The Path Partition Conjecture is True and its Validity Yields Upper Bounds for Detour Chromatic Number and Star Chromatic Number

G. Sethuraman

Department of Mathematics, Anna University
Chennai 600 025, INDIA
sethu@annauniv.edu

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Abstract

The detour order of a graph $G$, denoted $\tau(G)$, is the order of a longest path in $G$. A partition $(A, B)$ of $V(G)$ such that $\tau(V(A)) \leq a$ and $\tau(V(B)) \leq b$ is called an $(a, b)$-partition of $G$. A graph $G$ is called $\tau$-partitionable if $G$ has an $(a, b)$-partition for every pair $(a, b)$ of positive integers such that $a + b = \tau(G)$. The well-known Path Partition Conjecture states that every graph is $\tau$-partitionable. In [7] Dunber and Frick have shown that if every 2-connected graph is $\tau$-partitionable then every graph is $\tau$-partitionable. In this paper we show that every 2-connected graph is $\tau$-partitionable. Thus, our result settles the Path Partition Conjecture affirmatively. We prove the following two theorems as the implications of the validity of the Path Partition Conjecture.

**Theorem 1:** For every graph $G$, $\chi_s(G) \leq \tau(G)$, where $\chi_s(G)$ is the star chromatic number of a graph $G$. 
The $n^{th}$ detour chromatic number of a graph $G$, denoted $\chi_n(G)$, is the minimum number of colours required for colouring the vertices of $G$ such that no path of order greater than $n$ is mono coloured. These chromatic numbers were introduced by Chartrand, Gellar and Hedetniemi \cite{5} as a generalization of vertex chromatic number $\chi(G)$.

**Theorem 2:** For every graph $G$ and for every $n \geq 1$, $\chi_n(G) \leq \left\lceil \frac{\tau_n(G)}{n} \right\rceil$, where $\chi_n(G)$ denote the $n^{th}$ detour chromatic number. Theorem 2 settles the conjecture of Frick and Bullock \cite{9} that $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$, for every graph $G$, for every $n \geq 1$, affirmatively.

**Keywords:** Path Partition; Path Partition Conjecture; Star Chromatic Number; Detour Chromatic Number; Upper bound of chromatic number; Upper bound of Star Chromatic Number; Upper bound of Detour Chromatic Number.

1 Introduction

All graphs considered here are simple, finite and undirected. Terms not defined here can be referred from the book \cite{20}. A longest path in a graph $G$ is called a detour of $G$. The number of vertices in a detour of $G$ is called the detour order of $G$ and is denoted by $\tau(G)$. A partition $(A, B)$ of $V(G)$ such that $\tau(\langle A \rangle) \leq a$ and $\tau(\langle B \rangle) \leq b$ is called an $(a, b)$-partition of $G$. If $G$ has an $(a, b)$-partition for every pair $(a, b)$ of positive integers such that $a + b = \tau(G)$, then we say that $G$ is $\tau$-partitionable. The following conjecture is popularly known as the Path Partition Conjecture.

**Path Partition Conjecture:** Every graph is $\tau$-partitionable.

The Path Partition Conjecture was discussed by Lovasz and Mihok in 1981 in Szeged and treated in the theses \cite{13} and \cite{19}. The Path Partition Conjecture first appeared in the literature in 1983, in a paper by Laborde et al. \cite{14}. In 1995 Bondy \cite{2} posed the directed version of the Path Partition Conjecture. In 2004, Aldred and Thomassen \cite{11} disproved two stronger versions of the Path Partition Conjecture, known as the Path Kernel Conjecture \cite{4, 16} and the Maximum $P_n$-free Set Conjecture \cite{8}. Similar partitions were studied for other graph parameters too. Lovasz proved in \cite{15} that every graph is $\Delta$-partitionable, where $\Delta$ denotes the maximum degree (A graph $G$ is $\Delta$-partitionable if, for every pair $(a, b)$ of positive integers satisfying...
\[ a + b = \Delta(G) - 1 \], there exists a partition \((A, B)\) of \(V(G)\) such that \(\Delta(\langle A \rangle) \leq a \) and \(\Delta(\langle B \rangle) \leq b\). For the results pertaining to the Path Partition Conjecture and related conjectures refer to \([3, 4, 6, 7, 8, 9, 10, 13, 14, 16, 18, 19, 17]\).

An \(n\)-detour colouring of a graph \(G\) is a colouring of the vertices of \(G\) such that no path of order greater than \(n\) is monocoloured. The \(n^{th}\) detour chromatic number of graph \(G\), denoted by \(\chi_n\), is the minimum number of colours required for an \(n\)-detour colouring of a graph \(G\). It is interesting to note that for a graph \(G\), when \(n = 1\), \(\chi_1(G) = \chi(G)\). These chromatic numbers were introduced by Chartrand, Gellor and Hedetnimi \([5]\) in 1968 as a generalization of vertex chromatic number.

If the Path Partition Conjecture is true, then the following conjecture of Frick and Bullock \([9]\) is also true.

**Frick-Bullock Conjecture:** \(\chi_n(G) \leq \lceil \frac{\tau(G)}{n} \rceil \) for every graph \(G\) and for every \(n \geq 1\).

Recently, Dunbar and Frick \([7]\) proved the following theorem.

**Theorem 1.1** (Dunbar and Frick \([7]\)). If every 2-connected graph is \(\tau\)-partitionable then every graph is \(\tau\)-partitionable.

