Interrelation of rigidness of triangular cross-sections under bar torsion with conformal radii relation

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Abstract. The investigated problems of interrelation of the given geometrical rigidness of bars cross-sections of pure torsion with the geometrical characteristic of cross-sections – the relation of the internal conformal radius to the external conformal radius are covered in the article. Formulas for calculation of values of internal, external conformal radii and their relations for arbitrary, isosceles and rectangular triangles are given and the approximating functions for definition of these geometrical characteristics for triangles of any form are constructed. The graphic analysis of the obtained values shows that all variety of the relations of conformal radii presented graphically depending on one of the triangle corners falls into two subsets. One of them includes all variety of acute triangles, which is limited to isosceles acute triangles and rectangular triangles, and another one is limited with isosceles obtuse triangles and rectangular triangles. The graphic analysis of the whole variety of the known values of the given geometrical rigidness of triangular cross-sections at torsion, presented depending on one corner of a triangle, demonstrates that this geometrical characteristic is functionally connected with the relation of conformal radii; the corresponding approximating functions have linear character.

1. Introduction

Working out and development of the analytical methods the solution of the problem on the noncircular cross-section bar torsion is one of the main problems in the modern structural mechanics. The bars perceiving torsional de-formations are widely used in the construction industry and structural mechanics. When calculating the constructional elements for torsion primarily their geometrical rigidness is determined. This problem for common cross-sections (sphere, ellipse, rectangle, rectangular triangle) in the rigidness theory [1] is solved with the direct methods [2], which consists in integration of the corresponding differential equation. For bars of complicated type, the variation methods, the approximate methods, based on the membrane, the hydrodynamic and the electrodynamic analogies [3-5] and the approximate numerical methods [6,7] are used. The cross-sections in the triangular form are examined in the articles [8]. In recent years the geometrical methods – the isoperimetric method (ISPM) [9] and the interpolation method by form factor (IMFF) [10] were applied for solution of prismatic bar torsion problems. These methods are based on the isoperimetric properties of the geometrical characteristics of cross-sections – form factor, due to their usage the complicated physical problem on determining geometrical rigidness of cross-sections is reduced to the solution of the elementary geometrical problem [9-12]. The above-mentioned publications contain the
information that cross-section geometrical rigidness is functionally connected with a single argument -
form factor $K_f$. The error of the obtained results in most cases does not exceed $(3\text{-}4)\%$.

In the monograph [13] while investigating the plate stability problem the relation of the internal
conformal radii to the external conformal radii was used instead of geometrical argument $K_f$ for the
first time. This argument was used while investigating the problems of cross bending and plate free
oscillations [14,15]. The investigations proved that while using the mentioned relation of the
conformal radii, determination accuracy of maximum deflection and basic frequency of plate
oscillations. In this article for the first time, this relation is used for determination of geometrical
rigidness of cross-sections $I_k$ in the problems of pure torsion of prismatic bars.

2. Conformal radii determination and their relations for triangles

2.1. General provisions

Please follow these instructions as carefully as possible so all articles within a conference have
the same style to the title page. This paragraph follows a section title so it should not be indented.

The concepts of: internal and external conformal radii are used in the theory of the complex-variable
function and the conformal mapping of one plane region onto another one [16]. If the conformal
mapping of region $D$ is done onto the interior of some circle, then arbitrary point $a$ of region $D$ passes
into this circle center, which is characterized with internal conformal radius $r_g$. The maximum
possible meaning of this radius in mathematical physics [17] is expressed with symbol $r^\neq$. If in simply
connected region $D$ we take infinitely remote point then this region should be mapped onto the
external region of some circle, which is characterized with internal radius $r^\neq$. In the present article at
determining the normalized geometrical rigidness of cross-sections $I_k$ in the form of triangles, the
relation of the internal radius to the external conformal radius will be used $K = r^\neq/r^\neq$.

For the regions in the form of arbitrary triangle the values of the internal conformal radius is
determined from the formula [17]

\[ r^\neq = 4\pi \cdot f(\alpha) f(\beta) f(\gamma) \cdot \rho \]  

(1)

where $\alpha$, $\beta$, $\gamma$ – angle of a triangle (in radians); $f(x) = \frac{I}{\Gamma(x)} \left( \frac{x^x}{(1-x)^{1-x}} \right)^\Gamma(x) - \Gamma$ – function
($\gamma$-function); $\rho$ – radius of the described circle. The values of the external conformal radius are
obtained from the known [17] expression

\[ r^\neq = A/\pi r^\neq \]  

(2)

where $A$ – triangle area. The values of the conformal radii and their relations for some variety of
triangles of the arbitrary form are presented in table 1.

