Coherent states for quadratic Hamiltonians

Alonso Contreras-Astorga, David J Fernández C and Mercedes Velázquez
Department of Physics, Cinvestav, AP 14-740, 07000 Mexico DF, Mexico
E-mail: david@fis.cinvestav.mx

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Abstract
The coherent states for a set of quadratic Hamiltonians in the trap regime are constructed. A matrix technique which allows us to directly identify the creation and annihilation operators will be presented. Then, the coherent states as simultaneous eigenstates of the annihilation operators will be derived, and will be compared with those attained through the displacement operator method. The corresponding wavefunction will be found, and a general procedure for obtaining several mean values involving the canonical operators in these states will be described. The results will be illustrated through the asymmetric Penning trap.

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1. Introduction

As was shown by Schrödinger in 1926 for the harmonic oscillator, the quasi-classical states are important for the description of physical systems in the classical limit (see e.g. [1]). The catchy term coherent states (CS) was used for the first time by Glauber long after, when studying electromagnetic correlation functions [2, 3]. With this application it was realized that CS are also useful in the intrinsically quantum domain. Indeed, the CS approach is nowadays widely employed for dealing with quantum physical systems. According to Glauber there are three equivalent ways to construct the CS for the harmonic oscillator. The first is to define them as eigenstates of the annihilation operator. The second is to build the CS through the action of a displacement operator onto the ground state. The third is to consider them as quantum states having a minimum Heisenberg uncertainty relationship. These three properties can be used as definitions to build CS for systems different from the harmonic oscillator. However, it is noteworthy that each of them leads to sets of CS which do not coincide in general [4–13]. In fact, even for the harmonic oscillator the third definition does not produce just the standard CS, since it also includes the so-called squeezed states [14, 15].

Despite being known for many years, from time to time there are some advances which maintain interest in the subject. This is the case, e.g., with the recently discovered coherent
states for a charged particle inside an ideal Penning trap [16]. Since the corresponding Hamiltonian is quadratic in the position and momentum operators, one would expect that the CS appear as a generalized displacement operator acting onto the corresponding ground state. However, it is worth noting that the Penning trap Hamiltonian does not have any ground state at all, since it is not a positively defined operator. Despite that, it was possible to implement in a simple way the corresponding CS construction. Thus, we need to take into account this Hamiltonian property when studying the CS of general systems.

In this paper we are going to address the CS construction for systems characterized by a certain set of quadratic Hamiltonians. The CS will be built up as simultaneous eigenstates of the corresponding annihilation operators, and also by applying a generalized displacement operator onto an appropriate extremal state. We will see that in the case of a positively defined Hamiltonian this extremal state will coincide with the ground state. In order to perform the CS construction, we need to first find the annihilation and creation operators. This task will be performed using a matrix technique, which generalizes the one employed in [16] (see also [17–20]). In this way, we will simply and systematically identify the characteristic algebra of the Hamiltonians involved. Our procedure represents a generalization of the standard technique to dimensions greater than one to deal with the harmonic oscillator, which is closely related to the well-known factorization method (see, e.g., [21, 22]).

The paper is organized as follows. In section 2 we introduce a detailed recipe for systematically obtaining the annihilation and creation operators for quadratic Hamiltonians in the trap regime. The CS derivation is elaborated in section 3, while in section 4 we address the completeness of this set of CS, obtain the mean values of several important physical quantities and the time evolution of these states. We apply our general results to an asymmetric Penning trap in section 5, and our conclusions are presented in section 6.

2. Ladders operators for quadratic Hamiltonians

Throughout this work we are going to consider a general set of $n$-dimensional quadratic Hamiltonians of the form

$$ H = \frac{1}{2} \eta^T B \eta, $$

where $B$ is a $2n \times 2n$ real constant symmetric matrix, $\eta = (\vec{X}, \vec{P})^T$, and $\vec{X}, \vec{P}$ are the $n$-dimensional coordinate and momentum operators in the Schrödinger picture satisfying the canonical commutation relationships $[X_i, P_j] = i\hbar \delta_{ij}$ (note that a system of units such that $\hbar = 1$ will be used throughout this paper). The time evolution of the operator vector $\eta(t) = U(t)\eta U(t)$ in the Heisenberg picture is governed by

$$ \frac{d\eta(t)}{dt} = U(t)[iH, \eta]U(t) = U(t)\Lambda \eta U(t) = \Lambda \eta(t), $$

where $U(t)$ is the evolution operator of the system such that $U(0) = 1$, and $\Lambda = JB$, where $J$ is the well-known $2n \times 2n$ matrix

$$ J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, $$

satisfying

$$ J^T = -J, \quad J^2 = -I_{2n}, \quad \det(J) = 1, $$

in which $I_m$ represents the $m \times m$ identity matrix. The solution of equation (2) is given by

$$ \eta(t) = e^{\Lambda t} \eta(0) = e^{\Lambda t} \eta. $$

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In order to identify the annihilation and creation operators of $H$, we need to find the right and left eigenvectors of $\Lambda$. Since in general $\Lambda$ is non-Hermitian, its right and left eigenvectors are not necessarily adjoint to each other.

Let us consider first the $2n$th-order characteristic polynomial of $\Lambda$, $P(\lambda) = \det(\Lambda - \lambda) = \det(JB - \lambda)$. Using equation (4) and the fact that $B$ is a symmetric matrix, we obtain

$$P(\lambda) = \det(JB - \lambda) = \det[(JB - \lambda)^T] = \det(JB + \lambda) = P(-\lambda).$$

This means that if $\lambda$ is an eigenvalue of $\Lambda$, then so will $-\lambda$. Throughout this work we denote the eigenvalues of $\Lambda$ as $\lambda_k$ and $-\lambda_k$, taking $\lambda_k$ in the way $\text{Re}(\lambda_k) > 0$ or $\text{Im}(\lambda_k) > 0$ if $\text{Re}(\lambda_k) = 0$, $k = 1, \ldots, n$.

Let us label as $u_k^\pm$ and $f_k^\pm$ the right and left eigenvectors associated with the eigenvalues $\pm \lambda_k$, respectively, i.e.

$$\Lambda u_k^\pm = \pm \lambda_k u_k^\pm, \quad f_k^\pm \Lambda = \pm \lambda_k f_k^\pm.$$ (8)

Note that both $u_k^\pm$ and $f_k^\pm$ can be determined from equation (8) up to arbitrary factors. Part of this arbitrariness will be eliminated by imposing two requirements. The first is that the right and left eigenvectors be dual to each other, namely,

$$f_j^r u_k^r = \delta_{jk} \delta_{rr'},$$ (9)

with $j, k = 1, \ldots, n$ and $r, r' = +, -$. The second condition, which is needed in order to recover the standard annihilation and creation operators for the one-dimensional harmonic oscillator, is to ask that the left eigenvectors $f_k^\pm$ involved in the commutators

$$[f_k^- \eta, f_k^+ \eta] = \gamma_k, \quad \gamma_k \in \mathbb{C},$$ (10)

such that

$$|\gamma_k| = 1.$$ (11)

In this paper we only discuss the case in which there are no degeneracies in the eigenvalues $\pm \lambda_k$ so that the identity matrix $I_{2n}$ can be expanded as (see [23])

$$I_{2n} = \sum_{k=1}^{n} (u_k^+ \otimes f_k^+ + u_k^- \otimes f_k^-),$$ (12)

with $\otimes$ representing the tensor product. Then we obtain

$$\eta(t) = e^{\Lambda t} \left[ \sum_{k=1}^{n} (u_k^+ \otimes f_k^+ + u_k^- \otimes f_k^-) \right] \eta = \sum_{k=1}^{n} \left[ e^{\lambda_k t} u_k^+ \otimes f_k^+ \eta + e^{-\lambda_k t} u_k^- \otimes f_k^- \eta \right]$$

$$= \sum_{k=1}^{n} \left( e^{\lambda_k t} u_k^+ L_k^+ + e^{-\lambda_k t} u_k^- L_k^- \right),$$ (13)

where $L_k^\pm \equiv f_k^\pm \eta$.