In this paper we show that the Path Partition Conjecture is true for every 2-connected graph. Thus, Theorem \([11]\) and our result imply that the Path Partition Conjecture would imply the following Path Partition Theorem.

**Path Partition Theorem.** For every graph \(G\) and for every \(t\)-tuple \((a_1, a_2, \ldots, a_t)\) of positive integers with \(a_1 + a_2 + \cdots + a_t = \tau(G)\) and \(t \geq 1\), there exists a partition \((V_1, V_2, \ldots, V_t)\) of \(V(G)\) such that \(\tau(G(\langle V_i \rangle)) \leq a_i\), for every \(i, 1 \leq i \leq t\).

The Path Partition Theorem immediately implies that the Conjecture of Frick and Bullock is true. The validity of Frick and Bullock Conjecture naturally implies the classical upper bound for the chromatic number of a graph \(G\) that \(\chi(G) = \chi_1(G) \leq \tau(G)\) proved by Gallai\([11]\).

A star colouring of a graph \(G\) is a proper vertex colouring in which every path on four vertices uses at least three distinct colours. The star chromatic number of \(G\) denoted by \(\chi_s(G)\) is the least number of colours needed to star color \(G\). As a consequence of the Path Partition Theorem, we have obtained an upper bound for the star chromatic number. More precisely, we show that \(\chi_s(G) \leq \tau(G)\) for every graph \(G\).
2 Main Result

In this section we prove our main result that every 2-connected graph is \( \tau \)-partitionable.

We use Whitney’s Theorem on the characterization of 2-connected graph in the proof of our main result given in Theorem 2.2.

An ear of a graph \( G \) is a maximal path whose internal vertices have degree 2 in \( G \). An ear decomposition of \( G \) is a decomposition \( P_0, P_1, \ldots, P_k \) such that \( P_0 \) is a cycle and \( P_i \) for \( i \geq 1 \) is an ear of \( P_0 \cup P_1 \cup \cdots \cup P_i \).

**Theorem 2.1** (Whitney [21]). A graph is 2-connected if and only if it has an ear decomposition. Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition.

**Theorem 2.2.** Every 2-connected graph is \( \tau \)-partitionable.

*Proof.* Let \( G \) be a 2-connected graph. By Whitney’s Theorem there exists an ear decomposition \( S = \{P_0, P_1, \ldots, P_n\} \), where \( P_0 \) is a cycle and \( P_i \) for \( i \geq 1 \) is an ear of \( P_0 \cup P_1 \cup \cdots \cup P_i \). We prove that \( G \) is \( \tau \)-partitionable by induction on \(|S|\).

When \(|S| = 1\), \( S = \{P_0\} \). Then \( G = P_0 \). Thus, \( G \) is a cycle. As every cycle is \( \tau \)-partitionable, \( G \) is \( \tau \)-partitionable. By induction, we assume that if \( G \) is any 2-connected graph having an ear decomposition \( S = \{P_0, P_1, \ldots, P_k\} \), that is, with \(|S| = k\), then \( G \) is \( \tau \)-partitionable.

Let \( H \) be a 2-connected graph with an ear decomposition \( S = \{P_0, P_1, \ldots, P_{k-1}, P_k\} \). That is, \(|S| = k+1\). We claim that \( H \) is \( \tau \)-partitionable. Let \((a, b)\) be a pair of positive integers with \( a + b = \tau(H) \). Since \( H \) is having the ear decomposition \( S = \{P_0, P_1, \ldots, P_{k-1}, P_k\} \), \( H \) can be considered as a 2-connected graph obtained from the 2-connected graph \( G \) having the ear decomposition \( S' = \{P_0, P_1, \ldots, P_{k-1}\} \) by adding a new path (ear) \( P_k : x v_1 v_2 \ldots v_r y \) to \( G \), where \( x, y \in V(G) \) and \( v_1, v_2, \ldots, v_r \) are new vertices to \( G \). As \( G \) is a 2-connected graph having the ear decomposition \( S' = \{P_0, P_1, \ldots, P_{k-1}\} \) with \(|S| = k\), by induction \( G \) is \( \tau \)-partitionable. Let \((a_1, b_1)\) be a pair of positive integers such that \( a_1 \leq a, b_1 \leq b \) with \( \tau(G) = a_1 + b_1 \). Since \( G \) is \( \tau \)-partitionable, there exists an \((a_1, b_1)\) partition \((A', B')\) of \( V(G) \) such that \( \tau(G((A'))) \leq a_1 \) and \( \tau(G((B'))) \leq b_1 \). In order to prove our claim that \( H \) is \( \tau \)-partitionable, we define an \((a, b)\)-partition \((A, B)\) of \( V(H) \) from the \((a_1, b_1)\) partition \((A', B')\) of \( V(G) \) as well as using the path \( P_k : x v_1 v_2 \ldots v_r y \). The construction of an \((a, b)\)-partition \((A, B)\) of \( V(H) \) is...
given under three cases, depending on \( r = 0, r = 1 \) and \( r \geq 2 \), where \( r \) is the number of new vertices in the path \( P_k \).

**Case 1.** \( r = 0 \)

Then \( P_k : xy \), where \( x \) and \( y \) are the vertices of \( G \). Thus, \( H = G + xy \). This implies, \( V(H) = V(G) \).

