String-Inspired Gravity with Interacting Point Particles

Dongsu Bak and Domenico Seminara

Center for Theoretical Physics
Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139, U.S.A.

ABSTRACT

We reformulate two dimensional string-inspired gravity with point particles as a gauge theory of the extended Poincaré group. A non-minimal gauge coupling is necessary for the equivalence of the two descriptions. The classical one-particle problem is analyzed completely. In addition, we obtain the many-particle effective action after eliminating the gravity degrees of freedom. We investigate properties of this effective action, and show how to recover the geometrical description. Quantization of the gauge-theoretic model is carried out and the explicit one-particle solution is found. However, we show that the formulation leads to a quantum mechanical inconsistency in the two-particle case. Possible cures are discussed.

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I Introduction

There have been many attempts to describe gravity in terms of a gauge theory, since gauge theories enjoy a well-defined quantization procedure. These include the Ashtekar formulation of Einstein gravity, Poincaré gravity, Chern-Simons gravity and so on. Recently, string-inspired gravity was also reformulated as a gauge theory using the extended Poincaré group. Its quantization was carried out and it was found that the quantum solution of pure gravity is described by a rather trivial state characterized by two constant modes of the gravity fields — constants that can be interpreted as the black hole mass and the cosmological constant. The quantum equivalence to the geometric approach in has been proven by a direct comparison of wave functionals. A model in which point particles couple to gravity in a gauge invariant fashion, is available as well. Its explicit many-particle quantum solutions has been found. One unsatisfactory aspect of the solution is that the particles do not feel any gravitational interactions, even though gravity fields feel the presence of particles. The reason follows from the fact that in the model, particles couple to the gauge metric that is flat and differs from the original physical metric by a conformal factor, while the particles are massive and therefore not conformally invariant.

In this paper, we introduce a gauge-theoretic model, in which massive particles couple to the physical metric of the string-inspired gravity. In order to provide a gauge invariant description for matter particles, one introduces Poincaré coordinates as additional dynamical variables. We shall show that the model is classically equivalent to the geometric formulation of the string-inspired gravity with point particles by comparing equations of motion in a specific gauge called the “unitary” gauge. We solve the classical one-particle problem and discuss the equivalence at the level of the solution space. We also analyze the classical many-particle problem, and derive an effective Lagrangian for particles in which gravity variables are completely eliminated.
When pursuing quantum solutions of the model, one of our main concerns is whether
the enlargement of dynamical variables causes any problems in the equivalence between
the geometric approach and gauge-theoretic formulation. We shall see that in the present
model, inconsistencies at the quantum level do arise from the enlargement for the case of
more than one particle. We shall clarify the origin of the inconsistencies and discuss possible
cures in detail.

II Gauge formulation of lineal gravity

When the string-inspired gravity was first proposed, its geometrical action was taken to
be
\[ I = \frac{1}{4\pi G} \int d^2x \sqrt{-g} \rho e^{-2\phi} (R(g_\rho) + 4g^\mu_\nu \partial_\mu \phi \partial_\nu \phi - \lambda). \] (2.1)

where \( \lambda \) is the cosmological constant and \( \phi \) the dilaton field. Subsequently it was recognized
that the introduction of the new variables
\[ g^\mu_\nu = e^{-2\phi} g_\rho^\mu_\nu \quad \text{and} \quad \eta = e^{-2\phi} \] (2.2)
transforms the action (2.1) into a simpler expression
\[ I = \frac{1}{4\pi G} \int d^2x \sqrt{-g} (\eta R(g) - \lambda), \] (2.3)

which can be reformulated as a gauge theory. In fact a gauge theoretical formulation
of the action (2.3) can be given by using the 4-parameter extended Poincaré group in 1+1
dimensions, with the Lie algebra
\[ [P_a, P_b] = \epsilon_{ab} \mathbf{1}, \quad [P_a, J] = \epsilon^b_a P_b \] (2.4)
\[ [P_a, J] = [J, J] = 0, \]

\footnote{Notation: the signature of the metric tensor \( g^\mu_\nu \) is assumed to be \((1, -1)\). The Latin indices \( a, b, c \ldots \)
run over a tangent space where the flat Minkowski metric \( h_{ab} = \text{diag}(1, -1) \) is defined. The antisymmetric
symbol \( \epsilon^{ab} \) is normalized by \( \epsilon^{01} = 1 \).}
where \( I \) is the central element that modifies the conventional algebra of translation generators \( P_a \), while the Lorentz boost sector is unchanged. The extension arises naturally in two dimensions if one allows non-minimal gravitational coupling, as pointed out in Ref. [7].

The gauge field is now introduced as a connection one-form that takes values in the Lie algebra

\[
A_\mu = e^a_\mu P_a + \omega_\mu J + a_\mu I, \tag{2.5}
\]

where \( e^a \) and \( \omega \) are, respectively, the Zweibein and the spin connection. The potential \( a_\mu \) is related to the volume form [7]. The connection defined in (2.5) transforms according to the adjoint representation. In components the transformation reads

\[
\begin{align*}
(e^U)^a_\mu &= (\Lambda^{-1})^a_b (e^b_\mu + e^b_\mu \theta^c \omega_\mu + \partial_\mu \theta^b) \\
(\omega^U)_\mu &= \omega_\mu + \partial_\mu \alpha \\
(a^U)_\mu &= a_\mu - \theta^a \epsilon_{ab} e^b_\mu - \frac{1}{2} \theta^a \theta_\mu \theta_\mu + \frac{1}{2} \partial_\mu \theta^a \epsilon_{ab} \theta^b
\end{align*}
\]

where we have parameterized the gauge transformation as follows

\[
U = \exp(\theta^a P_a) \exp(\alpha J) \exp(\beta I) \tag{2.9}
\]

and \( \Lambda^a_b \) is the Lorentz transformation matrix

\[
\Lambda^a_b = \delta^a_b \cosh \alpha + \epsilon^a_b \sinh \alpha. \tag{2.10}
\]

The field strength associated to the connection (2.5) is now computed from its definition

\[
F = dA + [A, A] = (de^a + \epsilon^a_b \omega \wedge e^b)P_a + d\omega J + (da + \frac{1}{2} \epsilon_{ab} e^a \wedge e^b)I. \tag{2.11}
\]

To construct an invariant action linear in the curvature, we introduce a multiplet, \( \eta_A \equiv (\eta_a, \eta_2, \eta_3) \), that transforms according to the co-adjoint representation

\[
(\eta^U)_a = (\eta_b - \eta_3 \epsilon_{bc} \theta^c) \Lambda^b_a \tag{2.12}
\]
\[(\eta^U)_2 = \eta_2 - \eta_a \epsilon^a_b \theta^b + \frac{1}{2} \eta_3 \theta^a \theta_a \quad (2.13)\]
\[(\eta^U)_3 = \eta_3. \quad (2.14)\]

Note that \(\eta_a\) may be set to zero by a gauge transformation. The action is now simply formed by contracting \(\eta_A\) with \(\epsilon^{\mu \nu} F_{\mu \nu}^A\)

\[I_g = \frac{1}{4\pi G} \int d^2 x \epsilon^{\mu \nu} \left( \eta_a D_\mu e^a_\nu + \eta_2 \partial_\mu \omega_\nu + \eta_3 (\partial_\mu a_\nu + \frac{1}{2} \epsilon_{abc} e^a_\mu e^c_\nu) \right). \quad (2.15)\]

It is easy to show [7, 10] that this \(B\)-F theory is equivalent to the string-inspired gravity defined by the geometrical action (2.3) once we identify \(\eta_2 = 2 \eta\) and \(g_{\mu \nu} = e^a_\mu e^a_\nu\). The cosmological constant, \(\lambda\), is generated dynamically by the field \(\eta_3\), which is fixed to be a constant by the equations of motion.

