Private Counting of Distinct and $k$-Occurring Items in Time Windows

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Abstract

In this work, we study the task of estimating the numbers of distinct and $k$-occurring items in a time window under the constraint of differential privacy (DP). We consider several variants depending on whether the queries are on general time windows (between times $t_1$ and $t_2$), or are restricted to being cumulative (between times $1$ and $t_2$), and depending on whether the DP neighboring relation is event-level or the more stringent item-level. We obtain nearly tight upper and lower bounds on the errors of DP algorithms for these problems. En route, we obtain an event-level DP algorithm for estimating, at each time step, the number of distinct items seen over the last $W$ updates with error polylogarithmic in $W$; this answers an open question of Bolot et al. (ICDT 2013).

2012 ACM Subject Classification  Theory of computation → Theory of database privacy and security

Keywords and phrases  Differential Privacy, Algorithms, Distinct Elements, Time Windows

Digital Object Identifier  10.4230/LIPIcs.ITCS.2023.55

Related Version  Full Version: https://arxiv.org/abs/2211.11718

1 Introduction

Counting distinct elements is a fundamental algorithmic problem with numerous applications across different areas including data mining [42, 1, 29, 37], computational advertising [32, 13], computational biology [7, 4], graph analysis [38], network security [3, 23], query optimization [41, 40]. Another related problem is that of counting $k$-occurring items in which we wish to count the number of items that occur at least $k$ times in the data for some given parameter $k$. Estimating distinct and $k$-occurring elements over extended time periods capture natural use cases, where we wish to study these estimates over particular time windows.

A concrete application stems from advertising where these problems correspond to the so-called reach and frequency$^1$ estimates of an ad campaign, considered two of the most useful metrics [32, 13]. These estimates can then be used to train machine learning (ML) models. For example, it is common for advertisers to seek to optimize the reach and frequency of their campaigns subject to a fixed ad spend budget; they can train an ML model to forecast the reach and frequency histogram for any ad spend budget [12]. Constructing the training dataset for this task in turns entails estimating the reach and frequency for a given ad spend budget. Doing so runs the risk of leaking sensitive information about user activity on the publisher sites, which motivates the study of privately estimating reach and frequency [25].

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$^1$ Here, reach is the number of individuals (or households) exposed to an ad campaign, whereas the $i$th frequency $f_i$ denotes the number of individuals exposed to the ad campaign exactly $i$ times [47].
Increased user awareness and regulatory scrutiny have led to significant research on privacy-preserving algorithms. Differential privacy (DP) [16, 15] has emerged as a widely popular method for quantifying the privacy of algorithms. Loosely, DP dictates that the output of the (randomized) algorithm remains statistically indistinguishable when a dataset is replaced by a “neighboring” dataset (which differs on the contributions from a single user).

**Definition 1 (Differential Privacy (DP))**\[16, 15\]. For \(\epsilon, \delta \geq 0\), an algorithm \(\mathcal{M}\) is said to be \((\epsilon, \delta)\)-differentially private (i.e., \((\epsilon, \delta)\)-DP) if, for any neighboring input datasets \(D, D'\) and any set \(O\) of outputs, we have \(\Pr[\mathcal{M}(D) \in O] \leq e^\epsilon \cdot \Pr[\mathcal{M}(D') \in O] + \delta\).

For \(\delta = 0\), the algorithm is said to be pure-DP, for which we abbreviate \((\epsilon, \delta)\)-DP as just \(\epsilon\)-DP. Otherwise, when \(\delta > 0\), the algorithm is said to be approximate-DP.

For counting distinct elements, previous work on DP algorithms includes [43, 11, 25], all of which have focused on the case of static datasets, i.e., those on which the queries are evaluated once. However, in practice, datasets that are collected over an extended time period are more prevalent. More precisely, at each time step \(t\), a histogram \(S^t\) of universe elements (contributed from multiple users) is added to the dataset. This setting – of estimating queries over a varying dataset and over an extended time period – is the focus of our work.

**Our Contributions**

We first study the task of counting distinct elements (or “reach”) over an extended time period. We consider three natural types of queries depending on the underlying time intervals: *time-window* queries (spanning time steps \(i\) through \(j\) for all \(1 \leq i \leq j \leq T\), where \(T\) is a given fixed time horizon parameter), *cumulative* queries (with the restriction that \(i = 1\)), and *fixed-window* queries (with the restriction that \(j - i = W - 1\) for a fixed \(W\)). We moreover consider the two most natural DP neighboring relations: *item-level* DP (where both datasets are neighboring if they differ on a single universe element’s occurrences across all time steps), and *event-level* DP (where two datasets are neighboring if they differ on a single universe element’s occurrence for a single time step). In addition to the count distinct problem, we also study the closely related task of estimating the number of universe elements that appear at least \(k\) times in the input dataset; we call such an item \(k\)-occurring. Finally, we consider both the *singleton* setting where a single item arrives at each time step, as well as the *bundle* setting where no such restriction is enforced. For each combination of the previous choices, we prove nearly tight upper and lower bounds on the errors of both pure- and approximate-DP algorithms; see Table 1.

**2 Setting**

Let \([U]\) be a universe of *items*.\(^2\) Let \(T\) be the number of time steps (aka, *time horizon*). Our input dataset is \(S = (S^1, \ldots, S^T)\) where \(S^t\) is a multiset of items at time step \(t \in [T]\), i.e., \(S^t_j\) denotes the number of occurrences of item \(j\) at time step \(t\). Let \(S[t_1, t_2]\) denote \(S^{t_1} + \cdots + S^{t_2}\).

**2.1 Utility Definition**

All problems considered in this work can be viewed as a set \(Q\) of *queries*, where each query \(q \in Q\) maps each dataset \(S\) to a real number. The goal of the algorithm is then to output an estimate \(\tilde{q}_S\) of \(q(S)\). Following [6], we say that an algorithm satisfies \((\alpha, \beta)\)-utility if and only if for every query \(q \in Q\), \(\Pr[|\tilde{q}_S - q(S)| \leq \alpha] \geq 1 - \beta\).

\(^2\) Throughout, we write \([m]\) for a non-negative integer \(m\) as shorthand for \(\{1, \ldots, m\}\).
For simplicity of exposition, we sometimes only state the bound on $\alpha$ (referred to as the “error”) for a small constant $\beta$ (e.g., 0.1). It is simple to see that, in all the cases that we consider, we can get $\beta$ to be arbitrarily small at the cost of at most a $O(1/\beta)2^{\log(1/\beta)}$ multiplicative factor in $\alpha$.

### 2.2 Cumulative, Time-Window, and Fixed-Window Queries

Given a function $g : 2^{[U]} \rightarrow \mathbb{R}$, we consider three problem versions. The first is the *time-window* version, which requires us to compute estimates $e^g_{t_1,t_2}$ of $g(S_{t_1:t_2})$ for all $1 \leq t_1 \leq t_2 \leq T$. The second is the *cumulative* version, which restricts to the case where $t_1 = 1$. The third is the *fixed-window version*, which is the restriction of the time-window version to the case where $t_2 - t_1 = W - 1$ for some $W \in [T]$ (given beforehand). We view each tuple $(g,t_1,t_2)$ as a query for the utility definition above. Moreover, the algorithm should be DP when $e^g_{t_1,t_2}$'s are published simultaneously, for all $t_1$, $t_2$'s of interest.

For simplicity, we assume that the algorithm gets to see all the data (i.e., $S^1, \ldots, S^T$) before answering the queries. Another model (also considered in [6]) is the aforementioned continual release setting where $S^1, \ldots, S^T$ arrives in order and we have to answer each query as soon as we have sufficient information to do so (i.e., the algorithm has to answer $e^g_{t_1,t_2}$ immediately after receiving $S^{t_2}$). As explained below, our algorithms are obtained through reductions to range query problems; since there are continual release algorithms for the latter [18, 10, 20], these imply continual release algorithms for the problems considered in our work too. We omit the full statements and proofs for brevity.

### 2.3 Neighboring Notions

As we consider DP, we have to specify the neighboring notion under which our algorithm should be DP. There are two neighboring relations that we consider in this work:

- **Item-level DP**: Two datasets $(S^1, \ldots, S^T)$ and $(\bar{S}^1, \ldots, \bar{S}^T)$ are neighbors iff there exists $u \in [U]$ for which $S^t_u = \bar{S}^t_{u'}$ for all $u' \neq u$ and all $t \in [T]$. In other words, one results from adding/removing a single item $u \in [U]$ to/from (possibly multiple) multisets.

- **Event-level DP**: Two datasets $(S^1, \ldots, S^T)$ and $(\bar{S}^1, \ldots, \bar{S}^T)$ are neighbors iff $\sum_{t \in [T]} ||S^t - \bar{S}^t||_1 \leq 1$. In other words, one results from adding/removing a single element $u \in [U]$ to/from a single multiset.

We remark that our item-level DP notion coincides with user-level DP as in, e.g., [19].

### 2.4 Counting Distinct and $k$-Occurring Elements

The family of queries we consider is $\text{CntOcc}^{\geq k}(S) := |\{u \in [U] \mid S_u \geq k\}|$, i.e., the number of items that appear at least $k$ times in $S$.

Clearly, $\text{CntOcc}^{\geq 1}$ is counting the distinct elements. We also note that it is possible to define $k$-occurring queries that count the number of items that appear *exactly* $k$ times in $S$. It turns out that the bounds in our definitions above carry over with only constant multiplicative overheads. We omit the details in this version.

### 2.5 Singleton vs Bundle Setting

Several previous works in the literature (e.g., [18, 10, 6]) have considered the setting where $||S^t||_1 \leq 1$ for all $t \in [T]$. We refer to this setting as the *singleton* setting, whereas the setting where no such restriction is enforced will be referred to as the *bundle* setting.
3 Our Contributions

3.1 Highlights of our results

We derive tight utility bounds in all cases up to polylogarithmic factors in $T$ and the dependence on $k$. Our results are summarized in Table 1. The highlights of our results are:

1. **Polylogarithmic errors for all event-level DP problems.** In the event-level DP, we give algorithms for fixed-window and time-window with errors $O(k \log^{1.5} W)$ and $O(k \log^3 T)$ respectively. Note that a cumulative algorithm with $O(\log^{1.5} T)$ error was already presented in [6] and it works even in the item-level DP setting. On the other hand, the same work [6] raised as an open question getting an algorithm with error polylogarithmic in $W$ for the fixed window setting; our aforementioned algorithm answers this question in the affirmative.

2. **Dependence on $k$.** The above upper bound for cumulative does not depend on $k$ whereas those for fixed/time-window grow with $k$. We show that this polynomial dependence on $k$ is necessary.

3. **Lower bounds in other settings.** We show lower bounds of $T^{\Omega(1)}$ and $(T/W)^{\Omega(1)}$ for time-window and fixed-window in the item-level DP setting; we also show matching upper bounds. These lower bounds give separations between event-level and item-level DP.

