Bundle gerbes

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Abstract. Just as $\mathbb{C}^\times$ principal bundles provide a geometric realisation of two-dimensional integral cohomology; gerbes or sheaves of groupoids, provide a geometric realisation of three dimensional integral cohomology through their Dixmier-Douady class. I consider an alternative, related, geometric realisation of three dimensional cohomology called a bundle gerbe. Every bundle gerbe gives rise to a gerbe and most of the well-known examples examples of gerbes are bundle gerbes. I discuss the properties of bundle gerbes, in particular bundle gerbe connections and curvature and their associated Dixmier-Douady class.

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1. Introduction

In [1] Brylinski describes Giraud’s theory of gerbes. Loosely speaking a gerbe over a manifold $M$ is a sheaf of groupoids over $M$. Gerbes, via their Dixmier-Douady class, provide a geometric realisation of the elements of $H^3(M, \mathbb{Z})$ analogous to the way that line bundles provide, via their Chern class, a geometric realisation of the elements of $H^2(M, \mathbb{Z})$.

I want to introduce here another sort of object, which is not a sheaf, and which also gives rise to elements of $H^3(M, \mathbb{Z})$. For want of a better name I have called these objects bundle gerbes. A bundle gerbe over $M$ is a pair consisting of a fibration $Y \to M$ and a principal $\mathbb{C}^\times$ bundle $P$ over the fibre product $Y^[2]$. The bundle $P$ is required to have a product, that is a $\mathbb{C}^\times$ bundle morphism which on fibres is of the form:

$$P_{(x,y)} \otimes P_{(y,z)} \to P_{(x,z)}; \quad (1.1)$$

for any $x, y, z$ in the same fibre of $Y \to M$. From a bundle gerbe it is possible to construct a presheaf of groupoids and hence a gerbe. However not all gerbes arise in this way.

By considering a connection on $P$ compatible with the product (1.1) it is possible to construct a closed, integral three-form on $M$ analogous to the curvature of a line bundle. This three form is a representative for the image in $H^3(M, \mathbb{R})$ of the Dixmier-Douady class of the corresponding gerbe. Moreover every integral three class is represented by a closed, integral three form arising from a bundle gerbe $P \to Y^[2]$ where $Y \to M$ is the path fibration. This construction is analogous to the construction of a line bundle with given curvature two-form. It follows from the results in [1] that every isomorphism class of gerbes contains a bundle gerbe. If $Q \to Y$ is a $\mathbb{C}^\times$ bundle there is a bundle gerbe whose fibre at $(x, y)$ is $\text{Aut}_{\mathbb{C}^\times}(Q_x, Q_y)$. I shall call such a bundle gerbe trivial. The geometric interpretation of the Dixmier-Douady class of a bundle gerbe is that it is the obstruction to the bundle gerbe being trivial.

Many interesting gerbes are bundle gerbes. For instance if $\mathbb{C}^\times \to \hat{G} \to G$ is a central extension of groups it is well known that the obstruction to lifting a principal $G$ bundle over $M$ to a principal $\hat{G}$ bundle is a class in $H^3(M, \mathbb{Z})$ and Brylinski discusses the gerbe defined by such a principal bundle whose Dixmier-Douady class is this obstruction. This principal bundle also gives rise to a bundle gerbe in a natural way and this bundle gerbe is trivial precisely when the original bundle lifts to $\hat{G}$.

Having outlined the virtues of bundle gerbes I should mention two of their deficiencies. The first is that there are many bundle gerbes which are isomorphic as gerbes but not as bundle gerbes. As a consequence the theorem that isomorphism classes of gerbes are in bijective correspondence with $H^3(M, \mathbb{Z})$ does not hold for bundle gerbes. The second is that if a bundle gerbe $P \to Y^[2]$ is non-trivial then the fibres of $Y \to M$ must be infinite dimensional.
In outline the paper is as follows. Section 2 reviews the properties of $\mathbb{C}^\times$ groupoids and their relationship with line bundles. This is in preparation for Section 3 which introduces bundle gerbes. The motivating example of the lifting of a principal bundle for a central extension is discussed in Section 4 and this leads to the introduction of the Dixmier-Douady class of a bundle gerbe in Section 5. A de Rham representative for this class is provided by the theory of bundle gerbe connections and curvature introduced in Sections 6, 7 and 8. The relationship with gerbes is discussed in Section 9 as in the relationship of the bundle gerbe connection and curvature with Brylinski’s connective structure and curving. In Section 10 I show how to construct a bundle gerbe with given Dixmier-Douady class. Section 11 considers the Deligne cohomology class defined by a bundle gerbe with connection and in Section 12 the holonomy of a bundle gerbe connection over a two-sphere is defined. Finally in Section 13 I explain why the fibering $Y \to M$ has to have infinite dimensional fibres if the bundle gerbe is non-trivial.

I will assume, when talking about gerbes, some familiarity with Brylinski’s book [1]. The material on bundle gerbes however is intended to be self-contained.

2. $\mathbb{C}^\times$ groupoids

Denote by $\mathbb{C}^\times$ the group of non-zero complex numbers. If $P$ and $Q$ are principal $\mathbb{C}^\times$ bundles over a manifold $Z$ then it is possible to define a new principal $\mathbb{C}^\times$ bundle $P \otimes Q$ over $Z \times Z = Z^2$. This is called the contracted product [1]. If $P$ and $Q$ are the frame bundles of line bundle $L$ and $J$ respectively then $P \otimes Q$ is the frame bundle of the line bundle $L \otimes J$. To define $P \otimes Q$ take $P \times Q$ which is a principal $\mathbb{C}^\times \times \mathbb{C}^\times$ bundle over $Z^2$ and quotient by the ‘anti-diagonal’ copy of $\mathbb{C}^\times$ inside $\mathbb{C}^\times \times \mathbb{C}^\times$, that is, the subgroup of all pairs $(z, z^{-1})$. What makes this construction possible, of course, is the fact that $\mathbb{C}^\times$ is abelian. Consider now a manifold $X$ and inside $X^2 \times X^2$ define

$$X^2 \circ X^2 = \{((x, y), (y, z)) \mid x, y, z \in X\}.$$  \hspace{1cm} (2.1)

If $P$ is a principal $\mathbb{C}^\times$ bundle over $X^2$ define $P \circ P$ to be the restriction of $P \otimes P$ to $X^2 \circ X^2$.

Recall [2] that a groupoid is a category with every morphism invertible. Let us consider an equivalent definition that we will see below is easily generalised to define bundle gerbes. Define a $\mathbb{C}^\times$ groupoid to be a principal $\mathbb{C}^\times$ bundle $P$ over $X^2$ with a product, that is, a $\mathbb{C}^\times$ bundle morphism $P \circ P \to P, (p, q) \mapsto pq$, covering the map $((x, y), (y, z)) \mapsto (x, z)$. The product is required to be associative, that is $(pq)r = p(qr)$ whenever these products are defined.

