Sliding Mode Control of One-Sided Lipschitz Nonlinear Markovian Jump Systems With Partially Unknown Transition Rates

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ABSTRACT This paper investigates the problem of sliding mode control (SMC) for one-sided Lipschitz (OSL) nonlinear Markovian jump systems with partially unknown transition rates. Unmatched norm-bounded uncertainties of state matrices and output matrices are considered. First, a suitable integral-type sliding surface is proposed and a sufficient condition is given such that the sliding mode dynamics is stochastically stable with an $H_\infty$ performance level $\gamma$. Next, an SMC law is synthesized such that reachability of the specified sliding surface can be ensured. Finally, two simulation examples are provided to prove the effectiveness of the proposed approach.

INDEX TERMS Markovian jump systems, partially unknown transition rates, sliding mode control, one-sided Lipschitz.

I. INTRODUCTION

It is well known that most practical systems are nonlinear, so it is important to design controllers or observers for nonlinear systems. Most nonlinear systems can easily satisfy the Lipschitz condition globally, or at least locally [1]. Many important results on Lipschitz systems have been presented, such as observer design [2], robust passive control [3], and so on. Studies have shown that design methods based on traditional Lipschitz conditions are generally applicable to systems with a smaller Lipschitz constant. To overcome this shortcoming, the one-sided Lipschitz (OSL) condition and the quadratically inner-bounded condition are introduced to reduce the conservatism of traditional Lipschitz condition [4], [5]. At the same time, the one-sided Lipschitz constant is obviously less than the corresponding Lipschitz constant, which can reduce the influence of nonlinearities on the design of control system to some extent. Recently, one-sided Lipschitz systems have received considerable attention. A lot of results on controller or observer design approaches have been presented for OSL nonlinear systems [6]-[10]. The controllers are designed to guarantee the one-sided Lipschitz systems that are finite-time boundedness with a desired $H_\infty$ performance index in [6]. An $H_\infty$ output feedback controller is presented for one-sided Lipschitz systems, which consists of a state feedback control together with a nonlinear observer in [7]. The results of designing many kinds of observers for the one-sided Lipschitz systems have been developed, such as unknown input observer [8], reduced-order observer [9], $H_\infty$ observer [10].

Besides, Markovian jump systems (MJSs) have been paid much attention in the past decades since MJSs can be used to describe physical systems with random changes in both structure and parameters [11], [12]. It is very important to obtain the transition rates for MJSs since the stability and dynamic performance of the system are closely related to the transition rate matrix. However, in practical engineering, it is highly uneasy to accurately obtain all the information on the transition rate matrix. Thus the investigation of the MJSs with partially unknown transition rates is very significant. Recently, some controller or observer designs have been presented for one-sided Lipschitz MJSs [13]-[15]. In [13], a state feedback controller and an observer-based controller are designed for one-sided Lipschitz MJSs with partially unknown transition rates. In [14], an adaptive sliding mode observer for one-sided Lipschitz MJSs with general uncertain...
transition rates is designed. In [15], the design of a stochastic observer for one-sided Lipschitz MJSSs is studied. However, there are still several interesting and relevant problems to be addressed.

On the other hand, robust control theory and methods have been the focus of the control community in recent decades [16]–[19]. The sliding mode control technology is one of the powerful means of robust controller design. The sliding mode dynamics is a reduced-order system and completely insensitive to the matched uncertainties. For the system with the matched uncertainties, various sliding mode control methods are proposed. A sliding mode control technique for the stabilization of a class of uncertain and nonlinear systems with perturbation is presented in [20]. In [21], a chattering free optimal second-order sliding mode control method is proposed to stabilize nonlinear systems affected by uncertainties. The nonlinear optimal control strategy is based on the control Lyapunov function. The problem of sliding mode output regulation for nonlinear systems with unmodeled unknown time-varying disturbance is discussed in [22]. In [23], the problem of designing adaptive output feedback sliding mode controllers for a class of linear systems with matched external disturbances is proposed. However, in the practical system, we often encounter the uncertainties that cannot satisfy the matched condition. So the control system must consider the question of whether the controller can stabilize the controlled system and meet the desired performance metrics with unmatched uncertainties. To deal with unmatched uncertainties, many researchers have combined sliding mode control with $H_{\infty}$ control technology to achieve the ideal control effect for the control systems [24]–[27]. In [24], the robust $H_{\infty}$ sliding mode load frequency control of multi-area power system has been investigated. An integral SMC method with $H_{\infty}$ measure for systems with matched and unmatched perturbations has been proposed by Castanos and Fridman [25]. In [26], the problem of dynamic output feedback $H_{\infty}$-SMC has been discussed. The system has the mismatched uncertainty and external disturbance, once it is in the sliding mode. In [27], the $H_{\infty}$-integral sliding mode control for systems with mismatched parameter uncertainties along with disturbances and matched non-linear perturbations has been addressed. Very recently, the problem of $H_{\infty}$-integral sliding mode control for OSL nonlinear systems subject to unmatched uncertainties, external disturbances, and nonlinearities is studied in [28]. The control problem for uncertain OSL nonlinear systems subject to input nonlinearities is studied in [29]. For nonlinear MJSSs, the adaptive SMC problem is considered in [30]. In [31], the adaptive sliding mode controller design for a class of MJSSs with actuator faults has been investigated, where both completely known Markov transition rates and uncertain Markov transition rates have been considered. To the best of author’s knowledge, although the SMC design method has been widely applied for many kinds of systems, no one has studied the design of sliding mode controller for OSL nonlinear Markovian jump systems with partially unknown transition rates, not to mention the case when unmatched nonlinearities, unmatched uncertainties, and external disturbances are also involved simultaneously. Moreover, these existing works cannot be simply extended to the above systems. This motivated our research.

