Effective order reduction method based on parametrization of slow invariant manifolds

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Abstract. The method of integral manifolds is used to study the multidimensional systems of differential equations. This approach allows to solve an important problem of order reduction of differential systems. If a slow invariant manifold cannot be described explicitly then its parametrization is used for the system order reduction. In this case, either a part of the fast variables, or all fast variables, supplemented by a certain number of slow variables, can play a role of the parameters.

1. Introduction

Consider the system of differential equations

\[ \dot{x} = f(x, y, \varepsilon), \]
\[ \varepsilon \dot{y} = g(x, y, \varepsilon), \]

where \( x \in \mathbb{R}^n \), \( y \in \mathbb{R}^n \), \( \varepsilon \) is a small positive parameter, \( 0 < \varepsilon \ll 1 \), functions \( f \) and \( g \) are continuous with respect to \((x, y)\) for all \( x \in \mathbb{R}^n \), \( y \in D \subset \mathbb{R}^m \) \((D \subset \mathbb{R}^m)\).

We will consider the situation when the system (1), (2) has an integral manifold, that is, when the following conditions are fulfilled [1]–[5]:

1) the equation \( g(x, y, 0) = 0 \) has an isolated solution \( y = \psi_0(x) \) for \( x \in \mathbb{R}^n \);
2) the functions \( f \) and \( g \) are uniformly continuous and bounded together with partial derivatives with respect to all variables up to \((k + 2)\)-th order inclusive \((k \geq 0)\) in some region \( \Omega_0 = \{(x, y, \varepsilon) : x \in \mathbb{R}^n, \|y - \psi_0(x)\| < \rho, 0 \leq \varepsilon \leq \varepsilon_0\} \);
3) the eigenvalues of the matrix

\[ B(x) = \frac{\partial g}{\partial y}(x, \psi_0(x), 0) \]

satisfy the inequality

\[ \text{Re} \lambda_i(x) \leq -2\gamma < 0. \]

Recall, that the origins of the method of integral manifolds are found in the works of J. Hadamard [6], A. Lyapunov [7], H. Poincare [8] and O. Perron [9]. The essence of the method of integral manifolds was realized with amazing depth by A. Lyapunov [7] who developed a constructive procedure for problems of reduction on the stability of multidimensional systems
to systems of lower dimension. The possibility of the system order reduction is the essential aspect of the method of integral manifolds. The foundations of the theory were laid by N. Bogolyubov [10] and significant impact on the development of the method was provided by N. Bogolyubov and Yu Mitropolskii [11, 12] and J. Hale [13]. Various aspects of the theory of slow integral manifolds and the behavior of solutions in their neighborhood are presented in [14–25], see also references therein.

The degenerated system regarding to (1), (2) has a form

$$\dot{x} = f(x, y, 0),$$

$$0 = g(x, y, 0).$$

(3)

It should be noted that the equations in the system (3) can often be either transcendental or polynomials of a high degree with respect to $y$. In these cases a solution of the system cannot be found in explicit form as $y = \psi_0(x)$. For the system order reduction in these cases it is possible to use a parametric form for the representation of the slow invariant manifolds [26, 27, 28].

Three main cases, in which either fast variables, or only a fraction of the fast variables, or fast variables supplemented by a certain number of slow variables, play a role of the parameters are considered below.

2. The case $n = m$

Consider the case of the dimensions equality of the fast and slow variables. Suppose that the system (3) can be solved with respect to $x$ in the form $x = \varphi_0(y)$. In this case the fast vector–variable $y$ can play a role of a parameter for the representation of the slow invariant manifolds in the parametric form

$$x = \varphi(y, \varepsilon) = \varphi_0(y) + \varepsilon \varphi_1(y) + ... + \varepsilon^k \varphi_k(y) + ... .$$

(4)

The corresponding invariance equation is obtained by substituting (4) in (1):

$$\frac{\partial \varphi}{\partial y} g(\varphi, y, \varepsilon) = \varepsilon f(\varphi, y, \varepsilon).$$

