ON THE NON-EXISTENCE OF TENSOR PRODUCTS OF ALGEBRAIC CYCLES

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Abstract. Let $\otimes$ be the map which classifies the tensor product of two line bundles, an extension of this map to the space of all codimension 1 algebraic cycles is constructed. It is proved that this extension cannot exist in codimension greater than 1.

1. Introduction

Boyer, Lawson, Lima-Filho, Mann and Michelsohn settled the Segal conjecture in [1]. One of the fundamental results which motivated the proof is that there is a geometric construction which extends the map classifying the direct sum of vector bundles in $BU$ to the space $Z$ of all algebraic cycles, namely, the linear join $\#$ of cycles, i.e. the following diagram commutes:

\[
\begin{array}{ccc}
BU \times BU & \xrightarrow{\otimes} & BU \\
\downarrow c & & \downarrow c \\
Z \times Z & \xrightarrow{\#} & Z
\end{array}
\]

where $c$ is the total chern class map.

In [1] the authors mention that one would like to have a geometric construction on the space of algebraic cycles which extends the tensor product in the level of $BU$ (i.e. degree one cycles). Segal proved in [4] that $BU$ has an infinite loop space structure where the $H$-space structure is induced by the map classifying the tensor product of vector bundles. Therefore the construction requested by the authors of [1] would give yet another infinite loop space structure on $Z$. We will provide an idea of the extent to which such construction is possible:

Theorem 1.1. There is an algebraic biadditive pairing $\hat{\otimes}$ which extends the tensor product to all effective divisors:

\[
\mathcal{E}_d(P^n) \times \mathcal{E}_e(P^m) \xrightarrow{\otimes} \mathcal{E}_{de}(P^{mn+m+n})
\]

This product is constructed via an algebraic pairing in the corresponding rings of polynomials which may be of interest in its own right. The formula obtained

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\end{itemize}
from stabilizing and group-completing the pairing to the stabilized space \( \mathcal{Z}_1^1(\mathbb{P}^\infty) \) of algebraic cycles of codimension 1 and degree 0 yields a commutative diagram
\[
\begin{array}{ccc}
\mathcal{G}_1^1(\mathbb{P}^\infty) \times \mathcal{G}_1^1(\mathbb{P}^\infty) & \longrightarrow & \mathcal{G}_1^1(\mathbb{P}^\infty) \\
\downarrow c_1 & & \downarrow c_1 \\
\mathcal{Z}_0^1(\mathbb{P}^\infty) \times \mathcal{Z}_0^1(\mathbb{P}^\infty) & \xrightarrow{\otimes} & \mathcal{Z}_0^1(\mathbb{P}^\infty)
\end{array}
\]
which recovers the group structure in the second cohomology group given by the tensor product of line bundles
\[
c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in H^2(BU_1)
\]

The Hurewicz map is the main tool in proving that a general pairing does not exist. The following theorem calculates the classes pulled back by the Hurewicz map.

**Theorem 1.2.** The inclusion of the grassmannian \( \mathcal{G}_1^1(\mathbb{P}^n) \) of hyperplanes in \( \mathbb{P}^n \) into the space \( \mathcal{Z}_1^1(\mathbb{P}^n) \) of all cycles in \( \mathbb{P}^n \) factors through the free group \( \mathcal{Z}_1^1(\mathbb{P}^n) \):

\[
\begin{array}{cccc}
\mathcal{G}_1^1(\mathbb{P}^n) & \xrightarrow{i} & \mathcal{Z}_1^1(\mathbb{P}^n) & \xrightarrow{j} & \mathcal{Z}_1^1(\mathbb{P}^n) \\
\downarrow \cong \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
\prod_{j=0}^n K(\mathbb{Z}, 2j) & & K(\mathbb{Z}, 0) \times K(\mathbb{Z}, 2)
\end{array}
\]

With respect to the canonical product decomposition given in (\ref{can-decomp}), the map \( i \) classifies the cohomology class \( 1 \times \omega \times \cdots \times \omega^n \) where \( \omega \) is the multiplicative generator of \( H^2(\mathcal{G}_1^1(\mathbb{P}^n)) \) and the map \( j \) is homotopic to the projection \( \pi_0 \times \pi_1 \).

The pairing constructed for divisors in (\ref{pairing}) cannot be extended to a continuous biadditive pairing on the space of cycles of higher codimension, but it does admit an extension if we restrict the second factor of the pairing to the subgroup \( \mathcal{Z}_1^1(\mathbb{P}^{nm}) \) of cycles which are unions of hyperplanes (possibly with multiplicities).

**Theorem 1.3.** There is a continuous biadditive pairing \( \otimes \) which makes the following diagram commute
\[
\begin{array}{ccc}
\mathcal{G}_1^1(\mathbb{P}^n) \times \mathcal{G}_1^1(\mathbb{P}^m) & \otimes & \mathcal{G}_1^1(\mathbb{P}^{nm+n+m}) \\
\downarrow c \times h & & \downarrow c \\
\mathcal{Z}_0^1(\mathbb{P}^n) \times \mathcal{Z}_0^1(\mathbb{P}^m) & \xrightarrow{\otimes} & \mathcal{Z}_0^1(\mathbb{P}^{nm+n+m})
\end{array}
\]

The relevance of this diagram is twofold, on the one hand it provides a new way of calculating the formula for the total chern class of the tensor product of a vector bundle and a line bundle, namely

\[
c_i(E \otimes L) = \sum_{j=0}^i \binom{rk(E) - j}{i - j} c_j(E)c_1(L)^{i-j}
\]

and on the other hand it suggests a path for generalizing the Bott periodicity map which is related to the top arrow of this diagram (the problem for generalizing the Bott map is that there is no "orthogonal complement" in the space of cycles.)

