Comments on “Angular Momentum Transport in Quasi-Keplerian Accretion Disks”

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Subramanian, Pujari and Becker (2004) claim that the correct expression for the angular momentum transport in an accretion disc, which is proportional to $d\Omega/dR$, can be derived on the basis of the analysis of the epicyclic motion of gas parcels in adjacent eddies in the disc. We study their argument and show that their derivation contains several fundamental errors: 1) the biased choice of the desired formula from an infinite number of formulae; 2) the biased choice of parcel trajectories; and 3) confusion regarding the reference frames. Following 1) we could derive, for example, a (invalid) formula in which the angular momentum transport is proportional to $dv_\phi/dR$, and from 2) we could even prove that the angular momentum transport is either inward or null. We present the correct approach to the problem of angular momentum transport in an accretion disc in terms of mean free path theory.

§1. Introduction

Elucidating the mechanism(s) involved in the transport of angular momentum in accretion discs is a long-standing problem. Observations show that the gas in an accretion disc rotating around a compact object in a close binary does gradually accrete on the compact object. This can only happen if the angular momenta of the molecules/fluid parcels constituting the rotating gas are transported outwardly through the disc. (In the present paper we do not distinguish between molecules in kinetic theory and fluid parcels in turbulent motion, unless explicitly stated.) In recent years, the theory of angular momentum transport by magnetically induced turbulence is getting fashionable (e.g. Balbus and Hawley). Nevertheless, it is important to derive a correct viscosity formula in a rotating gas, and we concentrate on the viscosity due to either molecular motion or hydrodynamic turbulence in the present paper.

Consider a rotating gas flow represented by the velocity field $\mathbf{u} = (0, R\Omega(R))$ in cylindrical coordinates, where $R$ and $\Omega$ are the radial distance and angular velocity, respectively. Then the correct $R - \phi$ component of the viscous stress should be

$$\sigma_{R\phi} = -\eta R(d\Omega/dR),$$

where $\eta$ is the viscosity coefficient.

In order to prove that the viscous stress is proportional to $d\Omega/dR$, some textbooks apply a simple mean free path theory, in which a gas molecule/cell parcel jumps over a distance equal to the mean free path, carrying with it the angular momentum it possessed at the place where it originated (Frank, King and Raine, hereinafter referred to as FKR; Hartmann). However, Hayashi and Matsuda (hereinafter typeset using PTP\TeX.cls (Ver.0.9)
HM) pointed out that if the proof is carried out correctly, it should lead to the incorrect formula \( \sigma_R \propto -d(R^2 \Omega)/dR \) (see also Subramanian, Pujari and Becker,\(^5\) hereinafter referred to as SPB). This is clear when one applies a simple mean free path theory (see, for example, Vincenti and Kruger\(^6\)). In consideration of this, HM claimed, quoting Vincenti and Kruger,\(^6\) that the correct formula cannot be derived on the basis of the mean free path theory alone. They speculated that the correct formula can only be derived using the Boltzmann equation with the Coriolis force taken into account.

In 2004, however, Clarke and Pringle\(^7\) (hereinafter referred to as CP) showed that the correct viscosity formula for a rotating gas, Eq. (1), can indeed be derived on the basis of the mean free path theory, if one properly takes into account the thermal motion/spread of the molecules. They obtained the correct result in the inertial frame, thus obviating inclusion of the Coriolis force. They approximate molecular orbits by straight lines, which is acceptable under the assumption \( \lambda \ll R \), or, more precisely, under the assumption \( \lambda \ll H \ll R \), where \( H \) and \( R \) are the disc thickness and the radial distance. Note that \( H = c/\Omega \), where \( c \) is a typical molecular speed.

Developing CP’s idea further, Matsuda and Hayashi\(^8\) (hereinafter referred to as MH) showed that the results can be grasped more easily and calculated more simply when calculations are made in a rotating frame. Quoting MH’s conclusion: “... of the molecules originally present in the inner annulus, those whose angular momenta are larger than the average angular momentum of the outer annulus will preferentially be transported from the inner to the outer annulus. The contribution from those molecules that have smaller angular momenta and have reached the outer annulus is small. Conversely, of the molecules originally present in the outer annulus, those having angular momenta smaller than the average angular momentum of the inner annulus will preferentially be transported to the inner annulus. Accordingly, angular momentum indeed does flow against its gradients.”

