New Modified Adomin Decomposition Method for Boundary Value Problems of Higher-order Ordinary Differential Equation

Zainab Ali Abdu Al-Rabahi$^1$ and Yahya Qaid Hasan$^{1*}$

$^1$Department of Mathematics, Faculty of Education, Art and Science, Sheba Region University, Yemen.

Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2020/v16i330178

Editor(s): (1) Prof. Wei-Shih Du, National Kaohsiung Normal University, Taiwan.

Reviewers: (1) Francisco Bulnes, Tecnológico de Estudios Superiores de Chalco, Mexico.
             (2) Emad A. Az-Zo’bi, Mutah University, Jordan.

Complete Peer review History: http://www.sdiarticle4.com/review-history/54751

Received: 12 December 2019
Accepted: 18 February 2020
Published: 11 March 2020

Original Research Article

Abstract

This study will present a new modified differential operator for solving third-order boundary value problems into higher-order ordinary differential equation. We found the differential operator for new three inverse operator which can be applied for solving equations at more than one type in different conditions. We put a detailed plan for five non-linear examples from a high-order, we get dynamic and quickly to the exact solution.

Keywords: Boundary value problems; adomain decomposition method; boundary conditions; higher-order nonlinear ODE.

1 Introduction

This paper studies Boundary Value Problems of the form:

$$y^{(n+2)} = f(x, y, y', \ldots, y^{(n+1)}), n \geq 1,$$

(1.1)

*Corresponding author: E-mail: yahya217@yahoo.com;
with one of the following conditions
\[ y(0) = r_0, y'(0) = r_1, \ldots, y^{(n-m)}(0) = r_{n-m}, y^{(n+1)}(s) = k, \]
\[ y(a) = b_0, y'(a) = b_1, \ldots, y^{(n)}(a) = b_n, y^{(n+m)}(0) = d, \]
\[ y(c) = h_0, y'(c) = h_1, \ldots, y^{(n+1)}(c) = h_n. \]

Where \( f \) is a differential operator of linear or non-linear of order less than \((n + 2)\). And \( m = 0 \) or \( m = 1, a, b_0, b_1, \ldots, b_n, c, d, h_0, h_1, \ldots, h_n, r_0, r_1, \ldots, r_m, s, k, \) are real constant.

The Boundary Value Problems (BVPs) consider chemical reactions, heat transfer, gas dynamics a nuclear physics as models for example BVPs [1]. There are numerous techniques solutions for BVPs in literature [3-5,6].

The Adomain decomposition method (ADM) [7,8,9], has been studied by many scientists for solving differential and integral problems in many scientific and physical applications. It resolve the solution into the series which converges quickly. In this paper, a new modified of the Adomain decomposition method (MADM) is proposed to find a differential operator as well as its inverse operator in order to solve BVP. This paper aims to employ the new MADM which can be used for solution of higher-order BVPs under various kinds of different conditions to solve an equation at more than one condition. The present study analyzing method. A lot of numerical techniques commentary are illustrate in the following.

2 Analysis of the Method

To study the equation (1.1), we would suggest the new differential operator,
\[ L(.) = \frac{d^m}{dx^m}x^{m-1} \frac{d^{2-m}}{dx^{2-m}}x^{3-m} \frac{d}{dx} x^{m-2} \frac{d^{n-1}}{dx^{n-1}}(.), \]
now, can be written the equation (1.1) as,
\[ L y = f(x, y, y', y'', \ldots, y^{(n+1)}), \]
under one of the conditions (1.2), (1.3) and (1.4), for three inverse operators \( L^{-1} \) is given, respectively as
\[ L^{-1} = \int_a^b \int_a^b \int_a^b \ldots \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b (x \ dx_1 \ dx_2 \ dx_3 \ dx_4 \ dx_5 \ dx_6 \ dx_7 \ dx_8 \ dx_9), \]
\[ L^{-1} = \int_a^b \int_a^b \int_a^b \ldots \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b (x \ dx_1 \ dx_2), \]
\[ L^{-1} = \int_a^b \int_a^b \int_a^b \ldots \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b \int_a^b (x \ dx_1 \ dx_2 \ dx_3 \ dx_4 \ dx_5 \ dx_6 \ dx_7 \ dx_8 \ dx_9). \]

By applying \( L^{-1} \) on (2.2), we obtain
\[ y(x) = \phi(x) + L^{-1} f(x, y, y', \ldots, y^{(n+1)}), \]
where $\phi(x)$ represent the terms arising from using auxiliary conditions. The Adomain decomposition method represent the solution $y(x)$ and the non-linear function $f(x, y, y', y'', ..., y^{(n+1)})$ by infinite series

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (2.4)$$

and

$$f(x, y, y', y'', ..., y^{(n+1)}) = \sum_{n=0}^{\infty} A_n, \quad (2.5)$$

where the components $y_n(x)$ of the solution $y(x)$ will be determined recurrently by algorithm [10,11,12].

$A_n$ are the Adomain polynomials, which are obtain formula the following

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} \left[ F \left( \sum_{i=0}^{n} \lambda^i y_i \right) \right]_{\lambda=0}, \quad n = 0, 1, 2, ..., \quad (2.6)$$

Substituting eq.(2.4) and eq.(2.5) into eq.(2.3), we get

$$\sum_{n=0}^{\infty} y_n(x) = \phi(x) + L^{-1} \sum_{n=0}^{\infty} A_n, \quad (2.7)$$

we get the components $y_n$ can be specified as

$$y_0 = \phi(x), \quad y_{n+1} = L^{-1} A_n, n \geq 0, \quad (2.8)$$

From (2.6) and (2.8), we find the components $y_n(x)$, and hence the series solution of $y(x)$ in (2.7) can be directly obtained. For numerical aim, the $n-$ term approximate

$$\Psi(x) = \sum_{k=0}^{n-1} y_k$$

can be used to approximate the exact solution. The approach above can be support by testing it on a variety of several linear and nonlinear BVP.
3 Application of MADM

In this part, when \( n = 1, 2, 4 \), in a differential operator (2.1). We apply the proposed algorithm on two third order non-linear boundary value problems at \( m = 0 \) & \( m = 1 \), two fourth order non-linear boundary value problems at \( m = 0 \) & \( m = 1 \) and one sixth order non-linear boundary value problem at \( m = 0 \) & \( m = 1 \) and in every case three boundary conditions.