The formula \( \ref{pairing} \) does not describe completely the map induced in cohomology by the
Theorem 1.4. The pairing
\[ Z^p_0(P^n) \times \mathbb{Z}_G^1(P^m) \to Z^p(P^{nm+n+m}) \]
induces the following map in rational cohomology
\[ \hat{\otimes}^*(i_{2k}) = \sum_{j=0}^k \binom{p-j}{k-j} i_{2j} \otimes \hat{i}_{2(i-j)} \]
where \( i_{2k} \) is the fundamental class of the \( k \)-th factor of \( Z^p_0(P^n) \) and \( \hat{i}_{2l} \) is the fundamental class in the \( l \)-th factor of \( \mathbb{Z}_G^1(P^m) \).

Using theorems 1.2 and 1.4 the following theorem is proved

Theorem 1.5. There is no continuous biadditive pairing in the stabilized space of cycles which makes the following diagram commute

\[
\begin{array}{ccc}
BU \times BU & \hat{\otimes} & BU \\
\downarrow & & \downarrow \\
\mathbb{Z} \times \mathbb{Z} & \hat{\otimes} & \mathbb{Z}
\end{array}
\]

Remark 1.6. Burt Totaro used the projection formula in [5] to prove that the total Chern class map does not extend to a map of cohomology theories. This is also implied by the results in this paper.

2. Tensor Pairing for Divisors

In this section we will define a product
\[ \hat{\otimes} : \mathbb{C}^1(P^{m-1}) \times \mathbb{C}^1(P^{m-1}) \to \mathbb{C}^1(P^{mn-1}) \]
which is continuous and biadditive (each of the factors has the structure of a topological monoid). This pairing generalizes the pairing on cycles of degree 1 which classifies the tensor product of the universal quotient bundle.

Let \( \mathbb{C}[[x]]_d := \mathbb{C}[x_0, \ldots, x_{n-1}]_d \) denote the set of complex polynomials of degree \( d \) in the variables \( x_0, \ldots, x_{n-1} \). This set is a complex vector space of dimension \( N = \binom{n+d}{n} \). If we order the variables lexicographically, we get the following ordered basis for this vector space:
\[ \mathcal{B} := \{ x_0 x_0 \cdots x_0, \ldots, x_{j_1} x_{j_2} \cdots x_{j_{n-1}}, \ldots, x_{n-1} \cdots x_{n-1} \} \]
where \( j_1 \leq j_2 \leq \ldots \leq j_{n-1} \), i.e. the set of all monomials of degree \( d \) ordered lexicographically.

Definition 2.1. Let
\[ \Psi_{de} : \mathbb{C}[[x]]_d \times \cdots \times \mathbb{C}[[x]]_d \times \mathbb{C}[[\tilde{y}]]_e \times \cdots \times \mathbb{C}[[\tilde{y}]]_e \to \mathbb{C}[[\tilde{z}]]_{de} \]
be the multilinear homomorphism defined on the elements of the bases $B$ by

$$\Psi_{de}(x_{j_1} \cdots x_{j_k}, \ldots, x_{j_1} \cdots x_{j_k}, y_{k_1} \cdots y_{k_1}, \ldots, y_{k_1} \cdots y_{k_2}) = z_{j_1}k_1 \cdots z_{j_1}k_1' \cdots z_{j_1}k_2 \cdots z_{j_1}k_2'$$

and extended multilinearly.

Notice that this definition depends on the ordered bases $B$, in particular, a different order on the elements would yield a different homomorphism.

The function $\Psi_{1e}$ has a particularly nice expression when the degree 1 forms are monomials.

**Lemma 2.2.**

$$\Psi_{1e}(x_{i_1}, \ldots, x_{i_e}, g) = g(z_{i_1j_1}, \ldots, z_{i_1j_e})$$

**Proof.** Let $g = \sum b_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}$. Then, following the definition we get

$$\Psi(x_{i_1}, \ldots, x_{i_e}, g) = \Psi(x_{i_1}, \ldots, x_{i_e}, \sum b_{j_1 \cdots j_e} y_{j_1} \cdots y_{j_e}) = \sum b_{j_1 \cdots j_e} \Psi(x_{i_1}, \ldots, x_{i_e}, y_{j_1} \cdots y_{j_e}) = \sum b_{j_1 \cdots j_e} z_{i_1j_1} \cdots z_{i_1j_e} = g(z_{i_1j_1}, \ldots, z_{i_1j_e})$$

With the previous definition, we can now define the tensor product of divisors:

