On Generalized Pre-Continuous Fuzzy Proper Function from a Fuzzy Topological Space to Another Fuzzy Topological Space

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Abstract: The purpose of this paper is to introduce and study the concepts of fuzzy generalized pre-open sets, fuzzy generalized pre-closed sets and generalized pre-continuous fuzzy proper functions. Some of its properties have also been investigated. Relation between continuous fuzzy proper functions and generalized pre-continuous fuzzy proper functions has also been established.

Keywords: Fuzzy generalized pre-closed sets, Fuzzy proper function, Generalized pre-continuous fuzzy proper functions.

Introduction and preliminaries: The concept of fuzzy set and fuzzy set operations were first introduced by Zadeh's [1]. The notion of a fuzzy subsets naturally plays a significant role in the study of fuzzy topology was introduced by Change [2].

Fuzzy pre-semi-closed sets in fuzzy topological spaces (or f.t.s. in short) on X introduced and investigated their properties by Murugesan and Thangavelu [3].

In this paper generalization of fuzzy closed sets are based on topological notions which were introduced and studied by Leving [4], 1970.

In 1997, Balasubramanium and Sundaram [5] introduced the concept fuzzy generalized closed set and spaces. Recently EL-Shafei et al [6], 2007 introduced and study semi generalized continuous function in fuzzy topological space.

Kalaiselvi and Seenivasan [7] defined fuzzy generalized pre-closed set in fuzzy topological space (or f.t.s. in short) on X, I[X] is the collection of all mapping from X in to I=[0,1]. A fuzzy set A in f.t.s (X,T) is said to be quasi-coincident (or q-coincident in short) with a fuzzy set B, denoted by AqB, if there exists x ∈ X such that μ_A(x) + μ_B(x) > 1 [8]. If A,B ∈ I[X], μ_B(x) ≤ μ_A(x) ∀x ∈ X, then B is said to be a fuzzy subset of A and denoted by B ⊆ A [2]. A fuzzy point P_x in a set X is a fuzzy set with membership function μ_{P_x}(x), defined by:

μ_{P_x}(x) = r for x = x_p,

and μ_{P_x}(x) = 0 for x ≠ x_p.

where r ∈ (0,1), x_p is called the support of P_x, and r the value of P_x [9]. A fuzzy point P_x in X is called belong to a fuzzy set A in X (notation: P_x ∈ A) iff r ≤ μ_A(x), a fuzzy set A in X is the union of all its fuzzy points [10].

In the present paper we study the properties of generalized pre-continuous fuzzy proper functions and prove results about this concept.
1. Basic Concept of a Fuzzy Topological Space on Fuzzy Set $\tilde{A}$:

In this section, we present fuzzy topological space on fuzzy set with fundamental concepts in fuzzy topological space on fuzzy set, such as quasi-coincident, complement of fuzzy set, maximal fuzzy set, etc.

Remark (1.1)[11]: Let $\tilde{A} \in I^X$, $\tilde{C} \in I^Y$, $P(\tilde{A}) = \{ \tilde{B} \in I^X: \tilde{B} \subseteq \tilde{A} \}$ and $P(\tilde{C}) = \{ \tilde{D} \in I^Y: \tilde{D} \subseteq \tilde{C} \}$.

Definition (1.2)[11]: Let $\tilde{A}, \tilde{B}$, be two fuzzy sets in $X$ with $\tilde{B} \in P(\tilde{A})$ then the complement of $\tilde{B}$ relative to $\tilde{A}$, denoted by $(\tilde{B})^c$, is defined by:

$$\mu_{(\tilde{B})^c}(x) = \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x).$$

Definition (1.3)[10]: Let $P^c_x \subseteq \tilde{A}$ and $\tilde{B} \in P(\tilde{A})$ are said to be q-coincident relative to $\tilde{A}$ [written as $P^c_x q_{\tilde{A}} \tilde{B}$] if there exists $x \in X$, such that $r + \mu_{\tilde{B}}(x) > \mu_{\tilde{A}}(x)$. If $P^c_x$ is not q-coincident with $\tilde{B}$ in $\tilde{A}$, we denote this $P^c_x \notin q_{\tilde{A}} \tilde{B}$.