3.1 Example

The first case, when \( n = 1 \) and \( m = 0 \), we give example non-linear equation of third order:

\[
y'''(x) = y^2 - y - x^2(x^2 - 1),
\]

under one of the following conditions

\[
y(0) = 1, y'(0) = 0, y(1) = 0,
\]

\[
y(1) = 0, y'(1) = -2, y''(0) = 0,
\]

\[
y\left(\frac{1}{2}\right) = \frac{3}{4}, y'\left(\frac{1}{2}\right) = -1, y''\left(\frac{1}{2}\right) = -2.
\]

The exact solution is 

\[
y(x) = 1 - x^2.
\]

Can be written eq. (3.1.1), as

\[
Ly = y^2 - y - x^2(x^2 - 1),
\]

from an operator (2.1), give

\[
L(.) = x^{-1} d^2 dx^2 \int_{-1}^{1} x^3 d x^2 (.)
\]

for three inverse operators under one of the following conditions, respectively

\[
L^{-1}(.) = x^2 \int_{1}^{x} x^{-3} \int_{0}^{x} x(.) d x d x.
\]

\[
L^{-1}(.) = x^2 \int_{1}^{x} x^{-3} \int_{0}^{x} x(.) d x d x.
\]

\[
L^{-1}(.) = x^2 \int_{\frac{1}{2}}^{x} x^{-3} \int_{\frac{1}{2}}^{x} x(.) d x d x.
\]

Applying \( L^{-1} \) to both sides of (3.1.2) and using the boundary conditions, we obtain respectively

\[
y(x) = 1 - x^2 + L^{-1} y^2 - L^{-1} y - L^{-1} x^2(x^2 - 1),
\]

\[
y(x) = 1 - x^2 + L^{-1} y^2 - L^{-1} y - L^{-1} x^2(x^2 - 1),
\]

\[
y(x) = 1 - x^2 + L^{-1} y^2 - L^{-1} y - L^{-1} x^2(x^2 - 1),
\]

employing ADM for \( y^2(x) \), as yield for every one above

\[
\sum_{n=0}^{\infty} y_n(x) = 1 - x^2 - L^{-1} x^2(x^2 - 1) - L^{-1} y_n + L^{-1} \sum_{n=0}^{\infty} A_n, n \geq 0,
\]

the components for \( y_n(x) \), introduces the recursive relation, respectively

\[
y_0 = 1 - 1.0119 x^2 + 0.0166667 x^5 - 0.0047619 x^7,
\]
\[ y_0 = 1.0131 - 1.025 x^2 + 0.0166667 x^5 - 0.0047619 x^7, \]
\[ y_0 = 0.997433 + 0.0130208 x - 1.01771 x^2 + 0.0166667 x^5 - 0.0047619 x^7, \]
\[ y_{n+1} = -L^{-1} y_n + L^{-1} A_n, n \geq 0, \]

Applying Adomain polynomial \( A_n \), for the non-linear term \( y^2 \), when for \( n=0,1 \) gives
\[ A_0 = y_0^2, \]
\[ A_1 = 2y_0 y_1, \]

Then, we can proceed to compute the first few components respectively, as follows
\[ y_1 = 0.0119856 x^2 - 0.0168651 x^5 + 0.00487596 x^7 + \ldots + 0.0000534612 x^{10}, \]
\[ y_2 = -0.0000810072 x^2 + 0.00199759 x^5 + \ldots + 0.0000547322 x^{10}, \]
\[ y_1 = -0.0261226 + 0.128954 x - 0.236643 x^2 + 0.188956 x^5 + \ldots + 0.0000534612 x^{10}, \]
\[ y_2 = -0.0000810072 x^2 + 0.00199759 x^5 + \ldots + 0.0000547322 x^{10}, \]
\[ y_1 = -0.00256696 + 0.0130208 x - 0.0177083 x^2 + 0.0047619 x^7, \]

The first terms, the approximate is following, respectively
\[ y(x) = y_0 + y_1 + y_2 = 1 - 1. x^2 + 1.34678 10^{-6} x^5 + \ldots + 5.35508 10^{-14} x^{25}, \]
\[ y(x) = y_0 + y_1 + y_2 = 0.986771 + 0.128954 x - 1.26164 x^2 + \ldots + 1.19708 10^{-10} x^{20}, \]
\[ y(x) = y_0 + y_1 = 0.994866 + 0.0260417 x - 1.03542 x^2 + 0.0333333 x^5 - 0.00952381 x^7, \]

| x | Exact at the first condition | MADN at the second condition | Absolute Error | Exact at the third condition | MADN at the third condition | Absolute Error |
|---|-----------------------------|-----------------------------|----------------|-----------------------------|-----------------------------|----------------|
| 0.1 | 0.99 | 0.99 | 0.00 | 0.987219 | 0.002781 | 0.987116 | 0.002884 |
| 0.2 | 0.96 | 0.96 | 0.00 | 0.963443 | 0.003443 | 0.958668 | 0.001332 |
| 0.3 | 0.91 | 0.91 | 0.00 | 0.916419 | 0.006419 | 0.909570 | 0.000430 |
| 0.4 | 0.84 | 0.84 | 0.00 | 0.847052 | 0.007052 | 0.839942 | 0.000058 |
| 0.5 | 0.75 | 0.75 | 0.00 | 0.756144 | 0.006144 | 0.750000 | 0.000000 |
| 0.6 | 0.64 | 0.64 | 0.00 | 0.644279 | 0.004379 | 0.640066 | 0.000066 |
| 0.7 | 0.51 | 0.51 | 0.00 | 0.512316 | 0.002316 | 0.510559 | 0.000559 |
| 0.8 | 0.36 | 0.36 | 0.00 | 0.360398 | 0.000398 | 0.361958 | 0.001958 |
| 0.9 | 0.19 | 0.19 | 0.00 | 0.188956 | 0.001044 | 0.194744 | 0.004744 |
| 1.0 | 0.00 | 0.00 | 0.00 | -0.001791 | 0.001791 | 0.093006 | 0.093006 |

We see from Table 3.1, the error less than possible, and the first condition is the exact solution.

