CURVES IN $\mathbb{R}^n$ WITH FINITE TOTAL FIRST CURVATURE ARISING FROM THE SOLUTIONS OF AN ODE

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Abstract. We use the solution set of a real ordinary differential equation which has order $n \geq 2$ to construct a smooth curve $\sigma$ in $\mathbb{R}^n$. We describe when $\sigma$ is a proper embedding of infinite length with finite total first curvature.

1. Statement of results

Let $\sigma$ be an immersion of $\mathbb{R}$ into $\mathbb{R}^n$ for $n \geq 2$; if $\sigma$ is proper, then the length of $\sigma$ is then necessarily infinite. The first curvature $\kappa$ and the total first curvature $\kappa[\sigma]$ are given, respectively, by:

$$\kappa := \frac{||\dot{\sigma} \wedge \ddot{\sigma}||}{||\dot{\sigma}||^3} \quad \text{and} \quad \kappa[\sigma] := \int_{-\infty}^{\infty} \kappa ||\dot{\sigma}|| \, dx.$$  \hfill (1.a)

1.1. History. Fenchel [16] showed that a simple closed curve in $\mathbb{R}^3$ had $\kappa[\sigma] \geq 2\pi$. Fáry [15] and Milnor [18] showed the total curvature of any knot is greater than $4\pi$. Castrillón López and Fernández Mateos [6], and Kondo and Tanaka [17] have examined the global properties of the total curvature of a curve in an arbitrary Riemannian manifold. The total curvature of open plane curves of fixed length in $\mathbb{R}^2$ was studied by Enomoto [11]. The analogous question for $S^2$ was examined by Enomoto and Itoh [12, 13]. Enomoto, Itoh, and Sinclair [14] examined curves in $\mathbb{R}^3$. We also refer to related work of Sullivan [19]. Borisenko and Tenenblat [4] studied the problem in Minkowski space. Buck and Simon [5] and Diao and Ernst [9] studied this invariant in the context of knot theory, and Ekholm [10] used this invariant in the context of algebraic topology. Alexander, Bishop, and Ghrist [1] extended these notions to more general spaces than smooth manifolds. The total curvature also appears in the study of Plateau’s problem – see the discussion in Desideri and Jakob [8].

The literature on the subject is a vast one and we have only cited a few representative papers to give a flavor for the subject.

The papers cited above focused on closed curves, polygonal curves, knotted curves, curves with fixed endpoints, curves with finite length and pursuit curves. This present paper deals, by contrast, with properly embedded curves which arise from ordinary differential equations. We take as our starting point a real constant coefficient ordinary differential operator $P$ of degree $n \geq 2$ of the form:

$$P(y) := y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_0 y.$$
Let \( \mathcal{S} = \mathcal{S}_P \) be the solution set, let \( \mathcal{P} = \mathcal{P}_P \) be the associated characteristic polynomial, and let \( \mathcal{R} = \mathcal{R}_P \) be the roots of \( \mathcal{P} \):
\[
\mathcal{S} := \{ y : P(y) = 0 \}, \\
\mathcal{P}(\lambda) := \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_0, \\
\mathcal{R} := \{ \lambda \in \mathbb{C} : \mathcal{P}(\lambda) = 0 \}.
\]
We suppose for the moment that all the roots of \( \mathcal{P} \) have multiplicity 1 and enumerate the roots of \( \mathcal{P} \) in the form:
\[
\mathcal{R} = \{ s_1, \ldots, s_k, t_1, \hat{t}_1, \ldots, t_\ell, \hat{t}_\ell \} \text{ for } k + 2\ell = n
\]
where \( s_i \in \mathbb{R} \) for \( 1 \leq i \leq k \) and where \( t_j = a_j + \sqrt{-1}b_j \) with \( b_j > 0 \) for \( 1 \leq j \leq \ell \).
Since we have assumed that all the roots are distinct, the standard basis for \( \mathcal{S} \) is given by the functions
\[
\phi_1 := e^{s_1 x}, \quad \ldots, \quad \phi_k := e^{s_k x}, \\
\phi_{k+1} := e^{a_1 x} \cos(b_1 x), \quad \phi_{k+2} := e^{a_1 x} \sin(b_1 x), \quad \ldots, \\
\phi_{n-1} := e^{a_x x} \cos(b_\ell x), \quad \phi_{n} := e^{a_x x} \sin(b_\ell x).
\]
Of course, if all the roots are real, then \( k = n \) and we omit the functions involving \( \cos(\cdot) \) and \( \sin(\cdot) \). Similarly, if all the roots are complex, then \( k = 0 \) and we omit the pure exponential functions. We define the associated curve \( \sigma_P : \mathbb{R} \rightarrow \mathbb{R}^n \) by setting:
\[
\sigma_P(x) := (\phi_1(x), \ldots, \phi_n(x)).
\]

