SQUARE FUNCTION CHARACTERIZATION OF WEAK HARDY SPACES

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Abstract. We obtain a new square function characterization of the weak Hardy space \( H^{p,\infty} \) for all \( p \in (0, \infty) \). This space consists of all tempered distributions whose smooth maximal function lies in weak \( L^p \). Our proof is based on interpolation between \( H^p \) spaces. The main difficulty we overcome is the lack of a good dense subspace of \( H^{p,\infty} \) which forces us to work with general \( H^{p,\infty} \) distributions.

1. Introduction

In this work we extend Peetre’s \cite{20} characterization of Hardy spaces in terms of the Littlewood-Paley square function to the setting of weak Hardy spaces, \( H^{p,\infty} \).

Hardy spaces first appeared in the work of Hardy \cite{16} in 1914. Their study was based on complex methods and their theory was one-dimensional. Burkholder, Gundy and Silverstein \cite{3} proved that a complex function \( F = u + iv \) on the upper half space lies in \( H^p \) if and only if the nontangential maximal function of \( u \) lies in \( L^p(\mathbb{R}) \). This result inspired the extension of the theory of Hardy spaces to higher dimensions, in particular the celebrated work of Fefferman and Stein \cite{10} on this topic. A deep structural characterization of these spaces was given by Coifman \cite{4} and Latter \cite{18}, in terms of their atomic decomposition. The books of Lu \cite{19}, Uchiyama \cite{24}, and Triebel \cite{23} provide comprehensive expositions on the theory of Hardy spaces on Euclidean spaces. The theory of Hardy spaces has proved to be so rich and fruitful that has been extended to spaces of homogeneous type; we refer to the works of Coifman, and Weiss \cite{5}, Macías, and Segovia \cite{21}, Duong, and Yan \cite{7}, Han, Müller and Yang \cite{15} and Hu, Yang, Zhou \cite{17} for results and applications in this setting.

The Hardy-Lorentz spaces \( H^{p,q} \), \( 0 < p < \infty, 0 < q \leq \infty \) are defined as the spaces of all distributions whose smooth maximal function lies in the Lorentz space \( L^{p,q} \). These spaces were studied by Fefferman and Soria \cite{9}, Alvarez \cite{2}, and Abu-Shammala and Torchinsky \cite{1}. Fefferman, Rivi`ere and Sagher \cite{8} showed that the \( H^{p,q} \) spaces are intermediate spaces of Hardy spaces in the \( K \)-interpolation method. The interpolation result in \cite{8} was only proved for Schwartz functions, which is not a dense subspace of \( H^{p,q} \) when \( q = \infty \), a fact also observed in \cite{9}.

In this article we focus on the case \( q = \infty \) which presents difficulties due to the lack of a good dense subspace of it. We prove an interpolation theorem for weak Hardy spaces as intermediate spaces of Hardy spaces and we work with general tempered distributions and the grand maximal function to accomplish this; for this reason our
proof looks unavoidably complicated. As an application we obtain a new Littlewood-Paley square function characterization of weak Hardy spaces. This shows that $H^{p, \infty}$ is a natural extension of $L^{p, \infty}$ when $p \leq 1$, just like $H^{p}$ is a natural extension of $L^{p}$ for $p \leq 1$, in view of the Littlewood-Paley theorem on weak $L^{p}$. This characterization reveals the orthogonality of weak $L^{p}$ spaces for $p < 1$ (Corollary 1), which is crucial in the theory of multilinear paraproduct.

We now state our main result. We denote by $\Delta_{j}(f) = \Psi_{2^{-j}} \ast f$ the Littlewood-Paley operator of a distribution $f$, where $\Psi_{t}(x) = t^{-n} \hat{\Psi}(x/t)$.

**Theorem 1.** Let $\Psi$ be a radial Schwartz function on $\mathbb{R}^{n}$ whose Fourier transform is nonnegative, supported in $1 - \frac{1}{2} \leq |\xi| \leq 2$, and satisfies $\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j} \xi) = 1$ when $\xi \neq 0$. Let $\Delta_{j}$ be the Littlewood–Paley operators associated with $\Psi$ and let $0 < p < \infty$. Then there exists a constant $C = C_{n, p, \Psi}$ such that for all $f \in H^{p, \infty}(\mathbb{R}^{n})$ we have

\[
(1) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{j}(f)|^{2} \right)^{1/2} \right\|_{L^{p, \infty}} \leq C \| f \|_{H^{p, \infty}}.
\]

Conversely, suppose that a tempered distribution $f$ satisfies

\[
(2) \quad \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{j}(f)|^{2} \right)^{1/2} \right\|_{L^{p, \infty}} < \infty.
\]

Then there exists a unique polynomial $Q$ such that $f - Q$ lies in $H^{p, \infty}$ and satisfies

\[
(3) \quad \frac{1}{C} \| f - Q \|_{H^{p, \infty}} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_{j}(f)|^{2} \right)^{1/2} \right\|_{L^{p, \infty}}.
\]

The proof of this theorem is based on Theorem 1 discussed in Section 1.

2. Background

We introduce the weak Hardy space $H^{p, \infty}$ via the Poisson maximal function, following the classical definition of the Hardy space. So, we begin our study by listing a result containing the equivalence of quasinorms of several kinds of maximal functions, which also appear in the theory of Hardy spaces.

We denote by $\ell^{2}$ the space $\ell^{2}(\mathbb{Z})$ of all square-integrable sequences and by $\ell^{2}(L)$ the finite-dimensional space of all sequences of length $L \in \mathbb{Z}^{+}$ with the $\ell^{2}$ norm. We say that a sequence of distributions $\{f_{j}\}_{j}$ lies in $\mathcal{S}'(\mathbb{R}^{n}, \ell^{2})$ if there are constants $C, M > 0$ such that for every $\varphi \in \mathcal{S}(\mathbb{R}^{n})$ we have

\[
\left\| \{ \langle f_{j}, \varphi \rangle \}_{j} \right\|_{\ell^{2}} = \left( \sum_{j} |\langle f_{j}, \varphi \rangle|^{2} \right)^{1/2} \leq C \sup_{|\alpha|, |\beta| \leq M} |y^{\beta} \partial^{\alpha} \varphi(y)|.
\]

A sequence of distributions $\tilde{f} = \{ \tilde{f}_{j} \}_{j}$ in $\mathcal{S}'(\mathbb{R}^{n}, \ell^{2})$ is called bounded if

\[
\left\| \{ \varphi \ast \tilde{f}_{j} \}_{j} \right\|_{\ell^{2}} = \left( \sum_{j} |\varphi \ast \tilde{f}_{j}|^{2} \right)^{1/2} \in L^{\infty}(\mathbb{R}^{n})
\]

for every $\varphi$ in $\mathcal{S}(\mathbb{R}^{n})$.

Let $a, b > 0$ and let $\Phi$ be a Schwartz function on $\mathbb{R}^{n}$.
Definition 1. For a sequence $\vec{f} = \{f_j\}_{j \in \mathbb{Z}}$ of tempered distributions on $\mathbb{R}^n$ we define the smooth maximal function of $\vec{f}$ with respect to $\Phi$ as
\[
M(\vec{f}; \Phi)(x) = \sup_{t > 0} \left\| \left\{ (\Phi_t * f_j)(x) \right\}_j \right\|_{\ell^2}.
\]
We define the nontangential maximal function with aperture $a$ of $\vec{f}$ with respect to $\Phi$ as
\[
M^*_a(\vec{f}; \Phi)(x) = \sup_{t > 0} \sup_{y \in \mathbb{R}^n} \left\| \left\{ (\Phi_t * f_j)(y) \right\}_j \right\|_{\ell^2}.
\]
We also define the auxiliary maximal function
\[
M^{**}_b(\vec{f}; \Phi)(x) = \sup_{t > 0} \sup_{y \in \mathbb{R}^n} \frac{\left\| \left\{ (\Phi_t * f_j)(x - y) \right\}_j \right\|_{\ell^2}}{(1 + t^{-1}|y|)^b}.
\]

For a fixed positive integer $N$ we define the grand maximal function of $\vec{f}$ as
\[
\mathcal{M}_N(\vec{f}) = \sup_{\varphi \in \mathcal{F}_N} M^*_1(\vec{f}; \varphi),
\]
where
\[
\mathcal{F}_N = \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \mathfrak{N}_N(\varphi) \leq 1 \right\},
\]
and
\[
\mathfrak{N}_N(\varphi) = \int_{\mathbb{R}^n} (1 + |x|)^N \sum_{|\alpha| \leq N+1} |\partial^\alpha \varphi(x)| \, dx.
\]

If the function $\Phi$ is not assumed to be Schwartz but say $\Phi$ is the Poisson kernel, then the maximal functions $M(\vec{f}; \Phi), M^*_a(\vec{f}; \Phi),$ and $M^{**}_b(\vec{f}; \Phi)$ are well defined for sequences of bounded tempered distributions $\vec{f} = \{f_j\}_j$.

We note that the following simple inequalities
\[
M(\vec{f}; \Phi) \leq M^*_a(\vec{f}; \Phi) \leq (1 + a)^b M^{**}_b(\vec{f}; \Phi)
\]
are valid. We now define the vector-valued Hardy space $H^{p, \infty}(\mathbb{R}^n, \ell^2)$.

Definition 2. Let $\vec{f} = \{f_j\}_j$ be a sequence of bounded tempered distributions on $\mathbb{R}^n$ and let $0 < p < \infty$. We say that $\vec{f}$ lies in the vector-valued weak Hardy space $H^{p, \infty}(\mathbb{R}^n, \ell^2)$ vector-valued Hardy space if the Poisson maximal function
\[
M(\vec{f}; P)(x) = \sup_{t > 0} \left\| \left\{ (P_t * f_j)(x) \right\}_j \right\|_{\ell^2}
\]
lies in $L^{p, \infty}(\mathbb{R}^n)$. If this is the case, we set
\[
\|\vec{f}\|_{H^{p, \infty}(\mathbb{R}^n, \ell^2)} = \|M(\vec{f}; P)\|_{L^{p, \infty}(\mathbb{R}^n)} = \left\| \sup_{\varepsilon > 0} \left( \sum_j |P_{\varepsilon} * f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p, \infty}(\mathbb{R}^n)}.
\]

The next theorem provides a characterization of $H^{p, \infty}$ in terms of different maximal functions. Its proof is a copy of that for $H^p$ cases in [13].
Theorem 2. Let $0 < p < \infty$. Then the following statements are valid:

(a) There exists a Schwartz function $\Phi$ with integral 1 and a constant $C_1$ such that

\begin{equation}
\| M(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} \leq C_1 \| \tilde{f} \|_{H^{p,\infty}(\mathbb{R}^n, \ell^2)}
\end{equation}

for every sequence $\tilde{f} = \{f_j\}_j$ of tempered distributions.

(b) For every $a > 0$ and $\Phi$ in $\mathcal{S}(\mathbb{R}^n)$ there exists a constant $C_2(n, p, a, \Phi)$ such that

\begin{equation}
\| M^*_a(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} \leq C_2(n, p, a, \Phi) \| M(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)}
\end{equation}

for every sequence $\tilde{f} = \{f_j\}_j$ of tempered distributions.

