Mathematical Models to Determine Stable Behavior of Complex Systems

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Abstract. The paper analyzes a possibility to predict functioning of a complex dynamic system with a significant amount of circulating information and a large number of random factors impacting its functioning. Functioning of the complex dynamic system is described as a chaotic state, self-organized criticality and bifurcation. This problem may be resolved by modeling such systems as dynamic ones, without applying stochastic models and taking into account strange attractors. Key words: complex dynamic system, chaos, self-organized criticality, bifurcation, stochastic models, strange attractor, stochastic integral, Ito integral, Stratonovich integral, fractal dimension, Lyapunov theory, Fokker-Planck-Kolmogorov equation.

1. Introduction.
The problem of the paper is defined as follows: analysis and synthesis of complex systems (CSs) with considerations for mathematical modeling [1] and application of non-linear systems is a complex problem where no universal methods of synthesis and modeling exist, thus, impeding solution of a diverse range of applied problems, including those with possibility to control CS functioning.

To make it possible to control functioning of a CS, let us assume that it transforms input (control) signals \( U(t) \) into output signals \( X(t) \), describing the CS’s state at moment \( t \) with considerations of possible disturbances \( \xi(t) \).

Real-life CSs, including social ones, may have multiple inputs, outputs and disturbing influences. Constructing a formalized model of social CSs is difficult due to them having certain peculiarities [2]:

- stochasticity;
- non-linearity;
- time-dependence;
- deterministic nature at a small-scale time interval;
- non-stationary behavior;
- impossibility to give adequate description of a system being studied;
- indeterminateness.

Control of social CSs for partial problems may be described with non-linear systems. Thus, solution of partial problems linked to CS control with non-linear systems is a timely task, because many models of real-life CSs may be reduced to a certain partial problem.
2. Theoretical Part

CS functioning with non-linear systems with random inputs may be described with differential equations in the following form [3]:

\[
\frac{dX(t)}{dt} = f(t, X(t)) + \sigma(t, X(t))G(t),
\]

where \(X\) is a state vector;
\(f(t, x), \sigma(t, x)\) are a vector and a matrix function;
\(G(t)\) is a standard Gauss's white noise.

For differential equation (1), the initial state is described as a random vector, whose values have a known probability density. The Ito integral is often employed to find the stochastic integral; it is calculated as a limit of convergent integral sums in a mean-square sense:

\[
\sum_{i=1}^{N-1} \sigma(r_i, X(r_i))W(r_{i+1}) - W(r_i).
\]

where \(W(t)\) is a standard Wiener random process under an assumption that \(W(t_i) = 0, M[W(t_i)] = 0\) for all \(t > t_0\), while vector \(W(t)\) has a normal distribution, assuming that the process is uniform and has independent increments. As an alternative to the Ito integral, a Stratonovich integral may be used, as well as a generalized \(\theta\)-integral. The problem of analysis, synthesis and modeling of a CS control system is often reduced to a deterministic problem with preset limits [1,3]. Thus, the problem of CS output parameter analysis is based on determination of a given probability density \(p_0(x)\), vector \(f(t, x)\) and matrix \(a_i(t, x) = \sum_{j=1}^{n} \sigma_j(t, x) \cdot \sigma_j(t, x)\) and in finding the law describing changes in probability density \(p(t, x)\). Using the Fokker-Planck-Kolmogorov equation:

\[
\frac{\partial p(t, x)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[f_i(t, x)p(t, x)\right] + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j} \left[a_{ij}(t, x)p(t, x)\right].
\]

Equation (3) is a parabolic partial-derivative equation; application of different numerical or analytical methods, one may find the full information on the behavior of a dynamic CS.

CS control with modeling on the basis of non-linear systems [2,7] CS inputs is sets of functional uncertainties, determined by a measured deterministic disturbance and described with an equation in the form of:

\[
x = f(x) + G(x)(u + w(x, t) + \omega),
\]

where \(x \in \mathbb{R}^n\) is a state vector;
\(u\) is a control signal;
\(f(x)\) and \(G(x)\) are smooth vector functions and and matrix functions;
\(w(x, t)\) is an unknown vector function describing uncertain characteristics of CS's functioning.

For CS in the form of (4), it is possible to use a control model, where for a certain class of functional uncertainties and for some limited external disturbances, an asymptotic stability may be reached for some variables of the CS as described with non-linear models, which may be put into a normal canonical form.

The problem of state estimate of a hierarchic social system takes an important place in the analysis of such systems [4]. For example, in [5], such estimate may be implemented by applying the least square method with extension of state space. However, the least square estimation is performed on the basis of a large number of calculations, thus, lowering the accuracy.
For any CS, determining its parameters is a complex task due to a large volume of information circulating inside the CS, as well as a multitude of random factors influencing the functioning of the system.

