Meson Decay Constant Predictions of the Valence Approximation to Lattice QCD

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We evaluate $f_\pi/m_\rho$, $f_K/m_\rho$, $1/f_\rho$, and $m_\phi/(f_\rho m_\rho)$, extrapolated to physical quark mass, zero lattice spacing and infinite volume, for lattice QCD with Wilson quarks in the valence (quenched) approximation. The predicted $m_\phi/(f_\rho m_\rho)$ differs from experiment by less than its statistical uncertainty of approximately 15%. The other three constants are 10% to 20% below experiment, equivalent to between one and two times the corresponding statistical uncertainties.
In a recent paper [1] we presented lattice QCD predictions for the masses of eight low-lying hadrons, extrapolated to physical quark mass, zero lattice spacing, and infinite volume, using Wilson quarks in the valence (quenched) approximation. The masses we found were within 6% of experiment, and all differences between prediction and experiment were consistent with the calculation’s statistical uncertainty. We argued that this result could be interpreted as quantitative confirmation of the low-lying mass predictions both of QCD and of the valence approximation. It appeared to us unlikely that eight different valence approximation masses would agree with experiment yet differ significantly from QCD’s predictions including the full effect of quark-antiquark vacuum polarization.

We have now evaluated the infinite volume, zero lattice spacing, physical quark mass limit of $f_\pi/m_\rho$, $f_K/m_\rho$, $1/f_\rho$, and $m_\phi/(f_\phi m_\rho)$. To our knowledge there have been no previous systematic calculations of this physical limit of lattice meson decay constants. A review of earlier work is given in Ref. [2]. The predicted $m_\phi/(f_\phi m_\rho)$ differs from experiment by less than its statistical uncertainty of approximately 15%. The other three constants are 10% to 20% below experiment, equivalent to between one and two times the corresponding statistical uncertainties.

The decay constant predictions we obtain may appear to be marginally in conflict with our earlier mass results. A simple argument suggests, however, that valence approximation decay constants actually should lie at least some amount below the predictions of the full theory and therefore, presumably, below experiment [3]. The valence approximation may be viewed as replacing the momentum and frequency dependent color dielectric constant arising from quark-antiquark vacuum polarization with its zero-momentum limit [4]. The valence approximation might thus be expected to be fairly reliable for low-lying baryon and meson masses, which are determined largely by the low momentum behavior of the chromoelectric field. The effective quark charge used in the valence approximation may be thought of as the product of the zero-momentum dielectric constant and the quark’s true charge. The valence approximation’s effective quark charge at higher momenta can then be obtained from the low momentum charge by the Callan-Symanzik equation. The leading coefficient
\( \beta_0 \) entering the Callan-Symanzik equation in the valence approximation is larger than \( \beta_0 \) of the full theory, which is reduced by dynamical quark vacuum polarization. It follows that the quark charge in the valence approximation will fall faster with momentum than it does in the full theory. At sufficiently short distance the attractive quark-antiquark potential in the valence approximation will therefore be weaker than in the full theory. Meson wave functions in the valence approximation will then be pulled into the origin less than in the full theory and will be smaller at the origin. Since meson decay constants are proportional to quark-antiquark wave functions at the origin, decay constants in the valence approximation will be smaller than in the full theory.

Thus while Ref. [1] confirms the valence approximation for the low momentum behavior of the chromoelectric field governing the binding energy of low-lying hadrons, the present calculation shows that the valence approximation appears to become less reliable, as expected, for the higher momenta contributing to meson wave functions at the origin. Our result suggests that valence approximation predictions for the hadronic matrix elements entering weak decays will also tend to be somewhat smaller than their values in the full theory.

The calculations described here were done on the GF11 parallel computer at IBM Research [5] and use the same collection of gauge configurations and quark propagators generated for the mass calculations of Ref. [1]. The full set of mass and decay constant calculations took approximately one year to complete. GF11 was used in configurations ranging from 384 to 480 processors, with sustained speeds ranging from 5 Gflops to 7 Gflops. With the present set of improved algorithms and 480 processors, these calculations could be repeated in less than four months.

