is necessarily scalar on the irreducible \( \text{Ad} L \)-submodules \( n^j \). Since \( B(X_\phi, X_\phi) = 1 \), the eigenvalue of \( C_{p_a} \) on \( n^j \) is given by \( B(C_{p_a} X_\phi, X_\phi) \), for any \( \phi \in R_{n^j}^+ \). We shall write \( b_\phi^a \) for this eigenvalue, i.e.,

\[ b_\phi^a = B(C_{p_a} X_\phi, X_{-\phi}) \quad (3.14) \]

\[ C_{p_a} |_{n^j} = b_\phi^a I_{n^j}, \quad \forall \phi \in R_{n^j}. \quad (3.15) \]

Furthermore, the eigenvalue of \( C_{p_a} \) on \( n^j \), \( b_\phi^a \), must coincide with the eigenvalue of \( C_{p_a^c} \) on \( (n^j)^C \). Hence, we also have
\[ b^\phi_a = \text{Kill}(C^\phi_{p_a}E_{\phi}, E_{-\phi}). \]  

(3.16)

**Remark 3.1.** We observe that in previous Sections the notation \( n_j \) was used to denote \( \text{Ad K} \)-irreducible submodules of \( n \), while in here we use the similar notation \( n^j \) to denote \( \text{Ad L} \)-irreducible submodules, whereas \( n \) is \( \text{Ad K} \)-irreducible. Similarly, \( b^\phi_j \) was used before to denote the eigenvalue of \( C^\phi_{p_a} \), if this operator was scalar, on \( n_j \), while in here \( b^\phi_a \) is the eigenvalue of this same operator on \( n^j \). No confusion should arise from this since we shall use this second notation only when \( n \) is an irreducible \( \text{Ad K} \)-module.

The necessary condition for existence of an adapted Einstein metric on \( M \) given in Corollary 1.4 can now be rewritten as follows:

**Corollary 3.1.** If there exists on \( M \) an Einstein adapted metric, then there are positive constants \( \lambda_1, \ldots, \lambda_s \) such that

\[
\sum_{a=1}^{s} \lambda_a (b^\phi_{a1} - b^\phi_{a2}) = 0,
\]

for every \( \phi_1, \phi_2 \in \mathcal{R}_n \).

The condition in Corollary 3.1 shall play a fundamental role as a preliminary test for existence of Einstein adapted metrics. It is a very restrictive condition which is not satisfied by many of the spaces under study in this Chapter.

For any roots \( \phi \) and \( \alpha \) let \( \phi + n\alpha, p_{a\phi} \leq n \leq q_{a\phi} \), be the \( \alpha \)-series containing \( \phi \). By definition, the \( \alpha \)-series containing \( \phi \) is the set of all roots of the form \( \phi + n\alpha \) where \( n \) is an integer. It is known that \( \phi + n\alpha \) is an interrupted series ([16], Chap.III, §4). For roots \( \alpha \) and \( \phi \) the square of the structure constant \( N_{a\phi} \) is given by

\[
N_{a,\phi}^2 = \frac{q_{a\phi}(1 - p_{a\phi})}{2} \alpha(H_\alpha).
\]

(3.17)

**Proposition 3.1.** Suppose that \( \text{rank } L = \text{rank } G \). For every \( \phi \in \mathcal{R}_n \) and \( a = 1, \ldots, s \),

\[
b^\phi_a = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+_{p_a}} d_{a\phi} |\alpha|^2,
\]

where \( d_{a\phi} = q_{a\phi} - p_{a\phi} - 2p_{a\phi}q_{a\phi} \) and \( \phi + n\alpha, p_{a\phi} \leq n \leq q_{a\phi} \) is the \( \alpha \)-series containing \( \phi \).
Proof: By using (3.16) and (3.11) we obtain the following:

\[ b^\phi_a = \text{Kill}(C_{p^a} E_\phi, E_{-\phi}) \]
\[ = \sum_{\alpha \in \mathbb{R}_{p^a}} \text{Kill}([E_{-\alpha}; [E_\alpha, E_\phi]], E_{-\phi}) \]
\[ = \sum_{\alpha \in \mathbb{R}_{p^a}} N_{\alpha,\phi} \text{Kill}([E_{-\alpha}, E_{\phi+\alpha}], E_{-\phi}) \]
\[ = \sum_{\alpha \in \mathbb{R}_{p^a}} N_{\alpha,\phi} N_{-\alpha,\phi+\alpha} \text{Kill}(E_\phi, E_{-\phi}) \]
\[ = \sum_{\alpha \in \mathbb{R}_{p^a}} N_{\alpha,\phi} N_{-\alpha,\phi+\alpha} \]

From (3.4) we have \( N_{-\alpha,\phi+\alpha} = N_{\alpha,\phi} \) and we get

\[ b^\phi_a = \sum_{\alpha \in \mathbb{R}_{p^a}} N_{\alpha,\phi}^2 = \sum_{\alpha \in \mathbb{R}_{p^a}^+} (N_{\alpha,\phi}^2 + N_{-\alpha,\phi}^2). \]

Now let \( \phi + n\alpha, p_{\alpha \phi} \leq n \leq q_{\alpha \phi} \), be the \( \alpha \)-series containing \( \phi \). It is known that

\[ N_{\alpha,\phi}^2 = \frac{q_{\alpha \phi}(1 - p_{\alpha \phi})}{2} \alpha(H_\alpha), \]

as mentioned in (3.17).

On the other hand, to compute \( N_{-\alpha,\phi}^2 \) we need the \((-\alpha)\)-series containing \( \phi \). Clearly, this series is \( \phi - n'\alpha \), where \(-q_{\alpha \phi} \leq n' \leq -p_{\alpha \phi} \). Hence, we obtain the following:

\[ N_{-\alpha,\phi}^2 = \frac{-p_{\alpha \phi}(1 - (-q_{\alpha \phi}))}{2} (-\alpha)(H_{-\alpha}) = \frac{-p_{\alpha \phi}(1 + q_{\alpha \phi})}{2} \alpha(H_\alpha). \]

Hence,

\[ b^\phi_a = \sum_{\alpha \in \mathbb{R}_{p^a}^+} \left( \frac{q_{\alpha \phi}(1 - p_{\alpha \phi})}{2} - \frac{-p_{\alpha \phi}(1 + q_{\alpha \phi})}{2} \right) \alpha(H_\alpha), \]

which yields the required formula.

\[ \Box \]

Let us consider a decomposition of \( \mathfrak{k} \) into its center \( \mathfrak{k}_0 \) and simple ideals \( \mathfrak{k}_a \), for \( a = 1, \ldots, t \),

\[ \mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_t, \quad (3.18) \]

and let \( \gamma_a \) denote the eigenvalue of the Casimir operator of \( \mathfrak{k} \) on \( \mathfrak{k}_a \). We present a formula to compute the eigenvalues \( \gamma_a \)'s by making use of dual Coxeter numbers. We start by recalling some facts about roots. On this topic we refer to ([13], V.5) and ([17], 10.4).
There are at most two different lengths in a given irreducible root system, and the corresponding roots are designated by **long** and **short** roots. If there is only one length it is conventional to say that all the roots are long. If \( \alpha \) is a long root and \( \beta \) is short, then

\[
\frac{|\alpha|^2}{|\beta|^2} = 3, \quad \text{in the case of } G_2 \text{ and }
\]

\[
\frac{|\alpha|^2}{|\beta|^2} = 2, \quad \text{in the case of } B_n, C_n \text{ and } F_4.
\]

(3.19)

In the remaining cases, \( A_n, D_n, E_6, E_7 \) and \( E_8 \), there is only one length. These facts can be read off from the corresponding Dynkin diagrams.

We also recall that a length of a root \( \alpha \) is given by

\[
|\alpha|^2 = \alpha(H_\alpha) = \text{Kill}(H_\alpha, H_\alpha).
\]

The **dual Coxeter number** of a simple Lie algebra \( \mathfrak{g} \) is the number given by

\[
h^*(\mathfrak{g}) = \frac{1}{|\alpha|^2},
\]

where \( \alpha \) is a long root (see e.g. [35]). The dual Coxeter numbers of each irreducible root system are given in Table 3.1.

We may suppose that \( \mathfrak{h}_a = \mathfrak{h} \cap \mathfrak{k}_a \) is a Cartan subalgebra of \( \mathfrak{k}_a \) and thus a root of \( \mathfrak{k}_a \) can be viewed as a root for \( \mathfrak{g} \). Hence we can compare lengths of roots of \( \mathfrak{g} \) with lengths of roots of \( \mathfrak{k}_a \). So let \( \delta_a \) be the ratio of the square length of a long root for \( \mathfrak{g} \) to that of \( \mathfrak{k}_a \), i.e.,

\[
\delta_a = \frac{|\alpha|^2}{|\beta|^2}_{\mathfrak{g}} = \frac{\text{Kill}(H_\alpha, H_\alpha)}{\text{Kill}(H_\beta, H_\beta)},
\]

where \( \alpha \) is a long root of \( \mathfrak{g} \) and \( \beta \) is a long root of \( \mathfrak{k}_a \). Clearly, \( \delta_a = 1 \) if there exists only one length for \( \mathfrak{g} \) or if both \( \mathfrak{g} \) and \( \mathfrak{k}_a \) have two lengths. If \( \delta_a \neq 1 \), then, according to (3.19), \( \delta_a \) is equal to either 2 or 3. We recall the following result by D. Panyushev:

**Proposition 3.2.** [35] Suppose that \( \mathfrak{g} \) is simple. Then

\[
\gamma_a = \frac{h^*(\mathfrak{k}_a)}{\delta_a h^*(\mathfrak{g})}, \quad a = 1, \ldots, s,
\]

where \( h^*(\mathfrak{k}_a) \) and \( h^*(\mathfrak{g}) \) are the dual Coxeter numbers of \( \mathfrak{k}_a \) and \( h^*(\mathfrak{g}) \), respectively.

We observe that so far we used only the fact that \( L \) has maximal rank. Now if, in addition, \( F = K/L \) is a symmetric space, then we consider its DeRham decomposition
where, for \( a = 1, \ldots, s \), \( K_a \) is simple, as in Section 2.5. We observe that since \( L \) has maximal rank in the deRham decomposition of \( K/L \) the factor \( K_0/L_0 \) must be trivial. The Lie algebras of \( K_1, \ldots, K_s \), denoted by \( \mathfrak{t}_1, \ldots, \mathfrak{t}_s \), are some of the ideals in the decomposition (3.18). As explained at the beginning of Section 2.5 the irreducible \( \text{Ad} L \)-submodules \( p_1, \ldots, p_s \) are chosen as the symmetric reductive complements of \( \mathfrak{t}_a \) in \( \mathfrak{t}_a \), for \( a = 1, \ldots, s \). Hence, the constant \( \gamma_a \) defined throughout by the equality \( \text{Kill}_{\mathfrak{t}_a} |_{\mathfrak{t}_a \times \mathfrak{t}_a} = \gamma_a \text{Kill} |_{\mathfrak{p}_a \times \mathfrak{p}_a} \) in previous sections, is now just the eigenvalue of the Casimir operator of \( \mathfrak{t} \) on \( \mathfrak{t}_a \), and thus can be determined by the formula in Proposition 3.2.

In Appendix A we compute the eigenvalues \( b_\phi^a \)'s and \( \gamma_a \)'s using Propositions 3.1 and 3.2. Their values are indicated in Tables 3.4, 3.5, 3.6 and 3.7 in Section 3.6 for each bisymmetric triple.

### 3.2 Bisymmetric Triples of Maximal Rank - Classification

Let us consider a homogeneous fibration \( F \rightarrow M \rightarrow N \), for \( M = G/L \), \( N = G/K \) and \( F = K/L \), where \( G \) is a compact connected semisimple Lie group and \( L \subsetneq K \subsetneq G \) connected closed non-trivial subgroups such that \( F \) and \( N \) are symmetric spaces. We shall call such a fibration a **bisymmetric fibration**. In particular, the pairs \((G, K)\) and \((K, L)\) are symmetric pairs of compact type \([16]\). With a slight abuse of terminology, we shall also say that the pairs of Lie algebras \((\mathfrak{g}, \mathfrak{t})\) and \((\mathfrak{t}, \mathfrak{l})\) are symmetric pairs of compact type whenever the corresponding pairs \((G, K)\) and \((K, L)\) are.

**Definition 3.1.** A **bisymmetric triple** is a triple \((\mathfrak{g}, \mathfrak{t}, \mathfrak{l})\) where \( \mathfrak{g}, \mathfrak{t} \) and \( \mathfrak{l} \) are Lie algebras satisfying the following conditions:

(i) \( \mathfrak{l} \subsetneq \mathfrak{t} \subsetneq \mathfrak{g} \);

(ii) \((\mathfrak{g}, \mathfrak{t})\) and \((\mathfrak{t}, \mathfrak{l})\) are symmetric pairs of compact type.

A bisymmetric triple is said to be irreducible if \( \mathfrak{g} \) is a simple Lie algebra and said to be of maximal rank if \( \mathfrak{l} \) has maximal rank in \( \mathfrak{g} \), i.e., it contains a maximal toral subalgebra of \( \mathfrak{g} \).

Clearly, there is a one-to-one correspondence between bisymmetric fibrations, up to cover, and bisymmetric triples. All the bisymmetric triples \((\mathfrak{g}, \mathfrak{t}, \mathfrak{l})\) considered in this chapter are irreducible and of maximal rank, even when this is not explicitly stated. Consequently, any bisymmetric fibration
F → M → N here considered is such that L is a subgroup of maximal rank in G and N = G/K is an irreducible symmetric space.

**Definition 3.2.** A bisymmetric triple \((g, ℓ, l)\) is said to be of

(i) **Type I** if \(F\) is an isotropy irreducible symmetric space; equivalently, if \(p\) is an irreducible \(Ad L\)-module;

(ii) **Type II** if \(F\) is the direct product of two isotropy irreducible symmetric spaces; equivalently, if \(p = p_1 ⊕ p_2\), where \(p_1\) and \(p_2\) are nontrivial irreducible \(Ad L\)-modules.

A bisymmetric fibration \(F → M → N\) of is said to be of Type I or II if the corresponding bisymmetric triple \((g, ℓ, l)\) is either of Type I or II, respectively.

As we shall see any irreducible bisymmetric triple of maximal rank with \(g\) simple is of Type I or II.

Isotropy irreducible symmetric spaces have been classified and a classification can be found in [16]. By using this we obtain a list of all possible triples \((g, ℓ, l)\) such that \(l\) and \(ℓ\) are subalgebras of maximal rank of \(g\) and \((g, ℓ)\) and \((ℓ, l)\) are symmetric pairs of compact type. By inspection of the classification of symmetric pairs \((g, ℓ)\) of compact type in [16] we obtain that those of maximal rank are the pairs in Tables 3.2 and 3.3.

We observe that the cases when \(ℓ\) is the centralizer of a torus are only the cases \((\mathfrak{so}_6 ⊕ \mathbb{R}), (\mathfrak{so}_7, \mathfrak{so}_6 ⊕ \mathbb{R}), (\mathfrak{so}_{2n}, \mathfrak{u}_n), (\mathfrak{so}_n, \mathbb{R} ⊕ \mathfrak{so}_{n-2}), (\mathfrak{sp}_n, \mathfrak{u}_n)\) and \((\mathfrak{su}_n, \mathfrak{su}_p ⊕ \mathfrak{su}_{n-p} ⊕ \mathbb{R})\). This follows from the fact that these are the only subalgebras \(ℓ\) corresponding to painted Dynkin diagrams of the Dynkin diagram of \(g\) or as they are the only ones such that \(ℓ\) is not centerless. In all the other cases \(ℓ\) is semisimple. If \(ℓ\) is simple then \((ℓ, l)\) shall be an irreducible symmetric pair, i.e., \(p\) is an irreducible \(L\)-invariant subspace. Thus, \((g, ℓ, l)\) is of type I. In the cases where \(ℓ = ℓ_1 ⊕ \mathbb{R}\) with \(ℓ_1\) a simple ideal of \(ℓ\), since we require \(l\) to be of maximal rank, we have \(l = ℓ_1 ⊕ \mathbb{R}\), where \(ℓ_1\) is a subalgebra of \(ℓ\) with maximal rank and \((ℓ, l) \cong (ℓ_1, ℓ_1)\) is an irreducible symmetric pair. Thus, in this case, \(p = p_1\) is also an irreducible \(L\)-invariant subspace and \((g, ℓ, l)\) is of type I. In the cases where \(ℓ = ℓ_1 ⊕ ℓ_2\), with \(ℓ_1\) and \(ℓ_2\) simple ideals of \(ℓ\), we have \(l = ℓ_1 ⊕ ℓ_2\), where, for \(i = 1, 2\), \(ℓ_i\) is a subalgebra of \(ℓ\) of maximal rank. Clearly, one of the \(ℓ_i\)'s must be proper as we require that \(l\) is a proper subalgebra of \(ℓ\). If both \(ℓ_1\) and \(ℓ_2\) are proper, then \(p = p_1 ⊕ p_2\), where \(p_1\) and \(p_2\) are nonzero irreducible \(L\)-invariant subspaces. Hence, in this case, \((g, ℓ, l)\) is of type II. If exactly one of \(ℓ_i\)'s coincides with \(ℓ\), then \((ℓ, l) \cong (ℓ_j, ℓ_j)\) and \(p = p_j\), for that \(j\) satisfying \(ℓ_j = ℓ\), and once again \((g, ℓ, l)\) is of type I. Finally, we have the case of the spaces \((\mathfrak{su}_n, \mathfrak{su}_p ⊕ \mathfrak{su}_{n-p} ⊕ \mathbb{R})\),
Clearly, \( l \) must be of the form \( l = l_1 \oplus l_2 \oplus \mathbb{R} \), where \( l_1 \) and \( l_2 \) are maximal rank subalgebras of \( \text{su}_p \) and \( \text{su}_{n-p} \), respectively. We obtain a triple of type \( I \) if exactly one of the \( l_i \)'s is proper and a triple of type \( II \) if both \( l_1 \) and \( l_2 \) are proper. This proves the following:

**Lemma 3.1.** An irreducible bisymmetric triple of maximal rank \((g, \mathfrak{k}, l)\) such that \( g \) is simple is either of Type \( I \) or \( II \). Moreover, all such bisymmetric triples \((g, \mathfrak{k}, l)\) are those in Tables 3.4, 3.5, 3.6 and 3.7.

**Lemma 3.2.** For an irreducible bisymmetric triple of maximal rank \((g, \mathfrak{k}, l)\) of Type \( I \), let \( \gamma \) be the eigenvalue of the Casimir operator of \( \mathfrak{k} \) on \( p \) and \( b^\phi \)'s the eigenvalues of the Casimir operator of \( p \) on \( n \). For each bisymmetric triple of maximal rank these eigenvalues have the values listed in Tables 3.4 and 3.5.

**Lemma 3.3.** For an irreducible bisymmetric triple of maximal rank \((g, \mathfrak{k}, l)\) of Type \( II \), let \( \gamma_a \) be the eigenvalue of the Casimir operator of \( \mathfrak{k} \) on \( p_a \) and \( b^\phi \)'s the eigenvalues of the Casimir operator of \( p_a \) on \( n \), \( a = 1, 2 \). For each bisymmetric triple of maximal rank these eigenvalues have the values listed in Tables 3.6 and 3.7.

### 3.3 Einstein Adapted Metrics for Type I

In this Section we determine all the bisymmetric Riemannian fibrations \( F \to M \to N \) of maximal rank of Type \( I \) which admit an Einstein adapted metric. We recall that for Type \( I \), \( p \) is an irreducible \( \text{Ad} L \)-submodule. Moreover, since \( n \) is an irreducible \( \text{Ad} K \)-module, any adapted metric is binormal. As in addition \( F \) and \( N \) are symmetric spaces, we may apply Corollary 2.9. We recall that according to Corollary 2.9 there exists on \( M \) an Einstein (binormal) adapted metric if and only if

\[
(i) \text{ the Casimir operator of } p \text{ is scalar on } n \text{ and } \\
(ii) \Delta = 1 - 2\gamma(1 - \gamma + 2b) \geq 0. \tag{3.21}
\]

If these two conditions are satisfied, Einstein binormal metrics are, up to homothety, given by

\[
g_M = B|_{p\times p} \oplus XB|_{n\times n}, \text{ where } X = \frac{1 \pm \sqrt{\Delta}}{2\gamma}. \tag{3.22}
\]

We also recall that \( b \) is the eigenvalue of \( C_p \) on \( n \), in the case when this operator is scalar, and \( \gamma \) is the eigenvalue of the Casimir operator of \( \mathfrak{k} \) on \( p \). These constants are computed in Appendix A and their values are indicated in Tables 3.4 and 3.5.
as has been stated in Lemma 3.2. We recall that condition (i) translates into \( b^\phi = b \), for every \( \phi \in \mathfrak{R}_n \), for Type I, according to Corollary 3.1. Hence, the first test for existence of an adapted Einstein metric shall be to observe if there exists only one eigenvalue \( b^\phi \) in the corresponding columns of Tables 3.4 and 3.5.

**Theorem 3.1.** The bisymmetric fibrations \( F \to M \to N \) of Type I such that there exists on \( M \) an Einstein adapted metric are those whose bisymmetric triples are listed in Tables 3.8 and 3.9. For each case there are exactly two Einstein adapted metrics. Furthermore, these Einstein metrics are, up to homothety, given by

\[
g_M = B|_{\mathbf{p} \times \mathbf{p}} \oplus XB|_{\mathbf{n} \times \mathbf{n}},
\]

where \( X \) is indicated in the Tables mentioned above.

**Proof:** As explained in the discussion above, by Corollary 2.9 the existence of an adapted Einstein metric implies that the Casimir operator of \( \mathfrak{p} \) is scalar on \( \mathfrak{n} \). By inspection we conclude from Tables 3.5 and 3.4 that the only spaces satisfying this condition are those corresponding to the labels

\begin{align*}
\text{A.6}, \text{A.13}, \text{A.21}, \text{A.25}, \text{A.26}, \text{A.31}, \text{A.32}, \text{A.34}, \text{A.36}, \text{A.41}, \text{A.43}, \text{A.46}, \text{A.52}, \text{A.53}
\end{align*}

and

\begin{align*}
\text{A.1} & \text{ for } l = \frac{p}{2} \text{ with } p \text{ even; in this case, } b = \frac{\sqrt{p}}{2n}; \\
\text{A.5} & \text{ for } s = \frac{n-p}{2} \text{ with } n - p \text{ even; in this case, } b = \frac{n-p}{2(2n-1)}; \\
\text{A.10} & \text{ for } l = \frac{p}{2} \text{ with } p \text{ even; in this case, } b = \frac{p}{4(n-1)}; \\
\text{A.18} & \text{ for } l = \frac{p}{2} \text{ with } p \text{ even; in this case, } b = \frac{p}{4(2n-1)}; \\
\text{A.50} & \text{ for } p = 1; \text{ in this case } b = \frac{1}{9}; \\
\text{A.54} & \text{ for } p = 1; \text{ in this case } b = \frac{1}{8}.
\end{align*}

We compute \( \triangle \) given in formula 3.21 and the values obtained are as follows
\[
\begin{align*}
\Delta &> 0 \\
A.1 & \quad \left( \frac{n-p}{n} \right)^2 > 0 \\
A.5 & \quad \frac{n^2 + 8n - 4n + 5}{(2n-1)^2} > 0, \ \forall p = \lfloor \frac{\sqrt{4n-1}}{2} \rfloor, \ldots, n - 1 \\
A.6 & \quad \left( \frac{2p+1}{2n-1} \right)^2 > 0 \\
A.10 & \quad \frac{p^2 - (2n+1)p + n^2 + 1}{(n-1)^2} > 0 \\
A.13 & \quad \left( \frac{n-p}{n} \right)^2 > 0 \\
A.18 & \quad \frac{3p^2 + (3-4n)p + 2(n^2 + 1)}{2(n+1)^2} > 0 \\
A.21 & \quad \left( \frac{n-p}{n+1} \right)^2 > 0 \\
A.25 & \quad \frac{106 - 63p + 7p^2}{162} > 0, \ \text{iff} \ p = 1, 7 \\
A.26 & \quad \frac{49}{81} > 0 \\
A.31 & \quad \frac{1}{4} > 0 \\
A.32 & \quad \frac{11}{16} > 0 \\
A.34 & \quad \frac{7p^2 - 56p + 113}{225} > 0 \\
A.36 & \quad \frac{196}{225} > 0 \\
A.41 & \quad -\frac{2}{25} < 0 \\
A.43 & \quad \frac{64}{81} > 0 \\
A.46 & \quad \frac{164 - 60p + 5p^2}{324} > 0, \ p = 2, 4 \\
A.50 & \quad \frac{25}{81} > 0 \\
A.52 & \quad \frac{1}{3} > 0, \ p = 2; -\frac{1}{3}, \ p = 4 \\
A.53 & \quad \frac{25}{36} > 0 \\
A.54 & \quad \frac{1}{4} > 0
\end{align*}
\]

For \( A.25 \), since \( p = 1, 3, 5, 7, \) then \( \Delta > 0 \) for \( p = 1, 7 \) and \( \Delta < 0 \) otherwise.

For \( A.46 \), we have \( p = 2, 4, 6. \) Then, \( \Delta > 0 \) for \( p = 2, 4 \) and \( \Delta < 0 \) for \( p = 6. \)

For \( A.34 \), we have \( p = 1, \ldots, 4 \) and thus \( \Delta > 0 \) for every \( p. \)

For \( A.10 \), \( p = 1, \ldots, \lfloor \frac{n}{2} \rfloor. \) We have that \( \Delta \geq 0 \) if and only if

\[
p \in \left( -\infty, \frac{2n + 1 - \sqrt{4n - 3}}{2} \right) \cup \left( \frac{2n + 1 + \sqrt{4n - 3}}{2}, +\infty \right) \cap \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}
\]

We can show that \( \frac{2n + 1 - \sqrt{4n - 3}}{2} \geq \frac{n}{2} \) and thus \( \Delta > 0, \) for every \( p = 1, \ldots, \lfloor \frac{n}{2} \rfloor. \)

For \( A.5 \), \( p = 1, \ldots, n - 1. \) We show that \( \Delta > 0, \) if and only if

\[
p \in \left( -1 + \frac{\sqrt{4n - 1}}{2}, +\infty \right) \cap \{1, \ldots, n - 1\}
\]

Since, for every \( n, \) \( -1 + \frac{\sqrt{4n - 1}}{2} < n - 1 \) we conclude that \( \Delta > 0, \) if and only if

\[
p = \lfloor -1 + \frac{\sqrt{4n - 1}}{2} \rfloor + 1, \ldots, n - 1 = \lfloor \frac{\sqrt{4n - 1}}{2} \rfloor, \ldots, n - 1.
\]
Finally, we compute $X = \frac{1 \pm \sqrt{\triangle}}{2\gamma}$ for those cases when $\triangle > 0$. The values of $X$ are indicated in Table 3.8.

\[\square\]

**Remark 3.2.** For the bisymmetric triples of Type I such that $C_p$ is scalar on $n$ but $\triangle < 0$, there is still a complex non-real solution $X = \frac{1 \pm \sqrt{\triangle}}{2\gamma}$. So even in these cases we can conclude that there exists a complex Einstein adapted metric on $M$ as in (3.22). These are just the cases $A.5$ for $p \geq -1 + \sqrt{4n-1}$, $A.25$ for $p = 3, 5$ and $A.47$.

### 3.4 Einstein Adapted Metrics for Type II

In this Section we study the existence of Einstein adapted metrics on bisymmetric fibrations of Type II. Whereas for Type I any adapted metric was binormal this is clearly not true for Type II, since $p$ is not an irreducible $Ad L$-module. We shall classify all the bisymmetric triples which admit an Einstein binormal metric. Since for bisymmetric fibrations, in particular, $F$ is a symmetric space, we know from Corollary 2.11 that for any Einstein binormal metric $g_M$, $g_F$ is also Einstein. We shall also classify all the bisymmetric triples which admit an Einstein non-binormal adapted metric $g_M$ whose restriction $g_F$ is also Einstein.

Since for type II $p = p_1 + p_2$, where $p_i$, $i = 1, 2$, are irreducible $L$-modules and $n$ is an irreducible $K$-module, the existence of an Einstein adapted metric implies that there exist positive constants $\lambda_1$, $\lambda_2$, such that the operator $\lambda_1 C_{p_1} + \lambda_2 C_{p_2}$ is scalar on $n$, according to Corollary 1.4. As rephrased in Corollary 3.1, the condition above translates into

$$\lambda_1(b_1^{\phi_1} - b_1^{\phi_2}) + \lambda_2(b_2^{\phi_1} - b_2^{\phi_2}) = 0, \text{ for every } \phi_1, \phi_2 \in \mathbb{R}_n \quad (3.23)$$

By using Tables 3.6 and 3.7 we conclude the following:

**Lemma 3.4.** The only bisymmetric triples satisfying condition (3.23) are the cases $A.15$, $A.23$, $A.33$, $A.42$, $A.47$ and

- $A.5$ for $p = 2l, n - p = 2s$  
- $A.12$ for $p = 2l, n - p = 2s$  
- $A.16$ for $p = 2l$  
- $A.20$ for $p = 2l, n - p = 2s$  
- $A.24$ for $p = 2l$  
- $A.55$ for $p = 1$
For all other bisymmetric triples of Type II we can conclude that there exists no Einstein adapted metric on $M$.

Furthermore, for all the triples listed in Lemma 3.4, we observe that $C_{p_1}$ and $C_{p_2}$ are scalar on $n$ and thus $C_p$ is also scalar on $n$. We shall write

$$C_{p_i} |_n = b_i I_d_n, \ i = 1, 2 \quad (3.24)$$
$$C_p |_n = b I_d_n, \ \text{for } b = b_1 + b_2, \quad (3.25)$$

following the notation used in previous chapters.

