Non-abelian action of D0-branes from Matrix theory in the longitudinal 5-brane background

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ABSTRACT

We study one-loop effective action of Berkooz-Douglas Matrix theory and obtain non-abelian action of D0-branes in the background field produced by longitudinal 5-branes. Since these 5-branes do not have D0-brane charge and are not present in BFSS Matrix theory, our analysis provides an independent test for the coupling of D-branes to general weak backgrounds proposed by Taylor and Van Raamsdonk from the analysis of the BFSS model. The proposed couplings appear in the Berkooz-Douglas effective action precisely as expected, which suggests the consistency of the two matrix models. We also point out the existence of the terms which are not given by the symmetrized trace prescription in the Matrix theory effective action.

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1 Introduction

D-branes [1] have played crucial roles in understanding the string dualities, as well as in attempting to give a formulation of M-theory. In weakly-coupled string theory, D-branes appear as solitonic objects which allow a remarkably simple description: Dp-branes are defined as $(p + 1)$-dimensional hypersurfaces which support endpoints of open strings. The fact that D-branes couple to gravity via open-string closed-string interaction suggests that they must be considered as dynamical objects. The open-string massless scalar modes which live on the $(p + 1)$-dimensional world-volume corresponds to the collective coordinates of Dp-branes. A notable feature of D-branes is that when $N$ D-branes are coincident, the transverse motion is described by $N \times N$ matrices rather than just $N$ coordinates, due to the presence of extra massless scalars coming from the strings connecting different D-branes [2]. Understanding the dynamics of D-branes and especially, revealing the consequence of the non-commutativity of matrix-valued coordinates are undoubtedly important for the further clarification of the non-perturbative nature of string theory.

Effective action for a single D-brane is well-understood. A D$p$-brane is described by the $(p+1)$-dimensional Born-Infeld (BI) action plus Chern-Simons (CS) terms, in the low-acceleration limit (where the second derivatives of the fields are neglected). The $(9 - p)$ scalars describe the position of the brane and there are gauge fields corresponding to the U(1) symmetry. BI action is obtained from the condition of the conformal invariance on the string world-sheet and includes all the $\alpha'$ corrections associated to the open-string massless modes [3]. CS terms give the coupling of a D$p$-brane to Ramond-Ramond $(p+1)$-form potential and also to lower $(p - 1, p - 3, \ldots)$-form potentials in the presence of non-trivial configuration of U(1) gauge fields [4, 5].

For the action of multiple D-branes, we have only limited understanding at present. The leading terms of the low-energy effective action of $N$ D$p$-branes in flat space is given by the $D = (p + 1)\ U(N)$ supersymmetric Yang-Mills theory which is obtained by the dimensional reduction from the 10D SYM [2]. Contrary to the case of a single D-brane, non-commutativity of field strengths makes it difficult to obtain effective action to all orders in $\alpha'$, even in flat space. It was argued by Tseytlin that the part of the action independent of the commutator of the field strengths is given by a non-abelian generalization of BI action in which the trace for the gauge group is taken after symmetrizing the product [6]. However, full form of the action is not understood. The contribution at each order in $\alpha'$ should be determined from the analysis of scattering amplitude of the open-string massless states, as in refs.[7, 8]. Indeed, there are suggestions that there must be corrections to the Tseytlin’s action at the sixth order of field strengths [9].
How to couple multiple D-branes to curved background is further unclear. As mentioned above, the collective coordinates of D-branes are promoted to matrices, thus the background geometry should be regarded as a function of matrices. Principle for writing an action in such circumstances is obscure, despite some attempts \[10\]. In ref.\[11\], a generalization of the notion of the general coordinate transformation is discussed, but the constraint from that symmetry does not seem to be enough for determining the action unambiguously. An approach taken in refs.\[12, 18\] is to treat the background fields as an expansion around a point in spacetime, in which the coordinates in the expansion are replaced by matrices (‘non-abelian Taylor expansion’). Scattering amplitudes of a closed-string state and open-string states studied in refs.\[13, 14\] give some evidence for the consistency of the above approach, but the form of the action at higher orders of the expansion, and especially, how to order the matrices are not clear. Important observation about the action of multiple D-branes is that non-abelian version of CS terms should allow the coupling of D\(_p\)-branes to RR forms which are of higher degree than (\(p + 1\))-forms by non-commutative configurations of scalar fields \[12, 18\].

Matrix theory \[13\], which is the proposal for the exact definition of M-theory in the light-cone frame, provides an alternative way to study interactions of D-branes. In a series of papers by Kabat and Taylor \[17\] and by Taylor and Van Raamsdonk \[18, 19, 20\], detailed study of Matrix theory effective action was performed. In refs.\[17, 18\], by reinterpreting the one-loop effective potential of Matrix theory as the result of tree-level supergravity interactions, Matrix theory operators which couple to the supergravity fields were identified. Taylor and Van Raamsdonk proposed the Matrix theory action in general weak backgrounds using those couplings \[18\]. Further, D0-brane action in weak 10D background was obtained \[19\], following the scaling argument due to Seiberg \[21\] and Sen \[22\] which relates Matrix theory to 10D type IIA string theory. Applying T-duality, similar couplings for general D\(_p\)-brane were given in ref.\[20\]. Consistency of the couplings which were obtained in this way has been confirmed in several contexts: the D9-brane action is indeed 10-dimensional Lorentz covariant \[20\]; the coupling of D0-branes to the background field is consistent with the ones obtained from the matrix-regularization of the supermembrane in a curved background \[23\]; and absorption cross sections of dilaton partial waves by D3-branes which were evaluated by semi-classical gravity are reproduced by the gauge theory using the above couplings \[24\].

In this paper, we study a variant of Matrix theory which was proposed by Berkooz and Douglas as the definition of M-theory in the light-cone frame in the presence of longitudinal 5-branes \[16\]. As reviewed in section 2, Berkooz-Douglas (BD) Matrix theory has extra degrees of freedom compared to the original Matrix theory. We integrate them out at one-loop order and obtain effective action for the D0-brane degrees of freedom in the
background fields produced by the longitudinal 5-branes. Since the 5-branes which we are discussing have no D0-brane charge, they are not present in the ordinary Matrix theory. Thus, our analysis can be regarded as an independent test of the Taylor and Van Raamsdonk’s proposal for the D-brane action in weak background fields, and also as a check of the consistency between the two formulations of Matrix theory. We confirm that the couplings expected from the above proposal indeed exist in the effective action of BD Matrix theory. In addition to the proposed couplings, we find corrections involving extra commutators in the Matrix theory effective action. We also discuss the consistency of the effective action of BD Matrix theory and a proposal of Myers [12] for the D-brane action in curbed space.

This paper is organized as follows. In section 2, we review BD Matrix theory and set notations. In section 3, one-loop integration of the massive fields is performed. In section 4, we compare the effective action obtained in section 3 with Taylor and Van Raamsdonk’s proposal for the Matrix theory action in weak background fields. In section 5, we comment on the interpretation of our result from the perspective of 10D string theory. In section 6, we conclude and discuss directions for the future works.

2 Matrix theory in the longitudinal 5-brane background

According to the Matrix-theory conjecture of Banks, Fischler, Shenker and Susskind (BFSS) [15], M-theory in the infinite momentum frame (IMF) is defined by the large $N$ limit of the supersymmetric quantum mechanics with $U(N)$ gauge symmetry, which is the effective action of $N$ D0-branes in the low-energy limit. A D0-brane has a unit positive momentum in the longitudinal (11-th) direction and is a natural candidate for the basic constituent in the IMF. In ref. [16], Berkooz and Douglas proposed a formulation of M-theory in the presence of longitudinal 5-branes. Longitudinal 5-branes, which fill the 11-th direction and have zero longitudinal momentum in their ground state, are considered as non-trivial background in the IMF. Note that the ‘5-brane in the ground state’ does not have D0-brane charge and cannot be constructed in ordinary Matrix theory. Based on the philosophy that different vacua give rise to different Hamiltonians in the IMF in general, modification of Matrix theory was conjectured. Compared to the original BFSS Matrix theory, this theory has extra degrees of freedom, and has only half of the supersymmetries.

Precisely, the action of Berkooz-Douglas (BD) Matrix theory is the 0-0 and 0-4 string sectors of the SYM describing the D0-D4 bound state, which is given by the dimensional
reduction of the D=6, \( \mathcal{N} = 1 \) SYM. In the case of \( N \) D0-branes and \( N_4 \) D4-branes, the 0-0 sector fields, which are the degrees of freedom of the original Matrix theory, are in adjoint rep. of \( U(N) \). The 0-4 sector fields are the hypermultiplets of the 6D theory which consist of bosons with 4 real components and fermions with 8 real components, both of which transform as the bi-fundamental rep. of \( U(N) \times U(N_4) \). We consider the BD Matrix theory as M-theory in the presence of \( N_4 \) longitudinal 5-branes, which is compactified in the light-like direction \( x^- \) with total longitudinal momentum \( P_- = N/R \) (where \( R = g_s \ell_s \)), following the usual DLCQ interpretation of finite \( N \) Matrix theory [23].

