A Method to construct all the Paving Matroids over a Finite Set

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Abstract We give a characterization of a matroid to be paving, through its set of hyperplanes and give an algorithm to construct all of them.

Keywords simple matroid, paving matroid, sparse-paving matroid, lattice, hyperplanes of a matroid, circuits of a matroid

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1 Introduction

The study of sparse-paving and paving matroids helps to understand the behavior of matroids in general. Important examples of matroids, like the combinatorial finite geometries are indeed paving matroids. They also have an important role in Computer Science with the greedy algorithms and the matroid oracles among others.

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In 1959, Hartmanis [6] introduced the definition of paving matroid through the concept of \(d\)-partition in number theory, see also the works of Welsh [15] (1976), Oxley [13] and Jerrum [7]. In this paper, we will work with the lattice of all the subsets of a set, not only with the so called lattice of a matroid. For references of theory of lattices and theory of lattices of matroids, see [4], [6].

In this work, we give another characterization of a matroid to be paving, which leads us to an algorithm to construct all the paving matroids. Namely,

**Theorem.** Let \(M = (S, \mathcal{I})\) be a simple matroid of rank \(r \geq 2\). Let \(H\) be the set of hyperplanes of \(M\). Then

\[ M \text{ is a paving matroid if and only if } \forall X \neq Y \in H \text{ such that } |X|, |Y| \geq r. \]

Then \(|X \cap Y| \leq r - 2\).

The above result is a consequence of

**Theorem.** Let \(S = \{1, \ldots, n\}\) be a set, \(r \in \mathbb{N}\) and \(2 \leq r \leq n\).

Take \(\emptyset \neq H' \subseteq \mathcal{P}S\) such that \(\forall X \in H', r \leq |X| < n\) and \(\forall X, Y \in H'\).

\[ X \neq Y \implies |X \cap Y| \leq r - 2. \]

Let define by \(C_r := \{A \in \binom{S}{r-1}; \exists X \in H' \text{ with } A \subseteq X\}\) and \(H := H' \cup \binom{S}{r-1}; \forall D \in \binom{S}{r-1}; A \subset D \implies D \in \binom{S}{r-1}\}\).

Then \(H\) is the set of hyperplanes of a paving matroid on \(S\) of rank \(r\). With \(C_r\) its set of \(r\)–circuits and \(\mathcal{B} = \binom{S}{r-1}\) \(C_r\) its set of basis.

**Theorem.** Let \(\mathcal{M}_{n,r}\) be the set of the matroids on a set \(S\) with \(|S| = n\) and rank \(r\) and \(\mathcal{S}_{p,n,t}\) be the set of sparse-paving matroids on \(S\) of rank \(t\). Then

\[ \mathcal{M}_{n,r} \hookrightarrow \prod_{t=r}^{n-1} \mathcal{S}_{p,n,t}. \]

Therefore,

\[ |\mathcal{M}_{n,r}| \leq \prod_{t=r}^{n-1} |\mathcal{S}_{p,n,t}| \leq |\mathcal{S}_{p,n,2}|^{n-r}. \]

The material is organized as follows: In Section I, we give a characterization of paving and sparse-paving matroids by their sets of circuits. In section II, we give our main result: a construction of all the paving matroids using the so called \(d\)–partitions. In section III, we give an algorithm to construct the paving matroids on \(S\), \(|S| = n\) and rank \(r\).
2 Definitions and known results

We recall that a matroid $M = (S, \mathcal{I})$ consists of a finite set $S$ and a collection $\mathcal{I}$ of subsets of $S$ (called the independent sets of $M$) satisfying the following independence axioms:

(I1) The empty set $\emptyset \in \mathcal{I}$.

(I2) If $X \in \mathcal{I}$ and $Y \subseteq X$ then $Y \in \mathcal{I}$.

(I3) Let $U, V \in \mathcal{I}$ with $|U| = |V| + 1$ then $\exists x \in U \setminus V$ such that $V \cup \{x\} \in \mathcal{I}$.

A subset of $S$ which does not belong to $\mathcal{I}$ is called a dependent set of $M$. A basis [respectively, a circuit] of $M$ is a maximal independent [resp. minimal dependent] set of $M$. The rank of a subset $X \subseteq S$ is $rk_X := \max\{|A|; A \subseteq X, A \in \mathcal{I}\}$ and the rank of the matroid $M$ is $rk_M := rk_S := r$. A hyperplane $X \subseteq S$ is a maximal subset of rank $r - 1$.

