A dynamical model for Positive-Operator Valued Measures

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We tackle the dynamical description of the quantum measurement process, by explicitly addressing the interaction between the system under investigation with the measurement apparatus, the latter ultimately considered as macroscopic quantum object. We consider arbitrary Positive Operator Valued Measures (POVMs), such that the orthogonality constraint on the measurement operators is relaxed. We show that, likewise the well-known von-Neumann scheme for projective measurements, it is possible to build up a dynamical model holding a unitary propagator characterized by a single time-independent Hamiltonian. This is achieved by modifying the standard model so as to compensate for the possible lack of orthogonality among the measurement operators of arbitrary POVMs.

I. INTRODUCTION

A distinctive trademark of quantum mechanics is represented by the quantum measurements and by the randomness of their outcomes. The postulates of the theory dictate how to compute the associated statistics for quantum observables through projective measures, while no mechanism is provided to predict how the actual finally observed result comes about. In this respect the measurement process still represents an open field of research [1–8]. Actually, from the dawn of quantum theory, two main steps towards complementary directions have been performed. On the one hand a clear description of the measurement process entailing the definition of a time-independent interaction Hamiltonian between the system and an ultimately macroscopic apparatus has been provided by von Neumann [9], and fully characterized by Ozawa [10] several years later. On the other hand, the statistical description of quantum measurements has been extended to non-necessarily orthonormal measurement operators by the introduction of the so-called Positive Operator Valued Measures (POVMs) [1–3].

In this manuscript we unify these two approaches, introducing a dynamical description of arbitrary quantum measurements. We show that, in order to achieve a well-defined – i.e., completely positive trace preserving (CPT) – dynamical map, the lack of orthogonality of arbitrary measurement operators needs to be compensated by properly modifying the von Neumann-Ozawa (vN-O), time-independent Hamiltonian representation. In our analysis we rely on the Naimark extension theorem [25–28], which allows one to describe an arbitrary POVM performed on the system of interest, in terms of a projective measurement performed on an external probing system that was properly coupled with the latter. This provides a proper generalization of the von Neumann model to arbitrary measurements. We recall that addressing the actual dynamics behind the formal description of a quantum measurement not only helps us understanding fundamental aspects of the process, but it also gives relevant indication about the actual design of quantum-measurement experiments (see, e.g., Refs. [11–13]).

The paper is structured as follows: as a premise in Sec. II we introduce the notation and review few basic notions regarding POVMs and the vN-O construction for projective measurements. Section III contains the original part of the work: Here we rigorously define the problem we wish to address and present a solution for it; the fundamental element of our analysis is the explicit construction of a Naimark Hamiltonian discussed in Secs. III A – III B. Conclusion and final remarks are given in Sec. IV while Technical considerations are presented in the Appendix.

II. QUANTUM MEASUREMENTS

The minimal description of a quantum measurement requires two elements: a set of mutually distinguishable outcomes, \( \{ \mu_\gamma; \gamma = 1, \ldots, n \} \), and the corresponding probability distribution \( \{ p_\gamma \} \). Herein, without loss of generality, we will exclusively consider countable sets of outputs and hence discrete distributions. This process involves at least two players interacting with each other: the system \( S \), upon which the measurement is performed, and the apparatus \( \Xi \), from which one actually obtains the outcomes. Let \( \mathcal{H}_S \) be the Hilbert space of \( S \). Formally, a quantum measure on a state \( \rho_S^\text{in} \) is defined by a bijection from \( \{ \mu_\gamma \} \) into the set of positive operators \( \{ F_\gamma \} \) on \( \mathcal{H}_S \), called elements of the measure or effects, such that \( p_\gamma = \text{Tr}[\rho_S^\text{in} F_\gamma(\gamma)] \), \( \forall \gamma \). In order for \( \sum_\gamma p_\gamma = 1 \) to hold, it must be \( \sum_\gamma F_\gamma(\gamma) = \mathbb{1}_S \). As a process on \( S \), a single measurement acts on an input state \( \rho_S^\text{in} \), upon which we want to gain some information, and produces one output \( \mu_\gamma \) with probability \( p_\gamma \), as defined above. After the interaction with the apparatus \( \Xi \) and before the production of the outcomes, the system is described by the so-called post-measurement state \( \rho_S^\text{out} \) of \( S \), defined as

\[
\rho_S^\text{out} = \sum_\gamma M_\gamma(\gamma) \rho_S^\text{in} M_\gamma(\gamma)^\dagger = \sum_\gamma p_\gamma \rho_S^\text{out}_\gamma, \tag{1}
\]

\[
\rho_S^\gamma = \frac{1}{p_\gamma} M_\gamma(\gamma) \rho_S^\text{in} M_\gamma(\gamma)^\dagger, \quad p_\gamma = \text{Tr} \left[ M_\gamma(\gamma) \rho_S^\text{in} M_\gamma(\gamma)^\dagger \right]. \tag{2}
\]
In this expression the operators \( M^{(\gamma)}_S \), dubbed as measurement or detection operators, are defined by \( F^{(\gamma)}_S = \sum_{\gamma} \rho^{(\gamma)}_S M^{(\gamma)}_S \), a decomposition always allowed, due to the positiveness of the measurement operators. We will refer to the states \( \rho^{(\gamma)}_S \) as \( \gamma \)-detected states. Notice that, in general, \( n_{\Gamma} \) is not constrained by the dimension of the Hilbert space \( n_S = \dim \mathcal{H}_S \). This is because neither the elements \( F^{(\gamma)}_S \) of the POVM nor the measurement operators \( M^{(\gamma)}_S \) are required to satisfy any orthogonality constraint. This is actually the case for a more specific type of quantum measurement defined by a Projective-Valued Measure (PVM), or projective measure. The latter is characterized by a set of operators \( \Pi^{(\gamma)}_S \) being orthonormal projectors on \( S \), which implies that \( n_{\Gamma} \leq n_S \). A PVM \( \{\Pi^{(\gamma)}_S\} \) defines self-adjoint operators \( O_S = \sum_{\gamma} \sigma_{\gamma} \Pi^{(\gamma)}_S \), with \( \sigma_{\gamma} \) real \( \forall \gamma \) and in one-to-one relation with \( \mu_{\gamma} \) via an invertible calibration function \( f(\sigma_{\gamma}) = \mu_{\gamma} \). In fact, the usual formulation of the quantum-measurement postulate refers to the above operators as “observables” and assigns the probability \( p_{\gamma} = \text{Tr}(\rho^n_S \Pi^{(\gamma)}_S) \) to the eigenvalue \( \sigma_{\gamma} \). As for the \( \gamma \)-detected states, their definition as \( \{\rho^{(\gamma)}_S = \Pi^{(\gamma)}_S \rho^n_S \Pi^{(\gamma)}_S / p_{\gamma}\} \) is an integral part of the postulate for PVM, asserting that, after one single measurement process with output \( \mu_{\gamma} \), the system is in the state \( \rho^{(\gamma)}_S \) with absolute certainty. This gives the state \( \rho^{(\gamma)}_S \) the consistent meaning of statistical mixture of the detected states produced in a series of many identical repetitions of the measurement. When \( \text{rank}\{\Pi^{(\gamma)}_S\} = 1 \forall \gamma \), i.e. \( \Pi^{(\gamma)}_S = |\gamma\rangle_S \langle \gamma| \), the PVM is called ideal, and \( n_{\Gamma} = \dim \mathcal{H}_S \).