We notice in the Fig. 3.1 the convergence between the exact solution and the approximate solutions under the boundary conditions.
3.2 Example

In this case and at the same time m=1, below example non-linear of third order:

\[ y'''(x) = 60x^2 + x^{10}y - y^3, \quad (3.2.1) \]

under one of the following conditions

\[
\begin{align*}
y(0) &= 0, \quad y''(0) = 0, \quad y(1) = 1, \\
y\left(\frac{1}{2}\right) &= \frac{1}{32}, \quad y'(\frac{1}{2}) = \frac{5}{16}, \quad y''(0) = 0, \\
y\left(\frac{1}{2}\right) &= \frac{1}{32}, \quad y'(\frac{1}{2}) = \frac{5}{16}, \quad y''(\frac{1}{2}) = \frac{5}{2}.
\end{align*}
\]

The exact solution is \( y(x) = x^5 \). Can be written eq. (3.2.1), as

\[
Ly = 60x^2 + x^{10}y - y^3,
\]

(3.2.2)

from an operator (2.1), we get

\[ L(\cdot) = \frac{d}{dx}x^{-1} \frac{d}{dx}x^2 \frac{d}{dx}x^{-1}(\cdot), \]

for three inverse operators under one of the following conditions respectively,

\[
\begin{align*}
L^{-1}(\cdot) &= x \int_{1}^{x} x^{-2} \int_{0}^{x} x \int_{0}^{x} (\cdot) \, dx \, dx \\
L^{-1}(\cdot) &= x \int_{\frac{1}{2}}^{x} x^{-2} \int_{\frac{1}{2}}^{x} x \int_{0}^{x} (\cdot) \, dx \, dx \\
L^{-1}(\cdot) &= x \int_{\frac{1}{2}}^{x} x^{-2} \int_{\frac{1}{2}}^{x} x \int_{\frac{1}{2}}^{x} (\cdot) \, dx \, dx.
\end{align*}
\]

Applying \( L^{-1} \), to both sides of (3.2.2) and using the boundary conditions, give respectively

\[ y(x) = x + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3, \]

25
\[ y(x) = \frac{-1}{8} + \frac{5}{16}x + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3, \]
\[ y(x) = \frac{3}{16} - \frac{15}{16}x + \frac{5}{4}x^2 + L^{-1}60x^2 + L^{-1}x^{10}y - L^{-1}y^3, \]

employing ADM for \( y^3(x) \), as yield for every one above

\[ \sum_{n=0}^{\infty} y_n(x) = x + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0, \]
\[ \sum_{n=0}^{\infty} y_n(x) = \frac{1}{8} + \frac{5}{16}x + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0, \]
\[ \sum_{n=0}^{\infty} y_n(x) = \frac{3}{16} - \frac{15}{16}x + \frac{5}{4}x^2 + L^{-1}60x^2 + L^{-1}x^{10}y_n - L^{-1}\sum_{n=0}^{\infty} A_n, n \geq 0, \]

the components for \( y_n(x) \) introduces the recursive relation, respectively

\[ y_0 = x^5, \]
\[ y_0 = x^5, \]
\[ y_0 = x^5, \]
\[ y_{n+1} = L^{-1}x^{10}y_n - L^{-1}A_n, n \geq 0, \]

the first components respectively, as follows

\[ y_1 = 0, \]
\[ y_2 = 0, \]
\[ y_1 = 0, \]
\[ y_2 = 0, \]
\[ y_1 = 0, \]
\[ y_2 = 0, \]

This means that the solution in a series from is following

\[ y(x) = y_0 + y_1 + y_2 = \]
\[ y(x) = x^5. \]

Plainly, the previous example, we have the exact solution. Thus the good method and its effectiveness.

### 3.3 Example

The second case, we give example for non-linear of fourth order, at \( n=2, m=0 \)

\[ y^{(4)} = (y')^2 - y'' - 4x^2 + e^x(1 + x^2 - 4x), \quad (3.3.1) \]

under one of the following conditions

\[ y(0) = 1, y'(0) = 1, y''(0) = 3, y''(\frac{1}{2}) = 3.72, \]
\[ y(1) = 3.72, y'(1) = 4.72, y''(1) = 4.72, y''(0) = 3, \]
\[ y(\frac{1}{2}) = 1.9, y'(\frac{1}{2}) = 2.65, y''(\frac{1}{2}) = 3.65, y'''(\frac{1}{2}) = 1.65. \]
The exact solution is \( y(x) = e^x + x^2 \). Can be written eq.(3.3.1), gives

\[
Ly = (y')^2 - yy'' - 4x^2 + e^x (1 + x^2 - 4x),
\]

(3.3.2)

from an operator (2.1), we get

\[
L(.) = x^{-1} \frac{d^2}{dx^2} x^3 \frac{d}{dx} x^{-2} \frac{d}{dx} (.),
\]

for three inverse operators under one of the following conditions, respectively

\[
L^{-1}(.) = \int_0^x x^2 \int_0^x x^{-3} \int_0^x x(.) \, dx \, dx \, dx,
\]

\[
L^{-1}(.) = \int_1^x x^2 \int_0^x x^{-3} \int_0^x x(.) \, dx \, dx \, dx,
\]

\[
L^{-1}(.) = \int_1^x x^2 \int_1^x x^{-3} \int_1^x x(.) \, dx \, dx \, dx.
\]

