SHARP LOWER ERROR BOUNDS FOR STRONG APPROXIMATION OF SDES WITH DISCONTINUOUS DRIFT COEFFICIENT BY COUPLING OF NOISE

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Abstract. In the past decade, an intensive study of strong approximation of stochastic differential equations (SDEs) with a drift coefficient that has discontinuities in space has begun. In the majority of these results it is assumed that the drift coefficient satisfies piecewise regularity conditions and that the diffusion coefficient is globally Lipschitz continuous and non-degenerate at the discontinuities of the drift coefficient. Under this type of assumptions the best $L^p$-error rate obtained so far for approximation of scalar SDEs at the final time is $3/4$ in terms of the number of evaluations of the driving Brownian motion. In the present article we prove for the first time in the literature sharp lower error bounds for such SDEs. We show that for a huge class of additive noise driven SDEs of this type the $L^p$-error rate $3/4$ can not be improved.

For the proof of this result we employ a novel technique by studying equations with coupled noise: we reduce the analysis of the $L^p$-error of an arbitrary approximation based on evaluation of the driving Brownian motion at finitely many times to the analysis of the $L^p$-distance of two solutions of the same equation that are driven by Brownian motions that are coupled at the given time-points and independent, conditioned on their values at these points. To obtain lower bounds for the latter quantity, we prove a new quantitative version of positive association for bivariate normal random variables $(Y, Z)$ by providing explicit lower bounds for the covariance $\text{Cov}(f(Y), g(Z))$ in case of piecewise Lipschitz continuous functions $f$ and $g$. In addition it turns out that our proof technique also leads to lower error bounds for estimating occupation time functionals $\int_0^1 f(W_t) \, dt$ of a Brownian motion $W$, which substantially extends known results for the case of $f$ being an indicator function.

1. Introduction and main results

Consider a scalar autonomous stochastic differential equation (SDE)

$$dX_t = \mu(X_t) \, dt + \sigma(X_t) \, dW_t, \quad t \in [0, 1],$$

$$X_0 = x_0$$

with deterministic initial value $x_0 \in \mathbb{R}$, drift coefficient $\mu : \mathbb{R} \to \mathbb{R}$, diffusion coefficient $\sigma : \mathbb{R} \to \mathbb{R}$ and 1-dimensional standard Brownian motion $W$. Assume that the SDE (1) has a unique strong solution $X$. In this article we study $L^p$-approximation of $X_1$ by means of methods that use finitely many evaluations of the driving Brownian motion $W$ in the case when the drift coefficient $\mu$ may have discontinuity points.

SDEs with a drift coefficient that has discontinuities in space arise e.g. in mathematical finance, insurance and stochastic control problems. In the past decade, an intensive study of strong approximation of such SDEs has begun. All investigations carried out so far study the performance of classical numerical methods for such equations or present new numerical methods
and provide corresponding upper error bounds. See [5, 6] for results on convergence in probability and almost sure convergence of the Euler-Maruyama scheme and [3, 4, 7, 12, 13, 14, 17, 19, 20, 21] for results on $L_{\mu}$-approximation. In the present article we provide for the first time lower error bounds that are valid for any approximation of $X_1$ based on a finite number of evaluations of $W$ and are sharp for a huge class of additive noise driven SDEs of this type.

To be more precise, consider the following conditions on the coefficients $\mu$ and $\sigma$.

$(\mu 1)$ There exist $k \in \mathbb{N}$ and $-\infty = \xi_0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = \infty$ such that $\mu$ is Lipschitz continuous on $(\xi_{i-1}, \xi_i)$ for every $i \in \{1, \ldots, k+1\}$,

$(\sigma 1)$ $\sigma$ is Lipschitz continuous on $\mathbb{R}$ and $\sigma(\xi_i) \neq 0$ for every $i \in \{1, \ldots, k\}$.

If $\mu$ and $\sigma$ satisfy $(\mu 1)$ and $(\sigma 1)$, respectively, then the SDE (1) has a unique strong solution $X$, see [12]. $L_{\mu}$-approximation of $X_1$ under the assumptions $(\mu 1)$ and $(\sigma 1)$ has been studied in [12, 13, 14, 19, 21]. In particular, in [12, 13] the first numerical method has been constructed which achieves, under $(\mu 1)$ and $(\sigma 1)$, an $L_2$-error rate of at least $1/2$ in terms of the number of evaluations of $W$. This method is based on a suitable transformation of the strong solution $X$ into a strong solution of an SDE with Lipschitz continuous coefficients. Thereafter, in [21] an adaptive Euler-Maruyama scheme has been constructed, which achieves, under $(\mu 1)$ and $(\sigma 1)$, an $L_2$-error rate of at least $1/2$ in terms of the average number of evaluations of $W$ used by this method. Finally, in [19] it has been proven that, under $(\mu 1)$ and $(\sigma 1)$, the standard Euler-Maruyama scheme with $n$ equidistant steps in fact achieves for all $p \in [1, \infty)$ an $L_p$-error rate of at least $1/2$ in terms of the number $n$ of evaluations of $W$ as in the classical case of SDEs with globally Lipschitz continuous coefficients.

Recently in [17] the first higher-order method has been constructed for such SDEs, which achieves for all $p \in [1, \infty)$ an $L_p$-error rate $3/4$ if $\mu$ and $\sigma$ satisfy $(\mu 1)$ and $(\sigma 1)$ and additionally the following piecewise regularity assumptions

$(\mu 2)$ $\mu$ has a Lipschitz continuous derivative on $(\xi_{i-1}, \xi_i)$ for every $i \in \{1, \ldots, k+1\}$,

$(\sigma 2)$ $\sigma$ has a Lipschitz continuous derivative on $(\xi_{i-1}, \xi_i)$ for every $i \in \{1, \ldots, k+1\}$.

More precisely, in [17] the following theorem has been proven.

**Theorem 1.** Assume that $\mu$ satisfies $(\mu 1)$ and $(\mu 2)$ and that $\sigma$ satisfies $(\sigma 1)$ and $(\sigma 2)$. Then there exist a sequence of measurable functions $g_n : \mathbb{R}^n \to \mathbb{R}$, $n \in \mathbb{N}$, such that for every $p \in [1, \infty)$ there exists $c \in (0, \infty)$ such that for every $n \in \mathbb{N},$

$$\mathbb{E} \left[ |X_1 - g_n(W_{1/n}, W_{2/n}, \ldots, W_1)|^p \right]^{1/p} \leq c/n^{3/4}.$$

The approximations $g_n(W_{1/n}, W_{2/n}, \ldots, W_1)$ in Theorem 1 are obtained by applying a suitable transformation $G : \mathbb{R} \to \mathbb{R}$ to the strong solution $X$ of the SDE (1) such that the transformed solution $Y = (G(X_t))_{t \in [0,1]}$ is a strong solution of a new SDE with sufficiently regular coefficients. A Milstein-type scheme $\hat{Y}_n$ with $n$ equidistant steps is then used to approximate $Y_1$ and $G^{-1}(\hat{Y}_n)$ yields an approximation of $X_1$, which satisfies the upper error bound in Theorem 1. See [17, Section 4] for details. We add that in [20] it has been proven that if $\sigma = 1$ and $\mu$ is bounded and piecewise $C^2_b$ then the standard Euler-Maruyama scheme in fact achieves an $L_2$-error rate of at least $3/4$ in terms of the number of evaluations of $W$. Note that in the latter case the Euler-Maruyama scheme coincides with the Milstein scheme.
It is well known that in the classical case of globally Lipschitz continuous coefficients $\mu$ and $\sigma$, the Milstein scheme achieves for all $p \in [1, \infty)$ an $L_p$-error rate of at least $1$ in terms of the number of evaluations of $W$ under the additional regularity assumption that $\mu$ and $\sigma$ have bounded and Lipschitz continuous derivatives, see e.g. \cite{9}. It is therefore natural to ask whether there exists a method based on finitely many evaluations of $W$ that achieves under the assumptions ($\mu_1$), ($\mu_2$) and ($\sigma_1$), ($\sigma_2$) a better $L_p$-error rate than the rate $3/4$ guaranteed by Theorem \ref{thm:1}. To the best of our knowledge the answer to this question was not known in the literature up to now. In the present article we answer this question in the negative. More precisely, we show that no numerical method based on $n$ evaluations of $W$ can achieve an $L_p$-error rate better than $3/4$ in terms of $n$ if $\sigma = 1$ and $\mu$ satisfies, additionally to ($\mu_1$) and ($\mu_2$), the conditions

\begin{enumerate}
  \item ($\mu_3$) $\exists i \in \{1, \ldots, k\}$: $\mu(\xi_{i+}) \neq \mu(\xi_{i-})$,
  \item ($\mu_4$) $\mu$ is increasing,
  \item ($\mu_5$) $\mu$ is bounded.
\end{enumerate}

More formally, the main result of this article is the following theorem.

**Theorem 2.** Assume that $\mu$ satisfies ($\mu_1$) to ($\mu_5$) and that $\sigma = 1$. Then there exists $c \in (0, \infty)$ such that for all $n \in \mathbb{N}$,

\begin{equation}
\inf_{t_1, \ldots, t_n \in [0,1], \ g: \mathbb{R}^n \to \mathbb{R} \text{ measurable}} \mathbb{E}[\|X_1 - g(W_{t_1}, \ldots, W_{t_n})\|] \geq \frac{c}{n^{3/4}}.
\end{equation}

Note that Theorem \ref{thm:2} also shows that the $L_p$-error rate $3/4$ can in general not be improved even then when additionally to the assumptions ($\mu_1$), ($\mu_2$) and ($\sigma_1$), ($\sigma_2$) piecewise regularity assumptions are imposed on $\mu$ and $\sigma$. Not even the property of being piecewise infinitely often differentiable with uniformly bounded derivatives may help.

As an example consider the strong solution $X$ of \cite{11} with $\sigma = 1$ and $\mu = 1_{[0, \infty)}$. We then have by Theorem \ref{thm:1} and Theorem \ref{thm:2} that for all $n \in \mathbb{N}$,

$$
\frac{c_1}{n^{3/4}} \leq \inf_{t_1, \ldots, t_n \in [0,1], \ g: \mathbb{R}^n \to \mathbb{R} \text{ measurable}} \mathbb{E}\left[\int_0^1 1_{[0, \infty)}(X_s) \, ds - g(W_{t_1}, \ldots, W_{t_n})\right]^{1/p} \leq \frac{c_2}{n^{3/4}},
$$

where $c_1, c_2 \in (0, \infty)$ depend only on $p$.

We briefly discuss the additional conditions ($\mu_3$) to ($\mu_5$) used in Theorem \ref{thm:2}. First note that property ($\mu_1$) implies that the limits $\mu(\xi_{i-}) = \lim_{x \downarrow \xi_i} \mu(x)$ and $\mu(\xi_{i+}) = \lim_{x \uparrow \xi_i} \mu(x)$ exist for all $i \in \{1, \ldots, k\}$, see Lemma \ref{lem:1}. In the presence of ($\mu_1$) and ($\mu_2$), the condition ($\mu_3$) can not be waived in Theorem \ref{thm:2} if $\mu$ satisfies ($\mu_1$) and ($\mu_2$) and $\mu(\xi_i+) = \mu(\xi_i-)$ for every $i \in \{1, \ldots, k\}$ then $\mu_{\mu_1}(\xi_1, \ldots, \xi_k)$ has a Lipschitz continuous extension $\tilde{\mu}: \mathbb{R} \to \mathbb{R}$, which has a Lipschitz continuous derivative on $(\xi_{i-}, \xi_i)$ for every $i \in \{1, \ldots, k + 1\}$, and $X$ is also a strong solution of the SDE \cite{11} with $\mu$ replaced by $\tilde{\mu}$ and $\sigma = 1$. By \cite{17} Theorem 2 it then follows that the Milstein scheme achieves at least an $L_p$-error rate $1$ for approximation of $X_1$, which is in contradiction to the lower bound \ref{eq:lower-bound}. The condition ($\mu_4$) is of major importance for our proof of Theorem \ref{thm:2}, see the discussion of our proof strategy below, and it is unclear to us whether this condition could be weakened or even dropped. With respect to condition ($\mu_5$) we believe that its use in the proof of Theorem \ref{thm:2} could be avoided by fully exploiting the fact that under the
condition \((\mu 1)\) the drift coefficient \(\mu\) satisfies a linear growth condition, see Lemma 1, which in turn implies that \(X_1\) has finite moments of any order.

We add that lower error bounds for strong approximation of scalar SDEs at a single time are already provided in [8] and [16], but in the setting of Theorem 2 these bounds turn out to be much too small. In fact, if \(\sigma = 1\), \(\mu\) satisfies \((\mu 1)\) and if there exists an open interval \(I \subset \mathbb{R}\) and a time \(t_0 \in [0,1)\) such that \(\mu\) is three times continuously differentiable on \(I\), \(\mu' \neq 0\) on \(I\) and \(\mathbb{P}(X_{t_0} \in I) > 0\) then [8, Theorem 6] implies only that (2) holds with \(c/n^{3/4}\) replaced by \(c/n\). Note, however, that the lower bound \(c/n\) in [8, Theorem 6] is also valid for approximations of \(X_1\) that may use \(n\) sequential evaluations of \(W\) on average, while Theorem 2 only covers approximations that are based on evaluation of \(W\) at \(n\) fixed discretization sites. We conjecture that the lower bound in Theorem 2 does not hold anymore if one allows for sequential evaluation of \(W\) and that one can achieve under the assumptions \((\mu 1), (\mu 2)\) and \((\sigma 1), (\sigma 2)\) an \(L_p\)-error rate 1 by a method based on adaptive step-size control. The proof of this conjecture will be the subject of future work.

We turn to a sketch of our proof strategy for Theorem 2, which heavily differs from the techniques known from the literature that have been employed so far for establishing lower error bounds in the context of approximation of SDEs. To avoid technical details we restrict to the analysis of the \(L_2\)-error and we only study approximations of \(X_1\) that are based on equidistant evaluations of \(W\). Fix \(n \geq 2\) and put \(t_i = i/n\) for \(i \in \{0,1,\ldots,n\}\).

The central idea of our proof is to consider a second Brownian motion \(\tilde{W}\) such that \(W\) and \(\tilde{W}\) are coupled at the points \(t_1,\ldots,t_n\) but independent, conditioned on \(W_{t_1},\ldots,W_{t_n}\), and to study the mean square distance of the two corresponding strong solutions \(X\) and \(\tilde{X}\) of the SDE (1) at time 1. Formally, let \(\bar{W}\) denote the piecewise linear interpolation of \(W\) at the points \(t_0,\ldots,t_n\), let \(B = W - \bar{W}\) denote the corresponding piecewise Brownian bridge process, and define

\[
\tilde{W} = \bar{W} + \tilde{B},
\]

where \(\mathbb{P}^B = \mathbb{P}^{\tilde{B}}\) and \(W, \tilde{B}\) are independent. Then for all \(t \in [0,1]\),

\[
X_t = x_0 + \int_0^t \mu(X_s) \, ds + W_t, \quad \tilde{X}_t = x_0 + \int_0^t \mu(\tilde{X}_s) \, ds + \tilde{W}_t,
\]

and for every measurable function \(g: \mathbb{R}^n \to \mathbb{R}\) one has

\[
(3) \quad \mathbb{E}[|X_1 - g(W_{t_1},\ldots,W_{t_n})|^2]^{1/2} \geq \frac{1}{2} \mathbb{E}[|X_1 - \tilde{X}_1|^2]^{1/2},
\]

see Lemma 11. By the coupling of \(W\) and \(\tilde{W}\) we have

\[
X_{t_i} - \tilde{X}_{t_i} = X_{t_{i-1}} - \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds
\]
for all $i \in \{1, \ldots, n\}$, which yields

$$E[|X_1 - \bar{X}_1|^2] = 2 \sum_{i=1}^{n} E\left[ (X_{t_{i-1}} - \bar{X}_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\bar{X}_s)) \, ds \right]$$

(4)

$$= m_i$$

$$+ \sum_{i=1}^{n} E\left[ \left( \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\bar{X}_s)) \, ds \right)^2 \right].$$

(5)

Using the assumption that $\mu$ is increasing and the fact that pathwise uniqueness holds for equation (1), we obtain by a comparison theorem for SDEs that

$$m_i \geq 0$$

for all $i \in \{1, \ldots, n\}$, see Lemma 13.

