Covariant Origin of the 
$U(1)^3$ model for Euclidean Quantum Gravity

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Abstract

If one replaces the constraints of the Ashtekar-Barbero $SU(2)$ gauge theory formulation of Euclidean gravity by their $U(1)^3$ version, one arrives at a consistent model which captures significant structures of its $SU(2)$ version. In particular, it displays a non-trivial realisation of the hypersurface deformation algebra which makes it an interesting testing ground for (Euclidean) quantum gravity as has been emphasised in a recent series of papers due to Varadarajan et al.

The simplification from $SU(2)$ to $U(1)^3$ can be performed simply by hand within the Hamiltonian formulation by dropping all non-Abelian terms from the Gauss, spatial diffeomorphism, and Hamiltonian constraints respectively. However, one may ask from which Lagrangian formulation this theory descends. For the $SU(2)$ theory it is known that one can choose the Palatini action, Holst action, or (anti-)selfdual action (Euclidean signature) as starting point all leading to equivalent Hamiltonian formulations.

In this paper, we systematically analyse this question directly for the $U(1)^3$ theory. Surprisingly, it turns out that the Abelian analog of the Palatini or Holst formulation is a consistent but topological theory without propagating degrees of freedom. On the other hand, a twisted Abelian analog of the (anti-)selfdual formulation does lead to the desired Hamiltonian formulation.

A new aspect of our derivation is that we work with 1. half-density valued tetrads which simplifies the analysis, 2. without the simplicity constraint (which admits one undesired solution that is usually neglected by hand) and 3. without imposing the time gauge from the beginning. As a byproduct, we show that also the non-Abelian theory admits a twisted (anti-)selfdual formulation. Finally, we also derive a pure connection formulation of Euclidean GR including a cosmological constant by extending previous work due to Capovilla, Dell, Jacobson, and Peldan which may be an interesting starting point for path integral investigations and displays (Euclidean) GR as a Yang-Mills theory with non-polynomial Lagrangian.

1 Introduction

In our two companion papers [1] we studied the reduced phase space formulation of the $SU(2) \rightarrow U(1)^3$ truncation model for Euclidean signature General Relativity introduced in [2]. The motivation is to set up a reduced phase space quantisation as an alternative to the operator constraint quantisation much studied recently in [3]. The two quantisation methods have complementary advantages and disadvantages: The reduced approach allows to work directly with gauge-invariant variables, the physical Hilbert space and a physical time evolution while in the constrained approach these can only be extracted after having found the full solution space of the quantum constraints. On the other hand, the reduced approach in general uses suitable gauge fixings such that manifest covariance is lost and it requires non-trivial effort to prove that different gauges result in quantum theories that make identical predictions for gauge invariant objects, also gauge fixings may not be perfect and suffer from

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the existence of Gribov copies. In general, one expects that the two approaches enrich each other by providing complementary insight. In particular, the programme of [3] is designed to reduce quantisation ambiguities of the physical Lorentzian theory [4,5,6,7] in the framework of Loop Quantum Gravity (LQG), [8] using the simpler $U(1)^3$ model as a test laboratory and we aim to support that enterprise with our reduced phase space approach.

In our companion papers, [1] we focussed on the Hamiltonian formulation of the theory which is well suited as a starting point for a canonical quantisation of the reduced theory. In this paper, we consider the Lagrangian formulation of $U(1)^3$ theory where we have a subsequent path integral quantisation in mind. Note that the $U(1)^3$ theory is usually introduced as a truncation by hand of the Hamiltonian formulation of Euclidean General Relativity and not derived from an action principle. Thus the question arises whether such actions exist and how they look like.

The answer is not a priori clear: How should the gauge group of the Lagrangian be chosen? For Euclidean GR one can start with the $SO(4)$ Palatini-Holst action [9] and after a tedious constraint analysis involving second class constraints [2] and using the time gauge to fix the "boost" part of $SO(4)$ (which is one of the two copies of $SU(2)$ into which it factorises modulo sign issues) one arrives at the usual Hamiltonian $SU(2)$ formulation. Thus, a natural guess could be that a suitable covariant action starts from a suitably "abelianised" version of the $SO(4)$ Palatini Holst action, perhaps $SU(2) \times U(1)^3$ or $U(1)^6$. In [2], $U(1)^3$ itself was used to propose an action that should serve as a covariant origin of the Hamiltonian $U(1)^3$ model. We could not find a proof in the literature that this action indeed serves this purpose. In appendix [5] we show that the action of [2] has too many degrees of freedom (it has 6 rather than 4 propagating degrees of freedom). This is maybe not too surprising because up to the replacement of $SU(2)$ by $U(1)^3$, the action of [2] is the same as that of [13] where an explicit analysis also revealed 6 propagating degrees of freedom. We therefore have to look for a different covariant action. The motivation to look at $U(1)^6$ was that it is the natural compact Abelian group of the same dimension as compact $SO(4)$, analogous as $U(1)^3$ is the natural Abelian group of the same dimension as compact $SO(3)$ or $SU(2)$. Then boiling it down from $U(1)^6$ to $U(1)^3$ is supposed to happen in the time gauge.

These kinds of actions would be of "Palatini-Holst" type. Yet another very interesting possibility is that one can find a pure connection formulation of the theory without using tetrads at all. That such a possibility in principle exists in presence of a cosmological constant was pointed out in the context of the original self-dual theory for Lorentzian signature in works by Capovilla, Dell and Jacobson [11] and Peldan [12], however, no closed formula was provided. In fact, some recent and in-depth analysis of pure connection formulations after Peldan was promoted in [16,18]. Nevertheless, here our aim is not to derive the most general action for the $U(1)^3$ theory but rather give a concrete example where such a non-polynomial pure connection Lagrangian can be given explicitly. Such a Lagrangian would necessarily be rather non-polynomial in the curvature of the connection but spacetime diffeomorphism covariant and is interesting in its own right because it would put (Euclidean) General Relativity on equal footing with (Euclidean) Yang-Mills theory in the sense that only a connection is required to formulate the action.

The architecture of this article is as follows:

In section two we introduce the notation and collect the formulae for the Hamiltonian formulation of $SU(2)$ or $U(1)^3$ Euclidean General Relativity. The presentation will be brief, for more details we refer to [1] and references therein.

In section three we first consider the $U(1)^6$ Palatini action as a possible covariant origin of the $U(1)^3$ model. Surprisingly, this theory is topological, it has no propagating degrees of freedom. The technical reason for this striking difference with the $SO(4)$ theory is that in the Dirac constraint stability analysis in a crucial step the Abelian nature of the model forbids solving the equations for the existing Lagrange multipliers but rather results in additional secondary constraints. We then consider a one-parameter family of Lagrangians based on the gauge group $U(1)^3$. The parameter plays a role similar to but different from the Immirzi parameter and gives rise to a

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1It is worth mentioning that in the first reference of [17] the second-class constraints of the Holst action are solved in a manifestly Lorentz-covariant way and in the second one the authors performed the canonical analysis of the Holst action using an appropriate parametrisation of the tetrad and the connection. Finally, they arrived at the Hamiltonian formulation of the Holst action which involves only first-class constraints after integrating out some auxiliary fields. Their procedure avoids the introduction of second-class constraints.
“twisted self-dual” and Abelian connection. We show that in this case the Hamiltonian theory is indeed equivalent to the U(1)\(^3\) truncation of Euclidean General Relativity. As a byproduct, we show that the same action for SU(2) rather than U(1)\(^3\) also gives rise to the Hamiltonian formulation of Euclidean General Relativity which so far was only known for the Euclidean (anti-)self-dual theory (the parameter equals plus or minus unity). Note that despite starting from an SU(2) theory, this theory is different from the Husain-Kuchar theory [13] as our model has a non-trivial Hamiltonian constraint. As a technical advance, we perform the analysis using half densitised tetrads and avoiding the introduction of simplicity constraints which always have spurious solutions.

In section four we review the programme of [11, 12] and carry out the required computations to the end. We show that the final equation to be solved for the backward Legendre transform can be brought into a rather manageable and explicit form and can be stated as the roots of a quartic polynomial whose coefficients are simple polynomials in the spacetime covariant curvature. The roots are algebraically accessible using the Cardano-Ferrari theory [14] and will be analysed further in future publications.

In section five we conclude and point out directions for further work.

In appendix A we analyse the general solution of an equation that arises in the Dirac stability analysis of section three.

In appendix B we present the Hamiltonian analysis of the Somlin’s action [2] and conclude that it does not lead to the Hamiltonian U(1)\(^3\) model.

### Notation and review of the Hamiltonian U(1)\(^3\) model for Euclidean vacuum GR

We use \(\mu, \nu, \rho, \ldots \in \{0, 1, 2, 3\}\) as general spacetime tensor indices and \(A, B, C, \ldots \in \{t, 1, 2, 3\}\) as those adapted to an ADM slicing. Likewise we will use \(a, b, c, \ldots \in \{1, 2, 3\}\) as spatial ADM indices. Furthermore \(I, J, K, \ldots \in \{0, 1, 2, 3\}\) is a SO(4) index and \(i, j, k, \ldots \in \{1, 2, 3\}\) is an SU(2) or U(1)\(^3\) index.

By \(N, N^a, \Lambda^j\) we denote respectively lapse function, shift vector and Lagrange multiplier of the SU(2) or U(1)\(^3\) Gauss constraint. The canonically conjugate variables are \(A^a_j\) and \(E^a_j\). The difference between the U(1)\(^3\) truncation and the SU(2) model arise in the Gauss constraint and the curvature which for SU(2) read

\[
G_j = \partial_a E^a_j + \epsilon_{jkl} A^k_a E^l_q, \quad F_{ab}^j = 2\partial_{[a} A_{b]}^j + \epsilon_{jkl} A^k_a A^l_b
\]  

while for U(1)\(^3\) the terms proportional to \(\epsilon_{jkl}\) are missing.

The spatial diffeomorphism and Hamiltonian constraint (with cosmological constant) are identical for both models and read

\[
C_a = F_{ab}^j E^b_j, \quad C = [\det(E)]^{-1/2} [F_{ab}^j \epsilon_{jkl} E^a_k E^b_l + \Lambda \det(E)]
\]  

### Covariant U(1)\(^6\) theory

In this section we consider an Abelian analog of the Palatini or Holst action as a potential Lagrangian formulation of the U(1)\(^3\) theory. In the first subsection we recall the procedure that one follows in the non-Abelian case. We use it as a guideline to carry out the corresponding analysis in the Abelian case in the second subsection.

#### 3.1 Review: SO(4) Holst action

The non-Abelian SU(2) formulation can be obtained from the Palatini action or the Holst action for the gauge group SO(4). The two actions differ by a topological term multiplied by the Immirzi parameter [9]. The way this works is that the momentum conjugate to the SO(4) connection has to descend from a tetrad which we call tetrad constraint \(T^{ab}\), \(a, b = 1, 2, 3\), \(T^{[ab]} \equiv 0\). From the tetrad constraint one deduces the simplicity constraint \(S^{ab}\) which has the advantage that one can formulate it purely in terms of the momentum conjugate to the connection. Unfortunately the tetrad constraint and simplicity constraint are not equivalent: The simplicity constraint admits another solution inequivalent to the tetrad constraint which is usually neglected by hand. In order to avoid any doubtful reasoning, we therefore avoid the simplicity constraint and work instead with the proper tetrad constraint.
Nevertheless, requiring the dynamical stability of the primary simplicity constraint yields among others a secondary constraint which we call dynamical constraint $D^{ab}$. In the non-Abelian theory the Dirac algorithm stops here (there are no tertiary constraints), the constraints $S^{ab}, D^{ab}$ form a second class pair and the Lagrange multiplier of $S^{ab}$ within the primary Hamiltonian is therefore completely fixed by the stability condition. To arrive at the Hamiltonian formulation in terms of SU(2) one utilises the Lie algebra isomorphism $\text{so}(4) \cong \text{su}(2) \oplus \text{su}(2)$ and gauge fixes one copy of the decoupled $\text{su}(2)$ Gauss constraints using the time gauge $e^i_J = 0$ ($e^i_J$ is the tetrad with tensorial indices $A = t, a; a = 1, 2, 3$, and Lie algebra indices $I = 0, j; j = 1, 2, 3$). The Hamiltonian formulation then results from solving the second class constraints, half of the Gauss constraints and the time gauge and by passing to the corresponding Dirac bracket.

### 3.2 The U(1)$^6$ model

As we have just seen, in the non-Abelian case the Hamiltonian $\text{su}(2)$ formulation results from the Lagrangean $\text{so}(4)$ formulation by gauge fixing one of the $\text{su}(2)$ copies. In the Abelian case this suggests to start from the Lie algebra $\mathfrak{g} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{h}$ where $\mathfrak{h}$ is some Lie algebra such that $\mathfrak{g}$ admits a four dimensional representation (in which the tetrad transforms while the connection transforms in the adjoint). If we wish to stay as close as possible to the non-Abelian theory then $\mathfrak{g}$ should be six dimensional. If the gauge group corresponding to $\mathfrak{g}$ is supposed to be compact then the one corresponding to $\mathfrak{h}$ is also compact. Finally, if $\mathfrak{g}$ is supposed to be Abelian this fixes $\mathfrak{g}$ as direct sum of six $\mathfrak{u}(1)$ copies.

