On Inaudible Properties of Broken Drums –
Isospectral Domains with Mixed Boundary Conditions

Peter Herbrich

28th February 2012

Abstract

Since Kac raised the question “Can one hear the shape of a drum?”, various families of non-smooth
counterexamples have been constructed using the transplantation method, which we consider for
domains with mixed boundary conditions. We formulate this method in terms of graphs, which
allows for a computer-aided search for transplantable pairs, a characterisation of transplantability
in terms of induced representations, and the development of tools with which new pairs can be gen-
erated from given ones. At the end, we discuss inaudible properties, and present the first example
of a connected drum that sounds disconnected, and of a broken drum that sounds unbroken, which
shows that an orbifold can be Dirichlet isospectral to a manifold. The appendix lists transplantable
pairs among which there are 10 broken versions of Gordon-Webb-Wolpert drums.
1 Introduction

Inverse spectral geometry developed in consequence of Kac’s [Kac66] famous question “Can one hear the shape of a drum?”. We consider broken drums, that is, drums whose drumheads are only partially attached. The drumhead is modelled as a compact flat manifold $M$ with piecewise smooth boundary $\partial M$. We choose disjoint open smooth subsets $\partial_D M, \partial_N M \subseteq \partial M$, representing the attached and unattached parts of the drumhead, such that

$$\partial M = \partial_D M \cup \partial_N M.$$  

(1.1)

The drumhead’s displacement $u(x,t)$ obeys the wave equation $u_{tt} = -c^2 \Delta u$ with mixed boundary conditions

$$u = 0 \text{ on } \partial_D M \quad (\text{Dirichlet})$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \partial_N M \quad (\text{Neumann}),$$

where $\frac{\partial u}{\partial n}$ denotes the normal derivative. Separation leads to pure tones $u(x,t) = \varphi(x) \sin(c \sqrt{\lambda}(t-t_0))$ with audible frequencies $c \sqrt{\lambda}/2\pi$, such that

$$\Delta \varphi = \lambda \varphi \text{ on } M^o$$

$$\varphi = 0 \text{ on } \partial_D M$$

$$\frac{\partial \varphi}{\partial n} = 0 \text{ on } \partial_N M,$$  

(1.2)

where $M^o$ denotes the interior of $M$. The mixed eigenvalue problem (1.2) is known as the Zaremba problem, and it also occurs in string theory [Sie76, KW97]. We recall that integration with respect to the Riemannian measure gives rise to an inner product on $C^\infty(M)$, with Hilbert space completion $L^2(M)$. Henceforth, we only consider cases in which the following is satisfied.

Assumption A. The eigenvalues $\lambda$ for which there exists $\varphi \in C(M) \cap C^\infty(M^o \cup \partial_D M \cup \partial_N M)$ satisfying (1.2) are given as an unbounded sequence

$$0 \leq \lambda_1 \leq \lambda_2 \leq \ldots,$$  

(1.3)

and each eigenspace is finite dimensional. In addition, eigenspaces belonging to distinct eigenvalues are $L^2$-orthogonal, and $L^2(M)$ is the Hilbert direct sum of all the eigenspaces.

There is a rich literature on elliptic boundary value problems, and we refer to [Wig70, KMR97, KMR01] for the case of manifolds with corners, which covers all examples of the following sections. Whenever the imposed boundary conditions are unambiguous, the sequence (1.3) will be called the spectrum of $M$, and manifolds are called isospectral if their spectra coincide. The terms Dirichlet isospectral and Neumann isospectral refer to homogeneous boundary conditions, that is, $\partial_N M = \emptyset$ or $\partial_D M = \emptyset$. An inspection of the heat kernel shows that isospectral manifolds share basic properties such as dimension, volume, and difference of Neumann boundary volume and Dirichlet boundary volume [BCG90, HGKV96].

Prior to Kac’s question, Milnor [Mil64] found isospectral, non-isometric flat 16-dimensional tori, which were also isospectral (compare [FK05]). In subsequent years, counterexamples to variations of Kac’s question followed [Kac80, Vig80a, Vig80b, Ura82]. Motivated by results from number theory, Sunada [Sun85] developed a general technique to construct isospectral manifolds, which has been widely applied and extended [BT87, Ber92, Ber93, Pes96, BPBS09, PB10]. Using Börard’s extension to the orbifold setting, Gordon et al. [GWW92] answered Kac’s question by constructing the Dirichlet isospectral planar domains shown in Figure 1.1a.

Buser et al. [BCDS94] applied Sunada’s method to projective special linear groups. They found 17 pairs of planar domains, whose isospectrality could also be verified by using the transplantation method that had been developed earlier by Buser [Bus86]. Roughly speaking, one cuts each eigenfunction of one domain into pieces, and superposes these restrictions linearly on the blocks of the other domain in such a way that the resulting function is also an eigenfunction. This idea revealed that the isospectrality is independent of the shape of the underlying building block and only depends on the way in which its copies are glued together. Replacing the heptagonal blocks
in Figure 1.1a by triangles yields the so-called Gordon-Webb-Wolpert drums shown in Figure 1.1b. Their isospectrality has been confirmed experimentally using microwave cavities SK94 DMRSS03. This article continues the following recent developments. In 2001, Okada and Shudo OS01 used Buser’s Bus88 idea of encoding gluing patterns by edge-coloured graphs to perform a computer-aided search for transplantable pairs. In 2006, Levitin et al. LPP06 JLNP06 applied the transplantation method to planar domains with mixed boundary conditions, which motivated the work of Band et al. BPBS09 PB10 on isospectral quantum graphs and drums. The reader is referred to GT10 for a recent review on isospectrality.

2 The Transplantation Method

The transplantation method can be applied to domains which are obtained by successive reflections and gluings of some fundamental building block.

Definition 1. A building block is a compact Riemannian manifold $B$ with piecewise smooth boundary $\partial B$, which contains disjoint open smooth subsets $\partial^1_B, \partial^2_B, \ldots, \partial^C_B$, called reflecting segments, each of which has a neighbourhood in $B$ that is isometric to an open subset of Euclidean upper half space.

Note that two copies $B_1$ and $B_2$ of a building block $B$ can be glued along their reflecting segments $\partial^i_R B_1$ and $\partial^i_R B_2$ by identifying their closures $\partial^i_R B_1 \subset \partial B_1$ and $\partial^i_R B_2 \subset \partial B_2$. The resulting topological space $B_1 \cup_i B_2$ is compact, Hausdorff, and second countable. We only consider cases in which $B_1 \cup_i B_2$ is a Riemannian manifold. By assumption, each point of $\partial^i_R B$ lies in the image of a local isometry $\psi$ with Euclidean domain $(-l,l)^{d-1} \times (-\ell,0]$ for some $l > 0$. If $\psi_1$ and $\psi_2$ denote the corresponding charts of $B_1$ and $B_2$, then

$$\psi_1(x_1, \ldots, x_{d-1}, x_d) = \begin{cases} \psi_1(x_1, \ldots, x_{d-1}, x_d) & \text{for } x_d \leq 0 \\ \psi_2(x_1, \ldots, x_{d-1}, -x_d) & \text{for } x_d > 0 \end{cases}$$

(2.1)

shall be a local isometry of $B_1 \cup_i B_2$ with domain $(-l,l)^d$. For the sake of simplicity, we use Euclidean space locally, however, the theory can be easily extended, for instance, to other constant curvature spaces.

Definition 2. A tiled domain is a compact Riemannian manifold $M$ with piecewise smooth boundary $\partial M$, such that $M$ is composed of copies $B_1, B_2, \ldots, B_V$ of some building block $B$, where $B_1, B_2, \ldots, B_V$ are glued together along some of their reflecting segments. These inner segments belong to $M^\circ$, in contrast to outer segments which remain unglued and belong to $\partial M$. In addition, $M^\circ$ is the union of $B_1^\circ, B_2^\circ, \ldots, B_V^\circ$ and the inner segments, and Assumption A is satisfied for all choices of boundary conditions described as below.

Note that Definition 2 avoids interior singularities. We continue with the boundary conditions that we can impose on a given tiled domain $M$ having outer segments $(\partial^k_R B_i)_k$. Roughly speaking, each $\partial^k_R B_i$ allows a binary choice, whereas $\partial M \setminus \bigcup_k \partial^k_R B_i$ is determined by the underlying building block $B$ as follows. In compliance with (1.1), we assume that

$$\partial B = \bigcup_{i=1}^C \partial^i_R B \cup \partial_G B \cup \partial_N B,$$

(2.2)
∂₁B \cup \partial₃B ⊂ ∂₂B

(a) Building block

(b) The drums of Gordon et al. [GWW92] with new boundary conditions

Figure 2.1: Tiled domains with mixed boundary conditions. Solid lines indicate Dirichlet boundary conditions, dashed ones indicate Neumann boundary conditions, and dotted ones indicate reflecting segments.

where ∂₁B and ∂₃B are disjoint open smooth subsets of ∂B \ ∪ \bigcup^{C}_{i=1} \partial^{R}_{i}B as indicated in Figure 2.1a.

On each ∂ₖB_i, we impose either Dirichlet or Neumann boundary conditions to obtain the Zaremba problem

\[ \Delta \varphi = \lambda \varphi \quad \text{on } M^o \]
\[ \varphi = 0 \quad \text{on } \partial_D M \]
\[ \frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial_N M \]

with desired solutions \( \varphi \in C(M) \cap C^\infty(M^o \cup \partial_D M \cup \partial_N M) \), where \( \partial_D M \) denotes the union of the copies of \( \partial_D B \) and of those \( \partial^{R}_{i}B_i \) that carry Dirichlet boundary conditions, \( \partial_N M \) is defined analogously. Figure 2.1 exemplifies the construction.

**Definition 3.** Let \( M \) and \( \hat{M} \) be tiled domains each of which is composed of \( V \) copies of the building block \( B \). Then, each invertible real \( V \times V \) matrix \( T \) gives rise to a linear isomorphism \( T : L^2(M) \to L^2(\hat{M}) \), called transplantation, such that the transform \( \hat{\varphi} = T(\varphi) \) can be written as

\[ \hat{\varphi}_i = \sum^{V}_{j=1} T_{ij} \varphi_j \quad \text{almost everywhere}, \]

where \( \varphi_i \) and \( \hat{\varphi}_i \), denote the restrictions of \( \varphi \) and \( \hat{\varphi} \) to the building blocks of \( M \) and \( \hat{M} \), respectively.

The inverse of a transplantation matrix \( T \) gives rise to the inverse transplantation \( T^{-1} \). If \( \varphi \) solves some Zaremba problem on \( M \) with eigenvalue \( \lambda \), then any transplant \( \hat{\varphi} = T(\varphi) \) on \( \hat{M} \) will solve the Helmholtz equation \( \Delta \hat{\varphi} = \lambda \hat{\varphi} \) almost everywhere, but will, in general, neither be smooth nor satisfy the desired boundary conditions on \( \hat{M} \).

**Definition 4.** Two tiled domains with predefined boundary conditions are called transplantable if there exists a transplantation \( T \) such that \( T \) and \( T^{-1} \) convert solutions of the Zaremba problem on one domain into such on the other domain. In this case, \( T \) is said to be intertwining.

Note that if \( T : L^2(M) \to L^2(\hat{M}) \) is intertwining, then for any solution \( \varphi \) with eigenvalue \( \lambda \) of the Zaremba problem on \( M \)

\[ (\hat{\Delta} \circ T)(\varphi) = \lambda T(\varphi) = T(\lambda \varphi) = (T \circ \Delta)(\varphi). \]

Since \( T \) and \( T^{-1} \) map eigenspaces into eigenspaces, the spectra of \( M \) and \( \hat{M} \) coincide.

**Proposition 5.** Transplantable domains are isospectral.

In the style of [Bus88, OSU1], we encode each tiled domain by an edge-coloured graph each of whose loops carries one of the signs \( D \) and \( N \) indicating the boundary conditions. The correspondence is summarised in Table 1 and an example is shown in Figure 2.2b, where we used different types of lines (straight, zig-zag, wavy) instead of colours.

**Definition 6.** A loop-signed graph is a finite graph with vertices \( 1, 2, \ldots, V \) together with edge colours \( 1, 2, \ldots, C \) such that at each vertex there is exactly one incident edge or loop of each colour, and each loop carries one of the signs \( D \) or \( N \).
Table 1: Correspondence between a tiled domain and its graph representation.

| Tiled domain          | Loop-signed graph          |
|-----------------------|----------------------------|
| Building blocks       | Vertices                   |
| Glued reflecting      | Connecting Edges           |
| reflecting segments   | Loops                      |
| Unglued reflecting    | Edge colours               |
| reflecting segments   | Loop signs                 |

Figure 2.2: The graph representation of the tiled domains in Figure 2.1(b).

**Definition 7.** The loopless version of a loop-signed graph is the edge-coloured graph obtained by removing all loops. A loop-signed graph is called treelike if its loopless version is a tree.

Each loop-signed graph is determined by its adjacency matrices defined as follows.

**Definition 8.** The adjacency matrices \( (A^c)_{c=1}^C \) of a loop-signed graph with vertices 1, 2, ..., \( V \) and edge colours 1, 2, ..., \( C \) are the \( V \times V \) matrices with off-diagonal entries

\[
A^c_{ij} = \begin{cases} 
1 & \text{if vertices } i \text{ and } j \text{ are joined by a } c\text{-coloured edge} \\
0 & \text{otherwise,}
\end{cases}
\]

encoding connectivity, and diagonal entries

\[
A^c_i = \begin{cases} 
-1 & \text{if vertex } i \text{ has a } c\text{-coloured loop with sign } D \\
1 & \text{if vertex } i \text{ has a } c\text{-coloured loop with sign } N \\
0 & \text{otherwise,}
\end{cases}
\] (2.3)

encoding boundary conditions.

Note that adjacency matrices are symmetric signed permutation matrices with non-negative off-diagonal entries. Table 2 lists the adjacency matrices of the loop-signed graphs in Figure 2.2(b) with respect to the vertex numbering that is indicated in Figure 2.3. In general, a vertex renumbering gives rise to new adjacency matrices of the form

\[
P A^c P^{-1}
\]

for some permutation matrix \( P \). (2.4)

This corresponds to the trivial case of isometric tiled domains of the following central transplantability criterion.