Table 1 Summary of our results. We use $\tilde{O}(f)$ to hide factors polylogarithmic in $f$. For simplicity of the bounds, we assume that $\epsilon, \beta \in (0, 1]$ are constants, $T > W^2$, and $W > k^2$. Note that we also provide reductions from 1d-range query to cumulative and time-window CntOcc$^{\geq k}$ in event-level DP. However, no concrete bound is shown in the table because, to the best of our knowledge, no non-trivial bound for 1d-range query is known for the utility notion we use, although an $\Omega\left(\frac{\log M}{\epsilon} \right)$ lower bound is known for the stronger $\ell_\infty$ utility notion (see, e.g., [46]).

|               | Cumulative | Fixed-Window | Time-Window |
|---------------|------------|--------------|-------------|
|               | Upper      | Lower        | Upper       | Lower        | Upper       | Lower       |
| Event Level   |            |              |             |              |             |             |
| pure          | $O(k \log^{1.5} W)$ (Cor. 20) | $\Omega(\sqrt{k})$ (Lem. 23, Cor. 27) | $O(k \log^3 T)$ (Cor. 39) | $\Omega(\sqrt{k} + \log T)$ (Cor. 41, Cor. 46) |
| approx.       | $O(k \cdot \frac{\sqrt{T}}{W})$ (Cor. 15) | $\Omega(\sqrt{k} + \frac{\sqrt{T}}{W})$ (Cor. 27, 36) | $O(k \log^2 T)$ (Cor. 40) | $\Omega(\sqrt{k} + \frac{\sqrt{T}}{W})$ (Cor. 41, Cor. 46) |
| Item Level    |            |              |             |              |             |             |
| pure          | $O(\log^{1.5} T)$ (Cor. 15) | $\Omega(\sqrt{k})$ (Lem. 17) | $O(T)$ (Cor. 48) | $\Omega(T)$ (Cor. 55) |
| approx.       | $O(k \cdot \frac{\sqrt{T}}{W})$ (Cor. 31) | $\Omega(\sqrt{k} + \frac{\sqrt{T}}{W})$ (Cor. 27, 37) | $O(\sqrt{T})$ (Cor. 51) | $\Omega(\sqrt{T})$ (Cor. 57) |

We next proceed to describe the main ideas in our algorithms and reductions. For ease of presentation, we group the results together based on proof techniques and only explain the high-level approaches. All omitted details will be formalized later.

3.2 Algorithmic Overview: Event-Level DP Setting

We start our algorithmic overview with the event-level DP setting, which is our main contribution. Our results will establish reductions to the range query problem.
Definition 2 (Range Query). In the d-dimensional range query problem, the input consists of \(x^1, \ldots, x^n \in \{0, \ldots, M\}^d\). The goal is to output, for every \(y^1, y^2 \in \{0, \ldots, M\}^d\), \(r_{y^1,y^2}(x) := |\{j \in [n] \mid y^1 \preceq x^j \preceq y^2\}|\), where \(x \preceq y\) iff \(x_i \leq y_i\) for all \(i \in [d]\).

Here DP is with respect to adding/removing a single \(x^j\). DP algorithms for range query with errors polylogarithmic in \(M\) are known (see Section 4.2). Indeed, DP range query remain an active area of research (e.g., \([28, 33]\)); by reducing to/from range query, any future improvements there can be immediately combined with our reductions to yield better bounds for our problems as well.

To describe the reductions, let us also denote by \(t^u_k\) the time step that an item \(u \in [U]\) is reached for the \(k\)th time\(^3\). The reductions in each case proceed as follows:

### 3.2.1 Cumulative

The reduction was already implicit in [6], but we present it in our framework as a warm-up to the other results. It is simple to see that \(u\) should be counted in the \((1, t)\) query (i.e., the cumulative query ending at time \(t\)) if and only if \(t^u_k \leq t\). The reduction is now clear: for each item \(u\), we create a point \(t^u_k \in [T]\). Running the 1d-range query algorithm on these points then gives us estimates for the cumulative count distinct values.

### 3.2.2 Time-window

First we focus on the \(k = 1\) case. Observe that \(u\) should not be included in the \((t_1, t_2)\) query iff for some \(\ell\), it holds that \(t^u_{\ell} < t_1, t_2 < t^u_{\ell+1}\). This leads to the following reduction to the 2d-range query problem. For each item \(u \in [U]\), we create points \((t^u_{\ell} + 1, t^u_{\ell+1} - 1)\) for all \(\ell\) to answer a \((t_1, t_2)\) time-window query, we use a range query to find the number of points whose\(^4\) first coordinate is at most \(t_1\) and second coordinate is at least \(t_2\), which gives the number of items not included in the desired time query.

It is important to take the “complement” perspective in the above reduction. Specifically, if we were to use the fact that \(u\) should be included iff \(t_1 \leq t^u_\ell \leq t_2\) for some \(\ell\), this will lead to double counting because the condition can be satisfied by multiple values of \(\ell\). However, the condition given in the previous paragraph is satisfied by (at most) a single value of \(\ell\).

For the \(k > 1\) case, we can use a similar approach but change the condition to \(t^u_{\ell} < t_1, t_2 < t^u_{\ell+k}\) (i.e., replacing \(\ell + 1\) with \(\ell + k\)). However, this leads to double counting because it is possible that \(t_{\ell+k} > t_2\) and \(t_1, t_{\ell+1}\) are both less than \(t_1\). To solve this issue, we create another instance of 2d-range query with respect to the condition \(t^u_{\ell+1} < t_1, t_2 < t^u_{\ell+k}\). We then subtract the answer of the second instance from that of the first instance; it can be seen that this results in no double counting. Finally, note that a single event change results in at most \(O(k)\) changes to the instances because at most \(k + 1\) conditions \(t^u_{\ell} < t_1, t_2 < t^u_{\ell+k}\) are affected. By group privacy (Fact 6), this means we may apply the \((O(\epsilon/k), O(\delta/k))\)-DP 2d-range query algorithm and still get \((\epsilon, \delta)\)-DP in our answer.

### 3.2.3 Fixed-window

In the fixed-window case, we can do better than time-window by reducing to 1d- rather than 2d-range query. Again, let us start from the \(k = 1\) case. Recall from the above that an item \(u \in [U]\) should not be included in the fixed-window query \((t, t + W - 1)\) iff

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3 Note that in the actual algorithm, we need to handle the case where \(S^u_t > 1\) and also handle the boundaries. For a formal treatment of the modification required for handling this, see Section 4.

4 This corresponds to a range query in Definition 2 with \(y^1 = (-\infty, t_2), y^2 = (t_1, \infty)\).

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$t_u^w < t, t + W - 1 < t_{u+1}^w$. This condition can be further simplified as $t \in (t_u^w, t_{u+1}^w - W]$. Therefore, we may create two instances of the 1d-range query problem: if $t_{u+1}^w - t_u^w > W$, add $t_u^w$ to the first instance and $t_{u+1}^w - W$ to the second instance. To count the number of items $u$ such that $t \in (t_u^w, t_{u+1}^w - W]$, we query for the points of value at most $t$ in each instance and then subtract the first answer from the second answer. This gives us the number of items that should not be included, which we subtract from $U$.

For $k > 1$, we start from the condition $t \in (t_u^w, t_{u+k}^w - W]$. Naturally, this gives us the following algorithm: if $t_{u+k}^w - t_u^w > W$, add $t_u^w$ to the first instance and $t_{u+k}^w - W$ to the second instance. (Then, proceed as above.) Unfortunately, this again results in double counting when $t_{u+k}^w \geq t + W$ and $t_u^w, t_{u+1}^w$ are both less than $t$. To resolve this, we change the second point from $t_{u+k}^w - W$ to $\min\{t_{u+1}^w, t_{u+k}^w - W\}$.

The above ideas are sufficient for a bound of the form $k \cdot \log(O(1)) T$. To decrease log $T$ to log $W$, we make the following observation: since the window length is fixed to $W$, we may run the above algorithm for time steps $1, \ldots, 2W$, then run it again for $W + 1, \ldots, 3W$, then for $2W + 1, \ldots, 4W$ and so on. This suffices to answer any fixed-window query of length $W$. Furthermore, observe that each event-level change only affects two runs, meaning that we only need each run to be $(\epsilon/2, \delta/2)$-DP. Since each run now has time horizon only $2W$, we have decreased log $T$ to log $W$ as desired.

3.3 Algorithmic Overview: Item-Level DP Setting

We next briefly describe how to extend each result to the item-level DP case. First, we remark that the above cumulative algorithm actually works also with item-level DP because each item contributes to at most one point in the 1d-range query instance. As for the time-window and fixed-window settings, they are solved by reducing to multiple calls of easier problems and using composition theorems:

3.3.1 Time-window

We run the cumulative algorithm starting at each time step. Via the basic or advanced composition theorems (Theorems 3 and 4), it suffices for each cumulative algorithm to be $(\epsilon/T)$-DP and $O(\epsilon/\sqrt{T \log(1/\delta)})$-DP respectively, which yields the nearly tight bounds.

3.3.2 Fixed-window

This case is more subtle. We start by observing that the above event-level DP algorithm for $T = 2W$ in fact works even in the item-level DP setting: the reason is that the condition $t_{u+k}^w - t_u^w > W$ cannot occur for two different indices $t$ that are at least $k$ apart. (Otherwise, we would have $2W > 2W$, a contradiction.) This means that we can still solve each subinstance of time horizon $2W$ with error $k \cdot \log O(1) W$. Since there are $O(T/W)$ subinstances (in the reduction described above), we can use the basic/advanced composition over them. This gives us the final $(T/W)^{O(1)} \cdot \log O(1) W$ bounds.

Finally, we remark that we can sometimes obtain improvements in the singleton setting. We do so via a technique from [30] where the $T$ time steps are partitioned into $[T/T']$ grouped time steps, each consisting of $\leq T'$ consecutive time steps. The main observation is that we may run the bundle algorithm on the $[T/T']$ grouped time steps and incur an additional error of at most $T'$ due to the grouping. By picking $T'$ appropriately, this yields the bounds for our time-window algorithms in the singleton setting.
3.4 Overview of the Lower Bounds

At a high-level, we prove two types of lower bounds. First are the lower bounds from 1d-/2d-range query. These are done by essentially “reversing” our algorithms explained above. In fact, these reductions show that the cumulative, fixed-window, and time-window CutOcc_{≥k} are equivalent to 1d-, 1d-, and 2d-range query, respectively, up to a multiplicative factor of $O(k)$ in the values of $\epsilon$, $\delta$.

The second type of reduction is from (variants of) the 1-way marginal problem. Recall that in the 1-way marginal problem each user receives $x^j \in \{0, 1\}^d$ and the goal is to compute $\tilde{x} := \sum_{j \in [n]} x^j$. Lower bounds of $d^{Ω(1)}$ are well-known for this problem [27, 8].