Known: Any matroid $M = (S, \mathcal{I})$ is completely determined by its set of basis, $\mathcal{B}$. Namely, $\mathcal{I} = \{X \subseteq S; \exists B \in \mathcal{B} \text{ with } X \subseteq B\}$. And if $r = rk_M$ is the rank of $M$, then any circuit $X$ of $M$, has cardinality $|X| \leq rk_M + 1$. And any hyperplane $Y$ has cardinality $r - 1 \leq |Y| \leq n - 1$.

Let $M = (S, \mathcal{I})$ be a matroid of rank $r$. Denote by $M^* = (S, \mathcal{I}^*)$ the dual matroid of $M$ whose set of basis is $\mathcal{B}^* := S \setminus \mathcal{B}$, then the rank of $M^*$ is $|S| - r$.

A matroid is paving if it has no circuits of cardinality less than $rkM$. And a matroid $M$ is sparse-paving if $M$ and its dual $M^*$ are paving matroids.

Along the paper, a matroid means a simple matroid, that is, it has not circuits of cardinality 1. For general references of Theory of Matroids, see [10], [12], [13] and [15].

3 A description of the Paving and Sparse-paving matroids through their set of circuits

For any set $X$ and $m \in \mathbb{N}$, let define by $\binom{X}{m} = \{A \subseteq X; |A| = m\}$ the $m$–subsets of $X$.

Recall that $U_{n,r}$ is a uniform matroid on $S$ of cardinality $n$ and rank $r$ if $\mathcal{B} = \binom{S}{r}$. That is, $\mathcal{I} = \{X \subseteq S; |X| \leq r\}$. For example, any matroid of rank 0 or $n$ are uniform. Any uniform matroid is an sparse-paving matroid, therefore a paving one.

For any $n$ and $r = 1$, since we work with simple matroids, it follows that it must be $U_{n,1}$ the uniform matroid of rank 1.

Then we will work with $2 \leq r \leq n$.

For this section, see [10].

Definition. Let $M = (S, \mathcal{I})$ be a paving matroid on $S = \{1, 2, ..., n\}$ of rank $2 \leq r \leq n$. 
Let $B$ be the set of basis of $M$.

Let $C_r$ (resp. $C_{r+1}$) be the $r$-circuits ($r+1$-circuits), the set of the circuits of cardinality $r$ ($r+1$).

$N_1 = \{ X \in \binom{S}{r+1}; \exists C \in C_r \text{ and } \exists B \in B \text{ tal que } X = C \cup B \}$. Moreover, for $X \in N_1$, $\exists C \in C_r$ such that $C \subset X$.

$N_2 = \{ X \in \binom{S}{r}; \forall A \in \binom{S}{r} \text{, } A \subset X \text{ entonces } A \in C_r \}$. And $\forall X \in N_2$, $rk(X) = r - 1$.

Observation

In the lattice of all subsets of $S$, $\mathcal{L}_S$, we have that $(\binom{S}{r-1}) \subset I$, $(\binom{S}{r}) = B \cup C_r$ and $(\binom{S}{r+1}) = C_{r+1} \cup N_1 \cup N_2$.

Proposition [10] If $M = (S, I)$ is a paving matroid, then $M$ is a sparse-paving if and only if $N_2 = \emptyset$.

Moreover, we get a method and an algorithm to construct all the sparse-paving matroids. For possible interest, since the proof is using only circuits, we put the proof of the next theorem in the Appendix.

Theorem [10]. Let $S = \{ 1, \ldots, n \}$ be a set, $r \in \mathbb{N}$ and $r \leq n$.

Let take $C_r \subset \binom{S}{r}$ such that $\forall C, C' \in \binom{S}{r}$ with $C \neq C'$ then $|C \cap C'| \leq r - 2$ and let $B := (\binom{S}{r}) \setminus C_r$ be the set of basis of a matroid $M$ on $S$. Then $M$ is a sparse-paving matroid of rank $r$.