(5)

For all functions occurring in (5), we write the formal asymptotic expansions in powers of the small parameter $\varepsilon$:

$$f \left( \sum_{k \geq 0} \varepsilon^k \varphi_k(y, \varepsilon) \right) = \sum_{k \geq 0} \varepsilon^k f^{(k)}(\varphi_0, ..., \varphi_k, y),$$

$$g \left( \sum_{k \geq 0} \varepsilon^k \varphi_k(y, \varepsilon) \right) = g^{(0)}(\varphi_0, y) + B(y) \sum_{k \geq 1} \varepsilon^k \varphi_k + \sum_{k \geq 1} \varepsilon^k g^{(k)}(\varphi_0, ..., \varphi_{k-1}, y),$$

where $g^{(0)}(\varphi_0, y) = g(\varphi_0, y, 0)$ and the non-degenerate matrix $B(y) = g_x(\varphi_0, y, 0)$ [4, 29, 30].

Taking these expansions into account, the invariance equation (5) takes the form:

$$\sum_{k \geq 0} \varepsilon^k \frac{\partial \varphi_k}{\partial y} \left( g^{(0)} + B \sum_{k \geq 1} \varepsilon^k \varphi_k + \sum_{k \geq 1} \varepsilon^k g^{(k)} \right) = \varepsilon \sum_{k \geq 0} \varepsilon^k f^{(k)}.$$

Equating the coefficients of like powers of $\varepsilon$ in the last equation, we get the expressions, which uniquely define the coefficients in (4) if $\text{det} \left( \frac{\partial \varphi_0}{\partial y} \right) \neq 0$. 

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Indeed, for $\varepsilon^0$ we have $g(\varphi_0, y, 0) = 0$, which gives the function $\varphi_0(y)$.
For $\varepsilon^1$ we get

$$
\varphi_1 = \left( \frac{\partial \varphi_0}{\partial y} B \right)^{-1} \left( f^{(0)} - \frac{\partial \varphi_0}{\partial y} g^{(1)} \right).
$$

By similar way, for $\varepsilon^k$ we obtain:

$$
\varphi_k = \left( \frac{\partial \varphi_0}{\partial y} B \right)^{-1} \left[ f^{(k-1)} - \frac{\partial \varphi_0}{\partial y} g^{(k)} - \sum_{i=1}^{k-1} \frac{\partial \varphi_i}{\partial y} \left( B \varphi_i + g^{(k-i)} \right) \right].
$$

Thus, the parametric representation of the slow invariant manifold of (1), (2) is found in the form (4).

3. The case $n < m$

Consider the case when the number of fast variables in the system (1), (2) exceeds the number of slow variables. Then the system (3) contains $m$ equations and $n$ unknowns, where $n < m$.

We take all components of vector $x$ ($\dim x = n$) complemented by $(m-n)$ components of vector $y$, as an unknowns. Thereby the number of equations and unknowns in the system (3) will coincide.

Suppose that the solution of (3) can be written in the form

$$
x = \varphi_0(y_2), \quad y_1 = \psi_0(y_2),
$$

with a parameter $y_2$, where $y = (y_1, y_2)^T$, $\dim y_1 = m-n$, $\dim y_2 = n$.

The system (1), (2) in this case can be rewritten in more convenient form as

\begin{align*}
\dot{x} &= f(x, y_1, y_2, \varepsilon), \quad (6) \\
\varepsilon \dot{y}_1 &= g_1(x, y_1, y_2, \varepsilon), \quad (7) \\
\varepsilon \dot{y}_2 &= g_2(x, y_1, y_2, \varepsilon). \quad (8)
\end{align*}

We will find the slow integral manifold in the form

\begin{align*}
x &= \varphi(y_2, \varepsilon), \quad (9) \\
y_1 &= \psi(y_2, \varepsilon). \quad (10)
\end{align*}

