A new study of generalized Ma-Minda type class of meromorphic functions

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Abstract. In this paper, a new study of generalized Ma-Minda type class of meromorphic functions is introduced. Based on the studied classes and by giving some special values to the parameters involved, several results are obtained such as inclusion results, integral properties, coefficient estimates and convolution properties. Moreover, these results can be applied by considering meromorphic functions involving generalized hypergeometric functions.

1. Introduction
Let \( H \) be the set of all analytic functions defined in the unit disk \( U = \{ z : z \in C \text{ and } 0 \leq |z| < 1 \} \). We denote by \( A \) the class of normalized analytic functions
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
defined in \( U \).

Let the functions \( f \) and \( g \) be analytic in \( U \), then we say that \( f \) is subordinate to \( g \) in \( U \), and write \( f \prec g \); if there exists a Schwarz function \( w \) analytic in \( U \) such that \( |w(z)| < 1 \), \( z \in U \); and \( w(0) = 0 \) with \( f(z) = g(w(z)) \) in \( U \). Further, if \( g \) is univalent in \( U \); then \( f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \) and \( f(U) \subset g(U) \).

A function \( f \in A \) is starlike if \( z f'(z)/f(z) \) is subordinate to \( (1+z)/(1-z) \) and convex if \( 1+z f''(z)/f'(z) \) is subordinate to \( (1+z)/(1-z) \) see [1]. Ma and Minda [2] gave a unified presentation of these classes and introduced the classes
\[
S^*(h) = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} \right| < h(z) \right\},
\]
\[
C(h) = \left\{ f \in A : \left| \frac{1+zf''(z)}{f'(z)} \right| < h(z) \right\},
\]
where \( h \) is an analytic function with positive real part, \( h(0) = 1 \), and \( h \) maps the unit disk \( U \) onto a region starlike with respect to \( 1 \).

The convolution or the Hadamard product of two analytic functions \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) and
\[ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \]
is given as follows:

\[ (f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \] (2)

In term of convolution, a function \( f \) is starlike if \( f * \left( \frac{z}{1 - z} \right) \) is starlike, and convex if \( f * \left( \frac{z}{1 - z}^2 \right) \) is starlike. These ideas led to the study of the class of all functions \( f \) such that \( f * g \) is starlike for some fixed function \( g \) in \( A \). In this direction, Shanmugam [3] introduced and investigated various subclasses of analytic functions by using the convex hull method [4, 5] and the method of differential subordination. Ravichandran [6] introduced certain classes of analytic functions with respect to \( n \)-ply symmetric points, conjugate points, and symmetric conjugate points, and also discussed their convolution properties. Some other related studies were also made in [7, 8].

Let \( \Sigma \) signify analytic meromorphic functions’ class \( f(z) \) which is normalized by

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \] (3)
in the punctured disk \( U^* = \{ z : z \in C \text{ and } 0 < |z| < 1 \} = U \setminus \{0\} \). Also, \( \Sigma S^*(\beta) \) and \( \Sigma C(\beta) \) are employed, the subclasses of \( \Sigma \) that contain all meromorphic functions, respectively, include starlike of order \( \beta \) and convex of order \( \beta \) in \( U^* \), \( 0 \leq \beta \) which are defined by

\[ \Sigma S^*(h) = \left\{ f \in \Sigma \mid -\Re \left( \frac{zf'(z)}{f(z)} \right) > \beta \right\}, \]

\[ \Sigma C(h) = \left\{ f \in \Sigma \mid -\Re \left( \frac{1+zf''(z)}{f'(z)} \right) > \beta \right\}. \] (4)

Motivated by the results obtained by Wang et al. in [9], we involve this convolution structure in this study. In fact, several subclasses of meromorphic functions defined by means of convolution with a given fixed meromorphic function are introduced in next section. These new subclasses extend the classical classes of meromorphic starlike and convex functions given in 4 see for example [10, 11] and [12].

