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Positive Solutions for a Class of Integral Boundary Value Problem of Fractional $q$-Difference Equations

Shugui Kang 1, Yunfang Zhang 2, Huiqin Chen 1 and Wenying Feng 3,4,*

1 School of Mathematics and Statistics, Shanxi Datong University, Datong 037009, China
2 School of Mathematics and Computer Science, Shanxi Normal University, Taiyuan 030032, China
3 Department of Mathematics, Trent University, Peterborough, ON K9L 0G2, Canada
4 Department of Computer Science, Trent University, Peterborough, ON K9L 0G2, Canada
* Correspondence: wfeng@trentu.ca

Abstract: This paper studies a class of integral boundary value problem of fractional $q$-difference equations. We first give an explicit expression for the associated Green’s function and obtain an important property of the function. The new property allows us to prove sufficient conditions for the existence of positive solutions based on the associated parameter. The results are derived from the application of a fixed point theorem on order intervals.

Keywords: $q$-derivative; fractional $q$-difference equation; Green’s function; integral boundary value problem; fixed point theorem; positive solution

1. Introduction

It is known that fractional differential equations and their discrete analogues, fractional difference equations, have broad applications in many areas such as neural computing [1], transportation modelling [2], and dynamical systems [3,4]. Compared to the traditional integer ordered calculus, fractional ordered derivatives have less requirements for the smoothness of the functions [5]. This property provides some advantages in modeling computational algorithms and other applications [6,7]. From the perspectives of numerical analysis, fractional integral equations can be solved via various neural networks [8].

The concept of quantum calculus ($q$-calculus) was first developed by Jackson [9,10]. The theory has advanced applications in mathematical physics, impulsive waves, and signal analysis [11–15]. Fractional $q$-difference calculus is the counterpart of fractional integral and difference operators [16,17]. For a comprehensive introduction to this subject, we refer to the book [12].

The study of positive solutions for Boundary Value Problems (BVPs) is paramount due to its significant applications. For example, positive solutions of a BVP arising from chemical reactor theory [18] represent the temperature during the reaction. In the literature, existence of positive solutions for differential equations (DE) and fractional differential equations (FDE) have been extensively studied [19,20]. However, results for $q$-difference fractional boundary value problems are relatively rare [21,22].

The following nonlinear $q$-fractional BVP was considered in [21]:

$$\begin{align*}
(D_q^s y)(t) &= -f(x, y(x)), & 0 < x < 1, \\
y(0) &= D_q y(0) = 0, & D_q y(1) = \beta \geq 0.
\end{align*}$$

A common approach to prove the existence of positive solutions for BVPs is to convert the problem to an integral equation and then apply fixed point theorems. Results of [21] on sufficient conditions for the existence of positive solutions were obtained following this idea. The well-known Guo–Krasnoselskii fixed point theorem was applied.
Later, as a special case of the boundary conditions for BVP (1), when $\beta = 0$, the following parameter dependent fractional-difference BVP was examined in [23]:

$$\begin{cases}
(D_\alpha^q u)(x) + \lambda h(x)f(u(x)) = 0, & 0 < x < 1, \\
u(0) = D_q u(0) = D_q u(1) = 0.
\end{cases}$$

(BVP (2) is equivalent to a class of time-independent fractional-difference Schrödinger equations that simulate the evolution of the slowly varying amplitude of a nonlinear wave [23]. Again, applying the Guo–Krasnoselskii fixed point theorem, the existence of positive solutions depending on the values of the parameter $\lambda$ were obtained.

Recently, the uniqueness of solutions for BVP involving a parameter and the integral operator was obtained in [24]:

$$\begin{cases}
(D_\alpha^q u)(t) + f(t, u(t)) = 0, & t \in (0, 1), \\
u(0) = D_q u(0) = 0, & D_q u(1) = \lambda \int_0^1 u(s)d_q s.
\end{cases}$$

The approach of [24] relies on the iterative technique and a fixed point theorem that ensures uniqueness [25].

Motivated by the above studies, in this paper, we consider the fractional $q$-difference equation associated with a parameter and subject to integral boundary conditions:

$$\begin{cases}
(D_\alpha^q u)(t) + \lambda f(u(t)) = 0, & 0 < t < 1, \\
u(0) = D_q u(0) = 0, & D_q u(1) = [\alpha - 1]q \int_0^1 u(s)d_q s,
\end{cases}$$

where the parameters $\lambda > 0$, $2 < \alpha \leq 3$, and $f : [0, +\infty) \to [0, +\infty)$ is a continuous function. $D_q^\alpha$ denotes the $q$-fractional differential operator of order $\alpha$.

