Categories over quantum affine algebras and monoidal categorification

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Abstract: Let \( U'_q(\mathfrak{g}) \) be a quantum affine algebra of untwisted affine \( ADE \) type, and \( \mathcal{C}^0 \) the Hernandez-Leclerc category of finite-dimensional \( U'_q(\mathfrak{g}) \)-modules. For a suitable infinite sequence \( \tilde{w}_0 = \cdots s_{i_1}s_{i_2}s_{i_3} \cdots \) of simple reflections, we introduce subcategories \( \mathcal{C}_{a,b}^r \) of \( \mathcal{C}^0 \) for all \( a \leq b \in \mathbb{Z} \cup \{\pm \infty\} \). Associated with a certain chain \( \mathcal{C} \) of intervals in \( [a,b] \), we construct a real simple commuting family \( M(\mathcal{C}) \) in \( \mathcal{C}_{a,b}^r \), which consists of Kirillov-Reshetikhin modules. The category \( \mathcal{C}_{a,b}^r \) provides a monoidal categorification of the cluster algebra \( K(\mathcal{C}_{a,b}^r) \), whose set of initial cluster variables is \( [M(\mathcal{C})] \). In particular, this result gives an affirmative answer to the monoidal categorification conjecture on \( \mathcal{C}^0 \) by Hernandez-Leclerc since it is \( \mathcal{C}^{[-\infty,0]} \), and is also applicable to \( \mathcal{C}^0 \) since it is \( \mathcal{C}^{[-\infty,\infty]} \).

Key words: Monoidal categorification; quantum affine algebra; cluster algebra; Kirillov-Reshetikhin module; \( T \)-system.

1. Introduction. Let \( U'_q(\mathfrak{g}) \) be a quantum affine algebra. The category \( \mathcal{C}_a^0 \) of finite-dimensional integrable modules over \( U'_q(\mathfrak{g}) \) has been intensively studied due to its rich structure. For instances, every object \( M \) in \( \mathcal{C}^0 \) has its left \( M^* \) and right dual \( 'M \), and the \( q \)-characters of Kirillov-Reshetikhin modules in \( \mathcal{C}^0 \) provide a solution of the \( T \)-system, a system of differential equations appearing in solvable lattice models ([3,5,18,19]).

On the other hand, the cluster algebras were introduced by Fomin and Zelevinsky in [2] to investigate upper global bases and total positivity in an aspect of combinatorics.

Interestingly, it is proved in [6,7,9] that the Grothendieck rings \( K(\mathcal{C}) \) of monoidal subcategories \( \mathcal{C} = C_N \) (\( N \in \mathbb{Z}_{\geq 1} \)), \( \mathcal{C}_Q \), \( \mathcal{C}^0_a \) of \( \mathcal{C}^0 \) have cluster algebra structures \( \mathcal{A} \), and conjectured that every cluster monomial corresponds to the isomorphism class of a real simple module in \( \mathcal{C} \); that is, \( \mathcal{C} \) is expected to be a monoidal categorification of \( \mathcal{A} \). The conjectures for \( \mathcal{C}_N \) (\( N \in \mathbb{Z}_{\geq 1} \)) of untwisted affine \( ADE \) types are proved in [6,8,20] and [23]. Also, the conjecture for the subcategory \( \mathcal{C}_Q \subset \mathcal{C}^0 \), determined by a \( Q \)-data \( Q = (Q, \phi_Q) \) ([4,16]), is proved in [11] via the quantum affine Weyl-Schur duality functor \( F_Q \) ([10,12,17,21]) from the category \( \mathcal{C}_QH \) of finite-dimensional graded modules over the symmetric quiver Hecke algebra to \( \mathcal{C}_Q \). More precisely, the category \( \mathcal{C}_QH \) provides a monoidal categorification of the quantum cluster algebra \( A_n(\mathfrak{q}) \), the quantum unipotent coordinate algebra of finite simply-laced type ([1]). Since \( F_Q \) is an exact monoidal functor preserving simplicity, we can prove the conjecture for \( \mathcal{C}_Q \) in an indirect way. However, this method could not be applicable to other \( \mathcal{C} \) directly.

