Mass gap in quantum energy-mass spectrum of relativistic Yang-Mills fields

Alexander Dynin

Department of Mathematics, Ohio State University
Columbus, OH 43210, USA, dynin@math.ohio-state.edu

Abstract

A non-perturbative relativistic quantum Yang-Mills theory with a semisimple compact gauge Lie group on the four-dimensional Minkowski spacetime is set up in a Schroedinger representation with infinite-dimensional differential operators in the framework of sesqui-holomorphic nuclear Kree-Gelfand triples.

The linear quantum Yang-Mills energy-mass operator $H$ is defined as the anti-normal quantization of the non-linear Yang-Mills energy-mass functional of Cauchy data supported by a ball $B(R)$ with the center at the origin of $\mathbb{R}^3$ and the variable radius $R > 0$. The general global solution of the non-linear Yang-Mills system of partial differential equations (with no restrictions at infinity) is reduced to the solution of the initial value problems in the temporal gauge with the Cauchy data supported by the balls $B(R)$.

It is shown that $H$ dominates the number operator $N$. Since $0$ is the spectral infimum of $H$ and, simultaneously, the simple fundamental eigenvalue of $N$, variational spectral principles imply that $0$ is the simple fundamental eigenvalue of $N$ as well. Thus $H$ has a positive mass gap. The domination proof depends crucially on the magic of the Killing quadratic form that reveals a mass quadratic form in the Weyl symbol of $H$.

With a dimensional transmutation of the coupling constant, the mass gap is proportional to $1/R$ and the running coupling constant is proportional to $\sqrt{R}$. The inverse dependence demonstrates an asymptotic freedom for quantum Yang-Mills self-interaction at short distances.

The mathematically rigorous theory is non-perturbative and provides a solution for the 7th Clay Institute Millennium problem.

Key words: Second quantization; Gelfand nuclear triples; pseudo-differential operator calculus; operators with variational derivatives; infinite-dimensional manifolds; Killing quadratic form; Yang-Mills Millennium problem; asymptotic freedom.

2000 AMS Subject Classification: 81T13, 81T70, 30H20, 35S05, 49R50
1 Introduction

A mathematically rigorous solution is given for both parts of the 7th Millennium problem of Clay Mathematics Institute:

*Prove that for any compact (semi)simple global gauge group, a non-trivial quantum Yang-Mills theory exists on the four-dimensional Minkowski spacetime and has a positive mass gap* (cp. JAFFE-WITTEN[1]).

(This formulation is from WITTEN[32] p. 24).

Significantly the established quantum Yang-Mills theory is non-perturbative. Indeed such a quantum field theory has been in demand for quite some time:

The puzzling conflict between the successful predictions and the divergence of the (renormalized) perturbative series legitimate the need for a non-perturbative approach to the problem of combining of quantum mechanics and relativity, with the aim of either validating the foundation of quantum field theory or displaying the need for radical changes and new ideas. (STROCCHI[31] Page 2).

In 1954 GELFAND-MINLOS[15] proposed to solve variational quantum fields equations with variational derivatives via approximations by solutions of partial differential equations with large but finite number of independent variables (cp. BEREZIN[4, Preface]). However the convergence of such approximations so far has not been established until the present paper.

In accordance with Yukawa principle, a positive mass gap is an experimental fact for weak and strong subnuclear forces: *A limited force range suggests its positive mass* (cp. Yukawa 1949 Nobel lecture). Yukawa was inspired by the asymptotics of Geen function for Klein-Gordon equation. The same analogy leads to the conclusion that the mass grows to infinity at short distances (asymptotic freedom).

In the Yang-Mills case the asymptotic freedom is a relativistic quantum effect. Indeed, when the natural units of the light velocity $c$ and Planck constant $\hbar$ are set to 1, the relativistic Yang-Mills energy-momentum vector acquires the natural physical dimension of the reciprocal length $[L]^{-1}$. This breaks the conformal symmetry of the Yang-Mills action functional on Minkowski spacetime (see, e.g., GLASSEY-STRAUSS[17]) and transmutes the dimension $[L]$ to the naturally dimensionless Yang-Mills coupling constant (cp. FADDEEV[13]).

---

1 The official formulation of the Millennium Yang-Mills problem[1] is looking for a quantum Yang-Mills theory with axiomatic properties at least as strong as axioms of Wightman relativistic quantum field theory. However even modified Wightman axioms (see, e.g., BOGOLIUBOV et al[8, chapter 10] are in a serious conflict with the simplest cases of Gupta-Bleuler theory of quantum electromagnetic fields, as well as commonly used local renormalizable gauges (see, e.g. STROCCHI[30] Chapter 6 and Appendix A.2).

2 Higgs mechanism ascribes a mass origin to classical interaction with an ad hoc scalar field. The 2012 experimental discovery of a Higgs scalar did not verify the Higgs mechanism itself (cp.[7]).
1.1 Contents

SECTION 2 provides a mathematically rigorous context for a non-perturbative quantum field theory in nuclear Kree-Gelfand triples. It includes the following items.

- Infinite-dimensional symbolic calculus of operators with variational derivatives as a mathematically rigorous infinite-dimensional extension of AGARWAL-WOLF [2] and HOWE [20] calculus of partial pseudo-differential operators.

- Convergence of approximations of operators with variational derivatives by finite-dimensional pseudodifferential operators, along with the convergence of their symbolic calculi. This is a justification for Gelfand-Minlos method for solution [15] of equations with variational derivatives.

SECTION 3 is a setup for a Gelfand nuclear triple of Yang-Mills Cauchy data on euclidean balls:

- In the temporal gauge the Yang-Mills system of the second order partial differential equations for Yang-Mills fields on the four-dimensional Minkowski space is equivalent to Cauchy problem for the Schwinger semi-linear first order hyperbolic system with initial data subject to constraint equations on \( \mathbb{R}^3 \).

- By GOGANOV-KAPITANSKII [19], the hyperbolicity of the Schwinger-Yang-Mills system implies the reduction of the general Cauchy problem to Cauchy data with compact supports.

- Theorem 3.1 converts the non-linear manifold of constrained initial data with a compact support into a Gelfand triple of topological vector spaces.

SECTION 4 presents the anti-normal quantization of the Yang-Mills energy-mass functional in Gelfand-Kree triple. The anti-normal quantization of the nonnegative energy-mass functional is chosen because it produces a nonnegative energy-mass operator but the normal quantization does not (cp. GLIMM-JAFFE [18]).

The Main Theorem 4.1 affirms a positive Yang-Mills mass gap: The semi-simplicity of the gauge group reveals the mass quadratic form in the Weyl symbol of the energy-mass operator missing in its anti-normal symbol. This allows to verify that the energy-mass operator dominates the number operator so that, by variational spectral theory, the zero spectral value of the energy-mass operator is a simple eigenvalue because it is such for the number operator.

