An enhanced uncertainty principle for the Vaserstein distance

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Abstract

We improve some recent results of Sagiv and Steinerberger that quantify the following uncertainty principle: for a function $f$ with mean zero, then either the size of the zero set of the function or the cost of transporting the mass of the positive part of $f$ to its negative part must be big. We also provide a sharp upper estimate of the transport cost of the positive part of an eigenfunction of the Laplacian. This proves a conjecture of Steinerberger and provides a lower bound of the size of a nodal set of the eigenfunction.

Introduction

For a continuous function with mean zero, the Vaserstein distance between the measures corresponding to the positive and the negative parts of the function indicates how oscillatory the function is. If this Vaserstein distance is small, then the work required to move the positive mass to the negative mass is small and so we expect the positive and the negative parts of the function to be close together. Consequently, we would expect the function to oscillate significantly.

Our main result is an improvement of an uncertainty principle due to Sagiv and Steinerberger [7] showing that the zero set of a mean zero, continuous function and the Vaserstein distance between the positive and negative parts of the function cannot both be small at the same time. We prove this result for a function defined in the unit cube of $\mathbb{R}^d$. It easily extends to functions defined on a smooth, compact Riemannian manifold $M$ of dimension $d$.

Finally, we obtain an upper estimate for this Vaserstein distance in the case of high frequency eigenfunctions of the Laplacian in $M$ — by the previous uncertainty principle, this indicates that the nodal sets of these eigenfunctions should be large.

A continuous function $f$ on the unit cube $Q_0 = [0, 1]^d$ in $\mathbb{R}^d$ that has zero mean is decomposed into its positive part $f^+ = \max\{f, 0\}$ and its negative part $f^- = \max\{-f, 0\}$. The interface between the supports of these two functions is the zero set $Z(f) = \{x \in Q_0 : f(x) = 0\}$.

Thinking of $f^+$ as earth that is to be moved and of $-f^-$ as holes that need to be filled, then the earth-moving work that is required to fill the holes is the Vaserstein distance between the measures with densities $f^+$ and $f^-$. As mentioned earlier, if the earth mover’s distance is small, then any earth to be moved $f^+$ must be close to a hole that needs to be filled $f^-$, and so the interface between the two must be large. This is the intuition behind the following quantitative result of Steinerberger [12, Theorem 2] in dimension 2. With a minor abuse of notation, we write $W_1(f^+, f^-)$ for the Vaserstein distance between the measures on $Q_0$ with densities $f^+$ and $f^-$, respectively, relative to Lebesgue measure. We write $H^{d-1}(Z(f))$ for the $(d-1)$-dimensional Hausdorff measure of the zero set of $f$. Then, in dimension $d = 2$,

$$W_1(f^+, f^-) H^1(Z(f)) \|f\|_\infty \gtrsim \|f\|_1^2.$$

Received 13 March 2020; revised 8 June 2020; published online 27 July 2020.

2010 Mathematics Subject Classification 28A75, 35B05, 35P20 (primary), 49Q20, 58C40 (secondary).

The last two authors have been partially supported by the Generalitat de Catalunya (grant 2017 SGR 359) and the Spanish Ministerio de Ciencia, Innovación y Universidades (project MTM2017-83499-P).

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The Vaserstein distance between probability measures \( \mu \) and \( \nu \) on \( Q_0 \) is defined by

\[
W_1(\mu, \nu) = \inf_{\rho} \int_{Q_0 \times Q_0} |x - y| \, dp(x, y),
\]

where the infimum is over all admissible transport plans, that is over all probability measures \( \rho \) on \( Q_0 \times Q_0 \) with marginals \( \mu \) and \( \nu \). Such probability measures \( \rho \) are also referred to as couplings of \( \mu \) and \( \nu \). The monograph *Optimal Transport, Old and New* by Cedric Villani [13] has become a classic reference on optimal transport and includes a detailed exposition of the Vaserstein distance, also known as the ‘earth-mover’s distance’.

The \( p \)-Vaserstein distance is defined similarly but with \( |x - y| \) raised to the power \( p \). The \( 1 \)-Vaserstein distance has at least two advantages. One is that it has an equivalent Monge–Kantorovich dual formulation as

\[
W_1(\mu, \nu) = \sup_{h \in \text{Lip}_{1,1}(Q_0)} \left| \int_{Q_0} h \, d\mu - \int_{Q_0} h \, d\nu \right|.
\]

Here \( \text{Lip}_{1,1}(Q_0) = \{ h : Q_0 \to \mathbb{R} : |h(x) - h(y)| \leq |x - y|, \ x, y \in Q_0 \} \).

The other, and more important reason, is that the definition does not change if in (2) \( dp \) is replaced by \( d|\rho| \) and \( \rho \) is allowed to be a signed measure or transport plan on \( Q_0 \times Q_0 \) with marginals \( \mu \) and \( \nu \) (see [3]). This extra freedom allows us to construct transport plans that lead to better estimates, specifically in the course of proving Theorem 3.

The method of proof that Steinerberger uses to obtain the estimate (1) does not extend to higher dimensions in any obvious way. Using a different method, Sagiv and Steinerberger [7] prove that

\[
W_1(f^+, f^-) H^{d-1}(Z(f)) \left( \frac{\|f\|_\infty}{\|f\|_1} \right)^{4-1/d} \gtrsim \|f\|_1
\]

in dimension \( d \geq 3 \). By a modification of the ‘balanced/unbalanced cubes’ method of Sagiv and Steinerberger, we can reduce the power from \( 4 - 1/d \) to \( 2 - 1/d \).