Definition (1.4)[12]: If $\tilde{B} \in P(\tilde{A})$ then $\tilde{B}$ is said to be maximal if $\forall x \in X$, $\mu_{\tilde{B}}(x) = 0$, then $\mu_{\tilde{B}}(x) = \mu_{\tilde{A}}(x)$.

Definition (1.5)[13]: A collection $\tilde{T}$ of a fuzzy subsets of $\tilde{A}$ is said to be fuzzy topology on $\tilde{A}$, if:

(a) $\emptyset, \tilde{A} \in \tilde{T}$.
(b) If $\tilde{B}, \tilde{C} \in \tilde{T}$, then $\tilde{B} \cap \tilde{C} \in \tilde{T}$.
(c) If $\tilde{B}_j \in \tilde{T}, \forall j \in J$, where $J$ is any index set, then $\bigcup_{j \in J} \tilde{B}_j \in \tilde{T}$.

$(\tilde{A}, \tilde{T})$ is said to be a fuzzy topological space on fuzzy set $\tilde{A}$ (or f.t.s on $\tilde{A}$) and the members of $\tilde{T}$ are said to be fuzzy open sets of $\tilde{A}$. We denote $\tilde{T}^c$ the family of fuzzy closed sets of $\tilde{A}$, that is $\tilde{B} \in \tilde{T}^c$ if and only if $\mu_{\tilde{B}}(x) = \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x) \in \tilde{T}$.

Definition (1.6)[13]: Let $\tilde{B}$ be a fuzzy set in a f.t.s $(\tilde{A}, \tilde{T})$. The closure $\tilde{B}$ (or $\text{cl}(\tilde{B})$) and interior $\tilde{B}^o$ (or $\text{int}(\tilde{B})$) of $\tilde{B}$ are defined, respectively, by:

$$\text{cl}(\tilde{B}) = \cap \{ \tilde{F} : \tilde{B} \subseteq \tilde{F}, \tilde{F} \in \tilde{T} \}$$

$$\text{int}(\tilde{B}) = \cup \{ \tilde{G} : \tilde{G} \subseteq \tilde{B}, \tilde{G} \in \tilde{T} \}$$

2. Fuzzy Generalized pre-Open Set and Fuzzy Generalized pre-Closed Set in a Fuzzy Topological Space on fuzzy set:

In this section, we give definitions fuzzy generalized closed sets, fuzzy generalized open sets and fuzzy proper function with some properties.

Definition (2.1)[13]: A fuzzy set $\tilde{B}$ in a fuzzy topological space $(\tilde{A}, \tilde{T})$ is said to be

1) Fuzzy pre-open set (in short, f.p.o.s.) if $\tilde{B} \subseteq \text{int}(\text{cl}(\tilde{B}))$
2) Fuzzy pre-closed set (in short, f.p.c.s.) if $\text{cl}(\text{int}(\tilde{B})) \subseteq \tilde{B}$

Definition (2.2)[13]: Let $\tilde{B}$ be a fuzzy set in a f.t.s $(\tilde{A}, \tilde{T})$. The pre-closure $\tilde{B}$ (or $\text{p-cl}(\tilde{B})$) of $\tilde{B}$ is defined by:

$$\text{p-cl}(\tilde{B}) = \cap \{ \tilde{F} : \tilde{B} \subseteq \tilde{F} \cap \tilde{F} \text{ is a fuzzy pre-closed set in } \tilde{A} \}$$
Definition (2.3)[13]: Let $\tilde{B}$ be a fuzzy set in a f.t.s $(\tilde{A}, \tilde{T})$. The pre-interior $\tilde{B}^p$ (or p-int($\tilde{B}$)) of $\tilde{B}$ is defined by:

$$p\text{-int}(\tilde{B}) = \cup \{ \tilde{G} : \tilde{G} \subseteq \tilde{B}, \text{is a fuzzy pre-open set in } \tilde{A} \}$$

Definition (2.4)[14]: A fuzzy set $\tilde{B}$ in a fuzzy topological space $(\tilde{A}, \tilde{T})$ is said to be fuzzy generalized pre-closed set (in short, f.g.p.c.s.) if p-cl($\tilde{B}$) $\subseteq \tilde{G}$, whenever $\tilde{B} \subseteq \tilde{G}$ and $\tilde{G}$ is fuzzy open set in $(\tilde{A}, \tilde{T})$.