The action is given as follows.

\[
S = S_0 + S_5
\]

\[
S_0 = \frac{1}{g_s \ell_s} \int dt \text{Tr} \left\{ \frac{1}{2} D_0 X_i D_0 X_i + \frac{1}{4 \lambda^2} [X_i, X_j]^2 + \frac{i}{2} \Theta \Gamma^0 D_0 \Theta + \frac{1}{2 \lambda^2} \Theta \Gamma^i [\Theta, X_i] \right\}
\]

\[
S_5 = \int dt \left\{ (D_0 v_I)^\dagger D_0 v_I - \frac{1}{\lambda^2} v_I^\dagger (X_a - Y_a)^2 v_I - i \chi^\dagger D_0 \chi - \frac{1}{\lambda} \chi^\dagger \gamma^0 \gamma^a (X_a - Y_a) \chi
\]

\[
- \frac{1}{2 \lambda^2} \left( v_I^\dagger \left[ [\phi_1, \phi_1] + [\phi_2, \phi_2] \right] v_I - v_2^\dagger \left[ [\phi_1, \phi_1] + [\phi_2, \phi_2] \right] v_2
\]

\[
- 2v_2^\dagger ([\phi_1, \phi_2]) v_1 + 2v_1^\dagger ([\phi_1, \phi_2]) v_2 \right) \right\}
\]

where \( S_0 \) is the part containing only the 0-0 sector, which is the same as the BFSS action, and \( S_5 \) is the additional part containing 0-4 sector.

Let us explain the notations and conventions. We use the indices \( i, j = 1, \ldots, 9 \) for the spatial directions in the 10D; \( m, n = 1, \ldots, 4 \) for the spatial directions tangent to the 5-branes except for \( x^{10} \) (i.e. tangent to the D4-brane); \( a, b = 5, \ldots, 9 \) for the directions transverse to the 5-branes. Length scale is given by \( \lambda = 2 \pi \ell_s^2 \). The D0-brane fields \( X_i \) and \( \Theta \) are \( N \times N \) Hermitian matrices where \( \Theta \) satisfy the 10D Majorana-Weyl condition. Covariant derivatives for these fields are defined as \( D_0 X_i = \partial_0 X_i + i [A_0, X_i] \). We also use complex combinations of \( X_m \) which are defined as \( \phi_1, \phi_2 = (X_1 + i X_2, X_3 + i X_4) \) with \( \phi_1 = \phi_1^1 \) and \( \phi_2 = \phi_1^2 \). The 0-4 sector fields are complex bosons \( v_I \) \( (I = 1, 2) \) and complex fermions \( \chi \) which satisfy the 6D Weyl condition \( (\bar{\gamma} \chi = \chi \text{ where } \bar{\gamma} = \gamma_0 \gamma_5 \ldots \gamma_9) \) Covariant derivatives for the 0-4 fields are defined as \( D_0 v_I = \partial_0 v_I + i A_0 v_I \). We will put the indices for the bi-fundamental rep. of \( U(N) \times U(N_4) \) as \( v_I^{A\bar{A}} \) and \( \chi^{A\bar{A}} \) \( (A = 1, \ldots, N \text{ and } \bar{A} = 1, \ldots, N_4) \) when necessary. \( Y_a \) are \( N_4 \times N_4 \) matrices, i.e. singlets under \( U(N) \), which specifies the positions of the 5-branes. When 5-branes are coincident, which is the case treated in this paper, \( Y_a \) is proportional to identity matrix \( I_{N_4 \times N_4} \). In this case, \( Y_a \) can be absorbed into the definition of \( X_a \), so we set \( Y_a = 0 \) hereafter.

Note that we have not adopted a convention using the SU(2) Majorana spinors, which may be familiar in the literatures (such as refs. [16, 27]). It is because we prefer unconstrained complex spinors to perform the loop calculations. The fact that SO(4) symmetry
in the $X_m$ direction is not manifest in the above expressions is a consequence of that choice, but the result of the loop calculation can of course be written in SO(4) covariant way. Also note that we have explicitly written only the part of the action which is needed for the one-loop integration of $v$ and $\chi$. There are also the $v^4$-terms and the $v\Theta\chi$-terms. The $v^4$-terms are proportional to $g_s$ in our normalization, and give rise to higher-loop corrections. Half of the components of $\Theta$ (which have definite 6D chirality $\bar{\gamma}\Theta = -\Theta$) appear in the $v\Theta\chi$-terms. The action (2.1) is invariant under the SUSY transformation with a 6D Weyl spinor parameter ($\bar{\gamma}\eta = -\eta$), and the number of real supercharges is eight.

3 One-loop effective action

3.1 Method for the perturbative calculation

In this section, we calculate effective action of the D0-brane degrees of freedom $X_i$ in BD Matrix theory by integrating out $v$ and $\chi$ in eq.(2.3) at one-loop order. We use Euclidean version of the action by transforming $t \rightarrow -i\tau$, $A_0 \rightarrow iX_0/\lambda$ and $S \rightarrow -iS$. We evaluate the one-loop determinant

$$S_{\text{eff}} = S_0 - \delta^{(1)},$$

$$\delta^{(1)} = \int d\tau \left[ -\ln \det K_{\text{bos}} + \ln \det K_{\text{fermi}} \right]$$

where $K_{\text{bos}}$ and $K_{\text{fermi}}$ are kernels of quadratic terms of complex bosons $v$ and fermions $\chi$, respectively.

In this paper, we take the matrix background as

$$\Theta = 0$$

$$(X_0, X_m, X_a) = (\hat{X}_0, \hat{X}_m, r_a + \hat{X}_a)$$

where $\hat{X}_i$ are general time dependent matrices and $r_a$ are constants proportional to the identity matrix. We divide the part of the action which is quadratic in $v$ and $\chi$ into free and interaction part as follows

$$S_{\text{free}} = \int d\tau \left\{ v^\dagger \left( -\partial_\tau^2 + \frac{r^2}{\lambda^2} \right) v + \chi^\dagger \left( -\partial_\tau + \frac{1}{\lambda} \bar{\gamma}^a r_a \right) \chi \right\}$$

$$S_{\text{int}} = \int d\tau \left\{ \frac{1}{\lambda^2} v_1^\dagger (2r_a \hat{X}_a + \hat{X}_a^2 - 2i\lambda \hat{X}_0 \partial_\tau)v_1 + \frac{1}{2\lambda^2} \left[ v_1^\dagger (\phi_1, \bar{\phi}_1) + [\phi_2, \bar{\phi}_2] \right] v_1 - v_2^\dagger (\phi_1, \bar{\phi}_1) + [\phi_2, \bar{\phi}_2] v_2 
- 2v_2^\dagger [\bar{\phi}_1, \phi_2] v_1 + 2v_1^\dagger [\phi_1, \bar{\phi}_2] v_2 + \frac{1}{\lambda} \chi^\dagger (\bar{\gamma}^a \hat{X}_a - i\hat{X}_0) \chi \right\}$$

$$\equiv \int d\tau \left\{ \frac{1}{\lambda^2} v_I^\dagger V_{IJ}(\tau) v_J + \frac{1}{\lambda} \chi^\dagger (\bar{\gamma}^a \hat{X}_a - i\hat{X}_0) \chi \right\}.$$
Here we take $\gamma^a = \gamma^0 \gamma^a$ as $4 \times 4$ matrices acting on 6D Weyl spinors ($\overline{\gamma}_\chi = \chi$) which have 4 complex components. Also note

$$\gamma^{a_1 a_2 a_3 a_4 a_5} = \gamma \epsilon_{a_1 a_2 a_3 a_4 a_5} = \epsilon_{a_1 a_2 a_3 a_4 a_5}$$

(3.7)

with $\epsilon_{56789} = 1$.

We adopt a method of calculation which is conceptually most straightforward: we evaluate one-loop diagrams with suitable number of vertex insertions, treating the vertices as an expansion in derivatives. Our method closely follows that of ref. [18] where the one-loop integration of off-diagonal blocks in BFSS Matrix theory is performed. We will set $\tilde{X}_0 = 0$ in all the expressions in the following. Since our calculation preserves gauge invariance, we can recover the dependence on $\tilde{X}_0 (A_0)$ by simply replacing $\partial_r$ with $D_r$ in the result.

