Abstract

Alain Connes’ applications of non-commutative geometry to interaction physics are described for the purpose of model building.
The possibilities of the Yang-Mills-Higgs model building kit have been explored thoroughly in the last 30 years and the standard $SU(3) \times SU(2) \times U(1)$ model of strong and electroweak interactions has emerged quite uniquely as adequate description of high energy phenomena. On the mathematical side this kit relies on two ingredients: the differential forms on spacetime together with their structure of differential algebra and a Lie group represented on a finite dimensional internal space. These two ingredients are thrown together in a tensor product to yield a gauge invariant action. In Connes’ approach [1] both spacetime and internal space are described by involution algebras and their tensor product produces a special class of Yang-Mills actions with spontaneous symmetry breaking. In this class the fermionic mass matrix is naturally unified with the Dirac operator, the Higgs scalars with the gauge bosons and the Higgs potential with the Yang-Mills Lagrangian. While the introduction of Lie groups is ad hoc, involution algebras have a profound mathematical motivation in this context, non-commutative geometry. A model builder however who is willing to accept Connes’ rules can very well do with a minimum of mathematics to be introduced below.

Let us quickly review input and output of the Yang-Mills-Higgs kit. To get started we have to commit ourselves to the following choices:

- a (finite dimensional) real, compact Lie group $G$,
- a positive definite, bilinear invariant form on the Lie algebra $\mathfrak{g}$ of $G$. This choice is parametrized by a few positive numbers $g_i$, the coupling constants.
- a (faithful, unitary) representation $H_L$ for left handed fermions (spin $\frac{1}{2}$),
- a representation $H_R$ for right handed fermions,
- a representation $H_S$ for scalars (spin 0),
- an invariant positive polynomial of order 4 on the representation space of the scalars, this polynomial is denoted by $V(\phi), \phi \in H_S$, the Higgs potential,
- one complex number or Yukawa coupling $g_Y$ for every singlet in the decomposition of the representation

\[(H_L^* \otimes H_R \otimes H_S) \oplus (H_R^* \otimes H_L \otimes H_S). \quad (1)\]

With these ingredients the popular Yang-Mills-Higgs algorithm produces the following output:

- the particle spectrum, roughly one particle for each basis vector in

\[\mathfrak{g}^C \oplus H_L \oplus H_R \oplus H_S. \quad (2)\]

The basis elements of the complexified Lie algebra $\mathfrak{g}^C$ are the gauge bosons, they have spin 1.
• the particle masses,

• the interactions, e.g. charges, Kobayashi-Maskawa matrix...

Trial and error, that is guessing an input, calculating the output and comparing with experiment, has singled out the standard model of electroweak and strong interactions:

\[ G = SU(3) \times SU(2) \times U(1) \]  

(3)

with three coupling constants \( g_3, g_2, g_1; \)

\[ \mathcal{H}_L = [(1, 2, -1) \oplus (3, 2, \frac{1}{3})] \times 3 \]  

(4)

\[ \mathcal{H}_R = [(1, 1, -2) \oplus (3, 1, \frac{4}{3}) \oplus (3, 1, -\frac{2}{3})] \times 3 \]  

(5)

\[ \mathcal{H}_S = (1, 2, -1) \]  

(6)

where \((n_3, n_2, y)\) denotes a tensor product of an \(n_3\) dimensional representation under \(SU(3)\), an \(n_2\) dimensional representation under \(SU(2)\) and the one dimensional representation of \(U(1)\) with hypercharge \(y\):

\[ \rho(e^{i\theta}) = e^{i3y\theta}, \quad 3y \in \mathbb{Z}, \theta \in [0, 2\pi) \]  

(7)

\[ V(\phi) = \lambda(\phi^* \phi)^2 - \frac{\mu^2}{2}\phi^* \phi, \quad \phi \in \mathcal{H}_S, \lambda, \mu > 0. \]  

(8)

There are 27 Yukawa couplings in the input, enough to allow for an arbitrary fermionic mass matrix (fermion masses and Kobayashi-Maskawa matrix) and an arbitrary scalar mass in the output.

In a first step we only consider a particular subset of Connes’ model building kit, where spacetime and internal space come in a tensor product. This subset compares naturally with the Yang-Mills-Higgs model building kit. As before the input concerns only the (finite dimensional) internal space:

• an associative involution algebra \( \mathcal{A} \) with unit 1,

• two representations \( \mathcal{H}_L \) and \( \mathcal{H}_R \) of \( \mathcal{A} \),

• a mass matrix \( M \) i.e. a linear map \( M : \mathcal{H}_R \rightarrow \mathcal{H}_L \),

• a certain number of coupling constants depending on the degree of reducibility of \( \mathcal{H}_L \oplus \mathcal{H}_R \).
The data \((\mathcal{H}_L, \mathcal{H}_R, M)\) plays a fundamental role in non-commutative geometry where it generalizes the Dirac operator. It is called K-cycle.

On the output side we find a complete action of Yang-Mills-Higgs type with \(G\) the group of unitary elements in \(\mathcal{A}\):

\[
G = \{ g \in \mathcal{A}, \quad gg^* = g^*g = 1 \}
\]

or possibly a subgroup thereof. In other words in Connes’ approach the representation space \(\mathcal{H}_S\) of scalars sits at the output end together with the representation space \(g^\mathcal{G}\) of the gauge bosons. Likewise the quartic Higgs potential together with the Klein-Gordon Lagrangian for the scalars are found in the output accompanying the quartic Lagrangian for the gauge bosons.

In the following we first discuss Connes’ rules for a finite dimensional (internal) space. Thereby we avoid the difficulties of functional analysis, only prerequisites being linear algebra, e.g. matrix multiplication. The second chapter deals with a particular infinite dimensional algebra, the functions on spacetime, and the Dirac operator. Model builders are well acquainted with this part of mathematics which also motivates Connes’ rules. The third chapter is simply the tensor product of the first two. As we go along, the general rules are illustrated by an example. The simplest typical one we know strikingly resembles the Glashow-Salam-Weinberg model.

### 1 The internal space

An involution algebra \(\mathcal{A}\) is an associative algebra with unit 1 and an involution \(*\) i.e. an antilinear map from \(\mathcal{A}\) into itself,

\[
(a + b)^* = a^* + b^*, \quad a, b \in \mathcal{A},
\]

(10)

\[
(\lambda a)^* = \lambda^* a^*, \quad \lambda \in \mathbb{R} \text{ or } \mathbb{C},
\]

(11)

with the properties:

\[
a^{**} = a,
\]

(12)

\[
1^* = 1,
\]

(13)

\[
(ab)^* = b^* a^*.
\]

(14)

The two classical examples of involution algebra are:

1. \(M_n(\mathbb{C})\), the algebra of \(n \times n\) matrices with complex entries. The involution is transposition and complex conjugation.
2. The algebra of functions from a manifold (spacetime) into the complex numbers. The involution is just complex conjugation.

The second example is infinite dimensional and commutative, while the first one is finite dimensional and non-commutative. Note that in this context the word non-abelian is non-fashionable.

A representation $\rho$ of $\mathcal{A}$ over a Hilbert space $\mathcal{H}$ is a homomorphism from $\mathcal{A}$ into the operators on $\mathcal{H}$:

$$\rho : \mathcal{A} \longrightarrow \text{End}(\mathcal{H})$$

$$a \longmapsto \rho(a).$$

(15)

Here homomorphism refers to all given structures, addition, scalar multiplication, multiplication, unit and involution:

$$\rho(a + b) = \rho(a) + \rho(b),$$

(16)

$$\rho(\lambda a) = \lambda \rho(a),$$

(17)

$$\rho(ab) = \rho(a)\rho(b),$$

(18)

$$\rho(1) = 1,$$

(19)

$$\rho(a^*) = \rho(a)^*.$$  

(20)

All representations will be supposed faithful, i.e. injective.

Model builders should note that the two choices, algebra and representation, are much more restricted than the choice of a group and a representation. Not every group $G$ is the group of unitary elements in an algebra $\mathcal{A}$. The phenomenologically important $SU(3)$ is an example to which we shall have to come back. Every algebra representation yields a group representation of its group of unitaries, but most of these group representations cannot be obtained from an algebra representation. For example the group $SU(2)$ has unitary representations of any dimension, 1, 2, 3... while the algebra of quaternions whose group of unitaries is $SU(2)$ admits only one irreducible representation, the one of dimension 2. Indeed the tensor product of two algebra representations is not a representation because compatibility with the linear structure is lost. Also the popular singlet representation, “dark matter”, is not available.

A K-cycle $(\mathcal{H}, \mathcal{D}, \chi)$ of an algebra $\mathcal{A}$ consists of a (faithful) representation $\rho$ on a Hilbert space $\mathcal{H}$, of a self adjoint operator $\chi$ on $\mathcal{H}$, called chirality, satisfying

$$\chi^2 = 1$$

(21)

and of a self adjoint operator $\mathcal{D}$ on $\mathcal{H}$, the generalized Dirac operator. Furthermore we suppose that $\rho(a)$ is even:

$$\rho(a)\chi = \chi \rho(a)$$

(22)
for all $a \in A$ and that $\mathcal{D}$ is odd:

$$\mathcal{D} \chi = -\chi \mathcal{D}. \quad (23)$$

In the infinite dimensional case there will be additional conditions to be satisfied by $\mathcal{D}$.