**Theorem 3.2.** The bisymmetric fibrations $F \to M \to N$ of Type II such that there exists on $M$ an Einstein binormal metric are those whose bisymmetric triples are listed in Table 3.10. Furthermore, the binormal Einstein metrics are, up to homothety, given by

$$g_M = B |_{p \times p} \oplus X B |_{n \times n},$$

where $X$ is indicated in Table 3.10. In all the cases, $g_N$ and $g_F$ are also Einstein.

**Proof:** Binormal Einstein metrics are in this case given by Corollary 2.10. First we observe that in order to exist an Einstein binormal metric on $M$, $C_p$ must be scalar on $n$ and $C_t$ must be scalar on $p$. The triples which satisfy the first condition are those listed in Lemma 3.4. Furthermore, the second condition implies that $\gamma_2 = \gamma_1$. From the cases in Lemma 3.4 we conclude from Tables 3.6 and 3.7 that the spaces which satisfy the condition $\gamma_2 = \gamma_1$ are those listed below:

- **A.3** for $s = l = 2p$, $n = 4l$; in this case, $\gamma_1 = \gamma_2 = \frac{1}{2}$ and $b_1 = b_2 = \frac{1}{8}$;
- **A.12** for $s = l = 2p$, $n = 4l$, $l \geq 2$; in this case, $\gamma_1 = \gamma_2 = \frac{2l - 1}{4l - 1}$ and $b_1 = b_2 = \frac{l}{2(4l-1)}$;
- **A.15** for $n = 2p$, $p \geq 2$; in this case, $\gamma_1 = \gamma_2 = \frac{p - 1}{2p - 1}$ and $b_1 = b_2 = \frac{p - 1}{4(2p - 1)}$;
- **A.16** for $p = 2l$, $n = 4l$; in this case, $\gamma_1 = \gamma_2 = \frac{2l - 1}{4l - 1}$ and $b_1 = b_2 = \frac{l}{2(4l-1)}$;
- **A.20** for $s = l = 2p$, $n = 4l$; in this case, $\gamma_1 = \gamma_2 = \frac{2l + 1}{4l + 1}$ and $b_1 = b_2 = \frac{l}{4(4l+1)}$;
- **A.23** for $n = 2p$; in this case, $\gamma_1 = \gamma_2 = \frac{p + 1}{2p + 1}$ and $b_1 = b_2 = \frac{p + 1}{4(2p + 1)}$;
- **A.24** for $p = 2l$, $n = 4l$; in this case, $\gamma_1 = \gamma_2 = \frac{2l + 1}{4l + 1}$ and $b_1 = b_2 = \frac{l}{4(4l+1)}$.

Now Einstein binormal metrics are given by positive solutions of (2.32). Since $c_{t,n} = \frac{1}{2}$, we have that there exists an Einstein binormal metric if and only if

$$\triangle = 1 - 2\gamma(1 - \gamma + 2b) \geq 0, \quad (3.26)$$
where $\gamma = \gamma_1 = \gamma_2$ and $b$ is the eigenvalue of $C_p$ on $n$, i.e., $b = b_1 + b_2$. In such a case these Einstein metrics are given by homotheties of

$$g_M = B \mid_{p \times p} \oplus XB \mid_{n \times n}, \text{ where } X = \frac{1 \pm \sqrt{\Delta}}{2\gamma}.$$ 

We compute $\Delta$:

$$\begin{array}{c|c}
\Delta & \\
\hline
A.3 & 0 \\
A.12 & \frac{1}{(4l-1)^2} > 0 \\
A.15 & \frac{1}{2p-1} > 0 \\
A.16 & \frac{2l}{(4l-1)^2} > 0 \\
A.20 & \frac{4l^2+2l+1}{(4l+1)^2} > 0 \\
A.23 & -\frac{1}{2p-1} < 0 \\
A.24 & \frac{(2l-1)}{(4l+1)^2} > 0 \\
\end{array}$$

Except in the case $A.23$, there exists an Einstein adapted metric. The values for $X$ are indicated in Table 3.10.

$g_N$ is Einstein because $N$ is irreducible and $g_F$ is Einstein due to Corollary 2.11.$\square$

**Remark 3.3.** In the case $A.23$, where $C_p$ is scalar on $n$ but $\Delta = -\frac{1}{2p+1} < 0$, for every $p$, we can still consider the non-real complex solution $X = \frac{1+\sqrt{\Delta}}{2\gamma}$ which gives rise to a non-real complex Einstein binormal metric on $M$.

**Theorem 3.3.** The bisymmetric fibrations $F \to M \to N$ of Type II such that there exists on $M$ an Einstein adapted metric such that $g_F$ is Einstein are those with an Einstein binormal metric, as in Theorem 3.2 and Table 3.10 and the fibration corresponding to the bisymmetric triple

$$(\mathfrak{su}_{2(l+s)}, \mathfrak{su}_{2l} \oplus \mathfrak{su}_{2s} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathfrak{su}_s \oplus \mathfrak{su}_s \oplus \mathbb{R}^3),$$

whose Einstein adapted metric is, up to homothety, given by

$$g_M = \frac{2l}{l+s} B \mid_{p_1 \times p_1} \oplus \frac{2s}{l+s} B \mid_{p_2 \times p_2} \oplus B \mid_{n \times n}.$$ 

This metric is binormal if and only if $l = s$.

**Proof:** The only cases which may admit an Einstein adapted metric are those listed in Lemma 3.4 since they are the only that satisfy the necessary condition (3.23) for existence of such a metric. Furthermore, if we require that $g_F$ is also
Einstein, then we must have one of the two conditions in Corollary ?? . In the first case, \( \gamma = 1 = \gamma_2 \), \( g_M \) is a binormal metric according to Corollary ?? . These are the cases given by Theorem 3.2 and listed in Table 3.10. In the second case, we have \( \gamma_2 = 1 - \gamma_1 \). By inspection of the triples listed in Lemma 3.4, we conclude that this is possible only for the triple \( A.3 \) when \( p = 2l \) and \( n - p = 2s \). In this case \( \gamma_1 = \frac{l}{l+s} \), \( \gamma_2 = \frac{s}{l+s} = 1 - \gamma_1 \), \( b_1 = \frac{h}{4} \), and \( b_2 = \frac{\bar{h}}{4} = l - n/4 \). Using Corollary 2.13, we obtain \( D(\gamma_1) = 0 \) and thus \( X_1 = \frac{l+s}{2l} \), \( X_2 = \frac{l+s}{2s} \). □

For bisymmetric fibrations of Type II the classifications of all Einstein adapted metrics is a difficult problem due to the high complexity of the Einstein equations. The classification of Einstein binormal metrics and Einstein adapted metrics such that \( g_F \) is Einstein as well, as done above, is a way to restrict the problem in a way that the Einstein equations are manageable. It can be seen by Theorems 3.3 and 3.2 that no bisymmetric fibration of Type II in the exceptional admits neither an Einstein binormal metric nor an Einstein adapted metric with \( g_F \) Einstein. However, for this spaces of exceptional type is is possible to classify all Einstein adapted metrics with the help of Maple. Once again, since \( C_{p_1} \) and \( C_{p_2} \) must be scalar on \( n \), from Lemma 3.4 we know that the only cases of Type II and exceptional are the fibrations \( A.55 \) for \( p = 1 \), \( A.33 \), \( A.47 \) and \( A.42 \). The results obtained for these cases are synthetized in Theorem 3.4 and Table 3.6.

**Theorem 3.4.** The only bisymmetric fibrations \( F \to M \to N \) of Type II, such that \( G \) is an exceptional Lie group, which admit an Einstein adapted metric are those whose bisymmetric triples are listed in Table 3.6. None of these metrics is neither binormal nor such that \( g_F \) is Einstein.

**Proof:** As mentioned above, it follows from Lemma 3.4 that the only cases of Type II with \( g \) exceptional which admit an Einstein adapted metric are the cases \( A.55 \) for \( p = 1 \), \( A.33 \), \( A.47 \) and \( A.42 \). We recall that for each of these spaces any Einstein adapted metric is of the form

\[
g_M = \frac{1}{X_1} B \big|_{p_1 \times p_1} \oplus \frac{1}{X_2} B \big|_{p_2 \times p_2} \oplus B \big|_{n \times n},
\]

where \( X_1 \) and \( X_2 \) are positive solutions of the system of equations given in Theorem 2.5 which are as follows:

\[
2\gamma_1 X_1^2 X_2 + (1 - \gamma_1) X_2 - 2\gamma_2 X_1 X_2^2 - (1 - \gamma_2) X_1 = 0, 
\]

\[
2b_1 X_2 + 2b_2 X_1 - 2X_1 X_2 + 2\gamma_1 X_1^2 X_2 + (1 - \gamma_1) X_2 = 0.
\]
Also we recall that the eigenvalues $b_i$ and $\gamma_i$, for $i = 1, 2$, can be found in Table 3.6. To show the result we use Maple and so many details are omitted.

(A.33) For the bisymmetric triple $\Lambda^{(3)}$ the non-zero solutions of the equations (3.27) and (3.28) are of the form

$$X_1 = \alpha_i, \quad X_2 = -\frac{7}{4} \alpha_i^3 + 12\alpha_i^2 - \frac{5899}{36} \alpha_i + 19,$$

where $\alpha_i$ is a root of the polynomial

$$t(z) = 63z^4 - 432z^3 + 1088z^2 - 1224z + 513.$$

Since $X_1 = \alpha_i$ we are interested only in positive real roots of $t$. This polynomial has exactly two positive roots which are indicated below:

$$\alpha_1 = \frac{12\xi \beta}{7} + \frac{\beta^3}{126} - \frac{\sqrt{7} (320\xi^2 \beta^2 + 7\xi^4 \beta^2 - 25781\beta^2 + 43416\xi^3)}{\xi \beta} \frac{1}{2},$$

$$\alpha_2 = \frac{12\xi \beta}{7} + \frac{\beta^3}{126} + \frac{\sqrt{7} (320\xi^2 \beta^2 + 7\xi^4 \beta^2 - 25781\beta^2 + 43416\xi^3)}{\xi \beta} \frac{1}{2},$$

where $\xi = (17756 + 81\sqrt{7662443})^{\frac{1}{6}}$ and $\beta = (960\xi^2 - 42\xi^4 + 154686)^{\frac{1}{4}}$.

Simple calculations show that $\alpha_i$, for $i = 1, 2$, yields positive real values for $X_2$ as well. Approximations for the corresponding values of $X_1$ and $X_2$ are given below:

| $i$ | $X_1$   | $X_2$   |
|-----|---------|---------|
| 1   | 0.5526  | 3.6958  |
| 2   | 0.7432  | 4.7185  |

Hence, there are on $M$ exactly two Einstein adapted metrics.

(A.55, for $p = 1$) In this case the non-zero solutions of the equations (3.27) and (3.28) are given by

$$X_1 = \alpha_i, \quad X_2 = -\frac{156}{7} \alpha_i^3 + \frac{552}{7} \alpha_i^2 - \frac{571}{7} \alpha_i + \frac{176}{7},$$

where $\alpha_i$ is a root of the polynomial

$$t(z) = 234z^4 - 828z^3 + 993z^2 - 474z + 77.$$

The polynomial $t$ has exactly four positive roots which are indicated below:

$$\alpha_1 = \frac{23}{26} - \frac{\sqrt{2\beta}}{156} - \frac{1}{156} \left(\frac{-3664\xi \beta + 26\xi^2 \beta + 71786\beta + 26460\sqrt{2\xi}}{\xi \beta}\right)^{\frac{1}{2}},$$

$$\alpha_2 = \frac{23}{26} - \frac{\sqrt{2\beta}}{156} + \frac{1}{156} \left(\frac{-3664\xi \beta + 26\xi^2 \beta + 71786\beta + 26460\sqrt{2\xi}}{\xi \beta}\right)^{\frac{1}{2}},$$

$$\alpha_3 = \frac{23}{26} + \frac{\sqrt{2\beta}}{156} - \frac{1}{156} \left(\frac{-3664\xi \beta + 26\xi^2 \beta + 71786\beta + 26460\sqrt{2\xi}}{\xi \beta}\right)^{\frac{1}{2}},$$

$$\alpha_4 = \frac{23}{26} + \frac{\sqrt{2\beta}}{156} + \frac{1}{156} \left(\frac{-3664\xi \beta + 26\xi^2 \beta + 71786\beta + 26460\sqrt{2\xi}}{\xi \beta}\right)^{\frac{1}{2}}.$$
where \( \xi = (136819 + 36i\sqrt{1796295})^{\frac{1}{3}} \) and \( \beta = \left(\frac{13\xi^2 + 91\xi + 35893}{\xi}\right)^{\frac{1}{2}}. \)

Simple calculations show that \( \alpha_i \), for \( i = 1, \ldots, 4 \), yields positive real values for \( X_1 \) and \( X_2 \) whose approximations are given below:

| \( i \) | \( X_1 \) | \( X_2 \) |
|-------|-------|-------|
| 1     | 0.3702 | 4.6215 |
| 2     | 0.5345 | 0.6682 |
| 3     | 1.0499 | 0.6338 |
| 4     | 1.5838 | 5.2195 |

Hence, there exist exactly four Einstein adapted metrics on \( M \).

(A.47, for \( p = 2 \)) For the bisymmetric triple \( A.47 \), in the case \( p = 2 \), the non-zero solutions of the equations (3.27) and (3.28) are of the form

\[
X_1 = \frac{1}{2} \alpha_i, \quad X_2 = -\frac{140}{3} \alpha_i^3 + 148 \alpha_i^2 - \frac{681}{5} \alpha_i + \frac{184}{5},
\]

where \( \alpha_i \) is a root of the polynomial

\[
t(z) = 350z^4 - 1110z^3 + 1179z^2 - 492z + 69.
\]

The polynomial \( t \) has exactly four positive real roots which are

\[
\begin{align*}
\alpha_1 &= \frac{111}{140} - \frac{\beta}{140} - \frac{1}{140} \left(\frac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}} \\
\alpha_2 &= \frac{111}{140} - \frac{\beta}{140} + \frac{1}{140} \left(\frac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}} \\
\alpha_3 &= \frac{111}{140} + \frac{\beta}{140} - \frac{1}{140} \left(\frac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}} \\
\alpha_4 &= \frac{111}{140} + \frac{\beta}{140} + \frac{1}{140} \left(\frac{2634\xi\beta - 14\xi^2\beta - 64526\beta - 35250\xi}{\xi\beta}\right)^{\frac{1}{2}}
\end{align*}
\]

where \( \xi = (290727 + 500i\sqrt{53545})^{\frac{1}{3}} \) and \( \beta = \left(\frac{14\xi^2 + 131\xi + 64526}{\xi}\right)^{\frac{1}{2}}. \)

Simple calculations show that \( \alpha_i, \; i = 1, \ldots, 4 \), yields positive real values of \( X_1 \) and \( X_2 \) whose approximations are given below:

| \( i \) | \( X_1 \) | \( X_2 \) |
|-------|-------|-------|
| 1     | 0.3086 | 7.4890 |
| 2     | 0.4686 | 0.6737 |
| 3     | 0.9326 | 0.6496 |
| 4     | 1.4616 | 8.1878 |

Hence, there are on \( M \) exactly four Einstein adapted metrics.

(A.47, for \( p = 4 \)) For the bisymmetric triple \( A.47 \), in the case \( p = 2 \), the non-zero solutions of the equations (3.27) and (3.28) are given by
\[ X_1 = \alpha_i, \quad X_2 = -\frac{100}{3} \alpha_i^3 + 100 \alpha_i^2 - \frac{262}{3} \alpha_i + 26, \]

where \( \alpha_i \) is a root of the polynomial

\[ t(z) = 200z^4 - 600z^3 + 614z^2 - 264z + 39. \]

Since \( X_1 = \alpha_i \), only positive roots of \( t \) yield positive solutions of the equations above. The polynomial \( t \) has exactly two positive roots which are indicated below:

\[ \begin{align*}
\alpha_1 &= \frac{3\xi \beta}{4} + \frac{\beta^3}{60} - \frac{\sqrt{3} \cdot 122926 \xi \beta^2 + 14002 \beta^2 + 1151 \beta^2 + 12620 \xi^3}{60 \xi \beta}, \\
\alpha_2 &= \frac{3\xi \beta}{4} + \frac{\beta^3}{60} + \frac{\sqrt{3} \cdot 122926 \xi \beta^2 + 14002 \beta^2 + 1151 \beta^2 + 12620 \xi^3}{60 \xi \beta},
\end{align*} \]

where \( \xi = (109457 + 180\sqrt{416842})^\frac{1}{5} \) and \( \beta = (14\xi^2 + 1317\xi + 64526)^\frac{1}{4} \).

Simple calculations show that \( \alpha_i, i = 1, \ldots, 4 \), yields positive real values of \( X_1 \) and \( X_2 \) whose approximations are given below:

\[
\begin{array}{cccc}
\hline
i & X_1 & X_2 \\
\hline
1 & 0.3143 & 7.3931 \\
2 & 1.4375 & 8.0839 \\
\hline
\end{array}
\]

Therefore, there are on \( M \) exactly two Einstein adapted metrics.

**A.47, for \( p = 6 \)** For the bisymmetric triple **A.47**, in the case \( p = 6 \), the non-zero solutions of the equations (3.27) and (3.28) are of the form

\[ X_1 = 3\alpha_i, \quad X_2 = -\frac{2500}{3} z^3 + 820z^2 - 235z + 24, \]

where \( \alpha_i \) is a root of the polynomial

\[ t(z) = 1250z^4 - 1230z^3 + 415z^2 - 60z + 3. \]

This polynomial has exactly two positive roots which are

\[ \begin{align*}
\alpha_1 &= \frac{41 \times 3 \xi \beta}{1500} + \frac{5 \xi \beta}{22500} - \frac{5 \xi \beta}{22500} \sqrt{3} (3887\sqrt{35} \frac{1}{5} \xi \beta^2 + 125\sqrt{35} \xi^3 \beta^2 - 20875\sqrt{35} \xi^3 \beta^2 + 527553 \times 5 \xi^3 \beta^2), \\
\alpha_2 &= \frac{41 \times 3 \xi \beta}{1500} + \frac{5 \xi \beta}{22500} + \frac{5 \xi \beta}{22500} \sqrt{3} (3887\sqrt{35} \frac{1}{5} \xi \beta^2 + 125\sqrt{35} \xi^3 \beta^2 - 20875\sqrt{35} \xi^3 \beta^2 + 527553 \times 5 \xi^3 \beta^2),
\end{align*} \]

where \( \xi = (14027 + 18\sqrt{2} \sqrt{483323})^\frac{1}{5} \) and \( \beta = (3887 \times 5 \xi^2 - 250 \times 5 \xi^4 + 208750)^\frac{1}{4} \).

Simple calculations show that \( \alpha_i, i = 1, \ldots, 2 \), yields positive real values of \( X_1 \) and \( X_2 \) whose approximations are given below:

\[
\begin{array}{cccc}
\hline
i & X_1 & X_2 \\
\hline
1 & 0.3163 & 7.3606 \\
2 & 1.4292 & 8.0485 \\
\hline
\end{array}
\]
Hence, there are on $M$ exactly two Einstein adapted metrics.\[A.42\] In this case there is no Einstein adapted metric on $M$. We get that, in particular, $X_2$ would be a root of the polynomial

$$t(z) = 9z^4 - 195z^3 + 1198z^2 - 1395z + 464,$$

but $t$ does not admit any positive root. \[\Box\]

In the classical case, similar methods as those briefly exposed in the proof above may be attempted to obtain solutions for the Einstein equations. However, as it can be read from Table 3.7 the eigenvalues depend on parameters in general which would retrieve very complicated equations. Though for the bisymmetric triples which satisfy one of the conditions $\gamma_1 = \gamma_2$ or $\gamma_2 = 1 - \gamma_1$ it is possible to classify all the Einstein adapted metrics by using Corollaries 2.14 and 2.15 respectively. For these spaces, the Einstein adapted metrics $g_M$ whose restriction $g_F$ is also Einstein were classified in 3.3. So it remains to obtain all the other possible Einstein adapted metrics.

**Theorem 3.5.** Let $F \rightarrow M \rightarrow N$ be a bisymmetric fibration of Type II such that $\gamma_2 = \gamma_1$ or $\gamma_2 = 1 - \gamma_1$. If there exists on $M$ an Einstein adapted metric such that $g_F$ is not Einstein, then the corresponding bisymmetric triple $(g, l, 1)$ is one of the triples in Table 3.11.

**Proof:** In the only case when $\gamma_2 = 1 - \gamma_1$, \[A.3\] for $p = 2l$ and $n = 2(l + s)$, we have $\gamma_1 = \frac{l}{l+s}$, $\gamma_2 = \frac{s}{l+s}$, $b_1 = \frac{l}{4(l+s)} = \frac{s}{4}$ and $b_2 = \frac{s}{4(l+s)} = \frac{1 - \gamma_1}{4}$. The required metric should be given by Corollary 2.15 (ii). Simple calculation show that $D(\gamma_1) = \frac{-1}{2}\gamma_1(1 - \gamma_1) < 0$, for every $l, s$, since $0 < \gamma_1 < 1$. Hence in this case there are no other Einstein adapted metrics besides those found previously.

The cases such that $\gamma_2 = \gamma_1$ are those listed in the proof of Theorem 3.2. In this case, there exists an Einstein adapted metric such that $g_F$ is not Einstein if and only if $D(\gamma_1) \geq 0$, where

$$D(\gamma_1) = 4r^2(1 - \gamma_1) - 2\gamma_1(2b_2 + 1 - \gamma_1)(2b_1 + 1 - \gamma_1),$$

according to Corollary 2.14 (ii).

In the cases \[A.3\] for $s = l = 2p$, \[A.15\] for $n = 2p$, \[A.23\] for $n = 2p$, as in the proof of 3.10 we have $b_1 = b_2 = \frac{s}{4}$. Hence, we simplify the expression for $D(\gamma_1)$ given above as
\[ D(\gamma_1) = \frac{1}{2}(-\gamma_1^3 + 4\gamma_1^2 - 6\gamma_1 + 2) \] (3.29)

For \( A.16 \) with \( s = l = 2p, n = 4l \), we have \( \gamma_1 = \frac{1}{2} \) and \( D(\frac{1}{2}) < 0 \); for \( A.15 \) with \( n = 2p, p \geq 2 \), we have \( \gamma_1 = \frac{p-1}{2p-1} \) and \( D(\gamma_1) \geq 0 \) only for \( p = 2, \ldots, 6 \); for \( A.23 \) with \( n = 2p, \gamma = \frac{p+1}{2p+1} \) and \( D < 0 \), for every \( p \geq 1 \).

For the case \( A.12 \) with \( s = l = 2p, n = 4l, l \geq 2 \), as in the proof of \( B.10 \) we have \( \gamma_1 = \frac{2l-1}{4l-1} \) and \( b_1 = b_2 = \frac{l}{2(4l-1)} = \frac{1-\gamma_1}{4} \). For this case, we have

\[ D(\gamma_1) = \frac{1}{2}(3\gamma_1 - 1)(3\gamma_1 - 2). \]

Since \( \gamma \in (\frac{3}{7}, \frac{1}{7}) \), we have \( D(\gamma_1) < 0 \), for every \( \gamma \) and thus there is no positive solution.

For \( A.16 \) with \( p = 2l, n = 4l \), we have \( \gamma_1 = \frac{2l-1}{4l-1} \) and \( b_1 = b_2 = \frac{l}{4(4l-1)} = \frac{1-\gamma_1}{8} \). We rewrite \( D(\gamma_1) \) as follows:

\[ D(\gamma_1) = \frac{1}{2}(1 - \gamma_1)(3\gamma_1^2 - 6\gamma_1 + 2). \]

Since \( \gamma_1 = \frac{2l-1}{4l-1} \in (\frac{1}{3}, \frac{1}{2}) \), simple calculations show that \( 3\gamma_1^2 - 6\gamma_1 + 2 \geq 0 \) only for \( l = 1 \). So only for \( l = 1 \) exists a metric with the desired properties.

For \( A.20 \) with \( s = l = 2p, n = 4l \), we have \( \gamma_1 = \frac{2l+1}{4l+1} \) and \( b_1 = b_2 = \frac{l}{4(4l+1)} = \frac{1-\gamma_1}{8} \). Hence

\[ D(\gamma_1) = \frac{1}{8}(1 - \gamma_1)(25\gamma_1^2 - 25\gamma_1 + 8). \]

As \( 25\gamma_1^2 - 25\gamma_1 + 8 > 0 \), for every \( \gamma_1 \), there exists a metric with the required properties for every \( l \geq 1 \).

In the case \( A.24 \) for \( p = 2l, n = 4l \), we have \( \gamma_1 = \frac{2l+1}{4l+1} \) and \( b_1 = b_2 = \frac{l}{4(4l+1)} = \frac{1-\gamma_1}{8} \). Thus

\[ D(\gamma_1) = \frac{1}{4}(1 - \gamma_1)(5\gamma_1^2 - 10\gamma_1 + 4). \]

Since \( \gamma_1 \in (\frac{1}{2}, \frac{3}{5}) \), we conclude that \( 5\gamma_1^2 - 10\gamma_1 + 4 \geq 0 \), for every \( l \geq 3 \). Hence, there exists an adapted Einstein metric for every \( l \geq 3 \).

\( \square \)

**Remark 3.4.** For the triples in Lemma \( 3.4 \), where \( C_{pi} \) is scalar on \( n \), such that \( D(\gamma_1) < 0 \), we can still consider the non-real complex solutions \( X_1, X_2 \) which gives rise to a non binormal Einstein adapted metric on \( M \) with non-real complex coefficients. The spaces in these conditions are the cases \( A.3 \) for solutions of the
form $X_2 = \frac{1}{2X_1}$, and for solutions of the form $X_2 = \frac{X_1}{1 - X_1}$ we have the cases \( A, 3 \)

$s = p = 2l$, \( n = 4l \), for every \( l \), \( \text{A.12} \)

\( n = 2p \), for \( p \geq 5 \), \( \text{A.23} \)

\( n = 2p \), for every \( p \), \( \text{A.12} \), with \( s = p = 2l \), \( n = 4l \), for \( l \geq 2 \), \( \text{A.10} \)

\( p = 2l \), \( n = 4l \), for \( l = 1, 2 \).

### 3.5 Application to 4-symmetric Spaces

A homogeneous space $G/L$ is said to be a 4-symmetric space if there exists $\sigma \in \text{Aut}(G)$ such that

$$(G_\sigma)_0 \subset L \subset G_\sigma$$

and $\sigma$ has order 4. Compact simply connected irreducible 4-symmetric spaces have been classified by J.A.Jimenez in \[20\] following the previous work of V.Kač (see e.g. \[16\], Chap.X), J.A Wolf and A.Gray \[48\]. It is shown in \[20\] that any compact simply connected irreducible 4-symmetric space is the total space of a fiber bundle whose fiber and base space are symmetric spaces and the base is an isotropy irreducible space of maximal rank. These spaces are fully described in Tables III, IV and V in \[20\]. Hence, for each compact simply connected irreducible 4-symmetric space $M$ there is a bisymmetric fibration $F \to M \to N$ of maximal rank whose base space $N$ is isotropy irreducible. The bisymmetric triples $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ corresponding to 4-symmetric spaces of maximal rank must be some of Tables 3.4, 3.5, 3.6 and 3.7. Hence, a simple comparison between the Tables in this chapter and the classification in \[20\] allow us to easily conclude about the existence of Einstein metrics on 4-symmetric spaces.

From Theorem 3.1 and Tables 3.8 and 3.9 we conclude the following:

**Corollary 3.2.** Let $M = G/L$ be a compact simply connected irreducible 4-symmetric spaces of Type I and $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ a bisymmetric triple corresponding to $M$ such that $L \subset K$. If $M$ admits an Einstein adapted metric, then $(\mathfrak{g}, \mathfrak{k}, \mathfrak{l})$ is one of the triples listed below.

(i) In the exceptional case:

\[
\begin{align*}
(\mathfrak{f}_4, \mathfrak{so}_9 \oplus \mathbb{R}) , \\
(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2 \oplus \mathfrak{sp}_3 \oplus \mathbb{R}) , \\
(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}) , \\
(\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R}) , \\
(\mathfrak{e}_8, \mathfrak{so}_{16} \oplus \mathfrak{so}_{2p} \oplus \mathfrak{so}_{16-2p}) , p = 1, 3 , \\
(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathbb{R}) , \\
(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathfrak{so}_8 \oplus \mathfrak{so}_8 \oplus \mathbb{R}) , \\
(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_6 \oplus \mathbb{R}) , \\
(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_6 \oplus \mathfrak{su}_2) ,
\end{align*}
\]
(ii) in the classical case:

\[
\begin{align*}
\text{(so}_{2n+1}, \text{so}_{2p+1} \oplus \text{so}_{2(n-p)}, \text{so}_{2p+1} \oplus \mathfrak{u}_{n-p}), \\
\text{(so}_{2n}, \text{so}_{2p} \oplus \text{so}_{2(n-p)}, \mathfrak{u}_{p} \oplus \text{so}_{2(n-p)}), \\
\text{(sp}_{n}, \text{sp}_{p} \oplus \text{sp}_{n-p}, \mathfrak{u}_{p} \oplus \text{sp}_{n-p}).
\end{align*}
\]

From Theorem 3.3 and Tables 3.10 and 3.11 we obtain the following results for Type II.

**Corollary 3.3.** Let \( M = G/L \) be a compact simply connected irreducible 4-symmetric spaces of Type II and \((\mathfrak{g}, \mathfrak{k}, \mathfrak{l})\) a bisymmetric triple corresponding to \( M \) such that \( L \subset K \). If \( M \) admits an Einstein adapted metric \( g_M \) such that \( g_F \) is also Einstein, then \((\mathfrak{g}, \mathfrak{k}, \mathfrak{l})\) is either

\[
(\text{so}_{8l}, \text{so}_{4l} \oplus \text{so}_{4l} \oplus \text{so}_{2l} \oplus \text{so}_{2l} \oplus \text{so}_{2l})
\]

or

\[
(\text{su}_{2(l+s)}, \text{su}_{2l} \oplus \text{su}_{2s}, \text{su}_{l} \oplus \text{su}_{s} \oplus \mathfrak{r} \oplus \mathfrak{r} \oplus \mathfrak{r}).
\]

The Einstein metric is binormal in the first case and in the second for \( l = s \).

