Hausdorff Operators on Some Spaces of Holomorphic Functions on the Unit Disc

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Abstract

We introduce Hausdorff operators over the unit disc and give conditions for boundedness of such operator in Bloch, Bergman, and Hardy spaces on the disc. Approximation of the identity by Hausdorff operators is also considered.

Keywords Hausdorff operator · Bloch space · Bergman space · Hardy space · Approximation of the identity

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1 Introduction and Preliminaries

In [1] Hausdorff operators were defined over locally compact groups $G$ via their automorphism groups $\text{Aut}(G)$. But for the circle group $\mathbb{T}$ this definition leads to almost trivial Hausdorff operators due to the almost triviality of $\text{Aut}(\mathbb{T})$. The main idea of this work is as follows. If we want to fix the afore-mentioned problem we should consider spaces of functions on the boundary $\mathbb{T} = \partial \mathbb{D}$ of the unit disc $\mathbb{D}$ with holomorphic extension into the disc and employ the reach group $\text{Aut}_0(\mathbb{D})$ of involutive Möbius automorphisms of the disc instead of $\text{Aut}(\partial \mathbb{D})$.

The one-dimensional Hausdorff operators were introduced by Garabedian and independently by Rogosinski (see [2, p. 282]). The modern theory of Hausdorff operators was inspired by [3]; see surveys [4,5]. Garabedian and Rogosinski introduced their operators as a continuous analog of Hausdorff means; similarly, to get a mean of a holomorphic function $f$ on $\mathbb{D}$ it is natural to integrate $f$ over all natural holomorphic
mixing of the disc, i.e., over the elements of the group $\text{Aut}(\mathbb{D})$ of Möbius transformations of $\mathbb{D}$ (see Theorems 4 and 5 below). So, we arrive at the following

**Definition 1** For a holomorphic function $f$ on $\mathbb{D}$ the value of a Hausdorff operator at $f$ is as follows:

$$(\mathcal{H}_{K,\mu} f) (z) := \int_{\mathbb{D}} K(w) f(\varphi_w(z)) \, d\mu(w), \quad z \in \mathbb{D},$$

where $\mu$ is some fixed positive Radon measure on $\mathbb{D}$, $K$ is some fixed $\mu$-measurable function on $\mathbb{D}$, and $\varphi_w$ is an element of $\text{Aut}(\mathbb{D})$ of the form

$$\varphi_w(z) = \frac{w - z}{1 - \overline{w}z} \quad (w \in \mathbb{D}).$$

We shall denote the set of such elements by $\text{Aut}_0(\mathbb{D})$.

**Example** If $\mu$ is concentrated on a subsequence $\{w_n\} \subset \mathbb{D}$ we get a class of operators of the form

$$(\mathcal{H}_d f)(z) := \sum_{n=0}^{\infty} d_n f\left(\frac{w_n - z}{1 - \overline{w_n}z}\right)$$

(discrete Hausdorff operators over $\mathbb{D}$). If the sequence $\{w_n\}$ is finite this is a sort of so-called functional operators (see [6]).

**Remark** In [7] operators in Lebesgue spaces $L^p(dH)$ of the form

$$K f(z) = \int_{\mathbb{D}} K(w) f(\varphi_\zeta(w)) \, dH(w)$$

where $dH$ is the Möbius invariant area measure were introduced under the name of Hausdorff–Berezin operators (see also [8]). This class of operators is different from our due to the fact that $\varphi_\zeta(s)$ is analytic in $s$, but not in $\zeta$ and therefore $K$ does not act in spaces of analytic functions, in contrast to $\mathcal{H}_{K,\mu}$ (see, e.g., the previous example).

It should be noted that different operators of Hausdorff type on spaces of holomorphic functions in the disc or the half-plane were considered in [9–17] but none of them is a special case of the Definition 1.

### 2 Boundedness of Hausdorff operators

The following lemma is crucial for the proofs of our main results.

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1 In the works on the Hausdorff means for the Fourier or power series the Hausdorff operators are defined not on functions but on the sequences of coefficients.
Lemma 1 [1, Lemma 2] Let \((X; m)\) be a measure space, \(\mathcal{F}(X)\) some Banach space of \(m\)-measurable functions on \(X\), \((\Omega, \mu)\) a \(\sigma\)-compact quasi-metric space with positive Radon measure \(\mu\), and \(F(w, z)\) a function on \(\Omega \times X\). Assume that

(a) the convergence of a sequence in norm in \(\mathcal{F}(X)\) yields the convergence of some subsequence to the same function for \(m\)-a.e. \(z \in X\);
(b) \(F(w, \cdot) \in \mathcal{F}(X)\) for \(\mu\)-a.e. \(w \in \Omega\);
(c) the map \(w \mapsto F(w, \cdot) : \Omega \to \mathcal{F}(X)\) is Bochner integrable with respect to \(\mu\).

