Black Holes and Neutron Stars in Tensor-Vector-Scalar Theory

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Abstract. Bekenstein's Tensor-Vector-Scalar (TeVeS) theory has had considerable success as a relativistic theory of Modified Newtonian Dynamics. In the strong-field regime it makes many predictions that differ from General Relativity implying this regime provides an excellent testing ground for the theory using both current and future observations of neutron stars and black holes in both the electromagnetic and gravitational wave spectrum. I look at the current status of black hole and neutron star solutions in both the original TeVeS theory and its generalisation, including some key observational properties that differ from General Relativity.

1. Introduction
Modified Newtonian Dynamics (MoND) [1] provides an enticing alternative to the Cold Dark Matter paradigm by phenomenologically modifying Newton's law of gravity on galactic scales (for a review see Ref. [2]). MoND is nothing more than a toy model of gravity though as it is not a covariant theory. A fully relativistic theory generalising MoND has recently been proposed by Bekenstein [3], which includes the original tensor field of General Relativity coupled to an extra vector and scalar field, hence the theory is known by its acronym TeVeS.

In the weak acceleration limit, TeVeS has been shown to reproduce MoND implying the theory carries forth the various successes of that theory on galactic scales. In the Newtonian limit, the theory reproduces the parametrized post-Newtonian (PPN) coefficients consistently within limits imposed by solar system experiments (for a comprehensive review see Ref. [4]). More recently, TeVeS, when combined with a single sterile neutrino species of mass around 11 eV, can match the angular power spectrum of the cosmic microwave background [5] as well as the straight gravitational lensing arc of Abell 2390 [6] – two previous chinks in the TeVeS armour.

Meanwhile, the strong-field study of TeVeS is relatively under-developed. Giannios [7] first solved the TeVeS field equations for a spherically symmetric, static, vacuum spacetime. Sagi & Bekenstein [8] subsequently solved the field equations for charged spacetimes, thus deriving the Reissner-Nordström solution. Subsequently, in Lasky, Sotani & Giannios [9], we solved the spherically symmetric field equations in the presence of a perfect fluid, thus finding the equivalent TOV solution in order to study neutron star structure. The properties of these various solutions are reviewed in considerable detail below in sections 3 and 4.

The original formulation of TeVeS is, however, not without its shortcomings. In particular, Bekenstein’s formulation [3] has been shown to suffer from dynamical problems. Seifert [10] showed that Giannios’ Schwarzschild-TeVeS solution is unstable to linear perturbations for
various relevant values of the theory’s coupling parameters. Contaldi et al. [11] then showed that the vector field is prone to forming caustics under a variety of dynamical situations. Finally, Sagi [12] showed that the vector field is constrained by the cosmological value of the scalar field in such a way that it prevents the scalar field from evolving. Given these inconsistencies, Contaldi et al. [11] and Skordis [13] have provided a generalisation of TeVeS whereby the vector field action is not given by the Maxwellian form of the original TeVeS, but a more general Einstein-Aether form (the tensor and scalar fields are described equivalently to the original formulation).

Very little has been studied in the generalised version of TeVeS. Skordis [13] studied the cosmological equations of the theory and showed that they are identical to the original TeVeS equations up to a rescaling of Hubble’s constant. Sagi [8] looked at the PPN parameters and found them not to conflict with solar system experiments for suitable values of the coupling parameters. In [14], I looked at the strong-field regime of this theory and found that the vacuum and charged black hole solutions and also the TOV solution are identical in this theory as the original theory up to a rescaling of various vector field coupling parameters.

In the present article I review the non-charged vacuum and neutron star solutions in the generalised and original versions of TeVeS. The black hole solutions are those found for the original version of the theory by Giannios [7], and subsequently derived for the generalised version in Lasky [14]. Meanwhile, the results for the neutron star spacetimes are reviewed from Lasky, Sotani & Giannios [9]. I first provide a brief description of the relevant field equations for the generalised TeVeS theory.

