Filtrated Pseudo-Orbit Shadowing Property and Approximately Shadowable Measures

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Abstract: In this paper, it is proved that every diffeomorphism possessing the filtrated pseudo-orbit shadowing property admits an approximately shadowable Lebesgue measure. Furthermore, the $C^1$-interior of the set of diffeomorphisms possessing the filtrated pseudo-orbit shadowing property is characterized as the set of diffeomorphisms satisfying both Axiom A and the no-cycle condition. As a corollary, it is proved that there exists a $C^1$-open set of diffeomorphisms, any element of which does not have the shadowing property but admits an approximately shadowable Lebesgue measure.

Keywords: filtration; pseudo-orbit; shadowing property; shadowable measure; approximately shadowable measure; Axiom A; no-cycle condition; quasi-Anosov

MSC: 37C50; 37D20

1. Introduction

The notion of pseudo-orbits appears often in the several branches of the modern theory of dynamical systems; especially, the pseudo-orbit shadowing property usually plays an important role in the investigation of stability theory and ergodic theory. Let $(X, d)$ be a compact metric space, and let $f : X \to X$ be a homeomorphism. For $\delta > 0$, a sequence of points $\{x_i\}_{i=1}^{\infty} \subset X$ ($-\infty \leq a < b \leq \infty$) is called a $\delta$-pseudo-orbit of $f$ if $d(f(x_i), x_{i+1}) < \delta$ for all $a \leq i \leq b - 1$.

Denote by $f|_A$ the restriction of $f$ to a set $A \subset X$. Let $\Lambda \subset X$ be a closed set (not necessarily $f$-invariant). We say that $f|_A$ has the shadowing property if for every $\epsilon > 0$ there is $\delta > 0$ such that for any $n \in \mathbb{N}$ and $\delta$-pseudo-orbit $\{x_i\}_{i=0}^{n} \subset \Lambda$ of $f$ there is $y \in X$ $\epsilon$-shadowing the pseudo-orbit—that is, $d(f^i(y), x_i) < \epsilon$ for all $0 \leq i \leq n - 1$. Note that only $\delta$-pseudo-orbits of $f$ "contained in $\Lambda$" can be $\epsilon$-shadowed, but the shadowing point $y \in X$ is "not necessarily" contained in $\Lambda$. We say that $f$ has the shadowing property if $X = \Lambda$ in the above definition. Since $X$ is compact, it is not difficult to show that if $f|_A$ has the shadowing property, then every pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty} \subset \Lambda$ can be shadowed by some true orbit of $f$.

In [1], we introduced the notion of shadowable measures as a generalization of the shadowing property from the measure theoretical view point, and investigated the dynamics of diffeomorphisms satisfying the notion (in fact, the dynamics of the $C^1$-interior of the set of diffeomorphisms possessing the shadowable measures is characterized as uniform hyperbolicity—see [1], Theorems 1 and 2). Every dynamical system possessing the shadowing property admits shadowable measures, but the converse is not generally true. In fact, an example of a diffeomorphism $g$ is constructed on the 2-torus $\mathbb{T}^2$ such that $g$ does not have the shadowing property but admits a shadowable Lebesgue measure (see [1], Example 3).

In this paper, generalizing the dynamics and shadowable Lebesgue measure of this example on $\mathbb{T}^2$, we introduce the notion of the filtrated pseudo-orbit shadowing property and that of approximately shadowable measures, and we prove that every diffeomorphism...
having the property admits an approximately shadowable Lebesgue measure. Furthermore, the $C^1$-interior of the set of diffeomorphisms possessing the filtrated pseudo-orbit shadowing property is characterized as the set of diffeomorphisms satisfying both Axiom A and the no-cycle condition. Finally, by making use of a quasi-Anosov diffeomorphism, we construct a $C^1$-open set of diffeomorphisms, any element of which does not have the shadowing property but admits an approximately shadowable Lebesgue measure.