Then for \(m\)-a.e. \(z \in X\) one has

\[
\left( B \int_{\Omega} F(w, \cdot) d\mu(w) \right)(z) = \int_{\Omega} F(w, z) d\mu(w).
\]

2.1 The Bloch Space

The Bloch space \(\mathcal{B}\) of \(D\) is defined to be the space of analytic functions \(f\) on \(D\) such that

\[
\|f\|_{\mathcal{B}} := \sup \left\{ \left( 1 - |z|^2 \right) |f'(z)| : z \in \mathbb{D} \right\} < +\infty.
\]

Then \(\| \cdot \|_{\mathcal{B}}\) is a complete semi-norm on \(\mathcal{B}\), and \(\mathcal{B}\) can be turned into a Banach space by introducing the norm

\[
\|f\|'_{\mathcal{B}} := \|f\|_{\mathcal{B}} + |f(0)|.
\]

It is important that \(\| \cdot \|_{\mathcal{B}}\) is Möbius invariant, that is,

\[
\|f \circ \varphi_w\|_{\mathcal{B}} = \|f\|_{\mathcal{B}}
\]

for all \(\varphi_w \in \text{Auto}_0(\mathbb{D})\) [18, p. 101].

Theorem 1 If \(K(w)\) and \(K(w) \log(1 - |w|)\) belong to \(L^1(\mu)\), then the operator \(\mathcal{H}_{K, \mu}\) is bounded on \(\mathcal{B}\) and

\[
\|\mathcal{H}_{K, \mu}\|_{\mathcal{B} \to \mathcal{B}} \leq \int_{\mathbb{D}} |K(w)| \left( 1 + \frac{1}{2} \log \frac{1 + |w|}{1 - |w|} \right) d\mu(w).
\]

Proof Note that the conditions of Lemma 1 hold for \(X = \Omega = \mathbb{D}, m = A\), where \(dA(z) = dx dy / \pi\) stands for the normalized area (Lebesgue) measure on \(\mathbb{D}\), \(\mathcal{F}(X) = \mathcal{B}\), and \(F(w, z) = K(w) f(\varphi_w(z))\). Indeed,

(a) if \(f_n \in \mathcal{B}\) and \(f_n \to 0\) strongly, then \(f_n(0) \to 0\) and \(f_n' \to 0\) uniformly on \(\{z \in \mathbb{D} : |z| < r\}\) for every \(r \in (0, 1)\) and therefore for all \(z \in \mathbb{D}\)

\[
f_n(z) = \int_0^z f_n'(t) dt + f_n(0) \to 0 \quad (n \to \infty);
\]
(b) since \( \|f \circ \varphi_w\|_{B} = \|f\|_{B} \), this is obvious;
(c) this follows from the estimate
\[
\|F(w, \cdot)\|_{B}' \leq |K(w)| \left(1 + \frac{1}{2} \log \frac{1 + |w|}{1 - |w|}\right)
\]
(see below).

So, by Lemma 1 for \( dA\)-a.e. \( z \in \mathbb{D} \)
\[
\mathcal{H}_{K, \mu} f(z) = \left((B) \int_{\mathbb{D}} K(w) f \circ \varphi_w d\mu(w)\right)(z),
\]
the Bochner integral for \( B \). The right-hand side of this equality is continuous. To show that the left-hand side of (1) is continuous, too, we choose an arbitrary point \( z_0 \in \mathbb{D} \) and a compact neighborhood \( U \subset \mathbb{D} \) of \( z_0 \). By [18, Theorem 5.5] we have
\[
|f(w)| \leq |f(w) - f(0)| + |f(0)| \leq \|f\|_{B} \beta(0, w) + |f(0)|,
\]
where \( \beta(z, w) \) stands for the Bergman distance. Since \( \beta \) is Möbius invariant (see, e.g., [18]), it follows that one can find such \( z_1 \in U \) that for all \( z \in U \)
\[
|f(\varphi_w(z))| \leq \|f\|_{B} \beta(0, \varphi_w(z)) + |f(w)|
\]
\[
= \|f\|_{B} \beta(\varphi_w(w), \varphi_w(z)) + |f(w)|
\]
\[
= \|f\|_{B} \beta(w, z) + |f(w)| \leq \|f\|_{B}(\beta(w, 0) + \beta(0, z)) + |f(w)|
\]
\[
\leq \|f\|_{B}(\beta(w, 0) + \beta(0, z) + \|f\|_{B} \beta(0, w) + |f(0)|
\]
\[
\leq \|f\|_{B} \left(\log \frac{1 + |w|}{1 - |w|} + \beta(0, z_1)\right) + |f(0)|.
\]
Thus, for fixed \( f \) if \( z \in U \) we get a \( \mu \) integrable majorant for the left-hand side of (1) of the form
\[
K(w) \left(C_1 \log \frac{1 + |w|}{1 - |w|} + C_2\right).
\]

Since both sides of the equality (1) are continuous, it is valid for all \( z \in \mathbb{D} \), i.e.,
\[
\mathcal{H}_{K, \mu} f = (B) \int_{\mathbb{D}} K(w) f \circ \varphi_w d\mu(w).
\]
Since \( \|f \circ \varphi_w\|_{B} = \|f\|_{B} \) [18, p. 101], and \( f \circ \varphi_w(0) = f(w) \), it follows that
\[
\|\mathcal{H}_{K, \mu} f\|_{B}' \leq \int_{\mathbb{D}} |K(w)| \|f \circ \varphi_w\|_{B}' d\mu(w)
\]
\[
= \|f\|_{B} \int_{\mathbb{D}} |K(w)| d\mu(w) + \int_{\mathbb{D}} |K(w)| |f(w)| d\mu(w).
\]
As was mentioned above,
\[ |f(w)| \leq \| f \|_B \beta(0, w) + |f(0)|. \]

