Optimisation reinforced concrete slab with risk analysis according to the criterion of minimum cost

Ekaterina Filimonova
Moscow State University of Civil Engineering, Yaroslavskoe Shosse, 26, Moscow, 129337, Russia

E-mail: e.filimonova13@gmail.com

Abstract. The safety strategy based on the principles of forecasting and preventing technology related accidents and structural failures is gaining momentum at the present time. In this context it can’t be argued that estimation and accounting for the risk of structural failure in design is becoming more and more relevant. The method of calculation and optimisation of engineering structures can be significantly improved by wider application of risk analysis. Usage of probabilistic apparatus allows to define quantitatively the structure safety level and to design them by minimum cost criteria. To achieve this goal objective function is formed, taking as inputs all the costs and damage from possible accident conditions. Optimal design of reinforced concrete structures is characterised by non-linear objective function and some number of local extremums. Random search algorithm is used to solve similar problems. Suggested improved searching method allows to solve the optimisation choice of structure parameters problem both with non-linear constraints and non-linear objective function.

1. Introduction
To find the optimal design solution the method of forming some objective function with the requirement of its minimisation is used. Optimisation of reinforced concrete slabs is a complex problem and linked to resolving a range of different issues. The main issue is the choice of optimal criteria of taken decision. The most preferred criteria in process of structure optimisation is the minimum of the total costs.

There are 3 main groups of expected costs. They include:
1. direct costs for the designed structure
2. current maintenance costs depending on the parameters of designed structure and operating conditions.
3. expenses for mitigation of consequences of possible emergency situations

Objective function $F_1$, taking in account costs of materials and manufacturing is defined as [1]:

$$F_1 = k_c k_3 \left[ C_1 V_b + C_2 V_b^2 + \sum_r \left( C_3 V_s + C_4 V_s^2 \right) + \sum_i C_{i\text{add}} \right].$$ (1)
Objective function, taking into account the cost of the concrete slab and maintenance costs, is defined as:

\[
F_2 = k_C k_3 \left[ C_b V_b + C_b V_b^2 + \sum_{r} \left( C_s V_s + C_s V_s^2 \right) + \sum_i C_{add} + C_{mnt} \right].
\]  

(2)

where \( C_{mnt} \) – maintenance costs.

Current Russian and foreign regulations [2,3] advise to calculate and design the structure taking into account the additional load resulting from the damage of the separate structural elements in possible emergency situation. As a result, a sharp increase of the external impact on the structure can lead to an unexpected structural failure. In this case, taking into consideration the risk of the structural failure in calculations is the most appropriate measure to minimise the structural damage costs [4,5].

Risk as a separate event has 2 most important characteristics - probability and damage. Risk as a set of events has a range of its realisations, each one of them having its own probability and magnitude of damage.

Damage of load bearing structure (column, pylon) as a most typical emergency situation which might lead to progressive collapse of the structure is analysed. In this case, expenses for preventing the possible damage included in the function of cost.

Based on the above mentioned parameters optimisation group is formed, defining the slab characteristics based on the risk of structural failure. Optimisation is carried out with the involvement of the economic analysis of safety, cost-based accounting to ensure the safety and losses from potential accidents [6].

The objective function, taking into consideration concrete slab cost, maintenance expenses and risks, depending on the building importance level:

\[
F_3 = k_C k_3 \left[ C_b V_b + C_b V_b^2 + \sum_{r} \left( C_s V_s + C_s V_s^2 \right) + \sum_i C_{add} + \left( 1 - P_t \right) \cdot F_1 \cdot T_{lft} + R(t) \right].
\]  

(3)

gde \( C_1, C_3 \) – production cost of 1 m³ of concrete and 1 ton of reinforcement steel «in work»;

\( C_2, C_4 \) – production costs of 1 m³ of concrete and 1 ton of steel reinforcement taken depending on the material consumption;

\( V_b = \omega_h h^3 \) - volume of concrete in k-th segment plate;

\( V_s,r = \omega_k h_0 \left( \mu_1 + \mu_2 \right) \) - volume of reinforcement steel type r for k-th segment plate;

\( k_C, k_3 \) - coefficient of recount; coefficient of winter appreciations;

\( \sum_i C_{add} = k_{add} \cdot C_3 \cdot V_{s,r} \) - additional costs, including reinforcement steel for technological openings framing, overlap and anchorage;

\( P_t \) - reliability of the structure in current time period;

\( 1 - P_t \) - probability of structural failure during the year;

\( T_{lft} \) – actual operating lifetime of the structure;

\( R(t) \) – risk of loss in case of structural failure.

