CELLULAR ALGEBRAS ARISING FROM
HECKE ALGEBRAS OF TYPE $H_n$

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Abstract. We study a finite-dimensional quotient of the Hecke algebra of type $H_n$
for general $n$, using a calculus of diagrams. This provides a basis of monomials in a
certain set of generators. Using this, we prove a conjecture of C.K. Fan about the
semisimplicity of the quotient algebra. We also discuss the cellular structure of the
algebra, with certain restrictions on the ground ring.

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0. Introduction

There has been much recent interest in the Temperley–Lieb algebra and its
various generalisations. Graham [4] in his thesis studied a certain quotient, which
we will call $TL(X)$, of a Hecke algebra $\mathcal{H}(X)$ associated to a Dynkin diagram $X$.
In the case where $X$ is a Dynkin diagram of type $A$, this quotient was considered
by Jones [8], who pointed out that it is nothing other than the Temperley–Lieb
algebra, which first appeared in [12]. The Temperley–Lieb algebra has applications
in several areas of mathematics, including statistical mechanics and knot theory.

A remarkable feature of the algebras $TL(X)$ is that they can be finite dimen-
sional, even when $\mathcal{H}(X)$ is infinite dimensional. Graham [4] classified the finite
dimensional algebras $TL(X)$ into seven infinite families: $A, B, D, E, F, H$ and $I$.
(Contrast this to the classification of Hecke algebras associated to irreducible Cox-
eter systems, in which there are only finitely many algebras of types $E, F$ and
$H$.)

This paper is concerned with the infinite family of type $H$, in which case the
Hecke algebra $\mathcal{H}(H_n)$ is finite dimensional only for $n \leq 4$. The algebra $TL(H_n)$
was mentioned briefly by Fan in [1, §7.3], where it was conjectured that $TL(H_n)$
is generically semisimple. The dimensions of the generically irreducible modules
are also conjectured. In the course of the paper, we will prove these conjectures.

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Note that semisimplicity is obvious in the cases where $\mathcal{H}(X)$ is finite dimensional, because in this case $\mathcal{H}(X)$ is itself generically semisimple, but this argument fails if $\mathcal{H}(X)$ is infinite dimensional.

Our approach is first to realise $TL(H_n)$ as an algebra of diagrams arising from the category of “decorated tangles” which was introduced by the author in [6]. Diagram calculi have already been developed for algebras of some of the other types: for $TL(A_n)$ it is well known (see [13] or [5, §6]), types $B_n$ and $D_n$ were done in [6], and the infinite-dimensional “affine” Temperley–Lieb algebra $TL(\hat{A}_n)$ was tackled in [2]. This is interesting to do in its own right, since many natural questions about the algebras (such as the determination of the cells, dimensions and structure constants) have simple formulations in terms of the combinatorics of the associated diagrams. We will show that the diagrams may be adapted into a datum for a cellular algebra (in the sense of [5]), provided that the polynomial $x^2 - x - 1$ splits into distinct linear factors over the ground ring. Since the algebra $TL(H_2)$ is a $q$-analogue of a 9-dimensional quotient of the group algebra of the dihedral group of order 10, it seems that such a hypothesis cannot be usefully weakened.

The algebras $TL(X)$ of types $A, B, D, E$ and $F$ each have a basis consisting of monomials in the obvious set of algebra generators (which correspond to the Coxeter generators), and the structure constants with respect to this basis are positive in a natural sense. Furthermore, the product of two monomials is a scalar multiple of another monomial. In type $H$, the obvious basis of monomials does not have the positivity property, and it is not true that the product of two monomials is a scalar multiple of one other. This means that Fan’s techniques from [1] are unsuitable for analysing the algebra $TL(H_n)$.

In this paper, we overcome this problem by working with the basis of diagrams, which has much more convenient properties (e.g. positivity of structure constants and compatibility with cellular algebras). This basis is not obvious from the description of $TL(H_n)$ via generators and relations, but is very natural from the viewpoint of decorated tangles. We also show how the new basis elements can be expressed as monomials in a slightly larger set of algebra generators.

1. Preliminaries

1.1 Coxeter groups of type $H_n$.

Let $n \in \mathbb{N}$ be at least 2. The Coxeter group of type $H_n$ corresponds to the Coxeter graph shown in Figure 1.

**Figure 1.** Coxeter graph of type $H_n$

> \begin{center}
> \begin{tikzpicture}
>     \node (1) at (0,0) [circle,fill,inner sep=2pt]{1};
>     \node (2) at (1,0) [circle,fill,inner sep=2pt]{2};
>     \node (3) at (2,0) [circle,fill,inner sep=2pt]{3};
>     \node (4) at (3,0) [circle,fill,inner sep=2pt]{4};
>     \node (5) at (4,0) [circle,fill,inner sep=2pt]{5};
>     \node (6) at (5,0) [circle,fill,inner sep=2pt]{n-1};
>     \node (7) at (6,0) [circle,fill,inner sep=2pt]{n};
>     \draw (1) -- (2);
>     \draw (2) -- (3);
>     \draw (3) -- (4);
>     \draw (4) -- (5);
>     \draw[dashed] (5) -- (6);
>     \draw (6) -- (7);
> \end{tikzpicture}
> \end{center}

**Definition 1.1.1.** The Coxeter group $W(H_n)$ is given by generating involutions \( \{s_i : i \leq n\} \) and defining relations

\[
\begin{align*}
    s_is_j &= s_js_i \quad \text{if } |i - j| > 1, \\
    s_is_js_i &= s_js_is_j \quad \text{if } |i - j| = 1 \text{ and } \{i, j\} \neq \{1, 2\},
\end{align*}
\]

\(s_{1281281281} = s_{281281281} \).