**Transplantation Theorem.** Let \( M \) and \( \hat{M} \) be tiled domains with Zaremba problems that are described by adjacency matrices \( (A^c)_{c=1}^C \) and \( (\hat{A}^c)_{c=1}^C \), respectively. Then, a transplantation between \( M \) and \( \hat{M} \) with transplantation matrix \( T \) is intertwining if and only if

\[
\hat{A}^c = T A^c T^{-1}
\]

for all edge colours \( c \). (2.5)

In particular, \( M \) and \( \hat{M} \) are transplantable precisely if there exists a real matrix \( T \) satisfying (2.5).
According to Definition 2, we only have to consider inner segments. Using local charts of \( U \), we must show that 

\[
I(\alpha, \beta) = \int_\alpha^0 \int_{-l}^l \cdots \int_{-l}^l \left( \sum_{i=1}^d \frac{\partial \phi}{\partial x_i} - \lambda \psi \right) (x_1, \ldots, x_{d-1}, x_d) \ dx_1 \ldots dx_{d-1} \ dx_d,
\]

we must show that \( I(-l, l) = 0 \). If we apply Fubini’s theorem and integrate by parts for either 

\( \alpha = -l \) and \( \beta < 0 \) or \( \alpha > 0 \) and \( \beta = l \), we obtain

\[
I(\alpha, \beta) = \int_{-l}^l \cdots \int_{-l}^l \left( \frac{\partial \phi}{\partial x_d} \right) (x_1, \ldots, x_{d-1}, \beta) - \left( \frac{\partial \phi}{\partial x_d} \right) (x_1, \ldots, x_{d-1}, \alpha) \ dx_1 \ldots dx_{d-1},
\]

where we used that \( \psi \in C_0^\infty((-l, l)^d) \) and \( (\Delta - \lambda) \psi = 0 \) on \((-l, l)^d \setminus \{x_d = 0\}\). As \( \psi \in C^1((-l, l)^d) \) and \( \psi \in C_0^\infty((-l, l)^d) \), we obtain the desired result

\[
I(-l, 0) = \int_{-l}^l \cdots \int_{-l}^l \left( \frac{\partial \phi}{\partial x_d} \right) (x_1, \ldots, x_{d-1}, 0) \ dx_1 \ldots dx_{d-1} = -I(0, l).
\]
Reflection Principle. Let $B$ be a building block, and let $\partial_R B$ be one of its reflecting segments. If $\varphi \in C^\infty(B^\circ \cup \partial_R B)$, and satisfies $\Delta \varphi = \lambda \varphi$ on $B^\circ$ for some $\lambda \geq 0$, as well as either
\[ \frac{\partial \varphi}{\partial n}|_{\partial_R B} \equiv 0 \quad \text{or} \quad \varphi|_{\partial_R B} \equiv 0, \]
then $\varphi$ can be smoothly extended across $\partial_R B$ by itself (Neumann case) or by $-\varphi$ (Dirichlet case), respectively. That is, if we reflect $B$ in $\partial_R B$, and take $\pm \varphi$ on the pasted copy, then the resulting function is smooth on the interior of the resulting tiled domain consisting of two copies of $B$.

Proof. In view of the Regularity Theorem, it suffices to show that the resulting function is continuously differentiable near $\partial_R B$. If we argue as before, the claim reduces to the statement that if $\varphi \in C^\infty((-l,l)^{d-1} \times (-l,0])$ for some $l > 0$, and $\Delta \varphi = \lambda \varphi$ on $(-l,l)^{d-1} \times (-l,0)$, as well as either
\[ \frac{\partial \varphi}{\partial x_d}|_{x_d=0} \equiv 0 \quad \text{or} \quad \varphi|_{x_d=0} \equiv 0, \]
then $\varphi$ can be smoothly extended to $(-l,l)^d$ by defining
\[ \varphi(x_1, \ldots, x_{d-1}, x_d) = \pm \varphi(x_1, \ldots, x_{d-1}, -x_d) \quad \text{for } x_d > 0. \]
In the Neumann case, we immediately obtain that $\varphi \in C((-l,l)^d)$, as well as $\frac{\partial \varphi}{\partial x_i} \in C((-l,l)^d)$ for $i = 1, 2, \ldots, d - 1$. Since $\frac{\partial \varphi}{\partial x_d}$ vanishes on the hyperplane $\{x_d = 0\}$, we also have $\frac{\partial \varphi}{\partial x_d} \in C((-l,l)^d)$ as desired. In the Dirichlet case, $\varphi$ and therefore also $\frac{\partial \varphi}{\partial x_i}$ vanish on $\{x_d = 0\}$ for $i = 1, 2, \ldots, d - 1$.

The continuity of $\frac{\partial \varphi}{\partial x_d}$ follows from
\[ \frac{\partial \varphi}{\partial x_d}(x_1, \ldots, x_{d-1}, x_d) = \frac{\partial \varphi}{\partial x_d}(x_1, \ldots, x_{d-1}, -x_d) \quad \text{for } x_d > 0. \]

The following uniqueness theorem follows from a classical result by Aronszajn [Aro57].

Unique Continuation Theorem. Let $B$ be a connected building block, and let $\partial_R B$ be one of its reflecting segments. If $\varphi \in C^\infty(B^\circ \cup \partial_R B)$, and satisfies $\Delta \varphi = \lambda \varphi$ on $B^\circ$ for some $\lambda \geq 0$, then any extension of $\varphi$ across $\partial_R B$ to a smooth eigenfunction of $\Delta$ is unique. That is, if we reflect $B$ in $\partial_R B$, then there is at most one function on the pasted copy that extends $\varphi$ to a smooth eigenfunction of $\Delta$ on the interior of the resulting tiled domain consisting of two copies of $B$.

The preceding existence and uniqueness theorems are summarised in the following proposition, which will imply the Transplantation Theorem.

Proposition 10. Let $M$ be a tiled domain with connected building blocks $B_1, B_2, \ldots, B_V$. If $\varphi$ is a solution of the Zaremba problem on $M$ given by the adjacency matrices $(A^c)^V_{C=1}$, then for any $i = 1, 2, \ldots, V$, the restriction $\varphi_i$ of $\varphi$ to $B_i$ has a unique extension to a smooth eigenfunction of $\Delta$ across each reflecting segment $\partial_R B_i \neq \emptyset$, and this extension is the non-vanishing summand of $\sum_j A^c_{ij} \varphi_j$.

Proposition 10 justifies the sign convention (2.3), and it suggests that any transplantable pair of connected tiled domains should posses a transplantation matrix with entries 0 and ±1 only that satisfies (2.4).

Proof of the Transplantation Theorem. We may assume that the underlying building block is connected, otherwise we consider its components separately. We first show that a transplantation is intertwining if its transplantation matrix $T$ satisfies (2.5). In other words, we show that if $\varphi$ solves the Zaremba problem represented by $(A^c)^V_{C=1}$, then $\hat{\varphi} = T(\varphi)$ given by
\[ \hat{\varphi}_i = \sum_k T_{ik} \varphi_k \quad \text{(2.6)} \]
solves the Zaremba problem represented by \((\hat{A}^c)_{c=1}^C\), where \((\varphi_k)\) and \((\hat{\varphi}_i)\) denote the restrictions of \(\varphi\) and \(\hat{\varphi}\) to the blocks of \(M\) and \(\hat{M}\), respectively. We repeatedly regard \(\varphi\) and \(\hat{\varphi}\) as column vectors of their respective restrictions \((\varphi_k)_i\) and \((\hat{\varphi}_i)_i\), in particular, \(\hat{\varphi} = T \varphi\) as vectors.

In order to prove that \(\hat{\varphi} \in C(\hat{M}) \cap C^\infty(M^c \cup \partial_D \hat{M} \cup \partial_N \hat{M})\), it suffices to show that whenever two blocks of \(\hat{M}\) numbered \(i\) and \(j\) share an inner segment corresponding to some edge colour \(c\), that is, if \(\hat{A}^c_{ij} = 1\), then \(\hat{\varphi}_i\) and \(\hat{\varphi}_j\) are smoothly connected. Assumption (2.5) yields
\[
\hat{A}^c \hat{\varphi} = \hat{A}^c T \varphi = T A^c \varphi.
\]

Since \(\hat{A}^c_{ij}\) is the only non-vanishing entry of the \(i\)th row of \(\hat{A}^c\), we have
\[
\hat{\varphi}_j = \hat{A}^c_{ij} \hat{\varphi}_j = (\hat{A}^c \hat{\varphi})_j = (T A^c \varphi)_i = \sum_k T_{ik} \left( \sum_l A^c_{kl} \varphi_l \right).
\]

For each value of \(k\) in (2.6) and (2.7), Proposition 10 says that \(\sum_l A^c_{kl} \varphi_l\) is a smooth extension of \(\varphi_k\), showing that \(\hat{\varphi}_i\) and \(\hat{\varphi}_j\) are smoothly connected. In order to prove that \(\hat{\varphi}\) satisfies the boundary conditions given by \((\hat{A}^c)_{c=1}^C\), it suffices to consider outer segments of \(\hat{M}\). Assume that the \(i\)th block \(B_i\) has an outer reflecting segment \(\partial_D B_i\) carrying Neumann or Dirichlet boundary conditions, that is, \(A^c_{ii} = \pm 1\). Arguing as before, we obtain that
\[
\pm \hat{\varphi}_i = \hat{A}^c_{ii} \hat{\varphi}_i = (\hat{A}^c \hat{\varphi})_i = (T A^c \varphi)_i = \sum_k T_{ik} \left( \sum_l A^c_{kl} \varphi_l \right)
\]
extends \(\hat{\varphi}_i\) smoothly across \(\partial_D B_i\). As the resulting function is symmetric, respectively antisymmetric, with respect to reflection in \(\partial_D B_i\), we see that \(\frac{\partial \hat{\varphi}_i}{\partial n}\), respectively \(\hat{\varphi}_i\), has to vanish along \(\partial_D B_i\). Finally, note that since \(A^c = T^{-1} \hat{A}^c (T^{-1})^{-1}\) for all edge colours \(c\), the given arguments also apply to the inverse transplantation. Hence, the transplantation given by \(T\) is intertwining.

In the following, we show that an intertwining transplantation satisfies (2.5). If \(\varphi\) is a solution of the Zaremba problem represented by \((A^c)_{c=1}^C\), then \(\hat{\varphi} = T(\varphi)\) with restrictions \(\hat{\varphi}_i = \sum_k T_{ik} \varphi_k\) solves the Zaremba problem represented by \((\hat{A}^c)_{c=1}^C\). If we apply Proposition 10 twice, we obtain that \(\hat{\varphi}_i\) is smoothly extended by
\[
\sum_j \hat{A}^c_{ij} \hat{\varphi}_j\]
as well as
\[
\sum_k T_{ik} \left( \sum_l A^c_{kl} \varphi_l \right),
\]
which have to coincide according to the Unique Continuation Theorem. In particular,
\[
\hat{A}^c T \varphi = \hat{A}^c \hat{\varphi} = T A^c \varphi
\]
for any solution \(\varphi\) of the Zaremba problem on \(M\). We show that this implies that \(\hat{A}^c T = T A^c\).

Each row \((a_{11}, a_{12}, \ldots, a_{1V})\) of \(\hat{A}^c T - T A^c\) may be identified with an \(L^2\)-function \(\phi^i\), whose restriction to the interior of block \(j\) of \(M\) is equal to \(a_{ij}\). Then, (2.8) says that \(\phi^i\) is \(L^2\)-orthogonal to all solutions of the Zaremba problem on \(M\). Assumption A implies that \(\phi^i = 0\), hence, \(a_{i1} = a_{i2} = \ldots = a_{iV} = 0\), which completes the proof.

Intertwining transplantations do not superpose Dirichlet with Neumann boundary conditions in the following sense. If the \(4k\)th block of \(M\) and the \(4k\)th block of \(\hat{M}\) have outer segments that correspond to the same edge colour \(c\), but that carry different boundary conditions, that is, if \(A^c_{kk} A^c_{ii} = -1\), then \(T_{ik} = 0\) since (2.5) implies
\[
T_{ik} A^c_{kk} = (T A^c)_{ik} = (\hat{A}^c T)_{ik} = \hat{A}^c_{ii} T_{ik}.
\]
In [OS01], Okada and Shudo study the relation between isospectrality and isolegth spectrality, and obtain a special case of the following theorem.
Trace Theorem. Two loop-signed graphs with adjacency matrices \((A^c)_{c=1}^C\) and \((\hat{A}^c)_{c=1}^C\) are transplantable if and only if for all finite sequences \(c_1 c_2 \ldots c_t\) of edge colours
\[
\text{Tr}(\hat{A}^{c_1} \cdots \hat{A}^{c_t} \hat{A}^{c_t}) = \text{Tr}(A^{c_1} \cdots A^{c_t} A^{c_t}).
\] (2.9)

Proof. The proof in [OS01] extends to our setting. In order to show that (2.9) implies transplantability, we consider the groups
\[
G = \langle A^1, A^2, \ldots, A^C \rangle \quad \text{and} \quad \hat{G} = \langle \hat{A}^1, \hat{A}^2, \ldots, \hat{A}^C \rangle.
\] (2.10)
Since the empty word \(\emptyset\) is associated with the identity matrices \(I_V \in G\) and \(I_{\hat{V}} \in \hat{G}\), each of the graphs has \(V = \hat{V}\) vertices. Note that \(G\) and \(\hat{G}\) are finite as they act faithfully on
\[
\{e_1, e_2, \ldots, e_V, -e_1, -e_2, \ldots, -e_V\},
\]
where \(e_i\) denotes the \(i\)th standard basis vector of \(\mathbb{R}^V\). We let \(F^C\) denote the free group generated by the letters \(1, 2, \ldots, C\), and consider the surjective homomorphism \(\Phi : F^C \to G\) given by
\[
\Phi(c_1^{\pm 1} e_2^{\pm 1} \cdots e_t^{\pm 1}) = A^{c_1} \cdots A^{c_t} A^{c_t}.
\]
We define \(\tilde{\Phi} : F^C \to \hat{G}\) analogously. Since \(I_V\) is the only element with trace \(V\) in \(G\) and in \(\hat{G}\),
\[
\ker(\Phi) = \{w \in F^C \mid \text{Tr}(\Phi(w)) = V\} = \{w \in F^C \mid \text{Tr}(\tilde{\Phi}(w)) = V\} = \ker(\tilde{\Phi}).
\]
Hence, \(\mathcal{I} : G \to \hat{G}\) given by \(\mathcal{I}(\Phi(w)) = \tilde{\Phi}(w)\) for \(w \in F^C\) defines an isomorphism. We obtain two linear representations of \(G\),
\[
id_G : G \to GL(\mathbb{C}^V) \quad \text{and} \quad id_{\hat{G}} \circ \mathcal{I} : G \to GL(\mathbb{C}^\hat{V}),
\]
where \(id_G\) and \(id_{\hat{G}}\) are the identity maps on \(G\) and \(\hat{G}\), respectively. Their characters are the maps \(\Phi(w) \mapsto \text{Tr}(\Phi(w))\) and \(\tilde{\Phi}(w) \mapsto \text{Tr}(\tilde{\Phi}(w))\) for \(w \in F^C\), which are equal by (2.9). Thus, \(id_G\) and \(id_{\hat{G}} \circ \mathcal{I}\) are equivalent, and there exists \(T \in GL(\mathbb{C}^V)\) such that
\[
TA^c = T \Phi(c) = \tilde{\Phi}(c) T = \hat{A}^c T \quad \text{for } c = 1, 2, \ldots, C.
\]
For \(z \in \mathbb{C}\), let \(T(z) = \text{Re}(T) + z \text{Im}(T)\), and note that \(T(z) A^c = \hat{A}^c T(z)\) for \(c = 1, 2, \ldots, C\). Since \(\det(T(i)) \neq 0\), the mapping \(z \mapsto \det(T(z))\) defines a non-zero polynomial. Hence, we can choose \(r \in \mathbb{R}\) such that \(T(r) \in GL(\mathbb{R}^V)\), which completes the proof. \(\square\)