For the analysis of the terms $d_i$ we first show that for all $i \geq n/2 + 1$ the solutions $X$ and $\bar{X}$ may be replaced on $[t_{i-1}, t_i]$ by the processes

$$(X_{t_{i-1}} + W_s - W_{t_{i-1}})_{s \in [t_{i-1}, t_i]}$$

and

$$(X_{t_{i-1}} + \bar{W}_s - \bar{W}_{t_{i-1}})_{s \in [t_{i-1}, t_i]},$$

respectively, in the sense that

$$d_i \geq \frac{1}{4} E\left[ \left( \int_{t_{i-1}}^{t_i} (\mu(X_{t_{i-1}} + W_s - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \bar{W}_s - \bar{W}_{t_{i-1}})) \, ds \right)^2 \right] - \frac{c}{n^{5/2 + 1/16}}.$$

(6)

see Lemma 14. To obtain (6) we establish appropriate $L_p$-estimates for the differences $X_{t_{i-1}} - \bar{X}_{t_{i-1}}$, see Lemma 12, and $L_2$-estimates for the total time of $(X_s)_{s \in [t_{i-1}, t_i]}$ and $(X_{t_{i-1}} + W_s - W_{t_{i-1}})_{s \in [t_{i-1}, t_i]}$ lying on different sides of a fixed horizontal line in order to cope with the discontinuities of the drift coefficient $\mu$, see Lemma 10.

It remains to provide lower bounds for the terms $E[R_i]$ for $i \geq n/2 + 1$. Note that $W_s - W_{t_{i-1}} = n(s-t_{i-1})(W_{t_i} - W_{t_{i-1}}) + B_s$ and $\bar{W}_s - \bar{W}_{t_{i-1}} = n(s-t_{i-1})(W_{t_i} - W_{t_{i-1}}) + \bar{B}_s$, and therefore for $\mathbb{P}(X_{t_{i-1}} + W_{t_{i}} - W_{t_{i-1}})$-almost all $(x, \delta) \in \mathbb{R}^2$,

$$E[R_i | X_{t_{i-1}} = x, W_{t_i} - W_{t_{i-1}} = \delta] = 2 \int_{t_{i-1}}^{t_i} \int_{t_{i-1}}^{t_i} \text{Cov}(\mu(x+a_s \delta + B_s), \mu(x+a_t \delta + B_t)) \, ds \, dt,$$

(7)

where $a_s = n(s-t_{i-1})$ for all $s \in [t_{i-1}, t_i]$. Note further that $B_s$ and $B_t$ are nonnegatively correlated for all $s, t \in [t_{i-1}, t_i]$. It is well known that nonnegatively correlated, jointly normally distributed random variables are positively associated so that $\text{Cov}(\mu(x+a_s \delta + B_s), \mu(x+a_t \delta + B_t)) \geq 0$ for all $s, t \in [t_{i-1}, t_i]$ because $\mu$ is increasing. In Lemma 17 in the appendix we establish for bivariate normal random variables $(Z_1, Z_2)$ with nonnegative correlation and increasing, piecewise Lipschitz continuous functions $f_1, f_2: \mathbb{R} \to \mathbb{R}$ a lower bound for the covariance of $f_1(Z_1)$ and $f_2(Z_2)$ in terms of the jump sizes of $f_1$ and $f_2$ at their discontinuity points. We then apply these covariance bounds to the integrand in the right hand side of (7) and take
Lemma 15. Assume that
\[ \mathbb{E}[R_i] \geq (\mu(\xi_\ell^+) - \mu(\xi_\ell^-))^2 \cdot \frac{c}{n^{5/2}} \]
for all \( i \geq n/2 + 1 \) and \( \ell \in \{1, \ldots, k\} \), where \( c \in (0, \infty) \) does not depend on \( n \), see Lemma 3 and Lemma 15.

Combining (4), (5), (6), (7) and (8) yields the claimed lower bound in Theorem 2 for the \( L_2 \)-error in place of the \( L_1 \)-error.

Our proof technique also applies to obtain lower error bounds for estimating occupation time functionals of the Brownian motion \( W \).

Theorem 3. Assume that \( \mu \) satisfies the assumptions (\( \mu 1 \)) and (\( \mu 3 \)) and is increasing or decreasing. Then for every \( \varepsilon \in (0, \infty) \) there exists \( c \in (0, \infty) \) such that for all \( n \in \mathbb{N} \),
\[ \inf_{g: \mathbb{R}^n \rightarrow \mathbb{R}} \mathbb{E}\left[ \left( \int_0^1 \mu(W_s) \, ds - g(W_{t_1}, \ldots, W_{t_n}) \right)^p \right]^{1/p} \geq \begin{cases} \frac{c}{n^{1/4}}, & \text{if } p = 2, \\ \frac{c}{n^{3/4 + \varepsilon}}, & \text{if } p = 1. \end{cases} \]

Theorem 3 generalizes a result in [22], which establishes the lower bound \( c/n^{3/4} \) for the \( L_2 \)-error of the Riemann-sum estimator \( n^{-1} \sum_{i=1}^n 1_{[0, \infty)}(W_{(i-1)/n}) \) of the integral \( \int_0^1 1_{[0, \infty)}(W_s) \, ds \).

For that particular estimator, \( c/n^{3/4} \) is also an upper \( L_2 \)-error bound, see [22]. We conjecture that the latter result extends to our more general setting as well, in the sense that the \( L_\mu \)-error of the Riemann-sum estimator \( n^{-1} \sum_{i=1}^n \mu(W_{(i-1)/n}) \) of \( I_\mu(W) = \int_0^1 \mu(W_s) \, ds \) is bounded by \( c/n^{3/4} \) if \( \mu \) satisfies (\( \mu 1 \)) and (\( \mu 2 \)). We furthermore add that lower and upper error bounds for estimators of occupation time functionals \( I_\mu(W) \) for bounded \( \mu \) from fractional \( L_2 \)-Sobolev spaces are established in [1]. In particular, in the latter paper it is shown that for \( \mu \in L_2(\mathbb{R}) \) having the Fourier-transform \( u \mapsto (1 + |u|)^{-s-1/2} \), where \( s \in [0, 1] \), the lower bound \( c/n^{(1+s)/2} \) holds for the \( L_2 \)-error of any estimator of \( I_\mu(W) \) based on \( W_{i/n}, i = 1, \ldots, n \).

Obviously, strong approximation at time 1 of the solution \( X \) of the equation (1) with \( \sigma = 1 \) is closely related to estimating the occupation time functional \( I_\mu(X) = \int_0^1 \mu(X_s) \, ds \). The first problem requires approximation of \( I_\mu(X) \) based on \( n \) evaluations of the driving Brownian motion \( W \), while the second problem deals with approximation of \( I_\mu(X) \) based on \( n \) evaluations of the process \( X \). It is an open question to us whether these problems have the same complexity in the sense of identical smallest possible error rates in terms of \( n \).

Clearly, Theorem 3 is not a special case of Theorem 2 but it seems likely that both results are particular cases of a (yet to be shown) result on sharp lower error bounds for strong approximation of systems of SDEs with commutative noise and drift coefficients that satisfy suitable multivariate versions of the conditions (\( \mu 1 \)) to (\( \mu 5 \)). The condition of commutative noise stems from the fact that for strong approximation at a single time of systems of SDEs with non-commutative noise the \( L_2 \)-error rate 1/2 can in general not be improved by any approximation based on finitely many evaluations of the driving Brownian motion. See [2] [15] for details.

2. Proofs

We briefly outline the structure of this section. In Subsection 2.1 we provide properties of the drift coefficient \( \mu \) under the assumption (\( \mu 1 \)) and the lower bound-techniques that are crucial for
then lim and hence has a limit which completes the proof of the lemma.

Using the Lipschitz continuity of \( \mu \), the proof of (10).

Now observe that for all \( i \in \{1, \ldots, k+1 \} \) fix some \( x_i \in (\xi_{i-1}, \xi_i) \). By (µ1) there exists \( c \in (0, \infty) \) such that for all \( i \in \{1, \ldots, k+1 \} \) and all \( x \in (\xi_{i-1}, \xi_i) \),

\[
|\mu(x)| \leq |\mu(x) - \mu(x_i)| + |\mu(x_i)| \leq c|x - x_i| + |\mu(x_i)| \leq c|x| + c|x_i| + |\mu(x_i)|.
\]

Hence for all \( x \in \mathbb{R} \),

\[
|\mu(x)| \leq \max_{i=1, \ldots, k} |\mu(x_i)| + \max_{i=1, \ldots, k+1} (c|x_i| + |\mu(x_i)|) + c|x|,
\]

which proves that \( \mu \) satisfies a linear growth condition.

Next observe that

\[
\mathbb{R}^2 \setminus \bigcup_{i=1}^{k+1} (\xi_{i-1}, \xi_i)^2 = \bigcup_{i=1}^{k} D_i.
\]

Using the latter fact, the linear growth property of \( \mu \) and (µ1) we see that there exist \( c_1, c_2 \in (0, \infty) \) such that for all \( x, y \in \mathbb{R} \),

\[
|\mu(x) - \mu(y)| \leq c_1 |x - y| + c_1 (1 + |x| + |y|) \mathbb{1}_{\bigcup_{i=1}^{k} D_i}(x, y)
\]

\[
\leq c_1 |x - y| + c_2 \sum_{i=1}^{k} (1 + |x - \xi_i| + |y - \xi_i|) \mathbb{1}_{D_i}(x, y).
\]

Now observe that for all \( i \in \{1, \ldots, k \} \) and all \( x, y \in D_i \) we have \( |x - \xi_i| \leq |x - y| \), which finishes the proof of (10).

Finally, let \( i \in \{1, \ldots, k \} \) and let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in \( (\xi_{i-1}, \xi_i) \), which converges to \( \xi_i \). Using the Lipschitz continuity of \( \mu \) on \( (\xi_{i-1}, \xi_i) \) we obtain that \( (\mu(x_n))_{n \in \mathbb{N}} \) is a Cauchy-sequence and hence has a limit \( z \in \mathbb{R} \). If \( (\tilde{x}_n)_{n \in \mathbb{N}} \) is a further sequence in \( (\xi_{i-1}, \xi_i) \), which converges to \( \xi_i \), then \( \lim_{n \to \infty} (x_n - \tilde{x}_n) = 0 \) and by the Lipschitz continuity of \( \mu \) on \( (\xi_{i-1}, \xi_i) \) we conclude that \( \lim_{n \to \infty} (\mu(x_n) - \mu(\tilde{x}_n)) = 0 \). Thus the sequence \( (\mu(\tilde{x}_n))_{n \in \mathbb{N}} \) converges to \( z \) as well. This proves the existence of the limit \( \mu(\xi_i -) \in \mathbb{R} \). The existence of the limit \( \mu(\xi_i +) \) in \( \mathbb{R} \) is shown in the same manner. This completes the proof of the lemma. \( \square \)
The following two lemmas are crucial for the proof of both Theorem 2 and Theorem 3. Lemma 2 is an elementary consequence of the triangle inequality, see also [18, Lemma 3].

**Lemma 2.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \((\Omega_1, \mathcal{A}_1)\) and \((\Omega_2, \mathcal{A}_2)\) be measurable spaces and let \(V_1: \Omega \to \Omega_1, V_2, V_2': \Omega \to \Omega_2\) be random variables such that

\[
\mathbb{P}(V_1, V_2) = \mathbb{P}(V_1, V_2').
\]

Then for all \(p \in [1, \infty)\) and for all measurable mappings \(\Phi: \Omega_1 \times \Omega_2 \to \mathbb{R}\) and \(\varphi: \Omega_1 \to \mathbb{R}\),

\[
\mathbb{E}[|\Phi(V_1, V_2) - \varphi(V_1)|^p]^{1/p} \geq \frac{1}{2} \mathbb{E}[|\Phi(V_1, V_2) - \Phi(V_1, V_2')|^p]^{1/p}.
\]

Put

\[
(11) \quad \kappa = \frac{1}{16\pi} e^{-\frac{3}{2}} \int_0^{1/\sqrt{3}} \frac{1}{\sqrt{1-x^2}} e^{-\frac{24}{1-4x^2}} dx.
\]

**Lemma 3.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, let \(t \in (0, 1], B, B': [0, t] \times \Omega \to \mathbb{R}\) be Brownian bridges on \([0, t]\), let \(U, V: \Omega \to \mathbb{R}\) be random variables and assume that \(B, B', U, V\) are independent. Furthermore, let \(k \in \mathbb{N}\), let \(-\infty = \xi_0 < \xi_1 < \ldots < \xi_k < \xi_{k+1} = \infty\) and let \(h: \mathbb{R} \to \mathbb{R}\) satisfy

(i) \(h\) is increasing or decreasing,

(ii) \(h\) is Lipschitz continuous on the interval \((\xi_{i-1}, \xi_i)\) for all \(i \in \{1, \ldots, k+1\}\).

Then for every \(i \in \{1, \ldots, k\}\) it holds

\[
\mathbb{E}\left[\left|\int_0^t (h(U + sV + B_s) - h(U + sV + B'_s)) \, ds\right|^2\right] \\
\geq \kappa (h(\xi_{i+}) - h(\xi_{i-}))^2 t^2 \mathbb{P}(U \in [\xi_i, \xi_{i+} + \sqrt{t}]) \mathbb{P}(V \in [0, 1/\sqrt{t}]).
\]

**Proof.** Note that all of the limits \(h(\xi_{i+}), h(\xi_{i-}), i = 1, \ldots, k\), exist due to the assumption (ii), see Lemma 3. Put

\[
R = (R_s = U + sV + B_s)_{s \in [0, t]}, \quad R' = (R'_s = U + sV + B'_s)_{s \in [0, t]}
\]

and let

\[
D = \mathbb{E}\left[\left|\int_0^t (h(R_s) - h(R'_s)) \, ds\right|^2\right].
\]

Clearly, \(\mathbb{P}(R, R') = \mathbb{P}(R', R)\), and therefore

\[
D = 2 \int_0^t \int_0^t \mathbb{E}[h(R_r) h(R_s) - h(R_r) h(R'_s)] \, ds \, dr.
\]

Let \(\varphi: [0, t]^2 \times \mathbb{R}^2 \to \mathbb{R}\) be given by

\[
\varphi(s, r, u, v) = \mathbb{E}[h(B_r + u + rv) h(B_s + u + sv)] - \mathbb{E}[h(B_r + u + rv)] \mathbb{E}[h(B_s + u + sv)].
\]
Moreover, we show that for all $i$ with $1 \leq i \leq k$

Thus, $Z,Y,f,g$ and define

\[ (14) \]

The latter bound and the assumption that $h$ is increasing or decreasing yield (12).
Next, let $i \in \{1, \ldots, k\}$. Since $h$ is increasing or decreasing we conclude from (13) in particular, that
\begin{equation}
\varphi(s, r, u, v) \geq (h(\xi_i) - h(\xi_j))^2 \frac{1}{2\pi} e^{-\frac{a^2}{2}} \int_0^\rho \frac{1}{\sqrt{1 - x^2}} e^{-\frac{(b_i - b_j)^2}{2(1-x^2)}} \, dx.
\end{equation}

Let $s, r \in [t/4, t/2]$, $u \in [\xi_i, \xi_i + \sqrt{7}]$, $v \in [0, 1/\sqrt{7}]$. We then have
\[ \max(|a_i|, |b_i|) \leq \left(\sqrt{7} + \frac{1}{2} \frac{1}{\sqrt{7}}\right) \frac{\sqrt{7}}{\sqrt{4/7}} = 2\sqrt{3}. \]

Thus,
\[ e^{-\frac{a^2}{2}} \geq e^{-6}. \]

Moreover, for all $x \in [0, \rho]$,
\[ e^{-\frac{(b_i - b_j)^2}{2(1-x^2)}} \geq e^{-\frac{|a_i| + |b_i|}{2(1-x^2)}} \geq e^{-\frac{24}{1-x^2}}, \]
and
\[ \rho = \frac{\sqrt{\max(s,r)} \sqrt{\min(s,r)}}{\sqrt{\max(s,r)} \sqrt{\min(s,r)}} \geq \frac{\sqrt{7}}{\sqrt{4/7}} = \frac{1}{\sqrt{3}}. \]

Hence we conclude that
\[ \frac{1}{2\pi} e^{-\frac{a^2}{2}} \int_0^\rho \frac{1}{\sqrt{1 - x^2}} e^{-\frac{(b_i - b_j)^2}{2(1-x^2)}} \, dx \geq \frac{1}{2\pi} e^{-6} \int_0^{1/\sqrt{3}} \frac{1}{\sqrt{1 - x^2}} e^{-\frac{24}{1-x^2}} \, dx = 8\kappa. \]

The latter estimate together with (15) implies (13) and completes the proof of the lemma. \qed

2.2. Proof of Theorem 3. In the following let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W: [0,1] \times \Omega \rightarrow \mathbb{R}$ be a standard Brownian motion on $[0,1]$.

