INHOMOGENEOUS STRICHARTZ ESTIMATES FOR SCHRÖDINGER’S EQUATION

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ABSTRACT. Foschi and Vilela in their independent works ([3], [13]) showed that the range of \((1/r, 1/\tilde{r})\) for which the inhomogeneous Strichartz estimate

\[
\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds \|_{L_q^r L_x^s} \lesssim \| F \|_{L_{\tilde{q}}^{\tilde{r}} L_x^s}
\]

holds for some \(q, \tilde{q}\) is contained in the closed pentagon with vertices \(A, B, B', P, P'\) except the points \(P, P'\) (see Figure 1). We obtain the estimate for the corner points \(P, P'\).

1. Introduction

In this paper we consider the following Cauchy problem for the Schrödinger equation:

\[
\begin{cases}
i\partial_t u + \Delta u = F(x, t), \\
u(x, 0) = f(x),
\end{cases}
\]

where \((x, t) \in \mathbb{R}^n \times \mathbb{R}, n \geq 1\). By Duhamel’s principle, we have the solution

\[
u(x, t) = e^{it\Delta} f(x) - i \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds,
\]

where \(e^{it\Delta}\) is the free Schrödinger propagator defined by

\[
e^{it\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot \xi - it|\xi|^2} \hat{f}(\xi) d\xi.
\]

The Strichartz estimates for the solution play important roles in the study of well-posedness for nonlinear Schrödinger equations (cf. [1], [12]). They actually consist of two parts, homogeneous \((F = 0)\) and inhomogeneous \((f = 0)\) part. The homogeneous Strichartz estimate

\[
\| e^{it\Delta} f \|_{L_q^r L_x^s} \lesssim \| f \|_{L^2}
\]

holds if and only if \((r, q)\) is admissible pair, that is,

\[r, q \geq 2, \quad (n, r, q) \neq (2, \infty, 2) \quad \text{and} \quad n/r + 2/q = n/2\]

(see [11], [4], [9], [6] and references therein). But determining the optimal range of \((r, q)\) and \((\tilde{r}, \tilde{q})\) for which the inhomogeneous Strichartz estimate

\[
\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s) ds \|_{L_q^r L_x^s} \lesssim \| F \|_{L_{\tilde{q}}^{\tilde{r}} L_x^s}
\]

holds is not completed yet when \(n \geq 3\). It was observed that this estimate is valid on a wider range than what is given by admissible pairs \((r, q), (\tilde{r}, \tilde{q})\) (see [2], [5]).

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Foschi and Vilela in their independent works ([3],[13]) showed that the range of \((1/r, 1/\tilde{r})\) for which (1.1) is valid for some \(q, \tilde{q}\) is contained in the closed pentagon with vertices \(A, B, B', P, P'\) except the points \(P, P'\) (see Figure 1). The aim of this paper is to obtain (1.1) for the points \(P, P'\). Our result is the following.

**Theorem 1.1.** Let \(n \geq 3\). Then (1.1) holds when

\[
\left( \frac{1}{r}, \frac{1}{\tilde{r}} \right) = \begin{cases} 
(\frac{n-2}{2(n-1)}, \frac{(n-2)^2}{2n(n-1)}) & \text{if } \frac{n-2}{2(n-1)} \leq \frac{1}{q} = \frac{1}{\tilde{q}} < \frac{n}{2(n-1)}, \\
\left( \frac{n-2}{2n(n-1)}, \frac{n-2}{2(n-1)} \right) & \text{if } \frac{n-2}{2(n-1)} < \frac{1}{q} = \frac{1}{\tilde{q}} \leq \frac{n}{2(n-1)}. 
\end{cases}
\]

and when

\[
\left( \frac{1}{r}, \frac{1}{\tilde{r}} \right) = \begin{cases} 
(\frac{n-2}{2(n-1)}, \frac{(n-2)^2}{2n(n-1)}) & \text{if } \frac{n-2}{2(n-1)} \leq \frac{1}{q} = \frac{1}{\tilde{q}} < \frac{n}{2(n-1)}, \\
\left( \frac{n-2}{2n(n-1)}, \frac{n-2}{2(n-1)} \right) & \text{if } \frac{n-2}{2(n-1)} < \frac{1}{q} = \frac{1}{\tilde{q}} \leq \frac{n}{2(n-1)}. 
\end{cases}
\]

**Remark 1.2.** Since \(1/r + 1/\tilde{r} = (n-2)/n\), the condition \(q = \tilde{q}'\) follows from the scaling condition

\[
\frac{1}{q} + \frac{1}{\tilde{q}} + \frac{n}{2} \left( \frac{1}{r} + \frac{1}{\tilde{r}} \right) = \frac{n}{2}.
\]

The conditions \(1/q < n/2(n-1)\) and \((n-2)/2(n-1) < 1/\tilde{q}'\) when \((1/r, 1/\tilde{r}) = P\) and \(P'\) correspond to the known necessary conditions ([3],[13])

\[
\frac{1}{q} < \frac{n}{2} \left( 1 - \frac{2}{r} \right) \quad \text{and} \quad \frac{1}{q} < \frac{n}{2} \left( 1 - \frac{2}{\tilde{r}} \right),
\]

respectively. In Section 3 we also give new necessary conditions for (1.1).

