Noncommutative Kähler Structures on Quantum Homogeneous Spaces

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Abstract

Building on the theory of noncommutative complex structures, the notion of a noncommutative Kähler structure is introduced. In the quantum homogeneous space case many of the fundamental results of classical Kähler geometry are shown to follow from the existence of such a structure, allowing for the definition of noncommutative Lefschetz, Hodge, Kähler–Dirac, and Laplace operators. Quantum projective space, endowed with its Heckenberger–Kolb calculus, is taken as the motivating example. The general theory is then used to show that the calculus has cohomology groups of at least classical dimension.

1 Introduction

One of the most exciting new trends in noncommutative geometry is the search for a theory of noncommutative complex geometry [17, 1, 31]. It is motivated by the appearance of noncommutative complex structures in a number of areas of noncommutative geometry, such as the construction of spectral triples for quantum groups [20, 7, 2], geometric representation theory for quantum groups [16, 25, 17, 18], the interaction of noncommutative geometry and noncommutative projective algebraic geometry [17, 18, 1], the Baum–Connes conjecture for quantum groups [37, 38], and the application of topological algebras to quantum group theory [34, 33]. While there have been a number of occurrences of Kähler phenomena in the literature, the question of whether metrics have a role to play in noncommutative complex geometry remains largely unexplored. Given the richness and beauty of classical Kähler geometry, the idea that some of its structure might generalise to the noncommutative setting is an enticing one.

To date there has been just one proposed framework for noncommutative Kähler geometry [10]. It uses a Riemannian, as opposed to spin, approach to spectral triples, and takes as its motivating example the noncommutative torus. In this paper we instead take the quantum flag manifolds as a motivating family of examples, and adopt

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an approach based on Woronowicz’s notion of a differential calculus [40]. Classically
the flag manifolds play a central role in Kähler and parabolic geometry [5], and as such,
their $q$-deformations serve as outstanding candidates for noncommutative Kähler spaces.
Moreover, the subfamily of irreducible quantum flag manifolds comes endowed with a
differential calculus, the Heckenberger–Kolb calculus, which is uniquely characterised
by a simple set of natural axioms [13, 14]. These calculi have already served as the
motivating examples for the theory of noncommutative complex structures [1, 17, 31].

Metric phenomena have appeared a number of times in the literature on the noncom-
mutative geometry of the quantum flag manifolds. A Hodge map for the Podleš sphere
was defined by Majid in [25], and the induced Laplace and Dirac operators studied.
Hodge maps on the Podleš sphere and $\mathbb{C}_q[\mathbb{P}^2]$ were examined by Landi, Zampini, and
D’Andrea in the series of papers [8, 23, 43]. A Kähler–Dirac operator for the irreducible
quantum flag manifolds was introduced by Krähmer in [20] and used to give a commu-
tator presentation of the Heckenberger–Kolb calculus. This operator was reconstructed
by Dąbrowski, Landi, and D’Andrea in [6, 7] for the special case of $\mathbb{C}_q[\mathbb{P}^n]$ and was
shown to satisfy the properties of a spectral triple. Finally, a direct $q$-deformation of the
Kähler form of $\mathbb{P}^2$ was constructed in [8].

Inspired by such phenomena, this paper introduces a general framework for noncom-
mutative Kähler geometry on quantum homogeneous spaces and applies it to quantum
projective space. The manner in which this is done has three main sources of inspira-
tion: The first is Majid’s frame bundle approach to noncommutative geometry [7, 24, 25]
which also underpins the author’s earlier papers [30, 31]. The second is Kustermanns,
Murphy, and Tuset’s approach to noncommutative Hodge theor
y [22], and the third is
the presentation of classical Kähler geometry found in [39] and [15], which is both global
and algebraic in style.

The major obstacle to formulating a coherent construction of metrics in the noncom-
mutative setting is that the classical extension of metrics from 1-forms to higher forms
does not easily generalise. The classical extension uses anti-commutativity of forms in
a fundamental way, while the multiplicative relations for a differential calculus over a
noncommutative algebra will in general be much more badly behaved. In certain cases,
such as bicovariant calculi [40], or braided complex structures [21], one can formulate a
braided generalisation of the classical construction [12]. However, in practice the metrics
produced are not ideal [21].

For Kähler manifolds, however, we show that it is possible to reverse the usual order of
construction and build a Hodge map from a Kähler form and then use this to extend the
metric. Adopting this viewpoint in the noncommutative setting produces a simple set
of criteria for a 2-form, which when satisfied, gives a coherent system for constructing
Hodge maps, metrics, codifferentials, and Dirac operators. Moreover, it also produces
noncommutative generalisations of classical Kähler phenomena that have not before
appeared in the literature: Lefschetz decomposition, the Lefschetz identities, Hodge
decomposition, and the Kähler identities.

This is the first of a series of papers. In subsequent works we will enlarge the family
of examples [28, 29], investigate the analytic properties of some of the associated Dirac operators [29], and investigate how the classical rules of Schubert calculus behave under $q$-deformation.

The paper is organised as follows: In Section 2 some well-known material is introduced about Hopf algebras, quantum homogeneous spaces, and in particular quantum projective space. In Section 3 the theory of covariant differential calculi is recalled, as is the more recent notion of a complex structure. The construction of the Heckenberger–Kolb calculus for $\mathbb{C}_q[\mathbb{C}P^n]$ is also recalled, and some basic results about it presented.

In Section 4 symplectic and Hermitian structures are introduced. A noncommutative generalisation of Lefschetz decomposition is proved in Proposition 4.3 allowing for the definition of a Hodge map in Definition 4.11. A $q$-deformation of the fundamental form of the classical Fubini–Study metric for complex projective space is then constructed, and a generalised version for the irreducible quantum flag manifolds is conjectured.

In Section 5 the construction of positive definite metrics from Hermitian forms is presented. Adjointability of $G$-comodule maps with respect to such metrics is then established in Corollary 5.8, and presentations of the codifferential and dual Lefschetz operators in terms of the Hodge map given. Finally, in Proposition 5.13 a deformed version of the Lefschetz identities is proved.

In Section 6, Hodge decomposition with respect to the holomorphic and anti-holomorphic derivatives is established, and shown to imply an isomorphism between cohomology classes and harmonic forms just as in the classical case. A noncommutative generalisation of Serre duality is also established.

In Section 7 the definition of a noncommutative Kähler structure is given and some of the basic results of classical geometry generalised, most notably the Kähler identities in Theorem 7.5. Equality of the three Laplacians up to scalar multiple follows in Corollary 7.6, implying in turn that Dolbeault cohomology refines de Rham cohomology. Finally, a noncommutative generalisation of the hard Lefschetz theorem and the $\bar{\partial}\partial$-lemma is given. The Hermitian form $\mathbb{C}_q[\mathbb{C}P^n]$ is then observed to be Kähler, implying that the Heckenberger–Kolb calculus has cohomology groups of at least classical dimension. We finish with some spectral calculations and a conjecture about constructing spectral triples from Kähler structures for the irreducible quantum flag manifolds.

Throughout the paper we endeavour to present the derivation of all results as explicitly as possible, so as to make the paper accessible to a noncommutative geometry audience not necessarily familiar with classical complex geometry.

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2 Preliminaries on Quantum Homogeneous Spaces

In this section we introduce some well-known material about cosemisimple Hopf algebras, quantum homogeneous spaces, and Takeuchi’s categorical equivalence. The motivating example, quantum projective space, is also introduced.

2.1 Compact Quantum Group Algebras

Let $G$ be a Hopf algebra with comultiplication $\Delta$, counit $\varepsilon$, antipode $S$, unit 1, and multiplication $m$. Throughout, we use Sweedler notation, and denote $g^+ := g - \varepsilon(g)1$, for $g \in G$, and $V^+ = V \cap \ker(\varepsilon)$, for $V$ a subspace of $G$.

For any left $G$-comodule $(V, \Delta_L)$, its space of matrix elements is the coalgebra

$$C(V) := \text{span}_\mathbb{C}\{(id \otimes f)\Delta_L(v) \mid f \in \text{Lin}_\mathbb{C}(V, \mathbb{C}), v \in V\} \subseteq G.$$ 

A comodule is irreducible if and only if its coalgebra of matrix elements is irreducible, and, for $W$ another left $G$-comodule, $C(V) = C(W)$ if and only if $V$ is equivalent to $W$. Moreover, $C(V)$ decomposes as a left $G$-comodule into $\dim C(V)$ copies of $V$.

The notion of cosemisimplicity for a Hopf algebra will be essential in the paper and all Hopf algebras will be assumed to have the property. We present three equivalent formulations of the definition (a proof of their equivalence can be found in [19, §11.2.1]).

Definition 2.1. A Hopf algebra $G$ is called cosemisimple if it satisfies the following three equivalent conditions:

1. It holds that $G \simeq \bigoplus_{V \in \hat{G}} C(V)$, where summation is over all equivalence classes of left $G$-comodules.

2. Every comodule of $G$ is a direct sum of (necessarily finite) irreducible comodules.

3. There exists a unique linear map $h : G \to \mathbb{C}$, which we call the Haar functional, such that $h(1) = 1$, and

$$(id \otimes h)\Delta(g) = h(g)1, \quad (h \otimes id)\Delta(g) = h(g)1.$$ 

While the assumption of cosemisimplicity is enough for most of our requirements, we will need something stronger when discussing positive definiteness in §5.

Definition 2.2. A compact quantum group algebra is a cosemisimple Hopf $*$-algebra such that $h(aa^*) > 0$, for all $a \neq 0$.

The condition of $G$ being a compact quantum group algebra is equivalent to it being the dense Hopf algebra of a compact quantum group [41], Woronowicz’s celebrated structure in the $C^*$-algebraic approach to quantum groups [40].
For any compact quantum group algebra, an inner product is given by the map
\[ G \otimes G \to \mathbb{C}, \quad g \otimes f \mapsto h(f^* g). \] (1)

Moreover, with respect to this inner product, the decomposition \( G \cong \bigoplus_{V \in \hat{G}} \mathcal{C}(V) \) is orthogonal.

### 2.2 Quantum Homogeneous Spaces

For a right \( G \)-comodule \( V \) with coaction \( \Delta_R \), we say that an element \( v \in V \) is coinvariant if \( \Delta_R(v) = v \otimes 1 \). We denote the subspace of all coinvariant elements by \( V^G \), and call it the coinvariant subspace of the coaction. We also use the analogous conventions for left comodules.

**Definition 2.3.** For \( H \) a Hopf algebra, a homogeneous right \( H \)-coaction on \( G \) is a coaction of the form \((\text{id} \otimes \pi)\Delta\), where \( \pi : G \to H \) is a surjective Hopf algebra map. A quantum homogeneous space \( M := G^H \) is the coinvariant subspace of such a coaction.

In this paper we will always use the symbols \( G, H, \pi \) and \( M \) in this sense. As is easily seen, \( M \) is a subalgebra of \( G \). Moreover, if \( G \) and \( H \) are Hopf \(*\)-algebras, and \( \pi \) is a Hopf \(*\)-algebra map, then \( M \) is a \(*\)-subalgebra of \( G \).

Our assumption of cosemisimplicity for Hopf algebras implies that \( G \) is faithfully flat over \( M \) [35, Theorem 5.1.6]. Recall that \( G \) is said to be faithfully flat as a right module over \( M \) if the functor \( G \otimes_M - : M\text{Mod} \to G\text{Mod} \), from the category of left \( M \)-modules to the category of complex vector spaces, maps a sequence to an exact sequence if and only if the original sequence is exact. This is necessary in particular for the categorical equivalence of the next section to hold.

### 2.3 Takeuchi’s Categorical Equivalence

We now define the abelian categories \( G^M\text{Mod}_0 \) and \( H\text{Mod}_0 \). The objects in \( G^M\text{Mod}_0 \) are \( M \)-bimodules \( E \) (with left and right actions denoted by juxtaposition) endowed with a left \( G \)-coaction \( \Delta_L \) such that \( EM^+ \subseteq M^+ E \), and
\[ \Delta_L(mem') = m_{(1)}e_{(-1)}m'_{(1)} \otimes m_{(2)}e_{(0)}m'_{(2)}, \quad \text{for all } m, m' \in M, e \in E. \] (2)

The morphisms in \( G^M\text{Mod}_0 \) are those \( M \)-bimodule homomorphisms which are also homomorphisms of left \( G \)-comodules. The objects in \( H\text{Mod}_0 \) are left \( H \)-comodules \( V \) endowed with the trivial right \( M \)-action \((v, m) \mapsto \varepsilon(m)v \). The morphisms in \( H\text{Mod}_0 \) are the left \( H \)-comodule maps. (Note that \( H\text{Mod}_0 \) is equivalent under the obvious forgetful functor to \( H\text{Mod} \), the category of left \( H \)-comodules.)

If \( E \in G^M\text{Mod}_0 \), then \( E/(M^+ E) \) becomes an object in \( H\text{Mod}_0 \) with the left \( H \)-coaction
\[ \Delta_L[e] = \pi(e_{(-1)}) \otimes [e_{(0)}], \quad e \in E. \] (3)
where \([e]\) denotes the coset of \(e\) in \(\mathcal{E}/(M^+\mathcal{E})\). We define a functor

\[ \Phi : G^M\text{Mod}_0 \to H\text{Mod}_0 \]

as follows: \(\Phi(\mathcal{E}) := \mathcal{E}/(M^+\mathcal{E})\), and if \(g : \mathcal{E} \to \mathcal{F}\) is a morphism in \(G^M\text{Mod}_0\), then \(\Phi(g) : \Phi(\mathcal{E}) \to \Phi(\mathcal{F})\) is the map to which \(g\) descends on \(\Phi(\mathcal{E})\).

If \(V \in H\text{Mod}_0\), then the cotensor product of \(G\) and \(V\), defined by

\[ G \square_H V := \ker(\Delta_R \otimes \text{id} - \text{id} \otimes \Delta_L : G \otimes V \to G \otimes H \otimes V), \]

becomes an object in \(G^M\text{Mod}_0\) by defining an \(M\)-bimodule structure

\[ m\left( \sum_i g^i \otimes v^i \right) = \sum_i m g^i \otimes v^i, \quad \left( \sum_i g^i \otimes v^i \right)m = \sum_i g^i m \otimes v^i, \quad (4) \]

and a left \(G\)-coaction

\[ \Delta_L \left( \sum_i g^i \otimes v^i \right) = \sum_i g^i_{(1)} \otimes g^i_{(2)} \otimes v^i. \]

We define a functor \(\Psi : H\text{Mod}_0 \to G^M\text{Mod}_0\) as follows: \(\Psi(V) := G \square_H V\), and if \(\gamma\) is a morphism in \(H\text{Mod}_0\), then \(\Psi(\gamma) := \text{id} \otimes \gamma\).

**Theorem 2.4** [36, Theorem 1] An equivalence of the categories \(G^M\text{Mod}_0\) and \(H\text{Mod}_0\), which we call Takeuchi’s equivalence, is given by the functors \(\Phi\) and \(\Psi\) and the natural transformations

\[ C : \Phi \circ \Psi(V) \to V, \quad \left[ \sum_i g^i \otimes v^i \right] \mapsto \sum_i \varepsilon(g^i) v^i, \quad (5) \]

\[ U : \mathcal{E} \to \Psi \circ \Phi(\mathcal{E}), \quad e \mapsto e_{(-1)} \otimes [e_{(0)}]. \quad (6) \]

**Corollary 2.5** Takeuchi’s equivalence restricts to an equivalence of categories between \(G^M\text{mod}_0\) and \(H\text{mod}_0\), where \(G^M\text{mod}_0\) is the full subcategory of \(G^M\text{Mod}_0\) consisting of finitely generated left \(M\)-modules, and \(H\text{mod}_0\) is the full subcategory of \(H\text{Mod}_0\) consisting of finite-dimensional comodules.

**Proof.** We begin by recalling the well-known [36, §1] isomorphism

\[ G \otimes_M \mathcal{E} \to G \otimes \Phi(\mathcal{E}), \quad g \otimes_M e \mapsto ge_{(-1)} \otimes [e_{(0)}]. \]

This implies that, for any \(V \in H\text{mod}_0\), we have that \(G \otimes_M \Psi(V)\) is finitely generated as a left \(G\)-module. Now cosemisimplicity of \(H\) implies that there exists a projection \(\Phi(G) \to \Phi(M)\), and so, we have a projection \(\rho : G \to M\). The image of \(G \otimes_M \mathcal{E}\) under \(m(\rho \otimes \text{id})\) is isomorphic to \(\mathcal{E}\) which we now see to be finitely generated. The proof of the converse is elementary. \(\Box\)
We define the *dimension* of an object $\mathcal{E} \in G^M_{\text{mod} 0}$ to be the vector space dimension of $\Phi(\mathcal{E})$. Note that by cosemisimplicity of $G$, the abelian category $H_{\text{mod} 0}$ is semisimple, and so, $G^M_{\text{mod} 0}$ is semisimple.

For $\mathcal{E}, \mathcal{F}$ two objects in $G^M_{\text{mod} 0}$, we denote by $\mathcal{E} \otimes_M \mathcal{F}$ the usual bimodule tensor product endowed with the standard left $G$-comodule structure. It is easily checked that $\mathcal{E} \otimes_M \mathcal{F}$ is again an object in $G^M_{\text{mod} 0}$, and so, the tensor product $\otimes_M$ gives the category a monoidal structure. With respect to the obvious monoidal structure on $H_{\text{mod} 0}$, Takeuchi’s equivalence is given the structure of a monoidal equivalence (see [31, §4] for details) by the morphisms

$$\mu_{\mathcal{E}, \mathcal{F}} : \Phi(\mathcal{E}) \otimes \Phi(\mathcal{F}) \rightarrow \Phi(\mathcal{E} \otimes_M \mathcal{F}), \quad [e] \otimes [f] \mapsto [e \otimes_M f], \quad \text{for } \mathcal{E}, \mathcal{F} \in G^M_{\text{mod} 0}.$$ 

In what follows, this monoidal equivalence will be tacitly assumed.

