The influence of charge detection on counting statistics

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Abstract. We consider the counting statistics of electron transport through a double quantum dot with special emphasis on the dephasing induced by a nearby charge detector. The double dot is embedded in a dissipative environment, and the presence of electrons on the double dot is detected with a nearby quantum point contact. Charge transport through the double dot is governed by a non-Markovian generalized master equation. We describe how the cumulants of the current can be obtained for such problems, and investigate the difference between the dephasing mechanisms induced by the quantum point contact and the coupling to the external heat bath. Finally, we consider various open questions of relevance to future research.

Keywords: dissipative systems (theory), quantum dots (theory), stochastic processes (theory)
1. Introduction

The study of random fluctuations has a relevant role in many branches of physics [1]–[4]. Close to equilibrium, fluctuations are intimately connected with dissipative relaxation mechanisms according to the fluctuation-dissipation theorem, independently of the physical origin of the fluctuations, classical or quantum mechanical [5,6]. In contrast, far from equilibrium fluctuating quantities provide a unique insight into the internal properties of the system under consideration [7]. An immediate example is the evaluation of the quasi-particle charge of carriers through measurements of the current cumulants [8].

Many important phenomena can be characterized in terms of counted, elementary entities. The concept of particles, in the quantum realm, naturally defines what quantities should be counted. From this point of view one recognizes that counting problems constitute a quite general framework in which many different dynamical processes can be interpreted, also in the presence of complex quantum physics. The first application of the counting approach in quantum physics came from photon counting experiments, where the concept of full counting statistics (FCS) was originally developed [9]. Recently, this concept has attracted intensive theoretical [4] and experimental [10]–[12] attention within the field of electron transport. In the context of mesoscopic transport, FCS was introduced in order to characterize the noise properties of nanodevices [13]. Later, it was demonstrated also to be a sensitive diagnostic tool for detecting quantum-mechanical coherence, entanglement, disorder, and dissipation [4].

Mathematically, FCS encodes the complete knowledge of the probability distribution $P(n,t)$ of the number $n$ of transmitted entities during the measurement time $t$ or, equivalently, of all corresponding cumulants. The study of counting statistics for stochastic processes is generally of broad relevance for a wide class of problems. For example, non-zero higher-order cumulants provide a description of non-Gaussian behavior and contain information about rare events, whose study has become an important topic within non-equilibrium statistics in physics, chemistry, and biology [6,14,15]. Within the framework of master equations some important results were recently obtained. Bagrets and Nazarov [16] have shown that the cumulant generating function (CGF) corresponding to a Markovian master equation is determined by the dominating eigenvalue of the rate matrix, when counting fields are appropriately included. Some of us have shown that it in
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principle is possible to calculate arbitrary orders of cumulants using perturbation theory in the counting field, rather than solving the full eigenvalue problem for the dominating eigenvalue [17]. For non-Markovian systems described by generalized master equations (GME), we have shown that the CGF scales linearly with time [18,19], as in the case of Markovian processes, if the memory kernel has no power law tails. Moreover, the CGF can be calculated using a so-called non-Markovian expansion [18].

Recently, we developed a method which unifies and extends these earlier approaches to FCS within a GME formulation [19]. Due to their intrinsic analytic structures, the previous approaches were in practice limited to systems with only a few states [16,18], or only the first few current cumulants could be addressed [17]. In contrast, this recent advancement enables studies of a much larger class of problems, including the evaluation of zero-frequency current cumulants of very high orders for non-Markovian systems with many states [19]. We also showed how the method allows calculations of the finite-frequency current noise for non-Markovian transport processes [19,20]. A detailed account of these techniques will be given elsewhere [21]. In this paper we mainly want to restate the essential findings and address the open questions of relevance in future research.

In order to demonstrate the applicability of our methods, we consider the current fluctuations of charge transport through a coherently coupled double quantum dot, a charge qubit. We consider the effects of a nearby quantum point contact (QPC) charge detector and the coupling to a dissipative phonon bath. If the QPC barrier height is modulated by electrons on the double quantum dot, the fluctuations of the current through the QPC can be used to monitor the dynamics of the qubit. This introduces a qubit dephasing mechanism. As we shall see, current fluctuations can be useful for extracting information about the internal dynamics of the double-dot system. We concentrate on the transition between coherent and incoherent transport through the qubit, demonstrating the sensitivity of the cumulants to this transition.

The structure of the paper is as follows. In section 2 we summarize the general concepts of our method, while clearly identifying the essential steps for obtaining the FCS or the current cumulants for a system governed by a non-Markovian GME. We briefly sketch the derivation of the expressions for the first few current cumulants (current, noise, and skewness) used in the following section. In section 3 we describe the model of a dissipative qubit with a nearby QPC charge detector and show how the current cumulants yield information about the dynamics of the qubit. In section 4 we discuss various open questions and give an outlook for future research.

