A Type of Unifying Relations for Berends-Giele Currents in (A)dS

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Abstract

Unifying relations of amplitudes are elegant results in flat spacetime, but the research on these in (A)dS case is not very rich. In this paper, we discuss a type of unifying relations in (A)dS by using Berends-Giele currents. By taking the flat limit, we also get a semi-on-shell way to prove the unifying relations in the flat case. We also discuss the applications of our results in cosmology.

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I. INTRODUCTION

Scattering amplitudes are the most fundamental observables in quantum field theory (QFT), and a lot of on-shell methods at the tree level have been researched (see [1–4] and so on for a review). However, compared with the adequacy of the on-shell methods, other methods are less well studied. As a semi-on-shell method, Berends-Giele (BG) currents [5] are popular because of their recursive character. In recent years, many techniques based on BG currents have been developed [6–17]. However, the research of BG currents in (A)dS is still very scarce. This paper explores BG currents’ properties and potential applications in (A)dS.

In flat spacetime, there are many elegant results for tree-level amplitudes. Unifying relations [18] are such results. They reveal the implicit relations among different theories by some differential operators, which tells us that the theories that seem pretty different may have intrinsic relations. We hope these relations can be generalized to the tree-level correlation functions in (A)dS. It seems that some recursion relations or factorization methods [19–27] are helpful in the proof of the unifying relations in (A)dS, which are the methods we use in the flat case. However, in dS spacetime, only the factorization methods for 4-pt correlation functions are understood very well [28, 29], thus it is still unknown whether they can be applied to this proof. Instead, BG currents recursion, as a semi-on-shell (semi-on-boundary) method, stands out because it is clearer than factorization methods in (A)dS [8]. In this paper, we discuss the unifying relations between gluons and conformally coupling scalars (minimal coupled with gluons) by using the recursion properties of BG currents. We find that this type of unifying relations holds in (A)dS only for the “1+1” dimension or 4-pt correlation functions. By the way, we can also prove the unifying relations for flat spacetime by taking the flat limit. This paper explores BG currents in (A)dS. In such cases, the unifying relations hold for arbitrary dimensions and arbitrary point functions.

Based on our conclusion, we want to discover some potential cosmology applications. It is widely held that at the beginning of our universe, it underwent an exponential expansion called cosmic inflation, and its background spacetime can be viewed as the dS space. During the period of inflation, the quantum fluctuations for all possible fields provide a source to generate the non-gaussianity and seed the Large Scale Structure (LSS) at present. In other words, the fluctuations of particles imprint some signals in the sky today, which allows us to research the history of the primordial universe. Such fascinating information can be abstracted from the cosmological correlation functions [24, 30]. The 4-pt correlation functions are the most valuable ones in cosmic experiments, and it is complicated to calculate [23, 25, 26, 31–33]. In this paper, we present a new aspect to understand the relations among different 4-pt tree-level cosmological correlation functions by unifying relations.

The paper is organized as follows. In section II, we briefly review the remarkable unifying relations. In section III, we review the concepts of BG currents and introduce the methods to calculate them. In section IV, we give some examples of a type of unifying relations for
BG currents in (A)dS and then discuss the general form. In section V, we talk about some applications of our results in cosmology.

II. UNIFYING RELATIONS IN FLAT SPACETIME

The unifying relations are first discovered in flat spacetime [18] and are found to be consistent with Cachazo-He-Yuan (CHY) formalism [34]. Let us review the unifying relations in [18] very briefly.

There are many theories in QFT, such as Yang-Mills (YM) theory, the bi-adjoint scalar (BS) theory, the special Galileon (SG) theory, and the Yang-Mills scalar (YMS) theory. Moreover, they seem to be quite different from each other formally. However, their amplitudes at the tree level can be related by using some differential operators. These operators need to preserve on-shell kinematics and gauge invariance of the amplitudes. Here we write down the operators directly:

\[
T_{ij} = \partial_{k_i \epsilon_j}
\]

\[
T_{ijk} = \partial_{k_i} \epsilon_j - \partial_{k_j} \epsilon_i
\]

\[
T[\alpha] = T_{\alpha_1 \alpha_n} \cdot \prod_{i=2}^{n-1} T_{\alpha_{i-1} \alpha_i \alpha_n}
\]

\[
L_i = \sum_k k_i k_j \partial_{k_j \epsilon_i}
\]

\[
L = \prod_i L_i.
\]

(1)

Here \(k_i\) is the momentum of the \(i\)-th particle, and \(\epsilon_i\) is the polarization vector of the \(i\)-th particle. Using these operators, we can transform the tree-level amplitudes of some theories into other theories. These transformations are concluded as “unifying relations”. We can write down these unifying relations explicitly:

\[
A_{YMS} = T[i_1 j_1] T[i_2 j_2] \cdots T[i_n j_n] \cdot A_{YM}
\]

\[
A_{BS} = T[i_1 i_2 \cdots i_n] \cdot A_{YM}
\]

\[
A_{NLSM} = L \cdot T[i_1 i_n] \cdot A_{YM}.
\]

(2)

By replacing \(A_{YM}\) with BI amplitudes \(A_{BI}\) or extended gravity amplitudes \(A_{G}\), we will get other unifying relations:

\[
A_{DBI} = T[i_1 j_1] T[i_2 j_2] \cdots T[i_n j_n] \cdot A_{BI}
\]

\[
A_{NLSM} = T[i_1 i_2 \cdots i_n] \cdot A_{BI}
\]

\[
A_{SG} = L \cdot T[i_1 i_n] \cdot A_{BI}
\]

\[
A_{EM} = T[i_1 j_1] T[i_2 j_2] \cdots T[i_n j_n] \cdot A_{G}
\]

\[
A_{YM} = T[i_1 i_2 \cdots i_n] \cdot A_{G}
\]

\[
A_{BI} = L \cdot T[i_1 i_n] \cdot A_{G}.
\]

(3)
Here the “EM” is the Einstein-Maxwell theory. We need to point out that the particles denoted by indices in each $T[\alpha]$ are in the same trace after $T[\alpha]$ acting on a certain amplitude. By the way, $T[ij] = T_{ij}$ is called “trace operator”. For example,

$$A_{\text{YMS}}(\phi_1\phi_2, \phi_3\phi_4, \cdots, \phi_{2n-1}\phi_{2n}) = T[12]T[34]\cdots T[2n - 1, 2n] \cdot A_{\text{YM}}(g_1, g_2, \cdots, g_{2n-1}, g_{2n}).$$  \hspace{1cm} (4)

Another important thing is that we can also use this relation to get a mixed amplitude. For example,

$$A_{\text{YMS}}(\phi_1\phi_2, g_3, g_4) = T[12] \cdot A_{\text{YM}}(g_1, g_2, g_3, g_4).$$  \hspace{1cm} (5)

The proof of these relations is based on the on-shell factorization.

In this paper, we will not discuss the operator $L$, so we skip the discussion about that. For a more comprehensive introduction, see [18].

### III. BG CURRENTS AND RECURRENCE

In this section, we will talk about the BG currents and recursions both in the flat and (A)dS cases. They can both be derived from the classical equation of motion [12, 17] and are connected by taking the flat limit, which will be introduced in this section.

#### A. BG Currents in Flat Spacetime

In this subsection, we will review some properties of BG currents in flat spacetime. We will mainly follow the methods in [17] but with different normalizations.

The definition of $n$-pt BG currents is $n$-pt tree amplitudes with an external leg being off-shell. It is well-known that BG currents can be computed from the classical equation of motion. Here we give the calculation of the BG currents of YM theory. From now on, we will take the Lorenz gauge. First of all, we write down the Lagrangian of YM theory (here we use the matrix formalism):

$$\mathcal{L}_{\text{YM}} = \frac{1}{4} \text{Tr} (F_{\mu\nu} F^{\mu\nu}),$$  \hspace{1cm} (6)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ and $\nabla_\mu = \partial_\mu - i[A_\mu, \cdot]$. The equation of motion is

$$\Box A^\mu(x) = i[A_\mu(x), \partial^\nu A^\nu(x)] + i[A_\mu(x), F_{\mu\nu}(x)].$$  \hspace{1cm} (7)

We introduce the perturbative expansion ansatz first:

$$A_\mu := \sum_P A_\mu P \cdot e^{ik_P x} = \sum_i A^\mu_i T^a_i e^{ik_i x} + \sum_{i,j} A^\mu_{ij} T^a_i T^a_j e^{ik_{ij} x} + \cdots,$$

$$F_{\mu\nu} := i \sum_P F_{\mu\nu} P \cdot e^{ik_P x}.$$  \hspace{1cm} (8)
Here $P$ goes over all non-empty words. For $A_\mu$ is in adjoint representation, we have the following shuffle identities \cite{14, 35, 36}:

\[ A_\mu^\nu_{P \cup Q} = F^\mu_\nu_{P \cup Q} = 0, \quad P, Q \neq \emptyset. \] (9)

where $\cup$ means we sum over all the permutations of the labels in $P \cup Q$ with the ordering of labels in $P$ and $Q$ preserved respectively. Then one can understand the perturbation expansion ansatz (8) as “power series” of the structure constant $f_{abc}$ of the corresponding Lie algebra.