(c) For every $a > 0$, $b > n/p$, and $\Phi$ in $\mathcal{S}(\mathbb{R}^n)$ there exists a constant $C_3(n, p, a, b, \Phi)$ such that

\begin{equation}
\| M^{**}_b(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} \leq C_3(n, p, a, b, \Phi) \| M^*_a(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)}
\end{equation}

for every sequence $\tilde{f} = \{f_j\}_j$ of tempered distributions.

(d) For every $b > 0$ and $\Phi$ in $\mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \Phi(x) \, dx \neq 0$ there exists a constant $C_4(b, \Phi)$ such that if $N = \lfloor b \rfloor + 1$ we have

\begin{equation}
\| \mathcal{M}(\tilde{f}) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} \leq C_4(b, \Phi) \| M^{**}_b(\tilde{f}; \Phi) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)}
\end{equation}

for every sequence $\tilde{f} = \{f_j\}_j$ of tempered distributions.

(e) For every positive integer $N$ there exists a constant $C_5(n, N)$ such that every sequence $\tilde{f} = \{f_j\}_j$ of tempered distributions that satisfies $\| \mathcal{M}(\tilde{f}) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} < \infty$ is bounded and satisfies

\begin{equation}
\| \tilde{f} \|_{H^{p,\infty}(\mathbb{R}^n, \ell^2)} \leq C_5(n, N) \| \mathcal{M}(\tilde{f}) \|_{L^{p,\infty}(\mathbb{R}^n, \ell^2)} ,
\end{equation}

that is, it lies in the Hardy space $H^{p,\infty}(\mathbb{R}^n, \ell^2)$.

We conclude that for $\tilde{f} \in H^{p,\infty}(\mathbb{R}^n, \ell^2)$, the inequality in (11) can be reversed whenever $N = \lfloor n/p \rfloor + 1$. Moreover, fix $N = \lfloor a/p \rfloor + 1$, $a < b < \lfloor a/p \rfloor + 1 - \frac{n}{p}$, and $\Phi$ a Schwartz function with $\int_{\mathbb{R}^n} \Phi(x) \, dx = 1$. Then for bounded distributions $\tilde{f} = \{f_j\}_j$ the following equivalence of quasi-norms holds

\begin{equation}
\| \mathcal{M}(\tilde{f}) \|_{L^{p,\infty}} \approx \| M^*_b(\tilde{f}; \Phi) \|_{L^{p,\infty}} \approx \| M^*_a(\tilde{f}; \Phi) \|_{L^{p,\infty}} \approx \| M(\tilde{f}; \Phi) \|_{L^{p,\infty}}
\end{equation}

with constants that depend only on $\Phi, a, n, p$, and all the preceding quasi-norms are also equivalent with $\| \tilde{f} \|_{H^{p,\infty}(\mathbb{R}^n, \ell^2)}$.

3. Properties of $H^{p,\infty}$

The spaces $H^{p,\infty}$ have several properties analogous to those of the classical Hardy spaces $H^p$.

Theorem 3. Let $1 < p < \infty$. Then we have $L^p = H^{p,\infty}$ and $\| f \|_{L^p} \approx \| f \|_{H^{p,\infty}}$. 

Proof. Given \( f \in L^{p,\infty} \), then \( f \) is locally integrable, and we can define \( \varphi_t \ast f \) for a Schwartz function \( \varphi \) with \( \int \varphi \neq 0 \). By Proposition 4 which we will prove in section 4 we have

\[
\|f\|_{H^{p,\infty}} = \|\sup_{t>0} |(\varphi_t \ast f)(x)|\|_{L^{p,\infty}} \leq C\|M(f)(x)\|_{L^{p,\infty}} \leq C\|f\|_{L^{p,\infty}},
\]

where \( M(f) \) is the Hardy-Littlewood maximal function. This shows that \( f \) lies in \( H^{p,\infty} \).

Suppose now that \( f \in H^{p,\infty} \). By the weak*-compactness of \( L^{p,\infty} = (L^{p',1})' \), there exists a sequence \( t_j \to 0 \) and a function \( f_0 \in L^{p,\infty} \) such that \( (\varphi_{t_j} \ast f, g) \to (f_0, g) \) for all \( g \in L^{p',1} \). This implies that \( \varphi_{t_j} \ast f \to f_0 \) in \( S' \). By \( \varphi_t \ast \psi \to \psi \) in \( S \) we have \( \varphi_t \ast f \to f \) in \( S' \), so \( f \) is in \( L^{p,\infty} \). In view of the Lebesgue differentiation theorem we obtain that

\[
\|f\|_{L^{p,\infty}} \leq \|\sup_{t>0} |\varphi_t \ast f|\|_{L^{p,\infty}} = \|f\|_{H^{p,\infty}}.
\]

The preceding inequalities show that the spaces \( L^{p,\infty} \) and \( H^{p,\infty} \) coincide with equivalence of norms. \( \square \)

Next, we define a norm on Schwartz functions relevant in the theory of Hardy spaces:

\[
\mathfrak{M}_N(\varphi; x_0, R) = \int_{R^n} \left(1 + \left|\frac{x - x_0}{R}\right|\right)^N \sum_{|\alpha| \leq N+1} R^{|\alpha|} |\partial^\alpha \varphi(x)| \, dx.
\]

Note that \( \mathfrak{M}_N(\varphi; 0, 1) = \mathfrak{M}_N(\varphi) \).

Theorem 4. (a) For any \( 0 < p \leq 1 \), every \( \vec{f} = \{f_j\}_j \) in \( H^{p,\infty}(\mathbb{R}^n, \ell^2) \), and any \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) we have

\[
\left( \sum_j |\langle f_j, \varphi \rangle|^2 \right)^{1/2} \leq \mathfrak{M}_N(\varphi) \inf_{|z| \leq 1} \mathcal{M}(\vec{f})(z),
\]

where \( N = \left[\frac{n}{p}\right] + 1 \), and consequently there is a constant \( C_{n,p} \) such that

\[
\left( \sum_j |\langle f_j, \varphi \rangle|^2 \right)^{1/2} \leq \mathfrak{M}_N(\varphi) C_{n,p} \|\vec{f}\|_{H^{p,\infty}}.
\]

(b) Let \( 0 < p \leq 1 \), \( N = \left[\frac{n}{p}\right] + 1 \), and \( p < r \leq \infty \). Then there is a constant \( C(p, n, r) \) such that for any \( \vec{f} \in H^{p,\infty} \) and \( \varphi \in \mathcal{S}(\mathbb{R}^n) \) we have

\[
\left\| \left( \sum_j |f_j \ast \varphi|^2 \right)^{1/2} \right\|_{L^r} \leq C(p, n, r) \mathfrak{M}_N(\varphi) \|\vec{f}\|_{H^{p,\infty}}.
\]

(c) For any \( x_0 \in \mathbb{R}^n \), for all \( R > 0 \), and any \( \psi \in \mathcal{S}(\mathbb{R}^n) \) we have

\[
\left( \sum_j |\langle f_j, \psi \rangle|^2 \right)^{1/2} \leq \mathfrak{M}_N(\psi; x_0, R) \inf_{|z - x_0| \leq R} \mathcal{M}(\vec{f})(z).
\]
Proof. (a) We use that \( \langle f_j, \varphi \rangle = (\tilde{\varphi} * f_j)(0) \), where \( \tilde{\varphi}(x) = \varphi(-x) \) and we observe that \( \mathfrak{M}_N(\varphi) = \mathfrak{M}_N(\tilde{\varphi}) \). Then follows from the inequality

\[
\left( \sum_j |(\tilde{\varphi} * f_j)(y)|^2 \right)^{1/2} \leq \mathfrak{M}_N(\varphi) M_1 \left( \bar{f}; \tilde{\varphi} \frac{\tilde{\varphi}}{\mathfrak{M}_N(\varphi)} \right)(z) \leq \mathfrak{M}_N(\varphi) M_N(\bar{f})(z)
\]

for all \(|y - 0| < 1\), which is valid, since \( \tilde{\varphi}/\mathfrak{M}_N(\varphi) \) lies in \( \mathcal{F}_N \). We deduce as follows: set \( \lambda_0 = \inf_{|z| \leq 1} M_N(\bar{f})(z), \) then

\[
\left( \sum_j |\langle f_j, \varphi \rangle|^2 \right)^{p/2} \leq \mathfrak{M}_N(\varphi)^p \inf_{|z| \leq 1} M_N(f)(z)^p
\]

\[
\leq \mathfrak{M}_N(\varphi)^p \frac{1}{|B(0, 1)|} |B(0, 1)| \inf_{|z| \leq 1} M_N(\bar{f})(z)^p
\]

\[
\leq \mathfrak{M}_N(\varphi)^p \frac{1}{\nu_n} |\{ y \in \mathbb{R}^n : M_N(\bar{f})(y) > \lambda_0/2 \}| \lambda_0^p
\]

\[
\leq \mathfrak{M}_N(\varphi)^p C_{p,n}^p \| \bar{f} \|_{H^{0, \infty}}^p.
\]

(b) For any fixed \( x \in \mathbb{R}^n \) and \( t > 0 \) we have

\[
(\sum_j |(\varphi_t * f_j)(x)|^2)^{1/2} \leq \mathfrak{M}_N(\varphi) M_1 \left( \bar{f}; \varphi \frac{\varphi}{\mathfrak{M}_N(\varphi)} \right)(y) \leq \mathfrak{M}_N(\varphi) M_N(\bar{f})(y)
\]

for all \( y \) satisfying \(|y - x| \leq 1\). Restricting to \( t = 1 \) results in

\[
(\sum_j |(\varphi * f_j)(x)|^2)^{p/2} \leq \mathfrak{M}_N(\varphi)^p C_{p,n}^p \| \bar{f} \|_{H^{0, \infty}}^p
\]

by an argument similar to the preceding one using \( \lambda_0 \). This implies that

\[
\left\| \left( \sum_j |\varphi * f_j|^2 \right)^{1/2} \right\|_{L^\infty} \leq C_{p,n} \mathfrak{M}_N(\varphi) \| \bar{f} \|_{H^{0, \infty}}.
\]

Choosing \( y = x \) and \( t = 1 \) in (15) and then taking \( L^{p, \infty} \) quasinorms yields

\[
\left\| \left( \sum_j |\varphi * f_j|^2 \right)^{1/2} \right\|_{L^{p, \infty}} \leq C_{p,n} \mathfrak{M}_N(\varphi) \| \bar{f} \|_{H^{0, \infty}}.
\]

By interpolation we deduce

\[
\left\| \left( \sum_j |\varphi * f_j|^2 \right)^{1/2} \right\|_{L^r} \leq C_{p,n,r} \mathfrak{M}_N(\varphi) \| \bar{f} \|_{H^{0, \infty}}.
\]

when \( r < p \leq \infty \).

(c) To prove (15), given a Schwartz function \( \psi \) and \( R > 0 \), define another function \( \varphi \) by \( \varphi(y) = \psi(-Ry + x_0) \) so that \( \varphi(x) = \varphi(\frac{x_0 - x}{R}) = R^n \varphi_R(x_0 - x) \). In view of (16) we have

\[
(\sum_j |\langle f_j, \psi \rangle|)^{1/2} = R^n \left( \sum_j |(\varphi_R * f)(x_0)| \right)^{1/2} \leq R^n \mathfrak{M}_N(\varphi) \inf_{|z - x_0| \leq R} M_N(\bar{f})(z).
\]
Corollary 1. Convergence in $H^{p,\infty}$ implies convergence in $S'$.

