The Skyrme Energy Functional and Naturalness

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Abstract

Recent studies show that successful relativistic mean-field models of nuclei are consistent with naive dimensional analysis and naturalness, as expected in low-energy effective field theories of quantum chromodynamics. The non-relativistic Skyrme energy functional is found to have similar characteristics.

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In a recent article [1], evidence for quantum chromodynamic (QCD) and chiral symmetry scales in a relativistic point-coupling model of nuclei was found by examining the parameters for naturalness. Naturalness means that coefficients of terms in the lagrangian are of order unity after appropriate combinations of strong interaction scales are extracted. Subsequent analyses of more general relativistic point-coupling and meson-nucleon lagrangians support this result and give new insight into the phenomenological success of these models [2,3]. One might wonder if naturalness is apparent only in relativistic models. In this report, we analyze a long-established nonrelativistic point-coupling model: the Skyrme-force energy functional.

The Skyrme interaction [4] has been successfully used in nonrelativistic nuclear structure calculations for many years [5–9]. The Skyrme potential takes the form of zero-range ("point-coupling") terms representing an expansion in the nucleon density and momentum, and is designed for use in Hartree-Fock calculations. It is generally interpreted as parametrizing a density-matrix expansion of the in-medium G-matrix [6,10], although in practice the parameters are determined from direct fits to nuclear observables.

The Skyrme approach was originally proposed long before QCD and has never been associated with QCD or chiral symmetry. Nevertheless, it has a form consistent with chiral effective field theories of QCD, such as chiral perturbation theory (ChPT), in which the degrees of freedom are pions and nonrelativistic nucleons. In particular, when non-pionic degrees of freedom are integrated out, one expects contact terms built from powers and derivatives of the nucleon fields, as in the Skyrme interaction. While there are no explicit pions in the Skyrme force, direct pion contributions largely average to zero for the bulk properties of nuclei and the effects of pion loops can be approximately absorbed into a general density functional for the energy [11,12]. Thus the nucleon terms should dominate the physics of closed-shell nuclei.

The signature of the underlying short-range physics should be the size of the coefficients of the effective lagrangian. However, it is not obvious that a Hartree-Fock energy functional fit directly to finite nuclei should exhibit naturalness, because many-body correlation effects will also be absorbed into its coefficients. While the results from relativistic mean-field models are encouraging, their naturalness might rely on the large isoscalar scalar and vector mean fields, which leads to “Hartree dominance” [2].

We apply Georgi and Manohar’s naive dimensional analysis (NDA) [13,14], which predicts the size of the coefficient of any term in an effective lagrangian for the strong interaction. This NDA has been extended to effective hadronic lagrangians for nuclei, both for point-coupling [1] and meson-exchange [2] models. The basic assumption of “naturalness” is that once the appropriate dimensional scales have been extracted using NDA, the remaining overall dimensionless coefficients should all be of order unity. For the strong interaction, there are two relevant scales: the pion-decay constant \( f_\pi \approx 93 \text{ MeV} \) and a larger scale \( 0.5 \lesssim \Lambda \lesssim 1 \text{ GeV} \), which characterizes the mass scale of physics beyond Goldstone bosons.

The NDA rules prescribe how these scales should appear in a given term in the lagrangian density if it is to have a consistent loop expansion. For a model with only nucleon fields \( \psi \), the counting reduces to a factor of \( 1/f_\pi^2 \Lambda \) for every bilinear \( \psi^\dagger \psi \), a factor of \( 1/\Lambda \) for every...
gradient, and an overall factor of $f_\pi^2 \Lambda^2$\footnote{In this work, we follow Ref. \cite{footnote1} and do not include any explicit counting factors. Such factors were included in the analysis of meson models in Ref. \cite{footnote2} but were not needed in Ref. \cite{footnote3}.} Thus an individual term in the effective lagrangian can be written schematically as

$$c \left[ \frac{\psi^\dagger \psi}{f_\pi^2 \Lambda} \right]^n \left[ \frac{\nabla}{\Lambda} \right]^n f_\pi^2 \Lambda^2,$$

with $c$ a dimensionless constant of order unity if the term is natural. The appropriate mass for $\Lambda$ might be the nucleon mass $M$ or a non-Goldstone boson mass, so we expect $500\text{ MeV} < \Lambda < 1000\text{ MeV}$.

