Inferring the Intermediate-mass Black Hole Number Density from Gravitational-wave Lensing Statistics

Joseph Gais1, Ken K. Y. Ng2,3, Eungwang Seo1, Kaze W. K. Wong4, and Tjonne G. F. Li1,5,6

1Department of Physics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong; 1155138494@link.cuhk.edu.hk
2LIGO, Massachusetts Institute of Technology, Cambridge, MA 02139, USA; kenkyng@mit.edu
3Kavli Institute for Astrophysics and Space Research, Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
4Center for Computational Astrophysics, Flatiron Institute, New York, NY 10010, USA
5Institute for Theoretical Physics, KU Leuven, Celestijnenlaan 200D, B-3001 Leuven, Belgium
6Department of Electrical Engineering (ESAT), KU Leuven, Kasteelpark Arenberg 10, B-3001 Leuven, Belgium

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Abstract

The population properties of intermediate-mass black holes remain largely unknown, and understanding their distribution could provide a missing link in the formation of supermassive black holes and galaxies. Gravitational-wave observations can help fill in the gap from stellar mass black holes to supermassive black holes with masses between $\sim 100-10^4 M_\odot$. In our work, we propose a new method for examining lens populations through lensing statistics of gravitational waves, here focusing on inferring the number density of intermediate-mass black holes through hierarchical Bayesian inference. Simulating $\sim 200$ lensed gravitational-wave signals, we find that existing gravitational-wave observatories at their design sensitivity could either constrain the number density of $10^6$ Mpc$^{-3}$ within a factor of 10, or place an upper bound of $\lesssim 10^3$ Mpc$^{-3}$ if the true number density is $10^5$ Mpc$^{-3}$. More broadly, our method leaves room for incorporation of additional lens populations, providing a general framework for probing the population properties of lenses in the universe.

Unified Astronomy Thesaurus concepts: Intermediate-mass black holes (816); Gravitational waves (678); Gravitational lensing (670); Hierarchical models (1925)

1. Introduction

To date, we have detected dozens of black holes within the stellar mass range $O(1-100) M_\odot$ from binary black hole merger gravitational-wave emission (Abbott et al. 2019, 2021a; LIGO Scientific Collaboration et al. 2021) and X-ray binary observations (McClintock & Remillard 2003; Remillard & McClintock 2006), as well as supermassive black holes of mass $> O(10^5) M_\odot$, first identified from stellar orbits about the center of the Milky Way (Ghez et al. 2005) and now imaged by the Event Horizon Telescope (Akiyama et al. 2019a, 2019b, 2019c, 2019d, 2019e; E.H.T. Collaboration et al. 2019).

The least understood parameter space of black holes lies between these two ranges, the so-called intermediate-mass black holes (IMBHs) in the mass range $[10^2, 10^4] M_\odot$. Understanding the formation channels of supermassive black holes and galaxies themselves will require filling in the missing link of IMBHs.

IMBHs may soon be detected. Search methods include stellar and gas dynamical searches as well as accreting IMBHs within galactic nuclei suggest a number of tentative IMBH discoveries (see Greene et al. 2019 for a recent review). Recently, the first half of the LIGO–Virgo third observing run has detected the gravitational waves of a binary black hole merger with a remnant mass of $142 M_\odot$ (Abbott et al. 2020), the first-ever confirmed IMBH. However, ground-based observatories are unlikely to directly detect IMBHs of mass $\gtrsim 10^3 M_\odot$ through the emission of their coalescence as the gravitational-wave frequency of such binaries lies below the sensitive frequency band of the interferometers. In this Letter, we propose a method for observing the IMBH population with the effect of gravitational-wave lensing, which could detect IMBHs even at mass $\gtrsim 10^2 M_\odot$.

If a gravitational wave passes by an IMBH mass lens closely, the measured gravitational wave will have a frequency-dependent amplification factor altering the waveform (Takahashi & Nakamura 2003). From careful study of detected gravitational waves, we may determine the lens parameters, with recent work demonstrating the detection of mass of an IMBH lens (Lai et al. 2018) and how gravitational-wave lensing can constrain black hole populations (Diego 2020). Although no gravitational-wave event has yet been conclusively identified as being lensed (Hamuksela et al. 2019; McIsaac et al. 2020; Abbott et al. 2021b; Liu et al. 2021), tentative lensing rates estimates suggest aLIGO could detect $O(1)$ yr$^{-1}$ lensed events at design sensitivity (Li et al. 2018; Ng et al. 2018; Oguri 2018). The rate of microlensing is still unknown, however, and a robust estimate of this rate could be meaningful for the constraints on the relevant lenses.

Building off of Lai et al. (2018), we consider the lensing of gravitational waves by IMBHs as a means of inferring the IMBH comoving number density $n_{L,0}$ (which is assumed to be non-evolving in this study). We develop an analytical model verified by simulation results for the distribution of the single-lensing event parameters, the normalized impact parameter $\gamma$, and the redshifted lens mass $M_{L,\gamma}$. We then use a hierarchical Bayesian model for constraining possible $n_{L,0}$ values from a population of recovered $\gamma$ alongside our simulated distributions of impact parameter for different lens number densities. Since a priori we have no means of identifying a lensed gravitational wave, we conduct the parameter estimation on all gravitational-wave events, where the posterior of unlensed gravitational
waves should demonstrate significant support at large $y$ and little support at $y \lesssim O(1)$. In contrast, lensed gravitational waves with $y \lesssim 1$ should be recovered from the parameter estimation. For any gravitational-wave event, we conduct parameter estimation of the redshifted lens mass, $M_L$, and $y$. The set of lens parameter estimation allows us to build a distribution for the full population of $y$ values. In turn, we are able either to constrain the number density of IMBHs if no IMBH mass range lenses are present within the full population, or measure on the IMBH number density if IMBH lens events are detected.

