Spectral and Transport Properties of Quantum Wires with Bond Disorder

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Dedicated to E. Müller-Hartmann on the occasion of his 60th birthday.

Abstract

Systems with bond disorder are defined through lattice Hamiltonians that are of pure nearest neighbour hopping type, i.e. do not contain on-site contributions. They stand representative for the general family of disordered systems with chiral symmetries. Application of the Dorokhov-Mello-Pereyra-Kumar transfer matrix technique [P. W. Brouwer et al., Phys. Rev. Lett. 81, 862 (1998); Phys. Rev. Lett. 84, 1913 (2000)] has shown that both spectral and transport properties of quasi one-dimensional systems belonging to this category are highly unusual. Most notably, regimes with absence of exponential Anderson localization are observed, the single particle density of states exhibits singular structure in the vicinity of the band centre, and the manifestation of these phenomena depends in an apparently topological manner on the even- or oddness of the channel number. In this paper we re-consider the problem from the complementary perspective of the non-linear $\sigma$-model. Relying on the standard analogy between one-dimensional statistical field theories and zero-dimensional quantum mechanics, we will relate the problem to the behaviour of a quantum point particle subject to an Aharonov-Bohm flux. We will build on this analogy to re-derive earlier DMPK results, identify a new
class of even/odd staggering phenomena (now dependent on the total number of sites in the system) and trace back the anomalous behaviour of the bond disordered system to a simple physical mechanism, viz. the flux periodicity of the quantum Aharonov-Bohm system. We will also touch upon connections to the low energy physics of other lattice systems, notably disordered chiral systems in 0 and 2 dimensions and antiferromagnetic spin chains.

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I. INTRODUCTION

In 1979, Wegner and Oppermann [1] introduced a variant of the disordered lattice Anderson model they termed ‘sublattice system’. The sublattice system differs from the generic Anderson model in that its Hamiltonian does not contain on-site matrix elements, i.e. is of pure hopping type. For a long time this species of disordered electronic systems went largely unnoticed. The status rapidly changed [2–13] after two aspects became generally appreciated: first, models with sublattice structure occur in a number of physical applications. The random flux model, lattice QCD models [14], random antiferromagnets [15], models of gapless semiconductors [16] and effective models of transport in manganese oxides [17] are of sublattice type or at least acquire sublattice structure in limiting cases. Second, and contrary to naive expectations based on universality and insensitivity to details of the microscopic implementation of disorder, the low energy properties of the sublattice system differ drastically from those of the generic Anderson model:

- In contrast to the Anderson model, average spectral and transport properties of the sublattice system sensitively depend on the value of the Fermi energy, $E_F$. For Fermi energies far away from the centre of the tight binding band, $E_F = 0$, the sublattice system falls into the universality class of standard disordered electron systems (which simply follows from the fact that a Fermi energy $E_F \neq 0$ can be interpreted as a constant non-vanishing on-site contribution to the Hamiltonian.) However, in the vicinity of zero energy drastic deviations from standard behaviour occur:

- In dimensions $d \leq 2$, the average density of states (DoS), $\nu(E)$, exhibits singular behaviour upon $E$ approaching the band centre.

- Perturbative one-loop RG calculations [2] as well as the analysis of Ref. [3] indicate that right in the middle of the band the 2$d$ system is metallic, i.e. does not drive towards an exponentially localized phase.

- Several phenomena of apparent topological origin are observed. E.g., the DoS profile
of sublattice quantum dots (systems in the ’zero-dimensional’ limit) sensitively dep-
dends on the total number of sites of the host lattice being even or odd. Similarly,
the properties of quasi one-dimensional sublattice systems depend on the number of
channels, \( N \), being even or odd. For \( N \) even the transport behaviour is conventional
– conductance exponentially decreasing on the scale of a certain localization length \( \xi \)
– whereas the energy dependent DoS vanishes in a close to linear fashion. In contrast,
for \( N \) odd, the wire at \( E = 0 \) is metallic, i.e. the length-dependent conductance decays
only algebraically. At the same time, the DoS diverges upon approaching the band
centre.

A schematic summary of the band centre phenomenology of sublattice models is displayed
in table \[4\].

All these phenomena root in the fact that the sublattice Hamiltonian possesses a dis-
crete symmetry not present in the Anderson model: \([\sigma_3, \hat{H}]_+ = 0\), where \([ \cdot , \cdot \]_+\) is the
anti-commutator and \( \sigma_3 \) a site-diagonal operator that takes values \(+1/ -1\) on alternating
sites. The presence of this ’chiral’ symmetry implies that sublattice systems fall into sym-
metry classes different from the three standard Wigner-Dyson classes ’unitary’, ’orthogonal’
or ’symplectic’. To be specific, let us focus on the simplest case of a sublattice system
with broken time reversal invariance but good spin rotation symmetry (the analogue of the
Anderson model of unitary symmetry.) Chiral systems fulfilling these two extra symme-
try criteria belong to a symmetry class denoted \( A_{III} \) in the terminology of Ref. \[18\]. An
alternative denotation, coined in the paper \[20\], is ChGUE for ’Chiral GUE’.
As with conventional disordered electronic systems of Wigner-Dyson symmetry, universal transport and thermodynamic properties of systems of class AIII can be described in terms of effective low energy theories. E.g., the results for sublattice quantum wires summarized above have been obtained within a symmetry adapted variant of the Dorokhov-Mello-Kumar-Pereyra (DMPK) transfer matrix approach [4,11,10]. This theory differs from the standard cases of unitary symmetry in that the transfer matrices describing the propagation through the system take values in a different target space. Similarly, the general field theory approach to disordered electronic systems, the nonlinear $\sigma$-model, has been extended to systems of class AIII [2,5,13,6], too. Like its conventional relatives, the AIII variant of the $\sigma$-model is a matrix field theory whose internal structure depends on the specific implementation (boson replicas, fermion replicas or supersymmetry.) Previous studies of these models have focused on the two-dimensional case [2,9], where information on long range behaviour can be obtained from renormalization group calculations, or on the zero-dimensional case [21], where the model can be evaluated rigorously by full integration over the zero-mode manifold.

It is the purpose of the present paper to analyse the intermediate, quasi one-dimensional variant of the field theory. As mentioned above key aspects of the phenomenology of quasi one-dimensional sublattice quantum wires have been discussed previously within the framework of the DMPK approach, and it is near at hand to ask what motivates revisiting the problem. Referring for a more substantial discussion to section IA below, we here merely...
mention a lose collection of points. The field theory approach enables one to approach the problem from a comparatively broad perspective. Specifically, the one-dimensional variant of the model is but a representative of a larger family of ‘chiral’ \( \sigma \)-models. This makes possible to relate the behaviour of the one-dimensional system to the extensively studied zero and two dimensional cases. Further, the Green function oriented \( \sigma \)-model formalism enables one to ‘microscopically’ implement coupling operators connecting the wire to external leads. (Within previous DMPK formulations, the coupling has been treated somewhat implicitly, see however [22].) Unexpectedly, we will observe strong, albeit universal sensitivity to the modelling of the coupling and yet another class of staggering phenomena. The origin of these effects, and their connection to the channel number staggering mentioned before will be discussed below. Finally, the \( \sigma \)-model of the time reversal non-invariant sublattice is the by far most simple of all ten [18] nonlinear \( \sigma \)-models of disordered systems: it has only four degrees of freedom, two Grassmann and two ordinary integration variables, the minimal set needed to construct a supersymmetric matrix model. This makes it an ideal tutorial system on which generic properties of the field theory approach to disordered quantum wires can be studied. We have tried to pedagogically expose several of these aspects, particularly the analogy one-dimensional field theories vs. point-particle quantum mechanics which plays an all important role in the present context. Nonetheless, the analysis below will be at times technical. To make its results and various qualitative connections generally accessible, the following section contains a synopsis of the paper.

A. Summary of Results and Qualitative Discussion

Consider the system depicted in figure Fig. 2: a sequence of \( L \) sites (alternatingly designated by crosses and circles) each of which supports \( N \) electronic states, or orbitals (represented by the vertical stacks of ovals.) Nearest neighbour hopping is controlled by a regular tight binding contribution, diagonal in the orbital index, (the horizontal line segments) plus some bond randomness that connects different orbitals (the hatched areas). As in Refs. [4],
we allow for some ‘staggering’ in the tight binding amplitudes, i.e. the hopping amplitudes regularly alternate in strength (the alternating bond length.) At either end, a number of sites is coupled through some tunnelling barriers (horizontal hatched areas) to leads.

![Quasi one dimensional sublattice system](image)

**FIG. 2.** Quasi one dimensional sublattice system. The system is realized in terms of a sequence of sites, alternatingly designated by crosses and circles, each of which carries $N$ states (the ovals). Hopping between the sites is mediated by a regular Hamiltonian (the lines connecting the ovals) plus a weak random contribution (hatched areas). At either end, a number of sites is coupled to continuum states (the wavy lines) propagating in leads.

In this paper, three different regimes will be considered: (i) the ‘quantum dot’ case, defined through the criterion that the time to diffusively propagate through the system is shorter than the Golden rule escape time into the leads, (ii) a diffusive regime, where the order of these time scales is inverted but localization effects do not yet play a role, and (iii) the regime of long systems with pronounced Anderson localization.

Beginning with the quantum dot case (i), we find that both transport and spectral properties sensitively depend on the number of sites $L$ being even or odd. Specifically, the DoS exhibits a singular spike in the band centre if the number of sites is odd (and the coupling to the leads switched off). Away from zero energy, $\nu(\epsilon)$ is strongly suppressed up to values $\nu \sim N\Delta$, where $\Delta$ is the mean level spacing. The conductance equals $\gamma N/2$, where $\gamma$ is a measure for the strength of the lead coupling, as for conventional quantum dots. In contrast,
for an even number of sites, the DoS shows the spectral ‘microgap’ of width $\Delta$ characteristic for finite random systems with chiral symmetries [14]. Curiously, the conductance vanishes provided that only one site at either end is connected to the leads. The strong sensitivity to the total number of sites in the system disappears if more than one site in the terminal regions is coupled to leads (cf. Fig. 2.) Summarizing, the tendency of sublattice systems to exhibit staggering phenomena pertains to the zero-dimensional limit. However, unlike in the localized regime, the relevant integer control parameter is the number of sites $L$ and not the number of channels $N$. A qualitative interpretation of these phenomena will be given below.

In this paper, only limited attention will be payed to the intermediate diffusive regime (ii). It is likely that the staggering phenomena observed in the zero-dimensional regime will have interesting, albeit non-universal extensions into the diffusive regime. We here avoid the confrontation with these effects by connecting several sites to the leads. This leads to equilibrated behaviour with Ohmic conductance, as for ordinary wires. As for the density of states, we have explored the influence of spatially fluctuating diffusion modes on the spectral microgaps discussed above. (Semiclassically, the spectral gaps observed in chiral or superconductor systems can be interpreted as due to an accumulation of diffusion modes (see Ref. [5] for a detailed discussion of this point). For large energies $\epsilon \gg \Delta$, these modes can be treated perturbatively by standard diagrammatic methods.) Surprisingly, no corrections are found up to and including three loop order. This is a speciality of the one-dimensional case. For a two dimensional system diffusive modes would lead to a modulation of the spectrum on the scale of the Thouless energy.

In the localized regime we reproduce the results found earlier within the DMPK approach: for an even number of channels, the conductance decays exponentially on the scale of a localization length $\xi \propto Nl$, where $l$ is the mean free path. In contrast, for an odd number of channels algebraic scaling, $g \sim L^{-1/2}$, supported by one delocalized mode is observed [4]. A comparably strong even/odd dependence is observed in the behaviour of the DoS. For $N$ even, the DoS vanishes at zero energy as $\nu(\epsilon) \sim -|s| \ln(|s|)$, where $s = \pi \epsilon / \Delta \xi$, and
the characteristic scale of the gap, $\Delta_\xi$, is the level spacing of a system of length $L \sim \xi$. This behaviour has a simple interpretation: roughly speaking, a system with $L \gg \xi$ can be viewed as a sequence of $L/\xi$ decoupled 'localization volumes'. On small energy scales $\epsilon < \Delta_\xi$, dynamics within each of these is approximately ergodic. One would then expect the DoS to be gapped, as for the sublattice quantum dot, with $\Delta_\xi$ as the characteristic level spacing. Generalizing an earlier result [23,24] for the specific case $N = 1$, it has been found that for an odd number of channels the DoS diverges as $\nu(\epsilon) \sim 1/(|s| \ln^3 |s|)$ [11]. This accumulation of spectral weight can be interpreted in two different directions. First, it is near at hand to view the algebraically decaying conductance observed in the odd case as a resonant tunnelling phenomenon: the principal tendency to localize is outweighed by the high density of states in the vicinity of zero energy. (This picture was first suggested by V. Kravtsov.) Second, it is tempting to interpret the zero energy peak as an generalization of the singular spike found in the $L$-odd zero-dimensional case. Unfortunately, we are not aware of a qualitative picture explaining this analogy. At least technically, however, both phenomena can be traced back to a common origin. Finally, a periodic modulation, or staggering, of the hopping amplitudes can be employed to continuously interpolate between the $N$-even and $N$-odd case, respectively.

We next briefly outline the field theory route to exploring the above phenomenology. Within the fieldtheoretical approach, long range properties of the system are described by a functional integral with action

$$S = \int_0^L dr \left[ -\frac{\xi}{8} \text{str} (\partial T \partial T^{-1}) - \frac{N + f}{2} \text{str} (T \partial T^{-1}) - i \frac{s}{2} \text{str} (T + T^{-1}) \right] + S_{Gade} + S_T. \quad (1)$$

Here we have introduced a continuum variable $r \in [0, L]$ replacing the formerly discrete site index, $f$ is a parameter related to the staggering strength, $s = \epsilon \pi \nu$, where $\nu$ is the bulk metallic density of states. Further, $T$ is a matrix field taking values in $\text{GL}(1|1)$, i.e. the group of two-dimensional invertible supermatrices, and 'str' is the supertrace. Finally, $S_{Gade}[T] \equiv C \text{str}^2(T \partial T^{-1})$, where $C$ is some small constant and $S_T$ is a contribution describing the coupling to the leads.
The action $S[T]$ defines a one-dimensional representative of the general family of 'chiral' non-linear $\sigma$-models. In contrast to its well investigated zero- and two-dimensional relatives – much of the results summarized in table 1 have been derived within these models – the 1$d$ variant has not been explored so far.

