CONTROLLED EQUILIBRIUM SELECTION IN
STOCHASTICALLY PERTURBED DYNAMICS

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Abstract. We consider a dynamical system with finitely many equilibria and perturbed by small noise, in addition to being controlled by an ‘expensive’ control. We study the invariant distribution of the controlled process as the variance of the noise becomes vanishingly small. It is shown that depending on the relative magnitudes of the noise variance and the ‘running cost’ for control, one can identify three regimes, in each of which the optimal control forces the invariant distribution of the process to concentrate near equilibria that can be characterized according to the regime. We also show that in the vicinity of the points of concentration the density of invariant distribution approximates the density of a Gaussian, and we explicitly solve for its covariance matrix.

1. Introduction

The study of dynamical systems has a long and profound history. A lot of effort has been devoted to understand the behavior of the system when it is perturbed by an additive noise \[3,16,22\]. Small noise diffusions have found applications in climate modeling \[4,15\], electrical engineering \[9,27\], finance \[15\] and many other areas. In this article we consider a controlled dynamical system with small noise, which is modelled as a \(d\)-dimensional controlled diffusion \(X = [X_1, \ldots, X_d]^T\) governed by the stochastic integral equation

\[
X_t = X_0 + \int_0^t \left( m(X_s) + \varepsilon U_s \right) \, ds + \varepsilon \nu W_t, \quad t \geq 0. \tag{1.1}
\]

Here:

- \(m = [m_1, \ldots, m_d]^T: \mathbb{R}^d \to \mathbb{R}^d\) is a bounded \(C^\infty\) function with bounded derivatives,
- \(W \) is a standard Brownian motion in \(\mathbb{R}^d\),
- \(U\) is an \(\mathbb{R}^d\)-valued control process with measurable paths satisfying the nonanticipativity condition: for \(t > s\), \(W_t - W_s\) is independent of \(X_0, W_r, U_r, r \leq s\). As pointed out in \[10\] p. 18, we may, without loss of generality, consider \(U\) to be adapted to the natural filtration of \(X\). (The set of such ‘admissible’ controls is denoted by \(\mathcal{U}\).)

- \(0 < \varepsilon \ll 1\),
- \(\nu > 0\).

We view this as a perturbation of the o.d.e. (for ordinary differential equation)

\[
\dot{x}(t) = m(x(t)), \tag{1.2}
\]
perturbed by the ‘small noise’ $\varepsilon W_t$ (‘small’ because $\varepsilon \ll 1$) and the control $\varepsilon U_t$. In the case when the control $U_t \equiv 0$, Freidlin and Wentzell developed a general framework for the analysis of small noise perturbed dynamical systems in \cite{16} that is based on the theory of large deviations. The goal of this article is to study the effect of the additional control when the parameter $\varepsilon$ tends to 0. We assume that the set of nonwandering points of (1.2) consists of finitely many hyperbolic equilibria, and that these are contained in some bounded open set which is positively invariant under the flow. The objective is to minimize the long run average (or ergodic) cost

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] ,$$

over all admissible controls. Here $\ell : \mathbb{R}^d \to \mathbb{R}_+$ is a prescribed smooth, Lipschitz function satisfying the condition:

$$\lim_{|x| \to \infty} \ell(x) = \infty .$$

Since $\varepsilon$ is small, this implies that the control is expensive. Under a stochastic Lyapunov condition we introduce later, the cost is finite for $U \equiv 0$, ensuring in particular that the set of controls with finite cost is nonempty. It is quite evident from ergodic theory that for $U = 0$ the limit (1.3) is the moment of $\ell$ with respect to the invariant measure of (1.1). It is not hard to show that as $\varepsilon \searrow 0$, the collection of invariant measures is tight and concentrates on the set of equilibria of $m$. To find the actual support of the limit, in the case of multiple equilibria, one often looks at the large deviation properties of these invariant measures \cite{16}. There are several studies in literature that deal with the large deviation principle of invariant measures of dynamical systems. Among the most relevant to the present are \cite{14,24} which obtain a large deviation principle for invariant measures (more precisely, invariant densities) of (1.1) under the assumption that there is only one equilibrium point. This is generalized to the multiple equilibria in \cite{8}. A large deviation principle for invariant measures for a class of reaction-diffusion systems is established in \cite{13}.

None of the above mentioned studies have any control component in their dynamics. One of our motivations for this work is to add a control in the dynamics and study its effects on the noise in the selection of equilibria.

Remark 1.1. The vector field $m$ is assumed bounded for simplicity. The reader however might notice that the regularity results in \cite{3}, on which the characterization of optimality is based (see Theorem 2.2), the hypotheses in \cite{3} Section 4.6.1 permit $m$ to be unbounded as long as

$$\limsup_{|x| \to \infty} \frac{|m(x)|^2}{\ell(x)} < \infty .$$

Provided that this condition is satisfied, the assumption that the drift is bounded can be waived, and all the results of this paper hold unaltered, with the proofs requiring no major modification.

Let us now state the following result on existence of solutions to (1.1) whose proof is given in Appendix A.

Lemma 1.1. Under the condition $\mathbb{E}\left[ \int_0^t |U_s|^2 \, ds \right] < \infty$, $\forall t \geq 0$, the diffusion in (1.1) has a unique weak solution.
The qualitative properties of the dynamics are best understood if we consider the special case \( d = 1 \), and \( m = -\frac{dF}{dx} \) for some continuously differentiable \( F: \mathbb{R} \to \mathbb{R} \). Then the trajectory of (1.2) converges to a critical point of \( F \). In fact, generically (i.e., for \( x(0) \) in an open dense set) it converges to a stable one, i.e., to a local minimum. If one views the graph of \( F \) as a ‘landscape’, the local minima are the bottoms of its ‘valleys’. The behavior of the stochastically perturbed (albeit uncontrolled) version of this model, notably the analysis of where the stationary probability distribution concentrates, has been of considerable interest to physicists (see, e.g., [23, Chapter 8] or [16, Chapter 6]). Recent work on ‘stochastic resonance’ (see, e.g., [21]) introduces an additional external input to the dynamics that may be viewed as a control. This is chosen so that it ‘resonates’ with the noise in an appropriate sense and induces transitions between valleys. The model in (1.1) goes a step further and considers the full-fledged optimal control version of this, wherein one tries to induce a preferred equilibrium behavior through a feedback control. The reason the latter has to be ‘expensive’ is because this captures the physically realistic situation that one can’t tweak the dynamics but cannot replace it by something altogether different without incurring considerable expense. The function \( \ell \) captures the relative preference among different points in the state space. Let us also point out that this problem can also be viewed as a multi-scale diffusion problem as the control and noise are scaled differently.

The following hypothesis on the vector field \( m \) is in effect throughout the paper.

**Hypothesis 1.1.** The set \( S := \{ x \in \mathbb{R}^d : m(x) = 0 \} \) is finite and its elements are hyperbolic, i.e., the Jacobian matrix \( Dm(x) \) of \( m \) at each point \( x \in S \) has no eigenvalues on the imaginary axis. Also there exist a twice continuously differentiable function \( \bar{V}: \mathbb{R}^d \to \mathbb{R}^+ \), and a bounded set \( K \subset \mathbb{R}^d \) containing \( S \), with the following properties:

(H1) \( c_1|x|^2 \leq \bar{V}(x) \leq c_2|x|^2 \) for some positive constants \( c_1, c_2 \), and all \( x \in K^c \).

(H2) \( \nabla \bar{V} \) is Lipschitz and satisfies

\[
\langle m(x), \nabla \bar{V}(x) \rangle < -2\gamma |x| \tag{1.4}
\]

for some \( \gamma > 0 \), and all \( x \in K^c \).

We need some definitions.

**Definition 1.1.** Let \( S_s \subset S \) denote the set of stable equilibria of (1.2), i.e., the set of points \( z \in S \) for which the eigenvalues of \( Dm(z) \) have negative real parts.

Throughout the paper \( \beta_s^e \) denotes the optimal value of (1.3), \( \eta^e_s \) denotes the stationary probability distribution of the process \( X \) under an optimal stationary Markov control (which is denoted as \( v^e_s \)), and \( g^e_s \) its density (for existence and uniqueness see Theorem 2.2). We also define the ‘running cost’ \( R[v]: \mathbb{R}^d \to \mathbb{R} \) corresponding to a stationary Markov control \( v: \mathbb{R}^d \to \mathbb{R}^d \) by

\[
R[v](x) := \ell(x) + \frac{1}{2} |v(x)|^2 .
\]

We say that a set \( K \subset \mathbb{R}^d \) is stochastically stable (or that \( \eta^e_s \) concentrates on \( K \)) if it is compact, and \( \lim_{\epsilon \to 0} \eta^e_s(K^c) = 0 \). It is evident that the class \( \mathcal{S} \) of stochastically stable sets, if nonempty, is closed under intersections. Hence we define the minimal stochastically stable set \( S \) by \( S := \cap_{K \in \mathcal{S}} K \).
Definition 1.2. For a square matrix $M \in \mathbb{R}^{d \times d}$, let $\mathcal{E}_+ (M)$ denote the sum of its eigenvalues that lie in the open right half complex plane. For $z \in \mathcal{S}$, and with $M \equiv Dm(z)$, we let $\hat{Q}_z$ and $\hat{\Sigma}_z$ be the symmetric, nonnegative definite, square matrices solving the pair of equations

$$M^T \hat{Q}_z + \hat{Q}_z M = \hat{Q}_z^2,$$
$$\left( M - \hat{Q}_z \right) \hat{\Sigma}_z + \hat{\Sigma}_z \left( M - \hat{Q}_z \right)^T = -I. \tag{1.5}$$

By Lemma 3.1 which appears later in the paper there exists a unique pair $(\hat{Q}_z, \hat{\Sigma}_z)$ of symmetric positive semidefinite matrices solving (1.5). It is also evident by (1.5) that $\hat{\Sigma}_z$ is invertible.

The main results of the paper are summarized in Theorems 1.1–1.3 that follow.

Theorem 1.1. The minimal stochastically stable set $\mathcal{S}$ is a subset of $\mathcal{S}$ for all $\nu > 0$.

We define

$$Z_1 := \text{Arg min}_{z \in \mathcal{S}} \{ \ell(z) \}, \quad \mathcal{J}_1 := \min_{z \in \mathcal{S}} [\ell(z)],$$
$$Z_2 := \text{Arg min}_{z \in \mathcal{S}} \{ \ell(z) + \mathcal{E}_+ (Dm(z)) \}, \quad \mathcal{J}_2 := \min_{z \in \mathcal{S}} [\ell(z) + \mathcal{E}_+ (Dm(z))],$$
$$Z_3 := \text{Arg min}_{z \in \mathcal{S}_s} \{ \ell(z) \}, \quad \mathcal{J}_3 := \min_{z \in \mathcal{S}_s} [\ell(z)].$$

The set $\mathcal{S}$ and the optimal value $\beta^\varepsilon_s$ depend on $\nu$ as follows:

(i) For $\nu > 1$ (‘supercritical’ regime), $\mathcal{S} \subset Z_1$, and $\beta^\varepsilon_s = \mathcal{J}_1 + \mathcal{O}(\varepsilon^{2\nu - 2})$ if $\nu \in (1, 2)$, while $\mathcal{O}(\varepsilon^2) \leq \beta^\varepsilon_s - \mathcal{J}_1 \leq \mathcal{O}(\varepsilon^{2\nu - 2})$ for $\nu \geq 2$.

(ii) For $\nu < 1$ (‘subcritical’ regime), $\mathcal{S} \subset Z_3$, and $\mathcal{O}(\varepsilon^{2 - 2\nu}) \leq \beta^\varepsilon_s - \mathcal{J}_3 \leq \mathcal{O}(\varepsilon^{4\nu - 2})$ if $\nu \in (2/3, 1)$, while $\beta^\varepsilon_s = \mathcal{J}_3 + \mathcal{O}(\varepsilon^\nu)$ if $\nu \leq 2/3$.

(iii) For $\nu = 1$ (‘critical’ regime), $\mathcal{S} \subset Z_2$, $\lim_{\varepsilon \searrow 0} \beta^\varepsilon_s = \mathcal{J}_2$, and $\beta^\varepsilon_s \leq \mathcal{J}_2 + \mathcal{O}(\varepsilon^2)$.

To prove Theorem 1.1 we first identify the optimal control for $\varepsilon > 0$ from the associated Hamilton–Jacobi–Bellman equation (HJB) and establish a rigorous connection of the HJB studied in [2] with the ergodic control problem. It is not hard to show that the optimal invariant measures $\eta^\varepsilon_s$ concentrate on $\mathcal{S}$ as $\varepsilon \searrow 0$ (see Lemma 4.1). In Theorem 1.1 we actually identify three regimes, corresponding to different values of $\nu$ and characterize the limiting $\beta^\varepsilon_s$. For $\nu > 1$ one can find a control $U$ under which the invariant measure of the dynamics (1.1) concentrates on a point in $\mathcal{S}$. Construction of invariant measures with similar properties is also possible for $z \in \mathcal{S}_s$ when $\nu < 1$. The important difference is that for $\nu < 1$ the optimal invariant measure $\eta^\varepsilon_s$ cannot concentrate on $\mathcal{S} \setminus \mathcal{S}_s$ (see Lemma 4.3). To show this fact we construct a suitable Lyapunov function for the Morse-Smale dynamics (see Theorem 2.1). The analysis in the critical regime $\nu = 1$ turns out to be more subtle than other two regimes. To establish the results for this regime we study the ergodic control problem for LQG systems (Lemma 3.4). Depending on some moment results (Lemma 3.5) we scale the space suitably and show that the resulting invariant measures are also tight. In particular, we examine the asymptotic behavior of $\eta^\varepsilon_s$ and show that under an appropriate spatial scaling it ‘converges’ to a Gaussian distribution in the vicinity of the minimal stochastically stable set.
Theorem 1.2. Let \( \nu \in (0, 2) \). Suppose that for \( z \in S \) there exists a sequence \( \varepsilon_n \searrow 0 \), such that for some open neighborhood \( N \) of \( z \) whose closure does not contain any other elements of \( S \) it holds that \( \liminf_{\varepsilon_n \searrow 0} \eta_\nu^\varepsilon(N) > 0 \). Then along this sequence we have

\[
\frac{\varepsilon^{\nu d} \xi^\varepsilon e^{\varepsilon x + z}}{\eta_\nu^\varepsilon(N)} \xrightarrow{\varepsilon \searrow 0} \frac{1}{(2\pi)^{d/2} |\det \hat{\Sigma}_z|^{1/2}} \exp\left(-\frac{x^T \hat{\Sigma}_z^{-1} x}{2}\right),
\]

uniformly on compact sets, where \( \det \hat{\Sigma}_z \) denotes the determinant of \( \hat{\Sigma}_z \).

The following theorem shows that in the supercritical case, the set \( S \) contains only those points \( z \in Z_1 \) for which \( E_+(Dm(z)) \) is minimal. In particular, if \( Z_1 \cap S \neq \emptyset \), then \( S \subset Z_3 \).

Theorem 1.3. We assume \( \nu \in (1, 2) \). Define the subset \( Z_1^* \subset Z_1 \) by

\[
Z_1^* := \text{Arg min}_{z \in Z_1} \{E_+(Dm(z))\}.
\]

If \( z \in S \setminus Z_1^* \) and \( N \) is a neighborhood of \( z \) whose closure does not contain other elements of \( S \), then \( \eta_\nu^\varepsilon(N) \xrightarrow{\varepsilon \searrow 0} 0 \). Moreover, for \( z \in Z_1^* \), it holds that

\[
\beta_\varepsilon = \ell(z) + \varepsilon^{2\nu} E_+(Dm(z)) + o(\varepsilon^{2\nu - 2}).
\]

Remark 1.2. It is evident from Theorem 1.1 (i) that \( \nu = 2 \) is a critical value. While for \( \nu \in (1, 2) \) and a point \( z \in Z_1^* \setminus S \), we have \( \liminf_{\varepsilon \searrow 0} \varepsilon^{2-2\nu} (\beta_\varepsilon - \ell(z)) > 0 \), which means that the control effort \( \int_{\mathbb{R}^d} \frac{1}{2} |v_\varepsilon|^2 \, d\eta_\varepsilon \) exceeds \( \int_{\mathbb{R}^d} \ell \, d\eta_\varepsilon \) for all \( \varepsilon \) sufficiently small, the opposite can occur if \( \nu > 2 \). A simple example where this happens is the one-dimensional model with data \( m(x) = x \) and \( \ell(x) = (x + 1)^2 \). For this example, direct substitution shows that the solution of the HJB equation (see (2.7)) is

\[
V_\varepsilon(x) = \frac{1 - b(\varepsilon)}{b(\varepsilon)} (x + b(\varepsilon))^2,
\]

\[
\beta_\varepsilon = 1 + \varepsilon^{2\nu} \frac{1 - b(\varepsilon)}{b(\varepsilon)} - 2b(\varepsilon) (1 - b(\varepsilon)),
\]

where

\[
b(\varepsilon) := 2\varepsilon^2 \left(1 + 2\varepsilon^2 + \sqrt{1 + 2\varepsilon^2}\right)^{-1}.
\]

A simple calculation shows that if \( \nu \in (1, 2) \) then \( \liminf_{\varepsilon \searrow 0} \varepsilon^{2-2\nu} (\beta_\varepsilon - 1) > 0 \), while if \( \nu > 2 \), then \( \limsup_{\varepsilon \searrow 0} \varepsilon^{-2} (\beta_\varepsilon - 1) < 0 \). Therefore (1.6) does not hold for this example when \( \nu > 2 \).

Theorem 1.4 is the combination of Lemma 3.1, Lemma 3.3, Corollary 3.1, Corollary 3.2 and Theorems 3.1, 3.3 presented in Section 3. Theorem 1.2 follows from Lemmas 3.7 and 3.8 – see also Theorems 3.5 and 3.6 for related results. The proof of Theorem 1.3 can be found in Section 3.5.
1.1. Notation. The following notation is used in this paper. The symbols $\mathbb{R}$, and $\mathbb{C}$ denote the fields of real numbers, and complex numbers, respectively. Also, $\mathbb{N}$ denotes the set of natural numbers. The Euclidean norm on $\mathbb{R}^d$ is denoted by $| \cdot |$, and $\langle \cdot , \cdot \rangle$ denotes the inner product. A ball of radius $r > 0$ in $\mathbb{R}^d$ around a point $x$ is denoted by $B_r(x)$, or as $B_r$ if $x = 0$. For a compact set $K$, $B_r(K)$ denotes the open $r$-neighborhood of $K$. For a set $A \subset \mathbb{R}^d$, we use $A$, $A^c$, and $\partial A$ to denote the closure, the complement, and the boundary of $A$, respectively. We write $A \in B$ to indicate that $A \subset B$. We define $C_k^k(\mathbb{R}^d)$, $k \geq 0$, as the set of functions whose $i$-th derivatives, $i = 1, \ldots , k$, are continuous and bounded in $\mathbb{R}^d$ and denote by $C_k^k(\mathbb{R}^d)$ the subset of $C_k^k(\mathbb{R}^d)$ with compact support. The space of all probability measures on a Polish space $\mathcal{X}$ with the Prohorov topology is denoted by $\mathcal{P}(\mathcal{X})$. The density of the $d$-dimensional Gaussian distribution with mean $0$ and covariance matrix $\Sigma$ is denoted by $\rho_\Sigma$.

The symbols $\mathcal{O}(x)$ and $\mathfrak{o}(x)$ denote the sets of functions $f : \mathbb{R}^d \to \mathbb{R}$ having the property
\[
\limsup_{|x| \searrow 0} \frac{|f(x)|}{|x|} < \infty , \quad \text{and} \quad \limsup_{|x| \searrow 0} \frac{|f(x)|}{|x|} = 0 ,
\]
respectively. Abusing the notation, $\mathcal{O}(x)$ and $\mathfrak{o}(x)$ occasionally denote generic members of these sets. Also $\kappa_1, \kappa_2, \ldots$ are generic constants whose definition differs from place to place.

In Section 2 we present a basic property of gradient-like flows (Theorem 2.1), and characterize the optimal control problem via a HJB equation (Theorem 2.2). Section 3 is devoted to the study of the support of the limit, as $\varepsilon \searrow 0$, of the optimal stationary probability distribution $\eta_{\varepsilon}$.

Appendix A contains the more technical proofs of Lemma 1.1 and Theorem 2.2.

