Determination of charm quark mass from temporal moment of charmonium correlator with Mobius Domain Wall fermion

[arxiv:1606.01002]

NAKAYAMA Katsumasa (Nagoya Univ.)

S. Hashimoto, B. Fahy (KEK) for JLQCD Collaboration

2016/07/25@LATTICE2016
◇ Charm quark mass from the temporal moment

- Determination of charm quark mass with $\delta m_c < 1.0\%$

◇ New calculation with Domain-Wall fermion
  (at $\alpha$: 0.083 ~ 0.044 fm).

Previous works (HPQCD 2008-14)  
- Staggered fermion  
- Lattice spacings: 0.15 ~ 0.06 fm

This work  
- Domain-Wall fermion  
- 0.083 ~ 0.044 fm

$m_c(m_c) = 1.272(10)$ GeV

- Truncation of perturbative expansion is a significant source of uncertainty in moment method
Update since Lattice 2015

- Estimations of perturbative truncation error
- Consistency check with the experimental data (Vector)
  - Improved measurement with the $\mathbb{Z}_2$ noize source
    $\implies$ Less statistical error
  - Leading effect subtracted using an effective theory
    $\implies$ Flatter continuum extrapolation
What is the Moment?
diamond Correlator and moment

Current correlator

\[ q^2 \Pi(q^2) = i \int dx e^{i qx} \langle j_5(x) j_5(0) \rangle \]

Moment: Derivative in terms of \( q^2 \)

\[ g_{2n} = \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (q^2 \Pi(q^2))_{q^2=0} \]

Perturbative expansion available as a function of \( m_c^{\overline{\text{MS}}} \), \( \alpha^{\overline{\text{MS}}} \).

[K. G. Chetyrkin et al. (2006)]
[R. Boughezal et al. (2006)]
[A. Maier et al. (2009)]
**Moment on the lattice**

- **Coordinate space**

\[ i \int dx \frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n e^{iqt} \rightarrow a^4 \sum_x t^{2n} \]

- **Correlator** \( G(t) \) **on the lattice**

\[ G(t) = a^6 \sum_x (am_{0c})^2 \langle j_5(x)j_5(0) \rangle \]

- **Moment is easily calculated from** \( G(t) \)

\[ G_n = \sum_t \left( \frac{t}{a} \right)^n G(t) \]
What’s the Moment?

Typical energy scale of the moment $G_m \sim m/n$

$$G_n = \sum_t \left( \frac{t}{a} \right)^n G(t)$$

$am\eta_c \sim 0.6636$
What’s the Moment? (From comparison with experiment)

- Vector moment can be measured in the experiment.

\[
\frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (\Pi(q^2))_{q^2=0} = \frac{1}{12\pi Q_f^2} \int ds \frac{1}{s^{n+1}} R(s)
\]

- Moment is a weighted integral of the R-ratio.

\[
R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadron})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}
\]
What’s the Moment? (From comparison with experiment)

- **Vector** moment can be measured in the experiment.

- Moment is a weighted integral of the **R-ratio**.

\[
R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadron})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}
\]
Consistency with experiment (Vector)

$R_n / (1 + f_1 (m_u + m_d + m_s) / m_c)$

- **Exp+Pheno**
- **Lattice result**

$n = 6$

$n = 8$

- [B. Dehnadi et al. (2011)]
- [J. H. Kuhn et al. (2007)]
- [J. H. Kuhn et al. (2001)]
- [A. H. Hoang et al. (2004)]

◎ Good agreement with the experimental + phenomenological analysis.
\( m_c \) Extraction
Correspondence between lattice and continuum

\[ G_n^{(\text{Lat})} = g_n^{(\text{cont})} \left( m_c^{\overline{\text{MS}}} , \alpha_{\overline{\text{MS}}} \right) \bigg/ (am_c^{\overline{\text{MS}}})^{n-4} \]

Define the reduced moment \( R_n \)
with \( G_n \) and \( G_n^{(0)} \), the counterparts at the tree-level,