Theorem (2.5)[14]: In a fuzzy topological space $(\tilde{A}, \tilde{T})$ the complement of a fuzzy generalized pre-closed set is fuzzy generalized pre-open set.

Definition (2.6): Let $\tilde{C}$ be a fuzzy set in a f.t.s $(\tilde{A}, \tilde{T})$ and $P_x^G$ is a fuzzy point of $X$. Then $\tilde{C}$ is called:

(a) Fuzzy generalized pre-neighborhood (f.g.p.nbd) (resp. fuzzy neighborhood (f.nbd)) of $P_x^G$ if and only if there exists a fuzzy generalized pre-open set (resp. fuzzy open set) $\tilde{B}$ in $(\tilde{A}, \tilde{T})$, such that $\tilde{P}_x^G \subseteq \tilde{B} \subseteq \tilde{C}$.

(b) Fuzzy generalized pre-quasi neighborhood (f.g.p.q.nbd)(resp. fuzzy quasi neighborhood (f.q.nbd)) of $P_x^G$ if and only if there exists a fuzzy generalized pre-open set (resp. fuzzy open set) $\tilde{B}$ in $(\tilde{A}, \tilde{T})$, such that $\tilde{P}_x^G \tilde{q}\tilde{B} \subseteq \tilde{C}$.

Definition (2.7)[9]: A fuzzy subset $\tilde{f}$ of $X \times Y$ is said to be a fuzzy proper function from $\tilde{A} \in \mathcal{X}$ to $\tilde{B} \in \mathcal{Y}$ if

a. $\tilde{f}(x, y) \leq \min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(y)), \forall (x, y) \in X \times Y$.

b. $\forall x \in X, \exists y^* \in Y$ such that $\tilde{f}(x, y^*) = \mu_{\tilde{A}}(x)$ and $\tilde{f}(x, y) = 0$ if $y \neq y^*$.

Definition (2.8)[9]: Let $\tilde{f}: \tilde{A} \to \tilde{B}$ be a fuzzy proper function, define the correspondent F-proper function $\tilde{f}: P(\tilde{A}) \to P(\tilde{B})$ and its reverse F-proper function $\tilde{f}^{-1}: P(\tilde{B}) \to P(\tilde{A})$ by

i. $\tilde{f}: P(\tilde{A}) \to P(\tilde{B}), \mu_{\tilde{f}(C)}(y) = \sup\{\tilde{f}(x, y) \land \mu_{\tilde{C}}(x) : x \in X\}, \forall y \in Y \text{ and } \forall \tilde{C} \in P(\tilde{A})$.

ii. $\tilde{f}^{-1}: P(\tilde{B}) \to P(\tilde{A}), \mu_{\tilde{f}^{-1}(x)}(y) = \sup\{\tilde{f}(x, y) \land \mu_{\tilde{B}}(y) : y \in Y\}, \forall x \in X \text{ and } \forall \tilde{D} \in P(\tilde{B})$.

Proposition (2.9)[12]: Let $\tilde{f}: \tilde{A} \to \tilde{B}$ be a fuzzy proper function, if $\tilde{G} \in P(\tilde{B})$ is maximal, then $\tilde{f}^{-1}(\tilde{G}^c) = [\tilde{f}^{-1}(\tilde{G})]^c$.

Proposition (2.10)[12]: For fuzzy proper function $\tilde{f}: \tilde{A} \to \tilde{B}$. Then

i. $\tilde{C} \subseteq \tilde{f}^{-1}(\tilde{f}(\tilde{C})), \forall \tilde{C} \in P(\tilde{A})$.

ii. $\tilde{f}(\tilde{f}^{-1}(\tilde{D})) \subseteq \tilde{D}, \forall \tilde{D} \in P(\tilde{B})$.