Much of our analysis of this model will rely on the standard analogy between one-dimensional statistical field theory and zero-dimensional quantum mechanics: $S[T]$ can be interpreted as the quantum-mechanical action of a point particle propagating on the supersymmetric target manifold, in our case GL$(1|1)$. The first term of the action represents a kinetic energy, the third term a potential, and the second term, linear in the ‘velocity’, coupling to a constant vector potential of strength $(N + f)/2$. Quantum analogies of this type have been proven a powerful technical tool in previous analyses of the standard $\sigma$-models [25]. However, the present case is special in that the target manifold is so simple that an intuitive interpretation of the quantum picture becomes straightforward. Indeed, the fields $T$ have the explicit matrix representation $T = \begin{pmatrix} u & \rho \\ \sigma & v \end{pmatrix}$, where $\rho, \sigma(u, v)$ are Grassmann (commuting) variables, to be compared with the four or eight dimensional matrices entering the standard $\sigma$-models. Later on we will see that convergence criteria constrain the component $u$ to be positive real, while $v = \exp(i\phi)$ is a pure phase. Thus, temporarily leaving the Grassmann variables aside, our model describes quantum propagation on a (half)line and on a circle, respectively. Notice that the latter component is topologically non-trivial.

At this stage, the role of the vector potential contribution to the action becomes evident. While inessential in the non-compact sector of the theory, in the compact, circular sector, it describes the presence of an Aharonov-Bohm type magnetic flux. This analogy explains the presence of phenomena periodic in the channel number. An even number of channels translates to an integer number of flux quanta through the ring, which has no effect. However for $N$ odd or, alternatively, a finite staggering parameter $f$, a fractional flux pierces the system and this influences both, spectrum and dynamics of the quantum system. To develop the picture somewhat further, notice that the conductance, essentially
the transition probability through the system, maps onto the Green function of the quantum system evaluated at imaginary ‘time’ $L$. Imagining the latter represented through a spectral decomposition, the large $L$ behaviour depends on the low lying portions of the quantum spectrum, in particular the discrete, and flux periodic level structure of the compact sector of the theory. Later on we will see that for half integer flux (i.e. $N$ odd) there is a zero-energy level ($\to$ absence of localization) while for all other magnetic configurations the spectrum is gapped ($\to$ exponential localization.) This mechanism, and its relevance for the localization behaviour of the system was first analysed by Martin Zirnbauer \cite{26}. Similar but slightly more elaborate arguments can be used to understand the profile of the DoS.

We finally mention some intriguing parallels to the physics of the antiferromagnetic spin chain. According to Haldanes conjecture, a chain of spins with half-integer (integer) $S$ is in a long range ordered (disordered) phase \cite{27}. It has also been found (see Ref. \cite{28} for review) that for a chain with staggered hopping amplitude $j$, the strict integer/half-integer pattern is violated, e.g. an integer chain can be fine-tuned to an ordered phase. Technically, the system is described by a $\sigma$-model with a topological term not dissimilar to the one above. The differences are that in the spin case (a) the base manifold is 1 + 1-dimensional (a quantum chain), (b) the target manifold is the two-sphere $\simeq \text{SU}(2)/\text{U}(1)$, and (c) the topological term classifies field configurations according to the number of coverings of the sphere (instead of winding numbers around the circle, as in our case). In the spin case, the coupling constant of the topological term is given by $S + j$, i.e. spin replaces channel number and staggering plays a similar role as in our case. (In fact, the linear coupling of the topological term to the staggering amplitude follows, independently of the model, from parity-conservation arguments to be discussed below.) Beyond these apparent technical parallels, the connection between the quenched disordered sublattice and the spin chain, respectively, is not understood.

The rest of the paper is organized as follows. In section II we quantify the definition of the model. Its field theory representation is introduced in section III. In section IV the $\sigma$-model transfer matrix approach, i.e. the representation of observables in terms of the
quantum Green function is discussed. This formulation is then applied to the calculation of conductance (section V) and density of states (section VI). We conclude in section VII.

II. DEFINITION OF THE MODEL

We begin this section by upgrading the above qualitative introduction of the sublattice system to a more quantitative formulation. Quantum transport through the bulk of the system is described by the Hamiltonian

$$\hat{H} = \sum_{l\mu} c_{l\mu}^\dagger \left( t_{ll'} \delta_{\mu\mu'} + R_{ll'}^{\mu\mu'} \right) c_{l'\mu'},$$

where $c_{l\mu}^\dagger$ creates a spinless electron at site $l$ in state $\mu = 1, \ldots, N$, the sum extends over nearest neighbour sites, and $R_{ll'}$ are $N \times N$ Gaussian distributed random hopping matrices with moments

$$\langle R_{ll'}^{\mu\nu} \rangle = 0,$$

$$\langle R_{ll'}^{\mu\nu} R_{ll'}^{\nu'\mu'} \rangle = \frac{\lambda^2}{N} \delta_{\mu\nu} \delta_{\mu'\nu'}, \quad \lambda \ll 1.$$

Apart from the Hermiticity condition $R_{ll'} = R_{l'l}^\dagger$ matrices on different links are statistically independent. These random matrices compete with the regular contribution to the hopping matrix elements, $t_{ll'}$. To be specific, we set $t_{ll'} = 1 + (-)a$ if the smaller of the two neighbouring site indices $l$ and $l'$ is even (odd). The real parameter $a$ is a measure for the staggering strength. Notice that $t_{ll} = \mathcal{O}(1) \gg \lambda$ implies that we are dealing with a weakly disordered system.

At both ends, a number of sites is coupled to leads (see figure 2). Quantum propagation within these leads is assumed to be generic, i.e. governed by a Hamiltonian without sublattice symmetry. To describe the coupling, we add to our bulk Hamiltonian a tunnelling contribution

$$\hat{H}_T = \hat{H}_T^L + \hat{H}_T^R,$$
\[ \hat{H}_T^L = \sum_{l=1,2,\ldots} \left( c^\dagger_\alpha W^L_{\alpha a} d^L_a + \text{h.c.} \right), \]
\[ \hat{H}_T^R = \sum_{l=L,L-1,\ldots} \left( c^\dagger_\alpha W^R_{\alpha a} d^R_a + \text{h.c.} \right), \]

where \( d^{(R)}_a \) creates an electron propagating in the left (right) lead in a certain state \( a = 1, \ldots, M \gg 1 \) and we have introduced a composite index \( \alpha = (l, \mu) \) comprising site and orbital index of the bulk system. The coupling matrix elements \( W^L/R_{\alpha} \) are subject to the orthogonality relation \[29\]
\[ \sum_{\mu} W^L_{\alpha l, \mu} W^L_{\mu b} = f(l) \delta_{ab} \delta_{ll'}, \]
\[ \sum_{\mu} W^R_{\alpha,(L-l) \mu} W^R_{(L-l') \mu, b} = f(l) \delta_{ab} \delta_{ll'}, \] (3)

where \( f(x) \) is some envelope function decaying on a scale of \( O(1) \) and normalized through \( \sum_l f(l) = \gamma \ll 1 \). The function \( f \) and parameter \( \gamma \) describe profile and strength of the coupling, respectively. Why did we introduce the multi-site coupling operators (3) instead of connecting just the two terminal sites \( l = 1, L \) to the lead continuum? Modelling the coupling in a more general way is motivated by the presence of the site number staggering phenomena mentioned in the introduction. The above implementation of the coupling operator is sufficiently flexible to selectively probe these effects (sections V A and VI A) or to average over any boundary oscillatory structures (sections V B and VI B.)

To conclude the definition of the problem, let us introduce the Green function,
\[ G(z) = \left( z - \hat{H} + i\pi (\text{sgn \, Im \, z}) \sum_{C=L,R} \hat{W}^C \hat{W}^{CT} \right)^{-1}, \] (4)

where \( \hat{W}^C \hat{W}^{CT} = \{ \sum_{a} W^C_{\mu l,a} W^C_{a, \mu' l'} \} \) is an operator describing the escape of electrons from the bulk system into the leads \[29,30\]. Expressed in terms of these objects, the Landauer conductance of the system assumes the form
\[ g = (2\pi)^2 \sum_{ab=1}^{M} W^L_{\alpha a} W^R_{\beta b} W^R_{\beta' a'} W^L_{\alpha' a'} \left\langle G_{\alpha\beta}(0^+)G_{\beta'\alpha'}(0^-) \right\rangle. \] (5)

Our second quantity of interest, the density of states per site, is given by the standard expression
Finally, to prepare the field theory formulation, let us consider the symmetries of the problem. Expressed in the notation introduced above, the chiral symmetry assumes the form $[\hat{H}, \hat{\sigma}_3] = 0$, where $\hat{\sigma}_3 = \{(-)^l \delta_{ll'}\}$ and $\hat{H}$ denotes the bulk part of the Hamiltonian. (Coupling to a non-sublattice continuum breaks chirality.) The presence of this symmetry implies invariance under the two parameter family of transformations

$$
c_i^\dagger \rightarrow c_i^\dagger e^{-z_1}, \quad c_i \rightarrow e^{-z_2} c_i, \quad l \text{ even},
$$

$$
c_i^\dagger \rightarrow c_i^\dagger e^{+z_2}, \quad c_i \rightarrow e^{+z_1} c_i, \quad l \text{ odd},
$$

where $z_{1,2}$ are complex numbers. Depending on the choice of these parameters, (7) expresses the standard $U(1)$-invariance of a model with conserved charge ($z_1 = -z_2$), or the axial symmetry characteristic for chiral systems ($z_1 = z_2$). (For the time being we treat the transformation as purely formal i.e. ignore the fact that for a general choice $z_{1,2}$ the transformed operators are no longer mutually adjoint.) We will come back to discussing the role of these symmetries after the effective field theory of the system has been introduced.

### III. FIELD THEORY

The construction of the low energy effective field theory of the sublattice wire follows the standard route of deriving nonlinear $\sigma$-models of disordered fermion systems [31], there are no conceptually new elements involved. Referring to Appendices [4] and [5] for technical details of this derivation, we here discuss structure and key features of the resulting model.

As shown in the Appendices, the field theory representation of conductance and DoS is given by

$$
g = -\left(\frac{M\pi\gamma}{2}\right)^2 \left\langle (T - T^{-1})_{12}(0)(T - T^{-1})_{21}(L) \right\rangle,
$$

and
respectively. Here, the bulk DoS, $\nu_0 = \frac{N}{2\pi}$, $T(r)$ is a field taking values in the supergroup GL$(1|1)$ and the continuum variable $r \in [0, L]$ replaces the site index. The angular brackets stand for functional averaging

$$\langle \ldots \rangle \equiv \int \mathcal{D}T e^{-S[T]} \langle \ldots \rangle$$

over a functional integral with action $S \equiv \int_0^L dr \mathcal{L}$ and effective Lagrangian

$$\mathcal{L} \equiv \mathcal{L}_{\text{fl}} + \mathcal{L}_{\text{top}} + \mathcal{L}_T + \mathcal{L}_z + \mathcal{L}_{\text{Gade}}.\quad (10)$$

The individual contributions are given by

$$\mathcal{L}_{\text{top}} = -\frac{N + f}{2} \text{str} \left(T \partial_r T^{-1}\right),$$

$$\mathcal{L}_{\text{fl}} = -\frac{\xi}{16} \text{str} \left(\partial_r T \partial_r T^{-1}\right),$$

$$\mathcal{L}_z = -i \frac{\pi \nu_0}{2} \text{str} \left(T + T^{-1}\right),$$

$$\mathcal{L}_T = \frac{\pi M \gamma}{2} \text{str} \left(T(r) + T^{-1}(r)\right) \left[\delta(r) + \delta(L - r)\right],$$

$$\mathcal{L}_{\text{Gade}} \equiv C \left[\text{str} \left(T \partial_r T^{-1}\right)\right]^2,\quad (11)$$

where $C$ is a coupling constant that need not be specified other than that it is small, $C = \mathcal{O}(1) \ll (N, M)$ and the parameter $f \equiv \frac{2Na}{\lambda^2}$ measures the degree of staggering. (Notice our $f$ is identical to the control parameter $f$ defined in Ref. [4].) Finally, we have introduced a parameter

$$\xi \equiv \frac{N}{2\lambda^2}$$

which will later identify as the localization length of the system.

To prepare the further discussion of the functional expectation values, let us briefly discuss the internal structure of the field theory. We first note that save for the values of the coupling constants, the structure of the action (10) can be anticipated from inspection of Eq. (7) and its supersymmetric extension: the field theory approach starts out from a promotion
of the fermionic operator representation (2) to a supersymmetric formulation in terms of Bose and Fermi fields. Within the latter representation (cf. e.g. Eq. (A1)), the space of permissible invariance transformations is enlarged. The formerly complex parameters $e^{i\xi}$ are replaced by two-dimensional matrices $T_i$ acting on the bosonic and fermionic components of the theory. Any sensible subsequent manipulation done on the functional integral must respect this invariance property. On the level of the effective theory described by $S[T]$, the transformation acts as $T \rightarrow T_1 T T_2$ and indeed one verifies that the contributions $\mathcal{L}_{\text{top}}, \mathcal{L}_{\text{fl}}$ and $\mathcal{L}_\phi$ of (10) are invariant under this operation. Further, the two building blocks $\text{str} \left( T \partial T^{-1} \right)$ and $\text{str} \left( \partial T \partial T^{-1} \right)$ are the only operators with $\leq 2$ derivatives compatible with the global GL(1|1) symmetry. In other words, the gross structure of the bulk action readily follows from the invariance criterion. For finite energies or coupling to the leads chirality is broken and global U(1) remains the only symmetry of the Hamiltonian. Within the supersymmetry formulation, the set of allowed transformations is then reduced down to configurations with $T_1 = T_2^{-1}$ (the super-generalization of U(1)). The operator $\text{str} \left( T + T^{-1} \right)$ is the minimal positive (see below) choice compatible with the restricted symmetry. Summarizing, the three terms $\text{str} \left( T \partial T^{-1} \right), \text{str} \left( \partial T \partial T^{-1} \right)$, and $\text{str} \left( T + T^{-1} \right)$ exhaust the set of operators compatible with the global transformation behaviour of the model.