A $\mathbb{C}^\times$ groupoid actually has two other important algebraic structures, an identity and inverse which could have been included in the definition but in fact are a consequence of it. The identity is a section $e$ of the bundle $P$ over the diagonal in $X^2$ which satisfies
$pe = ep = p$. To define it note first that if $p \in P_{(x,y)}$ and $q \in P_{(y,z)}$ then $pq \in P_{(x,y)}$. Hence there is some $z \in C^\times$ such that $pq = pz$. Define $e = qz^{-1}$ so that $pe = p$. Because the product is a bundle automorphism $(pw)e = (pe)w = pw$ for all $w \in C^\times$ and hence $qe = q$ for every $q$ in $P$. To show that $ep = p$ we use associativity. Clearly $ep = pz$ for some $z \in C^\times$ and considering $(pe)p = p(ep)$ it follows that $z = 1$. To define the inverse notice that the equation $pq = e$ can be solved by by acting by $C^\times$. Then we have that $qp = ez$ for some $z$ and using associativity in the form of $p(qp) = (pq)p$ it follows that $pez = ep = pe$ and hence $z = 1$. The inverse will be denoted by $p \mapsto p^{-1}$. 

To understand the global structure of the inverse notice that it is possible to construct a bundle $P^*$ over $X^2$ by defining it to be the same set as $P$ but changing the $C^\times$ action to $pz = pz^{-1}$. If $P$ is the frame bundle of a line bundle $L$ then $P^*$ is the frame bundle of $L^*$. Because $C^\times$ is abelian this is still a right action. The inversion then defines a map $P \rightarrow P^*$ covering the map $X^2 \rightarrow X^2$ defined by $(x,y) \mapsto (y,x)$. The identity and the inverse behave as one would expect with respect to the product.

Given a $C^\times$ groupoid we can recover the definition in terms of categories [2] by taking $X$ as the set of objects and $P_{(x,y)}$ as the morphisms from $x$ to $y$.

A simple example of a $C^\times$ groupoid is constructed by taking a $C^\times$ bundle $Q$ on $X$ and defining $P_{(x,y)} = Aut_{C^\times}(Q_x,Q_y)$ where $Q_x$ is the fibre of $Q$ over $x$ and the subscript $C^\times$ indicates that these automorphisms commute with the $C^\times$ action. An alternative way to define the is to use the two projections $\pi_1$ and $\pi_2$ on the first and second factors of $X^2 = X \times X$ and define $P = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$. The composition is, of course, composition of automorphisms. What makes $C^\times$ groupoids uninteresting is that every $C^\times$ groupoid arises in this way! To see this let $P$ be a $C^\times$ groupoid and pick a basepoint $x$ in $X$. Then define a $C^\times$ bundle $Q$ on $X$ by $Q_y = P_{(x,y)}$, that is $Q$ is the pull-back of $P$ under the map $y \mapsto (x,y)$. Then the composition and inversion can be used to define a $C^\times$ bundle isomorphism

$$Q^*_y \otimes Q_z = P^*_{(x,y)} \otimes P_{(x,z)} \rightarrow P_{(y,z)}$$

by $(p,q) \mapsto p^{-1}q$. It is easy to see that this is a $C^\times$ bundle isomorphism and that, moreover, it preserves the composition. Hence it is an isomorphism of $C^\times$ groupoids.

Although we have just seen that the theory of $C^\times$ groupoids is nothing more than the theory of $C^\times$ bundles over pointed sets it is useful to develop the theory further as the next section of bundle gerbes is then a straightforward generalisation.

A connection $\nabla$ on the bundle $P \rightarrow X^2$ gives rise to a connection on $P \otimes P$ and hence on $P \circ P$. Call it a groupoid connection if it is mapped by the product to itself again. Such connections exist because we can identify $P$ with $\pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$ for some $Q \rightarrow X$ and pick a connection on $Q$ and pull it back to a connection on $P$. The curvature $F_\nabla$ of a
groupoid connection on $P$ constructed in this way has the form

$$F_{∇} = \pi_2^*(f) - \pi_1^*(f)$$

(2.3)

where $f$ is the curvature of the original connection on $Q \to X$. It is also possible to show (see Section 8 below) that if $F_{∇}$ is the curvature of any groupoid connection then there is unique two-form $f$ on $X$ which satisfies (2.3). Call this two-form $f$ the groupoid curvature.

It is well known that given an integral closed two-form $f/(2\pi i)$ over a 1-connected space $X$ it is possible to explicitly construct a $C^\infty$ bundle over $X$ with a connection whose curvature is $f$. It will be useful later to know how to repeat this construction for $C^\infty$ groupoids. In fact this construction is a little more natural in the groupoid context as, unlike the construction of the $C^\infty$ bundle over $X$, it does not require the choice of a base point.

Assume then that $X$ is a 1-connected manifold and that $f/(2\pi i)$ is a closed, integral two-form on $X$. We shall construct a $C^\infty$ groupoid on $X$ with a groupoid connection whose groupoid curvature is $f$. Let $PX$ be the space of all piecewise smooth paths in $X$ and let $PX(x,y)$ be the set of all paths beginning at $x$ and ending at $y$. We need piecewise smooth paths as we will want to compose them to define the groupoid product. Define an equivalence relation on $PX(x,y) \times C$ by saying that $(\gamma, z)$ is equivalent to $(\tilde{\gamma}, \tilde{z})$ if $z = \exp(\int_D f)\tilde{z}$

(2.4)

where $D$ is a map of the disk into $X$ with boundary the union of $\gamma$ and $\tilde{\gamma}$ and oriented by $\gamma$. Let $P_{(x,y)}$ denote the quotient space. The $C^\infty$ action is just that induced by $(\gamma, z)w = (\gamma, zw)$. If $\gamma(1) = \tilde{\gamma}(0)$ then define a new piecewise smooth path $\gamma \ast \tilde{\gamma}$ by running along $\gamma$ at twice the speed for $t \in [0, 1/2]$ and then running along $\tilde{\gamma}$ at twice the speed for $t \in [1/2, 1]$. A product on $PX \times C$ is then defined by $(\gamma, z) \ast (\tilde{\gamma}, \tilde{z}) = (\gamma \ast \tilde{\gamma}, z\tilde{z})$ and it is straightforward to check that this descends to a product on $P$ making it a $C^\infty$ groupoid.

