Determining the coefficient of rolling friction using hypocycloidal oscillations

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Abstract. The paper presents a method of finding the coefficient of rolling friction based on the hypo-cycloidal oscillation motion performed by a bearing ball into the race of an outer immobile bearing ring. The equation of motion of a ball oscillating on the inner surface of a cylinder is obtained assuming a linear dependence between the rolling friction torque and the normal force. The experimental device consists in an immobile metallic cylinder and a rotor made of two identical balls. The oscillations of the rotor were recorded and the angular amplitude decrease in time is found. For the steel balls a disagreement appears between theoretical and experimental results as the experimental data present a quasi-linear aspect in amplitude decrease only in the final stage of oscillations. The coefficient of rolling friction is established from the condition that the theoretical angular elongation graph must fit best the experimental data.

1. Introduction

In machine building there are few cases of solid parts that do not directly interact with other solid elements, for instance hydraulic suspensions, gas bearings and magnetic bearing. In most of the situations, the parts are contacting each other directly and moreover, often, a relative motion is required between the boundary surfaces of the contacting parts, motion which is set by the functional role of the assembly containing the pair [1]. Technological and cost requirements imposed the use as boundaries of the contact zones the surfaces of advanced manufacturing amid which the plane surfaces and the axi-symmetric surfaces are remarked [2-3]. Dynamic considerations [4] led to the idea to impose plane-parallel motions, namely executed parallel to a certain plane, to all the elements from a mechanical structure. The immediate dynamic effect of this constructive particularity consists in cancelling all gyroscopic torques from the bearings of the structure. For the plane-parallel motion case, the motion of the whole system can be studied in any section parallel to the reference plane, using the intersection curves between the boundary surfaces of the elements and the plane to analyze. For most of the examples from mechanical engineering these curves are circles and/or straight lines. When the contacting circles or lines have more than one common point placed on a spatial curve, the contact between the two curves is a non-conforming or Hertzian contact and therefore a higher pair is created between the two bodies [5-6]. A major characteristic of the higher pairs is their irreversibility. This is evidenced by the fact that, by keeping immobile one of the curves and letting the other one roll over it without sliding, the initial contact point describes a curve different from the one obtained when the roles of the two curves reverse. All the possible cases of contact without sliding between circles and straight lines are presented in figure 1, the curves generated by the initial contact point being named cycloidal curves [7].
The cycloidal motions are appropriate to various technical applications, from the study of the planet motions to the differential mechanisms. In the tribological domain [8-11] there are applications that use the motions of a cycloidal pendulum to establish the coefficient of rolling friction. An evolventic pendulum is used in [12] to the same purpose. In [13] is applied the hypocycloidal motion of a bearing ball on the race of the outer ring that performs uniform rotation motion and establishes a correlation between the position of the dynamic equilibrium and the torque of rolling friction.

The present paper presents a method of finding the coefficient of rolling friction based on the hypocycloidal oscillation motion performed by a bearing ball into the race of an outer immobile bearing ring. The ball and the outer-ring ball-race are presented in figure 2 for two different values of the ball radius together to the corresponding hypo-cycloidal curves on which the initial contact point moves.

![Cycloidal curves](image)

**Figure 1.** Cycloidal curves

2. Equations of motion for hypo-cycloidal oscillations of an axi-symmetric body

A ball of \( r \) radius is in contact with the lower side of a cylinder of radius \( R + r \) in a manner that the center of the ball is moving permanently on a circle of \( R \) radius, concentric to the fix circle. The assumption that at the initial moment the ball is in the lowest point of the immobile circle is made. Two systems of coordinates are used in order to characterize the position of the ball. The fixed system \( Oxyz \) with the center in the centre of the immobile circle and \( Ox \) axis is the downward vertical and \( Oy \) axis is horizontal, as seen in figure 3. The second frame \( Ox'y'z' \) is a mobile coordinate system, attached to the ball, with the origin in the center of the ball. Accepting that the motion of the ball is plane-parallel in the \( Oxy \) plane, the planes \( Oxy \) and \( O'x'y' \) are permanently coincident, and the position of the ball will be completely stipulated by the \( \theta \) angle formed between the vector of position \( r_0 = OG \) of the center of the ball and the \( Ox \) axis together to the \( \varphi \) angle between the \( Ox' \) and \( Ox \) axes. The contact point between the ball and the circle is denoted \( C \).