Substituting (9), (10) into (6) and (7) and taking into account (8), we obtain the invariance equations

\begin{align*}
\frac{\partial \varphi}{\partial y_2} g_2(\varphi, \psi, y_2, \varepsilon) &= \varepsilon f(\varphi, \psi, y_2, \varepsilon), \\
\frac{\partial \psi}{\partial y_2} g_2(\varphi, \psi, y_2, \varepsilon) &= g_1(\varphi, \psi, y_2, \varepsilon).
\end{align*}

For the functions $f(\varphi, \psi, y_2, \varepsilon)$, $g_1(\varphi, \psi, y_2, \varepsilon)$, $g_2(\varphi, \psi, y_2, \varepsilon)$, $\varphi(y_2, \varepsilon)$, and $\psi(y_2, \varepsilon)$ we write the formal asymptotic expansions:

\begin{align*}
\varphi(y_2, \varepsilon) &= \varphi_0(y_2) + \varepsilon \varphi_1(y_2) + \ldots + \varepsilon^k \varphi_k(y_2) + \ldots, \quad (11) \\
\psi(y_2, \varepsilon) &= \psi_0(y_2) + \varepsilon \psi_1(y_2) + \ldots + \varepsilon^k \psi_k(y_2) + \ldots. \quad (12)
\end{align*}
\[
\begin{align*}
  f \left( \sum_{k \geq 0} \varepsilon^k \varphi_k, \sum_{k \geq 0} \varepsilon^k \psi_k, y_2, \varepsilon \right) &= \sum_{k \geq 0} \varepsilon^k f^{(k)}(\varphi_0, \ldots, \varphi_k, \psi_0, \ldots, \psi_k, y_2), \\
  g_1 \left( \sum_{k \geq 0} \varepsilon^k \varphi_k, \sum_{k \geq 0} \varepsilon^k \psi_k, y_2, \varepsilon \right) &= g_1^{(0)}(\varphi_0, \psi_0, y_2) + G_1(y_2) \sum_{k \geq 1} \varepsilon^k \varphi_k \\
  &+ B_1(y_2) \sum_{k \geq 1} \varepsilon^k \psi_k + \sum_{k \geq 1} \varepsilon^k g_1^{(k)}(\varphi_0, \ldots, \varphi_{k-1}, \psi_0, \ldots, \psi_{k-1}, y_2), \\
  g_2 \left( \sum_{k \geq 0} \varepsilon^k \varphi_k, \sum_{k \geq 0} \varepsilon^k \psi_k, y_2, \varepsilon \right) &= g_2^{(0)}(\varphi_0, \psi_0, y_2) + G_2(y_2) \sum_{k \geq 1} \varepsilon^k \varphi_k \\
  &+ B_2(y_2) \sum_{k \geq 1} \varepsilon^k \psi_k + \sum_{k \geq 1} \varepsilon^k g_2^{(k)}(\varphi_0, \ldots, \varphi_{k-1}, \psi_0, \ldots, \psi_{k-1}, y_2),
\end{align*}
\]

where
\[
G_1(y_2) = \frac{\partial g_1}{\partial x}(\varphi_0, \psi_0, y_2, 0), \quad G_2(y_2) = \frac{\partial g_2}{\partial x}(\varphi_0, \psi_0, y_2, 0), \\
B_1(y_2) = \frac{\partial g_1}{\partial y_1}(\varphi_0, \psi_0, y_2, 0), \\
g_1^{(0)}(\varphi_0, \psi_0, y_2) = g_1(\varphi_0, \psi_0, y_2, 0), \quad g_2^{(0)}(\varphi_0, \psi_0, y_2) = g_2(\varphi_0, \psi_0, y_2, 0).
\]