2. Generalised TeVeS

As the name suggests, TeVeS is a theory built upon three background fields – a tensor, a vector and a scalar field. The action describing the tensor field is simply the Einstein-Hilbert action from General Relativity. This implies that when the vector field and scalar field coupling parameters tend to zero, the theory reduces exactly to General Relativity. Herein I provide only a brief introduction to the relevant TeVeS equations. For more details the reader is referred to Bekenstein [3] for the equations of original TeVeS and to Contaldi et al. [11] and Skordis [13] for the generalised version of the theory.

Given the three background fields, one requires a metric in which physical measurements are made. This so-called physical metric, denoted \( \tilde{g}_{\mu\nu} \), is related to the tensor field, \( g_{\mu\nu} \) (also known as the Einstein metric), the vector field, \( A^\mu \), and the scalar field, \( \varphi \), through the following relation

\[
\tilde{g}_{\mu\nu} = e^{-2\varphi} \left( g_{\mu\nu} + A_\mu A_\nu \right) - e^{2\varphi} A_\mu A_\nu.
\] (1)

The modified Einstein field equations are found by varying the Einstein-Hilbert action where the Riemann tensor is that associated with the Einstein metric. These are

\[
G_{\mu\nu} = 8\pi G \left[ \tilde{T}_{\mu\nu} + \left( 1 - e^{-4\varphi} \right) A^\alpha \tilde{T}_{\alpha(\mu} A_{\nu)} + \tau_{\mu\nu} \right] + \Theta_{\mu\nu},
\] (2)

where \( G_{\mu\nu} \) is the Einstein tensor associated with the Einstein frame, \( \tilde{T}_{\mu\nu} \) is the physical stress-energy tensor and \( \tau_{\mu\nu} \) & \( \Theta_{\mu\nu} \) are the effective stress-energy terms associated with the scalar and vector fields respectively. In particular

\[
\tau_{\mu\nu} := \frac{\mu}{kG} \left[ \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{2} g^{\alpha\beta} \nabla_\alpha \varphi \nabla_\beta \varphi g_{\mu\nu} - A^\alpha \nabla_\alpha \varphi \left( A_{(\mu} \nabla_{\nu)} \varphi - \frac{1}{2} A^\beta \nabla_\beta \varphi g_{\mu\nu} \right) \right] - \frac{\mathcal{F}(\mu)}{2k^2 \ell^2 G} g_{\mu\nu},
\] (3)

where \( k \) is the scalar field coupling constant, \( \mu \) is a function associated with the MoND acceleration scale, \( \mathcal{F}(\mu) \) is related to the interpolation function in MoND and hence is not \textit{a priori} predicted by the theory and \( \ell \) is a fixed length scale associated with the free function. For the remainder of this article we only consider the strong-field regime of the theory whereby
\( \mu = 1 \) is an excellent approximation (for more details see [3, 7, 8, 11]). In this case the function \( F \) does not contribute to the field equations [11], and is therefore ignored for the remainder of the article.

The generalised version of TeVeS has four vector field coupling components which describe the relative strengths of the individual terms in the vector field action. These are denoted by \( K, K_+, K_2 \) and \( K_4 \), where Bekenstein’s original version of the theory is regained by setting the last three parameters to zero. The vector field contribution to the effective stress-energy in the modified Einstein equations, \( \Theta_{\mu \nu} \), can be expressed as a sum of terms associated with each of the vector field coupling constants, plus a term associated with a Lagrange multiplier, \( \lambda \):

\[
\Theta_{\mu \nu} := \Theta^K_{\mu \nu} + \Theta^K^+_{\mu \nu} + \Theta^K_2_{\mu \nu} + \Theta^K_4_{\mu \nu} + \Theta^\lambda_{\mu \nu},
\]

where

\[
\Theta^K_{\mu \nu} := K \left( F_{\alpha \mu}F_{\alpha \nu} - \frac{1}{4}g_{\mu \nu}F_{\alpha \beta}F^{\alpha \beta} \right),
\]

\[
\Theta^K^+_{\mu \nu} := K_+ \left[ S_{\mu \alpha}S_{\nu}^\alpha - \frac{1}{4}g_{\mu \nu}S_{\alpha \beta}S^{\alpha \beta} + \nabla_A \left( A^\alpha S_{\mu \nu} - S^\alpha_{(\mu}A_{\nu)} \right) \right],
\]