**Case 1.1.** Suppose \( x \) and \( y \) are in different parts of the partition \((A', B')\) of \( V(G) \).

Then, as \( x \) and \( y \) are in different parts of the partition \((A', B')\) of \( V(G) \), the introduction of the new edge \( xy \) between the vertices \( x \) and \( y \) does not increase the length of any path either in \( G(\langle A' \rangle) \) or in \( G(\langle B' \rangle) \). Further, as \( V(H) = V(G) \), we have \( \tau(H(\langle A' \rangle)) = \tau(G(\langle A' \rangle)) \leq a_1 \leq a \) and \( \tau(H(\langle B' \rangle)) = \tau(G(\langle B' \rangle)) \leq b_1 \leq b \). Thus, \((A', B')\) is a required \((a, b)\)-partition of \( V(H) \).

**Case 1.2.** Suppose \( x \) and \( y \) are in the same part of the partition \((A', B')\) of \( V(G) \).

Without loss of generality, we assume that \( x \) and \( y \) are in \( A' \).

Suppose \( \tau(H(\langle A' \rangle)) \leq a \). Then, as \( \tau(H(\langle B' \rangle)) \leq b_1 \leq b \), the \((A', B')\) is a required \((a, b)\)-partition of \( V(H) \).

Suppose \( \tau(H(\langle A' \rangle)) > a \), then observe that the addition of the edge \( xy \) to \( G \) has increased the order of some of the longest paths (at least one longest path) in \( H(\langle A' \rangle) \) from \( a_1 \) to \( t = a_1 + k > a \), where \( k \geq 1 \). On the other hand, any path of order \( t > a \) in \( H(\langle A' \rangle) \) must contain the edge \( xy \) also.

Let \( P : u_1u_2u_3 \ldots u_iu_{i+1} \ldots u_a u_{a+1} \ldots u_t \) be any path of order \( t > a \). Then, note that the edge \( xy = u_ju_{j+1} \) for some \( j \), \( 1 \leq j \leq t-1 \) and \( t \leq 2a_1 \).

**Observation 2.1.** If we remove the vertex \( u_{a+1} \) from the path \( P \), then we obtain two subpaths \( u_1u_2 \ldots u_iu_{i+1} \ldots u_{a-1}u_a \), say \( P' \) and \( u_{a+2}u_{a+3} \ldots u_{t-1}u_t \), say \( P'' \) of \( P \). The number of vertices in \( P' \) is exactly \( a \) and the number of vertices in \( P'' \) is \( t - (a+1) \leq t - (a_1 + 1) \leq 2a_1 - a_1 - 1 = a_1 - 1 < a_1 \leq a \).

**Observation 2.2.** Consider the subpath \( Q : u_1u_2 \ldots u_{a-1}u_a u_{a+1} \) of \( P \). Then observe that the end vertex \( u_{a+1} \) of \( Q \) cannot be adjacent to any of the end vertices of any path of order \( b \) in the induced subgraph \( H(\langle B' \rangle) = G(\langle B' \rangle) \) in \( H \).

For, suppose \( u_{a+1} \) is adjacent to an end vertex of a path, say \( Z \) of order \( b \) in \( H(\langle B' \rangle) = G(\langle B' \rangle) \). Let \( Z = v_1v_2 \ldots v_b \). Without loss of generality, let \( u_{a+1} \) be adjacent to \( v_1 \). Then, there exists a path \( Q \cup Z : u_1u_2 \ldots u_{a-1}u_a v_1v_2 \ldots v_b \) of order \( a + b + 1 > a + b = \tau(H) \), a contradiction (Similar contradiction hold good if \( u_{a+1} \) is adjacent \( v_b \)).
Let \( \{ R_0, R_1, \ldots, R_t \} \) be the set of all paths in \( H(\langle A' \rangle) \) of order at least \( a+1 \). For \( 1 \leq i \leq t \), let \( u_{a+1}^i \) denote the terminus vertex of the subpath of \( R_i \) of order \( a+1 \) and having its origin as the origin of \( R_i \). Let \( \{ u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h} \} \) be the set of distinct vertices from the vertices \( u_{a+1}^1, u_{a+1}^2, \ldots, u_{a+1}^t \), where \( h \leq t \). Suppose \( \{ u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h} \} \) induces any path in \( H(\langle A' \rangle) \). Consider any such path \( X : u_{a+1}^{\beta_1} u_{a+1}^{\beta_2} \ldots u_{a+1}^{\beta_c} \), where \( \{ \beta_1, \beta_2, \ldots, \beta_c \} \subseteq \{ \alpha_1, \alpha_2, \ldots, \alpha_h \} \).

Then, for \( 1 \leq i \leq c \), any vertex \( u_{a+1}^{\beta_i} \) divides the path \( X \) into three subpaths \( u_{a+1}^{\beta_1} u_{a+1}^{\beta_2} \ldots u_{a+1}^{\beta_{i-1}} u_{a+1}^{\beta_i} u_{a+1}^{\beta_{i+1}} \ldots u_{a+1}^{\beta_c} \).

![Figure 1: Structures of various paths of order \( t \geq a+1 \) in \( A' \)](image)

**Claim 1.** For every \( i, 1 \leq i \leq h \), the vertex \( u_{a+1}^{\beta_i} \) cannot be adjacent to any of the end vertices of any path of order greater than or equal to \( b-q \) in \( H(\langle B' \rangle) \), where \( q = i-1 \) or \( c-i \).

First we ascertain \( b < q+1 \) in Observation 2.3 then we prove the Claim 1.