3. Generalized pre-Continuous Fuzzy Proper Functions:

In this section, a new type of fuzzy continuous functions which are called fuzzy generalized pre-continuous functions are defined and their properties are studied.

Definition (3.1): A fuzzy proper function $\tilde{f}$ from a f.t.s $(\tilde{A}, \tilde{T})$ to f.t.s $(\tilde{B}, \tilde{T})$ is said to be generalized pre-continuous if $\tilde{f}^{-1}(\tilde{G})$ is f.g.p.o.s. in $\tilde{A}$, for each fuzzy open set $\tilde{G}$ in $\tilde{B}$.

Theorem (3.2): If $\tilde{f}: (\tilde{A}, \tilde{T}_1) \to (\tilde{B}, \tilde{T}_2)$ is fuzzy continuous, then it is fuzzy generalized pre-continuous.

Proof: Let $\tilde{G}$ be a fuzzy open set in $\tilde{B}$. Since $\tilde{f}$ is fuzzy continuous function, then $\tilde{f}^{-1}(\tilde{G})$ is fuzzy open set in $\tilde{A}$, this implies $\tilde{f}^{-1}(\tilde{G}) = \text{int}(\tilde{f}^{-1}(\tilde{G}))$ ....(3.1)
Since \( f^{-1}(\bar{G}) \subseteq cl(f^{-1}(\bar{G})) \), then \( int(f^{-1}(\bar{G})) \subseteq int(cl(f^{-1}(\bar{G}))) \).

From (3.1), we have \( f^{-1}(\bar{G}) \subseteq int(cl(f^{-1}(\bar{G}))) \), thus \( f^{-1}(\bar{G}) \) is fuzzy pre-open set.

This implies \( p - int(f^{-1}(\bar{G})) = f^{-1}(\bar{G}) \) ...(3.2)

Let \( \bar{F} \subseteq f^{-1}(\bar{G}) \) such that \( \bar{F} \) is a fuzzy closed set in \( \bar{A} \), from (3.2) we have \( \bar{F} \subseteq p - int(f^{-1}(\bar{G})) \). hence \( f^{-1}(\bar{G}) \) is fuzzy generalized pre-open set in \( \bar{A} \), thus \( \bar{f} \) is fuzzy generalized pre-continuous function.

**Theorem (3.3):** Let \( \bar{f}: (\bar{A}, \bar{T}_1) \to (\bar{B}, \bar{T}_2) \) be a fuzzy proper function, then if \( \bar{G} \) is maximal fuzzy closed subset of \( \bar{B} \) then \( \bar{f} \) is fuzzy generalized pre-continuous function if and only if \( \bar{f}^{-1}(\bar{G}) \) is f.g.p.c.s., \( \forall \bar{G} \in \bar{T}_2^c \).

**Proof:** Let \( \bar{G} \) is maximal fuzzy closed set in \( \bar{B} \) with \( P_\bar{y}^r \in \bar{G} \), then \( \bar{G}^c \) is fuzzy open set. Since \( \bar{f} \) is fuzzy generalized pre-continuous function, then \( \bar{f}^{-1}(\bar{G}^c) \) is f.g.p.o.s. in \( \bar{A} \).

Since \( \bar{G} \) is maximal fuzzy set, thus \( \bar{f}^{-1}(\bar{G}^c) = [f^{-1}(\bar{G})]^c \) this implies \( [f^{-1}(\bar{G})]^c \) is f.g.p.o.s. in \( \bar{A} \).

Hence \( [\bar{f}^{-1}(\bar{G})]^c = f^{-1}(\bar{G}) \) is f.g.p.c.s. in \( \bar{A} \).

Conversely, Let \( \bar{G} \) be a maximal fuzzy closed set in \( \bar{B} \)

Then \( \bar{f}^{-1}(\bar{G}) \) is f.g.p.c.s. in \( \bar{A} \) and \( \bar{G}^c \) is fuzzy open set in \( \bar{B} \).