It may be noted that for \( f \in \Sigma \) and for \( z \in U^* \):

\[ f * \left( \frac{z}{1 - z} \right) = f \] (5)

and

\[ f * \left( \frac{z}{1 - z}^2 \right) = -zf'(z) = Df(z), \] (6)

where \( D : \Sigma \to \Sigma \) defines a linear derivative operator.

For functions \( f_j(z)(j = 1; 2) \) given below

\[ f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \] (7)

the Hadamard product (or convolution) for \( f_1(z) \) and \( f_2(z) \) can be expressed as follows:

\[ (f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \] (8)
Let us consider the function $\tilde{\phi}(\alpha, \beta; z)$ defined by

$$
\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left( \frac{\alpha}{\beta} \right)_{k+1} a_k z^k.
$$

(9)

where

$$
\left( \beta \in C \setminus Z_0^-; \alpha \in C \right),
$$

with

$$
Z_0^- = \{0, -1, -2, \cdots\} = Z^- \cup \{0\}.
$$

Here, and in the remainder of this paper, $(\lambda)_n$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$
(\lambda)_n := \frac{\Gamma (\lambda + n)}{\Gamma (\lambda)} = \begin{cases}
\lambda (\lambda + 1) \cdots (\lambda + n - 1) & (\kappa = n \in N; \, \lambda \in C) \\
1 & (\kappa = 0; \, \lambda \in C \setminus \{0\})
\end{cases}
$$

It is easy to see that, in the case when $a_k = 1$ ($k = 0, 1, 2, \cdots$), the following relationship holds true between the function $\tilde{\phi}(\alpha, \beta; z)$ and the Gaussian hypergeometric function [13]:

$$
\tilde{\phi}(\alpha, \beta; z) = \frac{1}{z} \, _2F_1(1, \alpha; \beta; z).
$$

(10)

where

$$
_2F_1(b, \alpha; \beta; z) = \sum_{n=0}^{\infty} \left( \frac{b}{\beta} \right)_n \frac{\alpha^n z^n}{n!}
$$

is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(\alpha, \beta; z)$, using the Hadamard product for $f(z) \in \Sigma$, we define a new linear operator $L(\alpha, \beta)$ on $\Sigma$ by

$$
L(\alpha, \beta) f(z) = \tilde{\phi}(\alpha, \beta; z) \ast f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \left( \frac{\alpha}{\beta} \right)_{n+1} a_n z^n.
$$

(11)

For a function $f \in L(\alpha, \beta) f(z)$ we define

$$
I^0 (L(\alpha, \beta) f(z)) = L(\alpha, \beta) f(z),
$$

and for $t = 1, 2, 3, \ldots$,

$$
I^t (L(\alpha, \beta) f(z)) = z \left( I^{t-1} L(\alpha, \beta) f(z) \right)' + \frac{2}{z}
$$

$$
= \frac{1}{z} + \sum_{n=1}^{\infty} n \left( \frac{\alpha}{\beta} \right)_{n+1} a_n z^n.
$$

(12)

We note that $I^t$ in (12) was studied by Ghanim and Darus, see for example [14] and [15].

Also, it follows from (12) that

$$
z (L(\alpha, \beta)f(z))' = \alpha L(\alpha + 1, \beta)f(z) - (\alpha + 1) L(\alpha, \beta)f(z).
$$

(13)
Also, from (12) and (13) we get
\[
z \left( I^L \alpha \beta f(z) \right)' = \alpha I^L \alpha + 1, \beta f(z) - (\alpha + 1) I^L \alpha, \beta f(z).
\]

Let \( \psi_k := \exp \left( \frac{2\pi i k}{N} \right) \) \((k \in N)\) then for some \( k \geq 2 \), the points \( \psi^k_k z \) \((v = 0, 1, 2, ..., k - 1; k \in N)\) called \( k \)-symmetric points. Let for some \( h \in \Sigma \) of the form:
\[
h(z) = \frac{1}{z} + \sum_{n=2}^{\infty} c_n z^n, \quad z \in U^*
\]
we define
\[
h_k(z) = \frac{1}{k} \sum_{v=0}^{k-1} \psi^v_k h(\psi^v_k z) = \left( \frac{1}{z(1-z^v)} \right) * h(z)
\]
\[
= \frac{1}{z} + \sum_{n=2}^{\infty} c_n d_n^{k} z^n \in \Sigma, \quad z \in U^*
\]
where
\[
d_n^{k} = \frac{1}{k} \sum_{v=0}^{k-1} \psi^v_k = \begin{cases} 1, & n = lk \ (l \in N) \\ 0, & n = lk + m \ (l \in N_0 = N \cup \{0\}, m = 1, 2, ..., k - 1, \ k \geq 2) \end{cases}
\]

