Strong Gravitational Lensing of Quasi-Kerr Compact Object with Arbitrary Quadrupole Moments

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Abstract

We study the strong gravitational lensing on the equatorial plane of a quasi-Kerr compact object with arbitrary quadrupole moments which can be used to model the super-massive central object of the galaxy. We find that, when the quadrupolar correction parameter $ξ$ takes the positive (negative) value, the photon-sphere radius $r_{ps}$, the minimum impact parameter $u_{ps}$, the coefficient $\bar{b}$, the relative magnitudes $r_m$ and the angular position of the relativistic images $θ_∞$ are larger (smaller) than the results obtained in the Kerr black hole, but the coefficient $\bar{a}$, the deflection angle $α(θ)$ and the angular separation $s$ are smaller (larger) than that in the Kerr black hole. These features may offer a way to probe special properties for some rotating compact objects by the astronomical instruments in the future.

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I. INTRODUCTION

In the framework of general relativity, the no-hair theorem [1] guarantees that a neutral rotating astrophysical black hole is uniquely described by the Kerr metric only with two parameters, the mass $M$ and the rotational parameter $a$. Observations of the weak gravitational systems agree well with the prediction of the general relativity. However, the hypothesis that the astrophysical black-hole candidates are described by the Kerr metric still lacks the direct evidence, and the general relativity has been tested only for weak gravitational fields. In the regime of strong gravity, the general relativity could be broken down and astrophysical black holes might not be the Kerr black holes as predicted by the no-hair theorem [2–5]. Several parametric deviations from the Kerr metric, i.e., multipole moments, have been suggested to study observational signatures including gravitational waves from extreme mass-ratio inspiral (EMRI) [6–10] and the electromagnetic spectrum emitted by the accreting disk around black holes [11, 12].

The general stationary axisymmetric neutral compact object can be described in terms of its mass, rotational parameter and multipole moments [13]. The multipole moments are consist of a set of mass multipole moment $M_l$ and current multipole moment $S_l$, here the subscript $l$ of them is labeled by the angular eigenvalue $l \geq 0$. The relation between the parameters of the multipole moments can be expressed as

$$M_l + iS_l = M(ia)^l + \delta M_l + i\delta S_l.$$  

For the Kerr black hole, the deviation $\delta M_l$ and $\delta S_l$ are equal to zero. Thus the relation (1) is unique to the Kerr black hole and is just a mathematical expression of the famous no-hair theorem. Therefore, for a general rotating spacetime, by measuring three independent multipole moments of the spacetime we could provide sufficient information for verifying whether or not the central body is a Kerr black hole. Now, four approaches for measuring the multipole moments or a perturbation of the Kerr black hole have been proposed to test the no-hair theorem. The first approach based on writing the general stationary axisymmetric metric in terms of multipole moments was proposed by Ryan and was extended by Barack et al. [6]. They demonstrated how the laser interferometer space antenna (LISA) detector could map compact object deviating from the Kerr metric by means of EMRI observations. In the second approach, Collins and Hughes [8] introduced “bumpy black hole” (a spacetime that slightly deviates from the exact black hole of general relativity). This approach was recently generalized to the case of a rotating black hole in an alternate theories of gravity [9, 14]. In the third one, based on the Manko-Novikov metric [15], Apostolatos et al. [7] showed that the appearance of
Birkhoff chains in the neighborhood of a resonant tori would lead to such a modification of a gravitational-wave measurement. Finally, in the fourth one, Glampedakis et al. proposed a quasi-Kerr metric by choosing the (dimensionless) quadrupole moment to be $q_{\text{Kerr}} - \xi$, where the quadrupolar correction parameter $\xi$ represents a potential deviation from the Kerr metric and $q_{\text{Kerr}} = -J^2/M^4 = -a^2/M^2$. And the quadrupole moment can be written as

$$Q = -M \left( a^2 + \xi M^2 \right).$$

Glampedakis et al calculated the periastron precession and constructed ‘kludge’ gravitational waveforms as a function of the parameter $\xi$. These waveforms can be significantly different from the expected Kerr signal even for small changes of the quadrupole moment. In a series of papers, Johannsen et al analyzed in detail the innermost stable circular orbit, the circular photon orbit and the electromagnetic spectrum of the quasi-Kerr spacetime to test the no-hair theorem.