First, we consider contribution from bosons $v$ to the effective action. Propagators are determined from $S_{\text{free}}$ as

$$\left(-\partial^2 + \frac{1}{\lambda^2} r^2\right) \langle v_{I, AA}(\tau) v_{J, BB}^\dagger(\tau') \rangle = \delta^{IJ} \delta^{AB} \delta \tilde{A} B \delta (\tau - \tau'),$$

$$\langle v_{I, AA}(\tau) v_{J, BB}^\dagger(\tau') \rangle = \delta^{IJ} \delta^{AB} \delta \tilde{A} B \int \frac{dk}{2\pi} \frac{e^{ik(\tau - \tau')}}{k^2 + r^2/\lambda^2} \equiv \delta^{IJ} \delta^{AB} \delta \tilde{A} B \Delta (\tau - \tau').$$

(3.8)

The bosonic part of $\delta^{(1)}$ is given as

$$\delta^{\text{bos}} = - \ln \det(K_{\text{bos}}^{(\text{free})} + V)$$

$$= \Gamma^{\text{bos}} - \ln \det K_{\text{bos}}^{(\text{free})}$$

(3.9)

where

$$\Gamma^{\text{bos}} = \Tr \sum_{n=1}^{\infty} \frac{(-1)^n}{n} [V(K_{\text{bos}}^{(\text{free})})^{-1}]^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{\lambda^{2n}} \int d\tau_1 \cdots d\tau_n \Tr[V_{I_1 I_2}(\tau_1) V_{I_2 I_3}(\tau_2) \cdots V_{I_n I_1}(\tau_n)]$$

$$\times \Delta (\tau_1 - \tau_2) \Delta (\tau_2 - \tau_3) \cdots \Delta (\tau_n - \tau_1)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{\lambda^{2n}} \sum_{D_i = 0}^{\infty} \int d\tau \Tr \left[V_{I_1 I_2}(\tau) V_{I_2 I_3}(\tau) \cdots V_{I_n I_1}(\tau)\right]$$

$$\times \prod_{i=1}^{n} \int \frac{dk_i}{2\pi} \prod_{i=2}^{n} \left(\frac{d\sigma_i}{D_i!} \right) \frac{e^{-ik_1 \sigma_2}}{k_1^2 + r^2/\lambda^2} \frac{e^{-ik_2 (\sigma_2 - \sigma_3)}}{k_2^2 + r^2/\lambda^2} \cdots \frac{e^{ik_n \sigma_n}}{k_n^2 + r^2/\lambda^2}.$$
To obtain the last expression, we rewrote the vertices $V_{I_{i_1}}(\tau_1)$ using Taylor expansion around a reference point $\tau_1$. The superscript $(D_i)$ means the $D_i$-th derivative in $\tau$ and $\sigma_i \equiv \tau_i - \tau_1$. Performing the integration over $\sigma_i$ and $k_i$ ($i = 2, \ldots, n$), $\Gamma^{\text{bos}}$ reads

$$
\Gamma^{\text{bos}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{D_i=0}^{\infty} \left( \prod_{i=2}^{n} \frac{1}{D_i!} \right) \int d\tau \ Tr \left[ V_{I_{i_2}}(\tau) V_{I_{i_3}}^{(D_2)}(\tau) \cdots V_{I_{i_nI_1}}^{(D_n)}(\tau) \right] \times \frac{\lambda^{D-1}}{r^{2n+D-1}} \int \frac{dk}{2\pi k^2 + 1} (i\partial_k)^{D_2} \left\{ \frac{1}{k^2 + 1} \left[ (i\partial_k)^{D_3} \frac{1}{k^2 + 1} \left( \cdots (i\partial_k)^{D_n} \frac{1}{k^2 + 1} \right) \right] \right\} \right) (3.11)
$$

where $D = \sum D_i$. Note that the terms with odd number of derivatives do not contribute to $\Gamma^{\text{bos}}$, for they are proportional to $\int dk k^{2n+1}/(k^2 + 1)^l$ which are vanishing.

The fermionic propagator is given by

$$
\langle \chi_{\alpha,\AA}(\tau) \chi_{\beta,\BB}^\dagger(\tau') \rangle = \delta^{AB} \delta^{\AA\BB} \int \frac{dk}{2\pi k^2 + r^2} (\gamma' + ik)_{\alpha\beta} = \delta^{AB} \delta^{\AA\BB} (\partial_\tau + \gamma')_{\alpha\beta} \Delta(\tau - \tau') \quad (3.12)
$$

where $r' = r/\lambda$ and $\gamma' = \gamma^a r_a/\lambda$. Contribution from the fermionic loop to the effective action is obtained in the same way as in the bosonic case.

$$
\delta^{\text{fermi}} = \ln \det(K^{(\text{free})}) + V = \Gamma^{\text{fermi}} + \ln \det(K^{(\text{free})}) \quad (3.13)
$$

$$
\Gamma^{\text{fermi}} = - \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{1}{\lambda^n} \sum_{D_i=0}^{\infty} \left( \prod_{i=2}^{n} \frac{1}{D_i!} \right) \int d\tau \ Tr \left[ \hat{X}_{a_2}(\tau) \hat{X}_{a_2}^{(D_2)}(\tau) \cdots \hat{X}_{a_n}^{(D_n)}(\tau) \right] \times \int \frac{dk}{2\pi} \ Tr \left\{ \gamma^a_1 \gamma' + ik \gamma^a_2 (i\partial_k)^{D_2} \left[ \gamma^a + ik \gamma^a_3 (i\partial_k)^{D_3} \cdots \gamma^a_{a_n} (i\partial_k)^{D_n} \frac{\gamma' + ik}{k^2 + r^2} \right] \right\}. \quad (3.14)
$$

where the trace in the first line of eq.(3.14) is for gauge indices and the one in the second line is for spinor indices.

As a consequence of the supersymmetry, one-loop determinant of the free propagator of bosons and fermions cancel each other ($-\ln \det(K^{(\text{free})}_{\text{bos}}) + \ln \det(K^{(\text{free})}_{\text{fermi}}) = 0$), thus the effective action is given by $\delta^{(1)} = \Gamma = \Gamma^{\text{bos}} + \Gamma^{\text{fermi}}$. We obtain effective action in the expansion with respect to the number of vertices and of derivatives.

The region where our expansion is good is when D0-branes are slowly moving and nearly coincident, as explained below. Firstly, the expansion in derivatives is justified when $\lambda \partial_\tau \hat{X}_i / r^2$ is small. It is because each derivative is associated with a factor $\lambda/r$ and each $\hat{X}_i$ is associated with $1/r$, as we see from eqs.(3.11) and (3.14). The expansion in $\hat{X}_i$ is good when $\hat{X}_i / r$ is small. The one-loop approximation is justified when $g_s \lambda^{3/2}/r^3$ is small: As we go to one higher loop, $\nu^4$ vertex is inserted once which contributes a
factor \( g_s \), and two extra propagators and one extra momentum integral are needed which contribute a factor \( \lambda^3/r^2 \).\[72x728\]

In the following, we present the result of the calculation for the terms containing up to two derivatives. Explicitly, up to fourth order in \( \hat{X}_i \) for \( D = 2 \) and to sixth order in \( \hat{X}_i \) for \( D = 0, 1 \). We denote by \( \Gamma((\hat{X}_a)^{N_a}(\hat{X}_m)^{N_m}, D) \) the term which contains \( N_a \) \( \hat{X}_a \)'s and \( N_m \) \( \hat{X}_m \)'s and \( D \) derivatives.

### 3.2 Potential terms

First, we consider terms with no derivatives. For the terms with only \( \hat{X}_a \)'s, we have

\[
\Gamma((\hat{X}_a)^{N_a}, D = 0) = \frac{N_4}{8\lambda r^3} \int d\tau \text{STr}[\hat{X}_{a_1}, \hat{X}_{a_2}]^2 \\
- \frac{3N_4 r_\alpha}{8\lambda r^5} \int d\tau \text{STr}(\hat{X}_a [\hat{X}_{a_1}, \hat{X}_{a_2}]^2) \\
- \frac{3N_4}{16\lambda r^3} \int d\tau \text{Tr}((\hat{X}_a)^2[\hat{X}_{a_1}, \hat{X}_{a_2}]^2) \\
+ \frac{15N_4 r_\alpha r_{\alpha_2}}{16\lambda r^7} \int d\tau \text{STr}(\hat{X}_a, \hat{X}_{a_2}[\hat{X}_{b_1}, \hat{X}_{b_2}]^2) \\
+ \frac{N_4}{8\lambda r^3} \int d\tau \text{Tr}([\hat{X}_{a_1}, \hat{X}_{a_2}][\hat{X}_{a_2}, \hat{X}_{a_3}][\hat{X}_{a_3}, \hat{X}_{a_1}]) \\
+ O((\hat{X}_a)^7)
\]

where \( \text{STr}(\cdots) \) stands for symmetrized trace, which means that trace operation is taken after symmetrizing the ordering of all \( [\hat{X}_{a_1}, \hat{X}_{a_2}], \hat{X}_a \) and \( \hat{X}_a \) in the parenthesis. Note that for the first two lines of the above equation, there is no difference between \( \text{Tr} \) and \( \text{STr} \). Vanishing of the terms containing fewer number of \( X_a \)'s can easily be proved.