1.2. The length of the curve \( \sigma_P \). Let \( \Re(\lambda) \) denote the real part of a complex number \( \lambda \). Define:
\[
r_+(P) := \max_{\lambda \in \mathcal{R}} \Re(\lambda) = \max(s_1, \ldots, s_k, a_1, \ldots, a_\ell), \\
r_-(P) := \min_{\lambda \in \mathcal{R}} \Re(\lambda) = \min(s_1, \ldots, s_k, a_1, \ldots, a_\ell).
\]
The numbers \( r_\pm(P) \) control the growth of \( ||\sigma_P'|| \) as \( x \rightarrow \pm \infty \). Section 2 is devoted to the proof of the following result:

**Theorem 1.1.** Assume that all the roots of \( \mathcal{P} \) are simple. If \( r_+(P) > 0 \), then \( \sigma_P \) is a proper embedding of \([0, \infty)\) into \( \mathbb{R}^n \) with infinite length. If \( r_-(P) < 0 \), then \( \sigma_P \) is a proper embedding of \((-\infty, 0] \) into \( \mathbb{R}^n \) with infinite length.

1.3. The total first curvature. If \( \sigma \) is a curve, then the first curvature \( \kappa \) of Equation (La) is a local invariant of the curve which does not depend on the parametrization. Let \( (\rho(x)) \) be the radius of the best circle approximating \( \sigma \) at \( x \).
We then have that \( \kappa = \rho^{-1} \). If we were to reparametrize \( \sigma \) by arc length, then \( \dot{\sigma} \) would be perpendicular to \( \dot{\sigma} \). In particular in that setting, \( \kappa \) vanishes identically if and only if \( \dot{\sigma} = 0 \) or equivalently if \( \sigma \) is a geodesic. Thus \( \kappa(\sigma) \) is also called the geodesic curvature and is a very important invariant of the curve.

Let \( \kappa_+ \) (resp. \( \kappa_- \)) be the total first curvature of \( \sigma_P \) on \([0, \infty)\) (resp. on \((-\infty, 0] \)):
\[
\kappa_+ := \int_0^\infty \frac{||\dot{\sigma}_P \wedge \bar{\sigma}_P||}{||\sigma_P||^2} \, dx \quad \text{and} \quad \kappa_- := \int_{-\infty}^0 \frac{||\dot{\sigma}_P \wedge \bar{\sigma}_P||}{||\sigma_P||^2} \, dx.
\]
We normalize the labeling of the roots so that
\[
s_1 > s_2 > \cdots > s_k \quad \text{and} \quad a_1 \geq \cdots \geq a_\ell.
\]
We then have \( r_+ = \max(s_1, a_1) \) and \( r_- = \min(s_k, a_\ell) \). Section 3 is devoted to the proof of the following result:

**Theorem 1.2.** Assume that all the roots of \( \mathcal{P} \) are simple, that \( r_+(P) > 0 \), and that \( r_-(P) < 0 \).

1. If \( s_1 > a_1 \), then \( \kappa_+ < \infty \); otherwise, \( \kappa_+ = \infty \).
(2) If \( s_k < a_\ell \), then \( \kappa_- < \infty \); otherwise \( \kappa_- = \infty \).

We note that if there are no complex roots, then \( s_1 > 0 \) and \( s_k < 0 \) and we may conclude that \( \kappa_+ \) and \( \kappa_- \) are finite. This is quite striking as these curves are, obviously, not straight lines. On the other hand, if there are no real roots, then \( a_1 > 0 \) and \( a_\ell < 0 \) and we may conclude that \( \kappa_+ \) and \( \kappa_- \) are infinite.

1.4. Examples. Section 4 treats several families of examples. We construct examples where \( \kappa_+ \) and \( \kappa_- \) are both finite, where \( \kappa_+ \) is finite but \( \kappa_- \) is infinite, where \( \kappa_+ \) is infinite but \( \kappa_- \) is finite, and where both \( \kappa_+ \) and \( \kappa_- \) are infinite.