Accordingly, the Lagrangian variables are a $U(1)^6$ connection $A^I_B$, $A^{(IJ)}_B = 0$ and a tetrad $e^A_I$ which transforms in the trivial representation of $U(1)^6$. The expectation is then that in analogy to the non-Abelian case three copies of $U(1)$ get gauge fixed in the course of the Dirac algorithm, either by hand as in the non-Abelian case or as a consequence of the stability conditions. In the first case it is clear that the time gauge cannot serve as gauge fixing condition since the tetrad is Gauss invariant. In the second case the time gauge could arise as a second class constraint.

Surprisingly, while consistent, the $U(1)^6$ model turns out to be topological. As in the non-Abelian case, the Dirac analysis stops at the secondary level, i.e. there are no tertiary constraints. The secondary constraints involve the Abelian analog of the dynamical constraint $D^{ab}$. However, precisely due to the Abelian nature of the model, the pair of constraints ($T^{ab}, D^{ab}$) is now first class rather than second class. By the usual naive counting, this leads to a reduction by 24 rather than 12 degrees of freedom. As the non-Abelian theory only has 4 propagating degrees of freedom, this means that the resulting theory is topological.

In what follows we provide the details of the analysis. We start from the action

$$S = \frac{1}{2} \int dt d^3x \ F_{AB} I J A B \hat{\sigma}_{IJ}^{AB}, \quad \hat{\sigma}_{IJ}^{AB} = \hat{\Sigma}_{IJ}^{AB} + \frac{1}{2} \gamma \epsilon_{IJ} \ K L \hat{\Sigma}_{KL}^{AB}, \quad \hat{\Sigma}_{IJ}^{AB} = \hat{e}^A_I \hat{e}^B_J \tag{3.1}$$

where Lie algebra indices are moved with the Kronecker symbol. Here we work with half density valued tetrads

$$\hat{e}^A_I = \det\{e^A_I\}^{1/2} e^A_I \tag{3.2}$$

in terms of which the action is polynomial. It is assumed that the tetrad is nowhere degenerate so that w.l.g. its determinant is everywhere positive. Here $\gamma$ is the Immirzi parameter. In this subsection we assume $\gamma \neq \pm 1$ because otherwise the connection is projected onto its (anti-)selfdual part. The values $\gamma = \pm 1$ will be treated in the next subsection as a special case. The analysis for $\gamma \neq \pm 1$ is qualitatively similar to the case $\gamma = 0$ so that we set $\gamma = 0$ for the rest of this subsection.

The 3+1 decomposition reveals

$$S = \int dt d^3x [F_{ta}^{IJ} \hat{\Sigma}_{IJ}^{ta} + \frac{1}{2} F_{ab}^{IJ} \hat{\Sigma}_{IJ}^{ab}] \tag{3.3}$$

Computing the momenta $\pi^B_{IJ}, \hat{P}_A^I$ conjugate to $A^I_B, \hat{e}^A_I$ we find the primary constraints

$$\pi^B_{IJ} = 0, \quad T_{IJ}^a := \pi^a_{IJ} - \hat{\Sigma}_{IJ}^{ta} = 0, \quad \hat{P}_A^I = 0 \tag{3.4}$$
and the Legendre transform of (3.3) yields with the velocities \( \dot{v}^A_{IJ}, \dot{\pi}^A_I \) that one cannot solve for the primary Hamiltonian

\[
H = \int d^3x \left\{ \dot{v}^I_{IJ} \pi^A_{IJ} + \dot{\pi}^A_I \dot{P}^I_A - L \right\}
\]

\[
= \int d^3x \left\{ \dot{v}^I_{IJ} T^a_{IJ} + \dot{\pi}^A_I \dot{P}^I_A + \dot{v}^I_{IJ} \pi^A_{IJ} + A^I_{t,a} \Sigma^a_{IJ} - \frac{1}{2} F^I_{ab} \Sigma^b_{IJ} \right\}
\]

(3.5)

Stability of \( \pi^A_{IJ} = 0 \) yields the U(1)⁶ Gauss secondary constraint

\[
G_{IJ} = \partial_a \Sigma^a_{IJ}
\]

(3.6)

Stability of \( \dot{P}^I_A = 0 \) yields with the Lagrange multiplier \( f^I_{aJ} := v^I_{aJ} - \partial_a A^I_{tJ} \) the 16 equations

\[
f^I_{aJ} \hat{e}^a_J = F^I_{ab} \hat{e}^b_J - f^I_{aJ} \hat{e}^a_J = 0
\]

(3.7)

We solve them by decomposing \( f^I_{aJ} = f^I_{aJ} \hat{e}^b_J \hat{\Sigma}^b_{IJ} \) where \( \hat{\Sigma}^b_{IJ} = \hat{e}^I_{aB} \hat{e}^J_{bC} \), \( \hat{e}^I_{aB} \hat{e}^J_{bC} = \delta^I_0 \delta^J_0 \), \( \hat{e}^I_{aB} \hat{e}^J_{bC} = \delta^I_0 \delta^J_0 \) and \( \hat{e}^I_{aB} \hat{e}^J_{bC} = \delta^I_0 \delta^J_0 \) and exploiting the relations

\[
\hat{\Sigma}^b_{IJ} \Sigma^c_{IJ} = \delta^b_0 \delta^c_0, \quad \hat{\Sigma}^b_{IJ} \Sigma^d_{IK} = \delta^b_0 \delta^d_0
\]

(3.8)

We find the following restriction on the Lagrange multipliers

\[
f^I_{aJ} = F^I_{bc} \hat{\Sigma}^b_{IJ}, \quad f^I_{aJ} = \epsilon^{bcd} s_{ad}, \quad s_{[ad]} = 0
\]

(3.9)

and the 4 secondary constraints

\[
C_a := F^I_{ab} \hat{\Sigma}^b_{IJ}, \quad C := F^I_{ab} \hat{\Sigma}^b_{IJ}
\]

(3.10)

That is, 12 of 18 Lagrange multipliers have been fixed while the 6 degrees of freedom encoded by the symmetric tensor \( s_{ad} \) remain free. The 16 relations (3.9) and (3.10) ensure stability of \( \dot{P}^I_A \).

Stability of \( T^a_{IJ} \) yields

\[
\partial_b \Sigma^b_{IJ} + \dot{v}^I_{aJ} \hat{e}^a_J + \hat{e}^I_{aJ} \hat{\Sigma}^a_{IJ} = 0
\]

(3.11)

We decompose \( \dot{\pi}^A_I = \dot{\pi}^B_I \hat{e}^B_I \) and find the restriction on the Lagrange multipliers

\[
\hat{\Sigma}^a_{IJ} \hat{\Sigma}^a_{IJ} + \dot{v}^I_{aJ} \hat{\Sigma}^a_{IJ} = 0
\]

(3.12)

and the 6 secondary constraints

\[
D^{ab} = \epsilon^{cd(a} \hat{\Sigma}^b_{IJ,c} \hat{\Sigma}^c_{IJd}
\]

(3.13)

That is, 12 of 16 Lagrange multipliers have been fixed while the 4 multipliers \( \hat{\pi}^A_I \) remain free. Altogether the 18 relations (3.12), (3.13) ensure stability of \( T^a_{IJ} = 0 \).

We now must stabilise \( G_{IJ}, C_a, C, D^{ab} \). This is most conveniently done as follows. We have modulo \( T^a_{IJ} = 0 \) that \( G_{IJ} = \partial_a \pi^a_{IJ} \). Since \( T^a_{IJ} \) is already stabilised, we may equivalently stabilise the constraint in the form \( \hat{G}_{IJ} := \pi^a_{IJ} = 0 \) in which it generates U(1)⁶ gauge transformations. Since \( H \) depends on \( A^I_{tJ} \) only through its curvature \( F^I_{ab} = 2\partial^b_{[a} A^I_{tJ]} \) the Gauss constraint \( \hat{G}_{IJ} \) is already stabilised. Next, modulo \( T^a_{IJ} = G_{IJ} = 0 \) we have \( C_a = F^I_{ab} \pi^b_{IJ} - A^I_{t,a} \hat{G}_{IJ} \) in which form it generates spatial diffeomorphisms on \( (A^I_{tJ}, \pi^a_{IJ}) \). Since \( \dot{P}^I_A \) has already been stabilised, we add terms linear in \( \dot{P}^I_A \) to \( C_a \) so that the resulting constraint \( \hat{C}_a \) generates spatial diffeomorphisms also on the variables \( \hat{\pi}^A_I, \hat{P}^I_A \) (in doing so explicitly, note the density weight \( \pm 1/2 \) respectively). Since the primary Hamiltonian only depends on constraints which are tensor densities, the constraint \( \hat{C}_a \) and thus \( C_a \) is already stabilised.
Unfortunately, these abstract arguments are not available for $C, D^{ab}$. We show some stages of the surprisingly tedious computation

\[
\{H, C(f)\} = \int d^3 x \left\{ v_a^{IJ} \pi_{IJ}^a + \dot{v}_A^I \dot{P}_A^I, C(f) \right\}
\]

\[
= 2 \int d^3 x f \left[ \sum_{bc} \partial_b v_c^{IJ} + \dot{v}_a^b e_c^L F_{bc}^{KL} \right]
\]

\[
= 2 \int d^3 x f \left[ \sum_{bc} \partial_b f_c^{IJ} + \dot{v}_a^b e_c^L F_{bc}^{KL} \right]
\]

\[
= 2 \int d^3 x f \left[ (f_b^{ab})_a - \sum_{IJ, a} f_b^{IJ} + \dot{v}_t^I C_b + \dot{v}_a^b f_0^a \right]
\]

\[
= 2 \int d^3 x f \left[ \sum_{IJ, a} - \sum_{IJ, a} f_b^{IJ} + \dot{v}_t^I C_b \right]
\]

\[
= 2 \int d^3 x f \left[ \sum_{IJ, a} - \sum_{IJ, a} f_b^{IJ} + \dot{v}_t^I C_b \right]
\]

where in the first step we isolated the contributing part of $H$, in the second step we carried out the Poisson brackets and integrated by parts, in the third step we noticed that the r.h.s. depends on $v_a^{IJ}$ only through $f_b^{IJ}$, in the fourth we inserted (3.9), (3.12), in the fifth we dropped $C_b$ and used $f_b^{ab} = 0$ and decomposed $f_b^{IJ}$, in the sixth we used again (3.9), (3.12), and in the seventh we cancelled terms, dropped $D^{ab}$ and used $C = f_0^a$. Accordingly $C$ is stable where the precise form of $f_0^a$, $\dot{v}_t^I$ and the secondary constraints $C_a, C, D^{ab}$ had to be used in a crucial way.

In order to carry out the analysis for $D^{ab}$ it is useful to rewrite it in the simpler form

\[
D^{ab} = \epsilon^{cd(a} w_{c(b)} d), \quad w_{c(d)} = \epsilon_{c(d} \epsilon_{I,d} \delta_d^b \quad (3.15)
\]

Using also the abbreviation $\sigma_b^a := \dot{v}_t^I \dot{v}_t^{I,b}$ it is not difficult to check that

\[
\dot{v}_b^a = \delta_b^a - [v_t^I + \sigma_c^a_d] \dot{v}_t^I, \quad \dot{v}_t^I = -w_{(a(b)} b)
\]

With this machinery we compute

\[
\{H, D^{ab}(g_{ab})\} = \int d^3 x \left\{ \dot{v}_t^I \dot{P}_A^I, D^{ab}(g_{ab}) \right\}
\]

\[
= -\int d^3 x g_{ab} \epsilon^{a\sigma_c\delta_d^b} \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} + \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} - \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d}
\]

where in the first step we isolated the contributing piece of $H$, in the second we carried out the Poisson bracket and integrated by parts and used above abbreviations, in the last step we realised that the piece $\times \delta_b^a$ in $\dot{v}_t^I$ drops out due to symmetry of $g_{ab}$. It is easy to verify

\[
\epsilon^{a\sigma_c\sigma_d} \delta_d^b = -\epsilon^{a\sigma_c\sigma_d} \delta_d^b \epsilon_{I,d} \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} \quad (3.18)
\]

and with (3.15) and decomposing into symmetric and antisymmetric piece wrt indices $a, b$

\[
\epsilon^{a\sigma_c\sigma_d} \epsilon_{I,d} \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} = D^{ae} + \epsilon^{a\sigma_c\sigma_d} \epsilon_{I,d} \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} \quad (3.19)
\]

Dropping the term $\times D^{ae}$ we can simplify

\[
\{H, D^{ab}(g_{ab})\} = \int d^3 x g_{ab} \left[ \sigma_b^a \epsilon^{a\sigma_c\sigma_d} \dot{v}_t^I \dot{v}_t^{I,c} \dot{v}_t^{I,d} - \dot{v}_t^b u^a \right]
\]

(3.20)
where
\[ w_i := w_i^a b + e_i^a e_{I,a}, \quad u^a := e^{acd} e_i^a e_{c,d} \] (3.21)

The final step consists in verifying that
\[ u^a = -\frac{1}{2} \epsilon^{abc} \hat{\Sigma}_{bc}^{IJ} G_{IJ}, \quad w_a = 2 \hat{\Sigma}_{ta}^{IJ} G_{IJ} \] (3.22)

so that \( D^{ab} \) is stable modulo \( D^{ab}, G_{IJ} \).