Applying \( L^{-1} \), to both sides of (3.3.2) and using the boundary conditions, we give respectively

\[
y(x) = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2 + L^{-1} (y')^2 - L^{-1} yy'/',
\]

\[
y(x) = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2 + L^{-1} (y')^2 - L^{-1} yy'/',
\]

\[
y(x) = 0.9998 + 1.03x + 1.415x^2 + 0.273x^3 + L^{-1} e^x (1 + x^2 - 4x)
\]

\[
= L^{-1} 4x^2 + L^{-1} (y')^2 - L^{-1} yy'/',
\]

employing ADM for \((y')^2 - yy''\), as yield for all above

\[
\sum_{n=0}^{\infty} y_n(x) = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2 +
\]

\[
L^{-1} \sum_{n=0}^{\infty} A_n - L^{-1} y_n y'_n, n \geq 0,
\]

\[
\sum_{n=0}^{\infty} y_n(x) = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2 +
\]

\[
L^{-1} \sum_{n=0}^{\infty} A_n - L^{-1} y_n y'_n, n \geq 0,
\]

\[
\sum_{n=0}^{\infty} y_n(x) = 0.9998 + 1.03x + 1.415x^2 + 0.273x^3 + L^{-1} e^x (1 + x^2 - 4x)
\]

\[
= L^{-1} 4x^2 + L^{-1} (y')^2 - L^{-1} yy'/',
\]

the components for \( y_n(x) \) introduces the recursive relation, respectively

\[
y_0 = 1 + x + 1.5x^2 + 0.853x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2,
\]

\[
y_0 = 1.07 + 0.86x + 1.5x^2 + 0.29x^3 + L^{-1} e^x (1 + x^2 - 4x) - L^{-1} 4x^2,
\]

\[
y_0 = 0.9998 + 1.03x + 1.415x^2 + 0.273x^3 + L^{-1} e^x (1 + x^2 - 4x)
\]

\[
= L^{-1} 4x^2 + L^{-1} (y')^2 - L^{-1} yy'/',
\]

\[
y_{n+1} = L^{-1} A_n - L^{-1} y_n y'_n, n \geq 0,
\]
We notice in the Fig. 3.3 the convergence between the exact solution and the approximate solutions under the boundary conditions. And we have access to the solution exact.

Table 3.3. The comparison between exact solution and MADM under three conditions

| x   | Exact solution at the first condition | MADM at the first condition | Absolute Error | Exact solution at the second condition | MADM at the second condition | Absolute Error | Exact solution at the third condition | MADM at the third condition | Absolute Error |
|-----|-------------------------------------|-----------------------------|----------------|----------------------------------------|------------------------------|----------------|---------------------------------------|-------------------------------|----------------|
| 0.0 | 1.000000                            | 1.000000                    | 0.000000       | 0.986000                               | 0.014000                    | 1.002500       | 0.002500                              |                               |                |
| 0.1 | 1.11517                             | 1.11509                     | 0.00082        | 1.005900                               | 0.10927                     | 1.11843        | 0.00326                               |                               |                |
| 0.2 | 1.26140                             | 1.26775                     | 0.00635        | 1.24689                                | 0.01451                     | 1.26500        | 0.00360                               |                               |                |
| 0.3 | 1.43986                             | 1.46058                     | 0.02072        | 1.42619                                | 0.01367                     | 1.44361        | 0.00375                               |                               |                |
| 0.4 | 1.65182                             | 1.69913                     | 0.04731        | 1.63987                                | 0.01195                     | 1.65568        | 0.00386                               |                               |                |
| 0.5 | 1.89872                             | 1.98727                     | 0.08855        | 1.88920                                | 0.00952                     | 1.90267        | 0.00395                               |                               |                |
| 0.6 | 2.18212                             | 2.32786                     | 0.14574        | 2.17546                                | 0.00666                     | 2.18618        | 0.00406                               |                               |                |
| 0.7 | 2.50375                             | 2.72259                     | 0.21884        | 2.49997                                | 0.00378                     | 2.50793        | 0.00418                               |                               |                |
| 0.8 | 2.86554                             | 3.171169                    | 0.3056         | 2.86422                                | 0.00132                     | 2.86998        | 0.00444                               |                               |                |
| 0.9 | 3.26960                             | 3.67369                     | 0.40409        | 3.27008                                | 0.00048                     | 3.27484        | 0.00524                               |                               |                |
| 1.0 | 3.71828                             | 4.22504                     | 0.50676        | 3.72000                                | 0.00172                     | 3.72579        | 0.00751                               |                               |                |

In this method, any assistance can be obvious calculated at any order. If we solve for the first term, the approximate is following, respectively

\[ y(x) = y_0 + y_1 + y_2 = \]

\[ y_0 = 1 + x + 1.5 x^2 + 0.840303 x^3 + 0.0416667 x^4 - 0.025 x^5 + ... + 5.85008 \times 10^{-10} x^{14}, \]

\[ y_1 = 0.166028 x^3 - 0.210076 x^4 - 0.5033485 x^5 - 0.0196187 x^6 + ... + 4.25391 \times 10^{-17} x^{25}, \]

\[ y_2 = -0.000676185 x^3 + 0.00276714 x^4 + 0.00136013 x^5 + ... + 2.34098 \times 10^{-35} x^{50}, \]

\[ y = 1.19399 + 0.640618 x + 1.5 x^2 + 0.387846 x^3 + ... + 1.9290110^{-6} x^{10}, \]

\[ y_1 = -0.177877 + 0.292906 x - 0.0631219 x^3 - 0.0986711 x^4 + ... + 3.44211 \times 10^{-17} x^{25}, \]

\[ y_2 = -0.0302707 + 0.0613275 x - 0.0558587 x^3 + 0.0128297 x^4 + ... + 4.5765 \times 10^{-35} x^{50}, \]

\[ y_0 = 0.996293 + 1.05407 x + 1.36146 x^2 + 0.2996 x^3 + 0.0416667 x^4 + ... + 5.85008 \times 10^{-10} x^{14}, \]

\[ y_1 = 0.00625454 - 0.0440354 x + 0.105108 x^2 - 0.0790648 x^3 + ... + 1.21385 \times 10^{-20} x^{28}, \]

\[ y_2 = -0.0000184692 + 0.000235623 x - 0.00123993 x^2 + ... + 1.40914 \times 10^{-12} x^{24}, \]
Fig. 3.3. Comparison between exact solution \( y(x) = e^x + x^2 \), and MADM at three conditions \( \sum_{n=0}^{2} y_n(x) \), respectively

3.4 Example

In this case and at the same time \( m=1 \), we give example non-linear of fourth-order [13]:

\[
y^{'''}(x) = e^{-x} y^2(x),
\]