Proposition 4.1 exhibits an asymptotic self-similarity of the energy-mass spectra on the running energy-mass scale. This entails an asymptotic freedom at short distances.
2 Variational pseudodifferential operators

2.1 Review of Kree-Gelfand triples [21, 22]

Consider a Gelfand triple of densely imbedded complex topological spaces with conjugation (see, e.g., GELFAND-VILENKIN [16])

\[ \mathcal{H} \subset \mathcal{H}^0 \subset \mathcal{H}^* , \]

where

- The space \( \mathcal{H}^0 \) is a Hilbert space with a Hermitian conjugation \( z \mapsto z^* \) and the Hermitian sesqui-linear form \( z^* w \):

- The space \( \mathcal{H} \) of elements \( z \) is a nuclear countably Hilbert space;

- The space \( \mathcal{H}^* \) of elements \( z^* \) is the anti-dual space of \( \mathcal{H} \) of continuous anti-linear functionals \( z^* w \) on \( \mathcal{H} \).

Kree-Gelfand nuclear triple \( \mathcal{N} \) (KREE [21] and [22]) is a sesqui-holomorphic second quantization of the Gelfand triple \( \mathcal{H} \),

\[ \mathcal{N} \subset \mathcal{N}^0 \subset \mathcal{N}^* \]

where

- The nuclear space \( \mathcal{N}^* \) is the locally convex space of entire holomorphic functionals \( \Psi(z) \) on \( \mathcal{N} \) with the topology of compact convergence;

- The space \( \mathcal{N}^0 \) is the Hilbert space of square integrable entire holomorphic functionals on \( \mathcal{N}^* \) with respect to the Gaussian probability measure \( d^\infty z d^\infty z e^{-z^* z} \) (see, e.g., GELFAND-VILENKIN [15]).

\[ \langle \Psi^* \mid \Phi \rangle := \int d^\infty z^* d^\infty z e^{-z^* z} \Psi^*(z) \Phi(z^*) , \quad \Psi^*, \Phi \in \mathcal{N} , \]

where \( \Psi^*(z) \) is the complex conjugate of \( \Psi(z^*) \).

The notation is ambiguous because there is no Lebesgue measure \( d^\infty z^* d^\infty z \) on the infinite-dimensional space \( \mathcal{N}^* \). However the integral is the convergent limit of finite-dimensional integrals over \( p(\mathcal{H}^*) \), where \( p : \mathcal{H}^* \to \mathcal{H} \) are selfadjoint (aka orthogonal) projectors of a finite rank \( r(p) \)

\[ (2\pi i)^{-r(p)} \int (d^r(p) z)^* (d^r(p) z) e^{-(p z^*)^*(p z)} \Psi^*(p z) \Phi(p z^*) , \]

where \( n \) is the complex conjugate of \( \Psi(z^*) \).

\[ \text{The notation is bracketless as, e.g., in BEREZIN [4].} \]

\[ \text{A functional is entire on a locally convex complex vector space if it is continuous and entire on every complex line in that space (see, e.g., COLOMBEAU [9]).} \]
as the projectors $p$ strongly converge in $H$ to the identity operator (cp. BEREZIN [4], Chapter 1, Section 2).

**Functionals** $\Psi^*(pz)$, $\Phi(pz^*)$ are cylindrical functionals of the rank of $p$. $\text{dim}_{\mathbb{C}}$. They form dense subspaces in $H$ and $H^*$.

- The space $\mathcal{K}$ is the space of continuous anti-linear functionals $\Psi^*$ on $\mathcal{K}^*$, in Dirac bra-ket notation

$$\Psi^*(\Phi) = \langle \Psi^* \mid \Phi \rangle, \quad \Phi \in \mathcal{K}.$$  

The nuclear countably Hilbert space $\mathcal{K}$ is isomorphic to the space of entire holomorphic functionals $\Psi(z^*)$ of exponential growth with respect to any continuous semi-norm on $\mathcal{K}^*$ (see e.g. COLOMBAU [9, Chapter 7]).

**Exponential states** $e^{\eta}(z^*)(\eta) := e^{\xi^*\eta}$ in $\mathcal{K}$ have the following well known basic properties:

- $\langle e^{\xi^*} \mid e^{\eta} \rangle = e^{\xi^*\eta}$ (2.5)

- The set of **normalized exponential states**

$$|\eta\rangle := e^{-(1/2)\eta^*\eta}e^{\eta}, \quad \eta \in \mathcal{K},$$  

is an overcomplete **continuous orthonormal basis** for the Kree-Gelfand triple, i.e. the **normalized Borel-Fourier transform** (see, e.g., COLOMBAU [9, Chapter 7])

$$\Psi^*(\xi) := \langle \Psi^* \mid \xi \rangle, \quad \Psi(\xi^*) := \langle \xi^* \mid \Psi \rangle,$$  

is a topological unitary automorphism of Kree-Gelfand triple (Plancherel-like property).

Kree-Gelfand triple of the sesqui-Hermitian direct products

$$\mathcal{H} \times \mathcal{H} \subset \mathcal{H}^0 \times \mathcal{H}^0 \subset \mathcal{H}^* \times \mathcal{H}^*$$  

(2.8)

carries the Hermitian conjugation

$$(z^*,w^*) := (w^*,z)$$  

(2.9)

The associated Kree-Gelfand triples of sesqui-holomorphic Grothendieck kernels consists of

$$\mathcal{H} \otimes \mathcal{H} \subset \mathcal{H}^0 \otimes \mathcal{H}^0 \subset \mathcal{H}^* \otimes \mathcal{H}^*$$  

(2.10)

where $\mathcal{H}^0 \otimes \mathcal{H}^0$ is the Hilbert space of Hilbert-Schmidt Grothendieck kernels.

The corresponding exponential functionals are

$$e^{\xi^*\eta}\left((z^*,w^*)\right) = e^{w^*\eta + \xi^*z}, \quad |\xi^*,\eta\rangle = e^{-\xi^*\eta-\eta^*\xi}e^{(z^*,\eta)}.$$  

(2.11)

\(^5\)For starters they are straightforward on cylindrical states and then, by strong limits, are extended to all states (cp. e.g. [11])
2.2 Pseudovariational operators

Kree-Gelfand triple has the linear representation by continuous linear transformations of \( \zeta \in \mathcal{K}^* \) and \( \zeta^* \in \mathcal{K} \) into the adjoint operators of creation and annihilation continuous operators of multiplication and complex directional differentiation in \( \mathcal{K} \)

\[
\hat{\zeta}\Psi(z^*) := (z^*\zeta)\Psi(z^*), \quad \hat{\zeta}^*\Psi(z^*) := \partial_{\zeta^*}\Psi(z^*), \quad (2.12)
\]
such that

1. Bosonic commutation relations hold

\[
[\hat{\zeta}^*, \hat{\eta}] = \zeta^*\eta; \quad (2.13)
\]

2. The exponentials \( e^{\eta^*} \), \( \eta^* \in \mathcal{K}^* \), and \( e^{\eta}, \eta \in \mathcal{K} \), are the eigenstates of the annihilation operators

\[
\hat{\zeta}^*e^\eta = (\zeta^*\eta)e^\eta, \quad \hat{\zeta}e^{\eta^*} = (\eta^*\zeta)e^{\eta^*}. \quad (2.14)
\]

Creators and annihilators generate strongly continuous abelian operator groups of quantum exponentials in \( \mathcal{K}^* \) parametrized by \( \zeta \) and \( \zeta^* \):

\[
e^{\hat{\zeta}}\Psi(w^*) = e^{w^*\zeta}\Psi(w^*), \quad e^{\hat{\zeta}^*}\Psi(w^*) = \Psi(w^* + \zeta^*). \quad (2.15)
\]

Thus the operator products \( e^{\hat{\theta}}e^{\hat{\eta}^*} \) and \( e^{\hat{\theta}^*}e^{\hat{\eta}} \) are continuous operators in \( \mathcal{K} \).

