An end-to-end Differentially Private Latent Dirichlet Allocation 
Using a Spectral Algorithm

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Abstract

We provide an end-to-end differentially private spectral algorithm for learning LDA, based on matrix/tensor decompositions, and establish theoretical guarantees on utility/consistency of the estimated model parameters. The spectral algorithm consists of multiple algorithmic steps, named as “edges”, to which noise could be injected to obtain differential privacy. We identify subsets of edges, named as “configurations”, such that adding noise to all edges in such a subset guarantees differential privacy of the end-to-end spectral algorithm. We characterize the sensitivity of the edges with respect to the input and thus estimate the amount of noise to be added to each edge for any required privacy level. We then characterize the utility loss for each configuration as a function of injected noise. Overall, by combining the sensitivity and utility characterization, we obtain an end-to-end differentially private spectral algorithm for LDA and identify the corresponding configuration that outperforms others in any specific regime. We are the first to achieve utility guarantees under the required level of differential privacy for learning in LDA. Overall our method systematically outperforms differentially private variational inference.

1 Introduction

Topic modeling has been used extensively in document categorization, social sciences, machine translation and so forth. Learning topic modeling involves projecting high dimensional observations (documents) to a lower dimensional latent structure (topics), and outputting a model parameter estimation that describes the generative process of observed documents. In this paper, we focus on a popular topic model, Latent Dirichlet Allocation (LDA) [5]. Popular methods to learning LDA, such as variational inference [5], optimizes over a lower bound of the likelihood and is susceptible to local optima due to the non-convexity and high-dimensionality of the likelihood bound.

To provide a guaranteed consistent learning algorithm for LDA, tensor decomposition (spectral method) using method of moments is proposed [1,3]. For linearly independent topics, the tensor decomposition spectral algorithm [1,3] guarantees consistent recovery of the topic-word distribution (i.e. LDA model parameters), if the third order data moment tensor, which denotes the expected co-occurrence of triplets of words in a document, is uniquely decomposed. A state-of-the-art tensor decomposition algorithm, called the simultaneous power method [23], has been proven to recover the true components of tensors with orthogonal components. Therefore, a whitening procedure that transforms
the third order data moment tensor into a tensor with orthogonal components, which in turn can be decomposed using the simultaneous power method, is used. The whitening procedure involves matrix decomposition on the second order data moment matrix, which denotes the expected co-occurrence of pairs of words in a document.

Overall, we have an end-to-end spectral learning algorithm for LDA, based on matrix/tensor decomposition. This algorithm is guaranteed to consistently recover LDA’s model parameters. We introduce an algorithmic flow graph that illustrates our spectral learning algorithm in figure [1] where each node corresponds to an intermediate objective required for a final output estimation and each edge \( e \in \{ e_i \}_{i=0}^9 \) denotes certain operation required as a step of the spectral learning algorithm.

Although the spectral algorithm enjoys a provable guarantee in learning LDA, the output of this method could leak sensitive information, which limits the applicability of LDA in legal, financial and medical domains. For instance, consider a situation in which a sensitive document corpus \( D \) is kept hidden, but an adversary can obtain the output of the spectral algorithm on \( D \). If an additional document \( d \) is added to the corpus, and the output changes for one topic \( t \), an adversary can then infer that \( d \) is related to \( t \). Differential privacy (DP) [10] is a general framework for quantifying leakage of private information from an algorithm. A generic method to convert an algorithm \( A \) to be differentially private is to add sufficient noise to \( A \)’s output.

The goal of this work is: (1) to introduce the first differentially private algorithm that is guaranteed to recover high quality estimates (guarantee consistency) of the LDA model parameters; and (2) to identify a mechanism that suffers least from utility loss under some level of differential privacy.

To achieve goal (1), we consider injecting noise to a subset \( E \) of edges \( \{ e_i \}_{i=0}^9 \) that separates the input and the output (a cut). When \( E \) is a cut, differentially privately releasing all nodes preceding the edges in \( E \) guarantees the overall differential privacy according to the composition theorem and the closure to post processing. For instance, adding noise to \( E = \{ e_0, e_2 \} \) guarantees no privacy as the non-private information could flow to the output through the path below. However adding noise to the output \( \{ e_9 \} \), for example, guarantees overall differential privacy as long as the global sensitivity of \( \{ e_9 \} \) is bounded\(^1\). We call such a subset of edges as a “configuration” if adding noise to all edges in this configuration guarantees differential privacy of the overall algorithm. Four configurations are identified as shown in Section 6 “Differentially Private Spectral Algorithm”.

The amount of noise needed to achieve same level of differential privacy is different across edges as the sensitivities of edges vary, and thus utility losses caused by injecting the noise vary across edges. Depending on which configuration we choose, we may have quite different utility measures. To obtain goal (2) and solve the problem of where to add the noise for least utility loss to guarantee \(( \epsilon, \delta )\)-differential privacy, we characterize the sensitivity of all the edges \( \{ e_i \}_{i=0}^9 \) and utility bounds for all configurations with respect to the input. We then identify corresponding configuration that outperforms others in any regime.

Related Works In Appendix 2 we provide context on popular LDA methods, and provide a primer on differential privacy and tensor decomposition.

Comparison of our method against differentially private variational inference. The proposed approach is advantageous over differentially private VI as it (1) retains consistency guarantees, (2) is computationally more efficient, (3)

\(^1\)The global sensitivity is not bounded for \( \{ e_9 \} \), unfortunately.
achieves higher accuracy in synthetic experiments, moreover, (4) does not require performing composition across multiple iterations. However, the proposed approach (1) is less data efficient and (2) does not report posterior uncertainty in the parameter estimates.

Summary of Contributions
(1) We illustrate the computation graph of our spectral algorithm for LDA in figure[1] based on which we list subsets of edges as configurations, on which noise could be added to guarantee differential privacy according to the composition theorem.
(2) We bound the sensitivity of releasing various intermediate quantities on the computational graph, which leads to a number of methods for achieving both the pure-$\epsilon$-DP and the approximate $(\epsilon, \delta)$-DP. In the cases when the global sensitivity is unbounded, we come up with a data-dependent DP that exploits the small local sensitivity.
(3) We achieve consistency guarantee of the algorithm by characterizing the utility loss guarantees between the true parameters and the differentially private parameters we obtain. This result is stronger than traditional utility loss between the non-differentially private parameters and the differentially private parameters.
(4) Overall, we introduce a systematic analysis of differentially private spectral algorithm for LDA and obtain the regimes under which adding noise to each configuration guarantees the best performance. Using this framework, we demonstrate multiple mechanisms, which permit differentially private algorithms whose utilities are advantageous in different regimes as listed in Remark[13][15] and[16].
(5) Empirical studies confirm that our method systematically outperforms differentially private variational inference.

2 A. Related Work

Wang et al. [25] establishes an impressive result of frequentist consistency and asymptotic normality of VB methods, in which the consistency is based on the assumption of achieving the optimal variational posterior. However the feasibility of achieving such optimal variational posterior (which requires global optimal solution over a non-convex ELBO objective) remains unclear as the global optimality is not necessarily guaranteed. This work focuses on LDA parameter estimation based on spectral algorithms which, unlike EM-based algorithms[18][17], guarantee parameter recovery if a mild set of assumptions are met [1][2]. The spectral estimation method relies on matrix decomposition and tensor decomposition methods. Thus, differentially private PCA and tensor decomposition are related to our objective.

Differentially private PCA is an established topic, and $(\epsilon, 0)$ differentially private PCA was achieved using the exponential mechanism in [7][15]. The algorithm in [15] provides guarantees but with complexity $O(d^6)$; in contrast, [7] introduces an algorithm that is near optimal but without an analysis of convergence time. Although $(\epsilon, \delta)$ differential privacy is a more loose definition of differential privacy, it leads to better utility. Comparative experimental results show that the $(\epsilon, \delta)$ PCA algorithm of [14] outperform $(\epsilon, 0)$ significantly, and [12] introduce a simple input perturbation algorithm which achieves near optimal utility. In our work, we follow the $(\epsilon, \delta)$ definition and use [12] to obtain a differentially private matrix decomposition when needed.

Differentially private tensor decomposition is studied in [24] with an incoherence basis assumption. It is not clear the extent to which such an assumption holds in topic modeling. Only utility bounds are proved for the top eigenvector in [24]. The authors exclude the possibility of input perturbation as that causes the privacy parameter to be lower bounded by the dimension $(\epsilon = \Omega(d))$ which is prohibitive. However, the same analysis on the tensor of a reduced dimension would conclude that $\epsilon = \Omega(k)$, which is acceptable for a reduced dimension whitened tensor as $k \ll d$.

3 Preliminaries and Notations

Latent Dirichlet Allocation is characterized by two model parameters: $\alpha$, the dirichlet parameter of the topic prior, and $\mu$, the topic word matrix. $\alpha$ parameterizes a dirichlet distribution, which determines the topic mixture in each document. $\mu$ controls the word distribution per topic. We provide a detailed explanation of LDA in Appendix[3]. We use $d$ to denote the number of distinct words in a vocabulary, $N$ to denote the total number of documents, $k$ to denote the number of topics. The topic prior Dirichlet distribution is parameterized by $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $\alpha_0 = \sum_{i=1}^{k} \alpha_i$. For each document $n$, topic proportion is $\theta_n$, document length is $l_n$, and word frequency vector is denoted as $c_n$. Word
tokens are denoted by \( x \). Let \( D, D' \) be two datasets. We say datasets \( D \) and \( D' \) are adjacent (denoted by \( D \sim D' \)) if we can form \( D' \) by replacing exactly one document from \( D \).

**Definition 1 ((\( \epsilon, \delta \))-Differential Privacy).** Let \( A : D \rightarrow Y \) be a randomized algorithm. If \( \forall D \sim D', \forall S \subseteq Y \Pr[A(D) \in S] \leq e^\epsilon \Pr[A(D') \in S] + \delta \), then \( A \) is \( (\epsilon, \delta) \)-differentially private.

**Definition 2 (Local / Global Sensitivity).** The local sensitivity \( \Delta_f(D) := \max_{D \sim D'} ||f(D) - f(D')|| \) and the global sensitivity \( \Delta_f := \max_D \Delta_f(D) \). \( \ell_p \) norm corresponds to \( \ell_p \) sensitivity.

In Appendix A, we review utility loss & error, the Gaussian mechanism and the composition theorem.

## 4 Differentially Private LDA Topic Model

The commonly used variational inference method, which optimizes over a likelihood lower bound, provides no consistency guarantee due to the non-convexity of the likelihood function. To achieve a guarantee on the utility for learning LDA, we use a spectral algorithm via matrix/tensor decompositions - the only existing method that provides a guarantee on the performance with enough documents. For the LDA model, we define the first, second, and third order LDA moments in Lemma 3. Using the properties of LDA, we achieve unbiased estimators of the LDA parameters by decomposing these moments into factors that correspond to each \( \mu_i \), formalized in Lemma 3. Therefore, as long as we empirically estimate the moments \( M_1 \), \( M_2 \), and \( M_3 \) without bias, we obtain the model parameters \( \alpha \) and \( \mu \) via tensor decomposition on the empirically estimated moments.

**Lemma 3 (LDA moments and Moment Decompositions Recover Model Parameters).** Let random variables \( x_1 \), \( x_2 \) and \( x_3 \) denote the first, second and third tokens in a document. Tokens are represented as one-hot encodings, i.e., \( x_1 = e_0 \) if the first token is the \( v \)-th word in the dictionary. We define the first, second, and third order moments of LDA \( M_1 \), \( M_2 \) and \( M_3 \) as \( M_1 \overset{\text{def}}{=} E[x_1], M_2 \overset{\text{def}}{=} E[x_1 \otimes x_2] - \frac{\alpha_0}{\alpha_0 + 1} E[x_1] \otimes E[x_1] \) and \( M_3 \overset{\text{def}}{=} E[x_1 \otimes x_2 \otimes x_3] + \frac{2\alpha_1^2}{(\alpha_0 + 1)(\alpha_0 + 2)} E[x_1] \otimes E[x_1] \otimes E[x_1] - \frac{1}{\alpha_0 + 2} \left( E[x_1 \otimes x_2 \otimes E[x_3]] + E[x_1 \otimes E[x_2] \otimes x_3] + E[E[x_1] \otimes x_2 \otimes x_3]\right) \). The LDA moments relate to the model parameters \( \alpha \) and \( \mu \) through matrix/tensor decomposition as follows:

\[
M_1 = \sum_i \frac{\alpha_i}{\alpha_0} \mu_i, \quad M_2 = \sum_i \frac{\alpha_i}{\alpha_0(\alpha_0 + 1)} \mu_i \otimes \mu_i, \\
M_3 = \sum_i \frac{2\alpha_i}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)} \mu_i \otimes \mu_i \otimes \mu_i. \tag{1}
\]

Proof is in Appendix C. Note that \( \alpha_0 \) is pre-specified and thus data-independent. Using the properties of LDA, the moments are decomposed as factors shown in Lemma 3 and the factors \( \mu_i \) correspond to the LDA model parameters we aim to estimate. According to Lemma 3, decomposing on matrix \( M_2 \) only will not result in correct recovery of \( \mu_i \) as there are no unique \( \mu_i \)’s unless \( \mu_i \perp \mu_j \) and \( \alpha_i \neq \alpha_j \). The word distributions under different topics are only linearly independent instead of orthogonal. However, tensor decomposition on \( M_3 \) will yield a unique decomposition [3].

**Method of Moments & Tensor Decomposition** Inspired by Lemma 3, we conclude that tensor decomposition on \( M_3 \) will result in consistent estimation of the LDA parameters \( \alpha \) and \( \mu_i \). We have no access to population moments \( M_1 \), \( M_2 \) and \( M_3 \), but do have access to word frequency vectors \( c_n \).

To solve this problem, we empirically estimate the moments \( M_1 \), \( M_2 \), \( M_3 \) as in Equations 3, 6, and 7 given the observations of word frequency vectors \( c_n \) and obtain the model parameters \( \alpha \) and \( \mu \) by implementing tensor decomposition on those empirically estimated moments. In Lemma 3 in Appendix C, we prove that the empirical moment estimators are unbiased.