**Remark 1.3.** Our proof can be easily modified to cover the range of \((r, q)\) and \((\tilde{r}, \tilde{q})\) obtained by Foschi and Vilela. But we have chosen to present the proof only for the points \(P, P'\) to keep the exposition as simple as possible. The case \(n \geq 4\) in Theorem 1.1 was already shown in [7] but the argument there does not suffice to obtain the same result in dimension \(n = 3\).
2. Proof of Theorem 1.1

Under the same conditions in Theorem 1.1, we will show
\[ \left\| \int_{-\infty}^{t} e^{i(t-s)\Delta} F(\cdot, s) ds \right\|_{L^q_t L^\infty_x} \lesssim \|F\|_{L^q_t L^\infty_x} \]  \hspace{1cm} (2.1)
which implies (1.1). Indeed, to obtain (1.1) from (2.1), first decompose the $L^q_t$ norm in the left-hand side of (1.1) into two parts, $t \geq 0$ and $t < 0$. Then the latter can be reduced to the former by changing the variable $t \mapsto -t$, and so it is only needed to consider the first part $t \geq 0$. But, since $[0, t) = (-\infty, t) \cap [0, \infty)$, applying (2.1) with $F$ replaced by $\chi_{[0, \infty)}(s)F$, one can bound the first part as desired.

Let $\phi$ be a cut-off function with $\phi(\xi) = 1$ if $|\xi| \leq 1$, $\phi(\xi) = 0$ if $|\xi| > 2$, and $0 \leq \phi(\xi) \leq 1$. Then it is enough to show that
\[ \left\| \int_{-\infty}^{t} \int_{\mathbb{R}^n} e^{ix\cdot\xi - i(t-s)|\xi|^2} |\phi(\xi)|^2 F(\cdot, s)(\xi) d\xi ds \right\|_{L^q_t L^\infty_x} \lesssim \|F\|_{L^q_t L^\infty_x}. \]  \hspace{1cm} (2.2)
Once we have this estimate, by the usual scaling we see that for all $j \geq 0$
\[ \left\| \int_{-\infty}^{t} \int_{\mathbb{R}^n} e^{ix\cdot\xi - i(t-s)|\xi|^2} |\phi(\xi/2^j)|^2 F(\cdot, s)(\xi) d\xi ds \right\|_{L^q_t L^\infty_x} \lesssim 2^{2j(\frac{1}{q} - \frac{1}{2} - \frac{1}{2} + \frac{1}{2}(1 - \frac{1}{q} - \frac{1}{2}))} \|F\|_{L^q_t L^\infty_x} \lesssim \|F\|_{L^q_t L^\infty_x}. \]
Here, for the last inequality, we used the scaling condition (1.2). Since we may assume that $F$ is contained in the Schwartz space on $\mathbb{R}^{n+1}$, by a limiting argument ($j \to \infty$), we now get (2.1) from the above estimate.

Now, for fixed $t$, we define
\[ T_t f(x) = \int e^{ix\cdot\xi - i(t-s)|\xi|^2} \phi(\xi) \hat{f}(\xi) d\xi \]
and note that its adjoint operator $T_t^*$ is given by
\[ T_t^* f(x) = \int e^{ix\cdot\xi + i(t-s)|\xi|^2} \overline{\phi(\xi)} \hat{f}(\xi) d\xi. \]
Then the desired estimate (2.2) can be rewritten as
\[ \left\| \int_{-\infty}^{t} T_t T_t^* F_s ds \right\|_{L^q_t L^\infty_x} \lesssim \|F\|_{L^q_t L^\infty_x} \]
where we use the notation $F_s$ to denote $F_s(\cdot) = F(\cdot, s)$. By duality we are now reduced to showing the bilinear form estimate
\[ \left| \int_{\mathbb{R}} \int_{-\infty}^{t} \langle T_t^* F_s, T_t^* G_t \rangle_{L^2_x} ds dt \right| \lesssim \|F\|_{L^q_t L^\infty_x} \|G\|_{L^q_t L^\infty_x}. \]  \hspace{1cm} (2.3)
under the same conditions in Theorem 1.1. To show (2.3), we will use the following lemma.