2. Definitions and Statement of the Results

Recall that $(X,d)$ is a compact metric space and $f : X \to X$ is a homeomorphism of $X$. For given points $x, y \in X$ and $\delta > 0$, we write $x \sim^\delta y$ if there is a $\delta$-pseudo-orbit $(x_i)_{i=0}^n$ of $f$ such that $x_0 = x$ and $x_n = y$ for some $n = n_\delta \in \mathbb{N}$. Write $x \Rightarrow^\delta y$ if $x \sim^\delta y$ and $y \sim^\delta x$. Finally, we write $x \Rightarrow y$ if $x \Rightarrow^\delta y$ for any $\delta > 0$. The chain recurrent set of $f$, denoted by $\mathcal{R}(f)$, is the set of points $x \in X$ such that $x \Rightarrow x$. The chain recurrent set is one of the main subjects to consider in the shadowing theory of dynamical systems. Clearly, $\Omega(f) \subset \mathcal{R}(f)$ by definition, where $\Omega(f)$ is the non-wandering set of $f$.

Let $X^n = X \times \cdots \times X$ (the $n$-times of direct product) be the sequences of points of $X$ with length $n \in \mathbb{N}$, and denote by $\mathcal{M}(X)$ the space of Borel probability measures of $X$. For any $\mu \in \mathcal{M}(X)$ (not necessarily $f$-invariant), let $\mu_n = \mu \times \cdots \times \mu$ (n-times) be the direct product measure of $X^n$. For any $\delta > 0$, denote by $\mathcal{PO}(\delta, n)$ the space of $\delta$-pseudo-orbits $\{x_i\}_{i=0}^{n-1} \in X^n$ of $f$, and for $\epsilon > 0$, denote by $\mathcal{SPO}(\delta, \epsilon, n) \subset \mathcal{PO}(\delta, n)$ the set of $\delta$-pseudo-orbits $\epsilon$-shadowed by some point.

We say that $\mu \in \mathcal{M}(X)$ is a shadowable measure of $f$ (or simply, $f$ is $\mu$-shadowable) if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$
\mu_n(\mathcal{SPO}(\delta, \epsilon, n)) = \mu_n(\mathcal{PO}(\delta, n))
$$

for any $n \in \mathbb{N}$ (if $A$ is a subset of $X$, then we define the shadowable measure for $f_A$ by the same manner). Observe that if $f$ has the shadowing property, then $f$ is $\mu$-shadowable for any $\mu \in \mathcal{M}(X)$. Denote by $\text{supp}(\mu)$ the support of $\mu \in \mathcal{M}(X)$. Then, since $X$ is compact, it is known that if $f$ is $\mu$-shadowable, then $f_{|\text{supp}(\mu)}$ has the shadowing property (see [1], Lemma 1).

In this paper, we generalize the notion of shadowable measures to describe the dynamics of the system such as ([1], Example 3) from the measure theoretical view point. Let $\mu \in \mathcal{M}(X)$, and let $Y \subset X$ be a Borel set. For any Borel set $A \subset X$, we put

$$
\tilde{\mu}(A) = \tilde{\mu}_Y(A) = \frac{\mu(A \cap Y)}{\mu(Y)} \in \mathcal{M}(X).
$$

We say that $\mu$ is approximately shadowable if for any $\epsilon > 0$ there exists a Borel set $Y$ of $X$ with $\mu(Y) > 1 - \epsilon$ and $f(Y) \subset Y$ such that $\tilde{\mu}$ is a shadowable measure.

Hereafter, let $M$ be a closed $C^\infty$ manifold, and let $d$ be a distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$. Denote by $\text{Diff}(M)$ the space of diffeomorphisms of $M$ endowed with the $C^1$-topology as usual. We say that a sequence

$$
\emptyset = M_0 \subset M_1 \subset \cdots \subset M_k = M
$$

of smooth compact submanifolds $M_k$ with boundary such that $\dim M_k = \dim M$ for $1 \leq k \leq K$ is a filtration adapted to $f \in \text{Diff}(M)$ if the following conditions (a) and (b) are met:

(a) The chain recurrent set $\mathcal{R}(f)$ of $f$ is composed of mutually disjoint closed $f$-invariant sets $\{\Lambda_k(f)\}_{k=1}^K (K \geq 1)$ of $f$—that is,

$$
\mathcal{R}(f) = \Lambda_1(f) \cup \cdots \cup \Lambda_K(f);
$$

(b) For any $1 \leq k \leq K$,
(b.1) \( f(M_k) \subseteq \text{int}M_k \);
(b.2) \( \Lambda_k(f) \subseteq \text{int}(M_k \setminus M_{k-1}) \);
(b.3) \( \Lambda_k(f) = \cap_{m \in \mathbb{Z}} f^m(M_k \setminus M_{k-1}) \).

Here \( \text{int}A \) denotes the interior of a set \( A \subseteq M \).