### A. The Naimark extension theorem

The Naimark extension theorem \[25–28\] establishes a formal connection between POVM and PVM. Specifically it states that any given POVM \( \{F^{(\gamma)}_S\} \) for \( S \) can be represented as a PVM \( \{\Pi^{(\gamma)}_S\} \) for an ancillary system \( A \), that has unitarily interacted with \( S \) prior to be tested. Let \( n_A \) be the dimension of the Hilbert space \( \mathcal{H}_A \) associated to the ancilla. Formally, the Naimark theorem requires that \( n_A \geq n_{\Gamma} \), allowing the choice \( n_A = n_{\Gamma} \) that entails an ideal PVM on \( A \), \( \Pi^{(\gamma)}_A = |\gamma\rangle_A \langle \gamma| \). It then states that there exists: i) a state \( |\psi_0\rangle_A, |\psi_0\rangle_A \in \mathcal{L}(\mathcal{H}_A) \), ii) a unitary operator \( V_{SA} \in \mathcal{L}(\mathcal{H}_{SA}) \) and iii) an ideal PVM \( \{\Pi^{(\gamma)}_A\} \) for \( A \) (see Fig. 1), such that

\[
F^{(\gamma)}_S = \text{Tr}_A \left( (I_S \otimes |\psi_0\rangle_A \langle \psi_0|) V^\dagger_{SA} (I_S \otimes |\gamma\rangle_A \langle \gamma|) V_{SA} \right),
\]

and

\[
p_{\gamma} = \text{Tr} \left[ (\rho^n_S \otimes |\psi_0\rangle_A \langle \psi_0|) \left( V^\dagger_{SA} (I_S \otimes |\gamma\rangle_A \langle \gamma|) V_{SA} \right) \right],
\]

which allows us to consistently write

\[
M^{(\gamma)}_S = A(\gamma |V_{SA}| \psi_0)_A.
\]

An explicit example of the above construction is presented in Sec. IIIA it should be stressed however that this is by no means the only possibility, as different choices for the Naimark operator \( V_{SA} \) are typically available for each given

\[
\text{POVM. It should also be noticed that, conversely, a unitary transformation of the state } \rho^n_S \otimes |\psi_0\rangle_A \langle \psi_0| \text{ into } V_{SA}(\rho^n_S \otimes |\psi_0\rangle_A \langle \psi_0|) V^\dagger_{SA} \text{, followed by an ideal PVM } \{\Pi^{(\gamma)}_A\} \text{ on } A, \text{ defines a proper PVM on } S. \text{ In this respect, the entrance of the ancilla is extremely valuable, as it provides the theoretical scheme with the versatility needed to describe diverse experimental situations, such as those where a physical mediator actually exists, and is ultimately responsible for the information transfer from } S \text{ to } \Xi \text{.}
\]

### B. Dynamical models for PVM

Dynamical models for quantum measurements are meant to describe how a measurement process takes place in time, in terms of a (time-independent) Hamiltonian coupling between the system \( S \) and an external environment \( \Xi \) playing the role of the apparatus which, at the end of the process will store the measurement outcomes. More specifically, in its simplest, yet completely general version, the von Neumann-Ozawa (vN-O) dynamical model for PVM \[9\] \[10\] \[16\] \[19\], assumes that the interaction between \( S \) and \( \Xi \) reads

\[
H_{S\Xi} := O_S \otimes O_\Xi, \tag{6}
\]

with \( O_S = \sum_{\gamma} \sigma_{\gamma} \Pi^{(\gamma)}_S \) an observable on \( S \) and \( O_\Xi \) a self-adjoint operator on \( \Xi \) which is canonically conjugated to what is typically referred to as “the pointer” observable \[7\]. Hence, the associated unitary evolution writes

\[
U_{S\Xi}(t) := e^{-itO_S \otimes O_\Xi} = \sum_{\gamma} \Pi^{(\gamma)}_S \otimes U^{(\gamma)}_\Xi(t), \tag{7}
\]

where \( U^{(\gamma)}_\Xi(t) = e^{-it\sigma_{\gamma}O_\Xi} \) in units \( h = 1 \). The model also assumes that \( \Xi \) is initially prepared in a pure state \( |D\rangle \) that is not an eigenstate of \( O_\Xi \). If the system \( S \) is initialized at \( t = 0 \) in the state \( \rho^n_S \), the unitary \[7\] makes the system \( S + \Xi \) evolve into the joint density matrix

\[
\rho_{S\Xi}(t) := \sum_{\gamma\gamma'} \Pi^{(\gamma)}_S \rho^n_S \Pi^{(\gamma')}_S \otimes |\Xi^{(\gamma')}(t)\rangle \langle \Xi^{(\gamma')}(t)|, \tag{8}
\]

which upon partial trace with respect to \( \Xi \), corresponds to the following local mapping