This implies that \( [\bar{f}^{-1}(\bar{G})]^c \) is f.g.p.o.s. in \( \bar{A} \).

But \( [\bar{f}^{-1}(\bar{G})]^c = f^{-1}(\bar{G}^c) \), we have \( f^{-1}(\bar{G}^c) \) is f.g.p.o.s. in \( \bar{A} \).

This implies \( \bar{f} \) is fuzzy generalized pre-continuous function.

**Theorem (3.4):** Let \( \bar{f}: (\bar{A}, \bar{T}_1) \to (\bar{B}, \bar{T}_2) \) be a fuzzy proper function, then the following statements are equivalent:

1. For each fuzzy point \( P_\bar{y}^r \) in \( \bar{A} \), for each \( y \in Y \) and for each fuzzy open set \( \bar{G} \) in \( \bar{B} \) with \( P_\bar{y}^r \in \bar{G} \), there exists f.g.p.o.s. \( \bar{H} \) in \( \bar{A} \), such that \( P_\bar{y}^r \in \bar{H} \) and \( \bar{f}(\bar{H}) \subseteq \bar{G} \).
2. For each fuzzy point \( P_\bar{y}^r \) in \( \bar{A} \), for each \( y \in Y \) and for each fuzzy neighborhood \( \bar{N} \) of \( P_\bar{y}^r \) in \( \bar{B} \), \( f^{-1}(\bar{N}) \) is fuzzy generalized pre-neighborhood of \( P_\bar{y}^r \) in \( \bar{A} \).

**Proof:** Let \( \bar{G} \) be a fuzzy open set in \( Y \) with \( P_\bar{y}^r \in \bar{G} \).

Let \( \bar{H} \) be f.g.p.o.s. in \( \bar{A} \), such that \( P_\bar{y}^r \in \bar{H} \) and \( \bar{f}(\bar{H}) \subseteq \bar{G} \).

Since \( P_\bar{y}^r \in \bar{G} \) and \( \bar{G} \) fuzzy open set, thus \( \bar{G} \) is f.nbd of \( P_\bar{y}^r \).

Since \( \bar{H} \) is f.g.p.o.s. in \( I^X \) and \( \bar{f}(\bar{H}) \subseteq \bar{G} \), then \( \bar{f}^{-1}(\bar{f}(\bar{H})) \subseteq \bar{f}^{-1}(\bar{G}) \Rightarrow \bar{H} \subseteq \bar{f}^{-1}(\bar{G}) \), hence \( P_\bar{y}^r \in \bar{H} \subseteq \bar{f}^{-1}(\bar{G}) \).

Since \( \bar{H} \) is f.g.p.o.s. in \( \bar{A} \), then \( \bar{f}^{-1}(\bar{G}) \) is f.g.p.nbd of \( P_\bar{y}^r \) in \( \bar{A} \).

(2 \( \Rightarrow \) 1) Suppose that \( \bar{G} \) is a fuzzy open set in \( \bar{B} \) with \( P_\bar{y}^r \in \bar{G} \), then \( \bar{G} \) is f.nbd of \( P_\bar{y}^r \Rightarrow \bar{f}^{-1}(\bar{G}) \) is f.g.p.nbd of \( P_\bar{y}^r \) in \( I^X \), this implies there exists f.g.p.o.s. \( \bar{H} \) in such that \( P_\bar{y}^r \in \bar{H} \subseteq \bar{f}^{-1}(\bar{G}) \).

Since \( \bar{H} \subseteq \bar{f}^{-1}(\bar{G}) \Rightarrow \bar{f}(\bar{H}) \subseteq \bar{G} \).

**Theorem (3.5):** Let \( \bar{f}: (\bar{A}, \bar{T}_1) \to (\bar{B}, \bar{T}_2) \) be a fuzzy proper function, then the following statements are equivalent:

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1. For each fuzzy point \( P^r_y \) in \( A \), for each \( y \in Y \) and for each f.nbd \( M \) of \( P^r_y \) in \( B \), \( \tilde{f}^{-1}(M) \) is f.g.p.nbd of \( P^r_x \) in \( A \).

