Hyers-Ulam stability of loxodromic Möbius difference equation

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Abstract

Hyers-Ulam of the sequence $\{z_n\}_{n \in \mathbb{N}}$ satisfying the difference equation $z_{i+1} = g(z_i)$ where $g(z) = \frac{az+b}{cz+d}$ with complex numbers $a$, $b$, $c$ and $d$ is defined. Let $g$ be loxodromic Möbius map, that is, $g$ satisfies that $ad - bc = 1$ and $a + d \in \mathbb{C} \setminus [-2, 2]$. Hyers-Ulam stability holds if the initial point of $\{z_n\}_{n \in \mathbb{N}}$ is in the exterior of avo\text{\textbf{d}}\text{\textbf{i}}\text{\textbf{d}}\text{\textbf{e}}\text{\textbf{d}}\text{\textbf{d}}\text{\textbf{i}}\text{\textbf{d}}\text{\textbf{k}} region, which is the union of the certain disks of $g^{-n}(\infty)$ for all $n \in \mathbb{N}$.

1 Introduction

Ulam [13] posed the the stability of group homomorphisms in 1940. Given a metric group $(G, \cdot, d)$ for given $\varepsilon > 0$, suppose that a function $f : G \to G$ which satisfies the inequality $d(f(xy), f(x)f(y)) \leq \varepsilon$ for all $x, y \in G$. The question is about the existence of a homomorphism $a : G \to G$ such that $d(a(x), f(x)) \leq \delta$ where $\delta$ depends only on $G$ and $\varepsilon$ for all $x \in G$. Hyers [4] gave the affirmative answer this question in 1941 for Cauchy additive equation in Banach spaces. Hyers-Ulam stability has been developed for functional equations for a few decades by various authors. For a few example, see [2, 4, 5, 12].

The difference equation has Hyers-Ulam stability if each terms of the sequence with the given relation has (small) error, this sequence is approximated by the sequence with same relation which has no error. For the
introduction of difference equation, for example, see [3]. Hyers-Ulam stability of difference equations, see [6, 7, 8, 9, 11]. Especially Pielou logistic difference equation has Hyers-Ulam stability in [8] only if the initial point of the sequence is contained in definite intervals. Hyers-Ulam stability is extended on the complex plane in [10] the difference equation as follows

\[ z_{i+1} = \frac{az_i + b}{cz_i + d} \]

over \( \mathbb{C} \) where \( ad - bc = 1 \), \( c \neq 0 \) and \( a + d \in \mathbb{R} \setminus [-2, 2] \) for complex numbers \( a, b, c \) and \( d \). In this paper, we generalize the result the case that \( a + d \in \mathbb{C} \setminus [-2, 2] \), that is, the map \( z \mapsto \frac{az + b}{cz + d} \) is the loxodromic Möbius map.

Möbius map

Linear fractional map on the Riemann sphere \( \hat{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) is called Möbius map or Möbius transformation

\[ g(z) = \frac{az + b}{cz + d} \]

where \( ad - bc \neq 0 \) for \( z \in \hat{\mathbb{C}} \).

The non-constant Möbius map \( g(z) = \frac{az + b}{cz + d} \) has the following properties.

- Without loss of generality, we may assume that \( ad - bc = 1 \).
- \( g(\infty) \) is defined as \( \frac{a}{c} \) and \( g\left(-\frac{d}{c}\right) \) is defined as \( \infty \).
- The composition of two Möbius maps is also a Möbius map.
- The map \( g \) is the linear map if and only if \( \infty \) is a fixed point of \( g \).
- The image of circle or line under Möbius map is circle or line.

The equation \( \frac{az + b}{cz + d} = \frac{pa z + pb}{pc z + pd} \) holds for all \( p \neq 0 \). We define the matrix representation of Möbius map \( z \mapsto \frac{az + b}{cz + d} \) as follows \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) where \( ad - bc = 1 \). Thus we assume that \( ad - bc = 1 \) throughout this paper. Denote the trace of the matrix representation of Möbius map \( g \) by \( \text{tr}(g) \) and we call \( \text{tr}(g) \) the trace of \( g \).
2 Möbius map with attracting and repelling fixed points

The trace of matrix is invariant under conjugation. Thus qualitative classification of Möbius map depends on the trace of matrix representation. For this classification, see [1].

Definition 2.1. If the matrix representation of the non-constant Möbius map \((\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})\) has its trace \(a + d\), say \(\text{tr}(g)\), is in the set \(\mathbb{C} \setminus [-2, 2]\), then the map \(g\) is called the loxodromic Möbius map. Moreover, if \(\text{tr}(g)\) is \(\mathbb{C} \setminus \mathbb{R}\), then map \(g\) is called purely loxodromic.

Denote the fixed points of \(g\) by \(\alpha\) and \(\beta\). If \(|g'(\alpha)| < 1\), then \(\alpha\) is called the attracting fixed point. If \(|g'(\beta)| > 1\), then \(\beta\) is called the repelling fixed point.

Lemma 2.2. Let \(g\) be the Möbius map such that \(g(z) = \frac{az + b}{cz + d}\) where \(ad - bc = 1\) and \(c \neq 0\). If \(g\) is loxodromic, then \(g\) has two different fixed points, one of which is the attracting fixed point and the other is the repelling fixed point.

Proof. The fixed points of \(g\) are the roots of the quadratic equation

\[cz^2 - (a - d)z - b = 0.\]

Denote the fixed points of \(g\) as follows

\[
\alpha = \frac{a - d + \sqrt{(a + d)^2 - 4}}{2c} \quad \text{and} \quad \beta = \frac{a - d - \sqrt{(a + d)^2 - 4}}{2c}. \tag{2.1}
\]

Observe that \(\alpha + \beta = \frac{a - d}{c}\) and \(\alpha \beta = \frac{b}{c}\). Thus we have the following equation

\[
(c\alpha + d)(c\beta + d) = c^2\alpha\beta + cd(\alpha + \beta) + d^2
= -bc + d(a - d) + d^2
= -bc + ad
= 1. \tag{2.2}
\]

Thus we obtain the following inequality using the equation (2.1)

\[
ca + d = \frac{a + d + \sqrt{(a + d)^2 - 4}}{2}. \tag{2.3}
\]
We claim that $|c\alpha + d| \neq 1$. Denote temporarily $a + d$ by $z$ and $c\alpha + d$ by $w$. Then we have $w = \frac{z + \sqrt{z^2 - 4}}{2}$. The straightforward calculation shows that

$$z = w + \frac{1}{w}.$$ 

Suppose that $|w| = 1$. Then

$$\text{tr}(g) = a + d = z = w + \frac{1}{w} = e^{i\theta} + e^{-i\theta} = 2 \cos \theta.$$ 

Then $\text{tr}(g)$ is the real number and $|\text{tr}(g)| \leq 2$. It contradicts to the fact that $g$ is the loxodromic Möbius map. Thus we may assume that $|c\alpha + d| > 1$. Since $g'(z) = \frac{1}{(cz+d)^2}$ and by the equations (2.2) and (2.3), we obtain that $|g'(\alpha)| = \frac{1}{|c\alpha + d|^2} < 1$ and $|g'(\beta)| = \frac{1}{|c\beta + d|^2} > 1$. Hence, $g$ has both attracting and repelling fixed points.

**Lemma 2.3.** Let $g$ and $h$ are Möbius map as follows

$$g(z) = \frac{az + b}{cz + d} \quad \text{and} \quad h(z) = \frac{z - \beta}{z - \alpha}$$

where $\alpha$ and $\beta$ are the fixed points of $g$ and $ad - bc = 1$. If $\alpha \neq \beta$, then $h \circ g \circ h^{-1}(w) = kw$ where $k = \frac{1}{(c\beta + d)^2}$. In particular, if $g$ is the loxodromic Möbius map and $\beta$ is the repelling fixed point, then $|k| > 1$.

**Proof.** The maps $g$ and $h$ are Möbius map. Thus so is $h \circ g \circ h^{-1}$. By the direct calculation, we obtain that $h^{-1}(w) = \frac{\alpha w - \beta}{w - 1}$. Observe that $h^{-1}(0) = \beta$, $h^{-1}(\infty) = \alpha$ and $h^{-1}(1) = \infty$. Then we have

$$h \circ g \circ h^{-1}(0) = h \circ g(\beta) = h(\beta) = 0$$

$$h \circ g \circ h^{-1}(\infty) = h \circ g(\alpha) = h(\alpha) = \infty$$

The points 0 and $\infty$ are fixed points of $h \circ g \circ h^{-1}$. So $h \circ g \circ h^{-1}(w) = kw$ for some $k \in \mathbb{C}$. Since $k = h \circ g \circ h^{-1}(1)$, the following equation holds by (2.1) and (2.2)

$$k = h \circ g \circ h^{-1}(1) = h \circ g(\infty) = h\left(\frac{a}{c}\right)$$

$$= \frac{a - \beta}{c - \alpha} = \frac{a - c\beta}{a - c\alpha}$$

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\[
\frac{a + d + \sqrt{(a + d)^2 - 4}}{a + d - \sqrt{(a + d)^2 - 4}} \quad \frac{c\alpha + d}{c\beta + d} = \frac{1}{(c\beta + d)^2}.
\]

If \(g\) is the loxodromic Möbius map, then \(|k| = \frac{1}{|c\beta + d|^2} > 1\) by the proof of Lemma 2.2.

**Corollary 2.4.** In Lemma 2.3, if \(g\) is the purely loxodromic map, then \(k\) cannot be a positive real number.

*Proof.* The proof of Lemma 2.2 implies that \(\text{tr}(g) = w + \frac{1}{w}\) where \(w = c\alpha + d\) is the complex number and \(|w| \neq 1\). Denote \(w\) by \(re^{i\theta}\) for some \(r \neq 1\). Then the trace of \(g\) satisfies that

\[
\text{tr}(g) = \left(r + \frac{1}{r}\right) \cos \theta + i \left(r - \frac{1}{r}\right) \sin \theta.
\]

Since \(g\) is purely loxodromic, we have \(\text{tr}(g) \in \mathbb{C} \setminus \mathbb{R}\), that is, \(\theta \neq 0\). Then \(c\alpha + d\) is a non real complex number. Hence, \(k = (c\alpha + d)^2\) cannot be any positive real number.