Based on the data in table 1 the graph of variation of value $K = r^\neq/r^\neq$ depending on angles $\alpha$ and $\beta$
of an arbitrary triangle (figure 1) is plotted, and the approximating function [7] is obtained:

\[ K = \frac{a + c \ln \alpha + e \ln \beta + \left( \ln \alpha \right)^i + \left( \ln \beta \right)^i + k \ln \alpha \ln \beta}{1 + b \ln \alpha + d \ln \beta + f \left( \ln \alpha \right)^i + \left( \ln \beta \right)^i + j \ln \alpha \ln \beta} \]  

(3)

where $a = -0.07119$; $b = -0.18777$; $c = 0.07191$; $d = -0.21962$; $e = 0.026978$; $f = 0.008523$;
ge $= -0.011417$; $h = 0.013928$; $i = -0.004845$; $j = 0.019253$; $k = -0.0021934$. The error of the function
(2) does not exceed $1.52\%$.

2.2. Isosceles triangles

For isosceles triangles ($\alpha = \beta$) expression (1) will have the following view:
\[ \dot{r} = 4\pi \cdot f^2(\alpha) f(\gamma) \cdot \rho \]  

(4)

Table 1. The values of the conformal radii \( \dot{r} \), \( \bar{r} \), and their relations for arbitrary triangles.

| \( \alpha \) | \( 10^\circ \) | \( 20^\circ \) | \( 30^\circ \) | \( 40^\circ \) | \( 50^\circ \) | \( 60^\circ \) | \( 70^\circ \) | \( 80^\circ \) | \( 90^\circ \) |
|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| \( \dot{r} \), \( a \) | 0.0538 | 0.0711 | 0.0797 | 0.0849 | 0.0885 | 0.0911 | 0.0932 | 0.0949 | 0.0963 |
| \( \bar{r} \), \( a \) | 0.2608 | 0.2658 | 0.2696 | 0.2730 | 0.2762 | 0.2795 | 0.2830 | 0.2868 | 0.2913 |
| \( K \) | 0.2063 | 0.2676 | 0.2957 | 0.3112 | 0.3204 | 0.3260 | 0.3293 | 0.3308 | 0.3308 |
| \( \dot{r} \), \( b \) | 0.1054 | 0.1259 | 0.1397 | 0.1499 | 0.1578 | 0.1644 | 0.1699 | 0.1749 | \n |
| \( \bar{r} \), \( b \) | 0.2747 | 0.2822 | 0.2891 | 0.2960 | 0.3033 | 0.3112 | 0.3203 | 0.3312 | \n |
| \( K \) | 0.3837 | 0.4462 | 0.4834 | 0.5064 | 0.5205 | 0.5281 | 0.5305 | 0.5281 | \n |
| \( \dot{r} \), \( a \) | 0.1567 | 0.1790 | 0.1965 | 0.2106 | 0.2226 | 0.2333 | 0.2431 | \n |
| \( \bar{r} \), \( a \) | 0.2608 | 0.2658 | 0.2696 | 0.2730 | 0.2762 | 0.2795 | \n |
| \( K \) | 0.2063 | 0.2676 | 0.2957 | 0.3112 | 0.3204 | 0.3260 | \n |
| \( \dot{r} \), \( b \) | 0.1054 | \n |
| \( \bar{r} \), \( b \) | 0.2747 | \n |
| \( K \) | 0.3837 | \n |

\( a \) and \( \beta \) – two angles of an arbitrary triangle (\( \alpha < 90^\circ; \beta < 90^\circ \)).

\( b \) – base (side) an arbitrary triangle (figure 1, b). As \( \alpha \) we accept the minor angle out of two angles (\( \alpha \leq \beta \)); dash “–” means, that such a triangle is already in the table.

