On Nilpotent Multipliers of Pairs of Groups and their Nilpotent Covering Pairs

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Abstract. In this paper, we determine the structure of the nilpotent multipliers of all pairs \((G, N)\) of finitely generated abelian groups where \(N\) admits a complement in \(G\). We also show that every nilpotent pair of groups of class at most \(k\) with non trivial \(c\)-nilpotent multiplier does not admit any \(c\)-covering pair, for all \(c > k\). Moreover, some inequalities for the nilpotent multiplier of a pair of finite groups and their factor groups are given.

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1. Introduction

Let \(G\) be a group with a free presentation \(1 \to R \to F \to G \to 1\). Then the \(c\)-nilpotent multiplier of \(G\) is defined to be

\[M^{(c)}(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, cF]}\].

It is easy to see that \(M^{(c)}(G)\) is independent of the choice of the free presentation of \(G\). In particular, \(M^{(1)}(G)\) is the well-known notion \(M(G)\), the Schur multiplier of \(G\), (see [5]).

The structure of the Schur multiplier of a finitely generated abelian group is obtained by I. Schur [5]. In 1997, M.R.R. Moghaddam and the third author [8] gave an implicit formula for the \(c\)-nilpotent multiplier of a finite abelian group.

The theory of the Schur multiplier was extended for pairs of groups by Ellis [2], in 1998. By a pair of groups \((G, N)\), we mean a group \(G\) with a normal subgroup \(N\). The Schur multiplier of a pair \((G, N)\) of groups is a
functorial abelian group $M(G, N)$ whose principal feature is a natural exact sequence
\[
H_3(G) \xrightarrow{\eta} H_3\left(\frac{G}{N}\right) \to M(G, N) \xrightarrow{\mu} M\left(\frac{G}{N}\right) \to \frac{N}{[N, G]} \to (G)^{ab} \xrightarrow{\alpha} \left(\frac{G}{N}\right)^{ab} \to 0
\]
in which $H_3(G)$ is the third homology of $G$ with integer coefficients. In particular, if $N = G$, then $M(G, G)$ is the usual Schur multiplier $M(G)$.

Let $(G, N)$ be a pair of groups. Ellis [1] showed that if $N$ admits a complement in $G$, then
\[
M(G, N) \cong \ker(\mu : M(G) \to M(G/N)). \tag{1.1}
\]

Let $F/R$ be a free presentation of $G$ and $S$ be a subgroup of $F$ with $N \cong S/R$. If $N$ admits a complement in $G$, then (1.1) implies that
\[
M(G, N) = \frac{R \cap [S, F]}{[R, F]},
\]
(see [2]). This fact suggests the definition of the $c$-nilpotent multiplier of a pair $(G, N)$ of groups as follows:
\[
M^{(c)}(G, N) = \frac{R \cap [S, cF]}{[R, cF]}.
\]
In particular, if $G = N$, then $M^{(c)}(G, G) = M^{(c)}(G)$ is the $c$-nilpotent multiplier of $G$.

In this paper, we study the $c$-nilpotent multiplier of a pair of groups. The paper is organized as follows. In Section 2, we present a formula for the $c$-nilpotent multipliers (and consequently for the Schur multipliers) of all pairs $(G, N)$ of finitely generated abelian groups where $N$ has a complement in $G$. In Section 3, we discuss on the notion of $c$-covering pairs of groups. It is well known that every finite group has at least a covering group (see [3]). But this fact is not true for the extended notion, $c$-covering group (see [4]). In 1998, Ellis [1] defined the concept of covering pair of groups and proved that every pair of finite groups has a covering pair and here we show that this fact is not true for the extended notion, $c$-covering pairs of groups. In fact, we prove that every nilpotent pair of groups of class at most $k$ with non trivial $c$-nilpotent multiplier does not admit any $c$-covering pair, for all $c > k$.

In the final section, we give some inequalities for the order, the exponent and the minimal number of generators of the $c$-nilpotent multiplier of a pair of finite groups and their factor groups.