As for the coupling constants of the theory – not specified by symmetry arguments but all derived in Appendix B – notice that contrary to the standard $\sigma$-models of systems with WD symmetry two, instead of just one second derivative operators appear in the action. Mathematically, the reason for the appearance of two contributions is that the target manifold of the theory, GL(1|1) is a reducible symmetric space; it decomposes into two irreducible factors, a point discussed in detail in Ref. [18]. Each of these factors can be endowed with its own metric which implies the existence of two independent second derivative operators in the theory. Physically, the presence of the non-standard operator $\sim \text{str}^2 \left( T \partial T^{-1} \right)$ has profound consequences for the behaviour of the 2d-version of the field theory [2]: the RG analysis of the model shows that the coupling constant of this operator grows under renormalization while driving the coupling of the energy operator $\sim \text{str} \left( T + T^{-1} \right)$ to large values. At the
same time the coupling constant of the standard gradient operator $\sim \text{str} (\partial T \partial T^{-1})$, essentially the conductance, remains un-renormalized. In one dimension, the situation is different. Contrary to what one might expect, the contribution $L_{Gade}$ is not remotely as important as in two dimensions, and it is another operator that drives the localization behaviour of the model. In parentheses we remark that the target space of the transfer matrix approach to the problem, \( \text{GL}(M)/U(M) \simeq \text{SL}(M)/\text{SU}(M) \times \mathbb{R}^+ \) factors into two components, too \[10\]. Accordingly, the Fokker-Planck equation governing the 'Brownian motion' on that space is controlled by two independent coupling constants, both determined by the microscopic definition of the model. The second of these contributions, essentially the analogue of our Lagrangian $L_\phi$, leads to non-universal corrections to the overall picture which perish in the limit of a large number of channels.

A second aspect discriminating the present model from its WD relatives is the appearance of a first order gradient operator in the action. In fact, the presence of this contribution might raise suspicion: although allowed by symmetry, \( \text{str} (T \partial_r T^{-1}) \) is not invariant under space reflection $r \rightarrow -r$, in contrast to the microscopic parent model (for $a = 0$.) The resolution of this puzzle lies in the fact that $S_{\text{top}} \equiv \int L$ is of topological origin and, although not manifestly so, does respect the space inversion property. We will discuss this point momentarily after the internal structure of the field manifold and the integration measure have been specified.

Functional integrals can only be defined on manifolds that are Riemannian, i.e. endowed with a positive metric. The supergroup $\text{GL}(1|1)$ (like all other supersymmetric spaces that appear in the context of field theories of disordered Fermi systems) does not fulfil this criterion, a point discussed in detail in Ref. \[18\]. However, it does contain a maximally Riemannian submanifold $\mathcal{M}$, defined as follows: the boson-boson block of $\mathcal{M}$, a one-dimensional manifold by itself, is isomorphic to the non-compact symmetric space $\text{GL}(1)/\text{U}(1) \simeq \mathbb{R}^+$, i.e. the positive real numbers. (This space is trivially Riemannian.) The fermion-fermion block is isomorphic to the compact symmetric space $\text{U}(1)$. No limitations in the Grassmann valued boson-fermion sectors of the theory are needed since the whole issue of convergence does not
arise here. Summarizing, $\mathcal{M} = \mathcal{R}^+ \times \text{U}(1)$, where the notation is symbolic, specifying the boson-boson and fermion-fermion sector, respectively.

A convenient field representation respecting these convergence criteria is given by

$$T = k a k^{-1}, \quad k = \exp \left( \frac{\eta}{\nu} \right), \quad a = \exp \left( \frac{x}{i y} \right), \quad x, y \in \mathcal{R}. \quad (12)$$

In this parameterization, the group integration $\int dT$ extends over the degrees of freedom $x, y, \eta, \nu$, without further constraints. The invariant group measure associated to the representation (12) are defined in Eqs. (C1) and (C2).

We are now in a position to discuss the role of the contribution $S_{\text{top}}$. First notice that for sufficiently strong lead coupling the boundary action $L_T$ projects the fields onto the group origin $T(x, y) = 1$, i.e. enforces $x(0) = x(L) = 0$ and $y(0) = 2\pi k, y(L) = 2\pi k'$, where $k, k'$ are integer. As a consequence, the first derivative operator can be written as

$$S_{\text{top}}[T] = -\frac{N + f}{2} \int dr \text{str} (T \partial_r T^{-1}) =$$

$$= \frac{N + f}{2} \int dr \partial_r \text{str} \ln T =$$

$$= \frac{N + f}{2} \text{str} (\ln T(L) - \ln T(0)) =$$

$$= i\pi (N + f)(k - k'). \quad (13)$$

This makes the topological nature of the term manifest: it counts winding numbers in the fermionic sector of the theory. The integer $k - k'$ is a topological invariant characterizing each field configuration $T(r)$. Further notice that for $f = 0$,

$$e^{-S_{\text{top}}[T]} = e^{+i\pi N(k - k')} = e^{-i\pi N(k - k')}.$$

Since it is the exponentiated action and not the action itself that matters, the last identity tells us that the global sign of the first gradient operator is irrelevant (all for $f = 0$). This settles the above raised issue of the behaviour of the model under space reflection: although the first order derivative is not invariant under $r \to -r$ the exponentiated action is.

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IV. TRANSFER MATRIX APPROACH

Principal aspects of the system properties we are going to analyse are non-perturbative, i.e. cannot be obtained as power series in the coupling constants of the action. Progress with such type of problems can be made by applying the $\sigma$-model transfer-matrix technique [25], an approach conceptually similar to the DMPK formalism.

We begin by defining the two functions

$$ Y_L(T_1, T_2, r) \equiv \int_{T(0)=T_1}^{T(r)=T_2} DT e^{-\int_0^r dr L[T]}, $$

$$ Y_R(T_1, T_2, r) \equiv \int_{T(r)=T_1}^{T(L)=T_2} DT e^{-\int_r^L dr L[T]}.$$

Expressed in terms of these objects, the DoS assumes the form

$$ \nu(\epsilon) = \frac{\nu_0}{2L} \text{Re} \int_0^L dr \int dT Y_L(\mathbb{1}, T, r) \left( T_{11} + T_{11}^{-1} \right) Y_R(T, \mathbb{1}, L - r), \quad (14) $$

where we have used that for sufficiently strong coupling to the leads, the boundary configurations $T$ are close to unity. (Throughout much of this paper we will consider the DoS of coupled systems. For sufficiently large systems, the choice of boundary conditions is inessential, a point to be verified below.) Similarly, the conductance obtains as

$$ g = - \left( \frac{\pi M \gamma}{2} \right)^2 \int dT dT' \times $$

$$ \times e^{-\frac{\pi M \gamma}{2} \text{str}(T + T^{-1})(T - T^{-1})_{12}} Y_L(T, T', L) \left( T' - T'^{-1} \right)_{21} e^{-\frac{\pi M \gamma}{2} \text{str}(T' + T'^{-1})} \quad (15) $$

From Eqs. (14) and (15) it is clear that the functions $Y_{L,R}$ encode the essential system properties we are interested in.

As a first step towards the computation of these objects let us explore how the symmetries of the action translate to symmetries of $Y_{L,R}$. We first consider the case $z = 0$, relevant to the analysis of the conductance, where the action is fully invariant under $\text{GL}(1|1)$-transformations. The invariance $L[T] = L[T_1 TT_2], \ T_{1,2} = \text{const.}$ then directly implies $Y_{L,R}(T, T', r) = Y_{L,R}((T_1 TT_2), (T_1 TT_2), r)$. From this identity one readily verifies that
\[ Y_R[T, T', r] = Y_R[TT'^{-1}, \mathbb{I}, r] = Y_R[T'^{-1}T, \mathbb{I}, r], \]

\[ Y_L[T, T', r] = Y_L[\mathbb{I}, T'T^{-1}, r] = Y_L[\mathbb{I}, T^{-1}T', r], \quad (z = 0). \]

In other words, for \( z = 0 \) the arguments of the heat kernels enter in an invariant product type form and it suffices to consider the reduced functions

\[ Y_R(T, r) \equiv Y_R(T, \mathbb{I}, r), \quad Y_L(T, r) \equiv Y_L(\mathbb{I}, T, r), \quad (16) \]

depending on a single argument only. The same invariance property (now evaluated for \( T_2 = T_1^{-1} \)) implies that \( Y_{L,R}(T, r) = Y_{L,R}(T_1TT_1^{-1}, r) \). Imagining the argument \( T \) to be represented in the polar decomposition (12) and setting \( T_1 = k^{-1} \) the argument can be reduced to the diagonal matrix of eigenvalues \( a \):

\[ Y_{L,R}(kak^{-1}, r) = Y_{L,R}(a, r). \quad (17) \]

For \( z \neq 0 \), the invariance of the theory collapses down to transformations with \( T_2 = T_1^{-1} \). However, the representation of the DoS above implies that we are solely interested in functions of the type (16), with second argument set to unity, anyway. Since transformations \( T \rightarrow T_1TT_1^{-1} \) are still permissible, these objects depend on the eigenvalues of the argument matrix only, as before for the case \( z = 0 \). Summarizing, in the analysis of both conductance and DoS, it is sufficient to consider functions \( Y_{L,R} \) depending on a single argument with invariance property (17).

Following the basic philosophy of the transfer matrix approach, we will compute the functions \( Y_{L,R}(T, r) \) iteratively, by asking how much they change under infinitesimal variation of the arguments \( r \rightarrow r + \epsilon \). Considering the function \( Y_L \) for definiteness, we first notice that, by definition,

\[ Y_L(T, r + \epsilon) = \int DT e^{-\int_T^{r+\epsilon} dT' L[T]} Y_L(T(r), r). \]

For sufficiently small \( \epsilon \), the action can be expanded and we obtain

\[ Y_L(T, r + \epsilon) - Y_L(T, r) = \int dT' e^{-\frac{1}{4\pi} \text{str}(TT'^{-1} + T'^{-1}T) \times}
\]

\[ \times e^{\frac{N+L}{2} \text{str}(TT^{-1}) + \frac{3N+2L}{2} \text{str}(T + T^{-1})} (Y_L(T', r) - Y_L(T, r)), \]

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where we have used that, due to supersymmetry, \( \int D T \exp(- \int r^+ T L[T]) \times 1 = 1 \). We have also temporarily set the coupling constant of the operator \( L_{\text{Gade}} \) to zero. As mentioned above, this term does not have much relevance in the present context. We will briefly discuss its role in section (V E).

From hereon, the derivation of an evolution equation for \( Y_L \) is conceptually straightforward: the exponential weight factor in the first line of the equation enforces that \( T' \) is close to \( T \), symbolically, \( T'T^{-1} = 1 + O(\epsilon) \). We should thus expand \( T' \) around \( T \) and evaluate the integral \( \int dT' \) perturbatively in \( \epsilon \). This expansion is most conveniently done in the polar coordinates introduced above (because the heat kernel depends on the radial degrees of freedom, only.) As a result of a calculation similar but much more simple than the one for the standard \( \sigma \)-models with their larger matrix fields, one obtains the Schrödinger type equation

\[
\left( \mp \partial_t - 4(D \pm A)^2 + V(x, y) \right) Y_{L,R}(a, r) = 0, \tag{18}
\]

where \( V(x, y) = -i s(\cosh(x) - \cos(y)) \) and we have introduced the dimensionless parameters

\[ t \equiv \frac{r}{\xi}, \quad s = \pi \nu \xi \epsilon. \]

Physically, \( t \) is the length coordinate measured in units of the localization length \( \xi \) and \( s \) the energy measured in units of the single particle level spacing \( \Delta \xi \equiv \xi \nu \) of a system of length \( \xi \). Further, the symbol \( D = (D_x, D_y)^T \) denotes a vector differential operator defined through

\[
D_x = \partial_x - \frac{1}{2} \coth \left( \frac{x - iy}{2} \right), \quad
D_y = \partial_y + \frac{i}{2} \coth \left( \frac{x - iy}{2} \right), \tag{19}
\]

where the constant vector

\[ A = \frac{N + f}{2} (1, -i)^T. \tag{20} \]

Finally, evaluation of (18) for asymptotically small times \( t \to 0 \) leads to the initial condition

\[
\lim_{t \to 0} Y_{L,R}(x, y, t) = \delta(x, y) \equiv \lim_{t \to 0} e^{-\frac{1}{8t}(x^2 + y^2)}. \tag{21}
\]
It is very instructive to interpret the structure of the evolution equation (18) in the light of the analogy between field theory and point particle quantum mechanics on GL(1|1) discussed in the introduction. Within this picture, the functions $Y_L[T, r]$ acquire the meaning of Green functions, i.e. transition amplitudes for the propagation between the origin of the manifold $T = 1$ at time 0 and a final configuration $T$ at time $t$. The evolution equations (18) becomes a time-dependent Schrödinger equation with quantum Hamiltonian, $H = -2(D - A)^2 + V(x, y)$. While the term $V(x, y) = -i\epsilon \cosh(x) - \cos(y) = -\frac{i\epsilon}{2} \text{str}(T + T^{-1})$ simply represents the potential inherited from the Lagrangian, the 'kinetic' operator is more interesting. The covariant structure $(D - A)^2$ describes minimal coupling to the constant vector potential where the unfamiliarly looking structure of the derivative operator $D$ is a consequence of the fact that our particle lives on a curved manifold. Indeed, it is straightforward to verify that for $A = 0$

$$D \cdot D = \sum_{i=x,y} J^{-1} J_i J_i,$$

where $J(x, y)$, specified in Eq. (C2), is the square root of the determinant of the metric tensor on GL(1|1). This structure identifies $D \cdot D$ as the radial part of the Laplace operator on GL(1|1). ('Radial', because the two coordinates $x$ and $y$, spanning a maximally commutative sub-algebra of the Lie algebra of GL(1|1).)