We say that \( f \) has the filtrated pseudo-orbit shadowing property if there exists a filtration \( \emptyset = M_0 \subset M_1 \subset \cdots \subset M_K = M (K \geq 1) \) adapted to \( f \) such that for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any filtrated \( \delta \)-pseudo-orbit \( \{x_i\}_{i=0}^n \subseteq \text{int}(M_k \setminus M_{k-1}) \) of \( f \) (\( n \in \mathbb{N} \)) there exists \( x \in M \) satisfying \( d(f^n(x), x_i) < \epsilon \) for all \( 0 \leq i \leq n \).

Denote by \( FS \) the set of \( f \in \text{Diff}(M) \) having the filtrated pseudo-orbit shadowing property. The first result of this paper is the following.

**Theorem 1.** Every \( f \in FS \) admits an approximately shadowable Lebesgue measure \( m \in \mathcal{M}(M) \).

**Remark 1.** Suppose \( m \in \mathcal{M}(M) \) is the normalized Lebesgue measure on \( M \). Let us emphasize at this point that this \( m \) is an approximately shadowable Lebesgue measure for \( f \in FS \). In fact, we will see that for any \( \epsilon > 0 \) there exists a set \( Y \subset M \) \((m(Y) > 1 - \epsilon)\) such that \( \tilde{m} \in \mathcal{M}(M) \) is shadowable and \( \text{supp}(\tilde{m}) = Y \).

Denote by \( \text{int}FS \) the \( C^1 \)-interior of the set \( FS \) in \( \text{Diff}(M) \); that is, \( f \in \text{int}FS \) if and only if there exists a \( C^1 \)-neighborhood \( U(f) \) of \( f \) in \( \text{Diff}(M) \) such that any \( g \in U(f) \) meets all the properties \((a), (b)\) with respect to \( g \) and has the filtrated pseudo-orbit shadowing property. More precisely, for any \( g \in U(f) \),

- \( \mathcal{R}(g) \) is composed of mutually disjoint closed \( g \)-invariant sets \( \{\Lambda_k(g)\}_{k=1}^{K_g} (K_g \geq 1) \)
  that is,
  \[
  \mathcal{R}(g) = \Lambda_1(g) \cup \cdots \cup \Lambda_{K_g}(g)
  \]
  and properties \((b.1)-(b.3)\) are met, and
- Any filtrated pseudo-orbit \( \{x_i\}_{i=0}^n \subseteq \text{int}(M^g_{k} \setminus M^g_{k-1}) \) of \( g \) (\( n \in \mathbb{N} \)) is \( g \)-shadowed—that is,
  \[
  \text{int}(M^g_{k} \setminus M^g_{k-1})
  \]
  has the shadowing property for \( 1 \leq k \leq K_g \).

Let \( \Lambda \subset M \) be a closed \( f \)-invariant set. The set \( \Lambda \) is hyperbolic if the tangent bundle \( T_\Lambda M \) has a \( Df \)-invariant splitting \( E^s \oplus E^u \) with constants \( C > 0 \) and \( 0 < \lambda < 1 \) such that
\[
\|Df^n|_{E^s}\| \leq C\lambda^n \quad \text{and} \quad \|Df^{-n}|_{E^u}\| \leq C\lambda^n
\]
for all \( x \in \Lambda \) and \( n \geq 0 \). Suppose that \( \Lambda \) is hyperbolic. Then it is well-known that \( f|_\Lambda \) has the shadowing property (see [2,3]).

The stable manifold of a point \( x \in \Lambda \) is the set
\[
W^s(x) = \{ y \in M : d(f^n(x), f^n(y)) \to 0 \text{ as } n \to \infty \}.
\]

The unstable manifold, \( W^u(x) \), of \( x \in \Lambda \) is also defined analogously for \( n \to -\infty \). It is also well-known that \( W^s(x) \) and \( W^u(x) \) are both immersed manifolds (see [3], among others).