For the proof of Theorem 3 it suffices to show that for every $\varepsilon \in (0, \infty)$ there exists $c \in (0, \infty)$ such that for all $n \in 2\mathbb{N}$ with $n \geq 6$ and all $t_1, \ldots, t_n \in [0,1]$ with
\begin{equation}
0 < t_1 < \ldots < t_n = 1
\end{equation}
and
\begin{equation}
2/n, 4/n, \ldots, 1 \in \{t_1, \ldots, t_n\}
\end{equation}
we have
\begin{equation}
\inf_{g: \mathbb{R}^n \rightarrow \mathbb{R} \text{ measurable}} \mathbb{E} \left[ \left| \int_0^1 \mu(W_s) \, ds - g(W_{t_1}, \ldots, W_{t_n}) \right|^p \right]^{1/p} \geq \begin{cases} \frac{\varepsilon}{n^{p/2}}, & \text{if } p = 2, \\ \frac{\varepsilon}{n^{p/4}}, & \text{if } p = 1. \end{cases}
\end{equation}

In the sequel we fix $n \in 2\mathbb{N}$ with $n \geq 6$ and $t_1, \ldots, t_n \in [0,1]$ with (16) and (17). Moreover, we put $t_0 = 0$. Let $\overline{W}: [0,1] \times \Omega \rightarrow \mathbb{R}$ denote the piecewise linear interpolation of $W$ on $[0,1]$ at the points $t_0, \ldots, t_n$, i.e.
\[ \overline{W}_t = \frac{t - t_{i-1}}{t_i - t_{i-1}} W_{t_i} + \frac{t - t_i}{t_{i+1} - t_i} W_{t_{i+1}}, \quad t \in [t_{i-1}, t_i], \]
for $i \in \{1, \ldots, n\}$, and put
\[ B = W - \overline{W}. \]
Assume that \((B_t)_{t \in [t_i, t_{i-1}]}\) is a Brownian bridge on \([t_{i-1}, t_i]\) for every \(i \in \{1, \ldots, n\}\). Furthermore, \((B_t)_{t \in [t_0, t_1]}, \ldots, (B_t)_{t \in [t_{n-1}, t_n]}\), \(\overline{W}\) are independent. Without loss of generality we may assume that \((\Omega, \mathcal{F}, \mathbb{P})\) is rich enough to carry for every \(i \in \{1, \ldots, n\}\) a Brownian bridge \((\tilde{B}_t)_{t \in [t_{i-1}, t_i]}\) on \([t_{i-1}, t_i]\) such that \((\tilde{B}_t)_{t \in [t_0, t_1]}, \ldots, (\tilde{B}_t)_{t \in [t_{n-1}, t_n]}\), \(W\) are independent. Put \(\tilde{B} = (\tilde{B}_t)_{t \in [0,1]}\) and define a Brownian motion \(\tilde{W}: [0,1] \times \Omega \to \mathbb{R}\) by

\[
\tilde{W} = \overline{W} + \tilde{B}.
\]

**Lemma 4.** Assume that \(\mu\) satisfies (\(\mu_1\)). Then for all measurable \(g: \mathbb{R}^n \to \mathbb{R}\) and all \(p \in [1, \infty)\),

\[
\mathbb{E} \left[ \left| \int_0^1 \mu(W_s) \, ds - g(W_{t_1}, \ldots, W_{t_n}) \right|^p \right]^{1/p} \leq \frac{1}{2} \mathbb{E} \left[ \left| \int_0^1 \mu(W_s) \, ds - \int_0^1 \mu(\tilde{W}_s) \, ds \right|^p \right]^{1/p}.
\]

**Proof.** For convenience of the reader we first show that \(\int_0^1 \mu(W_s) \, ds\) is well-defined and a random variable. By (\(\mu_1\)) the function \(\mu\) is piecewise continuous and therefore Borel-measurable. Thus, for every \(f \in C([0,1], \mathbb{R})\) the function \(\mu \circ f: [0,1] \to \mathbb{R}\) is Borel-measurable. Moreover, by Lemma 1 there exists \(c \in (0, \infty)\) such that for every \(t \in [0,1]\) we have \(|\mu(f(t))| \leq c(1 + |f(t)|) \leq c(1 + \|f\|_{\infty})\), which shows that \(\mu \circ f\) is bounded. Hence \(\int_0^1 \mu(f(s)) \, ds\) exists for every \(f \in C([0,1], \mathbb{R})\) and therefore, \(\int_0^1 \mu(W_s) \, ds\) defines a bounded Riemann-integrable function. Thus, the mapping \([0,1] \times C([0,1], \mathbb{R}) \ni (s, f) \mapsto f(s) \in \mathbb{R}\) is continuous. Hence, the mapping \([0,1] \times C([0,1], \mathbb{R}) \ni (s, f) \mapsto 1_{\mathcal{D}}(f(s)) \in \mathbb{R}\) is Borel-measurable, which implies the Borel-measurability of the mapping

\[
T: C([0,1], \mathbb{R}) \to \mathbb{R}, \quad f \mapsto \int_0^1 1_{\mathcal{D}}(f(s)) \, ds.
\]

Hence, \(T^{-1}(\{0\})\) is a Borel-subset of \(C([0,1], \mathbb{R})\). Let \(f \in T^{-1}(\{0\})\). Then \(\lambda(\{s \in [0,1]: f(s) \in \mathcal{D}\}) = 0\). By (\(\mu_1\)) we know that \(\{s \in [0,1]: \mu \circ f\ \text{is discontinuous in} \ s\} \subset \{s \in [0,1]: f(s) \in \mathcal{D}\}\). Hence \(\mu \circ f\) is a bounded Riemann-integrable function. Thus

\[
\forall f \in T^{-1}(\{0\}): \quad \int_0^1 \mu(f(s)) \, ds = \lim_{m \to \infty} R_m(f)
\]

with

\[
R_m: C([0,1], \mathbb{R}) \to \mathbb{R}, \quad f \mapsto \frac{1}{m} \sum_{i=1}^m \mu(f(i/m)).
\]

Clearly, the mappings \(R_m, m \in \mathbb{N}\), are Borel-measurable, and therefore the mappings

\[
S_m := R_m \cdot 1_{T^{-1}(\{0\})}: C([0,1], \mathbb{R}) \to \mathbb{R}, \quad m \in \mathbb{N},
\]

are Borel-measurable as well. Using (21) we obtain that the limit

\[
S := \lim_{m \to \infty} S_m: C([0,1], \mathbb{R}) \to \mathbb{R}
\]
exists, is Borel-measurable and satisfies \( S(f) = \int_0^1 \mu(f(s)) \, ds \) for all \( f \in T^{-1}([0]) \). Note that for \( V = W, \tilde{W} \),
\[
\mathbb{E} \left[ \int_0^1 1_{D}(V_s) \, ds \right] = \int_0^1 \mathbb{P}(V_s \in D) \, ds = 0.
\]

Hence \( \mathbb{P}(V \in T^{-1}([0])) = 1 \) and we conclude that \( S(V) = \int_0^1 \mu(V_s) \, ds \) almost surely. Thus, (20) holds for the Borel-measurable function
\[
\Phi : C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \rightarrow \mathbb{R}, \quad (f, g) \mapsto S(f + g).
\]

Let \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) be measurable. Clearly, there exists a measurable function \( \varphi : C([0, 1], \mathbb{R}) \rightarrow \mathbb{R} \) such that
\[
g(W_1, \ldots, W_n) = \varphi(W).
\]
Since \( W \) and \( B \) are independent, \( \bar{W} \) and \( \bar{B} \) are independent, and \( \mathbb{P}^B = \mathbb{P}^{\bar{B}} \), we have
\[
\mathbb{P}(\bar{W}, B) = \mathbb{P}(\bar{W}, \bar{B}).
\]
We may thus apply Lemma 2 with \( \Omega_1 = \Omega_2 = C([0, 1], \mathbb{R}) \), \( V_1 = \bar{W}, \ V_2 = B, \ \bar{V}_2 = \bar{B}, \ \Phi \) as in (20) and \( \varphi \) as in (22) to obtain (19) for every \( p \in [1, \infty) \).

Throughout the following we use \( c, c_1, c_2, \ldots \in (0, \infty) \) to denote positive constants that may change their values in every appearance but neither depend on \( n \) nor on the discretization points \( t_0, \ldots, t_n \).

Next, we provide an upper bound for the right hand side of (19) in the case \( p = 4 \).

**Lemma 5.** Assume that \( \mu \) satisfies \((\mu_1)\). Then for every \( \delta \in (0, \infty) \) there exists \( c \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \left| \int_0^1 (\mu(W_s) - \mu(\bar{W}_s)) \, ds \right|^4 \right]^{1/4} \leq \frac{c}{n^{3/4 - \delta}}.
\]

**Proof.** For all \( i \in \{1, \ldots, n\} \) put
\[
J_i = \int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(\bar{W}_s)) \, ds.
\]
First, we show that for all \( m \in \mathbb{N}, \ i \in \{1, \ldots, n\} \) and \( j_1, \ldots, j_m \in \{1, \ldots, n\} \setminus \{i\} \) we have
\[
\mathbb{E}[J_i \cdot J_{j_1} \cdots J_{j_m}] = 0.
\]
To this end let \( m \in \mathbb{N}, \ i \in \{1, \ldots, n\} \) and \( j_1, \ldots, j_m \in \{1, \ldots, n\} \setminus \{i\} \). We note that by the construction of \( \bar{W} \) and the independence of \( B, \bar{B}, \bar{W} \) we have for \( \mathbb{P}^{\bar{W}} \)-almost all \( y \in C([0, 1], \mathbb{R}) \),
\[
\mathbb{P}(\bar{W} = y|\bar{W}) = \mathbb{P}(y + B, \bar{y} + \bar{B}).
\]
Moreover, the processes
\[
(B_s, \bar{B}_s)_{s \in [t_{i-1}, t_i]} \quad \text{and} \quad (B_s, \bar{B}_s)_{s \in [0, 1] \setminus [t_{i-1}, t_i]} \quad \text{are independent.}
\]
Consequently, for $\mathbb{P}^W$-almost all $y \in C([0,1], \mathbb{R})$,
\[
\mathbb{E}[J_i \cdot J_j \cdots J_m | W = y] = \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (\mu(y_s + B_s) - \mu(y_s + \tilde{B}_s)) \, ds \right] \mathbb{E}\left[\prod_{\ell=1}^{m} \int_{t_{\ell-1}}^{t_\ell} (\mu(y_s + B_s) - \mu(y_s + \tilde{B}_s)) \, ds \right].
\]
Furthermore, since $\mathbb{P}^B = \mathbb{P}^{\tilde{B}}$ we have
\[
\mathbb{E}\left[\int_{t_{i-1}}^{t_i} (\mu(y_s + B_s) - \mu(y_s + \tilde{B}_s)) \, ds \right] = \int_{t_{i-1}}^{t_i} (\mathbb{E}[\mu(y_s + B_s)] - \mathbb{E}[\mu(y_s + \tilde{B}_s)]) \, ds = 0.
\]
Combining the latter two equalities and taking expectation with respect to $\mathbb{P}^W$ yields (25).

Next, put
\[
i^* = \min\{i \in \{1, \ldots, n\} : t_i > 2/n\}
\]
and for all $i \in \{i^*, \ldots, n-1\}$ put
\[
\ell_i = \max\{j \in \{i+1, \ldots, n\} : t_j \leq t_{i-1} + 4/n\}.
\]
Note that (17) and the assumption $n \geq 6$ imply that $i^* \leq n-1$ and $t_{i^*-1} = 2/n$. Moreover, (17) implies that $t_{i+1} - t_i \leq 4/n$ for all $i \in \{1, \ldots, n-1\}$, which shows that all numbers $\ell_i$ are well-defined.

Clearly,
\[
(28) \quad \mathbb{E}\left[\left| \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right|^4 \right] \leq 4 \mathbb{E}\left[\left| \int_0^{2/n} (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right|^4 \right] + 4 \mathbb{E}\left[\left( \sum_{i=i^*}^{n} J_i \right)^4 \right].
\]
Moreover, with the help of (25) we obtain
\[
\mathbb{E}\left[\left( \sum_{i=i^*}^{n} J_i \right)^4 \right] = \mathbb{E}\left[\sum_{i,j=i^*}^{n} J_i^2 J_j^2 \right] = \sum_{i=i^*}^{n} \mathbb{E}[J_i^4] + 2 \sum_{i=i^*}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[J_i^2 J_j^2]
= \sum_{i=i^*}^{n} \mathbb{E}[J_i^4] + 2 \sum_{i=i^*}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[J_i^2 J_j^2]
\leq \sum_{i=i^*}^{n} \mathbb{E}[J_i^4] + 2 \sum_{i=i^*}^{n-1} \sum_{j=i+1}^{n} \mathbb{E}[J_i^2 J_j^2].
\]
Since $\mu$ satisfies a linear growth condition, see Lemma 1 and $\sup_{s \in [0,1]} \mathbb{E}[|W_s|^4] < \infty$ we obtain that there exists $c \in (0, \infty)$ such that for all $0 \leq a \leq b \leq 1$,
\[
\mathbb{E}\left[\left( \int_a^b |\mu(W_s) - \mu(\tilde{W}_s)| \, ds \right)^4 \right] \leq c (b-a)^4.
\]
Employing the latter fact and Hölder’s inequality and observing (17) we conclude from (28) and (29) that there exist $c_1, c_2 \in (0, \infty)$ such that

$$
\mathbb{E} \left[ \left| \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right|^4 \right] 
\leq c_1 \left( (2/n)^4 + \sum_{i=i^*}^{n} (t_i - t_{i-1})^4 + \sum_{i=i^*}^{n-1} (t_i - t_{i-1})^2 \cdot (t_{i-1} + 4/n - t_i)^2 \right)
$$

(30)

$$
+ 8 \sum_{i=i^*}^{n-1} \sum_{j=\ell_{i+1}}^{n} \mathbb{E}[J_i^2 J_j^2] 
\leq \frac{c_2}{n^3} + 8 \sum_{i=i^*}^{n-1} \sum_{j=\ell_{i+1}}^{n} \mathbb{E}[J_i^2 J_j^2] 
$$

Note that for all $i \in \{i^*, \ldots, n - 1\}$ and $j \in \{\ell_i + 1, \ldots, n\}$ we have

(31)

$$
t_{j-1} \geq t_j - 2/n > t_{i-1} + 2/n.
$$

Below we show that there exists $c \in (0, \infty)$ such that for all $i \in \{i^*, \ldots, n - 1\}$ and $j \in \{\ell_i + 1, \ldots, n\}$,

(32)

$$
\mathbb{E}[J_i^2 J_j^2] \leq c \left( \frac{1}{n^5} + \frac{\ln(n+1)}{n^3} \cdot \frac{(t_i - t_{i-1})(t_j - t_{j-1})}{\sqrt{(t_i - 2/n)(t_j - 2/n - t_{i-1})}} \right).
$$

Employing (30) to (32) we conclude that there exist $c_1, c_2 \in (0, \infty)$ such that

$$
\mathbb{E} \left[ \left| \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right|^4 \right] 
\leq c_1 \left( \frac{1}{n^3} + \frac{\ln(n+1)}{n^3} \sum_{i=i^*}^{n-1} \sum_{j=\ell_{i+1}}^{n} \frac{(t_i - t_{i-1})(t_j - t_{j-1})}{\sqrt{(t_i - 2/n)(t_j - 2/n - t_{i-1})}} \right)
$$

$$
\leq c_1 \left( \frac{1}{n^3} + \frac{\ln(n+1)}{n^3} \sum_{i=i^*}^{n-1} \sqrt{t_i - 2/n} \int_{t_{i-1}+2/n}^{1} \frac{1}{\sqrt{y - 2/n - t_{i-1}}} \, dy \right)
$$

$$
\leq 2c_1 \left( \frac{1}{n^3} + \frac{\ln(n+1)}{n^3} \int_{2/n}^{1} \frac{1}{\sqrt{x - 2/n}} \, dx \right) \leq c_2 \frac{\ln(n+1)}{n^3}.
$$

The latter estimate clearly implies the statement of the lemma.

