An explicit combinatorial design

Xiongfeng Ma\textsuperscript{1,} and Xiaoqing Tan\textsuperscript{2,1,}'

\textsuperscript{1}Center for Quantum Information and Quantum Control, Department of Physics and Department of Electrical \\& Computer Engineering, University of Toronto, Toronto, Ontario, Canada

\textsuperscript{2}Department of Mathematics, College of Information Science and Technology, Jinan University, Guangzhou, Guangdong, P. R. China

Abstract

A combinatorial design is a family of sets that are almost disjoint, which is applied in pseudo random number generations and randomness extractions. The parameter, $\rho$, quantifying the overlap between the sets within the family, is directly related to the length of a random seed needed and the efficiency of an extractor. Nisan and Wigderson proposed an explicit construction of designs in 1994. Later in 2003, Hartman and Raz proved a bound of $\rho \leq e^2$ for the Nisan-Wigderson construction. In this work, we prove a tighter bound of $\rho < e^2$ with a larger parameter range by slightly refining the Nisan-Wigderson construction. Following the block idea used by Raz, Reingold, and Vadhan, we present an explicit weak design with $\rho = 1$.

\textsuperscript{*}Electronic address: xfma@qic.ca

\textsuperscript{†}Electronic address: ttanxq@jnu.edu.cn
I. INTRODUCTION

Combinatorial designs play an important role in pseudo random number generations [1] and randomness extractions [2]. Nisan and Wigderson propose a simple construction of designs (Nisan-Wigderson design) for pseudo random number generators [1], which is later applied to construct randomness extractors by Trevisan [2].

A combinatorial design is a family of subsets, drawn from the set, which have a same size, \( q \), and are almost disjoint. Consider a family of disjoint subsets, the size of the set, \( l \), grows linearly with the number of subsets, \( n \). Later, we will see that with a design, the size of the set only grows as \( \text{poly}(\log n) \).

One key parameter of a design, \( \rho \), is used to quantify the overlap between subsets in the family. Generally speaking, the smaller \( \rho \) is, the more disjoint the subsets are. This parameter is linked to the seed length and approximately indicates the ratio of randomness that can be extracted by Trevisan’s extractor [2, 3]. In the application of extractors, \( \rho \) is normally required to be close to 1. Furthermore, the size of the set, \( l \), is linked to the initial randomness input (as seed) required for Trevisan’s extractor. In general, the size \( (l) \) should be small compared to the number of subsets \( (n) \).

Hartman and Raz proved a bound of \( \rho \leq e^2 \) (\( e \) as the Euler’s number) for the Nisan-Wigderson design [4] when \( n \) is a power of a prime power number, \( q \) (subset size). By slightly refining the Nisan-Wigderson design, we give a better bound \( \rho < e \) for a wider range of \( n \) when \( n \) can be divided by the largest power of \( q \) no greater than \( n \). With the refined Nisan-Wigderson design, we also give a conjecture that \( \rho < e \) is true for all \( n \leq q^q \). Furthermore, we follow the block idea used by Raz, Reingold, and Vadhan to construct an explicit design with \( \rho = 1 \) and \( l = O(\log^3 n) \).

In Section II we review the definitions of combinatorial designs, the Nisan-Wigderson design and the Hartman-Raz bound. In Section III, we refine the Nisan-Wigderson design and show a better bound of \( \rho \). In Section IV we construct an explicit \( \rho = 1 \) design. We finally conclude with discussions in Section V.
II. PRELIMINARIES

A. Notations and Definitions

Notations: \( [l] = \{0, 1, 2, \ldots, l - 1\} \); \( \log \) is base 2; \( \ln \) is the natural logarithm; and \( e \) is the base of the natural logarithm or the Euler’s number.

Define a Galois (finite) field, \( GF(q) = [q] \) where \( q = p^r \), \( r \) is a positive integer, and \( p \) is a prime. Here, we represent an element, \( j \in [q] \), by a \( p \)-nary string. Define \( F_q \) to be the ring of polynomials over the field \( GF(q) \). For a polynomial \( \phi(x) \in F_q \), denote \( \lambda(\phi) \) to be its number of roots over \( GF(q) \). For the sake of simplicity, we use \( p = 2 \) in the following. We remark that our results apply to the case of a general prime \( p \) with minor modifications.