It is worth pointing out that, in the classical case, $L_k^\pm$ represent $c$-numbers which are related to the initial conditions. The time dependence of $\eta(t)$ is determined essentially by the $\lambda_k$-values, which are complex in general. If $\text{Re}(\lambda_k) \neq 0$ it can be seen that one of the two involved exponentials of the $k$th term in the previous relation diverges as $t$ increases and, thus, the classical motion will be in general unbounded (see [24–27]). The only way in which this does not happen is when all the eigenvalues are purely imaginary so that the corresponding exponentials will just induce oscillations in time, and therefore in this case the classical evolution of the vector $\eta(t)$ will always remain bounded.
On the other hand, in the quantum regime the \( L_\pm^k \) are linear operators in the canonical variables \( \vec{X}, \vec{P} \). It is straightforward to show that their commutators with \( H \) reduce to

\[
[H, L_\pm^k] = \mp i \lambda_k L_\pm^k. \tag{14}
\]

In addition, it turns out that

\[
[L_j^-, L_k^+] = [L_j^+, L_k^-] = 0, \quad [L_j^-, L_k^-] = 0, \quad k \neq j. \tag{15}
\]

However,

\[
[L_j^-, L_k^+] = \gamma_k \neq 0, \quad k = 1, \ldots, n. \tag{16}
\]

Equation (14) implies that \( L_\pm^k \) behaves, at least formally, as ladder operators for the eigenvectors of \( H \), changing its eigenvalues by \( \pm i \lambda_k \). However, this statement has to be managed carefully since it could happen that the action of \( L_\pm^k \) onto an eigenvector of \( H \) produces something which does not belong to the domain of \( H \), and in this case we would not obtain new eigenvectors of \( H \). In the next section we explore an interesting situation (the trap regime of our systems) for which the application of \( L_\pm^k \) onto an eigenvector of \( H \) produces a new one with different eigenvalue.

Let us point out that equations (14)–(16) imply that \( H \) can be expressed in a simple way in terms of \( \{L_\pm^k, k = 1, \ldots, n\} \) [24]. This is a consequence of the following general theorem:

**Theorem 1.** If \( \mathcal{L} \) is an irreducible algebra of operators generated by \( L_\pm^k \) which obey

\[
[L_j^+, L_j^+] = [L_j^-, L_j^-] = 0, \quad [L_j^-, L_j^+] = \gamma_i \delta_{ij} \quad \text{with} \quad |\gamma_i| = 1, \quad i, j = 1, \ldots, n,
\]

then an operator \( H \in \mathcal{L} \) which fulfills relation (14) can be written as

\[
H = \sum_{k=1}^{n} \left( -\frac{i \lambda_k}{\gamma_k} \right) L_+^k L_-^k + g_0, \tag{17}
\]

where \( g_0 \in \mathbb{C} \).

**Demonstration.** Actually, due to equations (14)–(16) it turns out that

\[
g_0 \equiv H - \sum_{k=1}^{n} \left( -\frac{i \lambda_k}{\gamma_k} \right) L_+^k L_-^k
\]

commutes with \( L_\pm^k \) for all \( k \), thus with any function of them, and so \( g_0 \) must be a c-number.

From now on we discard situations such that \( \text{Re}(\lambda_k) \neq 0 \) for some \( k = 1, \ldots, n \), restricting ourselves to cases in which the \( \lambda_k \) are purely imaginary for all \( k \), i.e. we stick to the trap regime of our systems.

2.1. Algebraic structure of \( H \) in the trap regime

Let us suppose that \( \lambda_k = i \omega_k \) with \( \omega_k > 0 \), \( k = 1, \ldots, n \). Hence, \( \lambda_k^* = -\lambda_k \), and since \( \Lambda \) is real, without losing generality we can choose

\[
f_k^- = (f_k^+)^*, \quad u_k^- = (u_k^+)^* \quad \text{for } k = 1, \ldots, n.
\]

Since \( L_\pm^k \) are linear combinations of the Hermitian components of \( \eta (X_1, \ldots, X_n, P_1, \ldots, P_n) \), it turns out that

\[
(L_\pm^k)^\dagger = L_\mp^k. \tag{18}
\]

Note moreover that \( \gamma_k^* = \gamma_k \), i.e. \( \gamma_k \in \mathbb{R} \), and the use of equation (11) implies that \( \gamma_k = \pm 1 \). Summarizing these results, equation (14) in this case becomes

\[
[H, L_k^\pm] = \pm i \omega_k L_k^\pm, \tag{20}
\]
i.e. the modifications suffered by the eigenvalues of $H$ through the action of the ladder operators $L_k^\pm$ are given by the real quantities $\pm \omega_k$. In addition, $H$ is factorized as (see equation (17))

$$H = \sum_{k=1}^{n} \gamma_k \omega_k L_k^+ L_k^- + g_0, \quad g_0 \in \mathbb{R}.$$ (21)

Note that the previous summation either involves terms for which $\gamma_k = 1$, which are of the oscillator kind since they are positively defined, or terms with $\gamma_k = -1$ which are of the antioscillator type since they are negatively defined. Thus, it is natural to define a global algebraic structure for our system (see, e.g., [16, 28]), which is independent of the spectral details but has to do with the fact that any quadratic Hamiltonian in the trap regime can be expressed in terms of several independent oscillators, some of them indeed being anti-oscillators (compare equation (21)). This global structure is characterized mathematically by identifying $n$ sets of number, annihilation and creation operators of the system, $\{ N_k, B_k, B_k^\dagger \}, k = 1, \ldots, n,$ in the way:

$$B_k = L_k^-, \quad B_k^\dagger = L_k^+, \quad \text{for} \quad \gamma_k = 1,$$

$$B_k = L_k^+, \quad B_k^\dagger = L_k^-, \quad \text{for} \quad \gamma_k = -1,$$

$$N_k = B_k^\dagger B_k, \quad k = 1, \ldots, n,$$ (22)

so that the standard commutation relationships are satisfied,

$$[B_j, B_k] = \delta_{jk}, \quad [B_j, B_k^\dagger] = [B_j^\dagger, B_k] = 0,$$

$$[N_k, B_k] = -B_k, \quad [N_k, B_k^\dagger] = B_k^\dagger, \quad j, k = 1, \ldots, n.$$ (23)