One might try to reformulate the Skyrme approach in the form of an effective lagrangian. Instead, we work here with the Skyrme energy functional, which is most directly connected to the nuclear input. We postulate that the size of coefficients in the functional should be consistent with NDA. That is, we assume that the dominant scales of the coefficients are determined by the short distance physics. A direct analysis of the Skyrme potential will be considered elsewhere. We echo the discussion in Ref. \cite{footnote1} and argue that refining the Skyrme approach by adding pion loops or a more complete set of terms will only change values of the coefficients in the effective lagrangian (and hence the energy functional) by factors of order unity. Once again, it is not at all obvious that many-body effects absorbed into parameters by fits to nuclei will not disrupt the power counting; here we test these assumptions empirically.

Therefore, we perform our analysis on the Skyrme energy density $H(r)$, which is derived by taking the expectation value of the Skyrme hamiltonian with respect to a Slater determinant of single-particle nucleon wave-functions for $N = Z$ nuclei \cite{footnote2, footnote4}. (The energy functional itself is $\int d^3 r \, H(r)$.) The result is

$$H(r) = \frac{1}{2M} \tau + \frac{3}{8} t_0 \rho^2 + \frac{1}{16} t_3 \rho^3 + \frac{1}{16} (3t_1 + 5t_2) \rho \tau + \frac{1}{64} (9t_1 - 5t_2) (\nabla \rho)^2 - \frac{3}{4} W_0 \rho \nabla \cdot \mathbf{J} + \frac{1}{32} (t_1 - t_2) \mathbf{J}^2,$$

where $\rho(r)$ is the nucleon density, $\tau(r)$ is the kinetic energy density, and $\mathbf{J}(r)$ is the so-called spin-orbit density \cite{footnote5}. Some other variations of the Skyrme interaction lead to fractional powers of $\rho$ in $H(r)$ \cite{footnote6} and will not be considered here. The coefficients in $H(r)$ are determined by fits to nuclear observables; it is the analog of the energy functionals in the relativistic mean-field analyses.

For the purpose of applying Eq. (1), we make the correspondences (neglecting irrelevant signs and spin matrices):

$$\rho \leftrightarrow \psi^\dagger \psi,$$

$$\tau \leftrightarrow \nabla \psi^\dagger \cdot \nabla \psi,$$

$$\mathbf{J} \leftrightarrow \psi^\dagger \nabla \psi.$$

Applying the scaling rules from (1) term by term to Eq. (2), we can rewrite $H(r)$ in terms of dimensionless coefficients $c_i$, which should be of order one if natural:
TABLE I. Parameter sets for some standard Skyrme interactions.

| Force       | $t_0$ (MeV-fm$^3$) | $t_1$ (MeV-fm$^5$) | $t_2$ (MeV-fm$^5$) | $t_3$ (MeV-fm$^6$) | $W_0$ (MeV-fm$^5$) | $x_0$ |
|-------------|---------------------|---------------------|---------------------|---------------------|---------------------|------|
| Skyrme 1    | $-1057.3$           | $235.9$             | $-100.0$            | $14463.5$           | $120$               | $0.56$ |
| Skyrme 2    | $-1169.9$           | $585.6$             | $-27.1$             | $9331.1$            | $105$               | $0.34$ |
| Skyrme 3    | $-1128.8$           | $395.0$             | $-95.0$             | $14000.0$           | $120$               | $0.45$ |
| Skyrme 4    | $-1205.6$           | $765.0$             | $35.0$              | $5000.0$            | $150$               | $0.05$ |
| Skyrme 5    | $-1248.3$           | $970.6$             | $107.2$             | $150$               | $150$               | $0.17$ |
| Skyrme 6    | $-1101.8$           | $271.7$             | $-138.3$            | $17000.0$           | $115$               | $0.58$ |

TABLE II. Dimensionless coefficients obtained for some conventional Skyrme interactions by applying Eq. 6.

| Force       | $c_1$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ | $c_7$ |
|-------------|-------|-------|-------|-------|-------|-------|-------|
| Skyrme 1    | $0.5$ | $-0.45$ | $1.09$ | $0.33$ | $1.05$ | $-2.31$ | $0.27$ |
| Skyrme 2    | $0.5$ | $-0.50$ | $0.70$ | $2.60$ | $2.17$ | $-2.02$ | $0.49$ |
| Skyrme 3    | $0.5$ | $-0.48$ | $1.05$ | $1.14$ | $1.62$ | $-2.31$ | $0.39$ |
| Skyrme 4    | $0.5$ | $-0.51$ | $0.38$ | $3.97$ | $2.70$ | $-2.89$ | $0.59$ |
| Skyrme 5    | $0.5$ | $-0.53$ | $0.00$ | $5.53$ | $3.29$ | $-2.89$ | $0.69$ |
| Skyrme 6    | $0.5$ | $-0.47$ | $1.28$ | $0.20$ | $1.26$ | $-2.21$ | $0.33$ |