Finally, the result below follows from Theorem 3.4 and Table 3.6.

**Corollary 3.4.** Let \( M = G/L \) be a compact simply connected irreducible 4-symmetric spaces of Type II, such that \( G \) is an exceptional Lie group, and \((\mathfrak{g}, \mathfrak{k}, \mathfrak{l})\) a bisymmetric triple corresponding to \( M \) such that \( L \subset K \). If \( M \) admits an Einstein adapted metric, then \((\mathfrak{g}, \mathfrak{k}, \mathfrak{l})\) is one of the following three cases:

\[
\begin{align*}
(\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_5 \oplus \mathfrak{r} \oplus \mathfrak{r}) \\
(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathfrak{so}_{10} \oplus \mathfrak{r} \oplus \mathfrak{r}) \\
(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathfrak{so}_{6} \oplus \mathfrak{so}_{6} \oplus \mathfrak{r}).
\end{align*}
\]

There are 4 Einstein adapted metrics for each of the first two cases and 2 for the last case. None of these metrics is binormal or such that the restriction to the fiber is Einstein.

**Remark 3.5.** We observe that for the 4-symmetric spaces corresponding to the first two cases the base space considered by Jimenez is different from the one indicated above. In this two cases, the bisymmetric triples considered in [20] are

\[
\begin{align*}
(\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathfrak{r}, \mathfrak{su}_5 \oplus \mathfrak{r} \oplus \mathfrak{r}) \\
(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathfrak{r}, \mathfrak{so}_{10} \oplus \mathfrak{r} \oplus \mathfrak{r}).
\end{align*}
\]

For these triples there is no Einstein adapted metric since the Casimir operator of \( \mathfrak{p} \) is not scalar.
### 3.6 Tables

#### Table 3.1: Dual Coxeter Numbers

| Coxeter group | Dual Coxeter number |
|---------------|---------------------|
| $A_n$         | $n + 1$             |
| $B_n$         | $2n - 1$            |
| $C_n$         | $n + 1$             |
| $D_n$         | $2n - 2$            |
| $E_6$         | 12                  |
| $E_7$         | 18                  |
| $E_8$         | 30                  |
| $F_4$         | 9                   |
| $G_2$         | 4                   |

#### Table 3.2: Symmetric pairs of compact type of maximal rank - Exceptional Spaces

| $\mathfrak{g}$ | $\mathfrak{k}$ | $\mathfrak{g}$ | $\mathfrak{k}$ |
|----------------|----------------|----------------|----------------|
| $\mathfrak{f}_4$ | $\mathfrak{sp}_3 \oplus \mathfrak{su}_2$ | $\mathfrak{e}_7$ | $\mathfrak{su}_8$ |
| $\mathfrak{f}_4$ | $\mathfrak{so}_9$ | $\mathfrak{e}_7$ | $\mathfrak{e}_6 \oplus \mathbb{R}$ |
| $\mathfrak{g}_2$ | $\mathfrak{su}_2 \oplus \mathfrak{su}_2$ | $\mathfrak{e}_7$ | $\mathfrak{so}_{12} \oplus \mathfrak{su}_2$ |
| $\mathfrak{e}_6$ | $\mathfrak{so}_{10} \oplus \mathbb{R}$ | $\mathfrak{e}_8$ | $\mathfrak{so}_{16}$ |
| $\mathfrak{e}_6$ | $\mathfrak{su}_6 \oplus \mathfrak{su}_2$ | $\mathfrak{e}_8$ | $\mathfrak{e}_7 \oplus \mathfrak{su}_2$ |

#### Table 3.3: Symmetric pairs of compact type of maximal rank - Classical Spaces

| $\mathfrak{g}$ | $\mathfrak{k}$ |
|----------------|----------------|
| $\mathfrak{so}_{2m}$ | $\mathfrak{u}_m$ |
| $\mathfrak{so}_n$ | $\mathfrak{so}_{2p} \oplus \mathfrak{so}_{n-2p}$ |
| $\mathfrak{sp}_n$ | $\mathfrak{u}_n$ |
| $\mathfrak{sp}_n$ | $\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$ |
| $\mathfrak{su}_n$ | $\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$ |
Table 3.4: Bisymmetric triples of type I and their eigenvalues - Exceptional spaces

| Appendix | g  | k  | l               | γ               | bφ          |
|----------|----|----|-----------------|-----------------|-------------|
| A.25     | f₄ | so₉| soₙ ⊕ so₉₋ₙ, p = 1, 3, 5, 7 | 7/9             | p(9−p)     |
| A.26     | f₄ | sp₂ ⊕ su₂| sp₃ ⊕ R       | 2/9             | 1/18         |
| A.27     | u₃ ⊕ su₂| sp₂ ⊕ su₂ ⊕ su₂ | 4/9             | 1/9, 1/18     |
| A.31     | g₂ | su₂ ⊕ su₂| R ⊕ su₂       | 1/2             | 1/8          |
| A.32     | su₂ ⊕ R     | 1/6            |              |
| A.33     | e₈ | so₁₆| so₂ₙ ⊕ so₁₆₋₂ₙ, p = 1, . . . , 4 | 1/5             | p(8−p)     |
| A.34     | u₈ | 1/5            |              |
| A.35     | e₈ | e₇ ⊕su₂| e₇ ⊕ R       | 1/15            | 1/15         |
| A.36     | e₆ | R ⊕ su₂       | 3/5            | 11/60, 9/20 |
| A.37     | so₁₂ ⊕ su₂ ⊕ su₂| 3/5            | 4/15, 1/5 |
| A.39     | su₄ ⊕ su₂| 4/9            |              |
| A.40     | e₇ | so₁₂ ⊕ su₂| so₁₂ ⊕ R     | 1/6             | 1/36         |
| A.41     | u₆ ⊕ su₂| 1/6            |              |
| A.42     | soₙ ⊕ so₁₂₋ₙ ⊕ su₂, p = 2, 4, 6 | 5/9            | p(12−p)     |
| A.43     | e₇ | e₆ ⊕ R       | 2/3             | 2/9, 1/9, 4/9 |
| A.44     | su₆ ⊕ su₂ ⊕ R | 2/3            |              |
| A.45     | e₇ | su₈ | suₙ ⊕ su₈₋ₙ ⊕ R, 1 ≤ p ≤ 4 | 4/9             |              |
| A.50     | su₈ | 4/9            |              |
| A.51     | e₆ | so₁₀ ⊕ R | u₅ ⊕ R | 2/3             | 2/9, 1/6, 1/4 |
| A.52     | soₙ ⊕ so₁₀₋ₙ ⊕ R, p = 2, 4 | 2/3             | p(10−p)     |
| A.53     | su₆ ⊕ su₂| su₆ ⊕ R | 1/6             | 1/24, p     |
| A.54     | suₙ ⊕ su₆₋ₙ ⊕ R ⊕ su₂ | 1/2             |              |
Table 3.5: Bisymmetric triples of type I and their eigenvalues - Classical spaces

| Appendix | \( g \) | \( \ell \) | \( l \) | \( \gamma \) | \( b^\phi \) |
|----------|--------|--------|-----------------|--------|-----------------|
| A.1      | \( \mathfrak{su}_n \) \( \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R} \) | \( \mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathbb{R} \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R} \) | \( \frac{p}{n} \) | \( \frac{p-l}{2n-l} \) |
| A.2      | \( \mathfrak{so}_{2n+1} \oplus \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)} \), \( p = 0, \ldots, n-1 \) | \( \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)} \) | \( \frac{2p-l}{2n-l} \) | \( \frac{4l+1}{4(n-l)} \) |
| A.5      | \( \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)} \) | \( \mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p} \) | \( \frac{2(n-p-s)}{2n-1} \) | \( \frac{s}{2n-1} \) |
| A.6      | \( \mathfrak{so}_{2n} \) \( \mathfrak{u}_n \) | \( \mathfrak{u}_p \oplus \mathfrak{u}_{n-p} \) | \( \frac{n}{2(n-1)} \) | \( \frac{n-p}{2(n-1)} \), \( \frac{p}{2(n-1)} \), \( \frac{n-2}{4(n-1)} \) |
| A.9      | \( \mathfrak{so}_{2n} \) \( \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)} \), \( p = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) | \( \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)} \) | \( \frac{p-l}{2(n-l)} \) | \( \frac{l}{2(n-1)} \) |
| A.10     | \( \mathfrak{so}_{2n} \) \( \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)} \), \( p = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor \) | \( \mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)} \) | \( \frac{p-1}{n-1} \) | \( \frac{p-l}{4(n-1)} \) |
| A.13     | \( \mathfrak{sp}_n \) \( \mathfrak{u}_n \) | \( \mathfrak{u}_p \oplus \mathfrak{u}_{n-p} \), \( p = 1, \ldots, n-1 \) | \( \frac{n}{2(n+1)} \) | \( \frac{n-p}{2(n+1)} \), \( \frac{p}{2(n+1)} \), \( \frac{p-1}{n+1} \), \( \frac{n+2}{2(n+1)} \) |
| A.17     | \( \mathfrak{sp}_n \) \( \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p} \) | \( \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_{n-p} \) | \( \frac{p+1}{n+1} \) | \( \frac{p-l}{4(n+1)} \) |
| A.18     | \( \mathfrak{sp}_n \) \( \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p} \) | \( \mathfrak{u}_p \oplus \mathfrak{sp}_{n-p} \) | \( \frac{p+1}{n+1} \) | \( \frac{p-l}{4(n+1)} \) |
| A.21     | \( \mathfrak{sp}_n \) \( \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p} \) | \( \mathfrak{u}_p \oplus \mathfrak{sp}_{n-p} \) | \( \frac{p+1}{n+1} \) | \( \frac{p-l}{4(n+1)} \) |
Table 3.6: Bisymmetric triples of type II and their eigenvalues - Exceptional spaces

| Appendix | g    | k    | l       | $\gamma_1$ | $\gamma_2$ | $b_1^\phi$ | $b_2^\phi$ |
|----------|------|------|---------|-------------|-------------|-------------|-------------|
| A.29     | $f_4$| $sp_3 \oplus su_2$| $u_3 \oplus R$| $\frac{4}{9}$| $\frac{2}{9}$| $\left(\frac{1}{3}, \frac{2}{9}\right)$| $\frac{1}{18}$|
| A.30     |      |     | $su_2 \oplus sp_2 \oplus R$| $\frac{4}{9}$| $\frac{2}{9}$| $\left(\frac{1}{3}, \frac{1}{9}\right)$| $\frac{1}{18}$|
| A.33     | $g_2$| $su_2 \oplus su_2$| $R \oplus R$| $\frac{1}{2}$| $\frac{1}{6}$| $\frac{1}{8}$| $\frac{1}{6}$|
| A.38     | $e_8$| $e_7 \oplus su_2$| $e_6 \oplus R \oplus R$| $\frac{3}{5}$| $\frac{1}{15}$| $\left(\frac{11}{60}, \frac{11}{60}, \frac{9}{20}\right)$| $\frac{1}{60}$|
| A.40     |      |     | $so_{12} \oplus su_2 \oplus R$| $\frac{3}{5}$| $\frac{1}{15}$| $\left(\frac{4}{15}, \frac{1}{5}\right)$| $\frac{1}{60}$|
| A.42     |      |     | $su_8 \oplus R$| $\frac{3}{5}$| $\frac{1}{15}$| $\frac{1}{4}$| $\frac{1}{60}$|
| A.45     | $e_7$| $so_{12} \oplus su_2$| $u_6 \oplus R$| $\frac{5}{9}$| $\frac{1}{9}$| $\left(\frac{5}{18}, \frac{1}{6}, \frac{5}{18}\right)$| $\frac{1}{36}$|
| A.47     |      |     | $so_p \oplus so_{12-p} \oplus R$, $p$ even| $\frac{5}{9}$| $\frac{1}{9}$| $\frac{1}{36}$| $\frac{p(12-p)}{144}$|
| A.55     | $e_6$| $su_6 \oplus su_2$| $su_p \oplus su_{6-p} \oplus R \oplus R$| $\frac{1}{2}$| $\frac{1}{6}$| $\frac{1}{24}$| $\frac{p+2}{24}$| $\frac{p}{8}$|
Table 3.7: Bisymmetric triples of type II and their eigenvalues - Classical spaces

| Appendix | $\mathfrak{g}$ | $\mathfrak{k}$ | $\mathfrak{l}$ | $\gamma_1$ | $\gamma_2$ | $b_1^0$ | $b_2^0$ |
|----------|----------------|----------------|---------------|------------|------------|----------|----------|
| A.3      | $\mathfrak{su}_n$ | $\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}$ | $\mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ | $\frac{p}{n}$ | $\frac{n-p}{n}$ | $\left( \frac{p-l}{2n-1}, \frac{l}{2n} \right)$ | $\left( \frac{n-p-s}{2n}, \frac{s}{2n} \right)$ |
| A.8      | $\mathfrak{so}_{2n+1}$ | $\mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}$ | $\mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}$ | $\frac{2p-1}{2n-1}$ | $\frac{2(n-p-1)}{2n-1}$ | $\left( \frac{p-l}{2n-1}, \frac{4l}{4(n-1)} \right)$ | $\left( \frac{n-p-s}{2n-1}, \frac{s}{2n-1} \right)$ |
| A.7      | $\mathfrak{so}_{2n}$ | $\mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}$ | $\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}$ | $\frac{p-1}{n-1}$ | $\frac{n-p-1}{n-1}$ | $\left( \frac{p-l}{2(n-1)}, \frac{4l}{4(n-1)} \right)$ | $\left( \frac{n-p-s}{2(n-1)}, \frac{s}{2(n-1)} \right)$ |
| A.12     | $\mathfrak{sp}_n$ | $\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$ | $\mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s}$ | $\frac{p+1}{n+1}$ | $\frac{n-p+1}{n+1}$ | $\left( \frac{p-l}{4(n+1)}, \frac{4l}{4(n+1)} \right)$ | $\left( \frac{n-p-s}{4(n+1)}, \frac{s}{4(n+1)} \right)$ |
| A.23     | $\mathfrak{sp}_n$ | $\mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}$ | $\mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s}$ | $\frac{p+1}{n+1}$ | $\frac{n-p+1}{n+1}$ | $\left( \frac{p-l}{4(n+1)}, \frac{4l}{4(n+1)} \right)$ | $\left( \frac{n-p-s}{4(n+1)}, \frac{s}{4(n+1)} \right)$ |
Table 3.8: Einstein Bisymmetric triples of type I - Exceptional spaces

| g   | ι   | l   | X            |
|-----|-----|-----|--------------|
| j₄  | sp₃ ⊕ su₂ | sp₃ ⊕ R | ½, 4        |
| j₄  | so₉ | so₈ | 1, 2/7       |
|     |     | so₇ ⊕ R | 9 ± √8       |
| g₂  | su₂ ⊕ su₂ | R ⊕ su₂ | ½, 3/2       |
|     |     | su₂ ⊕ R | 6 ± √27/2    |
| e₆  | so₁₀ ⊕ R | so₈ ⊕ R ⊕ R | 1, 1/2       |
| e₆  | su₆ ⊕ su₂ | R ⊕ su₅ ⊕ su₂ | ½, 3/2       |
|     |     | su₆ ⊕ R | 1/7, 11/7    |
| e₇  | so₁₂ ⊕ su₂ | so₁₂ ⊕ R | 17/2, 1/2    |
|     |     | R ⊕ so₁₀ ⊕ su₂ | ½, 13/10    |
|     |     | so₄ ⊕ so₈ ⊕ su₂ | 1, 4/5       |
| e₇  | su₈ | su₇ ⊕ R | ½, 7/4       |
| e₈  | c₇ ⊕ su₂ | c₇ ⊕ R | ½, 29/2      |
| e₈  | so₁₆ | so₂₀p ⊕ so₁₆₋₂p | 15 ± √7p² − 56p + 113/14 |
Table 3.9: Einstein Bisymmetric triples of type I - Classical spaces

| \( g \)     | \( \mathfrak{f} \)                     | \( \mathfrak{l} \)                     | \( X \)                                                                 |
|------------|----------------------------------------|----------------------------------------|-------------------------------------------------------------------------|
| \( \mathfrak{so}_{2n} \) | \( \mathfrak{so}_p \oplus \mathfrak{so}_{2(n-p)} \) | \( \mathfrak{so}_p \oplus \mathfrak{so}_p \oplus \mathfrak{so}_{2(n-p)} \), \( p \) even | \( n-1 \pm \sqrt{p^2-(2n+1)p+n^2+1} \) \( \frac{1}{2} \), \( \frac{n-p}{p-1} \) \( - \frac{1}{2} \) |
| \( \mathfrak{so}_{2n+1} \) | \( \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)} \) | \( \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{n-p} \oplus \mathfrak{so}_{n-p} \), \( n \) even | \( 2n-1 \pm \sqrt{4p^2+8p-4n+5} \) \( \frac{1}{2} \), \( \frac{n+p}{2(n-p-1)} \) |
| \( \mathfrak{sp}_n \) | \( \mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{n-2l} \) | \( \mathfrak{sp}_{l} \oplus \mathfrak{sp}_{l} \oplus \mathfrak{sp}_{n-2l} \) | \( n+1 \pm \sqrt{6l^2+(3-4n)l+n^2+1} \) \( \frac{1}{2} \), \( \frac{1}{2} + \frac{n-p}{p+1} \) |
| \( \mathfrak{su}_n \) | \( \mathfrak{su}_{2l} \oplus \mathfrak{su}_{n-2l} \oplus \mathfrak{R} \) | \( \mathfrak{su}_l \oplus \mathfrak{su}_l \oplus \mathfrak{R} \oplus \mathfrak{su}_{n-2l} \oplus \mathfrak{R} \) | \( \frac{1}{2} \), \( \frac{n}{2l} - \frac{1}{2} \) |
Table 3.10: Bisymmetric triples of type II with Einstein metric such that $g_F$ is also Einstein

### $g_M$ binormal

| $g$        | $\xi$        | $l$                | $X$                                      |
|------------|--------------|--------------------|------------------------------------------|
| $su_{4l}$  | $su_{2l} \oplus su_{2l} \oplus \mathbb{R}$ | $su_l \oplus su_l \oplus su_l \oplus su_l \oplus R^3$ | 1                                        |
| $so_{8l}$  | $so_{4l} \oplus so_{4l}$ | $so_{2l} \oplus so_{2l} \oplus so_{2l} \oplus so_{2l}$ | $\frac{4l-1+\sqrt{2l}}{2(2l-1)}$          |
| $so_{8l}$  | $so_{4l} \oplus so_{4l}$ | $so_{2l} \oplus so_{2l} \oplus u_{2l}$ | $\frac{2l-1+\sqrt{2l}}{2(l-1)}$          |
| $so_{4l}$  | $so_{2l} \oplus so_{2l}$ | $u_l \oplus u_l$, $l \geq 2$ | $\frac{2l+1+\sqrt{2l^2+2l+1}}{2(2l+1)}$ |
| $sp_{4l}$  | $sp_{2l} \oplus sp_{2l}$ | $sp_l \oplus sp_l \oplus sp_l \oplus sp_l$ | $\frac{4l+1+\sqrt{l(2l-1)}}{2(2l+1)}$ |
| $sp_{4l}$  | $sp_{2l} \oplus sp_{2l}$ | $sp_l \oplus sp_l \oplus u_{2l}$ | $\frac{4l+1+\sqrt{l(2l-1)}}{2(2l+1)}$ |

### $g_M$ non-binormal

| $g$        | $\xi$        | $l$                | $X_1$ | $X_2$ |
|------------|--------------|--------------------|-------|-------|
| $su_{2(l+s)}$ | $su_{2l} \oplus su_{2l} \oplus \mathbb{R}$ | $su_l \oplus su_l \oplus su_l \oplus su_l \oplus R^3$ | $\frac{l+s}{2l}$ | $\frac{l+s}{2s}$ |

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Table 3.11: All other Einstein adapted metrics for the bisymmetric triples of Type II which admit an EAM $g_M$ such that $g_F$ is Einstein.

| $g$    | $\mathfrak{r}$ | $l$       | $X_1$                                      | $X_2$                                      |
|--------|-----------------|-----------|-------------------------------------------|-------------------------------------------|
| $\mathfrak{so}_2\mathfrak{r}_l$ | $\mathfrak{so}_2\mathfrak{r}_l \oplus \mathfrak{so}_2\mathfrak{r}_l$ | $u_l \oplus u_l, l = 2, \ldots, 6$ | $\frac{2l(l-1) \pm \sqrt{(-l^4+7l^3-5l^2+l)/2}}{2(l-1)(3l-1)} \cdot \frac{4 \pm \sqrt{15}}{8}$ | $\frac{l}{2(l-1)} \cdot \frac{1}{X_1}$ |
| $\mathfrak{so}_{2l}$ | $\mathfrak{so}_{2l} \oplus \mathfrak{so}_{2l}$ | $\mathbb{R} \oplus \mathbb{R} \oplus u_2$ | $\frac{4l+1+\sqrt{13l^2+7l+4}}{5(2l+1)}$ | $\frac{l}{2l+1} \cdot \frac{1}{X_1}$ |
| $\mathfrak{sp}_{2l}$ | $\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$ | $\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus \mathfrak{sp}_l, l \geq 1$ | $\frac{2l(l+1) \pm \sqrt{4l^2-8l-1}}{5(2l+1)}$ | $\frac{l}{2l+1} \cdot \frac{1}{X_1}$ |
| $\mathfrak{sp}_{2l}$ | $\mathfrak{sp}_{2l} \oplus \mathfrak{sp}_{2l}$ | $\mathfrak{sp}_l \oplus \mathfrak{sp}_l \oplus u_{2l}, l \geq 3$ | $\frac{4l+1+\sqrt{13l^2+7l+4}}{5(2l+1)}$ | $\frac{l}{2l+1} \cdot \frac{1}{X_1}$ |
Table 3.12: Einstein bisymmetric fibrations of Type II with $g$ exceptional

| $g$       | $t$       | $l$       | Number of Einstein adapted metrics |
|-----------|-----------|-----------|-----------------------------------|
| $g_2$     | $su_2$ ⊕ $su_2$ | $R ⊕ R$  | 2                                 |
| $e_6$     | $su_6$ ⊕ $su_2$ | $su_5$ ⊕ $R ⊕ R$ | 4                                 |
| $e_7$     | $so_{12}$ ⊕ $su_2$ | $R ⊕ so_{10}$ ⊕ $R$ | 4                                 |
| $e_7$     | $so_{12}$ ⊕ $su_2$ | $so_4$ ⊕ $so_8$ ⊕ $R$ | 2                                 |
| $e_7$     | $so_{12}$ ⊕ $su_2$ | $so_6$ ⊕ $so_6$ ⊕ $R$ | 2                                 |
CHAPTER 4

In this Chapter we consider a fibration

$$\frac{\Delta^p G_0 \times \Delta^q G_0}{\Delta^n G_0} \to \frac{G^n_0}{\Delta^n G_0} \to \frac{G^p_0}{\Delta^p G_0} \times \frac{G^q_0}{\Delta^q G_0},$$

where $G_0$ is compact connected simple Lie group and $\Delta^m G_0$ is the diagonal subgroup in $G^m_0$, for $m = p, q, n$. The spaces $\frac{G^n_0}{\Delta^n G_0}$ are $n$-symmetric spaces and it has been proved by Kowalski that under some conditions they are not $k$-symmetric for $k < n$ (see e.g. [27]). Hence, throughout this Chapter we shall designate such a space by a Kowaslki $n$-symmetric space. McKenzie Y. Wang and Wolfgang Ziller have shown that these spaces are standard Einstein manifolds ([45], [36]). We obtain new Einstein metrics with totally geodesic fibers. In section 1 we describe the isotropy subspaces and compute the necessary eigenvalues to obtain the Ricci curvature of an adapted metric on $M$. In Section 2 we show that, for $n > 4$, there exists at least one non-standard Einstein adapted metric on $M$, which is binormal or such that the metric on the base space is also Einstein if and only if $p = q$. We prove that for $n = 4$ the standard metric is the only Einstein adapted metric. We remark that for $n = 4$ the fibration above is a bisymmetric fibration of non-maximal rank and whose base space is isotropy reducible in opposition to the cases studied in Chapter 3.

4.1 Kowalski N-Symmetric Spaces - The Isotropy
Representation and the Casimir Operators

Let $G_0$ be a compact connected simple Lie group and $\mathfrak{g}_0$ its Lie algebra. For any positive integer $m$ we denote by $G^m_0$ (or $\mathfrak{g}^m_0$) the direct product of $G_0$ ($\mathfrak{g}_0$, resp.) by itself $m$ times. By $\Delta^m G_0$ (or $\Delta^m \mathfrak{g}_0$) we denote the diagonal in $G^m_0$ (in $\mathfrak{g}^m_0$, resp.). Clearly, the Lie algebras of $G^m_0$ and $\Delta^m G_0$ are $\mathfrak{g}^m_0$ and $\Delta^m \mathfrak{g}_0$, respectively. Let $n, p, q$ be positive integers such that $p + q = n$ and $2 \leq p \leq q \leq n - 2$. Set $G = G^n_0$ and consider the following closed subgroups of $G$:

$$K = \Delta^p G_0 \times \Delta^q G_0.$$
\[
L = \triangle^n G_0 \subset K.
\]
The Lie algebras of \(G, K\) and \(L\) are, respectively,
\[
\mathfrak{g} = \mathfrak{g}_0^n,
\]
\[
\mathfrak{t} = \triangle^p \mathfrak{g}_0 \times \triangle^q \mathfrak{g}_0,
\]
\[
\mathfrak{l} = \triangle^n \mathfrak{g}_0.
\]
Following the notation of previous chapters, we write \(M = G/L\), \(N = G/K\) and \(F = K/L\). We consider the fibration \(F \to M \to N\). We note that \(N = \frac{G_0^n}{\triangle^n G_0} \times \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0}\) and thus this fibration is
\[
\frac{G_0^n}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0}
\]
with fiber
\[
F = \frac{\triangle^p G_0 \times \triangle^q G_0}{\triangle^n G_0}.
\]
Let \(\Phi_0\) be the Killing form of \(\mathfrak{g}_0\). Then, the Killing form of \(\mathfrak{g}\) is
\[
\Phi = \Phi_0 + \ldots + \Phi_0
\]
Following the notation of chapters 2 and 3, let \(n\) be the orthogonal complement of \(\mathfrak{t}\) in \(\mathfrak{g}\) and \(p\) be an orthogonal complement of \(l\) in \(\mathfrak{t}\), with respect to \(\Phi\). Then,
\[
\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p} \oplus \mathfrak{n}
\]
and we write \(m = p \oplus n\).

**Lemma 4.1.** (i) \(p = \{(\underbrace{qX, \ldots, qX}_{p}, \underbrace{-pX, \ldots, -pX}_{q}) : X \in \mathfrak{g}_0\}\) and \(p\) is an irreducible \(\text{Ad} L\)-module;

(ii) \(n = n_1 \oplus n_2\), where
\[
n_1 = \{(X_1, \ldots, X_p, 0, \ldots, 0) : X_j \in \mathfrak{g}_0, \sum_{j=1}^{p} X_j = 0\} \subset \mathfrak{g}_0^p \times 0_q
\]
\[
n_2 = \{(0, \ldots, 0, X_1, \ldots, X_q) : X_j \in \mathfrak{g}_0, \sum_{j=1}^{q} X_j = 0\} \subset 0_p \times \mathfrak{g}_0^q;
\]
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Proof: Let \((Z, \ldots, Z) \in \mathfrak{k}\) and \((X, \ldots, X, Y, \ldots, Y) \in \mathfrak{k}\), where \(X, Y\) and \(Z\) are arbitrary elements in \(\mathfrak{g}_0\).

\[
0 = \Phi((X, \ldots, X, Y, \ldots, Y), (Z, \ldots, Z)) = p\Phi_0(X, Z) + q\Phi_0(Y, Z) = \Phi_0(pX + qY, Z)
\]

Since \(\Phi_0\) is non-degenerate on \(\mathfrak{g}_0\) and \(Z\) is arbitrary, the identity above is possible if and only if \(pX + qY = 0\). Hence, we conclude that \(\mathfrak{p}\) is formed by elements of the form

\[
(gX, \ldots, gX, -pX, \ldots, -pX).
\]

where \(X \in \mathfrak{g}_0\). Moreover, it is clear that \(\text{Ad } L\)-submodules of \(\mathfrak{p}\) correspond to ideals of \(\mathfrak{g}_0\). Since \(\mathfrak{g}_0\) is simple we conclude that \(\mathfrak{p}\) is irreducible.

Clearly, we may write \(\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2\), where \(\mathfrak{n}_1\) is an orthogonal complement of \(\Delta^p\mathfrak{g}_0 \times 0\) in \(\mathfrak{g}_0^p \times 0\) and \(\mathfrak{n}_2\) is an orthogonal complement of \(0 \times \Delta^q\mathfrak{g}_0\) in \(0 \times \mathfrak{g}_0^q\), with respect to \(\Phi\).

