Global well-posedness and optimal large-time behavior of strong solutions to the non-isentropic particle-fluid flows

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Abstract
In this paper, we study the three-dimensional non-isentropic compressible fluid–particle flows. The system involves coupling between the Vlasov–Fokker–Planck equation and the non-isentropic compressible Navier–Stokes equations through momentum and energy exchanges. For the initial data near the given equilibrium we prove the global well-posedness of strong solutions and obtain the optimal algebraic rate of convergence in the three-dimensional whole space. For the periodic domain the same global well-posedness result still holds while the convergence rate is exponential. New ideas and techniques are developed to establish the well-posedness and large-time behavior. For the global well-posedness our methods are based on the new macro–micro decomposition which involves less dependence on the spectrum of the linear Fokker–Plank operator and fine energy estimates; while the proofs of the optimal large-time behavior rely on the Fourier analysis of the linearized Cauchy problem and the energy-spectrum method, where we provide some new techniques to deal with the nonlinear terms.

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1 Introduction
In this paper we study the global well-posedness and large time behavior of strong solutions for the three-dimensional fluid–particle flows, governed by the following Navier–Stokes...
The equations of compressible non-isentropic fluids coupled with the Vlasov–Fokker–Planck equation of particles [3,13,33]:

\begin{align}
\partial_t n + \nabla \cdot (nu) &= 0, \quad (1.1) \\
\partial_t (nu) + \nabla \cdot (nu \otimes u) - \mu \Delta u + \nabla p &= \mathcal{M}, \quad (1.2) \\
\partial_t (nE) + \nabla \cdot ((nE + p)u) - \kappa \Delta \tilde{\theta} &= \mathcal{F}, \quad (1.3) \\
\partial_t F + v \cdot \nabla_x F &= L_{u,\tilde{\theta}} F, \quad (1.4)
\end{align}

where, \( n = n(t,x) \geq 0, u = u(t,x) \in \mathbb{R}^3, p = p(t,x) \geq 0, E = E(t,x) \geq 0, \tilde{\theta} = \tilde{\theta}(t,x) \geq 0 \) for \((t, x) \in \mathbb{R}^+ \times \Omega\) denote the density, velocity, pressure, total energy, and temperature of the fluids, respectively; \( F = F(t,x,v) \geq 0 \) for \((t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3\) denotes the density distribution function of particles in the phase space; the spatial domain is \( \Omega = \mathbb{R}^3 \) or \( \mathbb{T}^3 \) (a periodic domain in \( \mathbb{R}^3 \)); and \( \mu, \kappa \) are the viscosity and heat conductivity constants. The total energy \( E \), internal energy \( e \), pressure \( p \), and temperature \( \tilde{\theta} \) satisfy the following relations: \( E = e + \frac{1}{2} |u|^2 \), \( p = R n \tilde{\theta} \), \( e = \frac{p}{(\gamma - 1) n} \), where \( \gamma > 1 \) is the adiabatic constant and \( R > 0 \) is constant. The Fokker–Planck operator is defined by

\[ L_{u,\tilde{\theta}} F = \text{div}_v (v - u) F + \tilde{\theta} \text{div}_v F, \]

which accounts for the friction force exerting on the particles by the surrounding fluids and the Brownian motion of the particles, that is, the friction force is assumed proportional to the relative velocity \( v - u \), and the Brownian motion depending on the temperature of the fluid induces diffusion with respect to the velocity variable. The two-phase flows have a disperse phase from a statistical viewpoint for particles, and a dense phase from continuum mechanics for fluids. The coupling terms \( \mathcal{M} \), \( \mathcal{F} \) depict the interaction between the disperse phase and the dense phase, and read

\[ \mathcal{M} = - \int_{\mathbb{R}^3} v L_{u,\tilde{\theta}} F \, dv = \int_{\mathbb{R}^3} (v - u) F \, dv, \]

\[ \mathcal{F} = - \int_{\mathbb{R}^3} \frac{|v|^2}{2} L_{u,\tilde{\theta}} F \, dv = \int_{\mathbb{R}^3} [v \cdot (v - u) - 3\tilde{\theta}] F \, dv, \]

indicating the momentum exchanges and the energy exchanges, respectively.

For smooth solutions of the compressible non-isentropic Navier–Stokes–Vlasov–Fokker–Planck system (1.1)–(1.4), the temperature \( \tilde{\theta} \) satisfies the following equation

\[ \partial_t \tilde{\theta} + n \cdot \nabla \tilde{\theta} + (\gamma - 1) \tilde{\theta} \text{div}_u - \frac{\kappa (\gamma - 1)}{n} \Delta \tilde{\theta} + \frac{\mu}{n} \Delta u \cdot u = \frac{\gamma - 1}{n} \int_{\mathbb{R}^3} [|v - u|^2 - 3\tilde{\theta}] F \, dv. \quad (1.5) \]

The fluid–particle flows have a wide range of applications from dynamics of sprays, combustion, pollution processes, waste water treatment, to biomedical flows; see [2–4,7,10,11,13,19,33,38–40,43,44] and the references therein for the discussions of applications and modeling issues. We remark that in the momentum equation (1.2), we only keep the shear viscosity and skip the bulk viscosity term for the sake of simplicity of presentations, since the bulk viscosity will not add any significant difficulty and we shall focus on the complexity caused by the heat conductivity and the Fokker–Planck operator.
The purpose of this paper is to establish the well-posedness of the system (1.1)–(1.4) near a global Maxwellian. Without loss of generality, we normalize the global Maxwellian as
\[ M = M(v) = \frac{1}{(2\pi)^{3/2}} \exp \left\{ - \frac{|v|^2}{2} \right\}. \]
For the Cauchy problem with the initial data
\[ (F, n, u, \tilde{\theta})|_{t=0} = (F_0(x), n_0(x), u_0(x), \tilde{\theta}_0(x)), \] (1.6)
we shall prove the global existence and uniqueness as well as the large-time behavior of the strong solution for the unknowns \((F, n, u, \tilde{\theta})\) near the global equilibrium state \((F, n, u, \tilde{\theta}) \equiv (M, 1, 0, 1)\).

There exist many different systems describing the kinetic-fluid for the physical regimes under consideration, such as the compressible or incompressible fluids, viscous or inviscid fluids, with or without thermal diffusion acting on the particles and so on. The mathematical analysis of such mathematical models is very difficult due to the nonlinear coupling of partial differential equations of different types.

We now give a brief review of works in literature on some kinetic-fluid models related to our system (1.1)–(1.4). The global existence of classical and weak solutions for the incompressible fluid–particle flows has been studied in many papers, see [6,9,14,15,20–22,28,30–32,42,45] and their references. For the compressible fluid, when the drag force exerted by the surrounding fluid is proportional to the relative velocity \(v - u\), the global weak solution and the asymptotic analysis were obtained in Mellet and Vasseur [36,37], the global classical solutions near an equilibrium and exponential decay were obtained in Chae, Kang, and Lee [16], and the dissipative quantities, equilibria and their stability were studied in Carrillo and Goudon [12].

When the drag force depends on both the relative velocity \(v - u\) and the density of the fluid, the local existence of classical solutions to the Euler–Vlasov system was obtained in Baranger and Desvillettes [5], and global strong solution near an equilibrium and large-time behavior to the isentropic compressible Navier–Stokes–Vlasov–Fokker–Planck system were established in Li et al. [29].

To the best of our knowledge, there are few rigorous mathematical results concerning the case of non-isentropic kinetic-fluid equations. In [8,23], some numerical analysis on the kinetic-fluid models with energy exchange involved was presented. In this paper, we shall address two problems for the non-isentropic system (1.1)–(1.4): (1) the global existence of strong solution in the framework of small perturbation of an equilibrium, (2) the asymptotic behavior to the given Maxwellian equilibrium. As far as we know, this is the first rigorous mathematical work which deals with the energy exchange between the disperse phase and the dense phase.

The perturbation of solutions to the Navier–Stokes–Vlasov–Fokker–Planck system (1.1)–(1.5) near the global equilibrium state \((F, n, u, \tilde{\theta}) \equiv (M, 1, 0, 1)\) satisfies the perturbation system (2.1)–(2.5). To prove the global existence of strong solution to the perturbation problem, we mainly use the fine energy estimates together with the local existence of strong solutions and continuum argument. It is well known that under a smallness condition on the perturbation, using fine energy estimates will lead to the global existence of strong solutions. This approach is in the spirit of the papers [26,27,34,35] for the Boltzmann, Landau and Navier–Stokes equations.

The fact that the unknowns of the Navier–Stokes–Vlasov–Fokker–Planck system do not depend on the same set of variables yields many technical difficulties, and the proof requires sharp estimates of solutions to both the kinetic equation and the fluid equations. We shall adopt
the techniques in the works of Guo [24–27] where the full coercivity of the linearized collision operator of the Boltzmann equation are crucial to obtain the global classical solution of the nonlinear kinetic equations near an equilibrium. In brief, the solution to the Boltzmann equation is decomposed into macro and micro components, and the dissipative effect through the microscopic H-theorem can be obtained for the microscopic component, which is important in order to use the energy method. The macro–micro decomposition is gained with the help of spectral structure of the linearized collision operator of the Boltzmann equation. Here, the linear Fokker–Planck operator $L$ and the collision operator of the Boltzmann equation have certain features in common, thus we can use the idea of the macro–micro decomposition of $f$ depending on the spectral structure of the operator. However, our Navier–Stokes–Vlasov–Fokker–Planck system is quite different from the pure Boltzmann equation or the coupled Boltzmann equations, therefore several new difficulties arise as described below.

For the Boltzmann equation, as mentioned in [30], the collision particles have the same mass and momentum as well as kinetic energy. As we mentioned earlier, there exists momentum and energy exchange between particles and the surrounding fluids. As a result, some linear terms appear in the kinetic equation and some coupling terms appear in the fluid equations, which leads to some new difficulties. Because we want to establish the global well-posedness in the case of small perturbation near an equilibrium, naturally these terms may be very small, but throughout our analysis, the main difficulties arise from these linear terms which are worse than the nonlinear terms. In order to handle these linear terms, we need certain dissipation effect and expect it from the linear Fokker–Planck operator. Unfortunately, as in Sect. 2.4 from the macro–micro decomposition only by the null space of $L$, we know that the dissipation induced by the linear Focker–Planck operator is only partial, which is not enough to control new linear terms, we eventually find a new macro–micro decomposition, and achieve the desired dissipative effect under the new decomposition. Achieving such control is one of the main new contributions of the present paper.

Due to the interaction between the particles and the fluids, no existing results on the Vlasov–Fokker–Planck system yield the regularity of $f$. Therefore, the new mixed estimates involving the derivatives of the particle velocity $v$ variable are necessary.

To obtain the uniform estimates, it is critical to note that the macroscopic part is bounded by its microscopic part (see precisely (2.8)), thus we need to pay special attention to the macroscopic equations of the particles. The macroscopic equations behave like elliptic so that it is straightforward to estimate $L^2$ norms of their derivatives for the macroscopic quantities of the particles. In periodic domains, the $L^2$ norms of the macroscopic quantities can be estimated by the Poincaré inequality. This leads to different decay rates in the whole space and the periodic domain as time tends to infinity.

In order to achieve the optimal decay rate in the whole space, the main ideas are based on the Fourier analysis to the linearized Cauchy problem of the perturbation system and the energy-spectrum method as in [14,16,18]. Our main difficulties arise from the strong coupling terms in the perturbation system, namely the nonlinear terms, because the Fourier transform of the product of functions is a convolution, which is difficult in our global time-decay analysis. Selecting subtly the nonhomogeneous source plays an important role to overcome this difficulty.

With the above new ideas and techniques, we shall be able to establish the global existence of strong solutions and optimal decay rates for the compressible non-isentropic Navier–Stokes–Vlasov–Fokker–Planck equations both in the whole space and in periodic domains. We remark that the methods introduced in the present work may be applied for other related non-isentropic kinetic-fluid models such as the drag force exerted by the fluid depending on the density of the fluid.
We organize the rest of the paper as follows. In Sect. 2, we reformulate the system (1.1)–(1.5) near the global equilibrium state, present the coercivity estimate of the linear part and the macro–micro decomposition, and state our main results. In Sect. 3, we first derive the uniform-in-time a priori estimates and then establish the existence of global strong solution. In Sect. 4, we prove the rate of convergence of solutions. In Sect. 5, we adapt our proof to the periodic domain case.

2 Preliminaries and main results

In this section, we reformulate the system (1.1)–(1.5) near the global equilibrium state, present the coercivity estimate of the linear part and the macro–micro decomposition, and state our main results.

2.1 Reformulation

We consider the solution \( (F, n, u, \tilde{\theta}) \) of the system (1.1)–(1.5) near the global equilibrium \((M, 1, 0, 1)\), i.e.,

\[
F = M + \sqrt{M} f,
\quad n = 1 + \rho,
\quad u = u,
\quad \tilde{\theta} = 1 + \theta.
\]

We shall take the constants \( \mu, \kappa, R \) to be one in this paper since their values do not play any role in the analysis. From the system (1.1)–(1.5), the perturbations \((f, \rho, u, \theta)\) satisfy the following equations:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + u \cdot \nabla_v f - \frac{1}{2} u \cdot vf - u \cdot v \sqrt{M} - (|v|^2 - 3) \sqrt{M} \theta &= L f + \theta M^{-\frac{1}{2}} \Delta_v(\sqrt{M} f), \\
\partial_t \rho + u \cdot \nabla \rho + (1 + \rho) \text{div} u &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{1 + \theta}{1 + \rho} \nabla \rho + \nabla_x \theta &= \frac{1}{1 + \rho} \left( \Delta u - u(1 + a) + b \right), \\
\partial_t \theta + u \cdot \nabla \theta + \theta \text{div} u + \text{div} u - \sqrt{6} \omega + 3 \theta &= \frac{1}{1 + \rho} \left( \Delta \theta + |u|^2 - 2u \cdot b + a|u|^2 - 3a \theta \right) - \frac{\rho}{1 + \rho} \left( \sqrt{6} \omega - 3 \theta \right).
\end{align*}
\]

Correspondingly, the initial data becomes

\[
(f, \rho, u, \theta)_{t=0} = \left( f_0(x, \cdot), \rho_0(x), u_0(x), \tilde{\theta}_0(x) \right)
= \left( \frac{F_0 - M}{\sqrt{M}}, n_0(x) - 1, u_0(x), \tilde{\theta}_0(x) - 1 \right),
\]

where \( F_0 = F(0, t, x), n_0 = n(0, t, x), u_0 = u(0, t, x), \tilde{\theta}_0 = \tilde{\theta}(0, t, x) \) is a small perturbation near the above equilibrium.

In (2.1)–(2.4), we denote the linearized Fokker–Planck operator \( L \) by

\[
L f = \frac{1}{\sqrt{M}} \nabla_v \cdot \left[ M \nabla_v \left( \frac{f}{\sqrt{M}} \right) \right].
\]
and \( a = a^f \), \( b = b^f \), \( \omega = \omega^f \) are defined by

\[
\begin{align*}
\quad a^f(t, x) &= \int_{\mathbb{R}^3} \sqrt{M} f(t, x, v) \, dv, \\
\quad b^f(t, x) &= \int_{\mathbb{R}^3} v \sqrt{M} f(t, x, v) \, dv, \\
\quad \omega^f(t, x) &= \int_{\mathbb{R}^3} \frac{|v|^2 - 3}{6} \sqrt{M} f(t, x, v) \, dv.
\end{align*}
\]

### 2.2 Notation

For \( \nu(v) = 1 + |v|^2 \), \( | \cdot |_v \) is the norm defined by

\[
|g|_v^2 := \int_{\mathbb{R}^3} \left( |\nabla_v g(v)|^2 + v(v)|g(v)|^2 \right) \, dv, \quad g = g(v).
\]

\( \langle \cdot, \cdot \rangle \) is the inner product of the space \( L^2_v \), namely,

\[
\langle g, h \rangle := \int_{\mathbb{R}^3} g(v)h(v) \, dv, \quad g, h \in L^2_v.
\]

In case of no confusion, we denote by \( \| \cdot \| \) the norm of \( L^2_v \) or \( L^2_{x, v} \) for simplicity. Define

\[
\|g\|_{L^2_v}^2 := \int_{\mathbb{R}^3} \int_{\Omega \times \mathbb{R}^3} \left( |\nabla_v g(x, v)|^2 + v(v)|g(x, v)|^2 \right) \, dx \, dv, \quad g = g(x, v).
\]

For \( q \geq 1 \), we also denote

\[
Z_q = L^2_v(L^q_{x, v}) = L^2(\mathbb{R}^3, L^2_{x, v}, L^q_{x, v}), \quad \|g\|_{Z_q}^2 = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} |g(x, v)|^q \, dx \right)^{\frac{1}{q}} \, dv.
\]

The norm \( \| (f, \rho, u, \theta) \|_{Z_q} \) is defined by

\[
\| (f, \rho, u, \theta) \|_{Z_q} = \| f \|_{Z_q} + \| (\rho, u, \theta) \|_{L^1},
\]

for \( f = f(x, v), (\rho, u, \theta) = (\rho(x), u(x), \theta(x)) \) and \( q \geq 1 \).

For an integrable function \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \), its Fourier transform \( \hat{g} = \mathcal{F}g \) is defined by

\[
\hat{g}(\xi) = \mathcal{F}g(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} g(x) \, dx, \quad x \cdot \xi = \sum_{j=1}^{3} x_j \xi_j,
\]

for \( \xi \in \mathbb{R}^3 \).

For multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta = (\beta_1, \beta_2, \beta_3) \), we denote by

\[
\partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}
\]

the partial derivatives with respect to \( x = (x_1, x_2, x_3) \) and \( v = (v_1, v_2, v_3) \). The length of \( \alpha \) and \( \beta \) are defined as \( |\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \) and \( |\beta| = \beta_1 + \beta_2 + \beta_3 \). Define the following norms:

\[
\|g\|_{H^s} := \sum_{|\alpha| \leq s} \|\partial^\alpha g\|, \quad \|g\|_{H^s_{x, v}} := \sum_{|\alpha| + |\beta| \leq s} \|\partial^\alpha g\|.
\]

We shall use the letter \( C \) to denote a generic positive (generally large) constant, \( \lambda \) a generic positive (generally small) constant; and use the symbol \( A \sim B \) to denote the relation \( \frac{1}{C} A \leq B \leq C A \) for some constant \( C > 0 \).
2.3 Coercivity estimate of the linear part in (2.1)

In this subsection, we first apply the similar coercivity estimate of the linear collision operator of Boltzmann equation to the linearized Fokker–Planck operator \( L \) by spectrum analysis, that is, (2.6) holds. Then, we find that the dissipative effect of \( L \) is partial and new difficulties have arisen.

