Abstract. In this paper we continue to explore the connection between tensor algebras and displacement structure. We focus on recursive orthonormalization and we develop an analogue of the Szegő type theory of orthogonal polynomials in the unit circle for several noncommuting variables. Thus, we obtain the recurrence equations and Christoffel-Darboux formulas for Szegő polynomials in several noncommuting variables, as well as a Favard type result. Also we continue to study a Szegő type kernel for the N-dimensional unit ball of an infinite dimensional Hilbert space.

Key words: Displacement structure, tensor algebras, Szegő polynomials

AMS subject classification: 15A69, 47A57

1. Introduction

In the first part of this paper, [5], we explored the connection between tensor algebras and displacement structure. The displacement structure theory was initiated in [13] as a recursive factorization theory for matrices whose implicit structure is encoded by a so-called displacement equation. This has been useful in several directions including constrained and unconstrained rational interpolation, maximum entropy, inverse scattering, $H^\infty$-control, signal detection, digital filter design, nonlinear Riccati equations, certain Fredholm and Wiener-Hopf equations, etc., see [14]. Aspects of the Szegő theory can be also revealed within the displacement structure theory. Our main goal is to develop an analogue for polynomials in several noncommuting variables of the Szegő theory of orthogonal polynomials on the unit circle. An analogue of the Szegő theory of orthogonal polynomials on the real line is being developed in the companion paper [6].

The paper is organized as follows. In Section 2 we review notation and several results from [5]. In this way, this paper can be read independently of [5]. In Section 3 we introduce orthogonal polynomials in several noncommuting variables associated to certain representations of the free semigroup and discuss their algebraic properties, mostly related to the recursions that they satisfy. In Section 4 we consider several positive definite kernels on the N-dimensional unit ball of an infinite dimensional Hilbert space. In particular, we prove a basic property of the Szegő type kernel studied in [5] by characterizing its Kolmogorov decomposition. In Section 5 we discuss the problem of recovering the representation from orthogonal polynomials and we prove a Favard type result. We plan a more detailed study of applications to multiscale systems in a sequel of this paper.
2. Preliminaries

We briefly review several constructions of the tensor algebra and introduce the necessary notation. We also review the connection with displacement structure theory as established in [7].

2.1. Tensor algebras. The tensor algebra over $\mathbb{C}^N$ is defined by the algebraic direct sum

$$\mathcal{T}_N = \bigoplus_{k\geq 0} (\mathbb{C}^N)^{\otimes k},$$

where $(\mathbb{C}^N)^{\otimes k}$ denotes the $k$-fold tensor product of $\mathbb{C}^N$ with itself. The addition is the componentwise addition and the multiplication is defined by juxtaposition:

$$(x \otimes y)_n = \sum_{k+l=n} x_k \otimes y_l.$$

If $\{e_1, \ldots, e_N\}$ is the standard basis of $\mathbb{C}^N$, then $\{e_{i_1} \otimes \ldots \otimes e_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq N\}$ is an orthonormal basis of $\mathcal{T}_N$. Let $\mathbb{F}_N^+$ be the unital free semigroup on $N$ generators $1, \ldots, N$ with lexicographic order $\prec$. The empty word is the identity element and the length of the word $\sigma$ is denoted by $|\sigma|$. The length of the empty word is 0. If $\sigma = i_1 \ldots i_k$ then we write $e_\sigma$ instead of $e_{i_1} \otimes \ldots \otimes e_{i_k}$, so that any element of $\mathcal{T}_N$ can be uniquely written in the form $x = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma e_\sigma$, where only finitely many of the complex numbers $c_\sigma$ are different from 0.

Another construction of $\mathcal{T}_N$ can be obtained as follows. Let $S$ be a unital semigroup and denote by $F_0(S)$ the set of functions $\phi : S \to \mathbb{C}$ with the property that $\phi(s) \neq 0$ for only finitely many values of $s$. This set has a natural vector space structure and $B_S = \{\delta_s \mid s \in S\}$ is a vector basis for $F_0(S)$, where $\delta_s$ is the Kronecker symbol associated to $s \in S$. Also $F_0(S)$ is a unital associative algebra with respect to the product

$$\phi * \psi = (\sum_{s \in S} \phi(s) \delta_s) * (\sum_{t \in S} \psi(t) \delta_t) = \sum_{s,t \in S} \phi(s) \psi(t) \delta_{st}. $$

It is readily seen that $F_0(\mathbb{F}_N^+)$ is isomorphic to $\mathcal{T}_N$. Since each element in $F_0(\mathbb{F}_N^+)$ can be uniquely written as a (finite) sum, $\phi = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma \delta_\sigma$, the isomorphism is the linear extension $\Phi_1$ of the mapping $\delta_\sigma \to e_\sigma$, $\sigma \in \mathbb{F}_N^+$.

Another copy of the tensor algebra is given by the algebra $\mathcal{P}_N^0$ of polynomials in $N$ noncommuting indeterminates $X_1, \ldots, X_N$ with complex coefficients. Each element $P \in \mathcal{P}_N^0$ can be uniquely written in the form $P = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma X_\sigma$, with $c_\sigma \neq 0$ for finitely many $\sigma$’s and $X_\sigma = X_{i_1} \ldots X_{i_k}$ where $\sigma = i_1 \ldots i_k \in \mathbb{F}_N^+$. The linear extension $\Phi_2$ of the mapping $\delta_\sigma \to X_\sigma$, $\sigma \in \mathbb{F}_N^+$, gives an isomorphism of $\mathcal{T}_N$ with $\mathcal{P}_N^0$.

Yet another copy of $\mathcal{T}_N$ inside the algebra of lower triangular operators allowed for the connection with displacement structure established in [7]. Thus, let $\mathcal{E}$ be a Hilbert
space and define: $E_0 = \mathcal{E}$ and for $k \geq 1$,
\begin{equation}
E_k = E_{k-1} \oplus \ldots \oplus E_{k-1}^{N\text{ terms}}.
\end{equation}

For $\mathcal{E} = \mathbb{C}$ we have that $\mathbb{C}_k$ can be identified with $(\mathbb{C}^N)^{\otimes k}$ and $T_N$ is isomorphic to the algebra $L_N^0$ of lower triangular operators $T = [T_{ij}] \in \mathcal{L}(\oplus_{k \geq 0} \mathbb{C}_k)$ with the property
\begin{equation}
T_{ij} = T_{i-1,j-1} \oplus \ldots \oplus T_{i,j-1} = T_{i-1,j-1}^{\oplus N\text{ terms}},
\end{equation}
for $i \leq j$, $i, j \geq 1$, and $T_{j0} = 0$ for all sufficiently large $j$'s.

