A BUILDING-THEORETIC APPROACH TO RELATIVE TAMAGAWA NUMBERS OF SEMISIMPLE GROUPS OVER GLOBAL FUNCTION FIELDS

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Abstract. Let $G$ be a semisimple group defined over a global function field $K$ of a rational curve, not anistropic of type $A_n$. We express the (relative) Tamagawa number of $G$ in terms of local data including the number $t_\infty(G)$ of types in one orbit of a special vertex in the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$ for some place $\infty$ and the class number $h_\infty(G)$ of $G$ at $\infty$.

1. Introduction

Let $C$ be a smooth, projective and irreducible algebraic curve defined over the finite field $\mathbb{F}_q$ and let $K = \mathbb{F}_q(C)$ be its function field. Let $G$ be a (connected) semisimple group defined over $K$. The Tamagawa number $\tau(G)$ of $G$ is defined as the covolume of the group $G(K)$ of $K$-rational points in the adelic group $G(\hat{\mathbb{A}})$ (embedded diagonally as a discrete subgroup) with respect to the volume induced by Tamagawa measure on $G(\hat{\mathbb{A}})$ (see [Weil], [Clo] and Section 4 below).

Let $\pi : G^{sc} \to G$ be the universal covering and let $F = \ker(\pi)$ be the fundamental group. We assume that $G$ is not an almost direct product of anisotropic almost simple groups of type $A_n$ and that $(\text{char}(K), |F|) = 1$. According to the Weil conjecture $\tau(G^{sc}) = 1$. By [Har1] the Weil conjecture is known to hold for split $G$. A geometric proof of Weil’s conjecture by Gaitsgory–Lurie has been announced in [Gai, 1.2.3], see also: [Lur].

In the present article we investigate the relative Tamagawa number $\frac{\tau(G)}{\tau(G^{sc})}$ from a building-theoretic point of view – in the situation that $G$ is locally isotropic everywhere. Let $\infty$ be some closed point of $K$ and let $\mathbb{A}_\infty := \hat{K}_\infty \times \prod_{p \neq \infty} \hat{\mathcal{O}}_p$ be the subring of $\{\infty\}$-integral ad` eles in the ad` ele ring $\mathbb{A}$. The double cosets set $\text{Cl}_\infty(G) := G(\mathbb{A}_\infty) \backslash G(\mathbb{A}) / G(K)$ is finite (1.9). This allows us to split off the class number $h_\infty(G) = |\text{Cl}_\infty(G)|$ and proceed by computing the co-volume of $G(K)$ in the trivial coset $G(\mathbb{A}_\infty)G(K)$ w.r.t. the Tamagawa measure by considering the natural action of $G(\mathbb{A}_\infty)$ on the Bruhat–Tits building of $G_\infty(\hat{K}_\infty)$, resulting in formula (13) below

$$\tau(G) = q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot i_\infty(G) \cdot \prod_p \omega_p(\mathcal{C}_p^0(\hat{\mathcal{O}}_p))$$

where for each $p$, $\omega_p$ is some multiplicative local Haar-measure, $\mathcal{C}_p^0$ stands for the connected component of the Bruhat–Tits $\hat{\mathcal{O}}_p$-model at some special point and $i_\infty(G)$ is an arithmetic invariant.

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related to \( G_\infty := G \otimes_K \hat{K}_\infty \). The key obstruction for using this formula is to determine a fundamental domain for the action of \( \mathcal{G}^0(O_{\{\infty\}}) \) on a \( G_\infty(\hat{K}_\infty) \)-orbit of the Bruhat–Tits building of \( G_\infty(\hat{K}_\infty) \) where \( \mathcal{G}^0 \) is a flat connected smooth and finite type model of \( G \) defined over the ring \( O_{\{\infty\}} \) of \( \{\infty\} \)-integers in \( K \) w.r.t. the one of \( \mathcal{G}^{sc}(O_{\{\infty\}}) \) associated to \( G^{sc} \).

In Proposition 5.3 below we will see that for computing the relative local volumes \( \frac{t_\infty(G)}{t_\infty(G^0)} \) it suffices to compare orbits under \( G^{sc}_\infty(\hat{K}_\infty) \) and \( G_\infty(\hat{K}_\infty) \), whose behavior is controlled by the number \( j_\infty(G) \) of types in the \( G_\infty(\hat{K}_\infty) \)-orbit of the fundamental special vertex and by the number \( j_\infty(G) \) expressing the comparison between the discrete subgroups \( \mathcal{G}^{sc}(O_{\{\infty\}}) \) and \( \mathcal{G}^0(O_{\{\infty\}}) \).

Altogether we then arrive at the following main result of our article. By \( K^s_\infty \) we denote the separable closure of \( \hat{K}_\infty \) with Galois group \( g_\infty = \text{Gal}(K^s_\infty/\hat{K}_\infty) \) and inertia subgroup \( I_\infty = \text{Gal}(K^s_\infty/K^{un}_\infty) \). Moreover, \( \sigma_\infty \) denotes a generator of \( g_\infty/I_\infty \), i.e., the map \( \sigma_\infty : x \mapsto x^{[k_\infty]} \) where \( k_\infty \) is the residue field of \( \hat{K}_\infty \). Let \( F_\infty := \ker[G^s_\infty \to G_\infty] \) and \( \hat{F}_\infty := \text{Hom}(F_\infty \otimes K^s_\infty, G_{m,K^s_\infty}) \).

**Main Theorem.** With these notations and assuming the Weil conjecture validity one has

\[
\tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)}.
\]

The number \( t_\infty(G) \) satisfies

\[
t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}_\infty^{s}))|_{|\sigma_\infty|} = |\hat{F}_\infty^{g_\infty}|.
\]

and:

\[
j_\infty(G) = \frac{|H^1_{\text{ét}}(O_{\{\infty\}}, \mathcal{F})|}{|\mathcal{F}(O_{\{\infty\}})|}
\]

where \( \mathcal{F} := \ker[\mathcal{G}^{sc} \to \mathcal{G}^0] \).