Finally, we note that, for any $\mathcal{E} \in G^M_{\text{mod} 0}$, the following decomposition exists:

$$\mathcal{E} \simeq G \square_H \Phi(\mathcal{E}) \simeq \bigoplus_{V \in \hat{G}} (\mathbb{C}(V) \square_H \Phi(\mathcal{E})) =: \bigoplus_{V \in \hat{G}} \mathcal{E}_V.$$

We call this the Peter–Weyl decomposition of $\mathcal{E}$.

### 2.4 Quantum Projective Space

We recall the definition of the well-known quantum coordinate algebras $\mathbb{C}_q[U_n]$ and $\mathbb{C}_q[SU_n]$, as well as the definition of quantum projective space, the motivating example considered throughout the paper. We finish with a discussion of weight decomposition for $\mathbb{C}_q[U_n]$-comodules, an important tool in what follows.

#### 2.4.1 The Quantum Groups $\mathbb{C}_q[U_n]$ and $\mathbb{C}_q[SU_n]$

We begin by fixing notation and recalling the various definitions and constructions needed to introduce the quantum unitary group and the quantum special unitary group. (Where proofs or basic details are omitted we refer the reader to [19, §9.2].)

For $q \in \mathbb{R}_{>0}$, let $\mathbb{C}_q[GL_n]$ be the quotient of the free noncommutative algebra $\mathbb{C}\langle u_{ij}, \det_n^{-1} \mid i, j = 1, \ldots, n \rangle$ by the ideal generated by the elements

$$u_i^k u_j^l - u_j^k u_i^l, \quad u_i^k u_j^l - q u_j^k u_i^l, \quad 1 \leq i < j \leq n, 1 \leq k \leq n;$$

$$u_i^j u_k^l - u_k^l u_i^j, \quad u_i^j u_k^l - u_k^j u_i^l - (q - q^{-1}) u_k^l u_i^l, \quad 1 \leq i < j \leq n, 1 \leq k < l \leq n;$$

$$\det_n \det_n^{-1} - 1, \quad \det_n^{-1} \det_n - 1,$$

where $\det_n$, the *quantum determinant*, is the element

$$\det_n := \sum_{\pi \in S_n} (-q)^{f(\pi)} u_{\pi(1)}^1 u_{\pi(2)}^2 \cdots u_{\pi(n)}^n,$$
with summation taken over all permutations \( \pi \) of the set \( \{1, \ldots, n\} \), and \( \ell(\pi) \) is the number of inversions in \( \pi \). As is well-known [19, §9.2.2], \( \det_n \) is a central element of the algebra.

A bialgebra structure on \( C_q[GL_n] \) with coproduct \( \Delta \), and counit \( \varepsilon \), is uniquely determined by \( \Delta(u_i^j) := \sum_{k=1}^n u_k^i \otimes u_j^k; \Delta(\det_n^{-1}) = \det_n^{-1} \otimes \det_n^{-1}; \) and \( \varepsilon(u_i^j) := \delta_{ij}; \varepsilon(\det_n^{-1}) = 1. \) The element \( \det_n \) is grouplike with respect to \( \Delta \) [19, §9.2.2]. Moreover, we can endow \( C_q[GL_n] \) with a Hopf algebra structure by defining

\[
S(\det_n^{-1}) := \det_n, \quad S(u_i^j) := (-q)^{i-j} \sum_{\pi \in S_{n-1}} (-q)^{\ell(\pi)} u_{\pi_1(i)}^{k_1} u_{\pi_2(i)}^{k_2} \cdots u_{\pi_{n-1}(i)}^{k_{n-1}} \det_n^{-1},
\]

where \( \{k_1, \ldots, k_{n-1}\} := \{1, \ldots, n\} \setminus \{j\} \), and \( \{l_1, \ldots, l_{n-1}\} := \{1, \ldots, n\} \setminus \{i\} \) as ordered sets. A Hopf \( * \)-algebra structure is determined by \( (\det_n^{-1})^* = \det_n \), and \( (u_i^j)^* = S(u_i^j) \). We denote the Hopf \( * \)-algebra by \( C_q[U_n] \), and call it the quantum unitary group of order \( n \). We denote the Hopf \( * \)-algebra \( C_q[U_n]/\langle \det_n - 1 \rangle \) by \( C_q[SU_n] \), and call it the quantum special unitary group of order \( n \).

### 2.4.2 Quantum Projective Space

Following the description introduced in [27, §3], we present quantum \( n \)-projective space as the subalgebra of coinvariant elements of a \( C_q[U_n] \)-coaction on \( C_q[SU_{n+1}] \). (This subalgebra is a \( q \)-deformation of the coordinate algebra of the complex manifold \( SU_{n+1}/U_n \). Recall that \( \mathbb{C}P^n \) is isomorphic to \( SU_{n+1}/U_n \).

**Definition 2.6.** Let \( \alpha_n : C_q[SU_{n+1}] \to C_q[U_n] \) be the surjective Hopf \( * \)-algebra map defined by setting \( \alpha_n(u_i^1) = \det_n^{-1}, \alpha_n(u_i^i) = \alpha_n(u_i^j) = 0, \) for \( i = 2, \ldots, n+1 \), and \( \alpha_n(u_i^j) = u_i^{j-1}, \) for \( i, j = 2, \ldots, n+1 \). Quantum projective \( n \)-space \( C_q[\mathbb{C}P^n] \) is defined to be the quantum homogeneous space of the corresponding homogeneous coaction \( (\text{id} \otimes \alpha_n) \circ \Delta \).

As is well known [19, §11.6], \( C_q[\mathbb{C}P^n] \) is generated as a \( \mathbb{C} \)-algebra by the set

\[
\{ z_{ab} := u_i^a S(u_i^b) \mid a, b = 1, \ldots, n \}.
\]

### 2.4.3 Weight Vectors for Objects in \( H_{\text{mod} 0} \)

Let \( \mathbb{C}[T^n] \) be the commutative polynomial algebra generated by \( t_k, t_k^{-1} \), for \( k = 1, \ldots, n \), satisfying the obvious relation \( t_k t_k^{-1} = 1 \). We can give \( \mathbb{C}[T^n] \) the structure of a Hopf algebra by defining a coproduct, counit and antipode according to \( \Delta(t_k) := t_k \otimes t_k, \varepsilon(t_k) := 1, \) and \( S(t_k) := t_k^{-1}. \) Moreover, \( \mathbb{C}[T^n] \) has a Hopf \( * \)-algebra structure defined by \( t_k^* := t_k^{-1} \). (Note that \( \mathbb{C}[T^n] \cong C[U_1]. \))

A basis of \( \mathbb{C}[T^n] \) is given by

\[
\{ t_\lambda := t_1^{l_1} \cdots t_n^{l_n} \mid \lambda = (l_1, \ldots, l_n) \in \mathbb{Z}^n \}.
\]
Since each basis element is grouplike, a $\mathbb{C}[T^n]$-comodule structure is equivalent to a $\mathbb{Z}^n$-grading. We call the homogeneous elements of such a grading weight vectors, and we call their degree their weight.

We are interested in $\mathbb{C}[T^n]$ because of the existence of the following map: Let $\tau : \mathbb{C}_q[U_n] \to \mathbb{C}[T^n]$ be the surjective Hopf $*$-algebra map defined by

$$\tau(\det^{-1}_n) := t^{-1}_\bullet, \quad \tau(u^i_j) := \delta_{ij} t_i,$$

where $t_\bullet := t_1 \cdots t_n$. For any left $\mathbb{C}_q[U_n]$-comodule $V$, a left $\mathbb{C}[T^n]$-comodule structure on $V$ is defined by $\Delta_{L,\tau} := (\tau \otimes \text{id}) \Delta_L$.

**Lemma 2.7** For any two objects $D, F \in G^M_{\mathcal{M}\text{-Mod}}$, and $d \in D, f \in F$ weight vectors of weight $w$ and $v$ respectively, then $d \otimes_M f \in D \otimes_M F$ is a weight vector of weight $w + v$.

**Proof.** This follows directly from

$$\Delta_L[e \otimes_M f] = \tau(d(-1)f(-1)) \otimes [d(0) \otimes_M f(0)] = t^{w+v} \otimes [d(0) \otimes_M f(0)].$$

\[ \square \]

3 Preliminaries and Basic Constructions on Differential Calculi and Complex Structures

In this section we recall some well-known definitions from the theory of differential calculi, including material on $*$-calculi, orientability, and integrals. Some more recent material on complex structures is also considered. Finally, a concise presentation of the Heckenberger–Kolb calculus for $\mathbb{C}_q[CP^n]$ is given, and some elementary results on weight space decomposition proved.

3.1 Complexes and Double Complexes

For $(S, +)$ a commutative semigroup, an $S$-graded algebra is an algebra of the form $A = \bigoplus_{s \in S} A^s$, where each $A^s$ is a linear subspace of $A$, and $A^s A^t \subseteq A^{s+t}$, for all $s, t \in S$. If $a \in A^s$, then we say that $a$ is a homogeneous element of degree $s$. A homogeneous mapping of degree $t$ on $A$ is a linear mapping $L : A \to A$ such that if $a \in A^s$, then $L(a) \in A^{s+t}$. We say that a subspace $B$ of $A$ is homogeneous if it admits a decomposition $B = \bigoplus_{s \in S} B^s$, with $B^s \subseteq A^s$, for all $s \in S$.

A pair $(A, d)$ is called a complex if $A$ is an $\mathbb{N}_0$-graded algebra, and $d$ is a homogeneous mapping of degree 1 such that $d^2 = 0$. A triple $(A, \partial, \overline{\partial})$ is called a double complex if $A$ is an $\mathbb{N}_0^2$-graded algebra, $\partial$ is homogeneous mapping of degree $(1, 0)$, $\overline{\partial}$ is homogeneous mapping of degree $(0, 1)$, and

$$\partial^2 = \partial \overline{\partial} = 0, \quad \overline{\partial} \partial = -\partial \overline{\partial}.$$

Note we can associate to any double complex $(A, \partial, \overline{\partial})$ three different complexes

$$\langle A, d := \partial + \overline{\partial} \rangle, \quad \langle A, \partial \rangle, \quad \text{and} \quad \langle A, \overline{\partial} \rangle.$$
where the \( \mathbb{N}_0 \)-grading on \( \mathcal{A} \) is given by \( \mathcal{A}^k := \bigoplus_{a+b=k} \mathcal{A}^{(a,b)} \).

For any complex \((\mathcal{A}, d)\), we call an element \( d \)-closed if it is contained in \( \text{ker}(d) \), and \( d \)-exact if it is contained in \( \text{im}(d) \). Moreover, the \( d \)-cohomology group of order \( k \) is the space
\[
H^k_d := \ker(d : \mathcal{A}^k \to \mathcal{A}^{k+1}) / \text{im}(d : \mathcal{A}^{k-1} \to \mathcal{A}^k).
\]

For a double complex \((\mathcal{A}, \partial, \bar{\partial})\) we define \( \partial \)-closed, \( \partial \)-exact, \( \bar{\partial} \)-closed, and \( \bar{\partial} \)-exact forms analogously. The \( \partial \)-cohomology group \( H^k_\partial \), and the \( \bar{\partial} \)-cohomology group \( H^k_{\bar{\partial}} \), are the cohomology groups of the complexes \((\mathcal{A}, \partial)\) and \((\mathcal{A}, \bar{\partial})\) respectively, where the gradings are the obvious ones.

### 3.1.1 Differential \(*\)-Calculi

A complex \((\mathcal{A}, d)\) is called a differential graded algebra if \( d \) is a graded derivation, which is to say, if it satisfies the graded Leibniz rule
\[
d(\alpha \beta) = d(\alpha) \beta + (-1)^k \alpha d \beta, \quad \text{for all } \alpha \in \mathcal{A}^k, \beta \in \mathcal{A}.
\]

The operator \( d \) is called the differential of the differential graded algebra.

**Definition 3.1.** A differential calculus over an algebra \( A \) is a differential graded algebra \((\Omega^\bullet, d)\) such that \( \Omega^0 = A \), and
\[
\Omega^k = \text{span}_C \{ a_0 da_1 \land \cdots \land da_k | a_0, \ldots, a_k \in A \}.
\]

We use \( \land \) to denote the multiplication between elements of a differential calculus when both are of order greater than 0. We call an element of a differential calculus a form. A differential map between two differential calculi \((\Omega^\bullet, d_\Omega)\) and \((\Gamma^\bullet, d_\Gamma)\), defined over the same algebra \( A \), is a bimodule map \( \varphi : \Omega^\bullet \to \Gamma^\bullet \) such that \( \varphi \circ d_\Omega = d_\Gamma \).

We call a differential calculus \((\Omega^\bullet, d)\) over a \(*\)-algebra \( A \) a \(*\)-differential calculus if the involution of \( A \) extends to an involutive conjugate-linear map on \( \Omega^\bullet \), for which \( (d\omega)^* = d\omega^* \), for all \( \omega \in \Omega \), and
\[
(\omega \land \nu)^* = (-1)^{kl} \nu^* \land \omega^*, \quad \text{for all } \omega \in \Omega^k, \nu \in \Omega^l.
\]

We say that a form \( \omega \in \Omega^\bullet \) is real if \( \omega^* = \omega \).
A differential calculus $\Omega^\bullet$ over a quantum homogeneous space $M$ is said to be covariant if $\Delta_L : M \to G \otimes M$ extends to a necessarily unique algebra map $\Delta_L : \Omega^\bullet \to G \otimes \Omega^\bullet$ such that

$$\Delta_L(mdn) = \Delta_L(m)((1) \otimes d \circ \Delta_L(n)) = m_{(1)}n_{(1)} \otimes m_{(2)}d_{n_{(2)}}, \quad m, n \in M.$$ 

In this paper, all covariant calculi will be assumed to be finite-dimensional and to satisfy $\Omega^\bullet M^+ \subseteq M^+\Omega^\bullet$, giving $\Omega^\bullet$ the structure of an object in $G^M_{\text{Mod}}$. This implies that a multiplication is defined on $\Phi(\Omega^\bullet)$ by $[\omega] \wedge [\nu] := [\omega \wedge \nu]$. It follows from (7) that every element of $\Phi(\Omega^k)$ is a sum of elements of the form $[\omega_1] \wedge \cdots \wedge [\omega_k]$, for $\omega_i \in \Omega^1$. When working with covariant calculi we usually use the convenient notation $V^\bullet := \Phi(\Omega^\bullet)$.

### 3.2 Orientability and Closed Integrals

We say that a differential calculus has total dimension $n$ if $\Omega^k = 0$, for all $k > n$, and $\Omega^n \neq 0$. If in addition there exists an $A$-$A$-bimodule isomorphism $\text{vol} : \Omega^n \simeq A$, then we say that $\Omega^\bullet$ is orientable. We call a choice of such an isomorphism an orientation. If $\Omega^\bullet$ is a covariant calculus over a quantum homogeneous space $M$ and $\text{vol}$ is a morphism in $G^M_{\text{Mod}}$, then we say that $\Omega^\bullet$ is covariantly orientable. Note all covariant orientations are equivalent up to scalar multiple. If $\Omega^\bullet$ is a $*$-calculus over a $*$-algebra, then a $*$-orientation is an orientation which is also a $*$-map. A $*$-orientable calculus is one which admits a $*$-orientation.

When the calculus is defined over a quantum homogeneous space, we define the integral, with respect to $\text{vol}$, to be the map which is zero on all $\Omega^k$, for $k < n$, and

$$\int : \Omega^n \to \mathbb{C}, \quad \omega \mapsto h(\text{vol}(\omega)),$$

where $h$ is the Haar functional. We say that the integral is closed if $\int d\omega = 0$, for all $\omega \in \Omega^{n-1}$.

**Lemma 3.2** For $\Omega^\bullet$ a covariant orientable calculus over a quantum homogeneous space $M$, the integral is closed if and only if $d\left(G(\Omega^{n-1})\right) = 0$.

**Proof.** Cosemisimplicity of $G$ guarantees that we can make a choice of complement $K \in G_{\text{Mod}}$ to $G(\Omega^{n-1})$ in $\Omega^{n-1}$. Since the map $\int \circ d : \Omega^{n-1} \to \mathbb{C}$ is a left $G$-comodule map, its restriction to $K$ must be the zero map. Hence if $d(G(\Omega^{n-1}) = 0$, then $\int d\omega = 0$, for all $\omega \in \Omega^{n-1}$.

Conversely, for any $\omega \in G(\Omega^{n-1})$, the fact that $d$ is a comodule map implies that $d\omega = \lambda \text{vol}^{-1}(1)$, for some $\lambda \in \mathbb{C}$. Moreover, $\int d\omega = \int \lambda \text{vol}^{-1}(1) = \lambda h(1) = \lambda$. Hence, if $\lambda \neq 0$, which is to say if $d\omega \neq 0$, then the integral is not closed. $\square$

**Corollary 3.3** The integral is closed if the decomposition of $V^{2n-1}$ into irreducible comodules does not contain the trivial comodule.

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Proof. If there exists a left $G$-coinvariant form $\omega \in \Omega^{n-1}$, then $\Phi(M\omega)$ is a trivial subcomodule of $V^{n-1}$. Hence, if no such subcomodule exists, there can be no coinvariant forms, and $d\left(\frac{G}{G}\Omega^{(n-1)}\right) = 0$ is satisfied vacuously. \qed

3.3 Complex Structures

In this subsection we recall the basic definitions and results of complex structures. For a more detailed introduction see [31].