1.5. Changing the basis. We have considered the standard basis for \( \mathcal{S} \) to define the curve \( \sigma_P \). More generally, let \( \Psi := \{ \psi_1, \ldots, \psi_n \} \) be an arbitrary basis for \( \mathcal{S} \). We define:

\[
\sigma_{\Psi, P}(x) := (\psi_1(x), \ldots, \psi_n(x)).
\]

In Section 5 we extend Theorem 4 and Theorem 12 to this setting and verify that the properties we have been discussing are properties of the solution space \( \mathcal{S} \) and not of the particular basis chosen:

**Theorem 1.3.** Assume that all the roots of \( \mathcal{P} \) are simple, that \( r_+(P) > 0 \), and that \( r_-(P) < 0 \). Then \( \sigma_{\Psi, P} \) is a proper embedding of \([0, \infty)\) and of \((-\infty, 0]\) into \( \mathbb{R}^n \) with infinite length.

1. If \( s_1 > a_1 \), then \( \kappa_+[\sigma_{\Psi, P}] < \infty \); otherwise, \( \kappa_+[\sigma_{\Psi, P}] = \infty \).
2. If \( s_k < a_\ell \), then \( \kappa_-[\sigma_{\Psi, P}] < \infty \); otherwise \( \kappa_-[\sigma_{\Psi, P}] = \infty \).

1.6. Roots with multiplicity greater than 1. If we have roots of multiplicity greater than 1, then powers of \( x \) arise. For example, if we consider the equation \( y^{(n)} = 0 \), then

\[
\mathcal{S} = \text{Span}\{1, x, \ldots, x^{n-1}\}.
\]

More generally, if \( s \) is a real eigenvalue of multiplicity \( \nu \geq 2 \), then we must consider the family of functions:

\[
\{ \phi_{s,0} := e^{sx}, \phi_{s,1} := xe^{sx}, \ldots, \phi_{s,\nu-1} := x^{\nu-1}e^{sx} \}
\]

while if \( t = a + \sqrt{-1}b \) for \( b > 0 \) is a complex root of multiplicity \( \nu \geq 2 \), then we must consider the family of functions:

\[
\{ \phi_{t,0} := e^{ax}\cos(bx), \phi_{t,1} := xe^{ax}\cos(bx), \ldots, \phi_{t,\nu-1} := x^{\nu-1}e^{ax}\cos(bx), \phi_{t,0} := e^{ax}\sin(bx), \phi_{t,1} := xe^{ax}\sin(bx), \ldots, \phi_{t,\nu-1} := x^{\nu-1}e^{ax}\sin(bx) \}
\]

**Theorem 1.4.** Assume that \( r_+(P) > 0 \) and that \( r_-(P) < 0 \).

1. If \( s_1 = r_+(P) \) and if the multiplicity of \( s_1 \) as a root of \( \mathcal{P} \) is larger than the corresponding multiplicity of any complex root \( t \) of \( \mathcal{P} \) with \( \Re(t) = s_1 \), then \( \kappa_+[\sigma_{t, P}] < \infty \); otherwise \( \kappa_+[\sigma_{t, P}] = \infty \).
2. If \( s_k = r_-(P) \) and if the multiplicity of \( s_k \) as a root of \( \mathcal{P} \) is larger than the corresponding multiplicity of any complex root \( t \) of \( \mathcal{P} \) with \( \Re(t) = s_k \), then \( \kappa_-[\sigma_{t, P}] < \infty \); otherwise \( \kappa_-[\sigma_{t, P}] = \infty \).

1.7. Indefinite signature inner products. In Section 7 we make some preliminary observations regarding the situation if the inner product on \( \mathbb{R}^n \) is non-degenerate with indefinite signature.

We hope that these families of examples, which arise quite naturally from the study of ordinary differential equations, help to shed light on questions of the finiteness of the total first curvature.
2. The proof of Theorem 1.1

Assume all the roots of $P$ are simple. It then follows from the definition that
\[ ||\sigma_P||^2 = \sum_{i=1}^{k} e^{2s_i x} + \sum_{j=1}^{\ell} e^{2a_j x}. \]
Thus $||\sigma_P||^2$ tends to infinity as $x \to \infty$ if and only if some $s_i$ or some $a_j$ is positive or, equivalently, if $r_+(P) > 0$. This implies that $\sigma_P$ is a proper map from $[0, \infty)$ to $\mathbb{R}^n$ and that the length is infinite. If $s_1 > 0$, then $\phi_1 = e^{s_1 x}$ is an injective map from $\mathbb{R}$ to $\mathbb{R}$ and consequently $\sigma_P$ is an embedding of $\mathbb{R}$ into $\mathbb{R}^n$. If $a_1 > 0$, then $e^{a_1 x} \cos(b_1 x), \sin(b_1 x))$ is an injective map from $\mathbb{R}$ to $\mathbb{R}^2$ and again we may conclude that $\sigma_P$ is an embedding. The analysis on $(-\infty, 0]$ is similar if $r_-(P) < 0$ and is therefore omitted in the interests of brevity. \(\square\)