Let us construct now a basis $\{| n_1, \ldots, n_n \rangle, n_j = 0, 1, 2, \ldots, j = 1, 2, \ldots, n \}$ of common eigenstates of $\{ N_1, \ldots, N_n \}$ (the Fock states)

$$N_j | n_1, \ldots, n_n \rangle = n_j | n_1, \ldots, n_n \rangle, \quad j = 1, \ldots, n,$$ (24)

starting from an extremal state $|0, \ldots, 0\rangle$, which is annihilated simultaneously by $B_1, \ldots, B_n$:

$$B_j |0, \ldots, 0\rangle = 0, \quad j = 1, \ldots, n.$$ (25)

If we assume that $|0, \ldots, 0\rangle$ is normalized, it turns out that

$$| n_1, \ldots, n_n \rangle = \frac{B_1^{n_1} \cdots B_n^{n_n} |0, \ldots, 0\rangle}{\sqrt{n_1! \cdots n_n!}}.$$ (26)

Moreover, $B_j$ and $B_j^\dagger$, $j = 1, \ldots, n$, act onto $| n_1, \ldots, n_n \rangle$ in a standard way:

$$B_j | n_1, \ldots, n_j-1, n_j, n_{j+1}, \ldots, n_n \rangle = \sqrt{n_j} | n_1, \ldots, n_j-1, n_j-1, n_{j+1}, \ldots, n_n \rangle,$$

$$B_j^\dagger | n_1, \ldots, n_j-1, n_j, n_{j+1}, \ldots, n_n \rangle = \sqrt{n_j+1} | n_1, \ldots, n_j-1, n_j+1, n_{j+1}, \ldots, n_n \rangle.$$ (27)

Now, in terms of the operators $\{ B_j, B_j^\dagger, j = 1, \ldots, n \}$ our Hamiltonian is expressed by

$$H = \sum_{k=1}^{n} \gamma_k \omega_k B_k^\dagger B_k + g_0.$$ (28)

It is clear that the Fock states $| n_1, \ldots, n_n \rangle$ are eigenstates of $H$ with eigenvalues $E_{n_1, \ldots, n_n} = \gamma_1 \omega_1 n_1 + \cdots + \gamma_n \omega_n n_n + g_0 = E(n_1, \ldots, n_n)$. In particular, the extremal state $|0, \ldots, 0\rangle$ has eigenvalue $E_{0, \ldots, 0} = g_0$. In case that $\gamma_j = 1$ for all $k$, then $H - g_0$ will be a positively defined operator, and the extremal state $|0, \ldots, 0\rangle$ will become the ground state for our system, associated with the lowest eigenvalue $E_{0, \ldots, 0} = g_0$ of $H$. On the other hand, if there is at least one index $j$ for which $\gamma_j = -1$, then $H - g_0$ will not be positively defined, since the
corresponding \( j \)th term is of inverted oscillator type, and the state \( |0, \ldots, 0\rangle \) will not be a ground state for our system (however it keeps its extremal nature since it is always annihilated by the \( n \) operators \( B_j, j = 1, \ldots, n \)).

Following [16, 28] it is straightforward to see that, besides the global algebraic structure, there is an intrinsic algebraic structure for our system, characterized by the existing relationship between the Hamiltonian \( H \) and the \( n \) number operators \( N_k \):

\[
H = E(N_1, \ldots, N_n) = \sum_{k=1}^{n} \gamma_k \omega_k N_k + g'_0. \tag{29}
\]

As in the examples discussed in [16, 28], it turns out that this intrinsic algebraic structure is responsible for the specific spectrum of our Hamiltonian. On the other hand, the global algebraic structure arises from the existence of the \( n \) independent oscillator modes for \( H \), each characterized by the standard generators \( \{ N_j, B_j, B^\dagger_j \} \), \( j = 1, \ldots, n \). This global behavior allows us to identify in a natural way the extremal state \( |0, \ldots, 0\rangle \equiv |0\rangle \), which plays the role of a ground state although it does not necessarily have a minimum energy eigenvalue. Moreover, the very existence of the extremal state \( |0\rangle \) is guaranteed by a theorem [24] ensuring that if the operators \( \{ B_1, \ldots, B_n \} \) obey the commutation relations given by equation (23), then the system of partial differential equations

\[
\langle \vec{x} | B_j | 0, \ldots, 0 \rangle = \langle \vec{x} | B_j | 0 \rangle = 0, \quad j = 1, \ldots, n, \tag{30}
\]

has the square integrable solution

\[
\phi_0(\vec{x}) = \langle \vec{x} | 0 \rangle = c e^{-\frac{1}{2}(\vec{a}^\dagger \vec{a})} = c e^{-\frac{1}{2}(\vec{x}^T \vec{a}^\dagger \vec{a})}, \tag{31}
\]

with \( c \) being a normalization factor. In this wavefunction, \( \vec{a} = (a_{ij}) \) represents a symmetric matrix whose complex entries are found by solving the system of equations (30), leading to

\[
\vec{a} \vec{\alpha}_j = \vec{\beta}_j, \quad j = 1, \ldots, n, \tag{32}
\]

where \( \vec{\alpha}_j \) and \( \vec{\beta}_j \) are obtained by expressing \( B_j \) and \( B^\dagger_j \) as

\[
B_j = i \vec{P} \cdot \vec{\alpha}_j + \vec{X} \cdot \vec{\beta}_j, \quad B^\dagger_j = -i \vec{\alpha}_j^\dagger \cdot \vec{P} + \vec{\beta}_j^\dagger \cdot \vec{X}, \quad j = 1, \ldots, n. \tag{33}
\]

The wavefunctions for the other Fock states can be found from equation (26).

3. Coherent states

Once our Hamiltonian has been expressed appropriately in terms of annihilation and creation operators, we can develop a similar treatment as for the harmonic oscillator to build up the corresponding CS. Here we are going to construct them either as simultaneous eigenstates of the annihilation operators of the system or as the ones resulting from acting the global displacement operator onto the extremal state.

