INFRARED DIVERGENCE OF A SCALAR QUANTUM FIELD MODEL ON A PSEUDO RIEMANNIAN MANIFOLD

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Abstract
A scalar quantum field model defined on a pseudo Riemannian manifold is considered. The model is unitarily transformed to the one with a variable mass. By means of a Feynman-Kac-type formula, it is shown that when the variable mass is short range, the Hamiltonian has no ground state. Moreover the infrared divergence of the expectation values of the number of bosons in the ground state is discussed.
1 Introduction

1.1 Preliminaries

Analysis of the infrared behavior in massless quantum field theory is an important issue. The infrared divergence is seen to arise as follows: the emission probability of massless boson becomes infinite with increasing wavelength. For some scalar quantum field model, which is the so-called Nelson model \[\text{Nel64},\] a sharp result concerning the relationship between the infrared behavior and the existence (or the absence) of ground states is known. The Nelson model describes a scalar field coupled to a quantum mechanical particle with external potential $V$ in such a way that the interaction is linear. Namely the Nelson model with mass $m_0 \geq 0$ is formally given by

$$H_N = \frac{1}{2}p^2 + V(q) + \frac{1}{2} \int \left( \pi(x)^2 + (\nabla \phi(x))^2 + m_0^2 \phi(x)^2 \right) dx + \int \phi(x) \chi(x - q) dx,$$

(1.1)

where $\chi$ denotes a cutoff function, $p$ and $q$ are the position operator and momentum operator of the particle, respectively, with bare mass 1, which satisfy $[p, q] = -i$, and $\pi(x)$ is the momentum field canonically conjugate to the scalar field $\phi(x)$, which satisfy $[\phi(x), \pi(y)] = i\delta(x - y)$. The dispersion relation for the Nelson model is given by

$$\hat{\omega}_N = \sqrt{\Delta + m_0^2},$$

(1.2)

in the position representation and the equation of motion is

$$\left( \Box + m_0^2 \right) \phi(x, t) = -\chi(x - q_t),$$

(1.3)

$$\partial_t^2 q_t = -\nabla_q V(q_t) - \nabla_q \phi(\chi(x - q_t)),$$

(1.4)

where $\Box = \partial_t^2 - \Delta_x$. It is established that $H_N$ with positive mass $m_0 > 0$ has a ground state but no ground state for $m_0 = 0$, and the expectation value of the number of bosons in the ground state diverges as $m_0 \to 0$.

While the Nelson model defined on a static Riemannian manifold is unitarily transformed to a model with a variable mass

$$v_m(x) = m(x)^2 \geq 0,$$

(1.5)

and the dispersion relation (1.2) is changed to

$$\hat{\omega} = \sqrt{-\Delta + v_m}.$$

(1.6)
By comparing (1.2) and (1.6), the variable mass is seen to intermediate between massive cases and massless cases, and furthermore the infrared behavior, as mentioned below, depends on the decay property of $v_m(x)$ as $|x| \to \infty$.

We consider in this paper a version of the Nelson model with variable masses. The Hamiltonian is formally given by

$$H_{\text{formal}} = \frac{1}{2}p^2 + V(q) + \frac{1}{2} \int (\pi(x)^2 + (\nabla \phi(x))^2 + v_m(x)\phi(x)^2) \, dx + \alpha \phi(\rho_q),$$  \hspace{1cm} (1.7)$$

where $p$ and $q$, and $\phi(x)$ and $\pi(y)$ satisfy the same canonical commutation relations as that of the Nelson model. The field operator $\phi(\rho_q) = \int \phi(x)\rho_q(x) \, dx$ is, however, a scalar field smeared by some function $\rho_q$ defined through $v_m$ and a given cutoff function $\chi$, and $\alpha$ a real coupling constant. Thus the equation of motion is given by

$$\Box \phi(x,t) = -\alpha \rho_q(x),$$  \hspace{1cm} (1.8)$$

$$\partial_t^2 q_t = -\nabla q V(q_t) - \alpha \nabla \phi(\rho_q).$$  \hspace{1cm} (1.9)$$

Here $\Box + v_m(x)$ appears in (1.8) instead of $\Box + m_0^2$. This is a unitary transformed version of a Klein-Gordon equation defined on a pseudo Riemannian manifold. See Section 2.5.

We are interested in investigating the infrared behavior of the Nelson model. In

\[
\begin{array}{c}
\begin{array}{c}
\hline \\
v_m = m_0^2 \\
\hline \\
0
\end{array}
\end{array}
\]

Figure 1: Positive constant mass

the case of constant mass $v_m(x) = m_0^2$ in (1.6), it is established that if $m_0 > 0$, the Nelson model has the unique ground state up to multiple constants (Fig.1), but if $m_0 = 0$ no ground state exists unless the infrared regularization is imposed. See e.g., [BFS98, BHLMS02, Che01, Ger00, HH06, Hk06, LMS02, Spo98] for detail. Here the infrared regular condition is defined by

$$\int_{\mathbb{R}^3} \frac{\chi(k)^2}{|k|^3} \, dk < \infty.$$  \hspace{1cm} (1.10)$$

Conversely

$$\int_{\mathbb{R}^3} \frac{\chi(k)^2}{|k|^3} \, dk = \infty$$  \hspace{1cm} (1.11)$$
IR divergence

is called the infrared singular condition. The singularity in \((1.11)\) comes from a neighborhood of \(k = 0\) if \(\chi\) has a compact support, since the dimension is three.

Our paper is motivated by extending constant mass cases to variable ones. Namely, going beyond the case of constant masses, we consider the infrared behavior of the Nelson model with variable masses. From the argument mentioned above it is expected that the Nelson model may have ground states if the variable mass decays sufficiently slowly in a neighborhood of origin (Fig. 2),

\[
\begin{array}{c}
\text{Figure 2: Long range variable mass} \\
0 \quad v_m
\end{array}
\]

but no ground state exists if it decays sufficiently fast (Fig. 3). Taking into account of

\[
\begin{array}{c}
\text{Figure 3: Short range variable mass} \\
0 \quad v_m
\end{array}
\]

this intuitive argument, as the first step, we consider two cases: (1) \(v_m\) is long range and (2) \(v_m\) is short range. In this paper we focus on (2) and prove that for a short range potential \(v \geq 0\) such that \(v_m(x) = \mathcal{O}(|x|^{-\beta})\) with \(\beta > 3\), \(H\) has no ground state in the Hilbert space unless the infrared regularization is imposed.

1.2 Strategy

It is proven that the functional integration is useful device to show the existence and non-existence of the ground state of the Nelson model with constant masses. It can be extended to the case of variable masses in this paper. The main tool used in this paper is functional integral representations of the semigroup \(e^{-tH}\) and an extension of the strategy developed in \([HLS02, LMS02]\) where the Nelson model with constant mass is discussed.
The Nelson model $H$ can be defined as a self-adjoint operator on some probability space. It is easily shown that

$$\varphi_g^T = \|e^{-TH}1\|^{-1}e^{-TH}1, \quad T > 0,$$

is a sequence approaching to a ground state of $H$ if a ground state exists. Conversely

$$\lim_{T \to \infty} (1, \varphi_g^T)^2 = a > 0,$$

implies the existence of the ground state of $H$, but the absence of ground state follows from

$$\lim_{T \to \infty} (1, \varphi_g^T)^2 = 0.$$

By making use of a modification of [LMS02] we show that (1.14) holds under the infrared singularity condition (1.11).

Throughout this paper we use the notation $E_\mu[\cdots]$ for $\int \cdots d\mu$ and $E_x^\nu[\cdots]$ for $\int \cdots d\nu_x$, where $\nu_x$ denotes a probability measure starting at $x$ on a path space. By using the functional integration, we have the bound

$$(1, \varphi_g^T)^2 \leq E_{\mu_T} \left[ e^{-\alpha^2 \int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|)} \right]$$

with some probability measure $\mu_T$ on the product configuration space $\mathbb{R}^3 \times C(\mathbb{R}; \mathbb{R}^3)$ and the so-called double potential $W = W(X_s, X_t, |s-t|)$ given by

$$W(X, Y, |t|) = \int \frac{\chi(k)^2}{2|k|} \frac{\Psi(k, X)}{\Psi(k, Y)} e^{-|t||k|} dk.$$

Here $\Psi(k, x)$ denotes the generalized eigenvector of $-\Delta + v_m$. By controlling the behavior of measures $\mu_T$ and $\int_{-T}^T ds \int_0^T dt W(X_s, X_t, |s-t|)$ as $T \to \infty$, we can show (1.14) under the infrared singular condition.

Next we consider the expectation values of the number of bosons in the ground state $\varphi_g$. Assume the infrared regular condition (1.10) and the existence of ground state. Let $N$ be the number operator. We can show that $(\varphi_g^T, e^{-\beta N} \varphi_g^T)$ can be analytically continued from $\beta \in [0, \infty)$ to the whole complex plane $\beta \in \mathbb{C}$. Then the moment $(\varphi_g^T, N^n \varphi_g^T)$ is given by

$$(\varphi_g^T, N^n \varphi_g^T) = (-1)^n \frac{d^n}{d\beta^n} (\varphi_g^T, e^{-\beta N} \varphi_g^T) \bigg|_{\beta=0}.$$

As an application we can show that the expectation value of the number of bosons in the ground state, $(\varphi_g, N \varphi_g)$, diverges as $\int_{\mathbb{R}^3} \frac{\chi(k)^2}{|k|^3} dk$ tends to infinity.
This paper is organized as follows: Section 2 is devoted to giving the definition of
the Nelson model with a variable mass. In Section 3 we discuss functional integration
in Euclidean quantum field theory. In Section 4 we prove the absence of ground state.
Finally in Section 5 we show the divergence of \((\varphi_g, N\varphi_g)\) in infrared singularity.

2 The Nelson model on a pseudo Riemannian manifold

2.1 Particle

We introduce the Schrödinger operator \(H_p\) by

\[
H_p = \frac{1}{2} p^2 + V, \tag{2.1}
\]

where \(p_\mu = -i \nabla_\mu\), \(p^2 = p \cdot p\), and \(V\) is an external potential. We say that \(V\) is
Kato-class if and only if

\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^3} \int_{|x - y| < r} \frac{|V(y)|}{|x - y|} dy = 0
\]

and \(V\) is local Kato-class if and only if \(1_K V\) is Kato-class for arbitrary compact set
\(K \subset \mathbb{R}^3\). If \(V = V_+ - V_-\) satisfies that \(V_+\) is local Kato-class and \(V_-\) Kato-class,
we say that \(V\) is Kato-decomposable. When \(V\) is Kato-class, \(V \in L^1_{\text{loc}}(\mathbb{R}^3)\) and \(V\) is
infinitesimally small with respect to \(p^2\) in the sense of form, furthermore when \(V = L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)\)
with \(p > 3/2\), \(V\) is Kato-class. In particular an arbitrary polynomial
is local Kato-class.

We introduce assumptions on external potential \(V\):

**Assumption 2.1 (Assumptions on \(V\))** We assume (1)-(3) below:

1. \(V = V_+ - V_-\) is Kato-decomposable with \(V_- \in L^p_{\text{loc}}(\mathbb{R}^3)\) for some \(p > 3/2\).
2. \(V\) is bounded from below and \(V(x) > C|x|^{2q}\) with some \(q > 0\) for \(x \in \mathbb{R}^3 \setminus M\) with
   some compact set \(M\).
3. The ground state of \(H_p\) is unique and strictly positive.

\(H_p\) is defined as a quadratic form sum. Since \(V\) is Kato-decomposable, \(H_p\) is closed on
\(Q(p^2) \cap Q(V_+)\) and bounded from below, where \(Q(T)\) denotes the form domain of \(T\). See
Moreover it follows that \( \sup_{x \in \mathbb{R}^3} \mathbb{E}_{P_W} \left[ e^{-\int_0^t V(B_s + x) ds} \right] < \infty \) for arbitrary \( t \geq 0 \), where \( (B_t)_{t \geq 0} \) denotes the 3-dimensional Brownian motion starting at zero on a probability space \((W, \mathcal{B}_W, P_W)\). By (2) of Assumption 2.1, \( V \to \infty \) as \(|x| \to \infty\). Then \( H_p \) has a compact resolvent. This can be proven by showing that \( \{ \psi \in Q(H_p) \mid \| \psi \| \leq 1, (\psi, H_p \psi) \leq 1 \} \) is compact in \( L^2(\mathbb{R}^3) \). See e.g., [RS78, Theorem XIII.67]. In particular the spectrum of \( H_p \) is purely discrete and the ground state \( \varphi_p \) exists. By assumptions, \( V_+ \in L^1_{\text{loc}}(\mathbb{R}^3) \) and \( V_- \in L^p(\mathbb{R}^3) \) with \( p > 3/2 \), and \( V(x) > C|x|^a \) for sufficiently large \(|x|\), it is known that \( \varphi_p(x) \) exponentially decays. We used this in Section 4.

