SPECTRA OF LATTICE DIRAC OPERATORS
IN NON-TRIVIAL TOPOLOGY BACKGROUNDS

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Abstract. Dirac operators in non-trivial topology backgrounds in a finite box are reviewed. We analyze how the formalism translates to the lattice, with special emphasis on uniform field backgrounds.

I. Introduction

Most of the numerical work performed within Lattice Gauge Theory makes use of periodic boundary conditions. This has the advantage of preserving translational invariance and homogeneity of the lattice. When the box size becomes large in physical units, correlation functions become independent of this size and boundary conditions. For smaller boxes and/or other observables the effect of the boundary conditions is non-negligible. This fact has to be taken into account when comparing lattice results with the continuum.

Topological aspects of the gauge fields are directly related to boundary conditions. There are well-known difficulties in extending topological notions to the lattice. For example, the space of three-dimensional (3D) non-abelian gauge transformations (or of 4D gauge fields) is connected on the lattice and disconnected on the continuum. In any case, it is more appropriate to relate topological properties of the lattice to those of continuum fields defined on the torus. For example, in trying to study individual instantons on the lattice there are, not only order a corrections, but also finite size corrections whose origin is in the continuum: there are no 4D self-dual solutions of topological charge 1 on the torus (without twist).

The topological properties of gauge fields are directly connected with the spectrum of the Dirac operator through index theorems. Within Lattice Gauge Theories important progress in this respect has arisen lately from
considerations of Ginsparg-Wilson symmetry\cite{1}, domain-wall fermions\cite{2} and the overlap method\cite{3}. It is therefore important to re-examine the study of the spectrum of lattice Dirac operators for non-trivial topology gauge field backgrounds. According to our previous reasoning we should consider the case of gauge fields in a box. In a recent paper\cite{4} the case of uniform (constant) field strength gauge field configurations in 2 and 4 dimensions has been analyzed in detail (See Ref. \cite{5} for an early work in this context). In this talk I will explain and develop certain aspects with greater depth. For simplicity and lack of space I will mostly concentrate in the two-dimensional case. We refer the reader to Ref. \cite{4} for other aspects not covered in this paper and a more complete list of references.

II. Two dimensions: Continuum

Charged matter fields on a 2-torus are sections of a U(1) bundle. Without sacrificing rigor we can view them as complex functions $\psi(x)$ defined on the plane and periodic up to gauge transformations:

$$
\psi(x + e_1) = \Omega_1(x) \psi(x) \\
\psi(x + e_2) = \Omega_2(x) \psi(x)
$$

(1)

where $e_1 = (l_1, 0)$ and $e_2 = (0, l_2)$ and $l_i$ are the torus lengths. The U(1) fields $\Omega_i(x) = \exp\{i \omega_i(x)\}$ are the transition functions, and must satisfy the following consistency conditions:

$$
(\omega_1(x + e_2) - \omega_1(x)) - (\omega_2(x + e_1) - \omega_2(x)) = 2\pi q .
$$

(2)

where $q$ is an integer characterizing the topology of the bundle. In mathematical terms this is the first Chern number of the U(1) bundle. The physical interpretation of this integer can be deduced by considering abelian gauge potentials on this torus. They must satisfy:

$$
A_j(x + e_i) = A_j(x) + \partial_j \omega_i(x)
$$

(3)

Now the total flux of the magnetic field $B = \epsilon_{ij} \partial_i A_j$ is given by

$$
\int dx_1 dx_2 B = \int dx_2 \partial_2 \omega_1(x) - \int dx_1 \partial_1 \omega_2(x) = 2\pi q
$$

(4)

