A FIBRATIONAL STUDY OF REALIZABILITY TOPOSES

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Chapter 1

Introduction

The present thesis is meant to be a contribution to the theory of ‘realizability toposes’, and more generally to ‘categorical realizability’ and ‘categorical logic’.

1.1 Context

1.1.1 Realizability

Realizability is a technique originating from proof theory and was originally devised by Kleene [33] to reason about metamathematical properties of formal systems. This is done by associating ‘realizers’ to logical formulas, which are usually finitary objects (integers, terms) viewed as approximations of proofs, using ideas from constructive mathematics.

In computer science, realizability is strongly related to typed lambda calculi via the proofs-as-programs correspondence. In this context, it is very instructive to compare realizability to typing à la Curry: typing constructs a binary relation between types and terms by induction on the term structure, whereas realizability constructs such a relation by induction on the structure of types. The realizability relation is generally bigger than the typing relation, and undecidable. This point of view on realizability is strongly related to normalization proofs of typed lambda calculi via ‘reducibility candidates’. Besides normalization, realizability can be used to reason about operational semantics.

Semantically, realizability can be viewed as a model construction for predicate logic. The basic idea here is that the set of realizers of a closed formula is viewed as its ‘truth value’: in the simplest case a truth value is ‘true’, if it is inhabited. This seems rather restricted as it admits (up to equivalence) only two truth values for closed formulas, but the structure diverges from the classical model as soon as we consider open formulas: these are interpreted by ‘predicates’, which are families of truth values, and a predicate is considered true whenever it has a uniform realizer, i.e. the intersection of all its truth values is inhabited.

1.1.2 The categorical approach

At the end of the 70ies, it was observed that the semantic aspect of realizability fits into a formalism described by Lawvere [36]: the above mentioned semantic predicates can be arranged into a hyperdoctrine, whereby it becomes apparent that the constructions thought out by proof theorists to interpret the logical connectives are characterized by universal properties. It was then realized that if the hyperdoctrine has enough structure, the model given by the hyperdoctrine
can be ‘internalized’ into a topos by performing a construction analogous to the construction of sheaves on a locale, providing an abstract way of turning realizability interpretations into models of higher order logic. The hyperdoctrines for which this construction is possible and results in a topos were called *triposes* by Hyland, Johnstone and Pitts [28].

The most well known and well studied of the toposes obtained this way is Hyland’s *effective topos* [27], which is the categorical incarnation of Kleene realizability. Kleene realizability uses natural numbers as realizers, but – as already noted in [28] –, the same construction works just as well when we replace the natural numbers by elements of an arbitrary partial combinatory algebra (pca) \( A \), giving rise to the *realizability topos* \( \text{RT}(A) \).

## 1.2 Motivation

The present work is not concerned with realizability in its proof theoretic sense at all, but only with its categorical abstraction in the form of partial combinatory algebras, triposes and toposes.

Broadly speaking, the intention of this work is to get a more abstract understanding of realizability toposes.

### 1.2.1 Questions

The ideas and constructions presented in this thesis can be motivated by the following two questions/challenges.

(i) Johnstone compared the state of the study of realizability toposes to ‘stamp-collecting’, calling for a Giraud [19] style ‘extensional’ characterization of realizability toposes.

(ii) The construction of realizability toposes is motivated by the analogy to Grothendieck toposes of sheaves on a locale, but realizability toposes themselves are not Grothendieck toposes. Nevertheless, we want to push the analogy further and try to adapt techniques from Grothendieck toposes to realizability. A more ambitious goal is to find an axiomatic framework generalizing Grothendieck toposes in such a way that it contains realizability toposes.

### 1.2.2 Approach

**Johnstone’s question**

Johnstone’s question for a Giraud style characterization is difficult to answer since it is not clear what precisely we want to call realizability. Clearly, the toposes constructed from pcas are realizability toposes, but we might also want to include toposes arising from modified realizability, relative realizability, and the Dialectica interpretation, to name just a few. Krivine’s [35] notion of realizability structure subsumes set-theoretical forcing, and if we want to adopt this liberal point of view we should also include a lot of Boolean valued models, and if we do not insist on classical logic, Heyting valued models as well. On the level of hyperdoctrines, a class of structures that contains all these examples and furthermore has good closure properties is the class of triposes, and in my opinion this is a reasonable candidate for an abstract framework for realizability.

---

1. Actually the definition of tripos is slightly stronger than what is necessary to construct a topos (see [51]), but the additional strength can naturally be viewed as a *smallness* condition.

2. At least as long as we are interested in realizability *toposes* – there are notions of realizability which do not give rise to impredicative models, and we will later consider corresponding types of hyperdoctrines. However, in
We do not know how to give an abstract characterization of toposes arising from triposes, but such a characterization can be given (as has been done by Pitts [50]) if we admit as additional structuring data the ‘constant objects functor’ \( \Delta : \text{Set} \to \text{Set} \) which embeds the category of sets into the topos.

**The constant objects functor**

The inclusion of the constant objects functor in the data can be justified by taking the point of view of ‘relative (Grothendieck) topos theory’: whenever we have a bounded geometric morphism \( (\Delta \dashv \Gamma : \mathcal{E} \to \mathcal{S}) \), \( \mathcal{E} \) is equivalent to a topos of sheaves on an internal site in \( \mathcal{S} \), whence bounded geometric morphisms into \( \mathcal{S} \) can be viewed as Grothendieck toposes relative to \( \mathcal{S} \). In the same way, functors \( \Delta : \mathcal{S} \to \mathcal{E} \) satisfying Pitts’ condition correspond to tripos-induced toposes relative to \( \mathcal{S} \) (the analogy makes sense since for triposes coming from complete Heyting algebras, the constant objects functor coincides with the inverse image functor of the geometric morphism). Now in the case of Grothendieck toposes relative to \( \text{Set} \), the geometric morphism can be constructed from the topos alone, but this is not the case for toposes coming from triposes, whence we have to include the functor in the data.

**The fibrational point of view**

A way to understand the relevance of the constant objects and direct image functors is via the gluing fibration: any regular functor \( \Delta : \mathcal{S} \to \mathcal{E} \) between toposes allows to view \( \mathcal{E} \) as a fibration (actually a stack) over \( \mathcal{S} \) by gluing (i.e. taking the pullback of the fundamental fibration of \( \mathcal{E} \) along \( \Delta \)), and one can argue that this is the real object of interest. Relative to \( \text{Set} \), this fibration is just the family fibration for Grothendieck toposes, while we obtain non-standard fibrations in case of tripos-induced toposes.

The fibrational point of view opens up an interesting new perspective: while from the non-fibered viewpoint the main structural difference between realizability toposes and Grothendieck toposes is that the former are not cocomplete, the gluing fibration of a realizability topos is fibrationally cocomplete, but not locally small, which implies for example that we can’t use Freyd’s adjoint functor theorems.

**Unifying realizability and Grothendieck toposes**

The previous deliberations suggest that a common framework for realizability and Grothendieck toposes is given by fibrations of toposes arising from gluing along regular functors \( \Delta : \text{Set} \to \mathcal{E} \) into toposes. To restrict this very general class of structures, it seems sensible to impose a boundness condition on the corresponding constant objects functors, which should correspond to the fact that the fibration can be constructed from a small, ‘site-like’ structure.

We do not study this question in detail or try to give an axiomatics, but some further speculations in this direction can be found Sections 4.12 and A.3.3. In particular, in Section A.3.3 we sketch a definition of the alluded site like structures which we call ‘uniform categories’, and the considerations of Section 4.12 suggest that on base toposes \( \mathcal{S} \) other than \( \text{Set} \) it seems to be necessary to postulate, in addition to the constant objects functor \( \Delta : \mathcal{S} \to \mathcal{E} \), a kind of ‘global sections functor’ \( \Gamma : \mathcal{E} \to \mathcal{S} \) (which is however not required to be adjoint to \( \Delta \) in general), in order to be able to reconstruct the generalized site from the fibration and the bound.

Section 4.12 we will see that those predicative classes of models are more difficult to handle on base categories other than \( \text{Set} \), and in particular iteration seems more difficult if possible at all, so there are good reasons to consider primarily the impredicative case.
In this work

As explained above, one might argue that when taking the fibrational point of view, and adopting a liberal notion of realizability, Johnstone’s question has already been answered by Pitts.

The main result of the present work is an analogous characterization for the smallest reasonable class of realizability toposes – those arising from pcas.

In order to achieve this, we develop a framework of ‘fibrational cocompletions’, which is manifested as a chain of biadjunctions between 2-categories of pre-stacks on a regular category $\mathbb{R}$. These biadjunctions are viewed as a fibrational generalization of the transition from a small finite-limit category $\mathbb{C}$ to the presheaf category $\hat{\mathbb{C}}$, and several intermediate steps.

The analogy being that finite-limit fibrations are generalized finite-limit categories on which we construct generalized presheaf categories, we will then concentrate on the posetal case, in particular on a class of posetal fibrations on $\text{Set}$ admitting a small representation – the uniform preorders. Uniform preorders provide an adequate framework for the analysis of realizability over pcas since they contain all triposes (in the presence of choice), as well as representations of pcas.

1.3 Overview

Chapter 2

In Chapter 2, we introduce the necessary parts of fibered category theory, including Moens’ theorem which clarifies the relationship between constant object functors and their gluing fibrations, and (pre-)stacks for the regular topology. In particular, we present the chain

$$\text{Lex}(\mathbb{R}) \leftrightarrow \text{Geo}(\mathbb{R}) \leftrightarrow \text{Pos}(\mathbb{R}) \leftrightarrow \text{Pretop}(\mathbb{R})$$ (1.3.1)

of 2-categories of fibered pretoposes, positive pre-stacks, geometric pre-stacks and finite-limit pre-stacks on a regular category $\mathbb{R}$ which form the basis of the developments in Chapter 3.

Chapter 3

Chapter 3 is about fibrational cocompletions – more precisely we construct left biadjoints to the chain of inclusion functors above. The fibrational cocompletions are meant to provide a common framework for presheaf-constructions and realizability constructions, the motivating examples are the following:

- Given a small finite-limit category $\mathbb{C}$, its family fibration $\text{fan}(\mathbb{C})$ is a finite-limit pre-stack.
  The fibered pretopos cocompletion of $\text{fan}(\mathbb{C})$ is $\text{fan}(\mathbb{Z})$ (the family fibration of the category of presheaves), and the geometric and positive prestack cocompletions are the subfibrations on families of subrepresentable presheaves, and coproducts of subrepresentable presheaves, respectively.
- For a pca $\mathcal{A}$, the posetal fibration of singleton valued predicates in $\mathcal{A}$ is a finite-limit pre-stack. Its geometric cocompletion is the realizability tripos, and its positive and pretopos cocompletions are the gluing fibrations of the category of assemblies and the realizability topos, respectively.

The reason for the use of pre-stacks is explained at the beginning of Section 2.3.

In Sections 3.1, 3.2, and 3.3, we present the constructions of the biadjoints in detail, and in Section 3.4 we take a second look on the cocompletions of finite-limit pre-stacks and geometric pre-stacks in the special case of posetal fibrations. This restriction simplifies the constructions considerably, and is sufficiently general for the treatment of realizability.
In Section 3.4.3, we slightly deviate from our main line of thought, to treat assemblies – as emphasized by Johnstone, realizability toposes constructed from pcas are much easier to work in than more general toposes constructed from triposes, the reason being that they can be presented as ex/reg completions of ‘concrete’ categories of assemblies. However, among the toposes constructed from triposes, those coming from pcas are not the only ones admitting such a presentation – other examples are given by relative realizability, and presheaves on meet-semilattices. This raises the question for a general criterion for when the construction via assemblies is possible. In Section 3.4.3, we show that for a topos \( \mathbb{Set}[\mathcal{P}] \) constructed from a tripos \( \mathcal{P} \) on \( \mathbb{Set} \) to be the ex/reg completion of a concrete subcategory of assemblies, it is sufficient that the embedding \( \delta : \text{sub}(\mathbb{Set}) \to \mathcal{P} \) of classical predicates into the tripos has a finite meet preserving left adjoint. Using classical logic, we can moreover show that in this case, the assemblies coincide with the \( \neg\neg \)-sheaves in \( \mathbb{Set}[\mathcal{P}] \).

Chapter 4

Although fibrational cocompletions work well, the framework that we present in Chapter 3 is a bit too general for the purpose of analyzing realizability – as hinted earlier, we want to impose boundedness conditions on the right side of the chain (1.3.1) (for positive pre-stacks and fibered pretoposes), and this should correspond to smallness conditions on the left side (for finite-limit pre-stacks and geometric pre-stacks).

To make this work, we need more structure in the base than that of a regular category – for example we want to express the transition from finite-limit pre-stacks to geometric pre-stacks entirely on the level of internal data, and to internalize the necessary constructions it is convenient to demand the base to be at least a topos. It turns out that not even that is enough – the only base category on which we can endow the chain of biadjunctions with size data in a straightforward manner is \( \mathbb{Set} \) (we can make it work on other base categories if we introduce an additional layer in the fibrations, as suggested in Section 4.12, but this is not worked out in detail in this thesis). As a further restriction we demand the fibrations on the small side of the scale (finite-limit pre-stacks and geometric pre-stacks) to be posetal from now on – as already pointed out, this is sufficient for the treatment of realizability, and leads us to the concept of uniform preorder.

Uniform preorders are representations of certain posetal fibrations on \( \mathbb{Set} \) conforming to a smallness condition – in the presence of choice the locally ordered category \( \mathbb{UOrd} \) of uniform preorders is equivalent to the full subcategory of posetal fibrations on \( \mathbb{Set} \) on those posetal fibrations which have a generic family of predicates. \( \mathbb{UOrd} \) is quite similar in structure to the category of preorders and has very good closure properties (in particular it is bicartesian closed and comes with a notion of distributor that makes it a locally posetal cartesian bicategory with duals [15]). This and the fact that it accommodates triposes and partial combinatory algebras in a natural way makes \( \mathbb{UOrd} \) an ideal framework to analyze realizability constructions, and enables us to prove our main results near the end of the chapter. These are:

- The identification of inclusions of typed pcas (corresponding to typed relative realizability) with relationally complete functional uniform preorders in Lemma 4.10.6. This identification can be specialized to characterizations of the untyped and non-relative cases by adding conditions.
- The characterization of realizability triposes and hyperdoctrines arising from (typed) (inclusions of) pcas in Theorem 4.11.1, using the previous result and a concept of ‘∃-primality’ (Definition 3.4.8) generalizing the notion of ‘completely join prime element’ in complete

3. Following Hofstra [23] we do not identify a pca with the corresponding realizability tripos, but with its subfibration on singleton valued predicates.
lattices.

- The characterization of the fibered (pre)toposes arising from (typed) (inclusions of) pcas by the fibered presheaf construction in Theorem 4.11.5, using the characterization of pcas and a fibrational generalization of the characterization of presheaf toposes in terms of indecomposable projectives.

- In the non-relative case, our characterization gives rise to a characterization of the non-fibered realizability categories/toposes, since the constant objects functor is right adjoint to the global sections functor in this case, and thus doesn’t give additional information.

In Section 4.12, we describe how the correspondence between uniform preorders and posetal fibrations can be expressed on base toposes other than Set, which gives an approach of how to generalize the the listed results to arbitrary base toposes.

Appendix

In Section A.1 we recall standard definitions from categorical realizability.

In Section A.2, we describe a decomposition result which is inverse to Pitts’ iteration theorem and is inspired by the idea that constant objects functors are ‘generalized geometric morphisms’ (since for geometric morphisms there are several such decompositions known).

Finally, Section A.3 contains outlines of unfinished work and ideas for future investigations.

1.4 Related work

The present work is based on a large body of work in categorical realizability that has been carried out throughout the last 20 years. This work can loosely be divided into two themes – exact completion and combinatorial structures:

- The connection between realizability toposes and exact completion was established by Robinson and Rosolini [52], who observed that realizability toposes are exact completions of their subcategories of partitioned assemblies. This inspired subsequent work by Birkedal, Carboni, Hofstra, Menni, Rosolini and Scott [12, 8, 45, 14, 46, 22], of which [14] is of particular importance for this thesis.

- The study of ‘combinatory structures’ generalizing pcas started with van Oosten’s [60] definition of ordered pcas, and ordered pcas were further developed by Hofstra and van Oosten [21, 25]. Longley generalized pcas by adding types [39], and Hofstra further generalized ordered pcas into basic combinatory objects (BCOs) [23, 24]. Recently, Longley [40] presented a vast generalization of his ordered pcas.

Of these works, [23] has been crucial to the development of this thesis – a lot of results about uniform preorders are just adoptions of Hofstra’s results about BCOs (we will indicate this in the text). However, BCOs correspond just to single-sorted uniform preorders. The idea to consider the many-sorted version was triggered by Streicher’s remark that the definition of uniform preorder (in its first, one-sorted version) resembled the definition of Longley’s C-structures [40]. Remarkably, some of our concepts that are based on intuitions about fibered preorders are similar to notions that Longley devised without these intuitions – most notably the definition of ‘relationally complete uniform preorder’ (Definition 4.10.1) is similar to Longley’s higher order C-structure.

Krivine’s realizability structures [35] can also be seen as generalizations of pcas, but their link to the structures mentioned above is not explored here.

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4. Longley’s thesis [38] should also be mentioned as it introduced the idea of organizing pcas into a locally ordered category, which was subsequently adopted for more general classes of combinatorial structures.
The idea to view realizability toposes as presheaf toposes can be traced to Hofstra’s observation that realizability triposes can be generated by freely adding existential quantification to the singleton fibrations (analogous to the fact that sheaves on a downset lattice are equivalent to presheaves on the generating preorder), the idea that this leads to an alternative reading of the ‘exact completion description’ of realizability toposes materialized during a discussion with Rosolini. This lead to a new approach to the question of finding a choice-free presentation, by taking inspiration from Bunge’s characterization of presheaf toposes over arbitrary bases. When adapting this viewpoint to realizability, the ‘partitioned assemblies’ which are important for exact completion become families of subterminal indecomposable projectives – the corresponding assembly is the internal sum of this family, and is only projective if the indexing set is, which explains the role of choice in a sense.

In November 2011, I learned from Naohiko Hoshino that he developed a framework for combinatory objects that somewhat resembled our uniform preorders. Hoshino defined his version of combinatory objects as monads in a more primitive locally ordered bicategory, a point of view that was influential for the ideas sketched in Appendix A.3.1.

Wouter Stekelenburg’s thesis on realizability categories is close in spirit to the present work in that it also aims towards a more abstract understanding of categorical realizability constructions. In comparison, Stekelenburg’s approach seems to be more ‘logical’, as opposed to the ‘geometric’ flavor of the present work which arises from emphasizing the analogy to Grothendieck toposes. Further analysis may be needed.

1.5 Conventions

1.5.1 Logical notation

In the present text, we make extensive use of notation in predicate logic, both informally and as ‘internal language’ in categories and posetal fibrations. We use the propositional connectives $\bot, \top, \wedge, \vee, \Rightarrow$ with the convention that $\wedge$ binds strongest of the binary connectives, and $\Rightarrow$ binds weakest, i.e. $\varphi \vee \psi \wedge \gamma \Rightarrow \psi$ has to be read as $(\varphi \vee (\psi \wedge \gamma)) \Rightarrow \psi$. Quantifiers $\forall, \exists$ have lowest precedence, in other words their scope stretches as far to the right as possible. For example, $\forall x. \varphi \Rightarrow \psi$ means $\forall x. (\varphi \Rightarrow \psi)$.

1.5.2 Internal logic

For an introduction to categorical logic we refer to [30] and [32, Part D], here we only review notational conventions and shorthands.

When reasoning in the internal logic of a category or a fibration we use the same symbols as in informal reasoning, but a bit more rigorously. We have three syntactic classes – terms, formulas, and judgments, which we write as follows:

| Type           | Syntax                           |
|----------------|---------------------------------|
| term           | $x_1:A_1 \ldots x_n:A_n \mid t$ |
| formula        | $x_1:A_1 \ldots x_n:A_n \mid \varphi$ |
| judgment       | $x_1:A_1 \ldots x_n:A_n \mid \varphi_1 \ldots \varphi_n \vdash \psi$ |

Note that all come with explicit contexts $x_1:A_1 \ldots x_n:A_n$ of typed (or rather ‘sorted’) variables, without which the expressions are meaningless. Similarly, bound variables always have types. Since the notation in this style is rather heavy, we will often omit types or even entire variable contexts – the types are normally clear from the context (in the non-technical sense of the word),
and if no variable context is given, we mean by convention simply all free variables that occur in
the expression.

Expressions in the internal language denote semantic entities – terms denote morphisms, formulas
 denote ‘predicates’, by which we mean either monomorphisms (in categories) or objects
in a fibration over the denotation of the context. If we assert the validity of a judgment, we
mean that a certain inequality holds in a subobject lattice or the fiber of a fibration (we will use
predicate logic only in posetal fibrations).

In general, we do not distinguish between a syntactic expression in the internal language and
its denotation since it is either clear from the context what we mean (if we refer to a formula as
a predicate, then we mean its denotation), or it doesn’t matter.

1.5.3 Reasoning with partial terms

When reasoning with partial terms (for example in pcas), we interpret function symbols and
primitive predicate symbols in the strict sense – for example if we assert \( s = t \) or \( s \in M \) where \( s \)
and \( t \) are partial terms, then this in particular means that \( s \) and \( t \) and all of their subterms have
to be defined\(^5\). This spares us from inserting ‘... then \( t \) is defined and ...’ in many definitions
and arguments.

We write \( t \downarrow \) for the proposition that a possibly partial term \( t \) is defined; in accordance with
our strictness convention this is equivalent to \( t = t \).

Formally, when doing first order logic with partial terms, we use \( E^+ \)-logic (see [59, Chap-
ter 2.2]).

We use the notations \( s \preceq t \) for \( s \downarrow \Rightarrow s = t \), and \( s \simeq t \) for \( (s \downarrow \lor t \downarrow) \Rightarrow s = t \); the second relation
is also called ‘strong equality’.

1.5.4 The axiom of choice

The proofs and developments in this thesis do not rely on the axiom of choice, unless explicitly
said otherwise. The abandonment of choice leads to a certain proliferation of mathematical
concepts, in particular in category theory – in the case of finite limit categories, for example, it
makes a difference whether we demand the existence of limiting cones for every finite diagram,
or whether we ask each such diagram to come with an explicit choice of such cone. Similarly, it
is sometimes desirable to have an explicit choice of cartesian liftings in fibrations. As is common
practice, we will generally assume that our categorical structures come with an explicit choice of
whichever structure we postulate.

---

\(^5\) Be aware, however, that we can not deduce that \( t \) is defined from \( \varphi(t) \) for general non-atomic formulas \( \varphi \) –
this goes wrong already for \( \neg(t = t) \).
Chapter 2

Fibrations

2.1 Basic theory

In the present work, fibrations form the central tool and formalism. We refer the reader to [57] for a general introduction, and to [30] and [31, Section B1.3] for fibrations in categorical logic and topos theory. Benabou’s original paper [5] gives a more philosophical account.

In the following, we recall some basic theory, mostly without proofs.

Definition 2.1.1 Let $C : B \to \mathbb{C}$ be a functor between categories $B, \mathbb{C}$.

(i) Let $u : J \to I$ in $B$ and $f : B \to A$ in $\mathbb{C}$ such that $C(f) = u$. We call $f$ cartesian (with respect to $C$), if for any $v : K \to J$ in $B$ and $g : C \to A$ in $\mathbb{C}$ such that $C(g) = uv$ there exists a unique $h : C \to B$ such that $C(h) = v$ and $fh = g$.

The domain of a fibration is called its total category, and the codomain its base category. To avoid having to come up with a new letter, we often denote the total category of a fibration $C$ by $|C|$. Given a fibration $C : |C| \to B$, we also say that $C$ is a fibration on $B$. If $C(A) = I$, or $C(f) = u$, we will say that $A$ is over $I$, and that $f$ is over $u$, respectively. Morphisms over identity morphisms in the base are called vertical. In diagrams we represent the ‘over’ relation by vertical alignment, as we already did in Diagram (2.1.1).

(ii) $C$ is a fibration, if for every $A \in \mathbb{C}$ and $u : J \to I$ in $B$ such that $C(A) = I$ there exists a cartesian lifting of $A$ along $u$, by which we mean a cartesian arrow $f : B \to A$ such that $C(f) = u$.

The domain of a fibration is called its total category, and the codomain its base category. To avoid having to come up with a new letter, we often denote the total category of a fibration $C$ by $|C|$. Given a fibration $C : |C| \to B$, we also say that $C$ is a fibration on $B$. If $C(A) = I$, or $C(f) = u$, we will say that $A$ is over $I$, and that $f$ is over $u$, respectively. Morphisms over identity morphisms in the base are called vertical. In diagrams we represent the ‘over’ relation by vertical alignment, as we already did in Diagram (2.1.1). Given $I \in B$, $C_I$ is the fiber of $C$ over $I$, which is the subcategory of $|C|$ on objects over $I$ and morphisms over $\text{id}_I$. We use the arrow symbol $\rightarrow$ for cartesian arrows.

A posetal fibration is a fibration $A : |A| \to B$ which is faithful as a functor, which is equivalent to the fact that all fibers $A_I$ for $I \in B$ are preorders. Since posetal fibrations can serve as models of first order logic, we refer to the objects in $|A|$ as predicates in this case. Specifically, given $\varphi \in A_I$, we call $\varphi$ a predicate on $I$. 

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Definition 2.1.2 Let $\mathcal{C}:|\mathcal{C}| \to \mathbb{B}$, $\mathcal{D}:|\mathcal{D}| \to \mathbb{B}$ be fibrations on a category $\mathbb{B}$.

(i) A fibered functor is a functor $F:|\mathcal{C}| \to |\mathcal{D}|$ which maps cartesian arrows in $\mathcal{C}$ to cartesian arrows in $\mathcal{D}$, and such that $\mathcal{D} \circ F = \mathcal{C}$.

(ii) Given two fibered functors $F, G: \mathcal{C} \to \mathcal{D}$, a fibered natural transformation is a natural transformation $\eta: F \to G$ such that all components $\eta_C$ for $C \in |\mathcal{C}|$ are vertical (or equivalently $\mathcal{D} \eta = \text{id}_\mathcal{C}$).

(iii) Fibrations, fibered functors, and fibered natural transformations on $\mathbb{B}$ form a 2-category which we denote by $\text{Fib}(\mathbb{B})$.

Since we don’t rely on the axiom of choice, it makes a difference whether we assume mere existence of cartesian liftings or an actual assignment of a lifting to each pair $(A \in \mathcal{C}_I, u: J \to I)$ in a fibration $\mathcal{C}:|\mathcal{C}| \to \mathbb{B}$. Such a choice of cartesian liftings is called a cleavage. Unless otherwise specified, we always assume that our fibrations are equipped with cleavages, which we leave implicit (we do, however, not require fibered functors to preserve chosen cartesian arrows). Using cleavages, we can employ a functorial notation for cartesian liftings: Given $u: J \to I$ and $A \in \mathcal{C}_I$, we denote the domain of the designated cartesian lifting of $A$ along $u$ by $u^*A$, and using the universal property of cartesian liftings, we can transport vertical maps $(f: A \to B) \in \mathcal{C}_I$ to vertical maps $(u^*f: u^*A \to u^*B) \in \mathcal{C}_J$.

\[
\begin{array}{ccc}
  u^*A & \Rightarrow & A \\
  \downarrow f & \Downarrow & \downarrow f \\
  u^*B & \Rightarrow & B \\
\end{array}
\]

From the universal property of cartesian liftings we can deduce that

- squares of the form (2.1.2) are always pullbacks,
- the construction gives rise to a functor $u^*: \mathcal{C}_I \to \mathcal{C}_J$, and
- the assignment $u \mapsto u^*: \mathcal{C}_I \to \mathcal{C}_J$ is functorial up to isomorphism and thus gives rise to a pseudofunctor of type

$$\mathbb{B}^{\text{op}} \to \text{Cat}$$

which we call the indexed category associated to $\mathcal{C}$.

We can show the following.

Lemma 2.1.3 The transition from a fibration to the associated indexed category gives a biequivalence of 2-categories $\text{Fib}(\mathbb{B}) \simeq [\mathbb{B}^{\text{op}}, \text{Cat}]$, where $[\mathbb{B}^{\text{op}}, \text{Cat}]$ is the 2-category indexed categories on $\mathbb{B}$, i.e., the 2-category of pseudofunctors, pseudo-natural transformations, and modifications.

To make this statement precise, we have to say something about relative sizes of the involved entities. It is easiest to assume that $\mathbb{B}$ and the fibrations in $\text{Fib}(\mathbb{B})$ are small relative to some universe, whereas $\text{Cat}$ is the large 2-category of small categories relative to the same universe.

The construction in the inverse direction from the one sketched above, i.e. the transition from an indexed category to a fibration, is known as the Grothendieck construction (see [31, Definition B1.3.1]). When defining fibrations, in particular posetal ones, we will often make implicit use of the Grothendieck construction, and only define the ordering in the fibers and the cartesian liftings.

In the spirit of Bénabou’s work [5], we view fibrations as generalized categories. To justify this point of view, we explain now how ordinary categories can be viewed as fibrations.
Definition 2.1.4 Let \( A \) be a category.

(i) Fam\((A)\) is the category of families in \( A \). Its objects are families \((A_i)_{i \in I}\), where \( I \) is a set and \( A_i \in A \) for \( i \in I \). A morphism from \((A_i)_{i \in I}\) to \((B_j)_{j \in J}\) is a pair \((u, (f_i)_{i \in I})\) of a function \( f : I \rightarrow J \) and a family of morphisms \( f_i : A_i \rightarrow B_{u(i)} \).

(ii) The family fibration of \( A \) is the functor \( \text{fam}(A) : \text{Fam}(A) \rightarrow \text{Set} \) which sends \((A_i)_{i \in I}\) to \( I \), and \((u, (f_i)_{i \in I})\) to \( u \).

Lemma 2.1.5 The assignment \( A \mapsto \text{fam}(A) \) gives rise to a 2-functor

\[ \text{fam} : \text{Cat} \rightarrow \text{Fib(Set)}, \]

which is a local equivalence.

A central theme in the ‘fibered category’ approach to fibrations is to take a possibly non-elementary property of categories, such as small completeness or local smallness, and to try to express it as an elementary property of the corresponding family fibration. A concept that fits in this pattern is ‘having internal sums’ in Definition 2.2.1-(i); for systematic treatments of this point of view see [5, 57].

2.1.1 Fibrations from (typed) pcas

To give examples of fibrations which are not given by the family construction, and since they are of central interest for this work, let us explain how to obtain posetal fibrations from (typed) pcas. The fibrations that we will now introduce are not the realizability triposes known from [28] and many subsequent works (we will present those in Definition A.1.7), but come from a more primitive construction whose importance was apparently first realized by Hofstra [23].

Even though we are not dealing with family fibrations in the previously defined sense, we are using similar notation and terminology. We do this in the hope that it will lead the intuition of the reader in the intended direction, which is to view pcas (or rather the associated uniform preorders – see Example 4.1.3-(ii)) as generalized preorders.

Definition 2.1.6 Let \((I, \mathcal{A})\) be a typed pca (Definition A.1.3). The uniform family fibration

\[ \text{ufam}(I, \mathcal{A}) : \text{UFam}(I, \mathcal{A}) \rightarrow \text{Set} \]

of \((I, \mathcal{A})\) is the posetal fibration defined as follows.

- Predicates on \( M \in \text{Set} \) are pairs \((i \in I, \varphi : M \rightarrow \mathcal{A}_i)\).
- The ordering on \( \text{ufam}(I, \mathcal{A})_M \) is defined by

\[ (i, \varphi) \leq (j, \psi) \iff \exists c : \mathcal{A}_{i \rightarrow j} \forall m. c \cdot \varphi(m) = \psi(m). \]

- Reindexing is given by precomposition.

We see that on a literal level the analogy to family fibrations makes sense, since the predicates in \( \text{ufam}(I, \mathcal{A}) \) really are families.

There is an obvious untyped analogue of the previous definition, which associates to each pca \( \mathcal{A} \) a posetal fibration

\[ \text{ufam}(\mathcal{A}) : \text{UFam}(\mathcal{A}) \rightarrow \text{Set}. \]

1. ‘Elementary’ here means roughly ‘first order axiomatizable’, in particular without references to set theoretic ‘size conditions’.
2.1.2 Finite-limit fibrations

Let us recall some basic facts about fibrations of finite limit categories from [57, Section 8].

Definition 2.1.7 A fibration of finite limit categories, or finite limit fibration on a base category \( \mathcal{B} \) is a fibration \( \mathcal{C} : |\mathcal{C}| \to \mathcal{B} \) where
- all fibers \( \mathcal{C}_I \) for \( I \in \mathcal{B} \) have finite limits, and
- reindexing preserves finite limits.

The following fact about finite limit fibrations is of central importance.

Lemma 2.1.8 Let \( \mathcal{C} : |\mathcal{C}| \to \mathcal{C} \) be a fibration on a category \( \mathcal{C} \) with finite limits. Then \( \mathcal{C} \) is a finite limit fibration iff \( |\mathcal{C}| \) has finite limits and \( \mathcal{C} \) preserves them.

Proof. See [57, Theorem 8.5].

This lemma highlights that we have to distinguish two kinds of limits in a finite limit fibration (as long as the base has finite limits, what we will always assume from now on) – the limits in the fibers (which we sometimes call ‘fiberwise’), and the ‘global’ limits in the total category. Furthermore, the lemma explains how they are connected:
- Since \( \mathcal{C} \) maps global limits to limits in the base, global limiting cones on vertical connected diagrams can be chosen vertical as well. This implies in particular that fiberwise connected limits are also limits in the total category.
- The fact that fiberwise connected limits are global limits implies that monomorphisms in the fibers are also monic in the total category, since monos can be characterized in terms of pullbacks.
- In general, we can express global limits in terms of limits in the fibers and in the base. For example, given a cospan \( B \to A \leftarrow C \) in \( |\mathcal{C}| \) over a cospan \( J \to I \leftarrow K \) in the base, we can take the pullback of \( f \) and \( g \) by first reindexing the cospan \( B \to A \leftarrow C \) into the fiber over the pullback of \( u \) and \( v \), and then taking the fiberwise pullback.

\[
\begin{array}{c}
B 	imes_A C \\
\downarrow \\
I
\end{array}
\]

Having both fiberwise and global limits in finite limit fibrations, we have to be careful to distinguish them notationally. In general we use global limits unless saying otherwise explicitly. For example, given \( C, D \in |\mathcal{C}| \), \( C \times D \) means their global product. For the fiberwise product of \( C, D \in \mathcal{C}_I \), on the other hand, we write \( C \times_I D \), in analogy to the common notation for pullbacks. More generally, given a cospan \( J \to I \leftarrow K \) in \( \mathcal{C} \), and \( B \in \mathcal{C}_J \), \( C \in \mathcal{C}_K \), we write \( B \times_I C \) for
the product of $B$ and $C$ relative to $u$ and $v$, in the sense of the following diagram.

Formally, this product is characterized as terminal among cones $J \xleftarrow{p} \bullet \xrightarrow{q} K$ in $|\mathcal{C}|$ such that $u\mathcal{C}(p) = v\mathcal{C}(q)$; it can equivalently be described as fiberproduct $B \times_1 C$ in $|\mathcal{C}|$.

Remarks 2.1.9  
- In posetal fibrations, we refer to fiberwise finite limits as finite meets, and we use the symbols $\top$ and $\wedge$ instead of $\mathbb{1}$ and $\times$.
- The uniform family fibrations $\text{ufam}(A)$ and $\text{ufam}(I,A)$ of (typed) pcas $A$ and $(I,A)$ have finite meets – in the typed case, if $\varphi : M \to A_i$ and $\psi : M \to A_j$ are predicates in $\text{ufam}(I,A)$, a greatest lower bound is the predicate $\varphi \wedge \psi : M \to A_{i\cap j}$ given by $m \mapsto \text{pair-}\varphi(m)\cdot\psi(m)$. In the untyped case, we can derive the existence of a pairing combinator from functional completeness – see e.g. [61, Section 1.1.1].

2.1.3 Localization and slicing

Definition 2.1.10 Given a functor $F : \mathcal{A} \to \mathcal{B}$ and $A \in \mathcal{A}$ we define the slice functor $F/A : \mathcal{A}/A \to \mathcal{B}/FA$ of $F$ over $A$ by

$$(f : X \to A) \mapsto (Ff : FX \to FA)$$

and in the obvious way for morphisms.

The following is easy to show.

Lemma 2.1.11 Given a fibration $\mathcal{C} : |\mathcal{C}| \to \mathcal{B}$, and $C \in \mathcal{C}_I$, $\mathcal{C}/C : |\mathcal{C}/C| \to \mathcal{B}/I$ is a fibration as well. If $\mathcal{C}$ is a finite limit fibration, then so is $\mathcal{C}/C$.

There is a construction that is somewhat similar to slicing of fibrations (see [57, Section 4]):

Definition 2.1.12 Given a fibration $\mathcal{C} : |\mathcal{C}| \to \mathcal{B}$ and $I \in \mathcal{B}$, we define the localization $\mathcal{C}/I$ of $\mathcal{C}$ to $I$ by the pullback

\[
\begin{array}{ccc}
|\mathcal{C}/I| & \rightarrow & |\mathcal{C}| \\
\mathcal{C}/I & \downarrow & \mathcal{C} \\
\mathcal{B}/I & \rightarrow & \mathcal{B}
\end{array}
\]

Lemma 2.1.13 If $\mathcal{C} : |\mathcal{C}| \to \mathcal{B}$ has terminal objects in the fibers which are stable under reindexing\(^2\), then localization is a special case of slicing. More precisely, given $I \in \mathcal{B}$, the localization $\mathcal{C}/I$ is equivalent to the slice fibration $\mathcal{C}/I_1$.

\(^2\) Such a fibration is called a fibration of categories with terminal objects in [57, Definition 8.2].
To develop an intuition, the best way to think about fibrations resulting from slicing and localization is as ‘fibered fibrations’ by ‘fibered fibration’ we mean a fibration on the total category of another fibration (see also the remarks at the beginning of Section 4.12).

Given an object \( I \in \mathcal{C} \) of a finite limit category, the projection \( \mathcal{C}/I \to \mathcal{C} \) is a fibration – it is the externalization (see [57, Section 4]) of the discrete internal category with \( I \) as set of objects. Now a fibration on a discrete category is the same thing as a family of categories – analogously we may regard a fibration on a discrete fibration on \( \mathcal{C} \) as a family of fibrations on \( \mathcal{C} \). From this point of view, the localization of \( \mathcal{C} \) at \( I \) as a fibered fibration

\[
|\mathcal{C}/I| \xrightarrow{\mathcal{C}/I} \mathcal{C}/I \to \mathcal{C}
\]

can be viewed as \( I \)-indexed family of fibrations on \( \mathcal{C} \) of value constant \( \mathcal{C} \), and for \( C \in \mathcal{C}/I \), the fibered fibration

\[
|\mathcal{C}|/C \xrightarrow{\mathcal{C}/C} \mathcal{C}/I \to \mathcal{C}
\]

can be viewed as \( I \)-indexed family of fibrations whose components are the (ordinary) slices of \( \mathcal{C} \) (viewed as generalized category) over the components of \( C \) (viewed as \( I \)-indexed family of objects in the generalized category).

This intuition is a helpful guideline to understand for example which properties of fibrations are preserved by localization and slicing – since localization corresponds simply to taking many copies of the same fibration, we can expect it to preserve all properties of fibrations that correspond to ‘reasonable’ properties of categories, such as having small (co)limits, local smallness, and well-poweredness. With slicing, we have to be a bit more careful – just as for ordinary categories, slicing of fibrations preserves the existence of finite limits, as well as the existence of (finite/small) coproducts, but not of products (neither finite nor small).

It seems that all classes of fibrations that we consider in this work are stable under localization and slicing – we give explicit proofs whenever we actually need this.

### 2.2 Internal sums

Internal sums in a fibration are an abstraction of infinite coproducts in categories, and of infinite joins in preorders. In this section, we introduce the general concept and then devote closer attention to two special cases – internal sums in posetal fibrations, and extensive sums.

We recall the definition of internal sums in fibrations from [57, Sections 6].

**Definition 2.2.1** Let \( \mathcal{C} : [\mathcal{C}] \to \mathcal{C} \) be a fibration on a finite limit category \( \mathcal{C} \).

(i) We say that \( \mathcal{C} \) has internal sums if

(a) in addition to being a fibration, \( \mathcal{C} \) is an opfibration in the sense that \( \mathcal{C}^{op} : [\mathcal{C}]^{op} \to \mathcal{C}^{op} \) is a fibration – in this case we call the cartesian arrows in \( \mathcal{C}^{op} \) cocartesian in \( \mathcal{C} \), and we use the arrow symbol \( \rightarrow \) for them, and

(b) the Beck-Chevalley condition (BCC) holds: cocartesian maps in \( [\mathcal{C}] \) are stable under pullbacks along cartesian maps.

(ii) If \( \mathcal{C} \) is a finite limit fibration with internal sums, then we say that internal sums in \( \mathcal{C} \) are stable, if cocartesian maps in \( [\mathcal{C}] \) are stable under pullback along arbitrary maps.

If \( A \) is a category, then \( \text{fam}(A) : \text{Fam}(A) \to \text{Set} \) has internal sums iff \( A \) has small coproducts. Internal sums in \( \text{fam}(A) \) are stable iff the same is true for the coproducts in \( A \).

Let us now consider the posetal case.

---

3. For this statement to make sense we don’t have to assume that \( \mathcal{C} \) is a finite limit fibration – for pullbacks along cartesian maps to exist we only need pullbacks in the base.
2.2.1 Existential fibrations and partial equivalence relations

Definition 2.2.2 An existential fibration is a posetal fibration \( \mathcal{X} : |\mathcal{X}| \to \mathcal{C} \) on a finite limit category \( \mathcal{C} \) which has finite meets and stable internal sums.

In an existential fibration, we can soundly interpret the \( \top, \land, \exists \) fragment of first order logic, where conjunctions are interpreted by meets in the fibers; and existential quantification is interpreted by internal sums (see [30, Section 4.2]). Because of this correspondence, we normally refer to internal sums in existential fibrations as existential quantification, and we denote existential quantification of a predicate \( \psi \in \mathcal{X}_J \) along a morphism \( u : J \to I \) in \( \mathbb{R} \) as \( \exists_u \psi \). Stability is known as Frobenius law in the posetal context, where it is normally expressed as

\[
\varphi \land \exists_u \psi \equiv \exists_u \varphi \land \psi \quad \text{for } \varphi \in \mathcal{X}_I \text{ and } \psi \in \mathcal{X}_J.
\]

Example 2.2.3 The subobject fibration \( \text{sub}(\mathbb{R}) \) of a regular category \( \mathbb{R} \) is an existential fibration.

A very useful construction on existential fibrations is the category of partial equivalence relations.

Definition 2.2.4 Let \( \mathcal{X} : |\mathcal{X}| \to \mathcal{C} \) be an existential fibration. The category \( \text{PER}(\mathcal{X}) \) is defined as follows.

- Objects are pairs \( (C, \rho) \) of an object \( C \in \mathcal{C} \) and a binary predicate \( \rho \in \mathcal{X}_{C \times C} \) which is a partial equivalence relation, i.e. the judgments
  
  \[
  \begin{align*}
  (\text{symm}) & \quad \rho(c, d) \vdash \rho(d, c) \\
  (\text{trans}) & \quad \rho(c, d), \rho(d, e) \vdash \rho(c, e)
  \end{align*}
  \]

  hold.

- Morphisms from \( (C, \rho) \) to \( (D, \sigma) \) are equivalence classes of binary predicates \( \phi \in \mathcal{X}_{C \times D} \) which are functional and total in a sense relative to \( \rho \) and \( \sigma \) – more precisely the judgments
  
  \[
  \begin{align*}
  (\text{strict}) & \quad \phi(c, d) \vdash \rho(c) \land \sigma(d) \\
  (\text{cong}) & \quad \phi(c, d), \rho(c, c'), \sigma(d, d') \vdash \phi(c', d') \\
  (\text{singval}) & \quad \phi(c, d), \rho(c, c') \vdash \sigma(d, d') \\
  (\text{tot}) & \quad \rho(c) \vdash \exists d . \phi(c, d)
  \end{align*}
  \]

  hold\(^6\). Two such predicates \( \phi, \phi' \in \mathcal{X}_{C \times D} \) are identified as morphisms in \( \text{PER}(\mathcal{X}) \) if they are logically equivalent, i.e. \( \phi(c, d) \dashv \vdash \phi'(c, d) \) holds.

- Composition of is given by relational composition, i.e. \( (\gamma \circ \phi)(c, e) \equiv \exists d . \phi(c, d) \land \gamma(d, e) \).

Associativity of composition follows from the Frobenius law, and it is easy to see that an identity morphism for \( (C, \rho) \) is given by \( \rho \) itself, thus \( \text{PER}(\mathcal{X}) \) is really a category. In the following, we want to show that \( \text{PER}(\mathcal{X}) \) is an exact category for any existential fibration \( \mathcal{X} \). This fact can probably be considered folklore; we will give a detailed proof since the occurring constructions will be important later.

The following lemma is easy to show.

\(^4\) In particular, we always assume the Beck-Chevalley condition when speaking about existential quantification.

\(^5\) The construction of \( \text{PER}(\mathcal{X}) \) will turn out to be the same as the construction of \( \mathcal{C}[\mathcal{X}] \) that we consider in Section 3.4.2.2, but for bootstrapping reasons, we use a different notation here.

\(^6\) For a partial equivalence relation \( \rho \), we often use \( \rho(c) \) as an abbreviation for \( \rho(c, c) \) – the ‘definedness’ part of the relation.
Lemma 2.2.5 Let $\mathcal{X} : |\mathcal{X}| \to \mathbb{C}$ be an existential fibration, and let $\phi : (C, \rho) \to (D, \sigma)$ be a morphism in $\text{PER}(\mathcal{X})$.
- If the judgment
  $$\sigma(d) \vdash \exists c. \phi(c, d) \tag{inj}$$
  holds, then $\phi$ is a monomorphism.
- If the judgment
  $$\phi(c, d), \phi(c', d) \vdash \rho(c, c') \tag{surj}$$
  holds, then $\phi$ is an cover, i.e. left orthogonal (Definition 2.2.6-(i)) to all monomorphisms.
- $\phi$ is an isomorphism iff (inj) and (surj) both hold.

Given an arbitrary morphism $\phi : (C, \rho) \to (D, \sigma)$ in $\text{PER}(\mathcal{X})$, we have a decomposition

$$
(C, \rho) \xrightarrow{\pi} (C, \pi) \xrightarrow{\phi} (D, \sigma) \xrightarrow{\sigma|_v} (D, \sigma)
$$

into an cover, an isomorphism, and a monomorphism. Here,

$$
\begin{align*}
\pi(c, c') & \equiv \rho(c) \land \rho(c') \land \exists d. \phi(c, d) \land \phi(c', d) \\
\phi(c, d) & \equiv \phi(c, d) \\
v(d) & \equiv \exists c. \phi(c, d) \\
\sigma|_v(d, d') & \equiv \sigma(d, d') \land v(d).
\end{align*}
$$

The predicate $v$ in this decomposition has a special relation to $\sigma$, which (following [61]) we call strictness:

Definition 2.2.6 Let $\mathcal{X} : |\mathcal{X}| \to \mathbb{C}$ be an existential fibration, and let $(C, \rho) \in \text{PER}(\mathcal{X})$. We call $\varphi \in \mathcal{X}_C$ strict with respect to $\rho$, if the judgments $\varphi(x) \vdash \rho(x)$ and $\varphi(x), \rho(x, y) \vdash \varphi(y)$ hold in $\mathcal{X}$.

The exactness of $\text{PER}(\mathcal{X})$ is now shown as follows.