\[
\Theta^K_2_{\mu \nu} := K_2 \left[ g_{\mu \nu} \nabla_A \left( A^\alpha \nabla_\beta A^\beta \right) - A_{(\mu} \nabla_\nu) \left( \nabla_A A^\alpha \right) - \frac{1}{2}g_{\mu \nu} \nabla_A A^\alpha \nabla_\beta A^\beta \right],
\]

\[
\Theta^K_4_{\mu \nu} := K_4 \left[ \dot{A}_{\mu} \dot{A}_{\nu} + \dot{A}_{A}(\mu \nabla_\nu)A^\alpha - \nabla_\alpha \left( \dot{A}_{A} A_{\mu} \nabla_\nu \right) - \frac{1}{2}g_{\mu \nu} \dot{A}_{A} A^\alpha \right],
\]

\[
\Theta^\lambda_{\mu \nu} := -\lambda A_{\mu} A_{\nu}.
\]

Here, \( F_{\mu \nu} := \nabla_\nu A_\mu \), \( S := \nabla_\nu A_\mu \) and \( \dot{A}_{\mu} := A^\alpha \nabla_\alpha A_\mu \).

Dynamics of the vector field are governed by vector field equation

\[
K \nabla_\alpha F^{\mu \alpha} + K_+ \nabla_\alpha S^\alpha_{\mu} + K_2 \nabla_\mu \left( \nabla_\alpha A^\alpha \right) + \lambda A^\mu + K_4 \left[ \nabla_\alpha \left( \dot{A}_{A} A^\alpha \right) - \dot{A}^\alpha \nabla_\mu A_\alpha \right] + \frac{8\pi m}{\hbar} A^\alpha \nabla_\alpha \varphi g^{\mu \beta} \nabla_\beta \varphi = 8\pi G \left( 1 - e^{-4\varphi} \right) g^{\mu \alpha} T_{\alpha \beta} A^\beta.
\]

Contraction of this equation with the vector field isolates the Lagrange multiplier. This subsequent equation can then be used in the modified Einstein equations, in particular the term expressed in equation (9), such that the system is fully determined.

The final field equation is that of the scalar field

\[
\nabla_\beta \left( \mu \left( g^{\alpha \beta} - A^\alpha A^\beta \right) \nabla_\alpha \varphi \right) = kG \left[ g^{\alpha \beta} + (1 + e^{-4\varphi}) A^\alpha A^\beta \right] T_{\alpha \beta}.
\]

3. Black Holes

Giannios [7] first solved the original TeVeS field equations [i.e. equations (2-11) with \( K_+ = K_2 = K_4 = 0 \)] for a static, spherically symmetric vacuum spacetime. He found two families of solutions dependent on the form of the vector field. When the vector field was constrained to be purely timelike, Giannios showed that the only black hole solution is exactly the Schwarzschild spacetime. Sagi & Bekenstein [8] subsequently generalised Giannios’ result to include charge, showing that the Reissner-Nordström spacetime is also a solution of the TeVeS field equations. In Lasky [14], I showed that the solutions derived by Giannios and Sagi & Bekenstein for the original TeVeS theory are also valid in the generalised version of the theory, where the vector field coupling parameter \( K \) of TeVeS gets replaced by the combination of parameters \( K + K_+ - K_4 \).

As mentioned, the only black hole solution when the vector field is temporally aligned is the Schwarzschild solution. However, there exists a larger class of solutions than this which exhibit exotic phenomena such as naked singularities. The solution found by Giannios [7], and
subsequently in Lasky [14], is usually represented in isotropic coordinates, which is for the physical metric

\[
\text{d}s^2 = - \left( \frac{1 - r_\text{e}/r}{1 + r_\text{e}/r} \right)^a \text{d}t^2 + \left( 1 + \frac{r_\text{e}}{r} \right)^{2+a} \left( 1 - \frac{r_\text{e}}{r} \right)^{2-a} \left( \text{d}r^2 + r^2 \text{d}\Omega^2 \right),
\]