Thas et al. [Tha06] [ST11] partially classified the isomorphism classes of groups appearing in (2.10) for transplantable treelike graphs with homogeneous loop signs. Recall that transplantable domains have equal heat invariants. For this reason, they share certain geometric properties [BGK99] [Dow05] [LPP06], some of which can be identified with expressions of the form (2.9):

- \(\text{Tr}(\mathcal{I})\) encodes the number of building blocks, that is, a tiled domain’s volume.
- \(\text{Tr}(A^c)\) encodes the difference of the Neumann boundary volume and the Dirichlet boundary volume coming from outer segments that correspond to \(c\)-coloured loops.
- For tiled domains \(M\) that are polygons, the quantity
\[
\sum_{DD} \frac{\pi^2 - \alpha^2}{\alpha} + \sum_{NN} \frac{\pi^2 - \alpha^2}{\alpha} - \frac{1}{2} \sum_{DN} \frac{\pi^2 + 2 \alpha^2}{\alpha}.
\] (2.11)
is a spectral invariant, where the sums are taken over all corners of \(\partial M\) formed by Dirichlet-Dirichlet (DD), Neumann-Neumann (NN), and Dirichlet-Neumann (DN) sides, respectively; in each case, \(\alpha\) is the corresponding angle [LPP06, Theorem 5.1]. Let \(e_1\) and \(e_2\) be two edge colours corresponding to neighbouring sides of the underlying building block, enclosing the angle \(\beta\). If \(C_{DD}, C_{NN}\) and \(C_{DN}\) denote the number of corners of \(M\) that
are formed by sides corresponding to $c_1$ and $c_2$, respectively, carrying Dirichlet-Dirichlet, Neumann-Neumann, and Dirichlet-Neumann boundary conditions, then the contribution of these corners to (2.11) is
\[
\frac{\pi^2}{\beta} \left( C_{DD} + C_{NN} - \frac{1}{2} C_{DN} \right) - \beta \left( C_{DD} + C_{NN} + C_{DN} \right).
\]
If the underlying graph is treelike, then this quantity is determined by
\[
\text{Tr}(A^{c_1} A^{c_2}) = C_{DD} + C_{NN} - C_{DN} \quad \text{and} \quad \text{Tr}(A^{c_1} A^{c_2} A^{c_1} A^{c_2}) = C_{DD} + C_{NN} + C_{DN}.
\]

The Trace Theorem allows for a computer-aided search for transplantable pairs. More precisely, one first creates one representative of each isomorphism class of loop-signed graphs with a given number of vertices and edge colours, and then sorts them according to finitely many expressions of the form (2.9).

**Proposition 11.** If $(A^{c_1})^C_{c=1}$ and $(\tilde{A}^{c_1})^C_{c=1}$ are the adjacency matrices of two loop-signed graphs each of which has $V$ vertices, then
\[
\text{Tr}(\tilde{A}^{c_1} \ldots \tilde{A}^{c_2} \tilde{A}^{c_1}) = \text{Tr}(A^{c_1} \ldots A^{c_2} A^{c_1}) \tag{2.12}
\]
holds for all words $c_1c_2\ldots c_l$ in the edge colours if it holds for all words with length $l \leq (2V)!$.

**Proof.** Recall that $G = \langle A^1, A^2, \ldots, A^C \rangle$ can be identified with a subgroup of the symmetric group on $2V$ elements. If $L > (2V)!$, then any word $c_1c_2\ldots c_L$ contains a subword $c_{k+1}c_{k+2}\ldots c_{k+l}$ with $0 < l < L$ such that $A^{c_{k+1}} \ldots A^{c_{k+l}} + A^{c_{k+2}} A^{c_{k+1}} = I_V$. Assuming that (2.12) holds for words up to length $L - 1$, we see that $\tilde{A}^{c_{k+1}} \ldots \tilde{A}^{c_{k+l}} \tilde{A}^{c_{k+1}} = I_V$. Hence, $c_1c_2\ldots c_L$ satisfies (2.12) if $c_1c_2\ldots c_kc_{k+1}c_{k+1}c_{k+2}\ldots c_L$ does, and the statement follows by induction.

The minimal upper bound on the necessary length of words is not known in general. The non-transplantable graphs in Figure 2.4 show that it can be greater than the number of vertices.
3 Transplantability and Induced Representations

In order to characterise transplantability in group-theoretic terms, we recall Sunada’s method \cite{Sun85}.

Definition 12. A triple \((G, H, \hat{H})\) consisting of a finite group \(G\) with subgroups \(H\) and \(\hat{H}\) is called a Gassmann triple if each conjugacy class \([g] \subset G\) satisfies
\[
| [g] \cap H | = | [g] \cap \hat{H} |.
\]

Theorem. \((\text{Sun85})\) If \((G, H, \hat{H})\) is a Gassmann triple such that \(G\) acts freely on some closed Riemannian manifold \(M\) by isometries, then \(M/H\) and \(M/\hat{H}\) are isospectral.

We recall that the Gassmann criterion \((3.1)\) is equivalent to the condition that the induced representations \(\text{Ind}^G_H(1_H)\) and \(\text{Ind}^G_{\hat{H}}(1_{\hat{H}})\) of the trivial representations \(1_H\) and \(1_{\hat{H}}\) of \(H\) and \(\hat{H}\) are equivalent \cite{Bro99}. Recently, Baud et al. \cite{BPBS09, PB10} extended the Sunada method to domains with mixed boundary conditions using non-trivial one-dimensional real representations. In essence, they showed that if \(G\) is a finite group of isometries of some tiled domain \(M\) mapping \(\partial_D M\) and \(\partial_N M\) to themselves, and if \(H\) and \(\hat{H}\) are subgroups of \(G\) with one-dimensional real representations \(R\) and \(\hat{R}\) such that \(\text{Ind}^G_H(R) \simeq \text{Ind}^G_{\hat{H}}(\hat{R})\), then \(R\) and \(\hat{R}\) give rise to mixed boundary conditions on \(M/H\) and \(M/\hat{H}\) turning them into isospectral manifolds. They considered the action of \(G\) on \(L^2(M)\), and showed that \(R\) and \(\hat{R}\) can be used to single out spaces of solutions of the Zaremba problem on \(M\) with a particular transformation behaviour with respect to crossings between fundamental domains of the actions of \(H\) and \(\hat{H}\) on \(M\). As an example, take the square \(S\) in Figure 3.1a carrying Dirichlet boundary conditions. Its isometry group is the dihedral group \(G = D_4 = \{ e, \sigma, \sigma^2, \sigma^3, \tau, \tau \sigma, \tau \sigma^2, \tau \sigma^3 \}\), where \(\tau\) and \(\sigma\) denote vertical reflection and rotation by \(\pi/2\), respectively. It has subgroups
\[
H = \{ e, \tau, \tau \sigma^2, \tau^2 \}, \quad \hat{H} = \{ e, \tau \sigma, \tau \sigma^3, \tau^2 \}
\]
with representations
\[
R : \{ e \mapsto 1, \tau \mapsto -1, \tau^2 \mapsto 1, \tau^3 \mapsto -1 \}, \quad \hat{R} : \{ e \mapsto 1, \tau \sigma \mapsto 1, \tau^3 \sigma \mapsto -1, \tau^2 \sigma \mapsto -1 \},
\]
such that \(\text{Ind}^G_H(R) \simeq \text{Ind}^G_{\hat{H}}(\hat{R})\). The domains \(S/R\) and \(S/\hat{R}\) in Figure 3.1a are fundamental domains for the actions of \(H\) and \(\hat{H}\) on \(S\), and their boundary conditions are determined by \(R\) and \(\hat{R}\) as follows. Using a slight generalisation of Proposition 10, one sees that each solution of the Zaremba problem on \(S/R\) gives rise to a solution of the Zaremba problem on \(S\) which transforms according to \(R\) and vice versa, that is, a solution which is horizontally symmetric (\(\tau^2 \mapsto 1\)) and vertically antisymmetric (\(\tau \mapsto -1\)); similarly for \(S/\hat{R}\) and \(\hat{R}\). Using Frobenius reciprocity, one can show that the existence of this mapping of solutions, together with \(\text{Ind}^G_H(R) \simeq \text{Ind}^G_{\hat{H}}(\hat{R})\), implies transplantability of \(S/R\) and \(S/\hat{R}\) \cite[Corollary 7.6]{BPBS09}. The isospectrality of these domains had been established by explicit computation beforehand \cite{LP09}. We complete these developments and derive the following characterisation of transplantability.

Theorem 13. Each pair of transplantable loop-signed graphs gives rise to a triple
\[
(G, ((H_i, R_i)_{i=1}^s, (\hat{H}_j, \hat{R}_j)_{j=1}^t))
\]
consisting of a finite group \(G\) together with finitely many pairs \((H_i, R_i)\) and \((\hat{H}_j, \hat{R}_j)\), where \(H_i\) and \(\hat{H}_j\) are subgroups of \(G\) of which is equipped with a one-dimensional real representation \(R_i\) and \(\hat{R}_j\), respectively, such that
\[
\bigoplus_i \text{Ind}^G_{H_i}(R_i) \simeq \bigoplus_j \text{Ind}^G_{\hat{H}_j}(\hat{R}_j).
\]

Moreover, if the graphs have adjacency matrices \((A^C)_{C=1}^s\) and \((\hat{A}^C)_{C=1}^t\), respectively, then they are isomorphic to the following unions of Schreier coset graphs defined as below
\[
\bigcup_i \Gamma^{\text{Cas}}(G, (A^C)_{C=1}^s) / R_i \quad \text{and} \quad \bigcup_j \Gamma^{\text{Cas}}(G, (\hat{A}^C)_{C=1}^t) / \hat{R}_j,
\]
that is, they can be recovered from \(G, ((H_i, R_i)_{i=1}^s, (\hat{H}_j, \hat{R}_j)_{j=1}^t)\) up to isomorphism.
In contrast to [BPBS09, PB10], we also consider triples of the form (3.3) where $G$ is not given as a group of isometries of some covering manifold. Instead, we use the group structure of $G$ to construct such a cover, and obtain quotients which are transplantable precisely if (3.1) holds.

In the following, let $\Gamma = \bigcup \Gamma_i$ and $\hat{\Gamma} = \bigcup \hat{\Gamma}_j$ be a pair of transplantable loop-signed graphs with edge colours $1, 2, \ldots, C$, vertices $1, 2, \ldots, V$, and connected components $(\Gamma_i)_i$ and $(\hat{\Gamma}_j)_j$, respectively. We let $(A^c)_{c=1}^C$ and $(\hat{A}^c)_{c=1}^C$ denote their $V \times V$ adjacency matrices, and consider the groups

$$G = \langle A^1, A^2, \ldots, A^C \rangle \quad \text{and} \quad \hat{G} = \langle \hat{A}^1, \hat{A}^2, \ldots, \hat{A}^C \rangle.$$

Note that each $\Gamma_i$ gives rise to an invariant subspace of the action of $G$ on $\mathbb{R}^V$, similarly for $(\hat{\Gamma}_j)_j$ and $\hat{G}$. By assumption, there exists a transplantation matrix $T \in GL(\mathbb{R}^V)$ such that

$$\hat{A}^c = TA^c T^{-1} \quad \text{for } c = 1, 2, \ldots, C.$$

In particular, the conjugation map $\mathcal{I} : G \to \hat{G}$ given by

$$\mathcal{I}(A^{e_1} \cdots A^{e_2} A^{e_3}) = \hat{A}^{e_1} \cdots \hat{A}^{e_2} \hat{A}^{e_3}$$

is an isomorphism. The main observation is that $G$ and $\hat{G}$ act on the following set of two-sets

$$\{\{e_1, -e_1\}, \{e_2, -e_2\}, \ldots, \{e_V, -e_V\}\},$$

where $e_i$ denotes the $i$th standard basis vector of $\mathbb{R}^V$. The point stabiliser subgroups of these actions will turn out to encode $\Gamma$ and $\hat{\Gamma}$, respectively. Let $(v_i)_i$ and $(\hat{v}_j)_j$ be the respective smallest vertices in $(\Gamma_i)_i$ and $(\hat{\Gamma}_j)_j$, in particular, $v_i$ is in $\Gamma_i$ and $\hat{v}_j$ is in $\hat{\Gamma}_j$ for each $i$ and $j$.

**Definition 14.** The associated pairs $((H_i, R_i)_i)$ and $((\hat{H}_j, \hat{R}_j)_j)_j$, consisting of subgroups $\{H_i\}_i$ and $\{\hat{H}_j\}_j$ of $G$ and respective one-dimensional representations $\{R_i\}_i$ and $\{\hat{R}_j\}_j$, are defined as

$$H_i = G_{\{e_{v_i}, -e_{v_i}\}} = \{ g \in G \mid g_{v_i, v_i} = \pm 1 \} \quad \quad R_i(g) = g_{v_i, v_i}$$

$$\hat{H}_j = \mathcal{I}^{-1}(\hat{G}_{\{e_{\hat{v}_j}, -e_{\hat{v}_j}\}}) = \{ g \in G \mid (\mathcal{I}(g))_{\hat{v}_j, \hat{v}_j} = \pm 1 \} \quad \quad \hat{R}_j(g) = (\mathcal{I}(g))_{\hat{v}_j, \hat{v}_j}.$$

Note that for any $v$ in $\Gamma$, with $v \neq v_i$, there exists $g \in G$ with $g_{v_i, v} = 1$, and $G_{\{e_{v_i}, -e_{v_i}\}} = g^{-1}H_i g$. In other words, the point stabiliser subgroups are conjugated in $G$. The orbit-stabiliser theorem implies that the index $[G : H_i]$ of $H_i$ in $G$ equals the number of vertices of $\Gamma_i$. Similar statements hold for $(\hat{\Gamma}_j)_j$ and $(\hat{H}_j)_j$.

**Definition 13** has the following geometrical motivation. Let $\Gamma^{DC}$ be the double cover of $\Gamma$ obtained by taking copies $\Gamma^+$ and $\Gamma^-$ of the loopless version of $\Gamma$, and joining their respective $i$th vertices with a $c$-coloured edge whenever $\Gamma$ has a $c$-coloured Dirichlet loop at its $i$th vertex. Note that $G$ entails a faithful permutation action on the vertices of $\Gamma^{DC}$, and we may interpret products of adjacency matrices of $\Gamma$ as walks on $\Gamma^{DC}$. According to the Reflection Principle, each solution.
The following theorem is the converse of the method of Band et al. \cite{BPS09, PB10}.

\textbf{Theorem 15.} Let $\Gamma = \bigsqcup_i \Gamma_i$ and $\tilde{\Gamma} = \bigsqcup_i \tilde{\Gamma}_j$ be transplantable loop-signed graphs as above.

1. The representation $id_G : G \to GL(C^\Gamma)$ is equivalent to $\bigoplus_i \text{Ind}_{H_i}^G(R_i)$.
2. The representation $id_{G'} \circ \mathcal{Z} : G \to GL(C^\Gamma)$ given by $A^c_1 \cdots A^c_2 A^c_1 \mapsto \tilde{A}^c_1 \cdots \tilde{A}^c_2 \tilde{A}^c_1$ is equivalent to $\bigoplus_j \text{Ind}_{H_j}^{G'}(\tilde{R}_j)$.  
3. We have
\[ \bigoplus_i \text{Ind}_{H_i}^G(R_i) \simeq \bigoplus_j \text{Ind}_{H_j}^{G'}(\tilde{R}_j) \]

Theorem 15 follows from Theorem 18 and Theorem 20 below. In order to reconstruct $\Gamma$ and $\tilde{\Gamma}$ from the data in (3.6), we consider the action of $G$ on its Cayley graph.