**Observation 2.3.** \( b \geq q+1 \)

For, suppose \( b < q+1 \). If \( q = c-i \), then consider the path,

\[
K = u_{a+1}^{\beta_i} u_{a+1}^{\beta_{i+1}} \ldots u_{a+1}^{\beta_c} u_{a+1}^{\beta_{c-1}} \ldots u_{a+1}^{\beta_2} u_{a+1}^{\beta_1}
\]

in \( H(\langle A' \rangle) \) having \( 1 + c - i + a = 1 + q + a \) vertices. As \( q+1 > b \), the path \( K \) has at least \( a + b + 1 \) vertices. This implies there exists a path of order
at least \(a + b + 1\) in \(H(\langle A' \rangle)\). A contradiction to the fact that \(\tau(H) = a + b\). Similarly, if \(q = i - 1\), then consider the path,

\[
K' = u_{a+1}^{\beta_1}u_{a+1}^{\beta_2} \ldots u_{a+1}^{\beta_i}u_{a+1}^{\beta_i}u_{a+1}^{\beta_1} \ldots u_{a+1}^{\beta_1}u_{a+1}^{\beta_1}
\]

in \(H(\langle A' \rangle)\) having \(i + a = 1 + q + a\) vertices. As \(q + 1 > b\), the path \(K'\) has at least \(a + b + 1\) vertices. This implies that there exists a path of order at least \(a + b + 1\) in \(H(\langle A' \rangle)\). A contradiction to the fact that \(\tau(H) = a + b\). Hence, \(b \geq q + 1\).

To prove Claim 1, we suppose \(u_{a+1}^{\beta_1}\), for some \(i, 1 \leq i \leq h\) is adjacent to an end vertex of a path of order \(l \geq b - q\) in \(H(\langle B' \rangle)\). Let \(Y = w_1w_2w_3 \ldots w_l\) be a path of order \(l \geq b - q\) in \(H(\langle B' \rangle)\) such that (without loss of generality) \(w_l\) is adjacent to the vertex \(u_{a+1}^{\beta_1}\).

**Case 1.2a.** \(q = c - i\)

Then consider the path \(S = w_1w_2 \ldots w_lu_{a+1}^{\beta_1}u_{a+1}^{\beta_i} \ldots u_{a+1}^{\beta_1}u_{a+1}^{\beta_c}u_{a+1}^{\beta_c} \ldots u_{a+1}^{\beta_c}u_{a+1}^{\beta_1}\), where \(u_{a+1}^{\beta_1}u_{a+1}^{\beta_i} \ldots u_{a+1}^{\beta_i}u_{a+1}^{\beta_c}\) is a subpath of \(R_{\beta_c}\) of order \(a + 1\) having the vertex \(u_{a+1}^{\beta_c}\), the origin of \(R_{\beta_c}\) as its origin. As \(Y : w_1w_2w_3 \ldots w_l\) is the path in \(H(\langle B' \rangle)\) such that \(w_l\) is adjacent to \(u_{a+1}^{\beta_1}\), it follows that \(S\) is a path in \(H\) having the order \(l + 1 + c - i + a \geq b - q + 1 + q + a = b + a + 1 > \tau(H)\), a contradiction.

**Case 1.2b.** \(q = i - 1\)

Then consider the path \(S' = w_1w_2 \ldots w_lu_{a+1}^{\beta_1}u_{a+1}^{\beta_i} \ldots u_{a+1}^{\beta_1}u_{a+1}^{\beta_i}u_{a+1}^{\beta_i} \ldots u_{a+1}^{\beta_1}u_{a+1}^{\beta_1}\), where \(u_{a+1}^{\beta_1}u_{a+1}^{\beta_i} \ldots u_{a+1}^{\beta_i}u_{a+1}^{\beta_i}\) is the subpath of \(R_{\beta_i}\) of order \(a + 1\) having the vertex \(u_{a+1}^{\beta_i}\), the origin of \(R_{\beta_i}\) as its origin. As \(Y : w_1w_2w_3 \ldots w_l\) is the path in \(H(\langle B' \rangle)\) such that \(w_l\) is adjacent to \(u_{a+1}^{\beta_1}\), it follows that \(S'\) is a path in \(H\) having the order \(l + 1 + i - 1 + a \geq b - q + 1 + q + a = b + a + 1 > \tau(H)\), a contradiction.

Hence the Claim 1.

Thus, it follows from the Claim 1 that

\[
\tau(H(\langle B' \cup \{u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h}\}\})) \leq b
\]

(1)

From Observation 1, it follows that

\[
\tau(H(\langle A' \setminus \{u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h}\}\})) \leq a
\]

(2)

Let \(A = A' \setminus \{u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h}\}\) and \(B = B' \cup \{u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h}\}\).

Then, from (1) and (2) it follows that \(\tau(H(\langle A \rangle)) \leq a\) and \(\tau(H(\langle B' \cup \{u_{a+1}^{\alpha_1}, u_{a+1}^{\alpha_2}, \ldots, u_{a+1}^{\alpha_h}\}\})) \leq b\).

Hence \((A, B)\) is a required \((a, b)\)-partition of \(H\).
Case 2. $r = 1$
Then $P_k : xv_1y$.

Case 2.1. Both $x$ and $y$ belong to the same partition $A'$ or $B'$.
Without loss of generality, we assume that $x, y \in B'$. That is, $x, y \notin A'$.
Then $(A' \cup \{v_1\}, B')$ is a required $(a, b)$-partition of $V(H)$.

