Variation of Bergman kernels of adjoint line bundles

Hajime TSUJI

November 13, 2005

Abstract

Let $f : X \to S$ be a smooth projective family and let $(L, h)$ be a singular hermitian line bundle on $X$ with semipositive curvature current. Let $K_s := K(X_s, K_{X_s} + L | X_s, h | X_s)(s \in S)$ be the Bergman kernel of $K_{X_s} + L | X_s$ with respect to $h | X_s$ and let $h_B$ the singular hermitian metric on $K_{X_s} + L$ defined by $h_B |_{X_s} := 1/K_s$. We prove that $h_B$ has semipositive curvature. This is a generalization of the recent result of Berndtsson ([B1]).

Using this result, we give a new proof of Kawamata’s semipositivity theorem for the direct image of relative multi canonical bundle.

Contents

1 Introduction 1

2 Preliminaries 4

2.1 Singular hermitian metrics 4

2.2 Analytic Zariski decompositions 5

2.3 $L^2$-extension theorem 7

3 Proof of Theorem 1.4 9

4 Proof of Theorem 1.7 9

4.1 Dynamical construction of an AZD 9

4.2 Dynamical construction of AZD as a family 12

4.3 Case of general type 13

1 Introduction

The theory of Bergman kernels was initiated by S. Bergman ([B]) in 1933. But the variation of Bergman kernels has not been studied until quite recently. In fact in 2004, F. Maitani and H. Yamaguchi proved the following theorem and initiated the study of the variation of Bergman kernels.

Theorem 1.1 ([M-Y]) Let $\Omega$ be a pseudoconvex domain in $\mathbb{C}_z \times \mathbb{C}_w$ with smooth boundary. Let $\Omega_t := \Omega \cap (\mathbb{C}_z \times \{t\})$ and Let $K(z, t)$ be the Bergman kernel function of $\Omega_t$.

Then $\log K(z, t)$ is a plurisubharmonic function on $\Omega$. □
Recently generalizing Theorem 1.1, B. Berndtsson proved the following higher dimension and twisted version of Theorem 1.1.

**Theorem 1.2** ([B1]) Let $D$ be a pseudoconvex domain in $\mathbb{C}^n \times \mathbb{C}^k_t$. And let $\phi$ be a plurisubharmonic function on $D$. For $t \in \Delta$, we set $D_t := \Omega \cap (\mathbb{C}^n \times \{t\})$ and $\phi_t := \phi |_{D_t}$. Let $K(z, t)$ be the Bergman kernel of the Hilbert space

$$A^2(D_t, e^{-\phi_t}) := \{ f \in \mathcal{O}(\Omega_t) \mid \int_{D_t} e^{-\phi_t} |f|^2 < +\infty \}. $$

Then $\log K(z, t)$ is a plurisubharmonic function on $D$. □

As mentioned in [B2], his proof also works for a pseudoconvex domain in a locally trivial family of manifolds which admits a Zariski dense Stein subdomain.

Using Theorem 1.2, he proved the following theorem.

**Theorem 1.3** ([B2, Theorem 1.1]) Let us consider a domain $D = U \times \Omega$ and let $\phi$ be a plurisubharmonic function on $D$. For simplicity we assume that $\phi$ is smooth up to the boundary and strictly plurisubharmonic in $D$. Then for each $t \in U$, $\phi_t := \phi(\cdot, t)$ is plurisubharmonic on $\Omega$. Let $A^2_t$ be the Bergman space of holomorphic functions on $\Omega$ with norm

$$\|f\|^2 := \int_{\Omega} e^{-\phi_t} |f|^2. $$

The spaces $A^2_t$ are all equal as vector spaces but have norms that vary with $t$. Then “infinite rank” vector bundle $E$ over $U$ with fiber $E_t = A^2_t$ is therefore trivial as a bundle but is equipped with a notrivial metric. Then $(E, \|\|)$ is strictly positive in the sense of Nakano. □

In Theorem 1.2 the assumption that $D$ is a pseudoconvex domain in the product space is rather strong. And in Theorem 1.3 Berndtsson also assumed that $D$ is a product.

In this paper we shall remove these assumptions and generalize Theorems 1.2, 1.3 to the case of adjoint line bundles smooth projective fibrations.

By using this generalization we can study nonlocally trivial algebraic fiber space.

To state our theorem, let us introduce the notion of the Bergman kernels of adjoint line bundles. Let $X$ be a complex manifold of dimension $n$ and let $(L, h)$ be a singular hermitian line bundle (cf. Definition 2.1) on $X$. Let $K_X$ denote the canonical line bundle on $X$. Let $A^2(X, K_X + L, h)$ be the Hilbert space defined by

$$A^2(X, K_X + L, h) := \{ \sigma \in H^0(X, \mathcal{O}_X(K_X + L)) \mid (\sqrt{-1})^n \int_X h \cdot \sigma \wedge \bar{\sigma} < +\infty \}, $$

where we have defined the inner product on $A^2(X, K_X + L, h)$ by

$$\langle \sigma, \tau \rangle := (\sqrt{-1})^n \int_X h \cdot \sigma \wedge \bar{\tau}. $$

We define the Bergman kernel $K(X, K_X + L, h)$ of the adjoint bundle $K_X + L$ with respect to $h$ by

$$K(X, K_X + L, h) := (\sqrt{-1})^n \sum_i \sigma_i \wedge \bar{\sigma}_i.$$
where \( \{\sigma_i\} \) is a complete orthonormal basis of the Hilbert space \( A^2(X, K_X + L, h) \).