The same as the work of [21,23,24], we will use the Riemann–Liouville type fractional derivative [26]. Different from methods applied in [21,23,24], we utilize a relatively new fixed point theorem on order intervals [27] that provides more information on the solutions.

In Section 2, we present some definitions and lemmas for fractional $q$-derivative and fractional $q$-integral. The Green’s function of the fractional integral BVP is obtained. Its properties are proved. In Section 3, new sufficient conditions for the existence of positive solutions are proved.

### 2. Preliminaries on $q$-Calculus

The following notations and definitions can be found in [21,28]. Suppose $q \in (0, 1)$ and define

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in R.$$  

If $a \in R$, then

$$(a - b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}.$$  

The $q$-gamma function is defined as follows:

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in R \setminus \{0, -1, -2, \ldots\}.$$  

The $q$-derivative of a function $f$ is defined by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$
The following four formulas will be used in the sequel:

\[ [a(t-s)]^{(a)} = a^a (t-s)^{(a)}, \]

\[ (a - b)^{(a)} = (a - bq^{-1}) (a - b)^{(a-1)}, \]

\[ tD_q (t-s)^{(a)} = [\alpha]_q (t-s)^{(a-1)}, \]

\[ \left( tD_q \int_0^t f(t,s) dq s \right)(t) = \int_0^t tD_q f(t,s) dq s + f(qt,t), \]

where \( tD_q \) denotes the derivative with respect to variable \( t \).

**Remark 1** ([22]). If \( \alpha > 0 \) and \( a \leq b \leq t \), then \( (t-a)^{(a)} \geq (t-b)^{(a)} \).

**Definition 1** ([17]). Assume that \( \alpha \geq 0 \) and \( f \) is a function defined on \([0,1]\). The fractional \( q \)-integral of the Riemann–Liouville type is defined by

\[ (I_q^a f)(t) = \frac{1}{\Gamma(a)} \int_0^t (t-qs)^{(a-1)} f(s) dq s, \quad \alpha > 0, \quad t \in [0,1]. \]

**Definition 2** ([14]). The fractional \( q \)-derivative of the Riemann–Liouville type of \( f \) is defined by

\[ (D_q^a f)(t) = (D_q^{a-p} I_q^{p} f)(t), \quad \alpha > 0, \]

where \( m \) is the smallest integer greater than or equal to \( \alpha \).

**Lemma 1** ([11,14]). Let \( \alpha, \beta \geq 0 \) and \( f \) be a function defined on \([0,1]\). We have the following formulas:

\[ (I_q^a I_q^b f)(t) = (I_q^{a+b} f)(t), \]

\[ (D_q^a I_q^b f)(t) = f(t). \]

**Lemma 2** ([22]). Assuming that \( \alpha > 0 \) and \( p \) is a positive integer, then

\[ (I_q^a D_q^p f)(t) = (D_q^{a-p} I_q^p f)(t) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} (D_q^{k} f)(0). \]

**Lemma 3.** Assume that parameter \( \lambda > 0, 2 < \alpha \leq 3, f : [0, +\infty) \rightarrow [0, +\infty) \) is a continuous function. Then, the solution of integral boundary value problem

\[
\begin{cases}
(D_q^a u)(t) + \lambda f(u(t)) = 0, & 0 < t < 1, \\
u(0) = D_q u(0) = 0, & D_q u(1) = [a-1]_q \int_0^1 u(s) dq s
\end{cases}
\]

is equivalent to the solution of

\[ u(t) = \lambda \int_0^1 G(t,q s) f(u(s)) dq s, \quad t \in [0,1], \]

where

\[ G(t,q s) = \begin{cases}
\frac{t^{a-1}(1-q s)^{(a-2)} ([\alpha]_q - (1 - sq^n)(1 - sq^{n-1})) - ([\alpha]_q - 1)(t - q s)^{(a-1)}}{([\alpha]_q - 1)\Gamma_q(\alpha)}, & 0 \leq q s \leq t \leq 1; \\
\frac{t^{a-1}(1-q s)^{(a-2)} ([\alpha]_q - (1 - sq^n)(1 - sq^{n-1}))}{([\alpha]_q - 1)\Gamma_q(\alpha)}, & 0 \leq t \leq q s \leq 1.
\end{cases} \]
**Proof.** In view of Definition 2 and Lemma 1, we obtain that

\[ (D_q^a u)(t) + \lambda f(u(t)) = 0 \Rightarrow (I_q^a D_q^3)^{-a} u(t) = -\lambda I_q^a f(u(t)). \]

