Sound propagation in elongated superfluid fermion clouds

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We use hydrodynamic equations to study sound propagation in a superfluid Fermi gas inside a strongly elongated cigar-shaped trap, with main attention to the transition from the BCS to the unitary regime. We treat first the role of the radial density profile in the quasi-one-dimensional limit and then evaluate numerically the effect of the axial confinement in a configuration in which a hole is present in the gas density at the center of the trap. We find that in a strongly elongated trap the speed of sound in both the BCS and the unitary regime differs by a factor $\sqrt{3/5}$ from that in a homogeneous three-dimensional superfluid. The predictions of the theory could be tested by measurements of sound-wave propagation in a set-up such as that exploited by M.R. Andrews et al. [Phys. Rev. Lett. 79, 553 (1997)] for an atomic Bose-Einstein condensate.

PACS numbers: 03.75.Kk, 03.75.Ss, 47.37.+q

Strong evidence for a superfluid state in ultracold Fermi gases near a Feshbach resonance has come from experimental studies using as probes ballistic expansion 4, 5, collective modes 6, 7, RF spectroscopy 8, and the generation of quantized vortices 9. Since the BCS weak-coupling limit and the unitary strong-coupling limit are characterized by the same collective-mode frequencies and expansion dynamics 10, attention has been drawn to the study of sound-wave propagation as allowing a clear identification of these two regimes. In particular Ho 11 and Heiselberg 12 have evaluated the speed of first and second sound in a homogenous Fermi superfluid as functions of the coupling regime. At low temperature the first sound velocity is given by $u_1 = v_F \sqrt{(1 + (2/\pi)\hbar k_F a)/3} \approx v_F/\sqrt{3}$ in the dilute BCS limit and by $u_1 \approx 0.37 v_F$ in the unitary limit. Here $v_F$ and $k_F$ are the Fermi velocity and wavenumber, while $a$ is the $s$-wave scattering length.

The main purpose of this Letter is to evaluate the propagation of density perturbations in a superfluid Fermi gas as a function of coupling strength in an experimentally realizable set-up. This simulates the set-up used in the experiments of Andrews et al. 13 on the propagation of sound pulses along the axis of an elongated cloud of Bose-Einstein condensed 23Na atoms. In their experiments a density perturbation is generated by turning on a laser at the center of the trap and two pulses propagate in opposite directions along the trap axis. We describe the dynamics of density fluctuations in the superfluid by hydrodynamic equations, namely the continuity equation

$$\partial_t n + \nabla \cdot (n\mathbf{v}) = 0 \tag{1}$$

and the Euler equation

$$m \partial_t \mathbf{v} + \nabla [\mu(n) + V(\mathbf{r}) + \frac{1}{2} n \mathbf{v}^2] = 0 \tag{2}$$

for the time-dependent density profile $n(\mathbf{r}, t)$ and velocity field $\mathbf{v}(\mathbf{r}, t)$ of the trapped gas. The equation of state enters through the density-dependent chemical potential $\mu(n)$, and $V(\mathbf{r})$ describes the external potentials. The laser beam is simulated in our study by a time-dependent density profile $n(\mathbf{r}, t)$, and integration in the $(x, y)$ plane yields the dispersion relation

$$\omega = q \left( \frac{1}{m} \int n_0 \partial_\mu/\partial n|_{n=n_0} \delta n d^2r_\perp / \int \delta n d^2r_\perp \right)^{1/2}. \tag{4}$$

The lowest collective mode of the gas is obtained by taking $\delta n$ as the lowest-energy solution of Eq. (4) with $q = 0$, i.e. $\delta n(r_\perp) \propto (\partial \mu/\partial n|_{n=n_0})^{-1}$. Therefore the sound velocity is

$$u_1 = \left( \frac{1}{m} \int n_0 d^2r_\perp / \int (\partial \mu/\partial n|_{n=n_0})^{-1} d^2r_\perp \right)^{1/2}. \tag{5}$$

Equation (5) applies to all Q1D gases which can be described by hydrodynamic equations.