If in particular \( G \) is quasi-split and the genus \( g \) of the curve \( C \) is 0 then \( j_\infty(G) = 1 \) and so

\[
\tau(G) = h_\infty(G) \cdot t_\infty(G) = h_\infty(G) \cdot |\hat{F}_\infty^{g_\infty}|.
\]

**Corollary 1.1.** (Cor. 7.10 below) If \( G \) is split and \( g = 0 \) then \( h_\infty(G) = 1 \) and \( \tau(G) = t_\infty(G) = |F| \).

**Corollary 1.2.** (Cor. 7.12 below) If \( G \) is adjoint (not necessarily split) and \( g = 0 \) then \( h_\infty(G) = 1 \) and \( \tau(G) = t_\infty(G) = |\hat{F}| \), where \( \hat{F} := \text{Hom}(F(K^s), G_{m,K^s}) \) and \( g := \text{Gal}(K^s/K) \).

Our method of proof is a combination of geometric group theory and cohomology. Our approach is independent of Prasad’s covolume formula described in [Pra2], but it is likely that with some effort it can be used to deduce our Main Theorem.

As an application in case the group \( G \) is quasi-split and the genus \( g \) of \( C \) is 0, we combine our result with [Ono1, Formula (3.9.1’)] and the techniques from [PR, § 8.2] in order to relate the cokernels of Bourqui’s degree maps \( \deg_{T^{sc}} \) and \( \deg_{T^T} \) from [Bou, Section 2.2], where \( T^{sc} \) and \( T \)
denote suitable Cartan subgroups of $G^{sc}$ and $G$ respectively; cf. Proposition 7.8. These concrete computations allow us to also provide a wealth of examples in Section 6 for which we compute the relative Tamagawa numbers. We also demonstrate the result in a case of a split group defined over the function field of an elliptic curve (Remark 7.11).

This article is organized as follows: In the preliminary Section 2 we fix the relevant notions from Bruhat–Tits theory. In Section 3 we compute volumes of parahoric subgroups over local fields, their maximal unramified extensions, and their valuation rings. In Section 4 we revise the definition of the Tamagawa number of semisimple $K$-groups and establish a decomposition of $G(\mathbb{A})/G(K)$ enabling us to express $\tau(G)$ in terms of a global invariant and a local one. In Section 5 we compute cohomology groups over rings of $S$-integers with $|S| = 1$, use Bruhat–Tits theory and Serre’s formula ([Ser1, p. 84], [BL, Corollary 1.6]) in order to derive the above-mentioned formula (13) for computing the Tamagawa number. In Section 6 we express the number $t_\infty(G)$ of types in the orbit of a special point in terms of $F_\infty$, accomplishing the proof of our Main Theorem. The final Section 7 addresses the above-mentioned application and examples.

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2. Basic notions from Bruhat–Tits theory

We retain the notation from Section 11 only we assume $G$ to be quasi-split. In Section 3 however, this assumption will be dropped. Since $K$ is a function field the valuations defined on $K$ are non-Archimedean. For any prime $p$ of $K$, let $K_p$ be the localization of $K$ at $p$, let $\hat{K}_p$ be its completion there and let $\mathcal{O}_p$ and $\hat{\mathcal{O}}_p$ be their ring of integers respectively. Let $k_p = \hat{\mathcal{O}}_p/p$ be the corresponding (finite) residue field. Then $G_p = G \otimes_K \hat{K}_p$ is semisimple. The second assumption ($\text{char}(K), |F| = 1$ says that $\pi$ is separable. $T_p = T \otimes_K \hat{K}_p$ is a Cartan subgroup in $G_p$. Let $(X^*(T_p), \Phi, X_*(T_p), \Phi^\vee)$ be the root datum of $(G_p, T_p)$ and let $W$ be the associated constant Weyl group. Let $B_p$ be the Bruhat–Tits building associated to the adjoint group of $G_p$ (cf. [BT1, Section 7], also [AB, Chapter 11]) and let $A$ be the apartment in $B_p$ corresponding to $T_p$.

We fix a special vertex $x \in A$, i.e., a vertex whose isotropy group in the setwise stabilizer of $A$ is isomorphic to $W$. Since the Bruhat–Tits building $B_p$ is locally finite, the stabilizer $P_x$ of $x$ in $G_p(\hat{K}_p)$ is a compact subgroup of $G_p(\hat{K}_p)$. Let $G_{\mathfrak{p}}$ be the Bruhat–Tits model associated to $P_x$,
i.e., such that $G_x(\hat{\mathcal{O}}_p) = P_x$. Denote by $\overline{G}_x$ the reduction modulo $p$ of $G_x$ and by $\overline{G}_x^0$ the open subscheme of $\overline{G}_x$ whose reduction is the identity component $G_x^0$ of $\overline{G}_x$. Let $T_p$ be the Néron–Raynaud $\hat{\mathcal{O}}_p$-model (shortly referred as NR-model) of $T_p$ which is of finite type, i.e., such that $T_p(\hat{\mathcal{O}}_p)$ is the maximal compact subgroup of $T_p(\hat{K}_p)$ (see Theorem 2 in [BLR] § 10.2 and [CY] § 3.2] for an explicit construction). Denote by $T_p^0$ its subscheme having a connected special fiber. $T_p^0(\hat{\mathcal{O}}_p)$ is the pointwise stabilizer of $A$ and is a subgroup of $G_p^0(\hat{\mathcal{O}}_p)$. Since $G_p$ is semisimple and the residue field $k_p$ is finite, the adjoint group of $G_p(\hat{K}_p)$ permutes transitively the special vertices (see [Tit, § 2.5]). If $\Phi$ is not reduced, we adapt the convention of Prasad in [Pra2, § 1.2] and Gross in [Gro] § 4 which is the following: for each component of the local Dynkin diagram of the type

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\ldots \Rightarrow \bullet \leftarrow \bullet \ldots
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we choose the special vertex at the right end of the diagram. Now $G_x$ is well defined up to isomorphism and would be denoted from now on by $G_p$. $x$ is called the fundamental special vertex of $B_p$.

