Knitted Complex Networks

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To a considerable extent, the continuing importance and popularity of complex networks as models of real-world structures has been motivated by scale free degree distributions as well as the respectively implied hubs. Being related to sequential connections of edges in networks, paths represent another important, dual pattern of connectivity (or motif) in complex networks (e.g., paths are related to important concepts such as betweenness centrality). The present work proposes a new supercategory of complex networks which are organized and/or constructed in terms of paths. Two specific network classes are proposed and characterized: (i) PA networks, obtained by star-path transforming Barabási-Albert networks; and (ii) PN networks, built by performing progressive paths involving all nodes without repetition. Such new networks are important not only from their potential to provide theoretical insights, but also as putative models of real-world structures. The connectivity structure of these two models is investigated comparatively to four traditional complex networks models (Erdős-Rényi, Barabási-Albert, Watts-Strogatz and a geographical model). A series of interesting results are described, including the corroboration of the distinct nature of the two proposed models and the importance of considering a comprehensive set of measurements and multivariated statistical methods for the characterization of complex networks.

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‘You can not travel the path until you have become the path itself.’ (Gautama Siddharta)

I. INTRODUCTION

Much of the interest in complex networks research (e.g. [1, 2, 3, 4, 5]) has been related to particularly interesting connectivity patterns arising in specific network models. A small world network, for instance, involves small shortest paths between its pairs of nodes, and high clustering coefficient. On the other hand, a scale free network presents a distribution of node degrees which follows a power law, enhancing the probability of hubs. These two types of structures are arguably the most important models in complex networks research, having been considered by the vast majority of related works. Interestingly, while small world networks are related to shortest paths, scale free structures are intrinsically linked to the concept of degree and hubs. It is perhaps not by chance that the two concepts characterizing these two principal types of networks have a rather distinct, dual, nature. As a matter of fact, a path is inherently sequential, while the concept of degree is associated to the star defined by the connections around a node. Another important issue potentially related to paths relates to the causality and transitivity of effects along the network (especially in the case of oriented networks). More specifically, a long oriented path in a network can be related to a sequence of causal effects. A star with particularly high degree has been called a hub. Figure 1 illustrates two simple networks, one formed by a single path (a) and another by a single hub. To a great extent, these two structures capture the essential features of sequential (e.g. Watts-Strogatz, which starts as a cycle involving all nodes) and centralized (e.g. Barabási-Albert) network models.

Although both structures in Figure 1 interconnect the same set of nodes and contain the same number of edges, they exhibit a completely distinct, dual, nature. For instance, the minimum number of edge crossings implied while visiting all the nodes can be easily verified to be equal to 7 and 12, respectively (note that the ratio between these values tends to 2 for large number of nodes). This feature corresponds to an inherent advantage of the path organization. However, in case the edge d in the structure in Figure 1(a) is removed (e.g. through attack or failure), half of the network (i.e. 1 to 4) will be rendered unaccessible to the other half (i.e. 5 to 8).
Contrariwise, a removal of any single edge in the network in Figure 1(b) will result in the isolation of just a single node. This fact therefore suggests that the centralized network is more resilient to edge attack, though it is particularly weak to node/hub attack (see, for instance, [7, 8, 9]). The distribution of shortest path lengths is also markedly different for each model: ranging between 1 and 7 in the path network and between 1 and 2 for the hub network. The path and star organizations differ in many other respects, including distribution of betweenness centrality and node correlations, often leading to opposite properties.

In the light of the above discussion, paths and stars can be considered dual structural elements (or motifs [10]) in complex networks. In particular, long paths would be duals to hubs. Therefore, it is expected that network models involving many long paths (e.g. Watts-Strogatz [18]) will in some way reflect the basic features of the path motif, i.e. their nodes can be effectively covered by relatively short path walks [14], but they will present low resilience to edge attack. Contrariwise, networks organized around hubs (e.g. Barabási-Albert structures) would be expected to be resilient to edge attack, but imply long path walks required to cover most of their nodes.

