Fractional analysis of induced resonance effects in the spherical motion of a rigid body with flywheels

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Abstract. The evolution of the motion of an asymmetric rigid body with flywheels can be explored by the methods of asymptotic analysis. It is shown, that the effect of small moments resulting from the displacement of the center of mass and of small control moments from the flywheels lead to the emergence of non-linear resonance evolutionary effects. The aim of this work is to investigate the resonance effects and analyze their influence on the change of the fast phase in the spherical motion of the asymmetrical rigid body with flywheels. To explore these resonance effects we applied the method of integral manifolds and the averaging method. The averaged equations showed that resonance effects can lead to the regular precession, to the formation of stationary points, or to the prolonged resonance. In addition, the characteristic features of the influence of secondary resonance effects on the change in the fast phase are revealed.

1. Introduction
The main difficulties involved in researching resonance phenomena in the perturbed rotation of a rigid body (RB) about a fixed point are the non-linearity of the system of equations of motion and the variability of the frequencies of the system [1-3]. In a number of problems of motion of a RB the method of integral manifolds can be used [4]. This method allows us to simplify the system of equations of motion and reduce its order. In these problems, dissipative external moments serve to attract the paths of motion of the system to the integral manifold [3]. At that, the motion with respect to the integral manifold is described by a single frequency system and nature of the motion of the RB is close to a precession. With the vanishing of a single frequency, a lowest-order resonance is observed in the system, having a significant effect on the change of the system variables. Application of the method of integral manifolds and the method of averaging can substantially simplify the system of equations of motion of the RB. The resulting averaged equations do not contain the fast phase and can be used for the analysis of resonance effects in the motion of the RB close to precession. In a number of well-known works, such as [4], cases of motion of a rigid body close to a regular precession have been examined to obtain the averaged equations of motion of the RB. Among early works on the subject of secondary resonance effects in the problems of motion of the RB, the paper [5-6] is known. It should be noted that the secondary resonance effects in these studies may take occur when changing the resonant frequency ratios in the denominators of the third approximation of the averaging method. The secondary resonant effect as a dynamic phenomenon was first named and described by Yu.A.
Sadov [7]. In a work [8] of the same author it had been shown that secondary resonance effects can occur in multi-frequency systems at the non-resonant cases. It is also known that this phenomenon has been described in the work [9] in the study of a rotational motion of the satellite with a magnetic damper. In [3] the secondary resonance effects have been studied for a case of rotation of the RB with a small asymmetry around a fixed point close to the Lagrange case. In this study, the secondary resonance effects were determined by the influence of a small moment resulting from a continuous displacement of the center of mass relative to the axis of dynamic symmetry and a small disturbing moment that is constant in the body-fixed coordinate system. From a mathematical point of view, the secondary resonance effects are explained by influence of the resonant frequency ratios contained in the denominators of higher approximations of averaged equations on the evolution of the slow variables of the system. One of the most obvious secondary resonance effects is the increase of the velocity of change of the slow variables when the resonant frequency ratio is close to zero value, to which the resonance corresponds. In practice, the secondary resonance effects can cause evolution of the variables of the system to the values of angular velocity, at which the prolonged resonance occurs. In particular, the mentioned effects were also examined in detail with regard to the problem of motion of the asymmetric RB in a resistant medium [9-10] and in a study of the disturbed rotational motion of the asymmetric RB with a strong magnet in the geomagnetic field [11].

The aim of this work is to investigate the secondary resonance effects and analyze their influence on the change of the fast phase in the spherical motion of the asymmetrical rigid body with flywheels. Applying the method of integral manifolds and obtaining second approximations of the averaging method, it is possible to explore these resonance effects in detail. It is shown that control of the flywheel moments can lead to the non-resonant evolution of variables of the system to the resonance values [10]. In addition, control of the flywheel moments can contribute to the slow transition of the body rotation to a mode of motion close to the regular precession.