In (2.1), we denote the important linear part by

\[
Lf = \mathcal{L}f + A = \mathcal{L}f + u \cdot v \sqrt{M} + (|v|^2 - 3) \sqrt{M} \theta.
\]

It is well-known from [1] that the classical linearized Fokker–Planck operator \( \mathcal{L} \) enjoys the following dissipative properties:

1. The null space of \( \mathcal{L} \) is the one-dimensional space \( \mathcal{N}_0 = \text{Span} \{ \sqrt{M} \} \).
2. Define the projection in \( L^2_{x,v} \) to the null space \( \mathcal{N}_0 \) by

\[
P_0 f := a f \sqrt{M}, \quad a f (t, x) = \int_{\mathbb{R}^3} \sqrt{M} f(t, x, v) \, dv.
\]

Using the integration by parts, we have

\[
- \int_{\mathbb{R}^3} f \mathcal{L} f \, dv = \int_{\mathbb{R}^3} \left| \nabla_v (I - P_0) f + \frac{v}{2} (I - P_0) f \right|^2 \, dv.
\]

(3) There exists a constant \( \lambda_0 > 0 \), such that the following coercivity estimate holds:

\[
- \int_{\mathbb{R}^3} f \mathcal{L} f \, dv \geq \lambda_0 \| (I - P_0) f \|^2_v, \quad \forall f = f(v).
\]

We hope that the linear part \( L \) also has a similar coercivity estimate. However, we notice that it is straightforward to make estimates on \( L - \mathcal{L} \) as

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u \cdot v \sqrt{M} + (|v|^2 - 3) \sqrt{M} \theta) f \, dx \, dv \leq C (\| u \| + \| \theta \|) \| (I - P_0) f \|,
\]

and from (2.6), one gets

\[
- \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f \mathcal{L} f \, dv \, dx \geq \lambda_0 \| (I - P_0) f \|^2_v.
\]

It is difficult to decide which of the two terms on the right hand side of the above inequalities is bigger. To this end, we decompose \( f \) into the macroscopic component \( P_0 f \) and the microscopic component \( (I - P_0) f \), however, the dissipative effect for \( (I - P_0) f \), i.e., (2.6), is not enough to deal with the linear part \( L \). It is nontrivial to get a coercivity estimate on the linear part \( L \). In order to control \( L \), we must extract part of dissipation of \( \mathcal{L} \) corresponding to the momentum component and the energy component respectively.

2.4 New macro–micro decomposition

In this subsection, we are concerned with the macro–micro decomposition of the solution into its macroscopic (fluid dynamic) and microscopic (kinetic) components. In view of the difficulties mentioned above, we want to extract a better dissipative effect, which is hard and
achieved based on the inspiration from the idea of spectrum analysis of the linear operator. In short, since the coupling terms in our system involve the momentum exchange and the energy exchange, we expand the original null space $N_0$ to the new space $N$ below. Then, by the macroscopic and microscopic projection on the new space, we can get the new macro–micro decomposition of the solution, as described below.

Denote the linear space $N$ by

$$N = \text{Span}\left\{ \sqrt{M}, v_1\sqrt{M}, v_2\sqrt{M}, v_3\sqrt{M}, |v|^2\sqrt{M} \right\},$$

and it has the following set of orthogonal basis

$$\begin{align*}
\chi_0 &= \sqrt{M}, \\
\chi_i &= v_i\sqrt{M}, \quad i = 1, 2, 3, \\
\chi_4 &= \frac{|v|^2 - 3}{\sqrt{6}}\sqrt{M}.
\end{align*}$$

Define the projector operator $P$ by

$$P : L^2 \rightarrow N, \quad f \mapsto \overset{\rightharpoonup}{P} f = \left\{ a f + b f \cdot v + \omega f \frac{|v|^2 - 3}{\sqrt{6}} \right\} \sqrt{M}.$$ 

We also introduce the projector $P_1, P_2$ respectively by

$$\begin{align*}
P_1 f &= b f \cdot v \sqrt{M}, \\
P_2 f &= \omega f \frac{|v|^2 - 3}{\sqrt{6}} \sqrt{M},
\end{align*}$$

so the projector $P$ can be also written as

$$P := P_0 \oplus P_1 \oplus P_2.$$

As usual, for fixed $(t, x)$, $f(t, x, v)$ can be uniquely decomposed as

$$\begin{align*}
f &= Pf + (I - P)f, \\
Pf &= \left\{ a f + b f \cdot v + \omega f \frac{|v|^2 - 3}{\sqrt{6}} \right\} \sqrt{M}, \tag{2.7}
\end{align*}$$

where $P f$ is called the macroscopic component of $f$, while $(I - P) f$ is called the corresponding microscopic component. Interestingly, our new decomposition is formally the same as that of the linearized collision operator of the Boltzmann equation [41], but our decomposition here comes not only from the spectral analysis of the linearized Fokker–Planck operator $L$, but also from the coupling term involving momentum and energy exchanges. This further reveals the internal relations and differences between the particle–fluid system and the Boltzmann equation or coupled Boltzmann equation.

According to new decomposition (2.7), the linearized Fokker–Planck operator $\mathcal{L}$ satisfies the following additional properties besides the above properties (1)–(3):

(4) $L f$ can be written as

$$\mathcal{L} f = \mathcal{L}(I - P) f + \mathcal{L}P f = \mathcal{L}(I - P) f - P_1 f - 2P_2 f.$$

(5) There exists a constant $\lambda > 0$, such that

$$\begin{align*}
\langle -\mathcal{L}(I - P) f, g \rangle &\geq \lambda |(I - P) f|^2, \\
\langle -\mathcal{L} f, f \rangle &\geq \lambda |(I - P) f|^2 + |b f|^2 + 2|\omega f|^2. \tag{2.8}
\end{align*}$$
2.5 Main results

We now state our main results. The first result is the global existence of classical solutions with small initial data and optimal algebraic rate of decay in the whole space.

**Theorem 2.1** Let \( \Omega = \mathbb{R}^3 \) and \((f_0, \rho_0, u_0, \theta_0)\) be the initial data such that \( F_0 = M + \sqrt{M} f_0 \geq 0 \), and there exists \( \varepsilon_0 > 0 \), \( \|f_0\|_{H^4_{x,v}} + \|\rho_0, u_0, \theta_0\|_{H^4} < \varepsilon_0 \). Then the Cauchy problem (2.1)–(2.5) admits a unique global solution \((f, \rho, u, \theta)\) satisfying \( F = M + \sqrt{M} f \geq 0 \) and

\[
 f \in C([0, \infty); H^4(\mathbb{R}^3 \times \mathbb{R}^3)), \quad (\rho, u, \theta) \in C([0, \infty); H^4(\mathbb{R}^3))^3,
\]

\[
 \sup_{t \geq 0} (\|f(t)\|_{H^4_{x,v}} + \|\rho, u, \theta\|(t)_{H^4}) \leq C \left( \|f_0\|_{H^4_{x,v}} + \|\rho_0, u_0, \theta_0\|_{H^4} \right),
\]

for some constant \( C > 0 \). Moreover, if we further assume that

\[
 \|(f_0, \rho_0, u_0, \theta_0)\|_{Z_1 \cap H^4} \leq \varepsilon_0,
\]

then

\[
 \|f(t)\|_{H^4_{x,v}} + \|(\rho, u, \theta)(t)\|_{H^4} \leq C (1 + t)^{-3/4} \|(f_0, \rho_0, u_0, \theta_0)\|_{Z_1 \cap H^4},
\]

for some constant \( C > 0 \) and all \( t \geq 0 \).

**Remark 2.1** Here the algebraic rate of decay in (2.10) is optimal in the sense that this rate coincides with that of the corresponding linear system.

The second result is concerned with the periodic spatial domain. Compared with the case of the whole space, here we know that the Poincaré inequality holds, thus the exponential convergence rate is obtained.

**Theorem 2.2** Let \( \Omega = T^3 \) and \((f_0, \rho_0, u_0)\) be the initial data such that \( F_0 = M + \sqrt{M} f_0 \geq 0 \), there exists \( \varepsilon_0 > 0 \), \( \|f_0\|_{H^4_{x,v}} + \|\rho_0, u_0\|_{H^4} < \varepsilon_0 \), and

\[
 \int_{T^3} a_0 \, dx = 0, \quad \int_{T^3} \rho_0 \, dx = 0,
\]

\[
 \int_{T^3} (b_0 + (1 + \rho_0)u_0) \, dx = 0,
\]

\[
 \int_{T^3} (1 + \rho_0) \left( \theta_0 + \frac{1}{2} |u_0|^2 \right) + \frac{\sqrt{6}}{2} \omega_0 \, dx = 0,
\]

where

\[
 a_0 = \int_{T^3} \sqrt{M} f_0(x, v) \, dv, \quad b_0 = \int_{T^3} v \sqrt{M} f_0(x, v) \, dv, \quad \omega_0 = \int_{T^3} \frac{|v|^2 - 3}{\sqrt{6}} \sqrt{M} f_0(x, v) \, dv.
\]

Then, the Cauchy problem (2.1)–(2.5) admits a unique global solution \((f, \rho, u)\) satisfying \( F = M + \sqrt{M} f \geq 0 \), and

\[
 f \in C([0, \infty); H^4(T^3 \times \mathbb{R}^3)), \quad (\rho, u, \theta) \in C([0, \infty); H^4(T^3))^3,
\]

\[
 \|f(t)\|_{H^4_{x,v}} + \|(\rho, u, \theta)(t)\|_{H^4} \leq C \left( \|f_0\|_{H^4_{x,v}} + \|\rho_0, u_0, \theta_0\|_{H^4} \right)e^{-\lambda t},
\]

with \( \lambda > 0 \) some constant, for all \( t \geq 0 \).

In the rest of this paper, we shall omit the integral domain \( \Omega \times \mathbb{R}^3 \) or \( \mathbb{R}^3 \) in the integrals for simplicity.
3 Global existence of classical solutions in the whole space

In this section, we shall establish the global existence of classical solutions to the problem (2.1)–(2.5) in the whole space $\mathbb{R}^3$. We first obtain the uniform a priori estimates. Then we construct the unique local solution by an iteration process, and obtain the global solution by the continuum argument.

3.1 A priori estimates

In this subsection, we will show the uniform-in-time a priori estimates in the space $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$. We assume that $(f, \rho, u, \theta)$ is a classical solution to the Cauchy problem (2.1)–(2.5) for $0 < t < T$ with a fixed $T > 0$, and

$$\sup_{0 < t < T} \left\{ \| f(t) \|_{H_{x,v}^1} + \| (\rho, u, \theta)(t) \|_{H_{x}^4} \right\} \leq \delta,$$

with $0 < \delta < 1$ sufficiently small constant.

The following lemma in [14] is useful for the forthcoming estimates.

Lemma 3.1 [14] There exists a constant $C > 0$ such that for any $f, g \in H^4(\mathbb{R}^3)$ and any multi-index $\gamma$ with $1 \leq |\gamma| \leq 4$,

$$\| f \|_{L^\infty(\mathbb{R}^3)} \leq C \| \nabla_x f \|_{L^2(\mathbb{R}^3)}^{1/2} \| \nabla_x^2 f \|_{L^2(\mathbb{R}^3)}^{1/2},$$

$$\| fg \|_{H^2(\mathbb{R}^3)} \leq C \| f \|_{H^2(\mathbb{R}^3)} \| \nabla_x g \|_{H^2(\mathbb{R}^3)},$$

$$\| \partial_\gamma f \|_{L^2(\mathbb{R}^3)} \leq C \| \nabla_x f \|_{H^1(\mathbb{R}^3)} \| \nabla_x g \|_{H^1(\mathbb{R}^3)}.$$  

The a priori estimates will be proved in the next two subsections, one for the pure $x$-variable part, and one for the mixed derivative part which involves both the $x$-variable and the particle velocity $v$-variable.

3.1.1 Energy estimates in the $x$-variable

Proposition 3.1 For classical solution of the system (2.1)–(2.5), we have

$$\frac{1}{2} \frac{d}{dt} \| (f, \rho, u, \theta)(t) \|^2 + \lambda (\| (I - P) f \|_v^2 + \| b - u \|^2 + \| \sqrt{2} \omega - \sqrt{3} \theta \|^2 + \| \nabla u, \nabla \theta \|^2) \leq C (\| (\rho, u, \theta) \|_{H^2} + \| \rho \|_{H^1} + \| u, \theta \|_{H^1}^2) \times \left( \| \nabla_x (a, b, \omega, \rho, u, \theta) \|^2 + \| u - b, \sqrt{2} \omega - \sqrt{3} \theta \|^2 + \| (I - P) f \|_v^2 \right),$$

for all $0 \leq t < T$ with any $T > 0$.

Proof Multiplying (2.1)–(2.4) by $f, \rho, u, \theta$ respectively, then taking integration and summation, we finally get

$$\frac{1}{2} \frac{d}{dt} \| (f) \|^2 + \| (\rho) \|^2 + \| (u) \|^2 + \| (\theta) \|^2 + \int (-\mathcal{L}(I - P) f, f) \, dx$$

$$+ \int \frac{\| \nabla u \|^2}{1 + \rho} \, dx + \int \frac{\| \nabla \theta \|^2}{1 + \rho} \, dx + \| b - u \|^2 + \| \sqrt{2} \omega - \sqrt{3} \theta \|^2$$

$$= \int u \left( \frac{1}{2} \{ f, v \} - \left\{ \frac{au}{1 + \rho}, u \right\} - \left\{ \frac{u \cdot b}{1 + \rho}, \theta \right\} \right) \, dx.$$
The estimates for Next, we make estimates of the terms on the right hand side of the equality (3.6).

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We first notice that and meanwhile by the Hölder and Sobolev inequalities, one gets

\[
\int \left( \frac{\rho}{1+\rho} - \frac{u}{1+\rho} \right) u \, dx + \int \frac{\nabla \rho}{(1+\rho)^2} \cdot u \, dx + \int \frac{\nabla \rho}{(1+\rho)^2} \cdot \nabla \theta \, dx - \int (\theta^2 + \rho^2) \text{div} u \, dx
\]

\[
+ \left( \frac{u-b}{1+\rho} \cdot u + a|u|^2 + \frac{\rho}{1+\rho} \theta \right) - \left( \frac{\rho}{1+\rho} (u-b) , u \right) - \left( \frac{\rho}{1+\rho} (\sqrt{2} \omega - 3\theta) , \theta \right). \quad (3.6)
\]

By (2.8), we have

\[
\langle -\mathcal{L}((I-P)f , f) \rangle \geq \lambda_0 \| (I-P)f \|_v^2.
\]

Next, we make estimates of the terms on the right hand side of the equality (3.6).

(1) The estimates for \( \int u \left( \frac{1}{2} v f , f \right) \, dx - \int \frac{au}{1+\rho} u \, dx - \int \frac{u}{1+\rho} \theta \, dx \) are obtained as follows.

We first notice that

\[
\int u \left( \frac{1}{2} v f , f \right) \, dx - \int a|u|^2 \, dx - \int u \cdot b \theta \, dx
\]

According to the macro–micro decomposition (2.7), we get

\[
\int u \left( \frac{1}{2} v f , f \right) \, dx = \int au \cdot (u-b) \, dx + \left\| \frac{\sqrt{3}}{3} \right\|_v \int (\sqrt{2} \omega - \sqrt{3}\theta) u \cdot b \, dx
\]

\[
+ \int u(vPf , (I-P)f) \, dx + \frac{1}{2} \int u(\theta , (I-P)f)^2 \, dx
\]

\[
\leq \left\| a \right\|_{L^6} \left\| u \right\|_{L^3} \left\| u - b \right\|_L^2 + \left\| b \right\|_{L^6} \left\| u \right\|_{L^3} \left\| \sqrt{2} \omega - \sqrt{3} \theta \right\|_L^2
\]

\[
+ C \left\| (a , b , \omega) \right\|_{L^6} \left\| u \right\|_{L^3} \left\| (I-P)f \right\|_L^2 + C \left\| u \right\|_{L^\infty} \left\| (I-P)f \right\|_v^2
\]

\[
\leq C \left( \left\| u \right\|_{H^1} + \left\| \nabla u \right\|_{H^1} \right) \left\| (I-P)f \right\|_v^2
\]

\[
+ C \left\| u \right\|_{H^1} \left( \left\| u - b \right\|_L^2 + \left\| \sqrt{2} \omega - \sqrt{3} \theta \right\|_L^2 + \left\| \nabla (a , b , \omega) \right\|_L^2 \right),
\]

and meanwhile by the Hölder and Sobolev inequalities, one gets

\[
\left\{ \frac{\rho}{1+\rho} au , u \right\} + \left\{ \frac{\rho}{1+\rho} u \cdot b , \theta \right\} \leq C \int \left\| \rho \right\| a \left\| u \right\|^2 \, dx + C \int \left\| \rho \right\| \theta \left\| u \right\| \left\| b \right\| \, dx
\]

\[
\leq C \left\| \rho \right\|_{H^1} \left( \left\| u \right\|_{H^1} + \left\| b \right\|_{H^1} \right) \left\| \nabla (a , b , u) \right\|_L^2.
\]

Thus, we eventually have

\[
\int u \left( \frac{1}{2} v f , f \right) \, dx - \int a|u|^2 \, dx - \int u \cdot b \theta \, dx
\]

\[
\leq C \left( \left\| u \right\|_{H^1} + \left\| \nabla u \right\|_{H^1} + \left\| \rho \right\|_{H^1} \left( \left\| u \right\|_{H^1} + \left\| b \right\|_{H^1} \right) \right)
\]

\[
\left\{ \left\| (I-P)f \right\|_v^2 + \left\| u - b \right\|_L^2 + \left\| \sqrt{2} \omega - \sqrt{3} \theta \right\|_L^2 + \left\| \nabla (a , b , u) \right\|_L^2 \right), \quad (3.7)
\]
(2) The estimates for \( \int \theta(\nabla v f - \frac{v}{2} f)(\nabla v f + \frac{v}{2} f) \, dx \, dv \) are obtained as follows. First, the term \( \int \theta(\nabla v f - \frac{v}{2} f)(\nabla v f + \frac{v}{2} f) \, dx \, dv \) can be rewritten as

\[
\begin{align*}
\int \theta(\nabla v f - \frac{v}{2} f)(\nabla v f + \frac{v}{2} f) \, dx \, dv &= \int \theta \left( |\nabla v f|^2 - \frac{|v|^2}{4} f^2 \right) \, dx \, dv \\
&= \int \theta \left( |\nabla v (I - P) f|^2 - \frac{|v|^2}{4} |(I - P) f|^2 \right) \, dx \, dv \\
&\quad + \int \theta \left( |\nabla v P f|^2 - \frac{|v|^2}{4} |P f|^2 \right) \, dx \, dv \\
&\quad + \int 2\theta \left( \nabla v (I - P) f \cdot \nabla v P f - \frac{|v|^2}{4} (I - P) f \cdot P f \right) \, dx \, dv \\
&= \int \theta \left( |\nabla v (I - P) f|^2 - \frac{|v|^2}{4} |(I - P) f|^2 \right) \, dx \, dv \\
&\quad + \int 2\theta \left( \nabla v (I - P) f \cdot \nabla v P f - \frac{|v|^2}{4} (I - P) f \cdot P f \right) \, dx \, dv - \sqrt{6} \int a\omega d\theta.
\end{align*}
\]

Thus, the following estimates hold

\[
\begin{align*}
&\quad - \int \theta(\nabla v f - \frac{v}{2} f)(\nabla v f + \frac{v}{2} f) \, dx \, dv - \left( \frac{3a\theta}{1 + \rho}, \theta \right) \\
&= \sqrt{3} \int a\theta(\sqrt{2}\omega - \sqrt{3}\theta) \, dx + \left( \frac{3a\theta}{1 + \rho}, \theta \right) \\
&\quad - \int \theta \left( |\nabla v (I - P) f|^2 - \frac{|v|^2}{4} |(I - P) f|^2 \right) \, dx \, dv \\
&\quad - \int 2\theta \left( \nabla v (I - P) f \cdot \nabla v P f - \frac{|v|^2}{4} (I - P) f \cdot P f \right) \, dx \, dv \\
&\leq C \|a\|_{L^6} \|\theta\|_{L^3} \|\sqrt{2}\omega - \sqrt{3}\theta\|_{L^2} + C \|a\|_{L^6} \|\rho\|_{L^6} \|\theta\|_{L^3}^2 \\
&\quad + C \|\theta\|_{L^\infty} \|\|I - P\| f\|_v^2 + C \|\|I - P\| f\|_v \|(a, b, \omega)\|_{L^6} \|\theta\|_{L^3} \\
&\leq C \left( \|\theta\|_{H^2} + \|\theta\|_{H^1}^2 \right) \\
&\quad \times \left( \|\|I - P\| f\|_v^2 + \|\nabla x (a, b, \omega)\|_{L^2}^2 + \|\sqrt{2}\omega - \sqrt{3}\theta\|_{L^2}^2 \right). \quad (3.8)
\end{align*}
\]