4 A description of the Paving Matroids through their set of hyperplanes

II.1. Welsh, D. J. A. in [15] characterizes the paving matroids in the following way:

If a paving matroid $M = (S, I)$ has rank $3 \leq d + 1 < |S|$, then its hyperplanes form a set system known as a $d$-partition. A family of two or more sets a $d$-partition if every set in $\mathcal{F}$ has size at least $d$ and every $d$-element subset of $\bigcup \mathcal{F}$ is a subset of exactly one set in $\mathcal{F}$. Conversely, if $\mathcal{F}$ is a $d$-partition, then it can be used to define a paving matroid on $E = \bigcup \mathcal{F}$ for which $\mathcal{F}$ is the set of hyperplanes. See also [6].

II.2. Proposition. Let $M = (S, I)$ be a paving matroid of rank $r \leq n$ and let $\mathcal{H}$ be its set of hyperplanes.

Then $\mathcal{H}$ has the following properties:

a) $\forall X, Y \in \mathcal{H}$ such that $X \neq Y$ and $|X|, |Y| \geq r$, we have $|X \cap Y| \leq r - 2$. 

b) \( M \) is sparse-paving if and only if \( \mathcal{H} \subset \binom{S}{r-1} \cup \binom{S}{r} \).

**Proof:** By Welsh (II.1), \( \mathcal{H} \) is an \((r-1)\)-partition. Then \( \forall A \in \binom{S}{r-1}, \exists X \in \mathcal{H} \) such that \( A \subseteq X \). Then

a) Let \( X, Y \in \mathcal{H} \) satisfy \( X \neq Y \) and \( |X|, |Y| \geq r \). Then \( (r-1) \cap \binom{Y}{r-1} \neq \emptyset \).

Therefore, \( |X \cap Y| \leq r - 2 \), (otherwise, if \( |X \cap Y| \geq r - 1 \) then \( \exists A \subset X \cap Y \) and \( |A| = r - 2 \), a contradiction).

b) By (I.2), \( M \) is sparse-paving if and only if \( \mathcal{N}_2 = \emptyset \)

if and only if \( \binom{S}{r+1} = C_{r+1} \cup \mathcal{N}_1 \) (That is by (I.1)), \( \forall \exists X \in \binom{S}{r+1}, \text{rk}X = r \)

if and only if \( \mathcal{H} \subset \binom{S}{r-1} \cup \binom{S}{r} \). ■

II.3. The next result is the construction of all paving matroids. Moreover, if there exists a hyperplane \( X \) of cardinality bigger than \( r \), then the matroid is paving no-sparse-paving.

**Theorem.** Let \( S = \{1, \ldots, n\} \) be a set, \( r \in \mathbb{N} \) and \( 2 \leq r \leq n \).

Take \( \emptyset \neq \mathcal{H} \subset \mathcal{P}S \) such that \( \forall X \in \mathcal{H}, r \leq |X| < n \) and \( \forall X, Y \in \mathcal{H} \).

\( X \neq Y \implies |X \cap Y| \leq r - 2 \).

Let define by \( C_r := \{ A \in \binom{S}{r} ; \exists X \in \mathcal{H} \text{ with } A \subseteq X \} \) and \( \mathcal{H} := \mathcal{H} \cup \{ A \in \binom{S}{r} ; \forall D \in \binom{\mathcal{H}}{r} ; A \subseteq D \implies D \in \binom{\mathcal{H}}{r} \} \).

Then \( \mathcal{H} \) is the set of hyperplanes of a paving matroid on \( S \) of rank \( r \). With \( \mathcal{C}_r \) its set of \( r \)-circuits and \( \mathcal{B} = \binom{\mathcal{H}}{r} \backslash \mathcal{C}_r \) its set of basis.

**Proof.** To prove \( \mathcal{H} \) is an \((r-1)\)-partition of \( S \).

Define by \( \mathcal{H}_r := \{ A \in \binom{S}{r-1} ; \forall D \in \binom{\mathcal{H}}{r} ; A \subseteq D \implies D \in \binom{\mathcal{H}}{r} \} \).

i) By construction \( \mathcal{H}_{r-1} = \binom{S}{r-1} \backslash \{ A \in \binom{S}{r-1} ; \exists X \in \mathcal{H} \text{ with } A \subseteq X \} \). Thus, \( \bigcup_{X \in \mathcal{H}} X = S \).

ii) To prove that \( \forall A \in \binom{S}{r-1}, \exists X \in \mathcal{H}, A \subseteq X \).

\[ \text{ii.a)} \] If \( A \in \mathcal{H}_{r-1} \) we have that for all \( Y \neq A \) such that \( A \subseteq Y \), \( \text{rk}Y = n \).

