Abstract. The $q$-electroweak theory obtained by replacing $SU(2)$ by $SU_q(2)$ in the Weinberg-Salam model is experimentally not distinguishable from the standard model at the level of the doublet representation. However, differences between the two theories should be observable when higher dimensional representations are taken into account. In addition the possibility of probing non-local structure may be offered by the $q$-theory.
Introduction.

The Weinberg-Salam model is obtained by gauging $SU(2)_L \times U(1)$, where $SU(2)_L$ is the chiral isotopic spin group and $U(1)$ is the weak hypercharge group.\footnote{1} Since $SU(2)$ is a degenerate form of $SU_q(2)$, it may be interesting to consider the model obtained by gauging $SU_q(2)_L \times U(1)$. It is reasonable to do this since $SU(2)$, unlike the Poincaré group, is a phenomenological group and $SU_q(2)$ may also be phenomenologically useful. In addition $SU_q(2)$ possibly offers a probe of the non-local (solitonic) structure of massive particles.

Although $SU_q(2)$ is discussed in the mathematical literature as a quasi triangular Hopf algebra,\footnote{2} we shall here make use of the usual language of Lie groups in order to make connections with the standard theory. We shall first summarize the necessary information about $SU_q(2)$.

2. Irreducible Representations of $SU_q(2)$.

The two-dimensional representation of $SL_q(2)$ may be defined by

\[ T \epsilon T^t = T^t \epsilon T = \epsilon \]  

where $t$ means transpose and

\[ \epsilon = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix}, \quad q_1 = q^{-1} \]  

Set

\[ T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \]  

Then

\[ \alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \alpha \delta = q \beta \gamma = 1 \]
\[ \delta \beta = q_1 \beta \delta, \quad \delta \gamma = q_1 \gamma \delta, \quad \delta \alpha - q_1 \beta \gamma = 1 \]  

\[ \beta \gamma = \gamma \beta \]  

If $q = 1$, Eqs. (2.4) are satisfied by complex numbers and $T$ is defined over a continuum, but if $q \neq 1$, then $T$ is defined only over this algebra–a non-commuting space.

A two-dimensional representation of $SU_q(2)$ may be obtained by going to a matrix representation of (2.4) and setting

\[ \gamma = -q_1 \bar{\beta}, \quad \delta = \bar{\alpha} \]  

where the bar means Hermitian conjugate. Then

\[ \alpha \beta = q \beta \alpha, \quad \alpha \bar{\alpha} + \beta \bar{\beta} = 1 \]
\[ \alpha \bar{\beta} = q \bar{\beta} \alpha, \quad \bar{\alpha} \alpha + q_1^2 \bar{\beta} \beta = 1 \]  

\[ \beta \bar{\beta} = \bar{\beta} \beta \]  

(A)
and $T$ is unitary:

$$\bar{T} = T^{-1}$$

If $q = 1$, $(A)$ may be satisfied by complex numbers and $T$ is a $SU(2)$ unitary-simplectic matrix. If $q \neq 1$, there are no finite representations of $(A)$ unless $q$ is a root of unity. We shall assume that $q$ is real and $q < 1$.

The irreducible representations of $SU_q(2)$ are as follows:

$$\mathcal{D}^j_{mm'}(\alpha, \bar{\alpha}, \beta, \bar{\beta}) = \Delta^j_{mm'} \sum_{s,t} \left\langle \frac{n_+}{s} \right| q_1^{t(n_1-s+1)}(-)^t \delta(s+t,n_+') \alpha^s \beta^{n_+} \bar{\alpha}^{n_-} \bar{\beta}^t \right| \frac{\langle n_0 \rangle_1}{\langle n_0 \rangle_1!} \langle n_0 \rangle_1! \langle n_s \rangle_1! \langle n_t \rangle_1!$$

where

$$n_\pm = j \pm m$$
$$n_\pm' = j \pm m'$$

$$\Delta^j_{mm'} = \left[ \frac{\langle n_+ \rangle_1! \langle n_- \rangle_1!}{\langle n_0 \rangle_1! \langle n_0 \rangle_1!} \right]^{1/2}$$

In the limit $q = 1$ $\mathcal{D}^j_{mm'}$ become the Wigner functions, $D^j_{mm'}(\alpha, \beta, \gamma)$, the irreducible representations of $SU(2)$.