Define \( P_d = \{ \phi_1, \phi_2, \ldots, \phi_{q^{d+1}} \} \subseteq F_q \) to be the set of all polynomials over \( GF(q) \) with the highest order no greater than \( d \in [q] \), hence, \( |P_d| = q^{d+1} \). We further divide the set \( P_d \) evenly into \( q \) disjoint subsets, \( \mathcal{N}_{d,j} \) with \( j \in GF(q) \),

\[
\mathcal{N}_{d,j} \triangleq \{ jx^d + \phi(x) | \phi(x) \in \mathcal{P}_{d-1} \}.
\] (1)

That is, the coefficient of \( x^d \) of each polynomial in \( \mathcal{N}_{d,j} \) is \( j \). It is not hard to see that

\[
\mathcal{P}_d = \bigcup_{j=0}^{q-1} \mathcal{N}_{d,j}
\] (2)

and hence for every \( j \in [q] \),

\[
|\mathcal{N}_{d,j}| = q^d.
\] (3)

For a polynomial set, \( \mathcal{M} \), define a function,

\[
\Lambda(\mathcal{M}) \triangleq \sum_{\phi \in \mathcal{M}} 2^{\lambda(\phi)}
\] (4)

In the summation on the right side, we assume that the number of roots of the trivial polynomial \( \phi \equiv 0 \) is zero. That is, \( \lambda(\phi) = 0 \) for every constant function \( \phi \).

B. Designs

A combinatorial design is a family (collection) of nearly disjoint subsets of a set \([l]\). Here are the three definitions of designs used in the literature.
**Definition II.1.** (Standard Design) A family of sets $S_0, S_1, \ldots, S_{n-1} \subseteq [l]$ is a standard $(n, q, l, \rho)$-design if

1. For all $i \in [n]$, $|S_i| = q$.
2. For all $i \neq j \in [n]$, $|S_i \cap S_j| \leq \log \rho$. \hfill (5)

**Definition II.2.** (Weak design) A family of sets $S_0, S_1, \ldots, S_{n-1} \subseteq [l]$ is a weak $(n, q, l, \rho)$-design if

1. For all $i \in [n]$, $|S_i| = q$.
2. For all $i \in [n]$, \[ \sum_{j<i} 2^{|S_i \cap S_j|} \leq (n-1)\rho. \] \hfill (6)

**Definition II.3.** (Modified weak design) A family of sets $S_0, S_1, \ldots, S_{n-1} \subseteq [l]$ is a modified weak $(n, q, l, \rho)$-design if

1. For all $i \in [n]$, $|S_i| = q$.
2. For all $i \in [n]$, \[ \sum_{j \neq i} 2^{|S_i \cap S_j|} \leq n\rho. \] \hfill (7)

Definition II.1 is originally used in the Nisan-Wigderson construction [1] that is applied in the Trevisan extractor [2]. Then, Raz et al. showed that a weaker version of design (Definition II.2) is sufficient for the use in the Trevisan extractor [3]. Later, Hartman and Raz proved a bound of $\rho$ of the Nisan-Wigderson construction for a modified version of the weak design (Definition II.3) [4].

A design can be treated as an $l \times n$ binary (or $p$-nary) matrix with the $i$-th row represents a subset $S_{i-1}$, for example, $n = 4$, $q = 2$, $l = 4$ and a binary matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}. \hfill (8)$$
Take \([l] = \{0, 1, 2, 3\}\), then the family of sets are \(S_1 = \{0, 2\}\), \(S_2 = \{1, 3\}\), \(S_3 = \{0, 3\}\) and \(S_4 = \{1, 2\}\). It is not hard to see that \(\rho = 2\) for the standard design from Eq. (5), while \(\rho = 5/4\) in the two weak design definitions from Eq. (6) and (7).

As pointed in the introduction, the objective of design construction is to minimize \(l\) and \(\rho\), given \(q\) and \(n\).

C. Nisan-Wigderson design

Without loss of generality, let the size of set (the length of the random seed in the application of Trevisan’s extractor), \(l\), be the square of a prime power number \((l = q^2\), if not, pick the smallest power of 2 which is greater than \(\sqrt{l}\)). Consider \([l]\) to be a \(q \times q\) 2-dimensional array, then every element of \([l]\) can be represented as a pair of elements in \(GF(q)\). The Nisan-Wigderson design is constructed as follows.

