CROSSING NUMBERS AND ROTATION NUMBERS OF CYCLES
IN A PLANE IMMERSED GRAPH

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Abstract. For any generic immersion of a Petersen graph into a plane, the number of crossing points between two edges of distance one is odd. The sum of the crossing numbers of all 5-cycles is odd. The sum of the rotation numbers of all 5-cycles is even. We show analogous results for 6-cycles, 8-cycles and 9-cycles. For any Legendrian spatial embedding of a Petersen graph, there exists a 5-cycle that is not an unknot with maximal Thurston-Bennequin number, and the sum of all Thurston-Bennequin numbers of the cycles is 7 times the sum of all Thurston-Bennequin numbers of the 5-cycles. We show analogous results for a Heawood graph. We also show some other results for some graphs. We characterize abstract graphs that has a generic immersion into a plane whose all cycles have rotation number 0.

1. Introduction

Let $G$ be a finite graph. We denote the set of all vertices of $G$ by $V(G)$ and the set of all edges by $E(G)$. We consider $G$ as a topological space in the usual way. Then a vertex of $G$ is a point of $G$ and an edge of $G$ is a subspace of $G$. A graph $G$ is said to be simple if it has no loops and no multiple edges. Suppose that $G$ has no multiple edges. Let $u$ and $v$ be mutually adjacent vertices of $G$. Then the edge of $G$ incident to both $u$ and $v$ is denoted by $uv$. Then $uv = vu$ as an unoriented edge. The orientation of $uv$ is given so that $u$ is the initial vertex and $v$ is the terminal vertex. Therefore $uv \neq vu$ as oriented edges. A cycle of $G$ is a subgraph of $G$ that is homeomorphic to a circle $S^1$. A cycle with $k$ edges is said to be a $k$-cycle. The set of all cycles of $G$ is denoted by $\Gamma(G)$ and the set of all $k$-cycles of $G$ is denoted by $\Gamma_k(G)$.

A spatial embedding of $G$ is an embedding of $G$ into the 3-space $\mathbb{R}^3$. The image of a spatial embedding is said to be a spatial graph. The set of all spatial embeddings of $G$ is denoted by $SE(G)$. A plane generic immersion of $G$ is an immersion of $G$ into the plane $\mathbb{R}^2$ whose multiple points are only finitely many transversal double points between edges. Such a double point is said to be a crossing point or a crossing. The image of a plane generic immersion together with the distinction of the crossing points and the image of the degree 4 vertices is said to be a plane immersed graph.

Let $f : G \to \mathbb{R}^2$ be a plane generic immersion of $G$. Let $H$ be a subgraph of $G$. We denote the number of crossings of the restriction map $f|_H : H \to \mathbb{R}^2$ by $c(f(H))$. The set of all plane generic immersion of $G$ is denoted by $PGI(G)$.

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Let $K_n$ be a complete graph on $n$ vertices and $K_{m,n}$ a complete bipartite graph on $m+n$ vertices. It is shown in [1] and [7] that for any spatial embedding $f : K_6 \to \mathbb{R}^3$, the sum of all linking numbers of the links in $f(K_6)$ is an odd number. Let $a_2(J)$ be the second coefficient of the Conway polynomial of a knot $J$. It is also shown in [1] that for any spatial embedding $f : K_7 \to \mathbb{R}^3$, the sum $\sum_{\gamma \in \Gamma(K_7)} a_2(f(\gamma))$ is an odd number. See also [4] for refinements of these results, and [9,13,17] etc. for higher dimensional analogues. An analogous phenomenon appears in plane immersed graphs. A self crossing is a crossing of the same edge. An adjacent crossing is a crossing between two mutually adjacent edges. A disjoint crossing is a crossing between two mutually disjoint edges. It is known that for $G = K_5$ or $G = K_{3,3}$, the number of all disjoint crossings of a plane generic immersion of $G$ is always odd. See for example [10, Proposition 2.1] or [13, Lemma 1.4.3]. Some theorems on plane immersed graphs are also stated in [13]. See also [8] for related results. As analogous phenomenon we show the following results.

Let $G$ be a finite graph with at least one cycle. The girth $g(G)$ of $G$ is the minimal lengths of the cycles of $G$. Namely every cycle of $G$ contains at least $g(G)$ edges and there is a $g(G)$-cycle of $G$. Let $G$ be a finite graph and $H,K$ connected subgraphs of $G$. The distance $d(H,K)$ of $H$ and $K$ in $G$ is defined to be the minimum number of edges of a path of $G$ joining $H$ and $K$. Then $d(H,K) = 0$ if and only if $H \cup K$ is connected. Let $d$ and $e$ be mutually distinct edges of $G$. We note that $d(d,e) = 0$ if and only if $d$ and $e$ are adjacent. Then $d(d,e) = 1$ if and only if $d$ and $e$ are disjoint and there exists an edge $x$ of $G$ adjacent to both of them. If $g(G) \geq 5$ then such $x$ is unique. Similarly $d(d,e) = 2$ if and only if $d$ and $e$ are disjoint, no edge of $G$ is adjacent to both of them and there exist mutually adjacent edges $x$ and $y$ of $G$ such that $x$ is adjacent to $d$ and $y$ is adjacent to $e$. Let $k$ be a natural number. Let $D_k(G)$ be the set of all unordered pairs $(d,e)$ of edges of $G$ with $d(d,e) = k$.

**Theorem 1.1.** Let $f : K_4 \to \mathbb{R}^2$ be a plane generic immersion. Then

$$\sum_{\gamma \in \Gamma(K_4)} c(f(\gamma)) \equiv 0 \pmod{2}.$$ 

**Theorem 1.2.** Let $f : K_{3,3} \to \mathbb{R}^2$ be a plane generic immersion. Then

$$\sum_{\gamma \in \Gamma(K_{3,3})} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma(K_{3,3})} c(f(\gamma)) \equiv 1 \pmod{2}.$$ 

We denote a Petersen graph by $PG$. A Petersen graph $PG$ and a plane generic immersion $g : PG \to \mathbb{R}^2$ of $PG$ is illustrated in Figure 1.1.

![Figure 1.1. A plane generic immersion of $PG$](image-url)
Theorem 1.3. Let $f : PG \to \mathbb{R}^2$ be a plane generic immersion. Then
\[ \sum_{(d,e) \in D_1(PG)} |f(d) \cap f(e)| \equiv 1 \pmod{2}. \]

Note that, for an edge $e$ of $PG$, the edges of $PG$ with distance 1 with $e$ forms an 8-cycle of $PG$, see Figure 1.2.

The modular equality in Theorem 1.3 has an integral lift to an invariant of spatial embeddings of $PG$ as stated in Theorem 1.4. This is a kind of spatial graph invariants called Reduced Wu and generalized Simon invariants. See [15] [16] [3].

We prepare some notions in order to state Theorem 1.4. Let $G$ be a finite graph and $f : G \to \mathbb{R}^2$ a plane generic immersion of $G$. Let $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ be a natural projection defined by $\pi(x, y, z) = (x, y)$. A spatial embedding $\varphi : G \to \mathbb{R}^3$ of $G$ is said to be a lift of $f$ if $f = \pi \circ \varphi$. The subset $f(G)$ of $\mathbb{R}^2$ together with the vertex information $f|_{V(G)}$ and over/under crossing information of $\varphi$ at each crossing of $f$ is said to be a diagram of $\varphi$ based on $f(G)$. Suppose $G$ is simple. Then a diagram of $\varphi$ restores $\varphi$ up to ambient isotopy of $\mathbb{R}^3$. Let $D$ be a diagram of $\varphi$. Suppose that each edge of $G$ is oriented. A crossing of $D$ is said to be a positive crossing or a negative crossing if it is as illustrated in Figure 1.3. Let $d$ and $e$ be mutually distinct edges of $G$. Let $c^+_D(d, e)$ be the number of positive crossings of $f(d) \cap f(e)$ and $c^-_D(d, e)$ the number of negative crossings of $f(d) \cap f(e)$. We set $\ell_D(d, e) = c^+_D(d, e) - c^-_D(d, e)$.