**Theorem 1.** Let \( f : Q_0 \to \mathbb{R} \) be a continuous function with zero mean. Let \( Z(f) \) be the nodal set \( Z(f) = \{ x \in Q : f(x) = 0 \} \). Let \( H^{d-1}(Z(f)) \) denote the \( (d - 1) \)-dimensional Hausdorff measure of \( Z(f) \). Then

\[
W_1(f^+, f^-) H^{d-1}(Z(f)) \left( \frac{\|f\|_\infty}{\|f\|_1} \right)^{2-1/d} \gtrsim \|f\|_1.
\]

The proof is based on a recursive selection, through a stopping time argument, of dyadic cubes \( Q \) where either the mass of \( |f| \) is irrelevant or \( \int_Q f^+ \) is much larger than \( \int_Q f^- \) (or the other way around). These cubes turn out to fill up the entire cube \( Q_0 \), up to a set of measure zero.

This proof extends to a somewhat more general setting. Let \( (M, g) \) be a \( d \)-dimensional, smooth, compact Riemannian manifold without boundary and let \( dV \) denote the volume form associated to \( g \). A function \( f : M \to \mathbb{R} \) has zero mean if \( \int_M f \, dV = 0 \).

In this setting, the Vaserstein distance between two probability measures \( \mu \) and \( \nu \) on \( M \) is then

\[
W_1(\mu, \nu) = \inf_{\rho} \int_{M \times M} d(x, y) \, dp(x, y)
= \sup_{h \in \text{Lip}_{1,1}(M)} \left| \int_M h \, d\mu - \int_M h \, d\nu \right|,
\]
where the infimum is over all admissible transport plans $\rho$ from $\mu$ to $\nu$. Here $d(x,y)$ stands for the distance induced by the metric $g$ and $\operatorname{Lip}_{1,1}(M) = \{ h : M \to \mathbb{R} : |h(x) - h(y)| \leq d(x,y), \ x, y \in M \}$.

**Theorem 2.** Let $(M, g)$ be a $d$-dimensional, smooth, compact Riemannian manifold without boundary. Let $f : M \to \mathbb{R}$ be a continuous function with zero mean and let $Z(f) = \{ x \in M : f(x) = 0 \}$. Then

$$W_1(f^+, f^-) H^{d-1}(Z(f)) \left( \frac{\|f\|_{L^\infty(M)}}{\|f\|_{L^1(M)}} \right)^{2-1/d} \gtrsim_{(M,g)} \|f\|_1. \quad (5)$$

We state this result for $M$ compact without boundary because of the application we have in mind (see Theorem 3), but it will be clear from the proof that the statement holds as well for $M$ compact with smooth boundary.

We also show by means of an example (see Proposition 5) that the power $2 - 1/d$ in (4) cannot be replaced by any power smaller than 1. In particular, Steinerberger’s estimate (1) in dimension 2 is best possible in this sense.

The uncertainty principle in Theorem 2 demonstrates that an upper estimate for the Vaserstein distance $W_1(f^+, f^-)$ implies a lower estimate on the size of the nodal set. In this context, we establish one direction of a conjecture of Steinerberger on the Vaserstein distance between the positive and negative parts of eigenfunctions of the Laplacian. Steinerberger in [11] posed the following conjecture:

**Conjecture.** Let $(M, g)$ be a smooth, compact Riemannian manifold without boundary. Is it true that if $\phi$ is an $L^2$-normalised eigenfunction of the Laplacian with eigenvalue $L$, so that $-\Delta \phi = L \phi$ on $(M, g)$, then

$$W_p(\phi^+, \phi^-) \asymp_{p, (M,g)} \frac{1}{\sqrt{\log L}} \|\phi\|_1^{1/p}.$$ 

Steinerberger proves that

$$W_1(\phi^+, \phi^-) \lesssim_{(M,g)} \frac{\log L}{L} \|\phi\|_1.$$ 

We obtain the conjectured upper bound for the case $p = 1$ and for all linear combinations of eigenfunctions with high frequencies. This formalises the intuition that for high frequency eigenfunctions it is ‘cheap’ to move from the positive to the negative part.

**Theorem 3.** Let $(M, g)$ be a $d$-dimensional, smooth, compact Riemannian manifold without boundary. Let $\{\phi_0, \phi_1, \ldots\}$ be an orthonormal basis of $L^2(M)$ consisting of eigenfunctions $-\Delta \phi_i = \lambda_i \phi_i$, and ordered in such a way that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $f = \sum_{k: \lambda_k \geq L} a_k \phi_k \in L^2(M)$, $a_k \in \mathbb{R}$. Then

$$W_1(f^+, f^-) \lesssim_{(M,g)} \frac{1}{\sqrt{L}} \|f\|_1.$$ 

The improvement by the factor $\sqrt{\log L}$ follows from the construction of a (signed) transport plan that is well concentrated on the diagonal.

There is nothing special about the Laplacian in the context of Theorem 3, in that the result holds for any elliptic operator with smooth coefficients in the manifold $M$. We only need certain estimates on a Bochner–Riesz type kernel that are known to hold for arbitrary elliptic operators, see [8].
Together Theorems 2 and 3 show that when $\phi$ is an eigenfunction of the Laplacian with eigenvalue $L$, 

$$H^{d-1}(Z(\phi)) \gtrsim \sqrt{L} \left( \frac{\|\phi\|_{L^1(M)}}{\|\phi\|_{L^\infty(M)}} \right)^{2-1/d}.$$ 

As such, it goes in the direction of Yau’s conjecture that, in a smooth compact Riemannian manifold without boundary and for an eigenfunction $\phi$ of the Laplacian with eigenvalue $L$, we have $H^{d-1}(Z(\phi)) \simeq \sqrt{L}$. The full lower bound in Yau’s conjecture, without terms involving $L^\infty$ and $L^1$ norms of $\phi$, that is $H^{d-1}(Z(\phi)) \simeq \sqrt{L}$, has already been proved by Logunov in [2].

We finally remark that our method seems to provide information only for the Vaserstein distance $W_1$. As mentioned, the definition of $W_1(\mu, \nu)$ does not change if the transport plan $d\rho$ is replaced by $d|\rho|$, where $\rho$ is a signed transport plan. This fails dramatically for $p > 1$.