Motivated by the above discussion, this paper study the problem of $H_{\infty}$-SMC for OSL nonlinear Markovian jump systems with partially unknown transition rates subject to unmatched uncertainties, external disturbances, and OSL nonlinearities. The main contributions in this paper are summarised as follows: (i) The problem of $H_{\infty}$-SMC for OSL nonlinear Markovian jump systems with partially unknown transition rates is unavailable in the literature and is addressed for the first time. (ii) A suitable integral sliding surface is established and a sufficient condition for stochastic stability with an $H_{\infty}$ performance level $\gamma$ of the sliding mode dynamics is presented in an LMI frame. (iii) An efficient SMC law is designed such that reachability of the specified sliding surface can be ensured.

The rest of this paper is organized in the following way. In Section II, the system description and some preliminaries are outlined. The main results are given in Section III. Section IV shows the simulation results and Section V concludes the paper.

Notation: $\mathbb{R}^n$ stands for the $n$ dimensional Euclidean space. $\| \cdot \|$ represents the Euclidean norm of a vector or induced matrix norm. $X^T$ is the transpose of the matrix $X$. $\lambda_{\max}(X)$ ($\lambda_{\min}(X)$) represents the maximum (minimum) eigenvalue of the matrix $X$. $(\cdot, \cdot)$ stands for the inner product. $X > (\leq) 0$ denotes the positive (negative) definite matrix $X$. $\text{Diag} \{s_1, s_2, \ldots, s_n\}$ denotes a diagonal matrix whose diagonal elements are $s_1, s_2, \ldots, s_n$, respectively. For a symmetric matrix, * denotes the symmetric elements. $I$ is the identity matrix with appropriate dimensions. For a matrix $X$, $\text{Sym}(X) = X^T + X$. $\mathbb{L}_2(0, \infty)$ denotes the space of square-integrable vector functions over $[0, \infty)$. $E$ denotes the mathematical expectation.

II. SYSTEM REPRESENTATION AND PRELIMINARIES

The following Markovian jump system is considered:

$$
\begin{align}
\dot{x}(t) &= (A_{r_t} + \Delta A_{r_t}(t))x(t) + f_{r_t}(x(t)) + B_{r_t}u(t) + E_{r_t}w(t), \\
y(t) &= (C_{r_t} + \Delta C_{r_t}(t))x(t) + D_{r_t}w(t),
\end{align}
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state vector, the control input vector, and the output vector, respectively. $A_{r_t} \in \mathbb{R}^{n \times n}$, $B_{r_t} \in \mathbb{R}^{n \times m}$, $C_{r_t} \in \mathbb{R}^{p \times n}$, $D_{r_t} \in \mathbb{R}^{p \times d}$, $E_{r_t} \in \mathbb{R}^{n \times q}$ are known constant matrices. $f_{r_t}(x(t)) \in \mathbb{R}^n$ with $f_{r_t}(0) = 0$ is the nonlinear function. $\{r_t, t \geq 0\}$ is a right-continuous Markov process taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Pi = (\pi_{ij})$ $(i, j \in S)$ given by

$$
P[r_{t+\Delta} = j | r_t = i] = \begin{cases} 
\pi_{ij}\Delta + o(\Delta), & j \neq i \\
1 + \pi_{ii}\Delta + o(\Delta), & j = i
\end{cases}
$$

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where $\Delta > 0$, $\lim_{\Delta \to 0} \frac{M(t)}{\Delta} = 0$; $\pi_{ij} \geq 0$ is the transition rate from mode $i$ to mode $j$ at time $t + \Delta$ if $i \neq j$, and $\pi_{ii} = -\sum_{j=1, j \neq i}^N \pi_{ij}$. Moreover, matrices $\Delta A_i(t)$ and $\Delta C_i(t)$ are the time-varying parametric uncertainties and satisfy $S_{ij}^k \cup S_{jk}^i$, where $\pi_{ij}$ is known, $S_{ij}^k = \{j : \pi_{ij} \text{ is known}\}$, and $S_{jk}^i = \{j : \pi_{ij} \text{ is unknown}\}$.

Next, the following definitions, assumptions, and lemmas are given for the sake of convenience.

**Definition 1** [5]: The nonlinear function $f_i(x)$ is said to satisfy the OSL condition, i.e., $\forall x_1, x_2 \in D_{\mathcal{A}_i}$, 
\[
\langle f_i(x_1) - f_i(x_2), x_1 - x_2 \rangle \leq \rho_i \|x_1 - x_2\|^2, \tag{7}
\]
where $\rho_i \in \mathbb{R}$ is the so-called OSL constant, $D_{\mathcal{A}_i}$ is a compact region and contains the origin.