2. Problem statement

Let us assume that a rigid body has a displacement of the center of mass relative to the origin $O$ of the RB-fixed coordinate system $OXYZ$: $\Delta \mathbf{r} = \Delta x \mathbf{i} + \Delta y \mathbf{j} + \Delta z \mathbf{k}$. It is expected that the following limitations are applied to the values of moments of inertia of the RB: $I = I_x = I_y > I_z$ and $I_{xy} = I_{yz} = I_{xz} = 0$. Since the movement of the RB about the $O$ point is considered to be close to the classical Lagrange case, therefore the values of the axial displacement of the center of mass are limited to: $\Delta z \gg \sqrt{\Delta x^2 + \Delta y^2}$. Analysis of the equations of the motion of the RB shows that in the following, without loss of generality of the results, it can be taken that $\Delta x = 0$ (with $\Delta y \neq 0$ and $\Delta z \neq 0$). It is assumed that the RB moving in a body-fixed coordinate system $OXYZ$ is under the influence of disturbing moments of three flywheels mounted on the RB so that their axes of rotation coincide with the axes $X, Y, Z$. The position of the coordinate system $OXYZ$ relative to the space-fixed coordinate system $OX_1Y_1Z_1$ is determined by means of three Euler angles: the nutation angle $\theta$, the precession angle $\psi$ and the angle of proper rotation $\phi$. The axis $OZ_1$ of the space-fixed coordinate system $OX_1Y_1Z_1$ is vertical in the case of spherical motion of the RB. The rotation of the RB is carried out in the neighborhood of the statically stable equilibrium $\theta=0$. The positional relationship of the coordinate systems $OX_1Y_1Z_1$, $OXYZ$ and the RB with flywheels is shown in Figure 1.

3. Mathematical model

The dynamic Euler equations in the case of rotational motion of the RB with flywheels about the fixed point $O$ (for $T_z = I_z / I$, $I_x = I_y = I$, $I_{xy} = I_{yz} = I_{xz} = 0$) are written as follows [3]:

\[
\frac{d\omega_x}{dt} - (1 - I_z)\omega_x \omega_z = \frac{M_x}{I},
\]

\[
\frac{d\omega_y}{dt} + (1 - I_z)\omega_y \omega_z = \frac{M_y}{I},
\]
Here $\omega_x, \omega_y, \omega_z$ are angular velocities of the RB, $M_x = \Delta y G_z - \Delta x G_y \cos \varphi + \Delta M_x + m^w \omega_x$, $M_z = G \cos \theta$, $G_a = G \sin \theta$, $G = \frac{m_b g}{I}$, $M_y = \Delta z G_x \sin \varphi + \Delta M_y + m^w \omega_y$, $M_z = -\Delta y G_z \sin \varphi + \Delta M_y + m^w \omega_z$, $m_b$ is the mass of the RB with flywheels, $g$ is the acceleration of gravity, $\Delta M_x, \Delta M_y, \Delta M_z$ are disturbing moments of flywheels acting in the direction of axes $X, Y, Z$; $m^w$ is a small coefficient of the damping moment.

The initial system of equations of motion of the body about a fixed point (1)-(6) is written in a form suitable for the application of the method of integral manifolds [3]:

$$\begin{align*}
\frac{d\theta}{dt} &= \omega_x \cos \varphi - \omega_y \sin \varphi, \\
\frac{d\psi}{dt} &= \omega_x \sin \varphi + \omega_y \cos \varphi, \\
\frac{d\phi}{dt} &= \omega_z - (\omega_x \sin \varphi + \omega_y \cos \varphi) \cot \theta.
\end{align*}$$

Figure 1. Positional relationship of the coordinate systems $O_XY Z_1$ and $OXYZ$.

$G_a = G \sin \theta$, $G = \frac{m_b g}{I}$, $M_y = \Delta z G_x \sin \varphi + \Delta M_y + m^w \omega_y$, $M_z = -\Delta y G_z \sin \varphi + \Delta M_y + m^w \omega_z$, $m_b$ is the mass of the RB with flywheels, $g$ is the acceleration of gravity, $\Delta M_x, \Delta M_y, \Delta M_z$ are disturbing moments of flywheels acting in the direction of axes $X, Y, Z$; $m^w$ is a small coefficient of the damping moment.