Lemma 2.2.7 Let $\mathcal{X} : |\mathcal{X}| \to \mathbb{C}$ be an existential fibration.

(i) For $(D, \sigma) \in \text{PER}(\mathcal{X})$, subobjects of $(D, \sigma)$ correspond to predicates in $\mathcal{X}_D$ which are strict with respect to $(D, \sigma)$. More precisely, monomorphisms into $(D, \sigma)$ can up to isomorphism be represented as $\sigma|_v : (D, \sigma|_v) \to (D, \sigma)$ where $v \in \mathcal{X}_D$ is strict with respect to $\sigma$.

(ii) $\text{PER}(\mathcal{X})$ has finite limits.

(iii) If $\phi : (C, \rho) \to (D, \sigma)$ is a monomorphism, then (inj) holds.

(iv) If $\phi : (C, \rho) \to (D, \sigma)$ is a regular epimorphism, then (surj) holds.

(v) $\text{PER}(\mathcal{X})$ is regular.

(vi) Covers with domain $(C, \rho)$ can up to isomorphism be represented as $\pi : (C, \rho) \to (C, \pi)$ where $\pi \in \mathcal{X}_{C \times C}$ is a partial equivalence relation satisfying $\rho(c, c') \vdash \pi(c, c')$ and $\pi(c) \vdash \rho(c)$.

(vii) $\text{PER}(\mathcal{X})$ is exact.

Proof. Ad (i). If we apply the decomposition (2.2.1) to a monomorphism, then by orthogonality the cover part will be an isomorphism. It is easy to see that the resulting $v$ is strict with respect to $\sigma$.

Ad (ii). A product of $(C, \rho), (D, \sigma)$ is given by $(C \times D, \rho \times \sigma)$ where

$$\rho \times \sigma(c, d, c', d') \equiv \rho(c, c') \land \sigma(d, d'),$$

Ad (iii). If we apply the decomposition (2.2.1) to a cover, then by orthogonality the monomorphism part will be an isomorphism. It is easy to see that the resulting $\phi$ is a monomorphism.

Ad (iv). If we apply the decomposition (2.2.1) to a regular epimorphism, then by orthogonality the isomorphism part will be a monomorphism. It is easy to see that the resulting $\sigma|_v$ is a monomorphism.

Ad (v). $\text{PER}(\mathcal{X})$ is regular if and only if the products in $\text{PER}(\mathcal{X})$ are regular.

Ad (vi). If we apply the decomposition (2.2.1) to a cover, then by orthogonality the isomorphism part will be an isomorphism. It is easy to see that the resulting $\pi$ is an isomorphism.

Ad (vii). If we apply the decomposition (2.2.1) to a regular epimorphism, then by orthogonality the isomorphism part will be an isomorphism. It is easy to see that the resulting $\pi$ is an isomorphism.
(1, T) is a terminal object, and an equalizer of φ, γ: (C, ρ) → (D, σ) is given by the subobject of (C, ρ) corresponding to the predicate \( \{ c \mid ∃ d. φ(c, d) ∧ γ(c, d) \} \).

Ad (iii). The kernel of φ is represented by the predicate \( \{ c, c' \mid ∃ d. φ(c, d) ∧ φ(c', d) \} \), and the diagonal subobject \( δ : (C, ρ) → (C, ρ) \) is represented by \( ρ \) itself. The claim follows since the kernel coincides with the diagonal for monomorphisms.

Ad (iv). We already know that PER(\( X \)) has finite limits and cover/mono factorizations. It remains to show that covers are stable under pullback. Using the previous construction of finite limits and characterization of covers, this is easy to verify.

Ad (vi). This follows from the factorization and orthogonality.

Ad (vii). We have to show that equivalence relations are effective, i.e. appear as kernel pairs. This follows since the binary predicates \( τ \) representing equivalence relations on (C, ρ) coincide exactly with those binary predicates representing quotients of (C, ρ) as in (vi).

2.2.2 Extensive fibrations and Moens’ theorem

We recall the definition of extensivity for internal sums from [57, Section 15].

Definition 2.2.8  
(i) If \( C \) is a finite limit fibration with internal sums, then we say that internal sums in \( C \) are disjoint, if for any cocartesian map \( s: A \to B \) in \( \mathcal{C} \), the canonical map \( δ: A → A × B \) is also cocartesian.

(ii) We call internal sums in a finite limit fibration extensive, if they are stable and disjoint. A finite limit fibration with extensive internal sums is also called a lextensive fibration.

(iii) \( \text{Lx}(\mathfrak{C}) \) is the 2-category of lextensive fibrations on \( \mathfrak{C} \). Its 1-cells are fibered functors preserving finite limits and internal sums, and its 2-cells are fibered natural transformations.

The following lemma describes two ways to construct lextensive fibrations.

Lemma 2.2.9  
(i) Let \( \mathfrak{C} \) be a finite limit category. Then the functor

\[
\text{cod}(\mathfrak{C}) : \mathfrak{C} ↓ \mathfrak{C} → \mathfrak{C},
\]

which sends every morphism to its codomain, is a lextensive fibration which we call the fundamental fibration of \( \mathfrak{C} \) (the fundamental fibration is also known as codomain fibration, e.g. in [30, 61]).

(ii) If \( \Delta: \mathfrak{C} → \mathfrak{D} \) is a finite limit preserving functor between finite limit categories, and \( \mathfrak{E} \) is a lextensive fibration on \( \mathfrak{D} \), then the (strict) pullback \( \Delta^*\mathfrak{E} \) of \( \mathfrak{E} \) along \( \Delta \) is a lextensive fibration on \( \mathfrak{C} \).

Remark 2.2.10 The fundamental fibration of a finite limit category \( \mathfrak{C} \) has an important subfibration – the subobject fibration

\[
\text{sub}(\mathfrak{C}) : \text{Sub}(\mathfrak{C}) → \mathfrak{C},
\]

which is defined as the full subfibration of cod(\( \mathfrak{C} \)) on those objects in \( \mathfrak{C} ↓ \mathfrak{C} \) which are monomorphisms in \( \mathfrak{C} \).
The combination of pullback and fundamental fibration is known as the **gluing construction**.

**Definition 2.2.11** Let $\Delta : C \to D$ be a finite limit preserving functor. The **gluing of $D$ along $\Delta$**, denoted by $\text{gl}_\Delta(D)$, is the fibration obtained by pulling back $\text{cod}(D)$ along $\Delta$.

$$
\begin{array}{c}
\text{Gl}_\Delta(D) \\
\Downarrow \text{gl}_\Delta(D) \\
\Delta \\
\Downarrow \text{cod}(D)
\end{array}
\xrightarrow{(2.2.2)}
\begin{array}{c}
C \\
\Downarrow \Delta \\
D
\end{array}
$$

Concretely, the total category $\text{Gl}_\Delta(D)$ of the gluing fibration is the comma category $D \downarrow \Delta$, and the functor $\text{gl}_\Delta(D)$ is the evident projection. Streicher [57] denotes the gluing along a functor simply by $\text{gl}(\Delta)$, but since essentially all of the functors of which we take the gluing are called $\Delta$, I chose a more informative notation.

Moens [47] observed that up to equivalence, all lextensive fibrations are obtained by gluing, which can be expressed as a biequivalence of 2-categories.

**Theorem 2.2.12** (**Moens’ theorem**) Given a finite limit category $C$, the assignment $$(\Delta : C \to D) \mapsto \text{gl}_\Delta(D)$$ from Diagram (2.2.2) gives rise to a biequivalence $$C \sslash \text{Lex} \simeq \text{Lxv}(C),$$ (2.2.3) where $\text{Lex}$ is the 2-category of finite limit categories, finite limit preserving functors and natural transformations, and $C \sslash \text{Lex}$ is the pseudo-co-slice\(^7\) 2-category of $\text{Lex}$ under $C$.

**Proof.** Given a lextensive fibration $\mathcal{E} : |\mathcal{E}| \to C$, the associated functor is given by $$\Delta_\mathcal{E} : C \to \mathcal{E}_1, \quad I \mapsto \Sigma_I 1_I,$$
and given a fibered functor $F : \mathcal{E} \to \mathcal{F}$ between extensive fibrations which preserves finite limits and internal sums, it follows from the preservation of internal sums that $F_1 \circ \Delta_\mathcal{E} \cong \Delta_\mathcal{F}$.

We refer to [57, Section 15] for details.

As a first application, we can deduce that the fundamental fibration $\text{cod}(C) : C \downarrow C \to C$ is bi-initial in $\text{Lxv}(C)$ since $\text{id}_C$ is bi-initial in $C \sslash \text{Lex}$. This means that for every lextensive fibration $\mathcal{E} : |\mathcal{E}| \to C$ there exists a unique (up to unique equivalence) lextensive fibered functor

$$\Delta : \text{cod}(C) \to \mathcal{E},$$ (2.2.4)

which we call $\Delta$ since it is the fibered analogue of the functor $\Delta : C \to \mathcal{E}_1$ defined in the proof above. Concretely, $\Delta$ is given by

$$\text{cod}(C)_I \ni (f : J \to I) \mapsto \Sigma_f 1_J \in \mathcal{E}_I.$$

---

\(^7\) The ‘pseudo’ here means that the triangles in the definition of morphism commute only up to specified isomorphism.
Remark 2.2.13 The biequivalence (2.2.3) can be generalized to a more general class of functors: given a fibered functor

\[ F : \mathcal{E} \to \mathcal{F} \]

between extensive fibrations \( \mathcal{E}, \mathcal{F} \) which preserves finite limits but not necessarily internal sums, the isomorphism \( F_1 \circ \Delta_\mathcal{E} \cong \Delta_\mathcal{F} \) becomes replaced by a natural transformation of type \( \Delta_\mathcal{F} \to F_1 \circ \Delta_\mathcal{E} \).

\[
\begin{array}{ccc}
\Delta_\mathcal{F} & \cong & \Delta_\mathcal{F} \\
\downarrow & & \downarrow \\
F_1 \circ \Delta_\mathcal{E} & \Rightarrow & F_1 \circ \Delta_\mathcal{E}
\end{array}
\]

Triangles of the form (2.2.5) form a 2-category \( (\mathcal{C}_{/}\text{Lex}) \) generalizing the pseudo-co-slice 2-category \( \mathcal{C}_{/\text{sslash}} \text{Lex} \) (which one may call ‘oplax co-slice 2-category’), and one can show that \( (\mathcal{C}_{/}\text{Lex}) \) is equivalent to the 2-category of extensive fibrations and finite limit preserving fibered functors. This is relevant in Section 4.8 where we treat ‘global sections functors’.

2.3 Regular pre-stacks and stacks

In this section, we introduce the classes of fibrations that we will use in Chapter 3 to give a framework for (pre)sheaf constructions and realizability. In this context, we will always require our fibrations to be pre-stacks on regular base categories. We have several reasons for insisting on the pre-stack condition.

- In his thesis [50], Pitts showed how iterated tripos constructions can be ‘composed’, provided the corresponding constant object functors are regular. This regularity requirement is equivalent to the pre-stack condition for the corresponding gluing fibrations (or triposes).
- In Chapter 4 we study uniform preorders, which are representations of fibered preorders. It turns out that a fibered preorder on \( \text{Set} \) can be represented by a uniform preorder if it is a pre-stack and has a generic family of predicates (see Lemma 4.2.5).
- Robinson and Rosolini [52] and Carboni [12] give construction of realizability toposes and presheaf toposes using exact completion. These constructions rely on the axiom of choice. The fibered presheaf construction (Section 3.1) captures these relationships without choice, giving a universal characterization in terms of a biadjunction between 2-categories of pre-stacks.

2.3.1 Definition and basic properties

In the present work, (pre-)stack always means ‘(pre-)stack for the regular topology on a regular category’.

We refer to [62, Definition 4.6] for the general definition of what it means for a fibration to be a (pre-)stack for a Grothendieck topology, and to [31, Example A2.1.11(a)] for the definition of the regular Grothendieck topology. In the following we give the instantiated definition of (pre-)stack for the regular topology.

Let \( \mathcal{C} : \{ \mathcal{C} \} \to \mathbb{R} \) be a fibration on a regular category \( \mathbb{R} \), and let \( e : J \hookrightarrow I \) be a regular epimorphism in \( \mathbb{R} \). We form the diagram

\[
\begin{array}{ccc}
J \times_1 J & J \times_1 J & J \times_1 J \\
\downarrow \partial_0, \partial_1, \partial_2 & \downarrow \partial_0, \partial_1 & \downarrow e \\
J \to J & \Rightarrow & J \to I
\end{array}
\]
of products and projection maps in $\mathbb{R}/I$. We employ notation coming from simplicial sets and write $\partial_i$ for the projection which omits the $i$-th component (see e.g. [20, Chapter I-1]). In particular, the simplicial identities [20, Equation I-(1.3)]

$$\partial_0 \partial_1 = \partial_0 \partial_0 \quad \partial_0 \partial_2 = \partial_1 \partial_0 \quad \partial_1 \partial_2 = \partial_1 \partial_1$$

are satisfied.

**Definition 2.3.1** The category $\text{Desc}(\mathcal{C}, e)$ of descent data over $e : J \rightarrow I$ is defined as follows.

- An object with descent data $((A_i), (p_i), (q_i))$ is a configuration

$$A_3 \xrightarrow{q_0, q_1, q_2} A_2 \xrightarrow{p_0, p_1} A_1$$

of objects and cartesian arrows in $|\mathcal{C}|$ over the truncated complex (2.3.1) satisfying the same simplicial identities

$$p_0 q_1 = p_0 q_0 \quad p_0 q_2 = p_1 q_0 \quad p_1 q_2 = p_1 q_1.$$  

- A morphism between objects with descent data

$$(f_i) : ((A_i), (p_i), (q_i)) \rightarrow ((B_i), (r_i), (s_i))$$

is a family of vertical maps $f_i : A_i \rightarrow B_i$

$$A_3 \xrightarrow{q_0, q_1, q_2} A_2 \xrightarrow{p_0, p_1} A_1$$

such that $f_1 p_1 = r_1 f_2$ and $f_2 q_1 = s_1 f_3$ for all appropriate $i$.  

The previous definition does not refer to a cleavage on $\mathcal{C}$. In some situations (in particular in the proof of Lemma 3.4.23) it is useful to rephrase the definition in a way that uses an explicit cleavage, in which case the so-called cocycle condition becomes visible:

**Lemma 2.3.2** The category $\text{Desc}(\mathcal{C}, e)$ of descent data is equivalent to the category given by

- objects: pairs $(A \in \mathcal{C}, \alpha : \partial_i^* A \rightarrow \partial_i^* A)$ such that the diagram

$$\begin{align*}
(\partial_1 \partial_1)^* A & \xrightarrow{\sim} \partial_1^* \partial_1^* A \xrightarrow{\partial_1^* \alpha} \partial_1^* \partial_1^* A \\
(\partial_0 \partial_1)^* A & \xrightarrow{\sim} \partial_0^* \partial_1^* A \xrightarrow{\partial_0^* \alpha} \partial_0^* \partial_1^* A \\
(\partial_0 \partial_2)^* A & \xrightarrow{\sim} \partial_0^* \partial_2^* A \xrightarrow{\partial_0^* \alpha} \partial_0^* \partial_2^* A \\
(\partial_1 \partial_2)^* A & \xrightarrow{\sim} \partial_1^* \partial_2^* A \xrightarrow{\partial_1^* \alpha} \partial_1^* \partial_2^* A \\
(\partial_0 \partial_2)^* A & = (\partial_1 \partial_0)^* A \\
\end{align*}$$
in \( \mathcal{C}_I \times \mathcal{J}_I \) commutes (the isomorphisms are those given by the universal property of cartesian liftings, and the equalities come from the simplicial identities), and

- morphisms from \((A, \alpha)\) to \((B, \beta)\): arrows \((f : A \to B) \in \mathcal{C}_I\) such that \(\partial^*_0 f \circ \alpha = \beta \circ \partial^*_1 f\).

**Proof.** [62, Section 4.1.2].

There is yet another definition of \(\text{Desc}(\mathcal{C}, e)\) in terms of sieves: the sieve \(\langle e \rangle\) generated by \(e : J \twoheadrightarrow I\) is a subfibration of the representable (discrete) fibration \(Y_I = \text{dom} : \mathbb{R}/I \to \mathbb{R}\), and one can show the following lemma.

**Lemma 2.3.3** We have an equivalence of categories \(\text{Desc}(\mathcal{C}, e) \simeq \text{Fib}(\mathbb{R})(\langle e \rangle, \mathcal{C})\).

**Proof.** [62, Proposition 4.5].

This allows us to embed \(\mathcal{C}_I\) into \(\text{Desc}(\mathcal{C}, e)\) via the chain

\[\mathcal{C}_I \simeq \text{Fib}(\mathbb{R})(Y I, \mathcal{C}) \to \text{Fib}(\mathbb{R})(\langle e \rangle, \mathcal{C}) \simeq \text{Desc}(\mathcal{C}, e)\,

(2.3.3)

where the first equivalence is the 2-Yoneda Lemma [62, Section 3.6.2] and the arrow is given by precomposition with the inclusion \(\langle e \rangle \subseteq Y_I\).

In order to effectively manipulate pre-stacks and stacks, we introduce some more terminology.

**Definition 2.3.4** Let \(\mathcal{C} : \mathbb{C} \to \mathbb{R}\) be a fibration on a regular category.

- \(\mathcal{C}\) is a **pre-stack**, if for each regular epimorphism \(e : J \twoheadrightarrow I\) in \(\mathbb{R}\), the embedding (2.3.3) is full and faithful.
- \(\mathcal{C}\) is a **stack**, if for each regular epimorphism \(e : J \twoheadrightarrow I\) in \(\mathbb{R}\), the embedding (2.3.3) is an equivalence of categories.

In order to effectively manipulate pre-stacks and stacks, we introduce some more terminology.

**Definition 2.3.5** Let \(\mathcal{C} : \mathbb{C} \to \mathbb{R}\) be a fibration on a regular category \(\mathbb{R}\).

(i) A **cover-cartesian** morphism in \(\mathbb{C}\) is a cartesian morphism over a regular epimorphism.

We will denote cover-cartesian morphisms by the arrow symbol \(\triangleright\).

(ii) A **collective epimorphism** in \(\mathbb{C}\) is a map \(e\) such that \(fe = ge\) implies \(f = g\) for vertical \(f, g\).

The following lemma gives a diagrammatic criterion for a fibration to be a pre-stack.

**Lemma 2.3.6** A fibration \(\mathcal{C} : \mathbb{C} \to \mathbb{R}\) is a pre-stack, iff for any configuration

\[
\begin{array}{ccc}
A_2 & \xrightarrow{p_0, p_1} & A_1 \\
| & \downarrow{f_2} & | \\
B_2 & \xrightarrow{r_0, r_1} & B_1 \\
\end{array}
\]

where \(e\) is a regular epi in the base with kernel pair \(\partial_0, \partial_1\), and the other maps are cartesian or vertical above in \(\mathbb{C}\) as indicated such that \(p_0p_1 = pp_1\) and \(rr_0 = rr_1\), if \(f_1p_0 = r_0f_2\) and \(f_1p_1 = r_1f_2\) then there exists a unique \(h\) such that \(hp = rf_1\).

**Lemma 2.3.7** Let \(\mathcal{C} : \mathbb{C} \to \mathbb{R}\) be a pre-stack. Cover-cartesian maps in \(\mathbb{C}\) are regular epimorphisms.
2.3.6

Proof. Let \( p : A_1 \rightrightarrows A \) be a cover-cartesian map. We show that \( p \) is the coequalizer of its kernel pair \( A_2 \rightrightarrows A_1 \rightrightarrows A \) (which is definable using only finite limits in the base). Let \( f : A_1 \to B \) such that \( fp_0 = fp_1 \). Then \( \mathcal{C}(f) \) factors uniquely through \( \mathcal{C}(p) = e \) since \( e \) is a regular epimorphism and \( \mathcal{C}(f) \) coequalizes its kernel pair \( \mathcal{C}(p_0) = \partial_0, \mathcal{C}(p_1) = \partial_1 \). Consider the following diagram.

\[
\begin{array}{ccc}
A_2 & \rightrightarrows & A_1 & \rightrightarrows & A \\
\downarrow k & & \downarrow p & & \downarrow 1 \\
C_2 & \rightrightarrows & C_1 & \rightrightarrows & C \\
\downarrow d & & \downarrow \gamma & & \downarrow \varepsilon \\
K & \rightrightarrows & I & \rightrightarrows & J
\end{array}
\]

Here \( k \) can be understood both as the cartesian lifting of \( fp_0 = fp_1 \), \( fp = e \partial_0 = \varepsilon \partial_1 \) and as the reindexing of \( f^* \) along either of \( \partial_0 \) and \( \partial_1 \). Now since the reindexing of \( f^* \) along both components of the kernel pair of \( e \) are equal and \( \mathcal{C} \) is a pre-stack by assumption, Lemma \ref{lem:equivalences} allows us to deduce that there exists a unique \( f' : A \to C \) such that \( f'p = df^* \). The map \( e f' \) provides the desired factorization of \( f \) through \( e \), uniqueness follows from uniqueness of \( f' \). □

A minimal structural requirement for pre-stacks to be of interest for us is to have finite limits. This leads us to the following definition.

**Definition 2.3.8**

(i) A **finite-limit pre-stack** is a pre-stack \( \mathcal{C} : |\mathcal{C}| \to \mathbb{R} \) on a regular category which is a finite limit fibration in the sense of Definition \ref{def:finitelimit}. 

(ii) \( \text{Lex}(\mathbb{R}) \) is the 2-category of finite-limit pre-stacks on \( \mathbb{R} \). Its 1-cells are finite limit preserving fibered functors, and its 2-cells are fibered natural transformations. ⊤

### 2.3.1.1 Weak equivalences

There is a class of fibered functors between pre-stacks which are almost, but not quite, equivalences – the **weak equivalences**. We recall the definition from [11].

**Definition 2.3.9** A fibered functor \( F : \mathcal{C} \to \mathcal{D} \) between pre-stacks on a regular category \( \mathbb{R} \) is called a **weak equivalence**, if it is full and faithful and for each \( D \in |\mathcal{D}| \) there exists a \( C \in |\mathcal{C}| \) and a cover-cartesian map \( e : FC \rightrightarrows D \). ⊤

**Remarks 2.3.10**

- A good way to understand the relevance of weak equivalences between pre-stacks is to note that if \( F \) is an externalization ([57, Section 4]) of an internal functor \( F_0 : C \to D \) between internal categories, then \( F \) is a weak equivalence if \( F_0 \) is a weak equivalence in the sense of the internal logic of \( \mathbb{R} \), meaning that the statement that \( F \) is full, faithful and essentially surjective holds in the internal logic, but the essential surjectivity is not necessarily witnessed by a choice of essential pre-image for each object of \( D \).

- Another intuition on weak equivalence is given by the fact that a fibered functor \( F : \mathcal{C} \to \mathcal{D} \) between pre-stacks \( \mathcal{C}, \mathcal{D} \) is a weak equivalence iff the induced functor \( \bar{F} : \bar{\mathcal{C}} \to \bar{\mathcal{D}} \) between the stack completions of \( \mathcal{C} \) and \( \mathcal{D} \) is an equivalence in the standard sense ([11, Corollary 2.12], we do not treat stacks and stack-completions here and refer to [11] and the references therein).

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In the same spirit as the weakening of the notion of equivalence, one can consider a weak notion of ‘having finite limits’ for regular pre-stacks by demanding that each diagram in the total category can be covered (in the sense of cover-cartesian maps) by a diagram having a limiting cone. This and similar considerations are important in [29]. It seems reasonable to assume that everything that we do with finite-limit pre-stacks in this work also works for pre-stacks that only have ‘weak’ finite limits (in the above sense).

I wrote ‘weak’ in quotes above since there is a clash with standard terminology – normally a weak limit for a diagram is a cone that satisfies the existence part, but not the uniqueness part of the universal property of a limiting cone. If a pre-stack has weak finite limits in the fibrational sense, then the total category has weak finite limits in the ordinary sense, but not necessarily vice versa.

2.3.2 Geometric pre-stacks

Definition 2.3.11 A pre-stack of regular categories on a regular category $\mathcal{R}$ is a pre-stack $\mathcal{C} : |\mathcal{C}| \to \mathcal{R}$ whose fibers are regular categories, and whose reindexing functors are regular functors.

Lemma 2.3.12 Let $\mathcal{C} : |\mathcal{C}| \to \mathcal{R}$ be a pre-stack of regular categories on a regular category $\mathcal{R}$. Vertical regular epimorphisms are closed under descent, i.e. if $e : J \to I$ is a regular epimorphism in $\mathcal{R}$, and $(f : X \to Y) \in \mathcal{C}_I$ such that $e^* f$ is regular epic in $\mathcal{C}_J$, then $f$ is already a regular epic in $\mathcal{C}_I$.

Proof. We show that $f$ is left orthogonal (see Definition 2.3.18-(i)) to monos in $\mathcal{C}_I$. Any square $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{m} & & \downarrow{e} \\ U & \xrightarrow{e^* f} & A \end{array}$ gives rise to a square $\begin{array}{ccc} e^* X & \xrightarrow{e^* m} & e^* U \\ \downarrow{h} & & \downarrow{e^* e} \\ e^* Y & \xrightarrow{e^* A} & e^* A \end{array}$ in $\mathcal{C}_I$ with mediator $\begin{array}{ccc} e^* X & \xrightarrow{e^* m} & e^* U \\ \downarrow{h} & & \downarrow{e^* e} \\ e^* Y & \xrightarrow{e^* A} & e^* A \end{array}$. Now the reindexings of $h$ along the two components of the kernel pair of $e$ coincide (since they mediate the same orthogonality square), and thus (by Lemma 2.3.6) $h$ descends to a mediator of the square in $\mathcal{C}_I$.

Definition 2.3.13 Let $\mathcal{C} : |\mathcal{C}| \to \mathcal{C}$ be a fibration of finite limit categories on a finite limit category $\mathcal{C}$.

(i) The fibered subobject fibration $\text{sub}(\mathcal{C}) : \text{Sub}(\mathcal{C}) \to |\mathcal{C}|$ of $\mathcal{C}$ is the posetal fibration on $|\mathcal{C}|$ whose predicates on $C \in |\mathcal{C}|$ in $\text{sub}(\mathcal{C})$ are vertical monomorphisms with codomain $C$, where entailment is given by inclusion of monomorphisms, and reindexing is given by pullback.

(ii) We say that $\mathcal{C}$ has internal unions, if $\text{sub}(\mathcal{C})$ admits left adjoints to reindexing along cartesian morphisms in $|\mathcal{C}|$ subject to the Beck-Chevalley condition for pullbacks along cartesian morphisms (i.e. for squares of cartesian morphism in $|\mathcal{C}|$ over pullbacks in $\mathcal{C}$).

(iii) We say that $\mathcal{C}$ has stable internal unions, if $\text{sub}(\mathcal{C})$ admits left adjoints to reindexing along cartesian morphisms in $|\mathcal{C}|$, subject to the Beck-Chevalley condition for pullbacks along arbitrary maps in $|\mathcal{C}|$.

8. For a long time I didn’t believe this to be relevant for realizability, but Wouter Stekelenburg pointed out to me that depending on the definition of pca, the category of partitioned assemblies over an internal pca does not necessarily have (strong) finite limits.
We leave it as an exercise to verify that for a finite limit category \( C \), \( \text{fam}(C) \) has internal unions iff the subobject lattices in \( C \) have small joins, and that internal unions in \( \text{fam}(C) \) are stable iff small joins of subobjects in \( C \) are stable under pullback.

**Definition 2.3.14**

(i) A **geometric pre-stack** is a pre-stack \( \mathcal{F} : |\mathcal{F}| \to \mathbb{R} \) of regular categories with stable internal unions.

(ii) \( \text{Geo}(\mathbb{R}) \) is the 2-category of geometric pre-stacks on \( \mathbb{R} \) – its 1-cells are regular fibered functors that preserve internal unions (‘geometric fibered functors’), and its 2-cells are fibered natural transformations.

**Remark 2.3.15** Johnstone [31, after Lemma A1.4.18] defines a **geometric category** to be a well-powered regular category with pullback-stable small joins of subobjects. It is easy to see that a small category \( S \) is geometric iff its family fibration \( \text{fam}(S) : \text{Fam}(S) \to \text{Set} \) is a geometric pre-stack in the sense of the preceding definition. The well poweredness condition is necessary for Johnstone to show that any geometric category is a Heyting category (since a monotone map between small cocomplete lattices has a right adjoint iff it preserves arbitrary joins), but it would be too restrictive for our purposes to make a similar assumption since we are interested in examples which do not have universal quantification in the fibers (see Remark 4.10.2-(iii)).

The nLab [49] doesn’t demand well poweredness either for geometric categories.

**Lemma 2.3.16** A pre-stack \( \mathcal{F} : |\mathcal{F}| \to \mathbb{R} \) of finite limit categories is a geometric pre-stack iff its fibered subobject fibration \( \text{sub}(\mathcal{F}) \) has existential quantification along arbitrary morphisms in \( |\mathcal{F}| \). In this case, \( \text{sub}(\mathcal{F}) \) also validates the Frobenius condition and thus is an existential fibration.

**Proof.** Let \( \mathcal{F} \) be a finite limit pre-stack such that \( \text{sub}(\mathcal{F}) \) has existential quantification. Clearly, if we have \( \exists \) along arbitrary maps, then we have it in particular along vertical and cartesian maps. Furthermore, the global Beck-Chevalley condition specializes to the same condition in the fibers, which implies that the fibers of \( \mathcal{F} \) are regular categories. To show that we are dealing with a fibration of regular categories, it remains to show that reindexing preserves regular epimorphisms. This follows from the global Beck-Chevalley condition for squares of the form

\[
\begin{array}{ccc}
\mathbf{D} & \xrightarrow{d} & D \\
\pi \downarrow & & \downarrow e \\
\mathbf{C} & \xrightarrow{c} & C
\end{array}
\]

with \( e \) (and thus \( \pi \)) vertical, since \( e \) is regular epic iff \( \exists_c \top \cong \top \), in which case we have \( \exists_{\pi} \top \cong \exists_{d^* \top} \cong e^* \exists_{c} \top \cong e^* \top \cong \top \), which means that \( \pi \) is also a regular epi. Thus, \( \mathcal{F} \) is a geometric pre-stack.

Conversely, assume that \( \mathcal{F} \) is a geometric pre-stack. Then we can existentially quantify in \( \text{sub}(\mathcal{F}) \) along vertical maps since the fibers are regular, and along cartesian maps by assumption. Since every map in the total category can be decomposed into vertical followed by cartesian part, we can thus quantify along all maps. It remains to check if the global Beck-Chevalley condition holds in this case. To this end consider a pullback square

\[
\begin{array}{ccc}
P & \xrightarrow{j} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{i} & C
\end{array}
\]
in $|\mathcal{V}|$. This square can be decomposed into a diagram of cartesian and vertical arrows as in

\[
\begin{array}{ccc}
P & \rightarrow & A \\
\downarrow & & \downarrow \\
\bullet & \rightarrow & \bullet,
\end{array}
\]

and to check the condition for the large square, it suffices to check it for the small ones. The condition for the upper left square follows since it holds in regular categories, for the two squares where we quantify along cartesian arrows it follows by assumption, and for the remaining one it is a consequence of the fact that reindexing preserves regular epimorphisms.

It remains to show the validity of the Frobenius condition. Consider the diagram

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow & & \downarrow \\
\varphi & \rightarrow & \varphi,
\end{array}
\]

We have to show that $\varphi \land \exists u\psi \cong \exists u\varphi \land \psi$. To show this, we can argue

\[
\varphi \land \exists u\psi \cong \exists \phi\varphi^* \exists u\psi \cong \exists \phi\exists \psi^* \psi \cong \exists u\exists \psi^* \psi \cong \exists u \psi \land \psi,
\]

which completes the proof.

Definition 2.3.17 Let $\mathcal{V}$ be a geometric pre-stack on $\mathbb{R}$. We call a morphism $f : A \rightarrow B$ in $|\mathcal{V}|$ a collective cover (or collectively covering), if $\exists f^* T \cong T$.

When thinking in terms of families, this means that the objects in the family $B$ are covered by (the unions of) the images of the objects in the family $A$.

We will now prove that collective covers and vertical monos form a stable factorization system on the total category of a geometric pre-stack. Let us first recall the definition from [9, Section 5].

Definition 2.3.18 Let $\mathcal{C}$ be a category.

(i) Let $f : A \rightarrow B$, $g : X \rightarrow Y$ in $\mathcal{C}$. We say that $f$ is left orthogonal to $g$ (or that $g$ is right orthogonal to $f$) and we write $f \perp g$, if for any commuting square

\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow \\
B & \rightarrow & Y
\end{array}
\]

there exists a unique $h : B \rightarrow X$ such that the two triangles commute.

(ii) A factorization system on $\mathcal{C}$ is a pair $(\mathcal{E}, \mathcal{M})$ of classes of morphisms of $\mathcal{C}$ such that

(a) $\mathcal{E}$ and $\mathcal{M}$ are both closed under composition and contain all isomorphisms,

(b) for all $e \in \mathcal{E}$ and $m \in \mathcal{M}$ we have $e \perp m$, and

(c) any $f : A \rightarrow B$ in $\mathcal{C}$ can be factorized as $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$.

(iii) If $\mathcal{C}$ has pullbacks, we call a factorization system $(\mathcal{E}, \mathcal{M})$ on $\mathcal{C}$ stable, if given $f : A \rightarrow B$ and $e : C \rightarrow B$ we have $e \in \mathcal{E} \Rightarrow f^* e \in \mathcal{E}^9$.

9. It follows from orthogonality that the class $\mathcal{M}$ is closed under pullbacks.
Lemma 2.3.19 Let $\mathcal{S}$ be a geometric pre-stack on $R$.

(i) Cover-cartesian maps are collectively covering.

(ii) Collective covers and vertical monos form a stable factorization system on $|\mathcal{S}|$.

(iii) The collective-cover/vertical-mono factorization system allows to express existential quantification in $\text{sub}(\mathcal{S})$. Concretely, given $f : B \to A$ and a vertical monomorphism $m : U \to B$ in $|\mathcal{S}|$, $\exists f m$ is given by the vertical mono part of the collective-cover/vertical-mono factorization of $f m$.

(iv) Collective covers are collectively epic, i.e. if $e$ is collectively covering and $fe = ge$ where $f$ and $g$ are vertical, then $f = g$.

Proof. Ad (i). Let $e : A \xleftarrow{e} B$ be cover-cartesian in $|\mathcal{S}|$. We have to show that $e$ doesn’t factor through any nontrivial vertical monomorphism. Consider the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{m} & \bullet \\
\downarrow & & \downarrow \exists f m \\
B & \xrightarrow{f} & A
\end{array}
\]

The existence of $h$ is equivalent to $e^*m \cong \top$, thus we have to show that $e^*m \cong \top \Rightarrow m \cong \top$. This follows from the fact that reindexing along cover-cartesian maps reflects inclusion of vertical subobjects, which is a consequence of the fact that $\mathcal{S}$ is a pre-stack.

Ad (ii). It is clear that collective covers as well as vertical monos are closed under composition and contain all isos. To show the orthogonality property, since $|\mathcal{S}|$ has pullbacks which preserve vertical monos it is sufficient to show that whenever a collective cover factors through a vertical mono, then this is already an iso. As we already saw in the proof of (i), this is just a rephrasing of the property of being collectively covering. The existence of factorizations is not difficult to see either – we can factor any $f : A \to B$ in $|\mathcal{S}|$ through the vertical mono given by $\exists f \top$, and it is easy to see that the left part of this factorization is collectively covering. Finally, pullback stability of collective covers follows from the Beck-Chevalley condition in the definition of stable internal unions.

Ad (iii). This follows because $\exists f m$ as well as the vertical-mono part of the factorization can be characterized as minimal vertical subobject of $A$ admitting a factorization of $f m$.

Ad (iv). This is since on the one hand, $e$ factors through the equalizer of $f$ and $g$, which is a vertical subobject of $B$ and on the other hand $\top = \exists e \top$ is the minimal subobject of $B$ admitting a factorization of $e$. 

\[\Box\]

2.3.3 Positive pre-stacks

Definition 2.3.20 Let $R$ be a regular category.

(i) A positive pre-stack on $R$ is a pre-stack $\mathcal{P} : |\mathcal{P}| \to R$ of regular categories with extensive internal sums.

(ii) $\text{Pos}(R)$ is the 2-category of positive pre-stacks on $R$. Its 1-cells are fibered functors which are fiberwise regular and preserve internal sums (‘positive fibered functors’), and its 2-cells are arbitrary fibered natural transformations.

\[\Diamond\]
Remark 2.3.21 The family fibration $\text{fan}(P) : \text{Fam}(P) \to \text{Set}$ of a category $P$ is a positive pre-stack iff $P$ is regular and has small extensive sums. This is an infinitary version of what Johnstone calls ‘positive coherent category’ [31, Section A1.4] (and close to what he calls ‘∞-positive geometric category’, though there is again the difference about well poweredness that we pointed out in Remark 2.3.15).

Observe that we do not demand any form of sums in the fibers, only ‘between’ the fibers. In particular, the fibers of a positive fibration are not necessarily positive in the sense of Johnstone.♦

Since positive pre-stacks are in particular lex extensive fibrations, Moens’ theorem applies and one may ask how the additional conditions on the fibration can be expressed in terms of the corresponding functor. We will answer this question after an auxiliary lemma.

Lemma 2.3.22 Let $\mathcal{C} : |\mathcal{C}| \to \mathbb{R}$ be a positive pre-stack. For any regular epimorphism $e : J \to I$ in $\mathbb{R}$ and $A \in \mathcal{C}_I$, the fibered codiagonal map $\sigma : \Sigma e^*A \to A$ is a regular epimorphism in $\mathcal{C}_I$.

Proof. By Lemma 2.3.12, it is sufficient to show that $e^*\sigma$ is a regular epimorphism. In fact, $e^*\sigma$ is even a split epimorphism since it is a leg of one of the triangle equalities of the adjunction $\Sigma e \dashv e^*$.

Lemma 2.3.23 The gluing construction gives rise to a biequivalence

$$\mathbb{R}/\mathbb{R} \simeq \text{Pos}(\mathbb{R})$$

where $\text{Reg}$ is the 2-category of regular categories, regular functors and natural transformations, and $\mathbb{R}/\mathbb{R}$ is the pseudo-co-slice 2-category of $\text{Reg}$ under $\mathbb{R}$.

Proof. We rely on Moens’ Theorem 2.2.12 and only show that additional conditions on one side imply additional conditions on the other side and vice versa.

Since the fundamental fibration of a regular category is a pre-stack, and pre-stacks are stable under pullback along regular functors, $\text{gl}_\Delta(Q)$ is a positive pre-stack whenever $\Delta : \mathbb{R} \to Q$ is a regular functor between regular categories. Conversely, assume that $\mathcal{P}$ is a positive pre-stack on $\mathbb{R}$. We have to show that the functor

$$I \mapsto \sum_I 1_I : \mathbb{R} \to \mathcal{P}_1$$

preserves regular epimorphisms. Given a regular epimorphism $e : J \to I$ in $\mathcal{C}$, the map $\sigma : \Sigma_1 1_J \to 1_I$ in $\mathcal{P}_1$ is regular epic by Lemma 2.3.22, and its image $\Sigma_I \sigma$ is regular epic in $\mathcal{P}_1$ since $\Sigma : \mathcal{P}_I \to \mathcal{P}_1$ preserves regular epimorphisms as a left adjoint. This shows the equivalence between fibrations and functors. The correspondence on the level of 1-cells is easy to see. ■

Remark 2.3.24 The following remark does not really fit into the flow of ideas, and in particular relies on concepts that will be introduced only later. We nevertheless present it here, since it seems to be the right place for the informed reader.

There is a connection between $\mathbb{R}/\text{Reg}$ and Longley’s $\nabla \Gamma$-categories. Recall from [38, Definition 1.4.2] that a $\nabla \Gamma$-category is a regular category $Q$ together with regular functors $\nabla : \text{Set} \to Q$ and $\Gamma : Q \to \text{Set}$ such that $\Gamma \dashv \nabla$ and $\Gamma \nabla \cong \text{id}_{\text{Set}}$; the archetypal example being the category $\text{Asm}(A)$ of assemblies over a partial combinatory algebra $A$. A $\nabla \Gamma$-functor from $(Q, \nabla, \Gamma)$ to $(Q', \nabla', \Gamma')$ is a regular functor $F : Q \to Q'$ such that $F \nabla \cong \nabla'$ and $\Gamma'F \cong \Gamma$, and Longley’s ‘equivalence theorem’ [38, Theorem 2.2.20] states that applicative morphisms between pcas correspond to $\nabla \Gamma$-functors between the corresponding categories of assemblies.

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Now it turns out that for $\nabla\Gamma$-functors between categories of assemblies, the condition $\Gamma' F \cong \Gamma$ is redundant, and thus in this case $\nabla\Gamma$-functors are the same things as 1-cells in $\text{Set}/\text{Reg}$. Therefore, Longley’s equivalence theorem can be read as stating an equivalence between applicative morphisms between pcas, and positive fibered functors between the associated positive fibrations of assemblies.

The following lemma clarifies the relation between $\text{Geo}(\mathbb{R})$ and $\text{Pos}(\mathbb{R})$.

**Lemma 2.3.25**

(i) Positive pre-stacks are geometric pre-stacks.

(ii) Let $\mathcal{P} : |\mathcal{P}| \to \mathbb{R}$ be a positive fibration. Let $f : A \to B$ be a map in $|\mathcal{P}|$ over $u : I \to J$ in $\mathbb{R}$. Then $f$ is cocartesian iff the judgments

\[ x, y : A \mid fx = fy \vdash x = y \quad (2.3.4) \]

\[ z : B \mid \exists x : A. fx = z \quad (2.3.5) \]

hold in $\text{sub}(\mathcal{P})$.

(iii) A fibered functor $F : \mathcal{P} \to \mathcal{Q}$ between positive pre-stacks $\mathcal{P}$ and $\mathcal{Q}$ is positive iff it is geometric.

**Proof.** Ad (i). Given $f : A \to B$ over $u : I \to J$ and a vertical mono $m : V \to A$, existential quantification $\exists fm$ of $m$ along $f$ is given by cocartesian lifting and image factorization as in the following diagram.

\[ \begin{array}{ccc}
V & \xrightarrow{\Sigma_{\mathcal{P}}} & \Sigma_{\mathcal{P}}V \\
m \downarrow & & \downarrow \\
A & \xrightarrow{f} & \exists m \, B \\
\end{array} \]

(2.3.6)

Stability follows from stability of internal sums in extensive fibrations.

Ad (ii). Assume that judgments (2.3.4) and (2.3.5) hold for $f : A \to B$. Let $g : A \to C$ over $u : I \to J$. We have to show that there exists a unique vertical $h : B \to C$ such that $hf = g$. The uniqueness follows since $f$ is a collective cover and thus collectively epic (Lemma 2.3.19-(iv)). For existence, observe that since $f$ and $g$ are over the same arrow $u : I \to J$, the map $(f, g) : A \to B \times C$ factors through $B \times_J C$ as $k : A \to B \times_J C$. Consider the following diagram.

\[ \begin{array}{ccc}
U & \xrightarrow{r} & \exists f \, B \times_C C \\
\downarrow & & \downarrow \\
A & \xrightarrow{k} & B \times_J C \\
\end{array} \]

$r \circ e$ is a collective-cover/vertical-mono factorization of $k$, and we want to show that $pr$ is an isomorphism (this gives us a vertical map from $B \to C$ via the span $B \xrightarrow{=} U \to C$). Since $\mathcal{P}_J$ is regular, it suffices to show that $pr$ is a mono and a cover. The image of $pr$ can be written as $\exists f r$ in $\text{sub}(\mathcal{P})$, which is equal to $\exists f \top$, and thus equivalent to $\top$ by (2.3.5). To see that $pr$ is monic, observe that from $u \mid \vdash \exists a . ea = u$ and $fa = fa' \vdash a = a'$ we can derive $pru = prad \vdash u = u'$.
Conversely, assume that \( f \) is cocartesian. Judgment (2.3.5) is just a fancy way of saying that \( f \) is a collective cover, which is true for cocartesian maps as can be easily seen from the proof of (i). For the other judgment, let’s make the denotations of both sides of the turnstile explicit. The predicate \( (x,y:A \mid fx = fy) \) is the vertical image (in the sense of the collective-cover/vertical-mono factorization system) of \( A \times_B A \rightarrow A \times A \), and the predicate \( (x,y:A \mid x = y) \) is the vertical image of \( A \rightarrow A \times A \). Let \( m \circ e \) be a collective-cover/vertical-mono factorization of \( A \times_B A \rightarrow A \times A \). The canonical map \( \delta : A \rightarrow A \times_B A \) is cocartesian – and thus collectively covering – since internal sums in \( \mathcal{P} \) are disjoint (see Definition 2.2.8–(i)).

\[
\begin{array}{c}
A \xrightarrow{+} A \times_B A \xrightarrow{+} B \\
\downarrow \delta \downarrow \downarrow \\
A \times A \xrightarrow{f \times f} B \times B
\end{array}
\]

This implies that \( m \circ e \delta \) is a collective-cover/vertical-mono factorization of \( A \rightarrow B \times B \), hence the two maps have the same vertical image, and the predicates are equivalent.

Ad (iii). If \( F \) is a positive fibered functor, then it preserves diagrams of the form (2.3.6), hence it also preserves existential quantification. Conversely, if \( F \) is geometric, then the induced transformation from \( \text{sub}(\mathcal{P}) \) to \( \text{sub}(\mathcal{Q}) \) preserves the validity of the judgments (2.3.4) and (2.3.5), whence it also preserves cocartesian maps.

2.3.4 Fibered pretoposes

**Definition 2.3.26** Let \( \mathbb{R} \) be a regular category.

(i) A pre-stack of exact categories on \( \mathbb{R} \) is a pre-stack whose fibers are exact categories, and whose reindexing functors are regular functors.

(ii) A fibered pretopos on \( \mathbb{R} \) is a pre-stack of exact categories with extensive internal sums.

(iii) \( \text{Pretop}(\mathbb{R}) \) is the 2-category of fibered pretoposes on \( \mathbb{R} \). Its 1-cells are positive fibered functors (Definition 2.3.20), and its 2-cells are arbitrary fibered natural transformations.

**Remarks 2.3.27**

- The family fibration \( \text{fam}(\mathcal{X}) : \text{Fam}(\mathcal{X}) \rightarrow \text{Set} \) of a category \( \mathcal{X} \) is a fibered pretopos iff the category is an \( \infty \)-pretopos.

- Contrary to Johnstone’s definition of ‘\( \mathcal{S} \)-indexed \( \infty \)-pretopos’ [31, Definition 3.3.9], we do not assume that the fibers of a fibered pretopos are pretoposes – the extensive sums in our case are ‘purely infinitary’, i.e. between the fibers.

Again, we have a specialization of Moens’ theorem:

**Lemma 2.3.28** The gluing construction gives rise to a biequivalence

\[ \mathbb{R} \downarrow \mathbb{U} \simeq \text{Pos}(\mathbb{R}), \]

where \( \mathbb{U} : \text{Ex} \rightarrow \text{Reg} \) is the forgetful 2-functor from exact to regular categories, and \( \mathbb{R} \downarrow \mathbb{U} \) is the evident pseudo comma category.
Proof. This follows directly from Lemma 2.3.23 since exact categories are stable under slicing.

As a consequence of the preceding lemma we see that a fibered pretopos is not only a pre-stack, but even a stack (since the fundamental fibration of an exact category is a stack, and stacks are stable under change of base along regular functors).
Chapter 3

Fibrational cocompletions

Given a regular category $\mathbb{R}$, the 2-categories of pre-stacks on $\mathbb{R}$ that we introduced in Section 2.3 can be arranged into a sequence

$$\text{Lex}(\mathbb{R}) \leftrightarrow \text{Geo}(\mathbb{R}) \leftrightarrow \text{Pos}(\mathbb{R}) \leftrightarrow \text{Pretop}(\mathbb{R})$$

(3.0.1)

corresponded by forgetful functors. In this section, we will prove the existence of, and give constructions for, left biadjoints for all forgetful functors in this sequence. Before diving into the details, let us make some general remarks.