**Remark 2.1.** If $G$ is either simply connected or adjoint, it is a finite product of restriction of scalars $R_{K_i/K}(G_i)$ where each $K_i$ is a separable extension of $K$ and $G_i$ is split and simple. If $G$ is also quasi-split, its Cartan subgroup $T$ being the product of the centralizers of the split tori of the $G_i$’s, is a maximal torus (see in the proof of [Spr, Prop. 16.2.2.]) being quasi-trivial, i.e. a finite product of Weil’s tori $R_{K_i/K}(\mathbb{G}_m)$. In this case $T_p$ is connected at any $p$ (see [NX, Prop. 2.4]).

### 3. Volumes of parahoric subgroups

We retain our restriction of $G$ to be an almost direct product of anisotropic almost simple groups of type $A_n$, implying that $G_p$ is $K_p$-isotropic (see [BT3] 4.3 and 4.4]). As $\hat{K}_p$ is locally compact its underlying additive group admits a Haar measure which is unique up to a scalar multiple. As in [Weil] § 2.1 we normalize such a measure $dx_p$ by $dx_p(\hat{\mathcal{O}}_p) = 1$. This induces a multiplicative Haar measure $\varpi_p$ on the locally compact group $G_p(\hat{K}_p)$, see [Weil] § 2.2]. Our choice of the Bruhat–Tits model in the preceding section allows us to easily compute the volume of the fundamental parahoric subgroup with respect to this Haar measure.

**Proposition 3.1.** Up to a multiplication by a scalar of $K_p^\times$, which is uniquely determined by the normalization of a multiplicative Haar measure on $G_p(\hat{K}_p)$, the volume of $G_p^0(\hat{\mathcal{O}}_p)$ w.r.t. this measure is $|G_p^0(k_p)| \cdot |k_p|^{-\dim G_p}$. 
Proof. Assume first that $G_p$ is quasi-split over $\hat{K}_p$. As $G_p^{0}$ is smooth and $\hat{O}_p$ is Henselian and therefore complete, the reduction of its group of $\hat{O}_p$-points is surjective (see [BLR] Proposition 2.3/5). Consider the exact sequence

$$1 \to G_p^0(\hat{O}_p) \to G_p^0(\hat{O}_p) \xrightarrow{\text{red}} G_p^0(k_p) \to 1.$$ 

$G_p^0(\hat{O}_p)$ is the reduction preimage of $1_d$ in $G_p^0(k_p)$, where $d = \dim G_p^0 = \dim G_p$. Since $T_p$ is maximal and $G_p$ is quasi-split, by [BT2] Corollary 4.6.7 $G_p^0(\hat{O}_p) = T_p^0(\hat{O}_p) \mathcal{X}(\hat{O}_p)$ where $\mathcal{X}(\hat{O}_p)$ is the group generated by the root subgroups each fixing an half apartment containing $x$. The preimage of $1_d$ in $G_p^0(\hat{O}_p)/T_p^0(\hat{O}_p)$ is isomorphic to $(1 + \mathfrak{p})^{\dim T_p}$, being homeomorphic to $\mathfrak{p}^{\dim T_p}$. Together, $\text{Lie}(G_p^0(\hat{O}_p)) \cong \mathfrak{p}^{\dim T_p} \cdot |\Phi| = \mathfrak{p}^d$. The normalization condition $dx_p(\hat{O}_p) = 1$ is equivalent to $dx_p(\mathfrak{p}) = |k_p|^{-1}$ implying that

$$\varpi_p(G_p^0(\hat{O}_p)) = \bigwedge_{i=1}^d dx_p(\mathfrak{p}^d) = |k_p|^{-d}.$$ 

Now from the exactness of the sequence we deduce

$$\varpi_p(G_p^0(\hat{O}_p)) = [G_p^0(\hat{O}_p) : G_p^0(\hat{O}_p)] \cdot \varpi_p(G_p^0(\hat{O}_p)) = [G_p^0(k_p)] \cdot |k_p|^{-d}.$$ 

If $G_p$ is not quasi-split this computation needs to be applied to an inner form of $G_p$ that is quasi-split, w.r.t. the twisted measure of $\varpi_p$, being again some scalar multiple of $\varpi_p$ (see [Gro] Prop. 4.7).

Remark 3.2. [BT2] 4.6.22 [If $G_p$ splits over an unramified extension, then $G_p^0(\hat{O}_p) = G_p^0(\hat{O}_p)$.]

Let $\pi_p : G_p^{sc} \to G_p$ be the universal covering of $G_p$. According to [BT2] 4.4.18(VI), the cover $\pi_p$ restricted to $T_p^{sc}$ extends to a homomorphism $T_p^{sc} \to T_p$ over Spec $\hat{O}_p$. Together with the associated root subgroups $\hat{O}_p$-scheme $\mathcal{X}$, which is equal for both $G_p^{sc}$ and $G_p$, this homomorphism over Spec $\hat{O}_p$ extends to a homomorphism $G_p^{sc} \to G_p$ of the Bruhat–Tits schemes. Let $E_p := \ker[G_p^{sc} \to G_p]$. It is finite, flat and its generic fiber is $F_p$. As $F_p$ is central, so is $E_p$, thus embedded in $T_p^{sc}$.

Let $\hat{K}_p^{un}$ be the maximal unramified extension of $\hat{K}_p$, i.e., the strict henselization of $\hat{K}_p$ with ring of integers $\hat{O}_p^{sh}$ and algebraically closed residue field $k_p^s$. Let $\hat{K}_p^s$ be a separable closure of $\hat{K}_p$ containing $\hat{K}_p^{un}$ and let $I_p = \text{Gal}(\hat{K}_p^s/\hat{K}_p^{un})$ be the inertia subgroup of $\mathfrak{g}_p = \text{Gal}(\hat{K}_p^s/\hat{K}_p)$. Let $\sigma_p$ be a generator of $\mathfrak{g}_p/I_p$, i.e., the map $\sigma_p : x \mapsto x^{1/k_p^s}$ where as above $k_p$ is the residue field of $\hat{K}_p$.

Proposition 3.3. Any separable isogeny $\pi_p : T_p \to T_p'$ of $\hat{K}_p$-tori can be extended to an isogeny $\Pi : T_p^{un} \to (T_p')^{un}$ over $\hat{O}_p^{sh}$, inducing a surjection $T_p^{un}(\hat{O}_p^{sh}) \to (T_p')^{un}(\hat{O}_p^{sh})$.