It remains to prove (32). We first show that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ we have

(33)

$$
\mathbb{P}(W_s \in [t_{i-1}, t_i] \cap (\tilde{W}_s) \in [t_{j-1}, t_j]) = \mathbb{P}(W_s \in [t_{i-1}, t_i] \cap (\tilde{W}_s) \in [t_{j-1}, t_j]).
$$
To this end we use \ref{26} and \ref{27} to obtain that for $\mathbb{P}^\mathcal{W}$-almost all $y \in C([0, 1], \mathbb{R})$,
\[ \mathbb{P}(W_s)_{s \in [t_{i-1}, t_i]}(\mathcal{W}_s)_{s \in [t_{j-1}, t_j]) = \mathbb{P}(y_s + B_s)_{s \in [t_{i-1}, t_i]}(y_s + B_s)_{s \in [t_{j-1}, t_j]} = \mathbb{P}(y_s + B_s)_{s \in [t_{i-1}, t_i]} \times \mathbb{P}(y_s + B_s)_{s \in [t_{j-1}, t_j]} = \mathbb{P}(y_s + B_s)_{s \in [t_{i-1}, t_i]}(y_s + B_s)_{s \in [t_{j-1}, t_j]} = \mathbb{P}(W_s)_{s \in [t_{i-1}, t_i]}(W_s)_{s \in [t_{j-1}, t_j]) \]
which clearly implies \ref{33}. Next, recall that $W$ and $\widetilde{W}$ coincide at the points $t_0, \ldots, t_n$. The latter fact and \ref{33} imply that for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$,
\[
\mathbb{E}[J_i^2 J_j^2] \leq 4 \mathbb{E}
\left[
\left(\int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(W_{t_{i-1}})) \, ds \right)^2 + \left(\int_{t_{i-1}}^{t_i} (\mu(\mathcal{W}_s) - \mu(\mathcal{W}_{t_{i-1}})) \, ds \right)^2
\right]
\times
\left[
\left(\int_{t_{j-1}}^{t_j} (\mu(W_s) - \mu(W_{t_{j-1}})) \, ds \right)^2 + \left(\int_{t_{j-1}}^{t_j} (\mu(\mathcal{W}_s) - \mu(\mathcal{W}_{t_{j-1}})) \, ds \right)^2
\right]
\]
\[
= 16 \mathbb{E}
\left[
\left(\int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(W_{t_{i-1}})) \, ds \right)^2 \left(\int_{t_{j-1}}^{t_j} (\mu(W_s) - \mu(W_{t_{j-1}})) \, ds \right)^2
\right].
\]
By \ref{10} in Lemma 1 and \ref{17} we see that there exists $c \in (0, \infty)$ such that for all $i \in \{1, \ldots, n\}$,
\[
\left|\int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(W_{t_{i-1}})) \, ds \right| \leq c \left(\int_{t_{i-1}}^{t_i} |W_s - W_{t_{i-1}}| \, ds + \int_{t_{i-1}}^{t_i} \sum_{\ell=1}^{k} 1_{D_\ell}(W_s, W_{t_{i-1}}) \, ds \right)
\]
as well as
\[
\int_{t_{i-1}}^{t_i} \sum_{\ell=1}^{k} 1_{D_\ell}(W_s, W_{t_{i-1}}) \, ds \leq \frac{c}{n}
\]
and
\[
\mathbb{E}
\left[
\left(\int_{t_{i-1}}^{t_i} |W_s - W_{t_{i-1}}| \, ds \right)^2
\right] \leq (t_i - t_{i-1})^3 \leq \frac{c}{n^3}.
\]
Using \ref{35} and \ref{36} we conclude that there exists $c \in (0, \infty)$ such that for all $i, j \in \{1, \ldots, n\}$,
\[
\left(\int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(W_{t_{i-1}})) \, ds \right)^2 \left(\int_{t_{j-1}}^{t_j} (\mu(W_s) - \mu(W_{t_{j-1}})) \, ds \right)^2
\]
\[
\leq c \left(\int_{t_{i-1}}^{t_i} |W_s - W_{t_{i-1}}| \, ds \right)^2 \left(\int_{t_{j-1}}^{t_j} |W_s - W_{t_{j-1}}| \, ds \right)^2
\]
\[
+ \frac{c}{n^2} \left(\int_{t_{i-1}}^{t_i} |W_s - W_{t_{i-1}}| \, ds \right)^2 + \left(\int_{t_{j-1}}^{t_j} |W_s - W_{t_{j-1}}| \, ds \right)^2
\]
\[
+ \frac{c}{n^2} \sum_{\ell,r=1}^{k} \int_{t_{i-1}}^{t_i} 1_{D_\ell}(W_s, W_{t_{i-1}}) \, ds \int_{t_{j-1}}^{t_j} 1_{D_r}(W_s, W_{t_{j-1}}) \, ds.
\]
Employing (37) and (38) and observing the fact that for all \( \ell \in \{1, \ldots, k\} \) and all \((u, v) \in D_\ell\)

\[ |u - v| = |(u - \xi_\ell) - (v - \xi_\ell)| \geq |v - \xi_\ell| \]

we obtain that there exist \(c_1, c_2 \in (0, \infty)\) such that for all \(i, j \in \{1, \ldots, n\}\) with \(i \neq j\),

\[
\mathbb{E}\left[ \left( \int_{t_{i-1}}^{t_i} (\mu(W_s) - \mu(W_{t_{i-1}})) \, ds \right) \left( \int_{t_{j-1}}^{t_j} (\mu(W_s) - \mu(W_{t_{j-1}})) \, ds \right) \right]^2 
\leq c_1 \mathbb{E}\left[ \left( \int_{t_{i-1}}^{t_i} |W_s - W_{t_{i-1}}| \, ds \right)^2 \right] \mathbb{E}\left[ \left( \int_{t_{j-1}}^{t_j} |W_s - W_{t_{j-1}}| \, ds \right)^2 \right] 
\leq \frac{c_1}{n^2} \sum_{\ell, r=1}^{k} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \mathbb{P}( |W_s - W_{t_{i-1}}| \geq |W_{t_{i-1}} - \xi_\ell|, |W_t - W_{t_{j-1}}| \geq |W_{t_{j-1}} - \xi_r| ) \, dt \, ds 
\leq \frac{c_2}{n^2} + \frac{c_1}{n^2} \sum_{\ell, r=1}^{k} \int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \mathbb{P}( |W_s - W_{t_{i-1}}| \geq |W_{t_{i-1}} - \xi_\ell|, |W_t - W_{t_{j-1}}| \geq |W_{t_{j-1}} - \xi_r| ) \, dt \, ds.
\]

Put \( \alpha_n = 2\sqrt{\ln(n+1)/n} \).

Observing (17) we obtain by standard estimates for Gaussian probabilities that for all \(\ell, r \in \{1, \ldots, k\}\), all \(i, j \in \{1, \ldots, n\}\) with \(i < j\) and all \(s \in [t_{i-1}, t_i], t \in [t_{j-1}, t_j]\) we have

\[
\mathbb{P}( |W_s - W_{t_{i-1}}| \geq |W_{t_{i-1}} - \xi_\ell|, |W_t - W_{t_{j-1}}| \geq |W_{t_{j-1}} - \xi_r| ) 
\leq \mathbb{P}( |W_s - W_{t_{i-1}}| \geq \alpha_n, |W_t - W_{t_{j-1}}| \geq \alpha_n ) 
\leq 4 \mathbb{P}( W_1 \geq \sqrt{n/2} \alpha_n ) + \frac{2\alpha_n^2}{\pi \sqrt{t_{i-1} (t_{j-1} - t_{i-1})}} 
\leq 4 \mathbb{P}( W_1 \geq \sqrt{2 \ln(n+1)} + \frac{8 \ln(n+1)}{\pi n \sqrt{t_{i-1} (t_{j-1} - t_{i-1})}} 
\leq \frac{4}{\sqrt{4\pi \ln(n+1)(n+1)}} + \frac{8 \ln(n+1)}{\pi n \sqrt{t_{i-1} (t_{j-1} - t_{i-1})}}.
\]

Using (17) and (31) we conclude from (40) that there exists \(c \in (0, \infty)\) such that for all \(\ell, r \in \{1, \ldots, k\}\), all \(i \in \{i^*, \ldots, n-1\}\) and \(j \in \{\ell+1, \ldots, n\}\) we have

\[
\int_{t_{i-1}}^{t_i} \int_{t_{j-1}}^{t_j} \mathbb{P}( |W_s - W_{t_{i-1}}| \geq |W_{t_{i-1}} - \xi_\ell|, |W_t - W_{t_{j-1}}| \geq |W_{t_{j-1}} - \xi_r| ) \, dt \, ds 
\leq \frac{c}{n^3} + \frac{c \ln(n+1)}{n} \mathbb{E}\left[ |W_{t_i} - W_{t_{i-1}}| \right]^{(t_i - t_{i-1})/(t_j - t_{j-1})} 
\leq \frac{c}{n^3} + \frac{c \ln(n+1)}{n} \sqrt{(t_i - 2/n)(t_j - 2/n - t_{i-1})}.\]
Finally, combining (34), (39) and (41) yields (32), which completes the proof of the lemma. □

We proceed by providing a lower bound for the right hand side of (19) for the case $p = 2$. 

**Lemma 6.** Assume that $\mu$ satisfies $(\mu 1)$, $(\mu 3)$ and is increasing or decreasing. Then there exists $c \in (0, \infty)$ such that

\[
E \left[ \left| \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right| ^2 \right] \geq \frac{c}{\pi^{3/2}}.
\]

**Proof.** Recall the definition (24) of $J_1, \ldots, J_n$ in the proof of Lemma 5. By (25) we have

\[
E \left[ \left| \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds \right| ^2 \right] = \sum_{i=1}^n E[J_i^2].
\]

Fix $i \in \{1, \ldots, n\}$, observe that for all $s \in [t_{i-1}, t_i]$ it holds

\[ W_{s} = W_{t_{i-1}} + \frac{s-t_{i-1}}{t_i-t_{i-1}} (W_{t_i} - W_{t_{i-1}}) \]

and put

\[ U = W_{t_{i-1}}, \quad V = \frac{1}{t_i-t_{i-1}} (W_{t_i} - W_{t_{i-1}}). \]

We then have

\[ J_i = \int_0^{t_{i-1} - t_{i-1}} (\mu(U + sV + B_{t_{i-1}+s}) - \mu(U + sV + \tilde{B}_{t_{i-1}+s})) \, ds. \]

Choose $\ell \in \{1, \ldots, k\}$ according to condition $(\mu 3)$, i.e. $\mu(\xi_{\ell}+) \neq \mu(\xi_{\ell}-)$. Applying Lemma 3 we may then conclude that

\[
E[J_i^2] \geq c_1 (t_i - t_{i-1})^2 P(W_{t_{i-1}} \in [\xi_\ell, \xi_{\ell} + \sqrt{t_i - t_{i-1}}]) P(W_{t_i} - W_{t_{i-1}} \in [0, \sqrt{t_i - t_{i-1}}]),
\]

where

\[ c_1 = \kappa (\mu(\xi_{\ell}+) - \mu(\xi_{\ell}-))^2 > 0 \]

and $\kappa$ is given by (11). Moreover,

\[
P(W_{t_i} - W_{t_{i-1}} \in [0, \sqrt{t_i - t_{i-1}}]) = \frac{1}{\sqrt{2\pi}} \int_0^{1} e^{-x^2/2} \, dx \geq \frac{1}{\sqrt{2\pi} e}.\]

Furthermore, if $t_{i-1} \geq 1/2$ then

\[
P(W_{t_{i-1}} \in [\xi_{\ell}, \xi_{\ell} + \sqrt{t_i - t_{i-1}}]) = \int_{\xi_{\ell}/\sqrt{t_i-t_{i-1}}}^{(\xi_{\ell}+\sqrt{t_i-t_{i-1}})/\sqrt{t_i-t_{i-1}}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \geq \frac{1}{\sqrt{2\pi}} e^{-((\xi_{\ell})^2+1)/2} \sqrt{t_i-t_{i-1}}.\]

Let $r \in \{1, \ldots, n\}$ satisfy $t_r = 1/2$. Using (44) to (46) we conclude that there exists $c \in (0, \infty)$ such that

\[
\sum_{i=1}^{n} E[J_i^2] \geq c \sum_{i=r+1}^{n} (t_i - t_{i-1})^{5/2}.
\]
By Hölder’s inequality,
\[
\frac{1}{2} = \sum_{i=r+1}^{n} (t_i - t_{i-1}) \leq n^{3/5} \cdot \left( \sum_{i=r+1}^{n} (t_i - t_{i-1})^{5/2} \right)^{2/5}.
\]
Thus,
\[
(48) \quad \sum_{i=r+1}^{n} (t_i - t_{i-1})^{5/2} \geq \frac{1}{2^{5/2} n^{3/2}}.
\]
Hence there exists \(c \in (0, \infty)\) such that
\[
\sum_{i=1}^{n} \mathbb{E}[J_i^2] \geq c \frac{1}{n^{3/2}}.
\]
Combining (43) and the latter inequality completes the proof of the lemma. \(\square\)

We are ready to establish the estimate (18). Clearly, the lower bound in (18) for the case \(p = 2\) is a consequence of (19) in Lemma 4 with \(p = 2\) and Lemma 6.

For the case \(p = 1\) put
\[
Z = \int_0^1 (\mu(W_s) - \mu(\tilde{W}_s)) \, ds
\]
and let \(\delta \in (0, \infty)\). Using Lemma 5 and Lemma 6 we obtain by Hölder’s inequality that there exist \(c_1, c_2 \in (0, \infty)\) such that
\[
(49) \quad c_1 n^{-3/2} \leq \mathbb{E}[Z^2] = \mathbb{E}[|Z|^{2/3} \cdot |Z|^{4/3}] \leq \mathbb{E}[|Z|^{2/3} \cdot \mathbb{E}[|Z|^{4/3}]}^{1/3} \leq \mathbb{E}[|Z|^{2/3} \cdot (c_2 n^{-3/2 + 4\delta})^{1/3}].
\]
Hence
\[
(50) \quad \mathbb{E}[|Z|] \geq c_1^{3/2} n^{-9/4} \cdot c_2^{-1/2} n^{3/2 - 2\delta} = c_1^{3/2} c_2^{-1/2} n^{-3/4 - 2\delta}.
\]
The latter estimate with \(\delta = \varepsilon/2\) and (19) in Lemma 4 with \(p = 1\) yield the lower bound in (18) for the case \(p = 1\), which completes the proof of (18) and hereby the proof of Theorem 3.

