Cuspidal representations in the \(\ell\)-adic cohomology of the Rapoport-Zink space for \(\text{GSp}(4)\)

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Abstract. In this paper, we study the \(\ell\)-adic cohomology of the Rapoport-Zink tower for \(\text{GSp}(4)\). We prove that the smooth representation of \(\text{GSp}_4(\mathbb{Q}_p)\) obtained as the \(i\)th compactly supported \(\ell\)-adic cohomology of the Rapoport-Zink tower has no quasi-cuspidal subquotient unless \(i = 2, 3, 4\). Our proof is purely local and does not require global automorphic methods.

1 Introduction

In [RZ96], M. Rapoport and Th. Zink introduced certain moduli spaces of quasi-isogenies of \(p\)-divisible groups with additional structures called the Rapoport-Zink spaces. They constructed systems of rigid analytic coverings of them which we call the Rapoport-Zink towers, and established the \(p\)-adic uniformization theory of Shimura varieties generalizing classical Čerednik-Drinfeld uniformization. These spaces uniformize the rigid spaces associated with the formal completion of certain Shimura varieties along Newton strata.

Using the \(\ell\)-adic cohomology of the Rapoport-Zink tower, we can construct a representation of the product \(G(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\), where \(G\) is the reductive group over \(\mathbb{Q}_p\) corresponding to the Shimura datum, \(J\) is an inner form of it, and \(W(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)\) is the Weil group of the \(p\)-adic field \(\mathbb{Q}_p\). It is widely believed that this realizes the local Langlands and Jacquet-Langlands correspondences (cf. [Rap95]). Classical examples of the Rapoport-Zink spaces are the Lubin-Tate space and the Drinfeld upper half space; these spaces were extensively studied by many people and many important results were obtained (cf. [Dri76], [Car90], [Har97], [HT01], [Dat07], [Boy09] and references therein). However, very little was known about the \(\ell\)-adic cohomology of other Rapoport-Zink spaces.

The aim of this paper is to study cuspidal representations in the \(\ell\)-adic cohomology of the Rapoport-Zink tower for \(\text{GSp}_4(\mathbb{Q}_p)\). Let us denote the Rapoport-Zink space for \(\text{GSp}_4(\mathbb{Q}_p)\) by \(\mathcal{M}\). It is a special formal scheme over \(\mathbb{Z}_{p^\infty} = W(\overline{\mathbb{F}}_p)\) in the sense of Berkovich [Berk96]. Let \(\mathcal{M}^{\text{rig}}\) be the Raynaud generic fiber of \(\mathcal{M}\), that is, the generic fiber of the adic space \(\mathfrak{t}(\mathcal{M})\) associated with \(\mathcal{M}\). Using level structures

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at $p$, we can construct the Rapoport-Zink tower
\[
\cdots \to \mathcal{M}_{m+1}^{\text{rig}} \to \mathcal{M}_m^{\text{rig}} \to \cdots \to \mathcal{M}_2^{\text{rig}} \to \mathcal{M}_1^{\text{rig}} \to \mathcal{M}_0^{\text{rig}} = \mathcal{M}_{\text{rig}},
\]
where $\mathcal{M}_m^{\text{rig}} \to \mathcal{M}_m^{\text{rig}}$ is an \'{e}tale Galois covering of rigid spaces with Galois group $\text{GSp}_4(\mathbb{Z}/p^m\mathbb{Z})$. We take the compactly supported $\ell$-adic cohomology (in the sense of [Hub98]) and take the inductive limit of them. Then, on $H^i_{RZ} := \lim_{\to} H^i_c(\mathcal{M}_m^{\text{rig}} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell)$ (here $\mathbb{Q}_p = \text{Frac} \mathbb{Z}_p$), we have an action of a product
\[
\text{GSp}_4(\mathbb{Q}_p) \times J(\mathbb{Q}_p) \times W(\mathbb{Q}_p/\mathbb{Q}_p),
\]
where $J$ is an inner form of $\text{GSp}_4$.

The main theorem of this paper is as follows:

**Theorem 1.1 (Theorem 3.2)** The $\text{GSp}_4(\mathbb{Q}_p)$-representation $H^i_{RZ} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$ has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.

For the definition of quasi-cuspidal representations, see [Bern84, 1.20]. Note that since $\mathcal{M}_m^{\text{rig}}$ is 3-dimensional for every $m \geq 0$, $H^i_{RZ} = 0$ unless $0 \leq i \leq 6$.

Our proof of this theorem is purely local. We do not use global automorphic methods. The main strategy of the proof is similar to that of [Mie10a], in which the analogous result for the Lubin-Tate tower is given; we construct the formal model $\mathcal{M}_m$ of $\mathcal{M}_m^{\text{rig}}$ by using Drinfeld level structures and consider the geometry of its special fiber. However, our situation is much more difficult than the case of the Lubin-Tate tower. In the Lubin-Tate case, the tower consists of affine formal schemes $\{\text{Spf} A_m\}_{m \geq 0}$, and we can associate it with the tower of affine schemes $\{\text{Spec} A_m\}_{m \geq 0}$. In [Mie10a], the second author defined the stratification on the special fiber of $\text{Spec} A_m$ by using the kernel of the universal Drinfeld level structure, and considered the local cohomology of the nearby cycle complex $R\psi A$ along the strata. On the other hand, our tower $\{\mathcal{M}_m\}_{m \geq 0}$ does not consist of affine formal schemes and there is no canonical way to associate it with a tower of schemes. To overcome this problem, we take a sheaf-theoretic approach. For each direct summand $I$ of $(\mathbb{Z}/p^n\mathbb{Z})^4$, we will define the complex of sheaves $\mathcal{F}_{m, I}$ on $(\mathcal{M}_m)^{\text{red}}$ so that the cohomology $H^i((\mathcal{M}_m)^{\text{red}}, \mathcal{F}_{m, I})$ substitutes for the local cohomology of $R\psi A$ along the strata defined by $I$ in the Lubin-Tate case. For the definition of $\mathcal{F}_{m, I}$, we use the $p$-adic uniformization theorem by Rapoport and Zink.

There is another difficulty; since a connected component of $\mathcal{M}$ is not quasi-compact, the representation $H^i_{RZ}$ of $\text{GSp}_4(\mathbb{Q}_p)$ is far from admissible. Therefore it is important to consider the action of $J(\mathbb{Q}_p)$ on $H^i_{RZ}$, though it does not appear in our main theorem. However, the cohomology $H^i((\mathcal{M}_m)^{\text{red}}, \mathcal{F}_{m, I})$ has no apparent action of $J(\mathbb{Q}_p)$, since $J(\mathbb{Q}_p)$ does not act on the Shimura variety uniformized by
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We use the variants of formal nearby cycle introduced by the second author in [Mie10b] to endow it with an action of $J(\mathbb{Q}_p)$. Furthermore, to ensure the smoothness of this action, we use a property of finitely generated pro-$p$ groups (Section 2). In fact, extensive use of the formalism developed in [Mie10b] make us possible to work mainly on the Rapoport-Zink tower itself and avoid the theory of $p$-adic uniformization except for proving that $\mathcal{M}_m$ is locally algebraizable. However, for the reader’s convenience, we decided to make this article as independent of [Mie10b] as possible.

The authors expect that the converse of Theorem 1.1 also holds. Namely, we expect that $H^i_{RZ} \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell$ has a quasi-cuspidal subquotient if $i = 2, 3, 4$. We hope to investigate it in a future work.

The outline of this paper is as follows. In Section 2, we prepare a criterion for the smoothness of representations over $\mathbb{Q}_\ell$. It is elementary but very powerful for our purpose. In Section 3, we give some basic definitions concerning with the Rapoport-Zink space for GSp(4) and state the main theorem. Section 4 is devoted to introduce certain Shimura varieties related to our Rapoport-Zink tower and recall the theory of $p$-adic uniformization. The proof of the main theorem is accomplished in Section 5. The final Section 6 is an appendix on cohomological correspondences. The results in the section are used to define actions of $\text{GSp}_4(\mathbb{Q}_p)$ on various cohomology groups.

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Notation Let $p$ be a prime number and take another prime $\ell$ with $\ell \neq p$. We denote the completion of the maximal unramified extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p^\infty$ and its fraction field by $\mathbb{Q}_p^\infty$. Let $\text{Nilp} = \text{Nilp}_{\mathbb{Z}_p^\infty}$ be the category of $\mathbb{Z}_p^\infty$-schemes on which $p$ is locally nilpotent. For an object $S$ of $\text{Nilp}$, we put $\overline{S} = S \otimes_{\mathbb{Z}_p^\infty} \overline{\mathbb{F}}_p$.

In this paper, we use the theory of adic spaces ([Hub94], [Hub96]) as a framework of rigid geometry. A rigid space over $\mathbb{Q}_p^\infty$ is understood as an adic space locally of finite type over $\text{Spa}(\mathbb{Q}_p^\infty, \mathbb{Z}_p^\infty)$.

Every sheaf and cohomology are considered in the étale topology. Every smooth representation is considered over $\mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$. For a $\mathbb{Q}_\ell$-vector space $V$, we put $V_{\mathbb{Q}_\ell} = V \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}}_\ell$.

2 Preliminaries: smoothness of representations of profinite groups

Let $G$ be a linear algebraic group over a $p$-adic field $F$. In this section, we give a convenient criterion for the smoothness of a $G(F)$-representation over $\mathbb{Q}_\ell$. The following theorem is essential:

Theorem 2.1 Let $K$ be a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$ and $(\pi, V)$ a finite-dimensional representation over $\mathbb{Q}_\ell$ of $K$ as an abstract group. Assume that there exists a $K$-stable $\mathbb{Z}_\ell$-lattice $\Lambda$ of $V$. Then this representation is automatically smooth.
In order to prove this theorem, we require several facts on pro-$p$ groups. Put $K_1 = K \cap (1 + pM_n(\mathbb{Z}_p))$, which is a pro-$p$ open subgroup of $K$.

**Lemma 2.2** The pro-$p$ group $K_1$ is (topologically) finitely generated.

*Proof.* By [DdSMS99, §5.1], the profinite group $\text{GL}_n(\mathbb{Z}_p)$ has finite rank. In particular, $K_1$, a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$, has finite topological generators.

**Lemma 2.3** Every subgroup of finite index of $K_1$ is open.

*Proof.* In fact, this is true for every finitely generated pro-$p$ group; this is due to Serre [Ser94, 4.2, Exercices 6)]. See also [DdSMS99, Theorem 1.17], which gives a complete proof.

**Remark 2.4** More generally, every subgroup of finite index of a finitely generated profinite group is open ([NS03], [NS07a], [NS07b]). It is a very deep theorem.

**Lemma 2.5** Let $G$ be a pro-$\ell$ group. Then every homomorphism $f : K_1 \to G$ is trivial.

*Proof.* Let $H$ be an open normal subgroup of $G$ and denote the composite $K_1 \xrightarrow{f} G \to G/H$ by $f_H$. By Lemma 2.3, Ker $f_H$ is an open normal subgroup of $K_1$. Thus $K_1/\text{Ker} f_H$ is a finite $p$-group. On the other hand, $G/H$ is a finite $\ell$-group. Since we have an injection $K_1/\text{Ker} f_H \to G/H$, we have $K_1/\text{Ker} f_H = 1$, in other words, $f_H = 1$. Therefore the composite $K_1 \xrightarrow{f} G \to \varprojlim H G/H$ is trivial. Hence we have $f = 1$, as desired.

*Proof of Theorem 2.1.* Since $K_1$ is an open subgroup of $K$, we may replace $K$ by $K_1$. Take a $K_1$-stable $\mathbb{Z}_p$-lattice $\Lambda$ of $V$. Then, $\Lambda/\ell \Lambda$ is a finite abelian group. Therefore, by Lemma 2.3, there exists an open subgroup $U$ of $K_1$ which acts trivially on $\Lambda/\ell \Lambda$. In other words, the homomorphism $\pi : K_1 \to \text{GL}(\Lambda) \subset \text{GL}(V)$ maps $U$ into the subgroup $1 + \ell \text{End}(\Lambda)$. Since $U$ is a closed subgroup of $1 + pM_n(\mathbb{Z}_p)$ and $1 + \ell \text{End}(\Lambda)$ is a pro-$\ell$ group, by Lemma 2.5, the homomorphism $\pi|_U : U \to 1 + \ell \text{End}(\Lambda)$ is trivial. Namely, $\pi|_U$ is a trivial representation.

**Lemma 2.6** Let $F$ be a $p$-adic field and $G$ a linear algebraic group over $F$. Then every compact subgroup $K$ of $G(F)$ can be realized as a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$ for some $n$.

*Proof.* Take an embedding $G \hookrightarrow \text{GL}_m$ defined over $F$. Since $G(F)$ is a closed subgroup of $\text{GL}_m(F)$, $K$ is also a closed subgroup of $\text{GL}_m(F)$. Therefore we have a faithful continuous action of $K$ on $F^m$. By taking a $\mathbb{Q}_p$-basis of $F$, we have a faithful continuous action of $K$ on $\mathbb{Q}_p^n$ for some $n$. Since $K$ is compact, it is well-known that there is a $K$-stable $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p^n$. Hence we have a continuous injection $K \hookrightarrow \text{GL}_n(\mathbb{Z}_p)$. Since $K$ is compact, it is isomorphic to a closed subgroup of $\text{GL}_n(\mathbb{Z}_p)$. 

4
Corollary 2.7 Let $F$ and $G$ be as in the previous proposition. Let $I$ be a filtered ordered set and $\{K_i\}_{i \in I}$ be a system of compact open subgroups of $G(F)$ indexed by $I$.

Let $(\pi, V)$ be a (not necessarily finite-dimensional) $\mathbb{Q}_\ell$-representation of $G(F)$ as an abstract group. Assume that there exists an inductive system $\{V_i\}_{i \in I}$ of finite-dimensional $\mathbb{Q}_\ell$-vector spaces satisfying the following:

- For every $i \in I$, $V_i$ is endowed with an action of $K_i$ as an abstract group.
- For every $i \in I$, $V_i$ has a $K_i$-stable $\mathbb{Z}_\ell$-lattice.
- There exists an isomorphism $\lim_{\rightarrow i \in I} V_i \rightarrow V$ as $\mathbb{Q}_\ell$-vector spaces such that the composite $V_i \rightarrow \lim_{\rightarrow i \in I} V_i \rightarrow V$ is $K_i$-equivariant for every $i \in I$.

Then $(\pi, V)$ is a smooth representation of $G(F)$.

Proof. Let us take $x \in V$ and show that $\text{Stab}_{G(F)}(x)$, the stabilizer of $x$ in $G(F)$, is open. There exists an element $i \in I$ such that $x$ lies in the image of $V_i \rightarrow V$. Take $y \in V_i$ which is mapped to $x$. By Theorem 2.1 and Lemma 2.6, $V_i$ is a smooth representation of $K_i$. Therefore $\text{Stab}_{K_i}(y)$ is open in $K_i$, hence is open in $G(F)$. Since $V_i \rightarrow V$ is $K_i$-equivariant, we have $\text{Stab}_{K_i}(y) \subset \text{Stab}_{K_i}(x) \subset \text{Stab}_{G(F)}(x)$. Thus $\text{Stab}_{G(F)}(x)$ is open in $G(F)$, as desired.

