WHEELED PROPS IN ALGEBRA, GEOMETRY AND QUANTIZATION

S.A. MERKULOV

Abstract. Wheeled props is one the latest species found in the world of operads and props. We attempt to give an elementary introduction into the main ideas of the theory of wheeled props for beginners, and also a survey of its most recent major applications (ranging from algebra and geometry to deformation theory and Batalin-Vilkovisky quantization) which might be of interest to experts.

1. Introduction

The theory of operads and props undergoes a rapid development in recent years; its applications can be seen nowadays almost everywhere — in algebraic topology, in homological algebra, in differential geometry, in non-commutative geometry, in string topology, in deformation theory, in quantization theory etc. The theory demonstrates a remarkable unity of mathematics; for example, one and the same operad of little 2-disks solves the recognition problem for based 2-loop spaces in algebraic topology, describes homotopy Gerstenhaber structure on the Hochschild deformation complex in homological algebra, and also controls diffeomorphism invariant Hertling-Manin’s integrability equations in differential geometry!

First examples of operads and props were constructed in the 1960s in the classical papers of Gerstenhaber on deformation theory of algebras and of Stasheff on homotopy theory of loop spaces. The notion of prop was introduced by MacLane already in 1963 as a useful way to code axioms for operations with many inputs and outputs. The notion of operad was ultimately coined 10 years later by P.May through axiomatization of properties of earlier discovered associhedra polytopes and the associated $A_\infty$-spaces by Stasheff and of the little cubes operad by Boardman and Vogt.

In this paper we attempt to explain the main ideas and constructions of the theory of wheeled operads and props and illustrate them with some of the most recent applications [Gr1, Gr2, Me1-Me7, MMS, MeVa, Mn, Str1, Str2] to geometry, deformation theory and Batalin-Vilkovisky quantization formalism of theoretical physics. In the heart of these applications lies the fact that some categories of local geometric and theoretical physics structures can be identified with the derived categories of surprisingly simple algebraic structures. The language of graphs is essential for the proof of this fact and permits us to reformulate it as follows: solution spaces of several important highly non-linear differential equations in geometry and physics are controlled by (wheeled) props which are resolutions of very compact graphical data, a kind of “genome”; for example, the “genome” of the species local Poisson structures is the prop of Lie 1-bialgebras built from two “genes”, $\bigtriangleup$ and $\bigtriangledown$, subject to the following engineering rules (see §2 for precise details),

\[
\begin{align*}
\bigtriangleup + \bigtriangledown + \bigtriangleup & = 0, \\
\bigtriangleup & = \bigtriangledown + \bigtriangleup + \bigtriangledown + \bigtriangleup + \bigtriangleup + \bigtriangleup.
\end{align*}
\]

We shall explain how a slight modification of the above rules by addition of two extra conditions,

\[
\begin{align*}
\bigtriangledown & = 0 \text{ and } \bigtriangledown = 0, \\
\bigtriangledown & = 0, \\
\bigtriangledown & = 0,
\end{align*}
\]

changes the resulting “species” dramatically: instead of the category of local Poisson structures one gets the category of quantum BV manifolds with split quasi-classical limit which, for example, naturally emerges in the study of quantum master equations [BaVi, Sc] for BF-type quantum field theories (see §5 for precise details). Moreover, in the homotopy theory...
sense, this category is as perfect as, for example, the nowadays famous category of $\mathcal{L}ie_{\infty}$-algebras: quasi-isomorphisms of quantum BV manifolds turn out to be equivalence relations.

It is yet to see how non-trivial topology can be incorporated into the current pro(p)file of local differential geometry, but it is worth stressing already now that this approach to geometry and physics turns space-time — “the background of everything” — into an ordinary observable, a certain function (representation) on a prop and hence unveils a possibility for a new architecture:

$$E_{\text{end}_{\infty}}$$

**A new architecture of geometry and physics:** a prop is the fundamental background for both a space-time and structures

In fact, some elements of this architecture have been envisaged long ago by Roger Penrose [Pe] in his “abstract index calculus”.

The paper is organized as follows. In section 2 we give a short but self-contained introduction into the theory of (wheeled) operads and props. Sections 3 and 4 aim to give an account of most recent applications of that theory to geometry and, respectively, deformation theory. In Section 5 we explain some ideas of Koszul duality theory and its relation to the homotopy transfer formulae and Batalin-Vilkovisky formalism.

A few words about notations. The symbol $S_n$ stands for the permutation group, i.e. for the group of all bijections, $[n] \to [n]$, where $[n]$ denotes (here and everywhere) the set $\{1, 2, \ldots, n\}$. Given a partition, $[n] = I_1 \sqcup \ldots \sqcup I_k$, the symbol $\sigma(I_1, \ldots, I_k)$ denotes the sign of the permutation $[n] \to \{I_1, \ldots, I_k\}$. If $V = \bigoplus_{i \in \mathbb{Z}} V^i$ is a graded vector space, then $V[k]$ is a graded vector space with $V[k]^i := V^{i+k}$. We work throughout over a field $\mathbb{K}$ of characteristic 0.

2. AN INTRODUCTION TO OPERADS, DIOPERADS, PROPERADS AND PROPS

2.1. Directed graphs. Let $m$ and $n$ be arbitrary non-negative integers. A directed $(m, n)$-graph is a triple $(G, f_{\text{in}}, f_{\text{out}})$, where $G$ is a finite 1-dimensional CW complex all of whose every 1-dimensional cells (“edges”) are oriented (“directed”), and

$$f_{\text{in}} : [m] \to \left\{ \begin{array}{l}
\text{the set of all 0-cells, } v, \text{ of } G \\
\text{which have precisely one adjacent edge directed from } v
\end{array} \right\}, \quad f_{\text{out}} : [n] \to \left\{ \begin{array}{l}
\text{the set of all 0-cells, } v, \text{ of } G \\
\text{which have precisely one adjacent edge directed towards } v
\end{array} \right\}$$

are injective maps of finite sets (called labelling maps or simply labellings) such that $\text{Im} f_{\text{in}} \cap \text{Im} f_{\text{out}} = \emptyset$. The set, $\mathfrak{G}_{\text{dir}}(m, n)$, of all possible directed $(m, n)$-graphs carries an action, $(G, f_{\text{in}}, f_{\text{out}}) \to (G, f_{\text{in}} \circ \sigma^{-1}, f_{\text{out}} \circ \tau)$, of the group $S_m \times S_n$ (more precisely, the right action of $S_m^p \times S_n$ but we declare this detail implicit from now). We often abbreviate a triple $(G, f_{\text{in}}, f_{\text{out}})$ to $G$. For any $G \in \mathfrak{G}_{\text{dir}}(m, n)$ the set,

$$V(G) := \{\text{all 0-cells of } G\} \setminus \{\text{Im} f_{\text{in}} \cup \text{Im} f_{\text{out}}\},$$

of all unlabelled 0-cells is called the set of vertices of $G$. The edges attached to labelled 0-cells, i.e. the ones lying in $\text{Im} f_{\text{in}}$ or in $\text{Im} f_{\text{out}}$ are called incoming or, respectively, outgoing legs of the graph $G$. The set

$$E(G) := \{\text{all 1-cells of } G\} \setminus \{\text{legs}\},$$

is called the set of (internal) edges of $G$. Legs and edges of $G$ incident to a vertex $v \in V(G)$ are often called half-edges of $v$; the set of half-edges of $v$ splits naturally into two disjoint sets, $\text{In}_v$ and $\text{Out}_v$, consisting of incoming and, respectively, outgoing half-edges. In all our pictures the vertices
of a graph will be denoted by bullets, the edges by intervals (or sometimes curves) connecting the vertices, and legs by intervals attached from one side to vertices. A choice of orientation on an edge or a leg will be visualized by the choice of a particular direction (arrow) on the associated interval/curve; unless otherwise explicitly shown the direction of each edge in all our pictures is assumed to go from bottom to the top. For example, the graph \[ \begin{array}{c}
1 \\
2
\end{array} \] \( \in \mathcal{G}(2, 2) \) has four vertices, four legs and five edges; the orientation of all legs and of four internal edges is not shown explicitly and hence, by default, flows upwards. Sometimes we skip showing explicitly labellings of legs (as in Table 1, for example). We set \( \mathcal{G} := \sqcup_{m,n \geq 0} \mathcal{G}(m, n) \). Note that elements of \( \mathcal{G} \) are not necessarily connected, e.g. \[ \begin{array}{c}
1 \\
2
\end{array} \] \( \in \mathcal{G}(2, 4) \).

2.2. Decorated directed graphs. Let \( E \) be an \( \mathbb{S} \)-bimodule, that is, a family, \( \{E(p, q)\}_{p,q \geq 0} \), of vector spaces on which the group \( \mathbb{S}_p \) acts on the left and the group \( \mathbb{S}_q \) acts on the right, and both actions commute with each other. We shall use elements of \( E \) to decorate vertices of an arbitrary graph \( G \in \mathcal{G} \) as follows. First, for each vertex \( v \in V(G) \) we construct a vector space \( E(Out_v, \text{In}_v) := (Out_v) \otimes_{\mathbb{S}_p} E(p, q) \otimes_{\mathbb{S}_q} (\text{In}_v) \), where \( \langle Out_v \rangle \) (resp., \( \langle \text{In}_v \rangle \)) is the vector space spanned by all bijections \( \#Out_v \to Out_v \) (resp., \( \#\text{In}_v \to \#\text{In}_v \)). It is (non-canonically) isomorphic to \( E(p, q) \) as a vector space and carries natural actions of the automorphism groups of the sets \( Out_v \) and \( \text{In}_v \). These actions make the following unordered tensor product over the set \( V(G) \) (of cardinality, say, \( k \)),

\[
\bigotimes_{v \in V(G)} E(Out_v, \text{In}_v) := \left( \bigoplus_{i: [k] \to V(G)} E(Out_{i(1)}, \text{In}_{i(1)}) \otimes \ldots \otimes E(Out_{i(k)}, \text{In}_{i(k)}) \right)_{\mathbb{S}_k},
\]

into a representation space of the automorphism group, \( \text{Aut}(G) \), of the graph \( G \) which, by definition, is the subgroup of the symmetry group of the 1-dimensional \( CW \)-complex underlying the graph \( G \) which fixes its legs. Hence with an arbitrary graph \( G \in \mathcal{G} \) and an arbitrary \( \mathbb{S} \)-bimodule \( E \) one can associate a vector space,

\[
G(E) := \left( \bigotimes_{v \in V(G)} E(Out_v, \text{In}_v) \right)_{\text{Aut}G},
\]

whose elements are called decorated (by \( E \)) graphs. For example, the automorphism group of the graph \( G = \begin{array}{c}
1 \\
2
\end{array} \) is \( \mathbb{Z}_2 \) so that \( G(E) = E(1, 2) \otimes_{\mathbb{Z}_2} E(2, 2) \). It is useful to think of an element in \( G(E) \) as of the graph \( G \) whose vertices are literally decorated by some elements \( a \in E(1, 2) \) and \( b \in E(2, 1) \) and subject to the following relations,

\[
\begin{align*}
\begin{array}{c}
a \\
b
\end{array} & = \begin{array}{c}
a \sigma^{-1} \\
\sigma b
\end{array}, \quad \sigma \in \mathbb{Z}_2, && \lambda \left( \begin{array}{c}
a \\
b
\end{array} \right) = \begin{array}{c}
\lambda a \\
\lambda b
\end{array}, && \forall \lambda \in \mathbb{K},
\end{align*}
\]

\[
\begin{array}{c}
a_1 + a_2 \\
b
\end{array} = \begin{array}{c}
a_1 \\
b
\end{array} + \begin{array}{c}
a_2 \\
b
\end{array}
\]

and similarly for \( b \).

