Two-Level Lipkin Model in Unconventional Boson Realization

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With the aim of applying to the Lipkin model in the case of open shell system, a possible form of the boson realization for the $\mathfrak{su}(2)$-algebra is proposed both in the Schwinger and the Holstein-Primakoff representation. The basic idea is borrowed from the Schwinger boson realization for the $\mathfrak{su}(4)$-algebra for many-quark system recently presented by the present authors. As the simplest approximation, a certain approximate form is given and the result is a natural generalization of the RPA result in the case of closed shell system.

§1. Introduction

It is well known that, with the aid of boson operators, we can describe various phenomena of nuclear and hadron physics successfully. Especially, the studies of microscopic structures of the boson operators trace back to the year 1960. In this year, Marumori, Arvieu & Veneroni and Baranger¹ proposed independently a theory, which is called as the quasi-particle random phase approximation. With the aid of this theory, we could understand microscopic structure of the boson operators describing the collective vibrational motion observed in the spherical nuclei. Further, the success of this theory has stimulated the studies of higher order corrections and one of the goal is the boson expansion theory: Belyaev & Zelevinsky, Marumori, Yamamura & Tokunaga, da Providência & Weneser and Marshalek.² We can find various further studies concerning the boson expansion theory in the review by Klein & Marshalek.³ Especially, this review concentrated on the boson realization of Lie algebra governing many-fermion system under consideration. The above is a rough sketch of the boson expansion theory at early stage. After these studies, too many papers have been published and it is impossible to follow them completely. Then, hereafter, we will narrow the discussion down to the boson realization of the $\mathfrak{su}(2)$-algebra.

We know two simple many-fermion models, which obey the $\mathfrak{su}(2)$-algebra: The single-level pairing model⁴ and the two-level Lipkin model⁵ Each is composed of three $\mathfrak{su}(2)$-generators $\tilde{S}_\pm$ and $\tilde{S}_0$, which are of the bilinear forms for the fermion operators. Further, each contains the total fermion number operator $\tilde{N}$. The operator $\tilde{N}$ commutes with the Hamiltonian which is widely adopted. In the pairing model, $\tilde{S}_0$ is a linear function of $\tilde{N}$. Therefore, the change of the eigenvalue of $\tilde{S}_0$ automatically leads to the change of the total fermion number. In the Lipkin model,
$\tilde{S}_0$ is proportional to the difference between the fermion number operators of the two levels. The total fermion number operator is the sum of both fermion number operators. Therefore, the change of the eigenvalue of $\tilde{S}_0$ corresponds to the change of the difference between the fermion numbers of the two levels. The eigenvalues $\tilde{S}_0$ and $\tilde{N}$ are independent from each other. The above is an essential difference between the two models.

The single-level pairing model can be completely formulated in the frame of the conventional boson realization of the $su(2)$-algebra, namely, the Schwinger\textsuperscript{[6]} and the Holstein-Primakoff\textsuperscript{[7]} boson realization. In the former, the magnitude of the $su(2)$-spin is treated as $q$-number. In the later, it is treated as $c$-number. Of course, the total fermion number operator can be expressed in terms of the boson operators which are used for the generators. However, the case of the two-level Lipkin model is different from the case of the single-level pairing model. As was already mentioned, the total fermion number operator cannot be expressed in the frame of the boson operators in the conventional boson realization of the $su(2)$-algebra. From the above reason, usually, the total fermion number is fixed to the value corresponding to the closed shell system. Then, formally, we can apply the conventional boson realization of the $su(2)$-algebra used in the pairing model. Of course, the roles of various quantities appearing in the two models are different from each other. The above consideration suggests us that we must add a special device in order to complete the boson realization for the Lipkin model in the case of any fermion number.

Main aim of this paper is to present a possible form of the boson realization of the $su(2)$-algebra which is effective to the two-level Lipkin model in the case of any fermion number. We call it as unconventional boson realization, because conventional realization is not useful for the present case. The hint can be found in the Schwinger boson realization of the $su(4)$-algebra for the Bonn model and its modification describing many-quark system.\textsuperscript{[8]} The idea of these works is based on the Schwinger boson realization for the $su(4)$-algebra presented by the present author with Kuriyama and Kunihiro,\textsuperscript{[9]} which was intended to apply to the high temperature superconductivity.\textsuperscript{[10]} The important conclusion is the following: In order to understand the quark-triplet formation as the important aspect of many-quark system, the symmetric representation is powerless. A certain non-symmetric representation, which is constructed under the extra degrees of freedom, should be adopted. Borrowing this idea, we can formulate the boson realization which makes our aim satisfy. We introduce the extra boson operators which do not contain in the conventional boson realization. The ideas in the $su(4)$- and the $su(2)$-algebra come from the Schwinger boson representation for the $su(M+1) \otimes su(N,1)$-algebra presented by the present authors with Kuriyama.\textsuperscript{[11]} There exists a viewpoint that the Lipkin model was proposed as a model which enables us to describe particle-hole pair type collective excitation schematically, and therefore, it may be enough to investigate only the case of the closed shell system. However, it may be interesting to investigate how the collective excitation varies from the closed to the open shell system and it may be important to establish a theory, with the aid of which the above problem can be describe.

After recapitulating the two-level Lipkin model, in §2, we will discuss the conven-
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In §§3 and 4, we will formulate the Schwinger (§3) and the Holstein-Primakoff (§4) realization in unconventional form. In §5, the coupling of two $su(2)$-spins will be treated for the Lipkin model. Section 6 will be devoted to discussing the simplest approximation and it will be shown that the result is a natural generalization from that shown in the closed shell system based on RPA. In §7, the isoscalar pairing model will be discussed. Finally, in §8, concluding remark will be given.

§2. Two-level Lipkin model and its conventional boson realizations

Many-fermion system investigated in this paper is called the Lipkin model, which consists of two single-particle levels with the same degeneracy $2j + 1 (= 2\Omega, \ j : \text{half integer})$. We denote the two single-particle levels as the p- and the h-level. The fermion operators are denoted as $(c_{pm}^*, c_{pm})$ and $(c_{hm}^*, c_{hm})$, respectively, where $m = -j, -j + 1, \cdots, j - 1, j$. For the above system, we can define the following operators:

$$
\tilde{S}^+ = \sum_m c_{pm}^* c_{hm}, \quad \tilde{S}^- = \sum_m c_{hm}^* c_{pm},
$$

$$
\tilde{S}_0 = \frac{1}{2} \sum_m (c_{pm}^* c_{pm} - c_{hm}^* c_{hm}).
$$

(2.1)

The set $(\tilde{S}_{\pm,0})$ obeys the $su(2)$-algebra:

$$
[\tilde{S}_+, \tilde{S}_-] = 2\tilde{S}_0, \quad [\tilde{S}_0, \tilde{S}_\pm] = \pm \tilde{S}_\pm.
$$

(2.2a)

The Casimir operator $\tilde{S}^2$ is given in the form

$$
\tilde{S}^2 = \tilde{S}_+ \tilde{S}_- + \tilde{S}^2_0 - \tilde{S}_0, \quad [\tilde{S}^2, \tilde{S}_{\pm,0}] = 0.
$$

(2.2b)

Further, the total fermion number operator $\tilde{N}$ is given as

$$
\tilde{N} = \sum_m (c_{pm}^* c_{pm} + c_{hm}^* c_{hm}), \quad [\tilde{N}, \tilde{S}_{\pm,0}] = 0.
$$

(2.3)

It should be noted that $\tilde{S}^2$ and $\tilde{N}$ commute with $\tilde{S}_{\pm,0}$, but, they are independent of each other. Therefore, an orthogonal set is specified by the eigenvalues of $\tilde{N}, \tilde{S}^2$ and $\tilde{S}_0$ for a given value of $\Omega$. This point is different from the single-level pairing model. The Hamiltonian of the present model $\tilde{H}$ is usually expressed in the form

$$
\tilde{H} = \epsilon \tilde{S}_0 - \chi (\tilde{S}_+^2 + \tilde{S}_-^2). \quad (\epsilon, \chi > 0)
$$

(2.4)

Since $\epsilon > 0$, energetically the p-level is higher than the h-level with the energy difference $\epsilon$. The parameter $\chi$ denotes the force strength. We also notice the relations $[\tilde{N}, \tilde{H}] = [\tilde{S}^2, \tilde{H}] = 0$, but, $[\tilde{S}_0, \tilde{H}] \neq 0$. 

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$$

(2.4)
It may be convenient to express the above operators in terms of the particle and the hole operator:

\[ c_{pm} = a_m, \quad c_{hm} = b^*_m (= (-)^{j-m} b^*_{-m}) . \] (2.5)

Then, \( \tilde{S}_{\pm,0} \) and \( \tilde{N} \) can be expressed as

\[ \tilde{S}_+ = \sum_m a^*_m b^*_m, \quad \tilde{S}_- = \sum_m b_m a_m , \]
\[ \tilde{S}_0 = \frac{1}{2} \sum_m (a^*_m a_m + b^*_m b_m) - \Omega , \]
\[ \tilde{N} = \sum_m (a^*_m a_m - b^*_m b_m) + 2\Omega . \] (2.6)

The forms (2.6) and (2.7) give us

\[ \tilde{S}_0 = \frac{1}{2} (\tilde{N}_p + \tilde{N}_h) - \Omega , \quad \tilde{N} = \tilde{N}_p - \tilde{N}_h + 2\Omega , \] (2.8)

conversely,

\[ \tilde{N}_p = \tilde{S}_0 + \frac{1}{2} \tilde{N} , \quad \tilde{N}_h = \tilde{S}_0 - \frac{1}{2} \tilde{N} + 2\Omega . \] (2.9)

Here, \( \tilde{N}_p \) and \( \tilde{N}_h \) denote the particle and the hole number operator, respectively:

\[ \tilde{N}_p = \sum_m a^*_m a_m , \quad \tilde{N}_h = \sum_m b^*_m b_m . \] (2.10)

The form (2.6) tells us that \( \tilde{S}_+ \) plays a role of the creation of the particle-hole pair coupled to the angular momentum \( J = 0 \). Further, it should be noted that in contrast with the single-level pairing model, the total fermion number is not the simple sum of \( \tilde{N}_p \) and \( \tilde{N}_h \), namely \( \tilde{N}_p + \tilde{N}_h \). This fact is in an important position in the present model.

