KERNELS OF DISCRETE CONVOLUTIONS AND SUBDIVISION OPERATORS

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Abstract. We consider kernels of discrete convolution operators or, equivalently, homogeneous solutions of partial difference operators and show that these solutions always have to be exponential polynomials. The respective polynomial space in connected directly though somewhat intricately to the multiplicity of the common zeros of certain multivariate polynomials, a concept introduced by Gröbner in the description of kernels of partial differential operators with constant coefficients. These results can are then used to determine the kernels of stationary subdivision operators as well.

1. Introduction

This paper considers a simple question: which sequences \( c : \mathbb{Z}^s \to \mathbb{R} \) can be kernels of convolution or subdivision operators. Recall that a convolution operator or filter based on a finite impulse \( h \in \ell_0(\mathbb{Z}^s) \) acts on as sequence \( c \) as

\[
(1.1) \quad c = h * c = \sum_{\alpha \in \mathbb{Z}^s} h(\alpha) c(\cdot - \alpha), \quad c \in \ell(\mathbb{Z}^s).
\]

Here and in what follows \( \ell(\mathbb{Z}^s) \) denotes all multi-infinite sequences, written as functions from \( \mathbb{Z}^s \to \mathbb{C} \) while \( \ell_0(\mathbb{Z}^s) \) stands for those with compact, i.e., finite, support: \( \#\{\alpha \in \mathbb{Z}^s : c(\alpha) \neq 0\} < \infty \).

Convolution operators can also be viewed as partial difference operators. Let \( \tau_j : c \mapsto c(\cdot + \epsilon_j) \) denote the forward partial shift operator and \( \epsilon_j \) the \( j \)th unit index in \( \mathbb{N}_0^s \) as well as \( \tau^\alpha := \tau_1^{\alpha_1} \cdots \tau_s^{\alpha_s} \), then

\[
\begin{align*}
 h * c &= \sum_{\alpha \in \mathbb{Z}^s} h(\alpha) \tau^{-\alpha} c = h^*(\tau^{-1}) c,
\end{align*}
\]

with the symbol

\[
 h^*(z) = \sum_{\alpha \in \mathbb{Z}^s} h(\alpha) z^\alpha, \quad z \in \mathbb{C}_x^s := (\mathbb{C} \setminus \{0\})^s,
\]

which associates to a finitely supported sequence \( h \in \ell_0(\mathbb{Z}^s) \) a Laurent polynomial. Therefore, the kernels of the convolution operators are the solution of the homogeneous difference equation \( h^*(\tau^{-1}) c = 0 \).

It is not hard to guess what these solutions should be when taking into account that for any exponential sequence \( e_\theta : \alpha \mapsto \theta^\alpha, \theta \in \mathbb{C}^s_\times \), we get

\[
(1.2) \quad h * e_\theta = \sum_{\alpha \in \mathbb{Z}^s} h(\alpha) \theta^{-\alpha} = e_\theta h^*(\theta^{-1}),
\]

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hence \( e_\theta \) belongs to \( \ker h \) if and only if \( h^*(\theta^{-1}) = 0 \). Therefore, the exponentials in the kernel of any finitely supported convolution operator encoded in the zeros of the symbol and it only remains to show that essentially no other sequences can be annihilated by convolution operators. It is also to be expected that the order of the zero at \( \theta^{-1} \) will affect the structure of the kernel and indeed, it will allow for some exponential polynomial sequences.

The following classical result for \( d = 1 \) is widely used in systems theory and stated, for example, in [9, p. 543ff] or, more as some type of “cooking recipe”, in [4].

**Theorem 1.1.** Let \( h \in \ell_0(\mathbb{Z}) \) whose symbol factors as

\[
h^*(z) = c z^m \prod_{\theta \in \Theta} (z - \theta^{-1})^{k_\theta}, \quad k_\theta \in \mathbb{N},
\]

then

\[
\ker(h \ast (\cdot)) = \bigoplus_{\theta \in \Theta} e_\theta \Pi_{k_\theta - 1}.
\]

Here, \( \Pi_k \) denotes the vector space of all polynomials of degree at most \( k \), hence the *multiplicity* of the zero at \( \theta^{-1} \) directly corresponds to the degree of the exponential polynomial space that belongs to the kernel of the convolution operator.

Our goal will be to give a complete analog of Theorem 1.1 in several variables, which, of course, will need a more careful treatment of the (common) zeros of polynomials and in particular of their multiplicities. Multiplicities of zeros of polynomial ideals have been considered for example in [2, 10], but the main results are already mentioned in [7], where Gröbner refers to his papers [5, 6], where not only the concept of multiplicities is introduced and clarified, but where he also solves the continuous counterpart of our question, describing the kernels of partial differential operators.

Based on Gröbner’s multiplicity theory, we will state and prove the counterpart of Theorem 1.1 for zero dimensional ideals in Section 2, while in Section 3 we briefly apply these results to also describe the kernels of stationary subdivision operators in several variables.