3.4.1 Polynomial in $T$ and $T/W$ Lower Bounds

Let us start with the simpler reductions for lower bounds of the form $(T/W)^{Ω(1)}$ and $(T)^{Ω(1)}$ for fixed-window and time-window queries in item-level DP. For the ease of presentation, let us consider the time-window bundle setting with $k = 1$ as an example. In this case, we let $d = T, U = n$, and $S^j_i = x^j_i$ for all $j \in [n], i \in [d]$ (i.e., “copying” each vector $x^j$ into the occurrence pattern of item $j$). It can be immediately seen that CutOcc_{≥k}(S^j_i) is exactly equal to $\tilde{x}_i$, which translates the lower bounds of $d^{Ω(1)} = T^{Ω(1)}$ from 1-way marginals to time-window bundle CutOcc_{≥k}.

We remark that our reductions bear similarities to those in [27, 8], although, given that different problems are studied in that work, the reductions and arguments are not the same.

3.4.2 Polynomial in $k$ Lower Bounds

Lower bounds of the form $k^{Ω(1)}$ are also shown via reductions from (variants of) 1-way marginals with $d = Θ(k)$. However, these are more challenging than the above as we have to handle the event-level DP setting. Notice that in the above reduction each user corresponds to a vector $x^j \in \{0, 1\}^d$ and changing this vector can cause the change in $S^j_i$ for all $i \in [T]$, which invalidates its use in the event-level DP setting. For event-level DP reductions, we have to somehow be able to encode the entire information of the $d$-dimensional vector $x^j \in \{0, 1\}^d$ while ensuring that changing this entire vector only causes a single event change.

The aforementioned challenges led us to the idea of creating multiple items for each possible value of $x^j_i$. In other words, we now create an item for each $(j, x) \in [n] \times \{0, 1\}^d$. Each of these items is “dormant”, meaning that it does not contribute to CutOcc_{≥k} queries at all. However, each item $(j, x)$ can be “turned on” by a single event change, after which it contributes to the queries in a pattern designated by the vector $x$. This is done by ensuring that for each $i \in [d]$ there are prespecified $t_i, t'_i$ such that $S^j_{(i, x)}$ is equal to $k - 1 + x_i$. The event-level change to turn on $(j, x)$ is to increase $S^j_{(i, x)}$ such that $t \in [t_i, t'_i]$ for all $i \in [d]$. (Note that, since we are aiming for event-level change, we need to use a single $t$ that satisfies $t \in [t_i, t'_i]$ simultaneously for all $i \in [d]$.) After turning on $(j, x)$ for all $j \in [n]$, we thus have $\tilde{x}_j = \text{CutOcc}_{≥k}(S^j_{(i, x)})$, allowing us to compute the 1-way marginal. This concludes the main idea of the reduction.

While the sketch above suffices for the bundle setting, it is not enough for the singleton case. This is because we created as many as $2^d = 2^{Ω(k)}$ dormant items, which requires $T > 2^d$ in the singleton case. This is too large to yield any meaningful results. Instead, we turn to the so-called linear queries problem, which may be viewed as a restriction of 1-way marginals where each $x_j$ belongs to a (predefined) small set of size $d^{Ω(1)}$. (See Section 4.4
for a formal definition.) A classic result of [14] showed that, even in this restricted version, any DP algorithm has to incur an error of $d^{R(1)}$. Since we now only have to create dormant items $(j, x)$ for $x$ that belongs to the set, we can now let $T$ be as small as $d^{O(1)}$.

### 3.5 Related Work

A closely related line of work is the continual release model of DP, which was first introduced in the work of Dwork et al. [18], and Chan et al. [10]. For the binary summation task in this model, they designed the binary tree mechanism, which achieves an additive error polylogarithmic in $T$; Dwork et al. [18] also showed that a logarithmic error is needed. The binary tree mechanism was extended to the task of estimating weighted sums with exponentially decaying coefficients [6], and sums of bounded real values [39]. The continuous release model of DP was later studied on graph data and statistics in [44, 24], and the aforementioned DP binary tree mechanism was used in the context of DP online learning [31, 26, 2]. Very recently, Jain et al. [30] show that tasks closely related to summation, namely requiring the selection of the largest of a set of sums (and previously studied by [9]), have to incur an error polynomial in $T$ in the continuous release model, and are fundamentally harder than in the standard setting. We remark that this separation implies a polynomial separation between cumulative queries and a single query (for their problems) in our terminologies. This is a different behavior compared to the CountDistinct and CntOcc problem studied in our work, as there is a polylogarithmic-error algorithm (Lemma 14). In fact, we use Jain et al.’s [30] technique in our upper bound for the item-level setting. Our lower bound reductions bear some similarities to theirs, but there are some fundamental differences. For example, we crucially use the structure of CntOcc when proving $k^{Ω(1)}$ lower bounds.

CountDistinct and CntOcc have also been studied in the more challenging pan-private model [19, 35], where the DP constraint is applied not only to the output but also to the internal memory of the algorithm (after any time step); hence, the utility guarantees are much weaker than the algorithms we present (and those in [6]). Mir et al. [35] prove a strong lower bound in this model: any pan-private algorithm for CountDistinct must incur an additive error of at least $Ω(\sqrt{U})$. Due to such a lower bound, [19, 35] design several algorithms for CountDistinct and CntOcc with guarantees that include both multiplicative and additive errors. This is in contrast to our algorithms, which do not require any multiplicative errors and have additive error bounds that are completely independent of $U$.

We also note that CountDistinct has been studied in the local, and shuffle settings of DP (see, [5, 11]), although these results focus on the single-query (i.e. $T = 1$) setting.

### 4 Preliminaries

In the algorithms below, we will assume that the input is appended at the beginning and at the end with multisets $S^0, S^{T+1}$ such that $S^0_u = S^{T+1}_u = k$ for all $u \in [U]$. This helps simplify the notation in the reductions below. For each item $u \in [U]$, it is also convenient to define the sequence $0 = t^{u}_1 \leq \cdots \leq t^{u}_{m_u} = T + 1$ of time steps $u$ is reached; this is the sequence where each $t \in \{0, 1, \ldots, T + 1\}$ appears $S^{t}_u$ times. We use $S^{≤t}$ for $S^1 + \cdots + S^t$.

### 4.1 Tools from Differential Privacy

We list below several tools that we will need from the DP literature. We refer the reader to the monographs [21, 46] for a thorough overview of DP.
4.1.1 Composition Theorems

We will use the following composition theorems throughout this paper.

▶ **Theorem 3 (Basic Composition).** For any \(\epsilon, \delta > 0, k \in \mathbb{N}\), an algorithm that is a result of running \(k\) mechanisms, each of which is \((\epsilon/k, \delta/k)\)-DP, is \((\epsilon, \delta)\)-DP.

▶ **Theorem 4 (Advanced Composition [22, 21]).** For any \(\epsilon, \delta \in (0, 1), k \in \mathbb{N}\), an algorithm that is a result of running \(k\) mechanisms, each of which is \(\left(\frac{\epsilon}{\sqrt{2k \ln(2/\delta)}}, \frac{\delta}{2k}\right)\)-DP, is \((\epsilon, \delta)\)-DP.

▶ **Theorem 5 (Parallel Composition [34]).** Let \(\varphi_1, \ldots, \varphi_t\) be deterministic functions that map a dataset to another dataset such that, for any neighboring datasets \(D, D'\), there exists \(j \in [t]\) for which \(\varphi_i(D) = \varphi_i(D')\) for all \(i \neq j\) and \(\varphi_j(D), \varphi_j(D')\) are neighbors. Then, an algorithm that is a result of running an \((\epsilon, \delta)\)-DP algorithm on each of \(\varphi_1(D), \ldots, \varphi_t(D)\) is \((\epsilon, \delta)\)-DP.

4.1.2 Group Privacy

Let \(\sim\) denote any neighboring relationship. For any \(k \in \mathbb{N}\), let \(\sim_k\) denote the neighboring relationship where \(D \sim_k D'\) if and only if there exists \(D = D_1, \ldots, D_k = D'\) such that \(D_i \sim D_{i+1}\) for all \(i \in [k-1]\). The following so-called group privacy bound is well-known.

▶ **Fact 6 (Group Privacy (e.g., [45])).** Let \(\epsilon, \delta > 0, k \in \mathbb{N}\). Any algorithm \(M\) that is \((\epsilon, \delta)\)-DP under a neighboring relationship \(\sim\) is \((k\epsilon, \frac{e^k-1}{e^k-1}\delta)\)-DP under the relationship \(\sim_k\).

4.2 Range Query

Here, we view each \(r_{y_1, y_2}\) as a query, and the definition of utility is the same as in Section 2. The following algorithms for range query are known from previous works:

▶ **Theorem 7 ([18, 10]).** For any \(\epsilon > 0\), there is an \(\epsilon\)-DP algorithm for the 1d-range query problem with \(O(\log^{1.5} M \cdot \log(1/\beta)/\epsilon, \beta)\)-utility.

▶ **Theorem 8 ([20]).** For any \(\epsilon > 0\), there is an \(\epsilon\)-DP algorithm for the 2d-range query problem with \(O(\log^3 M \cdot \log(1/\beta)/\epsilon, \beta)\)-utility.

Although the following was not stated explicitly in [20], it is not hard to derive, by simply replacing Laplace noise by Gaussian noise.

▶ **Theorem 9 ([20]).** For any sufficiently small constants \(\epsilon, \delta, \beta > 0\), there is an \((\epsilon, \delta)\)-DP algorithm for the 2d-range query problem with \(O(\log^2 M \cdot \sqrt{\log(1/\delta) \log(1/\beta)}/\epsilon, \beta)\)-utility.

As for the lower bound, we are not aware of any non-trivial lower bound for the 1d case for the utility notion we use, although lower bounds for \(\ell_\infty\)-error are known (see e.g. [46]). The lower bound for the 2d case is stated below.

▶ **Theorem 10 ([36]).** For any sufficiently small constants \(\epsilon, \delta, \beta > 0\), there is no \((\epsilon, \delta)\)-DP algorithm with \(o(\log M), \beta\)-utility for the 2d-range query problem even when the number of input points \(n\) is \(O(M^2)\).
4.3 1-Way Marginal

In the 1-way marginal problem, we are given $x^1, \ldots, x^n \in \{0,1\}^d$, and the goal is to output an estimate of $\bar{x} = x^1 + \cdots + x^n$. Here we view each $\bar{x}_j$ as a query, and the definition of utility is in Section 2.

▶ Theorem 11 ([27]). For any $\epsilon \in (0,1]$, there is no $\epsilon$-DP algorithm with $(o(\min\{n,d/\epsilon\}),0.01)$-utility for the 1-way marginal problem.