### 3. The Lie Algebra of $SU_q(2)$

The Lie algebra of $SU_q(2)$ may be obtained in the following way. The two-dimensional representation $T$ may be Borel factored:

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = e^{\beta \sigma_x} e^{\lambda \delta \sigma_3} e^{C \sigma_-} \quad (3.1)$$

The algebra of $(\alpha, \beta, \gamma, \delta)$ is then inherited by $(B, C, \theta)$ as

$$(B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C \quad \lambda = \ln q \quad (3.2)$$

The 2$j$+1 dimensional irreducible representation of $SU_q(2)$ shown in (2.6) may be rewritten in terms of $(B, C, \theta)$. Then by expanding to terms linear in $(B, C, \theta)$ one has

$$\mathcal{D}^j_{mm'}(B, C, \theta) = \mathcal{D}^j_{mm'}(0, 0, 0) + B(J^j_B)_{mm'} + C(J^j_C)_{mm'} + 2\lambda \theta (J^j_\theta)_{mm'} + \ldots \quad (3.3)$$

where the non-vanishing matrix coefficients $(J^j_B)_{mm'}$, $(J^j_C)_{mm'}$, and $(J^j_\theta)_{mm'}$ are

$$\langle m-1|J^j_B|m\rangle = (\langle j + m\rangle_1 | j - m + 1\rangle_1)^{1/2} \quad (3.4B)$$
$$\langle m+1|J^j_C|m\rangle = (\langle j - m\rangle_1 | j + m + 1\rangle_1)^{1/2} \quad (3.4C)$$
$$\langle m|J^j_\theta|m\rangle = m \quad (3.4\theta)$$
and where
\[ \langle n \rangle_1 = \langle n \rangle_{q^2_1} = \frac{q^{2n}_1 - 1}{q^2_1 - 1} \] (3.5)
is a basic integer corresponding to \( n \). The \((B,C,\theta)\) and \((J_B,J_C,J_\theta)\) are generators of two dual algebras satisfying the following commutation rules:
\[ (J_B,J_\theta) = -J_B \quad (J_C,J_\theta) = J_C \quad (J_B,J_C) = q^{2j-1}_1 [2J_\theta] \] (3.6)
\[ (B,C) = 0 \quad (\theta,B) = B \quad (\theta,C) = C \] (3.7)
Here
\[ [x] = \frac{q^x - q^{x}_1}{q - q_1} \] (3.8)

4. Fundamental and Adjoint Representations.

For comparison with the standard theory we need to know the fundamental and adjoint representations. In the fundamental representation \((j = 1/2)\) we have by (3.4)-(3.6):
\[ J_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad J_\theta = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix} \] (4.1)
and
\[ (J_B,J_C) = 2J_\theta \quad (J_B,J_\theta) = -J_B \quad (J_C,J_\theta) = J_C \] (4.2)
In the adjoint representation \((J = 1)\) we have by the same relations (3.4)-(3.6):
\[ J_B = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \quad J_C = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \quad J_\theta = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \] (4.3)
and
\[ (J_B,J_C) = x^2 J_B \quad (J_B,J_\theta) = -J_B \quad (J_C,J_\theta) = J_C \] (4.4)
where
\[ x = \langle 1|J_B|0 \rangle = \langle 2 \rangle_1^{1/2} \langle 1 \rangle_1^{1/2} \] (4.5)
Here \( \langle n \rangle_1 \) is given by (3.5). Then
\[ x = \langle 2 \rangle_1^{1/2} = (1 + q^2_1)^{1/2} \] (4.6)
and
\[ (J_B,J_C) = (1 + q^2_1)J_\theta \] (4.7)
In general we have by (3.4) for all representations:

\[(J_B, J_C) = q_1^{2j-1} \, [2J_\theta]_q\]  \hspace{1cm} (4.8)

where

\[ [2J_\theta]_q = \frac{q^{2J_\theta} - q_1^{2J_\theta}}{q - q_1} = \frac{e^{2\lambda J_\theta} - e^{-2\lambda J_\theta}}{q - q_1} \]  \hspace{1cm} (4.9)

and

\[ q = e^\lambda \]  \hspace{1cm} (4.10)