It also follows from the definition that \[ \begin{array}{c}
a \\
b
\end{array} \in \mathcal{G}(6, 12) \], \( (12) \in \mathbb{Z}_2 \).
2.2.1. Remark. If \( E = \{ E(p, q) \} \) is a differential graded (dg, for short) \( S \)-bimodule, i.e. if each vector space \( E(p, q) \) is a complex equipped with an \( S_p \times S_q \)-equivariant differential \( \delta \), then, for any graph \( G \in \mathcal{G}(m, n) \), the associated graded vector space \( \delta G(E) \) comes equipped with an induced \( S_m \times S_n \)-equivariant differential \( \delta G \) so that the collection, \( \{ \oplus_{G \in \mathcal{G}(m, n)} G(E) \}_{m,n \geq 0} \), is again a dg \( S \)-bimodule. We sometimes abbreviate \( \delta G \) to \( \delta \).

\[
\begin{array}{c}
\text{input legs} \\
m \end{array}
\begin{array}{c}
\text{output legs} \\
n \end{array}
\]

2.2.2. Remark. The one vertex graph \( C_{m,n} := \) is often called the \((m,n)\)-corolla. It is clear that for any \( S \)-bimodule \( E \) one has \( \delta G(E) = E(m, n) \).

2.3. Wheeled props. A wheeled prop is an \( S \)-bimodule \( P = \{ P(m, n) \} \) together with a family of linear \( S_m \times S_n \)-equivariant maps,

\[
\{ \mu_G : G(P) \to P(m, n) \}_{G \in \mathcal{G}(m, n)}, \quad m, n \geq 0,
\]

parameterized by elements \( G \in \mathcal{G}(m, n) \), which satisfy a “three-dimensional” associativity condition,

\[
\mu_G = \mu_{G/H} \circ \mu'_H,
\]

for any subgraph \( H \subset G \). Here \( G/H \) is the graph obtained from \( G \) by shrinking the whole subgraph \( H \) into a single internal vertex, and \( \mu'_H : G(E) \to (G/H)(E) \) stands for the map which equals \( \mu_H \) on the decorated vertices lying in \( H \) and which is identity on all other vertices of \( G \).

If the \( S \)-bimodule \( P \) underlying a wheeled prop has a differential \( \delta \) satisfying, for any \( G \in \mathcal{G}(m, n) \), the condition \( \delta \circ \mu_G = \mu_G \circ \delta_G \), then the wheeled prop \( P \) is called differential.

By Remark 2.2.2 the values of the maps \( \mu_G \) can be identified with decorated corollas, and hence the maps themselves can be visually understood as contraction maps, \( \mu_{G \in \mathcal{G}(m, n)} : G(P) \to C_{m,n}(P) \), contracting all the edges and vertices of \( G \) into a single vertex.

2.3.1. Remark. Strictly speaking, the notion introduced in § 2.3 should be called a wheeled prop without unit. A wheeled prop with unit can be defined as in §2.1.1 provided one enlarges \( \mathcal{G} \) by adding a family of graphs, \( \{ \uparrow \cdots \uparrow \circ \cdots \circ \} \), without vertices [MMS].

2.4. Props, properads, operads, etc. as \( \mathcal{G} \)-algebras. Let \( \mathcal{G} = \sqcup_{m,n} G(m, n) \) be a subset of the set \( \mathcal{G} \), say, one of the subsets defined in Table 1 below. A subgraph \( H \) of a graph \( G \in \mathcal{G} \) is called admissible if \( H \in \mathcal{G} \) and \( G/H \in \mathcal{G} \), i.e. a contraction of a graph from \( \mathcal{G} \) by a subgraph belonging to \( \mathcal{G} \) gives a graph which again belongs to \( \mathcal{G} \).

A \( \mathcal{G} \)-algebra is, by definition (cf. § 2.3), an \( S \)-bimodule \( P = \{ P(m, n) \} \) together with a family of linear \( S_m \times S_n \)-equivariant maps, \( \{ \mu_G : G(P) \to P(m, n) \}_{G \in \mathcal{G}(m, n)} \), parameterized by elements \( G \in \mathcal{G} \), which satisfy condition (1) for any admissible subgraph \( H \subset G \). Applying this idea to the subfamilies \( \mathcal{G} \subset \mathcal{G} \) from Table 1 gives us, in the chronological order, the notions of prop, operad, dioperad, properad, \( \frac{1}{2} \)-prop and their wheeled versions which have been introduced, respectively, in the papers [Mc, May, Ga, VA, Ko, Me] [MMS].

We leave it to the reader as an exercise to check that \( \mathcal{G} \)-algebra structures on an \( S \)-bimodule \( E \) with only \( E(1, 1) \) non-zero are precisely associative algebra structures on \( E(1, 1) \). This fact implies that, for any \( \mathcal{G} \)-algebra \( E = \{ E(m, n) \}_{m,n \geq 0} \), the space \( E(1, 1) \) is an associative algebra.

2.5. Basic examples of \( \mathcal{G} \)-algebras. (i) For any \( \mathcal{G} \) and any finite-dimensional vector space \( V \) the \( S \)-bimodule \( \epsilon nd_V = \{ \text{Hom}(V^{\otimes m}, V^{\otimes n}) \} \) is naturally a \( \mathcal{G} \)-algebra with contraction maps \( \mu_{G \in \mathcal{G}} \) being ordinary compositions and, possibly, traces of linear maps; it is called the endomorphism \( \mathcal{G} \)-algebra of \( V \). In the cases \( \mathcal{G} \neq \mathcal{G} \), the assumption of finite-dimensionality of \( V \) can be dropped (as the defining operations \( \mu_G \) do not employ traces).

(ii) With any \( S \)-bimodule, \( E = \{ E(m, n) \} \), there is associated another \( S \)-bimodule, \( \mathcal{F}^\mathcal{G} \langle E \rangle = \{ \mathcal{F}^\mathcal{G} \langle E \rangle(m, n) \} \) with \( \mathcal{F}^\mathcal{G} \langle E \rangle(m, n) = \bigoplus_{G \in \mathcal{G}(m, n)} G(E) \), which has a natural \( \mathcal{G} \)-algebra structure
with the contraction maps $\mu_G$ being tautological. The $\mathfrak{S}$-algebra $F^\mathfrak{S}(E)$ is called the free $\mathfrak{S}$-algebra generated by the $\mathfrak{S}$-bimodule $E$. We often abbreviate notations by replacing $F^\mathfrak{S}_G$ by $F^\mathfrak{S}$, $F^\mathfrak{S}_G$ by $F^{\mathfrak{S}}$, etc.

(iii) Definitions of $\mathfrak{S}$-subalgebras, $Q \subset P$, of $\mathfrak{S}$-algebras, of their ideals, $I \subset P$, and the associated quotient $\mathfrak{S}$-algebras, $P/I$, are straightforward. We omit the details.

| $\mathfrak{S}$ | Definition | $\mathfrak{S}$-algebra | Typical examples |
|---------------|------------|-------------------------|-----------------|
| $\mathfrak{S}^\bigcirc$ | All possible directed graphs | Wheeled prop | ![Diagram](image1)
| $\mathfrak{S}^\bigcirc_c$ | A subset $\mathfrak{S}^\bigcirc_c \subset \mathfrak{S}^\bigcirc$ consisting of all connected graphs | Wheeled properad | ![Diagram](image2)
| $\mathfrak{S}^\bigcirc_{\text{oper}}$ | A subset $\mathfrak{S}^\bigcirc_{\text{oper}} \subset \mathfrak{S}^\bigcirc_c$ consisting of graphs whose vertices have at most one output leg | Wheeled operad | ![Diagram](image3)
| $\mathfrak{S}^\uparrow$ | A subset $\mathfrak{S}^\uparrow \subset \mathfrak{S}^\bigcirc$ consisting of graphs with no wheels, i.e. with no directed closed paths of edges | Prop | ![Diagram](image4)
| $\mathfrak{S}^\uparrow_c$ | A subset $\mathfrak{S}^\uparrow_c \subset \mathfrak{S}^\uparrow$ consisting of all connected graphs | Properad | ![Diagram](image5)
| $\mathfrak{S}^\uparrow_{c,0}$ | A subset $\mathfrak{S}^\uparrow_{c,0} \subset \mathfrak{S}^\uparrow_c$ consisting of graphs of genus zero | Dioperad | ![Diagram](image6)
| $\mathfrak{S}^{\frac{1}{2}}$ | A subset $\mathfrak{S}^{\frac{1}{2}} \subset \mathfrak{S}^\uparrow_{c,0}$ consisting of all $(m,n)$-graphs with the number of directed paths from input legs to the output legs equal to $mn$ | $\frac{1}{2}$-Prop | ![Diagram](image7)
| $\mathfrak{S}^\Lambda$ | A subset $\mathfrak{S}^\Lambda \subset \mathfrak{S}^\uparrow_{c,0}$ consisting of graphs whose vertices have precisely one output leg | Operad | ![Diagram](image8)
| $\mathfrak{S}^\mid$ | A subset $\mathfrak{S}^\mid \subset \mathfrak{S}^\Lambda$ consisting of graphs whose vertices have precisely one input leg | Associative algebra | ![Diagram](image9)

2.6. Morphisms and resolutions of $\mathfrak{S}$-algebras. A morphism of $\mathfrak{S}$-algebras, $\rho : \mathcal{P}_1 \to \mathcal{P}_2$, is a morphism of the underlying $\mathfrak{S}$-bimodules such that, for any graph $G$, one has $\rho \circ \mu_G = \mu_G \circ (\rho \circ G)$, where $\rho \circ G$ is a map, $G(\mathcal{P}_1) \to G(\mathcal{P}_2)$, which changes decorations of each vertex in $G$ in accordance with $\rho$. A morphism of $\mathfrak{S}$-algebras, $\mathcal{P} \to \text{End}(V)$, is called a representation of the $\mathfrak{S}$-algebra $\mathcal{P}$ in a graded vector space $V$. 
A free resolution of a dg $\mathfrak{S}$-algebra $\mathcal{P}$ is, by definition, a dg free $\mathfrak{S}$-algebra, $(\mathcal{F}^\mathfrak{S}(E), \delta)$, together with a morphism, $\pi : (\mathcal{F}(E), \delta) \to \mathcal{P}$, which induces a cohomology isomorphism. If the differential $\delta$ in $\mathcal{F}(\mathcal{E})$ is decomposable with respect to compositions $\mu_G$, then it is called a minimal model of $\mathcal{P}$ and is often denoted by $\mathcal{P}_\infty$.

3. Applications to Algebra and Geometry

3.1. The operad of associative algebras. Let $A_0 = \{A_0(m, n)\}$ be an $\mathcal{S}$-bimodule with all $A_0(m, n) = 0$ except $A_0(1, 2) := \mathbb{K}[S_2]$. The associated free operad $\mathcal{F}^\mathcal{A}(A_0)$ can be identified with the vector space spanned by all connected planar graphs of the form $\begin{array}{c} 1 \\ \vdots \end{array}$ (1 2) (2 1). In particular, $\mathcal{F}^\mathcal{A}(A_0)(1, 2) \cong A_0(1, 2) \cong \text{span}(\begin{array}{c} 1 \\ \vdots \end{array})$. Let $I_0$ be an ideal of $\mathcal{F}^\mathcal{A}(A_0)$ generated by the following 6 planar graphs,

\begin{align*}
\sigma(1) & \sigma(2) \\
\sigma(1) & \sigma(2) \\
\sigma(1) & \sigma(3)
\end{align*} \in \mathcal{F}^\mathcal{A}(A_0)(1, 3), \forall \sigma \in S_3.