Definition 3.4. An almost complex structure for a differential $*$-calculus $\Omega^*$, over a $*$-algebra $A$, is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b) \in \mathbb{N}_0^2} \Omega^{(a,b)}$ for $\Omega^*$ such that

1. $\Omega^k = \bigoplus_{a+b=k} \Omega^{(a,b)}$, for all $k \in \mathbb{N}_0$,
2. $(\Omega^{(a,b)})^* = \Omega^{(b,a)}$, for all $(a, b) \in \mathbb{N}_0^2$.

We call an element of $\Omega^{(a,b)}$ an $(a, b)$-form. We say that an almost complex structure is factorisable if we have bimodule isomorphisms

$$\wedge : \Omega^{(a,0)} \otimes_A \Omega^{(0,b)} \simeq \Omega^{(a,b)} \quad \text{and} \quad \wedge : \Omega^{(0,b)} \otimes_A \Omega^{(a,0)} \simeq \Omega^{(a,b)}.$$ (8)

If the algebra is a quantum homogeneous space and $\Omega^*$ is a covariant calculus, then we say that the almost complex structure is covariant if $\Omega^{(a,b)}$ is a sub-object of $\Omega^*$ in $Q^G\text{mod}_0$, for all $(a, b) \in \mathbb{N}_0^2$. We say that an almost complex structure is of diamond type if whenever $a > n$, or $b > n$, then necessarily $\Omega^{(a,b)} = 0$. Note that any almost complex structure on the de Rham complex of a manifold is necessarily of diamond type. Moreover, any calculus which is not of diamond type can clearly be quotiented to a calculus of diamond type.

Let $\partial$ and $\overline{\partial}$ be the unique homogeneous operators of order $(1, 0)$, and $(0, 1)$ respectively, defined by

$$\partial|_{\Omega^{(a,b)}} := \text{proj}_{\Omega^{(a+1,b)}} \circ d,$$
$$\overline{\partial}|_{\Omega^{(n,n-1)}} := \text{proj}_{\Omega^{(a,b+1)}} \circ d,$$

where $\text{proj}_{\Omega^{(a+1,b)}}$, and $\text{proj}_{\Omega^{(a,b+1)}}$, are the projections from $\Omega^{a+b+1}$ onto $\Omega^{(a+1,b)}$, and $\Omega^{(a,b+1)}$, respectively. Assuming that the calculus is of total dimension $2n$, and that the almost complex structure is of diamond type, then $d$ restricts to $\partial$ on $\Omega^{(n-1,n)}$, and to $\overline{\partial}$ on $\Omega^{(n,n-1)}$. Hence closure of the integral is equivalent to

$$\int \partial \omega = \int \overline{\partial} \omega' = 0,$$

for all $\omega \in \Omega^{(n-1,n)}$, $\omega' \in \Omega^{(n,n-1)}$. (9)

As observed in [1, §3.1] the proof of the following lemma carries over directly from the classical setting [15, §2.6]. Since the formulation of the definition of an almost complex structure used here differs from that used in [1, §3.1] (see Remark 3.8 below) we include a proof.
Lemma 3.5 [1, §3.1] If $\bigoplus_{(a,b) \in \mathbb{N}^2} \Omega^{(a,b)}$ is an almost complex structure for a differential $*$-calculus $\Omega^*$ over an algebra $A$, then the following two conditions are equivalent:

1. $d = \partial + \overline{\partial}$,

2. the triple $\left( \bigoplus_{(a,b) \in \mathbb{N}^2} \Omega^{(a,b)}, \partial, \overline{\partial} \right)$ is a double complex.

Proof. Let us first show that 1 implies 2. For any $\omega \in \Omega^k$,

$$0 = d^2(\omega) = \partial^2(\omega) + (\overline{\partial} \circ \partial + \partial \circ \overline{\partial})(\omega) + \overline{\partial}^2(\omega).$$

Since each of the three summands on the right hand side lie in complementary subspaces of $\Omega^{k+2}(M)$, each must be zero.

Let us now show that 2 implies 1. Note first that, for $g \in A$,

$$0 = \text{proj}_{\Omega^{(2,0)}}(d^2 g) = \text{proj}_{\Omega^{(2,0)}}(d(dg + \overline{\partial}g))$$

$$= \text{proj}_{\Omega^{(2,0)}}(d(dg)) + \overline{\partial}^2 g$$

$$= \text{proj}_{\Omega^{(2,0)}}(d(dg)).$$

Thus, for any $f \in A$, the form $d(f \partial g) = df \wedge \partial g + f d(\partial g)$ is contained in $\Omega^{(2,0)} \oplus \Omega^{(1,1)}$, and so, $d(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$. An analogous argument, using instead the projection $\text{proj}_{\Omega^{(2,0)}}$, shows that $d(\Omega^{(1,0)}) \subseteq \Omega^{(2,0)} \oplus \Omega^{(1,1)}$. Since $\Omega^{(a,b)}$ is spanned by products of $a$ elements of $\Omega^{(1,0)}$, and $b$ elements of $\Omega^{(0,1)}$, it follows from the Leibniz rule that $d = \partial + \overline{\partial}$ as required.

Definition 3.6. When the conditions in Lemma 3.5 hold for an almost complex structure, then we say that it is integrable.

We usually call an integrable almost complex structure a complex structure, and the double complex $(\bigoplus_{(a,b) \in \mathbb{N}^2} \Omega^{(a,b)}, \partial, \overline{\partial})$ its Dolbeault double complex. An easy consequence of integrability is that

$$\partial(\omega^*) = (\overline{\partial} \omega)^*, \quad \overline{\partial}(\omega^*) = (\partial \omega)^*, \quad \text{for all } \omega \in \Omega^*. \quad (10)$$

Remark 3.7. The property of integrability for an almost complex structure has a number of other equivalent formulations in addition to the two presented above. For details see [31, Lemma 2.13].

Remark 3.8. For a discussion of the equivalence of the definition of an almost complex structure used here with the definition of Beggs and Smith in [1, Definition 2.6] see [31, Remark 2.16].

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3.4 The Heckenberger–Kolb Calculi for Quantum Projective Space

In this subsection we give a brief presentation of the Heckenberger–Kolb calculus over $\mathbb{C}_q[\mathbb{C}P^n]$. A more detailed presentation, in the notation of this paper, can be found in [31]. The calculi were originally introduced by Heckenberger and Kolb in [13] for the more general class of examples given by the irreducible quantum flag manifolds, as discussed in §4.5. Their maximal prolongations and complex structures were first presented in [14].

3.4.1 First-Order Calculi and Maximal Prolongations

In this subsection we recall some details about first-order differential calculi necessary for our presentation of the Heckenberger–Kolb calculus below. A first-order differential calculus over $A$ is a pair $(\Omega^1, d)$, where $\Omega^1$ is an $A$-$A$-bimodule and $d : A \to \Omega^1$ is a linear map for which the Leibniz rule, $d(ab) = a(db) + (da)b$, for $a, b, \in A$, holds and for which $\Omega^1 = \text{span}_\mathbb{C}\{abd | a, b \in A\}$. The notions of differential map, and left-covariance when the calculus is defined over a quantum homogeneous space $M$, have obvious first-order analogues, for details see [31, §2.4]. The direct sum of two first-order differential calculi $(\Omega^1, d_{\Omega})$ and $(\Gamma^1, d_{\Gamma})$ is the first-order calculus $(\Omega^1 \oplus \Gamma^1, d_{\Omega} + d_{\Gamma})$. Finally, we say that a left-covariant first-order calculus over $M$ is irreducible if it does not possess any non-trivial quotients by a left-covariant $M$-bimodule.

We say that a differential calculus $(\Gamma^*, d_{\Gamma})$ extends a first-order calculus $(\Omega^1, d_{\Omega})$ if there exists a bimodule isomorphism $\varphi : \Omega^1 \to \Gamma^1$ such that $d_{\Gamma} = \varphi \circ d_{\Omega}$. It can be shown [31, §2.5] that any first-order calculus admits an extension $\Omega^*$ which is maximal in the sense that there exists a unique differential map from $\Omega^*$ onto any other extension of $\Omega^1$. We call this extension the maximal prolongation of the first-order calculus.

3.4.2 Defining the Calculus

We present the calculus in two steps, beginning with Heckenberger and Kolb’s classification of first-order calculi over $\mathbb{C}_q[\mathbb{C}P^n]$, and then discussing the maximal prolongation of the direct sum of the two calculi identified.

Theorem 3.9 [13, §2] There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi of finite dimension over $\mathbb{C}_q[\mathbb{C}P^n]$. We call the direct sum of these two calculi the Heckenberger–Kolb calculus of $\mathbb{C}_q[\mathbb{C}P^n]$.

We denote these two calculi by $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$, and denote their direct sum by $\Omega^1$. For a proof of the following lemma see [30, Lemma 5.2].

Lemma 3.10 A basis of $V^{(1,0)} := \Phi(\Omega^{(1,0)})$ and $V^{(0,1)} := \Phi(\Omega^{(0,1)})$ is given respectively by

$$\{e_a^+ := [\partial z_{a+1,1}] | a = 1, \ldots, n\}, \quad \{e_a^- := q^{2(a+1)}[\overline{\partial z}_{a+1,1}] | a = 1, \ldots, n\}.$$
Moreover, \([\partial z_{ab}] = [\overline{\partial} z_{ab}] = 0\) for \(a, b \geq 2\) or \(a = b = 1\).

We call a subset \(\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}\) ordered if \(a_1 < \cdots < a_k\). For any two ordered subsets \(I, J \subseteq \{1, \ldots, n\}\), we denote

\[
e_i^+ \land e_j^{-} := e_{i_1}^+ \land \cdots \land e_{i_k}^+ \land e_{j_1}^- \land \cdots \land e_{j_k}^-.
\]

**Lemma 3.11** For \(\Omega^*\) the maximal prolongation of \(\Omega^1\), the space \(V^k := \Phi(\Omega^k)\) has dimension \(\binom{2n}{k}\). A basis is given by

\[
\{e_I^+ \land e_J^- | I, J \subseteq \{1, \ldots, n\}\} \text{ ordered subsets such that } |I| + |J| = k.
\]

A full set of generating relations of the algebra \(V^*\) is given in following lemma, for a proof see [31, Proposition 5.8].

**Proposition 3.12** The set of relations in \(V^*\) is generated by the elements

\[
e_i^- \land e_j^+ + q e_j^+ \land e_i^-,
\]

\[
e_i^- \land e_i^+ + q e_i^+ \land e_i^- + (q^2 - 1) \sum_{a=i+1}^{n} e_a^+ \land e_a^-,
\]

\[
e_i^- \land e_h^+ + q e_h^- \land e_i^-,
\]

\[
e_i^+ \land e_i^- + q e_i^- \land e_i^+,
\]

\[
e_i^+ \land e_j^+ + q e_i^+ \land e_j^-,
\]

\[
e_i^- \land e_i^-,
\]

for \(h, i, j = 1, \ldots, n\), \(i \neq j\), and \(h < i\).

### 3.4.3 Weight Space Decomposition of \(V^1\)

In this subsection we give an explicit description of the left module structure of \(\Phi(\Omega^1)\), as well as its weight space decomposition. A proof of the \(\mathbb{C}_q[U_n]\)-coaction formulae can be found in [31, Lemma 6.11]. The weight decomposition is novel, and so, we include a proof.

**Lemma 3.13** The left \(\mathbb{C}_q[U_n]\)-coactions on \(V^{(1,0)}\) and \(V^{(0,1)}\) are given by

\[
\Delta_L(e_i^+) := \sum_{k=1}^{n} u_k^i \det_n \otimes e_k^+,
\]

\[
\Delta_L(e_i^-) := \sum_{k=1}^{n} S(u_k^i) \det_n^{-1} \otimes e_k^-.
\]

**Corollary 3.14** The induced left \(\mathbb{C}_q[\mathbb{T}^n]\)-coactions on \(V^{(1,0)}\) and \(V^{(0,1)}\) are given by

\[
\Delta_{L,\tau}(e_i^+) = t_i \bigotimes e_i^+,
\]

\[
\Delta_{L,\tau}(e_i^-) = (t_i \bigotimes e_i^-).
\]

**Proof.** The first identity follows immediately from

\[
\Delta_{L,\tau}(e_i^+) = \sum_{k=1}^{n} \tau(u_k^i \det_n) \otimes e_k^+ = \tau(u_i^i) \bigotimes e_i^+ = t_i \bigotimes e_i^+.
\]

The second identity is established similarly. □
3.4.4 A Complex Structure

Finally, we come to the definition of a complex structure for the calculus. Denote

\[ V^{(a,b)} := \text{span}_C \{ e_I^+ \wedge e_J^- \mid I, J \subseteq \{1, \ldots, n\} \text{ ordered subsets with } |I| = a, |J| = b \}. \]

The decomposition \( V^k \cong \bigoplus_{(a+b=k)} V^{(a,b)} \), for all \( k \), follows immediately. For a proof of the following proposition see [31, §6, §7].

**Proposition 3.15** For the Heckenberger–Kolb calculus over \( \mathbb{C}[\mathbb{C}P^n] \), there is a unique covariant complex structure \( \Omega^{(\bullet, \bullet)} \) such that \( \Phi(\Omega^{(a,b)}) = V^{(a,b)} \). Moreover, the complex structure is factorisable.

**Lemma 3.16** Every zero weight vector of \( V^k \) is contained in \( \bigoplus_{k=1}^n V^{(k,k)} \).

**Proof.** The lemma follows from Corollary 3.14 and the multiplicativity of \( \Delta_{L,\tau} \). \( \square \)

**Corollary 3.17** The integral associated to any covariant orientation of \( \Omega^k \) is closed.

**Proof.** The lemma tells us that \( V^{(n-1,n)} \) and \( V^{(n,n-1)} \) contain no zero weights. Hence, they contain no elements coinvariant with respect to \( \Delta_L \). Closure of the integral now follows from Corollary 3.3. \( \square \)

4 Hermitian Structures and Hodge Maps

In this section we introduce symplectic and Hermitian forms and use them to prove a noncommutative generalisation of the Lefschetz decomposition. Motivated by Weil’s well-known classical formula [39, §1.2] relating the Hodge map with Lefschetz decomposition, we introduce a definition for a Hodge map associated to any Hermitian form.

Throughout this section \( \Omega^k \) denotes a differential calculus, over an algebra \( A \), of total dimension \( 2n \).

4.1 Almost Symplectic Forms

As a first step towards the definition of an Hermitian form, we present a direct noncommutative generalisation of the classical definition of an almost symplectic form.

**Definition 4.1.** An almost symplectic form for \( \Omega^k \) is a central real 2-form \( \sigma \) such that, with respect to the Lefschetz operator

\[ L : \Omega^k \to \Omega^k, \quad \omega \mapsto \sigma \wedge \omega, \]

isomorphisms are given by

\[ L^{n-k} : \Omega^k \to \Omega^{2n-k}, \quad \text{for all } 1 \leq k < n. \] (11)
Note that since $\sigma$ is a central real form, $L$ is an $A$-$A$-bimodule $*$-homomorphism. Moreover, if $\sigma$ is an almost symplectic form for a covariant calculus over a quantum homogeneous space $M$, then $L$ is a morphism in $G_M\text{mod}_0$ if and only if $\sigma$ is a left $G$-coinvariant form.

The existence of a symplectic form has important implications for the structure of a differential calculus. Crucial to understanding this structure is the notion of a primitive form, which directly generalises the classical definition of a primitive form [15, §1.2, §3.1].

**Definition 4.2.** For $L$ the Lefschetz operator of any almost symplectic form, the space of primitive $k$-forms is

$$P^k := \{ \alpha \in \Omega^k \mid L^{n-k+1}(\alpha) = 0 \}, \quad \text{if } k \leq n, \quad \text{and} \quad P^k := 0, \quad \text{if } k > n.$$ 

One of the main reasons primitive forms are so important is the following decomposition result. It shows that an almost symplectic form implies the existence of a further refinement of the $\mathbb{N}_0$-decomposition of a differential calculus.

**Proposition 4.3** For $L$ the Lefschetz operator of any almost symplectic form, we have the $A$-bimodule decomposition

$$\Omega^k \simeq \bigoplus_{j \geq 0} L^j(P^{k-2j}),$$

which we call the Lefschetz decomposition.

**Proof.** Let us assume that the decomposition holds for some $k \leq n - 2$. Consider the composition

$$\Omega^k \xrightarrow{L} \Omega^{k+2} \xrightarrow{L^{n-k-1}} \Omega^{2n-k}.$$ 

Since $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ is an isomorphism of $A$-$A$-bimodules, we have the $A$-$A$-bimodule decomposition

$$\Omega^{k+2} \simeq \ker(L^{n-k-1}|_{\Omega^{k+2}}) \oplus L(\Omega^k) = \ker(L^{n-(k+2)+1}|_{\Omega^{k+2}}) \oplus L(\Omega^k) = P^{k+2} \oplus L(\Omega^k) = P^{k+2} \oplus \left( \bigoplus_{j \geq 1} L^j(P^{k-2j}) \right)$$

$$= \bigoplus_{j \geq 0} L^j(P^{k+2-2j}).$$

Since $\Omega^0 = P^0$ and $\Omega^1 = P^1$, it now follows from an inductive argument that the proposition holds for each space of forms of degree less than or equal to $n$. 

