Boundary estimates and a Wiener criterion for the fractional Laplacian

Jana Björn
Department of Mathematics, Linköping University, SE-581 83 Linköping, Sweden
jana.bjorn@liu.se, ORCID: 0000-0002-1238-6751

Dedicated to Vladimir Maz’ya on his 85th birthday.

Abstract Using the Caffarelli–Silvestre extension, we show for a general open set \( \Omega \subset \mathbb{R}^n \) that a boundary point \( x_0 \) is regular for the fractional Laplace equation \( (-\Delta)^s u = 0, \) \( 0 < s < 1, \) if and only if \( (x_0, 0) \) is regular for the extended weighted equation in a subset of \( \mathbb{R}^{n+1} . \) As a consequence, we characterize regular boundary points for \( (-\Delta)^s u = 0 \) by a Wiener criterion involving a Besov capacity. A decay estimate for the solutions near regular boundary points and the Kellogg property are also obtained.

Key words and phrases: Besov capacity, Caffarelli–Silvestre extension, Dirichlet problem, fractional Laplacian, Kellogg property, regular boundary point, Wiener criterion.

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1. Introduction

We assume throughout the paper that \( 0 < s < 1 \) and that \( \Omega \subset \mathbb{R}^n, n \geq 2, \) is an open set such that its complement \( \mathbb{R}^n \setminus \Omega \) has positive Besov \( B^{s}_{2,2}(\mathbb{R}^n) \) capacity, as given in (1.3) and Definition 2.3. Note that we do not require any additional assumptions about the regularity of \( \Omega, \) which is allowed to be unbounded.

Consider the Dirichlet boundary value problem in \( \Omega \) for the fractional equation

\[
(-\Delta)^s u = 0.
\]

Recall that up to a multiplicative constant, the fractional Laplacian is given by the principle value integral

\[
(-\Delta)^s u(x) := C_{n, s} \text{ p. v. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy.
\]

Solutions of \( (-\Delta)^s u = 0 \) coincide with the so-called \( \alpha \)-harmonic functions (with \( \alpha = 2s \)), defined by means of balayage and associated with the Riesz potentials \( |x|^{\alpha-n} \), as in Bliedtner–Hansen [3, Chapter V.4], Hoh–Jacob [17] and Landkof [20, Chapter IV.5].

The fractional Laplacian is a nonlocal operator and hence the Dirichlet boundary data for (1.1) are prescribed on the complement \( \mathbb{R}^n \setminus \Omega, \) rather than on the boundary \( \partial \Omega. \) The above assumption that the complement has positive capacity is natural since otherwise the complement and the boundary data are not seen by the Besov space \( B^{s}_{2,2}(\mathbb{R}^n), \) associated with the fractional Laplacian.
We study the boundary behaviour and regularity of the solutions of (1.1), i.e. whether every solution of the Dirichlet problem for (1.1) in $\Omega$, with continuous boundary data $f$, attains its boundary value as the limit
\[
\lim_{\Omega \ni x \to x_0} u(x) = f(x_0) \quad \text{at } x_0 \in \partial \Omega.
\]

The following sufficient and necessary condition for regular boundary points is proved in Section 3.

**Theorem 1.1.** (Wiener criterion) A boundary point $x_0 \in \partial \Omega$ is regular for $\Omega$ with respect to the fractional equation (1.1) if and only if
\[
\int_0^1 \frac{\text{cap}_{B^s_r}(F_r, B(x_0, 2r))}{r^{n-2s}} dr = \infty,
\]
where $F_r = B(x_0, r) \setminus \Omega$ and $\text{cap}_{B^s_r}$ is the condensor capacity defined by a Besov seminorm on $\mathbb{R}^n$ as
\[
\text{cap}_{B^s_r}(F_r, B(x_0, 2r)) = \inf_{v} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,
\]
where the infimum is taken over all $v \in C_0^\infty(B(x_0, 2r))$ such that $v \geq 1$ on $F_r$.

For regular boundary points $x_0 \in \partial \Omega$, we obtain the pointwise decay estimate
\[
\text{osc}_{B(x_0, \rho)} u \leq \text{osc}_{F_2^\infty} f + \left( \text{osc}_{F_2^\infty} f \right) \exp \left( -C \int_0^R \frac{\text{cap}_{B^s_r}(F_r, B(x_0, 2r))}{r^{n-2s}} dr \right),
\]
for some constant $C > 0$ and all $0 < \rho \leq R < \infty$, see Proposition 3.3 for a more precise formulation. It follows that $u$ is Hölder continuous at $x_0$ whenever $\mathbb{R}^n \setminus \Omega$ has a corkscrew at $x_0$, in the sense that there are $0 < c < 1$ and $r_0 > 0$ such that for all $0 < r \leq r_0$, the set $F_r$ contains a ball of radius $cr$, cf. Lemma 2.4.

Decay estimates of the type (1.4) first appeared for solutions of $\Delta u = 0$ in Maz’ya [24] (and for nonlinear equations in Maz’ya [25]), where they were used to obtain the sufficiency part of the Wiener criterion for such equations, as well as sufficient conditions for the Hölder continuity of the solutions at the boundary.