Case 2.2. The vertices $x$ and $y$ belong to different partitions $A'$ and $B'$.
Without loss of generality, we assume that $x \in A'$ and $y \in B'$. If $x$ is not an end vertex of a path of order $a$ in $H(A')$, then $(A' \cup \{v_1\}, B')$ is a required $(a, b)$-partition of $V(H)$. If $x$ is an end vertex of a path of order $a$ in $H(A')$, then $y$ cannot be an end-vertex of a path of order $b$ in $H(B')$ (otherwise, $H$ would have a path of order $a + b + 1 > \tau(H)$). Therefore $(A', B' \cup \{v_1\})$ is a required $(a, b)$-partition of $V(H)$.

Case 3. $r \geq 2$
Colour all vertices of $A'$ with red colour and colour all the vertices of $B'$ with blue colour. Since the vertices $x, y \in V(G)$, they are coloured with either blue or red colour. Without loss of generality, we assume that $x \in A'$. Give $v_r$ the alternate colour to that of the vertex $y$. As $x$ is coloured with red colour, colour the vertex $v_1$ with blue colour. In general, for $2 \leq i \leq r - 1$, sequentially colour the vertex $v_i$ with the alternate colour to the colour of the vertex $v_{i-1}$. Then observe that $P_k$ contains no induced monochromatic subgraph of order greater than 2 and no monochromatic path in $A'$ or in $B'$ can be extended to include any of the vertices $v_1, v_2, \ldots, v_r$ of $P_k$.
Let $X_1$ be the set of all red coloured vertices of $P_k - \{x, y\}$ and let $X_2$ be the set of all blue coloured vertices of $P_k - \{x, y\}$. Then $H(A' \cup X_1) \leq a_1 \leq a$ and $H(B' \cup X_2) \leq b_1 \leq b$. Hence $(A' \cup X_1, B' \cup X_2)$ is a required $(a, b)$-partition of $H$.
Thus, $H$ is $\tau$-partitionable. This completes the induction. Hence every 2-connected graph is $\tau$-partitionable.

The following Corollary 2.1 is an immediate consequence of Theorem 1.1 and Theorem 2.2.

Corollary 2.1. Every graph is $\tau$-partitionable.

It is clear that Corollary 2.1 settles the Path Partition Conjecture affirmatively. Thus, "the Path Partition Conjecture is true".

The following Theorem 2.3 called "Path Partition Theorem" is a simple implication of Corollary 2.1.
Theorem 2.3 (Path Partition Theorem). For every graph $G$ and for every $t$-tuple $(a_1, a_2, \ldots, a_t)$ of positive integers with $a_1 + a_2 + \cdots + a_t = \tau(G)$ and $t \geq 1$, there exists a partition $(V_1, V_2, \ldots, V_t)$ of $V(G)$ such that $\tau(G(\langle V_i \rangle)) \leq a_i$, for every $i$, $1 \leq i \leq t$.

**Proof.** Let $G$ be a graph. Consider any $t$-tuple $(a_1, a_2, \ldots, a_t)$ of positive integers with $a_1 + a_2 + \cdots + a_t = \tau(G)$, and $t \geq 1$. Then by Corollary 2.1, for the pair of positive integers $(a, b)$ with $a + b = \tau(G)$, where $a = a_1$ and $b = a_2 + \cdots + a_t$, there exists a partition $(U_1, U_2)$ of $V(G)$ such that $\tau(G(\langle U_1 \rangle)) \leq a = a_1$ and $\tau(G(\langle U_2 \rangle)) \leq b = a_2 + \cdots + a_t$. Consider the graph $H = G(\langle U_2 \rangle)$. Then for the pair of positive integers $(c, d)$ with $c + d = \tau(H) = \tau(G(\langle U_2 \rangle))$, where $c = a_2$ and $d = a_3 + a_4 + \cdots + a_t$, by Corollary 2.1, there exists a partition $(U_{21}, U_{22})$ of $V(H)$ such that $\tau(H(\langle U_{21} \rangle)) \leq c = a_2$ and $\tau(H(\langle U_{22} \rangle)) \leq d = a_3 + a_4 + \cdots + a_t$. As $H(\langle U_{21} \rangle) = G(\langle U_{21} \rangle)$ and $H(\langle U_{22} \rangle) = G(\langle U_{22} \rangle)$, we have $\tau(G(\langle U_{21} \rangle)) \leq c = a_2$ and $\tau(G(\langle U_{22} \rangle)) \leq d = a_3 + a_4 + \cdots + a_t$. Similarly, if we consider the pair of positive integers $(x, y)$ with $x + y = \tau(Q)$, where $Q = G(\langle U_{22} \rangle)$, $x = a_3$ and $y = a_4 + a_5 + \cdots + a_t$, by Corollary 2.1, we get a partition $(U_{31}, U_{32})$ such that $\tau(G(\langle U_{31} \rangle)) \leq x = a_3$ and $\tau(G(\langle U_{32} \rangle)) \leq y = a_4 + a_5 + \cdots + a_t$. Continuing this process, finally we get a partition $(V_1, V_2, \ldots, V_t)$ of $V(G)$ such that $\tau(G(\langle V_i \rangle)) \leq a_i$, for every $i$, $1 \leq i \leq t$, where $V_1 = U_1$, $V_2 = U_{21}$, $V_3 = U_{31}$ and so on. This completes the proof.

**Corollary 2.2.** The $n^{th}$ detour chromatic number $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$ for every graph $G$ and for every $n \geq 1$.