3.1. Annihilation operator coherent states

In the first place let us look for the annihilation operator coherent states (AOCS) as common eigenstates of the \( B_j \)’s:

\[
B_j |z_1, \ldots, z_n\rangle = z_j |z_1, \ldots, z_n\rangle, \quad z_j \in \mathbb{C}, \quad j = 1, \ldots, n. \tag{34}
\]

Following a standard procedure, let us expand them in the basis \( \{|n_1, \ldots, n_n\rangle\} \):

\[
|z_1, \ldots, z_n\rangle = \sum_{n_1, \ldots, n_n=0}^{\infty} c_{n_1, \ldots, n_n} |n_1, \ldots, n_n\rangle. \tag{35}
\]
By imposing now that equation (34) is satisfied, the following recurrence relationships are obtained,
\[ c_{n_1,\ldots,n_j,\ldots,n_n} = \frac{z_j}{\sqrt{n_j}} c_{n_1,\ldots,n_j-1,\ldots,n_n}, \quad j = 1, \ldots, n, \tag{36} \]
which, when iterated, lead to
\[ c_{n_1,\ldots,n_j,\ldots,n_n} = \frac{z_j^{n_j}}{\sqrt{n_j!}} c_{n_1,\ldots,n_{j-1},\ldots,n_n}, \quad j = 1, \ldots, n. \tag{37} \]
Hence, it turns out that
\[ c_{n_1,\ldots,n_n} = \frac{z_1^{n_1} \cdots z_n^{n_n}}{\sqrt{n_1! \cdots n_n!}} c_{0,\ldots,0}, \tag{38} \]
where \( c_{0,\ldots,0} \) is to be found from the normalization condition. Thus the normalized AOCS become finally
\[ |z_1, \ldots, z_n\rangle = \exp \left( -\frac{1}{2} \sum_{j=1}^{n} |z_j|^2 \right) \sum_{n_1,\ldots,n_n=0}^{\infty} \frac{z_1^{n_1} \cdots z_n^{n_n} |n_1, \ldots, n_n\rangle}{\sqrt{n_1! \cdots n_n!}}, \tag{39} \]
up to a global phase factor.

3.2. Displacement operator coherent states
The displacement operator for the \( j \)th oscillator mode of the Hamiltonian reads
\[ D_j(z_j) = \exp(z_j B_j^\dagger - z_j^* B_j). \tag{40} \]
By using the BCH formula it turns out that
\[ D_j(z_j) = \exp \left( -\frac{|z_j|^2}{2} \right) \exp (z_j B_j^\dagger) \exp (-z_j^* B_j). \tag{41} \]
Now, the global displacement operator is given by
\[ D(\mathbf{z}) \equiv D(z_1, \ldots, z_n) = D_1(z_1) \cdots D_n(z_n), \tag{42} \]
where \( \mathbf{z} \) denotes the complex variables \( z_1, \ldots, z_n \) associated with the \( n \) oscillator modes.
Let us obtain now the displacement operator coherent states (DOCS) \( |\mathbf{z}\rangle \) by applying \( D(\mathbf{z}) \) onto the extremal state \( |0, \ldots, 0\rangle \equiv |0\rangle \):
\[ |\mathbf{z}\rangle = D(\mathbf{z})|0\rangle = \exp \left( -\frac{1}{2} \sum_{j=1}^{n} |z_j|^2 \right) \sum_{n_1,\ldots,n_n=0}^{\infty} \frac{z_1^{n_1} \cdots z_n^{n_n} |n_1, \ldots, n_n\rangle}{\sqrt{n_1! \cdots n_n!}}. \tag{43} \]
Note that the AOCS and the DOCS are the same (compare equations (39) and (43)).

3.3. Coherent state wavefunctions
In order to find the wavefunctions of the CS previously derived, we employ that \([z_j B_j^\dagger - z_j^* B_j, z_k B_k^\dagger - z_k^* B_k] = 0 \forall j, k\). Thus,
\[ D(\mathbf{z}) = \exp(z_1 B_1^\dagger - z_1^* B_1) \cdots \exp \left( z_n B_n^\dagger - z_n^* B_n \right) = \exp \left[ (z_1 B_1 + \cdots + z_n B_n^\dagger) - (z_1^* B_1 + \cdots + z_n^* B_n) \right]. \tag{44} \]
Using equation (33) we can now write
\[ D(\mathbf{z}) = e^{-i\vec{\tilde{p}} \cdot \vec{\tilde{x}}} = e^{-\frac{i}{2} \vec{\tilde{p}} \cdot \vec{\tilde{x}}} e^{\vec{\tilde{p}} \cdot \vec{\tilde{x}}} e^{-i\vec{\tilde{p}} \cdot \vec{\tilde{x}}} = e^{\frac{i}{2} \vec{\tilde{p}} \cdot \vec{\tilde{x}}} e^{-i\vec{\tilde{p}} \cdot \vec{\tilde{x}}} e^{i\vec{\tilde{p}} \cdot \vec{\tilde{x}}}, \tag{45} \]
where we have employed once again the BCH formula and have taken
\[
\hat{\Gamma} = 2 \text{Re}[z_n^* \hat{a}_1 + \cdots + z_n^* \hat{a}_n], \quad \hat{\Sigma} = -2 \text{Im}[z_n^* \hat{b}_1 + \cdots + z_n^* \hat{b}_n].
\] (46)

Now, it is straightforward to find the wavefunction for the coherent state \(|\mathbf{z}\rangle\),
\[
\phi_\mathbf{z}(\tilde{\mathbf{x}}) = (\mathbf{z}|D(\mathbf{z})|0\rangle = e^{-\frac{i}{2} \hat{\mathbf{y}} \cdot \hat{\mathbf{x}}} e^{\frac{i}{2} \hat{\mathbf{x}} \cdot \mathbf{z}} \phi_0(\tilde{\mathbf{x}}).
\] (47)

Since the operator \(\hat{\mathbf{P}}\) is the coordinate displacement generator [1], it turns out that
\[
\langle \tilde{\mathbf{x}} | e^{-i\hat{\mathbf{P}}} = \langle \tilde{\mathbf{x}} - \hat{\mathbf{P}} \rangle,
\] (48)

so that
\[
\phi_\mathbf{z}(\tilde{\mathbf{x}}) = e^{-\frac{i}{2} \hat{\mathbf{x}} \cdot \hat{\mathbf{y}}} e^{\frac{i}{2} \hat{\mathbf{x}} \cdot \mathbf{z}} \phi_0(\tilde{\mathbf{x}} - \hat{\mathbf{P}}).
\] (49)

A further calculation, using equation (31), finally leads to
\[
\phi_\mathbf{z}(\tilde{\mathbf{x}}) = e^{-\frac{i}{2} (\hat{\mathbf{y}} + i \hat{\mathbf{P}}) \cdot \tilde{\mathbf{x}}} \phi_0(\tilde{\mathbf{x}}).
\] (50)

Once again, the importance of the extremal state in our treatment becomes evident, since its wavefunction determines the corresponding wavefunction for any other CS. Moreover, as can be seen from equation (49), the position probability density for the CS \(|\mathbf{z}\rangle\) just becomes a displaced version of the corresponding one for the extremal state \(|0\rangle\).

### 4. Mathematical and physical properties

Let us next derive the completeness relationship for the previously derived CS. Note that, from the point of view of the analysis of states in the Hilbert space of the system, this is the most important property which our CS would have [9, 11, 29, 30]. This is the reason why several authors use it as the fourth coherent state definition, considering it the fundamental one which will survive in time (see e.g. [11]). We are also going to calculate some important physical quantities in these states.