These expansion is called “color-stripped” for the coefficients $A_\mu^\nu$ and $F^\mu_\nu$ do not have any color degrees of freedom. We can construct color-ordered amplitudes by these coefficients. After substituting the ansatz and shuffle identities to the equation of motion, we have

\[ -s_P A_\mu^\nu = \sum_{P=XY} [A_\mu^\nu (k_X \cdot A_Y) + A_X^\nu F^\mu_\nu - (X \leftrightarrow Y)] \]

\[ F^\mu_\nu = k_\mu^\nu A_\nu^\nu - k_\nu^\nu A_\mu^\nu - \sum_{Y=RS} (A_\mu^\nu A_\nu^\nu - A_\nu^\nu A_\mu^\nu). \] (10)

where $-1/s_P$ is the propagator of the off-shell leg and the sum over $P = JK$ represents the sum over divisions which includes all the order-preserving ways of splitting the $P$ into $JK$, like $P = 12345$ and $J = 123, K = 45$. The terms summed over once involve the information of 3-pt vertices, while the terms summed over twice, for example, $A_\mu^\nu (A_X \cdot A_S)$, involves 4-pt vertices since any two adjacent vertices of a binary tree has 4 lines connecting to other vertices so that it can be thought as a 4-pt vertex effectively. We impose the one-particle states as the initial conditions:

\[ A_\mu^\nu = \epsilon_\mu^\nu, \] (11)

where the polarization vectors $\epsilon_\mu^\nu$ satisfy the transversality condition $\epsilon_i^\mu \cdot k_i = 0$.

Note that the sum in (10) can be represented as the sum of all possible binary trees; see \cite{12}. In \cite{12}, they use a different notation. One of the advantages of this notation is that we can divide our currents into many binary trees, and each tree relates to a Lie monomial \cite{12}, which brings us many conveniences. This correspondence can be obtained by replacing the generators $T^a_i$ by the indices $i$. Then after substituting our ansatz, we will get some Lie monomials like $[[1,2],3]$ where 1 and 2 are two different branches originating from the vertex. Then 1 and 2, as a whole, together with 3 are also two different branches from a new vertex. These can be diagrammatically represented by the binary tree FIG. 1.
FIG. 1: Binary representations for Lie monomials. (a) is the diagrammatic representation for \([1,2,3]\) and (b) represents the Lie algebra structure for \([1,[2,3]]\).

We can construct amplitudes from BG currents, which can be realized by moving the off-shell leg to be on-shell. For YM color-ordered amplitudes \(A_{\text{YM}}\), we have

\[
A_{\text{YM}}(Pn) = s_P \mathcal{A}_P \cdot \mathcal{A}_n \quad (s_P \to 0). \tag{12}
\]

For BS theory, we can also obtain its BG currents:

\[
\Phi_{P|Q} = \frac{1}{s_P} \sum_{P=RS} \sum_{Q=TU} [\Phi_{R|T} \Phi_{S|U} - (R \leftrightarrow S)], \tag{13}
\]

and the initial conditions \(\Phi_{ij} = \delta_{ij}\). One can also see [6, 12] for a more compact formalism. The double color-ordered amplitude of BS theory can also be constructed by BG currents:

\[
m(Pn|Qn) = s_P \Phi_{P|Q} \Phi_{n|n} \quad (s_P \to 0). \tag{14}
\]

We should point out an interesting thing. By counting the number of \(\epsilon_i \cdot k_j\) and \(\epsilon_i \cdot \epsilon_j\), we find that

\[
\Phi_{P|Q} = \mathcal{T}[Qn] \cdot (\mathcal{A}_P \cdot \epsilon_n) \tag{15}
\]
is correct for every binary tree up to an overall constant. Here we require that the length of the word \(P\) is the same as \(Qn\). Note that here the \(n\)-th particle is still off-shell, and the vector \(\epsilon_n\) is only a formal polarization vector which is used for the convenience of allowing us to use the trace operator \(\mathcal{T}[Qn]\). Then (15) is just a relation for currents, and it is equivalent to the relation for corresponding amplitudes since the unifying relations for amplitudes can easily be derived from the unifying relations for currents and vice versa. This fact implies that we can use BG currents to prove some of the unifying relations for amplitudes by proving the relations for currents of every binary tree, which means we find a semi-on-shell way to prove the unifying relations for amplitudes. This is the starting point of our work.
B. BG Currents in (A)dS

Now we tend to focus on the BG currents in curved spacetime. Though BG currents themselves were proposed in flat spacetime [36] for the first time, we can also generalize it to (A)dS [8]. We can define BG currents as $n$-pt tree-level correlation functions with all but one external leg on the spacetime boundary. We need to point out that the radial coordinate is not Fourier transformed; hence if we want to construct correlation functions, we need to integrate the radial coordinate. Previous literature [8] has developed the recursions for BG currents in (A)dS by the classical equation of motion. We briefly review a bit about it and apply the recursions here to derive the unifying relations in (A)dS. One can also find other perspectives to understand the BG currents in curved spacetime [37, 38]. This section will mainly follow the notations in [8].

For convenience, we consider the AdS$_{d+1}$ in the Poincaré patch and the metric can be parameterized as

$$ds^2 = \frac{l^2}{z^2} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad (16)$$

where $l$ is the radius and $0 < z < \infty$. Here $\eta_{\mu\nu}$ is the flat boundary metric with Lorentzian signature and $\mu, \nu = 0, 1, \cdots, d - 1$. It should be noted that the dS$_{d+1}$ metric can be derived from (16) through analytical continuation $z \to -i\eta$ and $l \to -il$ after taking the boundary metric to be Euclidean. The equation of motion for Yang-Mills theory in AdS background is

$$g^{np} \partial_p F_{mn} = ig^{np}[A_p, F_{mn}] + J_m + g^{np}(\Gamma^q_{mp} F_{qn} + \Gamma^q_{np} F_{mq}), \quad (17)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m - i[A_m, A_n]$ is the field strength, $A_m$ is the Lie algebra valued for some gauge group with the generator $T^a$. And $J_m$ is the current coupled to other fields and is the model dependent.

The ansatz for multi-particle solutions of the Yang-Mills theory in AdS can be written as

$$A_\mu(x, z) = \frac{l}{z} \sum_I A_{I\mu}(z) T^{a_I} e^{ik_I \cdot x},$$

$$A_z(x, z) = \frac{l}{z} \sum_I \alpha_I(z) T^{a_I} e^{ik_I \cdot x},$$

$$J_m(x, z) = \frac{l}{z} \sum_I J_{Im}(z) T^{a_I} e^{ik_I \cdot x}, \quad (18)$$

where $I$ is a bunch of letters $I = i_1 \cdots i_s$ and each $i_s$ represents for a single particle state. Here $k_{I\mu} = k_{1\mu} + \cdots + k_{s\mu}$ is the total momentum for multi-particle states and $T^{a_I} = T^{a_{i_1}} \cdots T^{a_{i_s}}$. Now we let $U \cdot V = \eta_{\mu\nu} U^\mu V^\nu$. Substituting the ansatz (18) into the equation of motion (17),
we can obtain the multi-particle recursions:

\[
\frac{1}{z^2}(D^2 + d - 1)A_{\mu} = ik_{I\mu}(\partial_z + \frac{2 - d}{z})\alpha_I - \frac{l}{z}J_{I\mu} + \sum_{I=JKL} \{(k_{K\mu}\alpha_K + 2i\partial_z A_{K\mu})\alpha_J
\]

\[
+ k_{K\mu}(A_{J\cdot K}) + A_{K\mu}[i(\partial_z - \frac{d}{z})\alpha_J - 2(k_{K\cdot A_J})] - (J \leftrightarrow K)\}
\]

\[
+ \frac{l^2}{z^2} \sum_{I=JKL} \{[\alpha_J\alpha_K A_{L\mu}] + (A_{J\cdot A_K})A_{L\mu} - (K \leftrightarrow L)]
\]

\[
+ [\alpha_K\alpha_L A_{J\mu}] + (A_{K\cdot A_L})A_{J\mu} - (J \leftrightarrow K)\}
\]

(19)

and

\[
k^2_{I\alpha_I} = \frac{l}{z} \sum_{I=JKL} \{2\alpha_K(k_{K\cdot A_J}) - 2\alpha_J(k_{J\cdot A_K}) + i(A_{J\cdot \partial_z A_K}) - i(A_{K\cdot \partial_z A_J})\]

\[
+ \frac{l}{z}J_{I\alpha_I} + \frac{l^2}{z^2} \sum_{I=JKL} \{\alpha_K(A_{J\cdot A_L}) - \alpha_L(A_{J\cdot A_K}) + (J \leftrightarrow L),
\]

(20)

with \(A_{i\mu} = \phi_i(z)\epsilon_{i\mu}\) and \(\alpha_i = 0\). Here \(D^2 = D^2_{kl}\) is the d’Alembert operator in AdS:

\[
D^2_{kl} = z^2\partial_z^2 + (1 - d)z\partial_z - z^2k^2_{I\alpha_I},
\]

(21)

and \(\phi_i(z)\) satisfies \((D^2 - M^2)\phi_i(z) = 0\). We have also imposed the boundary transversal gauge: \(\eta^{\mu\nu}\partial_\mu A_\nu = 0\). The information of 4-pt vertices is now included in the terms with the sum over \(I = JKL\), which is equivalent to the double sum we have mentioned before since we have summed over all the possible binary trees. By the way, we should emphasize that gluons must be massless; however, the mass we consider here is the effective mass that comes from we take \((18)\) to get a locally flat patch.