(3.4.1)

under one of the following conditions

\[
y(0) = 1, y'(0) = 1, y''(0) = 1, y'(1) = 2.72,
\]

\[
y\left(\frac{1}{2}\right) = 1.65, y\left(\frac{1}{2}\right) = 1.65, y''\left(\frac{1}{2}\right) = 1.65, y''(0) = 1.
\]

\[
y(1) = 2.7, y'(1) = 2.7, y''(1) = 2.7, y'''(1) = 2.7,
\]

The exact solution is \( y(x) = e^x \). Can be written eq.(3.4.1), as

\[
Ly = e^{-x} y^2(x),
\]

(3.4.2)

from an operator (2.1), when \( m=1, n=2 \), we get

\[
L(\cdot) = \frac{d}{dx} x^{-1} \frac{d}{dx} x^2 \frac{d}{dx} x^{-1} \frac{d}{dx} (\cdot),
\]

for three inverse operators under one of the following conditions respectively,

\[
L^{-1}(\cdot) = \int_0^x \int_0^x x^{-2} x^2 \int_0^x (\cdot) \, dx \, dx \, dx.
\]

\[
L^{-1}(\cdot) = \int_0^x \int_{\frac{1}{2}}^x x^{-2} x^2 \int_0^x (\cdot) \, dx \, dx \, dx.
\]

\[
L^{-1}(\cdot) = \int_0^x \int_{\frac{1}{2}}^x x^{-2} x^2 \int_0^x (\cdot) \, dx \, dx \, dx.
\]

Applying \( L^{-1} \), to both sides of (3.4.2) and using the boundary conditions, we give respectively

\[
y(x) = 1 + x + 0.61x^2 + 0.167x^3 + e^{-x} y^2(x),
\]

\[
y(x) = 0.997 + 0.95x + 0.575x^2 + 0.167x^3 + L^{-1} e^{-x} y^2(x),
\]
employing ADM for $y^2(x)$, as yield for every one above

$$y(x) = 0.907 + 1.36x + 0.453x^3 + L^{-1}e^{-x}y^2(x),$$

$$
\sum_{n=0}^{\infty} y_n(x) = 1 + x + 0.61x^2 + 0.167x^3 + L^{-1}\sum_{n=0}^{\infty}e^{-x}A_n, n \geq 0,
$$

$$
\sum_{n=0}^{\infty} y_n(x) = 0.997 + 0.95x + 0.575x^2 + 0.167x^3 + L^{-1}\sum_{n=0}^{\infty}e^{-x}A_n, n \geq 0,
$$

$$
\sum_{n=0}^{\infty} y_n(x) = 0.907 + 1.36x + 0.453x^3 + L^{-1}\sum_{n=0}^{\infty}e^{-x}A_n, n \geq 0,
$$

the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + 0.61x^2 + 0.167x^3,$$

$$y_0 = 0.997 + 0.95x + 0.575x^2 + 0.167x^3,$$

$$y_0 = 0.907 + 1.36x + 0.453x^3,$$

$$y_{n+1} = L^{-1}e^{-x}A_n, n \geq 0,$$

the first components respectively, as follows

$$y_1 = -0.110759x^2 + 0.0416667x^4 + 0.00833333x^5 + 0.002x^6 + \ldots + 6.5830810^{-14}x^{20},$$

$$y_2 = 0.00164057x^2 - 0.000615327x^6 + 0.000035099x^8 + \ldots + 1.3681410^{-13}x^{20},$$

$$y_1 = -0.00947569x + 0.0499038x - 0.073157x^2 + 0.041417x^4 + \ldots + 6.6094410^{-14}x^{20},$$

$$y_2 = 0.00051099x - 0.000293793x + 0.000500661x^2 + \ldots + 3.9713610^{-13}x^{20},$$

$$y_1 = 0.9939159x - 0.358699x + 0.497474x^2 - 0.280518x^3 + \ldots + 4.8632710^{-13}x^{20},$$

$$y_2 = 0.000120243x - 0.000944067x + 0.00321515x^2 + \ldots + 7.9495810^{-10}x^{20},$$

Thus, respectively

$$y(x) = y_0 + y_1 + y_2 =$$

$$y(x) = 1 + x + 0.500882x^2 + \frac{x^3}{6} + 0.0416667x^4 + 0.00833333x^5 + 0.00138467x^6 +$$

$$0.000198413x^7 + 0.000017499x^8 + 6.8122710^{-6}x^9 + 1.1324510^{-6}x^{10} -$$

$$6.4911910^{-7}x^{11} + 7.6081310^{-8}x^{12} + 6.651410^{-9}x^{13} + \ldots + 3.8149910^{-27}x^{37},$$

$$y(x) = 0.987575 + 0.99961x + 0.502185x^2 + 0.167x^3 + 0.0406298x^4 +$$

$$0.00833908x^5 + 0.00138481x^6 + 0.000213127x^7 + 0.0000161673x^8 + 3.8459510^{-6}x^9 +$$

$$1.5613310^{-6}x^{10} - 6.4803510^{-7}x^{11} + 5.3312910^{-8}x^{12} + \ldots + 3.8379310^{-27}x^{37},$$

$$y(x) = 1.00104 + 1.00036x + 0.50069x^2 + 0.166327x^3 + 0.0413755x^4 + 0.00898998x^5 +$$

$$0.000558478x^6 + 0.000100613x^7 - 0.000571726x^8 + 0.0003373x^9 - 0.000141281x^{10} +$$

$$0.0000445961x^{11} - 0.0000101886x^{12} + 1.6194810^{-6}x^{13} + \ldots + 7.6602410^{-26}x^{37},$$
Table 3.4. The comparison between exact solution and MADM under three conditions