The method of moments uses the property of data moments of the LDA model (in Lemma 3) to estimate the parameters of topic model \( \alpha \) and \( \mu_i \), \( \forall i \in k \). The algorithm flow is depicted in Figure 1 and consists of the following steps: (1) Using \( c_n \) for document \( \forall n \in [N] \), estimate \( M_2 \) and \( M_3 \) using equation (6) (\( c_0 \) in Figure 1) and equation (7)
(c1 in Figure 1). (2) Apply SVD on \( \hat{M}_2 \) to obtain an estimation of the whitening matrix \( \hat{W} \equiv \hat{U}\hat{\Sigma}^{-\frac{1}{2}} \), where \( \hat{U} \) and \( \hat{\Sigma} \) are the top \( k \) singular vectors and singular values of \( M_2 \). (3) Whiten the tensor \( \hat{\mathcal{T}} = M_3(\hat{W}, \hat{W}, \hat{W}) \) using multilinear operations\(^2\) on \( M_3 \) with \( e_3 \) and \( e_4 \) in Figure 1. (4) Implement tensor decomposition on the whitened tensor \( \hat{\mathcal{T}} \) and denote the resulting eigenvectors as \( \hat{\mu}_i, \forall i \in [k] \). (5) Obtain the unwhitening matrix \( \hat{W}^\dagger = \sum_{i=0}^{k-1} \hat{\Sigma}^{-\frac{1}{2}} \hat{U} \) \( e_5 \) in Figure 1. (6) Un-whiten the singular vectors to obtain LDA parameters: \( \hat{\mu}_i \propto (\hat{W}^\dagger)\top \hat{\mu}_i \) and \( \hat{\alpha}_i, \forall i \in k \). \( e_7 \) and \( e_8 \) in Figure 1.

Our end-to-end spectral algorithm guarantees the correct learning of topic models (see Lemma 25).

**Differentially Private LDA Problem Statement** We assume that the corpus of data is held by a trusted curator and that an analyst will query for the parameters of the topic model. The curator has to output the model parameters \( \alpha_i, \mu_i \) in a differentially private manner with respect to the documents. While it is easy to achieve differential privacy, the challenge is in guaranteeing high utility. We will use the Gaussian mechanism described in Proposition 18 in this paper to achieve \((\epsilon, \delta)\)-differentially private topic modeling for each of the configurations. We will compute sensitivities of edges in each configuration in Section 5 to obtain the noise level that must be added to each edge. Our derived sensitivity and utility loss results are demonstrated in Section 6 “Differentially Private Spectral Algorithm”.

## 5 Sensitivity of Algorithms in Node Flow

We will use \( \Delta_x \) to denote the sensitivity of \( \hat{M}_2 \) if \( x = 2 \), \( \hat{M}_3 \) if \( x = 3 \), \( \hat{\mathcal{T}} \) if \( x = \hat{\mathcal{T}} \). A key challenge is where to add noise in the data flow shown in Figure 1. First, we calculate the sensitivities at the different nodes of the data flow graph. Then, we consider various options and establish the utilities for different possible noise addition configurations in Section 6 “Differentially Private Spectral Algorithm”.

In the following theorems, the exact forms of the sensitivity and the proofs are in appendix H. We note that the \( \ell_1 \) sensitivity bounds the \( \ell_2 \) sensitivity, similar to how \( \|x\|_2 \leq \|x\|_1 \) for any given vector \( x \). At times we only bound the \( \ell_1 \) sensitivity which in turn bounds \( \ell_2 \).

**Theorem 4** (Global sensitivity of second and third order LDA moments). Let \( \Delta_2 \) and \( \Delta_3 \) be the \( \ell_1 \) sensitivities for \( \hat{M}_2 \) and \( \hat{M}_3 \) respectively. Both \( \Delta_2 \) and \( \Delta_3 \) are upper bounded by \( \frac{k^{1.5}}{N} \).

**Theorem 5** (Local sensitivity of the whitened tensor \( \hat{\mathcal{T}} \)). The \( \ell_1 \) sensitivity of the whitened tensor \( \hat{\mathcal{T}} \), denoted as \( \Delta_{\hat{\mathcal{T}}} \), is upper bounded by \( \Delta_{\hat{\mathcal{T}}} = O\left(\frac{k^{1.5}}{N\sigma_k(M_2)^{1.5}}\right) \).

**Theorem 6** (Local sensitivity of the output of tensor decomposition \( \hat{\mu}_i, \hat{\alpha}_i \)). Let \( \hat{\mu}_1, \ldots, \hat{\mu}_k \) and \( \hat{\alpha}_1, \ldots, \hat{\alpha}_k \) be the results of tensor decomposition before unwhitening. The sensitivity of \( \hat{\mu}_i \), denoted as \( \Delta_{\hat{\mu}_i} \), and the sensitivity of \( \hat{\alpha}_i \), denoted as \( \Delta_{\hat{\alpha}_i} \), are both upper bounded by \( O\left(\frac{k^2}{\gamma_sN\sigma_{\hat{\mu}_i}(M_2)^{1.5}}\right) \), where \( \gamma_s = \min_{i \in [k]} \frac{\sigma_i(\hat{\mathcal{T}}) - \sigma_{i+1}(\hat{\mathcal{T}})}{4} \).

**Theorem 7** (Local sensitivity of the final output \( \hat{\mu}_i, \hat{\alpha}_i \)). The sensitivities \( \Delta_{\hat{\mu}_i} \) and \( \Delta_{\hat{\alpha}_i} \) of the final output are upper bounded by \( O\left(\frac{k^2\sqrt{\gamma_s\sigma_{\hat{\mu}_i}(M_2)}}{\gamma_{sN}\sigma_k\mu_{\hat{\alpha}_i}(M_2)}\right) \).

**Remark.** The sensitivities before the whitening are \( O\left(\frac{k^{1.5}}{N\sigma_k(M_2)^{1.5}}\right) \). The whitening step increases the sensitivity by \( O\left(\frac{k^{1.5}}{N\sigma_k(M_2)^{1.5}}\right) \). Further, the simultaneous power method for tensor decomposition increases the sensitivity by \( \frac{k^2}{\gamma_s} \), leading to \( O\left(\frac{k^2}{\gamma_sN(\sigma_k(M_2)^{1.5})}\right) \). The unwhitening increases the sensitivity by \( \sqrt{\sigma_1(M_2)} \), leading to \( O\left(\frac{k^2\sqrt{\gamma_s\sigma_{\hat{\mu}_i}(M_2)}}{\gamma_{sN}\sigma_k\mu_{\hat{\alpha}_i}(M_2)}\right) \).

\(^2\)The \((i, j, k)\)-th entry of the multilinear operation \( M_3(\hat{W}, \hat{W}, \hat{W}) \) is \( \sum_{m,n,l} [M_3]_{m,n,l}W_{m,i}W_{n,j}W_{l,k} \). Since \( \hat{W} \) is a \( d \times k \) matrix and \( M_3 \) is a \( d \times d \times d \) tensor, \( \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) \) is a \( k \times k \times k \) tensor.
Calibrating Noise

Using Lemma 8 and Lemma 9, we describe an algorithm that guarantees $(\sigma, 0)$-DP output of the configuration: $f$(DATA)

The sensitivities of $f$ are small, and release $\sigma_k(\hat{M}_2)$ with $\sigma_k$ in LS

\begin{align*}
5.1 \text{ Data-dependent Privacy Calibration}
\end{align*}

Theorem 5 and Lemma 6 are local sensitivities, which are functions of the input data set. It is well-known that adding noise proportional to the local sensitivity does not guarantee differential privacy as the local sensitivity may be sensitive to adding or removing of individuals from the dataset and lead to the identification of individuals.

Two seminal solutions to this problem include the smooth sensitivity framework [16] and the propose-test-release (PTR) framework [9]. The idea of the smooth sensitivity framework is to construct a smooth upper bound of the local sensitivity that is insensitive and to calibrate noise with a heavier tail that satisfies certain “dilation” and “shift” properties to achieve pure-DP. The PTR framework involves proposing bounds of the local sensitivity and testing its validity. If the test is passed, we calibrate the noise according to the proposed test. PTR is often easier to use but can only provide an $(\epsilon, \delta)$-DP with $\delta > 0$.

In our problem, the smooth sensitivity itself is unbounded, thus we cannot apply the smooth sensitivity framework naively. Instead, we use a variant of propose-test-release framework that releases a confidence bound of the local sensitivity in a differentially private manner, and calibrates noise accordingly, similar to the idea in [6] and a more recent example in the context of data-adaptive differentially private linear regression [26]. We formalize the idea using the following lemma.

\begin{lemma}
Let LS be the $\ell_p$ local sensitivity of a function $f$ on a fixed data set. Let LS obeys $(\epsilon_1, 0)$-DP and that $p(\text{LS} > \text{LS}) \leq \delta_1$ (where the probability is only over the randomness in releasing LS). Then the algorithm releases $f$(DATA) + $Z(\epsilon, \delta, \text{LS})$ that is $(\epsilon_1 + \epsilon, \delta_1 + \delta)$-DP, where $Z(\epsilon, \delta, \text{LS})$ is any way of calibrating the noise for privacy (for Gaussian mechanism $Z(\epsilon, \delta, \text{LS}) = N(0, \frac{21^2 \log(1.25/\delta)}{\epsilon_2})$).
\end{lemma}

The proof is in Appendix H.6. In our problem, the local sensitivities depend on the data only through $\sigma_k(\hat{M}_2)$ and $\gamma_s$. A natural idea would be to privately release $\sigma_k(\hat{M}_2)$ and $\gamma_s$ and construct a high-confidence upper bound of the local sensitivity through a high-confidence lower bound of $\sigma_k(\hat{M}_2)$ and $\gamma_s$. We will show the global sensitivities of $\sigma_k(\hat{M}_2)$ and $\sigma_i(\hat{T})$ are small, and release $\sigma_k(\hat{M}_2)$ and $\sigma_i(\hat{T})$ differentially privately.

\begin{lemma}
(Global Sensitivity of $\sigma_k(\hat{M}_2)$ and $\gamma_s$). The sensitivities of $\sigma_k(\hat{M}_2)$ and $\gamma_s$ are each $2/N$.
\end{lemma}

The proof is in Appendix H.7.

Calibrating Noise Using Lemma 8 and Lemma 9, we describe an algorithm that guarantees $(\epsilon_1 + \epsilon', \delta_1 + \delta')$-DP under local sensitivity LS in Procedure 1.
6 Differentially Private Spectral Algorithm

In Figure 1 each node corresponds to an intermediate objective required for a final output estimation and each edge denotes certain operation required as a step of the spectral learning algorithm. We consider injecting noise to a subset $E$ of edges $\{e_i\}_{i=0}^7$ that separates the input and the output (a cut). When $E$ is a cut, differentially privately releasing all nodes preceding the edges in $E$ guarantees the overall differential privacy according to the composition theorem and the closure to post processing. We call such a subset of edges as a “configuration” if adding noise to all edges in this configuration guarantees differential privacy of the overall algorithm.

In this section, we denote $\gamma_i = \frac{\sqrt{2\ln 2/\alpha_i}}{\epsilon_i}$. Further, $\delta_k$ and $\bar{\gamma}_s$ are determined by a choice of $\epsilon_i, \delta_1$ and $(\epsilon_1, \delta_1')$ according to Procedure 1 in what follows, if noise is added to edge $e_i$, then $e_i$ refers to the associated differential privacy parameter.

Four configurations are identified.

6.1 Config. 1 ($e_3, e_4, e_8$): Perturbation on $\hat{M}_2, \hat{M}_3$ for private $\hat{M}_3, \hat{W}, \hat{W}^\dagger$

In Config. 1, since ($e_3, e_4, e_8$) is a cut that separates the input and the output, we add Gaussian noise $\mathcal{N}(0, \frac{1}{\sqrt{2\gamma_i}}\tau_{e_i, \delta_i})$ on $\hat{M}_3$ to ensure ($e_3, \delta_3$)-DP $\hat{M}_3$ for edge $e_3$, noise $\mathcal{N}(0, \frac{1}{\sqrt{2\gamma_i}}\tau_{e_4, \delta_4})$ on $\hat{M}_2$ to ensure ($e_4, \delta_4$)-DP $\hat{W}$ for edge $e_4$, and also noise $\mathcal{N}(0, \frac{1}{\sqrt{2\gamma_i}}\tau_{e_8, \delta_8})$ on $\hat{M}_2$ to ensure ($e_8, \delta_8$)-DP $\hat{W}^\dagger$ for edge $e_8$.

Config. 1 has a bounded global sensitivity $O(1/N)$ and allows a pure-DP if we add Laplace noise using Laplacian mechanism.

6.2 Config. 2 ($e_6, e_8$): Perturbation on $\hat{T}$ and $\hat{M}_2$ for private $\hat{M}_2, \hat{W}^\dagger$

In Config. 2, since ($e_6, e_8$) is a cut that separates the input and the output, we add Gaussian noise $\mathcal{N}(0, \frac{k^3}{\sqrt{2\gamma_i}}\tau_{e_6, \delta_6})$ on $\hat{T}$ to ensure ($e_6, \delta_6$)-DP $\hat{T}$ for edge $e_6$, $\mathcal{N}(0, \frac{1}{\sqrt{2\gamma_i}}\tau_{e_8, \delta_8})$ on $\hat{M}_2$ to ensure ($e_8, \delta_8$)-DP $\hat{W}^\dagger$ for edge $e_8$.

In Config. 2 the whitening matrix results from a noiseless $\hat{M}_2$, but the pseudo-inverse results from a noisy $\hat{M}_2$. We add noise to a tensor of a smaller dimension, at the expense of an increased sensitivity by a factor of $\frac{k^{3/2}}{\sqrt{\gamma_i}}(\hat{M}_2)$.

To guarantee utility, we need $\epsilon_6 = \Omega(\frac{\sqrt{n_2 \gamma_1(T)k^{3/2}}}{N\gamma_i(\hat{M}_2)})$ and $\epsilon_8 = \Omega(\frac{\sqrt{d}}{(\sigma_k(\hat{M}_2) - \sigma_{k+1}(\hat{M}_2))N})$. The dependence on $\sqrt{d}$ still remains however, as it originates from adding noise to $\hat{M}_2$ which is still done for $\hat{W}^\dagger$.

6.3 Config. 3 ($e_7, e_8$): Perturbation on $\hat{\mu}_i, \alpha_i$ and $\hat{M}_2$ for private $\hat{\mu}, \hat{W}^\dagger$

In Config. 3, since ($e_7, e_8$) is a cut that separates the input and the output, we add Gaussian noise $\mathcal{N}(0, \frac{k^4}{\sqrt{2\gamma_i}}\tau_{e_7, \delta_7})$ on $\hat{\mu}_i$ and $\alpha_i$ to ensure ($e_7, \delta_7$)-DP $\hat{\mu}_i$ and $\alpha_i$ for edge $e_7$, and noise $\mathcal{N}(0, \frac{1}{\sqrt{2\gamma_i}}\tau_{e_8, \delta_8})$ on $\hat{M}_2$ to ensure ($e_8, \delta_8$)-DP $\hat{W}^\dagger$ for edge $e_8$.

This configuration adds noise to the output of the simultaneous tensor power method and thus the sensitivity after the output of the simultaneous power iteration increases by a factor of $\frac{1}{\gamma_i}$ compared to Config. 2. However, according to utility loss guarantees in Theorem 11 and 12 the dependence on $k$ in the last term drops from $k^{2.5}$ to $k^2$ compared to Config. 2. This is because although the previous configuration adds noise before the decomposition at a lower sensitivity, the error in the output grows by a factor of $\frac{\sqrt{\gamma_i}}{\gamma_i}$.

