Correlation length critical exponent as a function of the percolation radius for one-dimensional chains in bond problems

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Abstract. A one-dimensional lattice percolation model is constructed for the problem of constraints flowing along non-nearest neighbors. In this work, we calculated the critical exponent of the correlation length in the one-dimensional bond problem for a percolation radius of up to 6. In the calculations, we used a method without constructing a covering lattice or an adjacency matrix (to find the percolation threshold). The values of the critical exponent of the correlation length were obtained in the one-dimensional bond problem depending on the size of the system and at different percolation radii. Based on original algorithms that operate on a computer faster than standard ones (associated with the construction of a covering lattice), these results are obtained with corresponding errors.

1. Introduction
The classical and periodic literature on the theory of percolation describes the flow along the nearest neighbors [1] on two-dimensional or three-dimensional lattices, since on one-dimensional lattices the case of percolation along the nearest neighbors is trivial. However, it ceases to be such if we consider percolation over non-nearest neighbors.

Percolation theory is used to describe the elasticity of polymer gels, hopping conductivity in doped semiconductors at low temperatures, and a number of other problems. In addition, many new problems have appeared, which cover a number of related areas, in the solution of which percolation models are used [2].

The purpose of this work is to calculate the average values of the correlation length exponent in the bond problem for one-dimensional chains of different sizes and different percolation radii. In this case, we used data on the values of the percolation threshold with the corresponding minimum error in one-dimensional gratings of different lengths at different radii of percolation.

2. Building a mathematical model and algorithms
Lattice models of percolation theory are primarily of interest from a theoretical point of view; it is for them that a number of rigorous statements and relations have been proved. Let us dwell in more detail on the concept of the percolation threshold. The percolation threshold in the site problem is the largest
fraction of whole sites at which the connecting cluster disappears) [2]. In the bond problem, the percolation threshold is the largest fraction of whole bonds at which there is no connecting cluster [2].

Classical lattice problems in percolation theory are the site problem and the bond problem with the same probability for any connection to be either whole or blocked [3].

Let us consider the one-dimensional percolation model in the constraint problem in more detail. Sites (in the number of \( N \) pieces), which are located at the same distance from each other, represent a horizontal chain. The same number of links emerge from each node. The number of bonds does not depend on the location of the node itself in the lattice; the number of bonds depends only on the radius of flow. The probability of blocking an arbitrarily chosen link is the same, all bonds in the model are equal. The chain bonds are numbered from the left edge.

There can be only two types of connections between nodes: whole and blocked (broken, through which no current flows - in the problem of electrical conductivity). In the bond problem, only one bond is blocked at a time, and all nodes are considered whole. Two lattice sites are considered connected to each other if they are connected by a chain of whole bonds. A set of sites connected to each other is called a cluster [1, 4]. If the proportion of whole bonds is small, they form small separate clusters. With an increase in the number of integer bonds, the clusters grow, merge, and a so-called connecting (“infinite”) cluster arises, which permeates the entire lattice, along which the flow is carried out from one end of the lattice to the other.

The average maximum fraction of intact bonds at which there is no connecting cluster is the \( P_{cb} \) percolation threshold. It depends on the percolation radius, the dimension of the lattice, its type and the number of sites \( N \) [1-5].

Thus, the lattice is a horizontal one-dimensional chain, consisting of \( N \) sites, with a percolation radius \( R > 1 \).

The percolation threshold separates two phases: in one phase there are only finite clusters and there is no infinite (connecting) cluster, in the other phase there is a connecting cluster [6]. This phase is analogous to the ferromagnetic phase for a magnetic phase transition.

The emergence of an infinite cluster in these problems is in many respects similar to a second-order phase transition (more precisely, even a third-order phase transition according to Ehrenfest). The percolation threshold depends significantly on the type of percolation problems, but the critical exponents are the same for different problems and are determined only by the dimension of the space and the symmetry of the order parameter. The representations borrowed from the theory of continuous phase transitions of the second order make it possible to obtain relations connecting different critical exponents (scaling relations) [2-5].

In a finite system, the percolation threshold varies from experience to experience, i.e. is a random quantity. However, the values that this random variable takes with overwhelming probability fall into a certain region with a width of \( \delta(N) \), which is called the critical region [2].

With an increase in the number of nodes in the \( N \) system, the width of this region decreases according to a power law, so that at \( N \to \infty \) the percolation threshold acquires a clear meaning, turning from a random variable into a reliable value [2-5].