Now we consider the terms containing \( \hat{X}_m \). As we see from (3.3), \( \hat{X}_m \) appear only in the vertex for bosons which is of the form \( \sim v[\hat{X}_{m_1}, \hat{X}_{m_2}]v \). Thus,

\[
\Gamma((\hat{X}_m)^{2N_m+1}(\hat{X}_a)^{N_a}, D) = 0
\]

holds generally. Also,

\[
\Gamma((\hat{X}_m)^{2}(\hat{X}_a)^{N_a}, D) = 0
\]

for the contribution from two bosons cancel each other in this case, which is due to the relation \( V_{11} = -V_{22} \). Thus non-zero contribution are only from \( \Gamma((\hat{X}_m)^{2k}(\hat{X}_a)^{N_a}, D = 2d) \) where \( k = 2, 3, \ldots \) and \( d = 0, 1, \ldots \). The result is summarized as

\[
\Gamma((\hat{X}_m)^{N_m\neq0}(\hat{X}_a)^{N_a}, D = 0) = \Gamma((\hat{X}_m)^{N_m\neq0}(\hat{X}_a)^{N_a}, D = 0)_B \\
\Gamma((\hat{X}_m)^{N_m\neq0}(\hat{X}_a)^{N_a}, D = 0)_C + O((\hat{X}_i)^7).
\]

\[72x728\]These conditions are the same as the one for the perturbation in BFSS Matrix theory.\[72x728\]
where the first part contains no epsilon tensor

\[
\Gamma((\dot{X}_m)^{N_m\neq0}(X)^{N_a}, D = 0)_B = -\frac{N_4}{8\lambda} \frac{1}{r^3} \int d\tau STr[\dot{X}_{m_1}, \dot{X}_{m_2}]^2
\]

\[
+ \frac{3N_4}{8\lambda} \frac{r_a}{r^5} \int d\tau STr(\dot{X}_a [\dot{X}_{m_1}, \dot{X}_{m_2}])^2
\]

\[
+ \frac{3N_4}{16\lambda} \frac{1}{r^7} \int d\tau Tr((\dot{X}_a)^2 [\dot{X}_{m_1}, \dot{X}_{m_2}])
\]

\[
- \frac{15N_4}{16\lambda} \frac{r_{a_1} r_{a_2}}{r^7} \int d\tau STr(\dot{X}_{a_1}^{\perp} \dot{X}_{a_2} [\dot{X}_{m_1}, \dot{X}_{m_2}])
\]

\[
- \frac{N_4}{8\lambda} \frac{1}{r^5} \int d\tau Tr([\dot{X}_{m_1}, \dot{X}_{m_2}] [\dot{X}_{m_2}, \dot{X}_{m_3}] [\dot{X}_{m_3}, \dot{X}_{m_1}])
\]

(3.24)

and the second part has indices contracted using epsilon tensor.

\[
\Gamma((\dot{X}_m)^{N_m\neq0}(X)^{N_a}, D = 0)_C = \]

\[
+ \frac{3N_4}{16\lambda} \frac{r_a}{r^5} \int d\tau STr(\dot{X}_a [\dot{X}_{m_1}, \dot{X}_{m_2}]) \epsilon_{m_1 m_2 m_3 m_4}
\]

\[
+ \frac{3N_4}{32\lambda} \frac{1}{r^5} \int d\tau Tr((\dot{X}_a)^2 [\dot{X}_{m_1}, \dot{X}_{m_2}]) \epsilon_{m_1 m_2 m_3 m_4}
\]

\[
- \frac{15N_4}{32\lambda} \frac{r_{a_1} r_{a_2}}{r^7} \int d\tau STr(\dot{X}_{a_1}^{\perp} \dot{X}_{a_2} [\dot{X}_{m_1}, \dot{X}_{m_2}]) \epsilon_{m_1 m_2 m_3 m_4}
\]

\[
- \frac{N_4}{8\lambda} \frac{1}{r^5} \int d\tau Tr([\dot{X}_{m_1}, \dot{X}_{m_2}] [\dot{X}_{m_2}, \dot{X}_{m_3}] [\dot{X}_{m_3}, \dot{X}_{m_1}]) \epsilon_{m m_1 m_2 m_3}
\]

(3.29)

where \(\epsilon_{1234} = 1\).

### 3.3 Kinetic terms

We summarize the results of terms with two derivatives. For the terms with only \(\dot{X}_a\)'s,

\[
\Gamma((\dot{X}_a)^{N_a}, D = 2) = -\frac{N_4\lambda}{4} \frac{1}{r^3} \int d\tau STr(\dot{X}_a \dot{X}_a)
\]

\[
+ \frac{3N_4\lambda}{4} \frac{r_a}{r^5} \int d\tau STr(\dot{X}_a \dot{X}_b \dot{X}_b)
\]

\[
+ \frac{3N_4\lambda}{8} \frac{1}{r^7} \int d\tau Tr(\dot{X}_a \dot{X}_b \dot{X}_b \dot{X}_b)
\]

\[
- \frac{15N_4\lambda}{8} \frac{r_{a_1} r_{a_2}}{r^7} \int d\tau STr(\dot{X}_a \dot{X}_b \dot{X}_b [\dot{X}_{a_1}, \dot{X}_{b_2}])
\]

\[
+ \frac{3N_4\lambda}{8} \frac{1}{r^5} \int d\tau Tr(\dot{X}_a \dot{X}_b [\dot{X}_{a_1}, \dot{X}_{b_2}])
\]

\[
- \frac{N_4\lambda}{32} \frac{1}{r^5} \int d\tau Tr((\dot{X}_a)^2)
\]

\[
+ O(\dot{X}_a^5)
\]

(3.33)
where the ‘dot’ denotes derivative with $\tau$. For the terms including $\hat{X}_m$, we have

$$
\Gamma((\hat{X}_m)^N_m \neq 0, \hat{X}_a)^N_a, D = 2) = \frac{N_4 \lambda}{32} \frac{1}{r^5} \int d\tau \text{Tr}(\partial_\tau [\hat{X}_m, \hat{X}_a])^2 \tag{3.39}
$$

$$
+ \frac{N_4 \lambda}{64} \frac{1}{r^3} \int d\tau \text{Tr}(\partial_\tau [\hat{X}_{m_1}, \hat{X}_{m_2}] \partial_\tau [\hat{X}_{m_3}, \hat{X}_{m_4}]) \epsilon_{m_1 m_2 m_3 m_4} \tag{3.40}
$$

$$
+ O(\partial^2_\tau (\hat{X}_i))^5).
$$

### 3.4 Terms with one derivative

Now we deal with terms with one derivative. They only come from fermionic loops $\Gamma^{\text{fermi}}$, and as a result, there are no contribution from $\hat{X}_m$ for they do not appear in the vertices for fermions. The result for each number $N_a$ of $\hat{X}_a$’s is as follows.