Then \( K(X, K_X + L, h) \) is independent of the choice of the complete orthonormal basis \( \{\sigma_i\} \). In fact

\[
K(X, K_X + L, h)(x) = \sup\{ (\sqrt{-1})^{n^2} \sigma(x) \wedge \bar{\sigma}(x) \mid \| \sigma \| = 1 \}
\]

holds.

Now we shall state the main theorem in this paper.

**Theorem 1.4** Let \( f : X \to S \) be a smooth projective family of projective varieties over a complex manifold \( S \). Let \( (L, h) \) be a singular hermitian line bundle on \( X \) such that \( \Theta_h \) is semipositive on \( X \). Let \( K_s := K(X_s, K_X + L|_{X_s}, h|_{X_s}) \) be the Bergman kernel of \( K_{X_s} + (L|_{X_s}) \) with respect to \( h|_{X_s} \). Then the singular hermitian metric \( h_B \) of \( K_{X/S} + L \) defined by

\[
h_B|_{X_s} := K^{-1}_s
\]

has semipositive curvature on \( X \).

Theorem 1.4 follows from Theorem 1.2 by a simple trick as follows. We may assume that \( S \) is the unit open disk \( \Delta \) centered at \( O \). \( f : X \to S \) is not locally trivial. We shall embed \( X \) into the trivial family \( p : X \times \Delta \to \Delta, p(x, t) = x(x \in X, t \in \Delta) \) by

\[
i : X \to X \times \Delta
\]

defined by

\[
i(x) := (x, f(x)).
\]

Then \( i(X) \) is a hypersurface in \( X \times \Delta \) and not a domain in \( X \times \Delta \). So we shall thicken \( i(X) \) by replacing \( X_t (t \in \Delta) \) by \( f^{-1}(\Delta(t, \varepsilon)) \), where \( \Delta(t, \varepsilon) \) denotes the open disk of radius \( \varepsilon \) centered at \( t \). In this way we construct a thickened family

\[
f_\varepsilon : X(\varepsilon) \to \Delta(1/2)
\]

which is considered to be a pseudoconvex domain in the product family \( X \times \Delta(1/2) \) over \( \Delta(1/2) \), where \( \Delta(1/2) \) denotes \( \Delta(0,1/2) \). Then Theorem 1.2 is applicable to the family of Bergman kernels of the adjoint bundle of \( p^*(L, h) \) over \( \Delta(1/2) \). Letting \( \varepsilon \) tend to 0, with the rescaling constant \( \pi \varepsilon^3 \), we obtain

Theorem 1.4.

As a direct consequence, we can also generalize Theorem 1.3 as follows.

**Theorem 1.5** Let \( f : X \to S \) be a smooth projective family of projective varieties over a complex curve \( S \) of relative dimension \( n \). Let \( (L, h) \) be a hermitian line bundle on \( X \) such that \( \Theta_h \) is semipositive on \( X \). We define the hermitian metric \( h_E \) on \( E := f_*\mathcal{O}_X(K_{X/S} + L) \) by

\[
h_E(\sigma, \tau) := (\sqrt{-1})^{n^2} \int_{X_s} h \cdot \sigma \wedge \bar{\tau}.
\]

Then \( (E, h_E) \) is semipositive in the sense of Nakano. Moreover if \( \Theta_h \) is strictly positive, then \( (E, h_E) \) is strictly positive in the sense of Nakano. \( \Box \)
Theorem 1.6 ([Kan], p.57, Theorem 1) Let $f : X \to C$ be an algebraic fiber space over a projective curve $C$. Then $f_* \mathcal{O}_X(mK_{X/S})$ is a semipositive vector bundle on $S$ in the sense that for any quotient sheaf $Q$ of $f_* \mathcal{O}_X(mK_{X/S})$, $\deg_C Q \geq 0$ holds. □

Theorem 1.7 Let $f : X \to S$ be projective family such that $X$ and $S$ are smooth. Let $S^\circ$ be a nonempty Zariski open subset such that $f$ is smooth over $S^\circ$. Then $K_{X/S}$ has a relative AZD $h$ over $S^\circ$ such that $\Theta_h$ is semipositive on $X$.

And $F_m := f_* \mathcal{O}_X(mK_{X/S})$ carries a continuous hermitian metric $h_{F_m}$ with Nakano semipositive curvature in the sense of current on $S^\circ$.

Let $x \in S - S^\circ$ be a point and let $\sigma$ be a local holomorphic section of $F_m$ on a neighbourhood $U$ of $x$. Then $\sqrt{-1}\partial \bar{\partial} \log h_{F_m}(\sigma, \sigma)$ extends as a closed positive current across $(S - S^\circ) \cap U$. □

Corollary 1.8 Let $f : X \to S$ be a smooth projective family. Then $P_m(X_s) = \dim H^0(X_s, \mathcal{O}_{X_s}(mK_{X_s}))$ is independent of $s \in S$. □

After I completed writing this work, I have received a preprint of Bo Berndtsson [B3], which proved Theorem 1.4 under the assumption that $h$ is $C^\infty$. His proof is more computational than the one here and works also for smooth proper Kähler morphisms. But it looks quite different from his proof of Theorem 1.2 which is very ingenious and beautiful. Also it is not clear whether his proof works also for singular $h$ although it seems not to be difficult at least for projective morphisms.

The proof presented here is very simple and based on the beautiful proof of Theorem 1.2 in [B1]. I would like to thank Professor Ohsawa for stimulating discussion.