Regular points for the $\alpha$-fine potential theory ($\alpha = 2s$) associated with Riesz potentials were characterized by the Wiener criterion in Bliedtner–Hansen [3] Corollary V.4.17 and Landkof [20] Theorem 5.2. Proposition VII.3.1 in [3] connects such potential-theoretic regular points to regular boundary points for $\alpha$-harmonic functions in the sense of our Definition 3.1 and [3] Chapter VII.3.

We instead use the extension results from Caffarelli–Silvestre [8] to derive the Wiener criterion (1.2) for the nonlocal equation (1.1) from the Wiener criterion for local degenerate (weighted) divergence equations in $\mathbb{R}^{n+1}$. This is the content of the following theorem, proved in Section 3. As a consequence, we also obtain the Kellogg property, saying that the set of irregular boundary points for equation (1.1) has zero $B^s_2$-capacity, see Corollary 3.2.

**Theorem 1.2.** Assume that $\mathbb{R}^n \setminus \Omega$ has positive $B^s_2$-capacity. A boundary point $x_0 \in \partial \Omega$ is regular for $\Omega$ with respect to the fractional equation (1.1) if and only if $x_0 := (x_0, 0)$ is regular for $G$ with respect to the weighted equation (1.6).

Our condition (1.2) differs somewhat from the ones in [3] and [20] in that we use the variational (condensor) capacity (1.3) and that our sets $F_r$ are defined using balls rather than annuli, cf. [20] (5.1.7). In addition to the Caffarelli–Silvestre extension, our results are based on the following comparison between the Besov capacity and a weighted condensor capacity in $\mathbb{R}^{n+1}$, see Lemma 2.2 and Section 2.
Lemma 1.3. Let $E \subset \overline{B(x_0,r)} \subset \mathbb{R}^n$ be a Borel set and $z_0 = (x_0,0) \in \mathbb{R}^{n+1}$. Then for $p > 1$ and $0 < s < 1$,
\[
\text{cap}_{B_p}(E, B(x_0,2r)) \simeq \text{cap}_{p,|t|^{a}}(E \times \{0\}, B(z_0,2r)), \quad \text{where } a = p(1-s) - 1,
\]
and $\text{cap}_{p,|t|^{a}}$ is the condenser capacity associated with the weight $w(x,t) = |t|^a$ in $\mathbb{R}^n \times \mathbb{R}$.

Condition (1.2) is a fractional analogue of the famous Wiener criterion proved for the Laplace equation (i.e. (1.1) with $s = 1$) by Wiener [35] in 1924. The Wiener criterion has been extended to various linear and nonlinear elliptic equations by e.g. Littman–Stampacchia–Weinberger [22], Maz'ya [24], Gariepy–Ziemer [15], Dal Maso–Mosco [9, 10], Lindqvist–Martio [21] and Kilpeläinen–Malý [18]. A weighted version of the Wiener criterion for degenerate elliptic equations, which will be of great importance in this paper, was proved by Fabes–Jerison–Kesig [12], Heinonen–Kilpeläinen–Martio [16] and Mikkonen [30].

Similar sufficient conditions were obtained for the boundary continuity of solutions with zero boundary data for the nonhomogeneous polyharmonic equation
\[
(-\Delta)^m u = f \in C_0^\infty(\Omega)
\]
with some integer powers $m \geq 2$, see Maz'ya [25, 26] and Maz'ya–Donchev [29]. Condition (1.2) (with a slightly different capacity) was shown to guarantee boundary continuity of solutions with zero boundary data for the nonhomogeneous fractional Laplace equation
\[
(-\Delta)^s u = f \in C_0^\infty(\Omega) \quad \text{with } s \in (0,1) \cup \left(\frac{n}{2} - 1, \frac{n}{2}\right),
\]
see Eilertsen [11]. Estimates similar to (1.4) were also proved in the above papers [11, 25, 26] and [29] on fractional and higher order equations, but necessity does not seem to have been considered there. For an extensive exposition of results on boundary regularity and the Wiener criterion for a wide class of elliptic equations, see the monograph by Maz'ya [28].

In the literature, the Dirichlet problem for $(-\Delta)^s$ is often considered in smooth (or at least Lipschitz) domains with zero boundary data and for the nonhomogeneous equation $(-\Delta)^s u = f$. Formally and for sufficiently smooth data, this formulation and the one considered in this paper (with zero right-hand side and general boundary data) can be transformed into each other, see e.g. Hoh–Jacob [17, Section 5] and Ros-Oton [31, Section 7].

The boundary Harnack inequality for the fractional Laplacian was proved by Bogdan [4] and Bogdan–Kulczycki–Kwasnicki [3]. Optimal regularity up to the boundary was in sufficiently smooth bounded domains (Lipschitz or $C^{1,1}$) proved by Ros-Oton–Serra [32, 33] for solutions of $(-\Delta)^s u = f$ with zero boundary data and various right-hand sides $f$.

In this paper, we treat general open sets (with complements of positive capacity). Our approach to boundary regularity is based on the following result due to Caffarelli–Silvestre [8, Section 4]: The solution $u$ of (1.1) in $\Omega$ with $u = f$ on $F := \mathbb{R}^n \setminus \Omega$ coincides with the restriction
\[
u(x) := U(x,0)
\]
of the solution $U$ to the Dirichlet problem in
\[
G := \mathbb{R}^{n+1} \setminus (F \times \{0\}) \quad (1.5)
\]
for the weighted equation
\[
\text{div}(|t|^{-2s}\nabla U(x,t)) = 0 \quad (1.6)
\]
with boundary data \( f \) on \( \partial G = (F \times \{0\}) \cup \{\infty\} \).