The recursions for scalars in the adjoint representation of the gauge group can also be derived from what we have done above. Similarly, we can write down the ansatz for the multi-particle solution to the scalars equation of motion:

\[
\phi = \sum_I \Phi_I(z)T^{a_I}e^{ik_Iz}
\]

(22)

As we have said, the interaction between scalar and gauge field is through the current \(J\) presented in \((18)\). Generally speaking, the interaction should be model dependent. However, we focus on the minimal coupled interaction between scalars and gauge fields for simplicity. In this case the current should be \(J_m = [(i\partial_m\phi + [A_m, \phi]), \phi]\) in \((17)\). Specifically, the minimal coupled interaction terms in BG currents can be written as

\[
J_{I\mu} = \sum_{I=JKL} (-k_{J\mu}\Phi_J\Phi_K + k_{K\mu}\Phi_K\Phi_J)
\]

\[
+ \frac{l}{z} \sum_{I=JKL} [\tilde{A}_{J\mu}\Phi_K\Phi_L - \tilde{A}_{K\mu}\Phi_J\Phi_L - \Phi_J\tilde{A}_{K\mu}\Phi_L + \Phi_J\tilde{A}_{L\mu}\Phi_K]
\]

\[
+ \frac{l}{z} \sum_{I=JKL} [\tilde{\alpha}_J\Phi_K\Phi_L - \tilde{\alpha}_K\Phi_J\Phi_L - \Phi_J\tilde{\alpha}_K\Phi_L + \Phi_J\tilde{\alpha}_K\Phi_L],
\]

(23)
where the tilde is used to distinguish the pure gluon and scalar cases.

Similarly, we can also impose the boundary transversal gauge. Consequently, we can obtain the recursions for scalars minimally coupled to gauge fields:

\[
\frac{1}{z^2}(D_I^2 - M^2)\Phi_I = \frac{l}{z} \sum_{I=JK} [2\Phi_J(k_J \cdot \tilde{A}_K) - i(\Phi_J \partial_z \tilde{\alpha}_K + 2\tilde{\alpha}_K \partial_z \Phi_J + \frac{d-2}{z} \Phi_J \tilde{\alpha}_K) - (J \leftrightarrow K)] \\
+ \frac{l^2}{z^2} \sum_{I=JKL} [\tilde{A}_J \cdot \tilde{A}_K \Phi_L - \tilde{A}_J \cdot \tilde{A}_L \Phi_K + \tilde{\alpha}_J \tilde{\alpha}_K \Phi_L - \tilde{\alpha}_J \tilde{\alpha}_L \Phi_K + (J \leftrightarrow L)].
\]

Note that for the currents above, the flat limit can be realized by taking \( A_{I\mu}(\tilde{A}_{I\mu}, \tilde{J}_{Im}) \rightarrow \tilde{z} A_{I\mu}(\tilde{z} \tilde{A}_{I\mu}, \tilde{z} \tilde{J}_{Im}) \) and \( \phi_i(z)(\Phi_i(z)) = 1 \), then taking the limit \( l/z \rightarrow 1 \). Then \( \alpha_I(\tilde{\alpha}_I) \) will be zero, and the propagators and the ansatz (18) will be the flat ones. This process can be understood as follows: first restore \( z \) (or not) in the ansatz according to (18), then take the mode of flat Klein-Gordon equation to construct one-particle states and the metric to be flat. After we take this limit, the currents in (A)dS coincide with the flat currents.

As in the flat case, we can also construct tree-level correlation functions in (A)dS. We define them for gluons as

\[
A_{YM}(1, 2, \cdots, N) = -\frac{1}{N} \int \frac{dz}{z^{d+1}} A_N \cdot (D_{1\cdots N-1}^2 + d - 1)A_{1\cdots N-1} + \text{cyclic}(1, 2, \cdots, N),
\]

and for scalars as

\[
A_S(1, 2, \cdots, N) = -\frac{1}{N} \int \frac{dz}{z^{d+1}} \Phi_N(D_{1\cdots N-1}^2 - M^2)\Phi_{1\cdots N-1} + \text{cyclic}(1, 2, \cdots, N).
\]

Starting with the equation of motion for minimal coupling scalars and gauge fields, we have written down the recursion relations for both scalars and gauge fields in (A)dS background. In the following sections, we use these recursions to derive a type of unifying relations in (A)dS IV and calculate some specific cosmological correlators which are related to the cosmic observations V.

**IV. A TYPE OF UNIFYING RELATIONS IN (A)dS**

In section II, we have reviewed some of the unifying relations in flat spacetime. As a natural generalization, it is desirable to figure out whether the unifying relations are valid in curved spacetime. In this section IV, we use the BG currents recursions in (A)dS to bridge the correlation functions between pure YM theory and scalar theory minimal coupled with gluons by acting some trace operators \( T[ij] \) on YM side. To be familiar with the unifying relations we aim to prove, we take the four points IV A and six points IV B correlation functions as some examples. Furthermore, we can find the validity of unifying relations in...
such examples. In subsection IV C, we present a rigorous proof to unifying relations in (A)dS. In the final subsection, we discuss a more helpful type of unifying relations: the generalization of the relation in the subsection IV C. In this section, we will focus on the relations between an arbitrary binary tree on both sides, and we set $I = JK, J = AB, K = CD$ for a given binary tree. Then we write down the corresponding BG currents. We will not consider mixed currents of scalars and gluons until subsection IV D.

Before our proof, there are some comments. The scalar theory with a minimal coupling with gluons can be considered a degenerate YMS theory. Compared with YMS theory, minimal coupling scalar theory has only one color index for scalars; hence it only has single color-ordered correlation functions. Moreover, it has no $\phi^3$ and $\phi^4$ vertices. In this case, scalars with no gluon propagator between them are called “in the same trace”. Each trace can only have two scalars because of the lack of $\phi^3$ and $\phi^4$ vertices. Then we only need to let some $T[ij]$ operators act on the YM currents and prove that we will obtain the corresponding scalar currents. This section assumes that the spacetime background is AdS, while the analysis is also valid for the dS case. For convenience, we take $l = 1$, and the scalars are conformal coupling, i.e., $M^2 = 1 - d$, to make the propagators of gluons and scalars the same. In fact, whether the scalars are conformal coupling is not very important because we can establish a map between gluon and scalar propagators with different masses so that our relations are still correct.

The relations we want to prove are

$$T(D^2_I + d - 1)A_I \cdot \epsilon_n = (-1)^{|I|-1} (D^2_I + d - 1)\Phi_I,$$

where $T$ is the sum of operators corresponding to all possible trace structures of the corresponding correlators. For example, for $I = 123$, which corresponds to 4-pt correlation functions, we have $T = T[12]T[34] + T[13]T[24] + T[14]T[23]$.

### A. 4-points Unifying Relations in (A)dS

To convince the validity of unifying relations in AdS, let us consider 4-pt correlation functions first, which correspond to 3-pt BG currents. Without loss of generality, we only consider the following binary tree in this subsection (FIG. 2). In this case, we have $I = 123, J = 12, K = 3, A = 1, B = 2$ and there are no contributions from $C$ or $D$ terms. The corresponding Lie monomial of this binary tree is $[[1,2],3]$. For this binary tree, the only trace structure of the corresponding scalar correlator is $(12|34)$, and the corresponding operator
is $\mathcal{T}[12]\mathcal{T}[34]$. By using the recursions (19), it is not hard to find the gluon currents:

$$
\frac{1}{z^2} (D^2_{123} + d - 1) \mathcal{A}_{123\mu} = i k_{123\mu} \partial_z - \frac{2 - d}{z} \mathcal{A}_{123\mu} + \frac{1}{z^2} [2 i \partial_z \mathcal{A}_{3\mu} \alpha_{12} + k_{3\mu} (\mathcal{A}_{12} \cdot \mathcal{A}_3) \\
+ \mathcal{A}_{3\mu} (i (\partial_z - \frac{d}{z}) \alpha_{12} - 2 (k_3 \cdot \mathcal{A}_{12}) - k_{12\mu} (\mathcal{A}_3 \cdot \mathcal{A}_{12}) + 2 \mathcal{A}_{12\mu} (k_{12} \cdot \mathcal{A}_3))] \\
+ \frac{1}{z^2} [(\mathcal{A}_{2 \cdot \mathcal{A}_3}) \mathcal{A}_{1\mu} - (\mathcal{A}_1 \cdot \mathcal{A}_3) \mathcal{A}_{2\mu}],
$$