2 Preliminaries

2.1 Singular hermitian metrics

In this subsection $L$ will denote a holomorphic line bundle on a complex manifold $M$.

Definition 2.1 A singular hermitian metric $h$ on $L$ is given by

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$-hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(M)$ is an arbitrary function on $M$. We call $\varphi$ a weight function of $h$. □

The curvature current $\Theta_h$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\Theta_h := \Theta_{h_0} + \sqrt{-1}\partial \bar{\partial} \varphi,$$

where $\partial \bar{\partial}$ is taken in the sense of a current. The $L^2$-sheaf $\mathcal{L}^2(L, h)$ of the singular hermitian line bundle $(L, h)$ is defined by

$$\mathcal{L}^2(L, h) := \{ \sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{\text{loc}}(U) \},$$
where $U$ runs over the open subsets of $M$. In this case there exists an ideal sheaf $\mathcal{I}(h)$ such that
\[ L^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h) \]
holds. We call $\mathcal{I}(h)$ the **multiplier ideal sheaf** of $(L, h)$. If we write $h$ as
\[ h = e^{-\varphi} \cdot h_0, \]
where $h_0$ is a $C^\infty$ hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(M)$ is the weight function, we see that
\[ \mathcal{I}(h) = L^2(\mathcal{O}_M, e^{-\varphi}) \]
holds. For $\varphi \in L^1_{\text{loc}}(M)$ we define the multiplier ideal sheaf of $\varphi$ by
\[ \mathcal{I}(\varphi) := L^2(\mathcal{O}_M, e^{-\varphi}). \]

**Example 2.2** Let $\sigma \in \Gamma(X, \mathcal{O}_X(L))$ be the global section. Then
\[ h := \frac{1}{|\sigma|^2} = \frac{h_0}{h_0(\sigma, \sigma)} \]
is a singular hermitian metric on $L$, where $h_0$ is an arbitrary $C^\infty$ hermitian metric on $L$ (the right hand side is obviously independent of $h_0$). The curvature $\Theta_h$ is given by
\[ \Theta_h = 2\pi\sqrt{-1} (\sigma) \]
where $(\sigma)$ denotes the current of integration over the divisor of $\sigma$. \(\square\)

**Definition 2.3** $L$ is said to be **pseudoeffective**, if there exists a singular hermitian metric $h$ on $L$ such that the curvature current $\Theta_h$ is a closed positive current. Also a singular hermitian line bundle $(L, h)$ is said to be **pseudoeffective**, if the curvature current $\Theta_h$ is a closed positive current. \(\square\)

### 2.2 Analytic Zariski decompositions

In this subsection we shall introduce the notion of analytic Zariski decompositions. By using analytic Zariski decompositions, we can handle big line bundles like nef and big line bundles.

**Definition 2.4** Let $M$ be a compact complex manifold and let $L$ be a holomorphic line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an analytic Zariski decomposition, if the followings hold.

1. $\Theta_h$ is a closed positive current,
2. for every $m \geq 0$, the natural inclusion
\[ H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL)) \]
is an isomorphism. \(\square\)

**Remark 2.5** If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above. \(\square\)
Theorem 2.6 (\[T1, T2\]) Let $L$ be a big line bundle on a smooth projective variety $M$. Then $L$ has an AZD.

As for the existence for general pseudoeffective line bundles, now we have the following theorem.

**Theorem 2.7** (\[D-P-S, Theorem 1.5\]) Let $X$ be a smooth projective variety and let $L$ be a pseudoeffective line bundle on $X$. Then $L$ has an AZD.

**Proof of Theorem 2.7** Although the proof is in \[D-P-S\], we shall give a proof here, because we shall use it afterwards.

Let $h_0$ be a fixed $C^\infty$-hermitian metric on $L$. Let $E$ be the set of singular hermitian metric on $L$ defined by

$$E = \{ h; h : \text{lowersemicontinuous singular hermitian metric on } L, \Theta_h \text{ is positive}, \frac{h}{h_0} \geq 1 \}.$$ 

Since $L$ is pseudoeffective, $E$ is nonempty. We set

$$h_L = h_0 \cdot \inf_{h \in E} \frac{h}{h_0},$$

where the infimum is taken pointwise. The supremum of a family of plurisubharmonic functions uniformly bounded from above is known to be again plurisubharmonic, if we modify the supremum on a set of measure 0 (i.e., if we take the uppersemicontinuous envelope) by the following theorem of P. Lelong.

**Theorem 2.8** (\[L, p.26, Theorem 5\]) Let $\{ \varphi_t \}_{t \in T}$ be a family of plurisubharmonic functions on a domain $\Omega$ which is uniformly bounded from above on every compact subset of $\Omega$. Then $\psi = \sup_{t \in T} \varphi_t$ has a minimum uppersemicontinuous majorant $\psi^*$ which is plurisubharmonic. We call $\psi^*$ the uppersemicontinuous envelope of $\psi$.