The minimum weight state, which we denote as \(|m\rangle\), is introduced as a state satisfying the condition

\[ \tilde{S}_- |m\rangle = 0 , \quad \tilde{S}_0 |m\rangle = -s|m\rangle , \] (2.11a)
\[ \tilde{N} |m\rangle = N|m\rangle . \] (2.11b)

Therefore, the eigenstate of \( \tilde{S}_0 \) with the eigenvalue \( s_0 \) can be expressed in the form

\[ |\Omega, N; ss_0 \rangle = (\tilde{S}_+)^{s+s_0}|\Omega, N; s \rangle . \quad (|m\rangle = |\Omega, N; s \rangle) \] (2.12)

The quantity \( \Omega \) is not quantum number, but, the parameter characterizing the model. However, in our boson realization later we will discuss, \( \Omega \) is treated as the quantum number. Then, we add \( \Omega \) explicitly to the fermion states. It may be convenient to formulate the above result in terms of \( n_p \) and \( n_h \) which satisfy

\[ \tilde{N}_p |m\rangle = n_p |m\rangle , \quad \tilde{N}_h |m\rangle = n_h |m\rangle . \] (2.13)
The eigenvalue equation (2.13) is permitted from the relation (2.9). The relation (2.8) gives
\[ s = \Omega - \frac{1}{2}(n_p + n_h) \], \hspace{1cm} (2.14a)
\[ N = 2\Omega + (n_p - n_h) \]. \hspace{1cm} (2.14b)

For the case \( n_p = n_h \) (\( = n_0 \)), we have
\[ s = \Omega - n_0 \], \hspace{1cm} N = 2\Omega . \hspace{1cm} (2.15)\]

The relation \( N = 2\Omega \) shows us that the case \( n_p = n_h \) (\( = n_0 \)) corresponds to the case of the closed shell, where if no residual interaction, the h-level is completely occupied by the fermions in the ground state. The quantity \( n_0 \) denotes the number of the particle-hole pairs with the coupled angular momentum \( J \neq 0 \). Conventionally, the two-level Lipkin model has been investigated in this case. For this case, the Schwinger boson realization of the \( su(2) \)-algebra is applicable in the following form:
\[ \tilde{S}_+ \rightarrow \hat{S}_+ = \hat{a}^* \hat{b} \, , \quad \tilde{S}_- \rightarrow \hat{S}_- = \hat{b}^* \hat{a} \, , \quad \tilde{S}_0 \rightarrow \hat{S}_0 = \frac{1}{2}(\hat{a}^* \hat{a} - \hat{b}^* \hat{b}) \]. \hspace{1cm} (2.16)

Here, \( \hat{a}, \hat{a}^* \) and \( \hat{b}, \hat{b}^* \) denote two kinds of boson operators. In this case, the state \( |\Omega, N = 2\Omega; ss_0 \rangle \) corresponds to
\[ |\Omega, N = 2\Omega; ss_0 \rangle = (\hat{a}^* \hat{b})^{s+s_0}(\hat{b}^*)^{2s}|0\rangle \]
\[ = (\hat{a}^*)^{s+s_0}(\hat{b}^*)^{s-s_0}|0\rangle \, , \quad |m\rangle = (\hat{b}^*)^{2s}|0\rangle \]. \hspace{1cm} (2.17)

Here, \( |0\rangle \) denotes the vacuum of the bosons and, of course, \( s \) is given in the relation (2.15). With the aid of the relation (2.15), we can introduce the operator \( \hat{\Omega} \) expressing the degeneracy of the single-particle levels. The operator \( \hat{\Omega} \) commutes with \( (\hat{S}_\pm, \hat{S}_0) \), because the eigenvalue of \( \hat{\Omega} \) should not depend on individual eigenstates. Then, we may set up \( \hat{\Omega} \) in the form
\[ \hat{\Omega} = x + \frac{1}{2}y(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) \]. \hspace{1cm} (2.18)

Here, \( x \) and \( y \) denote constants to be determined. Operation of \( \hat{\Omega} \) on \( |m\rangle \) given in the relation (2.17) leads to
\[ \Omega = x + ys = x + y(\Omega - n_0) \, , \quad i.e., \quad x = (1 - y)\Omega + yn_0 \]. \hspace{1cm} (2.19)

Substituting the form (2.19) into the relation (2.18), we have
\[ \hat{\Omega} = (1 - y)\Omega + y \left( n_0 + \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) \right) \]. \hspace{1cm} (2.20)

It is not natural that \( \hat{\Omega} \) should depend on the eigenvalue. Therefore, we set \( y = 1 \), which leads to
\[ \hat{\Omega} = n_0 + \frac{1}{2}(\hat{a}^* \hat{a} + \hat{b}^* \hat{b}) \]. \hspace{1cm} (2.21)
The relation (2.21) shows that $n_0$ is a parameter, the value of which should be given from the outside. Total fermion number operator $\hat{N}$ is given as

$$\hat{N} = 2\Omega.$$  \hspace{1cm} (2.22)

The above is the Schwinger boson realization. We know another boson realization called the Holstein-Primakoff boson realization. With the use of one kind of boson ($\hat{A}, \hat{A}^*$), $\hat{S}_{\pm,0}$ corresponds to $\hat{S}_{\pm,0}(s)$, which is given in the form

$$\tilde{S}_+ \rightarrow \hat{S}_+(s) = \hat{A}^* \sqrt{2s - \hat{A}^* \hat{A}} \, , \quad \tilde{S}_- \rightarrow \hat{S}_-(s) = \sqrt{2s - \hat{A}^* \hat{A}} \, . \hspace{1cm} (2.23)$$

The state $|\Omega, N = 2\Omega; ss_0\rangle$ corresponds to

$$|\Omega, N = 2\Omega; ss_0\rangle \rightarrow |\Omega, N = 2\Omega; ss_0\rangle = \left(\hat{A}^* \sqrt{2s - \hat{A}^* \hat{A}}\right)^{s+s_0} |s\rangle \, , \quad |m\rangle = |s\rangle.$$  \hspace{1cm} (2.24)

Here, $|s\rangle$ denotes the boson vacuum satisfying $\hat{A}|s\rangle = 0$ and $\hat{S}_0(s)|s\rangle = -s|s\rangle$ is derived from the relation (2.23). The relation between both realizations will be discussed in §4.

From the above argument, we could learn that the conventional boson representations of the $su(2)$-algebra are only applicable to the case $N = 2\Omega$ in the two-level Lipkin model. This conclusion induces a problem how to treat the case $n_p \neq n_h$.

The relation (2.14) teaches us the following:

(i) The case $n_p \leq n_h$:

$$N = 2\Omega - (n_h - n_p) \, , \quad s = \frac{1}{2}N - n_p \, . \hspace{1cm} (2.26a)$$

Since $n_h - n_p \geq 0$, $N \geq 0$ and $s \geq 0$, the relation (2.26a) gives us the inequalities

$$0 \leq N \leq 2\Omega \, , \quad n_p \leq \frac{1}{2}N \, , \quad n_h - n_p \leq 2\Omega \, . \hspace{1cm} (2.26b)$$

(ii) The case $n_p \geq n_h$:

$$N = 2\Omega + (n_p - n_h) \, , \quad s = \frac{1}{2}(4\Omega - N) - n_h \, . \hspace{1cm} (2.27a)$$
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Since \( n_p - n_h \geq 0 \), \( N \leq 4 \Omega \) and \( s \geq 0 \), the relation (2.27a) gives us the inequalities

\[
2 \Omega \leq N \leq 4 \Omega , \quad n_h \leq \frac{1}{2}(4 \Omega - N) , \quad n_p - n_h \leq 2 \Omega . \tag{2.27b}
\]

In the next section, we discuss the boson realization including \( n_p \neq n_h \).

§3. Unconventional boson realization : Part (I)

The discussion in §2 suggests us that, in order to make the boson realization of the \( su(2) \)-algebra applicable to the Lipkin model including the case \( n_p \neq n_h \), we have to introduce extra degrees of freedom. For this purpose, we apply a form proposed by the present authors, in which the Schwinger boson representation of the \( su(M + 1) \)-algebra is formulated in terms of \( (M + 1)(N + 1) \) kinds of bosons. The case \( (M = 3, N = 1) \) has been applied to the Bonn model and its modification for many-quark system. This model obeys the \( su(4) \)-algebra. The case \( (M = 1, N = 0) \) corresponds to the \( su(2) \)-algebra adopted in §2. In this section, we treat the case \( (M = 1, N = 1) \), which contains four kinds of bosons. Under the notations appropriate to the present case, \( \hat{S}_{\pm,0} \) can be expressed in the form

\[
\hat{S}_+ = \hat{a}_p^* \hat{b}_h - \hat{a}_h^* \hat{b}_p , \quad \hat{S}_- = \hat{b}_h^* \hat{a}_p - \hat{b}_p^* \hat{a}_h ,
\]

\[
\hat{S}_0 = \frac{1}{2} \left[ (\hat{a}_p^* \hat{a}_p + \hat{a}_h^* \hat{a}_h) - (\hat{b}_p^* \hat{b}_p + \hat{b}_h^* \hat{b}_h) \right] . \tag{3.1}
\]