The identification of the duality between the path/star motif as well as between networks involving long paths and large hubs paves the way to several investigations in complex network research. These possibilities include but are not limited to: (i) comprehensive investigations of the topological properties of networks involving paths or hubs; and (ii) the definition of transformations between such motifs and networks and study of the respective effects on the networks properties. The latter possibility was preliminarily explored in a recent work [6], where the path-star and star-path transformations were proposed. That work considered the transformation of just one path, starting from a randomly chosen node. The effects of such transformation on the overall network connectivity were quantified in terms of ratios between measurements such as the average clustering coefficient and average shortest path length after and before the transformations. While the path-star transformation tended to increase the average clustering coefficient in Erdős-Rényi (ER) networks, it implied small changes in Barabási-Albert (BA) structures. Major decreases of the average shortest path length were observed for Watts-Strogatz (WS) networks.

The present work continues and extends the previous investigation in [6] by addressing the characterization of two networks grown in terms of paths, which are henceforth called knitted networks because of the similarity of their growth with the progressive incorporation of threads (i.e. paths). These two novel types of complex networks include: (a) networks obtained by the star-path transformations of all stars in BA networks; and (b) networks whose edges are obtained by randomly selecting each of the nodes just once. These two categories of networks, illustrated in Figure 2, are henceforth called Path-transformed BA (PA) and Path-regular (PN) networks. ER, BA, WS and a geographical type of networks have also been considered for comparative purposes.

Regarding the path-transformed BA (PA) model, it is obtained by transforming each of the stars in a BA network into a respective path. The transformation from star to path is performed in decreasing order of node degree, so that hubs are transformed first. The edges

![Path-transformed BA (PA) and Path-regular (PN) networks.](image-url)
removed from the stars in the original BA network are progressively transferred to the growing transformed network. Thus, each of the paths incorporated into the PA network corresponds to a respective star in the original BA structure. As such, the PA network has an embedded distribution of paths which exhibits the same power law as the degrees of the original BA network. For such a reason, this new kind of network can be thought as the path-dual of the BA model.

It is important to observe that the star-path transformation is not deterministic, in the sense that several slightly distinct networks can be obtained by transforming a star into a path (as explained in more detail in Section IIIA this is a consequence of the fact that there is no clear way in which to choose the sequence of nodes from the hub while generating the path, so that they are randomly selected.). On the other hand, the path-star transformation is deterministic.

The other considered knitted model, namely the path-regular (PN) structures, represents a simpler but nevertheless potentially interesting new model as it exhibits an almost perfect degree regularity (in the sense of most nodes exhibiting the same degree). As presented in this article, this network also resulted remarkably regular with respect to several other measurements. A PN network is obtained by connecting all the nodes (initially isolated) so that every node is selected only once, therefore defining a path in the growing network. This procedure can be repeated several times. The degrees of most nodes therefore result very close to twice the number of complete paths involving the whole set of nodes.

The two main reasons why it is interesting to consider new network models such as those outlined above are: (i) such models can provide theoretical insights about aspects related to their intrinsic nature or to the way in which they are grown; and (ii) it is useful to compare such models with specific real-world networks, in the sense that good similarity between them may contribute to understanding real-world problems. The proposal of novel types of networks thus immediately implies the important question of how such structures relate to other existing models, especially those more traditional such as uniformly random, scale-free, small world and geographical. It is therefore important to devise a reasonable methodology allowing such a problem to be properly tackled. The experimental methodology adopted in this article involves generating several realizations of each category of network, for two sizes (i.e. $N = 100$ and $N = 200$) and two average degrees (i.e. $\langle k \rangle = 6$ and $\langle k \rangle = 10$), and calculating a series of distinct measurements of their respective topology. The six networks types are them compared in terms of projections of these measurements, obtained by using canonical analysis, which projects the measurement space so as to maximize the separation of the six network categories (see [4]).

This article starts by presenting the basic concepts and methods in complex network, their growth, measurements, and multivariate methods for projection and analysis of the similarities between the models. It follows by presenting the obtained results and respective discussion.

II. BASIC CONCEPTS AND METHODOLOGY

This section describes the basic concepts and methods used in the present work, including the representation and measurement of networks, the traditional complex network models (ER, BA, WS and a geographical type of network), and multivariated statistical methods, which are described in the respective subsections. Additional information about network measurements and their analysis by using statistical methods can be found in [4].