2. Gradient-like flows and the optimal control problem

Recall the function $\bar{V}$ defined in Hypothesis 1.1. Since $\nabla \bar{V}$ is Lipschitz, $\Delta \bar{V}$ is bounded and thus $C^2(\mathbb{R}^d)$ implies that for
\[
\mathcal{L}_{\bar{V}} f(x) := \frac{\varepsilon^{2\nu}}{2} \Delta f(x) + \langle m(x), \nabla f(x) \rangle \quad \forall x \in \mathbb{R}^d , \quad f \in C^2(\mathbb{R}^d) ,
\]
we have
\[
\mathcal{L}_{\bar{V}} \bar{V}(x) \leq \gamma_0 - \gamma |x| \quad \forall \varepsilon \in (0, 1) ,
\]
for some constant $\gamma_0 > 0$. This is the ‘stochastic Lyapunov condition’ that implies in particular that the process $X$ with $U \equiv 0$ has a unique stationary probability distribution $\eta_{\varepsilon}^0$, and
\[
\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T |X_t| \, dt \right] = \int_{\mathbb{R}^d} |x| \eta_0^\varepsilon(dx) \leq \frac{\gamma_0}{\gamma} \quad \forall \varepsilon \in (0, 1) . \tag{2.1}
\]
In view of Theorem 2.2 and Proposition 2.3 of [19], this follows from the ergodic theory of Markov processes. Since $\ell$ is Lipschitz, (2.1) implies that there exists a constant $C$ independent of $\varepsilon$ such that $\int |m| \eta_{\varepsilon}^0 \leq C < \infty$. Moreover, from [8] there exists a unique Lipschitz continuous function $Z \geq 0$, such that $\min_{\mathbb{R}^d} Z = 0$, $Z(x) \to \infty$ as $|x| \to \infty$ and
\[
Z(x) = \inf_{\phi : \phi(t) \to x_i , \ x_i \in S} \left[ \frac{1}{2} \int_0^\infty |\phi(s) + m(\phi(s))|^2 \, ds + Z(x_i) \right] , \quad \phi(0) = x ,
\]
and, if $g_0^\epsilon$ denotes the density of $\eta_0^\epsilon$, then $-\epsilon^2 \ln g_0^\epsilon(x) \to Z(x)$ uniformly on compact subsets of $\mathbb{R}^d$ as $\epsilon \to 0$. The function $Z$ is generally referred to as the quasi-potential.

2.1. Gradient-Like Morse–Smale dynamical systems. It is well known from the theory of dynamical systems that if the set of nonwandering points of a flow on a compact manifold consists of hyperbolic fixed points, then the associated vector field is generically gradient-like (see Definition 2.1 and Theorem 2.1 below). This is also the case under Hypothesis 1.1, since the ‘point at infinity’ is a source for the flow of $m$.

The theorem below is well known [20, 25]. What we have added in its statement is the assertion that the energy function can be chosen in a manner that its Laplacian at critical points of positive index is negative.

Recall that a function $f : \mathbb{R}^d \to \mathbb{R}$ is called inf-compact if the set $\{ x \in \mathbb{R}^d : f(x) \leq C \}$ is compact (or empty) for every $C \in \mathbb{R}$. We start with the following definition.

Definition 2.1. We say that $V \in C^\infty(\mathbb{R}^d)$ is an energy function if it is inf-compact, and has a finite set $S = \{ z_1, \ldots, z_n \}$ of critical points, which are all nondegenerate. A $C^\infty$ vector field $m$ on $\mathbb{R}^d$ is called gradient-like relative to an energy function $V$ provided that the set of nonwandering points of its flow is $S$, that every point in $S$ is a hyperbolic singular point of $m$, and

$$m(x) \cdot \nabla V(x) < 0 \quad \forall x \in \mathbb{R}^d \setminus S.$$ 

If $m$ satisfies these properties, we also say that $m$ is adapted to $V$.

For a hyperbolic singular point $\hat{x}$ of a vector field $m$, we let $W_s(\hat{x})$ and $W_u(\hat{x})$ denote the stable and unstable manifolds of the flow. Recall that the index of $\hat{x}$ is defined as the dimension of $W_u(\hat{x})$.

Theorem 2.1. Suppose that a $C^\infty$ vector field $m$ in $\mathbb{R}^d$ satisfies (H1)–(H2) and

(a) The set of nonwandering points of its flow is a finite set $S = \{ z_1, \ldots, z_n \}$ of hyperbolic singular points.

(b) If $y$ and $z$ are singular points of $m$, then $W_s(y)$ and $W_u(z)$ intersect transversally (if they intersect).

Then $m$ is adapted to an energy function $V$ which satisfies

(i) $V(z_i) \neq V(z_j)$ for $i \neq j$.

(ii) $\Delta V(z) < 0$, for all $z \in S \setminus S_s$ where $S_s$, as defined earlier, denotes the set of singular points of index 0, i.e., the stable equilibria of the flow of $m$.

(iii) For each $z \in S$ there exists some open neighborhood $N_z$ of $z$ and a constant $C_0 > 0$ such that

$$-C_0^{-1} |x - z|^2 \leq \langle m(x), \nabla V(x) \rangle \leq -C_0 |x - z|^2 \quad \forall x \in N_z.$$

(iv) There exists some open neighborhood $Q$ of $S$ and a constant $C_0' > 0$ such that

$$-\frac{1}{C_0'} \langle m(x), \nabla V(x) \rangle \leq |\nabla V(x)|^2 \leq -C_0' \langle m(x), \nabla V(x) \rangle \quad \forall x \in Q.$$
Proof. Let $\dot{x}$ be a critical point of $m$ of index $q > 0$. Translating the coordinates we may assume that $\dot{x} \equiv 0$. Since $m(0) = 0$, then $m(x)$ takes the form

$$m(x) = Mx + o(x)$$

locally around $x = 0$, where $M = Dm(0)$ is the Jacobian of $m$ at 0. By hypothesis $M$ has exactly $q$ ($d - q$) eigenvalues in the open right half (left half) complex space. Therefore since the corresponding eigenspaces are invariant under $M$, there exists a linear coordinate transformation $T$ such that, in the new coordinates $\tilde{x} = T(x)$, the linear map $x \mapsto Mx$ has the matrix representation $\tilde{M} = TMT^{-1}$ and $\tilde{M} = \text{diag}(\tilde{M}_1, -\tilde{M}_2)$, where $\tilde{M}_1$ and $\tilde{M}_2$ are square Hurwitz matrices of dimension $d - q$ and $q$ respectively. By the Lyapunov theorem there exist positive definite matrices $\tilde{Q}_i$, $i = 1, 2$, satisfying

$$\tilde{M}_1^T \tilde{Q}_1 + \tilde{Q}_1 \tilde{M}_1 = -I_{d-q},$$
$$\tilde{M}_2^T \tilde{Q}_2 + \tilde{Q}_2 \tilde{M}_2 = -I_q,$$

(2.2)

where $I_{d-q}$ and $I_q$ are the identity matrices of dimension $d - q$ and $q$, respectively. Let $\theta > 1$ be such that

$$\theta \trace(T^T \text{diag}(0, \tilde{Q}_2)T) > \trace(T^T \text{diag}(\tilde{Q}_1, 0)T),$$

(2.3)

and define $V$ in some neighborhood of 0 by

$$V(x) := a + x^T T^T \text{diag}(\tilde{Q}_1, -\theta \tilde{Q}_2)Tx,$$

(2.4)

where $a$ is a constant to be determined later. By (2.3) we obtain $\Delta V(0) < 0$, and thus (ii) holds.

We have

$$\langle m(x), \nabla V(x) \rangle = x^T [M^T T^T \text{diag}(\tilde{Q}_1, -\theta \tilde{Q}_2)T + T^T \text{diag}(\tilde{Q}_1, -\theta \tilde{Q}_2)TM] x + o(|x|^2).$$

Expanding we obtain

$$T^T \text{diag}(\tilde{Q}_1, -\theta \tilde{Q}_2)TM = T^T \text{diag}(\tilde{Q}_1, -\theta \tilde{Q}_2)TT^{-1}\tilde{M}T$$
$$= T^T \text{diag}(\tilde{Q}_1 \tilde{M}_1, \theta \tilde{Q}_2 \tilde{M}_2)T.$$

By (2.2) we have

$$\langle m(x), \nabla V(x) \rangle = -x^T T^T \text{diag}(I_{d-q}, \theta I_q)Tx + o(|x|^2).$$

Therefore, since $\theta > 1$, we have

$$-|Tx|^2 + o(|x|^2) \leq \langle m(x), \nabla V(x) \rangle \leq -\theta |Tx|^2 + o(|x|^2),$$

thus establishing (iii).

Property (iv) follows by (iii) and (2.4).

As shown in [25] one can select $n$ distinct real numbers $a_i$ and define $V$ on $S$ by setting $V(z_i) = a_i$ in a consistent manner: if $z_i$ and $z_j$ are the $\alpha$- and $\omega$-limit points of some trajectory then $a_i > a_j$. Thus $V$ can be defined in nonoverlapping neighborhoods of the singular points by (2.4) so as to satisfy (i). This function can then be extended to $\mathbb{R}^d$ by the construction in [25].

The energy function $V$ can be constructed in a manner so that it agrees, outside some ball, with the Lyapunov function $\bar{V}$ in Hypothesis [4.1]. This is stated in the following lemma.
Lemma 2.1. Under the assumptions of Theorem 2.1, the energy function $V$ can be selected so that $V = \bar{V}$ on the complement of some ball which contains $\mathcal{S}$.

Proof. Recall the definition of $K$ in Hypothesis 1.1. It is evident that we can find radii $0 < R_1 < R_2 < R_3 < R_4$ and balls centered at the origin such that $\mathcal{S} \subset B_{R_1}$, $K \subset B_{R_3}$, and the following are satisfied

$$c_1 := \sup_{B_{R_1}} V < c_2 := \inf_{B_{R_1} \setminus B_{R_2}} V,$$

$$c_3 := \sup_{B_{R_2} \setminus B_{R_3}} V < c_4 := \inf_{B_{R_4}} V,$$

$$\tilde{c}_2 := \sup_{B_{R_2}} \bar{V} < \tilde{c}_3 := \inf_{B_{R_3}} \bar{V}.$$

Let $\psi: \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function such that $\psi(t) = t$ for $t \leq c_1$, $\psi(t) = c_4$ for $t \geq c_4$, and whose derivative is strictly positive on the interval $[c_1, c_3]$. Similarly let $\bar{\psi}: \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function such that $\bar{\psi}(t) = 0$ for $t \leq \tilde{c}_2 - c_4$ and $\bar{\psi}(t) = t$ for $t \geq \tilde{c}_3 - c_4$. Let $G := \psi \circ V + \bar{\psi} \circ (\bar{V} - c_4)$. By construction $G$ agrees with $V$ on $B_{R_1}$ and with $\bar{V}$ on $B_{R_4}$. It can also be easily verified that $\sup_{B_{R_3} \setminus B_{R_1}} m \cdot \nabla G < 0$. Replacing $V$ with $G$ we obtain a new energy function $\bar{V}$ such that $\nabla \bar{V}$ is Lipschitz. \qed

2.2. Existence of an optimal control and the HJB equation. Recall that a stationary Markov control is of the form $U_t = v(X_t)$ where $v: \mathbb{R}^d \to \mathbb{R}^d$ is a measurable map. Theorem 2.2 below establishes the existence of a stationary control that is optimal. Before stating this result we mention the occupation measure based formulation that is quite related to the ergodic control problem considered here 7,18. Define, for $f \in \mathcal{C}^2(\mathbb{R}^d)$,

$$\mathcal{L}^\varepsilon[f](x, u) := \frac{\varepsilon^{2\nu}}{2} \Delta f(x) + \langle m(x) + \varepsilon u, \nabla f(x) \rangle.$$

Now consider the following minimization problem

$$\inf_{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \left( \ell(x) + \frac{1}{2} |u|^2 \right) \pi(dx, du),$$

subject to:

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{L}^\varepsilon[f](x, u) \pi(dx, du) = 0 \quad \forall f \in \mathcal{C}_c^2(\mathbb{R}^d).$$

(2.5)

It is not hard to show that the value of this minimization problem does not exceed the optimal value $\beta^\varepsilon$ of the ergodic cost (1.3) under the controlled dynamics (1.1). Using the lower semi-continuity of the cost it is also easy to show that there exists a minimizer $\pi^* \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ which attains the optimal value of (2.5). In 18 it is shown that there exists a càdlàg stationary process $(X^*, v^\varepsilon(X^*))$ with distribution $\pi^*$ that satisfies the martingale problem with respect to the generator $\mathcal{L}^\varepsilon$. But it is not obvious that one can construct a weak solution to (1.1) with control $v^\varepsilon$. We adopt an analytic approach to find an optimal stationary control and characterize the optimal value $\beta^\varepsilon$.

Theorem 2.2. The HJB equation for the ergodic control problem, given by

$$\frac{\varepsilon^{2\nu}}{2} \Delta V^\varepsilon + \min_{u \in \mathbb{R}^d} \left[ \langle m + \varepsilon u, \nabla V^\varepsilon \rangle + \ell + \frac{1}{2} |u|^2 \right] = \beta^\varepsilon,$$

(2.6)
has a solution \((V^\varepsilon, \beta^\varepsilon)\) in \(C^2(\mathbb{R}^d) \times \mathbb{R}\), where \(V^\varepsilon\) is unique in the class of functions in \(C^2(\mathbb{R}^d)\) satisfying \(\lim_{|x| \to \infty} V^\varepsilon(x) = \infty\), \(V^\varepsilon(0) = 0\), and \(\beta^\varepsilon\) is uniquely specified as \(\beta^\varepsilon = \beta^\varepsilon_s\). Moreover, \(U_t = v^\varepsilon_s(X_t), \ t \geq 0\), where \(v^\varepsilon_s = -\varepsilon \nabla V^\varepsilon\) is an optimal control.

**Proof.** The proof is contained in Appendix A. \(\square\)

**Remark 2.1.** In view of the smoothness assumptions on the coefficients and the cost function, standard elliptic regularity theory allows us to improve the regularity of \(V^\varepsilon\) to \(C^k(\mathbb{R}^d)\), for any \(k \geq 2\).

From (2.10) it follows that

\[
\frac{\varepsilon^{2\nu}}{2} \Delta V^\varepsilon + m \cdot \nabla V^\varepsilon - \frac{\varepsilon^2}{2} |\nabla V^\varepsilon|^2 + \ell = \beta^\varepsilon_s. \tag{2.7}
\]

3. The Support of the Limit of the Stationary Probability Distribution

Throughout the rest of the paper \(\eta^\varepsilon_s\) denotes the stationary probability distribution under the optimal stationary control \(v^\varepsilon_s\) in Theorem 2.2. We start the analysis with the following lemma which states that \(\eta^\varepsilon_s\) concentrates on \(S\) as \(\varepsilon \searrow 0\).

**Lemma 3.1.** The family \(\{\eta^\varepsilon_s, \varepsilon \in (0, 1)\}\) is tight, and any sub-sequential limit as \(\varepsilon \searrow 0\) has support on \(S\).

**Proof.** Since \(\ell\) is inf-compact and \(V^\varepsilon\) is bounded below, it follows by (2.6) that the stationary control \(v^\varepsilon_s\) defined in Theorem 2.2 is stable. Let \(\eta^\varepsilon_s\) be the invariant probability measure of the SDE

\[
dX_t = (m(X_t) + \varepsilon v^\varepsilon_s(X_t)) \, dt + \varepsilon \nu \, dW_t.
\]

Also by Theorem 2.2 we have

\[
\int_{\mathbb{R}^d} \left(\ell(x) + \frac{1}{2} |v^\varepsilon_s(x)|^2\right) \eta^\varepsilon_s(dx) = \beta^\varepsilon_s.
\]

Recall that \(\eta^\varepsilon_0\) is the invariant probability measure of (1.1) under the control \(U \equiv 0\). Define

\[
\beta^\varepsilon_0 := \int_{\mathbb{R}^d} \ell(x) \eta^\varepsilon_0(dx).
\]

By (2.1) we have

\[
\int_{\mathbb{R}^d} \ell(x) \, \eta^\varepsilon_0(dx) \leq \beta^\varepsilon_0 \leq \frac{\gamma}{\gamma} \quad \forall \varepsilon \in (0, 1). \tag{3.1}
\]

Since \(\ell\) is inf-compact, (3.1) implies that \(\{\eta^\varepsilon_s, \varepsilon \in (0, 1)\}\) is tight. Let \(x(t)\) be the solution of (1.2). Therefore, if \(C_m\) denotes a Lipschitz constant of \(m\) and \(X_0 = x(0)\), we obtain

\[
|X_t - x(t)| \leq C_m \int_0^t |X_s - x(s)| \, ds + \varepsilon \int_0^t |v^\varepsilon_s(X_s)| \, ds + \varepsilon \nu |W_t|. \tag{3.2}
\]

Hence applying Gronwall’s inequality we obtain from (3.2) that

\[
|X_s - x(s)| \leq e^{C_m t} \left(\varepsilon \int_0^t |v^\varepsilon_s(X_s)| \, ds + \varepsilon \nu \sup_{s \leq t} |W_s|\right), \quad s \leq t. \tag{3.3}
\]
In turn, for any $\delta > 0$, (3.3) implies that
\[
\mathbb{P}_{x(0)}\left( |X_t - x(t)| \geq \delta \right) \leq \mathbb{P}_{x(0)}\left( \int_0^t |v_s^\varepsilon(X_s)| \, ds \geq \frac{\delta e^{-C_m t}}{2\varepsilon} \right) + \mathbb{P}_{x(0)}\left( \sup_{s \leq t} |W_s| \geq \frac{\delta e^{-C_m t}}{2\varepsilon \nu} \right),
\]
for $t > 0$. By Jensen’s inequality we have
\[
\mathbb{P}_{x(0)}\left( \int_0^t |v_s^\varepsilon(X_s)| \, ds \geq \frac{\delta e^{-C_m t}}{2\varepsilon} \right) \leq \mathbb{P}_{x(0)}\left( \int_0^t |v_s^\varepsilon(X_s)|^2 \, ds \geq \frac{\delta^2 e^{-2C_m t}}{4t\varepsilon^2} \right) \leq \frac{4t\varepsilon^2}{\delta^2} e^{2C_m t} \mathbb{E}_{x(0)}\left[ \int_0^t |v_s^\varepsilon(X_s)|^2 \, ds \right].
\]
Therefore for any compact set $K \subset \mathbb{R}^d$ we have
\[
\int_K \mathbb{P}_{x}\left( |X_t - x(t)| \geq \delta \right) \eta_s^\varepsilon(dx) \leq \frac{4t^2\varepsilon^2}{\delta^2} e^{2C_m t} \int_{\mathbb{R}^d} |v_s^\varepsilon(x)|^2 \eta_s^\varepsilon(dx) + \sup_{x \in K} \mathbb{P}_x\left( \sup_{s \leq t} |W_s| \geq \frac{\delta}{2\varepsilon \nu} e^{-C_m t} \right). \tag{3.4}
\]
It is clear that the right hand side of (3.4) tends to 0 as $\varepsilon \to 0$. Suppose that $\eta_s^\varepsilon \to \bar{\eta}$ along some subsequence as $\varepsilon \to 0$. We claim that for any $f \in \mathcal{C}_b(\mathbb{R}^d)$,
\[
\int_{\mathbb{R}^d} f_t(x) \bar{\eta}(dx) = \int_{\mathbb{R}^d} f(x) \bar{\eta}(dx), \tag{3.5}
\]
where $f_t(x) := f(x(t))$, and $x(t)$ is the solution of (1.2) with initial condition $x(0) = x$. Since the $\omega$-limit set of the trajectories of (1.2) is supported on $\mathcal{S}$, (3.5) shows that $\bar{\eta}$ has support on $\mathcal{S}$. Next we prove (3.5). It is enough to prove the claim for a bounded Lipschitz function $f$. Since $\eta_s^\varepsilon$ is an invariant probability measure we have
\[
\int_{\mathbb{R}^d} \mathbb{E}_x[f(X_t)] \eta_s^\varepsilon(dx) = \int_{\mathbb{R}^d} f(x) \eta_s^\varepsilon(dx),
\]
where $X$ solves (1.1) with control $v_s^\varepsilon$. Hence to prove (3.5) it is enough to show that
\[
\int_{\mathbb{R}^d} \mathbb{E}_x[f(X_t)] \eta_s^\varepsilon(dx) \xrightarrow{\varepsilon \to 0} \int_{\mathbb{R}^d} f_t(x) \bar{\eta}(dx)
\]
for all bounded, Lipschitz functions $f$. Since $f_t: \mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function it suffices to show that, for any compact set $K$, we have
\[
\int_K |\mathbb{E}_x[f(X_t)] - f_t(x)| \eta_s^\varepsilon(dx) \xrightarrow{\varepsilon \to 0} 0. \tag{3.6}
\]
But (3.6) follows by the Lipschitz property of $f$ and (3.4).

We next consider the three separate cases, i.e., the supercritical, subcritical and critical regimes. We use the following running example:

**Example 3.1.** Let $m$ be a vector field in $\mathbb{R}$ of the form $m = -\nabla F$ for $F$ a ‘double well potential’ given as follows: $F(x) := \frac{x^4}{4} - \frac{x^3}{3} - x^2$ on $[-10,10]$, with $F$ suitably extended so that it is globally Lipschitz and does not have any critical points outside the interval $[-10,10]$. Then $\nabla F$ vanishes at exactly three points: $-1, 0, 2$. Of these, $0$ is a local maximum, hence an unstable equilibrium for the o.d.e. $\dot{x}(t) = m(x(t))$, and both $-1$ and
2 are local minima, hence stable equilibria thereof. Let $\ell(x) = c|x|^2$ on $[-10, 10]$ for a suitable $c > 0$, modified suitably outside $[-10, 10]$ to render it globally Lipschitz. Note that $F(0) = 0$, $F(-1) = -\frac{5}{12}$, $F(2) = -\frac{8}{3}$. Thus $x = 2$ is the unique global minimum of $F$.