Summarizing, we have identified $H$ as the Hamiltonian of a charged particle on the group manifold GL(1|1). Our next task will be to compute its Green functions $Y_{L/R}$. We begin by considering the case of a free particle, $V = 0 \sim \epsilon = 0$. The solution of this problem will contain the information needed to compute the conductance.

V. CONDUCTANCE

In this section the general formalism developed above is employed to compute the conductance $g$ of the system. Our main objective will be to understand the impact of topology on the localization behaviour of the system. However, before embarking on this analysis
it is tempting to digress for a moment and to briefly consider the transport behaviour of short systems, specifically the aforementioned anomalous sensitivity to the coupling to the leads. Being not directly related to the mainstream of the paper, the technicalities of this discussion have been deferred to Appendix D and we here restrict ourselves to a summary of results.

**A. Digression: Conductance of Short Wires**

In the present context, the phrase ’short’ means that systems in the quantum dot regime $L < \xi/(M\gamma)$ are considered: the conductance is not so much determined by the bulk transport properties of the system but rather by the strength of the coupling to the leads. Moreover, and this is a speciality of the sublattice system, the parity of the coupling, i.e. the even- or oddness of the connecting sites, turns out to play a crucial role. More precisely we find that (a) for systems where at both ends a number of sites of alternating parity are connected – the setup considered in much of this paper – the short system conductance is given by

$$g = \frac{M\pi\gamma}{2},$$

in accord with the behaviour of conventional quantum dots. The same result obtains (b) for systems where only one site at either end is coupled and these sites have opposite parity (even/odd or odd/even). However, (c) for single site coupling with even/even or odd/odd connectors, the conductance vanishes. To heuristically understand the phenomenon, it is instructive to consider the toy-model case of a strictly one-dimensional clean sublattice system. For zero energy, the wavelength of current carrying excitations is commensurable with the lattice spacing. This means that the relevant quantum wave functions have nodes at alternating sites. E.g., for an even-even configuration a state entering from the right has zero quantum amplitude at the exit site on the right. This implies a total blockade of electric current. A less obvious fact is that this phenomenon survives generalization to many channels and disorder.
We repeat that all these results are obtained for short systems; field fluctuations, describing the propagation of spatially non-uniform diffusive excitations, are neglected. An interesting question, not considered in this paper, is how such modes would affect the conductance as the system size is increased.

B. Reduction of the Problem to (0-dimensional) quantum mechanics on GL(1|1)

We next turn back to the discussion of large systems (equilibrated coupling of type (a) understood.) Inspection of the basic expression (15) shows that the problem factorizes into doing the boundary integrals and analysing the bulk behaviour of the heat kernel, respectively.

We begin by discussing the boundary regions. Following a line of arguments developed in Ref. [32], we first notice that due to the exponential weights \( \sim \exp(-\frac{\pi M \gamma}{2} \text{str} (T + T^{-1})) \), the integrands are confined to the immediate vicinity of the group origin. This suggests to write \( T = \exp W \), and do the integrals over generators \( W \) in a Gaussian approximation. Setting \( W = \begin{pmatrix} u & \sigma \\ \rho & iv \end{pmatrix} \) and expanding in coefficients we arrive at integrals of the structure,

\[
g = \text{const.} \times \int dudvd\rho d\sigma e^{-\frac{\pi M \gamma}{4} (u^2 + v^2 + 2\sigma \rho) \sigma F(T(u, v, \sigma, \rho))},
\]

where the symbol \( F(T) \), represents the functional dependence of the heat kernel on the boundary field. Evaluation of the Gaussian superintegral leads to

\[
g = \text{const.} \times \frac{1}{\pi M \gamma} \partial_\rho F(T)|_{T=\mathbb{I}}.
\]

Doing the same procedure for the second boundary integral and fixing factors we obtain

\[
g = \partial_\rho \partial_\sigma Y_L(T, T', L)|_{T=T'=\mathbb{I}} = \partial_\rho \partial_\sigma Y_L(TT'^{-1}, L)|_{T=T'=\mathbb{I}},
\]

where the second expression contains the one-argument heat kernel introduced in Eq. (13). According to this expression, the conductance is obtained by second order expansion of the heat kernel around the origin. We next note that due to the invariance property (17) the expansion starts as
\[ Y_L(T = e^{\tilde{W}}, L) = 1 + c_1 \text{str}(\tilde{W}) + c_2 \text{str}(\tilde{W}^2) + c_3 [\text{str}(\tilde{W})]^2 + \ldots. \] \hspace{1cm} (22)

Now, in our case,

\[ \tilde{W} = \ln(TT_t^{-1}) = \ln \left( \exp \begin{pmatrix} 0 \\ \rho \end{pmatrix} \exp \begin{pmatrix} 0 & -\sigma' \\ \rho & \frac{1}{2} \sigma' \rho \end{pmatrix} \right) = \begin{pmatrix} \frac{1}{2} \sigma' \rho & -\sigma' \\ \rho & \frac{1}{2} \sigma' \rho \end{pmatrix}. \]

Substitution of this expression into (22) and differentiation leads to

\[ g = 2c_2, \] \hspace{1cm} (23)

i.e. the problem has been reduced to fixing the coefficients of the series expansion of \( Y_L \).

Notice that this series representation is totally determined by the features of the bulk system; all aspects related to the coupling to the leads have disappeared from the problem.

We now have to face up to the principal task, the calculation of the heat kernel. Its interpretation as the Green function of the problem suggests to begin by representing \( Y_L \) through a formal spectral decomposition: consider Eq. (18) at zero potential, \((-\partial_t - 4(D - A)^2) Y_L(x, y, t) = 0\), and suppose we had managed to find a set of eigenfunctions,

\[ -4(D - A)^2 \Psi_n^{(r)} = \epsilon_n \Psi_n^{(r)}. \] \hspace{1cm} (24)

Assuming completeness, we can then span the heat kernel as

\[ Y_L(x, y, t) = \sum_n c_n \Psi_n^{(r)}(x, y) e^{-\epsilon_n t}, \]

where the expansion coefficients are determined through the initial condition (21). Provided the spectrum is suitably structured (positive and gapped against some low lying levels), and keeping in mind that we are interested in asymptotically large values of \( t \), the sum may be restricted to a limited set of \( n \)'s. The problem thus reduces to (a) exploring the low-lying spectral content of the operator \( D - A \), and (b) fixing the expansion coefficients.
C. Spectrum and Eigenfunctions of the 'Kinetic Energy Operator' on GL(1|1)

In spite of the unfamiliarly looking coordinate representation of \((D - A)^2\) analytic progress with the problem is straightforward, the reason being that this operator is nothing but a plane wave operator in disguise. To make the hidden simplicity of the problem manifest, let us first remove the dependence of the differential operator on the (pure gauge) potential \(A\): transformation \(\Psi_n(x, y) \rightarrow e^{N/2(x - iy)} \tilde{\Psi}_n(x, y) \equiv \tilde{\Psi}_n(x, y)\) brings the eigenvalue equation (24) into the form

\[-4\Delta \tilde{\Psi}_n = \epsilon_n \tilde{\Psi}_n,\]

where \(\Delta = DD\) is the radial part of the Laplacian on GL(1|1). Notice that the structure of this equation does not imply that the vector potential has disappeared from the problem. It has merely been transferred from the differential operator to the boundary conditions attached to the differential equation. While irrelevant in the non-compact sector of the theory, changes of the boundary conditions in the compact sector generally cause qualitative effects. To appreciate this point, notice that the un-gauged Hilbert space of the problem has periodic boundary conditions, \(\Psi_n(x, y) = \Psi_n(x, y + 2\pi)\). Yet, after the gauge transformation, \(\tilde{\Psi}_n(x, y) = \tilde{\Psi}_n(x, y + 2\pi)(-)^{N+f}\), i.e. for \(N\) odd or non-zero staggering a transmutation to twisted boundary conditions has taken place. Needless to say that this change bears consequences for the spectral structure of the problem and, therefore, for the transport behaviour of the system.

To make further progress, we subject the eigenvalue problem to the similarity transformation

\[\tilde{\Psi} \rightarrow J^{1/2}\tilde{\Psi} \equiv \Psi,\]

\[\Delta \rightarrow J^{1/2} \Delta J^{-1/2} \equiv \tilde{\Delta} = \partial^2_x + \partial^2_y.\]

This change of representation entails an enormous simplification of the problem. The un-invitingly looking operator \(\Delta\) has become a flat two-dimensional Laplace operator [33]. However, notice that the transformation by \(J^{1/2}\) effects yet another change in the bound-
ary conditions: due to \( J^{1/2}(x,0) = -J^{1/2}(x,2\pi) \), the transformed states are subject to the condition

\[
\tilde{\Psi}_n(x,y) = \tilde{\Psi}_n(x,y + 2\pi)(-)^{N+f+1}.
\]

At this point, the solution of the eigenvalue problem has become a triviality. The equation

\[-4(\partial^2_x + \partial^2_y)\tilde{\Psi}_n(x,y) = \epsilon_n \tilde{\Psi}_n(x,y),\]

is solved by the exponentials \( \tilde{\Psi}_{k,l}(x,y) = e^{i p_k x + i p_l y} \), where \((k,l) \equiv n\) are two 'quantum numbers', \( p_{k,l} \) the associated momenta and the eigenvalues \( \epsilon_{kl} = 4(p_k^2 + p_l^2) \). From these states we obtain our un-gauged and un-transformed original wave functions as

\[
\Psi_{kl}(x,y) = \sinh \left( \frac{x - iy}{2} \right) e^{(i p_l - \frac{N+f}{2})x + i(p_k + \frac{N+f}{2})y}. \tag{25}
\]

To give this set of solutions some meaning, we need to specify the range of permissible \( k \)'s and \( l \)'s. In the compact sector the situation is clear – the circular boundary condition specified above enforces \( p_k = k - \frac{N+f}{2} \), with halfinteger \( k \). The conditions to be imposed in the non-compact sector are tightly linked to the integrability properties of our wave functions:

The space of radial functions on GL(1|1) is endowed with a natural scalar product, viz.

\[
\langle f, g \rangle \equiv \int_{-\infty}^{\infty} dx \int_0^{2\pi} dy J(x,y) f(x,y) g(x,y). \tag{26}
\]

We demand that the eigenfunctions contributing to the spectral decomposition of the heat kernel be square integrable w.r.t. \( \langle , \rangle \). Inspection of Eq. (25) shows that this requirement enforces \( p_l = l - i\frac{N+f}{2} \), where \( l \) may be arbitrary and real.

Finally, we need to specify a set of functions sufficiently complete to generate an expansion of the heat kernel. The present problem does not come with a natural Hermitian or symmetric structure (i.e. for finite \( A \) the kinetic energy neither symmetric nor Hermitian.) However, defining a 'fake complex conjugation' through \( \bar{\Psi}_{k,l} = \Psi_{-k,-l} \) it is straightforward to show that

\[
\langle \bar{\Psi}_{kl}, \Psi_{k'l'} \rangle = (2\pi)^2 \delta(l - l')\delta_{kk'}. \tag{27}
\]
We may thus attempt to represent sufficiently well behaved (for the cautious formulation, see below) functions as

\[ g(x, y) = \sum_k \int dl \ g_{kl} \Psi_{kl}(x, y), \quad g_{kl} = (2\pi)^{-2} \langle \bar{\Psi}_{kl}, g \rangle. \]  

(28)

Before applying this procedure to the heat kernel, let us summarize our main findings for clarity: the radial Laplacian on GL(1|1) is diagonalized by the set of functions

\[ \Psi_{kl}(x, y) = \sinh \left( \frac{x - iy}{2} \right) e^{i(\ell x + k y)}. \]  

(29)

where \( k \in \mathbb{Z} + 1/2, \ell \) real and the eigenvalues are given by

\[ \epsilon_{kl} = 4 \left[ \left( k - \frac{N + f}{2} \right)^2 + \left( \ell - i \frac{N + f}{2} \right)^2 \right]. \]  

(30)

(Notice that the appearance of an imaginary part proportional to the strength of the vector potential is due to the fact that for finite \( A \), the Hamiltonian of the theory is neither symmetric nor Hermitian.) The spectral decomposition of radial functions is defined through Eqs. (26), (27) and (28).

D. Computation of the Conductance

We now apply the machinery developed in the last section to the analysis of the heat kernel. In principle, the strategy seems to be prescribed by what was said above. We should determine the Fourier coefficients of the initial configuration (21), \( \delta_{kl} = \langle \bar{\Psi}_{lk}, \delta \rangle \), from where the heat kernel would follow as

\[ Y_L(x, y, t) = (2\pi)^{-2} \sum_k \int dl \ \delta_{kl} e^{-\epsilon_{kl}t} \Psi_{kl}(x, y). \]  

There is a problem with this procedure, viz. the expansion coefficients of the '\( \delta \)-distribution' \( \delta(x, y) \) vanish. Indeed one verifies that in the limit \( t \to 0 \), the support of the Gaussian in (21) shrinks to zero while its maximum remains limited by one. This readily implies \( \langle \Psi, \delta \rangle = 0 \). The reason for this pathological behaviour is that our radial theory memorizes that it derived from a supersymmetric parent theory. In a sense, supersymmetry can be interpreted as a theory on a zero-dimensional background, i.e. there is no singular 'volume factor' compensating...
for the vanishing support of the δ-function, as would be the case in spaces with positive dimension.