▶ Theorem 12 ([45]). For any $\epsilon \in (0,1)$ and $\delta \in (2^{-\Omega(n)}, 1/n^{1+\Omega(1)})$, there is no $(\epsilon,\delta)$-DP algorithm with $(o(\min\{n,\sqrt{d\log(1/\delta)/\epsilon}\},0.01)$-utility for the 1-way marginal problem.

4.4 Linear Queries

The linear query problem is parameterized by a set of (public) vectors $a_1, \ldots, a_d \in \{0,1\}^m$. The input to the algorithm is then a vector $x \in \{0,1\}^m$ and the goal is to estimate $\langle a_i, x \rangle$ for each $i \in [d]$. We view each $a_i$ as a query and utility is defined as in Section 2.

The neighboring relation for DP of linear queries is with respect to changing a single coordinate of $x$. For convenience, we may also think of the queries as the matrix $A \in \{0,1\}^{d \times m}$ where $a_i$ is the $i$th row of $A$. Under this notation, we have $(Ax)_i = \langle a_i, x \rangle$. We write “$A$-linear query” to signify the matrix $A$.

The following lower bound was shown in [17] and was an improvement over the classical lower bound of Dinur and Nissim [14].

▶ Theorem 13 ([17]). For any $m \in \mathbb{N}$ and any sufficiently small constants $\epsilon, \delta > 0$, there exists a matrix $A \in \{0,1\}^{d \times m}$ with $d = O(m)$ such that there is no $(\epsilon,\delta)$-DP algorithm with $(o(\sqrt{m}),0.01)$-utility for the $A$-linear query problem.

4.5 Assumptions on Parameters

Certain settings of parameters can lead to “degenerate” cases, for which there are uninformative algorithms that can get better errors. To avoid such scenarios, we will assume the following setting of parameters throughout the paper when we prove our lower bounds:

- $T, W \geq k \log k$. This is due to the fact that, in the singleton setting, the algorithm that outputs zero always gets an error of at most $T/k$ or $W/k$, which can be smaller than meaningful algorithms when $T, W$ are not much larger than $k$.
- $T \geq (1 + \Omega(1))W$. This is due to the fact that there are only $T - W + 1$ queries in the fixed-window setting, meaning that even, e.g., the Laplace mechanism would yield an error of $O(\epsilon(T - W))$. This can be small if $T - W$ is very small.

We stress that our upper bounds work for all settings of parameters (even those violating the above assumptions), but we assume the above for simplicity in the lower bound statements.

For simplicity of utility expressions, we assume throughout that $\epsilon \in (0,1], \delta \in [0,1/2)$ (including both in the upper and lower bounds). Our results can be extended to the $\epsilon > 1$ case but the utility expressions are more complicated because, e.g., the advanced composition theorem [22] has a more complicated expression in this case.

5 Cumulative Queries

We are now ready to prove our results, starting with algorithms and lower bounds for cumulative CntOcc$^{\geq k}$. 
5.1 Algorithm

As stated earlier, a similar algorithm was already derived in [6] but with less emphasis on
the relationship with 1d-range query. We provide a proof below both for completeness and
for providing a formal relationship with 1d-range query.

Lemma 14. Let \( k \in \mathbb{N} \) be any positive integer. If there exists an \((\epsilon, \delta)\)-DP algorithm
for 1d-range query with utility \((\alpha(M, \epsilon, \delta, \beta), \beta)\), then there exists an \((\epsilon, \delta)\)-DP algorithm for
cumulative \text{CntOcc} \geq k in the item-level DP and bundle setting with utility \((\alpha(T + 1, \epsilon, \delta, \beta), \beta)\).

Proof. Let \( M = T + 1 \) and \( n = d \). For each item \( u \in [d] \), we then define \( x_u := t_u^k \), i.e.,
the first time step before which (inclusive) \( u \) has appeared \( k \) times. It is not hard to see that
the prefix query \( r_{1,t} \) is exactly equal to \text{CntOcc} \geq k \((S_{\leq t})\). Therefore, we can run the \((\epsilon, \delta)\)-DP algorithm for 1d-range query, which yields the desired error. Finally, notice that each item
contributes to only one input point to the 1d-range query problem; therefore, the algorithm
remains \((\epsilon, \delta)\)-DP under the item-level neighboring notion.

Plugging the above into known algorithm for 1d-range query (Theorem 7) yields:

Corollary 15. For any \( k \in \mathbb{N} \) and \( \epsilon > 0 \), there is an \( \epsilon \)-DP algorithm for cumulative
\text{CntOcc} \geq k in the item-level DP and bundle setting with \( O(\log^{1.5} T \log(1/\beta)/\epsilon, \beta)\)-utility.

5.2 Lower Bounds

We now prove a lower bound showing a reverse reduction – from 1d-range query to cumulative
\text{CntOcc} \geq k – complementing our algorithm in Lemma 14. This shows that the two problems
are equivalent (up to a constant factor in the error). We remark that our lower bounds below
hold even in the more stringent event-level DP setting.

We start with a slightly simpler reduction in the bundle setting:

Lemma 16. Let \( k \) be any positive integer and \( \epsilon, \delta > 0 \). If there exists an \((\epsilon, \delta)\)-DP algorithm
for cumulative \text{CntOcc} \geq k in the event-level DP and bundle setting with \((\alpha(T, \beta), \beta)\)-utility,
then there exists an \((\epsilon, \delta)\)-DP algorithm for 1d-range query with \((2 \cdot \alpha(M, \beta/2), \beta)\)-utility.

Proof. Let \( T = M \) and let \( U \) be sufficiently large (i.e., larger than the dataset size of
the 1d-range query). Given an input \( x^1, \ldots, x^n \in [M] \) to the 1d-range query problem, we
construct an input to the cumulative \text{CntOcc} \geq k algorithm as follows:

- For all \( u \in [U] \), let \( S_u^1 = k - 1 \) and \( S_u^2 = \cdots = S_u^T = 0 \).
- For all \( j \in [n] \), increment \( S_{ij} \) by one.

It is simple to see that, for all \( t \in [T] \), \text{CntOcc} \geq k \((S_{\leq t}) = r_{0,t}(x) \). Therefore, we may answer
any range query \( r_{y_1, y_2}(x) \) by outputting \text{CntOcc} \geq k \((S_{\leq y_2}) - \text{CntOcc} \geq k \((S_{\leq y_1 - 1}) \). This is
indeed an \((\epsilon, \delta)\)-DP algorithm for 1d-range query with \((2 \cdot \alpha(M, \beta/2), \beta)\)-utility.

In the singleton setting, we cannot use the above reduction directly since the first step in the
previous reduction contains \( k - 1 \) copies of \( u \)’s at the same time step. Therefore, we have to “expand” this set into \((k - 1)n \) sets where \( n \) denote the number of input points in the
1d-range query problem. This results in a slight additive blow up of \((k - 1)n \) in the time horizon.
Similarly, the second step in the reduction can increment multiple values at the
same time step; to avoid this, we have to pay another multiplicative blow up of \( k - 1 \). These
are formalized below.
Lemma 17. Let $k$ be any positive integer and $\epsilon, \delta > 0$. If there exists an $(\epsilon, \delta)$-DP algorithm for cumulative $\text{CntOcc}^{\geq k}$ in the event-level DP and singleton setting with $(\alpha(T, \beta), \beta)$-utility, then there exists an $(\epsilon, \delta)$-DP algorithm for 1d-range query on at most $n$ input points with $(2 \cdot \alpha((M + k - 1)n, \beta/2), \beta)$-utility.

Proof. Let $T = (M + k - 1)n$ and let $U = n$. Given an input $x^1, \ldots, x^{n'} \in [M]$ to the 1d-range query problem where $n' \leq n$, we construct an input to the cumulative $\text{CntOcc}^{\geq k}$ algorithm as follows:

- For all $u \in [U]$, let $S_u^{(k-1)(u-1)+1} = \cdots = S_u^{(k-1)u} = 1$ and $S_u^{t} = 0$ for all $t \in [T] \setminus ((k - 1)(u - 1) + 1, \ldots, (k - 1)u)$.

- For all $j \in [n]$, increment $S_j^{n(k-1)+u(x^j-1)+j}$ by one.

It is not hard to see that this is a singleton instance. Similar to before, we can answer any range query $r_{y_1, y_2}(x)$ by outputting $\text{CntOcc}^{\geq k}(S^{\leq n(y_2-k-1)}) - \text{CntOcc}^{\geq k}(S^{\leq n(y_1+k-2)})$. This gives an $(\epsilon, \delta)$-DP algorithm for 1d-range query with $(2 \cdot \alpha((M + k - 1)n, \beta/2), \beta)$-utility.

6 Fixed-Window Queries

We next move on to prove our bounds for fixed-window queries. Since the bounds are different for the two DP notions, we start with event-level DP and then consider item-level DP.

6.1 Event-Level DP

6.1.1 Time Horizon Reduction: Proof of Lemma 18

We start with a lemma that allows us to reduce the time horizon $T$ to $2W$ in this setting. The proof of this lemma follows the overview discussed in Section 3.2.

Lemma 18. For any $k \in \mathbb{N}$, if there exists an $(\epsilon, \delta)$-DP algorithm for fixed-window $\text{CntOcc}^{\geq k}$ with $(\alpha(T, \epsilon, \delta, \beta), \beta)$-utility with event-level DP, there exists an $(\epsilon, \delta)$-DP algorithm with $(\alpha(2W, \epsilon/2, \delta/2, \beta), \beta)$-utility in the same setting.

Proof. Let $A$ denote the algorithm for the former and $\epsilon' = \epsilon/2, \delta' = \delta/2$. The new algorithm works as follows:

- For all $j \in \lfloor [T/W] \rfloor$, run a separate copy of the $(\epsilon', \delta')$-DP algorithm $A$ on $S^{(j-1)W+1}, \ldots, S^{(j+1)W}$.

- When we would like to answer the query for $i, i + W - 1$, use the $[i/T + 1]$th copy of $A$.

We can apply the parallel composition theorem (Theorem 5) on the copies with $j = 1, 2, \ldots$; this implies that the combined algorithm for such $j$’s is $(\epsilon', \delta')$-DP. Similarly, we have that the combined algorithm for $j = 2, \ldots$ is also $(\epsilon', \delta')$-DP. Then, applying basic composition ensures that the entire algorithm is indeed $(2\epsilon', 2\delta') = (\epsilon, \delta)$-DP.

The claimed accuracy follows immediately from definition.

6.1.2 Algorithm

Given Lemma 18, we will focus only on designing the algorithm for the $T = 2W$ case. Here we show a reduction to 1d-range query. We remark that the algorithm below works even in the item-level DP setting; indeed, we will also use it as a subroutine for item-level DP.