2.3. Properties of solutions of the equation \(dX_t = \mu(X_t) \, dt + dW_t\). Throughout this section we consider the scalar SDE
\[
(51) \quad dX_t = \mu(X_t) \, dt + dW_t
\]
and we provide properties of solutions \(X\) of (51), which are used in the proof of Theorem 2.

The following lemma provides upper and lower estimates for the probability of \(X_t\) taking values in bounded intervals.

Lemma 7. Assume that \(\mu\) is measurable and bounded, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, let \(W: [0, 1] \times \Omega \to \mathbb{R}\) be a standard Brownian motion, let \(x_0 \in \mathbb{R}\) and let \(X: [0, 1] \times \Omega \to \mathbb{R}\) be a strong solution of the SDE (51) on the time-interval \([0, 1]\) with driving Brownian motion \(W\) and initial value \(x_0\). Moreover, let \(\tau \in (0, 1]\) and \(M \in (0, \infty)\). Then there exist \(c_1, c_2 \in (0, \infty)\) such that for all \(t \in [\tau, 1]\) and all \(x, y \in \mathbb{R}\) with \(x \leq y\) it holds
\[
(52) \quad \mathbb{P}(X_t \in [x, y]) \leq c_1 (y - x)
\]
and for all \( t \in [\tau, 1] \) and all \( x, y \in [-M, M] \) with \( x \leq y \) it holds

\[
\mathbb{P}(X_t \in [x, y]) \geq c_2 (y - x).
\]  

(53)

Proof. It is well-known that the assumption that \( \mu \) is measurable and bounded implies that for every \( t \in (0, 1] \), the solution \( X_t \) has a Lebesgue density \( \rho_t: \mathbb{R} \to [0, \infty) \), which satisfies a two-sided Gaussian bound, i.e. there exist \( c_1, c_2, c_3, c_4 \in (0, \infty) \) such that for all \( t \in (0, 1] \) and all \( z \in \mathbb{R} \),

\[
c_1 \cdot \frac{1}{\sqrt{2\pi c_2 t}} \cdot e^{-\frac{(z-x_0)^2}{2c_2 t}} \leq \rho_t(z) \leq c_3 \cdot \frac{1}{\sqrt{2\pi c_4 t}} \cdot e^{-\frac{(z-x_0)^2}{2c_4 t}},
\]

see e.g. [27].

Let \( x, y \in \mathbb{R} \) with \( x \leq y \) and \( t \in [\tau, 1] \). Using the second inequality in (54) we obtain

\[
\mathbb{P}(X_t \in [x, y]) = \int_x^y \rho_t(z) \, dz \leq c_3 \cdot \frac{1}{\sqrt{2\pi c_4 t}} \cdot \int_x^y e^{-\frac{(z-x_0)^2}{2c_4 t}} \, dz \leq c_3 \cdot \frac{1}{\sqrt{2\pi c_4 t}} \cdot (y - x),
\]

which proves the upper bound (52).

Next assume that \( x, y \in [-M, M] \). Employing the first inequality in (54) we conclude

\[
\mathbb{P}(X_t \in [x, y]) = \int_x^y \rho_t(z) \, dz \geq c_1 \cdot \frac{1}{\sqrt{2\pi c_2 t}} \cdot \int_x^y e^{-\frac{(z-x_0)^2}{2c_2 t}} \, dz \geq c_1 \cdot \frac{1}{\sqrt{2\pi c_2 t}} \cdot e^{-\frac{(M + x_0)^2}{2c_2 t}} \cdot (y - x),
\]

which proves the lower bound (53) and completes the proof of the lemma. \( \square \)

Next, we provide an estimate for the expected occupation time of a neighborhood of an arbitrary point \( \xi \in \mathbb{R} \) by a strong solution of the SDE (51) with deterministic initial value.

**Lemma 8.** Assume that \( \mu \) is measurable and bounded, let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete probability space, let \( W: [0, 1] \times \Omega \to \mathbb{R} \) be a standard Brownian motion, and for every \( x \in \mathbb{R} \) let \( X^x: [0, 1] \times \Omega \to \mathbb{R} \) be a strong solution of the SDE (51) on the time-interval \([0, 1]\) with driving Brownian motion \( W \) and initial value \( x \). Then there exists \( c \in (0, \infty) \) such that for all \( x, \xi \in \mathbb{R} \), \( s \in [0, 1] \) and all \( \varepsilon \in (0, \infty) \),

\[
\mathbb{E} \left[ \int_0^s \mathbb{P}(|X^x_t - \xi| \leq \varepsilon) \, dt \right] \leq c \varepsilon \sqrt{s}.
\]

The proof of Lemma 8 is similar to the proof of Lemma 4 in [19]. For convenience of the reader we present the proof of Lemma 8 here.

**Proof.** Let \( x \in \mathbb{R} \). Clearly, \( X^x \) is a continuous semi-martingale with quadratic variation

\[
\langle X^x \rangle_t = t, \quad t \in [0, 1].
\]

(55)

For \( a \in \mathbb{R} \) let \( L^a(X^x) = (L^a_t(X^x))_{t\in[0,1]} \) denote the local time of \( X^x \) at the point \( a \). Hence, for all \( a \in \mathbb{R} \) and all \( t \in [0, 1] \),

\[
|X^x_t - a| = |x - a| + \int_0^t \text{sgn}(X^x_s - a) \mu(X^x_s) \, ds + \int_0^t \text{sgn}(X^x_s - a) \, dW_s + L^a_t(X^x),
\]
where \( \sgn(z) = 1_{(0,\infty)}(z) - 1_{(-\infty,0)}(z) \) for \( z \in \mathbb{R} \), see, e.g. [28] Chap. VI. Thus, for all \( a \in \mathbb{R} \) and all \( t \in [0,1] \),
\[
L_t^a(X^x) \leq |X_t^x - x| + \int_0^t |\mu(X_s^x)| ds + \left| \int_0^t \sgn(X_s^x - a) dW_s \right|
\leq 2 \int_0^t |\mu(X_s^x)| ds + |W_t| + \left| \int_0^t \sgn(X_s^x - a) dW_s \right|.
\]

Since \( \mu \) is bounded we may conclude that there exists \( c \in (0,\infty) \) such that for all \( x \in \mathbb{R} \), all \( a \in \mathbb{R} \) and all \( t \in [0,1] \),
\begin{equation}
(56) \quad \mathbb{E}[L_t^a(X^x)] \leq c \sqrt{t}.
\end{equation}

Using (55) and (56) we obtain by the occupation time formula that for all \( x, \xi \in \mathbb{R} \), all \( s \in [0,1] \) and all \( \varepsilon \in (0,\infty) \),
\[
\int_0^s \mathbb{P}(|X_t^x - \xi| \leq \varepsilon) dt = \mathbb{E} \left[ \int_0^s 1_{[\varepsilon-\varepsilon,\varepsilon+\varepsilon]}(X_t^x) dt \right] = \int_{\mathbb{R}} 1_{[\varepsilon-\varepsilon,\varepsilon+\varepsilon]}(a) \mathbb{E}[L_s^a(X^x)] da \leq 2c \varepsilon \sqrt{s},
\]
which completes the proof of the lemma.

The following lemma provides under the condition (\( \mu 1 \)) a common functional representation of arbitrary solutions of the SDE (51) as well as the transition probabilities and a comparison result for strong solutions of (51) with deterministic initial values.

**Lemma 9.** Assume that \( \mu \) satisfies (\( \mu 1 \)). Then strong existence and pathwise uniqueness hold for the SDE (51). In particular, for every \( T \in (0,\infty) \) there exists a Borel-measurable mapping 
\[
F: \mathbb{R} \times C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})
\]
such that for every complete probability space \( (\Omega,F,\mathbb{P}) \), every standard Brownian motion \( W: [0,T] \times \Omega \to \mathbb{R} \) and every random variable \( \eta: \Omega \to \mathbb{R} \) independent it holds that
(i) if \( X: [0,T] \times \Omega \to \mathbb{R} \) is a solution of the SDE (51) on the time-interval \([0,T]\) with driving Brownian motion \( W \) and initial value \( \eta \) then \( X = F(\eta,W) \) \( \mathbb{P} \)-almost surely,
(ii) \( F(\eta,W) \) is a strong solution of the SDE (51) on the time-interval \([0,T]\) with driving Brownian motion \( W \) and initial value \( \eta \).

Moreover, let \( (\Omega,F,\mathbb{P}) \) be a complete probability space, let \( T \in (0,\infty) \), let \( W: [0,T] \times \Omega \to \mathbb{R} \) be a standard Brownian motion, and for every \( x \in \mathbb{R} \) let \( X^x: [0,T] \times \Omega \to \mathbb{R} \) be a strong solution of the SDE (51) on the time-interval \([0,T]\) with driving Brownian motion \( W \) and initial value \( x \). Then
(iii) for all \( s \in [0,T] \) and \( \mathbb{P}^{X^x} \)-almost all \( x \in \mathbb{R} \) we have
\[
\mathbb{P}^{(X^x_t)_{t \in [s,T]}}(X^x_t = y) = \mathbb{P}^{X^x}(X^x_t = y),
\]
(iv) for all \( x, y \in \mathbb{R} \) with \( x \leq y \) we have
\[
\mathbb{P} (\forall t \in [0,T]: X^x_t \leq X^y_t) = 1.
\]
Proof. As a straightforward generalization of Lemma 7 and Lemma 8 in [19] one obtains that there exist Lipschitz continuous functions \( \bar{\mu}, \bar{\sigma} : \mathbb{R} \to \mathbb{R} \) and a strictly increasing, Lipschitz continuous bijection \( G : \mathbb{R} \to \mathbb{R} \) with a Lipschitz continuous inverse \( G^{-1} : \mathbb{R} \to \mathbb{R} \) such that for every \( T \in (0, \infty) \), every complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\), every standard Brownian motion \( W : [0, T] \times \Omega \to \mathbb{R} \) and every random variable \( \eta : \Omega \to \mathbb{R} \) such that \( W \) and \( \eta \) are independent it holds that

a) if \( X : [0, T] \times \Omega \to \mathbb{R} \) is a (strong) solution of the SDE

\[
dX_t = \bar{\mu}(X_t) \, dt + \bar{\sigma}(X_t) \, dW_t
\]

on the time-interval \([0, T]\) with driving Brownian motion \( W \) and initial value \( \eta \) then \( G^{-1} \circ X \) is a (strong) solution of the SDE \((51)\) on the time-interval \([0, T]\) with driving Brownian motion \( W \) and initial value \( G^{-1}(\eta) \),

b) if \( X : [0, T] \times \Omega \to \mathbb{R} \) is a (strong) solution of the SDE \((51)\) on the time-interval \([0, T]\) with driving Brownian motion \( W \) and initial value \( \eta \) then \( G \circ X \) is a (strong) solution of the SDE \((51)\) on the time-interval \([0, T]\) with driving Brownian motion \( W \) and initial value \( G(\eta) \).

By the Lipschitz continuity of \( \bar{\mu} \) and \( \bar{\sigma} \) strong existence and pathwise uniqueness hold for the SDE \((51)\). Using a) and b) it follows that strong existence and pathwise uniqueness hold for the SDE \((51)\) as well. For every \( T \in (0, \infty) \) the existence of \( F \) with the property (i) is now a consequence of Theorem 1 in [10]. Strong existence for the SDE \((51)\) and property (i) jointly imply that \( F \) has property (ii) as well.

We turn to the proof of (iii). Let \( s \in [0, T] \) and choose a Borel measurable \( F : \mathbb{R} \times C([0, T - s], \mathbb{R}) \to C([0, T - s], \mathbb{R}) \) according to the already proven part of the lemma. In particular, for all \( x \in \mathbb{R} \) we have \( (X^x_t)_{t \in [0, T - s]} = F(x, (W_t)_{t \in [0, T - s]}) \) almost surely. Let \( x \in \mathbb{R} \). The process \((X^x_{s+t})_{t \in [0, T - s]}\) is a solution of the SDE \((51)\) on the time-interval \([0, T - s]\) with driving Brownian motion \((W_{s+t} - W_s)_{t \in [0, T - s]}\) and initial value \(X^x_s\). By property (i) of \( F \) we thus have \((X^x_{s+t})_{t \in [0, T - s]} = F(x, (W_{s+t} - W_s)_{t \in [0, T - s]})\) almost surely. It follows that for \( \mathbb{P}^{X^x_s}\)-almost all \( y \in \mathbb{R} \) we have

\[
\mathbb{P}^{(X^x_{s+t})_{t \in [0, T - s]}}[X^x_t = y] = \mathbb{P}^{F(x, (W_{s+t} - W_s)_{t \in [0, T - s]})}[y] = \mathbb{P}^{(X^y_t)_{t \in [0, T - s]}}[y].
\]

Finally, we prove (iv). Let \( x, y \in \mathbb{R} \) with \( x \leq y \). Then \( G(x) \leq G(y) \). Using b) and a comparison result for SDEs with Lipschitz continuous coefficients, e.g. [11 Proposition 5.2.18], we thus obtain that \( \mathbb{P}(\forall t \in [0, T]: G(X^x_t) \leq G(X^y_t)) = 1 \). Since \( G^{-1} \) is increasing we furthermore have \( \{\forall t \in [0, T]: G(X^x_t) \leq G(X^y_t)\} \subset \{\forall t \in [0, T]: X^x_t \leq X^y_t\} \), which finishes the proof of (iv).

This completes the proof of the lemma.

Finally, we introduce a simple approximation of strong solutions of the SDE \((51)\) on a time-interval \([s, t]\) and provide an estimate of the mean squared total amount of time the solution and its approximation lie on different sides of a fixed horizontal line.

**Lemma 10.** Assume that \( \mu \) satisfies (\( \mu_1 \)) and (\( \mu_5 \)). Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space, let \( W : [0, 1] \times \Omega \to \mathbb{R} \) be a standard Brownian motion, and for every \( x \in \mathbb{R} \) let \( X^x : [0, 1] \times \Omega \to \mathbb{R} \) be a strong solution of the SDE \((51)\) on the time-interval \([0, 1]\) with driving Brownian motion...
W and initial value x. Then there exists $c \in (0, \infty)$ such that for all $\xi \in \mathbb{R}$, all $s, t \in [0, 1]$ with $s < t$ and all $x \in \mathbb{R}$,

$$E \left[ \left( \int_s^t 1 \{ (X_u^x - \xi) (X_s^x + W_u - W_s - \xi) \leq 0 \} \, du \right)^2 \right] \leq c (t - s)^3. \tag{58}$$

**Proof.** Fix $\xi \in \mathbb{R}$, $s, t \in [0, 1]$ with $s < t$ and $x \in \mathbb{R}$, and put

$$Y_u = X_s^x + W_u - W_s$$

for $u \in [s, t]$. By (µ5) we have for all $u \in [s, t]$,

$$|X_u^x - Y_u| = \left| \int_s^u \mu(X_v) \, dv \right| \leq \|\mu\|_{\infty} (u - s).$$

Hence,

$$\left( \int_s^t 1 \{ (X_u^x - \xi) (Y_u - \xi) \leq 0 \} \, du \right)^2 = \int_s^t \int_s^t 1 \{ (X_u^x - \xi) (Y_u - \xi) \leq 0 \} 1 \{ (X_v^x - \xi) (Y_v - \xi) \leq 0 \} \, du \, dv$$

$$\leq \int_s^t \int_s^t 1 \{ |X_u^x - \xi| \leq |X_v^x - Y_v| \} 1 \{ |X_v^x - \xi| \leq |X_u^x - Y_u| \} \, du \, dv$$

$$\leq \int_s^t \int_s^t 1 \{ |X_u^x - \xi| \leq \|\mu\|_{\infty} |t - s| \} 1 \{ |X_v^x - \xi| \leq \|\mu\|_{\infty} |t - s| \} \, du \, dv$$

$$= 2 \int_s^t \int_s^t 1 \{ |X_u^x - \xi| \leq \|\mu\|_{\infty} |t - s| \} \, du \, dv,$$

and therefore,

$$E \left[ \left( \int_s^t 1 \{ (X_u^x - \xi) (Y_u - \xi) \leq 0 \} \, du \right)^2 \right] \leq 2 \int_s^t \int_s^t E \left[ \int_0^1 1 \{ |X_v^x - \xi| \leq \|\mu\|_{\infty} |t - s| \} \, dv \right] X_v^x \, du \, dv. \tag{59}$$

Using Lemma [9][iii] and then Lemma [9] we obtain that there exists $c_1 \in (0, \infty)$, which only depends on $\mu$, such that for all $v \in [s, t]$ and $P^{X_v^x}$-almost all $y \in \mathbb{R}$,

$$E \left[ \int_v^t 1 \{ |X_u^x - \xi| \leq \|\mu\|_{\infty} |t - s| \} \, du \bigg| X_v^x = y \right]$$

$$= E \left[ \int_v^t 1 \{ |X_u^y - \xi| \leq \|\mu\|_{\infty} |t - s| \} \, du \right]$$

$$= \int_0^{t - v} P (|X_u^y - \xi| \leq \|\mu\|_{\infty} |t - s|) \, du \leq c_1 |t - s|^{3/2}. \tag{60}$$
Lemma 11. Assume that $X, Y : [0, 1] \times \Omega \to \mathbb{R}$ are strong solutions of the SDE (51) on the time-interval $[0, 1]$ with initial value $x_0$ and driving Brownian motion $W$. Then there exist $c_1, c_2 \in (0, \infty)$ such that for all $n \in 2\mathbb{N}$, all $t_1, \ldots, t_n \in [0, 1]$ with
\begin{equation}
0 < t_1 < \ldots < t_n = 1
\end{equation}
and
\begin{equation}
2/n, 4/n, \ldots, 1 \in \{t_1, \ldots, t_n\},
\end{equation}
and all measurable $g : \mathbb{R}^n \to \mathbb{R}$ we have
\begin{equation}
\mathbb{E}[\|X_1 - g(W_{t_1}, \ldots, W_{t_n})\|] \geq \frac{c_1}{n^{3/4}} (\max(0, (1 - c_2/n^{1/16})))^{3/2}.
\end{equation}

For the proof of Proposition 1 we fix $n \in 2\mathbb{N}$ as well as $t_1, \ldots, t_n \in [0, 1]$ with (63) and (64). Moreover, we put $t_0 = 0$.