Therefore
\[
\| \mathcal{H}_K \mu f \|_B' \leq \int_{\mathbb{D}} |K(w)| d\mu(w) \| f \|_B + \int_{\mathbb{D}} |K(w)| \beta(0, w) d\mu(w) \| f \|_B + \int_{\mathbb{D}} |K(w)| d\mu(w) \| f(0) \|
\]
\[
\leq \int_{\mathbb{D}} |K(w)|(1 + \beta(0, w)) d\mu(w) \| f \|_B.
\]

This completes the proof. \( \Box \)

### 2.2 Bergman Spaces

Let \( \alpha > -1 \) and \( dA_{\alpha}(z) := (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \). For \( p > 0 \) the Bergman spaces with standard weights are defined by

\[
L^p_{\alpha}(dA_{\alpha}) := H(\mathbb{D}) \cap L^p(dA_{\alpha})
\]

where \( H(\mathbb{D}) \) is the space of analytic functions in \( \mathbb{D} \) (see, e.g., [18, Chapter 4]).

In this subsection we shall consider the case \( d\mu = dA \), \( \| \cdot \|_{p, \alpha} \) denotes the norm in \( L^p_{\alpha}(dA_{\alpha}) \) induced from \( L^p(dA_{\alpha}) \).

**Theorem 2** Let \( p \geq 1 \) and the function \( K(w)/(1 - |w|)\frac{2+\alpha}{p} \) belong to \( L^1(dA) \). Then the operator \( \mathcal{H}_{K, A} \) is bounded on \( L^p_{\alpha}(dA_{\alpha}) \) and

\[
\| \mathcal{H}_{K, A} \|_{L^p_{\alpha} \to L^p_{\alpha}} \leq \int_{\mathbb{D}} |K(w)| \left( \frac{1 + |w|}{1 - |w|} \right)^{\frac{2+\alpha}{p}} dA(w).
\]

**Proof** According to [18, Proposition 4.3],

\[
\int_{\mathbb{D}} g \circ \varphi_w(z) dA_{\alpha}(z) = \int_{\mathbb{D}} g(z) \frac{(1 - |w|^2)^{2+\alpha}}{(1 - \overline{w}z)^{2(2+\alpha)}} dA_{\alpha}(z)
\]

for nonnegative \( g \). Putting here \( g = |f|^p \) where \( f \in L^p_{\alpha}(dA_{\alpha}) \) we get

\[
\| f \circ \varphi_w \|^p_{p, \alpha} = \int_{\mathbb{D}} |f \circ \varphi_w(z)|^p dA_{\alpha}(z) = \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |w|^2)^{2+\alpha}}{|1 - \overline{w}z|^{2(2+\alpha)}} dA_{\alpha}(z).
\]

Since \( |1 - \overline{w}z| \geq 1 - |w| \), this implies that

\[
\| f \circ \varphi_w \|^p_{p, \alpha} \leq \int_{\mathbb{D}} |f(z)|^p \frac{(1 - |w|^2)^{2+\alpha}}{(1 - |w|)^2(2+\alpha)} dA_{\alpha}(z) = \left( \frac{1 + |w|}{1 - |w|} \right)^{2+\alpha} \| f \|^p_{p, \alpha}. \tag{2}
\]
Note that all the conditions of Lemma 1 are satisfied for \((X, m) = (\mathbb{D}, dA_\alpha), (\Omega, \mu) = (\mathbb{D}, dA), \mathcal{F}(X) = L^p(\alpha),\) and \(F(w, z) = K(w) f(\phi_w(z))\) (indeed, (b) and (c) follow from the estimate (2); (a) is a consequence of a well known theorem of Riesz).

It follows in view of Lemma 1 that for \(dA_\alpha\) a.e. \(z \in \mathbb{D}\)

\[
\mathcal{H}_{K, \mu} f(z) = (B) \int_{\mathbb{D}} K(w) f \circ \phi_w d\mu(w) (z),
\]

the Bochner integral for \(L^p(\alpha)\). As in the proof of Theorem 1 to show that the equality (3) is valid for all \(z \in \mathbb{D}\) we shall prove the continuity of its left-hand side. To this end we shall prove that for every \(z_0 \in \mathbb{D}\) and for every compact neighborhood \(U \subset \mathbb{D}\) of \(z_0\) there is a \(dA\) integrable majorant for the integrand of the left-hand side of (3). First note that by [18, Proposition 4.13] there is a constant \(C > 0\) such that for \(f \in L^p(\alpha), z \in \mathbb{D}\)

\[
|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{2+\alpha}} \|f\|_{p, \alpha}. \tag{4}
\]