By Baker-Campbell-Hausdorff commutator formula and the canonical commutation relations \( (2.13) \),

\[
e^{\hat{\eta}}e^{\hat{\theta}^*} = e^{\hat{\eta} + \hat{\theta}^*}e^{\theta^*\eta/2}, \quad e^{\hat{\theta}^*}e^{\hat{\eta}} = e^{\hat{\eta} + \hat{\theta}^*}e^{-\theta^*\eta/2}. \quad (2.16)
\]

In particular, the operator \( e^{\hat{\eta} + \hat{\theta}^*} \) is also continuous in \( \mathcal{K} \).

As in AGARWAL-WOLF [2] in finite dimensions, one quantizes the sesqui-linear Borel-Fourier transform

\[
M(z^*, z) = \langle M(\zeta^*, \zeta) | e^{z^*\xi + \xi^*z} \rangle \quad (2.17)
\]
as continuous normal, Weyl, anti-normal pseudovariational operators from \( \mathcal{K} \) to \( \mathcal{K}^* \) defined by their matrix elements

\[
\langle z^* | \hat{M}_V | z \rangle := \langle M_V(\zeta^*, \zeta) | \langle z^* | e^{\zeta^*} | z \rangle \rangle, \quad (2.18)
\]
\[
\langle z^* | \hat{M}_\Theta | z \rangle := \langle M_\Theta(\zeta^*, \zeta) | \langle z^* | e^{\zeta^*} | z \rangle \rangle, \quad (2.19)
\]
\[
\langle z^* | \hat{M}_\alpha | z \rangle := \langle M_\alpha(\zeta^*, \zeta) | e^{-z^*} \langle z^* | e^{\zeta^*} | z \rangle \rangle. \quad (2.20)
\]

6The paper [2] is a formal operator algebra in terms of phase space c-equivalents of operator functions. The present papers provides, in particular, a rigorous functional analysis content (cp. DYNIN [11] for a preliminary version).
$M_\nu$, $M_\sigma$ and $M_\alpha$ are the normal and anti-normal co-Grothendieck kernels of the corresponding operators and, by (2.16), belong to $\mathcal{H}^* \hat{\otimes} \mathcal{H}^*$. As sesqui-holomorphic functionals they have convergent MacLaurin expansions into homogeneous polynomials (their $n$-th differentials), so that the operators

$$M_\nu = M_\nu(\hat{\zeta}^*, \zeta), \quad M_\sigma = M_\sigma(\hat{\zeta}^*, \zeta), \quad M_\alpha = M_\alpha(\hat{\zeta}^*, \zeta) \quad (2.21)$$

are strongly converging series of the corresponding quantizations of the $n$-th differentials (cp. Berezin generating functionals [4]).

By Grothendieck kernel theory, the nuclearity of the Kree-Gelfand triples implies that the locally convex vector spaces $\cdots \to \cdots$ of continuous linear operators are topologically isomorphic to the complete sesqui-linear tensor products (both spaces are endowed with the topology of compact uniform convergence).

$$\left(\mathcal{H} \to \mathcal{H}^*\right) \cong \mathcal{H}^* \hat{\otimes} \mathcal{H}^*, \quad (2.22)$$

$$\left(\mathcal{H} \to \mathcal{H}\right) \cong \mathcal{H}^* \hat{\otimes} \mathcal{H}, \quad (2.23)$$

where the operators are tame in the case of (2.23).

**Proposition 2.1** Any continuous linear operator $Q : \mathcal{H} \to \mathcal{H}^*$ has a unique normal co-Grothendieck kernel $M_\nu(\zeta, z) = \langle z | Q | z \rangle$.

In particular, if an operator $Q$ has a continuous polynomial co-Grothendieck kernel, than $Q$ is tame.

**Proof** Since

$$\langle z^* | e^{\hat{\zeta}^*} e^{\hat{\xi}^*} | z \rangle = e^{-\zeta z} \langle e^{\hat{w}^*} | e^{\hat{w}^*} e^{(w^*+\xi^*)z} \rangle = e^{-\zeta z} \langle e^{\hat{w}^*} | e^{w^*z} e^{\xi^*z} \rangle = e^{-\zeta z} \langle e^{\hat{w}^*} | e^{w^*z} e^{\xi^*z} \rangle,$$

so that

$$\langle z^* | \hat{M}_\nu | z \rangle = \langle M_\nu(\zeta^*, \zeta) | e^{-z^*z} e^{\zeta^*z} e^{\xi^*z} \rangle e^{-\zeta z} = M_\nu(\zeta^*, \zeta), \quad (2.24)$$

where $M_\nu(z^*, z)$ is the sesqui-linear Borel-Fourier transform of $M_\nu(\zeta^*, \zeta)$. Thus $M_\nu(z^*, z)$ is the co-Grothendieck kernel of $\hat{M}_\nu$.

By (2.10), any operator $Q : \mathcal{H} \to \mathcal{H}^*$ has a unique representation $\hat{M}_\nu$. QED

**Corollary 2.1** Any continuous operator $Q : \mathcal{H} \to \mathcal{H}^*$ has unique Weyl and anti-normal Grothendieck kernels $M_\sigma^0(\zeta^*, z)$ and $M_\alpha^0(\zeta^*, z)$. 

---

7 Cp. Berezin [5, Equation (1.7)]
Corollary 2.2 Any continuous operator \( Q : \mathcal{K} \rightarrow \mathcal{K}^* \) has a strongly convergent expansion into a power series of \( \hat{\zeta}^j \hat{\zeta}^k^*, j + k \geq 0 \) defined by a unique McLaurin series expansion of its sesqui-entire Grothendieck kernel.

Any polynomial \( Q \) is tame.