**Proposition 1.** Let $p > 1$ and let $\mu, \nu$ be two probability measures in the interval $I = [0, 1]$. We define

$$\tilde{W}_p^p(\mu, \nu) = \inf_{\rho} \int_{I \times I} |x - y|^p d|\rho|(x, y),$$

where the infimum is taken over all admissible signed transport plans, that is over all signed measures $\rho$ on $I \times I$ with marginals $\mu$ and $\nu$. Then $\tilde{W}_p^p(\mu, \nu) = 0$.

**Proof.** Consider first the case $\mu = \delta_0$ and $\nu = \delta_1$. Then we consider the sequence of transport plans $\rho_n$, which consist of $n$ negative Dirac deltas and $n + 1$ positive Dirac deltas located in points of $I \times I$ as in the figure:

On the white dots we place a positive Dirac delta and on the black dots a negative Dirac delta. More precisely we take $\rho_n$ to be

$$\rho_n = \delta_{(0,0)} + \sum_{j=1}^n \delta_{(j/n,1/(2n)+(j-1)/n)} - \sum_{j=1}^n \delta_{((j-1)/n,1/(2n)+(j-1)/n)}.$$
Clearly the marginals of $\rho_n$ are $\delta_0$ and $\delta_1$. For any of the Dirac deltas, whether positive or negative and located at a point $(x, y)$, we have that $|x - y| = 1/(2n)$, except for the Dirac delta at $(0,0)$. Thus,

$$
\int_{I \times I} |x - y|^p d|\rho_n|(x, y) = \sum_{j=1}^{n} 2 \left( \frac{1}{2n} \right)^p = (2n)^{1-p}.
$$

Thus,

$$
\tilde{W}_p^p(\delta_0, \delta_1) \leq \lim inf_{n} \int_{I \times I} |x - y|^p d|\rho_n|(x, y) = 0.
$$

This argument can be easily adapted to prove that $\tilde{W}_p^p(\delta_x, \delta_y) = 0$ for any pair $x, y \in [0, 1]$. Since linear combinations of Dirac deltas are weak*-dense in the space of probability measures, it follows that $\tilde{W}_p^p(\nu, \mu) = 0$ for any probability measures $\nu, \mu$. □

**Proof of Theorem 1**

Note that in general $f^+ dV$ and $f^- dV$ are not probability measures, which is the usual setting for the Vaserstein distance. However, the distance is well defined for measures with the same total mass. Alternatively, note that the zero mean condition implies that $2f^+ / \|f\|_1 dV$ and $2f^- / \|f\|_1 dV$ are probability measures, so we can define

$$
W_1(f^+, f^-) := \frac{\|f\|_1}{2} W_1 \left( \frac{2f^+}{\|f\|_1}, \frac{2f^-}{\|f\|_1} \right).
$$

In any case, replacing $f$ by $f / \|f\|_1$ if necessary, we may assume without loss of generality that $\|f\|_1 = 1$ and proceed to prove that there is a constant $C_d > 0$ such that

$$
W_1(f^+, f^-) H^{d-1}(Z(f)) \|f\|_\infty^{2-1/d} \geq C_d.
$$

If $H^{d-1}(Z(f)) = \infty$, the inequality (4) is trivially true, so we may assume that $H^{d-1}(Z(f)) < \infty$.

We shall use a dyadic decomposition of the cube $Q_0$ into cubes of different scales defined through a stopping time argument. The argument draws on constructions used by Steinerberger [10] and Sagiv and Steinerberger [7]. We need some definitions to describe this decomposition.

For any measurable set $A$ we denote its volume by $V(A)$. The side length of a subcube $Q$ of $Q_0$ is denoted by $l(Q)$, so $V(Q) = l(Q)^d$. We write

$$
V^+_f(Q) = V(Q \cap \{f > 0\}) \quad \text{and} \quad V^+_f(Q) = V(Q \cap \{f < 0\})
$$

and note that, since $H^{d-1}(Z(f)) < \infty$, $V(Q) = V^+_f(Q) + V^-_f(Q)$.

**DEFINITION 1.** We say that a cube $Q \subset [0, 1]^d$ is **unbalanced** if either

$$
V^+_f(Q) > 100 \|f\|_\infty V^-_f(Q)
$$

or

$$
V^-_f(Q) > 100 \|f\|_\infty V^+_f(Q).
$$

If this is not the case, that is if

$$
\frac{1}{100 \|f\|_\infty} \leq \frac{V^+_f(Q)}{V^-_f(Q)} \leq 100 \|f\|_\infty,
$$

we say that the cube is **balanced**.
DEFINITION 2. We say that a cube $Q \subseteq [0, 1]^d$ is full whenever $\int_Q |f| \geq V(Q)/10$. The empty cubes are those cubes $Q$ for which $\int_Q |f| < V(Q)/10$.

Consider now the standard dyadic partition of the unit cube: first split $Q_0$ into $2^d$ subcubes of length 1/2 and then, recursively, split each of the new subcubes into $2^d$ ‘descendants’ each with side length half that of the ‘parent’. For each $n \geq 1$, we have thus a generation $G_n$ of $2^d$ subcubes $Q$, each with side length $2^{-n}$, that form a partition of $Q_0$.

Now we shall associate to $f$ a collection of dyadic cubes, of different generations, by means of the following stopping time argument. We start our collection by taking the cubes $\tilde{Q}_i \in G_1$ such that either

(i) The cube $\tilde{Q}_i$ itself is empty;

(ii) One of the $2^d$ direct descendants $Q_i$ of $\tilde{Q}_i$ is full and unbalanced.

For each of the remaining cubes in the first generation (that is, those that are full and all $2^d$ direct descendants are either empty, or full and balanced) we repeat the process: we look at their $2^d$ descendants, which are now in $G_2$, and we add to our collection those cubes satisfying (i) or (ii) above.

Applying this process of subdiving the original cube recursively we obtain, after $n$ steps, a decomposition of $Q_0$ into three different families of cubes (of various generations): empty cubes $\tilde{Q}_i$; cubes $\tilde{Q}_i$ that are full and balanced, with at least one direct descendant that is full and unbalanced; and the remaining cubes (those that are not part of our collection at step $n$).