2. For each fuzzy point \( P^r_y \) in \( A \), for each \( y \in Y \) and for each f.nbd \( G \) of \( P^r_y \) in \( B \), there exists f.g.p.nbd \( H \) of \( P^r_x \) in \( A \), such that \( P^r_y \in H \) and \( f(H) \subseteq G \).

**Proof:** (1) \( \Rightarrow \) 2 Let \( G \) be f.nbd of \( P^r_y \) in \( B \), then \( \tilde{f}^{-1}(G) \) is f.g.p.nbd of \( P^r_x \) in \( A \) by proposition (2.2), \( \tilde{f}(\tilde{f}^{-1}(G)) \subseteq G \).

Let \( H = \tilde{f}^{-1}(G) \) then there exists f.g.p.nbd of \( P^r_x \) in \( A \) such that \( f(H) \subseteq G \).

(2) \( \Rightarrow \) 1 suppose that \( M \) is f.nbd of \( P^r_y \) in \( B \), then there exists f.g.p.nbd \( H \) of \( P^r_x \) in \( A \) such that \( f(H) \subseteq M \) this implies that \( H \subseteq \tilde{f}^{-1}(M) \).

Since \( H \) is f.g.p.nbd of \( P^r_x \) in \( A \), then \( \exists \) f.g.p.o.s. \( G \) in \( A \) such that \( P^r_y \in G \subseteq H \subseteq \tilde{f}^{-1}(M) \), hence \( P^r_x \in G \subseteq \tilde{f}^{-1}(M) \) thus \( \tilde{f}^{-1}(M) \) is f.g.p.nbd of \( P^r_x \) in \( A \).

**Theorem (3.6):** Let \( \tilde{f}: (\tilde{A}, \tilde{T}_1) \rightarrow (\tilde{B}, \tilde{T}_2) \) be a fuzzy proper function, then the following statements are equivalent:

1. For each fuzzy point \( P^r_x \) in \( \tilde{A} \), for each \( y \in Y \) and for each f.nbd \( G \) of \( P^r_y \) in \( \tilde{B} \), there exists f.g.p.nbd \( H \) of \( P^r_x \) in \( \tilde{A} \), such that \( P^r_Y \in H \) and \( f(H) \subseteq G \).

2. For each fuzzy point \( P^r_y \) in \( \tilde{A} \), for each \( y \in Y \) and for each fuzzy open set \( \tilde{M} \) in \( \tilde{B} \) such that \( P^r_y \in \tilde{M} \), there exists f.g.p.o.s. \( \tilde{N} \) in \( \tilde{A} \), such that \( P^r_y \in \tilde{N} \) and \( \tilde{f}(\tilde{N}) \subseteq \tilde{M} \).

**Proof:** (1) \( \Rightarrow \) 2 Let \( P^r_x \) be a fuzzy point in \( \tilde{A} \) and \( \tilde{M} \) be a fuzzy open set in \( \tilde{B} \) such that \( P^r_Y \in \tilde{M} \), then \( \mu_{\tilde{M}}(x_i) > \mu_{(P^r_Y)^c}(y) \) this implies \( (P^r_Y)^c \subseteq \tilde{M} \).

Since any fuzzy set is the union of all its fuzzy points, then \( (P^r_Y)^c = \bigcup_{i \in I} P^{r_{yi}} \)

Such that \( r_i = \mu_{\tilde{B}}(y_i) - r \), thus \( \bigcup_{i \in I} P^{r_{yi}} \subseteq \tilde{M} \subseteq \tilde{M} \), this implies that \( P^{r_{yi}} \subseteq \tilde{M} \), \( \forall i \)

Since \( \tilde{M} \) is a fuzzy open set, then \( \tilde{M} \) is a f.nbd of \( P^{r_{yi}} \) this implies that there exists f.g.p.nbd. \( \tilde{H} \) of \( P^{r_{yi}} \) in \( \tilde{A} \), such that \( P^{r_{yi}} \in \tilde{H} \), \( f(\tilde{H}) \subseteq \tilde{M} \) and \( r_i = \mu_{\tilde{A}}(x_i) - r \).