A gauge invariant description of the matter requires an introduction of a new variable, the Poincaré coordinate \(q^a\). In particular, for the case of point particles, a possible first-order invariant action is [3, 11]

\[I_m = \int d\tau \left[ p_a (D_\tau q)^a - \frac{1}{2} N (p^a p_a + m^2) \right] \quad (2.16)\]
\[(D_\tau q)^a \equiv \dot{q}^a + \epsilon^a_b \left( q^b \omega_\mu (X) - e^b_\mu (X) \right) \dot{X}^\mu, \quad (2.17)\]

where \(\tau\) is the trajectory parameter and the dot denotes derivative with respect to \(\tau\). [To avoid a cumbersome notation, we just use \(X\) when \(X^\mu (\tau)\) appears as argument of a function.]

The particle is described by the dynamical variables \(q^a (\tau), p^a (\tau)\) and \(X^\mu (\tau)\), while \(N (\tau)\) is a Lagrange multiplier that enforces the mass-shell condition. Under a gauge transformation the gauge variables, \(q^a (\tau)\) and \(p^a (\tau)\) transform according to the following rule

\[ (q^U)^a = (\Lambda^{-1})^a_b (q^b (\tau) + \epsilon^b_c \theta^c (X)) \quad (2.18)\]
\[ (p^U)^a = \Lambda^a_b p^b (\tau), \quad (2.19)\]

where all the gauge parameters are computed along the trajectory \(X^\mu\).
According to (2.18) the gauge, \( q^a = 0 \), is always available; when it is chosen, the usual geometrical action for a point particle

\[
I_m = \frac{1}{2} \int d\tau \left( \frac{\dot{X}^\mu g_{\mu\nu}(X) \dot{X}^\nu}{N} - Nm^2 \right)
\]  

(2.20)
is recovered by eliminating \( p^a \). (In a certain sense the \( q^a \) field looks like a Higgs field in a gauge theory with symmetry breaking: its presence insures the gauge invariance, but when the unitary gauge, \( q^a = 0 \), is chosen the physical content of the theory is exposed.)

Though the action in (2.16) appears natural in the gauge-theoretic approach, the particles do not follow geodesics of the physical metric \( g_{\mu\nu} \). Instead, \( I_m \) describes particles moving along geodesics of the gauge metric that differ from the physical one by the conformal factor \( \eta \). Since the particles are massive, such a conformal redefinition changes the dynamics. To make the particles move in the physical metric, a suitable modification of the action (2.16) is needed. In particular, an introduction of a non-minimal coupling with the \( \eta^4 \) fields is required.

### III The model and its classical solutions

In this section we shall investigate the string-inspired gravity coupled to point particles. In the geometric formulation, the action [3] is given by

\[
I = \frac{1}{4\pi G} \int d^2x \sqrt{-g} e^{-2\phi} (R(g_{\mu\nu}) + 4g_{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \lambda)
\]  

\[
+ \frac{1}{2} \int d\tau \left( \frac{\dot{X}^\mu \theta_{\mu\nu}(X) \dot{X}^\nu}{N} - Nm^2 \right)
\]  

(3.1)

where \( X^\mu(\tau) \) is the particle trajectory and \( m \) is the mass. In order to connect the geometrical approach with the gauge formulation of the previous section, we introduce the new variables defined in Eq. (2.2). The action (3.1) now reads

\[
I = \frac{1}{4\pi G} \int d^2x \sqrt{-g} (\eta R - \lambda) - \frac{1}{2} \int d\tau \left( \frac{\dot{X}^\mu \eta_{\mu\nu}(X) \dot{X}^\nu}{N\eta(X)} - Nm^2 \right). 
\]  

(3.2)
The gravity sector takes the same form of the gauge action in (2.3). The particle is coupled not only to the gauge metric $g_{\mu\nu}$, but also to the dilaton field $\eta$. This explains why the action (2.16) is not suitable to describe the matter sector of the action in (3.1).

A way of overcoming this difficulty is to consider the following gauge invariant action for the particle as well as the gravity

$$I = \frac{1}{4\pi G} \int d^2 x \epsilon^{\mu\nu} \left( \eta_a D_\mu e^a_\nu + \eta_2 \partial_\mu \omega_\nu + \eta_3 \left( \partial_\mu a_\nu + \frac{1}{2} \epsilon_{ab} e^a_\mu e^b_\nu \right) \right)$$

$$+ \int d\tau \left[ p_a (D_\tau q)^a - \frac{1}{2} N \left( \frac{q^A \eta_A(X)}{2} p^2 + m^2 \right) \right]$$

(3.3)

where $q^A \eta_A(X)$ is the gauge invariant combination $\eta_a q^a + \eta_2 + \frac{1}{2} \eta_3 q_a q^a$ computed along the trajectory $X^\mu$.

The equivalence between the two actions can be easily shown by comparing all the equations of motion in $q^a = 0$ gauge with those in the geometric formulation.

Let us now write down the equations of motion and look for the one body solution. For the gravity sector, we have

$$\delta \eta_a \rightarrow S^a_{\mu\nu} = -\pi G \epsilon_{\mu\nu} \int d\tau N p^2 q^a \delta^2(x - X)$$

(3.4)

$$\delta \eta_2 \rightarrow R_{\mu\nu} = -\pi G \epsilon_{\mu\nu} \int d\tau N p^2 \delta^2(x - X)$$

(3.5)

$$\delta \eta_3 \rightarrow \epsilon^{\mu\nu} (\partial_\mu a_\nu + \frac{1}{2} \epsilon_{ab} e^a_\mu e^b_\nu) = \frac{\pi G}{2} \int d\tau N p^2 q^2 \delta^2(x - X)$$

(3.6)

$$\delta e^a \rightarrow \partial_\mu \eta_a + \epsilon_{ab} \eta^b \omega_\mu + \epsilon_{ab} e^a_\mu \eta_3 = -4\pi G \epsilon_{ab} \epsilon_{\mu\nu} \int d\tau p^b \dot{X}^\nu \delta^2(x - X)$$

(3.7)

$$\delta \omega_\mu \rightarrow \partial_\mu \eta_2 + \epsilon_{ab} \eta^a e^b_\mu = -4\pi G \epsilon_{ab} \epsilon_{\mu\nu} \int d\tau p^a q^b \dot{X}^\nu \delta^2(x - X)$$

(3.8)

$$\delta a_\mu \rightarrow \partial_\mu \eta_3 = 0,$$

(3.9)

---

1. We notice that if we consider a particle with zero mass, the dependence on $\eta$ can be eliminated through a redefinition of the Lagrange multiplier $N(\tau)$. This is related to the fact that the light-like geodesics are preserved by conformal transformations of the metric.