Lemma 2.1. Let $n \geq 3$, and let $2 \leq r, \tilde{r} \leq \infty$ and $1 \leq q, \tilde{q} \leq \infty$. Define
\[ B_j(F, G) = \int_{\mathbb{R}} \int_{t-2^j+1}^{t-2^j} \langle T_t^* F_s, T_t^* G_t \rangle_{L^2_x} ds dt \]
and assume one of the following conditions for \((r, \bar{r}; q, \bar{q})\):

- i) \(\frac{1}{r} \leq \frac{n-2}{n} \frac{1}{\bar{r}} \) and \(\frac{1}{r} \leq \frac{1}{q} \leq \frac{1}{\bar{q}} \leq 1\),
- ii) \(\frac{n-2}{n} \geq \frac{1}{r} \leq \frac{1}{\bar{r}} \) and \(-\frac{n}{2} \frac{1}{r} - \frac{1}{\bar{r}} \leq \frac{1}{q} \leq \frac{1}{\bar{q}} \leq 1\),
- iii) \(\frac{1}{r} \leq \frac{1}{\bar{r}} \leq \frac{n-1}{n} \frac{1}{r} \) and \(0 \leq \frac{1}{q} \leq \frac{1}{\bar{q}} \leq 1 - \frac{n}{2} \frac{1}{r} - \frac{1}{\bar{r}}\),
- iv) \(\frac{n-1}{n} \geq \frac{1}{\bar{r}} \leq \frac{1}{r} \) and \(0 \leq \frac{1}{q} \leq \frac{1}{\bar{q}} \leq 1 - \frac{1}{r}\).

Then we have

\[
|B_j(F, G)| \lesssim 2^{j\beta(r, \bar{r}, q, \bar{q})} \|F\|_{L^\infty_r L^q_r} \|G\|_{L^{q'}_{\bar{r}} L^{q'}_{\bar{r}'}} ,
\]

where

\[
\beta(r, \bar{r}, q, \bar{q}) = \begin{cases} 
\frac{1}{q} + \frac{1}{q} + \frac{n-1}{n} - \frac{n}{r} & \text{if i) holds,} \\
\frac{1}{q} + \frac{1}{q} - \frac{n}{2} (1 - \frac{1}{r} - \frac{1}{\bar{r}}) & \text{if ii) or iii) holds,} \\
\frac{1}{q} + \frac{1}{q} + \frac{n-1}{n} - \frac{n}{r} & \text{if iv) holds.}
\end{cases}
\]

**Remark 2.2.** The ranges of \((1/r, 1/\bar{r})\) in i), ii), iii) and iv) correspond to the triangular regions \(OB'C', OAB', OAB\) and \(OBC\) in Figure 1 respectively.

**Proof of Lemma 2.1.** One can easily get the above lemma by interpolating the estimates in (2.4) in the following four cases:

- (a) \(r = \bar{r} = \infty\) (point \(O\)) and \(1 \leq \bar{q} \leq q \leq \infty\),
- (b) \(r = \bar{r} = 2\) (point \(A\)) and \(1 \leq \bar{q} \leq q \leq \infty\),
- (c) \(r = 2, \frac{2n}{n-2} \leq \bar{r} \leq \infty\) (segment \(BC\)) and \(2 \leq \bar{q} \leq q \leq \infty\),
- (d) \(\frac{2n}{n-2} \leq r \leq \infty, \bar{r} = 2\) (segment \(B'C'\)) and \(1 \leq \bar{q} \leq q \leq 2\).

The first and second ones, (a) and (b), were already shown in [7] (see Lemma 2.1 there). So we only need to show (c) and (d). For (c) we decompose \(F\) and \(G\) as \(F^k(x, s) = F(x, s)\chi_{\{2j \leq t < 2j+1\}}(s)\) and \(G^k(x, t) = G(x, t)\chi_{\{2j \leq t < 2j+1\}}(t)\) for fixed \(j\). Then we see that

\[
|B_j(F, G)| = \sum_{k \in \mathbb{Z}} \int_{t-2^j}^{t-2^{j+1}} \langle T^* F^k_s, T^*_t G^k_t \rangle_{L^r_x} ds dt \\
\leq \sum_{k \in \mathbb{Z}} \int_{t-2^j}^{t-2^{j+1}} \langle T^* F^k_s, T^*_t G^{k+1}_t \rangle_{L^r_x} ds dt + \sum_{k \in \mathbb{Z}} \int_{t-2^j}^{t-2^{j+1}} \langle T^* F^k_s, T^*_t G^{k+2}_t \rangle_{L^r_x} ds dt
\]

because \(t \in (s + 2^j, s + 2^{j+1})\). Using Hölder’s inequality in \(x\), we note that

\[
\sum_{k \in \mathbb{Z}} \int_{t-2^j}^{t-2^{j+1}} \langle T^* F^k_s, T^*_t G^{k+1}_t \rangle_{L^r_x} ds dt \leq \sum_{k \in \mathbb{Z}} \left\| T^* F^k_s ds \right\|_{L^r_x} \left\| T^*_t G^{k+1}_t dt \right\|_{L^{q'}_t}.
\]