| x   | Exact solution | MADM at the first condition | Absolute Error | MADM at the second condition | Absolute Error | MADM at the third condition | Absolute Error |
|-----|----------------|----------------------------|----------------|----------------------------|----------------|----------------------------|----------------|
| 0.0 | 1.00000        | 1.00000                    | 0.0000         | 0.987575                  | 0.012425      | 1.00104                    | 0.00104        |
| 0.1 | 1.10517        | 1.10518                    | 0.00001        | 1.09273                   | 0.012244      | 1.10625                    | 0.00108        |
| 0.2 | 1.22140        | 1.22144                    | 0.00004        | 1.20899                   | 0.01241       | 1.22253                    | 0.00113        |
| 0.3 | 1.34986        | 1.34994                    | 0.00008        | 1.33751                   | 0.01235       | 1.35105                    | 0.00119        |
| 0.4 | 1.49182        | 1.49197                    | 0.00015        | 1.47959                   | 0.01223       | 1.49309                    | 0.00127        |
| 0.5 | 1.64872        | 1.64894                    | 0.00022        | 1.63662                   | 0.01210       | 1.65006                    | 0.00134        |
| 0.6 | 1.82212        | 1.82244                    | 0.00032        | 1.81018                   | 0.01194       | 1.82353                    | 0.00141        |
| 0.7 | 2.01375        | 2.01418                    | 0.00043        | 2.00199                   | 0.01176       | 2.01525                    | 0.0015         |
| 0.8 | 2.22554        | 2.22610                    | 0.00056        | 2.21395                   | 0.01159       | 2.22711                    | 0.00157        |
| 0.9 | 2.45960        | 2.46031                    | 0.00071        | 2.44817                   | 0.01143       | 2.46125                    | 0.00165        |
| 1.0 | 2.71828        | 2.71916                    | 0.00088        | 2.70696                   | 0.01132       | 2.72000                    | 0.00172        |

![Graph](image)

Fig. 3.4. Comparison between exact solution $y(x) = e^x$, and MADM at three conditions $\sum_{n=0}^{2} y_n(x)$, respectively

We notice in the figure above the convergence between the exact solution and the approximate solutions under the boundary conditions. And we have access to the solution exact.

3.5 Example

In the last case, we will give example non-linear of sixth-order [13], at m=0,1,

$$\frac{d^6y}{dx^6} = e^{-x}y^2(x),$$

under one of the following condition

$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1, y''''(0) = 1, y'''''(1) = e,$

$y(1) = e, y'(1) = e, y''(1) = e, y'''(1) = e, y''''(1) = e, y'''''(1) = e, y''''''(0) = 1,$
The exact solution is
\[ y \left( \frac{1}{2} \right) = e^\frac{1}{2}, \quad y' \left( \frac{1}{2} \right) = e^\frac{1}{2}, \quad y'' \left( \frac{1}{2} \right) = e^\frac{1}{2}, \quad y''' \left( \frac{1}{2} \right) = e^\frac{1}{2}, \quad y'''' \left( \frac{1}{2} \right) = e^\frac{1}{2}. \]

The exact solution is \( y(x) = e^x \). Can be written eq.(3.5.1)

\[ Ly = e^{-x}y^2(x), \quad (3.5.2) \]

from an operator (2.1), when \( m=0,n=4 \), we obtain

\[ L() = x^{-1} \frac{d^2}{dx^2} x^3 \frac{d}{dx} x^{-2} \frac{d^6}{dx^3}( ), \]

for three inverse operators under one of the following conditions, respectively

\[ L^{-1}(.) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty x(.) \ dx dx dx dx, \]
\[ L^{-1}(.) = \int_1^\infty \int_1^\infty \int_1^\infty \int_1^\infty x(.) \ dx dx dx dx, \]
\[ L^{-1}(.) = \int_\frac{1}{2}^\infty \int_\frac{1}{2}^\infty \int_\frac{1}{2}^\infty \int_\frac{1}{2}^\infty x(.) \ dx dx dx dx. \]

Applying \( L^{-1} \), to both sides of (3.5.2) and using the boundary condition respectively, gives

\[ y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + L^{-1}e^{-x}y^2, \]
\[ y(x) = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5 + L^{-1}e^{-x}y^2, \]
\[ y(x) = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5 + L^{-1}e^{-x}y^2, \]

employing ADM for \( y^2 \), as yield

\[ \sum_{n=0}^{\infty} y_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5 + \]
\[ L^{-1} \sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0, \]

\[ \sum_{n=0}^{\infty} y_n(x) = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5 + \]
\[ L^{-1} \sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0, \]

\[ \sum_{n=0}^{\infty} y_n(x) = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5 + \]
\[ L^{-1} \sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0, \]

the components for \( y_n(x) \) introduces the recursive relation, respectively

\[ y_0 = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{30}x^5, \]
\[ y_0 = 1.03 + 0.943x + 0.545x^2 + 0.143x^3 + 0.0425x^4 + 0.0135x^5, \]
\[ y_0 = 0.997 + 1.021x + 0.473x^2 + 0.172x^3 + 0.0345x^4 + 0.0344x^5, \]
\[ y_{n+1} = L^{-1}e^{-x}A_n, n \geq 0, \]
applying Adomain polynomial $A_n$, for the non-linear term $y^2$, when for $n=0,1$, gives

$$A_0 = y^2,$$
$$A_1 = 2y_0 y_1,$$

the first components respectively, as follows

$$y_1 = -0.0036338 x^5 + 0.00138889 x^6 + 0.000198413 x^7 + ... + 2.15102 10^{-12} x^{20},$$
$$y_2 = 2.57621 10^{-7} x^5 - 2.18483 10^{-8} x^{11} + 4.17535 10^{-9} x^{12} + ... + 8.35363 10^{-13} x^{20},$$

$$y_1 = -0.00508562 + 0.0225422 x - 0.036792 x^2 + 0.0237785 x^3 + ... + 8.28423 x^{10} - 14 x^{20},$$
$$y_2 = 2.31144 10^{-6} - 0.0000113257 x + 0.0000214098 x^2 + ... + 1.99707 10^{-13} x^{20},$$
$$y_1 = 0.000034423 - 0.000396757 x + 0.00195246 x^2 + ... + 1.48474 10^{-12} x^{20},$$
$$y_2 = 1.61882 10^{-12} - 3.87397 10^{-11} x + 4.24644 10^{-10} x^2 + ... + 8.16771 10^{-13} x^{20},$$

The solution in a series form is following, respectively

$$y(x) = y_0 + y_1 + y_2 = 1 + x + 0.5 x^2 + 0.1667 x^3 + 0.04167 x^4 + ... + 4.38942 10^{-36} x^{47},$$
$$y(x) = y_0 + y_1 + y_2 = 1.02492 + 0.965531 x + 0.508229 x^2 + ... + 2.9159 10^{-37} x^{47},$$
$$y(x) = y_0 + y_1 + y_2 = 0.997033 + 1.0206 x + 0.474952 x^2 + ... + 4.82443 10^{-36} x^{47},$$