(3) For the estimates of all the remaining terms, using the Hölder and Sobolev inequalities as well as Lemma 3.1, we easily get

\[
\begin{align*}
&\int (u \cdot \nabla u) \cdot u \, dx + \int (u \cdot \nabla \rho) \cdot \rho \, dx + \int (u \cdot \nabla \theta) \cdot \theta \, dx \\
&\quad \leq C \|u\|_{L^3} \|\nabla (u, \nabla \rho, \nabla \theta)\|_{L^2} \|(u, \rho, \theta)\|_{L^6} \\
&\quad \leq C \|u\|_{H^1} \|\nabla (u, \nabla \rho, \nabla \theta)\|_{L^2}^2, \\
&\int \frac{1}{(1 + \rho)^{\frac{3}{2}}} \nabla \rho(u \cdot u + \nabla \theta \cdot \theta) \, dx \\
&\quad \leq C \|(u, \theta)\|_{H^2} \left( \|\nabla \rho\|^2 + \|\nabla u\|^2 + \|\nabla \theta\|^2 \right).
\end{align*}
\]
\[
\int \frac{\theta - \rho}{1 + \rho} \nabla \rho \cdot u \, dx \leq C \| (\rho, \theta) \|_{L^3} \| \nabla \rho \|_{L^2} \| u \|_{L^6} \\
\leq C \| (\rho, \theta) \|_{H^1} \left( \| \nabla \rho \|^2 + \| \nabla u \|^2 \right),
\]
\[
\int (\rho^2 + \theta^2) \text{div} u \, dx \leq C \| (\rho, \theta) \|_{L^3} \| \nabla u \|_{L^2} \| (\rho, \theta) \|_{L^6} \\
\leq C \| (\rho, \theta) \|_{H^1} \left( \| \nabla \rho \|, \| \nabla \theta \|, \| \nabla u \| \right),
\]
\[
\left( \frac{\rho}{1 + \rho} (b - u), u \right) + \left( \frac{u}{1 + \rho} (u - b), \theta \right) + \left( \frac{\rho}{1 + \rho} (\sqrt{6} \omega - 3 \theta), \theta \right) \\
\leq C \| (u, \theta) \|_{H^1} \left( \| (b - u, \sqrt{2} \omega - \sqrt{3} \theta) \| \right) \left( \| \nabla (\nabla \theta, \nabla \rho) \| \right),
\]
\[
\left( \frac{1}{1 + \rho} a |u|^2, \theta \right) \leq C \| a \|_{L^2} \| \theta \|_{L^6} \| u \|_{L^3}^2 \\
\leq C \| u \|_{H^4}^2 \left( \| \nabla \theta \|, \| \nabla \theta \| \right). \tag{3.9}
\]

Substituting all the above estimates (3.7), (3.8), (3.9) into (3.6) and using (3.1), we obtain (3.5).

\[
\square
\]

**Proposition 3.2** For smooth solutions of the problem (2.1)–(2.5), we have

\[
\frac{1}{2} \frac{d}{dt} \sum_{1 \leq |\alpha| \leq 4} \left\{ \| \partial^\alpha f \|_2^2 + \| \partial^\alpha \rho \|_2^2 + \| \partial^\alpha \theta \|_2^2 \right\} \\
+ \lambda \sum_{1 \leq |\alpha| \leq 4} \left\{ \| (I - P) \partial^\alpha f \|_2^2 + \| \partial^\alpha (b - u, \sqrt{2} \omega - \sqrt{3} \theta) \|_2^2 \right\} \\
\leq C \left\{ \| (\rho, u, \theta) \|_{H^4} + \| (\rho, u, \theta) \|_{H^3}^2 \right\} \cdot \left( 1 + \| \rho \|_{H^4}^4 \right)^2 \\
\times \left\{ \| \nabla \nabla (a, b, \omega, \rho, u, \theta) \|_{H^3}^2 + \sum_{1 \leq |\alpha'| \leq 4} \| (I - P) \partial^\alpha f \|_2^2 \right\}, \tag{3.10}
\]

for all \(0 \leq t < T\) with any \(T > 0\).

**Proof** Here, we follow the similar argument [29, Proposition 4.2]. Applying differentiation \(\partial^\alpha (1 \leq |\alpha| \leq 4)\) to (2.1)–(2.4) yields

\[
\partial_t (\partial^\alpha f) + v \cdot \nabla_x (\partial^\alpha f) + u \cdot \nabla_v (\partial^\alpha f) - \partial^\alpha u \cdot v M^{1 \over 2} - \{ v \}^2 - 3 \{ M \}^{1 \over 2} \partial^\alpha \theta - \mathcal{L} \partial^\alpha f \\
= \frac{1}{2} \partial^\alpha (u \cdot v f) + [- \partial^\alpha u \cdot \nabla v] f + \partial^\alpha (\theta \cdot M^{-1 \over 2} \Delta_v (M^{1 \over 2} f)), \tag{3.11}
\]

\[
\partial_t (\partial^\alpha \rho) + u \cdot \nabla \partial^\alpha \rho + (1 + \rho) \text{div} \partial^\alpha u = [- \partial^\alpha, \rho \nabla_x] u + [- \partial^\alpha, u \cdot \nabla_x] \rho, \tag{3.12}
\]

\[
\partial_t (\partial^\alpha u) + u \cdot \nabla (\partial^\alpha u) + \nabla \partial^\alpha \theta + \nabla \partial^\alpha \rho - \partial^\alpha \left( \frac{1}{1 + \rho} \Delta u \right) - \partial^\alpha (b - u) \\
= [- \partial^\alpha, u \cdot \nabla_x ] u + \partial^\alpha \left( \frac{\rho - \theta}{1 + \rho} \nabla \rho \right) - \partial^\alpha \left( \frac{1}{1 + \rho} u \right), \tag{3.13}
\]

\[
\partial_t (\partial^\alpha \theta) + \text{div} (\partial^\alpha u) - \partial^\alpha \left( \frac{1}{1 + \rho} \Delta \theta \right) - \sqrt{3} \left( \sqrt{2} \partial^\alpha \omega - \sqrt{3} \partial^\alpha \theta \right) \\
= - \partial^\alpha (u \cdot \nabla \theta + \theta \text{div} u) + \partial^\alpha \left( \frac{1}{1 + \rho} (u - b) \cdot u + a |u|^2 - u \cdot b \right). \]
\[ -\sqrt{3}\left(\frac{\rho}{1+\rho}\left(\sqrt{2}\alpha\omega - \sqrt{3}\beta\theta\right)\right) - \alpha\left(\frac{3}{1+\rho}\vartheta\cdot\omega\right), \tag{3.14} \]

where \([A, B]\) denotes the commutator \(AB - BA\) for two operators \(A\) and \(B\). Now, multiplying (3.11)–(3.14) by \(\bar{\alpha} f\cdot\cdot\cdot\bar{\alpha} u\cdot\bar{\alpha}\theta\) respectively, then taking integration and summation, we obtain

\[
\frac{1}{2}\frac{d}{dt}\left\{ \|\alpha f\|^2 + \|\alpha u\|^2 + \|\alpha^3 u\|^2 \right\} + \lambda \int \left( \|\nu(\alpha f)\|^2 + \|\alpha^3(\nu\alpha f)\|^2 \right) + \int \left( (\alpha f, [I - P]\alpha^3 f) + (\alpha u, [I - P]\alpha^3 f) \right) dx \\
+ \int \left( [-\alpha, u \cdot \nabla f] f, \alpha \alpha f \right) dx + \int \left( [-\alpha, u \cdot \nabla f] f, \alpha \alpha f \right) dx + \int \left( [-\alpha, u \cdot \nabla f] \rho \alpha^3 f \right) dx \\
+ \int \left( [-\alpha, u \cdot \nabla f] u \alpha^3 f \right) dx + \int \left( [-\alpha, \rho \nabla u] u \alpha^3 f \right) dx \\
- \int \left( \alpha (\frac{\alpha}{1+\rho}) \cdot \alpha u \alpha^3 f \right) dx - \int \left( \alpha (\frac{\alpha \cdot b}{1+\rho}) \alpha^3 f \right) dx - \int \left( \alpha (\frac{\alpha \cdot b}{1+\rho}) \alpha^3 f \right) dx \\
+ \int \left( \alpha (\frac{\alpha \cdot b}{1+\rho}) \alpha^3 f \right) dx - \int \left( \rho \div \alpha u \alpha^3 f \right) dx - \int \left( \alpha (\rho \div \alpha u) \alpha^3 f \right) dx \\
- \int \left( u \cdot \nabla \alpha^3 \rho \alpha^3 f \right) dx - \int \left( u \cdot \nabla \alpha u \cdot \alpha^3 f \right) dx - \int \left( \alpha (u \cdot \nabla f) \alpha^3 f \right) dx \\
- \sum_{1 \leq \beta \leq \alpha} c_{\alpha, \beta} \int \left( \frac{\alpha}{1+\rho} \right) \alpha^3 u \alpha^3 u \alpha^3 f \right) dx - \sum_{1 \leq \beta \leq \alpha} c_{\alpha, \beta} \int \left( \frac{\alpha}{1+\rho} \right) \alpha^3 u \alpha^3 f \right) dx \\
- \int \left( \alpha (\frac{\rho}{1+\rho} (b - u)) \cdot \alpha^3 f \right) dx - \sqrt{3} \int \left( \alpha (\frac{\rho}{1+\rho} (\sqrt{2} \omega - \sqrt{3} \theta)) \cdot \alpha^3 f \right) dx \\
+ \int \left( \alpha (\frac{1}{1+\rho} (u - b) \cdot u) \alpha^3 f \right) dx + \int \left( \alpha (\frac{1}{1+\rho} a |u|^2) \alpha^3 f \right) dx \\
:= \sum_{j=1}^{23} I_j, \tag{3.15} \]

where \(C_{\alpha, \beta}\) is constant depending only on \(\alpha\) and \(\beta\).

We need to estimate each term on the right hand side of (3.15). Firstly,

\[
I_1 \leq C \|\alpha^3 (u f)\| \|v^3 f\| \\
\leq C \|\nabla u\|_H^3 \|\nabla f\|_{L^2(H^3)} \|v^3 f\|, \\
I_2 = -\int \alpha \left( \theta (\nabla v f - \frac{v}{2} f) \right) \cdot \left( \nabla \alpha^3 f + \frac{v^3}{2} f \right) \\
\leq \|\alpha^3 (\theta (\nabla v f - \frac{v}{2} f))\| \|\nabla v \alpha^3 f + \frac{v^3}{2} f\| \\
\leq C \|\nabla \theta\|_H^3 \|\nabla v - \frac{v}{2} \nabla f\|_{L^2(H^3)} \|\nabla v + \frac{v}{2} \nabla f\|_{L^2(H^3)} \\
\leq C \|\nabla \theta\|_H^3 \|\nabla v - \frac{v}{2} \nabla f\|_{L^2(H^3)} \|\nabla v + \frac{v}{2} \nabla f\|_{L^2(H^3)} \\
\leq C \|\nabla \theta\|_H^3 \|\nabla v - \frac{v}{2} \nabla f\|_{L^2(H^3)} \|\nabla v + \frac{v}{2} \nabla f\|_{L^2(H^3)}.
\]
\[ \leq C \| \nabla \theta \|_{H^3} \left\{ \sum_{1 \leq |\alpha| \leq 4} \| (I - P) \partial^{\alpha} f \|_v + \| (\nabla (a, b, \omega)) \|^2_{H^3} \right\}, \]

where we have used the following fact

\[ \|(\nabla_v - \frac{v}{2})\nabla f\|_{L^2_v(H^3)} \]
\[ \leq \sum_{1 \leq |\alpha'| \leq 4} \left\| \left\{ (\nabla_v - \frac{v}{2}) \right\} \partial^{\alpha'} f \left\|_{L^2_v} \right. \]
\[ \leq \sum_{1 \leq |\alpha'| \leq 4} \left( \left\| \left\{ (\nabla_v - \frac{v}{2}) \right\} (I - P) \partial^{\alpha'} f \right\| + \left\| \left\{ (\nabla_v - \frac{v}{2}) \right\} P \partial^{\alpha'} f \left\| \right. \right. \]
\[ \leq \sum_{1 \leq |\alpha'| \leq 4} \left( \| (I - P) \partial^{\alpha'} f \|_v + \| \partial^{\alpha'} (a, b, \omega) \| \right), \]

and the similar estimate holds for \( \|(\nabla_v + \frac{v}{2})\nabla f\|_{L^2_v(H^3)} \).

Using the Hölder, Sobolev and Young inequalities, we get the following bounds

\[ I_3 = \int \langle \partial^{\alpha} (uf), \partial_v \partial^{\alpha} f \rangle \, dx \leq \| \partial^{\alpha} (uf) \| \| \nabla_v \partial^{\alpha} f \| \]
\[ \leq C \| \nabla u \|_{H^3} \| \nabla f \|_{L^2_v(H^3)} \| \nabla_v \partial^{\alpha} f \|, \]
\[ I_4 \leq C \| u \|_{H^4} \left( \| \nabla \rho \|_{H^3}^2 + \| \partial^{\alpha} \rho \| \right), \]
\[ I_5 \leq C \| u \|_{H^4} \left( \| \nabla u \|_{H^3}^2 + \| \partial^{\alpha} u \| \right), \]
\[ I_6 \leq C \| \rho \|_{H^4} \left( \| \nabla \rho \|_{H^3}^2 + \| \partial^{\alpha} \rho \| \right). \]
\[ + C \| (\rho, \theta) \|_{H^4} \| \partial^\alpha u \| \| \partial^\alpha \rho \| + C \| \rho \|_{H^4} \| (\rho, \theta) \|_{H^4} \| \partial^\alpha u \| \| \partial^\alpha \rho \| \]
\[ \leq \varepsilon \| \partial^\alpha \nabla u \|^2 + C \varepsilon (1 + \| \rho \|_{H^4}^4) \| (\rho, \theta) \|_{H^4} \left( \| \nabla \rho \|_{H^3}^2 + \| \partial^\alpha u \|^2 \right), \]
\[ I_{11} \leq \varepsilon \| \partial^\alpha \nabla u \|^2 + C \varepsilon \| \rho \|_{H^3}^2 \| \partial^\alpha \rho \|^2, \]
\[ I_{12} \leq \varepsilon \| \nabla^2 u \|_{H^3}^2 + C \varepsilon \| \theta \|_{H^4}^2 \| \partial^\alpha \theta \|^2, \]
\[ I_{13} \leq C \| u \|_{H^4} \| \partial^\alpha \rho \|^2, \]
\[ I_{14} \leq C \| u \|_{H^4} \| \partial^\alpha u \|^2, \]
\[ I_{15} \leq \varepsilon \| \nabla^2 \theta \|_{H^3}^2 + C \varepsilon \| u \|_{H^4}^2 \| \partial^\alpha \theta \|^2, \]

with \( \varepsilon > 0 \) a small constant, and
\[ I_{16} \leq C \| \nabla \rho \|_{H^2} \| \nabla \partial^\alpha u \| \| \partial^\alpha \rho \| + C \left( \| \nabla \rho \|_{H^3} + \| \nabla \rho \|_{H^4}^4 \right) \| \nabla u \|_{H^3}^2 \]
\[ \leq \varepsilon \| \nabla \partial^\alpha u \|^2 + C \varepsilon \left( \| \nabla \rho \|_{H^3} + \| \nabla \rho \|_{H^4}^4 \right) \| \nabla u \|_{H^3}^2, \]
\[ I_{17} \leq C \| \nabla \rho \|_{L^\infty} \| \nabla \partial^\alpha u \| \| \partial^\alpha \rho \| \]
\[ \leq \varepsilon \| \nabla \partial^\alpha u \|^2 + C \varepsilon \| \nabla \rho \|_{H^3}^2 \| \partial^\alpha u \|^2, \]
\[ I_{18} \leq \varepsilon \| \nabla \partial^\alpha \theta \|^2 + C \varepsilon \left( \| \nabla \rho \|_{H^3} + \| \nabla \rho \|_{H^4}^4 \right) \| \nabla \theta \|_{H^3}^2, \]
\[ I_{19} \leq \varepsilon \| \nabla \partial^\alpha \theta \|^2 + C \varepsilon \| \nabla \rho \|_{H^3}^2 \| \partial^\alpha \theta \|^2, \]
\[ I_{20} \leq \left\| \frac{\partial^\alpha (b - u)}{1 + \rho} \right\| \| \partial^\alpha \rho \| \leq \| \nabla \frac{\rho}{1 + \rho} \|_{H^3} \| \nabla (b - u) \|_{H^3} \| \partial^\alpha u \|
\leq C \left( 1 + \| \rho \|_{H^4}^4 \right) \| \rho \|_{H^4} \left( \sum_{1 \leq |\alpha| \leq 4} \| \partial^\alpha \rho (b - u) \| \right) \| \partial^\alpha u \|
\leq \varepsilon \sum_{1 \leq |\alpha| \leq 4} \| \partial^\alpha \rho (b - u) \|^2 + C \varepsilon \left( 1 + \| \rho \|_{H^4}^4 \right) \| \rho \|_{H^4}^2 \| \partial^\alpha u \|^2, \]
\[ I_{21} \leq \varepsilon \sum_{1 \leq |\alpha| \leq 4} \| \partial^\alpha \left( \sqrt{2} \omega - \sqrt{3} \theta \right) \|^2 + C \varepsilon \left( 1 + \| \rho \|_{H^4}^4 \right) \| \rho \|_{H^4}^2 \| \partial^\alpha \theta \|^2, \]
\[ I_{22} \leq \varepsilon \sum_{1 \leq |\alpha| \leq 4} \| \partial^\alpha \left( b - u \right) \|^2 + C \varepsilon \left( 1 + \| \rho \|_{H^4}^4 \right) \| \rho \|_{H^4}^2 \| \partial^\alpha \theta \|^2, \]
\[ I_{23} \leq C \left\| \frac{|u|^2}{1 + \rho} \right\|_{H^3} \| \nabla a \|_{H^3} \| \partial^\alpha \theta \|
\leq C \left( 1 + \| \rho \|_{H^4}^4 \right) \| u \|_{H^4}^2 \left( \| \nabla a \|_{H^3}^2 + \| \partial^\alpha \theta \|^2 \right). \]

Substituting all the above estimates on \( I_i (1 \leq i \leq 23) \) into (3.15) and taking summation over \( 1 \leq |\alpha| \leq 4 \), we complete (3.10).  \( \square \)

To estimate the energy dissipation of \((a, b, \omega)\), namely, \( \| \nabla \times (a, b, \omega) \|_{H^3} \), we need to deduce the equations satisfied by \((a, b, \omega)\). We first introduce the moment functional and the energy functional as
\[ \Gamma_{i,j} = \langle (v_i v_j - 1) \sqrt{M}, g \rangle, \quad Q_{i,j} = \left( \frac{1}{\sqrt{6}} v_i (|v|^2 - 3) M^{\frac{1}{2}}, g \right), \]
for any \( g = g(v), \ 1 \leq i, j \leq 3 \). It is easy to verify that the system of equations satisfied by \((a, b, \omega)\) is the following:
\[ \partial_t a + \text{div} b = 0, \]
(3.16)
\[
\partial_t b_i + \partial_t a + \frac{2}{\sqrt{6}} \partial_t \omega + \sum_{j=1}^{3} \partial_{x_j} \Gamma_{i,j}(\{I - P\} f) = -b_i + u_i + u_i a, \\
\partial_t \omega + \sqrt{2} (\sqrt{2} \omega - \sqrt{3} \theta) - \sqrt{6} a \theta + \frac{2}{\sqrt{6}} \text{div} b - \frac{1}{\sqrt{6}} u \cdot b \\
+ \sum_{i=1}^{3} \partial_{x_i} Q_i(\{I - P\} f) + \frac{1}{2} \sum_{i=1}^{3} u_i Q_i(\{I - P\} f) = 0, \\
\partial_j b_i + \partial_i b_j - (u_i b_j + u_j b_i) \\
- \frac{2}{\sqrt{6}} \delta_{ij} \left( \frac{2}{\sqrt{6}} \text{div} b - \frac{1}{\sqrt{6}} u \cdot b + \sum_{i=1}^{3} \partial_{x_i} Q_i(\{I - P\} f) + \frac{1}{2} \sum_{i=1}^{3} u_i Q_i(\{I - P\} f) \right) \\
= -\partial_i \Gamma_{i,j}(\{I - P\} f) + \Gamma_{i,j}(l + r + s), \\
\frac{5}{3} \left( \partial_t \omega + \omega u_i - \sqrt{6} \omega b_i \right) - \frac{2}{\sqrt{6}} \sum_{j=1}^{3} \partial_{x_j} \Gamma_{i,j}(\{I - P\} f) \\
= -\partial_i Q_i(\{I - P\} f) + Q_i(l + r + s),
\]

where \(l, r, s\) are defined by
\[
l := -v \cdot \nabla_v \{I - P\} f + \mathcal{L}(I - P) f, \\
r := -u \cdot \nabla_v \{I - P\} f + \frac{1}{2} u \cdot v(I - P) f, \\
s := \theta M^{-\frac{1}{2}} \text{div}_v \left( M^{\frac{1}{2}} (\nabla_v - \frac{v}{2})(I - P) f \right).
\]

In fact, multiplying (2.1) by \(\sqrt{M}\), \(v_i \sqrt{M}\) (\(1 \leq i \leq 3\), \(\frac{\|v\|^2 - 3}{\sqrt{6}} M^{\frac{1}{2}}\) respectively and then integrating with respect to velocity over \(\mathbb{R}^3\), we deduce that (3.16)–(3.18) hold.