2. Non-Markovian GME

For many nanoscopic systems it is convenient to consider the evolution of just a few degrees of freedom. The system evolution is then captured by the dynamic equation for the reduced density matrix corresponding to these degrees of freedom. This equation should contain the effects of all external forces driving the system and, at the same time, the effective dynamics due to the degrees of freedom that have been traced out. The dynamics of the reduced system is, in general, non-Markovian, and can for a large class of processes be described by a generic non-Markovian GME of the form [7,19,22]

$$\frac{d}{dt}\hat{\rho}(n,t) = \sum_{n'} \int_0^t dt' \mathcal{W}(n-n',t-t')\hat{\rho}(n',t')+\hat{\gamma}(n,t).$$

(1)
Here, the reduced density matrix $\hat{\rho}(n,t)$ of the system has been resolved with respect to the number of transferred charges $n$. The memory kernel $\mathcal{W}$ describes the influence of the environment on the dynamics of the system, while the inhomogeneity $\hat{\gamma}$ accounts for initial correlations between system and environment$^4$. For a given system, the derivation of an equation like equation (1) may be a difficult task, but many different examples can already be found in the literature [16, 23, 24]. Both $\mathcal{W}$ and $\hat{\gamma}$ decay with time, usually on comparable timescales. We consider systems where $\mathcal{W}$ and $\hat{\gamma}$ decay with time faster than any power law. With this condition it can be shown that the effects of the inhomogeneity vanish in the long-time limit. The inhomogeneity $\hat{\gamma}$ is consequently irrelevant for all statistical quantities in the long-time limit. For finite times, it may, however, play a crucial role [19].

The cumulant generating function $S(\chi, t)$ corresponding to $P(n, t)$ is defined as

$$e^{S(\chi, t)} = \sum_n P(n, t)e^{in\chi} = \sum_n \text{Tr}\{\hat{\rho}(n, t)\}e^{in\chi},$$

where $\chi$ is the so-called counting field. The second equality defines $P(n, t)$ as the trace of the $n$-resolved reduced density matrix. The $m$th cumulant $\langle\langle n^m\rangle\rangle(t)$ is directly connected to the Taylor coefficients of the CGF in equation (2) according to the definition $\langle\langle n^m\rangle\rangle(t) \equiv \partial^m S(\chi, t)/\partial(\chi)^m|_{\chi=0}$. We now derive a general expression for the CGF of a system described by a GME of the form given in equation (1). In Laplace space, defined by the transform $f(\chi, z) \equiv \int_0^\infty dt f(n, t)e^{i\chi t}$, the equation has the algebraic form

$$z\hat{\rho}(\chi, z) - \hat{\rho}(\chi, t = 0) = \mathcal{W}(\chi, z)\hat{\rho}(\chi, z) + \hat{\gamma}(\chi, z),$$

which can formally be solved for $\hat{\rho}(\chi, z)$ by introducing the resolvent $\mathcal{G}(\chi, z) \equiv [z - \mathcal{W}(\chi, z)]^{-1}$. On returning to the time domain by an inverse Laplace transformation, the CGF becomes

$$e^{S(\chi, t)} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} dz \text{Tr}\{\mathcal{G}(\chi, z)[\hat{\rho}(\chi, t = 0) + \hat{\gamma}(\chi, z)]\} e^{zt},$$

where $a$ is a real number, chosen such that all singularities of the integrand are situated to the left of the vertical line of integration. This expression constitutes a powerful formal result, but as we shall see in the following, it also leads to useful practical schemes.

As already mentioned above, the CGF scales linearly with time in the long-time limit for kernels that decay faster than any power law and is independent of the initial conditions [18]. For such systems we can define the zero-frequency cumulants of the current as $\langle\langle I^m\rangle\rangle = d\langle\langle n^m\rangle\rangle(t)/dt|_{t\to\infty}$ (with $e = 1$ in the following). With the counting field $\chi$ set to zero, the system tends exponentially to a unique stationary state determined by the $1/z$ pole of the resolvent $\mathcal{G}(\chi = 0, z)$. The stationary state is given by the eigenvector corresponding to the zero-eigenvalue of $\mathcal{W}_0 \equiv \mathcal{W}(\chi = 0, z = 0)$, i.e., $\lim_{t\to\infty} \hat{\rho}(\chi = 0, t) \equiv |0\rangle$, where $|0\rangle$ is the normalized solution to $\mathcal{W}_0|0\rangle = 0$. With finite values of $\chi$, an eigenvalue $\lambda_0(\chi, z)$ develops adiabatically from the zero-eigenvalue and the long-time behavior is still determined by the pole $1/[z - \lambda_0(\chi, z)]$ of $\mathcal{G}(\chi, z)$ close to zero. The particular pole $z_0(\chi)$ that solves the self-consistency equation

$$z_0 - \lambda_0(\chi, z_0) = 0,$$

$^4$ By ‘environment’ we mean all degrees of freedom that have been traced out.

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and goes to zero with $\chi$ going to zero, i.e., $z_0(0) = 0$, consequently determines the long-time limit of the CGF. We thus find $e^{S(\chi,t)} \rightarrow D(\chi)e^{z_0(\chi)t}$ for large $t$, where $D(\chi)$ is a time independent function depending on the initial conditions. The current cumulants then read $\langle I^m \rangle = \partial^m z_0(\chi)/\partial(i\lambda)^m |_{\lambda \rightarrow 0}$. In the Markovian limit for the kernel $W(\chi, z \rightarrow 0)$ we obtain $z_0(\chi) = \lambda_0(\chi, 0)$, consistently with previous results for the Markovian case [16, 17]. From equation (5), it is clear that calculations of current cumulants proceed in two steps. First, the dominating eigenvalue $\lambda_0(\chi, z)$ has to be determined. Secondly, the self-consistency equation must be solved for $z_0(\chi)$.