(i) On the right end of the sequence ($\text{Pretop}(\mathbb{R})$ and $\text{Pos}(\mathbb{R})$) we are dealing with extensive fibrations and are thus in the realm of Moens' theorem.

(ii) The forgetful functors $\text{Pretop}(\mathbb{R}) \to \text{Pos}(\mathbb{R})$ and $\text{Pos}(\mathbb{R}) \to \text{Geo}(\mathbb{R})$ are full on 1- and 2-cells and are thus local equivalences (for the second inclusion this follows from Lemma 2.3.25-(iii)), hence their left biadjoints are reflections. Concretely, this means for example that the fibered pretopos cocompletion of a geometric pre-stack $\mathcal{X}$ doesn’t add anything if $\mathcal{X}$ happens to be already a fibered pretopos.

(iii) The biadjunctions between finite limit pre-stacks and the three other types of pre-stacks are not reflections, but fulfill the weaker ‘lax idempotency’ condition $U\varepsilon \dashv \eta U$ between unit and counit ($U$ is the forgetful functor). This implies in particular that the induced pseudomonad is a KZ-monad \cite[Definition B1.1.11]{31}, which is characteristic of cocompletions.

One can recover a finite limit pre-stack (up to weak equivalence) from its (geometric/positive/pretopos)-cocompletion\footnote{see e.g. Lemma 3.1.16-(i) for the pretopos cocompletion}, something that is impossible for the cocompletions considered in (ii) because of idempotency.

To help the intuition of the reader we give some examples.

(i) Let $\mathcal{C}$ be a small finite limit category. The $\infty$-pretopos cocompletion of $\mathcal{C}$ is the presheaf topos $\hat{\mathcal{C}}$; the geometric and $\infty$-positive cocompletions are the subcategories of $\hat{\mathcal{C}}$ on sub-representables, and coproducts of sub-representables, respectively. This fits into our fibrational framework if we switch from categories to their family fibrations.

If $\mathcal{C}$ is not small but only locally small, the fibrational construction gives us small presheaves.

(ii) The $\infty$-pretopos cocompletion of a small geometric category $\mathcal{S}$ is the topos of sheaves for the canonical topology on $\mathcal{S}$. Again, this fits into the fibrational framework via the family construction.

1. see e.g. Lemma 3.1.16-(i) for the pretopos cocompletion
(iii) Given a pca $\mathcal{A}$, the uniform family fibration $\text{ufam}(\mathcal{A})$ is a finite limit pre-stack. The geometric cocompletion of $\text{ufam}(\mathcal{A})$ is the tripos $\text{rt}(\mathcal{A})$, the positive cocompletion is the gluing fibration of $\text{Asm}(\mathcal{A})$, and the fibered pretopos cocompletion is the gluing of the realizability topos $\text{RT}(\mathcal{A})$ (in each case along the ‘constant objects functor’).

The most ‘economic’ way to obtain left biadjoints to all functors (including compositions) in the sequence above would be to construct the left biadjoints only for the forgetful functors from one level to the next.

However, we will go a different way as it turns out to be most natural to construct first the left biadjoint of $\text{Pretop}(\mathbb{R}) \to \text{Lex}(\mathbb{R})$, and then to define the geometric and positive cocompletions of finite limit pre-stacks as suitable subfibrations.

In the same way, the left biadjoint of $\text{Pos}(\mathbb{R}) \to \text{Geo}(\mathbb{R})$ seems to be more fundamental than its 2-step decomposition (although in this case it is a bit arguable) whence we present it first.

3.1 Fibered presheaves

The ‘fibered presheaf construction’ is the left biadjoint to the forgetful functor

$$\text{Pretop}(\mathbb{R}) \leftrightarrow \text{Lex}(\mathbb{R}).$$

(3.1.1)

It is the starting point of our exploration of fibrational cocompletions. It is motivated by the following two facts.

- Robinson and Rosolini [52] observed that realizability toposes are exact completions of their subcategories of ‘partitioned assemblies’.
- Carboni [12] discovered that for a small category $\mathcal{C}$ with finite limits, the presheaf category $\hat{\mathcal{C}}$ is the exact completion of the category $\text{Fam}(\mathcal{C})$ of families.

Both statements are only true in presence of the axiom of choice. Now the motivation of the fibered presheaf construction is that it provides a common framework for the two observations, while managing to avoid the axiom of choice. The central observation is that both the category $\text{Fam}(\mathcal{C})$ of families of $\mathcal{C}$, and the category of partitioned assemblies over a pca $\mathcal{A}$ are total categories of certain fibrations – $\text{Fam}(\mathcal{C})$ is the total category of the family fibration $\text{fam}(\mathcal{C}) : \text{Fam}(\mathcal{C}) \to \text{Set}$, while the category of partitioned assemblies is the total category of the uniform family fibration $\text{ufam}(\mathcal{A}) : \text{UFam}(\mathcal{A}) \to \text{Set}$ defined after Definition 2.1.6 (this observation occurs to my knowledge first explicitly in Hofstra’s [23]). We now take the following point of view:

- Fibrations on $\text{Set}$ are ‘generalized categories/preorders’.
- In presence of choice, exact completion of the total categories followed by gluing along the appropriate inclusion functor is a cocompletion operation on fibrations which can be viewed as fibrational analogue of $\mathcal{C} \mapsto \hat{\mathcal{C}}$ for small $\mathcal{C}$ (more precisely, it is left adjoint to the forgetful functor (3.1.1)).

Without choice, the exact completion of the total category doesn’t do the job anymore, and we have to replace it by a construction that takes the structure of the fibration explicitly into account, while at the same time being closer intuitively to ideas about presheaves.

We start out by presenting an alternative reading of the exact completion of $\text{Fam}(\mathcal{C})$ ($\mathcal{C}$ small with finite limits), which makes the link to the presheaf construction directly visible.
The central idea is the fibration of sieves on $\mathcal{C}$, which is defined as the pullback of the subobject fibration of $\hat{\mathcal{C}}$ along $Y$.

\[
\begin{array}{ccc}
\text{Siev}(\mathcal{C}) & \longrightarrow & \text{Sub}(\hat{\mathcal{C}}) \\
\downarrow & & \downarrow \\
\text{siev}(\mathcal{C}) & \longrightarrow & \text{sub}(\hat{\mathcal{C}})
\end{array}
\]

We can reconstruct $\hat{\mathcal{C}}$ from $\text{siev}(\mathcal{C})$ in the following way. Given $F \in \hat{\mathcal{C}}$, we can cover it by representables $\Sigma Y(C_i) \rightarrow F$, and the kernel pair

\[
U_{i,j} \rightarrow U \rightarrow Y(C_i \times C_j)
\]

of this cover is determined by the sieves $U_{i,j} \rightarrow Y(C_i \times C_j)$ obtained by restricting to the summands.

The $U_{i,j}$ together represent an equivalence relation on $\Sigma_{i,j} Y(C_i \times C_j)$, and it turns out that we can express this fact without actually referring to the coproduct – we can write the conditions indexwise in the form

\[
\begin{align*}
i, j, k & \in I \Rightarrow U_{j,k} \circ U_{i,j} \subseteq U_{i,k} \\
i & \in I \Rightarrow \text{id}_{A_i} \subseteq U_{ii}.
\end{align*}
\]

Following a suggestion by Alex Simpson, we call a system $(U_{i,j})_{i,j \in I}$ of predicates that fulfills these conditions a heterogeneous equivalence relation (compare [3, Lemma 11.6]).

We now get our representation of $\hat{\mathcal{C}}$ by taking families $(C_i)_{i \in I}$ of objects in $\mathcal{C}$ equipped with heterogeneous equivalence relations $(U_{i,j})_{ij}$ in $\text{siev}(\mathcal{C})$ as objects, and an appropriate version of ‘heterogeneous functional relations’ as morphisms.

If we want to internalize our handling of families, we can express the heterogeneous relations as ordinary equivalence relations and functional relations in the fibration

\[
\text{Fam(siev}(\mathcal{C})) : |\text{Fam(siev}(\mathcal{C}))| \rightarrow \text{Fam}(\mathcal{C})
\]

whose predicates on a family $(C_i)_{i \in I}$ are just $I$-indexed families of sieves. Now the somewhat surprising observation is that in presence of choice, this fibration is equivalent to the posetal reflection of the fundamental fibration

\[
\text{cod(Fam}(\mathcal{C})) : \text{Fam}(\mathcal{C}), \text{Fam}(\mathcal{C}) \rightarrow \text{Fam}(\mathcal{C})
\]

- a morphism $(u, (f_{ij})) : (D_j)_{j} \rightarrow (C_i)_{i}$ can be viewed as associating to each $i$ a family of morphisms in $\mathcal{C}$ with codomain $C_i$, and these morphisms generate a sieve on $C_i$. Here, we get the link to the exact completion, which can be described as the category of equivalence relations and functional relations in the posetal reflection of the fundamental fibration for any category with finite limits.

---

2. I think I first heard about this fibration when Streicher explained Shulman’s ‘stack semantics’ [54] to me.
3.1.1 The fibration of sieves

**Definition 3.1.1** Let \( \mathcal{C} : [\mathcal{C}] \to \mathbb{R} \) be a finite limit pre-stack on a regular category \( \mathbb{R} \). The *fibration of sieves* on \( \mathcal{C} \) is the fibered preorder \( \text{siev}(\mathcal{C}) \) on \( [\mathcal{C}] \), where
- predicates on \( X \in [\mathcal{C}] \) are morphisms \( f : Y \to X \) in \( [\mathcal{C}] \)
- \( f \leq g \) over \( X \) if there exist \( e, k \) with \( e \) cover-cartesian such that

\[
\begin{array}{c}
Y' \\
\downarrow^k \\
Z \\
\downarrow_g \\
Y \\
\downarrow_f \\
X
\end{array}
\]

commutes.

**Remark 3.1.2**
- If \( g \) is a monomorphism in the previous definition, then \( f \leq g \) iff \( f \) factors through \( g \), since cover-cartesian maps in \( \mathcal{C} \) are regular epis by Lemma 2.3.7, and thus left orthogonal to monomorphisms.
- For a small finite limit category \( \mathcal{C} \), the fibration \( \text{siev} (\text{fam}(\mathcal{C})) \) is equivalent to the fibration \( \text{Fam}(\text{siev}(\mathcal{C})) \) from (3.1.3).

**Lemma 3.1.3** For any finite limit pre-stack \( \mathcal{C} \), \( \text{siev}(\mathcal{C}) \) is an existential fibration in the sense of Definition 2.2.2.

**Proof.** Conjunction is given by pullback and existential quantification by postcomposition.

**Remark 3.1.4** Cover-cartesian maps are surjective from the point of view of \( \text{siev}(\mathcal{C}) \). More precisely, if \( e : A \rightarrowtail B \) is cover-cartesian in \( [\mathcal{C}] \), then \( b \models \exists a. ea = b \) holds in \( \text{siev}(\mathcal{C}) \). This follows from the fact that existential quantification is given by postcomposition.

The following easy lemma will be very useful later.

**Lemma 3.1.5** Given a diagram

\[
\begin{array}{c}
A^* \\
\downarrow^e \\
A \\
\downarrow^h \\
B
\end{array}
\]

in \( [\mathcal{C}] \) where \( \mathcal{C} \) is a finite limit pre-stack and \( e \) is cover-cartesian, there exists a mediator \( h \) iff \( ex = ey \vdash fx = fy \) in \( \text{siev}(\mathcal{C}) \).

**Proof.** Since \( e \) is a regular epimorphism by Lemma 2.3.7, it suffices to show that the kernel of \( e \) is contained in the kernel of \( f \). This is equivalent to \( ex = ey \vdash fx = fy \) by Remark 3.1.2.

3.1.2 The fibered presheaf construction

Using the fibration of sieves, we can define presheaves on \( \mathcal{C} \) as ‘quotients of sums of representables’, or rather as formal quotients of objects in \( [\mathcal{C}] \) with respect to equivalence relations in \( \text{siev}(\mathcal{C}) \).

**Definition 3.1.6** Given a finite limit pre-stack \( \mathcal{C} : [\mathcal{C}] \to \mathbb{R} \), the category \( \mathbb{R}\{\mathcal{C}\} \) is the full subcategory of \( \text{PER}(\text{siev}(\mathcal{C})) \) on total equivalence relations.
Lemma 3.1.7 For any finite limit pre-stack $\mathcal{E} : |\mathcal{E}| \to R$, the category $\mathbb{R}\{\mathcal{E}\}$ is equivalent to PER(siev($\mathcal{E}$)) and therefore in particular exact.

Proof. We have to show that every partial equivalence relation in siev($\mathcal{E}$) is equivalent to a total one in the sense of PER(siev($\mathcal{E}$)). Given $(C, \rho)$ in the latter category, the predicate $(c \mid \rho(c, c)) \in$ siev($\mathcal{E}$)$_C$ is a morphism $f : D \to C$. If we define $\sigma \in$ siev($\mathcal{E}$)$_{D \times D}$ by $\sigma(x, y) = \rho(fx, fy)$ then $\sigma$ is a total equivalence relation, and furthermore $(D, \sigma)$ is isomorphic to $(C, \rho)$.

Thinking about presheaves as quotients of sums of representables, there is another way to represent morphisms besides functional relations in the fibration of sieves – thinking non-fibered for the moment, if $F$ and $G$ are presheaves covered by families of representables $(YA_i)_{i \in I}$ and $(YB_j)_{j \in J}$, then since representables are indecomposable and projective, the restriction $YA_i \to G$ of a morphism $f : F \to G$ to some $YA_i$ factors through some $YB_j$.

$$YA_i \hookrightarrow YB_j$$

This implies that in the presence of choice, morphisms from $F$ to $G$ can be represented by maps $u : I \to J$ and families $(f_i : A_i \to B_{u i})$ which are compatible with the equivalence relations. The next lemma does this in the fibrational setting. In the absence of choice in the base, however, while we can assert the existence of $j$ and $f_i : A_i \to B_j$ for a given $i \in I$, we can not actually choose one. This can be taken care of by working with spans $I \leftarrow \bullet \to J$ instead of maps $I \to J$; which motivates the following definition.

Definition 3.1.8 Let $\phi : (C, \rho) \to (D, \sigma)$ in $\mathbb{R}\{\mathcal{E}\}$. We call a span

$$C \leftarrow C^* \xrightarrow{f} D$$

with $c$ cover-cartesian a tracking family for $\phi$, if one of the following equivalent conditions holds in siev($\mathcal{E}$).

- $\phi(c, d) \vdash \exists x. ex = c \land \sigma(fx, d)$
- $\sigma(fx, d) \vdash \phi(ex, d)$
- $\phi(ex, d) \vdash \sigma(fx, d)$
- $\vdash \phi(ex, fx)$

A strict tracking family is a tracking family where the cover-cartesian part is an identity.

The first characterization shows how $\phi$ can be reconstructed from $(e, f)$. Thus, for fixed $\rho$ and $\sigma$, $(e, f)$ can be the tracking family of at most one morphism.

Lemma 3.1.9 (i) A span $C \leftarrow C^* \xrightarrow{f} D$ is a tracking family of some morphism from $(C, \rho)$ to $(D, \sigma)$ in $\mathbb{R}\{\mathcal{E}\}$ iff $\rho(ex, ey) \vdash (fx, fy)$ holds.

(ii) Two spans $(e, f)$, $(e', f')$ are tracking families of the same morphism from $(C, \rho)$ to $(D, \sigma)$ iff $\rho(ex, e'y) \vdash (fx, f'y)$ holds.

(iii) All morphisms in $\mathbb{R}\{\mathcal{E}\}$ have tracking families.

(iv) Morphisms of type $(C, \rho) \to (D, \sigma)$ (where the equivalence relation in the image is discrete), have strict tracking families, i.e., morphisms $f : C \to D$ such that $\rho(c, c') \vdash fc = fc'$.

The latter judgment is equivalent to the fact $f$ coequalizes the components of $\rho : R \to C \times C$.

3. Unless $G$ is the coproduct of the family $(YB_j)_{j \in J}$, see Lemma 3.1.9-(iv) below.
(v) For every morphism $\phi : (C, \rho) \to (D, \sigma)$ we can find an isomorphism $\iota : (C', \rho') \cong (C, \rho)$ such that $\phi \circ \iota$ has a strict tracking family.

**Proof.** Ad (i). Straightforward.

Ad (ii). Assume that $\rho(ex, e'y) \vdash \sigma(fx, fy)$ holds. We first show that $(e, f)$ (and therefore by symmetry also $(e', f')$) is a tracking family. Using (i), we have to show that $\rho(ex, ey) \vdash \sigma(fx, fy)$.

To do this, we argue informally in siev($\mathcal{E}$). Assume $\rho(ex, ey)$. By Remark 3.1.4, there exists $z$ in the domain of $e'$ such that $ex = e'z$. Since $\rho$ is total, we have $\rho(ex, e'z)$, which implies by assumption that $\sigma(fx, f'z)$. Substituting $e'z$ in $\rho(ex, ey)$, we have $\rho(e'z, ey)$, which implies (using symmetry) that $\sigma(f'z, fy)$. Together we have $\sigma(fx, fy)$.

The functional relation associated to a span $(e, f)$ is given by $(c, d \mid \exists x. ex = c \land \sigma(fx, d))$, therefore to show that $(e, f)$ and $(e', f')$ are equivalent, it remains to show that $\exists x. ex = c \land \sigma(fx, d) \vdash \exists y. e'y = c \land \sigma(f'y, d)$. Again, we reason informally in siev($\mathcal{E}$). Assume that $ex = c$ and $\sigma(fx, d)$. There exists $y$ such that $ex = c = e'y$. It remains to show that $\sigma(f'y, d)$, which by symmetry, transitivity and assumption is equivalent to $\sigma(fx, f'y)$. This follows from $\rho(ex, e'y)$, which holds because of totality of $\rho$.

Ad (iii). Given a functional relation $|\phi| \xrightarrow{\phi} C \times D$ between $(C, \rho)$ and $(D, \sigma)$, the totality judgment means precisely that there exist $e$ and $h$ making

\[
\begin{array}{ccc}
C^* & \xrightarrow{\phi} & D \\
\downarrow{h} & & \downarrow{\phi_2} \\
C & \xrightarrow{\phi_1} & D
\end{array}
\]

commute, and the desired span is obtained by setting $f = \phi_2 h$.

Ad (iv). If the span $(e, f)$ tracks a morphism from $(C, \rho)$ to $(D, =)$, we have $\rho(ex, ey) \vdash fx = fy$ by (i), thus in particular $ex = ey \vdash fx = fy$. By Lemma 3.1.5, there exists thus $h : C \to D$ such that $he = f$. Using (ii) one can verify that the span $(\text{id}_C, h)$ tracks the same morphism as $(e, f)$.

The fact that $\rho(c, c') \vdash fc = fc'$ iff $f\rho_1 = f\rho_2$ follows from Remark 3.1.2.

Ad (v). Given $\phi : (C, \rho) \to (D, \sigma)$ with tracking family $C \circ \eta \xrightarrow{\rho} D$, define an equivalence relation on $C^*$ by $\rho'(x, y) \equiv \rho(ex, ey)$. Then $(C^*, \rho') \cong (C, \rho)$, and $f$ is a strict tracking family of $\phi$ composed with the isomorphism.

$\mathbb{R}\{\mathcal{E}\}$ will be the fiber of the fibration of presheaves on $\mathcal{E}$ over the terminal object. To get the entire fibration, we have to define a functor along which we can glue.

**Definition 3.1.10** (i) The functor $\Delta : \mathbb{R} \to \mathbb{R}\{\mathcal{E}\}$ sends $I \in \mathbb{R}$ to $(I, =)$, i.e. the terminal object in $\mathcal{E}_I$ equipped with the discrete equivalence relation.

$u : I \to J$ in $\mathbb{R}$ is mapped to $(x:1_I, y:1_J \mid 1_u x = y)$ by $\Delta$, where $1_u : 1_I \to 1_J$ is the unique map over $u$ of this type in $|\mathcal{E}|$.

(ii) We define the fibration $\hat{\mathcal{E}}$ of presheaves on $\mathcal{E}$ by $\hat{\mathcal{E}} = \text{gl}_{\Delta}(\mathbb{R}\{\mathcal{E}\})$. 

\[
\begin{array}{ccc}
|\mathcal{E}| & \xrightarrow{\mathbb{R}\{\mathcal{E}\}} & \mathbb{R}\{\mathcal{E}\} \\
\downarrow{\phi} & & \downarrow{\text{cod}(\mathbb{R}\{\mathcal{E}\})} \\
\mathbb{R} & \xrightarrow{\Delta} & \mathbb{R}\{\mathcal{E}\}
\end{array}
\]
Lemma 3.1.11 $\Delta : \mathbb{R} \to \mathbb{R}\{\mathcal{C}\}$ is regular, full, and faithful and reflects regular epimorphisms.

Proof. It is clear that $\Delta$ preserves finite products. To see that it preserves equalizers, consider $u, v : I \to J$ in $\mathbb{R}$.

The equalizer of $\Delta u$, $\Delta v$ in $\mathbb{R}\{\mathcal{C}\}$ is represented by the predicate $(x:1_I \mid 1_u x = 1_v x)$ in $\text{siev}(\mathcal{C})$, which as a morphism in $\mathcal{C}$ is exactly the equalizer of $u$ and $v$. Thus it remains to check that the functor $1 : \mathbb{R} \to |\mathcal{C}|$, $I \mapsto 1_I$ preserves equalizers. This is easy to see.

To show that $\Delta$ is preserves and reflects regular epimorphisms, let $u : I \to J$ in $\mathbb{R}$. From Lemma 3.1.7 we know that $\Delta u$ is a regular epimorphism iff (surj) holds, and substituting definitions we see that this judgment is concretely given by $y:1_J \vdash \exists x:1_I . 1_u x = y$. Now the interpretation of the formula $(y:1_J \mid \exists x:1_I . 1_u x = y)$ is $1_u$ viewed as a predicate in $\text{siev}(\mathcal{C})_{1_J}$, and by the definition of the fibration of sieves, $1_u$ is equivalent to id$_J$ (the true predicate) iff there exists a cover-cartesian $e : X \to J$ which factors through $1_u$. This is in turn equivalent to the existence of an epimorphism $p$ into $J$ which factors though $u$, which is equivalent to $u$ being epic.

To see that $\Delta$ is faithful, let $u, f : I \to J$ in $\mathbb{R}$. We have $\Delta u = \Delta v$ iff $1_u x = 1_v x$ holds iff the equalizer of $1_u$ and $1_J$ is equivalent to id$_J$ in $\text{siev}(\mathcal{C})_{1_J}$. By Remark 3.1.2, this implies that the equalizer is an isomorphism.

To see that $\Delta$ is full, let $\phi : \Delta I \to \Delta J$. By Lemma 3.1.9-(iv), there exists a map $f : 1_I \to 1_J$ in $|\mathcal{C}|$ such that $x:1_I \vdash \phi(x, fx)$. The inverse image of $\phi$ is given by $\mathcal{C} f$.

We can express the fibers of $\mathcal{C}$ directly in terms of $\text{siev}(\mathcal{C})$, without using the gluing construction:

Lemma 3.1.12 (i) For $I \in \mathbb{R}$, we have an equivalence

$$\mathcal{C}_I \cong \frac{\mathbb{R}\{\mathcal{C}\}}{\Delta I} \simeq \frac{(\mathbb{R}/I)\{\mathcal{C}/I\}}{\sim},$$

of categories, where $\mathcal{C}/I$ is the localization of $\mathcal{C}$ to $I$, introduced in Definition 2.1.12.

(ii) More generally, given $C \in \mathcal{C}_I$, we have

$$\mathbb{R}\{\mathcal{C}\}/(C; =) \simeq \frac{(\mathbb{R}/I)\{\mathcal{C}/C\}}{\sim},$$

where $\mathcal{C}/C : |\mathcal{C}/C| \to \mathbb{R}/I$ is the slice of $\mathcal{C}$ over $C$ (see Lemma 2.1.11).

Proof. We only show the second claim, the first one is a special case by Lemma 2.1.13. Objects in $\mathbb{R}\{\mathcal{C}\}/\Delta C$ are equivalence relations $\sigma \in \text{siev}(\mathcal{C})_{D \times D}$ together with morphisms $\phi : (D, \sigma) \to (C, =)$, which by Lemma 3.1.9-(iv) correspond to morphisms $f : D \to C$ such that $f \sigma_1 = f \sigma_2$. Objects in $(\mathbb{R}/I)\{\mathcal{C}/C\}$, on the other hand, are equivalence relations in $\text{siev}(\mathcal{C}/C)$, where concretely the underlying object is a map $f : D \to C$ and the relation is a map $\sigma : S \to D \times C \times D$ satisfying the axioms. The key observation now is that an equivalence relation in $\text{siev}(\mathcal{C})$ on $D$ whose components are equalized by $f$ is the as an equivalence relation in $\text{siev}(\mathcal{C}/C)$ on $f$. We leave it to the reader to extend the correspondence to morphisms.

The preceding lemma gives a streamlined representation of the fibration $\mathcal{C}$, which we will use from now on. Let us spell it out for reference.

- Objects in $\mathcal{C}_I$ are given by triples $(u : J \to I, C \in \mathcal{C}_I, \rho : R \to C \times C)$ such that $\rho$ is an equivalence relation in $\text{siev}(\mathcal{C})$ and factors through $C \times C$.

- A morphism from $(u : J \to I, C, \rho)$ to $(v : L \to K, D, \sigma)$ over $w : I \to K$ is a span $C \triangleleft e \quad \bullet \xrightarrow{f} D$ such that $w u \mathcal{C}(e) = v \mathcal{C}(f)$ and $\rho(\sigma e, \sigma y) \vdash \sigma(f x, f y)$ in $\text{siev}(\mathcal{C})$. 41
This description allows us to give an easy definition of the Yoneda embedding.

Definition 3.1.13 The fibered Yoneda embedding \( Y : \mathcal{C} \to \hat{\mathcal{C}} \) is given by

\[
\mathcal{C}_I \ni C \mapsto (\text{id}_I, C, \_ \_ \_ ) \in \hat{\mathcal{C}}_I.
\]

Lemma 3.1.14  
(i) \( Y \) is full and faithful.

(ii) Given a fibered pretopos \( \mathcal{D} : |\mathcal{D}| \to \mathbb{R} \), precomposition by \( Y \) induces an equivalence

\[
\text{Lex}(\mathbb{R})(\mathcal{C}, \mathcal{D}) \simeq \text{Pretop}(\mathbb{R})(\hat{\mathcal{C}}, \mathcal{D}),
\]

where \( \text{Lex}(\mathcal{C}, \mathcal{D}) \) is the category of fibered finite limit preserving functors, and \( \text{Pretop}(\hat{\mathcal{C}}, \mathcal{D}) \) is the category of fibered regular functors preserving internal sums.

Proof. Ad (i). It is sufficient to show that \( Y \) is fiberwise full and faithful. For \( I \in \mathbb{R} \), \( Y_I \) can be decomposed as follows. 

\[
\mathcal{C}_I \cong (\mathcal{C} / I)_1 \xrightarrow{C \mapsto (C, \_ \_ \_ )} (\mathcal{R} / I)_{\{C / I\}}.
\]

The first part is an equivalence, thus it suffices to show that for arbitrary finite limit pre-stacks \( \mathcal{D} : |\mathcal{D}| \to \mathcal{S} \), the embedding \( \mathcal{D}_I \to \mathcal{S}(\mathcal{D}) \) is full and faithful. It is easy to see that the embedding is faithful; fullness is a consequence of Lemma 3.1.9-(iv), since every morphism between two objects in the terminal fiber is vertical.

Ad (ii). To see that precomposition with \( Y \) is faithful, consider fibered functors \( F, G : \hat{\mathcal{C}} \to \mathcal{D} \), which are fiberwise regular and preserve cocartesian morphisms. Assume that \( \eta, \theta : F \to G \) are fibered natural transformations such that \( \eta \circ Y = \theta \circ Y \). Let \( I \in \mathbb{R} \) and \( (u, C, \rho) \) in \( \mathcal{C}_I \).

We can cover \( (u, C, \rho) \) with an object in the image of \( Y \) as,

\[
YC = (\text{id}_I, C, \_ \_ \_ ) \xrightarrow{s \mapsto (u, C, \_ \_ \_ )} (u, C, \rho),
\]

where \( s \) is cocartesian and \( e \) is regular epic in \( \hat{\mathcal{C}}_I \). Applying \( \eta \) and \( \theta \) to this sequence, we obtain

\[
FYC \xrightarrow{\eta \circ Y} F(u, C, \rho) \xrightarrow{Fe} F(u, C, \rho)
\]

and

\[
GYC \xrightarrow{\theta \circ Y} G(u, C, \rho) \xrightarrow{Ge} G(u, C, \rho).
\]

Now the facts that \( Fs \) is cocartesian and the components of \( \eta \) and \( \theta \) are vertical imply that \( \eta_{(C, u, \_ \_ \_ )} = \theta_{(C, u, \_ \_ \_ )} \), and since \( Fe \) is regular epic in \( \mathcal{D}_I \), this furthermore implies that \( \eta_{(C, u, \rho)} = \theta_{(C, u, \rho)} \).

To see that precomposition by \( Y \) is full, consider \( F, G : \hat{\mathcal{C}} \to \mathcal{D} \) and \( \eta : FY \to GY \). We have to construct \( \tilde{\eta} : F \to G \) such that \( \tilde{\eta}Y = \eta \). Let \( (u, C, \rho) \in \mathcal{C}_I \) as before. Consider the following
Applying $F$, $G$, and $\eta$, we get:

\begin{equation}
\begin{array}{c}
\text{GYR} \xrightarrow{\eta_{w,R}} G(w,R,=) \\
\eta_{u,C} \xrightarrow{\eta_{w,R,C}} G(u,C,=)^2 \\
\eta_{u,C,\rho} \xrightarrow{\eta_{w,R,C}} G(u,C,\rho)
\end{array}
\end{equation}

Here, the arrows $\tilde{\eta}_{(w,R,=)}$, $\tilde{\eta}_{(u,C,=)^2}$, and $\tilde{\eta}_{(u,C,\rho)}$ are cocartesian liftings, $h$ exists since the regular epimorphism $Fp$ is left orthogonal to the monomorphism $Gm$ in $\mathcal{I}$, and $\eta_{(u,C,\rho)}$ exists since $\eta_{(u,C,\rho)}$ is compatible with the kernels $Fm$ and $Gm$ of $Fe$ and $Ge$, which is expressed by the existence of $h$.

To see that $\tilde{\eta}$ is natural, consider a morphism $\phi : (u,C,\rho) \to (v,D,\sigma)$ over $t : I \to K$ in $\mathcal{E}$. By 3.1.9-(iii), $\phi$ has a tracking family $C \xleftarrow{e} C^+ \xrightarrow{f} D$, and since $\phi$ is over $w$, we have $tu\mathcal{E}(e) = v\mathcal{E}(f)$. Set $w := u\mathcal{E}(e)$.

\begin{equation}
\begin{array}{c}
\text{C} \xleftarrow{e} C^+ \xrightarrow{f} D
\end{array}
\end{equation}
Defining $\rho^* \in \text{siev}(\mathcal{C})_{C \times C^*}$ to be the predicate $\rho^* = [x, y \mid \rho(ex, ey)]$, we have $(u, C, \rho) \cong (w, C^*, \rho^*)$. Thus, to show commutativity of the outer rectangle in the diagram

$$
\begin{align*}
F(u, C, \rho) & \xrightarrow{\eta} F(w, C^*, \rho^*) \xrightarrow{\eta} F(v, D, \sigma) \\
G(u, C, \rho) & \xrightarrow{\eta} G(w, C^*, \rho^*) \xrightarrow{\eta} G(v, D, \sigma)
\end{align*}
$$

in $|\mathcal{X}|$, it suffices to show that the left and right squares commutes, i.e. we have reduced the problem of showing naturality to checking the condition for morphisms which have tracking families with identity cover-cartesian part. Specifically, we show that the right square in the preceding diagram commutes, and the left one is analogous. In the diagram

$$
\begin{align*}
F(U, C, \rho) \xrightarrow{\eta} F(U, C^*, \rho^*) & \xrightarrow{\eta} F(V, D, \sigma) \\
G(U, C, \rho) \xrightarrow{\eta} G(U, C^*, \rho^*) & \xrightarrow{\eta} G(V, D, \sigma)
\end{align*}
$$

we know that the top face and the sides commute, and we have to show commutativity of the bottom face. Because $p$ is epic, the two paths around the bottom face are equal iff they are equal precomposed with $p$, and since these two composites have the same image under $\mathcal{X}$, it suffices to check their equality when moreover precomposed with $c$ (by the universal property of cocrarian arrows). This equality follows from the commutativity of the top and side faces.

This finishes the proof that precomposition with $Y$ is full.

Finally, we show that precomposition by $Y$ is essentially surjective. Let $F : \mathcal{C} \to \mathcal{X}$ be a finite limit preserving fibered functor. The idea of how to extend $F$ to $\tilde{F}$ along $Y : C \to \hat{C}$

$$
\begin{array}{c}
\mathcal{C} \\
\xymatrix{ \mathcal{C} \ar[r]^{Y} \ar[d] & \mathcal{X} \ar[d]^{\tilde{F}} \\
\mathcal{C} \ar[r]^{F} & \mathcal{X}
}
\end{array}
$$

is contained in diagram (3.1.4), which presents an object $(u, C, \rho)$ in $\mathcal{C}_{1}$ as a quotient of an internal sum of a representable, where the associated equivalence relation is itself the image (in the sense of image factorization) of the sum of a representable. Accordingly, we construct
\( \tilde{F}(u, C, \rho) \) as in
\[
\begin{array}{c}
FR \xrightarrow{\rho_1} \Sigma_u FR \\
\downarrow \rho_1 \\
FC \xrightarrow{r_2} \Sigma_u FC \\
\downarrow r_1 \\
K \xrightarrow{w} \tilde{F}(u, C, \rho) \\
\downarrow v_1 \downarrow v_2 \\
\end{array}
\]

where \( r = (r_1, r_2) \) is an equivalence relation obtained by cocartesian lifting and image factorization, and \( e \) is its quotient.

In the non-fibered case, categories of presheaves on small categories can be characterized as \( \infty \)-pretoposes having a small generating family of indecomposable projectives. We will prove the fibered analogue of this statement after defining fibrational versions of projectivity and indecomposability.

**Definition 3.1.15** Let \( \mathcal{P} : |\mathcal{P}| \to R \) be a positive pre-stack.

(i) We call \( P \in |\mathcal{P}| \) projective (with respect to \( \mathcal{P} \)), if given \( c, e, f \) as in the diagram

\[
\begin{array}{c}
P \xleftarrow{e} P^* \xrightarrow{f} X \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
Y \xrightarrow{g} \bullet \xrightarrow{d} \end{array}
\]

where \( c \) is cartesian and \( e \) is vertical and a regular epimorphism in its fiber, we can fill in \( d, g \) with \( d \) cover-cartesian such that the square commutes.

(ii) We call \( X \in |\mathcal{P}| \) indecomposable, if for every diagram

\[
\begin{array}{c}
X^* \xleftarrow{e} X \\
\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \\
Y \xrightarrow{m} S
\end{array}
\]

where \( c \) is cartesian and \( s \) is cocartesian, there exists a *unique* mediating arrow \( m \).

Observe that the definitions are given in a way which assures that indecomposables and projectives are closed under reindexing.

**Lemma 3.1.16**

(i) Let \( \mathcal{C} : |\mathcal{C}| \to R \) be a finite limit pre-stack.

(a) The objects in the image of \( Y : \mathcal{C} \to \hat{\mathcal{C}} \) are indecomposable projectives in \( \hat{\mathcal{C}} \).

(b) Given an indecomposable projective \( A \in |\hat{\mathcal{C}}| \), there exists a \( C \in |\mathcal{C}| \) and a cover-cartesian map \( e : YC \twoheadrightarrow \bowtie A \).

In other words, the co-restriction of \( Y \) to the subfibration of \( \hat{\mathcal{C}} \) on indecomposable projectives is a weak equivalence (Definition 2.3.9).
(ii) Let $\mathcal{X} : |\mathcal{X}| \to \mathbb{R}$ be a fibered pretopos. $\mathcal{X}$ is equivalent to the fibration of presheaves on its subfibration of indecomposable projectives if the latter is closed under finite limits, and every $X \in |\mathcal{X}|$ can be covered as

$$A \xleftarrow{\varepsilon} S \xrightarrow{\pi} X$$

where $A$ is indecomposable and projective, $s$ is cocartesian, and $e$ is vertical and a regular epimorphism in its fiber.

Proof. Ad (i)a. Since indecomposability is about cocartesian maps, we first have to understand what those look like in $\widehat{\mathcal{C}}$. It turns out that this is very easy – given $u : J \to I$, $(u, C, \rho) \in \widehat{\mathcal{C}}$, and $v : I \to H$, the internal sum of $(u, C, \rho)$ along $v$ is simply $(vu, C, \rho)$. Now consider $D \in \mathcal{C}_L$ and a morphism of type $(\text{id}_L, D, =) \to (vu, C, \rho)$, given by a map $w : L \to H$ and a span $\xymatrix{D \ar[r]_-{\varepsilon} & D^* \ar[r]^-{f} & C \ar[r]^-{p_1} & R \ar@/_/[l]_-{p_2}}$ such that $vuh = wp$ and $ex = ey \vdash \rho(fx, fy)$ in $\text{Siev}(\mathcal{E})$.

Lifting $(e, f)$ along $(u, C, \rho) \mapsto (vu, C, \rho)$ amounts to lifting $w$ along $v$, or equivalently extending $uh$ along $p$. This is possible if and only if the kernel pair of $p$ is contained in the kernel pair of $uh$, which follows from $ex = ey \vdash \rho(fx, fy)$ and $ur_1 = ur_2$.

For projectivity, recall that by 2.2.7-(vi), any regular epimorphism in $\widehat{\mathcal{C}}_I$ can up to isomorphism be represented as $(u, C, \rho) \mapsto (u, C, \sigma)$ where $u : J \to I$, $C \in \mathcal{C}_J$, and $\rho(x, y) \vdash \sigma(x, y)$. Consider $D \in \mathcal{C}_L$ and a morphism of type $(\text{id}_C, =) \to (u, C, \sigma)$ given by $w : L \to I$ and a span $(e, f)$

$$\xymatrix{D \ar[r]_-{\varepsilon} & D^* \ar[r]^-{f} & C \ar[r]^-{r_1} & K \ar@/_/[l]_-{r_2}}$$

where $ex = ey \vdash \sigma(x, y)$. The desired lifting in $\widehat{\mathcal{C}}$ is given in the following square.

$$\xymatrix{(\text{id}, D^*, =) \ar[r] & (u, C, \rho) \ar[d] \ar@/_/[l] \ar@/^/[l] \ar@{=>}[r] \ar@{=>}[r] \ar@{=>}[r] & (\text{id}, D, =) \ar[r] & (u, C, \sigma)}$$

Ad (i)b. Assume that $X \in \widehat{\mathcal{C}}_I$ is indecomposable and projective. Just like any object in $|\mathcal{C}|$, we can cover $X$ as $YC \xleftarrow{\varepsilon} S \xrightarrow{p} X$, and since $X$ is indecomposable and projective, there
exists \( e : X^* \rightarrow Y \) and \( f : X^* \rightarrow YC \) such that \( pcf = e \). Pulling \( pc \) back along \( e \), we obtain

\[
\begin{array}{c}
YC^* \xrightarrow{g} X^* \\
\downarrow f \downarrow \downarrow f_c \\
X^* \xrightarrow{e} Y \rightarrow X
\end{array}
\]

where \( g \) is induced by \( f \) and the pullback property and exhibits \( X^* \) as a retract of \( YC^* \). Since \( \mathcal{C} \) has finite limits and \( Y \) preserves them, the essential image of \( Y \) is closed under retracts in \( \hat{\mathcal{C}} \), which proves the claim.

**Ad (ii).** Clearly, the condition is necessary. Conversely, let \( \mathcal{C} \) be the subfibration of \( \mathcal{X} \) on indecomposable projectives and \( F : \mathcal{C} \rightarrow \mathcal{X} \) the inclusion. We have to show that the canonical map \( \tilde{F} : \mathcal{C} \rightarrow \mathcal{X} \) is an equivalence. By Moens’ theorem, it is sufficient to show this for the functor \( \tilde{F}_1 : R\{\mathcal{C}\} \rightarrow \mathcal{X}_1 \) between the terminal fibers. Consider the following diagram.

\[
\begin{array}{c}
\text{Siev}(\mathcal{C}) \xrightarrow{H} \text{Sub}(\mathcal{X}) \xrightarrow{K} \text{Sub}(\mathcal{X}_1) \\
\downarrow \text{siev}(\mathcal{C}) \downarrow \downarrow \text{sub}(\mathcal{X}) \downarrow \text{sub}(\mathcal{X}_1) \\
[\mathcal{C}] \rightarrow [\mathcal{X}] \xrightarrow{\Sigma} [\mathcal{X}_1] \\
\downarrow \mathcal{F} \downarrow \downarrow \downarrow \mathcal{X} \\
[\mathcal{R}] \rightarrow [\mathcal{X}] \xrightarrow{\Sigma} [\mathcal{X}_1]
\end{array}
\]

(3.1.6)

Here, \( H \) maps sieves on \( C \in \mathcal{C} \) onto the image of their cocartesian lifting

\[
\begin{array}{c}
D \xrightarrow{f} X \\
\downarrow \downarrow \downarrow \downarrow \downarrow Hf \downarrow \downarrow \downarrow \downarrow \downarrow \Sigma \\
J \xrightarrow{f} I \\
\end{array}
\]

and \( K \) is the action on vertical monomorphisms of the sum functor \( \Sigma \). \( K \) is well defined since vertical monomorphisms are monomorphisms in the total category, and \( \Sigma \) preserves finite limits (and thus monos) by [57, Lemma 15.7].

It follows from indecomposability and projectivity that \( H \) is fiberwise order-reflecting, and the assumption that every object in \( \mathcal{X} \) can be covered by an indecomposable projective implies that \( H \) is fiberwise essentially surjective. Thus, the left square in diagram (3.1.6) is a pullback. The right square is a pullback by extensivity of \( \mathcal{X} \). Furthermore, \( F \) trivially preserves finite limits, and \( \Sigma \) preserves finite limits by [57, Lemma 15.7].

Now the functor \( \tilde{F}_1 : R\{\mathcal{C}\} \rightarrow \mathcal{X}_1 \) can be expressed as applying \( KH \) to an equivalence relation in \( \text{siev}(\mathcal{C}) \), and then forming the quotient in the exact category \( \mathcal{X}_1 \). This functor is full and faithful, since \( H \) and \( K \) are parts of pullbacks as established previously, and thus preserve all logic, and furthermore \( \mathcal{X}_1 \simeq \text{PER}(\text{sub}(\mathcal{X}_1)) \). \( \tilde{F}_1 \) is essentially surjective since objects in \( \mathcal{X} \) can be covered by indecomposable projectives.

**Remarks 3.1.17** — As mentioned at the beginning of Section 3.1, the fibered presheaf construction is motivated by the desire to make certain constructions involving exact completions independent of the axiom of choice. Essentially the same question motivated Hofstra’s [22], but he uses a different approach: instead of working in a fibrational framework,
he constructs a left biadjoint to the forgetful functor

\[ \mathbb{R}/\text{Ex} \to \mathbb{R}/\text{Lex}, \]

where the objects of the 2-categories are regular and finite limit preserving functors, respectively, and in both cases the 1-cells are the respective commutative triangles. Via Moens’ theorem, this can of course be understood as a biadjunction between 2-categories of lextensive fibrations and fibered pretoposes (apart from the detail that Hofstra considers strict, not pseudo-slice categories), but taking this point of view there is still a discrepancy since in the present work we locate the transition from partitioned assemblies to realizability toposes in the left biadjoint to \( \text{Pretop}(\mathbb{R}) \to \text{Lex}(\mathbb{R}) \) and not in the left biadjoint to \( \text{Pretop}(\mathbb{R}) \to \text{Lxv}(\mathbb{R}) \) as a reading of Hofstra’s work through the glasses of Moens would suggest.

– In recent work \([55]\), Michael Shulman presents a notion of *exact completion of unary sites* (a unary site is a small category equipped with a Grothendieck topology which is generated by singleton covering families) that is related to the fibered presheaf construction: given a finite-limit pre-stack \( \mathcal{E} : |\mathcal{C}| \to \mathbb{R} \), the cover-cartesian maps in \( |\mathcal{C}| \) generate a unary topology, and the exact completion of the corresponding unary site is equivalent to \( \mathbb{R}\{\mathcal{E}\} \).

This was observed by Wouter Stekelenburg in the case of realizability over pcas.

\[\text{3.1.3 The fibered geometric cocompletion}\]

This section is about the left biadjoint to \( \text{Geo}(\mathbb{R}) \to \text{Lex}(\mathbb{R}) \). In the non-fibered case, the geometric cocompletion of a small finite limit category \( \mathcal{C} \) can be described as the full subcategory on sub-representables. In the fibered case, we can use essentially the same construction.

**Definition 3.1.18** Let \( \mathcal{E} : |\mathcal{C}| \to \mathbb{R} \) be a finite limit pre-stack on a regular category. The fibration \( D\mathcal{E} \) is defined to be the full subfibration of \( \hat{\mathcal{C}} \) on (vertical) subobjects of representables.

\[ \mathcal{C} \xhookrightarrow{Y} D\mathcal{E} \xrightarrow{y} \hat{\mathcal{C}} \]

We use a lowercase ‘\( y \)’ (read as ‘little Yoneda’) to denote the embedding of \( \mathcal{C} \) into \( D\mathcal{E} \), a convention that appears particularly natural in the posetal case which we will examine in more detail in Section 3.4.

By Lemma 2.2.7-(i), subobjects of \( Y(A) \) for \( A \in \mathcal{C}_I \) correspond to predicates in \( \text{siev}(\mathcal{E})_A \), i.e. to morphisms \( h : B \to A \) in \( |\mathcal{C}| \). This leads us to the following concrete description of the fibration \( D\mathcal{E} \).

– Objects in \( (D\mathcal{E})_I \) are morphisms \( h : B \to A \) in \( |\mathcal{C}| \) with \( A \in \mathcal{C}_I \).

– A morphism from \( h : B \to A \) to \( k : D \to C \) over \( u : I \to J \) is an equivalence class of spans

\[ B \xleftarrow{e} B^* \xrightarrow{f} D \]

with \( e \) cover-cartesian, such that \( \mathcal{E}(kf) = u\mathcal{E}(he) \) and

\[ x,y:B^* \mid h(ex) = h(ey) \vdash k(fx) = k(fy) \tag{3.1.7}\]

in \( \text{siev}(\mathcal{E}) \), where

– two spans \( B \xleftarrow{e} B^* \xrightarrow{f} D \) and \( B \xleftarrow{e'} B'^* \xrightarrow{f'} D \) represent the same morphism over \( u \), if

\[ x:B^*, y:B'^* \mid h(ex) = h(e'y) \vdash k(fx) = k(f'y) \]

holds in \( \text{siev}(\mathcal{E}) \).
Lemma 3.1.19 (Localization and slicing) Let $\mathcal{C}$ be a finite limit pre-stack on $\mathbb{R}$.

- Given $I \in \mathbb{R}$, we have an equivalence
  \[ D(\mathcal{C}/I) \simeq (D\mathcal{C})/I \]
of fibrations on $\mathbb{R}/I$.
- For $I \in \mathbb{R}$ and $A \in \mathcal{C}_I$, we have an equivalence
  \[ D(\mathcal{C}/A) \simeq (D\mathcal{C})/(yA) \]
of fibrations on $\mathbb{R}/I$.