6.4 Config. 4 ($e_9$): Perturbation on output $\mu_i, \alpha_i$ for private $\hat{\mu}$

The last option we consider is to add noise to the final output. In Config. 4, since ($e_9$) is a cut that separates the input and the output, we add Gaussian noise $\mathcal{N}(0, \frac{k^5}{\sqrt{2\gamma_i}}\tau_{e_9, \delta_9})$ on $\mu_i, \alpha_i$ to ensure ($e_9, \delta_9$)-DP $\hat{\mu}_i$, $\alpha_i$ for edge $e_9$. 
This method is arguably the simplest, as the previous configurations involve the composition of multiple differentially private outputs whereas this method only adds noise to one branch. Adding noise to $\epsilon_0$ instead of $\epsilon_T$ means that the noise vector increases in dimension from $k$ to $d$ which makes the utility loss larger.

Though it is possible to perform input perturbation, we exclude this option because the $l_2$ sensitivity is $\sqrt{2L}$ (where $L$ is the length of the longest document) which does not decay with the number of records. Therefore the utility of input perturbation is poor even with many records.

Figure 2: Error of our method under all configurations vs the differentially private VI over varying composite $\epsilon$ while fixing the composite $\delta = 10^{-7}$ using $N = 100k$ documents. vi-u and unnoised denote the non-differentially private version of variational inference and our spectral algorithm. Config. 3 overlaps with config. 4 and thus is hardly visible.

6.5 Utility Guarantees

For each configuration, we compute the noise needed to obtain $(\epsilon, \delta)$ differential privacy based on sensitivity, thereby characterizing the utility with necessary noise. The utility of each configuration is listed in Theorems 10, 11, 12 and 13. Proofs of all utility derivations are in Appendix I.

Theorem 10 (Config. 1 Utility Loss). The utility loss $\|\mu_i - \mu_i^{\text{DP}}\|$ using Config. 1 to guarantee $(\epsilon_3 + \epsilon_4 + \epsilon_5, \delta_3 + \delta_4 + \delta_5)$-DP is:

$$O(\sqrt{\frac{\sigma_1(M_2)^k}{\gamma_s(N \sigma_k(M_2))^{3/2}}} \tau_{e, \delta_1} + \sqrt{\frac{\sigma_1(M_2)d}{\sigma_k(M_2)N}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_v}{\sigma_k(M_2)\gamma_s(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s\gamma_v}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8}^3 + \sqrt{\frac{\sigma_1(M_2)\gamma_v}{\sigma_k(M_2)\gamma_s(N \sigma_k(M_2))}} \tau_{e, \delta_8}^3).$$

Theorem 11 (Config. 2 Utility Loss). The utility loss $\|\mu_i - \mu_i^{\text{DP}}\|$ using Config. 2 to guarantee $(\epsilon_3 + \epsilon_4 + \epsilon_5)$-DP is:

$$O(\sqrt{\frac{\sigma_1(M_2)k^2}{\gamma_s(N \sigma_k(M_2))^{3/2}}} \tau_{e, \delta_1} + \sqrt{\frac{\sigma_1(M_2)d}{\sigma_k(M_2)N}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_v}{\sigma_k(M_2)\gamma_s(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s\gamma_v}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8}).$$

Theorem 12 (Config. 3 Utility Loss). The utility loss $\|\mu_i - \mu_i^{\text{DP}}\|$ using Config. 3 to guarantee $(\epsilon_3 + \epsilon_4 + \epsilon_5)$-DP is:

$$O(\sqrt{\frac{\sigma_1(M_2)k^2}{\gamma_s(N \sigma_k(M_2))^{3/2}}} \tau_{e, \delta_1} + \sqrt{\frac{\sigma_1(M_2)d}{\sigma_k(M_2)N}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_v}{\sigma_k(M_2)\gamma_s(N \sigma_k(M_2))}} \tau_{e, \delta_8}).$$

Theorem 13 (Config. 4 Utility Loss). The utility loss $\|\mu_i - \mu_i^{\text{DP}}\|$ using Config. 4 to guarantee $(\epsilon_3 + \epsilon_4 + \epsilon_5)$-DP is:

$$O(\sqrt{\frac{\sigma_1(M_2)k^2}{\gamma_s(N \sigma_k(M_2))^{3/2}}} \tau_{e, \delta_1} + \sqrt{\frac{\sigma_1(M_2)d}{\sigma_k(M_2)N}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_s}{\sigma_k(M_2)\gamma_v(N \sigma_k(M_2))}} \tau_{e, \delta_8} + \sqrt{\frac{\sigma_1(M_2)\gamma_v}{\sigma_k(M_2)\gamma_s(N \sigma_k(M_2))}} \tau_{e, \delta_8}).$$

6.6 Comparison of Configurations

The utility loss is non-vacuous if the number of data points is $\geq \sqrt{\frac{1}{N}}$, as it depends on $O(\sqrt{1}/N)$. This sub-linear sample complexity dependence on dimension outperforms other methods. We present a pairwise comparison between the utilities of different configurations.
Remark 14. **Configuration 1 vs. 2:** The utility loss in config. 1 is high compared to config. 2 as the singular values of $\hat{M}_2$ are on the order of $\frac{1}{d}$. Config. 1 has a $\sqrt{d}$ factor higher utility loss compared to config. 2. Therefore, config. 2 is preferred over config. 1 in practice.

Remark 15. **Configuration 2 vs. 3:** The utility loss for config. 3 is lower than that of config. 2 by a factor of $k^{0.5}$ in the last term of the utility losses, assuming the same level of differential privacy. However, config. 3 has the extra requirement that $\Delta_3 \leq \frac{\gamma \sigma_k(\hat{T})}{2dK}$. Therefore the utility of config. 3 outperforms that of config. 2 only if the constraint is met. The advantage is enlarged when $k$ is large.

Remark 16. **Configuration 3 vs. 4:** The first two terms of utility loss in config. 3 are smaller than that in config. 4. In the regime of $N > \sqrt{d}$, config. 3 is preferred as the third term of config. 3 is smaller than that of config. 4.

7 Experiments

![Visualization of (a) the $k^{th}$ singular values of $\hat{M}_2$ and (b) the smallest singular value gap of $\hat{T}$ using 100k documents.](image)

Figure 3: Visualization of (a) the $k^{th}$ singular values of $\hat{M}_2$ and (b) the smallest singular value gap of $\hat{T}$ using 100k documents.

The main focus of the paper is on providing the first differentially private topic model with well understood and theoretically guaranteed utility. We simulate documents from an LDA model parameterized by varying choice of $\alpha$ and $\mu$ which are randomly sampled to ensure that a bursty use of a single word under certain topic is possible in our experiment. Therefore our setting covers a wide range of hyper-parameters and captures some common irregularities in distributional properties. Our synthetic setting allows for direct calculation of error on parameter recovery, which is not feasible in real data. We compare the empirical loss of each configuration in different settings. In addition, we compare all configurations of our spectral algorithm against differentially private variational inference [17] under the same settings. Our algorithm universally outperforms the state-of-the-art VI quantitatively.

Evaluation Metric: Our experiments evaluate the loss between the ground-truth $\mu$ and the estimated $\hat{\mu}$ via a $(\epsilon, \delta)$ differentially private algorithm across varying privacy parameters (composite $\epsilon$). For each edge set and a given composite $\epsilon$, we perform a grid search over the privacy parameters for each edge in the set to select the optimal combination. When working with real data (e.g., Wikipedia), a maximum likelihood criterion could be used to select this optimal configuration. We release only differentially private likelihoods by perturbing sufficient statistics, as described in [17]. See Appendix J for a more detailed explanation of this process.

VI vs Spectral: Figure 2 exhibits error for varying composite $\epsilon$ on different datasets. Under every configuration, our differentially private spectral algorithm universally outperforms differentially private variational inference, and has higher utility under the same level of privacy.

Small $\epsilon$ vs Large $\epsilon$: From figure 2a, config 2 performs better for low $\epsilon$, while Config. 1 performs better for larger $\epsilon$. Config. 3 and 4 perform universally worse than the other Configs as the number of topics is small.
Small $\alpha_0$ vs Large $\alpha_0$: In Figure 2c, config 1 performs the best. This is consistent with theoretical findings, as $\alpha_0$ is large and $N$ exceeds $d^2$. Config. 2 performs on par or worse than config. 1; this is partially due to $\hat{M}_2$ having small singular values for this dataset, driving up the amount of noise added.

Small Corpus vs Large Corpus: Figure 2b considers the limited data setting, $N = 1000$. Config 2 emerges as better when $\alpha_0$ is large.

$\hat{M}_2$’s singular values vs $\alpha_0$: The singular values of $\hat{M}_2$ are positively correlated with the $\alpha_0$ parameter. Figure 3a shows this correlation for large $N$. Consequently, we observe in figure 2c for $N = 100k$ and $\alpha_0 = 1000$, config. 2 has lower error. Low $\alpha_0$ yields documents polarized to single topics, while as $\alpha_0 \to \infty$, the topics mix. This yields increased variation among the singular values of $\hat{M}_2$, which drives down config 2’s sensitivity.

Singular value gap vs number of topics: The theoretical results posit that config 3 would likely be the best performing, but this isn’t the case. This is likely due to the small singular value gap of $\gamma_s$. However, as we increase $k$ shown in Figure 3b, the gap increases (Figure 3b), which lowers the sensitivity along config. 3, leading to lower noise addition.

7.1 Wikipedia Dataset

We verified good performance of our method on the wikipedia dataset. Preprocessing was minimal - removing all non-alphanumeric characters and lower-casing. We experienced results of differing quality as we changed $\epsilon$.

As shown in Figure 4, where the quantitative results (perplexity scores) on Wikipedia are compared with variational inference, our method suffers from less utility loss under the same privacy levels. As we observe in the Wiki results in Figure 4, performance of config.3 is improved under larger number of topics $k$, confirming our theory.

8 Conclusion

We have provided an end-to-end analysis of differentially private LDA model using a spectral algorithm. The algorithm involves a dataflow that permits different locations for injecting noise. We present a detailed sensitivity and utility analysis for different differentially private configurations.

We show that no configuration dominates and recommend configurations for different scenarios. Config 1 is preferable when $N \gg k$, or when the topics are highly polarized in documents (different documents do not have many topics in common). Config 2 is preferable when $N$ is small, and the topics are mixed in the dataset. Config 3
is preferable when \( k \) is large. Additionally, we identified an interesting correlation between the singular values of \( M_2 \) and \( \alpha_0 \), specifically that increasing the mixture of the topics in the documents leads to higher singular values.

The analysis that was used can be extended to other latent variable models where the parameters are estimated using similar spectral methods such as Gaussian mixtures and Hidden Markov Models [3].

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Appendix: An end-to-end Differentially Private Latent Dirichlet Allocation Using a Spectral Algorithm

A  B. Differential Privacy Review

Definition 17 (Utility Loss & Error). Let \( f : D \to Y \) be a random algorithm and \( f^{\text{DP}}(X) \) be the differentially private version of \( f \). For some value \( x \in D \), let \( y \in Y \) be the ground truth value. Then define \( \| f(x) - f^{\text{DP}}(X) \|_F \) as the utility loss for this input. Additionally, define \( \| y - f^{\text{DP}}(X) \|_F \) as the error for this input.

The Gaussian mechanism proposed in [8] makes a random algorithm differentially private by adding specifically designed Gaussian noise to the output.

Proposition 18. [Gaussian mechanism] Let \( f : D \to Y \) (\( Y \subset \mathbb{R}_k \)) be a random algorithm with \( \ell_2 \) sensitivity \( \Delta_f \).

Let \( g \in \mathbb{R}_k \) and each coordinate \( g_i \) be sampled i.i.d. from \( \mathcal{N}(0, \Delta_f^2) \), where \( \Delta_f, \epsilon, \delta \). Then the output \( f^{\text{DP}} = f + g \) is \((\epsilon, \delta)\) differentially private if \( 0 < \epsilon \leq 1 \).

The above bound is used for theoretical purposes only, a tighter and more general calibration of the Gaussian mechanism that does not require \( \epsilon \leq 1 \) was proposed in [4].

Composition theorem [11] provides insights on how the differential privacy is preserved under algorithm composition.

Proposition 19. [Composition theorem] Let \( f^{\text{DP}}_1(X), \ldots, f^{\text{DP}}_n(X) \) be \( n \) differentially private algorithms with privacy parameters \((\epsilon_1, \delta_1), \ldots, (\epsilon_n, \delta_n)\). Then \( g^{\text{DP}}(X) = f(f^{\text{DP}}_1(X), \ldots, f^{\text{DP}}_n(X)) \) is \((\epsilon_1 + \ldots + \epsilon_n, \delta_1 + \ldots + \delta_n)\) differentially private.

This is what we called a simple composition where epsilon increases linearly. There is an advanced composition where privacy loss for accessing for \( k \) times obey that \( \sqrt{k} \).

B  C. Latent Dirichlet Allocation

LDA, despite being a bag of words model, allows modeling of the mixed topics in a document to account for the more general case in which a document belongs to several different latent classes (topics) simultaneously. Latent Dirichlet Allocation has two major model parameters: topic prior \( \alpha \) and topic-word matrix \( \mu \). Topic prior \( \alpha \) determines the topic proportions and the topic-word matrix controls the word distribution per topic.

**Topic Proportions** The proportion of words in topics, known as *topic proportion* (denoted as \( \theta_n \) for document \( n \)), is drawn from a Dirichlet distribution (topic prior) parameterized by \( \alpha = (\alpha_1, \ldots, \alpha_k) \), with density \( P_n(\theta = \theta_n) = \frac{\Gamma(\alpha)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k \theta_n^{\alpha_i - 1} \), where \( \alpha_0 = \sum_{i=1}^k \alpha_i \).

**Topic-Word Matrix** Under a topic \( i \), tokens in the documents are assumed to be generated in a conditionally independent manner through \( \mu_i \), i.e., token \( x_1 \sim \text{Cat}(d, \mu_i) \) where \( \text{Cat}(d, \mu_i) \) denotes the categorical distribution. Under different topics, these conditional distributions \( \mu_i \) are linearly independent, \( \forall i \in [k] \).

With the definition of the two major parameters, we now describe the generative model of LDA topic model. The process involves generating topics first, followed by tokens.

**Topic Generation** LDA remains simple as each token in the corpus belongs to one of the \( k \) topics only, although tokens in the same document could belong to different topics. We denote the topic of token \( j \) in document \( n \) as \( z_{n,j} \).

Therefore, topics generated are categorical \( z_{n,j} \in [k] \) and distributed according to \( \theta_n \), i.e., \( z_{n,j} \sim \text{Cat}(k, \theta_n) \) where \( \text{Cat}(k, \theta_n) \) denotes the categorical distribution.
Word Generation Let $x$ denote the tokens. After determining the topic of the token $j$, $z_{n,j}$, token $j$ is generated conditionally independently through $\mu_{z_{n,j}}$, i.e., $x_j \sim \text{Cat}(d, \mu_{z_{n,j}})$. In a document $n$, if the $j^{th}$ token $x_{n,j'}$ is the $v$-th word in the dictionary, then $x_{n,j'} = e_v$, where $e_v$ is a one-hot encoding, i.e., $x_{n,j'}(j) = 0 \forall j \neq v$ and $x_{n,j'}(j) = 1$ if $j = v$. Let $l_n$ be the length of document $n$, random realizations of token $x$, i.e., $\{x_{n,j'}\}_{j'=1}^{l_n}$, are i.i.d.