Here, \((\hat{a}_p, \hat{a}_p^*), (\hat{a}_h, \hat{a}_h^*), (\hat{b}_p, \hat{b}_p^*), \text{ and } (\hat{b}_h, \hat{b}_h^*)\) denote four kinds of bosons. Associating with the \( su(2) \)-algebra, we can define the \( su(1,1) \)-algebra in the form

\[
\hat{T}_+ = \hat{a}_p^* \hat{b}_h^* + \hat{a}_h^* \hat{b}_p^* , \quad \hat{T}_- = \hat{b}_p \hat{a}_p + \hat{b}_h \hat{a}_h ,
\]

\[
\hat{T}_0 = \frac{1}{2} \left[ (\hat{a}_p^* \hat{a}_p + \hat{a}_h^* \hat{a}_h) + (\hat{b}_p^* \hat{b}_p + \hat{b}_h^* \hat{b}_h) \right] + 1 . \tag{3.2}
\]

The set \( \{ \hat{T}_{\pm,0} \} \) obeys

\[
[ \hat{T}_+ , \hat{T}_- ] = -2 \hat{T}_0 , \quad [ \hat{T}_0 , \hat{T}_{\pm} ] = \pm \hat{T}_{\pm} , \tag{3.3}
\]

\[
[ \text{any of } (\hat{T}_{\pm,0}) , \text{any of } (\hat{S}_{\pm,0}) ] = 0 . \tag{3.4}
\]

The Casimir operator \( \hat{T}^2 \) is given as

\[
\hat{T}^2 = -\hat{T}_+ \hat{T}_- + \hat{T}_0^2 + \hat{T}_0 . \tag{3.5}
\]

It is noted that the operator \( \hat{M} \) defined in the following commutes with any of \( (\hat{S}_{\pm,0}) \) and \( (\hat{T}_{\pm,0}) \):

\[
\hat{M} = (\hat{a}_p^* \hat{b}_p - \hat{b}_p^* \hat{a}_p) - (\hat{a}_h^* \hat{a}_h - \hat{b}_h^* \hat{b}_h) , \tag{3.6}
\]

\[
[ \hat{M} , \hat{S}_{\pm,0} ] = [ \hat{M} , \hat{T}_{\pm,0} ] = 0 . \tag{3.7}
\]

The above is an outline of the Schwinger boson representation of the \( su(2) \)-algebra in terms of four kinds of bosons. For this form, we must pay an attention to the
following: The above boson representation is not the boson realization of the Lipkin model as it stands, because the above does not contain the degeneracy operator $\hat{\Omega}$ and the total fermion number operator $\hat{N}$ which connect with the original many-fermion system.

As for the minimum weight state $|m\rangle$, which corresponds to $|m\rangle$, we postulate the following state:

$$|m\rangle = (\hat{b}_p^\dagger)^{n_p-n_h} (\hat{b}_h^\dagger)^{2\Omega-(n_p+n_h)-(n_p-n_h)} |0\rangle .$$  \hspace{1cm} (3.8)

Clearly, $|m\rangle$ satisfies

$$\hat{S}_-|m\rangle = 0 , \quad \hat{S}_0|m\rangle = -s|m\rangle , \quad s = \Omega - \frac{1}{2}(n_p + n_h) .$$  \hspace{1cm} (3.9)

The relation (3.9) corresponds to the relation (2.11a) with (2.14a). If $n_p = n_h = n_0$, the state (3.8) reduces to $|m\rangle = (\hat{b}_p^\dagger)^{2\Omega}|0\rangle (s = \Omega - n_0)$ and the $(s+s_0)$-time operation of $\hat{S}_0$ on $|m\rangle$ gives us $(\hat{a}_p^\dagger)^{s+s_0} (\hat{b}_h^\dagger)^{s+s_0} |0\rangle$ in the space spanned by four kinds of bosons. If $\hat{a}_p$ and $\hat{b}_h$ read $\hat{a}$ and $\hat{b}$, respectively, the above form reduces to the form (2.17). The above argument supports that $|m\rangle$ defined in the relation (3.8) may be regarded as the minimum weight state for our purpose. However, we must notice the connection of the state (3.8) to the $su(1,1)$-algebra (3.2). The state $|m\rangle$ satisfies the relation:

$$\hat{T}_-|m\rangle = 0 , \quad \hat{T}_0|m\rangle = (s+1)|m\rangle , \quad s = \Omega - \frac{1}{2}(n_p + n_h) .$$  \hspace{1cm} (3.10)

Next, we introduce the following state:

$$|n;m\rangle = (\hat{T}_+)^n|m\rangle , \quad (n = 0, 1, 2, \cdots)$$

$$|0;m\rangle = |m\rangle .$$  \hspace{1cm} (3.11)

The state (3.11) leads us to the same form as the relation (3.8):

$$\hat{S}_-|n;m\rangle = 0 , \quad \hat{S}_0|n;m\rangle = -s|n;m\rangle , \quad s = \Omega - \frac{1}{2}(n_p + n_h) .$$  \hspace{1cm} (3.12)

The above indicates that the present boson space is not in one-to-one correspondence with the original fermion space. Then, in order to guarantee the one-to-one correspondence, we require the condition that our minimum weight state is also the minimum weight state for the $su(1,1)$-algebra. The state $|m\rangle$ satisfies this condition:

$$\hat{T}_-|m\rangle = 0 , \quad \hat{T}_0|m\rangle = (s+1)|m\rangle , \quad s = \Omega - \frac{1}{2}(n_p + n_h) .$$  \hspace{1cm} (3.13)

Then, if $\Omega$, $n_p$ and $n_h$ are specified from the outside, $|m\rangle$ is given in the form (3.8), which corresponds to $|m\rangle$. Further, if the total fermion number is given as $N = 2\Omega + n_p - n_h$, the expression $s = \Omega - (n_p + n_h)/2$ gives us $|m\rangle$ in the form $|\Omega, N; s\rangle$ which corresponds to the state $|\Omega, N; s\rangle$ shown in the relation (2.12). The orthogonal state $|\Omega, N; ss_0\rangle$ is given in the form

$$|\Omega, N; ss_0\rangle = (\hat{S}_+)^{s+s_0} |\Omega, N; s\rangle .$$  \hspace{1cm} (3.14)
In §4, we will repeat the above discussion in a way slightly different from the above.

Our final task is to search $\hat{\Omega}$ and $\hat{N}$ which correspond to the degeneracy of the single-particle level $\Omega$ and the total fermion number operator $\hat{N}$. If we cannot search these two operators, our boson representation would not be permitted to call the boson realization of the Lipkin model. First, we treat $\hat{\Omega}$. Under the same viewpoint as that in §2, we set up the following form:

$$\hat{\Omega} = x + \frac{y}{2}(\hat{a}_p^* \hat{a}_p + \hat{a}_h^* \hat{a}_h + \hat{b}_p^* \hat{b}_p + \hat{b}_h^* \hat{b}_h) \quad .$$

(3-15)

With the aid of the relation $\hat{\Omega}|m\rangle = \Omega|m\rangle$, we obtain

$$x = (1 - y)\Omega + \frac{y}{2}(n_p + n_h) \quad .$$

(3-16)

Then, $\hat{\Omega}$ is expressed as the form

$$\hat{\Omega} = (1 - y)\Omega + \frac{y}{2}(n_p + n_h + \hat{a}_p^* \hat{a}_p + \hat{a}_h^* \hat{a}_h + \hat{b}_p^* \hat{b}_p + \hat{b}_h^* \hat{b}_h) \quad .$$

(3-17)

In the same idea as that adopted in §2, we have $y = 1$ and $\hat{\Omega}$ is expressed as

$$\hat{\Omega} = \frac{1}{2}(n_p + n_h) + \frac{1}{2}(\hat{a}_p^* \hat{a}_p + \hat{a}_h^* \hat{a}_h + \hat{b}_p^* \hat{b}_p + \hat{b}_h^* \hat{b}_h) \quad .$$

(3-18)

The above is a natural generalization of the form (2.21).

Our next task is to find the operator $\hat{N}$. For this aim, first, we pay an attention to the operators $\hat{N}_p$ and $\hat{N}_h$ in the form (2.10). The operators $\hat{N}_p$ and $\hat{N}_h$ have the following properties:

$$\hat{N}_p, \hat{N}_h \quad : \quad \text{positive definite},$$

(3-19)

$$[\hat{N}_p, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{N}_h, \hat{S}_\pm] = \pm \hat{S}_\pm,$$

(3-20)

$$\hat{N}_p|m\rangle = n_p|m\rangle, \quad \hat{N}_h|m\rangle = n_h|m\rangle \quad .$$

(3-21)

The relation (3.21) is a copy from the relation (2.13). We search $\hat{N}_p$ and $\hat{N}_h$ satisfying the relations which correspond to the relations (3.19)~(3.21). For this purpose, we define $\hat{N}_p$ and $\hat{N}_h$ in the form

$$\hat{N}_p = \hat{M}_p + x_p + y_p \hat{\Omega} \quad , \quad \hat{M}_p = \hat{a}_p^* \hat{a}_p - \hat{b}_p^* \hat{b}_p \quad ,$$

(3-22a)

$$\hat{N}_h = \hat{M}_h + x_h + y_h \hat{\Omega} \quad , \quad \hat{M}_h = \hat{a}_h^* \hat{a}_h - \hat{b}_h^* \hat{b}_h \quad .$$

(3-22b)

Here, $(x_p, y_p)$ and $(x_h, y_h)$ denote parameters to be determined. It should be noted that $\hat{M}_p$ and $\hat{M}_h$ satisfy

$$[\hat{M}_p, \hat{S}_\pm] = \pm \hat{S}_\pm, \quad [\hat{M}_h, \hat{S}_\pm] = \pm \hat{S}_\pm,$$

(3-23)

$$[\hat{M}_p, \hat{T}_\pm] = 0, \quad [\hat{M}_h, \hat{T}_\pm] = 0 \quad .$$

(3-24)