3. The proof of Theorem 1.2

Throughout our proof, we will let $C_i = C_i(P)$ denote a generic positive constant; we clear the notation after each case under consideration and after the end of any given proof; thus $C_i$ can have different meanings in different proofs or in different sections of the same proof. We shall examine $\sigma_P$ on $[0, \infty)$; the analysis on $(-\infty, 0]$ is similar and will therefore be omitted. We suppose $r_+ > 0$ or, equivalently, that $\max(s_1, a_1) > 0$. We also assume that all the roots of $P$ are simple. Suppose first that $s_1 > a_1$ or that there are no complex roots. Let
\[ \epsilon := \min_{\lambda \in \mathbb{R}, \lambda \neq s_1} (s_1 - \Re(\lambda)) = \min_{i > 1, j \geq 1} (s_i - s_1, s_1 - a_j) > 0. \]
This measures the difference between the exponential growth rate of $\phi_1$ and the growth (or decay) rates of the functions $\phi_i$ of Equation (1.1) for $i > 1$ as $x \to \infty$. We have
\[ ||\dot{\sigma}_P \wedge \ddot{\sigma}_P||^2 = \sum_{i<j} (\dot{\phi}_i \ddot{\phi}_j - \dot{\phi}_j \ddot{\phi}_i)^2. \quad (3.a) \]
Consequently, we may estimate:
\[
\begin{align*}
||\dot{\sigma}_P \wedge \ddot{\sigma}_P|| &\leq C_1 e^{(2s_1 - \epsilon) x}, \\
||\dot{\sigma}_P||^2 &\geq C_2 e^{2s_1 x} \text{ for } x \geq 0, \\
\frac{||\dot{\sigma}_P \wedge \ddot{\sigma}_P||}{||\dot{\sigma}_P||^2} &\leq C_3 e^{-\epsilon x} \text{ for } x \geq 0. \quad (3.b)
\end{align*}
\]
We integrate the estimate of Equation (3.b) to see $\kappa_+ < \kappa$.

Next suppose that $a_1 > 0$ and that $a_1 > s_1$ (or that there are no real roots). Then $e^{a_1 x}$ is the dominant term and we have
\[ ||\dot{\sigma}_P||^2 \leq C_1 e^{2a_1 x}. \quad (3.c) \]
The term $(\dot{\phi}_i \ddot{\phi}_j - \dot{\phi}_j \ddot{\phi}_i)^2$ in Equation (3.a) is maximized for $x \geq 0$ when we take $\phi_i = e^{a_1 x} \cos(b_1 x)$ and $\phi_j = e^{a_1 x} \sin(b_1 x)$. We have:
\[
\begin{align*}
\dot{\phi}_i &= e^{a_1 x} (a_1 \cos(b_1 x) - b_1 \sin(b_1 x)) \\
\ddot{\phi}_i &= e^{a_1 x} (a_1^2 - b_1^2) \cos(b_1 x) - 2a_1 b_1 \sin(b_1 x)) \\
\dot{\phi}_j &= e^{a_1 x} (a_1 \sin(b_1 x) + b_1 \cos(b_1 x))v_2, \\
\ddot{\phi}_j &= e^{a_1 x} (a_1^2 - b_1^2) \sin(b_1 x) + 2a_1 b_1 \cos(b_1 x)), \\
\dot{\phi}_i^2 + \ddot{\phi}_i^2 &= (a_1^2 + b_1^2) e^{2a_1 x}, \\
(\dot{\phi}_i \ddot{\phi}_j - \dot{\phi}_j \ddot{\phi}_i)^2 &= b_1^2 (a_1^2 + b_1^2) e^{4a_1 x}.
\end{align*}
\]
Since $b_1 \neq 0$, we may estimate:
\[ ||\dot{\sigma}_P \wedge \ddot{\sigma}_P|| \geq C_2 e^{2a_1 x}. \quad (3.d) \]
We use Equation (3.c) and Equation (3.d) to see

\[
\frac{||\sigma_\mathcal{P} \wedge \sigma_\mathcal{P}||}{||\sigma_\mathcal{P}||^2} = \frac{C_2}{C_1} > 0. \tag{3.e}
\]

We integrate the uniform estimate of Equation (3.e) to see \( \kappa_+ = \infty \).