Then

\[ [2J_\theta]_q = \frac{2 \sinh 2\lambda J_\theta}{q - q_1} = \frac{2}{q - q_1} \left[ 2\lambda J_\theta + \frac{(2\lambda J_\theta)^3}{3!} + \ldots \right] \]  \hspace{1cm} (4.12)

In the fundamental representation \( J_\theta \) is given by (4.1) and

\[ J_\theta^2 = \frac{1}{4} I \]  \hspace{1cm} (4.13)

Then by (4.12)

\[ [2J_\theta]_q = \frac{2J_\theta}{q - q_1} \left[ 2\lambda + \frac{2\lambda^3}{3!} + \ldots \right] \]  \hspace{1cm} (4.14)

\[ [2J_\theta]_q = \frac{4J_\theta}{q - q_1} \sinh \lambda = 2J_\theta \]  \hspace{1cm} (4.15)

Then one sees that (4.8), with the aid of (4.15) reduces to (4.2) in the fundamental representation.

In the adjoint representation \( J_\theta \) is given by (4.3) and

\[ J_\theta^3 = J_\theta \]  \hspace{1cm} (4.16)

Hence every odd power of \( J_\theta \) may be reduced as follows:

\[ J_\theta^{2p+1} = J_\theta^{3p-p+1} = J_\theta^{p-p+1} = J_\theta \]  \hspace{1cm} (4.17)
By (4.12) and (4.17)

\[
[2J_\theta]_q = \frac{2J_\theta}{q - q_1} \left[ 2\lambda + \frac{(2\lambda)^3}{3!} + \ldots \right]
\]

\[
= \frac{2J_\theta}{q - q_1} \sinh 2\lambda
\]

\[
= J_\theta \frac{q^2 - q_1^2}{q - q_1}
\]

\[
= (q + q_1)J_\theta
\]

Hence by (4.8) in the \(J = 1\) representation

\[
(J_B, J_C) = q_1(q + q_1)J_\theta
\]

\[
= (1 + q_1^2)J_\theta
\]

(4.19)

in agreement with (4.7).

Let us now change to the \((J_1, J_2, J_3)\) basis as follows:

\[
J_B = J_1 + i J_2
\]

\[
J_C = J_1 - i J_2
\]

\[
J_\theta = J_3
\]

(4.20)

In the fundamental representation, Eqn. (4.2) may be rewritten by (4.20) as

\[
(J_1, J_2) = i J_3
\]

\[
(J_2, J_3) = i J_1
\]

\[
(J_3, J_1) = i J_2
\]

(4.21a)

where as usual

\[
J_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

\[
J_2 = \frac{1}{2} \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}
\]

\[
J_3 = \frac{1}{2} \begin{pmatrix}
i & 0 \\
0 & -1
\end{pmatrix}
\]

(4.21b)

In the adjoint representation Eqs. (4.4) may be rewritten by (4.20) as

\[
(J_1, J_2) = i \frac{\langle 2 \rangle}{2} J_3
\]

\[
(J_2, J_3) = i J_1
\]

\[
(J_3, J_1) = i J_2
\]

(4.22a)

where

\[
J_1 = \frac{\langle 2 \rangle^{1/2}}{2} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

\[
J_2 = \frac{\langle 2 \rangle^{1/2}}{2} \begin{pmatrix}
0 & -i & 0 \\
i & 0 & -i \\
0 & i & 0
\end{pmatrix}
\]

\[
J_3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]

(4.22b)
Now write (4.21a) and (4.22a) in the following familiar way:

\[(t_a, t_b) = f_{ab}^m t_m \]  

(4.23)

The \( f_{ab}^m \) are the same for all representations of \( SU(2) \). For \( SU_q(2) \) however, the \( f_{ab}^m \) are different in the fundamental and adjoint representations, and in higher representations the commutator is no longer linear in the generators.