2.2. Displacement structure. We can now describe the displacement structure of the tensor algebra. We write this connection for $L_N^0$. Then it can be easily translated into any other realization of the tensor algebra. Let $F_k = [T_k^{ij}] \in \mathcal{L}(\oplus_{k \geq 0} \mathbb{C}_k)$, $k = 1, \ldots, N$, be isometries defined by the formulae: $T_{ij} = 0$ for $i \neq j + 1$ and $T_{i,j+1}$ is a block-column matrix consisting of $N$ blocks of dimension $\dim \mathbb{C}_i$, all of them zero except for the $k$th block which is the identity on $\mathbb{C}_k$. We have the following result noticed in [7].

**Theorem 2.1.** Let $T \in L_N^0$ and define $A = I - TT^*$. Then
\begin{equation}
A - \sum_{k=1}^N F_k AF_k^* = GJ_{11}G^*,
\end{equation}
where
\[
G = \begin{bmatrix}
1 & T_{00} \\
0 & T_{01} \\
\vdots & \vdots
\end{bmatrix} \quad \text{and} \quad J_{11} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]

The model $L_N^0$ of the tensor algebra is also useful in order to extend this algebra to some topological tensor algebras - see [12]. Here we consider only the norm topology and denote by $L_N$ the algebra of all lower triangular operators $T = [T_{ij}] \in \mathcal{L}(\oplus_{k \geq 0} \mathbb{C}_k)$ satisfying (2.2).

2.3. Multiscale processes. Multiscale processes are stochastic processes indexed by nodes on a tree. They became quite popular lately, see [1], [2], and have potential to model the self-similarity of fractional Brownian motion leading to iterative algorithms in computer vision, remote sensing, etc.

Here we restrict our attention to the case of the Cayley tree, in which each node has $N$ branches. The vertices of the Cayley tree are indexed by $F_N^+$.
Let \((X, \mathcal{F}, P)\) be a probability space and let \(\{v_\sigma\}_{\sigma \in \mathbb{F}_N^+} \subset L^2(P)\) be a family of random variables. Its covariance kernel is
\[
K(\sigma, \tau) = \int_X v_\sigma v_\tau dP,
\]
and assume that the process is stationary in the sense (considered earlier, e.g. [10]) that:
\[
K(\tau \sigma, \tau' \sigma') = K(\sigma, \sigma'), \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+,
\]
(2.4)
\[
K(\sigma, \tau) = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \alpha \tau \text{ or } \tau = \alpha \sigma.
\]
(2.5)

Conversely, by the invariant Kolmogorov decomposition theorem, see e.g., [15], Ch. II, there exists an isometric representation \(u \in \mathbb{F}_N^+\) on a Hilbert space \(K\) and a mapping \(v : \mathbb{F}_N^+ \to K\) such that
\[
K(\sigma, \tau) = \langle v(\tau), v(\sigma) \rangle_K,
\]
\[
u(\tau) v(\sigma) = v(\tau \sigma),
\]
for all \(\sigma, \tau \in \mathbb{F}_N^+\), the set \(\{v(\sigma) \mid \sigma \in \mathbb{F}_N^+\}\) is total in \(K\), and \(u(1), \ldots, u(N)\) are isometries with orthogonal ranges.

This class of multiscale processes would be suitable to model branching processes without "past". If a "past" should be attached to a process as above, we could try to consider processes indexed by the nodes of the tree associated to the free group on \(N\) generators \(1, \ldots, N\). As mentioned in the introduction, we plan to look at this matter in a sequel of this paper. Here we focus on processes with covariance kernel satisfying (2.4) and (2.5). It was shown in [3] that such a kernel has displacement structure. Also, it is clear that for all \(j, k \geq 1\),
\[
[K(\sigma, \tau)]_{|\sigma| = j, |\tau| = k} = ([K(\sigma', \tau')]_{|\sigma'| = j-1, |\tau'| = k-1})^{\otimes N},
\]
(2.6)
so that the kernel is determined by the elements \(s_\sigma = K(\emptyset, \sigma), \sigma \in \mathbb{F}_N^+\).

By Theorem 1.5.3 in [3], each positive definite kernel \(K\) on \(\mathbb{F}_N^+\) is uniquely determined by a family of contractions \(\{\gamma_{\sigma, \tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau\}\) such that \(\gamma_{\sigma, \sigma} = 0\), \(\sigma \in \mathbb{F}_N^+\), and otherwise \(\gamma_{\sigma, \tau} \in \mathcal{L}(\mathcal{D}_{\gamma_{\sigma+1, \tau}}, \mathcal{D}_{\gamma_{\sigma, \tau-1}})\) (for a contraction \(T\) between two Hilbert spaces \(D_T = (I - T^*T)^{1/2}\) denotes the defect operator of \(T\) and \(D_T\) is the defect space of \(T\) defined as the closure of the range of \(D_T\) – note that in our case \(\gamma_{\sigma, \tau}\) are just complex numbers and the condition \(\gamma_{\sigma, \tau} \in \mathcal{L}(\mathcal{D}_{\gamma_{\sigma+1, \tau}}, \mathcal{D}_{\gamma_{\sigma, \tau-1}})\) for \(\sigma < \tau\) encodes the fact that \(|\gamma_{\sigma+1, \tau}| = 1 \text{ or } |\gamma_{\sigma, \tau-1}| = 1\) implies that \(\gamma_{\sigma, \tau} = 0\); also, \(\tau - 1\) denotes the predecessor of \(\tau\) with respect to the lexicographic order \(<\) on \(\mathbb{F}_N^+\), while \(\sigma + 1\) denotes the successor of \(\sigma\). In addition, the positive definite kernel \(K\) satisfies (2.4) and (2.5) if and only if
\[
\gamma_{\tau \sigma, \tau' \sigma'} = \gamma_{\sigma, \sigma'}, \quad \tau, \sigma, \sigma' \in \mathbb{F}_N^+,
\]
(2.7)
\[
\gamma_{\sigma, \tau} = 0 \quad \text{if there is no } \alpha \in \mathbb{F}_N^+ \text{ such that } \sigma = \alpha \tau \text{ or } \tau = \alpha \sigma.
\]
(2.8)
We define \( \gamma_\sigma = \gamma_{0,\sigma}, \sigma \in \mathbb{F}_N^+ \) and we notice that \( \{ \gamma_{\sigma,\tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau \} \) is uniquely determined by \( \{ \gamma_\sigma \}_{\sigma \in \mathbb{F}_N^+} \) by the formula
\[
[\gamma_{\sigma,\tau}]_{|\sigma|=j,|\tau|=k} = ([\gamma_{\sigma',\tau'}]_{|\sigma'|=j-1,|\tau'|=k-1})^{\oplus N}, \quad j, k \geq 1.
\]

3. Szegö Polynomials

We introduce polynomials in several noncommuting variables orthogonal with respect to a positive definite kernel \( K \) satisfying (2.4) and (2.5). We extend some elements of the Szegö theory to this setting.