4. Examples

We now examine several specific cases. Since the eigenvalues are to be simple, we can just specify \( \mathcal{P} \) or equivalently \( \mathcal{R} \); the corresponding operator \( P \) is then:

\[
P = \mathcal{P} \left( \frac{d}{dx} \right) = \prod_{\lambda \in \mathcal{R}} \left\{ \frac{d}{dx} - \lambda \right\}.
\]

**Example 4.1.** Let \( \mathcal{P}(\lambda) = \lambda^n - 1 \). The roots of \( \mathcal{P} \) are the \( n \)\textsuperscript{th} roots of unity and all the roots have multiplicity 1. Since \( \mathcal{P}(1) = 0 \), 1 is always a root.

**Case I:** Suppose that \( n \) is odd. Then 1 is the only real root of \( \mathcal{P} \). The remaining roots are all complex. Thus \( k = 1 \) and it follows that \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty)\) to \( \mathbb{R}^n \). If \( \lambda^n = 1 \) and \( \lambda \neq 1 \), then necessarily \( \Re(\lambda) < 1 \). It now follows that \( \kappa_+ \) is finite. There exists a complex \( n \)\textsuperscript{th} root of unity with \( \Re(\lambda) < 0 \). Consequently, \( \sigma_\mathcal{P} \) is also a proper embedding of infinite length from \((\infty, 0] \) to \( \mathbb{R}^n \). Since there are no real roots with \( s_i < 0 \), we conclude \( \kappa_– \) is infinite.

**Case II:** Suppose that \( n \) is even. Then \( \pm 1 \) are the two real roots of \( \mathcal{P} \). It now follows that \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \). If \( \lambda^n = 1 \) and \( \lambda \) is not real, then \( -1 < \Re(\lambda) < 1 \). Consequently, \( \kappa_+ \) and \( \kappa_- \) are both finite.

**Example 4.2.** Let \( n \geq 3 \). Let \( \{1, \ldots, n - 2, -1 \pm \sqrt{-1} \} \) be the roots of \( \mathcal{P} \). Then \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \), \( \kappa_+ \) is finite, and \( \kappa_- \) is infinite.

**Example 4.3.** Let \( n \geq 3 \). Let \( \{-1, \ldots, 2 - n, 1 \pm \sqrt{-1} \} \) be the roots of \( \mathcal{P} \). Then \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \), \( \kappa_+ \) is infinite, and \( \kappa_- \) is finite.

**Example 4.4.** Let \( n \geq 2 \). Let \( \{1, \ldots, n - 1, -1 \} \) be the roots of \( \mathcal{P} \). Then \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \), \( \kappa_+ \) is finite, and \( \kappa_- \) is finite.

**Example 4.5.** Let \( n = 2k \geq 4 \) be even. Let

\[
\{1 \pm \sqrt{-1}, -1 \pm \sqrt{-1}, \ldots, -(k - 1) \pm \sqrt{-1}\}
\]

be the roots of \( \mathcal{P} \). Then \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \), \( \kappa_+ \) is infinite, and \( \kappa_- \) is infinite.

**Example 4.6.** Let \( n = 2k + 1 \geq 5 \) be odd. Let

\[
\{0, 1 \pm \sqrt{-1}, -1 \pm \sqrt{-1}, \ldots, -(k - 1) \pm \sqrt{-1}\}
\]

be the roots of \( \mathcal{P} \). Then \( \sigma_\mathcal{P} \) is a proper embedding of infinite length from \([0, \infty) \) to \( \mathbb{R}^n \) and from \((\infty, 0]\) to \( \mathbb{R}^n \), \( \kappa_+ \) is infinite, and \( \kappa_- \) is infinite.

5. The Proof of Theorem 1.3

Let \( \Phi = \{\phi_1, \ldots, \phi_n\} \) be the standard basis for \( \mathcal{S} \) given in Equation (1.1b) and let \( \Psi = \{\psi_1, \ldots, \psi_n\} \) be any other basis for \( \mathcal{S} \). Express

\[
\psi_i = \Theta_i^j \phi_j
\]
where we adopt the Einstein convention and sum over repeated indices. We use $\Theta^i_j$ to make a linear change of basis on $\mathbb{R}^n$ and to regard $\sigma_P = \Theta \circ \sigma; P$; correspondingly, this defines a new inner product $(\cdot, \cdot) := \Theta^i_j (\cdot, \cdot)$ on $\mathbb{R}^n$ so that

$$
||\hat{\sigma}_P|| = ||\hat{\sigma}_P||_\Theta \text{ and } ||\hat{\sigma}_P \wedge \hat{\sigma}_P, P|| = ||\hat{\sigma}_P \wedge \hat{\sigma}_P, P||_\Theta.
$$

(5.a)

Any two norms on a finite dimensional real vector space are equivalent. Thus

$$
C_1 ||v|| \leq ||v||_\Theta \leq C_2 ||v||.
$$

(5.b)

The desired result now follows from Theorem 1.1, Theorem 1.2, Equation (5.a), and Equation (5.b).