In the original approach to calculations of CGF for non-Markovian transport systems a so-called non-Markovian expansion was developed [18]. In this approach, the CGF is expressed as a series using only the Taylor expansion of the dominating eigenvalue around $z = 0$. Mathematically, the non-Markovian expansion is equivalent to the solution of the self-consistency equation. The method, however, requires an analytic solution for the dominating eigenvalue with its full dependence on the counting field [18]. This is not a feasible approach when the matrices involved are large or, equivalently, for systems with many degrees of freedom. However, if we are only interested in a finite number of cumulants an alternative route exists. To this end, we have developed a scheme for calculating finite orders of current cumulants, where both the eigenvalue problem and the self-consistency equation are solved using perturbation theory in the counting field. A particular strength of the scheme is that it is recursive, allowing for calculations of cumulants of very high orders [19, 21]. Here, we do not present all details of the derivation of the recursive scheme, but mainly focus on the final results for the first three current cumulants: mean current, noise, and skewness.

We consider an expansion of the dominating eigenvalue in $\chi$ and $z$, $\lambda_0(\chi, z) = \sum_{k,l=0}^{\infty} (i\chi)^k z^l c^{(k,l)}/(k! l!)$. Let us first suppose that the coefficients $c^{(k,l)}$ are known. This allows us to solve equation (5) for $z_0(\chi)$ to a given order in $\chi$. From the expansion $z_0(\chi) = \sum_{n=1}^{\infty} ((i\chi)^n/n!) \langle I^n \rangle$, we then extract the zero-frequency cumulants of the current. The results for the lowest cumulants are

$$\langle I^1 \rangle = c^{(1,0)},$$

$$\langle I^2 \rangle = c^{(2,0)} + 2c^{(1,0)}c^{(1,1)},$$

$$\langle I^3 \rangle = c^{(3,0)} + 3c^{(2,0)}c^{(1,1)} + 3c^{(1,0)} \left[ c^{(1,0)}c^{(1,2)} + 2(c^{(1,1)})^2 + c^{(2,1)} \right].$$

Higher-order cumulants are readily calculated in a recursive manner [19, 21]. The cumulants consist of contributions from the purely Markovian quantities $c^{(k,0)}$ and the non-Markovian terms $c^{(k,l)}$ for $0 < l$. We observe the general rule that the $n$th cumulant requires knowledge of non-Markovian terms $c^{(k,l)}$ of order $0 < l < n$ [18]. Consequently, the mean current is not sensitive to non-Markovian effects, whereas higher-order cumulants are. In general, the distinction made here between Markovian and non-Markovian systems is related to the existence of a finite memory in equation (1). In practice, the memory effects depend on the choice of degrees of freedom that are traced out. It may for example be convenient to trace out degrees of freedom of a Markovian system and describe the resulting, reduced system by a non-Markovian GME as was done in [24]. Of course, in such cases our method yields identical results regardless of the particular formulation, Markovian or non-Markovian.
We still need to calculate the expansion coefficients $c^{(k,l)}$ entering the expression $\lambda_0(\chi, z) = \sum_{k,l=0}^{\infty} (i\chi)^k z^l c^{(k,l)}/(k! l!)$. The eigenvalue problem for $\lambda_0(\chi, z)$ is defined by the equation

$$\mathcal{W}(\chi, z)|0(\chi, z)\rangle = [\mathcal{W}_0 + \mathcal{W}'(\chi, z)]|0(\chi, z)\rangle = \lambda_0(\chi, z)|0(\chi, z)\rangle,$$  

(9)

where we have written $\mathcal{W}(\chi, z)$ as the sum of an unperturbed part $\mathcal{W}_0$ and the perturbation $\mathcal{W}'(\chi, z) = \mathcal{W}(\chi, z) - \mathcal{W}_0$. The right eigenvector $|0(\chi, z)\rangle$ reduces adiabatically to the stationary state $|0\rangle$ with $\chi$ and $z$ going to zero and satisfies the normalization condition $\langle 0|0(\chi, z)\rangle = 1$. Here, the left eigenvector $\langle 0|$ is defined by the relation $\langle 0|\mathcal{W}_0 = 0$. Using Rayleigh–Schrödinger perturbation theory we can express the coefficients in terms of the Taylor coefficients of $\mathcal{W}'(\chi, z)$ denoted as $\mathcal{W}^{(k,l)}$, i.e., $\mathcal{W}'(\chi, z) = \sum_{k,l=0}^{\infty} (i\chi)^k z^l \mathcal{W}^{(k,l)}/(k! l!)$ with $\mathcal{W}^{(0,0)} = 0$ by definition, and the pseudoinverse of the kernel $\mathcal{R}$ defined as $\mathcal{R} = Q\mathcal{W}_0^{-1} Q$. Even if $\mathcal{W}_0$ is singular, the pseudoinverse is in fact well defined since the singular part of the kernel is projected away by the projector $Q \equiv 1 - |0\rangle\langle 0|$.\textsuperscript{5} We can now calculate the expansion coefficients and we report here the resulting expressions for a few of them:

$$c^{(1,0)} = \langle 0|\mathcal{W}^{(1,0)}|0\rangle;$$  

(10)

$$c^{(1,1)} = \langle 0|\mathcal{W}^{(1,1)} - \mathcal{W}^{(1,0)}\mathcal{R}\mathcal{W}^{(0,1)}|0\rangle;$$  

(11)

$$c^{(2,0)} = \langle 0|\mathcal{W}^{(2,0)} - 2\mathcal{W}^{(1,0)}\mathcal{R}\mathcal{W}^{(1,0)}|0\rangle.$$  

(12)

The expansion coefficients can also be calculated in a recursive manner as described in [19,21]. We note that the recursive scheme can be used both for analytic and for numerical calculations. Evaluation of the pseudoinverse $\mathcal{R}$ amounts to solving matrix equations, which is feasible even with very large matrices [25]. Numerically, the scheme is stable for very high orders of cumulants (>20) as we have tested on simple examples.