Now we construct a $C^\infty$ groupoid connection. Notice that $P$ is the quotient of a fibering

$$PX \times C \to P$$

(2.5)

where the fibres are defined by the equivalence relation (2.4). The tangent space to $PX \times C$ at $(\gamma, z)$ is the space of all pairs $(\xi, \alpha)$ where $\xi$ is a vector field along $\gamma$ and $\alpha$ is a complex number. A vector field along $\gamma$ means a continuous vector field which is smooth when $\gamma$ is smooth. The subspace of the tangent space that is tangent to the fibering (2.5) is

$$K_{(\gamma,z)} = \{ (\xi, \alpha) \mid \alpha = -\int_0^1 f(\gamma', \xi)dt, \xi(0) = 0 = \xi(1) \}$$

(2.6)
where $\gamma'$ is the tangent vector field along $\gamma$. Consider the map
\[ \text{ev}: PX \times [0, 1] \to X \] (2.7)
which maps $(\gamma, t) \mapsto \gamma(t)$. A one-form $\hat{A}$ on $PX \times [0, 1]$ is now defined by pulling back $f$ and integrating it over the $[0, 1]$ direction and then letting
\[ \hat{A} = dt + \int_0^1 \text{ev}^*(f). \] (2.8)

Notice that
\[ d\hat{A} = \text{ev}_1^*(f) - \text{ev}_0^*(f) \] (2.9)
where $\text{ev}_i(\gamma) = \gamma(t)$. The forms $\hat{A}$ and $d\hat{A}$ both annihilate vectors in the space (2.6) and hence $\hat{A}$ descends to a one-form $A$ on $P$ which defines a connection.

It remains to check that this connection is a groupoid connection. The product map defines a sum on tangent vectors. If $(\xi, \alpha)$ and $(\tilde{\xi}, \tilde{\alpha})$ are tangent to $(\gamma, z)$ and $(\tilde{\gamma}, \tilde{z})$ then $(\xi * \tilde{\xi}, \alpha + \tilde{\alpha})$ is tangent at $(\gamma, z) * (\tilde{\gamma}, \tilde{z})$ where $\xi * \tilde{\xi}$ is the obvious vector field along $\gamma * \tilde{\gamma}$. The essential point in the proof that $A$ preserves the product is that
\[ \int_0^1 f(\gamma', \xi)dt + \int_0^1 f(\tilde{\gamma}', \tilde{\xi})dt = \int_0^1 f((\gamma * \gamma)', \xi * \tilde{\xi})dt. \] (2.10)

It follows from (2.9) that
\[ dA = \pi_2^*(f) - \pi_1^*(f) \] (2.11)
and hence the curvature of this groupoid connection is $f$.

In the next section we are concerned with bundle gerbes which are fibrations whose fibres are $C^\times$ groupoids. Then it may not possible to choose basepoints continously and the constructions above become more interesting.

3. $C^\times$ bundle gerbes

Consider a fibration $\pi: Y \to M$. Define the fibre product $Y^{[2]}$ in the usual way, that is the subset of pairs $(y, y')$ in $Y \times Y$ such that $\pi(y) = \pi(y')$. Notice that the diagonal is inside $Y^{[2]}$ and that the map that transposes elements of $Y^2 = Y \times Y$ fixes $Y^{[2]}$. Denote by $\pi_i$ the restriction of the projection maps on $Y^2$ to $Y^{[2]}$. Denote by $Y^{[2]} \circ Y^{[2]}$ the intersection of $Y^{[2]} \times Y^{[2]}$ with $Y^2 \circ Y^2$. If $P$ and $Q$ are $C^\times$ bundles over $Y^{[2]}$ denote by $P \circ Q$ the restriction to $Y^{[2]} \circ Y^{[2]}$ of the bundle $\pi_1^{-1}(P) \otimes \pi_2^{-1}(Q)$ over $Y^2 \times Y^2$.

A bundle gerbe over $M$ is defined to be a choice of a fibration $Y \to M$ and a $C^\times$ bundle $P \to Y^{[2]}$ with a product, that is, a $C^\times$ bundle isomorphism $P \circ P \to P$ covering $((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3)$. The product is required to be associative whenever triple
products are defined. Just as for $\mathbb{C}^\times$ groupoids a bundle gerbe has an inverse and an identity denoted by the same symbols. Occasionally we shall denote a bundle gerbe as a triple $(P, Y, M)$.

**Example: 3.1** Let $Q \to Y$ be a principal $\mathbb{C}^\times$ bundle. Define $P_{(x,y)} = \text{Aut}_{\mathbb{C}^\times}(Q_x, Q_y) = Q^*_x \otimes Q_y$ Then this defines a bundle gerbe called the trivial bundle gerbe. We also have $P = \text{Aut}(\pi_1^{-1}Q, \pi_2^{-1}Q) = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$.

**Example: 3.2** An example of interest in [1] is to start with a fibration $Y \to M$ with 1 connected fibres and a two-form on $Y$ whose restriction to each fibre is closed and integral. Then we can apply the construction in Section 2 fibre by fibre to define a bundle gerbe $P \to Y^{[2]}$.

A morphism of bundle gerbes $(P, Y, M)$ and $(Q, X, N)$ is a triple of maps $(\alpha, \beta, \gamma)$. The map $\beta: Y \to X$ is required to be a morphism of the fibrations $Y \to M$ and $X \to N$ covering $\gamma: M \to N$. It therefore induces a morphism $\beta^{[2]}$ of the fibrations $Y^{[2]} \to M$ and $X^{[2]} \to N$. The map $\alpha$ is required to be a morphism of $\mathbb{C}^\times$ bundles covering $\beta^{[2]}$ which commutes with the product and hence also with the identity and inverse. A morphism of bundle gerbes over $M$ is a morphism of bundle gerbes for which $M = N$ and $\gamma$ is the identity on $M$.

Various constructions are possible with bundle gerbes. We can define a pull-back and product as follows. If $(Q, X, N)$ is a bundle gerbe and $f: M \to N$ is a map then we can pull back the fibration $X \to N$ to obtain a fibration $f^{-1}(X) \to M$ and a morphism of fibrations $f^{-1}: f^{-1}(X) \to X$ covering $f$. This induces a morphism $(f^{-1}(X))^{[2]} \to X^{[2]}$ and hence we can use this to pull back the $\mathbb{C}^\times$ bundle $Q$ to a $\mathbb{C}^\times$ bundle $f^{-1}(Q)$ say on $f^{-1}(X)$. It is easy to check that $(f^{-1}(Q), f^{-1}(X), M)$ is a bundle gerbe, the pull-back by $f$ of the gerbe $(Q, X, N)$. If $(P, Y, M)$ and $(Q, X, M)$ are bundle gerbes over $M$ then we can form a fibre product $Y \times_M X \to M$ and then form a $\mathbb{C}^\times$ bundle $P \otimes Q$ over $(Y \times_M X)^{[2]}$ which is the product of the bundle gerbes $(P, Y, M)$ and $(Q, X, M)$.