Lattice

\[ R_n = \frac{am_c \eta_c}{2am_c} \left( \frac{G_n}{G_n^{(0)}} \right)^{\frac{1}{n-4}} \]

Continuum

\[ r_n = \left( \frac{g_n}{g_n^{(0)}} \right)^{\frac{1}{n-4}} \]

\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_c^{\overline{\text{MS}}}} r_n \left( m_c^{\overline{\text{MS}}} , \alpha_{\overline{\text{MS}}} \right) \]
Lattice setup

- Mobius Domain Wall Fermion formalism

- $n_f = 2 + 1$ in the sea;

3 different lattice spacings for taking continuum limit.

| $\beta$ | $a^{-1}$ [GeV] | $L \times T$ | $L_5$ | $am_{ud}$ | $am_s$ | conffgs | $m_\pi$ [MeV] | $m_\pi L$ |
|--------|----------------|--------------|-------|-----------|--------|---------|---------------|----------|
| 4.17   | 2.4531(40)     | $32^3 \times 64$ ($L = 2.6$ fm) | 12    | 0.0035    | 0.040  | 300     | 230           | 3.0      |
|        |                |              |       | 0.007     | 0.030  | 300     | 310           | 4.0      |
|        |                |              |       | 0.007     | 0.040  | 300     | 310           | 4.0      |
|        |                |              |       | 0.012     | 0.030  | 300     | 400           | 5.2      |
|        |                |              |       | 0.012     | 0.040  | 300     | 400           | 5.2      |
|        |                |              |       | 0.019     | 0.030  | 300     | 500           | 6.5      |
|        |                |              |       | 0.019     | 0.040  | 300     | 500           | 6.5      |
|        |                | $48^3 \times 96$ ($L = 3.9$ fm) | 12    | 0.0035    | 0.040  | 401     | 230           | 4.4      |
| 4.35   | 3.6097(89)     | $48^3 \times 96$ ($L = 2.6$ fm) | 8     | 0.0042    | 0.0180 | 300     | 300           | 3.9      |
|        |                |              |       | 0.0042    | 0.0250 | 300     | 300           | 3.9      |
|        |                |              |       | 0.0080    | 0.0180 | 301     | 410           | 5.4      |
|        |                |              |       | 0.0080    | 0.0250 | 297     | 410           | 5.4      |
|        |                |              |       | 0.0120    | 0.0180 | 298     | 500           | 6.6      |
|        |                |              |       | 0.0120    | 0.0250 | 300     | 500           | 6.6      |
| 4.47   | 4.4961(92)     | $64^3 \times 128$ ($L = 2.8$ fm) | 8     | 0.0030    | 0.015  | 397     | 280           | 4.0      |
Construct moments from $G(t)$, \[ G_n = \sum_{t} \left( \frac{t}{a} \right)^n G(t) \]

Interpolate to the physical point by tuning \( (3m_{J/\psi} + m_{\eta_c})/4 \)

Chiral extrapolation
Extrapolation to continuum ($a = 0$)

$$R_n(a) = R_n(0) \left(1 + c_1(a m_c)^2\right) \times \left(1 + f_1 \frac{m_u + m_d + m_s}{m_c}\right)$$

| $R_n$   | (Stat.)($O(a^4)$) (Vol.) |
|---------|--------------------------|
| $R_6$   | 1.5048(5)(5)(4)          |
| $R_8$   | 1.3570(4)(22)(3)         |
| $R_{10}$| 1.2931(4)(27)(5)         |

Essentially flat (~1) term

\[R_6 = 1.5048(5)(5)(4)\]
\[R_8 = 1.3570(4)(22)(3)\]
\[R_{10} = 1.2931(4)(27)(5)\]
Possible systematic errors

\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_c^{\text{MS}}} r_n(m_c^{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \]

(1): Truncation error from perturbative expansion of \( r_n \)

(2): Input meson mass \( m_{\eta_c}^{\text{exp}} \) error

After correcting for...
(a) Electromagnetic effect,
(b) Disconnected diagram contributions.