2. This equivalence is more transparent if we rewrite the particle action in a second order formalism

$$I_m = \int d\tau \frac{1}{2} \left( \frac{D_\tau q^a D_\tau q_a}{N q^A \eta_A(X)} - N m^2 \right).$$

In fact, for $q^a = 0$ this action collapses immediately to the geometrical action. However, a reliable proof requires checking the equivalence of all the equations of motion.
Here $\delta^2(x - X)$ stands for $\delta(x^0 - X^0(\tau))\delta(x^1 - X^1(\tau))$, $S^{a\nu}_{\mu\nu}(\equiv D_{\mu}e^{a}_\nu - D_{\nu}e^{a}_\mu)$ denotes the torsion two-form and $R^{a\nu}_{\mu\nu}(\equiv \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu})$ is the curvature two-form in terms of the spin connection. The matter sector produces three independent equations

$$\delta N \rightarrow \frac{\eta A q^A}{2} p^2 + m^2 = 0 \quad (3.10)$$

$$\delta p^a \rightarrow D_{\tau} q^a - N p^\mu \frac{\eta A q^A}{2} = 0 \quad (3.11)$$

$$\delta q^a \rightarrow D_{\tau} p^a - \frac{N p^2}{4} (\eta^a + q^a \eta_3) = 0 \quad (3.12)$$

where $D_{\tau} p^a \equiv \dot{p}^a + e^a_b \omega_\mu \dot{X^\mu}$. The variation with respect to $X^\mu$ merely produces an equation of motion, which can be expressed as a linear combination of Eqs. (3.4-3.12). This linear dependence of the particle equations of motion is, in fact, a generic feature of theories with Poincaré invariance. It is straightforward to show that the geodesic equation of the physical metric is derived from Eqs. (3.10-3.12) when one chooses the unitary gauge, $q^a = 0$. This confirms the stated equivalence between the two formulations.

Classical solutions for the system may be obtained conveniently by choosing a gauge, but note, first, that Eq. (3.9) requires $\eta_3$ to be constant and we fix its value to be $\lambda$ to get agreement with the geometric description in (3.1). Upon choosing the gauge $\eta^a = 0$, a unique solution of Eq. (3.7) is

$$e^a_\mu(x) = -4\pi G \frac{p^a(\xi)}{\lambda} \frac{\dot{X}^\nu(\xi)}{X^0(\xi)} \delta(\sigma - X^1(\xi)) = -\frac{2\pi G}{\lambda} p^a(\xi) \partial_\mu \epsilon(\sigma - X^1(\xi)) \quad (3.13)$$

where $\sigma \equiv x^1$, $t \equiv x^0$, $\xi(t) \equiv (X^0)^{-1}(t)$ and $\epsilon(x)$ denotes the sign function of $x$. Note also that in this gauge, Eq. (3.8) becomes

$$\partial_\mu \eta_2 = 2\pi G \epsilon_{ab} q^a(\xi) p^b(\xi) \partial_\mu \epsilon(\sigma - X^1(\xi)) \quad (3.14)$$

The quantity $\epsilon_{ab} q^a p^b$ in the right side of the above equation is conserved in this gauge\footnote{In an arbitrary gauge one can show that the gauge invariant combination $B = \epsilon_{ab} (q^a + \frac{\omega^a(X^0)}{\eta}) p^b$ is conserved using particle equations of motion. The existence of this conservation law is related to the invariance under the Lorentz group. In the case of many particles, the conserved quantity is $\sum_i B_i$.}.
we call its constant value $\nu$. The solution of Eq. (3.14) reads

$$\eta_2(t, \sigma) = \frac{M}{2\lambda} + 2\nu \epsilon(\sigma - X^1(\xi)).$$  \hspace{1cm} (3.15)$$

where $M$ is a new integration constant. It can be shown that Eq. (3.4) is equivalent to Eq. (3.12) on the solution in (3.13) and that Eq. (3.6) simply fixes the form of the potential $a_\mu$ which does not play a role in the following. The remaining equation in (3.3) is solved by

$$\omega_0 = -\frac{\pi G}{2} p^2(\xi) \frac{N(\xi)}{X_0(\xi)} \epsilon(\sigma - X^1(\xi)), \hspace{1cm} (3.16)$$

where we choose $\omega_1 = 0$ by using the residual invariance under the Lorentz transformations.

All the equations of motion for the particle on the solutions in (3.14), (3.15) and (3.16) are reduced to

$$\frac{M + \lambda^2 q^2}{4\lambda} p^2 + m^2 = 0$$

$$q^a - \frac{N}{4\lambda} p^a(M + \lambda^2 q^2) = 0$$

$$p^a - \frac{N\lambda}{4} p^2 q^a = 0,$$

where the prescription, $\epsilon(0) = 0$ is used to specify $\eta_2(X)$ and $\omega_0(X)$. This excludes a possible self-interaction of the particle.

One striking fact at this point is that there is no equation for the position variable, $X^\mu$, hence it is an arbitrary function of time. The only requirement is that $\dot{X}^0(\tau)$ is positive definite, which is necessary for the existence of $\xi(\tau)$ in (3.13–3.16).

In the one-body problem, in order to reach the unitary gauge we first choose $X^a$ to be the same as $\epsilon^a_b q^b$ by using the arbitrariness of $X^a$, then we perform the gauge transformation generated by

$$\theta^a(x) = -x^a - \frac{2\pi G}{\lambda} \rho^a \epsilon(\sigma - X^1), \hspace{1cm} \alpha(x) = 0,$$  \hspace{1cm} (3.20)$$