\[
\rho_S(t) = \sum_{\gamma\gamma'} \Pi^{(\gamma)}_S \rho^n_S \Pi^{(\gamma')}_S \otimes (\Xi^{(\gamma')}(t) \otimes |\Xi^{(\gamma')}(t)|), \tag{9}
\]
for $S$. In the above expressions $|\Xi^\gamma(t)\rangle_S := U^\dagger_S(t)|D\rangle_S$ are pure states of $\Xi$ which encode the measurement outcomes $\gamma$ (see Fig.2 for a schematic representation of the process). The more distinguishable are such states, the larger is the information stored in $\Xi$ that allows one to distinguish between the different outcomes. In fact, the most favourable situation in terms of information transfer from $S$ to $\Xi$, corresponds to have the $|\Xi^\gamma(t)\rangle_S$ orthonormal. It is easily seen that when this condition holds, from Eq. (9) it follows that the matrix-representation of $\rho_S(t)$ on the basis of the $O_\Xi$ eigenstates is block diagonal, and viceversa, i.e. $\rho_S(t) = \sum_\gamma \Pi^\gamma_S(\rho^\gamma_S)\Pi^\gamma_S$, as required by (1) if $M^\gamma(\gamma) = \Pi^\gamma_S(\gamma) = F^\gamma(\gamma)$. This clarifies why decoherence plays such an important role in the quantum measurement process \[20\, 24\]. Therefore we say that the PVM $\{\Pi^\gamma_S\}$ can be successfully realized on $S$ only if, in the limit of a macroscopic apparatus \[23\, 24\], it exists a time $t_d$, typically referred to as decoherence time \[6\, 7\], such that for $t > t_d$ one has \[|\Xi^\gamma(t)\rangle_S = \delta_{\gamma\gamma'}, \quad (10)\]
or, at least, such that the above condition is approximately verified over some non trivial time interval preceding the data acquisition event (notice that although these scalar products are in principle periodic functions of time, in the limit of a macroscopic measuring device $\Xi$ one can safely take the time during which they stay approximately null much longer than the time necessary to perform the measurement).

**Fig. 2:** Schematic representation of the von Neumann-Ozawa dynamical scheme for projective measure: an observable $O_\Xi = \sum_\gamma o_\gamma \Pi^\gamma_S$ is measured on $S$ by letting it interact with the measurement apparatus $\Xi$, so as to encode the information on the states of the apparatus $|\Xi^\gamma(t)\rangle_S$.

### III. DYNAMICAL MODEL FOR ARBITRARY POVM

In this section we discuss how to generalize the vN-O construction for PVM to the case of arbitrary POVM, removing the constraint on the orthonormality of the measurement operators. More precisely we show how to modify Eqs. (6) and (7) in such a way that for times $t$ larger than a certain characteristic threshold time $t_d$, the interaction between $S$ and $\Xi$ will yield a joint density matrix of the form similar to Eq. (8), i.e. \[\rho_{SA}(t) = \sum_{\gamma\gamma'} M^\gamma_S(\gamma) \rho_{SA}^{in} M^\gamma_S(\gamma)\dagger \otimes |\Xi^\gamma(t)\rangle_S\langle\Xi^\gamma(t)|, \quad (11)\]

where $\{M^\gamma_S(\gamma) ; \gamma = 1, \ldots, n_F\}$ are elements of the POVM and where the vectors $\{|\Xi^\gamma(t)\rangle_S ; \gamma = 1, \ldots, n_F\}$ form a mutually orthonormal set as in Eq. (10).

Let us start by observing that at variance with the PVM scenario discussed in the previous section, we cannot expect Eq. (11) to apply at those times $t < t_d$ for which Eq. (10) does not hold. Indeed due to the lack of orthogonality of the operators $M^\gamma(\gamma)$ in this regime the resulting transformation would not be CPT in general, hence non physically implementable – see Appendix A. This of course does not imply that dynamical models cannot be found that describe a generic POVM: simply we need to replace the vN-O Hamiltonian coupling (6) with something else. The key ingredient for this construction is clearly provided by the Naimark extension theorem \[25\, 28\] we reviewed in Sec. II. A, which could be pictorially summarized as in Fig. 1. A tentative idea would be to work in a $S + A + \Xi$ scenario with a conventional vN-O couplings linking the apparatus $\Xi$ to $A$ or to $S + A$ ($A$ being the Naimark ancillary system). However, this approach, which we briefly review in Appendix B does not conclusively work as, while being able to reproduce the correct outcome probability distribution, it cannot yield a solution capable to approach Eq. (11) at some future time. On the contrary, a simpler and more effective way to construct a dynamical model for an arbitrary POVM is found by identifying the system environment $\Xi$ directly with $A$. Under this assumption we then look for a proper Hamiltonian coupling $H_{SA}$ generating an unitary evolution $U_{SA}(t) := e^{-itH_{SA}}$ which for all times $t$ larger than a certain critical time $t_d$ fulfills, at least approximately, the constraint \[U_{SA}(t) = V_{SA}, \quad (12)\]

where $V_{SA}$ being the unitary entering Eq. (5). Clearly due to the Stone theorem \[31\, 32\] such a Naimark Hamiltonian can always be identified. However, our goal is to produce an explicit construction for such a term, as we show in the following.

In order to construct our candidate for $H_{SA}$ we start with a first example that utilizes a small ancilla $A$, hence inducing a $S + A$ dynamics which is explicitly periodic: accordingly this model is capable to produce the same correlations as in Eq. (11) only for specific values of $t$, with cyclic recurrence that prohibits the possibility of maintaining such configuration indefinitely or at least for some non-zero time intervals. The second model, which is actually the central result of this manuscript, corrects this drawback adopting a much larger ancilla. A pictorial representation of the model is presented in Fig. 3 while the complete analytical derivation is presented in Sec. III B. In the specific, we introduce a degeneracy parameter $\ell = 1, \ldots, n_L$ for the ancilla Hilbert space $\mathcal{H}_A$ and define a coupling between $S$ and $A$ formally equivalent to first neighboring hopping terms, characterizing models for perfect state transfer \[33\, 34\]. Therefore, by increasing $n_L$ it is possible to extend the condition Eq. (11) over arbitrarily large (ideally infinitely long) time intervals, as shown in Fig. 4. Actually, our model allows us to translate (11) into\[\rho_{SA}(t) = \sum_{\gamma\gamma'} M^\gamma_S(\gamma) \rho_{SA}^{in} M^\gamma_S(\gamma)\dagger \otimes |A^\gamma(t)\rangle_A\langle A^\gamma(t)|, \quad (13)\]

where we have explicitly identified the state $|\Xi^\gamma(t)\rangle_S$ with the state of the enlarged ancilla $|A^\gamma(t)\rangle_A$. A crucial difference between $\{|\Xi^\gamma(t)\rangle_S\}$ and $\{|A^\gamma(t)\rangle_A\}$ is that the latter...
are orthogonal to each other for all times \( t \). This compensates for the possible lack of orthogonality of the measurement operators \( M^{(\gamma)}_S \), guaranteeing a posteriori the complete positivity of the unitary transformation \( U_{SA}(t) \).