In other words the representation $\rho$ is reducible and decomposes into a left handed and a right handed part $\rho_L$ and $\rho_R$ living on the left handed and right handed Hilbert spaces

$$\mathcal{H}_L := \frac{1 - \chi}{2} \mathcal{H}, \quad (24)$$

$$\mathcal{H}_R := \frac{1 + \chi}{2} \mathcal{H}. \quad (25)$$

For a finite dimensional Hilbert space we can pick a basis such that

$$\chi = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix}. \quad (26)$$

Then

$$\rho = \begin{pmatrix} \rho_L & 0 \\ 0 & \rho_R \end{pmatrix}, \quad (27)$$

$$\mathcal{D} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \quad (28)$$

with $M$ a matrix of size $\dim \mathcal{H}_L \times \dim \mathcal{H}_R$, the mass matrix.

**Example:** Let $A := M_2(\mathbb{C}) \oplus \mathbb{C}$ and denote its elements by $(a, b)$, $a$ being a $2 \times 2$ matrix and $b$ a complex number. We define a K-cycle by

$$\mathcal{H}_L := \mathbb{C}^2, \quad \mathcal{H}_R := \mathbb{C}^2, \quad (29)$$

$$\rho_L(a, b) := a, \quad \rho_R(a, b) := B \quad (30)$$

with

$$B := \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \quad (31)$$

and

$$M := \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}, \quad m_1, m_2 \in \mathbb{C}. \quad (32)$$

We shall always assume $|m_1| \neq |m_2|$. Later on it will be convenient to have $M$ of maximal rank. Should one of the $m_i$ vanish this can still be achieved by the following trick sometimes called leptonic reduction: Suppose $m_2 = 0$. We simply omit the second dimension of $\mathcal{H}_R$ which becomes one dimensional. Then $\rho(a, b)$ and $\mathcal{D}$ are $3 \times 3$ matrices and

$$M = \begin{pmatrix} m_1 \\ 0 \end{pmatrix}. \quad (33)$$
is reduced to a column vector of maximal rank.

Note the appearance of the complex conjugation in $\bar{b}$ which by equation (17) means that we must consider the algebra as defined over the real numbers.

Given an (involution) algebra and a K-cycle we now want to construct a differential algebra. Let us recall the axioms of a differential algebra $\Omega$. It is a graded vector space

$$\Omega = \bigoplus_{p \in \mathbb{N}_0} \Omega^p. \tag{34}$$

We denote its associative product by juxtaposition:

$$\Omega^p \times \Omega^q \longrightarrow \Omega^{p+q} \quad (\phi, \psi) \longmapsto \phi \psi. \tag{35}$$

Furthermore $\Omega$ is equipped with a differential $\delta$ that is a linear map

$$\delta : \Omega^p \longrightarrow \Omega^{p+1} \quad \phi \longmapsto \delta \phi \tag{36}$$

with two properties. It is nilpotent,

$$\delta^2 = 0 \tag{37}$$

and obeys a graded Leibniz rule,

$$\delta(\phi \psi) = (\delta \phi) \psi + (-1)^p \phi \delta \psi, \quad \phi \in \Omega^p. \tag{38}$$

The differential algebra is called (graded) commutative if in addition its product satisfies

$$\phi \psi = (-1)^{pq} \psi \phi, \quad \phi \in \Omega^p, \ \psi \in \Omega^q. \tag{39}$$

From our given algebra $\mathcal{A}$ we now construct first an auxiliary differential algebra $\hat{\Omega}A$, the so called universal differential envelope of $\mathcal{A}$:

$$\hat{\Omega}^0A := \mathcal{A}, \tag{40}$$

$\hat{\Omega}^1A$ is generated by symbols $\delta a, \quad a \in \mathcal{A}$ with relations

$$\delta 1 = 0 \tag{41}$$

$$\delta(ab) = (\delta a)b + a\delta b. \tag{42}$$

Therefore $\hat{\Omega}^1A$ consists of finite sums of terms of the form $a_0 \delta a_1$,

$$\hat{\Omega}^1A = \left\{ \sum_j a^j_0 \delta a^j_1, \quad a^j_0, a^j_1 \in \mathcal{A} \right\} \tag{43}$$
and likewise for higher $p$

\[ \hat{\Omega}^p \mathcal{A} = \left\{ \sum_j a^j_0 \delta a^j_1 \ldots \delta a^j_p, \quad a^j_q \in \mathcal{A} \right\}. \] (44)

The differential $\delta$ is defined by

\[ \delta(a_0 \delta a_1 \ldots \delta a_p) := \delta a_0 \delta a_1 \ldots \delta a_p. \] (45)

Two remarks: The universal differential envelop $\hat{\Omega} \mathcal{A}$ of a commutative algebra $\mathcal{A}$ is not necessarily graded commutative. The universal differential envelop of any algebra has no cohomology. This means that every closed form $\hat{\phi}$ of degree $p \geq 1$, $\delta \hat{\phi} = 0$, is exact, $\hat{\phi} = \delta \hat{\psi}$ for some $(p-1)$form $\hat{\psi}$.

The involution $^*$ can be extended from the algebra $\mathcal{A}$ to its universal differential envelop $\hat{\Omega}^1 \mathcal{A}$ by putting

\[ (\delta a)^* := \delta a^* =: \delta a^* \] (46)

and of course

\[ (\phi \psi)^* = \psi^* \phi^*. \] (47)

Note that Connes has a minus sign in the above relation which amounts to replacing $\delta$ by $i \delta$.

Our next step is to extend the representation $\rho$ from the algebra $\mathcal{A}$ to its universal differential envelop $\hat{\Omega} \mathcal{A}$. This extension is the central piece of Connes’ algorithm and deserves a new name:

\[ \pi: \hat{\Omega} \mathcal{A} \longrightarrow \text{End}(\mathcal{H}) \]

\[ \hat{\phi} \longmapsto \pi(\hat{\phi}) \] (48)

\[ \pi(a_0 \delta a_1 \ldots \delta a_p) := (-i)^p \rho(a_0)[\mathcal{D}, \rho(a_1)] \ldots [\mathcal{D}, \rho(a_p)]. \]

Note that in Connes’ notations there is no factor $(-i)^p$ on the rhs. A straight-forward calculation shows that $\pi$ is in fact a representation of $\hat{\Omega} \mathcal{A}$ as involution algebra, but the differential is not respected unless the generalized Dirac operator $\mathcal{D}$ satisfies

\[ [\mathcal{D}, [\mathcal{D}, \cdot]] \equiv 0. \] (49)

Otherwise there can be forms $\hat{\phi}$ with $\pi(\hat{\phi}) = 0$ and $\pi(\delta \hat{\phi}) \neq 0$. By dividing out these unpleasant forms we shall construct a new differential algebra $\Omega \mathcal{A}$, the real thing. It will of course depend on the chosen $K$-cycle.

The kernel of $\pi$ is a (bilateral) ideal in the image $\pi(\hat{\Omega} \mathcal{A})$. We turn it into a differential ideal $J$ ($J$ for junk):

\[ J := \ker \pi + \delta \ker \pi =: \bigoplus_p J^p \] (50)
with
\[ J^p = (\ker \pi)^p + \delta(\ker \pi)^{p-1} \] (51)
and divide it out:
\[ \Omega \mathcal{A} := \frac{\hat{\Omega} \mathcal{A}}{J}. \] (52)

The differential $\delta$ on $\hat{\Omega} \mathcal{A}$ passes to the quotient where it is again denoted by $\delta$. Degree by degree we have:
\[ \Omega^0 \mathcal{A} = \hat{\Omega}^0 \mathcal{A} \cong \rho(\mathcal{A}) \] (53)
because $\rho$ is faithful and $J^0 = (\ker \pi)^0 = 0$,
\[ \Omega^1 \mathcal{A} = \frac{\hat{\Omega}^1 \mathcal{A}}{(\ker \pi)^1} \cong \pi(\hat{\Omega}^1 \mathcal{A}) \] (54)
because $J^1 = (\ker \pi)^1$, and in degree $p \geq 2$
\[ \Omega^p \mathcal{A} \cong \frac{\pi(\hat{\Omega}^p \mathcal{A})}{\pi(\delta(\ker \pi)^{p-1})}. \] (55)

While $\hat{\Omega} \mathcal{A}$ has no cohomology $\Omega \mathcal{A}$ does in general. In fact let us anticipate, if $\mathcal{F}$ is the algebra of complex functions on a compact spin manifold $M$ of even dimensions and if the K-cycle is obtained from the Dirac operator then $\Omega \mathcal{F}$ is de Rham’s differential algebra of differential forms on $M$.

We come back to our finite dimensional case. Remember that the elements of the auxiliary differential algebra $\hat{\Omega} \mathcal{A}$ that we introduced for book keeping purposes only are abstract entities defined in terms of symbols and relations. On the other hand, with the above isomorphisms, the elements of $\Omega \mathcal{A}$, the “forms”, are operators on the Hilbert space $\mathcal{H}$ of the K-cycle, i.e. concrete matrices of complex numbers.