In view of the uniqueness of McLaurin coefficients, the sesqui-entire functionals are uniquely defined by their restrictions to the real diagonal
\[
\mathcal{R}(\mathcal{K}^* \times \mathcal{K}) = \{ (z^*, z) : z^* \text{ is the Hermitian conjugate of } z \}. \tag{2.25}
\]
Thus the normal symbol of the operator \( Q \)
\[
\sigma^Q_V(z) := M_V(z^*, z), \quad (z^*, z) \in \mathcal{R}(\mathcal{K}^* \times \mathcal{K}), \tag{2.26}
\]
exists and defines \( Q \) uniquely.

Similarly, the restrictions of \( M^Q_\Theta(z^*, z) \) and \( M^Q_\alpha(z^*, z) \) to the real diagonal \( \mathcal{R}(\mathcal{K}^* \times \mathcal{K}) \) define the Weyl and anti-normal symbols Weyl and anti-normal symbols \( \sigma^Q_\Theta(z) \) and \( \sigma^Q_\alpha(z) \) of \( Q \).

The symbols are real analytic functionals of \((\mathcal{R}z, \mathcal{I}z)\) on \( \mathcal{K} \) that have a unique extension to \( \mathcal{K}^* \times \mathcal{K} \) in \( \mathcal{K}^* \otimes \mathcal{K} \).

Since the Borel-Fourier transform intertwines the operators of differentiation \( \partial_z \) and \( \partial_{z^*} \) with operators of multiplication with the linear forms \( \zeta \) and \( \zeta^* \), the equations
\[
\sigma^Q_\Theta(z) = e^{-(1/2)\partial_z \partial_{z^*}} \sigma^Q_V(z), \tag{2.27}
\]
\[
\sigma^Q_\alpha(z) = e^{\partial_z \partial_{z^*}} \sigma^Q_V(z), \tag{2.28}
\]
\[
\sigma^Q_\Theta(z) = e^{-(1/2)\partial_z \partial_{z^*}} \sigma^Q_\alpha(z). \tag{2.29}
\]

Remark 2.1 The co-Grothendieck kernel extension of a real symbol is invariant under the complex conjugation, and so Grothendieck kernel is selfadjoint. The corresponding operator is symmetric on \( \mathcal{K} \). Furthermore, if a symbol is a continuous polynomial then and nonnegative then it has Friedrichs extension in \( \mathcal{K}^0 \).

2.3 Quantized Galerkin approximations

A Galerkin sequence \( p_n, j = 1, 2, ..., \) is an increasing sequence of selfadjoint projectors of rank \( n \) from \( \mathcal{K}^* \) to \( \mathcal{K} \) that strongly converge to the identity operator in \( \mathcal{K} \).

The finite dimensional projectors induce the quantized Galerkin sequence
\[
P_n \Psi(z^*) := \Psi(p_n z^*), \quad P_n \Psi^*(z) := \Psi(p_n z) \tag{2.30}
\]
of infinite dimensional projectors in the triple $\mathcal{H}$ onto cylindrical triples isomorphic to the pulled back sesqui-entire triples over the tautological finite dimensional triple $\mathbb{C}^n \subset \mathbb{C}^n \subset \mathbb{C}^n$.

By Proposition 2.1, the compressions of operators $Q_n := P_n Q P_n$ of $Q$ are cylindrical pseudodifferential operators with the exponential Grothendieck kernels,

$$\langle z^* | Q_n | e^w \rangle = \langle e^{P_n z^*} | Q | e^{P_n w} \rangle, \quad (2.31)$$

i.e. pullbacks from $\mathbb{C}^j$ of finite dimensional pseudodifferential operators of AGARWAL-WOLF [2].

**Theorem 2.1** Operator $Q$ is the strong limit of the cylindrical pseudodifferential operators $Q_n$ on $\mathcal{H}$.

**Proof** The matrix element $\langle \Psi^* | Q | \Phi \rangle$ is a separately continuous sesquilinear form on the Frechet space $\mathcal{H}$. By a Banach theorem (see, e.g., [26, v.1,Theorem V.7]), the sesquilinear form is actually continuous on $\mathcal{H}$. In particular, operator $Q$ is the weak limit of $Q_n$ in $\mathcal{H}$. Since $\mathcal{H}$ is a nuclear space, the weak convergence implies the strong one in the topology of $\mathcal{H}$. QED

As $n \to \infty$, the matrix elements

$$\langle z^* | Q_n | e^w \rangle = \langle e^{P_n z^*} | Q | e^{P_n w} \rangle \longrightarrow \langle z^* | Q | e^w \rangle, \quad (2.32)$$

so that symbols of the cylindrical $Q_\nu$ converge to the corresponding symbols of $Q$. This allows to extend to pseudovariational operators some results of finite-dimensionalf pseudodifferential theory. In particular, we get the following extensions for sufficient operator positivity tests of BEREZIN [5, Theorem 6] for anti-normal pseudodifferential operators and of HOWE [20, Theorem 3.2.1] for Weyl pseudodifferential operators.

**Theorem 2.2** An operator $Q$ in $\mathcal{H}$ is nonnegative and symmetric if its anti-normal symbol $\sigma^Q_\alpha$ or its Weyl symbol $\sigma^Q_\omega$ are nonnegative functionals on $\mathcal{H}$.

### 3 Yang-Mills theory

#### 3.1 Yang-Mills fields

The global gauge group $G$ of a Yang-Mills theory is a connected semi-simple compact Lie group with the Lie algebra $g$.

The Lie algebra carries the adjoint representation $\text{Ad}(g) X = g X g^{-1}$, $g \in G, X \in g$, of the group $G$ and the corresponding self-representation $\text{ad}(X) Y = [X, Y], \ X, Y \in \mathfrak{g}$.

---

8By Paley-Wiener theorem, Howe’s symbolic calculus is unitarily equivalent to the sesqui-holomorphic one in the case of finite dimensions (cp. FOLLAND [14] Chapter 2, Section 7)).
The adjoint representation is orthogonal with respect to the positive definite Ad-invariant scalar product

\[ X \cdot Y := -\text{Trace}(\text{ad}X \text{ad}Y), \]

(3.1)

the negative Killing form on \( g \).

There exists an orthonormal basis \( \{X_i\} \) in \( g \) such that

\[ [X_i, X_j] = c_{ijk} X_k, \]

(3.2)

with the structure constants \( c_{ijk} \) are skew-symmetric with respect to interchanges of all \( i, j, k \). Let the Minkowski space \( \mathbb{M} \) be oriented and time oriented with the Minkowski metric signature \((-1, 1, 1, 1)\). In a Minkowski coordinate system \( x^\mu, \mu = 0, 1, 2, 3 \) the metric tensor is diagonal. In the natural unit system, the time coordinate \( x^0 = t \).

The local gauge group \( \tilde{G} \) is the group of infinitely differentiable \( G \)-valued functions \( g(x) \) on \( \mathbb{M} \) with the pointwise group multiplication. The local gauge Lie algebra \( \tilde{g} \)-valued functions on \( \mathbb{M} \) with the pointwise Lie bracket. consists of infinitely differentiable \( g \)-valued functions on \( \mathbb{M} \) with the pointwise Lie bracket.

