Abstract

We illustrate, using a simple model, that in the usual formulation the time-component of the Klein-Gordon current is not generally positive definite even if one restricts allowed solutions to those with positive frequencies. Since in de Broglie's theory of particle trajectories the particle follows the current this leads to difficulties of interpretation, with the appearance of trajectories which are closed loops in space-time and velocities not limited from above. We show that at least this pathology can be avoided if one uses a covariant extension of the canonical formulation of relativistic point particle dynamics proposed by Gitman and Tyutin.
de Broglie’s pilot wave theory for the Klein-Gordon equation

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I. PARTICLE TRAJECTORY INTERPRETATIONS OF THE KLEIN-GORDON EQUATION

de Broglie first proposed a particle-trajectory interpretation of the Klein-Gordon equation in the period 1926-1927 [1]. He proceeded using the polar form of the scalar field,

\[ \phi = R \exp(iS) \]  

(1)

to decompose the wave equation (with \( \hbar = c = 1 \))

\[ (\partial^\mu + ieA^\mu) (\partial_\mu + ieA_\mu) \phi = -m_0^2 \phi \]  

(2)

into a continuity equation

\[ \partial^\mu (R^2 (\partial_\mu S + eA_\mu)) = 0 \]  

(3)

and a “Hamilton-Jacobi” equation

\[ (\partial_\mu S + eA_\mu) (\partial^\mu S + eA^\mu) - \frac{\Box R}{R} = m_0^2 \]  

(4)

thus casting the theory in the form of a “classical-particle” theory with canonical four-momentum \( \partial_\mu S \) and an additional “quantum potential” term \(-\frac{\Box R}{R}\) which de Broglie interpreted in terms of a variable rest mass \( \sqrt{m_0^2 + \frac{\Box R}{R}} \). Particles follow the streamlines determined by equation 3 with three-velocity defined by de Broglie as

\[ v^k_{dB} = \frac{S^k + eA^k}{S^0 + eA^0} \]  

(5)

with \( k = 1, 2, 3 \) and where \( S^\mu = \partial_\mu S \) and so on. However, the problem with the use of \( \partial_\mu S \) to define the flow lines, and hence particle trajectories, is that \( \partial_\mu S \) is not always a time-like four vector. Furthermore, as we demonstrate in a specific case below, the time component of the current, \( R^2 \partial_0 S \), is not always positive definite even if one restricts the allowed solutions to superpositions of those with positive frequencies only [2] (see also [4]). The above is often taken to imply the breakdown of the single-particle interpretation of the Klein-Gordon equation and to signal the unavoidable effects of negative energies and hence the need for anti-particles in the description.

A. Appearance of a negative time-component of the current

A general proof of the existence of a non positive-definite time-component of the current has been given in [2] and [4]. Here we illustrate this point by considering the following simple superposition of two positive frequency eigenstates of the Klein-Gordon equation
\[ \Phi(x, t) = \phi_1(x) e^{-i\omega_1 t} + \phi_2(x) e^{-i\omega_2 t} \]  

and let us assume that \( \phi_1 \) and \( \phi_2 \) are real functions of \( x \) and that \( \omega_1 \) and \( \omega_2 \) are positive. The time-component of the current is given by

\[ j^0 = -|\Phi|^2 \frac{\hbar}{m_0} \text{Im} \left[ \frac{\partial \Phi}{\partial t} \Phi \right] \]

\[ = \frac{\hbar}{m_0} \left[ \omega_1 |\phi_1|^2 + \omega_2 |\phi_2|^2 + \phi_1 \phi_2 (\omega_1 + \omega_2) \cos ((\omega_1 + \omega_2) t) \right] \]

and this may be backwards pointing in time if

\[ \cos ((\omega_1 + \omega_2) t) < - \left[ \frac{\omega_1 |\phi_1|^2 + \omega_2 |\phi_2|^2}{\phi_1 \phi_2 (\omega_1 + \omega_2)} \right] \]

and since \( \phi_1 \phi_2 \) can be negative the inequality can easily be satisfied.