Therefore, letting $E_n$ (for empty) and $F_n$ (for full) denote, respectively, the sets of indices of cubes of the first and second kind, we have the decomposition

$$Q_0 = [0, 1]^d = \left( \bigcup_{i \in E_n} \tilde{Q}_i \right) \cup \left( \bigcup_{i \in F_n} \tilde{Q}_i \right) \cup R_n,$$

where $R_n$ denotes the union of the remaining cubes, those of the third kind.

Note that the sets $\bigcup_{i \in E_n} \tilde{Q}_i$ and $\bigcup_{i \in F_n} \tilde{Q}_i$ are increasing in $n$, since we keep adding cubes to our collection as $n$ advances, and therefore the sets $R_n$ decrease with $n$.

Our final collection consists of all the cubes added at every possible step, that is, of $\{\tilde{Q}_i\}_{i \in E \cup F}$, where $E = \bigcup_{n \geq 1} E_n$ and $F = \bigcup_{n \geq 1} F_n$. This yields a decomposition

$$Q_0 = [0, 1]^d = \left( \bigcup_{i \in E} \tilde{Q}_i \right) \cup \left( \bigcup_{i \in F} \tilde{Q}_i \right) \cup R,$$

where $R = \cap_{n \geq 1} R_n$.

PROPOSITION 2. Assume $H^{d-1}(Z(f)) < \infty$. Then

(a) $V(R) = 0$;

(b) $\sum_{i \in F} \int_{\tilde{Q}_i} |f| \geq \frac{9}{10}$.

Proof. (a) We recall the following relative isoperimetric inequality (see [4–6]): for a cube $Q$ in $\mathbb{R}^d$ and $K \subset Q$,

$$\mathcal{H}^{d-1}(\partial K \cap \text{int}(Q)) \geq_d \left( \min\{V(K), V(Q \setminus K)\} \right)^{d-1}.$$

Denote by $Q_1^k, \ldots, Q_N^k$ the cubes in the $k$th generation, $Q_1^k \in G_k$, that are full and balanced. Then, for all $k \geq 1$,

$$R \subset \bigcup_{j=1}^{N_k} Q_j^k.$$
In all balanced cubes, the volumes of the positive and negative parts are comparable. Thus the isoperimetric inequality above guarantees that the Hausdorff \((d - 1)\)-measure of \(Z(f)\) in any of the cubes \(Q_k^i\) is at least a constant times \(2^{-k(d-1)}\), so that \(H^{d-1}(Z(f)) \gtrsim N_k \times 2^{-k(d-1)}\). Hence \(N_k \lesssim 2^k(d-1)\), and the volume of \(\bigcup_{j=1}^{N_k} Q_k^j\) is at most a constant times \(2^{-k}\). Since this is true for all generations \(k\), we see that \(V(R) = 0\), as desired.

(b) First let us note that the mass of \(f\) in the empty cubes \(\tilde{Q}_i\) cannot be very big:

\[
\sum_{i \in E} \int_{\tilde{Q}_i} |f| \leq \frac{1}{10} \sum_{i \in E} V(\tilde{Q}_i) \leq \frac{1}{10}.
\]

The mass of \(f\) in the complement of the empty cubes in our collection therefore satisfies

\[
\int_{Q_0 \setminus \bigcup_{i \in E} \tilde{Q}_i} |f| \geq \frac{9}{10}.
\]

The decomposition (9) and (a) then gives the result. \(\square\)

Remark 1. By Proposition 2(b), \(\sum_{i \in F} \int_{Q_i} |f| \geq 9\times 2^{-d}\). This is clear since, by the maximality of \(Q_i\), \(2^d \int_{Q_i} |f| \geq \int_{\tilde{Q}_i} |f|\).

Denote by \(F^+\) the set of indices of the full cubes \(\tilde{Q}_i\) in our decomposition whose chosen, maximal, full, unbalanced, direct descendant \(Q_i\) satisfies (6), that is \(V_f^-(Q_i) \preceq V_f^+(Q_i)\). Similarly, we denote by \(F^-\) the indices corresponding to those full cubes \(\tilde{Q}_i\) whose chosen, maximal, full, unbalanced, direct descendant \(Q_i\) satisfies (7), that is \(V_f^+(Q_i) \preceq V_f^-(Q_i)\). Then, \(F = F^+ \cup F^-\) and the union is disjoint.

Lemma 1. For \(i \in F^+\), and the corresponding full, unbalanced cubes \(Q_i\), each of the following estimates holds:

\[
\int_{Q_i} f^- \leq \frac{1}{9} \int_{Q_i} f^+ \quad \text{(11)}
\]

\[
\int_{Q_i} f^+ \geq \frac{9}{10} \int_{Q_i} |f| \quad \text{(12)}
\]

Analogous estimates hold for \(i \in F^-\).

Proof. If \(i \in F^+\), then, by (6),

\[
\int_{Q_i} f^- \leq ||f||_\infty V_f^-(Q_i) \leq \frac{1}{100} V_f^+(Q_i) \leq \frac{1}{100} V(Q_i).
\]

Since \(Q_i\) is full, we then have

\[
\int_{Q_i} f^+ = \int_{Q_i} |f| - \int_{Q_i} f^- \geq \frac{1}{10} V(Q_i) - \frac{1}{100} V(Q_i) = \frac{9}{100} V(Q_i).
\]
These estimates together imply (11). Finally,
\[ \int_{Q_i} f^+ = \int_{Q_i} |f| - \int_{Q_i} f^- \geq \int_{Q_i} |f| - \frac{1}{9} \int_{Q_i} f^+, \]
which leads to (12).

We are now ready to bound from below both the Hausdorff measure of the zero set and the Vaserstein distance between \( f^+ \) and \( f^- \). That the Hausdorff measure of the zero set cannot be small comes from the fact that the cubes \( \tilde{Q}_i \) are balanced. That the Vaserstein distance between \( f^+ \) and \( f^- \) cannot be small comes from the unbalanced sub-cubes \( Q_i \) of \( \tilde{Q}_i \). We first estimate from below the Hausdorff measure of \( Z(f) \) in each of the full cubes \( \tilde{Q}_i \) of our decomposition.