It follows that

\[
|f(\phi_w(z))| \leq \frac{C^{1/p}}{z(1 - |\phi_w(z)|^2)^{2+\alpha/p}} \|f\|_{p, \alpha}. \tag{5}
\]

On the other hand,

\[
1 - |\phi_w(z)|^2 = \frac{(1 - |w|^2)(1 - |z|^2)}{|1 - wz|^2} \geq \frac{(1 - |w|^2)(1 - |z|^2)}{(1 + |w|)^2} = \frac{(1 - |w|(1 - |z|^2)}{1 + |w|} \geq \frac{1}{2} \left(1 - |z|^2\right)(1 - |w|).
\]

Thus,

\[
|f(\phi_w(z))| \leq \frac{2^{2+\alpha} C^{1/p}}{(1 - |z|^2)^{2+\alpha/p}(1 - |w|)^{2+\alpha}} \|f\|_{p, \alpha}. \tag{6}
\]

This yields that for every compact neighborhood \(U \subset \mathbb{D}\) of \(z_0\) there is a constant \(C_U\) such that for all \(z \in U\)

\[
|K(w)||f(\phi_w(z))| \leq \frac{C_U}{(1 - |w|)^{2+\alpha}} |K(w)|,
\]

and the right-hand side here belongs to \(L^1(dA)\). Thus, the left-hand side of (3) is continuous, as well.
Then since $\| \cdot \|_{p,\alpha}$ is a norm, we have in view of (3) and (2) that

$$\| \mathcal{H}_K A f \|_{p,\alpha} \leq \int_D |K(w)| \| f \circ \varphi_w \|_{p,\alpha} dA(w)$$

$$\leq \int_D |K(w)| \left( \frac{1 + |w|}{1 - |w|} \right)^{\frac{2+\alpha}{p}} dA(w) \| f \|_{p,\alpha}.$$ 

This proves the theorem. 

\[ \square \]

### 2.3 Hardy Spaces

As is well known, for $0 < p < \infty$ the Hardy space $H^p = H^p(\mathbb{D})$ consists of analytic functions $f$ in the unit disc $\mathbb{D}$ such that

$$\| f \|_{H^p}^p := \sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$ 

Then $\| \cdot \|_{H^p}$ is a norm for $p \in [1, \infty)$. It is easy to verify also that this is a $p$-norm for $p \in (0, 1)$, in particular for $\| \cdot \|_{H^p}^p$ the triangle inequality holds. Every function $f(z) \in H^p(\mathbb{D})$ has boundary values $f(e^{i\theta}) \in L^p(\partial \mathbb{D})$ and the map $f(z) \mapsto f(e^{i\theta})$ is an isometrical isomorphism of $H^p(\mathbb{D})$ onto some closed subspace of $L^p(\partial \mathbb{D}, \frac{d\theta}{2\pi})$ (see, e.g., [18,19]).

Denote by $\tilde{\Lambda}_p$ the class of all holomorphic functions $g$ on $\mathbb{D}$ which are continuous on the closure of $\mathbb{D}$ and such that if $1/p \notin \{2, 3, \ldots \}$ and $n = [1/p]$ the derivative $g^{(n-1)}$ of the $2\pi$ periodic counterpart of the boundary function $g(e^{i\theta})$ belongs to the Lipschitz class $\Lambda_{1/p}$ on $\mathbb{R}$ (here $\{1/p\}$ stands for the fractional part of $1/p$) and if $1/p = n + 1$, then $g^{(n-1)} \in \Lambda_*$ where $\Lambda_*$ denotes the class of all continuous functions $\varphi$ on $\mathbb{R}$ such that there is a constant $A$ with the property

$$|\varphi(x + h) - 2\varphi(x) + \varphi(x - h)| < Ah$$

for all $x$ and for all $h > 0$. (The class $\Lambda_*$ is called Zygmund class).

**Theorem 3** Let the function $K(w)/(1 - |w|)^{1/p}$ belongs to $L^1(\mu)$.

1. If $1 \leq p < \infty$, then the operator $\mathcal{H}_{K,\mu}$ is bounded on $H^p(\mathbb{D})$ and

$$\| \mathcal{H}_{K,\mu} \|_{H^p \rightarrow H^p} \leq \int_D |K(w)| \left( \frac{1 + |w|}{1 - |w|} \right)^{1/p} d\mu(w).$$

2. Let $0 < p < 1$. The operator $\mathcal{H}_{K,\mu}$ is bounded on $H^p(\mathbb{D})$ if and only if for every $g \in \tilde{\Lambda}_p$

$$c_n = \int_D K(w) \left( \frac{g(0)}{w^n} + \text{res}_{z=w} \left( \frac{wz - 1}{z - w} \right)^n \frac{g(z)}{z} \right) d\mu(w), \quad n \in \mathbb{Z}_+ \quad (4)$$
is a sequence of Fourier coefficients of some function \( h \in \hat{\Lambda}_p \).