Simulating a catalog of $\approx 200$ events drawn from $n_{L,0} = \{10^3, 10^4\} \text{Mpc}^{-3}$ with a design sensitivity LIGO Hanford, LIGO Livingston (Aasi et al. 2015), and Virgo (Acernese et al. 2014) observatory network, we can confidently detect the density of IMBH lenses at $10^6 \text{Mpc}^{-3}$ or constrain to $\lesssim 10^4 \text{Mpc}^{-3}$ for a number density of $10^7 \text{Mpc}^{-3}$. IMBH densities inferred from gamma-ray burst observations (Paynter et al. 2021) suggest a density of $O(10^3) \text{Mpc}^{-3}$ for masses of $M_L \sim 10^{-5}$–$10^2 \text{M}_{\odot}$. Thus, our method could detect the IMBH number density in the near future with existing gravitational-wave observatories simply operating at design sensitivity.

We begin by describing the effect of a point-mass lens on a gravitational wave in Section 2. Then, in Section 3, we derive a hierarchical Bayesian model to infer the point-mass lens population from detected gravitational-wave events. In Section 4.1, we detail an analytical population model for IMBH lenses, validating our model against simulated results. We then conduct an injection campaign in the LIGO–Virgo detector network as described in Section 5. Finally, in Section 6, we present the recovered lens number density from our injections, and discuss our results and impact of improved detector networks on probing the IMBH population in Section 7.

2. Gravitational-wave Lensing

When a gravitational wave passes by a massive object, it is lensed in a manner similarly to electromagnetic waves. In the geometric optics regime, i.e., when the dimensionless frequency $w = 8\pi M_L f \gg 1$, where $M_L$ is the redshifted lens mass with gravitational frequency $f$ in the detector’s frame, the amplitude of the gravitational wave is either magnified or demagnified. However, in the wave-optics regime where $w \lesssim 1$, both the amplitude and phase of the gravitational wave are modulated in a frequency-dependent manner, yielding a rich structure in the lensed gravitational wave. Lensed gravitational waves could soon be detected (Ng et al. 2018; Oguri 2018; Hannuksela et al. 2019), with applications ranging from improved sky localization (Hannuksela et al. 2020), tests of the polarization of gravitational waves (Goyal et al. 2021), mapping galactic distributions (Xu et al. 2021), or probing dark matter (Urrutia & Vaskonen 2021).

Here, we focus on the case of a gravitational wave lensed by a single point mass. The details of the analytical calculation for the lensing amplification factor are outlined in Appendix A, resulting in an analytical solution for the isolated point mass (Nakamura 1998; Takahashi & Nakamura 2003),

$$F(w) = \exp \left\{ \frac{\pi w}{4} + i \frac{w}{2} \left[ \ln \left( \frac{w}{2} \right) - 2 \phi_{\text{gw}}(y) \right] \right\} \times \Gamma\left(1 - \frac{i}{2} w\right) F_{\frac{i}{2}}(\frac{i}{2} w, 1; \frac{i}{2} w^2),$$

where $w = 8\pi M_L f$ is the dimensionless frequency, $y$ is the impact parameter normalized by the lens’ Einstein radius, $M_L$ is the redshifted lens mass, $i F_1$ is the confluent hypergeometric function, and $\phi_{\text{gw}}(y) = (x_m - y)^2/2 - \ln x_m$ where $x_m = (y + \sqrt{y^2 + 4})/2$. The lensed waveform is then $\psi(f) = F(f) \psi_0(f)$, where $\psi_0(f)$ is the frequency-domain base waveform and $F(f)$ is the amplification factor.

Previous studies demonstrate that the gravitational-wave event parameters and lens parameters, $M_L$ and $y$, the impact parameter of the source–lens pair normalized by the lens’ Einstein radius, are detectable from Bayesian parameter estimation of the lensed gravitational wave for IMBHs (Lai et al. 2018). Following this example, we prepare a likelihood model for a lensed gravitational wave, from which one can infer the posterior on the lensing parameters. When $w y^2/2 \gg 1$, the amplification factor is highly oscillatory in the frequency domain, and the geometric optics approximation can be used. Using a dynamical lookup table in $(w, \ w y^2/2)$ for the evaluation of the hypergeometric function in $F(f)$, we are able to rapidly evaluate the amplification factor such that lensing parameter estimation is feasible, and use the geometric optics approximation for $w y^2 > 1000$ elsewhere.

3. Hierarchical Bayesian Analysis

In this section, we list the mathematical details of the hierarchical inference model for a generic lensing scenario. We seek to measure the number density of the lens population parameterized by $A_L$. Given a data set $d = \{d_i\}$ of $N$ detections and the properties of source population parameterized by $A_S$, we can compute the posterior of $A_L$, $p_A(A_L|d, A_S)$, by combining the measurement of waveform parameters $x$ of each detection,

$$\frac{p_A(A_L|d, A_S)}{p_A(A_S, A_L)} \propto \prod_{i=1}^{N} \int L_{\text{gw}}(d_i|x^i) \times p_A(x^i|A_S, A_L) \, dx^i,$$

where $L_{\text{gw}}(d_i|x^i)$ is the likelihood of the $i$th gravitational-wave detection, $p_A(x^i|A_S, A_L)$ is the distribution of waveform parameters given both the source and lens population properties, and $p_A(A_S, A_L)$ is the prior of $(A_S, A_L)$. While one can simultaneously infer $(A_S, A_L)$, we expect that the population properties of sources and lenses are weakly correlated and leave out $A_S$ for the rest of the Letter for simplicity. We list our choice of source population properties, such as the source mass spectrum and redshift evolution, in Appendix C. In the following, we separate the waveform parameters into $(y, \ x_S, \ x_L)$, in which $y$ can be thought of as the parameter characterizing the pairing of a source and a lens, $x_S = (z_S, \bar{x}_S)$ is the set of source parameters including source redshift $z_S$ and other parameters irrelevant to lensing, $\bar{x}_S$, and $x_L = (z_L, \bar{x}_L)$ is the set of lensing-relevant parameters including the lens redshift $z_L$ and the model-dependent parameters characterizing the internal properties of the lens, $\bar{x}_L$, which we simulate directly given hyperparameters $\Lambda$. The set of event parameters and population hyperparameters for both source and lens are detailed in Table 1.