The kernel \( K \) being given, we can introduce an inner product on \( F_0(\mathbb{F}_N^+) \) in the usual manner:
\[
\langle \phi, \psi \rangle_K = \sum_{\sigma, \tau \in \mathbb{F}_N^+} K(\sigma, \tau) \psi(\tau) \overline{\phi(\sigma)}.
\]
By factoring out the subspace \( \mathcal{N}_K = \{ \phi \in F_0(\mathbb{F}_N^+) \mid \langle \phi, \phi \rangle_K = 0 \} \) and completing with respect to the norm induced by (3.1) we obtain a Hilbert space denoted \( \mathcal{H}_K \). A similar structure can be introduced on \( \mathcal{P}_N^0 \). Let \( P = \sum_{\sigma \in \mathbb{F}_N^+} c_\sigma X_\sigma, Q = \sum_{\sigma \in \mathbb{F}_N^+} d_\sigma X_\sigma \) be elements in \( \mathcal{P}_N^0 \), then define:
\[
\langle P, Q \rangle_K = \sum_{\sigma, \tau \in \mathbb{F}_N^+} K(\sigma, \tau) d_\sigma \overline{c}_\tau.
\]
By factoring out the subspace \( \mathcal{M}_K = \{ P \in \mathcal{P}_N^0 \mid \langle P, P \rangle_K = 0 \} \) and completing with respect to the norm induced by (3.2) we obtain a Hilbert space denoted \( L^2(K) \). One can check that the map \( \Phi_2 \) defined by \( \delta_\sigma \rightarrow X_\sigma, \sigma \in \mathbb{F}_N^+ \), extends to a unitary operator from \( \mathcal{H}_K \) onto \( L^2(K) \).

From now on we assume that for any \( \alpha \in \mathbb{F}_N^+ \) the matrix \( [K(\sigma, \tau)]_{\sigma,\tau \leq \alpha} \) is invertible. This implies that \( \mathcal{M}_K = 0 \) and \( \mathcal{P}_N^0 \) can be viewed as a subspace of \( L^2(K) \). Also, for any \( \alpha \in \mathbb{F}_N^+, \{ X_\sigma \}_{\sigma \leq \alpha} \) is a linearly independent family in \( L^2(K) \). Then, the Gram-Schmidt procedure gives a family \( \{ \varphi_\sigma \}_{\sigma \in \mathbb{F}_N^+} \) of elements in \( \mathcal{P}_N^0 \) such that
\[
\varphi_\sigma = \sum_{\tau \leq \sigma} a_{\sigma,\tau} X_\tau, \quad a_{\sigma,\sigma} > 0;
\]
\[
\langle \varphi_\sigma, \varphi_\tau \rangle_K = 0, \quad \emptyset \leq \sigma < \tau.
\]
An explicit formula for the orthogonal polynomials \( \varphi_\sigma \) can be obtained in the same manner as in the classical (one variable) case. Define for \( \sigma \in \mathbb{F}_N^+ \),
\[
D_\sigma = \det [K(\sigma', \tau')]_{\sigma',\tau' \leq \sigma}
\]
and let \( \{ \gamma_\sigma \}_{\sigma \in \mathbb{F}_N^+} \) be the parameters associated to \( K \) as described in Section 2.2.3. Note that since all the matrices \( [K(\sigma, \tau)]_{\sigma,\tau \leq \alpha} \), \( \alpha \in \mathbb{F}_N^+ \), are assumed to be invertible, it follows that \( |\gamma_\sigma| < 1 \) for all \( \sigma \in \mathbb{F}_N^+ \).
Theorem 3.1. (1) $\varphi_\emptyset = 1$ and for $\emptyset < \sigma$,

$$\varphi_\sigma = \frac{1}{\sqrt{D_{\sigma-1}D_{\sigma}}} \det \begin{bmatrix} [K(\sigma', \tau')]_{\sigma' < \sigma; \tau' \leq \sigma} & \sum_{\tau < \sigma} a_{\sigma, \tau} X_\tau \end{bmatrix}.$$  

(3.6)

(2) For $\emptyset < \sigma = i_1 \ldots i_k$,

$$\varphi_\sigma = \frac{1}{\prod_{1 \leq j \leq k} (1 - |\gamma_{i_j \ldots i_k}|^2)^{1/2}} (X_\sigma + \text{lower order terms}).$$  

(3.7)

Proof. The proof is similar to the classical one. Thus, we deduce from the orthogonality condition (3.4) that $\langle \varphi_\sigma, X_\tau \rangle_K = 0$ for $\emptyset \leq \tau' \prec \sigma$, which implies that $\sum_{\tau \leq \sigma} a_{\sigma, \tau} K(\tau', \tau) = 0$ for $\emptyset \leq \tau' \prec \sigma$. Using the Cramer rules for the system

$$\begin{cases} \sum_{\tau \leq \sigma} a_{\sigma, \tau} K(\tau', \tau) = 0, & \emptyset \leq \tau' \prec \sigma, \\ \sum_{\tau \leq \sigma} a_{\sigma, \tau} X_\tau = \varphi_\sigma, \end{cases}$$

with unknowns $a_{\sigma, \tau}$, we deduce

$$a_{\sigma, \sigma} = \frac{\varphi_\sigma D_{\sigma-1}}{\det \begin{bmatrix} [K(\sigma', \tau')]_{\sigma' < \sigma; \tau' \leq \sigma} & \sum_{\tau < \sigma} a_{\sigma, \tau} X_\tau \end{bmatrix}}.$$

Therefore,

$$\varphi_\sigma = \frac{a_{\sigma, \sigma} D_{\sigma-1}}{D_{\sigma-1}} \det \begin{bmatrix} [K(\sigma', \tau')]_{\sigma' < \sigma; \tau' \leq \sigma} & \sum_{\tau < \sigma} a_{\sigma, \tau} X_\tau \end{bmatrix}.$$

We now compute $a_{\sigma, \sigma}$ and $D_{\sigma}$ in terms of the parameters $\{\gamma_\sigma\}_{\sigma \in \mathbb{N}}$ of $K$. First we notice that

$$\langle \det \begin{bmatrix} [K(\sigma', \tau')]_{\sigma' < \sigma; \tau' \leq \sigma} & \sum_{\tau < \sigma} a_{\sigma, \tau} X_\tau \end{bmatrix}, X_\sigma \rangle_K = D_{\sigma}$$

and since $X_\sigma = \frac{1}{a_{\sigma, \sigma}} \varphi_\sigma + \sum_{\tau < \sigma} c_\tau X_\tau$, we deduce

$$D_{\sigma} = \left(\frac{D_{\sigma-1}}{a_{\sigma, \sigma}}\right) \frac{1}{a_{\sigma, \sigma}} \varphi_\sigma + \sum_{\tau < \sigma} c_\tau X_\tau = \frac{D_{\sigma-1}}{a_{\sigma, \sigma}^2},$$

so that

$$\frac{1}{a_{\sigma, \sigma}^2} = \frac{D_{\sigma}}{D_{\sigma-1}}$$

which gives (3.6).