Since \( \tilde{H} \) is f.g.p.nbd of \( P^{r_{yi}} \), then there exists f.g.p.o.s \( \tilde{N} \) in \( \tilde{A} \) such that \( P^{r_{yi}} \in \tilde{N} \subseteq \tilde{H} \), from \( \tilde{N} \subseteq \tilde{H} \) we have \( \tilde{f}(\tilde{N}) \subseteq f(\tilde{H}) \subseteq \tilde{M} \), thus \( \tilde{f}(\tilde{N}) \subseteq \tilde{M} \), and from \( P^{r_{yi}} \in \tilde{N} \), \( \forall i \) we have \( \bigcup_{i \in I} P^{r_{yi}} \subseteq \tilde{N} \).

Since \( \bigcup_{i \in I} P^{r_{yi}} = (P^r_Y)^c \) this implies that \( (P^r_Y)^c \subseteq \tilde{N} \), therefore \( \mu_{\tilde{A}}(x_i) - r \leq \mu_{\tilde{N}}(x) \)

Thus \( P^r_Y \subseteq \tilde{N} \).

(2) \( \Rightarrow \) 1) Let \( \tilde{M} \) be a fuzzy open set in \( \tilde{B} \) such that \( P^r_Y \subseteq \tilde{M} \), then \( (P^r_Y)^c = \bigcup_{i \in I} P^{r_{yi}} \subseteq \tilde{M} \subseteq \tilde{M} \) this implies \( \tilde{M} \) is f.nbd of \( P^{r_{yi}} \).

Since \( \tilde{M} \) be a fuzzy open set in \( \tilde{B} \) such that \( P^r_Y \subseteq \tilde{M} \), then there exists f.g.p.o.s. \( \tilde{N} \) of \( P^r_Y \) such that \( P^r_Y \subseteq \tilde{N} \subseteq \tilde{M} \), from \( P^r_Y \subseteq \tilde{M} \) we have \( (P^r_Y)^c = \bigcup_{i \in I} P^{r_{yi}} \subseteq \tilde{N} \subseteq \tilde{N} \), hence \( \tilde{N} \) is f.g.p.nbd of \( P^{r_{yi}} \) and \( \tilde{f}(\tilde{N}) \subseteq \tilde{M} \).

**Theorem (3.7):** Let \( \tilde{f}: (\tilde{A}, \tilde{T}_1) \rightarrow (\tilde{B}, \tilde{T}_2) \) be a fuzzy proper function, then the following statements are
equivalent:

1. For each fuzzy point $P_x^f$ in $\tilde{A}$, for each $y \in Y$ and for each fuzzy open set $\tilde{G}$ in $\tilde{B}$ such that $P_y^f q \tilde{G}$, there exists f.g.p.o.s. $H$ in $\tilde{A}$ such that $P_x^f q H$ and $f(H) \subseteq \tilde{G}$.

2. For each fuzzy point $P_x^f$ in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $M$ of $P_x^f$, $f^{-1}(\tilde{M})$ is f.g.p.q.nbd. of $P_x^f$.

Proof: (1 $\Rightarrow$ 2) Let $P_x^f$ be a fuzzy point in $\tilde{A}$ and $M$ be a f.q.nbd. of $P_y^f$ in $\tilde{B}$.

Then there exists a fuzzy open set $\tilde{N}$ in $Y$ such that $P_y^f q \tilde{N} \subseteq \tilde{M}$.

Since $\tilde{N}$ is fuzzy open set in $\tilde{B}$ and $P_y^f q \tilde{N}$, then there exists f.g.p.o.s. $H$ in $\tilde{A}$ such that $P_x^f q H$ and $f(H) \subseteq \tilde{N} \subseteq \tilde{M}$ that is $\bar{f}(\tilde{H}) \subseteq \tilde{M}$ thus $H \subseteq f^{-1}(\tilde{M})$

This implies that $P_x^f q \tilde{H} \subseteq f^{-1}(\tilde{M})$, therefore $f^{-1}(\tilde{M})$ is f.g.p.q.nbd of $P_x^f$.

