Symmetry-Protected Topological relationship between $SU(3)$ and $SU(2) \times U(1)$ in Two Dimension

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(Dated: January 12, 2021)
Abstract

Symmetry-protected topological (SPT) phases are gapped short-range entangled states with symmetry $G$, which can be systematically described by group cohomology theory. $SU(3)$ and $SU(2) \times U(1)$ are considered as the basic groups of Quantum Chromodynamics and Weak-Electromagnetic unification, respectively. In two dimension ($2D$), nonlinear-sigma models with a quantized topological Theta term can be used to describe nontrivial SPT phases. By coupling the system to a probe field and integrating out the group variables, the Theta term becomes the effective action of Chern-Simons theory which can derive the response current density. As a result, the current shows a spin Hall effect, and the quantized number of the spin Hall conductance of SPT phases $SU(3)$ and $SU(2) \times U(1)$ are same. In addition, relationships between $SU(3)$ and $SU(2) \times U(1)$ which maps $SU(3)$ to $SU(2)$ with a rotation $U(1)$ will be given.

PACS numbers: 75.10.Jm, 73.43.Cd

In condensed matter physics, Quantum Chromodynamics ($QCD$) is a proper gauge theory to describe Strong interaction by investigating the relationship between the basic element ($quark$) and the gauge field $[1, 2]$. Gapped phases of quantum matter are naturally described by topological quantum field theories ($TQFTs$) at low energy and long distance $[3]$. In two dimension ($2D$), Abelian and non-Abelian Chern-Simons theories can be used to capture the topological properties of fractional quantum Hall conductance $[4, 5]$. And in the TQFTs, there is another interesting method, symmetry-protected topological (SPT) phases $[6, 8]$, which can be transformed to product states via local unitary ($LU$) transformations $[9, 11]$. In reference $[8, 12]$, the results for one-dimensional SPT phase are generalized to any dimensions. So far, the model of quark $[13, 15]$, construction of baryon $[16, 17]$, and normalized field $[18, 20]$ are constructed out of group $SU(3)$. The Weinberg-Salam model is a proper model with combinatorial group $SU(2) \times U(1)$, whose theoretical calculation results are consistent with the experiment (i.e. boson $W^+$, $W^-$ and $Z^0$ get quality, photons are massless) $[21, 23]$. This model can be used to describe Weak-Electromagnetic interaction, which consists of Weak interaction normalized group $SU(2)$ and the Electromagnetic interaction normalized group $U(1)$. If we split group $SU(3)$ into the product of two more basic groups as $SU(2) \times U(1)$, the QCD will become a new version. Using this method, some similar properties between group $SU(3)$ and $SU(2) \times U(1)$ can be easily founded.
In the work present here, we will introduce two kind of SPT phases—$SU(3)$ and $SU(2) \times U(1)$. If these phases couple to external probe field individually, the derived results will show that the spin Hall conductance are quantized, and the quantized number of $SU(3)$ and $SU(2) \times U(1)$ are same.

Principal chiral nonlinear sigma model (PCM) with a Theta term, which has action as \[24\]

$$S = \int_M d\tau d^2x \frac{1}{Q} Tr \left[ (g^{-1} \partial_\mu g) (g^{-1} \partial_\mu g) \right] + \frac{i \Theta}{24\pi^2} Tr \left[ \varepsilon^{\mu\nu\lambda} (g^{-1} \partial_\mu g) (g^{-1} \partial_\mu g) (g^{-1} \partial_\mu g) \right]$$  \hspace{1cm} (1)

where the second term on the right-hand side is the action of topological term, $g$ is a group element of $SU(n)$, $M$ the Euclidian space-time manifold, and $\Theta = 2\pi k$ with $k \in \mathbb{Z}$ meaning PCM is quantized. When $\Theta = \pi k$, the system also has two discrete symmetries, reflection varies as $x \rightarrow -x$ and time reversal varies as $i \rightarrow -i$, $t \rightarrow -t$ (consequently $\tau \rightarrow \tau$). It is easy to check that reflection will not affect the quantized number. According to \[25\], the discussion about the time reversal shows that the quantized number will not change, neither.

As we can see from the equation (1), if $Q$ flows to infinity, the action will flow to a fixed point where only the topological term remains. In spite of ignoring the first term of equation (1), the physical properties of the system will not change. In the rest part of this letter, gauge symmetry will be discussed in the fixed point condition.