Accordingly, there are no tertiary constraints. However, the reason for the absence of tertiary constraints is completely different as compared to the non-Abelian theory. There the absence of tertiary constraints was ensured by fixing Lagrange multipliers in the primary Hamiltonian while in the Abelian theory no Lagrange multipliers are fixed, the stability is ensured already by the secondary constraints.

This difference also leads to a crucial departure in the counting of the degrees of freedom\(^2\) between the Abelian and non-Abelian theories: In the U(1)\(^6\) theory we have altogether 2 \( \cdot (4 \cdot 6 + 4 \cdot 4) = 80 \) phase space degrees of freedom. We have 6 + 18 + 16 = 40 primary constraints \( \pi_{I,J}^a, T_{IJ}^a, \hat{P}_I^a \) and 6 + 4 + 6 = 16 secondary constraints \( G_{IJ}, \cal{C}_a, C, D^{ab} \). Altogether 12 of the 18 \( v_{a}^{IJ} \) and 12 of the 16 \( v_{I}^{A} \) go fixed in the course of the stability analysis while 6 of the \( v_{a}^{IJ} \) (encoded by \( s_{ab} \)) and 4 of the \( v_{I}^{A} \) (encoded by \( \hat{v}_{I}^{A} \)) remain free. Going through the same analysis as above one can verify that \( \hat{C} \) is a first class constraint where \( \hat{C} \) is the integrand of \( H \). This implies that \( \cal{C} \) is also first class. Furthermore, the fact that \( s_{ab}, \hat{v}_{I}^{A} \) remain free implies that the 6 + 4 constraints \( S^{ab}, \hat{P}_A \) multiplying them are also first class. Explicitly
\[ S^{ab} = \pi^{(a}{}_{IJ} \epsilon^{b)cd} \hat{\Sigma}_{cd}^{IJ}, \quad \hat{P}_I = \hat{P}_I^a \epsilon_{I}^a - \hat{P}_a^I \epsilon_{I}^a, \quad \hat{P}_a = \hat{P}_a^I \epsilon_{I}^a \] (3.23)

Finally \( D^{ab} \) has weakly vanishing Poisson brackets with all constraints except \( \hat{P}_A^a \). However, we may add a Term \( \propto T_{IJ}^a \) to \( D^{ab} \) such that the resulting \( D^{ab} \) has exactly vanishing Poisson brackets with all constraints. Specifically
\[ \hat{D}^{ab}(g_{ab}) = D^{ab}(g_{ab}) + T_{IJ}^a (\hat{\Sigma}_{IJ}^{AB} \delta_6^{AB}(g)), \quad h_a^{ca}(g) = -g_{ab} e^{abc} e_{I}^b \epsilon_{I,d}^c, \quad h_e^{ec}(g) = 2 e^{abc} w_{e,d}^b g_{ab} + e^{acd} g_{ac,d} \] (3.24)

To summarise: We have 12 second class pairs formed by 12 out of 18 \( T_{IJ}^a \) and 12 out of 16 \( \hat{P}_I^a \) and 6 + 6 + 4 + 6 + 4 + 6 = 32 first class constraints \( \pi_{I,J}^a, S^{ab}, \hat{P}_A, G_{IJ}, \cal{C}_a, C, D^{ab} \). Altogether these are 2 \( \cdot 12 + 32 = 56 = 40 + 16 \) constraints which reduce 24 + 2 \( \cdot 32 = 88 \) degrees of freedom. In the non-Abelian theory, the analogs of \( S^{ab}, D^{ab} \) form a second class pair. Thus there we have also 56 constraints but now we have 36 second class constraints and only 20 first class constraints thus reducing only 2 \( \cdot 20 + 36 = 76 \) degrees of freedom and leaving 4 propagating ones. Accordingly, the U(1)\(^6\) theory is consistent in the sense that the Dirac analysis does not lead to a contradiction, but it is topological in the sense that the constraint reduction leaves no local degrees of freedom. The U(1)\(^6\) theory therefore cannot be a Lagrangian origin of the Hamiltonian U(1)\(^3\) theory which has 4 propagating degrees of freedom.

### 3.3 The twisted selfdual model

The following analysis surprisingly applies to both U(1)\(^6\) and SO(4) simultaneously. We consider the action
\[ S = \frac{1}{2} \int dt \; d^3 x \; F_{IJ}^{AB} \hat{\Sigma}_{IJ}^{AB}, \quad \hat{\Sigma}_{IJ}^{AB} = \hat{e}_{I}^a \hat{e}_{J}^b \] (3.25)

but \( F_{IJ}^{AB} \) is not an U(1)\(^6\) or SO(4) curvature but rather a twisted selfdual U(1)\(^6\) or SO(4) curvature, that is,
\[ F^{0j} = F^{j} = \frac{1}{2 \gamma} \epsilon_{jkl} F^{kl} \] (3.26)

\(^2\)Note that a safe way to count degrees of freedom is to use Dirac’s algorithm which in particular ensures that the constraints encountered are algebraically independent. Counting at the Lagrangian level can be tricky: for example the action reveals 24 configuration degrees of freedom, 4 diffeomorphism gauges and 3 Yang-Mills type gauges. One cannot easily deduce from the Lagrangian that this theory has 3 propagating canonical pairs.
with $F^j = 2dA^j$ and $F^j = 2dA^j + \epsilon_{jkl}A^k \wedge A^l$ respectively a U(1)$^3$ and SU(2) curvature. Here $\gamma \neq 0$ is similar to but different from the Immirzi parameter: Note that condition (3.26) defines the (anti-)selfdual model only for $\gamma = \pm 1$. If one starts from the Holst action (3.11) with topological term and $\gamma = \pm 1$ one arrives at (3.25) because then the standard curvature of SO(4) or $U(1)^6$ is projected into the curvature of the (anti-)selfdual (i.e. SU(2) or $U(1)^3$ respectively) connection. However, when $\gamma \neq \pm 1$ the connection stays a genuine SO(4) or $U(1)^6$ connection and one does not arrive at (3.25). Thus for $\gamma \neq \pm 1$ the action is new. Still, as we will demonstrate below, it turns out to be equivalent to Euclidean GR or its $U(1)^3$ truncation respectively. The fact that the action contains a full co-tetrad $e^I_A$ rather than just three frame one forms $e^I_A$ makes it different from the SU(2) model defined by Husain and Kuchar [13] which has no Hamiltonian constraint although we also have only a SU(2) or $U(1)^3$ connection. It maybe puzzling how an SU(2) or $U(1)^3$ connection acts on $\mathbb{R}^4$ but this is by the same mechanism as a self-dual connection would act. Note that it is crucial that $\gamma \neq 0, \infty$ as we would otherwise end up with the action of [2] which is known to have too many degrees of freedom (see appendix B).

The subsequent analysis turns out to be much simpler than in the previous subsection. To anticipate the essential result for readers not interested in the details, we sketch the outcome of the analysis: This time among the primary constraints we find 9 tetrad constraints $T_j^a = \pi^a_j - \sigma_j^a$, 3 momentum constraints $\pi^t_j$ and the 16 momentum constraints $\hat{P}_A^I$ as before. The stability of $\pi^t_j$ enforces as secondary constraint the Gauss constraint $G_j$ for the corresponding gauge group. The stability of $\hat{P}_A^I$ leads to 7 secondary constraints $C_a, C, D_j$ where $C_a$ is the spatial diffeomorphism constraint, $C$ the Hamiltonian constraint and $D_j = \hat{c}_j^I$ is the time gauge constraint. That is, the time gauge is dynamically enforced in this model and not a convenient gauge choice. Furthermore all 9 Lagrange multipliers $v^a_I$ of $T_j^a$ get fixed in that process. Finally, stabilisation of $T_j^a$ fixes 9 of 16 Lagrange multipliers $\hat{v}_I^A$ of $\hat{P}_A^I$. Next, the constraints $G_j, C, C_a$ are already stable while stabilisation of $D_j$ fixes further 3 of 16 of the $\hat{v}_I^A$. This ends the stabilisation process.

All Lagrange multipliers but $4 = 16 - 9 - 3$ of the $\hat{v}_I^A$ and all 9 of the $v^a_I$ were fixed. The means that 4 of the $\hat{P}_A^I$, call them $\hat{P}_A$ are first class (they are linear combinations of the momenta conjugate to lapse and shift functions). Furthermore, $\pi^t_j, G_j, C, C_a$ are first class while 3 of the $\hat{P}_A^I$, call them $\hat{P}_j^I$ form second class pairs with $D_j$ while 9 of the $\hat{P}_A^I$, call them $\hat{P}_I^A$ form second class pairs with the $T_j^a$. Correspondingly we have $2 \cdot (3 + 9) = 24$ second class constraints $D_j, \hat{P}_j^I, T_j^a, \hat{P}_j^I$ and $3 + 3 + 4 + 4 = 14$ first class constraints $\pi^t_j, G_j, C, \hat{P}_A$. These reduces $24 + 2 \cdot 14 = 52$ of the $2 \cdot (12 + 16) = 56$ degrees of freedom $A_B^I, \pi_j^t, \pi_j^a, \hat{P}_A^I$ leaving the 4 propagating degrees of freedom of the Hamiltonian U(1)$^3$ or SU(2) theory respectively.

We now outline the details:

Plugging (3.26) into (3.25) results in the 3+1 decomposition

$$S = \int dt \, d^3x \left[ F^j_{la} \hat{\sigma}_j^a + \frac{1}{2} F_{ab} \hat{\sigma}_j^{ab} \right] - \frac{1}{4} \hat{\sigma}_j^{AB} := 2 \Sigma_{0j}^{AB} + \gamma \epsilon^{jkl} \hat{\sigma}_k^{AB}$$

(3.27)

The computation of the conjugate momenta leads to the primary constraints

$$\pi^t_j = 0, \quad T_j^a = \pi^a_j - \hat{\sigma}_j^a = 0, \quad \hat{P}_A^I = 0$$

(3.28)

and thus the primary Hamiltonian reads with the velocities $v^a_I, \hat{v}_I^A$

$$H = \int d^3x \left[ v^a_I \pi^a_I + \hat{v}_I^A \hat{P}_A^I - L \right]$$

$$= \int d^3x \left[ v^a_I \pi^a_I + v^a_I T_j^a + \hat{v}_I^A \hat{P}_A^I - A^I_j (\nabla_a \sigma_j^a) - \frac{1}{2} F^j_{ab} \sigma_j^{ab} \right]$$

(3.29)

where $\nabla$ denotes the U(1)$^3$ or SU(2) covariant derivative acting on Lie algebra indices only, that is, $\nabla_a T_j = \partial_a T_j$ and $\nabla_a T_j = \partial_a T_j + \epsilon_{jkl} A^k_a T_l$ respectively. Stabilisation of $\pi^t_j$ yields the Gauss constraint

$$G_j = \nabla_a \sigma_j^a$$

(3.30)
Stabilisation of \( \hat{\mathbf{P}}_A^I \) yields the condition

\[
\{ H, \hat{\mathbf{P}}_A^I (f_A^I) \} = \int d^3 x \left\{ \hat{\mathbf{P}}_A^I (f_A^I), v^a e^a_j \sigma^{ja}_j + \frac{1}{2} F^{ab}_{\mu\nu} \hat{\sigma}^{\mu\nu}_j \right\} = 0 \tag{3.31}
\]

for all \( f_A^I \). Isolating the i. \( A = t, I = 0 \), ii. \( A = t, I = i \), iii. \( A = c, I = 0 \), iv. \( A = c, I = i \) coefficients yields the following set of \( 1 + 3 + 3 + 9 = 16 \) conditions

\[
\begin{align*}
0 &= v^a e^a_j \\
0 &= v^i \hat{e}^a_0 + \gamma \epsilon^{ikl} v^k \hat{e}^a_l \\
0 &= -\hat{e}^t_0 v^j + F^{ab}_j \hat{e}^b_0 \\
0 &= v^i \hat{e}^t_0 + \gamma \epsilon^{ikl} v^k \hat{e}^t_l - [ F^i_{ab} \hat{e}^b_0 + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l ] 
\end{align*}
\tag{3.32}
\]

The general solution of the system (3.32) requires a detailed case by case analysis which is provided in the appendix \([A]\).

However, a physically motivated solution consists in the following 7 secondary constraints

\[
D_j := \hat{e}^t_0, \quad C_a := F_{ab}^j \hat{e}^b_j, \quad C := \epsilon_{jkl} F_{ab}^j \hat{e}^a_k \hat{e}^b_l \tag{3.33}
\]

and the following restriction on \( v^i_a \)

\[
v^i_a = \frac{1}{\epsilon^0} \left[ F^{kj}_{ab} \hat{e}^b_0 + \gamma \epsilon^{ikl} F^k_{ab} \hat{e}^b_l \right] \tag{3.34}
\]

where we have assumed \( \hat{e}^0_0 \neq 0 \). Indeed using \( D_j = 0 \) in the fourth equation of (3.32) results in (3.34). Using \( D_j = 0 \) in the third equation of (3.32) results in \( C_a = 0 \). Inserting (3.34) into \( \hat{e}^0_0 \) times the first equation of (3.32) yields \( \hat{e}^0_0 C_a + \gamma C = 0 \) i.e. \( C = 0 \). Finally inserting (3.34) into \( \hat{e}^0_0 \) times the second equation in (3.32) yields

\[
0 = \gamma^2 \epsilon^{ikl} \epsilon^{kmn} F_{ab}^{mn}_a \hat{e}^b_0 = \gamma^2 (-C_i \hat{e}^i_0 - F^i_{ab} \hat{e}^{ab}_0) = -\gamma^2 C_i
\tag{3.35}
\]

and is thus already satisfied. Here symmetry of \( \hat{e}^{ab}_0 = \delta^{ik} \hat{e}^i_j \hat{e}^k_l \) was used.

