Disjunctive domination in trees *

Wei Zhuang\textsuperscript{a}† Litao Guo\textsuperscript{a} Guoliang Hao \textsuperscript{b}

\textsuperscript{a} School of Applied Mathematics, Xiamen University of Technology, Xiamen Fujian 361024, P.R.China

\textsuperscript{b} College of Science, East China University of Technology, Nanchang Jiangxi 330013, P.R.China

Abstract

In this paper, we study a parameter that is a relaxation of arguably the most important domination parameter, namely the domination number. Given the sheer scale of modern networks, many existing domination type structures are expensive to implement. Variations on the theme of dominating sets studied to date tend to focus on adding restrictions which in turn raises their implementation costs. As an alternative route a relaxation of the domination number, called disjunctive domination, was proposed and studied by Goddard et al. A set $D$ of vertices in $G$ is a disjunctive dominating set in $G$ if every vertex not in $D$ is adjacent to a vertex of $D$ or has at least two vertices in $D$ at distance 2 from it in $G$. The disjunctive domination number, $\gamma_d^2(G)$, of $G$ is the minimum cardinality of a disjunctive dominating set in $G$. We show that if $T$ is a tree of order $n$ with $l$ leaves and $s$ support vertices, then $\frac{n-l+3}{4} \leq \gamma_d^2(T) \leq \frac{n-l+s}{4}$. Moreover, we characterize the families of trees which attain these bounds.

Keywords: Disjunctive dominating set, disjunctive domination number, tree.

1 Introduction

Over the last few decades, the scale of networks and the role of graphs as models for networks has changed, and in practical terms, many existing domination type structures

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†Corresponding author; E-mail: zhuangweixmu@163.com
are too expensive to implement. The majority of domination-type variants studied to date tend to focus on adding restrictions which in turn raises their implementation costs. As a result the idea of relaxing conditions on domination-type parameters is appealing. A relaxation of the domination number, called disjunctive domination, was proposed and studied in [2]. In this paper we continue the study of disjunctive domination in graphs.

A dominating set in a graph \(G\) is a set \(S\) of vertices of \(G\) such that every vertex in \(V(G) \setminus S\) is adjacent to at least one vertex in \(S\). The domination number of \(G\), denoted by \(\gamma(G)\), is the minimum cardinality of a dominating set of \(G\). A set \(D\) of vertices in a graph \(G\) is a disjunctive dominating set, abbreviated 2DD-set, in \(G\) if every vertex not in \(D\) is adjacent to a vertex of \(D\) or has at least two vertices in \(D\) at distance 2 from it in \(G\). We say a vertex \(v\) in \(G\) is 2D-dominated by the set \(D\), if \(N[v] \cap D = \emptyset\) or there exist at least two vertices in \(D\) at distance 2 from \(v\) in \(G\). The disjunctive domination number of \(G\), denoted by \(\gamma_d^2(G)\), is the minimum cardinality of a 2DD-set in \(G\). A disjunctive dominating set of \(G\) of cardinality \(\gamma_d^2(G)\) is called a \(\gamma_d^2\)-set. If the graph \(G\) is clear from the context, we simply write \(\gamma_d^2\)-set rather than \(\gamma_d^2(G)\)-set.

Every dominating set is a 2DD-set. The concept of disjunctive domination in graphs has been studied in [2–5] and elsewhere.

Let \(G = (V, E)\) be a graph with vertex set \(V\) of order \(n(G) = |V|\) and edge set \(E\) of size \(m(G) = |E|\), and let \(v\) be a vertex in \(V\). The open neighborhood of \(v\) is \(N(v) = \{u \in V | uv \in E\}\) and the closed neighborhood of \(v\) is \(N[v] = N(v) \cup \{v\}\). The degree of a vertex \(v\) is \(d(v) = |N(v)|\). For two vertices \(u\) and \(v\) in a connected graph \(G\), the distance \(d(u, v)\) between \(u\) and \(v\) is the length of a shortest \((u, v)\)-path in \(G\). The maximum distance among all pairs of vertices of \(G\) is the diameter of a graph \(G\) which is denoted by \(diam(G)\). A leaf of \(G\) is a vertex of degree 1 and a support vertex of \(G\) is a vertex adjacent to a leaf. Denote the sets of leaves and support vertices of \(G\) by \(L(T)\) and \(S(T)\), respectively. Let \(l(T) = |L(T)|\) and \(s(T) = |S(T)|\). A double star is a tree that contains exactly two vertices that are not leaves.

## 2 Main results

In this paper, we give a lower bound and an upper bound for the disjunctive domination number of a tree in terms of its order, the number of leaves and support vertices in the tree. Further, we provide the constructive characterization of trees that achieve equality in the two bounds. We state this formally as follows.

**Observation 2.1** [2] If \(T\) is a tree of order at least 3, then we can choose a \(\gamma_d^2\)-set of \(T\) contains no leaf.

**Corollary 2.2** Let \(T\) be a tree of order at least 3 and \(D\) be a \(\gamma_d^2\)-set of \(T\) contains no leaf, if a support vertex has degree two, then it belongs to \(D\).
By a weak partition of a set we mean a partition of the set in which some of the subsets may be empty. For our purposes, we define a *labeling* of a tree $T$ as a weak partition $S = (S_A, S_B, S_C, S_D)$ of $V(T)$ (This idea of labeling the vertices is introduced in [1]). We will refer to the pair $(T, S)$ as a labeled tree. The label or *status* of a vertex $v$, denoted $\text{sta}(v)$, is the letter $x \in \{A, B, C, D\}$ such that $v \in S_x$. Next, we ready to give two families $\mathcal{T}_1$ and $\mathcal{T}_2$, each member of which is obtained from the labeled trees $(P_3, S')$ and $(P_4, S'')$ respectively by a series of operations. Before this, we give two definitions. If a labeled tree $(T, S) \in \mathcal{T}_2$, the path $P_4$ (which comes from the labeled tree $(P_4, S'')$) is an induced path of $T$, and we call it the *basic path* of $T$. For a vertex $v \notin S(T)$, which has status $A$ and does not belong to the basic path, if there exists a vertex $u$ such that $vv_1v_2u$ is an induced path of $T$ and $\text{sta}(v_1) = C$, $\text{sta}(v_2) = D$, $\text{sta}(u) = B$, we call $u$ a *corresponding vertex* of $v$. In addition, for a vertex $u$, which has status $B$, if there exists a vertex $v$ such that $vv_1v_2u$ is an induced path of $T$ and $\text{sta}(v) = A$, $\text{sta}(v_1) = C$, $\text{sta}(v_2) = D$, we call $v$ a *corresponding vertex* of $u$.

![Diagram](a)

![Diagram](b)

![Diagram](c)

![Diagram](d)

Fig.1

In what follows, we give four operations as follows:

**Operation $O_1$:** Let $v$ be a vertex with $\text{sta}(v) = A$. Add a vertex $u$ and the edge $uv$. Let $\text{sta}(u) = C$. 
Operation $O_2$: Let $v$ be a vertex with $\text{sta}(v) = B$ that has a corresponding vertex of degree two. Add a path $u_1u_2$ and the edge $u_1v$. Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = C$.

Operation $O_3$: Let $v$ be a vertex with $\text{sta}(v) = C$ that has degree one. Add a path $u_1u_2u_3u_4$ and the edge $u_1v$. Let $\text{sta}(u_1) = D$, $\text{sta}(u_2) = B$, $\text{sta}(u_3) = A$, $\text{sta}(u_4) = C$.

Operation $O_4$: Let $v$ be a vertex not in the basic path that has status $A$ and has a corresponding vertex of degree two. Add a path $u_1u_2$ and the edge $u_1v$. Let $\text{sta}(u_1) = A$, $\text{sta}(u_2) = C$.

The three operations $O_1$, $O_2$, $O_3$ and $O_4$ are illustrated in Fig.1(a), (b), (c) and (d).