Figure 1. Graph \( K = f(\alpha; \beta) \) for arbitrary triangles.
The long radius we find applying the known from geometry formula [10]

$$\rho = h/2 \cdot \sin^2 \alpha$$  \hspace{1cm} (5)

where \( h \) – isosceles triangle height; \( \alpha \) – base angle. The expression (2) in the examined case has the following form:

$$r = \frac{\text{ctg} \alpha \cdot h^2}{\pi \hat{r}}$$  \hspace{1cm} (6)

Calculated by the formulas (4) and (6) values \( \hat{r}, \hat{r} \), and \( K \) are presented in table 2. Based on the data of this table the graph of variation of value \( K = \frac{\hat{r}}{\hat{r}} \) depending on the isosceles triangle base angle \( \alpha \) is plotted, and the approximating function is obtained [7]:

$$\frac{\hat{r}}{\hat{r}} = \frac{a + c \alpha + e \alpha^2 + g \alpha^3}{1 + b \alpha + d \alpha^2 + f \alpha^3 + h \alpha^4}$$ \hspace{1cm} (7)

where \( a = 4.88537 \cdot 10^6; \) \( b = -0.0128669; \) \( c = 0.0222194; \) \( d = -4.94719 \cdot 10^{-5}; \) \( e = -0.000455718; \) \( f = 1.13874 \cdot 10^6; \) \( g = 2.32039 \cdot 10^6; \) \( h = -3.53318 \cdot 10^9. \) Function error (7) does not exceed 0.09%.
Table 2. Values of conformal radii $\hat{r}$ and $\hat{r}'$, and their relations $K = \hat{r}/\hat{r}'$ for isosceles triangles and rectangular triangles.

| $\alpha$ | $\hat{r}$, $h^b$ | $\hat{r}'$, $h^b$ | $K$ | $\alpha'$ | $\hat{r}$, $c^c$ | $\hat{r}'$, $h^b$ | $K$ |
|----------|-----------------|-----------------|-----|----------|-----------------|-----------------|-----|
| 5°       | 0.6239          | 5.8315          | 0.1070 | 45°      | 0.2366          | 0.3363          | 0.7034 |
| 10°      | 0.6102          | 2.9584          | 0.2063 | 47.5°    | 0.2359          | 0.3361          | 0.7019 |
| 15°      | 0.5954          | 1.9952          | 0.2984 | 50°      | 0.2337          | 0.3353          | 0.6970 |
| 20°      | 0.5793          | 1.5097          | 0.3837 | 52.5°    | 0.2301          | 0.3341          | 0.6887 |
| 25°      | 0.5618          | 1.2151          | 0.4623 | 55°      | 0.2250          | 0.3323          | 0.6771 |
| 30°      | 0.5427          | 1.0159          | 0.5342 | 57.5°    | 0.2185          | 0.3301          | 0.6619 |
| 35°      | 0.5218          | 0.8712          | 0.5989 | 60°      | 0.2106          | 0.3272          | 0.6436 |
| 40°      | 0.4987          | 0.7607          | 0.6556 | 62.5°    | 0.2012          | 0.3240          | 0.6210 |
| 45°      | 0.4732          | 0.6727          | 0.7034 | 65°      | 0.1903          | 0.3203          | 0.5941 |
| 50°      | 0.4449          | 0.6003          | 0.7411 | 67.5°    | 0.1781          | 0.3159          | 0.5638 |
| 55°      | 0.4131          | 0.5395          | 0.7657 | 70°      | 0.1644          | 0.3111          | 0.5284 |
| 60°      | 0.3774          | 0.4874          | 0.7748 | 72.5°    | 0.1492          | 0.3059          | 0.4877 |
| 65°      | 0.3368          | 0.4407          | 0.7642 | 75°      | 0.1326          | 0.3001          | 0.4419 |
| 70°      | 0.2902          | 0.3992          | 0.7270 | 77.5°    | 0.1145          | 0.2937          | 0.3899 |
| 75°      | 0.2362          | 0.3611          | 0.6541 | 80°      | 0.0949          | 0.2868          | 0.3309 |
| 80°      | 0.1726          | 0.3252          | 0.5308 | 82.5°    | 0.0737          | 0.2795          | 0.2637 |
| 85°      | 0.0960          | 0.2901          | 0.3309 | 85°      | 0.0510          | 0.2710          | 0.1882 |
| 87.5°    | 0.0265          |                | 0.0265 | 87.5°    | 0.0265          |                | 0.1013 |

*a $\alpha$ – isosceles triangle base angle (for isosceles triangles) or a larger angle at rectangular triangle hypotenuse ($\alpha \leq 45°$),

$b$ $h$ – isosceles triangle height,

c $c$ – rectangular triangle hypotenuse.