#### 4.1. Completeness relationship

A straightforward calculation leads to
\[
\left(\frac{1}{\pi}\right)^n \int \cdots \int |\mathbf{z}\rangle \langle \mathbf{z}| d^2z_1 \cdots d^2z_n
\] 
\[
= \sum_{m_1,n_1,\ldots,m_n,n_n=0}^{\infty} |m_1,\ldots,m_n,n_1,\ldots,n_n| \frac{1}{\sqrt{m_1!n_1!\cdots m_n!n_n!}} \prod_{j=1}^{n} \left(\frac{1}{\pi} \int z_j^m z_{j+1}^n e^{-|z_j|^2} d^2z_j \right) = 1,
\] (51)

with 1 being the identity operator. Thus, the CS \(|\mathbf{z}\rangle\) form a complete set in the state space of the system (indeed they constitute an overcomplete set [31, 32]). This implies that any state can be expressed in terms of our CS, in particular, an arbitrary CS,
\[
|\mathbf{z}'\rangle = \left(\frac{1}{\pi}\right)^n \int \cdots \int |\mathbf{z}\rangle \langle \mathbf{z}|\mathbf{z}'\rangle d^2z_1 \cdots d^2z_n,
\] (52)

where the reproducing kernel \(\langle \mathbf{z}|\mathbf{z}'\rangle\) is given by
\[
\langle \mathbf{z}|\mathbf{z}'\rangle = \exp \left[ -\frac{1}{2} \sum_{j=1}^{n} (|z_j|^2 - 2z_j^* z_j' + |z_j'|^2) \right].
\] (53)

This means that, in general, our CS are not orthogonal to each other. Indeed, note that inside our infinite set of CS only the extremal state of the system, \(|0\rangle = |\mathbf{z} = 0\rangle = |0,\ldots,0\rangle\), is also an eigenstate of the Hamiltonian.
4.2. Mean values of some physical quantities in a CS

Now we can calculate easily the mean values \( \langle X_j \rangle_z \equiv \langle z|X_j|z \rangle \), \( \langle P_j \rangle_z \equiv \langle z|P_j|z \rangle \), \( j = 1, \ldots, n \), in a given CS \( |z \rangle = |z_1, \ldots, z_n \rangle \), as well as its mean square deviation in terms of the corresponding results for the extremal state \( |0 \rangle \). To do that, let us analyze first how the operators \( X_j, X_j^2, P_j, P_j^2 \) are transformed under \( D(z) \). By using equations (45) and (46) it is straightforward to show that

\[
D^\dagger(z)X_j^n D(z) = (X_j + \Gamma_j)^n, \quad D^\dagger(z)P_j^n D(z) = (P_j + \Sigma_j)^n, \quad n = 1, 2, \ldots
\]

where we have used that, for an operator \( A \) which commutes with \( [A, B] \), it turns out that \( e^A e^{-A} = B + [A, B] \) \( \Rightarrow e^A B^n e^{-A} = (B + [A, B])^n, \quad n = 1, 2, \ldots \)

Thus, a straightforward calculation leads to

\[
\langle X_j \rangle_z = \langle z|X_j|z \rangle = \langle 0|D^\dagger(z)X_j D(z)|0 \rangle = \langle X_j \rangle_0 + \Gamma_j.
\]

On the other hand,

\[
\langle X_j^2 \rangle_z = \langle X_j^2 \rangle_0 + 2\Gamma_j \langle X_j \rangle_0 + \Gamma_j^2.
\]

Hence,

\[
\langle \Delta X_j \rangle_z^2 = \langle X_j^2 \rangle_z - \langle X_j \rangle_z^2 = \langle X_j^2 \rangle_0 - \langle X_j \rangle_0^2 = \langle \Delta X_j \rangle_0^2.
\]

Working in a similar way for \( P_j \), it is obtained

\[
\langle P_j \rangle_z = \langle P_j \rangle_0 + \Sigma_j, \quad \langle P_j^2 \rangle_z = \langle P_j^2 \rangle_0 + 2\Sigma_j \langle P_j \rangle_0 + \Sigma_j^2.
\]

Then we also have that

\[
\langle \Delta P_j \rangle_z^2 = \langle \Delta P_j \rangle_0^2,
\]

i.e. the mean square deviations of \( X_j \) and \( P_j \) in the CS \( |z \rangle \) are independent of \( z \).

In order to finish this calculation, the mean values \( \langle X_j \rangle_0, \langle P_j \rangle_0, \langle X_j^2 \rangle_0, \) and \( \langle P_j^2 \rangle_0 \) for \( j = 1, \ldots, n \) are required. Let us describe now the procedure to find these \( 4n \) quantities. The first \( 2n, \langle X_j \rangle_0, \langle P_j \rangle_0, \) can be easily found by recalling the definitions of \( B_i, B_i^\dagger \) (see section 2.1) and using the fact that their mean values in the extremal state \( |0 \rangle \) always vanish for \( k = 1, \ldots, n \):

\[
\langle B_k \rangle_0 = \langle B_k^\dagger \rangle_0 = 0, \quad k = 1, \ldots, n.
\]

This is equivalent to the following linear system of \( 2n \) homogeneous equations

\[
f_k^\dagger (\eta)_0 = f_k^\dagger (\eta)_0 = 0, \quad k = 1, \ldots, n.
\]

Since the left eigenvectors \( f_k^\dagger, k = 1, \ldots, n \), are linearly independent, the only solution for the \( 2n \) unknowns \( \langle \eta \rangle_0 \) is the trivial one, i.e. \( \langle X_j \rangle_0 = \langle P_j \rangle_0 = 0 \). It is worth pointing out that this result simplifies equations (56), (57) and (59).

On the other hand, the mean values of the quadratic operators \( X_i^2, P_i^2, \quad i = 1, \ldots, n, \) in the extremal state \( |0 \rangle \) can be obtained from evaluating the corresponding quantities for the several non-equivalent products of pairs of annihilation \( B_j \) and creation \( B_j^\dagger \) operators. It is important to mention that these products should have the appropriate order to use the fact that \( B_j \) annihilates \( |0 \rangle \) and \( B_j^\dagger \) does with \( |0 \rangle \) (if the product involves one \( B_j \) it should be placed to the right while if it involves one \( B_j^\dagger \) it should be placed to the left). In general, we obtain \( n(2n + 1) \) non-equivalent products of pairs of operators \( B_i, B_j^\dagger, \quad i, j = 1, \ldots, n; \quad n(n + 1)/2 \) products of kind \( B_i^\dagger B_j, \quad j = 1, \ldots, n, \quad i \leq j; \quad n(n + 1)/2 \) products of kind \( B_i B_j^\dagger, \quad j = 1, \ldots, n, \quad i \leq j; \quad n^2 \) products of kind \( B_i^\dagger B_j^\dagger, \quad i, j = 1, \ldots, n \). The mean values in the extremal state \( |0 \rangle \) lead to an inhomogeneous systems of \( n(2n + 1) \) equations, with the same number of unknowns. When
solving this system we obtain \( \langle X_j^2 \rangle_0, \langle P_j^2 \rangle_0 \), \( j = 1, \ldots, n \), and the mean value of any other product of two canonical operators \( X_i, P_j \).