Lemma 19. If there is an $(\epsilon, \delta)$-DP algorithm for 1d-range query with $(\alpha(M, \epsilon, \delta, \beta), \beta)$-utility, then there is an $(\epsilon, \delta)$-DP algorithm for fixed-window $\text{CntOcc}^{\geq k}$ in the event-level DP and bundle setting when $T = 2W$ with $(2 \cdot \alpha(T + 1, \epsilon/2W, \delta/2, \beta))$-utility for every $k \in \mathbb{N}$.
Proof. Let $M = T + 1$. We will in fact create two instances of 1d-range query. For clarity, we will call the first instance $x$ and the second instance $x'$. The instances are as follows:

1. Recall the definition of $m_\alpha$ and $t_\alpha^1, \ldots, t_\alpha^{m_\alpha}$ from Section 4.

   For all $u \in [U]$ and $\ell = 1, \ldots, m_u - k$, do the following:
   
   a. If $t_\alpha^{\ell + k} - t_\alpha^{\ell} > W$, add $t_\alpha^{\ell}$ to $x$ and set $t_\alpha^{\ell + k} = W$ so that the time horizon is $T = 2W$.

   b. If $t_\alpha^{\ell + k} - t_\alpha^{\ell} < W$, add $t_\alpha^{\ell + k}$ to $x$ and do not change $t_\alpha^{\ell}$.

   c. If $t_\alpha^{\ell + k} - t_\alpha^{\ell} = W$, add both $t_\alpha^{\ell}$ and $t_\alpha^{\ell + k}$ to $x$.

2. We then run the $(\frac{T}{W}, \frac{W}{k})$ 1d-range query algorithms on both instances $x, x'$ to get estimates $\tilde{r}_{y_1,y_2}(x)$ and $\tilde{r}_{y_1,y_2}(x')$ for all $y_1, y_2 \in \{M\}$.

3. To answer $\text{CntOcc}^2(r_{[i,i+W-1]})$, we output $|U| - \tilde{r}_{1,i} + \tilde{r}_{1,i}'$.

Next, we claim that an item-level change can result in at most $2k$ changes to each of $x, x'$. To prove this, it suffices to show that any given element $u$ contributes to at most $k$ items added to each of $x, x'$. To see that the latter is true, let $\ell'$ be the smallest index for which $t_\alpha^{\ell' + k} - t_\alpha^{\ell'} > W$. Notice that this means $t_\alpha^{\ell' + k} \geq W + 1$. Since the time horizon is $T = 2W$, this means that $t_\alpha^{\ell' + k} - t_\alpha^{\ell'} \leq W$ for all $\ell > \ell' + k$. In other words, the condition in the loop cannot be satisfied for $\ell \notin \{\ell', \ldots, \ell' + k - 1\}$. Thus, the number of points added to each of $x, x'$ is at most $k$.

Given the above claim, we may apply group privacy (Fact 6) to conclude that the entire algorithm is $(\epsilon, \delta)$-DP as desired.

To see its correctness, for each $u \in [U]$, let $\ell^*(u, i)$ denote the last time step it is reached before $i$ (i.e., the largest $\ell$ such that $t_\alpha^{\ell} < i$). Notice that

\[
\text{CntOcc}^2(S^{[i,i+W-1]}) = |U| - |\{u \in U | S^{[i,i+W-1]} = k\}| = |U| - \sum_{u \in [U]} 1{[t_\alpha^{\ell^*(u,i)+k} > i + W - 1].}
\]

Observe that the two elements added in Step (1a) get canceled out for all $\ell \neq \ell^*(u,i)$ in $r_{1,i}(x) - r_{1,i}(x')$. For $\ell = \ell^*(u,i)$, they are not canceled if and only if $t_\alpha^{\ell^*(u,i)+k} - W > i$. Thus,

\[
r_{1,i}(x) - r_{1,i}(x') = \sum_{u \in [U]} 1{[t_\alpha^{\ell^*(u,i)+k} > i + W - 1].}
\]

By combining the above two equations, we then have \(\text{CntOcc}^2(S^{[i,i+W-1]}) = |U| - r_{1,i}(x) + r_{1,i}(x')\). The utility guarantee then follows from that of the 1d-range query algorithm.

Combining Lemma 18, Lemma 19, and Theorem 7 yields the following concrete bound.

**Corollary 20.** For any $k \in \mathbb{N}$ and $\epsilon > 0$, there is an $\epsilon$-DP algorithm for fixed-window CntOcc$^2$ in the event-level DP and bundle setting with $O(k \cdot \log W \log(1/\beta)/\epsilon, \beta)$-utility.

### 6.1.3 Lower Bounds

We prove two lower bounds for the problem, one based on the 1d-range query and the other based on linear queries. The latter shows that a polynomial dependence on $k$ is necessary.

#### 6.1.3.1 Range Query-Based Lower Bounds

We start with the former, which is just a reduction back to the cumulative case.

**Lemma 21.** Let $k$ be any positive integer. If there exists an $(\epsilon, \delta)$-DP algorithm for fixed-window CntOcc$^2$ in the event-level DP and bundle (resp. singleton) setting with $(\alpha(W, \beta), \beta)$-utility, then there exists an $(\epsilon, \delta)$-DP algorithm for cumulative CntOcc$^2$ in the event-level DP and bundle (resp. singleton) setting with $(\alpha(T, \beta), \beta)$-utility.
Private Counting of Distinct and $k$-Occurring Items in Time Windows

**Proof.** Given an input $S_1, \ldots, S_T$ for cumulative CntOcc$^{\geq k}$, we can create an instance $S'_1, \ldots, S'_{2T}$ for fixed-window CntOcc$^{\geq k}$ by letting $W = T$, $S'_1 = \cdots = S'_{2T} = \emptyset$ and $S'_{2T+t} = S_t$ for all $t \in [T]$. It is simple to see that CntOcc$^{\geq k}(S[i:T]) = $ CntOcc$^{\geq k}(S[i+1:t+W])$. Therefore, running the fixed-window algorithm on the new instance allows us to answer the old instance with the same utility.

Plugging the above into Lemma 16 and Lemma 17, we arrive at the following:

**Lemma 22.** Let $k$ be any positive integer and $\epsilon, \delta > 0$. If there exists an ($\epsilon, \delta$)-DP algorithm for fixed-window CntOcc$^{\geq k}$ in the event-level DP and bundle setting with $(\alpha(W, \beta), \beta)$-utility, then there exists an ($\epsilon, \delta$)-DP algorithm for 1d-range query on $2 \cdot \alpha(M, \beta/2)$-$\beta$-utility.

**Lemma 23.** Let $k$ be any positive integer and $\epsilon, \delta > 0$. If there exists an ($\epsilon, \delta$)-DP algorithm for fixed-window CntOcc$^{\geq k}$ in the event-level DP and singleton setting with $(\alpha(W, \beta), \beta)$-utility, then there exists an ($\epsilon, \delta$)-DP algorithm for 1d-range query on at most $n$ input points with $(2 \cdot \alpha((M + k - 1)n, \beta/2), \beta)$-utility.

### 6.1.3.2 Linear Query-Based Lower Bounds

Next, we proceed to prove that a polynomial dependence on $k$ is necessary, by a reduction from linear queries.

**Lemma 24.** Let $W, k$ be any positive integer and let $A$ be any $(d \times m)$ matrix such that $\min\{W, k\} \geq d + 2$. Then, if there exists an ($\epsilon, \delta$)-DP algorithm for fixed-window CntOcc$^{\geq k}$ in the event-level DP and bundle setting with $(\alpha, \beta)$-utility, then there exists an ($\epsilon, \delta$)-DP algorithm for $A$-linear query with the same utility.

**Proof.** Assume w.l.o.g. that $k = W = d + 2$ and $T = 2W$ and let $T = m$. Given an input $x \in \{0, 1\}^m$ to the $A$-linear query problem, we construct an input to the fixed-window CntOcc$^{\geq k}$ algorithm as follows:

- For all $u \in [m]$ and $i \in [d]$, let $S'_i = S'_{i+W-1} = A_{i,u}$.
- Furthermore, for all $u \in [m]$, let $S^{W-1}_u = x_u + k - 2 - \sum_{j \in [d]} A_{j,u}$.

(For all remaining pairs $(u, i) \in [m] \times [T]$ not mentioned above, $S'_u = 0$.)

Notice that, for all $i \in [d]$ and $u \in [m]$, we have

$$S^{i+W-1}_u = \sum_{i' = i}^{i+W-1} S'_{i'} = \left( \sum_{i' = i}^{d} S'_{i'} \right) + S^{W-1}_u + \left( \sum_{i' = i}^{i+W-2} S'_{i'} \right) + S^{i+W-1}_u$$

$$= \left( \sum_{i' = i}^{d} A_{i',u} \right) + \left( x_u + k - 2 - \sum_{j \in [d]} A_{j,u} \right) + \left( \sum_{i' = i}^{i+W-2} A_{i',W+1,u} \right) + A_{i,u}$$

$$= A_{i,u} + x_u + k - 2,$$

which is equal to $k$ if and only if $x_u = 1$ and $A_{i,u} = 1$, and is less than $k$ otherwise. Hence,

$$\text{CntOcc}^{\geq k}(S^{i+W-1}) = (Ax)_i.$$

Therefore, we may run the ($\epsilon, \delta$)-DP algorithm for CntOcc$^{\geq k}$, and compute $Ax$ with the same utility guarantee.

Plugging the above into the known lower bound for linear queries (Theorem 13 with $d, m = \Theta(\min\{k, W\})$), we get a concrete lower bound in terms of $k$:
ε, δ > 0 be any sufficiently small constant. Then, there is no (ε, δ)-DP algorithm for fixed-window \( \text{CntOcc}^{≥k} \) in the event-level DP and bundle setting with \( o(\sqrt{\min\{k, W/k\}}) \), 0.01)-utility.

The above reduction also works in the singleton setting where we have to “spread out” the changes so that each time step involves only a single element.

**Lemma 25.** Let \( W, k \) be any positive integer and let \( A \) be any \( (d \times m) \) matrix such that \( k \geq d + 2 \) and \( W \geq 4dm + 1 \). Then, if there exists an \((ε, δ)-DP\) algorithm for fixed-window \( \text{CntOcc}^{≥k} \) in the event-level DP and singleton setting with \((α, β)\)-utility, then there exists an \((ε, δ)-DP\) algorithm for \( A \)-linear query with the same utility.

**Proof.** Assume w.l.o.g. that \( k = d + 2, W = 4dm + 1 \) and \( T = 2W \). Let \( U = m \). Given an input \( x \in \{0, 1\}^m \) in the \( A \)-linear query problem, we construct an input to the cumulative \( \text{CntOcc}^{≥k} \) algorithm as follows:

- For all \( u \in [m] \) and \( i \in [d] \), let \( S^{mi+u}_u = S^{m(i-1)+u+W-1}_u = A_{i,u} \).
- Furthermore, for all \( u \in [m] \) and \( ℓ \in [d+1] \), let \( S^{W-1-mℓ+u}_u = 1 [ℓ ≤ u + k - 2 - \sum_{j \in [d]} A_{j,u}] \).