We say that \( f \) is Anosov when the whole space \( M \) is hyperbolic. At this moment, let us remark that any \( \mu \in \mathcal{M}(M) \) is shadowable if \( f \) is Anosov, and thus, every Anosov diffeomorphism admits a shadowable Lebesgue measure \( m \) such that
\[
\text{supp}(m) = M.
\]

Hereafter, let \( P(f) \) be the set of periodic points of \( f \), and recall that \( \Omega(f) \) is the set of non-wandering points of it. We say that \( f \) satisfy Axiom A if \( \Omega(f) \) is hyperbolic and \( \Omega(f) = P(f) \). Let \( f \) satisfies Axiom A. Then the non-wandering set has the so-called spectral decomposition—that is,
\[
\Omega(f) = \Lambda_1(f) \cup \cdots \cup \Lambda_L(f)
\]
composed of basic sets \( \{ \Lambda_i(f) \}_{i=1}^L \), and satisfies
\[
M = \bigcup_{i=1}^L W^s(\Lambda_i(f)),
\]
where
\[
W^s(\Lambda_i(f)) = \bigcup_{x \in \Lambda_i(f)} W^s(x) \quad (\sigma = s,u)
\]
for \( 1 \leq i \leq L \).

We say that \( f \) has a cycle if there is a subsequence \( \{ \Lambda_{i_j}(f) \}_{j=1}^L \) \((2 \leq l \leq L + 1)\) of the spectral decomposition such that
\[
\Lambda_{i_l}(f) = \Lambda_{i_j}(f) \quad \text{and} \quad W^u(\Lambda_{i_j}(f)) \cap W^s(\Lambda_{i_{l-1}}(f)) \neq \emptyset \quad (1 \leq j \leq l - 1).
\]
Note that if \( f \) satisfies the no-cycle condition, then \( \mathcal{R}(f) = \Omega(f) \) (see [4]).

The next result is the following.

**Theorem 2.** Let \( f \in \text{Diff}(M) \). Then \( f \in \text{int} \mathcal{F} \mathcal{S} \) if and only if \( f \) satisfies both Axiom A and the no-cycle condition.

We say that \( f \) is quasi-Anosov if for any \( v \in TM \setminus \{0\}, \{\|Df^n(v)\| : n \in \mathbb{Z}\} \) is unbounded. In [5], quasi-Anosov diffeomorphisms are characterized as diffeomorphisms satisfying both Axiom A and the no-cycle condition such that for any \( x \in M \),
\[
T_xW^s(x) \cap T_xW^u(x) = \{0\}.
\]

Every Anosov diffeomorphism is quasi-Anosov, but an example of quasi-Anosov, non-Anosov diffeomorphism is constructed by [6] (for more information, see [7]). In this paper, by making use of a quasi-Anosov diffeomorphism, we construct a \( C^1 \)-open set of \( \text{Diff}(M) \), any element of which does not have the shadowing property but admits an approximately shadowable Lebesgue measure (see Corollary 1).

For quasi-Anosov diffeomorphisms, the relationship to the shadowing property is considered in [8], and the following result is obtained therein.

**Theorem 3.** Let \( f \in \text{Diff}(M) \). Then \( f \) is quasi-Anosov possessing the shadowing property if and only if \( f \) is Anosov.

Since every quasi-Anosov diffeomorphism is in \( \text{int} \mathcal{F} \mathcal{S} \) by Theorem 2, the next result follows from Theorems 1 and 3.

**Corollary 1.** Let \( f \in \text{Diff}(M) \) be a quasi-Anosov diffeomorphism that is not Anosov. Then there is a \( C^1 \)-open set (a neighborhood of \( f \)), any \( g \) of which does not have the shadowing property but admits an approximately shadowable Lebesgue measure \( m \).

**Remark 2.** Since the example \( g \) on \( \mathbb{T}^2 \) constructed in ([1], Example 3) is in \( \text{int} \mathcal{F} \mathcal{S} \), every \( h \) \( C^1 \)-nearby \( g \) admits an approximately shadowable Lebesgue measure. However, it is easy to see that, for any \( C^1 \)-neighborhood \( \mathcal{V}(g) \) of \( g \), there is a \( h \in \mathcal{V}(g) \) possessing the shadowing property.

We close this section by pointing out an example which does not admit an approximately shadowable Lebesgue measure.

**Example 1.** Let \( S^1 = \{ e^{2\pi i \theta} : \theta \in \mathbb{R} \} \subset \mathbb{C} \). For \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \), let \( \rho_\alpha : S^1 \to S^1 \) be an irrational rotation map defined by \( \rho_\alpha(e^{2\pi i \theta}) = e^{2\pi i (\theta + \alpha)} \). Then the map \( \rho_\alpha \) does not admit an approximately shadowable Lebesgue measure since \( \rho_\alpha \) does not satisfy the shadowing property (see [1], Example 2) and we have \( \overline{Y} = S^1 \) for \( \emptyset \neq Y \subset S^1 \) with \( f(Y) \subset Y \).
3. Proofs of the Results

In this section, we give the proofs of Theorems 1 and 2 and Corollary 1.