**Remark 2.9** In the above theorem the equality $\psi = \psi^*$ holds outside of a set of measure 0 (cf.\[L, p.29\]).

By Theorem 2.3, we see that $h_L$ is also a singular hermitian metric on $L$ with $\Theta_h \geq 0$. Suppose that there exists a nontrivial section $\sigma \in \Gamma(X, O_X(mL))$ for some $m$ (otherwise the second condition in Definition 2.3 is empty). We note that

$$\log | \sigma |^{\frac{1}{m}}$$

gives the weight of a singular hermitian metric on $L$ with curvature $2\pi m^{-1}(\sigma)$, where $\langle \sigma \rangle$ is the current of integration along the zero set of $\sigma$. By the construction we see that there exists a positive constant $c$ such that

$$\frac{h_0}{| \sigma |^{\frac{1}{m}}} \geq c \cdot h_L$$

holds. Hence

$$\sigma \in H^0(X, O_X(mL) \otimes \mathcal{I}_\infty(h_L^m))$$
holds. Hence in particular
\[ \sigma \in H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^n_L)) \]
holds. This means that \( h_L \) is an AZD of \( L \).

\[ \square \]

**Remark 2.10** By the above proof we have that for the AZD \( h_L \) constructed as above
\[ H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^n_L)) \simeq H^0(X, \mathcal{O}_X(mL)) \]
holds for every \( m \).

\[ \square \]

It is easy to see that the multiplier ideal sheaves of \( h^n_m(m \geq 1) \) constructed in the proof of Theorem 2.2 are independent of the choice of the \( C^\infty \)-hermitian metric \( h_0 \). We call the AZD constructed as in the proof of Theorem 2.7 a canonical AZD of \( L \).

### 2.3 \( L^2 \)-extension theorem

**Theorem 2.11** ([O-T, p.200, Theorem]) Let \( X \) be a Stein manifold of dimension \( n \), \( \psi \) a plurisubharmonic function on \( X \) and \( s \) a holomorphic function on \( X \) such that \( ds \neq 0 \) on every branch of \( s^{-1}(0) \). We put \( Y := s^{-1}(0) \) and \( Y_0 := \{ x \in Y; ds(x) \neq 0 \} \). Let \( g \) be a holomorphic \((n-1)\)-form on \( Y_0 \) with
\[ c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g} < \infty, \]
where \( c_k = (-1)^{\frac{k(k-1)}{2}}(\sqrt{-1})^k \). Then there exists a holomorphic \( n \)-form \( G \) on \( X \) such that
\[ G(x) = g(x) \wedge ds(x) \]
on \( Y_0 \) and
\[ c_n \int_X e^{-\psi}(1+|s|^2)^{-2}G \wedge \bar{G} \leq 1620\pi c_{n-1} \int_{Y_0} e^{-\psi} g \wedge \bar{g}. \]

\[ \square \]

For a extension from an arbitrary dimensional submanifold, T. Ohsawa extended Theorem 2.11 in the following way.

Let \( M \) be a complex manifold of dimension \( n \) and let \( S \) be a closed complex submanifold of \( M \). Then we consider a class of continuous function \( \Psi : M \to [-\infty, 0) \) such that
1. \( \Psi^{-1}(-\infty) \supset S \),
2. if \( S \) is \( k \)-dimensional around a point \( x \), there exists a local coordinate \((z_1, \ldots, z_n)\) on a neighbourhood of \( x \) such that \( z_{k+1} = \cdots = z_n = 0 \) on \( S \cap U \) and
\[ \sup_{U \setminus S} |\Psi(z) - (n-k) \log \sum_{j=k+1}^n |z_j|^2| < \infty. \]
The set of such functions $\Psi$ will be denoted by $\sharp(S)$.

For each $\Psi \in \sharp(S)$, one can associate a positive measure $dV_M|\Psi$ on $S$ as the minimum element of the partially ordered set of positive measures $d\mu$ satisfying

$$
\int_{S_k} f \, d\mu \geq \lim_{k \to \infty} \frac{2(n-k)}{\sqrt{2n-2k-1}} \int_M f \cdot e^{-\Psi} \cdot \chi_{R(\Psi,t)} \, dV_M
$$

for any nonnegative continuous function $f$ with $\text{supp} f \subset M$. Here $S_k$ denotes the $k$-dimensional component of $S$, $v_m$ denotes the volume of the unit sphere in $\mathbb{R}^{m+1}$ and $\chi_{R(\Psi,t)}$ denotes the characteristic function of the set $R(\Psi,t) = \{ x \in M \mid -t - 1 < \Psi(x) < -t \}$.

Let $M$ be a complex manifold and let $(E, h_E)$ be a holomorphic hermitian vector bundle over $M$. Given a positive measure $d\mu_M$ on $M$, we shall denote $A^2(M, E, h_E, d\mu_M)$ the space of $L^2$ holomorphic sections of $E$ over $M$ with respect to $h_E$ and $d\mu_M$. Let $S$ be a closed complex submanifold of $M$ and let $d\mu_S$ be a positive measure on $S$. The measured submanifold $(S, d\mu_S)$ is said to be a set of interpolation for $(E \otimes K_M, h_E \otimes (dV_M)^{-1}, dV_M)$, if there exists a bounded linear operator $I : A^2(S, E|_S, h_E, d\mu_S) \to A^2(M, E, h_E, d\mu_M)$ such that $I(f)|_S = f$ for any $f$. $I$ is called an interpolation operator. The following theorem is crucial.

**Theorem 2.12** ([12], Theorem 4)](12) Let $M$ be a complex manifold with a continuous volume form $dV_M$, let $E$ be a holomorphic vector bundle over $M$ with $C^\infty$-fiber metric $h_E$, let $S$ be a closed complex submanifold of $M$, let $\Psi \in \sharp(S)$ and let $K_M$ be the canonical bundle of $M$. Then $(S, dV_M(\Psi))$ is a set of interpolation for $(E \otimes K_M, h_E \otimes (dV_M)^{-1}, dV_M)$, if the followings are satisfied.