(28)

where $\mathcal{A}_{12}$ and $\alpha_{12}$ are the boundary and $z$ components for gluon 2-pt current, and these 2-pt currents can also be solved out from the recursions relation (19):

$$
\frac{1}{z^2} (D^2_{12} + d - 1) \mathcal{A}_{12\mu} = i k_{12\mu} \partial_z - \frac{2 - d}{z} \mathcal{A}_{12\mu} + \frac{1}{z^2} [k_{2\mu} (\mathcal{A}_1 \cdot \mathcal{A}_2) - 2 \mathcal{A}_{2\mu} (k_2 \cdot \mathcal{A}_1) - (1 \leftrightarrow 2)] \\
\alpha_{12} = \frac{1}{z k^2_{12}} i (\epsilon_1 \cdot \epsilon_2) \phi_1 \leftrightarrow \phi_2
$$

(29)

Here we have denoted that $u \leftrightarrow v = u \partial_z v - v \partial_z u$ for simplicity. To show the relations we want to prove, we can act the operator $\mathcal{T}[12]\mathcal{T}[34]$ on the 3-pt current (28) directly. This operation will kill the terms that do not contain $\epsilon_3 \cdot \epsilon_4$. Then the remaining terms after substituting 2-pt currents (29) are:

$$
\frac{1}{z^2} (D^2_{123} + d - 1) \mathcal{A}_{123\mu} = \frac{1}{z^2} [2 i \partial_z \mathcal{A}_{3\mu} \alpha_{12} + \mathcal{A}_{3\mu} (i (\partial_z - \frac{d}{z}) \alpha_{12} - 2 (k_3 \cdot \mathcal{A}_{12}))] \\
= \frac{1}{z^2} [-2 \epsilon_{3\mu} \partial_z \phi_3 (\frac{1}{z k^2_{12}} (\epsilon_1 \cdot \epsilon_2) \phi_1 \leftrightarrow \phi_2) \\
- \epsilon_{3\mu} \phi_3 \{ (\partial_z - \frac{d}{z}) (\frac{1}{z k^2_{12}} (\epsilon_1 \cdot \epsilon_2) \phi_1 \leftrightarrow \phi_2) \\
+ 2 \frac{1}{z k^2_{12} + d - 1} [-k_3 \cdot k_{12} (\partial_z - \frac{d - 2}{z}) (\frac{1}{z k^2_{12}} (\epsilon_1 \cdot \epsilon_2) \phi_1 \leftrightarrow \phi_2) \\
+ \frac{1}{z} k_3 \cdot (k_2 - k_1) (\epsilon_1 \cdot \epsilon_2) \phi_1 \phi_2)]].
$$

(30)

The final result after we act $\mathcal{T}$ (equals to the result we act $\mathcal{T}[12]\mathcal{T}[34]$ since the sum of the contribution of the others is 0) on the 3-pt current (30) is

$$
\mathcal{T} (D_{123} + d - 1) \mathcal{A}_{123} \cdot \epsilon_4 = \mathcal{T}[12] \mathcal{T}[34] (D_{123} + d - 1) \mathcal{A}_{123} \cdot \epsilon_4 \\
= z [-2 \partial_z \phi_3 (\frac{1}{z k^2_{12}} \phi_1 \leftrightarrow \phi_2) - \phi_3 (\partial_z - \frac{d}{z}) (\frac{1}{z k^2_{12}} \phi_1 \leftrightarrow \phi_2) \\
+ 2 \phi_3 \frac{1}{D^2_{12} + d - 1} [k_3 \cdot k_{12} z^2 (\partial_z - \frac{d - 2}{z}) (\frac{1}{z k^2_{12}} \phi_1 \leftrightarrow \phi_2) \\
- z k_3 \cdot (k_2 - k_1) \phi_1 \phi_2]].
$$

(31)
FIG. 2: Binary tree [[1,2],3] for 3-point currents. One can understand the dashed line as the boundary of spacetime. In our following calculation, we focus on the AdS background, but this representation is also correct for the dS background.

For scalar currents, there is no gluon external leg hence $\tilde{A}^\mu$ must be zero. We can write down the 2-pt currents first by the recursions (23) and (24):

$$J_{12} = (k_2 - k_1) \Phi_1 \Phi_2$$

$$J_{1z} = -ik_1 \partial \Phi_2$$

$$\tilde{a}_{12} = -i k_{12} \Phi_1 \partial \Phi_2$$

$$\frac{1}{z^2} (D^2_{12} + d - 1) \tilde{A}_{12} = ik_1 \partial (\partial z + \frac{2 - d}{z}) \tilde{a}_{12} - \frac{1}{z} (k_2 - k_1) \Phi_1 \Phi_2$$

Apply the recursions (24) for scalar into 3-pt scalar correlation and take the 2-pt current (32) into consideration, we can obtain the 3-pt correlation function for the conformally coupled scalar:

$$(D^2_{123} + d - 1) \Phi_{123} = -2 \left( \frac{k_1 \cdot k_3}{k_{12}^2} (k_2^2 - k_1^2) - (k_2 - k_1) \cdot k_3 \right) z \Phi_3 (\frac{1}{D^2_{12} + d - 1} z \Phi_1 \Phi_2)$$

$$+ \frac{1}{k_{12}^2} ((k_2^2 - k_1^2) \Phi_1 \Phi_2 \Phi_3 + 2 \Phi_1 \partial \phi_2 \partial \phi_3 + 2(d - 2) \Phi_3 \frac{\Phi_1 \partial \Phi_2}{z})$$

(33)

After some derivation, we find that When $d = 1$, if we let $\Phi_i = \phi_i$ (from now on, we always take this condition), we will have

$$T[12]T[34](D^2_{123} + d - 1) A_{123} \cdot \epsilon_4 = -(D^2_{123} + d - 1) \Phi_{123}.$$  

(34)

In fact, in the general case,

$$(D^2_{123} + d - 1)(T A_{123} \cdot \epsilon_4 + \Phi_{123}) = \frac{2}{k_{12}^2} (d - 1) \phi_3 \frac{\phi_1 \partial \phi_2}{z}.$$  

(35)

We call the terms like these “anomalous terms”. Fortunately, when $d \neq 1$, the anomalous terms will be canceled by the other possible binary tree [1,[2,3]], so it will not enter the final correlation functions. In fact, consider [4,[1,2]] (we label the off-shell leg “4” in [1,[2,3]] and take cyclic transformation), now “3” is the off-shell leg, and the contribution of this binary
tree to correlation functions is the same as the above binary tree [[1,2],3] up to a minus sign after exchanging “3” and “4”. In the corresponding correlation function, these two terms cancel with each other exactly. Therefore, even though 3-pt currents do not satisfy the unifying relations for the general spacetime dimension, for 4-pt correlation functions, the relation always holds for correlation functions no matter what the spacetime dimension is.

Note that some other trace structures are included in the gluon current above. These terms, which come from the 4-pt vertices of gluons’ action, will lead to an extra 4-pt scalar contact terms after acting, say, the operator $\mathcal{T}|13\rangle\mathcal{T}|24\rangle$. These terms are the same as the flat case [18] and will be canceled here after summing all trace structures. In the higher point cases, these terms can be realized by the vertices of two scalars and two gluons.

**B. 6-points Unifying Relations in (A)dS**

Now we turn to the 6-pt correlation functions and consider the 5-pt currents. In this subsection, we choose the binary tree in (FIG. 3), and the trace structure of the corresponding scalar correlator is (12|36|45). Now we have $J = 123, K = 45, A = 12, B = 3, C = 4, D = 5$, and the corresponding Lie monomial is [[[1,2],3],[4,5]]. Since the binary tree of the 3-pt current we have computed in the previous subsection is the sub-tree of this 5-pt current case, we can use the results in the previous subsection to evaluate the recursion.

![FIG. 3: Binary tree [[[1,2],3],[4,5]] for 5-points currents. One can understand the dashed line as the boundary of spacetime. We focus on the AdS background in our following calculation, but this representation is also correct for the dS background.](image)

Similar to the 4-pt correlation function, we can take nearly the same procedure here to deal with the 6-pt correlation function. First, we can also write down the 5-pt scalar current from the scalar recursions (24).

$$
(D^2_{12345} + d - 1)\Phi_{12345} = z[2\Phi_{123}(k_{123} \cdot \tilde{A}_{45}) - i(\Phi_{123}\partial_z \tilde{A}_{45} + 2\tilde{A}_{45}\partial_z \Phi_{123} + \frac{d - 2}{z} \Phi_{123}\tilde{A}_{45})]
$$

$$
- \left[ (\tilde{A}_{45} \cdot \tilde{A}_{12})\Phi_{3} + \tilde{\alpha}_{12}\tilde{A}_{45}\Phi_{3} \right].
$$

(36)
For gluon currents, note that some terms have no contribution after acting the unifying operator $T[12]T[45]T[36]$ (for the same reason as 3-pt currents, $T$ is equivalent to $T[12]T[45]T[36]$). The terms which will be non-vanishing after acting $T[36]$ are

\[(D^2_{12345} + d - 1)A_{12345\mu} = -z[2i\partial_z A_{123\mu} \alpha_{45} + A_{123\mu}(i(\partial_z - \frac{d}{z})\alpha_{45} - 2(k_{123} \cdot A_{45}))]
- \left[(A_{12} \cdot A_{45})A_{3\mu} - \alpha_{12} \alpha_{45} A_{3\mu}\right].
\] (37)

After substituting lower point currents, one can easily find that when $d = 1$, we have

\[T(D^2_{12345} + d - 1)A_{12345} \cdot \epsilon_6 = (D^2_{12345} + d - 1)\Phi_{12345}\] (38)

and the anomalous terms for this binary tree are

\[(D^2_{12345} + d - 1)(T A_{12345} \cdot \epsilon_6 - \Phi_{12345}) = -2i\tilde{\alpha}_{45}\partial_3 \left(\frac{2}{k_{12}^2}(d - 1)\phi_3 \frac{\nabla_1 \phi_2}{z}\right)
- \frac{2}{k_{12}^2}(d - 1)\phi_3 \frac{\nabla_1 \phi_2}{z}[i(\partial_z - \frac{d}{z})\tilde{\alpha}_{45} - 2(k_{123} \cdot \tilde{A}_{45})].\] (39)

This time the anomalous terms cannot be canceled by other binary trees. So for $d \neq 1$, 6-pt correlation functions do not satisfy the unifying relation. In the following subsection, we will prove the unifying relation to any $n$-pt correlation functions for $d = 1$.