Proof. The claim about localization is straightforward. The proof of the claim about slicing involves one argument which is not purely formal. Let us just consider the objects. Let $u : J \to I$ in $\mathbb{R}$. An object in $D(\mathcal{C}/A)_u$ is a configuration

\[
\begin{array}{ccc}
C & \xrightarrow{g} & B \\
\downarrow K & & \downarrow J \\
\downarrow u & & \downarrow u \\
I & \xrightarrow{I} & I
\end{array}
\]

(3.1.8)

where $g$ is viewed as a sieve on $f \in (\mathcal{C}/A)_u$. An object in $((D\mathcal{C})/(yA))_u$, on the other hand, is given by a configuration

\[
\begin{array}{ccc}
C^* & \xrightarrow{e} & A \\
\downarrow C & & \downarrow \text{id} \\
\downarrow g & & \downarrow \text{id} \\
B & \xrightarrow{w} & I \\
\downarrow K^* & & \downarrow \text{id} \\
\downarrow p & & \downarrow \text{id} \\
J & \xrightarrow{u} & I
\end{array}
\]

such that $g(ex) = g(ey) : fx = fy$ in siev($\mathcal{C}$) (see (3.1.7)) – $g$ is represents a sieve on $A$, and $(e,f)$ represents a morphism from the corresponding sub-representable into $yA$. Now for reasons similar to Lemma 3.1.9-(iv) and since $yA$ is represented by $\text{id}_A$, we can represent the morphism given by $(e,f)$ without cover-cartesian part, yielding a simplified configuration

\[
\begin{array}{ccc}
C & \xrightarrow{g} & B \\
\downarrow K & & \downarrow J \\
\downarrow u & & \downarrow u \\
I & \xrightarrow{I} & I
\end{array}
\]

(3.1.9)

where $g$ is a sieve on $B$, and $h$ represents a map from the corresponding sub-representable into $yA$. To establish the equivalence, observe first that every configuration of the form (3.1.8) induces
a configuration of the form (3.1.9). To transform data of the latter into the former form, however, we have to make use of finite limit structure – the object in \( D(\mathcal{C} \mid A)_u \) corresponding to (3.1.9) is given by

\[
\begin{array}{ccc}
C & \xrightarrow{(y,h)} & B \times_I A \\
\downarrow & & \downarrow \pi_A \\
K & \xleftarrow{v} & J \\
\end{array}
\]

(3.1.10)

The verification that this construction induces the desired equivalence is purely technical. ■

**Lemma 3.1.20** Let \( \mathcal{C} : [\mathcal{C}] \to \mathbb{R} \) be a finite limit pre-stack on a regular category.

- \( D\mathcal{C} \) is a geometric pre-stack.
- Given a geometric pre-stack \( \mathcal{F} \), we have \( \text{Lex}(\mathcal{C} \mid \mathcal{F}) \simeq \text{Geo}(D\mathcal{C} \mid \mathcal{F}) \).

**Proof.** By Lemma 2.3.25?, \( \hat{\mathcal{C}} \) is a geometric pre-stack. The subfibration on sub-representables is closed under finite limits (since \( \mathcal{C} \) is), image factorization, and internal unions of subobjects (i.e., existential quantification in the fibered subobject fibration), hence it is geometric as well.

Given a second geometric pre-stack \( \mathcal{F} \), precomposition with \( y : \mathcal{C} \to \mathcal{D}\mathcal{C} \) induces a functor of type \( \text{Geo}(\mathcal{D}\mathcal{C} \mid \mathcal{F}) \to \text{Lex}(\mathcal{C} \mid \mathcal{F}) \), since \( y \) preserves finite limits. This functor is faithful since every object in \( \mathcal{D}\mathcal{C} \) is a vertical subobject of an object in the image of \( y \), and the functors in \( \text{Geo}(\mathcal{D}\mathcal{C} \mid \mathcal{F}) \) preserve monomorphisms. To see that precomposition is full, let \( F, G \in \text{Geo}(\mathcal{D}\mathcal{C} \mid \mathcal{F}) \), and \( \eta : F \circ y \to G \circ y \). We have to extend \( \eta \) to a natural transformation \( \eta_0 : F \to G \). Let \( h : B \to A \) in \( [\mathcal{C}] \) with \( A \in \mathcal{C}_1 \) represent an object \( h \in (\mathcal{D}\mathcal{C})_1 \). Then we have a decomposition of \( y(h) : y(B) \to y(A) \) in \( [\mathcal{D}\mathcal{C}] \) as

\[
y(B) \xrightarrow{e} h \xrightarrow{m} y(A)
\]

with \( e \) collectively covering and \( m \) vertical monic. Applying \( F \) and \( G \), we get

\[
\begin{array}{ccc}
F(y(B)) & \xrightarrow{F_e} & F(h) \\
\downarrow \eta_B & & \downarrow \eta_A \\
G(y(B)) & \xleftarrow{G_e} & G(h)
\end{array}
\]

Since \( F \) and \( G \) preserve vertical monomorphisms and collective covers, the lifting property for \( Fe \) and \( Gm \) gives us a unique candidate for \( \eta_0, h \). We leave it to the reader to verify that \( \eta_0 \) is well defined.

It remains to show that precomposition by \( y : \mathcal{F} \to D\mathcal{F} \) is essentially surjective. For this, let \( F : \mathcal{C} \to \mathcal{F} \) be a finite limit preserving fibered functor. We want to construct a geometric functor \( \tilde{F} : D\mathcal{C} \to \mathcal{F} \) such that \( \tilde{F} \circ y \cong \eta \). Let \( h : B \to A \) represent an object \( h \in (D\mathcal{C})_1 \) as before. We construct \( \tilde{F}h \in \mathcal{F}_1 \) by taking the collective-cover/vertical-mono factorization.

\[
y(B) \xrightarrow{\tilde{F}h} \tilde{F}h \xrightarrow{y(A)}
\]

We leave it to the reader to figure out how to extend this operation to morphisms, and to verify that \( \tilde{F} \) is a geometric extension of \( F \). ■
3.1.4 The fibered positive cocompletion

We now come to the left adjoint of \( \text{Pos}(\mathbb{R}) \to \text{Lex}(\mathbb{R}) \). Just as for the geometric cocompletion, we define the positive cocompletion of a finite limit pre-stack as a subfibration of the fibration of presheaves. Since positive pre-stacks are in the realm of Moens’ theorem, we define the fibration by giving the terminal fiber and gluing.

**Definition 3.1.21** Let \( \mathcal{C} : |\mathcal{C}| \to \mathbb{R} \) be a finite limit-pre-stack. The category \( \mathcal{A}(\mathcal{C})_1 \) is the full subcategory of \( \mathbb{R}\{\mathcal{C}\} \) on subobjects of objects of the form \((X,=)\).  

Similar as for the geometric cocompletion, subobjects of of objects \((X,=)\) correspond to predicates in \(\text{siev}(\mathcal{C})\) by Lemma 2.2.7-(i), which allows us to give the following more concrete description of the category \( \mathcal{A}(\mathcal{C})_1 \):

- Objects are morphisms \( h : Y \to X \) in \(|\mathcal{C}|\).
- A morphism from \( h : Y \to X \) to \( k : W \to Z \) is a span \( \begin{array}{c} Y \rightarrow e \cr \downarrow \cr X \rightarrow \end{array} f \rightarrow W \) such that \( \rho(ex,ey) \vdash \sigma(fx,fy) \) in \(\text{siev}(\mathcal{C})\), where \(\rho\) and \(\sigma\) are the kernel pairs of \(h\) and \(k\) viewed as equivalence relations in \(\text{siev}(\mathcal{C})\).
- Two morphisms from \( h \) to \( k \) given by spans \( \begin{array}{c} Y \rightarrow e \cr \downarrow \cr X \rightarrow f \rightarrow W \end{array} \) and \( \begin{array}{c} Y \rightarrow e' \cr \downarrow \cr X \rightarrow f' \rightarrow W \end{array} \) are identified as morphisms in \( \mathcal{A}(\mathcal{C})_1 \), if \( y:Y,w:W \mid \exists y^*:Y^*.ey^* = y \land \sigma(fy^*,w) \vdash \exists y^*:Y^*.e'y^* = y \land \sigma(f'y^*,w) \) holds in \(\text{siev}(\mathcal{C})\).

Since \( \mathbb{R}\{\mathcal{C}\} \) is an exact category and \( \mathcal{A}(\mathcal{C})_1 \) is a subcategory which is closed under products and subobjects, it is regular. Furthermore, the functor \( \Delta : \mathbb{R} \to \mathbb{R}\{\mathcal{C}\} \) factors through \( \mathcal{A}(\mathcal{C})_1 \), allowing us to make the following definition.

**Definition 3.1.22** The fibration \( \mathcal{A}(\mathcal{C}) : |\mathcal{A}(\mathcal{C})| \to \mathbb{R} \) is defined by \( \mathcal{A}(\mathcal{C}) = \text{gl}_\Delta(\mathcal{A}(\mathcal{C})_1) \).

Since \( \mathcal{A}(\mathcal{C})_1 \) is a regular category, and \( \Delta : \mathbb{R} \to \mathcal{A}(\mathcal{C})_1 \) is a regular functor, \( \mathcal{A}(\mathcal{C}) \) is a positive pre-stack. We state its universal property without proof.

**Lemma 3.1.23** Let \( \mathcal{C} \) be a finite limit pre-stack, and \( \mathcal{P} \) a positive pre-stack. Then we have an equivalence

\( \text{Lex}(\mathbb{R})(\mathcal{C}, \mathcal{P}) \simeq \text{Pos}(\mathbb{R})(\mathcal{A}(\mathcal{C}), \mathcal{P}) \)

of categories of fibered functors.

3.2 The fibered sheaf construction

Having treated cocompletions of finite limit pre-stacks in the preceding section, we now come to cocompletions of geometric pre-stacks. In this context, the fibered subobject fibration
(Definition 2.3.13) plays a similar role as the fibration of sieves for cocompletions of finite limit pre-stacks.

First, we describe the free fibered pretopos on a geometric fibration. We give a construction via gluing, in close analogy to what we did for the fibered presheaf construction.

In analogy to Definition 3.1.6, we define:

**Definition 3.2.1** Let \( \mathcal{S} : \mathcal{S} \to R \) be a geometric pre-stack. The category \( R[\mathcal{S}] \) is the category of (total) equivalence relations and functional relations in \( \text{sub}(\mathcal{S}) \) – in other words the full subcategory of \( \text{PER}(\text{sub}(\mathcal{S})) \) on total equivalence relations.

If we view objects in \( \mathcal{S} \) as families of objects, and predicates in \( \text{sub}(\mathcal{S}) \) as families of subobjects, we realize that the equivalence relations in \( R[\mathcal{S}] \) can be viewed as heterogeneous equivalence relations, just as we explained for the fibered presheaf construction in Section 3.1. In particular, given a geometric pre-stack \( \mathcal{S} \), we can form the categories \( R\{\mathcal{S}\} \) and \( R[\mathcal{S}] \), both of which can be viewed as categories of heterogeneous equivalence relations – the first one with respect to \( \text{siev}(\mathcal{S}) \), and the second one with respect to \( \text{sub}(\mathcal{S}) \).

Let me make a slightly deviating remark at this point. If \( S \) is a small geometric category, then \( \text{Set}\{\text{fam}(S)\} \) is equivalent to the category \( \hat{S} \) of presheaves on \( S \), and \( \text{Set}\{\text{fam}(S)\} \) is equivalent to the category \( \text{Sh}(S) \) of sheaves on \( S \) for the canonical topology. Furthermore, both categories are toposes and the latter is a subtopos of the former. For general geometric pre-stacks, the categories do not have to be toposes, nor does it generally seem to be the case that \( R[\mathcal{S}] \) is a localization of \( R\{\mathcal{S}\} \).

Interestingly, what is lacking for the localization is not the inverse but the direct image part – while the embedding \( \text{Sh}(S) \to \hat{S} \) is just the identity in the usual presentation, there is no generic method to construct a functor of type \( R[\mathcal{S}] \to R\{\mathcal{S}\} \).

An intuition as to what goes wrong is given by thinking about small sheaves and presheaves on large geometric categories. Already here the embedding of sheaves into presheaves does not have to be well defined since if a functor \( F : S^{\text{op}} \to \text{Set} \) is a small colimit of representables in the category of sheaves, there is no reason why it should also be such a small colimit in the category of presheaves. On the positive side, if the geometric pre-stack is a tripos, then the embedding can be constructed using impredicativity/weakly complete objects – see also the remark after Corollary 4.7.13.

Back to the main line of thoughts, we state a lemma analogous to Lemma 3.1.7.

**Lemma 3.2.2** If \( \mathcal{U} : \mathcal{U} \to R \) is a geometric fibration, then \( R[\mathcal{U}] \) is equivalent to \( \text{PER}(\text{sub}(\mathcal{U})) \) and thus in particular exact.

**Proof.** Again, we have to show that every partial equivalence relation is equivalent to a total one. This is evident for \( \text{sub}(\mathcal{U}) \) since predicates are vertical monomorphisms, and we can restrict any partial equivalence relation to its support. ■

**Definition 3.2.3** The functor \( \Delta : R \to R[\mathcal{U}] \) maps \( I \in R \) to \( (1_I, =) \) and \( u : I \to J \) to the predicate \( (x:1_I, y:1_J \mid 1_u(x) = y) \), where \( 1_u : 1_I \to 1_J \) is the unique function of this type over \( u \).

It is easy to see that \( \Delta \) is regular, which allows us to give the expected definition of the fibration of sheaves on \( \mathcal{U} \).

**Definition 3.2.4** For a geometric pre-stack \( \mathcal{U} : \mathcal{U} \to R \), the fibration \( \text{Sh}(\mathcal{U}) : \mathcal{U} \to R \) of sheaves on \( \mathcal{U} \) is defined as \( \text{Sh}(\mathcal{U}) = g|_{\Delta}(R[\mathcal{U}]) \).

4. in the sense of ‘reflective subcategory with finite limit preserving left adjoint’
In Lemma 3.2.8, we will show that $\text{Sh}(\mathcal{S})$ is the free fibered-pretopos completion of a geometric pre-stack $\mathcal{S}$, but before that we need a representation of the slices of $\text{Sh}(\mathcal{S})$ analogous to Lemma 3.1.12.

**Lemma 3.2.5** Let $\mathcal{S}$ be geometric pre-stack and $A \in \mathcal{S}$.

(i) $\mathcal{S}/A$ is a geometric fibration.

(ii) We have $R[\mathcal{S}]/(A,=) \simeq (R/I)[\mathcal{S}/A]$.

**Proof.** We already know that $\mathcal{S}/A$ has finite limits. Image factorizations and internal unions are inherited from $\mathcal{S}$ in the straightforward way, preserving all stability properties. Thus, $\mathcal{S}/A$ is a geometric pre-stack.

To show the claimed equivalence, we sketch constructions of functors in both directions. An object in $(R/I)[\mathcal{S}/A]$ is given by a map $f : B \to A$ in $\mathcal{S}$ and a vertical subobject $\rho : R \to B \times_A B$ which is reflexive and transitive as a predicate in $\text{sub}(\mathcal{S}/A)$. We can transform $\rho$ into a predicate on $B \times B$ by quantifying existentially along $m : B \times A B \to B \times B$, and the resulting $\exists_m \rho$ is an equivalence relation in $\text{sub}(\mathcal{S})$, whence $(B, \exists_m \rho)$ is an object in $R[\mathcal{S}]$. We claim that $f$ induces a functional relation of type $(B, \exists_m \rho) \to (A,=)$ — to show this, we have to verify that $f$ is compatible with the two relations, i.e. that we can fill in the dashed arrow in

![Diagram](image_url)

where the two lower horizontal squares (in the spatial sense) are pullbacks in the total category and the base, respectively, and the predicate $\exists_{A \times A}$ is (by definition) the equality predicate $=_{A \times A}$ in $\text{sub}(\mathcal{S})$. The validity of the claim can be seen by chasing along the backside of the cube, i.e., over $B \times A B \to A \to A \times A$; the fact that the squares are pullbacks is not essential. This shows how an object in $(R/I)[\mathcal{S}/A]$ can be transformed into an object in $R[\mathcal{S}]/(A,=)$.

In the other direction, take an object in $R[\mathcal{S}]/(A,=)$, i.e. a morphism $\phi : (B, \rho) \to (A,=)$. Let $m : U \to A \times B$ be the vertical subobject corresponding to the image of $\langle (\phi, \text{id}) : (B, \rho) \to (A \times B, = \times \rho) \rangle$. Then $(U, (\rho)_U)$ is isomorphic to $(B, \rho)$, and composing this isomorphism with $\phi$, we obtain an isomorphic representation of $\phi$ which is tracked by the projection $p : U \to A \times B \to A$. Since the projection is compatible with the equivalence relations, $= \times \rho$ factors through $U \times A U$, which induces an equivalence relation on $p$ in $\mathcal{S}/A$. This shows how to go from $R[\mathcal{S}]/(A,=)$ to $(R/I)[\mathcal{S}/A]$.

We leave it to the reader to verify that the sketched constructions are functorial and constitute an equivalence of categories. \[\blacksquare\]

The preceding lemma does in particular give us a representation of $\text{Sh}(\mathcal{S})$ as

$$\text{Sh}(\mathcal{S})_I \simeq (R/I)[\mathcal{S}/I].$$

In the following, we will identify $\text{Sh}(\mathcal{S})_I$ with $(R/I)[\mathcal{S}/I]$, as this is easiest to work with. Explicitely,
an object in $\text{Sh}(\mathcal{F})_I$ is thus given by a triple $(u, A, \rho)$ with $u : J \to I$, $A \in \mathcal{F}_I$, and $\rho$ a vertical subobject of $A \times_I A$ which is an equivalence relation in $\text{sub}(\mathcal{F}/I)$, and

- a morphism from $(u, A, \rho) \in \text{Sh}(\mathcal{F})_I$ to $(v, B, \sigma) \in \text{Sh}(\mathcal{F})_K$ over $w : I \to K$ is a vertical subobject of $A \times_K B$ which is functional with respect to the extension of $\rho$ to $A \times_K A$ and $\sigma$ in $\text{sub}(\mathcal{F}/K)$.

We also give explicit constructions of cartesian and cocartesian lifting and internal unions in $\text{Sh}(\mathcal{F})$ relative to the above representation, since we will need them later.

- Given $(u, A, \rho)$ in $\text{Sh}(\mathcal{F})_I$ and $v : K \to I$, the cartesian lifting of $(u, A, \rho)$ along $v$ is given by $(v^*u, v^*A, v^*\rho)$, where $v^*u$, $v^*A$, and $v^*\rho$ are defined as in the following diagrams.

\[
\begin{align*}
  v^*A & \hookrightarrow A \\
  v^*J & \hookrightarrow J \\
  v^*v & \downarrow \phantom{m} \, \, K \rightarrow I
\end{align*}
\]

- In the other direction, the cocartesian lifting of $(u, A, \rho)$ along $w : I \to L$ is given by $(vu, A, m \cap \rho)$, where $m : A \times_I A \to A \times_L A$ is the canonical embedding.

- For internal unions, recall that subobjects of $(u, A, \rho)$ in $\text{Sh}(\mathcal{F})_I$ correspond compatible predicates in $\text{sub}(\mathcal{F}/I(u,A))$, i.e., vertical subobjects of $A$ which are compatible with $\rho$. Now given $(u, A, \rho) \in \text{Sh}(\mathcal{F})_I$, $v : K \to I$, and a compatible subobject $m$ of $v^*A$ (as above), the subobject $n$ of $A$ corresponding to the internal union along $v$ is given by internal union along $u^*v$ in $\mathcal{F}$,

\[
\begin{align*}
  v^*A & \hookrightarrow A \\
  v^*J & \hookrightarrow J \\
  v^*v & \downarrow \phantom{m} \, \, K \rightarrow I
\end{align*}
\]

which is automatically strict with respect to $\rho$ (intuitively because sets which are compatible with an equivalence relation are closed under arbitrary unions).

Using this representation, we can define the embedding of $\mathcal{F}$ into $\text{Sh}(\mathcal{F})$, which we call $Z$, since it is close to the fibered Yoneda embedding $Y : \mathcal{C} \to \mathcal{C}$ defined in Lemma 3.1.13.

**Definition 3.2.6** For a geometric pre-stack $\mathcal{F} : |\mathcal{F}| \to \mathbb{R}$, we define

\[
Z : \mathcal{F} \to \text{Sh}(\mathcal{F})
\]

\[
\mathcal{F}_I \ni A \mapsto (\text{id}_I, A, =) \in \text{Sh}(\mathcal{F})_I.
\]

**Lemma 3.2.7** The previously defined $Z$ is full, faithful and geometric.

**Proof.** Fullness, faithfulness, and regularity are fiberwise properties, thus it suffices to show them for the functors $Z_I$, $I \in \mathbb{R}$. It suffices even to verify them for $Z_I : \mathcal{F}_I \to \mathbb{R}[\mathcal{F}]$, since $\mathcal{F}_I \cong (\mathcal{F}/I)_{\text{id}_I}$, and we can apply the same argument to $\mathcal{F}/I$. For $A, B \in \mathcal{F}_I$, a functional
relation from \((A, =)\) to \((B, =)\) in \(\operatorname{sub}(\mathcal{C})\) is just a functional relation between \(A\) and \(B\) in the (ordinary) subobject fibration of \(\mathcal{C}_1\), and we know that morphisms are in bijection with functional relations in regular categories, whence \(Z_1\) is full and faithful. A morphism \(\phi : (A, \rho) \to (B, \sigma)\) in \(\mathcal{R}[\mathcal{C}]\) is regular epic iff \(b \vdash \exists A. \phi(a, b)\) holds. Since \(\exists\) is given by collective-cover/vertical-mono factorization, which restricts to ordinary cover/mono factorization in \(\mathcal{C}_1\), it is easy to see that \(Z_1\) preserves regular epimorphisms.

Finally, the preservation of internal unions follows directly from the description of internal unions in \(\mathbf{Sh}(\mathcal{C})\) given before Definition 3.2.6.

\[\text{Lemma 3.2.8} \quad \mathbf{Sh}(\mathcal{C}) \text{ is characterized by the equivalence} \]
\[
\mathbf{Geo}(\mathcal{R})(\mathcal{C}, \mathcal{C}) \cong \mathbf{Pretop}(\mathcal{R})(\mathbf{Sh}(\mathcal{C}), \mathcal{C})
\]

\[\text{for fibered pretoposes } \mathcal{C}.\]

\[\text{Proof.} \quad \text{Since } Z : \mathcal{C} \to \mathbf{Sh}(\mathcal{C}) \text{ is a geometric fibered functor, precomposition induces a functor of type } \mathbf{Pretop}(\mathbf{Sh}(\mathcal{C}), \mathcal{C}) \to \mathbf{Geo}(\mathcal{C}, \mathcal{C}). \text{ We show that this functor is full, faithful, and essentially surjective.} \]

For faithfulness, assume that \(F, G \in \mathbf{Pretop}(\mathbf{Sh}(\mathcal{C}), \mathcal{C})\) and \(\eta, \theta : F \to G\) such that \(\eta Z = \theta Z\). Let \((u, A, \rho) \in \mathbf{Sh}(\mathcal{C})_I\) with \(u : J \to I\). This object can be covered by an object in the image of \(Z\) as \(ZA = (\operatorname{id}_J, A, =) \to (u, A, =) \to (u, A, \rho)\) where the first map is cocartesian, and the second is a vertical regular epi, whence both are collective covers, which are preserved by fibered geometric functors, and in particular fibered pretopos morphisms. Thus, applying \(F, G, \eta, \) and \(\theta\), we obtain
\[
\begin{align*}
F(\operatorname{id}_J, A, =) & \longrightarrow F(u, A, \rho) \\
\eta(\operatorname{id}_J, A, =) = \theta(\operatorname{id}_J, A, =) & \quad \quad \eta(u, A, \rho) = \theta(u, A, \rho)
\end{align*}
\]
in \(\mathcal{C}\), and we deduce that \(\eta(\operatorname{id}_J) = \theta(\operatorname{id}_J)\) since collective covers are collectively epic, and the components of \(\eta, \theta\) are vertical.

To see that precomposition by \(Z\) is full, let \(\mu : FZ \to GZ\) and consider again \((u, A, \rho) \in \mathbf{Sh}(\mathcal{C})_I\) covered by \((\operatorname{id}_J, A, =) \to (u, A, =) \to (u, A, \rho)\). Applying \(F\) and \(G\), we obtain
\[
\begin{align*}
FZA & \longrightarrow F(u, A, =) \\
\mu_A & \downarrow \quad \eta_A \downarrow \\
GZA & \longrightarrow G(u, A, =)
\end{align*}
\]
and we have to show that we can fill in the dashed arrows. The existence of the first one follows from the universal property of cocartesian liftings, for the second one we have to compare the kernels of the two vertical (horizontal in the diagram) epimorphisms. Now these two kernels can be collectively covered by the images of \(\rho : R \to A \times_I A\) under \(FZ\) and \(GZ\), respectively, and the desired inclusion of kernels can be deduced by considering the arrow \(\mu_R\) between their respective coverings. We leave it to the reader to verify that this construction give rise to a natural transformation between \(F\) and \(G\).

It remains to check that precomposition by \(Z\) is essentially surjective. Let \(F : \mathcal{C} \to \mathcal{C}'\) be a geometric fibered functor. We have to construct an extension \(\tilde{F} : \mathbf{Sh}(\mathcal{C}) \to \mathcal{C}'\) of \(F\). Consider \((u, A, \rho) \in \mathbf{Sh}(\mathcal{C})_I\). If we apply \(\tilde{F}\) to \(A\) and \(\rho\), we obtain \(FA \in \mathcal{C}_J\) and \(F\rho : FR \to \)
To construct \( \tilde{F}(u, A, \rho) \), we take the internal sum of \( FA \) along \( u \), and then push \( F\rho \) from \( FA \times I \) to \( \Sigma_u FA \times \Sigma_u FA \).

\[ \begin{aligned}
FA \times \Sigma_u FA & \xrightarrow{++} \Sigma_u FA \\
FA \times I FA \times \Sigma_u FA & \xrightarrow{++} \Sigma_u FA \times \Sigma_u FA \\
FA \times FA & \xrightarrow{++} \Sigma_u FA \times \Sigma_u FA
\end{aligned} \]

\( \rho' \) defined like this is an equivalence relation in \( \mathcal{X}_I \), and we take \( \tilde{F}(U, A, \rho) \) to be its quotient. ■

**Corollary 3.2.9** For a finite limit pre-stack \( \mathcal{C} : |\mathcal{C}| \to \mathbb{R} \), on a regular category, we have an equivalence

\[ \mathcal{C} \simeq \text{Sh}(D\mathcal{C}) \]

of fibrations. In particular, we have \( \mathbb{R}\{\mathcal{C}\} \simeq \mathbb{R}[D\mathcal{C}] \).

**Proof.** The universal property of the fibered presheaf construction says that it constitutes a left biadjoint to the forgetful functor \( \text{Pretop}(\mathbb{R}) \to \text{Lex}(\mathbb{R}) \) from fibered pretoposes on \( \mathbb{R} \) to finite limit pre-stacks on \( \mathbb{R} \). The universal properties of \( \text{Sh}(\mathcal{C}) \) and \( D \) characterize the two constructions as left biadjoints to the forgetful functors of type \( \text{Pretop}(\mathbb{R}) \to \text{Geo}(\mathbb{R}) \) and \( \text{Geo}(\mathbb{R}) \to \text{Lex}(\mathbb{R}) \), respectively. The claim follows from the fact that biadjunctions compose. ■

### 3.2.1 Discrete sheaves

In this section, we consider the left biadjoint to

\[ \text{Pos}(\mathbb{R}) \to \text{Geo}(\mathbb{R}). \]

We do this without going into details and without proofs. Abstractly, the positive cocompletion of a geometric fibration \( \mathcal{F} \) can be understood as the subfibration of \( \text{Sh}(\mathcal{F}) \) which is generated by the image of \( Z : \mathcal{F} \to \text{Sh}(\mathcal{F}) \) by closing under sums. Concretely, we give a construction analogous to the positive completion of a finite limit pre-stack in Section 3.1.4 by identifying an appropriate subcategory of \( \mathbb{R}[\mathcal{F}] \) and gluing. It turns out that this subcategory has a particularly simple description.

**Definition 3.2.10** Let \( \mathcal{F} : |\mathcal{F}| \to \mathbb{R} \) be a geometric pre-stack. The category \( \mathcal{P}(\mathcal{F})_1 \) is defined as the full subcategory of \( \mathbb{R}[\mathcal{F}] \) on discrete equivalence relations. In other words, the objects of \( \mathcal{P}(\mathcal{F})_1 \) are the objects of \( |\mathcal{F}| \), and the morphisms are functional relations in \( \text{sub}(\mathcal{F}) \).

Since discrete equivalence relations in \( \mathbb{R}[\mathcal{F}] \) are closed under finite products and subobjects, \( \mathcal{P}(\mathcal{F})_1 \) is a regular category. Moreover, the functor \( \Delta : \mathbb{R} \to \mathbb{R}[\mathcal{F}] \) (Definition 3.2.3) factors through \( \mathcal{P}(\mathcal{F})_1 \)

\[ \begin{array}{ccc}
\mathbb{R} & \xrightarrow{\Delta} & \mathcal{P}(\mathcal{F})_1 \\
\downarrow & \downarrow & \downarrow \\
\mathbb{R}[\mathcal{F}] & \xrightarrow{} & \mathcal{P}(\mathcal{F})_1
\end{array} \]
and we can define $\mathcal{P}(\mathcal{F})$ by

$$\mathcal{P}(\mathcal{F}) = gl_\Delta(\mathcal{P}(\mathcal{F})_1).$$

Since $\mathcal{P}(\mathcal{F})_1$ and $\Delta$ are both regular, $\mathcal{P}(\mathcal{F})$ is indeed a positive pre-stack. Furthermore, the characterizing equivalence holds.

**Lemma 3.2.11** Let $\mathcal{F}$ be a geometric pre-stack, and $\mathcal{P}$ a positive pre-stack on $\mathcal{R}$. Then we have

$$\text{Geo}(\mathcal{R})(\mathcal{F}, \mathcal{P}) \simeq \text{Pos}(\mathcal{R})(\mathcal{P}(\mathcal{F}), \mathcal{P})$$

### 3.3 Fibered pretoposes from positive fibrations

To finish our treatment of left biadjoints to the forgetful functors (3.0.1), it remains to construct the left biadjoint to $\text{Pretop}(\mathcal{R}) \to \text{Pos}(\mathcal{R})$, i.e., the construction of the free fibered pretopos on a positive pre-stack. This is simply done by fiberwise ex/reg completion.

One way to understand this is to consider instead the forgetful functor $\mathcal{R} \downarrow U \to \mathcal{R}/\text{sslash Reg}$ between the pseudo-co-slice 2-categories which are biequivalent to $\text{Pretop}(\mathcal{R})$ and $\text{Pos}(\mathcal{R})$ via the correspondence of Moens theorem (see Lemmas 2.3.23 and 2.3.28). It is easy to see that the ordinary ex/reg completion, which is left biadjoint to $U : \text{Ex} \to \text{Reg}$, lifts to a biadjunction between the pseudo-co-slice 2-categories.

### 3.4 The preordered case

The free constructions that we have developed in the preceding sections become easier when we consider fibered preorders instead of general fibrations.

In this section, we explore some of the consequences, with the goal of giving a characterization of when the category $\mathcal{R}\{A\}$ for a pre-stack $A : |A| \to \mathcal{R}$ of meet-semilattices is locally cartesian closed.

#### 3.4.1 Fibered frames

Fibered frames are posetal geometric fibrations. Since in the posetal case in the presence of greatest elements the existence of internal unions already implies the existence of internal sums, we can give the following equivalent definition.

**Definition 3.4.1** Let $\mathcal{R}$ be a regular category.

(i) A *fibered frame* is an existential fibration $\mathcal{X} : |\mathcal{X}| \to \mathcal{R}$ (Definition 2.2.2) which is a pre-stack.

(ii) $\text{FFrm}(\mathcal{R})$ is the locally ordered category of fibered frames on $\mathcal{R}$. Its morphisms are fibered monotone maps preserving finite meets and existential quantification.

Given a fibered frame $\mathcal{X} : |\mathcal{X}| \to \mathcal{R}$, we can use the existential quantification to ‘embed’ the subobject fibration of $\mathcal{R}$ into $\mathcal{X}$ via the fibered monotone map

$$\delta : \text{sub}(\mathcal{R}) \to \mathcal{X}, \quad (m : U \to I) \mapsto (\exists m \top \in \mathcal{X}_I). \quad (3.4.1)$$

We write ‘embed’ in quotes since $\delta$ is not necessarily order-reflecting (a counterexample is the terminal fibration). However, we can show the following statements.
Lemma 3.4.2 Let $\mathcal{X}$ be a fibered frame on $\mathcal{R}$.
- $\delta$ preserves conjunction and existential quantification.
- Let $m : U \to I$ be a monomorphism in $\mathcal{R}$. Then we have
  $$\mathcal{X}_U \simeq \{ \varphi \in \mathcal{X}_I \mid \varphi \leq \delta m \}.$$ 

Proof. The preservation of conjunction follows from the Beck-Chevalley condition and Frobenius reciprocity. Preservation of $\exists$ follows from the fact that $\mathcal{X}$ is a pre-stack.

For the second claim, use reasoning in the internal logic, making use of the facts that $\exists m^* \varphi = (i \mid \exists u. mu = i \land \varphi(mu))$ and $m^* \exists_m \psi = (u \mid \exists v. mv = mu \land \psi v)$ for $\varphi \in \mathcal{X}_I, \psi \in \mathcal{X}_U$, and that $mu = mv \vdash u = v$ holds in $\mathcal{X}$ since $m$ is a monomorphism. ■

3.4.1 Totally connected fibered frames

Definition 3.4.3 We call a fibered frame $\mathcal{X} : |\mathcal{X}| \to \mathcal{R}$ totally connected, if $\delta : \text{sub}(\mathcal{R}) \to \mathcal{X}$ has a finite meet preserving left adjoint $\pi : \mathcal{X} \to \text{sub}(\mathcal{R})$.

The term ‘totally connected’ comes from topos theory – a geometric morphism $\Delta \dashv \Gamma : \mathcal{E} \to \mathcal{S}$ is called totally connected if the fibered functor $\Delta : \text{cod}(\mathcal{S}) \to \text{gl}_\Delta(\mathcal{E})$ (see (2.2.4)) has a finite limit preserving left adjoint (which is automatically positive since it is a left adjoint). In this case, the adjunction is necessarily a reflection, since $\text{cod}(\mathcal{S})$ is bi-initial among fibered pretoposes on $\mathcal{S}$.

In the case of totally connected fibered frames, we can deduce that the adjunction is a reflection in a similar way.

Lemma 3.4.4 Let $\mathcal{X} : |\mathcal{X}| \to \mathcal{R}$ be a totally connected fibered frame. Then $\pi \dashv \delta$ is a reflection and $\delta$ is order reflecting.

Proof. Let $m : U \to I$ in $\mathcal{R}$. To show that $\pi \dashv \delta$ is a reflection, it suffices to show that $U \subseteq \pi \delta U$, which is equivalent to $m^* (\pi \delta U) \cong \top$. This follows from the fact that $m^* U \cong \top$, and that $\pi$ and $\delta$ commute with reindexing and preserve $\top$.

The second claim follows from the first one. ■

In Lemma 3.4.12 we will see that fibered frames of the form $DA$ for $A$ with finite meets are always totally connected; in Section 3.4.3 we will show that sheaves over totally connected fibered frames have well behaved subcategories of assemblies.

3.4.2 Cocompletions

In the following we revisit the constructions of fibered presheaves and sheaves, and the geometric completion $D$ for pre-stacks of preorders. We will do this in the converse order, since it appears more natural in the posetal case.

3.4.2.1 The $D$-construction

Since fibered frames are posetal geometric pre-stacks, we can expect to obtain a fibered frame when applying the $D$-construction (Definition 3.1.18) to a pre-stack of meet-semilattices.

Recall that for a finite limit pre-stack $\mathcal{E}$, the fibration $D\mathcal{E}$ is defined as the full subfibration of $\mathcal{E}$ on sub-representables, and can be characterized as the completion of $\mathcal{E}$ to a geometric pre-stack (Lemma 3.1.20). Now for pre-stacks of meet-semilattices, $D$ can alternatively be characterized as freely adjoining existential quantification. This point of view allows us to give a direct construction of $D$ without detour over fibered presheaves, which allows us to go in the
opposite direction and define fibered presheaves in terms of $D$ (following Corollary 3.2.9) without being circular.

Furthermore, the point of view on $D$ as adjoining existential quantification gives a construction that even makes sense in the absence of finite limits/meets:

**Definition 3.4.5** Let $A : |A| \to \mathbb{R}$ be a pre-stack of preorders. The fibration $DA$ on $\mathbb{R}$ is defined as follows.
- A predicate on $I \in \mathbb{R}$ is a pair $(u : J \to I, \varphi \in A_J)$
- $(u : J \to I, \varphi) \leq (v : K \to I, \psi)$ if there exists a span $\xymatrix{J \ar@{^(->}[r]^e & L \ar@{->>}[r]_f & K}$ with $e$ regular epic, $we = vf$, and $e^* \varphi \leq f^* \psi$.

The embedding $y : A \to DA$ is defined by
\[ A_I \ni \varphi \mapsto (\text{id}_I, \varphi) \in DA_I. \]

**Lemma 3.4.6** For pre-stacks $A$, the ordering in $DA$ can equivalently be defined in terms of total relations: we have $(u : J \to I, \varphi) \leq (v : K \to I, \psi)$ iff there exists a total relation $t : J \to K$ such that $vt = u$, and $\varphi(j), (\text{id})\langle j, k \rangle \models \psi(k)$ holds in $A$.

**Proof.** Any total relation gives a span via its two projections, the left leg is epic since the relation is total. Conversely, given a span $\xymatrix{J \ar@{^(->}[r]^e & L \ar@{->>}[r]_f & K}$ with left leg epic, we get a total relation by taking the image of $(e, f)$ in $J \times K$. \[\blacktriangleleft\]

**Lemma 3.4.7** Let $A : |A| \to \mathbb{R}$ be a pre-stack of preorders.

(i) $y : A \to DA$ is (monotone and) order-reflecting.
(ii) $DA$ has existential quantification.
(iii) $DA$ is generated under existential quantification by the image of $y : A \to DA$. More precisely, for every $\varphi \in DA_I$ there exists $\psi \in A_J$ and $u : J \to I$ with $\exists_y \psi(\psi) \equiv \varphi$.
(iv) Given a posetal pre-stack $B$ with existential quantification, pre-composition with $y : A \to DA$ induces an equivalence
\[ \text{PFib}(\mathbb{R})(A, B) \simeq \exists \text{-PFib}(\mathbb{R})(DA, B) \]

between preorders of fibered monotone maps, and fibered monotone maps commuting with existential quantification.

**Proof.** It is easy to see that $y$ is order-reflecting (the fact that $A$ is a pre-stack is necessary).

The existential quantification of $(u, \varphi)$ along $v : I \to K$ is given by $(vu, \varphi)$, and the Beck Chevalley condition follows from the pullback lemma. The fact that $DA$ is generated by the image of $A$ follows immediately from the description of existential quantification.

For the fourth claim, we have to show that precomposition by $y : A \to DA$ is order-reflecting and essentially surjective. Let $F, G : DA \to B$ be fibered monotone maps commuting with $\exists$, such that $F \circ y \leq F \circ y$. Then $F \leq G$ follows from the facts that $DA$ is generated by the image of $y$ under existential quantification, and that $F$ preserves existential quantification. To show that precomposition is essentially surjective, let $F : A \to B$ be a fibered monotone map. Then we define $\bar{F} : DA \to B$ by $\bar{F}(u, \varphi) = \exists_u F \varphi$. \[\blacktriangleleft\]

**Definition 3.4.8** Let $B : |B| \to \mathbb{R}$ be a posetal pre-stack with existential quantification subject to the Beck-Chevalley condition. We call $\pi \in B_I$ $\exists$-prime, if whenever we are given maps
Let \( \rho \) denote \( J \subseteq K \) and \( \theta \in \mathcal{B}_K \) such that \( u^*\pi \leq \exists_v\theta \), there exists a span \( J \xrightarrow{\rho} L \xleftarrow{w} K \) such that \( vw = e \) and \((ue)^*\pi \leq w^*\theta\).

\[
\begin{array}{ccc}
\pi & L & \theta \\
\downarrow & \swarrow & \searrow \\
I & J & K \\
\end{array}
\]

\[\diamondsuit\]

Remarks 3.4.9

Again, we can replace the span in the definition by a total relation since
we are in a posetal framework.
In terms of relations, \( \pi \) is prime iff for all \( I \subseteq J \subseteq K \) and \( \theta \in \mathcal{B}_K \) such that \( u^*\pi \leq \exists_v\theta \),
there exists a total relation \( t : J \rightarrow K \) such that \( v \circ t = \text{id}_J \) and
\[\pi(u), (\delta t)(j, k) \vdash \theta(k)\]
holds in \( \mathcal{B} \).

Let \((L, \leq)\) be a complete lattice. Then a family \( \varphi : I \rightarrow L \) is \( \exists \)-prime in \( \text{fam}(L) : \text{Fam}(L) \rightarrow \text{Set} \) iff all components \( \varphi(i) \) for \( i \in I \) are completely join prime in the usual order theoretic sense \([63]\).

\[\diamondsuit\]

Lemma 3.4.10 Let \( A : |A| \rightarrow \mathbb{R} \) be a posetal pre-stack.

(i) Given \( \varphi \in A_I \), \( y(\varphi) \) is \( \exists \)-prime in \( DA \).

(ii) Given an \( \exists \)-prime \( \pi \in (DA)_I \), there exists \( \psi \in A_K \) and \( e : K \rightarrow I \) regular epic such that
\[y(\psi) \cong e^*\pi.\]

(iii) Let \( B \) be a posetal pre-stack with existential quantification, and let \( A \) be its subfibration
on \( \exists \)-prime predicates. The fibered monotone map \( H : DA \rightarrow B \) induced by the inclusion
\( H : A \rightarrow B \) is order-reflecting. It is essentially surjective (and thus an equivalence) iff for
every \( \psi \in Bi \) there exists an \( \exists \)-prime predicate \( \pi \in B_J \) and a map \( u : J \rightarrow I \) such that
\[\psi \cong \exists_u\pi.\]

Proof. Ad (i). Assume that \( y(\varphi) \leq \exists_u(v, \theta) \) for \( (v, \theta) \in (DA)_J \), i.e., \( v : K \rightarrow J \) and \( \theta \in A_K \)
(since the image of \( y \) is closed under reindexing, we can omit the reindexing in the hypothesis of
the definition of primality). The definition of the ordering in \( DA \) provides us with the required
span.

Ad (ii). Assume that \( \pi \in (DA)_I \) is \( \exists \)-prime. By Lemma 3.4.7-(iii), there exist \( \varphi \in (DA)_J \)
and \( u : J \rightarrow I \) such that \( \varphi \) is in the image of \( y \) and \( \exists_y\varphi \cong \pi \). Since \( \pi \) is prime, there exists a span \( J \xrightarrow{\rho} K \xleftarrow{w} I \) with \( uw = e \) and \( e^*\pi \leq w^*\varphi \). If we can show that \( e^*\pi \cong w^*\varphi \) we are
done, since the essential image is closed under reindexing. We already know that the left hand
side is \( \leq \) than the right hand side; the other inequality is derived as follows.

\[\exists_y\varphi \leq \pi \quad \Rightarrow \quad \varphi \leq u^*\pi \quad \Rightarrow \quad w^*\varphi \leq w^*u^*\pi \cong e^*\pi\]

Ad (iii). Consider \((u : J \rightarrow I, \pi), (v : K \rightarrow I, \xi) \in (DA)_I \) such that \( \pi \) and \( \xi \) are \( \exists \)-prime in \( A \)
and \( H(u, \pi) = \exists_u\pi \leq \exists_v\xi = H(v, \xi) \). Form the pullback of \( u, v \).

\[
\begin{array}{ccc}
L & \xrightarrow{w} & K \\
\downarrow & \searrow & \swarrow \\
M & \xleftarrow{u} & J \\
\end{array}
\]

60
From the Beck-Chevalley condition it follows that \( \pi \leq \exists_x y \xi \), and since \( \pi \) is prime, we can find a span \((e, y)\) such that \( ey = e \) and \( e^* \pi \leq y^* w^* \xi \). The span \((e, wy)\) witnesses the inequality \((u, \pi) \leq (v, \xi)\), which shows that the canonical functor \(DA \to B\) is order reflecting. The condition on essential surjectivity is clear from the construction of \(\tilde{H}\) given in the proof of Lemma 3.4.7. \[\Box\]

**Lemma 3.4.11** If \(A\) is a pre-stack of meet-semilattices, then \(DA\) is its completion to a fibered frame. More precisely, we have

- \(DA\) has finite meets, compatible with \(\exists\) in the sense of Frobenius reciprocity.
- \(y : A \to DA\) preserves finite meets.
- For any fibered frame \(X\), pre-composition with \(y : A \to DA\) induces an equivalence
  \[\Lambda-PFib(\mathbb{R})(A,X) \simeq \mathcal{FFrm}(\mathbb{R})(DA,X)\]

between preorders of fibered monotone maps commuting with \(\wedge\) and fibered monotone maps commuting with \(\Lambda\) and \(\exists\).

Moreover, \(DA\) is equivalent to the completion of \(A\) to a geometric category from Definition 3.1.18, which justifies the use of the same notation.

**Proof.** Given predicates \((u : J \to I, \varphi)\), \((v : K \to I, \psi)\), their meet is given by \((y, x^* \varphi \wedge w^* \psi)\) with \(w, x, y\) as in the following pullback diagram.

\[
\begin{array}{ccc}
L & \xrightarrow{w} & K \\
\downarrow x & & \downarrow y \\
J & \xrightarrow{u} & I \\
\end{array}
\]

This description makes it clear that finite meets are preserved by \(y\); thus, in particular, pre-composition by \(y\) induces a monotone map of type \(\mathcal{FFrm}(DA,X) \to \Lambda-PFib(\mathbb{R})(A,X)\) which is order-reflecting as we showed in Lemma 3.4.7-(iii). It remains to check that if \(F : A \to X\) preserves finite meets, then so does \(\tilde{F} : DA \to X\). For this it suffices to show (for the binary case) that for predicates \(\gamma \in X_J, \delta \in X_J\) we have \((\exists u \gamma) \wedge (\exists u \delta) \cong \exists u x^* \gamma \wedge w^* \delta\), which is an easy exercise.

For the claim about the coincidence of the \(D\)-construction from this section and the geometric completion from Section 2.3.2, note that fibered frames are the same thing as posetal geometric pre-stacks, and fibered geometric functors in the posetal case coincide with fibered monotone maps commuting with \(\wedge\) and \(\exists\). Moreover, as we defined the geometric completion of a finite limit pre-stack \(\mathcal{G}\) as the subfibration of \(\mathcal{G}'\) on the sub-representables, it is easy to see that the geometric completion of a fibered frame is posetal, and thus a fibered frame. From this, it follows that the geometric completion in the posetal case has the same universal property that we just established for \(DA\). \[\Box\]

The fact that \(\exists\) in \(DA\) is free has an interesting consequence, described in the following lemma.

**Lemma 3.4.12** If \(A\) is a pre-stack of meet-semilattices. Then the fibered frame \(DA\) is totally connected in the sense of Definition 3.4.3.