\textbf{Definition 16.} The Cayley graph $\Gamma^\text{Cay}(G, (\gamma^c)_{c=1}^C)$ of a finite group $G$, which is generated by elements $(\gamma^c)_{c=1}^C$ of order 2, is the edge-coloured graph with edge colours 1, 2, \ldots, $C$ and $|G|$ vertices which are identified with the elements of $G$, and the vertices $g, g' \in G$ are joined by a $c$-coloured edge if and only if $g' = g \gamma^c$.

As an example, Figure 3.2a shows $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))$. The striking similarity with the square $S$, on which $D_4$ acts by isometries, originates from the action of $D_4$ on $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))$. More precisely, each $g \in D_4$ entails an isomorphism of $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))$ by mapping $h \in D_4$ to $gh$. Note that, in general, two vertices $g h_1, g h_2 \in G$ of $\Gamma^\text{Cay}(G, (\gamma^c)_{c=1}^C)$ are joined by a $c$-coloured edge if and only if the same holds for $h_1, h_2 \in G$ since $g h_1 = g h_2 \gamma^c$ if and only if $h_1 = h_2 \gamma^c$. In Figure 3.2a the actions of $\tau$ and $\tau \sigma^2$ on $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))$ correspond exactly to their former actions on $S$, that is, vertical and horizontal reflection. The pairs $(H, R)$ and $(\tilde{H}, \tilde{R})$ in (3.2) give rise to quotients of $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))$ defined as follows.

\textbf{Definition 17.} Let $G$ be a finite group generated by elements $(\gamma^c)_{c=1}^C$ of order 2, and let $H$ be a subgroup of $G$ with one-dimensional real representation $R$. The Schreier coset graph $\Gamma^\text{Cay}(G, (\gamma^c)_{c=1}^C)/R$ is the isomorphism class of loop-signed graphs with edge colours 1, 2, \ldots, $C$ and $|G : H|$ vertices which are identified with the right cosets of $H$. Moreover, the vertices $Hg$ and $Hg'$ are joined by a $c$-coloured edge if and only if $Hg' = Hg \gamma^c$, and $Hg$ has a $c$-coloured loop carrying a Neumann or a Dirichlet sign, respectively, if and only if $Hg = Hg \gamma^c$ and $\tilde{R}(g \gamma^c g^{-1}) = \pm 1$.

Although we defined Cayley and Schreier coset graphs only up to isomorphism, we use the notation $\Gamma^\text{Cay}(G, (\gamma^c)_{c=1}^C)/R$ to mean any of its isomorphic incarnations. Figure 3.2a shows $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))/R$ and $\Gamma^\text{Cay}(D_4, (\tau, \tau \sigma^3))/\tilde{R}$, where $R$ and $\tilde{R}$ are given by (3.2). These graphs underlie the transplantable broken square and triangle in Figure 3.11 which can be explained as follows.

\textbf{Theorem 18.} Let $\Gamma = \bigsqcup_i \Gamma_i$ and $\tilde{\Gamma} = \bigsqcup_i \tilde{\Gamma}_j$ be transplantable loop-signed graphs with $V$ vertices and adjacency matrices $(A^c)_{c=1}^V$ and $(\tilde{A}^c)_{c=1}^V$ generating $G$ and $\tilde{G}$, respectively. If $((H_i, R_i))$ and $((\tilde{H}_j, \tilde{R}_j))$ denote the associated pairs given by (3.6), then the graphs $\bigsqcup_i \Gamma^\text{Cay}(G, (A^c)_{c=1}^V)/R_i$ and $\bigsqcup_j \Gamma^\text{Cay}(G, (A^c)_{c=1}^V)/\tilde{R}_j$ are isomorphic to $\Gamma$ and $\tilde{\Gamma}$, respectively. More precisely, for each $i$ and $j$
\[ \Gamma_i \simeq \Gamma^\text{Cay}(G, (A^c)_{c=1}^V)/R_i \quad \text{and} \quad \tilde{\Gamma}_j \simeq \Gamma^\text{Cay}(G, (A^c)_{c=1}^V)/\tilde{R}_j. \]  

\textbf{Proof.} It suffices to prove (3.7). In order to simplify notation, we fix some $\Gamma_i$ and assume it has vertices $\{1, 2, \ldots, V_i\}$, in particular, $v_1 = 1$ and $H_i = \{g \in G \mid g v_1 = \pm 1\}$. The orbit-stabiliser theorem implies that $[G : H_i] = V_i$, and for each $k \in \{1, 2, \ldots, V_i\}$ there exists $g_k \in G$ such that $g_k e_1 = e_1$. The elements $(g_k)_{k=1}^{V_i}$ yield a system of representatives for the right cosets of $H_i$ since $H_i g_k \cap H_i g_l \neq \emptyset$ implies $g_k^{-1} g_l \in H_i = \{g \in G \mid g e_1 = \pm e_1\}$, that is, $g_k^{-1} e_1 = \pm g_k^{-1} e_1$, which
If the following theorem implies Theorem 13, and reveals the group-theoretic nature of transplantability. (that is bipartite with respect to \( \gamma \) renumbering of its vertices. The statement \( \hat{A} \) consisting of a subgroup or Dirichlet loop if and only if \( g_k A^c e_l = \pm e_1 \), that is, if \( A^c_{kl} = \pm 1 \). Moreover, the vertex \( H_i g_k \) has a \( c \)-coloured Neumann or Dirichlet loop if and only if \( g_k A^c g_k^{-1} \in H_i \) and \( R_i (g_k A^c g_k^{-1}) = \pm 1 \), which is equivalent to \( g_k A^c e_l = g_k A^c e_k = \pm e_1 \), that is, \( A^c_{kl} = \pm 1 \). Hence, the first \( V_i \times V_i \) block of \( A^c \) coincides with the adjacency matrix corresponding to the edge colour \( c \) of \( \Gamma^{Cay}(G, (A^c)_{C_i}) / R_i \) up to a renumbering of its vertices. The statement \( \hat{\Gamma}_j \simeq \Gamma^{Cay}(G, (A^c)_{C_i}) / \hat{R}_j \) is proved along the same lines using the isomorphism \( \hat{\sigma} \).

As an example, start with the transplantable graphs in Figure 3.2. Table 3 lists their adjacency matrices, as well as the associated permutations on \( \{e_1, e_2, -e_1, -e_2\} \).

| \( A^{\text{straight}} \) | \( A^{\text{zig-zag}} \) | \( T = 2T^{-1} \) | \( \hat{A}^{\text{straight}} \) | \( \hat{A}^{\text{zig-zag}} \) |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| \((-1, 0) \) \( (0, 1) \) | \((-1, 0) \) \( (1, 1) \) | \((0, 1) \) \( (1, 0) \) | \((-1, 0) \) \( (0, 1) \)
| \((1, 3)(2)(4) \) \( (1, 2)(3, 4) \) | \((1, 2)(3, 4) \) \( (1, 3)(2)(4) \) |
| \( \tau \) \( \tau^3 \) | \( \tau \) \( \tau^3 \) |

Table 3: The graphs in Figure 3.2 in terms of their adjacency matrices and associated permutations on \( \{e_1, e_2, -e_1, -e_2\} \).

As an example, start with the transplantable graphs in Figure 3.2. Table 3 lists their adjacency matrices, as well as the associated permutations on \( \{e_1, e_2, -e_1, -e_2\} \). For instance, \( A^{\text{straight}} \) interchanges \( e_1 \) and \( -e_1 \), and maps \( e_2 \) and \( -e_2 \) to themselves, hence, it gives rise to \((1, 3)(2)(4)\). If we label the midpoints of the sides of \( S \) as indicated in Figure 3.1a, then \((1, 3)(2)(4)\) can be identified with the reflection \( \tau \) of \( S \) as noted in the last line of Table 3. We see that the group generated by \( A^{\text{straight}} \) and \( A^{\text{zig-zag}} \) is \( D_4 \). Moreover, the associated pairs \((H_1, R_1)\) and \((\hat{H}_1, \hat{R}_1)\) given by (3.6) coincide with \((H, R)\) and \((\hat{H}, \hat{R})\) in (3.2). In accordance with Theorem 18 \( \Gamma^{Cay}(D_4, (\tau, \tau^3)) / R \) and \( \Gamma^{Cay}(D_4, (\tau, \tau^3)) / \hat{R} \) are the graphs we started with.

As we are solely interested in the spectrum of the Laplace operator on the trivial bundle, we make the following additional assumption.

**Definition 19.** Let \( G \) be a finite group generated by elements \((\gamma^c)_{C_i=1}\) of order 2. A pair \((H, R)\) consisting of a subgroup \( H \) of \( G \), and a one-dimensional real representation \( R \) of \( H \) is called bipartite with respect to \((\gamma^c)_{C_i=1}\) if the right cosets of \( H \) have a system of representatives \( \langle g_i \rangle_{i=1}^{[G:H]} \) such that whenever \( i \neq j \) and \( H g_i \gamma^c = H g_j \), we have \( R(g_i \gamma^c g_j^{-1}) = 1 \).

Note that the associated pairs in Definition 14 are bipartite with respect to the adjacency matrices generating \( G \) precisely because off-diagonal entries of adjacency matrices are non-negative. The following theorem implies Theorem 13 and reveals the group-theoretic nature of transplantability.

**Theorem 20.** If \( G \) is a finite group with generators \((\gamma^c)_{C_i=1}\) of order 2, and if \((H, R)\) is a pair that is bipartite with respect to \((\gamma^c)_{C_i=1}\), then the adjacency matrices of \( \Gamma^{Cay}(G, (\gamma^c)_{C_i=1}) / R \) yield a representation of \( G \) which is equivalent to \( \text{Ind}^G_H(R) \). In particular, if \(((H_i, R_i))_i\) and \(((\hat{H}_j, \hat{R}_j))_j\)
are tuples of pairs that are bipartite with respect to \((\gamma^c)^C_{c=1}\), then \(\bigcup_i \Gamma^{C_{c=1}}(G, (\gamma^c)^C_{c=1})/R_i\) and
\(\bigcup_j \Gamma^{C_{c=1}}(G, (\gamma^c)^C_{c=1})/\bar{R}_j\) are transplantable if and only if \(\bigoplus_i \text{Ind}^G_H(R_i) \simeq \bigoplus_j \text{Ind}^G_H(\bar{R}_j).

Proof. It suffices to prove the first statement. Let \(V = [G : H]\) and choose a system of representatives \((g_i)^V_{i=1}\) for the cosets of \(H\) such that whenever \(i \neq j\) and \(Hg_i \gamma^c = Hg_j\), then \(R(g_i \gamma^c g_j^{-1}) = 1\).

If we number the vertices of \(\Gamma^{C_{c=1}}(G, (\gamma^c)^C_{c=1})/R\) according to \((g_i)^V_{i=1}\), then its \(V \times V\) adjacency matrices \((A^c)^C_{c=1}\) read
\[
A^c_{ij} = \begin{cases} R(g_i \gamma^c g_j^{-1}) & \text{if } Hg_i \gamma^c = Hg_j \\ 0 & \text{otherwise}. \end{cases}
\]

In the following, we show that the map
\[
\Phi : \gamma^{c_1} \cdots \gamma^{c_l} \mapsto A^{c_1} \cdots A^{c_l}
\]
(3.8)
yields a well-defined homomorphism \(\Phi : G \rightarrow GL(\mathbb{R}^V)\), that is, a representation. More precisely, we prove by induction that for any \(p = \gamma^{c_1} \cdots \gamma^{c_l} \gamma^{c_1}\), definition (3.8) leads to
\[
\Phi(p)_{ij} = \begin{cases} R(g_i p g_j^{-1}) & \text{if } Hg_i p = Hg_j \\ 0 & \text{otherwise}. \end{cases}
\]
(3.9)
Assume that (3.9) holds for any \(p = \gamma^{c_1} \cdots \gamma^{c_l} \gamma^{c_1}\) with \(l \leq L\). For arbitrary \(\gamma^c\), we get
\[
\Phi(\gamma^c p)_{ij} = (\Phi(\gamma^c) \Phi(p))_{ij} = \sum_{k=1}^V \Phi(\gamma^c)_{ik} \Phi(p)_{kj}.
\]
Assume that \(\Phi(p)m_j\) is the non-vanishing entry in the \(j\)th column of \(\Phi(p)\), that is, \(\Phi(g_m p hazards Hg_j) \Phi(p)m_j = R(g_m p g_j^{-1})\). Then, \(\Phi(\gamma^c p)\) is if and only if \(A^{c_1} = \Phi(\gamma^c)\) \(\neq 0\), that is, if \(Hg_i \gamma^c = Hg_m\), and in this case \(\Phi(\gamma^c) = R(g_i \gamma^c g_m^{-1})\). Since \(g_m p g_j^{-1} \in H\), this is equivalent to
\[
g_i \gamma^c g_m^{-1} g_m p g_j^{-1} = g_i \gamma^c p g_j^{-1} \in H,
\]
and in this case
\[
\Phi(\gamma^c p)_{ij} = \Phi(\gamma^c)_{im} \Phi(p)m_j = R(g_i \gamma^c g_m^{-1}) R(g_m p g_j^{-1}) = R(g_i \gamma^c p g_j^{-1}),
\]
where we used that \(R\) is a homomorphism. Hence, (3.9) follows by induction on \(L\).

It remains to show that \(\Phi\) and \(\text{Ind}^G_H(R)\) have equal characters. For arbitrary \(p \in G\), let \(B^\pm(p) = \{i \in \{1, 2, \ldots, V\} \mid g_i p g_i^{-1} \in H\text{ and } R(g_i p g_i^{-1}) = \pm 1\}\).