Gravitational lensing caused by deflection of light rays in a gravitational field is an ordinary phenomenon in astronomical observations when the light pass through the massive compact objects such as black hole, quasars and supernova. In the last decades, the theory of gravitational lensing has been developed along two branches. The former takes the weak deflection limit as photon radius is much larger than the gravitational radius of the lens. The latter uses the strong deflection limit as the light ray loops around the massive compact object many times before it reaches to the observer. By this mechanism, two infinite series of relativistic images appears on each side of the lens. These relativistic images can provide us not only some important signatures about compact objects in the universe, but also profound verification of alternative theories of gravity in the strong field regime. Thus, the strong gravitational lensing is regarded as a powerful indicator of the physical nature of the central celestial objects and then has been studied extensively in various theories of gravity.

The main purpose of this paper is to study the strong gravitational lensing by the quasi-Kerr compact object and to see whether it can leave us the signature of the quadrupole moment parameter in the photon sphere radius, the deflection angle, the coefficients and the observable quantities of strong gravitational lensing. Moreover, we will explore how it differs from the Kerr black hole lensing.

The paper is organized as follows: In Sec. II, we will review briefly the metric of the quasi-Kerr compact object with the quadrupole moment parameter proposed by Glampedakis et al. and calculate the radius of photon sphere. In Sec. III, we study the physical properties of the strong gravitational lensing by quasi-
Kerr compact object and probe the effects of the quadrupole moment parameter on the deflection angle, the coefficients and the observable quantities for gravitational lensing in the strong field limit. We end the paper with a summary.

II. ROTATING QUASI-KERR SPACETIME AND RADIUS OF PHOTON SPHERE

With the help of the Hartle-Thorne spacetime \[37\], the rotating quasi-Kerr compact object with arbitrary quadrupole moments was proposed by Glampedakis et al \[16\]. This metric has three independent parameters, i.e., the mass \(M\), the spin \(a\) and the quadrupolar correction parameter \(\xi\). The parameter \(\xi\) describes the deviation of the quadrupole moments from the Kerr value. The Kerr metric \(g^K_{ab}\) in the standard Boyer-Lindquist coordinates can be expressed as

\[
ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \left(\frac{4Mar \sin^2 \theta}{\Sigma}\right) dtd\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2r \sin^2 \theta}{\Sigma}\right) \sin^2 \theta d\phi^2 \tag{3}\]

with

\[
\Delta \equiv r^2 - 2Mr + a^2, \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta. \tag{4}\]

The quadrupolar correction is introduced by choosing a quadrupole moment of the form \[2\] in the Hartle-Thorne metric. Then, the quasi-Kerr metric \(g^{QK}_{ab}\) in the Boyer-Lindquist coordinates is given by \[16\]

\[
g^{QK}_{ab} = g^K_{ab} + \xi h_{ab} + O(\delta M_{\geq 4}, \delta S_{\geq 3}), \tag{5}\]

with

\[
h^{tt} = (1 - 2M/r)^{-1} \left[(1 - 3 \cos^2 \theta) F_1(r)\right], \]

\[
h^{rr} = (1 - 2M/r) \left[(1 - 3 \cos^2 \theta) F_1(r)\right], \]

\[
h^{\theta\theta} = -\frac{1}{r^2} \left[(1 - 3 \cos^2 \theta) F_2(r)\right], \]

\[
h^{\phi\phi} = -\frac{1}{r^2 \sin^2 \theta} \left[(1 - 3 \cos^2 \theta) F_2(r)\right], \]

\[
h^{t\phi} = 0, \tag{6}\]

where the functions \(F_1(r)\) and \(F_2(r)\) are given in appendix A of the Ref. \[16\]. The radius of the event horizon of the quasi-Kerr black hole, obtained from the relation \(g^{QK}_{tt} - g_{tt}g_{\phi\phi} = 0\), increases with increasing positive values of the quadrupolar correction parameter \(\xi\) but decreases as the spin \(a\) increases \[17\]. However, if \(\xi\) is negative, the event horizon is absent and the quasi-Kerr spacetime exists a naked singularity.
The biggest real root external to the horizon of this equation is defined as the radius of the photon sphere with
condition $\lambda \theta$ quasi-Kerr compact object and the whole trajectory of the photon is limited on the same plane. Using the
condition $\theta = \pi/2$ and taking $2M = 1$, the metric (3) is reduced to