The problem can be circumvented by a cute trick \[32\]. Instead of Fourier expanding \( Y_L \), we consider the function \( Y_L - 1 \). Since unity by itself solves the heat equation, no harm has been done and all that has changed is the boundary condition: \( \lim_{t \to 0} (Y_L(x, y, t) - 1) = \delta(x, y) - 1 \), a function that equals minus unity almost everywhere save for the origin where it vanishes.

We thus represent the heat kernel as

\[
Y_L(x, y, t) = 1 - (2\pi)^{-2} \sum_k \int dl \, 1_{kl} \Psi_{kl}(x, y)e^{-\epsilon_{kl}t}, \tag{31}
\]

where

\[
1_{kl} = \langle \Psi_{kl}, 1 \rangle = \int dx \int dy \frac{e^{i(lx + ky)} \sinh \left( \frac{x - iy}{2} \right)}{l - ik} = \frac{4\pi i}{l - ik}.
\]

(To obtain the last equality, it is convenient to first do the \( x \)-integral. Closure of the integration contour in the upper/lower complex half plane for positive/negative \( l \) yields a semi-infinite sum over residues of the sinh-function, along with a \( y \)-integral that is of simple plane wave type. Doing sum and integral one obtains the result.)

Substitution of this result into the expansion of the heat kernel now yields

\[
Y_L(x, y, t) - 1 = -\frac{i}{\pi} \sum_k \int dl \, \frac{1}{l - ik} \sinh \left( \frac{x - iy}{2} \right) e^{i(lx + ky)} e^{-\epsilon_{kl}t} \frac{o(x, y)^2}{4\pi} \sum_k \int dl \left( \frac{1 + ik}{l - ik} (x - iy)^2 + (x^2 + y^2) \right) e^{-\epsilon_{kl}t} =
\]

\[
\frac{1}{4\pi} \sum_k \int dl \left( \frac{l + ik}{l - ik} (\text{str}(W))^2 + \text{str}(W^2) \right) \left( x - iy)^2 + (x^2 + y^2) \right) e^{-\epsilon_{kl}t},
\]

where in the last line we have switched back to a coordinate invariant representation. Comparison with Eqs. (22) and (23) finally leads to the identification

\[
g = \frac{1}{2\pi} \sum_{k \in \mathbb{Z} + 1/2} \int_{-\infty}^{\infty} d\xi e^{\frac{i\xi}{4\pi} \left( k - \frac{N+l}{2} \right)^2} = \frac{1}{2} \left( \frac{\xi}{4\pi L} \right)^{1/2} \sum_{k \in \mathbb{Z} + 1/2} e^{-\frac{i\xi}{4\pi}(k - \frac{N+l}{2})^2}.
\]
where we have inserted the explicit form of the eigenvalues. Notice that all manipulations leading from the original $\sigma$-model representation to the above Gaussian integral representation have been exact.

We next evaluate this result in the two limiting cases of physical interest, $L \ll \xi$ (Ohmic regime) and $L \gg \xi$ (localized regime.) Beginning with the Ohmic case, we first notice that for $L \ll \xi$ many terms contribute to the $k$ summation implying that the sum can be approximated by an integral. Thus,

$$g \overset{L \ll \xi}{\approx} \frac{1}{2} \left( \frac{\xi}{4\pi L} \right)^{1/2} \int dk e^{-\frac{4L}{\pi} \left(k - \frac{N-1}{2}\right)^2} = \frac{\xi}{16L}.$$  

As for any ordinary conductor, $g$ is inversely proportional to the system size; the parity of the channel number does not play a role.

In the opposite case, $L \gg \xi$, only those discrete indices that minimize the exponent contribute to the sum. Specifically, for an even channel number and no staggering,

$$g \overset{L \gg \xi}{\approx} \left( \frac{\xi}{4\pi L} \right)^{1/2} e^{-\frac{L}{\xi}}, \quad N \text{ even, } f = 0.$$  

In contrast, for $N$ odd and $f$ still zero, the exponent vanishes for the half integer $k = \frac{N}{2}$ and

$$g \overset{L \gg \xi}{\approx} \frac{1}{2} \left( \frac{\xi}{4\pi L} \right)^{1/2}, \quad N \text{ odd, } f = 0$$

depends algebraically on the system size. Finally, it is clear that for non-vanishing staggering, $f \neq 0$, intermediate types of behaviour are realized. E.g., an $N$ even chain with staggering $f = \pm 1$ behaves like a symmetric $N$ odd chain, etc.

E. The Role of the Gade Term

Before leaving this section let us briefly discuss the role of the, so far neglected, Gade operator $S_{\text{Gade}}[T]$. The inclusion of this term in the derivation of the heat equation is straightforward. As a result, the planar Laplacian $\tilde{\Delta} = \partial_x^2 + \partial_y^2$ gets replaced by

$$\tilde{\Delta} = \partial_x^2 + \partial_y^2 + \tilde{\eta}(\partial_x - i\partial_y)^2,$$
where $\tilde{\eta} \equiv \frac{16C}{\xi} \propto N^{-1} \ll 1$. This operator is still diagonalized by the plane waves discussed above. The eigenvalues change to

$$\epsilon_{kl} = 4 \left[ (1 - \tilde{\eta}) \left( k - \frac{N+f}{2} \right)^2 + (1 + \tilde{\eta}) \left( l - \frac{iN+f}{2} \right)^2 \right].$$

Recapitulating the computation of the conductance, one finds that the small dilatation introduced by finite values of $\tilde{\eta}$ does not affect the long range transport behaviour of the system.

All this is compatible with the structure of the DMPK transfer matrix approach to the problem. As mentioned above, the DMPK evolution equation is controlled by two coupling constants. One of these, in Ref. [10] denoted by $\eta$, is small, $\eta \sim O(M^{-1})$ and becomes inessential in the limit of a large number of channels. The analysis above suggests that the coupling constant of the Gade operator, $\propto \tilde{\eta}$, and the $\eta$ of the DMPK approach play the same role. This analogy is supported by the above discussed geometric structure of the target spaces of the two theories.

VI. DENSITY OF STATES

We now turn our attention to the low energy density of states of the sublattice system. As with the conductance, we will first consider the behaviour of short wires, and then discuss the localized regime.

A. Density of states of short wires

As in section V A we consider a short sublattice wire of length $L < \xi/(\gamma M)$. The spectrum of such systems exhibits structure on the scale of the mean level spacing. In order not to blur these fine structures, the coupling to the external leads will be switched off throughout this section. Following the logics of section V A, one would then conclude that the action reduces to $S[T] = \int dr \mathcal{L}_z[T] = -i \frac{s}{2} \text{str} (T + T^{-1})$, where the matrix $T$ parameterizes a zero mode configuration, $s = \frac{\nu}{\Delta}$ and $\Delta = (\nu L)^{-1}$ is the level spacing of the
isolated system. (Temporarily deviating from the convention of the rest of the paper, in this section $s$ measures the energy in units of the total level spacing and not the level spacing of a localization volume.) This presumption is almost but not quite correct. The point is that our so far discussion of the low energy action implicitly assumed that the number of sites of the system is even. In the opposite case, an extra contribution $S_{\text{top},2}[T] = N \text{str ln}(T(L))$, derived and discussed in section B1, appears. The structure of this term reflects the fact that due to the presence of one uncompensated site the global $\text{GL}(1|1)$-invariance of the model is lost. ($S_{\text{top},2}[T]$ is not invariant under $T \to T_L T_T$ even for constant $T_{L,R}$.) In the majority of cases this extra term is of little interest. However for the spectral properties of a short isolated system, the presence of $S_{\text{top},2}$ bears crucial effects. Indeed we will see that this term is responsible for the formation of staggering phenomena akin, and probably related to the effects discussed earlier in section V A.

Adding the two contributions to the action and using Eq. (9) we obtain

$$\nu(\epsilon) = \frac{1}{2\Delta} \int dT \left( T_{11} + T_{11}^{-1} \right) e^{i \frac{2}{\Delta}(T+T^{-1}) - N \text{str ln}(T)},$$

for the $\sigma$-model representation of the zero-dimensional DoS. The task thus is to integrate over a single copy of the target manifold $\text{GL}(1|1)$. For sufficiently small energies $\epsilon \sim \Delta$, the action is not large enough to confine the integrand to the origin of the group manifold implying that the integral has to be done non-perturbatively. Referring to Ref. [5] and Appendix [3] for technical details we here merely display the final result of this integration procedure,

$$\nu(s) = \frac{\pi \delta(s)}{2\Delta} \left[ J_0^2(s) + J_1^2(s) \right], \quad L \text{ even}$$

$$\nu(s) = N \delta(s) + \frac{\pi \delta(s)}{2\Delta} \left[ J_N^2(s) - J_{N+1}(s) J_{N-1}(s) \right], \quad L \text{ odd}$$

The structure of these DoS profiles is shown in Fig. 3 for the example $N = 3$.

Eqs. (32) have been obtained earlier within pure random matrix theory [19], and its supersymmetry implementation [33,21]. Studies of chiral random matrix ensembles were largely motivated by the relevance of the spectral structure of effectively zero dimensional
chiral systems in finite size lattice QCD. More generally, 'microgaps' of the type shown in Fig. 2 are an omnipresent side effect seen in the spectrum of generic chiral systems with finite mean level spacing $\Delta$.

![Density of states of the short sublattice system. Solid line: DoS of a system with $L$ even. Dashed line: DoS of a system with $L$ odd and $N = 3$. (The singular zero energy spike causing the spectral depletion up to values $s \sim 3$ is not shown.)](image)

FIG. 3. Density of states of the short sublattice system. Solid line: DoS of a system with $L$ even. Dashed line: DoS of a system with $L$ odd and $N = 3$. (The singular zero energy spike causing the spectral depletion up to values $s \sim 3$ is not shown.)

Qualitatively, origin and structure of these gaps can easily be understood. First, the chiral symmetry $[H, \sigma_3]_+ = 0$ entails that for any non-vanishing energy level $\epsilon_n$ its negative $-\epsilon_n$ is an eigenvalue, too. Disorder generated level repulsion prevents these states from coming close to each other (on the scale of the mean level spacing) which explains the presence of the spectral gap in the $L$-even case. For $L$ odd, this picture has to be modified. To understand what is happening, let us imagine the Hamiltonian as a block off-diagonal matrix in sublattice space:

$$H = \begin{pmatrix} Z \\ Z^\dagger \end{pmatrix}$$

Since $L$ is odd, the number of odd sites exceeds the number of even sides by one, i.e. the blocks $Z$ are rectangular with $N(L - 1)/2$ rows and $N(L + 1)/2$ columns. Now, any block-off diagonal matrix with $k$ rows and $l$ columns has $|k - l|$ eigenvalues 0. Applied to our system, this means that the $L$ odd system has $N$ zero modes for any realization of disorder (the $\delta$-function contribution in $[32]$.) Other levels repel from this concentrated accumulation.
of spectral weight which explains the bathtub type suppression of the DoS up to energies \( \sim N\Delta \).

Summarizing we have seen that the \( L \) even/odd staggering behaviour observed earlier in connection with the conductance pertains to spectral properties. Above we had argued that the vanishing of the conductance in the \( L \) even case was due to the peculiar spatial profile of zero energy wave functions. In view of (32) it is tempting to relate the same effect to the vanishing of spectral weight at zero energy, although we have not analysed this picture any further.

Keeping in mind the tendency towards buildup of zero energy spectral weight in systems with mis-matched sublattice structure we next turn back to the analysis of large systems.

B. Heat kernel for finite energies

The aim of this section is to compute the DoS of a large system with \( L \gg \xi \). To facilitate comparison with the behaviour of the conductance we will drop the assumption of isolatedness and again couple the system to leads. Notice that this stands complementary to the analysis of Ref. [11], where a closed system was discussed. That we will obtain identical scaling of the DoS is proof of the (in view of the existence of topological zero modes not entirely obvious) assertion [11] that boundary conditions do not affect the bulk spectrum.

We will not be able to compute the DoS \( \nu(\epsilon) \) for arbitrary \( \epsilon \). Instead, an asymptotic expression valid for low energies will be derived. Remembering the behaviour found for small systems we anticipate nonanalytic behaviour, \( \nu(\epsilon) \sim \ln^n(|\epsilon|)|\epsilon| \), where the logarithmic factor is due to potentially existent localization corrections to the zero mode behaviour. Our objective is to identify the most singular contribution of this type.

Starting point of the analysis is the transfer matrix representation (14) of \( \nu(\epsilon) \). To evaluate this expression we need to compute the functions \( Y^{L/R} \), now for finite potential \( V \). We will do this following a procedure developed in Ref. [36]. Starting out from a spectral decomposition of the type (31), we first notice that only zero energy eigenfunctions
contribute to long distance behaviour of $Y_{L/R}(r)$. This means that our ‘time’ dependent ‘Schrödinger equation’ (18) can effectively be replaced by the stationary form

$$t \gg \xi : \quad \left( -(\mathbf{D} \pm \mathbf{A})^2 + \eta(\cosh(x) - \cos(y)) \right) Y_{L/R}(x, y) = 0,$$

(33)

where we have substituted the explicit form of the potential and omitted the spatial argument of $Y_{L/R}$ for simplicity. For later convenience we have also analytically continued from real to imaginary energy arguments, $-is^+ \rightarrow 4\eta > 0$. Substitution of these functions into Eq. (14) yields the reduced representation

$$\nu(E) = \nu_0 \left( 1 + \text{Re} \frac{1}{2\pi} \int dx dy \frac{\cosh(x) - \cos(y)}{\sinh^2 \left( \frac{x-iy}{2} \right)} Y_L(x, y) Y_R(x, y) \right).$$

(34)

To obtain this equation we have expanded the pre-exponential sources in Grassmann variables and integrated over these. The constant contribution $\nu_0$ appears as a consequence of the Efetov-Wegner theorem. (The representation of the pre-exponential term in polar coordinates contains a purely non-Grassmann contribution. Integration over a term of this type obtains the integrand at the origin [31] which, in our case, equals unity.)