The explicit form of $\omega_i(x)$ is a matter of gauge choice. A convenient widely used choice is: $\omega_i(x) = \pi q \epsilon_{ij} x_j / l_j$. A particular gauge field satisfying these boundary conditions is $A_i^{(0)}(x) = - \frac{F}{2\pi} \epsilon_{ij} x_j$, where $F = \frac{2\pi q}{A}$ and $A = l_1 l_2$ is the area of the torus. The corresponding magnetic field strength is constant and equal to $F$. On a torus the field strength is not the only gauge invariant
quantity, one has also the Polyakov lines winding around non-contractible loops. Hence, given a pair of real constants \( v_i \) (defined modulo \( 2\pi/l_i \)), one can construct a whole family of gauge inequivalent gauge potentials \( A_i^{(0)}(x) = A_i^{(0)}(x) + v_i \) having the same constant field strength \( F \). The fields \( A_i^{(0)} \) play an important role in parameterizing the space of gauge fields compatible with the boundary conditions. Indeed, one can make use of Hodge theorem to write an arbitrary gauge field as:

\[
A_i = A_i^{(0)} + v_i + \partial_i \phi + \epsilon_{ij} \partial_j h
\]  

where both \( \phi(x) \) and \( h(x) \) are real periodic functions on the torus. We see that \( v_i \) parametrises the harmonic 1-forms on the torus. Setting \( \phi = 0 \) we have the expression of the gauge field in Coulomb gauge.

Coming back to matter fields, let us denote by \( \mathcal{H} \) the space of functions defined on \( \mathbb{R}^2 \) satisfying the boundary conditions Eq. (1). This has the structure of a Hilbert space. Therefore, given a complete set of commuting operators one obtains an explicit representation of this space. Covariant derivatives \( D_i = \partial_i - iA_i \) with respect to compatible gauge fields are anti-hermitian operators defined on \( \mathcal{H} \). A natural basis is obtained by selecting the covariant derivatives with respect to the uniform field strength potentials:

\[
D_i^{(0)} = \partial_i - iA_i^{(0)}
\]  

They define a Heisenberg algebra:

\[
[D_1^{(0)}, D_2^{(0)}] = -iF
\]  

One can choose (to diagonalize) one the two operators to provide a natural representation of the space \( \mathcal{H} \). However, for \( |q| \neq 1 \) this operator alone does not provide a complete set. There is an additional pair of operators \( K^{(i)} \) commuting with \( D_i^{(0)} \) and satisfying:

\[
K^{(1)} K^{(2)} = \exp\left\{ \frac{2\pi i}{q} \right\} K^{(2)} K^{(1)}
\]  

defining a Heisenberg group. The boundary conditions imply \( (K^{(i)})^q = I \) and the space decomposes into a direct sum of \( q \) subspaces. The presence of these \( q \) spaces corresponds to the finite degeneracy of Landau levels. In infinite space \( (q \to \infty) \) the degeneracy becomes infinite and the relation (8) leads to a new Heisenberg algebra relation. This is the quantum version of the canonical relations for the Hamiltonian system of a particle in a plane in an uniform magnetic field \( (P_i^\pm = p_i \pm \frac{F}{2} \epsilon_{ij} x_j \) define the two commuting Poisson bracket algebras).
Our considerations lead to the following parametrization for an arbitrary element of $H$:

$$\psi(x_1, x_2) = \exp\left\{i\pi q \frac{x_1 x_2}{A}\right\} \sum_{n=1}^{q} \sum_{s \in \mathbb{Z}} \exp\left\{2\pi i \frac{x_1}{l_1} (n + sq)\right\} h_n(x_2 + \frac{n + qs}{q} l_2)$$

(9)

where $h_n(y)$ are $q$ arbitrary functions of a single real variable $y$. This yields a dimensionally reduced description of our Hilbert space. The price to pay is that, in general, differential operators acting on $\psi(x)$ map onto integro-differential operators acting on $h_n(y)$. However, covariant derivatives with respect to gauge potentials depending only on the variable $x_1$ preserve their differential operator character.

As a particular case one can consider the Dirac operator $D$ in the background of a uniform magnetic field. The spectrum consists on $q$-dimensional spaces of eigenvectors corresponding to the eigenvalues 0 and $\pm i\sqrt{2}p$, where $p$ is a positive integer. We see that the eigenvalues only depend on the area of the torus and not on its shape. The corresponding eigenfunctions in the $h_n$ representation are also universal, given in terms of harmonic oscillator eigenfunctions. Notice, however, that shape parameters enter, through Eq. 9, in expressing them in the standard basis. All these results hold for any value of $v_i$ since (for $q \neq 0$) covariant derivatives $\partial_i - iA_i^{(0)}v_i(x)$ are unitarily equivalent.