Table 3.5.1.1. The comparison between Exact solution and MADM for under three conditions

| x    | Exact solution at the first condition | MADM Absolute Error | Exact solution at the second condition | MADM Absolute Error | Exact solution at the third condition | MADM Absolute Error |
|------|--------------------------------------|---------------------|----------------------------------------|---------------------|--------------------------------------|---------------------|
| 0.0  | 1.00000                              | 0.00000             | 1.02492                                | 0.02492             | 0.99703                              | 0.002967            |
| 0.1  | 1.10517                              | 0.00000             | 1.12672                                | 0.02155             | 1.10401                              | 0.00116             |
| 0.2  | 1.22140                              | 0.00001             | 1.23976                                | 0.01836             | 1.22156                              | 0.00016             |
| 0.3  | 1.34986                              | 0.00011             | 1.36518                                | 0.01532             | 1.35088                              | 0.00102             |
| 0.4  | 1.49182                              | 0.00046             | 1.50429                                | 0.01247             | 1.49332                              | 0.00015             |
| 0.5  | 1.64872                              | 0.00141             | 1.65850                                | 0.00978             | 1.65048                              | 0.00176             |
| 0.6  | 1.82212                              | 0.00352             | 1.82937                                | 0.00725             | 1.82418                              | 0.00206             |
| 0.7  | 2.01375                              | 0.00761             | 2.01865                                | 0.00490             | 2.01653                              | 0.00278             |
| 0.8  | 2.22554                              | 0.01484             | 2.22825                                | 0.00271             | 2.22999                              | 0.00445             |
| 0.9  | 2.45960                              | 0.02675             | 2.46025                                | 0.00065             | 2.46738                              | 0.00778             |
| 1.0  | 2.71828                              | 0.04530             | 2.71700                                | 0.00128             | 2.71394                              | 0.01366             |

We will study the same example at $m=1$, give under one of the following condition

$$y(0) = 1, y'(0) = 1, y''(0) = 1, y'''(0) = 1, y''''(0) = 1, y'''''(0) = 1, y''''''(0) = 1,$$
$$y(1) = e, y'(1) = e, y''(1) = e, y'''(1) = e, y''''(1) = e, y'''''(1) = e, y''''''(1) = e,$$
$$y( \frac{1}{2} ) = e^{ \frac{1}{2} }, y'( \frac{1}{2} ) = e^{ \frac{1}{2} }, y''( \frac{1}{2} ) = e^{ \frac{1}{2} }, y'''( \frac{1}{2} ) = e^{ \frac{1}{2} }, y''''( \frac{1}{2} ) = e^{ \frac{1}{2} }, y'''''( \frac{1}{2} ) = e^{ \frac{1}{2} }.$$

The exact solution is $y(x) = e^x$. Can be written eq.(3.5.1)

$$L_y = e^{-x} y^2(x).$$

(3.5.3)
from an operator (2.1), when \( m=1, n=4 \), we obtain

\[
L(\cdot) = \frac{d}{dx}x^{-1} \frac{d}{dx}x^2 \frac{d}{dx}x^{-1} \frac{d^3}{dx^3}(\cdot),
\]

for three inverse operators under one of the following conditions, respectively

\[
L^{-1}(\cdot) = \int_{x_0}^{x} \int_{x_0}^{x} \int_{x_0}^{x} \int_{x_0}^{x} \int_{x_0}^{x} \int_{x_0}^{x} dx dx dx dx dx dx.
\]

Applying \( L^{-1} \), to both sides of (3.5.3) and using the boundary condition respectively, gives

\[
y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.05075x^4 + \frac{1}{120}x^5 + L^{-1}e^{-x}y^2,
\]

\[
y(x) = 1.0119 + 0.947x + 0.596x^2 + 0.083x^3 + 0.0717x^4 + 0.0083x^5 + L^{-1}e^{-x}y^2,
\]

\[
y(x) = 1.01127 + 0.916x + 0.755x^2 + 0.0172x^3 + 0.206x^4 + 0.01375x^5 + L^{-1}e^{-x}y^2,
\]

employing ADM for \( y^2 \), as yield

\[
\sum_{n=0}^{\infty} y_n(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + 0.05075x^4 + \frac{1}{120}x^5 +
\]

\[
L^{-1} \sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,
\]

\[
\sum_{n=0}^{\infty} y_n(x) = 1.0119 + 0.947x + 0.596x^2 + 0.083x^3 + 0.0717x^4 + 0.0083x^5 +
\]

\[
L^{-1} \sum_{n=0}^{\infty} e^{-x}A_n, n \geq 0,
\]
the components for $y_n(x)$ introduces the recursive relation, respectively

$$y_0 = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + 0.05075 x^4 + \frac{1}{120} x^5,$$

$$y_0 = 1.0119 + 0.947 x + 0.596 x^2 + 0.083 x^3 + 0.0717 x^4 + 0.0083 x^5,$$

$$y_0 = 1.01127 + 0.916 x + 0.755 x^2 + 0.0172 x^3 + 0.206 x^4 + 0.01375 x^5,$$

$$x$$

$$y_{n+1} = L^{-1} e^{-x} A_n , n \geq 0,$$

The first components respectively, as follows

$$y_1 = -0.00009843 x^4 + 0.00138889 x^6 + 0.000198413 x^7 + ... + 3.27175 \times 10^{-19} x^{25},$$

$$y_2 = 3.35831 \times 10^{-6} x^4 - 1.2035 \times 10^{-7} x^{10} + ... + 1.13674 \times 10^{-18} x^{25},$$

$$y_1 = -0.0111946 + 0.0530443 x - 0.0976931 x^2 + 0.0845656 x^3 + ... + 1.46188 \times 10^{-17} x^{24},$$

$$y_2 = 0.000324587 - 0.00016184 x + 0.000321221 x^2 + ... + 9.59612 \times 10^{-17} x^{25},$$

$$y_1 = 0.000346808 - 0.000410713 x + 0.0020165 x^2 + ... + 8.15701 \times 10^{-18} x^{25},$$

$$y_2 = 1.70115 \times 10^{-12} - 4.06568 \times 10^{-11} x + 4.54143 \times 10^{-10} x^2 + ... + 4.57668 \times 10^{-16} x^{25},$$