Using the concept of subordination, for a convex (univalent) function \( \phi \in \mu \) satisfying the condition that \( \phi'(z) > 0 \) and \( \phi(U) \) is symmetric with respect to the real axis, we define a Ma-Minda type class by, (see for more details [2] and [16])
\[
\mu[\phi] = \{ p \in \mu : p(z) \prec \phi(z) \}.
\]
For \(-1 \leq B < A \leq 1, z \in U, \mu \left[ \frac{1+Bz}{1+Bz} \right] \equiv \mu(A,B) \) and \( \mu(1,-1) \equiv \mu \).

For the purpose of this paper, we introduce here a generalized Ma-Minda type class \( \Sigma(h, \phi, \lambda; \alpha, \beta) \) of meromorphic functions, which we define as follows:

**Definition 1:** A function \( f \in \Sigma \), is said to be in the class \( \Sigma(h, \phi, \lambda; \alpha, \beta) \) if for \( h \in \Sigma, 0 \leq \lambda \leq 1 \) and the linear operators \( D \) and \( I^L \) be defined, respectively, by (6) and (12), with \( (1-\lambda) h(z) + \lambda I^L h(z) \neq 0 \), \( z \in U^* \), it satisfies the following condition:
\[
\frac{D \left( (1-\lambda) f(z) + \lambda I^L f(z) \right)}{(1-\lambda) h(z) + \lambda I^L h(z)} \in \mu[\phi], \quad z \in U.
\]

We deem it worthwhile to point out some useful special cases of the class \( \Sigma(h, \phi, \lambda; \alpha, \beta) \). For instance, if we put \( \lambda = 0 \) in Definition 1 above, we may write \( \Sigma(h, \phi, 0; \alpha, \beta) = C(h, \phi, \alpha, \beta) \) and if \( \alpha = \beta \) and \( \lambda = 0 \) we have \( \Sigma(h, \phi, 0; \alpha, \beta) = C(h, \phi) \).

If we choose \( f_k \) in place of \( h \); then we denote \( \Sigma(f_k, \phi, \lambda; \alpha, \beta) \) by \( \Sigma_k(\phi, \lambda; \alpha, \beta) \), wherein, the function \( f \in \Sigma \) satisfies the condition that
\[
\frac{D \left( (1-\lambda) f(z) + \lambda I^L f(z) \right)}{(1-\lambda) f(z) + \lambda I^L f(z)} \in \mu[\phi], \quad z \in U.
\]

In this paper, we study a generalized Ma-Minda type class \( \Sigma(h, \phi, \lambda; \alpha, \beta) \) defined as above in Definition 1. Based on the classes, obtained from \( \Sigma(h, \phi, \lambda; \alpha, \beta) \) for special values of the parameters involved, several results such as inclusion results, integral representations, coefficient estimates and convolution properties are obtained. We specifically point out (briefly) how our results can be applied to certain familiar types of hypergeometric functions.
2. Inclusion and Main Results

In proving our main results, we need following lemmas:

**Lemma 1.** [17] Let $\phi$ be convex univalent in $U$ with $\phi(0) = 1$ and $\varphi$ be analytic in $U$ with $\Re(\varphi(z)) \geq 0$, $z \in U$. If $q$ is analytic in $U$ and $q(0) = \phi(0)$, then

$$q(z) + \varphi(z) z q'(z) \prec \phi(z), \quad z \in U$$

implies that $q(z) \prec \phi(z), z \in U$.