Let $\mathcal{T}_1$ be the minimum family of labeled trees that: (i) contains $(P_3, S')$ and $S'$ is the labeling that assigns to the two leaves of the path $P_3$ status $C$, and the central vertex status $A$; and (ii) is closed under the two operations $O_1$ and $O_3$ that are listed as above, which extend the tree $T'$ to a tree $T$ by attaching a tree to the vertex $v \in V(T')$.

Let $\mathcal{T}_2$ be the minimum family of labeled trees that: (i) contains $(P_4, S'')$ where $S''$ is the labeling that assigns to the two leaves of the path $P_4$ status $C$, and the remaining vertices status $A$; and (ii) is closed under the three operations $O_2$, $O_3$ and $O_4$ that are listed as above, which extend the tree $T'$ to a tree $T$ by attaching a tree to the vertex $v \in V(T')$.

We take an example to make it easier for reader to understand the family $\mathcal{T}_1$ and $\mathcal{T}_2$. The trees are depicted in Fig.2(a) and (b) belong to $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. In Fig.2(b), the induced path $v_1v_2v_3v_4$ is the basic path of the tree.

Let $(T, S) \in \mathcal{T}_1$ (or $\mathcal{T}_2$) be a labeled tree for some labeling $S$. Then there is a sequence of labeled trees $(T_0, S_0)$, $(T_1, S_1)$, $\cdots$, $(T_{k-1}, S_{k-1})$, $(T_k, S_k)$ such that $(T_0, S_0) = (P_3, S')$ (or $(P_4, S'')$), $(T_k, S_k) = (T, S)$. The labeled tree $(T_i, S_i)$ can be obtained from $(T_{i-1}, S_{i-1})$ by one of the operations $O_1$ and $O_3$ (or $O_2$, $O_3$ and $O_4$), where $i \in \{1, 2, \cdots, k\}$.
the number of terms in such a sequence of labeled trees that is used to construct \((T, S)\), the \textit{length} of the sequence. Clearly, the above sequence has length \(k\). We remark that a sequence of labeled trees used to construct \((T, S)\) is not necessarily unique.

Two main conclusions of our paper are listed as follows.

\textbf{Theorem 2.3} If \(T\) is a nontrivial tree of order \(n(T)\) with \(l(T)\) leaves, then \(\gamma_2^d(T) \geq \frac{n(T) - (l(T)) + 3}{4}\), with equality if and only if \((T, S) \in \mathcal{T}_1\) for some labeling \(S\).

\textbf{Theorem 2.4} If \(T\) is a nontrivial tree of order \(n(T)\) with \(l(T)\) leaves and \(s(T)\) support vertices, then \(\gamma_2^d(T) \leq \frac{n(T) + l(T) + s(T)}{4}\), with equality if and only if \((T, S) \in \mathcal{T}_2\) for some labeling \(S\).

Furthermore, we can slightly improve the upper bound of Theorem 2.4.

\textbf{Corollary 2.5} If \(T\) is a nontrivial tree of order \(n(T)\) with \(l(T)\) leaves and \(s(T)\) support vertices, then \(\gamma_2^d(T) \leq \frac{n(T) + 3s(T) - l(T)}{4}\).

\textbf{Proof.} Let \(T'\) be the tree obtained from \(T\) by deleting all but one leaf from each support vertex of \(T\). Then, \(n(T') = n(T) - \lfloor l(T) - s(T) \rfloor\), \(s(T') = s(T)\), \(l(T') = l(T)\) and \(\gamma_2^d(T) = \gamma_2^d(T')\). By Theorem 2.4, we have that \(\gamma_2^d(T) = \gamma_2^d(T') \leq \frac{n(T') + l(T') + s(T')}{4} = \frac{n(T) - \lfloor l(T) - s(T) \rfloor + 2s(T)}{4} = \frac{n(T) + 3s(T) - l(T)}{4}\). \(\square\)

\section{Proof of Theorem 2.3}

The following observation establishes properties of trees in the family \(\mathcal{T}_1\).

\textbf{Observation 3.1} If \((T, S) \in \mathcal{T}_1\), then \((T, S)\) has the following properties.

(a) Every support vertex of \(T\) has status \(A\) and every leaf has status \(C\).
(b) Let \(v\) be a vertex has status \(A\), then \(\text{sta}(u) \in \{B, C\}\) for \(u \in N(v)\).
(c) The set \(S_A\) is a 2DD-set of \(T\).
(d) The set \(S_A, S_B, S_C\) and \(S_D\) are independent sets.
(e) If \(\text{sta}(v) \neq A\), then \(d(v) \leq 2\).

\textbf{Lemma 3.2} If \(T\) is a tree of order \(n(T) \geq 3\) with \(l(T)\) leaves, and \((T, S) \in \mathcal{T}_1\) for some labeling \(S\), then \(\gamma_2^d(T) = |S_A| = \frac{n(T) - l(T) + 3}{4}\), and the set \(S_A\) is the unique \(\gamma_2^d\)-set of \(T\).

\textbf{Proof.} We proceed by induction on the length \(k\) of a sequence required to construct the labeled tree \((T, S)\). Let \(D\) be any \(\gamma_2^d\)-set of \(T\).

When \(k = 0\), \((T, S) = (P_3, S')\), \(\gamma_2^d(T) = |S_A| = 1\), the set \(S_A\) is the unique \(\gamma_2^d\)-set of \(T\). This establishes the base case. Let \(k \geq 1\) and assume that if the length of sequence used to construct a labeled tree \((T', S^*)\) \(\in \mathcal{T}_1\) is less than \(k\), then \(\gamma_2^d(T') = |S'_A| = \frac{n(T') - l(T') + 3}{4}\), \(S'_A\) is the unique \(\gamma_2^d\)-set of \(T'\). Now, \((T, S) \in \mathcal{T}_1\) and let \((T_0, S_0), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1})\),
(T_k, S_k) be a sequence of length k used to construct (T, S), where (T_0, S_0) = (P_3, S'), (T_k, S_k) = (T, S), (T_i, S_i) can be obtained from (T_{i-1}, S_{i-1}) by one of the operations G_i and G_3, i \in \{1, 2, \ldots, k\}. Let T' = T_{k-1} and S^* = S_{k-1}. Note that (T', S^*) \in \mathcal{F}. By the inductive hypothesis, γ^d(T') = |S^*_A| = \frac{n(T') - l(T') + 3}{4}, S^*_A is the unique γ^d-set of T'. (T, S) can be obtained from (T', S^*) by operation G_1 or G_3.

In the former case, we have that n(T) = n(T') + 1, l(T) = l(T') + 1, and |S_A| = |S^*_A|.

It follows Observation 3.1(c) that γ^d(T) ≤ |S_A| = |S^*_A| = \frac{n(T') - l(T') + 3}{4} = \frac{n(T) - l(T) + 1 + 1}{4}. On the other hand, assume that V(T) \setminus V(T') = \{v\}, and v is the support vertex of u. Take a set D' = (D \setminus (L(T) \cap N(v))) \cup \{v\} when (L(T) \cap N(v)) \cap D \neq \emptyset, otherwise, D' = D. D' is a 2DD-set of T'. That is, γ^d(T) ≥ γ^d(T') = |S^*_A| = |S_A|.

In summary, γ^d(T) = |S_A| = \frac{n(T) - l(T) + 3}{4}. By the inductive hypothesis, S^*_A is the unique γ^d-set of T'. Hence, D' = S^*_A. In addition, if u \in D, then v \notin D. It follows from (T, S) \in \mathcal{F} and Observation 3.1(a), (b) that v has status A, and the non-leaf neighbor of v, say w, has status B or C. From the choice of D' and D' = S^*_A, u is the unique vertex in D which is within distance two from w. It conclude that w is not 2D-dominated by D, a contradiction. Therefore, u \notin D. Similarly, all leaf-neighbors of v do not belong to D, and then D = D' = S^*_A = S_A.