Reliability of the structure is a complex quality indicator, demonstrating the ability of the structure to preserve planned performance characteristics during its defined lifetime.

Reliability of the structure \( P_t \) in the moment of time t is defined by non linear approximation of safety parameter and operating duration of the element with coefficients of composition of normal and second exponential distribution law.
Risk is calculated as a probability of structural failure with certain level of consequences during defined operating period, the value of risk in monetary terms is found according to the following formula [7]:

\[ R(t) = Q(t) \cdot C_{cum}, \]  

(4)

where \( Q(t) \) – risk of structural failure; 
\( C_{cum} \) – cumulative damage in case of structural failure.

Notion of damage in general terms can be viewed as a harm caused to physical health, property or environment. To achieve goal of optimisation Direct damage from a complete or partial slab destruction is calculated based on loss of its cost and minimum required expenses for repair and reconstruction:

\[ C_j = \sum_{i=1}^{n} (\Delta P^i \times K^i_d) + P_{min}, \]  

(5)

where \( \Delta P^i \) – reduction in cost of the slab as a result of the complete or partial destruction (defined by the area of collapse); 
\( K^i_d \) – amortising coefficient of the reinforced concrete slab; 
\( n \) – number of slab segments damaged or destroyed; 
\( P_{min} \) – minimal repair and other expenses, required for slab reconstruction.

To solve for the problem of optimisation with consideration of risk one needs to know the function \( Q(t) \) defining the probability of emergency situation in different periods of operating lifetime.

The structural collapse is interpreted as an exiting of random process of load bearing capacity in a negative field (area). The probability estimation of element failure in this case is written as [8]:

\[ Q(t) = \frac{\omega_q f_v(\beta)t}{\beta_o \sqrt{2\pi}}, \]  

(6)

where \( \omega_q \) - cumulative effective frequency of random process; 
\( f_v(\beta) \) - density of load bearing reserve distribution; 
\( \beta \) - safety parameter; 
\( t \) – operating lifetime; 
\( \beta_o \) - coefficient of random process broadbandness.

In case of applying one temporal random load formula of cumulative (effective) frequency is simplified:

\[ \omega_q = \frac{1}{\sqrt{1+K^2}} \omega_1, \]  

(7)

and broadbandness coefficient can be approximated as

\[ \beta_o = \beta_{12} = \beta \frac{\sqrt{1+K^2}}{K}, \]  

(8)

where \( K = \frac{\hat{X}_2}{\hat{X}_1} = \frac{\hat{\sigma}}{\hat{\sigma}_T} \).

Analysis of probabilistic calculation methods of structural elements shows that most evident collapse indicator is safety characteristic \( \beta \) [9,10]. This parameter is defined as the number of standards \( \sigma(X) \), that fit within the interval \([0, \bar{X}]\).

\[ \beta = \frac{\bar{X}}{\sigma(X)} = \frac{\bar{M}_R - \bar{M}_q}{\sqrt{\sigma^2(M_R) + \sigma^2(M_q)}}, \]  

(9)

where \( \bar{X} \) - mathematical expectation of load bearing capacity;
\[ \sigma(X) \] - standard of load bearing capacity;
\[ \mathcal{M}_R, \mathcal{M}_q \] - mathematical expectation of strength and load;
\[ \sigma^2(M_R); \sigma^2(M_q) \] - corresponding standards of strength and load.