2. Nilpotent multipliers of pairs of finitely generated abelian groups

In this section, we intend to find the structure of the $c$-nilpotent multiplier of a pair $(G, N)$ of finitely generated abelian groups, where $N$ has a complement in $G$. The proof relies on basic commutators and their properties.
Definition 2.1. ([2]) Let $X$ be an independent subset of a free group, and select an arbitrary total order for $X$. We define the basic commutators on $X$, their weight $wt$, and the ordering among them as follows:

1. The elements of $X$ are basic commutators of weight one, ordered according to the total order previously chosen.
2. Having defined the basic commutators of weight less than $n$, the basic commutators of weight $n$ are the $c_k = [c_i, c_j]$, where
   a. $c_i$ and $c_j$ are basic commutators and $wt(c_i) + wt(c_j) = n$, and
   b. $c_i > c_j$, and if $c_i = [c_s, c_t]$, then $c_j \geq c_t$.
3. The basic commutators of weight $n$ follow those of weight less than $n$. The basic commutators of weight $n$ are ordered among themselves lexicographically; that is, if $[b_1, a_1]$ and $[b_2, a_2]$ are basic commutators of weight $n$, then $[b_1, a_1] \leq [b_2, a_2]$ if and only if $b_1 < b_2$, or $b_1 = b_2$ and $a_1 < a_2$.

M. Hall [2] proved that if $F$ is the free group on a finite set $X$, then the basic commutators of weight $n$ on $X$ provide a basis for the free abelian group $\gamma_n(F)/\gamma_{n+1}(F)$. The number of these basic commutators is given by Witt formula.

Theorem 2.2. (The Witt formula [2]) The number of basic commutators of weight $n$ on $d$ generators is given by the following formula

$$\chi_n(d) = \frac{1}{n} \sum_{m|n} \mu(m) d^{m/n},$$

where $\mu(m)$ is the M"{o}bius function, which is defined to be

$$\mu(m) = \begin{cases} 1 & ; m = 1, \\ 0 & ; m = p_1^{\alpha_1} ... p_k^{\alpha_k} & \exists \alpha_i > 1, \\ (-1)^s & ; m = p_1 ... p_s, \end{cases}$$

where the $p_i$’s are distinct prime numbers.

Hereafter, let $G$ be a finitely generated abelian group with $G = N \oplus K$ where $N = \mathbb{Z}^{(l)} \oplus \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_s}$ and $K = \mathbb{Z}^{(s)} \oplus \mathbb{Z}_{r_{t+1}} \oplus \cdots \oplus \mathbb{Z}_{r_n}$, such that $r_i | r_{i+1}$, for all $1 \leq i \leq n - 1$. Put $m = l + s$, and $G_i \cong \mathbb{Z}$, for all $1 \leq i \leq m$, and $G_{m+j} \cong \mathbb{Z}_{r_j}$, for all $1 \leq j \leq n$. Let

$$1 \rightarrow R_i = 1 \rightarrow F_i = \langle y_i \rangle \rightarrow G_i \rightarrow 1$$

be a free presentation of the infinite cyclic group $G_i$, for all $1 \leq i \leq m$, and let

$$1 \rightarrow R_j = \langle x_j^{r_j} \rangle \rightarrow F_{m+j} = \langle x_j \rangle \rightarrow G_{m+j} \rightarrow 1$$

be a free presentation of $G_{m+j}$, for all $1 \leq j \leq n$.

Put $Y_1 = \{y_1, y_2, \ldots, y_l\}$, $Y_2 = \{y_{l+1}, y_{l+2}, \ldots, y_m\}$, $X_1 = \{x_1, \ldots, x_l\}$, $X_2 = \{x_{l+1}, \ldots, x_m\}$ and $Y = Y_1 \cup Y_2$, $X = X_1 \cup X_2$. Then it is easy to see that $G = N \oplus K$ has the following free presentation

$$1 \rightarrow R = T\gamma_2(F) \rightarrow F \xrightarrow{\theta} G \rightarrow 1,$$
where $F$ is the free group on $X \cup Y$, and $T = \langle x_1^{r_1}, \ldots, x_n^{r_n} \rangle^F$. Considering the natural map $\theta : F \to G$, we have $\theta^{-1}(N) = SR$ with $S = (Y_1 \cup X_1)^F$ and so $1 \to R \to SR \to N \to 1$ is a free presentation of $N$, which implies that

$$M^{(c)}(G, N) = \frac{R \cap [RS, cF]}{[R, cF]} = \frac{[RS, cF]}{[R, cF]}.$$ 

Hence we have

$$M^{(c)}(G, N) \cong \frac{[S, cF][T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)}{[T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)}. \quad (2.1)$$

To determine the structure of $M^{(c)}(G, N)$, we need suitable bases for the free abelian groups $[S, cF][T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)$ and $[T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)$. The authors have already obtained a basis for $[T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)$ as follows.