This is a direct corollary of (a) of Theorem 4.

Proposition 1. If $f_j \to f$ in $S'$, and $\|f_j\|_{H^p} \leq C$, then $\|f\|_{H^p} \leq C$. If $f_j \to f$ in $S'$, and $\|f_j\|_{H^{p,\infty}} \leq C$, then $\|f\|_{H^{p,\infty}} \leq C$.

Proof. Note that $f_j \to f$ in $S'$ implies that $\varphi_t * f_j \to \varphi_t * f$ pointwise for any Schwartz function $\varphi$ with integral 1. Then for any $t > 0$ we have

$$|\varphi_t * f| = \liminf_{j \to \infty} |\varphi_t * f_j| \leq \liminf_{j \to \infty} \sup_{s > 0} |\varphi_s * f_j|.$$

Taking the supremum over $t > 0$ on the left and applying Fatou’s lemma we prove this theorem.

Proposition 2. The following triangle inequality holds for all $f, g$ in $H^{p,\infty}$:

$$\|f + g\|_{H^{p,\infty}} \leq 2^p(\|f\|_{H^{p,\infty}}^p + \|g\|_{H^{p,\infty}}^p).$$

Moreover, for $0 < r < p$ we have

$$\|\{f_j\}\|_{H^{p,\infty}(R^n, \ell^r)} \approx \sup_{0 < |E| < \infty} |E|^{-\frac{1}{2} + \frac{1}{p}} \left( \int_{E} \sup_{t > 0} \|\{\varphi_t * f_j(x)\}_j\|_{\ell^r} dx \right)^{\frac{1}{r}}.$$

Proof. The first claim follows from the sequence of inequalities:

$$\|f + g\|_{H^{p,\infty}} = \sup_{\lambda > 0} \lambda^p \{x : \sup_{t > 0} |\varphi_t * (f + g)(x)| > \lambda\}$$

$$\leq \sup_{\lambda > 0} \lambda^p \{x : \sup_{t > 0} |\varphi_t * f(x)| > \frac{\lambda}{2}\} + \sup_{\lambda > 0} \lambda^p \{x : \sup_{t > 0} |\varphi_t * g(x)| > \frac{\lambda}{2}\}$$

$$= 2^p(\|f\|_{H^{p,\infty}}^p + \|g\|_{H^{p,\infty}}^p).$$

The second claim comes from the corresponding result of $L^{p,\infty}$, see [12], p. 13.

Proposition 3. $H^{p,\infty}(R^n, \ell^2(L))$ are complete quasi-normed spaces.

Proof. Consider first the case $L = 1$. Let $\{f_j\}$ be a Cauchy sequence in $H^{p,\infty}$; then $\{f_j\}$ is also Cauchy in $S'$ with limit $f$. We use the $\| \cdot \|_{H^{p,\infty}}$ norm, for which we know from Proposition 2 that $\| \cdot \|_{H^{p,\infty}}$ is sublinear if $r < p$ and $r \leq 1$. We choose a subsequence $\{f_{j_k}\}$ of $\{f_j\}$ with $\|f_{j_k+1} - f_j\|_{H^{p,\infty}} \leq 2^{-k}$, which gives us that $\|f_{j_k}\|_{H^{p,\infty}} \leq C$ and therefore $\|f\|_{H^{p,\infty}} \leq C$, similarly $\|f_{j_k} - f\| \leq \epsilon$ for any large $i$, hence $f_j \to f$ in $H^{p,\infty}$.

Now if $\{(f^{(j)}_k)_{k=1}^L\}_{j=1}^\infty$ is Cauchy in $H^{p,\infty}(R^n, \ell^2(L))$, then for each $k$ we have a limit $f_k$ in $H^{p,\infty}$, and $\{f_k\}_{k=1}^L \in H^{p,\infty}(R^n, \ell^2(L))$ since $L$ is finite. If we choose $j$ large enough, we would see that

$$\left\| \sup_{t > 0} \left( \sum_{k=1}^L \|((f_k - f^{(j)}_k) * \varphi_t)(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \leq \sum_{k=1}^L \|f_k - f^{(j)}_k\|_{H^{p,\infty}} \leq \epsilon,$$

thus $\{(f^{(j)}_k)_{k=1}^L\}_{j=1}^\infty$ converge to $\{f_k\}_{k=1}^L$ in $H^{p,\infty}(R^n, \ell^2(L))$ as $j \to \infty$. 

□
Next we show that Schwartz functions are not dense in $H^{p,\infty}$ for $p \geq 1$. To realize this, we investigate the decay of functions in $L^p$ and $L^{p,\infty}$ first.

**Lemma 1.** For $g \in L^p(\mathbb{R}^n)$, we have

\[
\lim_{M \to \infty} \frac{|\{ |x| \leq M : |g(x)| \geq |x|^{-\frac{n}{p}} \}|}{M^n} = 0,
\]

and

\[
\lim_{\delta \to 0} \frac{|\{ |x| \leq \delta : |g(x)| \geq |x|^{-\frac{n}{p}} \}|}{\delta^n} = 0.
\]

For $g \in L^{p,\infty}$, we have

\[
\lim_{M \to \infty} \frac{|\{ |x| < M : |g(x)| \geq |x|^{-\frac{n}{p_2}} \}|}{M^n} = 0,
\]

for all $p_2 > p$ and

\[
\lim_{\delta \to 0} \frac{|\{ |x| < \delta : |g(x)| \geq |x|^{-\frac{n}{p_1}} \}|}{\delta^n} = 0
\]

for all $0 < p_1 < p$.

**Proof.** We set $E = \{ x : |g(x)| \geq |x|^{-\frac{n}{p}} \}$. Suppose that (17) failed. Then there exists an $\epsilon > 0$ and a sequence $\{M_k\}$ such that $|E \cap \{ x : 1 < |x| < M_k \}| > \epsilon M_k^n$ for all $M_k$; moreover, we can take $M_k$ such that $\epsilon M_k^n > 2v_n M_{k-1}^n$. Observe

\[
|E \cap \{ x : M_{k-1} \leq |x| \leq M_k \}| > \epsilon M_k^n - v_n M_{k-1}^n > \frac{\epsilon}{2} M_k^n,
\]

therefore

\[
\int_{\mathbb{R}^n} |g(x)|^p dx = \sum_{k=2}^{\infty} \int_{\{M_{k-1} \leq |x| \leq M_k \}} |g(x)|^p dx \\
\geq \sum_{k=2}^{\infty} \int_{\{M_{k-1} \leq |x| \leq M_k \}} |x|^{-n} \chi_E dx \\
\geq \sum_{k=2}^{\infty} \int_{\{M_k - \frac{\epsilon}{2} M_k \leq |x| \leq M_k \}} |x|^{-n} dx \\
= C_n \sum_{k=2}^{\infty} \ln \left( \frac{1}{1 - \frac{\epsilon}{2}} \right) = \infty
\]

This is a contradiction and our claim is true. The proof of (18) is similar.

For an $L^{p,\infty}$ function $g$, we cut it as follows:

\[
g(x) = h(x) + k(x) = g(x) \chi_{\{|g| > \alpha\}} + g(x) \chi_{\{|g| \leq \alpha\}},
\]
where \( \alpha > 0 \). Fix \( p_1, p_2 \) with \( p_1 < p < p_2 \), then \( h \in L^{p_1} \) and \( k \in L^{p_2} \). Then

\[
\lim_{M \to \infty} \frac{|\{ x < M : g > |x|^{-\frac{\alpha}{p_2}} \}|}{M^n} \\
\leq \lim_{M \to \infty} \frac{|\{ 1 < x < M : 2|h| > |x|^{-\frac{\alpha}{p}} \}|}{M^n} + \lim_{M \to \infty} \frac{|\{ 1 < x < M : 2|k| > |x|^{-\frac{\alpha}{p}} \}|}{M^n} \\
= 0.
\]

This proves (19) and (20) can be proved in a similar way. 

**Theorem 5.** \( L^r \) is not dense in \( L^{p, \infty} \), whenever \( 0 < r \leq \infty \) and \( 0 < p < \infty \).

**Proof.** For simplicity we restrict ourselves to the case \( p = 1 \) and \( n = 1 \), and we note that the proof in this case contains the general idea.

For the case \( r \geq 1 = p \), we don’t need the decay we proved in Lemma 1. We will prove a stronger result: \( L^1_{loc} \) is not dense in \( L^{1, \infty} \). Suppose not, then in \( L^{1, \infty} \) the function \( f(x) = \frac{1}{x} \chi_{(0, 1)} \in L^{1, \infty} \) is in the closure of \( L^1_{loc} \). The fact that \( \| f - g \|_{L^{1, \infty}} \geq \| f - g \chi_{[0, 1]} \|_{L^{1, \infty}} \) for all \( g \in L^1_{loc} \) suggests us that \( f \) is also in the closure of \( L^1_{loc}[0, 1] = L^1[0, 1] \). Moreover step functions are dense in \( L^1 \) and \( \| h \|_{L^{1, \infty}} \leq \| h \|_{L^1} \), so \( f \) is also in the closure of set of step functions defined on \([0, 1] \). But such a step function \( g \) must be bounded by \( M \), then

\[
\| f - g \|_{L^{1, \infty}} = \| (f - M) \chi_{(0, \frac{1}{2M})} \|_{L^{1, \infty}} \geq \frac{1}{2}.
\]

This contradiction shows that \( L^1_{loc} \) is not dense in \( L^{1, \infty} \). In particular, \( L^r \) is not dense in \( L^{1, \infty} \) if \( r \in [1, \infty] \).

If \( 1 = p > r \), then we can take \( f = \frac{1}{x} \chi_{(1, \infty)} \in L^{1, \infty} \) with \( \| f \|_{L^{1, \infty}} = 1 \). Choose \( M \) large enough so that \( x^{-1} - x^{-\frac{1}{r}} > \frac{1}{2}x^{-1} \) for \( x \geq M \). Fix any \( g \in L^r \) and by Lemma 1 we can choose \( M' > 10M \) such that

\[
\frac{|\{ 1 < x < M' : |g(x)| > x^{-\frac{1}{r}} \}|}{M'} \leq \frac{1}{2},
\]

therefore we can estimate the difference of \( f \) and \( g \) as follows:

\[
\| f - g \|_{L^{p, \infty}} \geq \| (|f| - |g|) \chi_{(M, M')} \|_{L^{p, \infty}} \\
= \sup_{\alpha > 0} \left\{ \frac{|\{ M' > x > M : |g(x)| \leq x^{-\frac{1}{r}}, x^{-1} - x^{-\frac{1}{r}} > \alpha \}|}{x \alpha} \right\}^\frac{1}{p} \\
\geq \sup_{\alpha > 0} \left\{ \frac{|\{ M' > x > M : |g(x)| \leq x^{-\frac{1}{r}}, \frac{1}{2}x^{-1} > \alpha \}|}{x \alpha} \right\}^\frac{1}{p} \\
\geq \left( \frac{2}{5} M' \right) \alpha_0 = \frac{1}{10}
\]

where we took \( \frac{M'-1}{2} > \alpha_0 > \frac{M'-1}{4} \).