The above relation between (1.1) and (1.6) was used in [10] to derive the Harnack and boundary Harnack inequalities for (1.1). In particular, the local Hölder continuity of \( U \) in \( G \), proved in e.g. Fabes–Kenig–Serapioni [13] or Heinonen–Kilpeläinen–Martio [10, Theorem 6.6], directly yields interior regularity for the solutions of the fractional equation (1.1). Since then, the lift from (1.1) to (1.6) has become a standard tool in the analysis of the fractional Laplacian. It has been successfully exploited by many authors in various contexts, including interior regularity and free boundaries, see e.g. Aimar–Beltritti–Gómez [1], Barrios–Figalli–Ros-Oton [2], Caffarelli–Roquejoffre–Sire [6], Caffarelli–Salsa–Silvestre [7], Koch–Rüland–Shi [19] and Silvestre [34].

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2. **Weights, capacities and degenerate equations**

In this section we discuss properties of the degenerate equation (1.6) and its associated weighted capacity. More generally, we let \( p > 1 \) and consider the weight \( w(x, t) = |t|^a \) in \( \mathbb{R}^{n+1} \) with \(-1 < a < p - 1\). Even though we only need the case \( p = 2 \) in the rest of this paper, we state and prove the general results in this section for all \( p > 1 \), since the arguments are the same as for \( p = 2 \) and may be of independent interest.

It can be verified by a direct calculation that \( w \) is a Muckenhoupt \( A_p \) weight on \( \mathbb{R}^{n+1} \), i.e. it satisfies for all balls \( B \subset \mathbb{R}^{n+1} \),

\[
\int_B w(x, t) \, dx \, dt \left( \int_B w(x, t)^{1/(1-p)} \, dx \, dt \right)^{p-1} \leq C|B|^p,
\]

where \( |B| \) stands for the \((n+1)\)-dimensional Lebesgue measure of \( B \) and \( C > 0 \) is independent of \( B \). It follows that \( w \) is admissible for the theory of degenerate elliptic equations in the sense of Fabes–Jerison–Kenig [12] \((p = 2)\) or Heinonen–Kilpeläinen–Martio [16, Chapter 20] \((p > 1)\).

In [12], the arguments are restricted to a large fixed ball \( \Sigma = \{ z : |z| < R \} \). Since we later deal with the unbounded domain \( G \), given by (1.5), it will be convenient to use [10] even for \( p = 2 \). Equation (1.6) satisfies the assumptions (3.3)–(3.7) in [10] with \( p = 2 \) and the tools therein are therefore at our disposal.

Here and in what follows, we use the notation \( z = (x, t) \in \mathbb{R}^n \times \mathbb{R} = \mathbb{R}^{n+1} \) and define the measure \( \mu_a \) on \( \mathbb{R}^{n+1} \) by

\[
d\mu_a(z) = |t|^a \, dx \, dt.
\]

It follows from [10, p. 307] that \( \mu_a \) supports the \((p,p)\)-Poincaré inequality

\[
\int_B |v - v_B|^p \, d\mu_a \leq C r^p \int_B |\nabla v|^p \, d\mu_a,
\]

whenever \( B = B(z, r) \subset \mathbb{R}^{n+1} \) is a ball and \( v \in C^\infty(B) \) is bounded. Here

\[
v_B := \frac{1}{\mu_a(B)} \int_B v \, d\mu_a
\]

is the integral average of \( v \) and \( C > 0 \) is independent of \( B \). Since we consider balls both in \( \mathbb{R}^{n+1} \) and \( \mathbb{R}^n \), we adopt the convention that the dimension of a ball is determined by its centre, i.e. for \( z \in \mathbb{R}^{n+1} \) and \( x \in \mathbb{R}^n \),

\[
B(z, r) := \{ y \in \mathbb{R}^{n+1} : |y - z| < r \} \quad \text{and} \quad B(x, r) := \{ y \in \mathbb{R}^n : |y - x| < r \}.
\]
Unless specified otherwise, we consider open balls.

The following definition of capacity follows [16, Chapter 2].

**Definition 2.1.** Let $B \subset \mathbb{R}^{n+1}$ be a ball and $K \subset B$ be a compact set. The weighted variational (condenser) capacity of $K$ with respect to $B$ is

\[
\text{cap}_{p,|t|^s}(K,B) = \inf_{v} \int_{B} |\nabla v|^{p} \, d\mu_{a},
\]

where the infimum is taken over all $v \in C_{0}^{\infty}(B)$ such that $v \geq 1$ on $K$.

The weighted capacity $\text{cap}_{p,|t|^s}$ extends to all subsets of $B$ as a Choquet capacity and in particular, for all Borel sets $E \subset B$,

\[
\text{cap}_{p,|t|^s}(E,B) = \sup_{\text{compact } K \subset E} \text{cap}_{p,|t|^s}(K,B).
\]

A set $E \subset \mathbb{R}^{n+1}$ is said to be of zero $\text{cap}_{p,|t|^s}$-capacity if for all balls $B \subset \mathbb{R}^{n+1}$,

\[
\text{cap}_{p,|t|^s}(E \cap B, B) = 0.
\]

In [16], the variational capacity $\text{cap}_{p,|t|^s}$ is used to characterize boundary regularity for weighted equations of $p$-Laplace type and in particular for the equation (1.6).