The width of the critical area for volumetric problems (two-dimensional, three-dimensional, etc.) is also defined as for one-dimensional.

A small difference arises if you express the width \( \delta \) is not in terms of the size of the system \( L \), but in terms of the total number of sites \( N \). The fact is that

\[
N = \left( \frac{L}{a} \right)^{d},
\]

where \( a \) – lattice period and \( d \) – dimension of space [2].

Taking this into account, the following formula is valid:
\[ \delta(N) = \frac{C}{N^{\delta}}, \]  

(2)

where \( C \) – a numerical coefficient that cannot be determined from such simple considerations [2]. In the plane case (two-dimensional problems \( d = 2 \)), as a result of studying the dependence \( \delta(N) \) found on a computer, the exponent of the correlation radius of the plane problem was determined for the first time. It turned out that \( v_2 = 1.33 \) (Here and below, the number 2 shows that we are talking about the exponent of a two-dimensional system). For three-dimensional problems, the exponent \( v \) is different: \( v_3 = 0.8-0.9 \) (the number 3 means that the index refers to three-dimensional problems) [4].

Thus, considering the case of a one-dimensional system (one-dimensional chain), for which \( d = 1 \), formula (2) can be rewritten in the following form:

\[ \delta(N) = \frac{C}{N^{\nu}}, \]  

(3)

hence, the critical exponent of the correlation length will be found by the formula

\[ \nu = \frac{\ln \left( \frac{N_2}{N_1} \right)}{\ln \left( \frac{\delta_1}{\delta_2} \right)}. \]  

(4)

To calculate the correlation length exponent, it is necessary to know the percolation threshold for a given grating and the width of the critical region. First, all links are considered intact, then one link is randomly blocked in one step. The essence of the method for finding the percolation threshold in the problem of connections is to find reachable sites (this is a node that can be reached by integer connections from the left edge of the chain). If among them there is at least one of the last \( R \) sites, and a whole bond leaves from it to the right edge of the lattice, then the threshold is not reached, and the process of blocking bonds continues [7, 8]. In the process of running the program, the values of the percolation threshold were obtained for various configurations from the number of sites and radius [8]. Then, by averaging, we find the width of the critical region as the standard deviation of the percolation threshold. Using formula (4), we obtain the values of the critical exponent of the correlation length at different percolation radii. The calculation results table is shown in figure 1.

| N1 | N2 | R=2 | R=3 | R=4 | R=5 | R=6 |
|----|----|-----|-----|-----|-----|-----|
|    |    | \( \nu \) | \( \nu \) | \( \nu \) | \( \nu \) | \( \nu \) |
| 40 | 80 | 3.85 | 8.96 | 26.34 | 42.63 | 33.62 |
| 80 | 150| 3.11 | 5.76 | 11.47 | 12.46 | -    |
| 150| 250| 3.44 | 6.38 | 8.81 | 14.56 | 24.26 |
| 200| 300| 3.37 | 5.80 | 13.58 | 11.55 | 9.53 |
| 300| 500| 3.17 | 6.79 | 10.98 | 14.05 | 23.24 |
| Average value | \( \nu \) | 3.39 | 6.74 | 14.24 | 19.05 | 22.66 |

Figure 1. Values of the critical exponent of the correlation length in the bond problem up to \( R = 6 \).

It can be seen from the data obtained that the critical exponent of the correlation length \( \nu \) increases with an increase in the percolation radius in one-dimensional bond problems. This dependence will be true both for finite chains and for infinite one-dimensional chains, which is proved by the performed extrapolation.
For the critical exponent of the correlation length, the error is $3 - 16\%$ (with an increase in the lattice size, the error decreases).

3. Conclusion

Thus, for the first time, the critical exponent of the correlation length was calculated as a function of the percolation radius and the size of the system for a one-dimensional bond problem. It is shown that with an increase in the percolation radius, the correlation length exponent increases, and this index for the one-dimensional bond problem is significantly larger than for the three-dimensional and two-dimensional cases. This indicates that the dependence of the correlation length on temperature is stronger for the one-dimensional bond problem than for two-dimensional and three-dimensional space.

Within the framework of the proposed approach, it is possible to calculate other critical exponents and check the scaling relations for the bond problem in finite systems for an arbitrary percolation radius.

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