For a power-law form of the equation of state, $\mu(n) = \tilde{\mu} n^{\gamma}$, the equilibrium density profile takes the explicit form $n_0(r_\perp) = (\tilde{\mu}/C) \sqrt{1 - r^2/R_{TF}^2}$ where $R_{TF} = \sqrt{2\tilde{\mu}/m\omega^2_T}$ is the radial size of the cloud. Equation (5)
then reduces to

$$u_1^2 = \frac{\gamma \mu}{(1 + \gamma)m}. \quad (6)$$

For $\gamma = 1$ Eq. (5) gives the speed of sound pulses propagating along the axis of a Q1D condensate of bosonic atoms [14] or of molecular bosons formed by pairs of fermionic atoms. For a Fermi gas in the BCS ($k_F|a| \ll 1$) and the unitary ($k_F|a| \gg 1$) regime the equation of state is instead proportional to the Fermi energy $\frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}$, so that $\gamma = 2/3$ and $u_1 = 2\mu / 5m$. In the former limit at very low temperatures $\mu = m v_F^2 / 2$ and $u_1 = v_F / \sqrt{5}$, while in the latter $\mu = m(1 + \beta)v_F^2 / 2$ with $\beta$ being a constant that has been evaluated in various approaches [15]. Quantum Monte Carlo calculations find $\beta \simeq -0.57$ [16, 17] and then $u_1 \simeq 0.29v_F$. These velocities thus differ by a factor $\sqrt{5}$ from those in the homogeneous Fermi gas [12] and would be measured in a sound propagation experiment through the central portion of a strongly elongated trap. It is worth noticing that these results differ from what a 1D calculation would predict: in the latter it is straightforward to show that $u_1 = \sqrt{\gamma \mu / m}$, i.e. $u_1 = v_F$ for a strictly 1D hydrodynamic Fermi gas.

At intermediate couplings, when $k_F|a|$ is finite, $C$ depends on the density and several forms of the equation of state have been proposed [13, 18, 19]. A possible solution is to use again a power law with an effective exponent $\gamma$, which may be inferred from a measured collective mode frequency [18]. The effective value of $C$ can instead be determined from the cloud size. In this work we focus on a gas with a negative scattering length and choose the equation of state given by Heiselberg in the Galitskii approximation [15]. This is

$$\mu(n) = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \times \left[1 + \frac{c_1 n^{1/3}}{1 - c_2 n^{1/3}} \left(2 + \frac{c_2 n^{1/3}}{3(1 - c_2 n^{1/3})}\right)\right], \quad (7)$$

with $c_1 = 2a/(9\pi)^{1/3}$ giving the mean-field contribution and $c_2 = 6[11 - 2 \ln 2] (3\pi^2)^{1/3} a/(35\pi)$ the second-order contribution from the ladder diagrams, which takes into account the opening of the superfluid gap. Equation (7) reproduces the result $\mu = m[1 + 4ak_F/(3\pi)] v_F^2 / 2$ in the limit of a dilute system, and is approximately valid at intermediate densities in the unitary limit where it gives $m(1 + \beta)v_F^2 / 2$. The ratio between $c_1$ and $c_2$ in this case yields $\beta \simeq -0.67$. The results for the sound velocity as obtained from a self-consistent evaluation of the chemical potential and of the equilibrium density are shown in Fig. 1. In this calculation we have taken a linear density $n$ corresponding to the chemical potential of $2 \times 10^3$ fermions in a cigar-shaped confinement with trap frequencies $\omega_\perp = 100 s^{-1}$ and $\omega_z = 6.3 s^{-1}$ (see below). The BCS and unitary limits are recovered for $a \to 0^-$ and $a \to -\infty$, respectively. The dashed line in Fig. 1 shows the sound velocity obtained within the mean-field model for a dilute Fermi gas by setting $c_2 = 0$ in Eq. (7). The mean-field model well describes the BCS limit, but as the coupling increases it predicts a strong drop of the sound velocity heralding collapse. The location and width of the transition from the BCS to the unitary regime are determined by the system parameters.

**Effect of axial potentials.** A weak harmonic confinement along the $z$ direction induces a smooth inhomogeneity in the axial density profile, which can be treated within a local density approximation. Assuming again a power-law equation of state, Eq. (3) leads to the velocity field

$$v_1(z) = \left(\frac{\bar{\mu}}{m}(1 + \gamma)\right)^{1/2} \left(1 - \frac{z^2}{Z_{TF}^2}\right)^{1/2}, \quad (8)$$

with $Z_{TF} = \sqrt{2\mu/m\omega_z^2}$, which corresponds to an oscillatory motion with frequency $\omega = \omega_z / \sqrt{2 + 2/\gamma}$ and velocity $u_1$ at the center of the trap. Even though $u_1$ depends on $C$, the oscillation frequency is the same in the BCS and in the unitary limit, since the product between the wavenumber $q \propto 1/Z_{TF}$ and the velocity $u_1 \propto \sqrt{\mu}$ does not depend on the chemical potential.