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)d\phi^2 - 2D(r)dt\phi,$$

with

$$A(r) = 1 - \frac{1}{r} + \frac{5\xi}{16r^5} \left[2a^2(6r^2 + 3r - 1) + r^3(12r^3 - 18r^2 + 4r + 1)\right]$$

$$- \frac{5\xi}{16r^5} [6(2r - 1)^2r^5 + a^2(2r^2 - 1)] \ln\left(\frac{r}{r - 1}\right),$$

$$B(r) = \frac{r^2}{a^2 - r - r^2} + \frac{5\xi}{16(a^2 + (r - 1)r)^2} \left[12r^3 + 18r^2 - 4r - 1 + 12(r - 1)^2r^2 \ln\left(\frac{r}{r - 1}\right)\right],$$

$$C(r) = a^2 + \frac{a^2}{r} + r^2 + \frac{5\xi}{16(r - 1)^2r^2} \left[12r^3 + 18r^2 - 4r - 1 + 12(r - 1)^2r^2 \ln\left(\frac{r}{r - 1}\right)\right]$$

$$+ \frac{5\xi}{16(r - 1)^3r^3} \left[(r^3 + a^2(1 + r))^2(6r^2 - 3r + 1 + (6r^3 - 3r) \ln\left(\frac{r}{r - 1}\right))\right],$$

$$D(r) = a + \frac{5\xi}{r} \left[a^2(-12r^2 + 12r - 1) - 2a^2(6r^4 + 3r^3 - 7r^2 - 3r + 1)\right]$$

$$\frac{8r^5}{16(r - 1)r^5} + \frac{5\xi}{16(r - 1)^3r^3} \left[6r^4(2r^2 - 3r + 1 + a^2(2r^5 - 3r^3 + r)) \ln\left(\frac{r}{r - 1}\right)\right].$$

Then, the null geodesics for the metric (7) can be expressed as

$$\frac{dt}{d\lambda} = \frac{C(r) - LD(r)}{D(r)^2 + A(r)C(r)},$$

$$\frac{d\phi}{d\lambda} = \frac{D(r) + LA(r)}{D(r)^2 + A(r)C(r)},$$

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{C(r) - 2LD(r) - L^2A(r)}{B(r)C(r)\left[D(r)^2 + A(r)C(r)\right]},$$

where $\lambda$ is an affine parameter along the geodesics and $L$ is the angular momentum of the photon. With the
condition $\frac{dr}{d\lambda}|_{r=r_0} = 0$, we can obtain the impact parameter $u(r_0)$

$$u(r_0) = L(r_0) = -\frac{D(r_0) + \sqrt{A(r_0)C(r_0) + D^2(r_0)}}{A(r_0)}.$$

Moreover, the photon sphere is a time-like hyper-surface ($r = r_{ps}$) on which the deflect angle of the light
becomes unboundedly large as $r_0$ tends to $r_{ps}$. In this spacetime, the equation for the photon sphere reads

$$A(r)C'(r) - A'(r)C(r) + 2L[A'(r)D(r) - A(r)D'(r)] = 0.$$

The biggest real root external to the horizon of this equation is defined as the radius of the photon sphere
$r_{ps}$. Obviously, this equation is more complex than that in the background of a static and Kerr black holes
Therefore, it is impossible to get an analytical form for the photon sphere radius in this case. In Fig. 1 we present the variety of the photon-sphere radius $r_{ps}$ with the rotational parameter $a$ for different quadrupolar correction parameter $\xi$ numerically. The figure shows us that the photon sphere radius $r_{ps}$ decreases as the rotational parameter $a$ increases, but it increases as the quadrupolar correction parameter $\xi$ increases. Moreover, we also find that the photon sphere radius $r_{ps}$ exists only in the regime $\xi > -0.312$ when the quasi-Kerr compact object rotates in the same direction as the photon ($a > 0$). As the quadrupolar correction parameter $\xi$ becomes negative the radius of the photon sphere becomes smaller, which implies that the photons are more easily captured by the quasi-Kerr compact object with the negative quadrupolar correction parameter $\xi$ than that of the Kerr black hole.

\[
\alpha(r_0) = I(r_0) - \pi, \quad (17)
\]
with
\[ I(r_0) = 2 \int_{r_0}^{\infty} \frac{\sqrt{B(r)|A(r_0)||D(r) + LA(r)|}}{\sqrt{D^2(r) + A(r)C(r)} \sqrt{sgn(A(r_0))|A(r_0)C(r) - A(r)C(r_0) + 2L[A(r)D(r_0) - A(r_0)D(r)]}} \, dr, \] (18)

where \( sgn(X) \) gives the sign of \( X \).