According to (3.9), the character \(\chi_\Phi\) of \(\Phi\) satisfies
\[
\chi_\Phi(p) = \text{Tr}(\Phi(p)) = |B^+(p)| - |B^-(p)|.
\]
The character of \(\text{Ind}^G_H(R)\) reads [Ser77] Chapter 3, Theorem 12
\[
\chi_{\text{Ind}^G_H(R)}(p) = \frac{1}{|H|} \sum_{g \in G} \sum_{g p g^{-1} \in H} R(g p g^{-1}).
\]
Since any \(g \in G\) can be uniquely written as \(g = h g_i\) for some \(h \in H\) and \(i \in \{1, 2, \ldots, V\}\), the condition \(g p g^{-1} \in H\) is equivalent to \(g = Hg_i\) for some \(i \in B^+(p) \cup B^-(p)\). Moreover, if \(g = h g_i\) with \(h \in H\) and \(i \in B^+(p)\), then
\[
R(g p g^{-1}) = R(h g_i p g_i^{-1} h^{-1}) = R(g_i p g_i^{-1}) = \pm 1.
\]
Hence,
\[
\chi_{\text{Ind}^G_H(R)}(p) = \frac{1}{|H|} \left( \sum_{i \in B^+(p)} \sum_{g \in Hg_i} 1 + \sum_{i \in B^-(p)} \sum_{g \in Hg_i} -1 \right) = |B^+(p)| - |B^-(p)| = \chi_\Phi(p),
\]
which completes the proof.
4 Generating Tools

The Transplantation Theorem converts the problem of constructing transplantable and therefore isospectral manifolds into a combinatorial question involving linear algebra only. As a result, it allows to derive tools with which new transplantable pairs can be generated from given ones. In the following, let $\Gamma = \bigcup_i \Gamma_i$ and $\hat{\Gamma} = \bigcup_j \hat{\Gamma}_j$ be a pair of transplantable loop-signed graphs with edge colours $1, 2, \ldots, C$ and vertices $1, 2, \ldots, V$, where $(\Gamma_i)_i$ and $(\hat{\Gamma}_j)_j$ denote the connected components of $\Gamma$ and $\hat{\Gamma}$, respectively. We let $(A^c)^C_{c=1}$ and $(\hat{A}^c)^C_{c=1}$ denote their $V \times V$ adjacency matrices, and choose $T \in GL(\mathbb{R}^V)$ such that
\[ \hat{A}^c = T A^c T^{-1} \quad \text{for } c = 1, 2, \ldots, C. \] (4.1)

4.1 Dualisation

Dualisation refers to the process of changing loop signs of $\Gamma$ and $\hat{\Gamma}$ without affecting their transplantability. As a trivial example, if both $A^c$ and $\hat{A}^c$ are diagonal matrices for some edge colour $c$, then we can replace them by $-A^c$ and $-\hat{A}^c$ to obtain a new transplantable pair. For any choice of building block, the two pairs of graphs would give rise to the same tiled domains but with opposite boundary conditions on all outer segments corresponding to the edge colour $c$.

We generalise this idea. For arbitrary $S \subseteq \{1, 2, \ldots, C\}$, let $(\Gamma_{S,k})_k$ and $(\hat{\Gamma}_{S,l})_l$ be the connected components of the graphs obtained by removing all $S$-coloured edges from $\Gamma$ and $\hat{\Gamma}$, respectively. Assume we can assign each of $(\Gamma_{S,k})_k$ to either $+1$ or $-1$ such that whenever two of them were connected by an $S$-coloured edge in $\Gamma$, then they are assigned to opposite numbers, similarly for $(\hat{\Gamma}_{S,l})_l$. We encode these partitionings of the vertices of $\Gamma$ and $\hat{\Gamma}$ by diagonal matrices of the form
\[ P = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1) \quad \text{and} \quad \hat{P} = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1). \] (4.2)

We show that the graph obtained by swapping the signs of all $S$-coloured loops of $\Gamma$ has adjacency matrices $-P A^c P$ for all $c \in S$, and $P A^c P$ for all $c \notin S$, similarly for $\hat{\Gamma}$. Since $A^c$ and $\pm P A^c P$ have the same vanishing entries, it suffices to consider the non-vanishing ones. If $A^c_{ij} = 1$ with $i \neq j$, then $P_{ii} P_{jj} = \mp 1$ depending on whether $c \in S$ or $c \notin S$, which yields
\[ (\mp P A^c P)_{ij} = \mp P_{ii} A^c_{ij} P_{jj} = A^c_{ij}. \]

On the other hand, if $A^c_{ii} = 0$, then
\[ (\mp P A^c P)_{ii} = \mp P_{ii} A^c_{ii} P_{ii} = \mp A^c_{ii}. \]

Since $P$ and $\hat{P}$ are self-inverse, the transplantation matrix $\hat{P} T P$ satisfies
\[ (\mp \hat{P} \hat{A}^c \hat{P})(\hat{P} T P) = (\mp \hat{P} \hat{A}^c T P) = \mp \hat{P} T A^c P = (\hat{P} T P)(\mp P A^c P). \]

Hence, swapping the signs of all $S$-coloured loops of $\Gamma$ and $\hat{\Gamma}$ yields transplantable graphs. Note that we can choose $S = \{1, 2, \ldots, C\}$ if and only if the loopless versions of $\Gamma$ and $\hat{\Gamma}$ are bipartite.

**Definition 21.** Each pair of transplantable loop-signed graphs with bipartite loopless versions has a transplantable dual pair obtained by swapping all loop signs. A pair is called self-dual if it is isomorphic to its dual pair.

All pairs we encountered so far have dual pairs. Note that loop-signed graphs with bipartite loopless versions give rise to orientable tiled domains if the underlying building block $B$ is orientable. More precisely, if $\omega$ is a non-vanishing volume form on $B$, and if $P = \text{diag}(\pm 1, \pm 1, \ldots, \pm 1)$ is as in (4.2), then one obtains a non-vanishing volume form on the tiled domain by taking $P_{ii} \omega$ on its $i$th block. The different signs on neighbouring blocks compensate for the reflection coming from the gluing process.
that is, if the components $\Gamma$ affecting their transplantability. According to Theorem 18 and Theorem 20, the converse holds, parts of domains, this can sometimes be interpreted as regarding a component of a reflecting segment or

On the analogy of adding edge colours, we can add transplantable components to $\Gamma$ and

Alternatively, we can add an edge colour by setting $A^{C+1} = \hat{A}^{C+1} = \pm I_V$. On the level of tiled domains, this can sometimes be interpreted as regarding a component of a reflecting segment or parts of $\partial_D B$ or $\partial_N B$ in $\hat{A}$ as a new reflecting segment, respectively. On the other hand, one can trivially omit an edge colour which corresponds to undoing the respective gluings and imposing either Dirichlet or Neumann boundary conditions on all resulting outer reflecting segments as indicated in Figure 4.1.

On the analogy of adding edge colours, we can add transplantable components to $\Gamma$ and $\hat{\Gamma}$ without affecting their transplantability. According to Theorem 18 and Theorem 20 the converse holds, that is, if the components $\Gamma_k$ and $\hat{\Gamma}_{\tilde{l}}$ of $\Gamma$ and $\hat{\Gamma}$ are transplantable, then so are $\Gamma \setminus \Gamma_k = \bigcup_{i \neq k} \Gamma_i$ and $\hat{\Gamma} \setminus \hat{\Gamma}_{\tilde{l}} = \bigcup_{j \neq \tilde{l}} \hat{\Gamma}_j$. Note that the same statement holds for isospectrality, that is, domains remain isospectral if one removes isospectral components. As an example, we can omit the triangles with number 6 in Figure 4.1. The resulting pair was recently discovered by Band et al. Their search was motivated by Chapman’s two-piece band with homogeneous boundary conditions, which can be obtained similarly.

4.4 The Crossing Method

In the following, we use tensor products of linear maps. Their basis dependent matrix representations are known as Kronecker products.

**Definition 22.** The Kronecker product of an $m \times m$ matrix $A = (a_{ij})_{ij}$ and an $n \times n$ matrix $B = (b_{kl})_{kl}$ is the $mn \times mn$ block matrix

$$A \otimes B = \begin{pmatrix}
a_{11}B & a_{12}B & \ldots & a_{1m}B \\
a_{21}B & a_{22}B & \ldots & a_{2m}B \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}B & a_{m2}B & \ldots & a_{mm}B
\end{pmatrix}.$$ 

**Lemma 23.** For each pair $(m, n)$ of positive integers, there is an $mn \times mn$ permutation matrix $P^{(m, n)}$ such that for each $m \times m$ matrix $A$ and each $n \times n$ matrix $B$

$$(A \otimes B) P^{(m, n)} = P^{(m, n)} (B \otimes A).$$
Definition 24. Let \( \Gamma \) be an \( n \times n \) matrix such that 
\[
\Gamma = \sum_{c} c \times \text{symmetric permutation matrices},
\]
which justifies the following definition.

Lemma 23 is an easy exercise with bases. Note that Kronecker products of symmetric permutation matrices are symmetric permutation matrices, which justifies the following definition.

**Definition 24.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be loop-signed graphs without Dirichlet loops having \( V_1 \), respectively \( V_2 \), vertices and adjacency matrices \( (A^c_1)_{c=1}^{c_1} \), respectively \( (A^c_2)_{c=1}^{c_2} \). The crossing \( \Gamma_1 \otimes \Gamma_2 \) is the loop-signed graph with \( C_1 \) \( C_2 \) edge colours, \( V_1 \) \( V_2 \) vertices, and adjacency matrices \( A^{[c_1,c_2]} = A^{c_1}_1 \otimes A^{c_2}_2 \),

where \([c_1,c_2] = c_1 + (c_2 - 1) C_1 \) for \( c_1 = 1, 2, \ldots, C_1 \) and \( c_2 = 1, 2, \ldots, C_2 \).

Figure 4.2 provides an example of a crossing. Lemma 23 implies that for \( \Gamma_1 \) and \( \Gamma_2 \) as above, \( \Gamma_1 \otimes \Gamma_2 \) and \( \Gamma_2 \otimes \Gamma_1 \) only differ by a renumbering of their vertices and edge colours.

Let \(((\Gamma_i, \hat{\Gamma}_i))_{i=1,2} \) be two pairs of transplantable loop-signed graphs without Dirichlet loops given by adjacency matrices \( (A^c_i)_{c=1}^{c_i} \) and \( (\hat{A}^c_i)_{c=1}^{c_i} \), respectively. For \( i \in \{1, 2\} \), let \( T_i \) be a transplantation matrix such that

\[
\hat{A}^c_i = T_i A^c_i T_i^{-1} \quad \text{for } c = 1, 2, \ldots, C_i.
\]

Using the invertible matrix \( T_1 \otimes T_2 \), we obtain that for \( c_1 = 1, 2, \ldots, C_1 \) and \( c_2 = 1, 2, \ldots, C_2 \)

\[
\hat{A}^{[c_1,c_2]} (T_1 \otimes T_2) = \hat{A}^{c_1}_1 T_1 \otimes \hat{A}^{c_2}_2 T_2 = T_1 A^{c_1}_1 \otimes T_2 A^{c_2}_2 = (T_1 \otimes T_2) A^{[c_1,c_2]},
\]

which shows that \( \Gamma_1 \otimes \Gamma_2 \) and \( \hat{\Gamma}_1 \otimes \hat{\Gamma}_2 \) are transplantable.

### 4.5 The Substitution Method

Levitin et al. [LPP06] recently discovered the strikingly simple pair of transplantable domains shown in Figure 4.3, a broken square that sounds like a broken triangle. The matrix

\[
T = \begin{pmatrix}
-1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}
\]

gives an intertwining transplantation. We subdivide the triangles into smaller blocks as indicated in Figure 4.3, where we added notches for the sake of clarity. Note that the transplantation respects this subdivision in the sense that if \( \varphi \) solves the Zaremba problem on the punctured square and has restrictions \( (\varphi_i)_{i=1}^4 \), then the restrictions \( \hat{\varphi}_1 \) and \( \hat{\varphi}_2 \) of its transform \( \hat{\varphi} = T(\varphi) \) on the notched triangle depend on \( \varphi_1 \) and \( \varphi_2 \) only, similarly for \( \hat{\varphi}_3 \) and \( \hat{\varphi}_4 \). In other words, the subdivided domains can be regarded as a pair of transplantable tiled domains each of which consists of 4 blocks. With respect to the subdivision, the transplantation reads

\[
\begin{pmatrix}
-1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \otimes T. \quad (4.3)
\]
With respect to the graph representation, subdividing the blocks corresponds to substituting a whole graph into each of the vertices of the associated loop-signed graphs as indicated in Figure 4.3. We develop this method in detail. Let \( \Gamma \) and \( \Gamma_S \) be loop-signed graphs having \( C \), respectively \( C_S \), edge colours and \( V \), respectively \( V_S \), vertices. Let \( (A^c)_{c=1}^C \) and \( (S^\chi)_{\chi=1}^{C_S} \) denote their adjacency matrices. In order to assign Neumann loops of \( \Gamma_S \) to edge colours of \( \Gamma \), we let \( I_{\chi} \) be a set of vertices of \( \Gamma_S \) that have a \( \chi \)-coloured Neumann loop, that is, \( I_{\chi} \subseteq \{ i | S_{\chi}^{ii} = 1 \} \).

\[
S_{\chi}^i = S_{\chi} - \sum_{i \in I_{\chi}} S_{\chi}^i \quad \text{(4.4)}
\]
describes the \( \chi \)-coloured part of \( \Gamma_S \) that will reappear at each vertex of \( \Gamma \).

**Definition 25.** The substituted graph \( \Gamma_S \triangleright \Gamma \) is the loop-signed graph with edge colours \( \chi = 1, 2, \ldots, C_S \) and vertices \( 1, 2, \ldots, V \) given by the adjacency matrices

\[
A_{\chi}^c = S_{\chi}^c \otimes I_V + \sum_{c} \sum_{i \in I_{\chi}} S_{\chi}^i \otimes A^c, \quad \text{(4.5)}
\]

where \( I_V \) denotes the \( V \times V \) identity matrix.

We briefly justify why (4.5) yields adjacency matrices. One easily sees that all summands are symmetric, and that none has negative off-diagonal entries. Moreover, one can view \( A_{\chi}^c \) as a sum of \( V_S \times V_S \) block matrices where the \( V \times V \) blocks are given by multiples of \( I_V \) and \( A^c \). Since \( S_{\chi} \), \( I_V \) and \( (A^c)_{c=1}^C \) are signed permutation matrices, one can use (4.4) to show that each row and each column of \( A_{\chi}^c \) contains exactly one non-vanishing entry.

The graphs in Figure 4.3 exemplify the substitution method. The outward-pointing edges of \( \Gamma_S \) represent its Neumann loops that are assigned to edge colours of \( \Gamma \), namely,

\[
I_{\text{straight}} = I_{\text{ridged}} = \{ 1, 2 \} \quad I_{\text{zig-zag}} = I_{\text{curly}} = \{ 2 \} \quad I_{\text{wavy}} = \emptyset.
\]

For example, each curly loop of \( \Gamma \) is replaced by a zig-zag loop with the same sign, and the ridged edge of \( \Gamma \) is replaced by two straight edges. For the reader’s convenience, we give the decompositions of the adjacency matrices of \( \Gamma_S \triangleright \Gamma \):

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix} \otimes \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 1
\end{pmatrix} \otimes \begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\]
Proof. As before, let

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= \begin{pmatrix}
-1 & 0 \\
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\otimes \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 1
\end{pmatrix}
\otimes \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}
\]

Substitution Theorem. Let \( \Gamma \) and \( \Gamma_S \) be loop-signed graphs as above.

1. If \( \Gamma \) and \( \Gamma_S \) are connected and for each edge colour \( c \) of \( \Gamma \) there is a Neumann loop of \( \Gamma_S \) that is assigned to it, that is, if \( \sum_{\chi=1}^{C_S} |I_G^\chi| > 0 \), then \( \Gamma_S \bowtie \Gamma \) is connected.

2. If \( \Gamma \) and \( \Gamma_S \) are treelike and for each edge colour \( c \) of \( \Gamma \) there is exactly one Neumann loop of \( \Gamma_S \) that is assigned to it, that is, if \( \sum_{\chi=1}^{C_S} |I_G^\chi| = 1 \), then \( \Gamma_S \bowtie \Gamma \) is treelike.