This situation is best described by the schematic diagram in Fig. 1, in which the horizontal and the vertical axes represent the radial position \( R \) and the specific angular momentum of the rotating gas \( J \) in a Keplerian disc, respectively. Molecules/ fluid parcels in the inner annulus at \( R - \lambda/2 \) fly to the outer annulus at \( R + \lambda/2 \) while maintaining their angular momentum, and vice versa. The solid curve represents the radial distribution of the angular momentum of the gas. The arrows C and D represent the angular momentum transport considered by, say, Hartmann.\(^3\) In this picture, it is apparent that the angular momentum transport is inward. In order to have the correct picture, we must take into account the thermal spread of the angular momentum, which is represented by the vertical dotted lines in Fig. 1. If one calculates the mean angular momentum transfer correctly, it should be represented by the arrows A and B. In this picture the angular momentum is transported outward. The original claim of HM, that the correct formula (1) cannot be derived from the mean free path theory, turns out to be incorrect.

In the SPB paper, it is also claimed that the correct formula for the viscous torque (1) can be derived on the basis of the mean free path theory. According
Fig. 1. Schematic diagram showing the angular momentum transport in a Keplerian disc. The horizontal axis \( R \) represents the radial coordinate, and the vertical axis \( J \) the specific angular momentum of gas. Molecules or fluid parcels move from the inner annulus at \( R - \lambda/2 \) to the outer one at \( R + \lambda/2 \), where \( \lambda \) is the mean free path. The arrows A and B indicate the correct inward/outward angular momentum transport, while C and D represent the wrong ones adopted in a textbook. The vertical dotted lines represent the thermal spread of the angular momentum of molecules/parcels.

To SPB, the correct formula can be derived by properly taking into account the epicyclic motion of gas parcels. Because both CP and MH approximate the molecule trajectory as linear, their arguments are valid only for \( \lambda \ll H \), while SPB treat the case \( \lambda \sim H \). However, we find that the process of deriving the “correct” formula employed by SPB is incorrect. In §2 we briefly review SPB and point out their three major errors. A summary of our conclusions is given in §3.

§2. Process employed by SPB and their errors

2.1. Their first error: biased choice of the desired formula

SPB start from Fig. 2, which, together with its caption in quotation marks (but excluding the dotted arcs, which were added by the present authors), is shown here and may be self-explanatory. A fluid parcel A jumps from the outer annulus to the
Fig. 2. “Angular momentum transport in the disc involves the interchange of two parcels, A and B, initially at radii $R + \lambda/2$ and $R - \lambda/2$, respectively, around central mass M. They each participate in ballistic, epicyclic motion in eddies of radius $\lambda$, with A moving inwards and B moving outwards.” (Taken from SPB, with the radial distance labels rearranged.) The parcel A has a smaller angular momentum, $J_{\text{out}}$, while B has a larger angular momentum, $J_{\text{in}}$. Therefore, one can prove that the angular momentum flows outwards (SPB). However, there are other possible trajectories of, for example, parcels $A'$ and $B'$ having angular momenta $J'_{\text{out}} (= J_{\text{in}})$ and $J'_{\text{in}} (= J_{\text{out}})$. Thus we could equally prove that the angular momentum flows inwards. Or, if we take into account A, B, $A'$ and $B'$ altogether, we could prove that the angular momentum transfer is zero. Therefore, this figure alone proves nothing. (See the discussion in §2.2.)

The above step, at first glance, seems to be valid, provided that we accept Fig. 2.
(SPB’s original, i.e. without dotted arcs). However, this is proved to be incorrect in §2.2. Because \( J_{\text{in}} > J_{\text{out}} \), they are also to claim that the angular momentum flows outwards. For the moment, let us accept the above argument.