Such behavior of a dynamic CS corresponds to a state of chaos, bifurcation, self-organized criticality [5].

CS functioning in the state of chaos may be described if initial conditions are known, that is, the state of the CS largely depends on input parameters, which may be measured with a certain level of error. It is clear that prediction of CS functioning is not a simple task due to the fact that description of functioning of non-linear dynamic systems is challenging due to non-linearity and local instability of their dynamics; also, a small initial error is leading to a change in CS parameters which is exponential with respect to time, complicating long-term prognosis.

This issue may be partially alleviated for solution of simple autonomic dynamic systems without use of stochastic models [6] by applying strange attractors.

Regularity of chaotic behavior of a CS correlates (at different time scales) with fractal behavior (at different spacial scales). In both CS and fractals, characteristic uncertainty distributions behave following a power law and not the exponential one. It should be highlighted that CSs functioning as fractals [7] are changing at the edge of chaos and their functioning may be described with deterministic chaos, and in this case analysis on the basis of long-term forecasting is possible.

In absolutely chaotic CSs, forecasting is possible only over a time interval depending on Kolmogorov’s K-entropy:

\[ K = \lim_{\tau \to 0} \lim_{N \to \infty} \frac{1}{\sum_{i=0}^{N-1} P_{i0} \cdots iN} \ln P_{i0} \cdots iN. \]  

Study of dynamic CS behavior by studying fractal dimensions (behavior of dynamic systems with strange attractors) is conducted by determining systemic indicators of such CS [7], which is characterized with its random chaotic behavior even for a certain deterministic model.

Analysis of behavior of dynamic CSs with fractal dimensions is based on the analysis of two principal states: movement to an attractor and movement along the attractor.

Analysis of such CSs has shown that they will never reach a state of equilibrium while a transitional state from one metastable state to another is possible in case of a small-scale disturbance of the CS. A full description of CS dynamics with an attractor is possible only when the number of CS variables is the same as the dimensionality of the attractor; thus, fractal dimension of the attractor which is a characteristic of instability is also a characteristic of CS metastable state.

Fractal dimension of CS or dimension of its conformity is determined with a formula:

\[ d = \frac{\log_{r} N}{\log_{r} \frac{1}{1}}, \]  

where \( N \) is a number of equal sub-objects;

\( r \) is a coefficient of similarity.

In applied problems, instead of (6) the following equation is used:

\[ d = -\lim_{\varepsilon \to 0} \frac{\log N_{\varepsilon}(\varepsilon)}{\log_{\varepsilon} \varepsilon}, \]  

where \( N_{\varepsilon}(\varepsilon) \) is the minimal number of radius spheres;

\( \varepsilon \) is a radius necessary to cover set A.

Lyapunov’s theory is practical for prediction of change in CS parameters with the help of non-linear systems and terms of attractor dynamic dimension on the basis of the following formulas:

1) Kaplan-Yorke formula [7]
\[ d_L^{(1)} = j + \sum_{k=1}^{j} \frac{\lambda_k}{\lambda_{j+1}}, \quad (8) \]

where \( j \) are determined in accordance with condition \( \lambda_1 + \lambda_2 + \ldots + \lambda_j > 0 \) and \( \lambda_1 + \lambda_2 + \ldots + \lambda_j + \lambda_{j+1} < 0 \);

2) Young's formula

\[ d_L^{(2)} = m + \sum_{k=1}^{m} \frac{\lambda_k}{\lambda_{n}}, \quad (9) \]

where \( m \) is the number of non-negative Lyapunov exponents;

3) Mori formula

\[ d_L^{(3)} = m + \sum_{k=1}^{m} \frac{\lambda_k^+}{\lambda_k^-}, \quad (10) \]

where \( m \) is the number of non-negative;
\( p \) is the number of positive;
\( q \) is the number of negative Lyapunov exponents.

Quantitative calculations of attractor dimension [8] may be conducted on the basis of time series of observations. Analysis of CS functioning in fractal geometry terms may be conducted on the basis of the Fokker-Planck-Kolmogorov equation (3) with the right part which is linear with respect to the function being determined [1] and with the help of an instability criterion:

\[ \lim_{j} S_j = \begin{cases} 
2 < d_H < 3 \rightarrow f 3 - d_H, \\
1.5 < d_H < 2 \rightarrow f (d - 1) H, \\
d_H = 1.5 \rightarrow |\ln f|, \\
f \rightarrow 0, \ d_H = -d, \\
1 < d_H < 1.5 \rightarrow f 5 - 3 d_H, \\
0 < d_H < 1 \rightarrow f 1 - d_H. 
\end{cases} \quad (11) \]

Analysis of formula (11) allows one to conclude that a fractal with dimension in intervals \( 2 < d_H < 3 \); \( 1.5 < d_H < 2 \); \( 1 < d_H < 1.5 \); \( 0 < d_H < 1 \) is a flicker noise, and thus in CSs with such dimension, a phenomenon of self-organized criticality appears.