Our aim is to formulate the Wiener criterion in terms of a capacity associated with the Besov space $B_{p}^{s}(\mathbb{R}^{n})$ and the fractional equation (1.1). Following Maz'ya [27, p. 512], we define the Besov seminorm

\[
\|v\|_{B_{p}^{s}(\mathbb{R}^{n})} := \left( \int_{\mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \frac{|v(x) - v(y)|^{p}}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}, \quad 0 < s < 1 < p.
\]

Theorem 1 in [27, p. 512] asserts that for all $v \in C_{0}^{\infty}(\mathbb{R}^{n})$,

\[
\|v\|_{B_{p}^{s}(\mathbb{R}^{n})} \simeq \inf_{\tilde{v}} \left\| |t|^{1-s-1/p} \nabla \tilde{v} \right\|_{L^{p}(\mathbb{R}^{n+1})}, \tag{2.2}
\]

where the infimum is taken over all extensions $\tilde{v} \in C_{0}^{\infty}(\mathbb{R}^{n+1})$ of $v$. Moreover, it follows from the proof of [27, Theorem 1 on p. 512] that

\[
\|v\|_{B_{p}^{s}(\mathbb{R}^{n})} \simeq \left\| |t|^{1-s-1/p} \nabla V \right\|_{L^{p}(\mathbb{R}^{n+1})}, \tag{2.3}
\]

where the extension $V$ is defined by

\[
V(x,t) = \frac{1}{t^{n}} \int_{\mathbb{R}^{n}} \varphi\left( \frac{\xi - x}{t} \right) v(\xi) \, d\xi, \tag{2.4}
\]

with any $0 \leq \varphi \in C_{0}^{\infty}(\mathbb{R}^{n})$ vanishing outside $B(0,1)$, such that

\[
\int_{\mathbb{R}^{n}} \varphi(\xi) \, d\xi = 1 \quad \text{and} \quad \varphi(-\xi) = \varphi(\xi),
\]

see [27, (10.1.7) on p. 514]. Here, and in what follows, we use the notation $X \simeq Y$ if there is a positive constant $C$ independent of $X$ and $Y$ such that $X/C \leq Y \leq CX$. Similar one-sided inequalities are denoted $\lesssim$ and $\gtrsim$ in an obvious way.

The following lemma relates the weighted capacity $\text{cap}_{p,|t|^s}$ in $\mathbb{R}^{n+1}$ to a Besov type condenser capacity in $\mathbb{R}^{n}$, see Definition 2.3. For simpler notation, we identify $K \subset \mathbb{R}^{n}$ with $K \times \{0\} \subset \mathbb{R}^{n+1}$.

**Lemma 2.2.** Let $z_{0} = (x_{0},0) \in \mathbb{R}^{n+1}$ and $K \subset \overline{B(x_{0},r)} \subset \mathbb{R}^{n}$ be a compact set. Then

\[
\text{cap}_{p,|t|^s}(K,B(z_{0},2r)) \simeq \inf_{v} \|v\|_{B_{p}^{s}(\mathbb{R}^{n})}^{p}, \quad \text{where} \quad s = 1 - \frac{1}{p} - \frac{a}{p},
\]

and the infimum is taken over all $v \in C_{0}^{\infty}(B(x_{0},2r))$ such that $v \geq 1$ on $K$. 

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Proof. Note that $a/p = 1 - s - 1/p$ coincides with the exponent in (2.2) and (2.3). Let $\tilde{v} \in C^\infty_0(B(z_0,2r))$ be admissible in the definition of $\text{cap}_{p,[t]}(K, B(z_0, 2r))$ and set $v(x) := \tilde{v}(x, 0)$ for $x \in \mathbb{R}^n$. It then follows from (2.2) and the definition of $\mu_a$ that

$$\int_{B(z_0,2r)} |\nabla \tilde{v}|^p \, d\mu_a \gtrsim \|v\|_{B^p_\infty(\mathbb{R}^n)}.$$ 

Since $v \in C^\infty_0(B(x_0,2r))$ and $v \geq 1$ on $K$, taking infimum over all such functions $\tilde{v}$ gives the $\gtrsim$-inequality in the statement of the lemma.

For the reverse inequality, let $v \in C^\infty_0(B(x_0,2r))$ be such that $v \geq 1$. Since $\varphi$ in (2.3) vanishes outside $B(0,1)$, it is easily verified that the extension $V$, given by (2.4), satisfies

$$V(x, t) = 0 \quad \text{whenever} \quad |x - x_0| \geq 2r + t. \quad (5.5)$$

To estimate $\text{cap}_{p,[t]}(K, B(z_0, 2r))$, let $\eta \in C^\infty_0(B(z_0, 2r))$ be a cut-off function such that $0 \leq \eta \leq 1$ in $\mathbb{R}^{n+1}$, $\eta = 1$ in $B(z_0, r)$ and $|\nabla \eta| \leq 2/r$. We then have

$$\text{cap}_{p,[t]}(K, B(z_0, 2r)) \leq \int_{B(z_0,2r)} |\nabla (V\eta)|^p \, d\mu_a \leq 2^p \int_{B(z_0,2r)} |\nabla V|^p \, d\mu_a + \frac{4^p}{p^p} \int_{B(z_0,2r)} |V|^p \, d\mu_a. \quad (5.6)$$