Proof. Any $\hat{K}_p^{un}$-torus $T_p$ admits a decomposition, i.e., an exact sequence of $\hat{K}_p^{un}$-tori

$$1 \to T_{I,p} \to T_p \to T_{\alpha,p} \to 1$$

(1)
on which $T_{I,p}$ is the maximal subtorus of $T_p$ splitting over $K_{p}^{\text{un}}$ and $T_{a,p}$ is $I_p$-anisotropic, i.e., such that $X^*(T_{a,p})^{I_p} = \{0\}$.

We denote by $\mathcal{T}_{lft}^I$ the locally of finite type (lft) NR-model of $T_p$ defined over Spec $\hat{O}_{\text{sh}}$ (see [2]). Let $j_*$ be the functor taking algebraic $K_{p}^{\text{un}}$-tori to their lft-Néron models. Since $T_{un}^I$ is $K_{p}^{\text{un}}$-split, we have $R^1j_* = 0$ (cf. the beginning of the proof of III.C.10 in [Mil2]). Thus the exact sequence (1) can be extended to

$$1 \rightarrow \mathcal{T}_{lft}^I \rightarrow \mathcal{T}_{lft}^p \rightarrow \mathcal{T}_{lft}^a \rightarrow 1.$$ (2)

According to [LL, Proposition 4.2(b)] the groups of $k_{p}^{s}$-points of the connected components of the reductions of these models fit into the exact sequence

$$1 \rightarrow T_{I,p}^0(k_{p}^{s}) \rightarrow T_{p}^0(k_{p}^{s}) \rightarrow T_{a,p}^0(k_{p}^{s}) \rightarrow 1.$$ (3)

As $k_{p}^{s}$ is algebraically closed, this sequence implies the corresponding exact sequence of $k_{p}^{s}$-schemes

$$1 \rightarrow T_{I,p}^0 \rightarrow T_{p}^0 \rightarrow T_{a,p}^0 \rightarrow 1.$$ (4)

Now let $\pi_p : T_p \rightarrow T'_p$ be an isogeny of $\hat{K}_p$-tori. Denote by $T_{p}^{un}$ and $(T'_p)^{un}$ these tori tensored with $K_{p}^{un}$. Then applying the decomposition (3) on both $T_{p}^{un}$ and $(T'_p)^{un}$ results in the exact sequences

$$1 \rightarrow T_{I,p}^{un0} \rightarrow T_{p}^{un0} \rightarrow T_{a,p}^{un0} \rightarrow 1,$$

$$1 \rightarrow (T'_I)^{un0} \rightarrow (T'_p)^{un0} \rightarrow (T'_a)^{un0} \rightarrow 1.$$ (4)

If we show that the left-hand and right-hand groups in the upper sequence surject onto the corresponding groups in the lower one, then surjection of the middle groups will follow. On the left hand side $T_{I,p}^{un}$ and $(T'_I)^{un}$ are isogenous and $\hat{K}_p^{un}$-split. Then $\pi_I := \ker[T_I^{un} \rightarrow (T'_I)^{un}]$ is a finite $\hat{K}_p^{un}$-split group of multiplicative type. Thus the Kummer exact sequence of $\hat{K}_p^{un}$-schemes

$$1 \rightarrow \pi_I \rightarrow T_{I,p}^{un} \rightarrow (T'_I)^{un} \rightarrow 1$$

extends to the exact sequence of corresponding schemes over $\hat{O}_{p}^{\text{sh}}$

$$1 \rightarrow \pi_I \rightarrow \mathcal{T}_{I,p}^{un} \rightarrow (\mathcal{T}'_I)^{un} \rightarrow 1$$

showing the desired surjection on the left-hand side (notice that both $\mathcal{T}_{I,p}^{un}$ and $(\mathcal{T}'_I)^{un}$ split over $\hat{O}_{p}^{\text{sh}}$ thus connected, i.e. coincide with their identity component (see Remark 3.2).
Both groups $T^\text{un}_{a,p}$ and $(T'_{a,p})^\text{un}$ on the right-hand side of sequences (1) are $I_p$-anisotropic. Therefore their NR-models coincide with the finite type (classical) Néron model. In that case, according to [BLR, Section 7.3, Proposition 6], the $\hat{K}_p^\text{un}$-isogeny $T^\text{un}_{a,p} \to (T'_{a,p})^\text{un}$ extends to a $\hat{O}_p^\text{sh}$-isogeny $\overline{T^\text{un}}_{a,p} \to (\overline{T'}_{a,p})^\text{un}$, such that the surjection holds for the identity components, see Definition 4 of loc. cit. Hence we deduce the surjection $T^\text{un0}_p \to (\overline{T'}_{a,p})^\text{un0}$.

Further, as the degree of the latter $\hat{O}_p^\text{sh}$-isogeny is prime to char($\hat{K}_p$) (see [1]), its kernel $\overline{F}^\text{un}_p$ has a smooth reduction as well. Thus the exact sequence of the reduction groups over the algebraically closed residue field $k_p$

$$1 \to \overline{T^\text{un}}_{a,p}(k_p^a) \to \overline{T^\text{un0}}_{a,p}(k_p^a) \to (\overline{T'}_{a,p})^\text{un0}(k_p^a) \to 1$$

implies the exactness of the reduction preimage groups of $\hat{O}_p^\text{sh}$-points

$$1 \to \overline{F}^\text{un}_p(\hat{O}_p^\text{sh}) \to \overline{T^\text{un0}}_{a,p}(\hat{O}_p^\text{sh}) \to (\overline{T'}_{a,p})^\text{un0}(\hat{O}_p^\text{sh}) \to 1.$$  

\[\square\]