We also note that

\[
\|T_t f\|_{L^{q'}_t L^r_x} \lesssim \|f\|_{L^2}
\]

holds for \(r, q \geq 2\) and \(n/r + 2/j \leq n/2\). Indeed, by the stationary phase method (see p. 344 in [10]), we see \(\|T_t f\|_{L^{q'}_t L^r_x} \lesssim (1 + |t|)^{-n/2}\|f\|_{L^1}\). Then (2.5) follows directly.
from the abstract Strichartz estimates of Keel and Tao [6]. Using the dual estimate of (2.5),
\[
\left\| \int_{\mathbb{R}} T_t^* F_s ds \right\|_{L^2_{\tilde{t}}} \lesssim \| F \|_{L^q_t L^r_s},
\]
we now get
\[
\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left\| T_t^* F_s, T_t^* G_t^{k+1} \right\|_{L^2_{\tilde{t}}} d\tilde{t} dt \right| \lesssim \sum_{k \in \mathbb{Z}} \| F^k \|_{L^q_t L^r_s} \| G^{k+1} \|_{L^q_t L^r_s},
\]
where \( \frac{2n}{n-2} \leq \tilde{r} \leq \infty \). On the other hand, by Hölder’s inequality in time, it follows that
\[
\sum_{k \in \mathbb{Z}} \| F^k \|_{L^q_t L^r_s} \| G^{k+1} \|_{L^q_t L^r_s} \leq \sum_{k \in \mathbb{Z}} \left( 2^j \frac{1}{4 + i \frac{1}{4} - \frac{1}{2}} \right) \| F^k \|_{L^q_t L^r_s} \| G^{k+1} \|_{L^q_t L^r_s} \\
\leq 2 \left( \frac{1}{4 + i \frac{1}{4} - \frac{1}{2}} \right) \| F \|_{L^q_t L^r_s} \| G \|_{L^q_t L^r_s}
\]
if \( 2 \leq q \leq q \leq \infty \). Here, for the last inequality we used that
\[
\sum_n |A_n B_n| \leq \left( \sum_n |A_n|^p \right)^{\frac{1}{p}} \left( \sum_n |B_n|^q \right)^{\frac{1}{q}} \quad \text{if} \quad \frac{1}{p} + \frac{1}{q} \geq 1. \tag{2.6}
\]
Hence,
\[
\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left\| T_t^* F_s, T_t^* G_t^{k+1} \right\|_{L^2_{\tilde{t}}} d\tilde{t} dt \right| \leq 2 \left( \frac{1}{4 + i \frac{1}{4} - \frac{1}{2}} \right) \| F \|_{L^q_t L^r_s} \| G \|_{L^q_t L^r_s}.
\]
Similarly,
\[
\left| \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left\| T_t^* F_s, T_t^* G_t^{k} \right\|_{L^2_{\tilde{t}}} d\tilde{t} dt \right| \leq 2 \left( \frac{1}{4 + i \frac{1}{4} - \frac{1}{2}} \right) \| F \|_{L^q_t L^r_s} \| G \|_{L^q_t L^r_s}.
\]
Consequently, we get (c). The case (d) can be shown in a similar way as (c).

Now we return to (2.3). It suffices to show that
\[
\sum_{j \in \mathbb{Z}} |B_j(F, G)| \lesssim \| F \|_{L^q_t L^r_s} \| G \|_{L^q_t L^r_s}. \tag{2.7}
\]
We only consider the case \((1/r, 1/\tilde{r}) = P\) since the case \((1/r, 1/\tilde{r}) = P'\) follows from the same argument. Now let
\[
\left( \frac{1}{r}, \frac{1}{\tilde{r}} \right) = \left( \frac{n-2}{2(n-1)}, \frac{(n-2)^2}{4n(n-1)} \right) = P \quad \text{and} \quad \frac{n-2}{2(n-1)} \leq \frac{1}{q} = \frac{1}{q} < \frac{n}{2(n-1)}.
\]
Note that the point \( P \) lies on the segment \( OB \), and so we will use Lemma 2.1 under the conditions iii) and iv) (see Figure 1 and Remark 2.2). Since \( 1 - \frac{1}{4} \left( \frac{1}{2} - \frac{1}{2} \right) = 1 - \frac{1}{r} = \frac{n}{2(n-1)} > \frac{1}{4} \), if we choose \( \epsilon > 0 \) small enough so that \( \epsilon \leq \frac{1}{20} \left( \frac{n}{2(n-1)} - \frac{1}{q} \right) \), we can use Lemma 2.1 under the conditions iii) and iv) for \((a, b, q, \tilde{q})\) with all \((a, b) \in B((\frac{1}{r}, \frac{1}{\tilde{r}}), 10\epsilon)\), where
\[
B\left( \left( \frac{1}{r}, \frac{1}{\tilde{r}} \right), 10\epsilon \right) = \left\{ \left( \frac{1}{a}, \frac{1}{b} \right) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] : | \frac{1}{r} - \frac{1}{a}|, | \frac{1}{\tilde{r}} - \frac{1}{b}| < 10\epsilon \right\}.
\]
Now, using Lemma 2.1 we see that
\[
|B_j(F, G)| \lesssim 2 \|a, b, q, \tilde{q} \| F \|_{L^q_t L^r_s} \| G \|_{L^q_t L^r_s}. \tag{2.8}
\]
where