1. Find \(n\) distinct polynomials \(\{\phi_0(\cdot), \phi_1(\cdot), \ldots, \phi_{n-1}(\cdot)\}\) on \(GF(q)\) of degree at most \(d\).
   This can be done as long as \(n \leq q^{d+1}\) and \(d \in [q]\).

2. The nearly disjoint sets are given by
   \[
   S_i = \{< j, \phi_i(j) > \mid j \in GF(q)\} \tag{9}
   \]
   where \(< j, \phi_i(j) >\) presents an element in \([l]\).

The following facts can be easily verified \([1]\):

1. The size of each set is exactly \(q\), \(|S_i| = q\) for every \(i \in [q]\).

2. Any two sets intersect in at most \(d\) points.

3. There are at least \(q^{d+1}\) possible sets (the number of polynomials on \(GF(q)\) of degree at most \(d\)).

In the original proposal of the Nisan-Wigderson design, the polynomials (with a degree at most \(d\)) are chosen in an arbitrary manner. A natural way to choose these polynomials is to go from low order polynomials to higher ones, which results the highest order of polynomials to be \(d = \lceil \log n / \log q - 1 \rceil \leq \log n\). According to Definition \([1]\) it is straightforward to see that \(\rho \leq \log n\) as shown by Nisan and Wigderson \([1]\).
D. Hartman-Raz bound

Hartman and Raz proved that the Nisan-Wigderson design is an explicit modified weak $(n, q, l, \rho)$-design with $l = q^2$ and $\rho \leq e^2$ in Theorem 1 of ref. [4]. We remark that Hartman and Raz’s result is only proven to for the case when $n$ is a power of $q$.

III. NEW BOUND

Intuitively, the more sets the design has, the harder to make sets disjoint. Thus, one might conjecture that the parameter $\rho$ grows with $n$. Mathematically, this is not necessarily true, because the overlap is normalized by $n$, as shown in Eq. (7). In fact, one can find counter examples to this conjecture for the Nisan-Wigderson design. However, as we will prove in Lemma II.3 and II.4, $\rho$ does grow when $n$ increases by a large scale. It is not too hard to show that for a full matrix (where $n = q^q$), $\rho = (1 + q^{-1})q - (2/q)^q < e$ for the Nisan-Wigderson design. Therefore, we expect $\rho < e$ for all positive integers $n \leq q^q$ (see, Conjecture III.8).

A. Refined Nisan-Wigderson design

Here, we refine the Nisan-Wigderson design by choosing the $i$-th polynomial for Eq. (9) in the following manner:

$$\phi_i(x) = \sum_{k=0}^{d} (\lfloor i/q^k \rfloor \mod q) x^k$$

where $d = \lceil \log n / \log q - 1 \rceil$ (then, $q^d < n \leq q^{d+1}$) and the coefficients calculated by the modulo function ($\lfloor i/q^k \rfloor \mod q$) can be treated as elements of $GF(q)$. These polynomials form a set $\mathcal{M}_n = \{\phi_0, \phi_1, \ldots, \phi_{n-1}\}$. Each polynomial, $\phi_i$, in $\mathcal{M}_n$ corresponds to a set $S_i$ in the design in the form of Eq. (9). It is simple to verify the following facts.

1. The number of intersection elements $|S_i \cap S_j|$ equals to the number of roots of $\phi_i = \phi_j$.
2. For any $i > j$, $\phi_i - \phi_j = \phi_{i-j}$ is also in $\mathcal{M}_n$.
3. We can see that there are no roots for $\phi_i - \phi_j$ (which is a constant function) when $0 \leq i < j < q$, thus,

$$\Lambda(\mathcal{M}_q) = q.$$
4. Consider a set of polynomials that only differ from each other in the constant terms,

\[ S_{q,k} = \{ \phi_i | i = kq, kq + 1, \ldots, kq + q - 1 \}, \quad (12) \]

where \( \phi_i \) is defined in Eq. (10), the sum of the number of roots of each polynomial in \( S_{q,k} \) is \( q \),

\[ \sum_{\phi \in S_{q,k}} \lambda(\phi) = q, \quad (13) \]

for every integer \( 1 \leq k < q^{q-1} \). In fact, each element of \( GF(q) \) appears as a root of one of the polynomials exactly once.