We want to take the more general result we proved as a corollary.
Corollary 2. $L^p_{\text{loc}} \cap L^{p,\infty}$ is not dense in $L^{p,\infty}$.

We also want to remark that to prove this corollary we can also use the decay we proved in Lemma 1. More concretely we can consider the distance of $f(x) = x^{-1/p} \chi_{(0,1)}$ and any $L^p_{\text{loc}}$ function.

We have showed that Schwartz functions are not dense in $H^{p,\infty}$ for $p > 1$ since $L^{p,\infty} = H^{p,\infty}$. Unlike the situation for strong Hardy spaces where $H^1 \subset L^1$, the distribution $\delta_1 - \delta_{-1}$ is in $H^{1,\infty}$ but not in $L^{1,\infty}$. This shows that $H^{1,\infty}$ is not a subspace of $L^{1,\infty}$. Therefore we need a different method to show that Schwartz functions are not dense in $H^{1,\infty}$. Actually the preceding distribution can be used to show this.

Theorem 6. The space of Schwartz functions $\mathcal{S}$ is not dense in $H^{1,\infty}$. Moreover $L^1$ is not dense in $H^{1,\infty}$.

Proof. We will provide a constructive proof. More concretely, we will show that for any $\varphi \in \mathcal{S}$, $\|f - \varphi\|_{H^{1,\infty}} \geq \frac{1}{20}$, where $f = \delta_1 - \delta_{-1}$. An easy calculation shows that

$$f^+(x) = \sup_{t>0} |(\psi_t * f)(x)| \approx \begin{cases} \frac{|x|}{|x|^2} & x < 0, \\ \frac{|x|}{|x|^2} & x > 0. \end{cases}$$

By the symmetry of this function, we can consider only the part $x > 0$. So $f^+(x) = \frac{2}{(x+1)^2} \frac{1}{(1-x)^2}$ for $0 < x < 1$ and $\frac{2}{(x+1)^2} \frac{1}{(1-x)^2}$ for $x > 1$. If we cut this function in two parts, $f^+(x) = g(x) + h(x)$ with $g(x) = f^+(x) \chi_{|x| < a} + f^+(x) \chi_{x > b}$ for fixed small $a$ and large $b$, say $\frac{1}{2}$ and 100, then $h$ is bounded and compactly supported and lies in any $L^{p,\infty}$.

Now we consider $g(x)$. If $\alpha$ is large, then

$$|\{x : g(x) > \alpha\}| \leq |\{x > 0 : \frac{8}{9} \frac{1}{|1-x|} > \alpha\}| \leq \frac{16}{9\alpha},$$

therefore $\sup_{\alpha > \frac{1}{100}} \alpha^p \frac{16}{9\alpha} < \infty$ for all $p \leq 1$. Meanwhile, if $\alpha$ is small, then

$$|\{x : g(x) > \alpha\}| \leq 1 + |\{x > 100 : \frac{2}{7} \alpha > \alpha\}| \leq \sqrt{\frac{2}{\alpha} - 99},$$

and we have $\sup_{\alpha \leq \frac{1}{100}} \alpha^p (\sqrt{2/\alpha} - 99) < \infty$ for $p \geq \frac{1}{2}$. In conclusion, $g \in L^{p,\infty}$ for $p \in \left[\frac{1}{2}, 1\right]$ and $f \in H^{p,\infty}$ for the same range of $p$’s.

Next, we show that $\|f - \varphi\|_{H^{1,\infty}} > \frac{1}{20}$. We achieve this via the estimate

$$\|g - \varphi^+(\chi_{|x| < a} + \chi_{x > b})\|_{L^{p,\infty}} \leq \|f^+ - \varphi^+\|_{L^{p,\infty}} \leq \|f - \varphi\|_{H^{p,\infty}}.$$ 

For $1 \leq x \leq \frac{3}{2}$, we have $\frac{2}{(x+1)^2} \frac{1}{(1-x)^2} > \frac{2}{25} \frac{1}{(x-1)^2}$. So $\|\varphi^+\|_{L^{\infty}} \leq C$, and thus

$$\|g - \varphi^+(\chi_{|x| < a} + \chi_{x > b})\|_{L^{1,\infty}} \geq \sup_{\alpha > 100} \alpha |\{x \in (1, \frac{3}{2}) : \frac{4}{25(x-1)^2} - C > \alpha\}| > \frac{2}{25}.$$ 

This proves the required claim for Schwartz functions.

For functions in $L^1$ we argue as follows. Keep $f = \delta_1 - \delta_{-1}$ as before and let $g$ be any $L^1$ function. Actually $g$ is also in $H^{1,\infty}$ since

$$\|g\|_{H^{1,\infty}} = \|M(g; P)\|_{L^{1,\infty}} \leq \|M(g)\|_{L^{1,\infty}} \leq C\|g\|_{L^1},$$

where $M(g; P)$ denotes the modulation operator of $g$. Therefore $g$ is not dense in $H^{1,\infty}$.
where $M$ is the Hardy-Littlewood maximal operator. Now for any given $\epsilon > 0$, we can choose a Schwartz function $\varphi$ such that $\|\varphi - g\|_{L^1} \leq \epsilon$. Apply the previous discussion we deduce that

$$\|\varphi - g\|_{H^{1,\infty}} \leq C\epsilon.$$ 

By Proposition 2 $\|f - \varphi\|_{H^{1,\infty}} \leq 2(\|f - g\|_{H^{1,\infty}} + \|g - \varphi\|_{H^{1,\infty}})$, but $\|f - \varphi\|_{H^{1,\infty}} \geq \frac{1}{20}$, therefore

$$\|f - g\|_{H^{1,\infty}} \geq \frac{1}{2}\|f - \varphi\|_{H^{1,\infty}} - \|g - \varphi\|_{H^{1,\infty}} \geq 1/40 - C\epsilon \geq 1/80$$

if we choose $\epsilon$ small enough. 

4. Two interpolation results

The following version of the classical Fefferman-Stein vector-valued inequality \cite{1} will be necessary in our work. This result for upper Boyd indices $r$ of $L^{p,r}$, which is less than $\infty$, is contained in \cite{5} (page 85) and here we provide the proof for the case $r = \infty$ not contained in \cite{5}.

**Proposition 4.** If $1 < p, q < \infty$, then for all sequences of functions $\{f_j\}_j$ in $L^{p,\infty}(\ell^q)$ we have

$$\|\{M(f_j)\}\|_{L^{p,\infty}} \leq C_{p,q} \|\{f_j\}\|_{L^{p,\infty}},$$

where $M$ is the Hardy-Littlewood maximal function.

**Proof.** We know that $\|\{M(f_j)\}\|_{L^{p}} \leq C_{p,q} \|\{f_j\}\|_{L^{p}}$ for $1 < p, q < \infty$, see \cite{1}. Now fix $q$ and take $1 < p_1 < p < p_2$. Set $\vec{F} = \{f_j\}$ and $|\vec{F}| = (\sum_{j \in \mathbb{Z}} |f_j|^q)^{\frac{1}{q}}$. We split $\vec{F}$ at the height $\alpha > 0$, and define $\vec{F}_\alpha = \vec{F}\chi_{|\vec{F}| > \alpha}$ and $\vec{F}_\alpha = \vec{F} - \vec{F}_\alpha = \vec{F}\chi_{|\vec{F}| \leq \alpha}$. It is easy to verify that

$$\{|\vec{F}_\alpha| > \lambda\} = \begin{cases} \frac{d\vec{F}(\lambda)}{d\vec{F}(\alpha)} \lambda > \alpha \\ \frac{d\vec{F}(\lambda)}{d\vec{F}(\alpha)} \lambda \leq \alpha \end{cases}$$

and

$$\{|\vec{F}_\alpha| > \lambda\} = \begin{cases} 0 \quad \lambda > \alpha \\ \frac{d\vec{F}(\lambda)}{d\vec{F}(\alpha)} \lambda \leq \alpha \end{cases},$$

where $d\vec{F}(\lambda) = |\{|\vec{F}| > \lambda\}|$. Consequently, $\|\vec{F}_\alpha\|_{L^{p,\infty}}^p \leq \frac{p}{p-p_1} \alpha^{p_1-p} \|\vec{F}\|_{L^{p,\infty}}^p$ and $\|\vec{F}_\alpha\|_{L^{p,\infty}}^{p_2} \leq \frac{p_2}{p_2-p} \alpha^{p_2-p} \|\vec{F}\|_{L^{p,\infty}}^{p_2} - d\vec{F}(\alpha) \alpha^{p_2}$. For each $j$, split $f_j = f_j\chi_{|\vec{F}| > \alpha} + f_j\chi_{|\vec{F}| \leq \alpha}$. Then we have

$$\|\{M(f_j)\}\|_{e^q} > \lambda\}
\leq \|\{M(f_j)\chi_{|\vec{F}| > \alpha}\}\|_{e^q} + \|\{M(f_j)\chi_{|\vec{F}| \leq \alpha}\}\|_{e^q} > \frac{\lambda}{2}\}
\leq C(p_1, q) (\frac{2}{\lambda})^{p_1} \int_{\mathbb{R}^n} \{f_j\chi_{|\vec{F}| > \alpha}(x)\} dx + C(p_2, q) (\frac{2}{\lambda})^{p_2} \int_{\mathbb{R}^n} \{f_j\chi_{|\vec{F}| \leq \alpha}(x)\} dx
\leq C(p_1, q) (\frac{2}{\lambda})^{p_1} \frac{p}{p-p_1} \alpha^{p_1-p} \|\vec{F}\|_{L^{p,\infty}}^p + C(p_2, q) (\frac{2}{\lambda})^{p_2} \frac{p_2}{p_2-p} \alpha^{p_2-p} \|\vec{F}\|_{L^{p,\infty}}^p
\leq (\frac{p}{p-p_1} + \frac{p_2}{p_2-p}) 2^{p_2} C(p_1, q) \frac{p_2}{p_2-p_1} C(p_2, q) \frac{p_2}{p_2-p_1} \lambda^{-p} \|\vec{F}\|_{L^{p,\infty}}^p
where we set $\alpha = \lambda \gamma$, where $\gamma = \left( \frac{C(p_1,q)}{C(p_2,q)} \right)^{\frac{1}{p_2-p_1}}$.

Next we have the following result which has a lot of applications in this work. The scalar version of this theorem has been proved in [8], but it is incomplete in the case $q = \infty$ due to the fact that Schwartz functions are not dense in $H^{p,\infty}$; this is shown in Theorem 5 and Theorem 6. Here we complete this gap pushing further the approach [8] and combining it with ideas from chapter III of [22].

**Theorem 7.** (a) Let $J$ and $L$ be positive integers and let $0 < p_1 < p < p_2 < \infty$, moreover $p_1 \leq 1$. Let $T$ be a sublinear operator defined on $H^{p_1}(\mathbb{R}^n, \ell^2(L)) + H^{p_2}(\mathbb{R}^n, \ell^2(L))$. Assume that maps $H^{p_1}(\mathbb{R}^n, \ell^2(L))$ to $H^{p_1}(\mathbb{R}^n, \ell^2(J))$ with constant $A_1$ and $H^{p_2}(\mathbb{R}^n, \ell^2(L))$ to $H^{p_2}(\mathbb{R}^n, \ell^2(J))$ with constant $A_2$. Then there exists a constant $c_{p_1,p_2,p,n}$ independent of $J$ and $L$ such that

$$
\|T(\vec{F})\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(J))} \leq c_{p_1,p_2,p,n} A_1^{\frac{1}{p_1}} A_2^{\frac{1}{p_2}} \|\vec{F}\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(L))}
$$

for $\vec{F} \in H^{p,\infty}(\mathbb{R}^n, \ell^2(L))$.