**Proof** 1) The inequality

\[
|f(z)| \leq 2^\frac{1}{p} \| f \|_{H^p}(1 - |z|)^{-\frac{1}{p}} \quad (f \in H^p(\mathbb{D}), \, z \in \mathbb{D})
\]

(see [19, p. 36]) shows that the condition (a) of Lemma 1 is satisfied for \((X, m) = (\mathbb{D}, dA)\) and \((\Omega, \mu) = (\mathbb{D}, \mu)\). Conditions (b) and (c) are the consequences of the inequality

\[
\| f \circ \varphi_w \|_{H^p} \leq \left( \frac{1 + |w|}{1 - |w|} \right)^{1/p} \| f \|_{H^p}.
\]

This inequality follows from the Littlewood’s subordination theorem (see, e.g., [18, Theorem 11.12]), but we shall give a simple direct proof. If we as usual identify the function \( f(z) \in H^p \) with its boundary value \( f(e^{i\theta}) \), then

\[
\| f \circ \varphi_w \|_{H^p}^p = \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi_w(e^{i\theta}))|^p d\theta = \frac{1}{2\pi i} \int_0^{2\pi} |f(\varphi_w(e^{i\theta}))|^p \frac{d e^{i\theta}}{e^{i\theta}}.
\]

If we put in the last integral \( e^{it} = \varphi_w(e^{i\theta}) \), then \( e^{i\theta} = \varphi_w(e^{it}) \), and \( d e^{i\theta} = \frac{|w|^2 - 1}{(1 - w e^{-it})^2} e^{it} dt \). Thus,

\[
\| f \circ \varphi_w \|_{H^p}^p = \frac{1 - |w|^2}{2\pi} \int_0^{2\pi} \frac{|f(e^{it})|^p dt}{(w - e^{-it})(w - e^{it})} = \left( 1 - |w|^2 \right) \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(e^{it})|^p dt}{1 + |w|^2 - 2Re(we^{-it})} \leq \frac{1 + |w|}{1 - |w|} \| f \|_{H^p}^p.
\]

So, by Lemma 1 for \( dA \) a.e. \( z \in \mathbb{D} \)

\[
\mathcal{H}_{K, \mu} f(z) = (B) \int_{\mathbb{D}} K(w) f \circ \varphi_w d\mu(w)(z),
\]

the Bochner integral for \( H^p(\mathbb{D}) \). As in the proof of Theorem 1 to show that this equality holds for all \( z \in \mathbb{D} \) it suffices to prove the continuity of its left-hand side. To this end first note that by [18, Theorem 9.1]

\[
|f(\varphi_w(z))| \leq \frac{\| f \|_{H^p}}{(1 - |\varphi_w(z)|^2)^{1/p}}.
\]

On the other hand (see the proof of Theorem 2),

\[
1 - |\varphi_w(z)|^2 \geq \frac{1}{2} \left( 1 - |z|^2 \right) (1 - |w|).
\]
Thus,
\[
|f(\varphi_w(z))| \leq \frac{2^{1/p} \|f\|_{H^p}}{(1 - |z|^2)^{1/p}(1 - |w|)^{1/p}}.
\]

It follows, that for every compact neighborhood \(U \subset \mathbb{D}\)
\[
|K(w)||f(\varphi_w(z))| \leq C_U \frac{|K(w)|}{(1 - |w|)^{1/p}}
\]
is a \(\mu\) integrable majorant for the left-hand side. This proves the desired continuity.

Since \(\|\cdot\|_{H^p}\) is a norm, we have in view of the preceding inequality that
\[
\|\mathcal{H}_{K,\mu} f\|_{H^p} \leq \int_{\mathbb{D}} |K(w)||f \circ \varphi_w\|_{H^p} d\mu(w)
\]
\[
\leq \int_{\mathbb{D}} |K(w)| \left(\frac{1 + |w|}{1 - |w|}\right)^{1/p} d\mu(w) \|f\|_{H^p}.
\]

This completes the proof of the first statement.

2) Let \(0 < p < 1\). Note that \(K \in L^1(\mu)\) Since \(H^p(\mathbb{D})\) is an \(F\)-space and the dual \(H^p(\mathbb{D})^*\) separates the points of \(H^p(\mathbb{D})\) [19, p. 118], Theorem II.2.7 in [20] shows that \(\mathcal{H}_{K,\mu}\) is bounded on \(H^p(\mathbb{D})\) if and only if a linear functional \(l \circ \mathcal{H}_{K,\mu}\) belongs to \(H^p(\mathbb{D})^*\) for each \(l \in H^p(\mathbb{D})^*\).

We shall employ the general form of a linear functional on \(H^p(\mathbb{D})\) [19, Theorem 7.5]. By this theorem every bounded linear functional on \(H^p(\mathbb{D})\) has a unique representation of the form
\[
l_g(f) = \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(e^{-i\theta}) \, d\theta,
\]
where \(g \in \tilde{\mathcal{A}}_p\) (and vice versa).