\[ \text{Proof.} \quad \text{The set defined in Eq. (12) is equivalent to} \]

\[ S_{q,k} = \{ \psi + j | j = 0, 1, \ldots, q - 1 \}, \quad (14) \]

where \( \psi \) is an arbitrary non-constant polynomial in \( GF(q) \). Since any two distinct polynomials in \( S_{q,k} \) do not share same roots, the summation \( \sum_{\phi \in S_{q,k}} \lambda(\phi) \) is no larger than \( q \), the total number of possible roots. On the other hand, for any element \( h \in GF(q) \), one can find \( j \) such that \( \psi(h) + j = 0 \), hence it is a root of one of the polynomials in \( S_{q,k} \). Thus, the summation \( \sum_{\phi \in S_{q,k}} \lambda(\phi) \) is no less than \( q \), the size of \( GF(q) \). \( \square \)

5. By the definition of Eq. (1),

\[ N_{d,0} = \{ \phi_0, \phi_1, \ldots, \phi_{q^d-1} \} \subset \mathcal{M}_n \quad (15) \]

Note that item 2 is the key property of our refined design, and it is generally not satisfied when polynomials are chosen in an arbitrary manner as in the original Nisan-Wigderson construction. According to item 1 and 2, the design requirement, Eq. (17), can be rewritten as

\[ \sum_{j \neq i} 2^{|S_i \cap S_j|} = \sum_{\phi \in \mathcal{M}_n} 2^{\lambda(\phi)} \triangleq \Lambda(\mathcal{M}_n). \quad (16) \]

Now the question changes to how to find the roots of polynomials in \( \mathcal{M}_n \).

**Proposition III.1.** For any two sets defined in Eq. (1), \( N_{d,i} \) and \( N_{d,j} \) with \( ij \neq 0 \) and \( i, j \in GF(q) \), there exists a one-to-one map between them such that the two polynomials by the map have the same roots.
Proof. The map can be constructed by multiplying a scalar $i/j$ to the second set, since $ij \neq 0$ and $i/j \in GF(q)$.

We remark that the two polynomials not only have the same number of roots but also the values. According to the definition of $\Lambda(\cdot)$, Eq. (4), it is simple to see the following lemma.

**Lemma III.2.** The value of $\Lambda(N_{d,j})$ is the same for all $j \neq 0 \in GF(q)$.

**Proposition III.3.** For every positive integer $d$,

$$\Lambda(N_{d,0}) \leq \Lambda(N_{d,1}).$$

**Proof.** From Lemma 4 of ref. [5], we know that

$$\Lambda(N_{d,1}) = |N_{d,1}| \sum_{i=0}^{d-1} q^{-i} \binom{q}{i}. \tag{18}$$

With Eq. (3),

$$\Lambda(N_{d,0}) = \sum_{k=0}^{d-1} (q-1)\Lambda(N_{k,1}) + 1$$

$$\leq \left( \sum_{k=0}^{d-1} (q-1)|N_{k,1}| + 1 \right) \sum_{i=0}^{d-1} q^{-i} \binom{q}{i}$$

$$= \Lambda(N_{d,1}) \tag{19}$$

where the inequality comes from Lemma III.4.

**Lemma III.4.** Assume that $n$ is a power of $q$, $n = q^{d+1}$, then $\rho = \Lambda(M_n)/n$, as defined in Eq. (7), is an increasing function of $d \in [q]$, with $M_n$ constructed by Eq. (10).

**Proof.** This can be directly seen from Eq. (18).

From Lemma III.2 and III.3 we can show that

**Lemma III.5.** Assume that $n$ can be divided by $q^d$, $\rho = \Lambda(M_n)/n$, as defined in Eq. (7), is an increasing function of $n \in (q^d, q^{d+1})$, with $M_n$ constructed by Eq. (10).

**Proof.** In fact,

$$M_n = \bigcup_{j=0}^{n/q^d-1} N_{d,j} \tag{20}$$
and \( \{N_{d,j}\} \) are disjoint sets, which follows that

\[
\frac{1}{n}\Lambda(\mathcal{M}_n) = \frac{1}{n} \sum_{j=0}^{n/q^d-1} \Lambda(N_{d,j}) \\
= \left( \frac{1}{q^d} - \frac{1}{n} \right) \Lambda(N_{d,1}) + \frac{1}{n} \Lambda(N_{d,0}) \\
\leq \left( \frac{1}{q^d} - \frac{1}{n + q^d} \right) \Lambda(N_{d,1}) + \frac{1}{n + q^d} \Lambda(N_{d,0}) \\
= \frac{1}{n + q^d} \Lambda(\mathcal{M}_{n+q^d})
\]

where the second equation comes from Lemma III.2, the inequality comes from Eq. (17). \( \square \)

Note that Lemma III.4 can be treated as a special case of Lemma III.5.