**Lemma 2.3.** (Lemma 3.2 in [3]) Let $C_i$ be the set of all basic commutators of weight $c + 1$ on $\{x_i, \ldots, x_n, y_1, y_2, \ldots, y_m\}$. Then $[T, cF]\gamma_{c+2}(F)$ is a free abelian group with a basis $D = \bigcup_{i=1}^n D_i$, where

$$D_i = \{b^{r_i}\gamma_{c+2}(F) \mid b \in C_i \text{ and } x_i \text{ does not appear in } b\}.$$ 

Now we are ready to prove the main result of this section.

**Theorem 2.4.** With the previous notations and assumptions, the following isomorphism holds.

$$M^{(c)}(G, N) \cong \mathbf{Z}^{(f_0)} \oplus \mathbf{Z}^{(f_1)}_{r_1} \oplus \cdots \oplus \mathbf{Z}^{(f_t)}_{r_t+1} \oplus \cdots \oplus \mathbf{Z}^{(f_n-g_n)}_{r_n},$$

where $f_0 = \chi_{c+1}(m) - \chi_{c+1}(m-l)$, $f_i = \chi_{c+1}(m+n-i+1) - \chi_{c+1}(m+n-i)$, for $1 \leq i \leq n$, and $g_i = \chi_{c+1}(m+n-l-i+1) - \chi_{c+1}(m+n-l-i)$, for $t + 1 \leq i \leq n$.

**Proof.** In order to determine the structure of $M^{(c)}(G, N)$, we need to find a suitable basis for the free abelian group $[S, cF][T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)$. Let $E$ be the set of all basic commutators of weight $c + 1$ on $X \cup Y$ in which at least one of the $x_1, \ldots, x_t, y_1, \ldots, y_l$ does not appear. Put

$$\hat{E} = \{b^{r_i}\gamma_{c+2}(F) \mid b \in E\}.$$ 

Then $D \cup \hat{E}$ generates the free abelian group $[S, cF][T, cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)$. Recall that the distinct basic commutators are linearly independent (see [2]). Hence $\hat{E} = D' \cup \hat{E}$ is a basis for the mentioned free abelian group where $D'$ is the set of all elements $b^{r_i}\gamma_{c+1}(F)$ such that $b$ is a basic commutator of weight $c + 1$ on $X_2 \cup Y_2$ in which one of the elements of $X_2$ does not appear. In order to determine the structure of the group $M^{(c)}(G, N)$, we present $\hat{E}$ as follows:

$$\hat{E} = (A_1 - A_2) \cup (\bigcup_{i=1}^t B_i) \cup (\bigcup_{i=t+1}^n (B_i - N_i)) \cup (\bigcup_{i=t+1}^n H_i)$$

where

$A_1 = \{b^{r_i}\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on } Y\}$,

$A_2 = \{b^{r_i}\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on } Y_2\}$,

$B_i = \{b^{r_i}\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on }$
where \( B \) is a finitely generated abelian group in which the number of copies of \( H \)
and the obtained bases for the free abelian groups \( T, X \) and 1 have of finitely generated abelian groups such that
\( Y \) (without any extra condition).

Now putting \( f_0 = |A_1| - |A_2|, f_i = |B_i| \) and \( g_i = |N_i| \), for all \( 1 \leq i \leq t \) and it is
\( |B_i| - |N_i| \), for \( t + 1 \leq i \leq n \). On the other hand,
\( |A_1| = \chi_{c+1}(m), |A_2| = \chi_{c+1}(m - l), \\
|B_i| = \chi_{c+1}(m + n - i + 1) - \chi_{c+1}(m + n - i), \text{ for } 1 \leq i \leq n, \\
|N_i| = \chi_{c+1}(m + n - l - i + 1) - \chi_{c+1}(m + n - l - i), \text{ for } t + 1 \leq i \leq n.

Now putting \( f_0 = |A_1| - |A_2|, f_i = |B_i| \) and \( g_i = |N_i| \), for all \( 1 \leq i \leq n \), we have
\[
M^{(c)}(G, N) \cong Z^{(f_0)} \oplus Z_{r_1}^{(f_1)} \oplus \cdots \oplus Z_{r_t}^{(f_t)} \oplus Z_{r_{t+1}}^{(f_{t+1}-g_{t+1})} \oplus \cdots \oplus Z_{r_n}^{(f_n-g_n)}.
\]

Note that the extra condition \( r_i | r_{i+1} \) in the above theorem is not essential in the process of determining the structure of \( M^{(c)}(G, N) \). In fact without this condition the structure of \( M^{(c)}(G, N) \) is too complicated to state. Also, the mentioned condition helps us to state the proof of Theorem 2.4 more clear and understandable. The following example shows that the mentioned condition is not essential and the above theorem holds for all pairs \( (G, N) \) of finitely generated abelian groups such that \( N \) admits a complement in \( G \) (without any extra condition).