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Turning to forms of degree greater than \( n \), we see that, for \( k = 0, \cdots, n \),
\[
\Omega^{2n-k} \simeq L^{n-k}(\Omega^{k}) \simeq L^{n-k}\left( \bigoplus_{j \geq 0} L^{j}(P^{k-2j}) \right) = \bigoplus_{j \geq n-k} L^{j}(P^{2n-k-2j})
\]
where the last equality follows from the fact that, for \( j = 0, \ldots, n - k - 1 \), either \( 2n - k - 2j > n \) and \( P^{2n-k-2j} = 0 \) by definition, or \( k + 2 \leq 2n - k - 2j \leq n \), and so, we have \( L^{j}(P^{2n-k-2j}) = 0 \).

4.2 Closed, Central, and Symplectic Forms

In general, it can prove tedious to verify that a given 2-form is central. Assuming that
the form is d-closed, however, makes the task much easier.

**Lemma 4.4** A d-closed form is central if and only if it commutes with 0-forms.

**Proof.** If \( \sigma \) is a d-closed form which commutes with 0-forms, then
\[
\sigma \wedge da = d(\sigma a) = d(a\sigma) = da \wedge \sigma,
\]
for all \( a \in \Omega^0 \).

Thus \( \sigma \) commutes with all 1-forms, and hence with all forms. The proof in the other
direction is trivial. \( \Box \)

**Lemma 4.5** If \( \Omega^\bullet \) is a covariant calculus over a quantum homogeneous space \( M \), then
every left \( G \)-coinvariant form commutes with 0-forms.

**Proof.** With respect to the isomorphism \( U : \Omega^\bullet \simeq G \Box H V^\bullet \), any left \( G \)-coinvariant \( \omega \)
satisfies \( U(\omega) = 1 \otimes [\omega] \). That \( m \in M \) commutes with \( \omega \) is obvious from (4). \( \Box \)

**Corollary 4.6** Every left \( G \)-coinvariant d-closed form is central.

The following lemma gives us a sufficient criterion for a coinvariant form to be d-closed.
It should be noted, however, that the condition is not necessary.

**Lemma 4.7** If \( H(V^3) \) is trivial, then every left \( G \)-coinvariant 2-form is d-closed.

**Proof.** For a left \( G \)-coinvariant 2-form \( \omega \), covariance of the calculus implies that
\[
\Delta_L(da) = (\text{id} \otimes d)\Delta_L(\omega) = 1 \otimes d\omega.
\]
Hence, if \( d\omega \neq 0 \), the space \( G(\Omega^3) \) contains a non-trivial left \( G \)-coinvariant element. Since this cannot happen if \( H(V^3) \) is trivial, we
must conclude that \( d\omega = 0 \).

Motivated by this discussion of closed forms, we present the following noncommuta-
tive generalisation of the classical notion of a symplectic form \([15, \S 3.1]\).

**Definition 4.8.** A symplectic form is a d-closed almost symplectic form.
As we will see in §7, a Kähler form is a special type of symplectic form whose existence has many far-reaching consequences for the structure of a differential calculus.

### 4.3 Hermitian Structures and \( h \)-Hodge Maps

We begin by introducing the notion of an Hermitian structure for a differential \( * \)-calculus, which is essentially just a symplectic form interacting with a complex structure in a natural way. In the commutative case each such form is the fundamental form of a uniquely identified Hermitian metric [15, §3.1].

**Definition 4.9.** An Hermitian structure for a \( * \)-calculus \( \Omega \) is a pair \((\Omega^{(••)}, \sigma)\) where \( \Omega^{(••)} \) is a complex structure and \( \sigma \) is an almost symplectic form, called the Hermitian form, such that \( \sigma \in \Omega^{(1,1)} \).

When \( \Omega \) is a covariant \( * \)-calculus over a quantum homogeneous space, \( \Omega^{(••)} \) is a covariant complex structure, and \( \sigma \) is a left \( G \)-coinvariant form, then we say that \((\Omega^{(••)}, \sigma)\) is a covariant Hermitian structure. We omit the proof of the following lemma which is clear.

**Lemma 4.10** The existence of an Hermitian structure for a complex structure implies that it is of diamond type.

Taking our motivation from Weil’s well-known classical formula [39, §1.2] presenting the Hodge map in terms of the Lefschetz decomposition, we use Lemma 4.3 to introduce a noncommutative generalisation of the Hodge map. (Note that quantum integers are used instead of integers, as is discussed in the remark below.)

**Definition 4.11.** For \( h \in \mathbb{R}_{>0} \), the \( h \)-Hodge map associated to an Hermitian structure is the morphism uniquely defined by

\[
*_{h}(L^j(\omega)) = (-1)^{\frac{j(j+1)}{2}}^{a-b} \frac{[j]_h!}{[n-j-k]_h!} L^{n-j-k}(\omega), \quad \omega \in P(a,b) \subseteq P^k,
\]

where \([m]_h := h^{m-1} + h^{m-3} + \cdots + h^{-m+1}\) denotes the quantum integer of \( m \). We call \( h \) the Hodge parameter of the Hodge map.

As a first consequence of the definition, we establish direct generalisations of four of the basic properties of the classical Hodge map.

**Lemma 4.12** It holds that

1. \( *_{h}^2(\omega) = (-1)^{k} \text{id} \), for all \( \omega \in \Omega^k \),
2. \( *_{h} \) is an isomorphism,
3. \( *_{h}(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)} \),

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4. $*_{h}$ is a $*$-map.

Proof.

1. By Lefschetz decomposition it suffices to prove the result for a form of type $L^{j}(\alpha)$, for $\alpha \in P^{(a,b)} \subseteq P^{k}$, $j \geq 0$. From the definition of $*_{h}$ we have that

$$*_{h}^{2}(L^{j}(\alpha)) = (-1)^{k} \frac{k^{(k+1)}}{2} \frac{[j]!}{[n-j-k]!} *_{h}(L^{n-j-k}(\alpha))$$

$$= (-1)^{k} \frac{k^{(k+1)}}{2} \frac{[j]!}{[n-j-k]!} \left((-1)^{k} \frac{[n-j-k]!}{[n-(n-j-k)-k]!} L^{n-(n-j-k)-k}(\alpha)\right)$$

$$= (-1)^{k} \frac{[j]!}{[j]!} L^{j}(\alpha) = (-1)^{k} L^{j}(\alpha).$$

2. This follows immediately from 1.

3. This follows directly from the definition of $*_{h}$, the fact that $L$ is a degree $(1,1)$ map, and, as just established, the fact that $*_{h}$ is an isomorphism.

4. Again, it suffices to prove the result for a form of type $L^{j}(\alpha)$. Since $(P^{(a,b)})^{*} = P^{(b,a)}$, the definition of $*_{h}$ implies that

$$*_{h}((L^{j}(\alpha))^{*}) = *_{h}(L^{j}(\alpha^{*})) = (-1)^{\frac{k(k+1)}{2}} \frac{[j]!}{[n-j-k]!} a^{k-a}(L^{n-j-k}(\alpha^{*}))$$

$$= \left((-1)^{\frac{k(k+1)}{2}} \frac{[j]!}{[j]!} a^{k-a} L^{n-j-k}(\alpha)\right)^{*}$$

$$= ( *_{h} L^{j}(\alpha))^{*}. \quad \square$$

Corollary 4.13 With respect to a choice of Hermitian structure, a $*$-orientation, which we call the associated orientation, is given by $*_{h}$.

Corollary 4.14 If in addition we assume that $\Omega^{*}$ is a covariant calculus over a quantum homogeneous space, then the associated integral is closed if $V^{(1,0)}$, or equivalently $V^{(0,1)}$, does not contain the trivial corepresentation as a sub-comodule.

Proof. By Corollary 3.3 we know that the associated integral is closed if $V^{2n-1}$ does not contain the trivial comodule as a sub-comodule. But by the first part of the above lemma this is equivalent to $V^{1}$ not containing the trivial comodule as a submodule. Finally, we note that (despite not being a morphism in $U_{0}$) the $*$-map brings copies of the trivial comodule in $\Omega^{(1,0)}$ to copies of the trivial comodule in $\Omega^{(1,0)}$, and vice versa. Hence, one need only check either $V^{(1,0)}$ or $V^{(0,1)}$. \quad \square

Remark 4.15 Note that the Hermitian condition is not necessary for the existence of a $*$-orientation, one exists for any $*$-calculus containing a symplectic form. Moreover, by
dropping the factor $i^{a-b}$ from the definition of $\ast_h$, it is possible to define a Hodge map for an almost symplectic form which is not necessarily associated to a complex structure. For a discussion of such maps in the classical case see [3, 42].

**Remark 4.16** The use of quantum integers in the definition of the Hodge map is motivated by their appearance in the multiplicative structure of the Heckenberger–Kolb calculus in §4.4, in the Heckenberger–Kolb calculus of the quantum Grassmannians in [28], and in the calculus introduced for the full quantum flag manifold of $\mathbb{C}_q[SU_3]$ in [29]. It is worth stressing that the Hodge parameter need not depend on a deformation parameter: indeed, the definition of the Hodge map is well-defined for algebras which are not deformations and even for commutative algebras. As more examples of noncommutative Hermitian structures emerge, it can be expected that a more formal definition of the Hodge map will appear (see for example the recent paper on braided Hodge maps [26]).

### 4.4 An Hermitian Structure for the Heckenberger–Kolb Calculus over $\mathbb{C}_q[\mathbb{C}P^n]$

In this subsection we construct a covariant Hermitian $(\Omega^i(\cdot, \cdot), \kappa)$ structure for the Heckenberger–Kolb calculus over $\mathbb{C}_q[\mathbb{C}P^n]$. In the classical case, it follows from the classification of covariant metrics on complex projective space that $\kappa$ is equal, up to scalar multiple, to the fundamental form of the Fubini–Study metric.

Throughout this subsection we will, by abuse of notation, denote $\Phi(L), \Phi(\text{vol})$, and $\Phi(\ast_q)$, by $L, \text{vol}$, and $\ast_q$, respectively.

**Lemma 4.17** A left $G$-coinvariant closed form is given by

$$\kappa := i \sum_{k,l=1}^{n+1} q^{2k} \partial z_{kl} \wedge \bar{\partial} z_{lk} = U^{-1} \left( i \sum_{a=1}^{n} 1 \otimes e_a^+ \wedge e_a^- \right).$$

**Proof.** The fact that $\kappa$ is closed follows from Lemma 4.7 and Lemma 3.16. Left $G$-coinvariance of $\kappa$, and equality of the two given presentations, follow from

\[
U\left( \sum_{k,l=1}^{n+1} q^{2k} \partial z_{kl} \wedge \bar{\partial} z_{lk} \right) = \sum_{k,l=1}^{n+1} \sum_{a,b,c,d=1} q^{2k} u_a^k S(u_b^l) u_c^d S(u_d^h) \otimes [\partial z_{ab}] \wedge [\bar{\partial} z_{cd}] = \sum_{k=1}^{n+1} \sum_{a,b,d=1} q^{2k} u_a^k S(u_d^h) \otimes [\partial z_{ab}] \wedge [\bar{\partial} z_{bd}] = \sum_{a=2}^{n+1} 1 \otimes [\partial z_{a1}] \wedge (q^{2a} [\bar{\partial} z_{1a}]) = \sum_{a=1}^{n} 1 \otimes e_a^+ \wedge e_a^-.
\]

Note that in the penultimate line we have used the fact that $[\partial z_{kl}] = [\bar{\partial} z_{kl}] = 0$, for $k, l \geq 2$ or $k = l = 1$ (as presented in Lemma 3.10) and in the last line we have used the elementary identity $\sum_{k=1}^{n+1} q^{2k} u_a^k S(u_b^h) = \delta_{ad} q^{2a} 1$. \qed
Lemma 4.18 It holds that
\[ U(\kappa^l) := i^l (\text{mod } 2) [l]_q! \sum_{I \in O(l)} 1 \otimes e_I^+ \wedge e_I^-, \]
where \( O(l) \) is the set of all ordered subsets of \( \{1, \ldots, n\} \) with \( l \) elements.

Proof. Assuming that the proposition holds for \( l \), we have
\[ U(\kappa^{l+1}) = U(\kappa) \wedge \left( i^l (\text{mod } 2) [l]_q! \sum_{I \in O(l)} 1 \otimes e_I^+ \wedge e_I^- \right) \]
\[ = i^l (\text{mod } 2)^{l+1} [l+1]_q! \sum_{I \in O(l+1)} \sum_{a=1}^{l+1} 1 \otimes e_I^a_+ \wedge e_I^{-a}. \]
In order to re-express this identity, we introduce the following notation: For \( I \in O(l+1) \), denote by \( I_a \) and \( a_I \), the \((l+1)\)-tuples where the \( a \)th-entry has been bubbled through to the last, respectively first, position. Now, as a little thought will confirm, it holds that
\[ U(\kappa^{l+1}) = i^{l+1} (\text{mod } 2)^{l+1} [l]_q! \sum_{I \in O(l+1)} \sum_{a=1}^{l+1} 1 \otimes e_I^a_+ \wedge e_I^{-a}. \]
The commutation relations of the calculus imply that \( e_I^a_+ \wedge e_I^{-a} = (-1)^l q^{l-2(a-1)} e_I^+ \wedge e_I^- \).
Hence
\[ U(\kappa^{l+1}) = i^{l+1} (\text{mod } 2)^{l+1} [l]_q! \sum_{I \in O(l+1)} (q^l + q^{l-2} + \cdots + q^{-l+2} + q^{-l}) 1 \otimes e_I^+ \wedge e_I^- \]
\[ = i^{l+1} (\text{mod } 2)^{l+1} [l]_q! \sum_{I \in O(l+1)} 1 \otimes e_I^+ \wedge e_I^- \]
Finally, since the proposition clearly holds for \( l = 1 \), we can conclude that it holds for all \( l \in \mathbb{N} \).

Proposition 4.19 The pair \((\Omega(\bullet, \bullet), \kappa)\) is a covariant Hermitian structure for the Heckenberger–Kolb calculus over \( \mathbb{C}_q[\mathbb{C}P^n] \).

Proof. Since \( \kappa \) is closed, it follows from Lemma 4.6 that it is central. That \( \kappa \) is real follows from
\[ \kappa^* := -i \sum_{k,l=1}^n q^{2k} (\partial z_{kl} \wedge \overline{\partial} z_{kl})^* = i \sum_{k,l=1}^n q^{2k} (\overline{\partial} z_{kl})^* \wedge (\partial z_{kl})^* \]
\[ = i \sum_{k,l=1}^n q^{2k} \partial z_{kl} \wedge \overline{\partial} z_{kl} = \kappa. \]
It remains to show that $L^{n-k} : V^k \to V^{2n-k}$ is an isomorphism, for all $k = 0, \ldots, n-1$. Since Lemma 3.11 shows that $\dim(V^k) = \dim(V^{2n-k})$, we need only show that $L^{n-k}$ has zero kernel in $V^k$. To this end, consider the decomposition $V^k \simeq \bigoplus_{n \geq 0} V_r^k$ where

$$V_r^k := \{ e^+_i \wedge e^-_j \in V^k \mid |I \cap J| = r \}.$$  (12)

For $v \in V^k \cap \ker(L^{n-k})$, denote its decomposition with respect to (12) by $v = \sum_{r=1}^m v_r$, where without loss of generality we assume that $v_m \neq 0$. Since $L^{n-k+m}$ acts as zero on $V^k_r$, for $r < m$, we have

$$L^{n-k+m}(v) = \sum_{r=0}^m L^{n-r+m}(v_r) = L^{n-r+m}(v_m).$$

Hence, the proposition would follow if we could show that $L^{n-r+m}$ had trivial kernel in $V^k_m$. But this follows from the fact that, for any $e^+_i \wedge e^-_j \in V^k_r$, there exists a $m \in \mathbb{Z}$, such that

$$L^{n-k+r}(e^+_i \wedge e^-_j) = \pm q^m e^+_{I \cup (I \cap J) e} \wedge e^-_{J \setminus (I \cap J) e},$$

where $\cup$ denotes set union followed by reordering. \hfill \Box

**Corollary 4.20** Denoting $e^* := e^+_1 \wedge \cdots e^+_n \wedge e^-_1 \wedge \cdots e^-_n$, it holds that

$$\text{vol}(e^*) = i^{-n} \mod 2.$$

**Proof.** This follows from the calculation

$$1 = \text{vol}(*q(1)) = \text{vol}(\frac{1}{|n|_q} \kappa^n) = i^n \mod 2 \cdot \text{vol}(\frac{|n|_q e^*}{|n|_q}) = i^n \mod 2 \cdot \text{vol}(e^*).$$ \hfill \Box

**Lemma 4.21** Up to scalar multiple $\kappa$ is the unique coinvariant Hermitian form in $\Omega^{(1,1)}$.

**Proof.** Since the calculus is factorisable we have $V^{(1,1)} \simeq V^{(1,0)} \otimes V^{(0,1)}$. Using an elementary representation theoretic argument it can be shown that the decomposition of $V^{(1,0)} \otimes V^{(0,1)}$ into irreducible summands contains a unique copy of the trivial comodule $\mathbb{C}$. Thus, any other coinvariant $(1,1)$-form is a scalar multiple of $\kappa$. \hfill \Box

We will now look at Lefschetz decomposition and the associated Hodge map for some low dimensional examples.