6. The Proof of Theorem 1.2

We will assume that $\Psi$ is the standard basis for $S$ as the methods discussed in Section 5 suffice to derive the general result from this specific example. We shall deal with $[0, \infty)$ as the situation for $(-\infty, 0)$ is similar. The proof that $r_+(P) > 0$ implies $\sigma_P$ is a proper embedding of $[0, \infty)$ into $\mathbb{R}^n$ with infinite length is unchanged by any questions of multiplicity since $e^{ax}$ or $\{e^{ax} \cos(bx), e^{ax} \sin(bx)\}$ are still among the solutions of $P$ for suitably chosen $s$ or $(a, b).$ We adopt the notation of Equation (1.7) to define the functions $\phi_{s, \mu} = x^\mu e^{ax}$ for $s \in \mathbb{R}$ and we adopt the notation of Equation (1.3) to define the functions $\phi_{t, \mu} = x^\mu e^{ax} \cos(bx)$ and $\phi_{\ell, \mu} = x^\mu e^{ax} \sin(bx)$ for $t = a + b\sqrt{-1}.$ We divide our discussion of $\kappa_+$ into several cases:

Case 1: Suppose that $s_1 > a_1$ and that $s_1$ is a real root of order $\nu$. If $\nu = 1$, the proof of Theorem 1.2 extends to show $\kappa_+$ is finite; the multiplicity of the other roots plays no role as the exponential decay $e^{-ax}$ swamps any powers of $x$. We suppose therefore that the multiplicity $\nu(s_1) > 1$. We will show that there exists $x_0$ so that:

$$
||\hat{\sigma}_P||^2 \geq C_1 x^{2\nu - 2} e^{2s_1 x} \text{ for } x \geq x_0,
$$

(6.a)

$$
||\hat{\sigma}_P \wedge \hat{\sigma}_P, P|| \leq C_2 x^{2\nu - 4} e^{2s_1 x} \text{ for } x \geq x_0.
$$

(6.b)

It will then follow that

$$
\frac{||\hat{\sigma}_P \wedge \hat{\sigma}_P, P||}{||\hat{\sigma}_P||^2} \leq C_3 x^{-2} \text{ for } x \geq x_0.
$$

Since this is integrable on $[0, \infty)$, we may conclude $\kappa_+$ is finite as desired.

We establish Equation (6.a) by noting that we have the following estimate:

$$
||\hat{\sigma}_P||^2 = \sum_{i=1}^n |\dot{\phi}_i|^2 \geq |\dot{\phi}_{s_1, \nu - 1}|^2 = (s_1 x^{\nu - 1} + (\nu - 1) x^{\nu - 2}) e^{2s_1 x}.
$$

When dealing with $[0, \infty)$, we may take $x_0 = 1$. However, when dealing with $(-\infty, 0)$, we must take $x_0 < 0$ to ensure that the term $s_1 x^{\nu - 1}$ dominates the term $(\nu - 1) x^{\nu - 2}$ since these terms might have opposite signs and cancellation could occur.

We may compute that:

$$
||\hat{\sigma}_P \wedge \hat{\sigma}_P||^2 = \sum_{i<j} (\dot{\phi}_i \ddot{\phi}_j - \dot{\phi}_j \ddot{\phi}_i)^2.
$$

(6.c)