We expect that $\bar{x}_L$ and $x_L$ are independent of each other and hence their distributions are separable. We treat the constraint that a lens must be inside the volume within $z_S, z_L < z_S$ as a
condition imposed on the lens distribution in Bayes’ theorem. One can further marginalize over other irrelevant source parameters \( \hat{x}_S \). Putting these steps together, Equation (2) becomes

\[
\frac{p_{\lambda}(\lambda_L|d)}{\pi_{\lambda}(\lambda_L)} \propto \prod_{i=1}^{N} \int \int \int \int L_{gw}(d^i|\lambda_L) \pi_{\lambda}(y^i, x_L^i|z_S^i, \lambda_L, \mathcal{P}) \times \pi_{\lambda}(z_S^i, \hat{x}_S^i) \pi_{\lambda}(y^{ij}, x_L^{ij}|z_S^{ij}, x_L^{ij}, \mathcal{P}) \pi_{\lambda}(z_L^{ij}, \hat{x}_S^{ij}, x_L^{ij}, \mathcal{P}) \, dz_S^i \, dx_L^i, \tag{3}
\]

where \( \pi_{\lambda}(y, x_L|z_S, \lambda_L, \mathcal{P}) \) is the distribution of lens parameters given a source at redshift \( z_S \), and \( \pi_{\lambda} \) is the prior of the source of the lens. The conditional statement \( \mathcal{P} \) denotes the requirement of a source–lens pair having the strongest diffraction along the line of sight. We will explain the importance of this notion in Section 4.1.

To evaluate Equation (3), we can use importance sampling by recognizing that \( L_{gw}(d^i|\lambda_L)\pi(\lambda^i) = p_{gw}(\lambda^i|d^i) \), where \( \pi(\lambda) \) is the prior of waveform parameters used in the parameter-estimation algorithm that estimates the posterior of waveform parameters, \( p_{gw}(\lambda|x^i) \). We can reweight the samples drawn from the estimated posterior to evaluate the hierarchical likelihood,

\[
\frac{p_{\lambda}(\lambda_L|d)}{\pi_{\lambda}(\lambda_L)} \propto \prod_{i=1}^{N} \left\{ \frac{\pi_{\lambda}(z_S^i, \hat{x}_S^i)}{\pi_{\lambda}(y^{ij}, x_L^{ij}|z_S^{ij}, x_L^{ij}, \mathcal{P})} \right\}, \tag{4}
\]

where \( (\cdot)^i \) denote the \( i \)th sample drawn from \( \lambda^i \) posterior samples of the \( i \)th event.

Generically, in hierarchical Bayesian analysis of hyperparameters, the selection bias must be taken into account. For \( y \ll 1 \), the lensed waveform is greatly amplified (Nakamura 1998; Nakamura & Deguchi 1999; Takahashi & Nakamura 2003), resulting in higher signal-to-noise ratio \( (S/N) \) values. Selection of only those events above a certain threshold will then bias the recovered hyperparameter posterior toward higher lens number densities, as events with higher \( y \) values (and, thus, less of a lensing effect) are less likely to have a sufficiently high \( S/N \). However, for the physically motivated regime of number densities we consider, \( y \gg 1 \) in most events, resulting in magnifications very close to unity, and so the \( S/N \) of any event is hardly affected by lensing (and by extension the lens number density). Thus, the \( S/N \) selection is unlikely to bias our results and we ignore it for simplicity.

### 4. Distribution of the Nearest-effective Lenses

#### 4.1. Notion of the Nearest-effective Lens

We observe the population of the source–lens systems rather than the population of isolated lenses. One needs to cautiously account for this subtle difference when modeling \( \pi_L \) in Equation (4), which is no longer the intrinsic distribution of the lenses. We assume that a source is solely diffracted by a single lens, i.e., multiple lensing due to the next neighboring lenses is negligible. Since the size of the Einstein ring also affects the magnitude of \( y \), the nearest-neighbor lens (i.e., with the smallest value of \( \theta_0 = y/D_L \)) does not necessarily give rise to the strongest effect of diffraction. Instead, a source is the most diffracted by a lens whose parameters result in the smallest value of \( y \). We call such lenses the nearest-effective lenses. In terms of the lensing statistics, the statement \( \mathcal{P} \) is equivalent to the requirement of minimum \( y \) when pairing the lenses and sources. We can model the nearest-effective pairing by characterizing the distribution of neighboring lenses through a spatial Poisson process, which only depends on the spatial distribution among the lenses but not on the internal properties of the lenses. This is achievable by considering \( y \) as an effective distance between a source and its nearest-effective lens on the sky plane. Assuming the lenses are uniformly distributed on the sky plane, we can separate the joint distribution of \( y \) and \( x_L \) into

\[
\pi_L(y, x_L|z_S, \lambda_L, \mathcal{P}) = \pi_{\lambda_L}(z_S, \lambda_L) \pi_{\lambda}(z_S, x_L|z_S, \lambda_L, \mathcal{P}), \tag{5}
\]

where \( \pi_{\lambda} \) and \( \pi_{\lambda_L} \) are the distributions of \( y \) and \( x_L \) conditioned on the nearest-effective pairing between sources and lenses, respectively. In the following, we first derive \( \pi_{\lambda} \) and \( \pi_{\lambda_L} \) from the spatial Poisson process, then list out the mathematical details in the case of point-mass lenses, and validate the analytical model by comparing it to the direct simulation of the nearest-effective pairing of the source–lens systems.

The \( y \) distribution of effective lens distance \( \pi_{\lambda_L} \) is

\[
\pi_{\lambda_L}(z_S, \lambda_L) = 2 \pi y \Sigma(z_S, \lambda_L) e^{-\Sigma(z_S, \lambda_L)} y, \tag{6}
\]

where \( \Sigma(z_S, \lambda_L) \) is the number density of lenses in the plane, and details on the derivation are found in Appendix B. Conditioning the lens–source pair as that which minimizes the effective lens distance results in a bias factor of \( \beta_L^2 \) such that \( \pi_{\lambda_L}(z_S, \hat{x}_S|z_S, \lambda_L, \mathcal{P}) \propto \beta_L^2 \pi_{\lambda_L}(z_S, \hat{x}_S|z_S, \lambda_L) \). In Appendix D, we present the analytical form of \( \Sigma(z_S, \lambda_L) \), which we use in the rest of our calculations.