**Corollary 3.4.** The homomorphism of $\hat{O}_p$-schemes $G^\text{sc}_p \to \hat{O}_p^0$ is surjective.

**Proof.** Our assumption (char($K$), $F$) = 1 in Section [1] implies that the isogeny $\pi_p : T^\text{sc}_p \to T_p$ is separable at any $p$. As $G^0_p(\hat{O}_p^\text{sh}) = T^0_p(\hat{O}_p^\text{sh})\hat{X}(\hat{O}_p^\text{sh})$, the surjection of groups of $\hat{O}_p^\text{sh}$-points in Proposition 3.3 can be extended to $\overline{\pi} : \overline{G}^\text{sc}_p(\hat{O}_p^\text{sh}) \to \overline{G}_p^0(\hat{O}_p^\text{sh})$. As $G^\text{sc}_p$ is simply connected, $\overline{G}^\text{sc}_p$ has a connected special fiber (see [T], § 3.5.2). By [BT2], Proposition 1.7.6, we know that the coordinate ring representing $G^\text{sc}_p$ is

$$\hat{O}_p[G^\text{sc}_p] = \left\{ f \in \hat{K}_p[G_p] : f(G_p(\hat{O}_p^\text{sh})) \subset \hat{O}_p^\text{sh} \right\} \subset \hat{K}_p[G_p].$$

As $\overline{\pi}(\overline{G}^\text{sc}_p(\hat{O}_p^\text{sh})) = \overline{G}_p^0(\hat{O}_p^\text{sh})$, any function $f \in \hat{O}_p[\overline{G}^\text{sc}_p]$ satisfies

$$f \circ \overline{\pi}(\overline{G}^\text{sc}_p(\hat{O}_p^\text{sh})) \subset f(G_p(\hat{O}_p^\text{sh})) \subset \hat{O}_p^\text{sh},$$

thus $f \circ \overline{\pi} \in \hat{O}_p[\overline{G}^\text{sc}_p]$ yielding the surjection of the contravariant functor of schemes. \[\square\]

**Lemma 3.5.** $\overline{\pi}_p(\overline{G}^\text{sc}_p(\hat{O}_p)) = \overline{\pi}_p(\overline{G}_p^0(\hat{O}_p))$.

**Proof.** Consider the following exact sequences, obtained by the reduction of groups of points

$$1 \to \overline{G}^\text{sc1}_p(\hat{O}_p) \to \overline{G}^\text{sc0}_p(\hat{O}_p) \xrightarrow{\text{red}} \overline{G}_p^0(k_p) \to 1,$$

$$1 \to \overline{G}^1_p(\hat{O}_p) \to \overline{G}_p^0(\hat{O}_p) \xrightarrow{\text{red}} \overline{G}_p^0(k_p) \to 1.$$

Both kernels are homeomorphic to $p^{\dim G_p}$ (cf. the proof of Proposition 3.1) thus sharing both the same volume with respect to $\overline{\pi}_p$, namely $q^{-\dim G_p}$. Further, as the residue field $k_p$ is finite and the reductions $\overline{G}^\text{sc0}_p = \overline{G}^\text{sc0}_p$ and $\overline{G}_p^0$ are connected and $k_p$-isogeneous, they share the same number of rational $k_p$-points (see [Bor, § 16.8]). Now the claim follows from Proposition 3.1. \[\square\]
4. The Tamagawa number of semisimple groups

We return to the definition of $G$ over the global field $K$ as introduced in Section 1. Let $\omega$ be a differential $K$-form on $G$ of highest degree. It induces a Haar measure on the adelic group $G(\hat{A})$ of $G$, which is unique up to a scalar multiplication. Let $\omega_p$ be the multiplicative Haar measure induced locally by $\omega$ at $p$. The Tamagawa measure on $G(\hat{A})$ is defined as

$$\tau = q^{-(g-1)\dim G} \prod_p \omega_p$$

where $g$ is the genus of $C$.

Remark 4.1. Since the multiplicative Haar measure on $G_p(\hat{K}_p)$ at any $p$ is unique up to a scalar multiplication, there exists $\lambda_p \in \hat{K}_p^\times$ such that $\omega_p = \lambda_p \varpi_p$ (see notation in Section 3) and so Lemma 3.5 remains true after replacing $\varpi_p$ with $\omega_p$.

Due to the product formula the measure $\tau$ does not depend on the choice of $\omega$, i.e., for each $\lambda \in K^\times$ the volume forms $\omega$ and $\lambda \omega$ yield identical Haar measures (cf. [Weil, 2.3.1]). Therefore $\tau$ is well defined and the following quantity is meaningful. Identifying $K$ with its diagonal embedding in $\mathbb{A}$ and consequently $G(K)$ with its diagonal embedding in $G(\hat{A})$, we consider the following arithmetic invariant of $G$:

Definition 4.2. The Tamagawa number $\tau(G)$ of $G$ is the volume of $G(\hat{A})/G(K)$ with respect to the Tamagawa measure $\tau$.

Recall that all discrete valuations of $K$ are non-archimedean. For any finite set $S$ of primes of $K$, the ring of $S$-adeles is:

$$\mathbb{A}_S := \left\{ (x_p)_{p \notin S} : x_p \in \hat{O}_p \text{ for almost all } p \right\} \subset \prod_{p \notin S} \hat{K}_p.$$

We also define:

$$\mathbb{A}(S) := \prod_{p \in S} \hat{K}_p \times \prod_{p \notin S} \hat{O}_p.$$

For any prime $p$ let $G_p(\hat{O}_p)$ be the maximal subgroup of $G_p(\hat{K}_p)$ w.r.t. some special point $x$ as defined in [2]. With the notation as in Section 1 we set

$$G_S := \prod_{p \in S} G_p(\hat{K}_p), \quad G(\mathbb{A}(S)) := G_S \times \prod_{p \notin S} G_p(\hat{O}_p).$$

Definition 4.3 ([Kne, p. 187], [Pla]). We say that $G$ satisfies the strong approximation property w.r.t. a finite set of primes $S$, if the diagonal embedding $G(K) \hookrightarrow G(\mathbb{A}_S)$ is dense, or, equivalently, if $G_S \cdot G(K)$ is dense in $G(\hat{A})$. If $|S| = 1$ we call it the absolute strong approximation property.
Theorem 4.4 ([Pra1 Theorem A]). Let $G$ be a simply connected $K$-group. If the topological group $G_S$ is non-compact w.r.t. to a finite set of primes $S$, then $G_S \cdot G(K)$ is dense in $G(\mathbb{A})$.