**A. First implementation: periodic dynamics**

Our first step to tackle the problem is to explicitly write down a suitable candidate for the Naimark unitary \( V_{SA} \). We observe that Eq. (5) can be satisfied, e.g. by requiring that for all \( |\psi\rangle_S \) of \( S \) the following condition holds,

\[
V_{SA} |\psi\rangle_S \otimes |\psi_0\rangle_A = e^{i\alpha} \sum_{n=1}^{\infty} M_S^{(\gamma)} |\psi\rangle_S \otimes |\gamma\rangle_A, \tag{14}
\]

with \( \{ |\gamma\rangle_A ; \gamma = 1, 2, \cdots, n_\Gamma \} \) being the orthonormal set of vectors of \( A \) entering Eq. (5), the phase \( \alpha \) being absolutely irrelevant but being inserted for future reference (notice that the above requirement is fully consistent with the dimension \( n_\gamma \) of \( A \) being larger than the total number of measurements outcomes \( n_\Gamma \)). This transformation does not completely characterize \( V_{SA} \) on the full Hilbert space of \( S + A \), but does it only on a proper subspace of the latter – specifically the subspace associated with vectors having \( A \) into the input state \( |\psi_0\rangle_A \). By construction, at least on these vectors, it preserves the scalar product: hence it can be generalized to a global unitary acting on the full space of the system and of the ancilla. What we are going to do next is to explicitly construct such extension using a simplifying trick. Specifically, we assume the input vector \( |\psi_0\rangle_A \) of \( A \) to be orthogonal to all the elements of the orthonormal set \( \{ |\gamma\rangle_A ; \gamma = 1, 2, \cdots, n_\Gamma \} \), i.e.

\[
A \langle \psi_0 | \gamma \rangle_A = 0, \quad \forall \gamma = 1, 2, \cdots, n_\Gamma. \tag{15}
\]

This, of course, automatically implies that the dimension of \( A \) we are considering has to be at least larger than or equal to \( n_\Gamma + 1 \), i.e. slightly larger than the minimum value required by the Naimark theorem (i.e. \( n_\Gamma \)). Such small overhead turns out to be extremely useful as we now can decompose the matrix \( V_{SA} \) of (12) into a collection of \( 2 \times 2 \) independent blocks. Indeed, let us introduce an orthonormal basis \( \{ |j\rangle_A ; j = 1, \cdots, n_S \} \) for \( \mathcal{H}_A \). Expanding \( |\psi\rangle_S \) in such a basis we can then observe that the identity (14) gets replaced by

\[
V_{SA} |\xi^{(0)}\rangle_A = e^{i\alpha} |\xi^{(1)}\rangle_A, \tag{16}
\]

where for all \( j = 1, \cdots, n_S \) we defined the pure states

\[
|\xi^{(0)}_j\rangle_A := |j\rangle_A \otimes |\psi_0\rangle_A, \tag{17}
\]

\[
|\xi^{(1)}_j\rangle_A := \sum_{\gamma=1}^{n_s} M_S^{(\gamma)} |j\rangle_A \otimes |\gamma\rangle_A, \tag{18}
\]

that by construction are all mutually orthonormal, i.e.

\[
s_A \langle \xi^{(0)}_{j'} | \xi^{(0)}_j \rangle_A = \delta_{jj'}, \tag{19}
\]

with \( j, j' = 1, 2, \cdots, n_S \). They can be grouped in a collection of \( n_s \) mutually orthogonal, 2-dimensional subspaces

\[
\mathcal{H}_A^{(j)} := \text{Span} \{ |\xi^{(0)}_j\rangle_A, |\xi^{(1)}_j\rangle_A \}, \tag{20}
\]

labelled by \( j \) and spanned by the couple \( |\xi^{(0)}_j\rangle_A \) and \( |\xi^{(1)}_j\rangle_A \). According to (16) the unitary \( V_{SA} \) operates separately on each of the \( \mathcal{H}_A^{(j)} \) where, up to the global phase \( e^{i\alpha} \), it acts as the following effective Pauli transformations

\[
\sigma_A^{(j)} = |\sigma_A^{(j)}\rangle \langle \sigma_A^{(j)}| := |\xi^{(0)}_j\rangle_A \langle \xi^{(1)}_j|_A + |\xi^{(1)}_j\rangle_A \langle \xi^{(0)}_j|, \tag{21}
\]

leading to the identification

\[
V_{SA} = e^{i\alpha} \oplus_j \sigma_A^{(j)}, \tag{22}
\]

the direct sum being performed over all \( j = 1, \cdots, n_s \). Our first choice for the Naimark Hamiltonian is hence provided by the self-adjoint operator

\[
H_{SA} := \omega \sum_{j=1}^{n_s} \sigma_A^{(j)}, \tag{23}
\]

with \( \omega > 0 \) an arbitrary positive constant, which, using (17) and (18) can be equivalently expressed as

\[
H_{SA} := \omega \sum_{\gamma=1}^{n_S} \left( M_S^{(\gamma)} \otimes |\gamma\rangle_A \langle \psi_0| + \text{h.c.} \right). \tag{24}
\]

Its associated unitary evolution is periodic of period \( 2\pi/\omega \) and reads as

\[
U_{SA}(t) := e^{-iH_{SA}t} = \oplus_j e^{-i\omega t \sigma_A^{(j)}} = \oplus_j \left( \mathbb{I}^{(j)} \cos(\omega t) - i \sigma_A^{(j)} \sin(\omega t) \right), \tag{25}
\]

where we used the property

\[
\sigma_A^{(j)} \sigma_A^{(j')} = \delta_{jj'} \mathbb{I}^{(j)}, \tag{26}
\]

with

\[
\mathbb{I}^{(j)} := |\xi^{(0)}_j\rangle_A \langle \xi^{(0)}_j| + |\xi^{(1)}_j\rangle_A \langle \xi^{(1)}_j|, \tag{27}
\]
being the projection operator on $\mathcal{H}_S^{(j)}$. From Eq. (25) it then follows
\[
U_{SA}(t)|\psi\rangle_S \otimes |\psi_0\rangle_A = \cos(\omega t)|\psi\rangle_S \otimes |\psi_0\rangle_A - i \sin(\omega t) \sum_{\gamma=1}^{n_F} M_{SA}^{(\gamma)}|\psi\rangle_S \otimes |\gamma\rangle_A ,
\]
which yields Eq. (14) for $t = t_d = \pi/(2\omega)$, upon identifying the phase term $\alpha$ with $-\pi/2$.