Throughout this section we use $c, c_1, c_2 \in (0, \infty)$ to denote positive constants that may change their values in every appearance but neither depend on $n$ nor on the time-points $t_1, \ldots, t_n$.

Recall from Section 2.2 the definition and the properties of the processes $\tilde{W}, B, \tilde{B}, \tilde{W} : [0, 1] \times \Omega \to \mathbb{R}$ associated with the discretization (63). In particular, $\tilde{W}$ is a Brownian motion and for all $i \in \{0, \ldots, n\}$ we have
\begin{equation}
W_{t_i} = \tilde{W}_{t_i}.
\end{equation}

Lemma 11. Assume that $\mu$ satisfies (μ1), let $x_0 \in \mathbb{R}$ and let $X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R}$ be strong solutions of the SDE (51) on the time-interval $[0, 1]$ with initial value $x_0$ and driving Brownian motion $W$ and $\tilde{W}$, respectively. Then for all measurable $g : \mathbb{R}^n \to \mathbb{R}$ and all $p \in [1, \infty)$,
\begin{equation}
(\mathbb{E}[\|X_1 - g(W_{t_1}, \ldots, W_{t_n})\|^p])^{1/p} \geq \frac{1}{2} (\mathbb{E}[\|X_1 - \bar{X}_1\|^p])^{1/p}.
\end{equation}
Proof. By Lemma \([9](i)\) there exists a measurable function \(F : \mathbb{R} \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R})\) such that \(P\)-almost surely,

\[
X = F(x_0, W) \quad \text{and} \quad \tilde{X} = F(x_0, \tilde{W}).
\]

Hence there exist measurable functions \(\Phi : C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \to \mathbb{R}\) and \(\varphi : C([0, 1], \mathbb{R}) \to \mathbb{R}\) such that \(P\)-almost surely

\[
X_1 = \Phi(\tilde{W}, B), \quad \tilde{X}_1 = \Phi(\tilde{W}, \tilde{B}), \quad g(W_{t_1}, \ldots, W_{t_n}) = \varphi(\tilde{W}).
\]

Since \(\mathbb{P}(\tilde{W}, B) = \mathbb{P}(\tilde{W}, \tilde{B})\), we may apply Lemma \(2\) with \(\Omega_1 = \Omega_2 = C([0, 1], \mathbb{R})\), \(V_1 = \tilde{W}\), \(V_2 = B\), \(V_2' = \tilde{B}\) and \(\Phi, \varphi\) as in \(\text{(69)}\) to obtain \(\text{(67)}\).

In the analysis of the right hand side of \(\text{(67)}\) we will make use of the following upper bound on the \(L_p\)-distance between the two processes \(X\) and \(\tilde{X}\) at the time points \(t_0, \ldots, t_n\).

Lemma 12. Assume that \(\mu\) satisfies \((\mu1), (\mu2)\) and \((\mu5)\), let \(x_0 \in \mathbb{R}\) and let \(X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R}\) be strong solutions of the SDE \(\text{(51)}\) on the time-interval \([0, 1]\) with initial value \(x_0\) and driving Brownian motion \(W\) and \(\tilde{W}\), respectively. Then for every \(p \in [1, \infty)\) there exists \(c \in (0, \infty)\) such that

\[
\max_{i \in \{0, \ldots, n\}} \mathbb{E}[|X_{t_i} - \tilde{X}_{t_i}|^p]^{1/p} \leq \frac{c}{n^{3/4}}.
\]

Proof. Let \(i \in \{0, \ldots, n\}\). We have

\[
\mathbb{E}[|X_{t_i} - \tilde{X}_{t_i}|^p]^{1/p} \leq \mathbb{E}[|X_{\bar{t_i}} - \tilde{X}_{\bar{t_i}}|^p]^{1/p} + \mathbb{E}[|X_{t_i} - X_{\bar{t_i}} - \tilde{X}_{t_i} + \tilde{X}_{\bar{t_i}}|^p]^{1/p},
\]

where

\[
t_{\bar{i}} = \max \{\tau \in \{2j/n : j = 0, \ldots, n/2\} : t_j \geq \tau\}.
\]

Observing the fact that

\[
t_{i} - t_{\bar{i}} \leq \frac{2}{n}
\]

and employing \(\text{(66)}\) we obtain that

\[
|X_{t_i} - X_{t_{\bar{i}}} - \tilde{X}_{t_i} + \tilde{X}_{t_{\bar{i}}}| = \left| \int_{t_{\bar{i}}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds \right| \leq \frac{4\|\mu\|_{\infty}}{n}.
\]

Observing \((\mu5)\) we thus see that there exists \(c \in (0, \infty)\) such that

\[
\max_{i \in \{0, \ldots, n\}} \mathbb{E}[|X_{t_i} - X_{t_{\bar{i}}} - \tilde{X}_{t_i} + \tilde{X}_{t_{\bar{i}}}|^p]^{1/p} \leq \frac{c}{n}.
\]

Put \(k_n = n/2\) and let \(Y_{k_n} = (Y_{k_n, t})_{t \in [0, 1]}\) and \(\tilde{Y}_{k_n} = (\tilde{Y}_{k_n, t})_{t \in [0, 1]}\) denote the transformed time-continuous quasi-Milstein schemes from \(\text{[17], Section 4}\) that have step-size \(1/k_n\) and are associated to the SDE \(\text{(51)}\) on the time-interval \([0, 1]\) with initial value \(x_0\) and driving Brownian motion \(W\) and \(\tilde{W}\), respectively. Since \(\mu\) satisfies \((\mu1)\) and \((\mu2)\) we may apply Theorem 4 in \(\text{[17]}\) to obtain that

\[
\max_{j \in \{0, \ldots, k_n\}} \left( \mathbb{E}[|X_{j/k_n} - Y_{k_n, j/k_n}|^p]^{1/p} + \mathbb{E}[|\tilde{X}_{j/k_n} - \tilde{Y}_{k_n, j/k_n}|^p]^{1/p} \right) \leq \frac{c}{k_n^{3/4}}.
\]
By the definition of $Y_{k_n}$ and $\tilde{Y}_{k_n}$ we have for every $j \in \{0, \ldots, k_n\}$,

$$Y_{k_n,j/k_n} = g_j(W_{1/k_n}, \ldots, W_{j/k_n}), \quad \tilde{Y}_{k_n,j/k_n} = g_j(\tilde{W}_{1/k_n}, \ldots, \tilde{W}_{j/k_n})$$

with a measurable $g_j : \mathbb{R}^j \to \mathbb{R}$. Observing (60) we conclude by (72) and (73) that

$$\max_{i \in \{0, \ldots, n\}} \mathbb{E}[|X_{t_i} - \tilde{X}_{t_i}|^p]^{1/p} \leq \max_{i \in \{0, \ldots, n\}} \left( \mathbb{E}[|X_{t_i} - Y_{k_n,t_i}|^p]^{1/p} + \mathbb{E}[|\tilde{X}_{t_i} - \tilde{Y}_{k_n,t_i}|^p]^{1/p} \right)$$

$$= \max_{i \in \{0, \ldots, n\}} \left( \mathbb{E}[|X_{t_i} - Y_{k_n,t_i}|^p]^{1/p} + \mathbb{E}[|\tilde{X}_{t_i} - \tilde{Y}_{k_n,t_i}|^p]^{1/p} \right)$$

$$\leq \frac{c}{n^{3/4}}.$$ 

Combining the latter estimate with (70) and (71) yields the statement of the lemma. \hfill \Box

The following three lemmas are crucial to obtain a lower bound for the right hand side of (67) in the case $p = 2$.

**Lemma 13.** Assume that $\mu$ satisfies $(\mu 1)$ and $(\mu 4)$. Let $x_0 \in \mathbb{R}$ and let $X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R}$ be strong solutions of the SDE (51) on the time-interval $[0, 1]$ with initial value $x_0$ and driving Brownian motion $W$ and $\tilde{W}$, respectively. Then for all $i \in \{1, \ldots, n\}$ we have

$$\mathbb{E}\left[ (X_{t_{i-1}} - \tilde{X}_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds \right] \geq 0.$$

**Proof.** Fix $i \in \{1, \ldots, n\}$ and choose $F : \mathbb{R} \times C([0, t_i - t_{i-1}], \mathbb{R}) \to C([0, t_i - t_{i-1}], \mathbb{R})$ according to Lemma [9] Put

$$V = (V_t = W_{t_{i-1} + t} - W_{t_{i-1}})_{t \in [0, t_i - t_{i-1}]} \quad \text{and} \quad \tilde{V} = (\tilde{V}_t = \tilde{W}_{t_{i-1} + t} - \tilde{W}_{t_{i-1}})_{t \in [0, t_i - t_{i-1}]}.$$ 

Since the processes $(X_{t_{i-1}+t})_{t \in [0, t_i - t_{i-1}]}$ and $(\tilde{X}_{t_{i-1}+t})_{t \in [0, t_i - t_{i-1}]}$ are solutions of the SDE (51) on the time interval $[t_{i-1}, t_i]$ with initial value $X_{t_{i-1}}$ and driving Brownian motion $V$ and with initial value $\tilde{X}_{t_{i-1}}$ and driving Brownian motion $\tilde{V}$, respectively, we know by Lemma [9](i) that $\mathbb{P}$-almost surely

$$X_{t_{i-1}+t} \in [0, t_i - t_{i-1}] = F(X_{t_{i-1}}, V),$$

$$(\tilde{X}_{t_{i-1}+t})_{t \in [0, t_i - t_{i-1}]} = F(\tilde{X}_{t_{i-1}}, \tilde{V}).$$

Note that the random vector $(X_{t_{i-1}}, \tilde{X}_{t_{i-1}})$ is $\mathcal{G}/\mathcal{B}(\mathbb{R}^2)$-measurable, where $\mathcal{B}(\mathbb{R}^2)$ is the Borel $\sigma$-field in $\mathbb{R}^2$ and $\mathcal{G} \subset \mathcal{F}$ is the completion of the $\sigma$-field generated by $(W_t, \tilde{W}_t)_{t \in [0, t_i - t_{i-1}]}$, i.e.

$$\mathcal{G} = \sigma(\sigma((W_t, \tilde{W}_t) : t \in [0, t_i - t_{i-1}]) \cup \mathcal{N}),$$

where $\mathcal{N} = \{N \in \mathcal{F} : \mathbb{P}(N) = 0\}$. Furthermore, by definition of $\tilde{W}$ we have

$$(W_t, \tilde{W}_t)_{t \in [0, t_i - t_{i-1}]} \equiv \psi((W_t, \tilde{B}_t)_{t \in [0, t_i - t_{i-1}]}),$$

$$(V_t, \tilde{V}_t)_{t \in [0, t_i - t_{i-1}]} = \varphi((W_t - W_{t_{i-1}}, \tilde{B}_t)_{t \in [t_{i-1}, t_i]}))$$

for some measurable mappings $\psi : C([0, t_i - t_{i-1}], \mathbb{R}^2) \to C([0, t_i - t_{i-1}], \mathbb{R}^2)$ and $\varphi : C([t_{i-1}, t_i], \mathbb{R})^2 \to C([0, t_i - t_{i-1}], \mathbb{R}^2)$, which implies that $(W_t, \tilde{W}_t)_{t \in [0, t_i - t_{i-1}]}$ and $(V_t, \tilde{V}_t)_{t \in [0, t_i - t_{i-1}]}$ are independent.
As a consequence, the $\sigma$-fields $\mathcal{G}$ and $\sigma(V, \tilde{V})$ are independent, which in turn implies the independence of $(X_{t_{i-1}}, \tilde{X}_{t_{i-1}})$ and $(V, \tilde{V})$.