The initial system of equations of motion of the body about a fixed point (1)-(6) is written in a form suitable for the application of the method of integral manifolds [3]:

$$\begin{align*}
\frac{du}{dt} &= \varepsilon U(u, \varphi, \theta, \frac{d\theta}{dt}), \\
\frac{d\varphi}{dt} &= \varphi U(u, \theta), \\
\frac{d^2\theta}{dt^2} + F(u, \theta) &= \varepsilon [f_1(u, \varphi, \theta) + f_2(u, \theta) \frac{d\theta}{dt}] .
\end{align*}$$

Here $\varepsilon > 0$ is the small parameter, $u = (Q, \omega_z)$ is the vector of the slow variables, $Q$ is the projection of the angular moment of the RB on the axis $OZ_1$, assigned to the moment of inertia $I$, $\omega_z$ is the angular velocity of the RB relative to the axis $OZ_1$; $\varphi, \theta$ are fast variables; $\varepsilon U = (\varepsilon U_1, \varepsilon U_2)$, $\varepsilon U_1 = M_z I_z$, $\varepsilon U_2 = M_z I_z$, $F(u, \theta) = -M^0_{\varphi z} - F_1(u, \theta) F_2(u, \theta)$, $M_z = M \sin \varphi \sin \theta + M_y \cos \varphi \sin \theta + M_z \cos \theta$, $M_x = G \Delta y \cos \theta - G \Delta z \sin \theta + \Delta M_x + m^w \omega_x$, $M_y = G \Delta z \sin \theta \sin \varphi + \Delta M_y + m^w \omega_y$, $M_z = G \Delta y \cos \theta$. 

\[ \begin{align*}
\frac{d\theta}{dt} &= \omega_x \cos \varphi - \omega_y \sin \varphi, \\
\frac{d\psi}{dt} &= \omega_x \sin \varphi + \omega_y \cos \varphi, \\
\frac{d\phi}{dt} &= \omega_z - (\omega_x \sin \varphi + \omega_y \cos \varphi) \cot \theta.
\end{align*} \]
\[ M_z = -G\Delta y \sin \theta \sin \varphi + \Delta M_z, \quad M_{x0} = M_{x0}^0 + \Delta M_{x0}, \quad M_{y0} = -G\Delta z \sin \theta, \]
\[ \Delta M_{x0} = G\Delta y \cos \theta \cos \varphi + \Delta M_x \cos \varphi - \Delta M_y \sin \varphi + m^o \frac{d\theta}{dt}, \quad \Phi(u, \theta) = \omega_z - F_1(u, \theta) \cos \theta, \]
\[ F_1(u, \theta) = \left( Q - I_z \omega_z \cos \theta \right) / \sin^2 \theta, \quad F_1(u, \theta) = \left( Q \cos \theta - I_z \omega_z \right) / \sin \theta, \]
\[ \varepsilon f_1 = G\Delta y \cos \theta \cos \varphi + \Delta M_x \cos \varphi - \Delta M_y \sin \varphi, \quad \varepsilon f_2 = m^o. \]

The equation \( \frac{d\psi}{dt} = \frac{Q - T_z \omega_z \cos \theta}{\sin^2 \theta} \) does not depend on the angle \( \psi \). Therefore, it can be integrated separately. The small parameter \( \varepsilon \) in the equations (7) represents the value of displacement from the center of mass \( \Delta y \) as well as values of disturbing moments \( \Delta M_x, \Delta M_y, \Delta M_z \), and the value of the coefficient of damping moments \( m^o \). For the existence of an integral manifold, a number of conditions [12] must be fulfilled.