Table 3. Analysis of geometrical rigidness of torsion of cross-sections in the form of isosceles and rectangular triangles.

| No | Cross-section parameters | Values of geometrical characteristics of cross-section |
|----|--------------------------|-------------------------------------------------------|
|    | Cross-sections in the form of isosceles triangles |                                                                 |
| 1  | Base angle                | 20° | 30° | 40° | 45° | 50° | 60° | 70° | 80° |
| 2  | $i_t$ [1, 3]              | 0.0545 | 0.0775 | 0.0966 | 0.1042 | 0.1101 | 0.1155 | 0.1079 | 0.0774 |
| 3  | $\hat{r}'/\hat{r}$       | 0.3837 | 0.5342 | 0.6556 | 0.7034 | 0.7411 | 0.7748 | 0.7270 | 0.5308 |
| 4  | $i_t$ (11)                | 0.0549 | 0.0781 | 0.0966 | 0.1044 | 0.1102 | 0.1151 | 0.1080 | 0.0776 |
| 5  | Difference, %             | 0.74 | 0.77 | 0 | 0.19 | 0.09 | 0.03 | 0.09 | 0.26 |
|    | Cross-section in the form of rectangular triangles |                                                    |
| 1  | Apex angle (fig. 2)       | 45° | 60° | 67.5° | 75° | 87.5° |
| 2  | $i_t$ [1, 3]              | 0.1042 | 0.0955 | 0.0729 | 0.0638 | 0.0140 |
| 3  | $\hat{r}'/\hat{r}$       | 0.7034 | 0.6436 | 0.5041 | 0.4419 | 0.1013 |
| 4  | $i_t$ (11)                | 0.1044 | 0.0951 | 0.0735 | 0.0639 | 0.0140 |
| 5  | Difference, %             | 0.19 | 0.42 | 0.82 | 0.16 | 0 |

2.3. Rectangular triangles

Let's examine figure 2, which presents the transformation of a right isosceles triangle by means of affine pressing along vertical leg. The angle between leg and hypotenuse is marked $\alpha$. Under specified transformation this angle grows from 45°, tending to 90°.

For rectangular triangle, values $\hat{r}$ are calculated by the formula (1). The long radius $p = c/2$, where $c$ – rectangular triangle hypotenuse. The expression (2) for rectangular triangles has the form:
\[ \mathcal{P} = \frac{\sin 2\alpha \cdot c^2}{4\pi r} \]  

Then

\[ \frac{\hat{r}}{\mathcal{P}} = a + b\alpha + ca^2 + d\alpha^3 + ea^4 + f\alpha^5 + g\alpha^6 + ha^7 + ia^8 \]  

where

\[ a = 7.0481 \cdot 10^{-6}; \quad b = 0.044093; \quad c = -0.0016091; \quad d = 8.0776 \cdot 10^{-5}; \quad e = -4.2632 \cdot 10^{-6}; \quad f = 1.5714 \cdot 10^{-7}; \quad g = -3.55210^{-9}; \quad h = 4.3955 \cdot 10^{-11}; \quad i = -2.2763 \cdot 10^{-13}. \]

Function error (9) does not exceed 0.09%. Calculated by these formulas, values \( \hat{r}, \mathcal{P} \) and \( K \) are presented in table 2.

According to the data obtained for isosceles and rectangular triangles, the graphs are plotted (figure 3). In these graphs point 2 corresponds to the cross-section in the form of an isosceles triangle, point 1 – cross-section in the form if a right isosceles triangle, the curve 0-1 describes all values \( K \) for cross-sections in the form of isosceles obtuse triangles, and the curve 1-2-3 – for cross-sections in the form of isosceles acute triangles. Therefore, the region limited with the curves 1-3 and 1-2-3, includes values \( K \) for all variety of acute triangles, and the region limited with the curves 0-1 and 1-3 – values \( K \) for all variety of obtuse triangles.

3. Interconnection of the relation of conformal radii with the normalized geometrical rigidity of cross-sections at pure torsion

Further we examine the normalized geometrical rigidity \( i_k \) of triangular plates:

\[ i_k = I_k / A \]  

Known values \( i_k \) will be taken from the monographs [1,3] (see line 2 of table 3). According to the values in figure 4 we plotted the graph of dependence \( i_k - \hat{r}/\mathcal{P} \), which can be approximated with linear dependence.

\[ i_k = (154.8 \cdot K - 4.3) \cdot 10^{-3} \]  

Figure 4. Graph \( i_k = \hat{r}/\mathcal{P} \) for isosceles and rectangular triangles.