Thus, \(\mathcal{H}_{K,\mu}\) is bounded on \(H^p(\mathbb{D})\) if and only if for every \(g \in \tilde{\mathcal{A}}_p\) there is such \(h \in \tilde{\mathcal{A}}_p\) that
\[
l_g \circ \mathcal{H}_{K,\mu} = l_h.
\]

In other words, for every \(f \in H^p(\mathbb{D})\)
\[
\lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{D}} K(w) f(\varphi_w(re^{i\theta})) d\mu(w)\right) g(e^{-i\theta}) \, d\theta
\]
\[
= \lim_{r \to 1^-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) h(e^{-i\theta}) \, d\theta.
\]
Let $H_{K,\mu}$ is bounded on $H^p(D)$. Putting $f(z) = z^n$, $n \in \mathbb{Z}_+$ in (6) we get in view of the Fubini theorem that

$$
\lim_{r \to 1-0} \int_D K(w) \left( \frac{1}{2\pi} \int_0^{2\pi} \varphi_w(re^{i\theta})^n g(e^{-i\theta}) d\theta \right) d\mu(w)
= \lim_{r \to 1-0} \frac{1}{2\pi} \int_0^{2\pi} \left( re^{i\theta} \right)^n h(e^{-i\theta}) d\theta,
$$

(7)
or by the Lebesgue theorem ($\varphi_w$ and $g$ are bounded)

$$
\int_D K(w) \frac{1}{2\pi} \int_0^{2\pi} \left( \varphi_w(e^{i\theta}) \right)^n g(e^{-i\theta}) d\theta d\mu(w)
= \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} h(e^{-i\theta}) d\theta, \quad n \in \mathbb{Z}_+.
$$

(8)

Note that by the Cauchy theorem

$$
\frac{1}{2\pi} \int_0^{2\pi} \left( \varphi_w(e^{i\theta}) \right)^n g(e^{-i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{we^{it} - 1}{e^{it} - \overline{w}} \right)^n g(e^{it}) dt
= \frac{1}{2\pi i} \int_{|z|=1} \left( \frac{wz - 1}{z - \overline{w}} \right)^n \frac{g(z)}{z} dz = \frac{g(0)}{\overline{w}^n}
+ \operatorname{res}_{z=\overline{w}} \left( \frac{wz - 1}{z - \overline{w}} \right)^n \frac{g(z)}{z}.
$$

(9)

Therefore (8) is equivalent to

$$
\int_D K(w) \left( \frac{g(0)}{\overline{w}^n} + \operatorname{res}_{z=\overline{w}} \left( \frac{wz - 1}{z - \overline{w}} \right)^n \frac{g(z)}{z} \right) d\mu(w)
= \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} h(e^{i\theta}) d\theta, \quad n \in \mathbb{Z}_+.
$$

(10)

and the necessity follows.

To prove the sufficiency, for every $g \in \tilde{\Lambda}_p$ let $h \in \tilde{\Lambda}_p$ be such that for its Fourier coefficients $c_n$ (4) holds. Then formula (10) is valid. In view of (9) this implies (8). In turn, (8) implies (7) and then for every algebraic polynomial $q_n$ we get

$$
\int_D K(w) \left( \lim_{r \to 1-0} \frac{1}{2\pi} \int_0^{2\pi} q_n (\varphi_w(re^{i\theta})) g(e^{-i\theta}) d\theta \right) d\mu(w)
= \lim_{r \to 1-0} \frac{1}{2\pi} \int_0^{2\pi} q_n (re^{i\theta}) h(e^{-i\theta}) d\theta.
$$

In other words,

$$
\int_D K(w) (q_n \circ \varphi_w) d\mu(w) = l_h(q_n).
$$

(11)
Taking into account that polynomials are dense in $H^p$ (see, e.g., [18, Corollary 9.5]), for every $f \in H^p$ one can choose a sequence $q_n$ of polynomials that converges to $f$ in $H^p$. Since

$$l_H(q_n) \to l_H(f), \quad l_g(q_n \circ \varphi_w) \to l_g(f \circ \varphi_w) \quad \text{as} \quad n \to \infty,$$

and by Lemma 1

$$|l_g(q_n \circ \varphi_w)| \leq \|l_g\| \|q_n\|_p \left(\frac{1 + |w|}{1 - |w|}\right)^{1/p} \leq \text{const} \left(\frac{1 + |w|}{1 - |w|}\right)^{1/p},$$

formula (11) implies in view of the Lebesgue theorem that the property (5) is valid. This proves the sufficiency.

Putting $g(z) = z$ in the previous theorem we get the following

**Corollary 1** Let $0 < p < 1$ and let the function $K(w)/(1 - |w|)^{1/p}$ belongs to $L^1(\mu)$. If the operator $H_{K,\mu}$ is bounded on $H^p(D)$, then

$$c_n = n \int_D K(w) w^{n-1} \left(|w|^2 - 1\right) d\mu(w), \quad n \in \mathbb{Z}_+$$

is a sequence of Fourier coefficients of some function $h \in \tilde{\Lambda}_p$.

## 3 Approximation of the Identity by Hausdorff Operators

Although the class of operators we are considering differs from the class considered in [7], there are analogs of Theorem 14 from [7] that was devoted to approximation of the identity by Hausdorff–Berezin operators. Note that the question of approximation for Hausdorff operators was posed for the first time in [21].