(b) Suppose that $T$ is a sublinear operator defined on $H^{p_1}(\mathbb{R}^n, \ell^2(L)) + H^{p_2}(\mathbb{R}^n, \ell^2(L))$. Assume that maps $H^{p_1}(\mathbb{R}^n, \ell^2(L))$ to $L^{p_1}(\mathbb{R}^n, \ell^2(J))$ with constant $A_1$ and $H^{p_2}(\mathbb{R}^n, \ell^2(L))$ to $L^{p_2}(\mathbb{R}^n, \ell^2(J))$ with constant $A_2$. Then there exists a constant $C$ independent of $J$ and $L$ such that

$$
\|T(\vec{F})\|_{L^{p,\infty}(\mathbb{R}^n, \ell^2(J))} \leq C_{p_1,p_2,p,n} A_1^{\frac{1}{p_1}} A_2^{\frac{1}{p_2}} \|\vec{F}\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(L))}
$$

for all functions $\vec{F} \in H^{p,\infty}(\mathbb{R}^n, \ell^2(L))$.

**Lemma 2.** Let $0 < p_1 < p < p_2 < \infty$. Given $\vec{F} = \{f_k\}_{k=1}^{L} \in H^{p,\infty}(\mathbb{R}^n, \ell^2(L))$ and $\alpha > 0$, then there exists $\vec{G} = \{g_k\}_{k=1}^{L}$ and $\vec{B} = \{b_k\}_{k=1}^{L}$ such that $\vec{F} = \vec{G} + \vec{B}$ and

$$
\|\vec{B}\|_{H^{p_1}(\mathbb{R}^n, \ell^2(L))} \leq C\alpha^{p_1-p} \|\vec{F}\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(L))}^{p}
$$

and

$$
\|\vec{G}\|_{H^{p_2}(\mathbb{R}^n, \ell^2(L))} \leq C\alpha^{p_2-p} \|\vec{F}\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(L))}^{p}
$$

where $C = C(p_1,p_2,p,n)$, in particular is independent of $L$. 

Proof of theorem 7. Suppose that $T = \{T_j\}_{j=1}^J$. We apply Lemma 2 with $\alpha = \gamma \lambda$ where $\gamma = (A_2^{p_2} A_1^{-p_1})^{-\frac{1}{p_2 - 1}}$. We obtain

$$|\{x : \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{F}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} > \lambda \}|$$

$$\leq |\{x : \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{G}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} + \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{B}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} > \lambda \}|$$

$$\leq |\{x : \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{G}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} > \lambda/2 \}| + |\{x : \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{B}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} > \lambda/2 \}|$$

$$\leq (\frac{\lambda}{2})^{p_2} \| \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{G}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} \|_{L^{p_2}}^{p_2} + (\frac{\lambda}{2})^{p_1} \| \sup_{t>0} \left( \sum_{j=1}^J |T_j(\tilde{B}) * \psi_t(x)|^2 \right)^{\frac{1}{2}} \|_{L^{p_1}}^{p_1}$$

$$\leq A_2^{p_2} (\frac{\lambda}{2})^{p_2} \| \tilde{G} \|_{H^{p_2}(\mathbb{R}^n, \ell^2(L))} + A_1^{p_1} (\frac{\lambda}{2})^{p_1} \| \tilde{B} \|_{H^{p_1}(\mathbb{R}^n, \ell^2(L))}$$

$$\leq A_2^{p_2} (\frac{\lambda}{2})^{p_2} \| \tilde{F} \|_{H^{p_2}(\mathbb{R}^n, \ell^2(\mathbb{L}))}^{p} C (\gamma \lambda)^{p_2 - p} + A_1^{p_1} (\frac{\lambda}{2})^{p_1} \| \tilde{F} \|_{H^{p_1}(\mathbb{R}^n, \ell^2(\mathbb{L}))}^{p} C (\gamma \lambda)^{p_1 - p}$$

$$\leq C^{2p_2 + 1} (A_1^{-\theta} A_2^{p_1})^{p} \lambda^{-p} \| \tilde{F} \|_{H^{p_\infty}}^{p},$$

where $\theta = \frac{1}{p_2} - \frac{1}{p_1}$.

The proof of the second part is similar,
Proof of lemma. We introduce the notation

\[ \bar{F}^*(x) = M_N(\bar{F}(x)) = \sup_{N(x) \leq 1} \sup_{|x-y| \leq t} \left( \sum_{k=1}^L |(\phi_k \ast f_k)(y)|^2 \right)^{1/2} \]

for the grand maximal function and

\[ M_0(\bar{F})(x) = \sup_{t>0} \left( \sum_{k=1}^L |(\psi_k \ast f_k)(x)|^2 \right)^{1/2} \]

for a maximal function with respect to a fixed bump \( \psi \). It’s easy to check that \( \Omega_\alpha = \{ \bar{F}^*(x) > \alpha \} \) is open, so we can use the Whitney decomposition theorem to get a collection of cubes \( Q_j \) and functions \( \varphi_j \) such that

1. \( \bigcup_j Q_j = \Omega_\alpha \),
2. the \( Q_j \) are mutually disjoint,
3. \( \text{diam}(Q_j) < \text{dist}(Q_j, \Omega_\alpha) \leq 4 \text{diam}(Q_j) \), where \( \text{diam}(Q_j) \) is the diameter of \( Q_j \) which is \( \sqrt{n}l_j \) with \( l_j \) the length of \( Q_j \),
4. every point is contained in at most \( 12^n \) cubes of the form \( Q_j^* = aQ_j \) with \( a - 1 > 0 \) is fixed and small,
5. \( |(\frac{\partial f}{\partial x})^\beta \varphi_j(x)| \leq A_\beta l_j^{-|\beta|} \), where \( A_\beta \) is a constant independent of \( l_j \),
6. \( \text{supp} \varphi_j \subset Q_j^* = bQ_j \) (\( 1 < b < a \)) and there exists \( 0 < c < 1 \) depending on \( n \) such that for all \( j \), \( \varphi_j \geq c > 0 \) for \( x \in Q_j \).
7. \( \sum_j \varphi_j = \chi_{\Omega_\alpha} \).

Here \( aQ \) is a cube concentric with \( Q \) and of side length \( a \) times that of \( Q \). Next we will define \( b_j \) and show the corresponding estimates. Fix \( j \), define \( P_j^{(k)} \) as the polynomial of degree \( N \), where \( N \) is a fixed large integer to be chosen, such that

\[ \int_{\mathbb{R}^n} P_j^{(k)}(x-x_j)^\beta \varphi_j(x) dx = \langle f_k, (x-x_j)^\beta \varphi_j \rangle \quad \forall |\beta| \leq N, \]

where \( x_j \) is the center of \( Q_j \), and \( \langle f, \varphi \rangle \) is the action of \( f \) on \( \varphi \).

Take the norm of \( h \) in the Hilbert space of polynomials of degree \( \leq N \) as

\[ \|h\|^2 = \frac{\int_{\mathbb{R}^n} |h(x)|^2 \varphi_j(x) dx}{\int_{\mathbb{R}^n} \varphi_j(x) dx}. \]

We have an orthonormal basis \( \{e_m\} \) of this Hilbert space with each \( e_m \) is a polynomial of degree less than or equal to \( N \) and \( \| e_m \| = 1 \). Also we can write \( P_j^{(k)}(x) = \sum_m \langle f_k, e_m \rangle e_m(x) \), where \( \langle f, h \rangle \) is the inner product defined by \( \langle f, h \rangle = \frac{\langle f \varphi_j \rangle}{\langle \varphi_j \rangle} \). It’s not hard to check the following inequality by the method from [22],

\[ \sup_{x \in Q_j} |\partial^\beta P(x)| \leq C_\beta l_j^{-|\beta|} \left( \frac{\int_{Q_j} |P|^2}{|Q_j|} \right)^{1/2} \]

for any \( P \) with degree \( \leq N \).

To prove it, we can reduce this to the case \( Q_j \) is a unit cube and prove this case by that different norms of a finite dimension topological vector space are comparable.
Let’s notice also that \( \left( \frac{f_{Q_j'}|P|}{|Q_j'|} \right)^\frac{1}{2} \leq C \|P\| \). In particular, \( |e_m(x)| \leq C, \forall \ x \in Q^*_j \) and \( |\partial^\gamma e_m(y)| \leq C l_j^{-|\gamma|}, \forall \ y \in Q^*_j \). Now let’s restrict \( x \in Q^*_j \) and we have

\[
\left( \sum_{k=1}^L |P_j^{(k)}(x)|^2 \right)^\frac{1}{2} = \left( \sum_{k=1}^L \left| \left( f_k, \sum_{m} e_m(\cdot) e_m(\tilde{x}) \right) \right|^2 \right)^\frac{1}{2} \\
\leq C_N \inf_{|z-x| \leq 10\sqrt{n} l_j} \bar{F}_x(z) \leq C_N \alpha.
\]

Here we introduced the function \( \Phi(x, y) = \frac{\sum m e_m(y)e_m(x)\varphi_j(y)}{\varphi_j} \) for which the estimate below holds

\[
\mathfrak{R}_N(\Phi; x, 10\sqrt{n} l_j) \leq C_N \int_{Q_j^*} \sum_{|\gamma| \leq N+1} b_j^{(k)} l_j^{1-|\gamma|} dy \leq C_N.
\]

Let’s remark that this \( C_N \) is independent of \( L \). Now let’s define \( b_j^{(k)} = (f_k - P_j^{(k)})\varphi_j \) and consider \( \left( \sum_{k=1}^L |(b_j^{(k)} * \psi_t(x)|^2 \right)^\frac{1}{2} \), where \( \psi \) is a smooth function supported in \( B(0,1) \setminus B(0,\frac{1}{2}) \) with \( \int \psi \neq 0 \). If \( x \in Q^*_j \), then

\[
\left( \sum_{k=1}^L |(b_j^{(k)} * \psi_t(x)|^2 \right)^\frac{1}{2} \leq \left( \sum_{k=1}^L |((P_j^{(k)} \varphi_j) * \psi_t(x)|^2 \right)^\frac{1}{2} + \left( \sum_{k=1}^L |(f_k, \varphi_j(\cdot) \psi_t(x \cdot \cdot)^2 \right)^\frac{1}{2}.
\]

We know that the first term of the right hand side is controlled by \( C_N\alpha \) since so is \( \left( \sum_{k=1}^L |P_j^{(k)}(x)|^2 \right)^\frac{1}{2} \) for all \( x \in Q^*_j \). For the second term, if \( t \leq l_j \), then take \( \Phi(y) = \varphi_j(x - ty) \psi(y) \), and by Theorem 3 it is easy to check the following inequalities

\[
\left( \sum_{k=1}^L |(f_k, \varphi_j(\cdot) \psi_t(x \cdot \cdot)|^2 \right)^\frac{1}{2} \leq \mathfrak{R}_N(\Phi) \bar{F}_x(x) \leq C \bar{F}_x(x)
\]