Let \((X_1, \ldots, X_p, 0, \ldots, 0) \in \mathfrak{g}_0^p \times 0\) and \((Z, \ldots, Z, 0, \ldots, 0) \in \Delta^p\mathfrak{g}_0 \times 0\), where \(Z\) and \(X_j\) are arbitrary elements in \(\mathfrak{g}_0\).

\[
0 = \Phi((X_1, \ldots, X_p, 0, \ldots, 0), (Z, \ldots, Z, 0, \ldots, 0)) = \Phi_0(X_1, Z) + \ldots + \Phi_0(X_p, Z) = \Phi_0(\sum_{j=1}^{p} X_j, Z).
\]

Also by the nondegeneracy of \(\Phi_0\), we conclude that the identity above holds if and only if \(\sum_{j=1}^{p} X_j = 0\). This gives the required expression for \(\mathfrak{n}_1\). To prove the expression for \(\mathfrak{n}_2\) is similar.

\(\square\)

As usual we denote the Casimir operator of a subspace \(V\) with respect to \(\Phi\), the Killing form of \(\mathfrak{g}\), by \(C_V\) (see Definition 1.3). Also, we denote the identity map and the zero map of \(V\), by \(\text{Id}_V\) and \(0_V\), respectively.

**Proposition 4.1.** (i) \(C_{\mathfrak{g}} = \text{Id}_{\mathfrak{g}}\);

(ii) \(C_{\mathfrak{l}} = \frac{1}{n} \text{Id}_{\mathfrak{g}}\);

(iii) \(C_{\mathfrak{p}} = \frac{q}{np} \text{Id}_{\mathfrak{g}_0^p} \times \frac{p}{nq} \text{Id}_{\mathfrak{g}_0^q}\);
(iv) $C_t = \frac{1}{p} I_{d\varrho_0} \times \frac{1}{q} I_{d\varrho_0}$;

(v) $C_{n_1} = \left(1 - \frac{1}{p}\right) I_{d\varrho_0} \times 0_{\varrho_0}$ and $C_{n_2} = 0_{\varrho_0} \times \left(1 - \frac{1}{q}\right) I_{d\varrho_0}$.

**Proof:** Let $(u_i)_i$ and $(u'_i)_i$ be bases of $g_0$ dual with respect to $\Phi_0$. Then the Casimir operator of $g_0$ with respect to $\Phi_0$ is $C_{g_0} = \sum_i \text{ad}(u_i) \text{ad}(u'_i)$. We observe that since $g_0$ is a simple Lie algebra, $C_{g_0} = I_{g_0}$.

(i) \[ \{(u_i, 0, \ldots, 0), (0, u_i, \ldots, 0), \ldots, (0, 0, \ldots, u_i)\}_i \]

and \[ \{(u'_i, 0, \ldots, 0), (0, u'_i, \ldots, 0), \ldots, (0, 0, \ldots, u'_i)\}_i \]

are bases for $g$. We have

$$\Phi((0, \ldots, u_i), (0, \ldots, u'_j), (0, \ldots, 0)) = \delta_{kl} \Phi_0(u_i, u'_j) = \delta_{kl} \delta_{ij}.$$ 

Hence, the bases above are dual for $\Phi$ and thus we may write

$$C_{g} = \sum_i \text{ad}(u_i, 0, \ldots, 0) \text{ad}(u'_i, 0, \ldots, 0) + \ldots + \sum_i \text{ad}(0, \ldots, 0, u_i) \text{ad}(0, \ldots, 0, u'_i)$$

$$= (\sum_i \text{ad}(u_i) \text{ad}(u'_i), 0, \ldots, 0) + \ldots + (0, 0, \ldots, \sum_i \text{ad}(u_i) \text{ad}(u'_i))$$

$$= (C_{g_0}, \ldots, C_{g_0})$$

$$= (I_{d g_0}, \ldots, I_{d g_0})$$

$$= I_{d g}.$$

(ii) $\{(u_i, \ldots, u_i)\}_i$ and $\{(u'_i, \ldots, u'_i)\}_i$ are bases for $l$ and we have

$$\Phi((u_i, \ldots, u_i), (u'_j, \ldots, u'_j)) = n\Phi_0(u_i, u'_j) = n\delta_{ij}.$$ 

Hence, $\left\{\frac{1}{\sqrt{n}}(u_i, \ldots, u_i)\right\}_i$ and $\left\{\frac{1}{\sqrt{n}}(u'_i, \ldots, u'_i)\right\}_i$ are bases for $l$ dual with respect to $\Phi$. So we have
\[ C_t = \frac{1}{n} \sum_i \text{ad}_{(u_i, \ldots, u_i)} \text{ad}_{(u'_i, \ldots, u'_i)} \]
\[ = \frac{1}{n} (\sum_i \text{ad}_{u_i} \text{ad}_{u'_i}, \ldots, \sum_i \text{ad}_{u_i} \text{ad}_{u'_i}) \]
\[ = \frac{1}{n} (C_{g_0}, \ldots, C_{g_0}) \]
\[ = \frac{1}{n} (\text{Id}_{g_0}, \ldots, \text{Id}_{g_0}) \]
\[ = \frac{1}{n} \text{Id}_q. \]

(iii) \[ \{(qu_i, \ldots, qu_i, -pu_i, \ldots, -pu_i)\}_p \] and \[ \{(qu'_i, \ldots, qu'_i, -pu'_i, \ldots, -pu'_i)\}_q \] are bases of \( p \).

\[ \Phi((qu_i, \ldots, qu_i, -pu_i, \ldots, -pu_i), (qu'_i, \ldots, qu'_i, -pu'_i, \ldots, -pu'_i)) = q^2 p \Phi_0 (u_i, u'_j) + p^2 q \Phi_0 (u_i, u'_j) \]
\[ = (p + q) pq \delta_{ij} \]
\[ = np pq \delta_{ij}. \]

Hence dual bases of \( p \) for \( \Phi \) are as follows:

\[ \left\{ \left( \frac{q}{np} \right)^\frac{i}{2} u_i, \ldots, \left( \frac{q}{np} \right)^\frac{i}{2} u_i, - \left( \frac{p}{nq} \right)^\frac{i}{2} u_i, \ldots, - \left( \frac{p}{nq} \right)^\frac{i}{2} u_i \right\}_i \]

and \[ \left\{ \left( \frac{q}{np} \right)^\frac{i}{2} u'_i, \ldots, \left( \frac{q}{np} \right)^\frac{i}{2} u'_i, - \left( \frac{p}{nq} \right)^\frac{i}{2} u'_i, \ldots, - \left( \frac{p}{nq} \right)^\frac{i}{2} u'_i \right\}_i \].

Similar calculations as those done above show that

\[ C_p = \frac{q}{np} (C_{g_0}, \ldots, C_{g_0}) \times \frac{p}{nq} (C_{g_0}, \ldots, C_{g_0}) = \frac{q}{np} \text{Id}_{g_0^p} \times \frac{p}{nq} \text{Id}_{g_0^q}; \]

(iv)

\[ C_t = C_1 + C_p = \left( \frac{1}{n} + \frac{q}{np} \right) \text{Id}_{g_0^p} \times \left( \frac{1}{n} + \frac{p}{nq} \right) \text{Id}_{g_0^q} = \frac{1}{p} \text{Id}_{g_0^p} \times \frac{1}{q} \text{Id}_{g_0^q}. \]

(v)

\[ C_{n_1} + C_{n_2} = C_n = C_g - C_t = \left( 1 - \frac{1}{p} \right) \text{Id}_{g_0^p} \times \left( 1 - \frac{1}{q} \right) \text{Id}_{g_0^q}. \]

Since \([n_1, g] \subset g_0^p \times 0\) and \([n_2, g] \subset 0 \times g_0^q\), we conclude that
For the eigenvalues of the Casimir operators of \( l, t, p \) and \( n \) we use notation similar to that used in previous chapters. We recall that \( c_{l,p} \) is the eigenvalue of \( C_l \) on \( p \), \( c_{t,i} \) is the eigenvalue of \( C_t \) on \( n_i \). The Casimir operator of \( p \) is scalar on \( n_i \), as we can see from Corollary 4.1, and \( b^j \) denotes the eigenvalue of \( C_p \) on \( n_i \) for \( i = 1, 2 \). Also \( c_{n,p} \) and \( \gamma \) are the constants defined by

\[
\Phi(C_{n,\cdot \cdot \cdot}, \cdot |_{p\times p} = c_{n,p} \Phi |_{p\times p}, \quad i = 1, 2
\]

\[
\Phi(C_{t,\cdot \cdot \cdot}, \cdot |_{p\times p} = \gamma \Phi |_{p\times p}.
\]

**Corollary 4.1.** (i) \( c_{l,p} = \frac{1}{n} \);

(ii) \( C_p \) is scalar on \( n_j \), \( j = 1, 2 \) and \( b^1 = \frac{q}{np} \) and \( b^2 = \frac{p}{nq} \);

(iii) \( c_{t,1} = \frac{1}{p} \) and \( c_{t,2} = \frac{1}{q} \);

(iv) \( \gamma = \frac{q^2 + p^2}{n pq} \);

(v) \( c_{n,1,p} = \frac{(p - 1)q}{pn} \) and \( c_{n,2,p} = \frac{(q - 1)p}{qn} \).

**Proof:** The number \( c_{l,p} \) is the eigenvalue of the Casimir operator of \( l \) on \( p \). Thus, it follows from Proposition 4.1 (ii), that \( c_{l,p} = \frac{1}{n} \). Since \( n_1 \subset g_0^p \times 0 \) and \( n_2 \subset 0 \times g_0^q \), we conclude from Proposition 4.1 (iii) that \( C_p \) is scalar on \( n_1 \) and on \( n_2 \); moreover its eigenvalues on these two spaces are \( b^1 = \frac{q}{np} \) and \( b^2 = \frac{p}{nq} \), respectively. Similarly, it follows from Proposition 4.1 (iv) that the eigenvalues of \( C_t \) on \( n_1 \) and on \( n_2 \) are \( c_{t,1} = \frac{1}{p} \) and \( c_{t,2} = \frac{1}{q} \), respectively. Now to show (iv) we recall that \( \gamma \) is defined by the identity

\[
\Phi(C_{t,\cdot \cdot \cdot}, \cdot |_{p\times p} = \gamma \Phi |_{p\times p}.
\]

Let \( (qX, \ldots, qX, -pX, \ldots, -pX) \in p \).

\[
\Phi((qX, \ldots, qX, -pX, \ldots, -pX), (qX, \ldots, qX, -pX, \ldots, -pX))
\]

\[
= (q^2, p + p^2, q) \Phi_0(X, X)
\]

\[
= npq \Phi_0(X, X)
\]

By using Proposition 4.1 (iv) we get the following:
\[ \Phi(C_{qX, \ldots, qX, -pX, \ldots, -pX}, (qX, \ldots, qX, -pX, \ldots, -pX)) = \Phi((\frac{q}{p}X, \ldots, \frac{q}{p}X, -\frac{p}{q}X, \ldots, -\frac{p}{q}X), (qX, \ldots, qX, -pX, \ldots, -pX)) = \left(\frac{q^2}{p} + \frac{p^2}{q}\right)\Phi_0(X, X) = (p^2 + q^2)\Phi_0(X, X) \]

Therefore, \( \gamma = \frac{q^2 + p^2}{npq} \).

Finally, for \( j = 1, 2 \), the numbers \( c_{n_j, p} \) are defined by

\[ \Phi(C_{n_j} \cdot, \cdot) \mid_{p \times p} = c_{n_j, p} \Phi \mid_{p \times p}. \]

From Proposition 4.1 (v), we obtain the following:

\[ \Phi(C_{n_1} (qX, \ldots, qX, -pX, \ldots, -pX), (qX, \ldots, qX, -pX, \ldots, -pX)) = \Phi \left( \left(1 - \frac{1}{p}\right)qX, \ldots, \left(1 - \frac{1}{p}\right)qX, 0, \ldots, 0 \right), (qX, \ldots, qX, -pX, \ldots, -pX) \right) = \frac{(p-1)q^2}{p} p\Phi_0(X, X) = (p-1)q^2\Phi_0(X, X) \]

Therefore, \( c_{n_1, p} = \frac{(p-1)q^2}{npq} = \frac{(p-1)q}{np} \). Similarly, we show that \( c_{n_2, p} = \frac{(q-1)p}{qn} \).

\[ \square \]

### 4.2 Existence of Einstein Adapted Metrics

In this Section we investigate the existence of adapted metrics on \( M \) with respect to the fibration

\[ M = \frac{G_0^n}{\Delta^n G_0} \rightarrow \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0}, \]

as in previous Section. We shall consider adapted metrics of the form

\[ g_M = g_M(\lambda, \mu_1, \mu_2) \quad (4.3) \]

with respect to the decomposition \( m = p \oplus n_1 \oplus n_2 \), i.e., \( g_M \) is induced by the scalar product

\[ \lambda B \mid_{p \times p} \oplus \mu_1 B \mid_{n_1 \times n_1} \oplus \mu_2 B \mid_{n_2 \times n_2} \quad (4.4) \]
on \( \mathfrak{m} \), where \( B = -\Phi \), the negative of the Killing form. We observe that \( \mathfrak{n}_1 \) and \( \mathfrak{n}_2 \) are inequivalent \( Ad K \)-modules, but they are not irreducible, for \( n > 4 \). Hence, adapted metrics on \( M \) are not necessarily of the form \((4.3)\). However, throughout we shall focus only on adapted metrics of the form \( g_M = g_M(\lambda, \mu_1, \mu_2) \), unless it is explicitly stated otherwise.

We recall that it has been proved by McKenzie Y. Wang and Wolfgang Ziller \([15, 39]\) that the homogeneous space \( M = \frac{G_0}{\triangle^n G_0} \), \( n \geq 2 \), is a standard Einstein manifold. Hence, there exists at least one Einstein adapted metric of the form \((4.3)\). Indeed, by Corollary 1.2 and Corollary 4.1, the Ricci curvature of the standard metric is simply

\[
Ric = \frac{1}{2} \left( \frac{1}{2} + c_{t,m} \right) B = \left( \frac{1}{4} + \frac{1}{2n} \right) B.
\]  

\((4.5)\)

Consequently, the standard metric is Einstein. The standard metric is an example of a binormal metric on \( M \) with respect to the fibration \( M \to N \). Below we shall classify all the binormal Einstein metrics on \( M \).

Although the submodules \( \mathfrak{n}_1 \) and \( \mathfrak{n}_2 \) are not \( Ad K \)-irreducible for \( n > 4 \), the Casimir operators of \( \mathfrak{l}, \mathfrak{k} \) and \( \mathfrak{p} \) are always scalar on \( \mathfrak{n}_1 \) and on \( \mathfrak{n}_2 \). Hence, it is enough to consider one irreducible submodule in \( \mathfrak{n}_1 \) and one irreducible submodule in \( \mathfrak{n}_2 \), for effects of Ricci curvature. Therefore, according to Theorem 2.1 (2.19), there is an one-to-one correspondence, up to homothety, between binormal adapted Einstein metrics on \( M \) and positive solutions of the following set of equations:

\[
\delta_{12}^k (1 - X) = \delta_{12}^l
\]  

\((4.6)\)

\[
(\gamma + 2c_{t,p})X^2 - (1 + 2c_{t,j}) X + (1 - \gamma + 2b^j) = 0, \ j = 1, 2
\]  

\((4.7)\)

Given a positive solution \( X \), then binormal adapted Einstein metrics are given, up to homothety, by

\[
g_M = g_M(1, X),
\]

i.e., are induced by scalar products of the form \( <,> = B \ |_{\mathfrak{p} \times \mathfrak{p}} \oplus XB \ |_{\mathfrak{n} \times \mathfrak{n}} \) on \( \mathfrak{m} \).

**Theorem 4.1.** Let us consider the fibration

\[
M = \frac{G_0^m}{\triangle^n G_0} \to \frac{G_0^p}{\triangle^p G_0} \times \frac{G_0^q}{\triangle^q G_0} = N,
\]

where \( p + q = n \) and \( 2 \leq p \leq q \leq n - 2 \).
If $p \neq q$ or $n = 4$, then there exists on $M$ precisely one binormal Einstein metric, up to homothety, which is the standard metric. For $n > 4$ and $p = q$, then there are on $M$ precisely two binormal Einstein metrics, up to homothety, which are the standard metric and the metric induced by the scalar product

$$B |_{p \times p \oplus \frac{n}{4} B |_{n \times n}}.$$  

**Proof:** From Corollary 4.1 we obtain that $\delta_{12}^1 = c_{t1} - c_{t,2} = \frac{1}{p} - \frac{1}{q}$ whereas $\delta_{12}^1 = c_{l1} - c_{l,2} = \frac{1}{n} - \frac{1}{n} = 0$. Hence, Equation (4.6) implies that $X = 1$ or $p = q$. So if $p \neq q$, if there exists a binormal Einstein metric it must be the standard metric. This we already know it is Einstein by [36]. Therefore, if $p \neq q$, then there exists, up to homothety, exactly one binormal Einstein metric on $G/L$ which is the standard one.

By using Corollary 4.1 Equation (4.7) may be rewritten as

$$nX^2 - q(p + 2)X + pq + q - p = 0, \text{ for } j = 1 \quad (4.8)$$

and

$$nX^2 - p(q + 2)X + pq + p - q = 0, \text{ for } j = 2 \quad (4.9)$$

It is clear that $X = 1$ is actually a solution of both equations, and thus the standard metric is in fact Einstein.

Now suppose that $p = q$. As $n = p + q$, then $p = q = \frac{n}{2}$. Therefore, (4.8) and (4.9) become equivalent to

$$4X^2 - (n + 4)X + n = 0 \quad (4.10)$$

The polynomial above has two positive roots, 1 and $\frac{n}{4}$. Therefore, for $p = q$ and $n > 4$, there exist precisely two binormal Einstein metrics.

□

**Theorem 4.2.** Let $g_M$ be an Einstein adapted metric on $M$ of the form $g_M(\lambda; \mu_1, \mu_2)$. The projection $g_N$ onto the base space is also Einstein if and only if $p = q$ and $g_M$ is binormal.

**Proof:** By Theorem 2.2 we know that if $g_M$ and $g_N$ are Einstein then we must have the relation

$$\frac{r_1}{r_2} = \left(\frac{b_1}{b_2}\right)^{\frac{1}{2}}. \quad (4.11)$$
From Lemma 4.1 (ii) we obtain \( \left( \frac{k^I}{p} \right)^{\frac{1}{2}} = \frac{q}{p} \). Since \( [n_1, n_2] = 0 \), from Corollary 2.2 we get \( r_i = \frac{1}{2} \left( \frac{1}{2} + c_{ki} \right) \), \( i = 1, 2 \). Hence, \( r_{i_2} = \frac{(p+2)n}{(q+2)p} \), by using Lemma 4.1 (iii). Therefore, (4.11) is possible if and only if \( p = q \). Also from the proof of Theorem 2.2, if \( g_N \) and \( g_M \) are Einstein, then \( \frac{\rho}{\mu_1^{\frac{1}{2}}} = \frac{q}{p} = 1 \) and \( g_M \) is binormal. Conversely, if \( g_M \) is binormal and \( p = q = n/2 \), then by the above we also get

\[
\frac{r_1}{\mu_1} = \frac{r_2}{\mu_2}
\]

and \( g_N \) is Einstein.

\[\square\]

We observe that since \( p \) is an irreducible \( Ad L \)-submodule, \( g_F \) is always Einstein. Theorems 4.1 and 4.2 classify all Einstein binormal metrics on \( M \) and all Einstein adapted metrics such that \( g_N \) is also Einstein. It still remains to understand if there are other Einstein adapted metrics besides these. The Einstein equations in general for arbitrary \( p \) and \( q \) are extremely complicated. However with the help of Maple it is still possible to solve the problem in general. Next we shall classify all the Einstein adapted metrics on \( M \) of the form \( g_M(\lambda, \mu_1, \mu_2) \).

**Lemma 4.2.** Consider the fibration

\[
M = \frac{G^0_0}{\Delta^n G_0} \rightarrow \frac{G^0_0}{\Delta^p G_0} \times \frac{G^0_0}{\Delta^q G_0} = N,
\]

where \( p + q = n \) and \( 2 \leq p \leq q \leq n - 2 \). There is a one-to-one correspondence between Einstein adapted metrics on \( M \) of the form \( g_M = g_M(\lambda; \mu_1, \mu_2) \), up to homothety, and positive solutions of the following system of Equations:

\[
\begin{align*}
-2q^2X_1^2 + nq(p + 2)X_1 + 2p^2X_2^2 - np(q + 2)X_2 &= 0 \quad (4.12) \\
n^2 + q^2(p + 1)X_1^2 + p^2(q - 1)X_2^2 - np(p + 2)X_1 &= 0. \quad (4.13)
\end{align*}
\]

To a positive solution \((X_1, X_2)\) corresponds an Einstein metric of the form \( g_M = g_M(1; \frac{1}{X_1}, \frac{1}{X_2}) \).

**Proof:** Let \( g_M \) be an adapted metric on \( M \) of the form \( g_M(\lambda; \mu_1, \mu_2) \). We set

\[
X_i = \frac{\lambda}{\mu_i}, \quad i = 1, 2.
\]

Since the fiber \( F \) is irreducible we may use Proposition 2.1 to obtain the Ricci curvature. For \( X \in p \),

\[
Ric(X, X) = \left( \frac{1}{2} \left( c_{lp} + \frac{\gamma}{2} \right) + \frac{\lambda^2}{4} \sum_{j=1}^{n} \frac{c_{nj} \mu_j^2}{\mu_j^2} \right) B(X, X).
\]

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Hence by using the eigenvalues in Corollary 4.1 and the unknowns $X_1$ and $X_2$ defined in (4.14), we obtain

$$\frac{1}{2} \left( c_{i,p} + \frac{\gamma}{2} \right) = \frac{1}{2n} + \frac{q^2 + p^2}{4npq} = \frac{n}{4pq}$$

and

$$\text{Ric}(X, X) = \left( \frac{n}{4pq} + \frac{(p-1)q}{4pn} X_1^2 + \frac{(q-1)p}{4qn} X_2^2 \right) B(X, X). \quad (4.15)$$

For $Y \in \mathfrak{n}_k$, the Ricci curvature is given by

$$-\frac{\lambda}{2\mu_k} B(C_y Y, Y) + r_k B(Y, Y).$$

The Casimir operator of $\mathfrak{p}$ is scalar on $\mathfrak{n}_i$ with eigenvalues $\frac{q}{np}$, for $i = 1$, and $\frac{p}{nq}$, for $i = 2$, as we can see from Corollary 4.1. Since $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$, we use Corollary 2.2 to obtain $r_k$:

$$r_k = \frac{1}{2} \left( \frac{1}{2} + c_{i,k} \right).$$

Hence, from Corollary 4.1, we get

$$\text{Ric}(Y, Y) = \left( -\frac{q}{2np} X_1 + \frac{p + 2}{4p} \right) B(Y, Y), \; Y \in \mathfrak{n}_1 \quad (4.16)$$

$$\text{Ric}(Y, Y) = \left( -\frac{p}{2nq} X_2 + \frac{q + 2}{4q} \right) B(Y, Y), \; Y \in \mathfrak{n}_2. \quad (4.17)$$

Finally, as $C_{\mathfrak{n}}(\mathfrak{p}) \subset \mathfrak{k}$, for $i = 1, 2$, from Proposition 2.1 we conclude that $\text{Ric}(\mathfrak{p}, \mathfrak{n}) = 0$. Therefore, the Einstein equations for $g_M$ are just

$$\frac{n}{4pq} + \frac{(p-1)q}{4pn} X_1^2 + \frac{(q-1)p}{4qn} X_2^2 = \lambda E \quad (4.18)$$

$$-\frac{q}{2np} X_1 + \frac{p + 2}{4p} = \mu_1 E \quad (4.19)$$

$$-\frac{p}{2nq} X_2 + \frac{q + 2}{4q} = \mu_2 E \quad (4.20)$$

where $E$ is the Einstein constant. We obtain the system of equations stated in this lemma by eliminating $E$ from the system above and rearranging the resulting equations.

$\square$

**Theorem 4.3.** Let us consider the fibration

$$M = \frac{G^n_0}{\Delta^n G_0} \rightarrow \frac{G^p_0}{\Delta^p G_0} \times \frac{G^q_0}{\Delta^q G_0} = N,$$
where \( p + q = n \) and \( 2 \leq p \leq q \leq n - 2 \). If \( n > 4 \), there exist on \( M \) exactly two Einstein adapted metrics of the form \( g_M = g_M(\lambda, \mu_1, \mu_2) \) and one is the standard metric. For \( n = 4 \) the only Einstein adapted metric is the standard one. For \( p = q \) the non-standard Einstein adapted metric is binormal.

**Proof:** According to Lemma 4.2, the Einstein equations for \( g_M \) are as follows:

\[
-2q^2 X_1^2 + nq(p + 2)X_1 + 2p^2 X_2^2 - np(q + 2)X_2 = 0 \tag{4.21}
\]

\[
n^2 + q^2(p + 1)X_1^2 + p^2(q - 1)X_2^2 - nq(p + 2)X_1 = 0. \tag{4.22}
\]

By using Maple we obtain that the solutions of the system above are \( X_1 = X_2 = 1 \)

\[
X_1 = \alpha, \quad X_2 = \left( \frac{-q^2(p + 1)\alpha^2 + nq(p + 2)\alpha - n^2}{p^2(q - 1)} \right) \frac{1}{\sqrt{n}}, \tag{4.23}
\]

where \( \alpha \) is a root of the polynomial

\[
t(Z) = 4q^2 Z^3 - 4q(n + pq + 2)Z^2 + n(q(q + 2)(p + 1) + n + 8)Z - (q + 3)n^2.
\]

The solution \( X_1 = X_2 = 1 \) corresponds to a standard metric and we recover the result that \( M \) is an Einstein standard manifold. We are now interested in analysing the existence of other metrics. First we observe that from the expression for \( X_2 \) in (4.23) we conclude that,

\[
X_2 \in \mathbb{R} \text{ if and only if } \alpha \in \left( \frac{n}{q(p + 1)}, \frac{n}{q} \right).
\]

For this we compute the roots of the polynomial \(-q^2(p + 1)\alpha^2 + nq(p + 2)\alpha - n^2\), as in (4.23).

Simple calculations show that

\[
t\left( \frac{n}{q(p + 1)} \right) = \frac{p(q - 1)^2n^2}{q} > 0
\]

\[
t\left( \frac{n}{q} \right) = -\frac{p(p + 3)^2(q - 1)n^2}{q(p + 1)^3} < 0
\]

and thus \( t \) has at least one (positive) root in the interval \( \left( \frac{n}{q(p + 1)}, \frac{n}{q} \right) \), by the Bolzano Theorem. By the explained above, to this root corresponds an Einstein adapted metric on \( M \). Furthermore, we show that this root is unique and distinct from 1. From this we conclude that there exists a non-standard Einstein adapted
metric on $M$. We observe that the derivative of $t$, $\frac{dt}{dz}$ has no real zeros. Simple calculations show that the zeros of $\frac{dt}{dz}$ are

$$
n + pq + 2 \pm \sqrt{\delta} \over 3q,
$$

where

$$
\delta = (q + 1)^2 p^2 - (q - 1)(3q^2 + 4q - 8)p - (q - 1)(3q^2 + 8q + 16).
$$

We show that $\delta < 0$. For $p = q$, $\delta = -2q^4 - 2q^2 + 8q^2 - 16q + 16 < 0$, for every $q \geq 2$. So we suppose that $p < q$. In this case, since $p^2 \leq (q - 1)p$, we have

$$
\begin{align*}
\delta &\leq (q - 1)(- (2p + 3)q^2 - (2p + 8)q + (9p - 16)) \\
&< (q - 1)(-2p^2 - 5p^2 + p - 16) \\
&< 0,
\end{align*}
$$

for every $p \geq 2$.

With this we conclude that $\frac{dt}{dz}$ has no real zeros and thus the root of $t$ found above is the unique real root of $t$. Moreover, we must guarantee that this root does not yield the solution $X_1 = X_2 = 1$. If $X_1 = X_2 = 1$, then $\alpha = 1$ is a root of $t$. This may be possible since $1 \in \left( \frac{n}{q(p+1)}, \frac{n}{q} \right)$. Since

$$
t(1) = p(q + 2)(q - 1)(n - 4),
$$

and, consequently, $\alpha = 1$ is a root of $t$ if and only if $n = 4$. By using (4.23) we get that if $X_1 = 1$ when $n = 4$, then $X_2 = 1$ as well. Since non-standard Einstein adapted metrics are given by pairs of the form (4.23), with $\alpha \neq 1$, we conclude that there exists a unique non-standard Einstein adapted metric of the form $g_M(\lambda, \mu_1, \mu_2)$ if and only if $n > 4$; in the case $n = 4$, the standard metric is the unique Einstein adapted metric of the form $g_M(\lambda, \mu_1, \mu_2)$. Finally, we observe that, if $n = 4$, the subspaces $n^1$ and $n^2$ are irreducible $Ad L$-submodules. Hence, any adapted metric on $M$ is of the form $g_M(\lambda, \mu_1, \mu_2)$. Therefore, we conclude that, for $n = 4$, there exists a unique Einstein adapted metric on $M$ and it is the standard one.

Since there is a unique non-standard Einstein adapted metric on $M$, it follows from Theorem 4.1 that this metric is binormal if and only if $p = q$.

□

As it has been observed previously the modules $n_k$, $k = 1, 2$, are not irreducible $Ad K$-modules. From Lemma 4.1 we deduce a possible decomposition for $n_1$ and $n_2$ into irreducible $Ad K$-submodules:
Lemma 4.3. \( n_1 = \bigoplus_{j=1}^{p-1} n_{1,j} \) and \( n_2 = \bigoplus_{j=1}^{q-1} n_{2,j} \), where

\[
n_{1,j} = \{ (X, \ldots, X, -jX, 0, \ldots, 0) \in g_0^n : X \in g_0 \} \text{ for } j = 1, \ldots, p - 1
\]

\[
n_{2,j} = \{ 0, \ldots, 0, X, \ldots, X, -jX, 0, \ldots, 0 \} \in g_0^n : X \in g_0 \} \text{ for } j = 1, \ldots, q - 1.
\]

Furthermore, the \( n_{i,j} \)'s are irreducible \( \text{Ad } K \)-submodules.