Proposition 3. We have:
\[ H^{d-1}(Z(f)) \geq \frac{1}{\|f\|_\infty^{(d-1)/d}} \sum_{i \in \mathcal{F}} l(Q_i)^{d-1}. \] (13)

Proof. Observe that although \( Q_i \) is unbalanced, its parent cube \( \tilde{Q}_i \) is balanced, and thus the volumes in \( \tilde{Q}_i \) separated by \( Z(f) \) are comparable, up to a factor \( \|f\|_\infty \). In fact, since
\[ V_f^- (\tilde{Q}_i) \leq 100 \|f\|_\infty V_f^+ (\tilde{Q}_i) \]
and \( \|f\|_\infty \geq 1 \), we deduce from \( V(\tilde{Q}_i) = V_f^+ (\tilde{Q}_i) + V_f^- (\tilde{Q}_i) \) that
\[ V_f^+ (\tilde{Q}_i) \geq \frac{V(\tilde{Q}_i)}{101 \|f\|_\infty} \approx \frac{l(Q_i)^d}{\|f\|_\infty}. \]
Similarly, since \( V_f^- (\tilde{Q}_i) \leq 100 \|f\|_\infty V_f^+ (\tilde{Q}_i) \), we find that
\[ V_f^- (\tilde{Q}_i) \geq \frac{V(\tilde{Q}_i)}{101 \|f\|_\infty} \approx \frac{l(Q_i)^d}{\|f\|_\infty}. \]
Then, by the relative isoperimetric inequality (10),
\[ H^{d-1}(Z(f) \cap \text{int}(\tilde{Q}_i)) \geq \min \left\{ [V_f^+ (\tilde{Q}_i)]^{(d-1)/d}, [V_f^- (\tilde{Q}_i)]^{(d-1)/d} \right\} \]
\[ \geq \frac{l(Q_i)^{d-1}}{\|f\|_\infty^{(d-1)/d}}. \]
Since the cubes \( \tilde{Q}_i \) are disjoint, the estimate follows:
\[ H^{d-1}(Z(f)) \geq \sum_{i \in \mathcal{F}} H^{d-1}(Z(f) \cap \text{int}(\tilde{Q}_i)) \geq \sum_{i \in \mathcal{F}} \frac{l(Q_i)^{d-1}}{\|f\|_\infty^{(d-1)/d}}. \]

Now we are going to estimate the transport realized in each of the unbalanced, full cubes \( Q_i, i \in \mathcal{F} \).
Proposition 4. We have the following estimate of the Vaserstein distance between $f^+$ and $f^-$:

$$W_1(f^+, f^-) \gtrsim \frac{1}{\|f\|_{\infty}} \sum_{i \in F} \frac{(f_{Q_i}, |f|)^2}{l(Q_i)d-1}.$$  

Proof. By definition,

$$W_1(f^+, f^-) = \inf_{\rho} \int_{Q_0 \times Q_0} |x - y| d\rho(x, y),$$

where $\rho$ is a transport plan between $f^+$ and $f^-$, that is $\rho$ is a measure on $Q_0 \times Q_0$ such that for any measurable set $A \subset Q_0$,

$$\int_{A \times Q_0} d\rho(x, y) = \int_A f^+, \quad \int_{Q_0 \times A} d\rho(x, y) = \int_A f^-.$$

We need a uniform lower bound on the transport required for a general plan $\rho$. We have

$$W_1(f^+, f^-) \geq \inf_{\rho} \sum_{i \in F} \int_{Q_i \times Q_0} |x - y| d\rho(x, y)$$

$$\geq \inf_{\rho} \sum_{i \in F} \int_{Q_i \times Q_i^c} |x - y| d\rho(x, y)$$

$$\geq \inf_{\rho} \sum_{i \in F} \int_{Q_i \times Q_i^c} d(x, \partial Q_i) d\rho(x, y).$$

Here, $d(x, \partial Q_i)$ is the distance from $x \in Q_i$ to the boundary of the cube $Q_i$.

We now estimate the transport for each $Q_i$, $i \in F$. Assume $i \in F^+$, the case $i \in F^-$ being completely analogous. Given any transport plan $\rho$, write

$$\int_{Q_i \times Q_i^c} d(x, \partial Q_i) d\rho(x, y) = \int_{Q_i} d(x, \partial Q_i) d\nu(x),$$

where $\nu = \nu_{\rho, i}$ is the measure in $Q_i$ defined by $\nu(A) = \rho(A \times Q_i^c) = \int_{A \times Q_i^c} d\rho(x, y)$, for $A \subset Q_i$.

By definition, $\nu(A) \leq \rho(A \times Q_i) = \int_A f^+$, so $\nu \leq \chi_{Q_i} f^+ dV$. In particular

$$\nu(Q_i) \leq \int_{Q_i} f^+. \quad (15)$$

On the other hand

$$\nu(Q_i) = \rho(Q_i \times Q_i^c) = \rho(Q_i \times Q_0) - \rho(Q_i \times Q_i)$$

$$= \int_{Q_i} f^+ - \rho(Q_i \times Q_i).$$

Since, by (11),

$$\rho(Q_i \times Q_i) \leq \rho(Q_0 \times Q_i) = \int_{Q_i} f^- \leq \frac{1}{9} \int_{Q_i} f^+$$

we deduce, using (12), that

$$\nu(Q_i) \geq \frac{8}{9} \int_{Q_i} f^+ \geq \frac{4}{5} \int_{Q_i} |f|.$$

(16)
Next, writing the integral in terms of the distribution function,
\[
\int_{Q_i} d(x, \partial Q_i) d\nu(x) = \int_0^{l(Q_i)} \nu(\{x \in Q_i : d(x, \partial Q_i) \geq t\}) dt
\]
\[
= l(Q_i) \nu(Q_i) - \int_0^{l(Q_i)} \nu(\{x \in Q_i : d(x, \partial Q_i) < t\}) dt.
\]  \tag{17}