### B. Second implementation: non periodic dynamics

The main drawback of the previous example is that it exhibits a definite period $2\pi/\omega$, so that Eq. (28) reproduces Eq. (14) only at the precise instants $t_n = (2n+1)t_d$, where $n$ is an integer number. Hence it does not exactly fit into our requirement to enforce Eq. (11) for extended time interval after a given premearurement time $t_d$. Here we show however how one can easily modify the construction to explicitly fulfill this requirement too. The idea is to increase the dimension of the subspaces $\mathcal{H}_S^{(j)}$ of Eq. (20) and to equip the associated Hamiltonian with a richer frequency spectrum. For this purpose we replace the orthonormal set $\{|\gamma\rangle_A : \gamma = 1, \ldots, n_T\}$ entering the previous construction with a larger set of orthonormal vectors $\{|\gamma, \ell\rangle_A : \gamma = 1, \ldots, n_T\}$, the index $\ell$ being a degeneracy parameter which can take up to $n_L$ different values, i.e.
\[
A(\gamma, \ell, \gamma, \ell) = \delta_{\gamma,\gamma'}\delta_{\ell,\ell'} , \quad A(\psi_0|\gamma, \ell\rangle_A = 0 \quad \forall \gamma, \ell ,
\]
which implicitly dictates that now $A$ must have a dimension $n_A$, which is larger than or equal to $n_Tn_L + 1$. With that in mind we then replace Eq. (20) with the $n_L + 1$ dimensional spaces
\[
\mathcal{H}_{SA}^{(j)} := \text{Span}\{|\xi_j^{(0)}\rangle_{SA}, |\xi_j^{(1)}\rangle_{SA}, \ldots, |\xi_j^{(n_L)}\rangle_{SA}\} ,
\]
with $|\xi_j^{(0)}\rangle_{SA}$ still defined as in Eq. (17) and where, for $\ell = 1, \ldots, n_L$, $|\xi_j^{(\ell)}\rangle_{SA}$ are instead given by
\[
|\xi_j^{(\ell)}\rangle_{SA} := \sum_{\gamma=1}^{n_F} M_{SA}^{(\gamma)}|j\rangle_S \otimes |\gamma, \ell\rangle_A ,
\]
which still fulfill the orthogonality conditions (19). Define hence the new self-adjoint operators
\[
H_{SA}^{(j)} := \sum_{\ell=0}^{n_L-1} \omega_{\ell}|j\rangle_S \langle j| ,
\]
with $\omega_{\ell} > 0$ being frequency terms that play the role of free parameters in the model and where, for $\ell = 0, \ldots, n_L - 1$ the new Pauli operators $\sigma_{SA}^{(j,\ell)}$ are given by
\[
\sigma_{SA}^{(j,\ell)} = [\sigma_{SA}^{(j,\ell)}]^{\dagger} := |\xi_j^{(\ell)}\rangle_{SA} \langle \xi_j^{(\ell+1)}| + |\xi_j^{(\ell+1)}\rangle_{SA} \langle \xi_j^{(\ell)}| ,
\]
Notice that from the orthonormality conditions (19) it follows that, irrespectively from the values of $\ell$ and $\ell'$, the product of any two operators $\sigma_{SA}^{(j,\ell)}$ and $\sigma_{SA}^{(j',\ell')}$ with $j \neq j'$ vanishes, i.e.
\[
\sigma_{SA}^{(j,\ell)} \sigma_{SA}^{(j',\ell')} = 0 .
\]
Furthermore, the various $H_{SA}^{(j)}$ terms have exactly the same matrix form with respect to the associated canonical basis of the associated spaces $\mathcal{H}_S^{(j)}$, i.e.
\[
\langle \xi_j^{(\ell')}| H_{SA}^{(j)} |\xi_j^{(\ell)}\rangle_{SA} = \omega_{\ell}(\delta_{\ell,\ell'+1} + \delta_{\ell'+1,\ell'}) .
\]
Finally, we observe that $H_{SA}^{(j)}$ formally corresponds to the 1-excitation sector of a spin-1/2 chain Hamiltonian, with open boundary conditions, characterized by first neighbouring hopping terms, whose coupling terms are gauged by the frequencies $\omega_{\ell}$.

We hence introduce as the new Hamiltonian of the $S + A$ system the operator
\[
H_{SA} := \sum_{j=1}^{n_S} H_{SA}^{(j)} ,
\]
which, making use of Eqs. (31) and (17), can be equivalently recasted in the following compact form
\[
H_{SA} = \sum_{\gamma=1}^{n_F} M_{SA}^{(\gamma)} \otimes \Theta_A^{(\gamma)} + \sum_{\gamma,\gamma'=1}^{n_F} M_{SA}^{(\gamma)} M_{SA}^{(\gamma')} \otimes \Theta_A^{(\gamma\gamma')} + h.c .
\]
after defining the operators
\[
\Theta_A^{(\gamma)} := \omega_0|\gamma, 1\rangle_A \langle \psi_0| ,
\]
\[
\Theta_A^{(\gamma\gamma')} := \sum_{\ell=1}^{n_L-1} \omega_{\ell}|\gamma, \ell\rangle_A \langle \gamma', \ell + 1| .
\]
From Eqs. (34) and (35) we notice that, as in the case of Sec. III A $H_{SA}$ is block diagonal, with respect to the extended subspaces $\mathcal{H}_S^{(j)}$, with iso-spectral blocks . Hence it acts independently on each one of such subspaces, inducing on each one of them the same local unitary rotation, i.e.
\[
U_{SA}(t) = e^{-iH_{SA}t} = \odot_j e^{-iH_{SA}^{(j)}t} .
\]
If we now consider the evolution it induces on an input state of the form $|\psi\rangle_S \otimes |\psi_0\rangle_A$, where $|\psi\rangle_S$ is a generic vector of $S$, expanding the input state as a linear combination of the vectors $|\xi_j^{(0)}\rangle_{SA}$ we can write
\[
U_{SA}(t)|\psi\rangle_S \otimes |\psi_0\rangle_A = \sum_{j=1}^{n_S} \alpha_j |\xi_j^{(0)}(t)\rangle_{SA} ,
\]
$\alpha_j$ being the expansion coefficients of $|\psi\rangle_S$ with respect to the basis $\{|j\rangle_S : j = 1, \ldots, n_S\}$ and where the vector
\[
|\xi_j^{(0)}(t)\rangle_{SA} := e^{-iH_{SA}^{(j)}t}|\xi_j^{(0)}\rangle_{SA} ,
\]
is the evolution of $|\xi_j^{(0)}(t)\rangle_{SA}$ induced by the Hamiltonian component $H^{(j)}$ that is active on the subspace $H^{(j)}_{S,A}$. By construction $|\xi_j^{(0)}(t)\rangle_{SA} \in H^{(j)}_{S,A}$ so that we can write it as