**Lemma 2.5.** Let \(g\) be the loxodromic Möbius map on \(\hat{\mathbb{C}}\). Let \(\alpha\) and \(\beta\) be the attracting and the repelling fixed point respectively. Then

\[
\lim_{n \to \infty} g^n(z) \to \alpha \quad \text{as} \quad n \to +\infty
\]

for all \(z \in \hat{\mathbb{C}} \setminus \{\beta\}\).

*Proof.* By the classification of Möbius map, loxodromic Möbius map has both the attracting and the repelling fixed points. Let \(h\) be the linear fractional map as follows

\[
h(z) = \frac{z - \beta}{z - \alpha}.
\]

Then \(f = h \circ g \circ h^{-1}\) is the dilation with the repelling fixed point at zero, that is, \(f(w) = kw\) for \(|k| > 1\). Thus 0 is the repelling fixed point of \(f\).
Since $h$ is a bijection on $\hat{\mathbb{C}}$, the orbit, \( \{ g^n(z) \}_{n \in \mathbb{Z}} \) corresponds to the orbit, \( \{ f^n(h(z)) \}_{n \in \mathbb{Z}} \) by conjugation $h$. Observe that

\[ f^n(z) \to \infty \quad \text{as} \quad n \to +\infty \]

for all $z \in \hat{\mathbb{C}} \setminus \{0\}$. Hence,

\[ g^n(z) \to \alpha \quad \text{as} \quad n \to +\infty \]

for all $z \in \hat{\mathbb{C}} \setminus \{\beta\}$. \hfill \(\square\)

**Corollary 2.6.** Let $g$ be the map defined in Lemma 2.5. Then

\[ \lim_{n \to \infty} g^{-n}(z) \to \beta \quad \text{as} \quad n \to +\infty \]

for all $z \in \hat{\mathbb{C}} \setminus \{\alpha\}$.

**Proof.** Observe that $g^{-1}$ is also loxodromic Möbius map and $\beta$ and $\alpha$ are the attracting and the repelling fixed point under $g^{-1}$ respectively. Thus we apply the proof of Lemma 2.5 to the map $g^{-1}$. It completes the proof. \hfill \(\square\)

We collect the notions throughout this paper as follows

- The map $g$ is the purely loxodromic Möbius map and $g(z) = \frac{az + b}{cz + d}$ where $ad - bc = 1$ and $c \neq 0$.

- The Möbius map $h$ is defined as $h(z) = \frac{z - \beta}{z - \alpha}$ where $\alpha$ and $\beta$ are the attracting and the repelling fixed points of $g$.

- Without loss of generality, we may assume that the purely loxodromic Möbius map $g$ has the matrix representation where $\text{tr}(g)$ is the non real complex number.

- Since the trace of matrix is invariant under conjugation, we obtain that $\text{tr}(g) = \text{tr}(h \circ g \circ h^{-1})$. Then

\[ \text{tr}(g) = \text{tr} \begin{pmatrix} \sqrt{k} & 0 \\ 0 & \frac{1}{\sqrt{k}} \end{pmatrix} = \sqrt{k} + \frac{1}{\sqrt{k}}. \]
3 Image of circles under the conjugation

In this section, we show that the image of circles under the map \( h \) defined as \( h(z) = z - \beta z - \alpha \). Recall that the image of line or circle under Möbius map is line or circle. The map \( f = h \circ g \circ h^{-1} \) is the dilation defined as \( f(w) = kw \) where \( k = \frac{1}{(c\beta+d)^2} \) and \(|k| > 1\) by Lemma 2.3.

Lemma 3.1. Let \( h \) be the Möbius map defined as \( h(z) = \frac{z-\beta}{z-\alpha} \). Then the image of \(-\frac{d}{c}\) under \( h \) as follows

\[
\left. h \right|_{\left.-\frac{d}{c}\right.}=\frac{1}{k}, \quad -\frac{d}{c} = \frac{k\beta - \alpha}{k-1}\quad \text{and} \quad [c(\alpha - \beta)]^2 = \frac{(k - 1)^2}{k}.
\]

Proof. The map \( h \) is the conjugation from \( g \) to \( f \) and \( h(\infty) = 1 \). The fact that \( f \circ h = h \circ g \) implies that

\[
f \circ h \left(\frac{-d}{c}\right) = h \circ g \left(\frac{-d}{c}\right) = h(\infty) = 1 = f \left(\frac{1}{k}\right).
\]

Since \( f \) is a bijection on \( \mathbb{C} \), \( h(\frac{-d}{c}) = \frac{1}{k} \). Observe that the map \( h^{-1}(w) = \frac{\alpha w - \beta}{w - 1} \). Hence, we have

\[
h^{-1} \left(\frac{1}{k}\right) = -\frac{d}{c} = \alpha - \frac{k}{k-1} (\alpha - \beta). \quad (3.1)
\]

The equation (3.1) implies that \( c\alpha + d = \frac{k}{k-1} c(\alpha - \beta) \). Since \( k = (c\alpha + d)^2 \) by (2.2) and Lemma 2.3, we have

\[
[c(\alpha - \beta)]^2 = \frac{(k - 1)^2}{k}.
\]

\[\square\]

Let the circle \( \{z: |z - \beta| = r|z - \alpha|\} \) be \( C(r) \) for \( r > 0 \). In particular, denote \( C(1) \cup \{\infty\} \) by \( L_\infty \). Similarly, denote the region \( \{z: |z - \beta| \geq r|z - \alpha|\} \) by \( B(r) \) for \( r \geq 0 \). Observe that \( r_1 \geq r_2 \) if and only if \( B(r_1) \subset B(r_2) \).

Lemma 3.2. Let \( g \) be the Möbius map with two different fixed points \( \alpha \) and \( \beta \). Let \( h \) be the map \( h(z) = \frac{z-\beta}{z-\alpha} \). Then \( h(C(r)) = \{w: |w| = r\} \) for \( r \neq 1 \). In particular, \( h(L_\infty) = \{w: |w| = 1\} \).
Proof. Let $w = h(z)$. Then

$$|w| = \left| \frac{z - \beta}{z - \alpha} \right| = \left| \frac{r(z - \alpha)}{z - \alpha} \right| = |r| = r$$

for $r > 0$. Since Möbius transformation is bijective on \( \hat{\mathbb{C}} \), we have that

\[
h(C(r)) = \{ w : |w| = r \} \quad \text{for} \quad r \neq 1.
\]

In $r = 1$ case, $C(1)$ is the straight line rather than geometric circle and $h(\infty) = 1$. Then by the equation (3.2), $h(L_{\infty}) = \{ w : |w| = 1 \}$.

Denote $\{ w : |w| \geq r \}$ by $D(r)$ for $r \geq 0$.

**Lemma 3.3.** Let $g$ be the loxodromic map. Then $B(r)$ is invariant under $g$, that is, $g(B(r)) \subset B(r)$. Furthermore, $g(B(r)) = B(|k|r)$ where $k = \frac{1}{(c\beta + d)^2}$.

**Proof.** The map $f = h \circ g \circ h^{-1}$ where $f(w) = kw$ by Lemma 2.3. Observe that $f(D(r)) = D(|k|r)$. Moreover, $h(B(r)) = D(r)$ by the similar proof of Lemma 3.2. Thus

$$g(B(r)) = h^{-1} \circ f \circ h(B(r)) = h^{-1} \circ f(D(r)) = h^{-1}(D(|k|r)) = B(|k|r).$$

Since $|k| > 1$, $B(|k|r) \subset B(r)$. Hence, $B(r)$ is invariant under $g$.

Define the region $S(r)$ as follows

$$S(r) = \left\{ z \in \mathbb{C} : \left| z + \frac{d}{c} \right| > \frac{r}{|c|} \right\}$$

for $r > 0$.

**Proposition 3.4.** Let $g$ be the loxodromic Möbius map and $h$ be the map defined as $h(z) = \frac{z - \beta}{z - \alpha}$. Then

$$h(S(r)) = \left\{ w : \frac{\sqrt{|k|}}{r} \left| w - \frac{1}{k} \right| > |w - 1| \right\}$$

for $r > 0$.
Proof. The definition of $S(r)$ shows that $z \in S(r)$ if and only if $|cz + d| > r$.

Lemma 3.1 implies that $c\alpha + d = \frac{k}{k-1} c(\alpha - \beta)$ and $(k-1)^2 = [c(\alpha - \beta)]^2$.

Moreover, $w = \frac{z - \beta}{z - \alpha}$ if and only if $z = \alpha + \frac{\alpha - \beta}{w-1}$. Thus

$$cz + d = c\alpha + \frac{c(\alpha - \beta)}{w-1} + d$$

$$= \frac{k}{k-1} c(\alpha - \beta) + \frac{c(\alpha - \beta)}{w-1}$$

$$= c(\alpha - \beta) \left[ \frac{k}{k-1} + \frac{1}{w-1} \right]$$

$$= \frac{c(\alpha - \beta)}{k-1} \cdot \frac{k(w-1) + k - 1}{w-1}$$

$$= \frac{c(\alpha - \beta)}{k-1} \cdot \frac{kw - 1}{w - 1}$$

Then

$$|cz + d| = \left| \frac{c(\alpha - \beta)}{k-1} \right| \cdot \left| \frac{kw - 1}{w - 1} \right| = \frac{|k-1|/\sqrt{|k|}}{|k-1|} \cdot \frac{|k(w - \frac{1}{k})|}{|w - 1|} = \sqrt{|k|} \frac{|w - \frac{1}{k}|}{|w - 1|}$$

Then $|cz + d| > r$ implies that

$$\sqrt{|k|} \frac{|w - \frac{1}{k}|}{|w - 1|} > r \iff \sqrt{|k|} \left| w - \frac{1}{k} \right| > r|w - 1|$$

$$\iff \sqrt{|k|} \left| w - \frac{1}{k} \right| > |w - 1|.$$ 

Hence,

$$h(S(r)) = \left\{ w : \frac{\sqrt{|k|}}{r} \left| w - \frac{1}{k} \right| > |w - 1| \right\}$$

for $r > 0$. \qed

Denote the boundary of the set $S$ by $\partial S$ and the closure of the set $S$ by $\overline{S}$.