### III. Two dimensions: Lattice

Now let us examine the situation on a finite $L_1 \times L_2$ lattice. Transition functions do not appear in this formulation, so it seems at first unclear how one can make contact with the continuum formulation. However, if we consider compact U(1) lattice gauge fields in the standard way ($U_i(n) = \exp\{-iA_i\}$), it turns out that a unique decomposition similar to Eq. 5 holds on the lattice:

$$A_i = A_i^{(0)} + v_i + \Delta^+_i \phi + \epsilon_{ij} \Delta^-_j h \mod 2\pi \mathbb{Z}$$

(10)

where $\Delta^\pm$ are the forward/backward lattice derivatives, and $v_i$ are constants defined modulo $2\pi/L_i$. The functions $\phi$ and $h$ are periodic on the lattice and uniquely defined up to an additive constant. Finally $A_i^{(0)}$ is the lattice version of the constant field strength fields:

$$A_i^{(0)} = -\frac{F}{2} \epsilon_{ij} I_j(n)$$

(11)

where $I_1(n) = n_1$ for $1 \leq n_2 < L_2$ and $I_1(n_1, L_2) = (L_2 + 1)n_1$, and a similar definition for $I_2$. The lattice constant field strength $F = \frac{2\pi q}{A}$ with
\( A = L_1 L_2 \), contains information of the topological sector \( q \). Apparently we have managed to uniquely define a topology of the lattice fields. The problem is that the decomposition Eq. [10] is singular for fields having any plaquette on the lattice equal to \(-1\).

The lattice constant field \( A_i^{(0)} \) is periodic. Replacing \( I_i \) by \( n_i \) would lead to the naive non-periodic lattice discretization of the continuum field. The difference can be interpreted by considering that the lattice gauge potential includes the transition function \( w_i(n) = \pi q \epsilon_{ij} n_j / L_j \).

Lattice fermion fields \( \psi(n) \) are elements of a complex vector space of dimension \( 2A \). Usually one views these fields as being periodic on the lattice. However, fermion fields always appear in the action coupled to the gauge potential through covariant derivatives. Given the decomposition of the gauge potentials Eq. [10], one can alternatively consider the non-periodic gauge fields \( A_i^{(0)} \) explained in the previous paragraph and non-vanishing transition functions. In this case, the fermion fields are required to satisfy:

\[
\psi(n + e_i) = \exp\{i\pi q \epsilon_{ij} n_j \} \psi(n) \tag{12}
\]

where \( e_i \) take the same form as in the continuum but expressed in lattice units (\( a=1 \)). One might choose an appropriate basis on this space by requiring that a complete set of commuting operators (matrices) are diagonal. An important class of operators is given by the covariant shift operators \( T_i \). All versions of lattice Dirac operators are elements of the algebra generated by these operators and the Dirac matrices. As in the continuum case we will attribute a special role to the covariant shift operators for uniform fields \( T_i^{(0)} \). They satisfy:

\[
T_1^{(0)} T_2^{(0)} = \xi^{-q} T_2^{(0)} T_1^{(0)} \tag{13}
\]

where \( \xi = \exp\{2\pi i / A\} \) This relation defines a Heisenberg group. It has been studied in connection with twist-eaters [6]-[7]. One knows that the group acts irreducibly on a vector space of dimension \( A/(\gcd(q, A)) \). This implies that for \( q = 1 \) one can fix a basis of the vector space by diagonalizing any one of the two \( T_i^{(0)} \). If \( \gcd(q, A) > 1 \) there are operators commuting with these and allowing to define a complete set of commuting operators. If \( q \) divides \( L_i \) the formulas follow closely the continuum case. There is at least a \( q \)-fold degeneracy of all eigenvalues of any lattice Dirac operator. We refer to [7] for details.

For simplicity, let us restrict to the \( q = 1 \) case. The condition Eq. [13] has to be supplemented with the value of the Casimirs:

\[
(T_i^{(0)})^A = \exp\{-i\phi_i\} I \tag{14}
\]

All pairs of \( A \times A \) matrices satisfying Eqs. [13]-[14] are equal up a similarity transformation. In our case, we have \( \phi_i = A \bar{v}_i \), so that the spectrum only
depends on \(v_i\) up to integer multiples of \(2\pi/A\). This contrasts with the continuum where the spectrum is independent of \(v_i\).