The fact that the modules above are irreducible follows from the fact that \( g_0 \) is simple. Also by similar calculations to those in Lemma 4.1 we obtain the Casimir operators of the submodules \( n_{1,j} \) and \( n_{2,j} \):

\[
C_{n_{1,j}} = \left( \frac{1}{j(j+1)}, \ldots, \frac{1}{j(j+1)}, \frac{j}{j+1}, 0, \ldots, 0 \right), \text{ for } j = 1, \ldots, p - 1,
\]

\[
C_{n_{2,j}} = \left( 0, \ldots, 0, \frac{1}{j(j+1)}, \ldots, \frac{1}{j(j+1)}, \frac{j}{j+1}, 0, \ldots, 0 \right), \text{ for } j = 1, \ldots, q - 1.
\]

Proof: \( \{ (u_i, \ldots, u_i, -ju_i, 0, \ldots, 0) \} \) and \( \{ (u'_i, \ldots, u'_i, -ju'_i, 0, \ldots, 0) \} \) are bases of \( n_{1,j} \).

\[
= k\Phi_0(u_i, u'_j) + k^2\Phi_0(u_i, u'_j) = (k + 1)k\delta_{ij}
\]

Hence dual bases of \( n_{1,j} \) for \( \Phi \) are as follows:

\[
\{ \frac{1}{\sqrt{k(k+1)}}(u_i, \ldots, u_i, -ju_i, 0, \ldots, 0) \} \text{ and } \{ \frac{1}{\sqrt{k(k+1)}}(u'_i, \ldots, u'_i, -ju'_i, 0, \ldots, 0) \}.
\]

By using these bases we obtain the required expression for the Casimir operator of \( n_{1,j} \). For \( C_{n_{2,j}} \) is similar.

\[ \Box \]

Hence, we may consider on \( M \) an adapted metric of the form

\[
g_M = g_M(\lambda; \mu_{1,1}, \ldots, \mu_{1,p-1}, \mu_{2,1}, \ldots, \mu_{2,q-1}). \tag{4.24}
\]

Clearly, whereas the submodules \( n_{i,j} \) are \( \text{Ad } K \)-irreducible, the reader should note that they are not pairwise inequivalent. Hence, adapted metrics on \( M \) are not necessarily of the form (4.25).
Lemma 4.5. Let $g_M$ be an adapted metric on $M$ of the form

$$g_M(\lambda; \mu_1, \ldots, \mu_{1,p-1}, \mu_2, \ldots, \mu_{2,q-1}).$$

If $g_M$ is Einstein, then it is of the form

$$g_M(\lambda, \mu_1, \mu_2). \quad (4.25)$$

Proof: By Corollary 1.5, if exists on $M$ an Einstein adapted metric, then

$$\left( \sum_{j=1}^{p-1} \nu_{1,j} C_{n_1,j} + \sum_{j=1}^{q-1} \nu_{2,j} C_{n_2,j} \right)(p) \subset \mathfrak{k}, \quad (4.26)$$

for some $\nu_{1,j}, \nu_{2,j} > 0$. The inclusion (4.26) implies

$$\sum_j \nu_{i,j} C_{n,i,j}(p) \subset \mathfrak{k}, \quad (4.27)$$

for $i = 1, 2$, since $C_{n_1,j}(g) \subset g_0^p \times 0_q$ and $C_{n_2,j}(g) \subset 0_p \times g_0^q$. For $k = 1, \ldots, p$, let $P_k$ denote the $k$th component of $P = \sum_j \nu_{1,j} C_{n_1,j}$. By 4.1 (vi), we have

$$P_k = \frac{k-1}{k} \nu_{1,k-1} + \frac{1}{k(k+1)} \sum_{j=k}^{p-1} \nu_{1,k},$$

for $k = 2, \ldots, p-1$. We may then write

$$P_{k+1} = P_k + \frac{k-1}{k} (\nu_{1,k} - \nu_{1,k-1}). \quad (4.28)$$

On the other hand, the condition $P(p) \subset \mathfrak{k}$, implies that $P_k = P_{k+1}$. Hence, from (4.28) we obtain that $\nu_{1,k} = \nu_{1,k-1}$. Similarly we show that $\nu_{2,k} = \nu_{2,k-1}$. Now from the proof of Corollary 1.5, we can see that for an adapted Einstein metric on $M$ as in (4.3), then the constants $\nu_{1,k} = 1/\mu_{1,k}^2$ and $\nu_{2,k} = 1/\mu_{2,k}^2$ must satisfy (4.26). This concludes the proof.

Therefore, Theorem 4.2 may be understood as a classification of Einstein metrics of the form (4.25).

Corollary 4.2. Let us consider the fibration

$$M = \frac{G_0^p}{\Delta^p G_0} \times \frac{G_0^q}{\Delta^q G_0} = N,$$

where $p + q = n$ and $2 \leq p \leq q \leq n - 2$. Let

$$g_M(\lambda; \mu_1, \ldots, \mu_{1,p-1}, \mu_2, \ldots, \mu_{2,q-1}) \quad (4.29)$$
be an adapted metric on $M$, corresponding to the decomposition of $n$ given in Lemma 4.5. If $n > 4$, there exist on $M$ exactly two Einstein adapted metrics of the form (4.29) and one is the standard metric.
4.3 Closing Remarks

The main aim of this thesis was to establish conditions for existence of homogeneous Einstein metrics with totally geodesic fibers and to bring new existence and non-existence results for a significant class of homogeneous spaces. Necessary and sufficient conditions for existence of such metrics were obtained under some hypothesis and, for some classes of homogeneous fibrations, new Einstein metrics with totally geodesic fibers were found and in other cases all such metrics were classified. Nevertheless, some questions remain open for further research.

For irreducible bisymmetric fibrations of maximal rank all the Einstein adapted metrics were classified in the case when $G$ is an exceptional Lie group. If $G$ is a classical group, all the Einstein adapted metrics were classified for Type I, whereas for Type II we classified only those whose restriction to the fiber is still Einstein. For these bisymmetric fibrations which admit an Einstein adapted metric which satisfies this condition we can still classify all the other Einstein adapted metrics. However, in this classical case, it remains to obtain a general classification of Einstein adapted metrics.

Furthermore, the techniques and results for irreducible bisymmetric fibrations of maximal rank suggest that a similar research can be developed for homogeneous fibrations whose fiber and base space are $p$-symmetric spaces of higher order. For instance, the classification of compact simply-connected 3-symmetric spaces would allow us to consider fibrations whose fiber is a 3-symmetric space and the base is still an irreducible symmetric space.

For Kowalski $n$-symmetric spaces we have shown that there exists a non-standard Einstein adapted metric, if $n > 4$. For $n = 4$ we have proved that the standard metric is the only Einstein metric with totally geodesic fibers. It remains to classify all the Einstein adapted metrics, if $n > 4$. 
APPENDIX A

A classification of bisymmetric triples of maximal rank was given in Chapter 3 and they are listed in Tables 3.4, 3.5, 3.6 and 3.7. In this Appendix we determine the isotropy representation for each bisymmetric triple and compute the eigenvalues $\gamma_a$’s and $b_\phi^a$ of the Casimir operators $C_\ell$ and $C_{p_a}$ along the subspaces $p_a$ and $n$, respectively. We use the formulas presented in Chapter 3 in Propositions 3.1 and 3.2. We shall systematically use the roots systems and the dual Coxeter numbers. The roots systems used can be found in [16] and [34] and the dual Coxeter numbers are given in Table 3.1.

As introduced in Chapter 3, if $n = \bigoplus_j n_j$ is a decomposition of $n$ into irreducible $Ad L$-modules, we write

$$R_{n_j} = \{ \phi \in \mathcal{R} : E_\phi \in (n_j)^C \} \text{ and } n_j = \langle X_\phi, Y_\phi : \phi \in \mathcal{R}_{n_j}^+ \rangle.$$  \hspace{1cm} (A.1)

We recall that $b_\phi^a$, for $\phi \in R_{n_j}$, is the eigenvalue of $C_{p_a}$ on $n_j$:

$$C_{p_a} |_{n_j} = b_\phi^a I d_{n_j}.$$ 

If $p$ is $Ad L$-irreducible, we write simply $b_\phi$ for the eigenvalue of $C_p$ on $n_j$. These eigenvalues are given by the formula from Proposition 3.1

$$b_\phi^a = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_{p_a}^+} d_{\alpha \phi} |\alpha|^2,$$ \hspace{1cm} (A.2)

where

$$d_{\alpha \phi} = q_{\alpha \phi} - p_{\alpha \phi} - 2 p_{\alpha \phi} q_{\alpha \phi} \hspace{1cm} (A.3)$$

and $\phi + n\alpha$, $p_{\alpha \phi} \leq n \leq q_{\alpha \phi}$ is the $\alpha$-series containing $\phi$.

Also if $\mathfrak{k}_a$ is a simple ideal of $\mathfrak{k}$, $\gamma_a$ is the eigenvalue of $C_\ell$ on $\mathfrak{k}_a$:

$$C_\ell |_{\mathfrak{k}_a} = \gamma_a I d_{\mathfrak{k}_a}.$$ 

And finally to compute the $\gamma_a$’s, when $\mathfrak{k}_a$ is a simple ideal of $\mathfrak{k}$, we use Proposition 3.2.
\[ \gamma_a = \frac{h^*(t_a)}{\delta_a h^*(g)}, \quad a = 1, \ldots, s \tag{A.4} \]

where \( h^*(t_a) \) and \( h^*(g) \) are the dual Coxeter numbers of \( t_a \) and \( g \), respectively, and \( \delta_a = |\alpha|^2/|\beta|^2 \), for \( \alpha \) a long root of \( g \) and \( \beta \) a long root of \( t_a \). If there is only one root length on \( g \) or both \( g \) and \( t_a \) have two root lengths, \( \delta_a = 1 \). If \( \delta_a \neq 1 \) then \( \delta_a \) it is equal to 2 or 3. If \( p \) is \( Ad L \)-irreducible, then we write simply \( \gamma \) for the eigenvalue of \( C_t \) on the corresponding simple ideal of \( t \).

Since the computations of the eigenvalues \( \gamma_a \) and \( b^p_a \) consist of a systematic use of formulas (A.2) and (A.4) some details of these computations are omitted. For each simple Lie algebra \( g \) we present the set of roots of \( g \) and the corresponding length of the roots. We consider each symmetric pair of maximal rank \((g, t)\) and present the subsets of roots of each simple ideal \( t_a \) of \( t \), the corresponding value of \( \gamma_a \) and the subset of roots of the isotropy space \( n \). We recall that \( R_n = R - R_t \).

Finally, for each bisymmetric triple \((g, t, I)\), we indicate the subset of roots \( R_p \) for the symmetric complement \( p \) of the symmetric pair \((t, I)\) and the subset of roots of each irreducible \( Ad L \)-submodule \( n' \) of \( n \). For bisymmetric triples of Type I, we present the essential information to compute the eigenvalues \( b^p \) on each subspace \( n' \). Since \( n' \) is \( Ad L \)-irreducible, it suffices to choose any root \( \phi \) in \( R_{n'} \). Thus, we choose a root \( \phi \) in each \( n' \) and indicate all the roots \( \alpha \in R_{p}^+ \) such that the string \( \phi + n\alpha \) is not singular, i.e., it contains other roots besides \( \phi \); only for these \( \alpha \)'s the coefficients \( d_{a\phi} \) in formula (A.2) are non-zero. We indicate the elements in the non-singular string \( \phi + n\alpha \); only three cases occur: this string is formed either by (i) \( \phi, \phi + \alpha \), in which case \( p_{a\phi} = 0 \) and \( q_{a\phi} = 1 \); thus, \( d_{a\phi} = 1 \); (ii) \( \phi, \phi - \alpha \), in which case \( p_{a\phi} = -1 \) and \( q_{a\phi} = 0 \); thus, \( d_{a\phi} = 1 \); (iii) \( \phi, \phi \pm \alpha \), in which case \( p_{a\phi} = -1 \) and \( q_{a\phi} = 1 \); thus, \( d_{a\phi} = 4 \).

We indicate the length \( |\alpha|^2 \) of each root \( \alpha \) listed previously. Once all this information is obtained, we apply (A.2) to compute \( b^p \).

The reader may notice throughout that, in some cases, the root \( \alpha \) indicated is not a positive root. We observe that since \( d_{a\phi} = d_{-a\phi} \), we may choose the \( \alpha \)'s in \( R_p \) independently of the sign, as long as only one of \( \pm \alpha \) is chosen. This allows us to chose the necessary \( \alpha \)'s without any considerations about the order of the roots in the Lie algebra \( g \).

For bisymmetric triples of Type II, we obtain the eigenvalues \( b^p_a \) from the bisymmetric triples of Type I. For instance, \( b^p_1 \) and \( b^p_2 \) for \((g_2, su_2 \oplus su_2, R \oplus R)\) are given by \( b^p \) of \((g_2, su_2 \oplus su_2, R \oplus su_2)\) and by \( b^p \) of \((g_2, su_2 \oplus su_2, su_2 \oplus R)\), respectively.

In the cases of the classical Lie algebras, it is easy to understand whether the sum of two roots is a root, due to the simplicity of their root systems. However, in the
exceptional cases, this may be rather complicated, mainly in the cases of the Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. In these three cases, auxiliary lemmas are provided, where conditions under which the sum of two roots is a root are stated. These are the Lemmas A.1, A.2 and A.3.
A.1 \( A_{n-1} \)

In this Section we consider bisymmetric triples of the form \( (\mathfrak{su}_p, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathbb{R}) \), \( 1 \leq p \leq n - 1 \). We set \( t_1 = \mathfrak{su}_p, t_2 = \mathfrak{su}_{n-p} \) and \( t_0 = \mathbb{R} \).

For a root system of type \( A_{n-1} \) for \( g \) we take

\[
\mathcal{R} = \{ \pm(e_i - e_j) : i \leq j \leq n \}. \tag{A.5}
\]

In \( g \) there is only one root length and is

\[
|\alpha|^2 = \frac{1}{n}. \tag{A.6}
\]

**Symmetric Pair A.1.** \( (\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}), p = 1, \ldots, n - 1. \)

| \( i \) | \( \mathcal{R}_i \) | \( \gamma_i \) |
|---|---|---|
| \( \mathfrak{su}_p \) | \( \{ \pm(e_i - e_j) : 1 \leq j \leq p \} \) | \( \frac{2}{n} \), \( p \geq 2 \) |
| \( \mathfrak{su}_{n-p} \) | \( \{ \pm(e_i - e_j) : p + 1 \leq j \leq n \} \) | \( \frac{n-p}{n} \), \( p \leq n-2 \) |

\( \mathcal{R}_n = \{ \pm(e_i - e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n \} \)

**Bisymmetric Triple A.1.** \( (\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_{l-p} \oplus \mathbb{R} \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}), 1 \leq p \leq n - 1, 1 \leq l \leq p - 1. \) (Type I)

| \( \alpha \in \mathcal{R}_p^+ \) | \( \phi \) | \( \phi + n\alpha \) | \( d_{\alpha\phi} \) | No of \( \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 - e_n \) | \( e_1 - e_j, l + 1 \leq j \leq p \) | \( \phi, \phi - \alpha \) | 1 | \( p-l \) | \( \frac{1}{n} \) | \( \frac{p-l}{2n} \) |
| \( n^2 \) | \( e_p - e_n \) | \( e_j - e_p, 1 \leq j \leq l \) | \( \phi, \phi + \alpha \) | 1 | \( l \) | \( \frac{1}{n} \) | \( \frac{l}{2n} \) |

**Bisymmetric Triple A.2.** \( (\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_p \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathfrak{R} \oplus \mathbb{R}), 1 \leq p \leq n - 1, 1 \leq s \leq n - p - 1. \) (Type I)

| \( \alpha \in \mathcal{R}_p^+ \) | \( \phi \) | \( \phi + n\alpha \) | \( d_{\alpha\phi} \) | No of \( \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 - e_{p+s} \) | \( e_{p+s} - e_i, p + s + 1 \leq j \leq n \) | \( \phi, \phi + \alpha \) | 1 | \( n-p-s \) | \( \frac{1}{n} \) | \( \frac{n-p-s}{2n} \) |
| \( n^2 \) | \( e_p - e_n \) | \( e_i - e_n, p + 1 \leq j \leq p + s \) | \( \phi, \phi - \alpha \) | 1 | \( s \) | \( \frac{1}{n} \) | \( \frac{s}{2n} \) |
Bisymmetric Triple A.3. \((\mathfrak{su}_n, \mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathbb{R}, \mathfrak{su}_l \oplus \mathfrak{su}_{p-l} \oplus \mathfrak{su}_s \oplus \mathfrak{su}_{n-p-s} \oplus \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R})\), \(1 \leq p \leq n-1, 1 \leq l \leq p-1, 1 \leq s \leq n-p-1.\) (Type II)

\[
R_{p_1} = \{ \pm(e_i - e_j) : 1 \leq i \leq l, l + 1 \leq j \leq p\}
\]

\[
R_{p_2} = \{ \pm(e_i - e_j) : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n\}
\]

\[
n = n^1 \oplus n^2 \oplus n^3 \oplus n^4
\]

| \(R_{n^i}\) | \(b_1^\phi\) | \(b_2^\phi\) |
|-----------------|-----------------|-----------------|
| \{ \pm(e_i - e_j) : 1 \leq i \leq l, p + 1 \leq j \leq p + s\} | \(\frac{p-l}{2n}\) | \(\frac{n-p-s}{2n}\) |
| \{ \pm(e_i - e_j) : 1 \leq i \leq l, p + s + 1 \leq j \leq n\} | \(\frac{p-l}{2n}\) | \(\frac{n-l}{2n}\) |
| \{ \pm(e_i - e_j) : l + 1 \leq i \leq p, p + 1 \leq j \leq p + s\} | \(\frac{l}{2n}\) | \(\frac{n-p-s}{2n}\) |
| \{ \pm(e_i - e_j) : l + 1 \leq i \leq p, p + s + 1 \leq j \leq n\} | \(\frac{l}{2n}\) | \(\frac{n-l}{2n}\) |
**A.2** \( B_n \)

In this Section we consider all the bisymmetric triples of the form \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}), 0 \leq p \leq n - 1 \).

A root system for \( \mathfrak{g} \) is

\[ R = \{ \pm e_i : 1 \leq i \leq n; \pm e_i \pm e_j : 1 \leq i < j \leq n \} \quad (A.7) \]

and the length of a root is

\[ |\alpha|^2 = \begin{cases} \frac{1}{2(2n-1)}, & \alpha = \pm e_i \\ \frac{1}{2n-1}, & \alpha = \pm e_i \pm e_j \end{cases} \quad (A.8) \]

**Symmetric Pair A.2.** \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}), 0 \leq p \leq n - 1 \).

| \( \mathfrak{g} \)       | \( R_{\mathfrak{g}} \)                          | \( \gamma \) |
|-------------------------|-----------------------------------------------|--------------|
| \( \mathfrak{so}_{2p+1} \) | \{\pm e_i : 1 \leq i \leq p; \pm e_i \pm e_j : 1 \leq i < j \leq p\} | \frac{2n-1}{2n-1}, p \geq 1 |
| \( \mathfrak{so}_{2(n-p)} \) | \{\pm e_i \pm e_j : p + 1 \leq i < j \leq n\} | \frac{2(n-p-1)}{2n-1}, p \leq n - 2 |

\[ R_n = \{ \pm e_i \pm e_j : 1 \leq i \leq p + 1 \leq j \leq n \} \]

**Bisymmetric Triple A.4.** \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}), 0 \leq p \leq n - 1, 0 \leq l \leq p - 1 \). (Type I)

| \( n \) | \( \phi \) | \( \alpha \) | \( \phi + n \alpha \) | \( d_{\alpha \phi} \) | \( \text{No of } \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|---------|-----------|-------------|----------------------|------------------|------------------|-------------|------------|
| \( n^1 \) | \( e_n \) | \( e_i, l+1 \leq i \leq p \) | \( \phi, \phi + \alpha \) | 4 | \( p - l \) | \frac{1}{2(2n-1)} | \frac{p-l}{2n-1} |
| \( n^2 \) | \( e_p + e_n \) | \( e_i + e_p, 1 \leq i \leq l \) | \( \phi, \phi - \alpha \) | 1 | \( 2l \) | \frac{1}{2(2n-1)} | \frac{4l+1}{4(2n-1)} |

**Bisymmetric Triple A.5.** \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}), 0 \leq p \leq n - 1, 1 \leq s \leq n - p - 1 \). (Type I)

| \( n \) | \( \phi \) | \( \alpha \) | \( \phi + n \alpha \) | \( d_{\alpha \phi} \) | \( \text{No of } \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|---------|-----------|-------------|----------------------|------------------|------------------|-------------|------------|
| \( n^1 \) | \( e_{p+1} \) | \( e_{p+1} \pm e_i, p + s + 1 \leq i \leq n \) | \( \phi, \phi - \alpha \) | 1 | \( 2(n - p - s) \) | \frac{1}{2n-1} | \frac{n-p-s}{2n-1} |
| \( n^2 \) | \( e_n \) | \( \pm e_i + e_n, p + 1 \leq i \leq p + s \) | \( \phi, \phi + \alpha \) | 1 | \( 2s \) | \frac{1}{2n-1} | \frac{s}{2n-1} |
Bisymmetric Triple A.6. \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p+1} \oplus \mathfrak{u}_{n-p})\), \(0 \leq p \leq n-1\). \((\text{Type I})\)

\[ R_p = \{ \pm (e_i + e_j) : p + 1 \leq i < j \leq n \} \]

\(n\) irreducible \(\text{Ad } L\)-module

\[
\begin{array}{cccccc}
\phi & \in R_n & \alpha & \in R_p^+ & \phi + n\alpha & d_{\alpha\phi} & \text{No of } \alpha's & |\alpha|^2 & b^\phi \\
e_n & e_i + e_n, p + 1 \leq i \leq n - 1 & \phi, \phi - \alpha & 1 & n - p - 1 & \frac{1}{2n-1} & n - p - 1 & 2(2n-1) \\
\end{array}
\]

Bisymmetric Triple A.7. \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p})\), \(0 \leq p \leq n-1\), \(0 \leq l \leq p - 1\). \((\text{Type II})\)

\[ R_{p_1} = \{ \pm e_i : l + 1 \leq i \leq p, \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \} \]

\[ R_{p_2} = \{ \pm (e_i + e_j) : p + 1 \leq i < j \leq n \} \]

\(n = n^1 \oplus n^2\)

\[
\begin{array}{c|cc}
\mathcal{R}_n & b_1^\phi & b_2^\phi \\
\{ \pm e_i : p + 1 \leq i \leq n; \pm e_i \pm e_j : 1 \leq i \leq l, p + 1 \leq j \leq n \} & \frac{p - l}{2n - 1} & \frac{n - p - 1}{2(2n - 1)} \\
\{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq n \} & \frac{2l + 1}{2(2n - 1)} & \frac{n - p - 1}{2n - 1} \\
\end{array}
\]

Bisymmetric Triple A.8. \((\mathfrak{so}_{2n+1}, \mathfrak{so}_{2p+1} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l+1} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_2 \oplus \mathfrak{so}_{2(n-p-s)})\), \(0 \leq p \leq n-1, 0 \leq l \leq p - 1, 1 \leq s \leq n - p - 1\). \((\text{Type II})\)

\[ R_{p_1} = \{ \pm e_i : l + 1 \leq i \leq p, \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \}, \]

\[ R_{p_2} = \{ \pm e_i \pm e_j : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n \} \]

\(n = n^1 \oplus n^2 \oplus n^3 \oplus n^4\)

\[
\begin{array}{c|cc}
\mathcal{R}_n & b_1^\phi & b_2^\phi \\
\{ \pm e_i : p + 1 \leq i \leq p + s; \pm e_i \pm e_j : 1 \leq i \leq l, p + 1 \leq j \leq p + s \} & \frac{p - l}{2n - 1} & \frac{n - p - s}{2n - 1} \\
\{ \pm e_i : p + s + 1 \leq i \leq n; \pm e_i \pm e_j : 1 \leq i \leq l, p + s + 1 \leq j \leq n \} & \frac{p - l}{2n - 1} & \frac{s}{2n - 1} \\
\{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq p + s \} & \frac{2l + 1}{2(2n - 1)} & \frac{n - p - s}{2n - 1} \\
\{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + s + 1 \leq j \leq n \} & \frac{2l + 1}{2(2n - 1)} & \frac{2n - 1}{2n - 1} \\
\end{array}
\]
### A.3 $D_n$

In this Section we consider all the bisymmetric triples of the form $(\mathfrak{so}_{2n}, u_n, I)$ and $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p}, I)$, $1 \leq p \leq n - 1$.

A root system for $\mathfrak{g}$ is

$$\mathcal{R} = \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$$  \hspace{1cm} (A.9)

and the length of any root is

$$|\alpha|^2 = \frac{1}{2(n-1)}.$$  \hspace{1cm} (A.10)

#### Symmetric Pair A.3. $(\mathfrak{so}_{2n}, u_n)$.

| $\mathfrak{t}$ | $\mathcal{R}_\mathfrak{t}$ | $\gamma$ |
|----------------|----------------|----------|
| $u_n$ | $\{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$ | $\frac{n}{2(n-1)}$ |

$\mathcal{R}_n = \{\pm(e_i + e_j) : 1 \leq i < j \leq n\}$

#### Symmetric Pair A.4. $(\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2n-2p})$, $p = 1, \ldots, n - 1$.

| $\mathfrak{t}_i$ | $\mathcal{R}_{\mathfrak{t}_i}$ | $\gamma_i$ |
|-----------------|----------------|----------|
| $\mathfrak{so}_{2p}$ | $\{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ | $\frac{p}{n-1}$, $p \geq 2$ |
| $\mathfrak{so}_{2n-2p}$ | $\{\pm e_i \pm e_j : p + 1 \leq i < j \leq n\}$ | $\frac{n-p}{n-1}$, $p \leq n - 2$ |

$\mathcal{R}_n = \{\pm e_i \pm e_j : 1 \leq i \leq p, p + 1 \leq j \leq n\}$

#### Bisymmetric Triple A.9. $(\mathfrak{so}_{2n}, u_n, u_p \oplus u_{n-p})$, $0 \leq p \leq n - 1$. (Type I)

$\mathcal{R}_p = \{\pm(e_i - e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n\}$

$n^1 = n^1 \oplus n^2 \oplus n^3$

$\mathcal{R}_n^1 = \{\pm(e_i + e_j) : 1 \leq i < j \leq p\}$

$\mathcal{R}_n^2 = \{\pm(e_i + e_j) : p + 1 \leq i < j \leq n\}$

$\mathcal{R}_n^3 = \{\pm(e_i + e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n\}$

| $n^i$ | $\phi \in \mathcal{R}_n^i$, $\alpha \in \mathcal{R}_n^\alpha$ | $\phi + n\alpha$ | $d_{\alpha\phi}$ | No of $\alpha$'s | $|\alpha|^2$ | $b^\phi$ |
|-------|----------------|-----------------|-------------|----------------|--------------|-------------|
| $n^1$ | $e_1 + e_p$ | $e_1 - e_i, p + 1 \leq i \leq n$ | $\phi - \alpha$ | $1$ | $\frac{n-p}{n-1}$ | $\frac{1}{2(n-1)}$ |
|       | $e_p - e_i, p + 1 \leq i \leq n$ | $\phi + \alpha$ | $1$ | $\frac{n-p}{n-1}$ |
| $n^2$ | $e_{p+1} + e_n$ | $e_i - e_{p+1}, 1 \leq i \leq p$ | $\phi + \alpha$ | $1$ | $\frac{p}{2(n-1)}$ | $\frac{p}{2(n-1)}$ |
|       | $e_i - e_n, 1 \leq i \leq p$ | $\phi - \alpha$ | $1$ | $\frac{n-p}{n-1}$ |
| $n^2$ | $e_1 + e_n$ | $e_1 - e_i, p + 1 \leq i \leq n - 1$ | $\phi - \alpha$ | $1$ | $\frac{n-p-1}{n-1}$ | $\frac{1}{2(n-1)}$ |
|       | $e_i - e_n, 2 \leq i \leq p$ | $\phi + \alpha$ | $1$ | $\frac{n-p-1}{n-1}$ |
Bisymmetric Triple A.10. \( (\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p)}, 1 \leq p \leq n-1, 1 \leq l \leq p - 1. \) (Type I)

\[
\mathcal{R}_p = \{ \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \} \\
n = n^1 \oplus n^2 \\
\mathcal{R}_n^1 = \{ \pm e_i \pm e_j : 1 \leq i \leq l, p + 1 \leq j \leq n \} \\
\mathcal{R}_n^2 = \{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq n \}
\]

| \( n^1 \) | \( \phi \in \mathcal{R}_n^1 \) | \( \alpha \in \mathcal{R}_p^+ \) | \( \phi + n\alpha \) | \( d_{n\phi} \) | \( \text{No of } \alpha' s \) | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 + e_n \) | \( e_1 \pm e_i, l + 1 \leq i \leq p \) | \( \phi, \phi - \alpha \) | 1 | 2(p - l) | \( \frac{1}{2(n-1)} \) | \( \frac{p-1}{2(n-1)} \) |

| \( n^2 \) | \( e_p + e_n \) | \( \pm e_i + e_p, 1 \leq i \leq l \) | \( \phi, \phi + \alpha \) | 1 | \( p - l \) | \( \frac{1}{2(n-1)} \) | \( \frac{p-l}{2(n-1)} \) |