It is customary nowadays to group the mean values of the quadratic products of the operators \( X_i, P_j \) in the coherent state \( |z \rangle \) in a \( 2n \times 2n \) real symmetric matrix \( \sigma(z) \), called the covariance matrix, whose elements are given by (remember that \( \eta = (X_1, \ldots, X_n, P_1, \ldots, P_n)^T \)):

\[
\sigma_{ij}(z) = \frac{1}{2} \left( \langle \eta_i \eta_j + \eta_j \eta_i \rangle_z - \langle \eta_i \rangle_z \langle \eta_j \rangle_z \right), \quad i, j = 1, \ldots, 2n. \tag{63}
\]

A straightforward calculation leads to

\[
\sigma_{ij}(z) = \sigma_{ij}(0) = \frac{1}{2} \left( \langle \eta_i \eta_j + \eta_j \eta_i \rangle_0 \right) \equiv \sigma_{ij}, \tag{64}
\]

where we have used that \( \langle \eta_i \rangle_0 = 0 \), \( i = 1, \ldots, 2n \). The conclusion is that the covariance matrix in our coherent state \( |z \rangle \) is once again independent of \( z \) and depends only on the extremal state \( |0 \rangle \).

Note that the number of independent matrix elements \( \sigma_{ij} \) coincides with the number of unknowns which are determined from the set of \( n(2n+1) \) independent equations associated with the mean values of the quadratic products of \( B_j^\dagger B_k \) in the extremal state \( |0 \rangle \).

Once the covariance matrix is determined, the generalized uncertainty relation can be evaluated \[33–35\]

\[
\sigma_{ii} \sigma_{nn} + \sigma_{ni} \sigma_{ni} \geq \frac{1}{4}, \quad i = 1, \ldots, n, \tag{65}
\]

which coincides with the Robertson–Schrödinger uncertainty relation (see e.g. \[33\]).

Let us conclude this section by calculating the mean value of the Hamiltonian in a given CS \( |z \rangle \). Equation \( 28 \) leads to

\[
\langle H \rangle_z = \langle z | H | z \rangle = \sum_{k=1}^{n} \gamma_k \omega_k |z_k|^2 + g_0'. \tag{66}
\]

In order to obtain \( \langle H^2 \rangle_z \), let us note that

\[
H^2 = \sum_{j,k=1}^{n} \gamma_j \gamma_k \omega_j \omega_k B_j^\dagger B_k + \sum_{k=1}^{n} \omega_k^2 B_k^\dagger B_k + 2g_0' \sum_{k=1}^{n} \gamma_k \omega_k B_k^\dagger B_k + g_0^2. \tag{67}
\]

Thus we obtain

\[
\langle H^2 \rangle_z = \sum_{j,k=1}^{n} \gamma_j \gamma_k \omega_j \omega_k |z_j|^2 + \sum_{k=1}^{n} \omega_k^2 |z_k|^2 + 2g_0' \sum_{k=1}^{n} \gamma_k \omega_k |z_k|^2 + g_0^2. \tag{68}
\]

Hence,

\[
\langle \Delta H \rangle_z^2 = \sum_{k=1}^{n} \omega_k^2 |z_k|^2. \tag{69}
\]

Note that, for one-dimensional systems \( (n = 1) \), this expression reduces to the standard one for the harmonic oscillator (see e.g. \[1\]).

4.3. Time evolution of the CS

Suppose that at \( t = 0 \) our system is in a CS \( |z \rangle \). Thus, at a later time \( t > 0 \) the evolved state is found by acting on \( |z \rangle \) with the evolution operator of the system \( U(t) = \exp(-iHt) \). By making use of equation \( 29 \) it turns out that

\[
U(t) = e^{-i\gamma_0 t} \prod_{k=1}^{n} e^{-i\gamma_k \omega_k N_k}. \]
Hence,

\[ U(t)\ket{\mathbf{z}} = e^{-ig'0t}\ket{z_1(t), \ldots, z_n(t)} = e^{-ig'0t}\ket{\mathbf{z}(t)}, \]  

(70)

where \(z_j(t) = e^{-ig_j\omega_j t}z_j = |z_j|e^{i\theta_j - g_j\omega_j t}\). Equation (70) implies that a CS \(\ket{\mathbf{z}}\) evolves with time into a new CS \(\ket{\mathbf{z}(t)} = |z_1(t), \ldots, z_n(t)\rangle\), where the \(j\)th degree of freedom \(z_j(t)\) just rotates at its characteristic frequency \(\omega_j\) (clockwise if \(\gamma_j = 1\) and counterclockwise if \(\gamma_j = -1\)).

### 4.4. Gazeau–Klauder coherent states

At this point, it would be interesting to check if our CS belong to the class introduced recently by Gazeau and Klauder [36]. Using their notation, for a system with a Hamiltonian \(H\) such that the ground state energy is zero, the Gazeau–Klauder CS \(\{\ket{J, \theta}, J \geq 0, -\infty < \theta < \infty\}\) obey the following properties:

(a) Continuity: \((J', \theta') \rightarrow (J, \theta) \Rightarrow \ket{J', \theta'} \rightarrow \ket{J, \theta};\)

(b) Resolution of unity: \(1 = \int \ket{J, \theta} \bra{J, \theta} d\mu(J, \theta);\)

(c) Temporal stability: \(e^{-iHt}\ket{J, \theta} = \ket{J, \theta + \omega t}, \omega = \text{constant};\)

(d) Action identity: \(\bra{J, \theta} H \ket{J, \theta} = \omega J.\)

Concerning the first property, it is straightforward to check that our CS given in equation (43) are such that \(\ket{\mathbf{z}'} \rightarrow \ket{\mathbf{z}}\) as \(\mathbf{z}' \rightarrow \mathbf{z}\), i.e. they are continuous in \(\mathbf{z}\). As for the second and third properties, both were explicitly proven in sections 4.1 and 4.3, respectively. It remains just to analyze if the action identity given in (d) is valid. Let us first note that for each partial Hamiltonian \(H_k = \gamma_k\omega_kN_k\) of our system it turns out that

\[ \bra{\mathbf{z}} H_k \ket{\mathbf{z}} = \gamma_k\omega_k|z_k|^2,\]

which is time independent. Therefore, property (d) becomes valid for each degree of freedom separately and thus it is valid for our global system with the natural identification \(J_k = |z_k|^2, \theta_k = \text{arg}(z_k)\) so that

\[ \bra{\mathbf{z}} (H - g_0) \ket{\mathbf{z}} = \sum_{k=1}^n \gamma_k\omega_k J_k.\]

We conclude that our CS of equation (43) also become an \(n\)-dimensional generalization of the Gazeau–Klauder CS if we express each complex component \(z_k\) of \(\mathbf{z}\) in its polar form (the polar coordinates essentially coincide with the canonical action-angle variables for the corresponding classical system).