3. Dissipative double quantum dot with a QPC charge detector

We illustrate our method by considering a model of charge transport through a double quantum dot embedded in a dissipative environment; see figure 1(a). A QPC close to the double quantum dot is used as a charge detector [26–28]. The double dot is operated in the Coulomb blockade regime close to a charge degeneracy point, where maximally a single additional electron is allowed to enter and leave the double quantum dot. The Hamiltonian of the full setup is

$$\hat{H} = \hat{H}_{\text{DD}} + \hat{H}_{\text{QPC}} + \hat{H}_{\text{DD-QPC}} + \hat{H}_T + \hat{H}_L + \hat{H}_R + \hat{H}_B + \hat{H}_{\text{DD-B}},$$  

(13)

where the various terms are defined in the following.

The Hamiltonian of the double quantum dot is $\hat{H}_{\text{DD}} = (\varepsilon/2)\hat{s}_z + T_c\hat{s}_x$, introducing the pseudo-spin operators $\hat{s}_z \equiv |L\rangle\langle L| - |R\rangle\langle R|$ and $\hat{s}_x \equiv |L\rangle\langle R| + |R\rangle\langle L|$. The tunnel coupling between the two quantum dot levels $|L\rangle$ and $|R\rangle$ is denoted by $T_c$, while $\varepsilon$ is the energy detuning of the two levels. The pseudo-spin system is tunnel coupled to left (L) and right (R) leads via the tunnel Hamiltonian $\hat{H}_T = \sum_{k_\alpha} V_{k_\alpha} c_{k_\alpha}^{\dagger} |\alpha\rangle\langle \alpha| + \text{h.c.}$, with both leads described as non-interacting fermions, i.e., $\hat{H}_\alpha = \sum_{k_\alpha} \varepsilon_{k_\alpha} c_{k_\alpha}^{\dagger} c_{k_\alpha}, \alpha = L, R$.\textsuperscript{5}

\textsuperscript{5} Details of the super-operator notation used here can be found in [25].
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Figure 1. (a) Schematics of the double dot and the QPC charge detector. The state of the charge qubit modulates the transparency of the quantum point contact via the direct Coulomb interaction between them. (b) Mean current $\langle I \rangle / T_c$ on a log-scale versus level detuning $\varepsilon / T_c$; line styles correspond to different qubit regimes: coherent case ($\Gamma_d = \Gamma_R / 2$, solid), coupled only to the QPC ($\alpha = 0$, $\Gamma_d / T_c = 0.075$, dotted), coupled only to the phonon bath ($\Gamma_d = \Gamma_R / 2$, $\alpha = 6.25 \times 10^{-4}$) for decoupling (i) (dark gray) and decoupling (ii) (light gray), coupled to both QPC and phonon bath for decoupling (ii) (dot–dashed). Other parameters are $\Gamma_L / T_c = 10$, $\Gamma_L / T_c = 10$ and, for gray and dot–dashed curves only, $k_B T / T_c = 5$. See the text for descriptions of decouplings (i) and (ii).

We furthermore include a dissipative environment consisting of a reservoir of non-interacting bosons $\hat{H}_B = \sum_j \hbar \omega j \hat{a}_j^\dagger \hat{a}_j$. The heat bath couples to the $\hat{s}_z$ component of the pseudo-spin via the term $\hat{H}_{BD-B} = \hat{V}_B \hat{s}_z$ with $\hat{V}_B = \sum_j c_j (\hat{a}_j^\dagger + \hat{a}_j) / 2$, where $c_j$ is the electron–phonon coupling strength.

The Hamiltonian of the QPC detector is $\hat{H}_{QPC} = \sum_{k,k',j=L,R} \varepsilon_{kja} \hat{d}_{k,j}^\dagger \hat{d}_{k,j}$, where the first term models the QPC leads and the second term describes tunneling between them with a real energy independent tunneling matrix element $T_0$. The interaction between the QPC and the double quantum dot is given by $\hat{H}_{QPC-DD} = \sum_{k,k',j=R,L} \delta T_j \hat{d}_{k,j}^\dagger \hat{d}_{k',R}|j\rangle \langle j| + \text{h.c.}$, with $\delta T_j$ describing the variation of the QPC barrier opacity due to a localized electron occupying state $|j\rangle$, $j = L, R$. If $\delta T_j \equiv 0$, the current through the QPC at zero temperature is $I_{QPC}^{0} = 2\pi T_0^2 D_L D_R e^2 V / \hbar$ independently of the state of the qubit. Here $D_{L/R}$ are the densities of states of the QPC leads and $V$ is the bias across the QPC. If $\delta T_j \neq 0$ and the qubit is in state $|j\rangle$, the QPC current at zero temperature is $I_{QPC}^{j} = 2\pi (T_0 + \delta T_j)^2 D_L D_R e^2 V / \hbar$. For $\delta T_R \neq \delta T_L$ the QPC introduces a decoherence mechanism of the charge qubit, because it effectively ‘measures’ the right and left states of the qubit. In this paper we consider the weakly responding limit where the QPC current is only slightly modified by the charge state of the qubit, such that $|I_{QPC}^{j} - I_{QPC}^{0}| \ll I_{QPC}^{0}$ [29].