When (4.23) does hold, however, we also have

\[ \text{Tr}(t_a, t_b)t_d = f_{ab}^m \text{Tr} t_m t_d \]  

(4.24)

Introducing the “group metric”, namely

\[ g_{md} = \text{Tr} t_m t_d \]  

(4.25)

we have

\[ f_{abd} \equiv f_{ab}^m g_{md} = \text{Tr}(t_a, t_b)t_d \]  

(4.26)

\[ = \text{Tr}(t_d, t_a)t_b = -\text{Tr}(t_a, t_d)t_b \]

\[ = f_{dab} = -f_{adb} \]  

(4.27)

Then the \( f_{abd} \) are the completely antisymmetric structure constants. For the adjoint representation of \( SU_q(2) \) we have by (4.22b)

\[ g_{ab} = g_a \delta_{ab} \]  

(4.28a)

\[ g_1 = g_2 = 1 \]  

(4.28b)

\[ g_3 = 2 \]  

(4.28c)

The “structure constants” exhibited by (4.22) are

\[ f_{12}^3 = i \frac{(2)}{2} \]

\[ f_{23}^1 = f_{31}^2 = i \]

(4.29)

Then by (4.26), (4.28), and (4.29)

\[ f_{123} = f_{231} = f_{312} = i(2) \epsilon_{123} \]  

(4.30)

In the light of these properties of \( SU_q(2) \) let us now consider \( q \)-electroweak.
5. \textit{q-Electroweak.}

In the Weinberg-Salam model the Lagrangian density is\textsuperscript{4}

\begin{equation}
\mathcal{L} = \frac{1}{4}(G^{\mu\nu}G_{\mu\nu} + H^{\mu\nu}H_{\mu\nu}) + i(\bar{L}i\gamma L + \bar{R}i\gamma R) \\
+ (\overline{D\phi})(D\phi) - V(\bar{\phi}\phi) - \frac{m}{p_o}(\bar{L}\bar{\phi}R + \bar{R}\bar{\phi}L)
\end{equation}

(5.1)

where the covariant derivative is

\begin{equation}
D = \partial + ig\bar{\phi} + ig'W_o t_o
\end{equation}

(5.2)

Here $\bar{W}^\mu$ and $W_o^\mu$ are the connection fields of $SU(2)_L$ and $U(1)$, the chiral isotopic spin and hypercharge groups with independent coupling constants $g$ and $g'$, while $G$ and $H$ are the corresponding field strengths. The Lagrangian (5.1) also contains the contribution of one lepton doublet and the mass generating Hibbs doublet $\varphi$.

In (5.2), the expression for the covariant derivative, the matrices $\bar{t}$ and $t_o$ are the generators of the $SU(2)$ and $U(1)$ groups. If we now pass to $SU_q(2)$ without changing $U(1)$, Eqs. (5.2) will be unchanged since Eqs. (4.1) and (4.2) hold for both $SU(2)$ and $SU_q(2)$.

If no other changes are made in the standard theory, the expression for the covariant derivative may be rewritten as follows:

\begin{equation}
D = \partial + ig\bar{W} + ig'W_o t_o
\end{equation}

(5.3)

where $A$ and $Z$ are the Maxwell and Weinberg-Salam neutral fields while $W_+$ and $W_-$ are the charged vector fields and where by Weinberg-Salam:

\begin{align}
Q &= t_3 + t_o \\
Q' &= t_3 \cot \theta - t_o \tan \theta \\
e &= g \sin \theta = g' \cos \theta
\end{align}

(5.4) \hspace{1cm} (5.5) \hspace{1cm} (5.6)

The masses of the charged $W^\pm$ and neutral $Z$ are then determined by the Higgs mechanism as implemented in the unitary gauge by (5.1) and (5.3) to be

\begin{align}
M_W^2 &= \frac{1}{2}g^2 p_o^2 \\
M_Z &= \frac{M_W}{\cos \theta}
\end{align}

(5.7) \hspace{1cm} (5.8)
Eq. (5.8) has been checked by direct and independent measurements of the masses of the $W^\pm$ and $Z$ on the one hand and of the Weinberg angle $\theta$ on the other.

None of these relations is changed on passing from the algebra of $SU(2)$ to the algebra of $SU_q(2)$ since the three $\vec{t}$ matrices in the fundamental representation are the same for both. In particular (5.8) still holds and provides strong experimental confirmation of the Weinberg-Salam model. On the other hand higher dimensional representations, including already the adjoint representation, will be different for $SU_q(2)$ as one sees from (4.3) and (4.4) and also from the Clebsch-Gordan coefficients of $SU_q(2)$. Hence differences introduced by the $q$-theory are in principle detectable, although they would be difficult to quantitatively isolate from radiative corrections that are also present. Let us next see how differences in the adjoint representation manifest themselves in a more general $q$-gauge theory.