The terms with $N_a \leq 3$ vanish. For $N_a \geq 4$, we have

$$
\Gamma((\hat{X}_a)^4, D = 1) = \frac{3 N_4 r_a}{8} \frac{1}{r^5} \int d\tau \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4}) \epsilon_{a_1 a_2 a_3 a_4} \tag{3.41}
$$

$$
\Gamma((\hat{X}_a)^5, D = 1) = \frac{N_4 r_a}{4} \frac{1}{r^7} \int d\tau \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4} \hat{X}_{a_5})
$$

$$
\times (-2 r_{a_1} \epsilon_{a_2 a_3 a_4 a_5} + 2 r_{a_1} \epsilon_{a_2 a_3 a_4 a_5} + r_{a_2} \epsilon_{a_3 a_4 a_5 a_6} + r_{a_3} \epsilon_{a_2 a_4 a_5 a_6}) \tag{3.42}
$$

$$
\Gamma((\hat{X}_a)^6, D = 1) = \frac{N_4}{16} \frac{1}{r^7} \int d\tau \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4} \hat{X}_{a_5} \hat{X}_{a_6})
$$

$$
\times \left\{ \frac{35}{48} \frac{r_a}{r^9} (r_{a_1} r_{a_2} \epsilon_{a_3 a_4 a_5 a_6} + r_{a_3} r_{a_4} \epsilon_{a_1 a_2 a_3 a_4 a_6} + r_{a_1} r_{a_5} \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6}
$$

$$
+ r_{a_1} r_{a_2} \epsilon_{a_3 a_4 a_5 a_6} + r_{a_2} r_{a_3} \epsilon_{a_1 a_2 a_4 a_5 a_6} + r_{a_1} r_{a_3} \epsilon_{a_2 a_3 a_4 a_5 a_6}
$$

$$
- \frac{5}{16} \frac{r_a}{r^7} \epsilon_{a_1 a_2 a_3 a_4 a_5 a_6}
$$

$$
- \frac{5}{48} \frac{r_a}{r^7} (\delta_{a_1 a_2} \epsilon_{a_3 a_4 a_5 a_6} + \delta_{a_3 a_4} \epsilon_{a_1 a_2 a_5 a_6} + \delta_{a_4 a_5} \epsilon_{a_1 a_2 a_3 a_6})
$$

$$
+ \delta_{a_1 a_2} \epsilon_{a_3 a_4 a_5 a_6} + \delta_{a_2 a_3} \epsilon_{a_1 a_2 a_4 a_5 a_6} + \delta_{a_3 a_4} \epsilon_{a_1 a_2 a_3 a_5 a_6}) \right\} \tag{3.43}
$$

$$
- \frac{5 N_4 r_a}{16} \frac{1}{r^7} \int d\tau \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4} \hat{X}_{a_5} \hat{X}_{a_6})
$$

$$
\times (\delta_{a_1 a_2} \epsilon_{a_3 a_4 a_5 a_6} + \delta_{a_3 a_4} \epsilon_{a_1 a_2 a_3 a_5 a_6} - \delta_{a_1 a_4} \epsilon_{a_2 a_3 a_4 a_5 a_6}
$$

$$
- \delta_{a_2 a_3} \epsilon_{a_1 a_2 a_4 a_5 a_6} - \delta_{a_1 a_3} \epsilon_{a_2 a_3 a_4 a_5 a_6}) \tag{3.44}
$$

Note that the last two terms of $\Gamma((\hat{X}_a)^6, D = 1)$ have the same index structures. We have divided them for future purpose.
4 Consistency with the Taylor and Van Raamsdonk’s couplings

The effective action of BD Matrix theory which was obtained in the previous section gives the D0-brane (Matrix theory) action in the background of longitudinal 5-branes. We shall compare it with the couplings to general weak background fields which was proposed by Taylor and Van Raamsdonk [18] from the analysis of BFSS Matrix theory. In the first subsection, we explain the proposal of ref.[18] and show that BD Matrix theory effective action is consistent with it, at the leading order in the derivatives of backgrounds. In the second subsection, subleading (higher-moment) couplings are analyzed in detail. In the last subsection, we discuss the implication of our result for the consistency between BD and BFSS matrix models.

4.1 Agreement at the leading order

We have obtained one-loop effective action for the background matrices \( X \) of the form

\[
(X_m, X_a) = (\hat{X}_m, r_a + \hat{X}_a),
\]

as an expansion in the time-derivatives and in \( \hat{X}_i \). As a result of the decomposition of \( X_i \), the background fields produced by the 5-branes should appear as an expansion around a transverse position \( r_a \). We regard the following part of the effective action (rotated back to Minkowski signature) as the leading terms of the expansion.

\[
S = \frac{1}{g_s \ell_s} \int dt \text{Tr} \left\{ \frac{1}{2} \dot{\hat{X}}_m \dot{\hat{X}}_m + \frac{1}{2} (1 + \frac{k}{r^3}) \dot{\hat{X}}_a \dot{\hat{X}}_a + \frac{1}{4} k (1 - \frac{k}{r^3}) [\hat{X}_m, \hat{X}_n] [\hat{X}_m, \hat{X}_n] \\
+ \frac{1}{2} \lambda^2 [\hat{X}_a, \hat{X}_m] [\hat{X}_a, \hat{X}_m] + \frac{1}{4} \lambda^2 (1 + \frac{k}{r^3}) [\hat{X}_a, \hat{X}_b] [\hat{X}_a, \hat{X}_b] \right\} \\
- i \frac{3}{4} \frac{k r_a}{g_s \ell_s} \epsilon_{a_1 ... a_4} \int dt \text{Tr} \{ \hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4} \} \\
+ i \frac{3}{2} \frac{k r_a}{g_s \ell_s} \lambda^2 \epsilon_{m_1 ... m_4} \int dt \text{Tr} \{ \hat{X}_a \hat{X}_m \hat{X}_n \hat{X}_{m_1} \hat{X}_{m_2} \hat{X}_{m_3} \hat{X}_{m_4} \}
\]

where \( k = \pi N c g_s \ell_s^3 \).

To discuss consistency with the general form of couplings given in ref.[18], we first recall that the longitudinal 5-brane solution is given as

\[
ds^2 = H^{-1/3}(-dx^0 dx^0 + dx^{10} dx^{10} + \eta_{mn} dx^m dx^n) + H^{2/3} \delta_{ab} dx^a dx^b
\]

\[
F^{(4)}_{a_1 ... a_4} = -\epsilon_{a_1 ... a_4 b} \partial_b H
\]

where \( H \) is a harmonic function defined by

\[
H = 1 + \frac{k}{r^3}
\]

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with \( r^2 = (x^a)^2 \). Field strength \( F^{(4)} \) is equivalently expressed by its dual

\[
F^{(7)}_{0m_1...m_4a10} = -\epsilon_{m_1...m_4} \partial_a H^{-1}.
\]

The 5-brane is an electric source for the 6-form potential.

Taylor and Van Raamsdonk’s proposal for the Matrix theory action in a weakly curved background is given as follows [18]:

\[
S = S_0 + \int dt \left\{ \frac{1}{2} h_{MNT}T^{MN} + C^{(3)}_{MNP}J^{MNP} + C^{(6)}_{MNPQRS}M^{MNPQRS} \right\}
\]

\[
+ \sum_{n=1}^{\infty} \frac{1}{n!} \int dt \left\{ \frac{1}{2} \partial_{i_1} \ldots \partial_{i_n} h_{MNT;{i_1...i_n}} + \partial_{i_1} \ldots \partial_{i_n} C^{(3)}_{MNP}J^{MNP;i_1...i_n} \right\}
\]

\[
+ \partial_{i_1} \ldots \partial_{i_n} C^{(6)}_{MNPQRS}M^{MNPQRS;i_1...i_n} \right\} (4.4)
\]

where \( M, N = 0, 1, \ldots, 10 \) are the 11D indices. Here, \( T^{MN} \), \( J^{MNP} \) and \( M^{MNPQRS} \) are the energy-momentum tensor, ‘membrane current’ and ‘5-brane current’ of Matrix theory, respectively. They were identified by interpreting the one-loop effective potential between two diagonal blocks in BFSS Matrix theory as the result of tree-level interaction of DLCQ supergravity [17, 18]. Explicit forms of the bosonic part for the components which are needed for our discussion are

\[
T^{+} = \frac{1}{g_s^2} \text{STr} \left( \frac{1}{2} \dot{X}_i \dot{X}_i - \frac{1}{4\chi^2} [\dot{X}_i, \dot{X}_j][\dot{X}_i, \dot{X}_j] \right)
\]

\[
T^{ij} = \frac{1}{g_s^2} \text{STr} \left( \dot{X}_i \dot{X}_j - \frac{1}{\chi^2} [\dot{X}_i, \dot{X}_k][\dot{X}_k, \dot{X}_j] \right)
\]

\[
J^{a_1a_2a_3} = \frac{-i}{6g_s^2\lambda} \text{STr} \left( \dot{X}_{a_1}[\dot{X}_{a_2}, \dot{X}_{a_3}] + \dot{X}_{a_2}[\dot{X}_{a_3}, \dot{X}_{a_1}] + \dot{X}_{a_3}[\dot{X}_{a_1}, \dot{X}_{a_2}] \right)
\]

\[
M^{+m_1...m_4} = \frac{-1}{720g_s^2\lambda^2} \text{STr} \left( [\dot{X}_{m_1}, \dot{X}_{m_2}][\dot{X}_{m_3}, \dot{X}_{m_4}] + [\dot{X}_{m_1}, \dot{X}_{m_3}][\dot{X}_{m_4}, \dot{X}_{m_2}] \right.
\]

\[
+ [\dot{X}_{m_1}, \dot{X}_{m_4}][\dot{X}_{m_2}, \dot{X}_{m_3}] \right) (4.5)
\]

where we have changed the sign of \( J^{a_1a_2a_3} \) and the coefficient of \( M^{+m_1...m_4} \) from the ones of refs. [17, 18] to adjust to normalization of the antisymmetric tensor fields.