$$|\xi_j^{(0)}(t)\rangle_{SA} = \sum_{t=0}^{n_L} \beta_t(t)|\xi_j^{(t)}\rangle_{SA} . \quad (42)$$

In this expression the quantities $\beta_t(t)$ are (properly normalized) amplitude probabilities associated with the canonical orthonormal basis $|\xi_j^{(0)}\rangle_{SA}, |\xi_j^{(1)}\rangle_{SA}, \ldots, |\xi_j^{(n_L)}\rangle_{SA}$, whose explicit functional dependence on $t$ can be freely tailored by properly choosing the frequencies $\omega_1, \omega_2, \ldots, \omega_{n_L}$ of the model. The relevant observation here is the fact that due to the iso-spectral property (35), such coefficients do not bear any functional dependence upon the index $j$. Exploiting this fact and replacing Eq. (42) into (40) we can hence write

$$U_{SA}(t)|\psi\rangle_S \otimes |\psi_0\rangle_A = \beta_0(t)|\psi\rangle_S \otimes |\psi_0\rangle_A + \sqrt{1-|\beta_0(t)|^2} \sum_{\gamma=1}^{n_T} M_\gamma |\psi_\gamma\rangle_S \otimes |A^\gamma(t)\rangle_A , \quad (43)$$

where for $\gamma = 1, \ldots, n_T$,

$$|A^\gamma(t)\rangle_A := \frac{1}{\sqrt{1-|\beta_0(t)|^2}} \sum_{\ell=1}^{n_L} \beta_\ell(t)|\gamma, \ell\rangle_A , \quad (44)$$

form an orthonormal set of vectors of $A$ which are also orthogonal to $|\psi_0\rangle_A$, i.e. they fulfill the conditions

$$\langle A^\gamma(t)|A^\gamma(t)\rangle_A = \delta_{\gamma,\gamma'}, \quad \langle \psi_0|A^\gamma(t)\rangle_A = 0 . \quad (45)$$

As a consequence of Eq. (40), it follows that the evolved density matrix $\rho_{SA}(t) := \rho_{SA}(t)|\psi_0\rangle_A \langle \psi_0| \Upsilon_{SA}(t)$ of $S + A$ at time $t$ can be written as

$$\rho_{SA}(t) = |\beta_0(t)|^2 \rho^n_{\psi} \otimes |\psi_0\rangle_A \langle \psi_0| + |\beta_0(t)| \Delta_{SA}(t) + (1-|\beta_0(t)|^2) \sum_{\gamma\gamma'} M_\gamma^\gamma \rho^n_\gamma M_\gamma^\gamma \otimes |A^\gamma(t)\rangle_A \langle A^\gamma(t)| , \quad (46)$$

where we have define the bounded operator on $S + A$

$$\Delta_{SA}(t) = e^{-i\xi_0(t)} \sqrt{1-|\beta_0(t)|^2} \sum_{\gamma} M_\gamma |\psi_\gamma\rangle_s \otimes |A^\gamma(t)\rangle_A \langle \psi_0| + h.c. , \quad (47)$$

$e^{i\xi_0(t)}$ being the phase of $\beta_0(t)$.

The relevant quantity in Eq. (46) is the probability amplitude function $\beta_0(t)$: for $t = 0$ it is equal to 1, in agreement with the requirement that $\rho_{SA}(0) = \rho^n_{\psi} \otimes |\psi_0\rangle_A \langle \psi_0|$, but $\beta_0(t) \to 0$ in an extended time interval for large enough $n_L$, as shown in Fig. 4. Accordingly, in such time interval the above expression reduces to

$$\rho_{SA}(t) \cong \sum_{\gamma\gamma'} M_\gamma^\gamma \rho^n_\gamma M_\gamma^\gamma \otimes |A^\gamma(t)\rangle_A \langle A^\gamma(t)| , \quad (48)$$

which effectively achieves our target (11) by identifying $|A^\gamma(t)\rangle_A$ with $|\xi_\gamma(t)\rangle_S$.

IV. CONCLUSIONS

In this manuscript we discussed how to provide a comprehensive dynamical description for the quantum measurement process. For the case of projective measures an exhaustive well-established answer is provided by the von Neumann-Ozawa model hinging upon the decoherence induced by an ultimately macroscopic apparatus on the system under investigation. As far as the decoherence process takes place the states of the apparatus, on which the information about measurement outputs is encoded, progressively become orthogonal to each other. Once the decoherence process has taken place such states result to be perfectly distinguishable, thus allowing for an optimal encoding of the measurement results. We proved that this model cannot be directly applied to tackle the case of non-orthogonal measurements, as it could
induce a violation of the complete positivity requirement for such dynamical process before the decoherence is completed. We showed different strategies in order to overcome this hindrance. On the one hand, it results that it is possible to retrieve the correct probability distribution prescribed by an arbitrary POVM by extending the von Neumann description to an ancillary system and performing a joint projective measure on the system and the ancilla (Appendix B). However, this solution does not return the expression for the post-measurement state of the system prescribed by the definition of POVMs. In Section III we show that a possible solution to this problem can be realized by getting rid of such a net separation between the ancilla and the apparatus, and finally identifying the latter with a macroscopic ancilla. The key mechanism underlying our model consists in engineering a coupling between the system and the ancilla in terms of state transfer Hamiltonians acting on orthogonal eigenspaces of the global Hilbert space. By construction this allows to encode the information about the output results of an arbitrary POVM into the states of the ancilla which, at difference with the standard decoherence model, constitute an orthonormal set at all times. This allows us to retrieve not only the correct probability distribution for the output results, but also the correct expression for the post-measurement state of POVMs.