(12)

where \(\text{d}s^2 := \tilde{g}_{\alpha\beta} \text{d}x^\alpha \text{d}x^\beta\) and \(\text{d}\Omega^2 := \text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2\). The two constants in the physical metric, \(r_\text{e}\) and \(a\), are related to the various coupling parameters of the theory, including the “scalar mass”, \(m_\phi\), as well as the characteristic gravitational radius, \(r_g\), according to

\[
r_\text{e} = \frac{r_g}{4}\sqrt{\frac{1 + k}{\pi} \left( \frac{Gm_\phi}{r_g} \right)^2 - \frac{\mathcal{K}}{2}},
\]

(13)

\[
a = \frac{r_g}{2r_\text{e}} + k\frac{Gm_\phi}{4\pi r_\text{e}},
\]

(14)

where \(\mathcal{K} := K + K_+ - K_4\).

Giannios [7] showed that \(r_\text{e}\) is a black hole event horizon if and only if \(a = 2\). This is based on the requirement that the surface at \(r = r_\text{e}\) must have a finite surface area, and also the singularity must be removable. Evaluating the physical Ricci scalar shows that it diverges for values of \(a < 2\) and also \(2 < a < 4\). Meanwhile, as the surface area is proportional to the \(\tilde{g}_{rr}(r_\text{e})\) component of the metric, one finds that this diverges for \(a > 2\). One therefore has a black hole solution only when \(a = 2\). When \(0 < a < 2\) and \(2 < a < 4\) one has a naked singularity at \(r = r_\text{e}\), and when \(a \geq 4\), \(r_\text{e}\) represents a removable singularity with a divergent surface area.

It is interesting to note that the metric (12) is exactly equivalent to that of Brans ’type I’ metric [15] in Brans-Dicke scalar-tensor theory [16] with the assoiciation \(a = 2Q\) and \(\chi = 0\) from the notation of Ref. [17]. Note that in the Brans I solution, setting \(\chi = 0\) necessarily implies that the parameter \(Q\) is unity, and hence the Schwarzschild solution is recovered. This is because the Brans I solution has an extra algebraic equation which links \(Q\) and \(\chi\). The association with a Brans-Dicke solution is particularly interesting given the vast complexity of the field equations in TeVeS as compared with Brans-Dicke theory. However, the association is due to the simplifying assumption that the vector field is purely timelike. It was shown for the original version of TeVeS that black hole solutions with a non-zero radial component for the vector field produce significantly more complicated geometries than the case where the radial component vanishes [7]. This complexity will only increase when one analyses the black hole solutions with non-zero radial vector fields in the fully general theory.

Giannios [7] showed initial trepidation towards the above solution as the scalar field becomes negative close to the horizon, which therefore allows for superluminal propagation of scalar waves [3]. Sagi & Bekenstein [8] overcame this issue by showing that Giannios overlooked a branch of solutions where the scalar field was everywhere positive, although they did this only for the case where \(a = 2\). By rearranging equations (13) and (14) we can re-derive the extra solution of Ref. [8] with the generalisation that \(a\) remains arbitrary. In this case, one can show that the scalar field throughout the spacetime is given by

\[
\varphi(r) = \varphi_c + \delta_{\pm} \ln \left( \frac{1 - r_\text{e}/r}{1 + r_\text{e}/r} \right),
\]

(15)

where \(\varphi_c\) is the cosmological value of the scalar field and \(\delta_{\pm}\) can be found from equations (13) and (14) to be

\[
\delta_{\pm} := a \frac{(2 - \mathcal{K}) k/2 \pm \sqrt{2k [(k - a^2\pi)(2 - \mathcal{K}) + 8\pi]}}{(2 - \mathcal{K}) k + 8\pi},
\]

(16)
Note here that when \( a = 2 \), equation (16) reduces to exactly equation (67) of Ref. [8] (where \( \mathcal{K} ≡ K \) as they were working in the original TeVeS theory which has \( K_+ = K_2 = K_4 = 0 \)).