In view of (2.3), it therefore suffices to estimate the last term using the first integral on the right-hand side. To this end, we let $B = B(z_0,3r)$ and $V_B := \int_B V \, d\mu_a$. The Minkowski inequality then yields

$$\left(\int_B |V|^p \, d\mu_a\right)^{1/p} \leq \left(\int_B |V - V_B|^p \, d\mu_a\right)^{1/p} + |V_B|. \quad (5.7)$$

Note that, by the Hölder inequality and (2.4),

$$|V_B| \leq \int_B |V| \, d\mu_a \leq \left(\int_B |V|^p \, d\mu_a\right)^{1/p} \left(\frac{\mu_a(B \cap \text{supp} V)}{\mu_a(B)}\right)^{1-1/p} \leq \theta \left(\int_B |V|^p \, d\mu_a\right)^{1/p},$$

where the constant $0 < \theta < 1$ depends only on $n$, $p$ and $a$. Inserting the last estimate into (5.7) and subtracting the last term from the left-hand side, we get

$$(1 - \theta) \left(\int_B |V|^p \, d\mu_a\right)^{1/p} \leq \left(\int_B |V - V_B|^p \, d\mu_a\right)^{1/p}.$$ 

Together with (2.6) and the $(p,p)$-Poincaré inequality (2.1) for $\mu_a$, this implies that

$$\text{cap}_{p,[t]}(K, B(z_0, 2r)) \leq 2^p \int_{B(z_0,2r)} |\nabla V|^p \, d\mu_a + \frac{4^p C}{(1 - \theta)^p} \int_B |V|^p \, d\mu_a \lesssim \int_{\mathbb{R}^{n+1}} |\nabla V|^p \, d\mu_a.$$ 

The comparison (2.3) now concludes the proof.

In view of Lemma (2.2) and Definition (2.1), we make the following definition.
Definition 2.3. Let $B \subset \mathbb{R}^n$ be a ball and $K \subset B$ be a compact set. The variational (condenser) Besov capacity of $K$ with respect to $B$ is
\[
\text{cap}_{B^p}(K, B) = \inf_v \|v\|_{B^p([0,1])},
\]
where the infimum is taken over all $v \in C_0^\infty(B)$ such that $v \geq 1$ on $K$. For a Borel set $E \subset B$, we let
\[
\text{cap}_{B^p}(E, B) = \sup_{\text{compact } K \subset E} \text{cap}_{B^p}(K, B).
\]
We also say that a Borel set $E \subset \mathbb{R}^n$ has zero $B^p$-capacity if
\[
\text{cap}_{B^p}(E \cap B, B) = 0 \quad \text{for all balls } B \subset \mathbb{R}^n.
\]
Lemma 2.2 now implies that whenever $E \subset \overline{B(x_0, r)} \subset \mathbb{R}^n$ is a Borel set and $z_0 = (x_0, 0) \in \mathbb{R}^{n+1}$,
\[
\text{cap}_{p, |t|^n}(E, B(z_0, 2r)) \simeq \text{cap}_{B^p}(E, B(x_0, 2r)), \quad \text{where } s = 1 - \frac{1}{p} - \frac{a}{p} \quad (2.8)
\]
This proves Lemma 1.3. In particular, (2.8) holds for the sets $F_r$ in Theorem 1.1. Moreover, a Borel subset of $\mathbb{R}^n$ has zero $B^p$-capacity if and only if it has zero $\text{cap}_{p, |t|^n}$-capacity.

Lemma 2.4. Let $0 < \rho \leq r$, $z_0 = (x_0, 0)$ and $s = 1 - 1/p - a/p$. Then
\[
\text{cap}_{p, |t|^n}(B(z_0, \frac{1}{2}\rho), B(z_0, 2r)) \lesssim \text{cap}_{B^p}(B(x_0, \rho), B(x_0, 2r)) \lesssim \text{cap}_{p, |t|^n}(B(z_0, \rho), B(z_0, 2r)).
\]
In particular, $\text{cap}_{B^p}(\overline{B(x_0, cr)}, B(x_0, 2r)) \simeq r^{n-ps}$ with comparison constants depending only on $n$, $p$, $s$ and $0 < c \leq 1$.

Proof. Let $v \in C_0^\infty(B(x_0, 2r))$ be such that $v \geq 1$ on $\overline{B(x_0, \rho)}$. Then
\[
\|v\|_{B^p(\mathbb{R}^n)} \simeq \int_{\mathbb{R}^{n+1}} |\nabla V|^p d\mu_a,
\]
where the extension $V$ is given by (2.4). Note that as in (2.5), we have
\[
V(x, t) = 0 \text{ if } |x - x_0| \geq 2r + t \quad \text{and} \quad V(x, t) = 1 \text{ if } |x - x_0| \leq \rho - t.
\]
In particular, $V(x, t) = 1$ on $B(z_0, \frac{1}{2}\rho)$ and hence, as in the proof of Lemma 2.2,
\[
\text{cap}_{p, |t|^n}(B(z_0, \frac{1}{2}\rho), B(z_0, 2r)) \lesssim \int_{\mathbb{R}^{n+1}} |\nabla V|^p d\mu_a.
\]
Taking infimum over all functions $v$, which are admissible in the definition of $\text{cap}_{B^p}(B(x_0, \rho), B(x_0, 2r))$, proves the first inequality in the statement of the lemma. The second inequality follows from (2.8) and the inclusion $B(x_0, \rho) \times \{0\} \subset B(z_0, \rho)$, together with the equality
\[
\text{cap}_{p, |t|^n}(\overline{B(z_0, \rho)}, B(z_0, 2r)) = \text{cap}_{p, |t|^n}(B(z_0, \rho), B(z_0, 2r)),
\]
cf. [19] p. 32].