The assumption $s_1 > a_1$ shows that the maximal term in this sum occurs when $\phi_i = \phi_{s_1, \nu - 1}$ and $\phi_j = \phi_{s_1, \nu - 2}$ and thus

$$
||\hat{\sigma}_P \wedge \hat{\sigma}_P||^2 \leq \frac{n(n - 1)}{2} (\phi_{s_1, \nu - 1} \ddot{\phi}_{s_1, \nu - 2} - \dot{\phi}_{s_1, \nu - 2} \ddot{\phi}_{s_1, \nu - 1})^2 \text{ for } x \geq x_0.
$$
We have:

\[ \phi_{s_1, \nu - 1} = (s_1 x^{\nu - 1} + (\nu - 1)x^{\nu - 2})e^{s_1 x}, \]

\[ \phi_{s_1, \nu - 1} = (s_1^2 x^{\nu - 1} + 2s_1 (\nu - 1)x^{\nu - 2} + (\nu - 1)(\nu - 2)x^{\nu - 3})e^{s_1 x}, \]

\[ \phi_{s_1, \nu - 2} = (s_1 x^{\nu - 2} + (\nu - 2)x^{\nu - 3})e^{s_1 x}, \]

\[ \phi_{s_1, \nu - 2} = (s_1^2 x^{\nu - 2} + 2s_1 (\nu - 2)x^{\nu - 3} + (\nu - 2)(\nu - 3)x^{\nu - 4})e^{s_1 x}, \]

Consequently:

\[ \phi_{s_1, \nu - 1} \phi_{s_1, \nu - 2} - \phi_{s_1, \nu - 2} \phi_{s_1, \nu - 1} = \{ (s_1 x^{\nu - 1} + (\nu - 1)x^{\nu - 2}) \times (s_1^2 x^{\nu - 2} + 2s_1 (\nu - 2)x^{\nu - 3} + (\nu - 2)(\nu - 3)x^{\nu - 4}) \} e^{2s_1 x} = 0. \]

The leading terms cancel:

\[ \{ (s_1 x^{\nu - 1} s_1^2 x^{\nu - 2}) - (s_1 x^{\nu - 2} s_1^2 x^{\nu - 1}) \} e^{2s_1 x} = 0. \]

The remaining terms are \( O(x^{2\nu - 4}e^{2s_1 x}) \) as desired; Equation (6.14) now follows. This shows \( \kappa_+ \) is finite if \( s_1 > a_1 \).

**Case II:** Suppose \( a_1 > s_1 \). Choose the complex root \( t_1 = a_1 + b_1 \sqrt{-1} \) to have maximal multiplicity \( \nu \) among all the complex roots \( t \in \mathcal{R} \) with \( \Re(t) = a_1 \). The dominant term in Equation (6.6) occurs when \( \phi_i = \phi_{t_1, \nu - 1} \) and \( \phi_j = \phi_{t_1, \nu - 1} \). Differentiating powers of \( x \) lowers the order in \( x \) and give rise to lower order terms. Thus we may ignore these derivatives and use the computations performed in Section (3) to see:

\[ C_1 x^{2\nu - 2} e^{2s_1 x} \leq ||\phi_p||^2 \leq C_2 x^{2\nu - 2} e^{2s_1 x} \text{ for } x \geq x_0, \]

\[ (\phi_i \phi_j) - (\phi_j \phi_i) \geq C_3 x^{4(\nu - 1)} e^{4s_1 x} \text{ for } x \geq x_0. \]

We may now conclude that \( \kappa_+ = \infty \).

**Case III:** The difficulty comes when \( a_1 = s_1 \). If \( t_1 \) is a complex root of multiplicity at least as great as the multiplicity of \( s_1 \), the \( \{ \phi_{t_1, \nu - 1}, \phi_{t_1, \nu - 1} \} \) terms dominate the computation and the argument given above in Case II implies \( \kappa_+ \) is infinite. On the other hand, if all the complex roots with \( \Re(\lambda) = s_1 \) have multiplicity less than the multiplicity of \( s_1 \), then the \( \phi_{s_1, \nu - 1} \) terms dominate the computation and the argument given above in Case I shows that \( \kappa_+ \) is finite.

We conclude this section with an example where the multiplicity plays a crucial role and where our previous results are not applicable.

**Example 6.1.** Let \( P(y) = y^{(n)} \) for \( n \geq 2 \). Then \( \mathcal{R} = \{0\} \) and 0 is a root of multiplicity \( n \). We have \( \mathcal{S} = \text{Span}\{\phi_1 := 1, \phi_2 := x, \ldots, \phi_n := x^{n - 1}\} \). Since \( x \in \mathcal{S} \), \( \sigma_P \) is a proper map of infinite length on \([0, \infty)\) and on \((\infty, 0)\). We have:

\[ ||\phi_p||^2 \geq C_1 x^{2n - 2}, \text{ and } \sum_{i \neq j} (\phi_i \phi_j - \phi_j \phi_i) = \sum_{i \neq j} ((i - 1)(i - 2)(j - 1) - (j - 1)(j - 2)(i - 1))^2 x^{2(i + j - 3)} \]

\[ \leq C_2 x^{2(2n - 4)}. \]