If \( t > l_j \), we can use the same idea but \( \Phi(y) = \varphi_j(x - l_j y) \psi(\frac{y}{t}) \) to get that

\[
\left( \sum_{k=1}^L |(f_k, \varphi_j(\cdot) \psi_t(x \cdot \cdot)|^2 \right)^\frac{1}{2} \leq C \bar{F}_x(x).
\]

If \( x \in (Q_j^*)^c \), then there exists \( C \) such that \( \text{supp}\psi_t \cap Q_j^{**} = \emptyset \) if \( t \leq Cl_j \), from which we have \( (b_j^{(k)} * \psi_t(x) = 0 \) since \( b_j^{(k)} \) is a distribution supported in \( Q_j^{**} \). So we can assume \( t \geq Cl_j \) in the following discussion. Now let us fix \( x \) and write \( t^{-n} \psi(\frac{x - y}{t}) = P(y) + R(y) \) by Taylor’s formula, where

\[
P(y) = \sum_{|\beta| \leq N} \frac{\partial^n h(x_j)(y - x_j)^\beta}{\beta!}
\]
with \( h(y) = t^{-n} \psi((x - y)/t) \) is the Taylor polynomial of degree \( N \) at \( x_j \) and \( R \) is the remainder. Next we concern only \( y \) in the support of \( \varphi_j \) because, as we will see, \( y \) such that \( y \notin Q_j^{**} \) does not affect the following argument. It is easy to see that

\[
\left( \sum_{k=1}^{L} |(b^{(k)}_j * \psi_I)(x)|^2 \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{L} |(f_k - P^{(k)}_j) \varphi_j, \psi_I(x - \cdot)|^2 \right)^{\frac{1}{2}} = \left( \sum_{k=1}^{L} |(f_k - P^{(k)}_j) \varphi_j R|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k=1}^{L} |f_k, \varphi_j R|^2 \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{L} |P^{(k)}_j, \varphi_j R|^2 \right)^{\frac{1}{2}}.
\]

Since \( \sum_{k=1}^{L} |P^{(k)}_j(x)|^2 \leq C \alpha \) for \( x \in Q_j^{**} \), \( < P^{(k)}_j, \varphi_j R > \) can be written as an integral and \( \left( \sum_{k=1}^{L} |P^{(k)}_j, \varphi_j R|^2 \right)^{\frac{1}{2}} \leq t^{-n} C \alpha l_j^n \leq C \alpha (l_j / |x - x_j|)^{N+n+1} \).

We also have the estimate for \( y \in Q_j^{**} \)

\[
(21) \quad |\partial^n R(y)| \leq C l_j^{N+1-|\gamma|} / (|x - x_j|)^{N+n+1}.
\]

Indeed, for \( |\gamma| \leq N \), if we apply Taylor’s formula to \( \partial^n h \) again, we will have

\[
\partial^n h(y) = \partial^n P(y) + \sum_{|\beta| = N - |\gamma| + 1} \frac{\partial^{3+\gamma} h(\xi)(y - x_j)^{\beta}}{\beta!},
\]

where \( \xi \) is a point between \( x_j \) and \( y \). In other words,

\[
\partial^n R(y) = \partial^n (h - P)(y) = \sum_{|\beta| = N - |\gamma| + 1} \frac{\partial^{3+\gamma} h(\xi)(y - x_j)^{\beta}}{\beta!}.
\]

Notice that \( |y - x_j| \leq C l_j \) and \( |x - x_j| \leq C t \), then for \( |\beta| = N - |\gamma| + 1 \),

\[
|\partial^{3+\gamma} h(\xi)(y - x_j)^{\beta}| \leq t^{-n} t^{-N-1} |y - x_j|^{N-|\gamma|+1} \leq C l_j^{N+1-|\gamma|} |x - x_j|^{-N-n-1}.
\]

Consequently \( |\partial^n R(y)| \leq C l_j^{N+1-|\gamma|} |x - x_j|^{-N-n-1} \) if \( |\gamma| \leq N \).

Since \( |x - x_j| \leq |x - y| + |x_j - y| \leq |x - y| + C l_j \leq C |x - y| \), for the case \( N + 1 - |\gamma| \leq 0 \), we have

\[
|\partial^n R(y)| = |\partial^n h(y)| \leq t^{-n} t^{-|\gamma|} (\frac{t}{|x - y|})^{N+n+1} \leq C l_j^{N+1-|\gamma|} |x - x_j|^{-N-n-1}.
\]
Now let us take \( \Phi(z) = R(z) \varphi_j(z) \), and by Theorem 4 (c) we obtain

\[
\mathfrak{N}_N(\Phi; x_j, 10\sqrt{n}l_j) = \int_{\mathbb{R}^n} (1 + \frac{|z - x_j|}{10\sqrt{n}l_j})^N \sum_{|\gamma| \leq N+1} (10\sqrt{n}l_j)^{|\gamma|} |\partial^\gamma \Phi(z)|dz \\
\leq \int_{|z-x_j| \leq C_l_j} (1 + \frac{|z-x_j|}{10\sqrt{n}l_j})^N \sum_{|\gamma| \leq N+1} (10\sqrt{n}l_j)^{|\gamma|} |\partial^\gamma \Phi(z)|dz \\
\leq C\left( \frac{l_j}{|x-x_j|} \right)^{N+n+1}
\]

since \((10\sqrt{n}l_j)^{|\gamma|} |\partial^\gamma R(z) \partial^{\gamma-\beta} \varphi_j| \leq Cl_j^{|\gamma|} l_j^{N+1-|\beta|} |x-x_j|^{-N-n-1} l_j^{\beta-\gamma}\) by (21), and that \(|\{z \in \mathbb{R}^n : |z-x_j| \leq C_l_j\}| = Cl_j^n\).

If we take \( y_j \) as a point in \( \Omega^c_\alpha \) with \(|y_j-x_j| \leq 10\sqrt{n}l_j \) and apply the idea used in two previous cases again, we have

\[
\left( \sum_{k=1}^L |f_k, \varphi_j R >|^2 \right)^{\frac{1}{2}} \leq C\left( \frac{l_j}{|x-x_j|} \right)^{N+n+1} \inf_{|w-x_j| \leq 10\sqrt{n}l_j} \tilde{F}(w) \\
\leq C\left( \frac{l_j}{|x-x_j|} \right)^{N+n+1} \tilde{F}(y_j) \\
\leq C\alpha\left( \frac{l_j}{|x-x_j|} \right)^{N+n+1}.
\]

To summarize, \((\sum_{k=1}^L |(b^{(k)} \ast \psi_t)(x)|^2)^{\frac{1}{2}} \leq C\alpha\left( \frac{l_j}{|x-x_j|} \right)^{N+n+1} \) for \( x \in (Q_j^*)^c \).

If \( N \) is chosen such that \((N+n+1)p_1 > n\), then by \( p_1 \leq 1 \) we will get

\[
\int_{\mathbb{R}^n} \sup_{\tau > 0} \left( \sum_{k=1}^L \sum_{m_1 \leq j \leq m_2} |b^{(k)} \ast \psi_t(x)|^2 \right)^{\frac{p_1}{2}} dx \\
\leq \sum_{j=m_1}^{m_2} \left( \int_{Q_j^*} \tilde{F}(x)^{p_1} dx + C\alpha^{p_1} \int_{(Q_j^*)^c} \left( \frac{l_j}{|x-x_j|} \right)^{(N+n+1)p_1} dx \right) \\
\leq C \sum_{j=m_1}^{m_2} \int_{Q_j^*} \tilde{F}(x)^{p_1} dx \\
\leq C \int_{\Omega_\alpha} \tilde{F}(x)^{p_1} dx \\
\leq C\alpha^{p_1-n} \int_{\Omega_\alpha} \tilde{F}(x)^{p_1} dx \\
\leq C\alpha^{p_1-n} \|F\|_{L^{p,\infty}}^p.
\]

The penultimate inequality comes from an equivalent definition of \( L^{p,\infty} \) spaces, see 12 p. 13. When \( m_1 = 1 \), \( \int \sum_{j=1}^{m_2} \chi_{Q_j^*} \tilde{F}(x)^{p_1} dx \leq \int C\chi_{\Omega_\alpha} \tilde{F}(x)^{p_1} dx < \infty \)
by the decomposition of $\Omega_\alpha$, then apply the Lebesgue dominate convergence theorem, $\int \sum_{j=1}^{\infty} \chi_{Q_j}^* \mathcal{F}_\nu^*(x)^{p_1} \, dx < \infty$ and $\int \sum_{j=1}^{m_2} \chi_{Q_j}^* \mathcal{F}_\nu^*(x)^{p_1} \, dx$ can be arbitrary small if both $m_1$ and $m_2$ are large. Therefore $\{\sum_{1\leq j \leq m} b_j^{(k)}\}_m$ is Cauchy in $H^{p_1}(\mathbb{R}^n, \ell^2(\mathbb{L}))$. Since $H^{p_1}(\mathbb{R}^n, \ell^2(\mathbb{L}))$ is complete, the limit of the sequence $\{b^{(k)}\}_{k=1}^L$ exists and $\|\{b^{(k)}\}_k\|_{H^{p_1}(\mathbb{R}^n, \ell^2(\mathbb{L}))} \leq C\alpha^{p_1-p} \|\mathcal{F}_\nu^*\|_{L^{p_p}}$. This $C$ is independent of $L$.

Moreover, since for each $k$, $\sum_{j=1}^{m} b_j^{(k)} \rightarrow b^{(k)}$ in $S'$ as $m \rightarrow \infty$, we have

$$
\sup_{t>0} \left( \sum_{k=1}^{L} |b^{(k)} * \psi_t|^2 \right)^{\frac{1}{2}} \leq \sup_{t>0} \left( \sum_{k=1}^{L} |(b_j^{(k)} * \psi_t)(x)|^2 \right)^{\frac{1}{2}} \leq \sum_{j=1}^{\infty} \mathcal{F}_\nu^* \chi_{Q_j}^* + C\alpha \sum_{j=1}^{\infty} \left( \frac{\ell_j}{|x-x_j|} \right)^{N+n+1} \chi_{Q_j^*}^c \leq C\mathcal{F}_\nu^* \chi_{\Omega_\alpha} + C\alpha \sum_{j=1}^{\infty} \left( \frac{\ell_j}{|x-x_j|} \right)^{N+n+1},
$$

from which we have

$$
\lambda^p |\{\sup_{t>0} \left( \sum_{k=1}^{L} |b^{(k)} * \psi_t|^2 \right)^{\frac{1}{2}} > \lambda\}| \leq \lambda^p |\{\mathcal{F}_\nu^* > \frac{\lambda}{2}\}| + \lambda^p |\{C\alpha \sum_{j=1}^{\infty} \left( \frac{\ell_j}{|x-x_j|} \right)^{N+1} > \frac{\lambda}{2}\}| \leq C \|\mathcal{F}_\nu^*\|_{H^{p_1}(\mathbb{R}^n, \ell^2)} + \lambda^p \int_{\mathbb{R}^n} \sum_{j} \left( \frac{\ell_j}{|x-x_j|} \right)^{(N+1)p} \, dx \leq C \|\mathcal{F}_\nu^*\|_{H^{p_1}(\mathbb{R}^n, \ell^2)} + \lambda^p \sum_{j} |Q_j| \leq C \|\mathcal{F}_\nu^*\|_{H^{p_1}(\mathbb{R}^n, \ell^2)},
$$