6. General $q$-Gauge Theory.

The Lagrangian density for the standard non-Abelian gauge theory may be written in the following form:

$$\mathcal{L} = -\frac{1}{4} g_{ab} F^a_{\mu\nu} F^{b\mu\nu} + \mathcal{L}_m(\psi, \nabla_\nu \psi) \quad (6.1)$$

where

$$\nabla_\mu = \partial_\mu + A_\mu \quad (6.2)$$

$$A_\mu = igA^a_\mu t_a \quad (6.3)$$

$$F_{\mu\nu} = (\nabla_\mu, \nabla_\nu) \quad (6.4)$$

Here $\nabla_\mu$ is the covariant derivative. In (6.3) $g$ is the weak coupling constant and $t_a$ are the generators of $SU(2)$ in the adjoint representation. $\mathcal{L}$ is invariant under infinitesimal gauge transformations as follows:

$$\begin{pmatrix}
\delta F^a \\
\delta \nabla^a \\
\delta \varphi^a \\
\delta (\nabla \varphi)^a
\end{pmatrix} = i \epsilon \begin{pmatrix}
f^a_{\cd}
\end{pmatrix} \begin{pmatrix}
F^b \\
\nabla^b \\
\varphi^b \\
(\nabla \varphi)^b
\end{pmatrix} \quad (6.5)$$

We may pass from these relations for the standard theory to the corresponding relations for the $q$-standard theory by replacing the $\vec{t}$ of $SU(2)$ by the $\vec{t}$ of $SU_q(2)$ in the adjoint representation, and by replacing the structure constants of the $SU(2)$ algebra by the corresponding $f^a_{\cd}$ of $SU_q(2)$.
The invariance of scalar products like $g_{ab}F^a F^b$ with the transformation rule (6.5) still holds for $SU_q(2)$ since

$$\delta(g_{ab}F^a F^b) \sim \epsilon^{c} f_{ace} F^a F^e = 0 \quad (6.6)$$

where $f_{ace}$ is the completely antisymmetric symbol introduced in (4.26).

Let us now consider the self-coupling of the vector field. We have

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^a_{\ bc} A^b_\mu A^c_\nu \quad (6.7)$$

and

$$-\frac{1}{4} g_{mn} F^m_{\mu\nu} F^n_{\mu\nu} = -\frac{1}{4} g_{mn} (\partial_\mu A^m_\nu - \partial_\nu A^m_\mu + f^m_{\ bc} A^b_\mu A^c_\nu)(\partial^\nu A^m_{\nu'n} - \partial^\nu' A^m_{\nu n} + f^n_{\ k\ell} A^k_\mu A^\ell_{\nu'}) \quad (6.8)$$

The trilinear couplings are then

$$\sim g_{mn} f^m_{\ bc} A^b_\mu A^c_\nu (\partial^\nu A^m_{\nu'n} - \partial^\nu' A^m_{\nu n}) = f_{nbc} A^b_\mu A^c_\nu (\partial^\nu A^m_{\nu'n} - \partial^\nu' A^m_{\nu n}) \quad (6.9)$$

where

$$f_{nbc} = i(\langle 2 \rangle_1 \epsilon_{nbc} \quad \text{by (4.30)}$$

$$= i(1 + q_1^2) \epsilon_{nbc} \quad (6.10)$$

The weak coupling constant associated with the $SU_q(2)$ theory is therefore greater than the corresponding coupling constant associated with the $SU(2)$ theory by the ratio $\frac{1}{2}(1 + q_1^2)$, when $q_1 > 1$. Since the factor $(1 + q_1^2)$ can be absorbed into a new definition of the coupling constant, however, there is no observable difference between the two theories coming from the trilinear couplings.