3.1.1. Claim. There is a 1-1 correspondence between representations, $\rho : \text{Ass} \to \text{End}_V$, of the quotient operad, $\mathcal{A}_\mathfrak{S} := \mathcal{F}^\mathcal{A}(A_0)/\langle I_0 \rangle$, in a space $V$ and associative algebra structures on $V$.

Proof. The values of $\rho$ on arbitrary (equivalence classes of) planar graphs is uniquely determined by its value, $\rho(\begin{array}{c} 1 \\ \vdots \end{array}) \in \text{Hom}(V^\otimes 2, V)$, on one of the two generators. Denote this value by $\mu$. As $\rho$ sends any of the graphs $\begin{array}{c} 1 \\ \vdots \end{array}$ to zero, the multiplication in $V$ given by $\mu$ must be associative. \null \square

Thus the operad $\mathcal{A}_\mathfrak{S}$ can be called the operad of associative algebras. What could be a (minimal) free resolution of $\mathcal{A}_\mathfrak{S}$? By definition in §3.1.1 this must be a free operad, $\mathcal{F}^\mathcal{A}(A)$, generated by some $\mathcal{S}$-bimodule $A = \{A(n)\}_{n \geq 2}$ equipped with a differential $\delta$ and a projection $\pi : \mathcal{F}^\mathcal{A}(A) \to \mathcal{A}_\mathfrak{S}$ inducing an isomorphism, $H(\mathcal{F}^\mathcal{A}(A), \delta) \cong \mathcal{A}_\mathfrak{S}$, at the cohomology level. The latter condition suggests that we can choose $A(1, 2)$ to be identical to $A_0(1, 2)$ and set a differential $\delta$ to satisfy $\delta \begin{array}{c} 1 \\ \vdots \end{array} = 0$. Then the graphs $\begin{array}{c} 1 \\ \vdots \end{array}$ are cocycles in $\mathcal{F}^\mathcal{A}(A)(1, 3)$. In view of the cohomology isomorphism $\mathcal{F}^\mathcal{A}(A) \to \mathcal{A}_\mathfrak{S}$, we have to make them coboundaries, and hence are forced to introduce an $S_3$-module,

$$
A(1, 3) := \mathbb{K}[S_3][1] = \text{span} \left\{ \begin{array}{c} 1 \\ \vdots \end{array} \right\}, \\
\sigma(1) \sigma(2) \sigma(3) \quad \sigma \in S_3
$$

and set

$$
\delta \begin{array}{c} 1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3
\end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3
\end{array} \quad \sigma \in S_3.
$$

We get in this way a well-defined dg free operad together with a well-defined epimorphism, $(\mathcal{F}^\mathcal{A}(A), \sigma) \to (\mathcal{A}_\mathfrak{S}, 0)$, sending $(1, 3)$-corollas to zero. However, this epimorphisms fails to be a quasi-isomorphism as $\delta \begin{array}{c} 1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3 \\
1 \\ 2 \\ 3
\end{array} = 0$. To kill this cohomology class we have to introduce a new generating $(1, 4)$-corolla, $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\
1 \\ 2 \\ 3 \\ 4 \\
1 \\ 2 \\ 3 \\ 4 \\
1 \\ 2 \\ 3 \\ 4
\end{array}$, of degree $-2$ and set the value of the differential on it to be equal to the underbraced expression above. Again we get a well-defined dg free operad together with a natural homomorphism,
that representations of graphs ∈A

Proof. We need to explain only the subjective finite-dimensional, and that follows from the fact that representations of graphs ∈Ass ⊎ which have wheels involve traces. For example, the element ⟨⟩ ∈Ass ⊎(0, 1) gets represented in V as the image of the multiplication map \( \rho(\bigotimes) \in \text{Hom}(V \otimes V) \) under a natural trace map \( \text{Hom}(V \otimes V, V) \rightarrow \text{Hom}(V, \mathbb{K}) \).

\[ (\mathcal{F}(\bigotimes, \bigotimes, \bigotimes), \delta) \rightarrow \text{Ass} \], which, again, fails to be a quasi-isomorphism. To treat the new problem one has to introduce a new generating corolla of degree \(-3\) with 5 input legs and so on.

3.1.2. Theorem [Sta]. The minimal resolution of Ass is a dg free operad, \( \text{Ass}_\infty := (\mathcal{F}^\wedge(\mathcal{A}), \delta) \), generated by the \( \mathbb{S} \)-bimodule \( \mathcal{A} = \{ A(1, n) \} \),

\[ A(1, n) := \mathbb{K}[\mathbb{S}_n][n - 2] = \text{span} \left\{ \sigma(1) \sigma(2) \ldots \sigma(n) \right\}_{\sigma \in \mathbb{S}_n} \]

and with the differential given on the generators by

\[ \delta \sigma(1) \ldots \sigma(n) = \sum_{k=0}^{n-2} \sum_{l=2}^{n-k} (-1)^{k+(n-k-l)+1} \sigma(1) \ldots \sigma(k) \sigma(k+l+1) \ldots \sigma(n) \sigma(k+l) \]

3.1.3. Definition. Representations, \( \text{Ass}_\infty \rightarrow \text{End}_V \), of the dg operad \( \text{Ass}_\infty, \delta \) in a dg vector space \( V \) are called \( \text{Ass}_\infty \)-structures in \( V \).

3.1.4. Remark. We now suggest the reader to re-read Stasheff’s Theorem 3.1.2 from the end to the beginning: given an infinite dimensional graph complex, \( (\text{Ass}_\infty, \delta) \), spanned by all possible planar graphs (without wheels) built from \( (1, n) \)-corollas with \( n \geq 2 \) and equipped with differential \( \delta \), then its cohomology, \( H(\text{Ass}_\infty, \delta) \), is generated by only \( (1, 2) \)-corollas, i.e. it is surprisingly small. It is often impossible to obtain such a result by a direct computation. One of the main theorem-proving techniques in the theory of operads and props is called the Koszul duality theory, and a result of type 3.1.2 often requires a combination of ideas from homological algebra, algebraic topology, the theory of Cohen-Macaulay posets [Va2] and so on. Stasheff [Sta] proved Theorem 3.1.2 by constructing a remarkable family of polytopes called nowadays associahedra; in his approach the surprising smallness of \( H(\text{Ass}_\infty, \delta) \) gets nicely explained by the obvious contractibility of Stasheff’s polytopes as topological spaces. We shall review some theorem-proving techniques in §5 and continue this section with a list of examples which are most relevant to differential geometry.

3.2. The wheeled operad of finite-dimensional associative algebras. Theorem 3.1.2 has been obtained in the category of algebras over the family of graphs, \( \mathfrak{G}^\wedge \), which contain no closed directed paths of internal edges. What happens if we keep the same family of generators as in the case of \( \text{Ass} \),

\[ A_0(m, n) = \left\{ \begin{array}{ll} \mathbb{K}[\mathbb{S}_2] = \text{span} \left\{ 1 \otimes 2, 2 \otimes 1 \right\} & \text{for } m = 1, n = 2 \\ 0 & \text{otherwise} \end{array} \right. \]

the same family of relations [1], but enlarge the family of graphs we work over from \( \mathfrak{G}^\wedge \) to \( \mathfrak{G}^\wedge \wedge \). The associated quotient wheeled operad, \( \text{Ass}^\wedge := \mathcal{F}^\wedge(A_0)/(I_n) \), can be called the operad of finite-dimensional associative algebras. Indeed, one has the following

3.2.1. Claim. There is a 1-1 correspondence between representations, \( \rho: \text{Ass}^\wedge \rightarrow \text{End}_V \), of \( \text{Ass}^\wedge \) in a finite-dimensional vector space \( V \) and associative algebra structures on \( V \).

Proof. We need to explain only the subjective finite-dimensional, and that follows from the fact that representations of graphs ∈Ass ⊎ which have wheels involve traces. For example, the element \( \bigotimes \) ∈Ass ⊎(0, 1) gets represented in \( V \) as the image of the multiplication map \( \rho(\bigotimes) \in \text{Hom}(V \otimes V, V, V) \) under a natural trace map \( \text{Hom}(V \otimes V, V) \rightarrow \text{Hom}(V, \mathbb{K}) \).
It is easy to see that the straightforward analogue of Theorem 3.1.2 cannot hold true for the operad of finite-dimensional associative algebras as, for example, formula 3 implies

\[ \delta \begin{array}{c} 1 \\ 2 \\ \vdots \\ p \\ p+1 \\ n \end{array} = \begin{array}{c} 1 \\ 2 \\ 3 \\ \vdots \\ p+1 \\ n \end{array} - \begin{array}{c} 2 \\ 1 \\ 3 \\ \vdots \\ p+1 \\ n \end{array} - \begin{array}{c} 1 \\ 3 \\ \vdots \\ p+1 \\ n \end{array} = 0 \]

and hence provides us with a non-trivial cohomology class in \( H^{-1}(\mathcal{F}(A), \delta) \) which maps under the natural projection \( \mathcal{F}(A) \to \text{Ass}^\circ \) to zero. The correct analogue of Stasheff’s result for finite-dimensional associative algebras was found in [MMS].

3.2.2. Theorem. The minimal resolution of \( \text{Ass}^\circ \) is a dg free wheeled operad, \( (\text{Ass}^\circ)_{\infty} := (\mathcal{F}(A), \delta) \) generated by an \( S \)-bimodule \( \hat{A} = \{ \hat{A}(m, n) \} \),

\[
\hat{A}(m, n) := \begin{cases} 
\mathbb{K}[S_n][n-2] = \text{span} \left\{ \begin{array}{c} \sigma(1) \sigma(2) \sigma(n) \\ \sigma \in S_n \end{array} \right\} & \text{for } m = 1, n \geq 2 \\
\bigoplus_{p=1}^{n-1} \mathbb{K}[S_p]C_p \times C_{n-p}[n] = \text{span} \left\{ \begin{array}{c} \sigma(1) \sigma(2) \sigma(p) \sigma(p+1) \sigma(n) \\ \sigma \in S_n \end{array} \right\} & \text{for } m = 0, n \geq 2 \\
otherwise \end{cases}
\]

where \( C_p \times C_{n-p} \) is the subgroup of \( S_n \) generated by two commuting cyclic permutations \( \zeta := (1 \ldots p) \) and \( \xi := (p+1 \ldots n) \), and \( k[S_n]C_p \times C_{n-p} \) stands for coinvariants.

The differential is given on the generators of \( \hat{A}(1, n) \) by \( \delta \) and on the generators of \( \hat{A}(0, n) \) by

\[
\delta \begin{array}{c} 1 \\ 2 \\ \vdots \\ p \\ p+1 \\ n \end{array} = \oint_{(1 \ldots p)(p+1 \ldots n)} + \sum_{k=2}^{p} (-1)^{kn} \begin{array}{c} 1 \\ \vdots \\ k \\ p+1 \\ n \end{array} + \sum_{k=2}^{n-2} (-1)^{p+k(1+n-p)+1} \begin{array}{c} 1 \\ 2 \\ p+1 \\ p+k \\ n \end{array}
\]

where the symbol \( \oint \) stands for the cyclic skewsymmetrization of the indices \((i_1 \ldots i_k)\).

Thus the minimal resolution, \( (\text{Ass}^\circ)_{\infty} \), of the operad of finite-dimensional associative algebras is different from the naive “wheelification”, \( (\text{Ass}_{\infty})^\circ \), of the Stasheff’s minimal resolution of the operad, \( \text{Ass} \), of arbitrary associative algebras. A similar phenomenon occurs for the operad of commutative algebras [MMS]. In contrast, the operad, \( \text{Lie} \), of Lie algebras is rigid with respect to the wheelification:

3.2.3. Fact [McC]. \( (\text{Lie}^\circ)_{\infty} = (\text{Lie}_{\infty})^\circ \), i.e. wheeled \( L_{\infty} \)-algebras are exactly the same as ordinary finite-dimensional \( L_{\infty} \)-algebras.