Stabilisation of \( T^a_j \) leads to

\[
(\nabla_b \hat{e}^{ab}_j) + \hat{e}^t_0 \hat{e}^a_j - \hat{e}^t_0 \hat{e}^a_j + \hat{e}^t_0 \hat{e}^a_j + \epsilon^{ikl} \hat{e}^l_0 \hat{e}^i_0 = 0
\tag{3.36}
\]

which can be solved for \( \hat{e}^a_j \).

Before doing so, we consider the stabilisation of the secondary constraints. Obviously stabilisation of \( D_j \) enforces

\[
\hat{e}^t_0 = 0
\tag{3.37}
\]

so that (3.36) simplifies to

\[
(\nabla_b \hat{e}^{ab}_j) + \hat{e}^t_0 \hat{e}^{ab}_j + \hat{e}^t_0 \hat{e}^{ab}_j = 0
\tag{3.38}
\]

The Gauss constraint can be substituted by \( \hat{G}_j = \nabla_a \pi^a_j \) modulo \( T^a_j \) which is already stabilised. In this form it generates Gauss gauge transformations on the sector \((A^a_j, \pi^a_j)\). Since

\[
F_{ab} \sigma^{ab}_j = 2 \hat{e}^a_0 C_a + C
\tag{3.39}
\]

depends only on invariants, it Poisson commutes with this part of \( H \). In the Abelian case it also Poisson commutes with all other parts of \( H \) but in the non-Abelian case it does not Poisson commute with \( T^a_j \). In the non-Abelian case we add terms to \( \hat{G}_j \) linear in \( \hat{\mathbf{P}}_A^I \) which is already stabilised

\[
\hat{G}_j = \nabla_a \pi^a_j - \epsilon_{jkl} \hat{\mathbf{P}}_A^A \hat{e}^A_l
\tag{3.40}
\]

and now \( \hat{G}_j \) also generates SU(2) rotations in the \((\hat{e}^a_j, \hat{\mathbf{P}}_A^I)\) sector while it leaves the sector \((\hat{e}^0_0, \hat{\mathbf{P}}_A^A)\) invariant. In particular, it rotates \( T^a_j \) into itself. It follows that \( \hat{G}_j \) is stabilised and thus \( G_j \) is stabilised also in the non-Abelian theory.
Next, modulo $T_j^a, G_j$ which are already stabilised we can substitute $C_a$ by $\hat{C}_a = F_{ab}^j \pi_j^b - A_a^j \nabla_b \pi_j^b$ in whose form it generates spatial diffeomorphisms on the variables $A_a^j, \pi_j^b$. We add terms linear in the already stabilised $\hat{P}_A^I$ so that it generates spatial diffeomorphisms on all variables (taking the half-density weight into account so that $\hat{c}_I^j$ and $\hat{\epsilon}_I^j$ are respectively scalar and vector half-densities)

$$\hat{C}_a = F_{ab}^j \pi_j^b - A_a^j \nabla_b \pi_j^b + \frac{1}{2} (\hat{P}_t^I \hat{c}_I^t - \hat{P}_t^I \hat{\epsilon}_I^t + 2(\hat{P}_a^I \hat{c}_I^a)_b - \hat{P}_b^I \hat{c}_I^a + \hat{P}_b^I \hat{\epsilon}_I^a)$$

(3.41)

It follows that $\hat{C}_a$ and thus $C_a$ is already stabilised since $H$ is a linear combination of constraints and all constraints are tensor densities. As far as $C$ is concerned, we consider instead $\hat{C}$ which is the integrand of $H$ with the fixed expressions for $v_a^I, \hat{v}_J^A$. Then we compute

$$\{\hat{C}(f), \hat{C}(g)\} = \frac{1}{2} \int d^3x d^3y \left[ f(x) g(y) - f(y) g(x) \right] \{\hat{C}(x), \hat{C}(y)\}$$

(3.42)

Modulo constraints, only the contributions to the Poisson bracket which lead to derivatives of the $\delta$ distribution do not vanish in (3.42). The only derivatives within constraints in $\hat{C}$ come from the term $-\frac{1}{2} F_{ab}^j \hat{\sigma}_{ab}^j$ which has non vanishing Poisson brackets leading to derivatives of $\delta$ distributions only with the term $v_a^I T_j^a$. It follows

$$\{\hat{C}(f), \hat{C}(g)\} = -\frac{1}{2} \int d^3x d^3y \left[ f(x) g(y) - f(y) g(x) \right] v_a^I(x) \{\pi_j^b(x), F_{bc}^k(y)\} \sigma_k^c(y)$$

$$= \int d^3x \omega_b \left( F_{cd}^j + \gamma \epsilon^{klt} F_{cd}^k \hat{c}_I^l \right) (2\hat{c}_{0b}^c \hat{c}_I^c + \epsilon^{d_0 d b} \hat{c}_I^c \hat{c}_I^c)$$

$$= \int d^3x \omega_b \left( -C_d \hat{c}_I^c \hat{c}_I^c - (F_{cd}^j \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c - \gamma \hat{c}_I^c \hat{c}_I^c) C_d$$

$$+ \gamma \left[ \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c + \epsilon^{d_0 d b} \epsilon^{d_0 d b} \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \right]$$

$$= \gamma \int d^3x \omega_b \left( F_{cd}^k \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \right)$$

$$= \int d^3x \omega_b \left( F_{cd}^k \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \hat{c}_I^c \right)$$

(3.43)

where in the first step we isolated the relevant terms, in the second we dropped the term in $F_{ab}^j$ quadratic in the connection as its Poisson bracket is ultra-local (this step is avoided in the Abelian case), in the third we carried out the Poisson bracket and integrated by parts, in the fourth we defined $\omega_b := \frac{1}{\epsilon_0}(f g, b - f, b, g)$ to simplify the notation and explicitly made use of definition of $\hat{\sigma}_{ab}^j$ and (3.34), in the fifth we explicitly wrote out the six terms arising from the multiplication of the round brackets in the previous step, in the sixth we dropped the terms proportional to $C_d$ and the term in round brackets which vanishes due to antisymmetry and relabelled indices in the square bracket term and used summation identities, in the seventh we wrote out the square bracket term such that it manifestly symmetric in $c, d$ and used the definition of $C_d$ and $F_{cd}^j c_{0d} = 0$ and in the final step we used antisymmetry of $F_{cd}^k$. The identity (3.44) is thus a manifestation of the hypersurface deformation algebra. Choosing $f = 1$ shows that $\{H, \hat{C}(g)\}$ is weakly vanishing for all $g$ thus $\hat{C}$ is stabilised. Since $\hat{C} = C$ plus constraints that are already stabilised, it follows that $C$ itself is also stabilised.

Accordingly there are no tertiary constraints and all secondary constraints are stabilised. All $v_a^I$ and all $\hat{v}_J^A$ have been fixed while $\hat{c}_I^j$ remain free. We come to the classification of the primary constraints $\pi_j^b, T_j^a, \hat{P}_A^I$ and
secondary constraints $G_j, C, C_a, D_j$.

i. $\pi_j^a$

Since all constraints are independent of $A_j^B, \pi_j^t$ is first class.

ii. $\mathring{G}_j$

This constraint generates Gauss gauge transformations on $\pi_j^a, A_j^B, \mathring{c}_j^A, \mathring{P}_A^j$. All constraints either are invariant or covariant under Gauss transformations. Thus $\mathring{G}_j$ is first class.

iii. $\mathring{C}_a$

This constraint generates spatial diffeomorphisms on all variables and all constraints are tensor densities. Hence $\mathring{C}_a$ is first class.

iv. $\mathring{C}$

Since $\mathring{C}$ is the integrand of $H$ which stabilises all constraints, it follows that $\mathring{C}$ has weakly vanishing Poisson brackets with all constraints except possibly those whose Poisson brackets involve derivatives of $\delta$ distributions since $H = \mathring{C}(f = 1)$ rather than general $\mathring{C}(f)$. These are the brackets with $\mathring{C}(g)$ and with $T_j^a(g_j^t)$. The first bracket has been checked in (3.42) and the second yields the same result as with $H$ except that the integral of the resulting Poisson bracket also involves $f$ as an underived factor. It follows that $\mathring{C}$ is first class.

v. $\mathring{P}_0^0$

We only need to check its Poisson brackets with $\mathring{P}_0^B, T_0^a, D_j$. Clearly the brackets with $\mathring{P}_B^j, D_j$ vanish exactly while

$$\{\mathring{P}_A^0(f^A), T_a^j(g^t_j)\} = -\int d^3x \ f_A \ \frac{\partial \hat{\vartheta}_a^0}{\partial \hat{\vartheta}_0^a} = -\int d^3x \ g_a^t (f^0 \hat{\vartheta}_0^a - f^a D_j)$$

(3.44)

It follows that $\mathring{P}_0^0$ is first class. We substitute $\mathring{P}_t^0$ by $\mathring{P}_t^0 = \mathring{P}_t^0 - \frac{\delta}{\delta \hat{\vartheta}_t^0} \mathring{P}_a^0$. Note that this quantity is Gauss invariant and a tensor density hence we just need to check its Poisson brackets with $\mathring{P}_t^B, D_j, T_j^a$. With $\mathring{P}_t^B$ they vanish weakly and with $D_j$ exactly while

$$\{\mathring{P}_t^0(f), T_0^a(g_0^t)\} = \gamma \int d^3x \ f \ \frac{\delta}{\delta \hat{\vartheta}_t^0} g_0^t \ \delta_{jk} D_l$$

(3.45)

It follows that $\mathring{P}_t^0$ is also first class.

vi. $\mathring{P}_0^j, D_k$:

These obviously form a second class pair

$$\{\mathring{P}_i^0(x), D_k(y)\} = \delta_{jk} \delta(x, y)$$

(3.46)

vii. $T_j^a, \mathring{P}_0^k$.

These also form a second class pair

$$\{\mathring{P}_a^j(x), \mathring{P}_k^i(y)\} = \{\mathring{P}_0^k(y), \hat{\vartheta}_a^0(x)\} = \hat{\vartheta}_a^0 \delta(x, y) \delta_0^a \delta^i_j$$

(3.47)

modulo a term $\propto D_l$.

Accordingly, we arrive precisely at the constraint structure as anticipated above. It remains to solve the second class constraints and to compute the Dirac bracket. To solve $D_j = 0, P_A^j = 0$ is trivial. Note that for $D_j = 0$ we have $\sigma_j^a = \hat{\vartheta}_t^0 \hat{\vartheta}_a^0$ thus solving $T_j^a = 0$ is equivalent to $\pi_j^a = \hat{\vartheta}_t^0 \hat{\vartheta}_a^0$. Accordingly, we focus attention on the remaining variables $A_j^B, \pi_j^a, \mathring{P}_A^j, \mathring{c}_j^A$ and the spatial diffeomorphism and Hamiltonian constraint can equivalently be formulated as

$$C_a = F_{ij}^a \pi_j^i, \quad C = F_{ij}^a \epsilon_{jkl} \pi_k^a \pi_l^i$$

(3.48)

Due to (3.46) and (3.47) the Dirac bracket between phase space functions $U, V$ is given by

$$\{U, V\}^* = \{U, V\} \pm (\int d^3x \ \{U, \pi_t^j(x)\} \{D_j(x), V\} - U \leftrightarrow V) \pm (\int d^3x \ \frac{1}{\hat{\vartheta}_t^0(x)} \{U, T_j^a(x)\} \{\mathring{P}_a^j(x), V\} - U \leftrightarrow V)$$

(3.49)
If we restrict \( U, V \) to be functions of \( A^j_A, \pi^a_A, \dot{P}^0_A, \dot{e}^0_A \) then certainly \( \{ U, P^j_A \} = \{ V, P^j_A \} = 0 \). Hence on those functions, the Dirac bracket coincides with the Poisson bracket. Finally, since we set the second class constraints strongly to zero we have

\[
\dot{G}_j = G_j = \nabla_a \pi^a_j, \quad \dot{P}^0_t = \dot{P}^0_t \tag{3.50}
\]

and \( H \) is a linear combination of \( \dot{P}^0_A, C_a, G_j \). We note that when \( D_j = 0 \) we have in terms of lapse and shift functions

\[
e^0_j = \frac{1}{N}, \quad e^a_j = -\frac{N^a}{N} \tag{3.51}
\]

and \( e^a_j \) is invertible. Then \( \det(e^f_A) = N \det(e^a_j) \) and \( \dot{P}^0_A \) are essentially the momenta conjugate to lapse and shift (modulo a canonical transformation). Thus we arrive exactly at the Hamiltonian formulation of the \( U(1)^3 \) or \( SU(2) \) model (Euclidean GR) respectively independent of the value of \( \gamma \neq 0 \).