1. A specific example: the scalar potential box.

Although electromagnetic potentials are incapable of confining a relativistic particle, scalar potentials can do so. Using such a potential one can create a one-dimensional scalar potential “box” of length \( L \) such that \( V = 0 \) for \( 0 < x < L \) and \( V = \infty \) otherwise. The eigenstates are given by

\[ \phi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right) \exp (-i\omega_n t) \]

where \( n \) is an integer and

\[ \omega_n = c \sqrt{\left( \frac{n\pi}{L} \right)^2 + \left( \frac{m_0 c}{\hbar} \right)^2} \]

A simple superposition of states serves to illustrate the pathologies of de Broglie’s formulation. We consider the state

\[ \Phi(x, t) = \frac{1}{\sqrt{2}} [\phi_1 + \phi_2] \]

and for the purposes of the calculation we take \( L = \pi \) and \( \hbar = c = m_0 = 1 \).

In figure 1 we plot \(|\Phi|^2\) (dotted line), the de Broglie three-velocity, given by equation 5, (solid line) and the time component of the four velocity \( S_0 \) (dashed line) for the state given in equation 12 at \( t = 0.1 \). Clearly there are regions in which the three-velocity becomes superluminal (this is where the three velocity has a magnitude greater than unity in our chosen units). The de Broglie three-velocity is discontinuous where \( S_0 \) changes sign, for the values of the parameters chosen for the plot this occurs at \( x \simeq 2.086 \) and \( x \simeq 1.898 \) (to three decimal places). Figure 2 shows the space-time flows for this simple example. The flows are calculated from the four-velocity with
\[
\frac{\partial t(\tau)}{\partial \tau} = -S^0
\]
\[
\frac{\partial x(\tau)}{\partial \tau} = S^1
\]

One has closed loops in space-time which are difficult to interpret. As far as we know de Broglie never commented on this pathological situation. To consider this point more carefully one has to consider the meaning of Gauss’ theorem in four-dimensional space-time.

II. APPLICATION OF GAUSS’ THEOREM TO TIME-LIKE AND SPACE-LIKE FLOWS

Gauss’ theorem in four-dimensional space-time is

\[
\int_{V^4} \partial_{\mu} (u^\mu) \, d^4x = \int_{V^3} (u^\mu) \, \epsilon(n) \, n_\mu \, d^3x
\]

where \( \epsilon(n) = \pm 1 \) depending on the time-like or space-like character of \( n_\mu \). One can interpret this in terms of time-like or space-like conserved flows of \( u^\mu \) through hypersurfaces. For example, one can consider flow through a world-tube bounded by the hypersurface parallel to the flow and hypersurfaces normal to the flow at each end. (If the flow crosses a null hypersurface a limiting process must be used.)

Consider the application of Gauss’ theorem to the example previously discussed and illustrated in figure 2 where there are closed paths in space-time. In order to discuss the slicing of the flow tube normal to the direction of flow we consider the Fermi transport of an orthogonal dyad (in general a tetrad) of a time-like and a space-like vector around the closed loop. This provides a natural way of defining the orientation of the surfaces with respect to the flow as illustrated in figure 3. One should note that the orthogonal vectors maintain their orientation with respect to the time and the space axes. The full details of Fermi transport along time-like and space-like curves are given in reference [3]. An application of Gauss’ theorem now gives a conserved flow around the closed path in space-time. There remains the interpretative difficulty associated with backward-pointing time-like flows in regions where this is an invariant characterisation (e.g. the region of figure 3 bounded by the light-cone and containing the positive portion of the x-axis). In all other regions the space-like flows do not have an invariant characterisation in terms of orientation in time.

It seems, however, that a more careful formulation of the dynamics allows one to remove at least the pathology of the backward-pointing time-like flows, as we now proceed to demonstrate.

III. CLASSICAL RELATIVISTIC DYNAMICS REVISITED

In the foregoing we have followed de Broglie in formulating the theory in a particular reference frame. However, following the non-covariant treatment of Gitman and Tyutin [4], and [5], the de Broglie theory can be cast in covariant form in the following way.