**Proof.** Let $G$ be any graph. For every $n \geq 1$, consider the $\frac{\tau(G)}{n}$-tuple $(n, n, \ldots, n)$ if $\tau(G)$ is a multiple of $n$, while if $\tau(G)$ is not a multiple of $n$, then consider the $\left\lceil \frac{\tau(G)}{n} \right\rceil$-tuple $(n, n, \ldots, n, \nu)$, where $\nu = \tau(G) \pmod{n}$. Then, by Path Partition Theorem, there exist a partition $(V_1, V_2, \ldots, V_t)$, where

$$t = \begin{cases} \frac{\tau(G)}{n} & \text{if } \tau(G) \text{ is a multiple of } n \\ \left\lceil \frac{\tau(G)}{n} \right\rceil & \text{if } \tau(G) \text{ is not a multiple of } n \end{cases}$$

such that $\tau(G(\langle V_i \rangle)) \leq n$, for every $i$, $1 \leq i \leq t$. For each $i$, $1 \leq i \leq t$, assign the (distinct) colour $i$ to all the vertices in each $G(\langle V_i \rangle)$. Then every monochromatic path in $G$ has the order at most $n$. Thus, $\chi_n(G) \leq \left\lceil \frac{\tau(G)}{n} \right\rceil$. 

\[9\]
Corollary 2.2 essentially ascertains that “Frick-Bullock Conjecture is true”.

**Remark 1:** It is clear from the definition of $\chi_n(G)$, when $n = 1$, $\chi_1(G) = \chi(G)$. Thus, by Corollary 2.2 for a graph $G$, $\chi(G) = \chi_1(G) \leq \tau(G)$. This upper bound for the chromatic number of a graph $G$ that $\chi(G) \leq \tau(G)$ is the well known Gallai’s Theorem [11].

### 3 An Upper Bound for Star Chromatic Number

In this section we obtain an upper bound for star chromatic number as a consequence of path partition theorem.

**Theorem 3.1.** Let $G$ be a graph. Then the star chromatic number of $G$, $\chi_s(G) \leq \tau(G)$.

**Proof.** First we prove the result for connected graphs, then the result follows naturally for the disconnected graphs. Let $G$ be a connected graph.

**Claim 1:** There exists a proper $\tau(G)$-vertex colouring for $G$.

Consider $\tau(G)$. If $\tau(G)$ is even, say $2k$, for some $k \geq 1$, then consider the $k$-tuple $(2, 2, \ldots, 2)$ with $2 + 2 + 2 + \cdots + 2 = 2k = \tau(G)$. By Path Partition Theorem, there exists a partition $(V_1, V_2, \ldots, V_k)$ such that $\tau(G(V_i))) \leq 2$, for every $i$, $1 \leq i \leq k$. Therefore, every induced subgraph $G(V_i)$, for $i$, $1 \leq i \leq k$ is the union of a set of independent vertices and/or a set of independent edges. Thus, it is clear that, for $i$, $1 \leq i \leq k$, each $G(V_i)$ is proper 2-vertex colourable. Properly colour the vertices of each $G(V_i)$ with a distinct pair of colours $c_{i1}$ and $c_{i2}$, for $i$, $1 \leq i \leq k$. Consequently, this proper 2-vertex colouring of $G(V_i)$, for all $i$, $1 \leq i \leq k$ induces a proper $\tau(G)$-vertex colouring for the graph $G$. If $\tau(G)$ is odd, say $2k + 1$, for some $k \geq 1$, then consider the $k + 1$-tuple $(2, 2, \ldots, 2, 1)$ with $2 + 2 + 2 + \cdots + 1 = 2k + 1 = \tau(G)$. Then by Path Partition Theorem there exists a partition $(V_1, V_2, \ldots, V_k, V_{k+1})$ such that $\tau(G(V_i))) \leq 2$, for every $i$, $1 \leq i \leq k$ and $\tau(G(V_{k+1})) \leq 1$. Consequently, the vertices of each $G(V_i)$ can be properly coloured with a distinct pair of colours $c_{i1}$ and $c_{i2}$, for $i$, $1 \leq i \leq k$ and the vertices of $G(V_{k+1})$ are colored properly with a distinct color $c_{(k+1)1}$. Thus, this proper 2-vertex colouring of $G(V_i)$, for all $i$, $1 \leq i \leq k$ and the proper 1 colouring of $G(V_{k+1})$ induce a proper $\tau(G)$-vertex colouring for the graph $G$. Hence the Claim 1.
Claim 2: $\chi_s(G) \leq \tau(G)$

To prove Claim 2, we show that the vertices of every path of order four is either coloured with 3 or 4 different colours by the above proper $\tau(G)$-vertex colouring of $G$ or if there exists a bicoloured path of order four in $G$ by the above proper $\tau(G)$-vertex colouring of $G$, then those vertices of such a bicoloured path of order four are properly recoloured so that those vertices are coloured with at least three different colours after the recolouring.

Observation 3.1. As

$$\tau(G(V_i)) \leq \begin{cases} 2, & \text{for } 1 \leq i \leq k \\ 1, & \text{for } i = k + 1 \text{ and } \tau(G) \text{ is odd} \end{cases}$$

any path of order four in $G$ must contain vertices from at least two of induced subgraphs $G(V_i)$’s, where $1 \leq i \leq \alpha$, and $\alpha = k$ when $\tau(G)$ is even, while when $\tau(G)$ is odd, $\alpha = k + 1$ [Hereafter $\alpha$ is either $k$ or $k + 1$ depending on $\tau(G)$ is even or odd respectively]. If any path of order four of $G$ contains vertices from three or four of the induced subgraphs $G(V_i)$’s then such a path has vertices coloured with three or four colours by the proper $\tau(G)$-vertex colouring of $G$. Thus, we consider only those paths of order four in $G$ having vertices from exactly two of the induced subgraphs $G(V_i)$’s, where $1 \leq i \leq \alpha$, for recolouring if it is bicoloured.