Term-Document Matrix The term-document matrix $D \in \mathbb{N}^{d \times N}$. The $n^{th}$ column in $D$ is denoted by $c_n$, where its $j^{th}$ component $c_n(j) = \text{number of times word } j \text{ in the vocabulary appeared in document } n$. This means that $c_n = \sum_{j'=1}^{l_n} x_{n,j'}$ where $l_n$ is the number of words in document $n$. Clearly, $l_n = \sum_{j} c_n(j) = \|c_n\|_1$.

C D. Method of Moments for Latent Dirichlet Allocation

Empirical Moment Estimators The moments that we obtain are not the population moments but rather empirically estimated moments from the given data set. We list the forms of first, second, and third order empirical moment estimators for the single topic case as shown in [27]. Given a document $n$, the following quantities are calculated.

\[
\tilde{M}_1^n = \frac{c_n}{l_n}
\]

\[
\tilde{M}_2^n = \frac{1}{2(l_n^2)} (c_n \otimes c_n - \text{diag}(c_n))
\]

\[
\tilde{M}_3^n = \frac{1}{6l_n^3} \left( c_n \otimes c_n \otimes c_n + 2 \sum_{i=1}^{d} c_n(i)(e_i \otimes e_i \otimes e_i)ight)
\]

\[
- \sum_{i=1}^{d} \sum_{j=1}^{d} c_n(i)c_n(j)(e_i \otimes e_j + e_i \otimes e_j \otimes e_j + e_j \otimes e_i \otimes e_j)
\]

The empirically estimated moments are the averages of these quantities over the entire data set. Specifically,

Lemma 20. Single Topic Empirical Moment Estimators(Propositions 3 and 4 in [27])

\[
\hat{E}[x_1] = \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_1^n
\]

\[
\hat{E}[x_1 \otimes x_2] = \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_2^n
\]

\[
\hat{E}[x_1 \otimes x_2 \otimes x_3] = \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3^n
\]

Further these moments are unbiased, i.e.:  

\[
\mathbb{E}[\hat{E}[x_1]] = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_1^n\right] = \mathbb{E}[x_1]
\]

\[
\mathbb{E}[\hat{E}[x_1 \otimes x_2]] = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_2^n\right] = \mathbb{E}[x_1 \otimes x_2]
\]

\[
\mathbb{E}[\hat{E}[x_1 \otimes x_2 \otimes x_3]] = \mathbb{E}\left[\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3^n\right] = \mathbb{E}[x_1 \otimes x_2 \otimes x_3]
\]
Note that this lemma implies that: \( \mathbb{E}[\tilde{M}_1^n] = \mathbb{E}[x_1], \mathbb{E}[\tilde{M}_2^n] = \mathbb{E}[x_1 \otimes x_2], \) and that \( \mathbb{E}[\tilde{M}_1^n] = \mathbb{E}[x_1 \otimes x_2 \otimes x_3] \) for any sampled document \( n. \)

We extend the single topic moment estimators of [27] to the LDA case.

**Lemma 21.** Empirical Moment estimators for LDA

\[
\tilde{M}_1 = \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_1^n
\]

\[
\tilde{M}_2 = \frac{1}{N} \sum_{n=1}^{N} \left[ \tilde{M}_2^n \right] - \frac{a}{2} \left\{ \sum_{m, n=1}^{N} \tilde{M}_1^n \otimes \tilde{M}_1^m - \sum_{n=1}^{N} \tilde{M}_1^n \otimes \tilde{M}_1^n \right\}
\]

\[
\tilde{M}_3 = d \left[ \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3^n + B_1 + B_2 + B_3 + b \right]
\]

where

\[
B_1 \overset{\text{def}}{=} \frac{b}{2} \left( \sum_{n=1}^{N} \tilde{M}_2^n \right),
\]

\[
b \overset{\text{def}}{=} c \left( \sum_{n=1}^{N} \tilde{M}_1^n \right) \otimes \left( \sum_{n=1}^{N} \tilde{M}_1^n \right)
\]

\[
B_2 \text{ and } B_3 \text{ are formed by permuting, i.e., } [B_2]_{ij} = [B_1]_{ikj} \text{ and } [B_3]_{ijk} = [B_1]_{kij}. \text{ Further, } a = \frac{\alpha_n}{\alpha_n + 1}, b = \frac{-\alpha_n}{2(\alpha_n + 1)(\alpha_n + 2)}, c = \frac{2\alpha_n^2}{(\alpha_n + 1)(\alpha_n + 2)}, \text{ and } d = \frac{\alpha_n}{\alpha_n + 2}.
\]

Now we prove that these estimators are unbiased.

**Lemma 22 (The LDA Moment Estimators are Unbiased).** The estimators defined in definition [27] are unbiased, i.e.,

\[
\mathbb{E}[\tilde{M}_1] = M_1
\]

\[
\mathbb{E}[\tilde{M}_2] = M_2
\]

\[
\mathbb{E}[\tilde{M}_3] = M_3
\]

**Proof.** First order moment:

\[
\mathbb{E}[\tilde{M}_1] = \mathbb{E}\left[ \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_1^n \right] = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[\tilde{M}_1^n] = \frac{1}{N} \sum_{n=1}^{N} \mathbb{E}[\tilde{c}_n]
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} \frac{1}{l_n} \mathbb{E}[c_n] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{l_n} \mathbb{E}[\sum_{i=1}^{l_n} x_{n, i}] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{l_n} \sum_{i=1}^{l_n} \mathbb{E}[x_{n, i}]
\]

\[
= \frac{1}{N} \sum_{n=1}^{N} \frac{1}{l_n} \sum_{i=1}^{l_n} \mathbb{E}[x_{n, i}] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{l_n} l_n \mathbb{E}[x_{n, i}] = \frac{1}{N} \mathbb{E}[x_1] = \mathbb{E}[x_1] = M_1
\]

Second order moment: The first term of \( \tilde{M}_2 \) is actually the estimator the single-topic second order moment and
\[ \mathbb{E}\left[ \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_2^n \right] = \mathbb{E}[x_1 \otimes x_2] \text{ see proposition 3 in [27] and its appendix for the proof. Now we have:} \]

\[
\mathbb{E}\left[ \frac{a}{2(N^2)} \left( \sum_{m,n=1}^{N} \tilde{M}_1^m \otimes \tilde{M}_2^n - \sum_{n=1}^{N} \tilde{M}_1^n \otimes \tilde{M}_1^n \right) \right] = \mathbb{E}\left[ \frac{a}{2(N^2)} \left( \sum_{m=1}^{N} \tilde{M}_1^m \otimes \tilde{M}_1^m + \sum_{n=1}^{N} \tilde{M}_1^n \otimes \tilde{M}_1^n - \sum_{n=1}^{N} \tilde{M}_1^n \otimes \tilde{M}_1^n \right) \right] \\
= \mathbb{E}\left[ \frac{a}{2(N^2)} \sum_{m=1}^{N} \sum_{n=1, m \neq n}^{N} \tilde{M}_1^m \otimes \tilde{M}_1^n \right] \\
= \frac{a}{2(N^2)} \sum_{m=1}^{N} \sum_{n=1, m \neq n}^{N} \mathbb{E}[\tilde{M}_1^m] \otimes \mathbb{E}[\tilde{M}_1^n] \\
= \frac{a}{2(N^2)} \sum_{m=1}^{N} \sum_{n=1, m \neq n}^{N} \mathbb{E}[x_1] \otimes \mathbb{E}[x_1] \\
= \frac{a}{N(N-1)} N(N-1) \mathbb{E}[x_1] \otimes \mathbb{E}[x_1] \\
= \alpha_2 \mathbb{E}[x_1] \otimes \mathbb{E}[x_1]
\]

Thus, we have that:

\[ \mathbb{E}[\hat{M}_2] = \mathbb{E}[x_1 \otimes x_2] - \frac{\alpha_0}{\alpha_0 + 2} \mathbb{E}[x_1] \otimes \mathbb{E}[x_1] = M_2 \]

**Third order moment**: Similar to the second order moment, the first term of \( \hat{M}_3 \) is the estimator the single-topic second order moment and \( \mathbb{E}\left[ \frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3^n \right] = \mathbb{E}[x_1 \otimes x_2 \otimes x_3] \) as shown in proposition 4 in [27] and proved in its appendix. We need to prove that (1): \( \mathbb{E}[\hat{B}_1] = b \mathbb{E}[x_1] \otimes x_2 \otimes \mathbb{E}[x_3] \), note that \( \mathbb{E}[x_3] = M_1 \) and (2): \( \mathbb{E}[b] = c \mathbb{E}[x_1] \otimes \mathbb{E}[x_1] \otimes \mathbb{E}[x_1] = c M_1 \otimes M_1 \otimes M_1 \). Since \( \hat{B}_2 \) and \( \hat{B}_3 \) are permuted version of \( \hat{B}_1 \) their proofs follow from the proof of \( \hat{B}_1 \).

For \( \hat{B}_1 \) we simplify the expression and then show that the expectation of the resultant is equal to the desired
\[
\mathbb{E}[B_1] = \frac{b}{2^{(N/2)}} \mathbb{E} \left[ \left( \sum_{n=1}^{N} \tilde{M}_2^n \right) \otimes \left( \sum_{n=1}^{N} \tilde{M}_1^n \right) - \sum_{n=1}^{N} (\tilde{M}_2^n \otimes \tilde{M}_1^n) \right]
\]
\[
= \frac{b}{2^{(N/2)}} \mathbb{E} \left[ \sum_{m=1}^{N} \sum_{n=1}^{N} \left( \tilde{M}_2^n \otimes \tilde{M}_1^m \right) + \sum_{n=1}^{N} \left( \tilde{M}_2^n \otimes \tilde{M}_1^n \right) - \sum_{n=1}^{N} (\tilde{M}_2^n \otimes \tilde{M}_1^n) \right]
\]
\[
= \frac{b}{2^{(N/2)}} \sum_{m=1}^{N} \sum_{n=1}^{N} \mathbb{E} [\tilde{M}_2^n] \otimes \mathbb{E} [\tilde{M}_1^m]
\]
\[
= \frac{b}{2^{(N/2)}} \sum_{m=1}^{N} \sum_{n=1}^{N} \mathbb{E} [x_1 \otimes x_2] \otimes \mathbb{E} [x_3]
\]
\[
= \frac{b}{N(N-1)} N(N-1) \mathbb{E} [x_1 \otimes x_2] \otimes \mathbb{E} [x_3]
\]
\[
= b \mathbb{E} [x_1 \otimes x_2] \otimes \mathbb{E} [x_3]
\]
\[
= b \mathbb{E} [x_1 \otimes x_2 \otimes x_3]
\]

For identity \(\textbf{30}\) is applied, this leads to the following
\[
\mathbb{E}[b] = \frac{c}{6^{(N/3)}} \mathbb{E} \left[ \left( \sum_{n=1}^{N} (\tilde{M}_1^n) \otimes^3 + 3 \sum_{n=1,m=1}^{N,N} (\tilde{M}_1^n) \otimes^2 \tilde{M}_1^m + \sum_{n=1,m=1,p=1}^{N,N,N} \tilde{M}_1^n \otimes \tilde{M}_1^m \otimes \tilde{M}_1^p \right) 
\right.
\]
\[
- 3 \sum_{m=1}^{N} \left( \sum_{n=1}^{N} (\tilde{M}_1^n) \otimes^2 \left( \tilde{M}_1^m \right) \right) + 2 \sum_{n=1}^{N} (\tilde{M}_1^n) \otimes^3 \right) \right]
\]
\[
= \frac{c}{6^{(N/3)}} \mathbb{E} \left[ \sum_{n=1,m=1,p=1}^{N,N,N} \tilde{M}_1^n \otimes \tilde{M}_1^m \otimes \tilde{M}_1^p \right]
\]
\[
= \frac{c}{N(N-1)(N-2)} (N-1)(N-2) \mathbb{E} [\tilde{M}_1^n] \otimes \mathbb{E} [\tilde{M}_1^m] \otimes \mathbb{E} [\tilde{M}_1^p]
\]
\[
= c \mathbb{E} [x_1] \otimes \mathbb{E} [x_1] \otimes \mathbb{E} [x_1]
\]

Combing these results and plugging the values for \(a, b, \)and \(c\) we get:
\[
\mathbb{E} [\tilde{M}_3] = \mathbb{E} [x_1 \otimes x_2 \otimes x_3] - \frac{\alpha_0}{\alpha_0 + 2} \left( \mathbb{E} [x_1 \otimes x_2 \otimes \mathbb{E} [x_3]] + \mathbb{E} [x_1 \otimes \mathbb{E} [x_2] \otimes x_3] + \mathbb{E} [\mathbb{E} [x_1] \otimes x_2 \otimes x_3] \right)
\]
\[
+ \frac{2\alpha_0^2}{(\alpha_0 + 1)(\alpha_0 + 2)} \mathbb{E} [x_1] \otimes \mathbb{E} [x_1] \otimes \mathbb{E} [x_1] = M_3
\]
D E. Lemmas regarding Dirichlet Moments

This section introduces two lemmas regarding the moments of the dirichlet distribution that will be useful for the proof of

D.1 Dirichlet Moments

Lemma 23. The first, second and third moments of dirichlet distribution are

\[ \mathbb{E}[\theta] = \frac{1}{\alpha_0} \alpha \]

(13)

\[ \mathbb{E}[\theta \otimes \theta] = \frac{1}{\alpha_0(\alpha_0 + 1)}[\alpha \otimes \alpha + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t] \]

(14)

\[ \mathbb{E}[\theta \otimes \theta \otimes \theta] = \frac{1}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)}[\alpha \otimes \alpha \otimes \alpha + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes \alpha + \sum_{t=1}^{T} \alpha_t e_t \otimes \alpha e_t + 2 \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes e_t] \]

(15)

D.2 Raw Moments

Lemma 24.

\[ \mathbb{E}[x_1] = \mu \mathbb{E}[\theta] \]

(16)

\[ \mathbb{E}[x_1 \otimes x_2] = \mu \mathbb{E}[\theta \otimes \theta] \mu^\top \]

(17)

\[ \mathbb{E}[x_1 \otimes x_2 \otimes x_3] = \mathbb{E}[\theta \otimes \theta \otimes \theta](\mu, \mu, \mu) \]

(18)

Proof. First Order Moments Let us omit \( n \) and use \( x_1 \) to denote a token in any document, and we will use \( x_2 \) and \( x_3 \) to denote other two tokens in the same document. The the expectation of a token is

\[ \mathbb{E}[x_1] = \mathbb{E}[x_2] = \mathbb{E}[x_3] = \mathbb{E}[\mathbb{E}[x_1|\theta]] = \mu \mathbb{E}[\theta] \]

(19)

This is called the first order moment.