3.1. **Supercritical regime** ($\nu > 1$). Let $z \in \mathcal{S}$ and consider the control

$$U_t^\varepsilon = v(X_t) := -\frac{1}{\varepsilon}(m(X_t) + (X_t - z)), \quad t \geq 0.$$  

Then $X$ is given by

$$dX_t = -(X_t - z)\, dt + \varepsilon^{\nu}dW_t, \quad t \geq 0. \quad (3.7)$$  

Since (3.7) has a unique strong solution and $U$ is adapted to the family of sub-$\sigma$-fields generated by $X$, the control $U$ satisfies the nonanticipativity condition and is therefore admissible. Moreover, a standard argument using the Lipschitz property of $m$ and Gronwall’s inequality shows that, for some $K > 0$, we have

$$E\left[\int_0^t |U_s^\varepsilon|^2 \, ds\right] \leq \frac{K}{\varepsilon}E\left[1 + \int_0^t |X_s|^2 \, ds\right] < \infty \quad \forall t \geq 0,$$

as desired.

The diffusion in (3.7) has a Gaussian stationary distribution $\mu^\varepsilon$ with mean $z$ and variance $O(\varepsilon^{2\nu})$. In particular,

$$\int_{\mathbb{R}^d} |v|^2 \, d\mu^\varepsilon \leq 2 \int_{\mathbb{R}^d} (|m(x)|^2 + |x - z|^2)\varepsilon^{-2} \mu^\varepsilon(dx)
\leq \frac{C}{\varepsilon^2} \int_{\mathbb{R}^d} |x - z|^2 \mu^\varepsilon(dx)
\leq O(\varepsilon^{2\nu-2}) \quad (3.8)$$

for some $C > 0$, where the second inequality follows from the Lipschitz property of $m$ combined with the fact that $m(z) = 0$. It follows by (3.8) that the corresponding ergodic cost is $\ell(z) + O(\varepsilon^{2\nu-2}) \approx \ell(z)$ for sufficiently small $\varepsilon$. In particular, this in conjunction with Lemma 3.1 leads to:

**Theorem 3.1.** For $\nu > 1$, it holds that

$$\lim_{\varepsilon \searrow 0} \beta^\varepsilon_* = \min \{\ell(z) : z \in \mathcal{S}\}.$$  

and

$$\beta^\varepsilon_* \leq \min \{\ell(z) : z \in \mathcal{S}\} + O(\varepsilon^{2\nu-2}) \quad \forall \varepsilon \in (0, 1).$$

3.1.1. **An asymptotically optimal control that uses the energy function.** It is worth mentioning here that if $z \in \mathcal{S}$, then an asymptotically optimal control can be synthesized from an energy function. Let $V$ be an energy function that attains a unique global minimum at $z$ and such that $|\nabla V|$ is Lipschitz and inf-compact. Such a function exists by Theorem 2.1 and Lemma 2.1. Consider the control

$$v^\varepsilon(x) := -\frac{1}{\varepsilon}(m(x) + \nabla V(x)), \quad t \geq 0.$$

Then $X$ is given by

$$dX_t = -\nabla V(X_t)\, dt + \varepsilon^{\nu}dW_t, \quad t \geq 0.$$  

Let $\mu^\varepsilon$ denote its unique stationary probability distribution, and $\mathcal{L}^\varepsilon : = \frac{\varepsilon^{2\nu}}{2} \Delta + \nabla V \cdot \nabla$ its controlled extended generator. Since

$$\mathcal{L}^\varepsilon \mathcal{V} \leq \frac{\varepsilon^{2\nu}}{2} \|\nabla \mathcal{V}\|_\infty - |\nabla \mathcal{V}|^2,$$

it follows that

$$2 \int_{\mathbb{R}^d} |\nabla \mathcal{V}|^2 d\mu^\varepsilon \leq \varepsilon^{2\nu} \|\Delta \mathcal{V}\|_\infty.$$

Note that $\mu^\varepsilon$ has density $\rho^\varepsilon(x) = C(\varepsilon) e^{-\frac{\varepsilon^2 \mathcal{V}(x)}{2\nu}}$, where $C(\varepsilon)$ is a normalizing constant. Therefore we have

$$\int_{\mathbb{R}^d} |v^\varepsilon(x)|^2 \mu^\varepsilon(dx) \leq 2 \int_{\mathbb{R}^d} (|m(x)|^2 + |\nabla \mathcal{V}(x)|^2) \varepsilon^{-2} \mu^\varepsilon(dx)$$

$$\leq 2 \|m(x)\|^2 \mu^\varepsilon(dx) + \varepsilon^{2\nu-2} \|\Delta \mathcal{V}\|_\infty$$

$$\leq O(\varepsilon^{2\nu-2}) + \varepsilon^{2\nu-2} \|\Delta \mathcal{V}\|_\infty.$$

For the last inequality we used the fact that $m$ is bounded, $m(z) = 0$, and that $\mathcal{V}$ is locally quadratic around $z$.

In Example 3.1 $\ell$ attains its minimum over $\{-1, 0, 2\}$ at 0. Thus in the supercritical regime we have $\beta^\varepsilon \approx \ell(0) = 0$.

### 3.2. Subcritical regime ($\nu < 1$).

Recall that $\mathcal{S}_s$ is the collection of stable equilibrium points. The following lemma holds for any $\nu > 0$. It shows that if $\mathcal{V} \in \mathcal{S}_s$ then there exists a Markov stationary control with asymptotic cost $O(\varepsilon^n)$ for any $n \in \mathbb{N}$, that renders $\{z\}$ stochastically stable.

**Lemma 3.2.** For any $\nu > 0$ and $\mathcal{V} \in \mathcal{S}_s$ there exists a Markov control $v^\varepsilon$, for $\varepsilon \in (0, 1)$, with the following properties:

(a) With $\mu^\varepsilon$ denoting the invariant probability measure of (1.1) under the control $v^\varepsilon$, it holds that

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^d} |v^\varepsilon(x)|^2 \mu^\varepsilon(dx) = 0 \quad \forall n \in \mathbb{N}.$$

(b) If $\Sigma$ is the symmetric positive definite solution of the matrix Lyapunov equation

$$(Dm(z)) \Sigma + \Sigma (Dm(z))^T = -I,$$

and $C_\ell$ a Lipschitz constant for $\ell$, then

$$\limsup_{\varepsilon \searrow 0} \frac{1}{\varepsilon^n} \int_{\mathbb{R}^d} (\ell(x) - \ell(z)) \mu^\varepsilon(dx) \leq C_\ell \sqrt{\text{trace}(\Sigma)}.$$

(c) If $\nu \in (2/3, 1)$, then $\beta^\varepsilon \leq 3 + O(\varepsilon^{4\nu-2})$.

**Proof.** We let $M : = Dm(z)$. In order to simplify the notation, we translate the origin so that $z \equiv 0$. Let $R$ be the symmetric positive definite solution to the Lyapunov equation

$$M^T R + RM = -3R^2.$$  

(3.9)
We define the control \( v^\varepsilon \) by

\[
v^\varepsilon(x) := \begin{cases} 
\varepsilon^{-1}(Mx - m(x)) & \text{if } |x| \geq \varepsilon^{\nu/2}, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( L^\varepsilon_{v^\varepsilon} := \frac{\varepsilon^{2\nu}}{2} \Delta + (m(x) + \varepsilon v^\varepsilon(x)) \cdot \nabla \) denote the corresponding controlled extended generator. We apply the function \( F(x) := \varepsilon^{2\nu} \exp \left( \frac{x^T R x}{2\varepsilon^{\nu}} \right) \) to \( L^\varepsilon_{v^\varepsilon} \).

For \( |x| \geq \varepsilon^{\nu/2} \), using (3.10), we obtain

\[
L^\varepsilon_{v^\varepsilon} F(x) = (\varepsilon^{2\nu} \text{trace}(R) + |Rx|^2 + x^T (M^T R + RM^T) x) e^{x^T R x / \varepsilon^{\nu}} \leq (\varepsilon^{2\nu} \text{trace}(R) - 2|Rx|^2) e^{x^T R x / \varepsilon^{\nu}} \leq -|Rx|^2 e^{x^T R x / \varepsilon^{\nu}} \quad \forall \varepsilon \leq \left( \text{trace}(R) \| R^{-1} \|^2 \right)^{-1/\nu}.
\]

(3.10)

For \( |x| \leq \varepsilon^{\nu/2} \), we have

\[
L^\varepsilon_{v^\varepsilon} F(x) = (\varepsilon^{2\nu} \text{trace}(R) + |Rx|^2 + 2m(x) \cdot Rx) e^{x^T R x / \varepsilon^{\nu}} \leq (\varepsilon^{2\nu} \text{trace}(R) - 2|Rx|^2 + 2|Mx - m(x)||Rx|) e^{x^T R x / \varepsilon^{\nu}}.
\]

(3.11)

However, for some constant \( \kappa_0 > 0 \), it holds that \( |Mx - m(x)| \leq \kappa_0 |x|^2 \) for all sufficiently small \( |x| \). Therefore, for some \( \varepsilon_0 > 0 \) we have

\[
-2|Rx|^2 + 2|Mx - m(x)||Rx| \leq -|Rx|^2 \quad \text{for } |x| \leq \varepsilon^{\nu/2}, \quad \forall \varepsilon \leq \varepsilon_0.
\]

(3.12)

However, since

\[
\varepsilon^{2\nu} \text{trace}(R) - |Rx|^2 \leq 0 \quad \text{for } |x| \geq \varepsilon^{\nu} \| R^{-1} \|^2 \sqrt{\text{trace}(R)},
\]

it follows by (3.11)–(3.12) that there exist a positive constant \( \kappa_1 \), not depending on \( \varepsilon \), such that

\[
\sup \left\{ L^\varepsilon_{v^\varepsilon} F(x) : |x| \leq \varepsilon^{\nu/2} \right\} \leq \kappa_1 \varepsilon^{2\nu} \quad \forall \varepsilon \leq \varepsilon_0.
\]

(3.13)

By (3.10) and (3.13), for some \( \varepsilon_0' > 0 \), we obtain

\[
L^\varepsilon_{v^\varepsilon} F(x) \leq \kappa_1 \varepsilon^{2\nu} - |Rx|^2 e^{x^T R x / \varepsilon^{\nu}} \quad \forall \varepsilon \leq \varepsilon_0'.
\]

(3.14)

By (3.14) we obtain

\[
\exp \left( -\nu \| R^{-1} \|^{-2} \right) \int_{|x| \geq \varepsilon^{\nu/2}} |x|^2 \mu^\varepsilon(dx) \leq \int_{|x| \geq \varepsilon^{\nu/2}} |x|^2 e^{x^T R x / \varepsilon^{\nu}} \mu^\varepsilon(dx) \leq \| R^{-1} \| \kappa_1 \varepsilon^{2\nu} \quad \forall \varepsilon \leq \varepsilon_0',
\]

(3.15)

which implies part (a).

Consider the ‘scaled’ diffusion

\[
d\hat{X}_t = \hat{b}^\varepsilon(\hat{X}_t) dt + dW_t, \quad t \geq 0,
\]

where
where
\[ \hat{b}^\varepsilon := \frac{m(\varepsilon' x) + \varepsilon \varphi(\varepsilon' x)}{\varepsilon'}, \]
and let \( \hat{\mu}^\varepsilon \) denote its invariant probability measure. It \( \varphi^\varepsilon \) and \( \hat{\varphi}^\varepsilon \) denote the densities of \( \mu^\varepsilon \) and \( \hat{\mu}^\varepsilon \) respectively, then \( \varepsilon \varphi^\varepsilon(\varepsilon' x) = \hat{\varphi}^\varepsilon(x) \) for all \( x \in \mathbb{R}^d \). The (discontinuous) drift \( \hat{b}^\varepsilon \) converges uniformly to \( Mx \) as \( \varepsilon \downarrow 0 \). Therefore \( \hat{\mu}^\varepsilon \) converges to the Gaussian density \( \rho_\varepsilon \) as \( \varepsilon \downarrow 0 \), uniformly on compact sets. Moreover, \( \sup_{\varepsilon \in (0, 1)} \int_{\mathbb{R}^d}|x|^4 \hat{\mu}^\varepsilon(dx) < \infty \) by (3.15). Therefore, by uniform integrability, recalling that \( z \equiv 0 \), and the Cauchy-Schwartz inequality we obtain we obtain
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon'} \int_{\mathbb{R}^d} (\ell(x) - \ell(0)) \mu^\varepsilon(dx) \leq C_\ell \lim_{\varepsilon \to 0} \left( \int_{\mathbb{R}^d} |x|^2 \mu^\varepsilon(dx) \right)^{1/2}
\leq C_\ell \sqrt{\text{trace}(\Sigma)}.
\]

For part (c) apply the control \( \varphi^\varepsilon(x) = \varepsilon^{-1}(Mx - m(x)) \) for \( x \in \mathbb{R}^d \). Using the bound \( |Mx - m(x)| \leq \kappa_1 |x|^2 \) for some constant \( \kappa_1 \), we obtain
\[
\int_{\mathbb{R}^d} |\varphi^\varepsilon|^2 \, d\mu^\varepsilon \leq \int_{\mathbb{R}^d} \kappa_1^2 \varepsilon^{-2} |x|^4 \mu^\varepsilon(dx) \in O(\varepsilon^{4\nu - 2}).
\]

Also, using the Taylor series expansion of \( \ell \), and the fact that \( \mu^\varepsilon \) is Gaussian with mean 0 and covariance matrix \( \propto \varepsilon^{2\nu}I \), we have
\[
\int_{\mathbb{R}^d} (\ell(x) - \ell(0)) \mu^\varepsilon(dx) \in O(\varepsilon^{2\nu}). \tag{3.16}
\]

This completes the proof.

From Lemma 3.2 we see that we can always find a stable admissible control such that the corresponding invariant probability measure concentrates on a stable equilibrium point. Now we proceed to show that for \( \nu < 1 \), \( \eta^\varepsilon \) does not concentrate on \( S \setminus S_s \).

**Lemma 3.3.** Suppose \( \nu < 1 \). Then \( \eta^\varepsilon(S \setminus S_s) \to 0 \).

**Proof.** We argue by contradiction. If the statement of the theorem is not true then there exists \( \hat{x} \in S \setminus S_s \) and a decreasing sequence \( \{\varepsilon_n\} \), with \( \varepsilon_n \downarrow 0 \), such that
\[
\liminf_{n \to \infty} \eta^\varepsilon_n(B_r(\hat{x})) > 0 \tag{3.17}
\]
for all balls \( B_r(\hat{x}) \) of radius \( r > 0 \) centered at \( \hat{x} \). In the sequel all limits as \( \varepsilon \downarrow 0 \) are meant to be along this sequence \( \{\varepsilon_n\} \). Without loss of generality we may assume that \( V \) satisfies (i)–(iv) in Theorem 2.1.

Let \( \delta > 0 \) be such that the interval \( (V(\hat{x}) - 3\delta, V(\hat{x}) + 3\delta) \) contains no other critical values of \( V \) other than \( V(\hat{x}) \) (such a \( \delta \) exists by (i) of Theorem 2.1). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a smooth function such that

1. \( \varphi(V(\hat{x}) + y) = y \) for \( y \in (V(\hat{x}) - \delta, V(\hat{x}) + \delta) \);
2. \( \varphi' \in [0, 1] \) on \( (V(\hat{x}) - 2\delta, V(\hat{x}) + 2\delta) \);
3. \( \varphi' \equiv 0 \) on \( (V(\hat{x}) - 2\delta, V(\hat{x}) + 2\delta) \).

...
By Theorem 2.1 (ii) there exists \( r > 0 \) such that \( \sup_{x \in B_r(\hat{x})} \Delta V(x) < 0 \). We may also choose this \( r \) small enough so that

\[
B_r(\hat{x}) \subset \text{support}(\varphi(V(\cdot))) \subset B_{C_0}^c(S \setminus \{\hat{x}\}).
\]

Thus

\[
\int_{\mathbb{R}^d} \varphi'(V) \Delta V \, d\eta^\varepsilon = \int_{B_r(\hat{x})} \varphi'(V) \Delta V \, d\eta^\varepsilon + \int_{B_{C_0}^c(S)} \varphi'(V) \Delta V \, d\eta^\varepsilon. \tag{3.18}
\]

Since \( V \) is inf-compact, it follows that \( \varphi \circ V \) is constant outside a compact set. Therefore the support of \( \varphi'(V(\cdot)) \) is compact, and as a result \( \Delta V \) is bounded on this set. By (3.18), since \( \eta^\varepsilon(B_{C_0}^c(S)) \searrow 0 \) as \( \varepsilon \searrow 0 \), there exists \( \varepsilon_0 > 0 \) such that

\[
\int_{\mathbb{R}^d} \varphi'(V) \Delta V \, d\eta^\varepsilon < 0 \quad \forall \varepsilon < \varepsilon_0. \tag{3.19}
\]

By the infinitesimal characterization of an invariant probability measure we have

\[
\int_{\mathbb{R}^d} \mathcal{L}^\varepsilon[\varphi \circ V](x, v^\varepsilon(x)) \, \eta^\varepsilon \, (dx) = 0,
\]

which we write as

\[
\frac{\varepsilon^{2\nu}}{2} \int_{\mathbb{R}^d} \varphi'(V) \Delta V \, d\eta^\varepsilon + \varepsilon \int_{\mathbb{R}^d} \varphi'(v^\varepsilon \cdot \nabla V) \, d\eta^\varepsilon + \frac{\varepsilon^{2\nu}}{2} \int_{\mathbb{R}^d} \varphi''(V) |\nabla V|^2 \, d\eta^\varepsilon = - \int_{\mathbb{R}^d} \varphi'(V) \, (m, \nabla V) \, d\eta^\varepsilon. \tag{3.20}
\]

By the Cauchy–Schwarz inequality we have

\[
\left| \int_{\mathbb{R}^d} \varphi'(V)(v^\varepsilon \cdot \nabla V) \, d\eta^\varepsilon \right| \leq \left( \int_{\mathbb{R}^d} \varphi'(V) |\nabla V|^2 \, d\eta^\varepsilon \right)^{1/2} \left( \int_{\mathbb{R}^d} \varphi'(V) |v^\varepsilon|^2 \, d\eta^\varepsilon \right)^{1/2} \leq \sqrt{2/\beta^c} \sup_{\mathbb{R}^d} \varphi'(V) \left( \int_{\mathbb{R}^d} \varphi'(V) |\nabla V|^2 \, d\eta^\varepsilon \right)^{1/2}. \tag{3.21}
\]

Let

\[
\zeta^\varepsilon := \left( \int_{\mathbb{R}^d} \varphi'(V) |\nabla V|^2 \, d\eta^\varepsilon \right)^{1/2}. \tag{3.22}
\]

By (3.19)–(3.21) and Theorem 2.1 (iv) we obtain

\[
\frac{1}{C_0}(\zeta^\varepsilon)^2 - \varepsilon \frac{\sqrt{2/\beta^c}}{2} \sup_{\mathbb{R}^d} \varphi'(V) \zeta^\varepsilon - \frac{\varepsilon^{2\nu}}{2} \int_{\mathbb{R}^d} \varphi''(V) |\nabla V|^2 \, d\eta^\varepsilon \leq 0 \quad \forall \varepsilon < \varepsilon_0. \tag{3.23}
\]

Using the quadratic formula on (3.23) we obtain \( \zeta^\varepsilon \in \Theta(\varepsilon^{\nu}) \). On the other hand, since \( \varphi''(V) = 0 \) on some open neighborhood of \( S \), it follows that

\[
\int_{\mathbb{R}^d} \varphi''(V) |\nabla V|^2 \, d\eta^\varepsilon \xrightarrow{\varepsilon \searrow 0} 0.
\]

Therefore, since by (3.20)–(3.21) we have

\[
\frac{\varepsilon^{2\nu}}{2} \int_{\mathbb{R}^d} \varphi'(V) \Delta V \, d\eta^\varepsilon + \varepsilon \frac{\sqrt{2/\beta^c}}{2} \sup_{\mathbb{R}^d} \varphi'(V) \zeta^\varepsilon + \frac{\varepsilon^{2\nu}}{2} \int_{\mathbb{R}^d} \varphi''(V) |\nabla V|^2 \, d\eta^\varepsilon \geq 0,
\]
it follows that
\[
\liminf_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \varphi'(V) \Delta V \, d\eta_{V,\varepsilon} \geq 0.
\]
However, this contradicts (3.17), since \(\Delta V(\hat{x}) < 0\).

Lemmas 3.2 and 3.3 imply the following:

**Theorem 3.2.** If \(\nu < 1\), then
\[
\beta_{V,\varepsilon}^* \leq \left\{ \begin{array}{ll}
\min \{ \ell(z) : z \in S \} + O(\varepsilon^\nu) & \text{if } \nu \in (0, \frac{2}{3}) , \\
\min \{ \ell(z) : z \in S \} + O(\varepsilon^{4\nu-2}) & \text{if } \nu \in (\frac{2}{3}, 1). \end{array} \right.
\]

In Example 3.1, this leads to \(\beta_{V,\varepsilon}^* \approx \ell(-1) = c\).

### 3.3. Critical regime \((\nu = 1)\)

Recall that a square matrix is called *Hurwitz* if its eigenvalues lie in the open left half complex plane. We need the following definition.