In the following for the function $f$ on $D$ we put $f^\triangle(z) = f(-z)$ and for $0 < \varepsilon < 1$ consider operators of the form

$$\mathcal{H}_\varepsilon f(z) := \int_D K(w) (f \circ \varphi_{\varepsilon w})^\triangle(z) d\mu(w). \quad (12)$$

As in [7, Section 5] for the goals of approximation it is naturally to assume that

$$K \in L^1(d\mu), \quad \text{and} \quad \int_D K d\mu = 1. \quad (13)$$

**Theorem 4** Let the conditions of Theorem 2 be satisfied. Under the assumption (13) with $dA_\alpha$ instead of $d\mu$ the operators (12) with $dA_\alpha$ instead of $d\mu$ are approximations of the identity in $L^p(dA_\alpha)$, namely,

$$\lim_{\varepsilon \to 0} \|\mathcal{H}_\varepsilon f - f\|_{p,A_\alpha} = 0. \quad (14)$$
Proof As in the proof of Theorem 2 we have

\[ H_\varepsilon f = (B) \int_\mathbb{D} K(w) (f \circ \varphi_{\varepsilon w})^\Delta dA_\alpha(w), \tag{15} \]

the Bochner integral in \(L^p(dA_\alpha)\). It follows in view of (13) that

\[ \|H_\varepsilon f - f\|_{p,\alpha} \leq \int_\mathbb{D} |K(w)|| (f \circ \varphi_{\varepsilon w})^\Delta - f|_{p,\alpha} dA_\alpha(w). \tag{16} \]

We shall show that the operators \(H_\varepsilon\) are uniformly bounded when \(\varepsilon \in (0, 1/2)\), specifically,

\[ \|H_\varepsilon\| \leq 2^{2(2+\alpha)/p} \int_\mathbb{D} |K(w)| dA_\alpha(w). \tag{17} \]

Indeed, using [18, Proposition 4.3] with \(a = \varepsilon w\) we have

\[ \| (f \circ \varphi_{\varepsilon w})^\Delta - f\|_{p,\alpha} = \int_\mathbb{D} |f(z)|^p \frac{(1 - |\varepsilon w|^2)^{(2+\alpha)}}{|1 - \varepsilon wz|^{(2+\alpha)}} dA_\alpha(z) \leq \int_\mathbb{D} |f(z)|^p \frac{1}{(1 - \varepsilon)^{(2+\alpha)}} dA_\alpha(z) \leq 2^{2(2+\alpha)} \|f\|_{p,\alpha}. \tag{18} \]

Thus, (17) follows from (15) and (18).

In view of (17) to prove (14) it remains to check this equality on a dense subset of \(L^p(dA_\alpha)\). We shall use (16) to check (14) on the subset of bounded functions. Let \(|f(z)| \leq C\). Then

\[ \| (f \circ \varphi_{\varepsilon w})^\Delta - f\|_{p,\alpha} = \int_\mathbb{D} |(f \circ \varphi_{\varepsilon w})(-z) - f(z)|^p dA_\alpha(z) \to 0 \tag{19} \]

as \(\varepsilon \to 0\) by the Lebesgue’s dominated convergence theorem (\(|(f \circ \varphi_{\varepsilon w})(-z) - f(z)| \leq 2C\)). But formula (18) implies that

\[ \| (f \circ \varphi_{\varepsilon w})^\Delta - f\|_{p,\alpha} \leq \left(2^{2(2+\alpha)/p} + 1\right) \|f\|_{p,\alpha}. \]

It follows (again by the Lebesgue’s dominated convergence theorem) that the right-hand side of (16) vanishes as \(\varepsilon \to 0\). This completes the proof. \(\Box\)

Theorem 5 Let the conditions of Theorem 3 be satisfied. Under the assumption (13) the operators (12) are approximations of the identity in \(H^p(\mathbb{D})\), namely,

\[ \lim_{\varepsilon \to 0} \|H_\varepsilon f - f\|_{H^p} = 0. \tag{20} \]
The proof is similar to the proof of Theorem 4. Indeed, as in the proof of Theorem 3 we have

$$\mathcal{H}_\epsilon f = (B) \int_D K(w) (f \circ \varphi_{\epsilon w})^\Delta d\mu(w),$$  \hspace{1cm} (21)

the Bochner integral in $H^p(\mathbb{D})$. It follows in view of (13) that

$$\|\mathcal{H}_\epsilon f - f\|_{H^p} \leq \int_D |K(w)|| (f \circ \varphi_{\epsilon w})^\Delta - f\|_{H^p} d\mu(w).$$  \hspace{1cm} (22)

We shall show that the operators $\mathcal{H}_\epsilon$ are uniformly bounded when $\epsilon \in (0, 1/3)$, specifically,

$$\|\mathcal{H}_\epsilon\| \leq 2^{1/p} \int_D |K(w)| d\mu(w).$$  \hspace{1cm} (23)

Indeed, in the proof of Theorem 3 it was shown that

$$\| (f \circ \varphi_{\epsilon w})^\Delta \|_{H^p} \leq \left(\frac{1 + \epsilon |w|}{1 - \epsilon |w|}\right)^{1/p} \| f \|_{H^p}.$$

Thus,

$$\| (f \circ \varphi_{\epsilon w})^\Delta \|_{H^p} \leq 2^{1/p} \| f \|_{H^p}$$  \hspace{1cm} (24)

as $\epsilon \in (0, 1/3)$, and (23) follows from (21) and (24).