The application of the method of integral manifolds allows reducing the order of the system (7). The result is a system with a fast phase \( \varphi \), describing the motion of the RB with flywheels to an accuracy of the terms \( O(\varepsilon) \) with respect to the integral manifold:
\[ \frac{d\lambda}{dt} = \varepsilon \Lambda (\lambda, \varphi), \]
\[ \frac{d\varphi}{d\tau} = \Delta (\lambda) + \varepsilon \Omega (\lambda, \varphi). \] (8)

Here \( \lambda = (\omega_z, \theta) \) is the vector of slow variables \( \omega_z, \theta; \Lambda = (\Lambda_1, \Lambda_2), \)
\[ \Lambda_z = [\Delta y \sin \theta \sin \varphi + \Delta M_z + m^o \omega_z] / I_z, \quad \Lambda_z = -m^4 \cos(\varphi + \varphi_A) + m^o \omega_{z2} \sin \theta \left[ 3 \omega_{z2} - (1 + T_z) \omega_z \right] / F^{(0)} \]
\[ T_z = I_z / I; \quad F^{(0)} = \partial \bar{F} / \partial \theta; \quad \Omega (\lambda, \varphi) = \pm \frac{m^o \cot \theta}{21 \omega_0} \cos(\varphi + \varphi_B); \quad m^4, m^B \] are parameters, that characterize the magnitudes of the moments of the mass asymmetry and the magnitudes of flywheels moments; \( \varphi_A, \varphi_B \) are parameters, that determine the relative position of the moments of the mass asymmetry and the magnitudes of flywheels moments; \( m^A, m^B \) are parameters, that determine the relative position of the moments of the mass asymmetry and the magnitudes of flywheels moments; \( m^4 = \sqrt{(m^4_1)^2 + (m^4_2)^2} \),
\[ m^4_1 = \Delta M_{z2} [(1 + T_z) \omega_z - 3 \omega_{z2}] / IF^{(0)}, \quad m^4_2 = \frac{\Delta y G \cos \theta + \Delta M_x}{IF^{(0)}} [(1 + T_z) \omega_z - 3 \omega_{z2}] - \frac{\Delta y G \sin \theta \omega_{z2}}{IF^{(0)}}, \]
\[ \cos \varphi_A = -m^4_1 / m^4, \quad \sin \varphi_A = m^4_2 / m^4; \quad m^B = \sqrt{(m^B_1)^2 + (m^B_2)^2}, \quad m^B_1 = \Delta y G \cos \theta + \Delta M_x, \quad m^B_2 = \Delta M_y, \]
\[ \cos \varphi_B = m^B_1 / m^B, \quad \sin \varphi_B = m^B_2 / m^B. \]

The control moments from the flywheels are set as follows:
\[ \Delta M_j = k_i / \Delta^2 + k_j, \] (9)

In (9) the indices \( j \) are the following: \( j = x, y, z \). Here \( k_i, k_{2j} \) are small positive control coefficients;
\[ \Delta = \omega_z - \omega_{z2} \] is the resonant frequency ratio; \( \omega_{z2} = T_z \omega_z / 2 \pm \omega_b; \quad \omega_b = \sqrt{T_z^2 \omega_z^2 + \omega^2}; \quad \omega^2 = \frac{G\Delta z \cos \theta}{I}. \]

The equations of the system (8) contain a periodic dependence on the fast variable \( \varphi \) in their right sides, resulting in a substantial complication of the study of evolutions in the system. Using the known method of averaging equations in the non-resonance case [13] we obtain the equations for the averaged variable \( \bar{\omega}_z \) and \( \bar{\varphi} \) that do not depend on the fast phase \( \varphi \). Further, these averaged variables will be denoted as follows: \( \bar{\omega}_z, \bar{\varphi} \). Averaged equations for the slow variables \( \omega_z, \theta \) can be written taking into account the first two approximations as follows:
\[
\begin{align*}
\left\{ \frac{d\omega_z}{dt} \right\} &= \epsilon m^a \omega_z / I_z + \epsilon^2 \frac{G \Delta y m^a \omega^3 \cos \theta \cos \varphi_{\perp}}{2I_z \Delta} + \\
&+ \epsilon^2 \frac{G^2 \Delta y m^a \omega^2 \Delta \zeta \sin^2 \theta \cos \varphi_{\perp}}{4I_z I_{\omega\Delta}^2} \pm \epsilon^2 \frac{G \Delta y m^b \omega^3 \cos \theta \sin \varphi_{\perp}}{4I_z I_{\omega\Delta}^2}, \\
\left\{ \frac{d\theta}{dt} \right\} &= -\epsilon \left( \frac{\bar{m}^B}{2\Delta} \right)^2 \cdots \pm \epsilon \frac{G \Delta y m^a \omega^4 \eta \delta \varphi_{\perp}}{4I_{\omega\Delta}^2} + \\
&+ \epsilon \frac{G \Delta y m^a \omega^4 \delta \varphi_{\perp} \cos \varphi_{\perp} - \varphi_{\perp}}{4I_{\omega\Delta}^2}.
\end{align*}
\]  