**Definition 2** [5]: The nonlinear function $f_i(x)$ is said to satisfy the quadratically inner-bounded condition, i.e., $\forall x_1, x_2 \in D_{\mathcal{B}_i}$, 
\[
(f_i(x_1) - f_i(x_2))^T (f_i(x_1) - f_i(x_2)) \leq \beta_i \|x_1 - x_2\|^2 + \alpha_i \|x_1 - x_2\|^2, \tag{8}
\]
where $\beta_i, \alpha_i \in \mathbb{R}$ are known constants, $D_{\mathcal{B}_i}$ is a compact region and contains the origin.

**Remark 1**: Unlike the traditional Lipschitz constant must be positive, the one-sided Lipschitz constant can be positive, zero, or even negative. One-sided Lipschitz constants are usually smaller than the Lipschitz ones [5]. In addition, Lipschitz function must satisfy the OSL condition and the quadratically inner-bounded condition, but the converse is not true. For example, any local interval $(0, a)$, the function $1/\sqrt{x}$ is not Lipschitz but satisfies one-sided Lipschitz condition [33].

**Definition 3** [34]: Choosing $V(x(t), i)$ as the stochastic Lyapunov function for the system (5), and its weak infinitesimal operator satisfies 
\[
\mathcal{L}V(x(t), i) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} [\mathcal{E}\{V(x(t + \Delta), r_t + \Delta)|x(t), r_t = i\} - V(x(t), r_t)].
\]

**Definition 4** [35]: The system (5) with $u(t) = 0$ and $w(t) = 0$ is said to be stochastically stable if for any $x(0) \in \mathbb{R}^n$ and $r_0 \in S$, there exists a scalar $\tilde{M}(x(0), r_0)$
\[
\lim_{t \to \infty} \mathcal{E}\{\int_0^t x^T(s)x(s)ds|_{x(0), r_0}\} \leq \tilde{M}(x(0), r_0).
\]

**Definition 5** [35]: Given a scalar $\gamma > 0$, the system (5) with $u(t) = 0$ is said to be stochastically stable with an $H_\infty$ performance level $\gamma$ if it is stochastically stable with $w(t) = 0$ and under the initial condition, for nonzero $w(t) \in L_2[0, \infty)$, 
\[
\mathcal{E}\{\int_0^\infty y^T(s)y(s)ds\} < \gamma^2 \int_0^\infty w^T(s)w(s)ds.
\]

**Assumption 1**: For each mode $i \in S$, the nonlinear function $f_i(x)$ is OSL and quadratically inner-bounded.

**Assumption 2**: For each mode $i \in S$, the matrix $B_i$ has full column rank.

**Lemma 1 (Schur complement)** [27]: Let $X = X^T, Y = Y^T$, and $Z$ be real matrices of appropriate dimensions, then 
\[
X < 0, \quad Y - Z^TX^{-1}Z < 0
\]
is equivalent to 
\[
\begin{bmatrix}
X & Z \\
Z^T & Y
\end{bmatrix} < 0.
\]

**Lemma 2** [35]: Let $M, F, N, P$ be real matrices of appropriate dimensions with $P > 0, F^TF \leq I$ and a scalar $\epsilon > 0$. Then 
\[
MFN + NT^TF^TM \leq \epsilon MP^{-1}M^T + \frac{1}{\epsilon}N^TF^TPFN.
\]

**Remark 2**: From Lemma 2, when $F = I$ and $\epsilon = 1$, it follows that $MN + NT^TM \leq MP^{-1}MT + N^TPN$.

### III. MAIN RESULTS

In this section, a sliding surface is designed and the corresponding sliding motion is analyzed. Then sliding mode controllers are synthesized such that the closed-loop system has the desired performance.

**A. SLIDING SURFACE DESIGN**

For the system (5), consider the following integral sliding function:
\[
s(t) = G_i x(t) - \int_0^t G_i(A_i + B_iK_i)x(s)ds.
\]
where \( G_iB_i \) is nonsingular, \( G_i \) and \( K_i \) are real matrices to be designed. Following the method in [36], the matrix \( G_i \) can be designed by selecting \( G_i = B_i^TP_i \) with \( P_i > 0 \).

From SMC theory, when the system operates in the sliding mode, it follows that \( \dot{s}(t) = 0 \). Therefore, the equivalent control can be obtained by:

\[
u_{eq} = K_i x(t) - (G_iB_i)^{-1} G_i \{ \Delta A_i x(t) + f_i(x) + E_i \nu(t) \}.
\] (15)

Then, the equivalent control (15) is substituted into the system (5), and the sliding mode dynamics can be obtained as

\[
\begin{aligned}
\dot{x}(t) &= \bar{A}_i x(t) + \bar{G}_i f_i(x) + \bar{E}_i \nu(t), \\
y(t) &= \bar{C}_i x(t) + D_i \nu(t),
\end{aligned}
\] (16)

where

\[
\begin{aligned}
\bar{A}_i &= A_i + B_i K_i + \bar{G}_i \Delta A_i, \\
\bar{G}_i &= I - B_i (G_i B_i)^{-1} G_i, \\
\bar{E}_i &= \bar{G}_i E_i, \quad \bar{C}_i = C_i + \Delta C_i(t).
\end{aligned}
\]

First, a stochastic stability condition for sliding mode dynamics (16) will be developed.