3. If \( \Gamma \) and \( \hat{\Gamma} \) are transplantable loop-signed graphs, then \( \Gamma_S \bowtie \Gamma \) and \( \Gamma_S \bowtie \hat{\Gamma} \) are transplantable. Moreover, if \( T \) is an intertwining transplantation matrix for \( \Gamma \) and \( \hat{\Gamma} \), then \( I_{\Gamma_S} \otimes T \) intertwines \( \Gamma_S \bowtie \Gamma \) and \( \Gamma_S \bowtie \hat{\Gamma} \).

Proof. As before, let \( (A_c^e)_{e=1}^{C} \) and \( (S_X^c)_{X=1}^{C} \) denote the adjacency matrices of \( \Gamma \) and \( \Gamma_S \), respectively.

1. We have to show that any two vertices of \( \Gamma_S \bowtie \Gamma \) are connected by a path of edges. Again, we view \( A^X \) as a sum of \( V_S \times V_S \) block matrices with \( V \times V \) blocks given by multiples of \( I_V \) and \( A^c \). If we number the vertices of \( A^X \) by \((p - 1) V + q\) where \( 1 \leq p \leq V_S \) and \( 1 \leq q \leq V \), then edges coming from nonzero entries of the summand \( S_X^c \otimes I_V \), respectively \( S_X^c \otimes A^c \), connect vertices with equal values of \( q \), respectively \( p \). In order to connect the vertices \( (p_1 - 1) V + q_1 \) and \((p_2 - 1) V + q_2 \) in \( \Gamma_S \bowtie \Gamma \), one first chooses a sequence of edge colours \( c_1, c_2, \ldots, c_L \) of \( \Gamma \) describing a self-avoiding connecting path \( q_1 = i_0, i_1, \ldots, i_L = q_2 \). By assumption, the \( c_1 \)-connectivity is established by some edge colour of \( \Gamma_S \), that is, there is \( \chi_1 \in \{1, 2, \ldots, C_S\} \) with \( I_{G_1}^{\chi_1} \neq \emptyset \). We choose \( m_1 \in I_{G_1}^{\chi_1} \) and consider a self-avoiding path in \( \Gamma_S \) connecting \( p_1 \) with \( m_1 \). This gives rise to a path in \( \Gamma_S \bowtie \Gamma \) connecting \((p_1 - 1) V + q_1\) with \((m_1 - 1) V + q_1\), which is connected with \((m_1 + 1) V + i_1\) via a \( \chi_1 \)-coloured edge. In the next step, we choose \( m_2 \in I_{G_2}^{\chi_2} \) together with a self-avoiding path in \( \Gamma_S \) connecting \( m_1 \) with \( m_2 \). The rest follows by induction.

2. We have to show that any two vertices of \( \Gamma_S \bowtie \Gamma \) are connected by a unique self-avoiding path, that is, the self-avoiding paths constructed in 1. are the only connecting ones. This follows from the observation that a self-avoiding path in \( \Gamma_S \bowtie \Gamma \) gives rise to paths in \( \Gamma \) and \( \Gamma_S \), respectively, and non-uniqueness would contradict the assumption that these graphs are treelike.

3. We use that \( \hat{A}^c T = T A^c \) for \( c = 1, 2, \ldots, C \) to obtain

\[
\hat{A}^c \otimes (I_{\Gamma_S} \otimes T) = S_X^c \otimes T + \sum_c \sum_{i \in I_G^c} S_X^c \otimes \hat{A}^c T
= S_X^c \otimes T + \sum_c \sum_{i \in I_G^c} S_X^c \otimes T A^c = (I_{\Gamma_S} \otimes T) A_X^c.
\]
5 Examples of Transplantable Pairs

5.1 Treelike Pairs and Planar Domains

As Kac’s original question [Kac66] aimed at planar domains with homogeneous boundary conditions, transplantable treelike graphs with homogeneous loop signs have been studied the most [OS01, Tha06a, Tha06b, Tha06c, Tha07]. Note that the Substitution Theorem implies that there exist infinitely many such pairs, which answers a question raised in [GT10]. However, it could be the case that all such pairs arise from the 16 known ones in [BCDS94] by substitution. In [OS01], Okada and Shudo confirmed these pairs up to graphs with 13 vertices, and just missed the 7 pairs of graphs with 14 vertices that arise by substitution from the 3 known ones with 7 vertices per graph. Figure 5.1 shows one of these pairs. In the substitution, we altered the shape of the building block to avoid self-overlapplings which is not always possible. Using Dualisation and Theorem 13, one sees that transplantable treelike pairs with homogeneous loop signs always originate from Sunada’s original method [Sun85], that is, from Gassmann triples \((G,H,\hat{H})\). For each of the 7 pairs mentioned above, \(G\) is isomorphic to \(PSL_3(2) \times PSL_3(2)\), and its permutation action on the vertices is not 2-transitive, which gives a counterexample to a conjecture expressed in [ST11]. Using the SmallGroups Library of GAP [GAP08], we verified that the only Gassmann triple \((G,H,\hat{H})\) with \(|G| \leq 2000\) and \(|G| \neq 1024\) that yields treelike graphs is the one underlying the Gordon-Webb-Wolpert drums [GWW92].

\[
\begin{align*}
\Gamma &\rightarrow \Gamma_S \\
\hat{\Gamma} &\rightarrow \hat{\Gamma}_S
\end{align*}
\]

Figure 5.1: Dirichlet isospectral planar domains with 14 building blocks obtained by substitution. The pair \((\Gamma,\hat{\Gamma})\) belongs to the Gordon-Webb-Wolpert drums [GWW92] with Dirichlet boundaries.

5.2 Transplantable Tuples

We show how crossings and substitutions give rise to transplantable tuples, that is, tuples of pairwise transplantable and therefore isospectral domains. Assume first that \(\Gamma\) and \(\hat{\Gamma}\) are transplantable graphs without Dirichlet loops. Since graphs are trivially transplantable to themselves, we may apply the crossing method to the pairs \((\Gamma,\Gamma), (\Gamma,\hat{\Gamma}), (\hat{\Gamma},\hat{\Gamma})\) and \((\hat{\Gamma},\Gamma)\) to obtain the transplantable pairs \((\Gamma \otimes \Gamma,\Gamma \otimes \Gamma), (\Gamma \otimes \hat{\Gamma},\hat{\Gamma} \otimes \Gamma)\) and \((\hat{\Gamma} \otimes \hat{\Gamma},\hat{\Gamma} \otimes \Gamma)\). Using the transplantable tuple \((\Gamma \otimes \Gamma,\Gamma \otimes \Gamma,\hat{\Gamma} \otimes \Gamma)\) and the above pairs, we get the pairwise transplantable graphs \((\Gamma \otimes \Gamma) \otimes \Gamma, (\Gamma \otimes \Gamma) \otimes \hat{\Gamma}, (\Gamma \otimes \hat{\Gamma}) \otimes \Gamma, (\Gamma \otimes \hat{\Gamma}) \otimes \hat{\Gamma}, (\hat{\Gamma} \otimes \Gamma) \otimes \Gamma, (\hat{\Gamma} \otimes \Gamma) \otimes \hat{\Gamma}, (\hat{\Gamma} \otimes \hat{\Gamma}) \otimes \Gamma, (\hat{\Gamma} \otimes \hat{\Gamma}) \otimes \hat{\Gamma}\). Inductively, one obtains a sequence of transplantable tuples with exponentially growing lengths.

Figure 5.2a indicates a more general method allowing for mixed boundary conditions, whose basic idea goes back to [LPP06]. Note that we can regard \(M_{br}\) as being built out of copies of \(M_b\).
which gives transplantability with \( M_{tl} \). On the other hand, we can take \( M_r \) as the fundamental building block to show that \( M_{br} \) and \( M_{tr} \) are transplantable. Similar arguments yield that \( (M_{br}, M_{tl}, M_{tr}, M_{tl}) \) is a transplantable tuple.

In the same way, there can be several ways in which a given loop-signed graph can be interpreted as a substituted graph as indicated in Figure 5.2b. In order to generalise this idea, let \( \Gamma_1 \) and \( \Gamma_2 \) be loop-signed graphs having \( V_1 \), respectively \( V_2 \), vertices and adjacency matrices \( (A_1^e)_{e=1}^{C_1}, \) respectively \( (A_2^e)_{e=1}^{C_2} \). Let \( \Gamma_{1,S} \) be the graph that is obtained by adding a Neumann loop of each edge colour of \( \Gamma_2 \) to each vertex of \( \Gamma_1 \), that is, \( \Gamma_{1,S} \) has adjacency matrices \( (A_1^e)_{e=1}^{C_1+C_2} \), where \( A_1^{e+c} = I_{V_1} \) for \( e = 1, 2, \ldots, C_1 \). In Figure 5.2b the added loops are indicated by outward-pointing edges. In order to substitute \( \Gamma_{1,S} \) into \( \Gamma_2 \), we set \( I_2 = \emptyset \) for \( e = 1, 2, \ldots, C_1 \), that is, \( A_2^e = A_2^e \), and \( I_2^{e+c} = I_2^{e+c} = \{1, 2, \ldots, V_1\} \) for \( e = 1, 2, \ldots, C_2 \), that is, \( A_2^{e+c} = 0 \). According to Definition 25 the adjacency matrices of \( \Gamma_{1,S} \bowtie \Gamma_2 \) read

\[
A_{12}^{e} = A_1^e \otimes I_2 \quad \text{for} \quad e = 1, 2, \ldots, C_1, \quad \text{and} \\
A_{12}^{e+c} = I_{V_1} \otimes A_2^e \quad \text{for} \quad e = 1, 2, \ldots, C_2.
\]

Analogously, we let \( \Gamma_{2,S} \) be the graph with adjacency matrices \( (A_2^{e-c_1})_{e=1}^{C_1+C_2} \), where \( A_2^{e-c_1} = I_{V_2} \) for \( e = 1, 2, \ldots, C_1 \). If we substitute \( \Gamma_{2,S} \) into \( \Gamma_1 \) as above, we obtain \( \Gamma_{2,S} \bowtie \Gamma_1 \) with adjacency matrices \( A_{21}^{e} = I_{V_2} \otimes A_1^e \) for \( e = 1, 2, \ldots, C_1 \), and \( A_{21}^{e+c} = A_2^e \otimes I_{V_1} \) for \( e = 1, 2, \ldots, C_2 \). Thus, Lemma 23 implies that \( \Gamma_{1,S} \bowtie \Gamma_2 \) and \( \Gamma_{2,S} \bowtie \Gamma_1 \) are isomorphic, which is indicated in Figure 5.2b, where \( \Gamma_{br} = \Gamma_b \bowtie \Gamma_r \) and \( \Gamma_{lr} = \Gamma_l \bowtie \Gamma_r \).

We use this result to generate transplantable tuples. Let \( (\Gamma_1, \hat{\Gamma}_1) \) and \( (\Gamma_2, \hat{\Gamma}_2) \) be pairs of transplantable loop-signed graphs. Using the above notation, we see that \( (\Gamma_1,S \bowtie \Gamma_2, \Gamma_{2,S} \bowtie \Gamma_1, \Gamma_{1,S} \bowtie \Gamma_2, \hat{\Gamma}_{2,S} \bowtie \hat{\Gamma}_1) \) and \( (\Gamma_{1,S} \bowtie \hat{\Gamma}_2, \Gamma_{2,S} \bowtie \hat{\Gamma}_1) \) are pairs of isomorphic graphs. The Substitution Theorem shows that \( (\Gamma_1, S \bowtie \hat{\Gamma}_2, \Gamma_{1,S} \bowtie \hat{\Gamma}_2, \hat{\Gamma}_{1,S} \bowtie \hat{\Gamma}_2, \hat{\Gamma}_{2,S} \bowtie \hat{\Gamma}_1) \) and \( (\Gamma_{1,S} \bowtie \hat{\Gamma}_2, \Gamma_{1,S} \bowtie \hat{\Gamma}_2, \hat{\Gamma}_{1,S} \bowtie \hat{\Gamma}_2, \hat{\Gamma}_{2,S} \bowtie \hat{\Gamma}_1) \) are pairs of transplantable graphs. Hence, \( (\Gamma_1 \bowtie \hat{\Gamma}_2, \Gamma_1 \bowtie \hat{\Gamma}_2, \Gamma_{1,S} \bowtie \hat{\Gamma}_2, \hat{\Gamma}_{1,S} \bowtie \hat{\Gamma}_2) \) is a transplantable tuple, which, together with another transplantable pair \( (\Gamma_3, \hat{\Gamma}_3) \), can be used to generate \( 2^3 \) pairwise transplantable graphs. We proceed inductively, where in each step we could use the same transplantable pair \( (\Gamma_i, \hat{\Gamma}_i) = (\Gamma_1, \hat{\Gamma}_1) \), for instance, \( (\Gamma_r, \hat{\Gamma}_i) \) and \( (\Gamma_b, \hat{\Gamma}_i) \) in Figure 5.2b are isomorphic up to a permutation of their edge colours.

Figure 5.2: Transplantable tuple obtained as in [LPPD96]. Solid lines indicate Dirichlet boundary conditions, and dashed lines Neumann boundary conditions.
5.3 The Algorithm

Recall Proposition [11] which showed that the Trace Theorem allows for a computer-aided search for transplantable pairs, where one first creates one representative of each isomorphism class of loop-signed graphs with a given number of vertices and edge colours, and then sorts them according to expressions of vertices that appear when one performs a walk on the vertices of a loop-signed graph where the edges are used in the order of their colour. Doyle [Doy10] recently developed an algorithm to tackle the sorting part, which was motivated by the discovery of the graphs in Figure 5.3. Despite not being transplantable, these pairs are strongly isospectral in the sense that if \((A^c)_{c=1}^C\) and \((\hat{A}^c)_{c=1}^C\) denote their adjacency matrices, then for any \(z_1, z_2, \ldots, z_C \in \mathbb{C}\)

\[
\det \left( \sum_{c=1}^C z_c A^c \right) = \det \left( \sum_{c=1}^C z_c \hat{A}^c \right).
\]

The graphs in Figure 5.3a only differ in the order in which the edge colours straight and zig-zag appear along the outer 6-cycles, which suggests to interpret transplantability as a non-commutative version of strong isospectrality [Doy10]. More precisely, if \((A^c)_{c=1}^C\) and \((\hat{A}^c)_{c=1}^C\) are the adjacency matrices of a pair of transplantable graphs, then we obtain that for any square matrices \((Z^c)_{c=1}^C\) of equal size, and any \(k \in \mathbb{N}\)

\[
\text{Tr} \left( \sum_{c=1}^C A^c \otimes Z^c \right)^k = \text{Tr} \left( \prod_{i=1}^k A^{c_i} \otimes Z^{c_i} \right) = \prod_{i=1}^k \text{Tr} \left( A^{c_i} \right) \prod_{i=1}^k \text{Tr} \left( Z^{c_i} \right) \quad (5.1)
\]

\[
= \text{Tr} \left( \sum_{c=1}^C \hat{A}^c \otimes Z^c \right)^k. \quad (5.2)
\]

Note that \((5.1)\) contains all products of adjacency matrices of length \(k\). If the scalars \(\text{Tr}(\prod_{i=1}^k Z^{c_i}) \in \mathbb{C}\) were linearly independent over \(\mathbb{Q}\) up to the cyclic invariance of the trace, then one could read off the integer factors \(\text{Tr}(\prod_{i=1}^k A^{c_i})\). Unfortunately, this is not the case as can be seen by regarding the expressions \(\text{Tr}(\prod_{i=1}^k Z^{c_i})\) as homogeneous polynomials in the entries of \((Z^c)_{c=1}^C\). For instance, the graphs in Figure 5.3a satisfy \((5.2)\) for any 2 \(\times\) 2 matrices \((Z^c)_{c=1}^C\) despite not being transplantable.