The normalization we adopt for pseudoscalar and vector decay constants in continuum QCD is

\[
<0|J_{5\mu}|P(p,j)> = ip^\mu f_j \quad \text{and} \quad <0|J_{\mu}^\nu|V(p,\epsilon,j)> = \epsilon^\mu \epsilon^\nu \epsilon_{\mu\nu} f_j,
\]

for states normalized by

\[
<q|p> = (2\pi)^3\delta(\vec{p} - \vec{q}), \quad \text{where} \quad J_{\mu}^\nu \quad \text{and} \quad J_{5\mu} \quad \text{are vector and axial vector flavor-SU(3)} \quad \text{currents and} \quad j \quad \text{is an adjoint-representation flavor index running from 1 to 8. Assuming exact isospin symmetry, we have} \quad f_4 = f_\pi, \quad f_5 = f_K, \quad \text{for} \quad i = 1, \ldots, 3, \quad f_i = f_{\rho}, \quad \text{for} \quad i = 4, \ldots, 7, \quad \text{and} \quad F_i = m_\rho/f_\rho, \quad \text{for} \quad i = 1, \ldots, 3. \quad \text{In the valence approximation we have, in addition,} \quad F_8 = 3m_\phi/(\sqrt{2}f_\phi).\]
For simplicity, we will also use the names $F_\rho$ and $F_\phi$ for $F_1$ and $F_8$, respectively. Our normalization gives $f_\pi$ the value $(93.15 \pm 0.11)\text{MeV}$.

The hadron mass calculation of Ref. [1] was done using gaussian smeared quark and antiquark fields defined in Coulomb gauge. Smeared fields have, therefore, also been adopted for the present calculation. The smeared field $\phi_s(\vec{x}, t)$ is $\sum_y G_r(\vec{x} - \vec{y}) \psi(\vec{y}, t)$ for a meson with flavor $j$, assumed here to have $m_{\pi}$ as the critical hopping constant at which the pion's mass becomes zero. The decay constant $f$ is the value (93.15 ± 0.11)MeV. The gauge function $G$ of $(\sqrt{\pi}r)^{-1} \exp(-|\vec{z}|^2/r^2)$, and $\overline{\phi}_s(\vec{x}, t)$ is defined correspondingly. We take the smeared fields $\phi_0(x)$ and $\overline{\phi}_0(x)$ to be $\psi(x)$ and $\overline{\psi}(x)$, respectively. From these fields, define the composite field $J^5_{jr}$ and smeared lattice currents $J^\mu_{jr}$ and $J^{5\mu}_{jr}$ to be $\overline{\phi}_s \gamma^5 \lambda_j \phi_r$, $\overline{\phi}_s \gamma^\mu \lambda_j \phi_r$ and $\overline{\phi}_s \gamma^5 \gamma^\mu \lambda_j \phi_r$, respectively, where the $\lambda_j$ are orthonormal basis for the flavor-SU(3) Lie algebra with normalization $tr(\lambda_j \lambda_j) = 1/2$.

Define the constants $Z^{PP}_{jrr}$, $Z^{AP}_{jrr}$, and $Z^{VV}_{jrr}$ by the requirement that for a lattice with time direction period $L$ and a time $t$, $L \gg t \gg 1$, the correlation function $\sum_{\vec{x}} < [J^5_{jr}(\vec{x}, t)]^\dagger J^5_{jr}(0, 0) >$ approaches $Z^{PP}_{jrr} \exp(-m^P_j t)$, $\sum_{\vec{x}} < [J^{50}_{jrr}(\vec{x}, t)]^\dagger J^{50}_{jrr}(0, 0) >$ approaches $Z^{AP}_{jrr} \exp(-m^P_j t)$, and $\sum_{\vec{x}} < [J^{\mu}_{jr}(\vec{x}, t)]^\dagger J^{\mu}_{jr}(0, 0) >$ approaches $Z^{VV}_{jrr} \exp(-m^V_j t)$. Here $m^P_j$ and $m^V_j$ are pseudoscalar and vector masses, respectively.

Measured in units of the lattice spacing $a$ with any choice of the smearing radius $r$, the decay constant $f_j a$ is then given by $2^{5/2} (m_j a Z^{PP}_{jrr})^{-1/2} z^A_j Z^{AP}_{j0r}$, and $F_j a$ is given by $2^{1/2} (m_j a Z^{VV}_{jrr})^{-1/2} z^V_j Z^{VV}_{j0r}$. The coefficients $z^A_j$ and $z^V_j$ are finite renormalizations chosen so that the lattice currents $z^A_j J^{5\mu}_{jr}$ and $z^V_j J^\mu_{jr}$ approach the continuum currents $J^{5\mu}_{jr}$ and $J^\mu_{jr}$, respectively, as the lattice spacing approaches zero. A mean-field theory improved perturbation expansion [3] gives $z^A_j$ the value $(1 - 3k_j/4 k_c)[1 - 0.31 \alpha_{ms}(1/a)]$ and $z^V_j$ the value $(1 - 3k_j/4 k_c)[1 - 0.82 \alpha_{ms}(1/a)]$, where $k_j$ is the hopping constant corresponding to the mass of the quark and antiquark for a meson with flavor $j$, assumed here to have $m_q = m_{\pi}$, and $k_c$ is the critical hopping constant at which the pion’s mass becomes zero. The decay constant for a meson with $m_q \neq m_{\pi}$ will be discussed below. The renormalization constants $z^A_j$ and $z^V_j$ are often given the “naive” value $2k_j$. Even to leading order in perturbation theory, however, a better choice turns out to be $(1 - 3k_j/4 k_c)$ [4].