(3): Gluon condensate contribution
Error 1 Perturbative truncation

\[ R_n = \frac{m_{\eta c}^{\text{exp}}}{2m_{c}^{\overline{\text{MS}}}} r_n(m_{c}^{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \]

should depend on the renormalization scale \( \mu \) at all order of perturbative series.

\[(r_6)^2 = 1 + \left(3.9 + 2.0 \log \frac{m_{c}(\mu)^2}{\mu^2} \right) \frac{\alpha_s}{\pi} + \left(13.6 + 3.0 \log \frac{m_{c}(\mu)^2}{\mu^2} - 0.08 \left( \log \frac{m_{c}(\mu)^2}{\mu^2} \right)^2 \right) \left( \frac{\alpha_s}{\pi} \right)^2 + \left(13.2 + 14.2 \log \frac{m_{c}(\mu)^2}{\mu^2} + 1.03 \left( \log \frac{m_{c}(\mu)^2}{\mu^2} \right)^2 + 0.06 \left( \log \frac{m_{c}(\mu)^2}{\mu^2} \right)^3 \right) \left( \frac{\alpha_s}{\pi} \right)^3 \]

Estimate the truncation error from \( \mu \) dependence of \( \frac{r_n(\mu)}{m_{c}^{\overline{\text{MS}}}(\mu)} \)
Estimation with $\mu_m = \mu_\alpha$. 

Error is estimated in this region
Estimation with $\mu_m \neq \mu_\alpha$ as conservative one

|            | $\mu_m = \mu_\alpha$ | $\mu_m \neq \mu_\alpha$ |
|------------|-----------------------|--------------------------|
| $m_c(3\text{GeV})$ | 0.26%                 | 0.77%                    |
| $\alpha_s(3\text{GeV})$ | 1.9%                  | 4.7%                     |

Similar analysis with [B. Dehnadi et al. (2013),(2015)]

→ More conservative estimation

Error is estimated in this region
Result with three inputs.
(Gluon condensate is another free parameter)

\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_c^{\overline{\text{MS}}}} r_n(m_c^{\overline{\text{MS}}}, \alpha_s^{\overline{\text{MS}}}) \]
Result ($\mu_m \neq \mu_\alpha$)

- More conservative estimation of the perturbative error.

|                  | Lattice | $m_{\eta_c}$(negligible) |
|------------------|---------|--------------------------|
| $m_c(3\text{GeV})$ [GeV] | $1.0033(96)$ | $(77)$ \(1/2\) | $4$ | $O(a^4)$ | $30$ | $4$ | $(3)$ | $(4)$ | $(6)$ |
| $\alpha_s(3\text{GeV})$ | $0.2528(127)$ | $(120)$ | $32$ | $2$ | $(26)$ | $1$ | $(0)$ | $(0)$ | $(1)$ |

- Perturbative error is $\times 2$ larger than that of $\mu_m = \mu_\alpha$ and dominant source of the systematic error.

- ~1% precision is achieved for $m_c^{\overline{\text{MS}}}$. 

- $R_6/R_8, R_8, \& R_{10}$
### Result ($\mu_m \neq \mu_\alpha$)

|                              | This work       | PDG(2014)       |
|------------------------------|-----------------|-----------------|
| $m_c^{\overline{\text{MS}}} (\mu = 3 \text{ GeV})$ | 1.003(10) GeV   |                 |
| $m_c^{\overline{\text{MS}}} (\mu = m_c^{\overline{\text{MS}}})$ | 1.287(12) GeV   | 1.275(25) GeV   |
| $\alpha_{\overline{\text{MS}}} (\mu = 3 \text{ GeV})$ | 0.253(13)       | 0.2567(34)      |
| $\alpha_{\overline{\text{MS}}} (\mu = M_Z)$ | 0.1177(26)      | 0.1185(6)       |
| $\Lambda_{\overline{\text{MS}}}^{n_f=4}$ | 286(37) MeV     | 297(8) MeV      |
| $\Lambda_{\overline{\text{MS}}}^{n_f=5}$ | 205(32) MeV     | 214(7) MeV      |
Summary

- We extract $m_c^{\overline{\text{MS}}}$ and $\alpha_{\overline{\text{MS}}}$ from the temporal moments of charmonium current correlators.