### 2. Kernels of convolution operators

We begin by fixing some terminology. Let \( \Pi = \mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_s] \) denote the ring of polynomials in \( s \) variables over \( \mathbb{R} \), and let \( \deg f \) denote the *total degree* of \( f \in \Pi \). A polynomial \( f \in \Pi \) is called *homogeneous* if it can be written as

\[
f(z) = \sum_{|\gamma| = \deg f} f_\gamma z^\gamma, \quad z^\gamma := z_1^{\gamma_1} \cdots z_s^{\gamma_s},
\]

and we write \( \Pi^0 \) for all homogeneous polynomials, \( \Pi_k \) for all polynomials \( f \) with \( \deg f \leq k \) and \( \Pi^0_k \) for all homogeneous \( f \) with \( \deg f = k \), \( k \in \mathbb{N}_0 \). By \( \Lambda(f) \in \Pi^0_{\deg f - 1} \) we denote the homogeneous leading term of \( f \), defined by \( f - \Lambda(f) \in \Pi_{\deg f - 1} \). To a polynomial \( q \in \Pi \) we associated the constant coefficient partial difference operator

\[
q(D) = q \left( \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_s} \right) = \sum_{\alpha \in \mathbb{Z}^s} q_\alpha \frac{\partial^{||\alpha||}}{\partial z^\alpha} = \sum_{\alpha \in \mathbb{Z}^s} q_\alpha D^\alpha,
\]
and call a subspace $P$ of $\Pi$ $D$–invariant if $\Pi(D)P = P$, that is, $q(D)p \in P$, $p \in P$, $q \in \Pi$. Finally, we introduce an inner product $(\cdot, \cdot) : \Pi \times \Pi \to \mathbb{R}$ by setting

$$
(f, g) := (f(D)g)(0) = \sum_{\alpha \in \mathbb{N}^s_0} \alpha! f_{\alpha} g_{\alpha} .
$$

This inner product was used in [2] and also in the construction of the least interpolant, cf. [3]. I learned that it is sometimes called “Bombieri inner product” or “Fisher inner product” though unfortunately I cannot provide references; also, Charles Dunkl (private communication) mentioned that Calderon used this inner product in the context harmonic polynomials. For our purposes here it will turn out to be more useful than the “canonical” inner product $(f, g) = \sum f_{\alpha} g_{\alpha}$ that gives rise to Macaulay’s inverse systems, cf. [5, 7, 12].

2.1. $D$–invariant spaces. The identity

$$(p(D)f, g) = (f, pg), \quad f, p, g \in \Pi$$

is easily derived from (2.1) and directly yields the following observation.

**Lemma 2.1.** A subspace $Q \subseteq \Pi$ is $D$–invariant if and only if $Q^\perp = \{ f : (Q, f) = 0 \}$ is an ideal.

Based on Lemma 2.1 one can construct a homogeneous basis for the $D$–invariant space $Q$ by successively constructing bases for

$$
P_j = \{ f \in \Pi_j^0 : (f, \Lambda(Q^\perp)) = 0 \}, \quad j = 0, \ldots, \deg Q := \max\{ \deg q : q \in Q \},
$$

cf. [21]. Since $\bigoplus_j P_j \equiv \Pi/Q^\perp$, it follows that $P_0 + \cdots + P_{\deg Q} = Q$ and therefore $Q$ has a homogeneous basis which will be denoted by $Q$. Since $(f, g) = 0$ if deg $f \neq$ deg $g$, we can moreover assume that $Q$ is an orthonormal homogeneous basis, that is,

$$(q, q') = \delta_{q, q'}, \quad q, q' \in Q .
$$

Hence, any $f \in Q$ can be written as

$$
f = \sum_{q \in Q} (f, q) q = \sum_{q \in Q} (q(D)f)(0) q
$$

from which we can conclude for $x, y \in \mathbb{R}^s$ that

$$
f(x + y) = \sum_{q \in Q} (f(\cdot + y), q) q(x) = \sum_{q \in Q} (q(D)f)(y) q(x),
$$

hence, by symmetry,

$$
f(x + y) = \sum_{q \in Q} (q(D)f)(y) q(x) = \sum_{q \in Q} (q(D)f)(x) q(y).
$$

Note that (2.4) in particular implies that any $D$–invariant space is shift invariant.

2.2. Zero dimensional ideals. In several variables, a single convolution $h \ast c$ cannot be sufficient to have a finite dimensional kernel. Indeed, (1.2) shows that $h \ast e_0 = 0$ for any zero $\theta^{-1}$ of $h^*$, which can be a whole algebraic variety, hence usually not even a countable set. Therefore, we emerge from a finite set $H \subset \ell_{00}(\mathbb{Z}^s)$, consider the ideal

$$
\langle H^* \rangle = \left\{ \sum_{h \in H} f_h h^* : f_h \in \Pi \right\}
$$
generated by \( h^* \), \( h \in H \), and request that the ideal is zero dimensional, that is, there exists a finite set \( \Theta \subset \mathbb{C}^s \) such that
\[
H^* (\Theta^{-1}) = 0, \quad \text{i.e.,} \quad h^* (\theta^{-1}) = 0, \quad h \in H, \theta \in \Theta.
\]
Since \( h * c = 0 \) implies \( (g * h) * c = g * h * c = 0 \) with \( (g * h)^* = g^* h^* \) for any finite filter \( g \), the kernel does not depend on the generating set \( H \) but of the ideal \( \langle H^* \rangle \).

In Theorem 1.1 we have seen that multiplicities of the zeros play a fundamental role for the structure of the kernel. To extend this to the case of several variable, we recall the following classical description of the multiplicities of common zeroes, see also \[2\,10\].

**Theorem 2.2** \([5]\). \( I \subset \Pi \) is a zero dimensional ideal if and only if there exists a finite set \( Z \subset \mathbb{C}^s \) and \( D \)-invariant subspaces \( Q_\zeta \), \( \zeta \in Z \), such that
\[
f \in I \quad \iff \quad q(D) f(\zeta) = 0, \quad q \in Q_\zeta, \quad \zeta \in Z.
\]
In \([7]\), the dimension \( \dim Q_\zeta \) of \( Q_\zeta \) is called the multiplicity of the zero \( \zeta \), but it will be more appropriate here to work with the spaces \( Q_\theta \) themselves. The the dimension of \( Q_\zeta \) alone is not sufficient to fully describe the nature of the zero is easily seen from the the two examples
\[
Q_\zeta = \{1, x, y\}, \quad Q_\zeta = \{1, x + y, (x + y)^2\}
\]
of a triple zero in two variables.