It is obvious that the deflection angle increases as the parameter \( r_0 \) decreases. For a certain value of \( r_0 \) the deflection angle becomes \( 2\pi \), so that the light ray makes a complete loop around the lens before reaching the observer. If \( r_0 \) is equal to the radius of the photon sphere \( r_{ps} \), we can find that the deflection angle diverges and the photon is captured by the compact object.

In order to find the behavior of the deflection angle when the photon is close to the photon sphere, we use the evaluation method proposed by Bozza [22]. The divergent integral in Eq. (18) is first split into the divergent part \( I_D(r_0) \) and the regular one \( I_R(r_0) \), and then both of them are expanded around \( r_0 = r_{ps} \) with sufficient accuracy. This technique has been widely used in the study of the strong gravitational lensing for various black holes [18–36]. Let us now to define a variable
\[ z = 1 - \frac{r_0}{r}, \] (19)
and rewrite the Eq. (18) as
\[ I(r_0) = \int_0^1 R(z, r_0)f(z, r_0)dz, \] (20)

with
\[ R(z, r_0) = \frac{2r_0}{\sqrt{C(z)(1 - z)^2}} \frac{\sqrt{B(z)|A(r_0)||D(z) + LA(z)|}}{\sqrt{D^2(z) + A(z)C(z)}}, \] (21)
\[ f(z, r_0) = \frac{1}{\sqrt{sgn(A(r_0))|A(r_0) - A(z)\frac{C(r_0)}{C(z)} + \frac{2L}{C(z)}(A(z)D(r_0) - A(r_0)D(z))}}} \] (22)

Obviously, the function \( R(z, r_0) \) is regular for all values of \( z \) and \( r_0 \). However, the function \( f(z, r_0) \) diverges as \( z \) tends to zero, i.e., as the photon approaches the photon sphere. Thus, the integral (20) can be separated into two parts \( I_D(r_0) \) and \( I_R(r_0) \)
\[ I_D(r_0) = \int_0^1 R(0, r_{ps})f_0(z, r_0)dz, \]
\[ I_R(r_0) = \int_0^1 [R(z, r_0)f(z, r_0) - R(0, r_0)f_0(z, r_0)]dz. \] (23)

Expanding the argument of the square root in \( f(z, r_0) \) to the second order in \( z \), we have
\[ f_0(z, r_0) = \frac{1}{\sqrt{p(r_0)z + q(r_0)z^2}}. \] (24)
FIG. 2: Variation of the coefficients for the strong gravitational lensing with the parameter $a$ for different quadrupolar correction parameter $\xi$ in the rotating quasi-Kerr spacetime.
where

\[ p(r_0) = \frac{r_0}{C(r_0)} \left\{ A(r_0)C''(r_0) - A'(r_0)C(r_0) + 2L[A'(r_0)D(r_0) - A(r_0)D'(r_0)] \right\} , \]

\[ q(r_0) = \frac{r_0}{2C^2(r_0)} \left\{ 2 \left( C(r_0) - r_0 C'(r_0) \right) \left( A(r_0)C'(r_0) - A'(r_0)C(r_0) + 2L[A'(r_0)D(r_0) - A(r_0)D'(r_0)] \right) \right. \]

\[ + r_0 C(r_0) \left( A(r_0)C''(r_0) - A''(r_0)C(r_0) + 2L[A''(r_0)D(r_0) - A(r_0)D''(r_0)] \right) \right\} . \]  

(25)

From Eq. (25), we can find that if \( r_0 \) approaches the radius of photon sphere \( r_{ps} \) the coefficient \( p(r_0) \) vanishes and the leading term of the divergence in \( f_0(z, r_0) \) is \( z^{-1} \), which implies that the integral \( (20) \) diverges logarithmically. The coefficient \( q(r_0) \) takes the form

\[ q(r_{ps}) = \frac{\text{sgn}(A(r_{ps}))r_{ps}^2}{2C(r_{ps})} \left\{ A(r_{ps})C''(r_{ps}) - A''(r_{ps})C(r_{ps}) + 2L[A''(r_{ps})D(r_{ps}) - A(r_{ps})D''(r_{ps})] \right\} . \]