The forces acting upon the ball are the own weight \( G \) and the components of the reaction torsor from point \( C \), that is:

- the normal reaction \( N \), along the normal in the contact point and oriented towards the center of the fixed circle;
- the friction force $T$, tangent to the immobile circle, with unknown magnitude and sense when pure rolling exists in the contact point;
- the rolling friction torque $M_r$ oriented along $Oz$ axis, with the sense opposite to the angular velocity of the ball and magnitude proportional to the value of normal reaction.

$$M_r = -sN \, \text{sgn} \phi$$  \hspace{1cm} (1)

![Diagram of the ball in hypo-cycloidal oscillatory motion]

**Figure 3.** Scheme of the ball in hypo-cycloidal oscillatory motion

The pure rolling supposition accepted requires a relationship between the parameters $\theta$ and $\phi$, in this case is illustrated by the fact that the velocity of the contact point from the ball is zero

$$v_c = 0$$  \hspace{1cm} (2)

The relation looked for between the two parameters can be found by developing the vector equation (2). In the case analyzed is simpler to directly particularize the relation between the angular velocities of a differential mechanism. So, it can be considered the differential mechanisms form figure 3 consisting in the sun gear with $\omega^S$ angular velocity and the pitch circle:

$$d_w^S/2 = R + r$$  \hspace{1cm} (3)

The planetary wheel has the angular velocity $\omega^P = \dot{\phi}$ and the pitch circle:

$$d_w^P/2 = r$$  \hspace{1cm} (4)

and the satellite-arm $P$ is represented by the rod $OG$ with angular velocity $\omega^P = \dot{\phi}$. Between the three angular velocities the following equation exists [14]:

$$\frac{\omega^S - \omega^P}{\omega^S - \omega^P} = i_{SS}^P$$  \hspace{1cm} (5)

where $i_{SS}^P$ is the transmission ratio of the mechanism under the hypothesis of fixed satellite-arm:

$$i_{SS}^P = \frac{R + r}{r}$$  \hspace{1cm} (6)
Since the sun gear is immobile, \( \omega^s = 0 \) and from relations (5) and (6) it results:

\[
\phi = \frac{R}{r} \dot{\theta} \tag{7}
\]

A similar relation to the one between \( \theta \) and \( \varphi \) will exist between the angular accelerations \( \ddot{\theta} \) and \( \ddot{\varphi} \), when \( Ox' \) and \( OG \) coincide at the initial time.

In order to find the laws of motion and the reactions, the theorem of the center of mass [15] is applied:

\[
M\ddot{r}_c = G + N + T \tag{8}
\]

and the moment of momentum theorem, which presents projection only on the \( Oz \) axis:

\[
M\ddot{\varphi} = Tr - Mr \tag{9}
\]

The following vectors are introduced in order to establish the equation of motion of the mass center of the ball:

\[
n = i \cos \theta + j \sin \theta
\]

\[
t = -4 \sin \theta \varphi + j \cos \theta
\]

where \( i \) and \( j \) are the versor of \( Ox \) and \( Oy \) axes respectively. The relation (8) is written as:

\[
\frac{d^2}{dt^2} (Rn) = Mg i - Nn + Tt \tag{11}
\]