**Lemma III.6.** Assume that \( n \in (q^d, q^{d+1}], \ d \in [q], \) and \( q^d \) divides \( n, \ \rho = \Lambda(\mathcal{M}_n)/n, \) as defined in Eq. (7), is bounded by \( \rho \leq (1 + q^{-1})^q < e. \)

**Proof.** From Lemma III.4 and III.5 we can see that,

\[
\rho \leq \frac{\Lambda(\mathcal{M}_{q^d})}{q^d} = (1 + q^{-1})^q - \left( \frac{q}{2} \right)^q < e
\]

where the equality comes from Eq. (18) with \( d = q - 1. \) \( \square \)

**B. Main result**

**Theorem III.7.** For a positive integer \( n \) and a prime power number \( q, \) with \( n \in (q^d, q^{d+1}], \ d \in [q], \) and \( q^d \) divides \( n, \) there exists an explicit modified weak \( (n,q,l,\rho) \)-design with \( l = q^2 \) and \( \rho < (1 + q^{-1})^q < e. \)

**Proof.** Directly followed by Lemma III.6 \( \square \)

Comparing to the previous result by Hartman and Raz [4], \( \rho \leq e^2, \) which is only applied to the case \( n = q^{d+1}, \) here we present a better bound \( \rho < e \) with a larger parameter range \( n \in (q^d, q^{d+1}]. \)

We conjecture that Eq. (22) is true for every positive integer \( n \leq q^q. \) That is, Theorem III.7 is true for an arbitrary \( n. \)

**Conjecture III.8.** For a prime power number \( q, \) and every positive integer \( n \leq q^q, \) there exists an explicit modified weak \( (n,q,l,\rho) \)-design with \( l = q^2 \) and \( \rho < (1 + q^{-1})^q < e. \)
IV. DESIGN CONSTRUCTION

In Theorem III.7 we show that the design constructed by Eq. (10) can be bounded $\rho < e$. On the other hand, it not hard to see that $\rho > 2$ for the refined Nisan-Wigderson design (as constructed by Eq. (10)) in a reasonable regime of $n$ and $q$, e.g., $q \geq 16$ and $n > q^2$. Thus, our bound in Theorem III.7 is relatively tight.

In the application of extractors, such as [3], the value of $\rho$ roughly indicates the ratio of randomness that can be extracted. Thus, we need to achieve a $\rho$ that is close to 1. Then, we have to go beyond the Nisan-Wigderson design. In order to reduce the parameter $\rho$, one can extend the size of the set, from $[l]$ to $[l']$. Raz et al. proposed a block design idea to reduce $\rho$ [3, 4]. The basic idea is break the set $[l']$ into $b$ blocks (smaller sets), each of which has a size of $l$ (hence, $l' = lb$). That is, the $i$-th subset is $\{il+1, il+2, \ldots, (i+1)l\}$ and $i \in [b]$. The design sets are subsets of one of subsets. Obviously, the sets from different subsets are disjoint. Hartman and Raz show that with this technique (Lemma 17 of ref. [3]), $\rho$ can be reduced to 1 exponentially fast with the number of subsets grows. With this technique, we can reduce $\rho$ down to 1 with a finite number, $O(\rho \log(n\rho))$, of blocks by digging into details of the design constructed by Eq. (10).

Conjecture IV.1. Given the explicit modified weak $(n, q, l, \rho)$-design constructed by Eq. (10) with $l = q^2$ and $\rho > 1$, there exists an explicit weak $(n', q, l', 1)$-design with $n' = n\rho$, $l' = q^2b$ and

$$b = \left\lfloor \frac{\log n + \log \rho - \log q}{\log \rho - \log(\rho - 1)} \right\rfloor$$

$$= O(\log n)$$

(23)

as the number of blocks.

Conjecture IV.1 is a corollary of Conjecture III.8. Here we give a proof by assuming Conjecture III.8.