We now turn to the actual computation of the functions $Y_{L/R}$. The first step is the derivation of a set of matching, or boundary conditions relating the heat kernel to its $\eta \rightarrow 0$ asymptotics. To this end we evaluate the $\eta = 0$ spectral decomposition (31) in the limit $t \gg \xi$. Neglecting contributions that decay exponentially in $t/\xi$, it is straightforward to obtain the asymptotic expressions:

$$Y_{L/R}^\eta(x, y) \overset{\eta \rightarrow 0}{\rightarrow} 1, \quad N \text{ even},$$

$$Y_{L/R}^\eta(x, y) \overset{\eta \rightarrow 0}{\rightarrow} \frac{1}{2}(e^{\pm (x-iy)} + 1), \quad N \text{ odd},$$

where we have temporarily neglected the staggering parameter $f$. The structure of these functions will be motivated shortly. Turning to the case $\eta \neq 0$, we next gauge and transform Eq. (33) as in section [34]. Transformation $Y_{L}(x, y) \rightarrow J^{1/2}(x, y)e^{\mp \frac{1}{2}P_N(x-iy)}Y_{R}(x, y) \equiv \tilde{Y}_{L}(x, y)$, where $P_N \equiv (1 + (-)^N)/2$ brings the equation into the form

$$\left( \partial_x^2 + \partial_y^2 - \eta(\cosh(x) - \cos(y)) \right) \tilde{Y}_{L/R}(x, y) = 0,$$

(35)
while the transformed $\eta \to 0$ asymptotics read
\[
\tilde{Y}_{L/R}(x, y) \xrightarrow{\eta \to 0} \sinh^{-1}\left(\frac{x-iy}{2}\right) \xrightarrow{|x| \gg 1} 2 \text{sgn}(x)e^{-\text{sgn} x}\left(\frac{x-iy}{2}\right), \quad N \text{ even},
\]
\[
\tilde{Y}_{L/R}(x, y) \xrightarrow{\eta \to 0} \coth\left(\frac{x-iy}{2}\right) \xrightarrow{|x| \gg 1} \text{sgn}(x), \quad N \text{ odd}.
\]
Eq. (36)

Here we have anticipated that the dominant contribution to the above double integral representation of the DoS will come from large values of the non-compact variable $x$. Eqs. (35) and (36) have the nice feature of full separability. Writing $\tilde{Y}(x, y) = N Y_1(x) Y_2(y)$, where $N$ is a normalization factor and the subscript $L/R$ has been dropped for simplicity, Eq. (35) can be traded for the set of decoupled ordinary differential equations
\[
\left(\partial_x^2 - \frac{\eta}{2} e^x - \frac{P_N}{4}\right) Y_1(x) = 0 + O(\eta),
\]
\[
\left(\partial_y^2 + \frac{P_N}{4}\right) Y_2(y) = 0 + O(\eta).
\]
The second equation is trivially solved by $Y_2(y) \approx e^{\pm iP_N \frac{y}{2}}$. The first line is a Bessel equation. Its two solutions are given by $I_{\nu}(\eta) e^{\pm \text{sgn} x}$ and $K_{\nu}(\eta) e^{\pm \text{sgn} x}$. Discarding the exponentially divergent solutions $I_{\nu}$ and using that for small arguments, $K_{\nu}(\eta) \approx -\ln(\frac{\eta}{2})$, the normalization constants $N$ can now be fixed by matching to the zero energy asymptotics in the limit of small $\eta$. This obtains the approximate solution
\[
\tilde{Y}_{L/R}(x, y) \xrightarrow{|x| \gg 1} \left(8\eta\right)^{1/2} K_1\left((2\eta)^{1/2}e^{\pm |x|/2}\right) e^{i\text{sgn} x \frac{y}{2}}, \quad N \text{ even},
\]
\[
\tilde{Y}_{L/R}(x, y) \xrightarrow{|x| \gg 1} -\frac{2}{\ln \eta} K_0\left((2\eta)^{1/2}e^{\pm |x|/2}\right), \quad N \text{ odd}.
\]
It is now a straightforward matter to substitute this expression back into the above integral representation for $\nu$ and to integrate over coordinates. The $y$-integration, extending over a purely harmonic integrand, is trivially done. (Notice that the sinh^{-2}-factor in (34) cancels against the factor $(J^{1/2})^2$ from the similarity transformation). As for the $x$-integration, we note that due to the exponentially decaying asymptotics $K_{\nu}(z) \sim \exp(-z/2)$ for $|z| \gg 1$, the integral can be cut off at $(2\eta)^{1/2}e^{\pm |x|/2} \sim 1 \Rightarrow |x| \sim -\ln(2\eta)$. Within the domain of integration, the Bessel functions can be replaced by the small argument asymptotics specified above. Substituting these expressions it is then straightforward to obtain $\nu(\eta) \approx \nu_0 \text{Re} \eta \ln^2 \eta$. 

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(even $N$) and $\nu(\eta) \approx \nu_0 \Re (\eta \ln^2 \eta)^{-1}$ (odd $N$) for the small $\eta$-asymptotics of the DoS. Analytic continuation back to real energies finally leads to the result

$$\nu(s) \approx -\nu_0 |s| \ln |s|, \quad N \text{ even},$$
$$\nu(s) \approx -\frac{\nu_0}{|s| \ln^3 |s|}, \quad N \text{ odd},$$

(37)

for the low energy behaviour of the DoS. Eqs. (37) agree with the results found earlier in Ref. [11].

We finally discuss the extension of the above results to non-zero staggering. For non-vanishing $f$ and even $N$, the large argument asymptotics (36) generalize to

$$\tilde{Y}_{L/R}(x, y) \xrightarrow{\eta \to 0, x \gg 1} 2e^{\frac{i}{2}(\pm f + 1)(x - iy)}.$$  

(38)

One can now follow the same steps as in the non-staggered cases above to obtain the result

$$\nu = 2\nu_0 \frac{\Gamma(|f|)}{\Gamma(2 + |f|)} \frac{2^{-2(1-|f|)}}{|f|} \cos \left( \frac{\pi}{2} (1 - |f|) \right) |s|^{1-|f|}.$$  

(39)

For non-zero $f$, the DoS vanishes in a more singular manner as in the non-staggered case. This behaviour provokes the question, how matching with the diverging profile in the case $N$ odd, $f = 0$ might be obtained. The principal structure of the theory entails that $(N \text{ even}/f = 1)$ should be equivalent to $(N \text{ odd}/f = 0)$. On the other hand, the $f \not\to 1$ version of Eq. (39) certainly does not agree with the divergent result (37).

To resolve this paradox, it is helpful to re-interpret the asymptotic expressions (36) and (38) within the quantum mechanical picture of the theory. Focusing on the compact sector and temporarily ignoring the factor $J^{1/2}$ from the transformation to a flat Laplacian, these functions acquire the meaning of ground state wave functions $\Psi_0$ of a one dimensional ring subject to a gauge flux $N + f/2$. For $N$ even and zero $f$, an integer number of 'flux quanta' pierce the ring, and the ground state wave function carries zero 'momentum', $\Psi_0(y) \propto 1$ which, after multiplication with $J^{1/2}$ leads to the first line of Eq. (36). In contrast, for $N$ odd and $f$ still zero, a half integer flux pierces the ring. This is a special situation in the sense that the ground state wave function is two-fold degenerate, i.e. $\Psi_0(y) = c_+ e^{iy/2} + c_- e^{-iy/2}$. Our earlier
analysis has fixed the a priori un-determined constants $c_\pm$ to a symmetric configuration. Inclusion of the non-compact variable and multiplication with $J^{1/2}$ then leads to the second line of (36).

We can now understand what happens as $f$ is turned on for an $N$ even configuration: a flux $f/2$ is sent through the ring and the ground state wave function remains unique (cf. Eq. (38)) until $f$ comes close to the critical value 1. In the immediate vicinity of the degeneracy point, the fact must no longer be neglected that our one-dimensional system is subject to a weak potential $\eta \cos(y)$. For values of $f$ such that the level splitting $\sim 1 - f$ between the two nearly degenerate levels becomes comparable with the characteristic strength of the potential $\sim \eta$, the ‘true’ ground state configuration is given by the symmetric superposition of the two levels, as in the $N$ odd case. For these values of $f$ the ground state configuration is given by the second line of Eq. (36) and the DoS follows the $N$ odd asymptotics.

This qualitative argument predicts that for asymptotically small energies, the DoS scales as in Eq. (39). However, for larger values of the energy, $\eta \sim |1 - f|$, a crossover to the characteristics of the $N$ odd $f = 0$ DoS profile takes place.

VII. SUMMARY

In this paper, transport and spectral properties of weakly disordered quantum sublattice wires have been explored from a fieldtheoretical perspective. We re-derived results obtained previously within the DMPK transfer matrix formalism, observed a surprisingly strong sensitivity of system properties to the realization of the lead/device coupling, and found that conductance and DoS, at least of short systems, exhibit drastic dependence on the parity of the total site number in the sublattice chain. It is likely that both this phenomenon and the dependence of system properties on the parity of the channel number root in the same origin, i.e. the existence of zero energy states for effectively block off-diagonal Hamiltonians with rectangular (non-quadratic) block structure. Although we are not aware of an intuitive explanation for the channel number parity effects, this belief is supported by the observation
that all staggering phenomena are controlled by the same topological term $S_{\text{top}}$ in the action of the $\sigma$-model. From these findings one might expect that the delocalized band-centre behaviour exhibited by the two-dimensional sublattice model, is driven by the 2d-analogue of this operator. Curiously, this is not so. In the two-dimensional field theory, the so-called Gade term $S_{\text{Gade}}$, a two-gradient operator contribution with small and non-universal coupling constant, drives the system towards de-localization. The operator $S_{\text{top}}$ does have a generalization to two dimensions [5], but its role has not been investigated so far. Summarizing we find that comparable phenomenology (metallic behaviour and diverging DoS) in one and two dimensions is described by different operators in the $\sigma$-model action. This indicates that some ‘deeper’ physical principle, not understood at present, lies beyond the visible structure of the field theory.

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APPENDIX A: FIELD INTEGRAL FORMULATION

$\sigma$-model representations of zero- and two-dimensional systems with AIII-symmetry have been constructed in different contexts before [21,22]. That the following two sections discuss the construction of the field theory in some detail is motivated by non-generic features particular to the 1d-system, most notably the coupling operators and the existence of topological structures. In the present section, we will derive a representation of the model and its correlations functions in terms of a supersymmetric field integral. The projection of this, a priori exact representation onto its low energy sector will be the subject of the subsequent Appendix [3].

Consider the two correlation functions
\[ C_{\alpha\alpha'}^{(1)} \equiv \langle G_{\alpha\alpha'}(E^+) \rangle \quad \text{DoS} \]
\[ C_{\alpha\beta\alpha'\beta'}^{(2)} \equiv \langle G_{\alpha\beta}(0^+)G_{\alpha'\beta'}(0^-) \rangle \quad \text{conductance}, \]

relevant for the computation of DoS and conductance, respectively. To compute these objects, we follow the now standard supersymmetry scheme for disordered electronic systems \[31\] and represent the Green functions as

\[ G_{\alpha\alpha'}(z) = \frac{\delta}{\delta J_{\alpha'\alpha}^{\beta}} \bigg|_{\hat{J} = 0} \langle Z[\hat{J}] \rangle = - \frac{\delta}{\delta J_{\alpha'\alpha}^{\beta}} \bigg|_{\hat{J} = 0} \langle Z[\hat{J}] \rangle, \]

where

\[ Z[\hat{J}] = \int \mathcal{D}(\bar{\psi}, \psi) e^{i\bar{\psi}(z-H+i\pi\sum_c W c W^c \bar{c} \hat{J} \psi)}, \quad (A1) \]

and we have assumed that \( \text{sgn} \text{Im} z > 0 \). (In the opposite case, the sign of both the total action and the coupling operator change.) Here \( \psi = \{(S_{\alpha}, \chi_{\alpha})^T \} \) and \( \bar{\psi} = \{(\bar{S}_{\alpha}, \bar{\chi}_{\alpha})\} \) are two-component superfields where \( S_{\alpha} \) is the complex conjugate of \( S_{\alpha} \), while \( \chi_{\alpha} \) and \( \bar{\chi}_{\alpha} \) are independent Grassmann variables. The source field

\[ \hat{J} = \begin{pmatrix} \hat{J}^b & 0 \\ 0 & \hat{J}^f \end{pmatrix} \]

where \( \hat{J}^{b,f} = \{ \hat{J}_{\alpha\alpha'}^{b,f} \} \) are ordinary matrices in site and orbital space.