**Proof.** For a predicate \((u : J \to I, \varphi)\), \(\pi(u, \varphi)\) is the image of \(u\) as subobject of \(I\). It is easy to see that this is well defined, left adjoint to \(\delta\), and preserves finite meets. \[\Box\]
Example 3.4.13 If we apply the $D$-construction to ufam$(I, A)$ for a typed pca $(I, A)$, we obtain a fibration that is equivalent to the realizability hyperdoctrine $\mathcal{H}(I, A)$ (Definition A.1.8- (i)). This is not difficult to show by hand, but the nicest way to understand this is via uniform preorders – see Example 4.7.4.

In the same way, if $A$ is an untyped pca, the fibration $D(\text{ufam}(A))$ is equivalent to the realizability tripos $\text{rt}(A)$ (Definition A.1.7-(i)).

3.4.2.2 Fibered sheaves and assemblies

In 3.2.1, we defined the category $\mathbb{R}[\mathcal{S}]$ for a geometric pre-stack $\mathcal{S}$ on $\mathbb{R}$ as the category of equivalence relations and functional relations in $\text{sub}(\mathcal{S})$. For a fibered frame $\mathcal{X}$, given $\varphi \in \mathcal{X}_I$, a predicate in $\text{sub}(\mathcal{X})$ is simply a predicate $\psi \in \mathcal{X}_I$ such that $\psi \leq \varphi$, and it seems easier to express $\mathbb{R}[\mathcal{X}]$ directly in terms of $\mathcal{X}$ without making the additional layer in $\text{sub}(\mathcal{X})$ explicit. When taking this point of view, an equivalence relation in $\text{sub}(\mathcal{X})$ becomes simply a partial equivalence relation in $\mathcal{X}$, and we can easily show the following lemma.

Lemma 3.4.14 Given a fibered frame $\mathcal{X} : |\mathcal{X}| \rightarrow \mathbb{R}$, $\mathbb{R}[\mathcal{X}]$ is equivalent to the category $\text{PER}(\mathcal{X})$.

Let us recall that for geometric fibrations $\mathcal{S}$, we defined $\mathbb{R}[\mathcal{S}]$ as the category of total equivalence relations in $\text{sub}(\mathcal{S})$, and we showed in Lemma 3.2.2 that $\mathbb{R}[\mathcal{S}]$ is equivalent to $\text{PER}(\text{siev}(\mathcal{S}))$.

The lemma says that in the posetal case we can drop the siev($\cdot$).

In the following, we will always view $\mathbb{R}[\mathcal{X}]$ for $\mathcal{X}$ a fibered frame as category of partial equivalence relations. This is the point of view that is familiar from the tripos-to-topos construction. In particular, in this representation the diagonal functor $\Delta : \mathbb{R} \rightarrow \mathbb{R}[\mathcal{X}]$ from Definition 3.2.3 has the familiar description $I \mapsto (I, =)$, which is known as the ‘constant objects functor’. The following lemma is a direct consequence of Lemma 2.2.7-(i).

Lemma 3.4.15 Let $\mathcal{X}$ be a fibered frame on $\mathbb{R}$. Then we can reconstruct $\mathcal{X}$ from $\mathbb{R}[\mathcal{X}]$ and $\Delta$ as pullback

\[
\begin{array}{ccc}
|\mathcal{X}| & \rightarrow & \text{Sub}(\mathbb{R}[\mathcal{X}]) \\
\downarrow & & \downarrow \\
\mathcal{X} & \rightarrow & \text{sub}(\mathbb{R}[\mathcal{X}]). \\
\downarrow & & \downarrow \\
\mathbb{R} & \Delta & \rightarrow & \mathbb{R}[\mathcal{X}]
\end{array}
\]

In other words, $\mathcal{X}$ is equivalent to the subfibration of $\text{gl}_\Delta(\mathbb{R}[\mathcal{X}])$ on monomorphisms.

Assemblies have particularly good properties if $\mathcal{X}$ is totally connected, as we will explain in Section 3.4.3.
Remark 3.4.17 There are interesting analogies between the description and universal characterization of the categories $\text{Asm}(X)$ and $\mathbb{R}[X]$ for a fibered frame as given here, and work of Maietti and Rosolini [42].

Slightly paraphrasing, Maietti and Rosolini describe how to get the categories $\text{Asm}(X)$ and $\mathbb{R}[X]$ from a fibered frame (or rather an existential fibration) by a sequence of completion operations on fibrations. In particular, they introduce what we call the fibered subobject fibration, albeit with a different viewpoint: in our notation, they view $\text{sub}(X) : |\text{sub}(X)| \to |X|$ as the result of freely adding Lawvere comprehension to $X$, the coincidence of this with the fibered subobject fibration occurs only in the posetal case.

The major difference between the approach of [42] and the present work is that whereas both are about fibrational (co)completions, in the present work we adjoin categorical structure to fibrations on a constant base category whereas Maietti and Rosolini complete the base category with respect to the logical structure of the fibration. In particular, the categories $\text{Asm}(X)$ and $\mathbb{R}[X]$ occur as terminal fibers here, whereas they occur as base categories in [42].

A nice feature of Maietti and Rosolini’s approach is that it highlights the fact that the base category $\mathbb{R}$ is actually not essential for the construction of $\text{Asm}(X)$ and $\mathbb{R}[X]$, both of which only depend on the cartesian bicategory [15] of relations in $X$.

3.4.2.3 Fibered presheaves

We will now study the fibration of presheaves on a fibered meet-semilattice $A$, and give a criterion for $\mathbb{R}\{A\}$ to be locally cartesian closed. Since the direct manipulation of the fibration of sieves feels a bit unhandy and cumbersome in the posetal case, we will make use of the equivalence $\mathbb{R}\{A\} \cong \mathbb{R}[DA]$ from Corollary 3.2.9, and work explicitly only in the second category. An advantage of the first representation would be that we have tracking families (Definition 3.1.8, Lemma 3.1.9) available, but with a bit of care we can access them also in $\mathbb{R}[DA]$, as we will explain now.

Definition 3.4.18 Let $\mathcal{X} : [X] \to \mathbb{R}$ be a fibered frame, and let $\phi : (I, \rho) \to (J, \sigma)$ be a morphism in $\mathbb{R}[X]$, where $\rho$ and $\sigma$ are partial equivalence relations as in Lemma 3.4.14. A tracking relation for $\phi$ is a total relation $t : I \to J$ such that

\[
\rho(i), (\delta t)(i, j) \vdash \phi(i, j)
\]

holds in $\mathcal{X}$.

Lemma 3.4.19

(i) If $t : I \to J$ is a tracking relation of $\phi : (I, \rho) \to (J, \sigma)$, then

\[
\phi(i, j) \vdash \rho(i) \land \exists j'. (\delta t)(i, j') \land \sigma(i, j').
\]

thus $\phi$ is uniquely determined by the tracking relation and can be reconstructed from it.

(ii) A total relation $t : I \to J$ is a tracking relation of a morphism of type $(I, \rho) \to (J, \sigma)$ iff

\[
\rho(i, i'), (\delta t)(i, j), (\delta t)(i', j') \vdash \sigma(j, j')
\]

holds in $\mathcal{X}$.

(iii) Two total relations $t, u : I \to J$ are tracking relations of the same morphism of type $(I, \rho) \to (J, \sigma)$, iff

\[
\rho(i, i'), (\delta t)(i, j), (\delta u)(i', j') \vdash \sigma(j, j')
\]

holds in $\mathcal{X}$. 

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Proof. This is proved in a similar way as Lemma 3.1.9. ■

Lemma 3.4.20  (i) Let $X$ be a fibered frame, and $(I, \rho), (J, \sigma)$ two objects in $\mathbb{R}[X]$. If the existence part $(x \mid \rho(x, x)) \in (DA)_I$ of $\rho$ is $3$-prime, then every morphism $\phi : (I, \rho) \to (J, \sigma)$ has a tracking relation.

(ii) For a fibered meet-semilattice $A$, every object in $\mathbb{R}[DA]$ is isomorphic to an object with $3$-prime existence predicate.

Proof. Ad (i). Given $\phi : (I, \rho) \to (J, \sigma)$, the judgment $\rho(i) \vdash \exists j. \phi(i, j)$ holds since $\phi$ is total. Primality of $\{ i \mid \rho(i) \}$ implies that there exists a total relation $t : I \to J \times J$ such that $\rho(i), (\delta t)(i, i', j) \vdash \phi(i, j)$ and $\pi_I \circ t = \text{id}_I$. $\pi_I \circ t = \text{id}_I$ means that $t(i, i', j) \vdash i = i'$ holds in $\text{sub}(\mathbb{R})$, and thus that $(\delta t)(i, i', j) \vdash i = i'$ holds in $X$ since $\delta$ commutes with $\exists$. We claim that $\pi_J \circ t$ is a tracking relation for $\phi$. Since $\pi_J \circ t$ is just the relation $(i, j \mid t(i, i, j))$ (and in particular total since total relations are stable under composition), it suffices thus to show that $\rho(i), (\delta t)(i, i, j) \vdash \phi(i, j)$, which is just a substitution instance of the judgment following from primality.

Ad (ii). We could deduce this directly from Corollary 3.2.9, since the objects with $3$-prime existence predicate in $\mathbb{R}[DA]$ correspond to the objects in $\mathbb{R}[A]$. Instead, we give an explicit construction since it will be useful later. Let $(I, \rho) \in \mathbb{R}[DA]$. By Lemma 3.4.10 there exists a morphism $u : J \to I$ and a $3$-prime predicate $\pi \in (DA)_J$ such that $\exists u \pi \equiv \delta_I^u \rho$. Define a partial equivalence relation $\sigma \in (DA)_{I \times J}$ by $\sigma(i, i') \equiv \pi(i) \land \pi(i') \land \rho(u(i), u(i'))$. It is then easy to see that $(J, \sigma)$ has $\pi$ as existence predicate and is isomorphic to $(I, \rho)$. ■

Lemma 3.4.21 Let $E$ be a topos, and let $A : |A| \to E$ be a pre-stack of meet-semilattices. If $DA$ has implication and universal quantification, then $E[DA]$ is cartesian closed.

Proof. Let $(J, \sigma), (K, \tau) \in \mathbb{R}[DA]$, and assume without loss of generality that $\sigma$ has $3$-prime existence part. Denote by $t\text{Rel}(J, K) \subseteq P(\mathbb{R} \times J)$ the object of total relations from $J$ to $K$ in the sense internal to $E$. We claim that an exponential $(K, \tau)^{(J, \sigma)}$ is given by $(t\text{Rel}(J, K), \tau^\sigma)$, where the partial equivalence relation $\tau^\sigma$ is defined by

$$(\tau^\sigma)(t, u) \equiv \forall jj'kk'. \sigma(j, j') \land \delta((j, k) \in t) \land \delta((j', k') \in u) \implies \tau(k, k').$$

We leave it to the reader to verify that this is indeed a partial equivalence relation; the fact that the variables $t, u$ range over total relations is important.

To obtain the evaluation map associated to the exponential object, consider the relation $e : t\text{Rel}(J, K) \times J \to K$ which is just the appropriate transposition of the membership relation $(\in) \subseteq J \times K \times t\text{Rel}(J, K)$, i.e., $e(t, j, k) \Leftrightarrow (j, k) \in t$. The relation $e$ is total precisely because $t\text{Rel}(J, K)$ contains only total relations; to verify that it tracks a morphism

$$e : (t\text{Rel}(J, K), \tau^\sigma) \times (J, \sigma) \to (K, \tau),$$

we have to show (following Lemma 3.4.19-(ii)) that

$$(\tau^\sigma)(t, u), \sigma(j, j'), \delta((j, k) \in t), \delta((j', k') \in u) \vdash \tau(k, k'),$$

holds in $DA$, which is immediate from the definition of $\tau^\sigma$.

It remains to show how to construct exponential transposes. Consider $\phi : (I, \rho) \times (J, \sigma) \to (K, \tau)$, where we assume without loss of generality that $\rho$ has $3$-prime existence part. Since $3$-prime predicates in $DA$ are closed under finite meets (being the stack-completion of the $y$-image of $y : A \to DA$), $\rho \times \sigma$ has $3$-prime existence part as well, whence $\phi$ has a tracking relation.
s : I × J ↦ K. This relation corresponds to a unique function s' : I → P(J × K), which factors through tRel(J, K) ◁ P(J × K) as \( \tilde{s} : I \to t\text{Rel}(J, K) \) since s itself is total.

\[
\begin{array}{c}
I \xrightarrow{s} \text{tRel}(J, K) \\
\downarrow \quad \downarrow \\
P(J \times K)
\end{array}
\]

It is easy to see that \( \tilde{s} \) is a (functional) tracking relation of a morphism of type \((I, \rho) \to (t\text{Rel}(J, K), \tau^\sigma)\) which is an exponential transpose of \( \phi \).

\[\blacksquare\]

Remarks 3.4.22

(i) It is not possible to carry out the preceding proof entirely in the (non-replete) full subcategory of \( \mathcal{E}[DA] \) on objects with \( \exists \)-prime existence predicate, since the existence part of \( \tau^\sigma \) might not be \( \exists \)-prime.

(ii) The proof of the preceding lemma is impredicative – we make use of the totality \( \text{tRel}(J, K) \) of total relations between two objects which we define as a subobject of \( P(I \times J) \). In the presence of appropriate choice principles, however, the use of impredicativity can be avoided – for example, if regular epimorphisms split in \( \mathcal{E} \), we can use tracking functions instead of tracking relations, which allows us to take \( K^J \) instead of \( \text{tRel}(J, K) \) as underlying object of the exponential.

In general, all we need to make the proof work is a sufficient supply of total relations in \( \mathcal{E} \) – more precisely we need \( \mathcal{E} \) to be a regular category with universal quantification such that for every pair \( J, K \in \mathcal{E} \) of objects there exists an object \( T(J, K) \) parameterizing a family of total relations \( \varepsilon : E \Rightarrow T(J, K) \times J \times K \) from \( J \) to \( K \) (‘total’ meaning that \( t, j \models \exists k. \varepsilon(t, j, k) \) holds) such that for every \( I \)-indexed family \( r : R \Rightarrow I \times J \times K \) of total relations from \( J \) to \( K \) we have

\[
\forall i \exists j, k. \varepsilon(t, j, k) \Rightarrow r(i, j, k)
\]

This property holds for example in regular locally cartesian closed categories satisfying the internal axiom of choice; here \( T(J, K) \) is simply given by \( K^J \). This example is interesting because the internal axiom of choice is not sufficient to allow us to work entirely with tracking functions instead of tracking relations (we need epi-splitting for that), but in the construction of the exponential we can still use the function-object. To make this work, we have to modify the construction of the exponential transposes from the above proof a bit. Given \( s : I \times J \Rightarrow K \) as in the proof, the associated function \( s' : I \to P(J \times K) \) does not generally factor through \( K^J \Rightarrow P(J \times K) \) (it does only if \( s \) is already functional); instead we define the tracking relation \( \tilde{s} : I \Rightarrow K^J \) of the exponential transpose by

\[
\tilde{s}(i, f) :\iff \forall j, k. f j = k \Rightarrow s(i, j, k)
\]

and the totality follows from internal choice. \[\diamondsuit\]

In Theorem 3.4.25, we will show that in the situation of the preceding lemma, \( \mathcal{E}[DA] \) is even locally cartesian closed, but since the proof is a bit subtle we need two more lemmas.

---

5. As I learned from Benno van den Berg, this property is called fullness in constructive set theory (see [1, Definition 4.11]).
Lemma 3.4.23 Let $\mathcal{X}$ be an exact category, and $c : J \to I$ a regular epimorphism in $\mathcal{X}$. If $\mathcal{X}/J$ is cartesian closed, then $\mathcal{X}/I$ is cartesian closed.

In particular, to show that $\mathcal{X}$ is locally cartesian closed it is sufficient to show that every object $I$ can be covered by an object $J$ with cartesian closed slice $\mathcal{X}/J$.

Proof. By transitivity of slicing it is sufficient to show that if $\mathcal{X}/I$ is cartesian closed for an object $I$ with global support, then $\mathcal{X}$ is already cartesian closed.

The idea of the proof is as follows:

(i) Given $X \in \mathcal{X}$, the object $\pi_1 : I \times X \to I$ in $\mathcal{X}/I$ is the $I$-indexed family of objects which has value constant $X$.

(ii) Constant families in $\mathcal{X}/I$ can be identified with objects in $\mathcal{X}$.

(iii) Given two constant families in $\mathcal{X}$, their exponential is constant as well.

To make this precise, we identify the concept of ‘constant family’ by ‘object with descent data’ (Definition 2.3.1) which leaves us to show that the category $\mathbf{Desc}(\mathcal{X}, I \to 1)$ is cartesian closed, which is sufficient since the fundamental fibration $\text{cod}(\mathcal{X}) : \mathcal{X}/\mathcal{X} \to \mathcal{X}$ of an exact category is a stack for the regular topology (this is the formalization of (ii)).

For the proof we use the representation of $\mathbf{Desc}(\mathcal{X}, I \to 1)$ using an explicit cleavage from Lemma 2.3.2. Concretely, an object in $\mathbf{Desc}(\mathcal{X}, I \to 1)$ is a pair $(b : B \to I, \beta : \partial_i^* b \to \partial_0^* b)$ subject to the coherence axiom in the Lemma. Given two such objects, $(b, \beta), (c, \gamma)$, we construct a structure map on $\mathcal{E}$. The important step here is to realize that (as can easily be verified) pullback in finite limit categories preserves exponentials, in particular $\partial_i^* (\mathcal{E})$ is an exponential of $\partial_i^* b$ and $\partial_0^* c$ in $\mathcal{X}/(I \times I)$, independently of the question whether $\mathcal{X}/(I \times I)$ has all exponentials. This observation allows us to define a structure map on $\mathcal{E}$ via the following derivation.

\[
\begin{align*}
\pi_1^*(\mathcal{E}) & \to \pi_1^*(\mathcal{E}) \\
\pi_1^*(\mathcal{E}) \times \pi_1^*(b) & \to \pi_1^*(c) \\
\pi_1^*(\mathcal{E}) \times \pi_2^*(\mathcal{E}) & \to \pi_2^*(c) \\
\pi_1^*(\mathcal{E}) & \to \pi_2^*(\mathcal{E})
\end{align*}
\]

It remains to check that this structure map satisfies the coherence axiom, and that $\mathcal{E}$ with this structure map is an exponential of $(b, \beta), (c, \gamma)$ in $\mathbf{Desc}(\mathcal{X}, I)$. This follows again from pullback stability of exponentials, and from calculations $\mathcal{X}/I, \mathcal{X}/(I \times I)$ and $\mathcal{X}/(I \times I \times I)$ which are most easily carried out in $\lambda$-calculus.

Lemma 3.4.24 Let $\mathcal{A} : |\mathcal{A}| \to \mathcal{C}$ be a fibered meet-semilattice and $\varphi \in \mathcal{A}_I$. If $\mathcal{A}$ has implication and universal quantification, then so has $\mathcal{A}/\varphi$.

Proof. Recall that a predicate in $(\mathcal{A}/\varphi)_{\mathcal{A}}$ for $u : J \to I$ is a predicate $\psi \in \mathcal{A}_J$ such that $\psi \leq u^* \varphi$. To obtain implication and universal quantification in $\mathcal{A}/\varphi$, it is sufficient to perform the corresponding construction in $\mathcal{A}$ and then take the conjunction with the appropriate reindexing of $\varphi$.

Now we can prove the theorem.

Theorem 3.4.25 Let $\mathcal{E}$ be a topos, and let $\mathcal{A} : |\mathcal{A}| \to \mathcal{E}$ be a pre-stack of meet-semilattices. The following are equivalent.

- $DA$ has implication and universal quantification.

---

6. Actually all we need is a locally cartesian closed regular category where the fullness principle from Remark 3.4.22-(ii) holds in all slices.
– $\mathbb{R}[DA]$ is locally cartesian closed.

Proof. Since $DA$ can be represented by the pullback

$$
\begin{array}{ccc}
DA & \xrightarrow{\text{sub}(E[DA])} & E[DA] \\
\downarrow & & \downarrow \\
\mathcal{E} & \xrightarrow{\Delta} & E[DA]
\end{array}
$$

it is clear that existence of implication and universal quantification in $DA$ follow from local cartesian closedness of $E[DA]$.

In the converse direction, by Lemma 3.4.23 it is sufficient to show that for all $I \in \mathcal{E}$, $\varphi \in \mathcal{A}_I$, the slice categories $(E[DA])/(I,=_{y\varphi})$ are cartesian closed, since every object is isomorphic to one with $\exists$-prime existence predicate (Lemma 3.4.20-(ii)), which in turn can be covered by an object with the same existence predicate whose partial equivalence relation is sub-diagonal.

To see that $(E[DA])/(I,=_{y\varphi})$ is cartesian closed, consider the chain

$$(E[DA])/(I,=_{y\varphi}) \simeq (E/I)/[DA/y\varphi] \simeq (E/I)/[D(A/\varphi)]$$

of equivalences, where the first equivalence was proved in Lemma 3.2.5, and the second one follows from Lemma 3.4.21. To conclude cartesian closure, it suffices by Lemma 3.4.24 to show that $D(A/\varphi)$, or equivalently $DA/y\varphi$, has implication and universal quantification. This follows from Lemma 3.4.24. □

Remark 3.4.26 The preceding result, and in particular its proof via Lemma 3.4.23, is based on Carboni and Rosolini’s characterization of categories with locally cartesian closed exact completion [14, Theorem 3.3].

It would be nice to have a criterion for $\mathbb{R}\{\mathcal{E}\}$ to be locally cartesian closed for not necessarily posetal finite-limit pre-stacks $\mathcal{E}$. If regular epis split in the base, then by Carboni and Rosolini’s result $\mathbb{R}\{\mathcal{E}\}$ is locally cartesian closed if $[\mathcal{E}]$ is weakly cartesian closed in their sense, since then $\mathbb{R}\{\mathcal{E}\}$ is the exact completion of $[\mathcal{E}]$. It seems nontrivial, however, to rephrase this in a way such that it works without any choice principles, since projectivity in the fibrational sense (Definition 3.1.15) is more difficult to handle than projectivity of objects in regular categories\(^7\).

A careful study of [53] might give new clues. ◊

3.4.3 Assemblies

We introduced assemblies over fibered frames in Definition 3.4.16. Originally, assemblies were introduced in the case of realizability over pcas [13], and in this case they have particularly good properties – if $\text{rt}(A) : |A| \to \text{Set}$ is a realizability tripos (Definition A.1.7-(i)) over a pca $A$, then:

– $\text{Asm}(\text{rt}(A))$ can naturally be represented as a ‘concrete category’, by which we mean that the objects are sets with additional structure, morphisms are functions which are compatible in some sense with this structure, and equality of function is equality of functions (we emphasize this last condition since it does not hold e.g., for strict tracking families in the sense of Definition 3.1.8).

– $\text{Asm}(\text{rt}(A))$ is the category of separated objects for the $\sim$-topology on $\text{Set}[\text{rt}(A)] = \text{RT}(A)$, and furthermore $\text{Set}$ itself is the category of sheaves.

\(^7\) This is related to the fact that given a regular category $\mathbb{R}$, $f : P \to I$ is projective in $\mathbb{R}/I$ iff $P$ is projective in $\mathbb{R}$ whereas the same is not true for internal projectives. The link between internal projectives and projectives in fibrations is that $f$ is internal projective in $\mathbb{R}/I$ iff it is fibrationally projective in $\text{cod}(\mathbb{R})$
The aim of this section is to work out which requirements on a fibered frame $\mathcal{X}$ we need in order to have these properties.

The $\neg\neg$-topology is only definable if we have sufficient logical structure in $\mathcal{X}$ and $\mathbb{R}[\mathcal{X}]$, and we can only expect the base category to coincide with the $\neg\neg$-sheaves if its internal logic validates classical logic. However, it will turn out that rather weak conditions are sufficient to ensure that the assemblies coincide with separated objects for some topology – all we need for that is that the base category is an exact category, and with this localization comes a category of separated objects, which we can identify as the assemblies. If $\mathcal{X}$ is a tripos and the base is boolean, then the topology corresponding to the localization is precisely the $\neg\neg$-topology. Finally, the ‘concrete’ representation of $\text{Asm}(\mathcal{X})$ follows formally from facts about the localization.

**Lemma 3.4.27** If $\mathcal{X} : [\mathcal{X}] \to \mathcal{X}$ is a totally connected fibered frame on an exact category $\mathcal{X}$, then $\Delta : \mathcal{X}[\mathcal{X}] \to \mathcal{X}$ has a finite limit preserving left adjoint $\Pi \dashv \Delta$ such that the counit $\varepsilon : \Pi \Delta \to \text{id}_\mathcal{X}$ is a natural isomorphism.

**Proof.** $\pi : \mathcal{X} \to \text{sub(}\mathcal{X})$ preserves $\exists$ since it is a left adjoint. Since $\pi$ furthermore preserves finite meets by assumption, it preserves partial equivalence relations and functional relations, which allows us to construct a functor $\mathcal{X}[\mathcal{X}] \to \mathcal{X}[\text{sub(}\mathcal{X})]$, which is automatically regular. Moreover, since $\mathcal{X}$ is exact, we have $\mathcal{X}[\text{sub(}\mathcal{X})] \simeq \mathcal{X}$ – composing the former functor with the equivalence we obtain $\Pi : \mathcal{X}[\mathcal{X}] \to \mathcal{X}$. It is easy to see that $\Pi$ and $\Delta$ do indeed form a reflection $\Pi \dashv \Delta$. ■

The previous lemma says in particular that $\mathcal{X}$ is a localization of $\mathcal{X}[\mathcal{X}]$. For toposes, localizations correspond precisely to local operators (also known as Lawvere-Tierney topologies) on the subobject classifier. Since $\mathcal{X}[\mathcal{X}]$ does not necessarily have a subobject classifier, we can’t work with local operators here – however, we still have the corresponding universal closure operation $j : \text{sub(}\mathcal{X}[\mathcal{X}]\text{)} \to \text{sub(}\mathcal{X}[\mathcal{X}]\text{)}$. Let us recall the relevant concepts from [31, A4.3].

- Let $L : \mathcal{C} \to \mathcal{C}$ be a cartesian reflector\(^8\) on a category $\mathcal{C}$ with finite limits, and denote by $\mathbb{L}$ the corresponding replete reflective subcategory. We can define a fibered functor $j : \text{sub(}\mathcal{C}\text{)} \to \text{sub(}\mathcal{C}\text{)}$

by the construction given in the following diagram.

\[
\begin{array}{c}
U \to V \\
\downarrow m \quad \downarrow j \\
\mathcal{X} \quad \mathcal{X}[\mathcal{C}] \\
\downarrow \eta_X \quad \downarrow \eta_X \\
L \mathcal{X} \quad L \mathcal{X}[\mathcal{C}]
\end{array}
\]

(3.4.2)

$j$ is a universal closure operation, meaning that it is isotone and idempotent.

- Given a universal closure operation $j : \text{sub(}\mathcal{C}\text{)} \to \text{sub(}\mathcal{C}\text{)}$, call $m \in \text{sub(}\mathcal{C}\text{)}\mathcal{C}$ dense, if $jm \cong \top$, and closed if $jm \cong m$. Call $S \in \mathcal{C}$ $j$-separated, if given $m$ and $f$ as in

\[
\begin{array}{c}
U \to S \\
\downarrow m \quad \downarrow h \\
A \quad A
\end{array}
\]

with $m$ dense in $\text{sub(}\mathcal{C}\text{)}\mathcal{A}$, there exists at most one mediating $h$, and call $S$ a $j$-sheaf, if for any such $f$ and $m$ there exists exactly one $h$.

\(^8\) That is an idempotent monad which preserves finite limits.
Given a cartesian reflector \( L : \mathcal{C} \to \mathcal{C} \) with induced universal closure operation \( j : \text{sub}(\mathcal{C}) \to \text{sub}(\mathcal{C}) \), we have by [31, Lemma A4.3.6] that
- \( A \in \mathcal{C} \) is \( j \)-separated iff \( \eta_A : A \to LA \) is monic iff \( A \) is a subobject of an object in \( L \) iff the diagonal map \( A \to A \times A \) is closed.
- \( A \in \mathcal{C} \) is a \( j \)-sheaf iff \( \eta_A : A \to LA \) is an isomorphism iff \( A \in L \).

For universal closure operations on exact categories, we can furthermore show the following.

**Lemma 3.4.28** Let \( j : \text{sub}(\mathcal{X}) \to \text{sub}(\mathcal{X}) \) be a universal closure operation on an exact category \( \mathcal{X} \).

(i) \( j \)-separated objects in \( \mathcal{X} \) are closed under finite products and subobjects. Thus, the full subcategory of separated objects is closed under finite limits, regular, and an arrow is a regular cpi in the subcategory iff it is one in \( \mathcal{X} \).

(ii) The subcategory \( \text{Sep}_j(\mathcal{X}) \) of \( j \)-separated objects is reflective in \( \mathcal{X} \).

**Proof.** (i) The terminal object is separated since its diagonal predicate is already maximal in the lattice of subobjects. To see that the product of separate objects \( X, Y \in \mathcal{X} \) is separated, we have to verify the validity of

\[
x, x': X, y, y': E \mid j(x = x' \land y = y') \vdash x = x' \land y = y'.
\]

This holds since \( j \) commutes with conjunction. For subobjects, let \( m : U \to X \) be a monomorphism into a separated object. We have

\[
u, v : U \mid j(u = v) \iff j(mu = mv) \iff mu = mv \iff u = v.
\]

Ad (ii). Let \( X \in \mathcal{X} \). The closure \( j\delta_X \) of the diagonal predicate is an equivalence relation since \( j \) commutes with conjunction. For subobjects, let \( \rho : P \to X \) be a monomorphism into a separated object. We have

\[
u, v : U \mid j(u = v) \iff j(mu = mv) \iff mu = mv \iff u = v.
\]

The terminal object is separated since its diagonal predicate is already maximal.

In our case, the cartesian reflector is given by \( \Delta \Pi : \mathcal{X}[\mathcal{X}] \to \mathcal{X}[\mathcal{X}] \), and the corresponding reflective subcategory is equivalent to \( \mathbb{R} \). The associated universal closure operator \( j \) is best understood as an extension of \( \delta \) : \( \mathcal{X} \to \text{sub}(\mathcal{X}[\mathcal{X}]) \), an explicit description will be given in Lemma 3.4.30-(i).

We will now give a nice, ‘assembly style’ style representation of \( \text{Sep}_j(\mathcal{X}) \).

**Definition 3.4.29** Let \( \mathcal{X} \) be a totally connected fibered frame. We denote by \( \mathcal{X}_d \) (the dense part of \( \mathcal{X} \)) the subfibration of \( \mathcal{X} \) on the predicates \( \varphi \) such that \( \vdash_{\mathcal{X}} \pi \varphi \), or equivalently \( \vdash_{\mathcal{X}} \delta \pi \varphi \). \( \mathcal{X}_d \) is closed under finite meets since \( \pi \) preserves them, and we obtain an assembly-style presentation of the separated objects in \( \mathcal{X}[\mathcal{X}] \).

**Lemma 3.4.30** Let \( \mathcal{X} : [\mathcal{X}] \to \mathcal{X} \) be a totally connected fibered frame on an exact category.

(i) In terms of the representation of subobjects by strict predicates, the universal closure operation \( j \) on \( \mathcal{X}[\mathcal{X}] \) (see Diagram (3.4.2)) is given by \( \varphi \mapsto (c \mid \delta \pi \varphi c \land \rho c) \), where \( \varphi \in \mathcal{X}_C \) represents a subobject of \( (C, \rho) \).

(ii) \( \text{Sep}_j(\mathcal{X}[\mathcal{X}]) \simeq \mathcal{X}_d \) via the embedding \( \varphi \in \mathcal{X}_d(C) \mapsto (C, =|_\varphi) \).

(iii) \( \text{Sep}_j(\mathcal{X}[\mathcal{X}]) \) coincides up to equivalence with the terminal fiber \( \mathcal{P}(\mathcal{X})_1 \) of the positive completion of \( \mathcal{X} \) from Section 3.2.1.
Proof. Ad (i). Boring calculation.

Ad (ii). To see that the embedding is faithful, assume that \( \varphi \in X_C, \psi \in X_D \) are dense predicates. Since \( X \) is faithful, morphisms of type \( \varphi \to \psi \) in \( [X] \) can be identified with morphisms \( f : C \to D \) in \( X \) such that \( \varphi(x) \vdash \psi(fx) \). Take two such maps \( f, g : C \to D \) such that the induced maps of type \( (C, =|_\varphi) \to (D, =|_\psi) \) that are tracked by \( f \) and \( g \) are equal. Then we have \( \varphi_c, fc = d \vdash \psi gc = d \), and applying \( \pi \) gives \( fc = d \vdash \psi gc = d \) which implies that \( f = g \).

For fullness, observe that \( \Pi(C, =|_\varphi) \) is isomorphic to \( C \) if \( \varphi \) is dense (by construction of \( \Pi \) in the proof of 3.4.27). Given \( \phi : (C, =|_\varphi) \to (D, =|_\psi) \), a preimage of \( \phi \) can be obtained by composing \( \Pi \phi \) with these isomorphisms.

It remains to check that the essential image coincides with the separated objects. Let \( A \in \text{Sep}_j(X[X]) \). Then the monomorphism \( \eta_A : A \to \Delta \Pi A \sim (I, =) \) corresponds to a predicate \( \varphi \in X_I \) which is dense in \( X \) since \( \eta_A \) is dense and by (i).

Ad (iii). In Definition 3.2.10, \( \mathcal{P}(X)_1 \) is defined as full subcategory of \( X[X] \) on ‘discrete equivalence relations’; in the posetal case this means subobjects of constant objects \((I, =)\). \( j \)-separated objects, on the other hand, are precisely the subobjects of \( j \)-sheaves, and the proposition follows since constant objects coincide with sheaves.

3.4.3.1 Total connectedness and double negation

Lemma 3.4.31 Let \( \mathcal{P} : [\mathcal{P}] \to \mathcal{E} \) be a tripos on a topos. Then the embedding \( \delta : \text{sub}(\mathcal{E}) \to \mathcal{P} \) preserves \( \bot \).

Proof. Let \( ? : 0 \to I \) in \( \mathcal{E} \). We have to show that \( \exists ? \top \leq \bot \) in \( \mathcal{P}_I \), which is equivalent to \( \top \leq \bot \) in \( \mathcal{P}_0 \). But this follows from the fact that the predicates over 0 can be parametrized by \( \mathcal{E}(0, \text{Prop}) \), and there exists only one such map.

Lemma 3.4.32 Let \( \mathcal{P} : [\mathcal{P}] \to \mathcal{E} \) be a totally connected tripos on a topos. Then \( \delta : \text{sub}(\mathcal{E}) \to \mathcal{P} \) preserves implication.

Proof. Let \( U, V \subseteq I \) in \( \mathcal{E} \), \( \phi \in \mathcal{P}_I \). We have

\[
\begin{align*}
\phi & \leq \delta(U \Rightarrow V) & \text{iff} \\
\pi \phi & \leq U \Rightarrow V & \text{iff} \\
\pi \phi \land U & \leq V & \text{iff} \\
\pi \phi \land \pi \delta U & \leq V & \text{iff} \\
\pi(\phi \land \delta U) & \leq V & \text{iff} \\
\phi \land \delta U & \leq \delta V & \text{iff} \\
\phi & \leq \delta U \Rightarrow \delta V
\end{align*}
\]

The preceding proof works in general for strong monoidal reflections between monoidal closed categories. The lemma can also be seen as an analogue of the fact that for locally connected geometric morphisms \( \Delta : \mathcal{E} \to \mathcal{S} \) between toposes, the fibered functor \( \Delta : \text{cod}(\mathcal{S}) \to \text{gl}(\Delta(\mathcal{E})) \) preserves fiberwise cartesian closed structure [32, Proposition C3.3.1] (and actually the lemma can also be proved in the same way, without relying on the fact that \( \pi \) preserves finite meets).

Corollary 3.4.33 Let \( \mathcal{P} : [\mathcal{P}] \to \mathcal{S} \) be a totally connected tripos on an arbitrary topos. Then \( \delta : \text{sub}(\mathcal{E}) \to \mathcal{P} \) preserves negation.

Proof. This follows from the preservation of falsity and implication.
Let us recapitulate. If $\mathcal{P}$ is a totally connected regular tripos, then $\delta : \text{sub}(E) \to \mathcal{P}$ preserves $\exists, \land, \top$ for general reasons; furthermore it preserves $\lor$ since it is a right adjoint, and we just showed that it also preserves $\Rightarrow$ and $\bot$. This means that the only connective that is not preserved is disjunction $\lor$. Similarly, $\pi : \mathcal{P} \to \text{sub}(E)$ preserves $\land, \top$ by assumption, and $\exists, \bot, \lor$ since it is a left adjoint.

We can now show that the closure operation $\delta \pi$ on $\mathcal{P}$ coincides with double negation whenever the base is boolean.

**Theorem 3.4.34** Let $\mathcal{P} : [\mathcal{P}] \to E$ be a totally connected tripos on a boolean topos $E$. Then for any $A \in E$ and $\varphi \in \mathcal{P}_A$, we have $\delta \pi \varphi \equiv \neg \neg \varphi$.

**Proof.** Let $\varphi \in \mathcal{P}_A$ for $A \in E$. The first implication is shown as follows

\[
\begin{align*}
\varphi & \leq \delta \pi \varphi \\
\neg \neg \varphi & \leq \neg \neg \delta \pi \varphi \\
\neg \neg \varphi & \leq \delta \pi \varphi
\end{align*}
\]

and here is the proof of the second implication

\[
\begin{align*}
\neg \varphi \land \varphi & \leq \bot \\
\pi \neg \varphi \land \pi \varphi & \leq \bot \\
\pi \neg \varphi & \leq \neg \pi \varphi \\
\neg \varphi & \leq \delta \neg \pi \varphi \\
\neg \varphi & \leq \delta \pi \varphi \\
\neg \varphi \land \delta \pi \varphi & \leq \bot \\
\delta \pi \varphi & \leq \neg \neg \varphi
\end{align*}
\]

**Corollary 3.4.35** Let $\mathcal{P} : [\mathcal{P}] \to E$ be a totally connected tripos on a boolean topos $E$. Then we have

\[j(m) \equiv \neg \neg m \quad \text{for any } A \in E[\mathcal{P}] \text{ and } m \in \text{sub}(E[\mathcal{P}])_A,\]

where $j$ is the universal closure operation on $\text{sub}(E[\mathcal{P}])$ corresponding to the reflection $\Pi \dashv \Delta$.

**Proof.** Assume that $A \in E[\mathcal{P}]$ given by $(I, \rho)$. Relative to the representation of predicates in $\text{sub}(E[\mathcal{P}])_{(I, \rho)}$ by predicates in $\mathcal{P}_I$ which are strict with respect to $\rho$, we know by Lemma 3.4.30-(i) that $j$ is given by

\[\varphi \mapsto (i \mid (\delta \pi \varphi)(i) \land \rho(i)).\]

In the same way, $\neg \neg$ on $\text{sub}(E[\mathcal{P}])$ can be expressed in terms of $\neg \neg$ on $\mathcal{P}$ by

\[\varphi \mapsto (i \mid (\neg \neg \varphi)(i) \land \rho(i)).\]

The claim then follows from the theorem. 

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Chapter 4

Uniform preorders

Uniform preorders are representations of fibered preorders. More precisely, the category $\text{UOrd}$ of uniform preorders can be identified with a full subcategory of the locally ordered category $\text{PFib(Set)}$ of fibered preorders on $\text{Set}$. The definition of $\text{UOrd}$ is essentially a combination of ideas from Hofstra’s [23] work on basic combinatory objects (BCOs), and Longley’s [40] work on computability structures (C-structures).

Hofstra and Longley both introduce locally ordered categories of combinatory structures as frameworks for an abstract study of concepts from realizability (most importantly pcas, ordered pcas and typed pcas). $\text{UOrd}$ can be viewed as having Longley’s objects and Hofstra’s morphisms. We will see later that we can in fact recover Hofstra’s BCOs as a full subcategory, and Longley’s C-structures as a Kleisli-category of $\text{UOrd}$.

Retrospectively, Hofstra’s and Longley’s approach can be contrasted by saying that Longley works with relations, and Hofstra with (partial) functions. Our approach is to take relations as structuring data of the objects, and functions as morphisms. This choice is justified by the fact that $\text{UOrd}$ is equivalent to a subcategory of $\text{PFib(Set)}$ that can be characterized in a concrete way (see Lemma 4.2.2).

We compose relations like functions and denote their composition by $\circ$ or by juxtaposition, i.e. if $(x, y) \in r$ and $(y, z) \in s$, then $(x, z) \in sr$. In particular, we allow composition of relations with functions. For a relation $r$, $r^\circ$ denotes its opposite relation.

4.1 Definitions

**Definition 4.1.1** (i) A uniform preorder (or C-structure) $\mathfrak{A}$ is a triple $\mathfrak{A} = (I, A, R)$, where $A = (A_i)_{i \in I}$ is a family of sets, and $R = (R_{ij})_{i, j \in I}$, $R_{ij} \subseteq P(A_i \times A_j)$ is a family of sets of relations, subject to the following axioms.

(a) $i, j \in I$, $r \in R_{ij}$, $s \subseteq r \Rightarrow s \in R_{ij}$

(b) $i \in I \Rightarrow \text{id} \in R_{ii}$

(c) $i, j, k \in I$, $r \in R_{ij}$, $s \in R_{jk} \Rightarrow sr \in R_{ij}$

(ii) A monotone map between uniform preorders $(I, A, R)$, $(J, B, S)$ is a pair $(u : I \to K, (f_i : A_i \to B_{ui})_{i \in I})$ such that $r \in R_{ij}$ implies $f_j^* r f_i^\circ \in S_{ui, uj}$.

1. $f_j^* r f_i^\circ$ is just another way of writing $(f_i \times f_j)(r)$. We chose this representation since it is most natural in some calculations in Section 4.9.
(iii) Given \((u, f), (v, g) : (I, A, R) \to (J, B, S)\), we define \((u, f) \leq (v, g)\) iff for all \(i \in I\) we have \(\{ (f_i, a, g_o) \mid a \in A_i \} \in S_{i, a, o}\).

Uniform preorders and monotone maps form an order-enriched category \(\text{UOrd}\).

We call the set \(I\) of a uniform preorder \((I, A, R)\) its set of sorts. If \(I\) has exactly one element, we simply write \((A, R)\) for the uniform preorder. We sometimes refer to one-sorted uniform preorders as basic relational objects (BROs).

It is often convenient to describe a uniform preorder by giving only a generating system of relations. To this end, we introduce the concept of base.

**Definition 4.1.2** Let \(A = (A_i)_{i \in I}\) be a family of sets. A base for a uniform preorder structure on \(A\) is a family \((R_{ij})_{i,j \in I}\) with \((R_{ij} \subseteq P(A_i \times A_j))\) of sets of binary relations such that

\[
- i \in I \implies \exists r \in R_{ii}, \text{id}_{A_i} \subseteq r
- i, j, k \in I, \ r \in R_{ij}, \ s \in R_{jk} \implies \exists t \in R_{ik}, \ sr \subseteq t
\]

Given such a base, the family \(\downarrow R = (\downarrow R_{ij})_{ij}\) with \(\downarrow R_{ij} = \{ r \subseteq A_i \times A_j \mid \exists s \in R_{ij}. \ sr \subseteq t\}\) is a uniform preorder structure on \(A\), and we call \((I, A, \downarrow R)\) the uniform preorder generated by \((I, A, R)\).

Longley defines C-structures directly in terms of bases, without imposing the downward closed-ness condition. This has the advantage that it generalizes to predicative contexts, since predicatively it makes sense to talk about families of subsets, but not about downward closed such families. Products (Lemma 4.5.1-(i)) are also most easily defined in terms of bases, since we can avoid an additional downward closure operation. In the reconstruction of uniform preorders from fibered preorders in the proof of 4.2.5, on the other hand, the downward closure condition comes for free.

**Examples 4.1.3**

(i) Ordinary preorders can be viewed as uniform preorders. More precisely, given a preorder \((D, \leq)\), the singleton set \(\{ \leq \}\) is a base of a uniform preorder structure on \(D\) in the sense of Definition 4.1.2 (this follows directly from reflexivity and transitivity), whence \((D, \downarrow \{ \leq \})\) is a uniform preorder. Moreover, the assignment \((D, \leq) \mapsto (D, \downarrow \{ \leq \})\) extends to a 2-functor of type

\[
\text{Ord} \to \text{UOrd}
\]

which is a local equivalence.

(ii) (a) Given a pca \(A\), the partial functions of the form \((a \cdot -) : A \to A\) for \(a \in A\) form a base of a uniform preorder structure on \(A\) (the closure under composition follows from functional completeness). We denote the generated uniform preorder by \((A, R(A))\).

(b) Given a typed pca \((I, A)\), the partial functions of the form \((a \cdot -) : A_i \to A_j\) for \(a \in A_{\alpha} \subseteq A\) form a base of a uniform preorder structure on \((I, A)\). We denote the generated uniform preorder by \((I, A, R(A))\).

(c) Given an inclusion \(A_\# \subseteq A\) of pcas, the partial functions of the form \((a \cdot -) : A \to A\) for \(a \in A_\#\) form a base of a uniform preorder structure on \(A\). We denote the generated uniform preorder by \((A, R(A_\#))\).

(d) In the same way, an inclusion \((I, (A_\#, i \subseteq A_i)_{i \in I})\) of typed pcas induces a uniform preorder \((I, A, R(A_\#))\).

(iii) Denote by \(\text{Prim}\) the set of unary primitive recursive functions. Since \(\text{Prim}\) is closed under composition, it is a base of a uniform preorder structure on \(\mathbb{N}\). We call the induced uniform preorder \((\mathbb{N}, \downarrow \text{Prim})\) the primitive recursive uniform preorder.

We will see later that \((\mathbb{N}, \downarrow \text{Prim})\) has finite meets (Section 4.6), but is not relationally complete (Section 4.10). We can perform a construction analogous to the construction of
the effective topos using only primitive recursive functions, but the resulting category will only be a pretopos. \(\diamond\)

4.2 Uniform preorders and fibered preorders

**Definition 4.2.1** For any uniform preorder \(\mathfrak{A} = (I, A, R)\), we define a fibered preorder

\[
\text{ufam}(\mathfrak{A}) : \text{UFam}(\mathfrak{A}) \to \text{Set}
\]

as follows.

- a predicate on a set \(M\) is a pair \((i \in I, \phi : M \to A)\)
- given \((i, \phi), (j, \gamma) \in \text{ufam}(\mathfrak{A})_M\), we define

\[
(i, \phi) \leq (j, \gamma) \iff \{(\phi m, \gamma m) \mid m \in M\} \in R_{ij}
\]
- reindexing is given by precomposition \(\diamond\)

We will often omit the indices when talking about predicates in \(\text{ufam}(\mathfrak{A})\) and just write \(\phi\) instead of \((i, \phi)\).

**Lemma 4.2.2** The assignment \(\mathfrak{A} \mapsto \text{ufam}(\mathfrak{A})\) gives rise to a 2-functor

\[
\text{ufam}(-) : \text{UOrd} \to \text{PFib}(\text{Set})
\]

into posetal fibrations on \(\text{Set}\) which is a local equivalence. This 2-functor fits into a commutative (up to isomorphism) triangle

\[
\begin{align*}
\text{Ord} & \xrightarrow{\text{fam}(-)} \text{UOrd} \\
& \searrow \Downarrow \infty \nearrow \text{ufam}(-) \\
& \phantom{\searrow} \downarrow \Downarrow \infty \phantom{\nearrow} \uparrow \text{PFib}(\text{Set})
\end{align*}
\]

of local equivalences, where the horizontal arrow is defined in Example 4.1.3-(i), and the diagonal arrow is the posetal version of the family construction for categories from Definition 2.1.4.