It follows from Lemma 2 that the solution \( u \) of (9) is given by

\[ u(t) = c_1 t^{a-1} + c_2 t^{a-2} + c_3 t^{a-3} - \frac{\lambda}{\Gamma_q(a)} \int_0^t (t - q s)^{(a-1)} f(u(s)) d_q s, \]

where \( c_1, c_2, c_3 \in \mathbb{R} \). Since \( u(0) = 0 \), we obtain \( c_3 = 0 \).

Taking the derivative of both sides of Equation (10), with the help of (7) and (8), we obtain

\[ (D_q u)(t) = [\alpha - 1]_q c_1 t^{a-2} + [\alpha - 2]_q c_2 t^{a-3} - \frac{\lambda}{\Gamma_q(a)} \int_0^t [\alpha - 1]_q (t - q s)^{(a-2)} f(u(s)) d_q s. \]

Using the boundary conditions of \( D_q u(0) = 0 \) and \( D_q u(1) = [\alpha - 1]_q \int_0^1 u(s) d_q s \), we have \( c_2 = 0 \) and

\[ c_1 = -\frac{\lambda}{\Gamma_q(a)} \int_0^1 (1 - q s)^{(a-2)} f(u(s)) d_q s + \int_0^1 u(s) d_q s. \]

Thus, \( u \) can be calculated as

\[ u(t) = -\frac{\lambda}{\Gamma_q(a)} \int_0^t (t - q s)^{(a-1)} f(u(s)) d_q s + \frac{\lambda t^{a-1}}{\Gamma_q(a)} \int_0^1 (1 - q s)^{(a-2)} f(u(s)) d_q s + \int_0^1 u(s) d_q s. \]

Let \( C = \int_0^1 u(s) d_q s \). Integrating (11) with respect to \( t \) from 0 to 1, we obtain

\[ C = -\frac{\lambda}{\Gamma_q(a)} \int_0^1 d_q t \int_0^t (t - q s)^{(a-1)} f(u(s)) d_q s + \frac{\lambda}{\Gamma_q(a)} \int_0^1 t^{a-1} d_q t \int_0^1 (1 - q s)^{(a-2)} f(u(s)) d_q s + C \int_0^1 t^{a-1} d_q t. \]

By using equations \( \int_0^1 t^{a-1} d_q t = \frac{1}{[\alpha]_q} \) and

\[ \int_0^1 d_q t \int_0^t (t - q s)^{(a-1)} f(u(s)) d_q s = \frac{1}{[\alpha]_q} \int_0^1 (1 - q s)^{(a)} f(u(s)) d_q s, \]

we obtain

\[ C = -\frac{\lambda}{(\alpha)_q - 1} \int_0^1 (1 - q s)^{(a)} f(u(s)) d_q s + \frac{\lambda}{(\alpha)_q - 1} \int_0^1 (1 - q s)^{(a-2)} f(u(s)) d_q s. \]

In view of (6), substituting \( C \) into the formula (11), we deduce that
\[ u(t) = -\frac{\lambda}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} f(u(s)) ds + \frac{\lambda t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-2)} f(u(s)) ds \\
- \frac{([\alpha]_q - 1) t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-2)} f(u(s)) ds \\
+ \frac{([\alpha]_q - 1) t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-2)} ([\alpha]_q - (1 - sq^a)(1 - sq^{a-1})) f(u(s)) ds \\
= \lambda \int_0^t G(t,qs) f(u(s)) ds. \]

\[ \Box \]

**Lemma 4.** The above function \( G \) satisfies

(i) \( G(t,qs) \geq 0, \; t, s \in [0,1]; \)

(ii) \( t^{\alpha-1}G(1,qs) \leq G(t,qs) \leq G(1,qs), \; t, s \in [0,1]. \)