### 4 Pure Connection Formulation

As has been discovered in tandem with the Ashtekar-Barbero variables, there exist (almost) pure connection formulations for Lorentzian vacuum General Relativity. Without cosmological constant, it is possible to construct a polynomial action in terms of a self-dual \( SL(2, \mathbb{C}) \) connection and a density weighted volume form \([11]\), whereas with the cosmological constant it is possible to even remove the volume form and arrive at a non-polynomial pure connection formulation \([12]\). In this section we revisit these considerations for Euclidean signature and arbitrary \( \gamma \) parameter (twisted self-duality) and simultaneously for both \( U(1)^3 \) and \( SU(2) \). We follow closely \([12]\) but are able to go one step further in the following sense: In \([12]\) it was pointed out that imposing the Hamiltonian constraint as a primary constraint into the resulting action rather than a secondary constraint can be used \( in \ principle \) to arrive at a pure connection formulation. However, the equation to be solved was not written out in detail nor was it shown explicitly that it can be solved algebraically (it could be a polynomial equation in the Lagrange multiplier of higher than fourth order). In this section we show that the equation to be solved fortunately is just a quartic equation. We write it out explicitly in the form that can solved using the Cardano – Ferrari set of formulae \([14]\). The resulting Lagrangian is then a spacetime diffeomorphism covariant and pure connection Lagrangian for a \( SU(2) \) (or the \( U(1)^3 \) gauge theory that is equivalent to Euclidean General Relativity (or its \( U(1)^3 \) truncation) with cosmological constant which is quite remarkable as it brings GR much closer in language to Yang-Mills theory and opens new possibilities for path integral formulations. The difference with Yang-Mills theory is that the Lagrangian of GR is non-polynomial in the connection. Note that all considerations of this section also apply to Lorentzian signature, however then the curvatures appearing are genuinely complex valued.

Because the following calculations are involved, to avoid confusion, it is beneficial to state at the outset that the main result of this section is the achievement of the Lagrangian (4.39). As we will see in detail, in the expression of (4.39) the function \( \dot{D} \) depends merely on the connection \( A \) and \( \dot{\omega} \) is the real solution of the equation (4.38) all of whose coefficients are dependent only on \( A \), so is \( \dot{\omega} \) itself. Therefore, as a final result, the Lagrangian (4.39) provides the pure connection formulation of the theory.

We begin with the Hamiltonian of the previous section including a cosmological constant

\[
H = \int d^3x \ h, \quad h := -A^j_A \nabla_a E^a_j + N^a C_a - \frac{\gamma}{2} N \bar{C} - N \Lambda \left[ \det(E) \right]^{1/2},
\]

\[
C_a = F^b_{ac} E^c_j, \quad \bar{C} = \epsilon_{jkl} F^b_{ac} E^a_k E^c_l \left[ \det(E) \right]^{-1/2} \tag{4.1}
\]

where we have written out \( \dot{e}^0_A \) in terms of lapse and shift functions. To derive an action purely in terms of \( A^j_A \) we first perform the Legendre transform of (4.1) with respect to the momentum \( E^a_j \) conjugate to \( A^j_A \). This still leaves us with an expression that depends on \( N, N^a \). One then removes these by extremising that action with respect to \( N, N^a \) and resubstituting the respective solution into the action. As explained in \([12]\), this leads to an action from which \( C_a = 0 \) and \( C := \frac{1}{2} \bar{C} + \Lambda \left[ \det(E) \right]^{1/2} = 0 \) follow as primary constraints when passing again to the Hamiltonian formulation, rather than as secondary constraints as we deduced in the previous section.
\( H \) and \( \delta H \)

\[
\partial_t A^i_a = -\frac{\delta H}{\delta E^i_a} = \nabla_a A^i_a + N^b F^i_{ba} - \bar{N} \gamma_{ijkl} F^k_{ba} E^l_b - \bar{N} \frac{\Lambda}{2} \epsilon_{ijkl} \epsilon_{abc} E^k_b E^l_c
\]

(4.2)

where we have defined the lapse \( \bar{N} = N [\det(E)]^{-1/2} \) of density weight \(-1\) as an independent variable. By an abuse of notation we relabel \( C[\det(E)]^{1/2} \) as \( C \) again. Equation (4.2) can be rewritten as

\[
F^i_{ta} - N^b F^j_{ba} = N F^j_{na} = -\bar{N} \epsilon_{ijkl} \gamma F^k_{ba} E^l_b = \bar{N} \frac{\Lambda}{2} \epsilon_{ijkl} \epsilon_{abc} E^k_b E^l_c
\]

(4.3)

with the spacetime curvature \( F^j_{AB} \) and the normal \( n^t = \frac{1}{\bar{N}}, n^a = -\frac{N^a}{\bar{N}} \). Assuming the magnetic field \( B^a_j := \epsilon^{abc} F^j_{bc} / 2 \) to be non-degenerate we can decompose \( E^a_j = B^a_k \psi^j_k \) for some matrix \( \psi \) and can write (4.3) in the equivalent form (assuming the spatial metric to be non-degenerate, we also have \( \det(\psi) \neq 0 \))

\[
-\frac{N}{\bar{N} \det(B)} \epsilon^{ijkl} \delta^k_l = \gamma[\text{Tr}(\psi) \delta^j_k - \psi^j_k] + \Lambda (\psi^{-1})^j_k \det(\psi)
\]

(4.4)

To see which conditions are imposed on \( \psi \) when \( C = 0 \) hold we compute

\[
C_a = \epsilon_{abc} B^c_i B^k_k \psi^j_k \delta^j_i = 0, \quad C = \det(B) \left\{ \frac{\gamma}{2} (\text{Tr}(\psi))^2 - \text{Tr}(\psi^2) \right\} + \Lambda \det(\psi) = 0
\]

(4.5)

Thus, raising and lowering the internal indices with the Kronecker \( \delta \) it follows that \( \psi = \psi^T \) and that \( \text{Tr}(\psi^{-1}) = -\Lambda \gamma^{-1} \). We can use the antisymmetric part of \( \psi_{jk} \) to remove the antisymmetric part of the r.h.s. of (4.4). Hence in what follows we take \( \psi \) to be symmetric and thus (4.4) becomes

\[
-\frac{N}{\bar{N} \det(B)} \epsilon^{ijkl} \delta^k_l = \gamma[\text{Tr}(\psi) \delta^j_k - \psi^j_k] + \Lambda (\psi^{-1})^j_k \det(\psi)
\]

(4.6)

The l.h.s. is related to a symmetric, covariant spacetime scalar density one weighted matrix

\[
\kappa^{jk} := \frac{1}{4} \epsilon^{ABCD} F^j_{AB} F^k_{CD} = F^j_{ta} B^k_k \delta^j_l = N F^j_{na} B^k_k \delta^j_l
\]

(4.7)

and the scalar density of weight \(-1\)

\[
w := -\frac{1}{\bar{N} \det(B)}
\]

(4.8)

so that we find the density weight zero matrix identity

\[
\Omega := w \kappa = \gamma[\text{Tr}(\psi) 1_3 - \psi] + \Lambda \det(\psi) \psi^{-1}
\]

(4.9)

and the Lagrangian becomes

\[
L = E^a_j \partial_t A^i_a - h
\]

\[
= \bar{N} \det(B) \{ -\text{Tr}(\Omega \psi) + \frac{\gamma}{2} (\text{Tr}(\psi))^2 - \text{Tr}(\psi^2) \} + \Lambda \det(\psi)
\]

(4.10)

where in the second step we computed \( \text{Tr}(\Omega \psi) \) from (4.9). We did not yet impose \( C = 0 \) since we want to compare with the method followed in [12], so we postpone this to a later stage.

In order to organise the subsequent straightforward but tedious computations we define the scalars

\[
T := \text{Tr}(\Omega), \quad S := T^2 - \text{Tr}(\Omega^2), \quad D := \det(\Omega), \quad \tau := \text{Tr}(\psi), \quad \sigma := \tau^2 - \text{Tr}(\psi^2), \quad \delta := \det(\psi)
\]

(4.11)
We will also need the Caley-Hamilton identity in three dimensions
\[ \psi^3 = \delta 1_3 - \frac{\sigma}{2} \psi + \tau \psi^2 \] (4.12)
which in the present symmetric case can also be verified by elementary means by passing to the diagonal form.

The strategy of [12] is i. to derive three relations between \(T, S, D, \tau, \sigma, \delta\) from the master equation (4.9) by taking traces of its powers, ii. to solve \(\sigma, \delta\) in terms of \(T, D, S\), iii. to insert the solution into the Lagrangian (4.10), iv. to ask that the Lagrangian is stationary under variation of \(w\) which determines \(w\) in terms of \(\kappa\) and v. to insert that solution into (4.10). It is quite astonishing that one can get even to stage iii. since in stage ii. we obtain coupled algebraic equations of order three which when decoupling them may easily lead to polynomial equations of degree five or higher for which no algebraic solution can be found. Yet, it is possible to find an expression for \(L\) in terms of \(S, T, D\) in closed form. However, in stage iv. we encounter a quartic equation. While we still can provide a closed (Ferrari) formula for its solution (which however involves solving a cubic equation), its introduction in \(L\), which itself depends non-polynomially and not even rationally on \(w\), would fill several pages.

Thus, after having performed all steps up to iv. in order to illustrate the arising algebraic complexity and because a derivation was not provided in [12], we return to stage i. and solve \(C = 0\) already at that level. Now there are three relations between \(T, S, D\) and the two parameters \(\tau, \delta\) because by (4.5) \(C = 0\) is equivalent to \(\sigma = \frac{2}{\gamma} \Lambda \delta\). This means that there is a constraint among \(S, T, D\) which leads to a polynomial in \(w\) with \(\kappa\) dependent coefficients which is a depressed quartic equation. In this case, the Lagrangian depends rationally on \(w\) so that the final solution is of a lower complexity.

Stage i.
The following computations are drastically simplified by rewriting (4.9) in terms of the eigenvalues \(\lambda_j, \mu_j\) of \(\Omega, \psi\) respectively
\[ \lambda_j = \gamma [\tau - \mu_j] + \Lambda \frac{\delta}{\mu_j} \] (4.13)
where
\[ T = \lambda_1 + \lambda_2 + \lambda_3, \quad S = 2(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1), \quad D = \lambda_1 \lambda_2 \lambda_3 \] (4.14)
and similar for \(\tau, \sigma, \delta\).

Taking the trace of (4.9) we find
\[ T = 2\gamma \tau + \frac{\Lambda}{\gamma} \sigma \] (4.15)
Next, multiplying out the r.h.s. of \(S\) given in (4.14), we obtain after regrouping terms
\[ \frac{S}{2} = \gamma^2 [\tau^2 + \frac{\sigma}{2}] + \Lambda \gamma [3\delta - \sigma \tau + \tau^3 - \text{Tr}(\psi^3)] + \Lambda^2 \delta \tau \] (4.16)
Taking the trace of (4.12) we can write the r.h.s. just in terms of \(\tau, \sigma, \delta\)
\[ \frac{S}{2} = \gamma^2 [\tau^2 + \frac{\sigma}{2}] + \Lambda \gamma [3\delta + \frac{1}{2} \sigma \tau] + \Lambda^2 \delta \tau \] (4.17)
Note that the r.h.s. is only a quadratic polynomial. If that was not the case, we would not be able to complete even step ii.

Next, again multiplying out the r.h.s. of \(D\) given in (4.14) we obtain after regrouping terms
\[ D = \gamma^3 [-\delta + \frac{1}{2} \tau \sigma] + \gamma \Lambda^2 \delta \sigma + \Lambda^3 \delta^2 + \gamma^2 \Lambda [3\tau \delta + \frac{1}{2} \{[\text{Tr}(\psi^2)]^2 - \text{Tr}(\psi^4)\}] \] (4.18)
Multiplying (4.12) with \(\psi\) and taking the trace we find
\[ [\text{Tr}(\psi^2)]^2 - \text{Tr}(\psi^4) = \frac{1}{2} \sigma^2 - 4 \delta \tau \] (4.19)
so that (4.18) can again be written just in terms of \(\tau, \sigma, \delta\)
\[ D = \gamma^3 [-\delta + \frac{1}{2} \tau \sigma] + \gamma \Lambda^2 \delta \sigma + \Lambda^3 \delta^2 + \gamma^2 \Lambda [3\delta \tau + \frac{1}{4} \sigma^2] \] (4.20)
Again it is remarkable that the r.h.s. of (4.21) is only a quadratic polynomial.

Equations (4.15), (4.17) and (4.20) in principle allow to express $\tau, \sigma, \delta$ in terms of $T, S, D$. However, the fact that these are a coupled system of one linear and two quadratic polynomials still could forbid a simple algebraic solution. To see this, imagine expressing $\tau$ in terms of $T, \sigma$, using (4.15) in (4.17) and (4.20). Then we obtain a coupled system of two quadratic equations in terms of $\sigma, \delta$. We can solve (4.17) (which contains $\delta$ linearly) for $\delta$ in terms of $S, \sigma$ which is a fraction with a quadratic and linear polynomial in $\sigma$ in numerator and denominator respectively. Substituting that solution into (4.20) which contains the square of $\delta$ and after multiplying by the square of the denominator we obtain a quartic polynomial in $\sigma$ which in general is very complicated to solve.