**Theorem 1**: The sliding mode dynamics (16) is stochastically stable with \( w(t) = 0 \) if there exist scalars \( \epsilon_{1i} > 0, \epsilon_{2i} > 0, \mu_i > 0 \), and matrices \( P_i > 0, Q_i = Q_i^T \), such that for each \( i \in S \)

\[
\begin{bmatrix}
\Lambda_{11i} & \Lambda_{12i} \\
* & \Lambda_{22i}
\end{bmatrix} < 0,
\] (17)

\[
P_i - Q_i \geq 0, \quad \forall j \in S_i^k, j = i,
\] (18)

\[
Q_i - P_j \geq 0, \quad \forall j \in S_i^k, j \neq i,
\] (19)

where

\[
\begin{aligned}
\Lambda_{11i} &= \text{Sym}(P_i \bar{A}_i) + \sum_{j \in S_i^i} \pi_{ij} (P_j - Q_i) + 2 \epsilon_{1i} \rho_i I + 2 \epsilon_{2i} \beta_i I + \mu_i I, \\
\Lambda_{12i} &= P_i \bar{G}_i - \epsilon_{1i} I + \epsilon_{2i} \alpha_i I, \\
\Lambda_{22i} &= -2 \epsilon_{2i} I.
\end{aligned}
\]

**Proof**: Consider the following Lyapunov function

\[
V(x(t), i) = x^T(t) P_i x(t).
\] (20)

According to Definition 3, the weak infinitesimal operator is

\[
\begin{aligned}
&\mathcal{L} V(x(t), i) = 2x^T(t) P_i \{ \bar{A}_i x(t) + \bar{G}_i f_i(x) + \bar{E}_i \nu(t) \} \\
&\quad + \sum_{j=1}^{N} \pi_{ij} x^T(t) P_j x(t) 
\end{aligned}
\] (21)

Because \( \sum_{j=1}^{N} \pi_{ij} = 0 \), the following zero equation is introduced:

\[
- \sum_{j=1}^{N} \pi_{ij} x^T(t) Q_j x(t) = 0.
\] (22)

Adding the left-hand sides of (22) to the right-hand side of (21) yields

\[
\begin{aligned}
\mathcal{L} V(x(t), i) &= 2x^T(t) P_i \{ \bar{A}_i x(t) + \bar{G}_i f_i(x) + \bar{E}_i \nu(t) \} \\
&\quad + \sum_{j \in S_i^k} \pi_{ij} x^T(t) (P_j - Q_i) x(t) \\
&\quad + \sum_{j \in S_i^k} \pi_{ij} x^T(t) (P_j - Q_i) x(t).
\end{aligned}
\] (23)

For \( w(t) = 0 \), we have

\[
\begin{aligned}
\mathcal{L} V(x(t), i) &= \zeta^T \begin{bmatrix}
\text{Sym}(P_i \bar{A}_i) + \sum_{j \in S_i^i} \pi_{ij} (P_j - Q_i) \\
* & 0
\end{bmatrix} P_i \bar{G}_i \\
&\quad + \zeta^T \begin{bmatrix}
\sum_{j \in S_i^k} \pi_{ij} (P_j - Q_i) & 0 \\
* & 0
\end{bmatrix} \zeta.
\end{aligned}
\] (24)

where \( \zeta = [x^T(t) f_i^T(x(t))]^T \).

On the other hand, based on Assumptions 1, for \( x_1 = x \) and \( x_2 = 0 \), we can get

\[
\begin{aligned}
2 \epsilon_{1i} \rho_i \bar{x}^T - 2 \epsilon_{1f_i} \bar{x}(t) \bar{x} \geq 0, \\
2 \epsilon_{2i} \beta_i \bar{x}^T x + 2 \epsilon_{2a_i} \alpha_i \bar{x} f_i(x) - 2 \epsilon_{2f_i} f_i(x) f_i(x) \geq 0.
\end{aligned}
\] (25)

which further can be written as

\[
\begin{aligned}
\zeta^T \begin{bmatrix}
2 \epsilon_{1i} \rho_i I & - \epsilon_{1f_i} I \\
0 & 0
\end{bmatrix} \zeta \geq 0, \\
\zeta^T \begin{bmatrix}
2 \epsilon_{2i} \beta_i I & - \epsilon_{2f_i} I \\
0 & - \epsilon_{2f_i} I
\end{bmatrix} \zeta \geq 0.
\end{aligned}
\] (26)

By adding the left-hand side of (26) into the right-hand side of (24), we can obtain that

\[
\begin{aligned}
\mathcal{L} V(x(t), i) &\leq \zeta^T \Phi_{1i} \zeta + \zeta^T \Phi_{2i} \zeta,
\end{aligned}
\] (27)

where

\[
\Phi_{1i} = \begin{bmatrix}
\Lambda_{11i} - \mu_i I & \Lambda_{12i} \\
* & \Lambda_{22i}
\end{bmatrix},
\] (28)

\[
\Phi_{2i} = \begin{bmatrix}
\sum_{j \in S_i^k} \pi_{ij} (P_j - Q_i) & 0 \\
* & 0
\end{bmatrix}.
\] (29)

For the matrix \( \Phi_{2i} \), we argue in following two cases:

1) if \( j = i \), then \( \pi_{ij} < 0 \), thus (18) implies that \( \Phi_{2i} \leq 0 \),

2) if \( j \neq i \), then \( \pi_{ij} > 0 \), thus (19) implies that \( \Phi_{2i} \leq 0 \).