**Example 4.22.** For $\mathbb{C}_q[\mathbb{C}P^1]$, we have

$$*_h(e^+_1) = -ie^+_1, \quad *_h(e^-_1) = ie^-_1, \quad *_h(1) = \kappa = ie^+_1 \wedge e^-_1.$$

**Example 4.23.** For $\mathbb{C}_q[\mathbb{C}P^2]$, Lefschetz decomposition is trivial except for $V^{(1,1)}$, where it reduces to $V^{(1,1)} \simeq L(1) \oplus P^{(1,1)}$. The fact $e^+_1 \wedge e^-_2, e^+_2 \wedge e^-_1 \in P^{(1,1)}$ follows from

$$L(e^+_1 \wedge e^-_2) = e^+_1 \wedge \kappa \wedge e^-_2 = ie^+_1 \wedge (e^+_2 \wedge e^-_1 + e^+_2 \wedge e^-_2) \wedge e^-_2 = 0,$$
Theorem 4.24

of quantum homogeneous spaces. Moreover, Theorem 3.9 holds for this larger family

\[ \pi \]

\[ \delta \]

implies that \( e_1^+ \wedge e_1^- - q^{-2} e_2^+ \wedge e_2^- \in P^{(1,1)} \). Since the set

\[ \{ \kappa, e_1^+ \wedge e_2^-, e_2^+ \wedge e_1^-, e_1^+ \wedge e_1^- - q^{-2} e_2^+ \wedge e_2^- \} \]

is clearly a basis for \( V^{(1,1)} \), we must have that

\[ P^{(1,1)} = \text{span}_C \{ e_1^+ \wedge e_2^-, e_2^+ \wedge e_1^-, e_1^+ \wedge e_1^- - q^{-2} e_2^+ \wedge e_2^- \} \].

Setting the Hodge parameter equal to \( q \), the action of the associated Hodge map on \( V^1 \)

is given by

\[ *_q(e_1^+) = e_1^+ \wedge e_2^+ \wedge e_2^-, \quad *_q(e_2^+) = -q e_1^+ \wedge e_2^+ \wedge e_1^- \],

\[ *_q(e_1^-) = q^{-1} e_2^+ \wedge e_1^- \wedge e_2^-, \quad *_q(e_2^-) = -e_1^+ \wedge e_1^- \wedge e_2^- \].

The action of \( *_h \) on the primitive elements of \( V^2 \) is given by

\[ *_q|_{P(2,0)} = \text{id}, \quad *_q|_{P(1,1)} = -\text{id}, \quad *_q|_{P(0,2)} = \text{id} \].

4.5 A Conjectured Hermitian Structure for \( \mathbb{C}_q[G/L_S] \)

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra of rank \( r \) and \( U_q(\mathfrak{g}) \) the corresponding

Drinfeld–Jimbo quantised enveloping algebra [13, §6.1]. For \( S \) a subset of simple roots, denote by \( \pi_S : \mathbb{C}_q[G] \to \mathbb{C}_q[L_S] \) the Hopf algebra map dual to the inclusion \( U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g}) \), where

\[ U_q(\mathfrak{l}_S) := \langle K_i, E_j, F_j \mid i = 1, \ldots, r; \ j \in S \rangle \).

The quantum homogeneous space of this coaction is called the \textit{quantum flag manifold}

corresponding to \( S \), and is denoted by \( \mathbb{C}_q[G/L_S] \). (See [13, 14] for a more detailed

presentation of this definition.)

If \( S = \{ 1, \ldots, r \} \backslash \alpha_i \) where \( \alpha_i \) appears in any positive root with coefficient at most one, then we say that the quantum flag manifold is \textit{irreducible}. It follows that \( \mathbb{C}_q[CP^n] \) is an irreducible quantum flag manifold. Moreover, Theorem 3.9 holds for this larger family of quantum homogeneous spaces.

Theorem 4.24 [13, §2] There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi of finite dimension over \( \mathbb{C}_q[G/L_S] \).
In [14] the maximal prolongation of the direct sum of these two calculi is shown to have a unique covariant complex structure $\Omega(\bullet, \bullet)$. Using a representation theoretic argument directly analogous to that in Lemma 4.21, it can be shown that the each $\Omega^{(1,1)}$ contains a left-coinvariant form $\kappa$ that is unique up to scalar multiple.

**Conjecture 4.25.** For every irreducible quantum flag manifold $C_q[G_0/L_0]$, the pair $(\Omega^{\bullet, \bullet}, \kappa)$ is a covariant Hermitian structure for the Heckenberger–Kolb calculus.

## 5 Metrics, Inner Products, and Operator Adjoints

In the previous section, Hermitian structures were introduced as a noncommutative generalisation of the fundamental form of an Hermitian metric, and an associated Hodge map was defined through the classical Weil formula. In this section we bring this series of ideas full circle by defining a metric through the classical definition of the Hodge map. This allows for the construction of adjoint operators for all $G$-comodules maps, which is one of the principal motivations of the paper and an indispensable tool in §6 and §7. An interesting new phenomenon to emerge is a deformation of the classical Lefschetz identities to a representation of the quantised enveloping algebra of $\mathfrak{sl}_2$, see Corollary 5.14.

Throughout this section $\Omega^\bullet$ denotes a differential $*$-calculus of total dimension $2n$, and $(\Omega^{\bullet, \bullet}, \kappa)$ denotes an Hermitian structure for $\Omega^\bullet$.

### 5.1 Metrics

By reversing the classical order of definition, we use the Hodge map to associate a metric to any Hermitian structure.

**Definition 5.1.** The metric associated to the Hermitian structure $(\Omega^{\bullet, \bullet}, \kappa)$ is defined to be the map $g : \Omega^\bullet \otimes_M \Omega^\bullet \to M$ for which $g(\Omega^k \otimes_M \Omega^l) = 0$, for all $k \neq l$, and

$$g(\omega \otimes \nu) = \text{vol}(\omega \wedge *h(\nu^*)),$$

$\omega, \nu \in \Omega^k$.

The $N_0^2$-decomposition, and the Lefschetz decomposition, of the de Rham complex of a classical Hermitian manifold are orthogonal with respect to the metric [15, Lemma 1.2.24]. We now show that this carries over to the noncommutative setting. Moreover, we show as a consequence that the metric is conjugate symmetric.

**Lemma 5.2** It holds that

1. the $N_0^2$-decomposition of $\Omega^\bullet$ is orthogonal with respect to $g$,

2. the Lefschetz decomposition of $\Omega^\bullet$ is orthogonal with respect to $g$.

**Proof.**
1. The first part of Lemma 4.12 implies that, given any \( \omega \in \Omega^{(a,b)} \), \( \nu \in \Omega^{(a',b')} \), for which \( a + b = a' + b' \) but \( (a,b) \neq (a',b') \), then the product \( \omega \wedge \ast_h (\nu^*) \notin \Omega^{(a,n)} \). It now follows from Lemma 4.10 that \( g(\omega \otimes M \nu) = 0 \).

2. For \( \alpha \in P^k \), \( \beta \in P^l \), orthogonality of the \( N_0 \)-grading implies that \( g(L^j(\alpha) \otimes_M L^j(\beta)) \) is nonzero only if \( 2i + k = 2j + l =: m \). Assuming that \( \beta \in P^{(a,b)} \subseteq P^l \), we have

\[
g(L^{1/2(m-k)}(\alpha) \otimes_M L^{1/2(m-l)}(\beta)) = \text{vol}(L^{1/2(m-k)}(\alpha) \wedge \ast_h L^{1/2(m-l)}(\beta^*))
\]

\[
= \lambda \text{vol}(L^{1/2(m-k)}(\alpha) \wedge L^{n/2(m-l)}(\beta^*))
\]

where \( \lambda := (-1)^{(l+k+1)/2} a-b \frac{[j]_h!}{[n-k-j]_h!} \). Assuming now that \( k < l \), which is to say that \( l = k + r \), for some \( r \in 2\mathbb{N}_{>0} \),

\[
\lambda^{-1} g(L^{1/2(m-k)}(\alpha) \otimes_M L^{1/2(m-l)}(\beta)) = \text{vol}(L^{n-k+\frac{r}{2}}(\alpha) \wedge \beta^*).
\]

Since \( \alpha \in P^k \), we must have \( L^{n-k+\frac{r}{2}}(\alpha) = 0 \). The proof for \( k > l \) is analogous. \( \square \)

**Corollary 5.3** It holds that \( g(\omega \otimes M \nu) = (g(\nu \otimes M \omega))^* \), for all \( \omega, \nu \in \Omega^\bullet \).

**Proof.** By the above lemma, it suffices to prove the result for \( g(L^j(\alpha) \otimes_M L^j(\beta)) \), for some \( \alpha, \beta \in P^{(a,b)} \subseteq P^k \). This is done as follows

\[
g(L^j(\alpha) \otimes_M L^j(\beta)) = \text{vol}(L^j(\alpha) \wedge \ast_h (L^j(\beta^*))
\]

\[
= (-1)^{k(k+1)/2} b^{-a} \frac{[j]_h!}{[n-k-j]_h!} \text{vol}(L^j(\alpha) \wedge L^{n-k-j}(\beta^*))
\]

\[
= \left((-1)^{k(k+1)/2} b^{-a} \frac{[j]_h!}{[n-k-j]_h!}\right)(-1)^{k^2} \text{vol}(L^j(\beta) \wedge L^{n-k-j}(\alpha^*))^*
\]

\[
= \left((-1)^{k(k+1)/2} b^{-a} \frac{[j]_h!}{[n-k-j]_h!}\right) \text{vol}(L^j(\beta) \wedge L^{n-k-j}(\alpha^*))^*
\]

\[
= (g(L^j(\beta) \otimes_M L^j(\alpha)))^* \tag{13} \quad \square
\]

### 5.2 Inner Products and Operator Adjoints

In this subsection we specialise to the case where \( \Omega^\bullet \) is a covariant calculus over a quantum homogeneous space \( M \), and \( (\Omega^\bullet, \kappa) \) is a covariant Hermitian structure. Following the classical order of definition, we introduce an inner product by composing the associated metric with the Haar functional. In order for this to well-defined, however, we need to impose a positive definiteness condition on our Hermitian structure.

**Definition 5.4.** An Hermitian structure is said to be **positive definite** if an inner product is given by

\[
\langle \cdot, \cdot \rangle_V : V^{\otimes 2} \to \mathbb{C}, \quad [\omega] \otimes [\nu] \mapsto [g(\omega \otimes_M \nu)].
\]
Note that $\langle \cdot, \cdot \rangle$ is well-defined because of our assumption that $\Omega^\bullet \in \mathcal{G}_{\mu \mod 0}$. Another point, which is easily checked, is that positive definiteness of an Hermitian structure is independent of the choice of Hodge parameter.

In general it can be quite tedious to verify positive definiteness; the following lemma shows us that we need only do so for primitive elements.

**Lemma 5.5** For $\alpha, \beta \in P^\bullet$, and $(\begin{array}{c} a \\ b \end{array})_h := \frac{|a|_h!}{|b|_h! |a-b|_h!}$ the Gaussian binomial coefficient, it holds that

$$\langle L^j(\alpha), L^j(\beta) \rangle_V = \binom{n-j-k}{j}_h^{-1} \langle \alpha, \beta \rangle_V .$$

**Proof.** Assuming, without loss of generality, that $\beta \in P^{(a,b)} \subseteq P^k$, we have

$$\langle L^j(\alpha), L^j(\beta) \rangle_V = \text{vol}(L^j(\alpha) \wedge *_h(L^j(\beta^*)))$$

$$= (-1)^{\frac{k(k+1)}{2}} b^{-a} \frac{|j|_h!}{|n-j-k|_h!} \text{vol}(L^j(\alpha) \wedge L^{n-j-k}(\beta^*))$$

$$= \frac{|j|_h! |n-k|_h!}{|n-j-k|_h!} \text{vol}(\alpha \wedge *_h(\beta^*)) = \binom{n-j-k}{j}_h^{-1} \langle \alpha, \beta \rangle_V . \quad \square$$

**Corollary 5.6** If $\langle \cdot, \cdot \rangle_V$ restricts to an inner product on the space of primitive elements, then it is an inner product on all of $V^\bullet$.

We are now ready to introduce the inner product associated to an Hermitian structure and to establish the existence of adjoints with respect to this pairing.

**Lemma 5.7** For $*_h$ the Hodge map of a positive definite Hermitian structure, an inner product is given by

$$\langle \cdot, \cdot \rangle : \Omega^\bullet \otimes \Omega^\bullet \to \mathbb{C}, \quad \omega \otimes \nu \mapsto \int \omega \wedge *_h(\nu^*) = h(g(\omega \otimes_M \nu^*)). \quad (14)$$

Moreover, the Peter–Weyl decomposition of $\Omega^\bullet$ is orthogonal with respect to $\langle \cdot, \cdot \rangle$.

**Proof.** By Corollary 5.3, we need only establish positive definiteness. Let $\{ [\omega_k] \}_k$, for $\omega_k \in \Omega^\bullet$, be an orthonormal basis of $V^\bullet$ with respect to the inner product (13).

In what follows we denote $U(\omega) := \sum f_k \otimes [\omega_k]$, and tacitly assume the isomorphism $id \otimes \varepsilon = U^{-1} : G \boxtimes_H \Phi(M) \to M$. Noting that a morphism is given by $\overline{\sigma} := (id \otimes *)g$, positive definiteness of the bilinear form follows from

$$\langle \omega, \omega^* \rangle = h \circ (id \otimes \Phi(\overline{\sigma})) \circ U(\omega \otimes_M \omega^*) = \sum h(f_k f_l^*) [g(\omega_k \otimes_M \omega_l)]$$

$$= \sum h(f_k f_l^*) \in \mathbb{R}_{\geq 0} .$$

Orthogonality of the Peter–Weyl decomposition of $\Omega^\bullet$ is established similarly. \qed
Corollary 5.8 Any left $G$-comodule map $f : \Omega^\bullet \to \Omega^\bullet$ is adjointable with respect to $\langle \cdot, \cdot \rangle$. Moreover, if $f$ is self-adjoint, then it is diagonalisable, and commuting diagonalisable maps are simultaneously diagonalisable.

Proof. Since $f$ is a left $G$-comodule map $f(\Omega^*_V) \subseteq \Omega^*_V$, for all $V \in \hat{G}$. Adjointability of $f$ now follows from finite-dimensionality of $\Omega^*_V$ and the fact that Peter–Weyl decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle$. Analogously, $f$ can be shown to be diagonalisable whenever it is self-adjoint, and so, commuting diagonalisable maps can be shown to be simultaneously diagonalisable. □

Remark 5.9 In [22, §3] a calculus $\Omega^\bullet$ is defined to be non-degenerate if, whenever $\omega \in \Omega^k(M)$, and $\omega \wedge \nu = 0$, for all $\nu \in \Omega^{n-k}(M)$, then necessarily $\omega = 0$. Clearly, the existence of a positive definite Hermitian form for a $*$-calculus implies non-degeneracy.

5.3 Examples of Operator Adjoints

We now consider three explicit examples of adjointable operators: the Hodge map, the Lefschetz operator, and the differentials $d, \partial, \overline{\partial}$. The Hodge operator is shown to be unitary, while the adjoints of the other operators are shown to admit explicit descriptions in terms of the Hodge map. In the case of the Lefschetz map, this allows us to establish a $h$-deformation of the classical Lefschetz identities.

Note that throughout this subsection, we continue to assume that $\Omega^\bullet$ is a covariant $*$-calculus over a quantum homogeneous space $M$, and $(\Omega^\bullet, \kappa)$ is a covariant Hermitian structure. Moreover, $(\Omega^\bullet, \kappa)$ is assumed to be positive definite. To avoid confusion with the $*$-map, the symbol $^\dagger$ will be use to denote the adjoint of an operator.

5.3.1 Unitarity of the Hodge Map

Here we show that, just as in the classical case, $*_h$ is unitary. (Note that this property is assumed in the definition of the noncommutative Hodge map in [10, Definition 5.20].)

Lemma 5.10 For all values of the Hodge parameter $h$, the Hodge map is unitary.

Proof. For $\alpha, \beta \in P^{(a,b)} \subseteq P^k$, and $j \geq 0$, we have

$$\langle *_h(L^j(\alpha)), *_h(L^j(\beta)) \rangle = \int *_h(L^j(\alpha)) \wedge *_h^2(L^j(\beta^*))$$

$$= (-1)^{\frac{1}{2}(k+1)} \frac{j!h!}{[n-j-k]h!} \int L^{n-j-k}(\alpha) \wedge L^j(\beta^*)$$

$$= (-1)^{\frac{1}{2}(k+1)} \frac{j!h!}{[n-j-k]h!} \int L^j(\alpha) \wedge L^{n-j-k}(\beta^*)$$

$$= \int L^j(\alpha) \wedge *_h(L^j(\beta^*)) = \langle L^j(\alpha), L^j(\beta) \rangle.$$

The result now follows from orthogonality of the Lefschetz decomposition. □
5.3.2 The Dual Lefschetz Operator and the Lefschetz Identities

We now present an explicit formula for the adjoint of $L$ in terms of $\ast_h$, this is again a direct generalisation of a well-known classical formula [15, Lemma 1.2.2.3].

Lemma 5.11 It holds that $\Lambda := L^\dagger = \ast_h^{-1}$ $L \ast_h$.

Proof. For $\omega, \nu \in \Omega^k$, we have

$$\langle L(\omega), \nu \rangle = \int L(\omega) \wedge \ast_h(\nu^*) = \int \omega \wedge L \ast_h(\nu^*)$$

$$= \int \omega \wedge \ast_h\left(\ast_h^{-1} L \ast_h(\nu^*)\right) = \langle \omega, \ast_h^{-1} L \ast_h(\nu) \rangle . \quad \square$$

Classically the primitive forms are defined to be those contained in the kernel of $\Lambda$. The following corollary derives this as a consequence of our definition of primitive forms.