In the latter case, the tree T obtained from T' by attaching a path P_4 = u_1u_2u_3u_4 to a leaf v of T', where u_4 is a leaf in T. We have that n(T) = n(T') + 1, l(T) = l(T') and |S_A| = |S^*_A| + 1. It follows Observation 3.1(c) that γ^d(T) ≤ |S_A| = |S^*_A| + 1 = \frac{n(T') - l(T') + 3}{4} + 1 = \frac{n(T') - 3}{4} + 1 = \frac{n(T) - l(T) + 3}{4}. Let D' = (D \setminus \{u_4\}) \cup \{u_3\} when u_4 \in D and D' = D when u_4 \notin D, D'' = (D' \setminus \{u_1, u_2\}) \cup \{v\} when u_1 or u_2 belong to D', otherwise, D'' = D'. Then u_3 \in D and D'' \setminus \{u_3\} is a 2DD-set of T'. That is, γ^d(T) - 1 ≥ γ^d(T') = |S^*_A| = |S_A| - 1. In summary, γ^d(T) = |S_A| = \frac{n(T) - l(T) + 3}{4}. By the inductive hypothesis, S^*_A is the unique γ^d-set of T'. Hence, D'' \setminus \{u_3\} = S^*_A. If |\{u_1, u_2, u_3, u_4, v\} \cap D| ≥ 2, the set (D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\} is a 2DD-set of T'. More precisely, (D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\} is a γ^d-set of T'. By the uniqueness of γ^d-set of T', (D \setminus \{u_1, u_2, u_3, u_4\}) \cup \{v\} = S^*_A, a contradiction. Hence, |\{u_1, u_2, u_3, u_4, v\} \cap D| = 1. It implies that \{u_1, u_2, u_3, u_4, v\} \cap D = \{u_3\}. It is easy to see that D \setminus \{u_3\} is a γ^d-set of T'. By the uniqueness of γ^d-set of T', D \setminus \{u_3\} = S^*_A. So, D = S_A.

In what follows, we begin to prove Theorem 2.3.

**Proof.** The sufficiency follows immediately from Lemma 3.2. So we prove the necessity only. If \text{diam}(T) \leq 2, T is a star, γ^d(T) = 1 ≥ \frac{n(T) - l(T) + 3}{4}. Suppose that γ^d(T) = \frac{n(T) - l(T) + 3}{4}, it is easy to see that there exists a labeling S of the vertices of T such that (T, S) can be obtained from (P_3, S') by repeated applications of operation G_1. Hence, (T, S) \in \mathcal{F}. If \text{diam}(T) = 3, T is a double star, and then γ^d(T) = 2 > \frac{n(T) - l(T) + 3}{4}. So, we assume that \text{diam}(T) ≥ 4. The proof is by induction on n(T). The result is immediate for n(T) ≤ 5. For the inductive hypothesis, let n(T) ≥ 6. Assume that for every nontrivial tree T' of order less than n(T), we have that γ^d(T') ≥ \frac{n(T') - l(T') + 3}{4}, with equality only if
Let \( D \) be a \( \gamma_2^d \)-set of \( T \) which contains no leaf and \( P = v_1 v_2 \cdots v_t \) be a longest path in \( T \) such that \( d(v_3) \) as large as possible.

We now proceed with a series of claims that we may assume are satisfied by the tree \( T \), for otherwise the desired result holds.

**Claim 1.** Each support vertex in \( T \) has exactly one leaf-neighbor.

If not, assume that there is a support vertex \( u \) which is adjacent to at least two leaves. Deleting one of its leaf-neighbors, say \( u_1 \), and denote the resulting tree by \( T' \). Observe that \( n(T) = n(T') + 1 \), \( l(T) = l(T') + 1 \) and \( D = 2DD \)-set of \( T' \). That is, \( \gamma_2^d(T) \geq \gamma_2^d(T') \geq \frac{n(T')-l(T')+3}{4} = \frac{n(T)-l(T)+3}{4} = \frac{n(T)-l(T)+3}{4} \).

In particular, if \( \gamma_2^d(T) = \frac{n(T)-l(T)+3}{4} \), then \( \gamma_2^d(T') = \frac{n(T')-l(T')+3}{4} \). It means that \((T',S^*) \in \mathcal{R}_1 \) for some labeling \( S^* \). By Observation 3.1(a), \( u \) has status \( A \). Let \( S \) be obtained from \( S^* \) by labeling \( u_1 \) with label \( C \). Then \((T,S)\) can be obtained from \((T',S^*)\) by operation \( \mathcal{O}_1 \). Thus, \((T,S) \in \mathcal{R}_1 \). \( \square \)

By Claim 1, we can assume that \( d(v_2) = 2 \). And by Corollary 2.2, \( v_2 \in D \). Now, we consider the vertex \( v_3 \).

**Claim 2.** \( d(v_3) = 2 \).

Suppose that \( d(v_3) \geq 3 \). If \( v_3 \in D \), let \( T'' = T - \{v_1,v_2\} \). Clearly, \( D \setminus \{v_2\} \) is a \( 2DD \)-set of \( T' \). Note that \( n(T) = n(T') + 2 \), \( l(T) = l(T') + 1 \), then \( \gamma_2^d(T) \geq \gamma_2^d(T') + 1 \geq \frac{n(T')-l(T')+3}{4} + 1 = \frac{n(T)-2l(T)+1+3}{4} + 1 > \frac{n(T)-l(T)+3}{4} \). So we assume that \( v_3 \notin D \). If \( v_3 \) is adjacent to a support vertex outside \( P \), say \( v'_2 \). It follows from Claim 1 and Corollary 2.2 that \( v'_2 \in D \). Moreover, \((D \setminus \{v_1,v_2\}) \cup \{v_3\} \) is a \( 2DD \)-set of the tree \( T' \) obtained from \( T \) by removing all leaf-neighbors of \( v_2 \) and \( v'_2 \). Hence, \( \gamma_2^d(T) \geq \gamma_2^d(T') + 1 \geq \gamma_2^d(T'_2) + \frac{n(T')-l(T')}{4} + 1 = \frac{n(T)-l(T)+3}{4} + 1 > \frac{n(T)-l(T)+3}{4} \). Combining the assumption that \( d(v_3) \geq 3 \), \( v_3 \) is a support vertex of degree three of \( T \). We remove its leaf-neighbor, say \( u \), and \( D \) is still a \( 2DD \)-set of the resulting tree \( T' \) from \( u \notin D \). Hence, \( \gamma_2^d(T') \geq \gamma_2^d(T'_2) \geq \gamma_2^d(T') = \frac{n(T')-l(T')+3}{4} = \frac{n(T)-l(T)+3}{4} \). We show that in fact \( \gamma_2^d(T) > \frac{n(T)-l(T)+3}{4} \). Suppose to the contrary that \( \gamma_2^d(T) = \frac{n(T)-l(T)+3}{4} \). Then we have equality through the above inequality chain. In particular, \( \gamma_2^d(T) = \gamma_2^d(T') = \frac{n(T')-l(T')+3}{4} \). By the inductive hypothesis, \((T',S^*) \in \mathcal{R}_1 \) for some labeling \( S^* \). By Observation 3.1(a) and (b), the vertex \( v_3 \) has status \( B \) or \( C \) in \( S^* \). Since \( D \) contains no leaf, \( D \) is also a \( \gamma_2^d \)-set of \( T' \). On the other hand, by Lemma 3.2, \( S^*_A \) is the unique \( \gamma_2^d \)-set of \( T' \). So, \( D = S^*_A \). It implies that \( u \) cannot be \( 2D \)-dominated by \( D \), a contradiction. \( \square \)

**Claim 3.** \( d(v_4) = 2 \).

Assume that \( d(v_4) \geq 3 \) and \( v'_3 \) is a neighbor of \( v_4 \) outside \( P \). From Claim 1 and the choice of \( P \), one of the three cases as following holds:

1. \( v'_3 \) is adjacent to a support vertex, say \( v'_2 \), where \( v'_2 \) and \( v'_3 \) have degree two;
(2) $v'_3$ is a support vertex of degree two in $T$;
(3) $v'_3$ is a leaf.