Recently, in [13], the authors of the present paper (KKOP) developed \( Z \)-valued invariants \( \Lambda, \Lambda^\infty, \Lambda, \mathfrak{d} \) for pairs of modules in \( \mathcal{C}^0 \), which is extracted from distinguished \( U'_q(\mathfrak{g}) \)-module homomorphisms, called \( R \)-matrices. Furthermore, KKOP provided a criterion for a monoidal subcategory \( \mathcal{C} \subset \mathcal{C}^0 \) to become a monoidal categorification of a cluster algebra by using those invariants. This paper can be understood as a continuation of [13], since we will apply the above criterion to various subcategories \( \mathcal{C} \) of \( \mathcal{C}^0 \), including \( \mathcal{C}^0, \mathcal{C}^0_a \).
and \( \mathcal{C}_N \). We also give their initial monoidal seeds in a uniform manner.

Let \( \mathfrak{g}_0 \) be a finite-dimensional simple Lie algebra of \( ADE \) type with a Cartan matrix \( A = (a_{ij})_{i,j \in I_0} \), \( W \) the Weyl group generated by simple reflections \( s_i \) \((i \in I_0)\), \( \mathfrak{g} \) the untwisted affine Kac-Moody algebra associated with \( \mathfrak{g}_0 \), and \( U'_q(\mathfrak{g}) \) the quantum affine algebra associated with \( \mathfrak{g} \). In [6], Hernandez-Leclerc defined the full subcategory \( \mathcal{C}_\mathfrak{g}^0 \) of \( \mathcal{C}_\mathfrak{g} \). Since every simple module in \( \mathcal{C}_\mathfrak{g} \) is a tensor product of suitable parameter shifts of simple modules in \( \mathcal{C}_\mathfrak{g}^0 \), it is enough to consider subcategories of \( \mathcal{C}_\mathfrak{g}^0 \).

By extending a reduced expression \( s_{i_1} s_{i_2} \cdots s_{i_r} \) of the longest element \( w_0 \) of the Weyl group \( W \), we obtain an infinite sequence

\[
\hat{w}_0 = \cdots s_{i_{-1}} s_{i_0} s_{i_1} s_{i_2} \cdots
\]

of simple reflections satisfying properties (a) and (b) in Section 2, and then we define fundamental modules \( V[k]^{(a)} \) \((k \in \mathbb{Z}) \). For each interval \( [a, b] = \{k \in \mathbb{Z} \mid a \leq k \leq b \} \) with \( a \leq b \in \mathbb{Z} \cup \{\pm \infty\} \), we define the subcategory \( \mathcal{C}_{\mathfrak{g}}^{(a, b)} \) of \( \mathcal{C}_\mathfrak{g}^0 \), which is the smallest full monoidal subcategory containing \( V[k]^{(a)} \) for all \( k \in [a, b] \). Then \( \mathcal{C}_\mathfrak{g}^0 \) is nothing but \( \mathcal{C}_\mathfrak{g}^{(-\infty, +\infty)} \) and the subcategory \( \mathcal{C}_\mathfrak{g}^{(a, b)} \) introduced by Hernandez-Leclerc ([9]) can be identified with \( \mathcal{C}_\mathfrak{g}^{(a, b)} \) (Remark 2.3).

We say that an interval \( [a, b] \) is an \( i \)-box if \( i_a = i_b \). For each \( i \)-box \( [a, b] \), we define a simple module \( M[a, b] \), which can be understood as a quantum affine analogue of the determinantial tensor product of the longest element \( w \) of \( \mathcal{C} \). We take the algebraic closure \( \mathbb{C}(\mathfrak{g}) \) inside \( \bigcup_{m \geq 0} C(\{q^m\}) \) as the base field for \( U'_q(\mathfrak{g}) \). Recall that \( \mathcal{C}_\mathfrak{g} \) is the category of finite-dimensional integrable modules over \( U'_q(\mathfrak{g}) \), and \( \mathfrak{g} \) is simple for all \( k \in \mathbb{Z}_{\geq 1} \).