(26)

Therefore, the deflection angle in the strong field region can be expressed as

\[ \alpha(\theta) = -\bar{a} \log \left( \frac{\theta D_{OL}}{u_{ps}} - 1 \right) + \bar{b} + \mathcal{O}(u - u_{ps}), \]

(27)

with

\[ \bar{a} = \frac{R(0, r_{ps})}{2 \sqrt{q(r_{ps})}} \]

\[ \bar{b} = -\pi + b_R + \bar{a} \log \left\{ \frac{2q(r_{ps})C(r_{ps})}{u_{ps}|A(r_{ps})||D(r_{ps}) + u_{ps}A'(r_{ps})|} \right\} , \]

\[ b_R = I_R(r_{ps}), \]

\[ u_{ps} = \frac{-D(r_{ps}) + \sqrt{A(r_{ps})C(r_{ps}) + D^2(r_{ps})}}{A(r_{ps})} . \]  

(28)
where the quantity $D_{OL}$ is the distance between observer and gravitational lens. Making use of Eqs. (27) and (28), we can study the properties of strong gravitational lensing in the rotating quasi-Kerr spacetime. In Fig. 2 we plotted the changes of the coefficients $u_{ps}, \bar{a}$ and $\bar{b}$ with $a$ for a different quadrupolar correction parameter $\xi$. It is shown that the coefficients ($\bar{a}$ and $\bar{b}$) in the strong field limit are functions of the parameters $a$ and $\xi$. We can see that $u_{ps}$ has a similar behavior as $r_{ps}$ shown in Fig. 1. For a fixed $a$, the coefficient $\bar{a}$ decreases but coefficient $\bar{b}$ increases with increasing of the quadrupolar correction parameter $\xi$. With the help of the coefficients $\bar{a}$ and $\bar{b}$ we plotted the change of the deflection angles evaluated at $u = u_{ps} + 0.00326$ with $\xi$ in Fig. 3. It is shown that in the strong field limit the deflection angles have the similar properties of the coefficient $\bar{a}$ when the quadrupolar correction parameter $\xi$ takes a positive value. Moreover, it is interesting to note that, when the quadrupolar correction parameter $\xi$ takes a negative value, the deflection angles can increase to a peak value for a certain $a$. And the deflection angles of the quasi-Kerr compact object with the negative quadrupolar correction parameter $\xi$ is larger than the case of Kerr black hole. These results tell us that we could get the information of the compact object with quadrupole moment by means of strong gravitational lensing.

**B. Observable quantities of strong gravitational lensing**

Let us now to study the effect of the quadrupolar correction parameter $\xi$ on the observable quantities of strong gravitational lensing. Here we consider only the case in which the source, lens and observer are highly aligned so that the lens equation in strong gravitational lensing can be approximated well as \[ \gamma = \frac{D_{OL} + D_{LS}}{D_{LS}} \theta - \alpha(\theta) \mod 2\pi, \] (29)

where $D_{LS}$ is the lens-source distance and $D_{OL}$ is the observer-lens distance, $\gamma$ is the angle between the direction of the source and the optical axis, $\theta = u/D_{OL}$ is the angular separation between the lens and the image. Following ref. [23], we can find that the angular separation between the lens and the n-th relativistic image is

\[ \theta_n \simeq \theta_0^n \left( 1 - \frac{u_{ps} e_n (D_{OL} + D_{LS})}{\bar{a} D_{OL} D_{LS}} \right), \] (30)

with

\[ \theta_0^n = \frac{u_{ps}}{D_{OL}} (1 + e_n), \quad e_n = e^{\frac{\xi}{a} - 2\pi n}, \] (31)

where the quantity $\theta_0^n$ is the image positions corresponding to $\alpha = 2n\pi$, and $n$ is an integer. According to the past oriented light ray which starts from the observer and finishes at the source the resulting images stand
FIG. 4: Variation of the innermost relativistic image $\theta_\infty$, the relative magnitudes $r_m$ and the angular separation $s$ with the parameter $a$ for different $\xi$. Here, we set $2M = 1$.