In the first case, let $T'$ be a tree obtained from $T$ by removing $v_1, v_2, v_3$ and the leaf-neighbor of $v'_2$. We have that $n(T) = n(T') + 4$, $l(T) = l(T') + 1$ and $\gamma^d_2(T') \leq \gamma^d_2(T) - 1$.

In the latter two cases, let $T' = T - \{v_1, v_2, v_3\}$. We have that $n(T) = n(T') + 3$, $l(T) = l(T') + 1$ and $\gamma^d_2(T') \leq \gamma^d_2(T) - 1$. In either case, we always have $\gamma^d_2(T) > \frac{n(T) - l(T) + 3}{4}$ by an argument similar to the proof of Claim 2.

Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Note that $n(T) = n(T') + 4$, $\gamma^d_2(T') \leq \gamma^d_2(T) - 1$. In addition, $l(T) = l(T') + 1$ when $d(v_5) \geq 3$, and $l(T) = l(T')$ when $d(v_5) = 2$. Hence, we always have that $\gamma^d_2(T) \geq \gamma^d_2(T') + 1 \geq \frac{n(T') - l(T') + 3}{4} + 1 \geq \frac{n(T) - 4 - l(T) + 3}{4} + 1 = \frac{n(T) - l(T) + 3}{4}$. Suppose that $\gamma^d_2(T) = \frac{n(T) - l(T) + 3}{4}$, then we have equality throughout the above inequality chain. In particular, $d(v_5) = 2$ and $\gamma^d_2(T) - 1 = \gamma^d_2(T') = \frac{n(T') - l(T') + 3}{4}$. By the inductive hypothesis, $(T', S^*) \in \mathcal{F}_1$ for some labeling $S^*$. Since $v_5$ is a leaf in $T'$, by Observation 3.1(a), it has status $C$. Let $S$ be obtained from the labeling $S^*$ by labeling the vertices $v_1, v_2, v_3, v_4$ with label $C, A, B, D$, respectively. Then, $(T, S)$ can be obtained from $(T', S^*)$ by operation $\rho_3$. Thus, $(T, S) \in \mathcal{F}_1$.

4 Proof of Theorem 2.4

The following observation establishes properties of trees in the family $\mathcal{F}_2$.

**Observation 4.1** If $(T, S) \in \mathcal{F}_2$, then $(T, S)$ has the following properties.

(a) Every support vertex of $T$ has status $A$ and every leaf has status $C$.
(b) The set $S_A$ is a 2DD-set of $T$.
(c) Let $v$ be a vertex which has status $A$ or $B$, $v$ has at most one corresponding vertex. In particular, if there is no corresponding vertex of degree two of $v$ in $T$, then $d(v) = 2$.
(d) If $v$ is a support vertex, then $v$ has degree two.
(e) Let $v$ be a vertex of degree two which has status $C$, then it is adjacent to two vertices, say $u$ and $w$, which are labeled $A$ and $D$, respectively. In particular, if $d(u) = 2$, the component of $T - vw$ containing $v$, say $T'$, containing the basic path of $T$, and $(T', S^*) \in \mathcal{F}_2$ for some labeling $S^*$.

**Lemma 4.2** Let $T$ be a tree and $S$ be a labeling of $T$ such that $(T, S) \in \mathcal{F}_2$. Then, $\gamma^d_2(T) = \frac{n(T) + s(T) + l(T)}{4}$.

**Proof.** By Observation 4.1(b), $S_A$ is a 2DD-set of $T$ and $S_A = \frac{n(T) + s(T) + l(T)}{4}$ (We can obtain this conclusion by induction on $n(T)$, it is similar to the proof of Lemma 3.2, so we omit it). So, $\gamma^d_2(T) \leq \frac{n(T) + s(T) + l(T)}{4}$. Since $(T, S) \in \mathcal{F}_2$, $T = P_4$ when $n \leq 4$, and $\gamma^d_2(T) = 2 = \frac{n(T) + s(T) + l(T)}{4}$. So, we assume that $n(T) \geq 5$. Combining the definition of
In the form case, if \( d(v_3) = 2 \), then \( v_1v_2v_3v_4 \) is the basic path of \( T \), a contradiction.

If \( d(v_3) \geq 3 \), by Observation 4.1(d), \( v_3 \) is not a support vertex. From the choice of \( v_1 \) and the fact that \( \text{diam}(T) \geq 7 \), \( v_3 \) is adjacent to \( s \) support vertices of degree two other than \( v_2 \), where \( s \geq 1 \). These support vertices are labeled \( A \), and the leaf-neighbor of each of them is labeled \( C \). From the choice of \( D \) and Corollary 2.2, \( S(T) \cap N(v_3) \subseteq D \). \( v_4, v_5, v_6 \) have status \( D, C, A \), respectively, and \( d(v_4) = d(v_5) = 2 \). Moreover, there exists no a corresponding vertex of degree two of \( v_6 \) in \( T \), so \( d(v_6) = 2 \). Note that \( \{v_3, v_4, v_5, v_6\} \cap D \neq \emptyset \), then \( (D \setminus \{v_3, v_4, v_5\}) \cup \{v_6\} \) is also a \( \gamma^d_2 \)-set of \( T \). Hence, \( D' = D \setminus \{v_2\} \) is a 2DD-set of \( T' \) with order at most \( \gamma^d_2(T) - 1 \), where \( T' = T - \{v_1, v_2\} \). On the other hand, note that \( (T', S^*) \in \mathcal{F}_2 \) for some labeling \( S^* \), from the choice of \( T' \), \( \gamma^d_2(T') = \frac{n(T') + s(T') + l(T')}{4} = \frac{n(T) + s(T) + l(T)}{4} - 1 > \gamma^d_2(T) - 1 \). A contradiction.

If \( d(v_4) = 2 \), from the definition of \( \mathcal{F}_2 \), \( v_4 \) has status \( D \), and furthermore, \( v_5, v_6 \) have status \( C, A \), respectively. In particular, \( d(v_1) = d(v_3) = 2 \). Note that \( v_2 \in D \), and \( \{v_3, v_4, v_5, v_6\} \cap D \neq \emptyset \), so the set \( D' = (D \setminus \{v_3, v_4, v_5\}) \cup \{v_6\} \) is also a \( \gamma^d_2 \)-set of \( T \). Now, we distinguish two cases as follows.

**Case 1.** \( d(v_6) = 2 \).

The set \( D'' = D' \setminus \{v_2\} \) is a 2DD-set of \( T' \) with order at most \( \gamma^d_2(T) - 1 \), where \( T' = T - \{v_1, v_2, v_3, v_4\} \). On the other hand, from the choice of \( T \) and the fact that \( (T', S^*) \in \mathcal{F}_2 \) for some labeling \( S^* \), \( \gamma^d_2(T') = \frac{n(T') + s(T') + l(T')}{4} = \frac{n(T) + s(T) + l(T)}{4} - 1 > \gamma^d_2(T) - 1 \). A contradiction.