Fortunately, these complications can be avoided because we are only interested in the combination of $\tau, \sigma$ that appears in $L$. First let $y := \Lambda \delta + \frac{1}{2} \gamma \sigma$ then

$$ \frac{S}{2} = \gamma^2 (\tau^2 - \sigma) + 3 \gamma y + \Lambda \tau y, \quad D = -\gamma^3 \delta + \Lambda y^2 + \gamma^2 \tau y \quad (4.21) $$

Next with $z = \Lambda y + \gamma^2 \tau$

$$ \frac{S}{2} = \tau z + 3 \gamma y - \gamma^2 \sigma, \quad D = -\gamma^3 \delta + y z \quad (4.22) $$

whence

$$ \Lambda D + \frac{\gamma^2}{2} S = \gamma^3 l + z^2, \quad l = \frac{1}{2} \gamma \sigma + 2 \Lambda \delta \quad (4.23) $$

Recalling (4.10) and (4.15), we can now observe that

$$ L = w^{-1} l, \quad z = \frac{1}{2} [\Lambda \ l + \gamma T] \quad (4.24) $$

Consequently, we obtain the quadratic equation

$$ \Lambda D + \frac{\gamma^2}{4} [2S - T^2] = [\gamma^3 + \frac{1}{2} \gamma \Lambda T] \ l + \frac{\Lambda^2}{4} l^2 \quad (4.25) $$

with the two solutions

$$ l = -a \pm \sqrt{b + a^2}, \quad a = 2 \gamma^3 + \gamma \Lambda T, \quad b = \frac{4}{\Lambda^2} \{\Lambda D + \frac{\gamma^2}{4} [2S - T^2]\} \quad (4.26) $$

If we insist on a well defined $\Lambda \to 0$ limit, only the positive sign in front of the square root is allowed.

Equation (4.26) is as far as [12] went (modulo the fact that $\gamma = \pm i$ for Lorentzian (anti-)selfdual General Relativity). The possibility to remove the implicit appearance of $w$ within $L = w^{-1} l(w, \kappa)$ was mentioned in [12] but not carried out because of the tremendous complexity of the resulting equations. To see what would be required we push this a little further and define the dimension-free and $w$ independent quantities

$$ \hat{\kappa} = \frac{\kappa}{\Lambda^2}, \quad \hat{\omega} = \Lambda^3 \omega, \quad \hat{\Omega} = \Lambda \Omega = \hat{\omega} \ \hat{\kappa} \quad (4.27) $$

and the corresponding dimension-free and $\omega$ independent quantities

$$ \Lambda \ T = \hat{T} \ \hat{\omega}, \quad \Lambda^2 \ S = \hat{S} \ [\hat{\omega}]^2, \quad \Lambda^3 \ D = \hat{D} \ [\hat{\omega}]^3 \quad (4.28) $$

Then, the Lagrangian can be rewritten as

$$ L = w^{-1} l = [\hat{\omega}]^{-1} \Lambda^3 l = \Lambda \gamma \hat{l}, \quad \hat{l} = -\hat{a} \pm \sqrt{b + \hat{a}^2}, \quad \hat{a} = 2 \ \hat{\omega}^{-1} + \hat{T} \hat{b} = 4 \hat{D} \hat{\omega} + 2 \hat{S} - \hat{T}^2, \quad \hat{\omega} = \frac{\hat{\omega}}{\gamma^2} \quad (4.29) $$

The action (4.29) is stationary with respect to $\hat{\omega}$ when

$$ L' = \gamma \Lambda [\hat{l} \hat{a}' + \frac{\hat{b}' + 2 \hat{a} \hat{a}'}{2 \hat{W}}] = 0, \quad \hat{W} = \sqrt{\hat{b} + \hat{a}^2} \quad (4.30) $$

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where the prime denotes a derivative with respect to \( \hat{w} \). This can be written as

\[
2\hat{a}'[\hat{W} \mp \hat{a}] = \pm \hat{b}' = 2\hat{a}' \frac{\hat{b}}{W \pm \hat{a}} \tag{4.31}
\]

Solving for \( \hat{W} \) we find

\[
\pm \hat{W} = \hat{a} + \frac{\hat{b}'}{2\hat{a}'} = 2\frac{\hat{a}'\hat{b}}{\hat{b}'} - \hat{a} \tag{4.32}
\]
i.e.

\[
\hat{a}'(\hat{a}'\hat{b} - \hat{a}\hat{b}') = \left[ \frac{\hat{b}'}{2} \right]^2 \tag{4.33}
\]

Denoting \( \alpha = 4\hat{D}, \beta = 2\hat{S} - \hat{T}^2 \), we finally obtain

\[
\frac{2}{\hat{w}^2} \left[ \frac{2}{\hat{w}^2} (\alpha\hat{w} + \beta) + \alpha[2\hat{w} - 1 + \hat{T}] \right] = \left[ \frac{\alpha}{2} \right]^2 \tag{4.34}
\]

After multiplying by \( \hat{w}^4 \) this becomes a quartic equation in depressed form (no cubic term) which can be solved by Ferrari’s formula. However, since this requires implicitly solving a non-depressed (the quadratic term is non-vanishing) cubic, the explicit formula for the four roots is not only very lengthy, but also its insertion into (4.29) becomes extremely involved due to the appearance of the square root term.

Above procedure imposes the Hamiltonian constraint as a secondary constraint by extremising with respect to the Langrange multiplier \( \hat{w} \). We obtain of course an equivalent result by imposing \( C = 0 \), that is, \( \frac{\hat{a}^2}{2} + \Lambda\delta = 0 \), already in (4.15), (4.17) and (4.20) which makes the equation to be solved more transparent. First we find the simplified system

\[
\begin{align*}
D &= -\gamma^3 \delta \\
\frac{S}{2} &= \gamma^2 \tau^2 + 2\gamma\Lambda\delta \\
T &= 2\gamma\tau - \frac{\Lambda^2}{2\gamma}\delta
\end{align*} \tag{4.35}
\]

Eliminating \( \tau, \delta \) imposes the constraint on \( T, S, D \) given by

\[
\frac{S}{2} + 2\frac{\Lambda}{\gamma^2} D = \frac{1}{4} \left[ \frac{\Lambda^2}{\gamma^4} D - T \right]^2 \tag{4.36}
\]

which is equivalent to imposing \( C = 0 \) i.e. choosing \( \sigma \) appropriately. In terms of the dimension free quantities

\[
\hat{w} = \Lambda^3 w \gamma^{-2}, \quad \hat{T} = T \Lambda^{-2} w^{-1}, \quad \hat{S} = S \Lambda^{-4} w^{-2}, \quad \hat{D} = D \Lambda^{-6} w^{-3}, \tag{4.37}
\]

we find

\[
\frac{\hat{S}}{2} + 2\hat{w} \hat{D} = \frac{1}{4} \left[ \hat{w}^2 \hat{D} - \hat{T} \right]^2 \tag{4.38}
\]

which is again a quartic equation in depressed form for \( \hat{w} \). Note that the Lagrangian for \( C = 0 \) becomes simply

\[
L = -w^{-1} \Lambda \gamma^{-3} D = -w^2 \Lambda^7 \gamma^{-3} \hat{D} = -\hat{w}^2 \Lambda \gamma \hat{D} \tag{4.39}
\]

which looks deceptively simple but of course the challenge lies in solving the quartic equation (4.38) for \( \hat{w} \) in terms of \( \hat{T}, \hat{S}, \hat{D} \) which are traces of polynomials in the matrix \( \hat{\kappa} = \kappa\Lambda^{-2} \). We have access to the general solution of (4.38) using the Cardano-Ferrari theory. We do not display it here explicitly because the formulas are quite lengthy. However, we want to know whether among the four roots there are real ones and if yes, under which
The second task is to check that the Legendre transform of (4.39) delivers \( C_a = C = 0 \) as primary constraints. In order to make arrangements for the ensuing calculations which are rather tedious, we define

\[
\begin{align*}
\mathfrak{A} &:= -\frac{\gamma \dot{\omega}}{\Lambda(2 + \omega(T - D\dot{\omega}^2))}, \\
\mathfrak{B} &:= \frac{\gamma}{6(2 + \omega(T - D\dot{\omega}^2))}, \\
\mathfrak{M}^{ij} &:= \kappa^{ij} - \omega \dot{D}(\kappa^{-1})^{ij}, \\
\mathfrak{M} &:= \dot{T} + \dot{D}\omega^2, \\
\mathfrak{M} &:= 6 + 4\dot{T}\omega - 2\dot{D}\omega^3
\end{align*}
\]

Some of the quantities associated with the matrix \( \mathfrak{M}^{ij} \) that are needed below can be computed in terms of \( \dot{S}, \dot{T}, \dot{D}, \dot{\omega} \)

\[
\begin{align*}
\text{Tr}(\mathfrak{M}) & = \dot{T} - \frac{\dot{S}\dot{\omega}}{2} \\
\text{Tr}(\mathfrak{M}^2) & = \dot{T}^2 - \dot{S} - 6\dot{D}\dot{\omega} + \frac{\dot{S}^2\dot{\omega}^2}{4} - 2\dot{T}\dot{D}\dot{\omega}^2 \\
[\text{Tr}(\mathfrak{M})]^2 - \text{Tr}(\mathfrak{M}^2) & = \dot{S} + (6\dot{D} - \dot{S}\dot{T})\dot{\omega} + 2\dot{T}\dot{D}\dot{\omega}^2 \\
\text{det}(\mathfrak{M}) & = \dot{D} - \frac{1}{2}(\frac{\dot{S}^2}{2} - 4\dot{T}\dot{D})\dot{\omega} + \dot{D}(\dot{T}^2 - \dot{S})\dot{\omega}^2 - \dot{D}^2\dot{\omega}^3
\end{align*}
\]

where in order to obtain the second equation, we used \( \text{Tr}(\kappa^{-2}) = \frac{1}{2\dot{D}^2}\left(\frac{\dot{S}^2}{2} - 4\dot{T}\dot{D}\right) \) and for the last equation we made use of the following relation

\[
\text{det}(A + B) = \text{det}(A) + \text{det}(B) + \text{det}(B)\text{Tr}(AB^{-1}) + \text{det}(A)\text{Tr}(BA^{-1})
\]

which is valid for all \( 3 \times 3 \) invertible matrices \( A, B \).

Starting from the Lagrangian \( L = -\gamma \Delta \dot{D}\dot{\omega}^2 \), first we need to calculate the momentum conjugate to \( A^0_\mu \). Since the Lagrangian depends on \( \dot{\omega} \), the conjugate momentum depends on the variation of \( \dot{\omega} \) with respect to \( A^0_\mu \). Taking variation of both sides of (4.38) and isolating \( \delta \dot{\omega} \), one finds

\[
\delta \dot{\omega} = \frac{(\dot{T} - \dot{D}\dot{\omega}^2)(\delta \dot{T} - \omega^2\delta \dot{D} - 4\omega\delta D - \delta \dot{S})}{2\dot{D}(2 + \omega(T - D\dot{\omega}^2))} = \frac{\dot{T} - \dot{D}\dot{\omega}^2}{2\dot{D}(2 + \omega(T - D\dot{\omega}^2))}\delta \dot{T} + \frac{1}{2\dot{D}(2 + \omega(T - D\dot{\omega}^2))} \delta \dot{S} + \frac{-\omega^2(\dot{T} - \dot{D}\dot{\omega}^2) - 4\omega}{2\dot{D}(2 + \omega(T - D\dot{\omega}^2))} \delta \dot{D}
\]

(4.44)
Thus, the momentum is obtained as

\[
\Pi_j^a := \frac{\delta L}{\delta A^a_j} = -\gamma A \frac{\delta (\omega^2 D)}{\delta A^a_j} = -\gamma \Lambda \left(2 \dot{D} \frac{\delta \omega^2}{\delta A^a_j} + \omega^3 \frac{\delta \dot{D}}{\delta A^a_j}\right)
\]

\[
= - \frac{\gamma \Lambda}{2D(2 + \omega(T - \dot{D}\omega^2))} \left[2 \dot{D} \omega(T - \dot{D}\omega^2) \frac{\delta T}{\delta A^a_j} - 2 \dot{D} \frac{\delta \dot{S}}{\delta A^a_j} - 2 \omega \frac{\delta \dot{D}}{\delta A^a_j} - 4 \dot{D} \omega^2 \frac{\delta \dot{D}}{\delta A^a_j}\right]
\]

\[
= \frac{\gamma \Lambda \dot{\omega}}{\Lambda(2 + \omega(T - \dot{D}\omega^2))} \left[(\dot{T} - \dot{D}\omega^2)B_j^a - 2\dot{T}B_j^a - 2\kappa^{ij}B_i^a - 2\omega \dot{D}(\kappa^{-1})_{ji}B_i^a\right]
\]

\[
= \frac{\gamma \Lambda \dot{\omega}}{\Lambda(2 + \omega(T - \dot{D}\omega^2))} \left[2 \left(\kappa^{ij} - \omega \dot{D}(\kappa^{-1})_{ji}\right) B_i^a - (\dot{T} + \dot{D}\omega^2)B_j^a\right]
\]

\[
= \mathfrak{M} \left[2\mathfrak{M}^i B_i^a - \mathfrak{M} B_j^a\right]
\]  (4.45)

where from the third to the fourth line, we have used the following variations

\[
\left(\frac{\delta \tilde{k}}{\delta A^a_j}\right)^{ij} = \Lambda^{-2} \delta_i^i B^j_c,
\]

\[
\frac{\delta \tilde{T}}{\delta A^a_j} = \Lambda^{-2} B_i^a,
\]

\[
\frac{\delta \tilde{S}}{\delta A^a_j} = 2\dot{T} \frac{\delta \tilde{T}}{\delta A^a_j} - 2\tilde{k}^{ij} \left(\frac{\delta \tilde{k}}{\delta A^a_j}\right)^{ij} = \Lambda^{-2}(2\dot{T}B_i^a - 2\kappa^{ij}B_j^a)
\]

\[
\frac{\delta \tilde{D}}{\delta A^a_j} = \dot{D}(\kappa^{-1})_{ij} \left(\frac{\delta \tilde{k}}{\delta A^a_j}\right)^{ij} = \Lambda^{-2} \dot{D}(\kappa^{-1})_{ij} \delta_i^i B^j_c = \Lambda^{-2} D(\kappa^{-1})_{ij} B_j^a
\]  (4.46)

In the last line of (4.46), the Jacobi’s formula \(d(\det(M)) = \det(M)\text{Tr}(M^{-1}dM)\) has been used, which is valid for every invertible matrix \(M\).