Using (76) we thus have for $\mathbb{P}^{(X_{t_{i-1}}, \tilde{X}_{t_{i-1}})}$-almost all $(y, \tilde{y}) \in \mathbb{R}^2$ that

$$\mathbb{E}\left[(X_{t_{i-1}} - \tilde{X}_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds \bigg| (X_{t_{i-1}}, \tilde{X}_{t_{i-1}}) = (y, \tilde{y})\right]$$

$$= (y - \tilde{y}) \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (\mu(F(y, V)(s)) - \mu(F(\tilde{y}, \tilde{V})(s))) \, ds \bigg| (X_{t_{i-1}}, \tilde{X}_{t_{i-1}}) = (y, \tilde{y})\right]$$

$$= (y - \tilde{y}) \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (\mu(F(y, V)(s)) - \mu(F(\tilde{y}, V)(s))) \, ds \bigg| (X_{t_{i-1}}, \tilde{X}_{t_{i-1}}) = (y, \tilde{y})\right].$$

(76)

By Lemma 9(ii) we know that $F(y, V)$ and $F(\tilde{y}, V)$ are strong solutions of the SDE (51) on the time-interval $[0, t_i - t_{i-1}]$ with driving Brownian motion $V$ and initial value $y$ and $\tilde{y}$, respectively. Using Lemma 9(iv) and the assumption that $\mu$ is increasing we conclude that $\mathbb{P}$-almost surely

$$\forall s \in [t_{i-1}, t_i]: \quad (y - \tilde{y}) (\mu(F(y, V)(s)) - \mu(F(\tilde{y}, V)(s))) \geq 0.$$ 

Combining the latter fact with (76) we conclude that for $\mathbb{P}^{(X_{t_{i-1}}, \tilde{X}_{t_{i-1}})}$-almost all $(y, \tilde{y}) \in \mathbb{R}^2$

$$\mathbb{E}\left[(X_{t_{i-1}} - \tilde{X}_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds \bigg| (X_{t_{i-1}}, \tilde{X}_{t_{i-1}}) = (y, \tilde{y})\right] \geq 0,$$

(77)

which clearly implies (74). \qed

**Lemma 14.** Assume that $\mu$ satisfies $(\mu1)$ and $(\mu5)$. Let $x_0 \in \mathbb{R}$ and let $X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R}$ be strong solutions of the SDE (s1) on the time-interval $[0, 1]$ with initial value $x_0$ and driving Brownian motion $W$ and $\tilde{W}$, respectively. Then there exists $c \in (0, \infty)$ such that for all $i \in \{1, \ldots, n\}$ with $t_i > 1/2$ it holds

$$\mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds\right)^2\right]$$

$$\geq \frac{1}{4} \mathbb{E}\left[\left(\int_{t_{i-1}}^{t_i} (\mu(X_{t_{i-1}} + W_s - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}})) \, ds\right)^2\right]$$

$$- \frac{c}{n^{5/2+1/16}}.$$
Proof. For all $i \in \{1, \ldots, n\}$ put

$$A_i = \int_{t_{i-1}}^{t_i} \left( \mu(X_{t_{i-1}} + W_t - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) \right) dt,$$

$$B_i = \int_{t_{i-1}}^{t_i} \left( \mu(X_{t_{i-1}} + W_t - W_{t_{i-1}}) - \mu(X_t) \right) dt,$$

$$C_i = \int_{t_{i-1}}^{t_i} \left( \mu(X_t) - \mu(\tilde{X}_t) \right) dt,$$

$$D_i = \int_{t_{i-1}}^{t_i} \left( \mu(\tilde{X}_t) - \mu(\tilde{X}_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) \right) dt,$$

$$E_i = \int_{t_{i-1}}^{t_i} \left( \mu(\tilde{X}_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) \right) dt.$$

Clearly, $A_i = B_i + C_i + D_i + E_i$, which yields

$$E[A_i^2] \leq 4(E[B_i^2] + E[C_i^2] + E[D_i^2] + E[E_i^2]) = 4(2E[B_i^2] + E[C_i^2] + E[E_i^2]).$$

By Lemma 1 we see that there exists $c \in (0, \infty)$ such that for all $i \in \{1, \ldots, n\}$ and all $t \in [t_{i-1}, t_i]$,

$$|\mu(X_{t_{i-1}} + W_t - W_{t_{i-1}}) - \mu(X_t)|$$

$$\leq c \left( |X_{t_{i-1}} + W_t - W_{t_{i-1}} - X_t| + \sum_{j=1}^k 1\{X_{t_{i-1}} + W_t - W_{t_{i-1}} - X_t \leq 0\} \right).$$

For all $i \in \{1, \ldots, n\}$ and all $t \in [t_{i-1}, t_i]$ we furthermore have

$$|X_{t_{i-1}} + W_t - W_{t_{i-1}} - X_t| = \left| \int_{t_{i-1}}^{t} \mu(X_s) ds \right| \leq \|\mu\|_{\infty} (t - t_{i-1}).$$

Employing (5), Lemma 10 and (64) we thus obtain that there exist $c_1, c_2, c_3 \in (0, \infty)$ such that for all $i \in \{1, \ldots, n\},$

$$E[B_i^2] \leq c_1 E \left[ \left( (t_i - t_{i-1})^2 + \sum_{j=1}^k \int_{t_{i-1}}^{t_i} 1\{X_{t_{i-1}} + W_t - W_{t_{i-1}} - X_t \leq 0\} du \right)^2 \right]$$

$$\leq c_2 \left( (t_i - t_{i-1})^4 + \sum_{j=1}^k E \left[ \left( \int_{t_{i-1}}^{t_i} 1\{X_{t_{i-1}} + W_t - W_{t_{i-1}} - X_t \leq 0\} du \right)^2 \right] \right)$$

$$\leq c_3 \left( (t_i - t_{i-1})^4 + (t_i - t_{i-1})^3 \right) \leq \frac{16c_3}{n^3}.$$

Clearly, for all $i \in \{1, \ldots, n\},$

$$E_i^2 \leq 2\|\mu\|_{\infty} (t_i - t_{i-1}) \int_{t_{i-1}}^{t_i} \left| \mu(\tilde{X}_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}}) \right| ds,$$
and therefore
\[ E[E_t^2] \leq 2\|\mu\|_\infty (t_f - t_{i-1}) \int_{t_{i-1}}^{t_f} E[|\mu(X_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) - \mu(\tilde{X}_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}})|] \, dt. \]

Recall from the proof of Lemma 13 that for all \( i \in \{1, \ldots, n\} \), \((X_{t_{i-1}}, \tilde{X}_{t_{i-1}})\) and \((\tilde{W}_t - \tilde{W}_{t_{i-1}})_{t \in [t_{i-1}, t_i]}\) are independent. Hence, for all \( i \in \{1, \ldots, n\} \),
\[ \int_{t_{i-1}}^{t_i} E[|\mu(X_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}}) - \mu(\tilde{X}_{t_{i-1}} + \tilde{W}_t - \tilde{W}_{t_{i-1}})|] \, dt \]
\[ = \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} E[|\mu(X_{t_{i-1}} + u) - \mu(\tilde{X}_{t_{i-1}} + u)|] \, \mathbb{P}(\tilde{W}_t - \tilde{W}_{t_{i-1}}(du)) \, dt. \]

By Lemma 1, we know that there exists \( c \in (0, \infty) \) such that for all \( i \in \{1, \ldots, n\} \) and all \( u \in \mathbb{R} \),
\[ |\mu(X_{t_{i-1}} + u) - \mu(\tilde{X}_{t_{i-1}} + u)| \leq c \left( |X_{t_{i-1}} - \tilde{X}_{t_{i-1}}| + \sum_{j=1}^{k} 1_{\{(X_{t_{i-1}} + u - \xi_j) (\tilde{X}_{t_{i-1}} + u - \xi_j) \leq 0\}} \right). \]

Lemma 12 implies that there exists \( c \in (0, \infty) \) such that
\[ \max_{i \in \{1, \ldots, n\}} E[|X_{t_{i-1}} - \tilde{X}_{t_{i-1}}|] \leq \frac{c}{n^{3/4}}. \]

Moreover, for all \( i \in \{1, \ldots, n\} \), all \( u \in \mathbb{R} \), all \( j \in \{1, \ldots, k\} \) and all \( \gamma \in (0, \infty) \) we have
\[ E[1_{\{(X_{t_{i-1}} + u - \xi_j) (\tilde{X}_{t_{i-1}} + u - \xi_j) \leq 0\}}] = P((X_{t_{i-1}} + u - \xi_j) (\tilde{X}_{t_{i-1}} + u - \xi_j) \leq 0) \]
\[ \leq P(|X_{t_{i-1}} + u - \xi_j| \leq |X_{t_{i-1}} - \tilde{X}_{t_{i-1}}|) \]
\[ \leq P(|X_{t_{i-1}} + u - \xi_j| \leq \gamma) + P(|X_{t_{i-1}} - \tilde{X}_{t_{i-1}}| > \gamma). \]

Due to the assumptions \( n \in 2\mathbb{N} \) and (64) there exists \( r \in \{1, \ldots, n\} \) with \( t_r = 1/2 \). Using Lemma 7 with \( \tau = 1/2 \) we obtain that there exists \( c \in (0, \infty) \) such that for all \( j \in \{1, \ldots, k\} \), all \( u \in \mathbb{R} \) and all \( \gamma \in (0, \infty) \),
\[ \max_{i \in \{r+1, \ldots, n\}} P(|X_{t_{i-1}} + u - \xi_j| \leq \gamma) \leq c \gamma. \]

Furthermore, by Markov’s inequality and Lemma 12 there exists \( c \in (0, \infty) \) such that for all \( \gamma \in (0, \infty) \),
\[ \max_{i \in \{1, \ldots, n\}} P(|X_{t_{i-1}} - \tilde{X}_{t_{i-1}}| > \gamma) \leq \frac{c}{\gamma^3 \cdot n^{9/4}}. \]

Choosing
\[ \gamma = n^{-\frac{\alpha}{10}} \]
we conclude from (85) to (87) that there exists \( c \in (0, \infty) \) such that for all \( u \in \mathbb{R} \) and all \( j \in \{1, \ldots, k\} \),
\[ \max_{i \in \{r+1, \ldots, n\}} E[1_{\{(X_{t_{i-1}} + u - \xi_j) (\tilde{X}_{t_{i-1}} + u - \xi_j) \leq 0\}}] \leq c n^{-\frac{\alpha}{10}}. \]
Combining (81) to (84) and (88) and observing (64) we conclude that there exist \( c_1, c_2 \in (0, \infty) \) such that for all \( i \in \{ r + 1, \ldots, n \} \),

\[
\mathbb{E}[E_i^2] \leq c_1 (t_i - t_{i-1})^2 \frac{1}{n^{9/16}} \leq \frac{c_2}{n^{5/2+1/16}}.
\]

Inserting the estimates (80) and (89) into (79) we conclude that there exists \( c \in (0, \infty) \) such that for all \( i \in \{ r + 1, \ldots, n \} \),

\[
\mathbb{E}[A_i^2] \leq 4\mathbb{E}[C_i^2] + \frac{c}{n^{5/2+1/16}}
\]

which completes the proof of the lemma. \( \square \)

**Lemma 15.** Assume that \( \mu \) satisfies (\( \mu 1 \)), (\( \mu 4 \)) and (\( \mu 5 \)). Let \( x_0 \in \mathbb{R} \) and let \( X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R} \) be strong solutions of the SDE (51) on the time-interval \([0, 1]\) with initial value \( x_0 \) and driving Brownian motion \( W \) and \( \tilde{W} \), respectively. Then there exists \( c \in (0, \infty) \) such that for all \( \ell \in \{1, \ldots, k\} \) and all \( i \in \{1, \ldots, n\} \) with \( t_i > 1/2 \) it holds

\[
\mathbb{E}\left[ \left( \int_{t_{i-1}}^{t_i} (\mu(X_{t_{i-1}} + W_s - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}})) ds \right)^2 \right] 
\geq c (\mu(\xi_\ell^\ell) - \mu(\xi_{\ell-1}^\ell))^2 (t_i - t_{i-1})^{5/2}.
\]

**Proof.** We have for all \( i \in \{1, \ldots, n\} \) and all \( s \in [t_{i-1}, t_i] \),

\[
W_s - W_{t_{i-1}} = \frac{s - t_{i-1}}{t_i - t_{i-1}} (W_{t_i} - W_{t_{i-1}}) + B_s, \quad \tilde{W}_s - \tilde{W}_{t_{i-1}} = \frac{s - t_{i-1}}{t_i - t_{i-1}} (W_{t_i} - W_{t_{i-1}}) + \tilde{B}_s.
\]

Hence, for all \( i \in \{1, \ldots, n\} \),

\[
\int_{t_{i-1}}^{t_i} (\mu(X_{t_{i-1}} + W_s - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}})) ds 
= \int_0^{t_i - t_{i-1}} (\mu(X_{t_{i-1}} + \frac{s}{t_i - t_{i-1}} (W_{t_i} - W_{t_{i-1}}) + B_{t_{i-1}+s}) 
- \mu(X_{t_{i-1}} + \frac{s}{t_i - t_{i-1}} (W_{t_i} - W_{t_{i-1}}) + \tilde{B}_{t_{i-1}+s})) ds.
\]

Recall from the proof of Lemma \( \text{[***]} \) that for all \( i \in \{1, \ldots, n\} \), \( (X_{t_{i-1}}, \tilde{X}_{t_{i-1}}) \) and \((W_s - W_{t_{i-1}}, \tilde{W}_s - \tilde{W}_{t_{i-1}})_{s \in [t_{i-1}, t_i]} \) are independent. Hence, for all \( i \in \{1, \ldots, n\} \), \( X_{t_{i-1}}, W_{t_i} - W_{t_{i-1}}, \)

\( (B_{t_{i-1}+s})_{s \in [0, t_i - t_{i-1}]}, (\tilde{B}_{t_{i-1}+s})_{s \in [0, t_i - t_{i-1}]} \) are independent. We may thus apply Lemma \( \text{[***]} \) to obtain that for all \( \ell \in \{1, \ldots, k\} \) and all \( i \in \{1, \ldots, n\} \),

\[
\mathbb{E}\left[ \left( \int_{t_{i-1}}^{t_i} (\mu(X_{t_{i-1}} + W_s - W_{t_{i-1}}) - \mu(X_{t_{i-1}} + \tilde{W}_s - \tilde{W}_{t_{i-1}})) ds \right)^2 \right] 
\geq \kappa (\mu(\xi_\ell^\ell) - \mu(\xi_{\ell-1}^\ell))^2 (t_i - t_{i-1})^2 
\times \mathbb{P}(X_{t_{i-1}} \in [\xi_\ell, \xi_\ell + \sqrt{t_i - t_{i-1}}]) \mathbb{P}(W_{t_i} - W_{t_{i-1}} \in [0, \sqrt{t_i - t_{i-1}}]),
\]

where \( \kappa \) is given by (11).
Due to the assumptions $n \in 2\mathbb{N}$ and (63) there exists $r \in \{1, \ldots, n\}$ with $t_r = 1/2$. Using Lemma 7 with $\tau = 1/2$, $M = \max_{\ell = 1, \ldots, k} |\xi_\ell| + 1$ we see that there exists $c \in (0, \infty)$ such that for all $\ell \in \{1, \ldots, k\}$ and all $i \in \{r + 1, \ldots, n\}$,

$$\mathbb{P}(X_{t_{i-1}} \in [\xi_\ell, \xi_\ell + \sqrt{t_i - t_{i-1}}]) \geq c(t_i - t_{i-1})^{1/2}. \quad (93)$$

Combining (92) with (45) and (93) completes the proof of the lemma. \hfill \Box

We are ready to provide the appropriate lower bound for the right hand side of (67).

**Lemma 16.** Assume that $\mu$ satisfies (\mu1) to (\mu5). Let $x_0 \in \mathbb{R}$ and let $X, \tilde{X} : [0, 1] \times \Omega \to \mathbb{R}$ be strong solutions of the SDE (51) on the time-interval $[0, 1]$ with initial value $x_0$ and driving Brownian motion $W$ and $\tilde{W}$, respectively. Then there exist $c_1, c_2 \in (0, \infty)$ such that

$$\mathbb{E}|X_1 - \tilde{X}_1| \geq \frac{c_1}{n^{3/4}} \left(\max(0, (1 - c_2/n^{1/16}))\right)^{3/2}. \quad (94)$$

**Proof.** Put

$$\Delta_i = \mathbb{E}|X_{t_i} - \tilde{X}_{t_i}|^2$$

for $i \in \{0, \ldots, n\}$ and note that

$$\mathbb{E}|X_1 - \tilde{X}_1|^2 = \Delta_n. \quad (95)$$

Observing (64) we see that for all $i \in \{1, \ldots, n\}$,

$$\Delta_i = \mathbb{E}\left[|X_{t_{i-1}} - \tilde{X}_{t_{i-1}} + \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds|^2\right] = \Delta_{i-1} + 2m_i + d_i, \quad (96)$$

where

$$m_i = \mathbb{E}\left[(X_{t_{i-1}} - \tilde{X}_{t_{i-1}}) \int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds\right] \quad \text{and} \quad d_i = \mathbb{E}\left[\int_{t_{i-1}}^{t_i} (\mu(X_s) - \mu(\tilde{X}_s)) \, ds\right]^2.$$

Using Lemma 13 we have for all $i \in \{1, \ldots, n\}$ that $m_i \geq 0$. By (64) there exists $r \in \{0, \ldots, n\}$ such that $t_r = 1/2$. Combining Lemma 13 and Lemma 15 and observing property (\mu3) we conclude that there exist $c_1, c_2 \in (0, \infty)$ such that for $i \in \{r + 1, \ldots, n\}$,

$$d_i \geq c_1 (t_i - t_{i-1})^{5/2} - c_2 \frac{1}{n^{5/2+1/16}}. \quad (97)$$

Hence, observing (48) we obtain that there exists $c_3 \in (0, \infty)$ such that

$$\mathbb{E}|X_1 - \tilde{X}_1|^2 = 2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{n} d_i \geq \sum_{i=1}^{n} d_i \geq \sum_{i=r+1}^{n} d_i \geq c_1 \sum_{i=r+1}^{n} (t_i - t_{i-1})^{5/2} - \frac{c_2}{n^{3/2+1/16}} \geq \frac{c_3}{n^{3/2}} - \frac{c_2}{n^{3/2+1/16}} \geq \frac{c_3}{n^{3/2}} \cdot \left(1 - \frac{c_2}{c_3} \cdot \frac{1}{n^{1/16}}\right).$$
Employing Lemma 12 with \( p = 4 \) we may proceed similar to the end of the proof of Theorem 3, see (49) and (50), to conclude with the help of Hölder’s inequality that there exist \( c_1, c_2, c_3 \in (0, \infty) \) such that
\[
\frac{c_1}{n^{3/2}} \cdot \max(0, (1 - c_2/n^{1/16})) \leq \mathbb{E}[(X_1 - \tilde{X}_1)^2] \leq \mathbb{E}[(X_1 - \tilde{X}_1)^2]^{3/2} \frac{c_3}{n},
\]
which implies that there exist \( c_1, c_2 \in (0, \infty) \) such that
\[
\mathbb{E}[(X_1 - \tilde{X}_1)] \geq \frac{c_1}{n^{3/4}} (\max(0, (1 - c_2/n^{1/16})))^{3/2}
\]
and hereby finishes the proof of the lemma. □

Combining Lemma 11 with \( p = 1 \) and Lemma 16 yields Proposition 1 and hereby finishes the proof of Theorem 2.