**Lemma 3.5.** The center of the circle $h(\partial S(r))$ is $\frac{kr^2 - |k|}{k(r^2 - |k|)}$. The radius of $h(\partial S(r))$ is $\frac{r[k-1]}{\sqrt{|k|} r^2 - |k|}$ for $r > 0$. 

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Proof. $\partial S(r)$ are the concentric circles of which center is $-\frac{d}{c}$ for all $r > 0$. Thus any straight line which contains $-\frac{d}{c}$ meets all circles $\partial S(r)$ perpendicularly at the ends points of diameter of every circles. Let $\ell_C$ the straight line which contains $-\frac{d}{c}$ and $\alpha$, that is, $\ell_S = \{z: z = s\alpha - (1 - s)\frac{d}{c}, s \in \mathbb{R}\}$. Since $h$ is conformal, $h(\ell_S)$ and $h(\partial S(r))$ also meets orthogonally each other for all $r > 0$. Then $h(\ell_S) \cap h(\partial S(r))$ is the set of end points of the diameter of $h(\partial S(r))$ for each $r > 0$. Then the middle point of two points in $h(\ell_S) \cap h(\partial S(r))$ is the center of the circle $\partial S(r)$ and the half of the distance between two points in $h(\ell_S) \cap h(\partial S(r))$ is the radius of $\partial S(r)$.

Since $h \left(-\frac{d}{c}\right) = \frac{1}{k}$, $h(\alpha) = \infty$ and $h(\infty) = 1$, $h(\ell_S) \cup \{\infty\}$ is the extended straight line which contains $\frac{1}{k}$ and 1, that is,

$$h(\ell_S) \cup \{\infty\} = \left\{w: w = t + (1 - t)\frac{1}{k}, t \in \mathbb{R}\right\} \cup \{\infty\}.$$ 

Proposition 3.4 implies that

$$h(\partial S(r)) = \left\{w: \frac{\sqrt{|k|}}{r} \left|w - \frac{1}{k}\right| = |w - 1|\right\}.$$ 

Solve the equation for $t$ as follows to determine the points in $h(\ell_S) \cap h(\partial S(r))$. 

$$\frac{\sqrt{|k|}}{r} \left|t + (1 - t)\frac{1}{k} - \frac{1}{k}\right| = \left|t + (1 - t)\frac{1}{k} - 1\right|$$
Thus the values of $t$ for the above equation are $\frac{1}{1-\sqrt{|k|}}$ or $\frac{1}{1+\sqrt{|k|}}$. Then the end points of the diameter of $h(\partial S(r))$ are as follows

$$
\frac{r}{r - \sqrt{|k|}} + \left(1 - \frac{r}{r - \sqrt{|k|}}\right)\frac{1}{k} \quad \text{and} \quad \frac{r}{r + \sqrt{|k|}} + \left(1 - \frac{r}{r + \sqrt{|k|}}\right)\frac{1}{k}.
$$

The center of the circle $h(\partial S(r))$ is the arithmetic average of two end points as follows

$$
\frac{1}{2} \left\{ \frac{r}{r - \sqrt{|k|}} + \left(1 - \frac{r}{r - \sqrt{|k|}}\right)\frac{1}{k} \right\} + \left\{ \frac{r}{r + \sqrt{|k|}} + \left(1 - \frac{r}{r + \sqrt{|k|}}\right)\frac{1}{k} \right\}
$$

$$
= \frac{1}{2} \left\{ \frac{r}{r - \sqrt{|k|}} \left(1 - \frac{1}{k}\right) + \frac{r}{r + \sqrt{|k|}} \left(1 - \frac{1}{k}\right) + 2 \right\}
$$

$$
= \frac{1}{2} \left\{ \frac{2r^2}{r^2 - |k|} \left(1 - \frac{1}{k}\right) + \frac{2}{k} \right\}
$$

$$
= \frac{r^2}{r^2 - |k|} \cdot \frac{k - 1}{k} + \frac{1}{k}
$$

$$
= \frac{kr^2 - |k|}{k(r^2 - |k|)}.
$$

The radius of the circle $h(\partial S(r))$ is the half of the distance between two end points of the diameter

$$
\frac{1}{2} \left| \frac{r}{r - \sqrt{|k|}} + \left(1 - \frac{r}{r - \sqrt{|k|}}\right)\frac{1}{k} \right| - \left\{ \frac{r}{r + \sqrt{|k|}} + \left(1 - \frac{r}{r + \sqrt{|k|}}\right)\frac{1}{k} \right\}
$$

$$
= \frac{1}{2} \left| \frac{r}{r - \sqrt{|k|}} \left(1 - \frac{1}{k}\right) - \frac{r}{r + \sqrt{|k|}} \left(1 - \frac{1}{k}\right) \right|
$$

$$
= \frac{|r\sqrt{|k|} \cdot k - 1|}{r^2 - |k|}.
$$
Remark 3.1. If \( r = \sqrt{|k|} \), then \( h(\partial S(r)) \) is the straight line, which is perpendicular to the line segment between \( \frac{1}{k} \) and \( 1 \) and it contains \( \frac{1}{2} \left( 1 + \frac{1}{k} \right) \). The circle \( h(\partial S(r)) \) contains the origin if and only if \( r = 1/\sqrt{|k|} \). The modulus of the center of \( h\left(\partial S\left(1/\sqrt{|k|}\right)\right) \) is the same as the radius of the same circle. In this case, the radius is \( \frac{|k-1|}{|k|^2-1} \).

Corollary 3.6. The region \( h(S(1)) \) contains the following region
\[
\left\{ w : |w| > \frac{\sqrt{|k|} |k-1| + |k-|k||}{|k|(|k|-1)} \right\}
\]
and the bounds of radius is as follows
\[
\frac{1}{\sqrt{|k|}} \leq \frac{\sqrt{|k|} (|k-1| + |k-|k||)}{|k|(|k|-1)} \leq \left( \frac{\sqrt{|k|} + 1}{\sqrt{|k|(|k|-1)}} \right)^2.
\]

Proof. The origin is in the exterior of the circle \( h(\partial S(1)) \) because \( \frac{1}{\sqrt{|k|}} < 1 \). The maximum value of the distance between the origin and \( h(\partial S(1)) \) is the sum of the radius and the modulus of the center. Lemma 3.5 implies that the radius of \( h(\partial S(1)) \) is \( \frac{|k-|k||}{|k|(|k|-1)} \) and the modulus of the center is \( \frac{\sqrt{|k|} |k-1|}{|k|(|k|-1)} \). The triangular inequality implies that \( |k-1| \geq |k|-1 \). Then
\[
\frac{\sqrt{|k|} |k-1| + |k-|k||}{|k|(|k|-1)} \geq \frac{\sqrt{|k|} |k-1|}{|k|(|k|-1)} \geq \frac{\sqrt{|k|}}{|k|} \geq \frac{1}{\sqrt{|k|}}
\]
Moreover,
\[
\frac{\sqrt{|k|} |k-1| + |k-|k||}{|k|(|k|-1)} \leq \frac{\sqrt{|k|} (|k| + 1) + 2|k|}{|k|(|k|-1)} = \frac{\left( \sqrt{|k|} + 1 \right)^2}{\sqrt{|k|(|k|-1)}}.
\]
\[\square\]
Remark 3.2. Corollary 3.6 implies that if $k$ is the positive real number, that is, $g$ is the hyperbolic Möbius map, then $h(S(1))$ contains the region $\{w: |w| > \frac{1}{\sqrt{|k|}}\}$.

4 Hyers-Ulam stability on the region bounded by circle

Denote the set of natural numbers and zero, namely, $\mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$. Let $F$ be the function from $\mathbb{N}_0 \times \mathbb{C}$ to $\mathbb{C}$. Let a complex valued sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfies the inequality
\[ |a_{n+1} - F(n,a_n)| \leq \varepsilon \]
for a given positive number $\varepsilon$ for all $n \in \mathbb{N}_0$ where $|\cdot|$ is the absolute value of complex number. If there exists the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ which satisfies that
\[ b_{n+1} = F(n,b_n) \]
for each $n \in \mathbb{N}_0$, and $|a_n - b_n| \leq G(\varepsilon)$ for all $n \in \mathbb{N}_0$ where the positive number $G(\varepsilon)$ converges to zero as $\varepsilon \to 0$, then we say that the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ has Hyers-Ulam stability. Denote $F(n,z)$ by $F_n(z)$ if necessary.

The authors in [7] proved the following lemma. For the sake of completeness, we suggest the lemma and its proof.

Lemma 4.1. Let $F: \mathbb{N}_0 \times \mathbb{C} \to \mathbb{C}$ be a function satisfying the condition
\[ |F(n,u) - F(n,v)| \leq K|u - v| \quad (4.1) \]
for all $n \in \mathbb{N}_0$, $u, v \in \mathbb{C}$ and for $0 < K < 1$. For a given an $\varepsilon > 0$ suppose that the complex valued sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfies the inequality
\[ |a_{n+1} - F(n,a_n)| \leq \varepsilon \quad (4.2) \]
for all $n \in \mathbb{N}_0$. Then there exists a sequence $\{b_n\}_{n \in \mathbb{N}_0}$ satisfying
\[ b_{n+1} = F(n,b_n) \quad (4.3) \]
and

$$|b_n - a_n| \leq K^n|b_0 - a_0| + \frac{1 - K^n}{1 - K} \varepsilon$$

for \( n \in \mathbb{N}_0 \). If the whole sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) is contained in the invariant set \( S \subset \mathbb{C} \) under \( F \), then \( \{b_n\}_{n \in \mathbb{N}_0} \) is also in \( S \) under the condition, \( a_0 = b_0 \).