Bisymmetric Triple A.11. \( (\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2(n-p-s)}, 1 \leq p \leq n-1, 1 \leq s \leq n - p - 1. \) (Type I)

\[
\mathcal{R}_p = \{ \pm e_i \pm e_j : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n \} \\
n = n^1 \oplus n^2 \\
\mathcal{R}_n^1 = \{ \pm e_i \pm e_j : 1 \leq i \leq p, p + 1 \leq j \leq p + s \} \\
\mathcal{R}_n^2 = \{ \pm e_i \pm e_j : 1 \leq i \leq p, p + 1 \leq j \leq n \}
\]

| \( n^1 \) | \( \phi \in \mathcal{R}_n^1 \) | \( \alpha \in \mathcal{R}_p^+ \) | \( \phi + n\alpha \) | \( d_{n\phi} \) | \( \text{No of } \alpha' s \) | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 + e_{p+1} \) | \( e_{p+1} \pm e_i, p + s + 1 \leq i \leq n \) | \( \phi, \phi - \alpha \) | 1 | 2(n - p - s) | \( \frac{1}{2(n-1)} \) | \( \frac{n-p-s}{2(n-1)} \) |

| \( n^2 \) | \( e_1 + e_n \) | \( \pm e_i + e_n, p + 1 \leq i \leq p + s \) | \( \phi, \phi - \alpha \) | 1 | 2s | \( \frac{1}{2(n-1)} \) | \( \frac{s}{2(n-1)} \) |

Bisymmetric Triple A.12. \( (\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{so}_{2s} \oplus \mathfrak{so}_{2(n-p-s)}, 1 \leq p \leq n-1, 1 \leq l \leq p - 1, 1 \leq s \leq n - p - 1. \) (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \} \\
\mathcal{R}_{p_2} = \{ \pm e_i \pm e_j : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n \} \\
n = n^1 \oplus n^2 \oplus n^3 \oplus n^4
\]

| \( n^1 \) | \( \phi \in \mathcal{R}_n^1 \) | \( b^\phi \) | \( b^\phi \) |
|---|---|---|---|
| \( \{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq n \} \) | \( \frac{p-l}{2(n-1)} \) | \( \frac{p-1}{2(n-1)} \) |
| \( \{ \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \} \) | \( \frac{p-l}{2(n-1)} \) | \( \frac{n-p-1}{2(n-1)} \) |

Bisymmetric Triple A.13. \( (\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{u}_p \oplus \mathfrak{so}_{2(n-p)}, 1 \leq p \leq n-1. \) (Type I)

\[
\mathcal{R}_p = \{ \pm (e_i + e_j) : 1 \leq i < j \leq p \} \\
n \text{irreducible Ad } L \text{-module}
\]

| \( \phi \in \mathcal{R}_n \) | \( \alpha \in \mathcal{R}_p^+ \) | \( \phi + n\alpha \) | \( d_{n\phi} \) | \( \text{No of } \alpha' s \) | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|
| \( e_1 + e_n \) | \( e_1 + e_i, 2 \leq i \leq p \) | \( \phi, \phi - \alpha \) | 1 | p - 1 | \( \frac{1}{2(n-1)} \) | \( \frac{p-1}{4(n-1)} \) |
Bisymmetric Triple A.14. \((\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2p} \oplus \mathfrak{u}_{n-p}), 1 \leq p \leq n - 1.\) (Type I)

\[ \mathcal{R}_p = \{ \pm(e_i + e_j) : p + 1 \leq i < j \leq n \} \]

\(n\) irreducible \(Ad\ L\)-module

| \(\phi \in \mathcal{R}_n\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha \phi}\) | No of \(\alpha'\)'s | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|
| \(e_1 + e_n\) | \(e_i + e_n, p + 1 \leq i \leq n - 1\) | \(\phi, \phi - \alpha\) | 1 | \(n - p - 1\) | \(\frac{1}{2(n-1)}\) | \(\frac{n-p-1}{4(n-1)}\) |

Bisymmetric Triple A.15. \((\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{u}_{p} \oplus \mathfrak{u}_{n-p}), 1 \leq p \leq n - 1.\) (Type II)

\[ \mathcal{R}_{p_1} = \{ \pm(e_i + e_j) : 1 \leq i < j \leq p \} \]
\[ \mathcal{R}_{p_2} = \{ \pm(e_i + e_j) : p + 1 \leq i < j \leq n \} \]

\(n\) irreducible \(Ad\ L\)-module

| \(b_1^\phi\) | \(b_2^\phi\) |
|---|---|
| \(\frac{p-1}{4(n-1)}\) | \(\frac{n-p-1}{4(n-1)}\) |

Bisymmetric Triple A.16. \((\mathfrak{so}_{2n}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(n-p)}, \mathfrak{so}_{2l} \oplus \mathfrak{so}_{2(p-l)} \oplus \mathfrak{u}_{n-p}), 1 \leq p \leq n - 1, 1 \leq l \leq p - 1.\) (Type II)

\[ \mathcal{R}_{p_1} = \{ \pm(e_i + e_j) : 1 \leq i \leq l, l + 1 \leq j \leq p \} \]
\[ \mathcal{R}_{p_2} = \mathcal{R}_{p_2} = \{ \pm(e_i + e_j) : p + 1 \leq i < j \leq n \} \]

\(n = n_1 \oplus n_2\)

| \(\mathcal{R}_n\) | \(b_1^\phi\) | \(b_2^\phi\) |
|---|---|---|
| \(\{ \pm(e_i + e_j) : 1 \leq i \leq l, p + 1 \leq j \leq n \}\) | \(\frac{p-l}{2(n-1)}\) | \(\frac{n-p-1}{4(n-1)}\) |
| \(\{ \pm(e_i + e_j) : l + 1 \leq i \leq p, p + 1 \leq j \leq n \}\) | \(\frac{1}{2(n-1)}\) | \(\frac{n-p-1}{4(n-1)}\) |
A.4 \( C_n \)

We consider the bisymmetric triples of the form \((sp_n, u_n, l)\) and \((sp_n, sp_p \oplus sp_{n-p}, l)\), for \(1 \leq p \leq n-2\). The root system for \(g = sp_n\) is

\[
\mathcal{R} = \{ \pm 2e_i : 1 \leq i \leq n; \pm e_i \pm e_j : 1 \leq i < j \leq n \}. \tag{A.11}
\]

In \(g\) there are two root lengths:

\[
|\alpha|^2 = \begin{cases} 
\frac{1}{n+1}, & \alpha = \pm 2e_i \\
\frac{2}{(n+1)}, & \alpha = \pm e_i \pm e_j.
\end{cases} \tag{A.12}
\]

Symmetric Pair A.5. \((sp_n, u_n)\).

\[
\begin{array}{ccc}
t_i & \mathcal{R}_t & \gamma \\
u_n & \{ \pm (e_i - e_j) : 1 \leq i < j \leq n \} & \frac{n}{2(n+1)} \\
\end{array}
\]

\(\mathcal{R}_n = \{ \pm 2e_i, \pm (e_i + e_j) : 1 \leq i < j \leq n \}\)

Symmetric Pair A.6. \((sp_n, sp_p \oplus sp_{n-p})\), \(p = 1, \ldots, n-2\).

\[
\begin{array}{ccc}
t_i & \mathcal{R}_t & \gamma_i \\
s_{sp} & \{ \pm 2e_i : 1 \leq i \leq p; \pm e_i \pm e_j : 1 \leq i < j \leq p \} & \frac{p+1}{n+1} \\
s_{sp_{n-p}} & \{ \pm 2e_i : p + 1 \leq i \leq n; \pm e_i \pm e_j : p + 1 \leq i < j \leq n \} & \frac{n-p+1}{n+1} \\
\end{array}
\]

\(\mathcal{R}_n = \{ \pm e_i \pm e_j : 1 \leq i \leq p, p + 1 \leq j \leq n \}\)

Bisymmetric Triple A.17. \((sp_n, u_n, u_p \oplus u_{n-p})\), \(1 \leq p \leq n-1\). \((Type I)\)

\[
\begin{array}{ccc}
t_i & \mathcal{R}_t & \gamma_i \\
\end{array}
\]

\[
\begin{array}{c}
sp_p = \{ \pm (e_i - e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n \} \\
n = n^1 \oplus n^2 \oplus n^3 \\
\mathcal{R}_{n_1} = \{ \pm 2e_i, \pm (e_i + e_j) : 1 \leq i < j \leq p \}, \\
\mathcal{R}_{n_2} = \{ \pm 2e_i, \pm (e_i + e_j) : p + 1 \leq i < j \leq n \}, \\
\mathcal{R}_{n_3} = \{ \pm (e_i + e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n \}
\end{array}
\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n^1}\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{n\phi}\) | No of \(\alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(n^1\) | \(2e_1\) | \(e_1 - e_i, p + 1 \leq i \leq n\) | \(\phi, \phi - \alpha\) | 1 | \(n - p\) | \(\frac{1}{2(n+1)}\) | \(\frac{n-p}{4(n+1)}\) |
| \(n^2\) | \(2e_n\) | \(e_i - e_n, 1 \leq i \leq p\) | \(\phi, \phi + \alpha\) | 1 | \(p\) | \(\frac{1}{2(n+1)}\) | \(\frac{p}{4(n+1)}\) |
| \(n^3\) | \(e_1 + e_n\) | \(e_1 - e_i, p + 1 \leq i \leq n - 1\) | \(\phi, \phi - \alpha\) | 1 | \(n - p - 1\) | \(\frac{1}{2(n+1)}\) | \(\frac{n+2}{4(n+1)}\) |

\(\mathcal{R}_p = \{ \pm (e_i - e_j) : 1 \leq i \leq p, p + 1 \leq j \leq n \}\)
### Bisymmetric Triple A.18. \((\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_{n-p}), 1 \leq p \leq n-1, 1 \leq l \leq p-1.\) \((\text{Type I})\)

\[
\mathcal{R}_p = \{ \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \}
\]
\[
n = n^1 \oplus n^2
\]
\[
\mathcal{R}_{n_1} = \{ \pm e_i \pm e_j : 1 \leq i \leq l, p + 1 \leq j \leq n \}
\]
\[
\mathcal{R}_{n_2} = \{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq n \}
\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n_1}\) | \(\alpha \in \mathcal{R}^+_p\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(e_1 + e_n\) | \(e_1 \pm e_i, l + 1 \leq i \leq p\) | \(\phi, \phi - \alpha\) | 1 | \(p - l\) | \(\frac{1}{2(n+1)}\) | \(\frac{p-l}{4(n+1)}\) |
| \(\pm e_i \pm e_p, 1 \leq i \leq l\) | \(\phi, \phi - \alpha\) | 1 | \(l\) | \(\frac{1}{2(n+1)}\) | \(\frac{l}{4(n+1)}\) |

### Bisymmetric Triple A.19. \((\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_p \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s}), 1 \leq p \leq n-1, 1 \leq s \leq n-p-1.\) \((\text{Type I})\)

\[
\mathcal{R}_p = \{ \pm e_i \pm e_j : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n \}
\]
\[
n = n^1 \oplus n^2
\]
\[
\mathcal{R}_{n_1} = \{ \pm e_i \pm e_j : l \leq i \leq p, p + 1 \leq j \leq p + s \}
\]
\[
\mathcal{R}_{n_2} = \{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + s + 1 \leq j \leq n \}
\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n_1}\) | \(\alpha \in \mathcal{R}^+_p\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(e_1 + e_{p+1}\) | \(e_{p+1} \pm e_i, p + s + 1 \leq i \leq n\) | \(\phi, \phi - \alpha\) | 1 | \(n - p - s\) | \(\frac{1}{2(n+1)}\) | \(\frac{n-p-s}{4(n+1)}\) |
| \(\pm e_i \pm e_{p+1}, 1 \leq i \leq p + s\) | \(\phi, \phi - \alpha\) | 1 | \(s\) | \(\frac{1}{2(n+1)}\) | \(\frac{s}{4(n+1)}\) |

### Bisymmetric Triple A.20. \((\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{sp}_s \oplus \mathfrak{sp}_{n-p-s}), 1 \leq l \leq p-1, 1 \leq s \leq n-p-1.\) \((\text{Type II})\)

\[
\mathcal{R}_{p_1} = \{ \pm e_i \pm e_j : 1 \leq i \leq l, l + 1 \leq j \leq p \}
\]
\[
\mathcal{R}_{p_2} = \{ \pm e_i \pm e_j : p + 1 \leq i \leq p + s, p + s + 1 \leq j \leq n \}
\]
\[
n = n^1 \oplus n^2 \oplus n^3 \oplus n^4
\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n_1}\) | \(\alpha \in \mathcal{R}^+_p\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(\{ \pm e_i \pm e_j : 1 \leq i \leq l, p + 1 \leq j \leq p + s \}\) | \(b^\phi\) | \(b_1^\phi\) | \(b_2^\phi\) | \(\frac{p-l}{4(n+1)}\) | \(\frac{n-p-s}{4(n+1)}\) |
| \(\{ \pm e_i \pm e_j : 1 \leq i \leq l, p + s + 1 \leq j \leq n \}\) |  |  |  | \(\frac{l}{4(n+1)}\) | \(\frac{n-p-s}{4(n+1)}\) |
| \(\{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + 1 \leq j \leq p + s \}\) |  |  |  | \(\frac{s}{4(n+1)}\) | \(\frac{n-p-s}{4(n+1)}\) |
| \(\{ \pm e_i \pm e_j : l + 1 \leq i \leq p, p + s + 1 \leq j \leq n \}\) |  |  |  | \(\frac{s}{4(n+1)}\) | \(\frac{n-p-s}{4(n+1)}\) |

### Bisymmetric Triple A.21. \((\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{u}_p \oplus \mathfrak{sp}_{n-p}), 1 \leq p \leq n-1.\) \((\text{Type I})\)

\[
\mathcal{R}_p = \{ \pm 2e_i : 1 \leq i \leq p; \pm (e_i + e_j) : 1 \leq i < j \leq p \}
\]
\[
n \text{irreducible Ad } L\text{-module}
\]

| \(n^1\) | \(\phi \in \mathcal{R}_n\) | \(\alpha \in \mathcal{R}^+_p\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(e_1 + e_n\) | \(2e_i\) | \(\phi, \phi - \alpha\) | 1 | \(p - 1\) | \(\frac{1}{n+1}\) | \(\frac{p+1}{4(n+1)}\) |
Bisymmetric Triple A.22. \( (\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_p \oplus \mathfrak{u}_{n-p}), \, 1 \leq p \leq n-1. \) (Type I)

\[
\mathcal{R}_p = \{ \pm 2e_i : p + 1 \leq i \leq n; \pm (e_i + e_j) : p + 1 \leq i < j \leq n \}
\]

\[n \text{ irreducible } \operatorname{Ad} L\text{-module}\]

| \( \phi \in \mathcal{R}_n \) | \( \alpha \in \mathcal{R}_n^+ \) | \( \phi + n\alpha \) | \( d_{\alpha\phi} \) | No of \( \alpha' \)'s | | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|
| \( e_1 + e_n \) | \( 2e_n \) | \( e_i + e_n, \, p + 1 \leq i \leq n - 1 \) | \( \phi, \phi - \alpha \) | \( 1 \) | \( n - p - 1 \) | \( \frac{1}{2(n+1)} \) | \( \frac{n-p+1}{4(n+1)} \)

Bisymmetric Triple A.23. \( (\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{u}_p \oplus \mathfrak{u}_{n-p}), \, 1 \leq p \leq n-1. \) (Type II)

\[
\begin{align*}
\mathcal{R}_{p_1} &= \{ \pm 2e_i : 1 \leq i \leq p; \pm (e_i + e_j) : 1 \leq i < j \leq p \} \\
\mathcal{R}_{p_2} &= \{ \pm 2e_i : p + 1 \leq i \leq n; \pm (e_i + e_j) : p + 1 \leq i < j \leq n \}
\end{align*}
\]

\[n \text{ irreducible } \operatorname{Ad} L\text{-module}\]

\[
\begin{array}{c|c}
\mathcal{R}_n & b^\phi_1 \quad b^\phi_2 \\
\hline
\frac{p+1}{4(n+1)} & \frac{n-p+1}{4(n+1)}
\end{array}
\]

Bisymmetric Triple A.24. \( (\mathfrak{sp}_n, \mathfrak{sp}_p \oplus \mathfrak{sp}_{n-p}, \mathfrak{sp}_l \oplus \mathfrak{sp}_{p-l} \oplus \mathfrak{u}_{n-p}), \, 1 \leq p \leq n-1 \text{ and } 1 \leq l \leq p-1. \) (Type II)

\[
\begin{align*}
\mathcal{R}_{p_1} &= \{ \pm e_i \pm e_j : 1 \leq i \leq l, \, l + 1 \leq j \leq p \} \\
\mathcal{R}_{p_2} &= \{ \pm 2e_i : p + 1 \leq i \leq n; \pm (e_i + e_j) : p + 1 \leq i < j \leq n \}
\end{align*}
\]

\[n = n^1 \oplus n^2\]

\[
\begin{array}{c|c|c}
\mathcal{R}_n & b^\phi_1 & b^\phi_2 \\
\hline
\frac{p-l}{4(n+1)} & \frac{n-p+1}{4(n+1)} & \frac{l}{4(n+1)} & \frac{n-p+1}{4(n+1)}
\end{array}
\]

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In this section we analyze the bisymmetric triples of the form \((f_4, \mathfrak{so}_9, l)\) and \((f_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, l)\).

The root system for the simple Lie algebra \(f_4\) is

\[
\mathcal{R} = \{\pm e_i, \pm e_i \pm e_j, 1 \leq i < j \leq 4; \frac{1}{2} \sum_1^4 (-1)^{\nu_i} e_i\}, \quad (A.13)
\]

where \(e_1, \ldots, e_4\) is the canonical basis for \(\mathbb{R}^4\) and the signs are chosen independently. In \(f_4\), there are two root lengths. Roots of the form \(\pm e_i\) and \(\frac{1}{2} \sum_1^4 (-1)^{\nu_i} e_i\) are short, whereas those of the form \(\pm e_i \pm e_j\) are long, and we have

\[
|\alpha|^2 = \begin{cases} 
\frac{1}{18}, & \text{\(\alpha\) short} \\
\frac{1}{9}, & \text{\(\alpha\) long} 
\end{cases} \quad (A.14)
\]

Symmetric Pair A.7. \((f_4, \mathfrak{so}_9)\).

\[
\begin{array}{c|c|c}
\xi & \mathcal{R}_\xi & \gamma \\
\hline
\mathfrak{so}_9 & \{\pm e_i, \pm e_i \pm e_j, 1 \leq i < j \leq 4\} & \frac{7}{9} \\
\mathcal{R}_n = \{\frac{1}{2} \sum_1^4 (-1)^{\nu_i} e_i\} & & \\
\end{array}
\]

Symmetric Pair A.8. \((f_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2)\).

\[
\begin{array}{c|c|c}
\xi & \mathcal{R}_\xi & \gamma \\
\hline
\mathfrak{sp}_3 & \langle e_4, e_3 - e_4, \frac{1}{2}(e_1 - e_2 - e_3 - e_4) > & = \\
\{\pm e_3, \pm e_4, \pm e_3 \pm e_4, \pm (e_1 - e_2), \pm \frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)\} & \frac{4}{9} \\
\mathfrak{su}_2 & \{\pm (e_1 + e_2)\} & \frac{2}{9} \\
\mathcal{R}_n = \{\pm e_i, \pm e_i \pm e_j : i = 1, 2, j = 3, 4; \pm \frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\}, \text{ with signs chosen independently.} \\
\end{array}
\]

Bisymmetric Triple A.25. \((f_4, \mathfrak{so}_9, \mathfrak{so}_p \oplus \mathfrak{so}_{9-p}), p = 2l + 1, l = 0, 1, 2, 3. \ (Type I)\)

\[
\mathcal{R}_p = \{\pm 2e_i : p + 1 \leq i \leq n; \pm (e_i + e_j) : p + 1 \leq i < j \leq n\} \\
C_p \text{ scalar on } n
\]

\[
\begin{array}{c|c|c|c|c|c|c}
\phi \in \mathcal{R}_n & \alpha \in \mathcal{R}_p^+ & \phi + n\alpha & d_{\alpha \phi} & \text{No of } \alpha \text{'s} & |\alpha|^2 & b^\phi \\
\frac{1}{7} \sum_1^4 e_i & e_i, l + 1 \leq i \leq 4 & e_i + e_j, 1 \leq i \leq l, l + 1 \leq j \leq 4 & \phi, \phi - \alpha & 4 - l & \frac{4}{9} & \frac{p(9-p)}{42}
\end{array}
\]
Bisymmetric Triple A.26. \((f_4, sp_3 \oplus su_2, sp_3 \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{ \pm (e_1 + e_2) \}
\]

\[C_p\; scalar\; on\; n\]

| \(\phi \in \mathcal{R}_n\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|
| \(e_1\) | \(e_1 + e_2\) | \(\phi, \phi - \alpha\) | 1 | 1 | \(\frac{1}{3}\) | \(\frac{4}{15}\) |

Bisymmetric Triple A.27. \((f_4, sp_3 \oplus su_2, u_3 \oplus su_2)\). (Type I)

\[
\mathcal{R}_p = \{ \pm e_3, \pm e_3 \pm e_4, \pm(e_1 - e_2), \pm \frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4) \}
\]

\[n = n^1 \oplus n^2\]

\[\mathcal{R}_{n^1} = \{ \pm (e_1 + e_3), \pm (e_2 - e_3) \}\]

\[\mathcal{R}_{n^2} = \{ \pm e_1, \pm e_2, \pm (e_1 - e_3), \pm (e_1 + e_4), \pm (e_2 \pm e_3), \pm (e_2 \pm e_4), \pm \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \}\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n^1}\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(n^1\) | \(e_1 + e_3\) | \(\frac{1}{2}(e_1 - e_2 + e_3 \pm e_4)\) | \(e_3 \pm e_4, e_1 - e_2\) | \(\phi, \phi - \alpha\) | 1 | 3 | \(\frac{1}{18}\) | \(\frac{1}{2}\) |
| \(n^2\) | \(e_1\) | \(e_3\) | \(\phi, \phi \pm \alpha\) | 4 | 1 | \(\frac{1}{18}\) |
| \(\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)\) | \(\phi, \phi - \alpha\) | 1 | 1 | \(\frac{1}{9}\) | \(\frac{2}{9}\) |

Bisymmetric Triple A.28. \((f_4, sp_3 \oplus su_2, sp_2 \oplus su_2 \oplus su_2)\). (Type I)

\[
\mathcal{R}_p = \{ \pm e_3, \pm e_4, \pm \frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4) \}
\]

\[n = n^1 \oplus n^2\]

\[\mathcal{R}_{n^1} = \{ \pm e_i \pm e_j, i = 1, 2, j = 3, 4: \pm \frac{1}{2}(e_1 + e_2 \pm (e_3 + e_4)) \}\]

\[\mathcal{R}_{n^2} = \{ \pm e_1, \pm e_2, \pm \frac{1}{2}(e_1 \pm e_2 \pm (e_3 - e_4)) \}\]

| \(n^1\) | \(\phi \in \mathcal{R}_{n^1}\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(n^1\) | \(\frac{1}{2}(e_1 + e_2 + e_3 + e_4)\) | \(\frac{1}{2}(e_1 - e_2 - e_3 - e_4)\) | \(\phi, \phi + \alpha\) | 1 | 4 | \(\frac{1}{18}\) | \(\frac{1}{9}\) |
| \(\frac{1}{2}(e_1 - e_2 \pm (e_3 + e_4))\) | \(\phi, \phi - \alpha\) | 4 | 2 | \(\frac{1}{18}\) | \(\frac{5}{18}\) |
| \(n^2\) | \(e_1\) | \(e_3, e_4\) | \(\phi, \phi \pm \alpha\) | 4 | 2 | \(\frac{1}{18}\) | \(\frac{5}{18}\) |

Bisymmetric Triple A.29. \((f_4, sp_3 \oplus su_2, u_3 \oplus \mathbb{R})\). (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm e_3, \pm e_3 \pm e_4, \pm (e_1 - e_2), \pm \frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4) \}
\]

\[
\mathcal{R}_{p_2} = \{ \pm (e_1 + e_2) \}
\]

\[n = n^1 \oplus n^2\]

| \(\mathcal{R}_{n^1}\) | \(b_1^\phi\) | \(b_2^\phi\) |
|---|---|---|
| \(\{ \pm (e_1 + e_3), \pm (e_2 - e_3) \}\) | \(\frac{1}{3}\) | \(\frac{1}{18}\) |
| \(\{ \pm e_1, \pm e_2, \pm (e_1 - e_3), \pm (e_1 + e_4), \pm (e_2 \pm e_3), \pm (e_2 \pm e_4), \pm \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \}\) | \(\frac{1}{3}\) | \(\frac{1}{18}\) |
Bisymmetric Triple A.30. $(\mathfrak{f}_4, \mathfrak{sp}_3 \oplus \mathfrak{su}_2, \mathfrak{sp}_2 \oplus \mathfrak{su}_2 \oplus \mathbb{R})$. (Type II)

\[ R_{p_1} = \{ \pm e_3, \pm e_4, \pm \frac{1}{2}(e_1 - e_2 \pm (e_3 + e_4)) \} \]
\[ R_{p_2} = \{ \pm (e_1 + e_2) \} \]
\[ n = n^1 \oplus n^2 \]

\[ R_{n^1} \quad b_1^{\phi} \quad b_2^{\phi} \]
\[ \{ \pm e_i \pm e_j, i = 1, 2, j = 3, 4; \pm \frac{1}{2}(e_1 + e_2 \pm (e_3 + e_4)) \} \quad \frac{1}{9} \quad \frac{1}{18} \]
\[ \{ \pm e_1, \pm e_2, \pm \frac{1}{2}(e_1 + e_2 \pm (e_3 - e_4)) \} \quad \frac{1}{18} \quad \frac{1}{18} \]
A.6 \( g_2 \)

In this section we analyze all bisymmetric triples \((g_2, su_2 \oplus su_2, l)\). The symmetric pair \((g_2, su_2 \oplus su_2)\) corresponds to a flag manifold of \(G_2\) and thus is obtained by a painted Dynkin diagram. We observe that each factor \(su_2\) corresponds to a different root length. We set that \(t_1 = su_2\) corresponds to a long root and \(t_2 = su_2\) corresponds to a short root. If we consider the root system of \(g_2\),

\[
R = \{ \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(e_2 - e_3), \\
\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2) \},
\]

we can choose \(R_{t_1} = \{ \pm(2e_1 - e_2 - e_3) \}\) and \(R_{t_2} = \{ \pm(e_2 - e_3) \}\), the orthogonal of \(R_{t_1}\).

In \(g_2\) the length of a root is given by

\[
|\alpha|^2 = \begin{cases} 
\frac{1}{4} & \text{\(\alpha\) long} \\
\frac{1}{12} & \text{\(\alpha\) short}
\end{cases}, \tag{A.15}
\]

where roots of the form \(e_a - e_b\) are short and those of the form \(2e_a - e_b - e_c\) are long.

Symmetric Pair A.9. \((g_2, su_2 \oplus su_2)\).

| \(t_i\) | \(R_{t_i}\) | \(\gamma_i\) |
|-------|-------|------|
| \(su_2\) | \{ \pm(2e_1 - e_2 - e_3) \} | \frac{1}{7} |
| \(su_2\) | \{ \pm(e_2 - e_3) \} | \frac{1}{6} |

\(R_n = \{ \pm(e_1 - e_2), \pm(e_1 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2) \}\)

Bisymmetric Triple A.31. \((g_2, su_2 \oplus su_2, R \oplus su_2)\). (Type I)

\[
R_p = \{ \pm(2e_1 - e_2 - e_3) \}
\]

\(C_p\) scalar on \(n\)

| \(\phi \in R_n\) | \(\alpha \in R_p^+\) | \(\phi + n\alpha\) | \(d_{a,\phi}\) | No of \(\alpha\)’s | \(|\alpha|^2\) | \(b^\phi\) |
|---------|-------|---------|--------|-------------|--------|-------|
| \(e_1 - e_2\) | \(2e_1 - e_2 - e_3\) | \(\phi, \phi - \alpha\) | 1 | 1 | \(\frac{1}{4}\) | \(\frac{1}{6}\) |

Bisymmetric Triple A.32. \((g_2, su_2 \oplus su_2, su_2 \oplus R)\). (Type I)

\[
R_p = \{ \pm(e_2 - e_3) \}
\]

\(C_p\) scalar on \(n\)

| \(\phi \in R_n\) | \(\alpha \in R_p^+\) | \(\phi + n\alpha\) | \(d_{a,\phi}\) | No of \(\alpha\)’s | \(|\alpha|^2\) | \(b^\phi\) |
|---------|-------|---------|--------|-------------|--------|-------|
| \(e_1 - e_2\) | \(e_2 - e_3\) | \(\phi, \phi \pm \alpha\) | 4 | 1 | \(\frac{1}{12}\) | \(\frac{1}{6}\) |
Bisymmetric Triple A.33. \((\mathfrak{g}_2, \mathfrak{su}_2 \oplus \mathfrak{su}_2, \mathbb{R} \oplus \mathbb{R})\). (Type II)

\[ \begin{align*}
\mathcal{R}_{p_1} &= \{ \pm(2e_1 - e_2 - e_3) \} \\
\mathcal{R}_{p_2} &= \{ \pm(e_2 - e_3) \} \\
C_{p_i}, \text{ scalar on } n, \ i = 1, 2
\end{align*} \]

\[
\begin{array}{c|cc}
\phi_1 & \phi_2 \\
\hline
b_1^\phi & b_2^\phi \\
\frac{1}{g} & \frac{1}{g}
\end{array}
\]
A.7 $\epsilon_8$

In this section we consider the bisymmetric triples of the form $(\epsilon_8, \mathfrak{so}_{16}, l)$ and $(\epsilon_8, \epsilon_7 \oplus \mathfrak{su}_2, l)$. The root system for $\epsilon_8$ is

$$\mathcal{R} = \{ \pm e_i \pm e_j, 1 \leq i < j \leq 8; \pm \frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu_i} e_i, \sum_{i=1}^{8} \nu_i \text{ even } \}, \quad (A.16)$$

where $e_1, \ldots, e_8$ is the canonical basis for $\mathbb{R}^8$. In $\epsilon_8$ there is only one root length which is

$$|\alpha|^2 = \frac{1}{30}. \quad (A.17)$$

**Lemma A.1.** Let $\phi = \frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu_i} e_i \in \mathcal{R}$.

(i) Let $\alpha = \frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu_i} e_i \in \mathcal{R}$. The string $\phi + n \alpha$ is either singular, $\phi$, $\phi + \alpha$ or $\phi$, $\phi - \alpha$. So either $d_{\alpha \phi} = 0$ or 1, respectively.