Now let us define a unitary transformation. By (3) of Assumption 2.1 we can define the ground state transformation

\[
U_p : L^2(\mathbb{R}^3) \to \mathcal{H}_p = L^2(\mathbb{R}^3, \varphi_p^2 dx)
\]

by

\[
U_pf = \frac{1}{\varphi_p} f.
\]

Set

\[
L_p = U_p H_p U_p^{-1}
\]

and the probability measure \( \mu_p \) on \( \mathbb{R}^3 \) is defined by

\[
d\mu_p(x) = \varphi_p^2(x) dx.
\]

Thus the operator \( L_p \) acts on the probability space \( L^2(\mathbb{R}^3; d\mu_p) \). Formally \( L_p \) is given by

\[
L_pf = -\frac{1}{2} \Delta f + \frac{\nabla \varphi_p}{\varphi_p} \nabla f
\]

on \( L^2(\mathbb{R}^3; d\mu_p) \), it is of course not clear whether \( \varphi_p \in C^1(\mathbb{R}^3) \) or not. However by the Kolmogorov consistency theorem we can construct a continuous Markov process \( X = (X_t)_{t \in \mathbb{R}} \) associated with the semigroup \( e^{-tL_p} \). This process \( X \) is a formal solution of the stochastic differential equation:

\[
dX_t = dB_t + \frac{\nabla \varphi_p}{\varphi_p} (X_t) dt.
\]

We will discuss the Markov process \( X \) in Section 3.
2.2 Boson Fock space

The Boson Fock space over the one particle space $L^2(\mathbb{R}^3)$ is defined by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2_{\text{sym}}(\mathbb{R}^{3n}),$$

where $L^2_{\text{sym}}(\mathbb{R}^{3n})$ is the set of $L^2$ functions $f(k_1, \ldots, k_n)$, $k_j \in \mathbb{R}^3$, $j = 1, \ldots, n$, on $\mathbb{R}^{3n}$ such that it is symmetric with respect to $k_1, \ldots, k_n$ with $L^2_{\text{sym}}(\mathbb{R}^0) = \mathbb{C}$. The Fock vacuum $1 \oplus 0 \oplus 0 \oplus \cdots$ in $\mathcal{F}$ is denoted by $\Omega_\mathcal{F}$. The annihilation operators $a(f)$ smeared by $f \in L^2(\mathbb{R}^3)$ and the creation operators $a^\dagger(g)$ by $g \in L^2(\mathbb{R}^3)$ are defined in $\mathcal{F}$ and satisfy canonical commutation relations:

$$[a(f), a^\dagger(g)] = (\bar{f}, g)_{L^2(\mathbb{R}^3)}, \quad (2.6)$$

$$[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]. \quad (2.7)$$

Here $(f, g)_\mathcal{K}$ denotes the scalar product on a Hilbert space $\mathcal{K}$. We omit $\mathcal{K}$ unless confusion arises. Note that

$$(a(f))^* = a^\dagger(\bar{f})$$

and that $a^\dagger(f)$ and $a(f)$ are linear in $f$. We formally write $a(f) = \int a(k)f(k)dk$ and $a^\dagger(f) = \int a^\dagger(k)f(k)dk$. For a contraction operator $T : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$, define the contraction operator $\Gamma(T) : \mathcal{F} \rightarrow \mathcal{F}$ by $\Gamma(T)\Omega_\mathcal{F} = \Omega_\mathcal{F}$ and

$$\Gamma(T)a^\dagger(f_1) \cdots a^\dagger(f_n)\Omega_\mathcal{F} = a^\dagger(Tf_1) \cdots a^\dagger(Tf_n)\Omega_\mathcal{F}.$$

Note that $\Gamma(TS) = \Gamma(T)\Gamma(S)$ and $\Gamma(I) = I$. Then for a self-adjoint operator $h$ in $L^2(\mathbb{R}^3)$ there exists a unique self-adjoint operator $d\Gamma(h)$ in $\mathcal{F}$ such that

$$e^{i t d\Gamma(h)} = \Gamma(e^{i t h}), \quad t \in \mathbb{R}.$$

2.3 The Nelson model with variable mass

Let us assume that $-\Delta + v_m$ is a self-adjoint operator in $L^2(\mathbb{R}^3)$. Suppose that $-\Delta + v_m$ has generalized eigenfunctions $\Psi(k, x)$:

$$(-\Delta + v_m(x))\Psi(k, x) = |k|^2\Psi(k, x), \quad k \in \mathbb{R}^3. \quad (2.8)$$

We introduce the following assumptions.
**Assumption 2.2 (Assumptions on $\Psi(k,x)$)** The generalized eigenvectors satisfy that

1. $\sup_{k,x} |\Psi(k,x)| < \infty$,
2. $\Psi(k,x)$ is continuous in $x$ for almost every $k$,
3. the generalized Fourier transformation:

\[
(Ff)(k) = (2\pi)^{-3/2}\text{l.i.m.} \int f(x)\overline{\Psi(k,x)}dx
\]

is unitary on $L^2(\mathbb{R}^3)$.

By (3) above the inverse of $F$, $F^{-1}$, is given by

\[
(F^{-1}g)(x) = (2\pi)^{-3/2}\text{l.i.m.} \int g(k)\Psi(k,x)dk.
\]

Recall that $\tilde{\omega} = \sqrt{-\Delta + v_m}$. Then we have

\[
F\tilde{\omega}F^{-1} = \omega,
\]

where $\omega$ is the multiplication operator given by

\[
\omega(k) = |k|, \quad k \in \mathbb{R}^3.
\]

Let $\chi$ be a cutoff function. We define the field operator with the variable mass $v_m$ and the cutoff function $\chi$ by

\[
\hat{\Phi}(x) = \frac{1}{\sqrt{2}} \left( a^\dagger \left( \tilde{\omega}^{-1/2}\rho_x \right) + a \left( \tilde{\omega}^{-1/2}\rho_x \right) \right),
\]

where

\[
\rho_x(\cdot) = (2\pi)^{-3/2} \int \Psi(k,\cdot)\overline{\Psi(k,x)}\chi(k)dk.
\]

A physically reasonable choice of $\chi$ is

\[
\chi(k) = \frac{\chi_\Lambda(|k|)}{\sqrt{(2\pi)^3}}, \quad \Lambda > 0,
\]

where $\chi_\Lambda$ is an ultraviolet cutoff defined by $\chi_\Lambda(s) = \begin{cases} 0, & s \geq \Lambda \\ 1, & s < \Lambda \end{cases}$. If we take (2.15) as $\chi$, then $\rho_x \to \delta(\cdot - x)$ in $\mathcal{F}'$ as $\Lambda \to \infty$. 

Let us define the free Hamiltonian $\hat{H}_f$ by

$$\hat{H}_f = d\Gamma(\hat{\omega}).$$  \hspace{1cm} (2.16)

The total state space is defined by the tensor product of $\mathcal{H}_p$ and $\mathcal{F}$:

$$\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}.$$  \hspace{1cm} (2.17)

**Definition 2.3 (The Nelson model with variable mass)** The Nelson Hamiltonian with the variable mass $v_m$ is defined by

$$\hat{H} = L_p \otimes 1 + 1 \otimes \hat{H}_f + \alpha \hat{\Phi}$$  \hspace{1cm} (2.18)

on the Hilbert space $\mathcal{H}$, where $\hat{\Phi} = \int_{\mathbb{R}^3} \hat{\Phi}(x) dx$ under the identification $\mathcal{H} = \int_{\mathbb{R}^3} \mathcal{F} ds$.

Now we derive the equation of motion associated with $\hat{H}$. Let

$$\varphi(f) = \frac{1}{\sqrt{2}} \left( a^\dagger(\hat{\omega}^{-1/2}f) + a\left(\overline{\hat{\omega}^{-1/2}}f\right) \right)$$  \hspace{1cm} (2.19)

be the field operator smeared by $f$. Then $\hat{\Phi}(x) = \varphi(\rho_x)$. The time evolution of $\varphi(f)$ is given by

$$\varphi(f,t) = e^{it\hat{H}} \varphi(f) e^{-it\hat{H}}$$  \hspace{1cm} (2.20)

and that of $x$ by

$$q_t = e^{it\hat{H}} x e^{-it\hat{H}}.$$  \hspace{1cm} (2.21)

Since

$$[d\Gamma(\hat{\omega}), a(f)] = -a(\hat{\omega}f), \quad [d\Gamma(\hat{\omega}), a^\dagger(f)] = a^\dagger(\hat{\omega}f),$$

$\varphi(f,t)$ and $q_t$ satisfy that

$$\partial^2_t \varphi(f,t) + \varphi((-\Delta + v_m)f,t) = -\alpha(\rho_{qt}, f),$$  \hspace{1cm} (2.22)

$$\partial^2_t q_t = -\nabla V(q_t) - \alpha \varphi(\nabla \rho_{qt})$$  \hspace{1cm} (2.23)

on $\mathcal{H}$. Compare with (1.8) and (1.9).
2.4 Unitary transformation

In this subsection we unitarily transform the Nelson Hamiltonian to some self-adjoint operator \( H \). Let \( H_f \) be defined by

\[
H_f = d\Gamma(\omega)
\]  

and \( \Phi(x) \) by

\[
\Phi(x) = \frac{1}{\sqrt{2}} \int \left( \frac{\chi(k)}{\sqrt{\omega(k)}} \Psi(k, x) a^+(k) + \frac{\chi(k)}{\sqrt{\omega(k)}} \Psi(k, x) a(k) \right) dk.
\]  

Define \( H \) by

\[
H = L_p \otimes 1 + 1 \otimes H_f + \alpha \Phi,
\]

where \( \Phi = \int_{\mathbb{R}^3} \Phi(x) dx \). We introduce some assumption on cutoff function \( \chi \).

**Assumption 2.4 (Assumptions on \( \chi \))** Assume that \( \chi \) is real, \( \tilde{\chi} \geq 0 \) \((\neq 0)\), \( \chi/\sqrt{\omega} \in L^2(\mathbb{R}^3) \) and \( \chi/\omega \in L^2(\mathbb{R}^3) \), where \( \tilde{\chi} \) denotes the inverse Fourier transform of \( \chi \).

**Remark 2.5** Since the space dimension under consideration is three, from \( \tilde{\chi} \geq 0 \) in Assumption 2.4 it follows that \( \chi(0) > 0 \) and then it follows that

\[
\int \frac{\chi(k)^2}{\omega(k)^3} dk = \infty.
\]

The next proposition is standard.

**Proposition 2.6** Suppose Assumption 2.4 and (1) of Assumption 2.2. Then the Nelson Hamiltonian \( H \) (resp. \( \hat{H} \)) is self-adjoint on \( D(L_p) \cap D(H_f) \) (resp. \( D(L_p) \cap D(\hat{H}_f) \)) and bounded from below. Moreover \( H \) (resp. \( \hat{H} \)) is essentially self-adjoint on any core of \( L_p \otimes 1 + 1 \otimes H_f \) (resp. \( L_p \otimes 1 + 1 \otimes \hat{H}_f \)).

**Proof:** Since \( \Phi \) (resp. \( \hat{\Phi} \)) is infinitesimally small with respect to \( L_p \otimes 1 + 1 \otimes H_f \) (rep. \( L_p \otimes 1 + 1 \otimes \hat{H}_f \)), the proposition follows from the Kato-Rellich theorem. □

Let \( \mathcal{F}_b = \Gamma(\mathcal{F}) \) which is a unitary operator on \( \mathcal{F} \).

**Proposition 2.7** Suppose Assumption 2.4 and (1) of Assumption 2.2. Then

\[
H = (1 \otimes \mathcal{F}_b) \hat{H} (1 \otimes \mathcal{F}_b^{-1}).
\]  

(2.28)
**Proof:** Since
\[ \mathcal{F}\tilde{\omega}^{-1/2}\rho_x(\cdot) = \omega^{-1/2}(\cdot)\chi(\cdot)\Psi(\cdot, x) \]
and \( \mathcal{F}a^\dagger(\tilde{\omega}^{-1/2}\rho_x)\mathcal{F}^{-1}_b = a^\dagger(\tilde{\omega}^{-1/2}\rho_x) \) and \( \mathcal{F}a(\tilde{\omega}^{-1/2}\rho_x)\mathcal{F}^{-1}_b = a(\tilde{\omega}^{-1/2}\rho_x) \), it follows that \( \mathcal{F}\hat{\Phi}(x)\mathcal{F}^{-1}_b = \Phi(x) \) for each \( x \). By \( \mathcal{F}\tilde{\omega}\mathcal{F}^{-1} = \omega \) it also follows that \( \mathcal{F}\hat{H}_f\mathcal{F}^{-1}_b = H_f \). By a simple limiting argument we can complete the proof. \( \square \)

We give a remark on the relationship between \( H \) and the standard Nelson model \( H_N \) introduced in [Nel64]. Namely

\[ H_N = L_p \otimes 1 + 1 \otimes H_f + \alpha\Phi_N, \quad (2.29) \]

where \( \Phi_N = \int_{\mathbb{R}^3} \Phi_N(x)dx \) and

\[ \Phi_N(x) = \frac{1}{\sqrt{2}} \int \left( \frac{\chi(k)}{\sqrt{\omega(k)}} e^{-ikx}a^\dagger(k) + \frac{\chi(k)}{\sqrt{\omega(k)}} e^{+ikx}a(k) \right) dk. \]

Let \( v_m(x) \equiv m^2 \) be a nonnegative constant. Thus the generalized eigenfunction is \( \Psi(k, x) = e^{ikx} \) and \( \rho_x = \chi(\cdot - x) \). Then \( H \) covers \( H_N \).

### 2.5 Klein-Gordon equation on pseudo Riemannian manifold

In this subsection we give an example of a Klein-Gordon equation defined on a pseudo Riemannian manifold \( \mathcal{M} \) such that a short range potential \( v_m(x) = O(\langle x \rangle^{-\beta - 2}) \) appears, where \( \langle x \rangle = \sqrt{1 + |x|^2} \). See [FUL96] for details.