Consider any path $P$ of order four in $G$ having at least one vertex (at most three vertices) in $G(V_i)$ for each $i$, $1 \leq i \leq \alpha$ and at least one vertex (at most three vertices) in $G(V_j)$, for every $j$, $1 \leq i < j \leq \alpha$.

**Case 1.** Suppose a path $P$ of order four in $G$ has one vertex in $G(V_i)$ and three vertices in $G(V_j)$, for $i, j$, $1 \leq i < j \leq \alpha$

Then without loss of generality we assume that $u_{i_1}$ is one of the vertices of $P$ which is in $G(V_i)$ and we assume $w_{j_1}, w_{j_2}$ and $w_{j_3}$ are the other three vertices of $P$ which are in $G(V_j)$. Under this situation, in order that the path $P$ is to be a path of order four with the vertices $u_{i_1}, w_{j_1}, w_{j_2}, w_{j_3}$, two of the vertices from the three vertices $w_{j_1}, w_{j_2}$ and $w_{j_3}$ in $G(V_j)$ must be adjacent in $G(V_j)$. Since $V_{k+1}$ is an independent set of vertices, $j \leq k$. As the vertices of each induced subgraph $G(V_j)$ are properly coloured with two colours $c_{j_1}, c_{j_2}$, for $j$, $1 \leq j \leq k$ by the proper $\tau(G)$-vertex colouring, those two adjacent vertices from the three vertices $w_{j_1}, w_{j_2}$ and $w_{j_3}$ in $G(V_j)$ should have been coloured with two different colours $c_{j_1}, c_{j_2}$ by the $\tau(G)$-vertex colouring. In $G(V_i)$ each vertex is coloured with either $c_{i_1}$ or $c_{i_2}$ by...
the proper \(\tau(G)\) vertex colouring, the vertex \(u_{i_1}\) is coloured with either \(c_{i_1}\) or \(c_{i_2}\) in \(G(\langle V_i \rangle)\) by the proper \(\tau(G)\)-vertex colouring. This implies that the path \(P\) of order four having the vertices \(u_{i_1}, w_{j_1}, w_{j_2}\) and \(w_{j_3}\) are coloured with at least three different colours by the proper \(\tau(G)\)-vertex colouring of \(G\).

**Case 2** Suppose a path \(P\) of order four in \(G\) has exactly two vertices in 
\(G(\langle V_i \rangle)\) and has exactly two vertices in \(G(\langle V_j \rangle)\).

Let \(u_{i_1}\) and \(u_{i_2}\) be the two vertices of \(P\) in \(G(\langle V_i \rangle)\) and let \(w_{j_1}\) and \(w_{j_2}\) be the two vertices of \(P\) in \(G(\langle V_j \rangle)\).

**Case 2.1.** Suppose either \(u_{i_1}, u_{i_2}\) are coloured with two different colours 
\(c_{i_1}, c_{i_2}\) in \(G(\langle V_i \rangle)\) or \(w_{j_1}, w_{j_2}\) are coloured with two different colours 
\(c_{j_1}, c_{j_2}\) in \(G(\langle V_j \rangle)\) by the proper \(\tau(G)\)-vertex colouring.

Then the vertices of the path \(P\) of order four having the vertices \(u_{i_1}, u_{i_2}, w_{j_1}\) and \(w_{j_2}\) are coloured with three or four different colours by the proper \(\tau(G)\)-vertex colouring of \(G\).

**Case 2.2.** Suppose neither the vertices \(u_{i_1}, u_{i_2}\) received different colours in 
\(G(\langle V_i \rangle)\) nor the vertices \(w_{j_1}, w_{j_2}\) received different colours in 
\(G(\langle V_j \rangle)\) by the proper \(\tau(G)\)-vertex colouring.