Remark 2.8 Although we need the corollary above only for the case $F = \mathbb{Q}_p$, we proved it for a general $p$-adic field $F$ for the completeness.

3 Rapoport-Zink space for $GSp(4)$

3.1 The Rapoport-Zink space for $GSp(4)$ and its rigid analytic coverings

In this subsection, we recall basic definitions concerning with Rapoport-Zink spaces. General definitions are given in [RZ96], but here we restrict them to our special case.

Let $X$ be a 2-dimensional isoclinic $p$-divisible group over $\overline{F}_p$ with slope $1/2$, and $\lambda_0: X \rightarrow X^\vee$ a (principal) polarization of $X$, namely, an isomorphism satisfying $\lambda_0^\vee = -\lambda_0$. Consider the contravariant functor $\tilde{\mathcal{M}}: \text{Nilp} \rightarrow \text{Set}$ that associates $S$ with the set of isomorphism classes of pairs $(X, \rho)$ consisting of

- a 2-dimensional $p$-divisible group $X$ over $S$,
- and a quasi-isogeny (cf. [RZ96, Definition 2.8]) $\rho: X \otimes_{\mathbb{F}_p} S \rightarrow X \otimes_S \overline{S}$, such that there exists an isomorphism $\lambda: X \rightarrow X^\vee$ which makes the following
diagram commutative up to multiplication by $Q_p^\times$:

$$
\begin{array}{ccc}
X \otimes_{F_p} S & \xrightarrow{\rho} & X \otimes_{S} S \\
\downarrow_{\lambda_0 \otimes \text{id}} & & \downarrow_{\lambda \otimes \text{id}} \\
X^\vee \otimes_{F_p} S & \xleftarrow{\rho^\vee} & X^\vee \otimes_{S} S.
\end{array}
$$

Note that such $\lambda$ is uniquely determined by $(X, \rho)$ up to multiplication by $Z_p^\times$ and gives a polarization of $X$. It is proved by Rapoport-Zink that $\acute{\mathcal{M}}$ is represented by a special formal scheme (cf. [Berk96]) over $\text{Spf} Z_{p^\infty}$. Moreover, $\acute{\mathcal{M}}$ is neither quasi-compact nor $p$-adic.

We put $\bar{\mathcal{M}} = \acute{\mathcal{M}}_{\text{red}}$, which is a scheme locally of finite type and separated over $\mathbb{F}_p$. It is known that $\bar{\mathcal{M}}$ is 1-dimensional (for example, see [Vie08]) and every irreducible component of $\bar{\mathcal{M}}$ is projective over $\mathbb{F}_p$ [RZ96, Proposition 2.32]. In particular, $\bar{\mathcal{M}}$ has a locally finite quasi-compact open covering.

Let $D(X)_Q = (N, \Phi)$ be the rational Dieudonné module of $X$, which is a 4-dimensional isocrystal over $Q_{p^\infty}$. The fixed polarization $\lambda_0$ gives the alternating pairing $\langle \cdot, \cdot \rangle_{\lambda_0} : N \times N \rightarrow Q_{p^\infty}(1)$. We define the algebraic group $J$ over $Q_p$ as follows: for a $Q_p$-algebra $R$, the group $J(R)$ consists of elements $g \in \text{GL}(R \otimes_{Q_p} N)$ such that

- $g$ commutes with $\Phi$,
- and $g$ preserves the pairing $\langle \cdot, \cdot \rangle_{\lambda_0}$ up to scalar multiplication, i.e., there exists $c(g) \in R^\times$ such that $\langle gx, gy \rangle_{\lambda_0} = c(g) \langle x, y \rangle_{\lambda_0}$ for every $x, y \in R \otimes_{Q_p} N$.

It is an inner form of $G\text{Sp}(4)$, since $D(X)_Q$ is the isocrystal associated with a basic Frobenius conjugacy class of $G\text{Sp}(4)$.

In the sequel, we also denote $J(Q_p)$ by $J$. Every element $g \in J$ naturally induces a quasi-isogeny $g : X \rightarrow X$ and the following diagram is commutative up to $Q_p^\times$-multiplication:

$$
\begin{array}{ccc}
X & \xrightarrow{g} & X \\
\downarrow_{\lambda_0} & & \downarrow_{\lambda_0} \\
X^\vee & \xleftarrow{g^\vee} & X^\vee.
\end{array}
$$

Therefore, we can define the left action of $J$ on $\acute{\mathcal{M}}$ by $g : \acute{\mathcal{M}}(S) \rightarrow \acute{\mathcal{M}}(S)$; $(X, \rho) \mapsto (X, \rho \circ g^{-1})$.

We denote the Raynaud generic fiber of $\acute{\mathcal{M}}$ by $\acute{\mathcal{M}}^{\text{rig}}$. It is defined as $t(\acute{\mathcal{M}}) \setminus V(p)$, where $t(\acute{\mathcal{M}})$ is the adic space associated with $\acute{\mathcal{M}}$ (cf. [Hub94, Proposition 4.1]). As $\acute{\mathcal{M}}$ is separated and special over $Z_{p^\infty}$, $\acute{\mathcal{M}}^{\text{rig}}$ is separated and locally of finite type over $\text{Spa}(Q_{p^\infty}, Z_{p^\infty})$. Since $\acute{\mathcal{M}}$ has a locally finite quasi-compact open covering, $\acute{\mathcal{M}}^{\text{rig}}$ is taut by [Mie10b, Lemma 4.14]. Moreover, by using the period morphism $\text{RZ96, Chapter 5}$, we can see that $\acute{\mathcal{M}}^{\text{rig}}$ is 3-dimensional and smooth over $\text{Spa}(Q_{p^\infty}, Z_{p^\infty})$ (cf. [RZ96, Proposition 5.17]).
Next we will consider level structures. Let $\tilde{X}$ be the universal $p$-divisible group over $\mathcal{M}$ and $\tilde{X}^{\rig}$ be the associated $p$-divisible group over $\tilde{\mathcal{M}}^{\rig}$. Note that $\tilde{X}^{\rig}$ is an étale $p$-divisible group. Let us fix a polarization $\tilde{\lambda}: \tilde{X} \to \tilde{X}^\vee$ which is compatible with $\lambda_0$, i.e., satisfies the condition in the definition of $\mathcal{M}$. Let $S$ be a connected rigid space over $\mathbb{Q}_p\propto$ (i.e., a connected adic space locally of finite type over $\Spa(\mathbb{Q}_p\propto, \mathbb{Z}_p\propto)$), $S \to \tilde{\mathcal{M}}^{\rig}$ a morphism over $\mathbb{Q}_p\propto$ and $\tilde{X}_S^{\rig}$ the pull-back of $\tilde{X}^{\rig}$. Fix a geometric point $\overline{\tau}$ of $S$ and an isomorphism $T_p(\mu_{p\propto,S})\overline{\tau} = \mathbb{Z}_p(1) \cong \mathbb{Z}_p$. Then $\tilde{\lambda}$ induces an alternating bilinear form $\psi_{\overline{\tau}}$ on the $\pi_1(S,\overline{\tau})$-module $(T_p\tilde{X}_S^{\rig})_{\overline{\tau}}$.

$$\psi_{\overline{\tau}}: (T_p\tilde{X}_S^{\rig})_{\overline{\tau}} \times (T_p\tilde{X}_S^{\rig})_{\overline{\tau}} \to T_p(\mu_{p\propto,S})_{\overline{\tau}} \cong \mathbb{Z}_p.$$ 

Fix a free $\mathbb{Z}_p$-module $L$ of rank 4 and a perfect alternating bilinear form $\psi_0: L \times L \to \mathbb{Z}_p$. Put $K_0 = \text{GSp}(L, \psi_0)$, $V = L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and $G = \text{GSp}(V, \psi_0)$. Let $T(S, \overline{\tau})$ be the set consisting of isomorphisms $\eta: L \to (T_p\tilde{X}^{\rig})_{\overline{\tau}}$ which map $\psi_0$ to $\mathbb{Z}_p^\times$-multiples of $\psi_{\overline{\tau}}$. It is independent of the choice of $\tilde{\lambda}$ and $T_p(\mu_{p\propto,S})_{\overline{\tau}} \cong \mathbb{Z}_p$, since they are unique up to $\mathbb{Z}_p^\times$-multiplication. Obviously, the groups $K_0$ and $\pi_1(S,\overline{\tau})$ naturally act on $T(S, \overline{\tau})$.

For an open subgroup $K$ of $K_0$, a $K$-level structure of $\tilde{X}_S^{\rig}$ means an element of $(T(S, \overline{\tau})/K)^{\pi_1(S,\overline{\tau})}$. Note that, if we change a geometric point $\overline{\tau}$ to $\overline{\tau}'$, the sets $(T(S, \overline{\tau})/K)^{\pi_1(S,\overline{\tau})}$ and $(T(S, \overline{\tau}')/K)^{\pi_1(S,\overline{\tau}')}$ are naturally isomorphic. Thus the notion of $K$-level structures is independent of the choice of $\overline{\tau}$. The functor that associates $S$ with the set of $K$-level structures of $\tilde{X}_S^{\rig}$ is represented by a finite Galois étale covering $\tilde{\mathcal{M}}^{\rig}_{K_0} \to \tilde{\mathcal{M}}^{\rig}$, whose Galois group is $K_0/K$. Since $T(S, \overline{\tau})$ is a $K_0$-torsor, $\tilde{\mathcal{M}}^{\rig}_{K_0}$ coincides with $\tilde{\mathcal{M}}^{\rig}$. If $K'$ is an open subgroup of $K$, we have a natural morphism $\rho_{K,K'}: \tilde{\mathcal{M}}^{\rig}_{K'} \to \tilde{\mathcal{M}}^{\rig}_{K}$. Therefore, we get the projective system of rigid spaces $\{\tilde{\mathcal{M}}^{\rig}_{K}\}_{K}$ indexed by the filtered ordered set of open subgroups of $K_0$, which is called the Rapoport-Zink tower. Obviously, the group $J$ acts on the projective system $\{\tilde{\mathcal{M}}^{\rig}_{K}\}_{K}$.

Let $g$ be an element of $G$ and $K$ an open subgroup of $K_0$ which is enough small so that $g^{-1}Kg \subset K_0$. Then we have a natural morphism $\tilde{\mathcal{M}}^{\rig}_{K} \to \tilde{\mathcal{M}}^{\rig}_{g^{-1}Kg}$ over $\mathbb{Q}_p\propto$. If $g \in K_0$, then it is given by $\eta \mapsto \eta \circ g$; for other $g$, it is more complicated [RZ96 5.34]. In any case, we get a right action of $G$ on the pro-object "$\varprojlim \tilde{\mathcal{M}}^{\rig}_{K}$".

**Definition 3.1** We put $H^1_{\text{RZ}} = \varprojlim_K H^1_{\text{rig}}(\tilde{\mathcal{M}}^{\rig}_{K} \otimes_{\mathbb{Q}_p\propto} \overline{\mathbb{Q}_p\propto}, \mathbb{Q}_\ell)$.

Here $H^1_{\text{rig}}(\tilde{\mathcal{M}}^{\rig}_{K} \otimes_{\mathbb{Q}_p\propto} \overline{\mathbb{Q}_p\propto}, \mathbb{Q}_\ell)$ is the compactly supported $\ell$-adic cohomology of $\tilde{\mathcal{M}}^{\rig}_{K} \otimes_{\mathbb{Q}_p\propto} \overline{\mathbb{Q}_p\propto}$ defined in [Hub98]; note that $\tilde{\mathcal{M}}^{\rig}_{K}$ is separated and taut. By the constructions above, $G \times J$ acts on $H^1_{\text{RZ}}$ on the left (the action of $j \in J$ is given by $(j^{-1})^*$). Obviously the action of $G$ on $H^1_{\text{RZ}}$ is smooth. On the other hand, it is known that the action of $J$ on $H^1_{\text{RZ}}$ is also smooth. This is due to Berkovich (see [Far04 Corollaire 4.4.7]); see also Remark 5.12 where we give another proof of the smoothness. Hence we get the smooth representation $H^1_{\text{RZ}}$ of $G \times J$. 

Our main theorem is the following:
Theorem 3.2 (Non-cuspidality) The smooth representation $H^i_{\overline{RZ}, \overline{Q}_c}$ of $G$ has no quasi-cuspidal subquotient unless $i = 2, 3, 4$.

For the definition of quasi-cuspidal representations, see [Bern84, 1.20]. Theorem 3.2 is proved in Section 5.

3.2 An integral model $\tilde{M}_m$ of $\tilde{M}_{K_m}^{\text{rig}}$

For an integer $m \geq 1$, let $K_m$ be the kernel of $\text{GSp}(L, \psi_0) \to \text{GSp}(L/p^mL, \psi_0)$. It is an open subgroup of $K_0$. We can describe the definition of $K_m$-level structures more concretely. As in the previous subsection, we fix a polarization $\tilde{\lambda}$ of $\tilde{X}_{\text{rig}}$ which is compatible with $\lambda_0$. It induces the alternating bilinear morphism between finite étale group schemes $\psi_{\tilde{X}}: \tilde{X}_{\text{rig}}[p^m] \times \tilde{X}_{\text{rig}}[p^m] \to \mu_{p^m}$. Let $S \to \tilde{M}_{\text{rig}}^{\text{rig}}$ be as in the previous subsection. Then a $K_m$-level structure of $\tilde{X}_S$ naturally corresponds bijectively to an isomorphism $\eta: L/p^mL \cong \tilde{X}_S[p^m]$ between finite étale group schemes such that there exists an isomorphism $\mathbb{Z}/p^m\mathbb{Z} \cong \mu_{p^m,S}$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
L/p^mL \times L/p^mL & \xrightarrow{\psi_{\tilde{X}}} & \mathbb{Z}/p^m\mathbb{Z} \\
\eta \times \eta \cong & & \cong \\
\tilde{X}_S[p^m] \times \tilde{X}_S[p^m] & \xrightarrow{\psi_{\tilde{X}}} & \mu_{p^m,S}.
\end{array}
\]

For simplicity, we write $\tilde{M}_m^{\text{rig}}$ for $\tilde{M}_{K_m}^{\text{rig}}$ and $p_{mn}$ for $p_{K_mK_n}$. In this subsection, we construct a formal model $\tilde{M}_m$ of $\tilde{M}_m^{\text{rig}}$ by following [Man05, §6]. Let $S$ be a formal scheme of finite type over $\tilde{M}_{\text{rig}}$ and denote by $\tilde{X}_S$ the pull-back of $\tilde{X}$ to $S$. A Drinfeld $m$-level structure of $\tilde{X}_S$ is a morphism $\eta: L/p^mL \to \tilde{X}_S[p^m]$ satisfying the following conditions:

- the image of $\eta$ gives a full set of sections of $\tilde{X}_S[p^m]$,
- and there exists a morphism $\mathbb{Z}/p^m\mathbb{Z} \to \mu_{p^m,S}$ which makes the following diagram commutative:

\[
\begin{array}{ccc}
L/p^mL \times L/p^mL & \xrightarrow{\psi_{\tilde{X}}} & \mathbb{Z}/p^m\mathbb{Z} \\
\eta \times \eta & & \\
\tilde{X}_S[p^m] \times \tilde{X}_S[p^m] & \xrightarrow{\psi_{\tilde{X}}} & \mu_{p^m,S}.
\end{array}
\]

It is known that the functor that associates $S$ with the set of Drinfeld $m$-level structures of $\tilde{X}_S$ is represented by the formal scheme $\tilde{M}_m$ which is finite over $\tilde{M}$ (cf. [Man05, Proposition 15]). Note that, unlike the case of Lubin-Tate tower, $\tilde{M}_m$ is not necessarily flat over $\tilde{M}$. It is easy to show that $\tilde{M}_m$ gives a formal model of
\( \check{\mathcal{M}}^\text{rig}_m \), namely, the Raynaud generic fiber of \( \check{\mathcal{M}}_m \) coincides with \( \check{\mathcal{M}}^\text{rig}_m \). We denote \( (\check{\mathcal{M}}_m)_\text{red} \) by \( \check{\mathcal{M}}_m \), which is a 1-dimensional scheme over \( \overline{\mathbb{F}}_p \).