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Remark 1.1.2. The group $W(H_2)$ is isomorphic to the dihedral group of order 10 (i.e. $W(I_2)$), but it will be convenient to regard it as a Coxeter group of type $H$ in some of our proofs.

As explained in [7, §2], the groups $W(H_n)$ are finite for $n = 2, 3, 4$, where they have orders 10, 120 and 14400 respectively. These groups occur as the full symmetry groups of Platonic solids with pentagonal faces: $H_2$ corresponds to the pentagon, $H_3$ to the dodecagon and $H_4$ to a regular 120-sided solid in 4 dimensions. For $n > 4$, the group $W(H_n)$ is infinite, which is reminiscent of the fact that there is no analogue of the dodecagon in higher dimensions, the only Platonic solids being generalized tetrahedra, cubes and octahedra.

1.2 Hecke algebras of type $H_n$.

We now introduce the Hecke algebra and its quotient $TL(H_n)$.

Definition 1.2.1. The Hecke algebra $\mathcal{H}(H_n)$ is defined over the ring

$$\mathcal{A} := \mathbb{Z}[v, v^{-1}],$$

where $v = q^{1/2}$. It has a free $\mathcal{A}$-basis $\{T_w : w \in W(H_n)\}$, and the multiplication is defined by the rules

$$T_s T_w := \begin{cases} T_{sw} & \text{if } \ell(sw) > \ell(w), \\ qT_{sw} + (q - 1)T_w & \text{otherwise.} \end{cases}$$

Here, $\ell(w)$ is the length of $w$, i.e. the length of a shortest word in the $s_i$ which is equal to $w$.

Following Fan [1, §7.3] and Graham [4], we make the following definition.

Definition 1.2.2. Let $n \in \mathbb{N} \geq 2$. We define the associative, unital algebra $TL(H_n)$ over $\mathcal{A}$ via generators $E_1, E_2, \ldots E_n$ and relations

- $E_i^2 = [2]E_i$,
- $E_i E_j = E_j E_i$ if $|i - j| > 1$,
- $E_i E_j E_i = E_i$ if $|i - j| = 1$ and $i, j > 1$,
- $E_i E_j E_i E_j E_i = 3E_i E_j E_i - E_i$ if $\{i, j\} = \{1, 2\}$.

Here, $[2]$ denotes the Laurent polynomial $v + v^{-1}$.

Remark 1.2.3. The algebra $TL(H_n)$ is a quotient of $\mathcal{H}(H_n)$ which corresponds to the Coxeter graph in Figure 1. The quotient map takes the Kazhdan–Lusztig basis element $C'_s = v^{-1}T_e + v^{-1}T_s$, where $e$ is the identity and $s$ is of length 1, to $E_s$.

Later, we shall want to replace the base ring $\mathcal{A}$ with a field, but we will not be concerned with trying to generalise the results to characteristics 2, 3 or 5.

It is convenient for later purposes to define the following elements of $TL(H_n)$.

Definition 1.2.4. We define

$$\alpha := E_1 E_2 - 1,$$
$$\beta := E_2 E_1 - 1,$$
$$\varepsilon := E_1 E_2 E_1 - 2E_1$$
and
$$\zeta := E_2 E_1 E_2 - 2E_2.$$
Remark 1.2.5. Notice that we can rephrase the non-monomial relations in Definition 1.2.2 as the monomial relations $\varepsilon \beta = E_1$ and $\zeta \alpha = E_2$.

2. The diagram algebra $\Delta_n$

We define a calculus of diagrams which will be seen in §3 to describe the generalized Temperley–Lieb algebra of type $H$. A convenient way to explain this is via the category of "decorated tangles" which was introduced by the author in [6].

2.1 The category of decorated tangles.

Following [3], we define a tangle as follows.

Definition 2.1.1. A tangle is a portion of a knot diagram contained in a rectangle. The tangle is incident with the boundary of the rectangle only on the north and south faces, where it intersects transversely. The intersections in the north (respectively, south) face are numbered consecutively starting with node number 1 at the western (i.e. the leftmost) end.

Two tangles are equal if there exists an isotopy of the plane carrying one to the other such that the corresponding faces of the rectangle are preserved setwise.

We call the edges of the rectangular frame "faces" to avoid confusion with the "edges" which are the arcs of the tangle.

We extend the notion of a tangle so that each arc of the tangle may be assigned a nonnegative integer. (This is similar to the notion of "coloured" tangles in [3].) If an arc is assigned the value $r$, we represent this pictorially by decorating the arc with $r$ blobs. We also require some further restrictions, as explained in the following definition.

Definition 2.1.2. A decorated tangle is a crossing-free tangle in which each arc is assigned a nonnegative integer. Any arc not exposed to the west face of the rectangular frame must be assigned the integer 0.

Remark 2.1.3. This means that any decorated tangle consists only of loops and edges, none of which intersect each other.

Example 2.1.4. Figure 2 shows a typical example of a decorated tangle. We will tend to emphasise the intersections of the tangle with the frame rather than the frame itself, which is why each node (i.e. intersection point with the frame) is denoted by a disc. In this case, the only edges or loops exposed to the west wall are the three which already carry decorations.