Therefore, \( A \) is the unique hyperplane containing \( A \) itself.

\[ \text{ii.b)} \] If \( A \in \binom{S}{r-1} \backslash \mathcal{H}_{r-1} \). Then there exists \( C \in \mathcal{H}_r \cup \mathcal{C}_r \) such that \( A \subseteq C \),

where \( \mathcal{C}_r := \{ C \in \binom{S}{r} ; \exists X \in \mathcal{H}, |X| \geq r + 1 \text{ and } C \subset X \} \) and \( \mathcal{H}_r := \{ C \in \mathcal{H}, |X| = r \} \).

Subcases: \( C \in \mathcal{H}_r \) or \( C \in \mathcal{C}_r \).

**subcase**(ii.b.1): \( C \in \mathcal{H}_r \).

By (II.2), \( \forall X \in \mathcal{H} \setminus \{ C \}, |X \cap C| \leq r - 2 \), then (since \( |A| = r - 1 \), \( A \nsubseteq X \).)

**subcase**(ii.b.2): \( C \in \mathcal{C}_r \).

Then there exists \( X \in \mathcal{H} \) with \( |X| \geq r + 1 \) and \( A \subset C \subset X \). Again by (II.2), \( \forall Y \in \mathcal{H} \setminus \{ X \}, |A \cap Y| \leq |C \cap Y| \leq |X \cap Y| \leq r - 2 \), Thus \( A \nsubseteq Y \).

Therefore, \( \forall A \in \binom{S}{r-1}, \exists X \in \mathcal{H} \text{ such that } A \subseteq X \).
Therefore, $\mathcal{H}$ is the set of hyperplanes of a paving matroid on $S$. ■

**Corollary.** Let $M = (S, \mathcal{I})$ be a matroid of rank $r \leq n$ and let $\mathcal{H}$ be its set of hyperplanes. Then

$M$ is a paving matroid if and only if $\forall X \neq Y \in \mathcal{H}$ such that $|X|, |Y| \geq r$.

Then $|X \cap Y| \leq r - 2$.

## 5 An algorithm to construct the paving matroids

Let $n, r$ be natural numbers satisfying $r \leq n - 1$.

The algorithm below construct a maximal set of hyperplanes, $\mathcal{H}_t$, of cardinality $t \in \{r, \ldots, n - 1\}$ of a matroid $M = (S, \mathcal{I})$ of rank $r$.

The hyperplanes of cardinality $r - 1$, $\mathcal{H}_{r-1} = \binom{S}{r-1} \setminus \bigcup_{t=r}^{n-1} \mathcal{H}_t$

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**Algorithm 1:** Paving Matroids

**Input:** $r, n$

**Output:** $H$

$H = \emptyset$, $k = 0$;

$S = \{1, 2, \ldots n\}$;

**while** $k < \text{Bound}$ **do**

Choose $m \in [r, n - 1]$;

Choose $S_i \subset S$ such that $|S_i| = m$;

$flag = 0$;

**for** all $S_j \in \mathcal{H}$ **do**

**if** $|S_i \cap S_j| > r - 2$ **then**

$flag = 1$;

Break;

**end**

**end**

**if** $flag = 0$ **then**

$H = H \cup S_i$;

**end**

$k = k + 1$;

**end**
6 Appendix: Another construction of the Sparse-paving matroids, see [10]

Let \( S = \{1, \ldots, n\} \) be a set and \( 2 \leq r \leq n \).

Let take \( C_r \subseteq \binom{S}{r} \) such that \( \forall C, C' \in C_r, \ |C \cap C'| \leq r - 2 \). Then

**Theorem** [10]. Let \( S \) be a set of cardinality \( |S| = n \geq 3 \) and \( 2 \leq r \leq n - 1 \).

Let \( C \subseteq \binom{S}{r} \) be a set of \( r \)-subsets of \( S \), satisfying the following property

\[
\forall X, Y \in C \text{ with } X \neq Y \text{ then } |X \cap Y| \leq r - 2 \quad (**) \]

Define \( M := (S, \mathcal{I}) \) where \( \mathcal{B} := \binom{S}{r} \setminus C \) and \( \mathcal{I} := \{ X \subseteq S ; \exists B \in \mathcal{B} \text{ with } X \subseteq B \} \). Then, (A). \( M \) is a matroid of rk\( M = r \) and (B). \( M \) is sparse-paving.