Our first result is given by:

**Theorem 2.** Let $0 \leq \lambda \leq 1$ and for some $h \in \Sigma$ and $k \in N$, the function $h_k \in \Sigma$ be defined by (16) and the operator $I^h$ be defined by (12), then

$$\Sigma(h_k, \phi; \alpha, \beta) \subseteq C(h_k, \phi; \alpha, \beta)$$

provided (in case $\lambda \neq 0$) that

$$\Re \left( \frac{1}{\lambda} - 1 + \frac{I^h h_k(z)}{h_k(z)} \right) > 0, \quad z \in U.$$  

**Proof.** Let $f \in \Sigma(h_k, \phi; \alpha, \beta)$ then in view of Definition 1, we have

$$\frac{D \left( (1-\lambda)f(z) + \lambda I^h f(z) \right)}{(1-\lambda) h_k(z) + \lambda I^h h_k(z)} \prec \phi(z), \quad z \in U.$$  

If $\lambda = 0$, the result is obvious. If $\lambda = 0$, we set

$$q(z) = \frac{D f(z)}{h_k(z)}, \quad z \in U,$$

which on using the identity (14) gives $D f(z) = q(z) h_k(z)$ applying the operator $D$ defined by (6) in the resulting expression, we get

$$zq'(z) h_k(z) - q(z) D h_k(z) = D I^h f(z) - D f(z)$$

(19)

Now using (18), (19) and the identity (14) for $h_k \in \Sigma$, we get

$$\frac{D \left( (1-\lambda)f(z) + \lambda I^h f(z) \right)}{(1-\lambda) h_k(z) + \lambda I^h h_k(z)} = \frac{(1-\lambda)q(z) h_k(z) + \lambda(zq'(z) h_k(z) - q(z) D h_k(z) + D f(z))}{(1-\lambda) h_k(z) + \lambda I^h h_k(z)}$$

$$= \frac{(1-\lambda)q(z) h_k(z) + \lambda(zq'(z) h_k(z) - q(z) D h_k(z))}{(1-\lambda) h_k(z) + \lambda I^h h_k(z)} = q(z) + \frac{zq'(z)}{1 + \frac{I^h h_k(z)}{h_k(z)}} \prec \phi(z).$$

Thus by Lemma 1, we get the desired result. This proves Theorem 2. The following corollary can easily be obtained by taking $h(z) = \frac{1}{z}, z \in U^*$ in Theorem 2:

**Corollary 3.** Let $0 \leq \lambda \leq 1$ and $\alpha = \beta$, then

$$\Sigma(\phi, \lambda) = C(\phi).$$
Theorem 4. Let $0 \leq \lambda \leq 1$ and for some $h \in \Sigma$ and $k \in N$, the function $h_k \in \Sigma$ be defined by (16) and the operator $I^t$ be defined by (12), then

$$f \in \Sigma (h, \phi, \lambda; \alpha, \beta) \Rightarrow f_k \in \Sigma (h, \phi, \lambda; \alpha, \beta).$$

Further, $f_k \in \Sigma (h, \phi, \lambda; \alpha, \beta) \Rightarrow f_k \in C (h, \phi; \alpha, \beta)$, provided (in case $\lambda \neq 0$) that

$$\Re \left( \frac{1}{\lambda} - 1 + \frac{I^t h_k (z)}{h_k (z)} \right) > 0, \quad z \in U.$$

Proof. Let $f \in \Sigma (h_k, \phi, \lambda; \alpha, \beta)$, we have

$$\frac{D ((1 - \lambda) f (z) + \lambda f^t (z))}{(1 - \lambda) h_k (z) + \lambda I^t h_k (z)} \in \mu [\phi], \quad z \in U.$$

For $\psi_k := \exp \left( \frac{2\pi i}{k} \right), \psi_k^u z \in U, \; v = 0, 1, 2, ..., k - 1; \; k \in N$, we get

$$\frac{D ((1 - \lambda) f (\psi_k^u z) + \lambda I^t f (\psi_k^u z))}{(1 - \lambda) h_k (\psi_k^u z) + \lambda I^t h_k (\psi_k^u z)} \in \mu [\phi].$$