Notice that, unlike the case of $\mathbb{C}^\times$ groupoids, it is not clear that every bundle gerbe is trivial. The proof in Section 2 that a groupoid is the same as a bundle over a pointed set can only be applied fibre by fibre if $Y \to M$ has a section. We shall see in Section 5 that there is associated to a bundle gerbe a class in $H^3(M, \mathbb{Z})$, its Dixmier-Douady class, which is precisely the obstruction to the bundle gerbe being trivial.

4. **Central extensions**

A motivating example of a bundle gerbe is the bundle gerbe arising from a central extension of groups. Let

$$0 \to \mathbb{C}^\times \to \hat{G} \xrightarrow{\rho} G \to 0$$

(4.1)
be a central extension of groups and $Y \to M$ a principal $G$ bundle. Then it may happen that there is a principal $\hat{G}$ bundle $\hat{Y}$ and a bundle map $\hat{Y} \to Y$ commuting with the homomorphism $\hat{G} \to G$. In such a case $Y$ is said to lift to a $\hat{G}$ bundle. One way of answering the question of when $Y$ lifts to a $\hat{G}$ bundle is to present $Y$ with transition functions $g_{\alpha\beta}$ relative to a cover $\{U_\alpha\}$ of $M$. If the cover is sufficiently nice we can lift the $g_{\alpha\beta}$ to maps $\hat{g}_{\alpha\beta}$ taking values in $\hat{G}$ and such that $p(\hat{g}_{\alpha\beta}) = g_{\alpha\beta}$. These are candidate transition functions for a lifted bundle $\hat{Y}$. However they may not satisfy the cocycle condition $\hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\beta\gamma} = 1$ and indeed there is a $\mathbb{C}^\times$ valued map $e_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{C}^\times$. Defined by $\iota(e_{\alpha\beta\gamma}) = \hat{g}_{\alpha\beta}\hat{g}_{\beta\gamma}\hat{g}_{\beta\gamma}$. Because (4.1) is a central extension it follows that $e_{\alpha\beta\gamma}$ is a co-cycle and hence defines a class in $H^2(M, \mathbb{C}^\times)$. It is well-known that the coboundary map in the long exact sequence of cohomology induced by (4.1) defines an isomorphism

$$ H^2(M, \mathbb{C}^\times) \cong H^3(M, \mathbb{Z}). $$

The image of $e_{\alpha\beta\gamma}$ under this coboundary is the class in $H^3(M, \mathbb{Z})$ which is the obstruction to $Y \to M$ lifting to $\hat{G}$.

Another way to see when $Y \to M$ lifts to $\hat{G}$ is to construct a bundle gerbe which is trivial precisely when a lift is possible. To do this define $P \to Y^{[2]}$ by

$$ P_{(x,y)} = \{ h \in \hat{H} \mid xp(h) = y \}. $$

Assume now that $Y$ has a lift to a principal $\hat{G}$ bundle $\hat{Y}$ over $M$ so that there is a projection $q: \hat{Y} \to Y$ commuting with $p$ in the appropriate way. Then $\hat{Y} \to Y$ is a $\mathbb{C}^\times$ bundle over $Y$. Indeed let $g \in P_{(x,y)}$; then it defines a map $\hat{Y}_x \to \hat{Y}_y$ which, by centrality, commutes with the $\mathbb{C}^\times$ action. This defines an isomorphism

$$ P_{(x,y)} \cong Aut_{\mathbb{C}^\times}(\hat{Y}_x, \hat{Y}_y) $$

so that $P \simeq (\pi_1^*\hat{Y})^* \otimes \pi_2^*\hat{Y}$. On the other hand if the bundle gerbe $P$ is trivial, say isomorphic to $Aut(\hat{Y}, \hat{Y})$ for some $\mathbb{C}^\times$ bundle $\hat{Y} \to Y$ it is possible to define an action of $\hat{G}$ on $\hat{Y}$ and make it a lift of $Y$. To do this start with $g$ in $\hat{G}$ and the fibre $\hat{Y}_y$. Then define $x$ by $xp(g) = y$. Then $g \in P_{(x,y)} = Aut_{\mathbb{C}^\times}(\hat{Y}_x, \hat{Y}_x)$ so apply the corresponding automorphism to any element in $\hat{Y}_x$ to define the action of $g$. It can be checked that this defines a lift of $Y$.

This proves that $Y$ lifts to $\hat{G}$ if and only if the gerbe $P$ is trivial. In other words the bundle gerbe $P$ is trivial when the three class defined by $Y$ is zero. We shall see in the next
section that this examples generalises to all bundle gerbes. Every bundle gerbe defines a
degree three class, its Dixmier-Douady class, which is the obstruction to it being trivial.

5. The Dixmier-Douady class of a bundle gerbe.
Let $P \to Y^{[2]}$ be a bundle gerbe. Choose a cover $\{U_\alpha\}$ of $M$ such that over each $U_\alpha$ there is a section $s_\alpha$ of $Y$. Then on the overlap $U_\alpha \cap U_\beta$ we have a map

$$(s_\alpha, s_\beta): U_\alpha \cap U_\beta \to Y^{[2]}$$

(5.1)
defined by $(s_\alpha, s_\beta)(x) = (s_\alpha(x), s_\beta(x))$. Let $P_{\alpha \beta}$ be the pull-back of $P$ via this map. Notice that the product gives an isomorphism $P_{\alpha \beta} \otimes P_{\beta \gamma} \simeq P_{\alpha \gamma}$. Choose sections $\sigma_{\alpha \beta}$ of each $P_{\alpha \beta}$. Then using the product we define a $\mathbb{C}^\times$ valued function $g_{\alpha \beta \gamma}$ defined by

$$\sigma_{\alpha \beta} \sigma_{\beta \gamma} = \sigma_{\alpha \gamma} g_{\alpha \beta \gamma}.$$  

(5.2)

It is easy to check that $g$ defines a class in $H^2(M, \mathbb{C}^\times)$. The image of $g$ under this the isomorphism (4.3) is called the Dixmier-Douady class of the bundle gerbe $P$.

I claim that the Dixmier-Douady class is precisely the obstruction to a gerbe being trivial. To see this note first that if $P$ trivial, say $P = \pi_1^{-1}Q^* \otimes \pi_2^{-2}Q$ for some bundle $Q$ on $Y$ we can define $Q_\alpha = s_\alpha^*(Q)$ and we then have canonical isomorphisms

$$P_{\alpha \beta} = Q_\alpha^* \otimes Q_\beta$$

(5.3)
commuting with products. Hence in the construction of the cocycle in Section 4, if we choose $\delta_\alpha$ to be a section of $Q_\alpha$ and define $\sigma_{\alpha \beta} = (\delta_\alpha)^{-1} \otimes \delta_\beta$ we obtain a trivial cocycle $g$.