A. Network Representation and Measurements

Complex networks are discrete structures involving $N$ nodes and $E$ edges connecting those nodes. In this work we focus attention on undirected networks. This type of network can be represented by a symmetric adjacency matrix $K$ such that the presence of each edge $(i, j)$, where $i$ and $j$ are any network nodes and $i \neq j$, implies $K(i, j) = K(j, i) = 1$, with $K(i, j) = K(j, i) = 0$ otherwise.

A star in a network is henceforth understood as any node together with its respectively attached edges. Two edges $(i, j)$ and $(k, m)$ are adjacent whenever they share an extremity (i.e. $i = k$ or $j = k$ or $i = m$ or $j = m$). A sequence of adjacent edges constitutes a walk. Observe that a walk may go more than once over the same nodes or edges. A path is a walk which never re-visit any edge or node. A closed path is a cycle. The diameter of a network correspond to the length of the longest shortest path between any pair of nodes.

The immediate neighbors of a node $i$ are those which are distant by one edge from $i$. The degree of a node is equal to the number of its immediate neighbors. The clustering coefficient of a node $i$ quantifies how well the immediate neighbors of that node are interrelated. More specifically, if $n(i)$ is the number of immediate neighbors of node $i$, then its clustering coefficient can be calculated as:

$$cc(i) = \frac{2e(i)}{n(i)(n(i) - 1)}$$ \hspace{1cm} (1)

where $e(i)$ is the total number of undirected edges connecting the immediate neighbors of $i$. Though the degree and the clustering coefficient, which are traditionally adopted measurements, are defined for each node in a network, it is also interesting to consider the average and standard deviation of those values as global features of the whole network.

The clustering coefficient can be generalized to consider sets of nodes other than the immediate neighbors of a reference node [11, 12]. In this work we use the
second order clustering coefficient — second clustering coefficient, for short — of each node \( i \), \( cc_2(i) \), which considers the interconnectivity of the second neighbors of that node. The second neighbors are those nodes which are can be accessed from node \( i \) by a shortest path of length 2. The second clustering coefficient can be calculated as:

\[
cc_2(i) = \frac{2e_2(i)}{n_2(i)(n(i) - 1)}
\]

(2)

where \( e_2 \) is the number of undirected edges among the second neighbors of node \( i \) and \( n_2(i) \) is the number of those neighbors. Similarly to the degree and clustering coefficient, it is also interesting to consider the average and standard deviation of the second clustering coefficient.

Table I presents all the 9 measurements considered in the present work and their respective abbreviations.

| Measurement                        | Abbreviation |
|------------------------------------|--------------|
| Average Node degree                | \( \langle k \rangle \) |
| St. deviation of node degree       | \( \sigma_k \) |
| Average clustering coefficient     | \( \langle cc \rangle \) |
| St. deviation of clust. coeff.     | \( \sigma_{cc} \) |
| Average second clust. coeff.       | \( \langle cc_2 \rangle \) |
| St. deviation second clust. coeff. | \( \sigma_{cc_2} \) |
| Network diameter                   | \( D \) |
| Average shortest path length       | \( \langle sp \rangle \) |
| St. deviation short. path leng.    | \( \sigma_{sp} \) |

TABLE I: The measurements of the network connectivity considered in the present work and their respective abbreviations.

B. Traditional Complex Networks Models

In addition to the two theoretical models of networks proposed in this article (see Section III), four traditional models are also considered in order to provide comparison references. The basic procedure to obtain these three models — namely Erdős-Rényi (ER), Barabási-Albert (BA), Watts-Strogatz (WS) and a simple geographical model (GG) — are described as follows (see also \[1, 2, 3, 4, 5\]).

ER networks are obtained by starting with \( N \) nodes and establishing undirected connections with fixed probability \( \gamma \). The BA structures are grown from \( m0 \) initial, randomly connected nodes, through the progressive addition of new nodes with \( m \) edges each, which are connected with the previous nodes preferentially to their node degrees. The WS networks are obtained by starting with a cycle containing all the \( N \) nodes and then rewiring a proportion \( \alpha \) of the undirected edges. The geographical model adopted in this work, henceforth abbreviated as GG, involves distributing the \( N \) nodes with uniform random probability within a square two-dimensional space and then connecting each node (through undirected edges) to all nodes which are no longer than a maximum Euclidean distance \( d_{max} \). All considered networks are undirected and do not include self-connections. The also have the same number of nodes \( N \) and similar average degrees \( \langle k \rangle \) which are determined in terms of \( m \) (see, for instance, \[13, 14\]).