It is worthwhile to remark that generally the symbol \( h^* \) is not a polynomial but a Laurent polynomial, hence \( h^* = (\cdot)^\alpha f \) for some \( \alpha \in \mathbb{Z}^s \) and \( f \in \Pi \). Since it is easily seen that \( \ker H \) is a shift invariant space, we can always shift the impulse responses \( h \in H \) such that \( h^* \in \Pi \). However, one must keep in mind that a “spurious” zero of \( h^* \) at zero do not count when considering \( \ker H \); this is a well–known effect also in the context of smoothness analysis of refinable functions, see \([11]\).

**Definition 2.3.** A finite set \( H \subset \ell_{00}(\mathbb{Z}^s) \) of impulse responses is called zero dimensional if the ideal \( \langle H^* \rangle \) is zero dimensional or, equivalently, if there exist a finite subset \( \Theta \subset \mathbb{C}^s \) and finite dimensional \( D \)-invariant spaces \( Q_\theta \), \( \theta \in \Theta \), such that
\[
q(D) h^* (\theta^{-1}) = 0, \quad q \in Q_\theta, \quad \theta \in \Theta, \quad h \in H.
\]

### 2.3. Annihilation of exponential polynomials

In order to formulate the main results of this paper, we need some more terminology. The *partial difference operator* \( \Delta^\alpha \), acting on \( \ell(\mathbb{Z}^s) \) is recursively defined as
\[
\Delta^\alpha+\epsilon_j = (\tau^{\epsilon_j} - I) \Delta^\alpha \quad (\alpha \in \mathbb{N}_0^s, \quad j = 1, \ldots, s).
\]
We define an operator \( L : \Pi \to \Pi \) as
\[
Lf(x) = \sum_{|\gamma| \leq \deg f} \frac{1}{\gamma!} \Delta^\gamma f(0) x^\gamma
\]
and note that \( \Lambda(Lf) = \Lambda(f) \) as well as \( \deg Lf = \deg f \). This immediately leads to the following observation.

**Lemma 2.4.** \( L \) is a degree preserving linear isomorphism \( \Pi \to \Pi \) and \( \Pi_k \to \Pi_k \) for any \( k \in \mathbb{N} \). In particular, there exists an inverse \( L^{-1} \) on \( \Pi \) as well as on \( \Pi_k \), \( k \in \mathbb{N}_0 \).
Next, we introduce the scaling operator \( \sigma_\theta, \theta \in \mathbb{C}_\infty \), as
\[
\sigma_\theta f(z) := f(\theta z) := f(\theta z_1, \ldots, \theta z_s)
\]
with the abbreviation \( \sigma_- := \sigma_{(-1, \ldots, -1)} \). For \( \theta \in \mathbb{C}_\infty \) and a \( D \)-invariant subspace \( \mathcal{Q}_\theta \subset \Pi \), we define
\[
\hat{\mathcal{Q}}_\theta = \sigma_\theta \mathcal{Q}_\theta = \{ \sigma_\theta q : q \in \mathcal{Q}_\theta \},
\]
and note that \( \hat{\mathcal{Q}}_\theta \) is also \( D \)-invariant since
\[
p(D)q(\theta \cdot) = \sum_{\alpha \in \mathbb{N}_0} p_\alpha \theta^\alpha(D^\alpha q)(\theta \cdot), \quad p = \sum_{|\alpha| \leq \deg p} p_\alpha \cdot \alpha,
\]
and \( D^\alpha q \) can be expanded in terms of \( \mathcal{Q}_\theta \). Moreover, we introduce to \( \mathcal{Q}_\theta \) the space
\[
\mathcal{P}_\theta := \sigma_- L \hat{\mathcal{Q}}_\theta = \text{span} \{ L^{-1} \sigma_\theta q : q \in \mathcal{Q}_\theta \},
\]
where again \( \mathcal{Q}_\theta \) denotes an homogeneous orthonormal basis of \( \mathcal{Q}_\theta \).

**Example 2.5.** For the \( D \)-invariant space \( \mathcal{Q}_\theta = \text{span} \{ 1, (x + y), (x + y)^2 \} \) and \( \theta = (\theta_1, \theta_2) \) with \( \theta_1 \neq \theta_2 \) we get \( \hat{\mathcal{Q}}_\theta = \text{span} \{ 1, \theta_1 x + \theta_2 y, (\theta_1 x + \theta_2 y)^2 \} \neq \mathcal{Q}_\theta \). A straightforward computation yields
\[
L1 = 1
\]
\[
L(\theta_1 x + \theta_2 y) = \theta_1 x + \theta_2 y
\]
\[
L((\theta_1 x + \theta_2 y)^2) = (\theta_1 x + \theta_2 y)^2 + \theta_1^2 x + \theta_2^2 y,
\]
which shows that
\[
\mathcal{P}_\theta = \text{span} \{ 1, \theta_1 x + \theta_2 y, (\theta_1 x + \theta_2 y)^2 - (\theta_1^2 x + \theta_2^2 y) \}
\]
is not spanned by homogeneous polynomials, hence cannot be \( D \)-invariant as soon as \( \theta_1 \neq \theta_2 \). Moreover, \( \mathcal{P}_\theta \) is not \( \sigma_- \) invariant in that case.

Nevertheless, \( \mathcal{P}_\theta \) has a fundamental invariance property.