(For all remaining \( t \in [T] \) not mentioned above, \( S^t_u = 0 \).)

It is not hard to verify that this is indeed a singleton instance. Furthermore, notice that, for all \( i \in [d] \) and \( u \in [m] \), we have

\[
S^{[mi,mi+W-1]}_u = S^{[mi,mi+4md]}_u = \sum_{i'=1}^{i+4d-1} S^{mi'+u}_u = \sum_{i'=1}^{i+4d} S^{mi'+u}_u + \sum_{i'=i+4d}^{i+4d-2} S^{mi'+u}_u + S^{m(i-1)+u+W-1}_u
\]

\[
= \sum_{j=1}^{d} A_{i',u} + (x_u + k - 2 - \sum_{j \in [d]} A_{j,u}) + \sum_{i'=i+4d}^{i+4d-2} A_{i'-4d+1,u} + A_{i,u}
\]

which is equal to \( k \) if and only if \( x_u = 1 \) and \( A_{i,u} = 1 \), and is less than \( k \) otherwise. Thus,

\[
\text{CntOcc}^{≥k}(S^{[mi,mi+W-1]}_u) = (Ax)_i.
\]

Therefore, we may run the \((ε, δ)-DP\) algorithm for \( \text{CntOcc}^{≥k} \), and compute \( Ax \) with the same utility.

Again, plugging this into the known lower bound for linear queries (Theorem 13 with \( d, m = Θ(\min\{k, W/k\}) \)), we get a concrete lower bound in terms of \( k \):

**Corollary 27.** Let \( W, k \) be any positive integer such that \( W \geq k \) and ε, δ > 0 be any sufficiently small constant. Then, there is no \((ε, δ)-DP\) algorithm for fixed-window \( \text{CntOcc}^{≥k} \) in the event-level DP and singleton setting with \( o(\sqrt{\min\{k, W/k\}}) \), 0.01)-utility.

### 6.2 Item-Level DP

#### 6.2.1 Algorithms

We start with a time-horizon reduction similar to the event-level DP case. However, for item-level DP, we cannot apply the parallel composition theorem (because item-level change can affect all the \( O(T/W) \) subinstances) and instead have to apply basic/advanced composition, resulting in a reduction of \( (T/W)^{O(1)} \) privacy budget to each subinstance.
Lemma 28. Let \( k, T, W \in \mathbb{N} \) be any positive integers. Suppose that there exists an \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( 2W \) with \((\alpha(\epsilon, \delta, \beta), \beta)\)-utility. Then,

- There exists an \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( T \) with \((\alpha(\epsilon/(2T/W), \delta/(2T/W), \beta, \beta)\)-utility.
- There exists an \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( T \) with \((\alpha(\epsilon/(2T/W)), \delta/(2T/W), \beta, \beta)\)-utility.

Proof. We use exactly the same algorithm as in the proof of Lemma 18, except that this time we use basic or advanced composition theorems over the \([T/W] \leq 2T/W\) runs of the \((\epsilon', \delta')\)-DP algorithm. This ensures that the entire algorithm is \((\epsilon, \delta)\)-DP as long as we pick \( \epsilon' = \epsilon/(2T/W), \delta' = \delta/(2T/W) \) or \( \epsilon' = \frac{\epsilon}{2\sqrt{4\epsilon W \ln(2/\delta)}}, \delta' = \frac{\delta}{4T/W} \).

Combining the above with Lemma 19, we arrive at the following:

Lemma 29. Let \( k, T, W \in \mathbb{N} \) be any positive integers. Suppose that there exists an \((\epsilon, \delta)\)-DP algorithm for 1d-range query with \((\alpha(M, \epsilon, \delta, \beta), \beta)\)-utility. Then, there exists an

- \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( T \) with \((2W + 1, 2 \cdot \alpha(2W + 1, \epsilon/(8kT/W), \delta/(16kT/W), \beta/2), \beta)\)-utility.
- \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( T \) with \((2 \cdot \alpha(2W + 1, \frac{\epsilon}{8k\sqrt{4T/W \ln(2/\delta)}}, \frac{\delta}{16kT/W}, \beta/2), \beta)\)-utility.

Finally, plugging in the algorithmic bound for 1d-range query (Theorem 7) yields:

Corollary 30. For any \( T, W, k \in \mathbb{N} \), and \( \epsilon > 0 \), there is an \( \epsilon \)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with \( (O(k\cdot(T/W)\cdot\log^{1.5}W\log(1/\beta)/\epsilon, \beta)\)-utility.

Corollary 31. For any \( k \in \mathbb{N} \) and \( \epsilon, \delta > 0 \), there is an \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with \( (O(k \cdot \sqrt{T/W} \cdot \log(1/\delta) \log^{1.5} W \log(1/\beta)/\epsilon, \beta)\)-utility.

6.2.2 Lower Bounds

First, we note that the lower bounds based on \( k \) from the event-level DP case (Corollary 25 and Corollary 27) immediately translate to this case.

Next, we prove \( T^{\Omega(1)} \) lower bounds based on reductions from the 1-way marginal problem. We again start with the simpler bundle setting.

Lemma 32. Let \( T, W, d \) be positive integers such that \( T \geq W \cdot d \). Then, if there exists an \((\epsilon, \delta)\)-DP algorithm for fixed-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with fixed window \( W \) and time horizon \( T \) with \((\alpha, \beta)\)-utility, then there exists an \((\epsilon, \delta)\)-DP algorithm for the 1-way marginal problem with the same utility.

Proof. Let \( U = n \), Given an instance \( x^1, \ldots, x^n \in \{0, 1\}^d \) of the 1-way marginal problem, we construct the instance of fixed-window \( \text{CntOcc}^{\geq k} \) as follows: let

\[
S^i_j = \begin{cases} 
  k \cdot x^j_{(i-1)/W+1} & \text{if } W \mid (i-1) \text{ and } i \leq W \cdot d, \\
  0 & \text{otherwise}.
\end{cases}
\]

for all \( i \in [T], j \in [n] \).
Clearly, changing a single \(x^j\) leads to a change in only a single item. Furthermore, it is not hard to verify that \(\tilde{x}_t = \text{CntOcc}^{\leq k}(S^{(W^{(t-t')}+1,W)})\) for all \(t \in [d]\). Thus, by running the \((\epsilon,\delta)\)-DP algorithm for fixed-window \(\text{CntOcc}^{\leq k}\) on the above instance, we also get an estimate for the 1-way marginal with the same utility.

Plugging Lemma 32 into the known lower bounds for the 1-way marginal problem (Theorem 11 and Theorem 12) yields:

\begin{itemize}
  
  \item \textbf{Corollary 33.} Let \(T \geq W\) be any positive integers and \(\epsilon > 0\). Then, there is no \(\epsilon\)-DP algorithm for time-window \(\text{CntOcc}^{\leq k}\) in the item-level DP and bundle setting with \((o(T/W) / \epsilon), 0.01)\)-utility.
  
  \item \textbf{Corollary 34.} Let \(T \geq W\) be any positive integers and \(\epsilon, \delta > 0\) be such that \(\delta \in (2^{-\Omega(T)}, 1/T^{1+\Omega(1)})\). Then, there is no \((\epsilon, \delta)\)-DP algorithm for time-window \(\text{CntOcc}^{\leq k}\) in the item-level DP and bundle setting with \((o(\sqrt{T/W}) \log(1/\delta) / \epsilon), 0.01)\)-utility.
\end{itemize}

In the singleton case, we can use almost the same reduction as before, except that we now have to “spread out” the contribution across time step. This also means that we require an additional condition that the window is sufficiently large (i.e., \(W \geq kn\)) in the lemma below.

\begin{itemize}
  
  \item \textbf{Lemma 35.} Let \(T, W, d, k, n\) be positive integers such that \(T \geq W \cdot d\) and \(W \geq k \cdot n\). Then, if there exists an \((\epsilon, \delta)\)-DP algorithm for fixed-window \(\text{CntOcc}^{\leq k}\) in the item-level DP and singleton setting with fixed window \(W\) and time horizon \(T\) with \((\alpha, \beta)\)-utility, then there exists an \((\epsilon, \delta)\)-DP algorithm for 1-way marginal with the same utility.
\end{itemize}

**Proof.** Let \(U = n\). Given an instance \(x^1, \ldots, x^n \in \{0, 1\}^d\) of the 1-way marginal problem, we construct the instance of fixed-window \(\text{CntOcc}^{\leq k}\) as follows: let

\[
S^i_j = \begin{cases} 
  x^j_{(i-(i-1)W)+1} & \text{if } (i-k(j-1)) \mod W < k \text{ and } i \leq W \cdot d, \\
  0 & \text{otherwise},
\end{cases}
\]

for all \(i \in [T], j \in [n]\).

It is not hard to verify that \(\|S^i\|_1 \leq 1\) for all \(i \in [T]\), i.e., that this is a valid instance for the singleton case. Furthermore, changing a single \(x^j\) leads to a change in only a single item and \(\tilde{x}_t = \text{CntOcc}^{\leq k}(S^{(W^{(t-t')}+1,W)})\) for all \(t \in [d]\). Thus, by running the \((\epsilon, \delta)\)-DP algorithm for fixed-window \(\text{CntOcc}^{\leq k}\) on the above instance, we also get an estimate for the 1-way marginal with the same utility.

Similarly, plugging Lemma 35 to the known lower bounds for the 1-way marginal problem (Theorem 11 and Theorem 12 with \(n = \Theta(W/k), d = \Theta(T/W)\)) yields:

\begin{itemize}
  
  \item \textbf{Corollary 36.} Let \(T \geq W \geq k\) be any positive integers and \(\epsilon > 0\). Then, there is no \(\epsilon\)-DP algorithm for fixed-window \(\text{CntOcc}^{\leq k}\) in the item-level DP and singleton setting with \((o(\min\{W/k, (T/W) / \epsilon\}), 0.01)\)-utility.
  
  \item \textbf{Corollary 37.} Let \(T \geq W \geq k\) be any positive integers and \(\epsilon, \delta > 0\) be such that \(\delta \in (2^{-\Omega(W/k)}, 1/(W/k)^{1+\Omega(1)})\). Then, there is no \((\epsilon, \delta)\)-DP algorithm for fixed-window \(\text{CntOcc}^{\leq k}\) in the item-level DP and bundle setting with \((o\left(\min\left\{W/k, \sqrt{(T/W)} \log(1/\delta) / \epsilon\right\}, 0.01\right))\)-utility.
\end{itemize}

Again, we remark that achieving \(W/k\) error is trivial because in the singleton case, the answer to any fixed-window query is at most \(W/k\). Therefore, simply answering zero to all queries result in error at most \(W/k\).
7 Time-Window Queries

Finally, we will present our algorithms and lower bounds for time-window queries. Again, this section is divided into two subsections based on the DP notions.