Since \( \nu \leq f^+ x \chi_{Q_i} \, dV \) and \( f^+ \) is bounded, we have that
\[
\nu(\{x \in Q_i : d(x, \partial Q_i) < t\}) \leq \int f^+ x \chi_{Q_i \cap \{d(x, \partial Q_i) < t\}} \leq \|f\|_\infty V(Q_i \cap \{d(x, \partial Q_i) < t\})
\]
\[
\leq \|f\|_\infty t l(Q_i)^{d-1}.
\]

Then, by (15),
\[
\nu(\{x \in Q_i : d(x, \partial Q_i) < t\}) \leq \min \{\nu(Q_i), \|f\|_\infty t l(Q_i)^{d-1}\}.
\]

The crossover point where \( \nu(Q_i) \) dominates being when
\[
t = t_i = \frac{\nu(Q_i)}{\|f\|_\infty l(Q_i)^{d-1}},
\]
we have by (17) that
\[
\int_{Q_i} d(x, \partial Q_i) d\nu(x) \geq l(Q_i) \nu(Q_i) - \int_0^{t_i} \|f\|_\infty t l(Q_i)^{d-1} dt - \int_{t_i}^{l(Q_i)} \nu(Q_i) dt
\]
\[
= \nu(Q_i) t_i - \int_0^{t_i} \|f\|_\infty t l(Q_i)^{d-1} dt
\]
\[
= \frac{1}{2} \|f\|_\infty l(Q_i)^{d-1}.
\]

Going back to (14) and using the estimate (16) gives the estimate
\[
\int_{Q_i \times Q_i} d(x, \partial Q_i) d\rho(x, y) = \int_{Q_i} d(x, \partial Q_i) d\nu(x)
\]
\[
\geq \nu(Q_i)^2 \frac{\|f\|_\infty l(Q_i)^{d-1}}{\|f\|_\infty l(Q_i)^{d-1}} \geq \left(\frac{f_{Q_i}}{l(Q_i)^{d-1}}\right)^2
\]
which finishes the proof of Proposition 4. \(\Box\)

Finally, to conclude the proof of Theorem 1, we use first Proposition 4 and (13) to obtain
\[
W_1(f^+, f^-) H^{d-1}(Z(f)) \geq \frac{1}{\|f\|_\infty^{2-1/d}} \sum_{i \in \mathcal{F}} \left(\frac{f_{Q_i}}{l(Q_i)^{d-1}}\right)^2 \sum_{i \in \mathcal{F}} l(Q_i)^{d-1}
\]
By the Cauchy–Schwarz inequality for sums, applied in the opposite direction to usual, and by Remark 1 the result follows:
\[
W_1(f^+, f^-) H^{d-1}(Z(f)) \geq \frac{1}{\|f\|_\infty^{2-1/d}} \left(\sum_{i \in \mathcal{F}} f_{Q_i}\right)^2 \geq \frac{1}{\|f\|_\infty^{2-1/d}}.
\]
Proof of Theorem 2 (Sketch)
Partition $M$ into a finite number of cells $M^l$, $l = 1, \ldots, N = N(M)$ of similar size, in the sense that there exists $\delta > 0$ so that $\delta \leq \text{diam}(M^l) \leq 2\delta$, $l = 1, \ldots, N$. Assume further that $\delta$ is small enough so that there exist smooth diffeomorphisms

$$\varphi_l : Q_0 \rightarrow M^l, \quad l = 1, \ldots, N,$$

such that the pull-back metric $\varphi_l^* g$ and the Euclidean metric $G_E = \sum_{j=1}^d dx_j \otimes dx_j$ in $Q_0$ are comparable. Moreover, there exists $C = C(\delta) > 0$ such that for any unitary $u \in \mathbb{R}^d$,

$$\frac{1}{C} \leq \frac{\varphi_l^* g(u, u)}{G_E(u, u)} \leq C \quad l = 1, \ldots, N. \quad (18)$$

Informally, $\varphi_l$ is a small perturbation of a dilation of scale $\text{diam}(M_l)$, in the appropriate chart of $M^l$. Since $M$ is smooth and compact, this perturbation is uniformly bounded in $l$.

The dyadic partition $\{Q_j\}_j$ of $Q_0$ induces a ‘dyadic’ partition $\{M^l_j\}_j$ of $M^l$, just by taking $M^l_j := \varphi_l(Q_j)$. Note that, by $(18)$, if $Q_j$ is a cube in the $n$th generation of the dyadic partition of $Q_0$, then

$$\text{diam}(M^l_j) \simeq 2^{-n} \text{diam}(M^l)$$

and

$$V(M^l_j) \simeq 2^{-nd} V(M^l).$$

Now we start the stopping time process explained in Theorem 1 for all cells $M^l$ simultaneously. Here a cell $M^l_i$ is called balanced if

$$\frac{1}{100\|f\|_\infty} \leq \frac{V(M^l_i \cap \{f > 0\})}{V(M^l_i \cap \{f < 0\})} \leq 100\|f\|_\infty.$$

A cell $M^l_i$ is full whenever $\int_{M^l_i} |f| \, dV \geq V(M^l_i)/10$.

From here we follow the same arguments as in the proof of Theorem 1, mutatis mutandis, and taking care to replace $l(Q_l)$ by $\delta^l_i := \text{diam}(M^l_i)$.

Note that in the proof of the analogue of Proposition 4, the crossover point of the two estimates of $\nu(\{x \in M^l_i : d(x, \partial M^l_i) < t\})$ is not exactly $t_i = \frac{\nu(M^l_i)}{\|f\|_\infty l(\delta^l_i)^{d-1}}$, but the lower estimate is still valid with this choice of $t_i$.

An example

Next we show that the exponent $2 - 1/d$ in Theorem 1 cannot be replaced by any power smaller than 1. In particular, Steinerberger’s uncertainty principle $(1)$ in dimension 2 is best possible in this sense.