Let $u_i$ and $v_i$ be the vertices of a Petersen graph as illustrated in Figure 1.4 for $i = 1, 2, 3, 4, 5$. We consider the suffixes modulo 5. Namely $u_{-4} = u_1 = u_6$, $u_{-3} = u_2 = u_7$, $v_{-4} = v_1 = v_6$ and so on. Then $E(PG)$ consists of 15 edges $u_i u_{i+1}$, $u_i v_i$ and $v_i v_{i+2}$ for $i = 1, 2, 3, 4, 5$. We consider that these edges are oriented. We define a map $\varepsilon : D_1(PG) \to \mathbb{Z}$ by $\varepsilon(u_{i} u_{i+1}, u_{i+2} u_{i+3}) = 1$, $\varepsilon(u_{i} u_{i+1}, u_{i-1} v_{i-1}) = 1$, $\varepsilon(u_{i} u_{i+1}, u_{i+2} v_{i+2}) = -1$, $\varepsilon(u_{i} u_{i+1}, v_{i} v_{j+2}) = 1$, $\varepsilon(u_{i} v_{i}, u_{i+1} v_{i+2}) = -1$, $\varepsilon(u_{i} v_{i}, u_{i+2} v_{i+2}) = 1$, $\varepsilon(u_{i} v_{i}, v_{i+1} v_{i+3}) = -1$, $\varepsilon(v_{i} v_{i+2}, v_{i+1} v_{i+3}) = 1$ and $\varepsilon(v_{i} v_{i+1}, v_{i+2} v_{i+3}) = -1$ for $i, j \in \{1, 2, 3, 4, 5\}$. Let
Let $\varphi : PG \to \mathbb{R}^3$ be a spatial embedding that is a lift of a plane generic immersion $f : PG \to \mathbb{R}^2$. Let $D$ be a diagram of $\varphi$ based on $f(PG)$. We set

$$L(\varphi) = \sum_{(d,e) \in D_1(PG)} \varepsilon(d,e)\ell_d(d,e).$$

**Theorem 1.4.** Let $\varphi : PG \to \mathbb{R}^3$ be a spatial embedding of a Petersen graph $PG$ that is a lift of a plane generic immersion $f : PG \to \mathbb{R}^2$ of $PG$. Let $D$ be a diagram of $\varphi$ based on $f(PG)$. Then $L(\varphi)$ is a well-defined ambient isotopy invariant of $\varphi$ and we have

$$L(\varphi) \equiv \sum_{(d,e) \in D_1(PG)} |f(d) \cap f(e)| \equiv 1 \mod 2.$$ 

**Remark 1.5.** Let $G$ be a finite graph and $k$ a natural number. Let $S_k(G)$ be a subcomplex of a 2-dimensional complex $G \times G$ defined by

$$S_k(G) = \bigcup_{(d,e) \in D_k(G)} d \times e \cup e \times d.$$ 

It is known that $S_1(K_5)$ is homeomorphic to a closed orientable surface of genus 6 and $S_1(K_{3,3})$ is homeomorphic to a closed orientable surface of genus 4. Let $\varphi : G \to \mathbb{R}^3$ be a spatial embedding of $G$. Let $\tau_\varphi : S_k(G) \to S^2$ be a Gauss map defined by

$$\tau_\varphi(x,y) = \frac{\varphi(x) - \varphi(y)}{\|\varphi(x) - \varphi(y)\|}.$$ 

It is known that the mapping degree of $\tau_\varphi : S_1(G) \to S^2$ for $G = K_5$ or $K_{3,3}$ is equal to the Simon invariant of $\varphi$ up to sign. By a straightforward consideration we see that $S_1(PG)$ is homeomorphic to a closed orientable surface of genus 16, and for a spatial embedding $\varphi : PG \to \mathbb{R}^3$ we see that $L(\varphi)$ is equal to the mapping degree of $\tau_\varphi : PG \to S^2$ up to sign.

**Theorem 1.6.** Let $f : PG \to \mathbb{R}^2$ be a plane generic immersion. Then

$$\sum_{\gamma \in \Gamma_5(PG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(PG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_3(PG)} c(f(\gamma)) \equiv 1 \mod 2$$

and

$$\sum_{\gamma \in \Gamma_6(PG)} c(f(\gamma)) \equiv 0 \mod 4.$$

We denote a Heawood graph by $HG$. A Heawood graph $HG$ and a plane generic immersion $g : HG \to \mathbb{R}^2$ of $HG$ is illustrated in Figure 1.4.

**Figure 1.4.** A plane generic immersion of $HG$
Theorem 1.7. Let \( f : HG \to \mathbb{R}^2 \) be a plane generic immersion. Then
\[
\sum_{(d,e) \in D_2(HG)} |f(d) \cap f(e)| \equiv 1 \pmod{2}.
\]

Note that, for an edge \( e \) of \( HG \), the edges of \( HG \) with distance 2 with \( e \) forms an 8-cycle of \( HG \), see Figure 1.5

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1_5.png}
\caption{Distance 2 edges form an 8-cycle}
\end{figure}

We note that the modular equality in Theorem 1.7 also has an integral lift to an invariant of spatial embeddings of \( HG \). That is an invariant defined in [3, Theorem 3.16]. It is always an odd number. The invariant is an invariant of spatial embeddings of \( HG \).

Let \( u_1, u_2, u_3, u_4, u_5, u_6, u_7 \) and \( v_1, v_2, v_3, v_4, v_5, v_6, v_7 \) be the vertices of a Heawood graph as illustrated in Figure 1.4. We consider the suffixes modulo 7. Then \( E(HG) \) consists of 21 edges \( u_i v_i \), \( u_i v_{i-1} \) and \( u_i u_{i-2} \) for \( i = 1, 2, 3, 4, 5, 6, 7 \). We consider that these edges are oriented. We define a map \( \varepsilon : D_1(PG) \cup D_2(PG) \to \mathbb{Z} \) by \( \varepsilon(u_i v_i, u_{i+1} v_{i+1}) = 2 \), \( \varepsilon(u_i v_{i-1}, u_{i+2} v_{i+2}) = -2 \),
\[
\varepsilon(u_i v_1, u_{i+3} v_{i+3}) = -3, \quad \varepsilon(u_i v_1, u_{i+2} v_{i+2}) = \varepsilon(u_i v_1, u_{i-1} v_{i-2}) = 1,
\]
\[
\varepsilon(u_i v_1, u_{i+4} v_{i+4}) = \varepsilon(u_i v_1, u_{i+3} v_{i+3}) = -1, \quad \varepsilon(u_i v_1, u_{i+5} v_{i+5}) = 0,
\]
\[
\varepsilon(u_i v_1, u_{i-1} v_{i-1}) = \varepsilon(u_i v_{i-1}, u_{i-1} v_{i-2}) = 2, \quad \varepsilon(u_i v_{i-1}, u_{i+2} v_{i+2}) = -2,
\]
\[
\varepsilon(u_i v_{i-1}, u_{i-1} v_{i-1}) = \varepsilon(u_i v_{i-1}, u_{i-2} v_{i-2}) = 1, \quad \varepsilon(u_i v_{i-1}, u_{i-3} v_{i-3}) = -2,
\]
\[
\varepsilon(u_i v_{i-1}, u_{i+1} v_{i+1}) = \varepsilon(u_i v_{i-1}, u_{i+3} v_{i+3}) = 3, \quad \varepsilon(u_i v_{i-1}, u_{i+4} v_{i+4}) = 0,
\]
\[
\varepsilon(u_i v_{i-1}, u_{i+2} v_{i+2}) = 3, \quad \varepsilon(v_i u_{i-2}, v_{i+1} u_{i-1}) = \varepsilon(v_i u_{i-2}, v_{i-3} u_{i-3}) = 5,
\]
\[
\varepsilon(v_i u_{i-2}, v_{i+2} u_{i+2}) = \varepsilon(v_i u_{i-2}, v_{i-2} u_{i-2}) = 2, \quad \varepsilon(v_i u_{i-2}, v_{i+3} u_{i+3}) = 2 \text{ for } i = 1, 2, 3, 4, 5, 6, 7.
\]

Let \( \varphi : HG \to \mathbb{R}^3 \) be a spatial embedding of a Heawood graph \( HG \) that is a lift of a plane generic immersion \( f : HG \to \mathbb{R}^2 \) of \( HG \). Let \( D \) be a diagram of \( \varphi \) based on \( f(HG) \). We set
\[
\mathcal{L}(\varphi) = \sum_{(d,e) \in D_1(HG) \cup D_2(HG)} \varepsilon(d,e) \ell_D(d,e).
\]

Theorem 1.8. Let \( \varphi : HG \to \mathbb{R}^3 \) be a spatial embedding of a Heawood graph \( HG \) that is a lift of a plane generic immersion \( f : HG \to \mathbb{R}^2 \) of \( HG \). Let \( D \) be a diagram of \( \varphi \) based on \( f(HG) \). Then we have
\[
\mathcal{L}(\varphi) \equiv \sum_{(d,e) \in D_2(HG)} |f(d) \cap f(e)| \pmod{2}.
\]

Remark 1.9. The integral lift above involves disjoint edges of \( HG \) with distance 1. It is straightforward to check that there exists no integral lift involving only
disjoint edges of $HG$ with distance 2. As a related fact we see by a straightforward consideration that $S_2(HG)$ is homeomorphic to a closed non-orientable surface of non-orientable genus 30, and for any spatial embedding $\varphi : HG \to \mathbb{R}^3$, the $\mathbb{Z}/2\mathbb{Z}$-mapping degree of $\tau_\varphi : S_2(HG) \to S^2$ is equal to 1.