(10)

where \( \bar{m}^A = m^A / \omega^2, \bar{m}^B = m^B / \omega^2 \) are the dimensionless parameters.

In the equations (10)-(11) it is assumed that the disturbance moment caused by the flywheel with a rotational axis coinciding with the axis \( OZ \) is equal to zero: \( \Delta M_z = 0 \). Implementation of the equality \( \omega-z_1 = \omega-z_2 = 0 \) results in the occurrence of a lower order resonance. By solving the equation \( \omega-z_1 = \omega-z_2 = 0 \), the resonance value of the angular velocity is found:

\[
\omega^* = \pm \omega / \sqrt{1-\overline{I}_z}.
\]  

(12)

4. Analysis of resonant effects

A characteristic feature of the averaged equations (10)-(11) is the fact that they contain the resonance frequency ratios \( \Delta = \omega-z_1 - \omega-z_2 \) in the denominators of the second approximation. It is known [6-9] that a slow change of these resonance ratios in a non-small neighborhood of a resonance \( \Delta = 0 \) may lead to evolutionary phenomena called secondary resonance effects. We investigate these phenomena in a slow change of the perturbed rotation of an asymmetric RB with the flywheels. In order to achieve the secondary resonance effects, the first approximations of the averaging method are required to be equal to zero: \( A_1^0 = m^a \omega_z / I_z = 0 \) and \( A_1^0 = 0 \). The results of numerical simulation show that the value of the first approximation \( A_1^0 = m^a \omega_z / I_z \) has no significant impact on the occurrence of the secondary resonance effects at small \( m^a \) and small initial angular velocities \( \omega_z(0) \). In addition, the condition \( A_1^0 = 0 \) follows from the equation (11). By analogy with [11] it can be shown that the occurrence of secondary resonance effects suggests the fulfillment of the condition \( \sqrt{\epsilon} < |\Delta| < 1/ \epsilon \).

The analysis of the evolution of the slow variables of the system (10)-(11) makes it possible to identify three specific results obtained by using the control law (9) in the considered problem.

The first typical resonance effect reveals itself as follows: on approach of the angular velocity \( \omega_z \) to the resonance values \( \omega^* \) in the areas of non-resonant motion there is an increase of absolute values of rates of change of averaged variables \( \omega_z \) and \( \theta \). This phenomenon is explained by a decrease of absolute values of the ratios \( \Delta \) in the denominators of the equations (10)-(11). The first typical resonance effect has been studied in [3] with regard to the problem of motion of the RB \( \Delta M_x, \Delta M_y, \Delta M_z \) about a fixed point with a small asymmetry and small constant external moment in a case close to the Lagrange case. In the considered formulation of the problem in accordance with the expression (9) it is assumed that at the gradual decrease of the absolute value of frequency ratio \( \Delta \) a decrease in the magnitude of the moment \( \Delta M_x, \Delta M_y, \Delta M_z \) is observed. At the same time, two types of evolutions of the rotary motion of the RB occur. Firstly, if variation of the values \( \Delta M_x, \Delta M_y, \Delta M_z \) is not significant, the evolution \( \omega_z \) to the resonance values \( \omega^* \) is observed, accompanied by an increase in
absolute values of the averaged derivatives (10) and (11). In fact, this case is little different from the first characteristic resonance effect. Secondly, if within the considered time interval of the RB movement there is a decrease of the values $\Delta M_x, \Delta M_y$ which does not lead to the achievement of resonant values of the angular velocity $\omega_z$, then the derivatives (10) and (11) at small nutation angles may evolve to zero. Indeed, from the equations (10)-(11) it also follows that in the case of small nutation angles $\theta$ at $A_0^\infty \approx 0$, $A_0^\infty = 0$, $\Delta M_z = \Delta M_y = 0$, $\Delta M_x = -\Delta y G \cos \theta$ to the terms of the second order of smallness inclusively we get: $\langle d\omega_z / dt \rangle = 0$, $\langle d\theta / dt \rangle = 0$. The angular velocity $\omega_v = \omega_{z,2} / \cos \theta$ in this case takes the constant value and the RB performs the spherical movement close to the regular precession. This is the second typical resonance effect of a behavior of the system (10)-(11).