In view of (16), since

$$h_k (\psi_k^u z) = \psi_k^{-u} h_k (z) \quad \text{and} \quad I^t h_k (\psi_k^u z) = \psi_k^{-u} I^t h_k (z)$$

we find that for $v = 0, 1, 2, ..., k - 1$,

$$\frac{\psi_k^u D ((1 - \lambda) f (\psi_k^u z) + \lambda I^t f (\psi_k^u z))}{(1 - \lambda) h_k (\psi_k^u z) + \lambda I^t h_k (\psi_k^u z)} \in \mu [\phi], \quad z \in U,$$

Hence,

$$\frac{1}{k} \sum_{u=0}^{k-1} \left[ \frac{\psi_k^u D ((1 - \lambda) f (\psi_k^u z) + \lambda I^t f (\psi_k^u z))}{(1 - \lambda) h_k (\psi_k^u z) + \lambda I^t h_k (\psi_k^u z)} \right] \in \mu [\phi], \quad z \in U,$$

as $\phi$ is convex. Using (16), we conclude that

$$\frac{D ((1 - \lambda) f_k (z) + \lambda I^t f_k (z))}{(1 - \lambda) h_k (z) + \lambda I^t h_k (z)} \in \mu [\phi],$$

which proves that $f_k \in \Sigma (h, \phi, \lambda; \alpha, \beta)$. Further, we note that if $f_k \in \Sigma (h, \phi; \alpha, \beta)$ then we get by Theorem 2 that $f_k \in C (h, \phi; \alpha, \beta)$ provided (in case $\lambda \neq 0$) that

$$\Re \left( \frac{1}{\lambda} - 1 + \frac{I^t h_k (z)}{h_k (z)} \right) > 0, \quad z \in U,$$

which completes the proof of Theorem 4.

For $h_k = f_k$. Theorem 4 evidently gives the following result:

Corollary 5. Let $0 \leq \lambda \leq 1$ and for some $f \in \Sigma$ and $k \in N$, the function $f_k \in \Sigma$ be defined similar to (16), then

$$f \in \Sigma_k (\phi, \lambda; \alpha, \beta) \Rightarrow f_k \in \Sigma_k (\phi, \lambda; \alpha, \beta)$$
Further, \( f \in \Sigma_k (\phi, \lambda; \alpha, \beta) \Rightarrow f_k \in C_k (\phi; \alpha, \beta) \), provided (in case \( \lambda \neq 0 \)) that
\[
\Re \left( 1 + \frac{1}{\lambda} - \phi(z) \right) > 0, \quad z \in U.
\]

Some interesting results of Corollary 5 would follow if we involve a bilinear transformation. Thus, if we put \( \phi(z) = \left[ 1 + \frac{1}{1 + Bz} \right] \), \( z \in U \), in Corollary 5, we get

**Corollary 6.** Let \( 0 \leq \lambda \leq 1, f \in \Sigma \) and for some \( k \in \mathbb{N} \), the function \( f_k \in \Sigma \) be defined similar to (16) and the the linear operators \( D \) and \( I^t \), respectively, be defined by (6) and (12). If for \(-1 \leq B < A \leq 1\),
\[
\frac{D (1 - \lambda) f(z) + \lambda I^t f(z)}{(1 - \lambda) f_k(z) + \lambda I^t f_k(z)} < \frac{1 + Az}{1 + Bz}, \quad z \in U,
\]
then
\[
f_k \in S(A, B), \quad (20)
\]
provided (in case \( \lambda \neq 0 \)) that
\[
\Re \left( 1 + \frac{1}{\lambda} - \left( \frac{1 + Az}{1 + Bz} \right) \right) > 0, \quad z \in U.
\]

**Remark** From the known result [18], if \( B \neq -1 \) result (20) is equivalent to
\[
f_k \in S^* \left( \frac{1 - A}{1 - B} \right)
\]
and if \( B = -1 \) it is equivalent to
\[
f_k \in S^* \left( \frac{1 - A}{2} \right)
\]
Following results are direct consequences of Corollary 7 and above Remark.