Table [1] lists the lattice sizes, parameter values, sweeps skipped between gauge config-
urations, and number of configurations used in the ensembles from which decay constants were calculated. Gauge configurations were generated with the Cabbibo-Marinari-Okawa algorithm. A discussion of the algorithms by which quark propagators were found is given in Ref. [1]. Hadron masses were determined by fits to hadron propagators constructed from the quark propagators. The coefficients $Z_{j02}^{AP}$, $Z_{j22}^{PP}$, $Z_{j02}^{VV}$, and $Z_{j22}^{VV}$ were then extracted from fits to current-current and pseudoscalar-current correlation functions, and combined with both perturbative and naive renormalization to find two different sets of values for $f_\pi$ and $F_\rho$. Statistical errors on these quantities and all other parameters which we have determined were found by the bootstrap method. A more detailed discussion of our fits and error analysis will be given elsewhere [7].

Percentage changes in decay constants going from $16^3 \times 32$ to $24^3 \times 32$, at $\beta$ of 5.7, are given in Table II. These changes are the same for both perturbative and naive renormalization. All of the differences appear to be of marginal statistical significance and may therefore best be viewed as upper bounds on the volume dependence of our results. It appears quite likely that for the range of $k$, $\beta$, and lattice volume we have examined the errors in valence approximation decay constants due to calculation in a finite volume $L^3$ are bounded by an expression of the form $Ce^{-L/R}$, with a coefficient $R$ of the order of the radius of a hadron’s wave function. At $\beta$ of 5.7 for the $k$ we considered, $R$ is thus typically 3 lattice units or less. We therefore expect that the differences between decay constants on a $16^3$ volume and those on a $24^3$ volume are nearly equal to the differences between $16^3$ and true infinite volume limiting values.

At the largest $k$ on each lattice, the ratio $m_\pi/m_\rho$ is significantly larger than its experimentally observed value of 0.179. Thus to produce decay constants for hadrons containing only light quarks, our data has to be extrapolated to larger $k$ or, equivalently, to smaller quark mass. We did not calculate directly at larger $k$ both because the algorithms we used to find quark propagators became too slow and because the statistical errors we found in trial calculations became too large.

To extrapolate $F_\rho$ and $f_\pi$ to small quark mass, we first determined the critical hopping
constant $k_c$, for each lattice and $\beta$, at which $m_\pi$ becomes zero \[1\]. Defining the quark mass in lattice units $m_qa$ to be $1/(2k) - 1/(2k_c)$, we found $f_\pi a$ and $F_\rho a$, both for perturbative renormalization and for naive renormalization, to be nearly linear functions of $m_qa$ over the entire range of $k$ considered on each lattice. Figure 1 shows $f_\pi$ and $F_\rho$, given by perturbative renormalization, as functions of $m_q$. Data is shown from all lattices of Table 1 except $24^3 \times 32$ at $\beta = 5.7$. For convenience, we show all hadron masses in units of the physical rho mass, $m_\rho(m_n)$, given by $m_\rho$ evaluated at the “normal” quark mass $m_n$ which produces the physical value of $m_\pi/m_\rho$. The quark mass $m_q$ in Figure 1 is shown in units of the strange quark mass $m_s$. The value of $m_s$ for each lattice and $\beta$ is found in Ref. \[1\] by requiring $m_\pi[(m_n + m_s)/2]/m_\rho(m_n)$ to be equal to the physical value of $m_K/m_\rho$. The straight lines are fits to the $m_q$ dependence of $f_\pi$ and $F_\rho$ at the three smallest quark masses in each data set at fixed $\beta$. The fits in Figure 1 appear to be quite good and provide, we believe, a reliable method for extrapolating decay constants down to light quark masses. With naive renormalization, $f_\pi$ and $F_\rho$ fit straight lines in $m_q$ about as well as the perturbatively renormalized data of Figure 1.