Let us denote by \( \Psi \) the quiver whose set of vertices is

\[
\hat{T}_0 := \{ (i, k) \in I_0 \times \mathbb{Z} \mid k \equiv d(1, i) \bmod 2 \}
\]

and the arrows of \( \Psi \) consist of two types:

\[
(A) \quad (i, t) \rightarrow (j, s) \quad \text{with} \quad d(i, j) = 1 \quad \text{and} \quad s - t = 1,
\]

\[
(B) \quad (i, s + 2) \rightarrow (i, s).
\]

Here \((i, j)\) denotes the distance between the vertices \( i \) and \( j \) in the Dynkin diagram of \( \mathfrak{g}_0 \) and \( 1 \in I_0 \) is an arbitrary chosen element.

We say that an infinite sequence

\[
\hat{w}_0 = \cdots s_{i_{-1}} s_{i_0} s_{i_1} \cdots
\]

of simple reflections in the Braid group \( B(\mathfrak{g}_0) \) ([15]) of type \( \mathfrak{g}_0 \) is \textit{admissible} if

\begin{enumerate}
\item there exists a sequence \( \{t_k\}_{k \in \mathbb{Z}} \) of integers such that
\item \( t_k, t_k' \in \hat{T}_0 \),
\item \( t_{k'} = t_k + 2 \), and
\item \( t_k > t_k' \) if \( k > k' \) and \( d(i_k, i_{k'}) = 1 \).
\end{enumerate}
(b) $s_{i_k} \cdots s_{i_{k-1}} = w_0$ for all $k \in \mathbb{Z}$, where $\ell$ denotes the length of longest element $w_0 \in W$.

Here, for $k \in \mathbb{Z}$ and $j \in I_0$, we set

$$
\begin{align*}
&k^+ := \min\{p \mid k < p, \ i_k = i_p\}, \\
&k^- := \max\{p \mid p < k, \ i_k = i_p\}, \\
&k(j)^+ := \min\{p \mid k < p, \ i_p = j\}, \\
&k(j)^- := \max\{p \mid p < k, \ i_p = j\}.
\end{align*}
$$

**Remark 2.1.**

(i) We have $i_{k+1} = i_k^*$, where $^*$ denotes the involution on $I_0$ induced by $w_0$.

(ii) $\widehat{w}_0$ completely determines $\{(i_k, t_k)\}_{k \in \mathbb{Z}}$ up to an even translation.

(iii) For every $k \in \mathbb{Z}$, the reduced expression $s_{i_k} \cdots s_{i_{k-1}}$ in (b) is adapted to some Dynkin quiver $Q$ of type $\mathfrak{g}_0$. Conversely, for any Dynkin quiver $Q$ of type $\mathfrak{g}_0$, there exists a sequence $\widehat{w}_0$ satisfying (a) and (b) such that $s_{i_k} \cdots s_{i_1}$ is adapted to $Q$.

For each $k \in \mathbb{Z}$, we define the fundamental module

$$
V[k]^{\widehat{w}_0} := V(\varpi_k)^{(q)_k}.
$$

Then we have

$$
V[k]^+ v_0 \simeq V[k]^{\widehat{w}_0}, \quad V[k + \ell]^{\widehat{w}_0} = \mathcal{D}(V[k]^{\widehat{w}_0}),
$$

where $\mathcal{D}$ denotes the right dual functor.

**Definition 2.2.** For each interval $[a, b]$, we denote by $\mathcal{C}_q^{[a, b]}$ the smallest full subcategory of $\mathcal{C}$ satisfying the following conditions:

(i) it is stable under taking subquotients, extensions, tensor products and

(ii) it contains $V[k]^{\widehat{w}_0}$ for all $a \leq k \leq b$ and the trivial module $1$.

**Remark 2.3.** Many of known subcategories $\mathcal{C}$ of $\mathcal{C}_q$ can be identified with $\mathcal{C}_q^{[a, b]}$ by taking suitable $\widehat{w}_0$ and $[a, b]$:  

(1) $\mathcal{C}_{\mathfrak{g}}^{[-\infty, \infty]}$ coincides with the subcategory $\mathcal{C}_q^{0}$.