In view of (23) to prove (20) it remains to check this equality on a dense subset of $H^P(\mathbb{D})$. We shall use (22) to check (20) on the subset of polynomials. First note that since $|z_1^k - z_2^k| \leq (k + 1)|z_1 - z_2|$ for $z_1, z_2 \in \mathbb{D}, k \in \mathbb{N}$, we have

$$\left|\left(\frac{\epsilon w + z}{1 + \epsilon \overline{w} z}\right)^k - z^k\right| \leq (k + 1) \left|\frac{\epsilon w - \epsilon \overline{w} z^2}{1 + \epsilon \overline{w} z}\right| \leq \frac{2(k + 1)\epsilon}{1 - \epsilon},$$  \hspace{1cm} (25)

and therefore

$$\left\|\left(\frac{\epsilon w + z}{1 + \epsilon \overline{w} z}\right)^k - z^k\right\|_{H^p} \leq \frac{2(k + 1)\epsilon}{1 - \epsilon}.$$

It follows that

$$\| (f \circ \varphi_{\epsilon w})^\Delta - f\|_{H^p} \to 0$$
as $\varepsilon \to 0$ for every polynomial $f$. But formula (24) implies that

$$\| (f \circ \varphi_{\varepsilon w})^\Delta - f \|_{H^p} \leq \left(2^{\frac{1}{p}} + 1\right) \| f \|_{H^p}.$$  

Then the Lebesgue’s dominated convergence theorem shows that the right-hand side of (21) vanishes as $\varepsilon \to 0$. This completes the proof.

For our last theorem recall that the little Bloch space $B_0$ is the closed subspace of $B$ consisting of functions $f$ with

$$\lim_{z \to 1^{-}} \left(1 - |z|^2\right) f'(z) = 0.$$  

The space $B_0$ is Möbius invariant and it is the closure in $B$ of the set of polynomials (see e.g., [18, Corollary 5.10]).

**Theorem 6** Let the conditions of Theorem 1 be satisfied. Under the assumption (13) the operators (12) are approximations of the identity in $B_0$, namely, for all $f \in B_0$

$$\lim_{\varepsilon \to 0} \| \mathcal{H}_\varepsilon f - f \|_B = 0. \quad (26)$$

The proof is also similar to the proof of Theorem 4. First we shall show that the family $\mathcal{H}_\varepsilon$ is uniformly bounded in $B$ for $\varepsilon \in (0, 1/2)$. We have for such $\varepsilon$

$$\| f \circ \varphi_{\varepsilon w})^\Delta \|_B = \| f \circ \varphi_{\varepsilon w}) \|_B + |f(\varepsilon w)| \leq \| f \|_B + \| f \|_B \beta(0, \varepsilon w) + |f(0)|$$

$$\leq \| f \|_B (1 + \beta(0, \varepsilon w)) < \| f \|_B ' \left(1 + \frac{1}{2} \log 3\right).$$

Now formula (12) implies that

$$\| \mathcal{H}_\varepsilon f \|_B' \leq \int_{\mathcal{D}} |K(w)| \| (f \circ \varphi_{\varepsilon w}) \|_B d\mu(w) < \left(1 + \frac{1}{2} \log 3\right) \int_{\mathcal{D}} |K(w)| d\mu(w) \| f \|_B'.$$

So, the family $(\mathcal{H}_\varepsilon)_{\varepsilon \in (0, 1/2)}$ is uniformly bounded with respect to the operator norm in $L(B)$. Thus, it is sufficient to verify (26) on the subset of polynomials which is dense in $B_0$. To this end note that as in the proof of Theorem 5

$$\| \mathcal{H}_\varepsilon f - f \|_B \leq \int_{\mathcal{D}} |K(w)| \| (f \circ \varphi_{\varepsilon w})^\Delta - f \|_B d\mu(w).$$

Let $f(z) = z^k$ ($k \in \mathbb{Z}_+$). Then

$$\| (f \circ \varphi_{\varepsilon w})^\Delta (z) - f(z) \|_B = \| \varphi_{\varepsilon w}(z)^k - z^k \|_B + |\varepsilon w|^k$$

$$< \sup_{z \in \mathcal{D}} \left| (\varphi_{\varepsilon w}(z)^k - z^k) \right| + \varepsilon^k. \quad (28)$$
Formula (25) shows that $\varphi_{\varepsilon w}(z)^k - z^k \to 0$ uniformly in $w, z \in \overline{D}$ as $\varepsilon \to 0$. So, the right-hand side of (28) for every $w \in D$ tends to zero as $\varepsilon \to 0$ by the Weierstrass theorem. Straightforward calculation shows also, that this function is bounded in $w \in D$ and $\varepsilon \in (0, 1/2)$. Application of the Lebesgue dominated convergence theorem to the right-hand side of the formula (27) completes the proof.

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