Abstract. We prove (a weak version of) Arnold’s Chord Conjecture in \cite{2} using Gromov’s “classical” idea in \cite{1} to produce holomorphic disks with boundary on a Lagrangian submanifold.

Arnold’s Chord Conjecture. In this paper we prove the following theorem which was conjectured by Arnold in \cite{2}:

**Theorem 1.** For every closed, compact Legendrian submanifold in $S^{2n-1}$ with the standard contact structure and any contact form for this structure, there is a Reeb chord, i.e. an integral curve of the Reeb vector field which begins and ends on the Legendrian submanifold.

Theorem 1 will follow as a corollary from the main result of this paper, Theorem 4. In fact it can be applied to a more general situation:

**Theorem 2.** Let $(M, \xi)$ be a contact structure which arises as smooth boundary of a compact subcritical Stein manifold (see \cite{4} for a definition). Then for any Legendrian submanifold and any contact one form corresponding to $\xi$ there is a Reeb chord.

Our results include the existence of chords for Legendrians in the standard contact structure on $\mathbb{RP}^{2n-1}$ proved by Ginzburg and Givental \cite{8,7}, although it does not provide their statement of linear growth. They cover results by Abbas \cite{1} and Cieliebak \cite{6} who treat subcases of the problem on the sphere and on boundaries of subcritical Stein manifolds.

Lagrangian out of Legendrian embeddings. Consider a closed Legendrian submanifold $l \subset M^{2n-1}$ in a contact manifold. Given a contact one form $\alpha$ we will construct Lagrangian embeddings of the torus $l \times S^1$ into the symplectization $(M \times \mathbb{R}, d(e^s\alpha))$ of $(M, \xi)$ and study them. Denote by $\phi$ the flow of the Reeb vector field $R = R_\alpha$.

Assume $l$ has no Reeb chords of length at most $T > 0$. Then there is an embedding $\Phi : l \times (\mathbb{R} \times [0, T]) \hookrightarrow M \times \mathbb{R}$ such that $\Phi^*(d(e^s\alpha)) = e^sds \wedge dt$, $(s, t) \in \mathbb{R} \times [0, T]$ being the coordinates of the infinite strip. The map is, of course, constructed using the Reeb flow: $\Phi(x, s, t) := (\phi(t)(x), s)$. 

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The upshot of all this is, that any smooth embedded loop in the rectangle \([S,0] \times [0,T]\) defines a smooth Lagrangian embedding of \(l \times S^1\) into \(M \times [S,0]\). Here \(S < 0\) is any negative parameter. The integral of the primitive \(e^s\alpha\) over closed curves in such a Lagrangian depends only on the homology class of that curve. It vanishes on classes arising as closed loops in \(l\), since \(\alpha|_l \equiv 0\). On \(\{x\} \times l\) it is given by the area enclosed by the loop in the rectangle with respect to the volume form \(e^s ds dt\). Approximating the boundary of \([S,0] \times [0,T]\) by smooth embedded loops we obtain the following

**Proposition 3.** Assume the Legendrian \(l \subset M\) in a contact manifold has no Reeb chord of length smaller or equal to \(T\) with respect to a contact one form \(\alpha\). Then there exists a smooth Lagrangian embedding of \(l \times S^1\) into the symplectic manifold \((M \times [S,0], d(e^s\alpha))\) such that the integral of the primitive \(e^s\alpha\) over a smooth closed curve in the image is an integer multiple of some constant \(C > (1 - e^S)T\).

**Proof of Arnold’s Chord Conjecture.** The results described above will follow from the following

**Theorem 4.** Assume \(W\) is a symplectic manifold of bounded geometry, i.e. there is an almost complex structure \(J\) together with a hermitian metric \(g\) whose sectional curvature is globally bounded from above and injectivity radius bounded from below, such that \(\omega\) tames \(J\) uniformly w.r.t. \(g\): \(\omega(X,JX) \geq \text{const.}\|X\|^2_g\) for any tangent vector \(X \in TW\). Suppose \(W\) is monotone, i.e. any sphere in \(W\) has vanishing symplectic area. Let \((M \times [S,0], d(e^s\alpha)) \hookrightarrow (W,\omega)\) be a symplectic embedding of a finite cylinder in the symplectization of a contact manifold into \(W\), such that \(\pi_1(M)\) injects into \(\pi_1(W)\). Suppose the image of the embedding can be displaced by a (time-dependent) Hamiltonian flow with compact support and oscillation

\[
\|H\| := \int_0^1 \left( \max_{x \in W} H(t,x) - \min_{x \in W} H(t,x) \right) dt.
\]

Then any closed Legendrian \(l \subset M\) admits a Reeb chord of length not bigger than \(\|H\|/(1 - e^S)\).

**Proof.** Assume that there is a Legendrian embedding into \(M\) which admits no Reeb chord of length not bigger than \(T > 0\). Consider a Lagrangian embedding as constructed in the previous section. It will be displaced from itself by \(H\). Fix any almost complex structure \(J\) as in the theorem. Due to monotonicity of \(W\) there are no \(J\)-holomorphic spheres in \(W\). Then an argument given by Chekanov in [5] produces a non–constant \(J\)-holomorphic disk in \(W\) with boundary on \(L\) and (symplectic) area smaller than or equal to \(\|H\|\). But due to the injectivity of \(\pi_1(M) \hookrightarrow \pi_1(W)\) there is another disk completely lying in \(M\). Since \(W\) is monotone the symplectic areas of both disks agree. The latter is an integer
multiple of $C > T(1 - e^S)$. Non-constant holomorphic disks have positive symplectic area, hence $T \leq \|H\|/(1 - e^S)$.

Remarks. (1) Arnold’s conjecture follows from this result since every contact one form corresponding to the standard contact structure can be realized as the restriction of the primitive $\omega(z, .)$ of the standard symplectic structure in $\mathbb{C}^n$ to the smooth boundary of a star-shaped domain.

(2) Biran and Cieliebak observed in [4] that any compact set in a (complete) subcritical Stein manifold can be displaced from itself by a Hamiltonian isotopy with compact support. In particular, one can apply this to the strongly pseudo-convex domain in the Stein manifold within its completion to obtain Theorem 2.

(3) One could prove the result using Gromov’s Fredholm alternative for maps of the disk into $W$ with boundary on a Lagrangian satisfying a $\overline{\partial}$-equation with a non-zero right-hand side (see [3] and [4]). It should be an amusing and (non-trivial) exercise to produce exactly the same estimate. For our convenience we chose to use Chekanov’s result but feel obliged to point out that the result is a consequence of Gromov’s original ideas.

Problem. We have not answered Arnold’s original question yet, whether there always exists a Reeb orbit intersecting the Legendrian submanifold at least twice. We proved that there is a chord whose length is estimated by $S$ and the displacement energy. But this chord could be a closed orbit which intersects the Legendrian once. Notice that this question is bound to the sphere. E.g. pick the contact form $x_1d\theta + \frac{1}{2}(x_2dx_3 - x_3dx_2)$ on $S^1 \times S^2$ with coordinate $\theta \in S^1$ and $S^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \|x\| = 1\}$. Then $S^1 \times \{0\}$ is a Legendrian loop which intersects each of the (closed) Reeb orbits $\{\theta\} \times \{x_1 = 0\}$ only once.

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