Let \( \underline{x} = (t, x) = (x_0, x) \in \mathbb{R} \times \mathbb{R}^3 \). Let \( \mathcal{M} \) be the 4 dimensional pseudo Riemannian manifold equipped with the metric tensor:

\[ g(\underline{x}) = g(x) = \begin{pmatrix}
    e^{-\theta(x)} & 0 & 0 & 0 \\
    0 & -e^{-\theta(x)} & 0 & 0 \\
    0 & 0 & -e^{-\theta(x)} & 0 \\
    0 & 0 & 0 & -e^{-\theta(x)}
\end{pmatrix}. \quad (2.30) \]

Note that \( g \) depends on \( x \) but independent of \( t \). The line element associated with \( g \) is given by

\[ ds^2 = e^{-\theta(x)}dt \otimes dt - e^{-\theta(x)} \sum_j dx^j \otimes dx^j. \]

The Klein-Gordon equation on \( \mathcal{M} \) is

\[ \Box_g \phi + m^2 \phi = 0, \quad (2.31) \]
where the d’Alembertian operator is defined by
\[ \Box_g = e^{\theta(x)} \partial_t^2 - e^{2\theta(x)} \sum_j \partial_j e^{-\theta(x)} \partial_j. \]

Thus the Klein-Gordon equation \((2.31)\) is reduced to the equation
\[ \frac{\partial^2 \phi}{\partial t^2} = K_0 \phi, \tag{2.32} \]
where
\[ K_0 = e^{\theta(x)} \sum_j \partial_j e^{-\theta(x)} \partial_j - e^{-\theta(x)} m^2. \]

The operator \(K_0|_{C^\infty_c(\mathbb{R}^3)}\) is symmetric on the weighted \(L^2\) space \(L^2(\mathbb{R}^3; e^{-\theta(x)}dx)\). Now we transform the operator \(K_0\) to the one on \(L^2(\mathbb{R}^3)\). In order to do that, the unitary map \(U_0 : L^2(\mathbb{R}^3; e^{-\theta(x)}dx) \to L^2(\mathbb{R}^3)\) is introduced by \(U_0 f(x) = e^{-(1/2)^\theta(x)} f(x)\).

**Lemma 2.8** There exist functions \(\theta\) and \(v\) such that \(U_0 K_0 U_0^{-1} = \Delta - v\), \(v(x) = O(\langle x \rangle^{-\beta-2})\) for \(\beta \geq 0\), and \(-\Delta + v\) has no non-positive eigenvalues.

Hence the Klein-Gordon equation \((2.32)\) is transformed to the equation
\[ \frac{\partial^2 \phi}{\partial t^2} = \Delta \phi - v \phi \tag{2.33} \]
on \(L^2(\mathbb{R}^3)\). Although the proof of Lemma \(2.8\) is straightforward, we shall show this statement through a more general scheme in what follows.

Suppose that \(g = (g_{\mu\nu}), \mu, \nu = 0, 1, 2, 3,\) is a metric tensor on \(\mathbb{R}^4\) such that

1. \(g_{\mu\nu}(x) = g_{\mu\nu}(x),\) i.e., it is independent of time \(t,\)
2. \(g_{0j}(x) = g_{j0}(x) = 0, j = 1, 2, 3,\)
3. \(g_{ij}(x) = -\gamma_{ij}(x),\) where \(\gamma = (\gamma_{ij})\) denotes a 3-dimensional Riemannian metric.

Namely
\[ g = \begin{bmatrix} g_{00} & 0 \\ 0 & -\gamma \end{bmatrix}. \]

Let \(\mathcal{M}\) be a pseudo Riemannian manifold equipped with the metric tensor \(g\) satisfying (1)-(3) above. Then the line element on \(\mathcal{M}\) is given by
\[ ds^2 = g_{00}(x) dt \otimes dt - \sum_{ij} \gamma_{ij}(x) dx^i \otimes dx^j. \]
Let $g^{-1} = (g^{\mu\nu})$ denote the inverse of $g$. In particular $1/g_{00} = g^{00}$. We also denote the inverse of $\gamma$ by $\gamma^{-1} = (\gamma^{ij})$. The Klein-Gordon equation on the static pseudo Riemannian manifold $\mathcal{M}$ is generally given by

$$\Box_g \phi + (m^2 + \eta R) \phi = 0,$$

where $\eta$ is a constant, $R$ the scalar curvature of $\mathcal{M}$, and $\Box_g$ is given by

$$\Box_g = \sum_{\mu\nu} \frac{1}{\sqrt{|\det g|}} \partial_\mu g^{\mu\nu} \sqrt{|\det g|} \partial_\nu.$$

(2.35)

Let us assume that $g_{00}(x) > 0$. Then (2.34) is rewritten as

$$\frac{\partial^2 \phi}{\partial t^2} = K \phi,$$

(2.36)

where

$$K = g_{00} \left( \frac{1}{\sqrt{|\det g|}} \sum_{ij} \partial_j \sqrt{|\det g|} \gamma^{ij} \partial_i - m^2 - \eta R \right).$$

The operator $K|_{C_0^\infty(\mathbb{R}^3)}$ is symmetric on $L^2(\mathbb{R}^3; \rho(x)dx)$, where

$$\rho = \frac{\sqrt{|\det g|}}{g_{00}} = g^{-1/2}_{00} \sqrt{|\det \gamma|}.$$  

(2.37)

Now let us transform the operator $K$ on $L^2(\mathbb{R}^3; \rho(x)dx)$ to the one on $L^2(\mathbb{R}^3)$. Define the unitary operator $U: L^2(\mathbb{R}^3; \rho(x)dx) \to L^2(\mathbb{R}^3)$ by

$$Uf = \rho^{1/2} f.$$

Let $\rho_i = \partial_i \rho$ and $\partial_i \partial_j \rho = \rho_{ij}$ for notational simplicity. Furthermore we set $\alpha^{ij} = g_{00} \gamma^{ij}$ and $\partial_k \alpha^{ij} = \alpha_k^{ij}$. Since $U^{-1} \partial_j U = \partial_j + \frac{\rho_j}{\rho}$, we have as an operator identity

$$U^{-1} \left( \sum_{ij} \partial_i g_{00} \gamma^{ij} \partial_j \right) U = g_{00} \sum_{ij} \gamma^{ij} \partial_i \partial_j + V_1 + V_2,$$

where

$$V_1 = \sum_{ij} \left( \alpha_i^{ij} + \alpha_i^{ij} \frac{\rho_i}{\rho} \right) \partial_j,$$

$$V_2 = \frac{1}{4} \sum_{ij} \left( 2\alpha_i^{ij} \frac{\rho_j}{\rho} + 2\alpha_i^{ij} \frac{\rho_j}{\rho} - \alpha_i^{ij} \frac{\rho_i}{\rho} \rho \right).$$

(2.38)
Set $|\text{det} g| = G$ and $\partial_i G = G_i$. Hence we have
\[
V_1 = g_{00} \sum_{ij} \left( \gamma^i{}_{j} + \frac{G_i}{2G} \right) \partial_j,
\]
where $\gamma^i{}_{j} = \partial_i \gamma^{ij}$, and directly we can see that
\[
g_{00} \frac{1}{\sqrt{|\text{det} g|}} \sum_{ij} \partial_i \sqrt{|\text{det} g|} \gamma^{ij} \partial_j = V_1 + g_{00} \sum_{ij} \gamma^{ij} \partial_i \partial_j. \tag{2.39}
\]
Comparing (2.38) with (2.39) we obtain that
\[
U^{-1} \left( \sum_{ij} \partial_i g_{00} \gamma^{ij} \partial_j - V_2 \right) U = g_{00} \frac{1}{\sqrt{|\text{det} g|}} \sum_{ij} \partial_i \sqrt{|\text{det} g|} \gamma^{ij} \partial_j. \tag{2.40}
\]
Then we proved the lemma below.

**Lemma 2.9** It follows that
\[
UKU^{-1} = \sum_{ij} \partial_i g_{00} \gamma^{ij} \partial_j - v, \tag{2.41}
\]
where $v = g_{00}(m^2 + \eta \mathcal{R}) + V_2$.

By Lemma 2.9 (2.38) is transformed to the equation:
\[
\frac{\partial^2 \phi}{\partial t^2} = \left( \sum_{ij} \partial_i g_{00} \gamma^{ij} \partial_j - v \right) \phi \tag{2.42}
\]
on $L^2(\mathbb{R}^3)$.

**Proof of Lemma 2.8** Now we come back to the proof of Lemma 2.8. Set
\[
g_{\mu\nu}(x) = \begin{cases} 
e^{-\theta(x)}, & \mu = \nu = 0, \\ -e^{-\theta(x)}, & \mu = \nu = 1, 2, 3, \\ 0, & \mu \neq \nu. \end{cases}
\]
Then
\[
\rho = \frac{\sqrt{|\text{det} g|}}{g_{00}} = e^{-\theta}, \quad \alpha^{ij} = g_{00} \gamma^{ij} = \delta^{ij}, \tag{2.43}
\]
and $UKU^{-1} = \Delta - v$ follows by (2.41), where, inserting (2.43) to $v$, we have
\[
v = e^{-\theta}(m^2 + \eta \mathcal{R}) - \frac{\Delta \theta}{2} + \frac{|
abla \theta|^2}{4}. \tag{2.44}
\]
Taking $\eta = 0$, $m = 0$, and $\theta(x) = 2a\langle x \rangle^{-\beta}$, we obtain

$$v(x) = a\langle x \rangle^{-3\beta}(\beta - 1)|x|^2 - 3a\langle x \rangle^{-2\beta - 4} |x|^2. \quad (2.45)$$

In the case of $0 \leq \beta \leq 1$ and $a < 0$, we see that $v \geq 0$ and $v = \mathcal{O}(\langle x \rangle^{-\beta - 2})$. Furthermore $-\Delta + v$ has no non-positive eigenvalues. In the case of $\beta > 1$ and $a > 0$, we see that however $v \not\geq 0$. We can estimate the number of non-positive eigenvalues of $-\Delta + v$ by the Lieb-Thirring inequality [Lie73]:

$$\#\{\text{eigenvalues of } -\Delta + v \leq 0\} \leq C_{LT} \int |v_-(x)|^{3/2} dx, \quad (2.46)$$

where $v_-$ denotes the negative part of $v$ and $C_{LT}$ is a constant independent of $v$. This yields that $-\Delta + v$ has no non-positive eigenvalues for sufficiently small $a$. Thus the lemma holds.

\section{Functional integrations}

\subsection{Path measures for particles}

In order to construct a functional integral representation we introduce a probability measure $P^x$ with reference measure $\mu_p$ such that $(f, e^{-tL_p}g)$ can be expressed as

$$(f, e^{-tL_p}g) = \int d\mu_p(x) \mathbb{E}^x[f(X_0)g(X_t)]. \quad (3.1)$$

We already mention that formally $L_p$ is given by

$$L_p f = -\frac{1}{2} \Delta f + \nabla \varphi_p \nabla f. \quad (3.2)$$

Thus $X = (X_t)_{t \in \mathbb{R}}$ is the solution of the stochastic differential equation

$$dX_t = dB_t + \nabla \log \varphi_p(X_t) dt. \quad (3.3)$$

The regularity of ground state $\varphi_p$ is, however, unclear. So we construct the process $X$ through the Kolmogorov consistency theorem. Let us set $\bar{L}_p = L_p - \inf \sigma(L_p)$.

**Proposition 3.1** Suppose that Assumption 2.1 holds. Then there exists a probability space $(\Omega, \mathcal{F}, P^x)$ and an $\mathbb{R}^3$-valued continuous Markov process $X = (X_t)_{t \in \mathbb{R}}$ starting at $x$ such that for $t_0 \leq t_1 \leq \cdots \leq t_n$ and $f_0, f_n \in \mathcal{H}_p$ and $f_j \in L^\infty(\mathbb{R}^3)$, $j = 1, \ldots, n - 1$,

$$(f_0, e^{-(t_1 - t_0)\bar{L}_p}f_1 \cdots e^{-(t_n - t_{n-1})\bar{L}_p}f_n)_{\mathcal{F}_p} = \int d\mu_p(x) \mathbb{E}^x \left[ \prod_{j=0}^n f_j(X_{t_j}) \right]. \quad (3.4)$$
Proof: We show an outline of the proof. The proof is based on the Kolmogorov consistency theorem. For \( t_0 \leq t_1 \leq \cdots \leq t_n \) and \( A_j \in \mathcal{B}(\mathbb{R}^3), \ j = 0, 1, \ldots, n, \) where \( \mathcal{B}(\mathbb{R}^3) \) denotes the Borel \( \sigma \)-field, let

\[
\nu(A_0 \times \cdots \times A_n) = (1_{A_0}, e^{-(t_1-t_0)L_p}1_{A_1} \cdots e^{-(t_n-t_{n-1})L_p}1_{A_n})_{\mathcal{F}_p}.
\]

Thus \( \nu \) satisfies the consistency condition

\[
\nu(A_0 \times \cdots \times A_n \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3) = \nu(A_0 \times \cdots \times A_n).
\]

By the Kolmogorov consistency theorem there exists a measure \( \nu_\infty \) on \( (\mathbb{R}^3)^{(-\infty, \infty)} \) such that

\[
\nu(A_0 \times \cdots \times A_n) = \mathbb{E}_{\nu_\infty} \left[ \prod_{j=0}^n 1_{A_j}(X_{t_j}) \right],
\]

where \( X_t(\omega) = \omega(t) \) for \( \omega \in (\mathbb{R}^3)^{(-\infty, \infty)} \) the point evaluation. We note that by the Feynman-Kac formula \( E_{\nu_\infty} [|X_t - X_s|^{2n}] \) can be expressed in terms of Brownian motion \( (B_t)_{t \geq 0} \) on \( (W, \mathcal{B}_W, P_W) \) as

\[
\mathbb{E}_{\nu_\infty} [|X_t - X_s|^{2n}] = \int dx \mathbb{E}^x_{P_W} \left[ |B_{t-s} - B_0|^{2n} \varphi_p(B_0) \varphi_p(B_{t-s}) e^{-\int_0^{t-s} V(B_r)dr} \right] e^{(t-s)\inf \sigma(L_p)}.
\]