**Proof.** Let \( S \) be a set and take a subset \( C \subseteq \binom{S}{r} \) satisfying the property (**).

Take \( M = (S, I) \) with set of basis \( \mathcal{B} = \binom{S}{r} \setminus C \).

**A. To prove** \( M \) is a matroid of rank \( \text{rk} M = r \).

For this proof, we will use an equivalent definition of matroid, which says:

Let \( M = (S, \mathcal{I}) \) is a matroid if and only if \( \mathcal{I} \) satisfies (I1),(I2) as in the introduction and (I3)': let \( B_1, B_2 \in \mathcal{B} \) be two basis of \( M \) and \( x \in B_1 \setminus B_2 \). To prove \( \exists y \in B_2 \setminus B_1 \) such that \( (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B} \).

**Case a.** If \( |S| = 3 \) and \( \text{rk} M = 2 \), the possibilities for \( C \) to have property (**), are \( C = \emptyset \) or \( |C| = 1 \). In both cases, \( M \) is a matroid and it is sparse-paving.

**Case b.** \( |S| \geq 4 \).

(I1) To prove that \( \emptyset \) is an independent set. It is enough to prove that \( \mathcal{B} \) is not empty.

Since \( n \geq 4 \), \( 2 \leq r \leq n - 1 \) and \( S = \{1, \ldots, r+1, \ldots, n\} \). Take \( A_1 = \{1, \ldots, r+1\} \), \( A_2 = \{1, \ldots, r-1, r+1\} \) which are subsets of \( S \) with cardinality \( r \) and \( |A_1 \cap A_2| = r - 1 \). Then by (**), \( \exists i \in \{1, 2\} \) such that \( A_i \subseteq B \). Then \( \mathcal{B} \neq \emptyset \).

(I2) Let \( Y \subseteq X \subseteq S \) such that \( \exists B \in \mathcal{B} \) with \( X \subseteq B \). Then \( Y \subseteq B \), that is \( Y \) is independent, by definition.

(I3)' Now, let \( B_1, B_2 \in \mathcal{B} \) be two basis of \( M \) and \( x \in B_1 \setminus B_2 \). To prove \( \exists y \in B_2 \setminus B_1 \) such that \( (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B} \).

\( I3'1. \) Assume \( m := |B_2 \setminus B_1| = 1 \). That is, \( B_2 \cap B_1 = B_1 \setminus \{x\} \) and \( B_2 = (B_1 \setminus \{x\}) \cup \{y\} \) for some \( y \in S \). Then \( (B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B} \).

\( I3'2. \) Let define \( m := |B_2 \setminus B_1| \geq 2 \) and let \( B_2 = (B_1 \cap B_2) \cup \{y_1, y_2, y_3, \ldots, y_m\} \). Define \( A_i := (B_1 \setminus \{x\}) \cup \{y_i\} \) for \( i = 1, \ldots, m \). Since \( \forall i \neq j, |A_i \cap A_j| = r - 1 \) and \( m \geq 2 \), by (**), \( \exists A_{i_0} \in \mathcal{B} \). Therefore, \( (B_1 \setminus \{x\}) \cup \{y_i\} = A_{i_0} \in \mathcal{B} \), and \( M \) is a matroid.

Rank: By definition of \( M \), \( \text{rk} M = r \).

**B. To prove** \( M \) is a sparse-paving matroid.
B.1. First we will prove that $M$ is a paving matroid. Equivalently, to prove $\forall Z \subseteq S$ of $|Z| = \text{rk}M - 1$, $Z \in I$. This proof is similar to the one of (I1). Namely:

Let $\text{rk}M \leq n - 1$. Since $n \geq 3$ and $|Z| = \text{rk}M - 1$, we have $S = Z \cup \{x_1, x_2, \ldots, x_m\}$ with $m \geq 2$. Let denote $A_i := Z \cup \{x_i\}$ for $i = 1, 2, \ldots, m$. By (***) and $m \geq 2$, there exists $i_0 \in \{1, \ldots, m\}$ such that $(Z \subset A_{i_0}) \notin B$. Then $Z \in I$.

B.2. And by (1.2), $M$ is a sparse-paving matroid. ■

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