In order to compute $D_{\sigma}$ in terms of $\{\gamma_\sigma\}_{\sigma \in \mathbb{N}}$, we use Theorem 1.5.10 in [3] and the special structure of $D_{\sigma}$. Thus,

$$D_{\sigma} = \prod_{\emptyset < \sigma', \tau' \leq \sigma} (1 - |\gamma_{\sigma', \tau'}|^2)^{1/6}.$$
and for $\emptyset \prec \sigma = i_1 \ldots i_k$ we deduce
\[
\frac{1}{a_{\sigma, \sigma}^2} = \frac{D_\sigma}{D_{\sigma-1}} = \prod_{1 \leq j \leq k} (1 - |\gamma_{i_j \ldots i_k}|^2).
\]

Then,
\[
\varphi_\sigma = a_{\sigma, \sigma} X_\sigma + \sum_{\tau \prec \sigma} a_{\sigma, \sigma} c_\tau X_\tau
\]
\[
= \prod_{1 \leq j \leq k} \left(1 - \frac{1}{\Pi_{1 \leq j \leq k}(1 - |\gamma_{i_j \ldots i_k}|^2)^{1/2}} (X_\sigma + \text{lower order terms})\right),
\]
which gives (3.7).

We illustrate this result for $N = 2$. From now on it is convenient to use the notation
\[
d_{\sigma} = (1 - |\gamma_{\sigma}|^2)^{1/2}, \quad \sigma \in \mathbb{F}_N^+ - \{\emptyset\}.
\]

**Example.** Let $N = 2$ and assume the positive kernel $K$ satisfies the conditions in Theorem 3.1. We have $D_\emptyset = 1$ and the next three determinants are:
\[
D_1 = \det \begin{bmatrix} 1 & s_1 \\ \overline{s}_1 & 1 \end{bmatrix} = d_1^2;
\]
\[
D_2 = \det \begin{bmatrix} 1 & s_1 & s_2 \\ \overline{s}_1 & 1 & 0 \\ \overline{s}_2 & 0 & 1 \end{bmatrix} = d_1^2 d_2^2;
\]
\[
D_{11} = \det \begin{bmatrix} 1 & s_1 & s_2 & s_{11} \\ \overline{s}_1 & 1 & 0 & s_1 \\ \overline{s}_2 & 0 & 1 & 0 \\ \overline{s}_{11} & \overline{s}_1 & 0 & 1 \end{bmatrix} = d_1^4 d_2^2 d_{11}^2.
\]

Using Theorem 3.1 we can easily calculate the first four orthogonal polynomials of $K$. Thus, $\varphi_\emptyset = 1$ and then
\[
\varphi_1 = \frac{1}{d_1} \det \begin{bmatrix} 1 & s_1 \\ \overline{s}_1 & 1 \end{bmatrix} = -\frac{\gamma_1}{d_1} + \frac{1}{d_1} X_1;
\]
\[
\varphi_2 = \frac{1}{d_1^2 d_2} \det \begin{bmatrix} 1 & s_1 & s_2 \\ \overline{s}_1 & 1 & 0 \\ 1 & X_1 & X_2 \end{bmatrix} = -\frac{\gamma_2}{d_1 d_2} + \frac{\overline{\gamma}_1 \gamma_2}{d_1 d_2} X_1 + \frac{1}{d_2} X_2,
\]
where we used the fact that $s_2 = d_1 \gamma_2$. Then, after some calculations,
\[
\varphi_{11} = \frac{1}{d_1^3 d_2^2 d_{11}} \det \begin{bmatrix} 1 & s_1 & s_2 & s_{11} \\ \overline{s}_1 & 1 & 0 & s_1 \\ \overline{s}_2 & 0 & 1 & 0 \\ 1 & X_1 & X_2 & X_{1}^2 \end{bmatrix}
\]
\[
= -\frac{\gamma_{11}}{d_1 d_2 d_{11}} + \left(\frac{\gamma_1}{d_1 d_{11}} + \frac{\gamma_{11} \gamma_1}{d_1 d_2 d_{11}}\right) X_1 + \frac{\gamma_{11} \gamma_2}{d_2 d_{11}} X_2 + \frac{1}{d_{11} d_1} X_{1}^2.
\]

We establish that the orthogonal polynomials introduced above satisfy equations similar to the classical Szegö difference equations.
Theorem 3.2. The orthogonal polynomials satisfy the following recurrences: $\varphi_\emptyset = 1$ and for $k \in \{1, \ldots, N\}$, $\sigma \in \mathbb{F}_N^+$,

\begin{equation}
\varphi_{k\sigma} = \frac{1}{d_{k\sigma}}(X_k \varphi_\sigma - \gamma_k \varphi_{k\sigma-1}),
\end{equation}

where $\varphi_\emptyset = 1$ and for $k \in \{1, \ldots, N\}$, $\sigma \in \mathbb{F}_N^+$,

\begin{equation}
\varphi^\sharp_{k\sigma} = \frac{1}{d_{k\sigma}}(-\gamma_k X_k \varphi_\sigma + \varphi^\sharp_{k\sigma-1}).
\end{equation}

Proof. We deduce this result from similar formulae obtained for an arbitrary positive definite kernel. In this way we can show the meaning of the polynomials $\varphi^\sharp$, $\sigma \in \mathbb{F}_N^+$. Let $[t_{i,j}]_{i,j \geq 1}$ be a positive definite kernel on $\mathbb{N}$ and assume that each matrix $A^{(i,j)} = [t_{i,j}]_{1 \leq i \leq j}$, $1 \leq i \leq j$, is invertible. Also, assume $t_{k,k} = 1$ for all $k \geq 1$. Let $F_{i,j}$ be the upper Cholesky factor of $A^{(i,j)}$, so that $F_{i,j}$ is an upper triangular matrix with positive diagonal and $A^{(i,j)} = F^{*}_{i,j} F_{i,j}$. A dual, lower Cholesky factor is obtained as follows: define the symmetry of appropriate dimension,

\[ J = \begin{bmatrix}
0 & 0 & \cdots & 0 & I \\
0 & 0 & I & 0 \\
\vdots & \cdots \\
0 & I & 0 \\
I & 0 & 0 & 0
\end{bmatrix} \]

and then let $\tilde{F}_{i,j}$ denote the upper Cholesky factor of $B^{(i,j)} = JA^{(i,j)}J$. If $G_{i,j} = J\tilde{F}_{i,j}J$, then

\[ A^{(i,j)} = JB^{(i,j)}J = JF^{*}_{i,j} F_{i,j} J = G^{*}_{i,j} G_{i,j}, \]

and $G_{i,j}$ is a lower triangular matrix with positive diagonal, called the lower Cholesky factor of $A^{(i,j)}$. Let $P_{i,j}$ be the last column of $F_{i,j}^{-1}$ and let $P_{i,j}^\sharp$ be the first column of $G_{i,j}^{-1}$, that is