There is a natural left action of \( J \) on \( \check{\mathcal{M}}_m \) which is compatible with that on \( \check{\mathcal{M}}^\text{rig}_m \). On the other hand, the natural action \( K_0 \) on \( L/p^mL \) induces a right action of \( K_0 \) on \( \check{\mathcal{M}}_m \), which is compatible with that on \( \check{\mathcal{M}}^\text{rig}_m \).

We can also describe \( \check{\mathcal{M}}_m \) as a functor from \( \text{Nilp} \) to \( \text{Set} \); for an object \( S \) of \( \text{Nilp} \), the set \( \check{\mathcal{M}}_m(S) \) consists of isomorphism classes of triples \( (X, \rho, \eta) \), where \( (X, \rho) \in \check{\mathcal{M}}_m(S) \) and \( \eta : L/p^mL \to X[p^m] \) is a Drinfeld \( m \)-level structure of \( X \). By this description, the action of \( j \in J \) on \( \check{\mathcal{M}}_m \) is given by \( (X, \rho, \eta) \mapsto (X, \rho \circ j^{-1}, \eta) \). On the other hand, the action of \( g \in K_0 \) on \( \check{\mathcal{M}}_m \) is given by \( (X, \rho, \eta) \mapsto (X, \rho \circ g \circ \eta) \).

By [Man04, Lemma 7.2], \( \{ \check{\mathcal{M}}_m \}_{m \geq 0} \) forms a projective system of formal schemes equipped with the commuting action of \( J \) and \( K_0 \).

### 3.3 Compactly supported cohomology of \( \check{\mathcal{M}}_m \)

For \( m \geq 0 \), we denote the set of quasi-compact open subsets of \( \check{\mathcal{M}}_m \) by \( \mathcal{Q}_m \). It has a natural filtered order by inclusion.

**Definition 3.3** For an object \( \mathcal{F} \) of \( D^b(\check{\mathcal{M}}_m, \mathbb{Z}_\ell) \) or \( D^b(\check{\mathcal{M}}_m, \mathbb{Q}_\ell) \), we put

\[
H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) = \lim_{U \in \mathcal{Q}_m} H_c^i(U, \mathcal{F}|_U).
\]

Assume that \( \mathcal{F} \) has a \( J \)-equivariant structure, namely, for every \( g \in J \) an isomorphism \( \varphi_g : g^*\mathcal{F} \to \mathcal{F} \) is given such that \( \varphi_{gg'} = \varphi_g \circ g'^*\varphi_g \) for every \( g, g' \in J \). Then \( J \) naturally acts on \( H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) \) on the right. Therefore we get a left action of \( J \) on \( H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) \) by taking the inverse \( J \to J ; g \mapsto g^{-1} \).

**Theorem 3.4** Let \( \mathcal{F}^0 \) be an object of \( D^b_c(\check{\mathcal{M}}_m, \mathbb{Z}_\ell) \) and \( \mathcal{F} \) the object of \( D^b_c(\check{\mathcal{M}}_m, \mathbb{Q}_\ell) \) associated with \( \mathcal{F}^0 \). Assume that we are given a \( J \)-equivariant structure of \( \mathcal{F}^0 \) (thus \( \mathcal{F} \) also has a \( J \)-equivariant structure). Then \( H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) \) is a finitely generated smooth \( J \)-representation.

**Proof.** Let \( U \) be an element of \( \mathcal{Q}_m \). By [Far04, Proposition 2.3.11], there exists a compact open subgroup \( K^U \) of \( J \) which stabilizes \( U \). Then \( H_c^i(U, \mathcal{F}|_U) \) is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space endowed with the action of \( K^U \) and has the \( K^U \)-stable \( \mathbb{Z}_\ell \)-lattice \( \text{Im}(H_c^i(U, \mathcal{F}^0|_U) \to H_c^i(U, \mathcal{F}|_U)) \). Therefore \( H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) \) is a smooth \( J \)-representation by Corollary 2.7.

To prove that \( H_c^i(\check{\mathcal{M}}_m, \mathcal{F}) \) is finitely generated, we may assume \( m = 0 \), for \( H_c^i(\check{\mathcal{M}}_{m, 0}, \mathcal{F}) = H_c^i(\check{\mathcal{M}}_{0, p_0m, \mathcal{F}}) \). In this case, we can use the similar method as in [Far04, Proposition 4.4.13]. Let us explain the argument briefly. By [Far04, Théorème 2.4.13], there exists \( W \in \mathcal{Q}_0 \) such that \( \bigcup_{g \in J} gW = \check{\mathcal{M}}_0 \). We put \( K = \{ g \in J \mid gW = W \} \) and \( \Omega = \{ g \in J \mid gW \cap W \neq \emptyset \} \). As in the proof of [Far04, Proposition 4.4.13], \( K \) is a compact open subgroup of \( J \) and \( \Omega \) is a compact.
subset of \( J \). For \( \alpha = ([g_1], \ldots, [g_n]) \in (J/K)^n \), we put \( W_\alpha = g_1W \cap \cdots \cap g_nW \) and \( K_\alpha = \bigcap_{j=1}^n g_jKg_j^{-1} \). For an open covering \( \{gW\}_{g \in J/K} \), we can associate the Čech spectral sequence

\[
E_1^{r,s} = \bigoplus_{\alpha \in (J/K)^{-r+1}} H_c^s(W_\alpha, F|_{W_\alpha}) \Longrightarrow H^{r+s}(\mathcal{L}_0, F).
\]

Consider the diagonal action of \( J \) on \( (J/K)^{-r+1} \). The coset

\[ J \setminus \{ \alpha \in (J/K)^{-r+1} \mid W_\alpha \neq \emptyset \} \]

is finite; indeed, if \( W_\alpha \neq \emptyset \) for \( \alpha = ([g_1], \ldots, [g_{r-1}]) \in (J/K)^{-r+1} \), then \( g_{r-1}^{-1} \in \{ 1 \} \times \Omega/K \times \cdots \times \Omega/K \), which is a finite set.

Take a system of representatives \( \alpha_1, \ldots, \alpha_n \) of the coset above. Then there is a natural isomorphism \( \bigoplus_{\alpha_1} L_{J/K} H_c^s(W_{\alpha_1}, F|_{W_{\alpha_1}}) \cong c\text{-Ind}_{J/K} H^s_c(W_{\alpha_1}, F|_{W_{\alpha_1}}) \). Hence

\[
E_1^{r,s} \cong \bigoplus_j L_{J/K} H_c^s(W_{\alpha_j}, F|_{W_{\alpha_j}}) \text{ is a finitely generated } J\text{-module, since the cohomology } H^s_c(W_{\alpha_j}, F|_{W_{\alpha_j}}) \text{ is finite-dimensional for each } j. \]

By this and the fact that a finitely generated smooth \( J \)-module is noetherian \([\text{Bern4}]\), Remark 3.12], we conclude that \( H^i_c(\mathcal{L}_0, F) \) is finitely generated.

**Lemma 3.5** Let \( \mathcal{F} \) be an object of \( D^b_c(\mathcal{M}, \mathbb{Q}_\ell) \) with a \( K_0/K_m \)-equivariant structure. Let \( n \) be an integer with \( 0 \leq n \leq m \) and put \( \mathcal{G} = (p_{nm*} \mathcal{F})_{K_n/K_m} \). Then we have \( H^i_c(\mathcal{M}, \mathcal{F})_{K_n/K_m} = H^i_c(\mathcal{M}, \mathcal{G}) \).

**Proof.** Since the cardinality of \( K_n/K_m \) is prime to \( \ell \), \((-)^{K_n/K_m} \) commutes with \( H^i_c \).

Therefore, we have

\[
H^i_c(\mathcal{M}, \mathcal{F})_{K_n/K_m} = \lim_{U \in \mathcal{Q}_n} H^i_c(U, \mathcal{F}|_U)_{K_n/K_m} = \lim_{V \in \mathcal{Q}_n} H^i_c(p_{nm}^{-1}(V), \mathcal{F}|_{p_{nm}^{-1}(V)})_{K_n/K_m} = \lim_{V \in \mathcal{Q}_n} H^i_c(V, \mathcal{F}|_{p_{nm}^{-1}(V)})_{K_n/K_m} = H^i_c(\mathcal{M}, \mathcal{G}).
\]

**Definition 3.6** A system of coefficients over the tower \( \{\mathcal{M}_m\}_{m \geq 0} \) is the data \( \mathcal{F} = \{ \mathcal{F}_m \}_{m \geq 0} \) where \( \mathcal{F}_m \) is an object of \( D^b_c(\mathcal{M}_m, \mathbb{Q}_\ell) \) with a \( K_0/K_m \)-equivariant structure such that \( (p_{nm*} \mathcal{F})_{K_n/K_m} = \mathcal{F}_n \) for every integers \( m, n \) with \( 0 \leq n \leq m \). Then, by Lemma 3.5, we have \( H^i_c(\mathcal{M}_m, \mathcal{F}_m)_{K_n/K_m} = H^i_c(\mathcal{M}_n, \mathcal{F}_n) \). We put \( H^i_c(\mathcal{M}_\infty, \mathcal{F}) = \lim_{\mathcal{M}_m} H^i_c(\mathcal{M}_m, \mathcal{F}_m) \).

If each \( \mathcal{F}_m \) is endowed with a \( J \)-equivariant structure which commutes with the given \( K_0/K_m \)-equivariant structure, and for every \( 0 \leq n \leq m \) the \( J \)-equivariant structures on \( \mathcal{F}_m \) and \( \mathcal{F}_n \) are compatible under the identification \( (p_{nm*} \mathcal{F}_m)_{K_n/K_m} = \mathcal{F}_n \), then we say that we have a \( J \)-equivariant structure on \( \mathcal{F} \). Such a structure naturally induces the action of \( J \) on \( H^i_c(\mathcal{M}_\infty, \mathcal{F}) \).

By replacing “\( D^b_c(\mathcal{M}_m, \mathbb{Q}_\ell) \)” with “\( D^b_c(\mathcal{M}_m, \mathbb{Z}_\ell) \)”\), we may also define a system of integral coefficients \( \mathcal{F}_0 \) over \( \{\mathcal{M}_m\}_{m \geq 0} \), the cohomology \( H^i_c(\mathcal{M}_\infty, \mathcal{F}_0) \) and a \( J \)-equivariant structure on \( \mathcal{F}_0 \).
\section{Shimura variety and $p$-adic uniformization}

In this section, we introduce certain Shimura varieties (Siegel threefolds) related to our Rapoport-Zink tower. Let us fix a 4-dimensional $\mathbb{Q}$-vector space $V'$ and an alternating perfect pairing $\psi': V' \times V' \to \mathbb{Q}$. For an integer $m \geq 0$ and a compact open subgroup $K^p \subset \text{GSp}(V'_{\bar{k},p}) = \text{GSp}(V'_{\bar{k},p}, \psi'_{\bar{k},p})$, consider the functor $\text{Sh}_{m, K^p}$ from the category of locally noetherian $\mathbb{Z}_p$-schemes to the category of sets that associates $S$ with the set of isomorphism classes of quadruples $(A, \lambda, \eta^p, \eta_p)$ where

- $A$ is a projective abelian variety over $S$ up to prime-to-$p$ isogeny,
- $\lambda: A \to A'$ is a prime-to-$p$ polarization,
- $\eta^p$ is a $K^p$-level structure of $A$,
- $\eta_p : L/p^mL \to A[p^m]$ is a Drinfeld $m$-level structure

(for the detail, see [Kot92 §5]). Two quadruples $(A, \lambda, \eta^p, \eta_p)$ and $(A', \lambda', \eta'^p, \eta'_p)$ are said to be isomorphic if there exists a prime-to-$p$ isogeny from $A$ to $A'$ which carries $\lambda$ to a $\mathbb{Z}_p^\times$-multiple of $\lambda'$, $\eta^p$ to $\eta'^p$ and $\eta_p$ to $\eta'_p$. We put $\text{Sh}_{K^p} = \text{Sh}_{0, K^p}$. It is known that if $K^p$ is sufficiently small, $\text{Sh}_{m, K^p}$ is represented by a quasi-projective scheme over $\mathbb{Z}_p$ with smooth generic fiber. In the sequel, we always assume that $K^p$ is enough small so that $\text{Sh}_{m, K^p}$ is representable. We denote the special fiber of $\text{Sh}_{m, K^p}$ (resp. $\text{Sh}_{K^p}$) by $\text{Sh}_{m, K^p}$ (resp. $\text{Sh}_{K^p}$).

For a compact open subgroup $K^p$ contained in $K^p$ and an integer $m' \geq m$, we have the natural morphism $\text{Sh}_{m', K^p} \to \text{Sh}_{m, K^p}$. This is a finite morphism and is moreover étale if $m' = m$.

Next we recall the $p$-adic uniformization theorem, which gives a relation between $\mathcal{M}$ and $\text{Sh}_{K^p}$. Let us fix a polarized abelian surface $(A_0, \lambda_{A_0})$ over $\mathbb{F}_p$ such that $A_0[p^\infty]$ is an isoclinic $p$-divisible group with slope $1/2$. Note that such $(A_0, \lambda_{A_0})$ exists; for example, we can take $(A_0, \lambda_{A_0}) = (E^2, \lambda^2_E)$, where $E$ is a supersingular elliptic curve over $\mathbb{F}_p$ and $\lambda_E$ is a polarization of $E$. By definition, the rational Dieudonné module $D(A_0[p^\infty])_Q$ is isomorphic to $D(\mathbb{X})_Q$. Thus, by the subsequent lemma, there is an isomorphism of isocrystals $D(A_0[p^\infty])_Q \cong D(\mathbb{X})_Q$ which preserves the natural polarizations.

\begin{lemma}
We use the notation in [RR96 §1]. Let $d \geq 1$ be an integer.

i) Let $b$ be an element of $B(\text{GSp}_{2d})$ and $b'$ the image of $b$ under the natural map $B(\text{GSp}_{2d}) \to B(\text{GL}_{2d})$. Then $b$ is basic if and only if $b'$ is basic.

\end{lemma}
ii) The map $B(GSp_{2d})_{\text{basic}} \to B(GL_{2d})_{\text{basic}}$ induced from i) is an injection.

Proof. Note that the center of $GSp_{2d}$ coincides with that of $GL_{2d}$. Thus i) is clear, since $b$ (resp. $b'$) is basic if and only if the slope morphism $\nu_b: \mathbb{D} \to GSp_{2d}$ (resp. $\nu_{b'}: \mathbb{D} \to GSp_{2d} \hookrightarrow GL_{2d}$) factors through the center of $GSp_{2d}$ (resp. $GL_{2d}$).