#### 4.2. Lensing Statistics for Point-mass Lenses

IMBHs with masses of \( \sim \mathcal{O}(100-10^5) M_\odot \) may serve as point-mass lenses to diffract gravitational waves. The mass profile of a point-mass lens is entirely parameterized by its mass \( M_L \), i.e., \( \hat{x}_L = M_L \). Throughout the study, we assume the intrinsic lens-mass spectrum does not evolve with lens redshift, i.e., \( \pi_{M_L} = \pi_{M_L}(z_L) \propto M_L^{\alpha_M} \), in the domain \( [M_{L,\min} = 100 M_\odot, M_{L,\max} = 20000 M_\odot] \). For simplicity, we keep the lens...
number density constant in the comoving frame such that the prior of the lens redshift is \( \pi_{Lz}(z_L, z_S) \propto dV(z_L)/dz_L \) for \( z_L < z_S \). We note that one can relax the assumption of constant density to infer the lens-redshift evolution. As such, we only have two hyperparameters, \( \Lambda_L = (\alpha_L, \sigma_L) \).

Now, we write down the expressions for \( \pi_{Lz} \), \( \pi_{Mz} \), and \( \Sigma \). Including the lensing bias factor, \( \theta_{Lz}^2 \propto ML D_L / D_L \) at a fixed \( z_S \), we have

\[
\pi_{Lz}(z_L, z_S, \Lambda_L, \mathcal{P}) \propto \pi_{Lz} \cdot \frac{D_L}{D_L} \frac{D_{Lz}}{D_S} , \tag{7}
\]

\[
\pi_{Mz}(M_L, z_S, \Lambda_L, \mathcal{P}) \propto \pi_{Mz} \cdot (M_L, z_S, \Lambda_L) . \tag{8}
\]

Since \( F(f) \) only depends on \((y, M_L)\) and \( z_L \) is not directly measured, we further marginalize \( \pi_{Mz} \) over \( z_L \) to obtain the distribution of redshifted lens mass,

\[
\pi_{Mz}(M_L, z_S, \Lambda_L, \mathcal{P}) \propto \int_0^{z_S} \left( \frac{M_L}{1 + z_L} \right)^{1 - \alpha_L} \frac{D_{Lz}}{D_L} dV \frac{dz_L}{1 + z_L} , \tag{9}
\]

for \( M_L \in [M_{Lmin}, M_{Lmax}] \), and is zero otherwise. The extra factor of \((1 + z_L)^{-1}\) comes from the transformation of the differential \( dM_{Lz} = (1 + z_L) dM_L \). Throughout our analysis, we use Planck 18 cosmology (Aghanim et al. 2020) for the evaluation of cosmological distances.

4.3. Validation

Let us examine the behavior of \( \pi_{Lz} \). First, the inverse of the density parameter \((\Sigma \pi)^{-1}\) characterizes the scale of \( y \). In particular, the most probable value of \( y \) (or the peak of \( \pi_{Lz} \)) is \( y_p = (2 \Sigma \pi)^{-1} \). This can be understood physically by interpreting \((\Sigma \pi)^{-1}\) as the ratio of the mean cross-sectional area, \( \pi_{Lz}(t, \Lambda_L) \equiv 4 \pi / N_L \), to the mean area of lenses, \( \pi_{Lz}(t, \Lambda_L) \) (see Equation (B3)). Second, in the limit of \( y \to \infty \), the Gaussian term \( e^{-\Sigma \pi y^2} \) regulates the linear increase in \( \pi_{Lz} \) with \( ye^{-\Sigma \pi y^2} \to 0 \). The impact parameter cannot be arbitrarily large because the separation between adjacent lenses is characterized by the scale of \((\Sigma \pi)^{-1/2}\). Third, we consider the limit of \( 0 < y < y_{\text{max}} \) where \( y_{\text{max}} \) is the cutoff of \( y \) satisfying \( y_{\text{max}} < (\Sigma \pi)^{-1/2} \). In such a limit, sources are distributed uniformly around the vicinity of the nearest-effective lens, resulting in a linear distribution of \( y \). Indeed, the spatial Poisson piece, \( 2 \pi y \Sigma_{Lz} e^{-\Sigma \pi y^2} \), is well approximated by \( 2y / y_{\text{max}} \) for \( y^2 \ll (\Sigma \pi)^{-1} \) and independent of \( \Sigma \). Together with the lensing bias factor \( \theta_{Lz}^2 \), the asymptotic form of \( \pi_{Lz} \) for \( y \ll (\Sigma \pi)^{-1/2} \) is

\[
\begin{align*}
\pi^0_{Lz}(y, z_L, \tilde{z}_L, z_S, \Lambda_L, \mathcal{P}, y \ll (\Sigma \pi)^{-1/2}) \approx & \frac{2y}{y_{\text{max}}} \theta_{Lz}^2 \pi_{Lz}(z_L, z_S, \Lambda_L) ,
\end{align*}
\]

which, after the marginalization over \( z_L \), recovers the usual definition of the lensing optical depth (or the lensing probability) defined in the existing literature (Turner et al. 1984) for non-evolving point-mass lens distribution,

\[
\frac{d^2n}{dy dz_L} = \int_0^{z_S} 2y \pi_{Lz} \cdot \frac{D_L}{D_L} \frac{D_{Lz}}{D_S} dV \frac{dz_L}{1 + z_L} \times \pi_{Mz} (M_L, z_S, \Lambda_L) , \tag{11}
\]
distribution $\pi_{y}$ is more skewed toward smaller redshifts, as the bias factor $D_{LS}/D_{L}$ is maximized at smaller lens redshifts. Additionally, drawing independent samples from the bias-factored distributions, plotted in orange contours, we find that they match the simulated contours, indicating that the lensing parameters $(y, z_{L}, M_{L})$ are independent following selection of nearest-effective lens–source pairs.

Figure 2 shows the evolution of $\Sigma$ with source redshift. In particular, note that the effective density increases monotonically with source redshift, as more and more lenses are in the plane of the sky. As a result, the $y$ distribution shifts toward smaller values as $z_{S}$ increases, and Figure 3 plots the decreasing peak value of $p(y|z_{S}, n_{L})$ with $z_{S}$.