**Case 2.** \( d(v_6) \geq 3 \).

We have that \( \text{sta}(v_7) = A \) or \( B \). If \( \text{sta}(v_7) = B \), then all neighbors of \( v_6 \) outside \( P \) have status \( A \), and note that these neighbors are support vertices of degree two (From the choice of \( v_1 \) and the definition of \( \mathcal{F}_2 \)). We remove one of these support vertices, say \( u_1 \),
and its leaf-neighbor, say \( u_2 \), denote the resulting tree by \( T' \). Clearly, \((T', S^*) \in \mathcal{T}_2\) for some labeling \( S^* \). We know that \( v_2, v_6 \in D' \), and \( \{u_1, u_2\} \cap D' \neq \emptyset \), so \( D'' = D' \setminus \{u_1, u_2\} \) is a 2DD-set of \( T' \) with order at most \( \gamma^d_2(T) - 1 \), where \( T' = T - \{u_1, u_2\} \). On the other hand, from the choice of \( T \), \( \gamma^d_2(T) = \frac{n(T)+s(T)+l(T)}{4} - 1 > \gamma^d_2(T) - 1 \). A contradiction.

If \( \text{sta}(v_7) = A \), then one of the two cases as following holds:

1. There exists a neighbor of \( v_6 \) outside \( P \), say \( u_1 \), has status \( B \).
2. All neighbors of \( v_6 \) outside \( P \) have status \( A \).

In the former case, there exists a neighbor \( u_2 \) of \( u_1 \) which has status \( D \). Similarly, there exists a neighbor \( u_3 \) of \( u_2 \) which has status \( C \), and there exists a neighbor \( u_4 \) of \( u_3 \) which has status \( A \). Moreover, let \( u_5 \) be a neighbor of \( u_4 \) other than \( u_3 \), then \( u_5 \) has status \( A \) or \( B \). In either case, \( u_5 \) has degree at least two, which contradicts the choice of \( v_1 \).

In the latter case, we take any neighbor of \( v_6 \) outside \( P \), say \( u_1 \), and we have that \( u_1 \) has a neighbor which has status \( C \), say \( u_2 \). From the choice of \( v_1 \), \( u_2 \) is a leaf. By Observation 4.1(d), \( d(u_1) = 2 \). And we can obtain a contradiction by an argument similar to the case that \( \text{sta}(v_7) = B \) as above.

In summary, if \((T, S) \in \mathcal{T}_2 \). Then, \( \gamma^d_2(T) = \frac{n(T)+s(T)+l(T)}{4} \). \( \square \)

**Lemma 4.3** Let \( T \) be a tree and \( S \) be a labeling of \( T \) such that \((T, S) \in \mathcal{T}_2 \). Then for any leaf \( v \), there exists a set \( D \) with order \( \frac{n(T)+s(T)+l(T)}{4} - 1 \) such that each vertex of \( T \) is 2D-dominated by \( D \) except for \( v \), and the non-leaf neighbor of the support vertex of \( v \) belongs to \( D \).

**Proof.** Take any leaf \( v_1 \) of \( T \). We proceed by induction on the length \( k \) of a sequence required to construct the labeled tree \((T, S)\). When \( k = 0 \), \((T, S) = (P_4, S'')\), the result is immediate. Let \( k \geq 1 \) and assume that if the length of sequence used to construct a labeled tree \((T', S^*) \in \mathcal{T}_2 \) is less than \( k \), the result holds. Since \((T, S) \in \mathcal{T}_2 \), there exists always a sequence of length \( k \) used to construct \((T, S)\): \((P_4, S''), (T_1, S_1), \ldots, (T_{k-1}, S_{k-1}), (T, S)\).

First, we assume that \( v_1 \) is in the basic path of \( T \). Since \((T_{k-1}, S_{k-1}) \in \mathcal{T}_2 \), \( v_1 \) is still a leaf of \( T_{k-1} \). By the inductive hypothesis, there exists a set \( D' \) with order \( \frac{n(T_{k-1})+s(T_{k-1})+l(T_{k-1})}{4} - 1 \) such that each vertex of \( T_{k-1} \) is 2D-dominated by \( D' \) except for \( v_1 \), and \( v_3 \) belongs to \( D' \), where \( v_3 \) is the neighbor of the support vertex of \( v_1 \). We know that \((T, S)\) is obtained from \((T_{k-1}, S_{k-1})\) by one of the operations \( \Theta_2, \Theta_3 \) and \( \Theta_4 \). In the first or third case, let \( D \) be the set consisting of \( D' \) and the support vertex which belongs to \( V(T) \setminus V(T_{k-1}) \), and \( D \) is the desired set. In the second case, the tree \( T \) is obtained from \( T_{k-1} \) by adding a path \( u_1u_2u_3u_4 \) and joining \( u_1 \) to a leaf \( u \) of \( T_{k-1} \). Note that \( u \) has status \( C \), and by Observation 4.1(d), the neighbor of \( u \), say \( u' \), has degree two. By the inductive hypothesis, there exists a set \( D' \) with order \( \frac{n(T_{k-1})+s(T_{k-1})+l(T_{k-1})}{4} - 1 \) such that
each vertex of \( T_{k-1} \) is 2\(D\)-dominated by \( D' \) except for \( v_1 \), and \( v_3 \) belongs to \( D' \). Moreover, one of \( u \) and \( u' \) belongs to \( D' \). Let \( D \) be the set consisting of \( D' \) and the vertex \( u_3 \), and \( D \) is the desired set.

Next, we consider the case that \( v_1 \) is not in the basic path. Since \((T, S) \in \mathcal{R}_2 \), this leaf has status \( C \) and its support vertex \( v_2 \) is labeled \( A \). By Observation 4.1(d), \( v_2 \) has degree two. Let \( P = v_1v_2 \cdots v_4v \) be the path between \( v_1 \) and \( v \), where \( v \) is the vertex of basic path which has minimum distance from \( v_1 \). Note that the neighbor of \( v_2 \), say \( v_3 \), has status \( A \) or \( B \).

Next, we distinguish two cases as follows.

**Case 1.** \(\text{sta}(v_3) = A\).

If \( d(v_3) = 2 \), then it is easy to see that \( v_1v_2v_3v_4 \) is the basic path of \( T \), a contradiction.

If \( d(v_3) \geq 3 \), from the definition of \( \mathcal{R}_2 \) and the fact that a sequence of labeled trees used to construct \((T, S)\) is not necessarily unique, we have that there exists a sequence of length \( k \) used to construct \((T, S)\): \((P_1, S'''), (T'_1, S'_1), \ldots, (T'_{k-1}, S'_{k-1}), (T, S)\), such that \((T, S)\) is obtained from \((T'_{k-1}, S'_{k-1})\) by \( \mathcal{O}_4 \). That is, the tree \( T \) is obtained from \( T'_{k-1} \) by adding the path \( v_1v_2 \) and joining \( v_2 \) to a vertex \( v_3 \). Note that \( v_3 \) has a neighbor of degree two, say \( u \), which is labeled \( C \) (Otherwise, no vertex of \( T \) is the corresponding vertex of \( v_3 \)). By Observation 4.1(e), the component of \( T'_{k-1} - uu' \) containing \( u \), say \( T' \), containing the basic path, and \((T', S') \in \mathcal{R}_2 \) for some \( S^* \), where \( u' \) is the neighbor of \( u \) other than \( v_3 \). It implies that there always exists a sequence of length \( k \) used to construct \((T, S)\): \((P_1, S'''), (T''_1, S''_1), \ldots, (T''_{k-1}, S''_{k-1}), (T, S)\), satisfying the two conditions as follows:

1. \((T''_{k-1}, S''_{k-1}) = (T'_{k-1}, S'_{k-1})\);
2. There is a \( i \in \{1, 2, \ldots, k-2\} \) in this sequence such that \((T''_i, S''_i) = (T', S^*)\).