The last two lines of eq. (4.4) state that \( n \)-th derivative of background fields should couple to the matrix version of \( n \)-th moment of the currents. \( n \)-th moment is given from the above operators as follows.

\[
T^{MN;i_1...i_n} = \text{Sym}(T^{MN};\dot{X}_{i_1}, \ldots, \dot{X}_{i_n})
\]

\[
J^{MNP;i_1...i_n} = \text{Sym}(J^{MNP};\dot{X}_{i_1}, \ldots, \dot{X}_{i_n})
\]

\[
M^{MNPQRS;i_1...i_n} = \text{Sym}(M^{MNPQRS};\dot{X}_{i_1}, \ldots, \dot{X}_{i_n}) (4.6)
\]
where the RHS means the symmetrized trace is taken after inserting \( \hat{X}_i \) \( n \) times into the expressions inside the trace. (When symmetrizing the ordering, \([\hat{X}_i, \hat{X}_j]\) are treated as a single unit, as in the previous section.) We can also say that background fields enter the action as ‘non-abelian Taylor expansion’ around some point \( x_i = r_i \), where the coordinates in the series are replaced by matrices:

\[
\varphi = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{X}_{i_1} \cdots \hat{X}_{i_n} (\partial_{i_1} \cdots \partial_{i_n}) \varphi \bigg|_{x_i = r_i}.
\]  

(4.7)

The proposed couplings to \( C^{(3)}_{a_1a_2a_3} \) and \( C^{(6)}_{+i_1i_2i_3i_4} \) at the first few orders can be rewritten as

\[
\sum_{n=0}^{3} \frac{1}{n!} \partial_{i_1} \cdots \partial_{i_n} C^{(3)}_{a_1a_2a_3} J^{a_1a_2a_3;i_1 \ldots i_n} = -\frac{i}{4g_s \ell_s \lambda} \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4}) F^{(4)}_{a_1a_2a_3a_4} \\
- \frac{i}{30g_s \ell_s \lambda} \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4}) \left[ 4 \partial_{(a_1} F^{(4)}_{a_4)a_2a_3a_5} - 2 \partial_{a_2} F^{(4)}_{a_1a_4a_5a_1} \right] \\
- \frac{i}{36g_s \ell_s \lambda} \text{Tr}(\hat{X}_{a_1} \hat{X}_{a_2} \hat{X}_{a_3} \hat{X}_{a_4} \hat{X}_{a_5} \hat{X}_{a_6}) \\
\times \left[ \partial_{a_1} \partial_{a_2} F^{(4)}_{a_3a_4a_5a_6} + \partial_{a_4} \partial_{a_5} F^{(4)}_{a_3a_1a_2a_6} + \partial_{a_5} \partial_{a_1} F^{(4)}_{a_2a_3a_4a_6} \right],
\]

(4.8)

\[
\sum_{n=0}^{2} \frac{1}{n!} \partial_{i_1} \cdots \partial_{i_n} C^{(6)}_{+i_1i_2i_3i_4} M^{+i_1i_2i_3i_4;i_1 \ldots i_n} = -\frac{1}{10g_s \ell_s \lambda^2} \text{Tr}(\hat{X}_{i_1} \hat{X}_{i_2} \hat{X}_{i_3} \hat{X}_{i_4} \hat{X}_{i_6}) F^{(7)}_{+i_1i_2i_3i_4i_5} \\
- \frac{1}{12g_s \ell_s \lambda^2} \text{Tr}(\hat{X}_{i_1} \hat{X}_{i_2} \hat{X}_{i_3} \hat{X}_{i_4} \hat{X}_{i_6}) \partial_{i_6} F^{(7)}_{+i_1i_2i_3i_4i_5},
\]

(4.9)

where we have used the partial integration and the cyclic symmetry of the trace. Note that the zeroth moment \( J^{a_1a_2a_3} \) is a total derivative and \( M^{+m_1 \ldots m_4} \) vanishes for the cyclic symmetry of the trace, thus, the leading contributions are from \( J^{a_1a_2a_3a_4} \) and \( M^{+m_1 \ldots m_4; a} \). Also note that the terms involving the derivatives of field strengths, different expressions are also possible.

We can see that the part (1.1) of the BD Matrix theory effective action precisely agree with the lowest-moment contribution of eq. (4.4)

\[
S = S_0 + \int dt \left\{ \frac{1}{2} h_{MN} T^{MN} + J^{a_1a_2a_3a_4} \partial_{a_4} C^{(3)}_{a_1a_2a_3} + 30 M^{+m_1 \ldots m_4; a} \partial_{a} C^{(6)}_{+m_1 \ldots m_4} \right\}
\]

upon substitution of the longitudinal 5-brane background at the linear order

\[
h_{+-} = \frac{1}{3} \frac{k}{p^3}, \quad h_{mn} = -\frac{1}{3} \frac{k}{p^3} \delta_{mn}, \quad h_{ab} = \frac{2}{3} \frac{k}{p^3} \delta_{ab}
\]

\[
F^{(4)}_{a_1a_2a_3a_4} = 3 \frac{kr_a}{r^5} \epsilon_{a_1a_2a_3a_4a}, \quad F^{(7)}_{+m_1m_2m_3m_4a} = -3 \frac{kr_a}{r^5} \epsilon_{m_1m_2m_3m_4}.
\]
4.2 Subtleties for the higher-moment couplings

Now, we shall examine the terms which are of higher orders in $\hat{X}_i$. First, let us consider the coupling with the 11D metric. According to the Taylor and Van Raamsdonk's proposal, the coupling is given from the first two lines of (4.1) by replacing $k/r^3$ with the non-abelian Taylor expansion

$$\frac{k}{r^3} \to \frac{k}{r^3} - \frac{3k}{r^5} r^a \hat{X}_a + \frac{k}{2} \left( -\frac{3}{r^5} \hat{X}_a \hat{X}_a + 15 \frac{r_a r_b}{r^7} \hat{X}_a \hat{X}_b \right) + \cdots .$$

This part is to be compared with the part of the BD effective action $\Gamma((\hat{X}_m)^N a, D = 0)$, $\Gamma((\hat{X}_m)^N a, D = 0)_B$ and $\Gamma((\hat{X}_m)^N i, D = 2)$ which we have obtained in sections 3.2 and 3.3. Exact agreement goes through to the subleading order for this case. The terms (3.16), (3.27) and (3.34) in our result agree with the proposed first moment-couplings (eq.(4.1) evaluated at the first order of non-abelian Taylor expansions). However, we find discrepancies at the next order. The terms (3.17), (3.18), (3.26), (3.27), (3.35) and (3.36), in BD effective action have the coefficients expected from the proposed second-moment couplings, however, there are subtleties in the ordering of matrices. Some of the terms ((3.18), (3.27) and (3.36)) satisfy the symmetrized trace prescription, but others ((3.17), (3.26) and (3.35)) do not. Moreover, our result has corrections involving extra commutator of matrices (3.19), (3.28), (3.37) and (3.38), which are not present in the proposed action.

Next, we analyze the couplings to the 3-form potential. The $\Gamma((\hat{X}_a)^N a, D = 1)$ part of the BD effective action obtained in section 3.4 represent these couplings. Exact agreement holds at the subleading order, also for this case. The term (3.42) agree with the second term of eq.(4.8). At the next order (third-moment coupling), the proposed coupling (4.8) agrees with a part (3.43) of the BD effective action, however, the latter has extra contribution (3.44).

The couplings to the 6-form potential are given by the $\Gamma((\hat{X}_m)^N m (\hat{X}_a)^N a, D = 0)_C$ part in the BD Matrix theory effective action. For these couplings, we find disagreement already at the subleading order (second-moment coupling). Substituting the background into the second term of eq.(4.9), proposed coupling reads

$$\frac{3 N_4}{32 \lambda} \left( \delta_{ab} \frac{1}{r^5} - 5 \frac{r_a r_b}{r^7} \right) \text{STr}([\hat{X}_{m_1}, \hat{X}_{m_2}][\hat{X}_{m_3}, \hat{X}_{m_4}] \hat{X}_a \hat{X}_b) \epsilon_{m_1 m_2 m_3 m_4} .$$

The terms (3.30) and (3.31) in the BD effective action have the same coefficients as the first and the second terms of the above expression, respectively. However, there is a difference in the ordering for (3.30) (Tr, not STr). Our result also have a correction (3.32) containing extra $\hat{X}_m$’s in the form of commutators.
Finally, BD Matrix theory effective action have corrections to the kinetic terms of \( \hat{X}_m \) such as (3.39), which is not expected from the proposed action, also in the form of commutators.