The next steps they take, however, are problematic. In particular, they derive the “correct” formula \( J_{\text{in}} - J_{\text{out}} \propto R^2 d\Omega/dR \) as follows (SPB):

“The angular velocity \( \Omega(R) \) in a quasi-Keplerian accretion disc is very close to the Keplerian value, and therefore we can write

\[
\Omega(R) = \sqrt{\frac{GM}{R^3}}. \tag{25}
\]

It follows that

\[
R^2 \Omega \left( R + \frac{\lambda}{6} \right) = \sqrt{GM R} \left[ 1 - \frac{1}{4} \frac{\lambda}{R} \right]^{-3/2}, \tag{26}
\]

or, to first order in \( \lambda/R \),

\[
R^2 \Omega \left( R + \frac{\lambda}{6} \right) = \sqrt{GM R} \left[ 1 - \frac{1}{4} \frac{\lambda}{R} \right] + O \left[ \frac{\lambda^2}{R^2} \right]. \tag{27}
\]

This also implies that

\[
R^2 \Omega \left( R - \frac{\lambda}{6} \right) = \sqrt{GM R} \left[ 1 + \frac{1}{4} \frac{\lambda}{R} \right] + O \left[ \frac{\lambda^2}{R^2} \right]. \tag{28}
\]

Comparing equations (23) and (24) with equations (27) and (28), we find that to first order in \( \lambda/R \), the net angular momentum transfer is given by

\[
J_{\text{in}} - J_{\text{out}} = R^2 \Omega \left( R - \frac{\lambda}{6} \right) - R^2 \Omega \left( R + \frac{\lambda}{6} \right) \simeq - \frac{1}{3} \frac{\lambda^3}{R^2} \frac{d\Omega}{dR}. \tag{29}
\]

(a few lines omitted)

Hence we have demonstrated using a simple heuristic derivation that the viscous torque is indeed proportional to the gradient of the angular velocity in an accretion disc within the context of a mean free path, parcel-exchange picture.”

The steps above taken by SPB are incorrect. Firstly, why does \( \lambda/6 \) appear in Eqs. (26)-(29)? In fact, this has been chosen so as to derive \( \lambda/4R \) in Eqs. (23) and (24). In the following we show how the coefficient 1/6 can be obtained.

Let us assume that \( J_{\text{in}} - J_{\text{out}} \propto R^2 (d\Omega/dR) \) and attempt to obtain a relevant parameter \( \alpha \) to multiply \( \lambda \). First, we write down the following equation to obtain \( \alpha \):

\[
R^2 \Omega(R + \alpha \lambda) = \sqrt{GM R} \left[ 1 - \frac{1}{4} \frac{\lambda}{R} \right]. \tag{A}
\]

Then, assuming \( \lambda \ll R \), we can expand the left-hand side and retain the first term, obtaining
LHS \approx R^2(\Omega + \alpha \lambda \Omega d\Omega/dR) = R^2\left(\Omega + \alpha \lambda \left(-\frac{3\Omega}{2R}\right)\right) = \sqrt{GMR}\left(1 - \frac{3\alpha \lambda}{2R}\right). \quad (B)

Finally, equating the coefficients of the last terms in (A) and (B), we get \(\alpha = 1/6\). This value of \(\alpha\), together with that for Eq. (28), leads to Eq. (29) which they sought.

We now show that with their argument, we could derive any (invalid) formula we desire. First, let us assume the following (invalid) relation: \(J_{in} - J_{out} \propto R d\varphi/dR\), where \(\varphi = R\Omega\) is the azimuthal velocity. With this, we can prove the following relation:

\[
Rv_{\varphi}(R + \frac{1}{2}\lambda) = \sqrt{GMR}\left(1 - \frac{1}{4R}\right). \quad (27)^{'}
\]

The numerical coefficient \(1/2\) multiplying \(\lambda\) on the LHS of (27)^{'} is obtained by solving the following equation for \(\beta\):

\[
R(R + \beta\lambda)\Omega(R + \beta\lambda) = \sqrt{GMR}\left(1 - \frac{1}{4R}\right). \quad (C)
\]

To obtain \(\beta = 1/2\) is straightforward.

