A NOTE ON THE SCHRÖDINGER MAXIMAL FUNCTION

J. BOURGAIN

Abstract. It is shown that control of the Schrödinger maximal function
\[ \sup_{0 < t < 1} |e^{it\Delta} f| \quad \text{for} \quad f \in H^s(\mathbb{R}^n) \]
requires \( s \geq \frac{n}{2(n+1)} \).

1. Introduction

Recall that the solution of the linear Schrödinger equation
\[
\begin{aligned}
iu_t - \Delta u &= 0 \\
u(x,0) &= f(x)
\end{aligned}
\]  
(1.1)

with \((x,t) \in \mathbb{R}^n \times \mathbb{R}\) is given by
\[
e^{it\Delta} f(x) = (2\pi)^{-n/2} \int e^{i(x,\xi) \cdot t |\xi|^2} \hat{f}(\xi) d\xi.
\]  
(1.2)

Assuming \( f \) belongs to the space \( H^s(\mathbb{R}^n) \) for suitable \( s \), when does the almost convergence property
\[
\lim_{t \to 0} e^{it\Delta} f = f \quad \text{a.e.}
\]  
(1.3)

hold? The problem was brought up in Carleson’s paper \( \text{[C]} \) who proved convergence for \( s \geq \frac{1}{4} \) when \( n = 1 \). Dahlberg and Kenig \( \text{[D-K]} \) showed that this result is sharp. In higher dimension, the question of identifying the optimal exponent \( s \) has been studied by several authors and our state of knowledge may be summarized as follows. For \( n = 2 \), the strongest result to date appears in \( \text{[L]} \) and asserts \( 1.3 \) for \( f \in H^s(\mathbb{R}^2), s > \frac{3}{8} \). More generally, for \( n \geq 2 \) \( 1.3 \) was shown to hold for \( f \in H^s(\mathbb{R}^n), s > \frac{2n-1}{4n} \) (see \( \text{[H]} \)).

In the opposite direction, for \( n \geq 2 \) the condition \( s \geq \frac{n}{2(n+1)} \) was proven to be necessary (see \( \text{[L-R]} \) and also \( \text{[D-G]} \) for a different approach based on pseudo-conformal transformation). Here we show the following stronger statement.

Proposition 1. Let \( n \geq 2 \) and \( s < \frac{n}{2(n+1)} \). Then there exist \( R_k \to \infty \) and \( f_k \in L^2(\mathbb{R}^n) \) with \( f_k \) supported in the annulus \( |\xi| \sim R_k \), such that \( \|f_k\|_2 = 1 \) and
\[
\lim_{k \to \infty} R_k^{-s} \left\| \sup_{0 < t < 1} |e^{it\Delta} f_k(x)| \right\|_{L^1(B(0,1))} = \infty.
\]  
(1.4)

Date: September 20, 2016.

The author was partially supported by NSF grants DMS-1301619.
There is some evidence the exponent $\frac{n}{2(n+1)}$ could be the optimal one, though limited to multi-linear considerations appearing in [B]. Of course, the $n = 1$ case coincides with the [D-K] result, while for $n = 2$, the above Proposition leaves a gap between $\frac{4}{3}$ and $\frac{3}{2}$. It may be also worth to point out that for $n = 2$, in some sense, our example fits a scenario where the arguments from [B] require the $s > \frac{4}{3}$ condition.

2. An Example

Denote $x = (x_1, \ldots, x_n) = (x_1, x') \in B(0, 1) \subset \mathbb{R}^n$. Let $\varphi : \mathbb{R} \to \mathbb{R}_+, \Phi : \mathbb{R}^{n-1} \to \mathbb{R}_+$ satisfy $\text{supp } \hat{\varphi} \subset [-1, 1]$, $\text{supp } \hat{\Phi} \subset B(0, 1)$, $\hat{\varphi}, \hat{\Phi}$ smooth and $\varphi(0) = \Phi(0) = 1$. Set $D = R^{\frac{n+1}{2(n+2)}}$, and define

$$f(x) = e(R x_1) \varphi(R^{\frac{1}{2}} x_1) \Phi(x') \prod_{j=2}^n \left( \sum_{\frac{R}{2} < \ell_j < \frac{R}{2}} \varepsilon^{iD\ell_j x_j} \right) \quad (2.1)$$

where $\ell = (\ell_2, \ldots, \ell_n) \in \mathbb{Z}^{n-1}$. Hence

$$\|f\|_2 \sim R^{-\frac{n}{2}} \left( \frac{R}{D} \right)^{\frac{n-1}{2}}$$

and $\text{supp } \hat{f} \subset ||\xi| \sim R|$. (2.2)

Clearly, denoting $e(z) = e^{iz}$,

$$e^{it \Delta} f(x) = \int \int \hat{\varphi}(\lambda) \hat{\Phi}(\xi') \left\{ \sum_{\ell} e((R+\lambda R^{\frac{1}{2}}) x_1 + (\xi'+D\ell).x' + (R+\lambda R^{\frac{1}{2}})^2 t + |\xi'+D\ell|^2 t) \right\} d\lambda d\xi'.$$