**Examples:** Before continuing our example above let us mention a class of trivial examples that deserve the name *vector like models*. The algebra $\mathcal{A}$ is arbitrary, left and right representations are equal $\rho_L = \rho_R$, and the mass matrix appearing in the “Dirac” operator $\mathcal{D}$ is a multiple of the identity matrix $M = \lambda I$, $\lambda \in \mathbb{C}$. We shall see that these models will produce Yang-Mills theories with unbroken parity and unbroken gauge symmetry as electromagnetism and chromodynamics. Any vector like model has trivial differential algebra,
\[ \Omega^0 \mathcal{A} = \mathcal{A}, \] (56)
\[ \Omega^p \mathcal{A} = 0, \quad p = 1, 2, 3, \ldots \] (57)

Coming back to our serious example recall:
\[ \rho(a, b) = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}. \] (58)
We need the commutator
\[ [\mathcal{D}, \rho(a, b)] = \left( M^* \rho_L(a) - \rho_R(b) M \right) \]
\[ = (M^* a - BM^*) \]
\[ = \left( M^*(a - B) \right) \]
(59)
where in the last equality we have used that \( M \) and \( B \) are diagonal and commute. A general element in \( \Omega^1 A \) is of the form
\[ \pi((a_0, b_0) \delta(a_1, b_1)) = -i \left( \begin{array}{cc} 0 & -a_0(a_1 - B_1) M \\ M^* B_0(a_1 - B_1) & 0 \end{array} \right) \]
(60)
and as vector space
\[ \Omega^1 A = \left\{ i \left( \begin{array}{cc} 0 & h M \\ M^* \tilde{h}^* & 0 \end{array} \right), h, \tilde{h} \in M_2(\mathbb{C}) \right\}. \]
(61)
Likewise a general element in \( \pi(\hat{\Omega}^2 A) \) is
\[ \pi((a_0, b_0) \delta(a_1, b_1) \delta(a_2, b_2)) = \]
\[ \left( \begin{array}{cc} a_0(a_1 - B_1) MM^*(a_2 - B_2) & 0 \\ 0 & M^* B_0(a_1 - B_1)(a_2 - B_2) M \end{array} \right) = \]
\[ \left( \begin{array}{cc} \Sigma a_0(a_1 - B_1)(a_2 - B_2) + \Delta a_0(a_1 - B_1) \sigma_3(a_2 - B_2) & 0 \\ 0 & M^* B_0(a_1 - B_1)(a_2 - B_2) M \end{array} \right) \]
where we have used the decomposition
\[ MM^* = \left( \begin{array}{cc} |m_1|^2 & 0 \\ 0 & |m_2|^2 \end{array} \right) = \Sigma 1 + \Delta \sigma_3 \]
(63)
with
\[ \Sigma := \frac{1}{2} (|m_1|^2 + |m_2|^2), \]
(64)
\[ \Delta := \frac{1}{2} (|m_1|^2 - |m_2|^2), \]
(65)
\[ \sigma_3 := \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right). \]
(66)
A general element in \( (\ker \pi)^1 \) is a finite sum of the form
\[ \sum_j (a^j_0, b^j_0) \delta(a^j_1, b^j_1) \]
(67)
with the two conditions
\[ \left[ \sum_j a^j_0(a^j_1 - B^j_1) \right] M = 0, \]
(68)
\[ M^* \left[ \sum_j B_0^j (a_1^j - B_1^j) \right] = 0. \]  

(69)

Therefore the corresponding general element in \( \pi(\delta(\ker \pi)^1) \) is

\[
\left( \Sigma \sum_j (a_0^j - B_0^j) (a_1^j - B_1^j) + \Delta \sum_j (a_0^j - B_0^j) \sigma_3 (a_1^j - B_1^j) \right) \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

(70)

still subject to the two conditions. Recall that \( \Delta \neq 0 \) by assumption and we have the following inclusion

\[
\pi(\delta(\ker \pi)^1) \supset \left\{ \left( \begin{array}{cc} \Delta \sum_j a_0^j \sigma_3 a_1^j & 0 \\ 0 & 0 \end{array} \right), \quad \sum_j a_0^j a_1^j = 0 \right\}
\]

\[
= \left\{ \left( \begin{array}{cc} \Delta k & 0 \\ 0 & 0 \end{array} \right), \quad k \in M_2(\mathbb{F}) \right\}.
\]

(71)

To prove the last equality we note that the subspace is a bilateral ideal in the rhs. Furthermore the subspace contains the non-zero element with:

\[
a_0 := \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix},
\]

(72)

\[
a_1 := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},
\]

(73)

\[
a_0 a_1 = 0,
\]

(74)

\[
a_0 \sigma_3 a_1 = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}.
\]

(75)

The algebra \( M_2(\mathbb{F}) \) being simple the subspace is the entire algebra. Consequently

\[
\pi(\delta(\ker \pi)^1) = \left\{ \left( \begin{array}{cc} \Delta k & 0 \\ 0 & 0 \end{array} \right), \quad k \in M_2(\mathbb{F}) \right\}.
\]

(76)

Now we have to compute the quotient

\[
\Omega^2 A = \frac{\pi(\hat{\Omega}^2 A)}{\pi(\delta(\ker \pi)^1)}.
\]

(77)

We are tempted to conclude

\[
\Omega^2 A = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & M^* c M \end{bmatrix}, \quad c \in M_2(\mathbb{F}) \right\}.
\]

(78)

The problem with this conclusion is that a quotient of vector spaces consists of classes and is not canonically a subspace. The situation is simpler if our vector space comes equipped with a scalar product, in which case there is a privileged representative in each class, the one orthogonal to the subspace \( \pi(\delta(\ker \pi)^1) \). This canonical choice allows to consider the quotient as subspace.
Since the elements of $\pi(\hat{\Omega}.A)$ are operators on the Hilbert space $\mathcal{H}$, i.e. concrete matrices, they have a natural scalar product defined by

$$< \hat{\phi}, \hat{\psi} > := \text{tr}(\hat{\phi}^* \hat{\psi}), \quad \hat{\phi}, \hat{\psi} \in \pi(\hat{\Omega}.A)$$  \(79\)

for forms of equal degree and zero for the scalar product of two forms of different degree. With this scalar product $\Omega.A$ is a subspace of $\pi(\hat{\Omega}.A)$ and by definition orthogonal to $J = \ker \pi + \delta \ker \pi$. Now our conclusion in the example makes sense. As a subspace $\Omega.A$ inherits a scalar product which deserves a special name $\langle \ , \ \rangle$. It is given by

$$\langle \phi, \psi \rangle = \text{tr}(\phi^* P \psi), \quad \phi, \psi \in \Omega^p.A$$  \(80\)

where $P$ is the orthogonal projector in $\pi(\hat{\Omega}.A)$ onto the ortho-complement of $J$ and $\phi$ and $\psi$ are any representatives in their classes. Again the scalar product vanishes for forms with different degree.

Let us recapitulate our example:

$$\Omega^0.A = \left\{ \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \ a \in M_2(\mathfrak{g}), \ B = \begin{pmatrix} b & 0 \\ 0 & \bar{b} \end{pmatrix} \right\},$$  \(81\)

$$\Omega^1.A = \left\{ i \begin{pmatrix} 0 & hM \\ M^* \bar{h} & 0 \end{pmatrix}, \ h, \bar{h} \in M_2(\mathfrak{g}) \right\},$$  \(82\)

$$\Omega^2.A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & M^* c M \end{pmatrix}, \ c \in M_2(\mathfrak{g}) \right\}.$$  \(83\)

Since $\pi$ is a homomorphism of involution algebras the product in $\Omega.A$ is given by matrix multiplication followed by the projection

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$  \(84\)

and the involution is given by transposition complex conjugation. It is in order to calculate the differential $\delta$ that we need the complicated construction above:

$$\delta : \Omega^0.A \longrightarrow \Omega^1.A$$

$$\begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix} \longmapsto i \begin{pmatrix} 0 & (a - B)M \\ -M^*(a - B) & 0 \end{pmatrix},$$  \(85\)

$$\delta : \Omega^1.A \longrightarrow \Omega^2.A$$

$$i \begin{pmatrix} 0 & hM \\ M^* \bar{h} & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 0 & 0 \\ 0 & M^* (h + \bar{h}^*) M \end{pmatrix}.$$  \(86\)
Let us note that not every example can be calculated as explicitly as the above one. Just add a non-diagonal entry in the mass matrix $M$ of our example and $\delta$ will not have the simple form indicated.

We are now ready to make a first contact with gauge theories. Consider the vector space of antihermitian 1-forms

$$\{ H \in \Omega^1 A, H^* = -H \}. \quad (87)$$

A general element $H$ is of the form

$$H = i \begin{pmatrix} 0 & hM \\ M^* h^* & 0 \end{pmatrix}, \quad h : \mathcal{H}_L \to \mathcal{H}_R. \quad (88)$$

These elements are called Higgses or gauge potentials. In fact the space of gauge potentials carries an affine representation of the group of unitaries

$$G = \{ g \in A, gg^* = g^* g = 1 \} \quad (89)$$

defined by

$$H^g := \rho(g) H \rho(g^{-1}) + \rho(g) \delta \rho(g^{-1})$$
$$= \rho(g) H \rho(g^{-1}) + (-i) \rho(g) [D, \rho(g^{-1})]$$
$$= i \begin{pmatrix} 0 & h^g M \\ M^*(h^g)^* & 0 \end{pmatrix} \quad (90)$$

with

$$h^g = \rho_L(g)[h - 1] \rho_R(g^{-1}) + 1. \quad (91)$$

$H^g$ is the “gauge transformed of $H$”. As usual every gauge potential $H$ defines a covariant derivative $\delta + H$, covariant under the left action of $G$ on $\Omega A$:

$$^g \psi := \rho(g) \psi, \quad \psi \in \Omega A \quad (92)$$

which means

$$(\delta + H^g) ^g \psi = ^g [(\delta + H) \psi]. \quad (93)$$

As usual we define the curvature $C$ of $H$ by

$$C := \delta H + H^2 \in \Omega^2 A. \quad (94)$$

Note that here and later $H^2$ is considered as element of $\Omega^2 A$ which means it is the projection $P$ applied to $H^2 \in \pi(\hat{\Omega}^2 A)$. The curvature $C$ is a hermitian 2-form with homogeneous gauge transformations

$$C^g := \delta(H^g) + (H^g)^2 = \rho(g) C \rho(g^{-1}). \quad (95)$$
Finally we define the “Higgs potential” $V(H)$, a functional on the space of gauge potentials, by

$$V(H) := (C, C) = \text{tr}[(\delta H + H^2)P(\delta H + H^2)].$$

(96)