Hence, we have

\[
\begin{aligned}
\mathcal{L} V(x(t), i) &\leq \zeta^T \Phi_{1i} \zeta.
\end{aligned}
\] (30)

According to the inequality (17), we obtain that

\[
\begin{aligned}
\mathcal{L} V(x(t), i) &\leq \zeta^T \begin{bmatrix}
- \mu_i I & 0 \\
0 & 0
\end{bmatrix} \zeta \\
&\leq - \mu x^T(t) x(t),
\end{aligned}
\] (31)

where

\[
\mu \triangleq \min_{i \in S} \mu_i.
\]
Exploiting the Dynkin’s formula [37], we can derive that
\[ \mathcal{E}\left[ \int_0^{t'} x^T(s)x(s)\,ds\right] \leq \frac{1}{\mu} V(x(0), r_0). \] (32)

Setting
\[ \tilde{M}(x(0), r_0) = \frac{1}{\mu} V(x(0), r_0), \]
according to Definition 4, we can learn that the sliding mode dynamics (16) is stochastically stable. The proof is completed.

In the next theorem, a condition will be given under which the sliding mode dynamics of the considered system is guaranteed to be stochastically stable with $H_\infty$ performance. The proof of Theorem 2 is based on the results shown in Theorem 1.

**Theorem 2:** Given a scalar $\gamma > 0$, the sliding mode dynamics (16) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist scalars $\epsilon_{1i} > 0$, $\epsilon_{2j} > 0$, $\mu_i > 0$, $\gamma > 0$, and matrices $P_i > 0$, $Q_i = Q_i^T$, such that for each $i \in S$

\[
\begin{bmatrix}
\Lambda_{11i} & \Lambda_{12i} & P_i \bar{E}_i \bar{C}_i^T \\
* & \Lambda_{22i} & 0 \\
* & * & -\gamma^2 I & D_i^T
\end{bmatrix} < 0,
\]

\[
P_j - Q_i \geq 0, \quad \forall j \in S_i^j, \ j = i.
\]

\[
Q_i - P_j \geq 0, \quad \forall j \in S_i^j, \ j \neq i.
\]

**Proof:** From Theorem 1, we observed that (16) with $w(t) = 0$ is stochastically stable. We still choose the Lyapunov function $V(x(t), i)$ as one in Theorem 1, and consider the following performance index

\[ J = \mathcal{L}V(x(t), i) + y^T(t)y(t) - \gamma^2 w^T(t)w(t). \] (36)

Following a similar line as in the proof of Theorem 1, we can get that

\[ J < 0. \] (37)

Integrating from 0 to $\infty$ on both sides of (36), we can obtain that

\[ V(x(\infty), r_\infty) - V(x(0), r_0) + \int_0^{\infty} y^T(s)y(s)\,ds \]

\[ - \gamma^2 \int_0^{\infty} w^T(s)w(s)\,ds \leq 0. \] (38)

Considering the zero initial condition $x(0) = 0$ and $\lim_{t \to \infty} V(x(t), i) \geq 0$, we can derive that

\[ \mathcal{E}\left[ \int_0^{\infty} y^T(s)y(s)\,ds\right] < \gamma^2 \int_0^{\infty} w^T(s)w(s)\,ds. \] (39)

Hence according to Definition 5, this system is stochastically stable with an $H_\infty$ performance level $\gamma$. This completes the proof.

**Remark 3:** Theorem 2 gives a sufficient condition for stochastic stability with an $H_\infty$ performance level $\gamma$ of the sliding mode dynamics. But the condition is a set of nonlinear matrix inequalities. Generally speaking, it is difficult to solve.

In the following, a sufficient condition for stochastic stability of the sliding mode dynamics with an $H_\infty$ performance level $\gamma$ is presented in an LMI frame.

**Theorem 3:** Given a scalar $\gamma > 0$, the sliding mode dynamics (16) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist scalars $\epsilon_{1i} > 0$, $\epsilon_{2j} > 0$, $\mu_i > 0$, $\gamma > 0$, and matrices $Y_i$, $P_i > 0$, $Q_i = Q_i^T$ such that for each $i \in S$

\[
\begin{bmatrix}
\Theta_{1i} & * \\
* & \Theta_{3i}
\end{bmatrix} < 0,
\]

\[
P_j - Q_i \geq 0, \quad \forall j \in S_i^j, \ j = i,
\]

\[
Q_i - P_j \geq 0, \quad \forall j \in S_i^j, \ j \neq i,
\]

where

\[
\Theta_{1i} = \begin{bmatrix}
\bar{Q}_{1i} & \bar{Q}_{2i} & \bar{Q}_{3i}
\end{bmatrix},
\]

\[
\Theta_{2i} = \begin{bmatrix}
-P_iB_i & 0 & 0 \\
0 & P_iB_i & 0 \\
0 & 0 & M_i^T P_i B_i
\end{bmatrix},
\]

\[
\Theta_{3i} = \text{diag}\{-B_i^T P_i B_i, -B_i^T P_i B_i\},
\]