**Lemma 2.6.** The space \( \mathcal{P}_\theta \) is shift invariant.

**Proof.** Any \( p \in \mathcal{P}_\theta \) can be written as
\[
p(x) = \sum_{|\gamma| \leq \deg q} \frac{1}{\gamma!} \Delta^\gamma q(0)(-x)^\gamma = \sum_{|\gamma| \leq \deg q} (-1)^{|\gamma|} \frac{1}{\gamma!} \Delta^\gamma q(0)x^\gamma
\]
for some \( q \in \hat{\mathcal{Q}}_\theta \), hence, since \( \Delta^\gamma q(0) = 0 \) for \( |\gamma| > \deg q \),
\[
p(x + y) = \sum_{\gamma \in \mathbb{Z}^s} (-1)^{|\gamma|} \frac{1}{\gamma!} \Delta^\gamma q(0) \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} x^\beta y^{\gamma - \beta}
\]
\[
= \sum_{\gamma \in \mathbb{Z}^s} (-1)^{|\gamma|} \frac{1}{\gamma!} \sum_{\beta \leq \gamma} \binom{\gamma}{\beta} \Delta^\beta \Delta^\gamma q(0)x^\beta y^{\gamma - \beta}
\]
\[
= \sum_{\alpha, \beta \in \mathbb{Z}^s} \frac{1}{(\alpha + \beta)!} \binom{\alpha + \beta}{\beta} \Delta^\beta \Delta^\alpha q(0)(-x)^\beta (-y)^\alpha
\]
\[
= \sum_{\beta \in \mathbb{Z}^s} \frac{(-x)^\beta}{\beta!} \Delta^\beta \sum_{\alpha \in \mathbb{Z}^s} \frac{1}{\alpha!} \Delta^\alpha q(0)(-y)^\alpha = \sum_{\beta \in \mathbb{Z}^s} \frac{(-x)^\beta}{\beta!} \Delta^\beta \sigma q(0)
\]
\[
= \sigma_- L q_\theta(x)
\]
and since
\[ q_y := \sum_{|\alpha| \leq \deg q} \frac{(-y)^\alpha}{\alpha!} \Delta^\alpha q \]
belongs to \( \hat{Q}_\theta \) as this space is \( D \)-invariant, we can conclude that \( p(x + y) \in P_\theta \) as well. \( \square \)

Recalling an argument from [6], we note that the shift invariance of \( P_\theta \) implies that for any \( f \in P_\theta \) we have
\[ f(x + y) = \sum_{p \in P} g(y) p(x), \quad P_\theta = \sigma - L\sigma_0 Q, \]
and since, by symmetry, also \( g \in P_\theta \), we get that
\[ f(x + y) = \sum_{p, p' \in P} a_{p, p'}(f) p(x) p'(y), \quad a_{p, p'}(f) = a_{p', p}(f) \in \mathbb{R}. \]

In particular, any basis element \( p \in P_\theta \) can be written as
\[ p(x + y) = \sum_{p' \in P_\theta} g_{p, p'}(y) p'(x), \quad g_{p, p'} = \sum_{\tilde{p} \in P_\theta} a_{\tilde{p}, p'}(p) \tilde{p}, \]
or, in matrix notation, \( P_\theta (\cdot + y) = G(y) P_\theta \) with \( G(0) = I \); moreover, it was it was shown in [6] that \( \det G(y) = 1, \ y \in \mathbb{R}^s \). After defining the unimodular polynomial matrices
\[ \tilde{G} := [g_{q, q'} := g_{\sigma - L\sigma_0 q, \sigma - L\sigma_0 q'} : q, q' \in Q_\theta] \in \Pi^{Q_\theta \times Q_\theta} \]
and
\[ \hat{G} := [g_{p, q} := g_{p, \sigma - L\sigma_0 q} : p \in P_\theta, q \in Q_\theta] \in \Pi^{P_\theta \times Q_\theta} \]
which only differ in their way of indexing, we have all tools at hand to prove the next result.

**Proposition 2.7.** Let \( \theta \in \mathbb{C}_*^s \) and \( Q_\theta \) be a finite dimensional \( D \)-invariant subspace of \( \Pi \). Then the following statements are equivalent:

1. \( h^*(\mathcal{P}_\theta e_\theta) = 0 \), where \( \mathcal{P}_\theta \) is defined in (2.6).
2. \( q(D) h^*(\theta^{-1}) = 0, \ q \in Q_\theta. \)