The solution in a series from is following, respectively

$$y(x) = y_0 + y_1 + y_2 =$$

$$y(x) = 1 + x + 0.5x^2 + 0.1667 x^3 + 0.0416549 x^4 + 0.00833 x^5 + ... + 1.46391 \times 10^{-18} x^{25},$$

$$y(x) = 1.00074 + 0.999882 x + 0.540128 x^2 + 0.167252 x^3 + ... + 9.59612 \times 10^{-17} x^{25},$$

$$y(x) = 1.0113 + 0.915589 x + 0.757017 x^2 + 0.166765 x^3 + ... + 4.49511 \times 10^{-16} x^{25},$$

Table 3.5.1.2. The comparison between exact solution and MADM for under conditions

| x  | Exact solution at the first condition | MADN at the second condition | Absolute Error | MADM at the third condition | Absolute Error |
|----|-------------------------------------|-----------------------------|----------------|-----------------------------|----------------|
| 0.0| 1.00000                            | 1.00000                     | 0.00000       | 1.00074                    | 0.00074        |
| 0.1| 1.10517                            | 1.10517                     | 0.00000       | 1.1063                     | 0.00130        |
| 0.2| 1.22140                            | 1.22140                     | 0.00000       | 1.22372                    | 0.00232        |
| 0.3| 1.34986                            | 1.34986                     | 0.00000       | 1.35417                    | 0.00431        |
| 0.4| 1.49182                            | 1.49182                     | 0.00000       | 1.49888                    | 0.00706        |
| 0.5| 1.64872                            | 1.64872                     | 0.00000       | 1.65923                    | 0.01051        |
| 0.6| 1.82212                            | 1.82212                     | 0.00000       | 1.83669                    | 0.01457        |
| 0.7| 2.01375                            | 2.01375                     | 0.00000       | 2.03283                    | 0.01908        |
| 0.8| 2.22554                            | 2.22554                     | 0.00000       | 2.24936                    | 0.02382        |
| 0.9| 2.45060                            | 2.45060                     | 0.00000       | 2.48812                    | 0.02852        |
| 1.0| 2.71828                            | 2.71828                     | 0.00000       | 2.75110                    | 0.03282        |

35
We note the table above for the condition one, we got the exact solution and the another conditions, we got the approximate solutions for the exact solution. Therefore the method is very good and effective.

From the Tables 3.5.1.1, 3.5.1.2 and the Figs. 3.5.1.1, 3.5.1.2, we noticed the convergence and we obtain exact solutions, and thus the method is very useful and active to solve from high-order.

![Graph]

Fig. 3.5.1.2. Comparison between exact solution and MADM under three boundary conditions, respectively

4 Conclusion

This method is an active, useful and effective to get the approximate solutions. Through tables and figures are the previous illustrations of third-order boundary value problems into higher-order, we noticed that we reach the approximate solution and more than the exact solution. We found it highly efficient and it can be developed to be used to find solutions to develop the differential operator of the inverse operator by boundary conditions in general.

Acknowledgement

First of all, all praise is to Allah for all the blessings. He endows me with. In my research journey, I am blessed to be under the supervision of Professor Yahya Qiad Hassan. He has been so informative and cooperative giving me guidelines to make my work a piece of success. I also would like to thank my family for everything they do for me. Special thanks to my parents Baa Ali and Maa Akhlasmy for his help and support, for being so kind and for standing by my side whenever I need him. My husband, Nabeel, the source of hope, love and inspirations, I thank you so much for everything. I thank you for your constant prayers and for encouraging me to continue. I cannot forget my children who were so patient to endure life away from me. I am grateful to everyone who helped me in my research in any way. Thank you all.

Competing Interests

Authors have declared that no competing interests exist.
References

[1] Adomain G. Solving Frontier problems of Physics: The Decomposition Method. Kluwer Academic Dordrecht; 1994.

[2] Hasan YQ. The numerical solution of third order boundary value problems by the modified decomposition method. Advances in Intelligent Transportation Systems. 2012;1(3):71-74.

[3] Kasi Viswanadham KNS, Ballem S. Numerical solution of eighth-order boundary value problems by Galerkin Method with Quintic B-splines. International Journal of Computer Applications. 2014;89(15):7-13.

[4] Adomain G. A review of the decomposition method and some recent results for nonlinear equations. Math Comput Model. 1991;21(5):101-127.

[5] Adomain G. A review of the decomposition method in applied mathematics. Mathematical analysis and applications. 1988;135:501-544.

[6] Adomain G, Rach R. Inversion of nonlinear stochastic operators. J. Math Anal Appl. 1983;91:39-46.

[7] Chen J. Fast multilevel augmentation methods for nonlinear boundary value problems. Appl Math Comput. 2011;61:612-619.

[8] Deeba E, Khuri S, Xie S. An algorithm for solving boundary value problems. J Comput Physics. 2000;159:125-138.

[9] Hasan YQ. Modified Adomain decomposition method for solving first ordinary differential equations. Advances in Intelligent Transportation Systems. 2012;1(4):86-89.

[10] Hosseine SR, SheikholesIami M, Ghasemian M, Ganji DD. Nanofluid heat transfer analysis in a microchannel heat sink (MCHS) under the effect of magnetic field by means of KKL. model Powder Tech. 2018;324:36-47.

[11] Wazwaz AM. A new algorithm for calculating Adomian polynomials for non-linear operators. Appl Math comput. 2000;111:33-51.

[12] Liaqat A, Islam S, Gul T, Amiri IS. Solution of non-linear problems by a new analytical technique using Daftardar-Gejji and Jafari polynomials. Advances in Mechanical Engineering. 2019;11(12):1-10.

[13] Wazwaz AM. A First Course in Integral Equation. World Scientific Singapore; 1997.

© 2020 Al-Rabahi and Hasan; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)

http://www.sdiarticle4.com/review-history/54751