1. There exists a closed set $X \subset M$ such that

   (a) $X$ is locally negligible with respect to $L^2$-holomorphic functions, i.e., for any local coordinate neighbourhood $U \subset M$ and for any $L^2$-holomorphic function $f$ on $U \setminus X$, there exists a holomorphic function $f$ on $U$ such that $f|_{U \setminus X} = f$.

   (b) $M \setminus X$ is a Stein manifold which intersects with every component of $S$.

2. $\Theta_{h_E} \geq 0$ in the sense of Nakano,

3. $\Psi \in \sharp(S) \cap C^\infty(M \setminus S),$

4. $e^{-(1+\epsilon)\Psi} \cdot h_E$ has semipositive curvature in the sense of Nakano for every $\epsilon \in [0, \delta]$ for some $\delta > 0$.

Under these conditions, there exists a constant $C$ and an interpolation operator from $A^2(S, E \otimes K_M|_S, h \otimes (dV_M)^{-1}|_S, dV_M[\Psi])$ to $A^2(M, E \otimes K_M, h \otimes (dV_M)^{-1}, dV_M)$ whose norm does not exceed $C\delta^{-3/2}$. If $\Psi$ is plurisubharmonic, the interpolation operator can be chosen so that its norm is less than $2^4\pi^{1/2}$.
The above theorem can be generalized to the case that \((E, h_E)\) is a singular hermitian line bundle with semipositive curvature current (we call such a singular hermitian line bundle \((E, h_E)\) a pseudoeffective singular hermitian line bundle) as was remarked in [O].

**Lemma 2.13** Let \(M, S, \Psi, dV_M, dV_M[\Psi], (E, h_E)\) be as in Theorem 2.12. Let \((L, h_L)\) be a pseudoeffective singular hermitian line bundle on \(M\). Then \(S\) is a set of interpolation for \((K_M \otimes E \otimes L, dV_M^{-1} \otimes h_E \otimes h_L)\). \(\square\)

### 3 Proof of Theorem 1.4

Let \(f : X \to S\) be a projective family. Since the statement is local we may assume that \(S\) is the unit open disk \(\Delta\) in \(\mathbb{C}\). Let us consider the family

\[
X(\varepsilon)_t := f^{-1}(the \varepsilon\text{-neighbourhood of } t \text{ in } \Delta)
\]

where \(\varepsilon\) means the \(\varepsilon\text{-neighbourhood with respect to the Poincaré metric on } \Delta\). Then \(X(\varepsilon)\) is a pseudoconvex domain in \(X \times \Delta\). Then the Bergman kernel

\[
K(X(\varepsilon)_t, K_X + L, h_L),
\]

satisfies that

\[
\sqrt{-1} \partial \bar{\partial} \log K(X(\varepsilon)_t, K_X + L, h_L) \geq 0
\]

on \(X(\varepsilon)\) by Theorem 1.2. Since

\[
\lim_{\varepsilon \downarrow 0} (\pi \varepsilon^2) \cdot K(X(\varepsilon)_t, K_X + L, h_L) = K(X_t, K_{X_t} + L, h | X_t)
\]

holds.

In fact, if we consider the family

\[
\pi_{\varepsilon,t} : X(\varepsilon) \to \Delta(t, \varepsilon),
\]

where

\[
\Delta(t, \varepsilon) := \{ t' \in \mathbb{C} \mid |t' - t| < \varepsilon \}
\]

as a family over the unit open disk \(\Delta\) in \(\mathbb{C}\) by

\[
t' \mapsto \varepsilon^{-1}(t' - t),
\]

the limit as \(\varepsilon \downarrow 0\) is nothing but the trivial family \(X_t \times \Delta\).

This completes the proof of Theorem 1.4. \(\square\)

### 4 Proof of Theorem 1.7

#### 4.1 Dynamical construction of an AZD

Let \(X\) be a smooth projective variety and let \(K_X\) be the canonical line bundle of \(X\). Let \(n\) denote the dimension of \(X\). We shall assume that \(K_X\) is pseudoeffective. Then by Theorem , \(K_X\) admits an AZD \(h\).
Let $A$ be a sufficiently ample line bundle on $X$ such that for every pseudo-effective singular hermitian line bundle $(L, h_L)$
\[ \mathcal{O}_X(A + L) \otimes \mathcal{I}(h_L) \]
and
\[ \mathcal{O}_X(K_X + A + L) \otimes \mathcal{I}(h_L) \]
are globally generated. This is possible by [?, p. 667, Proposition 1].

Let $h_A$ be a $C^\infty$ hermitian metric on $A$ with strictly positive curvature. For $m \geq 0$, let $h_m$ be the singular hermitian metrics on $A + mK_X$ constructed as follows. Let $h_0$ be a $C^\infty$-hermitian metric $h_A$ on $A$ with strictly positive curvature. Suppose that $h_{m-1}$ ($m \geq 1$) has been constructed. We set
\[
K_m := K(X, m(K_X, h) + A, h_{m-1})
\]
\[
= \sup \{ |\sigma|^2 (\sqrt{-1})^{n^2} \int_X h_{m-1} \sigma \wedge \overline{\sigma} \leq 1, \sigma \in H^0(X, \mathcal{O}_X(mK_X + A) \otimes \mathcal{I}(h^m)) \}.
\]
And we define the singular hermitian metric $h_m$ on $A + mK_X$ by
\[
h_m := K_m^{-1}.
\]
It is clear that $K_m$ has semipositive curvature in the sense of currents. We note that for every $x \in X$
\[
K_m(x) = \sup \{ |\sigma|^2 (x); \sigma \in \Gamma(X, \mathcal{O}_X(A + mK_X \otimes \mathcal{I}(h^m))), \int_X h_{m-1} \cdot |\sigma|^2 = 1 \}
\]
holds by definition (cf. [K] p.46, Proposition 1.4.16).