Following the research \[25\], equation (1) is verified to be invariant under a symmetry $SU(n)_L \times SU(n)_R$, where $SU(n)_L$ and $SU(n)_R$ are left and right symmetry group, respectively. The group element $g$ varies as $g \rightarrow hg$ for $h \in SU(n)_L$, while varies as $g \rightarrow gh^{-1}$ for $h \in SU(n)_R$. It is easy to check that the equation (1) is invariable under symmetry group $SU(n)_L$, but no more invariant under symmetry group $SU(n)_R$.

In order to investigate the gauge symmetry of PCM, the Theta term should be couple to an external probe field $A$ by replacing every $g^{-1} \partial_\mu g$ term with $g^{-1} (\partial_\mu + A_\mu) g$.

At the fixed point, Theta term becomes

$$\frac{\Theta}{24\pi^2} \int_M Tr \left[ (g^{-1} (d + A) g) \right]^3 = \frac{\Theta}{24\pi^2} \int_M Tr \left[ (g^{-1} dg)^3 + A^3 + 3dg g^{-1} \wedge F + 3d (dg g^{-1} \wedge A) \right]$$  \hspace{1cm} (2)

where $F = dA + A \wedge A$ is the field strength of the probe field $A$. Focusing on the four terms on the right-hand side of the equation (2), we can obtain that the first term is the Theta
term in equation (1), the second term is the pure probe field function, and the rest two terms are functions of $A$ and $dgg^{-1}$. For simplicity, the Theta term can be reduced to the effective field theory of the external field $A$ by integrating out the group variables $g$. The effective action of probe field $A$ can be expressed as the Chern-Simons action

$$S_{\text{eff}}(A) = i \frac{\Theta}{8\pi^2} \int_M Tr \left( A \wedge F - \frac{1}{3} A^3 \right) = i \frac{\Theta}{16\pi^2} \int_M \varepsilon^{\mu\nu\lambda} Tr \left( A_\mu^a \partial_\nu A_\lambda^a + \frac{2}{3} \varepsilon_{abc} A_\mu^a A_\nu^b A_\lambda^c \right)$$

where the construction of $A$ depends on the different conditions, which will be discussed in the following sections.

Firstly, let us talk about the $SU(3)$ group. According to the reference [8], the $SU(3)$ SPT phases can be classified by group cohomology class $H^3[SU(3), U(1)] = \mathbb{Z}$ in 2D (the details of the calculation can be found in the Supplement Material Section A). This non-abelian probe field can be expressed as $A_\mu = \sum_a A_\mu^a T^a = \frac{1}{2} \sum_a A_\mu^a (2T^a)$, where $2T^a$ are the eight Gell-Mann matrixes that generate the Quantum Chromodynamics theory. The trace $Tr (T_a T_b) = \frac{1}{2} \delta^{ab}$ contributes an extra coefficient $\frac{1}{2}$ in the equation (3).

By calculating the variation of equation (3), the response current expressed as following

$$\mathcal{J}_\mu^a = \frac{\delta S_{\text{eff}}}{\delta A_\mu^a} = i \frac{\Theta}{8\pi^2} \varepsilon^{\mu\nu\lambda} \left( \partial_\nu A_\lambda^a + \varepsilon_{abc} A_\nu^b A_\lambda^c \right)$$

Without loss of generality, we assume that the probe field $A$ only contains $A^a$ component. Then the time component of the current in equation (4) can be expressed as

$$\mathcal{J}_t^a = i \frac{\Theta}{8\pi^2} \left( \partial_x A_y^a - \partial_y A_x^a \right)$$

and the space components can be expressed as

$$\mathcal{J}_x^a = i \frac{\Theta}{8\pi^2} \left( \partial_y A_t^a - \partial_t A_y^a \right)$$

$$\mathcal{J}_y^a = i \frac{\Theta}{8\pi^2} \left( \partial_t A_x^a - \partial_x A_t^a \right)$$

Equation (5)-(7) can be used to express the spin Hall effect, if the time and space components of the current representations are considered as magnetic and electric field, respectively. The spin Hall conductance is quantized as $\frac{\Theta}{8\pi^2}$. 
Secondly, in 2D, $SU(2) \times U(1)$ SPT phases are classified by group cohomology class $\mathcal{H}^3 [SU(2) \times U(1), U(1)] = \mathbb{Z}$ (For details, see the Supplement Material Section B). At the fixed point, the Theta term of the PCM of $SU(2) \times U(1)$ becomes

$$S = \frac{i \Theta}{24\pi^2} Tr \{ \varepsilon^{\mu \nu \lambda} [(gh)^{-1} \partial_\mu (gh)] [(gh)^{-1} \partial_\nu (gh)] [(gh)^{-1} \partial_\lambda (gh)] \}$$

$$= \frac{i \Theta}{24\pi^2} Tr \{ \varepsilon^{\mu \nu \lambda} [(g^{-1} \partial_\mu g) (g^{-1} \partial_\nu g) (g^{-1} \partial_\lambda g) + (h^{-1} \partial_\mu h) (h^{-1} \partial_\nu h) (h^{-1} \partial_\lambda h)] \}$$