\[
\bar{\Theta}_{1i} = \begin{bmatrix}
\bar{\Lambda}_{11i} & \bar{\Lambda}_{12i} & P_i \bar{E}_i \bar{C}_i^T \\
* & \bar{\Lambda}_{22i} & 0 \\
* & * & -\gamma^2 I & D_i^T
\end{bmatrix},
\]

\[
\bar{\Theta}_{2i} = \begin{bmatrix}
P_i M_i & \tau_1 N_i \bar{P}_1 & \tau_2 N_i \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & M_i^T
\end{bmatrix},
\]

\[
\bar{\Theta}_{3i} = \text{diag}\{-\tau_1, -\tau_1, -\tau_2, -\tau_2, -\tau_2\},
\]

\[
\bar{\Lambda}_{11i} = \text{Sym}(P_i A_i + Y_i) + \sum_{j \in S_i^j} \pi_j (P_j - Q_i) + 2 \epsilon_{1i} \beta_i I + 2 \epsilon_{2i} \beta_i I + \mu_i I,
\]

\[
\bar{\Lambda}_{12i} = P_i - \epsilon_{1i} I + \epsilon_{2i} \alpha_i I,
\]

\[
\bar{\Lambda}_{22i} = -2 \epsilon_{2i} I.
\]

Moreover, the parameter $K_i$ in (14) is given by

\[ K_i = B_i^T P_i^{-1} Y_i. \]

**Proof:** First, substituting the gain matrix $K_i$ into (40), and then using Lemma 1 and Lemma 2, we can derive that

\[
\begin{bmatrix}
\hat{\Theta}_{1i} & \hat{\Theta}_{2i} \\
* & \hat{\Theta}_{3i}
\end{bmatrix} < 0,
\] (43)
Then, the inequality (33) in the Theorem 2 can be obtained by using Lemma 1 and Lemma 2. This proof is done.

Remark 4: Theorem 3 gives a set of solutions to the parameters $K_i$ in the sliding mode function and also ensures that the sliding mode dynamics is stochastically stable with $H_\infty$ performance level $\gamma$. In practice, the $H_\infty$ performance level $\gamma$ is as small as possible. Based on Theorem 3, an optimization problem can be formulated as follows

$$
\min \gamma \epsilon_{i1} > 0, \epsilon_{2i} > 0, \tau_{1i} > 0, \tau_{2i} > 0, \\
\mu_i > 0, Y_i, P_i > 0, Q_i, \\
s.t. (40), (41), \text{ and } (42).
$$

The following conclusion is the direct deductions from Theorem 3. Because the proof process is similar to that of Theorem 1, it is omitted.

Corollary 1: Suppose that the transition rates are completely known. Given a scalar $\gamma > 0$, the sliding mode dynamics (12) is stochastically stable with an $H_\infty$ performance level $\gamma$ if there exist scalars $\epsilon_{1i} > 0, \epsilon_{2i} > 0, \tau_{1i} > 0, \tau_{2i} > 0, \mu_i > 0, \gamma > 0$, and matrices $Y_i, P_i > 0$ such that for each $i \in S$

$$
\left[ \begin{array}{cc}
\Xi_{1i} & \Xi_{2i} \\
* & \Xi_{3i}
\end{array} \right] < 0,
$$

where

$$
\Xi_{1i} = \left[ \begin{array}{ccc}
\Xi_{11i} & \Xi_{12i} & \Xi_{13i}
\end{array} \right], \\
\Xi_{11i} = \left[ \begin{array}{ccc}
\hat{X}_{1i} & X_{2i} & P_iE_i & C_i^T \\
* & \hat{X}_{22i} & 0 & 0 \\
* & * & -\gamma^2 I & D_i^T \\
* & * & * & -I
\end{array} \right], \\
\hat{X}_{11i} = \text{Sym}(P_i(A_i + B_iK_i)) + \sum_{j \in S_i^1} \pi_{ij}(P_j - Q_i), \\
-2\epsilon_{1i}\rho_iI + 2\epsilon_{2i}\beta_iI + \mu_iI,
$$

and there exist symmetric matrices $R_i$ for each $i \in S$ such that

$$
(G_iB_j)^{-1} - R_i \geq 0, \forall j \in S_i^1, j \neq i, \\
R_i - (G_iB_j)^{-1} \geq 0, \forall j \in S_i^1, j \neq i.
$$

\[ \text{sat}(s(t)) = \begin{cases} 
1, & s > 0.01 \\
100 * s, & |s| < 0.01 \\
-1, & s < -0.01.
\end{cases} \]

then the trajectories of the system (5) can be driven into the sliding surface $s(t) = 0$ in finite time and then maintain a sliding motion on it thereafter.

Proof: Consider the following Lyapunov function

$$
V_1(s(t), i) = \frac{1}{2} s^T(t)(G_iB_j)^{-1} s(t).
$$

Then, the weak infinesimal operator $\mathcal{L}$ of $V(s(t), i)$ is given by

$$
\mathcal{L}V_1(s(t), i) = s^T(t)(G_iB_j)^{-1} s(t) + \frac{1}{2} \sum_{j=1}^{N} \pi_{ij} s^T(t)(G_iB_j)^{-1} s(t).
$$
Different from the MJSs, semi-Markov jump systems (S-MJSs) are characterized by a fixed matrix of transition probabilities and a matrix of sojourn time probability density functions [38]. Therefore, S-MJSs have much broader applications than conventional MJSs. Extending the work of this paper to S-MJS will be a research topic in the future.