\[ P_{i,j} = F_{i,j}^{-1} E, \quad P_{i,j}^\sharp = G_{i,j}^{-1} J E, \]

where $E = [0 \ldots 0 I]^T$. Let $\{r_{i,j}\}_{1 \leq i \leq j}$ be the parameters associated to $[t_{i,j}]_{i,j \geq 1}$ by Theorem 1.5.3 in [3] and let $\rho_{i,j} = (1 - |r_{i,j}|^2)^{1/2}$. We have that

\begin{equation}
P_{1,n} = \frac{1}{d_{1,n}} \begin{bmatrix} 0 \\ P_{2,n} \end{bmatrix} - \frac{r_{1,n}}{d_{1,n}} \begin{bmatrix} P_{1,n}^\sharp \\ 0 \end{bmatrix},
\end{equation}

\begin{equation}
P_{1,n}^\sharp = -\frac{r_{1,n}}{d_{1,n}} \begin{bmatrix} 0 \\ P_{2,n} \end{bmatrix} + \frac{1}{d_{1,n}} \begin{bmatrix} P_{1,n}^\sharp \\ 0 \end{bmatrix}.
\end{equation}

These formulae are presumable known to the experts. For the sake of completeness we give a proof here based on results and notation from [3]. First we introduce the following elements. For $i < j$,

\begin{equation}
L^{(j)}_i = L(\{r_{i,k}\}_{k=i+1}^j) = [r_{i,i+1} \rho_{i,i+1} r_{i,i+1} \ldots \rho_{i,i+1} \ldots \rho_{i,j-1} r_{i,j}].
\end{equation}
\[ C_j^{(i)} = \begin{bmatrix} r_{j-1,j} \\ \vdots \\ r_{i+1,j} \rho_{i+2,j} \cdots \rho_{j-1,j} \\ r_{i,j} \rho_{i+1,j} \cdots \rho_{j-1,j} \end{bmatrix}, \]

and

\[ K_i^{(j)} = \begin{bmatrix} r_{i,i+1} \rho_{i,i+2} \cdots \rho_{i,j} \\ \vdots \\ r_{i,j} \rho_{i,j+1} \cdots \rho_{i,j} \end{bmatrix} = \begin{bmatrix} K_i^{(j-1)} \rho_{i,j} \\ r_{i,j} \end{bmatrix}. \]

Also, we define inductively: \( D_i^{(i+1)} = \rho_{i,i+1} \).

\[ D_i^{(j)} = D \left( \{ r_{i,k} \}_{k=i+1}^j \right) = \begin{bmatrix} D_i^{(j-1)} & -K_i^{(j-1)} r_{i,j} \\ 0 & \rho_{i,j} \end{bmatrix}. \] 

We also need to review the factorization of unitary matrices. This is an extension of Euler’s description of \( SO(3) \). First we define

\[ R_{j-i}(r_{i,k}) = I_{k-1-i} \oplus \begin{bmatrix} r_{i,k} & \rho_{i,k} \\ \rho_{i,k} & -r_{i,k} \end{bmatrix} \oplus I_{j-k-1}, \]

where \( I_{k-1-i} \) is the identity matrix of size \( k - 1 - i \). Then,

\[ R_{i,j} = R_{j-i}(r_{i,i+1}) \cdots R_{j-i}(r_{i,j}), \]

and

\[ U_{i,j} = R_{i,j}(U_{i+1,j} \oplus 1). \]

It turns out that any unitary matrix can be written as a matrix of the form of \( U_{i,j} \). The main idea for the proof of (3.10) is to use the identity

\[ U_{i,j} J G_{i,j} = F_{i,j}, \]

which follows from the relations (1.6.10), (6.3.8) and (6.3.9) in [3]. Thus, we notice that (3.14) implies

\[ P_{i,j}^z = F_{i,j} U_{i,j} E, \]

which is more tractable than the original definition of \( P_{i,j}^z \). This is seen from the following calculations. Using formula (1.5.7) in [3], the above definition of \( D_1^{(n)} \), and
the notation $D_1^{-(n)} = (D_1^{(n)})^{-1}$, we obtain that

$$P_{1,n} = \begin{bmatrix} 1 & -L_1^{(n)} D_1^{-(n)} \\ 0 & F_{2,n}^{-1} D_1^{-(n)} \end{bmatrix} \quad E = \begin{bmatrix} -L_1^{(n)} D_1^{-(n)} E \\ F_{2,n}^{-1} D_1^{-(n)} E \end{bmatrix}$$

$$= \begin{bmatrix} -L_1^{(n)} \left[ D_1^{-(n-1)} \frac{r_{1,n}}{\rho_{1,n}} D_1^{-(n-1)} K_1^{(n-1)} \right] \\ 0 \right] E \end{bmatrix}$$

$$= \begin{bmatrix} -L_1^{(n)} \left[ \frac{r_{1,n}}{\rho_{1,n}} D_1^{-(n-1)} K_1^{(n-1)} \right] \\ 0 \right] E \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\rho_{1,n}} \left[ \frac{1}{\rho_{1,n}} \right] \right]$$

The proof of formula (1.6.15) in [3] gives

$$L_1^{(n-1)} D_1^{-(n-1)} K_1^{(n-1)} + \rho_{1,2} \cdots \rho_{1,n-1} = \frac{1}{\rho_{1,2} \cdots \rho_{1,n-1}}$$

and using formula (1.5.6) in [3] we deduce

$$F_{2,n}^{-1} \begin{bmatrix} D_1^{-(n-1)} K_1^{(n-1)} \\ 0 \end{bmatrix} = \begin{bmatrix} F_{2,n-1}^{-1} D_1^{-(n-1)} K_1^{(n-1)} \end{bmatrix}.$$
therefore
\[ P_{1,n} = \frac{1}{\rho_{1,n}} \begin{pmatrix} 0 & 1 \\ P_{2,n} & 0 \end{pmatrix} - \frac{r_{1,n}}{\rho_{1,n}} \begin{pmatrix} 1 \\ -F_{2,n-1}^{-1}D_{1}^{-(n-1)}K_{1}^{(n-1)} \end{pmatrix}. \]

It remains to show that
\[ P_{1,n-1}^{\sharp} = \begin{pmatrix} 1 \\ -F_{2,n-1}^{-1}D_{1}^{-(n-1)}K_{1}^{(n-1)} \end{pmatrix}. \]