The operators $\hat{M}_p$ and $\hat{M}_h$ satisfy the same relation as that shown in the relation (3-20), but they are not positive definite. Then, we add the terms $(x_p + y_p \hat{\Omega})$ and
Since the relations (3.18), (3.22a) and (3.22b), we have

\[
\begin{align*}
(x_p - n_p - |n_p - n_h|) + y_p \Omega &= 0, \\
(x_h + n_p + |n_p - n_h|) + (y_h - 2) \Omega &= 0.
\end{align*}
\]  

(3.25)

For the relation (3.25), we require that \((x_p, y_p)\) and \((x_h, y_h)\) should not depend on \(\Omega\) which is an eigenvalue of \(\hat{\Omega}\). Under this requirement, we have

\[
\begin{align*}
x_p &= n_p + |n_p - n_h|, & y_p &= 0, \\
x_h &= -n_p - |n_p - n_h|, & y_h &= 2.
\end{align*}
\]  

(3.26)

Under the relation (3.20) and the explicit expressions of \(\hat{\Omega}, \hat{M}_p\) and \(\hat{M}_h\) shown in the relations (3.18), (3.22a) and (3.22b), we have

\[
\begin{align*}
\hat{N}_p &= n_p + |n_p - n_h| + \hat{a}_p^* \hat{a}_p - \hat{b}_p^* \hat{b}_p, \\
\hat{N}_h &= n_h - |n_p - n_h| + \hat{a}_h^* \hat{a}_h + 2\hat{a}_h^* \hat{a}_h + \hat{b}_h^* \hat{b}_h.
\end{align*}
\]  

(3.27)

Since \(\hat{N}_p(\hat{S}_+)^n|m\rangle = (n + n_p)(\hat{S}_+)^n|m\rangle\) and \(\hat{N}_h(\hat{S}_+)^n|m\rangle = (n + n_h)(\hat{S}_+)^n|m\rangle\), \(\hat{N}_p\) and \(\hat{N}_h\) are positive definite. Under the correspondence to the relation (2.7) for \(\hat{N}\), we define \(\hat{\tilde{N}}\) in the form

\[
\hat{\tilde{N}} = \hat{N}_p - \hat{N}_h + 2\hat{\Omega}.
\]  

(3.28)

Substituting the relations (3.18) and (3.27) into the definition (3.28), we have

\[
\hat{\tilde{N}} = 2(n_p + |n_p - n_h|) + (\hat{a}_p^* \hat{a}_p - \hat{b}_p^* \hat{b}_p) + (\hat{a}_h^* \hat{a}_h + \hat{b}_h^* \hat{b}_h).
\]  

(3.29)

With the use of \(\hat{M}\) defined in the relation (3.4), \(\hat{\tilde{N}}\) can be given as

\[
\hat{\tilde{N}} = 2(n_p + |n_p - n_h|) + \hat{M}.
\]  

(3.30)

The relation (3.7) leads to

\[
[\hat{\tilde{N}}, \hat{\hat{S}}_{\pm,0}] = [\hat{\tilde{N}}, \hat{T}_{\pm,0}] = 0.
\]  

(3.31)

Thus, we could formulate an unconventional boson realization for the Lipkin model including the case \(n_p \neq n_h\).

\section{Unconventional boson realization: Part (II)}

Main task of this section is to formulate the Lipkin model including the case \(n_p \neq n_h\) in the frame of the Holstein-Primakoff representation derived in relation to the Schwinger boson representation. For the preparation, first, we treat the case \(n_p = n_h (= n_0)\) discussed in §2. The interpretation is suggestive, but not strict. The strict interpretation has been given in Ref.\([12]\). In the present case, the operator introduced in the relation (2.21), \(\hat{\Omega}\), plays an essential role:

\[
\hat{\Omega} = n_0 + \hat{S}, \quad \hat{S} = \frac{1}{2} (\hat{a}_p^* \hat{a}_h + \hat{b}_p^* \hat{b}_h).
\]  

(4.1)
The operator $\hat S$ commutes with $\hat S_{\pm,0}$. Therefore, the irreducible representation is specified by the eigenvalue of $\hat S$ which we denote $s$. In the space specified by $s$, it may be permitted to set up

$$\frac{1}{2}(\hat a^*\hat a + \hat b^*\hat b) = s , \text{ i.e.} \quad \hat b^*\hat b = 2s - \hat a^*\hat a = \left(\sqrt{2s - \hat a^*\hat a}\right)^2 . \tag{4.2}$$

Of course, we have

$$\hat S = \Omega - n_0 . \tag{4.3}$$

Therefore, it should be noted that even if $(\hat a, \hat a^*)$ is boson, $(\hat b, \hat b^*)$ cannot be regarded as boson independent of $(\hat a, \hat a^*)$. Under the above consideration, we set up the relation (2.23).

Under the idea mentioned above, we will present the Holstein-Primakoff representation including the case $n_p \neq n_h$. As the operators playing the same role as $\hat S$ given in the relation (4.1), we introduce the following two operators:

$$\hat S_p = \frac{1}{2}(\hat a^*_p\hat a_p + \hat b^*_h\hat b_h) , \tag{4.4a}$$

$$\hat S_h = \frac{1}{2}(\hat a^*_h\hat a_h + \hat b^*_p\hat b_p) . \tag{4.4b}$$

They satisfy the relation

$$[\hat S_p , \hat S_{\pm,0} ] = [\hat S_h , \hat S_{\pm,0} ] = 0 . \tag{4.5}$$

Therefore, the irreducible representation is specified by the eigenvalues of $\hat S_p$ and $\hat S_h$, which are denoted by $s_p$ and $s_h$, respectively. The relations (3.18) and (3.29) lead to

$$\hat S_p = \frac{1}{2}\left(\hat \Omega + \frac{1}{2}\hat N\right) - \frac{1}{4}(3n_p + n_h) - \frac{1}{2}|n_p - n_h| , \tag{4.6a}$$

$$\hat S_h = \frac{1}{2}\left(\hat \Omega - \frac{1}{2}\hat N\right) + \frac{1}{4}(n_p - n_h) + \frac{1}{2}|n_p - n_h| . \tag{4.6b}$$

Therefore, we have

$$\hat S_p \rightarrow s_p = \frac{1}{2}\left(\hat \Omega + \frac{1}{2}\hat N\right) - \frac{1}{4}(3n_p + n_h) - \frac{1}{2}|n_p - n_h|$$

$$= \Omega - \frac{1}{2}(n_p + n_h) - \frac{1}{2}|n_p - n_h| , \tag{4.7a}$$

$$\hat S_h \rightarrow s_h = \frac{1}{2}\left(\hat \Omega - \frac{1}{2}\hat N\right) + \frac{1}{4}(n_p - n_h) + \frac{1}{2}|n_p - n_h|$$

$$= \frac{1}{2}|n_p - n_h| . \tag{4.7b}$$

Here, we used the relation (3.28):

$$\hat N \rightarrow N = n_p - n_h + \Omega . \tag{4.8}$$
The relation (4.7) corresponds to the relation (3.3). The relation (4.7) gives us
\[ s_p + s_h = \Omega - \frac{1}{2}(n_p + n_h) = s . \]  \hfill (4.9)
Here, we used the relation (2.14a).

In the same idea as the previous case, we set up the following relation which comes from the relation (4.4):
\[ \hat{b}_p^* \hat{b}_h = 2s_p - \hat{a}_p^* \hat{a}_p = \left( \sqrt{2s_p - \hat{a}_p^* \hat{a}_p} \right)^2 , \]  \hfill (4.10a)
\[ \hat{b}_p^* \hat{b}_p = 2s_h - \hat{a}_h^* \hat{a}_h = \left( \sqrt{2s_h - \hat{a}_h^* \hat{a}_h} \right)^2 . \]  \hfill (4.10b)
The form (3.1) gives us the idea for the relation
\[ \hat{S}_+ \rightarrow \hat{S}_+(s_p s_h) = \hat{A}_p^* \sqrt{2s_p - \hat{A}_p^* \hat{A}_p} - \hat{A}_h^* \sqrt{2s_h - \hat{A}_h^* \hat{A}_h} , \]
\[ \hat{S}_- \rightarrow \hat{S}_-(s_p s_h) = \sqrt{2s_p - \hat{A}_p^* \hat{A}_p} \hat{A}_p - \sqrt{2s_h - \hat{A}_h^* \hat{A}_h} \hat{A}_h , \]
\[ \hat{S}_0 \rightarrow \hat{S}_0(s_p s_h) = \hat{A}_p^* \hat{A}_p + \hat{A}_h^* \hat{A}_h - s . \]  \hfill (4.11)
Here, we used the relation (4.9). The form (1.1) is nothing but the Holstein-Primakoff representation. If \( n_p = n_h = \Omega \), \( s_h \) vanishes and \( s_p = s \). Then, if \( \hat{A}_p \) reads \( \hat{A} \), the form (4.11) is reduced to the form (2.23). This is in the same situation as the case of the Schwinger representation. The minimum weight state in the present case, which we denote as \( |\Omega, N; s\rangle \), is given as the vacuum of the bosons \( \hat{A}_p \) and \( \hat{A}_h \):
\[ \hat{A}_p |m\rangle = \hat{A}_h |m\rangle = 0 , \]  \hfill (4.12)
i.e., \( \hat{S}_-(s_p s_h) |m\rangle = 0 \), \( \hat{S}_0(s_p s_h) |m\rangle = -s |m\rangle . \)  \hfill (4.13)
Since \( s = \Omega - (n_p + n_h)/2 \) and \( N = 2\Omega + (n_p - n_h) \), \( |m\rangle \) can be also specified by \( |\Omega, N; s\rangle \). Then, the state \( |\Omega, N; ss_0\rangle \) is given as
\[ |\Omega, N; ss_0\rangle = (\hat{S}_+(s_p s_h))^{s+s_0} |\Omega, N; s\rangle . \]  \hfill (4.14)
The above is the Holstein-Primakoff boson realization of the Lipkin model including the case \( n_p \neq n_h \).