### 5. Asymmetric Penning trap coherent states

Let us apply now the previous technique to the asymmetric Penning trap. Such an arrangement can be used to control some quantum mechanical phenomena [37] as well as to perform high-precision measurements of the fundamental properties of particles. Moreover, it is a quite natural system to analyze the decoherence taking place due to the unavoidable interaction of the system with its environment [38, 39]. Since the asymmetric Penning trap becomes the ideal one when the asymmetry parameter vanishes [40–42], it will be straightforward to compare these results with those recently obtained for the ideal Penning trap [16] (see also [35, 43, 44]).
The Hamiltonian of a charged particle with mass $m$ and charge $q$ in an asymmetric Penning trap reads
\[
H = \frac{\hat{p}^2}{2m} + \frac{\omega_c}{2} (X P_x - Y P_y) + \frac{m}{2} (\omega_x^2 X^2 + \omega_y^2 Y^2 + \omega_z^2 Z^2),
\]
where $\omega_c = qB/m$ and $\omega_x$, $\omega_y$, $\omega_z$ being the cyclotron and axial frequencies respectively, and the frequencies $\omega_x$, $\omega_y$, $\omega_z$ are given by
\[
\omega_x^2 = \frac{\omega_x^2}{4} - \frac{\omega_c^2}{2} (1 + \epsilon), \quad \omega_y^2 = \frac{\omega_y^2}{4} - \frac{\omega_c^2}{2} (1 - \epsilon),
\]
where $|\epsilon| < 1$ is the real asymmetry parameter and we are denoting $\vec{P} = (P_x, P_y, P_z)^T$, $\vec{X} = (X, Y, Z)^T$. Without loss of generality \[16\], from now on we will assume that $m = 1$.

As was seen in section 2, the main role in our treatment is played by the matrix
\[
\Lambda = \begin{pmatrix}
0 & -\omega_c/2 & 1 & 0 & 0 & 0 \\
\omega_c/2 & 0 & 0 & 1 & 0 & 0 \\
-\omega_x^2 & 0 & 0 & -\omega_c/2 & 0 & 0 \\
0 & -\omega_y^2 & \omega_c/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\omega_z^2 & 0
\end{pmatrix}.
\]

The eigenvalues ($\lambda$) of $\Lambda$ are
\[
\lambda_1 = \frac{i\omega_c}{2} \sqrt{2 - \delta + R} = i\omega_1, \quad \lambda_2 = \frac{i\omega_c}{2} \sqrt{2 - \delta - R} = i\omega_2,
\]
\[
\lambda_3 = i\omega_c = i\omega_3, \quad R = \sqrt{4(1 - \delta) + 2\delta^2}, \quad 0 < \delta = \frac{2\omega_2^2}{\omega_c^2} < 1,
\]
and their corresponding complex conjugate. The right ($u$) and left ($f$) eigenvectors of $\Lambda$ become
\[
u_1^+ = s_1 \begin{pmatrix}
\frac{4}{\omega_c}(2 + \delta \epsilon + R) - i \omega_c \frac{2(1 - \delta) - \delta \epsilon + R}{\omega_1} & \frac{i\omega_c}{2\omega_1} \frac{2(1 - \delta) - \delta \epsilon + R}{\delta \epsilon + R}, 1, 0, 0
\end{pmatrix}^T,
\]
\[
u_2^+ = s_2 \begin{pmatrix}
\frac{4}{\omega_c}(\delta \epsilon - R) - i \omega_c \frac{2(1 - \delta) - \delta \epsilon - R}{\omega_2} & \frac{i\omega_c}{2\omega_2} \frac{2(1 - \delta) - \delta \epsilon - R}{\delta \epsilon - R}, 1, 0, 0
\end{pmatrix}^T,
\]
\[
u_3^+ = s_3 \begin{pmatrix}
0, 0, 0, 0, -i \omega_c, 1
\end{pmatrix}^T,
\]
\[
u_1^- = t_1 \begin{pmatrix}
\frac{\omega_c}{4}(R - \delta \epsilon), \frac{i\omega_c^2}{8\omega_1}[2(1 - \delta) + \delta \epsilon + R], -i\omega_c \frac{2(2 - \delta \epsilon + R)}{4\omega_1}, 1, 0, 0
\end{pmatrix},
\]
\[
u_2^- = t_2 \begin{pmatrix}
\omega_c \frac{(R - \delta \epsilon)}{4}, \frac{i\omega_c^2}{8\omega_2}[2(1 - \delta) + \delta \epsilon - R], -i\omega_c \frac{2(2 - \delta \epsilon - R)}{4\omega_2}, 1, 0, 0
\end{pmatrix},
\]
\[
u_3^- = t_3 \begin{pmatrix}
0, 0, 0, 0, i\omega_c, 1
\end{pmatrix},
\]
where $s_j, t_j \in \mathbb{C}, j = 1, 2, 3$. The requirement that the right and left eigenvectors be dual to each other implies
\[
s_1 = \frac{1}{4t_1} \begin{pmatrix}
1 + \frac{\delta \epsilon}{R}
\end{pmatrix}, \quad s_2 = \frac{1}{4t_2} \begin{pmatrix}
1 - \frac{\delta \epsilon}{R}
\end{pmatrix}, \quad s_3 = \frac{1}{2t_3}.
\]
On the other hand, up to some phase factors, the condition imposed by equations (10) and (11) leads to
\[ t_1 = \frac{1}{\sqrt{2}(f_{1a} f_{1c} - f_{1b})}, \quad t_2 = \frac{1}{\sqrt{2}(f_{2b} - f_{2a} f_{2c})}, \quad t_3 = \frac{1}{\sqrt{2} \omega_3}, \]
where we denote \( f_j^* = t_j(f_{1a}, f_{1b}, f_{1c}, 1, 0, 0), f_j = t_j(f_{2a}, f_{2b}, f_{2c}, 1, 0, 0) \) in order to simplify the notation (compare equation (75)). Moreover, the crucial signs for us to conclude that our asymmetric Penning trap Hamiltonian is not positively defined become
\[ \gamma_1 = 1, \quad \gamma_2 = -1, \quad \gamma_3 = 1. \] (78)

Thus, our annihilation operators take the form (see equation (22))
\[ B_1 = L_1 = t_1(f_{1a} X - f_{1b} Y - f_{1c} P_x + P_y), \]
\[ B_2 = L_2 = t_2(f_{2a} X + f_{2b} Y + f_{2c} P_x + P_y), \] (79)
\[ B_3 = L_3 = t_3(-\omega_3 Z + P_z). \]

From these operators and their Hermitian conjugates, it is straightforward to identify the \( \alpha_j \) and \( \beta_j, \; j = 1, 2, 3 \), which allow us to find the matrix \( A \) such that \( \alpha_j = \beta_j \). Its matrix elements \( a_{ij} \) now become
\[ a_{11} = -i \frac{f_{1a} - f_{2a}}{f_{1c} + f_{2c}}, \quad a_{12} = a_{21} = i \frac{f_{1b} + f_{2b}}{f_{1c} + f_{2c}}, \quad a_{22} = i \frac{f_{1c} f_{2b} - f_{2a} f_{1b}}{f_{1c} + f_{2c}}, \]
\[ a_{33} = \omega_3, \quad a_{31} = a_{32} = a_{33} = 0. \] (80)

It can be shown that \( a_{11}, a_{22}, a_{33} \in \mathbb{R}^+ \) while \( a_{12} \) is purely imaginary. Thus the extremal state wavefunction of equation (31) acquires the form
\[ \phi_{0}(\bar{\xi}) = c \exp \left( -\frac{1}{2} a_{11} x^2 - \frac{1}{2} a_{22} y^2 - a_{12} x y \right) \exp \left( -\frac{1}{2} a_{33} z^2 \right). \] (81)

The associated eigenvalue becomes \( E_{0,0} = (\omega_1 - \omega_2 + \omega_3)/2 \).