\[ \beta(a, b, q, \tilde{q}) = \begin{cases} 
\frac{1}{q} + \frac{1}{\tilde{q}} - \frac{n}{2}(1 - \frac{1 - a - \frac{1}{b}}{2}) & \text{if } \frac{1}{a} \leq \frac{n}{n - 2}, \\
\frac{1}{q} + \frac{1}{\tilde{q}} + \frac{n - 1}{a} - \frac{n}{2} & \text{if } \frac{1}{a} > \frac{n}{n - 2}.
\end{cases} \]

Next we decompose \( F \) and \( G \) using the following lemma whose proof can be found in \([E]\) (see Lemma 5.1 there):

**Lemma 2.3** (Atomic decomposition of \( L^p \)). Let \( 1 \leq p < \infty \). Then any \( f \in L^p \) can be written as

\[ f = \sum_{k=-\infty}^{\infty} c_k \chi_k \]

where each \( \chi_k \) is a function bounded by \( O(2^{-k/p}) \) and supported on a set of measure \( O(2^k) \) and the \( c_k \) are non-negative constants with \( \|c_k\|_{L^p} \lesssim \|f\|_{L^p} \).

By this lemma, we may write

\[ F_s(x) = \sum_{k \in \mathbb{Z}} f_k(s) \tilde{\chi}_{k,s}(x) \quad \text{and} \quad G_s(x) = \sum_{k \in \mathbb{Z}} g_k(t) \chi_{k,t}(x), \]

where \( \tilde{\chi}_{k,s}(x) \) is bounded by \( O(2^{-k/r'}) \) and supported on a set of measure \( O(2^k) \), and \( \chi_{k,t}(x) \) is bounded by \( O(2^{-k/r}) \) and supported on a set of measure \( O(2^k) \). Also, \( f_k \) and \( g_k \) satisfy

\[ \left( \sum_{k \in \mathbb{Z}} |f_k(s)|^{r'} \right)^{\frac{1}{r'}} \lesssim \|F_s\|_{L^{r'}} \quad \text{and} \quad \left( \sum_{k \in \mathbb{Z}} |g_k(t)|^{r'} \right)^{\frac{1}{r'}} \lesssim \|G_s\|_{L^{r'}.} \quad (2.9) \]

Combining (2.8) and this decomposition, we now get

\[ \sum_{j \in \mathbb{Z}} |B_j(F, G)| \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{j\beta(a, b, q, \tilde{q})} 2^{k(\frac{1}{a} - \frac{1}{b})} 2^{k(\frac{1}{\tilde{q}} - \frac{1}{\tilde{q}'})} \|f_k\|_{L^{r'}} \|g_k\|_{L^{r'}.} \quad (2.10) \]

If \( \frac{1}{a} \leq \frac{n}{n - 2}\), we use (2.10). But if \( \frac{1}{a} > \frac{n}{n - 2} \), we use (2.10) with \( 2^j \) replaced by \( 2^{-j} \). (Since \( j \in \mathbb{Z} \), we may replace \( 2^j \) by \( 2^{-j} \) in (2.10).) Then we conclude that

\[ \sum_{j \in \mathbb{Z}} |B_j(F, G)| \lesssim \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} H_{j,k,k}(a, b) \|f_k\|_{L^{r'}} \|g_k\|_{L^{r'}.}, \]

where

\[ H_{j,k,k}(a, b) = \begin{cases} 
2^{\left(2\frac{k}{2} - \frac{j}{2}\right) + \left(k - \frac{n}{2}\right)\left(\frac{1}{a} - \frac{1}{b}\right)} & \text{if } \frac{1}{a} \leq \frac{n}{n - 2}, \\
2^{\left(2\frac{k}{2} - \frac{j}{2}\right) + \left(k + (n - 1)\right)\left(\frac{1}{a} - \frac{1}{b}\right)} & \text{if } \frac{1}{a} > \frac{n}{n - 2}.
\end{cases} \]