C. Multivariated Statistical Methods

The set of \( M \) measurements obtained for a complex networks under analysis can be represented in terms of a feature vector in \( R^M \). The so-defined \( M \)-dimensional space is frequently called the measurement space adopted in a specific investigation. Therefore, each distinct network is mapped into a point, defined by the respective feature vector, in the measurement space. Observe that this mapping is not invertible because more than one network, though structurally distinct, may be mapped into identical feature vectors. Groups of points (also called clusters) appearing in such spaces indicate possible categories of networks exhibiting similar topological properties. Networks which are mapped into relatively distant points can be understood to present substantial differences in their topology (e.g. \[4\]).

Because the number of adopted measurements \( M \) is frequently larger than 2 or 3, it becomes impossible to visualize the distribution of the networks when mapped into the \( M \)-dimensional measurement space. Fortunately, it is possible to use stochastic projections in order to reduce the dimensionality of such spaces. In the present work we consider the canonical projection (e.g. \[4\]), where the projection is performed so as to maximize the distance between the categories of networks while simultaneously minimizing the dispersions inside each category.

III. KNITTED NETWORK MODELS

The two novel categories of networks proposed and investigated in this work are the path-transformed BA (PA) and the path-regular (PN) models. These are here understood to belong to a new supercategory of structures called knitted networks, characterized by being formed by paths. These two models are described in the next two subsections, respectively.

A. Path-Transformed BA Networks (PA)
Given a generic complex networks, it is possible to transform each of its stars into a path by using a methodology recently suggested \[\text{[6]}\]. Basically, given a node \(i\) and its connected edges (i.e. a star) in the original network, the following steps are performed: (i) the set of nodes \(S\) comprising the node \(i\) as well as its immediate neighbors is identified; (ii) one of these nodes, except \(i\), is randomly chosen as the beginning of the path; (iii) the edges interconnecting \(i\) to its immediate neighbors are randomly selected and removed from the original network and used to define the continuation of the growing path in the new network. Observe that the edges are transported from the original network into the growing network, i.e. two networks are considered in order to avoid transporting one edge more than once. In order to preserve the total number of edges, the transported edges are added to eventual edges already transported. Therefore, the transformed network is, in principle, a weighted structure (however, the present work considers all weights equal to one).

Because each star-path transformation depends on the choice of the sequence of nodes to be used along the path, it is important to investigate the intensity of the effects that such variations may have on the overall network structure. The average and standard deviation values of the clustering coefficients obtained for 100 path-transformations of the same initial BA network are: \(0.045 \pm 0.011\) for \(N = 100\) and \(m = 3\) (coefficient of variation \(\approx 0.24\)), \(0.0220 \pm 0.0048\) for \(N = 200\) (coefficient of variation \(0.22\)) and \(m = 3\), \(0.0497 \pm 0.0041\) for \(N = 100\) and \(m = 5\) (coefficient of variation \(0.08\)), and \(0.045 \pm 0.0048\) for \(N = 200\) and \(m = 5\) (coefficient of variation \(0.11\)). These results suggest that, at least for the considered configurations and especially for larger values of \(m\), the structural changes implied by the diverse choices of the nodes for the star-path transformation tend to lead to relatively small effects on the connectivity properties of the diverse PA networks obtained from the same BA structure.

### B. Path-Regular BA Networks (PN)

These networks are simply constructed by starting with \(N\) isolated nodes and incorporating \(M\) sequences of edges defined by choosing non-repeated, randomly chosen nodes (all nodes are taken). See Figure\[\text{[2]}\] for an illustration of a PN network built up by choosing the \(N\) nodes according to two random sequences. Thus, this type of networks is obtained, as hinted in the quotation in the beginning of this text, the networks are obtained from paths, not vice-versa, i.e. the paths and the network become one. Because each path enters and leaves most of the nodes 2 times, the average node degree in this category of networks tends to converges steadily to \(\langle k \rangle = 2M\). Therefore, this network is highly regular at least with respect to the node degree. After random networks such as those in the ER model, the PN networks are possibly those which are simplest to grow computationally.