The classical action for a relativistic, spinless particle can be written as
\[ S = -m \int L \, d\tau \quad (16) \]

where
\[ L = -m \sqrt{\dot{x}_\mu \dot{x}^\mu} \quad (17) \]
and \( \dot{x}_\mu = \frac{dx_\mu}{d\tau} \) with \( \tau \) a scalar parameter along the path of the particle. The canonical momenta \( \pi_\mu \) are then given by
\[ \pi_\mu = \frac{\partial L}{\partial \dot{x}_\mu} = -m \frac{\dot{x}_\mu}{\sqrt{\dot{x}_\mu \dot{x}^\mu}} \quad (18) \]
from which one has the primary constraint
\[ \pi_\mu \pi^\mu = m^2 \quad (19) \]
Decomposing \( \pi_\mu \) along a unit time-like four-vector \( n^\mu \) \( (n^0 > 0) \) and perpendicular to it one has
\[ \pi_\mu = \hat{\pi}_\mu + (\pi_\nu n^\nu) n^\mu \quad (20) \]
where \( \hat{\pi}_\mu \) is space-like and \( \hat{\pi}_\mu \hat{\pi}_\mu < 0 \). (In the following the same covariant decomposition notation will be used, the caret accent indicating the space-like part of the vector.) Therefore
\[ \pi_\mu \pi^\mu = \hat{\pi}_\mu \hat{\pi}_\mu + (\pi_\nu n^\nu)^2 = m^2 \quad (21) \]
Putting \( \pi_\nu n^\nu = \Pi \), the primary constraint becomes
\[ \Phi = \sqrt{\hat{\pi}_\mu \hat{\pi}_\mu + m^2 - |\Pi|} = 0 \quad (22) \]
which is a covariant formulation of the constraint. Resolving both sides of equation \( 18 \) along \( n^\mu \) and perpendicular to it one has
\[ \hat{\pi}_\mu = \frac{-m}{\sqrt{\dot{x}_\nu \dot{x}^\nu}} \hat{x}_\mu \quad (23) \]
\[ \Pi = \frac{-m}{\sqrt{\dot{x}_\nu \dot{x}^\nu}} \dot{X} \quad (24) \]
where \( \dot{X} = (\dot{x}_\mu n^\mu) \) and both \( \Pi \) and \( \dot{X} \) are scalars. Therefore
\[ \hat{x}_\mu = \hat{\pi}_\mu \frac{\dot{X}}{\Pi} \quad (25) \]
and since \( \Pi \) and \( \dot{X} \) are of opposite sign,
\[ \hat{x}_\mu = -\hat{\pi}_\mu \frac{X}{|\Pi|} = \frac{-\hat{\pi}_\mu}{\sqrt{-\hat{\pi}_\mu \hat{\pi}_\mu + m^2}} |\dot{X}| \quad (26) \]
The Hamiltonian \( H \) is given, as usual, by
\[ H = \pi_\mu \dot{x}^\mu - L \]  

which vanishes along the constraint surface. Following [4] one can choose a special gauge (in a covariant way) as

\[ \Phi^{(0)} = X - \zeta \tau \]

One can now switch to canonical variables \( x'^\mu, \pi'^\mu \) using a generating function \( W \) where

\[ W = x^\mu \pi'_\mu + \tau \|\Pi\| \]

\[ X' = X - \zeta \tau \]

\[ x'_\mu = x_\mu \]

\[ \pi_\mu = \pi'_\mu \]

The new Hamiltonian \( H' \) is

\[ H' = H + \frac{\partial W}{\partial \tau} = \|\Pi\| \]

on the constraint surface. Therefore

\[ H' = \sqrt{-\pi_\mu \pi'^\mu + m^2} \]

and since \( \dot{X} = \zeta \)

\[ \dot{x}_\mu = \frac{-\pi_\mu}{\sqrt{-\pi_\mu \pi'^\mu + m^2}} \theta \]

As in [4], \( \zeta \) is not fixed by the constraints and can be considered as another dynamical variable taking values \( \pm 1 \). In [4] it is shown that the two values of \( \zeta \) correspond to particle and antiparticle motion.