Theorem 4.5 ([Tha Thm. 3.2 3], [PR Prop. 8.8] in the number field case). Let $G$ be a connected reductive $K$-group such that the simply connected covering of the derived subgroup of $G$ has the strong approximation property w.r.t. a finite set of primes $S$. Then $G(\mathbb{A}(S))G(K)$ is a normal subgroup of $G(\mathbb{A})$ with finite abelian quotient, the $S$-class group $\text{Cl}_S(G) = G(\mathbb{A})/G(\mathbb{A}(S))G(K)$ of cardinality $h_S(G) = |\text{Cl}_S(G)|$.

We choose an arbitrary closed point $\infty$ of $C$ to be the point at infinity, and define:

$\mathbb{A}_\infty := \mathbb{A}(\{\infty\})$, $G(\mathbb{A}_\infty) := G(\mathbb{A}_\infty)(\hat{K}_\infty) \times \prod_{p \neq \infty} G_p(\hat{O}_p)$.

The following facts are now deduced from the preceding Theorems in the case of $S = \{\infty\}$:

**Definition 4.6.** There exists a finite set $\{x_1, ..., x_h\} \subset G(\mathbb{A})$ such that

$$G(\mathbb{A}) = \bigoplus_{i=1}^{h} G(\mathbb{A}_\infty)x_iG(K).$$

The finite number $h = h_\infty(G)$ is called the class number of $G$ (see [Beh Satz 7], [BP Prop. 3.9], also [BW proof of Theorem 2.1]).

**Remark 4.7.** As our group $G$ is assumed to be locally isotropic everywhere, by Theorem 4.4 in the case of $S = \{\infty\}$, $G^{sc}$ admits the absolute strong approximation property implying $h_\infty(G^{sc}) = 1$.

According to Theorem 4.5 together with Remark 4.7 $G(\mathbb{A}_\infty)G(K)$ is a normal subgroup of $G(\mathbb{A})$ and we may consider the natural epimorphism:

$$\varphi : G(\mathbb{A})/G(K) \twoheadrightarrow G(\mathbb{A})/G(\mathbb{A}_\infty)G(K) : \forall x \in G(\mathbb{A}) : xG(K) \mapsto xG(\mathbb{A}_\infty)G(K)$$

for which

$$\text{ker}(\varphi) = \{xG(K) : x \in G(\mathbb{A}_\infty)G(K)\} = G(\mathbb{A}_\infty)G(K)/G(K) \cong G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K).$$

Since all fibers of $\varphi$ are isomorphic to $\text{ker}(\varphi)$ we get a bijection of measure spaces

$$G(\mathbb{A})/G(K) \cong \text{im}(\varphi) \times \text{ker}(\varphi) \cong (G(\mathbb{A})/G(\mathbb{A}_\infty)G(K)) \times (G(\mathbb{A}_\infty)/G(\mathbb{A}_\infty) \cap G(K)) \cong \text{Cl}_\infty(G) \times (G(\mathbb{A}_\infty)/\Gamma)$$

on which the first factor cardinality is $h_\infty(G)$ and $\Gamma := G(\mathbb{A}_\infty) \cap G(K)$. We will next study the volume of the second factor.
5. On the cohomology of $\mathcal{O}_{\{\infty\}}$-schemes and relative local covolumes

The discrete group $\Gamma = G(K) \cap G(\mathbb{A}_\infty)$ consists only of elements over the ring of $\{\infty\}$-integers of $K$, namely:

$$\mathcal{O}_{\{\infty\}} = \{a \in K \mid v_p(a) \geq 0 \ \forall p \neq \infty\}.$$ 

So it would be natural to describe it using an $\mathcal{O}_{\{\infty\}}$-scheme. Consider its following construction:

For any $p$ let $\tilde{G}_p$ be the Bruhat-Tits model of $G_p$ defined over $\mathcal{O}_p$, i.e. such that:

(a) $\tilde{G}_p \otimes_{\mathcal{O}_p} \hat{K}_p = G_p$, and:

(b) $\tilde{G}_p \otimes_{\mathcal{O}_p} \mathcal{O}_p = G_p$.

According to Proposition D.4(a) in [BLR, § 6.2] the patch $(G_p, G_p, \tau)$, where $\tau$ is the canonical isomorphism $G_p \otimes_{\mathcal{O}_p} \hat{K}_p \cong G_p \otimes_{\mathcal{O}_p} \hat{K}_p$, corresponds uniquely to the $\mathcal{O}_p$-module $\tilde{G}_p$, in the sense that it covers it with a canonical descent datum. Now since $C$ is one dimensional, for any two distinct primes $p_1$ and $p_2$, the product $\mathcal{O}_{p_1} \otimes \mathcal{O}_{p_2}$ is isomorphic to $K$. Thus we may Zariski-glue all geometric fibers $\{\text{Spec} \mathcal{O}_p : p \neq \infty\}$ along the generic point $\text{Spec} K$, resulting in $\text{Spec} \mathcal{O}_{\{\infty\}}$. Then the aforementioned patches cover (with descent datum) a unique group scheme $G$ over $\text{Spec} \mathcal{O}_{\{\infty\}}$. Moreover, for any $p$, the localization $(\mathcal{O}_{\{\infty\}})_p$ is a base change of $\mathcal{O}_p$. Thus the bijection $\text{Spec} (\mathcal{O}_{\{\infty\}})_p \to \text{Spec} \mathcal{O}_p$ is faithfully flat (see [Liu, Thm. 3.16]). Hence $G$ extended to $\text{Spec} \hat{\mathcal{O}}_p$ is smooth by construction so that $G$ is smooth at $p$ by faithfully flat descent, see [EGAIV, 17.7.3]. Its generic fiber is $G$ and it satisfies:

$$G(\mathcal{O}_{\{\infty\}}) = G(K) \cap \prod_{p \neq \infty} G_p(\hat{\mathcal{O}}_p) = G(K) \cap G(\mathbb{A}_\infty).$$