Second Order Moments The second order moment is defined as

\[ \mathbb{E}[x_1 \otimes x_2] = \mathbb{E}[\mathbb{E}[x_1 \otimes x_2|\theta]] \]

(20)

\[ = \sum_{i,i'} \mathbb{E}[[x_1 \otimes x_2|z_{n,j} = e_i, z_{n,k} = e_{i'}]]P(z_{n,j} = e_i, z_{n,k} = e_{i'}) \]

(21)

\[ = \sum_{i,i'} \mathbb{E}[[x_1|z_{n,j} = e_i] \otimes \mathbb{E}[[x_2|z_{n,k} = e_{i'}]]P(z_{n,j} = e_i, z_{n,k} = e_{i'}) \]

(22)

\[ = \sum_{i,i'} \mu_{e_i} \otimes (\mu_{e_{i'}}) P(z_{n,j} = e_i, z_{n,k} = e_{i'}) \]

(23)

\[ = \mu \sum_{i,i'} e_i \otimes e_{i'} P(z_{n,j} = e_i, z_{n,k} = e_{i'}) \mu^\top \]

(24)

Third Order Moments The third order moment is defined as

\[ \mathbb{E}[x_1 \otimes x_2 \otimes x_3] = \mathbb{E}[\mathbb{E}[x_1 \otimes x_2 \otimes x_3|\theta]] = \mathbb{E}[\theta \otimes \theta \otimes \theta](\mu, \mu, \mu) \]

(26)

To clarify the notations, \( x \otimes y \) is a length(\( x \))-by-length(\( y \)) matrix which has entries \([x \otimes y]_{i,j} = x_i y_j\). And \( \mathbb{E}[\theta \otimes \theta \otimes \theta](\mu, \mu, \mu) \) is a tucker with core tensor \( \mathbb{E}[\theta \otimes \theta \otimes \theta] \) and projection \( \mu \) in all three modes. □
E F. Proof of Lemma 3

The lemma relates the LDA moments to the model parameters $\alpha$ and $\mu$.

**Proof.** In order to prove this relation, we combine Lemmas 23 and Lemma 24 to prove the forms of $M_1$, $M_2$ and $M_3$ in Lemma 3 as follows.

\[
M_1 = E[x_1] = \mu E[\theta] = \sum_{i=1}^{k} \frac{\alpha_i}{\alpha_0} \mu_i \tag{27}
\]

\[
M_2 = E[x_1 \otimes x_2] - \frac{\alpha_0}{\alpha_0 + 1} E[x_1] \otimes E[x_1] = E[\theta \otimes \theta](\mu, \mu) - \frac{1}{\alpha_0(\alpha_0 + 1)} M_1 \otimes M_1
\]

\[
= \sum_{i=1}^{k} \frac{\alpha_i}{\alpha_0(\alpha_0 + 1)} \mu_i \otimes \mu_i \tag{30}
\]

\[
M_3 = E[x_1 \otimes x_2 \otimes x_3] - \frac{1}{\alpha_0 + 2} (E[x_1 \otimes x_2 \otimes E[x_3]] + E[x_1 \otimes E[x_2] \otimes x_3]
+ E[\theta \otimes \theta \otimes \theta]) \mu, \mu, \mu
\]

\[
= \sum_{i=1}^{k} \frac{2\alpha_i}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)} \mu_i \otimes \mu_i \otimes \mu_i \tag{36}
\]

$\square$
F  G. Computing $\mathbb{E}[x_1]$, $\mathbb{E}[x_1 \otimes x_2]$, $\mathbb{E}[x_1 \otimes x_2 \otimes x_3]$

Let $c_n$ be the count vector.

$$\mathbb{E}[x_1] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{\Gamma(n)} c_n$$  \hspace{1cm} (37)

$$\mathbb{E}[x_1 \otimes x_2] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{2 \binom{D}{2}} [c_n \otimes c_n - \text{diag}(c_n)]$$  \hspace{1cm} (38)

$$\mathbb{E}[x_1 \otimes x_2 \otimes x_3] = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{6 \binom{D}{3}} [c_n \otimes c_n \otimes c_n - \sum_{i,j=1}^{D} c_n(i)c_n(j) e_i \otimes e_i \otimes e_j - \sum_{i,j=1}^{D} c_n(i)c_n(j) e_j \otimes e_i \otimes e_i - \sum_{i,j=1}^{D} c_n(i)c_n(j) e_j \otimes e_i \otimes e_i + 2 \sum_{i,j=1}^{D} c_n(i)e_i \otimes e_i \otimes e_i]$$  \hspace{1cm} (39)

G  H. Dirichlet Moments

We characterize the core tensor $\mathbb{E}[\theta \otimes \theta \otimes \theta]$, where $i, j, k$-th entry of the tensor is $\mathbb{E}[\theta_i \theta_j \theta_k]$. Now the moments of topic models are reduced to moments of topic proportions. Since topic proportions are dirichlet distributed, we characterize the dirichlet moments.

univariate moments for $i$-th coordinate of dirichlet variable $\theta_i$  We know that $\mathbb{E}[\theta_i^p] = \frac{\Gamma(\alpha_i + p)}{\Gamma(\alpha_i)} \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 + p)}$, therefore

$$\mathbb{E}[\theta_i] = \frac{\alpha_i}{\alpha_0}$$  \hspace{1cm} (40)

$$\mathbb{E}[\theta_i^2] = \frac{\alpha_i(\alpha_i + 1)}{\alpha_0(\alpha_0 + 1)}$$  \hspace{1cm} (41)

$$\mathbb{E}[\theta_i^3] = \frac{\alpha_i(\alpha_i + 1)(\alpha_i + 2)}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)}$$  \hspace{1cm} (42)

bivariate moments  We know that $\mathbb{E}[\theta_i^p \theta_j^q] = \frac{\Gamma(\alpha_i + p)}{\Gamma(\alpha_i)} \frac{\Gamma(\alpha_j + q)}{\Gamma(\alpha_j)} \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 + p + q)}$, therefore

$$\mathbb{E}[\theta_i \theta_j] = \frac{\alpha_i \alpha_j}{\alpha_0(\alpha_0 + 1)}$$  \hspace{1cm} (43)

$$\mathbb{E}[\theta_i^2 \theta_j] = \frac{\alpha_i(\alpha_i + 1) \alpha_j}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)}$$  \hspace{1cm} (44)

trivariate moments  We know that $\mathbb{E}[\theta_i \theta_j \theta_k] = \frac{\Gamma(\alpha_i + 1)}{\Gamma(\alpha_i)} \frac{\Gamma(\alpha_j + 1)}{\Gamma(\alpha_j)} \frac{\Gamma(\alpha_k + 1)}{\Gamma(\alpha_k)} \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_0 + 3)}$, therefore

$$\mathbb{E}[\theta_i \theta_j \theta_k] = \frac{\alpha_i \alpha_j \alpha_k}{\alpha_0(\alpha_0 + 1)(\alpha_0 + 2)}$$  \hspace{1cm} (45)
Table 1: Utility of different configurations that guarantee differentially private LDA. The table lists edges in Figure 1 on which to add Gaussian noise in order to achieve differentially private topic model using method of moments. We use \( \tau \) as defined in Proposition 18 to decompose the dependence of the noise variance on both the sensitivity and the privacy parameters \( \epsilon, \delta \), i.e. \( \sigma = \Delta \tau_{e_i, \delta_i} \), where \( \Delta \) is the sensitivity.

Therefore we obtain

\[
\mathbb{E}[\theta] = \frac{1}{\alpha_0} \alpha \tag{46}
\]

\[
\mathbb{E}[\theta \otimes \theta] = \frac{1}{\alpha_0 (\alpha_0 + 1)} \left[ \alpha \otimes \alpha + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \right] \tag{47}
\]

\[
\mathbb{E}[\theta \otimes \theta \otimes \theta] = \frac{1}{\alpha_0 (\alpha_0 + 1)(\alpha_0 + 2)} \left[ \alpha \otimes \alpha \otimes \alpha + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes e_t \right. \\
\left. + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes e_t + \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes e_t + 2 \sum_{t=1}^{T} \alpha_t e_t \otimes e_t \otimes e_t \right] \tag{48}
\]

Lemma 25 (Correctness of Method of Moments in Learning LDA [1]). Applying the method of moments over a corpus of \( N \) documents sampled iid. There exist universal constants \( C_1, C_2 \geq 0 \) such that if \( N > C_1 \left( \alpha_0 + 1 \right) / p_{\min}^2 \sigma_k(\mu)^2 \), then \( ||\mu - \tilde{\mu}||_2 \leq C_2 \frac{(\alpha_0 + 1)^2 k^3}{p_{\min}^2 \sigma_k(\mu)^2 N} \), where \( p_{\min} = \min_i \alpha_i / \alpha_0 \), \( \mu \) is a matrix of stacked word-topic vectors, i.e. \( \mu = [\mu_1 \ldots | \mu_k] \).

H 1. Sensitivity Proofs

In proving the sensitivities for \( \tilde{M}_2 \) and \( \tilde{M}_3 \) we rely on the fact that frequently in the calculations, we encounter probability vectors, matrices, and tensors where the elements sum to 1. This is identical to the stating that the \( l_1 \) norm equals 1. Further, we note the following Lemma which essentially states that taking the outer product of a vector with a probability vector or probability matrix does not increase the \( l_q \) norm of the vector and in fact keeps it the same if \( q = 1 \).
Lemma 26 (Multiplying by probabilities does not change the norm). Let \( v_p, M_p \) be a probability vector, matrix, respectively and let \( v, u \) be ordinary vectors, matrices, respectively. Then the following holds:

\[
\|uv_p^T\|_q \leq \|u\|_q, \text{ which is equal if } q = 1. \tag{49}
\]

If \( T = M_p \otimes u \), then \( \|T\|_q = \|M_p \otimes u\|_q \leq \|u\|_q \), which is equal if \( q = 1 \).

Proof.

\[
\|uv_p^T\|_q = \left( \sum_{i,j} \left| v_i v_p_j \right|^{q} \right)^{1/q} = \left( \sum_{i} \left| v_i \right|^q \sum_{j} \left| v_p_j \right|^{q} \right)^{1/q} = \|v\|_q \|u\|_q \leq \|u\|_q. \tag{51}
\]

Where we used the fact that \( \|x\|_1 \geq \|x\|_q \) for any \( q \geq 1 \) and that \( \|v_p\|_1 = 1 \). Thus the above inequality is tight if \( q = 1 \).

\[
\|T\|_q = \|M_p \otimes u\|_q = \left( \sum_{i,j,k} \left| M_{p,i,j} u_k \right|^{q} \right)^{1/q} = \left( \sum_{k} \left| u_k \right|^q \sum_{i,j} \left| M_{p,i,j} \right|^{q} \right)^{1/q} = \|u\|_q \|M_p\|_q \leq \|u\|_q. \tag{52}
\]

Where we used the fact that for any matrix \( M, \|M\|_1 \geq \|M\|_q \) for any \( q \geq 1 \) and that \( \|M_p\|_1 = 1 \). Thus the above inequality is tight if \( q = 1 \).

Proposition 27.

- \( \tilde{M}_1^n \) is a probability vector.
- \( \tilde{M}_2^n \) is a probability matrix.
- \( \tilde{M}_3^n \) is a probability tensor.

Proof. The proof is immediate as these moments correspond to join probability estimates [27], specifically:

\[
\tilde{M}_1^n(i) = \mathbb{P}[x_1 = i]
\]
\[
\tilde{M}_1^n(i,j) = \mathbb{P}[x_1 = i, x_2 = j]
\]
\[
\tilde{M}_1^n(i, j, k) = \mathbb{P}[x_1 = i, x_2 = j, x_3 = k]
\]

H.1 Proof for Theorem 4 (sensitivity for \( \hat{M}_2 \))

Let \( \Delta_2 \) be the \( \ell_1 \) sensitivity for \( \hat{M}_2 \), then \( \Delta_2 \) is \( \frac{2}{N} + \frac{\alpha m}{m + 1} = o\left(\frac{1}{N}\right) \).

Proof. Let \( \hat{M}_2 \) and \( \hat{M}'_2 \) be second order LDA moments generated from two neighboring corpora, WLOG assume the difference is in the \( n^{th} \) record, i.e. \( D = [c_1 | \ldots | c_{N-1} | c_N] \) and \( D' = [c_1 | \ldots | c_{N-1} | c_N'] \) then:

\[
\hat{M}_2 - \hat{M}'_2 = \frac{1}{N} (\tilde{M}_2^n - \tilde{M}'_2^n) - \frac{\alpha}{2} (\hat{M}_1^N \otimes \left( \sum_{n=1}^{N-1} \tilde{M}_1^n \right) + \left( \sum_{n=1}^{N-1} \tilde{M}_1^n \right) \otimes \hat{M}_1^N)
\]

\[
= \frac{1}{N} (\tilde{M}_2^n - \tilde{M}'_2^n) - \frac{\alpha}{2} \left( \left( \tilde{M}_1^n - \tilde{M}'_1^n \right) \otimes \left( \sum_{n=1}^{N-1} M_1^n \right) + \left( \sum_{n=1}^{N-1} M_1^n \right) \otimes \left( \tilde{M}_1^n - \tilde{M}'_1^n \right) \right)
\]

\[
= \frac{1}{N} (\tilde{M}_2^n - \tilde{M}'_2^n) - \frac{\alpha}{N} \left( \left( \tilde{M}_1^n - \tilde{M}'_1^n \right) \otimes \left( \frac{1}{N-1} \sum_{n=1}^{N-1} M_1^n \right) + \left( \frac{1}{N-1} \sum_{n=1}^{N-1} M_1^n \right) \otimes \left( \tilde{M}_1^n - \tilde{M}'_1^n \right) \right)
\]

*These norms are obtained by extending the vector definition to matrices or simply vectorizing the matrix and then calculating the norm.
Note that according to proposition (27) $\tilde{M}_1^N$ and $\tilde{M}_1'^N$ are probability vectors and $\tilde{M}_2^N$ and $\tilde{M}_2'^N$ are probability matrices. Further, $\left(\sum_{n=1}^{N-1} \tilde{M}_2^n\right)$ is also a probability matrix since it’s the normalized sum of probability matrices.

We upper bound the $l_1$ norm of the expression by applying the triangular inequality and using lemma (26) for the terms involving a tensor product. This leads to the following:

$$\left\|\tilde{M}_2 - \tilde{M}'_2\right\|_1 \leq \frac{2}{N} + \frac{2a}{N} + \frac{\alpha_0}{\alpha_0 + 1} \frac{4}{N} = O\left(\frac{1}{N}\right)$$

$a$ was replaced by its expression as in the above $a = \frac{\alpha_0}{\alpha_0 + 1}$ in the above.

\[\square\]

**H.2 Proof for Theorem 4 (sensitivity for $\tilde{M}_3$)**

Let $\Delta_3$ be the $l_1$ sensitivity for $\tilde{M}_3$, then $\Delta_3$ is $\frac{2}{N} + \frac{4a}{\alpha_0 + 1} + \frac{12a^2}{(\alpha_0 + 1)^2} \frac{(N-1)}{N(N-2)} = O\left(\frac{1}{N}\right)$.