We have that $\phi + \alpha$ is a root if and only if $\nu_i = \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 8\}$ and in this case, $\phi + \alpha = (-1)^{\nu_i} e_i + (-1)^{\nu_2} e_2$.

$\phi - \alpha$ is a root if and only if $\nu_i \neq \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 8\}$ and in this case, $\phi - \alpha = (-1)^{\nu_i} e_i + (-1)^{\nu_2} e_2$.

(ii) Let $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k \in \mathcal{R}$, $1 \leq j < k \leq 8$. The string $\phi + n \alpha'$ is either singular or $\phi$, $\phi - \alpha'$. $\phi - \alpha'$ is a root if and only if $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k$, for $1 \leq j < k \leq 8$. In this case, $\phi - \alpha' = \frac{1}{2} \left( \sum_{i=j, k}^{8} (-1)^{\nu_i} e_i + (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k \right)$.

**Proof:** Consider the roots $\phi = \frac{1}{2} \sum_{i=1}^{8} (-1)^{\nu_i} e_i$ and $\alpha = \frac{1}{2} \sum_{i=1}^{8} (-1)^{\mu_i} e_i$. Suppose that the string $\phi + n \alpha$ is not singular. Then, since $\phi + n \alpha$ is an uninterrupted string either $\phi + \alpha$ or $\phi - \alpha$ is a root. We have that

$$\phi + \alpha = \frac{1}{2} \sum_{i=1}^{8} ((-1)^{\nu_i} + (-1)^{\mu_i}) e_i \text{ and } \phi - \alpha = \frac{1}{2} \sum_{i=1}^{5} ((-1)^{\nu_i} - (-1)^{\mu_i}) e_i.$$ 

By observing the form of the roots in $\mathcal{R}$ given in (A.18) we conclude that $\phi + \alpha$ is a root if and only if $(-1)^{\nu_i} + (-1)^{\mu_i} = 0$, i.e., $\nu_i = \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 8\}$. We observe that this case is possible since $\sum_{i=1}^{8} \nu_i$ and $\sum_{i=1}^{8} \mu_i$ are even with 8 even. We thus obtain $\phi + \alpha = (-1)^{\nu_i} e_i + (-1)^{\mu_2} e_2$. Clearly $\phi + 2 \alpha$ is never a root. Similarly, $\phi - \alpha$ is a root if and only if $(-1)^{\nu_i} - (-1)^{\mu_i} \neq 0$, i.e., $\nu_i \neq \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 6\}$ and in this case, $\phi - \alpha = (-1)^{\nu_i} e_i + (-1)^{\mu_2} e_2$.

The element $\phi - 2 \alpha$ is never a root. Once again we observe that this case is possible.

Let $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}$. We have

\footnote{We may choose $-\alpha'$ in which case we obtain $\phi + \alpha'$ instead.}
\[ \phi - \alpha' = \frac{1}{2} \left( \sum_{i=1}^{8} (-1)^{\nu_i} e_i + ((-1)^{\nu_j} - 2(-1)^{\mu_j}) e_j + ((-1)^{\nu_k} - 2(-1)^{\mu_k}) e_k \right). \]

This element is a root if and only if \( \mu_j = \nu_j \) and \( \mu_k = \nu_k \) and, in this case,

\[ \phi - \alpha' = \frac{1}{2} \left( \sum_{i=1}^{8} (-1)^{\nu_i} e_i + (-1)^{\nu_j + 1} e_j + (-1)^{\nu_k + 1} e_k \right). \]

We observe that \( \phi - 2\alpha' \) is never a root.

\[ \square \]

Symmetric Pair A.10. \((\mathfrak{e}_8, \mathfrak{so}_{16})\).

| \(\mathfrak{e}\) | \(\mathcal{R}_\mathfrak{e}\) | \(\gamma\) |
|-----------------|-----------------|--------|
| \(\mathfrak{so}_{16}\) | \(\{ \pm e_i \pm e_j, 1 \leq i < j \leq 8 \}\) | \(\frac{7}{15}\) |

\(\mathcal{R}_n = \mathcal{R}_n = \{ \pm \frac{1}{2} \sum_1^8 (-1)^{\nu_i} e_i, \sum_1^8 \nu_i \text{ even} \}\)

Symmetric Pair A.11. \((\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2)\).

| \(\mathfrak{e}_i\) | \(\mathcal{R}_{\mathfrak{e}_i}\) | \(\gamma_i\) |
|-----------------|-----------------|--------|
| \(\mathfrak{e}_7\) | \(\{ \pm e_i \pm e_j, 1 \leq i < j \leq 6; \pm (e_7 - e_8); \pm \frac{1}{2} (e_7 - e_8 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \text{ odd} \}\) | \(\frac{7}{15}\) |
| \(\mathfrak{su}_2\) | \(\{ \pm (e_7 + e_8) \}\) | \(\frac{1}{3}\) |

\(\mathcal{R}_n = \{ \pm e_i \pm e_j, 1 \leq i \leq 6, j = 7, 8; \pm \frac{1}{2} (e_7 + e_8 + \sum_1^6 (-1)^{\nu_i} e_i), \sum_1^6 \nu_i \text{ even} \}\)

Bisymmetric Triple A.34. \((\mathfrak{e}_8, \mathfrak{so}_{16}, \mathfrak{so}_{2p} \oplus \mathfrak{so}_{2(8-p)})\), 1 \( \leq p \leq 4\). (Type I)

\(\mathcal{R}_p = \{ \pm e_i \pm e_j, 1 \leq i \leq p, p + 1 \leq j \leq 8 \}\)

\(C_p \) scalar on \(n\)

| \(\phi \in \mathcal{R}_n\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha' \text{'s}\) | \(|\alpha|^2\) | \(b^\phi\) |
|-----------------|-----------------|-----------------|--------|-----------------|--------|--------|
| \(\frac{1}{2} \sum_1^8 e_i\) | \(e_i + e_j, 1 \leq i \leq p, p + 1 \leq j \leq 8\) | \(\phi, \phi - \alpha\) | \(1\) | \(p(8 - p)\) | \(\frac{1}{30}\) | \(\frac{p(8-p)}{60}\) |
Bisymmetric Triple A.35. \((\varepsilon_8, \delta_0_{16}, u_8)\). \((\text{Type I})\)

\[
\mathcal{R}_p = \{\pm(e_i + e_j), 1 \leq i \leq 8\}
\]

\[
n = \bigoplus_{0,1,2} n^i
\]

\[
\mathcal{R}_p^i = \{\pm\frac{1}{2} \sum_1^8 (-1)^{\nu_j} e_j \mid 2i \text{ odd } \nu_j \}'s, i = 0, 1, 2
\]

| \(\phi \in \mathcal{R}_n^i\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \(\frac{1}{2} \sum_1^8 (-1)^{\nu_j} e_j\), 2i odd \(\nu_j\)'s, s.t. \(\nu_a = \nu_b\) \(\phi, \phi + (-1)^{\nu_a+1} \alpha\) | 1 | 4i^2 - 16i + 28(*) | \(\frac{1}{30}\) | \(\frac{2^2 - 4i + 7}{15}\) |

\(*)\) the number of possible pairs \((a, b)\), where \(a < b\) and \(\nu_a = \nu_b\), since \(2i \nu_j\)'s are odd and \(8 - 2i \nu_j\)'s are even, is

\[
\binom{2i}{2} + \binom{8 - 2i}{2}
\]

\[= 4i^2 - 16i + 28.\]
Bisymmetric Triple A.36. \((\epsilon_8, \epsilon_7 \oplus \mathfrak{su}_2, \epsilon_7 \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{\pm (e_7 + e_8)\}
\]

\(C_p\) scalar on \(n\)

| \(\phi \in \mathcal{R}_n\) | \(\alpha \in \mathcal{R}_n^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha 's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|
| \(\frac{1}{2}(e_7 + e_8 + \sum_{i=1}^6(-1)^\nu_i e_i)\), \(\sum_{i=1}^6 \nu_i \text{ even}\) | \(e_7 + e_8\) | \(\phi, \phi - \alpha\) | 1 | 1 | \(\frac{1}{36}\) | \(\frac{1}{60}\) |

Bisymmetric Triple A.37. \((\epsilon_8, \epsilon_7 \oplus \mathfrak{su}_2, \epsilon_6 \oplus \mathbb{R} \oplus \mathfrak{su}_2)\). (Type I)

\[
\mathcal{R}_p = \{\pm \epsilon_i \pm \epsilon_6, 1 \leq i \leq 5; \pm (e_7 - e_8); \pm \frac{1}{2}(e_8 - e_7 + e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ odd}\}
\]

\(n = n^1 \oplus n^2 \oplus n^3\)

\(\mathcal{R}_{n^1} = \{\pm \epsilon_i \pm \epsilon_j : 1 \leq i \leq 5, j = 7,8\},\)

\(\mathcal{R}_{n^2} = \{\pm (e_6 - e_7), \pm (e_6 + e_8), \pm \frac{1}{2}(e_8 + e_7 - e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ odd}\},\)

\(\mathcal{R}_{n^3} = \{\pm (e_6 + e_7), \pm (e_6 - e_8), \pm \frac{1}{2}(e_8 + e_7 + e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ even}\}\)

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
n^1 & \phi \in \mathcal{R}_{n^1} & \alpha \in \mathcal{R}_n^+ & \phi + n\alpha & d_{\alpha\phi} & \text{No of } \alpha 's & |\alpha|^2 & b^\phi \\
\hline
n^1 & e_1 + e_8 & \frac{1}{2}(e_8 - e_7 + e_6 + e_1 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ even} & \phi, \phi - \alpha & 1 & \frac{3}{2^3} & \frac{1}{36} & \frac{1}{60} \\
\hline
n^2 & e_6 - e_8 & e_1 + e_6, i \leq i \leq 5 & \phi, \phi - \alpha & 1 & \frac{5}{2^3} & \frac{1}{36} & \frac{1}{60} \\
\hline
n^2 & e_6 + e_8 & e_1 - e_6, i \leq i \leq 5 & \phi, \phi + \alpha & 1 & \frac{5}{2^3} & \frac{1}{36} & \frac{1}{60} \\
\hline
\end{array}
\]

Bisymmetric Triple A.38. \((\epsilon_8, \epsilon_7 \oplus \mathfrak{su}_2, \epsilon_6 \oplus \mathbb{R} \oplus \mathbb{R})\). (Type II)

\[
\mathcal{R}_{p_1} = \{\pm (e_7 + e_8)\}
\]

\[
\mathcal{R}_{p_2} = \{\pm \epsilon_i \pm \epsilon_6, 1 \leq i \leq 5; \pm (e_7 - e_8); \pm \frac{1}{2}(e_8 - e_7 + e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ odd}\}
\]

\(n = n^1 \oplus n^2 \oplus n^3\)

\[
\mathcal{R}_{n^1} = \{\pm \epsilon_i \pm \epsilon_j : 1 \leq i \leq 5, j = 7,8\},\)

\(\mathcal{R}_{n^2} = \{\pm (e_6 - e_7), \pm (e_6 + e_8), \pm \frac{1}{2}(e_8 + e_7 - e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ odd}\},\)

\(\mathcal{R}_{n^3} = \{\pm (e_6 + e_7), \pm (e_6 - e_8), \pm \frac{1}{2}(e_8 + e_7 + e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ even}\}\)

| \(\mathcal{R}_{n^1}\) | \(b_1^\phi\) | \(b_2^\phi\) |
|---|---|---|
| \{\pm \epsilon_i \pm \epsilon_j : 1 \leq i \leq 5, j = 7,8\} | \frac{11}{2^3} | \frac{1}{2^3} |
| \{\pm (e_6 - e_7), \pm (e_6 + e_8), \pm \frac{1}{2}(e_8 + e_7 - e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ odd}\} | \frac{11}{2^3} | \frac{1}{2^3} |
| \{\pm (e_6 + e_7), \pm (e_6 - e_8), \pm \frac{1}{2}(e_8 + e_7 + e_6 + \sum_{i=1}^5(-1)^\nu_i e_i), \sum_{i=1}^5 \nu_i \text{ even}\} | \frac{9}{20} | \frac{1}{20} |

Bisymmetric Triple A.39. \((\epsilon_8, \epsilon_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2)\). (Type I)

\[
\mathcal{R}_p = \{\pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^6(-1)^\nu_i e_i), \sum_{i=1}^6 \nu_i \text{ odd}\}
\]

\(n = n^1 \oplus n^2\)

\(\mathcal{R}_{n^1} = \{\pm \epsilon_i \pm \epsilon_j : 1 \leq i \leq j = 7,8\},\)

\(\mathcal{R}_{n^2} = \{\pm \frac{1}{2}(e_8 + e_7 + \sum_{i=1}^6(-1)^\nu_i e_i), \sum_{i=1}^6 \nu_i \text{ even}\}\)
Bisymmetric Triple A.40. \((\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathbb{R})\). (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm \frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), \sum_{i}^{6} \nu_i \text{ odd} \} \\
\mathcal{R}_{p_2} = \{ \pm (e_7 + e_8) \}
\]

\[
n = n^1 \oplus n^2
\]

\[
\mathcal{R}_{\phi} = \{ \pm e_i \pm e_j : 1 \leq i, j = 7, 8 \} \\
\{ \pm \frac{1}{2}(e_7 + e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), \sum_{i}^{6} \nu_i \text{ even} \}
\]

Bisymmetric Triple A.41. \((\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathfrak{su}_2 \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), 3 \text{ odd } \nu_i \}'s \}
\]

\[
n \text{ irreducible Ad } L \text{-module}
\]

\[
\phi \in \mathcal{R}_n \quad \alpha \in \mathcal{R}_p^+ \quad \phi + n\alpha \quad d_{\alpha\phi} \quad \text{No of } \alpha' \text{s} \quad |\alpha|^2 \quad b^\phi
\]

| \(n^1\) \(e_1 + e_7\) & \(\frac{1}{2}(e_7 - e_8 + e_1 + \sum_{i}^{6}(-1)^{e_i}e_i), \sum_{i}^{6} \nu_i \text{ odd} \) & \(\phi, \phi - \alpha\) & 1 & 2^4 & \(\frac{1}{30}\) & 4 \(\frac{1}{15}\) |
| \(n^2\) \(\frac{1}{2} \sum_{i=1}^{8} e_j\) & \(\frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), \sum_{i}^{6} \nu_i \text{ even} \) & \(\phi, \phi + \alpha\) & 1 & 6 & \(\frac{1}{30}\) & 1 \(\frac{1}{5}\) |

\[
\mathcal{R}_{p_1} = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), 3 \text{ odd } \nu_i \}'s \}
\]

\[
n \text{ irreducible Ad } L \text{-module}
\]

| \(\phi \in \mathcal{R}_n\) \(e_1 + e_7\) & \(\frac{1}{2}(e_7 - e_8 + e_1 + \sum_{i}^{6}(-1)^{e_i}e_i), 3 \text{ odd } \nu_i \)'s \) & \(\phi + n\alpha\) & \(d_{\alpha\phi}\) & \(\text{No of } \alpha' \text{s}\) & \(|\alpha|^2\) & \(b^\phi\) |
| \(\frac{1}{2} \sum_{i=1}^{8} e_j\) & \(\frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), \sum_{i}^{6} \nu_i \text{ even} \) & \(\phi, \phi - \alpha\) & 1 & \(\frac{5}{3}\) & \(\frac{1}{30}\) & \(\frac{1}{4}\) |

Bisymmetric Triple A.42. \((\mathfrak{e}_8, \mathfrak{e}_7 \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathbb{R})\). (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i}^{6}(-1)^{e_i}e_i), 3 \text{ odd } \nu_i \}'s \}
\]

\[
n \text{ irreducible Ad } L \text{-module}
\]

| \(b^\phi_1\) & \(b^\phi_2\) |
| \(\frac{1}{3}\) & \(\frac{1}{60}\) |

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A.8 $e_7$

In this Section we study the bisymmetric triples of the form $(\mathfrak{e}_7, \mathfrak{su}_8, 1)$, $(\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, 1)$ and $(\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, 1)$.

The root system for $\mathfrak{g} = \mathfrak{e}_7$ is

$$\mathcal{R} = \{ \pm e_i \pm e_j, 1 \leq i < j \leq 6; \pm (e_7 - e_8); \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i), \sum_{i=1}^{6} \nu_i \text{ odd } \},$$

where $e_1, \ldots, e_8$ is the canonical basis for $\mathbb{R}^8$. Throughout all the relations for the $\nu_i$'s are $\mod 2$. In $\mathfrak{e}_7$, all the roots have the same length which is

$$|\alpha|^2 = \frac{1}{18}.$$  \hfill (A.19)

**Lemma A.2.** Let $\phi = \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i) \in \mathcal{R}$.

(i) Let $\alpha = \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i) \in \mathcal{R}$. The string $\phi + n\alpha$ is either singular, $\phi$, $\phi + \alpha$ or $\phi$, $\phi$, $\phi - \alpha$. So either $d_{n\alpha} = 0$ or 1, respectively. We have that $\phi + \alpha$ is a root if and only if $\nu_i \neq \mu_i$, for every $i = 1, \ldots, 6$ and in this case, $\phi + \alpha = e_8 - e_7$; $\phi - \alpha$ is a root if and only if $\nu_i \neq \mu_i$, for two indices $i_1, i_2 \in \{1, \ldots, 6\}$ and in this case, $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$.

(ii) Let $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k \in \mathcal{R}$, $1 \leq j < k \leq 6$. The string $\phi + n\alpha'$ is either singular or $\phi$, $\phi$, $\phi$, $\phi'$. $\phi - \alpha'$ is a root if and only if $\alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k$, for $1 \leq j < k \leq 6$. In this case, $\phi - \alpha' = \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i + (-1)^{\nu_j+1} e_j + (-1)^{\nu_k+1} e_k)$ \hfill (A.18)

(iii) For $\alpha'' = e_7 - e_8 \in \mathcal{R}$, the string $\phi + n\alpha''$ is $\phi$, $\phi$, $\phi$, $\phi''$, with $\phi - \alpha'' = -\frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i)$.

**Proof:** $\mathcal{R}$ is a subsystem of roots of the root system for $\mathfrak{e}_8$. Hence, we use Lemma A.1.

For (i) let us consider the roots $\phi = \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i)$ and $\alpha = \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\mu_i} e_i)$. Suppose that the string $\phi + n\alpha$ is not singular. Since $\nu_7 = \mu_7$ and $\nu_8 = \mu_8$ at least two indices satisfy $\nu_i = \mu_i$. Hence, if for every $i \in \{1, \ldots, 6\}$, $\nu_i \neq \mu_i$, then $\phi + \alpha = e_8 - e_7$ is a root; if there is $i \in \{1, \ldots, 6\}$, $\nu_i = \mu_i$, then we obtain a root $\phi - \alpha$ if and only if $\nu_i = \mu_i$ for precisely two indices $i_1, i_2 \in \{1, \ldots, 6\}$.

In this case, $\phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2}$. We observe that both conditions on indices in $\{1, \ldots, 6\}$ are possible since $\sum_i^6 \nu_i$ and $\sum_i^6 \mu_i$ are odd and 6 is even.

(ii) and (iii) follow directly from (ii) in Lemma A.1.

$\square$

\footnote{We may choose $-\alpha'$ in which case we obtain $\phi + \alpha'$ instead.}
Symmetric Pair A.12. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2)\).

\[
\begin{array}{ccc}
\mathfrak{e}_i & \mathcal{R}_{\mathfrak{e}_i} & \gamma_i \\
\mathfrak{so}_{12} & \{\pm e_i \pm e_j, 1 \leq i < j \leq 6\} & \frac{5}{9} \\
\mathfrak{su}_2 & \{ \pm (e_7 \pm e_8) \} & \frac{1}{9}
\end{array}
\]

\[
\mathcal{R}_n = \left\{ \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i), \sum_{i=1}^{6} \nu_i \text{ odd} \right\}
\]

Symmetric Pair A.13. \((\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R})\).

\[
\begin{array}{ccc}
\mathfrak{e}_6 & \mathcal{R}_{\mathfrak{e}_6} & \gamma \\
\{\pm e_i \pm e_j : 1 \leq i < j \leq 5; \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : \sum_{i=1}^{5} \nu_i \text{ is even} \} & \frac{2}{3}
\end{array}
\]

\[
\mathcal{R}_n = \left\{ \pm e_i \pm e_6 : 1 \leq i \leq 5; \pm (e_7 - e_8) \pm \frac{1}{2}(e_7 - e_8 + e_6 \sum_{i=1}^{5} (-1)^{\nu_i} e_i), \sum_{i=1}^{5} \nu_i \text{ odd} \right\}
\]

Symmetric Pair A.14. \((\mathfrak{e}_7, \mathfrak{su}_8, \mathfrak{su}_p \oplus \mathfrak{su}_{8-p} \oplus \mathbb{R}), p = 1, \ldots, 4\).

\[
\begin{array}{ccc}
\mathfrak{s}_8 & \mathcal{R}_{\mathfrak{s}_8} & \gamma \\
\{\pm (e_i - e_j) : 1 \leq i < j \leq 6; \pm (e_7 - e_8); \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : 1 \text{ or } 5 \text{ odd } \nu_i's \} & \frac{4}{9}
\end{array}
\]

\[
\mathcal{R}_n = \left\{ \pm (e_i + e_j) : 1 \leq i < j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_i) : 3 \text{ odd } \nu_i's \right\}
\]

Bisymmetric Triple A.43. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_{12} \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{ \pm (e_7 - e_8) \}
\]

\[
C_p \text{ scalar on } n
\]

\[
\phi \in \mathcal{R}_n, \quad \alpha \in \mathcal{R}^+_p, \quad \phi + n \alpha, \quad d_{\alpha \phi}, \quad \text{No of } \alpha's, \quad |\alpha|^2, \quad b^0
\]

\[
\frac{1}{2}(e_7 - e_8 - e_1 \sum_{i=2}^{6} e_i) e_7 - e_8, \quad \phi - \alpha, \quad 1, \quad 1, \quad \frac{1}{18}, \quad \frac{1}{36}
\]

Bisymmetric Triple A.44. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{u}_6 \oplus \mathfrak{su}_2)\). (Type I)

\[
\mathcal{R}_p = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 6 \}
\]

\[
n = \oplus 0, 1, 2 \mathbb{N}
\]

\[
\mathcal{R}_n' = \{ \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_j) : 2i + 1 \nu_j's \text{ are odd} \}, i = 0, 1, 2
\]

\[
\phi \in \mathcal{R}_n', \quad \alpha \in \mathcal{R}^+_p, \quad \phi + n \alpha, \quad d_{\alpha \phi}, \quad \text{No of } \alpha's, \quad |\alpha|^2, \quad b^0
\]

\[
\frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{6} (-1)^{\nu_i} e_j), \quad \nu_a = \nu_b, \quad \nu_a = \nu_b, \quad \phi, \phi - (-1)^{\nu_a} \alpha, \quad 1, \quad 4i^2 - 8i + 10^{(*)}, \quad \frac{1}{18}, \quad 2i^2 - 4i + \frac{5}{18}
\]

\( (*) \) the number of possible pairs \((a, b)\), where \(a \neq b\) and \(\nu_a = \nu_b\), since \(2i + 1 \nu_j's \) are odd, is

\[
\left( \begin{array}{c} 2i + 1 \\ 2 \end{array} \right) + \left( \begin{array}{c} 6 - (2i + 1) \\ 2 \end{array} \right) = 4i^2 - 8i + 10.
\]
Bisymmetric Triple A.45. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{u}_6 \oplus \mathbb{R})\). (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 6 \}
\]
\[
\mathcal{R}_{p_2} = \mathcal{R}_{p_2} = \{ \pm (e_7 - e_8) \}
\]
\[
n = \oplus_{0,1,2} n^i
\]
\[
\{ \pm \frac{1}{2} (e_7 - e_8 + \sum_1^{6} (-1)^{e_j} e_j : 2i + 1 \nu_j's are odd \}^{(*)} \frac{2^2 - 4i + 5}{18} \frac{1}{30}
\]

\(*\) \(i = 0, 1, 2\) as in A.44

Bisymmetric Triple A.46. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathfrak{su}_2), p = 2, 4, 6\). (Type I)

\[
\mathcal{R}_p = \{ \pm e_i \pm e_j : 1 \leq i \leq p/2, p/2 + 1 \leq j \leq 6 \}
\]
\[
C_p \text{ scalar on } n
\]
\[
\phi \in \mathcal{R}_n \quad \alpha \in \mathcal{R}_p^+ \quad \phi + n\alpha \quad d_{\alpha, \phi} \quad \text{No of } \alpha's \quad |\alpha|^2 \quad b^\phi
\]

\[
\frac{1}{2} (e_7 - e_8 \sum_1^{6} e_i) \quad e_i - e_j, 1 \leq j \leq 6 \quad \phi, \phi - \alpha \quad 1 \quad \frac{p(12-p)}{4} \quad \frac{1}{18} \quad \frac{p(12-p)}{144}
\]

Bisymmetric Triple A.47. \((\mathfrak{e}_7, \mathfrak{so}_{12} \oplus \mathfrak{su}_2, \mathfrak{so}_p \oplus \mathfrak{so}_{12-p} \oplus \mathbb{R}), p = 2, 4, 6\). (Type II)

\[
\mathcal{R}_{p_1} = \{ \pm e_i \pm e_j : 1 \leq i \leq p/2, p/2 + 1 \leq j \leq 6 \}
\]
\[
\mathcal{R}_{p_2} = \{ \pm (e_7 - e_8) \}
\]
\[
C_p, \text{ scalar on } n, i = 1, 2
\]
\[
\frac{b_1^\phi}{b_2^\phi}
\]

\[
\frac{1}{30} \quad \frac{p(12-p)}{144}
\]
Bisymmetric Triple A.48. \((\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{so}_{10} \oplus \mathbb{R} \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{ \pm \frac{1}{2} (e_7 - e_8 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i, \sum_{i=1}^{5} \nu_i \text{ even} \}
\]

\(n = n^1 \oplus n^2 \oplus n^3\)

\(\mathcal{R}_{n^1} = \mathcal{R}_{n^1} = \{ \pm e_i \pm e_6 : 1 \leq i \leq 5 \},\)

\(\mathcal{R}_{n^2} = \{ \pm \frac{1}{2} (e_7 - e_8 + e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i, \sum_{i=1}^{5} \nu_i \text{ odd} \}\)

\(\mathcal{R}_{n^3} = \{ \pm (e_7 - e_8) \}\)

| \(n^i\) | \(\phi \in \mathcal{R}_{n^i}\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha\)'s | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(n^1\) | \(e_1 - e_6\) | \(\frac{1}{2} (e_7 - e_8 - e_6 + e_1 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i, \sum_{i=1}^{5} \nu_i \text{ even} \) | \(\phi, \phi - \alpha\) | 1 | 23 | \(\frac{1}{18}\) | \(\frac{2}{9}\) |
| \(n^2\) | \(\frac{1}{2} (e_7 - e_8 + e_6 - e_5 + \sum_{i=1}^{4} e_i)\) | \(\frac{1}{2} (e_7 - e_8 - e_6 + \sum_{i=1}^{4} e_i)\) | \(\phi, \phi + \alpha\) | 1 | 1 | \(\frac{1}{18}\) | \(\frac{1}{6}\) |
| \(n^3\) | \(e_7 - e_8\) | \(\frac{1}{2} (e_7 - e_8 - e_6 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i, \text{one odd } \nu_i\) | \(\phi, \phi - \alpha\) | 4 | 24 | \(\frac{1}{18}\) | \(\frac{4}{9}\) |

Bisymmetric Triple A.49. \((\mathfrak{e}_7, \mathfrak{e}_6 \oplus \mathbb{R}, \mathfrak{su}_6 \oplus \mathfrak{su}_2 \oplus \mathbb{R})\). (Type I)

\[
\mathcal{R}_p = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 5; \pm \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i), \text{ 2 } \nu_i \text{'s odd} \}
\]

\(n = n^1 \oplus n^2\)

\(\mathcal{R}_{n^1} = \{ \pm (e_i - e_6) : 1 \leq i \leq 5; \pm (e_7 - e_8) + \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i), \text{ 1 or 5 } \nu_i \text{'s odd} \}\)