- Take continuum limit by the data at $a^{-1} = 2.4, 3.6, 4.5$ GeV. Discretization effect is significant but controllable, and perturbative truncation is more important.

- In the vector channel, The moments are consistent with the experimental R-ratio.

\[
m_c^{\overline{\text{MS}}}(3 \ \text{GeV}) = 1.003(10) \ \text{GeV} \\
\alpha_{\overline{\text{MS}}}(3 \ \text{GeV}) = 0.253(13)
\]
Backup slides
Typical energy scale depends on the weight factor $n$

$$a^{-1} \gg (\text{Energy scale}) \gg \Lambda_{\text{QCD}} \quad \rightarrow \quad 6 \leq n \ll 20$$

**Window**

Energy scale

Contains Discretization effect

$\Lambda_{\text{QCD}} \sim 300 \text{ MeV}$

$am_{\eta_c} \sim 0.6636$
**Error 1** Input meson mass $m^{\text{exp}}_{\eta_c}$ error

- We use PDG value, $m^{\text{exp}}_{\eta_c} = 2.9836(7)$ GeV, after correcting for...
  - (a) Electromagnetic effect,
  - (b)Disconnected diagram contributions.

- Estimates from previous works (lattice, pheno):

  **Electromagnetic**

  \[
  m_{\eta_c} - m_{\eta_c}^{\text{no EM}} = -2.4(8) \text{ MeV} 
  \]

  [E. Follana, et al. (2007)]

  **Disconnected**

  \[
  m_{\eta_c} - m_{\eta_c}^{\text{no Disconnect}} = -2.6(13) \text{ MeV} 
  \]

  [C. T. H. Davis, et al. (2007)]
Error \( \mathbf{1} \) Input meson mass \( m_{\eta_c}^{\text{exp}} \) and error
Error \(\circledast\) Finite volume effect

- Prepare two ensembles (same setup except for the volume)

\[
\begin{align*}
R_n(L = 32) & \quad m_\pi L \sim 3.0 \\
R_n(L = 48) & \quad m_\pi L \sim 4.4
\end{align*}
\]

Finite volume error

\[
\delta_L R_n = |R_n(L = 48) - R_n(L = 32)|
\]
Error ❹ Gluon condensate

- Perturbative calculation does not contain gluon condensate.

It is known to 2-loop by OPE.

\[ \langle \left( \frac{\alpha_s}{\pi} \right) G^{\mu\nu} G_{\mu\nu} \rangle \frac{1}{n-4} \]

- We may extract it as a solution of the equations, \( n = 6, 8, \) & 10.

\[ \frac{\langle \left( \frac{\alpha_s}{\pi} \right) G^{\mu\nu} G_{\mu\nu} \rangle}{m^4} = -0.0006(78) \]
Dependence is almost canceled out.
Actually, we consider $\mu_\alpha \neq \mu_m (\alpha(\mu_\alpha), m_c(\mu_m))$, not only $\mu_\alpha = \mu_m$ [B. Dehnadi, A. H. Hoang, and V. Mateu (2015)]
Consistency with experiment (Vector)

\[ G^{(\text{Lat})}_n = g^{(\text{conti})}_n \left( \frac{m_c^{\text{MS}}}{am_c^{\text{MS}}} \right)^{n-4} \]

\[ \tilde{Z}_V(x) = Z_V + c_{-2}x^{-2} + c_4x^4 + c_6x^6 + O(x^8) \]

\[ Z_V(a^{-1} = 4.47 \, \text{GeV}) = 0.9651(46) \]
Vector current Moment

\[ R_n = \frac{m_{\eta_c}^{\text{exp}}}{2m_{c}^{\text{MS}}} r_n(m_{c}^{\text{MS}}, \alpha_{\text{MS}}) \]