**C. Proof of the Unifying relations in (A)dS**

Now we give the full proof for unifying relations for any number of points. First of all, we need to get the correct operator. We must take the operator as the sum of operators corresponding to all possible trace structures, and we denote it by $T$. The exact unifying relation is valid only when $d = 1$, which means gluons are massless. In this subsection, we always let $d = 1$. However, we will not substitute it to see how the currents depend on the boundary dimension $d$.

We give the following ansatz first and then prove the relations by induction:

\[T A_{I} \cdot \epsilon_n = (-1)^{|I|-1} \frac{|I|}{2} \Phi_{I} \quad (|I| \text{ odd})\]
\[T A_{I} \cdot k = (-1)^{|I|} \tilde{A}_{I} \cdot k \quad (|I| \text{ even})\]
\[T \alpha_{I} = (-1)^{|I|} \tilde{\alpha}_{I} \quad (|I| \text{ even})\] (40)

Here “$I \in \text{odd (even)}$” means the length of the word “$I$” is odd (even). Note that $T A_{I} \cdot \epsilon_n$ is invalid for even $I$ because there must be even number polarization vectors $\epsilon_i$ in the operator $T$. One can also find this because there is no scalar correlation function for odd-number
scalars. For $n$-pt gluon currents of a certain binary tree, we have ($I = JK, J = AB, K = CD$)

$$\frac{1}{z^2} (D_I^2 + d - 1) A_I \cdot \epsilon_n = ik_I \cdot \epsilon_n [\partial_z + (2 - d)/z] \alpha_I + \frac{1}{z^2} [(k_K \cdot \epsilon_n \alpha_K + 2i \partial_z A_K \cdot \epsilon_n) \alpha_J + k_K \cdot \epsilon_n (A_J \cdot A_K) + A_K \cdot \epsilon_n [(\partial_z - d/z) \alpha_J - 2k_K \cdot A_J] - (J \leftrightarrow K)] + \frac{1}{z^2} [\alpha_J \alpha_C A_D \cdot \epsilon_n + (A_J \cdot A_C) A_D \cdot \epsilon_n - (C \leftrightarrow D) + \alpha_K \alpha_B A_A \cdot \epsilon_n + (A_B \cdot A_K) A_A \cdot \epsilon_n - (A \leftrightarrow B)].$$  

(41)

We can also act the trace operator $T$ on the gauge field currents. From the experience of what we have learned from the 4-pt and 6-pt calculations, we know that there are only some effective terms, i.e., the terms will not be annihilated by $T[\epsilon_n]$. These terms are:

$$\frac{1}{z^2} (D_I^2 + d - 1) A_I \cdot \epsilon_n = \frac{1}{z} [2i \partial_z A_K \cdot \epsilon_n \alpha_J + A_K \cdot \epsilon_n [(\partial_z - d/z) \alpha_J - 2k_K \cdot A_J] - (J \leftrightarrow K)] + \frac{1}{z^2} [\alpha_J \alpha_C A_D \cdot \epsilon_n + (A_J \cdot A_C) A_D \cdot \epsilon_n - (C \leftrightarrow D) + \alpha_K \alpha_B A_A \cdot \epsilon_n + (A_B \cdot A_K) A_A \cdot \epsilon_n - (A \leftrightarrow B)].$$

(42)

The scalar currents can also be obtained from the scalar recursion (24):

$$\frac{1}{z^2} (D_I^2 + d - 1) \Phi_I = \frac{1}{z} [-2 \Phi_K(k_K \cdot \tilde{A}_J) + 2i \tilde{\alpha}_J \partial_z \Phi_K + i \Phi_K(\partial_z + (d - 2)/z) \tilde{\alpha}_J - (J \leftrightarrow K)] + \frac{1}{z^2} [(\tilde{A}_J \cdot \tilde{A}_C) \Phi_D + \tilde{\alpha}_J \tilde{\alpha}_C \Phi_D - (C \leftrightarrow D) + (\tilde{A}_K \cdot \tilde{A}_B) \Phi_A + \tilde{\alpha}_K \tilde{\alpha}_B \Phi_A - (A \leftrightarrow B)].$$

(43)

Recall that the interaction currents between gauge fields and conformally coupled scalars can be written as

$$\mathcal{J}_{I\mu} = -k_{I\mu} \Phi_J \Phi_K + k_{K\mu} \Phi_K \Phi_J + \frac{1}{z} [\tilde{A}_A \Phi_B \Phi_K - \tilde{A}_B \Phi_A \Phi_K - \Phi_J \tilde{A}_C \Phi_D + \Phi_J \tilde{A}_D \Phi_C]$$

$$\mathcal{J}_{Iz} = -i \Phi_J \leftrightarrow \partial_z \Phi_K + \frac{1}{z} [\tilde{\alpha}_A \Phi_B \Phi_K - \tilde{\alpha}_B \Phi_A \Phi_K - \Phi_J \tilde{\alpha}_C \Phi_D + \Phi_J \tilde{\alpha}_D \Phi_C].$$

(44)

For $z$ component of gauge field currents $\alpha_I$, we have

$$k_I^2 \alpha_I = \frac{1}{z^2} [2 \alpha_K (k_K \cdot A_J) + i (A_J \cdot \partial_z A_K) - (J \leftrightarrow K)] + \frac{1}{z^2} [\alpha_C (A_J \cdot A_D) - (C \leftrightarrow D) + \alpha_B (A_A \cdot A_K) - (A \leftrightarrow B)].$$

(45)

Note that $\Phi_I = 0$ for even $I$ has constrained that $\tilde{A}_{I\mu}, \tilde{\alpha}_I$ vanish for odd $I$ (one can also find this by induction), and we also have

$$\mathcal{T} (A_X \cdot A_Y) = (-1)^{|X|+|Y|/2} (-\Phi_X \Phi_Y + \tilde{A}_X \cdot \tilde{A}_Y).$$

(46)
Thus we have proven (for arbitrary second line of (J, C, D)
and T)
Then we have since if we take the effective mass of gluons in the curved spacetime to 0, we can regard
this as the flat case to some extent. However, it is not the real flat case because there are
still radial components in currents. In summary, the anomalous terms for $d \neq 1$ come from the effect of curved spacetime. In fact, by taking the flat limit, the anomalous terms will

for arbitrary $X, Y$ with $|X|, |Y| < |I|$. Thus we have

$$
\mathcal{T} k^2 I = \frac{1}{z^2} \left[ \frac{1}{z} \right] [2(-1)^{\frac{d}{2}} \alpha_k (k_k \cdot \tilde{A}_j) + (-1)^{\frac{d}{2}} i(\tilde{A}_j \cdot \partial_z \tilde{A}_K) + i(-1)^{\frac{d}{2}} \Phi_k \partial_z \Phi_K - (J \leftrightarrow K)]
$$

$$
+ \frac{1}{z^2} \left[ (-1)^{\frac{d}{2}} \alpha_C (\tilde{A}_j \cdot \tilde{A}_D - \Phi_j \Phi_D) - (C \leftrightarrow D) + (-1)^{\frac{d}{2}} \alpha_B (\tilde{A}_A \cdot \tilde{A}_K - \Phi_A \Phi_K) - (A \leftrightarrow B) \right]
$$

$$
= (-1)^{\frac{d}{2}} \frac{1}{z} [2\alpha_k (k_k \cdot \tilde{A}_j) + i(\tilde{A}_j \cdot \partial_z \tilde{A}_K) - (J \leftrightarrow K)]
$$

$$
- (-1)^{\frac{d}{2}} \frac{1}{z^2} [\alpha_C (\tilde{A}_j \cdot \tilde{A}_D) - (C \leftrightarrow D) + \alpha_B (\tilde{A}_A \cdot \tilde{A}_K) - (A \leftrightarrow B)] + (-1)^{\frac{d}{2}} \frac{1}{z} I_z
$$

$$
= (-1)^{\frac{d}{2}} k^2 I
$$

(47)