To that end we notice that using formula (1.5.8) in [3], the definition of \( U_{1,n-1} \), the fact that \( R_{1,n-1} \) is a unitary matrix, and the notation \( L_{1}^{*(n-1)} = (L_{1}^{(n-1)})^{*} \), we obtain
\[
P_{1,n-1}^{\sharp} = \begin{pmatrix} 0 & 1 \\ F_{2,n-1}^{-1} & -\frac{1}{\rho_{1,2}\cdots\rho_{1,n-1}}F_{2,n-1}^{-1}L_{1}^{-(n-1)} \end{pmatrix} R_{1,n-1}^{*}R_{1,n-1} \begin{pmatrix} U_{2,n-1} & 0 \\ 0 & 1 \end{pmatrix} E
\]
\[ = \begin{pmatrix} 1 \\ -\frac{1}{\rho_{1,2}\cdots\rho_{1,n-1}}F_{2,n-1}^{-1}L_{1}^{*(n-1)} \end{pmatrix}. \]

It follows that all we have to show is the equality
\[ F_{2,n-1}^{-1}D_{1}^{-(n-1)}K_{1}^{(n-1)} = \frac{1}{\rho_{1,2}\cdots\rho_{1,n-1}}F_{2,n-1}^{-1}L_{1}^{*(n-1)}. \]

Now this is a simple consequence of the formula \( T^{*}D_{T^{*}} = D_{T}T^{*} \) for the contraction \( T = L_{1}^{(n-1)} \). The formula (3.11) can be proved in a similar manner.

We rewrite (3.10) and (3.11) for a positive definite kernel \( K \) satisfying (2.4) and (2.5). We notice that \( P_{1,n-1}^{\sharp} \) is replaced by \( \varphi_{k_{\sigma}-1}^{\sharp} \) and then we have to show that \( P_{2,n} \) can be expressed in terms of \( \varphi_{\alpha} \). This follows by taking into account the relations (2.7) and (2.8) and using systematically (3.10). We can omit the details.

The previous recurrence equations look quite similar to the classical Szegö recursions, still they have a new component that is best understood when considering (3.10) and (3.11). This type of recurrence equations was also found in [4] in connection with some derivations on \( L_{N} \). It turns out that these derivations are related to those considered in [11], and later studied in [9].

We also notice a graded form of the recurrences (3.8) and (3.9). It is convenient to introduce the following notation: for \( n \geq 1 \), we use (3.12) and define
\[ g_{n} = L(\{\gamma_{\sigma}\}_{\sigma}|=n). \]
It was explained in [5] that \( g_n, \ n \geq 1, \) are the parameters associated to the kernel \( K \) in [10]. We also use (3.13) in order to introduce the notation
\[
H_n = D(\{\gamma_\sigma\}_{|\sigma|=n}), \quad n \geq 1.
\]

Let \( \sigma(n) \) be the largest word (with respect to the lexicographic order) of length \( n \), that is \( \sigma(n) = \underbrace{N \ldots N}_{N \text{ terms}} \).

**Corollary 3.3.** The Szegö polynomials satisfy the recurrences: for \( k \geq 1 \),
\[
(3.15) \quad \left[ \varphi_{|\sigma|=k} \right] = \left( \begin{bmatrix} X_1 & \cdots & X_N \end{bmatrix} \varphi_{|\sigma|=k-1}^{\oplus N} - \varphi_{\sigma(k)-1}^{\oplus N} g_k \right) H_k^{-1};
\]
\[
(3.16) \quad \varphi_{\sigma(k)}^{\sharp} = \prod_{|\tau|=k} d_{\tau}^{-1} \left( - \left[ \begin{bmatrix} X_1 & \cdots & X_N \end{bmatrix} \varphi_{|\sigma|=k-1}^{\oplus N} g_k^* + \varphi_{\sigma(k)-1}^{\oplus N} \right] \right).
\]

**Proof.** Both (3.15) and (3.16) follow by direct calculations from Theorem 3.2. \qed

## 4. Christoffel-Darboux Formula

A first consequence of the Szegö formula in the classical case is the Christoffel-Darboux formula. Here we find a similar formula in several noncommuting variables. To that end we introduce additional notation. Let \( E \) be a Hilbert space. In this paper \( E \) will always be infinite dimensional. The \( N \)-dimensional unit ball of \( E \) is defined by
\[
\mathcal{B}_N(E) = \{ Z = (Z_1 \ldots Z_N) \mid (Z \mid Z) < I_E \},
\]
where for two elements \( Z = (Z_1 \ldots Z_N) \) and \( W = (W_1 \ldots W_N) \) in \( \mathcal{L}(E)^N \) we define
\[
(4.1) \quad (Z \mid W) = \sum_{k=1}^{N} Z_k W_k^*.
\]

We also need a sort of Szegö kernel for \( \mathcal{B}_N(E) \). One suggestion was given in [4] to consider the following construction. For \( Z \in \mathcal{B}_N(E) \) define
\[
(4.2) \quad E(Z) = [Z_{|\sigma|=0}]^{\infty} \in \mathcal{L}(\oplus_{k \geq 0} E_k, E).
\]

Also we use the notation \( \text{diag}(S) \) to denote the diagonal operator in \( \mathcal{L}(\oplus_{k \geq 0} E_k) \) with diagonal \( S \). A Szegö type kernel on \( \mathcal{B}_N(E) \) is given by the formula
\[
K_S(Z, W) = E(Z) E(W)^*, \quad Z, W \in \mathcal{B}_N(E).
\]

The next result explain two important properties of \( K \).

**Lemma 4.1.** (a) For any \( T \in \mathcal{L}(E) \) and \( Z, W \in \mathcal{B}_N(E) \),
\[
E(Z) \text{diag}(T - \sum_{k=1}^{N} Z_k T W_k^*) E(W)^* = T.
\]

(b) The set \( \{E(W)^* E \mid W \in \mathcal{B}_N(E)\} \) is total in \( \oplus_{k \geq 0} E_k \).
Proof. (a) Using directly the definitions,
\[
E(Z) \text{diag}(T - \sum_{k=1}^{N} Z_k T W_k^*) E(W)^*
\]
\[
= T + \sum_{|\sigma|\geq 1} Z_\sigma T W_\sigma^* - \sum_{k=1}^{N} E(Z)) \text{diag}(Z_k T W_k^*) E(W)^*
\]
\[
= T + \sum_{|\sigma|\geq 1} Z_\sigma T W_\sigma^* - \sum_{k=1}^{N} \sum_{|\sigma|\geq 0} Z_\sigma Z_k T W_k^* W_\sigma^*
\]
\[
= T + \sum_{|\sigma|\geq 1} Z_\sigma T W_\sigma^* - \sum_{|\sigma|\geq 1} Z_\sigma T W_\sigma^*
\]
\[
= T.
\]

(b) Let \( e = \{e_\sigma\}_{\sigma \in \mathbb{F}_N^+} \) be an element of \( \bigoplus_{k \geq 0} \mathcal{E}_k \) orthogonal to the linear span of \( \{E(W)^* \mathcal{E} \mid W \in \mathcal{B}_N(\mathcal{E})\} \). Taking \( W = 0 \), we deduce that \( e_\emptyset = 0 \). Next, we claim that for each \( \sigma \in \mathbb{F}_N^+ - \{\emptyset\} \) there exist
\[
W_l = (W_1^l, \ldots, W_N^l) \in \mathcal{B}_N(\mathcal{E}), \quad l = 1, \ldots, 2|\sigma|,
\]
such that
\[
\text{range} \left[ \begin{array}{ccc}
W_{\sigma 1}^* & \cdots & W_{\sigma 2|\sigma|}^*
\end{array} \right] = \mathcal{E},
\]
and
\[
W_{\tau}^l = 0 \quad \text{for all} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|, \quad l = 1, \ldots, 2|\sigma|.
\]

Once this claim is proved, a simple inductive argument gives \( e = 0 \), so \( \{E(W)^* \mathcal{E} \mid W \in \mathcal{B}_N(\mathcal{E})\} \) is total in \( \bigoplus_{k \geq 0} \mathcal{E}_k \). Therefore we focus on the proof of the claim.