(2 $\Rightarrow$ 1) Suppose that $\tilde{G}$ be a fuzzy open set in $\tilde{B}$ such that $P_y^f q \tilde{G}$

Since $P_y^f q \tilde{G} \subseteq \tilde{G}$, then $\tilde{G}$ is f.q.nbd. of $P_y^f$ this implies that $f^{-1}(\tilde{G})$ is f.g.p.q.nbd of $P_x^f$ so $\exists$ f.g.p.o.s. $\tilde{H}$ in $\tilde{A}$ such that $P_x^f q \tilde{H} \subseteq f^{-1}(\tilde{G})$ thus $\bar{f}(\tilde{H}) \subseteq \tilde{G}.$

Theorem (3.8): Let $\tilde{f}: (\tilde{A}, \tilde{T}_1) \rightarrow (\tilde{B}, \tilde{T}_2)$ be a fuzzy proper function, then the following statements are equivalent:

1. For each fuzzy point $P_x^f$ in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $\tilde{G}$ of $P_x^f$, $f^{-1}(\tilde{G})$ is f.g.p.q.nbd. of $P_x^f$.

2. For each fuzzy point $P_x^f$ in $\tilde{A}$, for each $y \in Y$ and for each f.q.nbd. $\tilde{M}$ of $P_x^f$, there exists f.g.p.q.nbd. $\tilde{N}$ of $P_x^f$ such that $f(\tilde{N}) \subseteq \tilde{M}$.

Proof: (1 $\Rightarrow$ 2) Let $P_x^f$ be a fuzzy point in $\tilde{A}$ and $\tilde{M}$ be a f.q.nbd. of $P_y^f$ in $\tilde{B}$.

Then $f^{-1}(\tilde{M})$ is a f.g.p.q.nbd. of $P_x^f$, let $\tilde{N} = f^{-1}(\tilde{M})$ so $f(\tilde{N}) = f(\tilde{f}^{-1}(\tilde{M})) \subseteq \tilde{M}$.

(2 $\Rightarrow$ 1) Suppose that $\tilde{G}$ is f.q.nbd of $P_y^f$

Then there exists f.g.p.q.nbd $\tilde{H}$ of $P_x^f$ such that $f(\tilde{H}) \subseteq \tilde{G}$ so $\tilde{H} \subseteq f^{-1}(\tilde{G})$.

Since $\tilde{H}$ is f.g.p.q.nbd of $P_x^f$, then there exists f.g.p.o.s. $\tilde{E}$ in $\tilde{A}$ such that $P_x^f \tilde{E} \subseteq f^{-1}(\tilde{G})$ thus $f^{-1}(\tilde{G})$ is f.g.p.q.nbd of $P_x^f$.

Theorem (3.9): If $\tilde{f}: (\tilde{A}, \tilde{T}_1) \rightarrow (\tilde{B}, \tilde{T}_2)$ is fuzzy generalized pre-continuous function and $\tilde{g}: (\tilde{B}, \tilde{T}_2) \rightarrow (\tilde{C}, \tilde{T}_3)$ is fuzzy continuous function, then $\tilde{g} \circ \tilde{f}: (\tilde{A}, \tilde{T}_1) \rightarrow (\tilde{C}, \tilde{T}_3)$ is fuzzy generalized pre-continuous function.

Proof: Let $\tilde{G}$ be a fuzzy open set in $\tilde{Z}$.

Since $\tilde{g}$ is fuzzy continuous function, then $\tilde{g}^{-1}(\tilde{G})$ is fuzzy open set in $Y$.

Since $\tilde{f}$ is fuzzy generalized pre-continuous function, then $f^{-1}(\tilde{g}^{-1}(\tilde{G}))$ is f.g.p.o.s. in X, from $(\tilde{g} \circ \tilde{f})^{-1}(\tilde{G}) = f^{-1}(\tilde{g}^{-1}(\tilde{G}))$ we have $\tilde{g} \circ \tilde{f}: X \rightarrow Z$ is fuzzy generalized pre-continuous function.

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