By (1) of Assumption 2.1 we have

\[
\sup_{x \in \mathbb{R}^3} \mathbb{E}^x_{P_W} \left[ e^{-\int_0^{t-s} V(B_r)dr} \right] < \infty,
\]

and \( \mathbb{E}^x_{P_W} [|B_{t-s} - B_0|^{2n}] = C_{2n}|t-s|^n \) with some constant \( C_{2n} \). Then it can be shown that \( \mathbb{E}_{\nu_\infty} [|X_t - X_s|^{2n}] \leq C|t-s|^n \) with some constant \( C \) independent of \( s \) and \( t \). Then \( X = (X_t)_{t \in \mathbb{R}} \) has a continuous version \( \tilde{X} = (\tilde{X}_t)_{t \in \mathbb{R}} \). The image measure of \( \nu_\infty \) on \( \Omega = C(\mathbb{R}; \mathbb{R}^3) \) with respect to \( \tilde{X} \) is denoted by \( P \) and defined as the measure

\[
P^x(\cdot) = P(\cdot | \tilde{X}_0 = x)
\]

for \( x \in \mathbb{R}^3 \) on \( \Omega \). Then

\[
(1_{A_0}, e^{-(t_1-t_0)L_p}1_{A_1} \cdots e^{-(t_n-t_{n-1})L_p}1_{A_n})_{\mathcal{F}_p} = \mathbb{E}^x \left[ \prod_{j=0}^n 1_{A_j}(\tilde{X}_{t_j}) \right].
\]

---

\(^{1}\)Let \( \sigma(\tilde{X}_0) \) denote the \( \sigma \)-field generated by \( \tilde{X}_0 \). For \( Z \subset \Omega \), let \( P(Z | \sigma(\tilde{X}_0)) = \mathbb{E}_P[1_Z | \sigma(\tilde{X}_0)] \). Then \( P(Z | \sigma(\tilde{X}_0)) \) is \( \sigma(\tilde{X}_0) \)-measurable. Thus \( P(Z | \sigma(\tilde{X}_0)) \) is a function of \( \tilde{X}_0 \), i.e., \( P(Z | \sigma(\tilde{X}_0)) = G_Z(\tilde{X}_0) \) with some \( G_Z \). \( P(Z | \tilde{X}_0 = x) \) is defined by \( G_Z(\tilde{X}_0) \) with \( \tilde{X}_0 \) replaced by \( x \), i.e., \( P(Z | \tilde{X}_0 = x) = G_Z(x) \).
Here $\mathbb{E}^x = \mathbb{E}_{\mu^x}$. By a simple limiting argument, (3.4) can be proven. Finally we shall show the Markov property of $\tilde{X}$. Let

$$p_t(x, A) = \left( e^{-t\bar{L}_p}1_A \right)(x). \tag{3.7}$$

Then (3.6) is represented as

$$\int \prod_{j=0}^n 1_{A_j}(x_j) \prod_{j=1}^n p_{t_j-t_{j-1}}(x_{j-1}, dx_j) \varphi^2_p(x_0) dx_0.$$ 

Hence it is enough to show that $p_t(x, A)$ is a probability transition kernel. Note that $e^{-t\bar{L}_p}$ is positivity preserving. Then $0 \leq e^{-t\bar{L}_p}f \leq 1$ for all function $f$ such that $0 \leq f \leq 1$, and $e^{-t\bar{L}_p}1 = 1$ follow. Then it satisfies that

(a) $p_t(x, \cdot)$ is the probability measure on $\mathbb{R}^3$ with $p_t(x, \mathbb{R}^3) = 1$,

(b) $p_0(x, A) = 1_A(x)$,

(c) $\int p_s(y, A)p_t(x, dy) = p_{t+s}(x, A)$.

Hence $p_t(x, A)$ is a probability transition kernel. Then the process $\tilde{X}$ constructed above is Markov under the measure $\mathbb{P}^x$. \hfill \Box

By (3.4) it can be seen that $X$ is invariant with respect to any time shift, namely

$$\int d\mu_p(x)\mathbb{E}^x \left[ \prod_{j=0}^n f_j(X_{t_j}) \right] = \int d\mu_p(x)\mathbb{E}^x \left[ \prod_{j=0}^n f_j(X_{s+t_j}) \right]$$

for any $s \in \mathbb{R}$. The time reversal property also holds:

$$\int d\mu_p(x)\mathbb{E}^x \left[ \prod_{j=0}^n f_j(X_{t_j}) \right] = \int d\mu_p(x)\mathbb{E}^x \left[ \prod_{j=0}^n f_j(X_{-t_j}) \right].$$

Moreover $X_t$ and $X_{-s}$ for $-s \leq 0 \leq t$ are independent, since

$$\mathbb{E}^x[X_{-s}X_t] = \mathbb{E}^x[X_{-s}\mathbb{E}^x[X_t|\mathcal{B}_{[-s,0]}]] = \mathbb{E}^x[X_{-s}\mathbb{E}^{X_0}[X_t]] = \mathbb{E}^x[X_{-s}\mathbb{E}^x[X_t]],$$

where $\mathcal{B}_{[a,b]} = \sigma(X_r, a \leq r \leq b)$. 
3.2 Building of quantum fields and semigroups

The free Hamiltonian $H_f$ can be regarded as the infinite dimensional version of the harmonic oscillator $H_{osc} = \frac{1}{2}p^2 + \frac{1}{2}x^2 - \frac{1}{2}$. The process associated with $H_{osc}$ is the Ornstein-Uhlenbeck process $(q_t)_{t \in \mathbb{R}}$, and hence

$$\int dx \Psi(x)^2 \mathbb{E}[q_t q_s] = (x_\Psi e^{-(t-s)H_{osc}} x_\Psi) = e^{-|t-s|},$$

where $\Psi(x) = \pi^{-1/4} e^{-x^2/2}$ is the ground state of $H_{osc}$. There exists an infinite dimensional version of $q = (q_t)_{t \in \mathbb{R}}$.

Let $d = 1, 2, \ldots$ denote the dimension. Let $\Phi_d(f)$ be the Gaussian random process indexed by real-valued $f \in L^2(\mathbb{R}^d)$ on some probability space $(\mathcal{Q}_d, \mu_d)$ with mean zero and the covariance given by

$$\int_{\mathcal{Q}_d} \Phi_d(f) \Phi_d(g) d\mu_d = \frac{1}{2} (\hat{f}, \hat{g})_{L^2(\mathbb{R}^d)}.$$

The set of the linear hull of functions of the form $: \Phi_d(f_1) \cdots \Phi_d(f_n) :$ is dense in $L^2(\mathcal{Q}_d)$, where $: Z :$ denotes the Wick product of $Z$ inductively defined by $: \Phi_d(f) := \Phi_d(f)$ and

$$: \Phi_d(f) \Phi_d(f_1) \cdots \Phi_d(f_n) : =: \Phi_d(f_1) \cdots \Phi_d(f_n) - \frac{1}{2} \sum_{j=1}^n (\hat{f}, \hat{f}_j) : \Phi_d(f_1) \cdots \hat{\Phi_d(f_j)} \cdots \Phi_d(f_n) :,$$

where $\hat{\Phi_d(f_j)}$ denotes neglecting $\Phi_d(f_j)$. Note that

$$(: \Phi_d(f_1) \cdots \Phi_d(f_n) :, : \Phi_d(\rho_1) \cdots \Phi_d(\rho_m) :) = \delta_{nm} \frac{1}{2^n} \sum_{\sigma \in G_n} (f_1, \rho_{\sigma(1)}) \cdots (f_n, \rho_{\sigma(n)}).$$

For Hilbert spaces $A$ and $B$, let

$$\mathcal{C}(A, B) = \{ T : A \rightarrow B \| T \|_{A \rightarrow B} \leq 1 \}$$

be the set of contractions from $A$ to $B$, and

$$\mathcal{C}_0(A, B) = \{ T \in \mathcal{C}(A, B) | T \text{ is isometry} \}.$$

The second quantization $\Gamma$ is a functor:

$$\Gamma : \mathcal{C}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d'})) \rightarrow \mathcal{C}(L^2(\mathcal{Q}_d), L^2(\mathcal{Q}_{d'})).$$
and
\[ \Gamma : \mathcal{C}_0(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d'})) \rightarrow \mathcal{C}_0(L^2(\mathcal{D}_d), L^2(\mathcal{D}_{d'})), \]
and it is defined by \( \Gamma(T)1_{L^2(\mathcal{D}_d)} = 1_{L^2(\mathcal{D}_{d'})} \) and
\[ \Gamma(T) : \Phi_d(f_1) \cdots \Phi_d(f_n) : = : \Phi_{d'}(Tf_1) \cdots \Phi_{d'}(Tf_n) : . \] (3.8)

It satisfies the semigroup property:
\[ \Gamma(T)\Gamma(S) = \Gamma(TS), \] (3.9)
when \( S \in \mathcal{C}(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d'})) \) and \( T \in \mathcal{C}(L^2(\mathbb{R}^{d'}), L^2(\mathbb{R}^{d''})) \). Contraction operator \( \Gamma(T) \) depends on \( d \) and \( d' \), we do not, however, distinguish them, and simply write \( \Gamma(T) \).

\( \Gamma(e^{-itK}) \) for a self-adjoint operator \( K \) in \( L^2(\mathbb{R}^d) \) is one parameter unitary group on \( L^2(\mathcal{D}_d) \). Then its generator is denoted by \( d\Gamma(K) \), namely \( \Gamma(e^{-itK}) = e^{-itd\Gamma(K)} \).

Let \( h \geq 0 \) be a Borel measurable function on \( \mathbb{R}^d \). Define the family of isometries \( j_{d,h}(t) \in \mathcal{C}_0(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1})) \), \( t \in \mathbb{R} \), by
\[ j_{d,h}(t)f = \frac{e^{-itk_{d+1}}}{\sqrt{\pi}} \left( \frac{h(k)}{h(k)^2 + |k_{d+1}|^2} \right)^{1/2} \hat{f}(k), \quad k \in \mathbb{R}^d, \quad k_{d+1} \in \mathbb{R}. \] (3.10)

It satisfies that
\[ j_{d,h}(s)^* j_{d,h}(t) = e^{-|t-s|h(-i\nabla)}. \] (3.11)

For a given Borel measurable nonnegative functions \( h_1 \) on \( \mathbb{R}^3 \), \( h_2 \) on \( \mathbb{R}^4 \), \( h_3 \) on \( \mathbb{R}^5 \),..., we have a sequence
\[ L^2(\mathbb{R}^3) \xrightarrow{j_{3,h_1}(t)} L^2(\mathbb{R}^4) \xrightarrow{j_{4,h_2}(t)} L^2(\mathbb{R}^5) \xrightarrow{j_{5,h_3}(t)} \cdots. \] (3.12)

Each isometry in (3.12) satisfies (3.11). Define \( J_{d,h}(t) \in \mathcal{C}_0(L^2(\mathcal{D}_d), L^2(\mathcal{D}_{d+1})) \) by the second quantization of \( j_{d,h}(t) \in \mathcal{C}_0(L^2(\mathbb{R}^d), L^2(\mathbb{R}^{d+1})) \), namely \( J_{d,h}(t) = \Gamma(j_{d,h}(t)) \). Hence it follows that
\[ J_{d,h}(s)^* J_{d,h}(t) = \Gamma(e^{-|t-s|h(-i\nabla)}). \] (3.13)

Sequence (3.12) is inherited on \( L^2(\mathcal{D}_d) \) as
\[ L^2(\mathcal{D}_3) \xrightarrow{j_{3,h_1}(t)} L^2(\mathcal{D}_4) \xrightarrow{j_{4,h_2}(t)} L^2(\mathcal{D}_5) \xrightarrow{j_{5,h_3}(t)} \cdots. \] (3.14)

Let \( h \) and \( f \) be Borel measurable nonnegative functions on \( \mathbb{R}^d \). The crucial property is the intertwining property given by
\[ \Gamma(e^{-t(h(-i\nabla)^{\otimes 1})})J_{d,f}(s) = J_{d,f}(s)\Gamma(e^{-th(-i\nabla)}). \] (3.15)
IR divergence

Here $h(-i\nabla) \otimes 1 = h(-i\nabla) \otimes 1_{L^2(\mathbb{R})}$ is an operator on $L^2(\mathbb{R}^{d+1})$ under the identification $L^2(\mathbb{R}^{d+1}) \cong L^2(\mathbb{R}^d) \otimes L^2(\mathbb{R})$.

**Proposition 3.2** Let $h_j$, $j = 1, \ldots, N$, be Borel measurable nonnegative functions on $\mathbb{R}^3$. Let $H_j = d\Gamma(h_j(-i\nabla))$. Then

$$
\left( \Psi, \prod_{i=1}^{N} e^{-t_i H_i} \Phi \right)_{L^2(\mathcal{D}_i)} = \left( \prod_{i=N}^{1} J_{i+2, h_i^{\text{ex}}(0)} \Psi, \prod_{i=N}^{1} J_{i+2, h_i^{\text{ex}}(t_i)} \Phi \right)_{L^2(\mathcal{D}_{N+3})}.
$$

Here $\prod_{i=1}^{N} T_i = T_1 \cdots T_N$ and $\prod_{i=N}^{1} T_i = T_N \cdots T_1$ and $h_i^{\text{ex}}$ is an extension of $h$ to the nonnegative function on $L^2(\mathbb{R}^{2+i})$ defined by $h_i^{\text{ex}}(k, k_4, \ldots, k_{2+i}) = h_i(k)$ for $k \in \mathbb{R}^3$.

In order to construct a functional integral representation of the semigroup $e^{-tH}$ we take the Schrödinger representation instead of the Fock representation. In addition we need the Euclidean field. We set

$$
\mathcal{D} = \mathcal{D}_3, \quad \mu = \mu_3, \quad j_t = j_{3\omega}(t),
\mathcal{D}_E = \mathcal{D}_4, \quad \mu_E = \mu_4, \quad \xi_t = j_{4\omega}(t),
$$

where $I$ denotes the identity operator on $L^2(\mathbb{R}^4)$. It is well known that there exists an isomorphism between $\mathcal{F}$ and $L^2(\mathcal{D})$. By this isomorphism we can identify as $\Omega_{\mathcal{F}} \cong 1$, $H_t \cong d\Gamma(\omega(-i\nabla))$ and $\Phi(x) \cong \phi(\bar{\chi}(x))$, where

$$
\bar{\chi}(\cdot, x) = \left( \frac{\chi(\cdot)}{\sqrt{\omega(\cdot)}} \Psi(\cdot, x) \right)^\vee.
$$

Note that in the Schrödinger representation the test function is taken in the position representation while the momentum representation is used in the Fock representation.