It is a polynomial of degree 4 in $H$ with real, non-negative values. Furthermore it is gauge invariant,

$$V(H^g) = V(H)$$

(97)

because of the homogeneous transformation property of the curvature $C$ and because the orthogonal projector $P$ commutes with all gauge transformations

$$\rho(g)P = P\rho(g).$$

(98)

The transformation law for $H$ motivates the following change of variables

$$\Phi := i\begin{pmatrix}0 & \phi M \\ M^*\phi^* & 0\end{pmatrix} := H - iD.$$

(99)

In other words

$$\phi = h - 1.$$  

(100)

The new variable $\Phi$ transforms homogeneously

$$\Phi^g = \rho(g)\Phi\rho(g^{-1})$$

(101)

or

$$\phi^g = \rho_L(g)\phi\rho_R(g^{-1})$$

(102)

where the differential is of course considered gauge invariant

$$D^g = D.$$

(103)

In our example the Higgs $h$ is a complex $2 \times 2$ matrix, which in terms of $\phi$ decomposes under gauge transformations into two complex doublets, the two column vectors $\phi_1$ and $\phi_2$ of $\phi$. The curvature is readily calculated

$$C := \delta H + H^2 = \begin{pmatrix}0 & 0 \\ 0 & M^*cM\end{pmatrix}$$

(104)

with

$$c = h + h^* - h^*h = 1 - \phi^*\phi.$$  

(105)

The Higgs potential is

$$V(H) = \text{tr} \left[ C^2 \right] = \text{tr} \left[ (M^*(1 - \phi^*\phi)M)^2 \right] = |m_1|^4 + |m_2|^4 + |m_1|^4(\phi_1^*\phi_1)^2 + |m_2|^4(\phi_2^*\phi_2)^2 - 2|m_1|^4\phi_1^*\phi_1 - 2|m_2|^4\phi_2^*\phi_2 + 2|m_1|^2|m_2|^2(\phi_1^*\phi_2)(\phi_2^*\phi_1).$$

(106)

Its most interesting feature is that it breaks the gauge symmetry spontaneously. Indeed the only gauge invariant point in the space of gauge potentials or Higgses is $\phi = 0$. This point is not a minimum of the Higgs potential.
2 Spacetime

In this chapter our algebra is infinite dimensional, the algebra of differentiable, complex valued functions on spacetime $M$

$$\mathcal{F} := C^\infty(M).$$ (107)

The K-cycle is defined by the Dirac operator. We sketch how the differential algebra $\Omega\mathcal{F}$ reproduces the ordinary differential forms on $M$. For simplicity spacetime is taken flat, compact and Euclidean. To define the Dirac operator we also need a spin structure. We denote by $S$ the Hilbert space of the K-cycle. $S$ consists of square integrable spinors

$$\Psi = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \Psi_3(x) \\ \Psi_4(x) \end{pmatrix}. \quad (108)$$

The representation of $\mathcal{F}$ on $S$ is simply by multiplication and is denoted by $\cdot$,

$$(f\Psi)(x) := f(x)\Psi(x), \quad f \in \mathcal{F}, \ \Psi \in S. \quad (109)$$

The odd operator of the K-cycle is the (true) Dirac operator

$$\bar{\partial}\Psi := i \sum_{\mu=0}^{3} \gamma^{\mu} \frac{\partial}{\partial x^\mu}\Psi. \quad (110)$$

Our gamma matrices are self adjoint,

$$\gamma^0 := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \gamma^1 := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad (111)$$

$$\gamma^2 := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \gamma^3 := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}. \quad (112)$$

They satisfy the anticommutation relation

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2\eta^{\mu\nu}1 \quad (113)$$

with the flat Euclidean metric

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (114)$$
The chirality operator is by definition
\[ \gamma_5 := \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \] (115)

It is self adjoint, its square is one as postulated and since it anticommutes with all other gamma matrices
\[ \gamma^\mu \gamma_5 + \gamma_5 \gamma^\mu = 0, \] (116)
the Dirac operator is odd
\[ \partial \gamma_5 + \gamma_5 \partial = 0. \] (117)

The Dirac operator \( \partial \) has additional algebraic properties:

- The commutator \([\partial, f] \) is a bounded operator for all \( f \in \mathcal{F} \).

- The spectrum of \( \partial \) is discrete and the eigenvalues \( \lambda_n \) of \(|\partial|\) arranged in an increasing sequence are of order \( n^{(1/d)} \) for \( d \) dimensional manifolds \( M \), \( d \) even, for us \( d = 4 \).

These two properties are added to the axioms of an abstract K-cycle. They will be needed later to define a trace.

We denote by \( d \) the differential in the universal differential envelop \( \hat{\Omega} \mathcal{F} \) and by \( \pi_D \), \( D \) for Dirac, the algebra homomorphism
\[ \pi_D : \hat{\Omega} \mathcal{F} \longrightarrow \text{End}(\mathcal{S}) \]
\[ \hat{\phi} \longrightarrow \pi_D(\hat{\phi}). \] (118)
\[ \pi_D(f_0 df_1 \ldots df_p) := (-i)^p f_0 [\hat{\phi}, f_1 \ldots [\hat{\phi}, f_p]]. \] (119)

We need the commutator
\[ [\partial, f] \Psi = i \gamma^\mu \frac{\partial}{\partial x^\mu} (f \Psi) - if \gamma^\mu \frac{\partial}{\partial x^\mu} \Psi = i \left[ \gamma^\mu \frac{\partial}{\partial x^\mu} f \right] \Psi. \] (120)

(We use Einstein’s summation convention.) Therefore
\[ [\partial, f] = i \gamma^\mu \frac{\partial}{\partial x^\mu} f =: i \gamma (df) \] (121)
with
\[ df = \left[ \frac{\partial}{\partial x^\mu} f \right] dx^\mu. \] (122)
At this point already we see that the restriction to flat spacetime can be dropped. The Dirac operator on curved manifolds

\[ i\gamma^\mu (x) \left[ \frac{\partial}{\partial x^\mu} + \omega_\mu \right] \tag{123} \]

differs from the flat one in two respects, the gamma matrices are \( x \) dependent, no problem in the above commutator, and an additional algebraic term, a spin connection \( \omega = \omega_\mu dx^\mu \) valued in \( so(4) \) appears but drops out from the commutator. Since the Dirac operator only shows up in commutators Connes’ algorithm works on any Riemannian manifold.

The representation of functions by multiplication on spinors is faithful, of course, and

\[ \Omega^1 F \simeq \pi_D (\hat{\Omega}^1 F) \tag{124} \]

A general element of the rhs is a finite sum of terms

\[ \pi_D (f_0 df_1), \quad f_0, f_1 \in F. \tag{125} \]

It is identified with the differential 1-form on \( M \)

\[ f_0 df_1 \in \Omega^1 M. \tag{126} \]

For 2-forms the situation is less trivial, we must compute the junk \( \pi_D (d(\ker \pi_D)^1) \). Consider

\[ h^{-1} dh + h dh^{-1} \tag{127} \]

an element in \( \hat{\Omega}^1 F \) where \( h \in F \) is a non-vanishing function, \( h^{-1}(x) = 1/h(x) \). As \( \hat{\Omega} F \) is not graded commutative this element does not vanish!

\[ h^{-1} dh + h dh^{-1} \neq h^{-1} dh + (dh^{-1}) h = d(h^{-1} h) = d1 = 0. \tag{128} \]

Its image under \( \pi_D \) however does vanish

\[ \pi_D (h^{-1} dh + h dh^{-1}) = \gamma (h^{-1} dh + h dh^{-1}) = \gamma (h^{-1} dh + dh^{-1}) = 0. \tag{129} \]

Therefore the considered element is in \( (\ker \pi_D)^1 \) and the corresponding element in \( \pi_D (d(\ker \pi_D)^1) \) is

\[ \pi_D (dh^{-1} dh + dhdh^{-1}) = \gamma (dh^{-1}) \gamma (dh) + \gamma (dh) \gamma (dh^{-1}) \]

\[ = -\gamma^\mu \left( \frac{\partial}{\partial x^\mu} h^{-1} \right) \gamma^\nu \frac{\partial}{\partial x^\nu} h + \gamma^\nu \left( \frac{\partial}{\partial x^\nu} h \right) \gamma^\mu \frac{\partial}{\partial x^\mu} h^{-1} \]

\[ = [\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu] \left( \frac{\partial}{\partial x^\mu} h^{-1} \right) \frac{\partial}{\partial x^\nu} h \]

\[ = - \left( \frac{2}{h^2} \eta_{\mu\nu} \left( \frac{\partial}{\partial x^\mu} h \right) \frac{\partial}{\partial x^\nu} h \right) 1. \tag{130} \]
By linear combination

\[ \pi_D(d(\ker \pi_D)^1) = \{ f_1, \ f \in \mathcal{F} \} . \quad (131) \]

On the other hand

\[ \pi_D(\hat{\Omega}^2 \mathcal{F}) = \{ f_{\mu\nu} \gamma^\mu \gamma^\nu, \ f_{\mu\nu} \in \mathcal{F} \} \quad (132) \]

and

\[ \pi_D(df_1 df_2 + df_2 df_1) = \left( 2\eta_{\mu\nu} \frac{\partial}{\partial x^\mu} f_1 \frac{\partial}{\partial x^\nu} f_2 \right) 1 \quad (133) \]

After passage to the quotient \( \pi_D(df_1) \) and \( \pi_D(df_2) \) anticommute whereas they did not anticommute in \( \pi_D(\hat{\Omega}^2 \mathcal{F}) \) and we may now identify a general element

\[ \pi_D(f_0 df_1 df_2) \in \Omega^2 \mathcal{F} \quad (134) \]

with the differential 2-form on \( M \)

\[ f_0 df_1 df_2 \in \Omega^2 M. \quad (135) \]

As in chapter 1 we have treated the quotient space like a subspace which is legitimate only in presence of an appropriate scalar product. Again this scalar product will be defined in terms of the involution and a trace.