As for the last statement, it can be proved in the same way as in [16] Lemma 2.14 that the weighted condenser capacity of balls in $\mathbb{R}^{n+1}$ satisfies
\[
\text{cap}_{p, |t|^n}(B(z_0, cr), B(z_0, 2r)) \simeq \frac{1}{r^p} \int_{B(z_0, r)} |t|^n dx dt \simeq r^{n+1+a-p} = r^{n-ps}, \quad (2.9)
\]
where the comparison constants in $\simeq$ depend on $n$, $p$, $s$ and $c$, but are independent of $z_0$ and $r$. The first part of the lemma with $\rho = cr$ then concludes the proof. \[\Box\]
3. The Dirichlet problem and boundary regularity

In this section, we let \( p = 2 \) and \( a = 1 - 2s \), where \( 0 < s < 1 \). Note that \( s = \frac{1}{2} (1 - a) \) and so Lemmas 2.2 and 2.4 as well as the comparisons (2.8) and (2.9), apply with this choice of parameters. We write \( \text{cap}_{|t|} \) instead of \( \text{cap}_{|t|a} \).

The Dirichlet problem for \((-\Delta)^s u = 0\) was solved by the Perron method on general open sets \( \Omega \subset \mathbb{R}^n \) and for continuous boundary data vanishing at infinity by Bliedtner–Hansen \[3\] Chapter VII and Hoh–Jacob \[17\] Section 4. Solutions obtained as minimizers of energy integrals with sufficiently regular boundary data appear e.g. in Felsinger–Kassmann–Voigt \[14\]. Solutions defined using balayage were considered by Landkof \[20, Chapter IV.5\].

As mentioned in the introduction, it follows from Caffarelli–Silvestre \[8\], Section 4 that the Dirichlet problem for the fractional equation \((-\Delta)^s u = 0\) in \( \Omega \subset \mathbb{R}^n \) can be seen as a restriction of the Dirichlet problem for the weighted equation \[(1.6)\]

\[ G = \mathbb{R}^{n+1} \setminus (F \times \{0\}) \quad \text{where} \quad F = \mathbb{R}^n \setminus \Omega. \]

More precisely, assume that \( f \) is continuous on \( F \) and vanishes outside some bounded set. As before, for simpler notation, we identify \( F \) as in Heinonen–Kilpeläinen–Martio \[16, Definition 9.1\]. We point out that the solution \( U \) of the Dirichlet problem in \( G \) for the weighted equation

\[ \text{div}(|t|^{a} \nabla U(x, t)) = 0 \quad \text{(3.1)} \]

with boundary data \( f \) on \( \partial G \) then exists and can be obtained by the Perron method, as in Heinonen–Kilpeläinen–Martio \[16, Definition 9.1\]. We point out that the point at \( \infty \) in the Dirichlet problem for \( (3.1) \) on \( G \) is considered as a part of the boundary of \( G \) and that \( f \) is continuous also at \( \infty \), with \( f(\infty) = 0 \). Recall that we assume that \( f \) has positive \( B^s_2 \)-capacity. The estimate \( (2.8) \) then implies that also \( \text{cap}_{|t|a} (F \times \{0\}) > 0 \).

The upper Perron solution in \( G \) for an arbitrary bounded function \( f \) defined on \( \partial G \cup \{\infty\} \) is

\[ P_G f(z) := \inf_v v(z), \quad z \in G, \quad \text{(3.2)} \]

with the infimum taken over all lower semicontinuously regularized supersolutions \( v \) of \( (3.1) \) in \( G \), which are bounded from below and satisfy

\[ \liminf_{G \ni y \to z} v(y) \geq f(z) \quad \text{for all} \quad z \in \partial G \cup \{\infty\}, \]

see \[16\] Theorem 7.25 and Definition 9.1.

The lower Perron solution is defined similarly using upper semicontinuously regularized subsolutions of \( (3.1) \) or by \( P_G f := -P_G (-f) \). It follows directly from the definition of Perron solutions that if \( f_1 \leq f_2 \) on \( \partial G \) then the corresponding Perron solutions satisfy \( P_G f_1 \leq P_G f_2 \) and \( P_G f_1 \leq P_G f_2 \) in \( G \). Moreover, the comparison principle between regularized sub- and supersolutions \[16, p. 133\] yields that \( P_G f \leq P_G f \) for every \( f \).