Studies of dynamic CSs are then conducted in such case when only the type of time series of observations may be determined [9], while a formal description of functioning of such systems is
problematic. In this case, attractor dimension may be determined from time series for different parameters describing the CS functioning using Lyapunov exponents [10] or the Kolmogorov’s K-entropy.

Dimension of attractors may be found with Kolmogorov’s K-entropy from the following hypotheses:
1. Let us assume that the dynamic system may be described as a time series.
2. There is a global attractor in the dynamic system.
3. The system transits from one state to another on the attractor during measurement.

From the above-mentioned, the following conclusion may be made. Applying methods for determination of fractal dimension, it is possible to predict CS behavior using non-linear systems, determining an instability criterion from the fractal dimension of the attractor, i.e., in this case the CS transits into a deterministic chaos state, which allows determining CS’s behavior in a sustained stable state.

For a strange attractor, the set of its trajectories is not very large, so Takens theorem is inapplicable, but fractals with fractal Hausdorff dimension may be enclosed in space $R^d$, where $d$ is significantly large.

Let us assume that during the experiments the variables describing movement of the dynamic system was determined. In this case, correlation dimension will be determined in the following way.

Let us select a component of vector $x_j(t)$ out of $x(t) = (x_1(t), x_2(t), ..., x_n(t))$, which is a solution vector for a certain system of non-linear equations and form vector $\xi(t)$ of a form, allowing one to enclose the stream generated by the system in $R^{2m+1}$, where metrical properties of space $\{x(t)\}$ and $\{\xi(t)\}$ are the same:

$$\xi(t) = (x_i(t), x_j(t+\tau), ..., x_k(t+2m\tau)), \quad \tau > 0.$$  \hspace{1cm} (12)

During the experiment, the attractor dimension of the dynamic system is not known, so, the correlation dimension of the attractor shall be determined step-by-step for $m = 2, 3, ..., m = \infty$ etc., using the formula given below:

$$d_2^m = \lim_{\varepsilon \to 0} \frac{\ln C_m(\varepsilon)}{\ln \varepsilon}$$  \hspace{1cm} (13)

where

$$C_m(\varepsilon) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{i,j = 1}^{N} \Theta \left( e - |\xi_m(i) - \xi_m(j)| \right)$$  \hspace{1cm} (14)

is a generalized correlation integral.

In formula (13), $d_2^m$ and $m$ keep getting determined until $d_2^m$ significantly changes around $d_2^m$ at certain $m = m^*$ and $d_2^m$ is the correlation dimension of the attractor:

$$d_2 = d_2^{m^*}.$$  \hspace{1cm} (15)

It is clear that the right part of the system of non-linear equations was not involved in the calculation. Thus, sequences $\{x_j(i\tau)\}, \quad i = \frac{1}{\tau}$ may be described as a time series of observations and the right part of the system of non-linear equations may not be linked to any system of equations. Using such approach, the enclosing space may be characterized, making it possible to determine the correlation dimension of the attractor.
The prediction interval for parameters on the attractor may be determined with the help of time series of observations. The measure of predictability of parameters on the attractor is the sum of positive Lyapunov exponents, which determine the quantitative measure of the system’s divergence rate [10].

3. Practical Part

The process of education is described with a mathematical model of control, which is a social CS. The process of formation of prognostic parameters of education allows assessment of duration of the education. It is held that portions of education and time interval are equal.

For each student during the education process their individual characteristics were determined, i.e., fractal dimensions in the form of assessment of the number of variables in the education quality function, providing a criterion for the education quality with possibility to determine an amount of repeated study for a required portion of educational information in comparison with predicted values of the parameters.

The assessment will be performed for different values of $\gamma_1$ and $\gamma_2$ (individual features of students).

$n_{\text{мк}}$ is the mean duration of education, obtained experimentally;

$n$ is duration of education, obtained from formula (16):

$$n = N \left[ \ln \left( \frac{\ln \left( \frac{m - \sigma}{m} \right)}{\alpha \Delta t N} \right) \right] + 1$$

(16)

$n_{\text{мк}}$ is a factor showing by what-fold the duration of education is worse than the mean duration of education of the studied process.