\(\mathcal{R}_{n^2} = \{ \pm (e_i + e_6) : 1 \leq i \leq 5; \pm \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i), \text{ 3 } \nu_i \text{'s odd} \}\)

| \(n^i\) | \(\phi \in \mathcal{R}_{n^i}\) | \(\alpha \in \mathcal{R}_p^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(\alpha\)'s | \(|\alpha|^2\) | \(b^\phi\) |
|---|---|---|---|---|---|---|---|
| \(n^1\) | \(e_7 - e_8\) | \(\frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i), \text{ 2 } \nu_i \text{'s odd} \) | \(\phi, \phi - \alpha\) | 1 | \(\left(\begin{array}{c} 5 \\ 2 \end{array}\right)\) | \(\frac{5}{18}\) | \(\frac{5}{18}\) |
| \(n^2\) | \(e_1 + e_6\) | \(\frac{1}{2} (e_8 - e_7 - e_6 - e_1 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i), \text{ 1 odd } \nu_i\) | \(\phi, \phi - \alpha\) | 4 | \(\left(\begin{array}{c} 4 \\ 2 \end{array}\right)\) | \(\frac{1}{18}\) | \(\frac{2}{9}\) |
Bisymmetric Triple A.50. \((\mathfrak{sl}_6, \mathfrak{su}_8, \mathfrak{su}_p \oplus \mathfrak{su}_{8-p} \oplus \mathbb{R}), \ p = 1, \ldots, 4. \) (Type I)

\[ R_i = \{ \pm(e_i - e_j) : 1 < i < j \leq p \ or \ p + 1 \leq i < j \leq 6; \pm(e_7 - e_8); \ \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{p} e_i + \sum_{p+1}^{6} e_i) : 1 \ odd \ \nu_i; \]
\[ \pm \frac{1}{2}(e_7 - e_8 - \sum_{i=1}^{p} e_i - \sum_{p+1}^{6} e_i) : (5 - p) \ odd \ \nu_i \}\]

\[ R_p = \{ \pm(e_i - e_j) : 1 \leq i \leq p, p + 1 \leq j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{p} e_i + \sum_{p+1}^{6} e_i) : 1 \ odd \ \nu_i; \ \pm \frac{1}{2}(e_7 - e_8 + \sum_{p+1}^{6} e_i) : (p-1) \ odd \ \nu_i \}\]

\[ p = 1 \]

\[ R_p = \{ \pm(e_i - e_j) : 2 \leq j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + e_1 - \sum_{i=2}^{6} e_i); \ \pm \frac{1}{2}(e_7 - e_8 - e_1 + \sum_{i=2}^{6} e_i) \}\]

| No of \(a's\) | \(\|a\|^2\) | \(b^0\) |
|----------------|---------|---------|
| \(e_1 + e_2\) | \(e_1 - e_3\), \(3 \leq j \leq 6\) | \(\phi, \phi - \alpha\) | 1 | 4 | \(\frac{1}{18}\) | \(\frac{1}{9}\) |

\[ p = 2 \]

\[ R_p = \{ \pm(e_i - e_j) : 1 \leq i \leq 2, 3 \leq j \leq 6; \pm \frac{1}{2}(e_7 - e_8 + \sum_{i=1}^{2} (-1)^{\nu_i} \pm \sum_{i=3}^{6} e_i) : 1 \ odd \ \nu_i \}\]

\[ n = n^1 + n^2 \]

\[ R_{n^1} = \{ \pm(e_1 + e_2); \ \pm(e_i + e_j) : 3 \leq i < j \leq 6; \pm \frac{1}{2}(e_7 - e_8 - e_1 - e_2 + \sum_{i=3}^{6} (-1)^{\nu_i} e_i) : 1 \ odd \ \nu_i; \ \pm \frac{1}{2}(e_7 - e_8 + e_1 + e_2 + \sum_{i=3}^{6} (-1)^{\nu_i} e_i) : 3 \ odd \ \nu_i \}\]

\[ R_{n^2} = \{ \pm(e_i + e_j) : 4 \leq i < j \leq 6; \ \pm \frac{1}{2}(e_7 - e_8 \pm \sum_{i=1}^{3} e_i + \sum_{i=4}^{6} e_i) \}\]

| \(n^1\) | \(\phi \in R_{n^1}\) | \(\alpha \in R_{n^1}^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | No of \(a's\) | \(\|a\|^2\) | \(b^0\) |
|---------|-----------------|-----------------|----------------|-----------------|---------|---------|---------|
| \(n^1\) | \(e_1 + e_2\) | \(e_1 - e_j, i = 1, 2, 3 \leq j \leq 6\) | \(\phi, \phi - \alpha\) | 1 | 8 | \(\frac{1}{18}\) | \(\frac{2}{9}\) |

| \(n^2\) | \(e_1 + e_6\) | \(e_1 - e_j, 3 \leq j \leq 6\) | \(\phi, \phi - \alpha\) | \(\phi, \phi + \alpha\) | 1 | 1 | \(\frac{1}{18}\) | \(\frac{1}{9}\) |
\( p = 3 \)

\[ R_p = \{ \pm(e_i - e_j) : 1 \leq i \leq 3, 4 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i + \sum_1^6 e_i) : 1 \text{ odd } \nu_i; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i - \sum_1^6 e_i) : 2 \text{ odd } \nu_i/2 \} \]

\( n = n^1 \oplus n^2 \)

\[ R_{n^1} = \{ \pm(e_i + e_j) : 1 \leq i < j \leq 3 \text{ or } 1 \leq i \leq 3, 4 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i + \sum_1^6 e_i) : 2 \text{ odd } \mu_i/2 \text{ and } 1 \text{ odd } \nu_i; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i + \sum_1^6 e_i) : 1 \text{ odd } \mu_i/2 \text{ and } 2 \text{ odd } \nu_i \} \]

\[ R_{n^2} = \{ \pm(e_i + e_j) : 4 \leq i < j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4 e_i) \} \]

| \( n^1 \) | \( \phi \in R_{n^1} \) | \( \alpha \in R_p^+ \) | \( \phi + n\alpha \) | \( d_{\alpha\phi} \) | \( \text{No of } \alpha's \) | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 + e_2 \) | \( \frac{1}{2}(e_7 - e_8 + e_1 + e_2 - e_3 + \sum_1^6 e_i) \) | \( \phi, \phi - \alpha \) | 6 |
| \( n^1 \) | \( e_1 - e_2, i = 1, 2, j = 4, 5, 6 \) | \( \frac{1}{2}(e_7 - e_8 + e_1 + e_2 - e_3 + \sum_1^6 e_i) \) | \( \phi, \phi - \alpha \) | 1 |
| \( n^2 \) | \( e_4 + e_6 \) | \( \frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i + \sum_1^6 e_i), 1 \text{ odd } \nu_i \) | \( \phi, \phi - \alpha \) | 3 |
| \( n^2 \) | \( e_4 + e_6 \) | \( \frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i - \sum_1^6 e_i), 2 \text{ odd } \nu_i/2 \) | \( \phi, \phi + \alpha \) | 3 |

\( p = 4 \)

\[ R_p = \{ \pm(e_i - e_j) : 1 \leq i \leq 4, 5 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i + \sum_1^6 e_i) : 1 \text{ odd } \nu_i; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i - \sum_1^6 e_i) : 3 \text{ odd } \nu_i/2 \} \]

\( n = n^1 \oplus n^2 \oplus n^3 \)

\[ R_{n^1} = \{ \pm(e_i + e_j) : 1 \leq i < j \leq 4; \pm\frac{1}{2}(e_7 - e_8 + \sum_1^4(-1)^i e_i \pm e_5 \pm e_6) : 2 \text{ odd } \nu_i/2 \} \]

\[ R_{n^2} = \{ \pm(e_5 + e_6) \} \]

\[ R_{n^2} = \{ \pm(e_i + e_j) : 1 \leq i \leq 4, 5 \leq j \leq 6; \pm\frac{1}{2}(e_7 - e_8 - e_5 - e_6 + \sum_1^4(-1)^i e_i) : 1 \text{ odd } \nu_i; \pm\frac{1}{2}(e_7 - e_8 + e_6 + e_5 + \sum_1^4(-1)^i e_i) : 3 \text{ odd } \nu_i/2 \} \]
| $\mathbb{H}^1$ | $\phi \in \mathbb{R}_{\mathbb{H}^1}$ | $\alpha \in \mathbb{R}_{\mathbb{H}^3}$ | $\phi + n\alpha$ | $d_{\phi}$ | No of $\alpha$'s | $|\alpha|^2$ | $b_{\phi}$ |
|---|---|---|---|---|---|---|---|
| $\mathbb{H}^1$ | $e_1 + e_2$ | $\begin{cases} e_i - e_j, \ i = 1, 2, j = 5, 6 \\ \frac{1}{2}(e_7 - e_8 + e_6 + e_5 \mp e_4 \pm e_3 + e_2 + e_1) \\ \frac{i}{2}(e_7 - e_8 - e_6 - e_5 \mp e_4 \pm e_3 - e_2 - e_1) \end{cases}$ | $\phi, \phi - \alpha$ | $4$ | $1$ | $\frac{1}{18}$ | $\frac{2}{5}$ |
| $\mathbb{H}^2$ | $e_5 + e_6$ | $\begin{cases} e_i - e_j, \ i = 1, 2, 3, 4, j = 5, 6 \\ \frac{1}{2}(e_7 - e_8 + e_6 + e_5 + \sum_4^{i} (-1)^{\nu_i} e_i), 1 \ \mathrm{odd} \nu_i \\ \frac{i}{2}(e_7 - e_8 - e_6 - e_5 + \sum_4^{i} (-1)^{\nu_i} e_i), 3 \ \mathrm{odd} \nu_i's \end{cases}$ | $\phi, \phi + \alpha$ | $8$ | $1$ | $\frac{1}{18}$ | $\frac{4}{5}$ |
| $\mathbb{H}^3$ | $e_1 + e_6$ | $\begin{cases} e_1 - e_5 \\ \frac{1}{2}(e_7 - e_8 + e_6 + e_5 + e_1 + \sum_2^{i} (-1)^{\nu_i} e_i), 1 \ \mathrm{odd} \nu_i \\ \frac{i}{2}(e_7 - e_8 - e_6 - e_5 - e_1 + \sum_2^{i} (-1)^{\nu_i} e_i), 2 \ \mathrm{odd} \nu_i's \end{cases}$ | $\phi, \phi - \alpha$ | $3$ | $1$ | $\frac{1}{18}$ | $\frac{11}{36}$ | $\phi, \phi + \alpha$ | $3$ |
A.9 \( \mathfrak{e}_6 \)

In this Section we consider the bisymmetric triples of the form \( (\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{l}) \) and \( (\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{l}) \).

If \( e_1, \ldots, e_8 \) is the canonical basis for \( \mathbb{R}^8 \), we can write the root system for \( \mathfrak{e}_6 \) as follows:

\[
\mathcal{R} = \{ \pm e_i \pm e_j : 1 \leq i < j \leq 5; \pm \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : \sum_{i=1}^{5} \nu_i \text{ is even} \}.
\]

Throughout, all the relations for the \( \nu_i \)'s are \( \text{mod } 2 \).

We recall that on \( \mathfrak{e}_6 \) there is only one root length which is

\[
|\alpha|^2 = \frac{1}{12} \quad (A.20)
\]

**Lemma A.3.** Let \( \phi = \frac{1}{2} (e_8 - e_7 - e_6 \sum_{i=1}^{5} (-1)^{\nu_i} e_i) \in \mathcal{R} \).

(i) Let \( \alpha = \frac{1}{2} (e_8 - e_7 - e_6 \sum_{i=1}^{5} (-1)^{\mu_i} e_i) \in \mathcal{R} \). The string \( \phi + \alpha \) is either singular or \( \phi, \phi - \alpha \). So either \( d_{\alpha \phi} = 0 \) or \( 1 \), respectively. We have that \( \phi - \alpha \) is a root if and only if \( \nu_i \neq \mu_i \), for two indices \( i_1, i_2 \in \{1, \ldots, 5\} \) and in this case, \( \phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2} \).

(ii) Let \( \alpha' = (-1)^{\mu_j} e_j + (-1)^{\mu_k} e_k \in \mathcal{R}, 1 \leq j < k \leq 5 \). The string \( \phi + \alpha' \) is either singular or \( \phi, \phi - \alpha' \). \( \phi - \alpha' \) is a root if and only if \( \alpha' = (-1)^{\nu_j} e_j + (-1)^{\nu_k} e_k \), for \( 1 \leq j < k \leq 5 \). In this case, \( \phi - \alpha' = \frac{1}{2} (e_8 - e_7 - e_6 \sum_{i=1}^{5} (-1)^{\nu_i} e_i + (-1)^{\nu_j + 1} e_j + (-1)^{\nu_k + 1} e_k) \).

**Proof:** \( \mathcal{R} \) is a subsystem of roots of the root system for \( \mathfrak{e}_8 \) and thus we use Lemma [A.1] For (i), since \( \nu_i = \mu_i \), for \( i = 6, 7, 8 \), the case that \( \nu_i = \mu_i \) for precisely two indices in \( \{1, \ldots, 8\} \) never happens. Hence, \( \phi + \alpha \) is never a root. If \( \nu_i \neq \mu_i \) for two indices \( i_1, i_2 \in \{1, \ldots, 5\} \), then \( \phi - \alpha \) is a root and \( \phi - \alpha = (-1)^{\nu_{i_1}} e_{i_1} + (-1)^{\nu_{i_2}} e_{i_2} \).

This case may happen since \( \sum_{i=1}^{5} \nu_i \) and \( \sum_{i=1}^{5} \mu_i \) are even with 5 odd.

(ii) follows directly from (ii) in Lemma [A.1]

\[
\square
\]

**Symmetric Pair A.15.** \( (\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}) \).

\[
\begin{array}{cccc}
\mathfrak{t} & \mathcal{R}_\mathfrak{t} & \gamma \\
\mathfrak{so}_{10} & \{ \pm e_i \pm e_j : 1 \leq i < j \leq 5 \} & \frac{2}{3} \\
\mathcal{R}_\mathfrak{a} & \{ \pm \frac{1}{2} (e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : \sum_{i=1}^{5} \nu_i \text{ is even}^{(*)} \} \\
\end{array}
\]

\( (*) \) there are either 0, 2 or 4 negative signs.

\[\text{We may choose } -\alpha' \text{ in which case we obtain } \phi + \alpha' \text{ instead.}\]
Symmetric Pair A.16. \((\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2)\).

| \(\mathfrak{e}_i\) | \(\mathfrak{R}_{\mathfrak{e}_i}\) | \(\gamma_{\mathfrak{e}_i}\) |
|------------------|------------------|------------------|
| \(\mathfrak{su}_6\) | \(\{\pm(e_i - e_j) : 1 \leq i < j \leq 5; \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 4 \text{ odd } \nu_i's\}\) | \(\frac{1}{2}\) |
| \(\mathfrak{su}_2\) | \(\{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}\) | \(\frac{1}{6}\) |

\(\mathfrak{R}_n = \{\pm(e_i + e_j) : 1 \leq i < j \leq 5; \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 2 \text{ odd } \nu_i's\}\)

Bisymmetric Triple A.51. \((\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathfrak{R}, \mathfrak{u}_5 \oplus \mathfrak{R})\). (Type I)

\(\mathfrak{R}_\phi = \{\pm(e_i + e_j) : 1 \leq i < j \leq 5\}\)
\(n = n^0 \oplus n^2 \oplus n^4\)
\(\mathfrak{R}_{n^0} = \{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\}\),
\(\mathfrak{R}_{n^2} = \{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 2 \text{ odd } \nu_i's\}\),
\(\mathfrak{R}_{n^4} = \{\pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 (-1)^{\nu_i} e_i) : 4 \text{ odd } \nu_i's\}\)

| \(n^i\) | \(\phi \in \mathfrak{R}_{n^i}\) | \(\alpha \in \mathfrak{R}_\phi^+\) | \(\phi + n\alpha\) | \(d_{\alpha\phi}\) | \(\text{No of } \alpha's\) | \(|\alpha|^2\) | \(b^\phi\) |
|---------|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| \(n^0\) | \(\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^5 e_i)\) | \(\text{all}\) | \(\phi, \phi - \alpha\) | \(1\) | \(\frac{5}{2}\) | \(\frac{1}{12}\) | \(\frac{5}{12}\) |
| \(n^2\) | \(\frac{1}{2}(e_8 - e_7 - e_6 - e_1 - e_2 + \sum_3^5 e_i)\) | \(\frac{e_1 + e_2}{e_i + e_j}, 3 \leq i < j \leq 5\) | \(\phi, \phi + \alpha\) | \(1\) | \(\frac{1}{3}\) | \(\frac{1}{12}\) | \(\frac{1}{6}\) |
| \(n^4\) | \(\frac{1}{2}(e_8 - e_7 - e_6 + e_5 - \sum_1^4 e_i)\) | \(e_i + e_j, 1 \leq i < j \leq 4\) | \(\phi, \phi + \alpha\) | \(1\) | \(\frac{4}{2}\) | \(\frac{1}{12}\) | \(\frac{1}{4}\) |
Bisymmetric Triple A.52. \((\mathfrak{e}_6, \mathfrak{so}_{10} \oplus \mathbb{R}, \mathfrak{so}_p \oplus \mathfrak{so}_{10-p} \oplus \mathbb{R})\), \(p = 2, 4\). (Type I)

\[ \mathcal{R}_p = \{ \pm e_i \pm e_j : 1 \leq i \leq p/2, p/2 + 1 < j < 5 \} \]

\[ C_p \text{ scalar on } \mathfrak{n} \]

| \( \phi \in \mathbb{R}^n \) | \( \alpha \in \mathbb{R}^+ \) | \( \phi + n\alpha \) | \( d_{\alpha \phi} \) | No of \( \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|----------------|----------------|----------------|--------------------|----------------|-----------------|----------------|
| \( \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{\nu_i}e_i) \) | \( (-1)^{\nu_i}e_i - (-1)^{\nu_j}e_j, \) | \( 1 \leq i \leq p/2, p/2 + 1 \leq j \leq 5 \) | \( \phi, \phi - \alpha \) | 1 | \( \frac{p(10-p)}{4} \) | \( \frac{1}{12} \) | \( \frac{p(10-p)}{96} \) |

Bisymmetric Triple A.53. \((\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_6 \oplus \mathbb{R})\). (Type I)

\[ \mathcal{R}_p = \{ \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}e_i) \} \]

\[ C_p \text{ scalar on } \mathfrak{n} \]

| \( \phi \in \mathbb{R}^n \) | \( \alpha \in \mathbb{R}^+ \) | \( \phi + n\alpha \) | \( d_{\alpha \phi} \) | No of \( \alpha \)'s | \( |\alpha|^2 \) | \( b^\phi \) |
|----------------|----------------|----------------|--------------------|----------------|-----------------|----------------|
| \( \frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_{i=1}^{4}e_i) \) | \( \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}e_i) \) | \( \phi, \phi - \alpha \) | 1 | 1 | \( \frac{1}{12} \) | \( \frac{1}{24} \) |
Bisymmetric Triple A.54. \((\mathfrak{g}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathfrak{su}_2), p = 1, 2, 3\). (Type I)

\[ \mathcal{R}_p = \{ \pm (e_i - e_j) : 1 \leq i \leq 6-p, 7-p \leq j \leq 5; \pm \frac{1}{2}(e_8 - e_7 - e_6 - \sum_{i=1}^{5} e_i) \pm \frac{1}{2} \phi, \phi \} \]

\[ n = n^1 \oplus n^2, \text{ for } p = 2, 3 \text{ and } n \text{ irreducible } Ad\mathcal{L}\text{-module for } p = 1 \]

\[ b^\phi \]

\[ \frac{n^1 \oplus n^2}{p} \]

\[ p = 1 \]

\[ \mathcal{R}_{t_1} = \{ \pm (e_i - e_j) : 1 \leq i < j \leq 5 \} \]

\[ \mathcal{R}_{t_2} = \mathcal{R}_{t_2} = \{ \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} e_i) \} \]

\[ \mathcal{R}_p = \{ \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : 4 \text{ odd } \nu_i's \} \]

\[ n \text{ irreducible } Ad\mathcal{L}\text{-module} \]

\[ \begin{array}{cccccccc}
\phi & \in & \mathcal{R}_n & & & & & \\
\alpha & & & & & & & \\
\phi + n\alpha & & & & & & & \\
d_{\alpha\phi} & & & & & & & \\
No \text{ of } \alpha's & & & & & & & \\
|\alpha|^2 & & & & & & & \\
b^\phi & & & & & & & \\
\hline
\end{array} \]

\[ e_1 + e_2 \quad \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_1 - e_2 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i), \text{ 2 odd } \nu_i's \]

\[ \phi, \phi - \alpha \quad 1 \quad 3 \quad \frac{1}{17} \quad \frac{1}{7} \]

\[ p = 2 \]

\[ \mathcal{R}_{t_1} = \{ \pm (e_i - e_j) : 1 \leq i < j \leq 4; \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} e_i) \} \]

\[ \mathcal{R}_{t_2} = \{ \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} e_i) \} \]

\[ \mathcal{R}_p = \{ \pm (e_i - e_5) : 1 \leq i \leq 4; \pm \frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5} (-1)^{\nu_i} e_i) : 3 \text{ odd } \nu_i's \} \]

\[ \mathcal{R}_{n^1} = \{ \pm (e_i + e_j) : 1 \leq i < j \leq 4; \pm \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i) : 2 \text{ odd } \nu_i's \} \]

\[ \mathcal{R}_{n^2} = \{ \pm (e_i + e_5) : 1 \leq i \leq 4; \pm \frac{1}{2}(e_8 - e_7 - e_6 + e_5 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i) : 1 \text{ odd } \nu_i \} \]

\[ \begin{array}{cccccccc}
\phi & \in & \mathcal{R}_{n^1} & & & & & \\
\alpha & & & & & & & \\
\phi + n\alpha & & & & & & & \\
d_{\alpha\phi} & & & & & & & \\
No \text{ of } \alpha's & & & & & & & \\
|\alpha|^2 & & & & & & & \\
b^\phi & & & & & & & \\
\hline
\end{array} \]

\[ n^1 \quad e_1 + e_2 \quad \frac{1}{2}(e_8 - e_7 - e_6 - e_5 \pm e_4 - e_1 - e_2) \quad \phi, \phi - \alpha \quad 1 \quad 2 \quad \frac{1}{17} \quad \frac{1}{7} \]

\[ n^2 \quad e_1 + e_5 \quad \frac{1}{2}(e_8 - e_7 - e_6 - e_5 + \sum_{i=1}^{4} (-1)^{\nu_i} e_i), \text{ 2 odd } \nu_i's \quad \phi, \phi + \alpha \quad 1 \quad 3 \quad \frac{1}{17} \quad \frac{1}{7} \]
\( p = 3 \)

\[
\begin{align*}
\mathcal{R}_1 &= \{ \pm(e_i - e_j) : 1 \leq i < j \leq 3; \pm(e_4 - e_5) ; \pm\frac{1}{2}(e_8 - e_7 - e_6 \pm e_4 - \sum_1^3 e_i) \} \\
\mathcal{R}_2 &= \{ \pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_1^3 e_i) \} \\
\mathcal{R}_p &= \{ \pm(e_i - e_j) : i = 1, 2, 3, j = 4, 5; \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 (-1)^\nu_i e_i) : \text{exactly 2 odd } \nu_i's \} \\
\mathcal{R}_{n_1} &= \{ \pm(e_i + e_j) : 1 \leq i < j \leq 3 \text{ or } 1 \leq i \leq 3, j = 4, 5; \pm\frac{1}{2}(e_8 - e_7 - e_6 + e_5 + e_4 + \sum_1^3 (-1)^\nu_i e_i) : 2 \text{ odd } \nu_i's; \\
&\quad \pm\frac{1}{2}(e_8 - e_7 - e_6 \pm e_5 \mp e_4 - \sum_1^3 (-1)^\nu_i e_i) : 1 \text{ odd } \nu_i \} \\
\mathcal{R}_{n_2} &= \{ \pm(e_4 + e_5) ; \pm\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 e_i) \} \\
\end{align*}
\]

| \( n^1 \) | \( \phi \in \mathcal{R}_{n_1} \) | \( \alpha \in \mathcal{R}_p^+ \) | \( \phi + n\alpha \) | \( d_{\alpha \phi} \) | \( \text{No of } \alpha's \) | \( |\alpha|^2 \) | \( b^\phi \) |
|---|---|---|---|---|---|---|---|
| \( n^1 \) | \( e_1 + e_2 \) | \( e_i - e_j, i = 1, 2, j = 4, 5 \) | \( \phi, \phi - \alpha \) | 4 | 4 | \( \frac{1}{17} \) | \( \frac{5}{22} \) |
| \( n^2 \) | \( e_4 + e_5 \) | \( e_i - e_j, i = 1, 2, 3, j = 4, 5 \) | \( \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 + \sum_1^3 (-1)^\nu_i e_i), 2 \text{ odd } \nu_i's \) | \( \phi, \phi + \alpha \) | 3 | 6 | \( \frac{1}{17} \) | \( \frac{2}{27} \) |
Bisymmetric Triple A.55. \((\mathfrak{e}_6, \mathfrak{su}_6 \oplus \mathfrak{su}_2, \mathfrak{su}_p \oplus \mathfrak{su}_{6-p} \oplus \mathbb{R} \oplus \mathbb{R})\), \(p = 1, 2, 3\). (Type II)

\(\mathcal{R}_p = \{\pm(e_i - e_j) : 1 \leq i \leq 6 - p, 7 - p \leq j \leq 5; \pm\frac{1}{2}(e_8 - e_7 - e_6 - \sum_{1}^{5} e_i + \sum_{6-p}^{6} (-1)^{\nu_i} e_i) : (5 - p)\ odd\ \nu_i's\}\)

\(\mathcal{R}_{p_2} = \{\pm\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{1}^{5} e_i)\}\)

decomposition of \(n\ as\ in\ A.54\)

| \(n^i\) | \(b_1^\phi\) | \(b_2^\phi\) |
|--------|----------|----------|
| \(n^1\) | \(\frac{p+2}{24}\) | \(\frac{1}{24}\) |
| \(n^2\) | \(\frac{6}{24}\) | \(\frac{6}{24}\) |

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Notation List

Common notation:

\( A_g \) Conjugation by \( g \in G \) in a Lie group \( G \)
\( Ad_g \) Adjoint action of \( g \in G \) in the Lie algebra \( \mathfrak{g} \) of a Lie group \( G \)
\( ad_X \) Adjoint linear map of \( X \in \mathfrak{g} \)
\( Kill \) the Killing form of a Lie algebra \( \mathfrak{g} \)
\( B = -\text{Kill} \)
\( C_V \) the Casimir operator of a subspace \( V \) of a Lie algebra \( \mathfrak{g} \) with respect to the Killing form of \( \mathfrak{g} \)
\( c_{V,U} \) the eigenvalue of a Casimir operator \( C_V \) on a subspace \( U \)
\( R \) the curvature of a Riemannian metric
\( Ric \) the Ricci curvature of a Riemannian metric
\( K \) the sectional curvature of a Riemannian metric
\( h^*(\mathfrak{g}) \) the dual Coxeter number of a Lie algebra \( \mathfrak{g} \)
\( \mathcal{R} \) the system of roots of a Lie algebra \( \mathfrak{g} \)
\( \mathcal{R}_K \) the system of restricted roots to a subgroup of maximal rank \( K \)
\( S^+ \) the subset of positive roots of a set of roots \( S \)
\( \mathfrak{g}^C \) the complexification of a Lie algebra \( \mathfrak{g} \)
\( V^C \) the complexification of a vector space \( V \)
\( id_V \) the identity map of a vector space \( V \)
\( 0_V \) the null map of a vector space \( V \)

Notation for homogeneous fibrations:

\( G \) compact connected semisimple Lie group
\( K, L \) compact closed non-trivial subgroups of \( G \) such that \( L \subsetneq K \subsetneq G \)
\( \mathfrak{g}, \mathfrak{l}, \mathfrak{l} \) the Lie algebras of \( G, K \) and \( L \)
\( M = G/L \)
\( N = G/K \)
\( F = K/L \)
$g_M$ an adapted metric metric on $M$ with respect to a fibration $F \to M \to N$

$g_N$ the projection of an adapted metric $g_M$ onto the base space $N$ as above

$g_F$ the restriction of an adapted metric $g_M$ to the fiber space $F$ as above

$n$ $Ad K$-invariant complement of $\mathfrak{k}$ on $\mathfrak{g}$

$p$ $Ad L$-invariant complement of $\mathfrak{l}$ on $\mathfrak{k}$

$m = p \oplus n$ $Ad L$-invariant complement of $\mathfrak{l}$ on $\mathfrak{g}$

$n = n_1 \oplus \ldots n_n$ decomposition of $n$ into pairwise inequivalent irreducible $Ad K$-submodules

$n = n^1 \oplus \ldots n^{n'}$ decomposition of $n$ into pairwise inequivalent irreducible $Ad L$-submodules

$p = p_1 \oplus \ldots p_s$ decomposition of $p$ into pairwise inequivalent irreducible $Ad L$-submodules

$C_g, C_t, C_l$ the Casimir operator of $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{l}$, respectively

$C_{p_a}$ the Casimir operator of $p_a$, $a = 1, \ldots, s$

$C_{n_i}$ the Casimir operator of $n_i$, $i = 1, \ldots, n$

$c_{t,a}$ the eigenvalue of $C_l$ on $p_a$, $a = 1, \ldots, s$

$c_{t,p}$ the eigenvalue of $C_t$ on $p$, when $p$ is $Ad L$-irreducible

$c_{t,i}$ the eigenvalue of $C_t$ on $n_i$, $i = 1, \ldots, n$

$c_{t,n}$ the eigenvalue of $C_t$ on $n$, when $n$ is $Ad K$-irreducible

$c_{n_i,a}$ the constant defined by $\text{Kill}(C_{n_i}, \cdot |_{p_a \times p_a} = c_{n_i,a} \text{Kill} |_{p_a \times p_a}$, $a = 1, \ldots, s$

$\gamma_{a}$ the constant defined by $\text{Kill} |_{p_a \times p_a} = \gamma_{a} \text{Kill} |_{p_a \times p_a}$

$\gamma$ the constant defined by $\text{Kill} |_{p \times p} = \gamma \text{Kill} |_{p \times p}$, when $p$ is $Ad L$-irreducible

$b^i_{a}$ the eigenvalue of $C_{p_a}$ on $n_i$, $i = 1, \ldots, n$, when this eigenvalue exists (the indices are dropped in the case of irreducibility as above)

$b^i_{\phi}$ the constant defined by $b^i_{\phi} = B(C_{p_a}x_{\phi}, X_{-\phi}) = \text{Kill}(C_{p_a}E_{\phi}, E_{-\phi})$ for a root $\phi$; for $\phi \in \mathcal{R}_n$, it represents an eigenvalue of $C_{p_a}$ on $n$