If on the other hand $g$ is trivial, say

$$g_{\alpha \beta \gamma} = \rho_{\alpha \beta} \rho_{\beta \gamma} \rho_{\gamma \alpha}$$

(5.5)

where $\rho$ is $\mathbb{C}^\times$ valued, then we can divide $\sigma_{\alpha \beta}$ in equation (5.2) by $\rho_{\alpha \beta}$ and hence without loss of generality assume that $g$ is identically one. Let $Y_\alpha = \pi^{-1}(U_\alpha)$. Define a principal bundle $Q_\alpha$ over $Y_\alpha$ by defining its fibre at $y$ to be

$$(Q_\alpha)_y = P_{(y, s_\alpha(\pi(y)))}.$$  

(5.6)
The $\sigma_{\alpha \beta}$ are elements of

$$P_{(s_\alpha(\pi(y)), s_\beta(\pi(y)))} = P_{s_\alpha(\pi(y)), y}^* \otimes P_{s_\alpha(\pi(y)), y} = (Q_\alpha)_y^* \otimes (Q_\beta)_y.$$  

(5.7)
The $\sigma_{\alpha \beta}$ therefore define automorphisms between $Q_\alpha$ and $Q_\beta$ over $Y_\alpha \cap Y_\beta$. The standard clutching construction now defines a bundle $Q$ over all of $Y$ and it is straightforward to check that this trivialises the gerbe $P$ over $Y$. 


**Example 5.1:** If the fibration $Y \to M$ admits a global section then we can smoothly pick a base point in each fibre. The results of Section 2 can then be applied to show that the bundle gerbe is trivial. It is a trivial consequence of the definition in Section 4 that the Dixmier-Douady class is zero.

The Dixmier-Douady class behaves naturally with respect to the operations on bundle gerbes defined in section 3. The Dixmier-Douady class of a pull-back bundle gerbe is the pull-back of the Dixmier-Douady class and the Dixmier-Douady class of a product of two bundle gerbes is the product (sum) of the Dixmier-Douady classes of the individual bundle gerbes.

6. Connections on bundle gerbes.

Consider a connection $\nabla$ on a bundle gerbe $P \to Y^{[2]}$. Then it induces a connection $\nabla \circ \nabla$ on the bundle $P \circ P \to Y^{[2]} \circ Y^{[2]}$. The connection $\nabla$ is said to be a bundle gerbe connection if the image of $\nabla \circ \nabla$ under the product map is $\nabla$.

It is not clear that bundle gerbe connections exist. To see that they do note that if the bundle gerbe is trivial i.e. $P = Aut(\pi^{-1}_1 Q, \pi^{-1}_2 Q)$ for some principal bundle $Q$ on $Y$ then a connection $\nabla$ on $Q$ defines a bundle gerbe connection $\nabla^* \otimes \nabla$ on $P$. Now choose an open cover $\{U_\alpha\}$ of $M$ over which the fibration $Y \to M$ is trivial. Denote by $Y_\alpha$ the open subset of $Y$ which is the pre-image under the projection of $U_\alpha$. Then the bundle gerbe can be trivialised over $Y_\alpha$ and hence admits a bundle gerbe connection. Choose a partition of unity for the open cover $\{U_\alpha\}$ of $M$. This pulls-back to $Y^{[2]}$ to give a partition of unity for the open cover $\{Y^{[2]}_\alpha\}$ and can be used to patch together the bundle gerbe connections on the various open sets to give a bundle gerbe connection on $P$. It will follow from the results in Section 8 that the space of all bundle gerbe connections is an affine space for the vector space $\Omega^1(Y)/\pi^* (\Omega^1(M))$ of all one-forms on $Y$ modulo one-forms pulled back from $M$.

7. The curvature of a bundle gerbe connection.

A bundle gerbe connection $\nabla$ is a connection so it has a curvature $F_\nabla$ which is a two-form on $Y^{[2]}$. In the case that this is a trivial gerbe and the connection is the tensor product connection then the curvature can be written as

$$F_\nabla = \pi^*_2 f - \pi^*_1 f \quad (7.1)$$

where $f$ is the curvature of the connection on the bundle over $Y$ and the $\pi_i$ are the two projections $\pi_i: Y^{[2]} \to Y$. We shall see in the next section that it is always possible to find an $f$ satisfying equation (7.1). This is certainly true for a bundle gerbe connection constructed by a partition of unity argument as in Section 6. The choice of such an $f$ we will call a curvature for the gerbe connection.
Consider now $df$. In the case that this is a trivial gerbe we have, of course, that $df = 0$. More generally we have $dF = 0$ so that

$$
\pi_1^*df = \pi_2^*df.
$$

(7.2)

I claim that this means that $df = \pi^*(\omega)$ for some three-form $\omega$ on $M$. To see this note that a point of $Y[2]$ is a pair $(x, z)$ where $\pi(x) = \pi(z)$ and a tangent vector to $Y[2]$ is a pair $(X, Z)$ with $X \in T_xY$ and $Z \in T_zY$ and $\pi_*(X) = \pi_*(Z)$. Then equation (7.2) says that

$$
df(x)(X_1, X_2, X_3) = df(z)(Z_1, Z_2, Z_3)
$$

(7.3)

whenever $\pi_*(X_i) = \pi_*(Z_i)$ for $i = 1, 2, 3$. Hence if $m \in M$ and $\xi_i \in T_mM$ choose $x \in Y$ and $X_i \in T_xY$ such that $\pi(x) = m$ and $\pi_*(X_i) = \xi_i$ and define

$$
\omega(m)(\xi_1, \xi_2, \xi_3) = df(x)(X_1, X_2, X_3).
$$

(7.4)

Equation (7.3) shows that this definition is independent of the choice of $x$ and the $X_i$. Clearly $\pi^*(\omega) = df$ and moreover $\omega$ is closed. We call $\omega/(2\pi i)$ the Dixmier-Douady form of the pair $(\nabla, f)$.

It will follow from the results in Section 8 that the various choices in this construction do not change the cohomology class of the Dixmier-Douady form. We shall see in Section 11 that the de Rham cohomology class of the Dixmier-Douady form is the image in real cohomology of the Dixmier-Douady class defined in Section 5. This proves in particular that the Dixmier-Douady form is integral a fact that also follows from the discussion of holonomy in Section 12.

8. A complex with no cohomology

For a fibration $Y$ let $Y[p]$ denote the $p$th fibered product. There are projection maps $\pi_i: Y[p] \to Y[p-1]$ which omit the $i$th element for each $i = 1 \ldots p$. These define a map

$$
\delta: \Omega^q(Y[p-1]) \to \Omega^q(Y[p])
$$

(8.1)

by

$$
\delta(\omega) = \sum_{i=1}^{p} (-1)^i \pi_i^*(\omega).
$$

(8.2)

Clearly $\delta^2 = 0$ so that $\Omega^q(Y[*])$ is a complex. We wish to show that this complex has no cohomology. This will then settle the question of the existence of $f$ in Section 8 as we have $F_\nabla \in \Omega^2(Y[2])$ with $\delta(F_\nabla) = 0$ and hence $F_\nabla = \delta(f)$ for some $f \in \Omega^2(Y[1]) = \Omega^2(Y)$.