Proposition 5. Let $\varepsilon > 0$. There is a continuous function $f_\varepsilon : Q_0 \rightarrow \mathbb{R}$ such that

(i) $\|f_\varepsilon\|_\infty \simeq \varepsilon^{-1}$ and $\|f_\varepsilon\|_1 \simeq 1$;
(ii) $Z(f_\varepsilon) = \{x \in [0, 1]^d : x_d = 1/2\}$; hence $H^{d-1}(Z(f_\varepsilon)) = 1$;
(iii) $W_1(f_\varepsilon^+, f_\varepsilon^-) \simeq \varepsilon$.

Thus, the inequality

$$W_1(f_\varepsilon^+, f_\varepsilon^-) H^{d-1}(Z(f_\varepsilon)) \left(\frac{\|f_\varepsilon\|_\infty}{\|f_\varepsilon\|_1}\right)^\alpha \simeq \|f_\varepsilon\|_1,$$

does not hold in general for any exponent $\alpha < 1$. 

Proof. The construction is as follows. Write $x \in \mathbb{R}^d$ as $x = (x_{d-1}, x_d)$, where $x_{d-1} \in \mathbb{R}^{d-1}$. Take the function $f_\varepsilon(x) = h_\varepsilon(x_d)$ where the graph of $h_\varepsilon$ is as in the picture:

![Diagram](image)

Properties (i) and (ii) of the function $f_\varepsilon$ are then immediate.

The function $h_\varepsilon$ is symmetric about $x_d = 0.5$, so that $h_\varepsilon(1 - x_d) = -h_\varepsilon(x_d)$, $x_d \in [0, 1]$. For $x = (x_{d-1}, x_d) \in Q_0$, we write $\tilde{x} = \tilde{x}(x)$ for the point $(x_{d-1}, 1 - x_d) \in Q_0$, the reflection of $x$ in the hyperplane $x_d = 1/2$. Observe that $|x - \tilde{x}| = |1 - 2x_d|$. Then, $f_\varepsilon^+(\tilde{x}) = f_\varepsilon^-(x)$.

To prove the upper bound in (iii), consider the following transport plan $\rho(x, y) = f_\varepsilon^+(x) \delta_\varepsilon(y)$. Note that it has the correct marginals:

$$
\int_{y \in Q_0} d\rho(x, y) = f_\varepsilon^+(x) \int_{y \in Q_0} \delta_\varepsilon(y) = f_\varepsilon^+(x),
$$

$$
\int_{x \in Q_0} d\rho(x, y) = \int_{x \in Q_0} f_\varepsilon^+(x) \delta_\varepsilon(y) = f_\varepsilon^+(\tilde{y}) = f_\varepsilon^-(y).
$$

Therefore,

$$
W_1(f_\varepsilon^+, f_\varepsilon^-) \leq \iint_{Q_0 \times Q_0} |x - y| d\rho(x, y)
= \int_{x \in Q_0} f_\varepsilon^+(x) \int_{y \in Q_0} |x - y| \delta_\varepsilon(y)
= \int_{x \in Q_0} f_\varepsilon^+(x) |x - \tilde{x}| dV(x)
= 2 \int_{x \in Q_0} (x_d - \frac{1}{2}) f_\varepsilon^+(x) dV(x) \lesssim \varepsilon.
$$

For the lower bound, we use the Monge–Kantorovich duality lemma (see (3) or [13, Formula (6.3)]):
We denote by $V$ an interesting problem to determine the best constant in the equality (1).

Taking $g(x) = x_d - \frac{1}{2}$, we have

$$W_1(f_+, f_-) \geq \int_{Q_0} (x_d - \frac{1}{2}) (f_+ - f_-(x)) dV(x)$$

$$= \int_{Q_0} (x_d - \frac{1}{2}) f_+(x) dV(x) \geq \varepsilon. \quad \square$$

With a similar example one can check that in dimension 2, the inequality (1), that is $W_1(f^+, f^-) H^1(Z(f)) \|f\|_\infty \geq C \|f\|_2^2$, cannot hold with a constant $C$ greater than 1. It is an interesting problem to determine the best constant in the equality (1).

**Proof of Theorem 3 on eigenfunctions of the Laplacian**

We denote by $V$ the volume form on $M$ associated to $g$ and normalised so that $V(M) = 1$.

In order to construct a transport plan between $f^+$ and $f^-$, we consider an auxiliary kernel. Let $a : [0, 1] \rightarrow \mathbb{R}$ be a smooth decreasing function such that $a(t) \equiv 1$ in $[0, 1/4]$ and $a(t) \equiv 0$ in $[3/4, 1]$.

Observe that $\phi_0(x) = 1$ and therefore

$$\int_M \phi_i(x) dV(x) = \langle \phi_i, \phi_0 \rangle = 0, \quad i \geq 1. \quad (19)$$

For any $L > 0$, we write

$$B_L(x, y) = \sum_{\lambda_i < L} a(\lambda_i/L) \phi_i(x) \phi_i(y), \quad x, y \in M.$$ 

This is a kernel of Bochner–Riesz type. It is a smoothed out version of the Bergman kernel that gives the orthogonal projection from $L^2(M)$ to the span generated by the first eigenvector of the Laplacian, in the same spirit as the Riesz kernels are a smoothed version of the Dirichlet kernel on trigonometric sums. See [8, 9] for the basic properties of the kernel.