where the coefficient of the sign function is determined such that the transformed Zweibein does not involve any discontinuities and delta function singularities.
We note that (3.17-3.19), with the relation
\[ X^a = \epsilon^a_{\ b} q^b, \]
are the first-order formulation of the geodesic equation for a metric tensor
\[ g_{ab} = \frac{4h_{ab}}{M^2 - \lambda X_c X^c}, \tag{3.21} \]
which describes the two dimensional black-hole. The metric also agrees with the physical metric tensor felt at the particle position, namely \[ \bar{g}^P_{\mu\nu}(X) = 2\bar{e}_a^\mu(X)\bar{e}_\nu^a(X)/\eta_2(X). \]
The geodesics, solutions of Eqs. (3.17-3.19), are given by
\[ X^0 = -q^1 = \frac{M^2}{2\gamma} \left[ e^{\nu\tau} (\gamma \cosh(\gamma \tau) - \nu \sinh(\kappa \tau)) + e^{-\nu\tau} (\gamma \cosh(\kappa \tau) + \nu \sinh(\gamma \tau)) \right], \tag{3.22} \]
\[ X^1 = -q^0 = \frac{M^2}{2\gamma} \left[ e^{\nu\tau} (\gamma \cosh(\gamma \tau) - \nu \sinh(\gamma \tau)) - e^{-\nu\tau} (\gamma \cosh(\kappa \tau) + \nu \sinh(\gamma \tau)) \right], \tag{3.23} \]
where \( \gamma \equiv \sqrt{\nu^2 - 4m^2/\lambda}. \) The most general solutions are those obtained by applying a \( \tau \)-independent Lorentz transformations to the above solution. The integration constant \( M \) that is usually interpreted as a black hole mass, should be positive for the reality of the solution. The cosmological constant, \( \lambda, \) should also be positive to avoid a geodesic solution that describes a particle running into a naked singularity. For real \( \gamma, \) \( \nu^2 \geq 4m^2/\lambda, \) the geodesic represents a particle coming from infinity and falling into the black hole. On the other hand, the solution with imaginary \( \gamma, \) \( \nu^2 \geq 4m^2/\lambda, \) corresponds to a particle that comes out of the white hole and falls into the black hole.

Let us now turn to a many-body problem, described by an action
\[ I_N = \frac{1}{4\pi G} \int d^2 x \epsilon^{\mu\nu} \left( \eta_2 D_\mu e^a_\nu + \eta_2 \partial_\mu \omega_\nu + \eta_3 (\partial_\mu a_\nu + \frac{1}{2} \epsilon_{\ ab} e^a_\mu e^b_\nu) \right) + \sum_i \int d\tau \left[ p_{ia}(D_\tau q_i)^a - \frac{1}{2} N_i \left( \frac{q_i^A \eta_A(X_i)}{2} p_i^2 + m^2 \right) \right], \tag{3.24} \]
where \( i \) labels the \( i \)-th particle and \( X_i, q_i^a \) and \( p_i^a \) are, respectively, the position, Poincaré coordinates and Poincaré momenta of the \( i \)-th particle. The equations of motion for the gravity sector take the same form as Eqs. (3.4-3.9) except that the right sides of the equations are summed over all particles. The equations of motion for the \( i \)-th particle obtained from
variations with respect to \( \delta N_i \), \( \delta q_i \) and \( \delta p_i \) also take the same form as those for the one-particle in (3.10-3.12). However, contrary to the one-body problem, the \( \delta X_\mu^i \) variation gives linearly-independent equations of motion, which may be reduced to conditions

\[
(q_i^a - q_j^a) \delta^2(X_i - X_j) = 0 ,
\]

(3.25)

when one uses the other equations. Eq. (3.25) simply means that \( q_i = q_j \) when \( X_\mu^i = X_\mu^j \).

With the gauge choice \( \eta_a(x) = 0 \) and \( \omega_1 = 0 \), solutions for the gravity sector read

\[
e_a^\mu = -\frac{2\pi G}{\lambda} \sum_i p_i^a(\xi_i) \partial_\mu \epsilon(\sigma - X_1^i(\xi_i)) \] (3.26)

\[
\eta_2 = \frac{M}{2\lambda} + \sum_i \epsilon_{ab}q_i^a(\xi_i)p_i^b(\xi_i) \epsilon(\sigma - X_1^i(\xi_i)) \] (3.27)

\[
\omega_0 = -\frac{\pi G}{2} \sum_i p_i^2(\xi_i) \frac{N_i(\xi_i)}{X_0^i(\xi_i)} \epsilon(\sigma - X_1^i(\xi_i)). \] (3.28)

where \( \xi_i = (X_i^0)^{-1}(t) \). Taking these solutions into account, it is straightforward to show that all the remaining equations for the particles are derived from an effective Lagrangian,

\[
L = \sum_i \left( p_{ia} \dot{q}_i^a - \frac{N_i}{2}(\varphi_i p_i^2 + m^2) \right) - \frac{4\pi G}{\lambda} \sum_{i \neq j} \epsilon_{ab}q_i^a p_j^b \frac{d}{d\tau} \epsilon(X_i^1 - X_j^1) \] (3.29)

where

\[
\varphi_i = \frac{M + \lambda q_i^2}{2\lambda} + \frac{2\pi G}{\lambda} \sum_{i \neq j} \epsilon_{ab}q_i^a p_j^b \epsilon(X_i^1 - X_j^1) \] (3.30)

and all the dynamical variables \( (X_i, q_i, p_i \) and \( N_i \) ) are functions of \( \tau \). Note that the symplectic structure of this first order Lagrangian is not in a canonical form. To achieve this, we introduce a momentum \( \Pi_\mu \) conjugate to \( X_\mu^i \) and a new set of constraints

\[
M_i = \Pi_{i1} - \frac{4\pi G}{\lambda} \sum_{i \neq j} \epsilon_{ab}p_i^ap_j^b \delta(X_i^1 - X_j^1). \] (3.31)

Now the new Lagrangian reads

\[
L = \sum_i \left( p_{ia} \dot{q}_i^a + \Pi_\mu \dot{X}_\mu^i - \frac{N_i}{2}(\varphi_i p_i^2 + m^2) - U_i M_i \right) , \] (3.32)
where $U_i$ are the Lagrange multipliers that enforce the new constraints. It is interesting to notice that in the quantum case the particles will be described by the same first order Lagrangian, once we have solved gauge constraints.

As in the one-body case, the equations of motion do not determine the particle trajectories $X^\mu_i$. In addition, the equations of motion for $p^a_i$ and $q^a_i$ contain terms proportional to $\delta(X^1_i - X^1_j)$, which implies that the Poincaré coordinates $q^a_i$ must be discontinuous when particles meet (i.e. $X_i = X_j$). Therefore, the relation between the Poincaré coordinates and the particle trajectories is not as simple as in the one particle case.

All of these troubles disappear when one performs a gauge transformation generated by

$$\theta^a(x) = -x^a - \frac{2\pi G}{\lambda} \sum_i p^a_i \epsilon(\sigma - X^1_i), \quad \alpha = 0,$$

with a diffeomorphism gauge choice

$$X_i^a = \epsilon^a_b q^b_i + \frac{2\pi G}{\lambda} \sum_{j \neq i} p^a_j \epsilon(X^1_i - X^1_j).$$

As in the one-body case, this transformation leads to the unitary gauge. In (3.33), the coefficients of the sign function are chosen such that the transformed Zweibein does not involve any singularities and discontinuities. The identification in (3.34) is obtained simply from the invariance of the quantities $q^2_i + \eta(X_i)/\lambda$ under the above transformation.