**Definition 3.1.** Let \(A\) denote the collection of all \(d \times d\) Hurwitz matrices. For \(M \in \mathbb{R}^{d \times d}\) define
\[
\Lambda(M) := \min_{A \in A} \frac{1}{2} \text{trace} \left( (A - M) \Sigma_A (A - M)^T \right),
\]
where \(\Sigma_A\) is the (unique) symmetric solution of the Lyapunov equation
\[
A \Sigma_A + \Sigma_A A^T = -I.
\]

Suppose \(z \in S\), and without loss of generality assume that \(z = 0\). Consider a family of controls \(v_A^\varepsilon\) of the form
\[
v_A^\varepsilon(x) := \frac{Ax - m(x)}{\varepsilon},
\]
where \(A \in \mathbb{R}^{d \times d}\) is Hurwitz. The stationary probability distribution \(\eta_{A,\varepsilon}\) of the diffusion
\[
dX_t = AX_t \, dt + \varepsilon \, dW_t
\]
is Gaussian, with zero mean, and covariance matrix \(\varepsilon^{-2} \Sigma_A\), with \(\Sigma_A\) the solution of (3.25). Let \(M = Dm(0)\). Then for some constants \(\kappa_1 = \kappa_1(A)\) and \(\kappa_2\) we have
\[
|Ax - m(x)|^2 = |(A - M)x|^2 + \langle ((A - M)x, Mx - m(x)) + |Mx - m(x)|^2.
\]
Using the Taylor series for \(Mx - m(x)\) we write \(Mx - m(x) = f(x) + g(x)\), where \(f\) is an even quadratic function, and \(|g(x)| \leq \kappa|x|^3\). Using also the Taylor series for \(\ell\) as in (3.16), it follows that there exists a constant \(\kappa_3\) such that
\[
\beta_{V,\varepsilon}^* \leq \int_{\mathbb{R}^d} \left( \ell(x) + \frac{1}{2} |v_A^\varepsilon(x)|^2 \right) \eta_{A,\varepsilon}(dx) \leq \ell(0) + \Lambda(M) + \varepsilon^2 \kappa_3.
\]
Therefore, by (3.26) we obtain
\[
\limsup_{\varepsilon \searrow 0} \beta_{V,\varepsilon}^* \leq \min_{y \in S} \left[ \ell(y) + \Lambda(Dm(y)) \right] + O(\varepsilon^2).
\]
This establishes the second part of Theorem 1.1 (iii).

**Remark 3.1.** If \(y \in S\) then \(Dm(y)\) is Hurwitz, and by choosing \(A = Dm(y)\) in (3.24), it follows that \(\Lambda(Dm(y)) = 0\).
It is worthwhile at this point to present the following one-dimensional example, which shows how the limiting value of $\beta^\varepsilon_*$ bifurcates as we cross the critical regime.

**Example 3.2.** Let $d = 1$, $m(x) = Mx$, and $\ell(x) = \frac{1}{2}Lx^2$, with $M > 0$ and $L > 0$. Then the solution to (2.7) is:

$$V^\varepsilon = \frac{M + \sqrt{M^2 + L\varepsilon^2}}{2\varepsilon^2} x^2,$$
$$\beta^\varepsilon_* = \frac{\varepsilon^{2\nu-2}}{2} \left(M + \sqrt{M^2 + L\varepsilon^2}\right).$$

Note that $\beta^\varepsilon_* \to \ell(0) = 0$, $\beta^\varepsilon_* \to M$, and $\beta^\varepsilon_* \to \infty$, as $\varepsilon \searrow 0$, when $\nu > 1$, $\nu = 1$, and $\nu < 1$, respectively.

Recall from Theorem 1.1 that $E_+^1(M)$ denotes the sum of eigenvalues of a matrix $M$ that lie in the open right half complex plane. The following result is certainly not new, but since we could not locate it in this form in the literature, a proof is included.

**Lemma 3.4.** Suppose that $M \in \mathbb{R}^{d \times d}$ has no eigenvalues on the imaginary axis, and let $Q$ be the (unique) positive semidefinite symmetric solution of the matrix Riccati equation

$$M^TQ + QM = Q^2,$$  
(3.28)

satisfying

$$(M - Q)\Sigma + \Sigma(M - Q)^T = -I,$$  
(3.29)

for some symmetric positive definite matrix $\Sigma$. Let $\mathcal{U}^*_SM$ denote the class of stationary Markov controls $v$ with the following properties:

(i) $v: \mathbb{R}^d \to \mathbb{R}^d$ is continuous and has at most linear growth.

(ii) The diffusion $dX_t = (MX_t - v(X_t)) dt + dW_t$ is positive recurrent and its invariant probability distribution $\eta_v$ has finite second moments.

Let $\Lambda(M)$ be as defined in (3.24). Then we have the following properties:

(a) It holds that

$$\inf_{v \in \mathcal{U}^*_SM} \int_{\mathbb{R}^d} \frac{1}{2} |v(x)|^2 \eta_v(dx) = E_+(M) = \Lambda(M).$$  
(3.30)

(b) The Markov control $v(x) = -Qx$ is the unique control in $\mathcal{U}^*_SM$ which attains this infimum and it holds that $\Lambda(M) = \frac{1}{2} \text{trace}(Q)$.

**Proof.** It is well known that there exists at most one symmetric matrix $Q$ satisfying (3.28)-(3.29) [12, Theorem 3, p. 150]. For $\kappa > 0$, consider the ergodic control problem of minimizing

$$J_M(U) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \left( \kappa |X_s|^2 + \frac{1}{2} |U_s|^2 \right) ds \right]$$  
(3.31)

over $\mathcal{U}^*_SM$, subject to the linear controlled diffusion

$$X_t = X_0 + \int_0^t (MX_s + U_s) ds + W_t, \quad t \geq 0.$$  
(3.32)
As is well known, an optimal stationary Markov control for this problem takes the form

\[ U_t = -Q_\kappa X_t, \]

where \( Q_\kappa \) is the unique positive definite symmetric solution to the matrix Riccati equation (see \[12\] Theorem 1, p. 147)

\[
Q_\kappa^2 - M^T Q_\kappa - Q_\kappa M = 2 \kappa I.
\]  

(3.33)

Moreover, \( \Psi_\kappa(x) = \frac{1}{2} (x^T Q_\kappa x) \) is a solution of the associated HJB equation

\[
\frac{1}{2} \Delta \Psi_\kappa(x) + \min_{u \in \mathbb{R}^d} \left[ (Mx + u) \cdot \nabla \Psi_\kappa(x) + \frac{1}{2} |u|^2 \right] + \kappa |x|^2 = \frac{1}{2} \text{trace}(Q_\kappa).
\]  

(3.34)

The HJB equation (3.34) characterizes the optimal cost, i.e.,

\[
\inf_{U \in \mathcal{U}_M} J_M(U) = \frac{1}{2} \text{trace}(Q_\kappa).
\]

Since the stationary probability distribution of (3.32) under the control \( U_t = -Q_\kappa X_t \) is Gaussian, it follows by (3.31) that the matrix \( A_\kappa := M - Q_\kappa \) minimizes

\[
J_\kappa(A) := \kappa \text{trace}(\Sigma_A) + \frac{1}{2} \text{trace}\left( (A - M) \Sigma_A (A - M)^T \right)
\]

over all Hurwitz matrices \( A \), where \( \Sigma_A \) is as in (3.25) (compare with (3.24)). Combining this with (3.34) we have

\[
\inf_{A \in \mathcal{A}} J_\kappa(A) = \frac{1}{2} \text{trace}(Q_\kappa).
\]

(3.35)

It follows that \( \Lambda(M) \leq \frac{1}{2} \text{trace}(Q_\kappa) \) for all \( \kappa > 0 \). Note also that \( \kappa \mapsto \frac{1}{2} \text{trace}(Q_\kappa) \) is decreasing. It is straightforward to show that

\[
\Lambda(M) = \lim_{\kappa \searrow 0} \frac{1}{2} \text{trace}(Q_\kappa).
\]

(3.35)

It is also well known that \( Q_{\kappa'} - Q_\kappa \) is nonnegative definite if \( \kappa' \geq \kappa \). Therefore \( Q_\kappa \) has a unique limit \( Q \) as \( \kappa \searrow 0 \) which is nonnegative definite and satisfies (3.28).

Next we prove that \( M - Q \) is Hurwitz. Let \( T \) be a coordinate transformation such that \( \tilde{M} := TMT^{-1} = \text{diag}(\tilde{M}_1, -\tilde{M}_2) \), where \( \tilde{M}_1 \) and \( \tilde{M}_2 \) are square Hurwitz matrices of dimension \( d - q \) and \( q \) respectively. We may select \( T = [T_1, T_2]^T \) with \( T_1 \in \mathbb{R}^{(d-q) \times d} \) and \( T_2 \in \mathbb{R}^{q \times d} \) so that the columns of each submatrix \( T \) are orthonormal. By the orthonormality of the columns of \( T_1 \) and \( T_2 \) we have

\[
TT^T = \begin{pmatrix} I_{d-q} & T_1 T_2^T \\ T_2 T_1^T & I_q \end{pmatrix},
\]

(3.36)

where \( I_{d-q} \) and \( I_q \) are the identity matrices of dimension \( d - q \) and \( q \), respectively. Let \( \tilde{Q} := (T^T)^{-1} Q T^{-1} \). Then (3.28) takes the form

\[
\tilde{M}^T \tilde{Q} + \tilde{Q} \tilde{M} = \tilde{Q} TT^T \tilde{Q}.
\]

(3.37)

We write \( \tilde{Q} \) in the block form

\[
\tilde{Q} = \begin{pmatrix} \tilde{Q}_1 & R \\ R^T & \tilde{Q}_2 \end{pmatrix}
\]
with $\tilde{Q}_1 \in \mathbb{R}^{(d-q) \times (d-q)}$, $R \in \mathbb{R}^{(d-q) \times d}$ and $\tilde{Q}_2 \in \mathbb{R}^{q \times q}$. Block multiplication in (3.37) results in

$$\tilde{M}_1^T \tilde{Q}_1 + \tilde{Q}_1 \tilde{M}_1 = (\tilde{Q}_1 \ R) \ TT \begin{pmatrix} \tilde{Q}_1 \\ R^T \end{pmatrix},$$

and since $\tilde{M}_1$ is Hurwitz, $\tilde{Q}_1$ is positive semidefinite, and also the right-hand side is positive semidefinite, we must have $\tilde{Q}_1 = 0$. However this means that the right hand side of (3.38) equals $RR^T$, which implies that $R = 0$. Hence by (3.37) we obtain

$$\tilde{M}_2^T \tilde{Q}_2 + \tilde{Q}_2 \tilde{M}_2 = -\tilde{Q}_2^2.$$  

(3.39)

We claim that $\tilde{Q}_2$ is positive definite. Indeed, since by (3.39) the nullspace of $\tilde{Q}_2$ is invariant under $\tilde{M}_2$, if $\tilde{Q}_2$ is not invertible, then this implies that for some $y \in \mathbb{C}^q$, with $y \neq 0$, we have $\tilde{Q}_2 y = 0$, and $\tilde{M}_2 y = \lambda y$ for some $\lambda \in \mathbb{C}$. Since $\tilde{M}_2$ is Hurwitz, $\lambda$ must have negative real part. If we define $\tilde{y} := T^{-1}(0 \ y)$, with $(0 \ y) \in \mathbb{C}^d$, then we have

$$(M - Q)\tilde{y} = (T^{-1} \tilde{M}_T - T^T \tilde{Q} T) T^{-1} \begin{pmatrix} 0 \\ y \end{pmatrix}$$

$$= -\lambda \tilde{y}.$$  

(3.40)

However, $M - Q_\kappa$ is Hurwitz for all $\kappa > 0$ by (3.33), and thus, by continuity, $M - Q$ cannot have an eigenvalue with positive real part. This contradicts (3.40) and proves the claim.

If $M - Q$ is not Hurwitz, then for some nonzero vector $x \in \mathbb{C}^d$, and $\rho \in \mathbb{C}$ with nonnegative real part, we must have $(M - Q)x = \rho x$. The identity $(M - Q)_T Q + Q(M - Q) = -Q^2$ and the fact that $Q$ is nonnegative definite imply that $Qx = 0$, and therefore $Mx = \rho x$. Moreover, $Qx = 0$ implies that $\tilde{Q} x = 0$, and since $\tilde{Q}_2$ is invertible we must have $T_2^T x = 0$. Hence $T_2^T M x = 0$, and we obtain $\tilde{M}_1 T_1^T x = T_1^T M x = \rho T_1^T x$, which contradicts the fact that $\tilde{M}_1$ is Hurwitz. Therefore $M - Q$ must be Hurwitz. It follows that $Q$ satisfies (3.29) for some symmetric positive definite matrix $\Sigma$. Parenthetically, we mention that

$$\tilde{Q}_2 = \left( \int_0^\infty e^{\tilde{M}_2 t} e^{\tilde{M}_2^T t} dt \right)^{-1}.$$

It remains to calculate the trace of $Q$. By (3.36) we obtain

$$\text{trace}(Q) = \text{trace}(TT \diag(0, \tilde{Q}_2)T)$$

$$= \text{trace}(\diag(0, \tilde{Q}_2)TT^T)$$

$$= \text{trace}(\tilde{Q}_2).$$  

(3.41)

On the other hand, by (3.39) we have

$$\text{trace}(\tilde{Q}_2) = -\text{trace}(\tilde{M}_2^T + \tilde{M}_2)$$

$$= -2 \text{trace}(\tilde{M}_2).$$  

(3.42)

Thus by (3.35) and (3.41)–(3.42) we have shown that $E_+(M) = \Lambda(M) = \frac{1}{2} \text{trace}(Q)$ and that $v(x) = -Q x$ attains the infimum over all linear controls.
Now let $\hat{v} \in U^\ast_{SM}$ be any control. Since $\hat{v}$ is suboptimal for the problem associated with (3.34), then from (3.34) we obtain

$$\frac{1}{2} \Delta \Psi^\kappa(x) + (Mx + \hat{v}(x)) \cdot \nabla \Psi^\kappa(x) + \frac{1}{2} \hat{v}(x)^2 + \kappa |x|^2 = \frac{1}{2} \text{trace}(Q_\kappa) + f_\kappa(x),$$

where $f_\kappa(x) = \frac{1}{2} |Q_\kappa x - \hat{v}(x)|^2$. Applying Itô’s formula to (3.43), and using the fact that $\eta_0$ has finite second moments and $\Psi^\kappa$ is quadratic, a standard argument gives

$$\int_{\mathbb{R}^d} \left( \kappa |x|^2 - f_\kappa(x) + \frac{1}{2} \hat{v}(x)^2 \right) \eta_0(dx) = \frac{1}{2} \text{trace}(Q_\kappa).$$

Since $\int_{\mathbb{R}^d} |x|^2 \eta_0(dx) < \infty$, and $\lim_{\kappa \downarrow 0} \frac{1}{2} \text{trace}(Q_\kappa) = \Lambda(M)$, it follows by (3.44) that $\int_{\mathbb{R}^d} \frac{1}{2} \hat{v}(x)^2 \eta_0(dx) \geq \Lambda(M)$. Hence (3.30) holds. Suppose $\hat{v}$ is optimal. Then taking limits in (3.44) we must have

$$\lim_{\kappa \downarrow 0} \int_{\mathbb{R}^d} f_\kappa(x) \eta_0(dx) = 0.$$

Since $f_\kappa$ is nonnegative and locally equi-continuous, and $\eta_0$ has density, by Fatou’s lemma we deduce that $f_\kappa \to 0$ as $\kappa \downarrow 0$, uniformly on compacta. Therefore, $\hat{v}(x) = -Qx$ for all $x \in \mathbb{R}^d$.

We need the following lemma, which is valid for all $\nu$.

**Lemma 3.5.** For any bounded domain $G$, such that $S \subset G$, there exists a constant $\hat{\kappa}_0 = \hat{\kappa}_0(G, \nu)$ such that

$$\sup_{\varepsilon \in (0,1)} \int_G \left( \frac{\text{dist}(x,S)}{\varepsilon^{2(\nu+2)}} \right)^2 \eta_\nu(dx) \leq \hat{\kappa}_0 \quad \forall \nu > 0,$$

where $\text{dist}(x,S)$ denotes the Euclidean distance of $x$ from the set $S$.

**Proof.** Let $V$ be an energy function as in Theorem 2.1 which agrees with the Lyapunov function $\bar{V}$ in Hypothesis 1.1 on the complement of some ball containing $S$ (see Lemma 2.1). We fix some bounded domain $G$ which contains $S$, and choose some number $\delta$ such that $\delta \geq \sup_{x \in G} V(x)$. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a smooth function such that

(a) $\varphi(y) = y$ for $y \in (-\infty, \delta)$;
(b) $\varphi' \in (0,1)$ on $(\delta, 2\delta)$;
(c) $\varphi' \equiv 0$ on $[2\delta, \infty)$.

If $\nu \leq 1$, then the steps in the proof of Lemma 3.3 show that all the terms on the left hand side of (3.20) are $O(\varepsilon^{2\nu})$. Therefore, there exists a constant $C_0 > 0$ such that

$$\sup_{\varepsilon \in (0,1)} \int_{\{x : V(x) \leq \delta\}} \frac{\langle m, \nabla V \rangle}{\varepsilon^{2\nu}} \eta_\nu(dx) \geq -C_0.$$

Thus, by Theorem 2.1 (iii) we can find a constant $\hat{\kappa}_0$ such that

$$\sup_{\varepsilon \in (0,1)} \int_{\{x : V(x) \leq \delta\}} \left( \frac{\text{dist}(x,S)}{\varepsilon^{2\nu}} \right)^2 \eta_\nu(dx) \leq \hat{\kappa}_0.$$
Next we turn to the case $\nu > 1$. Define
\[ G_\epsilon := \int_{\mathbb{R}^d} \varphi'(V) |v_\epsilon^x|^2 \, d\eta_\epsilon^x. \]

By (3.21)–(3.22) we obtain
\[ \left| \int_{\mathbb{R}^d} \varphi'(V)(v_\epsilon^x \cdot \nabla V) \, d\eta_\epsilon^x \right| \leq \zeta_\epsilon \sqrt{G_\epsilon}. \]  
(3.45)

By (3.45) and Theorem 2.1 (iv), we have (compare with (3.23))
\[ \frac{1}{C_0} (\zeta_\epsilon)^2 - \epsilon \sqrt{G_\epsilon} \zeta_\epsilon + O(\epsilon^{2\nu}) \leq 0. \]  
(3.46)

Moreover, by Theorem 2.1 (iii) and Hypothesis 1.1 for some constant $\kappa_1 > 0$, it holds that
\[ \int_{\{x : V(x) \leq \delta\}} (\text{dist}(x, S))^2 \eta_\epsilon^x (dx) \leq \kappa_1 (\zeta_\epsilon)^2. \]  
(3.47)

Let $G$ be a bounded open set containing $S$ such that $\ell(x) > \min \{\ell(z) : z \in S\}$ for all $x \in G^c$. Since $\ell$ is Lipschitz, we have $\ell(x) - \min \{\ell(z) : z \in S\} \geq C_\ell \text{dist}(x, S)$ in $G$. Using the Cauchy–Schwarz inequality, we deduce from (3.47) that
\[ \int_{\mathbb{R}^d} \ell \, d\eta_\epsilon^x - \min \{\ell(z) : z \in S\} \geq -\kappa_2 \zeta_\epsilon \]  
(3.48)

for some constant $\kappa_2 > 0$. Thus by (3.48) and Theorem 3.1 we have
\[ -\kappa_2 \zeta_\epsilon \leq \beta_\epsilon^x - \min \{\ell(z) : z \in S\} \leq O(\epsilon^{2\nu - 2}) \]
which implies that
\[ G_\epsilon \leq \kappa_3 (\zeta_\epsilon \vee \epsilon^{2\nu - 2}) \]  
(3.49)

for some constant $\kappa_3 > 0$. Applying the quadratic formula to (3.46) and using (3.49) we obtain $G_\epsilon \in O(\epsilon^{\sqrt{G_\epsilon} \vee \epsilon^{\frac{1}{2}\nu} \vee \epsilon^\nu})$, which implies that
\[ \epsilon \sqrt{G_\epsilon} \in O(\epsilon^2 \vee \epsilon^{1+\nu/2} \vee \epsilon^\nu). \]
Therefore $\epsilon \sqrt{G_\epsilon} \in O(\epsilon^\nu)$ when $\nu \in (1, 2)$, and $\epsilon \sqrt{G_\epsilon} \in O(\epsilon^2)$ when $\nu \geq 2$. The result then follows by applying the quadratic formula to (3.46) and using (3.47). \[ \square \]

Note that above proof gives us $\zeta_\epsilon \in O(\epsilon^{2\nu - 2})$ for $\nu \in [1, 2]$. In the proof of Lemma 3.5 we have established the following useful fact.