Proof. By induction suppose that

$$|b_{n-1} - a_{n-1}| \leq K^{n-1}|b_0 - a_0| + \frac{1 - K^{n-1}}{1 - K} \varepsilon.$$ 

If \( n = 0 \), then trivially \( |b_0 - a_0| \leq \varepsilon \). Induction implies that

$$|b_n - a_n| \leq |b_n - F(n-1, a_{n-1})| + |a_n - F(n-1, a_{n-1})|$$

$$\leq |F(n-1, b_{n-1}) - F(n-1, a_{n-1})| + |a_n - F(n-1, a_{n-1})|$$

$$= K|b_{n-1} - a_{n-1}| + \varepsilon$$

$$\leq K \left( K^{n-1}|b_0 - a_0| + \frac{1 - K^{n-1}}{1 - K} \varepsilon \right) + \varepsilon$$

$$= K^n|b_0 - a_0| + \frac{1 - K^n}{1 - K} \varepsilon.$$ 

Moreover, if \( a_0 = b_0 \), then the sequence \( \{b_n\}_{n \in \mathbb{N}_0} \) satisfies the inequality (4.2) without error, namely \( \varepsilon = 0 \), under \( F \). Hence, \( \{b_n\}_{n \in \mathbb{N}_0} \) is contained in the invariant set \( S \).

The set \( S \) is called an invariant set under \( F \) (or \( S \) is invariant under \( F \)) only if \( s \in S \) implies that \( F(s) \in S \).

**Lemma 4.2.** Let \( g \) be the loxodromic Möbius map as \( g(z) = \frac{az + b}{cz + d} \) where \( a, b, c \) and \( d \) are complex numbers, \( ad - bc = 1 \) and \( c \neq 0 \). Let the set \( B_R \) be

$$B_R = \{ z : |z - \beta| \geq R|z - \alpha| \} \quad \text{for } R > 1$$

$$B_R = \{ z : |z - \beta| \geq R|z - \alpha| \} \cup \{ \infty \} \quad \text{for } 0 < R \leq 1.$$ 

Then \( g(B_R) \subset B_R \), that is, \( B_R \) is invariant under \( g \).
Proof. The map \( h(z) = \frac{z - \beta}{z - \alpha} \) and \( g = h^{-1} \circ f \circ h \) where \( f(w) = kw \). Recall that \(|k| > 1\). Thus

\[
\begin{align*}
  h(B_R) &= \{ w : |w| \geq R \} \cup \{ \infty \} \\
  f \circ h(B_R) &= \{ w : |w| \geq |k|R \} \cup \{ \infty \} \\
  h^{-1} \circ f \circ h(B_R) &= \{ z : |z - \beta| \geq |k|R \cdot |z - \alpha| \} \text{ for } |k|R > 1 \\
  h^{-1} \circ f \circ h(B_R) &= \{ z : |z - \beta| \geq |k|R \cdot |z - \alpha| \} \cup \{ \infty \} \text{ for } 0 < |k|R \leq 1.
\end{align*}
\]

Observe that \( h^{-1} \circ f \circ h(B_R) = g(B_R) = B_{|k|R} \). Since \( R < |k|R \), we have that \( f \circ h(B_R) \subset h(B_R) \). Hence, \( g(B_R) \subset B_R \), that is, \( B_R \) is invariant under \( g \).

Corollary 4.3. If \( z \in B_R \) for \( R > 0 \), then

\[
\left\{ z : |z - \beta| \leq \frac{R|k - 1|}{|c|(R + 1)\sqrt{|k|}} \right\} \subset \overline{C \setminus B_R}.
\]

Proof. The set inclusion

\[
\{ z : |z - \beta| \leq A \} \subset \{ z : |z - \alpha| \geq B \}
\]

holds if and only if \( A + B \leq |\alpha - \beta| \). Let \( B \) be \( \frac{A}{R} \). Then (4.4) holds if and only if \( A \leq \frac{R}{R+1} |\alpha - \beta| \). Moreover, \( B \geq \frac{1}{R+1} |\alpha - \beta| \). Then

\[
|z - \beta| \leq \frac{R}{R+1} |\alpha - \beta| \leq R|z - \alpha|.
\]

The inequality \(|z - \beta| \leq \frac{R}{R+1} |\alpha - \beta| \) implies that \( z \in \overline{C \setminus B_R} \). Lemma 3.1 implies that \(|\alpha - \beta| = \frac{|k-1|}{|c|\sqrt{|k|}} \). Hence,

\[
\left\{ z : |z - \beta| \leq \frac{R|k - 1|}{|c|(R + 1)\sqrt{|k|}} \right\} \subset \overline{C \setminus B_R}.
\]

\( \square \)
Proposition 4.4. Let $g$ be the loxodromic Möbius map. Let $B_R$ be the region defined in Lemma 4.2. Let a complex valued sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfies the inequality

$$|a_{n+1} - g(a_n)| \leq \varepsilon$$

for a given $\varepsilon > 0$ and for all $n \in \mathbb{N}_0$. For small enough $\varepsilon$, If $a_0 \in B_R$ for

$$R > \left(\frac{\sqrt{|k|+1}}{\sqrt{|k|(|k|-1)}}\right)^2,$$

then the whole sequence $\{a_n\}_{n \in \mathbb{N}_0}$ is also contained in $B_R$. Moreover, there exists the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ satisfying

$$b_{n+1} = g(b_n)$$

for each $n \in \mathbb{N}$ has Hyers-Ulam stability where $b_0 = a_0$.

Proof. The region $B_R$ is invariant under $g$ by Lemma 4.2. Define the distance between the circle, $\partial B_{r_1}$ and the disk (or the exterior of the disk), $B_{r_2}$ as follows

$$\text{dist}(B_{r_1}, \partial B_{r_2}) = \inf \{ |z - w| : z \in B_{r_1} \text{ and } w \in \partial B_{r_2} \}.$$ 

Recall that $h(B_R) \subset f \circ h(B_R)$. Thus $h(B_{|k|R})$ and $\partial h(B_R)$ are disjoint and

$$\text{dist}(h(B_{|k|R}), \partial h(B_R)) = (|k| - 1)R > 0.$$ 

Then since $h$ is a conformal isomorphism between $B_R$ and $h(B_R)$, the distance $\text{dist}(B_{|k|R}, \partial B_R)$ is also positive number. Choose $\varepsilon > 0$ satisfying $\varepsilon < \text{dist}(B_{|k|R}, \partial B_R)$. Suppose that $a_0$ is in $B_R$. Then $g(a_0) \in B_{|k|R}$ and $\{z : |z - g(a_0)| \leq \varepsilon\}$ is a subset of $B_R$, that is, $a_1 \in B_R$. By induction, the whole sequence $\{a_n\}_{n \in \mathbb{N}_0}$ is contained in $B_R$.

Recall that $g'(z) = \frac{1}{(cz+d)^2}$. The definition of $S(r)$ in (3.3) implies that $|g'(z)| < 1$ if and only if $z \in S(1)$. Corollary 3.6, the definition of $B_R$ in Lemma 4.2 and the condition

$$R > \left(\frac{\sqrt{|k|+1}}{\sqrt{|k|(|k|-1)}}\right)^2$$

implies that $h(S(1)) \supset h(B_R)$. Then $|g'(z)| < 1$ for all $z \in B_R$. Lemma 4.1 implies that

$$|b_n - a_n| \leq K^n|b_0 - a_0| + \frac{1 + K^n}{1 - K}\varepsilon$$

where $K = \inf \{|g'(z)|\} < 1$. Hence, the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ has Hyers-Ulam stability where $b_0 = a_0$. \qed
Avoided region

The map $g$ is the Möbius map with complex coefficients $g(z) = \frac{az + b}{cz + d}$ for $ad - bc = 1$ and $c \neq 0$. Since the point $\infty$ is not a fixed point of $g$, the preimage of $\infty$ under $g$, namely, $g^{-1}(\infty)$, is in the complex plane. For a given $\varepsilon > 0$, let $\{a_n\}_{n \in \mathbb{N}_0}$ be the sequence which satisfies the inequality $|a_{n+1} - g(a_n)| \leq \varepsilon$ for all $n \in \mathbb{N}$. If $\{a_n\}_{n \in \mathbb{N}_0}$ contains $g^{-1}(\infty)$, say $a_k$, then $|a_k + 1 - \infty|$ is not bounded where $|\cdot|$ is the absolute value of the complex number. In order to exclude where $|\cdot|$ is the absolute value of the complex number $g^{-1}(\infty)$ in the sequence $\{a_n\}_{n \in \mathbb{N}_0}$, the region $R_F$ is considered such that if the initial point of the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ is not in $R_F$, then the whole sequence $\{a_n\}_{n \in \mathbb{N}_0}$ cannot be in $R_F$. Let the forward orbit of $p$ under $F$ be the set $\{F(p), F^2(p), \ldots, F^n(p), \ldots\}$ and denote it by $\text{Orb}_N(p, F)$. For a given $\varepsilon > 0$, let $\{a_n\}_{n \in \mathbb{N}_0}$ be the sequence which satisfies the inequality $|a_{n+1} - g(a_n)| \leq \varepsilon$ for all $n \in \mathbb{N}$. Avoided region $R_F \subset \mathbb{C}$ is defined as follows:

1. $\mathbb{C} \setminus R_F$ is (forward) invariant under $F$, that is, $F(\mathbb{C} \setminus R_F) \subset \mathbb{C} \setminus R_F$.
2. For any given initial point $a_0$ in $\mathbb{C} \setminus R_F$, all points in the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfying $|a_{n+1} - g(a_n)| \leq \varepsilon$ are in $\mathbb{C} \setminus R_F$. If $R_F$ contains $\text{Orb}_N(p, F^{-1})$, where $p \in \mathbb{C}$, then it is called the avoided region at $p$ and is denoted by $R_F(p)$.

In the above definition, the avoided region does not have to be connected.