The quadlinear couplings are on the other hand

$$\sim g_{mn} f^m_{\ bc} f^n_{\ k\ell} A^b_\mu A^c_\nu A^k_\mu A^\ell_{\nu} \quad (6.11)$$

Here

$$g_{mn} f^m_{\ bc} f^n_{\ k\ell} = f_{nbc} f^m_{\ k\ell} \quad (6.12)$$

At this point there appears a real difference between the $SU(2)$ and $SU_q(2)$ theories coming from the $f^m_{\ k\ell}$ as shown by (4.29). This difference arises from the asymmetry in (4.22). It distinguishes one preferred direction in isotopic spin space, and in principle should be experimentally detectable.

Let us next consider the new degrees of freedom provided by the $q$-theories.
7. The Dual Algebra.

There are two $q$-algebras, $G_q$ and $g_q$, that are respectively deformations of the Lie group ($G$) and its algebra ($g$). The matrix elements of the representation matrices of $G$, $g$, and $g_q$ commute, but the same is not true for $G_q$ as one sees in (2.4) for example.

The deformation of the standard model that we have been discussing is based on $g_q$, the deformed Lie algebra and we have seen that it is a possibly acceptable modification of the standard model.

There is also a more speculative construction based on $G_q$, the deformed Lie group described in Section 2. In this construction we assume that all fields lie in the algebra of $SU_q(2)$. Thus a generic field may be expanded as follows:

$$\psi(x) = \sum \varphi_{mn}^j D_{mn}^j(\alpha|q)$$  \hspace{0.5cm} (7.1)

$$\bar{\psi}(x) = \sum \bar{\varphi}_{mn}^j D_{mn}^j(\alpha|q)$$  \hspace{0.5cm} (7.2)

where the $D_{mn}^j(\alpha|q)$ are the irreducible representations of $SU_q(2)$.

The partial fields $\varphi_{mn}^j(x)$ appearing in (7.1) may be expanded in Fock annihilation and creation operators

$$\varphi_{mn}^j(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d\vec{p}}{(2p_o)^{1/2}} \left[ e^{-ipx} a_{mn}^j(p) + e^{ipx} \bar{a}_{mn}^j(p) \right]$$  \hspace{0.5cm} (7.3)

To illustrate, consider the normal ordered Hamiltonian of a global scalar field

$$H^{(q)} = \frac{1}{2} \int \left[ \sum_0^3 \partial_k \bar{\psi} \partial_k \psi + m_o^2 \bar{\psi} \psi \right] : d\vec{x}$$  \hspace{0.5cm} (7.4)

where the symbol $h$ standing before the spatial integral denotes Woronowitz integration over the algebra $G_q$. Then by (7.1)-(7.3)

$$H^{(q)} = \int d\vec{p} p_o \sum_{j^m n^m} h(D_{mn}^j D_{m'n'}^{j'}) \frac{1}{2} : \left[ \bar{a}_{mn}^j(p) a_{m'n'}^{j'}(p) + a_{m'n'}^{j'}(p) \bar{a}_{mn}^j(p) \right] :$$  \hspace{0.5cm} (7.5)

The orthogonality of the $q$-irreducible representations is now expressed in terms of the following integral over the algebra:

$$h(D_{mn}^j D_{m'n'}^{j'}) = \delta^{jj'} \delta_{mm'} \delta_{nn'} \frac{q^{2n}}{[2j+1]_q}$$  \hspace{0.5cm} (7.6)

Hence

$$H^{(q)} |N(p); jmn\rangle = \frac{p_o q^{2n}}{[2j+1]_q} |N(p); jmn\rangle$$  \hspace{0.5cm} (7.7)
and the mass of a single field particle with “internal” quantum numbers \((jmn)\) is\(^7\)

\[
m_q^2 \frac{m_q^{2n}}{[2j + 1]_q}
\]

The spectrum resembles the square root of the spectrum of the \(q\)-H atom.

One may regard the mass \(m_o\) in (7.4) as generated by a Higgs type mechanism. Then by (7.8) the field particles have excited states that depend on the internal quantum numbers. The existence of excited states, impossible for a point particle, reveals non-local or solitonic structure. In the scenario sketched in this note, the dual algebras provide complementary pictures of the field particles: the \(G_q\) picture is microscopic and describes solitonic structure, while the \(g_q\) picture is phenomenological and approximates the standard theory. One may also speculate that \(SU(3)\) flavor and \(SU(3)\) color may be similarly related: that they may be based on the dual algebras of \(SU_q(3)\).

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