**Definition 3.3** (The Nelson model in Schrödinger representation)

In the Schrödinger representation the Nelson Hamiltonian is defined by

$$
\bar{L}_p \otimes 1 + 1 \otimes d\Gamma(\omega(-i\nabla)) + \alpha \int_{\mathbb{R}^3} \phi(\bar{\chi}(x)) dx
$$

on $\mathcal{H}_p \otimes L^2(\mathcal{D})$. Here we identify $\mathcal{H}_p \otimes L^2(\mathcal{D})$ as $\int_{\mathbb{R}^3} L^2(\mathcal{D}) d\mu_p$.

In what follows we write $[3.19]$ as $H_t$, $d\Gamma(\omega(-i\nabla))$ as $H_t$ and $\mathcal{H}_p \otimes L^2(\mathcal{D})$ as $\mathcal{H}$. 
The operator $d\Gamma(I)$ is called the number operator. The number operator on $L^2(\mathcal{D})$ (resp $L^2(\mathcal{D}_E)$) is denoted by $N$ (resp $N_E$). We define the specific families of isometries $J_t \in \mathcal{C}_0(L^2(\mathcal{D}), L^2(\mathcal{D}_E))$ and $\Xi_t \in \mathcal{C}_0(L^2(\mathcal{D}_E), L^2(\mathcal{D}_E))$ by

\[ J_t = \Gamma(j_t) = J_{3,\omega}(t), \]
\[ \Xi_t = \Gamma(\xi_t) = J_{4,I}(t) \] \hspace{1cm} (3.20)

for $t \in \mathbb{R}$. Thus it follows that

\[ J_s^* J_t = e^{-|t-s|H_t} \]
\[ \Xi_s^* \Xi_t = e^{-|t-s|N_E}. \] \hspace{1cm} (3.21)

Moreover we have

\[ e^{-\beta N_E} J_s = J_s e^{-\beta N}, \quad \beta \geq 0, \] \hspace{1cm} (3.22)

by the intertwining property (3.15).

**Example 3.4** From Proposition 3.2 it follows that

\[ (\Psi, e^{-\beta N} e^{-tH} \Phi)_{L^2(\mathcal{D})} = (\Xi_0 J_0 \Psi, \Xi_\beta J_t \Phi)_{L^2(\mathcal{D}_E)}. \] \hspace{1cm} (3.23)

### 3.3 Functional integral representations

Combining the functional integral representations of both $e^{-t\hat{L}_p}$ and $e^{-tH_t}$ stated in the previous sections, we can construct the functional integral representation of $e^{-tH}$

Let

\[ \phi_s(f) = \Phi_4(j_s f), \quad s \in \mathbb{R}. \]

It is the Gaussian random process indexed by real-valued functions $f \in L^2(\mathbb{R}^3)$ such that the mean is zero and the covariance is given by

\[ \int_{\mathcal{D}} \phi_s(f) \phi_t(g) d\mu_E = \int_{\mathbb{R}^3} \hat{f}(k) \hat{g}(k) e^{-|t-s|\omega(k)} dk. \] \hspace{1cm} (3.24)

Thus $(\phi_s(f))_{s \in \mathbb{R}}$ denotes the infinite dimensional version of the Ornstein-Uhlenbeck process. We note that $J_s : \phi(f_1) \cdots \phi(f_n) := \phi_s(f_1) \cdots \phi_s(f_n)$ and $J_s 1_{L^2(\mathcal{D})} = 1_{L^2(\mathcal{D}_E)}$. Combining the process $X_t$ in (3.4) and $J_t$ in (3.20) we obtain the theorem below.

**Theorem 3.5** Suppose Assumptions 2.1, 2.2 and 2.4. Let $F, G \in \mathcal{H}_0 \otimes L^2(\mathcal{D})$. Then

\[ (F, e^{-tH} G) = \int d\mu_0(x) E^x \left[ \left( J_0 F(X_0), e^{-\alpha \int_0^t \phi_s(\bar{x}(X_s)) ds} J_t G(X_t) \right)_{L^2(\mathcal{D}_E)} \right]. \] \hspace{1cm} (3.25)
**PROOF:** By the Trotter product formula

\[ e^{-tH} = s - \lim_{n \to \infty} \left( e^{-(t/n)L_p} e^{-(t/n)\alpha \phi(\tilde{\chi}(x))} e^{-(t/n)H_t} \right)^n, \]

the factorization formula (3.21), Markov property of \( E_t = J_t J_t^* \) and (3.4), we have

\[
(F, e^{-tH} G) = \lim_{n \to \infty} \int d\mu_p(x) E^x \left[ \left( J_0 F(X_0), e^{-\alpha \sum_{j=0}^n \frac{\beta}{n} \phi_{t_j/n}(\tilde{\chi}(X_{t_j/n}))} J_t G(X_t) \right)_{L^2(\mathcal{D}_E)} \right].
\]

(3.26)

Note that \( s \mapsto \tilde{\chi}(\cdot, X_s) \) is strongly continuous as the map \( \mathbb{R} \to L^2(\mathbb{R}^3) \) almost surely. Hence \( s \mapsto \phi_s(\tilde{\chi}(X_s)) \) is strongly continuous as the map \( \mathbb{R} \to L^2(\mathcal{D}_E) \). By a simple limiting argument we complete the proof. \( \square \)

Next let

\[ \phi_{s,t}(f) = \Phi_5(\xi_{s} f), \quad s, t \in \mathbb{R}. \]

It is also the Gaussian random process indexed by real-valued functions \( f \in L^2(\mathbb{R}^3) \) with mean zero and the covariance given by

\[
\int_{\mathcal{D}_E} \phi_{s,t}(f) \phi_{s',t'}(g) d\mu_E = \frac{1}{2} \int \hat{f}(k) \hat{g}(k) e^{-|s-s'|^2 \omega(k)} e^{-|t-t'|} dk.
\]

(3.27)

We see that \( \Xi_t : \phi_{s_1}(f_1) \cdots \phi_{s_n}(f_n) := \phi_{s_1,t}(f_1) \cdots \phi_{s_n,t}(f_n) : \) and \( \Xi_t 1_{L^2(\mathcal{D}_E)} = 1_{L^2(\mathcal{D}_E)}. \) Then we have the theorem.

**Theorem 3.6** Suppose Assumptions 2.1, 2.2 and 2.4. Let \( F, G \in \mathcal{H} \). Then

\[
(F, e^{-sH} e^{-\beta N} e^{-tH} G) = \int d\mu_p(x) E^x \left[ \left( \Xi_0 J_0 F(X_0), e^{-\alpha \int_0^s \phi_{\tau,0}(\tilde{\chi}(X_{\tau})) d\tau} e^{-\alpha \int_s^{s+t} \phi_{\tau,\beta}(\tilde{\chi}(X_{\tau})) d\tau} \Xi_0 J_t G(X_t) \right)_{L^2(\mathcal{D}_E)} \right]
\]

(3.28)

**PROOF:** Throughout this proof we set \( \prod_{j=0}^n T_j = T_0 T_1 \cdots T_n. \)

Simply we put \( \alpha \phi(\tilde{\chi}(x)) = \phi \). By the Trotter product formula we have

\[
(F, e^{-sH} e^{-\beta N} e^{-tH} G) = \lim_{n \to \infty} \lim_{m \to \infty} \left( F, \left( e^{-\frac{s}{n} L_p} e^{-\frac{s}{n} \phi} e^{-\frac{1}{n} H_t} \right)^n e^{-\beta N} \left( e^{-\frac{1}{m} L_p} e^{-\frac{1}{m} \phi} e^{-\frac{1}{m} H_t} \right)^m G \right).
\]

Inserting \( e^{-|T-S|H_t} = J_{T-S}^* J_S \) we have

\[
= \left( F, J_0^{n-1} \left( J_{t/n}^* e^{-\frac{s}{n} L_p} e^{-\frac{s}{n} \phi} J_{t/n}^* \right) J_s e^{-\beta N} J_s^* \right.
\]

\[
\left. \prod_{i=0}^{m-1} \left( J_{s+i/n}^* e^{-\frac{t-i}{m} L_p} e^{-\frac{t-i}{m} \phi} J_{s+i/n}^* \right) J_{s+t/G} \right).
\]

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Let $E_T = J_T J_T^*$. $E_T$ is the family of projection on $L^2(\mathcal{Q}_E)$. Since $J_T^* e^{\phi} J_T = E_T e^{\phi_T} E_T$ and by the intertwining property $J_s e^{-\beta N} J_s^* = J_s^* J_s e^{-\beta N_E} = E_s \Xi_0^* \Xi_\beta$, we have

$$\left( F, J_0^* \prod_{i=0}^{n-1} \left( E_{\Xi_0} e^{-\frac{L_{\phi}}{m} - \frac{\phi}{m} \Xi_{\Xi_0}} \right) E_s \Xi_0^* \Xi_\beta \right) \prod_{i=0}^{m-1} \left( E_{s+\frac{t}{m}} e^{-\frac{L_{\phi}}{m} - \frac{t}{m} \phi_{s+\frac{t}{m}}} \right) J_{s+t} G,$$

where $\phi_T = \alpha \phi_T(\tilde{\chi}(x))$. By the Markov property of $E_s$ we can neglect all $E_s$, then we have

$$\left( F, J_0^* \prod_{i=0}^{n-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{\phi}{m} \Xi_{\Xi_0}} \right) \Xi_0^* \Xi_\beta \prod_{i=0}^{m-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{t}{m} \phi_{s+\frac{t}{m}}} \right) J_{s+t} G \right).$$

Again we use the fact $\Xi_\beta e^{\phi_0} \Xi_\beta = E_\beta^\Xi e^{\phi_0, \beta} E_\beta^\Xi$, where $E_\beta^\Xi = \Xi_\beta^\Xi \Xi_\beta^*$ denotes the projection on $L^2(\mathcal{Q}_5)$. Hence we have

$$\left( \Xi_0 J_0 F, E_0^\Xi \prod_{i=0}^{n-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{\phi}{m} \Xi_{\Xi_0}} \right) \Xi_0^\Xi \Xi_\beta \prod_{i=0}^{m-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{t}{m} \phi_{s+\frac{t}{m}}} \right) J_{s+t} G \right).$$

Since by the Markov property of $E_s$ we can neglect $E_0^\Xi$ and $E_\beta^\Xi$, we can obtain

$$\left( \Xi_0 J_0 F, \prod_{i=0}^{n-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{\phi}{m} \Xi_{\Xi_0}} \right) \prod_{i=0}^{m-1} \left( e^{-\frac{L_{\phi}}{m} - \frac{t}{m} \phi_{s+\frac{t}{m}}} \right) \Xi_\beta J_{s+t} G \right),$$

where $\phi_{S,T} = \phi_{S,T}(\tilde{X}(x))$. By (3.4) and a limiting argument, we can prove the theorem. □

4 Infrared divergence and absence of ground states

4.1 Abstract theory of the absence of ground states

In this section we assume Assumptions 2.1, 2.2 and 2.4. By the functional integral representation obtained in Theorem 3.5, we can see that

$$(F, e^{-tH} G) > 0$$
IR divergence

for any $F \geq 0$ and $G \geq 0$ but $F \neq 0$ and $G \neq 0$. Thus $e^{-tH}$ is positivity improving. Then whenever a ground state $\varphi_g$ of $H$ exits, $\varphi_g > 0$ by the Perron-Frobenius Theorem. In particular the ground state is unique if it exists. Now we introduce a sequence approaching to the ground state. Let $1 = 1_H \otimes 1_{L^2(\mathcal{F})}$ and

$$\varphi_g^T = \|e^{-TH}1\|^{-1}e^{-TH}1, \quad T > 0. \quad (4.1)$$

Define

$$\gamma(T) = (1, \varphi_g^T)^2, \quad T > 0. \quad (4.2)$$

If $H$ has a ground state, then $\varphi_g^T$ converges to $\varphi_g$ strongly as $T \to \infty$. We can have a criteria on the existence and non-existence of the ground state.

**Proposition 4.1** (1) When $\lim_{T \to \infty} \gamma(T) = a > 0$, $H$ has a ground state. (2) When $\lim_{T \to \infty} \gamma(T) = 0$, $H$ has no ground state.

Note that

$$\gamma(T) = \frac{(1, e^{-TH}1)^2}{\|e^{-TH}1\|^2}. \quad (4.2)$$

Since $\phi_s(g)$ is a Gaussian random process, by means of the functional integral representation (3.25), we can see that

$$(1, e^{-TH}1) = \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2)(\int_0^T \phi_s(\chi(X_s))ds, \int_0^T \phi_t(\chi(X_t))dt)} \right]$$

$$= \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|)} \right],$$

where

$$W(X, Y, |t|) = \int \frac{\chi(k)^2}{2\omega(k)} \Psi(k, X)\Psi(k, Y)e^{-|t|\omega}dk. \quad (4.3)$$

Note that

$$\int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|) > 0 \quad (4.4)$$

follows, since the left hand side is expressed as $(\int_0^T \phi_s(\chi(X_s))ds, \int_0^T \phi_t(\chi(X_t))dt)$. While

$$\|e^{-TH}1\|^2 = \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T ds \int_0^T dt W(X_s, X_t, |s-t|)} \right]$$

$$= \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_{-T}^{T} ds \int_{-T}^{T} dt W(X_s, X_t, |s-t|)} \right].$$
by the shift invariance of \(X_t\). Then \(\gamma(T)\) can be expressed as

\[
\gamma(T) = \frac{\int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW(X_s, X_t | s-t)} \right]^2}{\int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW(X_s, X_t | s-t)} \right]}. \tag{4.5}
\]

Let \(\mu_T\) be the probability measure on \((\mathbb{R}^3 \times \Omega, \mathcal{B}(\mathbb{R}^3) \times \mathcal{B})\) defined by for \(A \times B \in \mathcal{B}(\mathbb{R}^3) \times \mathcal{B},\)

\[
\mu_T(A \times B) = \frac{1}{Z_T} \int d\mu_p(x) \mathbb{E}^x \left[ 1_{A \times B} e^{(\alpha^2/2) \int_0^T dt \int_0^T dW(X_s, X_t | s-t)} \right], \tag{4.6}
\]

where \(Z_T\) denotes the normalizing constant such that \(\mu_T\) becomes a probability measure.