**Proof of Theorem 1.** Suppose that $f \in \mathcal{F}S$, and let $m \in \mathcal{M}(M)$ be the normalized Lebesgue measure on $M$. Let $\mathcal{O} = M_0 \subset M_1 \subset \cdots \subset M_K = M$ be a filtration adapted to $f$ as in the definition of the filtrated pseudo-orbit shadowing property, and recall the conditions (a) and (b) that $f$ meets. We define a stable set for $\Lambda_k(f)$ ($1 \leq k \leq K$) by

$$W^s(\Lambda_k(f)) = \{ y \in M : \delta(f^n(y), \Lambda_k(f)) \rightarrow 0 \text{ as } n \rightarrow \infty \}.$$

Clearly, we have $f(W^s(\Lambda_k(f))) = W^s(\Lambda_k(f))$ and $W^s(\Lambda_k(f)) \cap W^s(\Lambda_1(f)) = \emptyset$ for $k \neq l$.

By (b.1)–(b.3) we have

1. $W^s(\Lambda_k(f)) \cap M_{k-1} = \emptyset$;
2. $\cap_{n=0}^{\infty} f^{-n}(M_k \setminus M_{k-1}) = W^s(\Lambda_k(f)) \cap M_k$;
3. $\bigcup_{k=1}^{K} W^s(\Lambda_k(f)) = M$.

If we set

$$W^n_k = W^s(\Lambda_k(f)) \cap f^{-n}(M_k)$$

for $1 \leq k \leq K$ and $n \geq 0$, then $\{W^n_k\}_{n=0}^{\infty}$ is an increasing sequence of closed sets satisfying that for $1 \leq k \leq K$

4. $f(W^n_k) = W^{n-1}_k \subset W^n_k$, $f^n(W^n_k) = W^n_k \subset M_k \setminus M_{k-1}$;
5. $W^n_k \cap W^n_l = \emptyset$ ($k \neq l$);
6. $\Lambda_k(f) \subset W^n_0 \subset W^n_1 \subset \cdots \subset W^n_k(f) = \bigcup_{n=0}^{\infty} W^n_k$.

Then, by (3) and (6), for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m\left(\bigcup_{k=1}^{K} W^N_k\right) > 1 - \epsilon.$$

In what follows, we put

$$Y = \bigcup_{k=1}^{K} W^N_k$$

for convenience (note that $f(Y) \subset Y$ by (4)).

Now, let us take $\delta_1 > 0$ such that if $x \in W^N_k$ and $y \in W^N_l$ for $k \neq l$, then $d(x, y) > \delta_1$ (see (5) above). Thus we have the following

Claim. There exists $0 < \delta_2 < \delta_1$ such that for any $\delta$-pseudo-orbit $\{x_i\}_{i=0}^{n} \subset Y$ of $f$ with $0 < \delta < \delta_2$ and $n \geq N + 1$, there is $1 \leq k \leq K$ such that $x_i \in \text{int}(M_k \setminus M_{k-1})$ for all $N + 1 \leq i \leq n$.

Indeed, let $\{x_i\}_{i=0}^{n} \subset Y$ be a given $\delta$-pseudo-orbit of $f$ with $0 < \delta < \delta_1$ and $n \geq N + 1$. Then it is easy to see that by the choice of $\delta_1 > 0$, there is $1 \leq k \leq K$ such that $\{x_i\}_{i=0}^{n} \subset W^N_k$. Since

$$f^{N+1}(W^N_k) \subset f(M_k) \subset \text{int}M_k,$$

by the uniform continuity of $f$, we can choose $0 < \delta_2 < \delta_1$ such that if $0 < \delta < \delta_2$, then $x_i \in \text{int}M_k$ for $N + 1 \leq i \leq n$. Moreover, by (4), we have $x_i \in \text{int}(M_k \setminus M_{k-1})$ for $N + 1 \leq i \leq n$, and thus the claim is proved.