Charm quark mass and Strong coupling…?
\[ R_n = \frac{m_n^{\text{exp}}}{2m_c^{\overline{\text{MS}}}} r_n(m_c^{\overline{\text{MS}}}, \alpha^{\overline{\text{MS}}}) \]
Error ❶ Input meson mass $m_{\eta_c}^{\text{exp}}$ error

- Finally we use...

\[
m_{\eta_c}^{\text{modified}} = 2983.6 + 2.4_{\text{Disc.}} + 2.6_{\text{EM}} \pm (0.7)_{\text{PDG}} \pm (0.8)_{\text{Disc.}} \pm (1.3)_{\text{EM}} \pm (2.3)_{\text{split}}
\]

◆ Note: All of these error sources are negligible.
Moment and R-ratio

Residue theorem (or Dispersion relation)

\[
\frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (\Pi(q^2))_{q^2=Q_0^2} = \int \frac{dq^2}{2\pi i} \frac{1}{(q^2 - Q_0^2)^{n+1}} \Pi(q^2)
\]

Contour integral

\[
= \int \frac{dq^2}{2\pi i} \frac{1}{(q^2 - Q_0^2)^{n+1}} 2i \text{Im} [\Pi(q^2)]
\]

Optical theorem

\[
= \int \frac{dq^2}{\pi} \frac{q^2}{(4\pi\alpha)^2 Q_f^2} \frac{1}{(q^2 - Q_0^2)^{n+1}} \sigma_{e^+ e^- \rightarrow \text{hadron}}(q^2)
\]

\[
= \frac{1}{12\pi Q_f^2} \int dq^2 \frac{1}{(q^2 - Q_0^2)^{n+1}} \frac{\sigma_{e^+ e^- \rightarrow \mu^+ \mu^-}(q^2)}{\sigma_{e^+ e^- \rightarrow \text{hadron}}(q^2)}
\]

Take \( Q_0 = 0 \)

\[
\frac{1}{n!} \left( \frac{\partial}{\partial q^2} \right)^n (\Pi(q^2))_{q^2=0} = \frac{1}{12\pi Q_f^2} \int ds \frac{1}{s^{n+1}} R(s)_{e^+ e^- \rightarrow \text{hadron}}
\]
Perturbative moment

| k | \( C_k^{(0)} \) | \( C_k^{(10)} \) | \( C_k^{(11)} \) | \( C_k^{(20)} \) | \( C_k^{(21)} \) | \( C_k^{(22)} \) | \( C_k^{(30)} \) | \( C_k^{(31)} \) | \( C_k^{(32)} \) | \( C_k^{(33)} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.3333 | 3.1111 | 0.0000 | 0.1154 | -6.4815 | 0.0000 | -1.2224 | 2.5008 | 13.5031 | 0.0000 |
| 2 | 0.5333 | 2.0642 | 1.0667 | 7.2362 | 1.5909 | -0.0444 | 7.0659 | -7.5852 | 0.5505 | 0.0321 |
| 3 | 0.3048 | 1.2117 | 1.2190 | 5.9992 | 4.3373 | 1.1683 | 14.5789 | 7.3626 | 4.2523 | -0.0649 |
| 4 | 0.2032 | 0.7128 | 1.2190 | 4.2670 | 4.8064 | 2.3873 | 13.3285 | 14.7645 | 11.0345 | 1.4589 |
| 5 | 0.1478 | 0.4013 | 1.1821 | 2.9149 | 4.3282 | 3.4971 | 16.0798 | 16.6772 | 4.4685 |
| 6 | 0.1137 | 0.1944 | 1.1366 | 1.9656 | 3.4173 | 4.4992 | 14.1098 | 19.9049 | 8.7485 |
| 7 | 0.0909 | 0.0500 | 1.0912 | 1.3353 | 2.2995 | 5.4104 | 10.7755 | 20.3500 | 14.1272 |
| 8 | 0.0749 | -0.0545 | 1.0484 | 0.9453 | 1.0837 | 6.2466 | 7.2863 | 17.9597 | 20.4750 |

n=4,6,8,10
**$Z_V$ factor extraction**

\[
\begin{array}{c|c}
\text{Input } Z_V & \text{Predict } R_n \\
\end{array}
\]

then invert it...