**Definition 2.3.** Given $f \in \mathbb{C}[x_0, \ldots, x_{n-1}]_d$ and $g \in \mathbb{C}[y_0, \ldots, y_{m-1}]_e$ we define $f \otimes g \in \mathbb{C}[\ldots, z_{jk}, \ldots]$ by

$$f \otimes g := \Psi_{de}(f, \underbrace{f, \ldots, f}_{e \text{ times}}, g, \underbrace{g, \ldots, g}_{d \text{ times}})$$

**Example 2.4.** Let $f = x_0^2 - 3x_1x_2$ and $g = y_5y_7$. Then

$$f \otimes g = \Psi(f, f, g, g) = \Psi(x_0^2 - 3x_1x_2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) = \Psi(x_0^2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) - 3\Psi(x_1x_2, x_0^2 - 3x_1x_2, y_5y_7, y_5y_7) = -3\Psi(x_1x_2, x_0^2, y_5y_7, y_5y_7) + 9\Psi(x_1x_2, x_1x_2, y_5y_7, y_5y_7) = 205 205 205 207 - 320 5 205 217 227 - 35215 225 207 207 + 9215 225 217 227$$

The next result is the first step towards proving that the pairing is indeed biadditive.

**Proposition 2.5.**

$$\Psi_{rm}(f_1, \ldots, f_m, g, \ldots, g) \Psi_{sm}(\phi_1, \ldots, \phi_m, g, \ldots, g) = \Psi_{(r+s)m}(f_1 \phi_1, \ldots, f_m \phi_m, g, \ldots, g)$$

**Proof.** Suppose that

$$\Psi_{rm}(f_1, \ldots, f_m, g, \ldots, g) \Psi_{sm}(\phi_1, \ldots, \phi_m, g, \ldots, g) = \Psi_{(r+s)m}(f_1 \phi_1, \ldots, f_m \phi_m, g, \ldots, g)$$

and
\[ \Psi_{rm}(F, f_2, \ldots, f_m, g, \ldots, g) \Psi_{sm}(\phi_1, \ldots, \phi_m, g, \ldots, g) = \Psi_{(r+s)m}(F\phi_1, f_2\phi_2, \ldots, f_m\phi_m, g, \ldots, g) \]

then it follows from the multilinearity of \( \Psi_{ij} \) that

\[ \Psi_{rm}(f_1 + cF, f_2, \ldots, f_m, g, \ldots, g) \Psi_{sm}(\phi_1, \ldots, \phi_m, g, \ldots, g) = \\
[\Psi_{rm}(f_1, f_2, \ldots, f_m, g, \ldots, g) + \Psi_{rm}(cF, f_2, \ldots, f_m, g, \ldots, g)] \Psi_{sm}(\phi_1, \ldots, \phi_m, g, \ldots, g) = \\
\Psi_{(r+s)m}(f_1\phi_1 + cF\phi_1, f_2\phi_2, \ldots, f_m\phi_m, g, \ldots, g) = \\
\Psi_{(r+s)m}(f_1 + cF)\phi_1, \ldots, f_m\phi_m, g, \ldots, g) \]

therefore, it suffices to prove the statement in the case that \( f_1 \) is a monic monomial. Analogously, it suffices to prove the statement in the case that every \( f_i \) is a monomial and \( \phi_i \) is a monomial. That is, we must show

\[ (3) \]
\[ \Psi_{rm}(x_{i_1} \cdot \cdot \cdot x_{i_m}, \ldots, x_{i_m}, g, \ldots, g) \Psi_{sm}(x_{k_1} \cdot \cdot \cdot x_{k_m}, \ldots, x_{k_m}, g, \ldots, g) = \\
\Psi_{(r+s)m}(x_{i_1} \cdot \cdot \cdot x_{i_1}, x_{k_1} \cdot \cdot \cdot x_{k_1}, \ldots, x_{i_m} \cdot \cdot \cdot x_{k_m}, \ldots, g, \ldots, g) \]

without loss of generality we may assume that \( x_{i_w} \leq x_{i_v} \) if \( a \leq b \) and \( x_{k_w} \leq x_{k_v} \) if \( a \leq b \). We will prove equation 3 by induction on \( r \) and \( s \).

**Base:** \( r = 1 \) and \( s = 1 \). Notice that by lemma 2.2

\[ \Psi(x_{i_1}, \ldots, x_{i_m}, g) = g(z_{i_1j_1}, \ldots, z_{i_mj_m}) \]

therefore,

\[ \Psi(x_{i_1}, \ldots, x_{i_m}, g) \Psi(x_{k_1}, \ldots, x_{k_m}, g) = \\
g(z_{i_1j_1}, \ldots, z_{i_mj_m})g(z_{k_1j_1}, \ldots, z_{k_mj_m}) \]