Consider first the case that the fibration is trivial, say $Y = M \times F$. The general case will follow by a partition of unity argument. In this case $Y[p] = M \times F^p$. Because the
notation is cumbersome at this point it will be convenient to denote a collection of \( q \) vectors 
\((X^1, \ldots, X^q)\) just by \( X \) and the action of a q form \( \tau \) on these vectors by \( \tau(X) \) rather than 
\( \tau(X^1, \ldots, X^q) \). When we are dealing with vectors tangent to \( F^{p+1} \) at \( f = (f_1, \ldots, f_{p+1}) \) 
then each of the \( X \) is a collection of vectors \( (X_1, \ldots X_{p+1}) \) where each \( X_j \) is a \( q \)-tuple of 
vectors in \( T_j(F) \). So, with these notational conventions we have for \( \omega \in \Omega^q(Y^{[p]}) \)
\[
\delta(\omega)(m, f)(\xi, (X_1, \ldots, X_{p+1})) = \sum_i (-1)^i \omega(m, f_1, \ldots, \hat{f}_i, \ldots, f_p)((\xi, (X_1, \ldots, \hat{X}_i, \ldots, X_{p+1})))
\]  
(8.3)

where \( \xi \) is a \( q \)-tuple of vector tangent to \( M \) at \( m \). Now fix a point \( f \) in \( F \) and a \( q \) tuple of 
vectors \( X \) in the tangent space at \( f \) and define \( \rho \in \Omega^q(Y^{[p-1]}) \) by
\[
\rho(m, f_1, \ldots, f_p)(\xi, X_1, \ldots, X_p) = \omega(m, f_1, \ldots, f_p, f)(\xi, X_1, \ldots, X_p, X).
\]  
(8.4)

It follows from the fact that \( \delta(\omega) = 0 \) that \( \delta(\rho) = (-1)^{p+1}\omega \). This proves the required result in the case that \( Y \) is the trivial fibration. The general case is proved by choosing an open cover \( U_\alpha \) such that \( Y \) is trivial over each \( U_\alpha \). Then let \( \psi_\alpha \) be a partition of unity subordinate to that cover. Let \( Y_\alpha \) by the part of \( Y \) sitting over \( U_\alpha \) and similarly for \( (Y^{[p]})_\alpha \). 

Note that \( (Y^{[p]})_\alpha = (Y_\alpha)^{[p]} \). There are projection maps for each \( Y^{[p]} \rightarrow M \) and we can pull the partition of unity back to any of these spaces. We will denote it by the same symbol. If we start with \( \omega \) in \( \Omega^q(Y^{[p]}) \) then \( \omega|_{U_\alpha} = \delta(\rho_\alpha) \) for some \( \rho_\alpha \). Hence we have
\[
\omega = \sum_\alpha \psi_\alpha \delta(\rho_\alpha) = \delta(\sum_\alpha \psi_\alpha \rho_\alpha) = \delta(\rho) \]  
(8.5)

where \( \rho = \sum_\alpha \psi_\alpha \rho \).

At \( p = 1 \) we define \( Y^{[0]} = M \) and let \( \delta: \Omega^q(M) \rightarrow \Omega^q(Y) \) be pull-back under \( \pi \). 

Exactness follows exactly as for the proof, in Section 7, that there exists an \( \omega \) such that \( \pi^*(\omega) = df \).

We can now confirm two facts stated earlier. The first is the affine space structure of the space of all bundle gerbe connections. If \( \nabla \) and \( \nabla' \) are two bundle gerbe connections they clearly differ by a one-form \( \eta \) on \( Y^{[2]} \) with \( \delta(\eta) = 0 \). So \( \eta = \delta(\mu) \). On the other hand \( \nabla + \delta(\mu) \) is a bundle gerbe connection for any \( \mu \) on \( Y \) and \( \delta(\mu) = 0 \) for such a \( \mu \) precisely when \( \mu \) is pulled back from \( M \). This gives the required result. The second fact is the independence of the class of the Dixmier-Douady form from various choices. The first is the choice of \( f \) satisfying \( \delta(f) = F\nabla \). If \( f' \) is another such then \( \delta(f - f') = 0 \) and hence \( f - f' = \pi^*(\rho) \) so that \( df - df' = \pi^*(\rho_0) \) and \( \omega - \omega' = d\rho \). The other choice is
the choice of bundle gerbe connection. If we have \( \nabla' = \nabla + \delta(\mu) \) then we can choose \( f' \) so that \( f = f' + d\mu \) and hence \( df = df' \) so that \( \omega = \omega' \).

9. Gerbes, connective structures and curvings

The relationship with the theory of gerbes discussed in [1] is as follows. For any open set \( U \subset M \) let \( C(U) \) be the set of all sections of \( Y \) which we want to think of as the objects in a category, in fact in a groupoid. If \( s \) and \( t \) are two such sections they define a section \((s, t)\) of \( Y[2] \) over \( U \) by \( m \mapsto (s(m), t(m)) \). The morphisms from \( s \) to \( t \) we define to be the sections of the bundle \((s, t)^{-1}P \) over \( U \). The composition is constructed from the composition on \( P \). This construction defines a pre-sheaf of groupoids. The sheafification of this presheaf gives rise to a gerbe in the sense of Brylinski [1].

In [1] Brylinski introduces the notion of connective structure and curving. We indicate here how these are related to the bundle gerbe connection and its curvature. Let \( \nabla \) be a bundle gerbe connection for the bundle gerbe \( P \) over \( Y \to M \). Let \( U \) be an open subset of \( M \) over which \( Y \) admits a section \( s: U \to Y \). Denote by \( \hat{s} \) the induced map \( Y|_U \to (Y|_U)[2] \) defined by \( y \mapsto (y, s(\pi(y))) \). Then we have an isomorphism \( P \simeq \hat{s}^{-1}P \otimes \hat{s}^{-1}P^* \) defined by the product

\[
P_{(p,q)} \to P^*_{(s(\pi(p)),p)} \otimes P_{(s(\pi(p)),q)}
\] (9.1)

which trivialises \( P \) over \( Y|_U \to U \). Consider the set \( Co(s) \) of all connections \( A \) on the bundle \( \hat{s}^{-1}P \) such that \( \delta(A) = \nabla \). This space of connections is an affine space for \( \Omega^1_U \) the space of all 1-forms on \( U \) and hence \( Co(s) \) defines a \( \Omega^1_U \) torsor. This torsor is a connective structure in the sense of Brylinski.