Let \( \{e_{ij}^n\}_{i,j=1}^n \) be the matrix units of the algebra \( M_n \) of \( n \times n \) matrices. Each \( e_{ij}^n \) is an \( n \times n \) matrix consisting of 1 in the \((i,j)\)th entry and zeros elsewhere. For a Hilbert space \( \mathcal{E}_1 \) we define \( E_{ij}^n = e_{ij}^n \otimes I_{\mathcal{E}_1} \) and we notice that
\[
E_{ij}^n E_{kl}^m = \delta_{jk} E_{il}^m, \quad E_{ji}^{*m} = E_{ij}^m.
\]

Let \( \mathcal{E} \) be infinite dimensional, so that \( \mathcal{E} = \mathcal{E}_1^{\otimes 2|\sigma|} \) for some Hilbert space \( \mathcal{E}_1 \). Let \( \sigma = i_1 \ldots i_k \). For \( s = 1, \ldots, N \), we define
\[
J_s = \{l \in \{1, \ldots, k\} \mid i_{k+1-l} = s\}
\]
and
\[
W_{s}^{*p} = \frac{1}{\sqrt{2}} \sum_{r \in J_s} E_{r+p-1,r+p}^{2|\sigma|}, \quad s = 1, \ldots, N, \quad p = 1, \ldots, |\sigma|.
\]

We show that for each \( p \in \{1, \ldots, |\sigma|\} \),
\[
W_{\sigma}^{*p} = \frac{1}{\sqrt{2}} E_{p,k+p}^{2|\sigma|},
\]
\[
W_{\tau}^p = 0 \quad \text{for} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|.
\]
Using (4.3), we deduce
\[
\sum_{s=1}^{N} W_{s}^{p} W_{s}^{p} = \frac{1}{2} \sum_{s=1}^{N} \sum_{r \in J_{s}} E_{r+p,r+p-1}^{2|\sigma|} E_{r+p-1,r+p}^{2|\sigma|}
\]
\[
= \frac{1}{2} \sum_{s=1}^{N} \sum_{r \in J_{s}} E_{r+p,r+p}^{2|\sigma|}
\]
\[
= \frac{1}{2} \sum_{r=1}^{k} E_{r+p,r+p}^{2|\sigma|} < I,
\]
hence \(W_{p} \in B_{N}(E)\) for each \(p = 1, \ldots, |\sigma|\). For each word \(\tau = j_{1} \ldots j_{k} \in \mathbb{F}_{N}^{+} - \{\emptyset\}\) we deduce by induction that
\[
W_{j_{k}}^{p} \ldots W_{j_{1}}^{p} = \frac{1}{\sqrt{2^{k}}} \sum_{r \in A_{r}} E_{r+p-1,r+p+k-1}^{2|\sigma|}, \tag{4.6}
\]
where \(A_{r} = \cap_{p=0}^{k-1}(J_{j_{k-p}} - p) \subset \{1, \ldots, N\}\) and \(J_{j_{k-p}} - p = \{l - p \mid l \in J_{j_{k-p}}\}\).

We show that \(A_{\sigma} = \{1\}\) and \(A_{\tau} = \emptyset\) for \(\tau \neq \sigma\). Let \(q \in A_{\tau}\). Therefore, for any \(p \in \{0, \ldots, k-1\}\) we must have \(q + p \in J_{j_{k-p}}\) or \(i_{k+1-q-p} = j_{k-p}\). For \(p = k-1\) we deduce \(j_{1} = i_{2-q}\) and since \(2 - q \geq 1\), it follows that \(q \leq 1\). Also \(q \geq 1\), therefore the only element that can be in \(A_{\sigma}\) is \(q = 1\), in which case we must have \(\tau = \sigma\). Since \(l \in J_{i_{k+1-l}}\) for each \(l = 1, \ldots, k-1\), we deduce that \(A_{\sigma} = \{1\}\) and \(A_{\tau} = \emptyset\) for \(\tau \neq \sigma\). Formula (4.6) implies (1.4). In a similar manner we can construct a family \(W_{p}, p = |\sigma| + 1, \ldots, 2|\sigma|\), such that
\[
W_{\sigma}^{p} = \frac{1}{\sqrt{2^{k}}} E_{p+k,p},
\]
and
\[
W_{\tau}^{p} = 0 \quad \text{for} \quad \tau \neq \sigma, \quad |\tau| \geq |\sigma|.
\]
Thus, for \(s = 1, \ldots, N\), we define
\[
K_{s} = \{l \in \{1, \ldots, k\} \mid i_{k} = s\}
\]
and
\[
W_{s}^{p} = \frac{1}{\sqrt{2}} \sum_{r \in K_{s}} E_{r+p-k,r+p-k-1}^{2|\sigma|}, \quad s = 1, \ldots, N, \quad p = |\sigma| + 1, \ldots, 2|\sigma|.
\]

Now,
\[
\begin{bmatrix}
W_{\sigma}^{1} & \ldots & W_{\sigma}^{2|\sigma|}
\end{bmatrix}
= \frac{1}{\sqrt{2^{k}}} \begin{bmatrix}
E_{1,k+1}^{2|\sigma|} & \ldots & E_{k,2k}^{2|\sigma|}
\end{bmatrix},
\]
whose range is \(E\). This concludes the proof. \(\Box\)

We note that the result given by Lemma 1.1(b) is not true in case \(E\) is finite dimensional. The meaning of the result is that in case \(E\) is infinite dimensional then \(E\) is really the Kolmogorov decomposition of the kernel \(K_{s}\).
We now let a polynomial \( P = \sum_{\sigma \in \mathbb{F}_N^+} c_{\sigma} X_{\sigma} \in \mathcal{P}^0_N \) take values on \( \mathcal{B}_N(\mathcal{E}) \) by the formula
\[
P(Z) = \sum_{\sigma \in \mathbb{F}_N^+} c_{\sigma} Z_{\sigma}, \quad Z \in \mathcal{B}_N(\mathcal{E}).
\]
(4.7)

Define the Cristoffel-Darboux kernel by the formula
\[
K_{CD}(Z, W) = E(Z)\text{diag}(\varphi_{\sigma(n)}^\sharp(Z)\varphi_{\sigma(n)}^\sharp(W)^*) - \sum_{|\tau|=n} \varphi_{\tau}(Z)\varphi_{\tau}(W)^*)E(W)^*,
\]
(4.8)
for \( Z, W \in \mathcal{B}_N(\mathcal{E}) \).