After sorting out pairs of possibly transplantable graphs via \((5.2)\), the existence of an intertwining transplantation \(T\) is determined by considering the action of the adjacency matrices on its entries as follows. If \((A^c)_{c=1}^C\) and \((\hat{A}^c)_{c=1}^C\) are \(V \times V\) adjacency matrices, and if \(A^c_{ij} \neq 0\) as well as \(\hat{A}^c_{ik} \neq 0\), then any intertwining \(T\) satisfies

\[
T_{ij} A^c_{ij} = (T A^c)_{ij} = (\hat{A}^c T)_{ij} = \hat{A}^c_{ik} T_{kl}. \quad (5.3)
\]

Thus, we consider the action of the group \(\langle (A^c, \hat{A}^c) \rangle_{c=1}^C\) on \(\{1, 2, \ldots, V\}^2 \times \{-1, 1\}\) given by

\[
(A^c, \hat{A}^c)((i, j), \pm 1) = ((k, l), \pm A^c_{ji} \hat{A}^c_{ik}),
\]

where \((k, l)\) is the unique pair such that \(A^c_{ij} \neq 0\) and \(\hat{A}^c_{ik} \neq 0\). If an orbit contains both \(((i, j), +1)\) and \(((i, j), -1)\) for some \(i\) and \(j\), then all entries of \(T\) with indices in that orbit must be zero. At the end, one checks whether the remaining orbits can be assigned to real numbers such that the resulting matrix \(T\) becomes invertible.

5.4 Pairs with 2 Edge Colours

The construction method in [LPP06, JLP06] corresponds to the case of connected tiled domains consisting of building blocks that have 2 reflecting segments only. In order to classify such pairs, let
Figure 5.3: Non-transplantable but strongly isospectral loop-signed graphs, where in each case the omitted loops have to be chosen to carry the same loop signs.

\[ \begin{array}{ccc}
(a) & \text{Graphs with Dirichlet loops} & (b) \text{Graphs with Dirichlet or Neumann loops} \\
(c) \text{Graphs with Dirichlet or Neumann loops satisfying} & \text{for any } 2 \times 2 \text{ matrices } (Z^c_{ij})_{n=1}^{m, n}
\end{array} \]

Figure 5.4: Transplantable non-isomorphic connected graphs with 2 edge colours.

\[ \begin{array}{ccc}
\Gamma & \cdots & \hat{\Gamma} \\
D & N & D & N
\end{array} \]

\[ \begin{align*}
\text{Tr}(\hat{A}^{c_1} \cdots \hat{A}^{c_2} \hat{A}^{c_1}) &= \text{Tr}(A^{c_1} \cdots A^{c_2} A^{c_1}) \\
(5.4)
\end{align*} \]

\[ \text{Tr}((A^2 A^1)^{l/2}) = \text{Tr}((A^1 A^2)^{l/2}) = \text{Tr}((\hat{A}^1 \hat{A}^2)^{l/2}) \]

The preceding arguments provide an alternative proof of the recent extension of [LPP06, Theorem 4.2] given in [BPBS09].

5.5 Pairs with 3 Edge Colours

Table 4 and Table 5 contain the results of the computer-aided search for transplantable pairs with 3 edge colours. For instance, there are 40 isomorphism classes of connected loop-signed graphs with 3 edge colours and 2 vertices, among which there are 9 transplantable pairs. If we identify pairs which differ only in a renumbering of their edge colours, then we obtain the 3 classes that are depicted in Figure 5.5. They arise from the pair with only 2 edge colours by adding or copying of the edge colour zig-zag, respectively. Note that, as a consequence of the Substitution Theorem, there are transplantable pairs for all even numbers of vertices per graph. Moreover, each transplantable tuple of length \( l \) contributes \( l (l - 1)/2 \) pairs.

The 32 pairs with 7 vertices per graph are listed in Appendix A.2. Their loopless versions first appeared in [OS01]. In particular, there exist 10 versions of broken Gordon-Webb-Wolpert drums [GWW92] shown in Figure 5.6. Pair 3 was recently discovered [PB10] using non-trivial

\[ \begin{array}{ccc}
\text{Figure 5.5: Transplantable connected pairs with 3 edge colours and 2 vertices per graph.}
\end{array} \]
Table 4: Transplantable isomorphism classes of connected loop-signed graphs with 3 edge colours. The last 2 columns contain the number of equivalence classes with respect to the relation generated by permutations of edge colours.

| Number of vertices | Loop-signed graphs (treelike) | Transplantable pairs (treelike) | Transplantable classes (treelike) |
|-------------------|-----------------------------|-------------------------------|-------------------------------|
| 2                 | 40 (30)                     | 9 (6)                         | 3 (2)                         |
| 3                 | 128 (96)                    | 0 (0)                         | 0 (0)                         |
| 4                 | 737 (472)                   | 118 (64)                      | 28 (18)                       |
| 5                 | 3 848 (2 304)               | 0 (0)                         | 0 (0)                         |
| 6                 | 24 360 (12 792)             | 957 (294)                     | 176 (56)                      |
| 7                 | 156 480 (73 216)            | 112 (112)                     | 32 (32)                       |
| 8                 | 1 076 984 (439 968)         | 13 349 (2 112)                | 2 343 (375)                   |
| 9                 | 7 625 040 (2 715 648)       | 0 (0)                         | 0 (0)                         |

Table 5: Transplantable isomorphism classes of connected loop-signed graphs with 3 edge colours and homogeneous loop signs. There are no such pairs with less than 7 vertices per graph.

| Number of vertices | Edge-coloured Dirichlet Neumann Treelike |
|-------------------|------------------------------------------|
|                   | graphs | trees | pairs classes | pairs classes | pairs classes |
| 7                 | 1 407  | 143   | 7         | 3             | 7             |
| 8                 | 6 877  | 450   | 64        | 16            | 28            |
| 9                 | 28 665 | 1 326 | 0         | 0             | 0             |
| 10                | 142 449| 4 262 | 0         | 0             | 0             |
| 11                | 681 467| 13 566| 34        | 9             | 70            |
| 12                | 3 535 172| 44 772| 2 362      | 440           | 42            |
| 13                | 18 329 101| 148 580| 26         | 9             | 26            |
| 14                | 99 531 092| 502 101| 342        | 77            | 798           |

With regard to Table 5, recall that transplantable pairs without Dirichlet loops originate from Gassmann triples. For instance, the 19 Neumann pairs with 11 vertices per graph come from $\text{PSL}_2(11)$. In accordance with [BdS02], there are no such pairs with 9 or 10 vertices per graph.

According to [Doy10], they were first discovered by John Conway.

Figure 5.6: Transplantable pairs of broken Gordon-Webb-Wolpert drums [GWW92]. By duality, one is free to choose whether solid lines represent Dirichlet boundary conditions and dashed ones Neumann boundary conditions or vice versa. The respective first (second) number refers to the case in which solid lines indicate Dirichlet (Neumann) conditions.
In the following, we demonstrate how the 6 self-dual pairs in Appendix A.1 give rise to 6 of the 8 Neumann pairs with 8 vertices per graph. The remaining 2 pairs can be found in Section 6.3. We consider the transplantable pair \((\Gamma, \hat{\Gamma})\) in Figure 5.7a and the self-dual pair \((\Gamma_S, \hat{\Gamma}_S)\) in Figure 5.7b, where outward-pointing edges indicate Neumann loops. According to the Substitution Theorem, \(\Gamma_S \triangleright \Gamma\) is transplantable to \(\Gamma_S \triangleright \hat{\Gamma}\), and \(\hat{\Gamma}_S \triangleright \Gamma\) is transplantable to \(\hat{\Gamma}_S \triangleright \hat{\Gamma}\), where the indicated loop assignments shall be used. Note that \(\Gamma_S \triangleright \hat{\Gamma}\) and \(\hat{\Gamma}_S \triangleright \hat{\Gamma}\) have an identical component with Dirichlet loops only. Omitting these components leaves us with the transplantable pair \((\Gamma_S, \hat{\Gamma}_S)\) we started with, hence, \(\Gamma_S \triangleright \hat{\Gamma}\) and \(\hat{\Gamma}_S \triangleright \hat{\Gamma}\) are transplantable for which reason \(\Gamma_S \triangleright \Gamma\) and \(\hat{\Gamma}_S \triangleright \Gamma\) are transplantable. Since these graphs have bipartite loopless versions, we can pass to the dual pair containing Neumann loops only. In the same way, some of the pairs with 14 vertices per graph and homogeneous loop signs come from graphs with 7 vertices and mixed loop signs.
6 Inaudible Properties

We finally turn to inaudible properties, and present isospectral domains with mixed or homogeneous boundary conditions that feature different topological or geometrical properties. In particular, there exists a connected drum that sounds disconnected, and a broken drum that sounds unbroken.

6.1 Self-Dual Pairs

One cannot hear which parts of a drum are broken.

In [JLNP06], Jakobson et al. considered the question whether two broken versions of a drum with drumheads attached exactly where the other version’s drumhead is free, can sound the same. They answered in the affirmative by showing that the half-disks in Figure 6.1a are transplantable and therefore isospectral. Note that these domains come from a self-dual pair, which in turn arises from the pair with just 2 edge colours and 2 vertices per graph as indicated in Appendix A.1. Figure 6.1b presents another self-dual pair giving rise to domains with a single Dirichlet boundary component. Similarly, one can use almost self-dual pairs, that is, pairs of transplantable graphs with bipartite loopless versions which differ from their dual pair by a permutation of edge colours only, to construct domains whose spectra are invariant under swapping Dirichlet with Neumann boundary conditions.

![Figure 6.1: Domains whose spectrum is invariant under swapping all boundary conditions.](image)

6.2 Fundamental Group

One cannot hear the fundamental group of a broken drum.

Isospectral manifolds with different fundamental groups first appeared in [Vig80a, Vig80b], which contains closed hyperbolic examples. Recently, planar examples with mixed boundary conditions were found [LPP06]. Similar domains can be constructed from the transplantable pairs in Appendix A.1. In contrast, transplantable connected graphs with homogeneous loop signs can be seen to have loopless versions with isomorphic fundamental groups. More precisely, if a connected loop-signed graph with either Dirichlet or Neumann loops only is given by the \( V \times V \) adjacency matrices \( (A^c)_{c=1}^C \), then the number \( V \) of vertices, and the number \( E \) of edges of its loopless version is determined by expressions of the form (2.9), namely,

\[
V = \text{Tr}(I_V) \quad \text{and} \quad E = \frac{1}{2} \sum_{c=1}^C (\text{Tr}(I_V) \pm \text{Tr}(A^c)).
\]

6.3 Orientability

One cannot hear whether a broken drum is orientable.

Doyle and Rossetti [DR08] recently showed that orientability of hyperbolic surfaces can be heard. In contrast, Bördard and Webb [BW95] constructed a pair of Neumann isospectral domains one of which is orientable while the other is not. Their example corresponds to the pairs in Figure 6.2.
Figure 6.2: Transplantable pairs. Only the respective first graph has a bipartite loopless version.

Figure 6.3: Numbering convention and corresponding transplantation matrix for the pairs in Figure 6.2. The parameters $\alpha$, $\beta$, $\gamma$ and $\delta$ can be chosen such that the matrix becomes invertible.

Since these pairs can be obtained from each other by braiding, it is possible to use the same transplantation matrix given in Figure 6.3. Table 6 provides a similar pair with 15 instead of 8 vertices per graph, and Figure 6.4 shows the first example of this kind with mixed boundary conditions. Note that these domains have different numbers of Dirichlet boundary components. Moreover, one can alter the building block continuously so that the first domain becomes planar. On the other hand, Doyle [Doy10] recently showed that connected graphs without Neumann loops can be transplantable only if their loopless versions are either both bipartite or non-bipartite.

We present his proof. Let $\Gamma$ and $\hat{\Gamma}$ be transplantable loop-signed graphs given by $V \times V$ adjacency matrices $(A^C)_{c=1}^C$ and $(\hat{A}^C)_{c=1}^C$ with non-positive diagonal entries. If $\Gamma$ is loopless, that is, $\text{Tr}(A^c) = 0$ for $c = 1, 2, \ldots, C$, then the Trace Theorem shows that $\hat{\Gamma}$ cannot have loops either. If, moreover, $\Gamma$ has an odd cycle with associated sequence of edge colours $c_1 c_2 \ldots c_{2n+1}$, then $0 < \text{Tr}(A^{c_2} A^1) = \text{Tr}(\hat{A}^{c_2} A^1)$, and $\hat{\Gamma}$ is non-bipartite as well. Hence, we may assume that both $\Gamma$ and $\hat{\Gamma}$ have loops. In the following, we consider the Markov chain with $V \times V$ transition matrix

$$P = \frac{1}{C} \sum_{c=1}^C |A^c|.$$

It represents a random walk on the vertices of $\Gamma$, where at each step one of the $C$ incident edges or loops is chosen with equal probability. Since $\Gamma$ is connected and has loops, $P$ is irreducible.
straight  
(1,8)(3,10)(5,12)(7,14)  
(1,2)(4,7)(8,14)(9,12)(10,15)(11,13)  
(1,6)(2,13)(3,11)(4,12)(5,10)(9,14)  
(1,12)(2,9)(3,5)(4,15)(7,10)(8,14)

wavy  
(1,2)(4,7)(8,14)(9,12)(10,15)(11,13)  
(1,6)(2,13)(3,11)(4,12)(5,10)(9,14)  
(1,12)(2,9)(3,5)(4,15)(7,10)(8,14)

zig-zag  
(1,8)(3,10)(5,12)(7,14)  
(1,2)(4,7)(8,14)(9,12)(10,15)(11,13)  
(1,6)(2,13)(3,11)(4,12)(5,10)(9,14)  
(1,12)(2,9)(3,5)(4,15)(7,10)(8,14)

Table 6: Transplantable graphs with 15 vertices all of whose loops carry Neumann signs. The table lists the edges of the loopless versions which are bipartite and non-bipartite, respectively.

aperiodic, and has the invariant distribution $V^{-1} (1, 1, \ldots, 1)$. Hence, we get convergence to equilibrium [Nor98, Theorem 1.8.3]

$$\lim_{k \to \infty} (P^k)_{ij} = V^{-1}.$$ 

If $\Gamma$ has a bipartite loopless version, then every product of its adjacency matrices with an even number of factors has non-negative entries on the diagonal. Hence,

$$1 = \text{Tr} \left( \lim_{k \to \infty} P^{2k} \right) = \lim_{k \to \infty} \frac{1}{C^{2k}} \sum_{1 \leq c_1, c_2, \ldots, c_{2k} \leq C} \text{Tr} \left( \prod_{i=1}^{2k} |A^{c_i}| \right) = \lim_{k \to \infty} \text{Tr} \left( \frac{1}{C} \sum_{c=1}^{C} A^c \right)^{2k}. \quad (6.1)$$

If, however, the loopless version of $\Gamma$ has an odd cycle, then we will see that there is $\varepsilon > 0$ such that for $k \to \infty$, the fraction of $2k$-cycles in $\Gamma$ with negative contribution to the trace in (6.1) is at least $\varepsilon$, which completes the proof. Note that for each vertex of $\Gamma$, we can prescribe two equally long cycles starting at that vertex, such that one uses an odd number of loops whereas the other uses an even number of loops. Now, if some cycle is concatenated to one of these preliminary ones, and the resulting path uses an even number of loops, then concatenating it with the other preliminary one yields a path with an odd number of loops. Hence, if $L$ is an upper bound on the lengths of the $2V$ preliminary cycles that we chose for the $V$ possible starting vertices, then the statement above holds for $\varepsilon = C^{-L/V}$.