From these formal expansions and the invariance equations we have
\[
\sum_{k \geq 0} \varepsilon^k \frac{\partial \varphi_k}{\partial y_2} \left( g_2^{(0)} + G_2 \sum_{k \geq 1} \varepsilon^k \varphi_k + B_2 \sum_{k \geq 1} \varepsilon^k \psi_k + \sum_{k \geq 1} \varepsilon^k g_2^{(k)} \right) = \varepsilon \sum_{k \geq 0} \varepsilon^k f^{(k)},
\]

\[
\sum_{k \geq 0} \varepsilon^k \frac{\partial \psi_k}{\partial y_2} \left( g_2^{(0)} + G_2 \sum_{k \geq 1} \varepsilon^k \varphi_k + B_2 \sum_{k \geq 1} \varepsilon^k \psi_k + \sum_{k \geq 1} \varepsilon^k g_2^{(k)} \right) = g_1^{(0)} + G_1 \sum_{k \geq 1} \varepsilon^k \varphi_k + B_1 \sum_{k \geq 1} \varepsilon^k \psi_k + \sum_{k \geq 1} \varepsilon^k g_1^{(k)}.
\]

Equating the coefficients of like powers of \( \varepsilon \) in the last equations, we get the expressions, which uniquely define the coefficients in (11) and (12) if \( \det \frac{\partial g_2}{\partial y_2} \neq 0 \) and \( \det \frac{\partial \psi_0}{\partial y_2} \neq 0 \).

Indeed, for \( \varepsilon^0 \) we have
\[
\frac{\partial \varphi_0}{\partial y_2} g_2^{(0)} = 0, \quad \frac{\partial \psi_0}{\partial y_2} g_2^{(0)} = g_1^{(0)},
\]

which imply
\[
g_1(\varphi_0, \psi_0, y_2, 0) = 0, \\
g_2(\varphi_0, \psi_0, y_2, 0) = 0,
\]

whose solution is the functions \( \varphi_0 = \varphi_0(y_2), \psi_0 = \psi_0(y_2) \).

For \( \varepsilon^1 \) we get
\[
\frac{\partial \varphi_0}{\partial y_2} \left( G_2 \varphi_1 + B_2 \psi_1 + g_2^{(1)} \right) = f^{(0)},
\]

\[
\frac{\partial \psi_0}{\partial y_2} \left( G_2 \varphi_1 + B_2 \psi_1 + g_2^{(1)} \right) = G_1 \varphi_1 + B_1 \psi_1 + g_1^{(1)},
\]


The case of (1), (2), we take all components of variables. We call attention to the degenerate subsystem (3). It contains unknowns, where

\[ x \]

as the parameters. Then a solution of the system (3) can be written in the parametric form

\[ \varphi = A_1^{-1} \left[ g_1^{(1)} + B_1 \left( \frac{\partial \psi_0}{\partial y_2} B_2 \right)^{-1} \frac{\partial \psi_0}{\partial y_2} g_2^{(1)} - \left( \frac{\partial \psi_0}{\partial y_2} B_2 - B_1 \right) \left( \frac{\partial \psi_0}{\partial y_2} \right)^{-1} \frac{\partial \psi_0}{\partial y_2} \left( \frac{\partial \psi_0}{\partial y_2} \right)^{-1} f^{(0)} \right], \]

\[ \psi = A_2^{-1} \left[ g_1^{(1)} + G_1 g_2^{(1)} - \left( \frac{\partial \psi_0}{\partial y_2} G_2 - G_1 \right) \left( \frac{\partial \psi_0}{\partial y_2} \right)^{-1} f^{(0)} \right], \]

where

\[ A_1 = B_1 \left( \frac{\partial \psi_0}{\partial y_2} B_2 \right)^{-1} \frac{\partial \psi_0}{\partial y_2} G_2 - G_1, \quad A_2 = G_1 G_2^{-1} B_2 - B_1. \]