To describe charge transport through the charge qubit and the QPC we follow the scheme outlined in figure 1 (see also [26]). Here, the variable $n$ ($m$) corresponds to the...
number of electrons counted in the right lead of the qubit (QPC). Instead of writing the equation of motion (EOM) for the reduced density matrix resolved with respect to the \((n, m)\) components, it is convenient to introduce the counting fields \(\chi\) and \(\phi\), corresponding to \(n\) (qubit) and \(m\) (QPC), respectively. We then trace out the leads of the qubit and the QPC, leading to an EOM for the reduced density matrix \(\hat{\sigma} = (\hat{\sigma}_{00}, \hat{\sigma}_{LL}, \hat{\sigma}_{RR}, \hat{\sigma}_{LR}, \hat{\sigma}_{RL})^T\) of the double dot and the bath of bosons. The elements \(\hat{\sigma}_{ij}\) are still operators in the Hilbert space of the boson bath. This approach is valid to all orders in the tunnel coupling \(T_c\) under the assumption of a large bias across the system and the QPC charge detector \([26, 30]\).

The Liouville equation is then
\[
\dot{\hat{\sigma}}(t) = \mathcal{L}(\chi, \phi)\hat{\sigma}(t) - i[H_B + \hat{V}_B \hat{s}_z, \hat{\sigma}(t)],
\]
where \(\mathcal{L}(\chi, \phi)\) describes the reduced dynamics of the charge qubit and the QPC, while the other contributions are given by the heat bath Hamiltonian \(\hat{H}_B\) and the interaction term \(\hat{H}_{DD-B} = \hat{V}_B \hat{s}_z\). The Liouville operator \(\mathcal{L}(\chi, \phi)\) for the combined qubit–detector system is
\[
\begin{pmatrix}
-\Gamma_L + D_0 h(\phi) & 0 & \Gamma_R e^{i\chi} & 0 & 0 \\
\Gamma_L & D_L h(\phi) & 0 & iT_c & -iT_c \\
0 & 0 & -\Gamma_R + D_R h(\phi) & -iT_c & iT_c \\
0 & iT_c & -iT_c & -\chi - \Delta_d(\phi) & 0 \\
0 & -iT_c & iT_c & 0 & \chi - \Gamma_d(\phi)
\end{pmatrix},
\]
with \(h(\phi) = (e^{i\phi} - 1)\) and the energy independent rates \(\Gamma_\alpha = 2\pi \sum_k |V_{k\alpha}|^2 \delta(\epsilon - \varepsilon_{k\alpha})\), \(\alpha = L, R\). These rates describe charges entering (leaving) the left (right) quantum dot from (to) the left (right) lead. The number of electrons \(n\) that have tunneled to the right lead is increased by tunnel processes from the right dot with rate \(\Gamma_R\), and the counting factor \(e^{i\chi}\) consequently enters the corresponding off-diagonal element of the matrix in equation (15).

The diagonal terms \(D_j(e^{i\phi} - 1)\) describe counting of tunneling events in the QPC charge detector with the tunneling rate \(D_j = I_j/e\) depending on the state of the qubit, \(|j\rangle\), \(j = 0, L, R\). The state \(|0\rangle\) corresponds to the double dot without an additional electron. The generalized dephasing rate \(\Gamma_d(\phi) = [\Gamma_R + (\sqrt{D_L} - \sqrt{D_R})^2 - 2\sqrt{D_L}D_R(e^{i\phi} - 1)]/2\) represents the decoherence of the off-diagonal terms \(\hat{\sigma}_{LR}\) and \(\hat{\sigma}_{RL}\), taking into account the dephasing induced by the QPC. For \(\phi = 0\) it yields \([\Gamma_R + (\sqrt{D_L} - \sqrt{D_R})^2]/2\), the dephasing rate expected from the coupling of the qubit to the right lead connected to the double dot and the nearby QPC charge detector \([26]\). If \(D_L = D_R\), the QPC does not detect the position of an electron on the double quantum dot. In that case, the qubit is not losing its coherence due to the QPC, but only due to the right lead, which contributes with the decoherence rate \(\Gamma_R/2\). The functional dependence on \(\chi\) and \(\phi\) contains information about correlations in the transport statistics of the combined QPC and qubit system. Such correlations will, however, not be explored in further detail in this work. Here, we limit ourselves to studies of the transport statistics of the qubit. We note that it is also possible to extend the formalism to cases where the thermal energy \(k_B T\) is comparable to the bias \(V\) across the QPC \([29]\).