6.4 Connectedness

One cannot hear whether a drum is connected.

In contrast to Neumann isospectral domains, mixed boundary conditions allow for isospectrality of domains with different numbers of components. An example is shown in Figure 6.5 which also presents the first known pair of Dirichlet isospectral domains with different numbers of components. This is also the first known pair of Dirichlet isospectral domains that are not Neumann isospectral. We could change the shape of the building block continuously such that the simply-connected domain becomes planar. Moreover, the transplantable graphs in Figure 6.4 give rise to the same domains if we use the building block shown in Figure 6.5b.

![Isospectral domains](image1)

(a) Isospectral domains

![Dirichlet isospectral domains](image2)

(b) Dirichlet isospectral domains

Figure 6.5: Isospectral domains with different numbers of components. For the second pair, an intertwining transplantation is given by the matrix in Figure 6.4.
The study of connectedness leads to implications of spectral theory to graph theory. For instance, two transplantable graphs without Neumann loops that have different numbers of components cannot both have bipartite loopless versions, for if they had, their dual pairs would give rise to Neumann isospectral domains with different numbers of components. Hence, it was to be expected that at least one of the graphs in Figure 6.5b has a non-bipartite loopless version. Similarly, if one omits all those edge colours $c$ of a transplantable pair for which there exist $c$-coloured loops with Dirichlet signs, then the resulting graphs must have the same number of components.

6.5 Brokenness and Isotropy Order

One cannot hear whether a drum is broken.

We finish with the transplantable triple shown in Figure 6.6. This is the first example of a connected domain with mixed boundary conditions that is isospectral to a connected domain with Dirichlet boundary. Note that the building block could be altered continuously such that $M_1$ becomes planar, whereas $M_2$ and $M_3$ contain a Möbius strip. In contrast, a connected domain with Neumann boundary conditions cannot be transplantable to a connected domain with mixed boundary conditions. More precisely, one can use the same technique as in Section 6.3 to show that if a connected loop-signed graph has a Dirichlet loop, then any connected graph it is transplantable to must have one. In the style of [GWW92], one can interpret $M_1$ as an orbifold with Dirichlet boundary by first gluing two copies of $M_1$ along their Neumann boundary parts, and then taking the quotient with respect to the involution given by interchanging of the copies. In conclusion, the triple proves that the number of Neumann boundary components is not spectrally determined, and that an orbifold can be Dirichlet isospectral to a manifold.

![Figure 6.6: A broken drum that sounds unbroken.](image-url)
7 Outlook

Combinatorial and Quantum Graphs

The results in [Tha07] imply that transplantable treelike graphs without Dirichlet signs have cospectral uncoloured loopless versions. In this way, the tools in Section 4 provide means of generating cospectral graphs. In contrast to combinatorial graphs, quantum graphs have edges which are identified with real intervals, from which they inherit their differential structure. In particular, the Laplace operator consists of the one-dimensional Laplacians on the edges, and is made essentially self-adjoint by appropriate boundary conditions at the vertices. For instance, a function is said to satisfy Neumann boundary conditions at the vertex \( v \) if its restrictions to the incident edges have the same value at \( v \), and the sum of their derivatives vanishes at \( v \). Recently, Shapira and Smilansky [SS06] showed that intertwining transplantations give rise to transplantable and therefore isospectral quantum graphs. As indicated in Figure 7.1 one translates the loop signs into boundary conditions on the vertices of the quantum graph with a single incident edge, and imposes Neumann boundary conditions at its other vertices. Remarkably, Gutkin and Smilansky [GS01] partially answered the inverse spectral question for quantum graphs in the affirmative. They showed that the spectrum of a finite quantum graph encodes its adjacency matrix and edge lengths, provided that the edge lengths are rationally independent and that there are no loops or parallel edges between vertices.

![Figure 7.1: Loop-signed graph and corresponding quantum graph with mixed boundary conditions.](image)

The interior vertices of the quantum graph carry Neumann conditions.

Open Questions

Kac’s question for smooth planar domains remains widely open. However, the results in [Zel00] suggest that one can hear the shape of convex drums. In view of Section 6.3 it appears natural to ask whether the Dirichlet spectrum encodes orientability. Moreover, we are interested in the additional information that is needed to distinguish transplantable domains. Considering quantum graphs, Band and Smilansky [BS07] recently conjectured that this can be achieved by counting the nodal domains of eigenfunctions.
A Graph Gallery

A.1 Pairs with 3 Edge Colours and 4 Vertices per Graph

There are 28 equivalence classes of transplantable pairs with 3 edge colours and 4 vertices per graph shown below, where second numbers refer to the corresponding dual pair. Most of the pairs can be reduced to pairs with fewer edge colours or vertices. In particular, pairs 13, 14 and 20 arise from the transplantable pair with 2 edge colours and 4 vertices per graph by adding or copying of an edge colour, respectively, and pairs 6 to 12 and 21 to 28 arise from the pair with 2 edge colours and 2 vertices per graph by substitution. In the latter cases, a suitable substituent is indicated, and with respect to the right vertex numberings, intertwining transplantations are given by \([1,3]\).
A.2 Pairs with 3 Edge Colours and 7 Vertices per Graph

Using the tools of Section 4, one can show that the 32 transplantable pairs below can be reduced to pair 1 and pair 3, which in turn arise from $S_4$-subgroups of $PSL_3(2)$ and their trivial and sign characters [PB10], respectively. The corresponding spaces of matrices satisfying (5.3) are two-dimensional in the former cases, and one-dimensional in the latter cases. The 3 loopless versions of the 32 pairs below first appeared in [OS01]. For each pair, the second number refers to its dual pair. Note that the graphs of each pair only differ by a swap of the edge colours straight and zig-zag, together with swaps of their loop signs in some cases. Pairs 1, 2, 13, 14, 25 and 26 are those found by Buser et al. [BCDS94], and pairs 3 to 12 represent the 10 versions of broken Gordon-Webb-Wolpert drums [GWW92].
Isospectral surfaces of small genus

[BT87] Robert Brooks and Richard Tse, Resolving the isospectrality of the dihedral graphs by counting nodal

On arithmetically equivalent number fields of small degree

[BdS02] Wieb Bosma and Bart de Smit, On arithmetically equivalent number fields of small degree, Algorithmic

number theory (Sydney, 2002), Lecture Notes in Comput. Sci., vol. 2369, Springer, Berlin, 2002, pp. 67–

[Bér92] Pierre Bérard, Transplantation et isospectralité. I, Math. Ann. 292 (1992), no. 3, 547–559.

[Bér93] Pierre Bérard, Transplantation et isospectralité. II, J. London Math. Soc. (2) 48 (1993), no. 3, 565–576.

[BG90] Thomas P. Branson and Peter B. Gilkey, The asymptotics of the Laplacian on a manifold with boundary., Comm. Partial Differential Equations 15 (1990), no. 2, 245–272.

[BGKV99] Thomas P. Branson, Peter B. Gilkey, Klaus Kirsten, and Dmitri V. Vassilevich, The asymptotics of the Laplacian on a manifold with boundary., Nuclear Phys. B 563 (1999), no. 3, 603–626.

[BPBS09] Ram Band, Ori Parzanchevski, and Gilad Ben-Shach, The isospectral fruits of representation theory: quantum graphs and drums, J. Phys. A 42 (2009), no. 17, 175202, 42.

[Bro90] Robert Brooks, The Sunada method, Tel Aviv Topology Conference: Rothenberg Festschrift (1998), Contemp. Math., vol. 231, Amer. Math. Soc., Providence, RI, 1999, pp. 25–35.

[BS07] Ram Band and Uzy Smilansky, Resolving the isospectrality of the dihedral graphs by counting nodal

domains, Eur. Phys. J. Special Topics 145 (2007), 171–179.

[BT87] Robert Brooks and Richard Tse, Isospectral surfaces of small genus, Nagoya Math. J. 107 (1987), 13–24.

[Bus86] Peter Buser, Isospectral Riemann surfaces, Ann. Inst. Fourier (Grenoble) 36 (1986), no. 2, 167–192.

[Bus88] Pierre Bérard and David Webb, On ne peut pas entendre l’orientabilité d’une surface, C. R. Acad. Sci.

Paris Sér. I Math. 320 (1995), no. 5, 533–536.

[CH95] John H. Conway and Tim Hsu, Quilts and T-systems, J. Algebra 174 (1995), no. 3, 856–908.

[Cha95] S. J. Chapman, Drums that sound the same, Amer. Math. Monthly 102 (1995), no. 2, 124–138.

[DG03] T. A. Driscoll and H. P. W. Gottlieb, Isospectral shapes with Neumann and alternating boundary conditions, Phys. Rev. E (3) 68 (2003), no. 1, 016702, 6.

[DMRSS03] Abhishek Dhar, D. Madhusudhana Rao, N. Udaya Shankar, and S. Sridhar, Isospectrality in chaotic billiards, Phys. Rev. E (3) 68 (2003), no. 2, 026208, 5.

[Dow05] J. S. Dowker, The hybrid spectral problem and Robin boundary conditions, J. Phys. A 38 (2005), no. 21, 4735–4754.

[Doy10] Peter G. Doyle, Private communication, 2010.

[DR08] Peter G. Doyle and Juan Pablo Rossetti, Isospectral hyperbolic surfaces have matching geodesics, New York J. Math. 14 (2008), 193–204.

[Eva10] Lawrence C. Evans, Partial differential equations, second ed., Graduate Studies in Mathematics, vol. 19, American Mathematical Society, Providence, RI, 2010.

[FK05] S. A. Fulling and Peter Kuchment, Coincidence of length spectra does not imply isospectrality, Inverse Problems 21 (2005), no. 4, 1391–1395.

[GAP08] GAP, GAP – Groups, Algorithms, and Programming, Version 4.4.12, The GAP Group, 2008.

[GS01] Boris Gutkin and Uzy Smilansky, Can one hear the shape of a graph?, J. Phys. A 34 (2001), no. 31, 6061–6068.

[GT10] Olivier Giraud and Koen Thas, Hearing shapes of drums: Mathematical and physical aspects of isospectrality, Rev. Mod. Phys. 82 (2010), no. 3, 2213–2255.

[GW09] C. Gordon, D. Webb, and S. Wolpert, Isospectral plane domains and surfaces via Riemannian orbifolds, Invent. Math. 110 (1992), no. 1, 1–22.

[Ike80] Akira Ikeda, On lens spaces which are isospectral but not isometric, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 3, 303–315.

[JLP06] Dmitry Jakobson, Michael Levitin, Nikolai Nadirashvili, and Iosif Polterovich, Spectral problems with mixed Dirichlet-Neumann boundary conditions: isospectrality and beyond, J. Comput. Appl. Math. 194 (2006), no. 1, 141–155.

[Kac66] Mark Kac, Can one hear the shape of a drum?, Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23.

[KMR97] V. A. Kozlov, V. G. Maz'ya, and J. Rossmann, Elliptic boundary value problems in domains with point singularities, Mathematical Surveys and Monographs, vol. 52, American Mathematical Society, Providence, RI, 1997.
[KMR01] Spectral problems associated with corner singularities of solutions to elliptic equations, Mathematical Surveys and Monographs, vol. 85, American Mathematical Society, Providence, RI, 2001.

[KW97] Ian I. Kogan and John F. Wheater, Neumann-Dirichlet tadpoles as new string states and quantum mechanical particle-wave duality from world-sheet t-duality, Phys. Lett. B 403 (1997), no. 1-2, 31–37.

[LPP06] Michael Levitin, Leonid Parnovski, and Iosif Polterovich, Isospectral domains with mixed boundary conditions, J. Phys. A 39 (2006), no. 9, 2073–2082.

[Mil64] J. Milnor, Eigenvalues of the Laplace operator on certain manifolds, Proc. Nat. Acad. Sci. U.S.A. 51 (1964), 542.

[Nor98] J. R. Norris, Markov chains, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 2, Cambridge University Press, Cambridge, 1998, Reprint of 1997 original.

[OS01] Yuichiro Okada and Akira Shudo, Equivalence between isospectrality and isolength spectrality for a certain class of planar billiard domains, J. Phys. A 34 (2001), no. 30, 5911–5922.

[PB10] Ori Parzanchevski and Ram Band, Linear representations and isospectrality with boundary conditions, J. Geom. Anal. 20 (2010), no. 2, 439–471.

[Pes96] Hubert Peix, Représentations relativement équivalentes et variétés riemanniennes isospectrales, Comment. Math. Helv. 71 (1996), no. 2, 243–268.

[Ser77] Jean-Pierre Serre, Linear representations of finite groups, Springer-Verlag, New York, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Sie76] Warren Siegel, Strings with dimension-dependent intercept, Nuclear Phys. B 109 (1976), 244–254.

[SK94] S. Sridhar and A. Kudrolli, Experiments on not "hearing the shape" of drums, Phys. Rev. Lett. 72 (1994), 2175–2178.

[SS06] Talia Shapira and Uzy Smilansky, Quantum graphs which sound the same, NATO Sci. Ser. II Math. Phys. Chem. 213 (2006), 17–29.

[ST11] Jeroen Schillewaert and Koen Thas, The 2-transitive transplantable isospectral drums, SIGMA 7 (2011), 080, 8 pp.

[Sun85] Toshikazu Sunada, Riemannian coverings and isospectral manifolds, Ann. of Math. (2) 121 (1985), no. 1, 169–186.

[Tha06a] Koen Thas, Kac’s question, planar isospectral pairs and involutions in projective space, J. Phys. A 39 (2006), no. 23, L385–L388.

[Tha06b] Kac’s question, planar isospectral pairs and involutions in projective space. II. Classification of generalized projective isospectral data, J. Phys. A 39 (2006), no. 42, 13237–13242.

[Tha06c] PSLn(q) as operator group of isospectral drums, J. Phys. A 39 (2006), no. 50, L673–L675.

[Zel00] S. Zelditch, Spectral determination of analytic bi-asymmetric plane domains, Geom. Funct. Anal. 10 (2000), no. 3, 628–677.