Assume now that we have chosen a two-form \( f \) on \( Y \) such that \( \delta(f) = F_\nabla \) where \( F_\nabla \) is the curvature of \( \nabla \). Then to any \( A \) in \( Co(P) \) we can define a two-form \( K(A) \) on \( U \) by

\[
\pi^*(K(A)) = F_A - f
\] (9.2)

where \( F_A \) is the curvature of \( A \). This equation makes sense because

\[
\delta(F_A - f) = F_{\hat{s}(A)} - \delta(f) = F_\nabla - F_\nabla = 0.
\] (9.3)

Finally notice that \( \pi^*(dK(A)) = -df \) so that \( dK(A) = -\omega \) so that, up to sign, this is the curvature of the bundle gerbe.

The definition of morphism of bundle gerbes on \( M \) in section 3 naturally gives rise to a notion of isomorphism. A fundamental result about gerbes is the theorem that the Dixmier-Douady class gives an exact correspondence between elements of \( H^3(M, \mathbb{Z}) \) and equivalence classes of gerbes [1]. This is not true for bundle gerbes on \( M \) and bundle gerbe isomorphism; there are bundle gerbes on \( M \) which are not isomorphic but which have the
same Dixmier-Douady class and hence define equivalent gerbes. Indeed it is not hard to show that if \((\alpha, \beta, 1_M)\) is a morphism of bundle gerbes \((P, Y, M)\) and \((Q, X, M)\) then the Dixmier-Douady classes of \((P, Y, M)\) and \((Q, X, M)\) are the same. In this example \(X\) and \(Y\) can be quite different. For example if \(Y\) admits a global section we can take \(X\) to be the image of that section and \(Q\) the restriction of \(P\) to \(X\). A bundle gerbe where the fibers are points clearly has Dixmier-Douady class zero and we have already seen that a bundle gerbe where the fibration has a section also has Dixmier-Douady class zero. This dependence of bundle gerbes on the choice of a fibration is nicely eliminated by the gerbe concept.

10. The tautological bundle gerbe

Let \(\omega/(2\pi i)\) be a form representing a class in \(H^3(M, \mathbb{Z})\) where \(M\) is 2 connected. We shall show how to construct a fibration of groupoids with \(\omega\) as its curvature. Recall that if \(\Sigma\) is an oriented two sphere in \(M\) the Wess Zumino Witten action is an element of \(C^\times\) associated to \(\Sigma\) by extending \(\Sigma\) to a ball \(B\) in \(M\) and defining

\[
\text{wzw}(\Sigma) = \exp\left(\int_B \omega\right).
\]  

(10.1)

Similarly if \(\Sigma\) and \(\Sigma'\) are two disks in \(M\) with common boundary denote by \(\text{wzw}(\Sigma, \Sigma')\) the Wess Zumino Witten action of the sphere formed by their union if it is given the orientation of the first disk.

Fix a base point for \(M\) and let \(Y \to M\) be the path-fibration. Then \(Y^{[2]}\) consists of all pairs of paths beginning at the basepoint and with the same endpoints. Define the fibre of \(P\) at such a point by taking all pairs consisting of a piecewise smooth surface with these two paths as boundary and a non-zero complex number and defining an equivalence relation

\[
(\Sigma, z) \sim (\Sigma', z')
\]  

if \(z = \text{wzw}(\Sigma, \Sigma')z'\). Denote equivalence classes by square-brackets. Then the set of all equivalence classes forms a principal \(C\times\) bundle over \(Y\). We need to show that it is a bundle gerbe by constructing a product.

The product map \(P_{(x,y)} \otimes P_{(y,z)} \to P_{(x,z)}\) is defined by

\[
[\Sigma, z] \otimes [\Sigma', z'] \to [\Sigma \cup \Sigma', zz'].
\]  

(10.2)

This makes sense because \(\Sigma\) and \(\Sigma'\) have half of each of their boundaries (the curve \(y\) ) in common.

We now show that this bundle gerbe has a bundle gerbe connection whose Dixmier-Douady form is \(\omega/(2\pi i)\). We could perform calculations analogous to those in Section 2.
however it is simpler to actually use those calculations as follows. Consider the evaluation map

\[ ev: Y \times [0, 1] \rightarrow M \]  \hspace{1cm} (10.3)

and use it to define a closed two-form \( f = \int_0^1 ev^*(\omega) \). Note that \( f/(2\pi i) \) is integral.

We can now repeat the constructions in Section 2 but restrict them to \( Y[2] \circ Y[2] \subset Y[2] \times Y[2] \). This defines the bundle gerbe with connection \( \nabla \) and curvature \( F_\nabla = \pi_2^*(f) - \pi_1^*(f) \). It is now an easy calculation to show that if \( \pi: Y \rightarrow M \) then \( df = \pi^*(\omega) \) as required.

11. Deligne cohomology

The Deligne cohomology of \( M \) that we are interested in is the total cohomology of the log-complex

\[ 0 \rightarrow C^\infty(M)^\times \rightarrow \Omega^1(M) \rightarrow \ldots \rightarrow \Omega^p(M) \rightarrow 0. \]  \hspace{1cm} (11.1)

Here the first non-zero map is the exterior derivative of the log or \( f \mapsto df/f \). If \( p = 1 \) then the elements of \( H^1 \) of this total cohomology are represented in Cech cohomology with respect to an open cover by pairs \( (A_\alpha, \sigma_{\alpha\beta}) \) subject to the condition that

\[ A_\alpha - A_\beta = \sigma_{\alpha\beta}^{-1} d\sigma_{\alpha\beta}. \]  \hspace{1cm} (11.2)

It is not hard to show that the elements of this cohomology are equivalence classes of \( \mathbb{C}^\times \) bundles with connection.

We shall show that in the case \( p = 2 \) that we can manufacture a class in this total cohomology from a gerbe with connection and curvature. A class in this cohomology will be a triple

\[ (f_\alpha, A_{\alpha\beta}, g_{\alpha\beta\gamma}). \]  \hspace{1cm} (11.3)

These have to satisfy

\[ A_{\alpha\beta} - A_{\beta\gamma} + A_{\gamma\alpha} = g_{\alpha\beta\gamma}^{-1} dg_{\alpha\beta\gamma} \]  \hspace{1cm} (11.4)

and

\[ f_\alpha - f_\beta = dA_{\alpha\beta}. \]  \hspace{1cm} (11.5)

We have already a candidate for \( g_{\alpha\beta\gamma} \). For the other two we first let \( \nabla_{\alpha\beta} \) be the pull-back connection

\[ \nabla_{\alpha\beta} = (s_\alpha, s_\beta)^* \nabla \]  \hspace{1cm} (11.6)

and then define

\[ A_{\alpha\beta} = \sigma_{\alpha\beta}^*(\nabla_{\alpha\beta}). \]  \hspace{1cm} (11.7)
We also define
\[ f_\alpha = s_\alpha^* f. \] (11.8)

The first relation follows from the fact that a bundle gerbe connection preserves the product. So
\[ \nabla_{\alpha \beta} \otimes \nabla_{\beta \gamma} = \nabla_{\alpha \gamma}. \] (11.9)

But the pull-back of \( \nabla_{\alpha \beta} \otimes \nabla_{\beta \gamma} \) with \( \sigma_{\alpha \beta} \otimes \sigma_{\beta \gamma} \) is
\[ A_{\alpha \beta} + A_{\beta \gamma}. \] (11.10)

On the other hand this is also the pull-back of \( A_{\alpha \gamma} \) with \( \sigma_{\alpha \gamma} \) or
\[ A_{\alpha \gamma} + g_{\alpha \beta \gamma}^{-1} d g_{\alpha \beta \gamma} \] (11.11)
and the result follows.