**Proof.** We first define two functions

\[ g_1(t,qs) = t^{\alpha-1}(1 - qs)^{(\alpha-2)} ([\alpha]_q - (1 - sq^a)(1 - sq^{a-1})) - ([\alpha]_q - 1)(t - qs)^{(\alpha-1)}, \]

\[ g_2(t,qs) = t^{\alpha-1}(1 - qs)^{(\alpha-2)} ([\alpha]_q - (1 - sq^a)(1 - sq^{a-1})). \]

In view of Remark 1, we have

\[ g_1(t,qs) = t^{\alpha-1}(1 - qs)^{(\alpha-2)} \left\{ ([\alpha]_q - 1) + [1 - (1 - sq^a)(1 - sq^{a-1})] \right\} - ([\alpha]_q - 1)(t - qs)^{(\alpha-1)} \]

\[ = t^{\alpha-1}(1 - qs)^{(\alpha-2)} ([\alpha]_q - 1) + [1 - (1 - sq^a)(1 - sq^{a-1})] - t^{\alpha-1}([\alpha]_q - 1)(1 - \frac{qs}{t})^{(\alpha-1)} \]

\[ \geq t^{\alpha-1}(1 - qs)^{(\alpha-2)} ([\alpha]_q - 1) + [1 - (1 - sq^a)(1 - sq^{a-1})] - t^{\alpha-1}([\alpha]_q - 1)(1 - qs)^{(\alpha-1)} \]

\[ = t^{\alpha-1}([\alpha]_q - 1)(1 - (1 - sq^a)(1 - sq^{a-1})) - (1 - qs)^{(\alpha-1)} \]

\[ + t^{\alpha-1}(1 - qs)^{(\alpha-2)}[1 - (1 - sq^a)(1 - sq^{a-1})] \geq 0. \]

Notice that \([\alpha]_q - 1 > 0\), thus

\[ g_2(t,qs) = t^{\alpha-1}(1 - qs)^{(\alpha-2)} \left\{ ([\alpha]_q - 1) + [1 - (1 - sq^a)(1 - sq^{a-1})] \right\} \]

\[ \geq t^{\alpha-1}(1 - qs)^{(\alpha-2)} [1 - (1 - sq^a)(1 - sq^{a-1})] \]

\[ \geq 0. \]

Therefore, \( G(t,qs) \geq 0. \) In addition, for fixed \( t \in [0,1], \)
\[ i D_q g_1(t, qs) = [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(\alpha - 1) q(t -qs)^{(a - 2)} \]
\[ = [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(\alpha - 1) q t^{\alpha - 2} (1 -qs^\alpha T)^{(a - 2)} \]
\[ \geq [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} \left\{ ([\alpha] q - 1) + [1 - (1 -sq^\alpha)(1 -sq^{a - 1})] \right\} \]
\[ - [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} ([\alpha] q - 1) \]
\[ = [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} [1 - (1 -sq^\alpha)(1 -sq^{a - 1})] \geq 0. \]

\[ i D_q g_2(t, qs) = [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} \left\{ ([\alpha] q - 1) + [1 - (1 -sq^\alpha)(1 -sq^{a - 1})] \right\} \]
\[ \geq [\alpha - 1] q t^{\alpha - 2} (1 -qs)^{(a - 2)} [1 - (1 -sq^\alpha)(1 -sq^{a - 1})] \]
\[ \geq 0. \]

i.e., \(g_1(t, qs)\) and \(g_2(t, qs)\) are increasing functions of \(t\), so \(G(t, qs)\) is an increasing function to \(t\) for fixed \(s \in [0, 1]\). If \(t \geq qs\), then

\[ G(t, qs) = \frac{t^a - 1 (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(t -qs)^{(a - 1)}}{(1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(1 -qs)^{(a - 1)}} \]
\[ \geq \frac{t^a - 1 (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(1 -qs)^{(a - 1)}}{(1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(1 -qs)^{(a - 1)}} \]
\[ = t^a. \]

If \(t \leq qs\), then

\[ G(t, qs) = \frac{t^a - 1 (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\}}{(1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(1 -qs)^{(a - 1)}} \]
\[ \geq \frac{t^a - 1 (1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\}}{(1 -qs)^{(a - 2)} \left\{ [\alpha] q - (1 -sq^\alpha)(1 -sq^{a - 1}) \right\} - ([\alpha] q - 1)(1 -qs)^{(a - 1)}} \]
\[ = t^a. \]

Thus, the proof is complete. \(\Box\)

3. Main Results

For convenience, we denote

\[ F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u}, \quad f_0 = \liminf_{u \to 0^+} \frac{f(u)}{u}, \]
\[ F_\infty = \limsup_{u \to +\infty} \frac{f(u)}{u}, \quad f_\infty = \liminf_{u \to +\infty} \frac{f(u)}{u}, \]
\[ M = \int_0^1 G(1, qs) dqs, \quad N = \int_0^1 G(1, qs) h(s) dqs, \]

where \(h(s) = s^{a - 1}\).

