On the formulae for the colored HOMFLY polynomials

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Abstract

We provide methods to compute the colored HOMFLY polynomials of knots and links with symmetric representations based on the linear skein theory. By using diagrammatic calculations, several formulae for the colored HOMFLY polynomials are obtained. As an application, we calculate some examples for hyperbolic knots and links, and we study a generalization of the volume conjecture by means of numerical calculations. In these examples, we observe that asymptotic behaviors of invariants seem to have relations to the volume conjecture.

Keywords: colored HOMFLY polynomials, volume conjecture, numerical calculations.

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1 Introduction

This article is devoted to formulae for the colored HOMFLY polynomials of knots and links and its application to the volume conjecture. In general, for a given knot or link, it is difficult to calculate the colored HOMFLY polynomial of it. Therefore, we provide some formulae for the colored HOMFLY polynomials with symmetric representations based on the linear skein theory. These formulae are useful to compute invariants of the knots and links whose diagram has twisted strands with opposite orientations. It is a generalization of the formula of the Jones polynomial [7]. As an application, we explicitly describe invariants of the 5_1 knot, the 6_1 knot, the Whitehead link and the twist knots. Similar invariants are obtained in [1, 12]. Furthermore, we take the limits of these invariants in the context of the volume conjecture by numerical calculations. The volume conjecture is first suggested by Kashaev, and formulated by H. Murakami and J. Murakami using the colored Jones polynomial [3, 10].

Conjectures 1 (Volume Conjecture). Let L be a hyperbolic link, and let \( J_N(L) = J_N(L; q) \) be the colored Jones polynomial associated with the \( N \) dimensional irreducible representation of \( U_q(\mathfrak{sl}(2, \mathbb{C})) \), and let \( q \) be \( \exp \frac{2\pi i}{N} \). Then

\[
2\pi \lim_{N \to \infty} \log \frac{J_N(L)}{N} = \text{vol}(L) + \sqrt{-1}\text{CS}(L),
\]

where \( \text{vol} \) is the hyperbolic volume of the complement of \( L \) in \( \mathbb{S}^3 \) and \( \text{CS} \) is the Chern-Simons invariant of the complement of \( L \) in \( \mathbb{S}^3 \), which is normalized by \( \text{CS}(L) = -2\pi cs(L) \mod \pi^2 \) [2, 11].

Now, the volume conjecture has been studied by many mathematicians. There are several extensions [9], and the numerical calculations are discussed in [11]. In the second half of this article, we consider another extension. Namely, since the Jones polynomial is extended to the HOMFLY polynomial, we discuss a generalization of the volume conjecture using the HOMFLY polynomials by numerical calculations. Here, according to the feature about the limit of the colored HOMFLY polynomials of the figure-eight knot [5], we calculate invariants of the 5_2 knot, the 6_1 knot, and the Whitehead link by numerical calculations. We observe that the asymptotic behaviors of these invariants are similar to that of the figure-eight knot, and that different behavior happens such that there exists limits which does not converge to the volume of corresponding knots and links.

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2 Preliminaries

Let \( a \) and \( q \) be variables in \( \mathbb{C} \). We define symbols by
\[
[n] = \frac{(q^n - q^{-n})}{(q - q^{-1})}, \quad [n; a] = \frac{aq^n - a^{-1}q^{-n}}{q - q^{-1}}, \quad \left[ \begin{array}{c} n \\ r \end{array} \right]_q = \frac{(1 - q^n)(1 - q^{n-1}) \cdots (1 - q^{n-r+1})}{(1 - q^r)(1 - q^{r-1}) \cdots (1 - q)}.
\]

The product is described by descending order with respect to the exponent, and it gives 1 if the product is not defined.

Let \( F \) be an oriented compact surface with \( 2n \) specified points on the boundary. The linear skein of \( F \) is the vector space of formal \( \mathbb{C} \)-linear sums of oriented arcs and link diagrams on \( F \). The arcs consist of \( n \) strands and the terminals of the arcs are connected to the \( 2n \) specified points on \( \partial F \). The linear skein satisfies the following conditions.

- regular isotopy,
- \( L \cup (a \text{ trivial closed curve}) = [0; a]L \), and \( \emptyset = 1 \),
- \( \overrightarrow{x} - \overrightarrow{x} = (q - q^{-1}) \overrightarrow{x} \) (the skein relation),
- \( \overrightarrow{\alpha} = a \to a^{-1} \).