Corollary 5.12 It holds that $P^k = \ker(\Lambda : \Omega^k \to \Omega^{k-2})$

Proof. For $\alpha \in P^{(a,b)} \subseteq P^k$, the inclusion $P^k \subseteq \ker(\Lambda : \Omega^k \to \Omega^{k-2})$ follows from

$$\Lambda(\alpha) = \ast_h^{-1} L \ast_h(\alpha) = (-1) \frac{k(k+1)}{2} i^{a-b} \frac{1}{[n-k]!} \ast_h L^{n-k+1}(\alpha) = 0.$$

For the opposite inclusion consider, for $j > 0$,

$$0 = \Lambda(L^j(\alpha)) = \ast_h^{-1} L \ast_h(L^j(\alpha)) = (-1) \frac{k(k+1)}{2} i^{a-b} \frac{[j]!}{[n-j-k]!} \ast_h L^{n-j-k+1}(\alpha).$$

Since $\ast_h$ is an isomorphism, we must have $L^{n-j-k+1}(\alpha) = 0$, and so, that $\alpha \in P^{k+j}$. Since $P^k \cap P^{k+j} = 0$, we must $\alpha = 0$.

Consider now the counting operators

$$H, K : \Omega^* \to \Omega^*, \quad H(\omega) = (k-n)\omega, \quad K(\omega) = h^{k-n}\omega, \quad \omega \in \Omega^k.$$

For a classical Hermitian manifold the operators $H$, $L$, and $\Lambda$, define a representation of $\mathfrak{sl}_2$ [15, Proposition 1.2.26]. We now show that in the noncommutative setting $H, L, \Lambda$, and $K$ give a representation of the quantised enveloping algebra of $\mathfrak{sl}_2$.

Proposition 5.13 We have the relations

$$[H, L]_{h^2} = [2]_h LK, \quad [L, \Lambda] = H, \quad [H, \Lambda]_{h^2} = -[2]_h K \Lambda,$$

where $[A, B]_{h^2} = AB - h^{2} BA$.
Proof. Beginning with the first relation, for $\omega \in \Omega^k$,

$$[H, L]_{h^{-2}}(\omega) = H L(\omega) - h^{-2} L H(\omega) = \left((k + 2 - n)h - h^{-2}(k - n)h\right)L(\omega)$$

$$= h^{k-n}[2]_h + h^{-2}(k - n)h - h^{-2}(k - n)h)L(\omega)$$

$$= h^{k-n}[2]_h L(\omega) = [2]_h L K(\omega).$$

Noting that $H$ and $K$ are self-adjoint operators, we see that the third relation is the operator adjoint of the first.

Coming finally to the second relation, for $\alpha \in P^{(a,b)} \subseteq P^k$, we have

$$LA(L^j(\alpha)) = L L^j(\alpha) = L L^j((-1)^{j(k+1)} (-1)^{j(a-b)} [j]_h! \frac{[j]_h!}{[n-j-k]_h!} L^{n-j-k}(\alpha))$$

$$= (-1)^{\frac{j(k+1)}{2} + k a-b} \frac{[j]_h!}{[n-j-k]_h!} L L^j(\alpha)$$

$$= (-1)^{\frac{j(k+1)}{2} + k a-b} \frac{[j]_h!}{[n-j-k]_h!} L L^j(\alpha)$$

$$= [j]_h [n-j-k+1]_h L^j(\alpha).$$

Similarly, it can be shown that

$$\Lambda L(L^j(\alpha)) = [j+1]_h [n-j-k]_h L^j(\alpha).$$

Hence,

$$[L, \Lambda] L^j(\alpha) = ([j]_h [n-j-k+1]_h - [j+1]_h [n-j-k]_h) L^j(\alpha)$$

$$= h^{-1}[j]_h [n-j-k]_h + h^{n-j-k}[j]_h$$

$$- h^{-1}[j]_h [n-j-k]_h - h^j [n-j-k]_h) L^j(\alpha)$$

$$= [2j + k - n]_h L^j(\alpha) = H L^j(\alpha).$$

Clearly, for $h = 1$, we get a representation of the Lie algebra $\mathfrak{sl}_2$. For the case of $h \neq 1$, we get a representation of the quantised universal enveloping algebra of $\mathfrak{sl}_2$ (where we use the conventions presented in [19, §3.1.1]).

**Corollary 5.14** A representation $\rho$ of $U_h(\mathfrak{sl}_2)$ is given by

$$\rho(E) = L, \quad \rho(K) = K, \quad \rho(F) = \Lambda.$$

**Proof.** It is clear that $\rho(K K^{-1}) = \rho(K^{-1} K) = 1$. Moreover,

$$\rho(K E K^{-1}) = K L K^{-1} = h^2 L = h^2 \rho(E),$$

and

$$\rho(K F K^{-1}) = K \Lambda K^{-1} = h^{-2} L = h^{-2} \rho(E).$$

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Finally, for $\omega \in \Omega^k$, we have
\[
\rho(\Delta, \Lambda)(\omega) = H(\omega) = [k - n](\omega) = \frac{h^{k-n} - h^{-(k-n)}}{h - h^{-1}}\omega.
\]
\[
= \frac{K - K^{-1}}{h - h^{-1}}(\omega) = \rho(\frac{K - K^{-1}}{h - h^{-1}})(\omega).
\]
Finally, we describe the irreducible representations of $U_h(\mathfrak{sl}_2)$. Note that by taking appropriate unions, the Lefschetz decomposition can be reproduced from this decomposition. In fact, this is how the Lefschetz decomposition is established classically.

**Lemma 5.15** The irreducible representations of $U_h(\mathfrak{sl}_2)$ are given by
\[
\bigoplus_{j \geq 0} L^j(\alpha), \quad \alpha \in P^{(a,b)}, \ (a, b) \in \mathbb{N}^2.
\]

### 5.3.3 Codifferential Operators

We call the adjoints of $d, \partial$, and $\partial\bar{\partial}$ the **codifferential**, **holomorphic codifferential**, and **anti-holomorphic codifferential**, respectively. Classically, these operators have expressions in terms of the Hodge operator analogous to the expression given above for the dual Lefschetz operator. The following lemma shows that this is also true in the non-commutative setting.

**Lemma 5.16** It holds that
\[
d^\dagger = - \ast_h d^\ast_h, \quad \bar{\partial}^\dagger = - \ast_h \bar{\partial}^\ast_h, \quad \bar{\partial}^\dagger = - \ast_h \partial^\ast_h.
\]

**Proof.** For $\omega \in \Omega^k, \nu \in \Omega^\ast$, the Leibniz rule and closure of the integral imply that
\[
0 = \int \bar{\partial}(\omega \wedge \nu) = \int \bar{\partial}\omega \wedge \nu + (-1)^k \int \omega \wedge \bar{\partial}\nu,
\]
and so, $\int \bar{\partial}\omega \wedge \nu = (-1)^{k+1} \int \omega \wedge \bar{\partial}\nu$. This in turn implies that
\[
\langle \omega, \ast_h \partial \ast_h (\nu) \rangle = \int \omega \wedge \ast_h ((\ast_h \partial \ast_h (\nu))^\ast) = \int \omega \wedge \ast_h (\ast_h \bar{\partial} \ast_h (\nu^\ast))
\]
\[
= (-1)^k \int \omega \wedge \bar{\partial} \ast_h (\nu^\ast) = - \int \bar{\partial}\omega \wedge \ast_h (\nu^\ast)
\]
\[
= - \langle \bar{\partial}\omega, \nu \rangle.
\]
Hence, $\bar{\partial}^\dagger = - \ast_h \partial^\ast_h$. The identities for $d^\dagger$ and $\bar{\partial}^\dagger$ are established similarly. \qed

**Corollary 5.17** For all $\omega \in \Omega^\ast$, it holds that
\[
d^\dagger(\omega^\ast) = (d^\dagger(\omega))^\ast, \quad \bar{\partial}^\dagger(\omega^\ast) = (\bar{\partial}^\dagger(\omega))^\ast, \quad \bar{\partial}^\dagger(\omega^\ast) = (\partial^\ast(\omega))^\ast. \quad (15)
\]

**Proof.** This follows from the given formulae for the codifferentials, the fact that the $\ast$-map commutes with the Hodge map, and identities given in (10). \qed
5.4 Positive Definiteness for the Heckenberger–Kolb Calculus

We begin by directly verifying positive definiteness of $\kappa$ in the two simplest cases $\mathbb{C}_q[\mathbb{C}P^1]$ and $\mathbb{C}_q[\mathbb{C}P^2]$. Throughout, by abuse of notation, we will write $\star_q$ for $\Phi(\star_q)$.

**Example 5.18.** In this example we will verify positiveness for $\mathbb{C}_q[\mathbb{C}P^1]$. By Lemma 5.5 we only need to show positiveness on non-trivial primitive elements. By definition these elements are all contained in $V^1$. For $V^{(0,1)}$ it holds that

$$\langle e_1^\top, e_1^- \rangle_{V} = q^4 \text{vol}(e_1^- \wedge \star_q(e_1^\top)) = -iq^4 \text{vol}(e_1^- \wedge e_1^\top) = iq^6 \text{vol}(e_1^\top \wedge e_1^-) = q^6.$$

Similarly, it can be shown that $\langle e_1^\top, e_1^+ \rangle_{V} = q^4$. Orthogonality of $e_1^\top$ and $e_1^-$ follows from $\langle e_1^\top, e_1^- \rangle_{V} = q^4$ and the analogous calculation for $\langle e_1^-, e_1^\top \rangle_{V}$. Hence $\langle \cdot, \cdot \rangle_{V}$ is indeed positive definite.

**Example 5.19.** We now turn to $\mathbb{C}_q[\mathbb{C}P^2]$. By Lemma 5.5 we only need to show positive definiteness on non-trivial primitive elements. By definition these elements are all contained in $V^1$ and $V^2$. For $V^1$, we have

$$\|e_1^\top\|_V := \langle e_1^\top, e_1^\top \rangle_{V} = \text{vol}(e_1^\top \wedge \star_q((e_1^\top)^*)) = q^{-4} \text{vol}(e_1^\top \wedge \star_q(e_1^-)) = q^{-5},$$

and similarly $\|e_2^\top\|_V = q^{-5}$, $\|e_1^-\|_V = q^7$, and $\|e_2^-\|_V = q^9$. Orthogonality of the spaces $V^{(1,0)}$ and $V^{(0,1)}$ follows from Lemma 5.2. For $e_1^\top, e_2^\top$, we have

$$\langle e_1^\top, e_2^\top \rangle_{V} = \text{vol}(e_1^\top \wedge \star_q(e_2^-)) = -i\text{vol}(e_1^\top \wedge \kappa \wedge e_2^-) = 0,$$

and similarly that $\langle e_1^\top, e_2^- \rangle_{V} = \langle e_2^\top, e_1^- \rangle_{V} = \langle e_2^-, e_1^- \rangle_{V} = 0.$

For $P^{(2,0)} = V^{(2,0)}$ and $P^{(0,2)} = V^{(0,2)}$ we have $\|e_1^\top \wedge e_2^\top\|_V = q^{-11}$ and $\|e_1^- \wedge e_2^-\|_V = q^{17}$.

Finally, for $P^{(1,1)}$ the basis elements $\{e_1^\top \wedge e_2^\top, e_2^\top \wedge e_1^- \wedge e_1^\top \wedge q^{-2} e_2^\top \wedge e_2^-\}$ are easily seen to be orthogonal. Moreover, we have

$$\langle e_1^\top \wedge e_2^- \wedge e_1^- \wedge e_2^- \rangle_{V} = \text{vol}(e_1^\top \wedge e_2^- \wedge \star_q((q^{-2} e_2^\top \wedge e_2^-)^*)) = -q^2 \text{vol}(e_1^\top \wedge e_2^- \wedge e_1^\top \wedge e_2^-)$$

$$= q^3 \text{vol}(e_1^\top \wedge e_2^- \wedge e_1^\top \wedge e_2^-) = q^3,$$

and $\|e_2^\top \wedge e_1^-\|_V = q$ and $\|e_1^\top \wedge e_1^- \wedge q^{-2} e_2^\top \wedge e_2^-\|_V = |2|_q$. Hence $\langle \cdot, \cdot \rangle_{V}$ does indeed give us an inner product.

From these examples it is easy to see that orthogonality of the basis elements $e_I$ with respect to $\langle \cdot, \cdot \rangle$ extends to the general $\mathbb{C}_q[\mathbb{C}P^n]$ case, as presented in the following lemma.

**Lemma 5.20** For $\mathbb{C}_q[\mathbb{C}P^n]$ the basis $\{e_I\}_I$ is orthogonal with respect to $\langle \cdot, \cdot \rangle_{V}$.
Directly extending positive definiteness to the general $\mathbb{C}_q[\mathbb{C}P^n]$ case proves more challenging and we postpone the technical details to a subsequent work. However, using a general argument, we can prove positive definiteness for the case of $q$ contained in a certain open interval in $\mathbb{R}$ around 1.

**Lemma 5.21** There exists an open real interval around 1, such that when $q$ is contained in this interval, the Hermitian structure $(\Omega^{\bullet,\bullet}, \kappa)$ is positive definite.

**Proof.** For $q = 1$, $V^\bullet$ is just the exterior algebra of $V$. Hence, we can extend the restriction $\langle \cdot, \cdot \rangle_V : V^1 \otimes V^1 \to \mathbb{C}$, to a positive definite bilinear pairing on $V^\bullet$ using the standard determinant formula. Moreover, by Weil’s formula it must coincide with $\langle \cdot, \cdot \rangle_V : V^\bullet \otimes V^\bullet \to \mathbb{C}$, which must then be positive definite.

For a general $q$, and some basis element $e_I$, consider the polynomial function in $q$ given by

$$g_I : \mathbb{R}_{>0} \to \mathbb{C}, \quad q \mapsto \langle e_I, e_I \rangle_V.$$

As is easily checked, $g_I$ has real coefficients, and so, by an elementary continuity argument, there exists an open interval in $\mathbb{R}$ around 1 on which it takes positive real values. Taking the (finite) intersection of these intervals of over all basis elements gives the required interval. \qed

### 6 Hodge Theory

In this section, Hodge decomposition with respect to $d$, $\partial$, and $\overline{\partial}$, is established and shown to imply an isomorphism between cohomology classes and harmonic forms, just as in the classical case. A noncommutative generalisation of Serre duality is also proved. These results give us some powerful tools with which to approach questions about cohomology. Most of the material in subsections 6.1 and 6.2 are generalisations to the quantum homogeneous space setting of results proven for quantum groups in [22].

Throughout this section, $\Omega^\bullet$ denotes a covariant $\ast$-calculus, of total dimension $2n$, over a quantum homogeneous space $M$. Moreover, $(\Omega^{\bullet,\bullet}, \kappa)$ denotes a positive definite covariant Hermitian structure such that the associated integral is closed.

#### 6.1 Laplacians and Harmonic Forms

Directly generalising the classical situation, we define the $d$-, $\partial$-, and $\overline{\partial}$-Laplacians to be, respectively,

$$\Delta_d := (d + d^\dag)^2, \quad \Delta_\partial := (\partial + \partial^\dag)^2, \quad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^\dag)^2.$$

Moreover, we define the space of $d$-harmonic, $\partial$-harmonic, and $\overline{\partial}$-harmonic forms to be, respectively,

$$\mathcal{H}_d := \ker(\Delta_d), \quad \mathcal{H}_\partial := \ker(\Delta_\partial), \quad \mathcal{H}_{\overline{\partial}} := \ker(\Delta_{\overline{\partial}}).$$
When $\Omega^\bullet$ is a covariant calculus over a quantum homogeneous space, $\Delta_d, \Delta_\partial,$ and $\Delta_{\overline{\partial}}$ are left $G$-comodule maps, and so, each space of harmonic forms is a left $G$-comodule. With respect to the $\mathbb{N}_0$-grading on the calculus, $\Delta_d$ is a homogeneous map of degree 0, implying the decomposition $H_d = \bigoplus_{k \in \mathbb{N}_0} H_d^k$. Moreover, $\Delta_\partial$ and $\Delta_{\overline{\partial}}$ are homogeneous maps of degree 0 with respect to the $\mathbb{N}_2^0$-grading, implying the decompositions $H_\partial = \bigoplus_{(a,b) \in \mathbb{N}_2^0} H_\partial^{a,b}$ and $H_{\overline{\partial}} = \bigoplus_{(a,b) \in \mathbb{N}_2^0} H_{\overline{\partial}}^{a,b}$. Note that $\Delta_d$ is not necessarily homogeneous with respect to the $\mathbb{N}_2^0$-grading, and so, such a decomposition is not guaranteed to exist (see Corollary 7.6).

6.2 The Hodge Decomposition

We now come to Hodge decomposition, the principal result of this section, which allows us to prove statements about the cohomology ring $H^\bullet$ which are independent of any choice of Hermitian structure. The fact that we can prove such statements is one of the principal justifications we provide for introducing Hermitian structures.

**Lemma 6.1** It holds that

1. $H_d \simeq \ker(d) \cap \ker(d^\dagger)$,
2. $H_\partial \simeq \ker(\partial) \cap \ker(\partial^\dagger)$,
3. $H_{\overline{\partial}} \simeq \ker(\overline{\partial}) \cap \ker(\overline{\partial}^\dagger)$.

**Proof.** Since $d + d^\dagger$ and $\Delta_d$ are commuting self-adjoint $G$-comodule maps, it follows from Corollary 5.8 that they are simultaneously diagonalisable, and in particular that their kernels coincide. Now since the codomains of $d$ and $d^\dagger$ are orthogonal, we must have $\ker(d + d^\dagger) = \ker(d) \cap \ker(d^\dagger)$, which proves that first identity. The proofs of the other two identities are analogous. \qed

**Theorem 6.2** The following decompositions are orthogonal with respect to $\langle \cdot, \cdot \rangle$

1. $\Omega^\bullet \simeq H_d \oplus d\Omega^\bullet \oplus d^\dagger \Omega^\bullet$,
2. $\Omega^\bullet \simeq H_\partial \oplus \partial\Omega^\bullet \oplus \partial^\dagger \Omega^\bullet$,
3. $\Omega^\bullet \simeq H_{\overline{\partial}} \oplus \overline{\partial}\Omega^\bullet \oplus \overline{\partial}^\dagger \Omega^\bullet$.