and

$$
\mathcal{T} \frac{1}{z^2} (D^2_J + d - 1) A_i \cdot k = (-1)^{\frac{d}{2}} ik_i \cdot k[\partial_z + (2 - d)/z]\alpha_I
$$

$$
+ (-1)^{\frac{d}{2}} \frac{1}{z} [(k_k \cdot k \alpha_K + 2i\partial_z \tilde{A}_K \cdot k)\alpha_J + k_K \cdot k(\tilde{A}_j \cdot \tilde{A}_K)
$$

$$
+ \tilde{A}_K \cdot k[i(\partial_z - d/z)\alpha_J - 2k_K \cdot \tilde{A}_j] - (J \leftrightarrow K)]
$$

$$
+ (-1)^{\frac{d}{2}} \frac{1}{z^2} [\alpha_j \tilde{A}_C \tilde{A}_D \cdot k + (\tilde{A}_j \cdot \tilde{A}_C) \tilde{A}_D \cdot k - (C \leftrightarrow D)
$$

$$
+ \alpha_K \tilde{A}_B \tilde{A}_A \cdot k + (\tilde{A}_B \cdot \tilde{A}_K) \tilde{A}_A \cdot k - (A \leftrightarrow B)]
$$

$$
- (-1)^{\frac{d}{2}} \frac{1}{z} k \cdot J_I
$$

$$
= (-1)^{\frac{d}{2}} \frac{1}{z^2} (D^2_J + d - 1) \tilde{A}_i \cdot k.
$$

(48)

Then we have

$$
\mathcal{T} \frac{1}{z^2} (D^2_J + d - 1) A_i \cdot \epsilon_n = (-1)^{\frac{d}{2}} \frac{1}{z} [(2i\partial_z \Phi_K)\alpha_J + \Phi_K[i(\partial_z - d/z)\alpha_J - 2k_K \cdot \tilde{A}_j] - (J \leftrightarrow K)]
$$

$$
+ (-1)^{\frac{d}{2}} \frac{1}{z^2} [\alpha_j \tilde{A}_C \Phi_D + (\tilde{A}_j \cdot \tilde{A}_C) \Phi_D - (C \leftrightarrow D) + \alpha_K \tilde{A}_B \Phi_A
$$

$$
+ (\tilde{A}_B \cdot \tilde{A}_K) \Phi_A - (A \leftrightarrow B)]
$$

$$
= (-1)^{\frac{d}{2}} \frac{1}{z^2} (D^2_J + d - 1) \Phi_I.
$$

(49)

Note that if $J, C, D$, for example, are all odd, then it will appear the term $\Phi_J \Phi_C \Phi_D$ in the second line of (49). However, it will be canceled by the corresponding term in ($C \leftrightarrow D$). Thus we have proven (40).

Our proof is for $d = 1$, and gluons and scalars are massless in this case. This makes sense since if we take the effective mass of gluons in the curved spacetime to 0, we can regard this as the flat case to some extent. However, it is not the real flat case because there are still radial components in currents. In summary, the anomalous terms for $d \neq 1$ come from the effect of curved spacetime. In fact, by taking the flat limit, the anomalous terms will
vanish, and our results are consistent with the flat case in any dimension, which means we have given a semi-on-shell proof for unifying relations in flat spacetime.

We can summarize that this relation is correct only when $d = 1$ or for 3-pt currents (corresponding to 4-pt correlation functions). However, during this proof, we can see how far the (A)dS case is from the flat case. The radial components of the gluon currents play a crucial role in the anomaly of the relations. In fact, we can obtain the anomalous terms for all currents if we are patient enough. An interesting problem is whether these terms have a closed form so that we can modify our relation and make it correct for any dimensions.

D. More on the Unifying Relations

In flat spacetime, we can get mixed amplitudes by acting $T[ij]$ on gluon amplitudes, like (5). We expect that in (A)dS, the correlation functions also have this property at least in $d = 1$. This subsection will discuss the unifying relations for mixed BG currents.

First, we need to show how to obtain BG currents for different types of external lines. In fact, this can be realized by modifying the initial conditions [5]. For example, for scalar currents of minimal coupling scalar theory, we need to keep the bulk leg scalar, and the boundary legs can be given different conditions: $\tilde{A}_i = 0, \Phi_i \neq 0$ for scalars and $\Phi_i = 0, \tilde{A}_i \neq 0$ for gluons. Then the BG currents we get will correspond to mixed correlation functions. Let us show some examples where we only consider the binary tree FIG. 2.

The mixed scalar currents for the “particle 2” scalar and other gluons should impose the following initial conditions:

$$\Phi_1 = \Phi_3 = 0, \quad \tilde{A}_2 = 0. \quad (50)$$

In other words, we do not switch off the gauge field source at the boundary and instead turn off the scalar source for “particle 1” and “particle 3” at the boundary. Then, the currents for these initial conditions can be calculated from the recursion relations (24):

$$(\mathcal{D}_{123}^2 + d - 1)\Phi_{123} = -4z\phi_3(k_{12} \cdot \epsilon_3)\frac{1}{\mathcal{D}_{12}^2 + d - 1}z\phi_2(k_2 \cdot \epsilon_1)\phi_1 - (\epsilon_1 \cdot \epsilon_3)\phi_1\phi_2\phi_3. \quad (51)$$

Exactly the same as $T[24](\mathcal{D}_{123}^2 + d - 1)A_{123} \cdot \epsilon_4$. Note that in this example, we have no anomalous terms.

We can also consider other initial conditions:

$$\Phi_1 = \Phi_2 = 0, \quad \tilde{A}_3 = 0. \quad (52)$$

Now we switch on the gauge fields source for “particle 1” and “particle 2” at the boundary,
and “particle 3” is a scalar. The scalar currents for such initial conditions can be written as

\[
(D_{123}^2 - M^2)\Phi_{123} = \Phi_3 \frac{1}{D_{12}^2 - M^2} \left[ \frac{2z^2 k_{12} \cdot k_3}{k_{12}^2} (\epsilon_1 \cdot \epsilon_2) \left( \frac{k_2^2 - k_1^2}{z} \right) \phi_1 \phi_2 \\
- 2z (k_2 \cdot k_3 - k_1 \cdot k_3) (\epsilon_1 \cdot \epsilon_2) \phi_1 \phi_2 + 4z (\epsilon_2 \cdot k_3) (k_2 \cdot \epsilon_1) \phi_2 \phi_1 \\
- 4z (\epsilon_1 \cdot k_3) (k_1 \cdot \epsilon_2) \phi_1 \phi_2 \right] - \Phi_3 \frac{1}{k_{12}^2} (\epsilon_1 \cdot \epsilon_2) \left( \frac{k_2^2 - k_1^2}{z} \right) \phi_1 \phi_2 + \frac{2(d - 2)}{z^2} \phi_1 \leftrightarrow \phi_2
\]

- \frac{2}{z k_{12}^2} (\epsilon_1 \cdot \epsilon_2) \phi_1 \leftrightarrow \phi_2 \phi_k \Phi_3.

This time we do have the anomalous terms. When \( d = 1 \), this result is the same as \( \mathcal{T}[34](D_{123}^2 + d - 1)A_{123} \cdot \epsilon_4 \). Fortunately, for 3-pt currents, anomalous terms would not enter into the correlation functions.

Next, we want to prove the unifying relations for mixed currents. For the BG currents whose initial conditions are \( \tilde{A}_i = 0 \) for a certain \( i \) and all the other boundary legs are gluons, we have the following ansatz:

\[
\mathcal{T}[in](D_I^2 + d - 1)A_I \cdot \epsilon_n = (D_I^2 + d - 1)\Phi_i,
\]

where \( \Phi_i \) denotes to the mixed currents with only the \( i \)-th particle scalar. We will prove this by induction. The effective terms of the left-hand side are

\[
(D_I^2 + d - 1)A_I \cdot \epsilon_n = z[2i \partial_z A_K \cdot \epsilon_n] \alpha_J + A_K \cdot \epsilon_n [i(\partial_z - d/z)\alpha_J - 2k_K \cdot A_J] - (J \leftrightarrow K)
\]

\[
+ [\alpha_J \alpha_C A_D \cdot \epsilon_n + (A_J \cdot A_C) A_D \cdot \epsilon_n - (C \leftrightarrow D) + \alpha_K \alpha_B A_A \cdot \epsilon_n
\]

\[
+ (A_B \cdot A_K) A_A \cdot \epsilon_n - (A \leftrightarrow B)].
\]

(55)

Without loss of generality, we can choose \( i \) in the word \( D \), then the effective terms are

\[
(D_I^2 + d - 1)A_I \cdot \epsilon_n = z[2i \partial_z A_K \cdot \epsilon_n] \alpha_J + A_K \cdot \epsilon_n [i(\partial_z - d/z)\alpha_J - 2k_K \cdot A_J]
\]

\[
+ [\alpha_J \alpha_C A_D \cdot \epsilon_n + (A_J \cdot A_C) A_D \cdot \epsilon_n]
\]

(56)