We can therefore define $\tilde{G} = \{g^{(k)}\}_{k=1}^{L}$ as $g^{(k)} = f_k - \sum_j b_j^{(k)}$ and obviously $\tilde{G}$ lies in $H^{p_\infty}(\mathbb{R}^n, \ell(\mathbb{L}))$. To estimate $(\sum_{j=1}^{L} |(g^{(k)} * \psi_t)(x)|^2)^{\frac{1}{2}}$, let’s consider first the case $x \notin \Omega_\alpha$. Then

$$
\mathcal{M}_0(\tilde{G})(x) = \sup_{t>0} \left( \sum_{k=1}^{L} |(g^{(k)} * \psi_t)(x)|^2 \right)^{\frac{1}{2}} \leq \sup_{t>0} \left( \sum_{k=1}^{L} |(f_k * \psi_t)(x)|^2 \right)^{\frac{1}{2}} + \sup_{t>0} \left( \sum_{k=1}^{L} |(\sum_j b_j^{(k)} * \psi_t)(x)|^2 \right)^{\frac{1}{2}} \leq C\mathcal{F}_\nu^* \chi_{\Omega_\alpha}(x) + \sum_{j=1}^{\infty} C\alpha \left( \frac{\ell_j}{|x-x_j|} \right)^{N+n+1}.
$$

We claim that this estimate is true for almost all $x$. 


Now let’s consider the case $x \in \Omega_\alpha$. There exists some $m$ such that $x \in Q_m$, and we can divide $\mathbb{N}$ into two sets $I$ and $\Pi$ with $j \in \Pi$ if $Q_j^* \cap Q_m^* \neq \emptyset$ and $j \in I$ otherwise.

$$\mathcal{M}_0(\{g^{(k)}\})(x) \leq \mathcal{M}_0(\{f_k - \sum_{j \in \Pi} b^{(k)}_j\})(x) + \mathcal{M}_0(\{\sum_{j \in I} b^{(k)}_j\})(x)$$

Since $x \notin Q^*_j$ for $j \in \Pi$, $\mathcal{M}_0(\{\sum_{j \in \Pi} b^{(k)}_j\})(x) \leq \sum_{j \in \Pi} C\alpha(\frac{l_j}{l_j + |x - x_j|})^{n + 1}$. We notice that

$$\mathcal{M}_0(\{f_k - \sum_{j \in I} b^{(k)}_j\})(x) \leq \mathcal{M}_0(\{f_k - \sum_{j \in I} f_k \varphi_j\})(x) + \mathcal{M}_0(\{\sum_{j \in I} P^{(k)}_j \varphi_j\})(x).$$

To estimate the second term, we have

$$\left(\sum_k \left|\sum_{j \in I} P^{(k)}_j \varphi_j\right|^2\right)^{\frac{1}{2}} \leq \sum_k \left(\sum_j |P^{(k)}_j|^2\right)^{\frac{1}{2}} \varphi_j \leq C\alpha,$$

and then $\mathcal{M}_0(\{\sum_{j \in I} P^{(k)}_j \varphi_j\})(x) \leq C\alpha$.

To estimate the other term, we notice that we need only to consider the case $t > cl_m$ ($c$ is independent of $m$), otherwise $(\sum_k ((f_k - \sum_{j \in I} f_k \varphi_j) * \psi_t)^2)^{\frac{1}{2}}(x) = 0$ since $\psi$ is supported in $B(0, 1)$, $f_k(1 - \sum_{j \in I} \varphi_j)$ is supported outside $Q^*_m$ and $x \in Q_m$. If $t < 10\sqrt{n}l_m$, then

$$\left(\sum_k \left|\sum_{j \in I} f_k \varphi_j\right|^2\right)^{\frac{1}{2}}(x) = \left(\sum_k \left|\langle f_k, \Phi \rangle\right|^2\right)^{\frac{1}{2}} \leq \mathfrak{N}_N(\Phi; x, \sqrt{n}l_m) \inf_{|z - x| \leq \sqrt{n}l_m} F^*(z) \leq C\alpha \leq C\alpha(\frac{l_m}{l_m + |x - x_m|})^{n + 1},$$

where $\Phi(y) = \psi_t(x - y)(1 - \sum_{j \in I} \varphi_j(y))$.

For $t > 10\sqrt{n}l_m$, $\Phi(y) = \psi_t(x - y)(1 - \sum_{j \in I} \varphi_j(y)) = \psi_t(x - y)$ since the support of $\sum_{j \in I} \varphi_j$ is contained in $B(x, 9\sqrt{n}l_m)$. We can check that $\mathfrak{N}_N(\Phi; x, t) \leq C$ with $C$ independent of $x$ and $t$. Therefore

$$\left(\sum_k \left|\sum_{j \in I} f_k \varphi_j\right|^2\right)^{\frac{1}{2}}(x) \leq \mathfrak{N}_N(\Phi; x, t) \inf_{|z - x| \leq t} F^*(z) \leq C\alpha.$$

To summarize, we have showed that

$$\mathcal{M}_0(\tilde{G})(x) \leq C\tilde{F}^*(x) \chi_{\Omega_\alpha}(x) + \sum_{j = 1}^\infty C\alpha(\frac{l_j}{l_j + |x - x_j|})^{n + 1} a.e.$$

This gives us that $\|\mathcal{M}_0(\tilde{G})\|_{L^p} \leq C\alpha(\frac{\mu_\Omega}{\mu_\alpha})^{\frac{p}{p^*}} \|F\|_{H^{p, \infty}}^{\frac{p}{p^*}}$ since $\|\sum_{j = 1}^\infty (\frac{l_j}{l_j + |x - x_j|})^{n + 1}\|_{L^p} \leq C(\Omega)^{\frac{1}{p^*}}$.

\hfill \Box

We have the following corollary.
Corollary 3. Let $0 < p < \infty$ and suppose that $\{K_j(x)\}_{j=1}^L$ is a family of kernels defined on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\sum_{j=1}^L |\partial^{\alpha} K_j(x)| \leq A|x|^{-n-|\alpha|} < \infty$$

for all $|\alpha| \leq \max\{[n/p] + 2, n + 1\}$ and

$$\sup_{\xi \in \mathbb{R}^n} \sum_{j=1}^L |\hat{K}_j(\xi)| \leq B < \infty.$$

Then for some $0 < p$ there exists a constant $C_{n,p}$ independent of $L$ such that

$$\left\| \sum_{j=1}^L K_j * f_j \right\|_{H^{p,\infty}(\mathbb{R}^n)} \leq C_{n,p}(A + B)\|\{f_j\}_{j=1}^L\|_{H^{p,\infty}(\mathbb{R}^n,L^2(L))}.$$

Proof. We pick $p_1 < p < p_2$ such that $p_1 \leq 1$ and $[n/p_1] + 1 = \max\{[n/p] + 2, n + 1\}$. Then (22) holds with $H^{p,\infty}$ replaced by both $H^{p_1}$ and $H^{p_2}$ in view of Theorem 6.4.14 in [13]. Using Theorem 7 we derive the required conclusion. \qed

5. Square function characterization of $H^{p,\infty}$

We discuss an important characterization of Hardy spaces in terms of Littlewood–Paley square functions. The vector-valued Hardy spaces and the action of singular integrals on them are crucial tools in obtaining this characterization.

We first set up the notation. We fix a radial Schwartz function $\Psi$ on $\mathbb{R}^n$ whose Fourier transform is nonnegative, supported in the annulus $1 - \frac{1}{7} \leq |\xi| \leq 2$, and satisfies

$$\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$$

for all $\xi \neq 0$. Associated with this bump, we define the Littlewood–Paley operators $\Delta_j$ given by multiplication on the Fourier transform side by the function $\hat{\Psi}(2^{-j}\xi)$, that is,

$$\Delta_j(f) = \Psi_{2^{-j}} * f.$$

We also define the function $\Phi$ by $\hat{\Phi}(\xi) = \sum_{j \leq 0} \hat{\Psi}(2^{-j}\xi)$ for $\xi \neq 0$ and $\hat{\Phi}(0) = 1$. Now we’re going to prove Theorem 11.

Proof of Theorem 11. Choose $f \in H^{p,\infty}$ and denote $f_M = \sum_{|j| \leq M} \Delta_j(f) = \Phi_{2^{-M}} * f - \Phi_{2^M} * f$ and $S(f) = (\sum_{|j| \leq M} |\Delta_j(f)|^2)^{\frac{1}{2}}$. We can see that $S(f + g) \leq S(f) + S(g)$ and $S(af) = |a|S(f)$. We also know from [13] that $S$ maps $H^{p_i}$ to $L^{p_i}$ ($i = 1, 2$) bounded by the square function characterization of Hardy spaces. Then by Theorem 7 it follows that $S$ maps $H^{p,\infty}$ to $L^{p,\infty}$ bounded for $p \in (p_1, p_2)$, so

$$\|S(f)\|_{L^{p,\infty}} \leq C\|f\|_{H^{p,\infty}}.$$
Applying Fatou’s lemma for $L^{p,\infty}$ spaces we have
\[
\left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \leq \liminf_{M \to \infty} \left\| \left( \sum_{|j| \leq M} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \leq C\|f\|_{H^{p,\infty}}.
\]

Assume that we have a distribution $f \in \mathcal{S}$ such that $\left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} < \infty$ and define $f_j = \Delta_j(f) = \Psi_{2^{-j}} * f$. We can show that $\{f_j\}_{j \in \mathbb{Z}} \in H^{p,\infty}(\mathbb{R}^n, \ell^2)$.

To prove this, let’s take $\varphi \in \mathcal{S}$ whose Fourier transform takes value 0 for $|\xi| \geq 2$ and 1 for $|\xi| \leq 1$. $\varphi \ast \Delta_j(f)(\xi) = \hat{\varphi}(t\xi)\hat{\Psi}(2^{-j}\xi)\hat{f}(\xi)$, so it’s just $\Delta_j * f$ if $\frac{1}{t} > 2^{j+1}$ and 0 if $\frac{1}{t} < \frac{3}{2} \cdot 2^j$. Therefore
\[
\sup_{t > 0} |\varphi_t \ast \Delta_j(f)| \leq |\Delta_j(f)| + \sup_{\frac{3}{2} \cdot 2^{-j} \geq t \geq 2^{-(j+1)}} |\varphi_t \ast \Delta_j(f)|.
\]

For $\frac{7}{3} \cdot 2^{-j} \geq t \geq 2^{-(j+1)}$, by lemma 6.5.3 of [13]
\[
|\varphi_t \ast \Delta_j(f)(x)| \leq C_N M(|\Delta_j(f)|^r)^{\frac{1}{r}}(x),
\]
where $M$ is the Hardy-Littlewood maximal function and $r < \min(2, p)$. Apply Proposition 1 to obtain
\[
\| \sup_{t > 0} \left( \sum_{j \in \mathbb{Z}} |\varphi_t \ast \Delta_j(f)|^2 \right)^{\frac{1}{2}} \|_{L^{p,\infty}}
\leq \| \left( \sum_{j \in \mathbb{Z}} (\sup_{t > 0} |\varphi_t \ast \Delta_j(f)|^2)^{\frac{1}{2}} \right) \|_{L^{p,\infty}}
\leq C_p \| \left( \sum_{j \in \mathbb{Z}} (|\Delta_j(f)|^2)^{\frac{1}{2}} \right) \|_{L^{p,\infty}} + C_p \| \left( \sum_{j \in \mathbb{Z}} (M(|\Delta_j(f)|^r)^{\frac{1}{r}})^{\frac{1}{2}} \right) \|_{L^{p,\infty}}
\leq C'_p \| \left( \sum_{j \in \mathbb{Z}} (|\Delta_j(f)|^2)^{\frac{1}{2}} \right) \|_{L^{p,\infty}}.
\]

The fact that $\| \sum_{k=1}^{\infty} g_k \|_{L^{p,\infty}} < \infty$ doesn’t imply that $\{\sum_{k=1}^{M} g_k\}_{M}$ is a Cauchy sequence in $L^{p,\infty}$, so we cannot apply the method used in $H^p$ case. But we still can use a new method which is also applicable to the $H^p$ case.