The calculation results by the formula (11) are listed in table 4 (line 4), and their deviations from the corresponding known solutions are listed in the same table (line 5). It is seen from the quoted results, solution error obtained by the formula (11), does not exceed one percent. For triangular sections with very sharp angles, the error increases significantly. Therefore, it is not recommended to use the formula (11) for such sections.
The attention should be paid to the fact, that this right line with exceptional accuracy describes the values of the normalized geometrical rigidity as for cross-sections in the form of isosceles triangles (acute and obtuse) as well as for cross-sections in the form of rectangular triangles. Inasmuch as all the variety of the values of the normalized geometrical rigidity of cross-sections in the form of arbitrary triangles is limited with the curves, which correspond to isosceles and rectangular triangles (see figure 3). An important conclusion follows from this: all the variety of values $i_k$ for triangular cross-sections of the arbitrary form is described with a singular formula (11).

4. Conclusions
The graphs (figure 3) of the dependences of relations of the conformal radii from the base angle (for isosceles triangles) and from the acute angle (for rectangular triangles) are plotted for isosceles and rectangular triangles. These graphs analysis displayed that the curve 0-1 describes all values $K$ for isosceles obtuse triangles, and the curve 1-2-3 – for isosceles acute triangles. The region, limited with the curves 1-3 and 1-2-3, includes values $K$ for all variety of acute triangles, and the region, limited with the curves 0-1 and 1-3, – values $K$ for all variety of obtuse triangles.

The analysis of the graphs in figure 4, where dependences $i_k = \tilde{r} / \tilde{r}$ for isosceles and rectangular triangles are plotted, permits to conclude that all the variety of values of the normalized geometrical rigidity of cross-sections in the form of arbitrary triangles is limited with the curves, which correspond to isosceles and rectangular triangles (fig. 3). All the values variety of parameter $i_k$ for triangular cross-sections of the arbitrary form is described with a singular formula (11).

The suggested approximating method of determination of the geometrical rigidity of the triangular cross-sections at the prismatic bars torsion with the conformal radii application can be extended to another cross-section type (rhombus, parallelogram, trapezoidal, sectorial, etc.).

5. References
[1] Varvak P M and Ryabov A F 1971 Elasticity theory reference book (for construction engineers) (Kiev: Budivelnik)
[2] Leybenzon L S 1951 Collected papers (Moscow: Pub. of the Academy of Scien. of the USSR)
[3] Arutyunan N X and Abramyan V L 1963 Elastic body torsion (Moscow: Fizmatgiz)
[4] Evstifeev V V, Teperin L L, Teperin L N 2011 TVFLXXXV 18
[5] Kazarina M V, Uskov V M, Chedrik A V, Chedrik V V 2015 Proc. of XI All-Russian congress on fundamental problems of theoretical and applied mechanics (Tomsk: Tomsk State University) p 1667
[6] Chen T 2001 Quart J. Mech. and Appl. Math. 54 227
[7] Warg C Y 1996 Mech. Struct. and Mach. 24 283
[8] Zonov D V 1998 News of Higher Educational Institutions. The North Caucasian region. Natural Sciences4 56
[9] Korobko V I 1997 Isoperimetric method in structural mechanics: Theoretical grounds of isoperimetric method (Moscow: ASV)
[10] Korobko A V 1999 Geometrical modeling with the region form in two-dimensional problems of the elasticity theory (Moscow: ASV)
[11] Korobko V I, Malykh S G 1986 News of Higher Educational Institutions. Mechanical Engineering3 2
[12] Korobko V I, Korobko A V, Chernyaev A A 2016 Proc. Eng.150 1648
[13] Korobko V I, Khustochkin A N 1994 Isoperimetric method in the plate stability problems (Rostov-upon-Don: The North Caucasian Scientific Center of the Higher education inst.)
[14] Chernyaev A A 2012 Struct. mech. of eng. constr. and facilities2 63
[15] Chernyaev AA 2012 Int. J. for Comp. Civil and Struct. Eng. 8 66
[16] Lavrentyev M A and Shabat B V 1987 Methods of complex variable theory (Moscow: Nauka)
[17] Polia G, Szego G 2006 Isoperimetric inequalities in mathematical physics (Moscow: KomKniga)