On the other hand, if

\[ g(y_1, \ldots, y_m) = \sum a_{j_1 \cdot \cdot \cdot j_r} y_{j_1} \cdot \cdot \cdot y_{j_r} \]

then

\[ (4) \]
\[ \Psi(x_{i_1} x_{k_1}, \ldots, x_{i_m} x_{k_m}, g, g) = \\
\Psi(x_{i_1} x_{k_1}, \ldots, x_{i_m} x_{k_m}, \sum a_{j_1 \cdot \cdot \cdot j_r} y_{j_1} \cdot \cdot \cdot y_{j_r}, \sum a_{j_1' \cdot \cdot \cdot j_r'} y_{j_1'} \cdot \cdot \cdot y_{j_r'}) = \\
\sum a_{j_1 \cdot \cdot \cdot j_r} a_{j_1' \cdot \cdot \cdot j_r'} \Psi(x_{i_1} x_{k_1}, \ldots, x_{i_m} x_{k_m}, y_{j_1} \cdot \cdot \cdot y_{j_r}, y_{j_1'} \cdot \cdot \cdot y_{j_r'}) = \\
\sum a_{j_1 \cdot \cdot \cdot j_r} a_{j_1' \cdot \cdot \cdot j_r'} z_{\sigma_{i_1}^{j_1}} \cdot \cdot \cdot z_{\sigma_{i_m}^{j_m}} \cdot \cdot \cdot z_{\tau_{i_1}^{j_1}} \cdot \cdot \cdot z_{\tau_{i_m}^{j_m}} \]

where

\[ \sigma_s^t = \begin{cases} 
  i_s^t, & \text{if } i_s^t \leq k_s^t \\
  k_s^t, & \text{if } i_s^t > k_s^t
\end{cases} \quad \text{and} \quad \tau_s^t = \begin{cases} 
  k_s^t, & \text{if } i_s^t \leq k_s^t \\
  i_s^t, & \text{if } i_s^t > k_s^t
\end{cases} \]

Now, notice that if we exchange the definition of \( \sigma \) and \( \tau \) the sum on the right hand side of 3 remains unchanged. This happens because we are taking two copies of \( g \).
Therefore we may assume without loss of generality that \( \sigma^i = i^i \) and \( \tau^i = k^i \). In this case, equation (5) becomes

\[
\Psi(x^i_1 x^{i_1}, \ldots, x^i_m x^{i_m}, g, g) = \sum a_{j_1 \ldots j_i} \cdots z_{i_1 j_1} \cdots z_{i_m j_m} = \left( \sum a_{j_1 \ldots j_i} \cdots z_{i_1 j_1} \cdots z_{i_m j_m} \right) = g(z^{i_1}_{j_1}, \ldots, z^{i_m}_{j_m}) g(z^{i_1}_{k_1}, \ldots, z^{i_m}_{k_m})
\]

and the base of the induction is proved.

**Inductive step:** Essentially the same argument proves that both sides of (3) are equal to

\[
g(z^{i_1}_{j_1}, \ldots, z^{i_m}_{j_m}) \cdots g(z^{i_1}_{j_1}, \ldots, z^{i_m}_{j_m})
\]

\[
\Psi(f, \ldots, f, g) = \Psi(f, \ldots, f, \sum a_{j_1 \ldots j_i} y_{j_1} \cdots y_{j_n}) = \sum a_{j_1 \ldots j_i} \Psi(f, \ldots, f, y_{j_1} \cdots y_{j_n}) = \sum a_{j_1 \ldots j_i} \Psi(f \circ y_{j_1} \cdots y_{j_n})
\]

This last expression is exactly what we are looking for, it says that we should substitute in the polynomial \( g \) the variable \( y_{j_i} \) with the polynomial \( f \circ y_{j_i} \) which in turn is equal to the polynomial \( f \) evaluated in the variables \( z \).
We recall that there is a one to one correspondence between homogeneous polynomials in the variables \(x_0, \ldots, x_s\) and the codimension 1 algebraic cycles in \(\mathbb{P}^s\). The correspondence is given in the following way: If \(f\) is a polynomial and it decomposes as a product \(f_1^{\alpha_1} \cdots f_t^{\alpha_t}\) where each \(f_k\) is irreducible, then the corresponding cycle \(\mathcal{C}(f)\) is given by \(\sum \alpha_i V(f_i)\) where \(V(f_i)\) is the (necessarily irreducible) variety defined by the polynomial \(f_i\). The disjoint union of all codimension cycles \(\mathcal{C}_1(P^s)\) forms a monoid with respect to the formal addition of cycles. The following theorem expresses the results of this section in terms of cycles.

**Theorem 2.9.** There is an algebraic pairing \(\hat{\otimes}\) in the space of codimension 1 cycles in projective space:

\[
\hat{\otimes} : \mathcal{C}_1(P^{n-1}) \times \mathcal{C}_1(P^{m-1}) \to \mathcal{C}_1(P^{mn-1})
\]

which satisfies the following properties

1. \(\hat{\otimes}\) coincides with the tensor product \(\otimes\) on linear cycles.
2. \(\hat{\otimes}\) is biadditive:
   \[
   \eta_1 \eta_2 \hat{\otimes} \xi = \eta_1 \hat{\otimes} \xi + \eta_2 \hat{\otimes} \xi
   \]
   and
   \[
   \eta \hat{\otimes} \xi_1 \xi_2 = \eta \hat{\otimes} \xi_1 + \eta \hat{\otimes} \xi_2
   \]
3. \(\hat{\otimes}\) stabilizes to a pairing
   \[
   \hat{\otimes} : \mathcal{C}_1(P^\infty) \times \mathcal{C}_1(P^\infty) \to \mathcal{C}_1(P^\infty)
   \]

Since the pairing is biadditive, it induces a pairing in the group completion

\[
\hat{\otimes} : \mathbb{Z}_1(P^\infty) \times \mathbb{Z}_1(P^\infty) \to \mathbb{Z}_1(P^\infty)
\]

In order to get a commutative diagram we construct the associated reduced pairing \(\hat{\otimes}\):

\[
\hat{\otimes}(\eta, \xi) := \eta \hat{\otimes} \xi + \eta \hat{\otimes} \xi_0 + \eta_0 \hat{\otimes} \xi
\]

where \(\xi_0\) and \(\eta_0\) are two fixed hyperplanes.