Let’s take a look at plotting of distribution curves of load bearing capacity of the slab as a difference of random values of strength and load. We will consider load bearing capacity as normally distributed. The emergency loads we will represent through Weibull law, wide usage of which is explained by its universality as it contains the additional alpha parameter. We will get normal, exponential and other types of distributions using respective values of parameters \( \alpha \) и \( \lambda \).

Density of Weibull distribution is described by following relationship:
\[
f(x) = \alpha \lambda e^{-\lambda x^\alpha},
\]
where \( \alpha \) – parameter of distribution curve shape; \( \lambda \) – parameter of scale; \( e = 2,71828 \) – base of natural logarithm.

For the difference of normal and weibull distribution we have:
\[
f(\beta) = \frac{D}{\sqrt{2\pi}} \int_0^Z \exp\left(-0.5E^2\right)dz,
\]
where
\[ E = \beta D - \frac{1}{C_v(F)} \left[ 1 - \left(-\ln Z\right)^{\frac{1}{\alpha}} \right]; \]
\[ \beta = \frac{X - \bar{X}}{\sigma(X)}; \]
\[ D = \sqrt{1 + \frac{p^2}{p}}; \]
\[ p = \frac{C_v(M_R)}{C_v(M_q)}; \]
\[ Z = \exp\left[-\Gamma^\alpha \left( -1 + \frac{C_v(M_q)}{C_v(M_R)} + \beta C_v(M_q) \sqrt{1 + p^2} \right) \right]. \]

Here \( \Gamma \) – is a gamma function of weibull distribution.

Example of resulting distribution of differences can be seen on the figure 1. Difference has a mode, shifted on the right regarding the center, however, with the increase of normal distribution influence and growth of \( p \) distribution of difference is quickly becoming normal.

![Figure 1. Distribution of random values of strength and loads differences.](image-url)
It is worth mentioning, that considerable advantage of this objective function is the connection of its components with variable parameters. Therefore, probability of structural failure is explained by load-bearing capacity which is defined by set of variable parameters: thickness of slab and protective layer, percentage of steel reinforcement, class of concrete and steel.

Value of structural system risk, impacted by external forces, is defined by value of structural damage and importance of the building. Levels of objects reliability are set depending on importance level of structures. That’s why it is required to establish level of critical condition, corresponding to the structural failure in relation to the size of risk and damage, caused by partial or complete destruction.

Relationship risk-damage $R\cdot S$ is most accurately approximated by exponential curve with correlation coefficient 0.973 (figure 2)

![Figure 2. Relationship of risk level and area of structural collapse.](image)

We’ll get the following risk relationships, determined by level of local collapse

First degree structural collapse ($100 \text{ m}^2 \leq S \leq 240 \text{ m}^2$), corresponds to $R = 10 \cdot 10^{-6}$:

$$R = 15 \cdot 10^{-5} \cdot e^{-(0.01 \cdot S)}.$$  \hspace{1cm} (12)

Second degree structural collapse ($240 \text{ m}^2 < S < 360 \text{ m}^2$), $R = 5 \cdot 10^{-6}$:

$$R = 10 \cdot 10^{-5} \cdot e^{-(0.01 \cdot S)}.$$ \hspace{1cm} (13)

3rd degree of structural collapse ($>360 \text{ m}^2$), $R = 1 \cdot 10^{-6}$:

$$R = 3.8 \cdot 10^{-5} \cdot e^{-(0.01 \cdot S)}.$$ \hspace{1cm} (14)

However, risk value shouldn’t exceed the acceptable level:

$$R(t) \leq [R(t)].$$ \hspace{1cm} (15)

Technical risk is considered acceptable in technological hazard situations if its value does not exceed $10^6$.

As a rule, the reduction of risk value leads to increase of construction costs. On the other hand, increase of risk level can result in structural collapse in a shorter time frame. Therefore to determine an acceptable level of risk is a very challenging task.

On the figure 3 relation of the cost levels and expected damage is presented. The cross hatched zone shows the area of acceptable values of damage and corresponding safety costs.
Figure 3. Relationship of damage and construction costs taking into account safety regulations.