In order to get (3.19)–(3.20), we can rewrite (2.1) as
\[
\partial_t P f + v \cdot \nabla_x P f + u \cdot \nabla_v P f - \frac{1}{2} u \cdot v P f - u \cdot v M^{\frac{1}{2}} - \theta(\|v\|^2 - 3) M^{\frac{1}{2}} \\
+ P f + 2 P f - \theta M^{-\frac{1}{2}} \text{div}_v \left( M^{\frac{1}{2}} (\nabla_v - \frac{v}{2}) P f \right) \\
= -\partial_t(\{I - P\} f) + l + r + s.
\]

Applying \(\Gamma_{ij}\) to (3.21) and combining (3.16) and (3.18), we get the Eq. (3.19). Similarly, applying \(Q_i\) to (3.21) and combining (3.17), we can obtain the Eq. (3.20).

Define a temporal functional
\[
\mathcal{E}_0(t) := \sum_{|\alpha| \leq 3} \sum_{i,j} \int_{\mathbb{R}^3} \partial^\alpha (\partial_j b_i + \partial_i b_j) \partial^\alpha \Gamma_{i,j}(\{I - P\} f) \ dx \\
+ \sum_{|\alpha| \leq 3} \sum_{i} \int_{\mathbb{R}^3} \partial^\alpha \partial_t \omega \partial^\alpha Q_i(\{I - P\} f) \ dx \\
+ \frac{2}{21} \sum_{|\alpha| \leq 3} \int_{\mathbb{R}^3} \partial_x^\alpha a \partial^\alpha \left( \frac{\sqrt{6}}{5} \sum_{i} \partial_t Q_i(\{I - P\} f - \text{div} b) \right) \ dx.
\]

We have the following estimate.
Proposition 3.3 For classical solution of the system (2.1)–(2.5), we have
\[
\frac{d}{dt} \mathcal{E}_0(t) + \lambda \| \nabla_x (a, b, \omega) \|^2_{H^3} \\
\leq C \left( \| (I - P) f \|^2_{L^2_t(H^\delta_x)} + \| u - b \|^2_{H^3} + \| \sqrt{2} \omega - \sqrt{3} \theta \|^2_{H^3} \right) \\
+ C \left( \| u, \theta \|_{H^3} + \| u, \theta \|^2_{H^3} \right) \\
\times \left( \| \nabla_x (I - P) f \|^2_{L^2_t(H^\delta_x)} + \| \nabla_x (a, b, \omega) \|^2_{H^3} \right), \tag{3.23}
\]
for all $0 \leq t < T$ with any $T > 0$.

**Proof** We divide this proof into three steps.

**Step 1** It follows from (3.19) that
\[
\sum_{i,j} \| a^\alpha (\partial_i b_j + \partial_j b_i) \|^2 - \sum_{i,j} \int \frac{2}{3} \delta_{ij} a^\alpha \text{div} b \cdot \partial^\alpha (\partial_i b_j + \partial_j b_i) \, dx \\
= - \frac{d}{dt} \sum_{i,j} \int a^\alpha (\partial_i b_j + \partial_j b_i) a^\alpha \Gamma_{i,j} ( (I - P) f ) \, dx \\
+ \sum_{i,j} \int a^\alpha (\partial_i \partial_j b_i + \partial_j \partial_i b_i) a^\alpha \Gamma_{i,j} ( (I - P) f ) \, dx \\
+ \sum_{i,j} \int a^\alpha (\partial_i b_j + \partial_j b_i) a^\alpha \left[ (u_i b_j + u_j b_i) + \frac{2}{\sqrt{6}} \delta_{ij} \left( \sum_k \partial_k Q_k (I - P) f \right) \right. \\
\left. \left. + \frac{1}{2} \sum_k u_k Q_k (I - P) f - \frac{1}{\sqrt{6}} u \cdot b \right) + \Gamma_{i,j} (l + r + s) \right] \, dx. \tag{3.24}
\]

Using (3.17), Lemma 3.1 and Young inequality, we get
\[
\sum_{i,j} \int a^\alpha (\partial_i \partial_i b_j + \partial_j \partial_i b_i) a^\alpha \Gamma_{i,j} ( (I - P) f ) \, dx \\
\leq \varepsilon \| \nabla_x (a, \omega) \|^2_{H^3} + C \| \nabla_x (I - P) f \|^2_{L^2_t(H^\delta_x)} \\
+ C \left( \| u - b \|^2_{H^3} + \| u \|^2_{H^3} \| \nabla_x a \|^2_{H^2} \right),
\]
with $\varepsilon > 0$ sufficiently small.

The estimate of the final term on the right hand side of (3.24) is the following:
\[
\sum_{i,j} \int a^\alpha (\partial_i b_j + \partial_j b_i) a^\alpha \left[ (u_i b_j + u_j b_i) + \frac{2}{\sqrt{6}} \delta_{ij} \left( \sum_k \partial_k Q_k (I - P) f \right) \right. \\
\left. \left. + \frac{1}{2} \sum_k u_k Q_k (I - P) f - \frac{1}{\sqrt{6}} u \cdot b \right) + \Gamma_{i,j} (l + r + s) \right] \, dx \\
\leq \frac{1}{4} \sum_{i,j} \| a^\alpha (\partial_i b_j + \partial_j b_i) \|^2 + C \sum_{i,j} \left( \| a^\alpha (u_i b_j + u_j b_i) \|^2 + \| a^\alpha \partial_k Q_k (I - P) f \|^2 \right. \\
\left. + \| a^\alpha (u_k Q_k (I - P) f) \|^2 + \| a^\alpha (u \cdot b) \|^2 + \| a^\alpha \Gamma_{i,j} (l + r + s) \|^2 \right) \\
\leq \frac{1}{4} \sum_{i,j} \| a^\alpha (\partial_i b_j + \partial_j b_i) \|^2 + C \| (I - P) f \|^2_{L^2_t(H^\delta_x)}
\]
By means of (3.18) and direct calculation, one gets

\[ + C \| (u, \theta) \|_{H^3}^2 (\| \nabla b \|_{H^3}^2 + \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2). \]

Here, we have used the following estimates

\[ \sum_{i,j} \| \partial^\alpha (u_i b_j + u_j b_i) \|^2 + \| \partial^\alpha (u \cdot b) \|^2 \leq C \| u \|_{H^3}^2 \| \nabla_x b \|_{H^3}^2, \]

\[ \| \partial^\alpha \partial_k Q_k (I - P) f \|_{L^2_x(H^3)}^2 \leq C \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2, \]

\[ \| \partial^\alpha (u_k Q_k (I - P) f) \|_{L^2_x(H^3)}^2 \leq C \| u \|_{H^3}^2 \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2, \]

\[ \sum_{i,j} \| \partial^\alpha \Gamma_{i,j} (l) \|^2 \leq C \| (I - P) f \|_{L^2_x(H^3)}^2, \]

\[ \sum_{i,j} \| \partial^\alpha \Gamma_{i,j} (r) \|^2 \leq C \| u \|_{H^3}^2 \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2, \]

\[ \sum_{i,j} \| \partial^\alpha \Gamma_{i,j} (s) \|^2 \leq C \| \theta \|_{H^3}^2 \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2. \]

Notice that

\[ \sum_{i,j} \| \partial^\alpha (\partial_i b_j + \partial_j b_i) \|^2 = 2 \| \nabla_x \partial^\alpha b \|^2 + 2 \| \nabla_x \cdot \partial^\alpha b \|^2, \]

\[ \sum_{i,j} \frac{2}{3} \delta_{ij} \int \partial^\alpha \text{div} b \partial^\alpha (\partial_i b_j + \partial_j b_i) \, dx = \frac{4}{3} \| \partial^\alpha \text{div} b \|^2. \]

By the above equality, we substitute the above estimates into (3.24), and then take the sum over \(|\alpha| \leq 3\) to obtain

\[ \frac{d}{dt} \sum_{|\alpha| \leq 3} \sum_{i,j} \int \partial^\alpha (\partial_i b_j + \partial_j b_i) \partial^\alpha \Gamma_{i,j} ((I - P) f) \, dx \]

\[ + \sum_{|\alpha| \leq 3} \left( \frac{3}{2} \| \nabla_x \partial^\alpha b \|^2 + \frac{1}{6} \| \nabla_x \cdot \partial^\alpha b \|^2 \right) \]

\[ \leq \varepsilon \| \nabla (a, \omega) \|_{H^3}^2 + C_\varepsilon \| (I - P) f \|_{L^2_x(H^3)}^2 + C \| u - b \|_{H^3}^2 \]

\[ + C \| (u, \theta) \|_{H^3}^2 \cdot \left( \| \nabla_x (I - P) f \|_{L^2_x(H^3)}^2 + \| \nabla_x (a, b) \|_{H^3}^2 \right), \]

(3.25)

with \( \varepsilon > 0 \) sufficiently small.

**Step 2** According to (3.20), through direct calculation, we obtain

\[ \| \partial^\alpha \partial_i \omega \|^2 = - \frac{3}{5} \frac{d}{dt} \int \partial^\alpha \partial_i \omega \partial^\alpha Q_i (I - P) f \, dx + \frac{3}{5} \int \partial^\alpha \partial_i \omega \partial^\alpha Q_i (I - P) f \, dx \]

\[ + \int \partial^\alpha \partial_i \omega \partial^\alpha (\omega u_i + \sqrt{6} \theta b_i) \, dx + \int \partial^\alpha \partial_i \omega \partial^\alpha Q_i (l + r + s) \, dx \]

\[ + \frac{\sqrt{6}}{5} \int \partial^\alpha \partial_i \omega \partial^\alpha \sum_j \partial_j \Gamma_{i,j} (I - P) f \, dx. \]

(3.26)

By means of (3.18) and direct calculation, one gets

\[ \int \partial^\alpha \partial_i \partial_j \omega \partial^\alpha Q_i (I - P) f \, dx \]
= \int \partial_i \partial^\alpha \left( \sqrt{6} a \theta - \sqrt{2} (\sqrt{2} \omega - \sqrt{3} \theta) - \frac{2}{\sqrt{6}} \text{div} b + \frac{1}{\sqrt{6}} u \cdot b \right) \\
- \sum_k \partial_k Q_k (I - P) f - \frac{1}{2} \sum_k u_k Q_k (I - P) f) \cdot \partial^\alpha Q_i (I - P) f \, dx \\
\leq \varepsilon \| \nabla b \|_{H^3}^2 + C_\varepsilon \| (I - P) f \|_{L^2_{\alpha}(H^4)}^2 + C \| \sqrt{2} \omega - \sqrt{3} \theta \|_{H^3}^2 \\
+ C \left( \| (u, \theta) \|_{H^3} + \| u \|_{H^3} \right) \cdot \left( \| \nabla (a, b) \|_{H^3}^2 + \| \nabla_x (I - P) f \|_{L^2_{\alpha}(H^4)}^2 \right).

For the remaining terms on the right hand side of (3.26), we have

\int \partial^\alpha \partial_i \omega \partial^\alpha \partial^\alpha Q_i (I - P) f \, dx + \int \partial^\alpha \partial_i \omega \partial^\alpha (\sum_j \partial_j \Gamma_{i,j} (I - P) f) \, dx \\
\leq \frac{2}{5} \| \partial^\alpha \partial_i \omega \|_{H^4}^2 + C \| (I - P) f \|_{L^2_{\alpha}(H^4)}^2 \\
+ C \left( \| (u, \theta) \|_{H^3} + \| (u, \theta) \|_{H^3} \right) \cdot \left( \| \nabla (b, \omega) \|_{H^3}^2 + \| \nabla_x (I - P) f \|_{L^2_{\alpha}(H^4)}^2 \right).

Thus, substituting the above estimates into (3.26), and then taking summation over \( |\alpha| \leq 3, 1 \leq i \leq 3 \), we obtain

\frac{d}{dt} \sum_{\alpha,i} \int \partial^\alpha \partial_i \omega \partial^\alpha Q_i (I - P) f \, dx + \sum_\alpha \| \partial^\alpha \nabla \omega \|_{H^3}^2 \\
\leq \varepsilon \| \nabla b \|_{H^3}^2 + C_\varepsilon \| (I - P) f \|_{L^2_{\alpha}(H^4)}^2 + C \| \sqrt{2} \omega - \sqrt{3} \theta \|_{H^3}^2 \\
+ C \left( \| (u, \theta) \|_{H^3} + \| (u, \theta) \|_{H^3} \right) \cdot \left( \| \nabla (b, \omega) \|_{H^3}^2 + \| \nabla_x (I - P) f \|_{L^2_{\alpha}(H^4)}^2 \right),

(3.27)

with \( \varepsilon > 0 \) sufficiently small.

**Step 3** Making full use of the Eqs. (3.17) and (3.20), by direct calculations, we obtain the following equality,

\| \partial^\alpha \partial_i a \|_{H^3}^2 = \frac{d}{dt} \int \partial^\alpha \partial_i a \partial^\alpha \left( \frac{\sqrt{6}}{5} Q_i (I - P) f - b_i \right) \, dx \\
- \int \partial^\alpha \partial_i a \partial^\alpha \left( \frac{\sqrt{6}}{5} Q_i (I - P) f - b_i \right) \, dx \\
+ \int \partial^\alpha \partial_i a \partial^\alpha \left( (u_i - b_i) + u_i a - \frac{2}{\sqrt{6}} \omega u_i - 2 \theta b_i \right) \\
- \frac{7}{5} \sum_j \partial_j \Gamma_{i,j} (I - P) f - \frac{\sqrt{6}}{5} Q_i (l + r + s) \, dx.

(3.28)

Using (3.16), one has

\[ - \int \partial^\alpha \partial_i a \partial^\alpha \left( \frac{\sqrt{6}}{5} Q_i (I - P) f - b_i \right) \, dx \]

\[ = \int \partial^\alpha \text{div} b \partial^\alpha \left( b_i - \frac{\sqrt{6}}{5} Q_i (I - P) f \right) \, dx \]
Applying the definition of $E_0$ for all $t$, we have

$$\leq \int \partial^\alpha \text{div} b \partial^\alpha \partial_t b_t \, dx + \frac{1}{4} \| \partial^\alpha \text{div} b \|^2 + C \| (I - P) f \|_{L^2_t(H^4)}^2.$$ 

For other terms, the following estimate is obtained:

$$\int \partial^\alpha \partial_t a \partial^\alpha ((u_i - b_i) + u_i a - \frac{2}{\sqrt{6}} \omega u_i - 2 \theta b_i)$$

$$- \frac{7}{5} \sum_j \partial_j \Gamma_{ij} (I - P) f - \frac{\sqrt{6}}{5} Q_t (l + r + s) \, dx$$

$$\leq \frac{1}{4} \| \partial^\alpha \partial_t a \|^2 + C \left( \| \partial^\alpha (u_i - b_i) \|^2 + \| \partial^\alpha (u_i a - \frac{2}{\sqrt{6}} \omega u_i - 2 \theta b_i) \|^2 \right)$$

$$+ \sum_j \| \partial^\alpha \partial_j \Gamma_{ij} (I - P) f \|^2 + \| \partial^\alpha Q_t (l + r + s) \|^2$$

$$\leq \frac{1}{4} \| \partial^\alpha \partial_t a \|^2 + C \| u - b \|^2_{H^3} + C \| (I - P) f \|^2_{L^2_t(H^4)}$$

$$+ C \| (u, \theta) \|^2_{H^3} \cdot \left( \| \nabla (a, b, \omega) \|^2_{H^3} + \| \nabla_x (I - P) f \|^2_{L^2_t(H^4)} \right).$$

Thus, substituting the above estimates into (3.28), and then taking summation over $|\alpha| \leq 3$, we have

$$\frac{d}{dt} \sum_{|\alpha| \leq 3} \int \frac{2}{21} \partial^\alpha a \partial^\alpha \left( \sum_k Q_k (I - P) f - \text{div} b \right) \, dx + \frac{1}{14} \sum_{|\alpha| \leq 3} \| \partial^\alpha \nabla a \|^2$$

$$\leq \frac{1}{6} \sum_{|\alpha| \leq 3} \| \partial^\alpha \text{div} b \|^2 + C \| u - b \|^2_{H^3} + C \| (I - P) f \|^2_{L^2_t(H^4)}$$

$$+ C \| (u, \theta) \|^2_{H^3} \cdot \left( \| \nabla (a, b, \omega) \|^2_{H^3} + \| \nabla_x (I - P) f \|^2_{L^2_t(H^4)} \right).$$

Integrating the estimates of the above three parts (3.25), (3.27) and (3.29), and applying the definition of $E_0(t)$, we conclude that (3.23) holds.

**Proposition 3.4** For classical solution of the system (2.1)–(2.5), we have

$$\frac{d}{dt} \sum_{|\alpha| \leq 3} \int \partial^\alpha u \cdot \partial^\alpha \nabla \rho \, dx + \lambda \| \nabla \rho \|^2_{H^3}$$

$$\leq C \left( \| u - b \|^2_{H^3} + \| \nabla u \|^2_{H^4} + \| \nabla \theta \|^2_{H^3} \right)$$

$$+ C \left( 1 + \| \rho \|^6_{H^3} \right) \left( \| \rho \|^2_{H^4} + \| (u, \theta) \|^2_{H^3} \right) \left( \| \nabla (a, \rho, u) \|^2_{H^3} + \| u - b \|^2_{H^3} \right)$$

(3.30)

for all $0 \leq t < T$ with any $T > 0$.