If for currents of scalar theory such as \( \tilde{A}_J, \tilde{\alpha}_J, \Phi_J \), all initial conditions are for gluons, then they are the same as the corresponding gluon currents and \( \Phi_J = 0 \). Also, note that \( \tilde{A}_{I\mu} \) (also \( \tilde{\alpha}^i \)) must be zero for there is no scattering process with odd number scalar external legs. Then we have

\[
\mathcal{T}[in](D_I^2 + d - 1)A_I \cdot \epsilon_n = z[(2i \partial_z \Phi_K) \alpha_J + \Phi_K [i(\partial_z - d/z)\alpha_J - 2k_K \cdot \tilde{A}_J]
\]

\[
+ [\alpha_J \tilde{\alpha}_C \Phi_D + (\tilde{A}_J \cdot \tilde{A}_C) \Phi_D]
\]

(57)

The last equality is correct only for \( d = 1 \). As usual, in \( d \neq 1 \), there exist some anomalous terms.
To extend these relations to the correlation functions, we also need to deal with the case of gluon currents \( \tilde{A}_{I \mu} \), whose bulk leg is gluon, in the minimal coupling scalar theory. We need to prove (for \( d = 1 \))

\[
\mathcal{T}_{ij}[(D_I^2 + d - 1)A_I \cdot \epsilon_n = -(D_I^2 + d - 1)\tilde{A}_I^{ij} \cdot \epsilon_n \tag{58}
\]

and

\[
\mathcal{T}_{ij}[k_i^2 \alpha_I] = -k_i^2 \tilde{\alpha}_I^{ij}. \tag{59}
\]

Here the labels \( "i, j" \) mean that all external legs are gluons except the \( i \)-th and the \( j \)-th. Obviously, the unifying relations hold for 3-pt currents. By induction, one can easily prove the case for \( i, j \in A \) (\( B, C, D \)) and \( i \in A, j \in B \) (\( i \in C, j \in D \)). For the case, without loss of generality, \( i \in A \) and \( j \in C \), we need to use (54) and then complete the proof.

Now we need to construct the mixed correlation functions. We define the 2-scalar-(\( n-2 \))-gluon correlation functions (with the first 2 particles scalar) as follows

\[
A_{\text{mixed}}(1^s, 2^s, 3^g, \cdots, n^g) = -\frac{1}{N} \int \frac{dz}{z^{d+1}} [\tilde{A}_n \cdot (D_{123\cdots n-1}^2 + d - 1)\tilde{A}_{123\cdots n-1}^{1,2} + \text{cyclic}(1^s, 2^s, 3^g, \cdots, N^g)]. \tag{60}
\]

Of course, the currents that appear in this definition need to sum over all possible binary trees. We need to explain this definition. The “cyclic” means the cyclic transformation of external legs, not just the label. This means there will appear scalar (\( n-1 \))-current, and in this case we need to replace

\[
-\tilde{A}_n \cdot (D_{123\cdots n-1}^2 + d - 1)\tilde{A}_{123\cdots n-1}^{1,2} \tag{61}
\]

with (for example)

\[
\Phi_1(D_{234\cdots n}^2 + d - 1)\Phi_{234\cdots n}^2. \tag{62}
\]

For example,

\[
A_{\text{mixed}}(1^s, 2^g, 3^s, 4^g) = -\frac{1}{N} \int \frac{dz}{z^{d+1}} [\tilde{A}_4 \cdot (D_{123}^2 + d - 1)\tilde{A}_{123}^{1,3} + \Phi_4(D_{124}^2 + d - 1)\Phi_{412}^1 - \tilde{A}_2 \cdot (D_{134}^2 + d - 1)\tilde{A}_{134}^{1,3} + \Phi_1(D_{234}^2 + d - 1)\Phi_{234}^1]. \tag{63}
\]

We need to explain the extra minus sign of the gluon current terms. The origin of this minus sign, together with the factor \( (-1)^{\frac{|j|-1}{2}} \) in subsection IV C, is that the 3-pt gluon current \([1^s, 2^s]\) has an opposite sign with the Feynman rules for this 3-pt vertex, while for scalar currents \([1^g, 2^g]\) and \([1^s, 2^s]\) the signs are the same as the Feynman rules.

Then we have proved the unifying relations for mixed amplitudes (for \( d = 1 \)):

\[
\mathcal{T}_{ij}A_{\text{YM}}(1, 2, 3, \cdots, n) = A_{\text{mixed}}(1^g, 2^g, \cdots, i^s, \cdots, j^s, \cdots, n^g). \tag{64}
\]

One can prove the unifying relations as above for the case that acting two or more \( \mathcal{T}_{ij} \) on the \( n \)-pt currents in \( d = 1 \) by using the methods we have used. Note that we still need
to sum over the trace operators for all possible trace structures of the scalars. It will be consistent with the flat case. The anomalous terms will also vanish in this case by taking the flat limit; hence one can also take the flat limit to get the proof of the unifying relations for flat amplitudes.

It is worth saying that we expect one can use our method to prove the unifying relations for the full YMS. In YMS, scalars have two color indices, which means we can distinguish each trace structure of the scalars after double color ordering. We hope that, in this case, we do not need to sum over the trace operators at all and can obtain the unifying relations for a certain trace operator instead. We will discuss this case in our future work.

V. APPLICATIONS IN COSMOLOGY

This section considers the applications for unifying relations in dS spacetime. During the period of inflation, our universe is dominated by dark energy, which means the primordial universe is dS spacetime. The interaction between particles and scalar fields has driven the cosmic expansion in the early universe will result in the anisotropy of Cosmic Microwave Background (CMB) at present. Therefore, the correlation functions between inflatons and other fields become a key to figuring out the history of our primordial universe. In this section, we use the propagators of dS spacetime and change AdS to dS by the method we have mentioned in subsection III B. For simplicity, we will take \( l = 1 \) and choose \( d = 3 \). Though our relations hold only when \( d = 1 \), for 4-pt correlation functions, the relations are independent of the dimension. Now we give some concrete examples to see the applications of the unifying relations in cosmology. In this subsection, we will focus on the 4-pt correlators \( \langle JOJO \rangle \) (see FIG. 4).

For \( s \)-channel (see FIG. 4 (a)), we have:

\[
\mathcal{T}[24]A_{YM}(1234)_{s} = -2i(k_{2} \cdot \epsilon_{1})(k_{4} \cdot \epsilon_{3}) \int \frac{d\eta}{\eta^{4}} \frac{1}{D_{34}^{2}} + 2 \eta \phi \phi_{4} + \eta \phi_{3} \phi_{4} \frac{1}{D_{12}^{2}} + 2 \eta \phi \phi_{2}^{2} \frac{1}{(k_{2} \cdot \epsilon_{1})(k_{4} \cdot \epsilon_{3})} \\
\sim \langle JOJO \rangle_{s}.
\]

(65)

Here we have stripped the contact terms, and \( s \) represents the \( s \)-channel \( (s^{2} = (k_{1} + k_{2})^{2}) \). For \( u \)-channel (see FIG. 4 (c)), there are more subtleties. In fact, this diagram is equivalent to \( \mathcal{T}[24]A_{YM}(1324) \), so we can calculate \( \mathcal{T}[34]A_{YM}(1234) \) first and then let 2 and 3 exchange.
can also be applied to higher dimension background spacetime. Besides, our calculations unifying relations can help us calculate the higher points correlation functions in the dS background after we are able to figure out all anomalous terms. Therefore, we can expect that the unifying relations can help us calculate the higher points correlation functions in the dS background after we are able to figure out all anomalous terms. Besides, our calculations can also be applied to higher dimension background spacetime.

First, we have

\[
\mathcal{T}[34] A_{YM} (1234)_s = -\frac{i}{k_{12}^2} (\epsilon_1 \cdot \epsilon_2) \int \frac{d^4 \phi_1}{\eta^4} \frac{d^4 \phi_2}{\eta^4} \frac{d^4 \phi_3}{\eta^4} \frac{d^4 \phi_4}{\eta^4} - \frac{i}{2} \left[ (\epsilon_1 \cdot \epsilon_2) \frac{2}{2} \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{k_{12}^2} + (\epsilon_1 \cdot \epsilon_2)(k_3 - k_4) \cdot (k_1 - k_2) + 4(k_3 \cdot \epsilon_2)(k_2 \cdot \epsilon_1) \right.
\]

\[
- 4(k_3 \cdot \epsilon_1)(k_1 \cdot \epsilon_2)] \cdot \eta \phi_4 + \eta \phi_3 \phi_4 \frac{1}{D_{12}^2} + \frac{1}{2} \eta \phi_1 \phi_2
\]

\[
= \frac{(\epsilon_1 \cdot \epsilon_2)(|k_1| - |k_2|)(|k_3| - |k_4|)}{k_{12}^2(|k_1| + |k_2| + |k_3| + |k_4|)} + \left[ (\epsilon_1 \cdot \epsilon_2) \frac{2}{2} \frac{(k_1^2 - k_2^2)(k_3^2 - k_4^2)}{k_{12}^2} + (\epsilon_1 \cdot \epsilon_2)(k_3 - k_4) \cdot (k_1 - k_2) + 4(k_3 \cdot \epsilon_2)(k_2 \cdot \epsilon_1) - 4(k_3 \cdot \epsilon_1)(k_1 \cdot \epsilon_2) \right]
\]

\[
\times \frac{1}{(|k_1| + |k_2| + s)(|k_3| + |k_4| + s)(|k_1| + |k_2| + |k_3| + |k_4|)}.
\]

Then we exchange 2 and 3 to get the \(u\)-channel mixed correlation function

\[
\langle JOJO \rangle_u \sim \frac{(\epsilon_1 \cdot \epsilon_3)(|k_1| - |k_3|)(|k_2| - |k_4|)}{k_{13}^2(|k_1| + |k_2| + |k_3| + |k_4|)} + \left[ (\epsilon_1 \cdot \epsilon_3) \frac{2}{2} \frac{(k_1^2 - k_3^2)(k_2^2 - k_4^2)}{k_{13}^2} + (\epsilon_1 \cdot \epsilon_3)(k_2 - k_4) \cdot (k_1 - k_3) + 4(k_2 \cdot \epsilon_3)(k_3 \cdot \epsilon_1) - 4(k_2 \cdot \epsilon_1)(k_1 \cdot \epsilon_3) \right]
\]

\[
\times \frac{1}{(|k_1| + |k_3| + s)(|k_2| + |k_4| + s)(|k_1| + |k_2| + |k_3| + |k_4|)}.
\]

We should emphasize that our results are the same as the previous results from cosmological bootstrap [23].