First we consider the cases where \( k \neq \frac{n}{2} \) and \( \tilde{k} \neq \frac{n}{2} \). Let us set

\[ U_1 = \{(j, \tilde{k}, k); \ k - \frac{n}{2} > 0, \ \tilde{k} - \frac{n}{2} > 0\}, \]
\[ U_2 = \{(j, \tilde{k}, k); \ k - \frac{n}{2} < 0, \ \tilde{k} - \frac{n}{2} > 0\}, \]
\[ U_3 = \{(j, \tilde{k}, k); \ k - \frac{n}{2} < 0, \ \tilde{k} - \frac{n}{2} < 0\}, \]
\[ U_4 = \{(j, \tilde{k}, k); \ k - \frac{n}{2} > 0, \ \tilde{k} - \frac{n}{2} < 0\}. \]
Then we may write

$$
\sum_{j \in \mathbb{Z}} |B_j(F, G)| \leq \sum_{(j, \tilde{k}, k) \in U_1 \cup U_2 \cup U_3} H_{j, \tilde{k}, k}(a,b) \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}} \\
+ \sum_{(j, \tilde{k}, k) \in U_4} H_{j, \tilde{k}, k}(a,b) \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}}. 
$$

(2.11)

For each $(j, \tilde{k}, k) \in U_1 \cup U_2 \cup U_3$, we choose $(\frac{1}{a}, \frac{1}{b}) \in B((\frac{1}{a}, \frac{1}{b}), 10\epsilon) \setminus B((\frac{1}{a}, \frac{1}{b}), \epsilon)$ with $\frac{1}{a} \leq \frac{n-2}{n}$ so that

$$
\sum_{U_1 \cup U_2 \cup U_3} 2^{(\tilde{k}-j)(\frac{1}{a}-\frac{1}{n})+(k-j)(\frac{1}{b}-\frac{1}{n})} \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}} \leq \sum_{U_1 \cup U_2 \cup U_3} 2^{-\epsilon(|\tilde{k}-j|+|k-j|)} \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}}.
$$

By summing in $j$, and using (2.6) and Young’s inequality since $K(\cdot) = (1+|\cdot|)2^{-\epsilon|\cdot|}$ is absolutely summable, we see that

$$
\sum_{U_1 \cup U_2 \cup U_3} 2^{-\epsilon(|\tilde{k}-j|+|k-j|)} \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}} \lesssim \sum_{k \in \mathbb{Z}} (1+|k-k|)2^{-\epsilon|\tilde{k}-k|} \|f_{\tilde{k}}\|_{L_t^p'} \|g_k\|_{L_t^{p'}} \\
\lesssim \left( \sum_{k \in \mathbb{Z}} \|f_{\tilde{k}}\|_{L_t^p'}^\theta \right)^{\frac{1}{\theta}} \left( \sum_{k \in \mathbb{Z}} \|g_k\|_{L_t^{p'}}^\theta \right)^{\frac{1}{\theta}}.
$$

\[\text{The line } OP \text{ intersects the regions } (b) \text{ and } (c) \text{ since its slope is } (n-2)/n \text{ (see Figure 2). Hence, if } (j, \tilde{k}, k) \in U_1, \text{ choose } (\frac{1}{a}, \frac{1}{b}) \text{ that lies above the line } OP \text{ in the region } (b). \text{ If } (j, \tilde{k}, k) \in U_2, \text{ choose } (\frac{1}{a}, \frac{1}{b}) \text{ in the region } (a). \text{ If } (j, \tilde{k}, k) \in U_3, \text{ choose } (\frac{1}{a}, \frac{1}{b}) \text{ that lies above the line } OP \text{ in the region } (c).\]
Since \( \tilde{q}' \geq \tilde{r}' \) and \( q' \geq r' \), by Minkowski's inequality and (2.9),
\[
\left( \sum_{k \in \mathbb{Z}} \|f_k\|_{L_{q'}^q}^q \right)^{\frac{1}{q}} \left( \sum_{k \in \mathbb{Z}} \|g_k\|_{L_{r'}^{r}}^r \right)^{\frac{1}{r}} \lesssim \left( \sum_{k \in \mathbb{Z}} \|f_k\|_{L_{q'}^q}^{q'} \right)^{\frac{1}{q'}} \left( \sum_{k \in \mathbb{Z}} \|g_k\|_{L_{r'}^{r}}^{r'} \right)^{\frac{1}{r'}} \lesssim \left\| \left( \sum_{k \in \mathbb{Z}} |f_k(s)|^{q'} \right)^{\frac{1}{q'}} \right\|_{L_{q'}^q} \left\| \left( \sum_{k \in \mathbb{Z}} |g_k(t)|^{r'} \right)^{\frac{1}{r'}} \right\|_{L_{r'}^{r'}} \lesssim \|F\|_{L_{q'}^q} \|G\|_{L_{r'}^{r'}}.
\]
(2.12)