(8)

where $g \in SU(2)$ and $h \in U(1)$. The two terms on right-side of second row represent the action of group $SU(2)$ and group $U(1)$, respectively. For group $SU(2)$, its construction can be referred to the same part of $SU(3)$ as equation (3), however probe fields satisfy $A_\mu = \sum_a A^a_\mu T^a = \frac{1}{2} \sum_a A^a_\mu (2T^a)$, where $2T^a$ are three Pauli matrixes. Do the same derivation as $SU(3)$, the time and space components of current can also be expressed as equations (5)-(7). So the quantized number of spin Hall effect is $\frac{\Theta}{8\pi}$. It is noted that $U(1)$ is an Abelian group and its construction is different from $SU(n)$. According to the reference [26], Abelian version of the Chern-Simons Lagrangian is

$$\mathcal{L}_{CS} = i \frac{\Theta}{8\pi^2} \varepsilon^{\mu \nu \lambda} Tr A_\mu \partial_\nu A_\lambda$$

(9)

Then the effective action and the response current density become

$$S_{\text{eff}} (A) = i \frac{\Theta}{8\pi^2} \int_M \varepsilon^{\mu \nu \lambda} Tr A_\mu \partial_\nu A_\lambda$$

(10)

$$J^\mu = \frac{\delta S_{\text{eff}}}{\delta A_\mu} = i \frac{\Theta}{8\pi^2} \varepsilon^{\mu \nu \lambda} Tr \partial_\nu A_\lambda$$

(11)

Combine equation (11) and (11), the response current density of the group $SU(2) \times U(1)$ is

$$J^a_\mu = i \frac{\Theta}{8\pi^2} \varepsilon^{\mu \nu \lambda} Tr \left( \partial_\nu A^a_\lambda + \varepsilon_{abc} A^b_\nu A^c_\lambda + \partial_\nu B_\lambda \right) = i \frac{\Theta}{8\pi^2} \varepsilon^{\mu \nu \lambda} Tr \left[ \partial_\nu \left( A^a_\lambda + B_\lambda \right) + \varepsilon_{abc} A^b_\nu A^c_\lambda \right]$$

(12)

where $A^a_\lambda \in SU(2)$ and $B_\lambda \in U(1)$. Assuming $A^a_\lambda + B_\lambda$ is a new probe field, it is easy to find that the response density construction of group $SU(2) \times U(1)$ is similar to group $SU(3)$. The group $SU(3)$ is non-Abelian and satisfies the commutation relations $[T^a, T^b] = f_{abc} T^c$, where
\( f_{abc} \) is the structural constant. The basic group of \( SU(2) \times U(1) \) satisfies \([T^a, T^b] = g_{abc}T^c\). The structural constant \( g_{abc} \) and \( f_{abc} \) are similar but not the same. Therefore, there should be a potential relationship between group \( SU(3) \) and \( SU(2) \times U(1) \).

In order to obtain the relationship, we give Gell-Mann matrixes and Pauli matrixes as following

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
\lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{align*}
\]

and

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{align*}
\]

Then, assuming group \( SU(2) \) maps to \( SU(3) \)

\[
\begin{align*}
\sigma_1 &\rightarrow \lambda_1, \lambda_4, \lambda_6, \\
\sigma_2 &\rightarrow \lambda_2, \lambda_5, \lambda_7, \\
\sigma_3 &\rightarrow \lambda_3, \lambda_8.
\end{align*}
\]

Picking some matrixes from the group \( SU(3) \), it is clear that there are some intrinsic relationships among them. The maps in (15) give three relationships. \( \lambda_1, \lambda_4 \) and \( \lambda_6 \) in the first relationship can be treated as the \( 3 \times 3 \) external matrixes of \( \sigma_1 \). These matrixes satisfy the anticommutation relationship \( \{\lambda_i, \lambda_j\} = \lambda_k \) (where \( i, j, k = 1, 4, 6 \)), which construct a new group. Considering matrixes \( \lambda_1, \lambda_4 \) and \( \lambda_6 \) are all off-diagonal and their elements are symmetrical about the diagonal, we take group \( U(1) \) as a rotation that rotates matrixes diagonally. Subsets of group \( SU(3) \) by the rotation \( U(1) \) can be shown in Fig [I]. The second relationship contains \( \lambda_2, \lambda_5 \) and \( \lambda_7 \). Unfortunately, there is neither commutation relationship...
\[ \alpha_1 = \alpha_2 = \alpha_3 = \frac{2\pi}{3} \]

\[ \beta_1 = \beta_2 = \beta_3 = \frac{2\pi}{3} \]

FIG. 1. In 2D, \( U(1) \) rotation relationship between the matrixes of \( SU(3) \). (a) matrixes \( \lambda_1, \lambda_4 \) and \( \lambda_6 \) form a new group, every vertex rotates to another by \( \frac{2\pi}{3} \). (b) matrixes \( \lambda_2, \lambda_5 \) and \( \lambda_7 \) form a subset, every vertex rotates to another by \( \frac{2\pi}{3} \).