Remark 5.2. The set $\mathbb{C} \setminus B_R$ in Proposition 4.4 for $R > \sqrt{|a|+1}$ is an avoided region at $\infty$. However, avoided region $\mathbb{C} \setminus B_R$ can be extended to some neighborhood of $\text{Orb}(\infty, g^{-1})$, which is denoted to be $R_R(\infty)$.

Definition 5.1. Let $F$ be the map on $\hat{\mathbb{C}}$ which does not fix $\infty$. Let $\{a_n\}_{n \in \mathbb{N}_0}$ be any sequence which satisfies $|a_{n+1} - g(a_n)| \leq \varepsilon$ for a given $\varepsilon > 0$. Avoided region $R_F \subset \mathbb{C}$ is defined as follows:

1. $\mathbb{C} \setminus R_F$ is (forward) invariant under $F$, that is, $F(\mathbb{C} \setminus R_F) \subset \mathbb{C} \setminus R_F$.
2. For any given initial point $a_0$ in $\mathbb{C} \setminus R_F$, all points in the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfying $|a_{n+1} - g(a_n)| \leq \varepsilon$ for a given $\varepsilon > 0$. Avoided region $R_F \subset \mathbb{C}$ is defined as follows:

In the above definition, the avoided region does not have to be connected.
Lemma 5.3. Let $f$ be the map $f(w) = kw$ for $|k| > 1$. Let $\{c_n\}_{n \in \mathbb{N}_0}$ be the sequence for a given $\delta > 0$ satisfying that

$$|c_{n+1} - f(c_n)| \leq \delta_0$$  \hspace{1cm} (5.1)

for all $n \in \mathbb{N}_0$. If $|c_j - \frac{1}{k^m}| > \frac{t\delta_0}{|k| - 1}$, then $|c_{j+1} - \frac{1}{k^{m-1}}| > \frac{t\delta_0}{|k| - 1}$ for each $j, m \in \mathbb{N}_0$ and for all $t \geq 1$.

Proof. The inequality $|c_{j+1} - f(c_j)| \leq \delta_0$ implies that

$$\delta_0 \geq |c_{j+1} - f(c_j)|$$

$$= \left| \left( c_{j+1} - \frac{1}{k^{m-1}} \right) - \left( kc_j - \frac{1}{k^{m-1}} \right) \right|$$

$$= \left| \left( c_{j+1} - \frac{1}{k^{m-1}} \right) - k \left( c_j - \frac{1}{k^m} \right) \right|$$

$$\geq \left| c_{j+1} - \frac{1}{k^{m-1}} \right| - |k| \left| c_j - \frac{1}{k^m} \right|$$

Then

$$\left| c_{j+1} - \frac{1}{k^{m-1}} \right| \geq |k| \left| c_j - \frac{1}{k^m} \right| - \delta_0 > |k| \frac{t\delta_0}{|k| - 1} - \delta_0 \geq \frac{|k| t\delta_0}{|k| - 1} - |k| - 1 - t\delta_0 = \frac{t\delta_0}{|k| - 1}$$

for $t \geq 1$. \hfill \Box

Let the region

$$D_n(\delta) = \left\{ w : \left| w - \frac{1}{k^n} \right| < \delta \right\}$$  \hspace{1cm} (5.2)

for $n \in \mathbb{N}_0$ and $\delta > 0$.

Lemma 5.4. Let $f$ be the map $f(w) = kw$ for $|k| > 1$. Let $\{c_n\}_{n \in \mathbb{N}_0}$ be the sequence for a given $\delta > 0$ satisfying that

$$|c_{n+1} - f(c_n)| \leq \delta_0$$

for all $n \in \mathbb{N}_0$. Let $\delta$ be a positive number $\frac{t\delta_0}{|k| - 1}$ for some $t > 1$. Define the set

$$\mathcal{D} = \bigcup_{n=1}^{\infty} D_n(\delta)$$

where $D_n$ is the closed disk defined in (5.2). If $c_0$ is contained in $\mathbb{C} \setminus \mathcal{D}$, then the sequence $\{c_n\}_{n \in \mathbb{N}_0}$ is contained in $\mathbb{C} \setminus \mathcal{D}$, that is, $\mathcal{D}$ an avoided region of $g$ at one.
Proof. The definition of $D_n(\delta)$ can be extended to the case that $n$ is integer. Thus $f(D_n(\delta)) = D_{n-1}(|k|\delta)$ for $n \in \mathbb{Z}$. Since $|k| > 1$, $f(D)$ is as follows

$$f(D) = f\left(\bigcup_{n=1}^{\infty} D_n(\delta)\right) = \bigcup_{n=1}^{\infty} f(D_n(\delta)) = \bigcup_{n=0}^{\infty} D_n(|k|\delta) \supset D$$

Then $\mathbb{C} \setminus D$ is invariant under $f$. Assume that $c_0 \in \mathbb{C} \setminus D$. Then we have

$$c_0 \in \left\{ w : \left| w - \frac{1}{k^m} \right| \geq \delta > \frac{t\delta_0}{|k| - 1} \right\}$$

for all $m \in \mathbb{N}$. Lemma 5.3 and the induction implies that for each $N \in \mathbb{N}_0$

$$c_N \in \left\{ w : \left| w - \frac{1}{k^m-N} \right| \geq \delta \right\}$$

for all $m \in \mathbb{N}$. Since

$$\mathbb{C} \setminus D \subset \bigcup_{m=1}^{\infty} \left\{ w : \left| w - \frac{1}{k^m-N} \right| \geq \delta \right\}$$

for every $N \in \mathbb{N}_0$, the set $D$ is an avoided region of $g$ at one. □

The origin is the accumulation point of the set $\left\{ \frac{1}{k^m} \right\}$ for $m \in \mathbb{N}_0$ and is the repelling fixed point of $f$. Then we may choose the avoided region of $f$ at
Proposition 5.5. Let $R_f(1)$ the avoided region defined in (5.3). For a given $\varepsilon > 0$, let $\{a_n\}_{n \in \mathbb{N}_0}$ be the sequence which satisfies the inequality

$$|a_{n+1} - g(a_n)| \leq \varepsilon$$

for all $n \in \mathbb{N}_0$ where $g$ is the loxodromic M"obius map. Then for sufficiently small $\varepsilon > 0$, an avoided region of $g$ at $\infty$ is $h^{-1}(R_f(1))$. The number $\varepsilon$ depends only on $|k|$, $\delta$ and $|c|$.

The proof of the above proposition requires the following lemma.

Lemma 5.6. Let $h$ be the map $h(z) = \frac{z - \beta}{z - \alpha}$. Let the disk of which center is $p$ and radius is $r$ be as follows

$$U(p, r) = \{z: |z - p| < r\}$$

Assume that $|\beta - p| - r \neq 0$. Then the radius of $h(U(p, r))$ is $\frac{r^2|\alpha - \beta|^2}{|\beta - p|^2 - r^2}$.

Proof. The equation $w = h(z)$ holds if and only if $z = \frac{\beta w - \alpha}{w - 1}$. Thus the following equivalent inequalities hold

$$|z - p| < r \iff \left| \frac{\beta w - \alpha}{w - 1} - p \right| < r$$

$$\iff |\beta w - \alpha - p(w - 1)| < r|w - 1|$$

$$\iff |(\beta - p)w - \alpha + p|^2 < r^2|w - 1|^2$$

$$\iff |\beta - p|^2 |w|^2 + \left( -\alpha + \overline{p} \right)(\beta - p)w + (\alpha + p) \left( \beta - p \right) \overline{w}$$

$$+ | - \alpha + p|^2 < r^2 \left( |w|^2 - w - \overline{w} + 1 \right)$$

$$\iff (|\beta - p|^2 - r^2) |w|^2 + \left\{ \left(-\alpha + \overline{p}\right)(\beta - p) + r^2 \right\} w$$

$$+ \left\{ \left(-\alpha + p\right)(\beta - p) + r^2 \right\} \overline{w} < r^2 - | - \alpha + p|^2$$
Suppose firstly that $|\beta - p| - r > 0$. Then

\[
\sqrt{|\beta - p|^2 - r^2} w - \frac{(-\alpha + p)(\beta - p) + r^2}{\sqrt{|\beta - p|^2 - r^2}} \frac{|(\alpha + p)(\beta - p) + r^2|}{|\beta - p|^2 - r^2} < \frac{|(\alpha + p)(\beta - p) + r^2|^2}{|\beta - p|^2 - r^2} + r^2 - | - \alpha + p|^2
\]

Then the radius is

\[
| - \alpha + p|^2 |\beta - p|^2 + r^2(-\alpha + p)(\beta - p) + r^2(-\alpha + p)(\beta - p) + r^4 + r^2|\beta - p|^2 - | - \alpha + p|^2|\beta - p|^2 + 4 - \alpha + p|^2|\beta - p|^2 + 4 - | - \alpha + p|^2
\]

\[
= r^2\{(-\alpha + p)(\beta - p) + (-\alpha + p)(\beta - p) + |\beta - p|^2 + | - \alpha + p|^2\}
\]

\[
= r^2(-\alpha \beta + \overline{\beta} \alpha + \overline{\alpha} p - |p|^2 - \alpha \overline{\beta} + \overline{\alpha} \beta - \alpha \overline{p} - |p|^2
\]

\[
+ |\beta|^2 - \alpha \beta - \alpha \overline{\beta} + |\beta|^2 + |\alpha|^2 - \alpha \overline{p} + \alpha |p|^2
\]

\[
= r^2(-\alpha \beta - \alpha \overline{\beta} + |\beta|^2 + |\alpha|^2)
\]

Then the radius is $\frac{r^2|\alpha - \beta|^2}{|\beta - p|^2 - r^2}$ only if $|\beta - p| - r > 0$. Similarly, if $|\beta - p| - r > 0$, then the radius is $\frac{r^2|\alpha - \beta|^2}{|\beta - p|^2 - r^2}$. Hence, the radius of $h(U(p, r))$ is

\[
\frac{r^2|\alpha - \beta|^2}{|\beta - p|^2 - r^2}.
\]

\[\square\]