We call a crossing positive or negative if it is the same crossing appearing in the first or second terms of the skein relation respectively. Let \( w(L) \) be the wriite of the oriented arcs and link diagrams \( L \) defined by the difference of the numbers of positive and negative crossings of \( L \). When we normalize a link diagram in the linear skein of \( S^2 \), with no specified points on the boundary, by \( a^{-w(L)}((a - a^{-1})/(q - q^{-1}))^{-1} \), we obtain the HOMFLY polynomial \( H(L; a, q) \). \( H(L; a, q) \) is characterized by
\[
\begin{align*}
aH(\overrightarrow{x}; a, q) - a^{-1}H(\overleftarrow{x}; a, q) &= (q - q^{-1})H(\overrightarrow{x}; a, q), \\
H(L; a, q) &= 1, \quad \text{where } L \text{ is a trivial knot.}
\end{align*}
\]

We remark that \( H(L; q, q^{-\frac{1}{2}}) \) is equal to the Jones polynomial \( V_L(q) \) or equivalently \( J_2(L; q) \).

An integer \( n \) beside the strand indicates \( n \)-parallel strands. For an integer \( n \geq 1 \), an \( n \)th \( q \)-symmetrizer, denoted by the white rectangle with \( n \), is inductively defined by
\[
\begin{align*}
\begin{array}{c}
\hline \\
\hline \end{array} & \quad \begin{array}{c}
\hline \\
\hline \end{array} \\
\begin{array}{c}
\hline \\
\hline \end{array} & = q^{-\frac{n+1}{n}} \begin{array}{c}
\hline \\
\hline \end{array}, \\
\begin{array}{c}
\hline \\
\hline \end{array} & = \frac{q^{-n+1}}{[n]} \begin{array}{c}
\hline \\
\hline \end{array} + \frac{1}{[n]} \begin{array}{c}
\hline \\
\hline \end{array} \quad (n \geq 2).
\end{align*}
\]

It is well-known that \( q \)-symmetrizers have useful properties, which are described by
\[
\begin{align*}
\begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array} & = q^n \begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array}, \\
\begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array} & = - \begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array}, \\
\begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array} & = \frac{1}{[n]} [n - 1; a] \begin{array}{c}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \end{array} \quad (1)
\end{align*}
\]

where \( k + l + m = n \), and the first and second equations hold even if the crossing and the \( l \)th \( q \)-symmetrizer appear in the left hand side of the \( n \)th \( q \)-symmetrizer. In what follows, when endpoints appear in a diagram, it means a local diagram.

**Lemma 2.1.** For positive integers \( m, n (m \geq n) \), the twisted strands can be resolved in the following way.

\[
\alpha_{m,n}^i(a, q) = (-1)^{i(a-1)}(q - q^{-1})^{i-1} \left[ \begin{array}{c} m \\ 1 \end{array} \right]_{q^{-2}} \cdots \left[ \begin{array}{c} m - i + 1 \\ 1 \end{array} \right]_{q^{-2}} \left[ \begin{array}{c} n \\ i \end{array} \right]_{q^{-2}}.
\]
Proof. We define $\alpha_{k,l}$ and $A_{k,l}$ by

$$\alpha_{k,l} = (-1)^l a^{-l} (q - q^{-1})^{l} q^{-l(2(n-k-l)-1)} \begin{bmatrix} m \cr 1 \end{bmatrix}_{q^{-2}} \cdots \begin{bmatrix} m - l + 1 \cr 1 \end{bmatrix}_{q^{-2}} \begin{bmatrix} k + l \cr l \end{bmatrix}_{q^{-2}}$$

Then it is sufficient to show

$$A_{0,0} = \sum_{n=k+l} \alpha_{k,l} A_{k,l} = \sum_{i=0}^{n} \alpha_{n-i,i} A_{n-i,i}. \tag{2}$$

The equation (2) holds for $k + l = 1$ by using the skein relation and the identities in (1). Assume $k + l \geq 2$. We remark that $A_{k,l}$ satisfies the following recursive formula.

$$A_{k,l} = A_{k+1,l} - a^{-1} (q - q^{-1})^{-2(n-k-l)} \begin{bmatrix} m - l \cr 1 \end{bmatrix}_{q^{-2}} A_{k,l+1}.$$  