Since \( \text{cap}_{|t|a} (\mathbb{R}^{n+1} \setminus G) > 0 \), every continuous function \( f \) on \( \partial G \cup \{\infty\} \) is resolutive, i.e. \( P_G f = P_G f \), see \[16\] Theorem 9.25. The Perron solution will therefore be denoted \( P_G f \). By \[8\], the restriction \( u(x) := P_G f(x, 0) \) satisfies the fractional equation

\[ (-\Delta)^s u = 0 \quad \text{(3.3)} \]

in \( \Omega \). Moreover, by the Kellogg property \[16\] Theorem 9.11 and (2.8),

\[ \lim_{y \to x} u(y) = \lim_{z \to (x, 0)} P_G f(z) = f(x) \]

holds for all \( x \in F \) outside a set of zero \( B^s_2 \)-capacity. This function \( u \) is a bounded solution of the Dirichlet problem for \((-\Delta)^s u = 0\) with boundary data \( f \) on \( F \).
Definition 3.1. A boundary point \( x_0 \in \partial \Omega \) is regular for \( \Omega \) with respect to the fractional equation (3.1) if for each \( f \in C(\mathbb{R}^n \setminus \Omega) \) vanishing outside some bounded set, the Perron solution \( u \) of (3.1) with boundary data \( f \) on \( \mathbb{R}^n \setminus \Omega \) satisfies
\[
\lim_{\Omega^n \ni x \to x_0} u(x) = f(x_0). \tag{3.4}
\]
A point is irregular if it is not regular.

Following [16, p. 171], we say that a point \( z_0 \in \partial G \) is regular for \( G \) with respect to the equation (3.1) if for each boundary data \( f \in C(\partial G \cup \{\infty\}) \), the Perron solution \( P_G f \) satisfies
\[
\lim_{G \ni z \to z_0} P_G f(z) = f(z_0). \tag{3.5}
\]
We are now ready to prove the equivalence between the above two notions of regular boundary points.

Proof of Theorem 1.2. Assume that \( z_0 \) is regular for \( G \) and let \( f \) be as in Definition 3.1. Then clearly, by the definition of \( u \) and by (3.5),
\[
\lim_{\Omega^n \ni x \to x_0} u(x) = \lim_{G \ni z \to z_0} P_G f(z) = f(x_0).
\]
Hence, \( x_0 \) is regular for \( \Omega \).

Conversely, assume that \( z_0 \) is not regular for \( G \). Then there exists a continuous function \( f \) on \( \partial G \cup \{\infty\} \) such that the corresponding Perron solution \( P_G f \) of the weighted equation (3.1) in \( G \) with boundary data \( f \) fails (3.5). By adding a constant and changing the sign, if needed, we can without loss of generality assume that \( f \geq 0 \) on \( \partial G \) and that
\[
0 \leq \liminf_{G \ni z \to z_0} P_G f(z) < f(z_0). \tag{3.6}
\]
Multiplying \( f \) by a continuous cut-off function with compact support, we can moreover assume that \( f \) vanishes outside some bounded set. To conclude the proof, it suffices to show that also (3.4) fails for the boundary data \( f \). So, assume for a contradiction that (3.4) holds and define
\[
\tilde{f}(z) = \begin{cases} f(z), & \text{if } z \notin G, \\ P_G f(z), & \text{if } z \in G. \end{cases}
\]
Let \( B \) be a ball centred at \( z_0 \) and consider the upper half-ball
\[
B_+ := \{(x, t) \in B : t > 0\}.
\]
Then \( \tilde{f} \) is a bounded function on \( \partial B_+ \), which is continuous at \( z_0 \), because of the assumption (3.4).

We claim that the restriction to \( B_+ \) of \( P_G f \) is the Perron solution for \( \tilde{f} \) in \( B_+ \). Indeed, if \( v \) is admissible in the definition (3.2) of \( P_G f = P_G f \), then by the definition of \( \tilde{f} \) and the continuity of \( P_G f \) in \( G \),
\[
\liminf_{B_+ \ni y \to z} v(y) \geq \begin{cases} f(z) = \tilde{f}(z), & \text{if } z \in \partial B_+ \cap \partial G, \\ \lim_{B_+ \ni y \to z} P_G(y) = \tilde{f}(z), & \text{if } z \in \partial B_+ \cap G. \end{cases}
\]
Hence
\[
v \geq P_{B_+} \tilde{f} \quad \text{in } B_+,
\]
and taking infimum over all such \( v \) shows that \( P_G f \geq P_{B_+} \tilde{f} \) in \( B_+ \). Similarly, \( P_G f \leq P_{B_+} \tilde{f} \) in \( B_+ \). Since also \( P_{B_+} \tilde{f} \leq P_{\Omega^n} \tilde{f} \), we see that the function \( \tilde{f} \) is resolutive and \( P_G f \) is the Perron solution of (3.1) in \( B_+ \) with boundary data \( \tilde{f} \).
Now, by the corkscrew condition [16, Theorem 6.31] (with respect to $\mathbb{R}^{n+1}$), $z_0$ is a regular boundary point for $B_+$ and (3.1). Since $\hat{f}$ is bounded on $\partial B_+$ and continuous at $z_0$, Lemma 9.6 in [16] implies that
\[
\lim_{B_+ \ni z \to z_0} P_G f(z) = \hat{f}(z_0) = f(z_0).
\]
A similar argument applied to $B_- := \{(x,t) \in B : t < 0\}$, together with the assumption that (3.4) holds, then gives
\[
\lim_{B_- \ni z \to z_0} P_G f(z) = f(z_0),
\]
which contradicts (3.6) and concludes the proof. \qed

We can now make use of the Wiener criterion for the weighted equation (3.1) in $G \subset \mathbb{R}^{n+1}$, provided by Heinonen–Kilpeläinen–Martio [16, Theorem 21.30] or Fabes–Jerison–Kenig [12].