**Proof.** It is easy to see that \(\text{ufam}(-)\) is functorial. To see that it is locally order-reflecting, consider monotone maps \((u, f), (v, g) : (I, A, R) \to (J, B, S)\) such that \(\text{ufam}(u, f) \leq \text{ufam}(v, g)\). Then we have in particular that for any \(i \in I\), \(\text{ufam}(u, f)(\text{id}_{A_i}) \leq \text{ufam}(v, g)(\text{id}_{A_i})\), which implies \((u, f) \leq (u, g)\).

To see that \(\text{ufam}(-)\) is essentially full, let \(F : \text{ufam}(I, A, R) \to \text{ufam}(J, B, S)\). For each \(i \in I\), \(F(i, \text{id}_{A_i})\) is a pair of an element \(u(i) \in J\) and a \(u(i)\)-valued predicate on \(A_i\), i.e. a map \(f_i : A_i \to B_{u(i)}\). We claim that the assignment \(i \mapsto u(i)\) together with the maps \((f_i)_{i \in I}\) gives the desired monotone map. Indeed, for \(h : M \to A_i\), we have \(F(i, h) = F((i, \text{id}_{A_i}) \circ h) \cong F(i, \text{id}_{A_i}) \circ h = (u(i), f_i) \circ h\), thus the action of \(F\) is given by postcomposition with \(F(\text{id}_{A_i})\).

It remains to show that \((u, f)\) is monotone. Let \(r \in R_{u'v'}\). Then we have projection mappings \(\pi_1 : r \to A_i, \pi_r : r \to A_v\) such that \(\pi_1 \leq \pi_r\), as predicates in \(\text{ufam}(I, A, R)\). By monotonicity of \(F\) we deduce that \(f_i \pi_1 \cong F(\pi_1) \leq F(\pi_r) \cong f_r \pi_r\), which means that \(\{(f_i a, f_r b) \mid (a, b) \in r\} \in S_{u', u'v'}\) as required.

The commutativity of the triangle of 2-functors is straightforward. \(\blacksquare\)

2. More precisely, it seems to be a list-arithmetic pretopos in the sense of [41].
Example 4.2.3 Given a typed pca \((I, A)\), the fibration ufam\((I, A, R(A))\) obtained by applying the family construction to the uniform preorder \((I, A, R(A))\) defined in Example 4.1.3-(ii)b is precisely the fibration ufam\((I, A)\) from Definition 2.1.6.

In the same way, for an untyped pca \(A\), ufam\((A, R(A))\) is equal to ufam\((A)\).

Definition 4.2.4 Let \(A : |A| \to \text{Set}\) be a fibered preorder. We way that a family \((\iota_i \in A_{A_i})_{i \in I}\) is a generic family of predicates, if for every set \(M\) and every predicate \(\phi \in A_M\) there exists \(i \in I\) and \(f : M \to A_i\) such that \(\phi \equiv f^*\iota_i\).

If a generic family comprises exactly one predicate, we call it a generic predicate.

Lemma 4.2.5 A fibered preorder \(A\) can up to equivalence be represented by a uniform preorder iff it has a generic family \((\iota_i \in A_{A_i})_{i \in I}\) of predicates and is a pre-stack with respect to the regular topology, which means that \(e^*\phi \leq e^*\psi\) implies \(\phi \leq \psi\) for surjective \(e : J \to I^3\).

Proof. Given a uniform preorder \((I, A, R)\), a generic family of predicates for ufam\((I, A, R)\) is given by the family of identity maps \((\text{id}_{A_i})_{i \in I}\).

Conversely, let \(A\) be a posetal pre-stack with generic family of predicates \(\iota_i \in P_{A_i}\). We define a uniform preorder structure on the family \((A_i)_{i \in I}\) by setting \(R_{ij} = \{r \subseteq A_i \times A_j \mid \pi^*_i r \leq \pi^*_j r\ \text{in} \ A_r\}\), where \(\pi_i : r \to A_i, \pi_j : r \to A_j\) are the projections. To see that ufam\((I, A, R) \simeq A\), it remains to show that for \(f : M \to A_i, g : M \to A_j\) we have \(gf^* \in R_{ij}\) iff \(f^*\iota_i + g^*\iota_j\). This can be seen by considering the epi-mono factorization \(M \to gf \hookrightarrow A_i \times A_j\) of \((f, g)\) and making use of the fact that \(A\) is a pre-stack.

The preceding lemma implies in particular that any regular tripos can be represented by a uniform preorder.

4.3 Functional uniform preorders and modesty

Definition 4.3.1 We call a uniform preorder \((I, A, R)\) functional if all relations \(r \in R_{ij}\) \((i, j \in I)\) are functional.

A uniform preorder given by a base (Definition 4.1.2) is of course functional whenever all relations in the base are functional. In particular, the uniform preorders induced by pcas (Example 4.1.3-(ii)) and the primitive recursive uniform preorder (Example 4.1.3-(iii)) are all functional.

Functionality is an ‘evil’ property, in that it is not closed under equivalence in the locally ordered category \(\text{UOrd}\). There is, however, a stronger condition which is closed under equivalence.

Lemma 4.3.2 Let \((2, \downarrow\{\leq\})\) be the uniform preorder associated to the preorder \(2 = \{0 \leq 1\}\), and let \((A, R)\) be a uniform preorder. Then every monotone map \(f : (2, \downarrow\{\leq\}) \to (B, S)\) factors (up to isomorphism \(^4\)) through the terminal uniform preorder.

Proof. This follows from the fact that the image \((f \times f)(\leq)\) of the order relation is functional since it is in \(R\).

The previous lemma shows how different functional uniform preorders are from ordinary preorders – one can explore the structure of a preorder by looking at maps from \(2\) (or any other linear order) into the preorder. If we do the same thing with a functional uniform preorder, we can’t see

3. The pre-stack condition is redundant in the presence of choice.
4. The ‘up to isomorphism’ is only necessary to make the property stable under equivalence as promised.
anything. The idea of studying preorders by sampling them with simple shapes (chains) is known
as the ‘nerve construction’ which associates a simplicial set to a given preorder (or category).
Functional uniform preorders have trivial nerves, and it does not seem to be possible to think
about them in a ‘geometric’ way.

Lemma 4.2.5 gives a criterion for a fibered preorder to be representable by a uniform preorder.
If we want the induced uniform preorder to be functional, we have to add an additional condition.

**Definition 4.3.3** Let $A : |A| \to \mathbb{R}$ be a posetal pre-stack. We call a predicate $\mu \in A_I$ modest,
if for any span \( J \xrightarrow{e} K \xrightarrow{f} I \) where $e$ is a regular epi, and any $\varphi \in A_J$ such that $e^* \varphi \leq f^* \mu$,
there exists a (necessarily unique) $h : J \to I$ such that $he = f$ and $\varphi \leq h^* \mu$.

**Remark 4.3.4** There is a concrete and an abstract intuition about modesty. The concrete one
is that in a realizability tripos over a pca, a predicate is modest iff distinct elements have disjoint
sets of reindexers (in particular, modest predicates are not stable under reindexing!). Abstractly
modesty is about functionality – the previous definition states that the relation induced by the
span $(e, f)$ is functional.

Retrospectively, the use of the word ‘modest’ for the above concept doesn’t seem to be such
a good idea after all – the reason is that in realizability (say over a pca $A$) it only coincides with
the intended meaning when applied to the fibration $ufam(A)$, not for the tripos $rt(A)$ – there
the empty truth value causes problems.

Hyland, Robinson and Rosolini [29] define a discrete object in the effective topos to be an
object whose terminal projection is right orthogonal (Definition 2.3.18-(i)) to $\Omega \to 1$ (which is
equivalent to being orthogonal to $\Delta^2 \to 1$), and define a modest object to be a discrete object
which is separated for the $\neg\neg$-topology. We give a generalization of their concept of discreteness
in Definition 4.11.4.

Summarizing, while the concepts of modest and discrete used in this work are related, their
relation differs from their relation in [29], which is mainly due to the fact that our use of ‘modest’
is a bit unfortunate.

In analogy to Lemma 4.2.5, we can now show the following.

**Lemma 4.3.5** A posetal pre-stack $A : |A| \to \textbf{Set}$ is induced by a functional uniform preorder iff
it has a generic family $(\iota_i \in A_{A_i})_{i \in I}$ of modest predicates.

**Proof.** Let $(I, A, R)$ be a functional uniform preorders. We have to show that the predicates $id_{A_i}$
for $i \in I$ are modest. Take a span $M \xrightarrow{e} N \xrightarrow{f} I$, and $\varphi : M \to A_j$ such that $e^* \varphi \leq f^* id_{A_i}$. Then
$(\{g(\varepsilon n), fn\} \mid n \in N) \subseteq A_j \times A_i$ is in $R_{ij}$ and thus functional, which implies that the
relation $\{(gen, fn) \mid n \in N\} \subseteq M \times A_i$ is functional as well. The second relation is furthermore
total since $e$ is surjective, and gives the desired mediator.

Conversely, assume that $A$ has a generic family $(\iota_i \in A_{A_i})_{i \in I}$ of modest predicates. The
uniform preorders structure on $(A_i)_{i \in I}$ is then given by $R_{ij} = \{r \subseteq A_i \times A_j \mid \pi^*_i \iota_i \vdash \pi^*_j \iota_j \in A_r\}.$
To see that modesty of $\iota_j$ implies that all $r \in R_{ij}$ are functional, consider the following diagram.

\[
\begin{array}{c}
\pi^*_i \iota_i \\
\downarrow e \\
\downarrow f \\
\iota_j \\
\end{array}
\]

\[
\begin{array}{c}
r \\
\downarrow h \\
\Downarrow A_j \\
\end{array}
\]

\[
\begin{array}{c}
r \\
\Downarrow A_i \\
\end{array}
\]
The existence of $h$ follows from the facts that $t_j$ is modest and that $(me^*) t_i \leq f^* t_j$. We can deduce that $e$ is an iso since $me$ and $f$ are jointly monic; and thus $r$ is functional. ■

4.4 Uniform preorders and BCOs

We know how to compare uniform preorders with Hofstra’s BCOs [23], since BCO as well as UOrd can be identified with full subcategories of PFib(Set). Given a BCO $(A, \leq, F)$, the relations $(\leq) \circ f$ with $f \in F$ generate a uniform preorder structure on $A$ which induces the same fibered preorder, thus BCO is a subcategory of UOrd. In the following, we lay out some deliberations that try to clarify the status of BCOs among uniform preorders.

**Lemma 4.4.1** Given a one sorted uniform preorder $(A, R)$, the relations $r \in R$ with id $\subseteq r$ are directed with respect to inclusion and their union is a preorder.

**Proof.** Given $r, s \supseteq \text{id}$ in $R$, an upper bound is given by $sr$. It is evident that the union is reflexive, for transitivity we have

$$\bigcup_{s \supseteq \text{id}} s \circ r \subseteq \bigcup_{r, s \supseteq \text{id}} sr \subseteq \bigcup_{r \supseteq \text{id}} r.$$ ■

**Definition 4.4.2** If the preorder $\leq$ from the previous lemma is contained in $R$, we call $(A, R)$ condensable, and we call $\leq$ its condensate. ♦

Discrete BCOs are condensable; their condensate is the identity. In general, however, the ordering on a BCO is not necessarily its condensate. For example, let $(A, \leq, F)$ be a BCO with least element $\bot$ for $\leq$, where $F$ consists of all total monotone functions. Then the constant $\bot$ function witnesses arbitrary inequalities, thus the BCO is biterminal, and its condensate is the indiscrete preorder.

If a uniform preorder $(A, R)$ contains a preorder $(\leq) \in R$, all relations $r \in R$ can be completed to order theoretic distributors $(\leq) \circ f : (A, \leq) \rightarrow (A, \leq)$ which are still in $R$. Since $r \subseteq (\leq) \circ f$, $R$ is generated by distributors. We call such a distributor $\phi$ partially functional, if for every $a \in A$, the upper set $\{ b \mid (a, b) \in \phi \}$ is either empty or representable. Partially functional distributors are precisely the distributors of the form $(\leq) f$ for partial monotone functions $f$ with downward closed domain, as explained in [6] for partial functors between categories. Using these techniques, we can characterize uniform preorders arising from BCOs.

**Lemma 4.4.3** A one sorted uniform preorder $(A, R)$ is induced by a BCO iff there exists a preorder $(\leq) \in R$ such that $R$ is generated by partially functional $\leq$-distributors.

**Proof.** Given a BCO $(A, \leq, F)$, the generators $(\leq) f$ for $f \in F$ of the uniform preorder structure are partially functional by the remarks preceding the lemma. Conversely, if a one sorted uniform preorder $(A, R)$ contains a preorder and is generated by partially functional distributors, then it is easy to see that the corresponding monotone partial functions form a BCO structure equivalent to $(A, R)$. ■

4.5 Closure properties of UOrd

**Lemma 4.5.1** (i) UOrd has small products.

(ii) UOrd has small coproducts.
(iii) **UOrd** is cartesian closed.

(iv) We can define an involution operation

\((-)_{\text{op}} : \text{UOrd}^{\text{co}} \to \text{UOrd}\),

that corresponds to taking the fiberwise opposite on the level of fibered preorders.

**Proof.** Ad (i). This is also true for one sorted uniform preorders, and to understand the proof it might be helpful to consider first the one sorted case to reduce the amount of indices.

Let \((I_l, A_l, R_l)_{l \in L}\) be a family of uniform preorders, and assume that for each \(l \in L\), we are given \(i_l, j_l \in I_l\) and \(r_l \in R_{i_l, j_l}\). Then the set

\[
\left\{ \left( (a_l), (b_l) \right) \in \prod_l A_{i_l} \times \prod_l A_{j_l} \mid \forall l. (a_l, b_l) \in r_l \right\}
\]

is a binary relation on \(\prod_l A_{i_l} \times \prod_l A_{j_l}\), and the set of all relations defined in this way is a basis for a uniform preorder structure on the family

\[
\left( \prod_l I_l, \left( \prod_l A_{i_l}, \left( \prod_l A_{j_l} \mid (i_l) \in \prod_l I_l \right) \right) \right).
\]

The product of the family \((I_l, A_l, R_l)_{l \in L}\) is the uniform preorder generated by this basis.

Ad (ii). The coproduct of a family \((I_l, A_l, R_l)_{l \in L}\) of uniform preorders is given by

\[
\left( \bigoplus_l I_l, \left( A_{i_l} \mid (l, i) \in \bigoplus_l I_l \right), R \right),
\]

where

\[
R_{(l, i), (m, j)} = \begin{cases} R_{i_l} & \text{if } l = j \\ \emptyset & \text{otherwise} \end{cases}
\]

Ad (iii). This is nontrivial and originally due to Longley [40], who showed that the locally ordered CSTRICT of computability structures is in his words ‘almost cartesian closed’ (it is not really cartesian closed, since it is a Kleisli category of **UOrd** as we show in Section 4.7.3, and cartesian closure is generally not preserved by Kleisli constructions). Our proof is an adaption of Longley’s to uniform preorders, expressed in terms of fibrations.

We show that the image of **UOrd** is an exponential ideal in **P Fib(Set)**. For general fibrations \(\mathcal{E}, \mathcal{D} \in \text{Fib(Set)}\), their exponential is given by \((\mathcal{D}^{\mathcal{E}})_{I} = \text{Fib(Set/I)}(\mathcal{E}/I, \mathcal{D}/I)\) (see [57, Section 4]). Instantiating by ufam(\(\mathfrak{A}\)), ufam(\(\mathfrak{B}\)) for uniform preorders \(\mathfrak{A} = (I, A, R), \mathfrak{B} = (J, B, S)\), we get the following description of the fibration ufam(\(\mathcal{B}\))_{ufam(\mathcal{A})}:

- Predicates on \(K\) are pairs \((u, f) = (u, (f_i))_i\) with \(u : I \to J\) and \(f_i : K \times A_i \to B_{u_i}\) for every \(i \in I\) such that

\[
\forall i, i' \in I, r \in R_{u_i}. \{ (f_i(k, a), f_{i'}(k, a')) \mid (a, a') \in r, k \in K \} \in S_{u_i, u_{i'}}.
\]

- \((u, f) \leq (v, g)\) if

\[
\forall i \in I. \{ (f_i(k, a), g_i(k, a)) \mid k \in K, a \in A_i \} \in S_{u_i, v_i}.
\]
By Lemma 4.2.5, it suffices to show that this fibration has a generic family of predicates (the pre-stack condition is always preserved by exponentiation).

Let \( u : I \to J \). An \( L \)-indexed family \( ((f^u_l)_{l \in L})_{l \in L} \) of monotone maps from \((I, A, R)\) to \((J, B, S)\) over \( u \) is called equimonotone, if

\[
\forall i, i' \in I \forall r \in R_{ij} \cdot \{(f^u_l(a), f^u_l(b)) \mid (a, b) \in R, l \in L\} \in S_{ui, ui'}\]

For each equimonotone family \( (L, ((f^u_l))_{l \in L}) \) over \( u \), we can define a predicate \((u, g) \in \left( \text{ufam}(B) \right)_{L} \) by

\[
g_i(l, a) = f^u_l(a),
\]

and it is easy to see that the collection of these predicates where \((u, g)\) ranges over all equimonotone families is jointly generic.

Ad (iv). The opposite of a uniform preorder \((I, A, R)\) is given by \((I, A, R^{op})\) where \( R^{op}_{ij} = \{ r^{op} \mid r \in R_{ij} \} \)

We observe that the construction of products restricts to the one sorted case, but not the construction of coproducts and exponentials. The construction of exponentials is particularly interesting, and an obvious question is which properties of uniform preorders are stable under exponentiation. What is the exponential of two pcas? What about relational completeness? I haven’t examined this at all yet.

### 4.6 Finitely complete uniform preorders

Hofstra observed that the 2-categorical approach gives a well behaved and useful notion of finite completeness for BCOs. We can do the same for uniform preorders.

**Definition 4.6.1** A uniform preorder \( \mathfrak{A} = (I, A, R) \) is called finitely complete, or a uniform (meet-)semilattice, if the maps

\[
\delta : \mathfrak{A} \to \mathfrak{A} \times \mathfrak{A} \quad \text{and} \quad ! : \mathfrak{A} \to 1
\]

have right adjoints

\[
\delta \dashv (*, \wedge) : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A} \quad \text{and} \quad ! \dashv (1, \top) : 1 \to \mathfrak{A}.
\]

Let us spell this out. Concretely, the existence of the right adjoints means that there exist maps and elements

\[
1 \in I \quad \quad (- * -) : I \times I \to I
\]

\[
\top \in A_i \quad (- \wedge_{ij} -) : A_i \times A_j \to A_{i+j} \quad \text{for } i, j \in I
\]

such that

(i) \((*, \wedge)\) is monotone:

\[
\forall i, j, k, l \forall r \in R_{ij} \forall s \in R_{kl} \cdot (\wedge \times \wedge)(r \otimes s) \in R_{ij,k+l} \quad (4.6.1)
\]

(ii) \(\delta \circ (*, \wedge) \leq \text{id}_{\mathfrak{A} \times \mathfrak{A}}\) (which is equivalent to \((*, \wedge) \leq \pi_i\) and \((*, \wedge) \leq \pi_j): \n
\[
\forall i, j \cdot \{(a \wedge_{ij} b, a) \mid a \in A_i, b \in A_j\} \in R_{i+j,i} \quad \text{and} \quad (4.6.2)
\]

\[
\forall i, j \cdot \{(a \wedge_{ij} b, b) \mid a \in A_i, b \in A_j\} \in R_{i+j,j} \quad (4.6.3)
\]

5. see Section 4.10

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(iii) \( \text{id}_A \leq (\ast, \land) \circ \delta: \)
\[
\forall i. \{(a, a \land_{ij} a) \mid a \in A_i\} \in R_{i,i+1}
\]
\[(4.6.4)\]

(iv) \( \text{id}_A \leq (1, \top) \circ !: \)
\[
\forall i. \{(a, \top) \mid a \in A_i\} \in R_{i,1}
\]
\[(4.6.5)\]

The conditions for the monotonicity of \((1, \top)\) and for \(! \circ (1, \top) \leq \text{id}_1\) are vacuous.

Since the embedding \(\text{ufam}(\ast) : \text{UOrd} \rightarrow \text{PFib}\) of uniform preorders into fibrations is a local equivalence (Lemma 4.2.2) and preserves finite products, \((I, A, R)\) is finitely complete iff \(\text{ufam}(I, A, R)\) is so, i.e. the fibers have chosen finite meets which are preserved up to isomorphism by reindexing. Hence, in particular uniform preorders induced by preorders (Example 4.1.3-(i)) are finitely complete iff the preorder has finite meets, as one would expect.

Moreover, the fibered monotone map \(\text{ufam}(u, f) : \text{ufam}(A) \rightarrow \text{ufam}(B)\) corresponding to a monotone map \((u, f) : A \rightarrow B\) preserves finite meets iff \((u, f)\) is compatible with the finite limit structure in the sense that the diagrams

\[
\begin{array}{ccc}
A \times B & \xrightarrow{(u,f) \times (u,f)} & B \times B \\
\downarrow \land & & \downarrow \land \\
A & \xrightarrow{(u,f)} & B
\end{array}
\]

commute up to isomorphism, in which case we will also say that \((u, f)\) preserves finite meets. This leads us to the following definition.

**Definition 4.6.2** \(\land\)-\text{UOrd} is the locally ordered category of finitely complete uniform preorders, and finite meet preserving monotone maps.

For functional uniform preorders, the functions \(\land_{ij} : A_i \times A_j \rightarrow A_{i \land j}\) can be viewed as a ‘recursive pairing functions’ — in particular we have:

**Lemma 4.6.3** If \(A = (I, A, R)\) is a finitely complete functional uniform preorder, then the functions \(\land_{ij} : A_i \times A_j \rightarrow A_{i \land j}\) are injective.

**Proof.** From (4.6.2) and (4.6.3) we know that \(\{(a \land_{ij} b, a) \mid a \in A_i, b \in A_j\} \in R_{i,j,i}\) and \(\{(a \land_{ij} b, b) \mid a \in A_i, b \in A_j\} \in R_{i,j,j}\). By assumption these relations are functional, which lets us deduce that \(a = a'\) and \(b = b'\) whenever \(a \land_{ij} b = a' \land_{ij} b'\).

**Examples 4.6.4**

(i) As remarked above, the uniform preorder \((D, \leq\)}\) associated to a preorder \((D, \leq)\) is finitely complete iff \((D, \leq)\) has finite meets.

(ii) Given a pca \(A\), the associated uniform preorder \((A, R(A))\) (Example 4.1.3-(ii)a) is finitely complete. This can be deduced from the fact that the associated fibered preorder \(\text{ufam}(A, R(A))\) can be identified with the subfibration of the realizability tripos \(rt(A)\) on singleton valued predicates, and those are closed under finite meets which are given by pairing.

The same argument generalizes to inclusions of pcas (Example 4.1.3-(ii)c) and (inclusions of) typed pcas (Example 4.1.3-(ii)b,(ii)d).

(iii) The uniform preorder \((N, \downarrow \text{Prim})\) from Example 4.1.3-(iii) is finitely complete, the meet \(\land : N \times N \rightarrow N\) is given by any primitive recursive pairing function.

Since \(\text{UOrd}\) has small products, we can also talk about infinite meets and joins in a uniform preorder. In particular, we call a uniform preorder \(A\) small (co)complete, if for all sets \(I\), the diagonal embedding \(A \rightarrow A^I\) has a right (left) adjoint. For preorders, small meets and joins allow to define quantification in the associated fibrations, but for uniform preorders the two concepts diverge. We explain how to handle quantification in Section 4.7.
4.6.1 Relational clones

By viewing morphisms $f: A_1 \times \cdots \times A_n \to B$ in a finite product category $\mathcal{C}$ as ‘multi’-morphisms with $n$ inputs and one output, any cartesian category may be viewed as a (cartesian [48]) multicategory. In the same way, any meet-semilattice may be viewed as a ‘multi’-ordering, allowing comparisons like $a_1, \ldots, a_n \leq b$, and it is not difficult to define a notion of ‘fibered multi-ordering’ which is induced by finitely complete uniform preorders. On the relational level, the corresponding structure is that of a (many-sorted) relational clone, which is most intuitive in the functional case, and will be helpful in Section 4.10.

Definition 4.6.5 A relational clone on a family of sets $(A_i)_{i \in I}$ is a family

$$(C_{i_1, \ldots, i_n, j} \subseteq P((A_{i_1} \times \cdots \times A_{i_n}) \times A_j) \mid n \in \mathbb{N}, i_1, \ldots, i_n, j \in I)$$

of sets of $(n+1)$-ary relations which

- contains all projections $\pi_l: A_{i_1} \times A_{i_n} \to A_{i_l}$ for $1 \leq l \leq n \in \mathbb{N}$ viewed as relations $\pi_l \in C_{i_1, \ldots, i_n, i_l}$, and
- is closed under composition in the sense that whenever $s \in C_{j_1, \ldots, j_n, k}$ and $r_l \in C_{i_1, \ldots, i_m, j_l}$ for $1 \leq l \leq n$, then the relation

$$s \circ (r_1, \ldots, r_n) \overset{\text{def}}{=} \{(a_1 \ldots a_m, c) \mid \exists b_1 \ldots b_n. s(b_1 \ldots b_n, c) \land \bigwedge_{1 \leq l \leq n} r_l(a_1 \ldots a_m, b_l)\}$$

is contained in $C_{i_1, \ldots, i_m, k}$.

Given a finitely complete uniform preorder $(I, A, R)$, the natural way to define a clone-like structure on $(A_i)_{i \in I}$ is by taking $C_{i_1, \ldots, i_n, j}$ to be the set of relations which is generated via downward closure by the relations

$$\{(a_1 \ldots a_n, b) \mid r(a_1 \land \cdots \land a_n, b)\} \quad \text{for} \quad r \in R_{i_1 \cdots i_n, j}. \quad (4.6.6)$$

The idea underlying this construction is due to Hofstra [23, Section 6], who uses finite limit structure on BCOs to talk about ‘computable’ partial functions in several variables.

The following lemma is purely technical.

Lemma 4.6.6 For a finitely complete uniform preorder $(I, A, R)$, the previously defined system $(C_{i_1, \ldots, i_n, j}i_1, \ldots, i_n, j)$ of sets of relations is a relational clone, and does not depend on the choice of $n$-ary meet maps

$$- \land \cdots \land - : A_{i_1} \times \cdots \times A_{i_n} \to A_{i_1 \cdots i_n}.$$ used to define the generators in (4.6.6).

Examples 4.6.7

- The relational clone associated to (the uniform preorder associated to) the first Kleene algebra $\mathcal{K}_1$ (Example A.1.2) consists of all partial subfunctions of $n$-ary partial recursive functions.
- The relational clone associated to the primitive recursive uniform preorder $(\mathbb{N}, \downarrow \text{Prim})$ (Example 4.1.3-(iii)) contains all partial subfunctions of $n$-ary primitive recursive functions.
- The relational clone associated to a meet-semilattice $A$ consists of all $n + 1$-ary relations $r \subseteq A^n \times A$ satisfying $a_1 \land \cdots \land a_n \leq b$ for all $(a_1 \ldots a_n, b) \in r$. 

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4.7 The quantification monads

This section is about the representation of the posetal D-construction from Section 3.4.2.1 in terms of uniform preorders.

This ‘uniform-preorder version’ was actually my starting point, it is based on the construction with the same name and characteristics used by Hofstra [23] in the context of BCOs.

**Definition 4.7.1** Let \((I,A,R)\) be a uniform preorder. For \(i,j \in I\) and \(r \in R_{ij}\), we define \([r] \subseteq PA_i \times PA_j\) by
\[
[r] = \{ (M,N) \mid \forall m \in M \exists n \in N . r(m,n) \}.
\]
The family \(\{ [r] \mid r \in R_{ij} \}_{ij \in I}\) is a base for a uniform preorder structure on \((I,\langle PA_i \rangle_{i \in I})\), and we denote the generated uniform preorder by \(D(I,A,R)\).

Let us clarify the connection between the D-construction on uniform preorders and the D-construction on fibered preorders:

**Lemma 4.7.2** For any uniform preorder \(\mathfrak{A} = (I,A,R)\), we have
\[
ufam(D\mathfrak{A}) \simeq D(ufam(\mathfrak{A})).
\]
In particular, ufam(D\(\mathfrak{A}\)) has existential quantification, and if \(\mathfrak{B}\) is a second uniform preorder such that ufam(\(\mathfrak{B}\)) has existential quantification, then we have
\[
UOrd(\mathfrak{A}, \mathfrak{B}) \simeq \exists\text{-}UOrd(D\mathfrak{A}, D\mathfrak{B}),
\]
where \(\exists\text{-}UOrd(D\mathfrak{A}, D\mathfrak{B})\) is the preorder of monotone maps whose associated fibered monotone maps preserve existential quantification.

**Proof.** Given a set \(M\), a predicate in ufam(D\(\mathfrak{A}\))\(_M\) is a function \(\varphi : M \to PA_i\), whereas a predicate in \(D(ufam(\mathfrak{A}))\)_\(M\) is span \(\mathcal{N} \xleftarrow{\psi} N \xrightarrow{\varphi} A_i\). Given a predicate \(\psi \in ufam(D\mathfrak{A})\)_\(M\) of the first form, we get a predicate in \((u,\psi) \in D(ufam(\mathfrak{A}))\)_\(M\) by taking the pullback
\[
\begin{array}{ccc}
N & \xrightarrow{m} & M \\
\downarrow & & \downarrow \varphi \\
A_i & \xleftarrow{\varepsilon_2} & PA_i
\end{array}
\]
where \(\varepsilon = (\varepsilon_1, \varepsilon_2) : E \hookrightarrow A_i \times PA_i\) is the membership predicate. Conversely, given a predicate \((u,\psi) \in D(ufam(\mathfrak{A}))\)_\(M\), we can define a predicate \(\varphi : M \to PA_i\) in ufam(D\(\mathfrak{A}\))\(_M\) by \(\varphi(m) = \{ \psi(n) \mid n \in u^{-1}(m) \}\). It is easy to see that these two operations give the desired equivalence.

The previous lemma implies that \(D\) is a left biadjoint of the forgetful functor \(\exists\text{-}UOrd \to UOrd\), and composing the two adjoints, we get a 2-monad
\[
D : UOrd \to UOrd
\]
for which we use the same name. Let us describe the morphism part, and unit and multiplication of \(D\) explicitly. A monotone map \((u,f) : (I,A,R) \to (J,B,S)\) is mapped to \((u,Df) : D(I,A,R) \to D(J,B,S)\) with \(Df(M) = \{ f_i m \mid m \in M \}\). Unit \((I,A,R) \to D(I,A,R)\) and multiplication \(DDD(I,A,R) \to D(I,A,R)\) are given by indexwise singleton-map and union, respectively. Observe that these definitions make \(D\) into a 2-monad, not merely a pseudo-monad. Given a uniform preorder \(\mathfrak{A}\), it is easily seen that we have
\[
\mu_\mathfrak{A} \dashv \eta_\mathfrak{A} : D\mathfrak{A} \to DD\mathfrak{A}
\]
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(one only has to check that \( \text{id}_{\mathcal{D}A} \leq \eta_{\mathcal{D}A} \circ \mu_{\mathcal{D}A} \)), which means that the monad is a \( \text{KZ-monad} \) [31, Definition B1.1.11]. Recall that for \( \text{KZ-monads} \), Eilenberg-Moore algebra structures coincide with left adjoints to the unit, whence algebras are not objects with supplementary structure, but rather objects with a property. In the case of \( D \), algebras give us an ‘internal’ way to talk about existential quantification, without mentioning the fibered preorder.

**Lemma 4.7.3** A uniform preorder \( \mathfrak{B} = (J, B, S) \) is a \( D \)-algebra iff \( \text{ufam}(\mathfrak{B}) \) has existential quantification.

**Proof.** This is not completely trivial, since we do not a priori know whether the adjunction between \( \text{UOrd} \) and \( \exists \text{-} \text{UOrd} \) is monadic (at least I don’t know of a generic argument to show this).

In the easy direction, if \( \mathfrak{B} \) has \( \exists \), we have \( \text{UOrd}(B, B) \cong \exists \text{-} \text{UOrd}(D, B) \) and the algebra map is the transpose of \( \text{id}_B \).

Conversely, a \( D \)-algebra structure \( \sqcup : D(B) \to B \) induces a fibered monotone map \( \sqcup : D(\text{ufam}(\mathfrak{B})) \cong \text{ufam}(D(B)) \xrightarrow{\text{ufam}(\sqcup)} \text{ufam}(\mathfrak{B}) \), and the quantification of \( \varphi : N \to B_j \) along \( u : N \to M \) is given by \( \sqrt{M}(u, \varphi) \).

**Example 4.7.4** Given a typed pca \( (I, A) \), the fibration \( \text{ufam}(D(I, A, R(A))) \) associated to the uniform preorder \( D(I, A, R(A)) \) precisely the realizability hyperdoctrine \( \mathcal{H}(I, A) \) from Definition A.1.8- (i).

Given an untyped pca \( A \), \( \text{ufam}(D(A, R(A))) \) is the realizability tripos \( \text{rt}(A) \) (Definition A.1.7- (i)).

**Remark 4.7.5** While not essential for this work, it is interesting to know that \( D \) is even a symmetric monoidal monad. This can most easily seen by remarking that \( D \) is given by the power set monad on underlying sets, which is known to be monoidal from [34]. Concretely, the monoidal structure is given by

\[ \phi_{A, B} = (\text{id}_{I \times J}, p) : D(A \times D(B) \to D(A \times B) \] 

with

\[ p_{i,j} : PA_i \times PA_j \to P(A_i \times A_j) \] 

given by

\[ (M, N) \mapsto M \times N, \]

where \( A = (I, A, R) \) and \( B = (J, B, S) \) are uniform preorders.

**Remark 4.7.6** Since a fibered preorder has universal quantification iff its opposite fibered preorder has existential quantification, and \( \text{UOrd} \) is closed under \( (\cdot)^{op} \), we can define a universal quantification monad \( U \) by simply dualizing the definition of \( D \).

Explicitly, given a uniform preorder \( (I, A, R) \), \( U(I, A, R) \) is defined by setting

\[ (r) = \{(M, N) \in PA_i \times PA_j \mid \forall n \in N \exists m \in M . r(m, n)\} \] 

for \( i, j \in I, r \in R_{ij} \),

and by setting \( U(I, A, R) \) the uniform preorder generated by the basis \( (I, (PA_i)_i, \{(r) \mid r \in R_{ij}\}_{ij}) \).

**4.7.1 Uniform frames**

**Definition 4.7.7** – A uniform frame is a uniform preorder \( \mathfrak{A} \) such that \( \text{ufam}(\mathfrak{A}) \) is a fibered frame (Definition 3.4.1).
\[ UFrm \] is the locally ordered category of uniform frames and monotone maps that preserve finite meets and existential quantification.

**Remark 4.7.8** If \( \mathfrak{A} \) is a uniform preorder, then \( ufam(\mathfrak{A}) \) has finite meets iff \( \mathfrak{A} \) is finitely complete, and \( ufam(\mathfrak{A}) \) has existential quantification iff \( \mathfrak{A} \) is a \( D \)-algebra. If we want to completely 'internalize' the definition of uniform frame (i.e. express it without referring to the family fibration), then we still have to internalize the Frobenius law. This can be done using the monoidal structure of \( D \) from Remark 4.7.5. Concretely, if \( \mathfrak{A} \) is a finitely complete uniform preorder with \( D \)-algebra structure \( \bigvee \dashv \eta_{D\mathfrak{A}} \), one can show that the Frobenius law holds in \( ufam(\mathfrak{A}) \) iff the two paths around the rectangle

\[ D\mathfrak{A} \times D\mathfrak{A} \xto{\phi_{\mathfrak{A},\mathfrak{A}}} D(\mathfrak{A} \times \mathfrak{A}) \xto{D(\cdot,\cdot)} D\mathfrak{A} \]
\[
\xleftarrow{\eta \times \text{id}}
\]
\[ \mathfrak{A} \times D\mathfrak{A} \xto{\text{id} \times \bigvee} \mathfrak{A} \times \mathfrak{A} \xto{(\cdot,\cdot)} \mathfrak{A} \]

are isomorphic.

By specializing Lemma 3.4.11 from fibered to uniform preorders, we can deduce that \( D \) is well behaved in relation to conjunction, which gives us a way to construct uniform frames.

**Lemma 4.7.9** If \( \mathfrak{A} \) is a finitely complete uniform preorder, then
- \( D\mathfrak{A} \) is a uniform frame,
- \( y: \mathfrak{A} \to D\mathfrak{A} \) preserves finite meets, and
- for any uniform frame \( \mathfrak{B} \), precomposition with \( y \) induces an equivalence

\[ \text{-}_\text{UOrd}(\mathfrak{A}, \mathfrak{B}) \simeq UFrm(D\mathfrak{A}, \mathfrak{B}) \]

of preorders.

**Proof.** This follows from Lemma 3.4.11, since the occurring categories of uniform preorders can be identified with full subcategories of the corresponding categories of posetal pre-stacks on \( \text{Set} \) in a way which preserves all relevant structure.

**Convention 4.7.10** When applying the constructions of Section 3.4 to the family fibrations of finitely complete uniform preorders \( \mathfrak{A} \) and uniform frames \( \mathfrak{B} \), we usually leave the \( ufam(\cdot) \) implicit. Thus,
- \( \text{Set}[\mathfrak{B}] \) is the category of partial equivalence relations in \( ufam(\mathfrak{B}) \), and \( \text{Sh}(\mathfrak{B}) \) is the corresponding gluing fibration
- \( \text{Set}\{\mathfrak{A}\} \) is the category of equivalence relations in \( \text{siev}(ufam(\mathfrak{A})) \), with corresponding gluing fibration \( \hat{\mathfrak{A}} \)

In particular, we have \( \text{Set}\{\mathfrak{A}\} \simeq \text{Set}[D\mathfrak{A}] \) by Corollary 3.2.9.

### 4.7.2 Preservation of logical structure by \( D \)

In Lemma 4.7.9, we showed that if a uniform preorder \( \mathfrak{A} \) has finite meets, then so does \( D\mathfrak{A} \), and furthermore \( y: \mathfrak{A} \to D\mathfrak{A} \) preserves them. Analogous statements are true for implication and universal quantification, as we show now.

**Lemma 4.7.11** Let \( \mathfrak{A} = (I, A, R) \) be a uniform preorder such that \( ufam(\mathfrak{A}) \) has universal quantification. Then \( ufam(D\mathfrak{A}) \) has universal quantification as well, and \( y: \mathfrak{A} \to D\mathfrak{A} \) preserves it.
Proof. We show that if $\mathfrak{A}$ has an algebra structure for the universal quantification monad $U$ from Remark 4.7.6, then so does $D\mathfrak{A}$. Assume that $(\pi, \wedge) : U(I,I) \to (I, I)$ is such an algebra structure. The algebra structure on $D(I, I)$ is given by $(\pi, \bar{\wedge}) : UD(I, I) \to D(I, I)$, where $\bar{\wedge}_i : PPA_i \to PPA_i$ is given by

$$\bar{\wedge}_i M = \{ \wedge_i U \ | \ U \subseteq \bigcup_{M} M, \forall M, \exists a \in M.a \in U \} \quad \text{for } M \subseteq PPA_i.$$ 

It is not difficult to see that $(\pi, \bar{\wedge})$ is monotone and right adjoint to $\eta : D\mathfrak{A} \to UD\mathfrak{A}$ (algebra structures are right adjoint to the unit in the case of $U$, since it is a dualized KZ-monad). Furthermore, $y : \mathfrak{A} \to D\mathfrak{A}$ is a strict $U$-algebra morphism with respect to the algebra structures $(\pi, \wedge)$ and $(\pi, \bar{\wedge})$, which follows from the construction of $\bar{\wedge}$. ■

Lemma 4.7.12 Let $\mathfrak{A} = (I, I)$ be a uniform preorder such that ufam($\mathfrak{A}$) has implication and universal quantification. Then ufam($D\mathfrak{A}$) has implication and universal quantification as well, and $y : \mathfrak{A} \to D\mathfrak{A}$ preserves both connectives.

Proof. Assume that $(I, I)$ has $\Rightarrow$ and $\forall$. It remains to show that $D(I, I)$ has $\Rightarrow$. For $i,j \in I$, the projections $A_i \leftarrow A_i \times A_j \xrightarrow{\pi_i \times \pi_j} A_j$ are predicates on $A_i \times A_j$. We denote the sort of $\pi_i \Rightarrow \pi_j$ by $i \Rightarrow j$, and thus we have a map $(\Rightarrow_{ij}) := (\pi_i \Rightarrow \pi_j) : A_i \times A_j \to A_{i \Rightarrow j}$.

Given predicates $\varphi : M \to PA_i$, $\psi : M \to PA_j$, we construct the implication of $\varphi$ and $\psi$ by

$$(\varphi \Rightarrow \psi)(m) = \left\{ \bigwedge_{i \Rightarrow j} \{ a \Rightarrow_{ij} b \ | \ (a, b) \in r \} \ | \ r \in tRel(\varphi(m), \psi(m)) \right\} \subseteq A_{\pi_{i \Rightarrow j}},$$

where $tRel(\varphi(m), \psi(m))$ is the set of total relations from $\varphi(m)$ to $\psi(m)$, and $(\pi, \bar{\wedge}) : U\mathfrak{A} \to \mathfrak{A}$ is the $U$-algebra structure representing universal quantification.

It is not difficult to see that this predicate has the desired property, and preservation of $\Rightarrow$ follows again from the definition. ■

Since regular triposes correspond to one-sorted uniform preorders $\mathfrak{A}$ such that ufam($\mathfrak{A}$) has finite meets, implication, and universal quantification, the previous lemma implies in particular that $D\mathcal{P}$ is a tripos for every regular tripos $\mathcal{P} : [\mathcal{P}] \to \text{Set}$. Moreover, we can show the following.

Corollary 4.7.13 Every regular tripos is a subtripo of a regular tripos with freely generated existential quantification.

Proof. Let $\mathfrak{A}$ be a uniform preorder representing a tripos. Since ufam($\mathfrak{A}$) has existential quantification, by Lemma 4.7.3 there exists $D$-algebra structure $\bigvee : D\mathfrak{A} \to \mathfrak{A}$ on $\mathfrak{A}$ which is given by transposing $\text{id}_\mathfrak{A}$ as in Lemma 4.7.2, and is left adjoint to $y : \mathfrak{A} \to D\mathfrak{A}$ since $D$ is a KZ-monad. Furthermore, it follows from Lemma 4.7.9 that $\bigvee$ preserves finite meets. Since algebra maps are left pseudoinverse to units by definition, we thus have a geometric inclusion

$$\bigvee y : \mathfrak{A} \to D\mathfrak{A},$$

where $D\mathfrak{A}$ is a tripos by the previous lemma. ■

Since inclusions of triposes induce inclusions of toposes via the tripos-to-topos construction, we thus obtain a subtopos inclusion

$$\text{Set}([\mathcal{P}]) \hookrightarrow \text{Set}(D\mathcal{P}) \simeq \text{Set}([\mathcal{P}])$$
which can be viewed as tripos-theoretic analogue of the statement that any Grothendieck topos is a subtopos of a presheaf topos. Every subtopos inclusion induces a Lawvere-Tierney topology on the larger topos, and a quasitopos of separated objects for this topology. In the case of the above inclusion, it can be shown that this quasitopos is equivalent to the q-topos associated to $P$ as defined in [17, Definition 5.1]. Thus, q-toposes constructed from regular triposes on $\text{Set}$ are always quasitoposes.

4.7.3 Computability structures and the monad $D_+$

Hofstra [23] defines, apart from the monad $D$, a monad $D_+$—the ‘nonempty downset monad’

This monad corresponds to existential quantification along surjective functions in the same way that $D$ corresponds to general existential quantification.

We can define an analogous monad for uniform preorders, and it turns out that its Kleisli category coincides almost precisely with Longley’s [40] category $\text{CSTRUCT}$.

Given a uniform preorder $(I, A, R)$, $D_+(I, A, R)$ is given by $(I, P_+A, D_+R)$ where $P_+$ is the non-empty power set and $D_+R$ is obtained by restricting the relations in $DR$ to $P_+A$.

Using $D_+$, we obtain the following description of $\text{CSTRUCT}$.

Lemma 4.7.14 $\text{CSTRUCT}$ is equivalent to the Kleisli 2-category of $D_+$ on those uniform preorders $(I, A, R)$ where all $A_i$ are inhabited.

The monad $D_+$ can also be used to define Longley’s applicative morphisms between pcas. Let us recall the definition.

Definition 4.7.15 (Longley) Let $A, B$ be pcas. An applicative morphism from $A$ to $B$ is a total relation $\gamma : A \rightarrow B$ such that there exists an $e \in B$ (the realizer of $\gamma$) such that

$$\gamma(a, b), \gamma(a', b'), aa' = a'' \vdash \gamma(fa'', ebb')$$

The following lemma describes applicative morphisms in terms of monotone maps between uniform preorders and $D_+$. Similar characterizations have been given by Hofstra and van Oosten [25] in terms of order-pcas (see also [61, Proposition 1.8.10]), and by Longley [40, Proposition 5.24] in terms of C-structures.

Lemma 4.7.16 Let $A, B$ be pcas. The following concepts are equivalent.

(i) applicative morphisms $\gamma : A \rightarrow B$

(ii) finite meet preserving monotone maps $f : (A, R(A)) \rightarrow D_+(B, R(B))$

(iii) finite meet preserving monotone maps $f : (A, R(A)) \rightarrow D(B, R(B))$

Proof. We start by showing the equivalence between the first two concepts. Total relations $\gamma : A \rightarrow B$ are in bijection with functions $f : A \rightarrow P_+B$ and $P_+B$ is the underlying set of $D_+(B, R(B))$, thus we have to check that the condition for applicative morphisms is equivalent to monotonicity and preservation of finite meets.

Let $\gamma : A \rightarrow B$ be an applicative morphism with realizer $e \in B$. To show that the corresponding $f : A \rightarrow P_+B$ constitutes a monotone map of type $(A, R(A)) \rightarrow D_+(B, R(B))$, we have to show that

$$\forall r \in A \exists s \in B. \forall a, a' \in A. ra = a' \Rightarrow \forall b \in f(a). sb \in f(b').$$

6. The definition can be traced back to Hofstra’s thesis [21] where he defines an analogous monad $T$ on ordered pcas.
For given \( r \in A \), let \( s_0 \in fr \). Without loss of generality, we can assume that \( es_0 \) is defined\(^7\), and setting \( s = es_0 \) the claim is immediate.

To see that \( f \) commutes with binary meets, we have to find an \( s \in B \) such that for every \( a, a' \in A \), we have

\[
b \in fa, b' \in fa' \vdash s(b \wedge b') \in f(a \wedge a').
\]

Let \( p \in A \) such that \( \forall a, a'. a \wedge a' = paa' \) and let \( q \in fp \). For \( a, a' \in A \), \( b \in fa \), and \( b' \in fa' \), we have \( eqb \in f(pa) \) and \( e(eqb)b' \in f(paa') = f(a \wedge a') \). The existence of an \( s \) such that \( s(b \wedge b') = e(eqb)b' \) follows from functional completeness.

Conversely, assume that \( f : (A, RA) \rightarrow D(B, RB) \) is a meet preserving monotone map. Let \( r \in A \) such that \( \forall a, a', a''a'' = a'' \Rightarrow r(a \wedge a') = a'' \). Since \( f \) is monotone, there exists an \( s \in B \) such that \( ra = a'', b \in fa \vdash sb \in fa'' \), which together with the first statement implies \( aa' = a'', b \in f(a \wedge a') \vdash sb \in fa'' \). Now since \( f \) preserves finite meets, there exists \( t \in B \) such that \( b \in fa, b' \in fa' \vdash t(b \wedge b') \in f(a \wedge a') \), and we can deduce \( aa' = a'', b \in fa, b' \in fa' \vdash s(t(b \wedge b')) \in fa'' \). By functional completeness there exists \( e \in B \) such that \( \forall b, b' \in B \), \( s(t(b \wedge b')) \leq cbb' \), and this \( e \) is the realizer of the applicative morphism \( \gamma : A \rightarrow B \) corresponding to \( f \).