This formula implies that the coefficient of $A_{k,l}$ is derived from those of $A_{k-1,l}$ and $A_{k,l-1}$. Starting $A_{0,0}$, we apply the recursive formula to obtain the coefficient of $A_{k,l}$, which is

$$1 \times \alpha_{k-1,l} - a^{-1} (q - q^{-1})^{-2(n-k-l)} \begin{bmatrix} m - l + 1 \cr 1 \end{bmatrix}_{q^{-2}} \times \alpha_{k,l-1}.$$  

This is equal to $\alpha_{k,l}$. Therefore the equation (2) holds for any $k + l \leq n$, and the assertion is obtained. \hfill \square

Lemma 2.2. Let $i,j$ be integers satisfying $1 \leq i \leq j \leq \min\{m,n\} - 1$. Then we have

$$\sum_{k=0}^{j-i} \beta_{i,j;m,n}^k (a,q) = \beta_{i,j;m,n}^k (a,q),$$

where $\beta_{i,j;m,n}^k (a,q)$ denotes

$$\beta_{i,j}^k = q^{k(i-j)} \frac{[m-j][m-j-1] \cdots [m-j-(k-1)][n-j][n-j-1] \cdots [n-j-(k-1)]}{[m][m-1] \cdots [m-(i-1)][n][n-1] \cdots [n-(i-1)]} \begin{bmatrix} i \cr k \end{bmatrix}_{q^2}$$

$$[m+n-j-k-1; a][m+n-j-k-2; a] \cdots [m+n-j-i; a].$$

Proof. When $i = 1$, we decompose the $m$th and the $n$th $q$-symmetrizers of the left hand side of (3) into the $(m-1)$th and the $(n-1)$th $q$-symmetrizers. By using the skein relation and the properties of (1), we obtain the middle of (3). We call its first and second terms deleting and slipping, respectively. We notice that the obtained slipping term contains the diagram below that we can decompose into deleting and slipping terms.
By decomposing the obtained slipping term into deleting and slipping terms iteratively, we obtain (4).

Next we assume that the assertion holds for $i-1 \geq 1$. The coefficient $\beta_{i,j}^k$ is derived from the term with coefficient $\beta_{i-1,j}^{k-1}$ by deleting and the term with coefficient $\beta_{i-1,j}^{k-1}$ by slipping, which is

$$
\frac{[m-(i-1)]}{[m-(i-1)[n-(i-1)]}
\frac{(m-(i-1))(n-(i-1))-(k-(i-1)+j-1)(i-1)}{[j-(i-1)+k]\times \beta_{i-1,j}^k} + \frac{[n-(i-1)]}{[m-(i-1)[n-(i-1)]}
$$

This agrees with the expression for $\beta_{i,j}^k$ written above. This concludes the proof. □

**Lemma 2.3.** Let $i,j,k$ be positive integers. We have the following formula in the linear skein of the annulus.

$$
C_{i,j,k} = \sum_{l=0}^{\min\{j,k\}} \gamma^l_{i,j,k}(a,q)
$$

where $\gamma^l_{i,j,k} = \gamma^l_{i,j,k}(a,q)$ denotes

$$
\gamma^l_{i,j,k} = q^{-l-1}\frac{[i][j+1]...[j+k]\times \beta_{i-1,j}^{k-1}}{[i+j][j+1]...[j+k-1][k+1]} \times \sum_{l=0}^{\min\{j,k\}} \gamma^l_{i,j,k}(a,q)
$$

The diagrams below describe the rule how the coefficients of $C_{i-1,j-1,k-1}$ and $C_{0,j-1,k-1}$ are obtained.

$$
C_{i,j,k} = C_{i-1,j,k} + C_{i,j-1,k} + C_{i,j-1,k-1} + C_{0,j,k} + C_{i-1,j-1,k-1} + C_{i-1,j-1,k-2} + C_{i-1,j-1,k-1} + C_{i-1,j-2,k-2} + C_{i-1,j-2,k-1} + C_{i-1,j-2,k-2} + C_{0,j-1,k-1} + C_{0,j-1,k-1} + C_{0,j-1,k-2} + C_{0,j-1,k-2} + C_{0,j-1,k-1} + C_{0,j-1,k-2}
$$