\[ H(\mathbf{r}) = \frac{\tau}{\Lambda} \left( \frac{\rho^2}{f_\pi^2} + \frac{\rho^3}{f_\pi^4 \Lambda} + \frac{\rho \tau}{f_\pi^2 \Lambda^2} + \frac{(\nabla \rho)^2}{f_\pi^2 \Lambda^2} + \frac{\rho \nabla \cdot \mathbf{J}}{f_\pi^2 \Lambda^2} \right) \]

The second line manifests the expansion and truncation of $H(\mathbf{r})$ in powers and derivatives of the nucleon fields, with expansion parameter $\rho/f_\pi \Lambda$.

There are many sets of parameters for the Skyrme force determined by different fits to nuclear observables such as experimental binding energies and radii of nuclei. Here we consider Skyrme 1 through Skyrme 6, which are parameters for models with energy densities of the form of $H(\mathbf{r})$ in Eq. (2). The coefficients $t_0$ through $t_3$ and $W_0$ are dimensional, but as usually presented there is little clue to the relevant scales that determine their size. In Table I, we list the coefficients for models 1 through 6, in the conventional units; the coefficients are certainly not natural as given!

Coefficients for Skyrme 1 through 6 scaled according to Eq. (6) are given in Table II, where we have used $\Lambda = M$. While this is likely an upper limit to $\Lambda$ in some cases, the coefficients for $500 \text{ MeV} < \Lambda < 1000 \text{ MeV}$ are not qualitatively different ($\Lambda = 1000 \text{ MeV}$ was used in Ref. [1]). However, the range in $\Lambda$ increases the uncertainty when estimating the size of omitted higher-order contributions (see below).

An obvious example of a natural coefficient is $c_1 = 1/2$, which follows since the scale of nucleon kinetic energy is the nucleon mass. However, the scaling of the other coefficients is non-trivial; if $M$ alone were extracted to define dimensionless coefficients they would be badly unnatural. For example, $c_3$ would be over $10^5$ for most of the forces. The coefficients are

\[ H(\mathbf{r}) = \frac{\tau}{\Lambda} \left( \frac{\rho^2}{f_\pi^2} + \frac{\rho^3}{f_\pi^4 \Lambda} + \frac{\rho \tau}{f_\pi^2 \Lambda^2} + \frac{(\nabla \rho)^2}{f_\pi^2 \Lambda^2} + \frac{\rho \nabla \cdot \mathbf{J}}{f_\pi^2 \Lambda^2} \right) \]
also unnatural if one expressed $H(r)$ as an expansion in $\rho(r)/\rho_0$, where $\rho_0$ is the saturation density of nuclear matter.

In contrast, the NDA scaling of Eq. (3) implies natural coefficients in essentially all cases. The “worst case” is $c_4$ in Skyrme 5, but we also note that this interaction is particularly unnatural by construction, since $c_3$ is taken to be zero.

Figure 1 shows the contributions to the nuclear matter energy per particle of the form $\rho^n$, evaluated at saturation density $\rho_0$. The crosses are estimates based on the assumption of natural coefficients given by Eq. (1) with $\psi^\dagger \psi \rightarrow \rho_0$, and the error bars show a range from 1/2 to 2 in the coefficients ($\Lambda = M$ is used in Fig. 1). The Skyrme contributions are consistent with naturalness, although a more systematic study of Skyrme-type energy functionals including higher powers of $\rho$ would be needed to be conclusive.

It is evident that naturalness implies a convergent density expansion for mean-field contributions to nuclear matter, with expansion parameter $\rho_0/f^2\pi \Lambda$ between 1/4 and 1/7 [15]. One can also anticipate good convergence for terms with gradients of the fields, since the nucleons are nonrelativistic, and gradients of the densities, since the relevant scale for derivatives in finite nuclei should be roughly the nuclear surface thickness $\sigma$, and so the predicted dimensionless expansion parameter is $1/\Lambda\sigma \leq 1/5$.

Nevertheless, the initial energy scale is large compared to the nuclear binding energy so that the $n = 3$ term is still important. (The size of the $n = 2$ contribution is discussed below.) The largeness of this “three-body” term is conventionally cited [7] as implying a strong density dependence to the microscopic effective interaction. While $\rho^5$ terms are unlikely to be relevant, the omitted $\rho^4$ contribution is estimated to be uncomfortably large (and would be larger with a smaller value of $\Lambda$) at nuclear saturation density.