Figure 2. A decorated tangle

We now define a category based on the set of decorated tangles, as follows.

Definition 2.1.5. The category of decorated tangles, $\mathcal{D}_T$, has as its objects the natural numbers (not including zero). The morphisms from $n$ to $m$ are the decorated tangles with $n$ nodes in the north face and $m$ in the south. The source of a
morphism is the number of points in the north face of the bounding rectangle, and the target is the number of points in the south face. Composition of morphisms works by concatenation of the tangles, matching the relevant south and north faces together.

Remark 2.1.6. Note that for there to be any morphisms from \( n \) to \( m \), it is necessary that \( n + m \) be even. Also notice that the asymmetric properties of the west face of the rectangle mean that we cannot introduce the tensor product of two morphisms by the lateral juxtaposition of diagrams as in [3].

The category-theoretic definition allows us to define an algebra of decorated tangles, as follows.

**Definition 2.1.7.** Let \( R \) be a commutative ring and let \( n \) be a positive integer. Then the \( R \)-algebra \( DT_n \) has as a free \( R \)-basis the morphisms from \( n \) to \( n \), where the multiplication is given by the composition in \( DT \).

**Definition 2.1.8.** The edges in a tangle \( T \) which connect nodes (i.e. not the loops) may be classified into two kinds: propagating edges, which link a node in the north face with a node in the south face, and non-propagating edges, which link two nodes in the north face or two nodes in the south face.

### 2.2 \( H \)-admissible diagrams.

We introduce the concept of an \( H \)-admissible diagram, which plays a key rôle in describing the diagram calculus relevant for \( TL(H_{n-1}) \).

**Definition 2.2.1.** An \( H \)-admissible diagram with \( n \) strands is an element of \( DT_n \) with no loops which satisfies the following conditions.

(i) No edge may be decorated if all the edges are propagating.

(ii) If there are non-propagating edges in the diagram, then either there is a decorated edge in the north face connecting nodes 1 and 2, or there is a non-decorated edge in the north face connecting nodes \( i \) and \( i + 1 \) for \( i > 1 \). A similar condition holds for the south face.

(iii) Each edge carries at most one decoration.

An example of an \( H \)-admissible diagram for \( n = 6 \) is shown in Figure 3.

**Figure 3.** An \( H \)-admissible diagram

The point of part (ii) in Definition 2.2.1 excludes situations like the one where Figure 4 appears as the top half of an element of \( DT_n \).

**Figure 4.** The north face of a diagram excluded by Definition 2.2.1 (ii)
Definition 2.2.2. The algebra $\Delta_n$ (over a commutative ring with identity) has as a basis the $H$-admissible diagrams with $n$ strands and multiplication induced from that of $\mathbb{D}T_n$ subject to the relations shown in Figure 5.

Remark 2.2.3. The meaning of the first relation in Figure 5 is that any undecorated loop can be removed and the resulting tangle multiplied by $\delta$. The second relation means that any tangle containing a loop with one decoration is equivalent to $0 \in \Delta_n$. The third relation means that any tangle $T$ containing an edge or loop $\varepsilon$ with $r$ decorations $r > 1$ is equivalent to the sum of two other tangles $T'$ and $T''$ which are the same as $T$ except that the edge or loop corresponding to $\varepsilon$ carries $r - 1$ (respectively, $r - 2$) decorations.

The second rule can be modified so that its removal corresponds to multiplication by a second parameter, $\delta'$. This would eventually lead to a two-parameter version of $TL(H_{n-1})$, but we do not pursue this here.

Lemma 2.2.4. The relations in Figure 5 allow the product of two elements of $\Delta_n$ to be expressed unambiguously as a linear combination of basis elements. This makes $\Delta_n$ into an associative algebra.

Proof. We observe that the product of two $H$-admissible diagrams can be expressed as a linear combination of others by using the reduction rules given.

A case by case check shows that the order in which the relations are applied is immaterial and that the end result can therefore be expressed unambiguously in terms of the basis of $H$-admissible diagrams.

Using these observations, associativity is inherited from the associativity of $\mathbb{D}T_n$, by consideration of the concatenation of three tangles $TT'T''$. \hfill \Box

3. Realisation of $TL(H_n)$ as an algebra of diagrams

3.1 Representation of $TL(H_n)$ by diagrams.

One of our main aims will be to show that $\Delta_{n+1}$ and $TL(H_n)$ are isomorphic. To do this, we show how to represent $TL(H_n)$ using the $H$-admissible diagrams.

Definition 3.1.1. The $H$-admissible diagram $U_i$, where $1 \leq i \leq n$, is the diagram all of whose edges are propagating and undecorated, except for those attached to
nodes $i$ and $i + 1$ in the north row, and nodes $i$ and $i + 1$ in the south row. These four nodes are connected in the pairs given, using decorated edges if $i = 1$, and using undecorated edges if $i > 1$.

Examples. When $n = 6$, the elements $U_1$ and $U_2$ are as shown in Figures 6 and 7.

**Figure 6.** The diagram $U_2$ for $n = 6$

![Diagram U2](image)

**Figure 7.** The diagram $U_1$ for $n = 6$

![Diagram U1](image)

From now on, we take the base ring for $\Delta_{n+1}$ to be $\mathcal{A}$, meaning that the parameter $\delta$ is $[2]$. More general results may be found by tensoring over a suitable ring.