The involution that \( \Omega M \) inherits from \( \Omega \mathcal{F} \) via the sketched isomorphism is with our conventions

\[ (f_0 df_1 df_2 \ldots df_p)^* = (-1)^{(1/2)p(p-1)} \bar{f}_0 df_1 df_2 \ldots df_p. \quad (136) \]

The definition of a trace is delicate because now our Hilbert space \( \mathcal{S} \) is infinite dimensional. For any bounded operator \( Q \) on \( \mathcal{S} \) we define the Dixmier trace \( \text{tr}_\omega \) by

\[ \text{tr}_\omega(Q|\mathcal{H}|^{-d}) := \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n \quad (137) \]

where \( d = \dim M = 4 \) and the \( \lambda_n \) are the eigenvalues of \( Q|\mathcal{H}|^{-d} \) arranged in a decreasing sequence discarding the zero modes of the Dirac operator. Now we proceed as in the finite dimensional case \( (d = 0) \) and define a scalar product on \( \pi_D(\hat{\Omega} \mathcal{F}) \) by

\[ \langle \hat{\phi}_D, \hat{\psi}_D \rangle := \text{tr}_\omega(\hat{\phi}_D^* \hat{\psi}_D|\mathcal{H}|^{-4}), \ \hat{\phi}_D, \hat{\psi}_D \in \pi_D(\hat{\Omega} \mathcal{F}). \quad (138) \]

Note that \( \hat{\phi}_D \) is bounded because \([\bar{\mathcal{H}}, \mathcal{F}]\) is. This scalar product can be computed to be

\[ \langle \hat{\phi}_D, \hat{\psi}_D \rangle = \frac{1}{32\pi^2} \int_M \text{tr}_4 \left[ \hat{\phi}_D^* \hat{\psi}_D \right] d^4x \quad (139) \]
independently of the four dimensional manifold $M$. $\operatorname{tr}_4$ denotes the trace over the gamma matrices. With this scalar product $\Omega F$ is a subspace of $\pi_D(\hat{\Omega} F)$ and by definition orthogonal to $J = \ker \pi_D + \operatorname{d} \ker \pi_D$. As subspace $\Omega F$ inherits a scalar product $(\cdot, \cdot)$ given by

$$ (\phi_D, \psi_D) = < \phi_D, P_D \psi_D >, \quad \phi_D, \psi_D \in \Omega^p F $$

(140)

where $P_D$ is the orthogonal projector in $\pi_D(\hat{\Omega} F)$ onto the ortho-complement of $J$ and $\phi_D$ and $\psi_D$ are any representatives in their classes. Thanks to well known results for $\operatorname{tr}_4 [\gamma^\mu_1 v_{\mu_1} \ldots \gamma^\mu_n v_{\mu_n}]$ this scalar product now vanishes for forms with different degree. By the above isomorphism between $\Omega F$ and $\Omega M$ the differential forms inherit a scalar product still denoted by $(\cdot, \cdot)$

$$ (\phi_D, \psi_D) = \frac{1}{8\pi^2} \int_M \phi_D^* \ast \psi_D, \quad \phi_D, \psi_D \in \Omega^p M $$

(141)

where the Hodge star $\ast$ should not be confused with the involution $\cdot^\ast$.

As an example let us consider the flat 4-torus, $M = T^4$ with all four circumferences measuring $2\pi$ and let us compute the eigenvalues of the Dirac operator

$$ \partial \Psi = \begin{pmatrix} i \frac{\partial}{\partial x^0} & 0 & -\frac{\partial}{\partial x^1} & -\frac{\partial}{\partial x^2} + i \frac{\partial}{\partial x^3} \\ 0 & i \frac{\partial}{\partial x^1} & -\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \\ i \frac{\partial}{\partial x^2} & i \frac{\partial}{\partial x^3} & 0 & -i \frac{\partial}{\partial x^0} \\ -\frac{\partial}{\partial x^3} & -i \frac{\partial}{\partial x^0} & i \frac{\partial}{\partial x^1} & 0 \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} = \lambda \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}. $$

(142)

After a Fourier transform

$$ \Psi_A = : \sum_{j_0, \ldots, j_3 \in \mathbb{Z}} c_A(j_0, \ldots, j_3) \exp(-ij_\mu x^\mu), \quad A = 1, 2, 3, 4 $$

(143)

this equation reads

$$ \begin{pmatrix} j_0 & 0 & ij_3 & ij_1 + j_2 \\ 0 & j_0 & ij_1 - j_2 & -ij_3 \\ -ij_3 & -ij_1 - j_2 & -j_0 & 0 \\ -ij_1 + j_2 & ij_3 & 0 & -j_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = \lambda \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}. $$

(144)

Its characteristic equation is

$$ \left[ \lambda^2 - \left( j_0^2 + j_1^2 + j_2^2 + j_3^2 \right)^2 \right]^2 = 0 $$

(145)

and for fixed $j_\mu$ each eigenvalue

$$ \lambda = \pm \sqrt{j_0^2 + j_1^2 + j_2^2 + j_3^2} $$

(146)

has multiplicity two. Therefore asymptotically for large $|\lambda|$ there are $4|\lambda|^4 B_4$ eigenvalues (counted with their multiplicity) whose absolute values are smaller than $|\lambda|$. $B_4 = \pi^2/2$ denotes the volume of the unit ball in $\mathbb{R}^4$. Let us arrange the absolute values of the eigenvalues in an increasing sequence. Taking due account of their multiplicities we have for large $n$

$$ |\lambda_n| \approx \left( \frac{n}{2\pi^2} \right)^{1/4} $$

(147)
and we can check the Dixmier trace in equation (139) for instance with \( \hat{\phi}_D = \hat{\psi}_D = 1 \)

\[
<1,1> = \text{tr}_\omega(|\hat{\phi}|^{-4}) = \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} |\lambda_n|^{-4} = \lim_{N \to \infty} \frac{1}{\log N} \int_1^{N} \frac{2\pi^2}{n} \, dn = 2\pi^2 = \frac{1}{32\pi^2} \int_M \text{tr}_4[1] d^4x.
\]

(148)

Let us come back to the general case. The group of unitaries is now infinite dimensional

\[
G = \{ g \in \mathcal{F}, \ g g^* = g^* g = 1 \} = \{ M \to U(1) \},
\]

(149)

the gauge group \( U(1) \). A gauge potential is simply a differential 1-form \( A \) with values in the Lie algebra \( u(1) \), the vector potential of electromagnetism. Its curvature \( F \) is the \( u(1) \) valued 2-form

\[
F := dA + A^2 = dA
\]

(150)

(the differential algebra \( \Omega \mathcal{F} \) being graded commutative \( A^2 \equiv 0 \)) and the Higgs potential is precisely the Maxwell action

\[
(F,F) = \int_M F \ast F.
\]

(151)

3 The tensor product

Remember the description of spinning particles in quantum mechanics. Particles with spin \( s \) come in \( 2s+1 \) dimensional unitary representations of the group \( SU(2) \). Position in space enters the picture via the tensor product of this finite dimensional Hilbert space with the space of square integrable functions. In this spirit we shall now turn the Higgses into genuine Higgs fields by tensorizing \( \mathcal{A} \) and \( \mathcal{F} \).

Let us denote this tensor product by

\[
\mathcal{A}_t := \mathcal{F} \otimes \mathcal{A}.
\]

(152)

The algebra \( \mathcal{A}_t \) admits a natural K-cycle \((\mathcal{H}_t, D_t, \chi_t)\), the tensor product of the K-cycles \((\mathcal{S}, \hat{\phi}, \gamma_5)\) on \( \mathcal{F} \) and \((\mathcal{H}, D, \chi)\) on \( \mathcal{A} \). The Hilbert space

\[
\mathcal{H}_t := \mathcal{S} \otimes \mathcal{H}
\]

(153)

carries the representation

\[
\rho_t := \gamma_5 \otimes \rho,
\]

(154)

the chirality operator is given by

\[
\chi_t := \gamma_5 \otimes \chi.
\]

(155)
The definition of the generalized Dirac operator

\[ D_t := \partial \otimes 1 + \gamma_5 \otimes D \]  

is well motivated from differential geometry. We denote by \( \delta_t \) the differential of the universal differential envelop \( \hat{\Omega}A_t \). This is only an auxiliary construction and we shall never need \( \delta_t \) expressed in terms of \( d \) and \( \delta \). After passage to the quotient the differential, still denoted by \( \delta_t \), will be the concrete operator \( -i [D_t, \cdot] \) which we shall have to calculate in terms of \( \partial / \otimes \) and \( D_t \).