**Proof.** We first note that the Newton formula for the Lagrange interpolant, [8, 15], yields for any polynomial \( f \in \Pi \) that
\[ f = \sum_{|\gamma| \leq \deg f} \frac{1}{\gamma!} \Delta^\gamma f(0)(\cdot)_\gamma, \quad (x)_\gamma = \prod_{j=1}^{s} \prod_{k=0}^{\gamma_j - 1} (x_j - k), \]
hence, for \( p \in \mathcal{P}_\theta \), \( p = \sigma_-p', p' \in L^{-1}\hat{Q}_\theta \), and \( \alpha \in \mathbb{Z}^* \),
\[
\begin{align*}
    h \ast (pe_\theta)(\alpha) &= \sum_{\beta \in \mathbb{Z}^*} h(\beta)p'(\beta - \alpha)\theta^{\alpha - \beta} \\
    &= \theta^\alpha \sum_{\beta \in \mathbb{Z}^*} h(\beta) \sum_{|\gamma| \leq \deg p} \frac{1}{\gamma!} \Delta^\gamma(\tau^{-\alpha}p')(0)(\beta)\theta^{-\beta} \\
    &= \theta^\alpha \sum_{|\gamma| \leq \deg p} \frac{1}{\gamma!} \Delta^\gamma(\tau^{-\alpha}p')(0)\theta^{-\gamma} \left( D^{\gamma} \sum_{\beta \in \mathbb{Z}^*} h(\beta)(\cdot)^\beta \right)(\theta^{-1}) \\
    &= \theta^\alpha (L\tau^{-\alpha}p')(\theta^{-1}D)h^*(\theta^{-1}).
\end{align*}
\]
By \( \text{(2.7)} \) it follows that
\[
\begin{align*}
    h \ast (pe_\theta)(\alpha) &= \theta^\alpha \sum_{p_1 \in \mathcal{P}_\theta} \sum_{q \in \hat{Q}_\theta} a_{p_1,\sigma_-L\sigma_\theta q}(p)p_1(\alpha)(LL^{-1}\sigma_\theta q)(\theta^{-1}D)h^*(\theta^{-1}) \\
    &= \theta^\alpha \sum_{q \in \hat{Q}_\theta} \left( \sum_{p_1 \in \mathcal{P}_\theta} a_{p_1,\sigma_-L\sigma_\theta q}(p) p_1(\alpha) \right) q(D)h^*(\theta^{-1}) \\
    &= \theta^\alpha \sum_{q \in \hat{Q}_\theta} p_q(\alpha) q(D)h^*(\theta^{-1}),
\end{align*}
\]
with
\[
p_q := \sum_{p_1 \in \mathcal{P}_\theta} a_{p_1,\sigma_-L\sigma_\theta q}(p) p_1 \in \mathcal{P}_\theta.
\]
Consequently, \( \text{(2)} \) implies \( \text{(1)} \) while for the converse we only need to set \( p := \sigma_-L^{-1}\sigma_\theta q \) for \( q \in \hat{Q}_\theta \) to get, according to \( \text{(2.8)} \),
\[
0 = h \ast (pe_\theta) = e_\theta \sum_{q' \in \hat{Q}_\theta} g_{q,q'} q'(D)h^*(\theta^{-1}), \quad q \in \hat{Q}_\theta,
\]
which gives \( 0 = e_\theta \bar{G} Q(D)h^*(\theta^{-1}) \) and since \( \det \bar{G} \equiv 1 \), we can conclude that \( \text{(1)} \) implies \( \text{(2)} \) as well.

**Remark 2.8.** Like in the univariate case, the local space to be annihilated is an exponential polynomial space \( \mathcal{P}_{\sigma_\theta} \), however, it is generally not the same as the multiplicity space \( \hat{Q}_\theta \), see Example \( \text{2.5} \).

### 2.4. Kernels of convolutions.

Now we have all the tools at hand to give the main result of this paper.

**Theorem 2.9.** If \( H \) is a zero dimensional set of impulse responses with zero set \( \Theta^{-1} \) and multiplicities \( \hat{Q}_\theta \), \( \theta \in \Theta \), respectively, then
\[
\text{ker } H = \mathcal{P}_\theta e_\theta, \quad \mathcal{P}_\theta := \sigma_-L^{-1}\sigma_\theta \hat{Q}_\theta.
\]

The proof of Theorem \( \text{2.9} \) is split into the following two propositions.

**Proposition 2.10.** With the assumptions of Theorem \( \text{2.9} \) we have that
\[
\text{ker } H \supseteq \bigoplus_{\theta \in \Theta} \mathcal{P}_\theta e_\theta.
\]
Proof. We identify \( p_\theta \in \mathcal{P}_\theta, \theta \in \Theta \), with its coefficient vector with respect to the basis \( P_\theta \), and write
\[
p_\theta = p_\theta^T P := \sum_{p \in P_\theta} p_{\theta,p}p.
\]
Expanding (2.11) further, we obtain
\[
h \ast \sum_{\theta \in \Theta} p_\theta e_\theta = \sum_{\theta \in \Theta} \sum_{p \in P_\theta} \sum_{q \in q_\theta} a_{p,q} \prod_{\sigma < q}(p_\theta) p q(D) h^*(\theta^{-1})
\]
\[
= \sum_{\theta \in \Theta} \sum_{p,p' \in P_\theta} \sum_{q \in q_\theta} p_{\theta,p'} a_{p,q} \prod_{\sigma < q}(p') p q(D) h^*(\theta^{-1}) = \sum_{\theta \in \Theta} e_\theta p_\theta^T \widehat{G}_\theta Q_\theta(D) h^*(\theta^{-1}).
\]
Since \( Q_\theta(D) h^*(\theta^{-1}) = 0 \) by assumption, (2.13) follows. \( \square \)

**Proposition 2.11.** With the assumptions of Theorem 2.9 we have that
\[
\ker H \subseteq \bigoplus_{\theta \in \Theta} \mathcal{P}_\theta e_\theta.
\]

Proof. We use induction on \#\( \Theta \) where the case \#\( \Theta = 1 \) is covered by Proposition 2.7.