We see that the EOM for \(\dot{\hat{\sigma}}(t)\) in equation (14) clearly is a Markovian GME defined on an (infinitely) large Hilbert space due to the inclusion of the bosonic heat bath. To reduce the dimensionality of the problem we trace out the boson degrees of freedom, and as we
The effect of the bath enters only via the dynamics of the off-diagonal elements $\hat{\sigma}_{LR}$ and $\hat{\sigma}_{RL}$. We do not show the similar solution for $\hat{\sigma}_{bath}$ coupling is so low that the state of the qubit does not affect the equilibrium of the heat fields $\chi$.

We derive an expression for the memory kernel using assumption (ii). The result can be written as

$$\Gamma(t) = \int_0^t d\tau e^{-\lambda_+(\phi)(t-\tau)} e^{-i\hat{H}_0(t-\tau)\sigma_{LL}(\tau) - \sigma_{RR}(\tau)} e^{i\hat{H}_0(t-\tau)},$$

where the term containing the initial condition $\hat{\sigma}_{LR}(0)$ eventually enters the inhomogeneity [19]. We will only be considering zero-frequency cumulants and can thus safely neglect this term. We do not show the similar solution for $\hat{\sigma}_{RL}(t)$, but it is important to note that $\hat{\sigma}_{RL}(t)$ is not simply the complex conjugate of $\hat{\sigma}_{LR}(t)$ due to the counting fields $\chi$ and $\phi$. Only in the limit $\chi, \phi \to 0$ is the standard relation between the off-diagonal elements re-established.

Substituting the solutions for $\hat{\sigma}_{LR}(t)$ and $\hat{\sigma}_{RL}(t)$ into equations (17) and (18), we can obtain a closed system of equations by performing a decoupling of the charge degrees of freedom and the boson bath. Two possible decouplings are considered:

(i) The standard Born factorization, where the system and the bath degrees of freedom are factorized as $\hat{\sigma}_{i} \simeq \rho_i \otimes \hat{\sigma}_\beta$ with $\hat{\sigma}_\beta \equiv e^{-\beta H_B} / \text{Tr}_B\{e^{-\beta H_B}\}$.

(ii) The so-called state dependent Born factorization [25], where the heat bath is assumed to equilibrate corresponding to the given charge state, such that $\hat{\sigma}_{LR/LR} \simeq \rho_{LR} \otimes \hat{\sigma}_\beta^{(\pm)/(-)}$ with $\hat{\sigma}_\beta^{(\pm)} \equiv e^{-\beta H_B^{(\pm)}} / \text{Tr}_B\{e^{-\beta H_B^{(\pm)}}\}$. This is equivalent to the standard Born approximation, after the qubit and heat bath have been decoupled via a polaron transformation at $T_c = 0$.

Here, $\beta = 1/k_BT$ is the inverse temperature. These decouplings are valid when the bath-assisted hopping rates $\Gamma_{B}^{(\pm)} (z)$ (proportional to $T_c^2$) are much smaller than $\Gamma_{L/R}$. Additionally, approximation (i) is only valid when the strength of the electron–phonon coupling is so low that the state of the qubit does not affect the equilibrium of the heat bath $\hat{\sigma}_\beta$. We derive an expression for the memory kernel using assumption (ii). The result corresponding to assumption (i) can easily be obtained via the substitution $\hat{\sigma}_\beta^{(\pm)} \to \hat{\sigma}_\beta$.

The memory kernel $W(\chi, \phi, z)$ for our model, with $\hat{\rho} = (\rho_0, \rho_L, \rho_R)^T$, is given in Laplace space as

$$\begin{pmatrix}
-G_L + D_0 h(\phi) & 0 & \Gamma_R e^{i\chi} \\
-G_L^{(+)}(z, \phi) + D_L h(\phi) & \Gamma_{B}^{(+)}(z, \phi) & \Gamma_{B}^{(0)}(z, \phi)
\end{pmatrix}.$$
functions are $\alpha$ and $J$ is the spectral density of the heat bath. Below, we consider the case of Ohmic dissipation $T$ ordering operator $\hat{V}$.

It is convenient to change the representation in equation (21), writing

$$
g^{(\pm)}(t) = \text{Tr}_B \{ e^{-i \hat{H}_B^{(\mp)} t} \sigma^{(\pm)}_\beta e^{i \hat{H}_B^{(\mp)} t} \}.
$$

Upon the substitution $\hat{\sigma}_\beta^{(\pm)} \rightarrow \hat{\sigma}_\beta$, the result for approximation (i) is obtained. In that case we have $\Gamma^{(+)}_B(z, 0) = \Gamma^{(-)}_B(z, 0)$.

We proceed by calculating the bath correlation functions $g^{(\pm)}(t)$ using assumption (i). It is convenient to change the representation in equation (21), writing

$$
g^{(\pm)}(t) = \text{Tr}_B \{ e^{-i \hat{H}_B^{(\mp)} t} \hat{\sigma}_\beta e^{i \hat{H}_B^{(\mp)} t} \} = \text{Tr}_B \{ \hat{U}^{(\pm)}_B(t) \hat{\sigma}_\beta \hat{U}^{(\mp)}_B(t) \},
$$