Taking $|t| < \frac{\xi}{R}, |x| < c$ for suitable constant $c > 0$, one gets

$$|e^{it \Delta} f(x)| \sim \int \hat{\varphi}(\lambda) \left\{ \sum_{\ell} e((R+\lambda R^{\frac{1}{2}}) x_1 + D\ell.x' + 2\lambda R^{\frac{3}{2}} t + D^2 |\ell|^2 t) \right\} d\lambda \sim \varphi \left( R^{\frac{1}{2}} (x_1 + 2Rt) \right) \left| \sum_{\ell} e(D\ell.x' + D^2 |\ell|^2 t) \right| \quad (2.3)$$

Specify further $t = -\frac{\pi}{2R} + \tau$ with $|\tau| < \frac{1}{R} R^{-\frac{3}{2}}$ in order to ensure that the first factor in (2.3) should be $\sim 1$. For this choice of $t$, the second factor becomes

$$\left| \sum_{\ell} e \left( D\ell.x' - \frac{D^2}{2R} |\ell|^2 x_1 + D^2 |\ell|^2 \tau \right) \right| = \prod_{j=2}^n \left| \sum_{\frac{R}{2} < \ell_j < \frac{R}{2}} \varepsilon^{ij'y_j + \ell_j^2(y_1 + s)} \right| \quad (2.4)$$

with

$$y' = D x'(mod 2\pi) \quad y_1 = -\frac{D^2}{2R} x_1 (mod 2\pi) \quad (2.5)$$

and where $s = D^2 \tau$ is subject to the condition

$$|s| \lesssim D^2 R^{-3/2} = R^{-\frac{n-1}{n+1}}. \quad (2.6)$$
We view $y = (y_1, y')$ as a point in the $n$-torus $\mathbb{T}^n$. Next, define the following subset

$$
\Omega = \bigcup_{q \sim R^\frac{n}{2(n+1)} a} \left\{ (y_1, y') : \left| y_1 - 2\pi \frac{a_1}{q} \right| < c R^{-\frac{n-1}{2(n+1)}} \quad \text{and} \quad \left| y' - 2\pi \frac{a'}{q} \right| < c \frac{D}{R} \right\} \quad (2.7)
$$

with $a = (a_1, a') \pmod{q}$ and $(a_1, q) = 1$.

Hence $|\Omega| \sim R^{\frac{n}{2(n+1)}} \frac{n-1}{2(n+1)} R^\frac{n-1}{2(n+1)} \left( \frac{q}{R} \right)^{n-1} \sim 1$ and we take $x \in B(0,1)$ for which $y$ given by (2.5) belongs to $\Omega$. Clearly this gives a set of measure at least $c_1 > 0$. We evaluate (2.4) for $y \in \Omega$. Let $q \sim R^{\frac{n}{2(n+1)}}$ and $(a_1, a') \pmod{q}$ satisfy the approximations stated in (2.7) and set $s = 2\pi \frac{a_1}{q} - y_1$ for which (2.6) holds.

Clearly for $j = 2, \ldots, n$, by the quadratic Gauss sum evaluation

$$
\left| \sum_{\not\equiv \ell_j < q} e(\ell_j y_j + s) \right| \sim \left| \sum_{\not\equiv \ell_j < q} e(2\pi \frac{a_1}{q} \ell_j + 2\pi \frac{a_1}{q} \ell_j^2) \right|
$$

$$
\sim R^{\frac{1}{n(n+1)}} \left| \sum_{\ell_j=0}^{q-1} e\left(2\pi \frac{a_1}{q} \ell_j + 2\pi \frac{a_1}{q} \ell_j^2\right) \right|
$$

$$
\sim R^{\frac{1}{n(n+1)}} q^\frac{n}{2} \sim R^{\frac{n+1}{4}}
$$

and

$$
(2.4) \sim R^{\frac{n+1}{4}} \quad (2.8)
$$

Recalling (2.2), we obtain for $x \in B(0,1)$ in a set of measure $c_1 > 0$ that

$$
\sup_{0 < t < 1} \left| e^{it\Delta} f(x) \right| \gtrsim R^{\frac{n-1}{2(n+1)}} \left( \frac{D}{R} \right)^{\frac{n-1}{2}} = R^\frac{n-1}{2(n+1)}.
$$

The claim in the Proposition follows.

References

[B] J. Bourgain, *On the Schrödinger maximal function in higher dimension*, Tr. Mat. Inst. Steklova 280 (2013). Ortogonalnye Ryady, Teoriya Priblizhenii i Smeznye Voprosy, 53–66; reprinted in Proc. Steklov Inst. Math. 280 (2013), no. 1, 46–60.

[C] L. Carleson, *Some analytic problems related to statistical mechanics*, Euclidean Harmonic Analysis (Proc. Sem., Univ. Maryland, College Park, Md. 1979), Lecture Notes in Math., Vol. 779, Springer, Berlin, 1980, pp. 5–45.

[D-G] C. Demeter, S. Guo, *Schrödinger maximal function estimates via the pseudoconformal transformation*, arXiv:1608.07640

[D-K] B.E.J. Dahlberg, C. E. Kenig, *A note on the almost everywhere behavior of solutions to the Schrödinger equation*, Harmonic Analysis (Minneapolis, Minn. 1981), Lecture Notes in Math., Vol. 908, Springer, Berlin, 1982, pp. 205–209.

[L] S. Lee, *On pointwise convergence of the solutions to Schrödinger equations in $\mathbb{R}^2$*, IMRN, Vol. 2006, 1–21.

[L-R] R. Luc, K. Rogers, *An improved necessary condition for the Schrödinger maximal estimate*, available on arXiv.