To advance the induction hypothesis, let \( \Theta' = \Theta \cup \{ \theta' \} \) and assume that the result has been proved for \#\( \Theta \). With
\[
\mathcal{I}_\Theta := \{ f \in \Pi : q(D)f(\theta^{-1}) = 0, q \in Q_\theta, \theta \in \Theta \}
\]
and any basis \( H^*_\Theta \) of \( \mathcal{I}_\Theta \) we get the quotient ideal representation
\[
\langle H^*_\Theta \rangle = \mathcal{I}_\Theta : \mathcal{I}_\Theta = \mathcal{I}_\Theta : \mathcal{I}_\Theta = \langle H^*_\Theta \rangle : \mathcal{I}_\Theta
\]
as well as (2.13). Let \( H^*_\Theta \) be any basis of \( \mathcal{I}_\Theta \), then (2.15) can be rephrased as
\[
h \ast c \in \ker H_\Theta, \quad h \in H_\Theta, c \in \ker H.
\]
Using the vector \( h = [h : h \in H_\Theta] \), this can be written by the induction hypothesis (2.13) as
\[
h \ast c = \sum_{\theta \in \Theta} p_\theta e_\theta, \quad p_\theta = [p_{\theta,h} : h \in H_\Theta] \in \mathcal{P}_\theta^H H_\Theta.
\]
We want to find \( s_\theta \in \mathcal{P}_\theta, \theta \in \Theta \), such that
\[
c = \sum_{\theta \in \Theta} s_\theta e_\theta
\]
satisfies (2.16). To that end, we use a special basis \( h \), consisting of the polynomials \( f_{\theta,q} \in \mathcal{I}_{\Theta} \). \( q \in Q_\theta, \theta \in \Theta \) such that
\[
\tilde{q}(D)f_{\theta,q}(\tilde{\theta}^{-1}) = \delta_{q,q}\delta_{\theta,\tilde{\theta}}, \quad \tilde{q} \in Q_{\tilde{\theta}}, \tilde{\theta} \in \Theta,
\]
and a basis \( H_{\Theta} \) of \( \mathcal{I}_{\Theta} \). The polynomials \( f_{\theta,q} \) exist since the associated Hermite interpolation problem is an ideal one, cf. [11] [13], and they are fundamental solutions for this problem. Since any element of \( \mathcal{I}_{\Theta} \) can be expressed as the sum of the Hermite interpolant and an element from \( \mathcal{I}_{\Theta} \), this is a proper basis for the ideal \( \mathcal{I}_{\Theta} \). Let us again write
\[
p_\theta = \sum_{p \in P_\theta} p_{\theta,p}p \quad \text{and} \quad s_\theta = \sum_{p \in P_\theta} s_{\theta,p}p, \quad p_{\theta,p} \in \mathbb{R}^H, \quad s_{\theta,p} \in \mathbb{R},
\]
as well as $s^T_\theta = [s_{\theta,p} : p \in P_\theta]$ and $p^T_{\theta,h} = [p_{\theta,h,p} : p \in P_\theta]$, respectively, for the row vectors of the coefficients, then the same computation as in the proof of Proposition 2.11 yields for $h \in H_\theta$

$$h \ast \sum_{\theta \in \Theta} s_{\theta} e_\theta = \sum_{\theta \in \Theta} e_\theta s^T_\theta \hat{G}_\theta Q_\theta(D) h^*(\theta^{-1}),$$

and also gives by symmetry, for $h, h' \in H_{\theta'}$,

$$h' \ast h \ast c = h' \ast \sum_{\theta \in \Theta} p_{\theta,h} e_\theta$$

$$= \sum_{\theta \in \Theta} e_\theta p^T_{\theta,h} \hat{G}_\theta Q_\theta(D) h'^*(\theta^{-1}) = \sum_{\theta \in \Theta} e_\theta p^T_{\theta,h} \hat{G}_\theta Q_\theta(D) h^*(\theta^{-1}),$$

that is,

$$(2.17) \quad p^T_{\theta,h} \hat{G}_\theta Q_\theta(D) h^*(\theta^{-1}) = p^T_{\theta,h} \hat{G}_\theta Q_\theta(D) h^*(\theta^{-1}),$$

which immediately gives that $p_{\theta,h} = 0$ whenever $Q_\theta(D) h^*(\theta^{-1}) = 0$, while for $h^* = f_{\theta,q}$ and $h'^* = f_{\theta,q'}$ we get

$$(2.18) \quad p^T_{\theta,h} \hat{G}_\theta \delta_{q'} = p^T_{\theta,h} \hat{G}_\theta \delta_q, \quad \delta_q := [\delta_{\theta,q'} : q' \in Q_{\theta}].$$

Setting $h^* = f_{\theta,q}$ and $P_{\theta} = [\sigma - L \sigma q : q \in Q_{\theta}]$, the requirement that the $s_\theta$ satisfy (2.10) can be expressed by (2.18) as

$$s^T_\theta \hat{G}_\theta = p^T_{\theta,h} \hat{G}_\theta P_{\theta}(0) = \sum_{q' \in Q_{\theta}} (\sigma - L \sigma q')(0) p^T_{\theta,h} \hat{G} \delta_{q'},$$

from which it follows that

$$(2.19) \quad s_\theta = \sum_{q' \in Q_{\theta}} (\sigma - L \sigma q')(0) p^T_{\theta,h} \hat{G} \delta_{q'},$$

guarantees is a solution for (2.10). Since any two solutions $c, c'$ of (2.16) must satisfy $h \ast (c - c') = 0$, it follows that

$$c - c' \in \ker H_{\theta'}, \quad i.e. \quad c - c' = s_{\theta'} e_{\theta'},$$

again by Proposition 2.14. In other words, $c \in \ker H$ implies that

$$c = \sum_{\theta \in \Theta'} s_{\theta'} e_{\theta'}, \quad s_{\theta} \in P_{\theta}, \theta \in \Theta',$$

which advances the induction hypothesis and completes the proof. \hfill \Box

Theorem 2.15 is the direct generalization of Theorem 1.14 to the case of several variables. The main difference is that the $D$–invariant space $Q_{\theta}$ that describes the multiplicity of the common zeros of the symbol is mapped to the shift invariant space $P_{\theta}$ that describes which polynomials to multiply to the exponential $e_\theta$. As Example 2.5 shows, these spaces need not coincide at all, though they have same dimension, hence the same scalar multiplicity. Nevertheless, the kernel space depends directly on the zeros and their multiplicity and the bases of the two spaces can even be chosen in such a way that they have they same homogeneous leading forms.
There is, however, an important special case, namely, when the $Q_\theta$ are spanned by monomials, more precisely, a lower set of monomials:

\[ (\cdot)^\alpha \in Q_\theta \quad \Rightarrow \quad (\cdot)^\beta \in Q_\theta, \quad \beta \leq \alpha. \]

In this case, $\hat{Q}_\theta = Q_\theta$ and $LQ_\theta = Q_\theta$, hence $P_\theta = Q_\theta$. This holds true in particular for the case of zeros of order $k$ or fat points, which is defined as $Q_\theta = \Pi_{k_\theta}$, $k_\theta \in \mathbb{N}_0$. Since in one variable multiplicities are always fat points, the discrepancy between $Q_\theta$ and $P_\theta$ is indeed a truly multivariate phenomenon.