(2) The subcategory $\mathcal{C}_Q^{[a, b]}$ associated to a Q-data $Q$ coincides with $\mathcal{C}_q^{[a, b]}$ for some interval $[a, b]$ with $|a - b| = b - a + 1 = \ell$.

(3) By taking $s_{i_k} \cdots s_{i_1}$ in $\mathcal{C}_q^{[a, b]}$ as adapted to the Dynkin quiver $Q$ with $\{1, 2\} \ni \phi_0(k) \equiv d(1, i_k)$ (mod 2) and $i_k = \phi_0(k) + 1$, $\mathcal{C}_N^a \mathcal{C}_N^b$ can be identified with $\mathcal{C}_q^{[a, b]}$, where $a = 1 - (N \times I_0)$, and $\mathcal{C}_N^a \mathcal{C}_N^b$ can be identified with $\mathcal{C}_q^{[-\infty, 0]}$. Those subcategories $\mathcal{C}_N^a \mathcal{C}_N^b$ of $\mathcal{C}_q^{[a, b]}$ are introduced in [6,9].

3. Real simple commuting family associated to an admissible chain of i-boxes. Let us fix an admissible sequence $\bar{w}_0$ and $\{t_k\}_{k \in \mathbb{Z}}$. We write $V[k]$ for $V[k]^{\bar{w}_0}$. We say that an interval $c = [a, b]$ is an i-box if $i_a = i_b$. For each i-box $[a, b]$, the module $M[a, b]$ in $\mathcal{C}_q^{[a, b]}$ is defined as follows:

$$
M[a, b] := \text{hd}(V[b^+] \otimes V[b^-] \otimes \cdots \otimes V[a^+] \otimes V[a]),
$$

where $\text{hd}(M)$ for $M \in \mathcal{C}_q^{[a, b]}$ denotes the head of $M$. In particular, $V[a] = M[a, a]$.

**Theorem 3.1.**

(i) $M[a, b]$ is a Kirillov-Reshetikhin module with a dominant extremal weight $\varpi_{i_a}$ where $s = |\{k \mid a \leq k \leq b, i_k = i_0\}|$.

(ii) For i-boxes $[a, b]$ and $[c, d]$, $M[a, b]$ and $M[c, d]$ commutes if either

$$
a^- < c \leq d < b^+ \quad \text{or} \quad c^- < a \leq b < d^+.
$$

(iii) For any i-box $[a, b]$, there exists an exact sequence in terms of $M[a, b]$’s as follows:

$$
0 \rightarrow \bigotimes_{d(i_{0,j}) = 1} D[a(j)^+, b(j)^-] \rightarrow M[a^+, b^-] \otimes M[a, b^+] \rightarrow 0,
$$

We call it a T-system.

**Remark 3.2.** For any reduced expression $w_0 = s_{j_1} \cdots s_{j_\ell}$ of $w_0$ (not necessarily adapted) and $[a, b]$ with $a = j_\ell$ and $b - a + 1 = \ell$, there exists a real simple module $D[a, b]^{\infty}$ in $\mathcal{C}_{QH}$ of type $\mathfrak{g}_0$, called the determinantial module, and there exists an exact sequence (called the T-system)

$$
0 \rightarrow \bigotimes_{d(i_{0,j}) = 1} D[a(j)^+, b(j)^-] \rightarrow D[a^+, b^-] \otimes D[a, b^+] \rightarrow 0
$$

in $\mathcal{C}_{QH}$, which is analogous to (3.1). More precisely, when $\bar{w}_0$ is adapted to some Dynkin quiver $Q$ of type $\mathfrak{g}_0$, quantum affine Weyl-Schur duality functor $\mathcal{F}_Q$ associated with some Q-data $Q = (Q, \phi_0)$ transforms the above exact sequence in $\mathcal{C}_{QH}$ to the T-system (3.1). Thus $M[a, b]$ can be understood as a quantum affine analogue of the determinantial module. (See [7, Proposition 4.1] and [11] for more detail.)