Given that \( 0 < \mathcal{K} < 2 \) and \( k > 0 \), one can trivially show that \( \delta_+ > 0 \). Moreover, after some extra work a condition is derived for \( \delta_- \) that is

\[
\delta_- < 0 \iff a^2 < \frac{8}{2 - \mathcal{K}},
\]

which further implies that \( \delta_- < 0 \) for all \( a \leq 2 \), whilst the opposite is not necessarily true.

The generic behaviour of the scalar field throughout the spacetime is of great importance for a forthcoming article [18] looking at the stability and quasinormal mode spectrum of black holes in generalised TeVeS and the relevance of superluminal propagation of scalar waves. Indeed, Bekenstein [3] showed that, in the eikonal approximation, scalar perturbations in the Newtonian limit of the theory travel with velocity \( v = \exp (-2\varphi_\circ) / \sqrt{2} \). This implies that scalar perturbations travel superluminally if and only if \( \varphi_\circ < 0 \). Equation (15) shows that the scalar field necessarily diverges as \( r \to r_\circ \). When \( \delta_\pm > 0 \), the scalar field diverges to positive infinity, and indeed the scalar field is everywhere positive (given \( \varphi_\circ > 0 \)). However, when \( \delta_\pm < 0 \), the scalar field diverges to negative infinity as one approaches the critical radius, \( r_\circ \), implying for some values of the radial coordinate the scalar field is negative. In Fig. 1 the scalar field is shown as a function of the radial coordinate for various values of the coupling parameters and also the spacetime parameter \( a \).

![Figure 1](image_url)

**Figure 1.** Scalar field, \( \varphi \), as a function of the radial coordinate with \( k = 0.003 \) and \( \varphi = 0.003 \) for the black hole spacetimes. The top panel shows various values of \( a \) with \( \mathcal{K} = 0.1 \). The lower panel varies the vector field coupling \( \mathcal{K} := K + K_+ - K_4 \) for \( a = 2 \). For both plots the thick (dashed) lines represent \( \delta_- (\delta_+) \). In the eikonal approximation the velocity of scalar field perturbations is proportional to \( \exp(-2\varphi_\circ) \), implying superluminal and subluminal propagation of perturbations for \( \varphi < 0 \) and \( \varphi > 0 \) respectively.

4. Neutron Stars

Neutron stars in TeVeS were first studied in Lasky, Giannios & Sotani [9] by solving the TOV equations of hydrodynamic equilibrium for a perfect fluid in spherical symmetry. In Lasky [14], I further showed that the neutron star solutions for the original TeVeS theory are also solutions of the generalised TeVeS field equations, with the same association that the coupling parameter \( K \) of TeVeS gets replaced by the combination of parameters \( K + K_+ - K_4 \). The following is a review of the results from [9, 14] on neutron star structure.

We now express the physical metric in Schwarzschild coordinates where the scalar field has been taken into account from equation (1)

\[
d\tilde{s}^2 = -e^{\nu + 2\varphi}dt^2 + e^{-2\varphi} \left( \frac{dr^2}{1 - 2m/r} + r^2d\Omega^2 \right).
\]
Here, the metric components, \( \nu \) and \( m \), and the scalar field are only functions of the radial coordinate. Conservation of stress-energy, prescribed in the physical frame, reduces simply to the radial Euler equation

\[
- \frac{d\tilde{P}}{dr} = - \left( \frac{1}{2} \frac{d\nu}{dr} + \frac{d\varphi}{dr} \right) \left( \tilde{\rho} + \tilde{P} \right).
\]

(19)

The scalar field equation (11) can be once integrated to give

\[
\frac{d\varphi}{dr} = \frac{kGM_{\varphi}e^{-\nu/2}}{4\pi r^2 \sqrt{1 - 2m/r}}, \quad M_{\varphi}(r) := 4\pi \int_0^r \frac{\left( \tilde{\rho} + 3\tilde{P} \right) e^{\nu/2 - 2\varphi} r^2 dr}{\sqrt{1 - 2m/r}}.
\]

(20)