### 2.5. Eigenvectors of convolutions.

A simple application of Theorem [2.9] is to find eigensequences of convolution operators. Suppose that $H \subset \ell_{\ell_0}(\mathbb{Z}^s)$ is again a finite set of impulse responses and assume that there exist $\lambda_h \in \mathbb{C}$ and $\alpha_h \in \mathbb{N}_0^s$ such that

\[ h \ast c = \lambda_h c(\cdot + \alpha_h), \quad h \in H. \]

This is equivalent to

\[ (h(\cdot + \alpha_h) - \lambda_h \delta) \ast c = 0, \quad h \in H \]

and thus depends on the zeros of the (Laurent) ideal

\[ \langle z^{-\alpha_h} h^*(z) - \lambda_h \rangle : h \in H \rangle = \langle h^*(z) - \lambda_h z^{\alpha_h} : h \in H \rangle. \]

Hence, also the eigensequences of convolution operators can be only exponential polynomials.

**Corollary 2.12.** If $\langle h^*(z) - \lambda z^{\alpha_h} : h \in H \rangle$ is zero dimensional with zeros $\Theta^{-1} \subset \mathbb{C}_\chi$ and respective multiplicities $Q_\theta$, then the solutions of (2.20) are $\otimes \theta P_\theta c_\theta$ and the conditions on $H^s$ are

\[ q(D)h^*(\theta^{-1}) = \lambda_h (q(D)(\cdot)^{\alpha_h})(\theta^{-1}), \quad q \in Q_\theta, \quad \theta \in \Theta, \quad h \in H. \]

The situation is particularly simple if no shifts are involved as then $q(D)(h^* - \lambda_h)(\theta^{-1})$ yields the conditions

\[
\begin{align*}
(2.21) \quad h^*(\theta^{-1}) &= \lambda_h, \\
q(D)h^*(\theta^{-1}) &= 0, \quad q \in Q_\theta, \quad \deg q > 0.
\end{align*}
\]

### 3. Kernels of subdivision operators

As a final application of Theorem [2.9] we have a brief look at the kernels of subdivision operators in several variables. To that end, let $\Xi \in \mathbb{Z}^{s \times s}$ be an expanding matrix which means that all eigenvalues of $\Xi$ are larger than one in modulus, or, equivalently, that $\|\Xi^{-k}\| \to 0$ as $k \to \infty$. A stationary subdivision operator $S_\alpha$ with scaling matrix $\Xi$ and finitely supported mask $\alpha \in \ell_{\ell_0}(\mathbb{Z}^s)$ acts on $\ell(\mathbb{Z}^s)$ in the convolution–like way

\[ S_\alpha c = \sum_{\alpha \in \mathbb{Z}^s} a(\cdot - \Xi \alpha) c(\alpha), \quad c \in \ell(\mathbb{Z}^s). \]

To analyze the kernels of such operators, we need a little bit more terminology. By $E_\Xi := \Xi[0, 1)^s \cap \mathbb{Z}^s$ we denote the set of coset representers for $\mathbb{Z}^s/\Xi \mathbb{Z}^s$, i.e.,

\[ \mathbb{Z}^s = \bigcup_{\xi \in E_\Xi} \xi + \Xi \mathbb{Z}^s. \]

Similarly, $E_\Xi' := \Xi^T[0, 1)^s \cap \mathbb{Z}^s$ stands for the representers of $\mathbb{Z}^s/\Xi^T \mathbb{Z}^s$. 


An important tool will be the subsymbols of $a$, defined as

$$a^*_\xi(z) = \sum_{\alpha \in \mathbb{Z}^s} a(\xi + \Xi \alpha)z^\alpha, \quad \xi \in E_\Xi, \; z \in \mathbb{C}_\xi^s.$$ 

It is easily seen that symbol and subsymbols are related via

(3.2) \[ a^*(z) = \sum_{\xi \in E_\Xi} z^\xi a^*_\xi(z^\Xi) \]

and

(3.3) \[ a^*_\xi(z^\Xi) = \frac{1}{|\det \Xi|} \sum_{\xi' \in E'_\Xi} e^{-2\pi i \xi^T \Xi^{-T} \xi'} a^*(e^{-2\pi i \Xi^{-T} \xi'} z), \]

cf. [14]. Here, $z^\Xi = (z^{\xi_1}, \ldots, z^{\xi_s})$, where the $\xi_j$ are the columns of $\Xi$, i.e., $\Xi = [\xi_1 \ldots \xi_s]$. Splitting the requirement $S_a c = 0$ modulo $\Xi$, we get

$$0 = S_a c(\xi + \Xi \alpha) = \sum_{\beta \in \mathbb{Z}^s} a(\xi + \Xi(\alpha - \beta))c(\beta) = a_\xi \ast c, \quad \xi \in \Xi,$$

from which the following conclusion can be drawn.

Corollary 3.1. Suppose that the ideal $\langle a^*_\xi : \xi \in E_\Xi \rangle$ is zero dimensional with zeros $\Theta^{-1}$ and respective multiplicities $Q_\theta$. Then $\ker S_a = \bigoplus_{\theta \in \mathbb{Q}} P_\theta e_\theta$.