on the eastern side of the black hole for direct photons ($a > 0$) and are described by positive $\gamma$. Retrograde photons ($a < 0$) have images on the western side of the compact object and are described by negative values of $\gamma$. In the limit $n \to \infty$, we can find that $e_n \to 0$, which means that the relation between the minimum
impact parameter $u_{ps}$ and the asymptotic position of a set of images $\theta_\infty$ can be simplified further as

$$u_{ps} = D_{OL} \theta_\infty.$$  \hspace{1cm} (32)

In order to obtain the coefficients $\bar{a}$ and $\bar{b}$, we needs to separate at least the outermost image from all the others. As in refs. \[22, 23\], we consider here the simplest case in which only the outermost image $\theta_1$ is resolved as a single image and all the remaining ones are packed together at $\theta_\infty$. Thus the angular separation between the first image and other ones can be expressed as \[22, 23, 25\]

$$s = \theta_1 - \theta_\infty = \theta_\infty e^{\frac{-2\pi}{\xi_0}} - \theta_\infty e^{\frac{-2\pi}{\xi}}.$$  \hspace{1cm} (33)

By measuring $s$ and $\theta_\infty$, we can obtain the strong deflection limit coefficients $\bar{a}$, $\bar{b}$ and the minimum impact parameter $u_{ps}$. Comparing their values with those predicted by the theoretical models, we can obtain information of the compact object.

The mass of the central object of our Galaxy is estimated recently to be $4.4 \times 10^6 M_\odot$ \[39\] and its distance is around $8.5kpc$, so that the ratio of the mass to the distance $M/D_{OL} \approx 2.4734 \times 10^{-11}$. Making use of Eqs. \[28\], \[32\] and \[33\] we can estimate the values of the coefficients and observable quantities for gravitational lens in the strong field limit. The numerical value for the angular position of the relativistic images $\theta_\infty$, the angular separation $s$ and the relative magnitudes $r_m$ are plotted in Fig. 3 and listed in table I. We find that with the increase of $\xi$, the angular position of the relativistic images $\theta_\infty$ decreases for both the direct photons

| $\xi$ | $\theta_\infty$ ($\mu$ arcsec) |
|-------|-------------------------------|
| $-0.10$ | 27.478 27.854 28.190 28.497 28.781 29.047 29.534 29.978 |
| $-0.05$ | 26.289 26.759 27.160 27.516 27.839 28.138 28.678 29.161 |
| $0$ | 24.905 25.581 26.085 26.510 26.885 27.224 27.827 28.356 |
| $0.05$ | 24.222 24.943 25.474 25.919 26.310 26.987 27.569 |
| $0.10$ | 23.682 24.403 24.946 25.403 26.167 26.808 |

| $\xi$ | $s$ ($\mu$ arcsec) |
|-------|------------------|
| $-0.10$ | 0.0405 0.0317 0.0262 0.0223 0.0196 0.0174 0.0144 0.0123 |
| $-0.05$ | 0.0610 0.0426 0.0330 0.0271 0.0230 0.0201 0.0161 0.0135 |
| $0$ | 0.1069 0.0631 0.0417 0.0332 0.0271 0.0223 0.0197 0.0153 |
| $0.05$ | 0.0806 0.0519 0.0360 0.0279 0.0197 0.0154 |
| $0.10$ | 0.0806 0.0519 0.0360 0.0279 0.0197 0.0154 |

| $\xi$ | $r_m$ (magnitudes) |
|-------|--------------------|
| $-0.10$ | 6.5018 6.8540 7.1120 7.3151 7.4821 7.6236 7.8543 8.0377 |
| $-0.05$ | 5.8734 6.4513 7.1120 7.3151 7.4821 7.6236 7.8543 8.0377 |
| $0$ | 3.9225 6.2569 6.7774 7.1022 7.5179 7.8018 |
| $0.05$ | 3.9744 6.2569 6.7774 7.1022 7.5179 7.8018 |
| $0.10$ | 4.8954 6.2569 6.7774 7.1022 7.5179 7.8018 |
(a > 0) and the retrograde photons (a < 0). The variation of the angular separation s and the relative magnitudes \( r_m \) with \( \xi \) is similar to that of the deflection angle \( \alpha(\theta) \) and the deflection angle coefficient \( \bar{b} \), respectively.