Appendix A: CPT conditions for the mapping \( S \)

If we force the mapping \( S \) to apply also when the projectors \( \Pi^{(\gamma)} \) are replaced by the element \( M^{(\gamma)}_S \) associated to a generic POVM, at local level on \( S \) this would induce the following transformation

\[
\rho^{in}_S \rightarrow \sum_{\gamma} M^{(\gamma)}_S \rho^{in}_S M^{(\gamma)\dagger}_S \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S .
\]  

(A1)

Notice that the scalar product \( \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S \) can be seen as the element \( \gamma, \gamma' \) of a positive semidefinite matrix \( Q(t) \) in a given orthonormal basis \( \{|\phi_\gamma\rangle\}_{\gamma=1}^{n_T} \) of a \( n_T \)-dimensional Hilbert space. Let then \( Q(t) = \sum_{j=1}^{n_T} q_j(t)|u_j(t)\rangle\langle u_j(t)| \) be the spectral decomposition of \( Q(t) \), with \( q_j(t) \geq 0 \) and

\[
\sum_{j=1}^{n_T} q_j(t) = \text{Tr}[Q] = \sum_{\gamma=1}^{n_T} \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S = n_T .
\]  

(A2)

Writing \( \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S \) in the eigenbasis of \( Q(t) \), we can then recast the mapping (A1) as

\[
\rho^{in}_S \rightarrow \sum_{\gamma} q_j(t) L^{(\gamma)}(t) \rho^{in}_S L^{(\gamma)\dagger}(t) ,
\]  

(A3)

where \( L^{(\gamma)}(t) = \sum_{\gamma=1}^{n_T} |u_j(t)\rangle\langle \phi_\gamma| \) are operators fulfilling the constraint

\[
\sum_{j=1}^{n_T} L^{(\gamma)}(t) \dagger L^{(\gamma)}(t) = \sum_{\gamma} M^{(\gamma)\dagger}_S M^{(\gamma)}_S = \mathbb{1}_S .
\]  

(A4)

It is then easy to verify that Eq. (A3) is CPT if and only if the following condition holds

\[
\sum_{j=1}^{n_T} q_j(t) L^{(\gamma)}(t) \dagger L^{(\gamma)}(t) = \sum_{\gamma} \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S M^{(\gamma)}_S \dagger M^{(\gamma)}_S = \mathbb{1}_S .
\]  

(A5)

The identity is trivially attained when the \( M^{(\gamma)}_S \) form a complete set of orthogonal projectors as in the case of POVMs. On the contrary, if this condition is not met then Eq. (A3) is in general in conflict with (A4) with the exception of the case when the \( q_j(t) \) are all equal to 1, forcing \( Q(t) \) to be the identity operator, and the vectors \( |\Xi^{\gamma}(t)\rangle \) to be orthonormal, i.e. \( \langle \Xi^{\gamma}(t) | \Xi^{\gamma}(t) \rangle_S = \delta_{\gamma,\gamma'} \).

Appendix B: \( S + A + \Xi \) approach to dynamical mapping

A reasonable, yet non completely satisfying approach, to produce a generic dynamical model for describing an arbitrary POVM follows by considering the extended \( SA \) system of the Naimark representation as the system of interest, and introducing an external environment \( \Xi \) which perform a POVM on it. First we notice that any POVM \( \{\Pi^{(\gamma)}\} \) on \( SA \) together with an arbitrary state \( |\psi_0\rangle \) defines a POVM on \( S \) with measurement operators \( \{F^{(\gamma)} = |\psi_0\rangle \langle \psi_0| \Pi^{(\gamma)}\} \). Actually, thanks to the Naimark theorem, the reverse statement is also true. Indeed, if we take an arbitrary POVM \( \{F^{(\gamma)}\} \) on \( S \), from Eqs. (3) and (4) we can define the projectors

\[
P^{(\gamma)} = V^{\dagger} \Pi^{(\gamma)} \otimes |\gamma\rangle_A \langle \gamma| V_S ,
\]  

(B1)

which form a complete orthonormal set in the space \( L(H_{SA}) \) of linear operators of \( S + A \). Let us now construct the von-O dynamical model for such POVM introducing the interaction \( O_{SA} \otimes O_{\Xi} \), with \( O_{SA} = \sum_{\gamma=1}^{n_T} a_\gamma \Pi^{(\gamma)}_A \), \( a_\gamma \in \mathbb{R} \), and \( O_{\Xi} \) self-adjoint; the corresponding propagator reads

\[
U_{SA\Xi}(t) := e^{-itO_{SA\Xi}} = \sum_{\gamma} P^{(\gamma)}_S \otimes U^{(\gamma)}_\Xi(t) ,
\]  

(B2)

with \( U^{(\gamma)}(t) = e^{-itO_{\Xi}} \). Subject to such unitary, an initial state \( \rho^{in}_S \otimes |\psi_0\rangle_A \langle \psi_0| \otimes |D\rangle_A \) evolves into

\[
\rho^{SA\Xi}_S(t) = \sum_{\gamma,\gamma'} P^{(\gamma)}_S \rho^{in}_S \otimes |\psi_0\rangle_A \langle \psi_0| P^{(\gamma')}_S \otimes |\Xi^{\gamma}(t)\rangle_S \langle \Xi^{\gamma'}(t) | \Xi^{\gamma}(t)\rangle_S ,
\]  

at a later time \( t \), and for \( t > t_d \) the density operator of the joint system \( SA \) will have a block-diagonal form with respect to the basis of the POVM \( \{P^{(\gamma)}_S\} \). From the viewpoint of the principal system \( S \), the composite system \( S + \Xi \) is however seen as a single measurement apparatus: In this perspective, if we expand \( \rho_{SA\Xi}(t) \) into an arbitrary basis \( \{|e_k\rangle_A\} \) of \( H_A \), we have

\[
\rho_{SA\Xi}(t) = \sum_{\gamma,\gamma'} \langle e_k | P^{(\gamma)}_S | \psi_0 \rangle_A \rho^{in}_S \langle \psi_0 | P^{(\gamma')}_S | e_{k'} \rangle_A |e_k\rangle_A \langle e_{k'}| \otimes |\Xi^{\gamma}(t)\rangle_S \langle \Xi^{\gamma'}(t) | \Xi^{\gamma}(t)\rangle_S ,
\]  