Example. Let \( \langle x_1 | x_1^2 \rangle \cong Z_m^2, \langle x_2 | x_2^3 \rangle \cong Z_{p^4}, \langle x_3 | x_3^3 \rangle \cong Z_{p^3}, \langle x_4 | x_4^5 \rangle \cong Z_{p^5}, \) and \( \langle y_i \rangle \cong Z_2 \) for all \( 1 \leq i \leq m \). Put \( G = N \oplus K \), where \( N = \langle y_1 \rangle \oplus \cdots \oplus \langle y_l \rangle \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle \) and \( \langle x_3 \rangle \oplus \langle x_4 \rangle \), Put \( X_1 = \{ x_1, x_2 \}, X_2 = \{ x_3, x_4 \}, X_1 \cup X_2, \) and \( X_2 = \{ y_{l+1}, \ldots, y_m \}, Y = \{ y_1, \ldots, y_l \}, Y_2 = \{ y_{l+1}, \ldots, y_m \}, Y = Y_1 \cup Y_2. \) Then \( 1 \to R = T \gamma_2(F) \to F \to G \to 1 \) is a free presentation of \( G \) and \( 1 \to R \to SR \to N \to 1 \) is a free presentation of \( N \) where \( F \) is the free group on \( X \cup Y, T = \langle x_1^2, x_2^2, x_3^3, x_4^5 \rangle F \) and \( S = \langle Y_1 \cup X_1 \rangle F \). Define
\[
A_1 = \{ b_{\gamma_{c+2}}(F) | b \text{ is a basic commutator of weight } c + 1 \text{ on } Y \}, \\
A_2 = \{ b_{\gamma_{c+2}}(F) | b \text{ is a basic commutator of weight } c + 1 \text{ on } Y_2 \}, \\
C_1 = \{ b_{\gamma_{c+2}}(F) | b \text{ is a basic commutator of weight } c + 1 \text{ on } \\
X \cup Y \text{ such that } x_1 \text{ does appear in } b \}, \\
C_2 = \{ b_{\gamma_{c+2}}(F) | b \text{ is a basic commutator of weight } c + 1 \text{ on } \\
\{ x_2, x_4 \} \cup Y \text{ such that } x_2 \text{ does appear in } b \}, \\
C_3 = \{ b_{\gamma_{c+2}}(F) | b \text{ is a basic commutator of weight } c + 1 \text{ on } \\
\}
\{x_2, x_3, x_4\} \cup Y \text{ such that } x_3 \text{ does appear in } b\},

C_4 = \{b\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on } \{x_4\} \cup Y \text{ such that } x_4 \text{ does appear in } b\},

N_3 = \{b\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on } \{x_3, x_4\} \cup Y_2 \text{ such that } x_3 \text{ does appear in } b\},

N_4 = \{b\gamma_{c+2}(F) \mid b \text{ is a basic commutator of weight } c + 1 \text{ on } \{x_4\} \cup Y_2 \text{ such that } x_4 \text{ does appear in } b\},

D_1 = \{b^2\gamma_{c+2}(F) \mid b\gamma_{c+2}(F) \in C_1\}

D_2 = \{b^3\gamma_{c+2}(F) \mid b\gamma_{c+2}(F) \in C_2\}

D_3 = \{b^4\gamma_{c+2}(F) \mid b\gamma_{c+2}(F) \in C_3\}

D_4 = \{b^5\gamma_{c+2}(F) \mid \gamma_{c+2}(F) \in C_4\}

H_3 = \{b^3\gamma_{c+2}(F) \mid b\gamma_{c+2}(F) \in N_3\}

H_4 = \{b^5\gamma_{c+2}(F) \mid b\gamma_{c+2}(F) \in N_4\}

Then using an argument similar to the proof of Theorem 2.4, one can obtain that \(E = (A_1 - A_2) \cup \{C_1 \cup C_2 \cup \{C_3 - N_3\} \cup \{C_4 - N_4\}\} \cup \{H_3 \cup H_4\}\) is a basis for \([S, \, fF][T, \, eF]\gamma_{c+2}(F)/\gamma_{c+2}(F)\) and \(D = D_1 \cup D_2 \cup \{D_3 - H_3\} \cup \{D_4 - H_4\}\) is a basis for \([T, \, eF]\gamma_{c+2}(F)/\gamma_{c+2}(F)\). Therefore by (2.1) we have