5. Gravitational-wave Lens Parameter Estimation

In order to effectively use lens parameter estimation to draw conclusions on the IMBH population, injected lens parameters should be recoverable in the parameter estimation. To conduct parameter estimation, we use the Bilby library (Ashton et al. 2019) with the Dynasty sampler (Speagle 2020). Figures 4 and 5 demonstrate typical results for the impact parameter of a lensed gravitational-wave injection, with an injected $y < 1$ and $y \gg 1$, respectively. In the case of $y < 1$ in Figure 4, the injected $y$ parameter is accurately recovered in the posterior of both $y$ and $M_{Lz}$, and the likelihood is only nonzero about the injected value. Thus, injections with $y < 1$ for IMBHs are clearly detectable.

In contrast to the small $y$ case, Figure 5 illustrates the posterior for a large injected value, $y \gg 1$. With a uniform in log prior, the posterior remains relatively flat, and the posterior is not localized about the injected value, as the effects of lensing on the waveform are too small to be detected, and the $M_{Lz}$ posterior is agnostic. However, the posterior has no support for $y \lesssim 1$, ruling out the parameter space where lensing effects are significant. In this way, the diffraction effects of a microlens can either be detected or ruled out.

At small $n_{L,0}$ values the typical $y$ value is large, with the $y$ distribution peaking at $y_{p} \sim 1/\sqrt{\Sigma}$. This could present a problem if multiple diffraction effects are combined, as the lens with the smallest $y$ value for the source could be large enough that other lenses have a similar $y$ value. However, as these parameter-estimation results show, the diffraction effects are still minimal at large $y$, and so an arbitrarily large $y$ value can be injected without consideration of possible contaminating effects from other source–lens pairings in a multiple-lensing scenario.

5.1. Generating the Injection Bank

Finally, for a fixed lens number density and lens-mass power law, we create an injection set to test our ability to recover the lens number density hyperparameter. For the lens parameters, the source position $\pi(y|M_{Lz}, z_{S})$ is sampled from Equation (B5), and the source parameters are sampled from the distributions discussed in Appendix C. For the base unlensed waveform, we use the IMRPhenomD approximate (Husa et al. 2016; Khan et al. 2016), which encompasses the inspiral, merger, and ringdown. The lensed waveform is then the product of the
amplification factor and the base waveform. We threshold sampled injections by $S/N$, selecting only those injections with network $S/N_{\rho_{\text{net}}}>12$ in a three-detector network consisting of the LIGO Livingston, LIGO Hanford, and Virgo observatories at design sensitivity.

For the hyperparameters, we fix $\alpha_L=1$, and generate injection sets with IMBH densities $n_L = \{10^3, 10^6\}$ Mpc$^{-3}$. At $n_L = \{10^3, 10^6\}$ Mpc$^{-3}$ the $S/N$ gain due to strong lensing is negligible, and so we neglect the selection effect.

6. Results of Hierarchical Analysis

Figures 6 and 7 show the recovered hierarchical likelihood for the cases of $10^3$ Mpc$^{-3}$ and $10^6$ Mpc$^{-3}$, respectively. At $10^3$ Mpc$^{-3}$, the recovered likelihood can constrain the hyperparameter to $\lesssim 10^3$ Mpc$^{-3}$ at 90% confidence. This upper constraint can improve with further unlensed detections.

For a density of $10^6$ Mpc$^{-3}$, the injected hyperparameter is recoverable with this network, with the likelihood of Figure 6 ruling out both $n_L \lesssim 10^3$ Mpc$^{-3}$ and $n_L \gtrsim 10^6$ Mpc$^{-3}$ at 90% confidence. Thus, even with just a three-detector network, the population properties of IMBH lenses are not only possible to constrain but even to detect. This is because $O(1)$ events in our injection set are lensed with recoverable $y$ injection parameters in the parameter estimation, ruling out smaller lens number densities.

With a more sensitive network the volume of detectable mergers grows, and since $\tau_L(y) \geq S_L, A_L, P$, increases monotonically with source redshift, the probability of encountering a significantly lensed event increases. Thus, lensed events by IMBH lenses could be detectable even at these relatively small redshifts, and the recovered likelihood for an injected $10^3$ Mpc$^{-3}$ hyperparameter may resemble a true measurement, rather than just an upper bound.

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With a more sensitive network the volume of detectable mergers grows, and since $\tau_L(y) \geq S_L, A_L, P$, increases monotonically with source redshift, the probability of encountering a significantly lensed event increases. Thus, lensed events by IMBH lenses could be detectable even at these relatively small redshifts, and the recovered likelihood for an injected $10^3$ Mpc$^{-3}$ hyperparameter may resemble a true measurement, rather than just an upper bound.

7. Discussion

We present a novel method of probing population distributions for lenses of gravitational waves, using the statistics of gravitational-wave lensing, assuming that multiple-lensing effects are negligible. Deriving population models for the lensing statistics of point-mass lenses distributed uniformly in comoving volume with a power-law mass distribution, we verify our models with direct simulations, and demonstrate a hierarchical Bayesian model for computing the likelihood of the lens density from successive observations. We then conduct an injection campaign with gravitational-wave samples, generating catalogs of lensed injections with network $S/N_{\rho_{\text{net}}}>12$ for densities of $\{10^3, 10^6\}$ Mpc$^{-3}$. Our results, shown in Figures 6 and 7, show that we may either constrain or directly detect the lens number density for $\{10^3, 10^6\}$ Mpc$^{-3}$, respectively.

In the specific case of IMBHs, our method can probe their relatively unknown population properties with just a three-detector network of already existing gravitational-wave observatories operating at design sensitivity. Since the effective lensing probability increases with source redshift, a more sensitive detector network could greatly improve our ability to
probe the IMBH population, detecting or constraining lower values of the lens number density. With the addition of a few more planned observatories, like LIGO-India or KAGRA, the IMBH number densities of $\sim 10^3 - 10^4 \text{ Mpc}^{-3}$ could be directly detected. Additionally, third-generation detectors like the Einstein Telescope (Punturo et al. 2010) or Cosmic Explorer (Abbott et al. 2017; Reitze et al. 2019) could probe extremely high source redshifts of $z_s > 30$, detect $\sim 10,000$ binary black hole mergers per month (Regimbau et al. 2017), and be sensitive to higher injected $y$ values, so that smaller IMBH densities would be detectable. Indeed, applying the third-generation population forecast discussed in Ng et al. (2020) with isolated galactic field formation, dynamical globular cluster formation, and Population III stars at high-redshift subpopulations, we find that $\sim 1$ event with $y < 1$ could be detected each month for a density of $n_L = 10^3 \text{ Mpc}^{-3}$. With sufficient events in third-generation detector networks, we could further extend our analysis to probe possible redshift dependences of the IMBH density.