**Proposition 3.1.2.** There is a homomorphism of $\mathcal{A}$-algebras from $TL(H_n)$ to $\Delta_{n+1}$ which takes $E_i$ to $U_i$ for each $i$.

**Proof.** This is simply a matter of checking that all the relations in Definition 1.2.2 hold, which presents no problems. □

### 3.2 Algebra generators for $\Delta_{n+1}$

In order to prove that $\rho$ is an isomorphism, we will first show that $\Delta_{n+1}$ is generated as an $\mathcal{A}$-algebra (with identity) by the elements $U_i$.

During the course of the proofs, it helps to understand the case $n = 2$, which was the motivation for Definition 1.2.4.

**Lemma 3.2.1.** The map $\rho$ is an isomorphism for $n = 2$. The basis of $H$-admissible diagrams consists of the images of the 9 elements

$$1, E_1, E_2, E_1E_2, E_2E_1, E_1\beta, E_2\alpha, E_1\zeta, E_2\varepsilon = \zeta E_1.$$}

Thus, $\Delta_3$ is generated as an $\mathcal{A}$-algebra with $1$ by $U_1$ and $U_2$.

**Proof.** This is another routine exercise using the diagram multiplication, which is instructive to carry out. □

To deal with the case for general $n$, it is convenient to introduce a number of “moves”, in which a diagram element is multiplied (on the left or on the right) by a monomial in the generators

$$G_n := \{1, U_1, \ldots, U_n, \rho(\alpha), \rho(\beta), \rho(\zeta)\}$$
to form another diagram element. It will eventually turn out that any \( H \)-admissible
diagram may be obtained as a suitable word in the generators \( G_n \).

In the next five lemmas, \( D \) is an \( H \)-admissible diagram. The proofs of the lemmas
are all immediate from the diagram multiplication.

**Lemma 3.2.2.** Assume \( D \) has a propagating edge, \( E \), connecting node \( p_1 \) in
the north face to node \( p_2 \) in the south face.

If nodes \( p_1 + 1 \) and \( p_1 + 2 \) in the north face are connected by a (necessarily
undecorated) edge \( E' \), then \( U_{p_1}D \) is the \( H \)-admissible diagram obtained by removing
\( E' \), disconnecting \( E \) from the north face and reconnecting it to node \( p_1 + 2 \) in
the north face, and installing a new undecorated edge between points \( p_1 \) and \( p_1 + 1 \) in
the north face. The edge corresponding to \( E \) retains its original decoration status.

**Lemma 3.2.3.** Assume that in the north face of \( D \), nodes \( i \) and \( i + 1 \) are connected
by a decorated edge, \( e_1 \), and nodes \( i + 2 \) and \( i + 3 \) are connected by an undecorated
edge, \( e_2 \). Assume also that \( i > 1 \). Then \( U_iU_{i+1}D \) is the \( H \)-admissible diagram
obtained from \( D \) by exchanging \( e_1 \) and \( e_2 \). This procedure has an inverse, since
\( D = U_{i+2}U_{i+1}U_iU_{i+1}D \).

**Lemma 3.2.4.** Assume that in the north face of \( D \), nodes 1 and 2 are connected
by a decorated edge, \( E \), and nodes 3 and 4 are connected by an undecorated edge.
Then the \( H \)-admissible diagram \( \rho(\alpha)D \) is that obtained from \( D \) by decorating the edge
connecting nodes 3 and 4.

**Lemma 3.2.5.** Assume that in the north face of \( D \), nodes 1 and 2 are connected
by a decorated edge, \( E \), and nodes 3 and 4 are connected by an undecorated edge.
Then the \( H \)-admissible diagram \( U_3\rho(\zeta)D \) is that obtained from \( D \) by removing the
decoration on \( E \).

**Lemma 3.2.6.** Assume that in the north face of \( D \), nodes \( i \) and \( i + 1 \) are connected
by an undecorated edge, \( e_1 \), and nodes \( j < i \) and \( k > i + 1 \) are connected by an edge,
\( e_2 \). Assume also that \( j \) and \( k \) are chosen such that \( |k - j| \) is minimal. Then \( D \) is
of the form \( U_iD' \), where \( D' \) is an \( H \)-admissible diagram which is the same as \( D \)
except as regards the edges connected to nodes \( j, i, i + 1, k \) in the north face. Nodes
\( j \) and \( i \) in \( D' \) are connected to each other by an edge with the same decoration as
\( e_2 \), and nodes \( i + 1 \) and \( k \) are connected to each other by an undecorated edge.

As an illustration of what is going on, we present diagrammatic versions of these
lemmas in Figure 8. A hollow circle indicates the site of an optional decoration.

**Remark 3.2.7.** The algebra \( \Delta_{n+1} \) has an anti-automorphism, \( * \), which reflects each
diagram in the east-west line. Therefore, all of the five previous lemmas have
 corresponding statements about the south faces.

Using these five lemmas, we can prove the following result.

**Proposition 3.2.8.** Any \( H \)-admissible diagram \( D \) with \( n + 1 \) strands \((n \geq 2)\) can
be written as a word in the images under \( \rho \) of the generating set \( G_n \).

**Proof.** The case \( n = 2 \) is done by Lemma 3.2.1.

Iteration of Lemma 3.2.6 reduces the consideration to diagrams \( D \) where all the
non-propagating edges connect adjacent points.