**Proof** Applying $\partial^\alpha$ ($|\alpha| \leq 3$) to (2.3), and carrying a direct calculation, we achieve

$$\| \nabla \partial^\alpha \rho \|^2 = - \int \nabla \partial^\alpha \rho \partial^\alpha \partial_t u \, dx + \int \nabla \partial^\alpha \rho \partial^\alpha \left( \frac{1}{1 + \rho} (b - u) - \nabla \theta \right) \, dx$$

$$+ \int \nabla \partial^\alpha \rho \partial^\alpha \left[ -u \cdot \nabla u + \frac{1}{1 + \rho} \Delta u + \frac{\rho - \theta}{1 + \rho} \nabla \rho - \frac{1}{1 + \rho} ua \right] \, dx$$

$$:= Y_1 + Y_2 + Y_3.$$

(3.31)
For $Y_i$ ($i = 1, 2, 3$), using (2.2) and the Hölder, Sobolev and Young inequalities yields

\[
Y_1 = \frac{-d}{dt} \int \nabla \partial^\alpha \rho \partial^\alpha u \, dx + \int \partial^\alpha \operatorname{div} u \partial^\alpha (1 + \rho) \operatorname{div} u + u \cdot \nabla \rho \, dx \geq \frac{-d}{dt} \int \nabla \partial^\alpha \rho \partial^\alpha u \, dx + C \|\partial^\alpha \operatorname{div} u\|_2 + C \|\rho\|_H^4 \|\nabla u\|_H^2.
\]

\[
Y_2 \leq \frac{1}{8} \|\nabla \partial^\alpha \rho\|^2 + C \|\rho\|^2_2 \|\nabla \theta\|^2_1 + C \|\rho\|^6_H \|b - u\|^2_H,
\]

\[
Y_3 \leq \frac{1}{8} \|\nabla \partial^\alpha \rho\|^2 + C \|\nabla u\|^2_H + C \left(1 + \|\rho\|^6_H\right) \|\rho, u, \theta\|^2_H \|\nabla (a, \rho, u)\|^2_H.
\]

With the help of (3.1), substituting the above estimates into (3.31), we obtain (3.30).

Now let us define $E_1(t)$ and $D_1(t)$ by

\[
E_1(t) := \|f\|^2 + \|\rho, u, \theta\|^2 + \sum_{1 \leq |\alpha| \leq 4} \left\{\|\partial^\alpha f\|^2 + \|\partial^\alpha \rho, \partial^\alpha u, \partial^\alpha \theta\|^2\right\}
\]

\[
+ \tau_1 E_0(t) + \tau_2 \sum_{|\alpha| \leq 3} \int \partial^\alpha u \cdot \partial^\alpha \nabla \rho \, dx,
\]

\[
D_1(t) := \|\nabla (a, b, \omega, \rho, u, \theta)\|^2_H + \|b - u, \sqrt{2} \omega - \sqrt{3} \theta\|^2_H
\]

\[
+ \sum_{|\alpha| \leq 4} \left(\|\partial^\alpha (I - P) f\|^2_\nu + \|\partial^\alpha (\nabla u, \nabla \theta)\|^2\right),
\]

where $E_1(t), D_1(t)$ denote the temporal energy functional and the corresponding dissipation rate respectively, $\tau_1$ and $\tau_2$ are sufficiently small constants and will be determined later. Reorganizing the above estimates in Propositions 3.1–3.4, we obtain

\[
\frac{d}{dt} E_1(t) + \lambda D_1(t)
\]

\[
\leq C \left(1 + \|\rho\|^8_H\right) \left(\|\rho, u, \theta\|^2_H + \|\rho, u, \theta\|^2_H\right) \left(\|\nabla (a, b, \omega, \rho, u, \theta)\|^2_H + \sum_{|\alpha| \leq 4} \|\partial^\alpha (I - P) f\|^2_\nu\right).
\]

(3.32)

### 3.1.2 Energy estimates for mixed space-velocity derivatives

In this subsection, we shall deal with the energy estimates for the mixed space-velocity derivatives of $f$, i.e., $\partial^\alpha \rho f$. Since $\|\partial^\alpha P f\| \leq C \|\partial^\alpha f\|$ for any $\alpha$ and $\beta$, thus, it is enough to estimate $\|\partial^\alpha (I - P) f\|$ below.

First, we have the following facts:

\[
(I - P)(g \cdot h) = g \cdot (I - P) f - P(g \cdot (I - P)f) + (I - P)(g \cdot Pf),
\]

\[
(I - P)(v \sqrt{M}) = 0, \quad (I - P)((|v|^2 - 3) \sqrt{M}) = 0,
\]

\[
(I - P)\mathcal{L} f = \mathcal{L}(I - P) f.
\]
then, using \([I - P]\) to the equality (2.1), one gets

\[
\begin{align*}
\partial_t (I - P)f + v \cdot \nabla_x (I - P)f + u \cdot \nabla_v (I - P)f - \frac{1}{2} u \cdot v (I - P)f \\
- \theta M^{-\frac{1}{2}} \Delta_v (M^{\frac{1}{2}} (I - P)f) - (I - P)f \\
= P \left[ v \cdot \nabla_x (I - P)f + u \cdot \nabla_v (I - P)f - \frac{1}{2} u \cdot v (I - P)f \\
- \theta M^{-\frac{1}{2}} \Delta_v (M^{\frac{1}{2}} (I - P)f) \right] - (I - P) \left[ v \cdot \nabla_x P f + u \cdot \nabla_v P f \\
- \frac{1}{2} u \cdot v P f - \theta M^{-\frac{1}{2}} \Delta_v (M^{\frac{1}{2}} P f) \right].
\end{align*}
\]

(3.33)

**Proposition 3.5** Let \(1 \leq k \leq 4\). Let \((f, \rho, u, \theta)\) is a smooth solution of the system (2.1)–(2.5), we have

\[
\frac{d}{dt} \sum_{|\beta| = k} \| \partial_\beta^\alpha (I - P)f \|^2 + \lambda \sum_{|\beta| = k} \| \partial_\beta^\alpha (I - P)f \|^2_v \\
\leq C \chi_{2 \leq k \leq 4} \sum_{1 \leq |\beta'| \leq k - 1} \| \partial_\beta^\alpha (I - P)f \|^2_v \\
+ C \| \nabla (b, \omega) \|_{H^{4-k}}^2 + \sum_{|\alpha'| \leq 4-k+1} \| \partial_\alpha^\alpha (I - P)f \|^2_v \\
+ C \| (\rho, \theta) \|_{H^3}^2 \sum_{|\alpha'| \leq 4-k+1} \| \partial_\alpha^\alpha (I - P)f \|^2_v \\
+ C (\| \rho \|_{H^3} + \| u \|_{H^3}^2) \sum_{|\alpha'| \leq 4} \| \partial_\alpha^\alpha (I - P)f \|^2_v \\
+ C \| (\rho, \theta) \|_{H^{4-k+1}}^2 \| \nabla (b, \omega) \|_{H^3}^2
\]

(3.34)

for all \(0 \leq t < T\) with any \(T > 0\). Here \(\chi_E\) is the characteristic function on the set \(E\).

**Proof** This proof is based on some ideas of [17, Lemma 4.3].

Fix \(k (1 \leq k \leq 4)\). Let \(\alpha\) and \(\beta\) satisfy \(|\beta| = k\) and \(|\alpha| + |\beta| \leq 4\). For (3.33), by the standard \(L^2\) energy estimate, we get

\[
\frac{1}{2} \frac{d}{dt} \| \partial_\beta^\alpha (I - P)f \|^2 + \int \langle -L \partial_\beta^\alpha (I - P)f, \partial_\beta^\alpha (I - P)f \rangle \, dx := \sum_{i=1}^7 R_i
\]

(3.35)

with

\[
R_1 = \int \langle -\partial_\alpha^\alpha [\partial_\beta^\alpha, v \cdot \nabla_x] (I - P)f, \partial_\beta^\alpha (I - P)f \rangle \, dx,
\]

\[
R_2 = \int \langle \partial_\alpha^\alpha [\partial_\beta^\alpha, -|v|^2] (I - P)f, \partial_\beta^\alpha (I - P)f \rangle \, dx,
\]

\[
R_3 = \int \langle -\partial_\alpha^\alpha (u \cdot \nabla_v (I - P)f), \partial_\beta^\alpha (I - P)f \rangle \, dx,
\]

\[
R_4 = \int \left( \frac{1}{2} \partial_\alpha^\alpha (u \cdot v (I - P)f), \partial_\beta^\alpha (I - P)f \right) \, dx,
\]
Here the fact \([\partial^{\beta}_v, \mathcal{L}] = [\partial^{\beta}_v, -|v|^2]\) has been used.

Now we make estimates for each term \(R_i\) in (3.35) as the following:

\(R_1 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| [\partial^{\beta}_v, v \cdot \nabla_x] \partial^{\alpha}_x (I - P) f \|^2 \)
\(\leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial^{\alpha}_x \nabla_x (I - P) f \|^2 \)
\(+ \chi_{2 \leq k \leq 4} C_\eta \sum_{1 \leq |\beta'| \leq k-1 \atop |\alpha| + |\beta| \leq 4} \| \partial^{\alpha'} (I - P) f \|^2 \)

\(R_2 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| [\partial^{\beta}_v, -|v|^2] \partial^{\alpha}_x (I - P) f \|^2 \)
\(\leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial^{\alpha'} (I - P) f \|^2 \)
\(+ \chi_{2 \leq k \leq 4} C_\eta \sum_{1 \leq |\beta'| \leq k-1 \atop |\alpha| \leq 4-k} \| \partial^{\alpha'} (I - P) f \|^2 \)

\(R_3 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial^{\alpha}_x (u \cdot \nabla_v \partial^{\beta}_v (I - P) f) \|^2 \)
\(\leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| u \|^2 \sum_{1 \leq |\beta'| \leq 4 \atop |\alpha| + |\beta| \leq 4} \| \partial^{\alpha'} (I - P) f \|^2 \)

\(R_4 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial^{\alpha}_x (u \cdot \partial^{\beta}_v (v(I - P) f)) \|^2 \)
\(\leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| u \|^2 \sum_{1 \leq |\beta'| \leq 4 \atop |\alpha| + |\beta| \leq 4} \| \partial^{\alpha'} (I - P) f \|^2 \)
\(+ C_\eta \| u \|^2 \sum_{|\alpha| \leq 4-k} \| \partial^{\alpha'} (I - P) f \|^2 \)

\(R_5 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial (I - P) f \|^2 \sum_{1 \leq |\beta'| \leq 4 \atop |\alpha| + |\beta| \leq 4} \| \partial^{\alpha'} (I - P) f \|^2 \)
\(+ C_\eta \| \partial (I - P) f \|^2 \sum_{|\alpha| \leq 4-k+1} \| \partial^{\alpha'} (I - P) f \|^2 \)

\(R_6 \leq \eta \| \partial^{\beta}_v (I - P) f \|^2 + C_\eta \| \partial^{\alpha}_v P(v \cdot \nabla_x (I - P) f) \|^2 \)
\(+ C_\eta \| \partial^{\alpha}_v P(u \cdot \nabla_v (I - P) f) \|^2 \)
Choosing some suitable constants \( C_k \) together with the above lemma, we have

\[
\frac{d}{dt} \sum_{1 \leq k \leq 4} C_k \sum_{|\beta|=k} \frac{d}{d\beta} \langle I-P \rangle f \| \| + \sum_{1 \leq |\beta| \leq 4} \langle I-P \rangle f \| \| \\
\leq C \left( \| \nabla (b, \omega) \|_{H^3} + \sum_{|\alpha| \leq 4} \| \partial^\alpha \langle I-P \rangle f \|_V \right) \\
+ C \left( \| u \|_{H^3} + \| \theta \|_{H^3} \right) \sum_{|\alpha| \leq 4} \| \partial^\alpha \langle I-P \rangle f \|_V \\
+ C \left( \| \theta \|_{H^3} + \| u \|_{H^3} \right) \sum_{1 \leq |\beta| \leq 4} \| \partial^\beta \langle I-P \rangle f \|_V \\
+ C \| (u, \theta) \|_{H^4} \| \nabla (b, \omega) \|_{H^3}.
\]

(3.36)

Now, let us define \( \mathcal{E}_2(t) \) and \( \mathcal{D}_2(t) \) as

\[
\mathcal{E}_2(t) := \sum_{1 \leq k \leq 4} C_k \sum_{|\beta|=k} \| \partial^\beta \langle I-P \rangle f \|_V^2, \\
\mathcal{D}_2(t) := \sum_{1 \leq |\beta| \leq 4} \| \partial^\beta \langle I-P \rangle f \|_V^2.
\]
Then the total energy functional $E(t)$ and the dissipation rate $D(t)$ can be defined by

$$
E(t) := E_1(t) + \tau_3 E_2(t),
$$
$$
D(t) := D_1(t) + \tau_3 D_2(t),
$$

where $\tau_3 > 0$ is very small and will be determined later.

According to the inequalities (3.32), (3.36) and (3.1), we have

$$
\frac{d}{dt}E(t) + \lambda D(t) \leq C(\delta + \delta^2)D(t),
$$

Thus, as long as $0 < \delta < 1$ is sufficiently small, the integration of (3.37) with respect to time gives

$$
E(t) + \lambda \int_0^t D(s)ds \leq E(0)
$$

for all $0 \leq t < T$. In addition, (3.1) can be proved by choosing $E(0) \sim \|f_0\|_{H^4_x}^2 + \|\rho_0, u_0, \theta_0\|_{H^4}^2$ sufficiently small.

### 3.2 Global existence

In this subsection, we will show that there exists a unique global-in-time solution to the problem (2.1)–(2.5). The proof is based on the uniform energy estimates for the iteration sequence of approximate solutions. The sequence $\{f^n, \rho^n, u^n, \theta^n\}_{n=0}^{\infty}$ satisfies the following system:

$$
\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} - \mathcal{L} f^{n+1} = -u^n \cdot \left( \nabla_v f^{n+1} - \frac{v}{2} f^{n+1} - v \sqrt{M} \right)
$$

$$
+ \theta^n M^{\frac{1}{2}} \Delta_v (M + \sqrt{M} f^{n+1}),
$$

$$
\partial_t \rho^{n+1} + u^n \cdot \nabla \rho^{n+1} + (1 + \rho^n) \text{div} u^{n+1} = 0,
$$

$$
\partial_t u^{n+1} - \frac{1}{1 + \rho^n} \Delta u^{n+1} = -u^n \cdot \nabla u^n - \nabla \theta^n - \frac{1 + \theta^n}{1 + \rho^n} \nabla \rho^n
$$

$$
+ \frac{1}{1 + \rho^n} (b^n - u^n - u^n a^n),
$$

$$
\partial_t \theta^{n+1} - \frac{1}{1 + \rho^n} \Delta \theta^{n+1} = -u^n \cdot \nabla \theta^n - \theta^n \text{div} u^n - \text{div} u^n + \frac{\sqrt{3}}{1 + \rho^n} (\sqrt{2} \omega^n - \sqrt{3} \theta^n)
$$

$$
+ \frac{1}{1 + \rho^n} \left( (u^n - b^n) \cdot u^n - u^n \cdot b^n + a^n |u^n|^2 - 3a^n \theta^n \right),
$$

where $n = 0, 1, 2, \ldots$, and $(f^0, \rho^0, u^0, \theta^0) = (f_0, \rho_0, u_0, \theta_0)$ is the starting value of iteration.
We try to find solutions in the following function space

\[
X(0, T; A) := \left\{ f \in C([0, T], H^4(\mathbb{R}^3 \times \mathbb{R}^3)), M + \sqrt{M} f \geq 0; \right. \\
\rho \in C((0, T], H^4(\mathbb{R}^3)) \cap C([0, T], H^3(\mathbb{R}^3)); \\
u \in C((0, T], H^4(\mathbb{R}^3)) \cap C([0, T], H^2(\mathbb{R}^3)); \\
\theta \in C((0, T], H^4(\mathbb{R}^3)) \cap C([0, T], H^2(\mathbb{R}^3)); \\
\sup_{0 \leq t \leq T} \left\{ \| f(t) \|_{H^4} + \| (\rho, u, \theta) \|_{H^4} \right\} \leq A; \\
\rho_1 = \frac{1}{2} (-1 + \inf \rho(0, x)) > -1; \\
\inf_{0 \leq t \leq T, x \in \mathbb{R}^3} \rho(t, x) \geq \rho_1. \right. 
\]

(3.43)

We now state the local existence theorem.

**Theorem 3.1** There exist \( A_0 > 0 \) and \( T^* > 0 \), such that if \( f_0 \in H^4(\mathbb{R}^3 \times \mathbb{R}^3), \rho_0 \in H^4(\mathbb{R}^3), u_0 \in H^4(\mathbb{R}^3), \theta_0 \in H^4(\mathbb{R}^3) \) with \( F_0 = M + \sqrt{M} f_0 \geq 0 \) and \( E(0) \leq A_0^2 \), then for each \( n \geq 1 \), \( (f^n, \rho^n, u^n, \theta^n) \) is well-defined with

\[
(f^n, \rho^n, u^n, \theta^n) \in X(0, T^*; A_0). 
\]

(3.44)

Further, we obtain:

1. \( (f^n, \rho^n, u^n, \theta^n)_{n \geq 0} \) is a Cauchy sequence in the Banach space \( C([0, T^*]; H^2(\mathbb{R}^3 \times \mathbb{R}^3)) \).
2. Let \( (f, \rho, u, \theta) \) be the limit function, and then \( (f, \rho, u, \theta) \in X(0, T^*; A_0) \).
3. \((f, \rho, u, \theta)\) satisfies the system (2.1)-(2.5),
4. \((f, \rho, u, \theta)\) is the unique solution of (2.1)-(2.5) in \( X(0, T^*; A_0) \).