As \(F^i_{ab} = \epsilon_{cab}B^c_i\), from (4.45) we immediately conclude that

\[
C_a = F^i_{ab} \Pi^b_i = \mathfrak{M} \epsilon_{cab} B^c_i \left[2\mathfrak{M}^i B_j^a - \mathfrak{M} B_j^a\right] = 0
\]  (4.47)

because the matrix \(\mathfrak{M}\) is symmetric. This simply shows that the vector constraint is a primary constraint. Achieving a similar result for the Hamiltonian constraint requires more calculations.

\[
C = \frac{\gamma}{2} \epsilon_{ijk} F^i_{ab} \Pi^a_j \Pi^b_k + \Lambda \det(\Pi)
\]

\[
= \frac{\gamma}{2} \epsilon_{ijk} \epsilon_{cab} B^c_j \Pi^a_j \Pi^b_k + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon_{cab} \Pi^a_j \Pi^b_k
\]

\[
= \left(\frac{\gamma}{2} B_i^c + \frac{\Lambda}{6} \Pi_i^a\right) \epsilon_{ijk} \epsilon_{cab} \Pi^a_j \Pi^b_k
\]

\[
= \frac{\gamma}{2} \left(B_i^c - \frac{\dot{\omega}}{3(2 + \omega(T - \dot{D}\omega^2))} \left[2 \left(\kappa^{li} - \omega \dot{D}(\kappa^{-1})_{li}\right) B_i^l - (\dot{T} + \dot{D}\omega^2)B_i^c\right]\right) \epsilon_{ijk} \epsilon_{cab} \Pi^a_j \Pi^b_k
\]

\[
= \frac{\gamma}{6(2 + \omega(T - \dot{D}\omega^2))} \left(6 + 4\dot{T}\omega - 2\dot{D}\omega^3\right) B_i^c - 2\dot{\omega} \left(\kappa^{li} - \omega \dot{D}(\kappa^{-1})_{li}\right) B_i^l \epsilon_{ijk} \epsilon_{cab} \Pi^a_j \Pi^b_k
\]

\[
= \mathfrak{M} \left(2\mathfrak{M} B_i^c - \omega \mathfrak{M}^i B_j^a\right) \epsilon_{ijk} \epsilon_{cab} \Pi^a_j \Pi^b_k
\]  (4.48)
Using (4.45), we calculate $\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\Pi_j^{\alpha}\Pi_k^{\beta}$ part of $C$ as

$$
\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}\Pi_j^{\alpha}\Pi_k^{\beta} = \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} A^2 \left[ 2\epsilon^{mnj} B_m^\alpha - \hat{\epsilon} B_n^\alpha \right] \left[ 2\epsilon^{nk}\epsilon^{\alpha\beta} B_k^\beta - \hat{\epsilon} B_n^\beta \right] \\
= \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} A^2 \left[ 4\epsilon^{mnj}\epsilon^{nk}\epsilon^{\alpha\beta} B_m^\alpha B_n^\beta - 4\epsilon^{mnj}\epsilon^{nk}\epsilon^{\alpha\beta} B_m^\alpha B_n^\beta + \hat{\epsilon}\epsilon^{nk}\epsilon^{\alpha\beta} B_m^\beta B_n^\beta \right] \\
= \epsilon_{ijk}\epsilon_{\alpha\beta\gamma} A^2 \det(B) (B^{-1})_c \left[ 4\epsilon^{mnj}\epsilon^{nk}\epsilon^{\alpha\beta} \epsilon_{\ell m n} - 4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + \hat{\epsilon}\epsilon^{nk}\epsilon^{\alpha\beta} \epsilon_{\ell m n} \right] \\
= A^2 \det(B) (B^{-1})_c \left[ 4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} \right] (2n^2 - 4n^2_j n_t) \delta^i_j \\
= A^2 \det(B) (B^{-1})_c \left[ 4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} \right] (2n^2 - 4n^2_j n_t) \delta^i_j \\
(4.49)

Finally, plugging (4.49) into (4.48) and simplifying, we have

$$
C = A^2 \det(B) (B^{-1})_c \left[ \hat{\omega} B^i_c - 2\omega n^i n_t B^i_c \right] \left[ 4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} \right] (2n^2 - 4n^2_j n_t) \delta^i_j \\
= A^2 \det(B) \left( \hat{\omega} [4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} + 3(2n^2 - 4n^2_j n_t) \right] \\
- 2\omega \left[ 4\epsilon^{nk}\epsilon_{\ell m n} \right] \right) \right) \\
= A^2 \det(B) \left( \hat{\omega} [4(\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} - 24 \det(\epsilon) + 4(\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n}) \right] \\
= A^2 \det(B) \left( 4(\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} \right) \right) \\
= A^2 \det(B) \left( 4\epsilon^{mnj}\epsilon^{nk}\epsilon_{\ell m n} + 4\epsilon^{nk}\epsilon_{\ell m n} \right) \right) \right) \\
(4.49)

Now one can employ the relations (4.42) and write the Hamiltonian constraint as a polynomial of $\hat{\omega}$

$$
C = A^2 \det(B) \left[ -12\hat{D}^3 \hat{\omega}^7 - 6\hat{D}^2 \hat{S} \hat{\omega}^6 + 12\hat{D}^2 \hat{T} \hat{\omega}^5 + 6(14\hat{D}^2 + 2\hat{D} \hat{S} \hat{T}) \hat{\omega}^4 + 6(12\hat{D} \hat{S} + 2\hat{D} \hat{T}^2) \hat{\omega}^3 + 6(2\hat{S}^2 + 20\hat{D} \hat{T}^2 - \hat{S} \hat{T}^2) \hat{\omega}^2 + 6(16 \hat{D}^2 \hat{S} \hat{\omega} + 4 \hat{D} \hat{S} \hat{T} - 2 \hat{T}^3 \hat{\omega} + 24 \hat{S} - 12 \hat{T}^2) \right] \\
(4.50)

It can be concluded that $C$ arises as a primary constraint, if the equations (4.50) and (4.38) have a common real solution. Although (4.50) seems too complicated to be solved, not surprisingly one can write it as the product of two polynomials

$$
C = -6A^2 \det(B) \left( 2\hat{D}^3 \hat{\omega}^3 + \hat{S} \hat{\omega}^2 + 2\hat{T} \hat{\omega} + 2 \right) \left( \hat{D}^2 \hat{\omega}^4 - 2\hat{D} \hat{T} \hat{\omega}^3 - 8\hat{D} \hat{\omega} + \hat{T}^2 - 2\hat{S} \right) = 0 \\
(4.51)

which vanishes because $\hat{D}^3 \hat{\omega}^4 - 2\hat{D} \hat{T} \hat{\omega}^3 - 8\hat{D} \hat{\omega} + \hat{T}^2 - 2\hat{S} = 0$ due to the equation (4.38). This ends checking that $C = C_o = 0$ arise as primary constraints.

5 Conclusion and Outlook

In this paper we have shown in a detailed analysis that the U(1)^3 model for canonical gravity has two Lagrangian formulations from which it derives: One is a Palatini-Holst type action which uses a tetrad, the other is a pure connection Lagrangian. Both Lagrangians may serve as possible starting point for a path integral formulation and can be put into the language of spin foams (see [15] and references therein). As the gauge group is Abelian, we expect that the resulting spin foam model can be much better controlled than their non-Abelian versions which thus could serve as an interesting test laboratory for the spin foam approach to LQG. Interestingly, our results
immediately generalise from $U(1)^3$ to $SU(2)$ so that one can also write a spin foam model for Euclidean General Relativity but with gauge group $SU(2)$ rather than $SO(4)$ which may also lead to major simplifications even in the non-Abelian context. We leave this for future work.

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A The general solution of the system (3.32)

In this appendix we obtain the general solution of the system (3.32).

Case 1: $\hat{e}_0^i \neq 0$

First, we use the last equation of (3.32) to obtain $v_a^i$. Since $\hat{e}_0^i \neq 0$, one uses its inverse and gets

$$v_a^i = \frac{1}{\hat{e}_0^i} (F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \gamma e^{ikl} v_a^k \hat{e}_l^i)$$  \hspace{1cm} (A.1)

This is not the desired solution because the r.h.s. still depends on $v_a^k$. To get rid of it, we just plug (A.1) into itself and simplify the expression

$$v_a^i = \frac{1}{\hat{e}_0^i} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_0^b + \gamma e^{kmn} F_{ab}^m \hat{e}_n^b - \gamma e^{kmn} v_a^m \hat{e}_n^b) \right)$$

$$= \frac{1}{\hat{e}_0^i} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_0^b + \gamma e^{kmn} F_{ab}^m \hat{e}_n^b) \right)$$

$$= \frac{1}{\hat{e}_0^i} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_0^b + \gamma e^{kmn} F_{ab}^m \hat{e}_n^b) \right)$$

where in the last step, we have used the third equation of (3.32). Now by moving the last term of the r.h.s. to the l.h.s., one can isolate $v_a^i$ and obtain

$$v_a^i = \frac{1}{\hat{e}_0^i} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_0^b + \gamma e^{kmn} F_{ab}^m \hat{e}_n^b) \right)$$

$$= \frac{1}{\hat{e}_0^i} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_0^b + \gamma e^{kmn} F_{ab}^m \hat{e}_n^b) \right)$$

With the assumption $\hat{e}_0^i = 0$, (A.2) reduces to (3.34). Now by substituting (A.2) in the second, third and first equations of (3.32), respectively, the following constraints arise

$$\tilde{C}_a := \left( 1 - \frac{\gamma^2 e_l^i e_l^i}{\hat{e}_0^l \hat{e}_0^l + \gamma e^{ikl} F_{ab}^l \hat{e}_l^i} \right) F_{ab}^i \hat{e}_0^b - \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^l \hat{e}_0^b + \gamma e^{kl} F_{ab}^k \hat{e}_l^i)$$

$$\tilde{C}_i := \frac{\gamma^2 e_l^i}{(\hat{e}_0^l \hat{e}_0^l + \gamma e^{ikl} F_{ab}^l \hat{e}_l^i)} \left( \frac{\hat{e}_0^l}{\hat{e}_0^i} F_{ab} (2 e_l^i e_l^i - e_l^i e_l^i) + F_{ab}^i e_l^i \hat{e}_l^i + \frac{\gamma e^{ikl}}{\hat{e}_0^l} (F_{ab}^k \hat{e}_l^i - 2 F_{ab}^l \hat{e}_l^i) \right)$$

$$\tilde{C} := \frac{\hat{e}_0^i}{(\hat{e}_0^l \hat{e}_0^l + \gamma e^{ikl} F_{ab}^l \hat{e}_l^i)} \left( F_{ab}^i \hat{e}_0^b + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i \hat{e}_a^i - \frac{\gamma e^{ikl}}{\hat{e}_0^l} F_{ab}^k \hat{e}_l^i \hat{e}_a^i \right)$$  \hspace{1cm} (A.3)

We would like to check whether under the assumption of $\hat{e}_0^j = 0$ the above constraints (A.3) reduce to (3.33).

We begin with $\tilde{C}_a = 0$. It is obvious that putting $\hat{e}_0^j = 0$ in this equation yields to $F_{ab}^i \hat{e}_0^b = 0$ which is the second constraint of (3.33), i.e. $C_a = 0$.

As most of the terms in $\tilde{C}_a$ are proportional to $\hat{e}_0^j$, when the latter equals to zero the constraint $\tilde{C}_i$ reduces to the simple equation $0 = F_{ab}^l \hat{e}_0^b \hat{e}_0^l = -C_b \hat{e}_a^i$ that is not an independent constraint because we have already $C_a = 0$. Note that when we assume $\hat{e}_0^j = 0$, we are, in fact, substituting three constraints $D_j = 0$ for the above constraint $\tilde{C}_j = 0$ and that is why in this case $\tilde{C}_j = 0$ do not give us new independent constraints.