It is proved in [8, Lemma 2.1] that the following pointwise estimates hold: For any $N > 0$, there exists $C_N > 0$ such that

$$|B_L(x, y)| \leq C_N \frac{L^{d/2}}{1 + \sqrt{L} d(x, y)}^N, \quad x, y \in M \quad (20)$$

Now we use a slightly different definition of the Vaserstein distance (see [3, Formula (43)]):

$$W_1(\mu, \nu) = \inf_{\rho} \int_{M \times M} d(x, y) d|\rho|(x,y),$$

where $\rho$ are now signed measures on $M \times M$ with marginals $\rho(\cdot, M) = \mu$, $\rho(M, \cdot) = \nu$. This follows from the estimate of the Vaserstein distance using the dual expression (3):

$$W_1(\mu, \nu) = \sup_{h \in \text{Lip}_1(M)} \left| \int_M h(w) (d\mu(w) dV(w) - d\nu(w) dV(w)) \right|.$$ 

A direct estimate yields, for any signed measure $\rho$ with marginals $\mu$ and $\nu$,

$$W_1(\mu, \nu) = \sup_{h \in \text{Lip}_1(M)} \left| \int_M h(w) \left[ \int_{y \in M} d\rho(w, y) - \int_{x \in M} d\rho(x, w) \right] \right|$$

$$\leq \sup_{h \in \text{Lip}_1(M)} \int_M \int_M |h(x) - h(y)| d|\rho|(x,y)$$

$$\leq \int_M \int_M d(x,y) d|\rho|(x,y).$$

The other inequality is trivial.
Let $\sigma$ be the pushforward of the measure $f^{-} dV$ by the diagonal map $F : M \to M \times M$ defined as $F(x) = (x, x)$, that is $\sigma = F_{*}(f^{-} dV)$. The measure $\sigma$ is supported on the diagonal $\mathcal{D} = \{(x, y) \in M \times M : x = y\}$. Define a signed measure on $M \times M$ by

$$
\rho_{L}(x, y) = B_{L}(x, y) f(x) dV(x) dV(y) + \sigma(x, y).
$$

We compute the marginals of $\rho_{L}$. It is straightforward that both marginals of $\sigma$ are $f^{-} dV$, so we are left with the computation of the marginals of the first term in $\rho_{L}$. Clearly

$$
\int_{y \in M} B_{L}(x, y) f(x) dV(x) dV(y) = f(x) dV(x) \int_{M} B_{L}(x, y) dV(y)
$$

and, by definition and by (19),

$$
\int_{M} B_{L}(x, y) dV(y) = \sum_{\lambda_{i} < L} a \left( \frac{\lambda_{i}}{L} \right) \phi_{i}(x) \int_{M} \phi_{i}(y) dV(y) = \phi_{0}(x) V(M) = 1.
$$

Hence, the marginal of the first term in $\rho_{L}$ with respect to $y \in M$ is $f(x) dV(x)$, and therefore

$$
\int_{y \in M} d\rho_{L}(x, y) = f(x) dV(x) + f^{-}(x) dV(x) = f^{+}(x) dV(x).
$$

For the other marginal, we use the orthogonality of $f$ to all $\phi_{i}$, $\lambda_{i} < L$, (since it is a linear combination of eigenfunctions of $-\Delta$ with eigenvalues $\lambda_{k} \geq L$). Thus,

$$
\int_{x \in M} B_{L}(x, y) f(x) dV(x) dV(y)
$$

$$
= \sum_{\lambda_{i} < L} a \left( \frac{\lambda_{i}}{L} \right) \phi_{i}(y) dV(y) \int_{M} \phi_{i}(x) f(x) dV(x) = 0,
$$

and the second marginal of $\rho_{L}$ reduces to that of $\sigma$, which is $f^{-}(y) dV(y)$.

Now that we have checked that $\rho_{L}$ has the correct marginals let us prove the inequality in the statement of Theorem 3.

Since $\sigma$ is supported on the diagonal, it does not contribute to this last integral. Using (20), we are led to

$$
W_{1}(f^{+}, f^{-}) \lesssim \int_{M} \int_{M} |f(x)| \frac{L^{d/2} d(x, y)}{\left[1 + \sqrt{L} d(x, y)\right]^{N}} dV(x) dV(y)
$$

$$
\leq \frac{\|f\|_{1}}{\sqrt{L}} \sup_{x \in M} \int_{M} \frac{L^{d/2} \sqrt{L} d(x, y)}{\left[1 + \sqrt{L} d(x, y)\right]^{N}} dV(y)
$$

$$
\leq \frac{\|f\|_{1}}{\sqrt{L}} \sup_{x \in M} \int_{M} \frac{L^{d/2}}{\left[1 + \sqrt{L} d(x, y)\right]^{N-1}} dV(y).
$$

We are still free to choose $N$. We pick $N > d + 1$ (the choice $N = d + 2$ works fine) and complete the proof of Theorem 3 by showing that there is a finite constant $C$ independent of $L$ such that

$$
\sup_{x \in M} \int_{M} \frac{L^{d/2}}{\left[1 + \sqrt{L} d(x, y)\right]^{N-1}} dV(y) \leq C.
$$

(21)
Writing the integral in terms of the distribution function and substituting \( t = (1 + \sqrt{L} s)^{-N+1} \), we obtain
\[
\int_M \frac{L^{d/2}}{[1 + \sqrt{L} d(x, y)]^{N-1}} dV(y) = L^{d/2} \int_0^1 V \left( \left\{ y : \left[1 + \sqrt{L} d(x, y) \right]^{-N+1} > t \right\} \right) dt \\
= (N - 1)L^{d/2} \int_0^\infty V(\{ y : d(x, y) < t \}) \frac{\sqrt{L} ds}{(1 + \sqrt{L} s)^N}.
\]
Since \( M \) is compact, the volume of a geodesic ball \( \{ y : d(x, y) < s \} \) is at most a (global) constant times \( s^d \). We deduce, finally, that
\[
\int_M \frac{L^{d/2}}{[1 + \sqrt{L} d(x, y)]^{N-1}} dV(y) \lesssim L^{d/2} \int_0^\infty s^d \frac{\sqrt{L} ds}{(1 + \sqrt{L} s)^N} \\
= \int_0^\infty \frac{u^d du}{(1 + u)^N} \lesssim 1,
\]
which proves (21) and completes the proof of Theorem 3.

Acknowledgements. We are thankful to Gian Maria Dall’Ara for helpful discussions. We also wish to thank the referee for a careful reading of the manuscript and for many thought-provoking suggestions which have resulted in a significant improvement of the text.

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