Let \(X = C[0, 1]\) be an ordered Banach space with the cone \(X_+ = \{ u \in X, u(t) \geq 0, t \in [0, 1] \}\); suppose \(u_0 \in X_+\) with \(\| u_0 \| \leq 1\), and define

\[ P_{u_0} = \{ x \in X_+, x \geq \| x \| u_0 \}; \]

then, \(P_{u_0}\) is a cone of \(X_+\).
Let \( u_0 = t^{a-1} \), define \( P = \{ x \in X_+, x \geq t^{a-1} \| x \| \} \), and define the operator \( H : P \to X_+ \) as following
\[
(Hu)(t) = \lambda \int_0^1 G(t, qs)f(u(s))ds.
\]

For any \( r > 0 \), we denote \( \Omega_r = \{ x \in X : \| x \| < r \} \) and \( \partial \Omega_r = \{ x \in X : \| x \| = r \} \).

**Theorem 1** ([27]). Assume that \( X \) is a Banach space with the cone \( X_+ \). Let \( 0 \leq u_0 \leq \varphi \) be such that \( \| u_0 \| \leq 1, \| \varphi \| = 1 \) satisfying the condition:

- If \( x \in X_+, \| x \| \leq 1 \), then \( x \leq \varphi \).
- If there exist positive numbers \( 0 < a < b \) such that \( T : P_{u_0} \cap (\overline{\Omega_b} \setminus \Omega_a) \to P_{u_0} \) is a completely continuous operator and the conditions:

\[
\| T(x) \|_{x \in [au_0, b\varphi]} \leq a \quad \text{and} \quad \| T(x) \|_{x \in [b\varphi, b\varphi]} \geq b
\]

or
\[
\| T(x) \|_{x \in [au_0, b\varphi]} \geq a \quad \text{and} \quad \| T(x) \|_{x \in [b\varphi, b\varphi]} \leq b
\]

are satisfied, then \( T \) has a fixed point \( x_0 \in [au_0, b\varphi] \).

**Lemma 5.** Assume \( f : [0, +\infty) \to [0, +\infty) \) is a continuous function; then, \( H : P \to P \) is a completely continuous operator.

**Proof.** First, it is easy to verify that \( H \in C(P, P) \).

Next, from Lemma 4(ii), we have
\[
(Hu)(t) = \lambda \int_0^1 G(t, qs)f(u(s))ds \\
\geq \lambda t^{a-1} \int_0^1 G(1, qs)f(u(s))ds \\
= t^{a-1} \| Hu \|.
\]

That is, \( H(P) \subset P \).

Let \( D \subset P \) be bounded. There exists a positive constant \( M > 0 \) such that \( \| u \| \leq M \) for each \( u \in D \). Assume \( L = \max_{\| u \| \leq M} |f(u)| + 1 \); then, in view of Lemma 4(ii), we have
\[
| (Hu)(t) | = | \lambda \int_0^1 G(t, qs)f(u(s))ds | \\
\leq \lambda \int_0^1 | G(t, qs)f(u(s)) | ds \\
\leq \lambda L \int_0^1 | G(1, qs) | ds \\
\leq \frac{|a|q\lambda L}{(|a|q - 1)\Gamma_q(a)} \int_0^1 (1 - qs)^{(a-2)} ds \\
\leq \frac{|a|q\lambda L}{(|a|q - 1)\Gamma_q(a)}
\]

This shows that, for each parameter \( \lambda > 0 \), \( T(D) \) is bounded.