We restrict ourselves to the nontrivial case where \( D \) has \( r > 0 \) non-propagating
edges.
First, we assume that $D$ has a propagating edge. We define the diagram $D_0$, depending on $D$, which is chosen from the eight diagrams of form

$$D_0 = GU_4 U_6 \cdots U_{2r-2} U_{2r},$$

where $G$ is one of the nonidentity diagrams for the case $n = 2$ (see Lemma 3.2.1), and $D_0$ and $D$ share the following three properties.

1. If $D$ has a propagating edge meeting node 1 in the north face, then so does $D_0$.
2. If $D$ has a propagating edge meeting node 1 in the south face, then so does $D_0$.
3. The leftmost propagating edges in $D$ and $D_0$ are both decorated, or both undecorated.

If $D$ does not have a propagating edge, we define

$$D_0 = U_1 U_3 \cdots U_n,$$

where $n$ is necessarily odd, and $D_0$ has no propagating edges.

We claim that $D = w_1 D_0 w_2$, where $w_1$ is a word in the generators obtained by the moves arising from lemmas 3.2.2 to 3.2.5 as stated, and $w_2$ is similar but arises from the reflected versions of these lemmas after applying $*$ (see Remark 3.2.7).

For reasons of symmetry, we concentrate on the word $w_1$, the other part being similar. To do this, we show that there is a diagram $D'$ whose top half is that of $D$ and whose bottom half is that of $D_0$, satisfying $D' = w_1 D_0$. If $D'$ has a propagating edge, then the leftmost one has the same decorated status as that of $D$ or $D_0$.

We start with the diagram $D_0$. The first stage is to move the propagating edge (if there is one) so that it meets the north face at the desired point. This is achieved by iterations of Lemma 3.2.2.
Next, we generate all the decorated, non-propagating edges we desire, using Lemma 3.2.4. (Note that if we have to do this, then \( D_0 \) has a decorated edge connecting nodes 1 and 2 in the north face.) After these edges are formed, we commute them out of the way to the east using Lemma 3.2.3. If we require nodes 1 and 2 in \( D \) to be connected by an undecorated edge, this can be arranged by using Lemma 3.2.5 once. (Note that the definition of \( H \)-admissible implies that \( D \) must have two other points connected by a non-decorated edge in this case, so this is possible.) We then reach the diagram \( D' \) by further applications of Lemma 3.2.3.

The proof now follows. □

3.3 \( \Delta_{n+1} \) as a cellular algebra.

In order to count the dimension of \( \Delta_{n+1} \), it helps first to understand its structure as a cellular algebra. One can then compare the sizes of the cells to those arising from \( TL(H_n) \) as in [1, Proposition 7.3.2].

It is convenient to introduce a dyadic notation for the \( H \)-admissible diagrams similar to that used for \( TL(A_n) \) in [13, §5].

Definition 3.3.1. Let \( D \) be an \( H \)-admissible diagram for \( \Delta_{n+1} \). Remove all the propagating edges from \( D \), then take the upper half of what remains and call it \( d_1 \). Invert the lower half of \( D \) in a horizontal line and call this \( d_2 \). Then \( D \) may be reconstituted from the ordered pair \((d_1, d_2)\) provided that we know whether \( D \) has a decorated propagating edge or not.

We write \( D = |d_1⟩⟨d_2| \) if \( D \) has no decorated propagating edge, and \( D = |d_1⟩⟨d_2|_\ast \) if \( D \) has a decorated propagating edge.

Lemma 3.3.2. Let \( R \) be an integral domain of characteristic different from 2, 3 or 5 in which the polynomial \( x^2 - x - 1 \) splits into distinct linear factors \((x - \gamma_1)(x - \gamma_2)\).

Writing \( \gamma \) for the image of \( x \) in the algebra \( \Gamma = R[x]/\langle x^2 - x - 1 \rangle \), we have

\[
(\gamma - \gamma_1)^2 = (1 - 2\gamma_1)(\gamma - \gamma_1) \neq 0,
\]

and a similar identity holds for \( \gamma - \gamma_2 \).

Proof. This is immediate. Note that \( 1 - 2\gamma_1 \neq 0 \) because we are not in characteristic 5. □

Definition 3.3.3. Let \( R \) satisfy the hypotheses of Lemma 3.3.2, and let \( |d_1⟩⟨d_2| \) be an \( H \)-admissible diagram. Then we define

\[
|d_1⟩⟨d_2|_1 := |d_1⟩⟨d_2|\ast - \gamma_1|d_1⟩⟨d_2| \\
|d_1⟩⟨d_2|_2 := |d_1⟩⟨d_2|\ast - \gamma_2|d_1⟩⟨d_2|.
\]

We recall the definition of a cellular algebra from [5]:

Definition 3.3.4. Let \( R \) be a commutative ring with identity. A cellular algebra over \( R \) is an associative unital algebra, \( A \), together with a cell datum \((\Lambda, M, C, \ast)\) where
1. Λ is a poset. For each \( \lambda \in \Lambda \), \( M(\lambda) \) is a finite set (the set of “tableaux” of type \( \lambda \)) such that
\[
C : \prod_{\lambda \in \Lambda} (M(\lambda) \times M(\lambda)) \to A
\]
is injective with image an \( R \)-basis of \( A \).

2. If \( \lambda \in \Lambda \) and \( S, T \in M(\lambda) \), we write \( C(S, T) = C_{\lambda}^{\lambda} \in A \). Then * is an \( R \)-linear involutory anti-automorphism of \( A \) such that \( (C_{\lambda}^{\lambda})^* = C_{T}^{\lambda} \).