Corollary 3.1. For $\nu \in [1, 2)$ we have $\int_{\mathbb{R}^d} |v_\epsilon^x|^2 \, d\eta_\epsilon^x \in O(\epsilon^{2\nu - 2})$, whereas if $\nu \geq 2$ then $\int_{\mathbb{R}^d} |v_\epsilon^x|^2 \, d\eta_\epsilon^x \in O(\epsilon^2)$. Moreover, $\left\{ \int_{\mathbb{R}^d} \ell \, d\eta_\epsilon^x - \min \{\ell(z) : z \in S\} \right\} \in O(\epsilon^{2\nu - 2})$. In particular, using Theorem 3.1 we obtain that
\[ \beta_\epsilon^x = \begin{cases} \min \{\ell(z) : z \in S\} + O(\epsilon^{2\nu - 2}) & \text{for } \nu \in (1, 2), \\ \min \{\ell(z) : z \in S\} + O(\epsilon^2) & \text{for } \nu \geq 2. \end{cases} \]

We need an estimate on the growth of $\nabla V_\epsilon$. First a definition.
Definition 3.2. For the rest of the paper \( \{ \mathcal{B}_z : z \in \mathcal{S} \} \) is some collection of nonempty, disjoint balls, with each \( \mathcal{B}_z \) centered around \( z \), and we define \( \mathcal{B}_S := \bigcup_{z \in \mathcal{S}} \mathcal{B}_z \). We let \( r : [0, 1] \to [0, 1] \) be a continuous increasing function with \( r(0) = 0 \) and such that

\[
\frac{r(\varepsilon)}{\varepsilon^{\nu}} \to \infty \quad \text{as} \quad \varepsilon \searrow 0.
\]

For \( z \in \mathcal{S} \), we define

\[
\hat{V}^z(x) := V^z(\varepsilon x + z),
\]

and analogously, the ‘scaled’ vector field and penalty by

\[
\hat{m}^z(x) := \frac{m(\varepsilon x + z)}{\varepsilon^{\nu}}, \quad \hat{\ell}^z(x) := \ell(\varepsilon x + z).
\]

We define the ‘scaled’ density \( \hat{\rho}^z(x) := \varepsilon^{\nu d} \rho^z(\varepsilon x + z) \), and denote by \( \hat{\eta}^z \) the corresponding probability measure in \( \mathbb{R}^d \). We also define

\[
\hat{\varphi}^z(x) := \begin{cases} \frac{\hat{\rho}^z(x)}{\hat{\eta}^z(B_{r(\varepsilon)}(z))} & \text{if} \; \varepsilon x + z \in B_{r(\varepsilon)}(z), \\ 0 & \text{otherwise}, \end{cases}
\]

\[
\hat{\varphi}^z(x) := \begin{cases} \frac{\hat{\rho}^z(x)}{\hat{\eta}^z(\mathcal{B}_z)} & \text{if} \; \varepsilon x + z \in \mathcal{B}_z, \\ 0 & \text{otherwise}. \end{cases}
\]

and \( \hat{\eta}^z(dx) = \hat{\varphi}^z(x) dx, \hat{\eta}^z(dx) = \hat{\varphi}^z(x) dx \).

Lemma 3.6. Assume \( \nu \in (0, 2] \), and let \( \hat{\varphi}^z := \varepsilon^{2} V^z \). There exists a constant \( c_0 \) such that

\[
|\nabla \hat{V}^z(x)| \leq c_0 (1 + |x|) \quad \forall \varepsilon \in (0, 1), \quad \forall x \in \mathbb{R}^d. \tag{3.50}
\]

Also, the same applies to \( \hat{V}^z := \varepsilon^{2(1-\nu)} \hat{V}^z \), with \( \hat{V}^z \) as in Definition 3.2.

Proof. The function \( f^\varepsilon := \varepsilon^{2-2\nu} V^\varepsilon \) satisfies

\[
\frac{\varepsilon^{2\nu}}{2} \Delta f^\varepsilon + m \cdot \nabla f^\varepsilon - \frac{\varepsilon^{2\nu}}{2} |\nabla f^\varepsilon|^2 = \varepsilon^{2-2\nu} \left( \beta^\varepsilon - \ell \right). \tag{3.51}
\]

Thus for \( \nu \in (0, 1] \), the result follows by applying \cite{[19]} Lemma 5.1 to (3.51) and using the assumptions on the growth of \( m \) and \( \ell \).

We next turn to the case \( \nu \in (1, 2] \). Let \( z \in \mathcal{Z}_1 \) (see Theorem 1.1). The function \( \hat{V}^z \) satisfies

\[
\frac{1}{2} \Delta \hat{V}^z(x) + \hat{m}^z(x) \cdot \nabla \hat{V}^z(x) - \frac{1}{2} |\nabla \hat{V}^z(x)|^2 = \varepsilon^{2(1-\nu)} \left( \beta^\varepsilon - \hat{\ell}^z(x) \right). \tag{3.52}
\]

Since \( \ell \) is Lipschitz, the map \( x \mapsto \varepsilon^{2(1-\nu)} \left( \hat{\ell}^z(x) - \ell(z) \right) \) is bounded on compact subsets of \( \mathbb{R}^d \), uniformly in \( \varepsilon \in (0, 1) \). By Theorem 1.1 (i), which is established in Corollary 3.1, the constants \( \varepsilon^{2(1-\nu)} \left( \beta^\varepsilon - \ell(z) \right) \) are uniformly bounded in \( \varepsilon \in (0, 1) \). Applying \cite{[19]} Lemma 5.1
to (3.52) it follows that \( \tilde{V}_\varepsilon \) satisfies (3.50). Therefore, we have
\[
\nabla_x V^\varepsilon(x + z) = \varepsilon^{-\nu} \nabla_y \tilde{V}_\varepsilon^{(y)}|_{y = \varepsilon^{-\nu} x}
\]
\[
= \frac{\varepsilon^{-\nu}}{\varepsilon^{2(1-\nu)}} c_0 (1 + |\varepsilon^{-\nu} x|)
\]
\[
= \frac{c_0}{\varepsilon^2} (\varepsilon^{\nu} + |x|),
\]
and the proof is complete. \( \square \)

We continue with a version of Lemma 3.5 for unbounded domains.

**Proposition 3.1.** Let \( \nu \in (0, 2) \). Then for any \( k \in \mathbb{N} \) and \( r > 0 \), there exist constants \( \varepsilon_0 > 0 \) and \( \hat{\kappa}_1 \) such that
\[
\sup_{\varepsilon \in (0, \varepsilon_0)} \int_{B_r^\varepsilon(S)} \frac{(\text{dist}(x, S))^2k}{\varepsilon^{2+2\nu}} \eta_\varepsilon^2(dx) \leq \hat{\kappa}_1.
\]

**Proof.** Let \( \tilde{V}_\varepsilon = \varepsilon^2 V^\varepsilon \). By Lemma 3.6 the function \( \tilde{V}_\varepsilon = \varepsilon^2 V^\varepsilon \) is locally bounded, uniformly in \( \varepsilon > 0 \). Applying the function \( \mathcal{V} \circ \tilde{V}_\varepsilon \) to the operator
\[
\mathcal{L}_\varepsilon^\varepsilon := \frac{\varepsilon^{2\nu}}{2} \Delta + \left(m + \varepsilon^2 \nabla V^\varepsilon \right) \cdot \nabla,
\]
and using the identity \( \mathcal{L}_\varepsilon^\varepsilon[\tilde{V}_\varepsilon] = \varepsilon^2 (\beta_\varepsilon^\varepsilon - \ell) - \frac{1}{2} |\nabla \tilde{V}_\varepsilon|^2 \), after some calculations we obtain
\[
\mathcal{L}_\varepsilon^\varepsilon[\mathcal{V}^{2k} \varepsilon] = \mathcal{V}^{2k} \varepsilon \left[ \varepsilon^2 (\beta_\varepsilon^\varepsilon - \ell) + k\varepsilon^{2\nu} \Delta \mathcal{V} \mathcal{V} - \frac{1 - \varepsilon^{2\nu}}{2} |\nabla \tilde{V}_\varepsilon + \frac{2k}{1 - \varepsilon^{2\nu}} \nabla \mathcal{V}|^2 \right]
+ 2k \frac{m \cdot \nabla \mathcal{V}}{\mathcal{V}} + k \left( (2k - 1)\varepsilon^{2\nu} + \frac{2k}{1 - \varepsilon^{2\nu}} \right) \frac{||\nabla \mathcal{V}|^2}{\mathcal{V}^2}. \tag{3.53}
\]

By Hypothesis 1.1 and parts (iii) and (iv) of Theorem 2.1 we can add a positive constant to \( \mathcal{V} \) so that
\[
2 \frac{m \cdot \nabla \mathcal{V}}{\mathcal{V}} + \left( (2k - 1)\varepsilon^{2\nu} + \frac{2k}{1 - \varepsilon^{2\nu}} \right) \frac{||\nabla \mathcal{V}|^2}{\mathcal{V}^2} \leq \frac{m \cdot \nabla \mathcal{V}}{\mathcal{V}} \quad \text{on } \mathbb{R}^d, \tag{3.54}
\]
for all sufficiently small \( \varepsilon \). Let \( \kappa_0 \) be a bound of \( \beta_\varepsilon^\varepsilon + k \frac{||\nabla \mathcal{V}|}{\mathcal{V}} \), and define
\[
G_0 := \ell + \frac{1 - \varepsilon^{2\nu}}{2\varepsilon^2} |\nabla \tilde{V}_\varepsilon + \frac{2k}{1 - \varepsilon^{2\nu}} \nabla \mathcal{V}|^2.
\]

Using (3.54), we write (3.53) as
\[
\frac{1}{\varepsilon^{2+2\nu}} \mathcal{L}_\varepsilon^\varepsilon[\mathcal{V}^{2k} \varepsilon](x) \leq \mathcal{V}^{2k}(x) \varepsilon \tilde{V}_\varepsilon^{(x)} \left[ \kappa_0 - \varepsilon^{2 - 2\lambda \nu} G_0(x) + k \frac{m(x) \cdot \nabla \mathcal{V}(x)}{\varepsilon^{2\lambda \nu} \mathcal{V}(x)} \right]. \tag{3.55}
\]

Let \( F^\varepsilon(x) \) denote the right-hand side of (3.55). Since \( \ell \) is inf-compact, there exists \( r_0 > 0 \) such that \( F^\varepsilon < 0 \) on \( B_{r_0}^\varepsilon \). On the other hand, since \( m(x) \cdot \nabla \mathcal{V}(x) \) is approximately quadratic near \( S \) and negative on \( S^c \), it follows that there exists \( \kappa > 0 \) and \( \varepsilon_0 > 0 \), such that
\[
\{ x : F^\varepsilon(x) > 0 \} \subset B_{\rho_0}(S) \quad \forall \varepsilon < \varepsilon_0,
\]
where $\rho(\varepsilon) := \kappa \varepsilon^{1+\nu}$. By Lemma 3.6, $\kappa_0 V^{2k}(x) e^{\widetilde{V}(x)}$ is locally bounded, uniformly in $\varepsilon > 0$. Let $\kappa_1$ be such a bound on $B_{\rho(\varepsilon)}(S)$. Then
\[
\frac{1}{\varepsilon^{2\lambda+2\nu}} \mathcal{L}_s^\varepsilon [V^{2k} e^{\widetilde{V}(x)}] (x) \leq \kappa_1 - \nu \left( \frac{m(x) \cdot \nabla V(x)}{\varepsilon^{2\lambda+2\nu} \partial V(x)} \right) \quad \text{for all } x \in \mathbb{R}^d.
\]
As a result of the strong maximum principle, $V^\varepsilon$ attains its infimum over $\mathbb{R}^d$ in the set $\{x \in \mathbb{R}^d : \ell(x) \leq \beta^\varepsilon\}$. Therefore, $\widetilde{V}^\varepsilon$ is bounded below in $\mathbb{R}^d$ by Lemma 3.6. Thus, from (3.56) we obtain
\[
\int_{B_{\rho(\varepsilon)}(S)} \frac{m(x) \cdot \nabla V(x)}{\varepsilon^{2\lambda+2\nu}} V^{2k-1}(x) \eta_s^\varepsilon(dx) \leq \frac{\kappa_2}{k \inf_{\mathbb{R}^d} e^V} \quad \forall \varepsilon \leq \varepsilon_0.
\]
for some constant $\kappa_2$. Since by Hypothesis 1.1, $V$ has strict quadratic growth and $|m \cdot \nabla V|$ has strict linear growth, then (3.57) implies that for some constant $\kappa_3$, we must have
\[
\int_{B_{\rho(\varepsilon)}(S)} \frac{1}{\varepsilon^{2\lambda+2\nu}} (\text{dist}(x, S))^4 \eta_s^\varepsilon(dx) \leq \kappa_3 \quad \forall \varepsilon \leq \varepsilon_0.
\]
This finishes the proof. \qed

As a corollary to the above result we obtain the following.

**Corollary 3.2.** For $\nu \in (0, 1)$, we have $\beta^\varepsilon - 3_3 \geq \mathcal{O}(\varepsilon^{\nu(1+\nu)})$.

**Proof.** Recall from Theorem 1.1 that $3_3 = \min \{\ell(z) : z \in S_\varepsilon\}$. Consider $z \in S \setminus S_\varepsilon$, and let $\varphi$, $B_{\varepsilon}(z)$, and $\zeta^\varepsilon$, be as in Lemma 3.3. Recall from the proof of Lemma 3.3 that $\sup_{B_{\varepsilon}(z)} \Delta V < 0$. Thus, using Theorem 2.1 (iv) and Young’s inequality, we deduce from (3.20) that
\[
-\frac{1}{2} \int_{B_{\varepsilon}(z)} \varphi'(V) \Delta V \, d\eta^\varepsilon_s + \frac{\kappa_1}{\varepsilon^{2\nu}} (\zeta^\varepsilon)^2 \leq \kappa_2 \varepsilon^{2-2\nu} \int_{\mathbb{R}^d} \varphi'(V) |v^\varepsilon|^2 \, d\eta^\varepsilon_s + \kappa_3 (\zeta^\varepsilon)^2
\]
for some positive constants $\kappa_i$, $i = 1, 2, 3$. It follows that for some $r_0 \leq r$ we have $\eta^\varepsilon_s(B_{r_0}(z)) \in \mathcal{O}(\varepsilon^{2-2\nu})$. Therefore there exists a neighborhood $B$ of $S \setminus S_\varepsilon$ such that $\eta^\varepsilon_s(B) \in \mathcal{O}(\varepsilon^{2-2\nu})$. Let $K \subset \mathbb{R}^d$ be a compact set satisfying $\ell(x) - 3_3 \geq 0$ for $x \notin K$, and define $G_1 := \cup_{s \in S} B_{r_0}(z) \setminus B$ and $G_2 := K \setminus (G_1 \cup B)$. We can choose $r_0$ and $B$ in such a way that $\{B_{r_0}(z) : z \in S\}$ are disjoint and $B \cap S_\varepsilon = \emptyset$. By Proposition 3.1 we have $\eta^\varepsilon_s(G_2) \in \mathcal{O}(\varepsilon^{2\nu})$. Therefore
\[
\beta^\varepsilon - 3_3 \geq \int_{\mathbb{R}^d} \ell \, d\eta^\varepsilon_s - \min \{\ell(z) : z \in S_\varepsilon\}
\]
\[
\geq -\kappa_4 \left( \eta^\varepsilon_s(B) + \int_{G_1} \text{dist}(x, S) \, d\eta^\varepsilon_s + \eta^\varepsilon_s(G_2) \right)
\]
\[
\geq -\mathcal{O}(\varepsilon^{2-2\nu}) - \mathcal{O}(\varepsilon^{\nu}) - \mathcal{O}(\varepsilon^{2\nu})
\]
\[
\geq -\mathcal{O}(\varepsilon^{\nu(1+\nu)}),
\]
for some constant $\kappa_4$, where we use Lemma 3.5 in the third inequality. \qed

We proceed to prove the converse inequality to (6.27).
Lemma 3.7. Assume \( \nu = 1 \), and let \( \{ \mathcal{B}_z : z \in S \} \) and \( r(\varepsilon) \) be as in Definition 3.2. Suppose that for some \( z \in S \) and some sequence \( \varepsilon_n \searrow 0 \) it holds that \( \tilde{\eta}^{\varepsilon_n}(\mathcal{B}_z) \to \xi_z > 0 \). All limits are assumed along this sequence. Normalize \( V^\varepsilon \) so that \( V^\varepsilon(z) = 0 \), and let \( M \equiv Dm(z) \).

Let \( \tilde{V}_z^\varepsilon \) be as in Definition 3.2. Then

(a) The sequence \( \tilde{V}_z^{\varepsilon_n} \) converges to some function \( \nabla \in C^2(\mathbb{R}^d) \), along a subsequence of \( \varepsilon_n \searrow 0 \) (also denoted as \( \{ \varepsilon_n \} \)), and \( \nabla \) satisfies

\[
\frac{1}{2} \Delta \nabla(x) + \min_{u \in \mathbb{R}^d} \left[ (Mx - u) \cdot \nabla(x) + \frac{1}{2} |u|^2 \right] + \ell(z) = \tilde{\beta}.
\]  

(b) It holds that \( \tilde{\beta} \geq \ell(z) + \varepsilon_+ (Dm(z)) \).

(c) The diffusion

\[
dX_t = (MX_t - \nabla \nabla(X_t)) \ dt + dW_t
\]  

is positive recurrent, and its invariant probability measure \( \bar{\eta} \) has finite second moments.

(d) The densities \( \tilde{\eta}_z^\varepsilon \) and \( \bar{\eta}_z^\varepsilon \) in Definition 3.2 converge to the density \( \bar{\eta} \) of \( \bar{\eta}_z \), uniformly on compact sets.

(e) It holds that

\[
\liminf_{\varepsilon_n \searrow 0} \int_{B_{r(\varepsilon_n)}(z)} \left( \ell(x) + \frac{1}{2} |v^{\varepsilon_n}(x)|^2 \right) \eta^{\varepsilon_n}(dx) \geq \xi_z \left( \ell(z) + \varepsilon_+ (Dm(z)) \right).
\]

Proof. By (2.7) we obtain

\[
\frac{1}{2} \Delta \tilde{V}_z^\varepsilon + \tilde{m}_z^\varepsilon \cdot \nabla \tilde{V}_z^\varepsilon - \frac{1}{2} |\nabla \tilde{V}_z^\varepsilon|^2 + \tilde{\beta}^\varepsilon = \beta^\varepsilon.
\]

By [19] Lemma 5.1 (see also Lemma 3.6) and the assumptions on \( m \) and \( \ell \), there exists a constant \( c_0 \) such that

\[
|\nabla \tilde{V}_z^\varepsilon(x)| \leq c_0 (1 + |x|) \quad \forall \varepsilon \in (0, 1), \quad \forall x \in \mathbb{R}^d.
\]

It then follows that \( \tilde{V}_z^\varepsilon \) is locally bounded in \( C^{2, \alpha}(\mathbb{R}^d) \) for any \( \alpha \in (0, 1) \).

Taking limits in (2.6) along \( \varepsilon_n \searrow 0 \) we obtain a function \( \nabla \in C^2(\mathbb{R}^d) \) and a constant \( \tilde{\beta} \) which satisfy (3.58). This proves part (a).

In order to show that the diffusion in (3.59) is positive recurrent, consider the diffusion

\[
dX_t = (\tilde{m}_z^\varepsilon(X_t) - \nabla \tilde{V}_z^\varepsilon(X_t)) \ dt + dW_t,
\]

Recall from Definition 3.2 that \( \tilde{\eta}_z^\varepsilon \) and \( \bar{\eta}_z^\varepsilon \) denote the invariant probability measure of (3.61) and its density, respectively. Let

\[
\tilde{\mathcal{L}}_z^\varepsilon := \frac{1}{2} \Delta + (\tilde{m}_z^\varepsilon - \nabla \tilde{V}_z^\varepsilon) \cdot \nabla
\]

denote the extended generator of (3.61). By Lemma 3.5 and the Markov inequality we obtain \( \eta^{\varepsilon_n}(\mathcal{B}_z \setminus B_{r\varepsilon}(z)) \leq \frac{\kappa_0}{\varepsilon^2} \) for all \( n \in \mathbb{N} \). It follows that \( \{ \tilde{\eta}^{\varepsilon_n} : n \in \mathbb{N} \} \) is a tight family of measures. Since, by the Markov inequality just mentioned, we have

\[
\eta^{\varepsilon_n}(B_{r\varepsilon}(z)) \geq \eta^{\varepsilon_n}(\mathcal{B}_z) - \frac{\kappa_0 \varepsilon^2}{r^2(\varepsilon)},
\]  

(3.62)
and \( \eta^\varepsilon_n(B_2) \to \xi > 0 \) as \( \varepsilon_n \downarrow 0 \) the family \( \{ \eta^\varepsilon_n : n \in \mathbb{N} \} \) is also tight. By the Harnack inequality the family \( \{ \tilde{\eta}^\varepsilon_n : n \in \mathbb{N} \} \) is locally bounded, and locally Hölder equi-continuous, and the same of course applies to \( \{ \tilde{\eta}^\varepsilon_n : n \in \mathbb{N} \} \). Moreover, the tightness of \( \{ \tilde{\eta}^\varepsilon_n : n \in \mathbb{N} \} \) implies the uniform integrability of \( \{ \tilde{\eta}^\varepsilon_n : n \in \mathbb{N} \} \). Select any subsequence, also denoted by \( \{ \varepsilon_n \} \) along which \( \tilde{\eta}^\varepsilon_n \) converges locally uniformly, and denote the limit by \( \tilde{\eta} \).