**proof of Proposition 5.5.** The invariance of $\mathbb{C} \setminus h^{-1}(R_f(1))$ under $g$ is equivalent to the fact that $g(h^{-1}(R_f(1))) \supset h^{-1}(R_f(1))$. Recall that $f \circ h = h \circ g$. Then since $R_f(1)$ is an avoided region of $f$, $f(R_f(1)) \supset R_f(1)$. Then

\[
g(h^{-1}(R_f(1))) = g \circ h^{-1}(R_f(1)) = h^{-1} \circ f(R_f(1)) \supset h^{-1}(R_f(1)).
\]

By the above set inclusion, $\mathbb{C} \setminus h^{-1}(R_f(1))$ is invariant under $g$. Recall that $h^{-1}(\mathbb{C} \setminus D(|k|\delta)) = B_{|k|\delta}$. We may choose an $\varepsilon > 0$ which satisfies that $\varepsilon < \text{dist}(\partial B_{|k|\delta}, B_{|k|^2 \delta})$. Thus if $a_0 \in B_{|k|\delta}$, the sequence $\{a_n\}_{n \in \mathbb{N}_0}$ is also
contained in $B|k|\delta$ by induction. Choose $a_0 \in \mathbb{C} \setminus h^{-1}(\mathcal{R}_f(1))$ and suppose that $a_n$ is contained in the same set. Since $|a_{n+1} - g(a_n)| \leq \varepsilon$, it suffice to show that

$$\{z: |z - g(a_n)| \leq \varepsilon\} \cap h^{-1}(\mathcal{R}_f(1)) = \emptyset$$

for small enough $\varepsilon > 0$. Observe that $a_n \in \mathbb{C} \setminus h^{-1}(\mathcal{R}_f(1))$ if and only if $h(a_n) \in \mathbb{C} \setminus \mathcal{R}_f(1)$. Lemma 5.4 implies that

$$\{w: |w - f(h(a_n))| \leq \delta_0\} \cap \mathcal{R}_f(1) = \emptyset.$$  

(5.4)

Denote the set $\{z: |z - g(a_n)| \leq \varepsilon\}$ by $U_n$ and $\{w: |w - f(h(a_n))| \leq \delta_0\}$ by $V_n$. Find the small $\varepsilon > 0$ which implies that $h(U_n) \subset V_n$. Lemma 5.6 implies that the radius of $g(U_n)$ is $\frac{\varepsilon^2|\alpha - \beta|^2}{|\beta - a_n^2 - \varepsilon^2}$. Lemma 3.1 implies that $|\alpha - \beta|^2 = \frac{|k - 1|^2}{|c|}$ and Corollary 4.3 implies that $|\beta - a_n| \geq \frac{\delta \sqrt{|k||k - 1|}}{|c|}$. Thus an upper bound of the radius of $g(U_n)$ is as follows

$$\frac{\varepsilon^2|\alpha - \beta|^2}{|\beta - a_n|^2 - \varepsilon^2} \leq \frac{\varepsilon^2}{\frac{\delta \sqrt{|k||k - 1|}}{|c|}^2} \cdot \frac{|k - 1|^2}{|c|}$$

where $\varepsilon < \frac{\delta \sqrt{|k||k - 1|}}{|c|}$. Moreover, since $h \circ g = f \circ h$, the center of $V_n$, namely, $f \circ h(a_n) = h \circ g(a_n)$ is contained in the interior of $h(U_n)$. Thus $h(U_n) \subset V_n$ if and only if

$$\text{diameter of } g(U_n) \leq \text{radius of } V_n.$$  

Then choose $\varepsilon > 0$ satisfying the inequality $\frac{\varepsilon^2}{\frac{\delta \sqrt{|k||k - 1|}}{|c|}^2} \cdot \frac{|k - 1|^2}{|c|} < \delta$ or equivalently, choose $\varepsilon < \frac{\delta \sqrt{|k||k - 1|^2}}{\{2|k - 1|^2 + 2|c|\delta \sqrt{|k|}\}^\frac{1}{2}|c|(|\delta|k| + 1)}$. Then take $\varepsilon > 0$ as follows

$$\varepsilon < \min \left\{ \text{dist} \left( \partial B|k|\delta, B|k|^2\delta \right), \frac{\delta \sqrt{|k||k - 1|^2}}{\{2|k - 1|^2 + 2|c|\delta \sqrt{|k|}\}^\frac{1}{2}|c|(|\delta|k| + 1)} \right\}.  

(5.5)
The equation (5.4) implies that \((V_n) \cap (\mathcal{R}_f(1)) = \emptyset\). Moreover, if we choose \(\varepsilon > 0\) in (5.5), then \(h(U_n) \subset V_n\).

\[
U_n \cap h^{-1}(\mathcal{R}_f(1)) \subset h^{-1}(V_n) \cap h^{-1}(\mathcal{R}_f(1)) = h^{-1}(V_n \cap \mathcal{R}_f(1)) = \emptyset
\]

Hence, by induction \(\{a_n\}_{n \in \mathbb{N}_0}\) is contained in \(\mathbb{C} \setminus \mathcal{R}_f(1)\), which is an avoided region of \(g\) at \(\infty\).

\[\square\]

**Remark 5.7.** The forward orbit of 1 under \(f^{-1}\) is \(\left\{ \frac{1}{k^n} : n \in \mathbb{N} \right\}\). The set \(\left\{ h^{-1} \left( \frac{1}{k^n} \right) : n \in \mathbb{N} \right\}\) is the forward orbit of \(\infty\) under \(g^{-1}\) because \(h^{-1}(1) = \infty\) and \(g^{-1} = h^{-1} \circ f^{-1} \circ h\). Then the avoided region \(\mathcal{R}_g(\infty)\) can be chosen as \(h^{-1}(\mathcal{R}_f(1))\).

## 6 Escaping time from the region

Let the sequence \(\{c_n\}_{n \in \mathbb{N}_0}\) satisfies the following

\[|c_{n+1} - f(c_n)| \leq \delta_0\]  

(6.1)

for all \(n \in \mathbb{N}_0\). For the given set \(S\), assume that \(c_0 \in S\). If the distance between \(c_n\) and the closure of \(S\) is positive for all \(n \geq N\), then \(N\) is called escaping time of the sequence \(\{c_n\}_{n \in \mathbb{N}_0}\) from \(S\) under \(f\). If the escaping time \(N\) is independent of the initial point \(c_0\) in \(S\), then \(N\) is called uniformly escaping time.

**Lemma 6.1.** Let \(\{c_n\}_{n \in \mathbb{N}_0}\) be the sequence defined in the equation (6.1) where \(f(w) = kw\) for \(|k| > 1\). Denote the set \((\mathbb{C} \setminus \mathcal{D}(R)) \setminus \mathcal{R}_f(1)\) by \(E_f\) where \(R > \frac{(\sqrt{|k|} + 1)^2}{\sqrt{|k|(|k| - 1)}}\). Then the (uniformly) escaping time \(N\) from the region \(E_f\) under \(f\) satisfies the following inequality

\[
N > \log \left\{ \frac{1}{\delta_0} \left( \frac{2(\sqrt{|k|} + 1)^2}{\sqrt{|k|(|k| - 1)} + 1} \right) + 1 \right\} / \log |k|
\]

for small enough \(\delta_0 > 0\).
Proof. By triangular inequality, we have
\[
|f^n(c_0) - c_0| \leq |f^n(c_0) - f^{n-1}(c_1)| + |f^{n-1}(c_1) - f^{n-2}(c_2)| + \cdots + |f^2(c_{n-2}) - f(c_{n-1})| + |f(c_{n-1}) - c_n| + |c_n - c_0|
\]
\[
= \sum_{j=1}^{n} |k|^{n-j} |f(c_{j-1}) - c_j| + |c_n - c_0|
\]
\[
\leq |k|^n - 1 \frac{\delta_0}{|k| - 1} |c_n - c_0|.
\]
Recall that we may choose \(\delta\) which satisfies that \(\delta \geq \frac{\delta_0}{|k| - 1}\) by Lemma 5.4 and \(D(\delta|k|) \subset R_f(1)\). Since \(c_0 \in E_f\), we have \(|c_0| \geq \frac{|k| \delta_0}{|k| - 1}\). Then we obtain that
\[
|c_n - c_0| \geq |f^n(c_0) - c_0| - \frac{|k|^n - 1}{|k| - 1} \delta_0
\]
\[
= (|k| - 1) |c_0| - \frac{|k|^n - 1}{|k| - 1} \delta_0
\]
\[
= (|k| - 1) \left( |c_0| - \frac{\delta_0}{|k| - 1} \right)
\]
\[
\geq (|k| - 1) \left( \frac{|k| \delta_0}{|k| - 1} - \frac{\delta_0}{|k| - 1} \right)
\]
\[
= (|k| - 1) \delta_0
\]
If \(|c_n - c_0|\) is greater than the diameter of \((\mathbb{C} \setminus D(R)) \setminus R_f(1)\) for all \(n \geq N\), then the escaping time is \(N\). Observe that \(\mathbb{C} \setminus D(R)\) is the disk with radius \(R\). Thus the diameter of \(\mathbb{C} \setminus D(R)\) is twice of the radius. Then the inequality
\[
(|k|^N - 1) \delta_0 > \frac{2 (\sqrt{|k|} + 1)^2}{\sqrt{|k|(|k| - 1)}}
\]
hold for the escaping time \(N\). Hence, \(N\) is the uniformly escaping time satisfying \(N > \log \left\{ \frac{1}{\delta_0} \left( \frac{2 (\sqrt{|k|} + 1)^2}{\sqrt{|k|(|k| - 1)}} + 1 \right) \right\} / \log |k|\).