Introducing an external electromagnetic field \( A_\mu \) one obtains as usual

\[ H = \left( -\left( \pi_\mu + eA_\mu \right) \left( \pi'^\mu + eA'^\mu \right) + m^2 \right)^{1/2} \]

\[ \frac{d{\hat{x}}^\mu}{d\tau} = -\left( \frac{\pi^\mu + e\hat{A}^\mu}{H} \right) \]

\[ \frac{d\pi'^\mu}{d\tau} = -\left( \frac{e\hat{A}^\nu,\mu \left( \pi'_\nu + e\hat{A}'_\nu \right)}{H} \right) \]

Calling \( X_0 = \zeta \tau \) and \( \psi^\mu = \zeta \pi_\mu \) physical time and momenta, respectively one finds

\[ \frac{d{\hat{x}}^\mu}{dX} = \frac{\theta^\mu - g\hat{A}^\mu}{\left( \theta^\mu - g\hat{A}^\mu \right)^2 + m^2} \]

\[ \frac{d\hat{\psi}^\mu}{dX} = \frac{-g\hat{A}^\nu,\mu \left( \hat{\psi}^\nu - g\hat{A}^\nu \right)}{\left( \hat{\psi}^\mu - g\hat{A}^\mu \right)^2 + m^2} \]
where \( g = \zeta e \). Therefore trajectories with \( \zeta = 1 \) correspond to a particle with charge \( e \) and those with \( \zeta = -1 \) to a particle with charge \( -e \). One also notes that, since \( X = x_\mu n^\mu \) with \( n^0 > 0 \), the scalar time parameter \( X \) is positive for both cases (or negative for both cases).

In a given frame with \( n^\mu \) coincident with the time axis \( \Pi = \pi_0 \). So following this approach in de Broglie’s particle theory, remembering the connection of \( S^\mu \) with the canonical momenta one has to take the magnitude of \( S_0 \) in calculating the three-velocity; yielding, rather than equation 5 in a given frame and in the absence of potentials

\[
v_{dB}^k = \frac{S^k}{|S^0|} \tag{41}
\]

For the particular case, discussed above, of the square well with the wavefunction \( \Phi \) defined by \( 12 \), this modified form of the velocity is plotted in figure 4 as the solid line, along with \( |\Phi|^2 \), as the dotted line, for comparison. The two forms of the de Broglie velocity coincide except where \( S^0 \) is positive. Figure 5 shows the space-time flows calculating using

\[
\frac{\partial t(\tau)}{\partial \tau} = |S^0| \tag{42}
\]
\[
\frac{\partial x(\tau)}{\partial \tau} = S^1 \tag{43}
\]

which obviously do not have space-time loops, although the flows still become space-like in certain regions.

### IV. TIME-LIKE TRAJECTORIES

Elsewhere \( 7, 8 \) we have given details of an approach to the definition of particle trajectories for the Klein-Gordon equation which yields time-like trajectories in all circumstances. Our Lorentz invariant description defines the flow of stress-energy-momentum, and hence particle trajectories which follow the flow, through the intrinsic natural four-vector provided by the matter field itself through the eigenvalue equation

\[
T^\mu_\nu W^\nu = \lambda W^\mu \tag{44}
\]

where \( T^\mu_\nu \) is the stress-energy-momentum tensor, \( \lambda \) the eigenvalue and \( W^\mu \) the eigenvector. Writing a solution \( \phi \) of the Klein-Gordon equation as

\[
\phi = \exp[P + iS] \tag{45}
\]

the stress-energy-momentum tensor, \( T^\mu_\nu \), of the field \( \phi \) is given by

\[
T^\mu_\nu = |\phi|^2[m^2_0 - (P^\alpha P_\alpha + S^\alpha S_\alpha)]\delta^\mu_\nu + 2|\phi|^2[(P^\mu P_\nu + S^\mu S_\nu)] \tag{46}
\]

Once the state \( \phi \) is given the stress-energy-momentum tensor can be calculated along with its eigenvalues and eigenvectors. As we showed in \( 6 \), one finds a pair of eigenvectors one of which is time-like and the other space-like. The time-like vector and its eigenvalue determine the flows of energy and the density respectively. Applying this approach in this simple, one-dimensional example yields the one-dimensional three-velocity
\[ v^1 = \frac{W^1}{W^0} = -\frac{(T_{00} - \lambda)}{T_{01}} \]  

(47)

where

\[ \lambda = \frac{T_{00} - T_{11}}{2} \pm \sqrt{\frac{(T_{00} - T_{11})^2}{4} - (T_{01})^2} \]  

(48)

More generally (in more than one dimension) the velocity is given by

\[ v^k = \frac{S^k \pm e^{\pm \theta} \nabla P}{\left( \frac{\partial S}{\partial t} \pm e^{\pm \theta} \frac{\partial P}{\partial t} \right)} \]  