First, we prove the following identity:

$$
C_{i,j,k} = \sum_{l_1,l_2=\text{constant}} c_{l_1,l_2} C_{i-1,j-1,k-1} + C_{i-1,j-2,k-2}
$$

where $c_{l_1,l_2}$ denotes

$$
c_{l_1,l_2} = q^{-l_1-l_2}\frac{[j][i-1][j-1][j-2+1][k][k-1]...[k-l_2+1]}{[i+j][i+j-1]...[i+j-l_1-1][j-k][j+k][j+k-1]...[j+k-l_1-l_2+1]} \times \frac{[i][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1][i-1]$$
This agrees with the equation \( \text{(6)} \) when \( l_1 + l_2 = 1 \). Next, assume \( i - l_1 \geq 1 \). From the left hand side of \( \text{(5)} \), the coefficient of \( C_{i-1,j-2,k-2} \) is derived from those of \( C_{i-1,j-2,k-2,1} \) and \( C_{i-1,j-2,k-2,2} \), which is equal to

\[
\frac{[j - l_2 + 1][k - l_2 + 1]}{[i + j - 1 - l_2 + 1][i + k - l_1 - l_2 + 1]} C_{i-1,t=2-1} + \frac{[i + j + k - l_1 - 2l_2; a][i - l_1 + 1]}{[i + j - l_1 - l_2 + 1][i + k - l_1 - l_2 + 1]} C_{i-1,t=1}.
\]

(7)

Since the equation \( \text{(7)} \) is equal to \( c_{i,t=2} \), the equation \( \text{(6)} \) holds. We continue \( \text{(6)} \) to obtain \( C_{i-1,j-2,k-2} \). This contribution is to multiply a scalar \( \frac{[j+k-2l_1]}{[j+1]}[1] \). Finally, we obtain the coefficient of \( C_{i-1,j-2,k-2} \). This agrees with \( \gamma'_{i,j,k} \).

\textbf{Lemma 2.4.} For integers \( m, n \geq 0 \), the following holds.

\[ \begin{array}{ccc}
\text{m} & \Rightarrow & \text{n} \\
\downarrow & \downarrow & \downarrow \\
S_{m,n}(a,q) & = & S_{m,n}(a,q) \\
\end{array} \]

where \( S_{m,n}(a,q) \) denotes

\[ S_{m,n}(a,q) = \begin{cases} 
\sum_{i=0}^{n} a^{i,m}(a^{-1},q^{-1}) [n-1,a][n-2,a] \cdots [i,a] \\
\sum_{i=0}^{m} a^{i,n}(a^{-1},q^{-1}) [n-1,a][n-2,a] \cdots [i,a] 
\end{cases} \left( \begin{array}{c} \text{if} \ m \geq n, \\
\text{if} \ n \geq m. 
\end{array} \right) \]

\textbf{Proof.} Since the diagram above has the twisted strands which are the mirror image of Lemma \( \text{2.3} \), we apply Lemma \( \text{2.3} \) by replacing \( a \rightarrow a^{-1} \) and \( q \rightarrow q^{-1} \). Next, we apply the third identity of \( \text{(1)} \) to each element of the sum derived from Lemma \( \text{2.3} \) iteratively. Then, we obtain the assertion.

\section{Examples of colored HOMFLY polynomials}

For an oriented link diagram \( L \) in \( S^2 \), a \((1,1)\) tangle of \( L \) is defined by cutting one component, and locating the end points in the top and bottom. Associated with the \((1,1)\) tangle of \( L \), we discuss the linear skein of the disk with \( 2n \) points such that we set \( n \)-parallels of the \((1,1)\) tangle of \( L \), insert the \( n \)-th \( q \)-symmetrizer along the \( n \)-parallels in each component, and normalize it by \( \{a^n q^{n(n-1)/2}\}^{-w(L)} \). Then, we obtain a scalar in \( \mathbb{C} \), which turns out to be an ambient isotopy invariant of the \((1,1)\) tangle of \( L \). We denote it by \( H_n(L; a, q) \), and call the \textit{colored HOMFLY polynomial} of \( L \).

\[ \begin{array}{ccc}
\text{L} & \Rightarrow & \{a^n q^{n(n-1)/2}\}^{-w(L)} \\
\downarrow & \downarrow & \downarrow \\
L \Rightarrow H_n(L; a, q) \\
\end{array} \]