By the inductive hypothesis, there exists a set \( D' \) with order \( \frac{n(T'') + s(T'') + l(T'')}{4} - 1 \) such that each vertex of \( T''_i \) is 2\(D\)-dominated by \( D' \) except for \( u \), and \( u'' \) belongs to \( D' \), where \( u'' \) is a neighbor of \( v_3 \) in \( T''_i \) other than \( u \). Then \( D_i = D' \cup \{v_3\} \) is a \( \gamma''_2 \)-set of \( T''_i \). For each \( j \in \{i, i + 1, \ldots, k-2\} \), we know that \((T''_{j+1}, S''_{j+1})\) is obtained from \((T''_j, S''_j)\) by one of the operations \( \mathcal{O}_2 \), \( \mathcal{O}_3 \) and \( \mathcal{O}_4 \). Let \( D_{j+1} = D_j \cup \{w\} \), where \( w \in V(T''_{j+1}) \setminus V(T''_j) \) and has status \( A \). It is easy to see that \( D_{j+1} \) is a \( \gamma''_2 \)-set of \( T''_{j+1} \), and moreover, \( D_{k-1} \) is the desired set.

**Case 2.** \(\text{sta}(v_3) = B\).

In this case, if \( d(v_3) \geq 3 \), there must be a neighbor of \( v_3 \), say \( u \), which has status \( A \). From the definition of \( \mathcal{R}_2 \), the component of \( T - v_3u \) containing \( v_3 \), say \( T' \), containing the basic path, and \((T', S^*) \in \mathcal{R}_2 \) for some \( S^* \). We can obtain the desired set by an argument similar to the case of \(\text{sta}(v_3) = A \) and \( d(v_3) \geq 3 \).

If \( d(v_3) = 2 \), then \( v_4, v_5, v_6 \) have status \( D, C, A \), respectively, and \( d(v_4) = d(v_5) = 2 \). If \( d(v_6) = 2 \), let \( T' = T - \{v_1, v_2, v_3, v_4\} \). Note that \((T', S^*) \in \mathcal{R}_2 \) for some \( S^* \). By the inductive hypothesis, there exists a set \( D' \) with order \( \frac{n(T')+s(T')+l(T')}{4} - 1 \) such that each vertex of \( T' \) is 2\(D\)-dominated by \( D' \) except for \( v_5 \), and \( v_7 \) belongs to \( D' \), then the set
$D' \cup \{v_3\}$ is the desired set. So we consider the case of $d(v_6) \geq 3$. From the definition of $\mathcal{F}_2$, there must exist a neighbor of $v_6$, say $u$, such that $\text{sta}(u) = A$ and the component of $T - v_6u$ containing $v_6$, say $T'$, containing the basic path, and $(T', S^*) \in \mathcal{F}_2$ for some $S^*$. By the inductive hypothesis, there exists a set $D'$ with order $\frac{n(T') + s(T') + l(T')}{4} - 1$ such that each vertex of $T'$ is $2D$-dominated by $D'$ except for $v_1$, and $v_3$ belongs to $D'$. We can obtain the desired set by an argument similar to the case of $\text{sta}(v_3) = A$ and $d(v_3) \geq 3$. □

In what follows, we begin to prove Theorem 2.3.

**Proof.** The sufficiency follows immediately from Lemma 4.2. So we prove the necessity only. If $\text{diam}(T) \leq 2$, $T$ is a star, and $\gamma_2^d(T) = 1 < \frac{n(T) + s(T) + l(T)}{4}$. If $\text{diam}(T) = 3$, $T$ is a double star, and then $\gamma_2^d(T) = 2 \leq \frac{n(T) + s(T) + l(T)}{4}$. Support that $\gamma_2^d(T) = \frac{n(T) + s(T) + l(T)}{4}$, it is easy to see that $T = P_4$, let $S$ be the labeling that assigns to the two leaves of the path $P_4$ status $C$, and the remaining vertices status $A$, then the label tree $(P_4, S) \in \mathcal{F}_2$. So we assume that $\text{diam}(T) \geq 4$. The proof is by induction on $n(T)$. The result is immediate for $n(T) \leq 4$. For the inductive hypothesis, let $n(T) \geq 5$. Assume that for every nontrivial tree $T'$ of order less than $n(T)$, we have that $\gamma_2^d(T') \leq \frac{n(T') + s(T') + l(T')}{4}$, with equality only if $(T', S^*) \in \mathcal{F}_2$ for some labeling $S^*$.

Let $D$ be a $\gamma_2^d$-set of $T$ which contains no leaf and $P = v_1v_2 \cdots v_t$ be a longest path in $T$ such that

(i) $d(v_3)$ as large as possible, and subject to this condition
(ii) $d(v_4)$ as large as possible, and subject to this condition
(iii) $d(v_3)$ as large as possible.

We now proceed with a series of claims that we may assume are satisfied by the tree $T$, for otherwise the desired result holds.

**Claim 1.** Each support vertex in $T$ has exactly one leaf-neighbor.

If not, assume that there is a support vertex $u$ which is adjacent to at least two leaves, say $u_1, u_2$. Deleting $u_1$, and denote the resulting tree by $T'$. Take a $\gamma_2^d$-set of $T'$ contains no leaf, say $D'$. It follows that $u$ is either contained in $D'$ or has at least two non-leaf neighbors in $D'$, and then $D'$ is also a $2DD$-set of $T$. That is, $\gamma_2^d(T) \leq \gamma_2^d(T')$. Observe that $n(T) = n(T') + 1$, $l(T) = l(T') + 1$ and $s(T) = s(T')$. We have that $\gamma_2^d(T) \leq \gamma_2^d(T') \leq \frac{n(T') + s(T') + l(T')}{4} = \frac{n(T) - 1 + s(T) + l(T) - 1}{4} < \frac{n(T) + s(T) + l(T)}{4}$. □

By Claim 1, we can assume that $d(v_3) = 2$. And by Corollary 2.2, $v_2 \in D$. Now, we consider the vertex $v_3$.

**Claim 2.** $v_3$ is not a support vertex.

In other words, all neighbors of $v_3$ are support vertices of degree two, except possibly the vertex $v_4$. If not, support that $v_3$ is a support vertex and $u$ is the leaf-neighbor. Let $T' = T - \{v_1, v_2\}$. Note that $n(T) = n(T') + 2$, $l(T) = l(T') + 1$ and $s(T) = s(T') + 1$,
then \( \gamma_d^2(T) \leq \gamma_d^2(T') + 1 \leq \frac{n(T') + s(T') + l(T')}{4} + 1 = \frac{n(T) - 2 + s(T) - 1 + l(T) - 1}{4} + 1 = \frac{n(T) + s(T) + l(T)}{4} \).

In particular, if \( \gamma_d^2(T) = \frac{n(T) + s(T) + l(T)}{4} \), then \( \gamma_d^2(T') = \frac{n(T') + s(T') + l(T')}{4} \). It means that \((T', S^*) \in \mathcal{Z}_2 \) for some labeling \( S^* \). By Lemma 4.3, there exists a 2\( \mathcal{D} \)D-set \( S \) of \( T' - \{u\} \) with cardinality \( \gamma_d^2(T') - 1 \), and the non-leaf neighbor of \( v_3 \) in \( T' \) belongs to \( S \). It is easy to see that \( S \cup \{v_2\} \) is a 2\( \mathcal{D} \)D-set of \( T \) with cardinality \( \gamma_d^2(T) \). That is, \( \gamma_d^2(T) \leq \gamma_d^2(T') \). Contradicting the fact that \( \gamma_d^2(T) = \gamma_d^2(T') + 1 \). Hence, we have that \( \gamma_d^2(T) < \frac{n(T) + s(T) + l(T)}{4} \). \( \square \)

Let \((S(T) \cap N(v_3)) \setminus \{v_4\} = \{w_1, w_2, \ldots, w_t\} \), where \( w_1 = v_2, t \geq 1 \).