**Theorem 2.10.** The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{G}_1(P^\infty) \times \mathcal{G}_1(P^\infty) & \longrightarrow & \mathcal{G}_1(P^\infty) \\
\downarrow & & \downarrow \\
\mathbb{Z}_1(P^\infty) \times \mathbb{Z}_1(P^\infty) & \hat{\otimes} & \mathbb{Z}_1(P^\infty)
\end{array}
\]

It is proved in \([3]\) that the space \(\mathbb{Z}_1(P^\infty)\) splits as \(\mathbb{Z} \times \mathbb{Z}_0(P^\infty)\), where \(\mathbb{Z}_0(P^\infty)\) is the subgroup of all cycles of degree zero.

Since we know that \(\text{deg}(\eta \hat{\otimes} \xi) = \text{deg}(\eta) \text{deg}(\xi)\) we only have to calculate what happens with the pairing \(\hat{\otimes}\) when we restrict it to cycles of degree 0. Lawson proved that \(\mathbb{Z}_0(P^\infty)\) is an Eilenberg-Maclane space of type \(K(\mathbb{Z}, 2)\). Using this fact we will show that the pairing \(\hat{\otimes}\) restricted to the subgroup of cycles of degree zero is nullhomotopic.

**Theorem 2.11.** Any continuous biadditive pairing

\[
\hat{\otimes} : \mathbb{Z}_0(P^\infty) \times \mathbb{Z}_0(P^\infty) \to \mathbb{Z}_0(P^\infty)
\]

is nullhomotopic.
Proof. Since the pairing is biadditive it factors through the smash product
\[ Z_1^0(\mathbb{P}^\infty) \wedge Z_1^0(\mathbb{P}^\infty) \to Z_0^1(\mathbb{P}^\infty) \]
homotopically this last function is equivalent to a map
\[ K(\mathbb{Z}, 2) \wedge K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2) \]
Now, notice that \( K(\mathbb{Z}, 2) \wedge K(\mathbb{Z}, 2) \) is a CW-complex with cells only in dimension 4 and higher, therefore, the pullback in the second cohomology groups of the fundamental class in \( K(\mathbb{Z}, 2) \) is zero. \( \square \)

This theorem allows us to calculate the class pulled back via the pairing \( \hat{\otimes} \).

**Corollary 2.12.** Let \( \hat{\otimes} \) be the pairing
\[ \hat{\otimes}(\eta, \xi) := \eta \hat{\otimes} \xi + \eta \hat{\otimes} \xi_0 + \eta_0 \hat{\otimes} \xi \]
and let \( i_2 \) be the fundamental class in \( H^2(\mathbb{Z}^1(\mathbb{P}^\infty); \mathbb{Z}) \). Then
\[ \hat{\otimes}^*(i_2) = i_2 \otimes i_0 + i_0 \otimes i_2 \]

**Proof.** The formula follows at once from pulling back the last two summands of the pairing \( \hat{\otimes} \), since by the previous theorem the first summand is nullhomotopic. \( \square \)

**Corollary 2.13.** Let \( L_1 \) and \( L_2 \) be two line bundles. Then
\[ c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \]
where \( c_1 \) denotes the first chern class.

**Proof.** Lawson and Michelsohn proved in [3] that the inclusion \( i \) in theorem 2.10 classifies the chern class of the universal quotient bundle. Therefore the formula follows from the previous corollary. \( \square \)

### 3. Topological Obstruction for a General Pairing

The pairing constructed in the last section might be considered as a hint for a pairing in higher codimensions. We will prove that the is a topological obstruction for the existence of such a pairing. The general strategy is to factor the inclusion of the grassmannian \( \mathcal{G}^p(\mathbb{P}^n) \) into the space \( \mathcal{Z}^p(\mathbb{P}^n) \) of all codimension \( p \) cycles. This inclusion factors through the free abelian group \( \mathbb{Z}\mathcal{G}^p(\mathbb{P}^n) \) generated by the points of the grassmannian. The existence of this factorization and the chern class formula for the tensor product of bundles will yield a contradiction if we assume the existence of a pairing.