For an interval $c := [a, b]$, we introduce i-boxes

$$
[a, b] := [a, b(i_a)^-], \quad [a, b] := [a(i_b)^+, b],
$$

$$
L(c) := [a - 1, b], \quad R(c) := [a, b + 1].
$$

**Definition 3.3.** A chain $\mathcal{C}$ of i-boxes

$$
(c_k = [a_k, b_k])_{1 \leq k \leq l} \quad (l \in \mathbb{Z}_{\geq 1} \sqcup \{\infty\})
$$

is called admissible if $\bar{c}_k = [\bar{a}_k, \bar{b}_k] := \bigcup_{1 \leq j \leq l} [a_j, b_j]$ satisfies $|\bar{c}_k| = k$ and one of the following two statements.

A monoidal seed in \( \mathcal{C} \) is a pair \( \mathcal{S} = (\{M_i\}_{i \in K}, \widetilde{B}) \) consisting of a commuting family \( \{M_i\}_{i \in K} \) of real simple objects in \( \mathcal{C} \) and a \( \mathbb{Z} \)-valued \( K \times K_{\text{ex}} \)-matrix \( \widetilde{B} = (b_{ij})_{(i,j) \in K \times K_{\text{ex}}} \) such that (i) for each \( j \in K_{\text{ex}} \), there exist finitely many \( i \in K \) such that \( b_{ij} \neq 0 \), (ii) the principal part \( B := (b_{ij})_{(i,j) \in K \times K_{\text{ex}}} \) is skew-symmetric. For \( i \in K \), we call \( M_i \) the \( i \)-th cluster variable module of \( \mathcal{S} \).

For a monoidal seed \( \mathcal{S} = (\{M_i\}_{i \in K}, \widetilde{B}) \), let \( \Lambda^\mathcal{S} = (\Lambda^\mathcal{S}_{ij})_{i,j \in K} \) be the skew-symmetric matrix given by \( \Lambda^\mathcal{S}_{ij} = \Lambda(\lambda_i, M_j) \) (see \([14] \)).

A monoidal seed \( \mathcal{S} = (\{M_i\}_{i \in K}, \widetilde{B}) \) is called \( \Lambda \)-admissible if

\[
(i) \quad (\Lambda^\mathcal{S})_{jk} = -2\delta_{jk} \text{ for } (j,k) \in K \times K_{\text{ex}}, \text{ and}
(ii) \quad \text{for each } k \in K_{\text{ex}}, \text{there exist a simple object } M'_k \text{ of } \mathcal{C} \text{ commuting with } M_i \text{ for any } i \neq k \text{ and an exact sequence in } \mathcal{C}
\]

\[
(4.1) \quad 0 \rightarrow \bigotimes_{b_{ij}>0} M_i^{\otimes b_{ij}} \rightarrow M_k \otimes M'_k \rightarrow \bigotimes_{b_{ij}<0} M_i^{\otimes (-b_{ij})} \rightarrow 0.
\]

Under the following two assumptions

\[ (4.2) \quad \text{(a) there exists a } \Lambda \text{-admissible monoidal seed } \mathcal{S} = (\{M_i\}_{i \in K}, \widetilde{B}) \text{ in } \mathcal{C}, \text{ and}
\]

\[ (b) \quad K(\mathcal{C}) \text{ is isomorphic to the cluster algebra } \mathcal{A}(\mathcal{S}), \]

KKOP ([13, Theorem 6.10]) proved that \( \mathcal{C} \) provides a monoidal categorification of \( \mathcal{A}(\mathcal{S}) \). Here \( \mathcal{S} := (\{M_i\}_{i \in K}, \widetilde{B}) \) is a seed in \( K(\mathcal{C}) \), and \( \mathcal{A}(\mathcal{S}) \) denotes the cluster algebra with the initial seed \( \mathcal{S} \).