After the differentiation with respect to time of the left member is calculated and applying the relations (10) the equations of projections are obtained:

\[
-MR \ddot{\theta} \sin \theta - MR \ddot{\varphi}^2 \cos \theta = Mg - T \sin \theta - N \cos \theta
\]

\[
MR \ddot{\varphi} \cos \theta - MR \ddot{\varphi}^2 \sin \theta = T \cos \theta + N \cos \theta \tag{12}
\]

The differential of relation (7) with respect to time is used in relation (9) and it results:

\[
-J \frac{R}{r} \ddot{\varphi} = Tr - sN \text{sgn} \left( -\frac{R}{r} \ddot{\theta} \right) \tag{13}
\]

The final system of the equations of motion has the form:

\[
\begin{align*}
-MR \ddot{\theta} \sin \theta - MR \ddot{\varphi}^2 \cos \theta &= Mg - T \sin \theta - N \cos \theta \\
MR \ddot{\varphi} \cos \theta - MR \ddot{\varphi}^2 \sin \theta &= T \cos \theta + N \cos \\
-J \frac{R}{r} \ddot{\varphi} &= Tr + sN \text{sgn} \ddot{\theta}
\end{align*}
\tag{14}
\]

The solution of the system (14) is:

\[
\ddot{\theta} = -\frac{g \sin \theta - \left[ R \ddot{\varphi}^2 + g \cos \theta \right]^s}{R \left[ 1 + \frac{J}{Mr} \right]}
\]

\[
N = MR \ddot{\varphi}^2 + Mg \cos \theta
\]

\[
T = M (R \ddot{\varphi} + g \sin \theta)
\]
As it can be remarked from the system (15), resolving the problem supposes solving the differential equation from the system (15), which is a non-homogenous differential equation [16]. It can be noticed that in the case of absence of rolling friction, \( s_r = 0 \), the equation turns into the law of motion of a physical pendulum [17].

Obviously, all the relations obtained after integration of system (15) are valid as long as the following condition:

\[
|T|/N < \mu
\]  

is satisfied, where \( \mu \) is the dynamic sliding friction coefficient and the ratio \( |T|/N \) signifies the friction coefficient.

3. Experimental set-up, tests and results

The principle from figure 3 is difficult to apply in practice since the ball in contact with the inner cylinder presents five DOF, namely it may perform tree DOF of the model- translations about \( Ox \) and \( Oy \) axes and a rotation around \( Oz \) together to two rotations, one about the normal \( n \) and the other one about the axis of versor \( t \). Cancellation of the last two motions imposes the existence of a spinning torque and a rolling friction torque, parallel to each of the two vectors, respectively. The spinning torque is generated by the friction forces occurring on the contact area due to contact pressure. Referring to the rolling friction torque, it has a small value and this fact requires an accurate orientation of the axis if the cylinder in an horizontal plane to avoid any parasite rolling about an axis normal to the axis of the cylinder. The above considerations led to the idea of using a rolling body obtained by sticking two identical balls as in figure 4. The rolling body makes two contacts \( C_1 \) and \( C_2 \) with the cylindrical surface. In these points act the normal components of reactions \( N_{1,2} \), and the tangential components, split into \( T_{1,2} \) parallel to the \( t \) versor and \( T_{1,2} \) parallel to the axis of the cylinder.

![Figure 4. The proposed rolling body](image)

Supposing that the rolling body would tend to rotate about the \( t \) versor, this fact is possible only if one of the contacts, for instance \( C_2 \), is disrupt. But now, the rotation tendency around \( C_1 \) is cancelled by the moment due to the weight of the ball 2. In a similar manner, the spinning motion about a parallel to the normal passing through \( C_1 \) is annulated by the moment due to friction force \( T_2 \). Consequently, the only possible remaining rotation is about \( C_1C_2 \), since none of the forces from figure 4 doesn’t produce a moment with respect to this axis. In addition the friction components \( T'_{1,2} \) oppose
to sliding tendency on the direction of the axis of the cylinder. The mentioned comments proof that the motion of the rolling body from figure 4 is a plane-parallel one. The experimental set-up is presented in figure 5. On the inner surface of an immobile metallic tube (1), with inner diameter 5in, is placed a rolling body (2) consisting in two identical balls of diameter 9/8in. The roughness parameters were measured with the Nanofocus profilometer and are presented in figure 6. A stamp is glued on the top of one ball, having marked the trace (a) of the central axis of the rolling body. A protractor (b) is placed on the frontal side of the cylinder, concentric to it, in order to establish the motion of the rotor. An oscillatory motion of imposed amplitude of the rolling body is initiated using a rod.