Let $\widehat{\eta}(\xi) = \widehat{\Psi}(\xi/2) + \widehat{\Psi}(\xi) + \widehat{\Psi}(2\xi)$, then by Corollary 4
\[
\left\| \sum_{|j| \leq M} \Delta_j(f) \right\|_{H^{p,\infty}} = \left\| \sum_{|j| \leq M} \Delta_j(f) \right\|_{H^{p,\infty}}
\leq C\|\{f_j\}\|_{H^{p,\infty}(\mathbb{R}^n, \ell^2(M))}
\leq C\|\sup_{t > 0} \left( \sum_{|j| \leq M} |\varphi_t \ast f_j|^2 \right)^{\frac{1}{2}} \|_{L^{p,\infty}}
\leq C\|\left( \sum_{|j| \leq M} \sup_{t > 0} |\varphi_t \ast f_j|^2 \right)^{\frac{1}{2}} \|_{L^{p,\infty}}
\leq C\|\left( \sum_{j \in \mathbb{Z}} (|\Delta_j(f)|^2)^{\frac{1}{2}} \right) \|_{L^{p,\infty}}.
\]

So $\{\sum_{|j| \leq M} \Delta_j(f)\}_{M}$ is a bounded sequence in $H^{p,\infty}$ uniformly in $M$ and we are able to use the following lemma.
Lemma 3. If \( \{f_j\} \) is bounded by \( B \) in \( H^{p,\infty} \) (or \( H^p \)), then there exists a subsequence \( \{f_{j_k}\} \) such that \( f_{j_k} \to f \) in \( S' \) for some \( f \) in \( H^{p,\infty} \) (or \( H^p \)) with \( \|f\|_{H^{p,\infty}} \leq B \) (or \( \|f\|_{H^p} \leq B \)).

Proof. \( V = \{ \varphi \in S : \mathfrak{N}_N(\varphi) < \frac{1}{2BC} \} \) is a neighborhood of 0 in \( S \) and

\[ |(f_j, \varphi)| \leq C \mathfrak{N}_N(\varphi) \|f_j\|_{H^{p,\infty}} \leq 1. \]

So by the separability of \( S \) we have the weak*-compactness of this sequence, which means that there exists a subsequence \( \{f_{j_k}\} \) such that \( f_{j_k} \to f \) in \( S' \). Therefore \( \|f\|_{H^{p,\infty}} \leq \liminf_{k \to \infty} \|f_{j_k}\|_{H^{p,\infty}} \leq B \). \( \Box \)

By the lemma we know that \( \sum_{|j| \leq M_k} \Delta_j(f) \to g \) in \( S' \) with

\[ \|g\|_{H^{p,\infty}} \leq C \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}}. \]

Moreover, we know that \( \sum_{|j| \leq M} \Delta_j(f) \to f \) in \( S'/\mathcal{P} \), so there is a unique polynomial \( Q \) such that \( g = f - Q \). \( \Box \)

Corollary 4. Let \( \Psi \) be a smooth bump whose Fourier transform is supported in an annulus that does not contain the origin and satisfies for some positive integer \( q \):

\[ \sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-jq} \xi) = 1, \quad \xi \in \mathbb{R}^d \setminus \{0\}. \]

Then for any \( 0 < p < \infty \) there is a constant \( C(p,n) \) (that also depends on \( \Psi \)) such that for all functions \( f \) in \( L^r \) with some \( r \in [1, \infty] \) and whose “lacunary” square function \( S_q^\Psi(f) = (\sum_{\ell \in \mathbb{Z}} |\Delta_q(\phi(f)|^2)^{1/2} \) lies in weak \( L^p \) we have

\[ \|f\|_{L^{p,\infty}} \leq C(p,n) \|S_q^\Psi(f)\|_{L^{p,\infty}}. \]

Proof. Let us prove the case that \( \Psi \) satisfies the assumptions of Theorem 1 and therefore \( q = 1 \). Since \( f \in L^r \), it is an element of \( S'(\mathbb{R}^n) \). The square function of \( f \) lies in weak \( L^p \), thus Theorem 1 yields the existence of a polynomial \( Q \) such that \( f - Q \) lies in \( H^{p,\infty} \). By the Lebesgue differentiation theorem it follows that for almost all \( x \in \mathbb{R}^n \) we have

\[ |f(x) - Q(x)| \leq C \sup_{r>0} |(\varphi_r \ast (f - Q))(x)|, \tag{24} \]

where \( \varphi \) is a smooth compactly supported function with \( \int_{\mathbb{R}^n} \varphi(x) \, dx = 1 \). Taking \( L^{p,\infty} \) norms in both sides of (24), and using Theorem 1 we obtain that

\[ \|f - Q\|_{L^{p,\infty}} \leq C' \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f - Q)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} = C' \left\| \left( \sum_{j \in \mathbb{Z}} |\Delta_j(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}}. \]
If $f \in L^r$ and $g = f - Q \in L^{p,\infty}$, then choose $p_2 > p$ and denote $m = \max(r, p_2)$. By Lemma 1 we will have

$$
\lim_{M \to \infty} \frac{|\{1 < x < M : |Q| > x^{-\frac{1}{p_2}}\}|}{M} \leq \lim_{M \to \infty} \frac{|\{1 < x < M : 2|g| > x^{-\frac{1}{p_2}}\}|}{M} + \lim_{M \to \infty} \frac{|\{1 < x < M : 2|f| > x^{-\frac{1}{p_2}}\}|}{M} = 0
$$

which implies that $Q = 0$ since $x^{-\frac{1}{m}} \to 0$ as $x \to \infty$.

For more general case, the support of $\hat{\Psi}(\xi)$ may intersect more supports of functions of the form $\hat{\Psi}(2^{-jq}\xi)$ and the number of intersection is finite since the support of $\hat{\Psi}$ is a compact annulus that does not contain 0. If we take $\varphi$ as in Theorem 1 then $\sup_{t > 0} |\varphi_t * \Delta_j(f)| \leq |\Delta_j(f)| + \sup_{a \geq l \geq b} \{\varphi_t * \Delta_j(f)\}$, where $a$ and $b$ are constants depending on the support of $\hat{\Psi}$. If we choose an appropriate $\eta$ satisfying that $\tilde{\eta}(\xi) = 1$ on the support of $\hat{\Psi}$, then there is no difficulty to apply Corollary 3 to show that

$$
\left\| \sum_{|j| \leq M} \Delta_j(f) \right\|_{H^{p,\infty}} = \left\| \sum_{|j| \leq M} \Delta_j^\eta(f) \right\|_{H^{p,\infty}} \\
\leq C \left\| \sup_{t > 0} \left( \sum_{|j| \leq M} |\varphi_t \ast f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \\
\leq C' \left( \sum_{j \in \mathbb{Z}} \left| (\Delta_j(f))^2 \right| \right)^{\frac{1}{2}}_{L^{p,\infty}},
$$

which gives that $\|f - Q\|_{H^{p,\infty}} \leq C(p, n)\|S^\Psi_q(f)\|_{L^{p,\infty}}$. The rest discussion then follows easily as we did in the case that $\Psi$ satisfies assumptions of Theorem 1.

The preceding corollary has applications in the theory of paraproducts. See [14]. Moreover, the following corollary can be proved similarly to the previous corollary.

**Corollary 5.** Fix $\Psi$ in $\mathcal{S}(\mathbb{R}^n)$ with Fourier transform supported in $\frac{6}{7} \leq |\xi| \leq 2$, equal 1 on the $1 \leq |\xi| \leq \frac{12}{7}$, and satisfy $\sum_{j \in \mathbb{Z}} \hat{\Psi}(2^{-j}\xi) = 1$ for $\xi \neq 0$. Fix $b_1, b_2$ with $b_1 < b_2$ and define a Schwartz function $\Omega$ via $\hat{\Omega}(\xi) = \sum_{j = b_1}^{b_2} \hat{\Psi}(2^{-j}\xi)$. Define $\Delta^\Omega_k(g)(\xi) = \hat{g}(\xi)\hat{\Omega}(2^{-k}\xi)$, $k \in \mathbb{Z}$. Let $q = b_2 - b_1 + 1$, $0 < p \leq 1$, and fix $r \in \{0, 1, \ldots, q - 1\}$. Then there exists a constant $C = C_{p, b_1, b_2, \Psi}$ such that for all $f \in H^{p,\infty}(\mathbb{R}^n)$ we have

$$
\left\| \left( \sum_{j = r \pmod{q}} |\Delta_j^\Omega(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} \leq C \|f\|_{H^{p,\infty}}. 
$$

Conversely, suppose that a tempered distribution $f$ satisfies

$$
\left\| \left( \sum_{j = r \pmod{q}} |\Delta_j^\Omega(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}} < \infty.
$$
Then there exists a unique polynomial $Q(x)$ such that $f - Q$ lies in the weak Hardy space $H^{p,\infty}$ and satisfies for some constant $C = C_{n,p,b_1,b_2,\Psi}$

\begin{equation}
\frac{1}{C}\|f - Q\|_{H^{p,\infty}} \leq \left\| \left( \sum_{j=r \mod q} |\Delta_j^\Psi(f)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p,\infty}}.
\end{equation}

Proof. We have proved the direction that $\frac{1}{C}\|f - Q\|_{H^{p,\infty}} \leq \left( \sum_{j=r \mod q} |\Delta_j^\Psi(f)|^2 \right)^{\frac{1}{2}} \|_{L^{p,\infty}}.$

\begin{equation}
\left( \sum_{j=r \mod q} |\Delta_j^\Psi(f)|^2 \right)^{1/2} \leq \sum_{k=1}^{q} \left( \sum_{j} |\Delta_j^\Psi(f_{q+k})(f)|^2 \right)^{1/2} \leq q \left( \sum_{j} |\Delta_j^\Psi(f)|^2 \right)^{1/2} \text{ comes from the fact that } \hat{\Omega}(\xi) = \sum_{j=b_1}^{b_2} \hat{\Psi}(2^{-j}\xi), \text{ which proves the other direction.}
\end{equation}

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