By uniform integrability, \( \tilde{\eta}^\varepsilon_n \) also converges in \( L^1(\mathbb{R}^d) \), as \( n \to \infty \), and hence \( \int_{\mathbb{R}^d} \tilde{\eta}(x) \, dx = 1 \). Therefore \( \tilde{\eta}(dx) := \tilde{\eta}(x) \, dx \) is a probability measure. Let \( f \) be a smooth function with compact support, and

\[
\mathcal{L} := \frac{1}{2} \Delta + (Mx - \nabla V(x)) \cdot \nabla.
\]

Then

\[
\left| \int_{\mathbb{R}^d} \mathcal{L}^\varepsilon_n f(x) \tilde{\eta}^\varepsilon_n(x) \, dx - \int_{\mathbb{R}^d} \mathcal{L} f(x) \tilde{\eta}(x) \, dx \right| \leq \int_{\mathbb{R}^d} \mathcal{L}^\varepsilon_n f(x) \left( \tilde{\eta}^\varepsilon_n(x) - \tilde{\eta}(x) \right) \, dx + \int_{\mathbb{R}^d} \left( \mathcal{L}^\varepsilon_n f(x) - \mathcal{L} f(x) \right) \tilde{\eta}(x) \, dx. \tag{3.63}
\]

Since \( \tilde{\eta}^\varepsilon_n \to \tilde{\eta} \) in \( L^1(\mathbb{R}^d) \), the first term on the right hand side of (3.63) converges to 0 as \( n \to \infty \). Similarly since \( \tilde{m}^\varepsilon_n(x) \to Mx \) and \( \tilde{V}^\varepsilon_n \to \tilde{V} \) uniformly on compacta, the second term also converges to 0. Since \( \tilde{\eta}^\varepsilon_n \) is an invariant probability measure of (3.64), by the definition of \( \tilde{\eta}^\varepsilon_n \) we have \( \int_{\mathbb{R}^d} \mathcal{L}^\varepsilon_n f(x) \tilde{\eta}^\varepsilon_n(x) \, dx = 0 \), for all large enough \( \varepsilon_n \), which implies that \( \int_{\mathbb{R}^d} \mathcal{L} f(x) \tilde{\eta}(x) \, dx = 0 \). Hence, \( \tilde{\eta} \) is an infinitesimal invariant probability measure of (3.64), and since the diffusion is regular, it is also an invariant probability measure. This proves part (d).

Since the diffusion in (3.59) has an invariant probability measure, it follows that it is positive recurrent. By Lemma 3.5 we have

\[
\sup_{\varepsilon \in (0,1)} \int_{\{\varepsilon^\nu x + z \in B_2\}} |x|^2 \tilde{\eta}^\varepsilon_z(dx) < \infty.
\]

By Fatou’s lemma we obtain \( \int_{\mathbb{R}^d} |x|^2 \tilde{\eta}(dx) < \infty \). This completes the proof of part (c).

Since \( \tilde{V} \) has at most quadratic growth by (3.60), we have \( \int_{\mathbb{R}^d} |\nabla \tilde{V}(x)| \tilde{\eta}(dx) < \infty \). Thus, by Lemma 4.12 in [17], if \( \mathbb{E}_x \) denotes the expectation operator for the process governed by (3.59), it is the case that \( \mathbb{E}_x [\nabla \tilde{V}(X_t)] \) converges as \( t \to \infty \). Integrating both sides of (3.59) with respect to \( \tilde{\eta} \), we deduce that

\[
\int_{\mathbb{R}^d} \frac{1}{2} |\nabla \tilde{V}(x)|^2 \tilde{\eta}(dx) = \tilde{\beta} - \ell(z). \tag{3.64}
\]

Since the Markov control \( v(X_t) = -\nabla \tilde{V}(X_t) \) is in general suboptimal for the problem in (3.59), then by Lemma 3.4 we must have \( \mathcal{E}_+(M) \leq \tilde{\beta} - \ell(z) \). This proves part (b).
By \(3.62\), we have \(\frac{\varepsilon_n}{\xi_n} \to \xi\) along \(\{\varepsilon_n\}\) as \(n \to \infty\), uniformly on compacts. Therefore, using Fatou’s lemma, we obtain by part (b) that

\[
\liminf_{\varepsilon_n \downarrow 0} \int_{B_{\nu}(\varepsilon_n)} \left( \ell(x) + \frac{1}{2} |v^zn(x)|^2 \right) \eta_n^\varepsilon_n (dx) = \liminf_{\varepsilon_n \downarrow 0} \int_{\{\varepsilon_n x + z \in B_{\nu}(\varepsilon_n)\}} \left( \hat{\ell}_n^\varepsilon_n (x) + \frac{1}{2} |\nabla \hat{V}_n^zn(x)|^2 \right) \hat{\eta}_n^\varepsilon_n (dx) \geq \xi \beta \geq (E_+(M) + \ell(z)). \tag{3.65}
\]

This proves part (e) and thus completes the proof. \(\square\)

The statement in Theorem 1.1 (iii) follows from the following result.

**Theorem 3.3.** Recall the definition of \(\hat{J}_2\) from Theorem 1.1. Provided \(\nu = 1\), it holds that

\[
\lim_{\varepsilon \downarrow 0} \beta_\varepsilon^\nu = \hat{J}_2.
\]

**Proof.** In view of \(3.27\), it is enough to show that \(\liminf_{\varepsilon \downarrow 0} \beta_\varepsilon^\nu \geq \hat{J}_2\). Choose any sequence \(\varepsilon_n \downarrow 0\) such that \(\eta_n^\varepsilon_n (B_z) \to \xi\), for all \(z \in S\). Let \(S_0 := \{z \in S : \xi > 0\}\). Thus \(\sum_{z \in S_0} \xi_z = 1\). Therefore, by Lemma 3.7 (e) we have

\[
\liminf_{n \to \infty} \beta_\varepsilon^\nu \geq \sum_{z \in S_0} \int_{B_{\nu}(\varepsilon_n)} \left( \ell(x) + \frac{1}{2} |v^zn(x)|^2 \right) \eta_n^\varepsilon_n (dx) \\
\geq \sum_{z \in S_0} \xi_z \left( \ell(z) + E_+(Dm(z)) \right) \\
\geq \hat{J}_2, \tag{3.66}
\]

and the proof is complete. \(\square\)

Recall the definition of \(Z_2\) from Theorem 1.1 and let the function \(r\) be as in Definition 3.2. The inequality in (3.66) suggests that the control effort concentrates in the vicinity of \(Z_2\). This is asserted in the theorem that follows.

**Theorem 3.4.** Suppose \(\nu = 1\). Then \(\eta_\varepsilon^\nu (B_{\nu}(\varepsilon_n) (Z_2)) \to 0\) as \(\varepsilon \downarrow 0\). Moreover,

\[
\lim_{\varepsilon \downarrow 0} \int_{B_{\nu}(\varepsilon_n) (Z_2)} |v^\varepsilon_n(x)|^2 \eta_\varepsilon^\nu (dx) = 0.
\]

**Proof.** The first claim follows by Lemma 3.5 and Theorem 3.3.

To prove the second claim, given given any sequence \(\varepsilon_n \downarrow 0\), we extract a subsequence also denoted by \(\varepsilon_n\) along which \(\lim_{n \to \infty} \eta_n^\varepsilon_n (B_z) \to \xi\) for all \(z \in S\). As in the proof of Theorem 3.3, let \(S_0 := \{z \in S : \xi > 0\}\). Then \(\sum_{z \in S_0} \xi_z = 1\), and \(S_0 \subset Z_2\) by Theorem 3.3. Then, by (3.65) we have

\[
\liminf_{n \to \infty} \int_{B_{\nu}(\varepsilon_n)(S_0)} \left( \ell(x) + \frac{1}{2} |v^zn(x)|^2 \right) \eta_n^z (dx) \geq \hat{J}_2,
\]

and hence, using Theorem 3.3 we obtain
\[
\liminf_{n \to \infty} \int_{B_{\hat{c}(\varepsilon_n)}(S)} |v^{\varepsilon_n}(x)|^2 \eta^{\varepsilon_n}(dx) = 0,
\]
and the proof is complete. □

We next show that, under the hypothesis of Theorem 1.2, the scaled density \( \hat{\eta}^{\varepsilon_n}(z) \) in Definition 3.2 converges to that of a Gaussian distribution as \( \varepsilon \downarrow 0 \). This establishes Theorem 1.2 for the critical regime.

**Theorem 3.5.** Let \( \nu = 1 \), and suppose that for some \( z \in S \) and a sequence \( \varepsilon_n \downarrow 0 \), we have \( \lim_{n \to \infty} \eta^{\varepsilon_n}(B_z) = \xi_z > 0 \). Set \( M \equiv Dm(z) \), and let \( (\bar{Q}_z, \bar{\Sigma}_z) \) be the pair of matrices which solves (1.5). The following hold:

(a) Provided that we normalize \( V^{\varepsilon_n}(z) \) by setting \( V^{\varepsilon_n}(z) = 0 \), it holds that \( V^{\varepsilon_n}(\varepsilon_n x + z) = \frac{1}{2} (x^T \bar{Q}_z x) \) as \( n \to \infty \), in \( C^{2,\alpha}(\mathbb{R}^d) \), \( \alpha \in (0, 1) \).

(b) The density \( \hat{\eta}^{\varepsilon_n} \) in Definition 3.2 converges as \( n \to \infty \) (uniformly on compact sets) to the density of a Gaussian with mean 0 and covariance matrix \( \bar{\Sigma}_z \).

**Proof.** Since \( \bar{\beta} \) in Lemma 3.7 satisfies \( \bar{\beta} \leq \limsup_{\varepsilon \downarrow 0} \beta^{\varepsilon} \), it follows by Theorem 3.3 that \( \beta = J_2 \). Therefore, by (3.61) we have
\[
\int_{\mathbb{R}^d} \frac{1}{2} |\nabla V(x)|^2 \hat{\eta}(dx) = \mathcal{E}_+(Dm(z)).
\]
That \( \nabla V(x) = \bar{Q}_z x \) for all \( x \in \mathbb{R}^d \), as claimed, follows by Lemma 3.4 (b). This proves the first part of the result.

Since \( \nabla V(x) = \bar{Q}_z x \), the invariant probability measure \( \hat{\eta} \) of (3.59) is Gaussian with covariance matrix \( \bar{\Sigma}_z \) as stated. By Lemma 3.7 (d), \( \bar{\gamma}^{\varepsilon_n} \rightarrow \bar{\gamma} \) and \( \bar{\varepsilon_n}^{\varepsilon_n} \rightarrow \xi_z \) as \( \varepsilon \downarrow 0 \), uniformly on compact sets. Since \( \hat{\eta}^{\varepsilon_n}(B_z) \rightarrow \xi_z \), the second part of the theorem follows and the proof is complete. □

It is interesting to note that part (a) of Theorem 3.5 holds for any \( z \in Z_2 \). We state this as follows.

**Theorem 3.6.** Suppose \( \nu = 1 \). Recall the definition of \( Z_2 \) from Theorem 1.1. Then for any \( z \in Z_2 \), and subject to the normalization \( V^{\varepsilon_n}(z) = 0 \), it holds that
\[
\hat{V}^{\varepsilon_n}(x) \xrightarrow{\varepsilon \to 0} \frac{1}{2} x^T \bar{Q}_z x,
\]
uniformly on compact sets.

**Proof.** Let \( M = Dm(z) \). Let \( \varepsilon_n \downarrow 0 \) be any sequence, and extract a subsequence, also denoted as \( \{\varepsilon_n\} \) along which \( \hat{V}^{\varepsilon_n} \rightarrow \nabla \in C^2(\mathbb{R}^d) \). Taking limits as \( \varepsilon_n \downarrow 0 \) in (2.6), and using Theorem 3.3 and the fact that \( \mathcal{E}_+(Dm(z)) = \frac{1}{2} \text{trace}(\bar{Q}_z) \), we obtain
\[
\frac{1}{2} \Delta \nabla(x) + \min_{u \in \mathbb{R}^d} \left[ (Mx + u) \cdot \nabla V(x) + \frac{1}{2} |u|^2 \right] = \frac{1}{2} \text{trace}(\bar{Q}_z).
\]
By the suboptimality of \( u(x) = -\nabla \hat{Q}_x x \) we obtain
\[
\frac{1}{2} \Delta \nabla (x) + (M x - \hat{Q}_x x) \cdot \nabla (x) + \frac{1}{2} \hat{Q}_x x^2 = \frac{1}{2} \text{trace}(\hat{Q}_x) + f(x),
\] (3.67)
for some nonnegative function \( f \). However, since \( \nabla \nabla \) has linear growth, then both \( f \) and \( \nabla \nabla \) have at most quadratic growth. Hence, repeating the argument that led to the derivation of (3.44), we obtain from (3.67) that \( \int_{\mathbb{R}^d} f(x) \rho(z) dx = 0 \), hence \( f \equiv 0 \). If \( \mathbb{E} \) denotes the expectation operator for the diffusion \( X_t \) with linear drift \( (M - \hat{Q}_x) X_t \) and identity diffusion matrix then the stochastic representation of the solution of (3.67) gives
\[
q(x) := \frac{1}{2} x^T \hat{Q}_x x = \liminf_{r \to 0} \mathbb{E}_x \left[ \int_0^{\tau_r} \frac{1}{2} \left( |\hat{Q}_x X_t|^2 - \text{trace}(\hat{Q}_x) \right) dt \right],
\]
where \( \tau_r \) is the first hitting time of the ball \( B_r [1] \) Theorem 3.7.12. Applying Dynkin’s formula to (3.67), and since \( f \equiv 0 \), we obtain \( \nabla \nabla \geq q \). Also with \( \bar{L} := \frac{1}{2} \Delta + (M x - \nabla \nabla) \cdot \nabla \), we have \( \bar{L} (\nabla - q) \leq 0 \). Since \( \nabla (0) = q(0) \), we have \( \nabla = q \) on \( \mathbb{R}^d \) by the comparison principle, and the proof is complete.

**Remark 3.2.** In summary, when \( \nu = 1 \), the control cost for having \( \eta^z \) concentrate around a point \( z \in \mathcal{B} \) is \( E^z(Dm(z)) \). Therefore a point \( z' \in \mathcal{B} \) will be chosen over a point \( z \in \mathcal{B} \), if and only if \( \ell(z') + E^z(Dm(z')) < \ell(z) + E^z(Dm(z)) \). Recall also that \( E^z(Dm(z)) = 0 \) for \( z \in \mathcal{B} \).

In Example 3.1 \( Dm(0) = 2 \). Therefore, applying Theorem 3.3 and Lemma 3.4 we conclude that in the critical regime the invariant probability distribution \( \rho^z \) concentrates at \( x = 0 \) if \( c > 2 \), and at \( x = -1 \) if \( c < 2 \).

### 3.4. The supercritical and subcritical regimes revisited

We return to the analysis of the supercritical and subcritical regimes, in order to determine the asymptotic behavior of the invariant probability distribution in the vicinity of the minimal stochastically stable set. In these regimes there are two scales. If we center the coordinates around a point is \( \mathcal{B} \), then we have \( V^\varepsilon(x) \in O(\varepsilon^{-2}) \), while \( -\log q^z(x) \in O(\varepsilon^{-2\nu}) \). We start with the subcritical regime.

#### 3.4.1. Subcritical Regime

Recall the notation introduced in Definition 3.2. Let \( z \in \mathcal{B} \) be such that for some sequence \( \varepsilon_n \searrow 0 \) it holds that \( \liminf_{n \to \infty} \eta^z \geq \mathcal{B} \). Let \( r \) be as in Definition 3.2.

Normalize \( V^\varepsilon \) by setting \( V^\varepsilon(z) = 0 \), and let \( M := Dm(z) \). We scale the space as \( 1/\varepsilon^\nu \), to obtain
\[
\frac{1}{2} \Delta \bar{V}^\varepsilon(x) + \bar{m}^\varepsilon(x) \cdot \nabla \bar{V}^\varepsilon(x) - \varepsilon^{2(1-\nu)} |\nabla \bar{V}^\varepsilon(x)|^2 + \bar{\ell}^\varepsilon(x) = \beta^\varepsilon.
\] (3.68)

By Lemma 3.6 \( \nabla \bar{V}^\varepsilon = \varepsilon^{2(1-\nu)} \nabla \bar{V}^\varepsilon \) is locally bounded and has at most linear growth. We write (3.68) in the form (3.65) which corresponds to the scaled diffusion
\[
d\bar{X}_t = \left( \hat{m}^\varepsilon(\bar{X}_t) - \varepsilon^{2(1-\nu)} \nabla \bar{V}^\varepsilon(\bar{X}_t) \right) dt + d\bar{W}_t,
\] (3.69)
and the associated HJB equation
\[
\frac{1}{2} \Delta \bar{V}^\varepsilon(x) + \min_{u \in \mathbb{R}^d} \left[ (\hat{m}^\varepsilon(x) + \varepsilon^{-1-\nu} u) \cdot \nabla \bar{V}^\varepsilon(x) + \frac{\varepsilon^{2(1-\nu)} |u|^2}{2} \right] = \varepsilon^{2(1-\nu)} (\beta^\varepsilon - \bar{\ell}^\varepsilon(x)).
\] (3.70)
Letting $\tilde{u} = \varepsilon^{1-\nu} u$ and taking limits in \((3.69) - (3.70)\) along some subsequence $\varepsilon_n \searrow 0$, we obtain a function $V \in C^2(\mathbb{R}^d)$ of at most quadratic growth satisfying
\[
d\tilde{X}_t = \left( M\tilde{X}_t - \nabla V(\tilde{X}_t) \right) dt + d\tilde{W}_t, \tag{3.71}
\]
and
\[
\frac{1}{2} \Delta V(x) + \min_{\tilde{u} \in \mathbb{R}^d} \left[ (Mx + \tilde{u}) \cdot \nabla V(x) + \frac{1}{2} |\tilde{u}|^2 \right] = 0.
\]
Following the proofs of Lemma 3.7 and Theorem 3.5, and using Lemma 3.5, we deduce that the densities $\tilde{g}_z^\varepsilon$ and $\tilde{g}_z^\varepsilon$ in Definition 3.2 converge as $\varepsilon \searrow 0$ to the density of a Gaussian distribution with covariance matrix $\Sigma_z$, uniformly on compact sets. Also, $V(x) = \frac{1}{2} x^T \hat{Q}_z x$. However, since $Dm(z)$ is Hurwitz, then $\hat{Q}_z = 0$ by Remark 3.1 and Lemma 3.4.

### 3.4.2. Supercritical Regime

We assume $\nu \in (1, 2)$, and we use the same scaling and definitions as in Section 3.4.1 for the subcritical regime, except that $z \in \mathcal{J}_1$.

It is clear that $\varepsilon^{2(1-\nu)}(\beta^\varepsilon_z - \ell(z)) \to 0$ as $\varepsilon \searrow 0$. By Corollary 3.1 the constants $\varepsilon^{2(1-\nu)}(\beta^\varepsilon_z - \ell(z))$ are bounded, uniformly in $\varepsilon \in (0, 1)$. Extract a subsequence $\varepsilon_n \searrow 0$ along which $\varepsilon_n^{2(1-\nu)}(\beta^\varepsilon_z - \ell(z))$ converges to a constant, and a further subsequence along which $\tilde{V}_z^\varepsilon$ converges to $V \in C^2(\mathbb{R}^d)$, uniformly on compact sets. We obtain
\[
\frac{1}{2} \Delta V(x) + \min_{\tilde{u} \in \mathbb{R}^d} \left[ (Mx + \tilde{u}) \cdot \nabla V(x) + \frac{1}{2} |\tilde{u}|^2 \right] = \hat{\beta}.
\]
With $\tilde{u} = \varepsilon^{1-\nu} u$, as defined earlier, and noting that the scaled diffusion in \((3.71)\) has invariant probability distribution $\tilde{\eta}_z^\varepsilon$, we obtain by \((3.70)\) that
\[
\int_{\mathbb{R}^d} |\tilde{u}(\varepsilon^\nu x + z)|^2 \tilde{\eta}_z^\varepsilon(dx) + \int_{\mathbb{R}^d} \tilde{V}_z^\varepsilon(x) - \ell(z) \tilde{\eta}_z^\varepsilon(dx) = \varepsilon^{2(1-\nu)}(\beta^\varepsilon_z - \ell(z)). \tag{3.72}
\]
Let $G$ be a bounded open set containing $\mathcal{S}$ such that $\ell(x) > \ell(z)$ for all $x \in G^c$. Since $z \in \mathcal{J}_1$ and $\ell$ is Lipschitz, we have $\ell(x) - \ell(z) \geq C_\ell \mathrm{dist}(x, \mathcal{S})$ in $G$. Hence using the fact that $2\nu - 2 \leq \nu$ together with the Cauchy–Schwartz inequality and Lemma 3.5 it follows that the second term in \((3.72)\) vanishes as $\varepsilon \searrow 0$. Therefore, following the arguments in Lemma 3.7 it follows that $\hat{\beta} \geq \mathcal{E}_+(Dm(z))$, which implies that
\[
\liminf_{\varepsilon \searrow 0} \int_{\varepsilon^\nu x + z \in \mathcal{B}_z} \varepsilon^{2(1-\nu)} 2 |u(x)|^2 \tilde{\eta}_z^\varepsilon(dx) \geq \mathcal{E}_+(Dm(z)).
\]
However, under the control $u^\varepsilon(x) = -\frac{1}{2}(m(x) - M(x - z) + \hat{Q}_z(x - z))$ applied to the non-scaled problem, and with $\mu^\varepsilon$ denoting the stationary distribution of the controlled process we obtain
\[
\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \varepsilon^{2(1-\nu)} 2 |u(x)|^2 \mu^\varepsilon(dx) = \mathcal{E}_+(Dm(z)).
\]
Therefore $\hat{\beta} = \mathcal{E}_+(Dm(z))$. Hence arguing as in Theorem 3.3 we can show that
\[
\lim_{\varepsilon \searrow 0} \int_{\varepsilon^\nu x + z \in \mathcal{B}_z} \varepsilon^{2(1-\nu)} 2 |u^\varepsilon(x)|^2 \tilde{\eta}_z^\varepsilon(dx) = \mathcal{E}_+(Dm(z)). \tag{3.73}
\]
Moreover, by the proof of Theorem 3.3 we obtain $V(x) = \frac{1}{2} x^T \hat{Q}_z x$, and that $\tilde{g}_z^\varepsilon$ converges to the density of a Gaussian distribution with covariance matrix $\Sigma_z$. 