Further, for \( \varepsilon \) we have

\[ \frac{\partial \psi_0}{\partial y_2} \left( G_2 \varphi_k + B_2 \psi_k + g_2^{(k)} \right) + \cdots + \frac{\partial \psi_{k-1}}{\partial y_2} \left( G_2 \varphi_1 + B_2 \psi_1 + g_2^{(1)} \right) = f^{(k-1)}, \]

\[ \frac{\partial \psi_0}{\partial y_2} \left( G_2 \varphi_k + B_2 \psi_k + g_2^{(k)} \right) + \cdots + \frac{\partial \psi_{k-1}}{\partial y_2} \left( G_2 \varphi_1 + B_2 \psi_1 + g_2^{(1)} \right) = G_1 \varphi_k + B_1 \psi_k + g_1^{(k)}, \]

from which it follows that

\[ \varphi_k = A_1^{-1} \left[ g_1^{(k)} + B_1 \left( \frac{\partial \psi_0}{\partial y_2} B_2 \right)^{-1} \frac{\partial \psi_0}{\partial y_2} g_2^{(k)} - \sum_{i=1}^{k-1} \frac{\partial \psi_i}{\partial y_2} \left( G_2 \varphi_{k-i} + g_2^{(k-i)} \right) \right], \]

\[ \psi_k = A_2^{-1} \left[ g_1^{(k)} + G_1 g_2^{(k)} - \sum_{i=1}^{k-1} \frac{\partial \psi_i}{\partial y_2} \left( G_2 \varphi_{k-i} + g_2^{(k-i)} \right) \right], \]

\[ - \left( \frac{\partial \psi_0}{\partial y_2} B_2 - B_1 \right) \left( \frac{\partial \psi_0}{\partial y_2} B_2 \right)^{-1} \frac{\partial \psi_0}{\partial y_2} \left( \frac{\partial \psi_0}{\partial y_2} \right)^{-1} \left( f^{(k-1)} - \sum_{i=1}^{k-1} \frac{\partial \psi_i}{\partial y_2} \left( G_2 \varphi_{k-i} + g_2^{(k-i)} \right) \right) \right], \]

Thus, it is possible to uniquely determine the coefficients in (11), (12).

4. The case \( n > m \)

Consider the case when the dimension of slow variables is greater than the dimension of fast variables. We call attention to the degenerate subsystem (3). It contains \( m \) equations and \( n \) unknowns, where \( n > m \). To find the parametric representation of the slow invariant manifold of (1), (2), we take all components of \( y \) complemented by \((n - m)\) components of the vector \( x \), as the parameters. Then a solution of the system (3) can be written in the parametric form

\( x_1 = \varphi(x_2, y), \) where \( x = (x_1, x_2)^T, \dim x_1 = m, \dim x_2 = n - m. \)

The system (1), (2) in this case can be rewritten in more convenient form as

\[ \dot{x}_1 = f_1(x_1, x_2, y, \varepsilon), \]

\[ \dot{x}_2 = f_2(x_1, x_2, y, \varepsilon), \]

\[ \varepsilon \dot{y} = g_2(x_1, x_2, y, \varepsilon). \]

We will find the slow integral manifold in the form

\[ x_1 = \varphi(x_2, y, \varepsilon). \]
The invariance equation
\[ \varepsilon \frac{\partial \varphi}{\partial x_2} f_2(\varphi, x_2, y, \varepsilon) + \frac{\partial \varphi}{\partial y} g(\varphi, x_2, y, \varepsilon) = \varepsilon f_1(\varphi, x_2, y, \varepsilon) \] (15)
is yielded from (13) and (14). From (15) and the formal series
\[ G = \sum \varepsilon^k \varphi_k(x_2, y), \]
where
\[ \varepsilon = \sum \varepsilon^k \varphi_k(x_2, y) + ... + \varepsilon^k \varphi_k(x_2, y) + ..., \] (16)
is the expression for \( \varphi(x_2, y, \varepsilon) = \varphi_0(x_2, y) + \varepsilon \varphi_1(x_2, y) + ... + \varepsilon^k \varphi_k(x_2, y) + ... \),