We end by noting that the common use of lensing optical depth in Equation (11) carries the notion of a signal being *lensed versus unlensed*, which is less well defined in the wave-optics scenario. The classification of the lensed signals relies on the choice of $y \leq y_{\text{max}}$ to down-select the data of the lensed-only population for further analysis. One has to build up detection statistics, e.g., the Bayes factor statistics from a large-scale injection campaign (Basak et al. 2021) or the mismatch from the waveform (Wang et al. 2021), for identifying the events that belong to the lensed population. Besides being inflexible, this approach depends on a number of artificial choices, such as the choice of prior and the threshold of detection statistics for a lensed signal. As a result, such process can be fuzzy for weak signals and may misidentify the lensed population in the data. On the other hand, our method makes full use of the parameterization of $y$ and does not require the binary notion of “lensed versus unlensed.” With the hierarchical approach, we can treat the data as a whole population to infer the lens properties robustly, given a detailed model of the source–lens systems.

The mathematical framework derived in Sections 3 and 4 also allows for a flexible extension to test other lens models, such as the singular isothermal sphere or Navarro–Frenk–White profile (Navarro et al. 1995, 1996, 1997), by considering the population as a mixture of different types of lenses. Notably, inclusion of galactic lenses could boost the detectability of $y$ as shown in Seo et al. (2021). For lenses that do not obtain circular symmetry, such as elliptical lenses, the presented formalism still holds, with two modifications: (1) including the dependence of the symmetry-breaking parameter (e.g., ellipticity or external shear) in $\mathbf{x}_L$ to calculate $F(f)$, and (2) redefining the normalization of $y$ that respects the notion of the nearest-effective lens, i.e., the effect of diffraction is stronger when $y$ is smaller, to evaluate $\Sigma$ and $\tau_L(y, x_L, z_L, A_L, \mathcal{P})$. We will leave these extensions in the future work.

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**Appendix A**

**Amplification Function in Wave Optics**

The background metric of a gravitational wave passing by a lens is given by

$$ds^2 = -(1 + 2U)dt^2 + (1 + 2U)dr^2 \equiv g^{(B)}_{\mu\nu} dx^\mu dx^\nu,$$

(A1)

with lens potential $U(r) \ll 1$. For a gravitational wave propagating against the lens background, we consider a linear perturbation against the background metric, where

$$g_{\mu\nu} = g^{(B)}_{\mu\nu} + h_{\mu\nu}.$$  

(A2)

Under an appropriate gauge choice and applying the Eikonal approximation, we can express the gravitational wave $h_{\mu\nu}$ as

$$h_{\mu\nu} = \phi e_{\mu\nu},$$  

(A3)

with polarization tensor $e_{\mu\nu}$ and scalar $\phi$. The change in the polarization tensor along the null geodesic is $O(U) \ll 1$ such that we hold the polarization fixed. We then consider the propagation of the scalar field as it interacts with the background lens potential, with propagation equation

$$\partial_\lambda (\sqrt{-g^{(B)}} g^{(B)\mu\nu} \partial_\nu \phi) = 0.$$  

(A4)

In the frequency domain $\tilde{\phi}(f, r)$, Equation (A4) satisfies

$$(\nabla^2 + \omega^2) \tilde{\phi} = 4\omega^2 U \tilde{\phi},$$  

(A5)

where $\omega = 2\pi f$. We define the amplification function as the ratio of the lensed and unlensed ($U = 0$) gravitational-wave amplitudes, such that

$$F(f) = \frac{\tilde{\phi}(f)}{\tilde{\phi}(f)}.$$  

(A6)

In the thin-lens approximation, we decompose the source’s wave into wavelets of all possible paths and integrate their contribution by Kirchhoff’s diffraction formula to obtain the amplification function (Nakamura 1998; Nakamura & Deguchi 1999; Takahashi & Nakamura 2003)

$$F(f) = \frac{D_S^2 c_s^2 (1 + z_L) f}{D_I D_LS} \frac{1}{i} \int d^2 x \exp[2\pi if_d(x, y)],$$  

(A7)

where $D_S$ and $D_L$ are the source’s and lens’s angular diameter distances from the observer, respectively, $z_L$ is the lens redshift, $D_LS$ is the angular diameter distance between the source and lens, $c_s$ is the Einstein radius, $x = \xi/\xi_0$ is the position of the wavelet on the lens plane, $y = (\eta / D_S) / (\xi_0 / D_L)$ is the normalized impact parameter (or the normalized source position), and $t_d$ is the arrival time of the wavelet at the observer. In the case of a point-mass lens, Equation (A7) may be analytically
integrated yielding the solution
\[
F(w) = \exp \left\{ \frac{\pi w}{4} + \frac{i w}{2} \left[ \ln \left( \frac{w}{2} \right) - 2 \phi_m(y) \right] \right\} 
\times \Gamma \left( 1 - \frac{i}{w} \right) F_1 \left( \frac{i}{2}, 1; \frac{i}{2} \right),
\] (A8)

where \( w = 8\pi M_{\odot} f \) is the dimensionless frequency, \( M_{\odot} \) is the redshift lens mass, \( iF_1 \) is the confluent hypergeometric function, and
\[
\phi_m(y) = \frac{(x_m - y)^2}{2} - \ln x_m, 
\] (A9)
\[
x_m = \frac{y + \sqrt{y^2 + 4}}{2}. 
\] (A10)

To improve computational efficiency at the limit of \( y \ll 1 \) or \( w \ll 1 \), we switch to the geometric approximation of the magnification,
\[
F_{\text{geo}}(w) = \sqrt{\mu_+} - i \sqrt{\mu_-} e^{i \Delta \tau}, 
\] (A11)
\[
\mu_+ = \frac{1}{2} \pm \frac{\sqrt{y^2 + 4}}{2y\sqrt{y^2 + 4}}. 
\] (A12)
\[
\Delta \tau = \frac{y \sqrt{y^2 + 4}}{2} + \ln \left( \frac{\sqrt{y^2 + 4} + y}{\sqrt{y^2 + 4} - y} \right) 
\] (A13)

where \( \mu_+ \) and \( \mu_- \) are the magnifications of the two geometric images, and \( \Delta \tau \) is the normalized time delay between the two images.