**Proof** Let \( T^* > 0 \) be a constant which will be fixed later. For simplicity, without loss of generality, \( (f^n, \rho^n, u^n, \theta^n) \) is assumed to be smooth enough; if not, we can consider the following regularized iterative system:

\[
\begin{align*}
\partial_t f^{n+1, \varepsilon} + v \cdot \nabla_x f^{n+1, \varepsilon} - \mathcal{L} f^{n+1, \varepsilon} &= -u^{n, \varepsilon} \cdot \nabla_v f^{n+1, \varepsilon} - \frac{v}{2} f^{n+1, \varepsilon} - v\sqrt{M} \\
+ \theta^{n, \varepsilon} M^{-\frac{1}{2}} \Delta_v (M + \sqrt{M} f^{n+1, \varepsilon}), \\
\partial_t \rho^{n+1, \varepsilon} + u^{n, \varepsilon} \cdot \nabla \rho^{n+1, \varepsilon} + (1 + \rho^{n, \varepsilon}) \mathrm{div} u^{n+1, \varepsilon} &= 0, \\
\partial_t u^{n+1, \varepsilon} - \frac{1}{1 + \rho^{n, \varepsilon}} \Delta u^{n+1, \varepsilon} &= -u^{n, \varepsilon} \cdot \nabla u^{n, \varepsilon} - \nabla \theta^{n, \varepsilon} - \frac{1}{1 + \rho^{n, \varepsilon}} \nabla \rho^{n, \varepsilon} \\
+ \frac{1}{1 + \rho^{n, \varepsilon}} (b^{n, \varepsilon} - u^{n, \varepsilon} - u^{n, \varepsilon} d^{n, \varepsilon}, \\
\partial_t \theta^{n+1, \varepsilon} - \frac{1}{1 + \rho^{n, \varepsilon}} \Delta \theta^{n+1, \varepsilon} &= -u^{n, \varepsilon} \cdot \nabla \theta^{n, \varepsilon} - \theta^{n, \varepsilon} \mathrm{div} u^{n, \varepsilon} - \mathrm{div} u^{n, \varepsilon} \\
+ \frac{1}{1 + \rho^{n, \varepsilon}} ((u(n, \varepsilon - b^{n, \varepsilon}) \cdot u^{n, \varepsilon} - u^{n, \varepsilon} \cdot b^{n, \varepsilon} + a^{n, \varepsilon} |u^{n, \varepsilon}|^2) \\
+ \frac{\sqrt{3}}{1 + \rho^{n, \varepsilon}} (\sqrt{3} \theta^{n, \varepsilon} - \sqrt{3} a^{n, \varepsilon} \theta^{n, \varepsilon}),
\end{align*}
\]

(\(f^{n+1, \varepsilon}, \rho^{n+1, \varepsilon}, u^{n+1, \varepsilon}, \theta^{n+1, \varepsilon})(0) = (f_0^\varepsilon, \rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)\),

for any \( \varepsilon > 0 \) with \( (f_0^\varepsilon, \rho_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon) \) a smooth approximation of \((f_0, \rho_0, u_0, \theta_0)\) and pass to the limit by letting \( \varepsilon \to 0 \).

Applying \( \partial_x^\alpha \) with \( |\alpha| \leq 4 \) to the Eq. (3.39), multiplying the result by \( \partial_x^\alpha f^{n+1} \) and then taking integration over \( \mathbb{R}^3 \), one has
\[
\frac{1}{2} \frac{d}{dt} \|\partial^\alpha f^{n+1}\|^2 + \int \langle -\mathcal{L} \partial^\alpha f^{n+1}, \partial^\alpha f^{n+1} \rangle \, dx \\
= - \int \int \partial^\alpha \left( u^M M^{-\frac{1}{2}} \nabla_v (M + \sqrt{M} f^{n+1}) \right) \partial^\alpha f^{n+1} \, dx \, dv \\
+ \int \int \partial^\alpha \left( \theta^M M^{-\frac{1}{2}} \Delta_v (M + \sqrt{M} f^{n+1}) \right) \partial^\alpha f^{n+1} \, dx \, dv \\
\leq C \| (u^n, \theta^n) \|_{H^2} \| f^{n+1} \|_{L^2_v (H^4)} \\
+ C \| u^n \|_{H^2} \| f^{n+1} \|_{L^2_v (H^4)} \| \partial^\alpha f^{n+1} \|_v \\
+ C \| \theta^n \|_{H^4} \left( \sum_{|\alpha| \leq 4} \| \partial^\alpha f^{n+1} \|_v \right) \| \partial^\alpha f^{n+1} \|_v. \\
\tag{3.45}
\]

Notice that
\[
\int \langle -\mathcal{L} \partial^\alpha f^{n+1}, \partial^\alpha f^{n+1} \rangle \, dx \geq \lambda \| (I - P_0) \partial^\alpha f^{n+1} \|^2_v.
\]

By adding \( \| P_0 \partial^\alpha f^{n+1} \|_v^2 \) to the inequality (3.45), we get the sum over \( |\alpha| \leq 4 \),
\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq 4} \| \partial^\alpha f^{n+1} \|^2_v + \lambda \sum_{|\alpha| \leq 4} \| \partial^\alpha f^{n+1} \|^2_v \\
\leq C \| f^{n+1} \|_{L^2_v (H^4)}^2 + C \| (u^n, \theta^n) \|_{H^4} \| f^{n+1} \|_{L^2_v (H^4)}^2 \\
+ C \| u^n \|_{H^4} \| f^{n+1} \|_{L^2_v (H^4)}^2 + C \| (u^n, \theta^n) \|_{H^4} \sum_{|\alpha| \leq 4} \| \partial^\alpha f^{n+1} \|^2_v.
\]

Similar argument gives, for any \( 0 \leq t \leq T \leq T^* \),
\[
\frac{1}{2} \frac{d}{dt} \| f^{n+1} \|_{H^4}^2 + \lambda \sum_{|\alpha| + |\beta| \leq 4} \| \partial^\alpha \partial^\beta f^{n+1} \|^2_v \\
\leq C \| f^{n+1} \|_{H^4}^2 + C \| (u^n, \theta^n) \|_{H^4}^2 + C \| u^n \|_{H^4} \| f^{n+1} \|_{H^4}^2 \\
+ C \| (u^n, \theta^n) \|_{H^4} \sum_{|\alpha| + |\beta| \leq 4} \| \partial^\alpha \partial^\beta f^{n+1} \|^2_v. \\
\tag{3.46}
\]

Next, from [34], the system (3.40)–(3.41) has a unique solution \((\rho^{n+1}, u^{n+1}, \theta^{n+1})\) such that \( \rho^{n+1} \geq \rho_1 \), and
\[
\rho^{n+1} \in C ([0, T], H^4 (\mathbb{R}^3)) \cap C^1 ([0, T], H^3 (\mathbb{R}^3)), \\
u^{n+1}, \theta^{n+1} \in C ([0, T], H^4 (\mathbb{R}^3)) \cap C^1 ([0, T], H^2 (\mathbb{R}^3)).
\]

Now we estimate \( \frac{d}{dt} \| (\rho^{n+1}, u^{n+1}, \theta^{n+1}) \|_{H^4}^2 \). Applying \( \partial^\alpha \) \( (|\alpha| \leq 4) \) to the system (3.40)–(3.42), multiplying the results by \( \partial^\alpha \rho^{n+1}, \partial^\alpha u^{n+1}, \partial^\alpha \theta^{n+1} \) respectively, and then taking integration and summation, we get
\[
\frac{1}{2} \frac{d}{dt} \| (\rho^{n+1}, u^{n+1}, \theta^{n+1}) \|_{H^4}^2 + \lambda \sum_{|\alpha| \leq 4} \int \left( |\nabla \partial^\alpha u^{n+1}|^2 + |\nabla \partial^\alpha \theta^{n+1}|^2 \right) \, dx \\
\leq C \left( 1 + \| \rho^n \|_{H^4}^2 + \| u^n \|_{H^4}^2 \right) \| \rho^{n+1} \|_{H^4}^2 + C \left( 1 + \| u^n \|_{H^4}^2 + \| \rho^n \|_{H^4}^8 \right) \| u^{n+1}, \theta^{n+1} \|_{H^4}^2 \\
+ C \left( 1 + \| \rho^n \|_{H^4}^6 \right) \left( 1 + \| u^n \|_{H^4}^2 \right) \| \rho^n \|_{H^4}^2 + C \left( 1 + \| f^n \|_{H^4}^2 + \| u^n \|_{H^4}^2 \right) \| u^n \|_{H^4}^2 \\
+ C \left( 1 + \| \rho^n \|_{H^4}^6 + \| u^n \|_{H^4}^2 \right) \| \theta^n \|_{H^4}^2 + C \left( 1 + \| \theta^n \|_{H^4}^2 + \| u^n \|_{H^4}^2 \right) \| f^n \|_{H^4}^2. \\
\tag{3.47}
\]
Adding up (3.46) and (3.47) gives
\[
\frac{1}{2} \frac{d}{dt} \left( \|f^{n+1}\|_{H^4_{H,v}}^2 + \|\rho^{n+1}\|_{H^4}^2 + \|u^{n+1}\|_{H^4}^2 + \|\theta^{n+1}\|_{H^4}^2 \right) \\
+ \lambda \sum_{|\alpha|+|\beta| \leq 4} \|\partial_\beta \partial_\alpha f^{n+1}\|_{V}^2 + \lambda \sum_{|\alpha| \leq 4} \left( \|\nabla \partial_\alpha u^{n+1}\|_{V}^2 + \|\nabla \partial_\alpha \theta^{n+1}\|_{V}^2 \right) \\
\leq C \left( 1 + \|u^n\|_{H^4_{H,v}}^2 \right) \|f^{n+1}\|_{H^4_{H,v}}^2 + C \left( 1 + \|\rho^n\|_{H^4}^2 + \|u^n\|_{H^4} \right) \|\rho^{n+1}\|_{H^4}^2 \\
+ C \left( 1 + \|u^n\|_{H^4}^2 + \|\rho^n\|_{H^4}^2 \right) \|\rho^{n+1}\|_{H^4}^2 \\
+ C \left( 1 + \|\rho^n\|_{H^4}^2 \right) \|\rho^n\|_{H^4}^2 + C \left( 1 + \|\rho^n\|_{H^4}^2 + \|\rho^n\|_{H^4}^2 \right) \|\theta^n\|_{H^4}^2 \\
+ C \left( 1 + \|\rho^n\|_{H^4}^2 + \|\rho^n\|_{H^4}^2 \right) \|\rho^n\|_{H^4}^2 + C \left( 1 + \|\rho^n\|_{H^4}^2 + \|\rho^n\|_{H^4}^2 \right) \|\theta^n\|_{H^4}^2 \\
+ C \|\left( \|\rho^n\|_{H^4} \right) + \|\rho^n\|_{H^4}^2 \|\|f^n\|_{H^4}^2 + C \left( 1 + \|\rho^n\|_{H^4}^2 + \|\rho^n\|_{H^4}^2 \right) \|\rho^n\|_{H^4}^2 \\
+ C \|\left( \|\rho^n\|_{H^4} \right) + \|\rho^n\|_{H^4}^2 \|\|f^n\|_{H^4}^2 \right) + \sum_{|\alpha|+|\beta| \leq 4} \|\partial_\beta \partial_\alpha f^{n+1}\|_{V}^2. \tag{3.48}
\]

Using induction, we may assume \(A_n(T) \leq A_0\) and \(A_n(0) \leq \frac{A_0}{2}\) for some \(A_0 > 0\) with
\[
A_n(T) := \sup_{0 \leq t \leq T} \left\{ \|\rho^n(t)\|_{H^4}^2 + \|u^n(t)\|_{H^4}^2 + \|\theta^n(t)\|_{H^4}^2 + \|f^n(t)\|_{H^4_{H,v}}^2 \right\}.
\]

Integrating (3.48) over \([0, T]\) yields
\[
A_{n+1}(T) + \lambda \int_0^T \left\{ \sum_{|\alpha| \leq 4} \left( \|\nabla \partial_\alpha u^{n+1}\|_{V}^2 + \|\nabla \partial_\alpha \theta^{n+1}\|_{V}^2 \right) + \sum_{|\alpha|+|\beta| \leq 4} \|\partial_\beta \partial_\alpha f^{n+1}\|_{V}^2 \right\} dt \\
\leq A_{n+1}(0) + C \left( 1 + A_{n+1}^2 + A_n^2 \right) A_{n+1}(T) + C A_n(T) (1 + A_n^2) T \\
+ C A_n^2 \int_0^T \sum_{|\alpha|+|\beta| \leq 4} \|\partial_\beta \partial_\alpha f^{n+1}\|_{V}^2 dt \\
\leq A_0 + C \left( 1 + A_0^4 \right) A_{n+1}(T) + C (A_0 + A_0^5) T \\
+ C A_0^4 \sum_{|\alpha|+|\beta| \leq 4} \int_0^T \|\partial_\beta \partial_\alpha f^{n+1}\|_{V}^2 dt. \tag{3.49}
\]

It is easy to obtain that for \(t \leq T^*\),
\[
\left( 1 - C (1 + A_0^4) T \right) A_{n+1}(T) \leq \frac{A_0}{2} + C (A_0 + A_0^5) T.
\]

Choosing \(T^*\) satisfying \(T^* \leq \frac{A_0}{2}\) with \(A_0\) small enough, one gets
\[
A_{n+1} \leq A_0.
\]

By the Eq. (3.39), \(F^{n+1} = M + \sqrt{M} f^{n+1}\) satisfies the following equation,
\[
\partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} + (u - v) \cdot \nabla_x F^{n+1} - (1 + \theta^n) \Delta_v F^{n+1} - 3 F^{n+1} = 0,
\]
then, on the basis of the maximum principle, one achieves
\[
F^{n+1} \geq 0.
\]

Now we show that \(\|f^{n+1}\|_{H^4_{H,v}}^2\) is continuous over \(0 \leq t \leq T^*\).
Actually, similarly to the proof of (3.46), the following inequality holds

$$\left| \| f^{n+1}(t) \|_{H^4_{x,v}}^2 - \| f^{n+1}(s) \|_{H^4_{x,v}}^2 \right|$$

$$= \left| \int_s^t \frac{d}{d\eta} \| f^{n+1}(\eta) \|_{H^4_{x,v}}^2 \, d\eta \right|$$

$$\leq C A_0^\frac{1}{2} \sum_{|\alpha| + |\beta| \leq 4} \int_s^t \| \delta_\beta^\alpha f^{n+1} \|_{V}^2 \, d\eta + C (A_0 + A_0^\frac{3}{2}) |t - s|.$$  (3.50)

Moreover, $\| \delta_\beta^\alpha f^{n+1} \|_{V}^2$ is integrable over $[0, T^*]$. Thus, (3.44) is true for $n + 1$, namely it follows that (3.44) holds for any $n \geq 0$.

By straightforward calculation, $(f^{n+1} - f^n, \rho^{n+1} - \rho^n, u^{n+1} - u^n, \theta^{n+1} - \theta^n)$ satisfies the following system:

$$\partial_t (f^{n+1} - f^n) + v \cdot \nabla_x (f^{n+1} - f^n) - \mathcal{L} (f^{n+1} - f^n)$$

$$= -u^n \cdot (\nabla_v - \frac{v}{2}) (f^{n+1} - f^n) - (u^n - u^{n-1}) \cdot (\nabla_v - \frac{v}{2}) f^n + M^\frac{1}{2} v \cdot (u^n - u^{n-1})$$

$$+ \theta^n M^{-\frac{1}{2}} \Delta_v \left( M^\frac{1}{2} (f^{n+1} - f^n) \right) + M^{-\frac{1}{2}} \Delta_v M (\theta^n - \theta^{n-1}),$$

$$\partial_t (\rho^{n+1} - \rho^n) + u^n \cdot \nabla (\rho^{n+1} - \rho^n)$$

$$= -(u^n - u^{n-1}) \cdot \nabla \rho^n - (1 + \rho^n) \text{div} (u^{n+1} - u^n) - (\rho^n - \rho^{n-1}) \text{div} u^n,$$

$$\partial_t (u^{n+1} - u^n) - \frac{1}{1 + \rho^n} \Delta (u^{n+1} - u^n)$$

$$= \left( \frac{1}{1 + \rho^n} - \frac{1}{1 + \rho^{n-1}} \right) \left( \Delta u^n + b^n - u^n - u^n a^n \right)$$

$$- \left( (u^n - u^{n-1}) \cdot \nabla u^n + u^{n-1} \cdot \nabla (u^n - u^{n-1}) \right) - \nabla \left( \theta^n - \theta^{n-1} \right)$$

$$+ \left( \frac{1 + \theta^n}{1 + \rho^n} \Delta \rho^n - \frac{1 + \theta^{n-1}}{1 + \rho^{n-1}} \Delta \rho^{n-1} \right)$$

$$+ \frac{1}{1 + \rho^n} \left( (b^n - b^{n-1}) - (u^n - u^{n-1}) - (u^n a^n - u^{n-1} a^{n-1}) \right),$$

$$\partial_t (\theta^{n+1} - \theta^n) - \frac{1}{1 + \rho^n} \Delta (\theta^{n+1} - \theta^n)$$

$$= -\left( (u^n - u^{n-1}) \cdot \nabla \theta^n + u^{n-1} \cdot \nabla (\theta^n - \theta^{n-1}) \right) + \text{div} (u^n - u^{n-1})$$

$$- (\theta^n - \theta^{n-1}) \text{div} u^n - \text{div} (u^n - u^{n-1}) \theta^n$$

$$+ \left( \frac{1}{1 + \rho^n} - \frac{1}{1 + \rho^{n-1}} \right) \left( \Delta \theta^n + \sqrt{6} \omega^n - 3 \theta^n - 3 \theta^n a^n \right)$$

$$+ \left( \frac{1}{1 + \rho^n} - \frac{1}{1 + \rho^{n-1}} \right) \left( |u^n|^2 - 2 u^n \cdot b^n + a^n |u^n|^2 \right)$$

$$+ \frac{1}{1 + \rho^n} \left( |u^n|^2 - |u^{n-1}|^2 - 2 u^n \cdot b^n + 2 u^{n-1} \cdot b^{n-1} + a^n |u^n|^2 - a^{n-1} |u^{n-1}|^2 \right)$$

$$+ \frac{1}{1 + \rho^n} \left( \sqrt{6} (\omega^n - \omega^{n-1}) - 3 (\theta^n - \theta^{n-1}) - 3 (a^n \theta^n - a^{n-1} \theta^{n-1}) \right).$$
Similarly to (3.48), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left\{ \| f^{n+1} - f^n \|_{H^3_{k,v}}^2 + \| \rho^{n+1} - \rho^n \|_{H^3}^2 + \| u^{n+1} - u^n \|_{H^3}^2 + \| \theta^{n+1} - \theta^n \|_{H^3}^2 \right\}
+ \lambda \sum_{|\alpha| + |\beta| \leq 3} \| \partial^\alpha \beta \left( f^{n+1} - f^n \right) \|_{V}^2 + \lambda \| \nabla (u^{n+1} - u^n), \nabla (\theta^{n+1} - \theta^n) \|_{H^3}^2
\leq C \left( \| f^{n+1} - f^n \|_{H^3_{k,v}}^2 + \| \rho^{n+1} - \rho^n \|_{H^3}^2 + \| u^{n+1} - u^n \|_{H^3}^2 + \| \theta^{n+1} - \theta^n \|_{H^3}^2 \right)
+ C \left( \| f^n - f^{n-1} \|_{H^3_{k,v}}^2 + \| \rho^n - \rho^{n-1} \|_{H^3}^2 + \| u^n - u^{n-1} \|_{H^3}^2 + \| \theta^n - \theta^{n-1} \|_{H^3}^2 \right),
\]
where \( C > 0 \) is a constant that depends on \( \| (\rho^n, \rho^{n-1}) \|_{H^2}, \| (u^n, u^{n-1}) \|_{H^2}, \| (\theta^n, \theta^{n-1}) \|_{H^2}, \| (f^n, f^{n-1}) \|_{H^2}, \text{ and } \sum_{|\alpha| + |\beta| \leq 3} \| \partial^\alpha \beta f^n \|_{V}^2 \). Due to \( \mathcal{E}(0), T^* \text{ and } A_0 \) being small enough, according to (3.49), we know that
\[
\sup_n \int_0^{T^*} \sum_{|\alpha| + |\beta| \leq 3} \| \partial^\alpha \beta f^n \|_{V}^2 \, ds
\]
is also small enough. Therefore, there exists \( \delta \in (0, 1) \) such that
\[
\sup_{0 < t \leq T^*} \left\{ \| f^{n+1} - f^n \|_{H^3} + \| \rho^{n+1} - \rho^n \|_{H^3} + \| u^{n+1} - u^n \|_{H^3} + \| \theta^{n+1} - \theta^n \|_{H^3} \right\}
\leq \delta \sup_{0 < t \leq T^*} \left\{ \| f^n - f^{n-1} \|_{H^3} + \| \rho^n - \rho^{n-1} \|_{H^3}
+ \| u^n - u^{n-1} \|_{H^3} + \| \theta^n - \theta^{n-1} \|_{H^3} \right\}.
\]
(3.51)

According to (3.51), we conclude that \( (f^n, \rho^n, u^n, \theta^n)_{n \geq 0} \) is a Cauchy sequence in the Banach space \( C([0, T^*], H^3(\mathbb{R} \times \mathbb{R}^3)) \times \left( C([0, T^*], H^3(\mathbb{R}^3)) \right)^3 \), and we denote the limit function of this sequence by \( (f, \rho, u, \theta) \). Then \( (f, \rho, u, \theta) \) satisfies the system (2.1)–(2.5) by letting \( n \to \infty \). The fact that \( F^n(t, x, v) \geq 0 \) and the Sobolev embedding theorem imply
\[
F(t, x, v) \geq 0, \quad \sup_{0 \leq t \leq T^*} \| f(t) \|_{H^3_{k,v}} \leq A_0.
\]

Similar argument to (3.50) yields that \( f \in C([0, T^*], H^3(\mathbb{R} \times \mathbb{R}^3)) \). Namely, we obtain that \( (f, \rho, u, \theta) \in X(0, T^*, A_0) \).