§5. The Lipkin model in the coupling of two kinds of the \( su(2) \)-spin

Needless to say, the Lipkin model obeys the \( su(2) \)-algebra. The conventional form can be treated in terms of one kind of the \( su(2) \)-spin. But, the present Lipkin model, which we called the unconventional form, is treated in terms of the addition of two kinds of the \( su(2) \)-spins. This will be later shown. The aim of this section is to give a possible interpretation of the present Lipkin model in the frame of the coupling of two kinds of the \( su(2) \)-spins.

First, we discuss the case of the Schwinger representation. The form (3.1) can be re-expressed in the following form:
\[ \hat{S}_{\pm,0} = \hat{S}_{\pm,0}(p) + \hat{S}_{\pm,0}(h) , \]  \hfill (5.1)
\[ \hat{S}_+(p) = \hat{a}^*_p \hat{b}_h, \quad \hat{S}_-(p) = \hat{b}^*_h \hat{a}_p, \quad \hat{S}_0(p) = \frac{1}{2}(\hat{a}^*_p \hat{a}_p - \hat{b}^*_h \hat{b}_h), \quad (5.2a) \]
\[ \hat{S}_+(h) = -\hat{a}^*_h \hat{b}_p, \quad \hat{S}_-(h) = -\hat{b}^*_p \hat{a}_h, \quad \hat{S}_0(h) = \frac{1}{2}(\hat{a}^*_h \hat{a}_h - \hat{b}^*_p \hat{b}_p). \quad (5.2b) \]

We can see that the set of the generators (\(\hat{S}_{\pm,0}\)) forms simple sum of the two sets of the \(su(2)\)-generators, each of which is identical with the form presented in §2. Therefore, our problem is reduced to the addition of the \(su(2)\)-spins. The coupling scheme in the Schwinger representation has been formulated in detail by the present authors in Ref.13) and we copy some formulae from Ref.13). Of course, the notations are changed from the original to the present ones. The eigenstate of \(\hat{S}^2\) and \(\hat{S}_0\) with the eigenvalues \(s(s + 1)\) and \(s_0\), respectively, is given in Ref.13):

\[ |s_p s_h; s s_0\rangle = (\hat{T}_+)^{s_p + s_h - s} (\hat{S}_+)^{s + s_0} |s_p s_h; s\rangle, \]
\[ |s_p s_h; s\rangle = (\hat{b}^*_p)^{s_h - s_p} (\hat{b}_h)^{s - s_h + s_p} |0\rangle. \quad (5.3) \]

The exponents of \(\hat{T}_+\), \(\hat{b}^*_p\) and \(\hat{b}^*_h\) should be positive and we have well known rule:

\[ |s_p - s_h| \leq s \leq s_p + s_h. \quad (5.4) \]

Of course, the eigenvalues of \(\tilde{S}(p)^2\) and \(\tilde{S}(h)^2\) are given by \(s_p(s_p + 1)\) and \(s_h(s_h + 1)\), respectively. The minimum weight state \(|m\rangle\) is expressed as

\[ |m\rangle = |s_p s_h; s\rangle. \quad (5.5) \]

We pay a special attention to the case

\[ s_p + s_h = s. \quad (5.6) \]

In this case, \(|m\rangle\) is reduced to

\[ |s_p s_h; s = s_p + s_h\rangle = (\hat{b}^*_p)^{2s_h} (\hat{b}_h)^{2s_p} |0\rangle. \quad (5.7) \]

Further, we have

\[ |s_p s_h; s = s_p + s_h, s_0\rangle = (\hat{S}_+)^{s + s_0} |s_p s_h; s\rangle. \quad (5.8) \]

Since the state (5.8) does not contain \(\hat{T}_+\), the state (5.8) satisfies

\[ \hat{T}_- |s_p s_h; s = s_p + s_h\rangle = 0. \quad (5.9) \]

Therefore, the state investigated in §3 is nothing but the state (5.8) with

\[ s_p = \Omega - \frac{1}{2}(n_p + n_h) - \frac{1}{2}|n_p - n_h|, \quad s_h = \frac{1}{2}|n_p - n_h|. \quad (5.10) \]

From the above argument, we can learn that the operation of \(\hat{T}_+\) is necessary for obtaining the states with \(|s_p - s_h| \leq s < s_p + s_h\). However, these states do not have any counterparts to the original fermion states.
Next, we treat the case of the Holstein-Primakoff representation. According to the authors’ knowledge, there is no precedent for the formalism in which the coupling scheme of two $su(2)$-spins is treated in the frame of the Holstein-Primakoff representation. Therefore, newly we have to present the idea. In the same way as the previous case, \( \hat{S}_{\pm,0}(s_p s_h) \) can be decomposed into the following form:

\[
\hat{S}_{\pm,0}(s_p s_h) = \hat{S}_{\pm,0}^{(p)}(s_p) + \hat{S}_{\pm,0}^{(h)}(s_h) ,
\]

\[\hat{S}_{\pm}^{(p)}(s_p) = \hat{A}_p^* \sqrt{2s_p - \hat{A}_p^* \hat{A}_p} , \quad \hat{S}_{-}^{(p)}(s_p) = \sqrt{2s_p - \hat{A}_p^* \hat{A}_p} \hat{A}_p , \]

\[\hat{S}_{0}^{(p)}(s_p) = \hat{A}_p^* \hat{A}_p - s_p , \] (5.11a)

\[\hat{S}_{\pm}^{(h)}(s_h) = -\hat{A}_h^* \sqrt{2s_h - \hat{A}_h^* \hat{A}_h} , \quad \hat{S}_{-}^{(h)}(s_h) = -\sqrt{2s_h - \hat{A}_h^* \hat{A}_h} \hat{A}_h , \]

\[\hat{S}_{0}^{(h)}(s_h) = \hat{A}_h^* \hat{A}_h - s_h . \] (5.11b)

Our final goal is to find the eigenstate \( |s_p s_h; ss_0\rangle \) in our present case, which is expressed as

\[ |s_p s_h; ss_0\rangle = (\hat{S}_+(s_p s_h))^{s+s_0} |s_p s_h; s\rangle . \] (5.12)

Here, \( |s_p s_h; s\rangle \) denotes the minimum weight state:

\[ \hat{S}_- |s_p s_h; s\rangle = 0 , \quad \hat{S}_0 |s_p s_h; s\rangle = -s |s_p s_h; s\rangle . \] (5.13)

A possible choice of \( |s_p s_h; s\rangle \) is given in the case \( s = s_p + s_h \): In this case, \( |s_p s_h; s(= s_p + s_h)\rangle \) is the vacuum of \( \hat{A}_p \) and \( \hat{A}_h \):

\[ |s_p s_h; s(= s_p + s_h)\rangle = |0\rangle , \quad \hat{A}_p |0\rangle = \hat{A}_h |0\rangle = 0 . \] (5.14)

Then, let us consider the minimum weight states in other cases. For this purpose, we introduce the following operators:

\[
\hat{S}_-(s_p s_h; k) = \sqrt{2s_p - k + 1 - \hat{A}_p^* \hat{A}_p} \hat{A}_p - \sqrt{2s_h - k + 1 - \hat{A}_h^* \hat{A}_h} \hat{A}_h , \]

\[\hat{R}_+(s_p s_h; k) = \hat{A}_p^* \sqrt{2s_p - k + 1 - \hat{A}_p^* \hat{A}_p} \hat{A}_p + \hat{A}_h^* \sqrt{2s_p - k + 1 - \hat{A}_h^* \hat{A}_h} , \] (5.15)

\[ k = 1, 2, \cdots , k_m (= s_p + s_h - |s_p - s_h|) . \] (5.17)

It should be noted that the case \( k = 1 \) gives us

\[ \hat{S}_-(s_p s_h; 1) = \hat{S}_-(s_p s_h) . \] (5.18)

The operators \( \hat{S}_-(s_p s_h; k) \) and \( \hat{R}_+(s_p s_h; k) \) are defined under the condition

\[ 2s_p - k \geq 0 , \quad 2s_h - k \geq 0 , \quad \text{i.e.,} \quad k \leq s_p + s_h - |s_p - s_h| . \] (5.19)

Therefore, the maximum value of \( k \) is given as \( k_m \) shown in the relation \( 5.17 \).

Direct calculation gives us that \( \hat{S}_-(s_p s_h; k) \) and \( \hat{R}_+(s_p s_h; k) \) obey the relation

\[ \hat{S}_-(s_p s_h; k) \hat{R}_+(s_p s_h; k) = \hat{R}_+(s_p s_h; k) \hat{S}_-(s_p s_h; k + 1) , \]

\[ [ \hat{S}_0(s_p s_h) , \hat{R}_+(s_p s_h; k) ] = \hat{R}_+(s_p s_h; k) . \] (5.21)
With the use of \( \hat{R}_+(s_p s_h; s) \), we can construct the minimum weight states in the form
\[
|s_p s_h; s (= s_p + s_h)\rangle = |0\rangle, \quad (5.22a)
|s_p s_h; s\rangle = \hat{R}_+(s_p s_h; 1)\hat{R}_+(s_p s_h; 2) \cdots \hat{R}_+(s_p s_h; k \equiv s_p + s_h - s)|0\rangle,
\]
for \( s = 1, 2, \cdots, k_m (= s_p + s_h - |s_p - s_h|) \). \quad (5.22b)

The case (5.22a) is self-evident. The proof for the case (5.22b) which is the minimum weight state is performed in the following way: With the successive use of the condition (5.20), we have
\[
\hat{S}_-(s_p s_h) \parallel s_p s_h; s \rangle \\\n= \hat{S}_-(s_p s_h; 1)\hat{R}_+(s_p s_h; 1)\hat{R}_+(s_p s_h; 2) \cdots \hat{R}_+(s_p s_h; k)\hat{S}_-(s_p s_h; k + 1)|0\rangle, \quad (|s_p - s_h| < s < s_p + s_h) \quad (5.23a)
\]
\[
\hat{S}_-(s_p s_h) \parallel s_p s_h; s \rangle \\\n= \hat{R}_+(s_p s_h; 1)\hat{R}_+(s_p s_h; 2) \cdots \hat{R}_+(s_p s_h; k - 1)\hat{S}_-(s_p s_h; k)\hat{R}_+(s_p s_h; k)|0\rangle. \quad (s = |s_p - s_h|) \quad (5.23b)
\]

For the case (5.23a), \( \hat{S}_-(s_p s_h; k + 1)|0\rangle = 0 \) and for the case (5.23b), \( \hat{S}_-(s_p s_h; k)\hat{R}_+(s_p s_h; k)|0\rangle = 0 \). The reason why we treated separately comes from the condition (5.19). Operation \( \hat{S}_0(s_p s_h) \) on \( |s_p s_h; s\rangle \) gives us
\[
\hat{S}_0(s_p s_h) \parallel s_p s_h; s \rangle = (k - (s_p + s_h)) \parallel s_p s_h; s \rangle \\\n= -s \parallel s_p s_h; s \rangle. \quad (5.24)
\]

Here, we use the condition (5.21) and \( k = s_p + s_h - s \). Thus, we could complete the coupling scheme of two kinds of the \( su(2)\)-spin. But, in the same meaning as the case of the Schwinger representation, the case \( s = s_p + s_h \) is the counterpart of the Lipkin model.