Suggested objective function gives the most unbiased estimation of designed objects costs considering possible emergency situations. This approach will allow significantly enhance the quality of design and level of objects safety.

When solving the problem of reinforced concrete structures optimisation, objective function and constraints are non-linear and convex, therefore optimum point is unique and lies on the border of feasible region. Set of constraints, defining its position, include limit states constraints, resource consumption (material, labour), design, technological and architectural constraints, and also reliability and acceptable risk level constraints depending on the structural collapse degree.

Consideration of simultaneous changes of objective function $F$ and boundary conditions in the process of moving to optimum is necessary when choosing the method of search optimisation. This requirement is satisfied by search method using $C$-$J$ procedures.

In the first phase movement is happening from some point $x_i^{(r)}$, which can occupy any position in space. Transition of point $x_i^{(r)}$ to point $x_i^{(r+1)}$ is done by simultaneous increments on a few variable parameters (table 1).

| Number of combination | Variable parameters | Objective function by increment |
|-----------------------|---------------------|--------------------------------|
| 0                     | $x_1^{(r)}$         | $x_2^{(r)}$ $x_3^{(r)}$ $\Phi(x_1^{(r)}; x_2^{(r)}; x_3^{(r)})$ |
| 1                     | $x_1^{(r)} + \Delta x_1$ $x_2^{(r)}$ $x_3^{(r)}$ $\Phi(x_1^{(r)} + \Delta x_1; x_2^{(r)}; x_3^{(r)})$ |
| 2                     | $x_1^{(r)}$ $x_2^{(r)} + \Delta x_2$ $x_3^{(r)}$ $\Phi(x_1^{(r)}; x_2^{(r)} + \Delta x_2; x_3^{(r)})$ |
| 3                     | $x_1^{(r)}$ $x_2^{(r)}$ $x_3^{(r)} + \Delta x_3$ $\Phi(x_1^{(r)}; x_2^{(r)}; x_3^{(r)} + \Delta x_3)$ |
| 4                     | $x_1^{(r)} + \Delta x_1$ $x_2^{(r)} + \Delta x_2$ $x_3^{(r)}$ $\Phi(x_1^{(r)} + \Delta x_1; x_2^{(r)} + \Delta x_2; x_3^{(r)})$ |
| 5                     | $x_1^{(r)} + \Delta x_1$ $x_2^{(r)}$ $x_3^{(r)} + \Delta x_3$ $\Phi(x_1^{(r)} + \Delta x_1; x_2^{(r)}; x_3^{(r)} + \Delta x_3)$ |
| 6                     | $x_1^{(r)}$ $x_2^{(r)} + \Delta x_2$ $x_3^{(r)} + \Delta x_3$ $\Phi(x_1^{(r)}; x_2^{(r)} + \Delta x_2; x_3^{(r)} + \Delta x_3)$ |
| 7                     | $x_1^{(r)} + \Delta x_1$ $x_2^{(r)} + \Delta x_2$ $x_3^{(r)} + \Delta x_3$ $\Phi(x_1^{(r)} + \Delta x_1; x_2^{(r)} + \Delta x_2; x_3^{(r)} + \Delta x_3)$ |
The starting position of $\bar{x}^{(r)}_i$ is defined by two parameters: the value of the objective function $F$ and the generalized residual $P$.

The value of the generalized residual $P_j$ is determined by the formulas:

$$
P_j = \varphi_j \quad \text{if} \quad \varphi_j > 0, \\
P_j = 0 \quad \text{if} \quad \varphi_j \leq 0.
$$

where $\varphi_j$ - residual on certain constraints, defined as:

$$
\varphi_j = \frac{g_j - [g]_j}{[g]_j} \leq 0, \quad j = 1, 2, ..., n.
$$

Current restrictions considerably decrease the area of optimum search. Constraint residuals form a few groups:

- group of residuals of ultimate limit state (ULS);
- group of residuals of serviceability limit state (SLS);
- resource consumption constraints;
- design, architectural, technological constraints, which have a considerable impact on the outer appearance of the structure, its shape and size of its components;
- renewal coefficient constraint in repair-and-renewal maintenance works;
- resistance of the slab to collapse and complete loss of serviceability state constraint.