Finally, let \( (\tilde{f}, \tilde{\rho}, \tilde{u}, \tilde{\theta}) \in X(0, T^*, A_0) \) be another solution to the Cauchy problem (2.1)–(2.5). Using similar process to (3.51), it follows that
\[
\sup_{0 < t \leq T^*} \left\{ \| f - \tilde{f} \|_{H^3} + \| \rho - \tilde{\rho} \|_{H^3} + \| u - \tilde{u} \|_{H^3} + \| \theta - \tilde{\theta} \|_{H^3} \right\}
\leq \delta \sup_{0 < t \leq T^*} \left\{ \| f - \tilde{f} \|_{H^3} + \| \rho - \tilde{\rho} \|_{H^3} + \| u - \tilde{u} \|_{H^3} + \| \theta - \tilde{\theta} \|_{H^3} \right\},
\]
for \( 0 < \delta < 1 \). Thus it is easy to conclude that \( f \equiv \tilde{f}, \rho \equiv \tilde{\rho}, u \equiv \tilde{u}, \theta \equiv \tilde{\theta} \), i.e., the uniqueness holds. \( \square \)

**Proof of Theorem 2.1** By the fact \( \mathcal{E}(0) \sim \| f_0 \|_{H^1_{k,v}}^2 + \| (\rho_0, u_0, \theta_0) \|_{H^4}^2 \), there exists \( \varepsilon_0 \), such that if \( \| f_0 \|_{H^1_{k,v}} + \| (\rho_0, u_0, \theta_0) \|_{H^4} < \varepsilon_0 \), there exists \( \varepsilon_0 \), such that if \( \| f_0 \|_{H^1_{k,v}} + \| (\rho_0, u_0, \theta_0) \|_{H^4} < \varepsilon_0 \), one gets \( \mathcal{E}(0) \leq \frac{A_0}{2} \). Next, due to Theorem 3.1, the uniform estimate (3.38) holds for the local solution. Finally, by the standard bootstrap arguments as in [17,24,34] we eventually conclude that the global existence and uniqueness part in Theorem 2.1 holds. \( \square \)
4 Large time behavior

In this section, our main concern is the optimal time-decay rates of global solutions to the problem (2.1)–(2.5). First, we shall study the time-decay of the solution to the linearized Cauchy equations with a nonhomogeneous source, with the help of the Fourier analysis, we can obtain the algebraic decay when time tends to infinity, that is, Theorem 4.1 holds. Then, decomposing the nonlinear terms subtly and choosing the appropriate functions as the nonhomogeneous source, together with Theorem 4.1 and the energy-spectrum method developed in [18], we finally conclude the time-decay rate (2.10). To this end, we assume that all conditions in Theorem 2.1 hold, and let \((f, \rho, u, \theta)\) be the solution to the system (2.1)–(2.5).

We first consider the linearized equations with a nonhomogeneous source, namely,

\[
\begin{aligned}
&\frac{\partial}{\partial t} f + v \cdot \nabla_x f - u \cdot v M \frac{1}{2} - \theta (|v|^2 - 3) M \frac{1}{2} = \mathcal{L} f + S_f, \\
&\frac{\partial}{\partial t} \rho + \text{div} u = 0, \\
&\frac{\partial}{\partial t} u - \Delta u + \nabla \theta + \nabla \rho + u - b = 0, \\
&\frac{\partial}{\partial t} \theta - \Delta \theta + \text{div} u + \sqrt{3} (\sqrt{3} \theta - \sqrt{2} \omega) = 0.
\end{aligned}
\]  

(4.1)

Here, \(S_f\) has the following form,

\[ S_f = \text{div}_v G - \frac{1}{2} v \cdot G + \varphi, \]

with \(G = (G_i), G_i = G_i(t, x, v) \in \mathbb{R}, 1 \leq i \leq 3,\) and \(\varphi = \varphi(t, x, v) \in \mathbb{R},\) and we assume that

\[
P_0 G_i = 0, \quad P_1 G_i = 0, \quad P \varphi = 0, \tag{4.2}
\]

for all \(t \geq 0, x \in \mathbb{R}^3.\)

For the linearized problem (4.1), we easily conclude that it is well-posed in \(L^2,\) i.e., the following lemma is true.

**Lemma 4.1** There is a well-defined linear semigroup \(E_t : L^2 \rightarrow L^2, t \geq 0,\) such that for any given \((f_0, \rho_0, u_0, \theta_0) \in L^2, E_t(f_0, \rho_0, u_0, \theta_0)\) is the unique distributional solution to (4.1) with \(S_f = 0.\) Further, for any \((f_0, \rho_0, u_0, \theta_0) \in L^2,\) then there is a unique distributional solution to (4.1) such that

\[
(f(t), \rho(t), u(t), \theta(t)) = E_t(f_0, \rho_0, u_0, \theta_0) + \int_0^t E_{t-\tau} (S_f(\tau), 0, 0, 0) \, d\tau. \tag{4.3}
\]

**Proof** Similar argument as the local existence theorem can give the well-posedness part, thus we do not repeat here, and the variation of constants formula (4.3) is again true by a direct computation. \(\square\)

We first quote two lemmas of [14] for later proofs.

**Lemma 4.2** [14] Given any \(0 < \beta_1 \neq 1 \quad \text{and} \quad \beta_2 > 1,\)

\[
\int_0^t (1 + t - s)^{-\beta_1} (1 + s)^{-\beta_2} \, ds \leq C (1 + t)^{-\min\{\beta_1, \beta_2\}}
\]

for all \(t \geq 0.\)
Lemma 4.3 \[14\] Let \( y > 1 \) and \( g_1, g_2 \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) with \( g_1(0) = 0 \). For \( A \in \mathbb{R}_+ \), define
\[
\mathcal{A} := \{ y \in C^0(\mathbb{R}_+, \mathbb{R}_+) \mid y \leq A + g_1(A) + g_2(A)y', \ y(0) \leq A \}.
\]
Then, there exists a constant \( A_0 \in (0, \min\{A_1, A_2\}) \) such that for any \( 0 < A < A_0 \),
\[
y \in \mathcal{A} \quad \implies \quad \sup_{t \geq 0} y(t) \leq 2A.
\]

Theorem 4.1 Let \( 1 \leq q \leq 2 \) and \((f_0, \rho_0, u_0, \theta_0) \in L^2\). For any \( \alpha, \alpha' \) with \( \alpha' \leq \alpha \) and \( m = |\alpha - \alpha'| \),
\[
\| \partial^\alpha E_t(f_0, \rho_0, u_0, \theta_0) \|_{L^2} \leq C (1 + t)^{-\sigma_{q,m}} \left( \| \partial^\alpha'(f_0, \rho_0, u_0, \theta_0) \|_{L^2} + \| \partial^\alpha(f_0, \rho_0, u_0, \theta_0) \|_{L^2} \right),
\]
and
\[
\| \partial^\alpha \int_0^t E_{t-\tau}(S_f, 0, 0, 0) \, d\tau \|_{L^2}^2 \leq C \int_0^t (1 + t - \tau)^{-2\sigma_{q,m}} \left( \| \partial^\alpha(G(\tau), v^{-\frac{1}{2}} \psi(\tau)) \|_{L^2}^2 + \| \partial^\alpha(G(\tau), v^{-\frac{1}{2}} \phi(\tau)) \|_{L^2}^2 \right) \, d\tau,
\]
hold for \( t \geq 0 \), where \( C \) is a positive constant depending only on \( m, q \) and
\[
\sigma_{q,m} = \frac{3}{2} \left( \frac{1}{q} - \frac{1}{2} \right) + \frac{m}{2}.
\]

**Proof**  Applying the Fourier transform to (4.1) in \( x \), we obtain
\[
\begin{align*}
\partial_t \hat{f} + i v \cdot \xi \hat{f} - \hat{u} \cdot v M^{\frac{1}{2}} - \hat{\theta}(|v|^2 - 3) M^{\frac{1}{2}} = \mathcal{L} \hat{f} + \nabla_v \cdot \hat{G} - \frac{1}{2} v \cdot \hat{G} + \hat{\psi}, \\
\partial_t \hat{\rho} + i \xi \cdot \hat{\rho} = 0, \\
\partial_t \hat{u} + |\xi|^2 \hat{u} + i \xi \cdot \hat{\theta} + i \xi \hat{\rho} + \hat{u} - \hat{\theta} = 0, \\
\partial_t \hat{\theta} + |\xi|^2 \hat{\theta} + i \xi \hat{\rho} + \sqrt{3}(\sqrt{3} \hat{\theta} - \sqrt{2} \hat{\omega}) = 0.
\end{align*}
\]
By taking the inner product of the equations in (4.6) with the conjugate of \( \hat{f} \) and integrating in \( v \), we obtain its real part as
\[
\begin{align*}
\frac{1}{2} \partial_t \| \hat{f} \|_{L^2_v}^2 + Re \int_{R^3} \langle -\mathcal{L}(I - P) \hat{f} | (I - P) \hat{f} \rangle \, dv + |\hat{b}|^2 + 2 |\hat{\omega}|^2 - Re \langle \hat{u} | \hat{b} \rangle - Re \langle \hat{\theta} | \sqrt{6} \hat{\omega} \rangle \\
= Re \int_{R^3} \langle \nabla_v \cdot G - \frac{1}{2} v \cdot \hat{G} | \hat{f} \rangle \, dv + Re \int_{R^3} \langle \hat{\psi} | \hat{f} \rangle \, dv \\
= Re \int_{R^3} \langle \nabla_v \cdot G - \frac{1}{2} v \cdot \hat{G} | (I - P) \hat{f} \rangle \, dv + Re \int_{R^3} \langle \hat{\psi} | (I - P) \hat{f} \rangle \, dv,
\end{align*}
\]
and on the basis of the assumptions (4.2), we can calculate
\[
\int_{R^3} \langle \nabla_v \cdot G - \frac{1}{2} v \cdot \hat{G} | (I - P) \hat{f} \rangle \, dv = 0, \quad \int_{R^3} \langle \hat{\psi} | (I - P) \hat{f} \rangle \, dv = 0.
\]
Similarly, from the last three equations in (4.1) we have
\[
\begin{align*}
\frac{1}{2} \partial_t |\hat{\rho}|^2 + Re \langle i \xi \hat{u} | \hat{\rho} \rangle = 0, \\
\frac{1}{2} \partial_t |\hat{u}|^2 + Re \langle i \xi \hat{\theta} | \hat{u} \rangle + Re \langle i \xi \hat{\rho} | \hat{u} \rangle + |\hat{u}|^2 + |\xi|^2 |\hat{u}|^2 - Re \langle \hat{b} | \hat{u} \rangle = 0,
\end{align*}
\]
\[
\frac{1}{2} \partial_t |\hat{\theta}|^2 + \text{Re}(i \xi \hat{u} \hat{\theta}) + 3 |\hat{\theta}|^2 + |\xi|^2 |\hat{\theta}|^2 - \text{Re}(\sqrt{6}\hat{\omega} \hat{\theta}) = 0.
\]

Then, combining these estimates, using the coercivity of \(-\mathcal{L}\) and the Cauchy–Schwarz inequality, we show
\[
\frac{1}{2} \partial_t \left( \|\hat{f}\|_{L^2_v}^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2 \right) + \lambda |\mathbf{I} - \mathbf{P}| \hat{f}^2_v + |\hat{u} - \hat{b}|^2 + |\sqrt{2}\hat{\omega} - \sqrt{3}\hat{\theta}|^2 + |\xi|^2 |\hat{u}|^2 + |\xi|^2 |\hat{\theta}|^2 \\
\leq C \left( \|\hat{G}\|^2 + \|\nu^{-\frac{1}{2}} \phi\|^2 \right).
\]

(4.7)

Next, we consider the estimates on \(a, b, \omega\). Similar to (3.16)–(3.20), for the system (4.1), one has
\[
\partial_t a + \text{div} b = 0,
\]
\[
\partial_t b_i + \partial_i a + \frac{2}{\sqrt{6}} \partial_i \omega + \sum_{j=1}^{3} \partial_{x_j} \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) f) = u_i - b_i,
\]
\[
\partial_t \omega + \sqrt{2}(\sqrt{2}\omega - \sqrt{3}\theta) + \frac{2}{\sqrt{6}} \text{div} b + \sum_{i=1}^{3} \partial_{x_i} Q_i((\mathbf{I} - \mathbf{P}) f) = 0,
\]
\[
\partial_j b_i + \partial_i b_j - \frac{2}{\sqrt{6}} \delta_{ij} \left( \frac{2}{\sqrt{6}} \text{div} b + \sum_{i=1}^{3} \partial_{x_i} Q_i((\mathbf{I} - \mathbf{P}) f) \right) \\
= -\partial_t \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) f) + \Gamma_{i,j}(l + S_f),
\]
\[
\frac{5}{3} \partial_t \omega - \frac{2}{\sqrt{6}} \sum_{j=1}^{3} \partial_{x_j} \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) f) = -\partial_t Q_i((\mathbf{I} - \mathbf{P}) f) + Q_i(l + S_f),
\]

where \(l\) is still expressed as
\[
l = -v \cdot \nabla_x (\mathbf{I} - \mathbf{P}) f + \mathcal{L}(\mathbf{I} - \mathbf{P}) f.
\]

In the same way, taking the Fourier transform in \(x\), we obtain
\[
\partial_t \hat{\omega} + i \xi \cdot \hat{b} = 0,
\]
\[
\partial_t \hat{\theta} + i \xi \hat{\theta} + \frac{2}{\sqrt{6}} i \xi_i \hat{\omega} + \sum_{j=1}^{3} i \xi_j \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) \hat{f}) = \hat{u}_i - \hat{b}_i,
\]
\[
\partial_t \hat{\theta} + \sqrt{2}(\sqrt{2}\hat{\omega} - \sqrt{3}\hat{\theta}) + \frac{2}{\sqrt{6}} i \xi \cdot \hat{b} + \sum_{i=1}^{3} i \xi_i Q_i((\mathbf{I} - \mathbf{P}) \hat{f}) = 0,
\]
\[
i \xi_j \hat{b}_i + i \xi_i \hat{b}_j - \frac{2}{\sqrt{6}} \delta_{ij} \left( \frac{2}{\sqrt{6}} i \xi \cdot \hat{b} + \sum_{i=1}^{3} i \xi_i Q_i((\mathbf{I} - \mathbf{P}) \hat{f}) \right) \\
= -\partial_t \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) \hat{f}) + \Gamma_{i,j}(\hat{l} + \hat{S}_f),
\]
\[
\frac{5}{3} i \xi_i \hat{\omega} - \frac{2}{\sqrt{6}} \sum_{j=1}^{3} i \xi_j \Gamma_{i,j}((\mathbf{I} - \mathbf{P}) \hat{f}) = -\partial_t Q_i((\mathbf{I} - \mathbf{P}) \hat{f}) + Q_i(\hat{l} + \hat{S}_f).
\]
By adopting the similar calculation method as in Proposition 3.3, we can conclude the following inequalities:

\[
\begin{align*}
\partial_t \text{Re} & \sum_{i,j} \langle i \xi_i \hat{b}_j + i \xi_j \hat{b}_i | \Gamma_{i,j}(\{I - P\} \hat{f}) \rangle + \lambda \left( |\xi|^2 |\hat{b}|^2 + |\xi \cdot \hat{b}|^2 \right) \\
& \leq \varepsilon |\xi|^2 (|\hat{\omega}|^2 + |\hat{\lambda}|^2) + C (1 + |\xi|^2) \|I - P\| \hat{f} \|_{L^2_x}^2 + C (|\hat{\omega} - \hat{b}|^2 + \|\hat{G}\|^2 + \|v^{-\frac{1}{2}} \hat{\phi}\|^2), \\
\partial_t \text{Re} & \sum_{i} \langle i \xi_i \hat{\omega} | Q_i(\{I - P\} \hat{f}) \rangle + |\xi|^2 |\hat{\omega}|^2 \\
& \leq \varepsilon |\xi|^2 |\hat{b}|^2 + C (1 + |\xi|^2) \|I - P\| \hat{f} \|_{L^2_x}^2 + C (|\hat{\omega} - \hat{b}|^2 + \|\hat{G}\|^2 + \|v^{-\frac{1}{2}} \hat{\phi}\|^2), \\
\partial_t \text{Re} & \langle \hat{\omega} | \frac{\sqrt{6}}{5} \sum_{j=1}^3 \xi_j Q_j(\{I - P\} \hat{f}) \rangle - i \xi \cdot \hat{b} \rangle + \frac{3}{4} |\xi|^2 |\hat{a}|^2 \\
& \leq \frac{5}{4} |\xi \cdot \hat{b}|^2 + C (1 + |\xi|^2) \|I - P\| \hat{f} \|_{L^2_x}^2 + C (|\hat{\omega} - \hat{b}|^2 + \|\hat{G}\|^2 + \|v^{-\frac{1}{2}} \hat{\phi}\|^2).
\end{align*}
\]

Choosing \(\kappa_1\) small sufficiently, and setting \(\tilde{\mathcal{E}}(\hat{f})\) as

\[
\tilde{\mathcal{E}}(\hat{f}) := \frac{1}{1 + |\xi|^2} \left\{ \sum_{i,j} \langle i \xi_i \hat{b}_j + i \xi_j \hat{b}_i | \Gamma_{i,j}(\{I - P\} \hat{f}) \rangle \\
+ \sum_{i} \langle i \xi_i \hat{\omega} | Q_i(\{I - P\} \hat{f}) \rangle \right\} + \kappa_1 \langle \hat{\omega} | \frac{\sqrt{6}}{5} \sum_{j=1}^3 \xi_j Q_j(\{I - P\} \hat{f}) - i \xi \cdot \hat{b} \rangle,
\]

one has

\[
\partial_t \text{Re} \tilde{\mathcal{E}}(\hat{f}) + \frac{\lambda |\xi|^2}{1 + |\xi|^2} (|\hat{\omega}|^2 + |\hat{b}|^2 + |\hat{\lambda}|^2) \leq C (\|I - P\| \hat{f} \|_{L^2_x}^2 + |\hat{\omega} - \hat{b}|^2 + \|\hat{G}\|^2 + \|v^{-\frac{1}{2}} \hat{\phi}\|^2).
\]