We denote by $G^0$ the subscheme of $G$ whose geometric fibers are $G_p^0$. The same construction for $G^\text{sc}$ is denoted by $G^\text{sc}$. The surjectivity at the geometric fibers $G^\text{sc}_p \to G^0_p$ (see Lemma 3.4) leads to the surjection $\pi_{\mathcal{O}_{\{\infty\}}} : G^\text{sc} \to G^0$ over $\text{Spec} \mathcal{O}_{\{\infty\}}$ as étale sheaves. As we assumed $\text{char}(K)$ to be prime to $|F|$, $F := \ker[G^\text{sc} \to G^0]$ is smooth as well and we have an exact sequence of $\mathcal{O}_{\{\infty\}}$-models:

$$1 \to F \to G^\text{sc} \to G^0 \to 1.$$ (6)

Let $\mathcal{T}$ be the subscheme of $G$ whose generic fiber is $T$, let $\mathcal{T}^0 := \mathcal{T} \cap G^0$ and let $\mathcal{T}^\text{sc}$ be its preimage under $\pi_{\mathcal{O}_{\{\infty\}}}$ in $G^\text{sc}$. Being central (as all its geometric fibers), $F$ is equal to the kernel of the corresponding $\mathcal{O}_{\{\infty\}}$-tori-models, fitting into the following exact sequence of $\mathcal{O}_{\{\infty\}}$-schemes:

$$1 \to F \to \mathcal{T}^\text{sc} \to \mathcal{T}^0 \to 1.$$ (7)

Lemma 5.1. $H^1_{\text{ét}}(\mathcal{O}_{\{\infty\}}, G^\text{sc}) = 1$. 


Recall that $O_C$ words, since the curve stands for the integral closure of $O_X$. The sequence (7) gives rise to the following sequence of multiplicative groups of $\mathbb{G}_m$. Consequently, its $O$ does not have poles at any place $p$. If in addition the curve $p$ zero at some place $p$, then we have:

$$H^1_{\text{ét}}(O\{∞\}, G) \to H^1(K, G^\text{sc}(K^s))$$

This gives us the first assertion.

Proof. According to Nisnevitch ([Nis 3.6.2]) we have the following exact sequence

$$1 \to \text{Cl}_∞(G^\text{sc}) \to H^1_{\text{ét}}(O\{∞\}, G^\text{sc}) \to H^1(K, G^\text{sc}(K^s))$$

on which in our case $\text{Cl}_∞(G^\text{sc})$ is trivial (see Remark 4.7) and the latter group in the sequence is trivial as well due to Harder’s result (see [Har2, Satz A]). The claim follows. □

Lemma 5.2. Let $π_{O\{∞\}} : G^\text{sc}(O\{∞\}) \to G^0(O\{∞\})$. Then:

$$j_∞(G) := \frac{|\text{coker}(π_{O\{∞\}})|}{|\ker(π_{O\{∞\}})|} = \frac{|H^1_{\text{ét}}(O\{∞\}, F)|}{|F(O\{∞\})|}.$$ 

If in particular $G$ is quasi-split and $C$ is of genus $g = 0$, then $j_∞(G) = 1$ which means that the discrete groups $Γ^\text{sc}$ and $Γ^0$ are bijective.

Proof. Since $F$ is smooth as well as its geometric fibers we have: $H^1_{\text{ét}}(O\{∞\}, F) = H^1_{\text{fppf}}(O\{∞\}, F)$. Due to Lemma 5.1, flat cohomology applied on sequence (9) gives rise to the following sequence of groups of $O\{∞\}$-points:

$$1 \to F(O\{∞\}) \to G^\text{sc}(O\{∞\}) \xrightarrow{π_{O\{∞\}}} G^0(O\{∞\}) \to H^1_{\text{ét}}(O\{∞\}, F) \to 1. \quad (8)$$

This gives us the first assertion.

If $G$ is quasi-split, then $T^\text{sc}$ is quasi-trivial, i.e. isomorphic to a finite product of Weil’s tori $R_{K_i/K}(\mathbb{G}_m) \times \cdots \times R_{K_n/K}(\mathbb{G}_m)$ where the $K_i$’s are finite separable extensions of $K$ (see Remark 2.1). Consequently, its $O\{∞\}$-model $T^\text{sc}$ is isomorphic to $R_{O\{∞\},1} \times \cdots \times R_{O\{∞\},n}$ where $O\{∞\},i$ stands for the integral closure of $O\{∞\}$ in $K_i$ for each $i$. By Shapiro’s formula for the flat topology, we have:

$$H^1_{\text{fppf}}(O\{∞\}, T^\text{sc}) \cong \bigoplus_{i=1}^n H^1_{\text{fppf}}(O\{∞\}, R_{O\{∞\},i}/O\{∞\})(\mathbb{G}_m) \cong \bigoplus_{i=1}^n H^1_{\text{fppf}}(O\{∞\},i, \mathbb{G}_m) \cong \bigoplus \text{Pic}(C^\text{af}).$$

If in addition the curve $C$ is of genus 0 we have $\text{Pic}(C^\text{af}) = 0$, and so flat cohomology applied on sequence (7) gives rise to the following sequence of multiplicative groups of $O\{∞\}$-points:

$$1 \to F(O\{∞\}) \to T^\text{sc}(O\{∞\}) \xrightarrow{π_{O\{∞\}}} T^0(O\{∞\}) \to H^1_{\text{ét}}(O\{∞\}, F) \to 1. \quad (9)$$

Recall that $O\{∞\} = K \cap \bigcap_{p \neq ∞} O_p$, i.e., $O\{∞\}$ consists of exactly those elements of $K$ that do not have poles at any place $p \neq ∞$. If $x \in O\{∞\}$ has a proper pole at $∞$, then it has a proper zero at some place $p \neq ∞$. Hence its inverse $x^{-1} \in K$ has a proper pole at that place and, thus, $x^{-1} \in K\{O\{∞\}\}$. We conclude that the only invertible elements of $O\{∞\}$ are the constants. In other words, since the curve $C$ is projective, its regular functions are exactly the constants. This means that $T^\text{sc}(O\{∞\}) = T^\text{sc}(\mathbb{F}_q)$ and $T(O\{∞\}) = T(\mathbb{F}_q)$ are finite groups.
As the reduction of all geometric fibers of \( T^{sc} \) and \( T^0 \) are smooth and connected, the specializations \( T^{sc} = T^{sc} \otimes_{\text{Spec} \mathcal{O}_\infty} \text{Spec} \mathbb{F}_q \) and \( T^0 = T^0 \otimes_{\text{Spec} \mathcal{O}_\infty} \text{Spec} \mathbb{F}_q \) are connected \( \mathbb{F}_q \)-schemes, where \( \text{Spec} \mathbb{F}_q \to \text{Spec} \mathcal{O}_{\infty} \) is the closed immersion of the special point. Thus the exact sequence (9) can be rewritten as:

\[
1 \to \mathcal{F}(\mathcal{O}_{\infty}) \to T^{sc}(\mathbb{F}_q) \xrightarrow{\pi} T^0(\mathbb{F}_q) \to H^1_{\acute{e}t}(\mathcal{O}_{\infty}, \mathcal{F}) \to 1. \tag{10}
\]