We set
\[
\nu := \lim_{m \to \infty} \frac{\log \dim H^0(X, \mathcal{O}_X(A + mK_X) \otimes \mathcal{I}(h^m))}{\log m}
\]
and call it the numerical Kodaira dimension of $(K_X, h)$.

**Proposition 4.1** (cf. [T2])

\[
K_\infty := \lim_{m \to \infty} \sqrt[m]{(m!)^{-\nu} K_m}
\]
exists and
\[
h_\infty := 1/K_\infty
\]
is an AZD of $K_X$. □

**Proof of Proposition 4.1** There exists a positive constant $C$ such that
\[
h^0(X, \mathcal{O}_X(mK_X + A) \otimes \mathcal{I}(h)) \leq C m^\nu
\]
holds. Let $dV$ be a fixed $C^\infty$ volume form on $X$. Then by the submeanvalue inequality of plurisubharmonic function, we see that by induction there exists a positive constant $C_1$ such that
\[
h_A \cdot K_m \leq C_1 m^\nu dV^m
\]
Let $h$ be an AZD of $K_X$.

$$K_m(x) = \sup \{|\sigma|^2 \langle x \rangle ; \sigma \in \Gamma(X, O_X(A + mK_X)), \int_X h_{m-1} \cdot |\sigma|^2 = 1\}.$$  

Let $x \in X$ be a point. Let $H$ be a sufficiently ample divisor on $X$. Let $H_1, \cdots, H_{n-\nu}$ be a general member of $|H|$ containing $x$. And let $V := H_1 \cap \cdots \cap H_{n-\nu}$. Then the restriction morphism

$$H^0(X, O_X(mK_X + A) \otimes \mathcal{I}(h^m)) \to H^0(V, O_V(mK_X + A) \otimes \mathcal{I}(h^m))$$

is injective. Hence $(K_X, h)|V$ is big. By Kodaira’s lemma, $h|V$ is dominated by a singular hermitian metric $h'$ of $L|V$ such that $\Theta_{h'}$ is strictly positive on $V$. For $0 < \varepsilon < 1$ we set

$$h_{V, \varepsilon} := (h|V)^{1-\varepsilon} \cdot (h')^\varepsilon.$$  

Then $h|V < h_{V, \varepsilon}$ holds.

Suppose that

$$h_A \cdot K_m \geq C(m)(ml)^{\nu} \cdot h_{V, \varepsilon}^{-m} \quad (2)$$

holds on $V$ for some positive constant $C(m)$.

Then by the $L^2$-extension theorem, there exists a positive constant $C$ such that

$$h_A \cdot K_{m+1} \geq C \cdot C(m)(ml)^{\nu} \cdot (m + 1)^{\nu} \cdot h_{V, \varepsilon}^{-(m+1)} \quad (3)$$

holds on $V$, where $C$ is a positive constant independent of $m$.

Here we have applied the $L^2$-extension theorem, first to the extension from a point $x \in V$ and the second to the extension from $V$ to $X$. The constant $(m + 1)^{\nu}$ appears simply because $h_{V, \varepsilon}$ is strictly positive, hence we can take local frame $e$ of $K_X$ around $x \in V$ and coordinate $z_1, \cdots, z_\nu$ so that

$$h_{V, \varepsilon}(e, e) = (1 - \|z\|^2)h(e, e)(x) + o(\|z\|^2)$$

holds (cf[11, p.105, (1,11)]).

Hence combining (2) and (3) we have that there exists a positive constant $C_2$ independent of $m$ such that

$$h_A \cdot K_m \geq C_2^m (ml)^{-\nu} \cdot h_{V, \varepsilon}^{-m} \quad (4)$$

holds on $V$.

By (1), (2), (3), and (4), moving $x$ and $V$ and letting $\varepsilon$ tend to 0, we have that

$$K_\infty := \lim_{m \to \infty} \sqrt{\frac{m!}{(m!)^{\nu}} K_m}$$

exists and

$$h_\infty := \frac{1}{K_\infty}$$

is an AZD of $K_X$.

This completes the proof of Proposition 4.1. \Box
4.2 Dynamical construction of AZD as a family

Let \( f : X \rightarrow \Delta \) be a smooth projective family such that \( K_X \) is pseudoeffective.

Let \( n \) be the relative dimension of \( f : X \rightarrow \Delta \). Let \( A \) be a sufficiently ample line bundle as in the last subsection and let \( h_A \) be a hermitian metric with strictly positive curvature on \( X \).

Let \( h_0 \) be an AZD of \( X_0 = f^{-1}(0) \). We define the sequence of Bergman kernels

\[
K_{m,0} := K(X_0, m(K_{X_0}, h_0) + A, h_{m-1,0})
\]

starting

\[
K_{0,0} = (h_A \mid X_0)^{-1}
\]

as in the last subsection. Let \( \{\sigma_0^{(m)}, \ldots, \sigma_n^{(m)}\} \) be a set of orthonormal basis of

\[
H^0(X, \mathcal{O}_X(mK_{X_0} + A \mid X_0) \otimes \mathcal{I}(h_0^m))
\]

with respect to the inner product

\[
(\sigma, \sigma') := (\sqrt{-1})^n \int_{X_0} h_{m-1} \cdot \sigma \wedge \overline{\sigma}.
\]