The dependence level was determined from the correlation coefficient between level $Q^*$ of theoretical knowledge and the level of practical knowledge.

Taking into account that $c \in \{0, 1, 2, 3\}$ has 4 values, while ranking correlation coefficient takes values on interval [0, 1], let us divide the interval into four parts and assign rank $q$ to each of them:

$$q = \begin{cases} 1, & \text{if } 0.8 < Q \leq 1 \\ 2, & \text{if } 0.5 < Q \leq 0.8 \\ 3, & \text{if } 0.2 < Q \leq 0.5 \\ 4, & \text{if } 0 < Q \leq 0.2 \end{cases}$$

(17)

In accordance with formula (17), tables (1), (2) are obtained:
Table 1. VSU: Number of point with rank \([q, c]\).

| \(q\) | 0  | 1  | 2  | 3  |
|-------|----|----|----|----|
| 1     | 0  | 1  | 0  | 0  |
| 2     | 1  | 4  | 1  | 0  |
| 3     | 1  | 2  | 5  | 1  |
| 4     | 0  | 0  | 3  | 2  |

Table 2. LSTU: Number of point with rank \([q, c]\).

| \(q\) | 0  | 1  | 2  | 3  |
|-------|----|----|----|----|
| 1     | 1  | 0  | 1  | 0  |
| 2     | 2  | 3  | 0  | 1  |
| 3     | 2  | 2  | 2  | 1  |
| 4     | 0  | 4  | 5  | 0  |

From the data in Tables 1 and 2, a link between grasp of theoretical material and the level of practical knowledge is determined with the help of Spearman's rank correlation and then these two variables are plotted against each other.

Taking into account that there are related ranks, it is possible to average them with the Spearman coefficient, using the formula:

\[
\rho_s = 1 - \frac{6\sum d_i^2}{n^3 - n},
\]

where \(n\) is sample size.

Ordered assessments for each group for all instructors and lecturers, total points, GPA, dispersion and standard deviation are given in Tables 3 and 4.

Table 3. Statistical assessment of a review work. (Voronezh State University).

| Group 1 (exposure) | Group 2 (control) |
|--------------------|--------------------|
| \(X_{1j}^{(1)}\)   | \(X_{2j}^{(1)}\)   | \(X_{1j}^{(2)}\)   | \(X_{2j}^{(2)}\)   |
| 4      | 3.5   | 2    | 1.5 |
| 2.5    | 4     | 3.5  | 1.5 |
| 3.5    | 2.5   | 4.5  | 3   |
| 3      | 4     | 4    | 4.5 |
| 4      | 3     | 3    | 2   |
| 2.5    | 2.5   | 1.5  | 2   |
| 3      | 3     | 3    | 3.5 |
| 3      | 3.5   | 3.5  | 3.5 |
| 4      | 3     | 2.5  | 4   |
| 2      | 4     | 3.5  | 2   |
| GPA    | 3.4   | 3.4  | 3.2 | 3.2 |
| \(\bar{X}\) | 3.15  | 3.2  | 3.1 | 2.75 |
| \(s_x^2\) | 0.86  | 0.92 | 0.788 | 0.71 |
| \(s_x\)  | 0.93  | 0.959 | 0.89 | 0.84 |
Table 4. Statistical assessment of a review work. (Lipetsk State Technical University).

| Group 1 (exposure) | Group 2 (control) |
|-------------------|-------------------|
| $X_i^{(1)}$       | $X_j^{(1)}$       | $X_i^{(2)}$       | $X_j^{(2)}$       |
| 4.5               | 2.5               | 3.5               | 3.5               |
| 3                 | 1.5               | 2.5               | 3                 |
| 2                 | 2                 | 3                 | 2.5               |
| 3                 | 3                 | 3                 | 3.5               |
| 3                 | 2.5               | 1.5               | 1.5               |
| 2.5               | 3                 | 2.5               | 2                 |
| 3                 | 3.5               | 2.5               | 3                 |
| 2.5               | 4                 | 2.5               | 3.5               |
| 3.5               | 3.5               | 2                 | 4.5               |
| 3.5               | 4                 | 4                 | 4                 |
| GPA               | 3.2               | 3.2               | 3.15              | 3.15              |
| $\bar{X}$         | 3.05              | 2.95              | 2.7               | 3.1               |
| $s_{X}^2$         | 0.89              | 0.87              | 0.7               | 0.92              |
| $s_{X}$           | 0.94              | 0.93              | 0.837             | 0.959             |

The table uses the following notation:

$$X_i^{(k)} = \frac{1}{n^{(i)}} \sum_{j=1}^{n^{(i)}} X_{ij}^{(k)}$$

is the GPA in the $i$-th group for the $k$-th instructor; $n^{(i)}$ is the number of students in the $i$-th group; $X_{ij}^{(k)}$ is the score of the $j$-th student from the $i$-th group given by the $k$-th instructor; $i=1,2; n^{(1)}=n^{(2)}=10; j=1,2; \ s_{X_i^{(k)}}^2$ is an unbiased estimate of dispersion for the $i$-th group and the $k$-th instructor.