We prove ii). By [RR96, Theorem 1.15], it suffices to show that the natural map $\pi_1(GSp_{2d}) \to \pi_1(GL_{2d})$ is injective. Take a maximal torus $T$ (resp. $T'$) of $GSp_{2d}$ (resp. $GL_{2d}$) such that $T \subset T'$. Then, since $Sp_{2d}$ (resp. $SL_{2d}$) is simply connected, $\pi_1(GSp_{2d})$ (resp. $\pi_1(GL_{2d})$) can be identified with the quotient of $X_*(T)$ (resp. $X_*(T')$) induced by $c: T \to \mathbb{G}_m$ (resp. $det: T' \to \mathbb{G}_m$), where $c$ denotes the similitude character of $GSp_{2d}$. In particular, both $\pi_1(GSp_{2d})$ and $\pi_1(GL_{2d})$ are isomorphic to $\mathbb{Z}$.

The commutative diagram

$$
\begin{array}{ccc}
GSp_{2d} & \xrightarrow{c} & \mathbb{G}_m \\
\downarrow & & \downarrow \scriptstyle{z \to z^d} \\
GL_{2d} & \xrightarrow{det} & \mathbb{G}_m
\end{array}
$$

induces the commutative diagram

$$
\begin{array}{ccc}
X_*(T) & \xrightarrow{d} & X_*(\mathbb{G}_m) \\
\downarrow & & \downarrow \scriptstyle{\times d} \\
X_*(T') & \xrightarrow{d} & X_*(\mathbb{G}_m)
\end{array}
= \pi_1(GSp_{2d}) \\
\pi_1(GL_{2d}).
$$

In particular, the natural map $\pi_1(GSp_{2d}) \to \pi_1(GL_{2d})$ is injective. 

Therefore, there is a quasi-isogeny $X \to A[p^\infty]$ preserving polarizations. If we replace $(X, \lambda_0)$ by the polarized $p$-divisible group $(A_0[p^\infty], \lambda_{A_0})$ associated with $(A_0, \lambda_{A_0})$, the $G$-representation $H_{RZ}$ remains unchanged. Thus, in order to prove Theorem 3.2 we may assume that $(X, \lambda_0) = (A_0[p^\infty], \lambda_{A_0})$. In the remaining part of this article, we always assume it. Moreover, we fix an isomorphism $H_1(A_0, A^{\infty,p}) \cong V^r_{A^{\infty,p}}$ preserving alternating pairings.

Denote the isogeny class of $(A_0, \lambda_{A_0})$ by $\phi$ and put $I^\phi = \text{Aut}(\phi)$. We have natural group homomorphisms $I^\phi \to J$ and $I^\phi \to \text{Aut}(H_1(A_0, A^{\infty,p})) = GSp(V^r_{A^{\infty,p}})$. These are injective.

Let $Y_{K^p}$ be the reduced closed subscheme of $\overline{Sh}_{K^p}$ such that $Y_{K^p}(\overline{F}_p)$ consists of triples $(A, \lambda, \eta^p)$ where the $p$-divisible group associated with $(A, \lambda)$ is isogenous to $(X, \lambda_0)$. It is the basic (or supersingular) stratum in the Newton stratification of $\overline{Sh}_{K^p}$. Note that $(A, \lambda, \eta^p) \in \overline{Sh}_{K^p}(\overline{F}_p)$ if and only if $(A, \lambda) \in \phi$ ([Far04, Proposition 3.1.8], [Kot92, §7]). We denote the formal completion of $\overline{Sh}_{K^p}$ along $Y_{K^p}$ by $(\overline{Sh}_{K^p})'/Y_{K^p}$.

Now we can state the $p$-adic uniformization theorem:
\[ \theta_{K^p} : I^\phi \backslash (\tilde{\mathcal{M}} \times \text{GSp}(V'_{h,\infty,\rho})/K^p) \cong (\text{Sh}_{K^p})_{/Y_{K^p}}. \]

In the left hand side, \( I^\phi \) acts on \( \tilde{\mathcal{M}} \) through \( I^\phi \mapsto J \) and acts on \( \text{GSp}(V'_{h,\infty,\rho})/K^p \) through \( I^\phi \mapsto \text{GSp}(V'_{h,\infty,\rho}) \).

The isomorphisms \( \{ \theta_{K^p} \}_{K^p} \) are compatible with change of \( K^p \). (It is also compatible with the Hecke action of \( \text{GSp}_4(V'_{h,\infty,\rho}) \), but we do not use it.)

Let us briefly recall the construction of the isomorphism \( \theta_{K^p} \). Take a lift \((\tilde{X}, \lambda_0)\) of \((X, \lambda_0)\) over \( \mathbb{Z}_{p^\infty} \) (such a lift is unique up to isomorphism). Then, by the Serre-Tate theorem, the lift \((\tilde{A}_0, \tilde{\lambda}_0)\) of \((A_0, \lambda_0)\) is canonically determined. Let \( S \) be an object of \( \text{Nilp} \), \((X, \rho) \in \tilde{\mathcal{M}}(S) \) and \([g] \in \text{GSp}(V'_{h,\infty,\rho})/K^p \). Then \( \rho \) extends uniquely to the quasi-isogeny \( \tilde{\rho} : \tilde{X} \times \mathbb{Z}_{p^\infty} S \rightarrow X \). We can see that there exist a polarized abelian variety \((A, \lambda)\) and a \( p \)-quasi-isogeny \( \tilde{A}_0 \times \mathbb{Z}_{p^\infty} S \rightarrow A \) preserving polarizations, such that the associated quasi-isogeny \( \tilde{A}_0[p^\infty] \times \mathbb{Z}_{p^\infty} S \rightarrow A[p^\infty] \) coincides with \( \tilde{\rho} \). The fixed isomorphism \( H_1(A_0, \mathbb{A}^{\infty,\rho}) \cong V'_{h,\infty,\rho} \) naturally induces a \( K^p \)-level structure \( \eta \) of \( A \). The morphism \( \theta_{K^p} \) is given by \( \theta_{K^p}((X, \rho), [g]) = (A, \lambda, \eta \circ g) \).

By composing the morphism \( \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{M}} \times \text{GSp}(V'_{h,\infty,\rho})/K^p ; x \mapsto (x, [id]) \), we get a morphism \( \tilde{\mathcal{M}} \rightarrow (\text{Sh}_{K^p})_{/Y_{K^p}} \), which is also denoted by \( \theta_{K^p} \). For \( U \in \mathcal{O}_0 \), we denote the image of \( U \) under \( \theta_{K^p} \) by \( Y_{K^p}(U) \). It is an open subset of \( Y_{K^p} \).

**Proposition 4.3** Let \( U \) be an element of \( \mathcal{O}_0 \). Then for a sufficiently small compact open subgroup \( K^p \) of \( \text{GSp}(V'_{h,\infty,\rho}) \), \( \theta_{K^p} \) induces an isomorphism \( U \cong Y_{K^p}(U) \).

Moreover, if we denote the open formal subscheme of \( \tilde{\mathcal{M}} \) (resp. \( \text{Sh}_{K^p} \)) whose underlying topological space is \( U \) (resp. \( Y_{K^p}(U) \)) by \( \tilde{\mathcal{M}}_{/U} \) (resp. \( \text{Sh}_{K^p} \)) then \( \theta_{K^p} \) induces an isomorphism \( \tilde{\mathcal{M}}_{/U} \cong (\text{Sh}_{K^p})_{/Y_{K^p}(U)} \).

**Proof.** The proof is similar to [Far04 Corollaire 3.1.4]. Put \( \Gamma_{K^p} = I^\phi \cap K^p \), where the intersection is taken in \( \text{GSp}(V'_{h,\infty,\rho}) \). It is known that \( \Gamma_{K^p} \) is discrete and torsion-free [RZ96]. By Theorem 4.2, \( \theta_{K^p} \) gives an isomorphism from \( \Gamma_{K^p} \backslash \tilde{\mathcal{M}} \) to an open and closed formal subscheme of \( (\text{Sh}_{K^p})_{/Y_{K^p}} \). By the same method as in [Far04 Lemme 3.1.2, Proposition 3.1.3], we can see that every element \( \gamma \in \Gamma_{K^p} \) other than 1 satisfies \( \gamma \cdot U \cap U = \emptyset \) if \( K^p \) is sufficiently small. For such \( K^p \), the natural morphism \( \tilde{\mathcal{M}}_{/U} \hookrightarrow \Gamma_{K^p} \backslash \tilde{\mathcal{M}} \) is an open immersion. Thus we have an open immersion \( \tilde{\mathcal{M}}_{/U} \hookrightarrow \Gamma_{K^p} \backslash \tilde{\mathcal{M}}_{/K^p} \cong (\text{Sh}_{K^p})_{/Y_{K^p}(U)} \), whose image is \( (\text{Sh}_{K^p})_{/Y_{K^p}(U)} \).

Next we consider the case with Drinfeld level structures at \( p \). Let \( Y_{m,K^p} \) be the closed subscheme of \( \overline{\text{Sh}}_{m,K^p} \) obtained as the inverse image of \( Y_{K^p} \) under \( \overline{\text{Sh}}_{m,K^p} \rightarrow \overline{\text{Sh}}_{K^p} \). By the construction of \( \theta_{K^p} \) described above, we have the following result:
Corollary 4.4 Let $m \geq 0$ be an integer. We can construct naturally a morphism $\theta_{m,K^p} : \tilde{\mathcal{M}}_m \to (\text{Sh}_{m,K^p})^{\wedge}_{Y_{m,K^p}}$ which makes the following diagram cartesian:

$$
\begin{array}{ccc}
\tilde{\mathcal{M}}_m & \xrightarrow{\theta_{m,K^p}} & (\text{Sh}_{m,K^p})^{\wedge}_{Y_{m,K^p}} \\
\downarrow & & \downarrow \\
\mathcal{M} & \xrightarrow{\theta_{K^p}} & (\text{Sh}_{K^p})^{\wedge}_{Y_{K^p}}.
\end{array}
$$

In particular, the similar result as Proposition 4.3 holds for $\theta_{m,K^p}$; that is, for $U \in \mathcal{Q}_m$, $\theta_{m,K^p}$ induces $(\tilde{\mathcal{M}}_m)/U \cong (\text{Sh}_{m,K^p})^{\wedge}_{Y_{m,K^p}(U)}$ if $K^p$ is sufficiently small.

5 Proof of the non-cuspidality result

5.1 The system of coefficients $\mathcal{F}^{[h]}$, $\mathcal{F}^{(h)}$

Definition 5.1 Let $m \geq 1$ and $0 \leq h \leq 2$ be integers. We denote by $\mathcal{S}_{m,h}$ the set of direct summands of $L/p^m L$ of rank $4-h$, and by $\mathcal{S}_{m,h}^{\text{coi}}$ the subset of $\mathcal{S}_{m,h}$ consisting of coisotropic direct summands (recall that $I \in \mathcal{S}_{m,h}$ is said to be coisotropic if $I^\perp \subset I$). Put $\mathcal{S}_m = \bigcup_{h=0}^2 \mathcal{S}_{m,h}$ and $\mathcal{S}_m^{\text{coi}} = \bigcup_{h=0}^2 \mathcal{S}_{m,h}^{\text{coi}}$.

For $I \in \mathcal{S}_{m,h}$, let $\overline{\text{Sh}}_{m,K^p,[I]}$ be the $\overline{\mathbb{F}}_p$-scheme defined by

$$
\overline{\text{Sh}}_{m,K^p,[I]}(S) = \{(A, \lambda, \eta^p, \eta_p) \in \overline{\text{Sh}}_{m,K^p,[I]}(S) \mid I \subset \text{Ker} \eta_p\}.
$$

Clearly it is a closed subscheme of $\overline{\text{Sh}}_{m,K^p}$. Similarly, we can define the closed formal subscheme $\tilde{\mathcal{M}}_m, [I]$ of $\mathcal{M} \otimes_{\mathbb{Z}_p} \overline{\mathbb{F}}_p$. Obviously, $\tilde{\mathcal{M}}_m, [I]$ is stable under the action of $J$ on $\tilde{\mathcal{M}}_m$.

We denote by $Y_{m,K^p,[I]}$ the closed subscheme of $\overline{\text{Sh}}_{m,K^p,[I]}$ obtained as the inverse image of $Y_{m,K^p}$. As Corollary 4.4 we have the following cartesian diagram of formal schemes:

$$
\begin{array}{ccc}
\tilde{\mathcal{M}}_m, [I] & \xrightarrow{\theta_{m,K^p}} & (\overline{\text{Sh}}_{m,K^p,[I]})^{\wedge}_{Y_{m,K^p,[I]}} \\
\downarrow & & \downarrow \\
\mathcal{M}_m & \xrightarrow{\theta_{K^p}} & (\overline{\text{Sh}}_{K^p})^{\wedge}_{Y_{K^p}}.
\end{array}
$$

Definition 5.2 For $I \in \mathcal{S}_m$, we put

$$
\overline{\text{Sh}}_{m,K^p,(I)} = \overline{\text{Sh}}_{m,K^p,[I]} \setminus \bigcup_{I' \in \mathcal{S}_m, I \subset I'} \overline{\text{Sh}}_{m,K^p,[I']},
$$

which is an open subscheme of $\overline{\text{Sh}}_{m,K^p,[I]}$, and thus is a subscheme of $\overline{\text{Sh}}_{m,K^p}$. Moreover, for an integer $h$ with $0 \leq h \leq 2$, we put $\overline{\text{Sh}}^{[h]}_{m,K^p} = \bigcup_{I \in \mathcal{S}_m} \overline{\text{Sh}}_{m,K^p,[I]}$ and
\(\Sh_{m,Kp}^{(h)} = \bigcup_{I \in S_{m,h}} \Sh_{m,Kp,(I)}^{(h)}\). The former is a closed subscheme of \(\Sh_{m,Kp}\), which is the scheme theoretic image of \(\bigsqcup_{I \in S_{m,h}} \Sh_{m,Kp,(I)}^{[I]} \longrightarrow \Sh_{m,Kp}\). The latter is an open subscheme of \(\Sh_{m,Kp}^{(h)}\), since \(\Sh_{m,Kp}^{(h)} = \Sh_{m,Kp}^{[h]} \setminus \Sh_{m,Kp}^{[h-1]}\) (if \(h = 0\), we put \(\Sh_{m,Kp}^{[-1]} = \emptyset\)).

**Lemma 5.3**  
1) Let \(x = (A, \lambda, \eta^p, \eta_p)\) be an element of \(\Sh_{m,Kp}(\mathbb{F}_p)\). Then, for \(I \in S_m, x \in \Sh_{m,Kp,(I)}^{(h)}(\mathbb{F}_p)\) if and only if \(I = \Ker \eta_p\). For an integer \(h\) with \(0 \leq h \leq 2\), \(x \in \Sh_{m,Kp}^{(h)}(\mathbb{F}_p)\) if and only if \(\text{rank}_{\mathbb{F}_p} A[p] \leq h\) (resp. \(\text{rank}_{\mathbb{F}_p} A[p] = h\)).

2) For every integer \(h\) with \(0 \leq h \leq 2\), we have \(\Sh_{m,Kp}^{(h)} = \bigsqcup_{I \in S_{m,h}} \Sh_{m,Kp,(I)}^{(h)}\) as schemes.