The lemmas in the previous section are useful to compute \( H_n(L; a, q) \) such that the diagram of \( L \) has twisted strands with opposite orientations. Such examples of knots and links are the \( 5_2 \) knot, the \( 6_1 \) knot, the Whitehead link \( WH \), and the twist knot with \( p \) half-twist \( K_p \) (\( p \in \mathbb{Z} \)). The \((1,1)\)-tangles of the \( 5_2 \) knot, the \( 6_1 \) knot, and the Whitehead link \( WH \), and \( K_p \) are given by

\[ 5_2 = \begin{array}{ccc}
\text{dotted region in WH is used in the proof of the following proposition.} \\
\text{The 5_2 knot and 6_1 knot correspond to K_p for p = 3, 4 respectively.} \\
\text{Invariants of the 5_2 knot, 6_1 knot, and the Whitehead link are obtained by the following proposition.} \\
\end{array} \]
Proposition 3.1. We have the following invariants.

\[ H_n(52; a, q) = \left\{ a^n q^{n(n-1)} \right\}^6 \]

\[
\left\{ \sum_{i=0}^{n} \sum_{j=0}^{i-j} \sum_{k=0}^{j} a^{-2(2n-2i+j)} q^{-2(2n^2-2n^2+2i+2ij-j^2-j)} \alpha^{i}_n(a, q) \alpha^{j}_n(a, q) \alpha^{k}_n(a, q) \frac{[n-1; a][n-2; a] \cdots [n-i+j+k; a]}{[n][n-1] \cdots [n+1-i+j+k]} \right\},
\]

\[ H_n(61; a, q) = \left\{ a^n q^{n(n-1)} \right\}^2 \]

\[
\left\{ \sum_{i=0}^{n} \sum_{j=0}^{i-j} \sum_{k=0}^{j} a^{-2(2n-2i+j)} q^{-2(2n^2-2n^2+2i+2ij-j^2-j)} \alpha^{i}_n(a^{-1}, q^{-1}) \alpha^{j}_n(a, q) \alpha^{k}_n(a, q) \frac{[n-1; a][n-2; a] \cdots [n-i+j+k; a]}{[n][n-1] \cdots [n+1-i+j+k]} \right\},
\]

\[ H_n(WH; a, q) = \left\{ a^n q^{n(n-1)} \right\}^2 \]

\[
\left\{ \left( \sum_{i=1}^{n-1} \sum_{j=0}^{i-j} \alpha^{i}_n(a, q) \gamma^{j}_n(a, q) S_{n,i-j}(a, q) S_{n,i-j}(a^{-1}, q^{-1}) \right) + \alpha^{n}_n(a, q) S_{n,n}(a, q) S_{n,n}(a^{-1}, q^{-1}) + \alpha^{0}_n(a, q) \frac{[n-1; a][n-2; a] \cdots [0; a]}{[n][n-1] \cdots [1]} \right\}.
\]
Proof. The diagrams below show how $H_n(6_1; a, q)$ is calculated by using Lemma 2.1.

By connecting the upper and lower terminals on the right hand side of each diagram, we obtain the element of the linear skein associated with the $(1, 1)$ tangle of the $6_1$ knot. By using the properties of (1), we obtain $H_n(6_1; a, q)$. For $H_n(5_2; a, q)$, via the Reidemeister moves, the $5_2$ knot is transformed into the diagram which differs from the $6_1$ knot in two crossings on the top. This difference makes that $\alpha_{n,n}^i(a^{-1}, q^{-1})$ turns into $\alpha_{n,n}^i(a, q)$. For $H_n(WH; a, q)$, we apply Lemma 2.1 to the dotted region in $WH$, and apply Lemma 2.3 and 2.4 to the remaining twisted strands.

The proof of Proposition 3.1 implies the following theorem about the invariant of $K_p$ for $p \geq 3$. The proof of the theorem is similar to the proof of Proposition 3.1 by applying Lemma 2.1 to the remaining twisted strands.
Theorem 3.2. For a twist knot $K_p(p \geq 3)$, we obtain
\[
H_n(K_p; a, q) = \{a^n q^{n(n-1)}\} \sum_{i=0}^{n} \sum_{j_1, \ldots, j_l \geq 0} \alpha^i_{n,n}(a^i, q^i)a^{-2(n-i)}q^{-2(n-i)(n-i-1)-4(n-i)i}
\]
\[
\prod_{l=1}^{p'} \alpha^i_{n,1-i_j,1-i_j-1}(a, q)a^{-2(n-i+j)}q^{-2(n-i+j)(n-i+j-1)-4(n-i+j)(i-j)}
\]
where $p', p'', \epsilon = \pm 1$ and $j_l$ are defined as follows. $p'$ is defined by $\lceil \frac{p+1}{2} \rceil$. If $p$ is even, then $p'' = p-2, \epsilon = -1$. If $p$ is odd, then $p'' = p+3, \epsilon = +1$. For non-negative integers $j_1, \ldots, j_l \geq 0$, we denote $\sum_{k=1}^{l} j_k$ by $j_l$, and we define $j_0 = 0$.