\( \tilde{G} \) acts via the pointwise adjoint action on \( \tilde{G} \) and correspondingly on \( \mathcal{A} \), the real vector space of gauge fields \( A = A_\mu(x) \in \tilde{g} \).

Gauge fields \( A \) define the covariant partial derivatives

\[ \partial A_\mu X := \partial_\mu X - \text{ad}(A_\mu)X, \quad X \in \tilde{G}. \]

(3.3)

This definition shows that in the natural units gauge connections have the mass dimension \( 1/|L| \).

Any \( \tilde{g} \in \tilde{G} \) defines the affine gauge transformation

\[ A_\mu \mapsto A^{\tilde{g}}_\mu := \text{Ad}(\tilde{g})A_\mu - (\partial_\mu \tilde{g})\tilde{g}^{-1}, \quad A \in \mathcal{A}, \]

(3.4)

so that \( A^{\tilde{g}_1 \tilde{g}_2} = A^{\tilde{g}_1 \tilde{g}_2} \).

Yang-Mills curvature tensor \( F(A) \) is the antisymmetric tensor

\[ F(A)_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]. \]

(3.5)

The curvature is gauge invariant:

\[ \text{Ad}(g)F(A) = F(A^g), \]

(3.6)

\[ ^9 \text{Summation over repeated indices is presumed throughout this section.} \]

\[ ^{10} \text{The dimensionless Yang-Mills coupling } g_{YM}^2 \text{ is set to 1.} \]
as well as Yang-Mills Lagrangian

\[(1/4)F(A)_{\mu\nu} \cdot F(A)_{\mu\nu}.\]  \hspace{1cm} (3.7)

The corresponding gauge invariant Euler-Lagrange equation is a 2nd order non-linear partial differential equation \(\partial_{\mu} F(A)_{\mu\nu} = 0\), called the \textit{Yang-Mills equation}

\[\partial_{\mu} F^\mu - [A_{\mu}, F^\mu] = 0.\]  \hspace{1cm} (3.8)

\textit{Yang-Mills fields} are solutions of Yang-Mills equation.

### 3.2 First order formalism

In the temporal gauge \(A_0(t, x^k) = 0\) \footnote{I. Segal theory \cite{27} of infinite-dimensional Sobolev Lie groups implies that for any infinitely differentiable gauge field on \(\mathbb{M}\) there is a unique infinitely differentiable gauge transformation to the temporal gauge.} the 2nd order Yang-Mills equation (3.8) is equivalent to the 1st order Schwinger hyperbolic system for the time-dependent \(A_j(t, x^k)\), \(E_j(t, x^k)\) on \(\mathbb{B}\) (see, e.g., GOGANOV-KAPITANSKII \cite{19}, Equation (1.3))

\[\partial_t A_k = E_k, \quad \partial_t E_k = \partial_j F^j_k - [A_j, F^j_k], \quad F^j_k = \partial^j A_k - \partial_k A^j - [A^j, A_k].\]  \hspace{1cm} (3.9)

and the \textit{constraint equations}

\[[A^k, E_k] = \partial^k E_k, \quad \text{i.e.} \quad \partial_{kA} E_k = 0.\]  \hspace{1cm} (3.10)

By GOGANOV-KAPITANSKII \cite{19}, the evolution system is a semilinear first order partial differential system with \textit{finite speed propagation} of the initial data, and the Cauchy problem for it with constrained initial data at \(t = 0\)

\[a_k(x) := A(0,x_k), \quad e_k(x) := E(0,x_k), \quad \partial^k e_k = [a_k, e_k] \]  \hspace{1cm} (3.11)

is \textit{globally and uniquely solvable} in local Sobolev spaces on the whole Minkowski space \(\mathbb{M}\) (with no restrictions at the space infinity.)

This fundamental theorem has been derived via Ladyzhenskaya 1949 method (see \cite{19}) by a reduction to the case of Cauchy data on 3-dimensional balls \(\mathbb{B} = \mathbb{B}(R) : |x| < R\).

If the constraint equations are satisfied at \(t = 0\), then, in view of the evolution system, they are satisfied for all \(t\) automatically. Thus the \textit{1st order evolution system along with the constraint equations for Cauchy data is equivalent to the 2nd order Yang-Mills system.} Moreover the constraint equations are invariant under \textit{time independent} gauge transformations.
Consider the chain of Hilbert spaces $\mathcal{A}_s$, $-\infty < s < \infty$, of (generalized) connections $a(x)$ that are completions of connections with compact supports in open balls $\mathbb{B}$ of radius $R$ with respect to the norms

$$|a|^2_s := \int_{\mathbb{B}} d^3x (a \cdot (1 - \triangle)^s a) < \infty.$$ (3.12)

They define the real Gelfand nuclear triple (cp., e.g., [15])

$$\mathcal{S} : \mathcal{S} := \bigcap \mathcal{A}_s \subset \mathcal{A}_0 \subset \mathcal{A}_* := \bigcup \mathcal{A}_s,$$ (3.13)

where $\mathcal{A}$ is a nuclear countably Hilbert space with the dual $\mathcal{A}_*$. Similarly we define the chain of Sobolev-Hilbert spaces $S_s$, $-\infty < s < \infty$, of (generalized) Lorentz scalar fields $u(x)$ on $\mathbb{B}$ with values in $\text{Ad G}$ and the Hilbert norms $|u|_s$. Let

$$\mathcal{I} : \mathcal{I} := \bigcap \mathcal{S}_s \subset \mathcal{S}_0 \subset \mathcal{S}_* := \bigcup \mathcal{S}_s$$ (3.14)

be the corresponding Gelfand triple.

Let $a \in \mathcal{A}_s^{s+3}, s \geq 0$. Then, by Sobolev embedding theorem $a$ is continuously $s + 2$-differentiable on $\mathbb{B}$ and, therefore, the following gauged vector calculus operators are continuous:

- **Gauged gradient** $\text{grad}^a : \mathcal{I}_s^{s+1} \to \mathcal{A}_s$,
  $$\text{grad}^a u := \partial_k u - [a_k, u].$$ (3.15)

- **Gauged divergence** $\text{div}^a : \mathcal{A}_s^{s+1} \to \mathcal{S}_s$,
  $$\text{div}^a b := \text{div} b - [a; b], \quad [a; b] := a_k b_k.$$ (3.16)

- **Gauged curl** $\text{curl}^a : \mathcal{A}_s^{s+1} \to \mathcal{A}_s$,
  $$\text{curl}^a b := \text{curl} b - [a \times b], \quad [a \times b]_i := \varepsilon_{ijk} [a_j, b_k].$$ (3.17)

- **Gauged Laplacian** $\triangle^a : \mathcal{I}_s^{s+2} \to \mathcal{S}_s$,
  $$\triangle^a u := \text{div}^a(\text{grad}^a u).$$ (3.18)

The adjoints of the gauged operators are

$$(\text{grad}^a)^* = -\text{div}^a.$$ (3.19)

**Lemma 3.1** If $a \in \mathcal{A}_s^{s+3}, s \geq 0$, then the operator $\text{div}^a : \mathcal{A}_s^{s+1} \to \mathcal{A}_s$ is surjective.
Let $S^{s+2}$, $s \geq 0$, denote the closure in $S^{s+2}$ of the space of $a$’s with compact support in the interior of $B$. The conventional Laplacian $\Delta^0: S^{s+2} \to S^s$ is an isomorphism (see, e.g., Agranovich[3]).