Remark 7: Although the finite-time asynchronous SMC scheme for MJSs in [32], the SMC problem OSL nonlinear Markovian jump systems has not been fully investigated. The transition rates and unmatched uncertainties are important factors affecting the control performance. The above paper does not consider how to deal with the control problem with partially unknown transition rates and unmatched uncertainties. The asynchronous sliding mode control problem of the OSL nonlinear Markovian jump systems with partially unknown transition rates subject to unmatched uncertainties, external disturbances will also be a research topic in the future.

Remark 8: The proposed method is compared with the method of [13], in which the controller design problem of one-sided Lipschitz Markovian jump systems with partially unknown transition probabilities subject to unmatched uncertainties, external disturbances has been investigated. As a nonlinear control strategy, SMC has made great theoretical development because of its strong robustness to the uncertainty and disturbance. Thus, the SMC problem is presented in this paper. And the uncertainty of the output matrix $\Delta C(t)$ has not been considered in [13]. It should be noted that the proposed SMC law depends on the transition rates which reflect the effect of Markovian switching, and prescribe the desired dynamic performance of the system. Compared with this paper, the designed controller is not depend on the transition rates directly in [13].

**IV. ILLUSTRATIVE EXAMPLES**

In this section, two examples are provided to demonstrate the effectiveness of the proposed approach.

*Example 1:* Consider a Markovian jump system with three modes and parameters as follows:

Mode 1: $A_1 = \begin{bmatrix} -1 & -2 \\ 1 & 1.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $D_1 = 0.01$, $E_1 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}$, $f_1(x) = \begin{bmatrix} -x_1(x_1^2 + x_2^2) \\ -x_2(x_1^2 + x_2^2) \end{bmatrix}$, $M_{11} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$, $M_{12} = 0.1$, $N_1 = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$.

Mode 2: $A_2 = \begin{bmatrix} -3 & 1 \\ -1 & 1 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $C_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $D_2 = 0.02$, $E_2 = \begin{bmatrix} 0.5 \\ 0.4 \end{bmatrix}$, $f_2(x) = \begin{bmatrix} \sin(x_1 - x_2) \\ 0 \end{bmatrix}$, $M_{21} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$, $M_{22} = 0.1$, $N_2 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$.
Mode 3: \[
A_3 = \begin{bmatrix} 1 & 0.5 \\ -0.4 & -2 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
C_3 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D_3 = 0.03, \quad E_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]
\[
f_3(x) = \begin{bmatrix} 0 \\ 0.333 \sin(x_2) \end{bmatrix}, \quad M_{31} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix},
\]
\[
M_{32} = 0.1, \quad N_3 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.
\]

For the OSL functions \(f_1(x), f_2(x),\) and \(f_3(x),\) we select the coefficients respectively \(\rho_1 = 0, \beta_1 = -30,\) and \(\alpha_1 = -20,\)
\(\rho_2 = \sqrt{2}, \beta_2 = 2, \alpha_2 = 0,\) \(\rho_3 = 0.333, \beta_3 = 0.112, \alpha_3 = 0.\)

The transition rate matrix is given as
\[
\Pi = \begin{bmatrix} -1.2 & ? & ? \\ 0.3 & ? & ? \\ ? & 0.5 & ? \end{bmatrix}.
\]

By Theorem 3 and Theorem 4, a feasible solution can be obtained
\[
P_1 = \begin{bmatrix} 0.0393 & 0.0327 \\ 0.0327 & 0.05 \end{bmatrix},
\]
\[
P_2 = \begin{bmatrix} 0.0662 & -0.0168 \\ -0.0168 & 0.0126 \end{bmatrix},
\]
\[
P_3 = \begin{bmatrix} 0.0348 & -0.0242 \\ -0.0242 & 0.0747 \end{bmatrix},
\]
\[
K_1 = \begin{bmatrix} -3.6622 & 14.4774 \\ -1.8869 & -24.8271 \end{bmatrix},
\]
\[
K_2 = \begin{bmatrix} -2.5134 & -5.3487 \end{bmatrix},
\]
\[
R_1 = 13.7605, \quad R_2 = 17.7524, \quad R_3 = 29.0864, \quad \gamma = 0.1651.
\]

Then, the sliding surface can be computed by (14). Let the adjustable parameter \(\nu = 0.2,\) then the SMC law can be obtained by (46). For simulation purposes, the initial condition is chosen as \(x(0) = \begin{bmatrix} 1.5 & -1 \end{bmatrix}^T\) and \(w(t) = e^{-t}\sin(t).\) Simulation results are shown in Figs.1-4. A switching signal is displayed in Fig. 1. Fig. 2 shows the state response of the closed-loop system. The SMC input and sliding function are shown in Figs. 3 and 4, respectively. As observed, the sliding mode is attained in finite-time, and after that, the system states have desirable behaviors as designed. The SMC system performs satisfactorily and has good performance characteristics. It is obvious that the proposed scheme works well and provides nice robustness qualities in the presence of unmatched uncertainties and nonlinearities.