Here, the scalar mass function, \( M_{\varphi} \), has been defined according to [3]. After much algebra (for details see [9]), the remaining field equations can be reduced to two coupled equations which are analogous to the TOV equations in General Relativity

\[
\frac{dm}{dr} \left( 1 - \frac{K}{2} \right) - \frac{Km}{2r} = 4\pi Ge^{-2\varphi} r^2 \left( \tilde{\rho} + 2K\tilde{P} \right) + \left( 1 - \frac{2m}{r} \right) \left[ \frac{2\pi r^2}{k} \left( \frac{d\varphi}{dr} \right)^2 - \frac{Km}{4} \right] \left( 1 + \frac{r}{4} \frac{d\nu}{dr} \right),
\]

(21)

\[
\frac{d\nu}{dr} \left( 1 + \frac{Km}{8} \frac{d\nu}{dr} \right) = \frac{8\pi Ge^{-2\varphi} \tilde{P} r^2 + 2m/r}{r (1 - 2m/r)} + \frac{4\pi r}{k} \left( \frac{d\varphi}{dr} \right)^2.
\]

(22)

It can be shown [9] that taking \( K = k = 0 \) leads to the familiar TOV equations of General Relativity.

To build neutron star models from the above equations one must assume an equation of state relating the density and pressure. For these purposes we used equation of state A from Arnett & Bowers [19]. In Figs. 2 and 3, the ADM mass as a function of the central density and radius for various values of the vector field coupling constant \( K \) is shown. One can immediately see that strong variations of the vector coupling result in large deviations from the General Relativistic predictions. In Lasky, Sotani & Giannios [9] we also showed plots with the effect of varying the scalar field coupling constant. This was shown to have significantly smaller effect on the structure of neutron stars than the vector field parameter.

In general, Figs. 2 and 3 show that the mass and radii of neutron stars in TeVeS and generalised TeVeS are generically smaller than in General Relativity. The most accurate measurements of neutron star masses are from timing observations of binary pulsars, although significant progress is also being made using binaries with white dwarf companions (see Lattimer & Prakash [20] and references therein). A conservative estimate puts neutron star masses as \( \gtrsim 1.5M_\odot \), although the true value is likely considerably higher. Taking this into account, together with variations in possibilities for the equation of state, in Ref. [9] we ruled out the parameter space \( K \gtrsim 1 \), although we noted that this was a conservative estimate.

5. Conclusions and Future Directions

The study of the strong-field regime of TeVeS and its generalisation is in its infancy. Whilst Sotani [21, 22, 23] has looked at oscillations and slow rotations of neutron stars in the original formulation of TeVeS, these results do not carry forth to the generalised version of the theory. Moreover, neutron stars do not provide the best testing ground for theories of gravity in the strong-field regime due to degeneracies associated with uncertainties in the equation of state and more detailed internal structure.

Black holes are, however, clean systems. The “No Hair” theorem of General Relativity implies that a black hole is characterized by only its mass and angular momentum. However,
Figure 2. Central density as a function of the ADM mass for varying values of the vector field coupling constant $K$. Here, the scalar field coupling constant is held constant at $k = 0.03$ and the cosmological values of the scalar field is $\varphi_c = 0.003$.

Figure 3. Mass-radius relation for various values of the vector coupling parameter $K$. Again, $k = 0.03$ and $\varphi_c = 0.003$. For small values of the vector coupling there is little difference in the mass radius relation from the predictions of General Relativity.

any violation of this implies a violation of General Relativity. The solutions exhibiting naked singularities discussed above would have observational differences to black hole spacetimes in General Relativity. A relevant future study is therefore to establish the end state of gravitational collapse to determine if such solutions are physically meaningful. If the end state of gravitational collapse in TeVeS is a black hole, then, under the assumption that the vector field is aligned only in the temporal direction, such a black hole would be indistinguishable from a General Relativistic Schwarzschild black hole. However, the perturbations of such spacetimes would differ to those in General Relativity due to the different field equations as well as the additional degrees of freedom associated with the extra fields. For example, a spherically symmetric black hole in TeVeS would be allowed to emit gravitational waves by way of breathing modes of radiation, a phenomena not possible in General Relativity due to Birkhoff’s theorem. In a forthcoming article [18], we look at perturbations of black holes in generalised TeVeS to determine both stability properties and also differences one would expect in gravitational wave observations of black holes in TeVeS as opposed to General Relativity.

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