**Claim 3.** \( d(v_4) = 2 \).

Assume that \( d(v_4) \geq 3 \), let \( T' \) be the component of \( T - v_3v_4 \) containing \( v_4 \). It follows from \( n(T) = n(T') + 1 + 2t \), \( l(T) = l(T') + t \) and \( s(T) = s(T') + t \) that \( \gamma_d^2(T) \leq \gamma_d^2(T') + t \leq \frac{n(T') + s(T') + l(T')}{4} + t = \frac{n(T) - 2 + s(T) - 1 + l(T) - 1}{4} + t < \frac{n(T) + s(T) + l(T)}{4} \). \( \square \)

**Claim 4.** \( d(v_5) = 2 \).

Assume that \( d(v_5) \geq 3 \) and \( v_4' \) be a neighbor of \( v_5 \) outside \( P \). If \( t = 2 \), from the choice of \( P \) and Claim 1, we only need to consider the two case as follows (In other cases, let \( T' = T - \{v_1, v_2, v_3, v_4\} \). We can always obtain a \( \gamma_d^2 \)-set of \( T' \) which contains a vertex \( u \in N[v_5] \cap V(T') \). It means that \( \gamma_d^2(T) \leq \gamma_d^2(T') + 1 \). Observe that \( n(T) = n(T') + 4 \), \( l(T) = l(T') + 1 \) and \( s(T) = s(T') + 1 \). We always have that \( \gamma_d^2(T) < \frac{n(T) + s(T) + l(T)}{4} \).

1. \( v_5 \) is not a support vertex, \( v_4' \) is adjacent to a support vertex \( v_3' \), where \( v_3' \) and \( v_4' \) have degree two.
2. \( v_5 \) is not a support vertex and \( v_4' \) is adjacent to \( h \) support vertices of degree two, where \( h \geq 2 \).

Let \( T' \) be the component of \( T - v_5v_4 \) containing \( v_5 \). In the former case, \( n(T) = n(T') + 3 \), \( l(T) = l(T') + 1 \), \( s(T) = s(T') + 1 \) and \( \gamma_d^2(T) \leq \gamma_d^2(T') + 1 \). In the latter case, note that it is possible that \( v_4' \) is a support vertex, then \( n(T') + 2h + 1 \leq n(T) \leq n(T') + 2h + 2 \), \( l(T') + h \leq l(T) \leq l(T') + h + 1 \), \( s(T') + h \leq s(T) \leq s(T') + h + 1 \) and \( \gamma_d^2(T) \leq \gamma_d^2(T') + h \). In either case, we conclude that \( \gamma_d^2(T) < \frac{n(T) + s(T) + l(T)}{4} \).

If \( t \geq 3 \), let \( T' \) be the component of \( T - v_4v_5 \) containing \( v_5 \). Observe that \( n(T) = n(T') + 2 + 2t \), \( l(T) = l(T') + t \) and \( s(T) = s(T') + t \) and \( \gamma_d^2(T) \leq \gamma_d^2(T') + t \). Analogous to the proof of Case 3, we have that \( \gamma_d^2(T) < \frac{n(T) + s(T) + l(T)}{4} \). \( \square \)

**Claim 5.** \( d(v_6) = 2 \) or all neighbors of \( v_6 \) outside \( P \) are support vertices of degree two.

First, we show that \( v_6 \) is not a support vertex. If not, it follows from Claim 1 that \( v_6 \) has one leaf-neighbor, and construct a tree \( T' \) which is obtained from \( T \) by removing the leaf-neighbor of \( v_6 \) and joining a new vertex to \( v_2 \). Let \( D' \) be a \( \gamma_d^2 \)-set of \( T' \) which contains no leaf, then \( N(v_3) \cap S(T) \subseteq D' \). We take a set \( D'' = (D' \setminus \{v_3, v_4, v_5\}) \cup \{v_6\} \) when \( D' \cap \{v_3, v_4, v_5\} \neq \emptyset \), and otherwise, \( D'' = D' \). Note that \( D'' \) is also a 2\( \mathcal{D} \)D-set of
\[ T, \text{ and moreover, } n(T) = n(T'), l(T) = l(T'), s(T) = s(T') + 1. \] Hence, \( \gamma_2^d(T) \leq \gamma_2^d(T') \leq \frac{n(T)+s(T)+l(T)}{4} \leq \frac{n(T)+s(T)+l(T)}{4} < \frac{n(T)+s(T)+l(T)}{4}. \]

Let \( u_1 \) be a leaf outside \( P \) that at maximum distance from \( v_6 \), and \( P_1 = u_1u_2 \cdots u_{s-1}u_s \) be the path between \( u_1 \) and \( v_6 \), where \( v_s = v_6 \). Clearly, \( s \leq 6 \).

If \( s = 4 \), then we have that \( u_3 \) is adjacent to a support vertices of degree two, where \( a \geq 1 \). Suppose that \( u_3 \) is not a support vertex, let \( T' \) be the component of \( T - u_3v_6 \) containing \( v_6 \). It follows from \( n(T) = n(T') + 2a + 1, l(T) = l(T') + a \) and \( s(T) = s(T') + a \) that \( \gamma_2^d(T') + 1 \leq \frac{n(T')+s(T')+l(T')}{4} + 1 = \frac{n(T)+2a+s(T)+1}{4} + 1 \leq \frac{n(T)+s(T)+l(T)}{4} \). In particular, if \( \gamma_2^d(T') = \frac{n(T)+s(T)+l(T)}{4} \), then \( \gamma_2^d(T') = \frac{n(T)+s(T)+l(T)}{4} \). It means that \( (T', S') \in \mathcal{P}_2 \) for some labeling \( S^* \). By Lemma 4.3, there exists a 2DD-set \( S \) of \( T' - \{ u \} \) with cardinality \( \gamma_2^d(T') - 1 \), and a non-leaf neighbor of \( u_3 \) in \( T' \) belongs to \( S \). It is easy to see that \( S \cup \{ u_2 \} \) is a 2DD-set of \( T \) with cardinality \( \gamma_2^d(T') \). That is, \( \gamma_2^d(T) \leq \gamma_2^d(T') \), Contradicting the fact that \( \gamma_2^d(T) = \gamma_2^d(T') + 1 \).

If \( s = 5 \), by an argument similar to that of Claim 1, Claim 2 and Claim 3, we have that \( d(u_2) = d(u_4) = 2, u_3 \) is not a support vertex and adjacent to a support vertices of degree two, where \( a \geq 1 \). Let \( T' \) be the component of \( T - u_4v_6 \) containing \( v_6 \) and \( D' \) be a \( \gamma_2^d \)-set of \( T' \) contains no leaf. If \( a \geq 2 \), Observe that \( D' \cup (S(T) \cap N(u_3)) \) is a 2DD-set of \( T' \). Combining the fact that \( n(T) = n(T') + 2a + 2, l(T) = l(T') + a, s(T) = s(T') + a \). We have that \( \gamma_2^d(T') + 1 \leq \frac{n(T)+s(T)+l(T)}{4} + 1 = \frac{n(T)+2a+s(T)+1}{4} + 1 \leq \frac{n(T)+s(T)+l(T)}{4} \). So we can assume that \( n[v_6] \cap D' = \emptyset \). If \( \{ v_3, v_4 \} \cap D' \neq \emptyset, \) then \( D' \setminus \{ v_3, v_4 \} \cup \{ v_5 \} \) is also a \( \gamma_2^d \)-set of \( T' \), and we are done. If \( v_3, v_4 \notin D' \), it follows from \( d(v_5) = 2 \) and \( N[v_6] \cap D' = \emptyset \) that \( v_5 \) is not 2DD-dominated by \( D' \), a contradiction.