3.2.4. Reminder on \( L_{\infty} \)-algebras and their homotopy classification. For future reference we recall here a few useful facts about Lie and \( L_{\infty} \)-algebras [Ko2]. The operad, \( \text{Lie} \), of Lie algebras is the quotient operad, \( \text{Lie} := \mathcal{F}_+/\langle L_0 \rangle/I \), of the free operad generated by an \( S \)-bimodule \( L_0 = \{ L_0(m, n) \} \),

\[
L_0(m, n) = \begin{cases} 
\text{span} \left\{ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n \\ \end{array} \right\} & \text{for } m = 1, n = 2 \\
0 & \text{otherwise}
\end{cases}
\]
modulo the ideal $I$ generated by the following relations

$$\begin{align*}
\delta_1 + \delta_2 + \delta_3 &= 0.
\end{align*}$$

Its minimal resolution, $\mathcal{L}e_{\infty}$, is a dg free operad, $\mathcal{F}^\wedge(L)$ generated by an $S_n$-bimodule

$$L(m, n) := \begin{cases} 
\text{sgn}_n[n - 2] = \text{span} \left\langle \begin{array}{c}
1 \\
2 \\
\vdots \\
\end{array} \right\rangle & \text{for } m = 1, n \geq 2, \\
0 & \text{otherwise}
\end{cases}$$

with the differential given by

$$\delta = \sum_{[n] = i_1 \sqcup i_2} (-1)^{\sigma(i_1, i_2) + (\# i_1 + 1) \# i_2} \delta_{i_1} \delta_{i_2}.$$ 

Here (and elsewhere) $\text{sgn}_n$ stands for the 1-dimensional sign representation of $S_n$.

With an arbitrary graded vector space $V$ one can associate a formal graded manifold, $\mathcal{M}_V$, whose structure sheaf, $\mathcal{O}_{\mathcal{M}_V}$, is, by definition, the completed graded cocommutative coalgebra $\widehat{\delta}(V[1])$; if $V$ is finite dimensional, then one can equivalently view $\mathcal{M}_V$ as a small neighbourhood of zero in the space $V[1]$ equipped with the algebra (rather than coalgebra), $\widehat{\delta}(V^*[1])$, of ordinary smooth formal functions. It is well known (see, e.g., [Ko2]) that $L_{\infty}$-structures in a dg space $V$, that is, representations $L_{\infty} \to \mathcal{E}nd_V$, are in one-to-one correspondence with degree 1 vector fields, $\delta$, on $\mathcal{M}_V$ which vanish at the distinguished point, $\delta \big|_{0 e^{\mathcal{M}_V}} = 0$, and satisfy the condition $[\delta, \delta] = 0$ (such vector fields are called cohomological). The pairs $(\mathcal{M}_V, \delta)$ are often called dg manifolds. This interpretation of $L_{\infty}$-structures permits us to use simple and concise geometric instruments to describe notions which, in the pure algebraic translation, look awkwardly large. For example, a morphism of $L_{\infty}$-algebras $V \to W$ is nothing but a smooth map, $f : \mathcal{M}_V \to \mathcal{M}_W$, of the associated formal manifolds such that $f_*(\delta_V) = \delta_W$.

A $L_{\infty}$ algebra $(\mathcal{M}_V, \delta)$, is called minimal if the first Taylor coefficient, $\delta_{(1)}$, of the homological vector field $\delta$ at the distinguished point $0 \in \mathcal{M}_V$ vanishes. It is called linear contractible if the higher Taylor coefficients $\delta_{(\geq 2)}$ vanish and the first one $\delta_{(1)}$ has trivial cohomology when viewed as a differential in $V$. According to Kontsevich [Ko2], any $L_{\infty}$-algebra (or, better, the associated dg manifold) is isomorphic to the direct product of a minimal and of a linear contractible one. This fact implies that quasi-isomorphisms in the category of $L_{\infty}$-algebras are equivalence relations. A dg manifold is called contractible if it is isomorphic to a linear contractible one.

### 3.3. Unimodular Lie algebras

Many important Lie algebras $\mathfrak{g}$ (e.g., all semisimple Lie algebras) have the additional property that, for any $g \in \mathfrak{g}$, the trace of the associated adjoint action

$$\text{Ad}_g : \mathfrak{g} \to \mathfrak{g}$$

vanishes. Lie algebras with this property are called unimodular. The wheeled operad, $\mathcal{U}Lie$, controlling unimodular Lie algebras is the quotient of the free wheeled operad, $\mathcal{F}^\wedge(L_0)$, generated by the $S$-bimodule $\mathcal{U}$ modulo the ideal generated by the Jacobi relations $\mathcal{U}$ and the unimodularity relation, $\mathcal{U}_{(1)} = 0$. Its minimal resolution has been found in [Gr1]:

...
3.3.1. Theorem. The operad $\mathcal{ULie}_{\infty}$ is a dg free operad, $\mathcal{F}^\otimes(\hat{\mathcal{L}})$ generated by the $\mathcal{S}$-bimodule,

$$\hat{\mathcal{L}}(m,n) := \begin{cases} 
\text{sgn}_n[n-2] & \text{for } m=1, n \geq 2, \\
\text{sgn}_n[n] = \text{span} \langle \begin{array}{ccc} 1 & 2 & \ldots \ & n-1 & n \end{array} \rangle & \text{for } m=0, n \geq 1, \\
0 & \text{otherwise}
\end{cases}$$

with the differential on the generators of $\hat{\mathcal{L}}(1,n)$ given by $(\tilde{\delta})$ and on the generators of $\hat{\mathcal{L}}(1,n)$ by

$$\delta = \sum_{\sigma(I_1,I_2) \neq 1 \geq 2, \#I_2 \geq 0} (-1)^{\sigma(I_1,I_2)+(#I_1+1)\#I_2} \begin{array}{ccc} 1 & 2 & \ldots \ & n-1 & n \end{array} + \begin{array}{ccc} \ldots & I_2 \end{array}.$$

Geometrically, unimodular $L_{\infty}$-structures in $V$ can be interpreted as pairs, $(\tilde{\delta}, \omega)$, where $\tilde{\delta}$ is a cohomological vector field and $\omega$ a $Q$-invariant section of the Berezinian bundle on $V^*[1]$ (see [Gr1]).

3.4. Lie 1-bialgebras and Poisson geometry. A Lie $n$-bialgebra on a graded vector space $V$ is a pair of linear maps,

$$\Delta \simeq \begin{array}{ccc} 1 & 2 \\
1 \\
\end{array} : V \to V \wedge V, \quad [\bullet] \simeq \begin{array}{ccc} 1 & 2 \\
1 \\
\end{array} : \wedge^2(V[-n]) \to V[-n]$$

making the space $V$ into a Lie coalgebra and the space $V[-n]$ into a Lie algebra and satisfying, for any $a,b \in V$, the compatibility condition

$$\Delta[a \cdot b] = \sum a_1 \otimes [a_2 \cdot b] + [a \cdot b_1] \otimes b_2 + (-1)^{|a||b|+n|a|+n|b|}([b \cdot a_1] \otimes a_2 + b_1 \otimes [b_2 \cdot a]),$$

Here $\Delta a =: \sum a_1 \otimes a_2$ and $\Delta b =: \sum b_1 \otimes b_2$. The case $n=0$ gives the notion of Lie bialgebra which was introduced by Drinfeld [Dr] in the context of quantum groups. The case $n=1$, as we shall see below, is relevant to Poisson geometry. In this case one has $\wedge^2(V[-1]) = (\odot^2V)[-2]$ so that the basic binary operations have the following symmetries, $\begin{array}{ccc} 1 & 2 & 2 \\
1 & 1 & 2 \\
\end{array} = - \begin{array}{ccc} 1 & 2 & 2 \\
1 & 1 & 2 \\
\end{array}$ and $\begin{array}{ccc} 1 & 2 & 2 \\
1 & 1 & 2 \\
\end{array} = \begin{array}{ccc} 1 & 2 & 2 \\
1 & 1 & 2 \\
\end{array}$. Thus the prop of Lie 1-bialgebras, $\mathcal{Lie}^1\mathcal{B}$, is the quotient of the free prop, $\mathcal{F}^1\langle B \rangle$, generated by an $\mathcal{S}$-bimodule,

$$B(m,n) := \begin{cases} 
\text{sgn}_n \otimes 1_1 = \text{span} \langle \begin{array}{ccc} 1 & 2 \\
1 \\
\end{array} \rangle & \text{if } m=2, n=1, \\
1_1 \otimes 1_2[-1] \equiv \text{span} \langle \begin{array}{ccc} 1 \\
1 & 2 \\
\end{array} \rangle & \text{if } m=1, n=2, \\
0 & \text{otherwise}
\end{cases}$$

modulo the ideal generated by Jacobi relations (7) and the following ones

$$\begin{array}{c}
\begin{array}{ccc} 1 & 2 & 3 \\
1 & 2 & 1 \\
\end{array} & + & \begin{array}{ccc} 1 & 3 & 2 \\
1 & 2 & 1 \\
\end{array} + \begin{array}{ccc} 1 & 2 & 3 \\
1 & 2 & 1 \\
\end{array} = 0, \\
\begin{array}{ccc} 1 & 2 & 1 \\
1 & 2 & 1 \\
\end{array} & - & \begin{array}{ccc} 1 & 3 & 1 \\
1 & 2 & 1 \\
\end{array} + \begin{array}{ccc} 1 & 2 & 1 \\
1 & 2 & 1 \\
\end{array} - \begin{array}{ccc} 1 & 2 & 1 \\
1 & 2 & 1 \\
\end{array} + \begin{array}{ccc} 2 & 1 & 1 \\
1 & 2 & 1 \\
\end{array} = 0.
\end{array}$$

Its minimal resolution, $\mathcal{Lie}^1\mathcal{B}_{\infty}$, has been computed in [Me3].
3.4.1. **Theorem.** (i) \( \mathcal{L}i e^1 \mathcal{B}_\infty \) is a dg free prop, \( \mathcal{F}^\dagger (X) \), generated by an \( \mathbb{Z} \)-bimodule,

\[
X(m, n)[-1] = \text{sgn}_m \otimes 1_n |m - 2| = \text{span} \begin{pmatrix} 1 & 2 & \cdots & m \\ 1 & 2 & \cdots & n \end{pmatrix} \quad \text{for} \quad m \geq 1, n \geq 1, m + n \geq 3
\]

and with the differential given on the generators as follows,

\[
\delta = \sum (-1)^{\sigma(I_1 \cup I_2) + |I_1|(|I_2| + 1)}...
\]

(ii) For any \( d \in \mathbb{N} \), there is a one-to-one correspondence between representations of the dg prop \( \mathcal{L}i e^1 \mathcal{B}_\infty \) in \( \mathbb{R}^d \) and formal Poisson structures, \( \pi \), on \( \mathbb{R}^d \) vanishing at the origin.