In the basis which diagonalizes \(T_1^{(0)}\) we have:

\[
T_1^{(0)} = Q = \exp\{-i\phi_1/A\} \text{ diag}\{1, \xi, \xi^2, \ldots \xi^{A-1}\}
\]

\[
T_2^{(0)} = P \quad \text{with} \quad P_{jk} = \delta_{k,j+1} \exp\{-i\phi_2/A\}
\]

Thus we see that, irrespectively of the lattice Dirac operator we are using, the spectrum does not depend on the shape of the box but only on the area \(A\) (and \(v_i \mod 2\pi/A\)). Indeed, the spectrum also coincides with that of a twisted reduced model for the group \(U(A)\) at weak coupling.

Now we will investigate the properties of the spectrum of the Wilson-Dirac hamiltonian \(H_{WD}(M,r) = \gamma_3 D_{WD}(M,r)\) where \(D_{WD}(M,r) = M + D_N + rW\). The naive lattice Dirac operator \(D_N\) and Wilson term \(W\) are given by:

\[
D_N = \frac{1}{2} \sum_i \gamma_i (T_i - T_i^\dagger)
\]

\[
W = \frac{1}{2} \sum_i (2 - (T_i + T_i^\dagger))
\]

We explicitly indicate the dependence of the Wilson-Dirac operator on the mass \(M\) and the Wilson parameter \(r\).

In analyzing the spectrum of \(H_{WD}(M,r)\) it is useful to make use of symmetries. They must correspond to symmetries of the algebra Eq. 13. For example the transformation \(T_i^{(0)} \rightarrow -T_i^{(0)}\) preserves Eq. 13. However, it can only be an exact symmetry if it preserves the Casimirs as well, and this only occurs for \(\phi_i = 0, \pi\). In this case, it can be combined with a similar rotation in spin space \((\gamma_i \rightarrow \epsilon_{ij}\gamma_j)\) to produce a unitary matrix \(U\) commuting with \(H_{WD}(M,r)\). The operation \(U\) can be interpreted as a \(\pi/2\) rotation of space, which happens to be symmetry even when \(L_1 \neq L_2\). Indeed, it generates a finite group of 4 elements \(U^4 = I\).

For even \(A\) one can construct additional symmetry operations associated to \(T_i^{(0)} \rightarrow -T_i^{(0)}\). Using them one can show that \(H_{WD}(M,r), H_{WD}(M + 4r, -r)\) and \(-H_{WD}(-M - 4r, r)\) are unitarily equivalent.

In Ref. [3] formulas were given that allow the computation of the eigenvalues of \(H_{WD}(M,r)\) to machine precision. One particularly interesting aspect is the balance between the number of positive \((N_+\) and negative \((N_-)\) eigenvalues of \(H_{WD}(M,r)\). Obviously we have \(N_+ + N_- = 2A\), while \(I = N_- - A\) provides a lattice definition of the index. It is precisely the
index of the Neuberger (overlap) operator. For large positive or negative values of the mass $M$ it is easy to show that $N_+ = N_-$ and the index vanishes. As we decrease the mass towards negative values some eigenvalues might move from positive to negative or vice versa generating a non-zero index. Jumps of the index take place at values of the mass $\bar{M}$ for which there exist $\psi_{\bar{M}}$ satisfying $H_{WD}(\bar{M}, r)\psi_{\bar{M}} = 0$. Multiplying by $\gamma_3$ the previous equation, one sees that $\psi_{\bar{M}}$ must be an eigenvector of real eigenvalue (equal to $-\bar{M}$) of $D_{WD}(0, r)$. Furthermore, the sign of $\psi_{\bar{M}}^\dagger \gamma_3 \psi_{\bar{M}}$ determines whether the index increases or decreases at this point. Thus, to analyze the index of $H_{WD}(M, r)$ it is enough to determine the real eigenvalues of $D_{WD}(0, r)$ and the corresponding eigenvectors. For $r = 1$ one can apply similar techniques to those used in Ref. [4] to derive expressions that allow the computation of this real eigenvalues to machine precision.