To describe the kernel of a subdivision scheme in terms of the symbol $a^*$ alone, we say that $\zeta \in \mathbb{C}_\xi^s$ is a symmetric zero of $a^*$ if

(3.4) \[ a^*(e^{-2\pi i \Xi^{-T} \xi'} \zeta) = 0, \quad \xi' \in E'_\Xi. \]

Symmetric zeros of $a^*$ are in one-to-one correspondence with common zeros of $a^*_\xi$.

Lemma 3.2. $\zeta$ is a symmetric zero of $a^*$ if and only if $\zeta^\Xi$ is a common zero of $a^*_\xi$, $\xi \in \Xi$.

Proof. The key to the proof are (3.2), (3.3) and the simple observation that

(3.5) \[ (e^{-2\pi i \Xi^{-T} \xi'})^\Xi = e^{-2\pi i \Xi^{-T} \Xi \xi'} = e^{-2\pi i \xi'} = (1, \ldots, 1). \]

Indeed, if $\zeta$ is a symmetric zero, then (3.3) immediately yields that $a^*_\xi(\zeta^\Xi) = 0$, $\xi \in \Xi$, while for the converse we use (3.2) and (3.5) to verify that

$$a^*(e^{-2\pi i \Xi^{-T} \xi'} \zeta) = \sum_{\xi \in E_\Xi} e^{-2\pi i \xi^T \Xi^{-T} \xi'} a^*_\xi(\zeta^\Xi) = 0$$

holds for $\xi' \in E'_\Xi$, hence $\zeta$ is a symmetric zero. \(\square\)

Therefore, we can describe the kernel of a subdivision operator in terms of its symmetric zeros.

Corollary 3.3. There exists a polynomial space $P_\theta$ with $P_\theta e_\theta \subseteq \ker S_a$ if and only if $\theta^{-\Xi^{-1}}$ is a symmetric zero of $a^*$.

This result can be extended to zeros with multiplicity provided that the structure of the multiplicity is simple enough. The following corollary can be understood as a characterization of vanishing moments of the associated synthesis filterbank, cf. [14].
Corollary 3.4. For a subdivision operator $S_a$ with mask $a \in \ell_{00}(\mathbb{Z})$ and $\Theta \subset \mathbb{C}_x^*$ the following statements are equivalent:

1. $\theta^{\Xi^{-1}}$ is a symmetric zero of $a^*$ of order $k_\theta$, $\theta \in \Theta$.
2. One has

$$\ker S_a = \bigoplus_{\theta \in \Theta} \Pi_\theta e_\theta.$$

Proof. The only thing left to prove is the issue of multiplicity. To that end, we note that

$$\nabla a^* \left( (\cdot)^{\Xi} \right) = A_{\Xi}^a(z) \left( \nabla a^* \right) \left( (\cdot)^{\Xi} \right),$$

where

$$A_{\Xi}^a(z) := \begin{bmatrix} z_1^{-1} & \cdots & \cdots & z_s^{-1} \\ \vdots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix} \Xi \begin{bmatrix} z^\xi_1 \\ \vdots \\ z^\xi_s \end{bmatrix}$$

is nonsingular for $z \in \mathbb{C}^s_x$. Turning to the total derivatives $\nabla^j = \left[ \frac{\partial^j}{\partial z^\alpha} : |\alpha| = j \right]$ of order $j \leq k$, we observe that that

$$(3.6) \quad \nabla^j a^* \left( (\cdot)^{\Xi} \right) = \sum_{\ell=0}^{j} A_{\ell, \ell, \Xi}^a(z) (\nabla^\ell a^*) \left( (\cdot)^{\Xi} \right),$$

where

$$A_{\ell, j, \Xi}^a(z) = A_{\Xi}^a(z) \otimes \cdots \otimes A_{\ell}^a(z)$$

is the $j$-fold Kronecker product of $A_{\Xi}^a$ with itself and thus nonsingular for any $z \in \mathbb{C}_x^s$. This follows from applying $\nabla$ to (3.6) which yields inductively

$$\nabla^{j+1} a^* \left( (\cdot)^{\Xi} \right) = \sum_{\ell=0}^{j} \nabla A_{\ell, j, \Xi}^a(z) \left( (\nabla^\ell a^*) \left( (\cdot)^{\Xi} \right) + A_{\ell, j, \Xi}^a(z) \nabla (\nabla^\ell a^*) \left( (\cdot)^{\Xi} \right) \right)$$

$$= A_{\ell, j, \Xi}^a(z) \left[ A_{\Xi}^a(z) \nabla \frac{\partial^j}{\partial z^\alpha a^*} : |\alpha| = j \right] \left( (\cdot)^{\Xi} \right) + \sum_{\ell=0}^{j} A_{\ell+1, j, \Xi}^a(z) \left( (\nabla^\ell a^*) \left( (\cdot)^{\Xi} \right) \right)$$

$$= \left( A_{\ell, j, \Xi}^a(z) \otimes A_{\Xi}^a(z) \right) (\nabla^{j+1} a^*) \left( (\cdot)^{\Xi} \right) + \sum_{\ell=0}^{j} A_{\ell+1, j, \Xi}^a(z) \left( (\nabla^\ell a^*) \left( (\cdot)^{\Xi} \right) \right).$$

Consequently, we have for any $\theta \in \mathbb{C}_x^s$ that $(\nabla^j a^* \left( (\cdot)^{\Xi} \right))(\theta^{-1}) = 0$, $j = 0, \ldots, k_\theta$ if and only if $(\nabla^j a^* \left( (\cdot)^{\Xi} \right))(\theta^{-1}) = 0$, $j = 0, \ldots, k$.

With this observation, the claim follows immediately from differentiating (3.6) and (3.3).

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