**Proof.** Following a similar setting as in [H.1] we have the two moments $\tilde{M}_3$ and $\tilde{M}_3'$ generated from two neighboring corpora. First we note that the expression of $\tilde{M}_3$ and $\tilde{M}_3'$ have the following form: $\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3^n + B_1 + B_2 + B_3 + b$. Effectively there are three kinds of terms: (a) $\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3$, (b) $B_1$, and (c)$B_2$. Since $B_2$ and $B_3$ are permuted versions of $B_1$ they have a similar behavior

(a) $\frac{1}{N} \sum_{n=1}^{N} \tilde{M}_3$: The first term difference between $\tilde{M}_3$ and $\tilde{M}_3'$ would result in $\frac{1}{N} (\tilde{M}_3^N - \tilde{M}_3'^N)$.

$$\left\|\tilde{M}_3^N - \tilde{M}_3'^N\right\|_1 \leq \frac{1}{N} \left( \left\|\tilde{M}_3^N\right\|_1 + \left\|\tilde{M}_3'^N\right\|_1 \right) \leq \frac{2}{N}$$

Note that both $\tilde{M}_3^N$ and $\tilde{M}_3'^N$ are probability tensors.

(b) $B_1$: Based on the minimized expression, the $B_1$ term difference between $\tilde{M}_3$ and $\tilde{M}_3'$ is equal to:

$$B_1 - B_1' = \frac{b}{2^{(1)2}} \left[ \tilde{M}_2^N \otimes \left( \sum_{n=1}^{N-1} \tilde{M}_1^n \right) + \left( \sum_{n=1}^{N-1} \tilde{M}_2^n \right) \otimes \tilde{M}_1^N \right. \left. - \tilde{M}_2'^N \otimes \left( \sum_{n=1}^{N-1} \tilde{M}_1^n \right) - \left( \sum_{n=1}^{N-1} \tilde{M}_2^n \right) \otimes \tilde{M}_1'^N \right]$$

$$= \frac{b}{N} \left[ \left( \tilde{M}_2^N - \tilde{M}_2'^N \right) \otimes \left( \frac{1}{N-1} \sum_{n=1}^{N-1} \tilde{M}_1^n \right) + \left( \frac{1}{N-1} \sum_{n=1}^{N-1} \tilde{M}_2^n \right) \otimes \left( \tilde{M}_1^N - \tilde{M}_1'^N \right) \right]$$

Note that $\frac{1}{N} \sum_{n=1}^{N-1} \tilde{M}_1^n$ and $\frac{1}{N} \sum_{n=1}^{N-1} \tilde{M}_2^n$ are probability vectors and matrices, respectively. Thus lemma (26) can be used to upper bound the $l_1$ norm, leading to the following:

$$\left\|B_1 - B_1'\right\|_1 \leq \left|\frac{|b|}{N}(2 + 2) = \frac{4|b|}{N} = \frac{4\alpha_0}{\alpha_0 + 2} \frac{1}{N}$$

23
(c) b: Based on the minimized expression, the b term difference between \( \hat{M}_3 \) and \( \hat{M}_4' \) is equal to:

\[
b - b' = \frac{c}{6(N-3)} \left[ \left( \hat{M}^{N}_{1} \otimes \left( \sum_{m=1, p=1}^{N-1} \hat{M}^{m}_{1} \otimes \hat{M}^{p}_{1} \right) + \left( \sum_{n=1, m=1}^{N-1} \hat{M}^{m}_{n} \otimes \hat{M}^{n}_{n} \otimes \hat{M}^{p}_{1} \right) \right) \right.
\]

\[
+ \left( \sum_{n=1, m=1}^{N-1} \hat{M}^{n}_{1} \otimes \hat{M}^{m}_{1} \otimes \hat{M}^{N}_{1} \right) \right]
\]

\[
- \left( \sum_{n=1, m=1}^{N-1} \hat{M}^{n}_{1} \otimes \hat{M}^{m}_{1} \otimes \hat{M}^{N}_{1} \right) \left] \right.
\]

\[
\left. \left. \frac{c(N-1)}{N(N-2)} \left( \hat{M}^{N}_{1} - \hat{M}^{N}_{1}' \right) \otimes \left( \frac{1}{(N-1)^2} \sum_{m=1, p=1}^{N-1} \hat{M}^{m}_{1} \otimes \hat{M}^{p}_{1} \right) \right.
\]

\[
+ \left( \frac{1}{(N-1)^2} \sum_{n=1, m=1}^{N-1} \hat{M}^{n}_{1} \otimes \left( \hat{M}^{N}_{1} - \hat{M}^{N}_{1}' \right) \otimes \hat{M}^{p}_{1} \right) \right]
\]

\[
\left. + \left( \frac{1}{(N-1)^2} \sum_{n=1, m=1}^{N-1} \hat{M}^{n}_{1} \otimes \hat{M}^{n}_{1} \otimes \left( \hat{M}^{N}_{1} - \hat{M}^{N}_{1}' \right) \right] \right]
\]

Similarly, we have probability tensors so we use lemma 26 to bound the \( l_1 \) norm. This results in:

\[
\| b - b' \|_1 \leq \frac{c(N-1)}{N(N-2)} (2 + 2 + 2) = \frac{6c(N-1)}{N(N-2)} = \frac{12\alpha_0^2}{\alpha_0 + 1} \frac{(N-1)}{N(N-2)}
\]

Combining the results from (a), (b) and (c), we have the following bound:

\[
\Delta_3 \leq \frac{2}{N} + \frac{4\alpha_0}{\alpha_0 + 2} \frac{1}{N} + \frac{12\alpha_0^2}{\alpha_0 + 1}(N-1) \frac{1}{N(N-2)} = O\left( \frac{1}{N} \right)
\]

H.3 Proof for Theorem 5 (sensitivity for \( \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) \))

As explained before, the whitened tensor is denoted as \( \hat{T} \) for simplicity. Therefore we denote the sensitivity of \( \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) \) as \( \Delta_{\hat{T}} \). Theorem 5 states that \( \Delta_{\hat{T}} = O\left( \frac{k^2}{N \sigma_{\hat{T}}(M_2)^{1/2}} \right) \).

We need the following Lemma to prove Theorem 5.

Lemma 28. \( \left\| \hat{W}' - \hat{W} \right\|_F \leq \frac{\sqrt{2k\Delta_3}}{\sigma_k(M_2)\sqrt{\frac{2}{\sigma_k(M_2) + \sigma_{k+1}(M_2)}}} \)

Proof. We follow an analysis similar to \[1\]. Note that the whitening matrix \( \hat{W} \) is defined such that:

\[ \hat{W}^{T}M_{2,k} \hat{W} = I. \]

Analogously for the neighboring corpus,

\[ \hat{W}'^{T}M'_{2,k} \hat{W}' = I. \]

Let \( E_{M_2} \) denote the perturbation introduced to \( \hat{M}_2 \) by changing a single record.
Because the spectral gap of the perturbation introduced by modifying a single record is small according to the condition, applying the original whitening matrix to the neighboring data base moment \( \hat{M}_{2} \) would lead to a rank \( k \) matrix of size \( k \times k \).

Therefore, \( \hat{W}^T \hat{M}_{2,k} \hat{W} \) is a rank \( k \) matrix of size \( k \times k \), which can be factorized as:

\[
\hat{W}^T \hat{M}_{2,k} \hat{W} = ADA^T
\]

where \( A \) are the singular vectors of \( \hat{W}^T \hat{M}_{2,k} \hat{W} \), and \( D \) is a diagonal matrix of the corresponding singular values of \( \hat{W}^T \hat{M}_{2,k} \hat{W} \). This also leads to \( \hat{W}' = \hat{W} AD^{\frac{1}{2}} A^T \). Using this, we observe:

\[
\| \hat{W}' - \hat{W} \| = \| \hat{W}' - \hat{W}' AD^{\frac{1}{2}} A^T \|
\]

\[
= \| \hat{W}' (I - AD^{\frac{1}{2}} A^T) \|
\]

\[
\leq \| \hat{W}' \| \| I - AD^{\frac{1}{2}} A^T \|
\]

(53)

Now we bound \( \| I - AD^{\frac{1}{2}} A^T \| \):

\[
\| I - AD^{\frac{1}{2}} A^T \| = \| A^T A - \hat{W}' AD^{\frac{1}{2}} A^T \|
\]

\[
= \| I - D^{\frac{1}{2}} \|
\]

\[
\leq \| (I - D^{\frac{1}{2}}) (I + D^{\frac{1}{2}}) \|
\]

\[
\leq \| (I - D) \|
\]

\[
= \| I - ADA^T \|
\]

\[
= \| \hat{W}^T \hat{M}_{2,k} \hat{W} - \hat{W}'^T \hat{M}'_{2,k} \hat{W}' \|
\]

\[
\leq \| \hat{W} \|^2 \| \hat{M}_{2,k} - \hat{M}'_{2,k} \|
\]

\[
\leq \| \hat{W} \|^2 \| E_{M_2} \|
\]

(56)

We know that

\[
\| \hat{W} \|^2 \leq \frac{1}{\sigma_k(M_2)}
\]

(57)

\[
\| \hat{W}' \| \leq \frac{1}{\sqrt{\sigma_k(M'_{2})}} \leq \frac{1}{\sqrt{\sigma_k(M_2) - \| E_{M_2} \|}} \leq \frac{1}{\sigma_k(M_2) \sqrt{\frac{1}{2} \sigma_k(M_2) + \sigma_k(M_2)}}
\]

(58)

Weyl’s theorem was used in the last bound in Equation (58). Bounding the Frobenius norm, would result in the following:

\[
\| \hat{W}' - \hat{W} \|_F \leq \sqrt{2k} \| \hat{W}' - \hat{W} \| \leq \sqrt{\frac{2k \| E_{M_2} \|}{\sigma_k(M_2) \sqrt{\sigma_k(M_2) - \| E_{M_2} \|}}} \leq \sqrt{\frac{\sqrt{2k} \Delta_2}{\sigma_k(M_2) \sqrt{\frac{1}{2} \sigma_k(M_2) + \sigma_k(M_2)}}},
\]

where we have used the fact that the \( l_1 \) norm upper bounds the spectral norm of a matrix, since it upper bounds the Frobenius.

Now we are ready to prove Theorem 5.
Proof. \( \hat{M}_3 = \hat{M}_3 + E_3 \).

\[
\| \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) - \hat{M}_3'(\hat{W}', \hat{W}', \hat{W}') \|_F = \| \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) - M_3^{DA}(\hat{W}', \hat{W}', \hat{W}') - E_3(\hat{W}', \hat{W}', \hat{W}') \|_F \\
\leq \| \hat{M}_3^{DA}(\hat{W}', \hat{W}', \hat{W}') \|_F + \| E_3(\hat{W}', \hat{W}', \hat{W}') \|_F \\
\leq \| \hat{M}_3 \|_F \| \hat{W} - \hat{W}' \|_F^3 + \| \Delta_3 \|_F \| \hat{W}' \|_F^3
\]  

(59)

(60)

(61)

We have used the fact that the Frobenius norm of the difference between the tensors is bounded above by the \( l_1 \) norm of the difference \( \Delta_3 \). To bound the \( l_1 \) norm of \( \hat{M}_3 \) we use an analysis similar to calculating \( \Delta_3 \). Again we note that the \( l_1 \) norm upper bounds the Frobenius norm:

\[
\| \hat{M}_3 \|_F \leq \| \hat{M}_2 \|_1 = 1 + \frac{6\sigma_0}{\alpha_0 + 2N} \frac{N}{N - 1} + \frac{6\alpha_0^2}{(\alpha_0 + 1)(\alpha_0 + 2)} \frac{N^3}{N(N - 1)(N - 2)}
\]

(62)

Combining all the expressions we get:

\[
\Delta_{\hat{F}} = \| \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) - \hat{M}_3'(\hat{W}', \hat{W}', \hat{W}') \|_F \\
\leq (1 + \frac{6\sigma_0}{\alpha_0 + 2N} \frac{N}{N - 1} + \frac{6\alpha_0^2}{(\alpha_0 + 1)(\alpha_0 + 2)} \frac{N^3}{N(N - 1)(N - 2)}) \frac{N^3}{(\sigma_N(\hat{M}_2))^{3/2}} + \frac{\Delta_3k^{3/2}}{(\sigma_k(\hat{M}_2))^{3/2}}
\]

(63)

(64)

(65)

(66)

We see that if \( N \) is larger than \( d^{3/2} \), then \( N\sigma_k(\hat{M}_2)^{3/2} \geq 1 \) as \( \sigma_i(\hat{M}_2) \) is in the order of \( 1/d \).

**H.4 Proof for Theorem 6 (sensitivity of the output of tensor decomposition \( \bar{\mu}_i, \bar{\alpha}_i \))**

Let \( \bar{\mu}_1, \ldots, \bar{\mu}_k \) and \( \bar{\alpha}_1, \ldots, \bar{\alpha}_k \) be the results of tensor decomposition before unwhitening. The sensitivity of \( \bar{\mu}_i \), denoted as \( \Delta_{\bar{\mu}} \), and the sensitivity of \( \bar{\alpha}_i \), denoted as \( \Delta_{\bar{\alpha}} \), are both upper bounded by \( \Delta_{\bar{\mu}} \leq O(\gamma \sigma_N(\sigma_k(\hat{M}_2)))^{3/2} \), where \( \gamma = \min_{i \in [k]} \frac{\sigma_i - \sigma_{i+1}}{4}, \sigma_i \) is the \( i \)th eigenvalue of \( \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) \).

**Proof.** The proof follows from the result of the simultaneous tensor power method (Theorem 1 in [23]). Replacing the original eigenvectors with those resulting from database \( D \) leads to tensor \( \hat{M}_3(\hat{W}, \hat{W}, \hat{W}) \), then the tensor resulting from corpus \( D' \) with one record changed yields \( \hat{M}_3'(\hat{W}', \hat{W}', \hat{W}') \) where the spectral norm of the error is upper bounded by \( \epsilon \), if \( \Delta_{\bar{\mu}} \) is sufficiently small \( \Delta_{\bar{\mu}} \leq \frac{\epsilon}{2\sigma_R} \). Therefore we get \( \| \bar{\mu}_i - \bar{\mu}'_i \|_2 \leq \frac{2\sqrt{\Delta_{\bar{\mu}}}}{\gamma} \) and \( |\bar{\alpha}_i - \bar{\alpha}'_i| \leq \frac{2\sqrt{\Delta_{\bar{\alpha}}}}{\gamma} \).