Example 2: Consider the model of the single link robot arm (see in Fig.5) that can be given as a Markovian jump system...
FIGURE 5. Schematic of an elastic robot.

system [13].

where \( \rho = 9.8, L = 0.3, D = 2, M = 1.5 \).

For the OSL functions

\[
 f_i(x) = \begin{bmatrix} 0 \\ -MgL/J_i \end{bmatrix} \sin(x_1(t)),
\]

we select the coefficients \( \rho_1 = 2.205, \beta_1 = 4.862, \alpha_1 = 0, \rho_2 = 1.837, \beta_2 = 3.376, \alpha_2 = 0, \rho_3 = 1.102, \beta_3 = 1.215, \alpha_3 = 0, \rho_4 = 4.410, \beta_4 = 19.448, \alpha_4 = 0 \), respectively.

We assume that the system is affected by uncertain parameters and external disturbances. Thus, the other parameters for the four modes are given as follows:

- **Mode 1**: \( D_1 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \ E_1 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} \),
  \( M_{11} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \ M_{12} = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} \),
  \( N_1 = \begin{bmatrix} 0.1 \ 0.2 \end{bmatrix}, \ J_1 = 1 \).
- **Mode 2**: \( D_2 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \ E_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \),
  \( M_{21} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \ M_{22} = \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix} \),
  \( N_2 = \begin{bmatrix} 0.2 \ 0 \end{bmatrix}, \ J_2 = 2 \).
- **Mode 3**: \( D_3 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \ E_3 = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix} \),
  \( M_{31} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \ M_{32} = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} \),
  \( N_3 = \begin{bmatrix} 0.1 \ 0 \end{bmatrix}, \ J_3 = 2. \)
- **Mode 4**: \( D_4 = \begin{bmatrix} 0.01 \\ 0.01 \end{bmatrix}, \ E_4 = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix} \),
  \( M_{41} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \ M_{42} = \begin{bmatrix} 0.25 \\ 0.25 \end{bmatrix} \),
  \( N_4 = \begin{bmatrix} 0.2 \ 0.1 \end{bmatrix}, \ J_4 = 0.5 \).

The transition rate matrix is given as

\[
 \Pi = \begin{bmatrix}
 ? & ? & 0.2 & 0.4 \\
 ? & -0.6 & ? & ? \\
 0.5 & 0.1 & -0.7 & 0.1 \\
 ? & ? & ? & ? 
\end{bmatrix}.
\]

By Theorem 3 and Theorem 4, a feasible solution can be obtained

\[
 P_1 = \begin{bmatrix} 9.1932 & -0.0011 \\
 -0.0011 & 5.2844 \end{bmatrix},
 P_2 = \begin{bmatrix} 3.5784 & -0.0046 \\
 -0.0046 & 2.1775 \end{bmatrix},
 P_3 = \begin{bmatrix} 4.4620 & 0.0002 \\
 0.0002 & 1.1719 \end{bmatrix},
 P_4 = \begin{bmatrix} 8.2590 & 0.0003 \\
 0.0003 & 1.7122 \end{bmatrix},
 K_1 = \begin{bmatrix} -37.2434 & -7.8824 \end{bmatrix},
 K_2 = \begin{bmatrix} -4.1130 & -17.1077 \end{bmatrix},
 K_3 = \begin{bmatrix} 11.5894 & -31.8812 \end{bmatrix},
 K_4 = \begin{bmatrix} -53.7896 & -39.2614 \end{bmatrix},
 R_1 = 0.4258, \ R_2 = 8.9471, \ R_3 = 0, \ R_4 = 1.7733,
 \gamma = 0.4126.
\]

Then, the sliding surface can be computed by (14). Let the adjustable parameter \( \nu = 0.01 \), then the SMC law can be obtained by (45). For simulation purposes, the initial condition is chosen as \( x(0) = \begin{bmatrix} 0.5 & -1 \end{bmatrix}^T \) and \( w(t) = 0.1 \sin(t) \). The results of the simulation are given in Fig. 6-9. Fig. 6 is shown the evolution of the system mode, and the state response of the closed-loop system is given in Fig. 7. The control variables and the sliding surface functions are shown in Figs. 8 and 9, respectively. The sliding mode is attained in finite-time, and after that, the system states have desirable behavior as designed. The SMC system performs satisfactorily and has good disturbance attenuation performance in despite of unmatched uncertainties, which further demonstrates the validity of the results.

FIGURE 6. Mode evolution.
of the sliding mode in finite time has also been designed. Finally, two numerical examples are provided to show the effectiveness of the proposed method.

In the future, we will consider the sliding mode control for OSL nonlinear S-MJSs with partially unknown transition rates subject to unmatched uncertainties, external disturbances. To enhance system anti-disturbance performance, we further consider the sliding mode controller for system (5).

V. CONCLUSION

In this paper, the problem of sliding mode control has been investigated for OSL nonlinear Markovian jump systems with partially unknown transition rates. A suitable integral-type sliding surface is designed such that the resulting sliding mode dynamics is stochastically stable with an $H_\infty$ performance level $\gamma$. A robust controller that ensures the occurrence of the sliding mode in finite time has also been designed. Finally, two numerical examples are provided to show the effectiveness of the proposed method.

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