It remains to show that meet preserving maps \( (A, RA) \rightarrow D(B, RB) \) are equivalent to meet preserving maps \( (A, RA) \rightarrow D(B, RB) \). In one direction, we can compose with the embedding \( D+(B, RB) \rightarrow D(B, RB) \) which preserves finite meets. For the other direction, let \( f : (A, RA) \rightarrow D(B, RB) \) be finite meet preserving monotone, and consider the component

\[
ufam(f)_1 : ufam(A, RA)_1 \rightarrow ufam(D(B, RB))_1
\]

of the corresponding fibered monotone map between the terminal fibers. Since all elements of \( ufam(A, RA)_1 \) are equivalent, and \( ufam(f)_1 \) preserves \( \top \) (as a particular finite meet), all \( fa \) are equivalent to \( \{\top\} \) in \( ufam(D(B, RB))_1 \) and thus in particular inhabited, meaning that \( f \) factors through \( D+(B, RB) \rightarrow D(B, RB) \).

\[\blacklozenge\]

**Remark 4.7.17** It might be argued that from a presheaf theoretic point of view and for general uniform preorders, the relevant objects are the meet-preserving maps of type \( A \rightarrow DB \), since they correspond to positive fibered functors between the associated fibrations of presheaves. The fact that such maps factor through \( D+(B, RB) \rightarrow DB \) is specific to pcas.

### 4.8 Global sections

This section is not a priori about uniform preorders, but I decided to put it in this chapter nevertheless, since it is strongly related to Section 4.12.

Given a Grothendieck topos \( \mathcal{E} \), the diagonal functor \( \Delta : Set \rightarrow \mathcal{E} \) is left adjoint to the global sections functor \( \Gamma = \mathcal{E}(1, -) : \mathcal{E} \rightarrow Set \).

For a (say bounded) geometric morphism \( \Delta \rightarrow \Gamma : \mathcal{E} \rightarrow S \), the fact that \( \Delta \) has a right adjoint is equivalent to the statement that the fibration \( gl_\Delta(\mathcal{E}) \) has small global sections, which is a special case of the fibrational property of being locally small (see [57, Sections 10, 16])\(^8\). Thus, it still makes sense to view \( \Gamma \) as a global sections functor relative to the base topos \( S \).

In general, \( \Gamma \) does not fit into the framework of Moens’ theorem as stated in Theorem 2.2.12 since for \( \Gamma \) to correspond to a 1-cell in \( (S/\text{Lex})(\Delta, id_S) \) we need \( \Gamma \circ \Delta \cong id_S \). However, we can still use the generalized version of the correspondence presented in Remark 2.2.13, allowing us to

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\(^7\) By functional completeness, we can replace \( e \) by an \( e' \) for which it is everywhere defined, and which still realizes \( f \) as an applicative morphism, see [38, Remark 2.1.2 (i)].

\(^8\) Having small global sections is also equivalent to having comprehension in the sense of Lawvere [36].
view $\Gamma$ as a fibered functor of type $\text{gl}_{\Delta}(E) \to \text{cod}(S)$ which does not necessarily preserve internal sums.

If $\Delta : R \to X$ is an arbitrary regular functor into an exact category, there is no reason for the existence of a right adjoint -- in other words the fibration $\text{gl}_{\Delta}(X)$ does not have to have small global sections in the fibrational sense. However, if $R = \text{Set}$ and $X$ is locally small, the ordinary global sections functor $\Gamma = X(1, -)$ in the inverse direction of $\Delta$ is always definable. Moreover, we have half of the structure of the adjunction, namely a natural transformation $\eta : \text{id}_{\text{Set}} \to \Gamma \Delta$, given by

$$M \cong \text{Set}(1, M) \to X(\Delta 1, \Delta M) \cong X(1, \Delta M) = \Gamma \Delta M.$$ 

If the regular category $X$ is of the form $\text{Set}[X]$ for a fibered frame $X$, the same story can be told directly on the level of fibered preorders. We can define $\gamma : X \to \text{sub}(\text{Set})$ by

$$X M \ni \varphi \mapsto \{ m \mid \top \vdash m^* \varphi \} \subseteq M.$$ 

Then it is straightforward that $\gamma$ preserves finite meets, and by a similar argument as the one above one can show $U \subseteq \gamma \delta U$ for $U \subseteq M$. Moreover, $\gamma$ can be reconstructed from $\Gamma$ and $\eta$ as the following lemma shows.

**Lemma 4.8.1** Let $X : [X] \to \text{Set}$ be a fibered frame, and $\varphi \in X M$. Then $\gamma(\varphi) \cong \eta^*_M \Gamma \tilde{\varphi}$, where $\tilde{\varphi}$ is the subobject of $\Delta M$ represented by $\varphi$.

$$
\begin{array}{ccc}
\Gamma(M, =_\varphi) & \xrightarrow{\gamma(\varphi)} & \Gamma \Delta M \\
\downarrow \gamma & & \downarrow \gamma \tilde{\varphi} \\
M & \xrightarrow{\eta_M} & \Gamma \Delta M
\end{array}
$$

**Proof.** $m_0 \in M$ is contained in $\eta^*_M \Gamma \tilde{\varphi}$ iff the global element of $\Delta M$ given by the singleton predicate $(m \mid m = m_0)$ factors through $\varphi$, which is equivalent to $\vdash \varphi(m_0)$. ■

We remark that $\gamma$ is definable not only for fibered frames, but for arbitrary fibered posets with $\top$. For uniform preorders $(I, A, R)$ with greatest element $\top \in A$, $\gamma$ is defined by

$$\text{ufam}(I, A, R)_M \ni (i, \varphi) \mapsto \{ m \mid \{ (\top, \varphi(m)) \} \in R_{1,i} \} \subseteq M,$$

the fact that we are talking about the pointwise and not the uniform ordering corresponds to the use of singleton relations $\{ (\top, \varphi(m)) \}$.

### 4.9 Calculus of distributors

**Definition 4.9.1** Given preorders $D$ and $E$, a (posetal) distributor $\phi : D \to E$ is a monotone function of type $E^{op} \times D \to 2$, or equivalently a relation $\phi \subseteq D \times E$ which is upward closed in $D$ and downward closed in $E$.

Preorders and posetal distributors form a compact closed locally ordered category $\text{PDist}$, and we have an embedding

$$\text{Ord} \hookrightarrow \text{PDist}$$

which identifies the category $\text{Ord}$ of preorders with the subcategory of $\text{PDist}$ on left adjoints. We can do something completely analogous for uniform preorders.

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Definition 4.9.2  
(i) A uniform distributor \( H : (I, A, R) \rightarrow (J, B, S) \) between uniform preorders \((I, A, R)\) and \((J, B, S)\) is a family of sets 

\[
H_{ji} \subseteq P(B_j \times A_i)
\]

of binary relations such that 

(a) \( h \in H_{ji}, k \subseteq h \Rightarrow k \in H_{ji} \)
(b) \( h \in H_{ji}, r \in R_{ji'} \Rightarrow rh \in H_{ji'} \)
(c) \( s \in S'_{j'i}, h \in H_{ji} \Rightarrow hs \in H_{j'i} \)

(ii) The composition of uniform distributors 

\[
(I, A, R) \xrightarrow{G} (J, B, S) \xrightarrow{H} (K, C, T)
\]

is defined by \( (H \circ G)_{ki} = \downarrow \{ gh \mid j \in J, g \in G_{ji}, h \in H_{kj} \} \).

(iii) The product of uniform distributors 

\[
(I, A, R) \xrightarrow{G} (K, C, T), \quad (J, B, S) \xrightarrow{H} (L, D, U)
\]

is the uniform preorder 

\[
(I, A, R) \times (J, B, S) \xrightarrow{G \times H} (K, C, T) \times (L, D, U)
\]

defined by \( (G \times H)_{klij} = \downarrow \{ g \times h \mid g \in G_{ki}, h \in H_{lj} \} \).

\[\diamondsuit\]

These definitions give rise to a locally ordered monoidal category \textbf{UDist}, where the order on the morphisms is componentwise inclusion\(^{10}\) and the identity on \((I, A, R)\) is \(R\).

Given a monotone map \((u, f) : (I, A, R) \rightarrow (J, B, S)\), we define uniform distributors 

\[
(u, f)^* : (I, A, R) \rightarrow (J, B, S) \quad \text{and} \quad (u, f)_* : (J, B, S) \rightarrow (I, A, R)
\]

by 

\[
((u, f)^*)_{ji} = \{ g \subseteq B_j \times A_i \mid f_i g \in S_{j,ui} \} \quad \text{and} \quad ((u, f)_*)_{ij} = \{ h \subseteq A_i \times B_j \mid h f_i^* \in S_{ui,j} \}.
\]

Lemma 4.9.3  
(i) \((I, A, R)^{op}\) is dual to \((I, A, R)\) in \textbf{UDist}.

(ii) For \((u, f) : (I, A, R) \rightarrow (J, B, S)\), we have \((u, f)^* \dashv (u, f)_*\).

(iii) Let \(G \dashv H : (J, B, S) \rightarrow (I, A, R)\) be an adjunction in \textbf{UDist}. Then the axiom of choice implies that there exists a monotone \((u, f) : (I, A, R) \rightarrow (J, B, S)\) such that \((u, f)^* = G\) and \((u, f)_* = H\).

\(^{10}\) Observe that since the ordering of morphisms is defined in terms of inclusions, \textbf{UDist} is enriched in \textit{posets}, and not just in preorders as most of our locally ordered categories.

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Proof. The counit \( \varepsilon_{(I,A,R)} : (I,A,R)^{op} \times (I,A,R) \to 1 \) has to be a family of sets of relations of type \( 1 \to A \times A \), which are closed under composition with structuring relations of \((I,A,R)^{op} \times (I,A,R)\) on the left. It is given by \( \varepsilon_{(I,A,R)} = R \). Dually, \( \eta_{(I,A,R)} : 1 \to (I,A,R) \times (I,A,R)^{op} \) is a family of sets of relations of type \( A \times A \), which is closed under composition with structuring relations of \((I,A,R) \times (I,A,R)^{op}\) on the right, and is given by \( R \) as well. The verifications of the triangle equalities are straightforward.

To show that \((u,f)^* = (u,f)\)\(_\ast\), we have to verify \( R \subseteq (u,f)(u,f)^* \) and \((u,f)^*(u,f) \subseteq S\) componentwise. Let \( r \in R_{ijl} \). We have to show \( j \in J \) and \( g \in (u,f)^* \) such that \( r \subseteq gh \). This is the case for \( j = u_i \), \( g = rf_i^* \), and \( h = f_j \). For the second inclusion, we have to show that \( g \in (u,f)^* \) and \( h \in ((u,f)_i^j)_{ij} \) implies \( hg \in S_{j,j'} \). This follows from \( f'_i g \in S_{j,ui} \) and \( h f_i^* \subseteq S_{ui,j'} \).

For the third claim, it follows from \( R \subseteq HG \) that for given \( i \in I \) there exist \( j \in J \), \( h_i \in H_{ij} \), \( g_i \in G_{ji} \) such that \( i \subseteq g_i h_i \), which means that \( \forall a : A_i \exists g_i h_i \). Let \( u : I \to J \) be a choice function for the association of \( j \) to \( i \), and for each \( i \in I \), let \( f_i : A_i \to B_{ui} \) be a choice function for the second part of the statement. Then in particular \( f_i \subseteq h_i \) and \( f_i^* \subseteq g_i \). We claim that \((u,f) = \) the desired monotone map.

To see that \((u,f)\) is monotone, let \( r \in R_{ijl} \). Since \( R \subseteq HG \), there exist \( j \in J \), \( h'_i \in H_{ij} \), and \( g'_i \in G_{ji} \) such that \( r \subseteq g'_i h'_i \). Therefore we can argue \( f'_i r f_i^* \subseteq h'_i g'_i h_i \subseteq (SS)_{ui,ui'} = S_{ui,ui'} \). To verify that \((u,f)\) induces the adjunction \( G \dashv H \), we show \( G \subseteq (u,f)^* \).

In Lemma 4.2.2, we showed that we can identify the 2-category \( \mathcal{UOrd} \) with a full subcategory of \( \mathcal{PFib}(\mathcal{Set}) \). In the following, we will do something similar for \( \mathcal{UDist} \), but in this case it turns out to be more convenient to work with indexed preorders instead of fibered ones.

Recall that an indexed preorder is a pseudo-functor of type \( \mathcal{Set}^{op} \to \mathcal{Ord} \), and that \( - \) if we ignore size issues – the locally ordered categories \( \mathcal{Ord}(\mathcal{Set}) \) and \( \mathcal{PFib}(\mathcal{Set}) \) of indexed and fibered preorders on \( \mathcal{Set} \) are biequivalent. Given a uniform preorder \( \mathfrak{A} \), we do not distinguish the associated indexed and fibered preorders notionally – we denote both by \( \mathcal{ufam}(\mathfrak{A}) \).

**Definition 4.9.4**

(i) Let \( \mathfrak{A}, \mathfrak{B} \) be indexed preorders. An indexed distributor \( \Phi : \mathfrak{A} \to \mathfrak{B} \) is a family \( \Phi_M : A_M \to B_M \) of posetal distributors such that for \( u : N \to M, \psi \in A_M \), and \( \theta \in B_M \), we have \( \Phi_M(\theta, \psi) = \Phi_N(u^* \theta, u^* \psi) \).

(ii) If \( \mathfrak{A}, \mathfrak{B} \) are indexed preorders satisfying the pre-stack condition, we say that an indexed distributor \( \Phi : \mathfrak{A} \to \mathfrak{B} \) is separated, if for any epimorphism \( e : N \to M \) and predicates \( \psi \in A_M \) and \( \theta \in B_M \), \( \Phi_N(e^* \theta, e^* \psi) = \Phi_N(\theta, \psi) \).

In the presence of choice, this condition is always satisfied.

Indexed distributors can be composed componentwise, but the componentwise composition of two separated indexed distributors does not need to be separated. If we want to define a category of separated indexed distributors, we therefore have to compose them differently.

**Definition 4.9.5** Let \( \mathfrak{A}, \mathfrak{B}, \) and \( \mathfrak{C} \) be indexed preorders satisfying the pre-stack condition, and let \( \Phi : \mathfrak{A} \to \mathfrak{B} \) and \( \Psi : \mathfrak{B} \to \mathfrak{C} \) be two separated indexed distributors. Their composition

\[
\Psi \circ \Phi : \mathfrak{A} \to \mathfrak{C}
\]

is defined by

\[
(\Psi \circ \Phi)_M(\gamma, \alpha) : = \exists u : N \to M, \beta \in B_N \cdot \Psi_N(u^* \gamma, \beta) \wedge \Phi_N(\beta, u^* \alpha)
\]

where \( \alpha \in A_M \) and \( \gamma \in C_M \).
Using this definition, it is easy to see that the composition of separated indexed distributors is separated, which allows us to make the following definition.

**Definition 4.9.6** $\text{IDist}$ is the locally ordered category of indexed preorders satisfying the pre-stack condition, and separated indexed distributors, where the ordering is given by componentwise inclusion.

Given a uniform distributor $G : (I, A, R) \rightarrow (J, B, S)$, we can define an indexed distributor

$$\text{ufam}(G) : \text{ufam}(I, A, R) \rightarrow \text{ufam}(J, B, S)$$

by

$$\text{ufam}(G)_M((j, \psi), (i, \phi)) :\Leftrightarrow \{ (\psi m, \phi m) \mid m \in M \} \in G_{ji}.$$

**Lemma 4.9.7** The previously defined operation gives rise to a 2-functor

$$\text{ufam}(\cdot) : \text{UDist} \rightarrow \text{IDist}.$$

In particular, given uniform distributors $A \xrightarrow{G} B \xrightarrow{H} C$ the indexed distributors $\text{ufam}(G)$ and $\text{ufam}(H)$ are separated, and we have $\text{ufam}(G \circ H) = \text{ufam}(G) \circ \text{ufam}(H)$ with the composition of separated indexed distributors defined in 4.9.5.

Moreover, $\text{ufam}(\cdot) : \text{UDist} \rightarrow \text{IDist}$ is a local equivalence.

**Proof.** Consider uniform distributors $A \xrightarrow{G} B \xrightarrow{H} C$, where $A = (I, A, R), B = (J, B, S), C = (K, C, T)$. It follows directly from the definition that $\text{ufam}(G)$ and $\text{ufam}(H)$ are separated. For compatibility with composition, let $M \in \text{Set}$, $\alpha : M \rightarrow A_i$ and $\gamma : C_k$ for some $i \in I$ and $k \in K$.

Instantiating definitions, we get

$$(\text{ufam}(H) \circ \text{ufam}(G))_M(\gamma, \alpha) \Leftrightarrow$$

$$\exists e : N \rightarrow M, j, \beta : N \rightarrow B_j . \{ (\gamma(en), \beta n) \mid n \in N \} \in H_{kj} \land \{ (\beta n, \alpha(en)) \mid n \in N \} \in G_{ji}$$

and

$$\text{ufam}(H \circ G)_M(\gamma, \alpha) \Leftrightarrow$$

$$\exists j, h \in H_{kj}, g \in G_{ji} \forall m : M \exists v : B_j . h(\gamma m, b) \land g(b, \alpha m)$$

It is easy to see that the first condition implies the second one. For the converse direction, we can set $N = \{ (m, b) \mid h(\gamma m, b) \land g(b, \alpha m) \}$ for fixed $h$ and $g$.

Clearly, $\text{ufam}(\cdot)$ is monotone on morphisms. To see that it is also order reflecting, let $G, H : A \rightarrow B$ such that $\text{ufam}(G) \leq \text{ufam}(H)$. Let $B_j \times A_i \supseteq g \in G_{ji}$, and let $\alpha : g \rightarrow A_i$ and $\beta : g \rightarrow B_j$ be the projections. Then we can reason

$$\text{ufam}(G)_g(\beta, \alpha) \Rightarrow \text{ufam}(H)_g(\beta, \alpha) \Rightarrow g \in H_{ji},$$

whence $G \leq H$.

Finally, let $\Phi : \text{ufam}(A) \rightarrow \text{ufam}(B)$ be a separated indexed distributor. We can construct a pre-image $G : A \rightarrow B$ under $\text{ufam}(\cdot)$ by setting

$$G_{ji} = \{ g \subseteq B_j \times A_i \mid \Phi_g(\beta, \alpha) \},$$

where $\beta : g \rightarrow B_j$ and $\alpha : g \rightarrow A_i$ are again the projections.

$\blacksquare$
**Remarks 4.9.8**  
(i) It is possible to define the subcategory of \( \text{UOrd} \) on 1-sorted uniform preorders as a category of monads and modules on a more primitive locally ordered category whose objects are sets, and whose morphisms are downward closed sets of binary relations. I learned this from Naohiko Hoshino, who, in unpublished work, defined a locally ordered category of combinatory objects similar to (but more general than) 1-sorted uniform preorders using this approach.

A similar suggestion has been made earlier by Paul-André Melliès, who strongly promoted the importance of the construction of monads and modules on several occasions and in different contexts (see e.g. [44]). However, I failed to see the relevance back then.

Tom Hirschowitz pointed out that it is possible to define many-sorted uniform preorders as quantaloid-enriched categories, which is a direct generalization of the monads-and-modules approach. We describe this in more detail in Section A.3.2.

(ii) So far, we have observed a remarkable similarity between the theory of ordinary and of uniform preorders – ordinary preorders often serve as a guiding principle to come up with constructions for the uniform case. In the context of distributors, there is however also a major difference: In the case of ordinary preorders, the locally ordered category of distributors is the Kleisli category of the downset monad. In the uniform case, however, \( \text{UDist} \) is not the Kleisli category of the \( D \)-monad, and it does not seem to be representable by a Kleisli construction at all. ♦

**4.10 Relationally complete uniform preorders**

Given a (say finitely complete) preorder \( A \), its preorder \( DA \) of downsets has small meets and \( \hat{A} \) is a topos and thus in particular locally cartesian closed.

For a finitely complete uniform preorder \( A \), \( DA \) does not in general have universal quantification and implication, and neither does \( A \) need to be locally cartesian closed – however, as we showed in Theorem 3.4.25, these two properties are equivalent. In this section, we study an equivalent combinatorial criterion for this to be the case, which we call **relational completeness**.

The development of this concept was initially motivated by Hofstra’s characterization of BCOs \( A \) such that \( \text{ufam}(DA) \) is a tripos as coming from ‘ordered pcas with a filter’ [23, Theorem 6.9], but curiously enough, the end result is quite close to Longley’s concept of ‘higher order C-structure’ [40].

**Definition 4.10.1** A finitely complete uniform preorder \( (I, A, R) \) is called **relationally complete**, if for each pair \( j, k \in I \) there exists \( j \Rightarrow k \in I \) and \( @_{j \Rightarrow k} \in R_{(j \Rightarrow k) \times j,k} \) such that for all \( i \in I \) and \( r \in R_{i \Rightarrow j,k} \) there exists \( \bar{r} \in R_{i \Rightarrow j,k} \) such that

\[
\forall a \in A_i \exists h \in A_{j \Rightarrow k} . \bar{r}(a, h) \land r(a \land -, -) \subseteq @_{j \Rightarrow k}(h \land -, -). \tag{4.10.1}
\]

**Remarks 4.10.2**  
(i) Precision: to conform with our convention regarding the axiom of choice (Section 1.5.4), in particular to be able to construct a choice of universal quantification in Lemma 4.10.3 below, we require a relationally complete uniform preorder \( (I, A, R) \) to come with a choice of sorts \( j \Rightarrow k \) and relations \( @^R_{j \Rightarrow k} \) for all \( j, k \in I \). For the relations \( \bar{r} \), on the other hand, mere existence is sufficient.

(ii) In the one sorted case, we call \( @ \) the **generic relation**. In the case of number realizability, the generic relation is the universal Turing machine, for a meet-semilattice, the ordering relation is the generic relation.
Relational completeness is not tautological. The archetypal example of a finitely complete uniform preorder which is not relationally complete is the primitive recursive uniform preorder $(\mathbb{N}, \downarrow_{\text{Prim}})$ from Example 4.1.3(iii) – here, relational completeness would imply the existence of a primitive recursive interpreter for primitive recursive functions, which is impossible because of diagonalization.

It is instructive to compare condition (4.10.1) to the statement that

$$r \subseteq \exists_{k} \circ \land \circ (\tilde{r} \times \text{id}_{A_{j}}),$$

a variant of which occurs in Longley’s definition of ‘higher order C-structure’ [40, Definition 5.22]. Spelled out, the former is equivalent to

$$\forall a \exists h. \tilde{r}(a, h) \land \forall b, c. r(a, b, c) \Rightarrow \exists_{k} (h \land b, c)$$

whereas the latter is equivalent to

$$\forall a, b, c. r(a, b, c) \Rightarrow \exists h. \tilde{r}(a, h) \land \exists_{k} (h \land b, c).$$

We see that relational completeness is stronger since the $\exists h$ is further on the left. Nevertheless, the concepts of ‘relationally complete uniform preorder’ and ‘higher order C-structure’ are remarkably similar.

**Theorem 4.10.3** Let $\mathfrak{A} = (I, A, R)$ be a finitely complete uniform preorder. The following are equivalent.

(i) $\mathfrak{A}$ is relationally complete.

(ii) $D\mathfrak{A}$ has implication and universal quantification.

(iii) Set $\{\mathfrak{A}\}$ is locally cartesian closed.

**Proof.** We already showed in Theorem 3.4.25 that (ii) and (iii) are equivalent; now we will show the equivalence of (i) and (ii).

Assume first that $D\mathfrak{A}$ has implication and universal quantification. Given $j, k \in I$, let $E \rightarrow A_{j} \times A_{k} \times P(A_{j} \times A_{k})$ be the membership predicate. Let $u : E \rightarrow P(A_{j} \times A_{k})$ be the first projection, and let $\varphi : E \rightarrow PA_{j}, \psi : E \rightarrow PA_{k}$ be the first and second projections postcomposed with the singleton maps $A_{j} \rightarrow PA_{j}$ and $A_{k} \rightarrow PA_{k}$. We choose $j \Rightarrow k \in I$ to be the sort of the predicate $\forall_{u} \varphi \Rightarrow \psi$ and we choose $\exists_{k} \in R_{i, j, k}$ to be a relation realizing the valid judgment

$$(u^{*} \forall_{u} \varphi \Rightarrow \psi) \land \varphi \vdash \psi \tag{4.10.2}$$

(To get a functional choice of $\exists_{k}$, we can just choose the minimal such relation, which is unique since the predicate on the right of $\vdash$ is singleton valued). Now let $i \in I$ and $r \in R_{i, j, k}$. Define

$$A_{i} \times A_{j} \times A_{k} \supseteq M = \{(a, b, c) \mid (a \land b, c) \in r\},$$

and let $v : M \rightarrow A_{i}$ be the first projection. Let $\beta : M \rightarrow PA_{j}, \gamma : M \rightarrow PA_{k}$ be the second and third projections postcomposed with the singleton maps, and let $i : A_{i} \rightarrow PA_{i}$ be the singleton map on $A_{i}$. Then $u^{*} i$ is the predicate on $M$ corresponding to the first projection, and we have $u^{*} i \land \beta \vdash \gamma$ realized by $r$. Doing $\Rightarrow$ and $\forall$ introduction, we obtain the valid judgment $i \vdash \forall_{u} \varphi \Rightarrow \gamma$.

11. It is not literally the same condition, since Longley’s ‘cartesian C-structures’ [40, Definition 5.19-(ii)] are defined differently from our finitely complete uniform preorders.
on $A_i$. We define $w : A_i \rightarrow P(A_j \times A_k)$ by $A_i \ni a \mapsto \{(b, c) \mid (a \land b, c) \in r\}$, which gives us a pullback square

$$
\begin{array}{ccc}
M & \xrightarrow{v} & A_i \\
\downarrow & & \downarrow w \\
E & \xrightarrow{u} & P(A_j \times A_k)
\end{array}
$$

and moreover $x^*\varphi = \beta$ and $x^*\psi = \gamma$, which implies using the Beck-Chevalley condition that $(\forall u \beta \Rightarrow \gamma) \equiv (w^*\forall u \varphi \Rightarrow \psi)$. Choose $\bar{r} \in r_{i,j \Rightarrow k}$ to be a realizer of

$$
i \vdash w^*\forall u \varphi \Rightarrow \psi. \tag{4.10.3}
$$

To show relational completeness, it remains to verify (4.10.1). Let $a \in A_i$. Since $\bar{r}$ realizes (4.10.3), there exists $r \in (\forall u \varphi \Rightarrow \psi)(wa)$ such that $\bar{r}(a, h)$, and it remains to show that for $b \in A_j$, $c \in A_k$ we have $r(a \land b, c) \Rightarrow \bar{r}_k(h \land b, c)$. But if $r(a \land b, c)$ then $(a, b, c) \in M$, and $\bar{r}_k(h \land b, c)$ follows from the validity of (4.10.2) at $a(b, a, c)$.

Conversely, assume that $\mathfrak{A}$ is relationally complete. Instead of constructing implication and universal quantification separately, we show how to define the ‘synthetic’ connective $\forall u \varphi \Rightarrow \psi$ for $u : X \rightarrow Y$ and $\varphi, \psi \in \text{ufam}(D\mathfrak{A})_X$. Implication and universal quantification can then be recovered by either replacing $u$ by the identity, or $\varphi$ by the true predicate. For $\varphi : X \rightarrow PA_{\bar{A}}$, $\psi : Y \rightarrow PA_{\bar{A}k}$, define $(\forall u \varphi \Rightarrow \psi) : Y \rightarrow PA_{\bar{A}j \Rightarrow k}$ by

$$(\forall u \varphi \Rightarrow \psi)(y) = \bigcap_{ax=y} \{ h \in A_{i,j \Rightarrow k} \mid \forall b \in (\varphi(x) \Rightarrow \psi(x)) \Rightarrow b,c \in \psi(h \land b, c) \}. $$

It is then easy to see that $\bar{r}_k(h) \Rightarrow u^*\forall u \varphi \Rightarrow \psi$, $\varphi \vdash \psi$; and if $\zeta : Y \rightarrow PA_{\bar{A}}$ such that the judgment $u^*\xi, \varphi \vdash \psi$ is realized by $r \in A_{i,j \Rightarrow k}$, then $\bar{r}$ realizes $\xi \vdash \forall u \varphi \Rightarrow \psi$. }

**Examples 4.10.4**

- If $\mathfrak{A}$ is a uniform preorder such that $\text{ufam}(\mathfrak{A})$ has $\land, \Rightarrow$, and $\forall$, then $\mathfrak{A}$ is relationally complete. This follows from Lemma 4.7.12.
- Uniform preorders that come from meet-semilattices are always relationally complete, since downset lattices are complete Heyting algebras. 

Relational completeness is particularly interesting for *functional* uniform preorders. Let us rephrase the definition in this case, using the language of *relational clones* from Section 4.6.1.

**Lemma 4.10.5** A finitely complete functional uniform preorder $\mathfrak{A} = (I, A, R)$ is relationally complete iff for every pair $j, k \in I$ there exists $j \Rightarrow k \in I$ and a partial function

$$(-, j, k) : A_{j \Rightarrow k} \times A_j \rightarrow A_k$$

such that for all $i_1, \ldots, i_n \in I$ and every partial function $f \in C_{i_1, \ldots, i_n, j \Rightarrow k}$ there exists a total function $\Lambda f \in C_{i_1, \ldots, i_n, j \Rightarrow k}$ such that

$$\Lambda f(a_1, \ldots, a_n) : b \geq f(a_1, \ldots, a_n, b)$$

for appropriately typed $a_1, \ldots, a_n, b$, where $(C_{i_1, \ldots, i_n, j \Rightarrow k})_{i_1, \ldots, i_n, j \in I}$ is the relational clone associated to $\mathfrak{A}$ defined before Lemma 4.6.6.

The notation $\Lambda f$ is inspired by the apparent analogy to abstraction in $\lambda$-calculus; we denote iterated abstraction by $\Lambda^nf$. If we abstract an $n$-ary function $f \in C_{i_1, \ldots, i_n, j \Rightarrow k}$ $n$ times, we obtain
A ‘total 0-ary function’, which is a singleton \( \{ \Lambda^n f \} \in C_{\otimes: i_1 \otimes \cdots \otimes i_n \Rightarrow j} \) (i.e. \( \Lambda^n f \in A_{i_1 \otimes \cdots \otimes i_n \Rightarrow j} \)) such that
\[
(\Lambda^n f): a_1 \cdots a_n \simeq \cdots \simeq (\Lambda f)(a_1, \ldots, a_{n-1}) a_n \simeq f(a_1, \ldots, a_n),
\]
where we have strong equality \( \simeq \) between all but the last two terms since the functions \( \Lambda^i(f) \) are total.

**Lemma 4.10.6** Every relationally complete functional uniform preorder \( \mathfrak{A} = (I, A, R) \) is induced (in the sense of Definition 4.1.3-(ii))d) by an inclusion of typed pcas.

**Proof.** Relational completeness provides us with the type constructors on \( I \). First, let us show that the partial binary application maps \((-\cdot-): A_{i \Rightarrow j} \times A_i \rightarrow A_j \) make \( (A_i)_{i \in I} \) a typed pca. For this we have to construct the combinators postulated in Definition A.1.3 and verify the axioms.

For \( s_{i,j,k} \), consider the partial function \( f_s : A_{i \Rightarrow j \Rightarrow k} \times A_{i \Rightarrow j} \times A_i \rightarrow A_k \) given by \( (x, y, z) \mapsto xz(yz) \), which is contained in the clone \( (C_{i_1, \ldots, i_n ; j_1, \ldots, j_n ; k_1, \ldots, k_n})_{i_1, \ldots, i_n \in I} \), since the application maps are and the clone is closed under composition. Three-fold abstraction gives us a singleton \( \Lambda^3 f_s = \{ s \} \subseteq A_{(i \Rightarrow j \Rightarrow k) \Rightarrow (i \Rightarrow j) \Rightarrow i \Rightarrow k} \), and it follows from the remarks preceding the lemma that \( s \) has the desired properties.

In the same way, the \( k \) combinator is obtained by double abstracting the function \( (x,y) \mapsto x \).

The combinator \( \text{pair} \) is given by abstracting \( (x,y) \mapsto x \wedge y \) twice. For the (say) first projection, observe that for given \( i, j \in I \) we have \( \pi_1 \wedge \pi_j \leq \pi_i \) in \( \text{ufam}(A)_{A_i \times A_j} \) (\( \pi_i \) and \( \pi_j \) being the first and second projection), which implies that the set \( p = \{ (x \wedge y, x) \mid x \in A_i, y \in A_j \} \) is contained in \( R_{i \times j} \), and therefore in \( C_{i \times j} \). We take \( \text{fst} \) to be \( \Lambda p \in C_{\otimes ; j \Rightarrow i} \). Then the verification of the corresponding axiom is not difficult:
\[
\text{fst}(\text{pair} x y) = \text{fst}(x \wedge y) = p(x \wedge y) = x,
\]
where the last equation holds since \( (x \wedge y, x) \in p \) by definition. The construction for \( \text{snd} \) is analogous, which finishes the construction a typed pca structure on \( (A_i)_{i \in I} \).

To get a typed sub-pca, we set for each \( i \in I \)
\[
A_{\#; i} = \{ a \in A_i \mid \{ a \} \in C_{\otimes ; i} \}. \tag{4.10.4}
\]

Then the combinators \( k, s, \text{fst}, \text{snd}, \text{pair} \) are contained in the substructure by construction, closure under application follows from closure under composition of the clone.

It remains to check that the inclusion \( (A_{\#; i} \subseteq A_i)_{i \in I} \) of typed pcas does indeed induce the uniform preorder \( \mathfrak{A} \). This is equivalent to the statement that the partial functions \( (a-): A_i \rightarrow A_j \) for \( a \in A_{\#; i \Rightarrow j} \) generate the sets \( C_{i \times j} = R_{i \times j} \) under down closure. In one direction, given \( a \in A_{\#; i \Rightarrow j} \), the function \( (a- \cdot) \) can be expressed as a composition of \( (-\cdot-) \) and \( \{ a \} \) which are both in the clone. In the other direction, every element of \( f \in C_{i \times j} \) is contained in \( (\Lambda f \cdot -) \).

In the same way, we can show the one sorted case.

**Corollary 4.10.7** Every one-sorted relationally complete functional uniform preorder \( (A, R) \) is induced by an inclusion \( A_{\#} \subseteq A \) of pcas.

If we want non-relative versions of the previous two lemmas, we only have to add one condition. We will do this using terminology of Pitts [50, page 11].

**Definition 4.10.8** Let \( (I, A, R) \) be a finitely complete uniform preorder, and let \( (C_{i_1, \ldots, i_n ; j_1, \ldots, j_n})_{i_1, \ldots, i_n \in I} \) be the associated relational clone. A designated truth value is an \( a \in A_i \) such that \( \{ a \} \in C_{\otimes ; i} \).
Lemma 4.10.9  

- A one sorted uniform preorder \((A, R)\) is induced by a pca via the construction from Example 4.1.3-(ii), iff \((A, R)\) is relationally complete, functional, and all elements are designated truth values.
- A uniform preorder \((I, A, R)\) is induced by a typed pca iff \((I, A, R)\) is relationally complete, functional, and all elements are designated truth values.

Proof. The construction of the (typed) sub-pca in (4.10.4) gives the entire (typed) pca iff all truth values are designated. ■

4.11 Characterizations of realizability triposes and toposes

In this section, we assemble all our technology to obtain characterizations of realizability triposes and hyperdoctrines (Definitions A.1.7, A.1.8), and of the associated realizability categories (Definition A.1.9) together with their constant objects functors.

4.11.1 Realizability hyperdoctrines and triposes

Theorem 4.11.1 A posetal pre-stack \(X : |X| \to \text{Set}\) is equivalent to a relative realizability hyperdoctrine (Definition A.1.8- (ii)) iff

(i) \(X\) models the logical connectives \(\top, \land, \Rightarrow, \exists, \forall\), and
(ii) there exists a family \((A_i)_{i \in I}\) of sets, and a family \((\pi_i \in X_{A_i})_{i \in I}\) of \(\exists\)-prime predicates such that
- the subfibration of \(A \subseteq X\) generated by the \((\pi_i)_{i \in I}\) is closed under finite meets,
- all \(\pi_i\) are modest in \(A\), and
- \(A\) generates \(X\) under existential quantification.

Proof. First assume that \(X = \mathcal{H}(I, A, A\#)\) is a relative realizability hyperdoctrine. We claim that the family of identities \((\text{id}_{A_i})_i\) (more precisely the corresponding singleton maps) has the desired properties. By Example 4.7.4 we have \(X = \text{ufam}(D(I, A, R(A)))\), and \(A = \text{ufam}(I, A, R(A))\). \(A\) generates \(X\) under existential quantification by Lemma 3.4.10-(iii), and predicates in \(A\) are prime in \(X\) by Lemma 3.4.10. We explained in Remark 2.1.9 that \(\text{ufam}(I, A, R(A\#))\) is finitely complete, and while we didn’t explicitly prove it, it can be deduced from Lemma 4.6.6 that \(\mathcal{H}(I, A, A\#)\) models the claimed connectives. Finally, the predicates \(\text{id}_{A_i}\) are modest in \(A\) by Lemma 4.3.5.

In the other direction, let \((I, A, R)\) be the uniform preorder presentation of \(A\). \(A\) is functional by Lemma 4.3.5, and \(X \simeq DA\) by Lemma 3.4.10-(iii). This allows us to deduce that \((I, A, R)\) is relationally complete by Theorem 4.10.3, and thus induced by an inclusion of typed pcas by Lemma 4.10.6. ■

It is straightforward to derive a characterization of relative realizability triposes from the theorem.

Corollary 4.11.2 A posetal pre-stack \(X : |X| \to \text{Set}\) is equivalent to a relative realizability tripos (Definition A.1.7- (ii)) iff it satisfies the conditions of the theorem in such a way that the family \((\pi_i)_{i \in I}\) can be chosen to comprise a single predicate.

To characterize non-relative realizability, we have to add one more condition. Remark that since relative realizability hyperdoctrines \(X\) can be constructed by freely adding existential quantification to fibered meet-semilattices, they are totally connected, i.e. \(\delta : \text{sub(Set)} \to X\) has a finite meet preserving left adjoint \(\pi : X \to \text{sub(Set)}\)

12. with quantification along all maps, not just along projections
Corollary 4.11.3  (i) A posetal pre-stack $\mathcal{X} : |\mathcal{X}| \to \text{Set}$ is equivalent to a realizability hyperdoctrine (Definition A.1.8- (i)) iff it satisfies the conditions of the theorem, and $\gamma^{13} \cong \pi : \mathcal{X} \to \text{sub(S}et)$. 

(ii) $\mathcal{X}$ is equivalent to a realizability tripos Definition A.1.7- (i)), if the family of predicates $(\pi_i)_{i \in I}$ can moreover be chosen to comprise a single predicate.

Proof. It is easy to see that for realizability hyperdoctrines, $\pi$ does indeed coincide with $\gamma$. Conversely, it is sufficient by Lemma 4.10.9 to show that all truth values in a uniform meet-semilattice $\mathfrak{A}$ are designated iff $\gamma \cong \pi : ufam(D\mathfrak{A}) \to \text{sub(S}et)$, which is easy to see as well. ■

4.11.2 Realizability categories and toposes

The ‘modesty’ condition in Theorem 4.11.1 is a bit awkward since it is not a property of the $\pi_i$ as predicates in the fibration $\mathcal{X}$, but in the subfibration $\mathcal{A}$. If we want to characterize the fibered pretoposes associated to realizability categories (Definition A.1.9), this condition becomes replaced by a somewhat nicer condition involving the concept of discreteness that we introduce now (see also Remark 4.3.4 for a comparison of the concepts of modest and discrete).

Definition 4.11.4 Let $P : |P| \to R$ be a positive pre-stack. We call $D \in P_1$ discrete, if in any configuration

$$
\begin{array}{c}
V \\
\downarrow^e \\
U \\
\downarrow^h \\
\Downarrow^f \\
\Rightarrow D
\end{array}
\quad
\begin{array}{c}
K \\
\downarrow^p \\
J \\
\downarrow^u \\
\Downarrow^I
\end{array}
$$

where $U$ is subterminal in its fiber, $e$ is cartesian over the regular epi $p$, and $f$ is over $u \circ p$, there exists an $h$ over $u$ such that $he = f$. (In this case, $h$ is necessarily unique since cover-cartesian maps are collectively epic in pre-stacks.) ♦

Theorem 4.11.5 Let $\Delta : \text{Set} \to \mathcal{X}$ be a regular functor into an exact category, and $\mathcal{X} = \text{gl} \Delta(\mathcal{X})$ the associated fibered pretopos obtained by gluing. $\Delta$ is up to equivalence of the form $\Delta : \text{Set} \to \text{RC}(I, A, A^#)$ (Definition A.1.9) for an inclusion $(I, A^# \subseteq A)$ of typed pcas, iff

(i) $\mathcal{X}$ is locally cartesian closed

(ii) there exists a family $(\pi_i : D_i \mapsto \Delta A)_{i \in I}$ of monomorphisms in $\mathcal{X}$ such that

(a) all $\pi_i$ are indecomposable and projective in $\mathcal{X}$

(b) all $D_i \mapsto \Delta 1$ are discrete in $\mathcal{X}$

(c) the posetal subfibration $\mathcal{A} \subseteq \mathcal{X}$ generated by the $\pi_i$ is closed under finite meets and every $X \in |\mathcal{X}|$ can be covered by a $\varphi \in |\mathcal{A}|$ as in

$$
\varphi \mapsto S \xrightarrow{\varphi} X,
$$

where $s$ is cocartesian and $e$ is a vertical epimorphism.

Proof. First, assume that we are dealing with a functor $\Delta : \text{Set} \to \mathcal{X}(I, A, A^#)$, and let $\mathfrak{A} = (I, A, R(A^#))$ be the uniform preorder associated to the inclusion $(I, A^# \subseteq A)$ of typed pcas. Then $\mathfrak{A}$ is relationally complete, and $\mathcal{X} = \text{Set}(\mathfrak{A})$ is locally cartesian closed by Theorem 4.10.3. Since $\Delta^*\text{sub}(\mathcal{X}) = ufam(D\mathfrak{A})$, we can define the $\pi_i$ as predicates in the latter fibration, and we

13. See Section 4.8
set \( \pi_i \in \text{ufam}(D\mathfrak{A})_{A_i} \) to be the the singleton map corresponding to \( \text{id}_{A_i} \), keeping the indexing set \( I \) from \( (I, A_{\#} \subseteq A) \). With this choice of generators we have \( A = \text{ufam}(\mathfrak{A}) \), and \((\text{ii})\)a and \((\text{ii})\)c follow from Lemma 3.1.16.

It remains to verify the condition about discreteness. \( D_i \) is given by \((A_i, =_{\pi_i})\), and by instantiating Definition 4.11.4 with \( gl_{\Delta}(\mathfrak{X}) \), we see that we have to verify the existence of a mediator \( h \) in diagrams of the form

\[
\begin{array}{ccc}
N & \xrightarrow{n} & V \\
\downarrow & & \downarrow \\
M & \xrightarrow{m} & U \xrightarrow{h} D_i
\end{array}
\]

which live in the subcategory \( \text{Asm}(D\mathfrak{A}) \subseteq \text{Set}([D\mathfrak{A}]) \simeq \text{Set}(\mathfrak{A}) \) of assemblies. Now by Lemma 3.4.12 \( D\mathfrak{A} \) is totally connected, which implies with Lemma 3.4.30-(\text{ii}) and -\text{(iii)} that \( \text{Asm}(D\mathfrak{A}) \simeq [\text{ufam}(D\mathfrak{A})] \) - the total category of the subfibration of \( \text{ufam}(D\mathfrak{A}) \) on dense predicates (Definition 3.4.29). By restricting to the support if necessary, we can assume without loss of generality that \( m \) and \( n \) are dense in the diagram, and since denseness of a predicate \( \varphi : M \to PA_i \) in \( \text{ufam}(D\mathfrak{A}) \) means that it factors through \( P_{\pi_i}(A_i) \), it remains to show that \( \varphi e \leq \pi_i f \) for dense \( \varphi \) and epic \( e \) implies that there exists \( h : M \to A_i \) such that \( he = f \) and \( \varphi \leq \pi_i f \) (observe that compared with Definition 4.11.4, the triangle now lives in the base, not the total category). This follows from the fact that any relation realizing \( \varphi e \leq \pi_i f \) is functional, which forces \( f \) to be constant on the fibers of \( e \) since distinct elements of \( A_i \) have disjoint images under \( \pi_i \).

Conversely, assume that \( \Delta : \text{Set} \to \mathfrak{X} \) is an exact functor having the specified properties. Since the fibered preorder structure \( \mathfrak{X} = (I, A, R) \) whose underlying family of sets are the underlying sets of the \( \pi_i \) (Lemma 4.2.5). Since \( A \) has finite meets, the same is true for \( \mathfrak{X} \). Conditions \((\text{ii})\)a and \((\text{ii})\)c imply together with Lemma 3.1.16-(\text{ii}) that \( \mathfrak{X} = \mathfrak{X} \simeq \mathfrak{Sh}(D\mathfrak{A}) \), in particular \( \text{Set}([D\mathfrak{A}]) \simeq \text{Set}(\mathfrak{A}) \simeq \mathfrak{X} \) which implies by Theorem 4.10.3 that \( \mathfrak{A} \) is relationally complete. Since \( D\mathfrak{A} \) is totally connected by Lemma 3.4.12, \( \text{Set}([D\mathfrak{A}]) \simeq \mathfrak{X} \) has well behaved assemblies, which allows us to derive modesty of the \( \pi_i \) in \( A \) from discreteness of the maps \( D_i \to \Delta 1 \) using arguments similar to the ones above. Lemma 4.3.5 allows us then to derive that \( \mathfrak{A} \) is functional, and finally Lemma 4.10.6 allows us to deduce that \( \mathfrak{A} \) is induced by an inclusion of typed pca.

**Remarks 4.11.6** By Lemma 3.1.16-(\text{ii})b, \( A \) is weakly equivalent to the subfibration of \( \mathfrak{X} \) on indecomposable projectives. To deduce that \( \mathfrak{A} \) is finitely complete in \( \text{UOrd} \), it is however important to assume that \( A \) (and not only the fibration of indecomposable projectives) is closed under finite meets. From closure of indecomposable projectives under finite meets we can only deduce that \( \mathfrak{A} \) is finitely complete in the locally ordered category of left adjoints in \( \text{UDist} \) - i.e. the meet map is given by a distributor which has a left adjoint, but is not necessarily induced by a family of functions.

As for the characterization of hyperdoctrines and triposes, we can deduce untyped and non-relative versions of the theorem as corollaries.

**Corollary 4.11.7** Let \( \Delta : \text{Set} \to \mathfrak{X} \) be a regular functor into an exact category.

\((\text{i})\) \( \Delta \) is up to equivalence of the form \( \Delta : \text{Set} \to \text{RT}(A, A_{\#}) \) for an inclusion \( A_{\#} \subseteq A \) of pca, iff \( \Delta \) satisfies the conditions of Theorem 4.11.5 in such a way that the family \( (\pi_i : D_i \to \Delta A_i)_{i \in I} \) can be chosen to comprise a single mono.