Set of residuals $P_j$ is forming the vector $\vec{P}$ with a vector norm:

$$
P = \sqrt{\sum_{i=1}^{n} P_i^2}.
$$

The transition from the point $\bar{x}^{(r)}_i$ to any neighbouring point is associated with increments of $F$ and $P$ values as follows:

$$
\Delta F = F(\bar{x}^{r+1}_i) - F(\bar{x}^r_i), \\
\Delta P = P(\bar{x}^r_i) - P(\bar{x}^{r+1}_i).
$$

We are interested in the direction of motion, in which $\Delta P$ is, if possible, more and $\Delta \Phi$ – less. We introduce additional factor $\Delta C$ which is sensitivity of change in residual to the change in objective function in the optimum point:

$$
\Delta C = \frac{\Delta P}{\Delta F}
$$

Comparing the values $\Delta C$ at points $x^{(r)}_i$ with the value $x^{(r)}_i$, we can find a parameter, for which the change in the value of $\Delta x_i$ gives the best approximation to the optimum point. The movement is realized by changing the variable $x^{(r)}_i$ and keeping constant all other variable parameters values.

Transition to the iteration $(r+1)$ is implemented by maximising the parameter $\Delta C$ according to the formula:

$$
x^{(r+1)}_i = x^{(r)}_i + \Delta x_i.
$$

The increment size $\Delta x_i$ is proportionate to the variable parameter order and computed as:

$$
\Delta x_i = s_{xi}\Delta C_{x_i}^k R_i^{-k},
$$

where $k$ - is the correcting multiplier, defining the speed of search convergence in the optimum point. Relatively fast convergence is achieved when $k = 2$;
$s_{si}$ is the variable step of the search.

But the $C$-procedure does not provide absolutely exact hit in a neighborhood of the optimum point. For example, when the initial search point is chosen close to the border, but far from the optimum point, then $C$-procedure does not have time to correct the trajectory of the search. To continue the movement repulsion in the unacceptable area $H$ is implemented and from there reverse movement towards $D$ starts using algorithm of $C$-procedure. A new objective function multiplier $\Delta J$ is introduced to enable such backward movement, derived from $\Delta C$ inversion.

$$\Delta J = -\Delta C^{-1} = \frac{\Delta F}{\Delta P}. \tag{23}$$

The procedure of repulsion is called $J$-procedure and implemented using formulas (16) – (23) substituting $\Delta C$ by $\Delta J$.

Every new value of objective function on the boarder of acceptable solutions is not bigger than the previous value, which is why anti gradient movement is stopped when

$$F(\bar{x}_{lim,k+1}) > F(\bar{x}_{lim,k}). \tag{24}$$

In this case the movement is stopped and parameter vector $\bar{x}_{zp,k}$ is accepted as a problem solution. If condition (24) is not satisfied then zig-zag movement continues until the condition is fulfilled (figure 4).

![Figure 4. Optimal slab solution depending on the type of objective function.](image)

Let’s consider the optimisation problem using the example of cast-in-situ girderless slab 6x4m with $q = 15$ kN/m² applied load. Thickness of the slab $h$, reinforcement ratio $\mu_x$, reinforcement ratio $\mu_y$ are taken as variable parameters.

We’ll be using the method to define the area of collapse in case of supporting structure failure [11]. This method is based on usage of equivalent static loads. The calculation is made with accordance to limit state design method, using the special combination of accidental impact, dead and live long term loads. Load factors and load combination factors are taken equal to 1.

For optimisation we use the resistance to collapse and complete loss of serviceability state of slab segment constraint.

$$Z_{max} \leq \begin{cases} Z_u, \\ h_{room} - h_{size}. \end{cases} \tag{25}$$
Firstly we consider 1st case of ultimate deformation:

\[ P_{u1} \leq q_{ud} \] (26)

As a result of this calculation condition (25) is not fulfilled, therefore we should consider 2nd case of ultimate deformation. The design model of the calculated segment has to change into cable-stayed structure.