**Theorem 1.10.** Let $f : HG \to \mathbb{R}^2$ be a plane generic immersion. Then
\[
\sum_{\gamma \in \Gamma_k(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_k(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_\infty(HG)} c(f(\gamma)) \equiv 1 \pmod{2},
\]
and
\[
\sum_{\gamma \in \Gamma_\infty(HG)} c(f(\gamma)) \equiv 0 \pmod{4},
\]

Let $f : G \to \mathbb{R}^2$ be a plane generic immersion of a finite graph $G$. Let $\gamma$ be a cycle of $G$. Suppose that $\gamma$ is given an orientation. Then $f(\gamma)$ is an oriented plane closed curve. We denote the rotation number of $f(\gamma)$ by $\operatorname{rot}(f(\gamma))$. We note that the parity of $\operatorname{rot}(f(\gamma))$ is independent of the choice of orientation of $\gamma$. It is easy to see that
\[
\operatorname{rot}(f(\gamma)) - c(f(\gamma)) \equiv 1 \pmod{2}.
\]
We denote the number of elements of a finite set $X$ by $|X|$. The number of $k$-cycles of $G$ for $G = K_4, K_{3,3}$ and $PG$ listed below is known and also easy to enumerate. The number of $k$-cycles of $HG$ listed below is also known. See for example [12, 4.2]. Since $|\Gamma(K_4)| = 7 \equiv 1 \pmod{2}$, $|\Gamma_4(K_{3,3})| = 9 \equiv 1 \pmod{2}$, $|\Gamma_6(K_{3,3})| = 6 \equiv 0 \pmod{2}$, $|\Gamma_6(PG)| = 12 \equiv 0 \pmod{2}$, $|\Gamma_8(PG)| = 10 \equiv 0 \pmod{2}$, $|\Gamma_8(HG)| = 20 \equiv 0 \pmod{2}$, $|\Gamma_8(HG)| = 28 \equiv 0 \pmod{2}$, $|\Gamma_8(HG)| = 21 \equiv 1 \pmod{2}$, $|\Gamma_10(HG)| = 84 \equiv 0 \pmod{2}$, $|\Gamma_12(HG)| = 56 \equiv 0 \pmod{2}$ and $|\Gamma_14(HG)| = 24 \equiv 0 \pmod{2}$, we have the following immediate corollaries.

**Corollary 1.11.** Let $f : K_4 \to \mathbb{R}^2$ be a plane generic immersion. Then
\[
\sum_{\gamma \in \Gamma(K_4)} \operatorname{rot}(f(\gamma)) \equiv 1 \pmod{2}.
\]

**Corollary 1.12.** Let $f : K_{3,3} \to \mathbb{R}^2$ be a plane generic immersion. Then
\[
\sum_{\gamma \in \Gamma_4(K_{3,3})} \operatorname{rot}(f(\gamma)) \equiv 0 \pmod{2}
\]
and
\[
\sum_{\gamma \in \Gamma_6(K_{3,3})} \operatorname{rot}(f(\gamma)) \equiv 1 \pmod{2}.
\]

**Corollary 1.13.** Let $f : PG \to \mathbb{R}^2$ be a plane generic immersion. Then
\[
\sum_{\gamma \in \Gamma_5(PG)} \operatorname{rot}(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(PG)} \operatorname{rot}(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_8(PG)} \operatorname{rot}(f(\gamma))
\]
\[
\equiv \sum_{\gamma \in \Gamma_6(PG)} \operatorname{rot}(f(\gamma)) \equiv 1 \pmod{2}.
\]

**Corollary 1.14.** Let $f : HG \to \mathbb{R}^2$ be a plane generic immersion. Then
\[
\sum_{\gamma \in \Gamma_6(HG)} \operatorname{rot}(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_8(HG)} \operatorname{rot}(f(\gamma)) \equiv 1 \pmod{2}.
and
\[ \sum_{\gamma \in \Gamma_4(HG)} \text{rot}(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_{14}(HG)} \text{rot}(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_{14}(HG)} \text{rot}(f(\gamma)) \equiv 0 \pmod{2}. \]

Furthermore we have the following theorem.

**Theorem 1.15.** Let \( G \) be a finite graph. Then the following conditions are equivalent.

1. There is a plane generic immersion \( f : G \to \mathbb{R}^2 \) such that \( \text{rot}(f(\gamma)) = 0 \) for every cycle \( \gamma \) of \( G \).
2. The graph \( G \) does not have \( K_4 \) as a minor.

Let \((\mathbb{R}^3, \xi_{\text{std}})\) be the 3-space with the standard contact structure. A Legendrian knot is a smooth knot in \( \mathbb{R}^3 \) that is tangent to the contact plane at each point. We consider Legendrian knots up to Legendrian isotopy. The Thurston-Bennequin number \( \text{tb}(K) \) and the rotation number \( \text{rot}(K) \) of a Legendrian knot \( K \) are Legendrian isotopy invariants. A Legendrian knot \( K \) is said to be a trivial unknot if it is a trivial knot as a classical knot and \( \text{tb}(K) = -1 \). It is shown in [5] that a finite graph \( G \) has a Legendrian embedding with all cycles trivial unknots if and only if \( G \) does not have \( K_4 \) as a minor. See also [14] for related results. It is known that a trivial unknot has rotation number 0. The rotation number of a Legendrian knot \( K \) coincides with the rotation number of the plane immersed circle that is the image of \( K \) under Lagrangian projection. Therefore the only if part is an immediate consequence of Corollary 1.11. Namely we have the following corollary.

**Corollary 1.16 ([5]).** Let \( f : K_4 \to \mathbb{R}^3 \) be a Legendrian embedding. Then there is a cycle \( \gamma \) of \( K_4 \) such that \( f(\gamma) \) is not a trivial unknot.

Let \( G \) be a finite graph and \( f : G \to \mathbb{R}^3 \) a Legendrian embedding. Then \( f \) is said to be a minimal embedding if \( f(\gamma) \) is a trivial unknot for every \( g(G) \)-cycle \( \gamma \) of \( G \). Then we immediately have the following corollaries.

**Corollary 1.17.** The Petersen graph \( PG \) has no minimal Legendrian embedding.

**Corollary 1.18.** The Heawood graph \( HG \) has no minimal Legendrian embedding.

Let \( G \) be a finite graph and \( f : G \to \mathbb{R}^3 \) a Legendrian embedding. The total Thurston-Bennequin number of \( f \) is defined in [6] to be
\[ TB(f) = \sum_{\gamma \in \Gamma(G)} \text{tb}(f(\gamma)). \]

The following is also defined for a natural number \( k \) in [6].
\[ TB_k(f) = \sum_{\gamma \in \Gamma_k(G)} \text{tb}(f(\gamma)). \]

It is shown in [6] that \( TB(f) \) is determined by \( TB_3(f) \) when \( G \) is a complete graph and by \( TB_3(f) \) when \( G \) is a complete bipartite graph. In this paper we extend these results to a Petersen graph and a Heawood graph.

**Theorem 1.19.** Let \( f : PG \to \mathbb{R}^3 \) be a Legendrian embedding. Then
\[ TB_6(f) = TB_5(f), \]
\[ TB_8(f) = 2TB_5(f), \]
and
\[ TB_9(f) = 3TB_5(f). \]

Therefore
\[ TB(f) = 7TB_5(f). \]
Theorem 1.20. Let \( f : HG \to \mathbb{R}^3 \) be a Legendrian embedding. Then

\[
\begin{align*}
TB_6(f) &= TB_6(f), \\
TB_{10}(f) &= 5TB_6(f), \\
TB_{12}(f) &= 4TB_6(f),
\end{align*}
\]

and

\[
TB_{14}(f) = 2TB_6(f).
\]

Therefore

\[
TB(f) = 13TB_6(f).
\]

2. Crossing numbers of cycles in a plane immersed graph

Proposition 2.1. Let \( G \) be a finite graph and \( \Lambda \) a set of subgraphs of \( G \). Let \( m \) be a positive integer. Suppose that the following (1) and (2) hold.