**Lemma 4.2** \(\int_{-T}^T ds \int_0^T dW(X_s, X_t, |s-t|)\) is real and it follows that

\[
\gamma(T) \leq \mathbb{E}_{\mu_T} \left[ e^{-\alpha^2 \int_0^T dt \int_0^T dW(X_s, X_t, |s-t|)} \right] \tag{4.7}
\]

**Proof:** The numerator of (4.5) can be estimated by the Schwartz inequality and the time shift of \(X\) as

\[
0 \leq \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW} \right] \leq \int d\mu_p(x) \left( \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW} \right] \right) \left( \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW} \right] \right) = \int d\mu_p(x) \left( \mathbb{E}^x \left[ e^{(\alpha^2/2) \int_0^T dt \int_0^T dW} \right] \right)^2.
\]

Since \(X_t\) and \(X_s\) for \(s \leq 0 \leq t\) are independent, we have

\[
\gamma(T) = \int d\mu_p(x) \mathbb{E}^x \left[ e^{(\alpha^2/2) (\int_0^T dt \int_0^T dW + \int_0^T dt \int_0^T dW)} \right].
\]

Moreover from \(\int_{-T}^0 f_{-T} f_0^T = \int_{-T}^T f_{-T} f_0^T - 2 \int_{-T}^0 f_0^T\) and (4.4), it follows that integral \(\int_{-T}^0 ds \int_0^T dt W(X_s, X_t, |s-t|)\) is real and

\[
\gamma(T) = \int d\mu_p(x) \mathbb{E}^x \left[ e^{-\alpha^2 \int_{-T}^T dt \int_0^T dW + \alpha^2 / 2 \int_{-T}^T dt \int_0^T dW} \right].
\]

Then the lemma follows.

We can compute \(W\) explicitly. Note that the operator \(e^{-|t|\sqrt{-\Delta + m^2}}\) has the integral kernel

\[
e^{-|t|\sqrt{-\Delta + m^2}}(X, Y) = 2 \left( \frac{m}{2\pi} \right)^{(d+1)/2} \frac{|t|}{(|X - Y|^2 + |t|^2)^{d+1}/2} K_{d+1} \left( m \sqrt{|X - Y|^2 + t^2} \right),
\]
where $K_\nu$ denotes the modified Bessel function of the third kind. In particular in the case of $d = 3$ and $m = 0$ we have
\[
e^{-|t|^\sqrt{-\Delta}}(X, Y) = \frac{1}{\pi^2 (|X - Y|^2 + |t|^2)^2} \quad (d = 3).
\]
Then
\[
W(x, y, |T|) = \frac{1}{2} \int_T^\infty d|t| \left( \Psi_x \chi, e^{-|t|\omega} \Psi_y \chi \right)
= \frac{1}{4\pi^2} \int dX \int dY \frac{(\Psi_x \chi)^\vee (X)(\Psi_y \chi)^\vee (Y)}{|X - Y|^2 + |s - t|^2}.
\]

We are in the position to state the main theorem. This is an abstract version of LMS02.

**Theorem 4.3** Let $A_T = \mathbb{R}^3 \times \{ \tau \in \Omega | |X_\tau(\tau)| \leq T^\lambda, |s| \leq T \}$ for some $\lambda$ such that
\[
\frac{1}{q + 1} < \lambda < 1, \quad (4.8)
\]
where $q$ is the positive constant given in Assumption 2.1. Suppose that there exists $\varrho(T)$ independent of $\tau \in \Omega$ such that
\[
1_{A_T} \int_{-T}^0 ds \int_0^T dt \int dX \int dY \frac{(\Psi_x \chi)^\vee (X)(\Psi_y \chi)^\vee (Y)}{|X - Y|^2 + |s - t|^2} \geq \varrho(T) \quad (4.9)
\]
and $\lim_{T \to \infty} \varrho(T) = \infty$. Then there is no ground states of $H$.

**Proof:** By Lemma 4.2 it is enough to show that

1. $\lim_{T \to \infty} \mathbb{E}_{\mu_T} \left[ 1_{A_T} e^{-\alpha^2 \int_0^T ds \int_0^T dt W(X_s, X_t, |s - t|) \right] = 0,$

2. $\lim_{T \to \infty} \mathbb{E}_{\mu_T} \left[ 1_{A_T^c} e^{-\alpha^2 \int_0^T ds \int_0^T dt W(X_s, X_t, |s - t|) \right] = 0.$

(1) follows from assumption (4.9). We shall prove (2). Note that
\[
\int_{-T}^0 ds \int_0^T dt e^{-|s - t|\omega} = \frac{1}{\omega^2} \left( e^{-T\omega} - 1 \right)^2 \quad (4.10)
\]
and
\[
\int_{-T}^T ds \int_{-T}^T dt e^{-|s - t|\omega} = \frac{2}{\omega^2} \left( e^{-2T\omega} - 1 + 2T\omega \right). \quad (4.11)
\]
Then
\[
\left| \int_{-T}^0 ds \int_0^T dt W(X_s, X_t, |s - t|) \right| \leq \frac{T}{2} \| \chi/\omega \|^2
\]
and

\[
\mathbb{E}_{\mu_T} \left[ 1_{A_T^c} e^{-\alpha^2 \int_{-T}^0 ds \int_{-T}^T dt W(X_s, X_t, |s-t|)} \right] 
\leq e^{\alpha^2(T/2)\|\chi/\omega\|^2} \int d\mu_p(x) \mathbb{E}_x \left[ 1_{A_T^c} e^{\alpha^2 \int_{-T}^0 ds \int_{-T}^T dt W(X_s, X_t, |s-t|)} \right] 
\leq e^{\alpha^2(T/2)\|\chi/\omega\|^2} \left( \int d\mu_p(x) \mathbb{E}_x \left[ e^{\alpha^2 \int_{-T}^0 ds \int_{-T}^T dt W(X_s, X_t, |s-t|)} \right] \right)^{1/2} 
\int d\mu_p(x) \mathbb{E}_x \left[ 1_{A_T^c} \right].
\]

Moreover by (4.11), there exists a constant \( \delta > 0 \) such that

\[
-T \delta \|\chi/\omega\|^2 \leq \int_{-T}^T ds \int_{-T}^T dt W(X_s, X_t, |s-t|) \leq T \delta \|\chi/\omega\|^2.
\]

Then we have

\[
\left( \int d\mu_p(x) \mathbb{E}_x \left[ e^{\alpha^2 \int_{-T}^0 ds \int_{-T}^T dt W(X_s, X_t, |s-t|)} \right] \right)^{1/2} \leq e^{\alpha^2 \delta T \|\chi/\omega\|^2}.
\]

The crucial part is to show that there exists an at most polynomially growth function \( \xi(T) \) such that

\[
\int d\mu_p(x) \mathbb{E}_x \left[ 1_{A_T^c} \right] \leq \xi(T) \exp \left( -c T^{\lambda(q+1)} \right).
\]

This is proven in Lemma 4.4 below. Combining (4.12), (4.14) and (4.15) we have

\[
\lim_{T \to \infty} \mathbb{E}_{\mu_T} \left[ 1_{A_T^c} \right] \leq \lim_{T \to \infty} \xi(T) e^{-c T^{\lambda(q+1)}} e^{\alpha^2 (\delta + 1/2) T \|\chi/\omega\|^2} = 0,
\]

since \( \frac{1}{q+1} < \lambda < 1 \). Then (2) follows.

It remains to show (4.15).

**Lemma 4.4** \( (4.12) \) holds. Explicitly \( \lim_{T \to \infty} \xi(T)/T^{1-2\lambda} < \infty \).

**Proof:** Recall that the external potential is supposed to be \( V(x) > |x|^{2q} \) for sufficiently large \( |x| \), and \( V_+ \in L^1_{\text{loc}}(\mathbb{R}^3) \) and \( V_- \in L^p(\mathbb{R}^3) \) with \( p > 3/2 \). Then by [Car78], the ground state \( \varphi_g \) of \( H_p \) exponentially decays. More explicitly there exist constants \( C > 0 \) and \( \delta > 0 \) such that

\[
\varphi_p(x) \leq C e^{-\delta |x|^{q+1}}.
\]
We divide the left hand side of (4.15) as
\[
\int_{\mathbb{R}^3} \mathbb{E}^x \left[ \sup_{|s|<T^\lambda} |X_s| > T^\lambda \right] \varphi_p(x)^2 \, dx = \int_{|x|<T^\lambda/2} + \int_{|x|\geq T^\lambda/2} = Q_1 + Q_2. \tag{4.18}
\]

Let \( D_a(n) = \{ aj/2^n | j = 0, 1, \ldots, 2^n \} \) be the set of diadic points. By [KV86, Lemma 1.12] it follows that
\[
\mathbb{E}^0 \left[ \sup_{0 \leq s \leq a, s \in D_a(n)} |f(X_s)| > b \right] \leq 3 b \sqrt{(f, f)} + a(\bar{L}_p^{1/2} f, \bar{L}_p^{1/2} f) \tag{4.19}
\]
for \( f \in D(\bar{L}_p^{1/2}) \), where \((f, g) = \langle f, g \rangle_{L^2(\mathbb{R}^3; \varphi_p(x)^2 \, dx)} \). The right-hand side above is uniformly bounded with respect to \( n \), and the indicator function \( 1_{\{ \sup_{|s|<a, s \in D_a(n)} |f(X_s)| > b \}} \) is monotonously increasing in \( n \) and \( X_t(\omega) \) is continuous in \( t \) for each path \( \omega \). Thus by the monotone convergence theorem, we have
\[
\lim_{n \to \infty} \mathbb{E}^0 \left[ \sup_{0 \leq s \leq a, s \in D_a(n)} |f(X_s)| > b \right] = \mathbb{E}^0 \left[ \lim_{n \to \infty} \sup_{0 \leq s \leq a, s \in D_a(n)} |f(X_s)| > b \right] = \mathbb{E}^0 \left[ \sup_{0 \leq s \leq a} |f(X_s)| > b \right].
\]

Hence
\[
\mathbb{E}^0 \left[ \sup_{|s|<a} |f(X_s)| > b \right] \leq 2 \frac{3}{b} \sqrt{(f, f)} + a(\bar{L}_p^{1/2} f, \bar{L}_p^{1/2} f) \tag{4.20}
\]
follows. We apply (4.20) to (4.18). Suppose that \( f \in C^\infty(\mathbb{R}^3) \) and
\[
f(x) = \begin{cases} |x|, & |x| \geq T^\lambda, \\ 0, & |x| \leq T^\lambda - 1. \end{cases}
\]

Moreover we assume that
\[
e^{-\delta/2|x|^{q+1}} f^2, \quad e^{-\delta/2|x|^{q+1}} \partial_\mu f \cdot f, \quad e^{-\delta/2|x|^{q+1}} \partial_\mu^2 f \cdot f \in L^2(\mathbb{R}^3), \quad \mu = 1, 2, 3, \tag{4.21}
\]
and the \( L^2 \) norm of each terms in (4.21) has a upper bound independent of \( T \). By (4.20) for \( T^\lambda + b > 0 \),
\[
\mathbb{E}^0 \left[ \sup_{|s|<a} |f(X_s)| > T^\lambda + b \right] = \mathbb{E}^0 \left[ \sup_{|s|<a} |X_s| > T^\lambda + b \right] \leq \frac{6}{T^\lambda + b} \sqrt{(f, f)} + a(f, \bar{L}_p f). \tag{4.22}
\]
Let $|x| < T^\lambda/2$. Thus we have

\[
\mathbb{E}^x \left[ \sup_{|s| < T} |X_s| > T^\lambda \right] = \mathbb{E}^0 \left[ \sup_{|s| < T} |X_s + x| > T^\lambda \right] \\
\leq \mathbb{E}^0 \left[ \sup_{|s| < T} |X_s| > T^\lambda - |x| \right] \leq \frac{6}{T^\lambda/2} \sqrt{(f, f) + T(f, L_p f)}.
\]

We estimate the right-hand side above. By (4.17) we have

\[
(f, f) = \int f(x)^2 \varphi_p(x)^2 \, dx \leq C^2 e^{-\delta T^\lambda(q+1)} \int f(x)^2 e^{-\delta |x|^{q+1}} \, dx := a_1 e^{-\delta T^\lambda(q+1)}.
\] (4.23)

While

\[
(f, \overline{L}_p f) = -\inf \sigma(L_p)(f, f) + \int \varphi_p(x)^2 \cdot f(x) \frac{1}{\varphi_p(x)} \left( -\frac{1}{2} \Delta + V(x) \right) \varphi_p(x) f(x) \, dx \\
= -\inf \sigma(L_p)(f, f) + \int \varphi_p(x)^2 f(x)^2 \Delta \varphi_p(x) \, dx - \frac{1}{2} \int \varphi_p(x) f(x) \Delta (f \varphi_p)(x).
\]

Then the first term on the right-hand side above is

\[
\int \varphi_p(x)^2 f(x)^2 \Delta \varphi_p(x) \, dx \leq C^2 e^{-\delta T^\lambda(q+1)} \int e^{-\delta |x|^{q+1}} f(x)^2 |x|^{2q} \, dx := a_2 e^{-\delta T^\lambda(q+1)}
\] (4.24)

and the second term is

\[
\int \varphi_p(x) f(x) \Delta (f \varphi_p)(x) \, dx \\
= \int \varphi_p(x) \cdot \left( f(x)^2 \Delta \varphi_p(x) + 2 f(x) \nabla \varphi_p(x) \cdot \nabla f(x) + \Delta f(x) \cdot f(x) \varphi_p(x) \right) \, dx \\
\leq C e^{-\delta/2} T^\lambda(q+1) \int e^{-\delta/2 |x|^{q+1}} |G(x)| \, dx = a_3 e^{-(\delta/2)T^\lambda(q+1)}. \] (4.25)

Hence

\[
Q_1 \leq \frac{12}{T^\lambda} \sqrt{|a_1 - \inf \sigma(L_p)| + T(a_2 + a_3) e^{-(\delta/4)T^\lambda(q+1)}}.
\] (4.26)

Moreover

\[
Q_2 \leq C^2 e^{-\delta T^\lambda(q+1)} \int e^{-\delta |x|^{q+1}} \, dx = a_4 e^{-\delta T^\lambda(q+1)}.
\] (4.27)

(4.26) and (4.27) yield that

\[
\mathbb{E}_{\mu_T} \left[ \mathbb{1}_{A_5} \right] \leq \xi(T) e^{-(\delta/4)T^\lambda(q+1)},
\] (4.28)

where $\xi(T) = \frac{12}{T^\lambda} \sqrt{|a_1 - \inf \sigma(L_p)| + T(a_2 + a_3) + a_4}$. This completes the proof.
4.2 Absence of ground state for short range potentials

In this subsection we give an example for a short range variable mass $v_m$. We introduce the assumption below:

**Assumption 4.5** Let $v_m$ be of the form $v_m = \kappa w$ with $\kappa > 0$, where $w : \mathbb{R}^3 \to \mathbb{R}$ is bounded, $-\Delta + w$ has no non-positive eigenvalues, and there exist positive constants $C, R$ and $\beta > 3$ such that $|w(x)| \leq C|x|^{-\beta}$.