On the other hand, for all \( u \in D \), when \( 0 < t_1 < t_2 < 1 \), there exists a sufficiently small \( \delta \), such that \( |t_2 - t_1| < \delta \). One has \( |Hu(t_2) - Hu(t_1)| < \epsilon \), that is to say, \( H(\Omega) \) is equicontinuous. In fact, when \( t_1 \to t_2 \), we have
\[ |Hu(t_2) - Hu(t_1)| \]
\[ \leq \lambda \int_0^1 |G(t_2, qs) - G(t_1, qs)|f(u(s))d_q s \]
\[ \leq \lambda L \int_0^1 |G(t_2, qs) - G(t_1, qs)|d_q s \]
\[ = \frac{\lambda L}{(\alpha \beta - 1)\Gamma(q(\alpha))} \left( \int_0^{t_1} |(1 - qs)^{(\alpha - 2)}([\alpha]_q - (1 - sq^n)(1 - sq^{a-1}))|d_q s \right. \]
\[ + \int_{t_1}^{t_2} t_2^{-1} |(1 - qs)^{(\alpha - 2)}([\alpha]_q - (1 - sq^n)(1 - sq^{a-1}))|d_q s \]
\[ - \int_{t_1}^{t_2} t_2^{-1} |(1 - qs)^{(\alpha - 2)}([\alpha]_q - (1 - sq^n)(1 - sq^{a-1}))|d_q s \]
\[ \leq \frac{\lambda L}{(\alpha \beta - 1)\Gamma(q(\alpha))} \left( \int_0^{t_1} 2\delta d_q s + \int_{t_1}^{t_2} (\delta + 1) d_q s + \int_{t_2}^{1} \delta d_q s \right) \rightarrow 0. \]

Therefore, \( T(D) \) is equicontinuous. By means of the Arzela-Ascoli theorem, \( T : P \rightarrow P \) is completely continuous. \( \square \)

It is well-known that \( u \in X \) is a solution of (4) if and only if \( u \) is a fixed point of \( H \).

**Theorem 2.** Assume that \( MF_0 < Nf_{\infty} \). Then, BVP (4) has at least one positive solution for \( \lambda \in \left( \frac{1}{Nf_{\infty}}, \frac{1}{MF_0} \right) \).

**Proof.** From the definition of the cone \( P \), we have \( u_0 = h(t), \varphi = 1 \). For \( u \in X_+, \|u\| = u(1) \). Therefore, \( u_0 \) and \( \varphi \) satisfy the conditions of Theorem 1.

Since \( \lambda < \frac{1}{MF_0} \), we can choose a small enough number of \( \varepsilon_1 > 0 \) such that

\[ \lambda(F_0 + \varepsilon_1)M < 1. \]

From \( F_0 = \limsup_{u \to 0^+} \frac{f(u)}{u} \), there exists \( \delta > 0 \) such that

\[ \frac{f(u)}{u} < F_0 + \varepsilon_1 \text{ for } u \in (0, \delta). \]

Let \( a = \frac{\delta}{2} \). Then, for \( u \in [a^{\alpha-1}, a] \), we can obtain

\[ \|H(u)\| = H(u)(1) = \lambda \int_0^1 G(1, qs)f(u(s))d_q s \]
\[ \leq \lambda(F_0 + \varepsilon_1) \int_0^1 G(1, qs)u(s)d_q s \]
\[ \leq a. \]

On the other hand, using the condition \( \lambda > \frac{1}{Nf_{\infty}} \), there exist \( c > 0 \) and \( \varepsilon_2 > 0 \) such that

\[ \lambda(f_{\infty} - \varepsilon_2) \int_c^1 G(1, qs)h(s)d_q s > 1. \]
Let \( K > 0 \) such that
\[
\frac{f(u)}{u} > f_\infty - \epsilon_2 \quad \text{for} \quad u \geq K.
\]

Set \( K_0 = \max\{K, c^{a-1}\delta\} \) and \( b = \frac{K_0}{c^{a-1}} \). Then, for \( u \in [bh^{a-1}, b] \) \((t \in [c, 1])\). We can obtain
\[
\|H(u)\| = H(u)(1) = \lambda \int_0^1 G(1, q)s f(u(s))ds \\
\geq \lambda(f_\infty - \epsilon_2) \int_1^c G(1, q)s u(s)ds \\
\geq \lambda(f_\infty - \epsilon_2)b \int_1^c G(1, q)s h(s)ds \\
\geq b.
\]

By Theorem 1, \( H \) has a fixed point \( u_\lambda \in [a^{t^{a-1}}, b] \). It is a positive solution of (4).

**Theorem 3.** Assuming that \( MF_\infty < NF_0 \) holds, then BVP (4) has at least one positive solution for \( \lambda \in \left( \frac{1}{NF_0}, \frac{1}{MF_\infty} \right) \).