(1) For every edge \( e \) of \( G \), the number of elements of \( \Lambda \) containing \( e \) is a multiple of \( m \).

(2) For every pair of edges \( d \) and \( e \) of \( G \), the number of elements of \( \Lambda \) containing both \( d \) and \( e \) is a multiple of \( m \).

Then for any plane generic immersion \( f \) of \( G \)

\[
\sum_{\lambda \in \Lambda} c(f(\lambda)) \equiv 0 \pmod{m}.
\]

Proof. Let \( x \) be a self crossing of \( f(G) \). Then by the condition (1) \( x \) is counted a multiple of \( m \) times in the sum. Let \( x \) be an adjacent crossing or a disjoint crossing of \( f(G) \). Then then by the condition (2) \( x \) is counted a multiple of \( m \) times in the sum. Therefore the sum is a multiple of \( m \). \( \square \)

Proof of Theorem 1.1. We note that \( \Gamma(K_4) = \Gamma_3(K_4) \cup \Gamma_4(K_4) \), \( |\Gamma_3(K_4)| = 4 \) and \( |\Gamma_4(K_4)| = 3 \). Every edge of \( K_4 \) is contained in 2 3-cycles and 2 4-cycles. The total 4 is a multiple of 2. Every pair of mutually adjacent edges of \( K_4 \) is contained in a 3-cycle and a 4-cycle. The total 2 is a multiple of 2. Every pair of disjoint edges of \( K_4 \) is contained in no 3-cycles and 2 4-cycles. The total 2 is a multiple of 2. Then by Proposition 2.1 we have the result. \( \square \)

We note that the phenomenon described in Theorem 1.1 widely appears on graphs with certain symmetries. We show two of them below. The proofs are entirely analogous and we omit them.

Theorem 2.2. Let \( f : K_5 \to \mathbb{R}^2 \) be a plane generic immersion. Then

\[
\sum_{\gamma \in \Gamma_3(K_5)} c(f(\gamma)) \equiv 0 \pmod{2}.
\]

Let \( m \) be a natural number. Let \( T(m) \) be a graph of 3 vertices and \( 3m \) edges such that each pair of vertices is joined by exactly \( m \) multiple edges.

Theorem 2.3. Let \( m \) be a natural number. Let \( f : T(m) \to \mathbb{R}^2 \) be a plane generic immersion. Then

\[
\sum_{\gamma \in \Gamma_3(T(m))} c(f(\gamma)) \equiv 0 \pmod{m}.
\]
It is known that any two plane generic immersions are transformed into each other up to self-homeomorphisms of $G$ and $\mathbb{R}^2$ by a finite sequence of local moves illustrated in Figure 2.1. These moves are called Reidemeister moves.

**Proposition 2.4.** Let $G$ be a finite graph and $\Lambda$ a set of subgraphs of $G$. Let $m$ be a positive integer. The following conditions (A) and (B) are mutually equivalent.

(A) For any plane generic immersions $f$ and $g$ of $G$,
\[ \sum_{\lambda \in \Lambda} c(f(\lambda)) \equiv \sum_{\lambda \in \Lambda} c(g(\lambda)) \quad (\text{mod } m). \]

(B) All of the following conditions (1), (2), (3) and (4) hold.

(1) For every edge $e$ of $G$, the number of elements of $\Lambda$ containing $e$ is a multiple of $m$.

(2) For every pair of edges $d$ and $e$ of $G$, twice the number of elements of $\Lambda$ containing both $d$ and $e$ is a multiple of $m$.

(3) For every pair of a vertex $v$ and an edge $e$ of $G$,
\[ \sum_{i=1}^{k} |\{\lambda \in \Lambda \mid \lambda \supset e \cup e_i\}| \equiv 0 \quad (\text{mod } m), \]
where $e_1, \ldots, e_k$ are the edges of $G$ incident to $v$.

(4) For every pair of mutually adjacent edges $d$ and $e$ of $G$, the number of elements of $\Lambda$ containing both $d$ and $e$ is a multiple of $m$.

**Remark 2.5.** If $\Lambda \subset \Gamma(G)$, then
\[ \sum_{i=1}^{k} |\{\lambda \in \Lambda \mid \lambda \supset e \cup e_i\}| = 2 \sum_{1 \leq i < j \leq k} |\{\lambda \in \Lambda \mid \lambda \supset e \cup e_i \cup e_j\}| \equiv 0 \quad (\text{mod } 2). \]
Therefore the condition (3) of (B) for $m = 2$ automatically holds.

**Proof of Proposition 2.4.** We set $\tau_{\Lambda}(h) = \sum_{\lambda \in \Lambda} c(h(\lambda))$ for a plane generic immersion $h$ of $G$. We note that a Reidemeister move $R4$ in Figure 2.1 is realized by a Reidemeister move $R4'$ in Figure 2.2, and Reidemeister moves $R2$ in Figure 2.1. Therefore $f$ and $g$ are transformed into each other by $R1$, $R2$, $R3$, $R4'$ and $R5$. A Reidemeister move $R1$ create or annihilate a self crossing. Then we see that $\tau_{\Lambda}$ (mod $m$) is invariant under $R1$ if and only if the condition (1) holds. A Reidemeister move $R2$ create or annihilate two crossings. Suppose that they are self crossings, then $\tau_{\Lambda}$ (mod $m$) is invariant if the condition (1) holds. Suppose that they are
both adjacent crossings or both disjoint crossings, then $\tau_\Lambda \pmod{m}$ is invariant if the condition (2) holds. If the condition (2) does not hold, then we can find $f$ and $g$ that differs by a Reidemeister move R2 such that $\sum_{\lambda \in \Lambda} c(f(\lambda)) - \sum_{\lambda \in \Lambda} c(g(\lambda))$ is not a multiple of $m$. A Reidemeister move R3 does not change the number of self crossings, adjacent crossings and disjoint crossings for every edge, pair of mutually adjacent edges and pair of disjoint edges respectively. Therefore $\tau_\Lambda$ is always invariant under R3. A Reidemeister move R4 create or annihilate crossings between an edge $e$ of $G$ and the edges of $G$ incident to $v$. Then we see that $\tau_\Lambda \pmod{m}$ is invariant under R4 if and only if the condition (3) holds. A Reidemeister move R5 create or annihilate an adjacent crossing. Then we see that $\tau_\Lambda \pmod{m}$ is invariant under R5 if and only if the condition (4) holds. This completes the proof. □

![Figure 2.2. R4'](image)

Proof of Theorem 1.2. Let $g : K_{3,3} \to \mathbb{R}^2$ be a plane generic immersion illustrated in Figure 2.3. We see that exactly one 4-cycle of $K_{3,3}$ has a crossing under $g$ and exactly three 6-cycles of $K_{3,3}$ has a crossing under $g$. Therefore

$$\sum_{\gamma \in \Gamma_4(K_{3,3})} c(g(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(K_{3,3})} c(g(\gamma)) \equiv 1 \pmod{2}.$$  

Next we will check that the conditions (1), (2), (3) and (4) of Proposition 2.4 for $G = K_{3,3}$, $\Lambda = \Gamma_4(K_{3,3})$ or $\Lambda = \Gamma_6(K_{3,3})$, and $m = 2$ hold. Each edge of $K_{3,3}$ is contained in exactly 4 4-cycles and 4 6-cycles of $K_{3,3}$. Therefore (1) holds. Since $m = 2$, (2) automatically holds. Since $\Lambda \subset \Gamma(K_{3,3})$ and $m = 2$, (3) holds by Remark 2.5. Let $d$ and $e$ be mutually adjacent edges of $K_{3,3}$. Then we see that there exist exactly 2 4-cycles and exactly 2 6-cycles containing both of them. Therefore (4) holds. Then by Proposition 2.4 we have

$$\sum_{\gamma \in \Gamma_4(K_{3,3})} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_4(K_{3,3})} c(g(\gamma)) \pmod{2}$$  

and

$$\sum_{\gamma \in \Gamma_6(K_{3,3})} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(K_{3,3})} c(g(\gamma)) \pmod{2}.$$  

Therefore we have

$$\sum_{\gamma \in \Gamma_4(K_{3,3})} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(K_{3,3})} c(f(\gamma)) \equiv 1 \pmod{2}.$$  