3) We have \((\Sh_{m,Kp})^{[2]}_{\text{red}} = (\Sh_{m,Kp})_{\text{red}}^{(h)}\) and \((\Sh_{m,Kp})^{[0]}_{\text{red}} = (Y_{m,Kp})_{\text{red}}\).

**Proof.** Let us prove i). Put \(X = A[p^\infty]\). Then there is an exact sequence \(0 \longrightarrow X_0 \longrightarrow X \longrightarrow X_{et} \longrightarrow 0\), where \(X_0\) is a connected \(p\)-divisible group and \(X_{et}\) is an étale \(p\)-divisible group. By [HT01, Lemma II.2.1], \(\Ker \eta_p\) is a direct summand of \(L/p^mL\) and \((L/p^mL)/\Ker \eta_p \longrightarrow X_{\et}[p^m]\) is an isomorphism. Thus \(\Ker \eta_p \in S_m\), where \(r = \text{rank}_{\mathbb{F}_p} X_{\et}[p^m] = \text{rank}_{\mathbb{F}_p} A[p] \leq 2\). By this, all the claims in i) are immediate.

By i), \(\Sh_{m,Kp}^{(h)}\) coincides with \(\bigsqcup_{I \in S_{m,h}} \Sh_{m,Kp,(I)}^{(h)}\) as a set; thus to prove ii) it suffices to show that \(\Sh_{m,Kp,(I)}^{(h)}\) is closed (hence open) in \(\Sh_{m,Kp}^{(h)}\) for every \(I \in S_{m,h}\).

It is clear from \(\Sh_{m,Kp,(I)}^{(h)} = \Sh_{m,Kp,(I)}^{[I]} \cap \Sh_{m,Kp}^{(h)}\).

The former equality in iii) follows immediately from i). We will prove the latter. For \(x = (A, \lambda, \eta^p, \eta_p) \in \Sh_{m,Kp}^{[0]}(\mathbb{F}_p), X = A[p^\infty]\) has no étale part by i). Since \(X^\vee \cong X\), \(X\) has no multiplicative part. Therefore \(X\) is isochinic of slope 1/2; indeed, if a Newton polygon with the terminal point \((4,2)\) has neither slope 0 part nor slope 1 part, then it is a line of slope 1/2. Thus, by Lemma 4.1, there is a quasi-isogeny \(X \longrightarrow X\) preserving polarizations; namely, \(x \in Y_{m,Kp}(\mathbb{F}_p)\). The opposite inclusion is clear. \(\blacksquare\)

**Remark 5.4** The latter part of iii) in Lemma 5.3 is the only place where the same argument does not work in the case \(GSp(2d)\) with \(d \geq 3\).

**Definition 5.5** Let \(m \geq 1\) be an integer. Fix a compact open subgroup \(K_p\) of \(GSp(V_{h,\infty,p})\). For \(I \in S_m\), denote the natural immersion \(\Sh_{m,Kp,(I)} \longrightarrow \Sh_{m,Kp}\) by \(j_{m,I}\). For an integer \(h\) with \(0 \leq h \leq 2\), denote the natural immersions \(\Sh_{m,Kp}^{(h)} \longrightarrow \Sh_{m,Kp}\) and \(\Sh_{m,Kp}^{(h)} \hookrightarrow \Sh_{m,Kp}\) by \(j_{m,h}^{[h]}\) and \(j_{m,h}\), respectively.

We define \(F_{m,I}^0, F_{m,I}^{(h)}, F_{m}^{[h]}, F_{m}^{(h)}\) and \(F_{m}^{(h)}\) as follows:

\[
F_{m,I}^0 = \theta_{m}^*(Rj_{m,I}R_j^{[I]}R_{\psi}Z_{\ell})|_{Y_{m,Kp}}, \\
F_{m,I}^{[h]} = \theta_{m}^*(Rj_{m,I}^{[h]}R_{\psi}Z_{\ell})|_{Y_{m,Kp}}, \\
F_{m}^{(h)} = \theta_{m}^*(Rj_{m}^{[h]}R_{\psi}Z_{\ell})|_{Y_{m,Kp}}.
\]

\[
F_{m,I} = \theta_{m}^*(Rj_{m,I}R_j^{[I]}R_{\psi}Q_{\ell})|_{Y_{m,Kp}}, \\
F_{m}^{[h]} = \theta_{m}^*(Rj_{m}^{[h]}R_{\psi}Q_{\ell})|_{Y_{m,Kp}}, \\
F_{m}^{(h)} = \theta_{m}^*(Rj_{m}^{[h]}R_{\psi}Q_{\ell})|_{Y_{m,Kp}}.
\]

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Here \( \theta_m : \mathcal{M}_m \to Y_{m,K^p} \) is the morphism induced from \( \theta_{m,K^p} \) in Corollary 4.4. These are independent of the choice of \( K^p \); indeed, for another compact open subgroup \( K^p \) contained in \( K^p \), the natural map \( \text{Sh}_{m,K^p} \to \text{Sh}_{m,K^p} \) is étale.

**Proposition 5.6** Let \( h \) be an integer with \( 1 \leq h \leq 2 \).

i) We have the following distinguished triangle:
\[
\mathcal{F}^{[h-1]}_m \to \mathcal{F}^{[h]}_m \to \mathcal{F}^{(h)}_m \to \mathcal{F}^{[h-1]}_m[1].
\]

ii) We have \( \mathcal{F}^{(h)}_m = \bigoplus_{I \in \mathcal{S}_{m,h}} \mathcal{F}_{m,I} \).

**Proof.** By the definition, i) is clear. ii) is also clear from Lemma 5.7 ii).

**Proposition 5.7** For \( I \in \mathcal{S}_{m,h} \setminus \mathcal{S}^{\text{co}}_{m,h} \), we have \( \mathcal{F}^\circ_{m,I} = \mathcal{F}_{m,I} = 0 \).

**Proof.** We will prove that \( Rj^*_{m,I}R\psi \mathbb{Z}_\ell = 0 \). Since the dual of \( Rj^*_{m,I}R\psi \mathbb{Z}_\ell \) is isomorphic to \( j^*_{m,I}R\psi \mathbb{Z}_\ell(3)[6] \), it suffices to show that, for every \( x \in \text{Sh}_{m,K^p,(I)}(\mathbb{F}_p) \), no point on the generic fiber of \( \text{Sh}_{m,K^p} \) specializes to \( x \). In other words, for every complete discrete valuation ring \( R \) with residue field \( \mathbb{F}_p \) which is a flat \( \mathbb{Z}_{p^\infty} \)-algebra, and every \( \mathbb{Z}_{p^\infty} \)-morphism \( \tilde{x} : \text{Spec} R \to \text{Sh}_{m,K^p} \), the image of the closed point of \( \text{Spec} R \) under \( \tilde{x} \) does not lie in \( \text{Sh}_{m,K^p,(I)} \). This is a consequence of the following lemma.

**Lemma 5.7** Let \( R \) be a complete discrete valuation ring with perfect residue field \( k \) and with mixed characteristic \((0,p)\), and \((X, \lambda)\) a polarized \( p \)-divisible group over \( R \). We denote the generic (resp. special) fiber of \( X \) by \( X_\eta \) (resp. \( X_\s \)). Then, for every \( m \geq 1 \), the kernel of the specialization map \( X_\eta[p^m] \to X_\s[p^m] \) is a coisotropic direct summand of \( X_\eta[p^m] \).

**Proof.** We shall prove that the kernel of the specialization map \( T_pX_\eta \to T_pX_\s \) is a coisotropic direct summand of \( T_pX_\eta \). Consider the exact sequence \( 0 \to X_\s,0 \to X_\s \to X_{\s,\et} \to 0 \) over \( k \). It is canonically lifted to the exact sequence \( 0 \to X_0 \to X \to X_{\et} \to 0 \) over \( R \), where \( X_{\et} \) is an étale \( p \)-divisible group (cf. [Mes72, p. 76]). Thus we have the following commutative diagram, whose rows are exact:
\[
\begin{array}{ccccccccc}
0 & \to & T_pX_{\s,0,\eta} & \to & T_pX_\eta & \to & T_pX_{\s,\et,\eta} & \to & 0 \\
0 & \to & 0 & \to & T_pX_\s & \to & T_pX_{\s,\et} & \to & 0. \\
\end{array}
\]

Hence the kernel of \( T_pX_\eta \to T_pX_\s \) coincides with \( T_pX_{\s,0,\eta} \). Therefore it suffices to show that the composite \( (T_pX_{\s,0,\eta})^\perp \to T_pX_\eta \to T_pX_{\s,\et,\eta} \) is 0.

On the other hand, by the polarization \( T_pX_\eta \xrightarrow{\cong} (T_pX_\eta)^\vee(1) \), \( (T_pX_{\s,0,\eta})^\perp \) corresponds to \( (T_pX_{\s,\et,\eta})^\vee(1) \cong T_pX_{\s,\et,\eta}^\vee \). Thus it suffices to prove that every Galois-equivariant homomorphism \( T_pX_{\s,\et,\eta}^\vee \to T_pX_{\s,\et,\eta} \) is 0. For this, we may replace the
Tate modules $T_pX^\vee_{\mathrm{et},n}$ and $T_pX_{\mathrm{et},n}$ by the rational Tate modules $V_pX^\vee_{\mathrm{et},n}$ and $V_pX_{\mathrm{et},n}$. These are crystalline representations and the corresponding filtered $\varphi$-modules are the rational Dieudonné modules $D(X^\vee_{\mathrm{et}})_Q$ and $D(X_{\mathrm{et}})_Q$, respectively. Since the slope of the former is 1 and that of the latter is 0, there is no $\varphi$-homomorphism other than 0 from $D(X^\vee_{\mathrm{et}})_Q$ to $D(X_{\mathrm{et}})_Q$. This completes the proof.

The following corollary is immediate from Proposition 5.6 ii) and Proposition 5.7.

**Corollary 5.9** For $h$ with $1 \leq h \leq 2$, we have $\mathcal{F}_m^{(h)} = \bigoplus_{I \in \mathcal{S}_m} \mathcal{F}_{m,I}$.

Let us consider the action of $K_0$. Since $K_0/K_m$ naturally acts on $\text{Sh}_{m,K_P}$ and the action of $g \in K_0/K_m$ maps $\overline{\text{Sh}}_{m,K_P,[l]}$ onto $\overline{\text{Sh}}_{m,K_P,[g^{-1}l]}$, the subschemes $\overline{\text{Sh}}^h_{m,K_P}$ and $\overline{\text{Sh}}_{m,K_P}$ are preserved by the action of $K_0/K_m$. Therefore, $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{(h)}$, and $\mathcal{F}_m^{(h)}$ have natural $K_0/K_m$-equivariant structures. Moreover, in the same way as in [Mie10a, Proposition 2.5], we can observe that $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{(h)}$, and $\mathcal{F}_m^{(h)}$ have natural $K_0/K_m$-equivariant structures. Moreover, in the same way as in [Mie10a, Proposition 2.5], we can observe that $\mathcal{F}_m^{[h]} = \{\mathcal{F}_m^{[h]}\}_{m \geq 1}$ and $\mathcal{F}_m^{(h)} = \{\mathcal{F}_m^{(h)}\}_{m \geq 1}$ form systems of coefficients (resp. integral coefficients) over $\{\mathcal{M}_m\}_{m \geq 1}$.

Thanks to [Mie10b, Corollary 5.9], we can define $J$-equivariant structures on the systems of coefficients introduced above.

**Proposition 5.10** The complexes $\mathcal{F}_m^{[h]}$, $\mathcal{F}_m^{(h)}$, $\mathcal{F}_m^{(h)}$, $\mathcal{F}_m^{(h)}$, and $\mathcal{F}_m^{(h)}$ have natural $J$-equivariant structures. These structures are compatible with the distinguished triangles and the direct sum decompositions in Proposition 5.6.

**Proof.** We will prove the proposition for $\mathcal{F}_m^{(h)}$; other cases are similar. Put

$$\overline{\text{Sh}}^{[h]}_{m,K_P} = (\text{Sh}_{m,K_P})_{Y_{m,K_P}} \times \text{Sh}_{m,K_P} \overline{\text{Sh}}^{[h]}_{m,K_P}, \quad \overline{\text{Sh}}^{(h)}_{m,K_P} = (\overline{\text{Sh}}^{[h]}_{m,K_P}, \overline{\text{Sh}}^{[h-1]}_{m,K_P}),$$

$$\mathcal{M}_m^{[h]} = \mathcal{M}_m \times (\text{Sh}_{m,K_P})_{Y_{m,K_P}} \overline{\text{Sh}}^{[h]}_{m,K_P}, \quad \mathcal{M}_m^{(h)} = (\mathcal{M}_m^{[h]}, \mathcal{M}_m^{[h-1]}).$$

Then, by [Mie10b, Proposition 3.11], we have the canonical isomorphism

$$(Rj_m^*[h] \circ R\psi_\ell)|_{Y_{m,K_P}} \cong R\psi_\ell \circ \overline{\text{Sh}}^{(h)}_{m,K_P} \overline{\text{Sh}}^{(h)}_{m,K_P} Q_\ell.$$

Moreover, since $\theta_{m,K_P}$ is étale (cf. Corollary 4.4), by [Mie10b, Proposition 3.14], we have the canonical isomorphism

$$\mathcal{F}_m^{(h)} \cong R\psi_\ell \circ \mathcal{M}_m^{(h)} Q_\ell.$$
It is easy to see that the actions defined in the previous proposition give $J$-equivariant structures on the systems of (integral) coefficients $\mathcal{F}^{\circ [h]}$, $\mathcal{F}^{\circ (h)}$, $\mathcal{F}^{\circ (h)}$ and $\mathcal{F}^{\circ (h)}$. Thus we get the smooth representations $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ of $K_0 \times J$ (cf. Corollary 3.7).

**Proposition 5.11** There exists an isomorphism $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[0]}) \cong H^i_{R\mathbb{Z}}$, which is compatible with the action of $K_0 \times J$.

**Proof.** Let $m \geq 1$ be an integer and $U \in \mathcal{Q}_m$. Then, by [Mie10b, Corollary 4.40] and Proposition 4.3, we have the $J$-equivariant isomorphism

$$H^i_c(U, \mathcal{F}^{[0]}|_U) \cong H^i_c((\mathcal{M}_m)^{\rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell).$$

Since this isomorphism is functorial, we have $K_0 \times J$-equivariant isomorphisms

$$H^i_c(\mathcal{M}_m, \mathcal{F}^{[0]}) \cong \lim_{U \in \mathcal{Q}_m} H^i_c((\mathcal{M}_m)^{\rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell),$$

$$H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[0]}) \cong \lim_m H^i_c(\mathcal{M}_m^{\rig} \otimes_{\mathbb{Q}_p} \overline{\mathbb{Q}}_p, \mathbb{Q}_\ell) = H^i_{R\mathbb{Z}}.$$

For the isomorphism of (*), we need [Hub98 Proposition 2.1 (iv)] and [Mie10b, Lemma 4.14].

**Remark 5.12** We can deduce from Proposition 5.11 and Corollary 3.7 that the action of $K_0 \times J$ on $H^i_{R\mathbb{Z}}$ is smooth.