Proof. Denote the number of subsets from $i$-th subset to be $n_i$. We construct the design in such a way that

$$n_i = (1 - \rho^{-1})^i n$$

$$n_b = n\rho - \sum_{i=0}^{b-1} n(1 - \rho^{-1})^i$$

$$= n\rho(1 - \rho^{-1})^b$$

(24)
where the first equation holds for \( i \in [b] \). It is not hard to verify that \( \sum_{i=0}^{b} n_i = n \rho \) and \( n_b \leq q \) with Eq. (23). Thus, we can use disjoint sets, \( \mathcal{M}_q \), for the last block, which has a \( \rho = 1 \) according to Eq. (11). Now, we can verify the conditions in Definition II.2. Condition 1 is obviously satisfied. For a set \( S_j \) in block \( i \in [b] \),

\[
\sum_{j' < j} 2^{|S_j \cap S_{j'}|} \leq \sum_{i=0}^{i-1} n_{j'} + \rho n_i = n \rho.
\]

(25)

For the last block, \( \sum_{j' < j} 2^{|S_j \cap S_{j'}|} = j \). Thus, it is a weak \((n', q, l', 1 + 1/n')\)-design. Since \( \lfloor (1 + 1/n')(n' - 1) \rfloor = n' - 1 \), it is also a weak \((n', q, l', 1)\)-design.

If we use the matrix representation of designs as shown in Eq. (8), then the new design matrix from a refined Nisan-Wigderson design matrix \( A_0 \) can be written as

\[
\begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
\vdots \\
A_{b-1} \\
A_b
\end{pmatrix},
\]

(26)

where all the off-diagonal blocks are 0. According to the block design idea, presented in Conjecture IV.1, \( A_i \) take first \( n_i \) rows of \( A_{i-1} \) for \( i = \{1, 2, \ldots, b\} \), where \( n_i \) is defined in Eq. (24).

We remark that one does not need to prove Conjecture III.8 in practice. In fact, as long as a design can be verified (say, numerically) to satisfy design conditions, as given in Eq. (7), one can use Conjecture III.8. Note that we have numerically verify Conjecture III.8 for various values of \( q \) and \( n \).

V. DISCUSSIONS

One interesting topic to investigate is to prove Conjecture III.8 which allows the number of subsets, \( n \in [q^n] \), to be arbitrary. We remark that the observation as shown in Eq. (13) might be useful to prove the conjecture. The main question is how the \( q \) roots are distributed over \( q \) polynomials in the set. With this observation, one might expect the inequality of Eq. (17) can be replaced by approximately equal when \( d \) is relatively large.
In the Nisan-Wigderson construction, $n$ is limited by $q^q$, which is not necessarily true for a general case. Let us extend the example of Eq. (8),

\[
\begin{pmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}
\]  

(27)

One can easily verify that this design has a $\rho < 2$ and $n = 6 > q^q = 4$. The key point is that one does not need to pick only one element from one block, as used in Eq. (9). In general, one might expect $n = O\left(\binom{l}{q}\right)$ or $l = O(\log n)$. If one can find such a design with a reasonable $\rho$, one can apply the block design idea as shown in Eq. (23) so that the seed length for the Trevisan extractor is $O(\log^2 n)$.

**Acknowledgments**

We thank H.-K. Lo, B. Qi, C. Rockoff, F. Xu and H. Xu for enlightening discussions. Financial supports from CFI, CIPI, the CRC program, CIFAR, MITACS, NSERC, OIT, QuantumWorks, National Natural Science Foundation of China (No.61003258), and Special Funds for Work Safety of Guangdong Province of 2010 from Administration of Work Safety of Guangdong Province of China are gratefully acknowledged. X. Q. Tan especially thanks H.-K. Lo for the hospitality during her stay at the University of Toronto.

[1] N. Nisan and A. Wigderson, J. Comput. Syst. Sci. 49, 149 (1994), ISSN 0022-0000, URL [http://dx.doi.org/10.1016/S0022-0000(05)80043-1](http://dx.doi.org/10.1016/S0022-0000(05)80043-1).

[2] L. Trevisan, Journal of the ACM 48, 2001 (1999).

[3] R. Raz, O. Reingold, and S. Vadhan, Journal of Computer and System Sciences 65, 97 (2002), ISSN 0022-0000, URL [http://www.sciencedirect.com/science/article/B6WJ0-475JJYD-5/2/159c66730d373faa25dd9ce40576](http://www.sciencedirect.com/science/article/B6WJ0-475JJYD-5/2/159c66730d373faa25dd9ce40576).

[4] T. Hartman and R. Raz, in In Proceedings of RANDOM (2003).
[5] V. Leont’ev, Mathematical Notes 80, 300 (2006).