Concerning the CS \( |z_1, z_2, z_3 \rangle \), the general treatment developed in section 3 is straightforwardly applicable, and their explicit expressions are given by equation (43) with \( n = 3 \). Their corresponding wavefunctions are given by
\[ \phi_{i}(\bar{\xi}) = \langle \bar{\xi}|x, y, z \rangle = e^{-i\bar{\xi} \cdot \vec{\Sigma}/2} e^{i \bar{\xi} \cdot \vec{\Sigma}} \phi_{0}(x - \Gamma_1, y - \Gamma_2, z - \Gamma_3), \] (82)
where
\[ \vec{\Gamma} = \begin{pmatrix} t_{11} & t_{12} \exp [z_1] - t_{22} \exp [z_2] \\ -t_{11} \exp [z_1] - t_{22} \exp [z_2] \\ -t_{22} \exp [z_3] \end{pmatrix}, \quad \vec{\Sigma} = \begin{pmatrix} t_{11} f_{1a} \exp [z_1] + t_{22} f_{2a} \exp [z_2] \\ -t_{11} f_{1b} \exp [z_1] + t_{22} f_{2b} \exp [z_2] \\ t_{22} \omega_3 \exp [z_3] \end{pmatrix}. \] (83)

The mean values \( \langle X_j \rangle_x, \langle P_j \rangle_x \), immediately follow from equations (56) and (59) with \( \langle X_j \rangle_0 = \langle P_j \rangle_0 = 0 \), i.e.
\[ \langle X_j \rangle_x = \Gamma_j, \quad \langle P_j \rangle_x = \Sigma_j, \quad j = 1, 2, 3. \] (84)

As for the mean values of the quadratic operators in the extremal state, we have solved the system of equations arising from the null mean values of the products of pairs of annihilation \( B_j \) and creation \( B_j^\dagger \) operators. We obtain
\[ \langle X^2 \rangle_0 = \frac{1}{2 a_{11}}, \quad \langle P_x^2 \rangle_0 = \frac{1}{2} \left( a_{11}^2 - a_{12}^2 a_{22} \right), \]
\[ \langle Y^2 \rangle_0 = \frac{1}{2 a_{22}}, \quad \langle P_y^2 \rangle_0 = \frac{1}{2} \left( a_{22}^2 - a_{12}^2 a_{11} \right), \]
\[ \langle Z^2 \rangle_0 = \frac{1}{2 a_{33}}, \quad \langle P_z^2 \rangle_0 = \frac{1}{2} a_{33}. \] (85)
and the crossed products
\[\langle XP_x \rangle_0 = \frac{i}{2}, \quad \langle XP_y \rangle_0 = \frac{a_{12}}{a_{11}}, \quad \langle XP_z \rangle_0 = 0,\]
\[\langle YP_x \rangle_0 = \frac{i}{2}, \quad \langle YP_y \rangle_0 = 0, \quad \langle YP_z \rangle_0 = 1,\]
\[\langle ZP_x \rangle_0 = 0, \quad \langle ZP_y \rangle_0 = 0, \quad \langle ZP_z \rangle_0 = \frac{i}{2}.\]

Therefore, using equations (58) and (60) we obtain the Heisenberg uncertainty relationships
\[(\Delta X)^2_0 (\Delta P_x)^2_0 = (\Delta Y)^2_0 (\Delta P_y)^2_0 = \frac{1}{4} \left(1 + \frac{|a_{13}|^2}{a_{11}a_{22}}\right) \geq \frac{1}{4},\]
\[(\Delta Z)^2_0 (\Delta P_z)^2_0 = \frac{1}{4},\]

(86)

while equation (69) with \(n = 3\) gives the mean square deviation for the Hamiltonian.

Once we have calculated the mean values of the quadratic products given in equations (85) and (86), it is straightforward to evaluate the covariance matrix elements of equation (64). With the ordering \(\eta = (X, Y, P_x, P_y, Z, P_z)^T\), the following is obtained:

\[
\sigma = \begin{pmatrix}
\frac{ia_{12}}{2a_{11}} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{ia_{12}}{2a_{22}} & 0 & 0 & 0 & 0 \\
0 & 0 & (\Delta P_y)^2_0 & 0 & 0 & 0 \\
\frac{ia_{12}}{2a_{11}} & 0 & 0 & (\Delta P_z)^2_0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

(88)

Note that this covariance matrix is non-diagonal. However, since \(\sigma_{13} = \sigma_{24} = \sigma_{56} = 0\), it turns out that the generalized uncertainty relations of equation (65) reduce to the Heisenberg uncertainty relations given in equation (87).

A plot of \((\Delta X)_x(\Delta P_x)_x\) as a function of the parameters \(\epsilon\) and \(\delta\) is given in figure 1. As can be seen from equations (80), (87) and from figure 1, the CS minimize the Heisenberg uncertainty relationship for \(\epsilon = 0\), which coincides with the results recently obtained for the ideal Penning trap [16]. However, for \(\epsilon \neq 0\) it turns out that \((\Delta X)_x(\Delta P_x)_x > 1/2\). Note that the same plot will appear for the uncertainty product \((\Delta Y)_x(\Delta P_y)_x\).

### 6. Concluding remarks

In this work we have proposed a systematic technique to find the CS for systems governed by quadratic Hamiltonians in the trap regime. To do this, we introduced a prescription to identify in a simple way the appropriate ladder operators which play the same role as the annihilation and creation operators for the one-dimensional harmonic oscillator. These operators allowed us to generate the eigenvectors and eigenvalues for the Hamiltonian departing from the extremal state, the analog of the ground state, although it is not necessarily an eigenstate associated with the lowest possible eigenvalue. The explicit expression for the extremal state wavefunction was also explicitly calculated.

For systems governed by this kind of Hamiltonian, the two algebraic CS definitions (either as simultaneous eigenstates of the annihilation operators or as resulting from the action of
the displacement operator onto the extremal state) lead to the same set of states. The explicit expression for the corresponding wavefunctions has also been derived.

We have calculated explicitly the mean values of the position and momentum operators in an arbitrary coherent state. Moreover, we have also provided a prescription to obtain algebraically, by solving a linear systems of equations, the mean values of the quadratic products of these operators in the CS.

Through this method we have found the asymmetric Penning trap CS and we have explored some of their physical properties. In particular, it is worth pointing out that, in general, they do not minimize the Heisenberg uncertainty relationship. The differences from the minimum are induced by the deviations of the axial symmetry which the ideal Penning trap has (measured by the asymmetry parameter $\varepsilon$).

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