### 5.2 $G$-action on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$, $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})$

In this subsection, we define actions of $G$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]})$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ by using the method in [Man05, §6]. Put $G^+ = \{g \in G \mid g^{-1} L \subset L\}$, which is a submonoid of $G$. For $g \in G^+$, let $e(g)$ be the minimal non-negative integer such that $\text{Ker}(g^{-1}: V/L \rightarrow V/L)$ is contained in $p^{-e(g)} L/L$. Since $\text{Ker}(g^{-1} = (gL+L)/L$, we have $gL \subset p^{-e(g)} L$.

In the sequel, we fix a compact open subgroup $K^p$ of $\text{GSp}(\mathcal{V}_{\ell, \mathbb{A}_\infty})$ and denote $\text{Sh}_{m, K^p, \overline{\mathcal{B}}_{m, K^p}, \overline{\mathcal{B}}_{m, K^p, [l]}, \ldots}$ by $\text{Sh}_{m, \overline{\mathcal{B}}_{m, [l]}, \ldots}$, respectively. Moreover, we fix $g \in G^+$ and denote $e(g)$ by $e$ for simplicity.

Assume that $m \geq e$. Let us consider the $\mathbb{Z}_{p}\infty$-scheme $\text{Sh}_{m, g}$ such that for a $\mathbb{Z}_{p}\infty$-scheme $S$, the set $\text{Sh}_{m, g}(S)$ consists of isomorphism classes of quintuples $(A, \lambda, \eta^p, \eta_p, \mathcal{E})$ satisfying the following.

- The quadruple $(A, \lambda, \eta^p, \eta_p)$ gives an element of $\text{Sh}_m(S)$.
- $\mathcal{E} \subset X[p^e]$ is a finite flat subgroup scheme of order $p^{e(\det g^{-1})}$, where we put $X = A[p^\infty]$. It is self-dual with respect to $\lambda$, and satisfies $\eta_p^g(\text{Ker} g^{-1}) \subset \mathcal{E}(S)$, where $\eta_p^g$ denotes the composite $p^{-m} L/L \times_{p^m} L/p^m L \rightarrow X[p^m].$
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- For \( \mathcal{E} \) as above, we have the following commutative diagram:

\[
\begin{array}{c}
p^{-m}L/L \xrightarrow{\eta_p} X[p^m] \xrightarrow{g} X \\
\downarrow g^{-1} \downarrow \downarrow \downarrow \\
p^{-m}g^{-1}L/L \xrightarrow{\eta_p} X[p^m]/\mathcal{E} \xrightarrow{g} X/\mathcal{E} \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
L/p^{m-\epsilon}L \xrightarrow{\cong} p^{-m+\epsilon}L/L \xrightarrow{\eta_p} (X/\mathcal{E})[p^{m-\epsilon}].
\end{array}
\]

We denote the composite of the lowest row by \( \eta_p \circ g \) and assume that it gives a Drinfeld \((m - \epsilon)\)-level structure.

We have the two natural morphisms

\[
\begin{align*}
\text{pr}: & \quad \text{Sh}_{m,g} \to \text{Sh}_m; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \mapsto (A, \lambda, \eta^p, \eta_p), \\
[g]: & \quad \text{Sh}_{m,g} \to \text{Sh}_{m-\epsilon}; (A, \lambda, \eta^p, \eta_p, \mathcal{E}) \mapsto (A/\mathcal{E}, \lambda, \eta^p, \eta_p \circ g).
\end{align*}
\]

It is known that these are proper morphisms, pr induces an isomorphism on the generic fibers, and \([g]\) induces the action of \( g \) on the generic fibers [Man05, Proposition 16, Proposition 17].

We can easily see that \( \{\text{Sh}_{m,g}\}_{m \geq \epsilon} \) form a projective system whose transition maps are finite. Obviously, pr and \([g]\) are compatible with change of \( m \).

Similarly we can define the formal scheme \( \hat{\mathcal{M}}_{m,g} \) and the morphisms \( \text{pr}: \hat{\mathcal{M}}_{m,g} \to \hat{\mathcal{M}}_m \) and \([g]: \hat{\mathcal{M}}_{m,g} \to \hat{\mathcal{M}}_{m-\epsilon} \). The former morphism induces an isomorphism on the Raynaud generic fibers and the composite \([g]^{\text{rig}} \circ (\text{pr}^{\text{rig}})^{-1}\) coincides with the action of \( g \). The group \( J \) naturally acts on \( \hat{\mathcal{M}}_{m,g} \) and two morphisms \( \text{pr} \) and \([g]\) are compatible with the action of \( J \). Moreover, if we denote by \( Y_{m,g} \) the inverse image of \( Y_m \subset \text{Sh}_m \) under \( \text{pr}: \text{Sh}_{m,g} \to \text{Sh}_m \), then we can construct a morphism \( \theta_{m,g}: \hat{\mathcal{M}}_{m,g} \to (\text{Sh}_{m,g})^{\wedge}_{Y_{m,g}} \) which makes the following diagrams cartesian:

\[
\begin{array}{ccc}
\hat{\mathcal{M}}_{m,g} & \xrightarrow{\theta_{m,g}} & (\text{Sh}_{m,g})^{\wedge}_{Y_{m,g}} \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
\hat{\mathcal{M}}_m & \xrightarrow{\theta_{m}} & (\text{Sh}_m)^{\wedge}_{Y_m} \\
\downarrow [g] & & \downarrow [g] \\
\hat{\mathcal{M}}_{m-\epsilon} & \xrightarrow{\theta_{m-\epsilon}} & (\text{Sh}_{m-\epsilon})^{\wedge}_{Y_{m-\epsilon}}.
\end{array}
\]

Now let \( h \) be an integer with \( 1 \leq h \leq 2 \) and \( I \in \mathcal{S}_{m,h} \). Then we can define the subschemes \( \overline{\text{Sh}}_{m,g,[I]} \), \( \overline{\text{Sh}}_{m,g}(I) \), \( \overline{\text{Sh}}_{m,g}^{[h]} \) and \( \overline{\text{Sh}}_{m,g}^{(h)} \) of \( \text{Sh}_{m,g} \) in the same way as \( \overline{\text{Sh}}_{m,[I]} \), \( \overline{\text{Sh}}_{m,I} \), \( \overline{\text{Sh}}_{m}^{[h]} \) and \( \overline{\text{Sh}}_{m}^{(h)} \). The following proposition is obvious:

**Proposition 5.13** We have the commutative diagrams below:

\[
\begin{array}{ccc}
\overline{\text{Sh}}_{m,g,[I]} & \xrightarrow{\text{pr}} & \overline{\text{Sh}}_{m,g} \\
\downarrow & & \downarrow \\
\overline{\text{Sh}}_{m,[I]} & \xrightarrow{\text{pr}} & \overline{\text{Sh}}_{m}.
\end{array}
\]
The rectangles in the left diagram is cartesian. The rectangles in the right diagram is cartesian up to nilpotent elements (namely, \( \overline{\text{Sh}}_{m,g}^{[h]} \to \overline{\text{Sh}}_m^{[h]} \times_{\overline{\text{Sh}}_m} \overline{\text{Sh}}_{m,g} \) induces a homeomorphism on the underlying topological spaces, and so on).

Let us consider how \( \overline{\text{Sh}}_{m,g,[l]} \) are mapped by \( [g] : \text{Sh}_{m,g} \to \text{Sh}_{m-e} \). For this purpose, let us introduce some notation.

**Definition 5.14** We denote by \( S_{\infty,h} \) the set of direct summands of \( L \) of rank \( 4 - h \) and by \( S_{\infty,h}^{\text{coi}} \) the subset of \( S_{\infty,h} \) consisting of coisotropic direct summands. We can identify \( S_{\infty,h} \) with the set of direct summands of \( V \) of rank \( 4 - h \); thus \( G \) naturally acts on \( S_{\infty,h} \) and \( S_{\infty,h}^{\text{coi}} \). Let \( g^{-1} : S_{m,h} \to S_{m-e,h} \) be the unique map which makes the following diagram commutative:

\[
\begin{array}{ccc}
S_{\infty,h} & \longrightarrow & S_{m,h} \\
\downarrow^{g^{-1}} & & \downarrow^{g^{-1}} \\
S_{\infty,h} & \longrightarrow & S_{m-e,h}.
\end{array}
\]

The existence of such \( g^{-1} \) follows from \( p^n L \subseteq p^r L \subseteq g^{-1} L \subseteq L \). Indeed, for direct summands \( I, I' \) of \( V \), we have

\[
I \cap L + p^m L = I' \cap L + p^m L \implies g^{-1} I \cap g^{-1} L + p^m L = g^{-1} I' \cap g^{-1} L + p^m L \\
\implies g^{-1} I \cap g^{-1} L \cap p^e L + p^m L = g^{-1} I' \cap g^{-1} L \cap p^e L + p^m L \\
\iff g^{-1} I \cap L + p^m L = g^{-1} I' \cap L + p^{m-e} L.
\]

Obviously \( g^{-1} : S_{m,h} \to S_{m-e,h} \) induces a map from \( S_{m,h}^{\text{coi}} \) to \( S_{m-e,h}^{\text{coi}} \).

**Proposition 5.15** i) For \( h \in \{1, 2\} \) and \( I \in S_{m,h} \), \( [g] \) induces morphisms

\[
\begin{align*}
\text{Sh}_{m,g,[l]} & \to \text{Sh}_{m-e,[g^{-1} l]}, \\
\text{Sh}_{m,g}^{[h]} & \to \text{Sh}_{m-e}^{[h]}, \\
\text{Sh}_{m,g} & \to \text{Sh}_{m-e}.
\end{align*}
\]

ii) The rectangles of the following commutative diagram is cartesian up to nilpotent elements:

\[
\begin{array}{ccc}
\text{Sh}_{m,g}^{[h]} & \longrightarrow & \text{Sh}_{m,g}^{[h]} \\
\downarrow & & \downarrow \\
\text{Sh}_{m-e}^{[h]} & \longrightarrow & \text{Sh}_{m-e}^{[h]} \\
\downarrow & & \downarrow \\
\text{Sh}_{m-e} & \longrightarrow & \text{Sh}_{m-e}.
\end{array}
\]

**Proof.** By the definition of \( [g] \), it is clear that \( [g] \) induces a morphism \( \text{Sh}_{m,g,[l]} \to \text{Sh}_{m-e,[g^{-1} l]} \) for \( I \in S_{m,h} \), and thus induces a morphism \( \text{Sh}_{m,g}^{[h]} \to \text{Sh}_{m-e}^{[h]} \). On the other hand, note that, for every \( (A, \lambda, \eta^p, \eta_{1,p}, \mathcal{E}) \in \text{Sh}_{m,g}(\mathbb{F}_p) \), the \( p \)-divisible groups \( A[p^\infty] \) and \( A[p^\infty]/\mathcal{E} \) are isogenous, and thus have the same étale heights.
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Therefore, by Lemma 5.13 i), the inverse image of $\overline{Sh}_{m-e}$ (resp. $\overline{Sh}_{m-e}$) under $[g]$ coincides with $\overline{Sh}_{m,g}$ (resp. $\overline{Sh}_{m,g}$) as sets. Therefore a morphism $\overline{Sh}_{m,g} \rightarrow \overline{Sh}_{m-e}$ is naturally induced and the rectangles in the diagram above are cartesian up to nilpotent elements. Finally, since $\overline{Sh}_{m,g}(I) = \overline{Sh}_{m,g}(I) \cap \overline{Sh}_{m,g}$ and $\overline{Sh}_{m-e,(g-1)I} = \overline{Sh}_{m-e,(g-1)I}$, $[g]$ induces a morphism $\overline{Sh}_{m,g}(I) \rightarrow \overline{Sh}_{m-e,(g-1)I}$.

By Proposition 5.13 and Proposition 5.15 we have the natural cohomological correspondence $\gamma_g$ from $\mathcal{F}^{[h]}_m$ (resp. $\mathcal{F}^{(h)}_m$) to $\mathcal{F}^{[h]}_m$ (resp. $\mathcal{F}^{(h)}_m$); see Lemma 5.16 This cohomological correspondence induces a homomorphism $\gamma_g$ from $H^i_c(\mathcal{M}_m, \mathcal{F}^{[h]}_m)$ (resp. $H^i_c(\mathcal{M}_m, \mathcal{F}^{(h)}_m)$) to $H^i_c(\mathcal{M}_m, \mathcal{F}^{[h]}_m)$ (resp. $H^i_c(\mathcal{M}_m, \mathcal{F}^{(h)}_m)$). Indeed, for $U \in \mathcal{Q}_m$, we can take $U' \in \mathcal{Q}_m$ which contains $pr([g]^{-1}(U))$. Then $\gamma_g$ induces $H^i_c(U, \mathcal{F}^{[h]}_m) \rightarrow H^i_c(U', \mathcal{F}^{[h]}_m)$, and therefore induces $H^i_c(\mathcal{M}_m, \mathcal{F}^{[h]}_m) \rightarrow H^i_c(\mathcal{M}_m, \mathcal{F}^{(h)}_m)$. It is easy to see that this homomorphism is compatible with change of $m$; hence we get the endomorphism $\gamma_g$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}_m)$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)}_m)$.

Lemma 5.16 The endomorphism $\gamma_g$ commutes with the action of $J$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}_m)$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)}_m)$.

Proof. We will only consider $\gamma_g$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}_m)$, since the other case is similar. Let $U \in \mathcal{Q}_m$ and $U' \in \mathcal{Q}_m$ be as above and put $W = [g]^{-1}(U), W' = pr^{-1}(U')$. It suffices to show the commutativity of the following diagram for $j \in J$:

\[
\begin{array}{ccc}
H^i_c(jU, \mathcal{F}^{[h]}_m) & \xrightarrow{\gamma_g} & H^i_c(jW, \mathcal{F}^{[h]}_m|W) \\
\downarrow j & & \downarrow j \\
H^i_c(U, \mathcal{F}^{[h]}_m|U) & \xrightarrow{\gamma_g} & H^i_c(W, \mathcal{F}^{[h]}_m|W)
\end{array}
\]

By the construction of the $J$-actions, the left and the middle rectangles are commutative. On the other hand, since $pr$ is proper and induces an isomorphism on the generic fiber, $pr_*$ is an isomorphism and its inverse is $pr^*$. As $pr^*$ commutes with the $J$-action, the right rectangle above is also commutative. This concludes the proof.

Lemma 5.17 i) For $g, g' \in G^+$, $\gamma_{gg'} = \gamma_g \circ \gamma_{g'}$.
ii) For $g \in K_0$, $\gamma_g$ coincides with the action of $K_0$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{[h]}_m)$ or $H^i_c(\mathcal{M}_\infty, \mathcal{F}^{(h)}_m)$, which we already introduced.
iii) The endomorphism $\gamma_{p^{-1} \cdot id}$ an isomorphism (in fact, it coincides with the action of $p^{-1} \cdot id \in J$).

Proof. i) follows from Corollary 5.3, ii) and iii) are consequences of Proposition 16, Proposition 17] and the analogous properties for the Rapoport-Zink spaces (cf. Proposition 7.4 (4), (5)).
Note that $G$ is generated by $G^+$ and $p \cdot \text{id}$ as a monoid. Therefore, by the lemma above, we can extend the actions of $K_0$ on $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ and $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ to whole $G$. Together with Lemma 5.16, we have a smooth $G \times J$-module structures on $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ and $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)})$. We can observe without difficulty that the isomorphism in Proposition 5.11 is in fact compatible with the action of $G$:

**Proposition 5.18** The isomorphism $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{[0]}) \cong H_{KZ}^1$ in Proposition 5.11 is an isomorphism of $G \times J$-modules.