**Theorem 4.2.** For any \( Z, W \in \mathcal{B}_N(\mathcal{E}) \),
\[
K_{CD}(Z, W) = \sum_{0 \leq |\tau| < n} \varphi_{\tau}(Z)\varphi_{\tau}(W)^*.
\]

**Proof.** From (3.8) and (3.9) we deduce
\[
\varphi_{\sigma(k)}^\sharp(Z)\varphi_{\sigma(k)}^\sharp(W)^* - \varphi_{\sigma(k)}(Z)\varphi_{\sigma(k)}(W)^* = \varphi_{\sigma(k)}^\sharp(Z)\varphi_{\sigma(k)}^\sharp(W)^* - Z_k \varphi_{\sigma(k)}(Z)\varphi_{\sigma(k)}(W)^*W_k^*,
\]
for any \( k \in \{1, \ldots, N\}, \sigma \in \mathbb{F}_N^+ \), and \( Z, W \in \mathcal{B}_N(\mathcal{E}) \). Adding all these relations for \( k \in \{1, \ldots, N\} \) and \( 0 \leq |\sigma| \leq n-1 \), we deduce
\[
\varphi_{\sigma(n)}^\sharp(Z)\varphi_{\sigma(n)}^\sharp(W)^* - \sum_{0 \leq |\sigma| \leq n} \varphi_{\sigma}(Z)\varphi_{\sigma}(W)^* = \sum_{k=1}^{N} \sum_{0 \leq |\sigma| \leq n-1} Z_k \varphi_{\sigma}(Z)\varphi_{\sigma}(W)^*W_k^*.
\]
This relation and Lemma 4.1 give
\[
K_{CD}(Z, W)
\]
\[
= E(Z)(\sum_{0 \leq |\sigma| < n} \varphi_{\sigma}(Z)\varphi_{\sigma}(W)^*) - \sum_{k=1}^{N} Z_k(\sum_{0 \leq |\sigma| < n} \varphi_{\sigma}(Z)\varphi_{\sigma}(W)^*)W_k^*)E(W)^*
\]
\[
= \sum_{0 \leq |\tau| < n} \varphi_{\tau}(Z)\varphi_{\tau}(W)^*.
\]

\[\square\]

We can show one more application of Lemma 4.1. For a formal power series
\[
f = \sum_{\sigma \in \mathbb{F}_N^+} c_{\sigma} X_{\sigma},
\]
in \( N \) noncommuting variables \( X_1, \ldots, X_N \), we denote by \( T_f \) the lower triangular infinite matrix associated to \( f \) as described in Section 2.1. We denote by \( \mathcal{S}_N \) the Schur class of those formal power series \( f \) with the property that \( T_f \) is a contraction in \( \mathcal{L}(\oplus_{k \geq 0} C_k) \). If \( \mathcal{E} \) is an infinite dimensional Hilbert space then we can define \( f(Z) \) for \( Z \in \mathcal{B}_N(\mathcal{E}) \) as in (1), by the formula
\[
f(Z) = E(Z)(T_f \otimes I_\mathcal{E})/\mathcal{E}.
\]
(4.9)
We notice that this definition is consistent with (1,7). We extend a familiar characterization of the Schur class to the setting of this paper.
Theorem 4.3. The formal power series $f$ belongs to $S_N$ if and only if
$$C_f(Z, W) = E(Z) \text{diag}(I - f(Z)f(W)^*)E(W)^*, \quad Z, W \in B_N(E),$$
is a positive definite kernel on $B_N(E)$.

Proof. Using Lemma 3.1 in [4] we deduce that for $Z, W \in B_N(E),
$$E(Z)(I - (T_f \otimes I_E)(T_f \otimes I_E)^*)E(W)^*
= E(Z)E(W)^* - E(Z)(T_f \otimes I_E)(T_f \otimes I_E)^*)E(W)^*
= E(Z)E(W)^* - E(Z)\text{diag}(f(Z))\text{diag}(f(W)^*)E(W)^*
= E(Z)\text{diag}(I - f(Z)f(W)^*)E(W)^* = C_f(Z, W).$$
This relation implies that if $f \in S_N$ then $C_f$ is a positive definite kernel on $B_N(E)$. For the converse implication we have to use in addition Lemma 4.1. \qed

5. INVERSE PROBLEMS

In this brief section we prove a Favard type result for orthogonal polynomials in several noncommuting variables.

Theorem 5.1. Let $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}^+_N}$ be a family of complex numbers with $\gamma_\emptyset = 0$ and $|\gamma_\sigma| < 1$ for $\sigma \in \mathbb{F}_N^+ - \{\emptyset\}$. Then there exists a unique positive definite kernel $K$ satisfying (2.4) and (2.5) such that the polynomials $\varphi_\sigma, \sigma \in \mathbb{F}_N^+$, defined by the recursions:
$$\varphi_\emptyset = 1,$$
$$\varphi^*_\emptyset = 1,$$
$$\varphi_k\sigma = \frac{1}{d_k\sigma}(X_k\varphi_\sigma - \gamma_k\varphi^*_k\sigma_{-1}),$$
$$\varphi^*_k\sigma = \frac{1}{d_k\sigma}(-\gamma_k X_k \varphi_\sigma + \varphi^*_k\sigma_{-1}),$$
are orthogonal with respect to $K$.

Proof. Once again we rely on some results that are known for arbitrary positive definite kernels on the set of integers. In this way, the proof is quite straightforward. Let $\{\gamma_{\sigma, \tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau\}$ be the family of complex numbers associated to $\{\gamma_\sigma\}_{\sigma \in \mathbb{F}_N^+}$ by (2.9). Let $K$ be the positive definite kernel associated to $\{\gamma_{\sigma, \tau} \mid \sigma, \tau \in \mathbb{F}_N^+, \sigma \leq \tau\}$ by Theorem 1.5.3 in [3]. By Theorem 3.2, the polynomials $\varphi_\sigma, \sigma \in \mathbb{F}_N^+$, defined by the recursions:
$$\varphi_\emptyset = 1,$$
$$\varphi^*_\emptyset = 1,$$
$$\varphi_k\sigma = \frac{1}{d_k\sigma}(X_k\varphi_\sigma - \gamma_k\varphi^*_k\sigma_{-1}),$$
$$\varphi^*_k\sigma = \frac{1}{d_k\sigma}(-\gamma_k X_k \varphi_\sigma + \varphi^*_k\sigma_{-1}),$$
must be the orthogonal polynomials of $K$. \qed
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