### IV. RESULTS AND DISCUSSION

A total of 50 simulations were performed for the PA, PN, ER, BA, and WS models with respect to each of the four following configurations: (a) \(N = 100\) and \(m = 3\); (b) \(N = 100\) and \(m = 5\); (c) \(N = 200\) and \(m = 3\) and (d) \(N = 200\) and \(m = 5\). The obtained results are presented and discussed in the following sections with respect to several progressive combinations of measurements and projections.

Only the largest connected component of the original network is considered (note that, because of the relatively high average degrees considered in this work, the largest component often corresponds to the whole original network).

#### A. Node Degree and Clustering Coefficient

The average values of the node degree and clustering coefficient have been traditionally used in order to quantify the topological properties of complex networks. We start our investigation of the structural relationship between the new PA and PN networks with the traditional ER, BA, WS and GG models by considering these two measurements. The distributions of the several instances of each network category along this two-dimensional measurement space, considering the four adopted configurations, are shown in Figure\[\text{[3]}\]

A series of interesting facts can be inferred from these figures. First, note that, as expected, all nodes have average degrees near \(2m\) (i.e. \(\langle k \rangle \approx 6\) for \(m = 3\) and \(\langle k \rangle \approx 10\) for \(m = 5\)). Observe that, because of finite size effects, the average degrees of the BA networks are smaller than the expected theoretical value for \(m = 3\), but tends to approach \(2m\) for the larger value \(m = 5\). A similar tendency can be observed for the PA model, which is derived from the BA counterparts. Rather distinct dispersions of the average node degree are observed for each model, with the WS resulting in the smallest variation, while the GG model accounts for the largest dispersion. Interestingly, the PA model exhibited degree dispersion very similar to that of the BA model. The average values of clustering coefficients presented much less dispersion than the average degree. As expected, each type of network resulted with typical and characteristic average clustering coefficient values. The smallest clustering coefficients were obtained for the ER, PA and PN models, while the WS and GG models presented the highest values. Intermediate values of clustering coefficient were obtained for the BA networks. The dispersions of both the average degree and average clustering coefficient tend to decrease with \(N\).
FIG. 3: The measurement spaces involving the average node degree and average clustering coefficient obtained for the 50 instances of each of the 5 network models considered in this article with respect to the four configurations: (a) $N = 100$ and $m = 3$; (b) $N = 100$ and $m = 5$; (c) $N = 200$ and $m = 3$; and (d) $N = 200$ and $m = 5$. See text for discussion.

To judge from the measurement spaces defined by the average degree and average clustering coefficient, the PN novel type of network tends to be similar to the ER model (however, the ER networks present substantially higher node degree variance). At the same time, the PA model presented average degree similar to the BA, but smaller average clustering coefficients (the average clustering coefficient of the PA networks is comparable to those of the ER structures). Generally, if only these two measurements are considered, we can conclude that the PN and PA novel networks are reasonably similar to the ER and BA types of networks, respectively. As a matter of fact, because such differences are mostly related to the average degree (which tends to become more similar for large values of $N$), these four networks could ultimately be understood (incorrectly, as it will become clear soon) to exhibit little structural differences.

Figure 4 shows the distribution of the networks mapped in measurements spaces defined by the standard deviations of the respective node degrees and clustering coefficients. Observe that such standard deviations are calculated considering the degrees and clustering coefficients of each individual node in each of the network realizations. Though naturally implied, such measurements have been rarely used in the complex network area.

It is clear from these scatterplots that a substantially more distinct separation between the 6 complex network types is now obtained. A series of interesting trends can be discerned from Figure 4. First, as before, the overall dispersion of the clusters tended to decrease with $N$, for a fixed $m$. Also, the BA networks presented, in all cases, the largest standard deviations of node degrees (a fact implied by the wider distribution of node degrees in that model, with presence of hubs), while the GG model yielded the largest standard deviations of clustering coefficients in all situations (an interesting effect related to the distribution of nearest distances in systems of uniformly distributed points). As expected, the WS struc-
FIG. 4: The measurement spaces involving the standard deviations of the node degree and clustering coefficient obtained for the 50 instances of each of the 5 network models considered in this article with respect to the four configurations: (a) $N = 100$ and $m = 3$; (b) $N = 100$ and $m = 5$; (c) $N = 200$ and $m = 3$; and (d) $N = 200$ and $m = 5$.