The sign function in (3.34) removes discontinuous parts from the Poincaré coordinates, so that $X^a_i$ describes a continuous particle trajectory. Moreover, with the identification in (3.34), the conditions in (3.25) are trivially satisfied.

In this unitary gauge, using Eq. (3.34) we can rewrite the effective Lagrangian (3.29) in terms of $X_i^a$ and $P_i^a \equiv \epsilon^a_b p^b_i$

$$L = \sum_i \left( P_{ia} X^a_i - \frac{N_i}{2}(\tilde{\varphi}_i P_i^2 - m^2) \right),$$

where

$$\tilde{\varphi}_i = \frac{M}{2\lambda} - \frac{\lambda}{2} \left( X_i^a - \frac{2\pi G}{\lambda} \sum_{j \neq i} \epsilon^a_b P_i^b \epsilon(X^1_i - X^1_j)(X_{ia} - \frac{2\pi G}{\lambda} \sum_{j \neq i} \epsilon_{ac} P_i^c \epsilon(X^1_i - X^1_j)) \right)$$
\[ -\frac{2\pi G}{\lambda} \sum_{i \neq j} \epsilon_{a b} \left( X^a_j - \frac{2\pi G}{\lambda} \sum_{k \neq j} \epsilon^a_c P^c_k \epsilon(X^1_j - X^1_k) \right) P^b_j \epsilon(X^1_i - X^1_j). \]  

(3.36)

We present here the Lagrangian in a reparametrization invariant form, so it should be noted that all the dynamical variables in the Lagrangian are functions of \( \tau \).

This Lagrangian does not include any delta terms and, therefore, the problem arising from the delta terms in (3.29) disappears. It provides us with a clear picture of the system where all the gauge degrees of freedom have been eliminated. The geometry is now hidden in the factor \( \phi_i \). In fact, the action for each particle can be considered as the geodesic action for the metric \( h_{ab}/\phi_i \).

### IV Quantization

#### One-body problem

We first quantize the one-body problem. It turns out that for this case, one can consistently solve the constraints to find the most general wave-functional.

Owing to the gauge symmetry, the Lagrangian contains Gauss law constraints, which we proceed to solve first in order to find the most general gauge-invariant wave functional. Since the theory possesses also a general coordinate invariance, we encounter a Hamiltonian that consists of constraints only. Thus the quantization involves solely solving the constraints, by which one may find physical wave functionals.

Let us begin with recording the first order Lagrange density with a reparametrization gauge, \( t \equiv x^0 = X^0(\tau) \), which reads

\[
\mathcal{L} = \frac{1}{4\pi G} (\eta_a \dot{e}^a_1 + \eta_2 \dot{\omega}_1 + \eta_3 \dot{a}_1) + \epsilon^a_0 G_a + \omega_0 G_2 + a_0 G_3 \\
+ \left( p_a q^a + p^a \epsilon_{ab}(q^b \omega_1(X) - e^b_1(X)) \dot{X}^1 - N \left( \frac{1}{2} \eta_A q^A p^2 + m^2 \right) \right) \delta(\sigma - X^1(\tau))
\]

(4.1)

where dot/dash signifies time/space derivatives. The Gauss generators are

\[
G_a(\sigma) = \frac{1}{4\pi G} (\eta^a_1 + \epsilon_{ab} \eta^b_1 \omega_1 + \eta_3 \epsilon_{ab} e^b_1) + \epsilon_{ab} \delta(\sigma - X^1(\tau))
\]

(4.2)
\begin{align}
G_2(\sigma) &= \frac{1}{4\pi G}(\eta_2' + \eta^a\epsilon_{ab}\epsilon_1^b) - q^a\epsilon_{ab}p^b\delta(\sigma - X^1(\tau)) \quad (4.3) \\
G_3(\sigma) &= \frac{1}{4\pi G}\eta_3'. \quad (4.4)
\end{align}

Since the symplectic structure for the dynamical variable $X^1$ is not in a standard form, we introduce one more constraint,

$$
M(X) \equiv \Pi - p^a\epsilon_{ab}(q^b\omega_1(X) - e_1^b(X))
$$

with help of a Lagrange multiplier $U$. Thus, the Lagrangian is equivalently presented as

$$
\mathcal{L} = \frac{1}{4\pi G}(\eta_0\epsilon_0^a + \eta_2\omega_1 + \eta_3\dot{a}_1) + (p_0\dot{q}^a + \Pi\dot{X}^1)\delta(\sigma - X^1) \\
+ \epsilon_0^a G_a + \omega_0 G_2 + a_0 G_3 - \{NH(X) + UM(X)\}\delta(\sigma - X^1)
$$

where $H(X)$ denotes the mass-shell constraint $\frac{1}{2}\eta^A(X)q_Ap^2 + m^2$ in the background geometry $2\eta_{\mu\nu}/\eta_Aq^A$. The terms in the first line of the Lagrangian describe the symplectic structure of the theory, while the remaining terms work as constraints. When implementing the Dirac procedure, we first need to check whether there are any secondary constraints by commuting primary constraints with one another. The Gauss law generators simply produce the Lie algebra of the $ISO(1,1)$ group. The algebra is

\begin{align}
[G_a(\sigma), G_b(\sigma')] &= i\epsilon_{ab}G_3(\sigma)\delta(\sigma - \sigma') \quad (4.7) \\
[G_a(\sigma), G_2(\sigma')] &= i\epsilon_{ab}G^b(\sigma)\delta(\sigma - \sigma') \quad (4.8) \\
[G_A(\sigma), G_3(\sigma')] &= 0 \quad (4.9)
\end{align}

In addition, the momentum and mass-shell constraints are gauge invariant, i.e.

$$
[G_A(\sigma), H(X)] = [G_A(\sigma), M(X)] = 0
$$

and finally the commutation between mass-shell and momentum constraints is

$$
[M(X), H(X)] = -i2\pi Gq^A(p^2G_A(X)).
$$
Hence, the algebra of the constraints is closed, so the constraints are all first-class. (When we discuss two-particle problem, the closure of the constraint algebra will be at issue again, and in fact it will be shown that the algebra does not close.)

For each dynamical variable, we may use either coordinate representation or momentum representation: the two are related by an appropriate Fourier transform. In our case, we shall use momentum representations for gravitational and Poincarè variables, $A_1^A \equiv (e_1^a, \omega_1, a_1) = (i4\pi G \delta/\delta \eta_1, i4\pi G \delta/\delta \eta_2, i4\pi G \delta/\delta \eta_3)$, $q_a = i\partial/\partial p^a$, while we shall use the usual coordinate representation, $\Pi = \partial/\partial \eta^1$ for the particle position, $X^1$. Accordingly the wave function or functional is a function or functional of $\eta^A, p^a$ and $X^1$.