\[ f_1 \left( \sum_{k=0}^{\infty} \varepsilon^k \varphi_k(x_2, y, \varepsilon) \right) = \sum_{k=0}^{\infty} \varepsilon^k f_1^{(k)}(\varphi_0, \varphi_1, \varphi_2, x_2, y), \]
\[ f_2 \left( \sum_{k=0}^{\infty} \varepsilon^k \varphi_k(x_2, y, \varepsilon) \right) = \sum_{k=0}^{\infty} \varepsilon^k f_2^{(k)}(\varphi_0, \varphi_1, \varphi_2, x_2, y), \]
\[ g \left( \sum_{k=0}^{\infty} \varepsilon^k \varphi_k(x_2, y, \varepsilon) \right) = g^{(0)}(\varphi_0, x_2, y) + B(x_2, y) \sum_{k=1}^{\infty} \varepsilon^k \varphi_k + \sum_{k=1}^{\infty} \varepsilon^k g^{(k)}(\varphi_0, \varphi_1, \varphi_2, x_2, y), \]
where \( G(x_2, y) = \frac{\partial g}{\partial y}(\varphi_0, x_2, y, 0) \), we get

\[ \varepsilon \sum_{k=0}^{\infty} \varepsilon^k \frac{\partial \varphi_k}{\partial x_2} \sum_{k=0}^{\infty} \varepsilon^k f_2^{(k)} + \sum_{k=0}^{\infty} \frac{\partial \varphi_k}{\partial y} \left( g^{(0)} + G \sum_{k=1}^{\infty} \varepsilon^k \varphi_k + \sum_{k=1}^{\infty} \varepsilon^k g^{(k)} \right) = \varepsilon \sum_{k=0}^{\infty} \varepsilon^k f_1^{(k)}. \]

Equating the coefficients of like powers of \( \varepsilon \) in the last equation, we get the expressions, which uniquely define the coefficients in (16) if \( \det \left( \frac{\partial \varphi_0}{\partial y} G \right) \neq 0 \).

For \( \varepsilon^0 \) we have
\[ \frac{\partial \varphi_0}{\partial y} g(\varphi_0, y, 0) = 0, \]
which defines the function \( \varphi_0 = \varphi_0(x_2, y) \).

For \( \varepsilon^1 \) we have
\[ \frac{\partial \varphi_0}{\partial x_2} f_2^{(0)} + \frac{\partial \varphi_0}{\partial y} \left( G \varphi_1 + g^{(1)} \right) = f_1^{(0)}, \]
from which we find
\[ \varphi_1 = \left( \frac{\partial \varphi_0}{\partial y} G \right)^{-1} \left( f_1^{(0)} - \frac{\partial \varphi_0}{\partial x_2} f_2^{(0)} - \frac{\partial \varphi_0}{\partial y} g^{(1)} \right). \]

The expression for \( \varepsilon^k \) has the form:
\[ \frac{\partial \varphi_0}{\partial x_2} f_2^{(k-1)} + ... + \frac{\partial \varphi_{k-1}}{\partial x_2} f_2^{(0)} + \frac{\partial \varphi_0}{\partial y} \left( G \varphi_k + g^{(k)} \right) + ... + \frac{\partial \varphi_{k-1}}{\partial y} \left( G \varphi_1 + g^{(1)} \right) = f_1^{(k-1)}. \]
Hence
\[ \varphi_k = \left( \frac{\partial \varphi_0}{\partial y} G \right)^{-1} \left[ f_1^{(k-1)} - \sum_{i=0}^{k-1} \frac{\partial \varphi_i}{\partial x_2} f_2^{(k-i-1)} - \sum_{i=1}^{k-1} \frac{\partial \varphi_i}{\partial y} \left( B \varphi_{k-i} + g^{(k-i)} \right) - \frac{\partial \varphi_0}{\partial y} g^{(k)} \right]. \]
Thus, formula (16) defines the slow integral manifold of the system in the parametric form.
5. Michaelis–Menten Model

As an example, consider the Michaelis–Menten kinetic model [31]:

\[ \dot{x} = -x + (x + k - \lambda)y, \]  
\[ \varepsilon \dot{y} = x - (x + k)y. \]  

(17)  
(18)

In this case, the number of fast variables coincides with the number of slow variables. Putting \( \varepsilon = 0 \), we obtain a degenerate system

\[ \dot{x} = -x + (x + k - \lambda)y, \]  
\[ 0 = x - (x + k)y. \]  