**H.5 Proof for Theorem 7 (sensitivity of the final output \( \mu_i, \alpha_i \))**

We now prove the sensitivity of the final output \( \mu_i, \alpha_i \): \( \Delta_{\mu} = O(\frac{k^2 \sqrt{\sigma_3(M_2)}}{\gamma \sigma_N(\sigma_k(\hat{M}_2)))^{3/2}} \).

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Proof. We point out a number of things. Tensor decomposition outputs are: \( \tilde{\mu}_i, \tilde{\alpha}_i, i \in [k] \), where \( \tilde{\alpha}_i = \frac{2\sqrt{\alpha_i \alpha_0}}{(\alpha_i + 2)/(\alpha_i + 1)\alpha_0}. \) In order to recover the desired word topic vector \( \mu \), we have to “unwhiten” to get the \( \mu_i \) and \( \alpha_i \) before whitening, i.e. \( \mu_i = \frac{1}{\sqrt{\alpha_i}}(W^T)^\dagger \tilde{\mu}_i \), where \( \frac{1}{\sqrt{\alpha_i}} = \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \tilde{\alpha}_i \). The sensitivity would be:

\[
\max_{D,D'} \| \mu_i - \mu_i' \| \lesssim \max_{D,D'} \left\{ \left\| \frac{1}{\sqrt{\alpha_i}}(W^T)^\dagger \tilde{\mu}_i - \frac{1}{\sqrt{\alpha_i'}}(W^T')^\dagger \tilde{\mu}_i' \right\|_2 \right\}
\leq \max_{D,D'} \left\{ \frac{1}{\sqrt{\alpha_i}} \left\| (W^T)^\dagger \right\| \| \tilde{\mu}_i - \tilde{\mu}_i' \| + \frac{1}{\sqrt{\alpha_i}} \left\| W^\dagger - (W')^\dagger \right\| + \left\| (W')^\dagger \right\| \frac{1}{\sqrt{\alpha_i}} - \frac{1}{\sqrt{\alpha_i'}} \right\}
\]

We note the following:

1. \( \max_{D,D'} \frac{1}{\sqrt{\alpha_i}} - \frac{1}{\sqrt{\alpha_i'}} = \max_{D,D'} \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \tilde{\alpha}_i - \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \tilde{\alpha}_i' \leq \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \max_{D,D'} | \tilde{\alpha}_i - \tilde{\alpha}_i' | \leq \frac{2\sqrt{\Delta_f \gamma_s^k}}{\sqrt{\sigma_1(\hat{M}_2)}} \), where the above follows from the simultaneous power iteration method.

2. \( \max_{i \in [k]} \frac{1}{\sqrt{\alpha_i}} \leq \frac{1}{2\sqrt{(\alpha_i + 1)\alpha_0}} \max_{i \in [k]} \tilde{\alpha}_i = \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \sigma_1(\hat{T}) \)

3. \( \max \left\| ((W^T)^\dagger) \right\| \leq \sqrt{\sigma_1(\hat{M}')} \leq \sqrt{\sigma_1(\hat{M}_2) + \Delta_2} \)

4. Following an analysis similar to that in [28] we obtain \( \| W^\dagger - (W')^\dagger \| \leq \frac{\sqrt{\sigma_1(\hat{M}_2)} \Delta_2}{\sigma_k(\hat{M}_2)} \).

Combining all of this together leads to the following

\[
\max_{D,D'} \| \mu_i - \mu_i' \| \leq \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \sigma_1(\hat{T}) \sqrt{\sigma_1(\hat{M}_2) \Delta_2} \frac{2\sqrt{\Delta_f \gamma_s^k}}{\gamma_s} + \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \sigma_1(\hat{T}) \sigma_1(\hat{M}_2) \sqrt{\Delta_2} \frac{2\sqrt{\Delta_f \gamma_s^k}}{\gamma_s} + \frac{(\alpha_i + 2)}{2\sqrt{(\alpha_i + 1)\alpha_0}} \sigma_1(\hat{M}_2) + \Delta_2 \frac{2\sqrt{\Delta_f \gamma_s^k}}{\gamma_s} = O\left( \frac{k^2 \sqrt{\sigma_1(\hat{M}_2)}}{\gamma_s N \sigma_k^3(\hat{M}_2)} \right)
\]

\[\square\]

H.6 Proof for Lemma [8]

Proof. Let \( x, x' \) be two adjacent data sets and the overall output be \( O := f(\text{DATA}) + Z(\epsilon_2, \delta_2, \hat{L}\hat{S}). \) Let \( S_1 \subseteq \text{Dom}(O), S_2 \subseteq \text{Dom}(LS) \) be any measurable sets.

Let \( E \) be the measurable set of LS that represents the event that \( \hat{L}\hat{S} \geq LS \).

\[
p(O, \hat{L}\hat{S}) \in S_1 \times S_2 | x] = p(O, \hat{L}\hat{S}) \in S_1 \times (S_2 \cap E) | x] + p(O, \hat{L}\hat{S}) \in S_1 \times S_2 \cap E^c | x] \leq p(O, \hat{L}\hat{S}) \in S_1 \times (S_2 \cap E) | x] + \delta_3 \leq e^{\epsilon_1 + \epsilon_2} p(O, \hat{L}\hat{S}) \in S_1 \times (S_2 \cap E) | x'] + \delta_1 + \delta_2 + \delta_3 \leq e^{\epsilon_1 + \epsilon_2} p(O, \hat{L}\hat{S}) \in S_1 \times S_2 | x'] + \delta_1 + \delta_2 + \delta_3
\]

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The fourth line holds due to the fact that under event the $E$, $\tilde{L}S$ is always a valid upper bound of the local sensitivity, therefore, conditioning on the $\sigma$-field induced by $E \cap S_2$ for any $S_2, O$ is an $(\epsilon, \delta)$-DP release. By the simple composition Theorem of $(\epsilon, \delta)$-DP \cite{Theorem B.1}, by taking the measurable set of interest to be $S_1 \times (S_2 \cap E)$, we have that

$$p[(O, \tilde{L}S) \in S_1 \times (S_2 \cap E)|x] \leq e^{\epsilon_1+\epsilon_2}p[(O, \tilde{L}S) \in S_1 \times (S_2 \cap E)|x'] + \delta_1 + \delta_2$$

which wraps up the proof.

\section{H.7 Proof for Sensitivity of singular values $\sigma_k(\tilde{M}_2)$ (Lemma 9)}

\textit{Proof.} We first prove that the global sensitivity of $\sigma_k(\tilde{M}_2)$ is $1/n$. By Weyl’s lemma \cite{Theorem 1}, for any matrix $X$, any $i$, the singular value $|\sigma_i(X) - \sigma_i(X + E)| \leq \|E\|_2$. In our case, $E$ is coming from adding or removing one data point and we know that $\|E\|_2 \leq \|E\|_F \leq \|E\|_{1,1} \leq 2/n$, hence the bound.

Now we prove that the global sensitivity of $\gamma_s = \min_{i \in [k]} \sigma_i(\tilde{T}) - \sigma_i(\tilde{T} + \mathcal{E})$. For any tensor $\tilde{T}$, we consider a polyadic form or the so called tensor decomposition form, and denote the singular values as the amplitude of the components in the polyadic form. As shown in Section \textbf{H.2} $|\sigma_i(\tilde{T}) - \sigma_i(\tilde{T} + \mathcal{E})| \leq \|\mathcal{E}\| \leq 1$, where $\mathcal{E}$ comes from adding or removing one data point.

\section{I J. Utility Proofs}

Before starting the utility proofs, we point out a number of things. Tensor decomposition outputs $\bar{\mu}_i, \bar{\alpha}_i, i \in [k]$. Where, $\bar{\alpha}_i = \frac{2\sqrt{(\alpha_0+1)\alpha_0}}{\sqrt{\alpha_i}}$. In order to recover the desired word topic vector $\mu$, we have to ‘reverse whiten’, i.e.

$$\mu_i = \frac{1}{\sqrt{\alpha_i}}(W^T)^\dagger \bar{\mu}_i, \text{ where } \frac{1}{\sqrt{\alpha_i}} = \frac{2\sqrt{(\alpha_0+2)\alpha_0}}{2\sqrt{(\alpha_0+1)\alpha_0}} \bar{\alpha}_i.$$ We need to establish the distance between the non-differentially private output and the differentially private output, i.e.

$$||\mu_i - \mu_i^{DP}||.$$ This can be upper bounded similar to \textbf{H.5} by the following:

$$||\mu_i - \mu_i^{DP}|| \leq \frac{1}{\sqrt{\alpha_i}} ||(W^T)^\dagger || ||\bar{\mu}_i - \bar{\mu}_i^{DP}|| + \frac{1}{\sqrt{\alpha_i}} ||W^\dagger - (W^{DP})^\dagger|| + ||(W^{DP})^\dagger|| ||\frac{1}{\sqrt{\alpha_i}} - \frac{1}{\sqrt{\alpha_i,D_P}}||.$$ (67)

For this we frequently need to bound the following:

- $||\bar{\mu}_i - \bar{\mu}_i^{DP}||$
- $||W^\dagger - (W^{DP})^\dagger||$
- $||W^{DP}||$
- $\frac{1}{\sqrt{\alpha_i}} - \frac{1}{\sqrt{\alpha_i,D_P}}$
- $|\bar{\alpha}_i - \bar{\alpha}_i^{DP}|$

We point out the following facts before preceding:

- $\frac{1}{\sqrt{\alpha_i}} - \frac{1}{\sqrt{\alpha_i,D_P}} \leq \frac{(\alpha_0+2)}{2\sqrt{(\alpha_0+1)\alpha_0}} \bar{\alpha}_i - \frac{(\alpha_0+2)}{2\sqrt{(\alpha_0+1)\alpha_0}} \bar{\alpha}_i^{DP} \leq \frac{\alpha_0+2}{2\sqrt{(\alpha_0+1)\alpha_0}} \bar{\alpha}_i - \bar{\alpha}_i^{DP} |$

- $||W^\dagger|| \leq \sigma_1(\tilde{M}_2)$

- $\frac{1}{\sqrt{\alpha_i}} = \frac{(\alpha_0+2)}{2\sqrt{(\alpha_0+1)\alpha_0}} \bar{\alpha}_i \leq \frac{(\alpha_0+2)}{2\sqrt{(\alpha_0+1)\alpha_0}} \sigma_1(\tilde{T})$
I.1 Perturbation on $\hat{M}_2$, $\hat{M}_3$ Config. 1 ($e_3, e_4, e_8$): Proof for Theorem 10

Similar to the perturbation on $(e_6, e_8)$. We have that

$$\|W^\dagger - (W^{DP})^\dagger\| \leq \frac{\sqrt{\sigma_1(\hat{M}_2)} \|E_{8,G}\|}{\sigma_k(\hat{M}_2)}$$ (68)

$$\|W^{DP}\| \leq \sqrt{\sigma_1(\hat{M}_2)} + \|E_{8,G}\|$$ (69)

Now the perturbed tensor can be represented as $\hat{M}_3^{DP} = \hat{M}_3 + E_{3,G}$, where $E_{3,G}$ is symmetric Gaussian noise that has been added to the original tensor. Similar to the sensitivity analysis for the whitened tensor, we have that the error $\Phi$ can be bounded as follows:

$$\|\Phi\|_2 = \|\hat{M}_3(W, \hat{W}, \hat{W}) - \hat{M}_3^{DP}(W^{DP}, W^{DP}, W^{DP})\|_2$$ (70)

$$\leq \|\hat{M}_3\| \|W - W^{DP}\|^3 + \|E_{3,G}\| \|W^{DP}\|$$ (71)

Following an analysis similar to bounding $\|W^\dagger - (W^{DP})^\dagger\|$, we get that $\|W^\dagger - (W^{DP})^\dagger\| \leq \frac{\|E_{3,G}\|}{\sigma_k(\hat{M}_2)\sqrt{\|\hat{M}_3\|}}$. According to [36] we have that with high probability $\|E_{3,G}\| = O(\sqrt{d\Delta_3\tau_{e_3, e_4}})$. We note the following $\|\hat{m}_i - \hat{m}_i^{DP}\|_2 \leq \frac{\sqrt{E}\|\Phi\|}{\gamma_e}$. Similarly we have $|\hat{a}_i - \hat{a}_i^{DP}| \leq \frac{\sqrt{E}\|\Phi\|}{\gamma_e}$ and that $|\frac{1}{\sqrt{\gamma_e}} - \frac{1}{\sqrt{\gamma_e^{DP}}}| \leq \frac{(\alpha_0 + 2)\sqrt{E}\|\Phi\|}{2\sqrt{\tau\gamma_e^{DP}}}$. This leads to $\|\hat{m}_i - \hat{m}_i^{DP}\|_2 \leq \frac{(\alpha_0 + 2)}{2\sqrt{\tau\gamma_e}}\sigma_1(\hat{T})\sqrt{\sigma_1(M_2)}\sqrt{\frac{E}{\gamma_e}}\|\hat{m}_i - \hat{m}_i^{DP}\|_2\gamma_e + \frac{(\alpha_0 + 2)}{2\sqrt{\tau\gamma_e}}\sigma_1(\hat{T})\sqrt{\sigma_1(M_2)}\|E_{8,G}\|\|\hat{M}_3\|\gamma_e$.

Based on the bound on $\|\Phi\|$ we have with high probability $\|\hat{m}_i - \hat{m}_i^{DP}\|_2 = O(\sqrt{\frac{\sqrt{\sigma_1(M_2)}\gamma_e}{N\gamma_e}}(\frac{\sqrt{\sigma_1(M_2)}\gamma_e}{N\gamma_e})^{\frac{3}{2}}$\). If $\Delta_2 \leq \sigma_k(\hat{M}_3) - \sigma_{k+1}(\hat{M}_3)$. Then, given the spectral gap requirement, the sensitivity of the whitened tensor is $\Delta_2$.

I.2 Perturbation on $\hat{T}$ and $\hat{M}_2$ Config. 2($e_6, e_8$): Proof for Theorem 11

This configuration has two properties: the noise level introduced is low because the whitening step reduces the tensor dimension from $M_3 \in \mathbb{R}^{d \times d \times d}$ to $\hat{T} = \hat{M}_3(W, \hat{W}, \hat{W}) \in \mathbb{R}^{k \times k \times k}$. However, even though the dimension of the tensor is reduced, unless the whitening tensor (resulting from eigendecomposition over $\hat{M}_3$) is stable, the sensitivity of the whitened tensor is not necessarily low.