\[ M^{(c)}(G, N) \cong Z^{(I A_1 - A_2)} \oplus Z^{(D_1)} \oplus Z^{(D_2)} \oplus Z^{(D_4 - H_4)} \],

and hence

\[ M^{(c)}(G, N) \cong Z^{(f_0)} \oplus Z^{(f_1)} \oplus Z^{(f_2 - g_3)} \oplus Z^{(f_3)} \oplus Z^{(f_4 - g_4)} \],

where \(f_0 = \chi_{c+1}(m) - \chi_{c+1}(m-l), \ f_i = \chi_{c+1}(m + 4 - i + 1) - \chi_{c+1}(m + 4 - i), \) for \(1 \leq i \leq 4\), and \(g_i = \chi_{c+1}(m - l + 4 - i + 1) - \chi_{c+1}(m - l + 4 - i)\), for \(3 \leq i \leq 4\).

3. \(c\)-Covering pair

Let \(G\) and \(E\) be two groups with an action of \(G\) on \(E\). Then the \(G\)-commutator \([e, g]\) is the element \(e^{-1}g^{-1}e^g\) of \(E\), where \(e^g\) is the action of \(g\) on \(e\), for all \(g \in G, e \in E\). We also recall the definitions of the subgroups \(Z_c(E, G)\) and \([E, \, eG]\), for all \(c \geq 1\), as follows:

\[ Z_c(E, G) = \{e \in E \mid [e, g_1, \ldots, g_c] = 1, \ \text{for all } g_1, \ldots, g_c \in G\}, \]

\[ [E, \ eG] = \{[e, g_1, \ldots, g_c] \mid e \in E, \ g_1, \ldots, g_c \in G\}, \]

where \([e, g_1, \ldots, g_c] = [\ldots [[e, g_1], g_2], \ldots, g_c]\), inductively.

Moreover, we recall that a pair \((G, N)\) is called nilpotent of class \(k\), if \([N, \ kG] = 1\) and \([N, \ k-1G] \neq 1\), for some positive integer \(k\).

The notion of covering pairs was defined by Ellis [1] and here we extend this concept for the variety of nilpotent groups. For this we need to recall the definition of relative \(c\)-central extensions.
Definition 3.1. (4) Let $(G, N)$ be a pair of groups. A relative $c$-central extension of the pair $(G, N)$ is a group homomorphism $\varphi : E \to G$ together with an action of $G$ on $E$ such that

(i) $\varphi(E) = N$;
(ii) $\varphi(e^g) = g^{-1}\varphi(e)g$, for all $g \in G$, $e \in E$;
(iii) $e^e(e') = e^{-1}e'e$, for all $e, e' \in E$;
(iv) $\ker \varphi \subseteq Z_c(E, G)$.

Definition 3.2. Let $(G, N)$ be a pair of groups. A relative $c$-central extension $\varphi : N^* \to G$ of the pair $(G, N)$ is said to be a $c$-covering pair for $(G, N)$ if there exists a subgroup $A$ of $N^*$ such that

(i) $A \subseteq Z_c(N^*, G) \cap [N^*, cG]$;
(ii) $A \cong M^{(c)}(G, N)$;
(iii) $N \cong N^*/A$.

In particular, if $c = 1$, then a $c$-covering pair is the usual covering pair defined by Ellis [4]. Note that Ellis [4] proved that every pair of finite groups has at least a covering pair.

Here we are going to study the existence of $c$-covering pairs for a pair $(G, N)$ of groups. It is clear that if $M^{(c)}(G, N) = 1$, then $i : N \to G$ is a $c$-covering pair for $(G, N)$. Therefore we should focus on the case $M^{(c)}(G, N) \neq 1$. The following theorem gives a class of pairs of groups which do not admit any $c$-covering pair.

Theorem 3.3. Let $(G, N)$ be a nilpotent pair of groups of class $k$ such that $M^{(c)}(G, N) \neq 1$. Then $(G, N)$ has no $c$-covering pair, for all $c > k$.