Remark 6.2. Observe that the inequality \(\frac{2 (\sqrt{|k|} + 1)^3}{\sqrt{|k|(|k| - 1)}} + 1 \leq \left( \frac{\sqrt{|k|} + 1}{\sqrt{|k| - 1}} \right)^3\) for \(|k| > 1\). This inequality suggest a sufficient condition of the uniformly escaping time
\[
N > \log \left\{ \frac{1}{\delta_0} \left( \frac{\sqrt{|k|} + 1}{\sqrt{|k| - 1}} \right)^3 + 1 \right\} / \log |k|.
\]
The sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) is defined as the set each of which element \( a_n = h(c_n) \) for every \( n \in \mathbb{N}_0 \) where \( \{c_n\}_{n \in \mathbb{N}_0} \) is defined in (6.1). Recall that \( f \) is the map \( h \circ g \circ h^{-1} \). Denote the radius of the ball \( h^{-1}(B(c_j, \delta)) \) by \( \varepsilon_j \) for \( j \in \mathbb{N}_0 \). Then the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) as follows

\[
|a_{n+1} - g(a_n)| \leq \varepsilon_n. \quad (6.2)
\]
corresponds \( \{c_n\}_{n \in \mathbb{N}_0} \) by the conjugation \( h \). Then the escaping time of \( \{a_n\}_{n \in \mathbb{N}_0} \) from \( h^{-1}(E_f) \) under \( g \) is the same as that of \( \{c_n\}_{n \in \mathbb{N}_0} \) from \( E_f \) under \( f \) in Lemma 6.1. Furthermore, since \( h \) is uniformly continuous on the closure of the set, \( h^{-1}((\mathbb{C} \setminus D(R)) \setminus \mathcal{R}_f(1)) \) for \( R > \frac{\sqrt{|k|+1}^2}{\sqrt{|k|(|k|-1)}} \) under Euclidean metric, there exists \( \varepsilon > 0 \) such that \( h(B(a_j, \varepsilon)) \subset B(c_j, \delta) \) for \( j = 1, 2, \ldots N_1 \) for all \( c_j \in E_f \). Thus we obtain the following Proposition.

**Proposition 6.3.** Let \( \{c_n\}_{n \in \mathbb{N}_0} \) be the sequence satisfying

\[
|c_{n+1} - f(c_n)| \leq \delta_0
\]

where \( f(w) = kw \) for \( |k| > 1 \) on \( E_f \) defined in Lemma 6.1. Let \( N \) be the (uniformly) escaping time from \( E_f \) under \( f \). Let \( \{a_n\}_{n \in \mathbb{N}_0} \) be the sequence
satisfying \( a_n = h(c_n) \) for every \( n \in \mathbb{N}_0 \). Then there exists \( \varepsilon > 0 \) such that if \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfies that

\[
|a_{n+1} - g(a_n)| \leq \varepsilon
\]

(6.3)
on \( h^{-1}(E_f) \) for \( n = 1, 2, \ldots, N - 1 \), then the escaping time from \( h^{-1}(E_f) \) under \( g \) is also \( N \).

**Remark 6.4.** The set \( h^{-1}(E_f) \) is as follows

\[
h^{-1}((\mathbb{C} \setminus D(R)) \setminus \mathcal{R}_f(1)) = (\mathbb{C} \setminus B_R) \setminus h^{-1}(\mathcal{R}_f(1)) = (\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty).
\]

## 7 Hyers-Ulam stability on the complement of the avoided region

Hyers-Ulam stability of loxodromic Möbius map on the region \( B_R \) where \( R > \frac{(\sqrt{|k|+1})^2}{\sqrt{|k|(|k|-1)}} \) is proved in Proposition 4.4. In this section, we show Hyers-Ulam stability on the region \( (\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty) \) where \( \mathcal{R}_f(\infty) \) is the avoided region defined in Section 3 for the finite time bounded above by the uniformly escaping time. Then combining the stability on complement region and Proposition 4.4 implies Hyers-Ulam stability of \( g \) on the set \( \hat{\mathbb{C}} \setminus \mathcal{R}_g(\infty) \).

**Lemma 7.1.** Let \( D_1 \) be the disk \( \{w: |w - \frac{1}{k}| < \delta\} \). Then the disk \( h^{-1}(D_1) \) is \( \{z: |z + \frac{d}{k|z|} - \frac{k^2}{k-1}|z - \alpha| < \delta\} \). Moreover, the center of the disk \( h^{-1}(D_1) \) is

\[
\frac{|k - 1|^2}{|k - 1|^2 - \delta^2|k|^2} \left( -\frac{d}{c} \right) + \frac{-\delta^2|k|^2 \alpha}{|k - 1|^2 - \delta^2|k|^2}
\]

and the radius is

\[
\frac{\delta|k|\sqrt{|k|}|k - 1|}{|c|||k - 1|^2 + \delta^2|k|^2|}
\]

for small enough \( \delta > 0 \).
Proof. The map \( h(z) = \frac{z - \beta}{z - \alpha} \). Recall that \((c\alpha + d)(c\beta + d) = 1\) by equation (2.2) in Lemma 2.2 and \(k = \frac{1}{(c\beta + d)^2}\) by Lemma 2.3. Lemma 3.1 implies that \([c(\alpha - \beta)]^2 = \frac{(k-1)^2}{k}\). Thus

\[
\frac{w - 1}{k} = \frac{z - \beta - c\beta + d}{z - \alpha - c\alpha + d} = \frac{(z - \beta)(c\alpha + d) - (z - \alpha)(c\beta + d)}{(z - \alpha)(c\alpha + d)} = \frac{(cz + d)(\alpha - \beta)}{(z - \alpha)(c\alpha + d)} = \frac{z - \left(\frac{-d}{c}\right)}{z - \alpha} \cdot \frac{c(\alpha - \beta)}{c\alpha + d}.
\]

Thus the following equation holds

\[
\left| w - \frac{1}{k} \right| = \left| \frac{z - \left(\frac{-d}{c}\right)}{z - \alpha} \right| \left| \frac{k - 1}{|k|} \right|.
\]

Hence, \( |w - \frac{1}{k}| < \delta \) implies that

\[
h^{-1}(D_1) = \left\{ z : \left| z + \frac{d}{c} \right| < \delta \left| \frac{k}{k - 1} \right| \left| z - \alpha \right| \right\}.
\]

The straight line \( \ell(t) = \left\{ t\alpha + (1-t)\left(\frac{-d}{c}\right) : t \in \mathbb{R} \right\} \) goes through the center of \( h^{-1}(D_1) \). Thus the intersection points in \( \ell(t) \) and \( h^{-1}(\partial D_1) \) is the endpoints of the diameter of \( h^{-1}(D_1) \). Solve the following equation

\[
\left| t\alpha + (1-t)\left(\frac{-d}{c}\right) + \frac{d}{c} \right| = \delta \left| \frac{k}{k - 1} \right| \left| t\alpha + (1-t)\left(\frac{-d}{c}\right) - \alpha \right| = \delta \left| \frac{k}{k - 1} \right| |1-t| \left| \alpha + \frac{d}{c} \right| = \delta \left| \frac{k}{k - 1} \right| \left| 1-t \right| \left( \alpha + \frac{d}{c} \right).
\]

Thus \( t = 1 - \frac{|k - 1|}{|k - 1| + \delta|k|} \). Then \( \ell(t) \cap h^{-1}(D_1) \) the set of the following two points

\[
\left\{ \begin{array}{lcl}
\frac{|k - 1|}{|k - 1| + \delta|k|} \left( \frac{-d}{c} \right) + \frac{\delta|k|}{|k - 1| + \delta|k|} \alpha \\
\frac{|k - 1|}{|k - 1| - \delta|k|} \left( \frac{-d}{c} \right) - \frac{\delta|k|}{|k - 1| - \delta|k|} \alpha.
\end{array} \right.
\]
Since the points in (7.1) are the endpoints of the diameter of the disk $h^{-1}(D_1)$, the center is the arithmetic average of them. The radius is the half of the distance between two end points. Then the center is

$$\frac{1}{2} \left\{ \left( \frac{|k-1|}{|k-1|+\delta|k|} + \frac{|k-1|}{|k-1|-\delta|k|} \right) \left( -\frac{d}{c} \right) \right. $$

$$+ \left. \left( \frac{\delta|k|}{|k-1|+\delta|k|} - \frac{\delta|k|}{|k-1|-\delta|k|} \right) \alpha \right\}$$

$$= \frac{|k-1|^2}{|k-1|^2 - \delta^2|k|^2} \left( -\frac{d}{c} \right) + \frac{-\delta^2|k|^2}{|k-1|^2 - \delta^2|k|^2} \alpha$$

Assume that $\delta < |k^{-1}|$. The radius is

$$\frac{1}{2} \left| \left( \frac{|k-1|}{|k-1|+\delta|k|} - \frac{|k-1|}{|k-1|-\delta|k|} \right) \left( -\frac{d}{c} \right) \right. $$

$$+ \left. \left( \frac{\delta|k|}{|k-1|+\delta|k|} + \frac{\delta|k|}{|k-1|-\delta|k|} \right) \alpha \right|$$

$$= \frac{\delta|k||k-1|}{|k-1|^2 - \delta^2|k|^2} \left| \frac{d}{c} + \alpha \right|$$

$$= \frac{\delta|k|\sqrt{|k||k-1|}}{|c|(|k-1|^2 + \delta^2|k|^2)}.$$

Lemma 7.2. The avoided region $h^{-1}(R_f(1))$ for small enough $\delta > 0$ contains the disk

$$D_{\varepsilon_0} = \left\{ z : \left| z + \frac{d}{c} \right| < \varepsilon_0 \right\}$$

where $\varepsilon_0 \leq \frac{\delta|k|^2}{2|k-1|^2} |\alpha - \beta|.$

Proof. The avoided region $R_f(1)$ contains the disk $D_1 = \{ w : |w - \frac{1}{k}| < \delta \}$ by the definition of the avoided region. Since $h^{-1}(\frac{1}{k}) = -\frac{d}{c}$, the avoided region $h^{-1}(R_f(1))$ contains a disk of which center is $-\frac{d}{c}$. Moreover, $D_{\varepsilon_0} \subset h^{-1}(D_1)$ if and only if

$$\varepsilon_0 + \text{dist}(\text{center of } D_1, \text{ center of } D_{\varepsilon_0}) \leq \text{radius of } D_1.$$