where we have introduced the operators $\hat{U}^{(\pm)}_B(t) = e^{-i \hat{H}_B^{(\mp)} t} e^{i \hat{H}_B^{(\pm)} t}$, and moreover used the fact that $\hat{\sigma}_\beta$ does not evolve with the stationary Hamiltonian $\hat{H}_B$. It is easy to demonstrate that the EOM for the operators $\hat{U}^{(\pm)}_B(t)$ in this representation is $i \partial_t \hat{U}^{(\pm)}_B(t) = \mp i \hat{V}_B(t) \hat{U}^{(\pm)}_B(t)$ with $\hat{V}_B(t) = e^{-i \hat{H}_B t} \hat{V}_B e^{i \hat{H}_B t}$. We can thus identify these operators with the evolution operators in the interaction picture. The solution of the time dependent differential equations is $\hat{U}^{(\pm)}_B(t) = \hat{T} [ \hat{e}^{\mp i \int_0^t d\tau \hat{V}_B(\tau)} ]$ with $\hat{T} [ \cdot ]$ being the time-ordering operator. Equation (22) can then be written as

$$
g^{(\pm)}(t) = \text{Tr}_B \{ \hat{T} [ \hat{e}^{\mp i \int_0^t d\tau \hat{V}_B(\tau)} ] \rho_3 \hat{T} [ \hat{e}^{\mp i \int_0^t d\tau \hat{V}_B(\tau)} ] \}
$$

where we have introduced the time-antiordering operator $\hat{T} [ \cdot ]$ and the Keldysh contour ordering operator $T_K [ \cdot ]$. The second equality expresses the bath correlation function in terms of a Keldysh propagator with a branch dependent interaction potential $\tau_3 \hat{V}_B(t')$ with $\tau_3 = +(-)$ for the forward (backward) Keldysh branch. This expression can be evaluated using perturbation theory in the electron–phonon couplings $c_j$ and we obtain $g^{(\pm)}(t) = 1 - \int_0^\infty d\omega J(\omega) \{ [1 - \cos(\omega t)] \coth(\beta \omega/2 \} / \omega^2 + O(|c_j|^4)$, where $J(\omega) = \sum_j |c_j|^2 \delta(\omega - \omega_j)$ is the spectral density of the heat bath. Below, we consider the case of Ohmic dissipation $J(\omega) = 2 \alpha \omega e^{-\omega/\omega_c}$. We note that $g^{(+)}(t) = g^{(-)}(t) \equiv g(t)$. The Laplace transform of $g(t)$, to leading order in $1/\beta \omega_c$, reads

$$
g(z) = \frac{1}{z} \left\{ 1 - 2 \alpha \left[ \ln \left( \frac{\beta z}{2 \pi} \right) - \Psi \left( \frac{\beta z}{2 \pi} \right) - \frac{\pi}{\beta z} - D \left( \frac{z}{\omega_c} \right) \right] \right\},
$$

where $\Psi(x)$ is the digamma function and $D(x) = x \cos(x) + \sin(x) \sin(x) / x$ with the sine and cosine integrals defined as $\sin(x) = \int_0^x dt \sin(t)/t$ and $\cos(x) = -\int_x^\infty dt \cos(t)/t$.

For decoupling (ii) we can calculate the bath correlation functions exactly to all orders in $\alpha$ using standard many-body techniques [21]. The results for the bath correlations functions are $g^{(\pm)}(t) = e^{-W(\mp)t}$, where

$$
W(t) = \int_0^\infty d\omega \frac{J(\omega)}{\omega^2} \left\{ [1 - \cos(\omega t)] \coth \left( \frac{\beta \omega}{2} \right) + i \sin(\omega t) \right\}.
$$
In the weak coupling limit, we can expand this expression to first order in $\alpha$. The bath correlation functions $\tilde{g}^{(\pm)}(z)$ for an Ohmic spectral density, to leading order in $1/\beta\omega_c$, are $\tilde{g}^{(\pm)}(z) = \tilde{g}(z) \pm i2\alpha F(z/\omega_c)/z$ where $\tilde{g}(z)$ is given in equation (24) and $F(x) = ci(x)\sin(x) + \cos(x)[\pi/2 - si(x)]$. We see that the difference between the bath correlation functions obtained within the two decoupling schemes is proportional to the imaginary part of equation (25). This difference is the main reason that decoupling (i) does not lead to any asymmetry of the cumulants between the emission and absorption sides, as we shall see. We note that the bath correlation function is $\tilde{g}(z) = 1/z$ for $\alpha = 0$. This corresponds to the coherent regime of the qubit and the $z$ dependence of $\Gamma_B^{(\pm)}(z, \phi)$ describes the ‘effective’ memory due to the off-diagonal elements that have been traced out. Furthermore, for $z \to 0$ the rates become $\Gamma_B^{(\pm)}(0, \phi) = 4T_L^2\Gamma_d(\phi)/[\Gamma_d(\phi)^2 + 4\epsilon^2]$, and for $\phi = 0$ we obtain the standard expressions for incoherent tunneling rates in a double dot [32].