Let us start by observing that the Dold-Thom theorem implies that the free abelian group \( \mathbb{Z}\mathcal{G}^1(\mathbb{P}^n) \) is homotopically equivalent to the product \( \prod_{i=0}^{n} K(\mathbb{Z}, 2i) \). Also, if we consider the subgroup \( \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \) which is the kernel of the degree homomorphism i.e. the subgroup of 0-dimensional cycles of degree 0, we get the following homotopy equivalence
\[ \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \simeq \prod_{i=1}^{n} K(\mathbb{Z}, 2i) \]

Dold and Thom also proved that the inclusion
\[ i : \mathcal{G}^1(\mathbb{P}^n) \hookrightarrow \mathbb{Z}_0\mathcal{G}^1(\mathbb{P}^n) \]
induces the Hurewicz map when the \( \pi_i \) functors are applied. The next theorem calculates the class pulled back in cohomology by the inclusion \( i \).
Theorem 3.1. The Hurewicz map

\[ h : \mathcal{G}^1(\mathbb{P}^n) \rightarrow \mathbb{G}^1(\mathbb{P}^n) \simeq \prod_{i=0}^{n} K(\mathbb{Z}, 2i) \]

induces the following map in cohomology

\[ h^*(i_{2k}) = \omega^k \]

where \(i_{2k}\) is the generator of \(H^{2k}(K(\mathbb{Z}, 2k), \mathbb{Z})\) and \(\omega\) is the generator of \(H^2(\mathcal{G}^1(\mathbb{P}^n), \mathbb{Z})\).

Proof. By induction on \(n\).

Base: The case \(n = 1\) is a result of Lawson and Michelsohn in [3]. Namely, they prove that the inclusion \(i : \mathcal{G}^1(\mathbb{P}^1) \rightarrow \mathbb{G}^1(\mathcal{G}^1(\mathbb{P}^1))\) classifies the total Chern class of the universal quotient bundle, i.e. that \(i \simeq 1 \times \omega\). But in this case \(\mathbb{G}^1(\mathbb{P}^1) \simeq \mathbb{Z}\) and \(i\) is the Hurewicz map \(h\). Thus \(h \simeq 1 \times \omega\).

Inductive Step: Notice that \(\mathcal{G}^1(\mathbb{P}^n) \simeq \mathbb{P}^n\), so we will substitute throughout \(\mathcal{G}^1(\mathbb{P}^n)\) with \(\mathbb{P}^n\). Suppose that \(h^*(i_{2k}) = \omega^k\) for \(h : \mathbb{P}^n \rightarrow \mathbb{Z}\mathbb{P}^n\). The inclusion of \(\mathbb{P}^n \rightarrow \mathbb{P}^{n+1}\) is a cofibration and the quotient \(\mathbb{P}^{n+1}/\mathbb{P}^n\) is homeomorphic to the sphere \(S^{2(n+1)}\). Dold and Thom proved in [2] that a cofibration sequence induces a quasifibration sequence when taking the free abelian group functor. Hence we have the following commutative diagram:

\[
\begin{array}{ccc}
\mathbb{P}^n & \overset{j}{\rightarrow} & \mathbb{P}^{n+1} \\
\downarrow{h} & & \downarrow{h} \\
\mathbb{Z}\mathbb{P}^n & \overset{p}{\rightarrow} & \mathbb{Z}S^{2(n+1)}
\end{array}
\]

where \(j\) is a cofibration, \(p\) is a quasifibration and each \(h\) is the corresponding Hurewicz map. This diagram is homotopically equivalent to the following:

\[
\begin{array}{ccc}
\prod_{i=0}^{n} K(\mathbb{Z}, 2i) & \overset{j}{\rightarrow} & \prod_{i=0}^{n+1} K(\mathbb{Z}, 2i) \\
\downarrow{h} & & \downarrow{h} \\
\prod_{i=0}^{n+1} K(\mathbb{Z}, 2i) & \overset{p_{n+1}}{\rightarrow} & K(\mathbb{Z}, 2(n+1))
\end{array}
\]

The induction hypothesis implies that the vertical arrow on the left satisfies the condition \(h^*(i_{2k}) = \omega^k\). Therefore we are only concerned with what happens to the pullback of \(i_{2(n+1)}\) in the middle vertical arrow, but this is determined by the Hurewicz map on the far right of the diagram. \(\square\)

Theorem 3.2. For \(p > 1\) there is no continuous biadditive pairing

\[ \hat{\otimes} : \mathcal{Z}^1(\mathbb{P}^n) \times \mathcal{Z}^p(\mathbb{P}^m) \rightarrow \mathcal{Z}^p(\mathbb{P}^{nm+n+m}) \]

such that \(\eta \hat{\otimes} \xi = \eta \otimes \xi\) where \(\eta\) and \(\xi\) are linear spaces and \(\otimes\) is the map which classifies the tensor product of bundles via the universal quotient bundle.