**Appendix B**

**Spatial Poisson Process**

With a source centered at the origin, the probability that there are \( k \) lenses within an effective distance \( y \) is
\[
Poisson(k|\Sigma) = \frac{(\Sigma \pi y^2)^k}{k!} e^{-\Sigma \pi y^2}, 
\] (B1)

where \( \Sigma \pi \) is the effective density parameter of lenses within the volume of \( z_S \) projected on the sky. The differential probability of finding the nearest-effective lens inside an infinitesimal ring between \( y \) and \( y + dy \) is the product of the probability that there is no lens within the circle of radius \( y \), Poisson\((0|\Sigma) = e^{-\Sigma \pi y^2}\), and the probability of a lens lying inside the ring, \( 2\Sigma \pi y^2 dy \). Dividing this probability by \( dy \), the probability density function of the nearest-effective lens locating at \( y \) is
\[
p(y) = 2\Sigma \pi y e^{-\Sigma \pi y^2}. 
\] (B2)

Since \( y \) is the dimensionless ratio of the angular separation between the source and the lens to the angular size of the lens Einstein ring, the effective density parameter can be interpreted as the mean fractional area of all lenses within \( z_S \) relative to the full sky plane (or, equivalently, the inverse of the mean of \( y^2 \)), i.e.,
\[
\Sigma(z_S, \Lambda_L) \pi = N_L(z_S) \frac{\Theta^2_0}{4\pi}, 
\] (B3)

where \( N_L(z_S) = \int_0^{z_S} n_L(z_S) dV_c(z_S) \) is the total number of lenses within the comoving volume \( V_c(z_S) \) for an arbitrary number density evolution of lenses \( n_L(z_S) \), and
\[
\langle \Theta^2_0 \rangle_{\Lambda_L} = \int \theta^2_0(z_L, \tilde{x}_L | z_S) \times \pi^2_L(z_L, \tilde{x}_L | z_S, \Lambda_L) d\xi_L d\tilde{x}_L 
\] (B4)

is the mean area enclosed by the Einstein rings, with \( \langle \cdot \rangle_{\Lambda_L} \) being the mean quantity over the intrinsic lens distribution parameterized by \( \Lambda_L \) and \( \pi^2_L(z_L, \tilde{x}_L | z_S, \Lambda_L) \) being the joint distribution of redshift and mass of the intrinsic lens population (i.e., regardless of the pairing with the sources). Thus, the term \( \Sigma \pi y^2 \) in the exponent of Equation (B2) is equivalent to the mean number of lenses within the area \( \pi \theta^2_0 \). The desired \( \pi_y \) is then
\[
\pi_y(y \mid z_S, \Lambda_L, \mathcal{P}) = 2\pi y \Sigma(z_S, \Lambda_L) e^{-\Sigma \pi y^2}. 
\] (B5)

The pairing requirement, \( \mathcal{P} \), favors a source–lens system with the largest \( \theta^2_0 \) to minimize the value of \( y \). One can think of the pairing condition as choosing the lens with the largest area, \( \pi \theta^2_0 \). As a result, the final distribution of lens parameters in the source–lens systems has an additional lensing bias factor proportional to \( \theta^2_0 \) for sources at the same \( z_S \). Mathematically, the distribution of \( x_L \) after the nearest-effective pairing is
\[
\pi_{x_L}(z_L, \tilde{x}_L | z_S, \Lambda_L, \mathcal{P}) \propto \theta^2_0 \pi^2_L(z_L, \tilde{x}_L | z_S, \Lambda_L), 
\] (B6)

which is indeed the integrand of Equation (B4). Note that we do not consider the effects of multiple lensing, as the second-nearest-effective lens neighbor, which scales as \( y^4 e^{-\Sigma \pi y^2} \) is unlikely to drive a significant lensing effect for the IMBH densities we consider. For instance, with \( n_L = 10^7 \text{Mpc}^{-3} \), a back of the envelope estimate for the rates of two lenses with \( y \leq 1 \) is \( 4 \times 10^{-4} \), so that even in our most dense hyperparameter less than 1 in 2000 events will have significant multiple-lensing effects.

**Appendix C**

**Source Distribution**

The parameters of the source distribution from which we sample are as follows. For the mass distribution of the component source masses, we sample from the power law + peak model from population studies of GWTC-2 (Abbott et al. 2021). The source redshift distribution is drawn from the phenomenological fit to the population synthesis rate (Belczynski et al. 2016; Ng et al. 2020).
\[
p(z_S) \propto \frac{dV_c}{dz_S} \frac{(1 + z_S)^{1.57}}{1 + \left( \frac{1 + z_S}{3.36} \right)^{3.83}}. 
\] (C1)

The rest of the parameters, including the sky position, polarization angle, cosine of orbital inclination angle, and aligned spins, are distributed uniformly. After sampling the source parameters from the above distribution, we simulate the gravitational-wave signals in the presence of detectors’ noise, calculate the network S/N, and only select the signals with \( S/N \geq 12 \).
Appendix D
Number Density of Lenses in Plane

The expression of $\Sigma$ for $\pi_y$ is

$$\Sigma(z_S, \Lambda_L) = \frac{4n_L0\chi_S^3}{3D_S} \langle M_L \rangle_{\Lambda_L} \left( \frac{D_{LS}}{D_L} \right)_{\Lambda_L},$$  \hspace{1cm} (D1)

where $\chi_S$ is the comoving distance at $z_S$, $\langle M_L \rangle_{\Lambda_L}$ is the mean lens mass,

$$\langle M_L \rangle_{\Lambda_L} = \begin{cases} 
\frac{M_{L,max} - M_{L,min}}{\ln(M_{L,max}/M_{L,min})} & \text{for } \alpha_L = 1 \\
\frac{M_{L,max}^2 - M_{L,min}^2}{2} & \text{for } \alpha_L = 2 \\
1 - \alpha_L M_{L,max}^{-\alpha_L} - M_{L,min}^{-\alpha_L} & \text{otherwise},
\end{cases}$$  \hspace{1cm} (D2)

and $\langle D_{LS}/D_L \rangle_{\Lambda_L}$ is the mean distance factor given by

$$\left( \frac{D_{LS}}{D_L} \right)_{\Lambda_L} = \int_0^{z_S} \frac{D_{LS}}{D_L} d\chi_L.$$  \hspace{1cm} (D3)