Proof. Let $\varphi : N^* \to G$ be a $c$-covering pair for $(G, N)$. Then there exists a subgroup $A$ of $N^*$ such that $A \subseteq Z_c(N^*, G) \cap [N^*, cG]$, $A \cong M^{(c)}(G, N)$ and $N \cong N^*/A$. Since $(G, N)$ is nilpotent of class $k$, we have $[N^*, kG] \subseteq A$. Then the assumption $c > k$ implies that $[N^*, cG] \subseteq [N^*, kG] \subseteq A$. On the other hand $A \subseteq [N^*, cG]$. Thus we have

\[ [N^*, cG] = [N^*, kG] = A. \]  \hfill (3.1)

Also, $A \subseteq Z_c(N^*, G)$ implies that $[A, cG] = [N^*, c+kG] = 1$. Using (3.1), we replace $[N^*, cG]$ by $[N^*, kG]$ and we have $[N^*, 2kG] = 1$. If $2k \leq c$, we have $[N^*, cG] = 1$ and if $2k > c$, then we replace $[N^*, cG]$ by $[N^*, kG]$, again. By repeating this process we obtain $A = [N^*, cG] = 1$, which is a contradiction. This implies that $(G, N)$ does not admit any $c$-covering pair.

Corollary 3.4. Let $c > 1$ and $(G, N)$ be a pair of finitely generated abelian groups such that $N$ admits a complement in $G$. Then the following are equivalent.

(i) $N$ is trivial.
(ii) $M^{(c)}(G, N) = 1$.
(iii) $(G, N)$ admits a $c$-covering pair.
Proof. Theorem 2.4 implies that \( M^{(c)}(G, N) = 1 \) if and only if \( N \) is a trivial subgroup of \( G \). Then the result follows immediately by Theorem 3.3. \( \square \)

4. Some inequalities for the \( c \)-nilpotent multiplier of a pair of groups

In this section, we give some inequalities for the order, the exponent and the minimal number of generators of the \( c \)-nilpotent multiplier of a pair of finite groups and their factor groups. We recall the following lemmas that we need them in the sequel.

**Lemma 4.1.** (Theorem 2.2 in [10]) Let \((G, N)\) be a pair of finite groups and, \( K \) be a normal subgroup of \( G \) contained in \( N \). Then
\[
(i) \quad |\frac{K[N, cG]}{K, cG}||M^{(c)}(G, N)| = |M^{(c)}(\frac{G}{K}, \frac{N}{K})||M^{(c)}(G, K)|;
(ii) \quad d(M^{(c)}(G, N)) \leq d(M^{(c)}(\frac{G}{K}, \frac{N}{K})) + d(M^{(c)}(G, K));
(iii) \quad \exp(M^{(c)}(G, N)) \leq \exp(M^{(c)}(\frac{G}{K}, \frac{N}{K}))\exp(M^{(c)}(G, K));
(iv) \quad d(M^{(c)}(\frac{G}{K}, \frac{N}{K})) \leq d(M^{(c)}(G, N)) + d(\frac{K[N, cG]}{K, cG});
(v) \quad \exp(M^{(c)}(\frac{G}{K}, \frac{N}{K})) \divides \exp(M^{(c)}(G, N))\exp(\frac{K[N, cG]}{K, cG}),
\]
where \( d(X) \) is the minimal number of generators of the group \( X \).

**Lemma 4.2.** (Lemma 22 in [7]). Let \( F/R \) be a free presentation of a group \( G \) and \( B \) a normal subgroup of \( G \), with \( B = S/R \). Then there exists the following epimorphism
\[
\otimes^{c+1}(B, G/B) \rightarrow \frac{[S, cF]}{[R, cF][S, c+1F] \prod_{i=2}^{c+1} \gamma_{c+1}(S, F)_i},
\]
in which for all \( 2 \leq i \leq c, \gamma_{c+1}(S, F)_i = [D_1, D_2, \ldots, D_{c+1}] \) such that \( D_1 = D_2 = S \) and \( D_j = F \), for all \( j \neq 1, i \), and \( \otimes^{c+1}(B, G/B) = B \otimes G/B \otimes \ldots \otimes G/B \) involves \( c \) copies of \( G/B \).

**Lemma 4.3.** (Lemma 3.1 in [10]). Let \( H \) and \( N \) be subgroups of a group \( G \) and \( N = N_0 \supseteq N_1 \supseteq \cdots \) a chain of normal subgroups of \( N \) such that \( [N_i, G] \subseteq N_{i+1} \), for all \( i \in \mathbb{N} \). Then \( [N_i, [H, jG]] \subseteq N_{i+j+1} \), for all positive integers \( i, j \).

Now we state the first result of this section that is an extension of Corollary 2.3 in [10].