**Proof.** Since both $dd^\dagger$ and $d^\dagger d$ are self-adjoint left $G$-comodule maps, Corollary 5.8 implies that they are diagonalisable. Moreover, since $(dd^\dagger)(d^\dagger d) = 0 = (d^\dagger d)(dd^\dagger)$, they commute and hence are simultaneously diagonalisable.

Denoting the simultaneous eigenbasis by $\{b_i\}_{i \in I}$, let $\lambda_i$ and $\mu_i$ be the eigenvalues determined by $dd^\dagger b_i = \lambda_i b_i$ and $d^\dagger d b_i = \mu_i b_i$. Since $(dd^\dagger)(d^\dagger d) = 0$, we have $\lambda_i \mu_i = 0$, for all $i \in I$. If $\lambda_i \neq 0$, then $b_i = d(d^\dagger(\lambda_i^{-1} b_i)) \in d\Omega^\bullet$. Similarly, if $\mu_i \neq 0$, then...
\[ b_i = d^i(d(\mu_i^{-1}b_i)) \in d^i\Omega^* \]. Finally, if \( \lambda_i = \mu_i = 0 \), then \( b_i \in H^*_d \). This implies that 
\[ \Omega^* = H_d + d(\Omega^*) + d^i(\Omega^*) \].

We now show that this is an orthogonal decomposition. Since 
\[ \langle d\omega, d^i\nu \rangle = \langle d^2\omega, \nu \rangle = 0 \], 
the spaces \( d\Omega \) and \( d^i\Omega \) are orthogonal. Orthogonality of 
\( H_d \) and \( d\Omega \oplus d^i\Omega \) follows from
\[ \langle d\omega + d^i\nu, \rho \rangle = \langle \omega, d^i\rho \rangle + \langle \nu, d\rho \rangle = 0, \quad \omega, \nu \in \Omega^*, \rho \in H_d. \]
The other two isomorphisms are established analogously. □

**Corollary 6.3** It holds that
\[ \ker(d) \simeq H_d \oplus d\Omega^*, \quad \ker(\partial) \simeq H_\partial \oplus d\Omega^*, \quad \ker(\overline{\partial}) \simeq H_{\overline{\partial}} \oplus d\Omega^*, \]
and so, we have the isomorphisms
\[ H^k_d \to H^k_d, \quad H^{(a,b)}_\partial \to H^{(a,b)}_\partial, \quad H^{(a,b)}_{\overline{\partial}} \to H^{(a,b)}_{\overline{\partial}}. \]

**Proof.** For any \( \omega \in \Omega^* \) such that \( dd^i\omega = 0 \), we have \( 0 = \langle dd^i\omega, \omega \rangle = \langle d^i\omega, d\omega \rangle \), and so, by positive definiteness \( d^i\omega = 0 \), implying that \( \ker(d) \cap d^i\Omega^* = 0 \). Hodge decomposition and Lemma 6.1 now imply that \( \ker(d) \simeq H_d \oplus d\Omega^* \). The other two isomorphisms are established analogously. □

**Corollary 6.4** Any linear map \( A : \Omega^* \to \Omega^* \) which commutes with the Laplacian \( \Delta_d \) induces a unique map on \( H^* \) for which the following diagram is commutative:

\[
\begin{array}{ccc}
H^*_d & \overset{\simeq}{\longrightarrow} & H^*_d \\
A \downarrow & & A \downarrow \\
H^*_d & \overset{\simeq}{\longrightarrow} & H^*_d
\end{array}
\]

Moreover, if \( A \) restricts to an isomorphism \( H^k_d \to H^l_d \), for some \( k, l \in \mathbb{N}_0 \), then the corresponding map \( H^k_{\partial} \to H^l_{\partial} \) is also an isomorphism. The analogous results hold for \( \Delta_\partial \) and \( \Delta_{\overline{\partial}} \).

**Proof.** If \( A \) commutes with the Laplacian, then clearly it maps harmonic forms to harmonic forms, and so, by Hodge decomposition it induces a map on \( H^* \). Since the map \( A^{-1} : \Omega^l \to \Omega^k \) must also commute with the Laplacian, we must have an inverse \( A^{-1} : H^l_d \to H^k_d \). The proofs for \( \Delta_\partial \) and \( \Delta_{\overline{\partial}} \) are analogous. □

Using this corollary, we show that the Hodge map and the \( * \)-map induce isomorphisms on the cohomology ring of \( \Omega^* \), and present some easy but interesting consequences.

**Lemma 6.5** The Hodge map \( \ast_h \), and the \( * \)-map, commute with the Laplacian \( \Delta_d \), and so, induce isomorphisms on \( H^*_d \).
Proof. The fact that the ∗-map commutes with the ∆_d follows directly from Corollary 5.17. For the Hodge map, note that, for any ω ∈ Ω^k,

\[ [\ast_h, \Delta_d](\omega) = \ast_h (d(d^1 + d^1)\omega) - (d(d^1 + d^1)\ast_h(\omega) \]
\[ = - \ast_h d \ast_h d(\omega) - \ast_h^2 d \ast_h d(\omega) + d \ast_h d \ast_h(\omega) + \ast_h d \ast_h d \omega \]
\[ = (-1)^{2n-k}d \ast_h d(\omega) - (-1)^k d \ast_h d(\omega) = 0. \]

\[ \square \]

Corollary 6.6 If the cohomology ring \( H^*_d \) has finite dimension, then

\[ \dim(H^{2k+1}_d) \in 2\mathbb{N}_0, \quad \text{for all } k = 0, \ldots, n-1. \]

Proof. Since the ∗-map induces an isomorphism between \( H^{(a,b)}_d \) and \( H^{(b,a)}_d \), it implies that they have equal dimension. Evenness of \( \dim(H^{2k+1}_d) \) now follows from

\[ \dim(H^{2k+1}_d) = \sum_{i=0}^{2k+1} \dim(H^{2k+1-i,i}_d) = 2 \sum_{i=0}^{k} \dim(H^{(2k+1-i,i)}_d) \in 2\mathbb{N}_0. \]

\[ \square \]

Corollary 6.7 It holds that \( H^{2n}_d \neq 0 \).

Proof. Since \( \Delta_d(1) = 0 \), we have \( H^0_d \neq 0 \). The result now follows from the isomorphism \( \ast_h : H^0_d \to H^{2n}_d \).

\[ \square \]

6.3 Serre Duality

We finish the section with a proof of Serre duality for Dolbeault cohomology, following the standard proof in [15, §3.2]. (See also [32] for a discussion of Serre duality from a noncommutative algebraic geometry point of view.)

Proposition 6.8 Non-degenerate pairings are given by

\[ H^{(a,b)}_\partial \times H^{(n-a,n-b)}_\partial \to \mathbb{C}, \quad ([\alpha], [\beta]) \rightarrow \int \alpha \wedge \beta, \]

and the analogous pairing for \( \mathcal{H}^*_\partial \).

Proof. Recalling that \( \alpha \) and \( \beta \) are \( \partial \)-closed forms and that \( \int \) is assumed to be closed, the fact the pairing is well-defined follows from

\[ \int (\alpha + \overline{\partial} \omega) \wedge (\beta + \overline{\partial} \nu) = \int \alpha \wedge \beta + \int \overline{\partial} \omega \wedge \beta + \int \alpha \wedge \overline{\partial} \nu + \int \overline{\partial} \omega \wedge \overline{\partial} \nu \]
\[ = \int \alpha \wedge \beta + \int \overline{\partial}(\omega \wedge \beta) + (-1)^{a+b} \int \overline{\partial}(\alpha \wedge \nu) + \int \overline{\partial}(\omega \wedge \overline{\partial} \nu) \]
\[ = \int \alpha \wedge \beta. \]

Next note that for any nonzero \( \omega \in \mathcal{H}^{(a,b)}_\partial \), the form \( \ast_h(\omega^*) \) is an element of \( \mathcal{H}^{(n-a,n-b)}_\partial \). Since \( \int \omega \wedge \ast_h(\omega^*) = \langle \omega, \omega \rangle \), the pairing must be non-degenerate.

\[ \square \]
Corollary 6.9 If $Ω^\bullet$ has finite dimensional $∂$- and $\overline{∂}$-cohomology groups, then
$$H^a_b(Ω) \simeq (H^a_b)^*,$$  
$$H^a_b(Ω) \simeq (H^a_b)^*.$$  

7 Noncommutative Kähler Structures

In this section the definition of a noncommutative Kähler structure is introduced and some of the basic results of classical Kähler geometry generalised, most notably the Kähler identities. Equality up to scalar multiple of the three Laplacians $Δ_d, Δ_∂,$ and $Δ_{\overline{∂}},$ then follows, implying in turn that Dolbeault cohomology refines de Rham cohomology. A noncommutative generalisation of the hard Lefschetz theorem and the $∂\overline{∂}$-lemma is then given. The Hermitian structure of $C_q[CP^n]$ is observed to be Kähler, implying that the calculus has cohomology groups of at least classical dimension. Finally, we finish with some spectral calculations and a conjecture about constructing spectral triples for $C_q[G/L_S].$

Throughout this section, $Ω^\bullet$ denotes a covariant $*$-calculus, of total dimension $2n,$ over a quantum homogeneous space $M.$ Moreover, $(Ω^\bullet, κ)$ denotes a positive definite covariant Hermitian structure such that the associated integral is closed.

7.1 Kähler Structures and the First Set of Kähler Identities

Building on the definition of an Hermitian structure, we define the notion of a Kähler structure. In the classical case this reduces to the fundamental form of a uniquely defined Kähler metric [15, §3.1].

Definition 7.1. A Kähler structure for a differential $*$-calculus is an Hermitian structure $(Ω^\bullet, κ)$ such that the Hermitian form $κ$ is d-closed. We call such a $κ$ a Kähler form.

Every 2-form in a $*$-calculus with total dimension 2 is obviously d-closed. Hence, just as in the classical case [15, §3.1], with respect to any choice of complex structure, every $κ \in Ω^{(1,1)}$ is a Kähler form.

We now prove the first set of Kähler identities. While they follow more or less immediately from the closure of the Kähler form, they have important implications throughout the remainder of the paper.

Lemma 7.2 For any Kähler structure $(Ω^\bullet, κ),$ we have the following relations
$$[∂, L] = 0, \quad [\overline{∂}, L] = 0, \quad [∂^\dagger, Λ] = 0, \quad [\overline{∂}^\dagger, Λ] = 0.$$  

Proof. By definition a Kähler form satisfies $∂κ = 0,$ and so,
$$[∂, L](α) = ∂(κ ∧ α) − κ ∧ ∂α = ∂κ ∧ α + κ ∧ ∂α − κ ∧ ∂α = 0.$$
Analogously, $[\overline{\partial}, L] = 0$. The remaining two identities are the adjoints of the first two.

□

**Corollary 7.3** For every nonzero $\alpha \in P^k$, there exist unique forms $\alpha_0^+, \alpha_0^- \in P^{k+1}$, $\alpha_1^+, \alpha_1^- \in P^{k-1}$ such that

\[
\partial \alpha = \alpha_0^+ + L(\alpha_1^+), \quad \overline{\partial} \alpha = \alpha_0^- + L(\alpha_1^-).
\]

**Proof.** Using the Lefschetz decomposition, $\partial \alpha \in \Omega^{k+1}$ can be written as

\[
\partial \alpha = \sum_{j \geq 0} L_j(\alpha_j), \quad \alpha_j \in P^{k+1-2j}.
\]

Since $L$ commutes with $\partial$ and $L^{n-k+1}(\alpha) = 0$, we must have $0 = \sum_{j \geq 0} L^{n-k+1+j}(\alpha_j)$. Moreover, since the Lefschetz decomposition is a direct sum decomposition, we have $L^{n-k+j+1}(\alpha_j) = 0$, for all $j \geq 0$.

Now it is only for $j \leq 1$ that $\alpha_j$ can be contained in $\ker(L^{n-k+j+1})$. Hence $\alpha_j = 0$ for all $j > 2$, and the required identity for $\partial$ follows. Uniqueness of $\alpha_0^+$ is clear. Uniqueness of $\alpha_1^+$ follows from it being a form of degree at most $n-1$ and $L$ having trivial kernel in the space of such forms. The proof for the case of $\overline{\partial}$ is analogous.

7.2 The Second Set of Kähler Identities

In this section we prove the second set of Kähler identities and use them to generalise to the noncommutative setting one of the most important results of Kähler geometry, namely that Dolbeault cohomology is a refinement of de Rham cohomology. Throughout this subsection we adopt the useful convention $L^j = 0$, when $j$ is a negative integer.

**Lemma 7.4** For $\alpha \in P^k$, it holds that $\Lambda L^j(\alpha) = [j]_h[n-j-k+1]_h L^{j-1}(\alpha)$, for $j > 0$.

**Proof.** Assuming without loss of generality that $\alpha \in P^{(a,b)} \subseteq P^k$, the result follows from

\[
\Lambda L^j(\alpha) = *_{h^{-1}} L *_{h} L^j(\alpha) = (-1)^{\frac{k(k+1)}{2}} h^{a-b} \frac{[j]_h}{[n-j-k]_h} *_{h^{-1}} L^{n-j-k+1}(\alpha) = [j]_h[n-j-k+1]_h L^{j-1}(\alpha).
\]

**Theorem 7.5** The four identities

\[
[L, \partial] = i\overline{\partial}, \quad [L, \partial^\dagger] = -i\partial, \quad [\Lambda, \partial] = i\overline{\partial}^\dagger, \quad [\Lambda, \overline{\partial}] = -i\partial^\dagger
\]

hold in both of the following cases:

1. the Hodge parameter is fixed at $h = 1$,  

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Moving now to the right-hand side of the proposed identity, we see that
\[ h \in \{ \alpha \in P^{(a,b)} : h \subseteq \mathbf{P}^k \} \]
\[ \text{in the first case, that is when } h = 1, \text{ we have} \]
\[ \Lambda \partial (L^j(\alpha)) = \Lambda(L^j(\partial \alpha)) = \Lambda L^j(\alpha_0^+ + L(\alpha_1^+)) = \Lambda L^j(\alpha_0^+) + \Lambda L^j(\alpha_1^+) \]
\[ = [j]_h[n - j - (k + 1) + 1]_h L^j(\alpha_0^+) \]
\[ + [j + 1]_h[n - (j + 1) - (k - 1) + 1]_h L^j(\alpha_1^+) \]
\[ = [j]_h[n - j - k]_h L^j(\alpha_0^+) + [j + 1]_h[n - j - k + 1]_h L^j(\alpha_1^+). \]

It follows from the above lemma that
\[ \partial \Lambda (L^j(\alpha)) = [j]_h[n - j - k + 1]_h (L^j(\alpha_0^+) + L^j(\alpha_1^+)). \]

Putting these two result together gives
\[ [\Lambda, \partial] (L^j(\alpha)) = ([j]_h[n - j - k]_h - [j]_h[n - j - k + 1]_h) L^j(\alpha_0^+) \]
\[ + ([j + 1]_h[n - j - k + 1]_h - [j]_h[n - j - k + 1]_h) L^j(\alpha_1^+). \]

Moving now to the right-hand side of the proposed identity, we see that
\[ i\partial^j (L^j(\alpha)) = - i \ast h \partial \ast h (L^j(\alpha)) \]
\[ = (-1)^{i(k+1)} \frac{i^a - i^{b-1}}{a - b} \frac{[j]_h^1}{[n - j - k]_h^1} \ast h (L^{n-j-k}(\partial(\alpha))) \]
\[ = (-1)^{i(k+1)} \frac{i^a - i^{b-1}}{a - b} \frac{[j]_h^1}{[n - j - k]_h^1} \ast h (L^{n-j-k}(\alpha_0^+) + L^{n-j-k-1}(\alpha_1^+)) \]
\[ = - [j]_h L^j(\alpha_0^+) + [n - j - k + 1]_h L^j(\alpha_1^+). \]

We are now ready to show that the third identity holds in both cases considered above. In the first case, that is when \( h = 1 \), we have
\[ [\Lambda, \partial] (L^j(\alpha)) = (j(n - j - k) - j(n - j - k + 1)) L^j(\alpha_0^+) \]
\[ + ((j + 1)(n - j - k + 1) - j(n - j - k + 1)) L^j(\alpha_1^+) \]
\[ = - j L^{j-1}(\alpha_0^+) + (n - j - k + 1) L^j(\alpha_1^+) \]
\[ = i\partial^j (L^j(\alpha)). \]

In the second case, that is when \( j = 0 \), we have
\[ [\Lambda, \partial] (\alpha) = [n - k + 1]_h (\alpha_1^+) = i\partial^j (\alpha). \]

We now move on to the fourth identity, starting with the case of \( h = 1 \). It follows from the third identity that
\[ ( - i\partial^* (\omega)^* = i\partial^j (\omega^*) = [\Lambda, \partial] (\omega^*) = ([\Lambda, \partial] (\omega))^*, \quad \omega \in \Omega^*. \]
Hence, \([\Lambda, \partial] = -i\partial^*\) as required. The second case, that is when \(j = 0\), is proved analogously using the fact that \(P^*\) is closed under the \(*\)-map.