7.1 Event-Level DP

7.1.1 Algorithm

In the time-window setting, we reduce the problem to 2d-range query:

Lemma 38. If there exists an \((\epsilon, \delta)\)-DP algorithm for 2d-range query with \((\alpha(M, \epsilon, \delta, \beta), \beta)\)-utility, then there exists an \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\(^{\geq k}\) in the event-level DP and bundle setting with \(\left(2 \cdot \alpha \left(T + 1, \frac{\delta}{2(2k+1)}, \frac{\beta}{4(2k+1)}, \beta/2\right), \beta\right)\)-utility for every \(k \in \mathbb{N}\).

Proof. Let \(M = T + 1\). We will create two instances of 2d-range query. For clarity, we will call the first instance \(\mathbf{x}\) and the second instance \(\mathbf{x}'\). The instances are created as follows:

1. Recall the definition of \(m_u\) and \(t_u^1, \ldots, t_u^{m_u}\) from Section 4.
   a. For all \(u \in [U]\) and \(\ell = 1, \ldots, m_u - k\), do the following:
      i. Add \((t_u^\ell + 1, t_u^{\ell+k} - 1)\) to \(\mathbf{x}\).
      ii. Add \((t_u^{\ell+1} + 1, t_u^{\ell+k} - 1)\) to \(\mathbf{x}'\).

2. We then run the \(\left(\frac{\delta}{2(2k+1)}, \frac{\beta}{4(2k+1)}\right)\) 2d-range query algorithms on both instances \(\mathbf{x}, \mathbf{x}'\), to get estimates \(\tilde{r}_{u,v}(\mathbf{x})\) and \(\tilde{r}_{u,v}(\mathbf{x}')\) for all \(u, v \in [M]^2\).

3. To answer CntOcc\(^{\geq k}(S^{[i,j]}))\), we output \(|U| - \tilde{r}_{(1,j), (i,M)}(\mathbf{x}) + \tilde{r}_{(1,j), (i,M)}(\mathbf{x}')\).

To prove DP, consider any two neighboring datasets \(\mathbf{S} = (S^{1}, \ldots, S^{T})\) and \(\mathbf{S}'\) where \(\mathbf{S} \in (\tilde{S}^{1}, \ldots, \tilde{S}^{T})\) results from removing \(u\) from \(S^{t}\) for some \(u \in [U], t \in [T]\). To avoid confusion, when we refer to \(m_u, t_u^1, \ldots, t_u^{m_u}\), we will be explicit about whether they are corresponding to \(\mathbf{S}\) or \(\mathbf{S}'\). Let \(\ell'\) denote the largest index with \(t = t_u^{\ell'}(\mathbf{S})\). Note that we have \(m_u(\mathbf{S}) = m_u(\mathbf{S}') - 1\) and

\[
 t_u^{\ell'}(\mathbf{S}) = \begin{cases} 
 t_u^{\ell'}(\mathbf{S}) & \text{if } \ell < \ell', \\
 t_u^{\ell'+1}(\mathbf{S}) & \text{if } \ell \geq \ell', 
\end{cases}
\]

for all \(\ell \in [m_u(\mathbf{S})]\).

This implies the following:

- For all \(\ell \leq \ell'\) the \(\ell\)-th loop of the first step remains exactly the same for both \(\mathbf{S}\) and \(\mathbf{S}'\).
- For all \(\ell \geq \ell' + k\) the \(\ell\)-th loop of the first step for \(\mathbf{S}\) is exactly the same as the \((\ell + 1)\)-th loop for \(\mathbf{S}'\).

As a result, there can be at most \(2k + 1\) changes to each of \(\mathbf{x}, \mathbf{x}'\) between the two datasets \(\mathbf{S}, \mathbf{S}'\). Thus, group privacy (Fact 6), implies that the algorithm is \((\epsilon, \delta)\)-DP as desired.

To see its correctness, for each \(u \in [U]\), let \(\ell^u(u, i)\) denote the last time step it is reached before \(i\) (i.e., largest \(\ell\) such that \(t_u^\ell < i\)). Notice that

\[
 \text{CntOcc}^{\geq k}(S^{[i,j]}) = |U| - |\{u \mid S_u^{[i,j]} < k\}| = |U| - \sum_{u \in [U]} 1[\ell^u(u, i)] + k > j].
\]

Now, observe that the elements added gets canceled out for all \(\ell \neq \ell^u(u, i)\) in \(r_{(1,j), (i,M)}(\mathbf{x}) - r_{(1,j), (i,M)}(\mathbf{x}')\). For \(\ell = \ell^u(u, i)\), they are not canceled iff \(t_{u^\ell+k} > j\). To summarize, we have

\[
 r_{(1,j), (i,M)}(\mathbf{x}) - r_{(1,j), (i,M)}(\mathbf{x}') = \sum_{u \in [U]} 1[\ell^u(u, i)] + k > j].
\]

By combining the above two equations, we then have CntOcc\(^{\geq k}(S^{[i,j]}) = |U| - r_{(1,j), (i,M)}(\mathbf{x}) + r_{(1,j), (i,M)}(\mathbf{x}').\) The error guarantee follows from that of the 2d-range query algorithm. ◀
Plugging the above into Theorems 8 and 9, we arrive at the following bounds.

**Corollary 39.** For any \( k \in \mathbb{N}, \epsilon > 0 \), there is an \( \epsilon \)-DP algorithm for time-window CntOcc\( ^{\geq k} \) in the event-level DP and bundle setting with \( O(k \cdot \log^3 T \cdot \log(1/\beta)/\epsilon) \)-utility.

**Corollary 40.** For any \( k \in \mathbb{N}, \epsilon, \delta \in (0,1) \), there is an \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\( ^{\geq k} \) in the event-level DP and bundle setting with \( O(k \cdot \log^2 T \cdot \sqrt{\log(k/(\epsilon \delta))} \cdot \log(1/\beta)/\epsilon) \)-utility.

### 7.1.2 Lower Bounds

Once again, there are two lower bounds here, one based on 2d-range query (which is polylogarithmic in \( T \)) and one based on the 1-way marginal problem. In fact, the latter follows immediately from the lower bound in the fixed-window case since time-window includes fixed-window with \( W = 1 \) and \( W = T/2 \) as special cases. Specifically, the lower bounds from Corollary 25 and Corollary 27 yields the following:

**Corollary 41.** Let \( T, k \) be any positive integer and \( \epsilon, \delta > 0 \) be any sufficiently small constant. Then, there is no \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\( ^{\geq k} \) in the event-level DP and bundle setting with \( \alpha(\sqrt{\min(k, T)}), 0\)-utility.

**Corollary 42.** Let \( T, k \) be any positive integer such that \( T \geq 2k \) and \( \epsilon, \delta > 0 \) be any sufficiently small constant. Then, there is no \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\( ^{\geq k} \) in the event-level DP and singleton setting with \( \alpha(\sqrt{\min(k, T/k)}), 0\)-utility.

The lower bound for 2d-range query works by “reverse engineering” the reduction above. We start by describing this for the case of bundle.

**Lemma 43.** Let \( k \) be any positive integer. If there exists an \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\( ^{\geq k} \) in the event-level DP and bundle setting with \( \alpha(T, \epsilon, \delta, \beta), \beta \)-utility, then there exists an \((\epsilon, \delta)\)-DP algorithm for 2d-range query with \( (\alpha(2M + 1, \epsilon/2, \delta/4, 4), \beta) \)-utility.

**Proof.** Let \( T = 2M + 1 \) and let \( U \) be sufficiently large (i.e., larger than the dataset size of the 1d-range query). Given an input \( x^1, \ldots, x^n \in [M]^2 \) in the 2d-range query problem, we construct an input to the time-window \((\epsilon/2, \delta/4)\)-DP CntOcc\( ^{\geq k} \) algorithm as follows:

- For all \( u \in [U] \), let \( S_u^{M+1} = k-1 \) and \( S_u^{M} = 0 \) for all \( i \in \{T \} \setminus \{M + 1\} \).
- For all \( j \in [n] \), increment each of \( S_j^{M+1-x^j_1} \) and \( S_j^{M+1+x^j_2} \) by one.

It is not hard to verify that, for all \( y \in [M]^2 \), CntOcc\( ^{k} \left( S^{(M+1-x_1^y), M+1+y_2^y} \right) = n - r_{(1,1)}(x) \). Therefore, we may answer any range query \( r_{y^1,y^2}(x) \) by outputting an estimate to

\[
\begin{align*}
&- \text{CntOcc}^{\ge k}\left( S^{(M+1-y_1^y), M+1+y_2^y} \right) - \text{CntOcc}^{\ge k}\left( S^{(M+2-y_1^y), M+y_2^y} \right) \\
&\quad + \text{CntOcc}^{\ge k}\left( S^{(M+1-y_1^y), M+y_2^y} \right) + \text{CntOcc}^{\ge k}\left( S^{(M+2-y_1^y), M+1+y_2^y} \right),
\end{align*}
\]

where the estimate of each term is computed via the time-window \((\epsilon/2, \delta/4)\)-DP CntOcc\( ^{\ge k} \) algorithm.

By group DP (Fact 6), this is indeed an \((\epsilon, \delta)\)-DP algorithm for 2d-range query. Its utility claim follows trivially by definition.

Plugging the above into Theorem 10 gives the following lower bound in terms of \( T \):

**Corollary 44.** For any sufficiently small constants \( \epsilon, \delta, \beta > 0 \), there is no \((\epsilon, \delta)\)-DP algorithm for time-window CntOcc\( ^{\ge k} \) in the event-level DP and bundle setting with \( \alpha(\log T), \beta \)-utility.
Again, our standard “spreading out” technique also works with the above reduction with a usual blow up on the value of $T$.

\textbf{Lemma 45.} Let $T, k, n, M$ be any positive integers such that $T \geq 2nM + kn$. If there exists an $(\epsilon, \delta)$-DP algorithm for time-window CntOcc$^{\geq k}$ in the event-level DP and singleton setting with $(\alpha(T, \epsilon, \delta, \beta), \beta)$-utility, then there exists an $(\epsilon, \delta)$-DP algorithm for 2d-range query with $(4 \cdot \alpha(2M + 1, \epsilon/2, \delta/4, \beta/4), \beta)$-utility.