**Theorem 4.4.** Let \((G, N)\) be a pair of finite groups and \( K \) be a central subgroup of \( G \) contained in \( N \). Let \( F/R \) be a free presentation of \( G \) and \( T \) be a normal subgroup of the free group \( F \) such that \( K = T/R \). Then
\[
(i) \quad |\frac{K[N, cG]}{K, cG}||M^{(c)}(G, N)||M^{(c)}(\frac{G}{K}, \frac{N}{K})||\otimes^{c+1}(K, \frac{G}{K})|\frac{\prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i}{[R, cF]};
(ii) \quad d(M^{(c)}(G, N)) \leq d(M^{(c)}(\frac{G}{K}, \frac{N}{K})) + d(\otimes^{c+1}(K, \frac{G}{K})) + d(\frac{\prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i}{[R, cF]});
(iii) \quad \exp(M^{(c)}(G, N)) \divides \exp(M^{(c)}(\frac{G}{K}, \frac{N}{K}))\exp(\otimes^{c+1}(K, \frac{G}{K}))\exp(\frac{\prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i}{[R, cF]}).
Proof. (i) Since $K$ is a central subgroup of $G$, we have $[T, F] \leq R$. Then Lemma 4.2 implies the epimorphism

$$\otimes^{c+1}(K, \frac{G}{K}) \rightarrow \frac{[T, cF]}{[R, cF] \prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i}.$$ 

On the other hand, we have

$$\left| (R \cap [T, cF]) / [R, cF] \right| \prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i = \left| [T, cF] / [R, cF] \prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i \right|$$

Therefore

$$\left| [M^{(c)}(G, K)] \right| = \left| \otimes^{c+1}(K, \frac{G}{K}) \right| = \left| [R, cF] \prod_{i=2}^{c+1} \gamma_{c+1}(T, F)_i \right|.$$

Hence the result holds by Lemma 4.1. The proof of (ii) and (iii) are similar.

The following theorem generalizes Theorem 3.2 in [10] and also Theorem C in [7].

**Theorem 4.5.** Let $(G, N)$ be a pair of finite nilpotent groups of class $t$. Then

(i) If $t \geq c + 1$, then

$$|[N, t-1 G]| |M^{(c)}(G, N)|$$

divides $|M^{(c)}(\frac{G}{[N, t-1 G]}, \frac{N}{[N, t-1 G]})) \otimes^{c+1}([N, t-1 G], \frac{Z_{t-1}(N, G)}{G})|$;

(ii) If $t < c + 1$, then

$$|[N, c] G||M^{(c)}(G, N)|$$

divides $|M^{(c)}(\frac{G}{[N, t-1 G]}, \frac{N}{[N, t-1 G]})) \otimes^{c+1}([N, t-1 G], \frac{Z_{t-1}(N, G)}{G})|$;

(iii) $\text{exp}(M^{(c)}(G, N)) \leq \text{exp}(M^{(c)}(\frac{G}{[N, t-1 G]}, \frac{N}{[N, t-1 G]})) \text{exp}(\otimes^{c+1}([N, t-1 G], \frac{Z_{t-1}(N, G)}{G})).$

Proof. Let $G \cong F / R$ be a free presentation of $G$. Let $N \cong S / R$ and $Z_{t-1}(N, G) \cong T_i / R$, for all $0 \leq i \leq t$. Consider the following chain

$$S = T_0 \supseteq \ldots \supseteq T_k \supseteq \ldots \supseteq T_{t-1} \supseteq T_t = R \supseteq [R, F] \supseteq \ldots \supseteq [R, c F].$$

Since $[T_k, F] \subseteq T_{k+1}$, we have $[T_i, [S, t-1 F]] \subseteq [R_i F]$ by Lemma 4.3. This inclusion induces the following epimorphism.

$$\otimes^{c+1}(\frac{[S, t-1 F] R}{R}, \frac{F}{T_{t-1}}) \rightarrow \frac{[S, t-1 F] R c F}{[R, c F]}$$

$$s [R, c F] \otimes x_1 T_{t-1} \otimes \ldots \otimes x_c T_{t-1} \mapsto [s, x_1, \ldots, x_c] [R, c F]$$

So we have

$$\left| [S, t-1 F] R c F \right| / \left| [R, c F] \right| \otimes^{c+1} (\frac{[S, t-1 F] R}{R}, \frac{F}{T_{t-1}})$$

On the other hand, considering $K = [N, t-1 G]$ in Lemma 4.1, we have if $t \geq c + 1$, then

$$|[N, t-1 G]| |M^{(c)}(G, N)| = |M^{(c)}(\frac{G}{[N, t-1 G]}, \frac{N}{[N, t-1 G]})) | \left| [S, t-1 F] R c F \right| / \left| [R, c F] \right|,$$
Lemma 4.1, we have

\[
[N_c G]|M^{(c)}(G, N) = |M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right)||\left[\frac{S_{t-1} F}{R_c F}\right]_R.
\]

Now (i) follows by (4.1). One can obtain (ii) and (iii) similarly. □

The next result gives another upper bound for the order, the exponent and the minimal number of generators of the \(c\)-nilpotent multiplier of a pair of finite groups. This theorem is an extension of Theorem 3.4 in [10] and also generalizes Theorem B in [7].