To alleviate notations we shall often omit the \( \otimes \) and write e.g. \( \delta_t f a \) for \( \delta_t (f \otimes a) \). Again the good differential algebra \( \Omega A_t \) is obtained as a quotient via the homomorphism

\[ \pi_t (f_0 a_0 \delta_t f_1 a_1 \ldots \delta_t f_p a_p) := (-i)^p \rho_t(f_0 a_0) [D_t, \rho_t(f_1 a_1)] \ldots [D_t, \rho_t(f_p a_p)]. \]  

Our basic variable \( H_t \) is an antihermitian 1-form,

\[ H_t \in \Omega^1 A_t, \quad H_t^* = -H_t. \]  

Its curvature is the hermitian 2-form

\[ C_t := \delta_t H_t + H_t^2 \]  

used to calculate the functional

\[ V_t(H_t) := (C_t, C_t) \]  

where the scalar product now involves the Dixmier trace in \( S \) and the trace in \( H \). The main miracle of Connes’ recipe can be summarized in the

**Theorem:** We have the following decomposition

\[ H_t = A + H, \quad A \in \Omega^1(M, \rho(g)) \mapsto \Omega^1 F \otimes \Omega^0 A, \]

\[ H^* = -H \in \Omega^0(M, \Omega^1 A) \cong \Omega^0 F \otimes \Omega^1 A \]  

and

\[ C_t = F + C - D\Phi \gamma_5 \]  

with \( g \) the Lie algebra of the group of unitaries

\[ g := \{ X \in \mathcal{A}, X^* = -X \}, \]  

with the field strength

\[ F := dA + A^2 = dA + \frac{1}{2}[A, A] \in \Omega^2(M, \rho(g)), \]

and the covariant derivative

\[ D\Phi := d\Phi + [A\Phi - \Phi A] \in \Omega^1(M, \Omega^1 A). \]
Recall $\Phi := H - iD$. Finally the generalized Higgs potential reads

$$V_t(A + H) = \int_M \text{tr}(F \star F) + \int_M \text{tr}(D\Phi^* \star D\Phi) + \int_M *V(H) - \int_M *V_0(H)$$  \hspace{1cm} (166)

with

$$V_0(H) := \langle \alpha C, \alpha C \rangle$$  \hspace{1cm} (167)

and $\alpha$ is the linear map

$$\alpha : \Omega^2 A \longrightarrow \left[ \rho(A) + \pi(\delta(\ker \pi)^1) \right]^G$$  \hspace{1cm} (168)

determined by the two equations

$$\langle R, C - \alpha C \rangle = 0 \quad \text{for all } R \in \rho(A)^G,$$  \hspace{1cm} (169)

$$\langle K, \alpha C \rangle = 0 \quad \text{for all } K \in \pi(\delta(\ker \pi)^1)^G.$$  \hspace{1cm} (170)

The scalar product is the finite dimensional one of chapter 1, the $x$ dependence of $C$ can be ignored.

We have two basic variables. $A$ is a genuine gauge potential (non-abelian if $A$ is non-commutative) that is a differential 1-form on $M$ with values in the Lie algebra $\mathfrak{g}$ of the group of unitaries of $A$ represented on $\mathcal{H}$. $H$ is as before, however now with a differentiable $x$ dependence, i.e. $H$ is a multiplet of genuine scalar fields. The generalized Higgs potential reproduces the complete bosonic action of a Yang-Mills-Higgs model, namely the Yang-Mills action, the covariant Klein-Gordon action and the integral of the modified Higgs potential $V - V_0$. The modified potential is still a non-negative polynomial of fourth order in the scalar fields, in fact, as we shall see

$$V - V_0 = \text{tr}[(C - \alpha C)^2].$$  \hspace{1cm} (171)

The entire action is gauge invariant. An element $g$ of the group of unitaries of $A_t$ is a differentiable function from spacetime into the finite dimensional group $G = \{ g \in A, \; gg^* = g^*g = 1 \}$. Therefore these elements are genuine gauge transformations. Under the gauge group our fields transform as

$$A^a = \rho(g) A \rho(g^{-1}) + \rho(g) d \rho(g^{-1}),$$  \hspace{1cm} (172)

$$H^a = \rho(g) H \rho(g^{-1}) + \rho(g) \delta \rho(g^{-1}).$$  \hspace{1cm} (173)

Before proving the theorem let us compute the modified Higgs potential $V_0$ for our example $A = M_2(\mathfrak{g}) \oplus \mathfrak{c}$. Recall the generic elements

$$R = \begin{pmatrix} a & 0 \\ 0 & B \end{pmatrix}, \; \; a \in M_2(\mathfrak{g}), \; B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \; b_1, b_2 \in \mathfrak{c},$$  \hspace{1cm} (174)
Note that the entries $b_1$ and $b_2$ of $B$ are now unrelated due to the complexification $\rho(A)^\mathbb{C}$.

$$K = \begin{pmatrix} \Delta k & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathbb{C}),$$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & M^*cM \end{pmatrix}, \quad c \in M_2(\mathbb{C}).$$

Therefore

$$\alpha C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & |m_1|^2c_{11} & 0 \\ 0 & 0 & 0 & |m_2|^2c_{22} \end{pmatrix},$$

and the modified Higgs potential reads

$$V - V_0 = 2|m_1|^2|m_2|^2(\phi_1^*(\phi_1)\phi_2^*(\phi_2)).$$

Through the process of tensorizing we have lost the precious property of spontaneous symmetry breaking. Indeed, the gauge invariant point $\Phi = 0$ is minimum of the modified potential. However, this loss of symmetry breaking is particular to our example and not a typical feature. In fact a slight modification of the example namely adding more families remedies the evil. Consider the same algebra $A = M_2(\mathbb{C}) \oplus \mathbb{C}$ with a new Hilbert space consisting of $N$ copies of the old one,

$$\mathcal{H} := (\mathbb{C}^2 \oplus \mathbb{C}^2) \otimes \mathbb{C}^N.$$
has now entries $m_j$, non-degenerate complex $N \times N$ matrices which should be thought of as mass matrices of the quarks of charge $2/3$ and $-1/3$. In other words, the basis is ordered differently here:

\[(u_L, c_L, d_L, s_L, \cdots), \quad N = 2.\]  

(184)

The formulas of chapter one generalize naturally to the new situation,

\[MM^* = \begin{pmatrix} m_1m_1^* & 0 \\ 0 & m_2m_2^* \end{pmatrix} = 1 \otimes \Sigma + \sigma_3 \otimes \Delta\]  

(185)

with

\[\Sigma := \frac{1}{2}(m_1m_1^* + m_2m_2^*),\]  

(186)

\[\Delta := \frac{1}{2}(m_1m_1^* - m_2m_2^*).\]  

(187)

The generic elements $R \in \rho(\mathcal{A})^\mathcal{Q}$, $K \in \pi(\delta(\ker \pi)^\mathcal{I})$ and $C \in \Omega^2 \mathcal{A}$ become

\[R = \begin{pmatrix} a \otimes 1 & 0 \\ 0 & B \otimes 1 \end{pmatrix}, \quad a \in M_2(\mathcal{Q}), \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1, b_2 \in \mathcal{Q},\]  

(188)

\[K = \begin{pmatrix} k \otimes \Delta & 0 \\ 0 & 0 \end{pmatrix}, \quad k \in M_2(\mathcal{Q}),\]  

(189)

\[C = \begin{pmatrix} \tilde{c} \otimes \Sigma' & 0 \\ 0 & M^*(c \otimes 1)M \end{pmatrix}\]  

(190)

with

\[c := 1 - \phi^* \phi,\]  

(191)

\[\tilde{c} := 1 - \phi \phi^*,\]  

(192)

\[\Sigma' := \Sigma - \frac{\text{tr}(\Delta \Sigma)}{\text{tr}(\Delta^2)} \Delta.\]  

(193)

A straight-forward calculation yields

\[\alpha C = \begin{pmatrix} \tilde{c} \otimes Y & 0 \\ 0 & z \otimes 1 \end{pmatrix}\]  

(194)

with

\[Y := \frac{\text{tr} \Sigma'}{N - ((\text{tr} \Delta)^2/\text{tr} \Delta^2)} \left[1 - \frac{\text{tr} \Delta}{\text{tr} \Delta^2} \Delta\right],\]  

(195)
\[
 z := \frac{1}{N} \begin{pmatrix}
 \text{tr}(m_1^* m_1) c_{11} & 0 \\
 0 & \text{tr}(m_2^* m_2) c_{22}
 \end{pmatrix}
 \]  
(196)

and the (modified) Higgs potential

\[
 V - V_0 = \text{tr}(\Sigma' - Y)^2 \left[ (1 - \phi_1^* \phi_1)^2 + (1 - \phi_2^* \phi_2)^2 + 2(\phi_1^* \phi_2)(\phi_2^* \phi_1) \right]
 + \left[ \text{tr}(m_1^* m_1)^2 - \frac{1}{N}(\text{tr} m_1^* m_1)^2 \right] (1 - \phi_1^* \phi_1)^2
 + \left[ \text{tr}(m_2^* m_2)^2 - \frac{1}{N}(\text{tr} m_2^* m_2)^2 \right] (1 - \phi_2^* \phi_2)^2
 - 2\text{tr}(m_1^* m_1 m_2^* m_2)(\phi_1^* \phi_2)(\phi_2^* \phi_1)
 \]  
(197)

does break the symmetry spontaneously unless there is a numerical coincidence:

\[
 Y = \Sigma' \quad \text{and} \quad z \otimes 1 = \begin{pmatrix}
 c_{11} m_1^* m_1 & 0 \\
 0 & c_{22} m_2^* m_2
 \end{pmatrix}.
 \]  
(198)

**Proof of the theorem:**

Our conventions are summarized in table 1. In the following we compute only $\Omega^1 A_t$ and $\Omega^2 A_t$ for the particular case at hand. Details of the general case can be found in reference [2]. To get started, we need the commutator $[D_t, f \rho(a)]$. Using

\[
 D_t = \begin{pmatrix}
 \phi & \gamma_5 M \\
 \gamma_5 M^* & \phi^*
 \end{pmatrix}
 \]  
(199)

we obtain

\[
 [D_t, f \rho(a)] = i\gamma(d f)\rho(a) + \gamma_5 f [D, \rho(a)]
 \]  
(200)

and

\[
 \pi_t(f_0 a_0 \delta_t f_1 a_1) = \pi_D(f_0 d f_1)\rho(a_0 a_1) + f_0 f_1 \gamma_5 \pi(a_0 \delta a_1).
 \]  
(201)

Therefore $\Omega^1 A_t$ decomposes into a direct sum

\[
 \Omega^1 A_t = \Omega^1 F \otimes \rho(A) \oplus F \otimes \Omega^1 A
 \]  
(202)

and its general element can be put under the form

\[
 H_t = A + H,
 \]  
(203)

\[
 A = \pi_t(f_0 a_0 \delta_t f_1 1) \in \Omega^1 F \otimes \rho(A)
 \]  
(204)

is a 1-form on spacetime with values in $\rho(A)$ and

\[
 H = \pi_t(f a_0 \delta_1 a_1) \in F \otimes \Omega^1 A
 \]  
(205)

is an internal 1-form as in chapter 1 but now with a differential $x$ dependence. Later we shall impose that $H_t$ be antihermitian, in particular that $A$ take values in the Lie algebra $\mathfrak{g}$

\[
 \mathfrak{g} := \{ X \in A, \quad X^* = -X \}
 \]  
(206)
represented on $\mathcal{H}$ and $H$ is antihermitian.