Now, we have found that the numerical value for the angular position of the innermost relativistic images \( \theta_\infty \) of quasi-Kerr compact object is \( 23 - 29 \mu\text{arcsec} \). In principle, such a optical resolution is reachable by very long baseline interferometry (VLBI) projects, but we may be aware that relativistic images are difficult to detect and the resolution of the most powerful modern instruments is currently insufficient to perform such high precision astrometry. However, innovative near-infrared interferometry instruments are now under development at the very large Telescope and Keck, i.e., advanced radio interferometry between space and Earth (ARISE). These instruments have been conceived to achieve an astrometric accuracy of \( 10 - 100 \mu\text{arcsec} \) in combination with milli-arcsec angular-resolution imaging \cite{40}. Thus we can observe relativistic images within a not so far future.

The relativistic images obtained by means of strong gravitational lensing in our paper is different from the images of the quasi-Kerr spacetime obtained by the use of a ray-tracing algorithm in ref. \cite{17}. However, Johannsen et al showed that images of the quasi-Kerr compact object become oblate or prolate depending on the sign and value of the parameter \( \xi \), which has a link with the properties of the minimum impact parameter \( u_{ps} \) and the radius of photon sphere \( r_{ps} \) in our paper, i.e., when the parameter \( \xi \) takes the positive (negative) value, the photon-sphere radius \( r_{ps} \) and the minimum impact parameter \( u_{ps} \) are larger (smaller) than the results obtained in the Kerr black hole. Johannsen et al use the degree of asymmetry and displacement of the photon ring as a direct measure of the violation of the no-hair theorem, while we use observables (the angular position of the relativistic images \( \theta_\infty \), the angular separation \( s \) and the relative magnitudes \( r_m \)) to measure the relativistic images. It is interesting to note the angular size of the diameter of the photon ring \( \theta_{\text{ring}} = 53 \mu\text{arcsec} \) of a slowly rotating Kerr black hole obtained in ref. \cite{17} is approximatively equal to the size of the angular diameter of the innermost relativistic images \( \theta_\infty \) for the case \( a = 0, \xi = 0 \) in table II of our paper (noted that the data listed in table II is the radius of \( \theta_\infty \)). Thus, the observation of both the images of black hole in ref. \cite{17} and the relativistic images caused by strong gravitational lensing in our paper require astronomical instruments have high optical resolution. Based on the above discussion we can conclude that, with the electromagnetic spectrum proposed in ref. \cite{17} and the relativistic images of our paper, we can comprehensively observer how astronomical compact object deviates from the Kerr black hole.
IV. SUMMARY

In this paper, in order to test how astronomical compact object deviates from the Kerr black hole, we investigate the features of light propagation on the equatorial plane of the rotating quasi-Kerr spacetime proposed by Glampedakis et al. Assuming that the massive compact object at the center of our galaxy can be described by quasi-Kerr spacetime, we obtain the photon radius, the coefficients and observable quantities of the strong gravitational lensing. We find that the photon-sphere radius exist only when the quadrupolar correction parameter takes the value $\xi > -0.302$ for the photons in the prograde orbit ($a > 0$). Moreover, as the quadrupolar correction parameter $\xi$ becomes negative the radius of the photon sphere becomes smaller, which implies that the photons are more easily captured by the quasi-Kerr compact object with the negative quadrupolar correction parameter $\xi$ than that of the Kerr black hole. It is interesting to note that, when the quadrupolar correction parameter $\xi$ takes the positive (negative) value, the photon-sphere radius $r_{ps}$, the minimum impact parameter $u_{ps}$, the coefficient $\tilde{b}$, the relative magnitudes $r_m$ and the angular position of the relativistic images $\theta_\infty$ are larger (smaller) than the results obtained in the Kerr black hole, but the coefficient $\tilde{a}$, the deflection angle $\alpha(\theta)$ and the angular separation $s$ are smaller (larger) than that of the Kerr black hole. Based on the above results, we come to the conclusion that there are some significant effects of the quadrupolar correction parameter $\xi$ on the coefficients and observable parameters of the strong gravitational lensing. These results, in principle, may provide a possibility to test how astronomical black holes with arbitrary quadrupole moments deviate from the Kerr black hole in the future astronomical observations.

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