□

Proof of Theorem 1.3. We set

$$\kappa(f) = \sum_{(d,e) \in D_1(PG)} |f(d) \cap f(e)|$$
for a plane generic immersion $f : PG \to \mathbb{R}^2$ of $PG$. Let $g : PG \to \mathbb{R}^2$ be a plane generic immersion illustrated in Figure 1.1. We note that each crossing of $g(PG)$ is a disjoint crossing between two edges of $PG$ with distance 1. Therefore we have

$$\kappa(g) = \sum_{(d,e) \in D_1(PG)} |g(d) \cap g(e)| = 5 \equiv 1 \pmod{2}.$$ 

Next we will show that $\kappa(f)$ (mod 2) is invariant under Reidemeister moves. Since $\kappa$ do not count self crossings, it is invariant under R1. The change of $\kappa$ under R2 is $\pm 2$ or $0$ and therefore $\kappa(f)$ (mod 2) is invariant under R2. The third Reidemeister move R3 do not change $\kappa$ itself. Let $e$ be an edge of $PG$ that is involved in a fourth Reidemeister move R4. As we saw before, the edges of $PG$ with distance 1 with $e$ forms an 8-cycle of $PG$. Therefore we see that the change of $\kappa$ under R4 is $\pm 2$ or $0$ and therefore $\kappa(f)$ (mod 2) is invariant under R4. Since $\kappa$ do not count adjacent crossings, it is invariant under R5. Since $f$ and $g$ are transformed into each other by Reidemeister moves, we have

$$\kappa(f) \equiv \kappa(g) \equiv 1 \pmod{2}.$$ 

□

Proof of Theorem 1.4. The proof that $\mathcal{L}(\varphi)$ is a well-defined ambient isotopy invariant taking its value in a odd number is entirely analogous to that of Simon invariants [15] and reduced Wu and generalized Simon invariants [3]. The modular equality immediately follows from the definitions. □

We prepare the following lemma for the proof of Theorem 1.6.

Lemma 2.6.

(1) For any vertices $x$ and $y$ of $PG$, there exists an isomorphism of $PG$ that maps $x$ to $y$.

(2) For any edges $x$ and $y$ of $PG$, there exists an isomorphism of $PG$ that maps $x$ to $y$.

(3) For any pairs of mutually adjacent edges $x,y$ and $z,w$ of $PG$, there exists an isomorphism of $PG$ that maps $x \cup y$ to $z \cup w$.

(4) Let $x,y,z$ and $w$ be edges of $PG$ with $d(x,y) = d(z,w) = 1$. Then there exists an isomorphism of $PG$ that maps $x \cup y$ to $z \cup w$.

(5) Let $x,y,z$ and $w$ be edges of $PG$ with $d(x,y) = d(z,w) = 2$. Then there exists an isomorphism of $PG$ that maps $x \cup y$ to $z \cup w$. 

□
Lemma 2.7.

Let $p : PG \rightarrow PG$ be an isomorphism defined by $p(u_i) = u_{i+1}$ and $p(v_i) = v_{i+1}$ for $i = 1, 2, 3, 4, 5$. Let $q : PG \rightarrow PG$ be an isomorphism defined by $q(u_i) = u_{i-1}$ and $q(v_i) = v_{i-1}$ for $i = 1, 2, 3, 4, 5$. Let $r : PG \rightarrow PG$ be an isomorphism defined by $r(u_i) = v_{2i}$ and $r(v_i) = u_{2i}$ for $i = 1, 2, 3, 4, 5$. Let $s : PG \rightarrow PG$ be an isomorphism defined by $s(u_i) = u_1$, $s(u_2) = u_2$, $s(u_3) = u_5$, $s(v_1) = v_1$, $s(u_3) = v_2$, $s(u_4) = v_5$, $s(v_2) = u_3$, $s(v_5) = u_4$, $s(v_1) = v_4$ and $s(v_4) = v_3$. Then we see that all isomorphisms requested in (1) are generated by $p$ and $r$. We note that $q$ exchanges an edge $u_5u_1$ for an edge $u_5u_4$ and $s$ exchanges an edge $u_2v_2$ for an edge $u_2u_3$. Then by combining $p$ and $r$ we have all isomorphisms requested in (2) and (3). Suppose $d(x, y) = d(z, w) = 1$. Then there is an edge $e$ of $PG$ adjacent to both $x$ and $y$. By (2) we map $e$ to an edge $u_1u_2$. We note that $q$ maps an edge $u_2u_3$ to $u_3u_2$. Then by combining other isomorphisms we have an isomorphism that maps $u_1u_2$ to $u_2u_1$. Then combining $s$ and these isomorphisms if necessary, $x$ and $y$ is mapped to $u_2u_3 \cup u_3u_1$. Similarly $z \cup w$ is mapped to $u_2u_3 \cup u_5u_1$. This implies that $x \cup y$ is mapped to $z \cup w$ and (4) holds. Suppose $d(x, y) = d(z, w) = 2$. Then there are mutually adjacent edges $d$ and $e$ of $PG$ such that $d$ is adjacent to $x$ and $e$ is adjacent to $y$. By (3) we map $d \cup e$ to $u_1u_2 \cup u_3u_1$. Then we see that $x \cup y$ is mapped to $u_2u_2 \cup u_3u_4$ or $u_2u_3 \cup u_5u_5$. Similarly $z \cup w$ is mapped to $u_2u_2 \cup u_3u_4$ or $u_2u_3 \cup u_5u_5$. Since $s$ exchanges $u_2u_2 \cup u_3u_4$ for $u_2u_3 \cup u_3v_3$, we see that (5) holds. \(\square\)

Let $G$ be a finite graph and $k$ a natural number. Let $e$ be an edge of $G$. We set $\alpha_k(e, G) = \{ [\gamma \in \Gamma_k(G) \mid \gamma \supset e] \}$. Let $d$ be another edge of $G$. We set $\alpha_k(d \cup e, G) = \{ [\gamma \in \Gamma_k(G) \mid \gamma \supset d \cup e] \}$. Suppose that $d$ and $e$ are mutually disjoint and oriented. Let $\gamma$ be a cycle of $G$ containing $d \cup e$. The cycle $\gamma$ is said to be coherent with respect to $d$ and $e$ if the orientations of $d$ and $e$ are coherent in $\gamma$. Otherwise $\gamma$ is said to be incoherent with respect to $d$ and $e$. Let $C_k(d \cup e, G)$ be the set of all cycles of $\Gamma_k(G)$ containing $d \cup e$ coherent with respect to $d$ and $e$. Let $I_k(d \cup e, G)$ be the set of all cycles of $\Gamma_k(G)$ containing $d \cup e$ incoherent with respect to $d$ and $e$. Then $\alpha_k(d \cup e, G) = |C_k(d \cup e, G)| + |I_k(d \cup e, G)|$. We set $\beta_k(d \cup e, G) = |C_k(d \cup e, G)| - |I_k(d \cup e, G)|$.

Lemma 2.7.

(1) Let $e$ be an edge of $PG$. Then $\alpha_5(e, PG) = 4$, $\alpha_6(e, PG) = 4$, $\alpha_8(e, PG) = 8$ and $\alpha_9(e, PG) = 12$.

(2) Let $d$ and $e$ be mutually adjacent edges of $PG$. Then $\alpha_5(d \cup e, PG) = 2$, $\alpha_6(d \cup e, PG) = 2$, $\alpha_8(d \cup e, PG) = 4$ and $\alpha_9(d \cup e, PG) = 6$.

(3) Let $d$ and $e$ be mutually disjoint oriented edges of $PG$ with $d(d, e) = 1$. Then $\alpha_5(d \cup e, PG) = 1$, $\alpha_6(d \cup e, PG) = 1$, $\alpha_8(d \cup e, PG) = 4$ and $\alpha_9(d \cup e, PG) = 7$. Suppose that the 5-cycle of $PG$ containing $d \cup e$ is coherent with respect to $d$ and $e$. Then $\beta_5(d \cup e, PG) = 1$, $\beta_6(d \cup e, PG) = 1$, $\beta_8(d \cup e, PG) = 2$ and $\beta_9(d \cup e, PG) = 3$.

(4) Let $d$ and $e$ be mutually disjoint oriented edges of $PG$ with $d(d, e) = 2$. Then $\alpha_5(d \cup e, PG) = 0$, $\alpha_6(d \cup e, PG) = 2$, $\alpha_8(d \cup e, PG) = 4$, $\alpha_9(d \cup e, PG) = 8$ and $\beta_5(d \cup e, PG) = \beta_6(d \cup e, PG) = \beta_8(d \cup e, PG) = \beta_9(d \cup e, PG) = 0$.