(19)

The equation (19) is solvable with respect to \( x \):

\[ x = \frac{ky}{1 - y}. \]

To find the integral manifold in the parametric form

\[ x = \varphi(y, \varepsilon), \]  

(20)

we take \( y \) as a parameter. The invariance equation obtained from (17), (18) and (20) has the form

\[ \frac{\partial \varphi(y, \varepsilon)}{\partial y} (\varphi(y, \varepsilon) - (\varphi(y, \varepsilon) + k)y) = \varepsilon (-\varphi(y, \varepsilon) + (\varphi(y, \varepsilon) + k - \lambda)y). \]

From (20) and the formal asymptotic expansion

\[ \varphi(y, \varepsilon) = \varphi_0(y) + \varepsilon \varphi_1(y) + \varepsilon^2 \varphi_2(y) + O(\varepsilon^3), \]

we have

\[ \left( \frac{\partial \varphi_0}{\partial y} + \varepsilon \frac{\partial \varphi_1}{\partial y} + \varepsilon^2 \frac{\partial \varphi_2}{\partial y} + \ldots \right) \left( \varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots - (\varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots + k)y \right) \]

\[ = \varepsilon \left( -\varphi_0 - \varepsilon \varphi_1 - \varepsilon^2 \varphi_2 - \ldots + (\varphi_0 + \varepsilon \varphi_1 + \varepsilon^2 \varphi_2 + \ldots + k - \lambda)y \right). \]

Equating the coefficients of \( \varepsilon^0 \) in the last equation, we get

\[ \frac{\partial \varphi_0}{\partial y} (\varphi_0 - (\varphi_0 + k)y) = 0. \]

When the condition \( \frac{\partial \varphi_0}{\partial y} \neq 0 \) is satisfied, this equation gives the function

\[ \varphi_0 = \frac{ky}{1 - y}. \]

By similar way, for \( \varepsilon^1 \) we have

\[ \frac{\partial \varphi_0}{\partial y} (\varphi_1 - \varphi_1y) + \frac{\partial \varphi_1}{\partial y} (\varphi_0 - (\varphi_0 + k)y) = -\varphi_0 + (\varphi_0 + k - \lambda)y \]

or

\[ \varphi_1 = \left( \frac{\partial \varphi_0}{\partial y} (1 - y) \right)^{-1} (-\varphi_0 + (\varphi_0 + k - \lambda)y). \]
After substituting the expression for the function $\varphi_0$, from here we have

$$\varphi_1 = \frac{\lambda y(y - 1)}{k}.$$ 

Further, for $\varepsilon^2$ we have

$$\frac{\partial \varphi_0}{\partial y}(\varphi_2 - \varphi_2 y) + \frac{\partial \varphi_1}{\partial y}(\varphi_1 - \varphi_1 y) + \frac{\partial \varphi_2}{\partial y}(\varphi_0 - (\varphi_0 + k)y) = -\varphi_1 + \varphi_1 y,$$

which implies

$$\varphi_2 = -\left(\frac{\partial \varphi_0}{\partial y}(1 - y)\right)^{-1}\left(\frac{\partial \varphi_1}{\partial y} + 1\right)(1 - y)\varphi_1$$

or

$$\varphi_2 = \frac{\lambda y(1 - y)^3(k + \lambda(2y - 1))}{k^3}.$$ 

As a result, we obtain the slow invariant manifold in the form

$$x = \frac{ky}{1 - y} + \varepsilon\frac{\lambda y(y - 1)}{k} + \varepsilon^2\frac{\lambda y(1 - y)^3(k + \lambda(2y - 1))}{k^3} + O(\varepsilon^3).$$

6. Conclusions

The paper attempted to give an overview on the approaches for the parametrization of slow invariant manifolds of the multidimensional systems of differential equations. The algorithms for the construction of the slow invariant manifolds in the case with different dimensions of the fast and slow variables was derived. It should be noted that the slow invariant manifolds are used for the order reduction of differential systems.

7. References

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