From this representation, the one-point correlation function obtains as \( C_{\alpha\alpha'}^{(1)} = \delta_{\alpha'\alpha} \bigg|_{\hat{J} = 0} \langle Z[\hat{J}] \rangle \) for \( z = E^+ \). However, at first sight it looks like two Gaussian field integrals \( (A1) \) are needed to compute the two-particle correlator \( C^{(2)} \), one for the Green function \( G(0^+) \) the other for \( G(0^-) \). Fortunately, this is not so, a direct consequence of the chiral symmetry of the Hamiltonian: the relation \([\sigma_3, \hat{H}]_+ = 0 \) implies that

\[ \hat{G}(z) = -\sigma_3 G(-z) \sigma_3 \]

and thus \( G_{\alpha\beta}(0^+) = (-)^{l+k+1} G_{\alpha\beta}(0^-) \), where \( l \) and \( k \) are the site indices carried by the composite variables \( \alpha \) and \( \beta \), respectively. In other words, the retarded and the advanced Green function are not independent and we can obtain the correlation function \( C^{(2)} \) as

\[ C_{\alpha\beta\alpha'\beta'}^{(2)} = (-)^{l' + k'} \frac{\delta^2}{\delta J_{\beta\alpha'}^{b} \delta J_{\beta'\alpha}^{f}} \bigg|_{\hat{J} = 0} \langle Z[\hat{J}] \rangle \]

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from the comparatively simple generating functional of the one-point function (evaluated for \( z = 0^+ \)).

**APPENDIX B: DERIVATION OF THE FIELD THEORY**

In this Appendix we derive the effective Lagrangian \((A1)\) from the basic representation \((A1)\). To keep the discussion simple, we will suppress the source-field dependence of the partition function in much of what follows. The final results for the correlation functions \( C^{(1,2)} \), obtained by straightforward expansion of the action to first and second order in \( \hat{J} \) are displayed in the final Eq. \((B1)\).

The derivation of the effective action essentially follows the standard \([31]\) construction route of field theories of disordered electronic systems. We begin by averaging the partition function over the Gaussian distribution of the random hopping matrices \( R \):

\[
Z[0] = \int \mathcal{D}(\bar{\psi}, \psi) e^{i\bar{\psi}(z - \hat{H}_0 + i\pi \sum C W^C W^{CT})\psi - \frac{\lambda^2}{4} \sum_{\text{str}} \cdot \cdot \cdot },
\]

where \( \hat{H}_0 = \{t_{\alpha\alpha'} \} \) is the clean part of the Hamiltonian. Next, the quartic contribution is decoupled by means of two auxiliary fields:

\[
Z[0] = \int \mathcal{D}(Q, P) e^{-N \sum_i \text{str}(Q^2 + P^2)} \int \mathcal{D}(\bar{\psi}, \psi) e^{i\bar{\psi}(z - \hat{H}_0 + i\pi \sum C W^C W^{CT})\psi \cdot \cdot \cdot } \chi e^{i\lambda \sum_{i, \text{even}} \bar{\psi}_i [Q^3_{l,i+1} + Q^3_{l,i-1}] \psi_{i-1} \cdot \cdot \cdot },
\]

where \( Q^\pm = Q \pm iP \), and \( Q \) and \( P \) are two-component supermatrix fields (reflecting the two-component matrix structure of the dyadic products \( \bar{\psi}\psi \)) living on the non-directed links of our system. Both, internal structure and symmetry properties of these fields will be discussed momentarily. At this stage, we merely anticipate that the field configurations relevant to the long range behaviour of the system will be smooth. The structure of the action then suggests to define \( Q^{\pm}_l \equiv \frac{1}{2}(Q^\pm_{l,i+1} + Q^\pm_{l,i-1}) \) as a new field variable, which now again lives on the sites of our system.
To account for the staggered even/odd structure of the theory we next 'double the unit cell' and define a two component field

\[
\Psi_j = \begin{pmatrix}
\psi_{2j+1} \\
\psi_{2j}
\end{pmatrix} = \begin{pmatrix}
\Psi_{1,j} \\
\Psi_{2,j}
\end{pmatrix}
\]

Here we have introduced a new counting index \(j = 0, \ldots, L/2\) enumerating the doubled unit cells of our system. To avoid confusion, we will systematically designate the index \(0, \ldots, L\) of the 'primitive' sites by \(l, l', \ldots\) and the new index by \(j, j', \ldots\). Expressed in terms of \(\Psi\) the functional integral assumes the form

\[
Z[0] = \int \mathcal{D}(Q, P) e^{-2N \sum_j \text{str} (Q_j^2 + P_j^2)} \int \mathcal{D}(\bar{\Psi}, \Psi) \times
\]

\[
\times \exp \left( i \bar{\Psi} \left( z + 2\lambda(Q + iP\sigma_3) + i\pi \sum_{C=L,R} W^C W^{CT} + \sigma_+ H_0^{12} + \sigma_- H_0^{21} \right) \right).
\]

Here, \(\sigma_{\pm} = \sigma_1 \pm \sigma_2\) where \(\sigma_i, i = 1, 2, 3\) are Pauli matrices acting in the space defined through the two-component structure of \(\Psi\). In lack of better terminology, we will refer to this space as the 'chiral space'. The second line of the expression above, purely off diagonal in the chiral space, contains the clean Hamiltonian. The explicit lattice structure of the blocks \(H_0^{21} = (H_0^{12})^\dagger\) is given by

\[
H_{0, ll'}^{12} = \delta_{ll'}(1 + a) + \delta_{ll'-1}(1 - a),
\]

\[
H_{0, ll'}^{21} = \delta_{ll'}(1 + a) + \delta_{ll'+1}(1 - a),
\]

where \(a\) is the staggering parameter introduced in (2). Finally, notice that the chiral matrix structure of the coupling operator is given by

\[
W_j^C W_j^{CT} = \begin{pmatrix}
W_{2j+1}^C W_{2j+1}^{CT} \\
W_{2j}^C W_{2j}^{CT}
\end{pmatrix}.
\]

At this stage, the superfield \(\Psi\) can be integrated out and we arrive at

\[
Z[0] = \int \mathcal{D}(Q, P) e^{-2N \sum_j \text{str} (Q_j^2 + P_j^2)} \times
\]

\[
\times \exp \left( -N \text{str} \ln \left( z + i\pi \sum_C W^C W^{CT} + 2\lambda(Q + iP\sigma_3) + \sigma_+ H_0^{12} + \sigma_- H_0^{21} \right) \right).
\]
To make further progress, we subject the functional integral to a saddle point analysis. Following the standard scheme \[31\], we seek for saddle point configurations $\bar{Q}$ and $\bar{P}$ that are matrix-diagonal and spatially uniform. Further, we temporarily set $a = W = \text{Re} \, z = 0$ and neglect boundary effects due to the finite extent of the system. Making the ansatz $\bar{Q} = iq \cdot \mathbb{1}$, $\bar{P} = p \cdot \mathbb{1}$, where $q$ and $p$ are complex numbers and $\mathbb{1}$ is the two-dimensional unit matrix in superspace, a variation of the action w.r.t. $\bar{Q}$ and $\bar{P}$ then generates the set of equations

\[
q = \frac{i\lambda}{2} \text{tr} \left( \hat{G}_0^{-1} + 2i\lambda(q + p\sigma_3) \right)_{jj}^{-1},
\]

\[
p = \frac{i\lambda}{2} \text{tr} \left[ \left( \hat{G}_0^{-1} + 2i\lambda(q + p\sigma_3) \right)_{jj}^{-1} \sigma_3 \right],
\]

where $G_0 \equiv (i\delta + \sigma_+ H_0^{12} + \sigma_- H_0^{21})^{-1}|_{a=0}$ and the trace extends over the two chiral components of the operators (but not over superspace). To evaluate the trace, we Fourier transform to momentum space. Defining the transform through

\[
f(k) = \left( \frac{2}{L} \right)^{1/2} \sum_j e^{i2kj} f_j,
\]

\[
f_j = \left( \frac{2}{L} \right)^{1/2} \sum_k e^{-i2kj} f(k)
\]

(where the factor of two in the exponent serves as a mnemonic indicating that we have doubled the unit cell of our system) one finds that the Green function $G_0$ is diagonal in momentum space with $G_0(k) = \left( i\delta + (1 + e^{2ik})\sigma_+ + (1 + e^{-2ik})\sigma_- \right)^{-1}$. It is now straightforward to verify that our equations are solved by $p = 0$ and $q$ determined through the self consistency equation

\[
q = 2\lambda^2 \frac{2}{L} \sum_k \frac{1}{(2\lambda q)^2 + 2(1 - \cos(2k))}.
\]

Replacing the sum by an integral, $\sum_k \rightarrow \frac{L}{2\pi} \int_0^\pi dk$, one finds that the equation is solved by $q = \lambda/2$.

The presence of the chiral symmetry implies that the configuration $\bar{Q} = \frac{\lambda}{2} \mathbb{1}$ is but a particular representative of a whole manifold of solutions. To explore the morphology of that manifold, we notice that our restricted ($z = W = 0$) action is invariant under the transformation
\[\bar{\Psi}_1 \to \bar{\Psi}_1 T_1^{-1}, \quad \bar{\Psi}_2 \to \bar{\Psi}_2 T_2,\]
\[\Psi_1 \to T_2^{-1} \Psi_1, \quad \Psi_2 \to T_1 \Psi_2,\]
\[Q + iP \to T_1 (Q + iP) T_2, \quad Q - iP \to T_2^{-1} (Q - iP) T_1^{-1},\]

where \(T_1, T_2 \in \text{GL}(1|1)\). This symmetry, the super-generalization of the fermion symmetry (7), states that the model has \(\text{GL}(1|1) \times \text{GL}(1|1)\) as a global invariance group. Applying the transformation to the diagonal stationary phase solution discussed above we find that
\[\frac{i\lambda}{2} \to \frac{i\lambda}{2} \begin{pmatrix} T & \end{pmatrix} T^{-1},\]

where \(T = T_1 T_2\). Arguing in reverse, we conclude that any matrix \(T\) defines a solution of the mean field equations, i.e. \(\text{GL}(1|1)\) is the Goldstone mode manifold of the model. This is the celebrated mechanism of chiral symmetry breaking (see Ref. [14] for review): field theory implementations of models with a discrete chiral symmetry on the microscopic level, possess continuous factor groups \(G \times G\) as symmetry manifolds. (In our case \(G = \text{GL}(1|1)\).) This symmetry is spontaneously broken by the saddle point configurations of the model. What remains is a Goldstone mode isomorphic to a single factor \(G\).

Combining these results, we parameterize our field manifold as
\[\begin{pmatrix} Q + iP \\ Q - iP \end{pmatrix} = \frac{i\lambda}{2} \begin{pmatrix} UT \\ T^{-1} U \end{pmatrix},\]

where both \(T, U \in \text{GL}(1|1)\). Here, the matrices \(T\) span the Goldstone model manifold whereas the \(U\)'s, incompatible with the global symmetry of the model, represent massive modes. The next logical step in the construction of the field theory is to substitute these configurations back into the action and to expand in (i) energy arguments \(z\) and matrix elements \(W\), (ii) long-ranged spatial fluctuations of the Goldstone modes, and (iii) massive modes.

1. Goldstone mode fluctuations

We begin with the second element of the program formulated above, the expansion of the action in long ranged spatial fluctuations of the Goldstone mode. Temporarily setting
\( z = W = 0, U = 1 \), we re-organize the 'str ln' of the action according to

\[
X \equiv N \text{str} \ln \begin{pmatrix}
\lambda^2 T & H_0^{12} \\
H_0^{21} & i\lambda T^{-1}
\end{pmatrix} = N \text{str} \ln \begin{pmatrix}
i\lambda^2 & H_0^{12} \\
TH_0^{21}T^{-1} & i\lambda
\end{pmatrix},
\]

where \( T \) is a slowly fluctuating field of Goldstone modes. Writing \( TH_0^{21}T^{-1} = H_0^{21} + T[H_0^{21}, T^{-1}] \) and noticing that, due to the slow fluctuation of \( T \), the commutator is small, we expand as

\[
X = N \text{str} \left( \tilde{G}_{0}^{12}T[H_0^{21}, T^{-1}] \right) - \frac{N}{2} \text{str} \left( \tilde{G}_{0}^{12}T[H_0^{21}, T^{-1}]\tilde{G}_{0}^{12}T[H_0^{21}, T^{-1}] \right) + \ldots,
\]

where we have defined \( \tilde{G}_{0} = (G_{0} + i\lambda)^{-1} \) and the ellipses stand for infrared irrelevant higher order commutator terms. One next Fourier transforms these expressions, substitutes the explicit momentum representation \( G_{0}(k) \) and uses that the characteristic momentum \( q \) carried by the transforms \( T(q) \) is small. The subsequent integral over the 'fast momentum' \( k \) is most economically done by noticing that full integration over \( k \) amounts to integrating the characteristic phases \( \exp(i2k) \) once over the complex unit circle. This integral has a simple pole inside the integration contour, whose residues depend on the 'small' momentum \( q \). Expanding the residues to lowest non-vanishing order in \( q \) and transforming back to coordinate space one obtains \( X = S_{\text{top}} + S_{fl} \), where the two contributions are displayed in Eq. (11) and a continuum limit \( \sum_{j} \to \frac{1}{2} \int_{0}^{L} dr \) has been taken.

Notice that the above discussion implicitly assumed that the number of sites of our system is even: the operator \( X \) had a structure where each \( T \) (living at an odd site) came with a partner \( T^{-1} \) at the neighbouring even site. For a system with an odd number of sites \( L \), however, there remains one uncompensated degree of freedom \( T((L - 1)/2) \) at the terminating site. Neglecting gradients, the action due to this extra contribution reads \( S_{\text{top},2} = N \text{str} \ln T(L) \). In principle, this contribution is as relevant as the 'bulk' contribution to the topological action: it is local in space but, on the other hand, does not contain derivatives. Indeed, the structure of \( S_{\text{top},2} \) is closely related to that of \( S_{\text{top}} \) as can be seen by representing the latter as a boundary action (cf. the third line of (13).) Yet in the majority of cases, the extra contribution \( S_{\text{top},2} \) does not play much of a role, wherefo
have largely ignored it in main analysis. E.g., for a system coupled to the outside world, $S_{\text{top,}2} = N \text{str} \ln(T(L)) = N2\pi i$, evaluates to a phase of no physical effect (cf. the related discussion around Eq. (13).) There is, however, one exception to that rule, viz. the physics of isolated short systems with an odd number of sites discussed in section VI A, where the presence of the extra contribution is of key relevance.