§6. The simplest approximate diagonalization of the Hamiltonian of the Lipkin model

In our present boson representation, we can diagonalize the Hamiltonian of the Lipkin model exactly. But, it may be important to show an approximate solution on the same level as that obtained in the conventional random phase approximation. The Holstein-Primakoff representation may be suitable for this aim, because generally the operators describing the system under investigation are expressed in terms of the power series for the bosons playing a role of the fluctuations around the equilibrium.

The present diagonalization is based on the following relation:
\[
\hat{S}_+(s_p s_h) \approx \hat{A}^*_p \sqrt{2s_p} - \hat{A}^*_h \sqrt{2s_h},
\]
\[ \hat{S}_-(s_p s_h) \approx \sqrt{2s_p} \hat{A}_p - \sqrt{2s_h} \hat{A}_h , \]
\[ \hat{S}_0(s_p s_h) = \hat{A}_p^* \hat{A}_p + \hat{A}_h^* \hat{A}_h - s . \quad (s = s_p + s_h) \quad (6.1) \]

The approximation (6.1) indicates that the effects of \( \hat{A}_p^* \hat{A}_p \) and \( \hat{A}_h^* \hat{A}_h \) are negligibly small compared with \( 2s_p \) and \( 2s_h \), respectively, in the square root of the relation (4.11):
\[ \sqrt{2s_p} - \hat{A}_p^* \hat{A}_p \approx \sqrt{2s_p} , \quad \sqrt{2s_h} - \hat{A}_h^* \hat{A}_h \approx \sqrt{2s_h} . \quad (6.2) \]

The relation (6.2) shows that the existence of the equilibrium is presupposed. For the relation (6.1), we introduce the boson operators
\[ \hat{B}^* = \sqrt{\frac{s_p}{s}} \hat{A}_p^* - \sqrt{\frac{s_h}{s}} \hat{A}_h^* , \quad \hat{B} = \sqrt{\frac{s_p}{s}} \hat{A}_p - \sqrt{\frac{s_h}{s}} \hat{A}_h , \]
\[ \hat{C}^* = \sqrt{\frac{s_h}{s}} \hat{A}_p^* + \sqrt{\frac{s_p}{s}} \hat{A}_h^* , \quad \hat{C} = \sqrt{\frac{s_h}{s}} \hat{A}_p + \sqrt{\frac{s_p}{s}} \hat{A}_h . \quad (6.3a) \]

Conversely,
\[ \hat{A}_p^* = \sqrt{\frac{s_p}{s}} \hat{B}^* + \sqrt{\frac{s_h}{s}} \hat{C}^* , \quad \hat{A}_p = \sqrt{\frac{s_p}{s}} \hat{B} + \sqrt{\frac{s_h}{s}} \hat{C} , \]
\[ \hat{A}_h^* = -\sqrt{\frac{s_h}{s}} \hat{B}^* + \sqrt{\frac{s_p}{s}} \hat{C}^* , \quad \hat{A}_h = -\sqrt{\frac{s_h}{s}} \hat{B} + \sqrt{\frac{s_p}{s}} \hat{C} . \quad (6.3b) \]

Then, we have
\[ \hat{S}_+(s_p s_h) = \sqrt{2s} \hat{B}^* , \quad \hat{S}_-(s_p s_h) = \sqrt{2s} \hat{B} , \]
\[ \hat{S}_0(s_p s_h) = \hat{B}^* \hat{B} + \hat{C}^* \hat{C} - s . \quad (6.4) \]

Hereafter, the symbol \( \approx \) is replaced with the equal sign.

Under the relation (6.3), the Hamiltonian (2.4) is approximated to
\[ \hat{H} = \epsilon (\hat{B}^* \hat{B} + \hat{C}^* \hat{C} - s) - 2 \chi s (\hat{B}^2 + \hat{B}^2) . \quad (6.5) \]

Fig. 1. \( \omega/\epsilon \) for \( n_p = n_h = 0 \) (solid curve) and \( n_p = n_h = 1 \) (dashed curve) with \( N = 12 \) are shown.
Two-Level Lipkin Model in Unconventional Boson Realization

Fig. 2. The ground state energy for $n_p = n_h = 0$ (solid curve) and exact ground state energy (dashed curve) are shown in the case of the closed shell system with $N = 12$.

The Hamiltonian (6.5) is easily diagonalized in the form

\[
\hat{H} = E_0 + \epsilon \hat{C}^* \hat{C} + \omega(s) \hat{D}^* \hat{D},
\]

\[
E_0 = -\frac{1}{2}(\epsilon - \omega(s)) - \epsilon s,
\]

\[
\omega(s) = \sqrt{\epsilon^2 - (4\chi s)^2}.
\]

The above result is reduced to that based on the random phase approximation for the closed shell system, if $s$ is equal to $\Omega$, i.e., $n_p = n_h = 0$. The eigenstate and the eigenvalue are given as follows:

\[
|\lambda \mu \rangle = (\hat{D}^*)^\lambda (\hat{C}^*)^\mu |\phi \rangle, \quad \hat{D} |\phi \rangle = \hat{C} |\phi \rangle = 0,
\]

\[
E_{\lambda \mu} = E_0 + \lambda \omega(s) + \mu \epsilon.
\]
Fig. 4. The ground state energy for \( n_p = 0, \ n_h = 1 \) (solid curve) and exact ground state energy (dashed curve) are shown in the case of the open shell system with \( N = 11 \) and \( \Omega = 6 \).

Fig. 5. The energy difference between ground state and the first excited state for \( n_p = 0, \ n_h = 1 \) (solid curve) and exact results (dashed curve) are shown in the case of the open shell system with \( N = 11 \) and \( \Omega = 6 \).

However, it must be noticed that the result (6.10) contains a problem to be investigated. The result (6.10) is derived for the Hamiltonian (6.5) and it must be checked if the result (6.10) is derived for the Hamiltonian (2.4) or not. For this problem, we note the operator \( \hat{R} + (s_p s_h; k) \) defined in the relation (5.16). Under the same spirit as that for the relation (6.2), \( \hat{R} + (s_p s_h; k) \) may be approximated as

\[
\hat{R} + (s_p s_h; k) \approx \hat{A}_p^{\dagger} \sqrt{2s_h} + \hat{A}_h^{\dagger} \sqrt{2s_p} = \sqrt{2s} \hat{C}^*.
\]  

Therefore, under the present approximation, successive operation of \( \hat{R} + (s_p s_h; k) \) may be equivalent to that of \( \hat{C}^* \). On the other hand, the operation of \( \hat{S} + (s_p s_h) \) may be equivalent to that of \( \hat{B}^* \). Any state obtained by operation of \( \hat{C}^* \) does not have the counterpart in the original fermion space. Thus, we have the following conclusion: Only the part related to \( \mu = 0 \) should be selected such as

\[
|\lambda\mu = 0\rangle = (\hat{D}^*)^{\lambda} |\phi\rangle, \quad E_{\lambda\mu=0} = E_0 + \lambda\omega(s).
\]  

(6-12)
Fig. 6. The ground state energy for $n_p = 0, n_h = 2$ (solid curve) and exact ground state energy (dashed curve) are shown in the case of the open shell system with $N = 10$ and $\Omega = 6$.

Fig. 7. The energy difference between ground state and the first excited state for $n_p = 0, n_h = 2$ (solid curve) and exact results (dashed curve) are shown in the case of the open shell system with $N = 10$ and $\Omega = 6$.

Of course, the above result is applicable to the region

$$4\chi s \leq \epsilon .$$

(6.13)

The phase transition occurs at $\chi = \epsilon / 4s$. The relation (6.13) shows that the force strength $\chi$ at the phase transition point obtained in the case $s < \Omega$ becomes larger than $\chi$ obtained in the conventional random phase approximation for the case $s = \Omega$. From the above consideration, we can understand that our present approximation is a natural generalization from the conventional random phase approximation. Next, we will show some numerical results.

In Fig. 1 $\omega/\epsilon$ in Eq. (6.8) is depicted as a function of the force strength $\chi$ divided by $\epsilon$ in the cases $n_p = n_h = 0$ (solid curve) and $n_p = n_h = 1$ (dashed curve), respectively, with the fermion number $N = 12$.

First, let us consider the closed shell system with $n_p = n_h = 0$. In Fig. 2 the ground state energy for $n_p = n_h = 0$ (solid curve) is compared with the exact ground state energy (dashed curve) as a function of the force strength $\chi$ in the case $N = 12$.
Fig. 8. The ground state energy for \( n_p = 0, \ n_h = 9 \) (solid curve) and exact ground state energy (dashed curve) are shown in the case of the open shell system with \( N = 3 \) and \( \Omega = 6 \).