3. If \( \lambda \in \Lambda \) and \( S, T \in M(\lambda) \) then for all \( a \in A \) we have
\[
a.C_{\lambda}^{\lambda} \equiv \sum_{S' \in M(\lambda)} r_a(S', S)C_{S', T}^{\lambda} \mod A(< \lambda),
\]
where \( r_a(S', S) \in R \) is independent of \( T \) and \( A(< \lambda) \) is the \( R \)-submodule of \( A \) generated by the set
\[
\{C_{S''}^{\mu}, T'' : \mu < \lambda, S'' \in M(\mu), T'' \in M(\mu)\}.
\]

We now define our versions of the sets in the above definition.

Let \( \Lambda \) be the set of symbols \( \{0\} \cup \{1, 2, \ldots, k, 1^*, 2^*, \ldots, k^*\} \), where \( k \) is a natural number such that \( k < (n + 1)/2 \), together with the symbol \( (n + 1)/2 \) if \( n \) is odd.

We put a partial order < on these symbols by declaring that \( i < j \) if \( |i| > |j| \), where \( |i| = i \) if \( i \) is a natural number, and \( |i^*| = i \).

If \( \lambda \in \Lambda \), the set \( M(\lambda) \) has elements parametrised by the half-diagrams \( |d_1| \) arising from \( H \)-admissible diagrams with \( |\lambda| \) non-propagating edges in each half of the diagram.

The antiautomorphism * corresponds to top-bottom inversion of an \( H \)-admissible diagram.

The map \( C \) takes elements \( d_1 \) and \( d_2 \) from \( M(\lambda) \) and produces the element \( C(d_1, d_2) \) which is defined to be
\[
|d_1\rangle\langle d_2|_1
\]
if \( \lambda \) is a natural number or
\[
|d_1\rangle\langle d_2|_2
\]
otherwise, unless \( \lambda = 0 \) or \( \lambda = (n + 1)/2 \), in which case \( C(d_1, d_2) \) is given by
\[
|d_1\rangle\langle d_2|.
\]

Note that the identity element appears in the image of \( C \).

**Theorem 3.3.5.** Let \( R \) be a ring satisfying the hypotheses of Lemma 3.3.2. Then the algebra \( \Delta_{n+1} \) over the ring \( R[v, v^{-1}] \) has a cell datum \((\Lambda, M, C, *)\), where the sets are given as above.

**Proof.** The proof is largely straightforward. The fact that \( \gamma_1 \) and \( \gamma_2 \) are distinct ensures that the image of \( C \) is a basis for \( \Delta_{n+1} \).

The only other nontrivial part is the verification of axiom 3. Consider the product of two basis elements \( B_1 \) and \( B_2 \) parametrised by the respective elements \( \lambda \) and \( \lambda' \) of \( \Lambda \). The only difficulty arises when \( 0 < \lambda, \lambda' < \frac{n+1}{2} \), so we concentrate on
this case. It is convenient to think of each of the diagrams \( B_1 \) and \( B_2 \) as having a propagating edge decorated by one of the elements \( \gamma - \gamma_1 \) or \( \gamma - \gamma_2 \) of \( \Gamma \), where an ordinary decorated edge is thought of as being decorated by \( \gamma \in \Gamma \), and an undecorated one as being decorated by \( 1 \in \Gamma \). (Note that the third relation in Figure 5 corresponds to the equation \( \gamma^2 = \gamma + 1 \).)

We will assume that \(|\lambda'| \geq |\lambda|\); the other case is similar. Let us define \( \gamma_i \) by saying that the propagating edge of \( B_2 \) carries the element \( \gamma - \gamma_i \) (where \( i \in \{1, 2\} \)). If the product \( B = B_1 B_2 \) is a tangle with strictly fewer propagating edges than \( B_2 \) (and therefore fewer than \( B_1 \), since \(|\lambda'| \geq |\lambda|\)) then it is clear that \( B \) is a linear combination of basis elements corresponding to elements \( r \in \Lambda \) with \( r < \lambda' \), so axiom 3 holds.

The other possibility is that the product \( B \) has the same number of propagating edges as \( B_2 \) and, furthermore, that the leftmost propagating edge, \( E \), of \( B \) contains (as a segment) the leftmost propagating edge of \( B_2 \). The edge \( E \) therefore carries the generalized decoration \( \gamma - \gamma_i \), and possibly other decorations of various kinds.

Lemma 3.3.2 shows that if we multiply together all the decorations on the edge \( E \) (where an ordinary decoration corresponds to \( \gamma \), as before), we obtain a (possibly zero) multiple of \( \gamma - \gamma_i \). If we obtain zero then the product \( B \) is zero and there is nothing more to prove. Otherwise, \( B \) is a linear combination of basis elements whose leftmost propagating edges all carry \( \gamma - \gamma_i \), namely a combination corresponding to the element \( \lambda' \in \Lambda \). Since the structure constants are not affected by the pattern of non-propagating edges in the south face of \( B_2 \), axiom 3 follows. □

### 3.4 Faithfulness of the diagram representation.

Using the results of §3.3, we can enumerate the number of \( H \)-admissible diagrams of various types.

**Lemma 3.4.1.** The size of the set \( M(\lambda) \) associated with the algebra \( \Delta_{n+1} \) is equal to

\[
\binom{n+1}{|\lambda|} - 1,
\]

unless \(|\lambda| = 0\) in which case the set has size 1.