The gauged Laplacian $\Delta^a$ differs from the usual Laplacian $\Delta^0$ by first order differential operators, and, therefore is a Fredholm operator of zero index from $S^{s+2}$ to $S^s$, $s \geq 0$.

If $\Delta^a u = 0$ then then $(\Delta^a u)^* u = (\text{grad}^a u)^* (\text{grad}^a u)$, so that $\text{grad} u = [a, u]$. The computation

\[(1/2)\partial_k (u \cdot u) = (\partial_k u \cdot u) = [a_k, u] \cdot u = -\text{Trace}(a_k uu - ua_k u)] = 0 \quad (3.20)\]

shows that the solutions $u \in S^{s+2}$ are constant. Because they vanish on the ball boundary, they vanish on the whole ball. Since the index of the Fredholm operator $\Delta^a$ is zero, its range is a closed subspace with the codimension equal to the dimension of its null space. Thus the operator $\text{div}^a \text{grad}^a$ is surjective, and so is $\text{div}^a$.

QED

3.3 Transversal quasi gauge

Consider the bundles $E^s, s \geq 0$ of constraint Cauchy data with the base $A$ and the null spaces $E_a^{s+1}$ of the operators $\text{div}^a: E^{s+1} \to E^s$ as fibers over $a \in A$.

Their intersection $E$ is a bundle of nuclear countably Hilbert spaces over the nuclear countably Hilbert base $A$. Together with the unions of the dual spaces $E^{-s}$ they form a bundle of nuclear Gelfand triples $E$ over the same base.

**Theorem 3.1** The bundle $E$ is smoothly trivial, so that the total space of $E$ is smoothly isomorphic to the direct product of its base $A$ and the fiber $E_{a=0}$, the nullspace of the operator $\text{div}$ in $E$.

**Proof** For $0 \leq s \leq \infty$ consider the mapping

\[f : A^{s+2} \times E^{s+1} \to A^s, \quad f(a, e) := \text{div}_a(e) \quad (3.21)\]

Sobolev imbedding theorem shows that the mapping is continuous. Lemma 3.1 implies that the continuous partial Frechet derivatives $\partial_a f(a, e)$ are bounded linear operators onto a fixed Hilbert space $T(s)$, the orthogonal complement of constant $a$’s. continuously dependent on the parameter $a \in A^{s+2}$. The restrictions of $\partial_a f(a, e)$ to the orthogonal complements of the null spaces of $\text{div}_a$ are one-to-one. By the implicit function theorem on Hilbert spaces (see, e.g., [23]), this implies that the explicit solutions $e = e(a)$ of the equation $f(a, e) = 0$ provide infinitely smooth local trivializations of Hilbert bundles $E^s$.

Their intersection $E = \cap E^s$ is a locally trivial $C$-bundle over $A$ with the associated locally trivial bundle of smooth $\ast$-orthonormal frames in the fibers.

---

12In this paper, smooth = infinitely differentiable.
Since \( \mathcal{A} \) is a Frechet space, its smooth homothety retraction to the origin \( a = 0 \) has a homotopy lifting to the frame space. Thus the bundle \( \mathcal{C} \) is trivial, so that the total set of constraint Cauchy data carries the global chart \( \mathcal{A} \times \mathcal{C}_{a=0} \). QED

Let \( \mathcal{A}^s \) and \( \mathcal{E}^s \) denote the nullspaces of the operator \( \text{div} \) in \( \mathcal{A}^s \) and \( \mathcal{E}^s \).

By DELL’ANTONIO-ZWANZIGER [10], the closures of smooth gauge orbits in \( \mathcal{H}^0 := \mathcal{A}^0 \) intersect \( \mathcal{E}^0 \). These closures are the orbits of the Sobolev group, the closure in Sobolev space \( W^{1,2}(\mathbb{B}) \) of the group of smooth gauge transformations. (This Sobolev group is a topological group of continuous transformations in \( \mathcal{A}^0 \).)

Thus \( \mathcal{H}^0 := \mathcal{A}^0 \times \mathcal{E}^0 \) is a quasi-gauge for the orbifold of the direct product of the parallel transports (i.e. every \( (a, e) \in \mathcal{H}^0 \) is on an orbit but some orbits may intersect \( \mathcal{H}^0 \) more than once (cp. SINGER [29] and NARASIMHAN-RAMADAS [24]).

The Gelfand triple

\[
\mathcal{H} : \mathcal{H} := \mathcal{A} \times \mathcal{E} \subset \mathcal{H}^0 := \mathcal{A}^0 \times \mathcal{E}^0 \subset \mathcal{H}^* := \mathcal{A}^* \times \mathcal{E}^*
\]

is the direct product of the Gelfand triples \( \mathcal{A} \) and \( \mathcal{E} \).

4 Yang-Mills mass gap theorem

4.1 Yang-Mills energy-mass functional

Energy-mass functional of smooth Yang-Mills Cauchy data (cp. GLASSEY-strauss [17], Section 3] on \( \mathbb{B} \) is

\[
M(a, e) := (1/2) \int_{\mathbb{B}} d^3 x \left( (\text{curl} a \cdot (\text{curl} a) + e \cdot e. \right. \tag{4.1}
\]

The density \( \text{curl} a - [a \times a] \cdot (\text{curl} a - [a \times a]) \) is the scalar gauge curvature of \( a \) and, as such, is invariant under the gauge parallel transport but the density \( e \cdot e \) is not.

At the same time the density \( e \cdot e \) is invariant under the flat isometric parallel transport provided by Theorem 3.1. Thus the energy-mass functional \( M \) is constant on smooth orbits of the direct product of both parallel transports so that the energy-mass functional is uniquely defined by its restriction to smooth transversal \( (a, e) \).

From now on we consider only transversal \( (a, e) \) only.

The transversality \( \text{div} a = 0 \) implies that \( a \) has a matrix vector potential on \( \mathbb{B}(R) \) so that \( \text{curl} a \) is zero everywhere. Thus on transversal \( (a, e) \) the functional (4.1) becomes

\[
M(a, e) := (1/2) \int_{\mathbb{B}} d^3 x \left( [a \times a] \cdot [a \times a] + e \cdot e \right). \tag{4.2}
\]

By tensor computation with Levi-Civita epsilon and Kronecker delta, the commutator
part

\[
[a \times a] \cdot [a \times a] = \varepsilon_{ijk} \varepsilon_{klm} [a_j, a_k] \cdot [a_l, a_m] = (\delta_i \delta_{jm} - \delta_{im} \delta_{ji}) [a_j, a_k] \cdot [a_l, a_m] = [a_i, a_j] \cdot [a_i, a_j] - [a_i, a_j] \cdot [a_j, a_i] = 2 [a_i, a_j] \cdot [a_i, a_j].
\]

Let \( \alpha^k \) be an orthonormal basis for \( g \) with respect to the constant basis (3.2) with the anti-symmetric structure constants \( c_{ijk} \).