**Corollary 7.** Let \( f \in \Sigma \) and for some \( k \in \mathbb{N} \), the function \( f_k \in \Sigma \) be defined similar to (16) and the the linear operators \( D \) and \( I^t \), respectively, be defined by (6) and (12). If for \(-1 \leq B < A \leq 1\),
\[
\frac{DI^t f(z)}{I^t f_k(z)} < \frac{1 + Az}{1 + Bz}, \quad z \in U,
\]
then
\[
f_k \in S(A, B),
\]
or
\[
f_k \in S^* \left( \frac{1 - A}{1 - B} \right) \quad \text{if} \quad B \neq -1
\]
and
\[
f_k \in S^* \left( \frac{1 - A}{2} \right) \quad \text{if} \quad B = -1.
\]
3. Concluding Remarks and Observations

In this paper, a new generalized Ma-Minda type class of meromorphic functions is introduced. Based on the studied classes and by giving some special values to the parameters involved, several results are obtained such as inclusion results, integral properties, coefficient estimates and convolution properties. Moreover, these results can be applied by considering meromorphic functions involving generalized hypergeometric functions.

References

[1] Haji Mohd M, Ali R M, Keong L S and Ravichandran V 2009 Subclasses of Meromorphic Functions Associated with Convolution Journal of Inequalities and Applications 2009 190–291
[2] Ma W C and Minda D 1992 A unified treatment of some special classes of univalent functions Proceedings of the Conference on Complex Analysis 157–169
[3] Shanmugam T N 1989 Convolution and differential subordination International Journal of Mathematics and Mathematical Sciences 12(2) 333–340
[4] Barnard R W and Kellogg C 1980 Applications of convolution operators to problems in univalent function theory The Michigan Mathematical Journal 27(1) 81–94
[5] Ruscheweyh S and Sheil-Small T 1973 Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture Commentarii Mathematici Helvetici 48 119–135
[6] Ravichandran V 2004 Functions starlike with respect to n-ply symmetric, conjugate and symmetric conjugate points The Journal of the Indian Academy of Mathematics 26(1) 35–45
[7] Padmanabhan K S and Parvatham R 1985 Some applications of differential subordination Bulletin of the Australian Mathematical Society 32(3) 321–330
[8] Parvatham R and Radha S 1986 On α-starlike and α-close-to-convex functions with respect to asymmetric points Indian Journal of Pure and Applied Mathematics 17(9) 1114–1122
[9] Wang Z G, Jiang Y P and Srivastava H M 2009 Some Subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function Comput. Math. Appl. 57 571–586
[10] Bansal S K, Dziok J and Goswami P 2010 Certain results for a subclass of meromorphic multi-valent functions associated with the Wright function Eur. J. Pure Appl. Math. 3(4) 633–640
[11] Ghanim F 2017 Certain properties of classes of meromorphic functions defined by a linear operator and associated with the Hurwitz-Lerch Zeta function Advanced Studies in Contemporary Mathematics (ASCM) 27(2) 175–180
[12] Liu J and Srivastava H M 2004 Classes of meromorphically multivalent functions associated with the generalized hypergeometric function Math. Comput. Modelling 39 21–34
[13] Rainville E D 1960 Special Functions (New York: Macmillan Company)
[14] Ghanim F and Darus M 2011 A new class of meromorphically analytic functions with applications to generalized hypergeometric functions Abstract and Applied Analysis Online article http://www.hindawi.com/journals/aaa/2011/159405/
[15] Ghanim F and Darus M 2013 New result of analytic functions related to Hurwitz-Zeta function The Scientific World Journal 2013 Article ID 475643 doi:10.1155/2013/475643
[16] Raina R K and Sharma P 2013 On a generalized Ma-Minda type class of multivalent meromorphic functions Proceedings of the Jangjeon Mathematical Society 16(4) 489–502
[17] Eenigenburg P, Miller S S, Mocanu P T and Reade M O 1984 On a Briot-Bouquet differential subordination Rev. Roumaine Math. Pures Appl. 29 567–573
[18] Silverman H and Silvia E M 1985 Subclasses of starlike functions subordinate to convex functions Canad. J. Math. 37 48–61