**Proof.** If we generalized the \( H \)-admissible diagrams by excluding parts (i) and (ii) of Definition 2.2.1, then it would follow from [10, Proposition 2] that there would be \( \binom{n+1}{k} \) half diagrams with \( k \) non-propagating edges. If \( k > 0 \) then the force of part (ii) of Definition 2.2.1 is to exclude just one element: the one with an undecorated edge connecting points 1 and 2 and a decorated edge connecting points \( 2m + 1 \) and \( 2m + 2 \) for \( m < k \) (see Figure 4). This proves the assertion for \(|\lambda| > 0\), and the assertion for \(|\lambda| = 0\) is trivial. □

**Theorem 3.4.2.** Working over \( A \), the ranks of \( \Delta_{n+1} \) and \( TL(H_n) \) are identical. Therefore, \( \rho \) is an isomorphism.

**Proof.** Stembridge [11, §3.4] proves that the number of “fully commutative” elements in a Coxeter group of type \( H_n \) is given by

\[
1 + \sum_{\lambda \in \Lambda, |\lambda| > 0} \binom{n+1}{|\lambda| - 1}^2 = \binom{2n+2}{n+1} - 2^{n+2} + n + 3.
\]
Graham [4, Theorem 6.2] shows that these fully commutative elements index a basis for $TL(H_n)$. It follows from Lemma 3.4.1 that the rank of $\Delta_{n+1}$ is the same as the rank of $TL(H_n)$. The fact that $\rho$ is an isomorphism follows from Proposition 3.1.2 and Proposition 3.2.8. □

Remark 3.4.3. The fact that $TL(H_n)$ is cellular if $x^2 - x - 1$ splits has been observed by Graham [4, Remark 9.8], although a cell datum is not explicitly given.

Remark 3.4.4. The similarity with the blob algebra of [10] which is touched upon in the proof of Lemma 3.4.1 goes further. If conditions (i) and (ii) of Definition 2.2.1 are dropped, then the resulting algebra is isomorphic to the algebra of [10], although the isomorphism is not canonical.

4. Applications

We now examine some applications of theorems 3.3.5 and 3.4.2.

4.1 Positivity Properties.

We have seen how the diagram basis for $TL(H_n)$ can be expressed as monomials in a certain set of algebra generators (this follows from Proposition 3.2.8 and Theorem 3.4.2). We now show that this diagram basis has a positivity property.

Proposition 4.1.1. Assume we are working over the ring $A$.

Any basis element occurring with nonzero coefficient in the product of two basis elements $D_1D_2$ associated with the respective elements $\lambda_1$ and $\lambda_2$ of $\Lambda$ occurs with coefficient $c[2]^k$, where $c$ is a positive integer and $k \leq \max(|\lambda_1|, |\lambda_2|)$.

In particular, the structure constants of the basis of diagrams are polynomials in $\mathbb{N}[v, v^{-1}]$.

Proof. Note that the simplification process given by the rules in Figure 5 preserves positivity, and that $[2]$ is a polynomial in $\mathbb{N}[v, v^{-1}]$.

It is immediate that there cannot be more loops forming in the diagram multiplication than there were non-propagating edges in each half of either $D_1$ or $D_2$, which proves the assertion about the number $k$. □

Note that any basis obtained from monomials in the original set of generators cannot have this property: consider the monomial $E_1E_2E_1E_2E_1$.

4.2 Semisimplicity.

We recall from the theory of cellular algebras in [5, §2] that there is a bilinear form $\phi_\lambda(d_1, d_2) = \langle d_1, d_2 \rangle$ on the cell module $W(\lambda)$. (Recall that the module $W(\lambda)$ has a basis parametrised by the elements of $M(\lambda)$ for a fixed $\lambda$.) This form is defined from the equation

$$C(e_1, d_1)C(d_2, e_2) = \langle d_1, d_2 \rangle C(e_1, e_2) \mod A(< \lambda).$$

This is independent of the choice of $e_1$ and $e_2$, where $e_1, e_2, d_1, d_2$ are all elements of $M(\lambda)$ for the same $\lambda$.

The following result proves [1, Conjecture 7.3.1].
Theorem 4.2.1. Let $R$ be a field satisfying the hypotheses of Lemma 3.3.2. Then the algebra $\text{T}_L(H_n)$ over $R$ is semisimple, and all the cell modules $W(\lambda)$ are irreducible and pairwise inequivalent.

Proof. It is enough by [5, Theorem 3.8] to prove that $\phi_\lambda$ is nondegenerate for each $\lambda$.

Choose an element $\lambda \in \Lambda$ and two elements $d_1, d_2 \in M(\lambda)$, where possibly $d_1 = d_2$. Now consider $v^{-|\lambda|}\phi_\lambda(d_1, d_2) = v^{-|\lambda|}(d_1, d_2)$.

It follows from Proposition 4.1.1 that $v^{-|\lambda|}(d_1, d_2)$ is a polynomial in $v^{-1}$; furthermore, the constant term of the polynomial is zero unless $d_1 = d_2$. To see this, we use the fact that loops carrying a single blob result in annihilation of the associated diagram, and loops which carry 0 or 2 blobs both correspond to multiplication by [2]. In the case where $d_1 = d_2$, Lemma 3.3.2 shows that the constant term of the polynomial is nonzero and equal to $(1 - 2\gamma_1)$ or $(1 - 2\gamma_2)$, depending on the $\lambda$ which we are considering.