The linear fits of Figure 1 permit the determination of $f_K$ and $F_\phi$ in addition to $f_\pi$ and $F_\rho$. For a pion composed of a quark and antiquark with mass $m_q \neq m_\pi$, Figure 1 suggests $f_\pi = \alpha_q m_q + \alpha_\pi m_\pi + \beta$. Charge conjugation invariance then gives $\alpha_q = \alpha_\pi$. It follows that the kaon, which is a pion with, say, $m_q = m_s$ and $m_\pi = m_n$, will have the same decay constant as a pion composed of a single type of quark and antiquark with $m_q = m_\pi = (m_s + m_n)/2$. On the other hand, the linear fits of Figure 1 permit $F_\rho$ to be extrapolated to the point $m_q = m_\pi = m_s$ which, in the valence approximation, gives $F_\phi$.

The ratios $f_\pi/m_\rho$, $f_K/m_\rho$, $F_\rho/m_\rho$ and $F_\phi/m_\rho$ for physical quark masses we then extrapolated to zero lattice spacing. The physical lattice volume was held nearly fixed as the lattice spacing was taken to zero. For Wilson fermions the leading lattice spacing dependence in mass ratios is expected to be linear in $a$. Figure 2 shows decays constants with perturbative renormalization along with linear fits to $m_qa$. The quantity $m_qa$ may be viewed as the lattice spacing $a$ measured in units of the physical rho Compton wavelength, $1/m_\rho$. The vertical
bars at $m_\rho a$ of 0, offset slightly for visibility, are the extrapolated predictions’ uncertainties determined by the bootstrap method, and the horizontal bars are corresponding experimentally observed values. The calculated points in Figure 3 are for the lattices $16^3 \times 32$, $24^3 \times 36$ and $30 \times 32^2 \times 40$. The values of $\beta$ for these lattices were chosen so that the physical volume in each case is nearly the same. For lattice period $L$, the quantity $m_\rho L$ is respectively, $9.08 \pm 0.13$, $9.24 \pm 0.19$ and, averaged over three directions, $8.67 \pm 0.12$. Extrapolations to zero lattice spacing similar to those shown in Figure 3 were also done for naive renormalization.

The continuum ratios we found in finite volume were then corrected to infinite volume by an adaptation of the method used in Ref. [1] to correct finite volume continuum mass ratios to infinite volume. The difference between a unitless decay ratio found on the lattice $16^3 \times 32$, at $\beta$ of 5.7, and the same ratio found on the lattice $24^3 \times 32$, at $\beta$ of 5.7, we took as a finite lattice spacing approximation to the difference between the continuum decay ratio in a box with period having $m_\rho L$ of 9 and the continuum decay ratio in infinite volume. The error in this procedure, for perturbatively renormalized decay constants, can be estimated to be 4% or less as follows. For perturbative renormalization, the value of each decay ratio on the lattice $16^3 \times 32$ at $\beta$ of 5.7 differs from its continuum limit by at most 75% of the continuum limit. Moreover, as we argued earlier, the change in each decay ratio, at $\beta$ of 5.7, between $16^3 \times 32$ and $24^3 \times 32$ should be nearly the same as the corresponding change between $16^3 \times 32$ and infinite volume. Combining these two pieces of information, we expect that with a relative error of 75% or less, the change in any decay ratio between $16^3 \times 32$ and $24^3 \times 32$ at $\beta$ of 5.7 should be the same as the change between the continuum decay ratio in a box with period having $m_\rho L$ of 9 and the corresponding continuum ratio in infinite volume. Since the changes we found in decay ratios, for physical quark mass, between $16^3 \times 32$ and $24^3 \times 32$ are all less than 5%, the error in using these differences as estimates of corresponding continuum differences between $m_\rho L$ of 9 and infinite volume should of the order of 75% of 5%, which is 3.75%. The corresponding uncertainty for naive renormalization is 10.0% due to a larger change in these values between $16^3 \times 32$ at $\beta$ of 5.7 and the continuum limit.
The ratios $f_\pi/m_\rho$, $f_K/m_\rho$, $F_\rho/m_\rho$ and $F_\phi/m_\rho$, for both naive and perturbative renormalization, extrapolated to zero lattice spacing with $m_\rho L$ fixed at 9, and then corrected to infinite volume are shown in Table III. The errors shown for infinite volume ratios are statistical only and do not include the estimates we have just given for the systematic error in our procedure for making infinite volume corrections. Most of the perturbatively renormalized results lie below the corresponding ratios with naive renormalization. For all ratios except $F_\phi/m_\rho$ in finite volume, however, the two different renormalizations give results which are statistically consistent. Where there is a disagreement, the perturbatively renormalized ratios are more reliable estimates of the true infinite volume continuum limit of the valence approximation. For extrapolations done from lattice spacings which are sufficiently small, the two different renormalization procedures should give the same limiting results. The agreement between the two different methods for three of four parameters tends to support the reliability of our extrapolations to the continuum limit.