Nor anticommutation between \( \lambda_2, \lambda_5 \) and \( \lambda_7 \), in spite of they seem as the \( 3 \times 3 \) external matrixes of \( \sigma_2 \). If \( \lambda_2, \lambda_5 \) and \( \lambda_7 \) are treated as the subset of \( SU(3) \), rotation \( U(1) \) will still works in structure(Fig [1](b)). The last relationship contains \( \lambda_3 \) and \( \lambda_8 \), which are invariant by the rotation \( U(1) \) because they are both diagonal. For \( \lambda_3 \), from the anticommutation relationship between \( \lambda_3 \) and other matrixes of \( SU(3) \) except \( \lambda_8 \),

\[ \{ \lambda_3, \lambda_1 \} = 0, \{ \lambda_3, \lambda_2 \} = 0, \{ \lambda_3, \lambda_4 \} = \lambda_4, \]
\[ \{ \lambda_3, \lambda_5 \} = \lambda_5, \{ \lambda_3, \lambda_6 \} = -\lambda_6, \{ \lambda_3, \lambda_7 \} = -\lambda_7 \]

there should be a relationship between every two matrixes(Fig [2]). However, matrix except \( \lambda_8 \) anticommutes \( \lambda_8 \) will obtain 2 times or negative of the matrix itself,

\[ \{ \lambda_8, \lambda_1 \} = 2\lambda_1, \{ \lambda_8, \lambda_2 \} = 2\lambda_2, \{ \lambda_8, \lambda_3 \} = 2\lambda_3, \{ \lambda_8, \lambda_4 \} = -\lambda_4, \]
\[ \{ \lambda_8, \lambda_5 \} = -\lambda_5, \{ \lambda_8, \lambda_6 \} = -\lambda_6, \{ \lambda_8, \lambda_7 \} = -\lambda_7 \]

In this way, \( \lambda_8 \) can be considered as a constraint condition which makes the eight matrixes form a group completely.

In summary, we study PCM actions of \( SU(3) \) and \( SU(2) \times U(1) \), which both have a Theta term. Coupling the system to a probe field and integrating out the group variables, the results can be considered as effective action of Chern-Simons theory. As a consequence, the
Our work might be useful to investigate the further significant information of QCD by above method. The current results do work in 2D condition, the work in higher dimension will be carried out in

spin Hall conductance of $SU(3)$ and $SU(2) \times U(1)$ both are quantized as $\frac{e}{8\pi^2}$. Furthermore, in $SU(3)$, we calculate anticommutation relationships which show that there is an intrinsic connection between every two matrixes. In order to simplify the $SU(3)$, we give a mapping from $SU(2)$ to $SU(3)$ under the rotation $U(1)$. This rotation rotates matrixes diagonally, which classifies $SU(3)$ into three categories. Every category is invariant under the rotation $U(1)$, which makes a simpler model $SU(2) \times U(1)$ to represent $SU(3)$. In addition, there are three more subgroups in $SU(3)$ by the anticommutation relationships, which can be used to investigate the intrinsic constructions and properties of the quark. Our work might be useful to investigate the further significant information of QCD by above method.

FIG. 2. (color online). Every two vertexes have a relationship. The start vertex of the solid arrow is the matrix which can construct a new one; the end vertex is the matrix which is constructed by another. (a) Dotted line is the anticommutate relationships between $\lambda_3$ and other, and two endpoints of dotted line across $\lambda_3$ anticommutate each other. Every vertex in triangle $\lambda_1, \lambda_4, \lambda_6$ has two incoming edges and two outgoing edges, which makes $\lambda_1, \lambda_4, \lambda_6$ be a new group. (b) The red shaded area shows $\lambda_2, \lambda_4, \lambda_7$ construct a group; the green shaded area shows $\lambda_1, \lambda_5, \lambda_7$ construct a group; the blue shaded area shows $\lambda_2, \lambda_5, \lambda_6$ construct a group.
future.

This work is supported by the National Natural Science Foundation of China(Project No. 11374193).

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