Then, by SIMON [28, page 217],

\[
[a_i, a_j] \cdot [a_i, a_j] = \sum_k (\alpha^k_i \alpha^k_j c_{ijk})^2.
\]

(4.3)

Thus the transversal restriction (4.2) of the Yang-Mills energy-mass functional becomes

\[
M(a, e) = (1/2) \int_B d^3 x (2 \sum_k (\alpha^k_i \alpha^k_j c_{ijk})^2 + e \cdot e).
\]

(4.4)

### 4.2 Yang-Mills energy-mass operator

Convert the transversal quasi-gauge triple (3.22) into the complex Gelfand triple \( \mathcal{H} \) with conjugation where the real and imaginary parts are the direct factors

\[
\Re \mathcal{H} := \mathcal{A}, \quad \Im \mathcal{H} := \mathcal{E}.
\]

(4.5)

In particular,

\[
z := (a + ie)/\sqrt{2}, \quad z^* := (a - ie)/\sqrt{2}, \quad \partial^2 / \partial z^* \partial z = (1/2)(\partial a^2 + \partial e^2).
\]

(4.6)

Let the polynomial energy-mass functional \( M(z, z^*) := M(a, e) \) (ref: eq:Noether) be the \textit{anti-normal symbol} of the quantum Yang-Mills energy-mass operator

\[
H := \hat{M}_{\alpha} : \mathcal{H} \to \mathcal{H}.
\]

(4.7)

By Remark 2.1, it has a unique nonnegative selfadjoint Friedrichs extension \( H \) in the transversal \( \mathcal{H}^0 \) (the notation does not change).

**Theorem 4.1** The fundamental spectral value of the quantum energy-mass operator \( H \) is the simple zero eigenvalue with the vacuum eigenvector. In particular, the quantum Yang-Mills spectrum has a positive mass gap.

**Proof**

\[\text{There is an obvious misprint in the Simon formula: squarings are missing.}\]
• Separate the variables in the Yang-Mills energy-mass functional (4.4)

\[ M(a, e) = M_1(a) + M_2(e). \]  

(4.8)

By (2.29), the Weyl symbol \( \sigma_{\alpha}^{H} \) of the anti-normal Yang-Mills energy-mass operator \( H \) is

\[ (1 + \frac{1}{2})\partial^2 / \partial \alpha^2 + (1/2)^2(\partial^2 / \partial \alpha^2)^2)M_1(a) \]

(4.9)

\[ + (1 + (1/2)^2 / \partial \alpha^2 + (1/2)^2(\partial^2 / \partial \alpha^2)^2)M_2(e). \]

The differential operator \( \partial^2 / \partial \alpha^2 \) is invariant under the orthogonal transformation from \( a(x) \) to \( \alpha_i^k(x) \), so that, by SIMON (28, page 217),

\[ \partial^2 / \partial \alpha^2 M_1(a) = \int d^3x (1/2)\partial^2 / \partial (\alpha_i^k(x))^2 2 \sum_k (\alpha_i^k \alpha_j^k c_{ijk})^2(x). \]

(4.10)

The skew-symmetry of \( c_{ijk} \) implies that \( \sum_k \alpha_i^k \alpha_j^k c_{ijk} \) does not contain \( (\alpha_i^k)^2 \). Then, by a Leibniz formula,

\[ \frac{\partial^2}{\partial (\alpha_i^k)^2} \sum_k (\alpha_i^k \alpha_j^k c_{ijk})^2 = 2(\partial^2 / \partial (\alpha_i^k)^2) \sum_k (\alpha_i^k \alpha_j^k c_{ijk}) (\sum_k (\alpha_i^k \alpha_j^k c_{ijk}) \]

\[ + 2(\partial / \partial \alpha_i^k) \sum_k (\alpha_i^k \alpha_j^k c_{ijk}) (\partial / \partial \alpha_i^k) \sum_k (\alpha_i^k \alpha_j^k c_{ijk}) \]

\[ = 2 \sum_{ijk} \alpha_i^k c_{ijk} \alpha_j^k c_{ijk}(x) = 2a(x) \cdot a(x). \]  

(See SIMON (28 page 217)).

Thus

\[ \partial^2 / \partial \alpha^2 M_1(a) = \alpha^* a, \text{ and then } (\partial^2 / \partial \alpha^2)^2 M_1(a) = 2. \]

(4.11)

Besides,

\[ \partial^2 / \partial e^2 M_2(e) = 1, \quad (\partial^2 / \partial e^2(x))^2 M_2(e) = 0. \]

(4.12)

Equations (4.9), (4.11), (4.12) show that the Weyl symbol of \( \hat{M}_a \) is

\[ M(a, e) = (1/2)(\alpha^* a + 3/2 = (1/2)(\alpha^* a + 3/2) \]

\[ > (1/2)(\alpha^* a + e^* e) \]

\[ = z^* z. \]  

(4.13)

• The tame quadratic operator

\[ N = \partial^e \partial^* : \mathcal{H} \to \mathcal{H}, \quad \sigma_{\alpha}^{N} = z^* z, \]

(4.14)

is the number operator.
Lemma 4.1  Power states $z^n, n = 0, 1, \ldots$, form an orthogonal set in $\mathcal{H}^0$ and are eigenvectors of $N$:

$$N(z^n) = nz^n, n = 0, 1, 2, \ldots, \quad (4.15)$$

with the dense linear span in $\mathcal{H}^0$.

In particular, the number operator has the unique self-adjoint Friedrichs extension $H$ (the notation is preserved) with the spectrum consisting of the eigenvalues $0, 1, 2, \ldots$.

Proof  The monomials are cylindrical states in the Bargman-Segal space $\mathcal{H}^0$ of co-rank 1. Then the Lemma holds because it is true in the Bargman-Segal space on the complex line $\mathbb{C}$ (see, e.g., Faddeev-Slavnov [12, Chapter II, Section 2]).