3.5. On Theorem [1.3] We summarize the conclusions of Sections 3.4.1–3.4.2 in the following lemma.

Lemma 3.8. Assume \( \nu \in (0, 2) \) and let \( \{ B_z : z \in S \} \) be as in Definition 3.2. Suppose that for some \( z \in S \) and some sequence \( \varepsilon_n \downarrow 0 \) it holds that \( \liminf_{n \to \infty} \eta_{\nu n}^*(B_z) > 0 \). Let \( i = 1, 2, \) or \( 3 \), according to as \( \nu > 1, \nu = 1, \) or \( \nu < 1 \), respectively. Then, the density \( \hat{\hat{\rho}}_{\nu n}^* \) in Definition 3.2 converges as \( n \to \infty \) (uniformly on compact sets) to the density of a Gaussian with mean 0 and covariance matrix \( \hat{\Sigma}_z \).

The ergodic control problem in (1.1) and (1.3) is of course equivalent to minimizing
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2\varepsilon^2} |\tilde{U}_s - m(X_s)|^2 \right) ds \right]
\]
on all admissible controls \( \tilde{U} \), subject to
\[
X_t = X_0 + \int_0^t \tilde{U}_s ds + \varepsilon^\nu W_t, \quad t \geq 0.
\]
It follows that if \( v_*^\varepsilon \) is an optimal stationary Markov control for (1.1), (1.3), then
\[
\tilde{v}_*^\varepsilon := m(x) + \varepsilon v_*^\varepsilon
\]
is an optimal control for (3.31), (3.24), and vice-versa.

Suppose \( z \in S \). We define the ‘running cost’ \( \tilde{R}_z[v](x) \) by
\[
\tilde{R}_z[v](x) := \ell(x) - \ell(z) + \frac{1}{2\varepsilon^2} |v(x) - m(x)|^2.
\]
Then
\[
\beta_*^\varepsilon = \ell(z) + \int_{\mathbb{R}^d} \tilde{R}_z[\tilde{v}_*^\varepsilon](x) \eta_{\nu}^*(dx).
\]
Suppose \( \nu \geq 1 \). Using the suboptimal control \( \tilde{v}_z(x) = (M - \tilde{Q}_z)(x - z) \), with \( M = Dm(z) \), instead of \( \tilde{v}_*^\varepsilon \) we obtain
\[
\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} \frac{\ell(x) - \ell(z)}{\varepsilon^{2\nu}} \rho_{\tilde{\Sigma}_z}^\varepsilon(x) dx = \frac{1}{2} \text{trace}(\tilde{\Sigma}_z D^2 \ell(z)). \quad (3.74)
\]
where \( \rho_{\tilde{\Sigma}_z}^\varepsilon \) denotes the Gaussian density with covariance matrix \( \varepsilon^{2\nu} \tilde{\Sigma}_z \). We write
\[
|\tilde{v}(x) - m(x)|^2 = |\tilde{Q}_z(x - z)|^2 + R_z(x), \quad (3.75)
\]
where
\[
R_z(x) := (M(x - z) - m(x)) \cdot ((M - \tilde{Q}_z)(x - z) - m(x))
\]
Since \( |R_z(x)| \leq C|x - z|^3 \) for some constant \( C \), and the third moments of \( \rho_{\tilde{\Sigma}_z}^\varepsilon \) are 0, it follows by Lemma 3.4 and (3.75) that
\[
\int_{\mathbb{R}^d} \frac{1}{2\varepsilon^{2\nu}} |\tilde{v}_z(x) - m(x)|^2 \rho_{\tilde{\Sigma}_z}^\varepsilon(x) dx - \mathcal{E}_+(Dm(z)) \in \mathcal{O}(\varepsilon^{2\nu}). \quad (3.76)
\]
By (3.74) and (3.76), we obtain
\[
\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon^{2\nu} - 2} \int_{\mathbb{R}^d} \tilde{R}_z[\tilde{v}_z](x) \rho_{\tilde{\Sigma}_z}^\varepsilon(x) dx = \mathcal{E}_+(Dm(z)).
\]
We are now ready for the proof of the theorem.

**Proof of Theorem 1.3.** Suppose first that \( \nu \in (1, 2) \). Select the collection of balls \( \{ B_z : z \in S \} \) so as to satisfy

\[
\inf_\beta \beta > \min_\beta \beta \quad \forall z \in S \setminus Z_1.
\]

Let \( \varepsilon_n \searrow 0 \) be any sequence. Extract some subsequence also denoted by \( \{ \varepsilon_n \} \) along which \( \eta_\varepsilon^n(B_z) \) converges to \( \xi_z \in [0, 1] \) for all \( z \in Z_1 \). Let \( Z_1' := \{ z \in Z_1 : \xi_z > 0 \} \). Also define

\[
\delta(\varepsilon) := \sup_{\varepsilon \leq \varepsilon_n} \max_{z \in S \setminus Z_1'} \eta_\varepsilon^n(B_z).
\]

By Theorem 1.1 (i) and the definition of \( Z_1' \) we have \( \delta(\varepsilon) \searrow 0 \). In the rest of the proof all limits or estimates are meant to be along the subsequence \( \{ \varepsilon_n \} \). By (3.73) we have

\[
\int_{B_z} \overline{R}_z[\tilde{v}_\varepsilon](x) \eta_\varepsilon^n(x) \, dx \geq \varepsilon^{2\nu-2} \eta_\varepsilon^n(B_z) (\mathcal{E}_+(Dm(z)) - \delta_1(\varepsilon)) \quad \forall z \in Z_1',
\]

where \( \delta_1(\varepsilon) \to 0 \) as \( \varepsilon \searrow 0 \). Let \( z^* \in Z_1' \). By the Cauchy–Schwartz inequality and Lemma 3.5 we obtain

\[
\sup_{\varepsilon \in (0, 1)} \int_{B_z} \frac{|\ell(x) - \ell(z)|}{\varepsilon^{2\nu}} \eta_\varepsilon^n(dx) \leq \sup_{\varepsilon \in (0, 1)} \left( \int_{B_z} \frac{(\ell(x) - \ell(z))^2}{\varepsilon^{2\nu}} \eta_\varepsilon^n(dx) \right)^{1/2} < \infty.
\]

It follows by (3.77), (3.78)–(3.79), and Proposition 3.1 that

\[
\beta_\varepsilon - \ell(z^*) \geq \sum_{z \in Z_1} \int_{B_z} \overline{R}_z[\tilde{v}_\varepsilon](x) \eta_\varepsilon^n(dx) + \sum_{z \in Z \setminus Z_1} \int_{B_z} (\ell(x) - \ell(z)) \eta_\varepsilon^n(dx)
\]

\[
+ \int_{B_z} (\ell(x) - \ell(z)) \eta_\varepsilon^n(dx)
\]

\[
\geq \varepsilon^{2\nu-2} \sum_{z \in Z_1'} \eta_\varepsilon^n(B_z) (\mathcal{E}_+(Dm(z)) - \delta_1(\varepsilon)) + O(\varepsilon') + O(\varepsilon^2).
\]

On the other hand, with \( \tilde{v}_{z^*}(x) = (M - \hat{Q}_z)(x - z^*) \), we have by (3.76) that

\[
\int_{R^d} \overline{R}_{z^*}[\tilde{v}_{z^*}](x) \rho_{\Sigma_{z^*}}^\varepsilon(x) \, dx \leq \varepsilon^{2\nu-2} (\mathcal{E}_+(Dm(z^*)) + O(\varepsilon^2)).
\]

Combining (3.80)–(3.81) and since \( \tilde{v}_{z^*} \) is suboptimal, we obtain

\[
\sum_{z \in Z_1'} \eta_\varepsilon^n(B_z) \mathcal{E}_+(Dm(z)) \leq \mathcal{E}_+(Dm(z^*)) + \delta_2(\varepsilon),
\]

with \( \delta_2(\varepsilon) \searrow 0 \). By the definition of \( \delta \) we have

\[
\sum_{z \in Z_1'} \eta_\varepsilon^n(B_z) \geq 1 - \delta(\varepsilon).
\]
Therefore, by (3.82)–(3.83) we obtain
\[
\sum_{z \in Z'_1} \eta^\varepsilon(B_z) \left[ \mathcal{E}_+(Dm(z)) - \mathcal{E}_+(Dm(z^*)) \right] \xrightarrow{\varepsilon \searrow 0} 0,
\]
which together with (3.83) and the definition of \(Z'_1\) implies that
\[
\eta^\varepsilon(B_z) \xrightarrow{\varepsilon \searrow 0} 0 \quad \forall z \in Z_1 \setminus Z'_1.
\]

Also, by (3.80) and (3.83) we have
\[
\beta^\varepsilon - \mathcal{I}_1 \geq \varepsilon^{2
u-2} \mathcal{E}_+(Dm(z^*)) + o(\varepsilon^{2
u - 2}) \quad \forall z^* \in Z'_1,
\]
which together with the upper bound in (3.81) implies (1.6), and the proof is complete. □

4. Concluding remarks

In general, Morse–Smale flows may contain hyperbolic closed orbits, and it would be desirable to extend the results of the paper accordingly. An energy function \(\mathcal{V}\) as in Theorem 2.1 may be constructed to account for critical elements that are closed orbits [20, 25]. Note that under the control used in Section 3.1.1 the optimal stationary probability distribution concentrates on the minimum of \(\mathcal{V}\). In the case that \(z \in \mathbb{R}^d\) belongs to a stable periodic orbit with period \(T_0\), we can construct \(\mathcal{V}\) so that it attains its minimum on this closed orbit. In this manner, if \(\phi_t\) denotes the flow of the vector field \(m\), then under the control used in Section 3.1.1 we obtain
\[
\int_{\mathbb{R}^d} \ell(x) \mu^\varepsilon(dx) \xrightarrow{\varepsilon \searrow 0} \frac{1}{T_0} \int_0^{T_0} \ell(\phi_t(z)) \, dt.
\]
The same can be done in the subcritical regime, by modifying the proof of Lemma 3.2. We leave it up to the reader to verify that Lemma 3.1 still holds if the set of critical elements \(S\) contains hyperbolic closed orbits. Let us define
\[
\ellbar(z) := \frac{1}{T_0} \int_0^{T_0} \ell(\phi_t(z)) \, dt,
\]
when \(z\) belongs to a closed orbit, and \(\ellbar(z) = \ell(z)\), when \(m(z) = 0\). Then, provided \(\text{Arg min}_{z \in S} \ellbar(z)\) contains only stable critical elements, then the support of the limit of the stationary probability distribution lies in \(S_s\), and this is true in any of the three regimes. However, the full analysis when unstable closed orbits are involved, seems to be much more difficult.

Appendix A. Proofs of Lemma 1.1 and Theorem 2.2

Proof of Lemma 1.1. Let \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) be a complete probability space, and \(\{W_t\}\) be a \(d\)-dimensional standard Brownian motion defined on it. Consider any admissible control \(U\). Let
\[
\tau_N := \inf \left\{ t \geq 0 : \int_0^t |U_s|^2 \, ds \geq N \varepsilon^{-1} \right\}, \quad N \geq 1.
\]
Since
\[
\mathbb{E}\left[ \int_0^T |U_s|^2 \, ds \right] < \infty \quad \forall T \in (0, \infty),
\]
it follows that $\tau_N \uparrow \infty$ a.s., as $N \to \infty$. Define

$$
\Lambda_N(t) := \exp \left( -\varepsilon^1 - \nu \int_0^{t \wedge \tau_N} \langle U_s, dW_s \rangle - \frac{\varepsilon^2 - 2\nu}{2} \int_0^{t \wedge \tau_N} |U_s|^2 \, ds \right), \quad t \in (0, \infty).
$$

Let $P_0$ and $E_0[\cdot]$ denote the law of the process with $U \equiv 0$ and the expectation operator under this law, respectively. In turn $P_N$ and $E_N$ denotes the measure defined by $dP_N/dP_0 = \Lambda_N(T)$ and the corresponding expectation, respectively. Then the laws of $(X \cdot \wedge \tau_N, U \cdot \wedge \tau_N)$ under $P_N$ and $P$ coincide. Therefore

$$
E_0[\Lambda_N(T) \ln(\Lambda_N(T))] = E_N[\ln(\Lambda_N(T))]
$$

$$
= E_N \left[ -\varepsilon^1 - \nu \int_0^{T \wedge \tau_N} \langle U_s, dW_s \rangle - \frac{\varepsilon^2 - 2\nu}{2} \int_0^{T \wedge \tau_N} |U_s|^2 \, ds \right]
$$

$$
= \frac{1}{2} E_N \left[ \varepsilon^2 - 2\nu \int_0^{T \wedge \tau_N} |U_s|^2 \, ds \right]
$$

$$
\leq \frac{1}{2} E \left[ \varepsilon^{-2\nu} \int_0^T |U_s|^2 \, ds \right]
$$

$$
< \infty.
$$

The third equality follows from the fact that under $(\Omega, \mathcal{F}_t \wedge \tau_N, P_N)$,

$$
W_t + \int_0^{t \wedge \tau_N} \varepsilon^{-\nu} U_s \, ds, \quad t \geq 0,
$$

is a Brownian motion stopped at $\tau_N$. Since $x(\ln(x))^+ \leq x \ln(x) + e^{-1}$, it follows that

$$
\sup_N E_0[\Lambda_N(T)(\ln(\Lambda_N(T)))^+] < \infty.
$$

By [11, Theorem 1.3.4, p. 10], $\{\Lambda_N(T \wedge \tau_N)\}$ are uniformly integrable and converge a.s. and in $L^1$ to $\Lambda(T)$, which then satisfies $E_0[\Lambda(T)] = 1$. Defining the probability measure $P'$ by $dP'/dP_0 = \Lambda(T)$, we obtain

$$
W_t + \int_0^t \varepsilon^{-\nu} U_s \, ds, \quad t \geq 0,
$$

as a $(\Omega, \mathcal{F}_t, P')$ Brownian motion. This gives a weak solution for the control $U_s$ over $[0, T]$ under the probability measure $P_0$. To show the weak uniqueness we see that given two solutions we can always define $P'$ as above and get two weak solutions of (1.1) with $U = 0$ under the law of $P'$. But we have weak uniqueness in the latter. Hence weak uniqueness follows.

The rest of this section is devoted to the proof of Theorem 2.2. Without loss of generality we fix $\varepsilon = 1$. We consider the equation

$$
\frac{1}{2} \Delta V + \min_{u \in \mathbb{R}^d} \left[ (m + u, \nabla V) + \ell + \frac{1}{2} |u|^2 \right] = \beta.
$$

If $V$ is regular then the above equation can be rewritten as

$$
\ell - \frac{1}{2} |\nabla V|^2 = \beta - \frac{1}{2} \Delta V - m \cdot \nabla V.
$$

(A.1)
It is shown in [3] that there exists a unique pair \((V, \beta) \in C^2(\mathbb{R}) \times \mathbb{R}\) satisfying (A.1) such that \(V(x) \to \infty \) as \(|x| \to \infty\). The existence of this solution is established as a limit of the solution \(V_\alpha\) to the discounted problem [2, p. 175]

\[
\ell - \frac{1}{2} |\nabla V_\alpha|^2 = \alpha V_\alpha - \frac{1}{2} \Delta V_\alpha - m \cdot \nabla V_\alpha. \tag{A.2}
\]

In [2, Theorem 4.18, p. 177] it is proved that (A.2) has a solution in \(W^{2,s}_{loc}(\mathbb{R}^d), 1 \leq s < \infty\).

Then using standard elliptic pde results we can improve the solution so that \(V_\alpha \in C^2(\mathbb{R}^d)\).

The solution \(V_\alpha\) to (A.2) is obtained in [2] as a strong \(W^{1,2}_{loc}(\mathbb{R}^d)\) limit of the solution to the Neumann boundary value problem

\[
\ell - \frac{1}{2} |\nabla v_R|^2 = \alpha v_R - \frac{1}{2} \Delta v_R - m \cdot \nabla v_R, \quad \text{in } B_R(0), \tag{A.3}
\]

\[
\frac{\partial v_R}{\partial n} = 0 \quad \text{on } \partial B_R(0),
\]

as \(R \to \infty\) along some sub-sequence. Here \(n\) denotes the unit outward normal on \(\partial B_R(0)\).

In fact, there exists \(v_R \in L^\infty(B_R(0)) \cap W^{1,2}(B_R(0))\) solving the Neumann problem. In the following lemma we obtain an estimate of the growth of \(V_\alpha\).

**Lemma A.1.** For every \(\alpha > 0\), the solution \(V_\alpha\) of (A.2), which is obtained as a limit of \(v_R\) as \(R \to \infty\), is bounded below and has at most linear growth.

**Proof.** The proof mimics the calculations in [3]. By the proof of [3, Theorem 4.18] there exist constants \(\kappa_0\) and \(R_0\) which do not depend on \(\alpha\) such that

\[
\alpha v_R(x) \geq -\kappa_0 \quad \forall x \in B_R(0), \quad \forall R > R_0, \tag{A.4}
\]

and for all \(\alpha \in (0, 1)\). We divide the proof in a number of steps.

**Step 1.** The calculation here is based on the computations done in [3, pp. 179–180] (mentioned as Local bound from above). Choose \(0 < 4r < 1\). Consider a smooth function \(\psi\) such that \(\psi = 1\) on \(B_{2r}(x_0)\), \(\psi = 0\) outside of \(B_{4r}(x_0)\) and \(0 \leq \psi \leq 1\). It is clear that we can choose \(\psi\) such that \(|\nabla \psi| \leq c_r\), where \(c_r\) does not depends on \(x_0\) as we can translate this \(\psi\) to any point. Now we choose \(x_0\) in \(B_R(0)\) such that \(|x_0| + 4r < R\).

Multiplying (A.3) with \(\psi^2\) and integrating on \(B_R(0)\) we have (see [2, p. 179])

\[
\alpha \int_{B_R} v_R \psi^2 + \int_{B_R} (\nabla v_R \cdot \nabla \psi) \psi - \int_{B_R} (m \cdot \nabla v_R) \psi^2 + \frac{1}{2} \int_{B_R} |\nabla v_R|^2 \psi^2 = \int_{B_R} \ell \psi^2,
\]

which can also be written as

\[
\alpha \int_{B_{4r}(x_0)} v_R \psi^2 + \int_{B_{4r}(x_0)} (\nabla v_R \cdot \nabla \psi) \psi - \int_{B_{4r}(x_0)} (m \cdot \nabla v_R) \psi^2 + \frac{1}{2} \int_{B_{4r}(x_0)} |\nabla v_R|^2 \psi^2 = \int_{B_{4r}(x_0)} \ell \psi^2. \tag{A.5}
\]
Applying Young’s inequality to the second and third terms of (A.5) and using (A.4) we obtain
\[
\int_{B_{2r}(x_0)} |\nabla v_R|^2 \leq \int_{B_{4r}(x_0)} |\nabla v_R|^2 \psi^2
\]
\[
\leq 4 \left( \int_{B_{4r}(x_0)} \ell \psi^2 + \int_{B_{4r}(x_0)} |\nabla \psi - m\psi|^2 - \alpha \int_{B_{4r}(x_0)} v_R \psi^2 \right)
\]
\[
\leq \kappa_1(r) \left( 1 + \int_{B_{4r}(x_0)} \ell \psi^2 \right) \quad \forall \alpha \in (0, 1),
\]
and all \( R \) sufficiently large, where \( \kappa_1(r) \) is a constant depending on \( \|m\|_\infty, \kappa_0, \) and \( r \). Since \( \ell \) is nonnegative and Lipschitz continuous we have
\[
\ell(x) \leq \kappa_2(1 + |x|) \leq \kappa_2(1 + |x_0| + |x - x_0|).
\]
Therefore for some constant \( \kappa_3(r) \), independent of \( R \), we have
\[
\int_{B_{2r}(x_0)} |\nabla v_R|^2 \leq \kappa_3(r) \left( 1 + |x_0| \right) \quad \forall \alpha \in (0, 1), \quad (A.6)
\]
and all \( R \) sufficiently large.