Proof. There are 6 pairs of mutually disjoint 5-cycles of $PG$. They are $(u_1u_2u_3u_4u_1, v_1v_3v_5v_2v_4)$ and $(u_iu_{i+1}v_{i+1}v_{i+4}v_{i+4}u_i, v_iv_{i+2}u_{i+2}u_{i+3}v_{i+3}v_i)$ for $i = 1, 2, 3, 4, 5$. Then $F_5(PG)$ consists of these 12 5-cycles. The 10 6-cycles $u_1u_1v_{i+1}v_{i+1}v_{i+4}v_{i+4}u_i$ and $u_iu_{i+1}v_{i+1}v_{i+4}v_{i+2}u_i$ for $i = 1, 2, 3, 4, 5$ are the elements of $F_6(PG)$. 

Proof.
Since \( \alpha \) and \( \sum \alpha \) \( \sum \) \( 2.4 \) we have
\[ m \]
Remark 2.5. Let \( \alpha \) \( \alpha \) \( \alpha \) \( \alpha \) \( \sum \) \( \sum \) \( \sum \) 
Proof of Theorem 1.6. Let \( g : PG \to \mathbb{R}^2 \) be a plane generic immersion illustrated in Figure 1.1. Since each crossing of \( g(PG) \) is a crossing between two edges of \( PG \) with distance 1 and \( \alpha_5(d \cup e, PG) = 1 \) for edges \( d \) and \( e \) of \( PG \) with \( d(d, e) = 1 \), we have
\[ \sum_{\gamma \in \Gamma_5(PG)} c(g(\gamma)) = 5 \equiv 1 \pmod{2}. \]
Since \( \alpha_6(d \cup e, PG) = 1 \) and \( \alpha_9(d \cup e, PG) = 7 \) for edges \( d \) and \( e \) of \( PG \) with \( d(d, e) = 1 \), we have
\[ \sum_{\gamma \in \Gamma_6(PG)} c(g(\gamma)) = 5 \equiv 1 \pmod{2} \]
and
\[ \sum_{\gamma \in \Gamma_9(PG)} c(g(\gamma)) = 35 \equiv 1 \pmod{2}. \]
Next we will check that the conditions (1), (2), (3) and (4) of Proposition 2.4 for \( G = PG, \Lambda = \Gamma_5(PG), \Gamma_6(PG) \) or \( \Gamma_9(PG) \) and \( m = 2 \) hold. Since \( \alpha_5(e, PG) = \alpha_6(e, PG) = \alpha_9(e, PG) = 4 \) and \( \alpha_9(e, PG) = 12 \) for every edge \( e \) of \( PG \), we see that (1) holds. For \( m = 2 \), (2) automatically holds. Since \( \Lambda \subset \Gamma(PG) \) (3) holds for \( m = 2 \) by Remark 2.5. Let \( d \) and \( e \) be mutually adjacent edges of \( PG \). Then \( \alpha_5(d \cup e, PG) = \alpha_6(d \cup e, PG) = 2 \) and \( \alpha_9(d \cup e, PG) = 6 \). Therefore (4) holds. Then by Proposition 2.4 we have
\[ \sum_{\gamma \in \Gamma_5(PG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_5(PG)} c(g(\gamma)) \pmod{2}, \]
\[ \sum_{\gamma \in \Gamma_6(PG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_6(PG)} c(g(\gamma)) \pmod{2}, \]
and
\[ \sum_{\gamma \in \Gamma_9(PG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_9(PG)} c(g(\gamma)) \pmod{2}. \]
Therefore we have
\[ \sum_{\gamma \in \Gamma_9(PG)} c(f(\gamma)) \equiv 1 \pmod{2}. \]

We note that \( \alpha_8(e, PG) = 8 \) for every edge \( e \) of \( PG \) and \( \alpha_8(d \cup e, PG) = 4 \) for any distinct edges \( d \) and \( e \). Then by Proposition 2.1 we have
\[ \sum_{\gamma \in \Gamma_8(PG)} c(f(\gamma)) \equiv 0 \pmod{4}. \]

\[ \Box \]

Example 2.8. Another plane generic immersion \( f : PG \to \mathbb{R}^2 \) of \( PG \) is illustrated in Figure 2.4. The crossing number \( c(f(PG)) = 2 \) is known to be minimal among all plane generic immersions of \( PG \). Note that the upper crossing of \( f(PG) \) in Figure 2.4 is a crossing between distance 2 edges of \( PG \) and the lower crossing is a crossing between distance 1 edges of \( PG \). Then we see that \( f \) satisfies all modular equalities in Theorem 1.3 and Theorem 1.6.

![Figure 2.4. Another plane generic immersion of PG](image)

Proof of Theorem 1.7. We set
\[ \kappa(f) = \sum_{(d,e) \in D_2(HG)} |f(d) \cap f(e)| \]
for a plane generic immersion \( f : HG \to \mathbb{R}^2 \) of \( HG \). Let \( g : HG \to \mathbb{R}^2 \) be a plane generic immersion illustrated in Figure 1.4. We note that \( g(HG) \) has 7 crossings between distance 1 edges and 7 crossings between distance 2 edges. Therefore we have
\[ \kappa(g) = \sum_{(d,e) \in D_2(HG)} |g(d) \cap g(e)| = 7 \equiv 1 \pmod{2}. \]

Next we will show that \( \kappa(f) \pmod{2} \) is invariant under Reidemeister moves. Since \( \kappa \) do not count self crossings, it is invariant under R1. The change of \( \kappa \) under R2 is \( \pm 2 \) or \( 0 \) and therefore \( \kappa(f) \pmod{2} \) is invariant under R2. The third Reidemeister move R3 do not change \( \kappa \) itself. Let \( e \) be an edge of \( HG \) that is involved in a fourth Reidemeister move R4. As we saw before, the edges of \( HG \) with distance 2 with \( e \) forms an 8-cycle of \( HG \). Therefore we see that the change of \( \kappa \) under R4 is \( \pm 2 \) or \( 0 \) and therefore \( \kappa(f) \pmod{2} \) is invariant under R4. Since \( \kappa \) do not count adjacent crossings, it is invariant under R5. Since \( f \) and \( g \) are transformed into each other by Reidemeister moves, we have
\[ \kappa(f) \equiv \kappa(g) \equiv 1 \pmod{2}. \]
Proof of Theorem 2.8 We note that
\[ \ell_D(d,e) = c^+_D(d,e) - c^+_D(d,e) \equiv c^+_D(d,e) + c^+_D(d,e) \equiv |d \cap e| \pmod 2. \]
We also note that \( \varepsilon(d,e) \) is an odd number if \( d(e) = 1 \) and it is an even number if \( d(e) = 2 \). Therefore we have the modular equality. □

Lemma 2.9.
1. For any vertices \( x \) and \( y \) of \( HG \), there exists an isomorphism of \( HG \) that maps \( x \) to \( y \).
2. For any edges \( x \) and \( y \) of \( HG \), there exists an isomorphism of \( HG \) that maps \( x \) to \( y \).
3. For any pairs of mutually adjacent edges \( x, y, z, w \) of \( HG \), there exists an isomorphism of \( HG \) that maps \( x \cup y \) to \( z \cup w \).
4. Let \( x, y, z \) and \( w \) be edges of \( HG \) with \( d(x,y) = d(z,w) = 1 \). Then there exists an isomorphism of \( HG \) that maps \( x \cup y \) to \( z \cup w \).
5. Let \( x, y, z \) and \( w \) be edges of \( HG \) with \( d(x,y) = d(z,w) = 2 \). Then there exists an isomorphism of \( HG \) that maps \( x \cup y \) to \( z \cup w \).

Proof. A Heawood graph has the following symmetries. It is 7-periodic with respect to the isomorphism \( \alpha : HG \to HG \) defined by \( \alpha(u_i) = u_{i+1} \) and \( \alpha(v_i) = v_{i+1} \). It is 2-periodic with respect to the isomorphism \( \beta : HG \to HG \) defined by \( \beta(u_i) = v_{i-1} \) and \( \beta(v_i) = u_{i-1} \). It is 3-periodic with 2 fixed vertices with respect to the isomorphism \( \gamma : HG \to HG \) defined by \( \gamma(u_1) = u_1, \gamma(v_1) = v_1, \gamma(v_3) = v_3, \gamma(u_2) = u_3, \gamma(u_5) = u_3, \gamma(u_4) = u_2, \gamma(v_4) = v_2, \gamma(v_5) = v_2 \) and \( \gamma(u_6) = u_6, \gamma(v_6) = u_6 \). We note that \( \gamma \) cyclically maps the edges incident to \( u_1 \). Then we see that these isomorphisms generate all isomorphisms requested in (1). We note that \( \gamma \) cyclically maps the edges incident to \( u_1 \).