ORCID iDs

Joseph Gais @ https://orcid.org/0000-0003-0572-1221
Ken K. Y. Ng @ https://orcid.org/0000-0003-3896-2259
Eungwang Seo @ https://orcid.org/0000-0002-8588-4794
Kaze W. K. Wong @ https://orcid.org/0000-0001-8432-7788
Tjonnge F. Li @ https://orcid.org/0000-0003-4297-7365

References

Aasi, J., Abbott, B., Abbott, R., et al. 2015, CQGra, 32, 074001
Abbott, B., Abbott, R., Abbott, T. D., et al. 2019, PhRvX, 9, 031040
Abbott, B. P., Abbott, R., Abbott, T. D., et al. 2017, CQGra, 34, 044001
Abbott, R., Abbott, T. D., Abraham, S., et al. 2020, PhRvL, 125, 101102
Abbott, R., Abbott, T. D., Abraham, S., et al. 2021a, PhRvX, 11, 021053
Abbott, R., Abbott, T. D., Abraham, S., et al. 2021b, ApJ, 923, 14
Abbott, R., Abbott, T. D., Abraham, S., et al. 2021c, SoftX, 13, 100658
Acernese, F. A., Agathos, M., Agatsuma, K., et al. 2014, CQGra, 32, 024001
Aghanim, N., Akrami, Y., Arroja, F., et al. 2020, A&A, 641, A1
Akiyama, K., Alberdi, A., Alef, W., et al. 2019a, ApJL, 875, L2
Akiyama, K., Alberdi, A., Alef, W., et al. 2019b, ApJL, 875, L3
Akiyama, K., Alberdi, A., Alef, W., et al. 2019c, ApJL, 875, L4
Akiyama, K., Alberdi, A., Alef, W., et al. 2019d, ApJL, 875, L5
Akiyama, K., Alberdi, A., Alef, W., et al. 2019e, ApJL, 875, L6
Ashton, G., Hübnner, M., Lasky, P. D., et al. 2019, ApJS, 241, 27
Basak, S., Ganguly, A., Haris, K., et al. 2022, ApJL, 926, L28
Belczynski, K., Holz, D. E., Bulik, T., & O’Shaughnessy, R. 2016, Natur, 534, 512
Diego, J. M. 2020, PhRvD, 101, 123512
E.H.T. Collaboration, Akiyama, K., Alberdi, A., et al. 2019, ApJL, 875, L1
Ghez, A., Salim, S., Hornstein, S. D., et al. 2005, ApJ, 620, 744
Goyal, S., Haris, K., Mehta, A. K. & Ajith, P. 2021, PhRvD, 103, 024038
Greene, J. E., Strader, J., & Ho, L. C. 2020, ARA&A, 58, 257
Hannuksela, O., Haris, K., Ng, K., et al. 2019, ApJL, 874, L2
Hannuksela, O. A., Collett, T. E., Çalıṣkan, M., & Li, T. G. 2020, MNRAS, 498, 3395
Husa, S., Khan, S., Hannam, M., et al. 2016, PhRvD, 93, 044006
Khan, S., Husa, S., Hannam, M., et al. 2016, PhRvD, 93, 044007
Lai, K.-H., Hannuksela, O. A., Herrera-Martín, A., et al. 2018, PhRvD, 98, 083005
Li, S.-S., Mao, S., Zhao, Y., & Lu, Y. 2018, MNRAS, 476, 2220
LIGO Scientific Collaboration, Virgo Collaboration, KAGRA Collaboration, et al. 2021, arXiv:2111.03606
Liu, X., Hernandez, I. M., & Creighton, J. 2021, ApJ, 908, 97
Madau, P., & Dickinson, M. 2014, ARA&A, 52, 415
McClintock, J. E., & Remillard, R. A. 2003, arXiv:astro-ph/0306213
McIsaac, C., Keitel, D., Collett, T., et al. 2020, PhRvD, 102, 084031
Nakamura, T. T. 1998, PhRvL, 80, 1138
Nakamura, T. T., & Deguchi, S. 1999, PThPS, 133, 137
Navarro, J. F., Frenk, C. S., & White, S. D. 1995, MNRAS, 275, 720
Navarro, J. F., Frenk, C. S., & White, S. D. 1997, ApJ, 490, 493
Navarro, J. F., Frenk, C. S., & White, S. D. M. 1996, ApJ, 462, 563
Ng, K. K., Vitale, S., Farr, W. M., & Rodriguez, C. L. 2021, ApJL, 913, L5
Ng, K. K., Wong, K. W., Broadhurst, T., & Li, T. G. 2018, PhRvD, 97, 023012
Oguri, M. 2018, MNRAS, 480, 3842
Paynter, J., Webster, R., & Thrane, E. 2021, NatAs, 5, 560
Punturo, M., Abernathy, M., Acernese, F., et al. 2010, CQGra, 27, 194002
Regimbau, T., Evans, M., Christensen, N., et al. 2017, PhRvL, 118, 151105
Reitze, D., Adhikari, R. X., Ballmer, S., et al. 2019, BAAS, 51, 35
Remillard, R. A., & McClintock, J. E. 2006, ARA&A, 44, 49
Seo, E., Hannuksela, O. A., & Li, T. G. 2021, arXiv:2110.03308
Speagle, J. S. 2020, MNRAS, 493, 3132
Takahashi, R., & Nakamura, T. 2003, ApJ, 595, 1039
Turner, E. L., Ostriker, J. P., & Gott, J. R., III 1984, ApJ, 284, 1
Urrutia, J., & Vaskonen, V. 2022, MNRAS, 509, 1538
Wang, Y., Lo, R. K., Li, A. K., & Chen, Y. 2021, PhRvD, 103, 104055
Xu, F., Ezquiaga, J. M., & Holz, D. E. 2022, ApJ, 929, 9