### 4.3 On the consistency between BFSS and BD matrix models

As described in the previous subsections, we have confirmed that the couplings which are expected from the proposal of Taylor and Van Raamsdonk indeed exists in the effective action of BD Matrix theory. We also found subtleties in the ordering of matrices, that is, there are corrections to the above proposal involving extra commutators.

To discuss consistency between BFSS and BD matrix models, we first mention that the ordering problems similar to the ones which we have found is also present in the effective action of BFSS Matrix theory. In ref. [17], effective action was obtained using the ‘quasi-static approximation’ including the contributions of the higher moments of arbitrary orders. (In ref. [18], which uses the similar method of approximation as ours, only the zeroth and first moments were calculated explicitly.) It was noted in ref. [17], that the effective action is not given by the symmetrized trace in the usual sense, when there is a contraction of indices between two of the matrices which had been inserted to construct the moments. This is precisely the same situation as the one for the terms such as (3.17), (3.26) and (3.35). In the BFSS effective action, some of the terms will be interpreted as interactions between two (block diagonal) symmetrized trace operators but others will not.

The agreement of the symmetrized trace part of the BD effective action with the Taylor and Van Raamsdonk’s couplings suggests the consistency between the two matrix models. To examine the consistency between BFSS and BD matrix models further, it must be helpful to extend the analysis of ref. [18] to the next order and study the contribution from the second moment operators to the BFSS effective action.

### 5 Interpretation from the 10D perspective

Before going to the conclusion, we shall briefly discuss our results from the 10D perspective. The action of BD Matrix theory is the SYM for the D0-D4 system, which consists of only the lowest modes of open strings. Thus, the effective action resulting from the integration of \( \nu \) and \( \chi \) (lowest modes of 0-4 strings) is guaranteed to be valid when their mass are smaller than the mass of the higher modes of open strings, that is, when the distance between D0-branes and D4-branes is smaller than the string scale \( r \ll \ell_s \). If we
want to discuss the long distance interaction, the open-string cylinder amplitudes between D0-brane and D4-brane must be studied.

In this section we compare BD Matrix theory effective action with a proposal for the D-brane action in curved space due to Myers [12], and point out some agreement. We must emphasize that the region of validity of the two action is different, for we cannot expect that the Born-Infeld like action introduced in the following is valid at short distance.

Myers proposed the following form of the D-brane action in curved space, motivated by the consistency with a single D9-brane action when the T-duality in variance is assumed [12]. For D0-branes, it reads

\[ S_{\text{Myers}} = \int dt (\mathcal{L}_{\text{BI}} + \mathcal{L}_{\text{CS}}^{(3)} + \mathcal{L}_{\text{CS}}^{(5)}) \]

where the Born-Infeld (BI) part is given as

\[ \mathcal{L}_{\text{BI}} = -\frac{1}{g_s \ell_s} \text{STr} \left( e^{-\phi} \sqrt{-(E_{00} + (Q^{-1})^{jk} D_j X^k D_t X^l E_{ij})} \sqrt{\text{det}(Q^{jk})} \right) \]

(5.1)

where

\[ E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}, \quad Q^{jk} = \delta^{jk} + i \frac{1}{\lambda} [X^j, X^i] E_{ik}. \]

The Chern-Simons (CS) terms which are relevant to the coupling with D4-branes are given as

\[ \mathcal{L}_{\text{CS}}^{(3)} = -\frac{i}{2g_s \ell_s \lambda} \text{STr} \left( C_{ijk}^{(3)} [X^k, X^j] \partial_t X^i \right), \]

(5.2)

\[ \mathcal{L}_{\text{CS}}^{(5)} = -\frac{1}{8g_s \ell_s \lambda^2} \text{STr} \left( C_{0ijkl}^{(5)} [X^j, X^i][X^l, X^k] \right). \]

(5.3)

In the above action, background fields are prescribed to be given as the non-abelian Taylor expansion (4.7). Also note that the static gauge \( x^0 = t \) is assumed.

We consider the part of this action which is the leading terms in \( \alpha' \to 0 \) limit when \( \hat{X}_i / \alpha' \) and the background (including the series of the non-abelian Taylor expansion) are fixed.\footnote{It is a well-known fact that the SYM result for the \( (\partial_t \hat{X}_a)^2 \)-term in the abelian case is valid for \( r \gg \ell_s \) as well. Contributions from massive modes of D0-D4 open strings cancel for this term [23].} Substituting the D4-brane solution in the string frame metric

\[ ds^2 = H^{-1/2}(-dx^0 dx^0 + \delta_{mn} dx^m dx^n) + H^{1/2} \delta_{ab} dx^a dx^b, \quad e^{\phi} = H^{-1/4} \]

\[ F_{0m_1 m_2 m_3 m_4 a}^{(6)} = \epsilon_{m_1 m_2 m_3 m_4} \partial_a (H^{-1}), \quad F_{a_1 a_2 a_3 a_4}^{(4)} = -\epsilon_{a_1 a_2 a_3 a_4} \partial_a H, \quad H = 1 + \frac{k}{r^3} \]

\[ \parallel \]

This part is the one which allow an interpretation in terms of DLCQ M-theory. Following the argument of Seiberg and Sen, DLCQ M-theory is given from 10D type IIA string theory by an infinite boost in the compactified 11-th direction with a rescaling of the length scale by an infinite factor. (See ref. [19] for explicit transformation rules.) This part of D0-brane action remain non-vanishing after the transformation.
into the action, the $\alpha'$-leading part is given as follows. The Born-Infeld part reduces to

$$ S_{BI} \rightarrow T_0 \int dt S Tr \left\{ \frac{1}{2} \dot{\hat{X}}_m \ddot{\hat{X}}_m + \frac{1}{2} H \dot{\hat{X}}_a \ddot{\hat{X}}_a \\
+ \frac{1}{\lambda^2} \left( \frac{1}{4} H^{-1} [\hat{X}_m, \hat{X}_n]^2 + \frac{1}{2} [\hat{X}_m, \hat{X}_a]^2 + \frac{1}{4} H [\hat{X}_a, \hat{X}_b]^2 \right) \right\} \quad (5.4) $$

where we have taken the $A_0 = 0$ gauge. The CS terms (5.2) and (5.3) are of the same order as eq.(5.4).

At the linearized level of the background (up to the part linear in $k \propto N_4$), the terms (5.2), (5.3) and (5.4) agree with the Taylor and Van Raamsdonk's action as we see from eqs.(4.4), (4.5), (4.8) and (4.9). Thus the consistency (up to subtleties in the ordering of matrices) of our result with Myers action is as explained in section 4. There, we pointed out the existence of the terms which do not satisfy symmetrized trace prescription in the Matrix theory effective action. It will be appropriate to mention here a possibility for a difference in the long distance behavior depending on the ordering of matrices: It may be the case that only terms with the symmetrized ordering are protected by SUSY and are allowed to be interpreted as long distance interactions. Of course, to test the above statement, we must develop non-renormalization theorems in the gauge theory or study cylinder amplitudes between the multiple branes. These are important issues for the future study.

In addition, Myers’ action predicts that there are corrections for $[\hat{X}_m, \hat{X}_n]^2$ term and for the coupling to 5-form (6-form in 11D) potential coming from the expansion of $H^{-1}$ in $k$. These non-linear effects of the background fields are expected to be reproduced in BD Matrix theory by higher-loop effects.

6 Discussions

In this paper, we studied one-loop effective action of Berkooz-Douglas Matrix theory. BD Matrix theory is a proposal for the definition of M-theory in the presence of longitudinal 5-branes, and it has extra degrees of freedom (which are the lowest modes of D0-D4 string) compared to the original BFSS Matrix theory. The result of integrating out the extra fields gives the effective action for the D0-brane degrees of freedom in the background field produced by the 5-branes.

Since the 5-branes which we are dealing with have no D0-brane charge and cannot be realized in the BFSS Matrix theory, our analysis provides a non-trivial check for the coupling of D0-branes to general weak background fields proposed by Taylor and Van Raamsdonk [18, 19] from the Matrix theory analysis. We have confirmed that the couplings given by inserting the longitudinal 5-brane solution (at the linearized level) in the
above proposal appear in the effective action of BD Matrix theory precisely as expected. We also found that there are corrections involving extra commutators, which violates the proposed symmetrized trace prescription. However, the similar subtleties for the ordering of matrices are also present in the effective action of BFSS Matrix theory itself. The exact agreement between the symmetrized trace terms in the BD effective action and the Taylor and Van Raamsdonk's proposal can be regarded as a suggestion for the consistency between the BD and BFSS matrix models. To discuss the consistency in more detail, it should be helpful to extend the analysis of the BFSS effective action performed in ref. [18] to the higher order terms (second moment couplings).