(49)

where

\[ \sinh \theta = \frac{P^\mu S_\mu - S^\mu S_\mu}{2P^\mu S_\mu} \]  

(50)

For our illustrative example, the three-velocity, given by equation [49] with the state [12] is plotted in figure 6, again for the same parameter values used in the previous plots. The particle is moving slowly throughout the whole region, with no extremes as exist in the de Broglie velocities for this situation. Figure 7 shows the associated trajectories. In the case of the unmodified de Broglie velocity the region of this plot contained the space-time vortex, here we see perfectly regular flow through this region governed by [49]

V. CONCLUSION

We have shown that if one wishes to formulate an interpretation of the Klein-Gordon equation based on individual particle trajectories then this can be done in a consistent manner but only by going beyond the approach originally proposed by de Broglie. In de Broglie’s approach one has a non positive-definite time-component of the current, closed loops in the space-time flows, space-like particle motions and a rest mass which may become imaginary. The situation can be improved by formulating the theory in a covariant manner as one consequence of doing this properly is to remove the possibility of backward-pointing time-like flows. However, this modification of de Broglie’s theory does not overcome the problems associated with space-like trajectories. A particle-trajectory interpretation which uses the flows defined by the stress-stress-energy-momentum tensor of the Klein-Gordon field to determine the particle trajectories does not suffer from the pathologies discussed in de Broglie’s theory. Adopting the latter approach enables a consistent interpretation of the simple quantum phenomena associated with relativistic particles in terms of well-defined particle trajectories. At least the obstacles to the construction of a consistent particle interpretation of the Klein-Gordon equation that we have discussed here can be removed, and a consistent theory developed. The problems that we have discussed in this paper are often cited as reasons for the rejection of a particle-trajectory interpretation of the Klein-Gordon equation and the compulsion to adopt a basic field ontology for bosons as suggested by Bohm. We have shown that if one wishes to reject the particle ontology in favour of the field ontology then the justification must be sought elsewhere.
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Figure Captions

1. FIG. 1. $|\Phi|^2$ (dotted line), the de Broglie three-velocity, given by equation 3 (solid line), the time component of the four velocity $S_0$ (dashed line) plotted across the box, for the state given by equation 12 with $L = \pi$, $t = 0.1$.

2. FIG. 2. The flows in space-time determined from the de Broglie equations 13 and engendered by the state given in equation 12 with $L = \pi$. The integration was performed from the set of initial conditions given by $(x_0, \tau_0) = [(1.9, -0.04), (1.9, -0.1), (2.4, -0.4), (2.3, -0.4), (2.0, -0.4)]$ chosen to display the pathology around the minimum in $|\Phi|^2$ shown in figure 1.

3. FIG. 3. Fermi transport of an orthogonal dyad around a closed path in space-time. The solid arrow is the time-like vector and the outlined arrow the space-like vector, together they form an orthogonal dyad. The time-like vector maintains its orientation with respect to the $t$ axis and similarly the space-like vector with respect to the $x$ axis.

4. FIG. 4. $|\Phi|^2$ (dotted line) and the modified form of the de Broglie three-velocity, given by equation 41 plotted across the box, for the state given by equation 12 with $L = \pi, t = 0.1$. for comparison with FIG 1.

5. FIG. 5. The flows in space-time determined from equations 42 and engendered by the state given by equation 12 with $L = \pi$. The integration was performed from the set of initial conditions given by $(x_0, \tau_0) = [(1.9, -0.04), (1.9, -0.1), (2.4, -0.4), (2.3, -0.4), (2.0, -0.4)]$ as for figure 2.

6. FIG. 6. The three-velocity associated with the time-like eigenvector of the stress-energy-momentum tensor for the state given by equation 12 with $L = \pi, t = 0.1$. This velocity is perfectly regular, unlike either of the de Broglie velocities defined in equations 3 or 11.

7. FIG. 7. The flow lines governed by equation 49 derived from the eigenvectors of the stress-stress-energy-momentum tensor. The integration was performed from the set of initial conditions given by $(x_0, \tau_0) = [(1.94, -0.4), (2.0, -0.4), (2.14, -0.4), (2.3, -0.4), (2.4, -0.4)]$. 

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