Note that the sensitivity of $\hat{M}_2$ falls with $\frac{1}{\sqrt{N}}$ (Theorem 3). Therefore, we expect the sensitivity of $\hat{M}_2(W, \hat{W}, \hat{W})$ to drop with an increasing number of records. As Theorem 3 states, $\Delta_2 = O(\frac{\sqrt{\sigma_1(M_2)}\gamma_e}{N\gamma_e})$, if $\Delta_2 \leq \sigma_k(\hat{M}_3) - \sigma_{k+1}(\hat{M}_3)$. $\hat{M}_2$ is used to generate both the whitening and unwhitening matrix, and unlike input perturbation, the sensitivity over $\hat{M}_2$ and $\hat{M}_3$ falls as the dataset size increases (Theorem 3). However, an issue with this configuration is that adding noise to $\hat{M}_3$ leads to higher noise build up prior to the tensor decomposition. Note that by [36] w.h.p the norm of the error is $O(\sqrt{\sigma})$, with $\sigma$ being the variance of the noise (this bound would be $\sqrt{\sigma}$ if the noise is added to a symmetric tensor of size $k$). Tensor decomposition methods, in particular [24] require the spectral norm of the perturbation to the tensor to be lower than a certain threshold. Following arguments similar to [24], the spectral norm of the error is $O(\frac{\sqrt{\sigma}}{N\delta_3})$ and should be below $\frac{\sqrt{\sigma}}{\gamma_e\sigma_k(\hat{T})}$. Thus $\delta_3$ should satisfy $\delta_3 = \Omega(\frac{\sqrt{\sigma}}{\gamma_e\sigma_k(\hat{T})}N)$. Following similar arguments, this time using the bound on the spectral norm of the noisy matrices, to guarantee utility, the differentially private whitening $W$ and pseudo-inverse $W^\dagger$ should be close to their non-differentially private private values, which requires both $\epsilon_4$ and $\epsilon_8$ to be $\Omega(\frac{\sqrt{\sigma}}{\gamma_e(\sigma_k(\hat{T}))N})$. Although, the privacy parameters have a lower bound of $\frac{1}{\sqrt{N}}$, the bound also falls with $\frac{1}{\sqrt{N}}$. 29
The spectral norm of the noise added to $\hat{M}_2$ can be bounded by $\frac{E_8, G}{\sigma_k(M_2)}$ with high probability. Now, if we have $N = \Omega\left(\frac{\sqrt{\sigma_k(M_2) - \sigma_{k+1}(M_2)}}{\sigma_k(M_2)}\right)$, then with w.h.p we have that $\|E_8, G\| \leq \frac{\sigma_k(M_2) - \sigma_{k+1}(M_2)}{2}$, where $E_8, G$ is the spectral norm of the Gaussian matrix. This condition enables us to bound $\|W^\dagger - (W^{DP})^\dagger\|$, in a manner similar to establishing the bounds between $\|W - W'\|$ in 28 Following a similar analysis, given that

$$W^T(\hat{M}_2)W = I,$$
$$W^{T, DP}(\hat{M}_2 + E_8, G)W^{DP} = I,$$
$$W^T(\hat{M}_2 + E_8, G)W = ADA^T,$$

we have that $\|W^\dagger - (W^{DP})^\dagger\| \leq \|W^\dagger\| \|I - D\|$. We know that $\|W^\dagger\| \leq \frac{1}{\sqrt{\sigma_k(M_2)}}$ and $\|I - D\|$ can be bounded as follows:

$$\|I - D\| \leq \|I - ADA^T\|$$
$$\leq \|W^T(\hat{M}_2)W - W^T(\hat{M}_2 + E_8, G)W\|$$
$$\leq \|W^\dagger\| \|\hat{M}_2 - (\hat{M}_2 + E_8, G)\|$$
$$\leq \|W^\dagger\| \|E_8, G\|$$

This leads to $\|W^\dagger - (W^{DP})^\dagger\| \leq \frac{\sqrt{\sigma_1(\hat{M}_2)\|E_8, G\|}}{\sigma_k(M_2)}$.

Moreover, it is immediate by Weyl’s theorem that $\|W^{DP}\| \leq \sqrt{\sigma_1(\hat{M}_2 + E_8, G)} \leq \sqrt{\sigma_1(\hat{M}_2) + \|E_8, G\|}$.

Finally, by the results of simultaneous power iteration (with an argument similar to Theorem 6), if $N$ is sufficiently large, we have that $\|\hat{\mu}_i - \hat{\mu}_i^{DP}\| \leq \frac{2\sqrt{k\|E_8, G\|}}{\gamma_0}$ where $E_8, G$ is the Gaussian tensor added to the whitened tensor $\Delta_T$.

An identical bound is established for the eigenvalues, i.e. $|\hat{\alpha}_i - \hat{\alpha}_i^{DP}| \leq \frac{2\sqrt{k\|E_8, G\|}}{\gamma_0}$.

Now we can state the utility:

$$\|\mu_i - \mu_i^{DP}\| \leq \frac{(\alpha_0 + 2\sqrt{(\alpha_0 + 1)\alpha_0})}{2\sqrt{(\alpha_0 + 1)\alpha_0}}\sqrt{\sigma_1(\hat{M}_2)}\|E_8, G\|\gamma_0$$

We note that w.h.p we have the following bounds on spectral norms of noisy Gaussian matrix and noisy Gaussian tensor. In particular, $\|E_6, G\| = O\left(\frac{k^2}{\sqrt{\gamma_0}}\tau_{\delta_k, \delta_k}\right)$ and $\|E_8, G\| = O\left(\frac{\sqrt{d}}{\gamma_0}\tau_{\delta_k, \delta_k}\right)$. This leads to the following utility

$$\|\mu_i - \mu_i^{DP}\| = O\left(\frac{\sqrt{\sigma_1(\hat{M}_2)k^{2.5}}}{\gamma_0\sqrt{N\delta_k^{3/2}}}\tau_{\delta_k, \delta_k} + \frac{\sqrt{\sigma_1(\hat{M}_2)\alpha_0}}{\sigma_k(M_2)}\gamma_0\tau_{\delta_k, \delta_k} + \sqrt{\sigma_1(\hat{M}_2) + \frac{\sqrt{d}}{\gamma_0}\tau_{\delta_k, \delta_k}}\frac{k^{2.5}\tau_{\delta_k, \delta_k}}{\gamma_0\sqrt{N\delta_k^{3/2}}}\right).$$
I.3 Perturbation on the output of tensor decomposition $\hat{\mu}_i, \hat{\alpha}_i$ and $\hat{M}_2$ Config. 3 ($\epsilon_7, \epsilon_8$): Proof for Theorem [12]

This configuration shares edge 8 with the previous. This enables us to borrow the same bounds for the pseudo-inverse $W^\top$. Specifically, we have:

$$\|W^\top - (W^{DP})^\top\| \leq \frac{\sqrt{\sigma_1(M_2)\|E_{8,G}\|}}{\sigma_k(M_2)}$$

$$\|(W^{DP})^\top\| \leq \sqrt{\sigma_1(M_2) + \|E_{8,G}\|}$$

In this method, noise is added directly to the eigenvectors and eigenvalues resulting from the tensor decomposition. Therefore, we have:

$$\hat{\mu}_i^{DP} = \hat{\mu}_i + Y_i, \quad Y_i \sim N(0, \Delta_{i,\tau} I_k)$$

$$\hat{\alpha}_i^{DP} = \hat{\alpha}_i + n_i, \quad n_i \sim N(0, \Delta_{i,\tau} I_k)$$

where $\Delta_{i,\tau} = \frac{\sqrt{\pi} \tau_i}{\gamma_k} \tau_{i,\tau}$, with $\tau_{i,\tau}$. This leads to the following bound:

$$\|\hat{\mu}_i - \mu_i^{DP}\| \leq \frac{(\alpha_0 + 2)}{2(\alpha_0 + 1)\alpha_0} \sigma_1(\hat{T}) \sqrt{\sigma_1(M_2)\|Y\|} + \frac{(\alpha_0 + 2)}{2(\alpha_0 + 1)\alpha_0} \sigma_1(\hat{T}) \sqrt{\sigma_1(M_2)\|E_{8,G}\|} + \frac{(\alpha_0 + 2)}{2(\alpha_0 + 1)\alpha_0} \sqrt{\sigma_1(M_2) + \|E_{8,G}\||n_i|}$$

As before w.h.p $\|E_{0,G}\| = O(\frac{\sqrt{2 \log N}}{N\tau_{i,0}})$. The following bounds hold on $\|Y\|$ and $|n_i|$, because they are a Gaussian vector and variable. In particular, w.h.p. $\|Y\| = O(\frac{\sqrt{N^{5/2}}}{\gamma_k} \tau_{i,\tau})$ and $|n_i| = O(\frac{\sqrt{N}}{\gamma_k \tau_{i,\tau}} \tau_{i,\tau})$. This leads to the following utility: $O(\frac{\sqrt{\sigma_1(M_2)k^2}}{\gamma_k N \sigma_k^{5/2}} \tau_{i,\tau} + \sqrt{\sigma_1(M_2)\|\hat{T}\|} \tau_{i,\tau} + \sqrt{\sigma_1(M_2) + \|E_{8,G}\||\tau_{i,\tau} + \frac{\sqrt{\gamma_k \tau_{i,\tau}}}{\gamma_k} \frac{k^2 \tau_{i,\tau}}{\gamma_k N \sigma_k^{5/2}})$.

I.4 Perturbation on the final output $\mu_i, \alpha_i$ Config. 4 ($\epsilon_9$): Proof for Theorem [13]

In this configuration, we add noise proportional to the output’s sensitive

$$\mu_i^{DP} = \mu_i + Z_i, \quad Z_i \sim N(0, \Delta_{i,\tau} I_k)$$

where $\Delta_{i,\tau} = \Delta_{i,\tau} \tau_{i,\tau}$, with $\tau_{i,\tau} = \frac{\sqrt{2 \log N}}{\gamma_k}$. Similar to the previous analysis, since $Z$ is Gaussian, then w.h.p. $\|Z\| = O(\frac{\sqrt{\sigma_1(M_2)k^2}}{N \gamma_k \sigma_k^{5/2}} \tau_{i,\tau})$. We have the utility $O(\frac{\sqrt{\sigma_1(M_2)k^2}}{N \gamma_k \sigma_k^{5/2}} \tau_{i,\tau})$.

J. Evaluation

Recall the grid search process used to choose the best configuration for a particular $\epsilon$. For a particular edge set, we vary the $\epsilon$ parameter across each edge (making sure they still sum to the same composite $\epsilon$) and choosing the combination that yields the maximum likelihood. We do this across every possible configuration, and choose the configuration which yields the highest likelihood to the original output. However, this approach leaks the data likelihood. In order to make the max likelihood computation differentially private, we perturb the sufficient statistics of the data. The
K  L. Experiments on Wikipedia Dataset

| Topic # | Top Words |
|---------|-----------|
| 1       | 'air', 'th', 'force', 'squadron', 'special', 'operations', 'test', 'august', 'december', 'mission' |
| 2       | 'music', 'composition', 'kaufmann', 'works', 'dieter', 'vienna', 'acoustic', 'electro', 'des', 'president' |
| 3       | 'river', 'barnstaple', 'yeo', 'taw', 'flooding', 'tributary', 'rivers', 'south', 'devon', 'flows' |

Table 2: Top words in Wikipedia Dataset recovered from our DP algorithm with \( \epsilon = 1 \). Note that the topics shown here are not cherry picked but randomly selected.

L  M. Some Useful Identities and Theorems

Identity 29 (Square of Sum).

\[
\left( \sum_{i=1}^{N} a_i \right)^2 = \sum_{i=1}^{N} a_i^2 + \sum_{i=1,j=1}^{N,N} a_i a_j
\]

Identity 30 (Cube of Sum).

\[
\left( \sum_{i=1}^{N} a_i \right)^3 = \sum_{i=1}^{N} a_i^3 + 3 \sum_{i=1,j=1}^{N,N} a_i^2 a_j + \sum_{i=1,j,k=1}^{N,N,N} a_i a_j a_k
\]

Theorem 31 (Weyl’s theorem; Theorem 4.11, p. 204 in [19]). Let \( A, E \) be given \( m \times n \) matrices with \( m \geq n \), then

\[
\max_{i \in [n]} |\sigma_i(A) - \sigma_i(A+E)| \leq \|E\|_2
\]

Theorem 32 (Wedin’s theorem; Theorem 4.11, p. 204 in [19]). Let \( A, E \in \mathbb{R}^{m,n} \) with \( m \geq n \) and \( \hat{A} = A + E \), the following be the singular value decomposition of \( A \)

\[
\begin{bmatrix}
U_1^T \quad U_2^T
\end{bmatrix}
A
\begin{bmatrix}
V_1 \\
V_2
\end{bmatrix}
= \begin{bmatrix}
\Sigma_1 & 0 \\
0 & \Sigma_2
\end{bmatrix}
\]

Let \( \hat{A} \) have a similar decomposition, with \( (U_1, \hat{U}_2, \hat{U}_3, \Sigma_1, \Sigma_2, \hat{V}_1, \hat{V}_2) \). And let \( \Phi \) is the matrix of canonical angels between \( \text{range}(U_1) \) and \( \text{range}(\hat{U}_1) \) and \( \Theta \) is the matrix of canonical angels between \( \text{range}(V_1) \) and \( \text{range}(\hat{V}_1) \). If there exists an \( \delta, \alpha > 0 \) such that \( \min_i |\sigma_i(\Sigma_1) - \sigma_i(\Sigma_2)| \geq \alpha + \delta \) and \( \max_i |\sigma_i(\Sigma_2)| \), then

\[
\max(\|\Phi\|_2, \|\Theta\|_2) \leq \frac{\|E\|_2}{\delta}.
\]

Theorem 33 (Bound on the norm of a Gaussian Random Variable). Let \( n \) be a Gaussian \( \mathcal{N}(0, \sigma) \). Then \( \mathbb{P}(|n| \leq t) \geq 1 - 2e^{-\frac{t^2}{2\sigma^2}} \)

Theorem 34 (Bound on the norm of a Gaussian Vector). Let \( Y \sim \mathcal{N}(0, \sigma I_k) \), then \( \mathbb{P}(|\|Y\|^2_2 \geq \sigma^2(k + 2\sqrt{k}t + 2t)| \leq e^{-t} \).
Proof. The proof is immediate from Theorem 2.1 in [13] with $A = I, \mu = 0$. 

**Theorem 35** (Bound on the spectral norm of a Gaussian Matrix [21]). Let $E \in \mathbb{R}^{d \times d}$ be a symmetric Gaussian matrix with elements sampled iid from $\mathcal{N}(0, \sigma)$, then $\mathbb{P}[\|E\|_2 = O(\sqrt{d}\sigma)] \geq 1 - \text{negl}(d)$.

**Theorem 36** (Bound on the spectral norm of a Gaussian Tensor [22]). Let $E$ be a $K$th order tensor with each $E_{i_1, \ldots, i_K}$ be sampled i.i.d. from a Gaussian $\mathcal{N}(0, \sigma)$, then $\mathbb{P}[\|E\|_2 \leq \sqrt{8\sigma^2(\sum_{i=1}^{K} d_i) \ln(2K/K_0) + \ln(2/\delta)}] \geq 1 - \delta$, where $K_0 = \ln(3/2)$. Note by extension the bound also holds if the tensor is symmetric as well.