On level two we have

$$\pi_t(f_0 a_0 \delta_t f_1 a_1 \delta_t f_2 a_2) = \pi_D(f_0 d f_1 d f_2) \rho(a_0 a_1 a_2) + \frac{f_0 f_1 f_2}{2} \pi(a_0 \delta a_1 \delta a_2) + i \gamma (f_0 f_1 f_2) \gamma_5 \rho(a_0) [\mathcal{D}, \rho(a_1)] \rho(a_2) - i \gamma (f_0 f_1 f_2) \gamma_5 \rho(a_0 a_1) [\mathcal{D}, \rho(a_2)].$$

Consequently

$$\pi_t(\hat{\Omega}^2 \mathcal{A}_t) = \left[ \pi_D(\hat{\Omega}^2 \mathcal{F}) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \pi(\hat{\Omega}^2 \mathcal{A}) \right] \oplus \pi_D(\hat{\Omega}^1 \mathcal{F}) \otimes \pi(\hat{\Omega}^1 \mathcal{A}).$$

Remember from last chapter that spacetime zero- and two-forms mix before division by the junk. This entails that the sum in the above bracket is not direct. Next we compute the tensor junk. Consider a general element of $\Omega^1 \mathcal{A}_t$

$$\pi_t(f_0 a_0 \delta_t f_1 1) + \pi_t(f a_0 \delta_t 1 a_1)$$

Its pre-image $p$ belongs to $(\ker \pi_t)^1$ if and only if $\pi_D(f_0 d f_1) = 0$ and $\pi(a_0 \delta a_1) = 0$ in which case

$$\pi_t(\delta_t p) = \pi_D(d f_0 d f_1) \rho(a) + \underline{f} \pi(\delta a_0 \delta a_1).$$

It follows that

$$\pi_t(\delta_t (\ker \pi_t)^1) = \pi_D(d (\ker \pi_D)^1) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \pi(\delta (\ker \pi)^1)$$

where again the sum is not direct and by division we get

$$\Omega^2 \mathcal{A}_t = \left[ \Omega^2 \mathcal{F} \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \Omega^2 \mathcal{A} \right] \oplus \Omega^1 \mathcal{F} \otimes \Omega^1 \mathcal{A}.$$
\[ +\pi_D(f_0 df_1)\gamma_5\pi(aa_0\delta a_1) - \pi_D(ff_0 df_1)\gamma_5\pi(a_0\delta a_1a) \]
\[ = : A^2 + \tilde{H}^2 + [A\tilde{H} - \tilde{H}A] \gamma_5. \tag{217} \]

All together
\[ C_t = dA - \delta A\gamma_5 + \delta\tilde{H} + d\tilde{H}\gamma_5 + A^2 + \tilde{H}^2 + [A\tilde{H} - \tilde{H}A] \gamma_5 \]
\[ = F + C + (i[D, A] + d\tilde{H} + [A\tilde{H} - \tilde{H}A]) \gamma_5 \]
\[ = F + C - D\Phi \gamma_5 \tag{218} \]

with the curvatures
\[ F := dA + A^2, \tag{219} \]
\[ C := \delta\tilde{H} + \tilde{H}^2. \tag{220} \]

The covariant derivative
\[ D\Phi := d\Phi + [A\Phi - \Phi A] \in \Omega(M, \Omega^1 A) \tag{221} \]

makes sense because of the homogeneous transformation law of \( \Phi := \tilde{H} - iD. \) Note the purely algebraic term \( \delta A = -i[D, A] \) in the total curvature. This term generates the masses of the gauge bosons in the Lagrangian and does so by means of the fermionic mass matrix \( M \) in \( D. \)

Finally we have to work out the scalar product in \( \Omega^2 A_t, \) the space where the curvatures live. As usual we start with the auxiliary scalar product \(<\cdot,\cdot>\) in \( \pi_t(\tilde{\Omega} A_t) \) defined by
\[ <\hat{\phi}_D, \hat{\psi}_D \otimes \hat{\psi}> := <\hat{\phi}_D, \hat{\psi}_D> <\hat{\phi}, \hat{\psi}> = tr_\omega(\hat{\phi}^* \hat{\psi} D|\theta|^{-4}) tr(\hat{\phi}^* \hat{\psi}), \]
\[ \hat{\phi}_D, \hat{\psi}_D \in \pi(D(\tilde{\Omega}F), \hat{\phi}, \hat{\psi} \in \pi(\tilde{\Omega} A). \tag{222} \]

and define the scalar product on \( \Omega A_t \) by
\[ (\phi_t, \psi_t) := <\phi_t, P_t \psi_t>, \tag{223} \]

\( \phi_t \) and \( \psi_t \) are any representatives of their classes and \( P_t \) is the orthogonal projector on \( (\pi_t(\delta_t(\ker \pi_t))^\perp. \)

For our purpose it is sufficient to know this projector in \( \pi_t(\tilde{\Omega}^2 A_t) \) where it is already non trivial, \( P_t \neq P_D \otimes P. \) Let us come back to the decomposition
\[ \pi_t(\tilde{\Omega}^2 A_t) = [\pi_D(\tilde{\Omega}^2 F) \otimes \rho(A) + F \otimes \pi(\tilde{\Omega}^2 A)] \oplus \pi_D(\tilde{\Omega}^1 F) \otimes \pi(\tilde{\Omega}^1 A) \tag{224} \]

with general elements in the three subspaces
\[ \pi_t(f_0 a_0 \delta_1 f_1 f_2) = \pi_D(f_0 df_1 df_2) \rho(a_0), \tag{225} \]
\[ \pi_t(f_0 a_0 \delta_1 a_1 \delta_1 a_2) = f_0 \pi(a_0 \delta a_1 \delta a_2), \tag{226} \]
\[ \pi_t(f_0a_0\delta_1a_1\delta_1f_2) = -\pi_D(f_0d_1f_2)\gamma_5\pi(a_0\delta a_1). \] (227)

Our first conclusion is that the above direct sum is also an orthogonal sum because the trace in equation (139) is over \( \gamma_5 \) multiplied by an odd number of proper gamma matrices and vanishes. Therefore by equation (211) \( P_t \) leaves the third subspace untouched and we concentrate on the restriction \( P_1 \) of \( P_t \) to the first two subspaces. Let us introduce the following short hands:

\[ U := \pi_D(\hat{\Omega}^2F) \otimes \rho(A), \] (228)

\[ U_0 := \pi_D(d(\ker \pi_D)^1) \otimes \rho(A) \cong F \otimes \rho(A) \] (229)

and \( U_\perp \) is the ortho-complement of \( U_0 \) in \( U \),

\[ U_\perp := (U_0)^U \cong \Omega^2M \otimes \rho(A). \] (230)

Likewise

\[ W := F \otimes \pi(\hat{\Omega}^2A), \] (231)

\[ W_0 := F \otimes \pi(\delta(\ker \pi)^1) \] (232)

and \( W_\perp \) is the ortho-complement of \( W_0 \) in \( W \),

\[ W_\perp := (W_\perp)^W = F \otimes \Omega^2A \] (233)

and \( P_1 \) is the orthogonal projector in \( U + W \) on \( (U_0 + W_0)^\perp \neq U_0^\perp + W_0^\perp \). Next we remark that \( U_\perp \) is orthogonal to the other three subspaces

\[ U_\perp \cap (U_0 + (W_0 \oplus W_\perp)) = 0, \] (234)

\[ U_\perp \perp (U_0 + (W_0 \oplus W_\perp)) \] (235)

because \( \text{tr}_4[\gamma^\mu\gamma^\nu] = \text{tr}_4[\gamma^\nu\gamma^\mu] \). This means that also \( V_\perp \) decouples and it remains to compute the restriction \( P_2 \) of \( P_t \) to

\[ U_0 + (W_0 \oplus W_\perp) = F \otimes \left[ \rho(A) + (\pi(\delta(\ker \pi)^1) \oplus \Omega^2A) \right]. \] (236)

In this space now the \( x \) dependence is trivial and can be ignored reducing the calculation to a finite dimensional one in \( [\rho(A) + (\pi(\delta(\ker \pi)^1) \oplus \Omega^2A)] \) with scalar product defined by \( \text{tr} \), the trace over \( \mathcal{H} \). What we need is \( P_2C, C \in \Omega^2A \). Since \( P_2 \) is an orthogonal projector onto \( (U_0 + W_0)^\perp \), \( C - P_2C \) is perpendicular to \( (U_0 + W_0)^\perp \),

\[ C - P_2C \equiv \alpha C \in (U_0 + W_0)^\perp = (U_0 + W_0)^\Omega \] (237)
or

\[ P_2 C = C - \alpha C. \] \tag{238}

Finally by Pythagoras

\[
< C, P_1 C > =
\]

\[
< C, P_2 C > =< C, C - \alpha C > = (C, C) - < C, \alpha C >
\]

\[
= (C, C) - < P_2 C + \alpha C, \alpha C >
\]

\[
= (C, C) - < \alpha C, \alpha C > .
\] \tag{239}

4 Remarks

Instead of conclusions we offer a list of remarks.