Next we investigate the $G$-module structure of $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)})$ for $h \in \{1, 2\}$. Let us fix an element $\tilde{I}(h)$ of $S_{\text{col}, h}$ and denote its image under the natural map $S_{\text{col}, h} \rightarrow S_{\text{m}, h}$ by $\tilde{I}(h)_m$. Put $P_h = \text{Stab}_G(\tilde{I}(h))$, which is a maximal parabolic subgroup of $G$. Then we can identify $S_{\text{col}, h}$ with $G/P_h = K_0/(P_h \cap K_0)$ and $S_{\text{m}, h}$ with $K_m \backslash G/P_h = K_m\backslash K_0/(P_h \cap K_0)$. For $g \in G^+$ and an integer $m$ with $m \geq e := e(g)$, $g^{-1} : S_{\text{m}, h} \rightarrow S_{\text{m}, e, h}$ is identified with the map $K_m \backslash G/P_h \rightarrow K_{m-e} \backslash G/P_h$.

**Definition 5.19** We put $H_c^1(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)_m}) = \lim_{m \rightarrow \infty} H_c^1(\mathcal{M}_m, \mathcal{F}_{m, \tilde{I}(h)_m})$. Here the transition maps are given as follows: for integers $1 \leq m \leq m'$,

$$
H_c^1(\mathcal{M}_m, \mathcal{F}_{m, \tilde{I}(h)_m}) \rightarrow H_c^1(\mathcal{M}_m', \mathcal{F}_{m', \tilde{I}(h)_m}) \rightarrow \bigoplus_{l' \in S_{\text{col}, h}} H_c^1(\mathcal{M}_{m'}, \mathcal{F}_{m', l'})
$$

$$
\rightarrow H_c^1(\mathcal{M}_{m'}, \mathcal{F}_{m', \tilde{I}(h)_{m'}}).
$$

It is easy to see that $H_c^1(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$ has a structure of a smooth $P_h \times J$-module (use Theorem 3.4 and Proposition 5.13 i)). For each $m \geq 1$ we have the homomorphism

$$
H_c^1(\mathcal{M}_m, \mathcal{F}^{(h)}) = \bigoplus_{I \in S_{\text{col}, h}} H_c^1(\mathcal{M}_m, \mathcal{F}_{m, I}) \rightarrow H_c^1(\mathcal{M}_m, \mathcal{F}_{m, \tilde{I}(h)_m}),
$$

which induces the homomorphism $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)}) \rightarrow H_c^1(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$. By Proposition 5.15 i), we can prove that this is a homomorphism of $P_h \times J$-modules.

**Proposition 5.20** We have an isomorphism $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)}) \cong \text{Ind}_{P_h}^G H_c^1(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$ of $G \times J$-modules.

**Proof.** By the Frobenius reciprocity, we have a $G$-homomorphism $H_c^1(\mathcal{M}_\infty, \mathcal{F}^{(h)}) \rightarrow \text{Ind}_{P_h}^G H_c^1(\mathcal{M}_\infty, \mathcal{F}_{\tilde{I}(h)})$. We shall observe that this is bijective. For an integer $m \geq 1$, we have

$$
H_c^1(\mathcal{M}_m, \mathcal{F}^{(h)}) = \bigoplus_{I \in S_{\text{col}, h}} H_c^1(\mathcal{M}_m, \mathcal{F}_{m, I}) \cong \bigoplus_{g \in K_0/(P_h \cap K_0)} H_c^1(\mathcal{M}_m, \mathcal{F}_{m, g^{-1}\tilde{I}(h)_m})
$$

$$
\cong \text{Ind}_{(P_h \cap K_0)/(P_h \cap K_0)}^G H_c^1(\mathcal{M}_m, \mathcal{F}_{m, \tilde{I}(h)_m}),
$$

where $\text{Ind}$ denotes the induced representation.
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where the last isomorphism, due to \([\text{Boy99], Lemme 13.2}\), is an isomorphism as \(K_0\)-modules. By taking the inductive limit, we have isomorphisms

\[
H^i_c(\overline{M}_\infty, F^{(h)}) \xrightarrow{\cong} \text{Ind}_{P_h \cap K_0}^{K_0} H^i_c(\overline{M}_\infty, F_{I(h)_m}) \xleftarrow{\cong} \text{Ind}^G_{P_h} H^i_c(\overline{M}_\infty, F_{I(h)_m})
\]

(the second isomorphism follows from the Iwasawa decomposition \(G = P_h K_0\)). By the proof of \([\text{Boy99], Lemme 13.2}\], it is easy to see that the first isomorphism above is nothing but the \(K_0\)-homomorphism obtained by the Frobenius reciprocity for \(P_h \cap K_0 \subset K_0\). Therefore the composite of the two isomorphisms above coincides with the \(G\)-homomorphism introduced at the beginning of this proof. Thus we conclude the proof.

5.3 Proof of the main theorem

We begin with the following result on non-cuspidality:

**Theorem 5.21** For every \(i \in \mathbb{Z}\) and \(h \in \{1, 2\}\), the \(G\)-module \(H^i_c(\overline{M}_\infty, F^{(h)})_{\mathbb{Q}_\ell}\) has no quasi-cuspidal subquotient.

By Proposition 5.20 and \([\text{Bern84], 2.4}\], it suffices to show the following proposition:

**Proposition 5.22** Let \(h \in \{1, 2\}\). The unipotent radical \(U_h\) of \(P_h\) acts trivially on \(H^i_c(\overline{M}_\infty, F^{(h)})_{\mathbb{Q}_\ell}\).

To prove Proposition 5.22, we need some preparations. In the sequel, let \(G\) and \(H\) be connected reductive groups over \(\mathbb{Q}_p\), \(P\) a parabolic subgroup of \(G\) and \(U\) the unipotent radical of \(P\). We put \(P = P(\mathbb{Q}_p)\), \(H = H(\mathbb{Q}_p)\) and \(U = U(\mathbb{Q}_p)\).

**Lemma 5.23** Let \(A\) be a noetherian \(\mathbb{Q}\)-algebra and \(V\) an \(A\)-module with a smooth \(P\)-action. Assume that \(V\) is \(A\)-admissible in the sense of \([\text{Bern84], 1.16}\]. Then \(U\) acts on \(V\) trivially.

**Proof.** First assume that \(A\) is Artinian. Then we can prove the lemma in the same way as \([\text{Boy99], Lemme 13.2.3}\] (we use length in place of dimension).

For the general case, we use noetherian induction. Assume that the lemma holds for every proper quotient of \(A\). Take a minimal prime ideal \(p\) of \(A\). Then \(A_p\) is Artinian and \(V_p\) is an \(A_p\)-admissible representation of \(P\) (note that \((V_p)^K = (V^K)_p\) for every compact open subgroup \(K\) of \(P\)). Therefore \(U\) acts on \(V_p\) trivially. Let \(V'\) (resp. \(V''\)) be the kernel (resp. image) of \(V \rightarrow V_p\). Note that \(V'\) and \(V''\) are \(A\)-admissible representations of \(P\), for \(A\) is noetherian.

Consider the following commutative diagram:

\[
\begin{array}{c}
0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0 \\
\downarrow^{(1)} \downarrow^{(2)} \downarrow^{(3)} \\
0 \rightarrow V'_U \rightarrow V_U \rightarrow (V_p)_U \\
\end{array}
\]
It is well-known that the functor taking $U$-coinvariant $V \mapsto V_U$ is an exact functor; thus the lower row in the diagram above is exact. On the other hand, the arrow labeled $(3)$ is injective, since it is the composite of $V'' \hookrightarrow V_p \xrightarrow{\cong} (V_p)^U$. Therefore, by the snake lemma, the injectivity of $(2)$ is equivalent to that of $(1)$. In other words, we have only to prove that the action of $U$ on $V'$ is trivial.

On the other hand, by the definition, $V'$ is the union of $V_s := \{x \in V \mid sx = 0\}$ for $s \in A \setminus p$. Since $V_s$ can be regarded as an admissible $A/(s)$-representation, $U$ acts on $V_s$ trivially by the induction hypothesis. Hence $U$ acts on $V'$ trivially. 

**Proposition 5.24** Let $V$ be a smooth representation of $P \times H$ over $\overline{\mathbb{Q}}_l$ and assume that for every compact open subgroup $K$ of $P$, $V^K$ is a finitely generated $H$-module. Then $U$ acts on $V'$ trivially.

**Proof.** Since $\overline{\mathbb{Q}}_l$ and $\mathbb{C}$ are isomorphic as fields, we may replace $\overline{\mathbb{Q}}_l$ in the statement by $\mathbb{C}$. Let $\mathfrak{Z}$ be the Bernstein center of $H$ [Bern84]. It is decomposed as $\mathfrak{Z} = \prod_{\theta \in \Theta} \mathfrak{Z}_\theta$, where $\Theta$ denotes the set of connected components of the Bernstein variety of $H$. For $\theta \in \Theta$, we denote the $\theta$-part of $V$ by $V_\theta$. Then we have the canonical decomposition $V = \bigoplus_{\theta \in \Theta} V_\theta$, which is compatible with the action of $P \times H$. Therefore, by replacing $V$ with $V_\theta$, we may assume that the action of $\mathfrak{Z}$ on $V$ factors through $\mathfrak{Z}_\theta$ for some $\theta \in \Theta$.

By the assumption and [Bern84, Proposition 3.3], for every compact open subgroup $K$ of $P$, $V^K$ is a finitely generated $H$-module. Namely, for every compact open subgroup $K$ (resp. $K'$) of $P$ (resp. $H$), $V^{K \times K'}$ is a finitely generated $\mathfrak{Z}_\theta$-module. In other words, for every compact open subgroup $K$ of $H$, $V^K$ is a finitely generated $\mathfrak{Z}_\theta$-module. Since $\mathfrak{Z}_\theta$ is a finitely generated $\mathbb{C}$-algebra, $U$ acts trivially on $V^K$ by Lemma 5.23. Therefore $U$ acts trivially on $V'$ also.

**Proof of Proposition 5.22** By Proposition 5.24, we have only to prove that, for every $m \geq 1$, $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^{P_h \cap K_m}_p$ is a finitely generated $J$-module (recall that a finitely generated $J$-module is noetherian [Bern84, Remarque 3.12]). As a $J$-module, it is a direct summand of $(\text{Ind}^{G}_{P_h} H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)}))^K_m \cong H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^{K_m}$. On the other hand, by Corollary 3.7, $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^{K_m}$ is a finitely generated $J$-module. Thus $H^i_c(\mathcal{M}_\infty, \mathcal{F}_{(h)})^{P_h \cap K_m}_p$ is also finitely generated.

**Proposition 5.25** Let $i$ be an integer. If $i \geq 5$, then $H^i_c(\mathcal{M}_\infty, \mathcal{F}[2]) = 0$. On the other hand, if $i \leq 1$, then $H^i_c(\mathcal{M}_\infty, \mathcal{F}[0]) = 0$.

**Proof.** By the definition, it suffices to show that for every $m \geq 1$ and every $U \in Q_m$ we have $H^i_c(U, \mathcal{F}_m[1]|_U) = 0$ for $i \geq 5$ and $H^i_c(U, \mathcal{F}_m[0]|_U) = 0$ for $i \leq 1$. Thus the claim is reduced to the following lemma.

**Lemma 5.26** Let $S$ be the spectrum of a strict henselian discrete valuation ring and $X$ a separated $S$-scheme of finite type. We denote its special (resp. generic)
fiber by $X_s$ (resp. $X_n$). Let $Z$ be a closed subscheme of $X_s$ and denote the natural closed immersion $Z \rightarrow X$ by $i$. Assume that $X_n$ is smooth of pure dimension $d$ and $Z$ is purely $d'$-dimensional.

Then we have $H^n(Z, i^* R^i R\psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R\psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$ and $H^n_c(Z, R^i i^* R\psi_X \mathbb{Q}_\ell) = 0$ for $n < d - d'$.

**Proof.** First note that $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$ if $n > 2 \dim Z$ or $n > 2 \dim(\text{supp } R^k \psi_X \mathbb{Q}_\ell)$. By [BBD82, Proposition 4.4.2], for each $k \geq 0$ we have $\dim(\text{supp } R^k \psi_X \mathbb{Q}_\ell) \leq d - k$; therefore if $n + k > d + d'$ then we have

$$n > d' + (d - k) \geq \dim Z + \dim(\text{supp } R^k \psi_X \mathbb{Q}_\ell)$$

and thus $H^n(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R^k \psi_X \mathbb{Q}_\ell) = 0$. By the spectral sequence, we have $H^n(Z, i^* R\psi_X \mathbb{Q}_\ell) = H^n_c(Z, i^* R\psi_X \mathbb{Q}_\ell) = 0$ for $n > d + d'$.

On the other hand, by the Poincaré duality, we have

$$H^n_c(Z, R^i i^* R\psi_X \mathbb{Q}_\ell) = H^{-n}(Z, D_Z(R^i i^* R\psi_X \mathbb{Q}_\ell))^{\vee} = H^{-n}(Z, i^* R\psi_X D_X, \mathbb{Q}_\ell)^{\vee}$$

$$= H^{-n}(Z, i^* R\psi_X \mathbb{Q}_\ell(d)[2d])^{\vee} = H^{2d-n}(Z, i^* R\psi_X \mathbb{Q}_\ell)^{\vee}(-d),$$

where $D_Z$ (resp. $D_X$) denotes the dualizing functor with respect to $Z$ (resp. $X_n$). Therefore it vanishes if $2d - n > d + d'$, namely, $n < d - d'$.

Now we can prove our main theorem.

**Proof of Theorem 5.22.** By Proposition 5.11 and Proposition 5.25 we have $H^i_{RZ} = 0$ for $i \leq 1$. Therefore we may assume that $i \geq 5$.

By Proposition 5.6(i), we have the exact sequence of smooth $G$-modules

$$H^{i-1}_c(\mathcal{M}_\infty, \mathcal{F}(h))_{\mathbb{Q}_\ell} \rightarrow H^i_c(\mathcal{M}_\infty, \mathcal{F}[h^{-1}])_{\mathbb{Q}_\ell} \rightarrow H^i_c(\mathcal{M}_\infty, \mathcal{F}[h])_{\mathbb{Q}_\ell}$$

for every $h$ with $1 \leq h \leq 2$. Moreover, $H^i_c(\mathcal{M}_\infty, \mathcal{F}(h))_{\mathbb{Q}_\ell}$ has no quasi-cuspidal subquotient by Theorem 5.21. Thus, starting from $H^i_c(\mathcal{M}_\infty, \mathcal{F}[2])_{\mathbb{Q}_\ell} = 0$ (Proposition 5.23), we can inductively prove that $H^i_c(\mathcal{M}_\infty, \mathcal{F}[h])_{\mathbb{Q}_\ell}$ has no quasi-cuspidal subquotient; indeed, the property that a representation has no quasi-cuspidal subquotient is stable under sub, quotient and extension (use the canonical decomposition in [Bern84, 2.3.1]). In particular, $H^i_c(\mathcal{M}_\infty, \mathcal{F}[0])_{\mathbb{Q}_\ell} \cong H^i_{RZ, \mathbb{Q}_\ell} (\text{cf. Proposition 5.18})$ has no quasi-cuspidal subquotient. This completes the proof.