**Theorem 4.6.** Let \((G, N)\) be a pair of finite nilpotent groups of class at most \(t (t \geq 2)\). Then

(i) \(|[N_c G]|M^{(c)}(G, N)| \text{ divides } |M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right)|\prod_{i=1}^{t-1} |\otimes^{c+1}([N_i G], [N_i G])|;

(ii) \(d(M^{(c)}(G, N)) \leq d(M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right)) + \sum_{i=1}^{t-1} d(\otimes^{c+1}([N_i G], [N_i G]));

(iii) \(\exp(M^{(c)}(G, N)) \text{ divides } \exp(M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right))\prod_{i=1}^{t-1} \exp(\otimes^{c+1}([N_i G], [N_i G])).

**Proof.** (i) Let \(F, S, R\) be as in Theorem 4.5. Considering \(K = [N, G]\) in Lemma 4.1, we have

\[
|[N_c G]|M^{(c)}(G, N)| = |M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right)||M^{(c)}(G, [N, G])|[N_{c+1} G]|.
\]

On the other hand,

\[
|[N_{c+1} G]|M^{(c)}(G, [N, G])| = \left|\frac{[S_{t+1} F]_R}{R} \left|\frac{(R \cap [S, F_c F])_R}{R_c F}\right| \right| = \left|\frac{[S, F]_R}{R_c F}\right| = \left|\frac{[S, i F]_R}{R_c F}\right| \prod_{i=1}^{t-1} \left|\frac{[S, i+1 F]_R}{R_c F}\right|.
\]

By the assumption of theorem, 1 = \([N, t G] = ([S, t F] R) / R\) and hence \([S, t F] R, cF] = \(R, cF\). Therefore

\[
|[N, c G]|M^{(c)}(G, N)| = |M^{(c)}\left(\frac{G}{[N, G]}, \frac{N}{[N, G]}\right)\prod_{i=1}^{t-1} \left|\frac{[S, i F]_R}{R_c F}\right|.
\]

On the other hand, for all \(1 \leq i \leq t - 1\),

\[
\prod_{j=2}^{t} \gamma_{c+1}(|[S, i F]_R, F]_j | [S, i F] R, c+1 F] \leq |[S, i+1 F] R, c F].
\]

Considering this inequality, Lemma 4.2 implies that

\[
\left|\frac{[S, i F]_R}{R_c F}\right| \leq \left|\otimes^{c+1}([N, i G], [N, i G])\right|,
\]

and if \(t < c + 1\), then

\[
|[N, c G]|M^{(c)}(G, N)| = |M^{(c)}(G, [N, G])||[N_{c+1} G]|.
\]
and hence the assertion follows.

(ii) Let \( r(G) \) be the special rank of \( G \). Using 4.1 and 4.2, we obtain

\[
d(M^{(c)}(G, N)) \leq d(M^{(c)}(\frac{G}{[N,G]}, \frac{N}{[N,G]})) + d(M^{(c)}(G, [N,G]))
\]

\[
\leq d(M^{(c)}(\frac{G}{[N,G]}, \frac{N}{[N,G]})) + r(R \cap \frac{[S,F_c F]}{[R_c F]})
\]

\[
\leq d(M^{(c)}(\frac{G}{[N,G]}, \frac{N}{[N,G]})) + r([S,F]R_c F)
\]

\[
\leq d(M^{(c)}(\frac{G}{[N,G]}, \frac{N}{[N,G]})) + \sum_{i=1}^{t-1} r([S_i,F]R_c F)
\]

\[
\leq d(M^{(c)}(\frac{G}{[N,G]}, \frac{N}{[N,G]})) + \sum_{i=1}^{t-1} d(\otimes^{c+1}([N_i,G], \frac{G}{[N_i,G]})).
\]

The last inequality holds because \( r(A) = d(A) \) for any finite abelian group \( A \).

(iii) It can be proved easily by an argument similar to the proof of (i). \( \square \)

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