APPENDIX

It is well-known that the components of a bivariate normal random variable \((Z, Y)\) with \( \text{Cov}(Z, Y) \geq 0 \) are positively associated, whence, in particular, \( \text{Cov}(f(Z), g(Y)) \geq 0 \) holds for all increasing \( f, g: \mathbb{R} \rightarrow \mathbb{R} \) such that \( \text{Cov}(f(Z), g(Y)) \) exists. See, e.g. [29, Theorem 5.1.1]. The following lemma strengthens this result in the case when \( f \) and \( g \) are piecewise Lipschitz continuous.

**Lemma 17.** Let \( \rho \in [0, 1] \) and \((Z, Y) \sim \mathcal{N}(0, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})\). Moreover, let \( k, l \in \mathbb{N} \) and \(-\infty = a_0 < a_1 < \ldots < a_k < a_{k+1} = \infty \) and \( -\infty = b_0 < b_1 < \ldots < b_l < b_{l+1} = \infty \), and let \( f, g: \mathbb{R} \rightarrow \mathbb{R} \) satisfy

(i) \( f, g \) are both increasing or both decreasing,

(ii) \( f \) is Lipschitz continuous on the interval \((a_{i-1}, a_i)\) for all \( i \in \{1, \ldots, k+1\} \) and \( g \) is Lipschitz continuous on the interval \((b_{j-1}, b_j)\) for all \( j \in \{1, \ldots, l+1\} \).

Then it holds
\[
\mathbb{E}[f(Z)g(Y)] - \mathbb{E}[f(Z)]\mathbb{E}[g(Y)] \geq \sum_{i=1}^k \sum_{j=1}^l (f(a_i) - f(a_{i-1}))(g(b_j) - g(b_{j-1})) \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_0^\rho \frac{1}{\sqrt{1-u^2}} e^{-\frac{(y-b_{j-1})^2}{2(1-u^2)}} du.
\]

**Proof.** Clearly, we may assume that \( f \) and \( g \) are both increasing. Employing the assumption (ii) and Lemma 11 we see that there exists \( c \in (0, \infty) \) such that for all \( x \in \mathbb{R} \),
\[
|f(x)| + |g(x)| \leq c \cdot (1 + |x|),
\]
and that all of the limits \( f(a_i), f(a_i), g(b_j), g(b_{j+1}) \), \( i \in \{1, \ldots, k\}, j \in \{1, \ldots, l\} \) exist and are finite. Hence all of the expected values on the left hand side of (99) are well-defined and finite, and the right hand side of (99) is well-defined as well.

Clearly, (99) holds if \( \rho = 0 \). Next, assume that \( \rho \in (0, 1) \). We proceed similar to the proof of [29, Theorem 5.1.1]. For \( u \in [0, \rho] \) let
\[
(Z_u, Y_u) \sim \mathcal{N}(0, \begin{pmatrix} 1 & u \\ u & 1 \end{pmatrix})
\]
and define \( \psi : [0, \rho] \to [0, \infty) \) by

\[
\psi(u) = \mathbb{E}[f(Z_u) g(Y_u)].
\]

Then

(101)

\[
\mathbb{E}[f(Z) g(Y)] - \mathbb{E}[f(Z)] \mathbb{E}[g(Y)] = \psi(\rho) - \psi(0).
\]

Using the well-known fact that for all \( u \in [0, \rho] \) the conditional distribution of \( Y_u \) given \( Z_u \) satisfies \( \mathbb{P}^{Y_u|Z_u} = N(zu, 1 - u^2) \) for \( \mathbb{P}^{Z_u} \)-almost all \( z \in \mathbb{R} \), we obtain for all \( u \in [0, \rho] \) that

\[
\psi(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) g(y) h(z, y, u) \varphi(z) \, dy \, dz,
\]

where the functions \( h: \mathbb{R}^2 \times [0, \rho] \to \mathbb{R} \) and \( \varphi: \mathbb{R} \to \mathbb{R} \) are defined by

\[
h(z, y, u) = \frac{1}{\sqrt{2\pi(1-u^2)}} e^{\frac{(y-zu)^2}{2(1-u^2)}}, \quad \varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.
\]

Let \( z, y \in \mathbb{R} \). For all \( u \in [0, \rho] \),

\[
\frac{\partial}{\partial u} h(z, y, u) = \left( \frac{u}{1-u^2} - \frac{(y-zu)(yu-z)}{(1-u^2)^2} \right) h(z, y, u).
\]

Since

\[
\sup_{u \in [0, \rho]} e^{-\frac{(y-zu)^2}{2(1-u^2)}} \leq \sup_{u \in [0, \rho]} e^{-\frac{y^2}{2}} \leq e^{-\frac{1}{2}(1-\rho^2) + \frac{2\rho}{2}}
\]

we obtain that

\[
\sup_{u \in [0, \rho]} \left| \frac{\partial}{\partial u} h(z, y, u) \right| \leq \left( 1 + \frac{|y| + |z|}{1-\rho^2} \right) \frac{1}{\sqrt{2\pi(1-\rho^2)}} e^{-\frac{1}{2}(1-\rho^2) + \frac{2\rho}{2}}.
\]

By the latter fact and (100) we may apply the dominated convergence theorem to conclude that \( \psi \) is continuously differentiable with

(102)

\[
\psi'(u) = \int_{\mathbb{R}} \int_{\mathbb{R}} f(z) g(y) \frac{\partial}{\partial u} h(z, y, u) \varphi(z) \, dy \, dz, \quad u \in [0, \rho].
\]

Below we show that for all \( u \in (0, \rho] \),

(103)

\[
\psi'(u) = \sum_{i=1}^{k} \sum_{j=1}^{l} (f(a_i+) - f(a_i-)) (g(b_j+) - g(b_j-)) \varphi(a_i) h(a_i, b_j, u).
\]

The latter estimate, the equality

\[
\psi(\rho) - \psi(0) = \int_{0}^{\rho} \psi'(u) \, du
\]

and (101) imply (102).

It remains to prove (103). Straightforward calculations show that for all \( z, y \in \mathbb{R} \) and all \( u \in [0, \rho] \),

\[
\frac{\partial}{\partial z} h(z, y, u) = \frac{u(y-zu)}{1-u^2} h(z, y, u), \quad \frac{\partial^2}{\partial z^2} h(z, y, u) = \left( \frac{u^2(y-zu)^2}{(1-u^2)^2} - \frac{u^2}{1-u^2} \right) h(z, y, u),
\]

and (101) imply (102).
and therefore for all \( z, y \in \mathbb{R} \) and all \( u \in (0, \rho) \),
\[
\frac{\partial}{\partial u} h(z, y, u) = -\frac{1}{u} \left( \frac{\partial^2}{\partial z^2} h(z, y, u) - z \frac{\partial}{\partial z} h(z, y, u) \right).
\]

Observing (102) we may thus conclude that for all \( u \in (0, \rho) \),
\[
(104) \quad \psi'(u) = -\frac{1}{u} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(z) \varphi(z) \left( \frac{\partial^2}{\partial z^2} h(z, y, u) - z \frac{\partial}{\partial z} h(z, y, u) \right) dz dy.
\]

Let \( f_0: (-\infty, a_1] \to \mathbb{R}, f_1: [a_1, a_2] \to \mathbb{R}, \ldots, f_k: [a_k, \infty) \to \mathbb{R} \) denote the continuous extensions of \( f_1(-\infty,a_1), f_1(a_1,a_2), \ldots, f_1(a_k,\infty) \) on \((-\infty, a_1), [a_1,a_2], \ldots, [a_k,\infty)\), respectively. The assumption (ii) implies that for every \( i \in \{0, \ldots, k\} \) the function \( f_i \) is Lipschitz continuous on its domain and therefore has a Lebesgue density \( f_i' \). Applying the integration by parts formula and observing that \( \varphi'(z) = -z \varphi(z) \) for all \( z \in \mathbb{R} \) as well as
\[
\lim_{z \to -\infty} f(z) \varphi(z) \frac{\partial}{\partial z} h(z, y, u) = \lim_{z \to \infty} f(z) \varphi(z) \frac{\partial}{\partial z} h(z, y, u) = 0 \quad \text{for all} \quad y \in \mathbb{R} \quad \text{and} \quad u \in (0, \rho),
\]
we therefore obtain that for all \( y \in \mathbb{R} \) and all \( u \in (0, \rho) \),
\[
(105) \quad \int_{\mathbb{R}} f(z) \varphi(z) \frac{\partial^2}{\partial z^2} h(z, y, u) dz
\]
\[
= \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} f_i(z) \varphi(z) \frac{\partial^2}{\partial z^2} h(z, y, u) dz
\]
\[
= \sum_{i=1}^{k} (f(a_i) - f(a_i +)) \varphi(a_i) \frac{\partial}{\partial z} h(a_i, y, u) - \sum_{i=1}^{k} \int_{a_i}^{a_{i+1}} (f_i \cdot \varphi)'(z) \frac{\partial}{\partial z} h(z, y, u) dz
\]
\[
= \sum_{i=1}^{k} (f(a_i) - f(a_i +)) \varphi(a_i) \frac{\partial}{\partial z} h(a_i, y, u) - \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} f_i'(z) \varphi(z) \frac{\partial}{\partial z} h(z, y, u) dz
\]
\[
+ \int_{\mathbb{R}} f(z) \varphi(z) z \frac{\partial}{\partial z} h(z, y, u) dz.
\]

Using (104) and (105) we see that for all \( u \in (0, \rho) \),
\[
(106) \quad \psi'(u) = v(u) + w(u),
\]
where the functions \( v, w: (0, \rho] \to \mathbb{R} \) are defined by
\[
v(u) = \sum_{i=1}^{k} (f(a_i) - f(a_i +)) \varphi(a_i) \frac{1}{u} \int_{\mathbb{R}} g(y) \frac{\partial}{\partial z} h(a_i, y, u) dy,
\]
\[
w(u) = \sum_{i=0}^{k} \int_{a_i}^{a_{i+1}} f_i'(z) \varphi(z) \left( \frac{1}{u} \int_{\mathbb{R}} g(y) \frac{\partial}{\partial z} h(z, y, u) dy \right) dz.
\]

Note that for all \( z, y \in \mathbb{R} \) and \( u \in [0, \rho] \),
\[
(107) \quad \frac{\partial}{\partial z} h(z, y, u) = -u \frac{\partial}{\partial y} h(z, y, u).
\]
Let \( g_0: (-\infty, b_1] \to \mathbb{R}, g_1: [b_1, b_2] \to \mathbb{R}, \ldots, g_l: [b_l, \infty) \to \mathbb{R} \) denote the continuous extensions of \( g((\infty, b_1), \ldots, g((b_l, \infty)) \) on \((\infty, b_1], [b_1, b_2], \ldots, [b_l, \infty)\), respectively. The assumption (ii) implies that for every \( j \in \{0, \ldots, l\} \) the function \( g_j \) is Lipschitz continuous on its domain and therefore has a Lebesgue density \( g_j' \). Using (107), applying the integration by parts formula and observing that
\[
\lim_{y \to -\infty} g(y) h(z, y, u) = \lim_{y \to \infty} g(y) h(z, y, u) = 0
\]
for all \( z \in \mathbb{R} \) and all \( u \in [0, \rho] \), we therefore obtain that for all \( z \in \mathbb{R} \) and all \( u \in (0, \rho] \),
\[
\frac{1}{u} \int_{\mathbb{R}} g(y) \frac{\partial}{\partial z} h(z, y, u) \, dy = -\int_{\mathbb{R}} g(y) \frac{\partial}{\partial y} h(z, y, u) \, dy = -\sum_{j=0}^{l} \int_{b_j}^{b_{j+1}} g_j(y) \frac{\partial}{\partial y} h(z, y, u) \, dy
\]
\[
= \sum_{j=1}^{l} (g(b_j+) - g(b_j-)) h(z, b_j, u) + \sum_{j=0}^{l} \int_{b_j}^{b_{j+1}} g_j'(y) h(z, y, u) \, dy.
\]
Since \( f \) and \( g \) are both increasing we may assume that \( f_i', g_j' \geq 0 \) for all \( i \in \{0, \ldots, k\} \) and \( j \in \{0, \ldots, l\} \) and we conclude that for all \( u \in (0, \rho] \),
\[
v(u) \geq \sum_{i=1}^{k} \sum_{j=1}^{l} (f(a_i+) - f(a_i-)) (g(b_j+) - g(b_j-)) \varphi(a_i) h(a_i, b_j, u)
\]
and
\[
w(u) \geq \sum_{j=1}^{l} (g(b_j+) - g(b_j-)) \sum_{i=0}^{a_{i+1}} \int_{a_i}^{a_{i+1}} f_j'(z) \varphi(z) h(z, b_j, u) \, dz \geq 0.
\]
The latter two estimates together with (106) yield (103) and complete the proof of the lemma in the case \( \rho \in (0, 1) \).

Finally, assume that \( \rho = 1 \). Then \( Z = Y \mathbb{P}\text{-a.s.} \). Let \( U \sim N(0, 1) \) be independent of \( Z \) and for \( s \in [0, 1) \) put
\[
V_s = s Z + \sqrt{1 - s^2} U.
\]
Observe that
\[
(Z, V_s) \sim N\left(0, \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}\right)
\]
for all \( s \in [0, 1) \) and that \( g \) has at most finitely many discontinuity points. Hence, \( \mathbb{P}\text{-a.s.} \),
\[
\lim_{s \to 1} g(V_s) = g(Y).
\]
Observing (100) we may thus apply the dominated convergence theorem to conclude
\[
\mathbb{E}[f(Z) g(Y)] - \mathbb{E}[f(Z) \mathbb{E}[g(Y)]] = \lim_{s \to 1} \left( \mathbb{E}[f(Z) g(V_s)] - \mathbb{E}[f(Z) \mathbb{E}[g(V_s)]] \right).
\]
Applying (99) with \( Y = V_s \) for \( s \in [0, 1) \) and using the fact that for all \( a, b \in \mathbb{R} \),
\[
\lim_{s \to 1} \int_{0}^{s} \frac{1}{\sqrt{1 - u^2}} e^{-\frac{(b-aw)^2}{2(1-u^2)}} \, du = \int_{0}^{1} \frac{1}{\sqrt{1 - u^2}} e^{-\frac{(b-aw)^2}{2(1-u^2)}} \, du
\]
finishes the proof of (99) in the case \( \rho = 1 \) and completes the proof of the lemma. \( \square \)
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