2nd case of ultimate deformation check is fulfilled. The slab will not collapse, but the damaged segment of the slab above the column has to be replaced.

The width of the actively deformed zone is calculated from formulas:

\[ \cos \frac{\pi x_0}{2l_1} = \frac{5 \epsilon_{el}}{\epsilon_u}, \quad \cos \frac{\pi y_0}{2l_2} = \frac{5 \epsilon_{el}}{\epsilon_u}. \] (27)

Total area of collapse within one unit 6x4 is (figure 5):

\[ \omega_0 = x_0 \cdot y_0 = 2.85 \cdot 4.28 = 12.22 m^2. \]

![Figure 5. Borders of slab reinforcement deformation in plastic stay cables stage.](image)

Direct damage defined by area of collapse is calculated according to formula (4).

Comparison results of 3 objective function types are given in table 2.

| Zone | \( F_1 \), ths rub. | \( F_2 \), ths rub. | \( F_3 \), ths rub. |
|------|---------------------|---------------------|---------------------|
| \( Z_1 \) | 12.285 | 15.455 | 19.319 |
| \( Z_2 \) | 7.443 | 9.430 | 16.886 |
As a result of optimisation, we’ll get the total cost of slab 72,446 thousand roubles according to risk-based objective function.

The cost of structure taking into consideration reliability requirements and risk tolerance level increases accordingly. This cost is optimal among other possible solutions in comparison with values obtained from progressive collapse analysis, the cost savings are 20-25%.

It is worth mentioning, that risk based structure cost are considerably higher than direct costs. Under normal operating conditions during lifetime of the structure the risk of underutilization arises. In that case the strength reserves help to increase the durability and operational reliability of the structure and as a result, to reduction in maintenance costs.

References

[1] Tamrazyan A and Filimonova E 2014 Searching method of optimization of bending reinforced concrete slabs with simultaneous assessment of criterion function and the boundary conditions Applied Mechanics and Materials 467 pp 404-409
[2] Recommendations for the protection of high-rise buildings against progressive collapse (Moscow: MRDITED)
[3] ASCE 7 – 02 2002 Minimum Design Loads for Buildings and Other Structures (Reston, VA: American Society of Civil Engineers)
[4] Ayyub B M 2014 Risk analysis in engineering and economics (College Park: CRC Press)
[5] Nieto-Morote A and Francisco Ruz-Vila. 2011 A fuzzy approach to construction project risk assessment International Journal of Project Management 29.2 pp 220-231
[6] Ellingwood B R 2006 Mitigating risk from abnormal loads and progressive collapse. Journal of Performance of Constructed Facilities 20(4) pp 315-323.
[7] Tamrazyan A and Filimonova E 2014 Optimizacija zhelezobetonnoj plity perekrytija po kriteriju minimal"noj stoimosti s uchetom analiza riska Promyshlenoe i grazhdanskoie stroitel'stvo 6(12) pp 19-22
[8] Pichugin S F, Semk, A V and Makhin’ko A V 2005 K opredeleniyu koeffitsienta nadezhnosti po naznacheniyu s uchetom riskov v stroitel’stve Izv. vuzov. stroitel’stvo./News of Higher Education Institutions. Civil Engineering/ 11-12 pp 105-109
[9] Lychev A S 2008 Nadezhnost’ stroitel’nyh konstrukcij (Moskva: ASV) p 184
[10] Mahutov N A, Gadenin M M, Chernjavskij A O and Shatov M M 2012 Analiz riskov otkazov pri funkcionirovanii potencial’no opasnnyh ob’ektov Problemy analiza riska T. 9(3) pp 4-21
[11] Rastorguev B S Raschet nesushhih konstrukcij monolitnyh zhelezobetonnyh zdanij na progressirujushhee razrushenie s uchetom dinamicheskix jeffeektov Sbornik nauchnyh trudov Instituta stroitel’stva i arhitektury MGSU 2008 V.1 pp 68-75