**Proof.** From the definition of the cone \( P \), we have \( u_0 = b(t), \varphi = 1 \). For \( u \in X_+ \), \( \|u\| = u(1) \). Therefore, \( u_0 \) and \( \varphi \) satisfy the conditions of Theorem 1.

Since \( \lambda > \frac{1}{NF_0} \), we can choose a small enough number \( \epsilon_1 > 0 \), such that
\[
\lambda(f_\infty - \epsilon_1)N > 1.
\]

Let \( \delta > 0 \) such that
\[
\frac{f(u)}{u} > f_0 - \epsilon_1 \quad \text{for} \quad u \in (0, \delta),
\]

Let \( a = \frac{\delta}{2} \). Then, for \( u \in [a^{t^{a-1}}, a] \), we can obtain
\[
\|H(u)\| = H(u)(1) = \lambda \int_0^1 G(1, q)s f(u(s))ds \\
\geq \lambda(f_\infty - \epsilon_1) \int_0^{a} G(1, q)s u(s)ds \\
\geq \lambda(f_\infty - \epsilon_1)a \int_0^{a} G(1, q)s h(s)ds \\
\geq a.
\]

On the other hand, using the condition \( \lambda < \frac{1}{MF_\infty} \), there exists \( \epsilon_2 > 0 \) such that
\[
\lambda(F_\infty + \epsilon_2)M < 1.
\]

Let \( K > 0 \) such that \( \frac{f(u)}{u} < F_\infty + \epsilon_2 \) for \( u \geq K \).

Set \( K_0 = \max\{K, h^{a-1}\delta\} = bh^{a-1} \) \((t \in [0, 1])\); then, for \( u \in [bh^{a-1}, b] \) \((t \in [0, 1])\), we have
\[
\|H(u)\| = H(u)(1) = \lambda \int_0^1 G(1, q)s f(u(s))ds \\
\leq \lambda(F_\infty + \epsilon_2) \int_0^1 G(1, q)s u(s)ds \\
\leq b.
\]

By Theorem 1, \( H \) has a fixed point \( u_\lambda \in [a^{t^{a-1}}, b] \). It is a positive solution of (4).
Setting $\alpha = \frac{5}{2}$, $q = \frac{1}{2}$, a simple calculation shows that $M \approx 0.3868$, $N \geq 0.1159$.

**Example 1.** Consider the following fractional $q$-difference boundary value problem

\[
\begin{aligned}
(D_{q}^{\alpha}u)(t) + \lambda \left( \frac{u}{2} + \frac{4u^3}{1 + u^2} \right) &= 0, \quad 0 < t < 1; \\
u(0) &= D_q u(0) = 0, \quad D_q u(1) = [\alpha - 1]_q \int_0^1 u(s) d_qs,
\end{aligned}
\]

where $F_0 = 0.5$, $f_\infty = 4.5$. Applying Theorem 2, if $\lambda \in (1.92, 5.17) \subseteq \left( \frac{1}{N f_\infty}, \frac{1}{MF_0} \right)$, then the above boundary value problem has at least one positive solution.

**Example 2.** Consider the following fractional $q$-difference boundary value problem

\[
\begin{aligned}
(D_{q}^{\alpha}u)(t) + \lambda u \left( \frac{1}{2} + \frac{2}{1 + u^2} \right) &= 0, \quad 0 < t < 1; \\
u(0) &= D_q u(0) = 0, \quad D_q u(1) = [\alpha - 1]_q \int_0^1 u(s) d_qs,
\end{aligned}
\]

where $f_0 = 2.5$, $F_\infty = 0.5$. Applying Theorem 3, if $\lambda \in (3.45, 5.17) \subseteq \left( \frac{1}{N f_0}, \frac{1}{MF_\infty} \right)$, then the above boundary value problem has at least one positive solution.

### 4. Conclusions

We study a new class of fractional $q$-difference boundary value problem associated with a parameter and subject to boundary conditions that involve the integral operator. After the explicit expression for the associated Green’s function is given in Lemma 3, two important properties are proved in Lemma 4. In particular, the property (ii) of Lemma 4 shows that the Green’s function does not satisfy the strong positivity condition that is necessary for some popular methods in studying nonlinear boundary value problems [20]. The obtained results on existence of positive solutions are new. Applications are shown by two examples.

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