2. Finite energies and coupling to the leads

To obtain the action associated to finite $z$ and $WW^T$, we organize the action as,

$$
\text{str} \ln \left( \frac{z + i\pi WW^T + i\lambda^2 T}{H_0^{12}} \right) \approx \text{str} \ln \left[ \frac{(z + i\pi WW^T)^T}{H_0^{12}} \right] + \frac{i\lambda^2}{i\lambda^2} \left( \frac{H_0^{12}}{i\lambda^2} \right) \approx \sum_j \text{str} \left( (z + i\pi (WW^T)_{2j+1})T^{-1}_I \tilde{G}_{jj}^{11} + (z + i\pi (WW^T)_{2j})T_I \tilde{G}_{jj}^{22} \right) = -iz\pi \nu_0 \sum_j \text{str} (T + T^{-1}) + \frac{\pi}{2} \sum_j \text{str} \left( (WW^T)_{2j+1}T^{-1}_I + (WW^T)_{2j}T_I \right),
$$

where we have used that (from the saddle point equation) $\nu = i\frac{N}{\pi} \tilde{G}_{jj}^{11} = i\frac{N}{\pi} \tilde{G}_{jj}^{22} = \frac{N}{\pi}$ and $\nu$ is the DoS per site evaluated for energies far away from the middle of the band. We next assume that in the coupling region to the leads, i.e. the region where the envelope function $f$ defined through (3) is non-vanishing, fluctuations of the Goldstone modes are negligible. Application of the orthogonality relation (3) then directly leads to

$$
\frac{\pi M \gamma}{2} \sum_{j=0,L/2} \text{str} (T_j + T_j^{-1})
$$

for the contribution of the coupling term. Combining terms and taking the continuum limit, we finally obtain the two expressions $S_z$ and $S_T$ of Eq. (11) for the contribution of energy and coupling operators to the effective action, respectively.
3. Integration over massive modes

We finally turn to the discussion of the role played by the massive modes $U$. First, notice that due to the presence of a weight term $\sim \exp(-\frac{N}{2} \sum \text{str}(Q^2 + P^2))$ in the action and

$$N \text{str}(Q^2 + P^2) = N \text{str}((Q + iP)(Q - iP)) = -\frac{N\lambda^2}{8} \text{str}(U^2)$$

fluctuations of these fields are strongly inhibited. Starting out from an ansatz $U = \exp iW$, where $W$ is some generator, we may thus perturbatively expand the action around $W = 0$. The actual realization of this program is cumbersome (cf. [5] for a concrete example.) Since the result, an extra Goldstone mode operator weakly coupled to the action, will not play much of a role in the present analysis we restrict ourselves to a schematic outline of the calculation.

Perturbative expansion of the action in powers of $W$ obtains an expression like

$$Z[0] = \int DTe^{-S[T]} \langle e^{-S^{(1)[T,W]} + S^{(2)[T,W]} + \ldots} \rangle_W \approx \int DTe^{-S[T]} e^{-\langle S^{(2)[T,W]} \rangle w + \frac{1}{2} \langle (S^{(1)[T,W]})^2 \rangle w}$$

where $\langle \ldots \rangle_W \equiv \int DW e^{-\frac{\lambda^2}{8} \text{str}(W^2)}$ and $S^{(n)}[T,W]$ denotes the expansion of the action to $n$-th order in $W$. The ellipses stand for contributions of higher order in $W$ which can safely be neglected (due to the large overall factor $N \gg 1$). It is straightforward to verify that the explicit evaluation of the operators $S^{(1)[W,T]}$ obtains contributions of the structure $c_1 N \text{str}(W\Phi)$ and $c_2 N \text{str}(W\Phi W\Phi)$, where $\Phi \equiv T\partial T^{-1}$ and the coupling constants $c_{1,2}$ are proportional to powers of $\lambda$. Substituting these expressions back into the action and performing the Gaussian integration over $W$, we arrive at

$$\langle \text{str}(S^{(1)[T,W]})^2 \rangle_W \propto N c_1^2 \text{str}(\Phi^2) = N c_1^2 \text{str}(\partial T \partial T^{-1}),$$

$$\langle \text{str}(S^{(2)[T,W]}) \rangle_W \propto c_2 \left[ \text{str}(\Phi) \right]^2 = c_2 \left[ \text{str}(T \partial T^{-1}) \right]^2.$$
derivatives. The first of these is already contained in the action (cf. Eq. (11)) with a coupling constant parametrically larger than the constant $Nc_1^2$ obtained above. Thus, the first of the two contributions coming from the massive mode integration is irrelevant. In contrast, the second contribution must be taken seriously and, after integration over spatial coordinates, gives the term $S_{\text{Gade}}$ of Eq. (11).

This completes our derivation of the effective action of the model. Finally, to compute the correlation functions $C^{(1,2)}$ defined in the text, we have to add the source field $\hat{J}$ to the partition function, expand to first or second order and differentiate. The structure of the resulting expressions depends on the index configuration of the correlation functions. For the correlators relevant to the computation of conductance and DoS, respectively, we obtain

\[
C^{(1)}_{\alpha\alpha'} = -\frac{i}{2}\delta_{\mu\mu'} \begin{cases} 
\langle T_{\frac{1}{2},11} \rangle, & l \text{ even} \\
\langle T_{-\frac{1}{2},11}^{-1} \rangle, & l \text{ odd}
\end{cases},
\]

\[
C^{(2)}_{\alpha\alpha',\alpha',\alpha} = -\frac{1}{4}\delta_{\mu\mu'} \begin{cases} 
\langle T_{\frac{1}{2},12} \rangle \frac{\mu'}{T_{\frac{1}{2},21}} \rangle, & l \text{ even, } l' \text{ even} \\
-\langle T_{\frac{1}{2},12}^{-1} \rangle \frac{\mu'}{T_{\frac{1}{2},21}}^{-1} \rangle, & l \text{ even, } l' \text{ odd} \\
-\langle T_{\mu'-1,21}^{(l-1)} \rangle \frac{\mu'}{T_{\frac{1}{2},21}}^{-1} \rangle, & l \text{ odd, } l' \text{ even} \\
\langle T_{\mu'-1,21}^{(l-1)} \rangle \frac{\mu'}{T_{\frac{1}{2},21}}^{-1} \rangle, & l \text{ odd, } l' \text{ odd}
\end{cases},
\]

where the angular brackets stand for the functional average.

**APPENDIX C: GEOMETRY OF GL(1|1)**

The canonical metric on the supergroup GL(1|1) derives from the differential two-form $\omega \equiv -\text{str} (dTdT^{-1})$. $\omega$ is not a positive two-form, but its restriction to the sub-manifold $M \subset \text{GL}(1|1)$ defined in the text is; $M$ is the maximally Riemannian subset of $\text{GL}(1|1)$. Parameterizing the group manifold in terms of some coordinates, $T = T(x_1, \ldots, x_4)$ (half of which are anti-commuting), the metric two-form assumes the form

\[
\omega = \sum_{ij} g_{ij} dx_i dx_j,
\]
where \( \{ g_{ij} \} \) defines the metric tensor (represented in the basis \( \{ x_i \} \).) As with non-super Riemannian manifolds, the metric tensor determines the geometry of the manifold. Specifically, the invariant group integral is defined through

\[
\int dT = \int \prod_i dx_i \, J(x_1, \ldots, x_4),
\]

where the Jacobian \( J \propto \text{sdet} \, g \). Similarly, the Laplacian has the standard structure

\[
\Delta = \frac{1}{g^{1/2}} \partial_i g^{ij} g^{1/2} \partial_j,
\]

where \( g^{ij} \) is the inverse of \( g \), \( g^{ij} g^{jk} = \delta^k_i \).

We next wish to represent the metric tensor, the invariant measure, and the Laplacian in the polar decomposition, \( T = k a k^{-1} \) defined in Eq. (13). Due to the manifest invariance of \( \omega \) under transformation \( T \to k_0 T k_0^{-1} \), \( k_0 \) a fixed rotation matrix, it is sufficient to evaluate the metric tensor at \( k = I \). Substitution of the decomposition \( T = k a k^{-1} \) into the two form then yields the covariant structure

\[
\omega = - \text{str} \left( d(k a k^{-1}) d(k a k^{-1}) \right) \bigg|_{k = I} = - \text{str} \left( ([dk,a] + da) ([dk,a^{-1}] + da^{-1}) \right) =
- \text{str} \left( [dk,a][dk,a^{-1}] + da da^{-1} \right).
\]

Using that

\[
dk = \begin{pmatrix}
d\eta \\
d\nu
\end{pmatrix},\quad
da = \begin{pmatrix}
e^x dx \\
ie^y dy
\end{pmatrix},
\]

it is straightforward to verify by elementary matrix manipulation that the metric bilinear form assumes the form

\[
g = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & -4 \sinh^2 \left( \frac{x - iy}{2} \right) \\
+ 4 \sinh^2 \left( \frac{x - iy}{2} \right) & 0
\end{pmatrix},
\]

where \( g = \{ g_{ij} \} \), a vectorial structure \( d\mathbf{x} = (dx, dy, d\eta, d\nu) \) is understood and empty blocks are filled with zeros. The associated superdeterminant,

\[
g = \frac{1}{\left( 4 \sinh^2 \left( \frac{x - iy}{2} \right) \right)^2}.
\]
From $g$ derives the unit-normalized group integral

$$\int dT f(T) = \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} dy \int d\eta d\nu J(x, y) f(x, y, \eta, \nu), \tag{C1}$$

with Jacobian

$$J(x, y) = \sinh^{-2}\left(\frac{x - iy}{2}\right). \tag{C2}$$

Further, the radial part of the Laplacian reads as

$$\Delta = \sum_{i=x,y} g^{-1/2} \partial_i g^{1/2} \partial_i.$$

**APPENDIX D: CONDUCTANCE OF SHORT SYSTEMS**

Consider a short sublattice system of length $L < \xi/(M\gamma)$. This is the quantum dot regime, where the conductance is not Ohmic but rather determined by the coupling of the system to the leads. For a normal, non-sublattice system, $g \sim M\gamma$, reflecting that each of the $M$ channels contributes to transport with an efficiency set by the coupling. We here wish to explore to which extent this behaviour generalizes to the sublattice system. Specifically, three different cases will be discussed: (a) smoothened coupling where a set of sites at each end is connected – the type of coupling considered in the text, (b) coupling through a single site on either end where the two terminal sites are of the same parity, odd/odd, say. (c) One of the two terminating sites is even, the other odd,

That the system is short means that the functional integral is controlled by spatially uniform configurations $T = \text{const.}$; the stiffness introduced by the gradient term is too strong to allow for significant fluctuations. (More precisely, fluctuating field configurations lead to relative corrections of $O(LM\gamma/\xi)$ which we are not going to consider.)

Let us begin by considering case (a). Evaluation of the functional expectation value (8) for a spatially uniform field configuration leads to the expression

$$g^{(a)} = -\left(\frac{M\pi\gamma}{2}\right)^2 \int dT (T - T^{-1})_{12}(T - T^{-1})_{21} e^{-\pi M\gamma \text{str}(T + T^{-1})}.$$
We next need to do the group integral. Inspection of the exponent shows that the integration is dominated by configurations close to the group origin, $T = 1$. This suggests to represent our $T$’s as $T = \exp W$ and to integrate over the generators $W$ in a Gaussian approximation:

$$g^{(a)} \approx - (M \pi \gamma)^2 \int dWW_{12}W_{21}e^{-\pi M \gamma \text{str} W^2},$$

where we have used that close to the group origin the integration measure is flat. Doing the integral over the components of $W$ one finds

$$g^{(a)} = \frac{M \pi \gamma}{2},$$

in agreement with the behaviour of a non-sublattice system. The situation in case (b) is not much different. Noticing that fields $T$ sit at the odd sites of our system, while $T^{-1}$’s are attached to even sites, we find that the conductance is expressed as

$$g^{(b)} = (M \pi \gamma)^2 \int dTT_{12}(T^{-1})_{21}e^{-\pi M \gamma \text{str}(T+T^{-1})}.$$

(The disappearance of the combination $T - T^{-1}$ in the pre-exponent reflects the fact that only single sites of different parity are coupled to the continuum. Again the integral is dominated by the group origin and similar reasoning as above leads to

$$g^{(b)} \approx - \left( \frac{M \pi \gamma}{2} \right)^2 \int dWW_{12}W_{21}e^{-\pi M \gamma \text{str} W^2} = \frac{M \pi \gamma}{2}.$$

In case (c), however, the situation is different. Owing to the identical parity of the terminating sites, the integral representation of the conductance now assumes the form

$$g^{(c)} = - (M \pi \gamma)^2 \int dTT_{12}T_{21}e^{-\pi M \gamma \text{str}(T^2)}.$$

This is an unpleasant expression: the exponential weight no longer projects onto the group origin, in fact does not even have a stable saddle point. To compute the conductance we therefore have to integrate over the full group manifold, a task that is most efficiently done in polar coordinates. Using Eqs. (12) and (C1) and integrating out Grassmann components it is straightforward to verify that the radial part of the integral assumes the form

$$g^{(c)} = (M \pi \gamma)^2 \int_{-\infty}^{\infty} dx \int_{0}^{2\pi} \frac{(e^x - e^{iy})^2}{\sinh^2 \left( \frac{x-iy}{2} \right)} e^{-(e^x - e^{iy})} = 0.$$
To understand the vanishing of the integral, notice that $\int dy = \frac{1}{2\pi i} \oint \frac{dz}{z}$ can be transformed into the integral of the complex variable $z = e^{iy}$ over the unit circle. One verifies that, regardless of the value of $x$, the integrand is analytic and void of singularities inside the integration contour. Cauchies theorem then implies vanishing of the integral.
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