\[
\begin{array}{c|c}
\text{Input } R_n & \text{Predict } Z_V \\
\end{array}
\]

- Moment is known perturbatively (and experimentally).

\[
\begin{array}{c|c}
\text{(Input Experiment)} & \delta Z_V \sim 1\% \\
\end{array}
\]

\[
\begin{array}{c|c}
\text{Input Perturbation} & \delta Z_V \sim 3\% \\
\end{array}
\]

(or 2% with PDG $\alpha_{\text{MS}}$)
TABLE IX. Numerical results for $m_c(\mu)$ (top panel), $\alpha_s(\mu)$ (mid panel) and $<\alpha/\pi G^2>$ (bottom panel). The scale dependent quantities, $m_c(\mu)$ and $\alpha_s(\mu)$, are renormalized at $\mu = 3$ GeV. The results are listed for different choices of three input quantities out of $R_6$, $R_8$, $R_{10}$ and $R_6/R_8$. In addition to the central values with combined errors, the breakdown of the error is presented. They are the estimated errors from the truncation of perturbative expansion, the input value of $t_{1/2}$, statistical, discretization error of $O(a^4)$ (or $O(\alpha_s a^2)$), finite volume, experimental data for $m_{\eta_c}^{\text{exp}}$, disconnected contribution, electromagnetic effect, in the order given. The total error is estimated by adding the individual errors in quadrature.

| inputs            | $m_c(\mu)$ [GeV] | pert $t_{1/2}$ | stat $O(a^4)$ | vol $m_{\eta_c}^{\text{exp}}$ | disc EM |
|-------------------|------------------|----------------|---------------|------------------------------|---------|
| $R_6$, $R_8$, $R_{10}$ | $1.0032(98)$     | (82)           | (51)          | (5)                          | (16)    |
| $R_6$, $R_6/R_8$, $R_{10}$ | $1.0031(194)$   | (176)          | (78)          | (6)                          | (18)    |
| $R_6/R_8$, $R_8$, $R_{10}$ | $1.0033(96)$    | (77)           | (49)          | (4)                          | (30)    |

| inputs            | $\alpha_s(\mu)$ | pert $t_{1/2}$ | stat $O(a^4)$ | vol $m_{\eta_c}^{\text{exp}}$ | disc EM |
|-------------------|------------------|----------------|---------------|------------------------------|---------|
| $R_6$, $R_8$, $R_{10}$ | $0.2530(256)$   | (213)          | (134)         | (12)                         | (38)    |
| $R_6$, $R_6/R_8$, $R_{10}$ | $0.2528(127)$   | (120)          | (33)          | (2)                          | (25)    |
| $R_6/R_8$, $R_8$, $R_{10}$ | $0.2528(127)$   | (120)          | (32)          | (2)                          | (26)    |

| inputs            | $<\alpha/\pi G^2>$ | pert $t_{1/2}$ | stat $O(a^4)$ | vol $m_{\eta_c}^{\text{exp}}$ | disc EM |
|-------------------|-------------------|----------------|---------------|------------------------------|---------|
| $R_6$, $R_8$, $R_{10}$ | $-0.0005(99)$     | (85)           | (45)          | (4)                          | (23)    |
| $R_6$, $R_6/R_8$, $R_{10}$ | $-0.0006(144)$   | (133)          | (49)          | (4)                          | (23)    |
| $R_6/R_8$, $R_8$, $R_{10}$ | $-0.0006(78)$    | (68)           | (29)          | (3)                          | (22)    |