Fig. 9. The energy difference between ground state and the first excited state for \( n_p = 0, \ n_h = 9 \) (solid curve) and exact results (dashed curve) are shown in the case of the open shell system with \( N = 3 \) and \( \Omega = 6 \).

and \( \Omega = 6 \), where all quantities are scaled by the single particle energy \( \epsilon \). It is shown that this approximation is rather good in the region with small force strength compared with the phase transition point \( \chi = \epsilon/4s \). In Fig.8 the energy difference between the first excited state and the ground state is depicted compared with the exact energy difference. It is seen that the goodness of the approximation is similar to that for the ground state energy.

Next, let us consider the open shell system in which we take \( n_p = 0 \) and \( n_h \neq 0 \). Figure 4 shows the ground state energy (solid curve) in \( \Omega = 6 \) and \( n_p = 0 \) and \( n_h = 1 \), which leads to \( N = 11 \), and the exact energy eigenvalue (dashed curve). In our framework, it is shown that we can describe the open shell system well as for the ground state energy. In Fig.5 the energy difference between the first excited state and the ground state is depicted with the same parameter set as that in Fig.4.

The similar results are derived in the case \( n_p = 0 \) and \( n_h = 2 \) with \( \Omega = 6 \) which leads to \( N = 10 \). Figures 6 and 7 show the calculated results of the ground state energies and the energy differences between the first excited state and the ground state, respectively, for our approximated treatment and the exact diagonalization.

Further, let us consider the cases with small particle number in the open shell
Fig. 10. The ground state energy for \( n_p = 0, \ n_h = 8 \) (solid curve) and exact ground state energy (dashed curve) are shown in the case of the open shell system with \( N = 4 \) and \( \Omega = 6 \).

Fig. 11. The energy difference between ground state and the first excited state for \( n_p = 0, \ n_h = 8 \) (solid curve) and exact results (dashed curve) are shown in the case of the open shell system with \( N = 4 \) and \( \Omega = 6 \).

system. Figures 8 and 9 show the ground state energy and the energy difference between the first excited and ground state, respectively, in the case \( N = 3 \) with \( \Omega = 6, \ n_p = 0 \) and \( n_h = 9 \). The energy difference between the first excited state and the ground state does not depend on the force strength \( \chi \) in the exact result. Except for this situation, the behavior is similar to the case with large \( N \) in Figs. 2, 7. Also, in the case \( N = 4 \) with \( \Omega = 6, \ n_p = 0 \) and \( n_h = 8 \), the similar results are obtained as is shown in Figs. 10 and 11.

In conclusion, we can treat the open shell system by using of our unconventional boson realization method developed in this paper, adding to the case of the closed shell system. As a result, in spite of the approximation in Eq. (6.2), in which it is assumed that the effects expressed by the power series of \( \hat{A}_p \hat{A}_p \) and \( \hat{A}_h \hat{A}_h \) appearing in \( \hat{S}_\pm(s_p s_h) \) are small compared with \( 2s_p \) and \( 2s_h \), respectively, the obtained results are rather good, especially, in the region with small force strength, while the approximation is not so good when the force strength approaches to the transition point.
§7. The isoscalar pairing model

In addition to the Lipkin model, we know a many-fermion system consisting of the two single-particle levels and obeying the su(2)-algebra: The isoscalar pairing model. In this model, the two single-particle levels, which we call the p- and the n-level, are occupied by protons and neutrons, respectively. Of course, the degeneracies are the same as each other: \( \Omega = j + 1/2 \) (\( j \) : half-integer). Building block of this model is the proton-neutron pair coupled in the isoscalar type, which obeys the su(2)-algebra. Therefore, the isoscalar model is in a near relation to the Lipkin model. In this sense, it may be interesting to investigate both models comparatively. In this connection, the isovector pairing model obeys the so(5)-algebra and if both pairing models are combined with each other, we have the su(4)-algebra. The passage has been discussed in the high temperature superconductivity\(^3\)\(^4\).

We denote the proton and the neutron operator as \( \tilde{p}_m, p^*_m \) and \( \tilde{n}_m, n^*_m \), respectively. Here, of course, \( m = -j, -j+1, \ldots, j-1, j \). In this model, we can introduce a set of the operators \( \tilde{\sigma}_{\pm,0} \) defined as

\[
\tilde{\sigma}_+ = \sum_m \theta(m) p^*_m n^*_m, \quad \tilde{\sigma}_- = \sum_m \theta(m) n_m p_m, \quad \tilde{\sigma}_0 = \frac{1}{2} \sum_m (p^*_m p_m + n^*_m n_m) - \Omega.
\]  

Here, \( \theta(m) = m/|m| \), i.e., \( \theta(m) = 1 \) for \( m > 0 \) and \( \theta(m) = -1 \) for \( m < 0 \). The operator \( \tilde{\sigma}_+ \) is expressed in a form of a certain linear combination of \( (p^*n^*)_{J=odd,M=0} \). If \( \theta(m) = 1 \) for all \( m \), \( \tilde{\sigma}_+ = (p^*n^*)_{J=0,M=0} \) and in this case, including \( (p^*p^*)_{J=0,M=0} \), \( (n^*n^*)_{J=0,M=0} \), they form the so(5)-algebra. It is easily verified that \( (\tilde{\sigma}_{\pm,0}) \) defined in the relation \( (7.1) \) obeys

\[
[ \tilde{\sigma}_+, \tilde{\sigma}_- ] = 2\tilde{\sigma}_0, \quad [ \tilde{\sigma}_0, \tilde{\sigma}_\pm ] = \pm\tilde{\sigma}_\pm.
\]  

Further, we have

\[
[ \text{any of } (\tilde{\sigma}_{\pm,0}), \text{any of } (\tilde{\sigma}_{\pm,0}) ] = 0.
\]  

Here, \( (\tilde{\sigma}_{\pm,0}) \) denotes isospin operator:

\[
\tilde{\tau}_+ = \sum_m p^*_m n_m, \quad \tilde{\tau}_- = \sum_m n^*_m p_m, \quad \tilde{\tau}_0 = \frac{1}{2} \sum_m (p^*_m p_m - n^*_m n_m).
\]  

The relation \( (7.3) \) tells us that \( (\tilde{\sigma}_{\pm,0}) \) is isoscalar. Different from the case of the Lipkin model, the operator \( \tilde{\sigma}_0 \) is expressed in terms of the total nucleon number \( \tilde{N} \):

\[
\tilde{N} = \tilde{N}_p + \tilde{N}_n, \quad \tilde{N}_p = \sum_m p^*_m p_m, \quad \tilde{N}_n = \sum_m n^*_m n_m.
\]  

This fact is important. It gives us a possible boson realization of the su(2)-algebra which is different from the Lipkin model treated in §§3 and 4. The relations \( (7.1), (7.4) \) and \( (7.5) \) give us

\[
\tilde{N}_p = \Omega + \tilde{\sigma}_0 + \tilde{\tau}_0, \quad \tilde{N}_n = \Omega + \tilde{\sigma}_0 - \tilde{\tau}_0.
\]
By replacing the index \( h \) with \( n \) in the relation (3.1), we postulate the counterpart of \( \tilde{\sigma}_{\pm,0} \), which we denote \( \tilde{\sigma}_{\pm,0} \), in the following form:

\[
\tilde{\sigma}_+ = \hat{\sigma}_p^* \hat{b}_n - \hat{\sigma}_n^* \hat{b}_p , \quad \tilde{\sigma}_- = \hat{b}_n^* \hat{a}_p - \hat{b}_p^* \hat{a}_n , \quad \tilde{\sigma}_0 = \frac{1}{2} \left[ (\hat{\sigma}_p^* \hat{a}_p + \hat{\sigma}_n^* \hat{a}_n) - (\hat{b}_p^* \hat{b}_p + \hat{b}_n^* \hat{b}_n) \right]. \tag{7.7}
\]

The \( su(1,1) \)-generators are given as

\[
\tilde{T}_+ = \hat{\sigma}_p^* \hat{b}_n + \hat{\sigma}_n^* \hat{b}_p , \quad \tilde{T}_- = \hat{b}_n^* \hat{a}_p + \hat{b}_p^* \hat{a}_n , \quad \tilde{T}_0 = \frac{1}{2} \left[ (\hat{\sigma}_p^* \hat{a}_p + \hat{\sigma}_n^* \hat{a}_n) + (\hat{b}_p^* \hat{b}_p + \hat{b}_n^* \hat{b}_n) \right] + 1. \tag{7.8}
\]

Of course, the above expression comes from the relation (3.2). Further, we postulate the following form for the counterpart of \( (\tilde{\tau}_{\pm,0}) \):

\[
\tilde{\tau}_+ = \hat{\tau}_p^* \hat{a}_n - \hat{\tau}_n^* \hat{a}_p , \quad \tilde{\tau}_- = \hat{\tau}_n^* \hat{a}_p - \hat{\tau}_p^* \hat{a}_n , \quad \tilde{\tau}_0 = \frac{1}{2} \left[ (\hat{\tau}_p^* \hat{a}_p + \hat{\tau}_n^* \hat{a}_n) - (\hat{\tau}_p^* \hat{b}_p + \hat{\tau}_n^* \hat{b}_n) \right]. \tag{7.9}
\]

The set \( \{\tilde{\tau}_{\pm,0}\} \) obeys the \( su(2) \)-algebra and commutes with \( (\tilde{\sigma}_{\pm,0}) \) and \( (\tilde{T}_{\pm,0}) \). The relation (7.4) permits us to set up the relation

\[
\tilde{N}_p \to \tilde{N}_p = \hat{\Omega} + \tilde{\sigma}_0 + \tilde{\tau}_0 = \hat{\Omega} + \hat{\sigma}_p^* \hat{a}_p - \hat{\tau}_p^* \hat{b}_p , \quad \tilde{N}_n \to \tilde{N}_n = \hat{\Omega} + \tilde{\sigma}_0 - \tilde{\tau}_0 = \hat{\Omega} + \hat{\tau}_n^* \hat{b}_n - \hat{\sigma}_n^* \hat{a}_n . \tag{7.10}
\]

The operator \( \hat{\Omega} \) is determined in the framework of the form

\[
\hat{\Omega} = x + \frac{y}{2}(\hat{\sigma}_p^* \hat{a}_p + \hat{\sigma}_n^* \hat{a}_n + \hat{\tau}_p^* \hat{b}_p + \hat{\tau}_n^* \hat{b}_n) . \tag{7.11}
\]

Of course, \( x \) and \( y \) should be determined.