For $A \geq 3$ there are always four real eigenvalues, two in the interval $[-2, 0)$ and two in the interval $(-4, -2)$ (For even $A$ there is symmetry around $M = -2$). As we decrease the value of $M$ from positive values the index follows the sequence $0 \rightarrow 1 \rightarrow 0 \rightarrow -1 \rightarrow 0$. The physical region at which the index takes its continuum value (equal to one) has an upper edge given by $-0.63397, -0.58578, -0.29449, -0.12210, -0.0311755$ for $A = 3, 4, 10, 25, 100$. For large torus sizes it approaches $-Fr/2$. The lower edge is $-0.94203$ for $A = 3$ and decreases very fast towards $-2$ as the area increases. One can monitor the $M$ dependence of the smallest (in absolute value) eigenvalue of $H_{WD}(M, r)$. The interval $(-4, 0)$ is split into 4 regions, one for each of the eigenvectors of real eigenvalue $\psi_{\bar{M}}$. Within each region the behaviour is essentially linear (in $M$) with a slope determined by $\psi_{\bar{M}}^\dagger \gamma_3 \psi_{\bar{M}}$.

Other aspects are studied in Ref. [4]. For example, we numerically computed the eigenvalues of Neuberger’s operator in a uniform background gauge field, for several torus sizes up to $24 \times 24$ and several values of $M$. Eigenvalues and eigenvectors are quite close to the continuum results.

IV. Four dimensions

Much of the previous construction generalizes to the case of non-abelian gauge groups in 4 dimensions. For SU($N$) gauge groups the topology of gauge fields is indexed by the instanton number $Q$ (and twist sectors). To make connection with the two-dimensional U(1) formulation of the previous sections, we make use of the fact that (except for SU(2) and odd $Q$) it is possible to construct transition functions living in the maximal abelian subgroup U(1)$_{N-1}$[9]-[7]. Then the boundary conditions for the fermion fields apply for each of the color components independently. This reduces the problem from SU($N$) to a U(1) field in 4 dimensions. The fluxes of these
U(1) fields over the 2-planes encodes the information of the topology of the original field. Choosing appropriate transition functions and a basis of the 4 dimensional space, the problem decouples into two 2-planes. This allows to obtain the spectrum of the continuum Dirac operator and the lattice naive Dirac operator in terms of the two-dimensional result. The Wilson-Dirac and Neuberger operators, however, couple the 4 dimensions in a more complicated way. We refer to Ref. [4] for details.

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References

1. Hasenfratz P., Lalena V. and Niedermayer F., (1998) The index theorem in QCD with a finite cut-off, Phys.Lett. B427 pp. 125–131.
2. D. B. Kaplan, (1992) Phys. Lett. B288 342.
3. R. Narayanan and H. Neuberger, (1993) Phys. Lett. B302 62; (1993) Phys. Rev. Lett. 71 3251; (1995) Nucl. Phys. B443 305. H. Neuberger, (1998) Phys. Rev. D57 5417; (2001) Exact chiral symmetry on the lattice, Annu. Rev. Nucl. Phys. Part. Sci., Vol 51.
4. Giusti L., González-Arroyo A., Hoelbling C., Neuberger H. and Rebbi C. (2002) Fermions on tori in uniform abelian fields, Phys. Rev., D65:074506.
5. J. Smit and J. Vink, (1987) Nucl. Phys. B286, 485.
6. J. Groeneveld, J. Jurkiewicz and C. P. Korthals Altes, (1981) Phys. Scr. 23 1022; J. Ambjorn and H. Flyvberg, (1980) Phys. Lett. 97B 241; B. van Geemen and P. van Baal, (1986) J. Math. Phys. 27 455; D.R. Lebedev and M.I. Polikarpov, (1986) Nucl. Phys. B269 285.
7. A. González-Arroyo, (1998) Gauge fields on the four-dimensional torus Proceedings of the Peñiscola 1997 Advanced School on Non-Perturbative Quantum Field Physics, World Scientific.
8. A. González-Arroyo and M. Okawa (1983), Phys. Lett. 120B 174; (1983) Phys. Rev. D27 2397; T. Eguchi and R. Nakayama, (1983) Phys. Lett. 122B 59; A. González-Arroyo and C.P. Korthals Altes, (1983) Phys. Lett. 131B 396
9. P. van Baal, (1982) Commun. Math. Phys. 85 529.