It is valuable to comment more about the applications of unifying relations in cosmology. As we have presented above, in 4-pt correlation functions, unifying relations can exactly reproduce the results from cosmological bootstrap [23]. Therefore, we can expect that the unifying relations can help us calculate the higher points correlation functions in the dS background after we are able to figure out all anomalous terms. Besides, our calculations can also be applied to higher dimension background spacetime.
VI. CONCLUSION AND OUTLOOK

The remarkable unifying relations in flat spacetime show a connection between amplitudes in different theories. However, unlike the proof of unifying relations in the flat case where we use factorization, here in (A)dS, we use BG currents to study the unifying relations.

We first explicitly calculate the 4-pt $\text{IV A}$ and 6-pt $\text{IV B}$ correlation functions in both gauge and scalar theories. Then we prove that after the action of some trace operators $T[ij]$, the correlation functions of two theories are exactly the same when $d = 1$ or in the 4-pt case, while for other dimensions and higher point case we get some anomalous terms. Similarly, we also prove the unifying relations for mixed correlation functions and find that the unifying relations are also only correct in $d = 1$ or for 4-pt correlation functions.

Since our relations are correct for 4-pt correlation functions, which are the most valuable in cosmology so far, we explore some applications in cosmology. We calculate some channels of 4-pt cosmological correlators by using unifying relations and finally succeed.

In summary, we have discussed a type of unifying relations for BG currents in (A)dS. Moreover, we show some potential applications for our unifying relations in cosmological correlators calculation. However, it is desirable to mention that there are still ambiguities we have not illustrated well. First, the application for unifying relations in cosmology should be careful since there may be some anomalous terms at higher points. We wonder if there exists a method to modify these anomalous terms. Second, we discuss only one type of unifying relations in this work. In principle, there are still several unifying relations for which we do not give rigorous proof, which may be discussed in future work. It is necessary to mention that the correlation functions at the loop level can also be constructed by BG currents [7]. And we are looking forward to deriving the relations among 1-loop level correlation functions [39, 40] in future work.

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VIII. APPENDIX

This appendix focuses on the analytical calculation of the time integral in the mixed correlation function. Before the explicit calculation, let us explain the notations in the correlation functions. And we should also note that in the following discussion since we only focus on the time integral where we do not talk about the recursions, the notations for
momentum do not cause any ambiguities. Thus we denote $|k|$ as $k$ for simplicity. First, the
inverse of the d’Alembert operator is defined as

$$(D_1^2 - M^2)^{-1}O(\eta) = -i \int \frac{d\eta}{\eta} G_I(\eta, \eta') O(\eta'),$$

(68)

where $O$ is an arbitrary operator and $G_I(\eta, \eta')$ is the bulk-to-bulk propagator which satisfy the
following equation:

$$(D_1^2 - M^2)G_I = i\eta^4 \delta(\eta - \eta').$$

(69)

In section V, we need to evaluate the bulk-to-bulk propagator for gluon at $d = 3$. When we
are ready to solve the bulk-to-bulk propagator, we should impose some initial conditions.

In our following calculation, we impose the Bunch-Davis (BD) vacuum, representing a non-
particle state at the past infinity. Thus, the bulk-to-bulk propagator for color-stripped gluon

$$G_{ks}(\eta_1, \eta_2) = \frac{\pi(\eta_1 \eta_2)^{3/2}}{4} \left( H^{(2)}_{\frac{d}{2}}(-k_s \eta_1) H^{(2)}_{\frac{d}{2}}(-k_s \eta_2) + H^{(1)}_{\frac{d}{2}}(-k_s \eta_1) H^{(2)}_{\frac{d}{2}}(-k_s \eta_2) \right) \theta(\eta_1 - \eta_2)$$

$$+ \frac{\pi(\eta_1 \eta_2)^{3/2}}{4} \left( H^{(2)}_{\frac{d}{2}}(-k_s \eta_2) H^{(2)}_{\frac{d}{2}}(-k_s \eta_1) + H^{(1)}_{\frac{d}{2}}(-k_s \eta_2) H^{(2)}_{\frac{d}{2}}(-k_s \eta_1) \right) \theta(\eta_2 - \eta_1),$$

(70)

where $s$ denotes the $s$ channel and $H^{(1)}, H^{(2)}$ are separately the first and second type Hankel
function. Recall that the equation of motion for conformally coupled scalars is

$$(D_1^2 + 2)\phi(\eta, k) = 0.$$  

(71)

Also, impose the BD vacuum, we can write down the mode function for conformally coupled scalars:

$$\phi(\eta, k) = -\sqrt{\frac{\pi}{2}} (-\eta)^{3/2} k^{1/2} H^{(2)}_{\frac{d}{2}}(-k \eta).$$

(72)

Now we can carry out the time integral in the mixed correlation function. First, in $s$-channel, we have to evaluate the following integral:

$$\mathcal{I}_s = \int \frac{d\eta_1}{\eta_1^3} \phi(\eta_1, k_1) \phi(\eta_1, k_2) \frac{1}{D_1^2 + 2} \eta_2 \phi(\eta_2, k_3) \phi(\eta_2, k_4)$$

$$= -i \int \frac{d\eta_1}{\eta_1^3} \frac{d\eta_2}{\eta_2^3} \phi(\eta_1, k_1) \phi(\eta_1, k_2) G_{ks}(\eta_1, \eta_2) \phi(\eta_2, k_3) \phi(\eta_2, k_4).$$

(73)

Note that in the propagator, we have time ordering dependence. We consider the two
different time-ordering separately, and for simplicity but without loss of any generality, we
focus on the positive time ordering, $\eta_1 > \eta_2$. Then the time integral can be carried out
analytically:

\[ I_{s+} = -i \int \frac{d\eta_1}{\eta_1} \frac{d\eta_2}{\eta_2} \frac{\pi^2}{4} (\eta_1 \eta_2)^3 \sqrt{k_1 k_2 k_3 k_4} H_{\frac{1}{2}}^{(2)}(-k_1 \eta_1) H_{\frac{1}{2}}^{(2)}(-k_2 \eta_2) H_{\frac{1}{2}}^{(2)}(-k_3 \eta_2) H_{\frac{1}{2}}^{(2)}(-k_4 \eta_2) \]

\[ \times \frac{\pi (\eta_1 \eta_2)^{3/2}}{4} \left( H_{\frac{1}{2}}^{(2)}(-k_s \eta_1) H_{\frac{1}{2}}^{(2)}(-k_s \eta_2) + H_{\frac{1}{2}}^{(1)}(-k_s \eta_1) H_{\frac{1}{2}}^{(2)}(-k_s \eta_2) \right) \theta(\eta_1 - \eta_2) \]

\[ = \frac{2i}{(k_1 + k_2 + k_3 + k_4)(k_1 + k_2 + k_3 + k_4 + 2k_s)}. \tag{74} \]

Together with the negative time ordering contribution, we can obtain the full integral result:

\[ I_s = \frac{2i}{(k_1 + k_2 + k_3 + k_4)(k_1 + k_2 + k_3 + k_4 + 2k_s)}. \tag{75} \]

The second time integral is much simpler, which looks like the contact interaction and does not have the bulk-to-bulk propagator contribution.

\[ I_c = i \int \frac{d\eta}{\eta^4} \phi_1 \nabla \phi_2 \nabla \phi_3 \nabla \phi_4 \]

\[ = i \int \frac{d\eta}{\eta^4} [\phi(\eta, k_1) \partial_\eta \phi(\eta, k_2) - \phi(\eta, k_2) \partial_\eta \phi(\eta, k_1)] \frac{(\phi(\eta, k_3) \partial_\eta \phi(\eta, k_4) - \phi(\eta, k_4) \partial_\eta \phi(\eta, k_3))}{k_1 + k_2 + k_3 + k_4} \]

\[ = \frac{(k_1 - k_2)(k_3 - k_4)}{k_1 + k_2 + k_3 + k_4}. \tag{76} \]

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