\((\text{ii})\) \( \Delta \) is up to equivalence of the form \( \Delta : \text{Set} \to \text{RC}(I, A) \) for a typed pca \( (I, A) \), iff \( \Delta \) satisfies the conditions of the theorem, and moreover is right adjoint to the global sections functor \( \Gamma = \mathfrak{X}(1, -) : \mathfrak{X} \to \text{Set} \).
(iii) Δ is up to equivalence of the form Δ : Set → RT(A) for a pca A, iff Δ satisfies the conditions of the theorem in such a way that the family (π_i : Δ_i → ΔA_i)_{i ∈ I} can be chosen to consist of a single mono, and moreover Δ is right adjoint to the global sections functor Γ = X(1, −) : X → Set.

**Proof.** The only non-obvious part is (ii). The fact that Γ ⊣ Δ is well known for realizability over untyped pcas, and the relevant parts of the theory carry over to the typed case without change.

It remains to show that Γ ⊣ Δ : Set → RT(I, A, A#) implies A# = A. From Lemma 3.4.27 we know that Δ : Set → Set[X] has a finite limit preserving left adjoint Π whenever X is totally connected, thus we only have to show that Γ = Π implies A# = A.

In the proof of Corollary 4.11.3 showed that γ ⊣ π on the level of fibered posets implies A# = A, and it follows from Lemma 4.8.1 that γ ⊣ π whenever Γ = Π.

**Remark 4.11.8** Although our general approach is to characterize the categories together with their constant objects functors, we see that in the non-relative cases (ii) and (iii), the corollary gives us characterizations of the bare categories, since the constant objects functor is already determined by the fact that it is right adjoint to Γ in this case.

### 4.12 And on arbitrary bases?

The definition of uniform preorder can be internalized in any topos S (and with a bit of care even in predicative metatheories). However, on base categories other than Set, uniform preorders most naturally do not embed into posetal fibrations on S as one might naively expect, but rather into so-called fibered fibrations. The concept of ‘fibered fibration’ can be attributed to Bénabou, who realized that fibrations compose and more importantly that a fibration on a total category of another fibration can be viewed as ‘fibered fibration’ in the sense of ‘generalized category internal to another generalized category’ (the precise technical statement can be found in [57, Theorem 4.1]). In the following we are interested in the case where the ‘base fibration’ is a fundamental fibration.

**Definition 4.12.1** Let C be a category with finite limits. A fibered fibration on C is a fibration C ↓ C.

The following example is paradigmatic of our use of fibered fibrations.

**Definition 4.12.2** To any fibration E : |E| → Set on Set we can associate a fibered fibration ˜E : |E| → Set ↓ Set by setting

\[ ˜E(m : M \to L) = \prod_{l \in L} E_{M_l}. \]

(Note that this is not the fibered family construction from [57, Definition 6.2]).

**Lemma 4.12.3** Let E : |E| → Set be a fibration on Set.

(i) E has left/right adjoints to reindexing along vertical maps in cod(Set) iff E has left/right adjoints to reindexing along arbitrary maps in Set.

In this case the adjoints to reindexing in E satisfy the Beck Chevalley condition iff the adjoints to reindexing along vertical maps in E satisfy the Beck Chevalley condition for pullbacks along vertical maps.

(ii) E has left/right adjoints to reindexing along cartesian maps in cod(Set) iff the fibers of E have small (co)products.
In this case the adjoints to reindexing along cartesian maps in $\tilde{C}$ satisfy the Beck Chevalley condition for pullbacks along vertical maps iff the small (co)products in the fibers of $\tilde{C}$ are stable under pullback along arbitrary maps.

The Beck Chevalley condition in $\tilde{C}$ for pullbacks along cartesian maps is always satisfied. ■

Definition 4.12.4 Let $\mathfrak{A} = (I, A, R)$ be a uniform preorder internal to a topos $\mathcal{S}$. The fibered fibration $\text{ufam}(\mathfrak{A}) : |\mathfrak{A}| \to \mathcal{S} \downarrow \mathcal{S}$ is defined as follows.

- predicates on $(m : M \to L) \in \mathcal{S} \downarrow \mathcal{S}$ are commutative squares

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & A \\
\downarrow m & & \downarrow n \\
L & \xrightarrow{\psi} & I
\end{array}
\]

- $(u, \varphi) \leq (v, \psi)$ over $(m : M \to L)$ iff

$$\forall l : L. \{(\varphi m, \psi m) | m \in M_l\} \in R_{ul,vl}$$

in the internal logic. ◦

Remark 4.12.5 If we apply the construction from Definition 4.12.2 to the (ordinary) uniform family fibration $\text{ufam}(\mathfrak{A}) : |\text{ufam}(\mathfrak{A})| \to \text{Set}$ of a uniform preorder in $\text{Set}$, we obtain the uniform family fibration in the sense of Definition 4.12.4.

The intuition about $\text{ufam}(\mathfrak{A}) : |\text{ufam}(\mathfrak{A})| \to \mathcal{S} \downarrow \mathcal{S}$ for a uniform preorder $\mathfrak{A}$ in $\mathcal{S}$ is that the order on predicates over $m : M \to L$ is pointwise in $L$, but uniform in the fibers of $m$. In the case of realizability this means that for each $l$ there exists a realizer that works uniformly over $M_l$. ◦

Definition 4.12.6 Let $A : |A| \to \mathcal{S} \downarrow \mathcal{S}$ a fibered posetal fibration on $\mathcal{S}$.

- We say that $\leq$ is horizontally definable in $A$, if for every $\varphi, \psi \in A_{M \twoheadrightarrow L}$ there exists a greatest subobject $m : U \rightarrowtail L$ of $L$ such that $\varphi|_U \leq \psi|_U$.

\[
\begin{array}{ccc}
\psi|_U & \rightarrowtail & \psi \\
\forall l & & \\
\varphi|_U & \rightarrowtail & \varphi \\
\downarrow m & & \downarrow m \\
M & \xrightarrow{\varphi} & L
\end{array}
\]

- We call $A$ a fibered posetal pre-stack, if $e^*\varphi \leq e^*\psi$ implies $\varphi \leq \psi$ for every vertical epimorphism $e$ in $\mathcal{S} \downarrow \mathcal{S}$.

Lemma 4.12.7 The locally ordered category $\text{UOrd}(\mathcal{S})$ of uniform preorders internal to $\mathcal{S}$ is biequivalent to the locally ordered category of fibered posetal pre-stacks on $\mathcal{S}$ with horizontally definable $\leq$ and a generic predicate.
Proof. First of all we have to show that fibered fibrations of the form \( \text{ufam}(\mathfrak{A}) \) for internal uniform preorders \( \mathfrak{A} = (I, A, R) \) are fibered pre-stacks with horizontally definable \( \leq \). The pre-stack condition is immediate from the definition of \( \text{ufam}(\mathfrak{A}) \). For the horizontal definability of \( \leq \) let \( (u, \varphi), (v, \psi) \in \text{ufam}(\mathfrak{A}) \). 

\[
\begin{array}{c}
\begin{array}{c}
M \\
u
\end{array} \to \begin{array}{c}
A \\
\varphi
\end{array} \quad \begin{array}{c}
\begin{array}{c}
L \\
v
\end{array} \to \\
u
\end{array}
\end{array}
\]

Then the greatest subobject \( M \subseteq L \) such that \( \varphi|_U \leq \psi|_V \) is given by 

\[
L \supseteq M = \{ l \mid \{ (\varphi m, \psi m) \mid m \in M_l \} \in R_{u,l,v,l} \}.
\]

The proof that \( \mathfrak{A} \mapsto \text{ufam}(\mathfrak{A}) \) is a local equivalence is analogous to the proof of Lemma 4.2.2: Let \( \mathfrak{A} = (I, A, R) \) and \( \mathfrak{B} = (J, B, S) \) be uniform preorders internal to \( S \). To see that the construction is locally order reflecting, assume that \( (u, f), (v, g) : \mathfrak{A} \to \mathfrak{B} \) such that \( \text{ufam}(u, f) \leq \text{ufam}(v, g) : \text{ufam}(\mathfrak{A}) \to \text{ufam}(\mathfrak{B}) \). Then we have in particular \( \text{ufam}(u, f)_{A \to I}(\text{id}_I, \text{id}_A) \leq \text{ufam}(v, g)_{A \to I}(\text{id}_J, \text{id}_A) \), and since \( \text{ufam}(u, f) \) and \( \text{ufam}(v, g) \) act on predicates by postcomposition, and taking into account the definition of the ordering on monotone maps, this implies \( (u, f) \leq (v, g) \). To see that \( \text{ufam}(\neg) \) is essentially full, let \( F : \text{ufam}(\mathfrak{A}) \to \text{ufam}(\mathfrak{B}) \). An essential pre-image is given by \( F_{A \to I}(\text{id}_A, \text{id}_I) \).

Finally, we have to show that \( \text{ufam}(\neg) \) is bi-essentially surjective. Let \( A : |A| \to S \downarrow S \) be a fibered posetal pre-stack with horizontally definable \( \leq \) and generic predicate \( t \in A_{A \downarrow S} \).

Let \( P(a \times a) \to I \times I \) be the power object of \( A \times A \to A \times a \to I \times I \) in \( S/(I \times I) \), and let \( E \mapsto (A \times A) \times (I \times I) P(a \times a) \) be the associated membership predicate. Consider the diagram

\[
\begin{array}{c}
\begin{array}{c}
\pi_{i}|_{R} \leq \pi_{i}|_{E} \quad \pi_{e}|_{R} \quad \pi_{e}|_{E}
\end{array} \\
\begin{array}{c}
R \supseteq P(a \times a) \xrightarrow{p} A \quad I
\end{array}
\end{array}
\]

where the parallel horizontal pairs are the evident projections, and \( R \) is the maximal subobject of \( P(a \times a) \) such that \( p^*|_R \leq q^*|_R \). Then it is straightforward to check that \( (A, R) \) is an internal uniform preorder and that the associated fibered fibration is equivalent to \( A \).

Given an uniform preorder \( \mathfrak{A} \) in \( \text{Set} \), we saw earlier that the logical structure of \( \text{ufam}(\mathfrak{A}) \) can be characterized directly in \( \text{UOrd} - \mathfrak{A} \) has finite meets iff the diagonal and terminal projection maps of \( \mathfrak{A} \) have right adjoints, and it has quantification if \( \mathfrak{A} \) is an algebra for the corresponding monad \(^{14}\). The treatment of the propositional connectives generalizes straightforwardly to internal uniform preorders and fibered fibrations, but the quantifiers require attention: analogous to Lemma 4.12.3, the existence of a \( D \)-algebra structure on an internal uniform preorder \( \mathfrak{A} \) in a topos \( S \) corresponds to \( \text{ufam}(\mathfrak{A}) : |\text{ufam}(\mathfrak{A})| \to \text{Set} \) having existential quantification along

\(^{14}\) Actually we didn’t treat implication – to get a fibration-free treatment here we need \( \text{UDist} \). But in the following we are mainly concerned about quantification anyway.
vertical maps in $\text{cod}(S)$ -- quantification along cartesian maps would correspond to cocomplete fibers in the simply fibered case over $\text{Set}$, a property that is not important in the present work.

The emergence of the additional layer in the fibrations seems a bit frightening technically, especially since one can imagine situations where by iteration the layers get stacked up even further. Fortunately this can be avoided in an important special case as we will see now.

Let $\mathcal{A} : |\mathcal{A}| \to S \downarrow S$ be a fibered posetal fibration where $\leq$ is horizontally definable and where the fibers have greatest elements $\top$ which are stable under reindexing. The fibration $\mathcal{A}^{(1)} : |\mathcal{A}^{(1)}| \to S$ is the pullback of $\mathcal{A}$ along the functor

$$(M \mapsto (M \to 1)) : S \to S \downarrow S.$$  

We can define a fibered monotone map $\gamma : \mathcal{A}^{(1)} \to \text{sub}(\mathcal{A})$. The image of a predicate $\varphi \in \mathcal{A}^{(1)}_{M} = \mathcal{A}_{(M \to 1)}$ under $\gamma$ is given by first reindexing $\varphi$ in $\mathcal{A}$ onto $\text{id}_M$, and then taking the greatest subobject of $U \because M$ such that the restriction of the reindexing to $U$ is entailed by $\top$.

If $\mathcal{B} : |\mathcal{B}| \to \text{Set}$ is a fibered poset with pullback stable greatest elements, then the result of applying the previous construction to the fibration $\mathcal{B} : |\mathcal{B}| \to \text{Set} \downarrow \text{Set}$ from Definition 4.12.2, is precisely the transformation $\gamma$ defined in Section 4.8. Thus the coincidence of notation is justified, and moreover we see that the construction from Section 4.8 can be understood as a kind of comprehension principle after all.

Now let $(A, R)$ be a one-sorted uniform preorder with $\top, \land, \Rightarrow, \forall$ in a topos $S$ (over $\text{Set}$, these are exactly the regular triposes). We observe that since we only have one sort, the predicates in $\text{ufam}(A, R)_{M \to L}$ are in bijection with the predicates in $\text{ufam}(A, R)_{M \to 1}$ (both are simply morphisms $\varphi : M \to A$), and furthermore we have the following lemma.

**Lemma 4.12.8** Let $(A, R)$ be a one-sorted uniform preorder with $\top, \land, \Rightarrow, \forall$ in a topos $S$. Let $m : M \to L$ and $\varphi, \psi : M \to A$. Then $\varphi \leq \psi$ in $\text{ufam}(A, R)_m$ iff $\top \leq (\forall m (\varphi \Rightarrow \psi))$ in $\text{ufam}(A, R)^{(1)}$.

**Proof.** Consider the cube

$$
\begin{array}{ccc}
M & \to & L \\
\downarrow & & \downarrow \\
1 & \to & 1 \\
\end{array}
$$

which is a pullback square in $S \downarrow S$ (since both horizontal faces are). To avoid confusion, we denote the maps $\varphi, \psi$ by $\varphi_0, \psi_0$ when regarding them as predicates on $M \to 1$, and by $\varphi, \psi$ when regarding them as predicates on $M \to L$. Now we have of course that $(\text{id}_M, 1)^* \varphi_0 = \varphi$ and
(\text{id}_M, !_L)^* \psi_0 = \psi$, and we can argue

\begin{align*}
\varphi \leq \psi & \quad \text{in } \text{ufam}(A, R) \quad \text{iff} \\
\top \leq \varphi \Rightarrow \psi & \quad \text{in } \text{ufam}(A, R) \quad \text{iff} \\
\top \leq (\text{id}_M, !_L)^* (\varphi_0 \Rightarrow \psi_0) & \quad \text{in } \text{ufam}(A, R) \quad \text{iff} \\
\top \leq \forall (m, \text{id}_1) ((\text{id}_M, !_L)^* (\varphi_0 \Rightarrow \psi_0)) & \quad \text{in } \text{ufam}(A, R) \quad \text{iff} \\
\top \leq \gamma (\forall m (\varphi_0 \Rightarrow \psi_0)) & \quad \text{in } \text{ufam}(A, R)^{(1)}
\end{align*}

where the two last steps follow from the Beck-Chevalley condition and the definition of $\gamma$, respectively. 

Thus, all the ordering structure of $\text{ufam}(A, R)$ can be encoded in terms of $\text{ufam}(A, R)^{(1)}$ and $\gamma$. 

In order to get a characterization of one-sorted internal uniform preorders with $\top$, $\land$, $\Rightarrow$, $\forall$ in terms of fibered preorders and $\gamma$, it remains to characterize the fibered monotone maps $\gamma$ that arise in from the construction described before the lemma. For the purposes of this section, we use the following definitions.

**Definition 4.12.9** Let $S$ be a topos.

(i) A fibered tripos on $S$ is a fibered posetal pre-stack $X : |\mathcal{P}| \to S \downarrow S$ with horizontally definable $\leq$ whose fibers are pre-Heyting algebras, which has universal quantification along vertical maps (subject to BC along arbitrary maps), and a generic predicate $tr \in X(\text{Prop} \rightarrow 1)$.

(ii) A ‘tripos with $\gamma$’ on $S$ is a posetal pre-stack $P : |\mathcal{P}| \to S$ which models $\top$, $\land$, $\Rightarrow$, $\forall$ and has a generic predicate $tr \in P_{\text{Prop}}$, together with a finite meet preserving fibered monotone map $\gamma : P \to \text{sub}(S)$ satisfying

\[ \top \leq \gamma(p) \Rightarrow \top \leq p \quad \text{for } p \in P_1. \]

Observe that every regular tripos on $\text{Set}$ is uniquely a ‘tripos with $\gamma$’ – the condition that $\top$ is reflected over 1 already forces $\gamma$ with to coincide with the transformation defined in Section 4.8.

**Theorem 4.12.10** Let $S$ be a topos. The following concepts are equivalent.

- internal one-sorted uniform preorders with $\top$, $\land$, $\Rightarrow$, $\forall$
- fibered triposes on $S$
- triposes with $\gamma$ on $S$

**Proof.** The equivalence of internal uniform preorders with the specified properties and fibered triposes follows from Lemma 4.12.7 and our remarks about propositional connectives and quantification in fibered fibrations.

We described before Lemma 4.12.8 how to obtain the fibration $X^{(1)}$ from a fibered fibration $X$, and how to construct $\gamma$ using the horizontal definability of $\leq$.

Conversely, to construct a fibered tripos $X$ from a tripos $\mathcal{P}$ with $\gamma$, we take predicates in $X_{(M \Rightarrow L)}$ to be predicates in $P_M$, and define the ordering by

\[
\varphi \leq \psi \iff \top \leq \gamma (\forall m (\varphi \Rightarrow \psi))
\]

as in Lemma 4.12.8.

We leave the numerous verification necessary to establish that the constructions are well defined and mutually inverse to the reader. ■
Remarks 4.12.11  

(i) We could have phrased the previous result as a biequivalence of locally ordered categories, but it doesn’t really matter which morphisms we consider – the statement only depends on the concept of ‘equivalence’ of fibrations.

(ii) I find it remarkable that the logical structure that is necessary to make the theorem work is exactly that of a tripos. For me that is a strong indication that the above concept of ‘tripos with $\gamma$’ might be a good alternative to the usual definition of tripos on base toposes other than $\text{Set}$, especially when envisioning a ‘geometric theory of triposes’ which views constant objects functors as generalizations of geometric morphisms.

Example 4.12.12  
On base toposes $\mathcal{S}$ other than $\text{Set}$, it is possible that the same tripos $\mathcal{P}$ on $\mathcal{S}$ can be equipped with different maps $\gamma: \mathcal{P} \to \text{sub}(\mathcal{S})$ corresponding to different uniform preorders and giving rise to non-equivalent fibered triposes. We demonstrate this using as tripos the subobject fibration $\text{sub}(\widehat{2})$ of the Sierpinski topos $\widehat{2}$ ($\widehat{2} = (0 \to 1)$).

On the one hand, $\text{sub}(\widehat{2})$ is the externalization of the internal preorder $\Omega$, which can be viewed as uniform preorder via the construction from Example 4.1.3-(i). The corresponding transformation $\gamma : \text{sub}(\widehat{2}) \to \text{sub}(\widehat{2})$ is just the identity, and the order on predicates in the associated fibered fibration is given by

$$U \leq V \text{ over } A \to I \text{ iff } U \subseteq V$$

for subobjects $U, V \subseteq A$, in other words the fibered fibration is just given by the fibered family construction ([57, Definition 6.2])

$$\begin{array}{ccc}
\text{Sub}(\widehat{2}) & \xrightarrow{\gamma} & \text{sub}(\widehat{2}) \\
\downarrow & & \downarrow \\
\widehat{2} & \to & \text{sub}(\widehat{2}), \\
\downarrow & & \downarrow \\
\partial_1 & & \partial_1
\end{array}$$

where $\partial_1$ is the projection on the domain.

A different transformation $\gamma' : \text{sub}(\widehat{2}) \to \text{sub}(\widehat{2})$ satisfying the conditions of Definition 4.12.9-(ii) is given by

$$\begin{array}{ccc}
U_0 & \xleftarrow{m_0} & U_1 \\
\gamma \downarrow & & \downarrow \gamma' \\
A_0 & \xleftarrow{id} & A_1 \\
\downarrow & & \downarrow \\
A_0 & \xleftarrow{m_1} & A_1
\end{array}$$

To give an explicit description of the associated internal uniform preorder, remark that a uniform preorder structure on $\Omega$ is a subobject $R \subseteq P(\Omega \times \Omega)$, which amounts to a set of binary relations on $\Omega$ for $R_1$ and a set of binary relations on $\Omega_0$ for $R_0$, such that the obvious inclusion holds. In our case, $R_1$ is the set of subrelations of the implication relation on $\Omega$, and $R_0$ consists of all relations on $\Omega_0 \times \Omega_0$.

Finally, the fibered fibration associated to $\gamma'$ is given by $U \leq V \text{ over } A \to I \text{ iff } (\forall_a(U \Rightarrow}$
$V \cup V) = \top$ for $U, V \subseteq A$ as in the following diagram,

which concretely means that $U_1 \subseteq V_1$, and $U_0 \cap a_0^{-1}(I \leq I_1)) \subseteq V_0$. As two extreme cases, the ordering of predicates coincides with the ordering by inclusion whenever $I \leq$ is surjective, and the ordering collapses if $I_1$ is empty.

\diamondsuit
Appendix

A.1 Partial combinatory algebras and triposes

In this appendix, we recall classical concepts of categorical realizability which don’t have a natural place in the main text.

A.1.1 Partial combinatory algebras

**Definition A.1.1** A (weak) partial combinatory algebra (pca) is a set $A$ together with a partial binary operation $(-·-): A \times A \rightarrow A$ such that there exist $k, s \in A$ satisfying for all $x, y, z \in A$ the conditions

\[
\begin{align*}
  k \cdot x \cdot y &= x \\
  s \cdot x \cdot y &\downarrow \\
  s \cdot x \cdot y \cdot z &\succeq x \cdot z \cdot (y \cdot z). 
\end{align*}
\]

We refer to Section 1.5 for the notations $t \downarrow$ and $s \succeq t$. We use the usual convention that the binary application associates to the left; thus, for example, $s \cdot x \cdot y \cdot z$ should be read as $(s \cdot x) \cdot y \cdot z$. We usually omit the dot and write application simply by juxtaposition. The notion of the above definition is usually called weak pca, ‘strong’ pcas being those where the last condition is replaced by the strong equality $s \cdot x \cdot y \cdot z \simeq x \cdot z \cdot (y \cdot z)$. However, as we deal exclusively with the weak version in this text, we will simply call them pcas.

**Example A.1.2** The archetypal example of a pca is the so-called first Kleene algebra $K_1$ which has as underlying set the set $\mathbb{N}$ of natural numbers, and where the application operation is given by

\[n \cdot m = \phi_n(m),\]

where $(\phi_n)_{n \in \mathbb{N}}$ is an effective enumeration of partial recursive functions (see [61, Section 1.4.1]).

Next, we introduce Longley’s typed pcas [39], whose relation to ordinary pcas is analogous to the relation between typed and untyped λ-calculus. As for pcas, there is a ‘weak’ and a ‘strong’ version and we use the weak one.

**Definition A.1.3** A typed partial combinatory algebra (typed pca) is a pair $(I, A) = (I, (A_i)_{i \in I})$ consisting of

(i) a set $I$ of ‘types’, equipped with binary operations

\[
(- \ast -), (- \Rightarrow -): I \times I \rightarrow I,
\]

(ii) a family $(A_i)_{i \in I}$ of sets, with for each pair $i, j \in I$ of types a partial ‘application’ map

\[(- \cdot ij -): A_{i \Rightarrow j} \times A_i \rightarrow A_j,\]

where $(\phi_n)_{n \in \mathbb{N}}$ is an effective enumeration of partial recursive functions (see [61, Section 1.4.1]).

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such that for all $i, j, k \in I$ there exist elements

$$k_{ij} \in A_{i \Rightarrow j \Rightarrow i}$$

$$s_{ijk} \in A_{(i \Rightarrow j \Rightarrow k) \Rightarrow (i \Rightarrow j) \Rightarrow i \Rightarrow k}$$

$$\text{pair}_{ij} \in A_{i \Rightarrow j \Rightarrow (i \ast j)}$$

$$\text{fst}_{ij} \in A_{(i \ast j) \Rightarrow i}$$

$$\text{snd}_{ij} \in A_{(i \ast j) \Rightarrow j}$$

satisfying

$$k_{xy} = x$$

$$\text{fst}(\text{pair}_{xy}) = x$$

$$s_{xy} \downarrow$$

$$s_{xyz} \succeq xz(yz)$$

$$\text{snd}(\text{pair}_{xy}) = y$$

for all appropriately typed $x, y, z$.

As usual, the type constructor ($- \Rightarrow -$) for function spaces associates to the right.

We do not go into the details of the theory of pcas – for the untyped case we refer the reader to [61], and the typed case is not much different (certain constructions, such as fixed point combinators, don’t work in the typed case).

Definition A.1.4 Let $A$ be a pca. A sub-pca of $A$ is a subset $A\# \subseteq A$ which is closed under application in the sense that

$$a, b \in A\#, a \cdot b \downarrow \Rightarrow a \cdot b \in A\#,$$

such that the combinators $k, s$ for $A$ can be chosen in $A\#$.

In the same way, a typed sub-pca of a typed pca $(I, A)$ is a family $(A\#, i) \subseteq A_i) \in I$ of subsets which is closed under application in the sense that

$$a \in A\#, i, b \in A\#, i \Rightarrow j, a \cdot b \downarrow \Rightarrow a \cdot b \in A\#, j$$

and such that all the combinators for $(A_i)_{i \in I}$ can be chosen in the subsets.

A (typed) pca together with a (typed) sub-pca is also called an inclusion of pcas. Inclusions of pcas are important in relative realizability and occur naturally in our reconstruction. We denote inclusions of pcas and of typed pcas by $(A\# \subseteq A)$, and $(I, A\# \subseteq A)$, respectively.

A.1.2 Triposes

Informally, triposes – introduced in [28] and studied further in [50, 51] – are fibrational models of higher order intuitionistic logic.

In one sentence, a tripos on a cartesian closed category $C$ is a complete fibered Heyting pre-algebra $P$ with a generic predicate. In a bit more detail, this means the following.

Definition A.1.5 A tripos on a cartesian closed category $C$ is a fibered preorder

$$P : [P] \rightarrow C$$

such that

1. called elementary sub-pca in [61]
2. It is possible to define triposes on categories having only finite limits [50], or even finite products [51, 61, 17], but this is not relevant here.
(i) all fibers $\mathcal{P}_C$ for $C \in \mathcal{C}$ are Heyting pre-algebras, and the Heyting pre-algebra structure is preserved by reindexing.

(ii) $\mathcal{P}$ has internal products in the sense of [57, Section 7], and

(iii) generic predicate $tr \in \mathcal{P}_{\text{Prop}}$. ♦

Using the postulated structure, we can interpret the $\forall, \Rightarrow, \wedge, \top$ fragment of predicate logic in a tripos, and one can use the generic predicate and higher order encodings to interpret the remaining connectives $\exists, \forall, \bot$. If the base category is regular and the tripos is a pre-stack, it is thus in particular a fibered frame in the sense of Definition 3.4.1. We will refer to a tripos which is a pre-stack as a regular tripos.

It seems reasonable to only work with regular triposes, since all known constructions seem to give rise to triposes satisfying the pre-stack condition, and moreover Pitts’ [50] important iteration theorem depends on it. On the other hand, the condition is not vacuous – we will now state an example (due to Streicher) of a tripos which is not a pre-stack.

Example A.1.6 (Streicher) Let $\text{Set}^{• \rightarrow •}$ be the topos of non-reflexive graphs. Define the fibered preorder $\mathcal{P}$ on $\text{Set}^{• \rightarrow •}$ by the pullback

$$
\begin{array}{ccc}
\text{Set}^{• \rightarrow •} & \xrightarrow{\Gamma} & \text{Set} \\
\mathcal{P} & \xleftarrow{\text{Sub}(\text{Set})} & \text{Sub}(\text{Set})
\end{array}
$$

where $\Gamma$ is the global sections functor which sends each graph to its set of loops. $\mathcal{P}$ interprets full first order logic since $\text{sub}(\text{Set})$ does and the relevant structure is stable under change of base along finite limit preserving functors. Given a graph $G$, a predicate in $\mathcal{P}_G$ is a subset of $\Gamma(G)$, i.e. a set of loops. This intuition allows us to construct a generic predicate — $\text{Prop}$ is the graph with one vertex and two loops, and $tr$ singles out one of the two loops. Thus $\mathcal{P}$ is a tripos. To see that $\mathcal{P}$ is not a pre-stack, take $I$ to be the graph $(\bullet \rightarrow \bullet)$ with two vertices connected by one edge. Its terminal projection $!: I \rightarrow 1$ is an epimorphism, but $\top_1 \not\vDash \exists_I \top_I$, which contradicts the pre-stack property. ♦

Having defined pcas and triposes, we will now explain how to construct triposes from pcas. These so-called realizability triposes traditionally belong to the central concepts of categorical realizability.

Definition A.1.7 (i) Let $\mathcal{A}$ be a pca. The realizability tripos $\text{rt}(\mathcal{A}) : |\text{rt}(\mathcal{A})| \rightarrow \text{Set}$ is defined as follows

- Predicates on a set $M$ are functions $\varphi : M \rightarrow P(\mathcal{A})$ into the power set of $\mathcal{A}$.
- For predicates $\varphi, \psi : M \rightarrow P(\mathcal{A})$ the ordering is defined by $\varphi \leq \psi : \Leftrightarrow \exists c : \mathcal{A} \forall m : M \forall a \in \varphi(m). ea \in \psi(m)$.
- Reindexing is given by precomposition.

(ii) Let $(\mathcal{A}_\# \subseteq \mathcal{A})$ be an inclusion of pcas. The relative realizability tripos $\text{rt}(\mathcal{A}, \mathcal{A}_\#) : |\text{rt}(\mathcal{A}, \mathcal{A}_\#)| \rightarrow \text{Set}$ is defined as follows.

- Predicates on a set $M$ are functions $\varphi : M \rightarrow P(\mathcal{A})$. 108
For predicates $\varphi, \psi : M \to P(A)$, the ordering is defined by
$$\varphi \leq \psi : \iff \exists e : A_\# \forall m : M \forall a \in \varphi(m). ea \in \psi(m)$$

Reindexing is given by precomposition.

We can do analogous constructions for typed pcas, and this has been done in [37, Definition 3.1-(iv)], but we can not expect the result to be a tripos anymore.

**Definition A.1.8**

(i) Let $(I, A)$ be a typed pca. The realizability hyperdoctrine $\mathcal{H}(I, A) : \mathcal{H}(I, A) \to \text{Set}$ is defined as follows.

- Predicates on a set $M$ are pairs $(i \in I, \varphi : M \to PA_\#)$.
- For predicates $(i, \varphi), (j, \psi)$ on $M$, the ordering is defined by
  $$(i, \varphi) \leq (j, \psi) : \iff \exists e : A_{\#,i \to j} \forall m : M \forall a \in \varphi(m). ea \in \psi(m).$$

- Reindexing is given by precomposition.

(ii) Let $(I, A_\# \subseteq A)$ be an inclusion of typed pcas. The relative realizability hyperdoctrine $\mathcal{R}(I, A, A_\#) : \mathcal{H}(I, A, A_\#) \to \text{Set}$ is defined as follows.

- Predicates on a set $M$ are pairs $(i \in I, \varphi : M \to PA_\#)$.
- For predicates $(i, \varphi), (j, \psi)$ on $M$, the ordering is defined by
  $$(i, \varphi) \leq (j, \psi) : \iff \exists e : A_{\#,i \to j} \forall m : M \forall a \in \varphi(m). ea \in \psi(m).$$

- Reindexing is given by precomposition.

All (relative) realizability triposes and hyperdoctrines interpret the $(\top, \land, \Rightarrow, \exists, \forall)$-fragment of first order logic, in particular they are fibered frames. Thus, we can construct their categories of partial equivalence relations, for which we use the following terminology.

**Definition A.1.9**

- Given a pca $A$, the realizability topos $\text{RT}(A)$ is the category $\text{Set}[\text{rt}(A)]$ of partial equivalence relations in $\text{rt}(A)$.
- For an inclusion $A_\# \subseteq A$ of pcas, the relative realizability topos $\text{RT}(A, A_\#)$ is the category $\text{Set}[\text{rt}(A, A_\#)]$.
- Given a typed pca $(I, A)$, the realizability category $\text{RC}(I, A)$ is the category $\text{Set}[\mathcal{H}(I, A)]$.
- Given an inclusion $(I, A_\# \subseteq A)$ of typed pcas, the relative realizability category $\text{RC}(I, A, A_\#)$ is the category $\text{Set}[\mathcal{H}(I, A, A_\#)]$.

**Remark A.1.10** Given an inclusion $(I, A_\# \subseteq A)$ of typed pcas, it can be deduced from Theorem 3.4.25 and the fact that $\mathcal{H}(I, A, A_\#) \simeq D(\text{ufam}(I, A, A_\#))$ (Example 3.4.13) that $\text{RC}(I, A, A_\#)$ is locally cartesian closed. However, in general $\text{RC}(I, A, A_\#)$ does not seem to be a pretopos since it doesn’t have finite coproducts. Longley [39] writes:

"[...] since RC(A) doesn’t automatically have binary coproducts. But under a mild extra condition that we can “simulate” the booleans within $A$, we do."

**A.2 A first factorization result**

Remark 4.12.11-(ii) alluded to a possible ‘geometric theory of triposes’ which views constants objects functors associated to triposes in analogy to geometric morphisms. From Pitts’ iteration theorem we know that these functors compose (I expect this to generalize to ‘triposes with $\gamma$”).
Following the analogy to geometric morphisms, the natural question to ask is whether they can also be decomposed, paralleling the known factorization theorems for geometric morphisms (see [31, 32]).

Here I present a first such result.

**Definition A.2.1** Let $(\mathcal{P} : \mathcal{P} \to \mathcal{S}, \gamma : \mathcal{P} \to \text{sub}(\mathcal{S}))$, be a tripos with $\gamma$ (Definition 4.12.9).
- $(\mathcal{P}, \gamma)$ is called **realizability-like** 3, if $\gamma \dashv \delta$
- $(\mathcal{P}, \gamma)$ is called **localic**, if $\delta \dashv \gamma \phantom{x}$

**Remark A.2.2** It is clear that any internal locale $X$ in $\mathcal{S}$ gives rise to a localic tripos with $\gamma$.
Conversely, if $(\mathcal{P}, \gamma)$ is a localic tripos with $\gamma$ then the constant objects functor $\Delta : \mathcal{S} \to \mathcal{S}[\mathcal{P}]$ has a right adjoint and is thus a local geometric morphism, which shows that the terminology makes sense.

In the following we want to show that any tripos on $\mathbf{Set}$ can be decomposed into a realizability-like part and a localic part. To this end, we first recall a result from Birkedal’s thesis [7] about categories of assemblies. In [7, Definition 3.3.1] Birkedal defines a category $\mathbf{Asm}(X)$ (which we will call $B\mathbf{-Asm}(X)$ to distinguish it from the assemblies of Section 3.4.3) for any existential fibration $X : [\mathcal{X}] \to \mathcal{C}$ on a finite product category $\mathcal{C}$. For the moment we only consider the case with base category $\mathbf{Set}$ as it spares us to spell out some subtleties. For good measure, we shall also assume that the existential fibration satisfies the pre-stack condition.

**Definition A.2.3** Let $X : [\mathcal{X}] \to \mathbf{Set}$ be a fibered frame.
- The category $B\mathbf{-Asm}(X)$ of Birkedal assemblies has
  1. pairs $(M, \varphi)$ as objects, where $\varphi \in X_M$ such that $\gamma(\varphi) \equiv \top$ (as defined in Section 4.8), and
  2. functions $f : M \to N$ such that $\varphi \leq f^* \psi$ as morphisms from $(M, \varphi)$ to $(N, \psi)$.
- The functor $\Delta : \mathbf{Set} \to B\mathbf{-Asm}(\mathcal{P})$ is given by $M \mapsto (M, \top)$.

Birkedal shows that for any existential fibration $X$, $B\mathbf{-Asm}(X)$ is regular and $\Delta$ is right adjoint to the global sections functor $\Gamma = B\mathbf{-Asm}(X)(1, -)$ (in particular it preserves finite meets); if $X$ is a fibered frame then $\Delta$ is moreover regular. In [7, Theorem 3.5.1], Birkedal shows that if $X$ models $\Rightarrow, \forall$ in addition to being a fibered frame, then $B\mathbf{-Asm}(X)$ is locally cartesian closed.

**Theorem A.2.4** Let $\mathcal{P} : [\mathcal{P}] \to \mathbf{Set}$ be a regular tripos. Then $\Delta : \mathbf{Set} \to \mathbf{Set}[\mathcal{P}]$ can be factorized into two functors, where the first one is the constant objects functor of a realizability-like tripos and the second one is the constant objects functor of a localic tripos.

**Proof.** We define a fibered poset $\mathcal{Q} : [\mathcal{Q}] \to \mathbf{Set}$ by taking the pullback

$$
\begin{array}{ccc}
[\mathcal{Q}] & \xrightarrow{\text{Sub}(B\mathbf{-Asm}(\mathcal{P}))} & \text{Sub}(B\mathbf{-Asm}(\mathcal{P})) \\
\downarrow & & \downarrow \\
\mathbf{Set} & \xrightarrow{\Delta} & B\mathbf{-Asm}(\mathcal{P})
\end{array}
$$

of $\text{sub}(B\mathbf{-Asm}(\mathcal{P}))$ along $\Delta : \mathbf{Set} \to B\mathbf{-Asm}(\mathcal{P})$. Since $B\mathbf{-Asm}(\mathcal{P})$ is regular and locally cartesian closed and $\Delta$ is regular, $\mathcal{Q}$ is a fibered frame with $\Rightarrow, \forall$. Concretely, $\mathcal{Q}$ is given as follows.

- Predicates on $M$ are pairs $(U \subseteq M, \varphi \in \mathcal{P}_M)$ such that $\gamma(\varphi) \equiv \top$.

3. Suggestions for better terminology welcome – maybe ‘shallow’?
4. ‘regular fibration’ in his terminology
– $(U, \varphi) \leq (V, \psi)$ iff $U \subseteq V$ and $\varphi \leq \psi |_U$.
– Reindexing is given by pullback and reindexing in $\mathcal{P}$.

To show that $\mathcal{Q}$ is a tripos it remains to construct a generic predicate. The underlying object is given by $\text{Prop}_{\mathcal{Q}} = \widehat{\gamma (\text{tr}_\mathcal{P})}$ – the partial map classifier of the subobject $\gamma (\text{tr}_\mathcal{P}) \subseteq \text{Prop}_\mathcal{P}$, and $\text{tr}_\mathcal{Q}$ is given by the inclusion $\gamma (\text{tr}_\mathcal{P}) \hookrightarrow \widehat{\gamma (\text{tr}_\mathcal{P})}$ together with the predicate $\text{tr}_\mathcal{P} |_{\gamma (\text{tr}_\mathcal{P})}$. Thus, $\mathcal{Q}$ is a tripos, and it is straightforward to verify that $\mathcal{Q}$ is realizability-like.

We now construct a geometric morphism between $\mathcal{Q}$ and $\mathcal{P}$. Define $f^* : \mathcal{Q} \to \mathcal{P}$ by

$$\mathcal{Q}_M \ni (U, \varphi) \mapsto (\exists U \varphi) \in \mathcal{P}_M,$$

where $\exists U'$ is a shorthand for existential quantification along the inclusion $U \hookrightarrow I$. Define $f_* : \mathcal{P} \to \mathcal{Q}$ by

$$\mathcal{P}_M \ni \varphi \mapsto (\gamma (\psi), \varphi |_{\gamma (\psi)}) \in \mathcal{Q}_m.$$

It is easy to see that the fibered monotone maps $f^*$ and $f_*$ constitute a geometric morphism of triposes and thus by [61, Theorem 2.5.8] give rise to a localic geometric morphism $F^* \dashv F_* : \text{Set}[\mathcal{P}] \to \text{Set}[\mathcal{Q}]$ of toposes with $F^*$ preserving constant objects, or in other words $F^* \circ \Delta_{\mathcal{Q}} \cong \Delta_{\mathcal{P}}$.

\[\blacksquare\]

**Remarks A.2.5**

(i) The presented decomposition is not unique. This can be seen from relative realizability. Given an inclusion $\mathcal{A}_\# \subseteq \mathcal{A}$, the relative realizability topos $\text{RT}(\mathcal{A}, \mathcal{A}_\#)$ is localic over $\text{RT}(\mathcal{A}_\#)$ (see e.g. the introduction of [2]), which gives a decomposition of $\Delta : \text{Set} \to \text{RT}(\mathcal{A}, \mathcal{A}_\#)$ into a realizability-like followed by a localic part. However, this is not the factorization given by the above construction – the tripos $\mathcal{Q}$ obtained by applying the construction to the tripos $\text{RT}(\mathcal{A}, \mathcal{A}_\#)$ is equivalent to the subfibration of $\text{RT}(\mathcal{A}, \mathcal{A}_\#)$ on predicates $\varphi : M \to \mathcal{P}_A$ satisfying

$$m : M \mid \exists a : A. a \in \varphi (m) \vdash \exists a : A_\#. a \in \varphi (m),$$

which looks quite different from $\text{rt}(\mathcal{A}_\#)$.

One should search for strengthenings of the concepts ‘realizability-like’ and ‘localic’, which make the decomposition unique up to equivalence.

(ii) Given a tripos $\mathcal{P}$, we have $\text{B-Asm}(\mathcal{P}) \approx \text{Asm}(\mathcal{Q})$ ($\mathcal{Q}$ is the tripos constructed in the proof), which reconciles the definitions of assemblies of Birkedal and van Oosten in a certain sense.

(iii) Using ‘triposes with $\gamma$’, I hope that the factorization works over arbitrary bases. The necessity of $\gamma$ is clear from the definition of $\text{B-Asm}(\mathcal{P})$.

\[\Diamond\]

### A.3 Bits and pieces, open ends

In the following, I present some open questions and ideas that have not been fully explored yet.

#### A.3.1 Uniform preorders as internal preorders

A small presheaf on a locally small category $\mathcal{C}$ is a small colimit of representable presheaves in $\text{Set}^{\mathcal{C}^{\text{op}}}$. Following [16], we denote the category of small presheaves on $\mathcal{C}$ by $\mathcal{P}\mathcal{C}$. $\mathcal{P}\mathcal{C}$ is locally small and can abstractly be characterized as the small colimit cocompletion of $\mathcal{C}$. If $\mathcal{C}$ is small, then $\mathcal{P}\mathcal{C} = \hat{\mathcal{C}}$.  

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The category \( \mathcal{P} \text{-} \mathbf{Set} \) of small presheaves on \( \mathbf{Set} \) is a locally cartesian closed \( \infty \)-pretopos, but it is not a topos (Giraud’s theorem fails since we do not have a small cogenerating family, and the truth value object \( \Omega : \mathbf{Set}^{\mathbf{Set}} \to \mathbf{Set} \) is definable, but not small). Using the axiom of choice one can show that uniform preorders (and thus fibered preorders with a generic family of predicates) are equivalent to preorders internal to \( \mathcal{P} \text{-} \mathbf{Set} \), which means that the theory of uniform preorders is order theory in a predicative framework.\(^5\)

This point of view is a rich source of intuitions, and explains for example why the uniform distributors of Section 4.9 can not be represented as a Kleisli category in the same way as ordinary distributors (the monad applied to 1 would yield the truth value object of \( \mathcal{P} \text{-} \mathbf{Set} \) which does not exist). As another example, given a uniform frame \( \mathfrak{A} \) it is possible to view \( \mathbf{Set}[\mathfrak{A}] \) as a subcategory of the category of internal presheaves on \( \mathfrak{A} \) in \( \mathcal{P} \text{-} \mathbf{Set} \) – intuitively it is only a subcategory since the category of internal presheaves contains coproducts of elements of \( \mathfrak{A} \) over arbitrary indexing objects in \( \mathcal{P} \text{-} \mathbf{Set} \) whereas \( \mathbf{Set}[\mathfrak{A}] \) only contains the coproducts with respect to indexing objects in the image of \( Y : \mathbf{Set} \to \mathcal{P} \text{-} \mathbf{Set} \).

If we want to do all this without relying on the axiom of choice, we have to replace \( \mathcal{P} \text{-} \mathbf{Set} \) by a category of ‘small sheaves for the regular topology’. The problem is that it is not entirely clear how to define this category. One approach would be to take the subcategory of \( \mathcal{P} \text{-} \mathbf{Set} \) on sheaves for the regular topology, but is not clear whether this category is closed under colimits. Another approach would be to take small colimits of representables in the subcategory of \( \mathbf{Set}[\mathfrak{A}] \) on sheaves, but here we run into similar problems. For me, it appears to be the safest option to take a larger universe \( \mathbf{SET} \) of sets with respect to which \( \mathbf{Set} \) is small, and then to take the category of large sheaves on \( \mathbf{Set} \) with respect to the regular topology, that is the category of functors \( F : \mathbf{Set}^{\mathfrak{A}} \to \mathbf{SET} \) which are sheaves for the regular topology. This category is then a topos, and we can define the category of ‘small regular sheaves’ as small colimits of representables in ‘large regular sheaves’.

### A.3.2 Uniform preorders as enriched categories

Tom Hirschowitz made the remarkable observation that uniform preorders can be defined as categories enriched in a quantaloid (see e.g. [58]). Recall that a quantaloid is a category which is enriched in cocomplete lattices, thus can be viewed as a locally ordered 2-category. Bénabou [4] defined a category enriched in a bicategory to be a lax functor from an indiscrete category into a bicategory, and it is in this sense that ‘quantaloid-enriched’ has to be read here.

The quantaloid \( \mathcal{R} \) of interest is defined as follows.

- objects are sets
- a morphism from \( M \) to \( N \) is a downward closed set \( R \subseteq P(M \times N) \) of relations from \( M \) to \( N \)
- the ordering on \( \mathcal{R}(M,N) \) is given by inclusion
- composition is given by \( S \circ R = \downarrow \{ s \circ r \mid s \in S, r \in R \} \) for \( M \xrightarrow{R} N \xrightarrow{S} O \)

The reader is invited to verify that a category enriched in \( \mathcal{R} \) is exactly the same thing as a uniform preorder. Furthermore, \( \text{UDist} \) is the category of enriched profunctors, but – as Isar Stubbe pointed out – \( \text{UOrd} \) is more general than the category of enriched functors as defined in [58].

As pointed out earlier, Hirschowitz’s observation can be viewed as a generalization of Hoshino’s approach in the one-sorted case (Remark 4.9.8-(i)). It remains to be clarified if and how this

---

\(^5\) It is not surprising that fibered preorders are preorders internal to presheaves – the interesting part is the relation between smallness and the generic family of predicates.
approach can be reconciled with the presentation of uniform preorders as internal preorders from Section A.3.1.

### A.3.3 The non-posetal case

One of the motivating questions of this work was to find a common framework for Grothendieck toposes and toposes induced by triposes. This goal has not been achieved, but the theory developed here can nevertheless shed some light on the question. A first observation is that one-sorted uniform preorders with $\top, \land, \Rightarrow, \forall$ are more ‘site-like’ objects than triposes since they are small objects ‘inside a category’ instead of fibrations on it, which is already a step in the right direction.

The description of uniform preorders as preorders internal to small regular sheaves from Section A.3.1 tells us how to define ‘uniform categories’, namely as categories internal to small regular sheaves on $\text{Set}$ (it is possible – while not quite as nice as for the preorders – to give an internal/fibration-free presentation of these objects). With this in mind, I think a ‘non-posetal tripos’ should be at least a geometric uniform category $\mathcal{C}$ such that $\text{Set}[\mathcal{C}]$ is a topos.

To find the right conditions, one should also have a second look at Theorem 4.12.10. Since the conditions on a uniform preorder $\mathbb{A}$ that are necessary to make the theorem work are exactly those which make $\text{ufam}(\mathbb{A})$ into a tripos, it would be illuminating to have an analogous result for the tentative ‘uniform categories’.
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