Assumption 4.5 yields that there exists a generalized eigenfunction $\Psi_\kappa(k, x)$ satisfying $(-\Delta + v_m)\Psi_\kappa(k, x) = |k|^2\Psi_\kappa(k, x)$ and the Lippman-Schwinger equation

$$
\Psi_\kappa(k, x) = e^{ikx} - \frac{\kappa}{4\pi} \int e^{i|k||x-y|} w(y) \frac{\Psi_\kappa(k, y)}{|x-y|} dy
$$

(4.29)

by [Ik60].

**Lemma 4.6** Suppose Assumption 4.5. Then

1. $\Psi_\kappa(k, x)$ is continuous in $x$ for each $k$ but $k \neq 0$;

2. the generalized Fourier transformation $\mathcal{F}$ define by (2.9) with $\Psi_\kappa$ is unitary on $L^2(\mathbb{R}^3)$;

3. there exist positive constants $\kappa_0 > 0$ and $C_0 > 0$ such that, for any $\kappa \leq \kappa_0$,

$$
\sup_{k \in \mathbb{R}^3} |e^{ikx} - \Psi_\kappa(k, x)| \leq \kappa C_0 \langle x \rangle^{-1};
$$

(4.30)

4. $\sup_{x,k} |\Psi_\kappa(k, x)| < \infty$ uniformly for sufficiently small $\kappa$.

In particular $v_m$ satisfying Assumption 4.5 fulfills Assumption 2.2.

**Proof:** (1) follows from [Ik60], and (2) again from [Ik60] since there exist no non-positive eigenvalues for $-\Delta + \kappa w$. We prove (3). In general there exists a constant $c$ such that

$$
\int_{\mathbb{R}^n} \frac{1}{|x-y|^a(y)^b} dy \leq c \frac{1}{\langle x \rangle^a},
$$

if $0 < a < n < b$. Then by the assumption $\beta > 3$, we have

$$
\int_{\mathbb{R}^3} \frac{1}{|x-y|^{\beta}} dy \leq c' \frac{1}{\langle x \rangle},
$$
IR divergence

with some constant $c'$. Iterating (4.29), we have
\[ e^{ikx} - \Psi(k,x) = \sum_{n=1}^{\infty} \left( \frac{\kappa}{4\pi} \right)^n \int \cdots \int \frac{e^{ik|\sum_{j=1}^{n} |y_j-y_j-1|}}{\prod_{j=1}^{n} |y_j-y_j-1|} dy_1 \cdots dy_n \] (4.31)
with $y_0 = x$. Note that
\[ \int \frac{|w(y)|}{|x-y|} dy \leq \sup_{y \in \mathbb{R}^3} |w(y)|^\beta |\int \frac{1}{|x-y|^{\beta}} dy \leq C|x|^{-1} \]
with some constant $C$. The right hand side of (4.31) absolutely converges for sufficiently small $\kappa > 0$. By (4.31) it follows that
\[ |\Psi_\kappa(k,x) - e^{ikx}| \leq \sum_{n=1}^{\infty} \left( \frac{\kappa C}{4\pi} \right)^n \langle x \rangle^{-1} = \frac{\kappa C}{4\pi - \kappa C} \langle x \rangle^{-1}. \]
This completes (3). (4) is derived from (3). The proof is complete. \qed

Henceforth, we denote $\Psi_\kappa$ simply by $\Psi$. We define $W_N$ by $W$ with $\Psi$ replaced by $e^{ikx}$, i.e.,
\[ W_N(x,y,|t|) = \int \frac{\chi(k)^2}{2\omega(k)} e^{-|t|\omega} e^{-ik(x-y)} dk. \] (4.32)
Then
\[ W_N(x,y,|t|) = \frac{1}{4\pi^2} \int dX \int dY \frac{\tilde{x}(X)\tilde{\chi}(Y)}{|(X-x)-(Y-y)|^2 + |t|^2}. \] (4.33)
Note that, if \( \int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty \), then
\[ 0 \leq \sup_T \int_{-T}^{0} ds \int_{0}^{T} dt W_N(x,y,|s-t|) < \frac{1}{2} \int \frac{\chi(k)^2}{\omega(k)^3} dk \]
by (4.10). It is however not the case when \( \int \frac{\chi(k)^2}{\omega(k)^3} dk = \infty \). This proves the following:

**Theorem 4.7** Suppose Assumptions 2.1, 2.4 and 4.5. Assume $\kappa \leq \kappa_0$ and
\[ \frac{1}{q+1} + \kappa C_0(\kappa C_0 + 2) < 1, \] (4.34)
where $\kappa_0$ and $C_0$ are given in Lemma 4.6. Then H has no ground state.

**Proof:** Note that, by (4.34), one can take $0 < \lambda < 1$ such that
\[ \frac{1}{q+1} < \lambda < 1 - \kappa C_0(\kappa C_0 + 2). \]
It is enough to show (4.9), namely there exists $\rho(T)$ such that

$$
1_{A_T} \int_{-T}^{0} ds \int_{0}^{T} dt \int dX \int dY \frac{(\Psi_{X,t}\chi)^\vee(X)(\Psi_{X,t}\chi)^\vee(Y)}{|X-Y|^2 + |s-t|^2} > \rho(T) \quad (4.35)
$$

and $\rho(T) \to \infty$ as $T \to \infty$. By (4.30) it follows that

$$
\sup_{x,y,k} |\Psi(k,x)\Psi(k,y) - e^{-ikx}e^{iky}| \leq \kappa C_0 (\kappa C_0 + 2).
$$

Then

$$
W(X_s, X_t, |s-t|) \geq W_N(X_s, X_t, |s-t|) - \kappa C_0 (\kappa C_0 + 2) W_0(|t-s|),
$$

where

$$
W_0(|T|) = \int \frac{\chi(k)^2}{2\omega(k)}e^{-|T|\omega(k)} dk.
$$

By [LMS02] on $A_T$,

$$
\int_{-T}^{0} ds \int_{0}^{T} dt W_N(X_s, X_t, |s-t|)
\geq \frac{1}{4\pi^2} \int dX dY \tilde{\chi}(X)\tilde{\chi}(Y) \log \left(\frac{8T^{2\lambda} + |X+Y|^2 + T^2}{8T^{2\lambda} + 2|X+Y|^2}\right). \quad (4.36)
$$

Note that $\tilde{\chi} \geq 0$. While $\int_{-T}^{0} ds \int_{0}^{T} dt W_0(|t-s|)$ can be computed as

$$
\int_{-T}^{0} ds \int_{0}^{T} dt W_0(|t-s|)
= \frac{1}{4\pi^2} \int dX \int dY \tilde{\chi}(X)\tilde{\chi}(Y) \log \left(\frac{(|X-Y|^2 + T^2)^2}{|X-Y|^2(|X-Y|^2 + 4T^2)}\right)
+ \frac{1}{\pi^2} \int dX \int dY \tilde{\chi}(X)\tilde{\chi}(Y) \frac{T}{|X-Y|} \left(\arctan \frac{2T}{|X-Y|} - \arctan \frac{T}{|X-Y|}\right).
$$

The second term on the right hand side above is uniformly bounded by some constant $K$ with respect to $T$. Then

$$
\kappa C_0 (\kappa C_0 + 2) \int_{T}^{0} ds \int_{0}^{T} dt W_0(|t-s|)
\leq \frac{1}{4\pi^2} \int dX \int dY \tilde{\chi}(X)\tilde{\chi}(Y) \log \left(\frac{(|X-Y|^2 + T^2)^2}{|X-Y|^2(|X-Y|^2 + 4T^2)}\right)^{\kappa C_0 (\kappa C_0 + 2)} + K. \quad (4.37)
$$
By (4.36) and (4.37) we obtain
\[
W \geq \frac{1}{4\pi^2} \int dX \int dY \tilde{\chi}(X) \tilde{\chi}(Y) \log \left(\frac{\frac{8T^2|X| + |X^2 + Y^2|}{8T^2|X| + 2|Y^2|} |X^2 + Y^2|^{\kappa C_0(\kappa C_0 + 2)}}{\frac{(|X-Y|^2 + T^2)^2}{|X-Y|^2(|X-Y|^2 + 4T^2)}}}\right) - \kappa C_0(\kappa C_0 + 2) K.
\] (4.38)

Since \(\lambda < 1\),
\[
\log \left(\frac{\frac{8T^2|X| + |X^2 + Y^2|}{8T^2|X| + 2|Y^2|} |X^2 + Y^2|^{\kappa C_0(\kappa C_0 + 2)}}{\frac{(|X-Y|^2 + T^2)^2}{|X-Y|^2(|X-Y|^2 + 4T^2)}}}\right) \sim \log T^{2(1 - \lambda - \kappa C_0(\kappa C_0 + 2))}
\]
as \(T \to \infty\), and \(\lambda + \kappa C_0(\kappa C_0 + 2) < 1\), the right hand side of (4.38) diverges. Then the theorem follows. \(\square\)

5 The number of bosons in ground state

In this section we suppose Assumptions 2.1, 2.2 and 2.4, but we do not assume \(\tilde{\chi} \geq 0\). Moreover we suppose the following assumption holds:

Assumption 5.1 Suppose that (1) \(\int \frac{\tilde{\chi}(k)^2}{\omega(k)^2} dk < \infty\) and (2) \(H\) has a ground state \(\phi_g\) such that \(\phi_g > 0\).

Under Assumption 5.1 it follows that \(\phi_T^g \to \phi_g\) strongly as \(T \to \infty\). We have the proposition below.

Proposition 5.2 It follows that
\[
(\phi_T^g, e^{-\beta N} \varphi_T^g) = \mathbb{E}_{\mu_T} \left[ e^{-\alpha^2(1-\beta) \int_{-T}^T dW(X_t,X_0,t)} \right]. \tag{5.1}
\]

Proof: By Theorem 3.6 we have
\[
(\phi_T^g, e^{-\beta N} \varphi_T^g) = \frac{1}{Z_T} \int d\mu_p(x) \mathbb{E}_{\mu_T} \left[ e^{(\alpha^2/2) \| f_{-T} \phi_{t,\alpha}(\tilde{\chi}(X_s)) ds + \int_{-T}^T \phi_{t,\beta}(\tilde{\chi}(X_s)) dt \|^2} \right].
\]
Since
\[
(\phi_{s,\alpha}(f), \phi_{t,\beta}(g)) = \frac{1}{2} e^{-\beta} \int e^{-|t-s| \omega \int f(k) g(k) dk},
\]
we have
\[
\left\| \int_{-T}^{0} \phi_{r,0}(\tilde{\chi}(X_r))dr + \int_{0}^{T} \phi_{r,\beta}(\tilde{\chi}(X_r))dr \right\|^2
\]
\[
= \int_{-T}^{0} ds \int_{-T}^{0} dt W + \int_{0}^{T} ds \int_{0}^{T} dt W + e^{-\beta} \left( \int_{-T}^{0} ds \int_{0}^{T} dt W + \int_{0}^{T} ds \int_{-T}^{0} dt W \right)
\]
\[
= \int_{-T}^{T} ds \int_{-T}^{T} dt W + 2(e^{-\beta} - 1) \int_{-T}^{0} ds \int_{0}^{T} dt W.
\]
Then the proposition follows. \( \square \)

Note that
\[
\int_{-T}^{0} ds \int_{0}^{T} dt W(X_s, X_t, |s - t|) \leq \frac{1}{2} \int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty. \tag{5.2}
\]

Let \( g(\beta) = (\varphi_g^{T}, e^{-\beta N} \varphi_g^{T}) \). Thus we have a lemma below:

**Lemma 5.3** For each \( 0 < T \). (1) \( g \) can be analytically continued to the hole complex plane \( \mathbb{C} \); (2) \( \varphi_g^{T} \in D(e^{+\beta N}) \) for all \( \beta \in \mathbb{C} \); (3) \( \equation{5.1} \) holds true for all \( \beta \in \mathbb{C} \).

**Proof:** The proof is parallel with \[H03\]. Let \( \Pi_+ = \{ z \in \mathbb{C} \mid \Re z > 0 \} \) and \( \Pi_- = \mathbb{C} \setminus \Pi_+ \).