Lemma 2.10.
1. Let \( e \) be an edge of \( HG \). Then \( \alpha_0(e,HG) = 8, \alpha_8(e,HG) = 8, \alpha_{10}(e,HG) = 40, \alpha_{12}(e,HG) = 32 \) and \( \alpha_{14}(e,HG) = 16 \).
2. Let \( d \) and \( e \) be mutually adjacent edges of \( HG \). Then \( \alpha_0(d \cup e,HG) = 4, \alpha_6(d \cup e,HG) = 4, \alpha_{10}(d \cup e,HG) = 20, \alpha_{12}(d \cup e,HG) = 16 \) and \( \alpha_{14}(d \cup e,HG) = 8 \).
(3) Let $d$ and $e$ be mutually disjoint oriented edges of $HG$ with $d(d, e) = 1$. Then $\alpha_0(d \cup e, HG) = 2$, $\alpha_0(d \cup e, HG) = 2$, $\alpha_0(d \cup e, HG) = 18$, $\alpha_1(d \cup e, HG) = 16$ and $\alpha_1(d \cup e, HG) = 12$. Suppose that the 6-cycle of $HG$ containing $d \cup e$ is coherent with respect to $d$ and $e$. Then $\beta_0(d \cup e, HG) = 2$, $\beta_0(d \cup e, HG) = 2$, $\beta_1(d \cup e, HG) = 10$, $\beta_1(d \cup e, HG) = 8$ and $\beta_1(d \cup e, HG) = 4$.

(4) Let $d$ and $e$ be mutually disjoint oriented edges of $HG$ with $d(d, e) = 2$. Then $\alpha_0(d \cup e, HG) = 0$, $\alpha_0(d \cup e, HG) = 2$, $\alpha_0(d \cup e, HG) = 4$, $\alpha_0(d \cup e, HG) = 4$ and $\alpha_0(d \cup e, HG) = 8$. Suppose that the 6-cycle of $HG$ containing $d \cup e$ is coherent with respect to $d$ and $e$. Then $\beta_0(d \cup e, HG) = 1$, $\beta_0(d \cup e, HG) = 1$, $\beta_1(d \cup e, HG) = 5$, $\beta_1(d \cup e, HG) = 4$ and $\beta_1(d \cup e, HG) = 2$.

Proof. We note that $HG$ is a bipartite graph on 14 vertices and $g(HG) = 6$. Therefore

$$\Gamma(HG) = \bigcup_{k=3}^7 \Gamma_k(HG).$$

There are 28 6-cycles of $HG$. They are

$$(u_i u_{i+1} u_i v_i u_{i+1} v_i v_{i+1} u_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i) = 1, 2, 3, 4, 5, 6, 7.
$$

There are 21 8-cycles of $HG$. They are

$$(u_i u_{i+1} u_i v_i u_{i+1} v_i v_{i+1} u_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i) = 1, 2, 3, 4, 5, 6, 7.
$$

There are 34 10-cycles of $HG$. They are

$$(u_i u_{i+1} u_i v_i u_{i+1} v_i v_{i+1} u_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i) = 1, 2, 3, 4, 5, 6, 7.
$$

There are 56 12-cycles of $HG$. They are

$$(u_i u_{i+1} u_i v_i u_{i+1} v_i v_{i+1} u_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i) = 1, 2, 3, 4, 5, 6, 7.
$$

There are 24 14-cycles of $HG$.

They are

$$(u_i u_{i+1} u_i v_i u_{i+1} v_i v_{i+1} u_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i u_{i+1} v_i u_{i+2} u_i u_{i+2} v_i u_{i+1} v_i) = 1, 2, 3, 4, 5, 6, 7.
$$
Then by Lemma 2.3 (2) we see that counting cycles for a particular edge $e$ of $HG$ will show (1). Similarly we have (2), (3) and (4) by Lemma 2.9 (3), (4) and (5) respectively. □

Summarizing the statements in Lemma 2.10 we have Table 2.

**Table 2. number of $k$-cycles of $HG$**

| $k$ | $|\Gamma_k(HG)|$ | $|\Gamma_k(HG)| \cdot k$ | $\alpha_k(e, HG)$ | $\alpha_k(d \cup e, HG)$ | $\beta_k(d \cup e, HG)$ |
|-----|-----------------|---------------------------|----------------|------------------|------------------|
| 6   | 28              | 168                       | 8              | 4                | 2                |
| 8   | 21              | 168                       | 8              | 4                | 2                |
| 10  | 84              | 840                       | 40             | 2                | 3                |
| 12  | 56              | 672                       | 32             | 16               | 20               |
| 14  | 24              | 936                       | 16             | 8                | 10               |

Proof of Theorem 1.10. Let $g : HG \to \mathbb{R}^2$ be a plane generic immersion illustrated in Figure 1.3. We see that $g(HG)$ contains 7 disjoint crossings between distance 1 edges and 7 disjoint crossings between distance 2 edges. Since $\alpha_6(d \cup e, HG) = 2$ for edges $d$ and $e$ of $HG$ with $d(d, e) = 1$ and $\alpha_6(d \cup e, HG) = 1$ for edges $d$ and $e$ of $HG$ with $d(d, e) = 2$, we have

$$\sum_{\gamma \in \Gamma_6(HG)} c(g(\gamma)) = 2 \cdot 7 + 7 = 21 \equiv 1 \pmod{2}.$$  

Similarly we have

$$\sum_{\gamma \in \Gamma_8(HG)} c(g(\gamma)) = 2 \cdot 7 + 3 \cdot 7 = 35 \equiv 1 \pmod{2}$$  

and

$$\sum_{\gamma \in \Gamma_{10}(HG)} c(g(\gamma)) = 18 \cdot 7 + 17 \cdot 7 = 245 \equiv 1 \pmod{2}.$$  

Next we will check that the conditions (1), (2), (3) and (4) of Proposition 2.4 for $G = HG$, $\Lambda = \Gamma_6(HG)$, $\Gamma_8(HG)$ or $\Gamma_{10}(HG)$ and $m = 2$ hold. Since $\alpha_6(e, HG) = 8$ and $\alpha_{10}(e, HG) = 40$ for every edge $e$ of $HG$, we see that (1) holds. For $m = 2$, (2) automatically holds. Since $\Lambda \subset \Gamma(HG)$ (3) holds for $m = 2$ by Remark 2.8. Let $d$ and $e$ be mutually adjacent edges of $HG$. Then $\alpha_6(d \cup e, HG) = \alpha_8(d \cup e, HG) = 4$ and $\alpha_{10}(d \cup e, HG) = 20$. Therefore (4) holds. Then by Proposition 2.4 we have

$$\sum_{\gamma \in \Gamma_6(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_8(HG)} c(g(\gamma)) \pmod{2},$$

$$\sum_{\gamma \in \Gamma_8(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_8(HG)} c(g(\gamma)) \pmod{2}$$

and

$$\sum_{\gamma \in \Gamma_{10}(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_{10}(HG)} c(g(\gamma)) \pmod{2}.$$  

Therefore we have

$$\sum_{\gamma \in \Gamma_6(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_8(HG)} c(f(\gamma)) \equiv \sum_{\gamma \in \Gamma_{10}(HG)} c(f(\gamma)) \equiv 1 \pmod{2}.$$  

We note that $\alpha_{12}(e, HG) = 32$ for every edge $e$ of $HG$, $\alpha_{12}(d \cup e, PG) = 16$ for any adjacent edges $d$ and $e$, $\alpha_{12}(d \cup e, PG) = 16$ for any edges $d$ and $e$ with $d(d, e) = 1$.
and \( \alpha_{12}(d \cup e, PG) = 20 \) for any edges \( d \) and \( e \) with \( d(d, e) = 2 \). Then by Proposition 2.1 we have
\[
\sum_{\gamma \in \Gamma_{12}(HG)} c(f(\gamma)) \equiv 0 \pmod{4}.
\]
We note that \( \alpha_{14}(e, HG) = 16 \) for every edge \( e \) of \( HG \), \( \alpha_{14}(d \cup e, PG) = 8 \) for any adjacent edges \( d \) and \( e \), \( \alpha_{14}(d \cup e, PG) = 12 \) for any edges \( d \) and \( e \) with \( d(d, e) = 1 \) and \( \alpha_{14}(d \cup e, PG) = 10 \) for any edges \( d \) and \( e \) with \( d(d, e) = 2 \). Then by Proposition 2.1 we have
\[
\sum_{\gamma \in \Gamma_{14}(HG)} c(f(\gamma)) \equiv 0 \pmod{2}.
\]
\[\square\]

3. Rotation numbers of cycles in a plane immersed graph

In this section we give a proof of Theorem 1.15.