...features provided reasonably small dispersions of both degree and clustering coefficient, therefore defining a well-separated cluster in all scatterplots in Figure 4. Recall that we are referring to the coordinate values expressing the standard deviations, not the intrinsic dispersion of the clusters in the scatterplots.

Unlike in the previous analysis involving the average degree and average clustering coefficient, the novel PN model resulted well-separated from all other network categories, presenting the overall smallest standard deviations of both degree and clustering coefficient values in all scatterplots. This fact suggests an intense regularity of the PN model (please refer to [15] for a discussion on the generalization of the concept of regularity to topological measurements other than the node degree), in the sense that most nodes in that type of network present similar degree and clustering coefficient values. Unlike in our previous analysis involving the average values of degree and clustering coefficient, the new PA model now resulted well away from the BA model and more distinct from the ER model (except for the case $N = 100$ and $m = 3$ in Figure 4). Such results clearly indicate that the PN and PA models do exhibit specific structural features which distinguish them from the other traditionally used measurements. In particular, the PN stands out as being particularly regular with respect to both node degree and clustering coefficient. On the other hand, the PA networks tend to be similar to the ER model, particularly for smaller values of $N$ and $m$. However these two types of networks tend to separate one another for larger values of those parameters. For the values of $N$ and $m$ considered in this work, it is reasonable to say that the ER is the model which is closest to the PA as far as the standard deviations of the degree and clustering coefficient are concerned.
B. All Together Now

In order to gain further insights about the possible structural relationships between the novel PN and PA networks and the four traditional models considered in this work, we now resource to all the measurements in Table 1. Observe that a single value of diameter is obtained for the whole network, so that no average or standard deviation values are considered for this global feature. Because of the high dimensions of the the measurement spaces implied by these several measurements, it is no longer possible to visualize the distribution of the networks in the respective measurement spaces, as done in the previous section. In order to circumvent this problem, in the following we apply the canonical projection methodology to those measurement spaces. In addition, because the PN and PA models have already been found to be considerably distinct from the WS and GG models, these two traditional theoretical models are no longer considered in the following analysis.

Figure 5 shows the scatterplots obtained while considering all the network measurements in Table 1. The axes in these plots refer to the two main canonical variables $v_1$ and $v_2$, which are linear combinations of all the considered measurements, weighted so as to lead to maximum dispersion between the classes and minimum dispersion inside each of them. The reader should notice that, because these axes are obtained from eigenvectors, their orientation is arbitrary and may reverse from one case to another. It is clear from Figure 5 that the PN, PA, ER and BA categories of networks yielded well-defined, neat respective clusters with relative small inter-cluster dispersion, combined with a strong dispersion among the different classes. In other words, these results corroborate the fact that these four models have connectivity structures which are markedly different one another. The PN tends to present the smallest overall dispersion of measurements (specially for large values of $N$ and $m$), while the PA accounts for the largest dispersion.

V. CONCLUDING REMARKS

Two of the most important connectivity patterns (or motifs) underlying complex networks are paths and stars. While much of the attention from the complex networks community has been focused in star organizations (especially hubs), investigations of the distribution of paths in networks have been particularly incipient (except mainly for studies related to betweenness centrality). The present work has extended a recent previous investigation involving the duality/transformation between paths and stars in order to propose a new supercategory of complex networks, namely the so-called knitted nets, corresponding to networks organized in terms of paths. Two classes of such networks have been proposed and investigated: (i) PA, derived from BA networks through the star-path transformation; and (ii) PN, obtained by randomly selecting all the network nodes without repetition. The two main contributions of the present work include:

Proposal of a new superclass of complex networks (knitted): Because paths can be understood as the dual concept of the star motif which underlies important models such as scale-free networks (organized in terms of hubs), it is important to consider complex networks underlain by paths, which have been called knitted networks in the present work. Each of the two novel types of networks proposed here present distinctive features. The PN model, obtained by performing progressive path-walks involving all nodes (without repetition), has been found to present remarkable uniformity of measurements at the individual node level. This type of network therefore corresponds to an interesting theoretical model integrating stochasticity and uniformity. The second type of proposed networks, namely the PA nets obtained by path-transformations of BA counterparts, is potentially interesting because they are formed by a scale-free distribution of path lengths (recall that, in principle, the path-transformation produces weighted networks). In this respect, PA models can be understood as duals of the BA networks.