- **The standard model:** The first question is of course: can the standard model be accommodated in Connes’ approach? The answer is yes, but this requires three new items to be added [1,3,4] to Connes’ model building kit as reviewed so far. In our example the group of unitaries, \( U(2) \times U(1) \), is too big by one \( U(1) \) factor to describe electroweak interactions and there are two complex Higgs doublets, one too many. Both points are cured readily by replacing the internal algebra by \( \mathcal{A} = \mathbb{H} \oplus \mathbb{C} \), \( \mathbb{H} \) being the quaternions, keeping the K-cycle unchanged. Straightening out the hypercharges requires a first new item, the so called unimodularity condition which reduces the group of unitaries by a purely algebraic restriction. But then one has to start from a bigger algebra which is anyhow needed to include strong interactions. Since quantum chromodynamics is a vector like theory its inclusion is easy except for two points. First its group, \( SU(3) \), is the group of unitaries of no algebra \( \mathcal{A} \). This problem is taken care of by the unimodularity condition. Second the quarks come in a representation, that is a tensor product, \((3, 2)\), and therefore not available so far, a problem solved by introducing bimodules, the second item. As explained in our example the Higgs and gauge boson masses are determined by the fermion mass matrix. Also it is obvious that the coupling constants in the bosonic action \( V_t \) are related. In the complete standard model à la Connes these relation are [5]

\[
g_3 = \frac{1}{2} \left( \frac{4 - 2x}{1 - x} \right)^{1/2} g_2 \] \tag{240}

\[
\sin^2 \theta_W = \frac{3(1 - x/2)}{8 - 2x} \] \tag{241}

and, neglecting all fermion masses against the top mass \( m_t \):

\[
m_W = \frac{1}{2} \left( \frac{(1 - x)}{1 - x/2} \right)^{1/2} m_t \] \tag{242}
and

\[ m_H = \left( 3 - \frac{3}{2} \cdot 3x^2 - 8x + 5 \right)^{1/2} m_t. \]  

(243)

The parameter \( x \) ranges from \(-1\) to \(+1\). It is the relative weight of the traces in the Hilbert spaces \( \mathcal{H} \) of the leptons and of the quarks. Numerically

\[
\begin{array}{cccccc}
 x & -1 & 0 & \frac{1}{2} & 0.99 & 1 \\
(g_3/g_2)^2 & \frac{3}{4} & 1 & \frac{3}{2} & 50.5 & \infty \\
\sin^2 \theta_W & \frac{9}{20} & \frac{3}{8} & \frac{9}{28} & 0.2252 & \frac{1}{4} \\
m_t/m_W & \sqrt{3} & 2 & \sqrt{6} & 14.2 & \infty \\
m_H/m_W & 2.65 & 3.14 & 3.96 & 24.5 & \infty \\
\end{array}
\]

Note that the ratio \( m_H/m_t \) shows little variation from 1.53 to \( \sqrt{3} \). Of course these classical relations are unstable under quantum corrections [6] and there are at least two possible attitudes with respect to this dilemma. The first says we do not know yet what quantum field theory is in this new context. The other attitude is the third item: The trace in the Hilbert space \( \mathcal{H} \) that we have used to define the scalar product is not unique (up to normalisation) and taking the most general traces produces the standard model without the constraints on masses and coupling constants.

**- Other generalisations:** We have only considered a particular case of Connes’ algorithm. In general the total algebra \( A_t \) is not a tensor product of \( \mathcal{F} \) and a finite dimensional algebra \( A \). Also the basic variable \( H_t \) can live in a more general space than \( \Omega^1 A_t \), it can be a connection on any hermitian finite projective module over \( A_t \). These generalisations do not seem to help with the above mentioned problems.

**- Other algebras:** Up to now only few algebras \( A \) have been explored besides the standard model. It is most intriguing that the simplest typical example we know so far is already quite involved and reproduces most features of electroweak interactions. The other algebras considered in this context are \( M_5(\mathbb{Q}) \) [7] and \( Cliff(10) \) [8] in order to reproduce the \( SU(5) \) and \( SO(10) \) grand unified theories. The first one fails because the fermions in the \( SU(5) \) model sit in a \( \bar{5} + 10 \), the 10 comes from a tensor product and does not fit Connes’ rules. The second model is left- right symmetric and it is difficult to obtain the complicated Higgs sector necessary in \( SO(10) \). Finally a smaller left-right symmetric model with gauge group \( U(2) \times U(2) \) has been worked out [9]. As in the \( SO(10) \) model, the gauge symmetry is broken spontaneously and parity remains unbroken.

**- Splinters:** It does not look easy to get rid of imaginary time. In non compact, pseudo Riemannian spacetime traces and scalar products are ill defined. The Dirac operator has continuous spectrum, action integrals diverge and there is certainly more to do than simply invoking Wick rotation. There is no convincing motivation for the time integral in \( S = L^2(M, \mathcal{C}^4) \).
- **Other non commutative schemes:** Connes’ algorithm to produce a differential algebra $\Omega A$ starting from an algebra $A$ — such that the algebra $F$ of functions on a manifold $M$ reproduces de Rham’s differential algebra $\Omega M$ — is not unique. In fact already in 1988 Dubois-Violette [10] introduced a different such algorithm using derivations. This algorithm is successfully used by the southern Paris group to obtain particle models [11]. These models share the attractive features of Connes’ theory, they also unify gauge and Higgs bosons. In their scheme however the Higgs bosons transforms according to the adjoint representation of the group of unitaries irrespective of fermion representations. Their scheme has been generalized by Balakrishna, Gürsey & Wali [12] to include other Higgs representations. Another approach due to Coquereaux [13] takes immediately the differential algebra $\Omega A_t$ as starting point. The approach is thereby more transparent and less rigid. Many of its physical features have been worked out by the Marseille-Mainz group [14].

- **Non-commutative algebras and quantum physics:** The non-commutative models are clearly inspired by the mathematics of quantum mechanics, operator algebras. There are two main differences between non commutative and quantum. Ad one, the passage from classical to quantum mechanics can be considered as replacing the commutative algebra of functions on phase space, the ”observables”, by a non-commutative algebra [15]. In the above models not phase space but space time is rendered non-commutative. Ad two, in quantum mechanics the non-commutative algebra is God given and contains a dimensionful parameter, Planck’s constant. A nice example illustrating the interplay of quantum mechanics and non-commutative geometry is given by Madore [16]. We repeat that a generalisation of quantum field theory to the non-commutative setting is still lacking.

- **Parallel universes:** The first example studied by Connes and Lott [1] was $A = \mathbb{C} + \mathbb{C}$, $\mathcal{H} = \mathbb{C} + \mathbb{C}$. It has a nice geometric interpretation in terms of a Riemannian manifold $M = M_L + M_R$, disjoint union of two identical Riemannian manifolds $M_L$ and $M_R$ separated by the constant distance $(|m_1|^2 + |m_2|^2)^{-1/2}$. The left handed fermions live on $M_L$, the right handed fermions on $M_R$. This model can also be interpreted as a Kaluza-Klein theory with a discrete fifth dimension consisting of two points. The Kaluza-Klein analogy is present in all non-commutative models. It has been worked out in detail by the southern Paris group [17] and served as initial motivation for Coquereaux’s scheme [13].

- **Gravity:** A question of fundamental importance is: can the Einstein-Hilbert action be fit into the non commutative frame? Again the answer is not unique. A first proposal is due to the Zürich group [18]. Starting from Einstein-Cartan’s theory they arrive at a tensor scalar theory. The scalar has a geometric interpretation as the now variable distance between parallel universes. In a recent paper Chamseddine and Fröhlich [19] have coupled this scalar to the standard model and after addition of some effective Coleman-Weinberg potential they obtain a striking prediction for the top and Higgs masses,

\[ 146.2 \leq m_t \leq 147.4 \text{ GeV}, \]  

(245)
Connes [20] has found a more intrinsic way to incorporate gravity: he computes the Dixmier trace or more properly the Wodzicki residue of the (true) Dirac operator to the power minus two and obtains the Einstein-Hilbert action.

- **Unification**: Grand unification was based on the attractive idea to replace a direct product of groups by a simple group. In the same spirit and in order to reconcile particle interactions and gravity it seems attractive to look for an algebra $\mathcal{A}_t$, that is not just a tensor product of the algebra $\mathcal{F}$ of functions on spacetime and an internal algebra, but that is still sufficiently close to $\mathcal{F}$ to allow spacetime to subsist in some form. A 2-dimensional example of such an algebra is the non-commutative torus which plays an intriguing role in solid state physics.

We are indebted to Daniel Kastler who introduced us to the miraculous non-commutative world, he is a most pleasant guide. It is also a pleasure to acknowledge stimulating discussions with Robert Coquereaux, Gilles Esposito-Farèse, Bruno Iochum and Daniel T*.
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