Then without loss of generality, we assume that \(u_{i_1}, u_{i_2}\) received the same colour \(c_{i_1}\) in \(G(\langle V_i \rangle)\) and without loss of generality, we assume that \(w_{j_1}, w_{j_2}\) received the same colour \(c_{j_1}\) in \(G(\langle V_j \rangle)\) by the \(\tau(G)\)-vertex colouring. As the vertices of \(G\) are properly coloured, the vertices \(u_{i_1}\) and \(u_{i_2}\) should be non-adjacent in \(G(\langle V_i \rangle)\) as well as the vertices \(w_{j_1}\) and \(w_{j_2}\) should also be non-adjacent in \(G(\langle V_j \rangle)\). Since for every \(h, 1 \leq h \leq \alpha, \tau(G(\langle V_h \rangle)) \leq 2, G(\langle V_h \rangle)\) is the union of independent vertices and / or independent edges, every vertex in each \(G(\langle V_h \rangle)\) is of degree either 0 or 1. Suppose either \(u_{i_1}\) or \(u_{i_2}\) is of degree 0 in \(G(\langle V_i \rangle)\). Then without loss of generality, we assume that \(u_{i_1}\) is of degree 0 in \(G(\langle V_i \rangle)\). Since \(u_{i_1}\) is not adjacent to any vertex in \(G(\langle V_i \rangle)\), recolour the vertex \(u_{i_1}\) with the colour \(c_{i_2}\) [Since vertices of \(G(\langle V_i \rangle)\) are properly coloured with either \(c_{i_1}\) or \(c_{i_2}\) colours, this recolouring is possible]. Thus, after this recolouring, the vertices \(u_{i_1}, u_{i_2}, w_{j_1}\) and \(w_{j_2}\) of the path \(P\) have received three different colours. Hence, we assume neither \(u_{i_1}\) nor \(u_{i_2}\) is of degree 0 in \(G(\langle V_i \rangle)\). Therefore, the degree of each of the vertices \(u_{i_1}\) and \(u_{i_2}\) must be of degree 1 in \(G(\langle V_i \rangle)\). As vertices of each \(G(\langle V_i \rangle)\) are properly coloured for \(i, 1 \leq i \leq \alpha\) and as the vertices \(u_{i_1}\) and \(u_{i_2}\) are coloured with the same colour \(c_{i_1}\) in \(G(\langle V_i \rangle)\), the vertices \(u_{i_1}\) and \(u_{i_2}\) must be non-adjacent in \(G(\langle V_i \rangle)\). Since the \(\text{deg}(u_{i_1}) = 1\) in \(G(\langle V_i \rangle)\), the vertex \(u_{i_1}\) should have an adjacent vertex \(u_{i_1}'\) in \(G(\langle V_i \rangle)\) and it should have been coloured with the colour \(c_{i_2}\) in \(G(\langle V_i \rangle)\) by
the proper $\tau(G)$-vertex colouring. For each $h$, $1 \leq h \leq k$, $\tau(G((V_i))) \leq 2$, the edge $u_i, u'_i$ must be an independent edge in $G((V_i))$. Exchange the colours of $u_i$ and $u'_i$. Thus, after this recolouring (this exchange), the vertex $u_i$ is coloured with $c_{i_2}$. Therefore, after the recolouring the vertices $u_i$ and $w_{i_2}$ received two different colours $c_{i_2}$ and $c_{i_1}$ respectively in $G((V_i))$. As a result, the vertices $u_i$, $u_{i_2}$, $w_{j_1}$ and $w_{j_2}$ have received three different colours in $G((V_i \cup V_j))$. Hence the path $P$ of order four having the four vertices $u_i$, $u_{i_2}$, $w_{j_1}$, $w_{j_2}$ are coloured with three different colours in $G$ after the recolouring.

**Case 3** Suppose the path $P$ of order four in $G$ has three vertices in $G((V_i))$ and the remaining one vertex in $G((V_j))$, for $i, j$, $1 \leq i < j \leq \alpha$.

Without loss of generality, we assume that $w_{j_1}$ is one of the vertices of $P$ which is in $G((V_j))$ and we assume $u_i$, $u_{i_2}$ and $u_{i_3}$ are the other three vertices of $P$ which are in $G((V_i))$. Then as seen in Case 1, two of the vertices from the three vertices $u_i$, $u_{i_2}$ and $u_{i_3}$ should have received two different colours $c_{i_1}$, $c_{i_2}$ by the proper $\tau(G)$-vertex colouring. Consequently, the vertices of the Path $P$ should have received three or four different colours in $G$.

Thus, every path $P$ of order four in $G$ is either coloured with at least three different colours by the proper $\tau(G)$-vertex colouring of $G$ or else if they are bicoloured by the proper $\tau(G)$-vertex colouring, then the vertices of such a path $P$ can be recoloured as done in the above recolouring process so that the vertices of $P$ are coloured with at least three different colours.

Thus there exist a $\tau(G)$-star colouring for $G$. Hence, $\chi_s(G) \leq \tau(G)$. Hence Claim 2.

If $G$ is a disconnected graph with $t \geq 2$ components $G_1, G_2, \ldots, G_t$. Then by Claim 2, $\chi_s(G_i) \leq \tau(G_i)$, for $i$, $1 \leq i \leq t$. Let max $\chi_s(G_i) = \chi_s(G_k)$, for some $k$, $1 \leq k \leq t$. Since $\chi_s(G) = \max_{1 \leq i \leq t} \chi_s(G_i)$, we have $\chi_s(G) = \chi_s(G_k) \leq \tau(G_k) \leq \max_{1 \leq i \leq t} \tau(G_i) = \tau(G)$. Thus, $\chi_s(G) \leq \tau(G)$. This completes the proof.

An acyclic colouring of $G$ is a proper vertex colouring of $G$ such that no cycle of $G$ is bicoloured. Acyclic chromatic number of a graph $G$, denoted $a(G)$ is the minimum of colours which are necessary to acyclically colour $G$.

**Corollary 3.1.** Let $G$ be any graph. Then the acyclic chromatic number of $G$, $a(G) \leq \tau(G)$.

**Proof.** For every graph $G$, $a(G) \leq \chi_s(G)$. By Theorem 3.1 we have $\chi_s(G) \leq \tau(G)$ for any graph $G$. Thus, $a(G) \leq \tau(G)$, for any graph $G$. □
4 Discussion

Path Partition Theorem is a beautiful and natural theorem and it significantly helped to get the upper bounds for chromatic number, star chromatic number and detour chromatic number. We believe that Path Partition Theorem can be significantly used for obtaining upper bounds of other different coloring related parameters too. In a general approach, understanding the following question will be interesting and significant too.

What are the other graph parameters for which such partitions (like $\tau$-partition) can be obtained?

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