**Proof.** The proof of (i) is straightforward (see, e.g., [MeVa, Me8, Va1]) once one uses rather non-straightforward Koszul duality theory for dioperads, [GiKa, Ga], and Kontsevich’s ideas of \( \frac{1}{2} \)-props and path filtrations [Ko1, MaVo]. We shall discuss some of these ideas in §5 and show here only the proof of (ii). Since \( \mathbb{R}^p \) is concentrated in degree zero, an arbitrary representation \( \rho : \mathcal{L}i e^1 \mathcal{B}_\infty \rightarrow \mathcal{E}nd_{\mathbb{R}^p} \) can have non-zero values only on \( (m, n) \)-corollas with \( m = 2 \). Denote these values, \( \rho \left( \begin{array}{c} 1 \\ 2 \\ \vdots \\ m \\ 2 \\ \vdots \\ n \end{array} \right) \in \text{Hom}(\circ^m \mathbb{R}^p, \wedge^2 \mathbb{R}^p), \) by \( \pi \). As the tangent space, \( T_0 \), to \( \mathbb{R}^p \) at zero can be identified with \( \mathbb{R}^p \) itself, we can identify the total sum \( \pi := \sum_{n \geq 1} \pi_n \in \text{Hom}(\circ^{n+1} \mathbb{R}^p, \wedge^2 T_0) \) with a formal bi-vector field on \( \mathbb{R}^p \). Then the equation \( \rho \circ \delta = \delta \circ \rho \) becomes precisely the Poisson equation, \( [\pi, \pi]_S = 0 \), where \([ , ]_S \) is the Schouten bracket. \( \square \)

It is worth pointing out that the vanishing condition \( \pi|_{0 \in \mathbb{R}^p} = 0 \) in Theorem 3.4.1(ii) is no serious restriction: given an arbitrary formal or analytic Poisson structure \( \pi \) on \( \mathbb{R}^p \) (not necessarily vanishing at 0 \( \in \mathbb{R}^p \)), then, for any parameter \( \lambda \) viewed as a coordinate on \( \mathbb{R} \), the product \( \lambda \pi \) is a Poisson structure on \( \mathbb{R}^{p+1} = \mathbb{R}^p \times \mathbb{R} \) vanishing at zero \( 0 \in \mathbb{R}^{n+1} \) and hence is a representation of the prop \( \mathcal{L}i e^1 \mathcal{B}_\infty \).

3.4.2. **Bi-Hamiltonian geometry.** The prop profile of a pair of compatible Poisson structures (which is an important concept in the theory of integrable systems) has been computed by Strohmayer in [Str2] with the help of an earlier result of Dotsenko and Khoroshkin [DoKh].

3.4.3. **Wheeled Poisson structures**? Theorem 3.4.1 says that the minimal resolution,

\[
\mathcal{L}i e^1 \mathcal{B}_\infty = (\mathcal{F}^\dagger (X), \delta),
\]

of the prop, \( \mathcal{L}i e^1 \mathcal{B} \), of arbitrary Lie 1-bialgebras controls the category of local (formal) smooth Poisson structures. What can be said about a minimal resolution, \( (\mathcal{L}i e^1 \mathcal{B}^\circ)_\infty \), of the wheeled prop, \( \mathcal{L}i e^1 \mathcal{B}^\circ \), of finite dimensional Lie 1-bialgebras whose representations can, in view of Theorem 3.4.1(ii), be called wheeled Poisson structures? Note that \( \mathcal{L}i e^1 \mathcal{B}^\circ \) has the same generators and relations as \( \mathcal{L}i e^1 \mathcal{B} \), the only difference being that graphs now might have wheels. As in the case of associative algebra, the naive wheelification,

\[
(\mathcal{L}i e^1 \mathcal{B}_\infty)^\circ := (\mathcal{F}^\circ (X), \delta),
\]

creates new non-trivial cohomology classes, as e.g. this one [Me8]

\[
(12)
\]
which map under the natural projection \((\text{Lie}^1\mathcal{B}_\infty)_{\circ} \to \text{Lie}^1\mathcal{B}_\infty\) to zero. Thus the set of generators of a minimal resolution, \((\text{Lie}^1\mathcal{B}_\infty)_{\infty}, \text{of Lie}^1\mathcal{B}_\infty\) must be larger than the set \(\{1\}\), and at present its computation is beyond reach. All we can say now about mysterious wheeled Poisson structures on a graded formal manifold \(M\) is that (i) they are Maurer-Cartan elements of a certain \(L_\infty\)-algebra extension of the ordinary Schouten bracket on \(M\) which involves divergence operators (in fact, graph \((12)\) gives us a glimpse of the \(\mu_3\) composition in that \(L_\infty\)-algebra), and (ii) they can be deformation quantized in exactly the same sense as ordinary Poisson structures; moreover, it is proven in \([\text{Me4}]\) with the help of the theory of wheeled props that there exist universal formulae for deformation quantization of wheeled Poisson structures which involve only rational numbers \(\mathbb{Q}\).

3.5. Pre-Lie algebras, Nijenhuis geometry and contractible dg manifolds. A pre-Lie algebra is a vector space together with a binary operation, \(\circ : V^\otimes 2 \to V\), satisfying the condition
\[(a \circ b) \circ c + a \circ (b \circ c) = (-1)^{|b||c|}(a \circ c) \circ b + (-1)^{|b||c|}a \circ (c \circ b) = 0\]
for any \(a, b, c \in V\). Any pre-Lie algebra is naturally a Lie algebra with the bracket, \([a, b] := a \circ b - (-1)^{|a||b|}b \circ a\). Let us consider the following extension of this notion: a pre-Lie\(^2\) algebra is a pre-Lie algebra \((V, \circ)\) equipped with a compatible Lie bracket in degree 1, i.e. with a linear map \(\wedge^2(\bullet) : \wedge^2(V[-1]) \to V[-1]\) satisfying the Jacobi identities and the following compatibility condition,
\[[a \bullet b] \circ c + (-1)^{|a||b|}a \circ [b \bullet c] + (-1)^{|b||c|}b \circ [a \bullet c] =
(-1)^{|b||c|+|c|}((a \circ c) \bullet b) + (-1)^{(|a|+1)(|b|+|c|)+|a|}((b \circ c) \bullet a),\quad \forall a, b, c \in V\]
This compatibility condition can be understood as follows. The vector space \(V \oplus V[-1]\) is naturally a complex with trivial cohomology. If we write elements of \(V \oplus V[-1]\) as \(a + \Pi b\), where \(a, b \in V\) and \(\Pi\) is a formal symbol of degree 1, then the natural differential in \(V \oplus V[-1]\) is given by \(d(a + \Pi b) = 0 + \Pi a\). Given two arbitrary binary operations,
\[\circ : V \otimes V \to V,\quad (\bullet) : \otimes^2 V \to V[1],\]
define a degree zero map, \([, , ] : \wedge^2(V \oplus V[-1]) \to V \oplus V[-1]\) by setting,
\[[a, b] := a \circ b - (-1)^{|a||b|}b \circ a,\quad [\Pi a, b] := (-1)^{|a|}[a \bullet b] + \Pi a \circ b,\quad [\Pi a, \Pi b] := \Pi [a \bullet b].\]

3.5.1. Proposition \([\text{Me4}]\). The data \((V \oplus V[-1], d, [\ , , ] )\) is a (contractible) dg Lie algebra if and only if \((V, \circ, [\bullet] )\) is a pre-Lie\(^2\) algebra.

Rather surprisingly, the minimal resolution, pre-Lie\(^2\), of the operad of pre-Lie\(^2\)-algebras has much to do with the famous Nijenhuis integrability condition in differential geometry. The following result is based on the works \([\text{ChLi}, \text{Me4}, \text{Str2}]\).

3.5.2. Theorem. (i) The operad pre-Lie\(^2\) is a free operad, \(\mathcal{F}^\wedge(N)\), generated by an \(S\)-bimodule \(N\) with all \(N(n,m) = 0\) except the following ones,
\[N(1, n) := \bigoplus_{p=1}^n \text{Ind}_{S_p \times S_{n-p}}^{S_n} \mathbb{I}_p \otimes \text{sgn}_{n-p}[n - p - 1] = \text{span}
\begin{pmatrix}
symmetric \\ \ldots \\ i_2 \\ i_1 \\ \ldots \\ i_{p+1} \\ i_p \\ \ldots \\ i_n 
\end{pmatrix}
\]
and equipped with a differential given on the generators by
\[d = \sum_{J_1 \cup J_2 \subseteq \{1, \ldots, n\}, \# J_2 \geq 1, \# J_1 + \# J_2 \geq 1, \# J_2 \geq 1} (-1)^{\# J_2 + \# (J_1 \cup J_2)}\]

\(^1\)Equivalently, a linear map \([\bullet] : \otimes^2 V \to V[1].\)
(ii) For any $d \in \mathbb{N}$, there is a one-to-one correspondence between representations of $\text{pre-Lie}_2^{\infty}$ in $\mathbb{R}^d$ and endomorphisms, $J : T_{\mathbb{R}^d} \to T_{\mathbb{R}^d}$, of the tangent bundle on the affine space $\mathbb{R}^d$ satisfying the Nijenhuis integrability condition, $N_J = 0$, and the vanishing condition $J|_{\mathbb{R}^d} = 0$.

We recall that the Nijenhuis tensor of an endomorphism, $J : T_M \to T_M$, of the tangent bundle of an arbitrary smooth manifold $M$ (in particular, of $\mathbb{R}^m$) can be defined as a map

$$N_J : \wedge^2 T_{\mathbb{R}^d} \longrightarrow T_{\mathbb{R}^d},$$

$$X \otimes Y \longrightarrow N_J(X,Y) := [JX, JY] + J^2[X,Y] - J[X,JY] - J[JX,Y],$$

and that its beauty is hidden in the far from being obvious fact that it is not linear only over $\mathbb{R}$ but also over arbitrary smooth functions, $f \in \mathcal{O}_M$, on $M$, that is, $N_J(fX,Y) = N_J(X,fY) = fN_J(X,Y)$.

A representation of this dg operad in an arbitrary graded vector space $V$ might be called a graded or extended Nijenhuis structure on $V$ (viewed as a formal manifold). Interestingly, the category of these extended Nijenhuis manifolds is almost identical (see [Me4]) to the category of contractible dg manifolds which we first met in [3.2.24] when discussing Kontsevich’s homotopy classification of dg manifolds. Proposition 3.5.1 above is in fact one of the simplest manifestations of this more general phenomenon.

3.6. Gerstenhaber algebras and Hertling-Manin geometry. We conclude this section with an example which was actually the first one to reveal strong interconnections between derived (via minimal resolutions) categories of rather simple algebraic structures and solution sets of highly non-linear diffeomorphism covariant differential equations on ordinary smooth manifolds.

A Gerstenhaber algebra is, by definition, a graded vector space $V$ together with two linear maps, $\circ : \wedge^2 V \to V$ and $[\cdot] : \wedge^2 V \to V[1]$ such that $(V, \circ)$ is a graded commutative algebra, $(V[-1], [\cdot])$ is a graded Lie algebra, and the compatibility equation,

$$[(a \circ b) \cdot c] = a \circ [b \cdot c] + (-1)^{|b||c|+1}[a \cdot c] \circ b, \quad \forall a, b, c \in V,$$

holds. The operad of Gerstenhaber algebras is often denoted by $\mathcal{G}$. Its minimal resolution, $\mathcal{G}_\infty$, has been computed in [Ge Ja]; it is one of the most important operads in mathematics which found many applications in homological algebra, algebraic topology and deformation quantization. It was shown in [He Ma] that $\mathcal{G}_\infty$ has also a differential geometric dimension:

3.6.1. Theorem. For any $d \in \mathbb{N}$, there is a one-to-one correspondence between representations of the dg operad $\mathcal{G}_\infty$ in $\mathbb{R}^d$ (concentrated in degree 0) and morphisms of sheaves, $\mu : \wedge^2 T_{\mathbb{R}^d} \to T_{\mathbb{R}^d}$, making the tangent sheaf, $T_{\mathbb{R}^d}$, into a commutative and associative algebra and satisfying the Hertling-Manin integrability condition, $R_\mu = 0$, and the vanishing condition $\mu|_{\mathbb{R}^d} = 0$.