Consequently, we bound the first term in the right-hand side of (2.11) as desired. To bound the second term, we first write
\[
\sum_{U_4} H_{j, \tilde{k}, k}(a, b) = \sum_{j \in \mathbb{Z}} \sum_{k < \tilde{k} \in \mathbb{Z}} \sum_{k > \tilde{k} \in \mathbb{Z}} 2^{k(\frac{1}{q} - \frac{1}{r}) + (k + (n-1))\left(\frac{1}{q} - \frac{1}{r}\right)} \|f_k\|_{L_{q'}^q} \|g_k\|_{L_{r'}^r}
\]
\[
= \sum_{j \in \mathbb{Z}} 2^j (n-1)\left(\frac{1}{q} - \frac{1}{r}\right) \sum_{k < \tilde{k} \in \mathbb{Z}} 2^{k(\frac{1}{q} - \frac{1}{r})} \|f_k\|_{L_{q'}^q} \sum_{k > \tilde{k} \in \mathbb{Z}} 2^{k(\frac{1}{q} - \frac{1}{r})} \|g_k\|_{L_{r'}^r}
\]

and note that
\[
\sum_{k < \frac{\tilde{k}}{\tilde{r}}} 2^{k(\frac{1}{q} - \frac{1}{r})} \|f_k\|_{L_{q'}^q} \sum_{k > \frac{\tilde{k}}{\tilde{r}}} 2^{k(\frac{1}{q} - \frac{1}{r})} \|g_k\|_{L_{r'}^r}
\]
\[
\leq \left( \sum_{k < \frac{\tilde{k}}{\tilde{r}}} 2^{k(\frac{1}{q} - \frac{1}{r})} \right)^{\frac{1}{q'}} \left( \sum_{k > \frac{\tilde{k}}{\tilde{r}}} \|f_k\|_{L_{q'}^q}^{q'} \right)^{\frac{1}{q}} \left( \sum_{k > \frac{\tilde{k}}{\tilde{r}}} \|g_k\|_{L_{r'}^r}^{r'} \right)^{\frac{1}{r'}} \lesssim 2^j \sum_{k < \frac{\tilde{k}}{\tilde{r}}} 2^{k(\frac{1}{q} - \frac{1}{r})} \|f_k\|_{L_{q'}^q} \|G\|_{L_{r'}^{r'}}.
\]
Here we used (2.12) for the last inequality. Hence we get
\[
\sum_{U_4} H_{j, \tilde{k}, k}(a, b) \lesssim \sum_{j \in \mathbb{Z}} 2^j \left(\frac{1}{q} - \frac{1}{r}\right) 2^{\frac{n-1}{q}} \left(\frac{1}{q} - \frac{1}{r}\right) \|F\|_{L_{q'}^q} \|G\|_{L_{r'}^{r'}}.
\]

Now we choose \((\frac{1}{a}, \frac{1}{b}) \in B((\frac{1}{q}, \frac{1}{r}), 10\epsilon) \setminus B((\frac{1}{q}, \frac{1}{r}), \epsilon)\) with
\[
\frac{1}{a} > \frac{n}{n-2} \frac{1}{b}, \quad -10\epsilon < \frac{1}{r} - \frac{1}{a} < -9\epsilon, \quad 2\epsilon > \frac{1}{r} - \frac{1}{b} > \epsilon
\]
when \(j \geq 0\), and with
\[
\frac{1}{a} > \frac{n}{n-2} \frac{1}{b}, \quad -2\epsilon < \frac{1}{r} - \frac{1}{a} < -\epsilon, \quad 10\epsilon > \frac{1}{r} - \frac{1}{b} > 9\epsilon
\]
when \(j < 0\). Then we get the desired bound
\[
\sum_{U_4} H_{j, \tilde{k}, k}(a, b) \lesssim \|F\|_{L_{q'}^q} \|G\|_{L_{r'}^{r'}}.
\]
Consequently, we get (2.7).

Finally we consider the cases where \(k = \frac{\tilde{k}}{\tilde{r}} j\) or \(\tilde{k} = \frac{j}{\tilde{r}} j\). When \(k = \frac{\tilde{k}}{\tilde{r}} j\), we note
\[
\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \sum_{\tilde{k} \in \mathbb{Z}} H_{j, \tilde{k}, k}(a, b) \|f_k\|_{L_{q'}^q} \|g_k\|_{L_{r'}^r} = \sum_{k \in \mathbb{Z}} \sum_{\tilde{k} \in \mathbb{Z}} \sum_{\tilde{k} \in \mathbb{Z}} 2^{(\tilde{k}-k)(\frac{1}{q} - \frac{1}{r})} \|f_k\|_{L_{q'}^q} \|g_k\|_{L_{r'}^r}.
\]

\footnote{Choose \((\frac{1}{a}, \frac{1}{b})\) in the region (d) when \(j \geq 0\) and in the region (e) when \(j < 0\) (see Figure 2).}
where $\frac{1}{a} \leq \frac{n-1}{n-2} b$. Hence if we choose $(a, b)$ in the region $(a)$ or $(c)$ with $\frac{1}{a} \leq \frac{n-1}{n-2} b$, we see
\[
\sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} 2^{(\tilde{k} - k) \left(\frac{1}{r} - \frac{1}{b}\right)} \| f_k \|_{L^q_t} \| g_{k'} \|_{L^{q'}_t} \leq \sum_{k \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}} 2^{-\epsilon(\tilde{k} - k)} \| f_k \|_{L^q_t} \| g_{k'} \|_{L^{q'}_t}.
\]
From this, we get the desired bound as before. The other cases follow easily in a similar way.