Proof. Suppose that such a pairing exists. Then it must necessarily satisfy the following relation in the degrees

\[ \deg(\eta \hat{\otimes} \xi) = \deg(\eta) \deg(\xi) \]
This is because it is biadditive and continuous and it maps the degree one effective cycles into the degree one effective cycles. Thus it induces a continuous pairing in the subgroup $Z_0$ of cycles of degree zero:

$$Z_0^1(P^n) \times Z_0^p(P^m) \rightarrow Z_0^1(P^{nm+n+m})$$

Let $\mu : Z_0^1(P^n) \times Z_0^p(P^m) \rightarrow Z_0^1(P^{nm+n+m})$ be the function defined by

$$\mu(\eta, \xi) = \eta \circ \xi + \eta_0 \circ \xi + \eta \circ \xi_0$$

Then the following diagram commutes:

$$
\begin{array}{ccc}
G^1(P^n) \times G^p(P^m) & \xrightarrow{\otimes} & G^p(P^{nm+n+m}) \\
\downarrow{i \times c} & & \downarrow{c} \\
Z_0^1(P^n) \times Z_0^p(P^m) & \xrightarrow{\mu} & Z_0^p(P^{nm+n+m})
\end{array}
$$

where the vertical maps are the inclusions mapping a linear space $\eta$ into $\eta - \eta_0$ where $\eta_0$ is a fixed subspace. Lawson and Michelsohn proved in [3] that this inclusion classifies the total chern class map of the universal quotient bundle. Now, notice that we can restrict the pairing $\mu$ on the first factor to the subspaces $Z_0^1 G^1$ of cycles generated by the points of the grassmannian, that is, to the cycles which are formal sums of linear hypersurfaces with coefficients adding up to zero. Let $\rho$ be the restriction, then we have the following commutative diagram

$$
\begin{array}{ccc}
G^1(P^n) \times G^p(P^m) & \xrightarrow{\otimes} & G^p(P^{nm+n+m}) \\
\downarrow{i \times c} & & \downarrow{c} \\
Z_0^1 G^1 \times Z_0^p(P^m) & \xrightarrow{\rho} & Z_0^p(P^{nm+n+m}) \\
\downarrow{j \times id} & & \downarrow{id} \\
Z_0^1(P^n) \times Z_0^p(P^m) & \xrightarrow{\mu} & Z_0^p(P^{nm+n+m})
\end{array}
$$

where $i$ is the same map as before, $i(L) = L - L_0$ and $j$ is just the natural inclusion. The previous theorem gives a description of what $i$ and $j$ are in terms of the homotopy equivalences with the products of Eilenberg-MacLane spaces, namely $i$ is the Hurewicz map and $j$ is the projection onto the first factor. Hence we have the following diagram:

$$
\begin{array}{ccc}
G^1(P^n) \times G^p(P^m) & \xrightarrow{\otimes} & G^p(P^{nm+n+m}) \\
\downarrow{h \times c} & & \downarrow{c} \\
\prod_{i=1}^n K(Z, 2i) \times \prod_{i=1}^p K(Z, 2i) & \xrightarrow{\rho} & \prod_{i=1}^{nm+n+m} K(Z, 2i) \\
\downarrow{\pi_1 \times id} & & \downarrow{id} \\
K(Z, 2) \times \prod_{i=1}^p K(Z, 2i) & \xrightarrow{\mu} & \prod_{i=1}^{nm+n+m} K(Z, 2i)
\end{array}
$$

We will prove the case $p = 2$, the general proof is analogous. The chern class formula for the tensor product of a line bundle and a 2-dimensional bundle yields:

$$c_2(L \otimes E) = c_1^2(L) + c_1(E) c_1(L) + c_2(E)$$
The vertical arrows in the diagram induce isomorphisms in rational cohomology. So the chern class formula implies that in the 4-th cohomology groups $\rho$ should induce the following map:

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + ai_4 \otimes 1 + bi_2^2 \otimes 1$$

where each $i_k$ is the generator of $H^k(K(\mathbb{Z}, 2k); \mathbb{Q})$ and $a + b = 1$.

We claim that $a = 1$ and $b = 0$. This claim is the content of proposition 3.3.

The argument to prove the theorem is then the following:

The existence of the product $\hat{\otimes}$ implies the existence of the function $\mu$ which in turn implies the existence of the restriction $\rho$. But then, the claim implies that the diagram cannot commute!

This is because diagram implies that

$$\rho^*(i_4) = (\pi_1 \times id)^* \mu^*(i_4)$$

but there is no element in $H^4(K(\mathbb{Z}, 2) \times \prod_{i=1}^n K(\mathbb{Z}, 2i); \mathbb{Q})$ which gets pulled back to $i_4 \otimes 1$ in $H^4(\prod_{i=1}^n K(\mathbb{Z}, 2i) \times \prod_{i=1}^n K(\mathbb{Z}, 2i); \mathbb{Q})$ because $\pi_1$ is the projection into the first factor:

$$\pi_1 : \prod_{i=1}^n K(\mathbb{Z}, 2i) \to K(\mathbb{Z}, 2)$$

therefore we can only pullback elements of the form $ai_2^p \otimes j$ where $i_2$ is the generator of $H^2(K(\mathbb{Z}, 2); \mathbb{Q})$ and $i_2$ is the generator of $H^2(K(\mathbb{Z}, 2); \mathbb{Q})$. (These last equalities being a classical result of Serre).

Now we prove the claim mentioned in theorem 3.2.