As in [10], the most general solution of the gauge constraints in the momentum representation is

$$\Phi = e^{i(\Omega(\eta)+4i\pi G p^a \eta_a(X))} \delta(\eta_3') \delta((\eta^A \eta_A)' - 2\pi G \delta(\sigma - X) q^a \epsilon_{ab} p^b) \Psi(p^a, M, \lambda, X^1) \quad (4.12)$$

where $\Omega$ is an integral of the Kirillov–Kostant one-form

$$\Omega = \int \eta^a \epsilon_{ab} d\eta^b / \eta_3 \quad (4.13)$$

and $M$ and $\lambda$ are, respectively, constant parts of $\eta^A \eta_A$ and $\eta_3$. Note that the second delta function in (4.12) includes an operator $q^a \epsilon_{ab} p^b$, which could be diagonalized if $\psi$ is an eigenfunction see below.

As is usual in gauge theory, imposing the Gauss law constraint on the wave functional ensures gauge invariance of the “coordinate” representation wave functional. However, in the “momentum” representation, one expects that the wave functional is gauge-invariant up to a phase factor. This may be understood as follows. The momentum representation is related with the coordinate representation by a functional Fourier transform, $\Phi(\eta) = \int DA e^{-i \int d \eta^A A^A} \Phi(A)$. Since the measure $DA$ and $\Phi(A)$ are gauge invariant but $\int d \eta^A A^A \eta_A$ is not, the wave functional in the momentum representation is not gauge invariant. In
fact, it has a structure of gauge invariant functional multiplied by phase exponential of the integrated Kirillov–Kostant one–form, which is denoted by $\Omega$ above.

When one applies to $\Phi$ the remaining constraints, the mass–shell and the momentum, one obtains new constraints on $\Psi(p^a, M, \lambda, X^1)$. The new momentum constraint reads

$$\Pi \Psi = \frac{\partial}{i\partial X^1} \Psi = 0,$$  

which implies that the wave function does not depend on the particle coordinate $X^1$. On the other hand, the mass-shell constraint takes the form,

$$\left[\frac{\lambda}{4} (q^2 + \frac{M}{\lambda^2}) p^2 + m^2\right] \Psi(p^a) = 0.$$  

The equation may be interpreted as describing a particle moving in a background geometry with a metric,

$$\bar{g}_{\mu\nu} = \frac{4\eta_{\mu\nu}}{\lambda(q^2 + M/\lambda^2)}.$$  

The metric $\bar{g}_{\mu\nu}$ is not affected by the particle state since its form is determined by the contribution from the pure gravity sector. Thus, there are essentially no self-interactions.

For the ordering convention of the noncommuting operators, $p^a$ and $q^a$, we follow the same ordering as that in the Laplace-Beltrami operator with a metric $\bar{g}_{\mu\nu}$. With this ordering, when one takes the $m = 0$ limit, the constraint is reduced to $p^2 \Psi(p) = 0$. This is the expected result because for $m = 0$, the theory possesses a conformal invariance and the metric $\bar{g}_{\mu\nu}$ is conformally flat.

Since the boost generator $q^a \epsilon_{ab} p^b$ commutes with the mass-shell constraint, we diagonalize it:

$$q^a \epsilon_{ab} p^b \left(\frac{p^0 + p^1}{p^0 - p^1}\right) \frac{\hat{\Psi}}{\hat{\Psi}} = \nu \left(\frac{p^0 + p^1}{p^0 - p^1}\right) \frac{\hat{\Psi}}{\hat{\Psi}},$$  

where $\nu$ is a real eigenvalue of the boost generator ranging from $-\infty$ to $\infty$. 

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Finally, the solution of the mass-shell constraint is given by
\[ \Psi(p) = \left( \frac{p^0 + p^1}{p^0 - p^1} \right) \frac{1}{p^2} \psi_\nu(-p^2), \] (4.18)
where \( \psi_\nu(z) \) satisfies the differential equation:
\[ \left[ \frac{d^2}{dz^2} + \frac{d}{zd} + \frac{1}{z^2} \left( \frac{\nu^2}{4} - \frac{m^2}{\lambda} - \frac{M}{4\lambda^2 z} \right) \right] \psi_\nu(z). \] (4.19)
This is solved by the Bessel function
\[ \psi_\nu(z) = Z_{i\gamma}(\beta z^{1/2}), \] (4.20)
where \( \gamma = \sqrt{\nu^2 - 4m^2/\lambda}, \beta = M^{1/2} \) and \( Z_{i\gamma}(x) \) denotes the Bessel function of imaginary argument \( K_{i\gamma}(x) \) or \( I_{i\gamma}(x) \). (Here, we assume that the cosmological constant \( \lambda \) and the black hole mass \( M \) are positive.)

Note that the asymptotic behavior of the Bessel functions are
\[ I_{i\gamma}(z) \to \frac{e^z}{\sqrt{2\pi z}}, \quad K_{i\gamma}(z) \to \frac{e^{-z}}{\sqrt{2\pi z}}. \] (4.21)
Therefore, one has to exclude \( I_{i\gamma} \), since a wave function in the position space for \( I_{i\gamma} \) defined by a Fourier transform does not exist.

The meaning of this one-body wave function is clearly seen from the Poincaré coordinate space. The Fourier transform of the wave function with \( e^{iq^a p_a} \) gives a coordinate-space wave function depending solely on the combination \( \rho^a(X) = q^a + \frac{\eta^a(X)}{\lambda} \) [cf. (4.12)]. Thus, in the unitary gauge \( q^a = 0 \), choosing the diffeomorphism gauge \( \frac{\eta^a(X)}{\lambda} = \epsilon_a^b X^b \), we obtain a usual interpretation in terms of the position variable \( X \). It is a wave function describing a particle moving in the geometry characterized by the metric in (3.21).

Two-Body problem

Let us now turn to the many-body problem, which is more involved than the one-particle case since there are interactions between particles through the coupling of gravity.
For simplicity, we concentrate on the two-body problem in the following. The strategy of solving the constraints are the same as in the one-particle problem.

The gauge constraints read

\[
G_a(\sigma) = \frac{1}{4\pi G} (\eta_a' + \epsilon_{ab} \eta_b' \omega_1 + \eta_b' \epsilon_{ab} e_1^b) + \sum_i \epsilon_{ab} p_i^b \delta(\sigma - X_i^1(\tau)),
\]

\[
G_2(\sigma) = \frac{1}{4\pi G} (\eta_2' + \eta^a \epsilon_{ab} e_1^b) - \sum_i \eta_i^a \epsilon_{ab} p_i^b \delta(\sigma - X_i^1(\tau)),
\]

\[
G_3(\sigma) = \frac{1}{4\pi G} \eta_3'.
\]

As one sees from the above, the Gauss law constraint consists of a sum of the contributions from each particle. Mass-shell and momentum constraints are, respectively,

\[
M_i(X_i) \equiv \Pi_i - p_i^0 \epsilon_{ab} (q_i^b \omega_1 - e_i^b),
\]

\[
H_i(X_i) \equiv \frac{1}{2} \eta_A q_i^A p_i^2 + m^2.
\]