3. Necessary Conditions

It was shown in [3] that
\[
\frac{n-2}{r} - \frac{2}{q} < \frac{n}{r} \quad \text{and} \quad \frac{n-2}{r} - \frac{2}{q} \leq \frac{n}{r},
\]
are the necessary conditions for which the inhomogeneous estimate
\[
\left\| \int_0^t e^{i(t-s)\Delta} F(\cdot, s)ds \right\|_{L^q_t L^{q'}_x} \lesssim \| F \|_{L^q_t L^{q'}_x}
\]
holds. (We refer the reader to [3, 13, 8] for other necessary conditions.) Compared with (3.1), we give here the following new necessary condition:
\[
\frac{n-2}{r} - \frac{2}{q} \leq \frac{n}{r}, \quad \frac{n-2}{r} - \frac{2}{q} \leq \frac{n}{r}.
\]
The first condition in (3.3) is stronger than the first one in (3.1) when $1/r \leq 1/\tilde{r}$, and the second condition in (3.3) is stronger than the second one in (3.1) when $1/r \geq 1/\tilde{r}$.

Proof of (3.3). If (3.2) holds with a pair $(r, q)$ on the left and a pair $(\tilde{r}, \tilde{q})$ on the right, then it must be also valid when one switches the roles of $(r, q)$ and $(\tilde{r}, \tilde{q})$. By this duality relation, we only need to show the second condition $(n-2)/r - 2/q \leq n/r$ in (3.3). For this, we first write
\[
I(F) := \int_0^t e^{i(t-s)\Delta} F(\cdot, s)ds = (4\pi)^{-\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} |t-s|^\frac{1}{r} e^{i\frac{|x-y|^2}{4(t-s)}} F(y, s)dyds.
\]
Let $0 < \epsilon < 1/2$ and $F(y, s) = \chi_{\{0 < |s| < \epsilon, |y| < \epsilon\}}$. Since
\[
\frac{|x-y|^2}{4(t-s)} = \frac{|x|^2}{4t} + \frac{t(|x-y|^2 - |x|^2) + s|x|^2}{4t(t-s)},
\]
for $0 < s < \epsilon^2, |y| < \epsilon, 10 < t < 11$ and $| |x| - \frac{1}{2} | < \epsilon$, we see that
\[
\frac{t(|x-y|^2 - |x|^2) + s|x|^2}{4t|t-s|} < \frac{11 \cdot 3 + \epsilon^2 \cdot \frac{2}{9}}{4 \cdot 10 \cdot 9} < \frac{1}{2}.
\]
Hence, if $10 < t < 11$ and $| |x| - \frac{1}{2} | < \epsilon$,
\[
|I(F)(x, t)| \geq \left| (4\pi)^{-\frac{n}{2}} \int_0^t \int_{\mathbb{R}^n} |t-s|^\frac{1}{r} e^{i\frac{|x-y|^2}{4(t-s)}} F(y, s)dyds \right|
\geq \int_0^t \int_{|y|<\epsilon} dyds
\geq \epsilon^{n+2}.
\]
This implies that
\[ \|I(F)\|_{L^q_t L^r_x} \geq \|I(F)\|_{L^q_t(10 < t < 11)L^r_x(|x| - \frac{4}{7} < \epsilon)} \]
\[ \geq \epsilon^{n+2} \left((1/\epsilon + \epsilon)^n - (1/\epsilon - \epsilon)^n\right)^{1/r} \]
\[ \geq \epsilon^{n+2} \left((\epsilon^{-n} + (1 + \epsilon^2)^n - (1 - \epsilon^2)^n\right)^{1/r} \]
\[ \geq \epsilon^{n+2} \epsilon^{-n+2}. \]

On the other hand, \( \|F\|_{L^q_t L^r_x} \sim \epsilon^{\frac{2}{\tilde{q}}} \epsilon^{\frac{n}{r}}. \) Now the estimate (3.2) leads us to
\[ \epsilon^{n+2} \epsilon^{-n+2} \lesssim \epsilon^{\frac{2}{\tilde{q}}} \epsilon^{\frac{n}{r}}. \] By letting \( \epsilon \to 0, \) we conclude that \( 2/\tilde{q} + n/r \geq (n-2)/r. \) 

\[\square\]

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