**Figure 5.** Experimental set-up (left); actual rolling bodies : 1-steel balls; 2- computer mouse balls (right)

The motion of the rotor is video recorded and afterwards the motion is analyzed frame by frame using a software. The moments when the maximum angular elongations are reached are identified and the numerical dependence of amplitude damping versus time is obtained. For the two bodies shown on figure 5, the first made of two steel balls and the second made of two computer-mouse balls, the experimental amplitude variations with time are plotted in figures 7 and 8.

**Figure 6.** Profiles and roughness parameters obtained with Nanofocus profilometer for the steel ball and aluminium cylinder (inner surface)
Analyzing the plots from figures 7 and 8 it is revealed that for the steel rotor the amplitude decrease is not linear, as expected from theoretical studies upon dry damped oscillatory motion [18]. There are several causes of this non-concordance and some of these are evidenced in [10].
Given that the coefficient of rolling friction is found based on the condition that the experimental points are interpolated by the graph of angular elongation obtained as solution of integrating the differential equation (15), the fragment of the plot that is not rectilinear must be neglected. The equation (15) is integrated numerically [19] for different values of the coefficient of rolling friction and the solution that best matches the experimental data is chosen. In figures 9 and 10 there are presented the plots for the two rotors. In figure 9 is seen that only the experimental data revealing a quasi-linear decrease of amplitude were considered. Finally, in figures 11 and 12 the dependencies $T/N$ are presented for pure rolling confirmation. From the relations 15, both $T$ and $N$ depend on the angular elongation $\theta$ and its derivatives; as a consequence, the ratio $T/N$ should present a periodical behaviour.

4. Conclusions
The present work proposes a test-rig and method for finding the coefficient of rolling friction, starting from the results obtained in a previous paper.

In the first part of the paper, the equation of motion of a ball oscillating freely inside a cylinder is deduced under the hypothesis of linear dependency between the rolling friction torque and the normal force.

The experimental set-up has simple construction and consists in a fixed metallic circular cylinder and the rolling body. The rotor is made of two identical balls attached to each other, solution based on the requirements of elimination of ball’s spin motion and rolling about the axis of the cylinder.

The oscillations of the rotor were filmed and the film was split into frames for finding the angular amplitude decrease in time. The coefficient of rolling friction is found from the condition that the theoretical angular elongation graph should approximate the experimental data as well as possible.

The experimental tests were carried out for two cases: first, the rolling body is made from two identical bearing balls and second, the rotor is made from two computer-mouse balls. For the steel balls a discrepancy exists between theoretical and experimental results: while the theoretical model predicts for the dry friction situation a linear decrease of oscillation amplitude, the experimental data show a more important amplitude decrease at the start of the motion and only towards the end of oscillations, for small amplitudes, the decrease has a quasi-linear aspect. From this reason, the experimental data obtained in the immediate vicinity of the launching were eliminated and for the interpolation with the theoretical signal only the results from the final oscillation interval, where a linear amplitude damping can be accepted, were used.

The values of the coefficient of friction vary periodically and the remark that the amplitude decreases linearly can be made.

The method is simple and expedite. Another advantage of the technique consists in the fact that the tribological parameters found are characteristic to a very narrow region, of order of millimetres, compared to other methods like the inclined plane method where the coefficient of rolling friction is established as a mean value of the tribological behaviour on the entire length of the plane.

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