We recall that the Hertling-Manin tensor, $R_\mu$, of an arbitrary commutative and associative product, $\mu : T_M \otimes T_M \to T_M$, on the tangent sheaf of an arbitrary smooth manifold $M$ is a map [He Ma]

$$R_\mu : \wedge^3 T_M \longrightarrow T_M,$$

$$X \otimes Y \otimes Z \otimes W \longrightarrow R_\mu(X,Y,Z,W)$$

where

$$R_\mu(X,Y,Z,W) = \mu(X,Y,\mu(Z,W)) - \mu(\mu(X,Y),Z,W) - \mu(Z,\mu(X,Y),W) - \mu(X,\mu(Y,Z),W) + \mu(X,\mu(Z,W),Y) + \mu(X,\mu(Z,\mu(Y,W)))$$

$$+ \mu(\mu(X,Y,Z),W) + \mu([X,Z],\mu(Y,W)) + \mu([X,W],\mu(Y,Z)).$$
A remarkable fact is that this map is linear not only over $\mathbb{R}$ but also over arbitrary smooth functions, $f \in \mathcal{O}_M$, on $M$, that is, $R_\mu(fX,Y,Z,W) = fR_\mu(X,Y,Z,W)$, $R_\mu(X,fY,Z,W) = fR_\mu(X,Y,Z,W)$, etc. One can view the Hertling-Manin integrability equation as a diffeomorphism covariant version of the WDVV equation [HeMa, HMT].

4. Applications to deformation theory

4.1. From minimal resolutions to $L_\infty$-algebras. One of the advantages of knowing a dg free resolution, $\mathcal{P}_\infty$, of a $\mathfrak{g}$-algebra controlling a mathematical structure $\mathcal{P}$ is that $\mathcal{P}_\infty$ paves a direct way to the deformation theory of $\mathcal{P}$-structures. In the heart of this approach to the deformation theory of many algebraic and geometric structures is observation 4.1.2 (see below) which was proven in [MeVa] in several ways. For its precise formulation we need the following notion.

4.1.1. Definitions. A $L_\infty$-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n \mathfrak{g} \to \mathfrak{g}[2-n]\}_{n \geq 1})$ is called filtered if $\mathfrak{g}$ admits a non-negative decreasing Hausdorff filtration,

$$\mathfrak{g}_0 = \mathfrak{g} \supseteq \mathfrak{g}_1 \supseteq \cdots \supseteq \mathfrak{g}_i \supseteq \cdots,$$

such that $\text{Im} \mu_n \subset \mathfrak{g}_n$ for all $n \geq n_0$ beginning with some $n_0 \in \mathbb{N}$. In this case it makes sense to define the associated set, $\mathcal{MC}(\mathfrak{g})$, of Maurer-Cartan elements as a subset of $\mathfrak{g}$ consisting of degree 1 elements $\Gamma$ satisfying the equation $\sum_{n \geq 1} \frac{1}{n!} \mu_n(\Gamma, \ldots, \Gamma) = 0$.

A very useful fact is that to every Maurer-Cartan element, $\Gamma \in \mathcal{MC}(\mathfrak{g})$, of a filtered $L_\infty$-algebra $(\mathfrak{g}, \{\mu_n : \wedge^n \mathfrak{g} \to \mathfrak{g}\}_{n \geq 1})$ there corresponds a $\Gamma$-twisted $L_\infty$-algebra structure, $\{\mu_n^\Gamma : \wedge^n \mathfrak{g} \to \mathfrak{g}\}_{n \geq 1}$, on $\mathfrak{g}$. If one thinks of the original $L_\infty$ algebra as of a dg manifold $(\mathcal{M}_\mathfrak{g}, \partial)$ (see [1112]), then the set $\mathcal{MC}(\mathfrak{g})$ can be identified with the zero set of the homological vector field $\partial^\Gamma$ on $\mathcal{M}_\mathfrak{g}$ which is obtained from $\partial$ by the translation diffeomorphism, $x \to x + \Gamma$, $\forall x \in \mathcal{M}_\mathfrak{g}$.

4.1.2. Theorem [MeVa]. Let $(\mathcal{F}^\mathfrak{g}(E), \delta)$ be a dg free $\mathfrak{g}$-algebra (see Table 1) generated by an $\mathfrak{S}$-bimodule $E$, and let $(\mathcal{Q}, \delta_\mathcal{Q})$ be an arbitrary dg $\mathfrak{g}$-algebra. Then

(i) the graded vector space, $\mathfrak{g} := \text{Hom}_\mathcal{Q}(E, \mathcal{Q})[1]$, is canonically a filtered $L_\infty$-algebra;

(ii) the set of all morphisms, $\{\mathcal{F}^\mathfrak{g}(E) \to \mathcal{Q}\}$, of dg $\mathfrak{g}$-algebras is canonically isomorphic to the Maurer-Cartan set, $\mathcal{MC}(\mathfrak{g})$, of the $L_\infty$-algebra in (i).

Proof. As an illustration we show an elementary proof of the theorem in the simplest case $\mathfrak{g} = \mathfrak{S}^l$ (see Table 1), i.e. in the case when $\mathcal{F}^\mathfrak{g}(E)$ is the free associative algebra, $\otimes^* E$, generated by a graded vector space $E$ and $(\mathcal{Q}, \delta_\mathcal{Q})$ is an arbitrary dg associative algebra (we refer the reader to [MeVa] for all other cases from Table 1 except $\mathfrak{S}^l$ and to [Gr2] for the case $\mathfrak{S}^C$). With these data we shall associate a cohomological vector field, $\partial$, on the space $\mathfrak{g}[1] = \text{Hom}(E, \mathcal{Q}) = \mathcal{Q} \otimes E^*$, and we shall do it in local coordinates by assuming further (only for simplicity of sign factors in formulae) that the graded vector spaces $E$ and $\mathcal{Q}$ are free modules over some graded commutative ring, $R = \bigoplus_{i \in \mathbb{Z}} R^i$, with degree $\theta$ generators $\{e_\alpha\}_{\alpha \in J}$ and, respectively, $\{e_\alpha\}_{\alpha \in J}$. Then the differentials in $\otimes^* E$ and $\mathcal{Q}$, as well as multiplication $\circ$ in $\mathcal{Q}$, have, respectively, the following coordinate representations,

$$\delta e_\alpha = \sum_{\alpha_1, \ldots, \alpha_k \neq \alpha} \delta^\alpha_\alpha e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}, \quad \delta_{\mathcal{Q}} e_\alpha = \sum_{\beta \in J} Q^\beta_\alpha e_\beta, \quad e_\alpha \circ a_\beta = \sum_{\gamma \in J} \mu_\alpha^\gamma a_\beta e_\gamma,$$

for some coefficients $\delta^\alpha_\alpha \in R^1$, $Q^\beta_\alpha \in R^1$ and $\mu_\alpha^\gamma \in R^0$. The vector space of all $\mathcal{R}$-linear maps, $\text{Hom}(E, \mathcal{Q})$, is naturally graded, $\text{Hom}(E, \mathcal{Q}) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}^i(E, \mathcal{Q})$, with $\text{Hom}^i(E, \mathcal{Q})$ denoting the space of all homogeneous linear maps of degree $i$. In the chosen bases a generic element $\gamma \in \text{Hom}^i(E, \mathcal{Q})$ gets a coordinate representation, $\gamma(e_\alpha) = \sum_{\alpha \in J} \gamma_\alpha^\gamma e_\alpha$, for some coefficients $\gamma_\alpha^\gamma \in R^i$. The family of parameters $\{\gamma_\alpha^\gamma\}_{\alpha \in I, \gamma \in J, i \in \mathbb{Z}}$ provides us with a coordinate system on
the formal manifold \( M_\mathfrak{g} \approx \text{Hom}(E, Q) \). In these coordinates the required homological vector field on \( M_\mathfrak{g} \), that is, a \( L_\infty \)-structure on \( \text{Hom}(E, Q)[-1] \), is given explicitly by

\[
\partial = \left( \sum_{a, b, a, i} Q^a_{\beta} \frac{\alpha}{\alpha a(i)} \right) \frac{\partial}{\partial \gamma_{\alpha a(i)}}
\]

where, for \( k \geq 2 \),

\[
\gamma_{\alpha a_1 a_2 \ldots a_k(i)} = \sum_{\beta \in J} \mu_{\beta}^{\gamma_1} \gamma_2 \gamma_{\beta a_1(1)} \gamma_{\beta a_2(2)} \gamma_{\beta a_k(k)}.
\]

The equation \([\partial, \partial] = 0\) follows straightforwardly from the assumptions that \( \delta^2 = 0 \) and \( \delta_Q^2 = 0 \). This proves (i).

The Maurer-Cartan set \( MC(\mathfrak{g}) \) is precisely the set \( \{ \gamma \in \text{Hom}(E, Q) : \partial |_{\gamma} = 0 \} \) and, therefore, consists of all points in \( \text{Hom}(E, Q) \) which have all the coordinates \( \{ \gamma_{\alpha a(i)} \}_{a \neq 0} \) vanishing, and the coordinate \( \gamma_{\alpha a(0)} \) satisfying the equation,

\[
\sum_{\beta \in J} Q^\alpha_{\beta} \gamma_{\alpha a(0)} - \sum_{a_1, \ldots, a_k \in I} \delta_{\alpha a_1 \ldots a_k} \gamma_{\alpha a_1 \ldots a_k(0)} = 0.
\]

Which just says that the associated to \( \gamma_{\alpha a(0)} \) map of associative algebras, \( \varnothing \circ E \to Q \), commutes with the differentials \( \delta \) and \( \delta_Q \) defining thereby a morphism of \( dg \) algebras. This proves claim (ii).

4.2. Deformation theory. The theory of operads and props gives a universal approach to the deformation theory of many algebraic and geometric structures and provides us with a conceptual explanation of the well-known “experimental” observation that a deformation theory is controlled by a differential graded Lie or, more generally, a \( L_\infty \)-algebra. What happens is the following:

(I) an algebraic or a (germ of) geometric structure, \( s \), in a vector space \( V \) (which is an object in the corresponding category, \( \mathfrak{S} \), of algebraic or geometric structures) can often be interpreted as a representation, \( \alpha_s : s \to \text{End}_V \), of a \( \mathfrak{S} \)-algebra \( s \) uniquely associated to the category of \( s \)-structures;

(II) a dg resolution, \( \pi : S_\infty = (F^\bullet(E), \partial) \to s \), of the \( \mathfrak{S} \)-algebra \( s \) gives rise, by Theorem 4.1.2 to a filtered \( L_\infty \)-algebra on the vector space \( g = \text{Hom}_E(E, \text{End}_V)[-1] \) whose Maurer-Cartan elements correspond to all possible representations, \( S_\infty \to \text{End}_V \); in particular, our original algebraic or geometric structure \( s \) defines a Maurer-Cartan element \( \Gamma_s := \alpha_s \circ \pi \) in \( MC(g) \);

(III) the \( \Gamma_s \)-twisted \( L_\infty \)-algebra structure on \( g \) is precisely the one which controls, in Deligne’s sense, the deformation theory of \( s \).

For example, if \( s \) is the structure of associative algebra on a vector space \( V \), then,

(i) there is an operad, \( \mathfrak{Ass} \), uniquely associated to the category of associative algebras such that \( s \) corresponds to a morphism, \( \alpha_s : \mathfrak{Ass} \to \text{End}_V \), of operads (see §2.1);

(ii) there is a unique minimal resolution (see Theorem 3.1.2, \( \mathfrak{Ass}_\infty \), of \( \mathfrak{Ass} \) which is generated by the \( \mathbb{S} \)-module \( E = \{ \mathbb{K}[S_n] \mid n - 2 \} \) and whose representations, \( \pi : \mathfrak{Ass}_\infty \to \text{End}_V \), in a dg space \( V \) are in one-to-one correspondence with Maurer-Cartan elements in the Lie algebra,

\[
(\mathcal{G} := \text{Hom}_E(E, \text{End}_V)[{-1}] = \oplus_{n \geq 1} \text{Hom}_E(V^\otimes n, V)[1 - n], [\cdot, \cdot], \{\cdot\})
\]

where \([\cdot, \cdot], \{\cdot\}\) is the Gerstenhaber bracket.