The predicted infinite volume ratios in Table III are statistically consistent with the finite volume ratios. The main consequence of the correction to infinite volume is an increase in the size of the statistical uncertainty in each prediction.

Three of the four finite volume volume continuum ratios obtained by perturbative renormalization lie below experiment by between one and two standard deviations. The larger physical volume over which infinite volume meson wave functions can spread should give infinite volume decay constants some amount smaller than those calculated in finite volume. Thus the true disagreement between infinite volume $f_\pi/m_\rho$, $f_K/m_\rho$, and $F_\rho/m_\rho$ and experiment should be at least as great than the disagreement we find for the finite volume predictions. The statistical significance of the disagreement between our infinite volume results themselves and experiment, however, is weaker than for the finite volume predictions as a result of the large size of the error bars on the infinite volume predictions. The larger error bars of the naive renormalized decay constants also weakens the significance of their disagreement with experiment.

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FIGURES

FIG. 1. The decay constants $F_\rho$ and $f_\pi$, with perturbative renormalization, in units of the physical rho mass $m_\rho(m_n)$, as functions of the quark mass $m_q$, in units of the strange quark mass $m_s$.

FIG. 2. Perturbatively renormalized decay constants as functions of the lattice spacing $a$, in units of $1/m_\rho$. The error bars near zero lattice spacing are uncertainties in the extrapolated ratios, and the horizontal lines represent experimentally observed values.
## Tables

### Table I. Configurations analyzed.

| lattice       | β   | k         | skip | count |
|---------------|-----|-----------|------|-------|
| $16^3 \times 32$ | 5.7 | 0.1600 - 0.1675 | 2000 | 219   |
| $24^3 \times 32$ | 5.7 | 0.1600 - 0.1675 | 4000 | 58    |
| $24^3 \times 36$ | 5.93| 0.1543 - 0.1581 | 4000 | 217   |
| $30 \times 32^2 \times 40$ | 6.17| 0.1500 - 0.1532 | 6000 | 219   |

### Table II. Changes in perturbatively renormalized decay constants from $16^3 \times 32$ to $24^3 \times 32$ at $\beta = 5.7$.

| decay | k         | change          |
|-------|-----------|-----------------|
| $f_\pi$ | 0.1600   | $3.5_{-2.3}^{+3.2}$% |
|        | 0.1650   | $-1.1_{-5.4}^{+3.3}$% |
|        | 0.1663   | $-3.1_{-5.0}^{+4.1}$% |
|        | 0.1675   | $-6.2_{-7.1}^{+4.7}$% |
| $f_\rho$ | 0.1600   | $7.1_{-4.9}^{+1.4}$% |
|        | 0.1650   | $4.9_{-7.8}^{+3.0}$% |
|        | 0.1663   | $4.0_{-7.8}^{+1.6}$% |
|        | 0.1675   | $-2.3_{-3.9}^{+1.6}$% |
| decay  | renorm. | finite volume | infinite volume | obs. |
|--------|---------|---------------|-----------------|------|
| $f_\pi/m_\rho$ | perturb. | $0.102^{+0.021}_{-0.026}$ | $0.091^{+0.028}_{-0.036}$ | 0.121 |
|          | naive   | $0.097^{+0.030}_{-0.032}$ | $0.082^{+0.041}_{-0.051}$ |      |
| $f_K/m_\rho$  | perturb. | $0.119^{+0.017}_{-0.015}$ | $0.116^{+0.021}_{-0.024}$ | 0.148 |
|          | naive   | $0.130^{+0.024}_{-0.022}$ | $0.127^{+0.030}_{-0.032}$ |      |
| $F_\rho/m_\rho$ | perturb. | $0.177^{+0.017}_{-0.020}$ | $0.173^{+0.039}_{-0.031}$ | 0.199 |
|          | naive   | $0.191^{+0.033}_{-0.031}$ | $0.201^{+0.035}_{-0.059}$ |      |
| $F_\phi/m_\rho$ | perturb. | $0.217^{+0.014}_{-0.022}$ | $0.253^{+0.017}_{-0.051}$ | 0.219 |
|          | naive   | $0.270^{+0.021}_{-0.029}$ | $0.319^{+0.030}_{-0.070}$ |      |

**TABLE III.** Calculated values of meson decay constants extrapolated to zero lattice spacing in finite volume, then corrected to infinite volume, compared with observed values.