Inductively we extend each \( \sigma_i^{(m)} \) to

\[
\tilde{\sigma}_i^{(m)} \in H^0(X, \mathcal{O}_X(mK_X + A) \otimes \mathcal{I}((\tilde{h}_{m-1}))
\]

with the estimate

\[
\| \tilde{\sigma}_i^{(m)} \|^2 = (\sqrt{-1})^n \int_X \tilde{h}_{m-1} \tilde{\sigma}_i^{(m)} \wedge \overline{\tilde{\sigma}_i^{(m)}} \leq C,
\]

where \( C \) is a positive constant independent of \( m \) and \( i \) and \( \tilde{h}_{m-1} \) is defined inductively by

\[
\tilde{h}_0 = h_A
\]

and

\[
\tilde{h}_j := \frac{1}{K_j}
\]

where

\[
\tilde{K}_j := \sum_{i=0}^{N(j)} |\tilde{\sigma}_i^{(j)}|^2.
\]

Then by the same argument as in Section 4.1, we see that

\[
K_\infty := \lim_{m \to \infty} \sqrt[2n]{(m!)^{-n}} K_{m,0}
\]

and

\[
\tilde{K}_\infty := \lim_{m \to \infty} \sqrt[n]{(m!)^{-n}} \tilde{K}_m
\]

exists and both nonzero. Hence we see that an AZD of \( K_X \) restricts to an AZD of \( K_{X_0} \).

**Proposition 4.2** Let \( f : X \rightarrow \Delta \) be a smooth projective family such that \( K_X \) is pseudoeffective. Then for an AZD \( h \) of \( K_X \). The restriction \( h \mid X_t \) is an AZD of \( K_{X_t} \). □
Remark 4.3 As in [T3], by using Theorem 2.12, Proposition 4.2 implies the invariance of plurigenra, Corollary 1.8. □

Now we shall prove Theorem 1.7.

Let \( f : X \rightarrow \Delta \) be a smooth projective family over the unit open disk \( \Delta \) with center \( O \). Let \( A \) be a sufficiently ample line bundle on \( X \) and let \( h_A \) be a \( C^\infty \) hermitian metric on \( A \). Let \( h \) be an AZD of \( K_X \) constructed as in Theorem 2.7. Then by Proposition 4.2, we have that the restriction of \( h \) to \( X_t := f^{-1}(t) \) is an AZD of \( K_{X_t} \) for every \( t \in \Delta \). Let \( \nu_t \) be the numerical Kodaira dimension of \((K_{X_t}, h | X_t)\). By Proposition 4.2, we see that \( \nu_t \) is independent of \( t \in \Delta \). Hence we shall denote \( \nu_t \) simply by \( \nu \).

Let us perform the dynamical construction of AZD as in Section 4.1. Namely for every \( t \in \Delta \), we start from \( h_A | X_t \), by induction we define the hermitian metric \( h_{m,t} \) as in Section 4.1. Then

\[
h_{\infty,t} := \lim inf_{m \to \infty} \sqrt[m]{(m!)^\nu h_{m,t}}
\]

is an AZD of \( K_{X_t} \). By Theorem 1.4, we see that the singular hermitian metric \( h_m | X_t = h_{m,t} | X_t \) has semipositive curvature in the sense of current on \( X_t \). Then by the construction the singular hermitian metric \( h_{\infty} \) on \( K_{X/\Delta} \) defined by

\[
h_{\infty} | X_t = h_{\infty,t}
\]

has semipositive curvature in the sense of current on \( X_t \). Hence by the construction \( h_{\infty} \) is an AZD of \( K_{X/\Delta} \). This completes the proof of Theorem 1.7 except the last assertion.

The last assertion follows from the fact that the neighbourhood of the singular fiber is a manifold. Hence the proof of Theorem 1.4 implies the assertion. □.

4.3 Case of general type

Let \( m \) be a positive integer. For a section \( \eta \in H^0(X, \mathcal{O}_X(mK_X)) \) we define a nonnegative number \( \| \eta \|_m \) by

\[
\| \eta \|_m = \int_X |(\eta \wedge \bar{\eta})|^m .
\]

Then \( \eta \mapsto \| \eta \|_m \) is a continuous pseudonorm on \( H^0(X, \mathcal{O}_X(mK_X)) \), i.e., it is a continuous and has the properties:

1. \( \| \eta \|_m = 0 \iff \eta = 0, \)
2. \( \| \lambda \eta \|_m = |\lambda| \cdot \| \eta \|_m \) holds for all \( \lambda \in \mathbb{C} \).

But it is not a norm on \( H^0(X, \mathcal{O}_X(mK_X)) \) except \( m = 1 \). We define a continuous section \( K_m \) of \( (K_X \otimes K_X)^\otimes m \)

\[
K_m(x) := \sup \{ |\eta(x)|^2 : \| \eta \|_m = 1 \} \quad (x \in X),
\]
where $|\eta(x)|^2 = \eta(x) \otimes \bar{\eta}(x)$. We call $K^{NS}_m$ the $m$-th Narashimhan-Simha potential of $X$. We define the singular hermitian metric $h^{NS}_m$ on $H^0(X, \mathcal{O}_X(mK_X))$ by

$$h^{NS}_m := \frac{1}{K^{NS}_m}.$$  

We call $h^{NS}_m$ the Narashimhan-Simha metric on $mK_X$. This metric is introduced by Narashimhan and Simha for smooth canonically polarized variety to study the moduli space of canonically polarized variety \([\text{N-S}]\). We note that the singularities of $h_m$ is located exactly on the support of the base locus of $|mK_X|$. Hence in the case of canonically polarized variety, $h_m$ is a nonsingular continuous metric on $mK_X$ for every sufficiently large $m$. Since $K^{NS}_m$ is locally the supremum of the family of powers of absolute value of holomorphic functions we see that the curvature

$$\Theta_{h^{NS}_m} := \sqrt{-1} \partial \bar{\partial} \log K^{NS}_m$$

is a closed positive current.