Finally, let us show that every pseudo-orbit $\{x_i\}_{i=0}^{n} \subset Y$ is shadowed by a true orbit of $f$. For $\epsilon > 0$, by the uniform continuity of $f$, we can choose $0 < \delta' < \delta_2$ such that every $\delta'$-pseudo-orbit of $f$ with length less than $N + 1$ can be $\epsilon'$-shadowed by a true orbit of $f$. Thus, it is not difficult to show that any $\delta'$-pseudo-orbit $\{x_i\}_{i=0}^{n} \subset Y$ of $f$ can be
shadowed by a true orbit of $f$ reducing $\delta'$ if necessary. Therefore, for the set $Y$, if we define $\tilde{m} \in M(M)$ as
\[
\tilde{m}(A) = \frac{m(A \cap Y)}{m(Y)}
\]
for any Borel set $A \subset M$, then $\tilde{m}$ is shadowable, $m(Y) > 1 - \epsilon$ and $f(Y) \subset Y$, and thus Theorem 1 is proved. \hfill \Box

We need a lemma to prove Theorem 2. Remark that if $f$ possesses the filtrated pseudo-orbit shadowing property, then $f_{\mid R(f)}$ has the shadowing property by definition. The following lemma proved in ([9], Proposition 2.3) will be used in the proof of the “only if” part of Theorem 2.

**Lemma 1.** If $f_{\mid R(f)}$ has the shadowing property, then the shadowing point can be taken from $\Omega(f)$ for any pseudo-orbit in $\Omega(f)$.

**Proof of Theorem 2.** To prove the if part of the theorem, suppose that $f$ satisfies both Axiom A and the no-cycle condition. Then
\[
R(f) = \Omega(f) = \Lambda_1(f) \cup \cdots \cup \Lambda_K(f)
\]
for some $K \geq 1$, and there is a filtration $\emptyset = M_0 \subset M_1 \subset \cdots \subset M_K = M$ with respect to $f$. Here $\Lambda_k(f)$ is a hyperbolic basic set for $1 \leq k \leq K$ (see [3,4]). To prove the filtrated pseudo-orbit shadowing property for $f$, we note that for any $1 \leq k \leq K$ there is a neighborhood $U_k$ of $\Lambda_k(f)$ with the property that for all $\delta > 0$ there exists $\delta > 0$ such that for any $\delta$-pseudo-orbit $\{x_i\}_{i=0}^n \subset U_k$ there exists $x \in M$ satisfying $d(f^i(x), x_i) < \delta$ for all $0 \leq i \leq n$ since $\Lambda_k(f)$ is hyperbolic (see [2,3]). By (b,3), there is $m_k > 0$ such that
\[
\left( \bigcap_{m=-m_k} f^m(M_k \setminus M_{k-1}) \right) \subset U_k.
\]
To prove the only if part, let us denote by $\Omega.S$ the set of diffeomorphisms such that
- $f : \Omega(f) \to \Omega(f)$ has the shadowing property; and
- The shadowing point can be taken from $\Omega(f)$.

It was shown in ([10], Proposition 1) that any $f$ in the C1-interior of $\Omega.S$ satisfies both Axiom A and the no-cycle condition. Thus, to get the conclusion, it is enough to show that if $f \in \Omega.S$, then $f$ is in $\Omega.S$. Suppose that $f \in \Omega.S$. Then $f_{\mid R(f)}$ has the shadowing property, and thus, by Lemma 1, $f$ is in $\Omega.S$ since the shadowing point can be taken from $\Omega(f)$. Thus, Theorem 2 is proved. \hfill \Box

**Proof of Corollary 1.** Let $f$ be a quasi-Anosov diffeomorphism, so that $f$ satisfies both Axiom A and the no-cycle condition. Since $f \in \text{int} FS$ by Theorem 2, every $g$ C1-nearby $f$ admits an approximately shadowable Lebesgue measure by Theorem 1.

It is proved that every $g$ C1-nearby $f$ is also quasi-Anosov by ([5], Lemma 1.6), and that $f$ is Anosov if and only if $W^s(p)$ is the same dimension for all $p \in P(f)$ by ([5], Corollary 1).
Suppose further that \( f \) is not Anosov. Then, since there are hyperbolic periodic points \( p, q \in P(f) \) with different indices, that is, \( \dim W^s(p) \neq \dim W^s(q) \), every \( g \) \( C^1 \)-nearby \( f \) also has periodic points with different indices. Thus, \( g \) is quasi-Anosov but not Anosov by ([5], Corollary 1), so that \( g \) does not have the shadowing property by Theorem 3.  

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