**Step 2.** Let us first consider \( d = 1 \). Then \( v_R \) is absolutely continuous. Since \( v_R \to V_\alpha \) strongly in \( W^{1,2}_{loc}(\mathbb{R}) \) there exists a point, say \( 0 \), such that \( v_R(0) \to V_\alpha(0) \) as \( R \to \infty \). Thus \( |v_R(0)| \) is bounded. Now let \( x \in \mathbb{R} \) and choose \( R \) such that \( |x| + 4r < R \). Let \( N \) be such that \( 4rN > |x| > 4r(N - 1) \). We have
\[
|v_R(x)| \leq |v_R(0)| + \left| \int_0^x v'_R(s) \, ds \right|
\]
\[
\leq |v_R(0)| + \int_0^{4rN} |v'_R(s)| \, ds
\]
\[
= |v_R(0)| + \sum_{i=0}^{N-1} \int_{4ri}^{4r(i+1)} |v'_R(s)| \, ds
\]
\[
\leq |v_R(0)| + \sqrt{4r} \sum_{i=0}^{N-1} \left( \int_{4ri}^{4r(i+1)} |v'_R(s)|^2 \, ds \right)^{1/2}.
\]
So if we use the estimate (A.6) then it follows that
\[
|v_R(x)| \leq |v_R(0)| + \sqrt{4rN} \left( \kappa_3(r)(2 + |x|) \right)^{1/2}
\]
\[
\leq |v_R(0)| + \frac{\sqrt{4r(|x| + 4r)}}{4r} \left( \kappa_3(r)(2 + |x|) \right)^{1/2}
\]
\[
\leq \kappa_4(r)(1 + |x|^2).
\]
Here, $\kappa_4(r)$ does not depend on $R$. So in the limit we obtain $|V_\alpha(x)| \leq \kappa_4(r)(1 + |x|^2)$ and that $V_\alpha$ is bounded from below. Now rewriting (A.2) as

$$\frac{1}{2} \Delta V_\alpha + \min_{u \in \mathbb{R}^d} \left[ (m + u, \nabla V_\alpha) + \ell + \frac{1}{2} |u|^2 \right] = \alpha V_\alpha,$$

and using the fact that the control $U \equiv 0$ is sub-optimal, we obtain

$$V_\alpha(x) \leq \mathbb{E}_x \left[ \int_0^\ell e^{-\alpha s} \ell(X_0^x) \, ds \right] + \mathbb{E}_x \left[ e^{-\alpha t} V_\alpha(X_t^0) \right]. \quad (A.7)$$

In (A.7), $X^0$ denotes the solution to (1.1) for $U \equiv 0$. It is straightforward to show using Hypothesis 1.1 that

$$\mathbb{E}_x \left[ \int_0^\infty e^{-\alpha s} \ell(X_0^x) \, ds \right] \leq \kappa_5(1 + |x|),$$

for a constant $\kappa_5$ which depends on $\alpha$, and that $\mathbb{E}_x [|X_t^0|^2 e^{-\alpha t}] \to 0$ as $t \to \infty$. Therefore from (A.7), using also the fact that $V_\alpha$ is bounded from below, we conclude that $V_\alpha$ has at most linear growth.

**Step 3.** Let $d \geq 2$. We can use the Green function in this case, as used in [3, p. 179]. Again we consider $x_0 \in B^R(0)$ such that $|x_0| + 4r < R$. We consider a Green function $G^{x_0}(x)$ on $B_{2r}(x_0)$. We also use the estimates in [3, (4.83), p. 179] for Green functions where the constants depend only on $r$. Now we consider a test function $\eta$ such that $\eta = 1$ on $B_r(x_0)$, $\eta = 0$ outside of $B_{3r/2}(x_0)$, and $0 \leq \eta \leq 1$.

Now we test (A.3) with $\eta^2 G$ and we have (the first expression on [3, p. 180])

$$v_R(x_0) + \alpha \int_{B_{2r}(x_0)} v_R \eta^2 G + \frac{1}{2} \int_{B_{2r}(x_0)} \nabla v_R \cdot \nabla \eta^2 G + \frac{1}{2} \int_{B_{2r}(x_0)} \text{div}(v_R \nabla \eta^2 G)$$

$$- \int_{B_{2r}(x_0)} m \cdot \nabla v_R \eta^2 G = \frac{1}{2} \int_{B_{2r}(x_0)} |\nabla v_R|^2 \eta^2 G + \int_{B_{2r}(x_0)} \ell \eta^2 G.$$

At this point we observe that because of the choice of $r$, $G$ is positive on $B_{2r}(x_0)$ (for $d = 2$ we use that $2r < 1$, see [3, p. 73]). Also $m \cdot \nabla v_R \leq |m|^2 + \frac{1}{2} |\nabla v_R|^2$. Therefore

$$v_R(x_0) + \alpha \int_{B_{2r}(x_0)} v_R \eta^2 G + \frac{1}{2} \int_{B_{2r}(x_0)} \nabla v_R \cdot \nabla \eta^2 G + \frac{1}{2} \int_{B_{2r}(x_0)} \text{div}(v_R \nabla \eta^2 G)$$

$$\leq \int_{B_{2r}(x_0)} |m|^2 \eta^2 G + \int_{B_{2r}(x_0)} \kappa_2(1 + |x_0| + 2r) \eta^2 G.$$

Now we observe that $\nabla \eta$ is nonzero for $|x - x_0| \geq r$ and $x \in B_{3r/2}(x_0)$ where we can bound $G$ (from (4.83) of [3]). Again using the lower bound for $v_R$ and the gradient estimate in (A.6) and the $L^1$ bound of $G$ (see [3, (4.38)]) we have

$$v_R(x_0) \leq \kappa_6(r)(1 + |x_0|).$$

Here, the constant $\kappa_6(r)$ does not depend on $R$ and $x_0$ (Observe that one can translate $\eta$ and $G$ so that the bounds on $\nabla \eta$, $\nabla^2 \eta$, and $G$ do not change as $x_0$ varies). Hence passing to the limit as $R \to \infty$, we obtain $V_\alpha(x) \leq \kappa_6(r)(1 + |x|)$ and that $V_\alpha$ is bounded from below. This concludes the proof. □
Lemma A.2. For every $\alpha > 0$, $V_\alpha(x) \to \infty$ as $|x| \to \infty$. Moreover, there exists a compact set $B$ such that $\min_{\mathbb{R}^d} V_\alpha = \min_B V_\alpha$ for all $\alpha \in (0, 1)$.

Proof. Since $V_\alpha$ is bounded from below we may add a suitable constant to both sides of (A.2) and assume that $V_\alpha \geq 0$ ($\ell$ is translated by a constant accordingly). Define for $x_0 \in \mathbb{R}^d$,

$$\zeta(x) := c_1(1 - |x - x_0|^2), \quad \text{for } |x - x_0| \leq 1,$$

where $c_1$ is a constant to be chosen later. Let $\theta(x) = V_\alpha(x) - \zeta(x)$ in $B_1(x_0)$. Then in $B_1(x_0)$ we have

$$-\frac{1}{2} \Delta \theta - m \cdot \nabla \theta + \alpha \theta = -\frac{1}{2} \Delta V_\alpha - m \cdot \nabla V_\alpha + \alpha V_\alpha + \frac{1}{2} \Delta \zeta + m \cdot \nabla \zeta - \alpha \zeta$$

$$= -\frac{1}{2} |\nabla V_\alpha|^2 + \ell + \frac{1}{2} c_1(-2d) - 2c_1m \cdot (x - x_0) - \alpha \zeta.$$

Therefore in $B_1(x_0)$

$$-\frac{1}{2} \Delta \theta - m \cdot \nabla \theta + \frac{1}{2} (\nabla V_\alpha + \nabla \zeta) \cdot \nabla \theta + \alpha \theta = -\frac{1}{2} |\nabla \zeta|^2 + \ell - dc_1 - 2c_1m \cdot (x - x_0) - \alpha \zeta$$

$$\geq \ell - 2c_1^2 - M_0 c_1 - \alpha c_1,$$

where $M_0$ is a constant that depends on the bound of $m$. Since $\ell(x) \to \infty$ as $|x| \to \infty$ we can choose $c_1 = c_1(|x_0|)$ such that $c_1(|x_0|) \to \infty$ as $|x_0| \to \infty$ and

$$\ell(x) - 2c_1^2(x) - M_0 c_1(x) - \alpha c_1(x) > 0 \quad \forall x \in B_1(x_0).$$

Therefore

$$-\Delta \theta - m \cdot \nabla \theta + \frac{1}{2} (\nabla V_\alpha + \nabla \zeta) \cdot \nabla \theta + \alpha \theta > 0 \quad \text{in } B_1(x_0).$$

Since $\theta \geq 0$ on $\partial B_1(x_0)$ and $\alpha > 0$ applying the maximum principle we have $\theta \geq 0$ in $B_1(x_0)$. In particular, $V_\alpha(x_0) \geq c_1(|x_0|)$. This proves that $V_\alpha(\hat{x}) \to \infty$ as $|x| \to \infty$.

In particular, $V_\alpha$ attains its minimum at some $\hat{x} \in \mathbb{R}^d$. Also $\Delta V_\alpha(\hat{x}) \geq 0$. Thus from (A.2) we obtain

$$\ell(\hat{x}) = \alpha V_\alpha(\hat{x}) - \frac{1}{2} \Delta V_\alpha(\hat{x}) \leq \alpha V_\alpha(\hat{x}). \quad (A.8)$$

Let $\eta_0$ be the invariant probability measure corresponding to the control $U \equiv 0$. Then $\int_{\mathbb{R}^d} \ell \, d\eta_0 \leq \kappa$ for some constant $\kappa$. Applying Lemma A.3 we obtain

$$V_\alpha(\hat{x}) \leq \int V_\alpha \, d\eta_0 \leq \int_{\mathbb{R}^d} \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha s} \ell(X_s^0) \, ds \right] \, d\eta_0 \leq \frac{\kappa}{\alpha}.$$

Therefore using (A.8) we have $\ell(\hat{x}) \leq \kappa$. The result follows using the fact that $\ell(x) \to \infty$ as $|x| \to \infty$. \qed

Lemma A.3. For all $x \in \mathbb{R}^d$,

$$V_\alpha(x) = \inf_{U \in \mathcal{A}} \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right],$$

where $\mathbb{E}_\alpha$ denotes the expectation under the law of $(X, U)$.
Proof. From Lemma A.1 we have \(|V_\alpha(x)| \leq M_1 + M_2|x|\) and \(V_\alpha\) is bounded from below. Let us write (A.2) in the form of
\[
\frac{1}{2} \Delta V_\alpha + \min_{u \in \mathcal{U}} \left[ (m + u, \nabla V_\alpha) + \ell + \frac{1}{2} |u|^2 \right] = \alpha V_\alpha.
\] (A.9)

Let \(U \in \mathcal{U}\) be any admissible control such that
\[
\mathbb{E}_x^U \left[ \int_0^\infty e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) ds \right] < \infty,
\]
and
\[
X_t = x + \int_0^t m(X_s) ds + \int_0^t U_s ds + W_t, \quad t \geq 0.
\]
Since \(\mathbb{E}_x^U \left[ \int_0^t |U_s|^2 ds \right]\) is finite it is easy to see that
\[
\mathbb{E}_x^U \left[ \sup_{0 \leq s \leq t} |X(s)| \right] \leq C \left( |x| + t + \mathbb{E}_x^U \left[ \int_0^t |U_s| ds \right] \right) < \infty,
\]
Therefore multiplying both sides by \(e^{-\alpha t}\) we get
\[
e^{-\alpha t} \mathbb{E}_x^U [|X_t|] \leq C(|x| + t)e^{-\alpha t} + C e^{-\frac{\alpha}{2}t} \mathbb{E}_x^U \left[ \int_0^t e^{-\frac{\alpha}{2}s} |U_s| ds \right]
\]
\[
\leq C(|x| + t)e^{-\alpha t} + C \sqrt{t} e^{-\frac{\alpha}{2}t} \mathbb{E}_x^U \left[ \left( \int_0^t e^{-\alpha s} |U_s|^2 ds \right)^{1/2} \right]
\]
\[
\leq C(|x| + t)e^{-\alpha t} + C \sqrt{t} e^{-\frac{\alpha}{2}t} \mathbb{E}_x^U \left[ \int_0^t e^{-\alpha s} |U_s|^2 ds \right]^{1/2} \rightarrow 0. \quad t \to \infty.
\]
Now applying Itō’s formula to (A.9) we have
\[
V_\alpha(x) - e^{-\alpha t} \mathbb{E}_x^U [V_\alpha(X_t)] \leq \mathbb{E}_x^U \left[ \int_0^t e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) ds \right].
\]
Hence using Lemma A.1 we have
\[
V_\alpha(x) \leq \mathbb{E}_x^U \left[ \int_0^\infty e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) ds \right]. \quad (A.10)
\]
Since (A.10) holds for all \(U \in \mathcal{U}\), the desired upper bound to \(V_\alpha\) is established.

To show the converse inequality let \(v_\alpha\) be a measurable selector from the minimizer of (A.9). With \(\tau_R\) denoting the first exit time from a ball of radius \(R\) we then have
\[
V_\alpha(x) = \mathbb{E}^v_\alpha \left[ \int_0^{\tau_R} e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |v_\alpha(X_s)|^2 \right) ds \right] + \mathbb{E}^v_\alpha \left[ e^{-\alpha(\tau_R)} V_\alpha(X_{\tau_R}) \right]. \quad (A.11)
\]
We claim that \(\tau_R \to \infty\), \(\mathbb{P}^v_\alpha\)-a.s. Suppose that \(\tau_R \not\to \infty\) as \(R \to \infty\) in \(\mathbb{P}^v_\alpha\). Then there exists \(t_0 > 0\) and \(\varepsilon_0 > 0\) such that \(\mathbb{P}^v_\alpha(\tau_R < t_0) > \varepsilon_0\) for all \(R > 0\). Let \(R_0 > 0\) be such that
\[
\min_{\partial B_{R_0}} V_\alpha > \frac{2}{\varepsilon_0} e^{\alpha t_0} V_\alpha(x).
\]
Such $R_0$ exists by Lemma A.2. Since $\ell$ is nonnegative, we obtain
\[
\mathbf{E}_x^{\nu_\alpha} \left[ e^{-\alpha(t_0 \wedge \tau_{R_0})} V_\alpha(X_{t_0 \wedge \tau_{R_0}}) \right] > \varepsilon_0 e^{-\alpha t_0} \min_{\beta B_{R_0}} V_\alpha \\
> 2V_\alpha(x),
\]
which contradicts (A.11), since $\ell$ is nonnegative. Therefore $\tau_R \to \infty$, as $R \to \infty$, in probability, and therefore also $\mathbf{E}_x^{\nu_\alpha}$-a.s., since it is monotone in $R$.

Taking limits in (A.11) as $R \to \infty$, and applying Fatou’s lemma, we obtain
\[
V_\alpha(x) \geq \mathbf{E}_x^{\nu_\alpha} \left[ \int_0^T e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} \left| V_\alpha(X_s) \right|^2 \right) ds \right] + \mathbf{E}_x^{\nu_\alpha} \left[ e^{-\alpha T} V_\alpha(X_T) \right],
\]
and taking limits once more as $t \to \infty$, using the fact that $V_\alpha$ is bounded below, we have
\[
V_\alpha(x) \geq \mathbf{E}_x^{\nu_\alpha} \left[ \int_0^\infty e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} \left| V_\alpha(X_s) \right|^2 \right) ds \right]. \tag{A.12}
\]
By (A.10) we must have equality in (A.12). Moreover (A.10) and (A.12) imply that every measurable selector from the minimizer of (A.9), and in particular the control $U_t = -\nabla V_\alpha(X_t)$, is optimal for the $\alpha$-discounted problem and that $V_\alpha$ is the associated value function. Since the solution of (1.1) under the control $U_t = -\nabla V_\alpha(X_t)$ is regular, and since the drift $m - \nabla V_\alpha$ is locally Lipschitz, it follows that (1.1) under the optimal $\alpha$-discounted control $U_t = -\nabla V_\alpha(X_t)$ has a unique strong solution. \hfill \square

The following is a special case of the Hardy–Littlewood theorem [26, Theorem 2.2].

**Lemma A.4.** Let $\{a_n\}$ be a sequence of non-negative real numbers. Then
\[
\limsup_{\beta > 1} (1 - \beta) \sum_{n=1}^{\infty} \beta^n a_n \leq \limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a_n.
\]

Concerning the proof of Lemma A.4, note that if the right hand side of the above display is finite then the set $\{\frac{a_n}{n}\}$ is bounded. Therefore $\sum_{n=1}^{\infty} \beta^n a_n$ in finite for every $\beta < 1$. Hence we can apply [26, Theorem 2.2] to obtain the result.

Continuing, we define
\[
\beta_* := \inf_{U \in \mathcal{U}} \limsup_{T \to \infty} \frac{1}{T} \mathbf{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} \left| U_s \right|^2 \right) ds \right].
\]
To be more precise we should define $\beta_*$ as a function of the initial condition $x$. But we shall see that the infimum does not depend on the initial condition.

**Lemma A.5.** Let $\beta$ be given by (A.1). Then $\beta \leq \beta_*$. 

**Proof.** Fix $x \in \mathbb{R}^d$. Let $U \in \mathcal{U}$ be such that
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbf{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} \left| U_s \right|^2 \right) ds \right] < \infty.
\]
It is easy to see that
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbf{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} \left| U_s \right|^2 \right) ds \right] = \limsup_{N \to \infty} \frac{1}{N} \mathbf{E}_x \left[ \int_0^N \left( \ell(X_s) + \frac{1}{2} \left| U_s \right|^2 \right) ds \right],
\]
where \( N \) runs over the set of natural numbers. Define
\[
a_n := \mathbb{E}_x \left[ \int_{n-1}^n \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right], \quad n \geq 1.
\]

Therefore applying Lemma A.4 we have
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \limsup_{\beta \nearrow 1} (1 - \beta) \sum_{n=1}^{\infty} \beta^n a_n. \quad (A.13)
\]

For \( \alpha > 0 \), define \( \beta = e^{-\alpha} \) and thus we have
\[
\sum_{n=1}^{\infty} \beta^n a_n \geq e^{-\alpha} \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{n-1}^n e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] = e^{-\alpha} \mathbb{E}_x \left[ \int_0^{\infty} e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right].
\]

Therefore combining (A.13) with Lemma A.3 we obtain
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \limsup_{\alpha \searrow 0} (1 - e^{-\alpha}) e^{-\alpha} V_\alpha(x).
\]

Since \( \alpha V_\alpha(x) \to \beta \) as \( \alpha \searrow 0 \) from [2] we obtain from above that
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \beta.
\]

Therefore applying Lemma A.4 we have
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \limsup_{\beta \nearrow 1} (1 - \beta) \sum_{n=1}^{\infty} \beta^n a_n. \quad (A.13)
\]

For \( \alpha > 0 \), define \( \beta = e^{-\alpha} \) and thus we have
\[
\sum_{n=1}^{\infty} \beta^n a_n \geq e^{-\alpha} \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{n-1}^n e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] = e^{-\alpha} \mathbb{E}_x \left[ \int_0^{\infty} e^{-\alpha s} \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right].
\]

Therefore combining (A.13) with Lemma A.3 we obtain
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \limsup_{\alpha \searrow 0} (1 - e^{-\alpha}) e^{-\alpha} V_\alpha(x).
\]

Since \( \alpha V_\alpha(x) \to \beta \) as \( \alpha \searrow 0 \) from [2] we obtain from above that
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_s) + \frac{1}{2} |U_s|^2 \right) \, ds \right] \geq \beta.
\]

\( U \in \mathcal{U} \) being arbitrary this concludes the proof. \( \square \)

**Proof of Theorem 2.2.** From Lemma A.5 we have \( \beta \leq \beta^* \) where \( \beta \) is given by (A.1). It is easy to see that \( v^* = -\nabla V \) is a minimizing selector of
\[
\frac{1}{2} \Delta V + \min_{u \in \mathbb{R}^d} \left[ \langle m + u, \nabla V \rangle + \ell + \frac{1}{2} |u|^2 \right] = \beta,
\]
and the HJB takes the following form
\[
\frac{1}{2} \Delta V + \langle m, \nabla V \rangle - \frac{1}{2} |\nabla V|^2 + \ell = \beta.
\]

By [19, Lemma 5.1] we have
\[
|\nabla V(x)| \leq C_1 + C_2 |x|
\]
for some constants \( C_1, C_2 \). Since \( v_* \) is locally Lipschitz with at most linear growth there exists a unique strong solution to (1.1) with \( U = v_* \). Applying Itô’s formula to \( V \) we have
\[
\mathbb{E}_x \left[ V(X_{T \wedge \tau_R}) \right] - V(x) = \mathbb{E}_x \left[ \int_0^{T \wedge \tau_R} \left( \beta - \ell(X_s) - \frac{1}{2} |\nabla V(X_s)|^2 \right) \, ds \right],
\]
where \( \tau_R \) denotes the exit time from the ball of radius \( R > 0 \) around 0. Since \( V \) is bounded from below and \( \tau_R \to \infty \) as \( R \to \infty \) we have
\[
\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x \left[ \int_0^T \left( \ell(X_t) + \frac{1}{2} |\nabla V(X_t)|^2 \right) \, dt \right] \leq \beta.
\]
Hence $\beta = \beta_*$. Also this proves that $v_*$ is an optimal stable Markov control and by the ergodic theorem

$$\beta = \int_{\mathbb{R}^d} \left( \ell(x) + \frac{1}{2} |\nabla V(x)|^2 \right) d\eta_* ,$$

where $\eta_*$ is the invariant probability measure corresponding to the control $v_*$. \hfill \Box

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