Remark 3.3. S. Nawata points out that our formulae are useful to compute the colored HOMFLY polynomial for the Borromean rings with three independent colors, where a color means an integer along each component.

4 Numerical calculations

In this section, we examine asymptotic behaviors of invariants obtained in Proposition 3.1 by numerical calculations. Numerical calculations are performed by PARI/GP [13]. Visualizations are performed by Mathematica [8]. Let $\epsilon$ be the integer $N-1$. For an integer $M \geq 2$, set $q = \exp(\frac{2\pi \sqrt{-1}}{2(M+N-2)})$ and $a = q^M$. Then, we have $[N-2; a] = [M+N-2] = 0$. Let $H_{M,N}(5_2), H_{M,N}(6_1)$, and $H_{M,N}(WH)$ be $H_{N-1}(5_2; q^M, q), H_{N-1}(6_1; q^M, q)$, and $H_{N-1}(WH; q^M, q)$, respectively. We remark that $H_{N-1}(L; q^2, q)$ corresponds to $J_N(L; q^2)$.

First, we review the invariant $H_{M,N}(A_1)$ for the figure-eight knot $A_1$ [5]. $H_{M,N}(A_1)$ and its integral representation as the limit of $H_{M,N}(A_1)$ are explicitly given by

\[
H_{M,N}(A_1) = \sum_{i=0}^{N-1} \left\{ 2 \sin \left( \frac{M-1}{M+N-2} \pi \right) \cdots 2 \sin \left( \frac{M+i-2}{M+N-2} \pi \right) \right\}^2
\]
\[
2\pi \lim_{N \to \infty} \frac{\log H_{M,N}(A_1)}{N} = \begin{cases}
4 \int_{\theta_M}^{\frac{5}{6}\pi} \log(2 \sin t) \, dt & (0 \leq \theta_M \leq \frac{5}{6}\pi), \\
0 & \text{otherwise},
\end{cases}
\]
where $\theta_M$ denotes
\[
\theta_M = \pi \lim_{N \to \infty} \frac{M-1}{M+N-2}.
\]

By replacing $\theta_M \to \pi x$, we define $f(x)$ by
\[
f(x) = 4 \int_{\pi x}^{\frac{5}{6}\pi} \log(2 \sin t) \, dt, \quad (0 \leq x \leq \frac{5}{6}).
\]
It is known that $f(0)$ is equal to the hyperbolic volume of complement of the figure-eight knot in $\mathbb{S}^3$. The integral representation is illustrated by

Figure 11 gives the following suggestions. If $M$ is finite, then $\theta_M \to 0$. Hence, the limit converges to the volume of the figure-eight knot. If $M$ diverges keeping a ratio $\theta_M$ when $N$ diverges, then the limit converges $f(\theta_M)$ depending on $M$. Namely, there exists a sequence $\{(M_i, N_i)\}$ such that the limit converges. The
function $f(x)$ is almost convex upward on the interval $(0, \frac{\pi}{2})$, and it has a minimal value at $x = \frac{\pi}{2}$. Since the parameter $x$ has a relation with $M$, it is notable that $H_{M,N}(4_1)$ and its limit are considered as a continuous function with respect to $M$ and $N$, subject to the condition $[M + N - 2] = 0$. We observe that similar phenomena occur for the $5_2$ knot, the $6_1$ knot, and the Whitehead link.