**Lemma 3.1.** Let \( G \) be a 2-connected loopless graph. Let \( u \) and \( v \) be distinct vertices of \( G \) such that the graph obtained from \( G \) by adding an edge joining \( u \) and \( v \) does not have \( K_4 \) as a minor. Then there exists a plane generic immersion \( f : G \to \mathbb{R}^2 \) with the following properties.

1. It holds that \( \text{rot}(f(\gamma)) = 0 \) for every cycle \( \gamma \) of \( G \).
2. Let \( h : \mathbb{R}^2 \to \mathbb{R} \) be a height function defined by \( h(x, y) = y \). Then \( h(f(G)) = [h(f(v)), h(f(u))] \). If \( P \) is a path of \( G \) joining \( u \) and \( v \), then \( h \circ f \) maps \( P \) homeomorphically onto the closed interval \([h(f(v)), h(f(u))]\).

**Proof.** The following is a proof by induction on the number of vertices of \( G \).

First suppose \( |V(G)| = 2 \). Then \( G \) is a \( \theta_n \)-curve graph and a plane generic immersion described in Figure 3.1 where the case \( n = 5 \) is illustrated, satisfies (1) and (2).

Let \( k \) be a natural number greater than or equal to 2. Suppose that the claim is true if \( |V(G)| \leq k \). Suppose \( |V(G)| = k + 1 \). Let \( X_1, \ldots, X_n \) be the connected components of the topological space \( G \setminus \{u, v\} \) and \( H_1, \ldots, H_n \) the closures of them in \( G \). Namely \( H_i = X_i \cup \{u, v\} \) and \( H_i \) is regarded as a subgraph of \( G \) for \( i = 1, \ldots, n \). Let \( H'_i \) be a graph obtained from \( H_i \) by adding an edge \( e_i \) joining \( u \) and \( v \). By the assumption on \( G \) we see that \( H'_i \) is a 2-connected loopless graph that does not have \( K_4 \) as a minor.

We will show that \( n \) is greater than 1. Suppose to the contrary that \( n = 1 \). Then \( G = H_1 \). Let \( P \) be a path of \( H_1 \) joining \( u \) and \( v \). Suppose that there exists another path \( Q \) of \( H_1 \) joining \( u \) and \( v \) such that \( P \cap Q = \{u, v\} \). Since \( X_1 = H_1 \setminus \{u, v\} \) is connected there is a path \( R \) of \( H_1 \) away from \( u \) and \( v \) joining a vertex of \( P \) and a vertex of \( Q \). Then we see that \( H'_1 \) has \( K_4 \) as a minor. Therefore there are no such path of \( H_1 \). Since \( H_1 = G \) is 2-connected, \( H_1 \) has no cut vertices. Therefore \( \deg(u) \geq 2 \). Then there exists a path \( Q \) of \( H_1 \) joining \( u \) and a vertex \( w \) of \( P \) such that \( P \cap Q = \{u, w\} \). We may suppose that \( w \) is closest to \( v \) on \( P \) among all choices of such path \( Q \). Then we have either \( w \) is a cut vertex of \( H_1 \) or \( H'_1 \) has \( K_4 \) as a minor. See Figure 3.2 for the latter case. We note that the second vertex from bottom in the right graph of Figure 3.2 may be \( v \). Both contradict to the assumption. Thus we have \( n \geq 2 \).

Since \( |V(G)| \geq 3 \) at least one of \( H_1, \ldots, H_n \), say \( H_1 \), contains at least 3 vertices. Suppose that all other \( H_i \) contains exactly two vertices. Then \( H_i \) contains exactly one edge that joins \( u \) and \( v \) for \( i = 2, \ldots, n \). Then by an argument similar to the previous one we see that \( H_1 \) has a cut vertex \( w \). Let \( H_{1,1} \) and \( H_{1,2} \) be connected subgraphs of \( H_1 \) such that \( H_1 = H_{1,1} \cup H_{1,2} \) and \( H_{1,1} \cap H_{1,2} = \{w\} \). We may suppose without loss of generality that \( H_{1,1} \) contains \( u \) and \( H_{1,2} \) contains \( v \). Let
Let $H'_{1,1}$ be a graph obtained from $H_{1,1}$ by adding an edge $e_{1,1}$ joining $u$ and $w$. Let $H'_{1,2}$ be a graph obtained from $H_{1,2}$ by adding an edge $e_{1,2}$ joining $w$ and $v$. Then $H'_{1,1}$ and $u$ and $w$ satisfy the induction hypothesis and $H'_{1,2}$ and $w$ and $v$ also satisfy the induction hypothesis. Let $f_1 : H'_{1,1} \to \mathbb{R}^2$ and $f_2 : H'_{1,2} \to \mathbb{R}^2$ be plane generic immersions that satisfy (1) and (2). By a translation we may assume $f_1(w) = f_2(w)$. Let $f_0 : H_1 \to \mathbb{R}^2$ be a plane generic immersion defined by $f_0|_{H_{1,1}} = f_1|_{H_{1,1}}$ and $f_0|_{H_{1,2}} = f_2|_{H_{1,2}}$. Then by a construction similar to that illustrated in Figure 3.1 we have a plane generic immersion $f : G \to \mathbb{R}^2$ with $f|_{H_1} = f_0$ that satisfies (1) and (2). See for example Figure 3.3 where the case $n = 3$ is illustrated.

Next suppose that another $H_i$, say $H_2$, contains at least three vertices. Then by a construction similar to that illustrated in Figure 3.3, we have a plane generic immersion of $G$ that satisfies (1) and (2). See for example Figure 3.3. This completes the proof. □

**Figure 3.1.** All cycles have rotation number 0

**Figure 3.2.** $n$ is greater than 1

**Proof of Theorem 1.15.** Suppose that $K_4$ is a minor of $G$. Since $K_4$ is 3-regular, $G$ contains a subgraph that is homeomorphic to $K_4$. Then by Corollary 1.14 we see that every plane generic immersion of $G$ contains a cycle with non-zero rotation number. Therefore (1) implies (2). Suppose that $G$ does not have $K_4$ as a minor. We will show that there exists a plane generic immersion $f : G \to \mathbb{R}^2$ such that $\text{rot}(f(\gamma)) = 0$ for every cycle $\gamma$ of $G$. By considering the block decomposition, it is sufficient to show the case that $G$ is 2-connected and loopless. Then by Lemma 3.1 we have such a plane generic immersion. This completes the proof. □
4. The total Thurston-Bennequin number of a Legendrian embedding of a finite graph

**Lemma 4.1.** Let $G$ be a finite graph and $f : G \to \mathbb{R}^3$ a Legendrian embedding of $G$. Let $j$ and $k$ be natural numbers. Suppose that there exists a rational number $q$ such that the following holds.

1. For any edge $e$ of $G$, $\alpha_k(e, G) = q \cdot \alpha_j(e, G)$.
2. For any pair of mutually adjacent edges $d$ and $e$ of $G$, $\alpha_k(d \cup e, G) = q \cdot \alpha_j(d \cup e, G)$. 

**Figure 3.3.** A construction

**Figure 3.4.** An example
(3) For any pair of mutually disjoint oriented edges $d$ and $e$ of $G$, $\beta_k(d \cup e, G) = q \cdot \beta_j(d \cup e, G)$.

Then

$$TB_k(f) = q \cdot TB_j(f).$$

Proof. Let $\gamma$ be a cycle of $G$. Let $D = D(f(\gamma))$ be a Lagrangian projection of a Legendrian knot $f(\gamma)$. Let $w(D)$ be the writhe of $D$. It is known that $tb(f(\gamma)) = w(D)$.

We note that each crossing of $D$ is a self crossing, an adjacent crossing or a disjoint crossing. A self crossing of an edge $e$ of $G$ contribute $\pm \alpha_j(e, G)$ to $TB_j(f)$ and $\pm \alpha_k(e, G)$ to $TB_k(f)$. Similarly an adjacent crossing of edges $d$ and $e$ contribute $\pm \alpha_j(d \cup e, G)$ to $TB_j(f)$ and $\pm \alpha_k(d \cup e, G)$ to $TB_k(f)$. A disjoint crossing of edges $d$ and $e$ contribute $\pm \beta_j(d \cup e, G)$ to $TB_j(f)$ and $\pm \beta_k(d \cup e, G)$ to $TB_k(f)$. By taking the sum of these numbers we have the result. $\square$

Proof of Theorem 1.19. By Lemma 1.1 and Lemma 2.7 we have the result. $\square$

Proof of Theorem 1.20. By Lemma 1.1 and Lemma 2.10 we have the result. $\square$

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