(iii) the particular associative algebra structure \( s \) on \( V \) gives, therefore, rise to the associated Maurer-Cartan element \( \gamma_s := \alpha_s \circ \pi \) in \( \mathcal{G} \); twisting \( \mathcal{G} \) by \( \gamma_s \) gives the Hochschild dq Lie algebra, \( \mathcal{G}_s = (\oplus_{n} \text{Hom}_E(V^\otimes n, V)[1 - n], [\cdot, \cdot], \{\cdot\}, d_H := [\gamma_s, \cdot], \{\cdot\}) \) which indeed controls the deformation theory of \( s \).
This is a classical example illustrating how the machine works. For some new applications of this approach to deformation theory (e.g., to the proof of Deligne’s conjecture or to the deformation theory of associative bialgebras) we refer to [KoSo, MeVa] and to many references cited there.

5. Koszul duality theory, quantum BV manifolds and effective BF-actions

5.1. Quadratic $\mathfrak{g}$-algebras and their Koszul duals. Koszul duality theory of quadratic $\mathfrak{g}$-algebras is one of the most powerful theorem-proving techniques in the theory of (wheeled) operads and properads and their applications.

What is a quadratic $\mathfrak{g}$-algebra? Every family of graphs, $\mathfrak{g}$, from Table 1 has a uniquely defined subfamily, $\mathfrak{g}_{\text{gen}}$, of generating graphs, which, by definition, is the smallest subset of $\mathfrak{g}$ with the defining property that, for every $G \in \mathfrak{g}$ and any $\mathfrak{g}$-algebra $P$, the associated “contraction” composition $\mu_G : G(P) \to P$ can be represented as an iteration (in the sense of $\mathbf{1}$) of compositions $\mu_{G_i}$ for some $G_i \in \mathfrak{g}_{\text{gen}}$, $i \in I$. For example,

$$\mathfrak{g}^\lambda_{\text{gen}} = \{ \ldots \}, \quad \mathfrak{g}^\nu_{\text{gen}} = \{ \ldots \}$$

and

$$\mathfrak{g}^{(2)}_{\text{gen}} = \{ \ldots \}.$$  

5.1.1. Weight gradation. Let $\mathfrak{g}$ be a family of graphs from Table 1. For any genus $q$ graph $G \in \mathfrak{g}$ with $p$ vertices we set $||G|| := p + q$ if the family $\mathfrak{g}$ contains wheels and set $||G|| := p$ otherwise. This number is called the weight of $G$. Thus $\mathfrak{g}_{\text{gen}} \subset \mathfrak{g}$ consists precisely of graphs of weight 2.

For an $\mathfrak{S}$-bimodule $E$ let $F^{\mathfrak{g}}(E)$ stand for a subspace of the free $\mathfrak{g}$-algebra $F^{\mathfrak{g}}(E)$ spanned by decorated graphs of weight $\lambda$. Operadic compositions in $F^{\mathfrak{g}}(E)$ are homogeneous with respect to the weight gradation.

5.1.2. Definition. A $\mathfrak{g}$-algebra, $P$, is called quadratic if it is the quotient, $F^{\mathfrak{g}}(E)/(R)$, of a free $\mathfrak{g}$-algebra (generated by an $\mathfrak{S}$-bimodule $E$) modulo the ideal generated by a subspace $R \subset F^{\mathfrak{g}}(E) = \bigoplus_{G \in \mathfrak{g}_{\text{gen}}} G(E)$. It comes equipped with an induced weight gradation, $P = \bigoplus_{\lambda \geq 1} P(\lambda)$, where $P(\lambda) = F^{\mathfrak{g}}(\lambda)(E)/(R)$. In particular, $P(1) = E$ and $P(2) = F^{\mathfrak{g}}(2)(E)/(R).

5.2. Koszul duality. Let $\mathfrak{g}$ be any family of connected graphs from Table 1. In this case one can associate to any quadratic $\mathfrak{g}$-,algebra $P$ its Koszul dual $\mathfrak{g}$-,coalgebra $P$. We omit technical details (refer to [GK, GelJo, GaVa, NMS, Me]) and explain just the working scheme:

(i) The notion of $\mathfrak{g}$-,coproperad is obtained by an obvious dualization of the notion of $\mathfrak{g}$-,algebra (see §2.3): this is an $\mathfrak{S}$-bimodule $P = \{ P(m,n) \}$ together with a family of linear $S_m \times S_n$-equivariant maps,

$$\{ \Delta_G : P(m,n) \to G(P) \}_{G \in \mathfrak{g}_{\text{gen}}, m,n \geq 0},$$

which satisfy the coassociativity condition, $\Delta_G = \Delta_H \circ \Delta_{G/H}$, for any subgraph $H \subset G$ which belongs to the family $\mathfrak{g}$. Here $\Delta_H : (G/H)(E) \to G(E)$ is the map which equals $\Delta_H$ on the distinguished vertex of $G/H$ and which is the identity on all other vertices of $G$.

(ii) There exists a pair of adjoint exact functors

$$B : \text{the category of dg } \mathfrak{g}_{\text{c}}\text{-algebras} \rightleftharpoons \text{the category of dg } \mathfrak{g}_{\text{c}}\text{-coalgebras} : B^c$$

such that, for any dg $\mathfrak{g}_{\text{c}}$-algebra $P$ the composition $B^c(B(P))$ is a dg free resolution of $P$. The differential $\partial_P$ in $B(P)$ encodes both the differential and all the generating contraction compositions, $\{ \mu_G : G(P) \to P \}_{G \in \mathfrak{g}_{\text{gen}}}$, in the $\mathfrak{g}_{\text{c}}$-algebra $P$ (and similarly for $\partial_Q$).
(iii) As a vector space \( B(\mathcal{P}) \) is isomorphic to the free \( \mathfrak{S}_c \)-algebra, \( \mathcal{F}^\mathfrak{S}_c(\mathcal{P}) \), generated by an \( \mathfrak{S} \)-bimodule \( \mathcal{P} \) which is linearly isomorphic to \( \mathcal{P} \) and hence comes equipped with an induced weight gradation. The subspace \( B(\mathcal{P}(1)) := \mathcal{F}^\mathfrak{S}_c(\mathcal{P}(1)) \) of \( B(\mathcal{P}) \) is obviously a sub-cooperad. On the other hand, \( B(\mathcal{P}) \) has its own “outer” weight gradation, \( B(\mathcal{P}) = \bigoplus_{\mu \geq 1} B(\mu)(\mathcal{P}) \), induced from the weight gradation of the free algebra, \( B(\mu)(\mathcal{P}) := \mathcal{F}_c^\mathfrak{S}(\mathcal{P}) \); the cobar differential \( \partial_P \) has weight \(-1\) with respect to this outer weight gradation.

### 5.2.1. Definition

Given a quadratic \( \mathfrak{S}_c \)-algebra \( \mathcal{P} \), the \( \mathfrak{S}_c \)-coalgebra \( \mathcal{P}^! = \bigoplus_{\mu \geq 1} \mathcal{P}^!(\mu) \) with \( \mathcal{P}^!(\mu) := B(\mu)(\mathcal{P}(1)) \cap \text{Ker} \partial_P \subset B(\mathcal{P}) \) is called Koszul dual to \( \mathcal{P} \).

The beauty of this notion is that \( \mathcal{P}^! \) is again quadratic and, moreover, can often be easily computed directly from generators and relations, \( E \) and \( \mathcal{R} \), of \( \mathcal{P} \).

### 5.2.2. Definition

A quadratic \( \mathfrak{S}_c \)-algebra \( \mathcal{P} \) is called Koszul, if the associated inclusion of dg coproperads, \( \iota : (\mathcal{P}^!, 0) \to (B(\mathcal{P}), \partial_P) \), is a quasi-isomorphism.

As the cobar construction functor \( B^c \) preserves quasi-isomorphisms between connected \( \mathfrak{S}_c \)-coalgebras, the composition

\[
\pi : \mathcal{P}_\infty := B^c(\mathcal{P}^! \cap \text{Ker} \partial_P) \xrightarrow{\iota} B^c(B(\mathcal{P})) \xrightarrow{\text{natural projection}} \mathcal{P}
\]

is a quasi-isomorphism if and only if \( \mathcal{P} \) is Koszul; in this case the dg free \( \mathfrak{S}_c \)-algebra \( \mathcal{P}_\infty \) gives us a minimal resolution of the quadratic algebra \( \mathcal{P} \). Almost all minimal resolutions listed in §2 have been obtained in this way.

### 5.3. Homotopy transfer formulæ

If \( \mathcal{P}_\infty \) is a minimal resolution of some \( \mathfrak{S} \)-algebra \( \mathcal{P} \), and \( (V, d) \) is a complex carrying a \( \mathcal{P} \)-structure, then one might expect that the associated cohomology space, \( H(V, d) \), carries an induced structure of \( \mathcal{P}_\infty \)-algebra. In the case when \( \mathcal{P} \) is an operad of associative algebras, existence of such induced \( \text{Ass}_{\mathfrak{S}_c} \)-structures was proven by Kadeishvili in [Ka] and the first explicit formulæ have been shown in [Mc1]. Later Kontsevich and Soibelman [KoSo] have nicely rewritten these homotopy transfer formulæ in terms of sums of decorated graphs. In fact, it is a general phenomenon that the homotopy transfer formulæ can be represented as sums of graphs. The required graphs are precisely the ones which describe the image of the natural inclusion \( \iota : (\mathcal{P}^!, 0) \to (B(\mathcal{P}), \partial_P) \), and apply to any quadratic \( \mathfrak{S} \)-algebra, not necessarily the Koszul one [Mc7].

### 5.4. Example: unimodular Lie 1-bialgebras versus quantum BV manifolds

The wheeled prop, \( \mathcal{U} \text{Lie}^1\mathcal{B} \), of unimodular Lie 1-bialgebras was defined in [Mc7] (cf. [K1]) as the quotient, \( \mathcal{F}_{\mathfrak{S}}^\mathfrak{S}(\mathcal{B})/\langle \mathcal{R} \rangle \), of the free wheeled properad generated by \( \mathfrak{S} \)-bimodule \( \mathfrak{B} \) modulo the ideal generated by relations [7], [11] and the following ones,\( \mathcal{H} = 0, \mathcal{L} = 0 \), expressing unimodularity of both binary operations. This is a quadratic wheeled properad so that one can apply the above general machinery to compute its Koszul dual cooperad, \( \mathcal{U} \text{Lie}^1\mathcal{B}^! \), and then the dg properad \( \mathcal{P}_\infty := B^c(\mathcal{U} \text{Lie}^1\mathcal{B}^!) \) which turns out to be a free wheeled properad, \( \mathcal{F}_{\mathfrak{S}}^\mathfrak{S}(\mathcal{Z}) \), generated by an \( \mathfrak{S} \)-bimodule,

\[
Z(m, n) := \bigoplus_{a \geq 0} \text{sgn}_m \otimes 1_n [m - 2 - 2a] = \text{span} \left\{ \begin{array}{c}
\begin{array}{c}
1 \ 2 \\
\vdots \\
n \\
\end{array}
\begin{array}{c}
m \\
\vdots \\
1 \\
\end{array}
\end{array} \right\}.
\]

and equipped with the following differential
It is not known at present whether or not $\mathcal{ULie}^1\mathcal{B}$ is Koszul, i.e. whether or not the above free properad is a (minimal) resolution of the latter. In any case, $\mathcal{ULie}^1\mathcal{B}_\infty$ gives us an approximation to that minimal resolution, and has, in fact, a geometrically meaningful set, $\{\mathcal{ULie}^1\mathcal{B}_\infty \to \mathcal{End}_\mathcal{V}\}$, of all possible presentations. To describe this set let us recall a few notions from the Schwarz model [Sc] of the Batalin-Vilkovisky quantization formalism [BaYa].