Finally, by putting $\hat{e}_0^j = 0$ in the equation $\tilde{C} = 0$, we come to the conclusion that $0 = F_{ab}^l \hat{e}_0^b \hat{e}_0^l = -C_b \hat{e}_a^i + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i \hat{e}_a^i = -C_b \hat{e}_a^i + \gamma e^{ikl} F_{ab}^k \hat{e}_l^i \hat{e}_a^i$ whose first term already vanishes. Therefore, $\gamma e^{ikl} F_{ab}^k \hat{e}_l^i \hat{e}_a^i = 0$ serves as an independent constraint which is the third constraint in (3.33), i.e. $C = 0$.  

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Case 2: $\dot{e}_0 = 0$

In this case, in order to prevent degeneracy, at least one of $\dot{e}_i$ must be non-zero. Without loss of generality we can assume that $\dot{e}_i \neq 0$. A consequence of this choice is separation of quantities with index $i = 1$ and those with $i = \alpha \in \{2, 3\}$. Greek alphabet is used to indicate indices $i \neq 1$.

The last equation of (3.32) is employed to obtain $v_a^i$ as

$$v_a^i = \frac{1}{\gamma e_1^i}(\gamma v_a^i e_\alpha - \epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b])$$

and the third equation can be solved for $v_a^1$

$$v_a^1 = \frac{1}{e_1^1} (F_{ab} e_j^b - \dot{e} e_a^a)$$

Now inserting (A.4) in (A.5) gives $v_a^1$ completely in terms of the canonical variables

$$v_a^1 = \frac{\dot{e}_1^1}{e_1^1} e_1^1 \left( F_{ab} e_j^b + \epsilon_\alpha e_1^1 \epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b] \right)$$

and plugging (A.6) in (A.4) finishes the restrictions on Lagrange multipliers $v_a^i$, as

$$v_a^i = \frac{\dot{e}_a^i}{e_1^i} e_1^i \left( F_{ab} e_j^b + \epsilon_\alpha e_1^1 \epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b] \right) - \frac{\epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b]}{e_1^i} e_a^a$$

What is left to do is just substitution of $v_a^i$ into the rest of the equation of (3.32) to attain the constraints. These equations are of the form

$$0 = v_a^1 e_a^0 + \gamma e^{\alpha \beta} v_a^\alpha e^\beta$$

$$0 = v_a^0 e_a^0 - \gamma e^{\alpha \beta} v_a^\alpha e^\beta + \gamma e^{\beta} v_a^\beta e^\alpha$$

$$0 = \dot{e}_0 v_a^1 + \dot{e} e_a^a$$

where the first one is just the $i = 1$ component of the last equation of (3.32). After simplifying, the constraints acquire the following forms

$$\dot{C}_a := -\frac{\dot{e}_a^1}{e_1^i} [F^0_{ab} e_0 + \gamma e^{\alpha \beta} F^k_{ab} e^b]$$

$$\dot{C}_1 := -\frac{\dot{e}_1^1}{e_1^i} e_1^i \left( F_{ab} e_j^b + \epsilon_\alpha e_1^1 \epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b] \right) e_a^0 + \gamma e^{\alpha \beta} v_a^\alpha e^\beta$$

$$\dot{C}_\alpha := -\frac{\dot{e}_a^i}{e_1^i} e_1^i \left( F_{ab} e_j^b + \epsilon_\alpha e_1^1 \epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b] \right) - \frac{\epsilon_{\alpha \beta} [F^0_{ab} e_0 + \gamma e^{\beta kl} F^k_{ab} e^b]}{e_1^i} e_a^a$$

To summarise, solving 16 equations (3.32) to ensure that $\dot{P}^I_A$ is stabilised leads to fixing 9 Lagrange multipliers $v_a^i$ and $v_a^\alpha$ through (A.6) and (A.7), respectively, in addition to 7 secondary constraints $\dot{C}_a, \dot{C}_1, \dot{C}_\alpha, \dot{C}$. This ends the analysis.
B Hamiltonian analysis of the action of [2]

In this appendix, we discuss the Hamiltonian analysis of the action introduced in [2], that is

\[ S = \frac{1}{2} \int d^4x \, e^{\mu_\alpha \alpha \beta} e^{ij} e_{ij} F_{ab}^{ij} \]  

(B.1)

where \( \mu, \nu, \cdots \in \{0, 1, 2, 3\} \) are spacetime indices and \( i, j, \cdots \in 1, 2, 3 \) are Lie algebra indices. The 3+1 decomposition of (B.1) is given by

\[ S = \int dt \, \int d^3x \, e^{abc} (e^{ij} e_{ij} F_{bc}^{ij} + e^{ij} e_{ij} F_{tc}^{ij}) \]  

(B.2)

The conjugate momenta corresponding the configuration variables can easily be computed as

\[ P_t := \frac{\delta S}{\delta e_t^i} = 0, \quad P_t^a := \frac{\delta S}{\delta e_a^i} = 0, \quad \pi_{ij} := \frac{\delta S}{\delta A_t^a} = 0, \quad \pi_{ij}^a := \frac{\delta S}{\delta A_t^a} = e^{abc} e_b e_c \]  

(B.3)

that lead to the primary constraints

\[ P_t = 0, \quad P_t^a = 0, \quad \pi_{ij} = 0, \quad T_{ij}^a := \pi_{ij}^a - e^{abc} e_b e_c = 0 \]  

(B.4)

The Legendre transform of (B.4) yields

\[ H = \int d^3x \, \{ v^i P_t + v^a P_t^a + v^i \pi_{ij} + v^i \pi_{ij}^a - L \} \]  

\[ = \int d^3x \, \{ v^i P_t + v^a P_t^a + v^i \pi_{ij} + v^a T_{ij}^a - e^{abc} e_b e_c F_{bc}^{ij} + e^{abc} e_b e_c A_{ij}^a \} \]  

(B.5)

(B.6)

where \( v^i, v^a, \pi_{ij}, v_{ij}^a \) are the Lagrange multipliers and \( L \) is the Lagrangian associated with the action (B.2).

Stability of \( \pi_{ij} = 0 \) leads to the Gauss secondary constraint

\[ G_{ij} = e^{abc} \partial_c (e_{a}^{ij}) \]  

(B.7)

Stability of \( P_t = 0 \) yields 3 secondary constraints

\[ C_t = e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

(B.8)

Stability of \( P_t^a = 0 \) is obtained if and only if the following 9 equations are satisfied

\[ 0 = 2(A_{t,ij} - v_{ij}^c) e_{ab} e_{ij} - e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

\[ := f_{ij} e_{ab} e_{ij} - e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

\[ = f_{ij} e_{ab} e_{ij} - e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

\[ = f_{ij} e_{ab} e_{ij} - e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

(B.9)

that are equivalent to

\[ f_{ij} \varepsilon^{kl} = e^{abc} e_{a}^{ij} F_{bc}^{ij} \]  

(B.10)

Since \( f_{ijk} = 2(A_{t,ij} - v_{ij}^c) e_{k}^c \) is antisymmetric in \( i, j \) we may write it as \( f_{ijk} = \varepsilon_{ijm} g_{km}^m \) for some matrix \( g_{km}^m \). Thus (B.10) can be rewritten as

\[ g_{km}^m \delta^i_l - g_{li}^l = M_i^l \]  

(B.11)

whose trace amounts to \( 2g_{lm}^m = M^m_m \). Plugging this into (B.11), we get

\[ g_{li}^l = \frac{1}{2} M^m_m \delta^i_l - M_i^l \]  

(B.12)
meaning that all of the $f_{ijk}$ are fixed and $P_{ia}^a = 0$ is stabilised.
Stability of $T_{ij}^a = 0$ yields the equation
\[ e^{cbo} \partial_b \left( e_{c}^{[i} e_{c}^{j]} \right) + e^{abc} v_b^{[i} e_c^{j]} = 0 \]  
\[ \text{(B.13)} \]

Defining $v^b_a := v^b_d e^d_a$, we observe
\[ e^{abc} v_b^{[i} e_c^{j]} = -e^{cbo} \partial_b \left( e_{c}^{[i} e_{c}^{j]} \right) =: M_{ij}^a \]  
\[ \text{(B.14)} \]

Multiplying both sides in $e_{c}^{[i} e_{c}^{j]}$ results in
\[ e^{abf} v_b^{[i} e_c^{j]} = M_{ij}^a e_c^{[i} e_c^{j]} \]  
\[ \text{(B.15)} \]

and again by multiplying it in $\epsilon_{acd}$ we get
\[ 2\delta_{d}^{[i} v_c^{j]} = \epsilon_{acd} M_{ij}^a e_{c}^{[i} e_{c}^{j]} \]  
\[ \text{(B.16)} \]

from which the solution follows as
\[ v_b^{d} = \frac{1}{2} M_{ij}^a \epsilon_{abc} e_{c}^{[i} e_{c}^{j]} \]  
\[ \text{(B.17)} \]

Hence by fixing all the $v^b_a$, the stability of $T_{ij}^a = 0$ is obtained.
Therefore, all the primary constraints are stabilised. Now, we want to check stability of the secondary ones.

Stabilisation of $G_{ij}$ is already achieved, since $T_{ij}^a$ and $\pi_{ij,c}^a$ are stable. Modulo $T_{ij}^a = 0$ the Gauss constraint is equivalent to $G_{ij} = \pi_{ij,c}^a$ generating $U(1)^3$ gauge transformations.

To check the stability of $C_i$, first note that it is equivalent to $C_{a} := e_{a}^{i} C_{i} = e^{abc} e_{d}^{i} e_{c}^{j} F_{bc}^{ij} = e^{abc} e_{[a}^{i} e_{c]}^{j} F_{bc}^{ij}$. 
Looking at $\text{[B.4]}$, one immediately conclude that, modulo $T_{ij}^a = 0$, the equation $\epsilon_{abc} \pi_{ij}^a = 2\epsilon_{[b}^{i} e_{c]}^{j}$ holds. Therefore, modulo $T_{ij}^a = 0$, we have
\[ C_{a} = e^{abc} e_{[a}^{i} e_{c]}^{j} F_{bc}^{ij} = \frac{1}{2} \epsilon^{abc} \epsilon_{edc} \pi_{ij}^{e} F_{bc}^{ij} = -\frac{1}{2} \left( \delta_{d}^{[i} \delta_{e}^{j]} - \delta_{d}^{i} \delta_{e}^{j} \right) \pi_{ij}^{e} F_{bc}^{ij} = F_{ab}^{ij} \pi_{ij}^{a} \]  
\[ \text{(B.18)} \]

Next, modulo $T_{ij}^a = G_{ij} = 0$ we have $C_{a} = F_{ab}^{ij} \pi_{ij}^{b}$, since $\hat{P}_{a}^{i}, P_{i}$ have already been stabilised, we can add terms linear in $\hat{P}_{a}^{i}, P_{i}$ to $C_{a}$ so that the resulting constraint $C_{a}$ generates spatial diffeomorphisms also on the variables $e_{a}^{i}, e_{a}^{a}, e_{a}^{i}$, respectively. Since all the constraints are tensor densities, the constraint $C_{a}$ and thus $C_{a}$ is already stabilised.

Classification of the constraints:
Since all constraints are independent of $A_{ij}^{a}, e_{i}^{a}$, both $\pi_{ij}, P_{i}$ are first class constraints.
On the other hand, since all constraints either are invariant or covariant under Gauss transformations and spatial diffeomorphisms generated by $\hat{G}_{ij}$ and $\hat{C}_{a}$ respectively, both $G_{ij}$ and $\hat{C}_{a}$ are first class.
The constraints $P_{i}^{a}, T_{ij}^{a}$ are second class. To see this we need to show that the matrix
\[ \Delta_{ij}^{ab} := \{ P_{i}^{a}, T_{j}^{b} \} = -\epsilon^{abc} \epsilon_{ijk} e_{c}^{k} \]  
\[ \text{(B.19)} \]
is invertible as a symmetric (under $(a, i) \leftrightarrow (b, j)$) $9 \times 9$ matrix, where $T_{ij}^{b} := T_{j}^{b} e_{j}^{kl}/2$. It is easy to verify that
\[ \Delta_{ij}^{ab} = -\frac{3}{4} \epsilon_{ijk} \epsilon_{abc} e_{c}^{k} + \frac{1}{2} \det(e)^{-1} \epsilon_{i}^{a} e_{b}^{j} \]  
\[ \text{(B.20)} \]
is the inverse of $\Delta_{ij}^{ab}$, i.e., $\Delta_{ij}^{ab} \Delta_{jk}^{bc} = \delta_{i}^{[a} \delta_{j}^{b]}$ and hence $P_{i}^{a}, T_{ij}^{a}$ form a second class pair. Therefore, we have $48 - (2 \cdot 12 + 18) = 6$ degrees of freedom. This means that the action of $[2]$ cannot be a Lagrangian origin of the Hamiltonian $U(1)^3$ theory which has 4 propagating degrees of freedom. Note that just as in $[13]$ the Hamiltonian constraint $C = F_{ab}^{ij} e_{jkl}^{a} F_{k}^{a} e_{l}^{i}$ does not appear as a secondary constraint.
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