As in the one-particle problem, the Gauss law generators, \( G^A \), satisfies Lie algebra in (4.9), and \( M_i(X_i) \) and \( H_i(X_i) \) are also gauge invariant: \( M_i(X_i) \) and \( H_i(X_i) \) commute with the Gauss law generators and \( [M_1, M_2] = [H_1, H_2] = 0 \). The remaining commutation relations are

\[
[M_1 + M_2, H_i] = -i2\pi G q_i^A p_i^2 G_A(X_i),
\]

\[
[M_1 - M_2, H_1] = -i2\pi G q_1^A p_1^2 G_A(X_1) + i4\pi G (q_1 - q_2)^a \epsilon_{ab} p_2^b \delta(X_1^1 - X_2^1) p_1^2,
\]

\[
[M_1 - M_2, H_2] = i2\pi G q_2^A p_2^2 G_A(X_2) + i4\pi G (q_1 - q_2)^a \epsilon_{ab} p_1^b \delta(X_1^1 - X_2^1) p_2^2.
\]

where the fact that \( p_i^2 \) commutes with \( G_A \), has been used. Thus the constraint algebra is not closed unless \( X_1^1 \) is different from \( X_2^1 \). One might think that this problem is restricted to the zero measure subset of the whole phase space, so that one may ignore the problem, for example, by an appropriate boundary condition [13]. However, this cannot be achieved consistently in this case, as we shall see in the following.
At this point, one may try to find all possible secondary constraints by commuting repeatedly the induced constraint with the primary constraints. However, we shall not determine the secondary constraints, rather we show from just the primary constraints, that the theory does not admit any solutions.

Let us begin by solving the gauge constraints, where the most general solution is

$$\Phi = e^{i\Omega(\eta)} \sum_i 4\pi G p^a_i \eta_a(X_i) \delta(\eta_i) \delta(\eta^A \eta_A) - \sum_i 4\pi G \lambda \delta(\sigma - X_i^1) q^a_i \epsilon_{ab} p^b_i \delta(X_i^1 - X_i^1) \Psi(p_i^0, M, \lambda, X_i^1)(4.30)$$

Operating the mass-shell constraints and the momentum constraints on (4.30), we obtain a simplified version of the constraints on $\Psi$,

$$\check{M}_1 = \Pi_1 - \frac{4\pi G}{\lambda} p^a_i \epsilon_{ab} p^b_i \delta(X^1_1 - X^1_1),$$

$$\check{M}_2 = \Pi_2 + \frac{4\pi G}{\lambda} p^a_1 \epsilon_{ab} p^b_2 \delta(X^1_1 - X^1_2),$$

$$\check{H}_1 = \frac{\lambda}{4} \left( q^2_1 + \frac{M}{\lambda^2} + \frac{4\pi G}{\lambda} q^2_1 \epsilon_{ab} p^b_2 \epsilon(X^1_1 - X^1_2) \right) p^2_1 + m^2,$$

$$\check{H}_2 = \frac{\lambda}{4} \left( q^2_2 + \frac{M}{\lambda^2} - \frac{4\pi G}{\lambda} q^2_1 \epsilon_{ab} p^b_1 \epsilon(X^1_1 - X^1_2) \right) p^2_2 + m^2. \quad (4.31)$$

Note that this set of constraint can be obtained directly from the effective Lagrangian (3.32) for the case of two particles. As is seen in the mass-shell constraints, the particles interact via gravitation: for example, the metric of the first particle depends on the motion of the second particle. It is found that analyzing the constraints in a new coordinate system, $X_+ = \frac{X^1_1 + X^1_2}{2}$ and $X_- = X^1_1 - X^1_2$, is particularly convenient. Accordingly, we define $M_+ \equiv M_1 + M_2 = \partial / i \partial X_+$ and $M_- \equiv \frac{1}{2}(M_1 - M_2) = \partial / i \partial X_- - \frac{4\pi G}{\lambda} \epsilon_{ab} p^a_1 p^b_2 \delta(X_-)$. The total momentum constraint, $M_+$ commutes with all the other constraints, and merely implies that the wave function $\Psi$ does not depend on the coordinate $X_+$.

The remaining three require more careful analysis. Above all, the expressions involve $\delta(x)$ and the sign function $\epsilon(x)$, which need to be regularized to deal with, for example, $\delta(x) \epsilon^2(x)$ at $x = 0$. First, we take a following regularization defining

$$\epsilon_A(x) \equiv \tanh \Lambda x,$$
\[ \delta_{\Lambda}(x) \equiv \frac{d}{2dx}e_{\Lambda}(x). \]  

(4.32)

To get a representation of \( e(x) \), we take the limit in which \( \Lambda \) goes to infinity. Of course, during a specific computation, we do the computation with a generic \( \Lambda \) and in the end take the \( \Lambda \to \infty \) limit.

With the above regularization, we have the following rules of computation:

\[ \frac{d}{dx}e_{\Lambda}^{n}(x) = n\epsilon_{\Lambda}^{n-1}(x)\delta_{\Lambda}(x), \]  

(4.33)

\[
\lim_{\Lambda \to \infty} \int_{X} \frac{1}{2(n+1)} \frac{d}{dx}e_{\Lambda}^{n+1}(x)f(x)dx = -\lim_{\Lambda \to \infty} \int_{X} \frac{1}{2(n+1)} e_{\Lambda}^{n+1}(x) \frac{d}{dx}f(x)dx
\]

\[ = \frac{1}{2(n+1)} \left[ f(0^+) - (-1)^{n+1}f(0^-) \right] \]

\[ = \begin{cases} 
  0 & n = \text{odd} \\
  \frac{f(0)}{n+1} & n = \text{even}
\end{cases} \]  

(4.34)

where the continuity of the test function \( f(x) \) is assumed and \( n > 0 \). Hence,

\[ \delta(x)e^{n}(x) = \begin{cases} 
  0 & n = \text{odd} \\
  \frac{\delta(x)}{n+1} & n = \text{even}
\end{cases} \]  

(4.35)

Obviously one has to compute with arbitrary \( \Lambda \) and then take the limit in the end. We shall follow this way to solve our problem.

The regulated equations one has to solve to find the two-body wave function are

\[ \hat{M}\psi = \left( i\frac{\partial}{\partial X_-} - \frac{4\pi G}{\lambda} \epsilon_{ab}q_{1}^{a}q_{2}^{b}\delta_{\Lambda}(X_-) \right)\psi, \]  

(4.36)

\[ \hat{H}_{1}\psi = \left( q_{2}^{2} - \frac{M}{\lambda^{2}} + \frac{4\pi G}{\lambda} \epsilon_{ab}q_{1}^{a}q_{2}^{b}\epsilon_{\Lambda}(X_-) \right)\psi, \]  

(4.37)

\[ \hat{H}_{2}\psi = \left( q_{1}^{2} - \frac{M}{\lambda^{2}} - \frac{4\pi G}{\lambda} \epsilon_{ab}q_{1}^{a}q_{2}^{b}\epsilon_{\Lambda}(X_-) \right)\psi, \]  

(4.38)

where \( \psi(p_{1}^{a},p_{2}^{b}) = p_{1}^{a}p_{2}^{b}\Psi(p_{1}^{a},p_{2}^{b}) \).