(B3)
and
\[ ρ_{SA}(t) = \sum_{γ} P_{SA}^{(γ)}(t) ρ_{SA}^{in} \otimes |ψ_0⟩_A⟨ψ_0| P_{SA}^{(γ)} = \sum_{γ} p_γ ρ_{SA}^{(γ)} = ρ_{SA}^{out}. \]  

(B4)

Therefore, the system experiences a decoherence process which takes place in \( n_T \) (\( n_A \)-dimensional) subspaces spanned by \( \{|e_k⟩_A|Σ^γ(t)⟩_S\}_{k=1,\ldots,n_A} \) of \( \mathcal{H}_{SA} \). Indeed, since just the vectors \( \{|Σ^γ(t)⟩_S\} \) evolve in time, being \( Σ \) the actual macroscopic part of the apparatus, the effective decoherence process will emerge only with respect to the label \( γ \). As we are aiming at a dynamical model for the original POVM on \( S \), the relevant question is: what happens at the level of the principal system \( S \)? Clearly, no matter which subsystem we are going to identify as the apparatus (say \( Σ \) or \( Σ + A \)) Eq. (B3) has not the form we are aiming to, not yielding to something like (B1) even after the orthogonalization of the \( |Σ^γ(t)⟩_S \). As for the probability distribution \( \{p_γ\} \), the outcomes statistics generated by \( \{F^{(γ)}\} \) via \( p_γ = \text{Tr}[ρ^{in}_S F^{(γ)}] \) is nevertheless the same as that entering Eq. (B4), as can be easily seen by Eqs. (2) and (3).

As for the state of \( S \), by inserting the explicit expression for \( P_{SA}^{(γ)} \) into (B1) and tracing over the ancilla, we get
\[ ρ_S(t) = \sum_{γ,k} N^{(k)γ}_{S} \left(M^{(γ)}_{S} ρ^{in}_S M^{(γ)†}_{S}\right) N^{(k)γ†}_{S} = \sum_{γ} p_γ \sum_{k} N^{(k)γ}_{S} \rho^{(γ)}_{S} N^{(k)γ†}_{S}. \]  

(B5)

where \( N^{(k)γ}_{S} := ⟨e_k|V^{†}_{SA}|γ⟩_A \). Therefore, \( ρ_S(t) \) does not coincide with the post-measurement state \( ρ^{out}_S \) defined in Eq. (B1). The only exception is represented by the case in which \( \mathcal{H}_S = n_T = n_A \) and \( V_{SA} \) coincides with the swap operator \( S_{SA} := \sum_{γ,k} |γ⟩_S ⟨γ| \otimes |γ⟩_A ⟨γ| \). In this case it results \( P^{(γ)}_{SA} = |γ⟩_S ⟨γ| \otimes I_A \), which pulls back to the vN-O model for the ideal POVM \( \{|γ⟩_S ⟨γ| \} \) on \( S \). The operator \( S_{SA} \) is a unitary self-adjoint transformation such that for all operators \( Θ_{S} ∈ \mathcal{L}(\mathcal{H}_S) \) and \( Υ_{A} ∈ \mathcal{L}(\mathcal{H}_A) \) gives \( S_{SA}(Θ_{S} \otimes Υ_{A})S_{SA}^† = Θ_{A} \otimes Υ_{S} \).

However, we can push forward. Let us observe that for any fixed \( γ \) the set of operators \( \{N^{(k)γ}_{S}\} \) returns a resolution of the identity, i.e \( \sum_k N^{(k)γ}_{S} N^{(k)γ†}_{S} = I_S \). From this, two facts follow: The first is that \( ρ_S(t) \) reads as the post-measurement state of a double-labeled POVM \( \{F^{(γ,k)}_{S}\} \), with measurement operators \( \{M^{(γ,k)} := N^{(k)γ}_{S} M^{(γ)†}_{S}\} \). Such POVM accounts for \( n_T n_A \) possible outcomes, and the associated probability distribution \( p_{γk} = \text{Tr}[ρ^{in}_S F^{(γ,k)}] \) is related to that of the original POVM \( \{F^{(γ)}_{S}\} \)
\[ \sum_k p_{γk} = p_γ. \]  

(B6)

This means that, if we gather the \( n_T n_A \) outcomes \( μ_{γk} \) of the POVM \( \{F^{(γ,k)}_{S}\} \) in \( n_T \) sets \( O_γ \), each bearing \( n_A \) elements, \( O_γ = \{μ_{γ1}, μ_{γ2}, \ldots, μ_{γn_A}\} \) (see Figure 5), the probability for each set is the sum of the probabilities for the outcomes it collects. This is consistent with the fact that, as observed through Eq. (B3), from the viewpoint of the principal system, the decoherence process emerges in the form of \( n_T \) subspaces (one for each \( γ \)) in \( \mathcal{H}_{SA} \).

The second fact following from the condition \( \sum_k N^{(k)γ}_{S} N^{(k)γ†}_{S} = I_S \) is that for all \( γ \), the set \( \{N^{(k)γ}_{S}\} \) itself defines a POVM on \( \mathcal{H}_S \). This represents a meaningful result, as it tells us that the state \( \{F^{(γ)}_{S}\} \) prior to the output production is the statistical mixture, with the original POVM’s probability distribution \( \{p_γ\} \), of the \( n_T \) output states of a set of non-selective measurements, each labeled by \( γ \) and defined by the set of measurement operators \( \{N^{(k)γ}_{S}\} \), performed upon the respective \( γ \)-detected state \( ρ^{(γ)}_{S} \) resulting from the action of the original POVM on \( ρ^{in}_S \).

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