**Proposition 3.3.** Using the notation of theorem 3.2 we have the formula

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + i_4 \otimes 1$$

**Proof.** The chern class formula for the tensor product of bundles and the commutativity of $\hat{\otimes}$ implies that

$$\rho^*(i_4) = 1 \otimes i_4 + i_2 \otimes i_2 + a(i_4 \otimes 1) + b(i_2^2 \otimes 1)$$

with $a + b = 1$. Consider the diagram

$$
\begin{array}{ccc}
G^1(\mathbb{P}^n) \times G^1(\mathbb{P}^n) & \otimes 1 & \times \ottimes 2 & \otimes 1 & \times \ottimes 2 \rightarrow & G^2(\mathbb{P}^{mn+n+m}) & \times \ottimes 2 \rightarrow & G^2(\mathbb{P}^{mn+n+m}) \\
\phi \times c & & c \otimes c & & c \otimes c & & c \otimes c \\
Z_0^1(\mathbb{P}^n) \times Z_0^2(\mathbb{P}^m) & \otimes 1 & \times \ottimes 2 & \otimes 1 & \times \ottimes 2 \rightarrow & Z_0^2(\mathbb{P}^{mn+n+m}) \\
\end{array}
$$

where

- $\phi : G^1(\mathbb{P}^n) \times G^1(\mathbb{P}^n) \rightarrow Z_0^1(\mathbb{P}^n)$ is given by
  $$\phi(L_1, L_2) = (L_1 - L_0) + (L_2 - L_0)$$

where $L_0$ is a fixed linear space.
\( \otimes_1 \times \otimes_2 \) is given by
\[
(\otimes_1 \times \otimes_2)(L_1, L_2, E) = (L_1 \otimes E, L_2 \otimes E)
\]

- \( \phi \) is given by
\[
(c + c)(E_1, E_2) = (E_1 - L_0 \otimes E_0) + (E_2 - L_0 \otimes E_0)
\]
- \( \rho' = \rho + \tau \) where
\[
\tau(\eta, \xi) = L_0 \hat{\otimes} \xi
\]

We will verify that diagram 9 commutes:
\[
(c + c)(\otimes_1 \times \otimes_2)(L_1, L_2, E) = (c + c)(L_1 \otimes E, L_2 \otimes E) =
(L_1 \otimes E - L_0 \otimes E_0) + (L_2 \otimes E - L_0 \otimes E_0)
\]

On the other hand:
\[
\rho'(\phi \times c)(L_1, L_2, E) = \rho'((L_1 - L_0) + (L_2 - L_0), E - E_0) =
\rho((L_1 - L_0) + (L_2 - L_0), E - E_0) + \tau((L_1 - L_0) + (L_2 - L_0), E - E_0) =
\rho((L_1 - L_0) + (L_2 - L_0), E - E_0) + L_0 \hat{\otimes} (E - E_0) =
((L_1 - L_0) + (L_2 - L_0)) \hat{\otimes} (E - E_0) + 2(L_0 \hat{\otimes} (E - E_0)) + ((L_1 - L_0) + (L_2 - L_0)) \hat{\otimes} E_0 =
L_1 \hat{\otimes} E + L_2 \hat{\otimes} E - 2(L_0 \hat{\otimes} E_0)
\]

Now recall that the space \( \mathbb{Z}_0^2(\mathbb{P}^{n+m+n}) \) on the lower right corner of diagram 9 is homotopically equivalent to \( K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4) \). We will compute the pull-back through the whole diagram of the generator \( i_4 \) of the cohomology group \( H^4(K(\mathbb{Z}, 4); \mathbb{Q}) \), considered as a subgroup
\[
H^4(K(\mathbb{Z}, 4); \mathbb{Q}) \subset H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 4); \mathbb{Q}) = \mathbb{Q}i_4 \oplus \mathbb{Q}i_2^2
\]

To simplify the notation we will denote by \( L_1, L_2 \) and \( E \) the universal quotient bundles on \( \mathbb{G}^1 \), \( \mathbb{G}^1 \) and \( \mathbb{G}^2 \) correspondingly. Then the chern class formula for the tensor product and the fundamental result of 3 compute the composition \( (\otimes_1 \times \otimes_2)^*(c + c)^* \):
\[
(\otimes_1 \times \otimes_2)^*(c + c)^*(i_4) = (\otimes_1 \times \otimes_2)^*(c_2(E) \otimes c_2(E)) =
\]
\[c_1(L_1)^2 + c_1(L_1)c_1(E) + c_2(E) + c_1(L_2)^2 + c_1(L_2)c_1(E) + c_2(E)
\]

Now, notice that theorem 1.2 implies that in rational cohomology
\[
\phi^*(i_2) = \omega \otimes 1 + 1 \otimes \omega \quad \text{and} \quad \phi^*(i_4) = \omega^2 \otimes 1 + 1 \otimes \omega^2
\]

Hence, using the previous equation and the description that we have for \( \rho \) we get
\[
(\phi \times id)^*(\rho')^*(i_4) = (\phi \times id)^*(i_2 \otimes i_2 + 1 \otimes i_4 + b(i_2 \otimes 1) + a(i_4 \otimes 1) + 1 \otimes i_4) =
c_1(L_1)c_1(E) + c_1(L_2)c_1(E) + c_2(E) +
a(c_1(L_1) + c_1(L_2))^2 + c_1(L_1)^2 + c_1(L_2)^2 + c_2(E)
\]

Setting equal the compositions 10 and 12 we get that \( b = 0 \) and therefore \( a = 1 \), since there is no term \( 2c_1(L_1)c_1(L_2) \) in 10.

\[\square\]
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