Set
\[
g(\beta) = \mathbb{E}_{\mu_T} \left[ e^{-\alpha^2 (1-e^{-\beta}) \int_{-T}^{0} ds \int_{0}^{T} dt W(X_s, X_t, |s - t|)} \right].
\]

It is easily seen that \( g(\beta) \) can be analytically continued into the hole complex plane \( \mathbb{C} \) in \( \beta \). We denote its analytic continuation by \( \tilde{g} \). Let \( \beta_0 \in \Pi_+ \) be such that \( \Re \beta_0 = \epsilon \) with some \( \epsilon > 0 \). Fix an arbitrary \( R \) such that \( R > \epsilon \). We see that
\[
\tilde{g}(\beta) = \sum_{n=0}^{\infty} (\beta - \beta_0)^n b_n(\beta_0) \tag{5.3}
\]
for \( \beta \in U := \{ z \in \mathbb{C} \mid |\beta_0 - z| < R \} \), and \( \equation{5.3} \) absolutely converges. Let \( \nu(d\rho) \) denote the spectral projection of \( N \) with respect to \( \varphi_g^{T} \). Note that \( g(\beta) \) is analytic in the interior of \( \Pi_+ \). Then
\[
g(\beta) = \int_{0}^{\infty} e^{-\beta \rho} \nu(d\rho) = \sum_{n=0}^{\infty} (\beta - \beta_0)^n \frac{1}{n!} \int_{0}^{\infty} (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho) \tag{5.4}
\]
for \( \beta \) so that \( |\beta - \beta_0| < \epsilon \). Since \( g(\beta) = \tilde{g}(\beta) \) for \( \beta \) such that \( |\beta - \beta_0| < \epsilon \), we see together with \( \equation{5.4} \) that
\[
b_n(\beta_0) = \frac{1}{n!} \int_{0}^{\infty} (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho). \tag{5.5}
\]
Substituting (5.5) into the expansion of $\tilde{g}$ in (5.3), we have

$$\tilde{g}(\beta) = \sum_{n=0}^{\infty} (\beta_0 - \beta)^n \frac{1}{n!} \int_{0}^{\infty} (-\rho)^n e^{-\beta_0 \rho} \nu(d\rho)$$

(5.6)

for $\beta \in U$. In particular the right-hand side of (5.6) absolutely converges for $\beta \in U$, and $U \cap \Pi_- \neq \emptyset$ by $R > \epsilon$, and, for $\beta \in \mathbb{R} \cap U \cap \Pi_-$, by Fatou’s lemma we have for any $M > 0$,

$$\int_{0}^{M} e^{-\beta \rho} \nu(d\rho) \leq \sum_{n=0}^{\infty} |\beta_0 - \beta|^n \frac{1}{n!} \int_{0}^{\infty} \rho^n e^{-\beta_0 \rho} \nu(d\rho) < \infty.$$ 

Thus $\int_{0}^{\infty} e^{-\beta \rho} \nu(d\rho) < \infty$ follows for $\beta \in \mathbb{R} \cap U \cap \Pi_-$. This implies that $\varphi_g \in D(e^{-\beta/2} N)$ and (5.1) holds for $\beta \in \mathbb{R} \cap U \cap \Pi_-$. Since $R$ is an arbitrary large number, we get (5.1) for all $\beta \in \mathbb{C}$.  

By this proposition the moment $(\varphi_g, N^m \varphi_g)$ can be derived by

$$(\varphi_g^T, N^m \varphi_g^T) = (-1)^m \frac{d^m}{d\beta^m}(\varphi_g^T, e^{-\beta N} \varphi_g^T) |_{\beta=0}.  \quad (5.7)$$

Lemma 5.4 (Pull through formula) It follows that

$$(\varphi_g, N \varphi_g) = \frac{\alpha^2}{2} \int dk \frac{\chi(k)^2}{\omega(k)} \left( \Psi(k, \cdot) \varphi_g, (\overline{H} + \omega(k))^{-2} \Psi(k, \cdot) \varphi_g \right), \quad (5.8)$$

where $\overline{H} = H - \inf \sigma(H)$.

**Proof:** From

$$(\varphi_g^T, N \varphi_g^T) = \mathbb{E}_{\mu_T} \left[ \alpha^2 \int_{-T}^{0} ds \int_{0}^{T} dt W(X_s, X_t, |s-t|) \right] \quad (5.9)$$

it follows that

$$(\varphi_g^T, N \varphi_g^T) = \frac{\alpha^2}{2} \int dk \frac{\chi(k)^2}{\omega(k)} \int_{-T}^{0} ds \int_{0}^{T} dt e^{-|t-s|\omega} \mathbb{E}_{\mu_T} \left[ \overline{\Psi}(k, X_s) \Psi(k, X_t) \right].$$

Generally it can be obtained that for bounded $f$ and $g$,

$$\mathbb{E}_{\mu_T} [f(X_s)g(X_t)] = (e^{-sH} \varphi_g^T, f e^{-(t-s)H} g e^{tH} \varphi_g^T), \quad t \geq s. \quad (5.10)$$
This can be proven directly by the Trotter product formula. Then since
\[ \mathbb{E}_{\mu_T} \left[ \psi(k, X_s) \psi(k, X_t) \right] = (\Psi(k, \cdot)e^{-s H} \varphi^T_g, e^{-(t-s) H} \Psi(k, \cdot)e^{+t H} \varphi^T_g), \]  
we have
\[ (\varphi^T_g, N\varphi^T_g) = \frac{\alpha^2}{2} \int dk \frac{\chi(k)^2}{\omega(k)} \int_0^T ds \int_0^T dt e^{-|t-s|\omega} \left( \Psi(k, \cdot)e^{-s H} \varphi^T_g, e^{-(t-s) H} \Psi(k, \cdot)e^{+t H} \varphi^T_g \right). \]

Since (5.9) yields that
\[ \| N^{1/2} \varphi^T_g \| \leq \frac{\alpha^2}{2} \int \frac{\chi(k)^2}{\omega(k)^3} dk < \infty, \]
there exists a subsequence \( T' \) such that
\[ s - \lim_{T' \to \infty} N^{1/2} \varphi^T_g = N^{1/2} \varphi_g. \]  

Let us reset \( T \) for \( T' \). By (5.11)
\[ \left| \left( \Psi(k, \cdot)e^{-s H} \varphi^T_g, e^{-(t-s) H} \Psi(k, \cdot)e^{+t H} \varphi^T_g \right) \right| \leq \sup_{k, x} |\Psi(k, x)|^2 < \infty \]
and
\[ \lim_{T \to \infty} \left( \Psi(k, \cdot)e^{-s H} \varphi^T_g, e^{-(t-s) H} \Psi(k, \cdot)e^{+t H} \varphi^T_g \right) = \left( \Psi(k, \cdot)\varphi_g, e^{-(t-s)H} \Psi(k, \cdot)\varphi_g \right). \]

By the dominated convergence theorem we have
\[ \lim_{N \to \infty} \int dk \frac{\chi(k)^2}{2 \omega(k)} \int_{-T}^0 ds \int_0^T dt e^{-|t-s|\omega} \left( \Psi(k, \cdot)e^{-s H} \varphi^T_g, e^{-(t-s) H} \Psi(k, \cdot)e^{+t H} \varphi^T_g \right) \]
\[ = \int dk \frac{\chi(k)^2}{2 \omega(k)} \int_{-\infty}^0 ds \int_0^\infty dt e^{-|t-s|\omega} \left( \Psi(k, \cdot)\varphi_g, e^{-(t-s)H} \Psi(k, \cdot)\varphi_g \right). \]  
\[ (5.13) \]

The right hand side above is identical with
\[ \int dk \frac{\chi(k)^2}{2 \omega(k)} \left( \Psi(k, \cdot)\varphi_g, (H + \omega(k))^{-2} \Psi(k, \cdot)\varphi_g \right). \]

By (5.12) and (5.13) the lemma follows.

**Theorem 5.5** Set \( R = \int \frac{\chi(k)^2}{\omega(k)^3} dk \). Suppose that \( (\Psi(0, \cdot)\varphi_g, \varphi_g) \neq 0 \). Then
\[ \lim_{R \to \infty} (\varphi_g, N\varphi_g) = \infty. \]  
(5.14)
**Example 5.6** Assume that \( v_m = \kappa w \) satisfies Assumption 4.5. Then \(|1 - \Psi(0, x)| \leq \kappa C_0\) holds by Lemma 4.6. It yields that

\[ |(\Psi(0, \cdot)\varphi_g, \varphi_g) - 1| \leq \kappa C_0. \]

Thus \((\Psi(0, \cdot)\varphi_g, \varphi_g) \neq 0\) holds for sufficiently small \( \kappa \).

**Proof of Theorem 5.5**

By Lemma 5.4 we have

\[ (\varphi_g, N\varphi_g) = \frac{\alpha^2}{2} \int \frac{\chi(k)^2}{\omega(k)^3} \left( \Psi(k, \cdot)\varphi_g, \omega(k)^2 (\bar{H} + \omega(k))^{-2} \Psi(k, \cdot)\varphi_g \right) \quad \text{(5.15)} \]

We can see that

\[ \lim_{|k| \to 0} \left| (\Psi(k, \cdot)\varphi_g, \omega(k)^2 (\bar{H} + \omega(k))^{-2} \Psi(k, \cdot)\varphi_g) - (\Psi(0, \cdot)\varphi_g, \omega(k)^2 (\bar{H} + \omega(k))^{-2} \Psi(0, \cdot)\varphi_g) \right| = 0. \]

Let \( P_g \) (resp. \( P_g^\perp \)) denote the projection to the ground state \( \ker \bar{H} \) (resp. the orthogonal complement \( (\ker \bar{H})^\perp \) of \( \ker \bar{H} \)). We have

\[ (\Psi(0, \cdot)\varphi_g, \omega(k)^2 (\bar{H} + \omega(k))^{-2} P_g \Psi(0, \cdot)\varphi_g) = |(\varphi_g, \Psi(0, \cdot)\varphi_g)|^2 \quad \text{(5.16)} \]

Set \( A = |(\Psi(0, \cdot)\varphi_g, \varphi_g)|^2 > 0 \). Then

\[ A - \delta < (\Psi(k, \cdot)\varphi_g, \omega(k)^2 (\bar{H} + \omega(k))^{-2} \Psi(k, \cdot)\varphi_g) \]

for \(|k| < \epsilon\) with some sufficiently small \( \epsilon > 0 \). Then we have the bound

\[ (A - \delta) \frac{\alpha^2}{2} \int_{|k| < \epsilon} \frac{\chi(k)^2}{\omega(k)^3} dk + \frac{\alpha^2}{2} \int_{|k| \geq \epsilon} \frac{\chi(k)^2}{\omega(k)^3} dk \leq (\varphi_g, N\varphi_g) \quad \text{(5.17)} \]
IR divergence

with some positive $b$. Thus as $R \to \infty$, $(\varphi_g, N\varphi_g)$ goes to infinity. Then the proof is complete.

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\textbf{References}

[AHH99] A. Arai, M. Hirokawa and F. Hiroshima, On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoff, \textit{J. Funct. Anal.} 168 (1999), 470–497.

[BFS98] V. Bach, J. Fröhlich and I. M. Sigal, Quantum electrodynamics of confined non-relativistic particles, \textit{Adv. Math.} 137 (1998), 299–395.

[BHLMS02] V. Betz, F. Hiroshima, J. Lőrinczi, R. A. Minlos and H. Spohn, Ground state properties of the Nelson Hamiltonian — a Gibbs measure-based aproach, \textit{Rev. Math. Phys.}, 14 (2002), 173–198.

[Car78] R. Carmona, Pointwise Bounds for Schrödinger Eigenstates, \textit{Commun. Math. Phys.} 62 (1978), 97–106.

[Che01] T. Chen, Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED, mp-arc 01-301, preprint, 2001.

[DG03] J. Dereziński and C. Gérard, Scattering theory of infrared divergent Pauli-Fierz Hamiltonians, preprint, 2003.

[FUL96] S. A. Fulling, \textit{Aspects of Quantum Field Theory in Curved Space-Time}, London Mathematical Society Student Texts 17, Cambridge University Press, 1996.

[Ger00] C. Gérard, On the existence of ground states for massless Pauli-Fierz Hamiltonians, \textit{Ann. Henri Poincaré} 1 (2000), 443–459, and A remark on the paper: “On the existence of ground states for Hamiltonians”, mp-arc 06-146 (2006).

[HH06] D. Hassler and I. Herbst, Absence of ground states for a class of translation invariant models in nonrelativistic QED, preprint, 2006.

[Hk06] M. Hirokawa, Infrared catastrophe for Nelson’s model-non-existence of ground state and soft-boson divergence-, \textit{Publ. RIMS, Kyoto Univ.} 42, (2006), 897–922.

[H03] F. Hiroshima, Analysis of ground states of atoms interacting with a quantized radiation field, \textit{Topics in the Schrödinger operators} (H.Araki and H.Ezawa eds.), World Scientific, 2004, 145-273.

[Ik60] T. Ikebe, Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, \textit{Arch. Rational Mech. Anal.} 5 (1960), 1–34.

[JK79] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, \textit{Duke Math. J.}, 46 (1979), 583–611.
[KV86] C. Kipnis and S. R. S. Varadhan, Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions, *Commun. Math. Phys.* **104** (1986), 1–19.

[Lie73] Bounds on the eigenvalues of the Laplacian and Schrödinger operators, *Bull. AMS* **82** (1976), 751–753.

[LMS02] J. Lörinczi, R. A. Minlos and H. Spohn, The infrared behaviour in Nelson’s model of a quantum particle coupled to a massless scalar field, *Ann. Henri Poincaré* **3** (2002), 1–28.

[Nel64] E. Nelson, Interaction of nonrelativistic particles with a quantized scalar field, *J. Math. Phys.* **5** (1964), 1190–1197.

[RS78] M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic press, 1978.

[Sim82] B. Simon, Schrödinger semigroup, *Bulletin AMS* **7** (1982), 447–526.

[Spo98] H. Spohn, Ground state of quantum particle coupled to a scalar boson field, *Lett. Math. Phys.* **44** (1998), 9–16.