The above argument clearly shows that their result, (29), is not unique, as it demonstrates that we can also derive \(J_{in} - J_{out} \propto R d\varphi/dR\), which is obviously incorrect. Indeed, this derivation suggests that we can derive further similar relations. We may in fact assume the following general formula:

\[
J_{in} - J_{out} \propto R^n d(R^{2-n}\Omega)/dR \propto R^n d(R^{1/2-n})/dR, \quad 1/2 < n. \quad (D)
\]

This general formula indicates that there are an infinite number of formulae satisfying \(1/2 < n\). One such “false” formula, \(J_{in} - J_{out} \propto Rd(R\Omega)/dR\), which we presented above, corresponds to the case \(n = 1\). The correct formula corresponds to the case \(n = 2\).

SPB’s argument is similar to deriving an equation from a solution, a procedure which is generally invalid.

2.2. Second error in SPB: biased choice of parcel trajectories

Secondly, in Fig. 2, the thick solid arcs starting at A and B are not the only possible trajectories. We accept SPB’s approximation of the epicycle as circular, (although, in fact, it should be an ellipse having a major axis to minor axis ratio of 2:1 if the central gravity is taken into account). In Fig. 2 we add examples of other possible trajectories of parcels A’ and B’ in dotted arcs. As apparent from the figure, A’ and B’ start from the outer and inner annulus, respectively, carrying their angular momenta \(J'_{out}\) and \(J'_{in}\). In this case \(J'_{out} (= J_{in})\) is larger than \(J'_{in} (= J_{out})\), just contrary to \(J_{out} < J_{in}\) derived in SPB. Thus we can similarly prove that the angular momentum flows inwards rather than outwards. Or, if we take into account A, B, A’ and B’ altogether, we could prove that the angular momentum transport is zero. The parcel trajectories chosen by SPB are thus just those favorable to their argument. Figure 2 alone does not prove that the angular momentum flows outwards.
In order to obtain the correct conclusion, that the angular momentum flows outwards, one has to take into account the velocity distribution or thermal spread of molecules at the ejecting point. Then we find that particles ejected in the forward direction, carrying larger angular momenta than those carried by the mean flow at the ejection point, are preferentially transported outwards. This was shown by MH in the limit of a short mean free path. The same consideration should also apply to the case of a larger mean free path.

We now raise another question with regard to Fig. 2, which was originally presented by SPB. Let us specify the molecular trajectories, rather than a vague notion of the parcel motion of turbulent eddies. In kinetic theory, the mean free path, \( \lambda \), and the radius of the epicycle, \( \sim H \), are completely different concepts. The former is defined as \( \lambda = 1/n\sigma \), where \( n \) and \( \sigma \) are the number density and the cross section of the molecules, respectively. Contrastingly, \( H \) is defined as \( c/\Omega \), where \( c \) is the thermal molecular velocity. SPB confuses these two different concepts.

2.3. Third error in SPB: reference frames

Let us comment on a criticism of HM made by SPB. In their §3.3, where they discuss the derivation of the viscous torque by FKR, they write (here again the equation numbers are theirs) the following:

"... They (HM) assert that the linear velocity of the plasma at \( R - \lambda/2 \) as viewed by an observer at radius \( R \) should instead be given by

\[
v_{\text{rel}} \left( R - \frac{\lambda}{2} \right) = \left( R - \frac{\lambda}{2} \right) \Omega \left( R - \frac{\lambda}{2} \right) - R\Omega(R) + \Omega(R)\frac{\lambda}{2}.
\]

(13)

By employing this expression for \( v_{\text{rel}} \) and following the same procedure used to obtain equation (12), they find that to first order in \( \lambda \),

\[
G = L_{\text{in}} - L_{\text{out}} = -2\pi R^3 \beta \nu \Sigma \frac{d\Omega}{dR}.
\]

(14)

This expression for \( G \) does indeed have the correct dependence on \( d\Omega/dR \), but nonetheless we claim that equation (13) for the relative velocity used by HM is incorrect. The correct expressions for the relative velocities are in fact (see, e.g., Mihalas & Binney 1981)

\[
v_{\text{rel}} \left( R - \frac{\lambda}{2} \right) = \left( R - \frac{\lambda}{2} \right) \Omega \left( R - \frac{\lambda}{2} \right) - R\Omega(R)
\]

\[
v_{\text{rel}} \left( R + \frac{\lambda}{2} \right) = \left( R + \frac{\lambda}{2} \right) \Omega \left( R + \frac{\lambda}{2} \right) - R\Omega(R),
\]