If \( s = 6 \), from Claim 1, Claim 2 and the choice of \( T \), we have that \( d(u_2) = d(u_4) = d(u_5) = 2, \) and \( u_3 \) is not a support vertex and adjacent to a support vertices of degree two, where \( a \leq t \). Let \( T' \) be the component of \( T - v_5v_6 \) containing \( v_6 \) and \( D_1 \) be a \( \gamma_2^d \)-set of \( T' \) contains no leaf. Note that \( S(T) \cap N(u_3) \subseteq D_1 \). Take a set \( D' = (D_1 \setminus \{ u_3, u_4, u_5 \}) \cup \{ v_6 \} \) when \( \{ u_3, u_4, u_5 \} \cap D_1 \neq \emptyset \), and otherwise, \( D' = D_1 \). Notice that \( D' \cup \{ w_1, w_2, \cdots, w_t \} \) is a 2DD-set of \( T' \). Combining the fact that \( n(T) = n(T') + 2t + 3, l(T) = l(T') + t, s(T) = s(T') + t \). We have that \( \gamma_2^d(T') + t \leq \frac{n(T)+s(T)+l(T)}{4} + t = \frac{n(T)+2t+s(T)+1}{4} + t \leq \frac{n(T)+s(T)+l(T)}{4}. \)

We assume that \( |N(v_6) \setminus \{ v_5, v_7 \}| = a, \) then \( a \geq 0 \). In addition, by the claims as above, we have that \( d(v_2) = d(v_4) = d(v_6) = 2, v_3 \) is not a support vertex and adjacent to \( t \) support vertices of degree two, where \( t \geq 1 \).
If \( a = 0 \), then \( d(v_6) = 2 \). Let \( T' \) be the component of \( T - v_1v_5 \) containing \( v_5 \) and \( D' \) be a \( \gamma_2^d \)-set of \( T' \) contains no leaf. Observe that \( v_6 \in D' \) and \( D' \cup \{ w_1, w_2, \ldots, w_t \} \) is a 2DD-set of \( T \). It follows from \( n(T) = n(T') + 2t + 2 \), \( l(T) = l(T') + t - 1 \) and \( s(T) = s(T') + t - 1 \) that 

\[
\gamma_2^d(T) \leq \gamma_2^d(T') + t \leq n(T') + s(T') + l(T') - t + 1 + l(T') - t + 1 \leq n(T) + s(T) + l(T).
\]

Suppose that \( \gamma_2^d(T) = \frac{n(T') + s(T') + l(T')}{4} \), then we have equality throughout the above inequality chain. In particular, \( \gamma_2^d(T') = \frac{n(T') + s(T') + l(T')}{4} \). By the inductive hypothesis, \((T', S^*) \in \mathcal{T}_2\) for some labeling \( S^* \). Since \( v_5 \) is a leaf in \( T' \), by Observation 4.1(a), it has status \( C \), and then \( v_6 \) has status \( A \). Let \( S \) be obtained from the labeling \( S^* \) by labeling the vertices \( v_3, v_4 \) with label \( B, D \), respectively. And moreover, labeling \( w_1, w_2, \ldots, w_t \) with label \( A \), and label their leaf-neighbors with label \( C \). Then, \((T, S)\) can be obtained from \((T', S^*)\) by doing the operation \( \mathcal{O}_2 \) for \( t - 1 \) times. Thus, \((T, S) \in \mathcal{T}_2\).

Next we consider the case of \( a \geq 1 \). Let \( u_1, u_2, \ldots, u_a \) be all neighbors of \( v_6 \) outside \( P \) and \( u_i' \) be the leaf-neighbor of \( u_i \) \((i = 1, 2, \ldots, a)\). Let \( T' = T - \{ u_1, u_2, \ldots, u_a, u_1', u_2', \ldots, u_a' \} \) and \( D' \) be a \( \gamma_2^d \)-set of \( T' \) contains no leaf. Note that \( v_6 \) has degree two in \( T' \), and \( D' \cup \{ u_1, u_2, \ldots, u_a \} \) is a 2DD-set of \( T \). It follows from \( n(T) = n(T') + 2a \), \( l(T) = l(T') + a \) and \( s(T) = s(T') + a \) that 

\[
\gamma_2^d(T) \leq \gamma_2^d(T') + a \leq \frac{n(T') + s(T') + l(T')}{4} + a = n(T) - 2a + s(T) - a + l(T) - a + a = \frac{n(T) + s(T) + l(T)}{4}.
\]

Suppose that \( \gamma_2^d(T) = \frac{n(T) + s(T) + l(T)}{4} \), then we have equality throughout the above inequality chain. In particular, \( \gamma_2^d(T') = \frac{n(T') + s(T') + l(T')}{4} \). By the inductive hypothesis, \((T', S^*) \in \mathcal{T}_2\) for some labeling \( S^* \).

If \( t \geq 2 \), by Lemma 4.3, there exists a set \( D_1 \) with order \( \frac{n(T') + s(T') + l(T')}{4} - 1 \) such that each vertex of \( T' \) is 2D-dominated by \( D_1 \) except for \( v_1 \), and \( v_3 \) belongs to \( D_1 \). Since leaf-neighbor of each \( w_i \) \((i = 2, 3, \ldots, t) \) is 2D-dominated by \( D_1 \), without loss of generality, we can assume that each \( w_i \) \((i = 2, 3, \ldots, t) \) belongs to \( D_1 \). Note that \( d(v_3) = d(v_5) = d(v_6) = 2 \) in \( T' \) and \( \{ v_4, v_5, v_6, v_7 \} \cap D_1 \neq \emptyset \), we construct a set \( D_2 = (D_1 \setminus \{ v_4, v_5, v_6 \}) \cup \{ v_7 \} \), each vertex of \( T' \) is 2D-dominated by \( D_2 \) except for \( v_1 \) and \( |D_2| \leq |D_1| \). Let \( D_3 \) be a set which is obtained from \( D_2 \) by deleting \( v_3 \), and adding all neighbors of \( v_6 \) outside \( P \) and \( v_2 \). It is easy to see that \( D_3 \) is a 2DD-set of \( T \), and \( |D_3| \leq \frac{n(T) + s(T) + l(T)}{4} - 1 \), it is impossible.

If \( t = 1 \), the vertices \( v_1 \) and \( v_2 \) have status \( C \) and \( A \), respectively, in \( S^* \). And so, \( v_3 \) has status \( A \) or \( B \).

In the former case, it follows from \( d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2 \) and the definition of \( \mathcal{T}_2 \) that \( v_1v_2v_3v_4 \) is the basic path of \( T' \), and then \( v_4 \) has status \( C \). Moreover, \( v_5, v_6 \) have status \( D, B \), respectively. Let \( S \) be obtained from the labeling \( S^* \) by labeling each \( u_i \) with label \( A \), and each \( u_i' \) with label \( C \). Then, \((T, S)\) can be obtained from \((T', S^*)\) by doing the operation \( \mathcal{O}_2 \) for \( a \) times. Thus, \((T, S) \in \mathcal{T}_2\).

In the latter case, from the definition of \( \mathcal{T}_2 \), \( v_4, v_5, v_6 \) have status \( D, C, A \), respectively. And \( v_7 \) has status \( A \) or \( B \). Assume that \( \text{sta}(v_7) = A \). If \( d(v_7) = 2 \), we have that \( v_5v_6v_7v_8 \) is the basic path of \( T' \). Let \( S_1^* \) be obtained from \( S^* \) by changing the status \( v_3, v_4, v_5, v_6 \) to \( A, C, D, B \), respectively, and clearly, \((T', S_1^*) \in \mathcal{T}_2 \). Let \( S \) be obtained from the labeling \( S_1^* \) by labeling each \( u_i \) with label \( A \), and each \( u_i' \) with label \( C \). Then, \((T, S)\) can be obtained
from $(T', S_1^*)$ by doing the operation $\mathcal{O}_2$ for $a$ times. Thus, $(T, S) \in \mathcal{R}_2$. If $\text{sta}(v_7) = A$ and $d(v_7) \geq 3$, or $\text{sta}(v_7) = B$, let $S$ be obtained from the labeling $S^*$ by labeling each $u_i$ with label $A$, and each $u'_i$ with label $C$. Then, $(T, S)$ can be obtained from $(T', S^*)$ by doing the operation $\mathcal{O}_4$ for $a$ times. Thus, $(T, S) \in \mathcal{R}_2$. □

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