We think that the coupling of matrix fields to the background supergravity fields, and in particular, the ordering problem for the product of matrices are issues which need further analyses. Most direct way to obtain the action of D-branes should be the calculation of string scattering amplitudes. (See refs. [13, 14, 30] among others.) It may be necessary to perform thorough study of the scattering amplitudes for the operators of all possible orderings at the order of interest.

Finally, we shall list other problems to be studied.

1) Direct extension of this work is the study of the fermionic part of the one-loop effective action. It allows the following interesting consistency check of the action which was originally proposed in ref. [19]: By interpreting the effective action as the Matrix theory action on the longitudinal 5-brane background, we may evaluate the quantum effective action for two diagonal blocks, for example, starting from that action. It should correspond to the supergravity interaction between two objects evaluated on that background.

In addition, it is an important problem to find a gauge invariant (\(\kappa\)-symmetric) action of multiple D-branes which reduces to Matrix theory effective action after gauge fixing. In ref. [31], ambitious attempt was made to define a generalization of \(\kappa\)-symmetry which have non-Abelian parameters. However, the construction does not seem successful, for the action is not consistent with string amplitudes [32]. It may be more suitable to introduce \(\kappa\)-symmetry only for the U(1) (center of mass) part, as in ref. [33].

2) Studies toward establishing the validity of BD Matrix theory as a fundamental theory are definitely important. Firstly, cylinder amplitudes between D0-branes and D4-branes should be analyzed in detail. We want to clarify in what cases open string massive modes cancel and the SYM is able to describe long distance physics. Also, most interesting problem is whether BD Matrix theory can reproduce the non-linear supergravity fields of the 5-branes, as mentioned in section 5. Those kinds of analyses for this version of Matrix theory with half the maximal SUSY will give implications on the connection between matrix models and gravity.

3) An interesting physical phenomenon which is expected to occur in the D4-brane
background is the so-called Myers effect. As we have seen, multiple D0-branes can couple to the 4-form field strength produced by D4-brane. This coupling should give rise to a stable non-commutative configuration of D0-branes which has the shape of 2-sphere, following the qualitative argument first done by Myers [12]. In ref. [25], one of the present authors performed a detailed analysis of this problem. It was shown that a certain spherical configuration is indeed a solution of the equation of motion of the Myers’ action for D0-branes in the D4-brane background, by taking a special coordinate system where some of the coordinates are assumed to be commutative. Furthermore, a spherical configuration which exhibits exactly the same kinematical behavior as the point-like D0-branes was found. Similar result is likely to be reached in the framework of BD Matrix theory. However, the effective action which was obtained in this paper is not suitable for describing the configuration studied in ref. [25], for it will require all orders of the expansion in the derivatives of background fields. We hope to study quantum corrections around the background matrices of the form of the configuration of ref. [25] directly and discuss its stability in BD Matrix theory. (In a recent paper [27], non-commutative configurations of D0-branes with open topology ending on D4-branes were studied in BD Matrix theory coupled to external supergravity fields. What we mean here is configurations with closed topology in the theory without additional external fields.) The search for a finite-sized stable configuration of a collection of $N$ fundamental degrees of freedom must be important in regards of its possible connection to the holographic principle.

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References

[1] J. Polchinski, ‘Dirichlet-Branes and Ramond-Ramond Charges,’ Phys. Rev. Lett. 75 (1995) 4724, hep-th/9510017.

[2] E. Witten, ‘Bound States Of Strings And p-Branes,’ Nucl. Phys. B460 (1996) 335, hep-th/9510135.

[3] R.G. Leigh, ‘Dirac-Born-Infeld action from Dirichlet sigma model,’ Mod. Phys. Lett. A4 (1989) 2767.
[4] M. Li, ‘Boundary States of D-Branes and Dy-Strings,’ Nucl. Phys. B460 (1996) 351, hep-th/9510161.

[5] M.R. Douglas, ‘Branes within Branes,’ hep-th/9512077.

[6] A.A. Tseytlin, ‘On non-abelian generalisation of Born-Infeld action in string theory,’ Nucl. Phys. B501 (1997) 41, hep-th/9701123.

[7] D. Gross and E. Witten, ‘Superstring modification of Einstein’s equations,’ Nucl. Phys. B277 (1986) 1.

[8] A.A. Tseytlin, ‘Vector field effective action in the open superstring theory,’ Nucl. Phys. B276 (1986) 391, Erratum-ibid. B291 (1987) 876.

[9] A. Hashimoto and W. Taylor, ‘Fluctuation Spectra of Tilted and Intersecting D-branes from the Born-Infeld Action,’ Nucl. Phys. B503 (1997) 193, hep-th/9703217.

[10] M.R. Douglas, ‘D-branes and matrix theory in curved space,’ Nucl. Phys. Proc. Psuppl. 68 (1998) 381, hep-th/9707228.

[11] J. de Boer and K. Schalm, ‘General covariance of the non-abelian DBI-action,’ hep-th/0108161.

[12] R.C. Myers, ‘Dielectric-Branes,’ JHEP 9912 (1999) 022, hep-th/9910053.

[13] M.R. Garousi and R.C. Myers, ‘World-Volume Interactions on D-Branes,’ Nucl. Phys. B542 (1999) 73, hep-th/9809100.

[14] M.R. Garousi and R.C. Myers, ‘World-Volume Potentials on D-branes,’ JHEP 0011 (2000) 032, hep-th/0010122.

[15] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, ‘M Theory As A Matrix Model: A Conjecture,’ Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[16] M. Berkooz and M.R. Douglas, ‘Five-branes in M(atrix) Theory,’ Phys. Lett. B395 (1997) 196, hep-th/9610236.

[17] D. Kabat and W. Taylor, ‘Linearized supergravity from Matrix theory,’ Phys. Lett. B426 (1998) 297, hep-th/9712188.

[18] W. Taylor and M. Van Raamsdonk, ‘Supergravity currents and linearized interactions for Matrix Theory configurations with fermionic backgrounds,’ JHEP 9904 (1999) 013, hep-th/9812239.
[19] W. Taylor and M. Van Raamsdonk, ‘Multiple D0-branes in Weakly Curved Backgrounds,’ Nucl. Phys. B558 (1999) 63, hep-th/9904095.

[20] W. Taylor and M. Van Raamsdonk, ‘Multiple Dp-branes in Weak Background Fields,’ Nucl. Phys. B573 (2000) 703, hep-th/9910052.

[21] N. Seiberg, ‘Why is the Matrix Model Correct?,’ Phys. Rev. Lett. 79 (1997) 3577, hep-th/9710009.

[22] A. Sen, ‘D0 Branes on T^n and Matrix Theory,’ Adv. Theor. Math. Phys. 2 (1998) 51, hep-th/9709220.

[23] A. Dasgupta, H. Nicolai and J. Plefka, ‘Vertex Operators for the Supermembrane,’ JHEP 0005 (2000) 007, hep-th/0003280.

[24] I. Klebanov, W. Taylor and M. Van Raamsdonk, ‘Absorption of dilaton partial waves by D3-branes,’ Nucl. Phys. B560 (1999) 207, hep-th/9905174.

[25] M. Asano, ‘Non-commutative branes in D-brane backgrounds,’ hep-th/0106253.

[26] L. Susskind, ‘Another Conjecture about M(atrix) Theory,’ hep-th/9704080.

[27] M. Van Raamsdonk, ‘Open Dielectric Branes,’ hep-th/0112081.

[28] K. Becker, M. Becker, J. Polchinski and A. Tseytlin, ‘Higher Order Graviton Scattering in M(atrix) Theory,’ Phys. Rev. D56 (1997) 3174, hep-th/9706072.

[29] M.R. Douglas, D. Kabat, P. Pouliot and S.H. Shenker, ‘D-branes and Short Distances in String Theory,’ Nucl. Phys. B485 (1997) 85, hep-th/9608024.

[30] Y. Okawa and H. Ooguri, ‘Energy-momentum Tensors in Matrix Theory and in Noncommutative Gauge Theories,’ hep-th/0103124.

[31] E.A. Bergshoeff, M. de Roo and A. Sevrin, ‘Non-abelian Born-Infeld and kappa-symmetry,’ hep-th/0011018.

[32] E.A. Bergshoeff, A. Bilal, M. de Roo and A. Sevrin, ‘Supersymmetric non-abelian Born-Infeld revisited,’ JHEP 0107 (2001) 029, hep-th/0105274.

[33] D. Sorokin, ‘Coincident (Super)-Dp-Branes of Codimension One,’ JHEP 0108 (2001) 022, hep-th/0106212.