**Theorem 4.4 ([T4, Main Theorem])** Let $f : X \to S$ be a flat projective family of varieties with only canonical singularities over a complex manifold $S$. Let $h^{NS}_m$ be the Narashimhan-Simha singular hermitian metric on $mK_{X/S}$.

Then $h^{NS}_m$ has semipositive curvature in the sense of current on $X$. □

The dynamical construction of AZD in Section 4.1 works without an ample line bundle $A$ when the manifold is of general type and we can make the construction canonical.

The construction is as follows. Let $m_0$ be a sufficiently large positive integer such that $|m_0K_X|$ gives a birational embedding of $X$. Let $h^{NS}_{m_0}$ be the Narashimhan-Simha metric on $m_0K_X$. Then starting $h^{NS}_{m_0}$ we may construct an AZD as in Section 4.1. In fact one may easily seen that for a sufficiently large $m_0$, the movable part of $|m_0K_X|$ dominates an ample divisor $A$ used in Section 4.1.

Let $f : X \to S$ be a smooth projective family of manifolds of general type. Then by Theorem 4.4 we may construct an AZD of $K_{X/S}$ which is canonical. Hence we obtain the following theorem.

**Theorem 4.5** Let $f : X \to S$ be a smooth projective family of manifolds of general type. Then for every positive integer $m$, there exists a functorial $C^0$ hermitian metric $h_{F_m}$ on the vector bundle $F_m := f_*\mathcal{O}_{X/S}(mK_{X/S})$ with semipositive curvature current in the sense of Nakano.

Here “functorial” means that $h_{F_m}$ only depends on the birational moduli map $\text{birmod} : S \to \mathcal{M}_{\text{bir}}$, where $\mathcal{M}_{\text{bir}}$ denote the (set theoretic) moduli space of the birational equivalence classes of projective varieties of general type (after fixing $m_0$). □

As in [T4], using Theorem 4.5 we may prove the quasiprojectivity of the moduli space of canonically polarized varieties.

**References**

[B] Bergman, S., Über die kernfunction ein bereiches und ihr verhalten am runde, J. für Reine Angew. Math. 169(1933), 1-42.
[B1] Berndtsson, B., Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains, math.CV/0505469 (2005).

[B2] Berndtsson, B., Curvature of vector bundles and subharmonicity of vector bundles, math.CV/050570 (2005).

[B3] Berndtsson, B., Curvature of vector bundles associated to holomorphic fibrations, math.CV/0511225 (2005).

[BGS] Bismut, J.M., Gillet, H., Soulé, C.: Analytic torsion and holomorphic determinant bundles I, II, III. Commun. Math. Phy. 115 49–78 (1988); 115 79–126 (1988); 115 301–351 (1988).

[D-P-S] Demailly, J.P.-Peternell, T.-Schneider, M.: Pseudo-effective line bundles on compact Kähler manifolds, math. AG/0006025 (2000).

[Ka1] Kawamata, Y.: Kodaira dimension of Algebraic fiber spaces over curves, Invent. Math. 66 (1982), pp. 57-71.

[Ka2] Kawamata, Y., Minimal models and the Kodaira dimension, Jour. für Reine und Angewande Mathematik 363 (1985), 1-46.

[Ka4] Kawamata, Y.: Deformation of canonical singularities, Jour. of A.M.S. 12 (1999), 85-92.

[Kr] Krantz, S.: Function theory of several complex variables, John Wiley and Sons (1982).

[L] Lelong, P.: Fonctions Plurisousharmoniques et Formes Differentielles Positive, Gordon and Breach (1968).

[M-Y] Maitani, and Yamaguchi, S., Variation of Bergman metrics on Riemann surfaces, Math. Ann. 330 (2004) 477-489.

[N] Nadel, A.M.: Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. 132 (1990),549-596.

[N-S] Narasimhan, M.S. and Simha, R.R.: Manifolds with ample canonical class, Invent. math. 5 (1968), 120-128.

[O] Ohsawa, T.: On the extension of $L^2$ holomorphic functions V, effects of generalization. Nagoya Math. J. 161(2001) 1-21.

[O-T] Ohsawa, T and Takegoshi K., $L^2$-extension of holomorphic functions, Math. Z. 195 (1987),197-204.

[Ti] G. Tian, On a set of polarized Kähler metrics on algebraic manifolds, Jour. Diff. Geom. 32 (1990),99-130.

[T1] Tsuji H.: Analytic Zariski decomposition, Proc. of Japan Acad. 61(1992), 161-163.

[T2] Tsuji, H.: Existence and Applications of Analytic Zariski Decompositions, Trends in Math. Analysis and Geometry in Several Complex Variables, (1999), 253-272.
[T3] Tsuji, H.: Deformation invariance of plurigenera, Nagoya Math. J. 166 (2002), 117-134.

[T4] Tsuji, H.: Refined semipositivity and Moduli of canonical models, preprint (2005).

Author’s address
Hajime Tsuji
Department of Mathematics
Sophia University
7-1 Kioicho, Chiyoda-ku 102-8554
Japan
e-mail address: tsuji@mm.sophia.ac.jp