In Figure 2 we compare the typical Skyrme result (Skyrme 3 is used) to results from general relativistic point-coupling models fit to nuclear observables [3]. Contributions from individual terms to two relativistic models (labeled FZ4 and VA4) are shown as unfilled circles and squares while the net contributions are shown as filled symbols. The multiple contributions for each $n$ in the relativistic models are of the form $\rho_s^i \rho_v^j$ with $i + j = n$, where $\rho_s$ is the scalar density and $\rho_v$ is the vector (baryon) density. The naturalness of the relativistic models implies an expansion that can be truncated at $n = 4$ with an error of order 1 MeV, which is easily absorbed by slight adjustments of the other parameters.

The strong cancellation between the $\rho_s^2$ and $\rho_v^2$ terms is characteristic of relativistic point-coupling models [3]. A nonrelativistic reduction of the point-coupling model would incorporate this cancellation and therefore can be anticipated in the Skyrme energy. Indeed, the Skyrme $n = 2$ energy is consistent with the net $n = 2$ contribution from the relativistic models, which is just marginally natural because of the cancellations. For higher-order terms,

$$^2$$There are many sources of such terms in a nonrelativistic effective lagrangian, including relativistic effects [16,17].

$$^3$$Absolute values are plotted in the figure.

$$^4$$The correspondences between the Skyrme energy functional and relativistic mean-field models has been discussed by Reinhard and collaborators [18].
however, the net contribution is comparable to individual contributions, so one cannot rely on further cancellations to improve the convergence of the nonrelativistic expansion. Thus if the NDA estimates are used to anticipate contributions in a complete nonrelativistic point-coupling model, it would appear that $n = 4$ contributions are still significant at nuclear saturation density.

Note that this conclusion does not contradict the conventional wisdom from few-body calculations that four-body contributions are quite small [19,15], because the effective densities involved are significantly lower. Furthermore, while $n = 4$ terms in relativistic mean-field meson models are important for achieving good fits to bulk nuclear observables [3], very good fits can be obtained in point-coupling models with a truncation at $n = 3$. (In both cases the best fits require $n = 4$.) The $n \leq 3$ coefficients are able to adjust to absorb to a large degree the higher-order contributions. Thus it is not surprising that the Skyrme energy functional in its usual form is successful in reproducing nuclear observables.

Because the Skyrme energy functional includes only a limited set of terms, our results here are not by themselves definitive. But in the context of the other more complete investigations of relativistic models they are quite encouraging. In future work we will make a more extensive evaluation of Skyrme-like forces using the same approach applied to relativistic models [2,4]. This means considering all possible nonredundant terms (consistent with symmetries) in the energy functional, organized according to NDA. The goal is to constrain the parameters using a wide range of observables, instead of minimizing the number of parameters to improve predictability. The connection between naturalness in an effective Skyrme-like lagrangian and naturalness in the implied nonrelativistic Hartree-Fock energy functional (and beyond) will also be explored.

In summary, we have examined the nonrelativistic Skyrme energy functional in the context of low-energy effective field theories of QCD. As was found for relativistic point-coupling and mesonic models, Skyrme parameters are natural after applying naive dimensional analysis. This implies that QCD scales are relevant in analyzing the physics of nuclei, despite the complicated many-body physics and subtle dynamics of nuclear saturation that are absorbed into the parameters of the energy functional. The NDA provides a new organizational principle for Skyrme-like models at the mean-field level that suggests that current models are truncated prematurely. These results encourage the further application of effective field theory methods to finite density nuclear systems.

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FIG. 1. Contributions to the energy per particle in nuclear matter from terms of the form $\rho^n$, evaluated at saturation density $\rho_0$ for a variety of Skyrme interactions. The crosses are estimates based on Eq. (1) with $\Lambda = 939$ MeV. The arrow indicates the total binding energy $\epsilon_0 = 16.1$ MeV.

FIG. 2. Contributions to the energy per particle in nuclear matter at saturation density from terms of the form $\rho^n$ for the Skyrme 3 model and $\rho_i^j \rho_i^j$ with $i + j = n$ for two relativistic point-coupling models from Ref. [3] (see text). The crosses are estimates based on Eq. (1) with $\Lambda = 770$ MeV. The arrow indicates the total binding energy $\epsilon_0 = 16.1$ MeV.