Similarly, choosing \(\kappa_2\) small sufficiently, and setting \(\tilde{\mathcal{E}}(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta})\) as

\[
\tilde{\mathcal{E}}(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) := \tilde{\mathcal{E}}(\hat{f}) + \kappa_2 \frac{1}{1 + |\xi|^2} (\hat{\rho} | i \xi \hat{\rho}),
\]

then we have

\[
\partial_t \text{Re} \tilde{\mathcal{E}}(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) + \frac{\lambda |\xi|^2}{1 + |\xi|^2} (|\hat{\omega}|^2 + |\hat{b}|^2 + |\hat{\lambda}|^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2) \leq C (\|I - P\| \hat{f} \|_{L^2_x}^2 + |\hat{\omega} - \hat{b}|^2 + \|\hat{G}\|^2 + \|v^{-\frac{1}{2}} \hat{\phi}\|^2).
\]

Now, we define the functional \(\mathcal{E}_x(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta})\) by

\[
\mathcal{E}_x(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) := \|\hat{f}\|_{L^2_x}^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2 + \kappa_3 \text{Re} \tilde{\mathcal{E}}(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}),
\]

where a small constant \(\kappa_3 > 0\) is chosen such that
With the help of Gronwall’s inequality, we have

\[ \mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) \sim \| \hat{f} \|^2_{H^2} + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2 \]

\[ \sim \| (I - P) \hat{f} \|^2_{L^2} + |\hat{a}|^2 + |\hat{b}|^2 + |\hat{\omega}|^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2. \]

Finally, the linear combination (4.7) + \kappa_3 \times (4.8) gives

\[
\partial_t \mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) + \lambda \left( \| (I - P) \hat{f} \|^2_{L^2} + |\hat{a}|^2 + |\hat{b}|^2 + |\hat{\omega}|^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2 \right) \\
+ \frac{\lambda |\xi|^2}{1 + |\xi|^2} \left( |\hat{a}|^2 + |\hat{b}|^2 + |\hat{\omega}|^2 + |\hat{\rho}|^2 + |\hat{u}|^2 + |\hat{\theta}|^2 \right) \\
\leq C \left( \| \hat{G} \|^2 + \| v^{-\frac{1}{2}} \phi \|^2 \right),
\]

which further implies

\[
\partial_t \mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) + \frac{\lambda |\xi|^2}{1 + |\xi|^2} \mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) \leq C \left( \| \hat{G} \|^2 + \| v^{-\frac{1}{2}} \phi \|^2 \right).
\]

With the help of Gronwall’s inequality, we have

\[
\mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) \leq e^{-\frac{\lambda |\xi|^2}{1 + |\xi|^2} t} \mathcal{E}_X(\hat{f}, \hat{\rho}, \hat{u}, \hat{\theta}) + \int_0^t e^{-\frac{\lambda |\xi|^2}{1 + |\xi|^2} (t-\tau)} \left( \| \hat{G} \|^2 + \| v^{-\frac{1}{2}} \phi \|^2 \right) d\tau.
\]

In order to obtain the desired time-decay estimates (4.4) and (4.5), the same proof can be adopted as in [17, Theorem 3.1], and we omit the details.

\[ \square \]

**Proof of the large time behavior in Theorem 2.1** By the definitions of \( \mathcal{E}(t) \), \( \mathcal{D}(t) \) in the previous section, it follows that

\[
\mathcal{E}(t) \leq C \left( \| (I - P) f \|^2_{H^4} + \| (a, b, \omega) \|^2_{H^4} + \| (\rho, u, \theta) \|^2_{H^4} \right) \\
\leq C \left( \mathcal{D}(t) + \| f \|^2_{L^2} + \| (\rho, u, \theta) \|^2_{L^2} \right).
\]

(4.9)

From (3.37), one obtains

\[
\frac{d}{dt} \mathcal{E}(t) + \lambda \mathcal{D}(t) \leq 0,
\]

which, together with (4.9), yields

\[
\frac{d}{dt} \mathcal{E}(t) + \lambda \mathcal{E}(t) \leq C \left( \| f \|^2_{L^2} + \| (\rho, u, \theta) \|^2_{L^2} \right).
\]

According to the Gronwall inequality, it follows that

\[
\mathcal{E}(t) \leq e^{-\lambda t} \mathcal{E}(0) + C \int_0^t e^{-\lambda (t-s)} \left( \| f(s) \|^2_{L^2} + \| (\rho(s), u(s), \theta(s)) \|^2_{L^2} \right) ds.
\]

(4.10)

Next, the system (2.1)–(2.4) can be written as

\[
( f(t), \rho(t), u(t), \theta(t) ) = E_t ( f_0, \rho_0, u_0, \theta_0 ) + \int_0^t E_{t-s} ( S_f(s), S_\rho(s), S_u(s), S_\theta(s) ) ds,
\]

with

\[
S_\rho = -\text{div}(\rho u), \\
S_f = -u \cdot \nabla v f + \frac{1}{2} u \cdot v f + \theta M^{-\frac{1}{2}} \text{div}_v \left( M^\frac{1}{2} (\nabla v f - \frac{v}{2} f) \right),
\]

\[ \square \]
\[ S_u = -u \cdot \nabla u + \frac{\rho - \theta}{1 + \rho} \nabla \rho + \frac{\rho}{1 + \rho} (u - b) - \frac{1}{1 + \rho} au - \frac{\rho}{1 + \rho} \Delta u, \]

\[ S_\theta = -u \cdot \nabla \theta - \theta \text{div} u + \frac{\sqrt{3} \rho}{1 + \rho} (\sqrt{3} \theta - \sqrt{2} \omega) + \frac{1}{1 + \rho} \left( (1 + a)|u|^2 - 3a \theta - 2u \cdot b - \rho \Delta \theta \right), \]

where \( S_f \) can be decomposed as

\[ S_f := \nabla_v \cdot G - \frac{v}{2} v \cdot G + \varphi + au \cdot vM^{\frac{1}{2}} - u \cdot bM^{\frac{1}{2}} + a \theta (|v|^2 - 3)M^{\frac{1}{2}} \]

with

\[ G := -u (I - P_0 - P_1) f, \quad \varphi := \theta \cdot (I - P_2) M^{-\frac{1}{2}} \text{div}_v \left( M^{\frac{1}{2}} (\nabla_v f - \frac{v}{2} f) \right). \]

Therefore, \((f(t), \rho(t), u(t), \theta(t))\) can be rewritten as the sum of six terms,

\[
(f(t), \rho(t), u(t), \theta(t)) = E(t, f_0, \rho_0, u_0, \theta_0) + \int_0^t E_{t-s} (\nabla_v \cdot G - \frac{v}{2} v \cdot G + \varphi, 0, 0, 0) \, ds \\
+ \int_0^t E_{t-s} (au \cdot vM^{\frac{1}{2}} - u \cdot bM^{\frac{1}{2}} + a \theta (|v|^2 - 3)M^{\frac{1}{2}}, 0, 0, 0) \, ds \\
+ \int_0^t E_{t-s} (0, s_\rho(s), 0, 0) \, ds + \int_0^t E_{t-s} (0, 0, s_u(s), 0) \, ds \\
+ \int_0^t E_{t-s} (0, 0, 0, s_\theta(s)) \, ds \\
= U_1 + U_2 + U_3 + U_4 + U_5 + U_6.
\]

Applying directly (4.4) to \( U_1 \), one has

\[ \|U_1(t)\|_{L^2} \leq C (1 + t)^{-\frac{3}{2}} \|(f_0, \rho_0, u_0, \theta_0)\|_{Z^1 \cap L^2}. \]

With the help of the Hölder and Sobolev inequalities, using (4.5) to \( U_2 \), we deduce

\[
\|U_2(t)\|_{L^2}^2 \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \left( \|u \cdot (I - P_0 - P_1) f\|_{Z^1 \cap L^2}^2 \\
+ \|v^{-\frac{1}{2}} \cdot (I - P_2) M^{-\frac{1}{2}} \text{div}_v (M^{\frac{1}{2}} (\nabla_v f - \frac{v}{2} f))\| \right) \, ds \\
\leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \|E^2\|_s \, ds + C \int_0^t (1 + t - s)^{-\frac{3}{2}} \|\theta\|_{H^4} \|I - P\| f \|_v^2 \, ds \\
\leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \|E^2\|_s \, ds + C \int_0^t (1 + t - s)^{-\frac{3}{2}} \|\mathcal{E}(s)\| d(s) \, ds.
\]

Similarly, for \( U_i \) \((3 \leq i \leq 6)\), by means of the Hölder and Sobolev inequalities, we can apply (4.4) to them to compute

\[
\|U_3(t)\|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \left( \|au \cdot vM^{\frac{1}{2}} - u \cdot bM^{\frac{1}{2}}, a \theta (|v|^2 - 3)M^{\frac{1}{2}}\|_{Z^1 \cap L^2} \, ds \\
\leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \|\mathcal{E}(s)\| \, ds,
\]

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\[ \|U_4(t)\|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \| (\text{div } u \rho, u \cdot \nabla \rho) \|_{L^1 \cap L^2} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) \, ds, \]
\[ |U_5(t)|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \| S_u(s) \|_{L^1 \cap L^2} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) \, ds, \]
\[ |U_6(t)|_{L^2} \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \| S_{\theta}(s) \|_{L^1 \cap L^2} \, ds \]
\[ \leq C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) \, ds. \]

Therefore, it follows that
\[ \| (f(t), \rho(t), u(t), \theta(t)) \|_{L^2}^2 \leq 2 \sum_{i=1}^6 \| U_i \|_{L^2}^2 \]
\[ \leq C (1 + t)^{-\frac{3}{2}} \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1 \cap L^2}^2 + C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}^2(s) \, ds \]
\[ + C \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) D(s) \, ds + C \left( \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) \, ds \right)^2. \quad (4.11) \]

Define
\[ \mathcal{E}_\infty(t) := \sup_{0 \leq s \leq t} (1 + s)^{\frac{3}{2}} \mathcal{E}(s). \quad (4.12) \]

Using (4.11) and the fact that \( \mathcal{E}(t) \) and \( \mathcal{E}_\infty(t) \) are non-increasing in time, with the aid of Lemma 4.2, we obtain
\[ \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) \, ds \]
\[ = \int_0^t (1 + t - s)^{-\frac{3}{2}} \left( \mathcal{E}(s) \right)^{\frac{2}{3} + \gamma} \left( \mathcal{E}(s) \right)^{\frac{1}{3} - \gamma} \, ds \]
\[ \leq C \left( \mathcal{E}_\infty(t) \right)^{\frac{2}{3} + \gamma} \left( \mathcal{E}(0) \right)^{\frac{1}{3} - \gamma} \int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2} (\frac{2}{3} + \gamma)} \, ds \]
\[ \leq C (1 + t)^{-\frac{3}{2}} \left( \mathcal{E}_\infty(t) \right)^{\frac{2}{3} + \gamma} \left( \mathcal{E}(0) \right)^{\frac{1}{3} - \gamma}, \]
\[ \int_0^t (1 + t - s)^{-\frac{3}{2}} (\mathcal{E}(s))^2 \, ds \]
\[ \leq \mathcal{E}_\infty(t) \mathcal{E}(0) \int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} \, ds \]
\[ \leq C (1 + t)^{-\frac{3}{2}} \mathcal{E}_\infty(t) \mathcal{E}(0), \]
\[ \int_0^t (1 + t - s)^{-\frac{3}{2}} \mathcal{E}(s) D(s) \, ds \]
\[ \leq \mathcal{E}_\infty(t) \int_0^t (1 + t - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} D(s) \, ds \]
\[ \leq \mathcal{E}_\infty(t)(1 + t)^{-\frac{3}{2}} \int_0^t \mathcal{D}(s) \, ds \]

\[ \leq C(1 + t)^{-\frac{3}{2}} \mathcal{E}_\infty(t) \mathcal{E}(0), \]

with \( 0 < \gamma < \frac{1}{3} \). From this, \( \| (f(t), \rho(t), u(t), \theta(t)) \|_{L^2} \) can be further estimated as

\[ \| (f(t), \rho(t), u(t), \theta(t)) \|_{L^2}^2 \leq C(1 + t)^{-\frac{3}{2}} \{ \mathcal{E}_\infty(t) \mathcal{E}(0) + (\mathcal{E}_\infty(t))^{\frac{4+2\gamma}{3}} (\mathcal{E}(0))^{\frac{2}{3} - 2\gamma} \}. \]

Substituting this inequality into the right hand side of (4.10), and multiplying the resulting inequality by \( (1 + t)^{\frac{3}{2}} \), we obtain

\[ (1 + t)^{\frac{3}{2}} \mathcal{E}(t) \leq e^{-\lambda t} (1 + t)^{\frac{3}{2}} \mathcal{E}(0) + C \int_0^t e^{-\lambda(t-s)} \left( \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1} \right) e^{-\lambda t} \mathcal{E}(t) \mathcal{E}(0) + (\mathcal{E}_\infty(t))^{\frac{4+2\gamma}{3}} (\mathcal{E}(0))^{\frac{2}{3} - 2\gamma} \, ds \]

\[ \leq C \left( \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2 + \mathcal{E}_\infty(t) \mathcal{E}(0) + \left( \mathcal{E}_\infty(t) \right)^{\frac{4+2\gamma}{3}} \left( \mathcal{E}(0) \right)^{\frac{2}{3} - 2\gamma} \right). \]

Thus, by the definition (4.12), we further obtain

\[ \mathcal{E}_\infty(t) \leq C \left( \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2 + \mathcal{E}_\infty(t) \mathcal{E}(0) + \left( \mathcal{E}_\infty(t) \right)^{\frac{4+2\gamma}{3}} \left( \mathcal{E}(0) \right)^{\frac{2}{3} - 2\gamma} \right). \]

Since \( \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2 \) and \( \mathcal{E}(0) \sim \| (f_0, \rho_0, u_0, \theta_0) \|_{H^4}^2 \) are small enough, and \( 1 < 2(\frac{5}{3} + \gamma) < 2 \), one has

\[ y(t) \leq A(1 + y(t)) + C^{1-2(\frac{5}{3} + \gamma)} A^2(\frac{5}{3} - \gamma) y^2(t), \]

for all \( t \geq 0 \), with \( y(t) = \mathcal{E}_\infty(t) \), \( A = C \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2 \), due to Lemma 4.3, which implies

\[ \mathcal{E}_\infty(t) \leq 2A = 2C \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2, \]

namely,

\[ \mathcal{E}(t) \leq C \| (f_0, \rho_0, u_0, \theta_0) \|_{L^1}^2 (1 + t)^{-\frac{3}{2}}. \]

The proof of Theorem 2.1 is completed. \qed

5 The periodic case

In this Section we deal with the spatial periodic domain \( \Omega := T^3 \). A straightforward calculation can deduce the following conservation laws:

\[ \frac{d}{dt} \iint F \, dx \, dv = 0, \quad \frac{d}{dt} \left\{ \int n u \, dx + \iint v F \, dx \, dv \right\} = 0, \]

\[ \frac{d}{dt} \int n \, dx = 0, \quad \frac{d}{dt} \left\{ \int n E \, dx + \iint \frac{|v|^2}{2} - F \, dx \, dv \right\} = 0, \]
and by the assumption of Theorem 2.2, we obtain
\[
\int a \, dx = 0, \quad \int \rho \, dx = 0, \quad \int (b + (1 + \rho)u) \, dx = 0, \\
\int (1 + \rho) \left( \theta + \frac{1}{2} |u|^2 \right) \, dx + \frac{\sqrt{6}}{2} \omega \, dx = 0. \tag{5.1}
\]
for all \( t \geq 0 \).

Now, we give a brief proof of Theorem 2.2.

**Proof of Theorem 2.2** Here, we only give the proof of the global a priori estimates. It follows from the Poincaré inequality and the conservation laws (5.1) that
\[
\|a\|_{L^2} \leq C \|\nabla a\|_{L^2}, \quad \|\rho\|_{L^2} \leq C \|\nabla \rho\|_{L^2}, \tag{5.2}
\]
\[
\|u + b\|_{L^2} \leq \|b + u + \rho u\|_{L^2} + \|\rho u\|_{L^2} \\
\leq C \|\nabla (b + u + \rho u)\|_{L^2} + \|u\|_{L^\infty} \|\rho\|_{L^2} \\
\leq C \|\nabla (b, u)\|_{L^2} + C \|u\|_{H^2} \|\nabla \rho\|_{L^2} + C \|\rho\|_{H^2} \|\nabla u\|_{L^2}. \tag{5.3}
\]
\[
\left\| \frac{\sqrt{6}}{2} \omega + \theta \right\|_{L^2} \leq \left\| (1 + \rho) (\theta + \frac{1}{2} |u|^2) + \frac{\sqrt{6}}{2} \omega \right\|_{L^2} + \|\rho \theta\|_{L^2} + \|\rho \omega\|_{L^2} \\
\leq C \left( \|\theta\|_{H^2} + \|u\|_{L^2}^2 \right) (\|\theta\|_{L^2} + \|\nabla \theta\|_{L^2} + \|u\|_{H^2} \|\nabla u\|_{H^2}) \\
+ C (1 + \|\rho\|_{H^2}) \left( \|\sqrt{6}/2 \omega + \theta\right\|_{L^2} + \|\sqrt{2} \omega - \sqrt{3} \theta\|_{L^2}). \tag{5.4}
\]
Let the energy functionals \( E_1(t) \) and \( E_2(t) \) and the corresponding dissipation rate functional \( D_1(t) \) and \( D_2(t) \) be defined in the same way as in the case \( \Omega := \mathbb{R}^3 \). Similarly, we conclude that
\[
\frac{d}{dt} E_1(t) + \lambda D_1(t) \leq C (E_1^\frac{1}{2} + E_1^2) D_1(t), \tag{5.5}
\]
\[
\frac{d}{dt} E_2(t) + \lambda D_2(t) \leq C D_1(t) + C (E_1 + E_1^2) D_1(t) + C (E_1^\frac{1}{2} + E_1^2) D_2(t). \tag{5.6}
\]
Define
\[
D_{\tau,1}(t) := D_1(t) + \tau_3 (\|a\|_{H^2}^2 + \|\rho\|_{L^2}^2) + \tau_4 \|b + u\|_{L^2}^2 + \tau_5 \|\sqrt{6}/2 \omega + \theta\|_{L^2},
\]
where \( \tau_3, \tau_4 \) and \( \tau_5 \) are sufficiently small. Notice
\[
D_{T,1}(t) \sim \sum_{|\alpha| \leq 4} \|[(I - P) \partial^\alpha f]_T^2 + (a, b, \omega, \rho, u, \theta)\|_{H^4}^2. \tag{5.7}
\]
Combining (5.2), (5.3) and (5.5) together, we conclude that
\[
\frac{d}{dt} E_1(t) + \lambda D_{T,1}(t) \leq C (E_1^\frac{1}{2} + E_1^2) D_{T,1}(t). \tag{5.8}
\]
Define the functionals \( E(t) \) and \( D(t) \) as
\[
E(t) := E_1(t) + \tau_6 E_2(t), \quad D(t) := D_{T,1}(t) + \tau_6 D_2(t),
\]
where \( \tau_6 \) is sufficiently small. The inequality (5.6) together with (5.8) yields that
\[
\frac{d}{dt} E(t) + \lambda D(t) \leq C (E_1^\frac{1}{2} (t) + E^2(t)) D(t).
\]
Based on the fact that $E(t)$ is small enough and uniformly in time, and $E(t) \leq CD(t)$, it follows that

$$\frac{d}{dt} E(t) + \lambda E(t) \leq 0,$$

for all $t \geq 0$. By Gronwall’s inequality, it is easy to obtain the exponential decay, and we finish the proof of Theorem 2.2.

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