We next carry out a numerical study of sound propagation using the hydrodynamic equations (11) and (12) in a set-up simulating that used in the experiments of Andrews et al. [12]. In addition to radial and axial confinements provided by a strongly elongated harmonic trap, we switch on a hole-digging potential $U(z)$ at the center of the trap. The atoms are thereby expelled into two density perturbations that travel towards the ends of the trap. Since the perturbation velocity is mainly in the $z$ direction and imaging techniques will provide col-
umns densities, we focus on the evolution of the radially-integrated density profile \( \tilde{n}(z) \). The numerical calculations are carried out for a system of \( N = 2 \times 10^3 \) atoms equally distributed in two spin-polarized states, with trap frequencies \( \omega_z = 100 \text{ s}^{-1} \) and \( \omega_\perp = 6.3 \text{ s}^{-1} \). The potential \( U(z) \) has a width \( w \) equal to 5% of the axial width of the density profile and an amplitude \( U_0 \) of magnitude up to \( \bar{\omega}_z \). In Fig. 2 we illustrate the time evolution of \( \tilde{n}(z) \) for three values of the scattering length, \( a = -80, -10^5 \), and \(-10^6\) Bohr radii. The size of the cloud decreases on increasing the coupling strength and at the same time the speed of the density perturbation also decreases.

From the evolution of the integrated density we calculate the mean velocity \( \langle v_z \rangle \) of the perturbation in a time interval \( \Delta t = 5/\omega_\perp \). The results for \( \langle v_z \rangle \) as a function of the coupling strength and for two amplitudes of the perturbing potential are shown in Fig. 3. The comparison is also illustrated between the case with \( \lambda \equiv \omega_z/\omega_\perp \approx 0.06 \) and that with \( \lambda = 0.01 \) at the same zero-coupling Fermi velocity. For weak perturbations (for example \( U_0 = 0.16\bar{\omega}_z \)), full symbols in Fig. 3, giving \( \delta n \) of the order of a few percent of the equilibrium density at the trap center, the speed of sound for both values of \( \lambda \) is found to be in good agreement with the theoretical prediction given in Eq. (5) for the Q1D gas (thick solid line). The slight difference seen in Fig. 3 may be accounted for by the effect of the weak confinement causing the speed of sound to decrease while the density perturbation moves away from the central region of the trap (see Eq. 5). Beyond the perturbative regime (empty symbols in Fig. 3) referring to \( U_0 = 0.8\bar{\omega}_z \) and \( \delta n/n_0 \approx 20\% \) we find an upward shift of the mean velocity as a result of the increase in the local density. The overall behavior of \( \langle v_z \rangle \) is still the same as in the perturbative regime. The ratio between the values at zero and large \(|a|\) is about 0.61, to be compared with \( \sqrt{1 + \beta} \approx 0.57 \) for both the Q1D gas (\( \lambda = 0 \)) and the homogeneous gas.

Finally, we compare in Fig. 4 the sound velocity for \( \lambda \approx 0.06 \) in a superfluid Fermi gas with that in a normal Fermi gas at a temperature \( T = 0.2 T_F \), \( T_F \) being the Fermi temperature at zero coupling. The dynamics of the normal gas has been studied by the Vlasov-Landau kinetic equations as described in Ref. 24. Our method involves binary collisions between the particles and includes Pauli-blocking effects. At low coupling the normal Fermi mixture is in the collisionless zero-sound regime and sound propagation is associated with an anisotropic deformation of the Fermi surface, the zero-sound velocity being close to \( v_F \) as shown in Fig. 4. With increasing attractions the velocity of the perturbation drops. This drop occurs at about \( a \simeq -10^4 a_0 \), and can be attributed to both the transition towards the first-sound velocity \( v_F/\sqrt{5} \) and the instability of the approaching collapse of the normal state. The transition to the superfluid state restores stability.

In summary, we have studied sound propagation in a superfluid Fermi gas confined inside a strongly elongated harmonic trap. We have found that the sound velocity diminishes as the superfluid goes from the BCS to the unitary regime on increasing the strength of the attractions between the atoms, and that in both limits the speed of sound differs by a factor \( \sqrt{3/5} \) from that in the superfluid three-dimensional gas. The numerical study of the superfluid gas dynamics in a weak axial confinement shows that the value of the speed of sound around the central region of the trap is close to that in the fully quasi-one-dimensional gas. However, in situations of experimental relevance, a density perturbation of or-
der 20% could give rise to upward shifts of the speed of sound by 10–12%. These effects could be observed in experimental set-ups of trapped superfluid Fermi gases and thereby provide further insight in their superfluid state.

This work was partially supported by an Advanced Research Initiative of Scuola Normale Superiore di Pisa.

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