The second relation follows from the equation
\[ F = \pi_1^* f - \pi_2^* f \] (11.12)
by pulling-back both sides with \((s_\alpha, s_\beta)\). On the LHS we get \( d A_{\alpha \beta} \) and on the RHS \( f_\alpha - f_\beta \).

It now follows readily that \( df_\alpha \) is just the restriction of \( \omega \) to \( U_\alpha \) and moreover by standard double complex arguments it also follows that the class defined by \( \omega/(2\pi i) \) is the same as the class defined by \( g_{\alpha \beta \gamma} \). So the class defined by the Dixmier-Douady form is the image in \( H^3(M, \mathbb{R}) \) of the Dixmier-Douady class in \( H^3(M, \mathbb{Z}) \).

**12. Holonomy of a bundle gerbe connection over a two-sphere**

If we calculate the Deligne cohomology for a bundle gerbe with connection and curvature whose base-manifold is a two-sphere we can show that it is \( U(1) \). The resulting number is a generalisation of holonomy. A simple way of understanding what this is is to first consider the case that the base manifold is not a two-sphere but in fact has \( \pi_2(M) = 0 \). In that case consider an embedded two-sphere \( \Sigma \) in \( M \) which is the boundary of some ball \( B \) in \( \Sigma \). If \( \omega/(2\pi i) \) is the Dixmier-Douady class then
\[ \text{wzw}(\Sigma) = \exp(\int_B \omega) \] (12.1)
is the holonomy of the bundle gerbe connection over \( \Sigma \). In general this is not a satisfactory solution as it analogous to it is like defining holonomy for a connection on a \( C^\infty \) bundle by integrating curvature over spanning disks. In general we want an answer intrinsic to the two-sphere in question.
To define such the holonomy choose a point in the two-sphere $\Sigma$ and think of it as a disk with boundary identified. It is possible to lift this disk to a disk $D$ in $Y$ whose boundary lies entirely in one fibre. If we fix a point $y$ in this fibre we define a map of the fibre into $Y^{[2]}$ by $y' \mapsto (y', y)$. We can then calculate the holonomy of the connection on $P$ around the image of the boundary of $D$ under this map. Call this $\text{hol}(\partial D, \nabla)$. In addition we can integrate $f$ over $D$ and form

$$\text{hol}(\partial D, \nabla)^{-1} \exp(\int_D f).$$  \tag{12.2}$$

We need to show that this is independent of various choices. Let us leave the base point fixed for a moment and consider a second lift $D'$. Then we can define a disk $\tilde{D}$ inside $Y^{[2]}$ by

$$\tilde{D} = \{(x, y) \mid x \in D, y \in D'\}. \tag{12.3}$$

From the elementary properties of holonomy we have

$$1 = \text{hol}(\partial \tilde{D}, \nabla)^{-1} \exp(\int_{\tilde{D}} F_{\nabla}) \tag{12.4}$$

and from the equation satisfied by $f$, (7.1) we conclude that

$$\text{hol}(\partial D, \nabla)^{-1} \exp(\int_D f) = \text{hol}(\partial D', \nabla)^{-1} \exp(\int_{D'} f). \tag{12.5}$$

To show that the holonomy is independent of the base point is a similar type of calculation but more involved. Let $D_1$ and $D_2$ be two lifts of $\Sigma$ with different basepoints. Let $D_3$ denote the map of the cylinder into $Y$ which covers $\Sigma$ with the two basepoints removed and at each basepoint is coincides with either the boundary of $D_1$ of $D_2$. Now by considering each pair of $D_i$ respectively we define subsets of $Y^{[2]}$ by

$$D_{ij} = \{(x, y) \mid x \in D_i, y \in D_j\}. \tag{12.6}$$

Notice that topologically the union of these three cylinders is a cylinder with each end in the diagonal inside $Y^{[2]}$. On the diagonal the connection $\nabla$ is flat so it follows that the integral of curvature $F$ over the union of the $D_{ij}$ is $2\pi i$ times an integer. Expanding out this integral as before gives the required result.

It is straightforward to check that if $B$ is a three ball in $M$ then we have

$$\text{hol}(\partial B, \nabla) = \exp(\int_B \omega). \tag{12.7}$$

We can use (12.7) to prove that the Dixmier-Douady form is integral. Indeed if $X \subset M$ is any three dimensional submanifold of $M$ consider a family of three balls $B_r$ inside $X$
shrinking to a point as \( r \to 0 \). Then the integral of \( \omega \) over \( X - B_r \) is the holonomy over the boundary of \( B_r \) but as \( B_r \) shrinks to a point this has limit 1 and hence the exponential of the integral of \( \omega \) over all of \( X \) is 1. So the Dixmier-Douady form, \( \omega/(2\pi i) \), is integral.

13. Topology of bundle gerbes

So far we have ignored any properties that the fibering \( Y \to M \) must satisfy for there to be a non-trivial bundle gerbe \( P \to Y \). However the examples we have considered all have \( Y \to M \) having infinite dimensional fibres and we shall show now that this is a necessary condition. We use the result from [3] that if \( Y \to M \) is does not have infinite dimensional fibres then any smooth choice of closed \( p \)-form on the fibres is the restriction of a closed \( p \)-form on \( Y \). If this is true then consider the two-form \( f \) on \( Y \). Its restriction to each fibre is closed and hence by the theorem in [3] there exists a closed two-form \( \rho \) on \( Y \) such that \( f - \rho \) is vertical. But \( f - \rho \) and \( d(f - \rho) \) are both vertical so we can find a two-form \( \mu \) on \( M \) such that \( \pi^*(\mu) = f - \rho \). Finally \( \pi^*(\omega) = df = df - d\rho = \pi^*(d\mu) \) so that \( \omega - d\mu \) and the bundle gerbe has trivial Dixmier-Douady class.

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