(15)

where \( v_{\text{rel}}(R - \lambda/2) \) denotes the velocity of a plasma parcel at \( R - \lambda/2 \) as seen by an observer at \( R \), and \( v_{\text{rel}}(R + \lambda/2) \) denotes the velocity of a plasma parcel at \( R + \lambda/2 \) as seen by an observer at \( R \). ..."

We point out here that HM’s choice of the relative velocity in Eq. (13) is based on the (co-moving) rotating frame at radial position \( R \). By contrast, SPB’s choice of Eq. (15) implies that the relative velocity is observed in the (co-moving) non-rotating
frame. Therefore, it is not a matter of “correct” or “incorrect”. As explained in MH (p. 360), any frame can and should, with the succeeding calculations performed correctly, yield the same result. There are four typical frames:

1. The rest frame, which is an inertial frame, with respect to the central mass. This frame was employed by Hartmann.\(^3\) In this frame there is conservation of angular momentum. The angular momentum computed in this frame is called the absolute angular momentum, which is conserved. The velocity of fluid at \(R - \lambda/2\) is observed as \((R - \lambda/2)\Omega(R - \lambda/2)\).

2. The rotating frame, in which an observer is rotating with the angular velocity \(\Omega\) in the above rest frame. This frame was employed by FKR, and the fluid velocity is seen as \((R - \lambda/2)\Omega(R - \lambda/2) + \Omega(R)\lambda/2\).

3. The co-moving non-rotating frame employed by SPB.

4. The (co-moving) rotating frame (see Eq. (13)) suggested by HM for modification of the approach of FKR in order to obtain a seemingly correct result.

In any frame, if one does not properly take into account the thermal motion of molecules, but otherwise carries out the calculation correctly, one inevitably reaches the incorrect conclusion that the viscous stress is proportional to \(\frac{d}{dR}(R^2\Omega)/dR\). Here, by “correctly” we mean that the convected variable is taken as the absolute angular momentum. On the other hand, if one calculates the angular momentum using the apparent fluid velocity, \(v_{rel}\), one reaches the conclusion that the viscous stress is proportional to \(\frac{d}{dR}(R\Omega)/dR\) in frames 2 and 3 and to \(d\Omega/dR\) in frame 4. The last result seems to be the correct one. However, this conclusion is derived on the basis of the invalid assumption that the “apparent” angular momentum is conserved. Only the absolute angular momentum is conserved. We therefore cannot claim that such a simple manipulation leads to the correct answer. This is the point made by HM, and it is misinterpreted by SPB in the above quotation.

§3. Conclusions

1. Subramanian, Pujari and Becker\(^5\) (SPB) claim to have derived a formula according to which the angular momentum transport is proportional to \(R^2d\Omega/dR\), where \(R\) and \(\Omega\) are the radial distance and the angular velocity of the rotating disc, respectively. However, their derivation assumes, as its starting point, the formula to be derived, and it is therefore ill-founded from a mathematical point of view. Following their procedure, we could also derive, for example, a (invalid) formula according to which the angular momentum transport is proportional to \(Rdv_\phi/dR\), where \(v_\phi\) is the azimuthal velocity.

2. They chose parcel (molecule) trajectories favorable to their desired result. We could in the same way prove that the angular momentum flows inwards by choosing another set of trajectories. Moreover, they confused the concepts of the mean free path and the radius of an epicycle.

3. Their criticism of the formula for the relative velocity given in our previous paper (HM) is irrelevant. We chose a rotating frame, while they chose a non-
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rotating frame. This choice is not a matter of correctness or incorrectness. We studied four typical frames in the present paper. In any frame, if one carries out the calculation properly but ignores the thermal spread/motion of molecules, one inevitably ends up with the incorrect conclusion that the viscous stress is proportional to \(d(R^2 \Omega)/dR\).

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