### 6 Appendix: Complements on cohomological correspondences

In this section, we recall the notion of cohomological correspondences (cf. [SGA5, Exposé III], [Fuj97]) and give some results on them. These are used to define the action of $G$ on $H^i_c(\mathcal{M}_\infty, \mathcal{F}[h])$ and $H^i_c(\mathcal{M}_\infty, \mathcal{F}(h))$. 

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In this section, we change our notation. Let $k$ be a field and $\ell$ a prime number which is invertible in $k$. We denote one of $\mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$ by $\Lambda$. Let $X_1$ and $X_2$ be schemes which are separated of finite type over $k$, and $L_i$ an object of $D^b_c(X_i, \Lambda)$ for $i = 1, 2$ respectively. A cohomological correspondence from $L_1$ to $L_2$ is a pair $(\gamma, c)$ consisting of a separated $k$-morphism of finite type $\gamma : \Gamma \rightarrow X_1 \times X_2$ and a morphism $c : \gamma_1^*L_1 \rightarrow R\gamma_2^!L_2$ in the category $D^b_c(\Gamma, \Lambda)$, where we denote $\text{pr}_i \circ \gamma$ by $\gamma_i$. For simplicity, we also write $c$ for $(\gamma, c)$, if there is no risk of confusion. If we are given a cohomological correspondence $(\gamma, c)$ where $\gamma_1$ is proper, then we have the associated morphism $R\Gamma_c(\epsilon) : R\Gamma_c(X_1, L_1) \rightarrow R\Gamma_c(X_2, L_2)$ by composing

$$R\Gamma_c(X_1, L_1) \xrightarrow{\gamma_1^*} R\Gamma_c(\Gamma, \gamma_1^*L_1) \xrightarrow{R\Gamma_c(\epsilon)} R\Gamma_c(\Gamma, R\gamma_2^!L_1) = R\Gamma_c(X_2, R\gamma_2^!L_2) \xrightarrow{\text{adj}} R\Gamma_c(X_2, L_2).$$

We can compose two cohomological correspondences. Let $X_3$ be another scheme which is separated of finite type over $k$ and $L_3 \in D^b_c(X_3, \Lambda)$. Let $(\gamma', c')$ be a cohomological correspondence from $L_2$ to $L_3$. Consider the following diagram

$$
\begin{array}{ccc}
\Gamma \times X_2 & \xrightarrow{\text{pr}_2} & \Gamma' \\
\downarrow \gamma_1 & & \downarrow \gamma_2' \\
\Gamma & \xrightarrow{\gamma_2} & X_2 \\
\downarrow \gamma_1 & & \downarrow \gamma_2 \\
X_1 & & \end{array}
$$

Let $\gamma''$ be the natural morphism $\Gamma \times X_2 \Gamma' \rightarrow X_1 \times X_3$ and $c'' : \gamma''_1^*L_1 \rightarrow R\gamma''_2^!L_3$ the composite of

$$\gamma''_1^*L_1 = \text{pr}_1^* \gamma_1^*L_1 \xrightarrow{\text{pr}_1^*(c)} \text{pr}_1^* R\gamma_2^!L_2 \xrightarrow{b.c.} R \text{pr}_2 \gamma_1^*L_2 \xrightarrow{R \text{pr}_2^*(c)} R \text{pr}_2^* R\gamma_2^!L_3 = R\gamma''_2^!L_3,$$

where $b.c.$ denotes the base change morphism. We call the cohomological correspondence $(\gamma'', c'')$ the composite of $(\gamma, c)$ and $(\gamma', c')$, and denote it by $c' \circ c$. It is not difficult to see that if $\gamma_1$ and $\gamma_1'$ are proper, then $\gamma''_1$ is also proper and $R\Gamma_c(c' \circ c) = R\Gamma_c(c') \circ R\Gamma_c(\epsilon)$.

Let us recall some operations for cohomological correspondences. Let $X_1$, $X_2$, $X'_1$ and $X'_2$ be schemes which are separated of finite type over $k$, and $\gamma : \Gamma \rightarrow X_1 \times X_2$ and $\gamma' : \Gamma' \rightarrow X'_1 \times X'_2$ separated $k$-morphisms of finite type. Assume that the following commutative diagram is given:

$$
\begin{array}{ccc}
X'_1 & \xleftarrow{\gamma'_1} & \Gamma' \\
\downarrow a_1 & & \downarrow a \\
X_1 & \xleftarrow{\gamma_1} & \Gamma \\
\downarrow a_2 & & \downarrow a_2 \\
X_2 & \xrightarrow{\gamma'_2} & X'_2 
\end{array}
$$
First assume that every vertical morphism is proper. Let $L'_i$ be an object of $D^b_c(X'_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma', c')$ a cohomological correspondence from $L'_1$ to $L'_2$. Then we can define the cohomological correspondence $(\gamma, a_*c')$ from $R\Lambda_1, L'_1$ to $R\Lambda_2, L'_2$ by

$$
\gamma'_1 Ra_1, L'_1 \xrightarrow{b.c.} Ra_* \gamma'_1^* L'_1 \xrightarrow{Ra_*(c')} Ra_* R\gamma'_2^* L'_2 = Ra_1 R\gamma'_2^* L'_2
$$

The cohomological correspondence $(\gamma, a_*c')$ is called the push-forward of $(\gamma', c')$ by $a$. It is easy to see that push-forward is compatible with composition. Moreover, we have the following lemma whose proof is also immediate:

**Lemma 6.1** In the above diagram, assume that $X_1 = X'_1$, $X_2 = X'_2$, $a_1 = a_2 = id$ and $a$ is proper. Let $L_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have $R\Gamma_c(a_* c') = R\Gamma_c(c')$.

Next we assume that the right rectangle in the diagram above is cartesian. Let $L'_i$ and $(\gamma', c')$ be as above. Then we have the cohomological correspondence $(\gamma, a_* c')$ from $R\Lambda_1, L'_1$ to $R\Lambda_2, L'_2$ by

$$
\gamma'_1 Ra_1, L'_1 \xrightarrow{b.c.} Ra_* \gamma'_1^* L'_1 \xrightarrow{Ra_*(c')} Ra_* Ra_2, L'_2 \xrightarrow{b.c.} R\gamma'_2^* Ra_2, L'_2.
$$

On the other hand, let $L_i$ be an object of $D^b_c(X_i, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma, a_* c)$ from $a_1^* L_1$ to $a_2^* L_2$ by

$$
\gamma_1^* a_1^* L_1 = a_1^* \gamma_1^* L_1 \xrightarrow{a_1^*(c)} a_1^* R\gamma_2^* L_2 \xrightarrow{b.c.} R\gamma_2^* a_2^* L_2.
$$

Finally assume that the left rectangle in the diagram above is cartesian. Let $L_i$ and $(\gamma, c)$ be as above. Then we have the cohomological correspondence $(\gamma, a_* c)$ from $Ra_1^* L_1$ to $Ra_2^* L_2$ by

$$
\gamma'_1 Ra_1^* L_1 \xrightarrow{b.c.} Ra_1^* \gamma'_1^* L_1 \xrightarrow{Ra_1^*(c)} Ra_1^* Ra_2^* L_2 \xrightarrow{b.c.} R\gamma_2^* Ra_2^* L_2.
$$

These constructions are also compatible with composition.

Next we recall the specialization of cohomological correspondences. Let $S$ be the spectrum of a strict henselian discrete valuation ring on which $\ell$ is invertible. For an $S$-scheme $X$, we denote its special (resp. generic) fiber by $X_s$ (resp. $X_\eta$).

Let $X_1, X_2$ be schemes which are separated of finite type over $S$ and $\gamma: \Gamma \rightarrow X_1 \times_S X_2$ a separated $S$-morphism of finite type. Let $L_i$ be an object of $D^b_c(X_i, \eta, \Lambda)$ for each $i = 1, 2$ and $(\gamma, c)$ a cohomological correspondence from $L_1$ to $L_2$. Then we have the cohomological correspondence $(\gamma, R\psi(c))$ from $R\psi L_1$ to $R\psi L_2$ by

$$
\gamma'_{1,s} R\psi L_1 \xrightarrow{R\psi(c)} R\psi R\gamma'_{1,\eta} L_1 \xrightarrow{R\psi(R\gamma'_2^* L_2)} R\gamma'_2 L_2.
$$
It is easy to see that this construction is compatible with composition and proper push-forward (cf. [Fuj97, Proposition 1.6.1]).

Now we will give the main result in this section. Let $X_i$, $\gamma_i$, $L_i$ be as above and $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_{1,s}$. Assume that $\gamma_{i,s}^{-1}(Y_i) = \gamma_{2,s}^{-1}(Y_2)$ and $\gamma_{i,s}^{-1}(Z_i) = \gamma_{2,s}^{-1}(Z_2)$ as subschemes of $\Gamma_s$, and denote the former by $\Gamma_Y$ and the latter by $\Gamma_Z$. Then we have the following diagrams whose rectangles are cartesian:

\[\begin{array}{ccc}
Y_1 & \leftarrow & Y_2 \\
\gamma_{1,s}^{-1} & \downarrow & \gamma_{2,s}^{-1} \\
X_{1,s} & \rightarrow & X_{2,s}.
\end{array}\]

\[\begin{array}{ccc}
Z_1 & \leftarrow & Z_2 \\
\gamma_{1,s}^{-1} & \downarrow & \gamma_{2,s}^{-1} \\
X_{1,s} & \rightarrow & X_{2,s}.
\end{array}\]

Therefore, for a cohomological correspondence $(\gamma_i, c)$ from $L_1$ to $L_2$, the cohomological correspondence $i^*j_*j^!R\psi(c)$ from $i^*Rj_{1!}Rj_1^!R\psi L_1$ to $i_2^*Rj_2, Rj_2^!R\psi L_2$ is induced. If moreover we assume that $\gamma_1$ is proper, then we have

\[R\Gamma_c(i^*j_*j^!R\psi(c)) : R\Gamma_c(X_{1,s}, i_1^*Rj_{1!}Rj_1^!R\psi L_1) \longrightarrow R\Gamma_c(X_{2,s}, i_2^*Rj_2, Rj_2^!R\psi L_2).\]

**Proposition 6.2** The morphism $R\Gamma_c(i^*j_*j^!R\psi(c))$ depends only on the cohomological correspondence $(\gamma_i, c)$. More precisely, if another $S$-morphism $\gamma' : \Gamma' \rightarrow X_1 \times_S X_2$ has the same generic fiber as $\gamma$ and satisfies the conditions that $\gamma_{i,s}^{-1}(Y_i) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{i,s}^{-1}(Z_i) = \gamma_{2,s}^{-1}(Z_2)$ and $\gamma'_i$ is proper, then the morphism $R\Gamma_c(i^*j'_*j'^!R\psi(c))$ induced from $\gamma'$ is equal to $R\Gamma_c(i^*j_*j^!R\psi(c))$ (here $i'$ and $j'$ are defined in the same way as $i$ and $j$).

**Proof.** Since $\Gamma$ and $\Gamma'$ have the same generic fiber, there is the “diagonal” in the generic fiber of $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Let $\Gamma''$ be the closure of it in $\Gamma \times_{X_1 \times_S X_2} \Gamma'$. Then $\Gamma''$ has the same generic fiber as $\Gamma$. We have natural morphisms $\Gamma'' \rightarrow \Gamma$ and $\Gamma'' \rightarrow \Gamma'$, which are proper since $\gamma$ and $\gamma'$ are proper. Therefore $\gamma'' : \Gamma'' \rightarrow X_1 \times_S X_2$ also satisfies the same conditions as $\gamma$ and $\gamma'$. By replacing $\gamma'$ by $\gamma''$, we may assume that there exists a proper morphism $a : \Gamma' \rightarrow \Gamma$ such that $a \circ a = \gamma'$. Then, it is easy to see that the push-forward of the cohomological correspondence $(\gamma'_i, i'^*j'_*j'^!R\psi(c))$ by $a_s$ coincides with $(\gamma_i, i^*j_*j^!R\psi(c))$. Therefore the proposition follows from Lemma 6.1. \[\blacksquare\]

**Corollary 6.3** Let $X_1$, $X_2$ and $X_3$ be schemes which are separated of finite type over $S$, $Y_i$ (resp. $Z_i$) a closed (resp. locally closed) subscheme of $X_i$, and $L_i$ an object of $D^b_{c}(X_i, \eta, \Lambda)$ for each $i = 1, 2, 3$. Let $\gamma_i : \Gamma \rightarrow X_1 \times_S X_2$ (resp. $\gamma' : \Gamma' \rightarrow X_2 \times_S X_3$, resp. $\gamma'' : \Gamma'' \rightarrow X_1 \times_S X_3$) be an $S$-morphism such that $\gamma_1$ (resp. $\gamma'_1$, resp. $\gamma''_1$) is proper, and $(\gamma_i, c)$ (resp. $(\gamma'_i, c')$, resp. $(\gamma''_i, c'')$) a cohomological correspondence from $L_1$ to $L_2$ (resp. from $L_2$ to $L_3$, resp. from $L_1$ to $L_3$). Moreover we assume that $\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)$, $\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)$, $\gamma_{1,s}^{-1}(Y_2) = \gamma_{2,s}^{-1}(Y_3)$, $\gamma_{1,s}^{-1}(Z_2) = \gamma_{2,s}^{-1}(Z_3)$,
\(\gamma_{1,s}^{-1}(Y_1) = \gamma_{2,s}^{-1}(Y_2)\) and \(\gamma_{1,s}^{-1}(Z_1) = \gamma_{2,s}^{-1}(Z_2)\). Then, as above, the morphisms

\[
R\Gamma_c(i^*j_*^! R\psi(c)) : R\Gamma_c(X_{1,s}, i^*Rj_{1,s} Rj_{1}^! R\psi L_1) \to R\Gamma_c(X_{2,s}, i^*Rj_{2,s} Rj_{2}^! R\psi L_2),
\]

\[
R\Gamma_c(i^*j_*^! R\psi(c')) : R\Gamma_c(X_{2,s}, i^*Rj_{2,s} Rj_{2}^! R\psi L_2) \to R\Gamma_c(X_{3,s}, i^*Rj_{3,s} Rj_{3}^! R\psi L_3),
\]

\[
R\Gamma_c(i^*j_*^! R\psi(c'')) : R\Gamma_c(X_{1,s}, i^*Rj_{1,s} Rj_{1}^! R\psi L_1) \to R\Gamma_c(X_{3,s}, i^*Rj_{3,s} Rj_{3}^! R\psi L_3)
\]

are induced. Assume that the composite of \((\gamma_\eta, c)\) and \((\gamma_\eta', c')\) coincides with \((\gamma_\eta'', c'')\). Then we have \(R\Gamma_c(i^*j_*^! R\psi(c')) \circ R\Gamma_c(i^*j_*^! R\psi(c)) = R\Gamma_c(i^*j_*^! R\psi(c''))\).

**Proof.** By Proposition 6.2, we may replace \(\gamma''\) by \(\Gamma' \times X_2 \Gamma' \to X_1 \times S \times X_3\). Then the equality is clear, since all the operations for cohomological correspondences are compatible with composition.

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