Finally, we come to the first two identities. For the case of \(h = 1\), they are obtained as the adjoints of the first two with respect to the associated inner product. For the case of \(j = 0\), note that the explicit formulae calculated above for the action of \(i\partial^\dagger\) and \([\Lambda, \partial]\) on \(L^j(\alpha)\) imply that \(\partial^\dagger, [\Lambda, \partial], \) and \([\Lambda, \partial]\), each map \(P^\dagger\) to itself. Since the Lefschetz decomposition is orthogonal with respect to the inner product, this means that, for \(j = 0\), the first and second formulae can also be obtained by taking adjoints. □

**Corollary 7.6** When the Hodge parameter is fixed at \(h = 1\), it holds that

\[
\partial^\dagger \partial + \partial \partial^\dagger = 0, \quad \partial^\dagger \partial + \partial \partial^\dagger = 0, \quad \Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}.
\]

**Proof.** The first identity follows from

\[
i(\partial^\dagger + \partial \partial^\dagger) = \partial[\Lambda, \partial] + [\Lambda, \partial] = \partial^2 \Lambda - \partial \Lambda^2 - \partial \Delta \partial = 0.
\]

The second identity is the operator adjoint of the first.

Moving on to the third identity, we note first that

\[
\Delta_d = dd^\dagger + d^\dagger d = (\partial + \partial)(\partial^\dagger + \partial^\dagger) + (\partial^\dagger + \partial)(\partial + \partial^\dagger)
\]

\[
= (\partial \partial^\dagger + \partial^\dagger \partial) + (\partial \partial^\dagger + \partial^\dagger \partial) + (\partial \partial^\dagger + \partial^\dagger \partial) + (\partial \partial^\dagger + \partial^\dagger \partial)
\]

\[
= \Delta_\partial + \Delta_{\overline{\partial}}.
\]

It remains to show that \(\Delta_\partial = \Delta_{\overline{\partial}}\)

\[
-i\Delta_\partial = -i(\partial^\dagger + \partial \partial^\dagger) = \partial[\Lambda, \partial] + [\Lambda, \partial] = \partial^2 \Lambda - \partial \Lambda^2 - \partial \Delta \partial = 0.
\]

Proportionality of the Laplacians obviously implies equality of harmonic forms:

\[
\mathcal{H}_d^k \cong \bigoplus_{a+b=k} \mathcal{H}_{\partial}^{(a,b)} = \bigoplus_{a+b=k} \mathcal{H}_{\overline{\partial}}^{(a,b)}.
\]

Hence, Corollary 6.3 implies the following decomposition of cohomology classes.

**Corollary 7.7** De Rham cohomology is refined by Dolbeault cohomology, which is to say,

\[
\mathcal{H}_d^k \cong \bigoplus_{a+b=k} \mathcal{H}_{\partial}^{(a,b)} \cong \bigoplus_{a+b=k} \mathcal{H}_{\overline{\partial}}^{(a,b)}.
\]

Moreover, the decomposition is independent of the choice of Kähler form.
Proof. We just need to show independence of the decomposition. For $\kappa'$ another Kähler form, denote by $\mathcal{H}_\partial^{(a,b)}(\kappa')$ the corresponding space of harmonic forms. For $\omega \in \mathcal{H}_\partial^{(a,b)}$, let $\nu$ be the corresponding element in $\mathcal{H}_\partial^{(a,b)}(\kappa')$ with respect to the commutative diagram:

$$\begin{array}{ccc}
\mathcal{H}_\partial^{(a,b)} & \simeq & \mathcal{H}_\partial^{(a,b)}(\kappa') \\
\downarrow & & \downarrow \\
H^{(a,b)} & \simeq & H^k \simeq H^k_d.
\end{array}$$

We want to show that $\omega = \nu + d\rho'$, for some $\rho' \in \Omega^{(a-1,b-1)}$. We note first that $\omega = \nu + \partial\rho'$, for some $\rho \in \Omega^{(a-1,b)}$. Moreover, $\partial\rho'$ is d-closed because $d(\partial\rho') = d(\omega - \nu) = 0$. By Hodge decomposition with respect to $d$, this means $\partial\rho'$ is the sum of a harmonic form and a d-exact form. But Corollary 6.3 tells us that $\partial\rho'$ is contained in a complementary subspace to $\mathcal{H}^\bullet$. Hence, it must be d-exact and independence of the decomposition follows. \qed

7.3 Harmonic Forms and the Hodge Parameter

We begin by showing that the Lefschetz and dual Lefschetz operators commute with the Laplacian $\Delta_d$, and hence that they induce operators on the space of harmonic forms.

Lemma 7.8 When the Hodge parameter is fixed at $h = 1$,

$$[L, \Delta_d] = [\Lambda, \Delta_d] = 0.$$  

Proof. Using the Kähler identity $L\overline{\partial}^j = \overline{\partial}^j L - i\partial$, and proportionality of the Laplacians, we see that

$$\frac{1}{2}L\Delta_d = L\Delta\overline{\partial} = L(\overline{\partial}\partial^j + \overline{\partial}^j) = \overline{\partial}L\partial^j + (\overline{\partial}^j L - i\partial)\partial
= \overline{\partial}(\overline{\partial}^j L - i\partial) + \overline{\partial}^j\partial L + i\partial\partial = (\overline{\partial}\partial^j + \overline{\partial}^j L)
= \Delta_\partial L = \frac{1}{2}\Delta_d L.$$  

The second relation is the adjoint of the first. \qed

Up to this point we have avoided the question of whether the the space of harmonic forms depends on the Hodge parameter. The following lemma and its corollary provides an answer to this question.

Lemma 7.9 When the Hodge parameter is fixed at $h = 1$, a form $\omega$ with Lefschetz decomposition $\omega = \sum_j \Lambda^j(\alpha_j)$, for $\alpha_j \in \mathbb{P}^\bullet$, is harmonic if and only if $\alpha_j$ is d-closed, for all $j$.  

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Proof. Lemma 7.4 implies that
\[ \Lambda^m(\omega) = \Lambda^m \left( \sum_{j=1}^{m} L_j(\alpha_j) \right) = \Lambda^m L^m(\alpha_m) = \left( \prod_{j=1}^{m} j(n-j-k+1) \right) \alpha_m. \]

By the above lemma, \( \mathcal{H}^\bullet \) is closed under \( L \) and \( \Lambda \), and so, if \( \omega \in \mathcal{H}^\bullet \) then \( \alpha_m \in \mathcal{H}^\bullet \). This in turn implies that
\[ \omega - L^m(\alpha_m) = \sum_{j=1}^{m-1} L_j(\alpha_j) \in \mathcal{H}^\bullet. \]

Repeated applications of this argument show that if \( \omega \in \mathcal{H}^\bullet \) then \( \alpha_j \in \mathcal{H}^\bullet \), for all \( j \).

It remains to show that a primitive form is harmonic if and only if it is d-closed. Clearly, we need only show that d-closure implies harmonicity. This follows from Lemma 6.1 and the fact that, for \( \alpha \in P^{a,b} \subseteq P^k \), we have
\[ d^*\alpha = *_1 d *_1 (\alpha) = (-1)^{k(k+1)/2} i^{a-b} \frac{1}{(n-k)!} *_1 d(L^{n-k}(\alpha)) \]
\[ = (-1)^{k(k+1)/2} i^{a-b} \frac{1}{(n-k)!} *_1 L^{n-k}(d\alpha) = 0. \]

The proposition shows us that, for \( h = 1 \), the space of harmonic forms is completely determined by the d-closed primitive forms. The following corollary tells us that this is also the case when \( h \neq 1 \).

Corollary 7.10 For any choice of Hodge parameter \( h \in \mathbb{R}_{>0} \), it holds that
\[ \mathcal{H}^\bullet_0 = \mathcal{H}^\bullet_{\partial} = \mathcal{H}^\bullet_{\partial^*} = \text{span}_\mathbb{C} \{ L_j(\alpha) \mid j \in \mathbb{N}_0, \alpha \in P^\bullet \cap \ker(d) \}. \]

Proof. The fact that \( L_j(\alpha) \in \mathcal{H}^\bullet_0, \mathcal{H}^\bullet_{\partial}, \) and \( \mathcal{H}^\bullet_{\partial^*} \) for any value of \( h \), is shown just as in the \( h = 1 \) case. Hence
\[ \text{span}_\mathbb{C} \{ L_j(\alpha) \mid j \in \mathbb{N}_0, \alpha \in P^\bullet \cap \ker(d) \} \subseteq \mathcal{H}^\bullet_0, \mathcal{H}^\bullet_{\partial}, \mathcal{H}^\bullet_{\partial^*}. \]

Since the cohomology groups \( \mathcal{H}^\bullet_0, \mathcal{H}^\bullet_{\partial}, \) and \( \mathcal{H}^\bullet_{\partial^*} \) are defined independently of \( h \), Corollary 6.3 now implies that these inclusions are equalities.

7.4 The Hard Lefschetz Theorem

Lemma 7.8 and Corollary 6.4 imply that \( L \) and \( \Lambda \) induce maps on \( \mathcal{H}^\bullet \). This allows us to make the following definition.
Definition 7.11. For a Kähler structure, the \((a, b)\)-primitive cohomology group is the vector space

\[
H_{\text{prim}}^{(a,b)} := \ker \left( L^{n-(a+b)+1} : H^{(a,b)} \to H^{(n-b+1,n-a+1)} \right).
\]

Moreover, we denote \(H_{\text{prim}}^k := \bigoplus_{a+b=k} H_{\text{prim}}^{(a,b)}\).

This definition, together with Proposition 7.9 and Lemma 6.3, gives us the following noncommutative generalisation of the classical hard Lefschetz theorem [15, Proposition 3.3.13]. As a corollary we prove a generalisation Corollary 6.7 to the case of \(H^{2k}\), for all \(k = 0, \ldots, n\).

Theorem 7.12 Let \((\Omega^{\bullet, \bullet}, d)\) be a Kähler structure, then it holds that

1. \(L^k : H^{n-k} \to H^{n+k}\) is an isomorphism, for \(k = 0, \ldots, n\),
2. \(H^k \cong \bigoplus_{i \geq 0} L^i H_{\text{prim}}^{(a,b)}\).

Corollary 7.13 For a covariant differential \(*\)-calculus endowed with a covariant Kähler structure, it holds that

\[
\dim(H^{2k}) = \dim(H^0) \geq 1, \quad \text{for all } k = 0, \ldots, n.
\]

Proof. This is a direct consequence of the second statement of above theorem and the fact that \(d1 = 0\).

\[\square\]

7.5 The \(\partial\overline{\partial}\)-Lemma

We finish our study of the general theory of Kähler structures with a result known in the classical case as the \(\partial\overline{\partial}\)-lemma. While it may look like an innocent technical result, in the classical case it is crucial for many important results, such as formality for Kähler manifolds [15, §3.A].

Lemma 7.14 Let \(\Omega^*\) be a covariant differential \(*\)-calculus admitting a covariant Kähler structure. Then for a \(d\)-closed form \(\omega \in \Omega^{(a,b)}\), the following conditions are equivalent:

1. \(\omega\) is \(d\)-exact,
2. \(\omega\) is \(\partial\)-exact,
3. \(\omega\) is \(\overline{\partial}\)-exact,
4. \(\omega\) is \(\partial\overline{\partial}\)-exact.
We will prove the theorem by introducing a fifth equivalent condition: \( \omega \) is orthogonal to \( H_{(a,b)} \) for some choice of Kähler form.

Using Hodge decomposition, we see the fifth condition is implied by any of the other four conditions. Moreover, the fourth condition implies both the first, second, and third conditions. Thus, it suffices to show the fifth condition implies the fourth.

Since by assumption \( \omega \) is d-closed (and hence \( \partial \)-closed) and orthogonal to the space of harmonic forms, then Hodge decomposition with respect to \( \partial \) yields \( \omega = \partial \nu \), for some \( \nu \in \Omega^{(a-1,b)} \). Applying Hodge decomposition with respect to \( \bar{\partial} \) to \( \nu \) yields \( \nu = \bar{\partial} \nu' + \bar{\partial} \nu'' + \nu''' \), for some harmonic \( \nu''' \). Returning to original form \( \omega \), we now see that \( \omega = \partial \partial \nu' \). By assumption \( \partial \omega = 0 \), and so, fixing \( h = 1 \), Corollary 7.6 implies that

\[
0 = \bar{\partial} \omega = \bar{\partial} \partial \partial \nu' + \bar{\partial} \partial \bar{\partial} \nu' = -\bar{\partial} \partial \nu'.
\]

Since \( 0 = \langle \bar{\partial} \partial \nu', \partial \nu' \rangle = \langle \bar{\partial} \nu', \bar{\partial} \partial \nu' \rangle \), this means that \( \bar{\partial} \partial \nu' = 0 \). Thus \( \omega = \partial \partial \nu' \). \( \square \)

### 7.6 The Heckenberger–Kolb Calculus

The next result follows directly from Lemma 4.17 and Proposition 4.19.

**Lemma 7.15** The Hermitian structure \((\Omega^{(\bullet,\bullet)}, \kappa)\) for \( \mathbb{C}P^n \) is a Kähler structure.

The operator \( \bar{\partial} + \bar{\partial} \) is a direct \( q \)-deformation of the Dirac–Dolbeault operator of \( \mathbb{C}_q[\mathbb{C}P^n] \). Deformations of this operator have previously appeared in the literature [7] in the context of spectral triples [11, Chapter 10]. As an initial investigation of the spectrum of \( \bar{\partial} + \bar{\partial} \), we calculate the first non-zero eigenvalue of the Laplacian \( \Delta_{\bar{\partial}} \), for \( \mathbb{C}_q[\mathbb{C}P^n] \) in the following lemma and corollary.

**Lemma 7.16** For \( X, Y : \mathbb{C}_q[\mathbb{C}P^1] \to \mathbb{C} \) the linear functionals uniquely defined by

\[
X(m)e^+ + Y(m)e^- := [m^+] ,
\]

it holds that

\[
\Delta_{\bar{\partial}}(m) = -X(m_{(2)})Y(m_{(3)})m_{(1)} , \quad m \in \mathbb{C}_q[\mathbb{C}P^1] = \Omega^{(0,0)}.
\]

**Proof.** Suppressing explicit reference to \( U \), we see that

\[
\Delta_{\bar{\partial}}(m) = * q \circ \partial \circ * q \circ \partial (m)
\]

\[
= * q \circ \partial \circ * q (m_{(1)} X(m_{(2)}) \otimes e_1^-)
\]

\[
= -i * q \circ \partial (m_{(1)} X(m_{(2)}) \otimes e_1^-)
\]

\[
= -i * q (m_{(1)} Y(m_{(2)}) X(m_{(3)}) \otimes e_1^+ \wedge e_1^-)
\]

\[
= -m_{(1)} Y(m_{(2)}) X(m_{(3)}),
\]

where we have used [31, Lemma 5.5] to calculate the actions of \( \partial \) and \( \bar{\partial} \). \( \square \)
Corollary 7.17 It holds that, \( \Delta_{\nabla}(z_{ij}) = q[2]qz_{ij} \), for \( i \neq j \).

Proof. From the above lemma, we have that

\[
\Delta_{\nabla}(z_{ij}) = -X((z_{ij})^{(2)})Y((z_{ij})^{(3)})(z_{ij})^{(1)} = -\sum_{a,b,x,y=1}^{2} X(u_{a}^{b}S(u_{x}^{y}))Y(u_{1}^{b}S(u_{y}^{1}))u_{a}^{b}S(u_{x}^{y}).
\]

Using the formulae presented in [31, Proposition 3.3], it is easily calculated that the scalars \( X(u_{a}^{b}S(u_{x}^{y}))Y(u_{1}^{b}S(u_{y}^{1})) \) are non-zero only in the following cases

\[
X(u_{1}^{b}S(u_{x}^{y}))Y(u_{1}^{b}S(u_{y}^{1})) = q^{2}, \quad X(u_{1}^{1}S(u_{x}^{1}))Y(u_{1}^{1}S(u_{y}^{1})) = 1.
\]

Hence \( \Delta_{\nabla}(z_{ij}) = q^{2}u_{1}^{b}S(u_{x}^{1}) - u_{2}^{b}S(u_{x}^{2}) \). Finally, our assumption that \( i \neq j \) implies that \( u_{2}^{b}S(u_{y}^{1}) = -u_{1}^{b}S(u_{y}^{1}) \), and so,

\[
\Delta_{\nabla}(z_{ij}) = q^{2}u_{1}^{b}S(u_{x}^{1}) + u_{1}^{b}S(u_{x}^{1}) = q(q + q^{-1})z_{ij} = q[2]qz_{ij}. \quad \square
\]

Recall now \((\Omega^{\bullet,\bullet}, \kappa)\) the conjectured Hermitian structure for \( C_{q}[G/L_{S}] \) introduced in §4.5. Using a direct generalisation of Lemma 4.17, it can be shown that \( \kappa \) must be d-closed, and so, we have the following lemma.

Lemma 7.18 If the pair \((\Omega^{\bullet,\bullet}, \kappa)\) is an Hermitian structure for \( C_{q}[G/L_{S}] \), then it is a Kähler structure.

We finish with a conjecture about the completion of these conjectured Kähler structures to spectral triples for the irreducible quantum flag manifolds. The case of \( C_{q}[\mathbb{C}P^{n}] \) is treated in [9].

Conjecture 7.19. Denoting by \( L^{2}(\Omega^{(0,\bullet)}) \) the completion of the subcomplex \( \Omega^{(0,\bullet)} \) of the Heckenberger–Kolb calculus of \( C_{q}[G/L_{S}] \) with respect to the inner product associated to \( \kappa \), a spectral triple is given by

\[
(C_{q}[G/L_{S}], L^{2}(\Omega^{(0,\bullet)}), \bar{\partial} + \bar{\partial}^{*}).
\]

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