\textbf{Proof.} Let $U = n$. For convenience, also let $Q = nM + 1, R = Q + (k - 1)n$. Given an input $x^1, \ldots, x^n \in [M]^2$ to the 2d-range query problem, we construct an input to the time-window $(\epsilon/2, \delta/4)$-DP CntOcc$^{\geq k}$ algorithm as follows:

- For all $u \in [U]$, let $S^u_0 = 0, S^u_1 = \epsilon, S^u_2 = \epsilon, \ldots, S^u_{u - 1} = \epsilon, S^u_u = k - 1$ and $S^u_i = 0$ for all $i \in [T] \setminus \{Q + (k - 1)(u - 1) + 1, \ldots, Q + (k - 1)u\}$.
- For all $j \in [n]$, increment each of $S^{nx^1_j + u}_j$ and $S^{nx^2_j + u}_j$ by one.

It is not hard to verify that, for all $y \in [M]^2$, CntOcc$^y(S^{Q - ny_1, R + ny_2 - 1}) = n - r_{(1,1),y}(x)$. Therefore, we may answer any range query $r_{y_1,y_2}(x)$ by outputting an estimate to

\[
- \text{CntOcc}^{\geq k}(S^{Q - ny_1^1 + 1, R + ny_2^1 - 1}) - \text{CntOcc}^{\geq k}(S^{Q - ny_1^2, R + ny_2^2 - 1}) \\
+ \text{CntOcc}^{\geq k}(S^{Q - ny_1^2, R + ny_2^1 - 1}) + \text{CntOcc}^{\geq k}(S^{Q - ny_1^1, R + ny_2^2 - 1})
\]

where the estimate of each term is computed via the time-window $(\epsilon/2, \delta/4)$-DP CntOcc$^{\geq k}$ algorithm.

By group DP (Fact 6), this is indeed an $(\epsilon, \delta)$-DP algorithm for 2d-range query. Its utility claim follows trivially by definition. \hfill \textasteriskcentered

Plugging the above into Theorem 10 with $M = \frac{\sqrt{T}}{k}, n = \Theta(M^2)$ gives the following lower bound in terms of $T/k$:

\textbf{Corollary 46.} For any sufficiently small constants $\epsilon, \delta, \beta > 0$, there is no $(\epsilon, \delta)$-DP algorithm for time-window CntOcc$^{\geq k}$ in the event-level DP and singleton setting with $(o(\log(T/k)), \beta)$-utility.

\subsection{7.2 Item-Level DP}

\subsection*{7.2.1 Algorithms}

The algorithm for Item-Level DP is based on a rather simple approach of running the cumulative CntOcc$^{\geq k}$ algorithm with different starting times, and then applying the composition theorems to account for the privacy budget. This is formalized below.

\textbf{Lemma 47.} Let $k, T \in \mathbb{N}$ be any positive integer. Suppose that there exists an $(\epsilon, \delta)$-DP algorithm for cumulative CntOcc$^{\geq k}$ in the item-level DP and bundle setting with time horizon $T$ with $(\alpha(T, \epsilon, \delta, \beta), \beta)$-utility. Then,

- There exists an $(\epsilon, \delta)$-DP algorithm for time-window CntOcc$^{\geq k}$ in the item-level DP and bundle setting with time horizon $T$ with $(\alpha(T, \epsilon, \delta, \beta), \beta)$-utility.

\textbf{Proof.} We simply run the $(\epsilon', \delta')$-DP algorithm for cumulative CntOcc$^{\geq k}$ starting at time $1, \ldots, T$. Since we run the $(\epsilon', \delta')$-DP algorithm a total of $T$ times on the input dataset, the basic and advanced composition theorems respectively ensure that the entire algorithm is $(\epsilon, \delta)$-DP as long as we pick $\epsilon' = \epsilon/T, \delta' = \delta/T$ and $\epsilon' = \frac{\epsilon}{2\sqrt{T} \ln(2/\delta)} , \delta' = \frac{\delta}{2T}$, respectively. \hfill \textasteriskcentered
Combining the above lemma with our cumulative algorithm from Corollary 15, we immediately arrive at the following:

**Corollary 48.** For any \( k \in \mathbb{N} \) and \( \epsilon > 0 \), there is an \( \epsilon \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with \( O(T \cdot \log \epsilon^5 \cdot T \log(1/\beta)/\epsilon, \beta) \)-utility.

**Corollary 49.** For any \( k \in \mathbb{N} \) and \( \epsilon, \delta \in (0, 1) \), there is an \( (\epsilon, \delta) \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with \( O(\sqrt{T} \log(T/\delta) \cdot \log \epsilon^5 \cdot T \log(1/\beta)/\epsilon, \beta) \)-utility.

For the singleton setting, we can get an improved bound by “time compression” technique of [30] as formalized below.

**Lemma 50.** Let \( T' \) be any positive integer and \( \epsilon, \delta > 0 \). If there exists an \( (\epsilon, \delta) \)-DP algorithm for \( \text{CntOcc}^{\geq k} \) with utility \( (\alpha(T, \beta), \beta) \) in the bundle setting with item-level DP, then there exists an \( (\epsilon, \delta) \)-DP algorithm for \( \text{CntOcc}^{\geq k} \) with \( (\alpha([T/T'], \beta) + T', \beta) \)-utility in the singleton setting (both for item-level DP and event-level DP).

**Proof.** Let \( S^1, \ldots, S^T \) denote the input to the singleton setting. We construct an input to the bundle setting by, for all \( i \in [T/T'] \) letting \( \tilde{S}^i := S^{T(t-1)} + \cdots + S^{\min(T'(t+1), T)} \). When we want to answer the query for the time from \( t_1 \) to \( t_2 \) in the original instance, we instead use the answer for the query from time \( [t_1/T'] \) to \( [t_2/T'] \) in the new instance. From the fact that the original instance has only a single item per time step, it is not hard to see that this only introduces at most \( T' \) additional error. The claimed DP guarantee is immediate.

By applying Lemma 50 to the above corollaries (with \( T' = \sqrt{T/\epsilon} \) and \( T' = \sqrt{T/\epsilon^2} \) respectively), we get:

**Corollary 51.** For any \( k \in \mathbb{N} \) and \( \epsilon > 0 \), there is an \( \epsilon \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and singleton setting with \( O(\sqrt{T/\epsilon} \cdot \log \epsilon^5 \cdot T \log(1/\beta), \beta) \)-utility.

**Corollary 52.** For any \( k \in \mathbb{N} \) and \( \epsilon, \delta \in (0, 1) \), there is an \( (\epsilon, \delta) \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and singleton setting with \( O(\sqrt{T/\epsilon^2} \cdot \sqrt{\log(T/\delta)} \cdot \log \epsilon^5 \cdot T \log(1/\beta), \beta) \)-utility.

### 7.2.2 Lower Bounds

Since time-window includes fixed-window with \( W = 1 \) and \( W = kn \) as special cases, the lower bounds from Lemma 32 and Lemma 35 immediately translate to the following:

**Lemma 53.** Let \( T, d \) be positive integers such that \( T \geq d \). Then, if there exists an \( (\epsilon, \delta) \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and bundle setting with time horizon \( T \) with \( (\alpha, \beta) \)-utility, then there exists an \( (\epsilon, \delta) \)-DP algorithm for 1-way marginal with the same utility.

**Lemma 54.** Let \( T, d, k, n \) be positive integers such that \( T \geq kn \). Then, if there exists an \( (\epsilon, \delta) \)-DP algorithm for time-window \( \text{CntOcc}^{\geq k} \) in the item-level DP and singleton setting with time horizon \( T \) with \( (\alpha, \beta) \)-utility, then there exists an \( (\epsilon, \delta) \)-DP algorithm for 1-way marginal with the same utility.

Plugging Lemma 53 into the known lower bounds for the 1-way marginal problem (Theorem 11 and Theorem 12) yields:
Corollary 55. Let $T$ be any positive integer and $\epsilon > 0$. Then, there is no $\epsilon$-DP algorithm for time-window $\text{CntOcc}^{\geq k}$ in the item-level DP and bundle setting with $(o(T/\epsilon), 0.01)$-utility.

Corollary 56. Let $T$ be any positive integer and $\epsilon, \delta > 0$ be such that $\delta \in (2^{-O(T)}, 1/T^{1+\Omega(1)})$. Then, there is no $(\epsilon, \delta)$-DP algorithm for time-window $\text{CntOcc}^{\geq k}$ in the item-level DP and bundle setting with $(o(T \log(1/\delta)/\epsilon), 0.01)$-utility.

Similarly, plugging Lemma 54 to the known lower bounds for 1-way marginal (Theorem 11 with $n = \Theta(T/(k\epsilon)), d = \Theta(\epsilon T/k)$ and Theorem 12 with $n = \Theta(\sqrt{T \log(1/\delta)/\epsilon^2 k})$) yields:

Corollary 57. Let $T, k$ be any positive integers and $\epsilon > 0$ such that $T \geq k/\epsilon$. Then, there is no $\epsilon$-DP algorithm for time-window $\text{CntOcc}^{\geq k}$ in the item-level DP and singleton setting with $(o(\sqrt{T/\epsilon^2}), 0.01)$-utility.

Corollary 58. Let $T$ be any positive integer and $\epsilon, \delta > 0$ be such that $\delta \in (2^{-\Omega(T)}, 1/T^{1+\Omega(1)})$ and $T \geq k \sqrt{T \log(1/\delta)/\epsilon}$. Then, there is no $(\epsilon, \delta)$-DP algorithm for time-window $\text{CntOcc}^{\geq k}$ in the item-level DP and bundle setting with $(o(\sqrt{T \log(1/\delta)/\epsilon^2 k}), 0.01)$-utility.

All of the lower bounds are tight to within polylogarithmic factors and the dependence on $k$ compared to the algorithms given in the previous subsection. (Note that the lower bounds in terms of $k$ still carries over from the event-label DP case above, i.e., Corollary 41 and Corollary 42.)

8 Conclusions and Open Questions

In this work, we consider several variants of the tasks of counting distinct and $k$-occurring elements in time windows, including cumulative/time-window/fixed-window queries, user/item-level DP, and bundle/singleton settings. In each setting, we determine optimal error bounds that are tight up to polylogarithmic factors in $T$ or $W$ and the dependence on $k$. As mentioned earlier, our work is closely related to the continual release model [18, 10]. In fact, it is simple to modify our algorithms to work in the following setting: at time step $t$, the algorithm receives $S^t$ and must immediately answer all queries $(t_1, t_2)$ such that $t_2 = t$. This can be done by using the 1d-/2d-range query algorithms that work in the similar setting [18, 10, 20].

An interesting research direction is to close the gaps in our results. For example, can we get a bound of the form $O(k + \log^{1.5} W)$ in the fixed-window event-level DP setting or a bound of the form $O(k + \log^3 T)$ in the time-window event-level DP setting? We conjecture that answering these questions requires “white-box” solutions beyond directly reducing to/from range query or linear queries.

Finally, we note that in this work, we have not considered memory constraints on the algorithm. It is an interesting direction for future work to consider the trade-off between memory, privacy, and utility (e.g., similarly to [43]), which is an important aspect both in theory and in practice.

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