As referred to [2, 13], the volume conjecture is equivalent to the following equation.

$$2\pi \lim_{N \to \infty} \log(J_{N+1}(L)/J_N(L)) = \text{vol}(L) + \sqrt{-1}\text{CS}(L). \tag{8}$$

According to the equation (8), we define invariants $(x_{M,N}(L), y_{M,N}(L))$ of $L$ by

$$x_{M,N}(L) + \sqrt{-1}y_{M,N}(L) = 2\pi \log(H_{M,N+1}(L)/H_{M,N}(L)).$$

Then, first, we demonstrate sequences $(x_{M,N}, y_{M,N})$ in the cases such that $M$ is fixed to 2, 3, 4, 5, 6, and we observe that they converge to the volume and the Chern-Simons invariant. Second, from the suggestion that $H_{M,N}(L)$ is considered as a continuous function, although we assume $M \geq 2$, we perform numerical calculations for $H_{M,N}(L)$ when the condition $[M + N - 2] = 0$ holds for non integer $M$. In our calculations, we set $M = M_1/M_2$ for $M_2 = 10$ and $M_1 = 1, 3, 5, 7, 9, 11, 12, 13, 15, 17$. Hence, we demonstrate in the cases of $M = 0.1, 0.3, 0.5, 0.7, 0.9$ and $M = 1.1, 1.2, 1.3, 1.5, 1.7$. They are arranged in two rows. In Figures 2, 3, 4, 6, 7, 8, 10, 11, 12 the origin stands for approximations of the volume and the Chern-Simons invariant of the corresponding knot or link. Finally, we consider an analogue of the integral representation. For $N = 75, 125, 175$ and $k = 1, 2, \ldots, 11$, we define $M = M_k$ by

$$\frac{M_k - 1}{M_k + N - 2} = \frac{k}{12}.$$

We consider that $k/12$ for $k = 1, 2, \ldots, 11$ are discrete points of $x$ ($0 \leq x \leq 1$). For each $k/12$ and $N = 75, 125, 175$, we calculate $x_{M_k,N}$ and display the three sequences $(k/12, x_{M_k,N})$ according to $N$. Here we consider the sequence connected by lines according to same $N$ as the analogue of the integral representation.

### 4.1 The $5_2$ knot

The origin stands for $(2.82812, -3.02413)$ in Figures 2, 3 and 4. In Figure 2 for each $M = 2, 3, 4, 5, 6$, we display the five sequences. The sequence connected by lines indicates $(x_{M,N}, y_{M,N})$ from $N = 80$ to 175 in 5 steps. The point $(x_{M,80}, y_{M,80})$ is most further from the origin, and the point $(x_{M,175}, y_{M,175})$ is most nearest to the origin. We observe that the sequences converge to the origin for each $M$ when $N$ gets larger. Figures 3 and 4 present sequences for non-integers $M$. For each $M = 0.1, 1, 3, \ldots, 1.7$, sequences are connected by lines from $N = 80$ to 175 in 5 steps. The point for $N = 80$ is most further from the origin, and the point for $N = 175$ is most nearest to the origin. We also observe that the sequences converge to the origin for each $M$ when $N$ gets larger.

Figure 5 presents the analogue of the integral representation. A sequence is connected by lines according to the evaluation of $H_{M,N}(L)$.
to $k/12 = 1/12, \ldots, 11/12$. Three sequences correspond to the cases of $N = 75, 125, 175$. The sequence for $N = 75$ is located uppermost. The sequences for $N = 125, 175$ are almost overlapped on each other.

![Figure 5: Three graphs ($k/12, x_{M,N}$) for $N = 75, 125, 175$.](image)

### 4.2 The 6_1 knot

The origin stands for $(3.16396, -6.79074)$ in Figures 6, 7 and 8. In Figure 6 we plot data for the cases of $M = 2, 6$ because when we plot all data for $M = 2, 3, 4, 5, 6$, they are overlapped on each other. Sequences for $M = 0.1, \ldots, 1.7$ are presented in Figures 7, 8. The sequence connected by lines indicates $(x_{M,N}, y_{M,N})$ from $N = 80$ to 175 in 5 steps. For each sequence, the point for $N = 80$ is most further from the origin, and the point for $N = 175$ is most nearest to the origin.

The analogue of the integral representation is presented in Figure 9. The three sequences for $N = 75, 125, 175$ are almost overlapped on each other.
3.2

3.3

3.4

3.5

3.6

3.7

3.8

3.9

-6.79

-6.78

-6.77

-6.76

-6.75

M = 2

❄

-6.79

-6.78

-6.77

-6.76

-6.75

M = 6

❄

Figure 6: Two graphs \((x_{M,N}, y_{M,N})\) for \(M = 2, 6\).

Figure 7: Five graphs \((x_{M,N}, y_{M,N})\) for \(M = 0.1, 0.3, 0.5, 0.7, 0.9\).

Figure 8: Five graphs \((x_{M,N}, y_{M,N})\) for \(M = 1.1, 1.2, 1.3, 1.5, 1.7\).