We have now constructed an almost orthogonal basis (i.e. orthogonal modulo the span of strictly negative powers of $v$) for the module $W(\lambda)$ with respect to $\phi_\lambda$. It follows that $\phi_\lambda$ is nondegenerate, as required. □

Note that Lemma 3.4.1 now tells us the dimensions of the irreducible modules. This confirms [1, Conjecture 7.3.3].

4.3 Branching rules.

In this section, we continue to assume that the base ring of $\text{T}_L(H_n)$ is a field in which $x^2 - x - 1$ splits into distinct linear factors, although we do not assume that $\text{T}_L(H_n)$ is semisimple. The diagram calculus we have developed allows us to study the behaviour of the cell modules $W(\lambda)$ for $\text{T}_L(H_n)$ upon restriction to $\text{T}_L(H_{n-1})$ (assuming $n$ is at least 3). The embedding of $\text{T}_L(H_{n-1})$ into $\text{T}_L(H_n)$ is the natural one arising from the identification of the algebra generators, or the addition of a vertical edge on the east of the diagram. To describe the branching rules, it is convenient to make the following definition.

Definition 4.3.1. Let $\lambda \in \Lambda = \Lambda(H_n)$, and suppose that $0 < |\lambda| < \frac{n+1}{2}$. We define an element $\lambda - 1 \in \Lambda(H_{n-1})$ as follows:

$$
\lambda - 1 := \begin{cases} 
0 & \text{if } |\lambda| = 1, \\
(i - 1)^* & \text{if } |\lambda| \neq 1, \lambda = i^* \text{ where } i \in \mathbb{N}, \text{ and } i - 1 \neq n/2, \\
i - 1 & \text{otherwise.}
\end{cases}
$$

Proposition 4.3.2. Let $W(\lambda, n)$ be a cell module for $\text{T}_L(H_n)$. Then, after restriction to $\text{T}_L(H_{n-1})$, $W(\lambda, n)_{n-1}$ has a filtration by the cell modules $W(\lambda', n - 1)$ of $\text{T}_L(H_{n-1})$ described as follows.

If $\lambda = 0$ then restriction gives the trivial module corresponding to the poset element 0.

If $|\lambda| = 1$ then the composition factors occurring correspond to the poset elements 0 and $\lambda$, each with multiplicity 1.

If $\lambda = \frac{n+1}{2}$ then the composition factors occurring correspond to the poset elements 0, $\frac{n-1}{2}$ and $(\frac{n-1}{2})^*$, each with multiplicity 1.
For other values of $\lambda$, the composition factors occurring correspond to the poset elements $0$, $\lambda - 1$ and $\lambda$, each with multiplicity 1.

Proof. We first tackle the fourth case, dealing with the general value of $\lambda$.

The key observation, which is familiar from the diagram calculi of other types, is as follows. The half-diagrams in $M(\lambda)$ which have a non-connected point at the eastern extreme form a submodule for $TL(H_{n-1})$ on restriction, corresponding to removal of the easternmost point. This is canonically isomorphic to the module corresponding to $\lambda$ in $\Lambda(H_{n-1})$.

The quotient module associated with this submodule is obtained by taking the other elements of $M(\lambda)$ and, for each one, removing the easternmost point and the edge connected to it. However, this is not the same as one of the cell modules for $TL(H_{n-1})$, because an inadmissible half-diagram occurs. (All the edges in this are decorated, except for the one connecting points 1 and 2.) The reason that this arises is that in the original diagram, the two easternmost points could have been connected by an undecorated edge, making the half-diagram admissible, but this edge was removed in the restriction process. The admissible diagrams arising from this procedure span a cell module isomorphic to that parametrised by $\lambda - 1$. The appearance of the inadmissible diagram corresponds to a top quotient isomorphic to the trivial module.

The cases $|\lambda| = 0$ and $|\lambda| = 1$ can be obtained by degenerate versions of this technique.

The case $\lambda = \frac{n+1}{2}$ is the most subtle. To deal with it, it is convenient to modify the basis for the cell module $W(\lambda, n)$. One of the half-diagrams in $M(\lambda)$ becomes inadmissible once the rightmost point and its associated edge have been removed; we denote this half-diagram by $d_0$.

The other half-diagrams fall naturally into pairs as we now describe. First, note that any edge of a half-diagram of $M(\lambda)$ (for $\lambda = \frac{n+1}{2}$) is exposed to the west face, since there are no propagating edges involved. There is therefore an involution on $M(\lambda) \setminus \{d_0\}$ given by changing the decorated status of the edge connected to the rightmost point. The orbits are all of size 2.

If $d$ and $d^\bullet$ are two elements in the same orbit (where $d^\bullet$ carries the extra decoration), we define new basis elements $d_1$ and $d_2$ for $W(\lambda, n)$ by

$$d_1 := d^\bullet - \gamma_1 d$$

and

$$d_2 := d^\bullet - \gamma_2 d.$$

Analysis of the diagrams now shows that the diagram $d_0$ corresponds to a top quotient of $W(\lambda, n)$ isomorphic to the trivial module. Furthermore, the submodule of $W(\lambda, n)$ spanned by the new basis elements $d_i$ breaks up as a direct sum: the span of the elements $d_1$ is canonically isomorphic to $W\left(\frac{n-1}{2}, n - 1\right)$, and the span of the elements $d_2$ is canonically isomorphic to $W\left(\left(\frac{n-1}{2}\right)^\bullet, n - 1\right)$. □

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