From (4.13) we get the inequality of the Weyl symbols

$$\sigma_{\hat{a}\hat{a}}^H(z^*, z) > z^*z^* - \frac{1}{2} \quad (4.16)$$

The positive difference

$$\sigma_{\hat{a}\hat{a}}^H - \sigma_{\hat{a}\hat{a}}^N = (1/2)[a^* a] : |a^* a| + 2$$

is the positive Weyl symbol of a tame polynomial operator $Q$. It follows, by 2.2, that $Q$ is positive so that

$$H > N. \quad (4.17)$$

• The functional $M(z^*, z)$ is a polynomial with zero constant term. Therefore, its anti-normal quantization $H$ is spanned by monomials $\hat{\zeta}^j \hat{\xi}^k$, $j + k > 0$. For the constant vacuum state $1$

$$\hat{\zeta}^* 1 = \partial_{\zeta} 1 = 0, \quad \hat{\xi}^* 1 = (1^* \xi) 1 = 0, \quad (4.18)$$

so that $H1 = 0$. This implies, since $H$ is nonnegative, that the vacuum state 1 is a fundamental eigenstate with zero eigenvalue.

The vacuum state is the a fundamental eigenstate with zero eigenvalue for the number operator $N$. Moreover it is simple, i.e., the corresponding eigenspace of $N$ is one-dimensional, and the spectral gap of $N$ is

Then, by Glazman lemma ((see, e.g., Berezin-Shubin [6] Appendix 1, Lemma 3.1)), the operator inequality (4.13) implies that the the spectrum of the operator $H$, just as of $N$, may contain only eigenvalues in the open interval $-\infty < \lambda < 1$ and the sum of their multiplicities cannot be greater than such sum for $N$. The latter sum is equal to 1. Since the interval $(-\infty, 1)$ already contains the zero spectral value of $H$, it follows that this spectral value is the simple fundamental eigenvalue. Thus $H$ has a positive spectral gap $\geq 1$.  

17
The cutoff of the energy-mass functional \( M(a, e) \) to a ball \( B(R) \) is associated with the following spectral renormalization:

**Proposition 4.1** *The spectra of cut-off quantum Yang-Mills energy-mass operators are self-similar in the inverse proportion to the radius \( R \) of the ball \( B = B(R) \).*

**Proof** The scaling transformation

\[
\tilde{x} := x/R, \quad \tilde{a} := a/R, \quad \tilde{e} := e/R^2
\]  

(4.19)

converts the energy-mass functional \( \mathcal{M} \) over \( B(R) \) into the scaled energy-mass functional over \( B(1) \).

Moreover, the scaling transformation is canonical: it preserves the symplectic form \( \mathcal{S}(z_1^* z_2) \) so that the quantum canonical relations are conserved under the scaling.

QED

**References**

[1] *Quantum Yang-Mills theory*, [http://www.claymath.org/prizeproblems/index.htm](http://www.claymath.org/prizeproblems/index.htm)

[2] Agarwal, C. S., and Wolf, E., *Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics*, Physical Rev. D **2**, 2161-2225, 1970.

[3] Agranovich, M., S., *Partial differential operators. IX: Elliptic Boundary Problems*, (Encyclopaedia of Mathematical Sciences), Springer, 2010.

[4] Berezin, F. A., *The Method of Second Quantization*, Academic Press, New York, 1966.

[5] Berezin, F. A., Covariant and contravariant symbols, Math. USSR Izvestia **6** (1972), 1117-1151.

[6] Berezin, F. A., and Shubin, M. A., *The Schrödinger equation*, Kluwer Academic Publishers, 1991.

[7] J. Beringer et al (Particle Data Group), *Higgs bosons: theory and searches*, *Phys. Rev.*, **D86**, 010001, 2012.

[8] Bogoliubov, N. N., Logunov. A. A., Oksak, A. I., and Todorov, I. T., *General Principles of Quantum Field Theory*, Kluwer, 1990.

[9] Colombeau, J., *Differential Calculus and Holomorphy*, North-Holland Mathematics Studies, **84**, 1982.
[10] Dell’Antonio, G. and Zwanziger, D., Every gauge orbit passes inside the Gribov horizon, Comm. Math. Phys., 138 (1991), 259-299.

[11] Dynin, A., Feynman integral for functional Schrödinger equations, Partial Differential Equations: M. Vishik Seminar (American Mathematical Society Translations-Series 2, (Vishik, M.I, et al, eds), 206( 2002), 65-80.

[12] Faddeev, L. D., and Slavnov, A. A., Gauge Fields, Introduction to Quantum Theory, Addison-Wesley, 1991.

[13] Faddeev, L. D., Mass in quantum Yang-Mills theory (comment on a Clay Millennium Problem, Perspectives in Analysis, Math. Phys. Studies, 27. Springer, 2005. (arXiv:0911.1013 [math-ph])

[14] Folland, G. B, Harmonic Analysis in Phase Space, Princeton University Press, 1989.

[15] Gelfand, I. M., and Minlos, R. A., Solution of quantum field equations, reprinted in Gelfand, I. M., Collected Papers, VI, 462-465, Springer, 1987.

[16] Gelfand, I., Vilenkin, N.. Generalized functions, 4. Academic Press, 1964.

[17] Glassey, R. T., and Strauss, W. A., Decay of Classical Yang-Mills fields, Comm.. Math. Phys., 65 (1979), 1-13.

[18] Glimm, J. and Jaffe, A., An infinite renormalization of the hamiltonian is necessary, J. Math. Phys., 10 (1969), 2213-2214.

[19] Goganov, M. V., and Kapitanskii, L. V., Global solvability of the Cachy problem for Yang-Mills-Higgs equations, Zapiski LOMI, 147 (1985), 18-48; J. Sov. Math., 37 (1987), 802-822.

[20] Howe, R., Quantum mechanics and partial differential equations, J. Funct. Anal., 38 (1980),188254.

[21] Kree, P., Calcule symbolique et second quantification, C. R. Sc., Serie A, 284 (1978), 25-28.

[22] Kree, P., Methodes holomorphe et methodes nucleaires en analyse de dimension infinie et la theorie quantique des champs, Lecture notes in mathematics, 644 (1978), 212-254.

[23] Lang, S., Differential and Riemannian manifolds, Springer, 1995.

[24] Narasimhan, M.S., Ramadas, T.R., Geometry of SU(2) gauge fields, Communications in Mathematical Physics, 67 (1979), 121-136.
[25] Pitch, A., *Nuclear locally convex spaces*, Springer, 1972.

[26] Reed, M., Simon, B., *Methods of Modern Mathematical Physics*, Academic Press, 1972.

[27] Segal, I., *The Cauchy Problem for the Yang-Mills Equations*, Journal of Functional Analysis, 33(1979), 175-194.

[28] Simon, B., *Some Quantum Operators with Discrete Spectrum but Classically Continuous Spectrum*, Annals of physics, 146 (1983), 209-220.

[29] Singer, I. M., *Some remarks on the Gribov ambiguity*, Communications in Mathematical Physics, 60 (1978), 7-12.

[30] Strocchi, F. S., *Selected Topics of the General properties of Quantum Field Theory*, World Scientific, 1993.

[31] Strocchi, F. S., *An Introduction to Non-Perturbative Foundations of Quantum Field Theory*, Oxford University Press, 2013.

[32] Witten, E., *Physical law and the quest for mathematical understanding*, Bulletin of American Mathematical Society, 40(2002), no.1, 21-29.