**Proof of Theorem 1.2.** By Theorem 1.2, the regularity of $x_0$ with respect to the fractional equation $(-\Delta)^{s} u = 0$ is equivalent to the regularity of $z_0$ with respect to the weighted equation (3.1). This is in turn, by the Wiener criterion [16, Theorem 21.30 and (6.17)] with $p = 2$, equivalent to the condition
\[
\int_0^1 \frac{\text{cap}_{[t]}(F_r, B(z_0, 2r))}{\text{cap}_{[t]}(B(z_0, r), B(z_0, 2r))} \, \frac{dr}{r} = \infty.
\]
Here we have used the monotonicity of $\text{cap}_{[t]}$, together with estimates similar to the proof of [16, Lemma 2.16], to replace $\partial G \cap B(z_0, r)$ by the compact set $F_r = B(x_0, r) \setminus \Omega$ in the above integral. Finally, the estimates (2.8) and (2.9) (with $p = 2$) conclude the proof. \qed

We conclude the paper with two additional properties of regular boundary points for the fractional equation $(-\Delta)^s u = 0$.

**Corollary 3.2** (Kellogg property). The set of irregular boundary points for the fractional equation $(-\Delta)^{s} u = 0$ has $B_2^s$-capacity zero.

**Proof.** This follows immediately from Theorem 1.2, together with (2.8) and the Kellogg property [16, Theorem 9.11] for the weighted equation (3.1). \qed

**Proposition 3.3.** Assume that $x_0 \in \partial \Omega$ is regular for $(-\Delta)^{s} u = 0$ and that $f \in C(\mathbb{R}^n \setminus \Omega)$ vanishes outside some bounded set. Let $u$ be the Perron solution of $(-\Delta)^{s} u = 0$ in $\Omega$ with boundary data $f$ on $\mathbb{R}^n \setminus \Omega$. Then for all $0 < \rho \leq R < \infty$,
\[
\sup_{\Omega \cap B(z_0, \rho)} u \leq \sup_{F_{2R}} f + \left( \sup_{\mathbb{R}^n \setminus \Omega} f - \sup_{F_{2R}} f \right) \exp \left( -C \int_{\rho}^{R} \frac{\text{cap}_{B_2^s}(F_r, B(x_0, 2r))}{r^{n-2s}} \, dr \right),
\]
where $F_r = B(x_0, r) \setminus \Omega$ and $C$ depends only on $n$ and $s$.

**Proof.** We shall use [16, Theorem 6.18], where a similar decay estimate is proved for the weighted equation (3.1) in bounded domains and with continuous Sobolev boundary data. Recall that $u$ is the restriction to $\Omega$ of the Perron solution $P_G f$ for (3.1) in $G = \mathbb{R}^{n+1} \setminus ((\mathbb{R}^n \setminus \Omega) \times \{0\})$. Since $G$ is unbounded and the Perron solution in general only belongs to a local Sobolev space, we proceed as follows.

Let $\varepsilon > 0$, $m_\varepsilon = \sup_{F_{2R+n}} f$ and $M = \sup_{\mathbb{R}^n \setminus \Omega} f$. The Perron solution $P_G f$ of (3.1) clearly satisfies $P_G f \leq M$. Find $\hat{f}_\varepsilon \in C^\infty(\mathbb{R}^{n+1})$ such that
\[
\hat{f}_\varepsilon = m_\varepsilon = \min_{\mathbb{R}^{n+1}} \hat{f}_\varepsilon \text{ in } B(z_0, 2R) \quad \text{and} \quad \hat{f}_\varepsilon = M = \max_{\mathbb{R}^{n+1}} \hat{f}_\varepsilon \text{ outside } B(z_0, 2R + \varepsilon).
\]
Then \( \hat{f}_\varepsilon \geq f \) on \( \mathbb{R}^{n+1} \setminus G \). Theorem 6.18 in [10], applied to \( \hat{f}_\varepsilon \) and the bounded set \( G \cap B(z_0, 2r + \varepsilon) \), gives as in the proof of Theorem [11] that

\[
\sup_{\partial B(z_0, \rho)} u \leq \sup_{\partial B(z_0, \rho)} P_G f \leq \sup_{\partial B(z_0, \rho)} P_G \hat{f}_\varepsilon \\
\leq \hat{f}_\varepsilon(z_0) + \text{osc}_E \hat{f}_\varepsilon + (M - m_\varepsilon) \exp\left(-C \int_0^R \frac{\text{cap}_{t=0}(F_\varepsilon, B(z_0, 2r))}{\text{cap}_{t=0}(B(z_0, r), B(z_0, 2r))} \, dr\right).
\]

Since \( \text{osc}_{F_\varepsilon} \hat{f}_\varepsilon = 0 \) and \( \hat{f}_\varepsilon(z_0) = m_\varepsilon \to \sup_{F_\varepsilon} f \) as \( \varepsilon \to 0 \), the estimates (2.8) and (2.9) (with \( p = 2 \)) conclude the proof. \( \square \)

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