(7.2)
Thus the distance between $-\frac{d}{c}$ and the center of $h^{-1}(D_1)$ is as follows

$$\begin{align*}
&\left| \frac{|k - 1|^2}{|k - 1|^2 - \delta^2|k|^2} \left( -\frac{d}{c} \right) \right| + \frac{-\delta^2|k|^2}{|k - 1|^2 - \delta^2|k|^2} \frac{-\delta}{c} - \left( -\frac{d}{c} \right) \\
&= \left| \frac{\delta^2|k|^2}{|k - 1|^2 - \delta^2|k|^2} \right| \frac{d}{c} + \alpha \\
&= \frac{\delta^2|k|^2 \sqrt{|k|}}{|c|(|k - 1|^2 - \delta^2|k|^2)^2}. \tag{7.3}
\end{align*}$$

Recall that $|c||\alpha - \beta| = \frac{|k - 1|}{\sqrt{|k|}}$. The inequality in (7.2) and the radius of $h^{-1}(D_1)$ in Lemma 7.1 implies that

$$\varepsilon_0 \leq \frac{\delta|k| \sqrt{|k|}}{|c|(|k - 1|^2 + \delta^2|k|^2)} \left( \frac{\delta^2|k|^2 \sqrt{|k|}}{|c|(|k - 1|^2 - \delta^2|k|^2)^2} \right)$$

$$= \frac{\delta|k| \sqrt{|k|}}{|c|(|k - 1| + \delta|k|)}.$$ 

Moreover, since we may choose $\delta < \frac{|k - 1|}{k}$, the disk $D_{\varepsilon_0}$ is contained in $h^{-1}(D_1)$ where $\varepsilon_0 \leq \frac{\delta|k|^2}{2|k - 1|^2} |\alpha - \beta|$. \Box

Denote $h^{-1}(R_f(1))$ by $R_g(\infty)$, which is called the avoided region of $g$ at $\infty$.

**Corollary 7.3.** For every $z \in \mathbb{C} \setminus R_g(\infty)$, the following inequality holds

$$|g'(z)| \leq \frac{4|k - 1|^2}{\delta^2|k|^3}.$$ 

**Proof.** $R_g(\infty)$ contains $D_{\varepsilon_0}$ by Lemma 7.2. Recall that $g'(z) = \frac{1}{(cz + d)^2}$. In the region $\mathbb{C} \setminus D_{\varepsilon_0}$, the inequality $|cz + d| \geq |c|\varepsilon_0$ holds. The set inclusion
\( D_\varepsilon \subset R_g(\infty) \) holds even if we choose the maximum value of \( \varepsilon_0 \). Thus the upper bound of \(|g'|\) is as follows

\[
|g'(z)| = \frac{1}{|cz + d|^2} \leq \frac{1}{|c|^2 \varepsilon_0^2} = \frac{4|k - 1|^4}{|c|^2 \delta^2 k^4 |\alpha - \beta|^2}
\]

by Lemma 7.2. Recall the equation \(|c||\alpha - \beta| = \frac{|k - 1|}{\sqrt{|k|}}\). Hence,

\[
\frac{4|k - 1|^4}{|c|^2 \delta^2 k^4 |\alpha - \beta|^2} = \frac{4|k - 1|^4}{\delta^2 |k|^4}, \quad \frac{|k|}{|k - 1|^2} = \frac{4|k - 1|^2}{\delta^2 |k|^3}.
\]

The following is the mean value inequality for holomorphic function.

**Lemma 7.4.** Let \( g \) be the holomorphic function on the convex open set \( B \) in \( \mathbb{C} \). Suppose that \( \sup_{z \in B} |g'| < \infty \). Then for any two different points \( u \) and \( v \) in \( B \), we have

\[
\frac{|g(u) - g(v)|}{u - v} \leq 2 \sup_{z \in B} |g'|.
\]

*Proof.* The complex mean value theorem implies that

\[
\text{Re}(g'(p)) = \text{Re} \left( \frac{g(u) - g(v)}{u - v} \right) \quad \text{and} \quad \text{Im}(g'(q)) = \text{Im} \left( \frac{g(u) - g(v)}{u - v} \right)
\]

where \( p \) and \( q \) are in the line segment between \( u \) and \( v \). Hence, the inequality

\[
|\text{Re}(g'(p)) + i \text{Im}(g'(q))| \leq |\text{Re}(g'(p))| + |\text{Im}(g'(q))| \leq 2 \sup_{z \in B} |g'|
\]

completes the proof. \( \square \)

**Proposition 7.5.** Let \( g \) be the loxodromic Möbius map. For a given \( \varepsilon > 0 \), let a complex valued sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfies the inequality

\[
|a_{n+1} - g(a_n)| \leq \varepsilon
\]
for all $n \in \mathbb{N}_0$. Suppose that $a_0 \in (\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty)$ where $\mathcal{R}_g(\infty)$ is the avoided region of $g$ at $\infty$ defined as $h^{-1}(\mathcal{R}_f(1))$ for $\frac{\left(\sqrt{|k|}+1\right)^2}{\sqrt{|k||(|k|-1)|}}$. Then there exists the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ defined as

$$b_{n+1} = g(b_n)$$

for each $n = 0, 1, 2, \ldots, N$ which satisfies that

$$|a_N - b_N| \leq \frac{M^N - 1}{M - 1} \varepsilon$$

where $N$ is the uniformly escaping time from the region $(\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty)$ and $M = \frac{8|k-1|^2}{\delta^2|k|^3}$.

Proof. $\frac{M}{2}$ is an upper bound of $|g'|$ in $\mathbb{C} \setminus \mathcal{R}_g(\infty)$ by Corollary 7.3. The triangular inequality and Lemma 7.4 implies that

$$|a_N - b_N| \leq |a_N - g(a_{N-1})| + |g(a_{N-1}) - g(b_{N-1})| + |g(b_{N-1}) - b_N| \quad (7.5)$$

$$\leq \varepsilon + M |a_{N-1} - b_{N-1}| \quad (7.6)$$

where $M \geq \sup_{z \in \mathbb{C} \setminus \mathcal{R}_g(\infty)} 2|g'|$. The region $B_R \subset S(1)$ by Corollary 3.6. $|g'(z)| > 1$ if and only if $z \in S(1)$. Thus $M > 1$. The inequality $(7.5)$ implies that

$$|a_N - b_N| + \frac{\varepsilon}{M - 1} \leq M \left(|a_{N-1} - b_{N-1}| + \frac{\varepsilon}{M - 1}\right).$$

Then $|a_N - b_N|$ is bounded above by the geometric sequence with rate $M$, that is,

$$|a_N - b_N| \leq M^N \left(|a_0 - b_0| + \frac{\varepsilon}{M - 1}\right) - \frac{\varepsilon}{M - 1}$$

$$= M^N |a_0 - b_0| + \frac{M^N - 1}{M - 1} \varepsilon.$$  

Hence, if we choose $b_0 = a_0$, then

$$|a_N - b_N| \leq \frac{M^N - 1}{M - 1} \varepsilon.$$
We show the Hyers-Ulam stability of loxodromic Möbius map outside the avoided region.

**Theorem 7.6.** Let $g$ be a loxodromic Möbius map. For a given $\varepsilon > 0$, let a complex valued sequence $\{a_n\}_{n \in \mathbb{N}_0}$ satisfies the inequality

$$|a_{n+1} - g(a_n)| \leq \varepsilon$$

for all $n \in \mathbb{N}_0$. Suppose that a given point $a_0 \in \mathbb{C} \setminus \mathcal{R}_g(\infty)$ where $\mathcal{R}_g(\infty)$ is the avoided region defined in Section 5. For a small enough number $\varepsilon > 0$, there exists the sequence $\{b_n\}_{n \in \mathbb{N}_0}$

$$b_{n+1} = g(b_n)$$

satisfies that $|a_n - b_n| \leq H(\varepsilon)$ for all $n \in \mathbb{N}_0$ where the positive number $H(\varepsilon)$ converges to zero as $\varepsilon \to 0$.

**Proof.** Suppose first that $a_0 \in B_R$. Then by Proposition 4.4, we have the inequality

$$|b_n - a_n| \leq \frac{1 - K^n}{1 - K} \varepsilon$$

(7.7)

for some $K < 1$ where $b_0 = a_0$. Assume also that $a_0 \in (\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty)$ and $N \geq n$ where $N$ is the escaping time. Then by Proposition 7.5

$$|b_n - a_n| \leq \frac{M^n - 1}{M - 1} \varepsilon$$

(7.8)

where $M = \frac{8|k-1|^2}{\delta^2|k|^2}$. Suppose that $a_0 \in (\mathbb{C} \setminus B_R) \setminus \mathcal{R}_g(\infty)$ but $n > N$ for the last case. Then we combine the first and second case as follows

$$|b_n - a_n| \leq \left( \frac{M^n - 1}{M - 1} + \frac{1 - K^{n-N}}{1 - K} \right) \varepsilon$$

(7.9)

where $K$ and $M$ are the numbers used in the inequality (7.7) and (7.8).

**Conclusion and final note**

Let $\{b_n\}_{n \in \mathbb{N}_0}$ be the sequence satisfying the equation $b_{n+1} = g(b_n)$ for all $n \in \mathbb{N}_0$ where $g$ is the loxodromic Möbius map on $\hat{\mathbb{C}}$. Then the sequence has
Hyers-Ulam stability on the exterior of some neighborhood of $g^{-n}(\infty)$ for all $n \in \mathbb{N}$. Since any hyperbolic Möbius map is also loxodromic, the results and proofs in this paper could be also applied to the difference equation with the hyperbolic Möbius map. All figures in the paper are based on the purely loxodromic Möbius map

$$g(z) = \frac{1.64z - 25 + 11.07i}{0.04z + 0.27i}.$$ 

Thus the fixed points of $g$ are $\alpha = 25 + 12i$ and $\beta = 16 - 18.75i$. The trace of $g$ is $1.64 + 0.27i$ and $k = 1.5625 + 1.5i$.

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