Consequently \( |\kappa| \leq C_3 x^{2n - 4} \) for \( |x| \geq 1 \). This is integrable so \( \kappa_+ < \infty \) and \( \kappa_- < \infty \).
7. Higher signature geometry

According to modern physics, we do not live in a Riemannian universe, but rather in a Lorentzian universe or even in the higher signature setting according to some string theories. Indefinite signatures also appears in supergravity theory (see, for example, Abounasr, Belhaj, Rasmussen, and Saidi [2]). Walker metrics have indefinite signatures and are important as well (see, for example, the discussion in Davidov, Diaz-Ramos, Garcia-Rio, Matsushita, Muskarov, and Vazquez-Lorenzo [7]). Thus it is natural to consider the setting where \( \langle \cdot, \cdot \rangle \) is a non-degenerate symmetric bilinear form of signature \((p, q)\) on the underlying vector space \(\mathbb{R}^n\). The geometry here is very different and warrants investigation in its own right; we shall content ourselves with just a preliminary remarks in this current paper and postpone until a subsequent paper a more detailed investigation.

The existence of null directions (i.e. vectors \( 0 \neq v \) with \( \langle v, v \rangle = 0 \)) causes some additional technical complications that are at the heart of the matter; thus, for example, it is not possible to use Equation (1.a) to define the first curvature \( \kappa \) where the tangent vector \( \dot{\kappa} \) is null.

For the remainder of this section, we shall work in signature \((1,1)\). We consider the operator \( P(y) := y^{(2)} - y \) so that \( S = \text{Span}\{e^x, e^{-x}\} \) and \( \sigma P = (e^x, e^{-x}) \). This curve is a proper embedding. We let \( \langle \cdot, \cdot \rangle \) be the inner product on \(\mathbb{R}^2\) defined by the symmetric matrix:

\[
G := \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.
\]

This means that:

\[
\langle (w_1, w_2), (z_1, z_2) \rangle = \alpha w_1 z_1 + \beta (w_1 z_2 + w_2 z_1) + \gamma z_2 w_2.
\]

We suppose \( \det(G) < 0 \) to ensure \( G \) has signature \((1,1)\). We have:

\[
\sigma P = (e^x, -e^{-x}),
\]

\[
\langle \sigma P, \sigma P \rangle = \alpha e^{2x} + \gamma e^{-2x} - 2\beta,
\]

\[
\sigma P = (e^x, e^{-x}),
\]

\[
\langle \sigma P, \sigma P \rangle = 2\{(1,0) \wedge (0,1)\},
\]

\[
\langle \sigma P \wedge \sigma P, \sigma P \wedge \sigma P \rangle = 4(\alpha \gamma - \beta^2).
\]

Since the induced metric on \( \Lambda^2(\mathbb{R}) \) is negative definite, \( \alpha \gamma - \beta^2 \) is negative and it is natural to take the absolute value when extracting the square root:

\[
\kappa ds := \frac{2|\alpha \gamma - \beta^2|^{1/2}}{|\alpha e^{2x} + \gamma e^{-2x} - 2\beta|} dx.
\]

**Example 7.1.** Let \( \alpha = \gamma = 0 \) and \( \beta = -1 \). Then \( \sigma P \) is spacelike, of infinite length on \([0, \infty)\) and on \((-\infty, 0]\), \( \kappa_+ = \infty \), and \( \kappa_- = \infty \).

**Example 7.2.** Let \( \alpha = \gamma = \epsilon > 0 \) and \( \beta = -1 \) where \( \epsilon \) is small. Then \( \sigma P \) is spacelike, of infinite length on \([0, \infty)\) and on \((-\infty, 0]\), \( \kappa_+ < \infty \), and \( \kappa_- < \infty \).

**Example 7.3.** Let \( \alpha = \epsilon, \beta = -1, \) and \( \gamma = 0 \) where \( \epsilon \) is small. Then \( \sigma P \) is spacelike, of infinite length on \([0, \infty)\) and on \((-\infty, 0]\), \( \kappa_+ < \infty \), and \( \kappa_- = \infty \).

**Example 7.4.** Let \( \alpha = 0, \beta = -1 \), and \( \gamma = \epsilon \) where \( \epsilon \) is small. Then \( \sigma P \) is spacelike, of infinite length on \([0, \infty)\) and on \((-\infty, 0]\), \( \kappa_+ = \infty \), and \( \kappa_- < \infty \).

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