We now consider the current cumulants of the charge qubit for $\phi = 0$, where the effect of the QPC is captured by the dephasing rate $\Gamma_d$. Evaluating equations (6) and (8) using the kernel in equation (20), we find the analytic expressions for the cumulants. The expressions for the current and noise with $\alpha = 0$ coincide with known results for the coherent case [30, 31], however, with the total dephasing rate being $\Gamma_d = [\Gamma_R + (\sqrt{D_L} - \sqrt{D_R})^2]/2$ rather than just $\Gamma_R/2$. Considering only the Markovian contributions $c^{(k,0)}$, we obtain the well-known analytic expressions for sequential tunneling [32]. We next compare different regimes with various strengths of the QPC dephasing rate $\Gamma_d$ and different electron–phonon couplings $\alpha$. In figure 1(b), we show the mean current as function of the level detuning $\epsilon$. The solid black line corresponds to the coherent regime, where the peak is symmetric around $\epsilon = 0$ and the width of the peak is proportional to $\Gamma_R/2$. The dotted line shows the effect of the dephasing due to the QPC charge detector, without any contributions from the boson bath ($\alpha = 0$). For $D_R \neq D_L$, the peak width is given by the total dephasing rate, i.e., the sum of the intrinsic contribution $\Gamma_R/2$ and the contribution from the QPC $(\sqrt{D_L} - \sqrt{D_R})^2/2$. The width is thus larger than in the coherent case. The peak, however, remains symmetric. The gray curves show the mean current when the double dot is coupled to the heat bath at finite temperature, but with the QPC charge detector not detecting the position of an electron on the double dot ($D_R = D_L$). The two gray lines correspond to the two different decouplings, decoupling (i) shown with light gray and decoupling (ii) shown with dark gray. We note that the curve corresponding to decoupling (ii) is asymmetric around $\epsilon = 0$. Finally, the dot–dashed curve corresponds to the mean current with contributions to the dephasing from both the QPC and the heat bath.

While the mean current studied so far only reveals little information about the dephasing mechanisms, we expect more information to be contained in the higher-order cumulants. In figure 2 we show the Fano factor $\langle\langle I^2 \rangle\rangle/\langle\langle I \rangle\rangle$ and the normalized skewness $\langle\langle I^3 \rangle\rangle/\langle\langle I \rangle\rangle$ as functions of the level detuning $\epsilon$. Line styles and parameters are the same as in figure 1(b). Generally, we observe that the two cumulants to a higher degree than the mean current discriminate between various regimes. In particular, the coherent regime has strongly super-Poissonian behavior with both of the normalized cumulants reaching values larger than the Poissonian limit of 1. In contrast, for the sequential tunneling regime, given by the Markovian contributions to the cumulants, only sub-Poissonian behavior is observed [32].
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If a dephasing mechanism is introduced, either due to the QPC charge detector or to the heat bath, the super-Poissonian behavior is gradually reduced towards the sequential tunneling regime with increasing dephasing rate. The rate of dephasing $\Gamma_d$ due to the QPC can be modified via the coupling between the charge qubit and the QPC. On the other hand, the dephasing induced by the heat bath changes with the bath temperature, which can also induce the transition between coherent and incoherent transport. Such a transition was recently observed in shot noise measurements of transport through a double dot at low temperatures [32]. Comparing the scales of the vertical axes of the second and third cumulants in figure 2, we conjecture that such a transition may be more visible in the third cumulant and that even higher-order cumulants in general could be more sensitive to such a transition.

We now study how the symmetry of the cumulants changes in the various regimes. Without coupling to the QPC or the heat bath the cumulants are symmetric around $\epsilon = 0$. If the qubit is coupled only to the QPC, this symmetry remains intact. However, with non-zero coupling to the heat bath, asymmetry around $\epsilon = 0$ is observed (see the dark gray and dot–dashed lines). The asymmetry occurs due to the asymmetry in emission and absorption of bosons at low temperatures. Phonon absorption and emission dominate for $\epsilon < 0$ and $\epsilon > 0$, respectively. We note that the curves obtained using decoupling (i) do not capture this essential physics, not even to first order in $\alpha$. Indeed, in all the regimes for which the cumulants are symmetric, the rates have the property $\Gamma^{(+)}(z) = \Gamma^{(-)}(z)$. This particular property is not fulfilled in decoupling (ii) due to the imaginary part of equation (25), and we believe that decoupling (ii) should be correct to any order in $\alpha$. What is then the essential difference between the dephasing induced by the QPC charge detector and the bath-induced dephasing in the model that we are considering here? The QPC charge detector induces dephasing with little influence on the dynamics of the system and does not destroy the symmetry in the coherent regime. In contrast, the bath-induced dephasing is generated with emission and absorption of bosons, and the charge qubit is influenced by the intrinsic asymmetry of this process. We conclude the analysis of this model for now and postpone a further analysis to future research aimed at an improved understanding of these dephasing mechanisms.

Figure 2. (a) Fano factor $\langle I^2 \rangle / \langle I \rangle$ as a function of the level detuning $\epsilon / T_c$. (b) Normalized skewness $\langle I^3 \rangle / \langle I \rangle$ versus $\epsilon / T_c$. Parameters corresponding to the different line styles are given in the caption of figure 1.
4. Outlook and open questions

We have shown how it is now possible to calculate cumulants, also of high orders, for a wide class of counting problems. In this work we have applied the method to investigate how the dephasing mechanism of a QPC charge detector differs from that of an external heat bath. The method is, however, applicable to a large class of non-Markovian counting problems, also from outside the field of physics. The general approach outlined here is also suitable for calculations of the finite-frequency noise spectrum [19]. It is relevant to ask whether it is possible to extend this approach to frequency dependent cumulants of higher orders, similar to the results already obtained for Markovian systems [33]. Another interesting issue concerns systems with very long memory time (for example, Levy-flight processes). Such cases would require us to reconsider some of the assumptions underlying the theory presented here.

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