Comprehensive characterization of new network models: Though several investigations of the structure of complex networks have considered just a few measurements such as the average node degree, clustering coefficient and shortest path length, such features are often unable to comprehensively characterize the connectivity of the different network types. This again became clear in Section IV A where the consideration of only the average degree and average clustering coefficient did not lead to significative differences between the PN and PA models and the ER and BA types of networks. However, by considering additional measurements, with emphasis on the standard deviations of traditional measurements as well as the second clustering coefficient (a hierarchical or concentric kind of measurement), the two new models PN and PA were shown to be considerably distinct not only one from the other, but also from all the four traditional theoretical models of complex networks presently considered. The necessity of dimensionality reduction, accomplished through the multivariated statistical method of canonical projections, also again confirmed itself essential for making visual sense of distributions on complete networks in high dimensional spaces.

The concepts and results described in the present work have paved the way to a number of related future investigations, which include but are not limited to:

Maximum path investigations: By emphasizing the importance of paths as basic motifs in complex networks, the present work motivates additional investigations not only on networks underlain by such motifs, but also in paying greater attention to the path-structure in all complex networks models. One point of particular importance which has been relatively overlooked concerns the statistics of the longest path between nodes in com-
FIG. 5: The measurement spaces involving all the measurements in Table II for the 50 instances of each of the 5 network models considered in this article with respect to the four configurations: (a) $N = 100$ and $m = 3$; (b) $N = 100$ and $m = 5$; (c) $N = 200$ and $m = 3$; and (d) $N = 200$ and $m = 5$. See text for discussion.

plex networks. As a matter of fact, while great attention has been placed on the analysis of shortest paths, the study of maximal paths has received scant attention. Though the identification of maximal paths is known to be an NP-complete problem, the statistics of their lengths in complex networks is poised to provide valuable insights about diverse theoretical and real networks.

Considerations of Hierarchical measurements:
A series of hierarchical/concentric measurements [11] have been proposed in addition to the second clustering coefficient. It would be interesting to consider how the proposed models PN and PA differ among themselves and with respect to other theoretical models as far as such hierarchical measurements are concerned.

Additional types of knitted networks:
Though two classes of knitted networks have been suggested in this work, there are many other possible structures organized in terms of paths and cycles. It would be also interesting to consider networks formed by walks, instead of paths, allowing nodes to be visited more than once.

Consideration of directed networks:
Because a great deal of the attention in complex network research has been placed on undirected networks, the extension of such approaches — including that described in the present article — to directed complex networks represents a particularly promising area for future investigations. Indeed, the PN growing method can be immediately modified in order to provide directed networks (the edges would be oriented along the order of node visitation during the progressive path-walks).

Transformation of other network models:
Though the current work concentrated on star-path transformations, it would be interesting to consider several other possibilities of transformations between networks models involving motif-transformations. For instance, it would be particularly interesting to consider the path-star transformations described in [6] applied to all paths in the original network.

Comparison with real-world models:
Though the current work concentrated in comparisons between the PN/PA structures with traditional theoretical models, it would be of particular interest to consider such new networks as putative models of real-world structures. After all, one of the main motivations for complex networks research has been their potential for modeling real-world structures. Going back to the scatterplots in Figure 5...
it would also be very interesting to conceive other network models capable of filling in the gaps left between the existing models in those spaces.

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[18] It is important to observe that though Watts-Strogatz networks are characterized by small average shortest path length, they do involve at least one long path related to the initial configuration from which these networks are obtained (i.e. a cycle involving all nodes).
[19] A path-walk is a walk which never visits any of the nodes or edges more than once. Such walks can also be called self-avoiding walks.