The commutation relations between \( \hat{M} \) and \( \hat{H}_{1}/\hat{H}_{2} \) are now given by

\[ [\hat{M},\hat{H}_{1}] = i\frac{8\pi G}{\lambda} \left[ \epsilon_{ab}(q_{1} - q_{2})^{a}p_{2}^{b}\delta_{\Lambda}(X_-) + \frac{2\pi G}{\lambda}\delta_{\Lambda}(X_-)\epsilon_{\Lambda}(X_-)p_{1} \cdot p_{2} \right], \]  

(4.39)

\[ [\hat{M},\hat{H}_{2}] = i\frac{8\pi G}{\lambda} \left[ \epsilon_{ab}(q_{1} - q_{2})^{a}p_{1}^{b}\delta_{\Lambda}(X_-) - \frac{2\pi G}{\lambda}\delta_{\Lambda}(X_-)\epsilon_{\Lambda}(X_-)p_{1} \cdot p_{2} \right]. \]  

(4.40)
Since \([\hat{M}, \hat{H}_1] \psi = [\hat{M}, \hat{H}_2] \psi = 0\) and \(\delta_\Lambda(x)\) and \(\delta_\Lambda(x)\epsilon_\Lambda(x)\) are independent functions of \(x\) for all \(\Lambda\), we conclude that

\[
\epsilon_{ab}(q_1 - q_2)^a p_2^b \psi = \epsilon_{ab}(q_1 - q_2)^a p_1^b \psi = p_1 \cdot p_2 \psi = 0 .
\]  

(4.41)

The last equality implies that \(\psi \propto \delta(p_1 \cdot p_2)\) and the first two give that \(\epsilon_{ab} p_1^a p_2^b \delta'(p_1 \cdot p_2) = 0\). The only possible solution, \(\psi \propto \delta(p_1)\delta(p_2)\), is excluded from the consideration of \(H_1 \Psi = 0\). Thus the solution of the constraints does not exist.

When we solve \(\hat{M} \psi = \hat{H}_1 \psi = \hat{H}_2 \psi = 0\), we may use another regularization scheme.

For example, a regularization may be consistently defined by \(\delta(x)\epsilon(x) = 0\), \(\frac{d}{dx} \epsilon(x) = 2 \delta(x)\) and \(\epsilon_n(x) = \begin{cases} \epsilon(x) & n = \text{odd} \\ 1 & n = \text{even} \end{cases}\). In addition, all differentiation with respect to \(x\) should be applied after the third equation is used completely. With this regularization, though it is complicated, one may show again that a solution of the three constraints does not exist.

The obvious question is why the theory does not lead to any consistent solutions. As shown in the classical analysis, equivalence between the geometric approach and the gauge theoretic formulation provides the relations \(X^a = \epsilon^a q^b\) and (3.34) for the \(N\)-body problem. For the one-body case, this relation is used to reinterpret the wave function. For the two-body case, the wave function depends on both the Poincaré coordinates and \(X^\mu_1\).

Consequently, without this relation, equations for the Poincaré momentum and the position variables put separately too much restrictions. [For example, the conditions in (3.25) corresponds to new constraints in (4.39-4.40) upon quantization, which cause troubles in finding solutions, whereas they become trivial once the relation \(X^a = \epsilon^a q^b\) is used]. As a result, a consistent solution for the two-body problem does not exist.

One obvious resolution of this problem is as follows. First, using the relation (3.34), identify the classical phase space before quantization, which leads to the effective Lagrangian in (3.29). Next we quantize the system solving constraints, which is now perfectly well defined and consistent.
Another possibility is to look for a reasonable modification of the constraint that makes them compatible. For example one can consider $\tilde{M} = X_{-}\dot{M}$ instead of $\dot{M}$. The algebra that is formed by $\tilde{M}$, $\dot{H}_1$ and $\dot{H}_2$ is now closed and solvable. In fact using the result of the one body problem, we can easily show that the most general solution is

$$\psi = \left(\frac{p_1^0 + p_1^0}{p_1^0 - p_1^0}\right)^{\nu_1} \left(\frac{p_2^0 + p_2^0}{p_2^0 - p_2^0}\right)^{\nu_2} \left[(1 + \epsilon(X_-))K_{\nu_1} \left(\frac{\sqrt{M + 2\pi G\nu_2}}{\lambda} \sqrt{\rho_1 \rho_1}\right)\right]$$

$$\times K_{\nu_2} \left(\frac{\sqrt{M - 2\pi G\nu_1}}{\lambda} \sqrt{\rho_2 \rho_2}\right) + (1 - \epsilon(X_-))(\nu_1, \nu_2 \rightarrow -\nu_1, -\nu_2)$$

where $\gamma_i = \sqrt{\nu_i^2 - 4m^2/\lambda}$. The change considered here corresponds to modifying the contact interaction when the particles meet. However, one should note that there is no clear reason for such a modification.

V Conclusions

In this paper, we present the gauge theoretical model for the string-inspired gravity with point particles. For a gauge invariant description of particles we enlarge the phase space by introducing the Poincaré coordinate. We also use a non-minimal gauge interaction with $\eta_A$ fields in order to make particles couple to the physical metric.

In the investigation of the classical problem, we first show the equivalence to the geometric approach by comparing the equations of motion in the unitary gauge and obtain one-body solutions that describe a particle in the black-hole metric. For the many-body case, we derive the effective action for the Poincaré coordinates and the particle positions in which the gravity variables are eliminated completely. Transforming the action into the unitary gauge, we are led to the new effective action for the particle position only, from which one may see clearly its geometrical implications.

After solving the wave functional of the gravity part, the quantum one-body problem is reduced to the Klein-Gordon equation with the black hole metric, which implies that the classical picture for the geometry is preserved at the quantum level.
On the other hand, in the many-body case, the enlargement of the phase space makes the constraint algebra open at the points where particle coordinates coincide. Consequently, we prove that it is impossible to solve the constraints consistently. Thus one may conclude that the classical equivalence between the two formulations is lost upon quantization.

A way of overcoming the problem is to reduce the phase space first by adopting the unitary gauge and then to quantize the theory within the reduced phase space. This procedure essentially leads to the problem of quantizing the effective action in (3.35) that provides a closed constraint algebra and a well-defined symplectic structure. Detailed quantum analysis of the action might be enlightening in understanding the mutual effects of geometry, particles and gravity interaction.

A simpler problem may be an investigation of the one-body wave functional near the singularity and the horizon, and effects of the particle state to the geometry.

It is also interesting to pursue a direct quantization of the geometric model with point particles, so that, for example, one may see a natural diffeomorphism gauge choice in the geometric theory by solving constraints.

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