Figure 9: Three graphs \((k/12, x_{M_k,N})\) for \(N = 75, 125, 175\).

4.3 The Whitehead link

The origin in Figures 10, 11 and 12 represents \((3.66386, 2.46742)\). The sequence connected by lines indicates \((x_{M,N}, y_{M,N})\) from \(N = 80\) to \(175\) in 5 steps. For each sequence, the point for \(N = 80\) is most further from the origin, and the point for \(N = 175\) is most nearest to the origin.

The analogue of the integral representation is presented in Figure 13. The sequence for \(N = 75\) is located uppermost. The sequence for \(N = 175\) is located downmost. When \(k = 1, 2, 3\), the three sequences are almost overlapped on each other.
4.4 Conclusions

These examples show that $H_{M,N}(L)$ or equivalently, $x_{M,N}(L)$ and $y_{M,N}(L)$ have a relation to the volume and the Chern-Simons invariant if $N$ goes to infinity subject to that $M$ is finite. It converges more rapidly when $1.2 < M < 1.3$ in these examples. For the analogue of the integral representation, the real part $x_{M,k,N}$ is almost convex upward, and it has a minimum at the neighborhood of $11/12$. Unfortunately, the graph $(k/12, y_{M,N})$ derived from the imaginary part is complicated to describe it. The author do not understand the meaning of the graph $(k/12, y_{M,N})$.

From these observations, We conjecture that $x_{M,N}(L)$ and $y_{M,N}(L)$ have a relation to the volume the Chern-Simons invariant of $L$ for the hyperbolic knots and links $L$ subject to that $M$ is finite. We also conjecture that there exists the integral representation of $x_{M,k,N}(L)$ when $N$ goes to infinity, and the integral representation is almost convex upward. It is notable that $x_{M,N}(L)$ and $y_{M,N}(L)$ are considered as two variable continuous functions because if $H_{M,N}(L)$ is derived from the quantum groups, $M$ is a fixed positive integer($\geq 2$). Therefore, it is interesting whether there exists another geometric structure or meaning of
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References

[1] M. Aganagic and C. Vafa, *Large N Duality, Mirror Symmetry, and a Q-deformed A-polynomial for Knots*, arXiv:1204.4709.

[2] D. Coulson, O. A. Goodman, C. D Hodgson, and W. D. Neumann. “Computing arithmetic invariants of 3-manifolds.” *Experimental Mathematics* 9:1 (2000), 127–152.

[3] R. M. Kashaev. “The hyperbolic volume of knots from the quantum dilogarithm.” *Letters in Mathematical Physics* 39:3 (1997), 269–275.

[4] L. H. Kauffman. *Knots and physics*. Series on Knots and Everything, 1. World Scientific Publishing Co., Inc., River Edge, NJ, 1991.

[5] K. Kawagoe. “Limits of the HOMFLY polynomials of the figure-eight knot.” *Intelligence of low dimensional topology 2006*, 143–150, Ser. Knots Everything, 40, World Sci. Publ., Hackensack, NJ, 2007.

[6] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, 175. Springer-Verlag, New York, 1997.

[7] Gregor Masbaum, “Skein-theoretical derivation of some formulas of Habiro”, *Algebraic & Geometric Topology*, 3 (2003), 537-556

[8] Wolfram. “Mathematica.” 1986, [http://www.wolfram.com/](http://www.wolfram.com/)

[9] H. Murakami. “An introduction to the volume conjecture and its generalizations.” *Acta Mathematica Vietnamica* 33:3 (2008) 219–253.

[10] H. Murakami and J. Murakami. “The colored Jones polynomials and the simplicial volume of a knot.” *Acta Mathematica* 186:1 (2001), 85–104.

[11] H. Murakam, J. Murakam, M. Okamoto, T. Takata and Y. Yokota. “Kashaev’s conjecture and the Chern-Simons invariants of knots and links.” *Experimental Mathematics* 11:3 (2002), 427–435.

[12] S. Nawata, P. Ramadevi, Zodinmawia and X. Sun, *Super-A-polynomials for Twist Knots*, arXiv:1209.1409

[13] “PARI/GP.” 2003, [http://pari.math.u-bordeaux.fr/](http://pari.math.u-bordeaux.fr/)

[14] J. R. Weeks, “SnapPea.” 1999, [http://www.geometrygames.org/SnapPea/](http://www.geometrygames.org/SnapPea/)

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