It is known [9] that the presence in the denominators of the right parts $\langle d\omega_z / dt \rangle$ and $\langle d\theta / dt \rangle$ of frequency ratios $\Delta$ and $\Delta^2$ can lead to the formation of stationary points, in which the averaged variables $\omega_z, \theta$ evolve, in a neighborhood of the resonance values. Third secondary resonance effect is observed both at constant values $\Delta M_x, \Delta M_y, \Delta M_z$ and at values of the disturbing moments changing in accordance with the expression (9). From the equations (10)-(11) it follows that this effect occurs only at significant value of the nutation angle $\theta$. Indeed, for small $\theta$ in equations (10)-(11) the values of terms proportional to $1/\Delta^2$ are at least one order of smallness higher compared to value of the terms contained in the denominators $\Delta$. As a result, this resonance effect does not occur at low angles of nutation. In contrast, at considerable angles $\theta$ and decreasing values of $\Delta M_x, \Delta M_y$, the resonant effect can lead to an occurrence of stationary (in average) values of the variables $\omega_z, \theta$, observed in the divergent time points of the RB movement (third typical resonance effect). In this case, the motion of the RB will be close to regular precession. The study of the evolution of the averaged variables $\omega_z$ and $\theta$ with decreasing values $\Delta M_x, \Delta M_y$ shows that the formation of stationary points in the third typical resonance effect is less common compared to regular precession in the second resonance effect.

5. Analysis the change of the fast phase
The noted secondary resonance effects characterize the influence of the main resonance $\omega_z - \omega_{z,2} = 0$ on the evolution of slow variables $\omega_z$ and $\theta$ in the considered neighborhood of the resonance zone of order $\sqrt{\epsilon} < |\Delta| < 1/\epsilon$. Let us consider the evolution of the averaged fast phase $\phi$ in the cases of the realization of the indicated types of secondary resonance effects. From the analysis of three secondary resonance effects, it follows that we receive three characteristic cases of the evolution of the fast phase $\phi$.

Case 1. At the first secondary resonance effect, there is a simultaneous change in the slow variables $\omega_z$ and $\theta$. The magnitude of the frequency ratio $\Delta = \omega_z - \omega_{z,2}$ depends only on the change in these slow variables. Thus, with the first secondary resonance effect, we should expect the greatest increase in the rate of change of the fast phase $\phi$ as the angular velocity $\omega_z$ approaches to the resonance value.

Case 2. On the contrary, we observe stabilization of the slow variables in the second secondary resonance effect. In this case, the stabilization of the frequency ratio $\Delta = \omega_z - \omega_{z,2} = a \neq 0$. In this case, the fast phase changes according to a linear function $\phi = \phi_0 + at$. Here $a$ is the derivation of the fast phase at $t = 0$.

Case 3. When implementing the third secondary resonance effect, we observe the stabilization of only one of the slow variables $\omega_z$ and $\theta$. In this case, there is no stabilization of the frequency ratio
\[ \Delta = \omega_z - \omega_{z,2} \] (as in the second case). Thus, the fast phase does not change according to a linear function. It should be noted that the modulus of the rate of change of the fast phase \( \varphi \) in the third case is somewhat less compared with the first resonant effect.