5.5. **Formal quantum BV manifolds.** Let $\{x^a, \psi_a, h\}_{1 \leq a \leq n}$, $n \in \mathbb{N}$, be a set of formal homogeneous variables of degrees $|x^a| + |\psi_a| = 1$ and $|h| = 2$, and let $\mathcal{O}^h_{x,\psi} := \mathbb{K}[x^a, \psi_a, h]$ be the associated free graded commutative ring which we view from now on as a $\mathcal{K}$-module. The degree $-1$ Lie bracket,

$$\{f \bullet g\} := (-1)^{|f|} \Delta(fg) - (-1)^{|f|} \Delta(fg) - f \Delta(g), \quad \forall f, g \in \mathcal{O}^h_{x,\psi}$$

make $\mathcal{O}^h_{x,\psi}$ into a Gerstenhaber $\mathbb{K}[h]$-algebra (see (4.3)). Here and elsewhere $\Delta := \sum_{a=1}^{n} (-1)^{|x^a|} \partial^2 \frac{\partial^2 \mathcal{F}(x^a)}{\partial x^a \partial \psi_a}$. A quantum master function is, by definition, a degree 2 element $\Gamma \in \mathcal{O}^h_{x,\psi}$ satisfying a so-called quantum master equation

$$\hbar \Delta \Gamma + \frac{1}{2} \{\Gamma \bullet \Gamma\} = 0. \quad (13)$$

Such an element makes the $\mathbb{K}[h]$-module $\mathcal{O}^h_{x,\psi}$ differential with the differential $\Delta \Gamma := \hbar \Delta + \{\Gamma \bullet \}. \quad \text{Note that this differential does not respect the algebra structure in } \mathcal{O}^h_{x,\psi} \text{ but respects the Poisson brackets.}$

Consider a group of $\mathbb{K}[h]$-algebra automorphisms, $F : \mathcal{O}^h_{x,\psi} \to \mathcal{O}^h_{x,\psi}$, preserving the Lie brackets, $F(\{f \bullet g\}) = \{F(f) \bullet F(g)\}$ (but not necessarily the operator $\Delta$); this group is uniquely determined by a collection, $\mathcal{N} := \{x^a, \psi_a\}_{1 \leq a \leq n}$, of $2n$ integers and is denoted by $\text{Symp}_\mathcal{N}$. It is often called a group of symplectomorphisms of the Gerstenhaber algebra $(\mathcal{O}^h_{x,\psi}, \bullet \)$. A remarkable fact [Kh] is that $\text{Symp}_\mathcal{N}$ acts on the set of quantum master functions by the formula,

$$e^{\frac{F(\Gamma)}{\hbar}} := \text{Ber} \left( \begin{array}{cc} \frac{\partial F(x^a)}{\partial x^b} & \frac{\partial F(x^a)}{\partial \psi_b} \\ \frac{\partial F(\psi_a)}{\partial x^b} & \frac{\partial F(\psi_a)}{\partial \psi_b} \end{array} \right) \frac{1}{\hbar} e^{\frac{\Gamma(x, \psi, h)}{\hbar}}. \quad (14)$$

5.5.1. **Definition.** An equivalence class of pairs, $(\mathcal{O}^h_{x,\psi}, \Gamma)$, under the action of the group $\text{Symp}_\mathcal{N}$ is called a formal quantum BV manifold $M$ of dimension $\mathcal{N}$. A particular representative, $(\mathcal{O}^h_{x,\psi}, \Gamma)$, of $M$ is called a Darboux coordinate chart on $M$.

In geometric terms, $M$ is a formal odd symplectic manifold equipped with a special type semi-density [Kh [Sc]. We need an extra structure on $M$ which we again define with the help of a Darboux coordinate chart. Notice that the ideals, $I_x$ and $I_\psi$, in the $\mathbb{K}[h]$-algebra $\mathcal{O}^h_{x,\psi}$ generated, respectively, by $\{x^a\}_{1 \leq a \leq n}$ and $\{\psi_a\}_{1 \leq a \leq n}$, are also Lie ideals; geometrically, they define a pair of transversally intersecting Lagrangian submanifolds of $M$. A quantum BV manifold $M$ is said to have split quasi-classical limit (or, slightly shorter, $M$ is quasi-classically split) if it admits a Darboux coordinate chart in which the master function, $\Gamma(x, \psi, h) = \sum_{n \geq 0} \Gamma_n(x, \psi) \hbar^n$, satisfies the following two boundary conditions,

$$\Gamma_0 \in I_x I_\psi, \quad \Gamma_1 \in I_x + I_\psi.$$
In plain terms, these conditions mean that \( \Gamma(x, \psi, \hbar) \) is given by a formal power series of the form,

\[
\Gamma(x, \psi, \hbar) = \sum_{a, b} (x^a \psi^b) + \sum_{p, a_1, \ldots, a_p} \frac{1}{p!q!} \Gamma^{b_1 \ldots b_q}_{a_1 \ldots a_p} x^{a_1} \cdots x^{a_p} \psi^{b_1} \cdots \psi^{b_q} \hbar^n
\]

for some \( \Gamma^{b_1 \ldots b_q}_{a_1 \ldots a_p} \in K \). Quantum master equation (15) immediately implies that \( \{ \Gamma_0, \Gamma_0 \} = 0 \) so that \( \bar{\partial} := \{ \Gamma_0 \bullet \} \) is a differential in the Gerstenhaber algebra \( \mathcal{O}_{x, \psi}^\hbar \). Then master equation (13) for a quasi-classically split quantum master function can be equivalently rewritten in the form,

\[
\bar{\partial}\Gamma + \hbar \Delta \Gamma + \frac{1}{2} \{ \Gamma \bullet \Gamma \} = 0,
\]

where \( \mathbf{\Delta} \) is an element of \( \mathcal{O}_{x, \psi}^\hbar \) of polynomial order of at least 3 (here we set, by definition, the polynomial order of generators \( x \) and \( \psi \) equal to 1 and the polynomial order of \( \hbar \) equal to 2). The differential \( \bar{\partial} \) induces a differential on the tangent space, \( \mathcal{T}_x M \), to \( M \) at the distinguished point; we denote it by the same letter \( \bar{\partial} \). Such a quantum BV manifold is called minimal if \( \bar{\partial} = 0 \) and contractible if there exists a Darboux coordinate chart in which \( \mathbf{\Gamma} = 0 \) (i.e. \( \Gamma = \bar{\partial} \)) and the tangent complex \( (\mathcal{T}_x M, \bar{\partial}) \) is acyclic. An important class of so called BF field theories (see, e.g., \cite{CaRo} and references cited there) have associated quantum BV manifolds which do satisfy the split quasi-classical limit condition.

5.5.2. Proposition. For any dg vector space \( V \), there is a one-to-one correspondence between representations, \( \mathcal{H}Lie^1 B_{\infty} \rightarrow \mathcal{E}nde_{V} \), and structures of formal quasi-classically split quantum BV manifold on \( M_{V \oplus V^* [1]} \), the formal manifold associated to \( V \oplus V^*[1] \).

5.6. Morphisms of quantum BV manifolds \cite{Me7}. The above Proposition together with the Koszul duality approach to the homotopy transfer outlined in \cite{5.7} provide us with highly non-trivial formulae for constructing quantum BV manifold structures out of dg unimodular Lie 1-bialgebras. We would like to have a category of quantum BV manifolds in which such homotopy transfer formulae can be interpreted as morphisms. This can be achieved via the following definitions.

5.6.1. (i) A morphism of quasi-classically split quantum BV manifolds, \( F : M \rightarrow \hat{M} \), is, by definition, a morphism of dg \( \mathbb{K}[\hbar] \)-modules,

\[
F : (\mathcal{O}_M \simeq \mathcal{O}_{x, \psi}^\hbar, \Delta_f) \rightarrow (\mathcal{O}_\hat{M} \simeq \mathcal{O}_{x, \psi}^\hbar, \Delta_{\hat{f}}),
\]

inducing in the classical limit \( \hbar \rightarrow 0 \) a morphism of algebras, \( F|_{\hbar=0} : \mathcal{O}_{x, \psi} \rightarrow \mathcal{O}_{x, \psi} \) which preserves the ideals, \( F|_{\hbar=0}(\langle x \rangle) \subset \langle x \rangle \) and \( F|_{\hbar=0}(\langle \psi \rangle) \subset \langle \psi \rangle \), of the distinguished Lagrangian submanifolds in \( \mathcal{M}|_{\hbar=0} \) and \( \hat{M}|_{\hbar=0} \).

(ii) If \( F : M \rightarrow \hat{M} \) is a morphism of quantum BV manifolds, then \( dF|_{\hbar=0} \) induces in fact a morphism of dg vector spaces, \( (\mathcal{T}_x M, \bar{\partial}) \rightarrow (\mathcal{T}_x \hat{M}, \bar{\partial}) \); \( F \) is called a quasi-isomorphism if the latter map induces an isomorphism of the associated cohomology groups.

5.6.2. Theorem \cite{Me7}. Every quantum quasi-classically split BV manifold is isomorphic to the product of a minimal one and of a contractible one. In particular, every such manifold is quasi-isomorphic to a minimal one.

5.7. Homotopy transfer of quantum BV-structures via Feynman integral. Homotopy transfer formulae of \( P_\infty \)-structures given by Koszul duality theory are given by sums of decorated graphs which resemble Feynman diagrams in quantum field theory. This resemblance was made a rigorous fact in \cite{Min} for the case of the wheeled operad of unimodular Lie algebras (see \cite{3.3}). Given any complex \( V \) and a dg Lie 1-bialgebra structure on \( V \) with degree 0 Lie cobrackets \( \Delta_{\text{CoLie}} : V \rightarrow \Lambda^2 V \) and degree 1 Lie brackets \( [\bullet \bullet] : \mathcal{O}^2 V \rightarrow V [1] \), the associated by Koszul duality theory homotopy formulae transfer this rather trivial quantum BV manifold structure on \( V \) to a highly non-trivial quantum master function on its cohomology \( H(V) \); the same formulae can
also be described [Mc7] by a standard Batalin-Vilkovisky quantization [BaVi] of a BF-type field theory on the space $V \oplus V^*[1]$ with the action given by,

$$S : V \oplus V^*[1] \rightarrow \mathbb{K}$$

$$p \oplus \omega \rightarrow S(p, \omega) := \langle p, d\omega \rangle + \frac{1}{2}\langle p, [\omega, \omega] \rangle + \frac{1}{8}\langle p \bullet p, \omega \rangle,$$

where $\langle , , \rangle$ stands for the natural pairing, and $[ , , ] : \odot^2(V^*[1]) \rightarrow V^*[2]$ for the dualization of $\Delta_{\text{CoLie}}$. Thus at least in some cases the Koszul duality technique for homotopy transfer of $\infty$-structures is identical to the Feynman diagram technique in theoretical physics. The beauty of the latter lies in its combinatorial simplicity (due to the Wick theorem), while the power of the former lies in its generality: the Koszul duality theory applies to any (non-commutative case including) quadratic $\mathfrak{g}$-algebras.

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SERGEI A. MERKULOVA: DEPARTMENT OF MATHEMATICS, STOCKHOLM UNIVERSITY, 10691 STOCKHOLM, SWEDEN

E-mail address: sm@math.su.se