Uniformity of the empirical data obtained was checked with standard methods [11].

From the experiment conducted, it was proven that the two samples from the same group are samples from the same population, i.e., in each group corresponding samples may be united into a sample of a larger size and two collectives (exposure and control) may then be compared. Testing of the statistical hypothesis has shown that both collectives are samples of different populations. It is worth highlighting, that the average values of the samples differ from each other in exposure group $\bar{X}_1 = 3.4$ and $\bar{X}_1 = 3.05$ in the control $\bar{X}_2 = 3.1$ and $\bar{X}_2 = 2.9$, that is, the average value in the exposure group is higher than in the control one, meaning that the process of education is more efficient in the exposure group than in the control group [1,7,12].

Let us compare the result of testing on a three-point scale (0, 1 and 0.5 points ) in the two groups. In total, $S=50$ marks were given. On the assumption that the samples may be consolidated, let us determine the average grade for different instructors in both groups and find relative $W_i$. The results are given in Tables 5 and 6.

Assessment of changes in the quality of knowledge [11,12] will be conducted with the help of a coefficient, giving a quantitative characteristic to different educational practices.

$$K = \frac{K_0 + K_1}{2}$$ (19)
where \( K_0 = \frac{W_0(2)}{W_0(1)} \) is an indicator characterizing reduction of unfamiliarity; \( K_1 = \frac{W_1(1)}{W_1(2)} \) is an indicator characterizing an increase in knowledge.

Table 5. Number of 1, 0.5 and 0 scores in exposure and control groups.
(Voronezh State University).

| Group | Instructor | Score | Score | Score |
|-------|------------|-------|-------|-------|
| 1     | 1          | 12    | 13    | 25    |
|       | 2          | 9     | 18    | 23    |
| 2     | 1          | 10    | 16    | 24    |
|       | 2          | 12    | 21    | 17    |

GPA | 10.5 | 15.5 | 24 |
Relative score | 0.21 | 0.31 | 0.48 |

GPA | 11 | 17.5 | 20.5 |
Relative score | 0.22 | 0.35 | 0.41 |

Table 6. Number of 1, 0.5 and 0 scores in exposure and control groups.
(Lipetsk State Technical University).

| Group | Instructor | Score | Score | Score |
|-------|------------|-------|-------|-------|
| 1     | 1          | 9     | 19    | 22    |
|       | 2          | 9     | 22    | 19    |
| 2     | 1          | 11    | 24    | 15    |
|       | 2          | 11    | 16    | 23    |

GPA | 9 | 20.5 | 20.5 |
Relative score | 0.18 | 0.41 | 0.41 |

GPA | 11 | 20 | 19 |
Relative score | 0.22 | 0.4 | 0.38 |
The calculations gave the following results for the VSU:

\[ K_0 = \frac{0.22}{0.21} = 1.05; \quad K_1 = \frac{0.48}{0.41} = 1.17. \]  
(20)

The level of long-term knowledge has the value of \( K = 1.11 \) :

\[ \text{LSTU: } K_0 = \frac{0.22}{0.18} = 1.22; \quad K_1 = \frac{0.41}{0.38} = 1.08. \]  
(21)

The level of long-term knowledge has the value of \( K = 1.15 \).

Thus, use of the thesis research method led to an increase of the quality of education by the factor of 1.11 in the VSU and by the factor of 1.15 in the LSTU. The results of the experiment have shown that the number of Excellent marks has grown, while the number of Failed marks has fallen.

4. Conclusion

Dynamics of changes in a CS with initial uncertainty were studied. With the application of deterministic chaos, a stochastic behavior of dynamic system trajectories in the phase space was determined together with a possibility for a deterministic chaos to arise due to unstable dynamics. A method was developed to determine whether certain state parameters of a CS fall into the required region, using time series of observations without need to use non-linear systems.

Dynamics of changes in educational behavior with initial uncertainty was studied experimentally. A method was developed to determine whether the state parameters of a single given student fall into a required region using time series of observations and without need to use any non-linear systems; a portion of educational information necessary for a given student was determined.

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