Set \( \tilde{I}_0 := \tilde{I}_0 \cap (I_0 \times \mathbb{Z}_{\leq 0}) \) and let \( \Psi^- \) be the full subquiver of \( \Psi \) whose set of vertices is \( \tilde{I}_0 \). In [9], Hernandez-Leclerc proved that \( \mathcal{C}^- := K(\mathcal{C}^-) \) has a cluster algebra structure whose initial cluster variable modules \( \{M_{(i,t)}\}_{(i,t) \in \tilde{I}_0} \) consist of certain KR-modules. For a suitable choice of \( \tilde{\omega}_0 \) (Remark 2.3), we have \( \mathcal{C}^- = \mathcal{C}^-_{\tilde{\omega}_0} \) and \( \{M_{(i,t)}\}_{(i,t) \in \tilde{I}_0} \) can be described as \( M(\mathcal{C}^-) \) for the following admissible chain \( \mathcal{C}^- \) of \( \tilde{i} \)-boxes:

\[
\mathcal{C}^- = (0, \mathcal{S} = (\{L, L, L, \ldots\}))
\]

More precisely, for \( (i,t) = (i_a, t_a) \) \( (a \leq 0) \), we have

\[
M_{(i,t)} = M[a,0).
\]

The following theorem gives an affirmative answer for the conjecture on \( \mathcal{C}^- \):

**Theorem 4.1.** The monoidal seed
The envelope $\mathcal{E}$ is $\Lambda$-admissible, where $\mathcal{B}$ is the matrix associated to $\Psi$. Hence $\mathcal{E}$ provides a monoidal categorification of $\mathcal{A}$.

Now we shall generalize the above theorem to an arbitrary $\mathcal{E}_{\theta}^{[a,b]}$.

**Proposition 4.2.** Let $\mathcal{C} = (c_k)_{1 \leq k \leq l}$ be an admissible chain of $i$-boxes with the range $[a, b]$ and the envelope $\{c_k\}_{1 \leq k \leq l}$. Assume that $\mathcal{C} := \mathcal{C}_{\theta}^{[a,b]}$ provides a monoidal categorification of $K(\mathcal{C})$ with a $\Lambda$-admissible monoidal seed $(M(\mathcal{C}), \mathcal{B})$. Let $c_l$ be a movable $i$-box of $\mathcal{C}$ and set $\mathcal{E} = B_s(\mathcal{C})$. If $c_{l+1} \neq c_{l+1}$, then $M(\mathcal{C})$ is equal to $M(\mathcal{C})$ up to a permutation. If $c_{l+1} = c_{l+1}$, then $(M(\mathcal{C}), \mathcal{B})$ is the monoidal mutation of $M(\mathcal{C})$ at $s$. Moreover the corresponding exact sequence (4.1) is given by the $T$-system (3.1).

The above proposition and Theorem 3.4 show that all $M(\mathcal{C})$ with the same range are mutation equivalent.

Now we state our main theorem:

**Theorem 4.3.** For any admissible chain $\mathcal{C} = (c_k)_{1 \leq k \leq l}$ for $l \in \mathbb{Z}_{>0}$ with the range $[a, b]$ and $a \leq b \in \mathbb{Z} \cup \{\pm \infty\}$, there exists a $\Lambda$-admissible monoidal seed $\mathcal{A}$ of $\mathcal{E}_{\theta}^{[a,b]}$ such that

(i) its set of cluster variable modules is $M(\mathcal{C})$,

(ii) its set of frozen variable modules is $\{M[a(i)^+], b(i)^-\} | i \in I_0, -\infty < a(i)^+ \leq b(i)^- < +\infty\}$, and

(iii) $K(\mathcal{C}_{\theta}^{[a,b]})$ has a cluster algebra structure with the initial seed $[\mathcal{A}]$, and $\mathcal{C}_{\theta}^{[a,b]}$ provides a monoidal categorification of $\mathcal{A}(\mathcal{A})$.

By Remark 2.3, we have the following

**Corollary 4.4.** The Grothendieck ring $K(\mathcal{E}_{\theta}^{[a,b]})$ has a cluster algebra structure, and $\mathcal{E}_{\theta}^{[a,b]}$ provides a monoidal categorification of $K(\mathcal{E}_{\theta}^{[a,b]})$.

**Remark 4.5.** We can generalize the above results to an arbitrary quantum affine algebra $U_q(\mathfrak{g})$ by applying a similar framework with the results in [12,17,21,22].

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