### 6. Numerical results

In the numerical simulation of the behavior of the slow variables \( \omega_z, \theta \) in the case of non-resonant motion the original equations (1)-(6), non-linear "low-frequency" equations (7), averaged equations (10)-(11) and the expression for the resonant values of the angular velocity (12) were used. Some results of numerical simulation of resonance effects are shown in Figures 2-5. The values of the mass-geometrical parameters of a rigid body with flywheels, the initial conditions of integration used in drawing up of Figures 2-5 are as follows: \( \Delta x = 0, \Delta y = 10^{-3} m, \Delta z = 0.05 m, m_b = 0.1 kg, I_x = I_y = 2.5 \cdot 10^{-4} kgm^2, I_z = 5 \cdot 10^{-5} kgm^2, \omega_z(0) = 0.25 s^{-1}, \theta(0) = \pi / 20 rad, \varphi(0) = \pi / 6 rad \). In Figures 2-3, the coefficients in (13) have the following values: \( k_{1j} = 0, k_{2j} = 10^{-4} Nm \).

Figure 2 illustrates the first typical resonance effect. Here, the resonance effect leads to a non-resonant increase in the absolute value of derivative of the angular velocity \( \omega_z \) (the two upper curves) on approaching the resonant value \( \omega_{z,1} \) (the lower curve). Thickened upper curve in Figure 2 is drawn by means of a numerical integration of the system (7). To construct the thin upper curve in Figure 2, the averaged equation (10) was used. In addition, Figures 2-3 characterizes the first case of the influence of the secondary resonance effect on the change in the fast phase.

From Figure 3 it follows that in the first typical resonance effect there is a steady increase of the average value of the nutation angle from a low initial value to a value of 90 degrees, where the latter is achieved in a near-resonant area of motion. When building the curve \( \theta(t) \) in Fig. 3, the averaged equation (11) was used. As can be seen in Figures 4-5 the resonance effect leads to a simultaneous approximation of the average variables \( \omega_z, \theta \) to steady-state values.

The motion of a rigid body in the said steady state is close to the regular precession (the second typical effect). In contrast to previous Figures, in Figures 4-5 the control coefficients have the following values: \( k_{1j} = 10^{-3} kgm^2, k_{2j} = 0 \). The upper curve in Figure 4 describes the evolution of the angular velocity \( \omega_z \). The lower curve shows value of the resonance angular velocity \( \omega_{z,1} \). Note that Figures 4-5 characterize the second case of the influence of the secondary resonance effect on the change in the fast phase.

![Figure 2](image_url). Evolution of angular velocity upon reaching the resonance value.
Figure 3. Slow increase of the nutation angle to $\pi/2$ radians.

Figure 4. Evolution of the angular velocity during transition to the steady-state value.

Figure 5. Evolution of the nutation angle during the transition to the steady-state value.
7. Conclusion
In general, the first typical resonance effect can lead to the non-resonant evolution of variables of the system to the resonance values. The second and third typical resonance effects can contribute to the slow transition of the body rotation to a mode of motion close to a regular precession. The numerical results illustrate the behavior of the slow variables of a dynamical system during the realization of the considered resonance effects. Among the three considered resonant effects the first typical resonance effect has the most significant impact on the motion of a RB about a fixed point. From a theoretical point of view, the secondary resonance effects characterize the peculiarities of the evolution of the slow variables of the system (of the average values of the angular velocity and the nutation angle) induced by the resonance \( \Delta = \omega_z - \omega_z' = 0 \) in the areas of non-resonant motion of the RB with flywheels. Results of the study of the resonance effects in the problem of spherical motion of a rigid body have not only theoretical, but some practical value. A small displacement of the center of mass of a rigid body is a design feature that can occur during the operation of the actual gyroscope or spacecraft. It has been shown that the installation in the RB of flywheels acting on the body through small moments (9) results in evolution of the variables of the system to the point where a resonance is achieved. Furthermore, if the occurrence of a resonance mode is an invalid mode of motion of the RB, that the choice of coefficients in the control law (9) may stabilize the average values of the angular velocity and the nutation angle. Note, that the paper contains results on the study of the influence of secondary resonance effects on the non-resonant change in the fast phase.

To sum up, it should be noted that resonance phenomena continue to find widespread use in modern technology. For this reason, it is appropriate to continue studies of new aspects related to control of the spherical motion of a rigid body subject to a variety of resonance phenomena.

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