For the minimum weight state \( |m\rangle \), we require the relations

\[
\tilde{\sigma}_- |m\rangle = \tilde{T}_- |m\rangle = 0 , \quad \tilde{N}_p |m\rangle = n_p |m\rangle , \quad \tilde{N}_n |m\rangle = n_n |m\rangle , \quad \hat{\Omega} |m\rangle = \Omega |m\rangle . \tag{7.12, 7.13}
\]

The relations (7.12) and (7.13) gives us

\[
\hat{\Omega} = \frac{1}{2}(n_p + n_n) + \frac{1}{2}(\hat{\sigma}_p^* \hat{a}_p + \hat{\sigma}_n^* \hat{a}_n + \hat{\tau}_n^* \hat{b}_n) , \tag{7.14}
\]

\[
|m\rangle = (\hat{b}_p^*)^{\Omega-n_p} (\hat{b}_n^*)^{\Omega-n_n} |0\rangle . \tag{7.15}
\]

Then, we have

\[
\tilde{N}_p = \frac{1}{2}(n_p + n_n) + \frac{1}{2}(3\hat{\sigma}_p^* \hat{a}_p + \hat{\sigma}_n^* \hat{a}_n + \hat{\tau}_p^* \hat{b}_p + \hat{\tau}_n^* \hat{b}_n) , \tag{7.16}
\]

\[
\tilde{N}_n = \frac{1}{2}(n_p + n_n) + \frac{1}{2}(\hat{\sigma}_p^* \hat{a}_p + 3\hat{\sigma}_n^* \hat{a}_n + \hat{\tau}_p^* \hat{b}_p - \hat{\tau}_n^* \hat{b}_n) . \tag{7.16}
\]
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Namely,

\[ \hat{N} = n_p + n_n + 2(\hat{a}_p^* \hat{a}_p + \hat{a}_n^* \hat{a}_n) \]  \hspace{1cm} (7.17)

The state \( |m\rangle \) is also the eigenstate for \( \hat{\sigma}_0 \) with the eigenvalue

\[ \sigma = \Omega - \frac{1}{2}(n_p + n_n) \]  \hspace{1cm} (7.18)

We can see that \( \hat{\Omega} \) and \( \sigma \) are formally identical to those in the Lipkin model, but, \( \hat{N} \) is different from that in the Lipkin model. It is nothing but the expectation.

Next task is to discuss the orthogonal set for the present model. The minimum weight state \( |m\rangle \) shown in the form (7.5) satisfies the relation

\[ \hat{\sigma}_0 |m\rangle = -\sigma |m\rangle, \quad \hat{T}_0 |m\rangle = \tau |m\rangle, \quad \hat{\tilde{T}}_0 |m\rangle = T |m\rangle \]  \hspace{1cm} (7.19)

\[ \sigma = \Omega - \frac{1}{2}(n_p + n_n), \quad \tau_0 = \frac{1}{2}(n_p - n_n), \quad T = \Omega + 1 \]  \hspace{1cm} (7.20)

Further, for the Casimir operators \( \hat{\sigma}^2, \hat{\tau}^2 \) and \( \hat{T}^2 \), we have

\[ \hat{\sigma}^2 |m\rangle = \hat{\tau}^2 |m\rangle = \hat{T}^2 |m\rangle = \sigma(\sigma + 1) |m\rangle \]  \hspace{1cm} (7.21)

Then, \( |m\rangle \) can be rewritten in the form

\[ |\Omega; \sigma\sigma_0\tau_0 \rangle = (\hat{\sigma}_0^+)^{\sigma+\sigma_0} (\hat{\tau}_0^+)^{\sigma+\tau_0} (\hat{b}_p^*)^{2\sigma} |0\rangle \]  \hspace{1cm} (7.22)

Of course, including the phase factor, the normalization is arbitrary. Therefore, the eigenstate of \( \hat{\sigma}_0 \) with the eigenvalue \( \sigma_0 \) is given in the form

\[ |\Omega; \sigma_0\sigma_0\tau_0 \rangle = (\hat{\sigma}_0^+)^{\sigma+\sigma_0} (\hat{\tau}_0^+)^{\sigma+\tau_0} (\hat{b}_p^*)^{2\sigma} |0\rangle \]  \hspace{1cm} (7.23)

Since \( [\hat{\sigma}_+, \hat{\tau}_+] = 0 \), the state \( |\Omega; \sigma_0\sigma_0\tau_0 \rangle \) can be expressed in various forms, for example,

\[ |\Omega; \sigma_0\sigma_0\tau_0 \rangle = (\hat{\tau}_0^+)^{\sigma+\tau_0} (\hat{\sigma}_0^+)^{\sigma+\sigma_0} (\hat{b}_p^*)^{2\sigma} |0\rangle \]  \hspace{1cm} (7.24)

With the use of the relation (7.17), the eigenvalue of \( \hat{N}, \hat{N} \), for the state \( |\Omega; \sigma_0\sigma_0\tau_0 \rangle \) is expressed as

\[ \hat{N} = 2\Omega - 2\sigma_0 \quad \text{i.e.,} \quad \sigma_0 = \Omega - \frac{N}{2} \]  \hspace{1cm} (7.25)

Therefore, we have

\[ \sigma = \Omega - \frac{1}{2}(n_p + n_n), \quad \sigma_0 = \Omega - \frac{N}{2}, \quad \tau_0 = \frac{1}{2}(n_p - n_n) \]  \hspace{1cm} (7.26)

Since \( n_p \geq 0, n_n \geq 0, -\sigma \leq \sigma_0 \leq \sigma, n_p, n_n \) and \( \hat{N} \) obey the inequality

\[ 0 \leq n_p \leq \Omega, \quad 0 \leq n_n \leq \Omega, \quad n_p + n_n \leq \hat{N} \leq 4\Omega - (n_p + n_n) \]  \hspace{1cm} (7.27)
In this way, we know that $|\Omega; \sigma \sigma_0 \tau_0\rangle$ is characterized by $\Omega$, $N$, $n_p$ and $n_n$ governed by the relation (7.27). Of course, $n_p$ and $n_n$ determine the minimum weight state.

Finally, we will sketch the idea for constructing the Holstein-Primakoff boson realization. We introduce the operators $\hat{\sigma}_p$ and $\hat{\sigma}_n$ defined as

$$\hat{\sigma}_p = \frac{1}{2}(\hat{a}^*_p \hat{a}_p + \hat{b}^*_p \hat{b}_p) , \quad \hat{\sigma}_n = \frac{1}{2}(\hat{a}^*_n \hat{a}_n + \hat{b}^*_n \hat{b}_n) . \tag{7.28}$$

Since $[\hat{\sigma}_p, \hat{\sigma}_\pm, 0] = [\hat{\sigma}_n, \hat{\sigma}_\pm, 0] = 0$, we have

$$\hat{\sigma}_p |\Omega; \sigma \sigma_0 \tau_0\rangle = \sigma_p |\Omega; \sigma \sigma_0 \tau_0\rangle , \quad \sigma_p = \frac{1}{2}(\Omega - n_p) ,$$

$$\hat{\sigma}_n |\Omega; \sigma \sigma_0 \tau_0\rangle = \sigma_n |\Omega; \sigma \sigma_0 \tau_0\rangle , \quad \sigma_n = \frac{1}{2}(\Omega - n_n) . \tag{7.29}$$

Therefore, we have

$$\sigma = \sigma_p + \sigma_n . \tag{7.30}$$

The above consideration enables us to construct the Holstein-Primakoff boson realization in parallel with the Lipkin model.

§8. Concluding remark

In this paper, concentrating on the $su(2)$-algebraic many-fermion theory, a new framework of the boson realization for the $su(2)$-algebra was formulated. New boson realization developed in this paper is called the unconventional boson realization. In order to be able to treat the open shell system in the $su(2)$-algebraic model, the Schwinger and Holstein-Primakoff boson realizations were formulated in the unconventional form.

The unconventional boson realization method developed in this paper was applied to the two-level Lipkin model whose Hamiltonian can be expressed in terms of the generators of the $su(2)$-algebra. One can deal with the closed shell system only by using the conventional boson realization such as the Schwinger and the Holstein-Primakoff boson realization. However, it was shown that, in our formalism with the unconventional boson realization, the open shell system with $N \neq 2\Omega$ can be described. As an concrete example, the ground state energy and the energy difference between the ground state and the first excited state were investigated by using the Holstein-Primakoff-type unconventional boson realization under a certain approximation. This approximation corresponds to the random phase approximation describing the closed shell system. It was demonstrated that the calculated results in the region with small force strength are rather good in comparison with the exact results, while the approximated results are not so good when the force strength approaches the phase transition point. The behavior of the approximate solution mentioned above may be conjectured beforehand, because the approximation adopted in §6 is on the same level as that in the case of the closed shell system, i.e., the random phase approximation. In order to interpret the behavior near the
phase transition point, various ideas have been proposed. We can apply these ideas to the present system, but, this investigation is a problem to be solved in future.

Further, adding to the two-level Lipkin model, we discussed the isoscalar pairing model in our unconventional boson realization comparatively. It may be interesting to extending this model to the isovector pairing model governed by the $so(5)$-algebra and, also, the combined model with the isoscalar and the isovector pairing model governed by the $su(4)$-algebra.

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