The surjectivity of \( T^{sc} \to T^0 \) implies the one of \( T^{sc} \to T^0 \). These schemes are isogenous, connected and defined over \( \mathbb{F}_q \), so they share the same number of \( \mathbb{F}_q \)-points. Then the exactness of \( (10) \) implies that \( |\mathcal{F}(\mathcal{O}_{\infty})| = |H^1_{\acute{e}t}(\mathcal{O}_{\infty}, \mathcal{F})| \). Returning back to the exact sequence (8) we get the claim. \( \square \)

The group \( G(\mathbb{A}) \) admits a natural action on the product \( B = \prod_p B_p \) of the Bruhat–Tits buildings, and its subgroup \( G(\mathbb{A}_{\infty}) \) fixes the fundamental special vertex of each building \( B_p \) with \( p \neq \infty \). Identifying \( B_{\infty} \) with its product with these fundamental special vertices therefore yields an action of \( G(\mathbb{A}_{\infty}) \) on \( B_{\infty} \). Let:

\[
G^0(\mathbb{A}_{\infty}) = G_{\infty}(\hat{K}_{\infty}) \times \prod_{p \neq \infty} G^0_p(\hat{\mathcal{O}}_p), \quad \Gamma^0 = G^0(\mathbb{A}_{\infty}) \cap G(K) \subset \Gamma. \]

Notice that as \( G^{sc} \) is simply connected \( \Gamma^{sc} := G^{sc}(\mathbb{A}_{\infty}) \cap G(K) = (\Gamma^{sc})^0 \).

Consider the following compact subgroups:

\[
U^{sc} = \prod_p G^{sc}_p(\hat{\mathcal{O}}_p) \subset G^{sc}(\mathbb{A}_{\infty}), \quad U = \prod_p G^0_p(\hat{\mathcal{O}}_p) \subset G(\mathbb{A}_{\infty}).
\]

Let \( Y^{sc} \) and \( Y \) be the sets of representatives respectively for the double cosets sets:

\[
\Gamma^{sc}\setminus G^{sc}(\mathbb{A}_{\infty})/U^{sc} \cong (\Gamma^{sc} \cap G^{sc}(\hat{K}_{\infty}))/G^{sc}_\infty(\hat{K}_{\infty})/G^{sc}_\infty(\hat{\mathcal{O}}_{\infty}), \quad \Gamma^0\setminus G(\mathbb{A}_{\infty})/U \cong (\Gamma^0 \cap G(\hat{K}_{\infty}))/G_{\infty}(\hat{K}_{\infty})/G^0_{\infty}(\hat{\mathcal{O}}_{\infty}). \tag{11}
\]

For any \( y \in Y^{sc} \), \( yU^{sc}y^{-1} \) is compact and \( \Gamma^{sc} \) is discrete thus their intersection is finite. More precisely, by the isomorphism above any such \( y \) may represents a non-trivial double coset only at its \( \infty \)-component, whence \( yU^{sc}y^{-1} \subset G^{sc}(\mathbb{A}_{\infty}) \) and therefore:

\[
yU^{sc}y^{-1} \cap \Gamma^{sc} = yU^{sc}y^{-1} \cap (G^{sc}(K) \cap G^{sc}(\mathbb{A}_{\infty})) = yU^{sc}y^{-1} \cap G^{sc}(K).
\]

But conjugation by \( y \) on the \( \infty \)-component of \( U \) is a shift to the stabilizer of \( yx \) in \( G_{\infty}(\hat{K}_{\infty}) \):

\[
yU^{sc}y^{-1} = G^{sc}_{\infty, yx}(\hat{\mathcal{O}}_{\infty}) \times \prod_{p \neq \infty} G^{sc}_p(\hat{\mathcal{O}}_p).
\]

Thus \( yU^{sc}y^{-1} \cap G^{sc}(K) \) admits an underlying group scheme \( \tilde{G}^{sc} \) having only global sections on \( K \), i.e. defined over \( \text{Spec} \mathbb{F}_q \) (recall that \( C \) is projective). We denote by \( \tilde{G}_{\pi(y)} \) the resulting \( \mathbb{F}_q \)-group for the same construction for \( G \) with \( \pi(y) \) (here again: \( \pi(y)U \pi(y)^{-1} = G^0_{\infty, \pi(y)x}(\hat{\mathcal{O}}_{\infty}) \times \prod_{p \neq \infty} G^0_p(\hat{\mathcal{O}}_p) \)).

The surjectivity of \( G^{sc}_{\infty, yx} \to G^0_{\infty, \pi(y)x} \xrightarrow{\gamma} G^0_p \forall p \neq \infty \) and \( G^{sc} \to G \) implies the one of
\( \tilde{G}_y \to \tilde{G}_{\pi(y)} \) having a finite kernel as well. So the groups \( \tilde{G}_y \) and \( \tilde{G}_{\pi(y)} \), being isogeneous, connected and of finite dimension, defined over the finite field \( \mathbb{F}_q \), share the same finite number of \( \mathbb{F}_q \)-points, i.e.:

\[
\forall y \in Y^\text{sc} : |y U^\text{sc} y^{-1} \cap \Gamma^\text{sc}| = |\pi(y) U \pi(y)^{-1} \cap \Gamma^0|.
\] (12)

As \( G(\mathbb{A}_\infty) \) is unimodular by [Mar, Corollary I.2.3.3] we get to Serre’s formula ([Ser1, p. 84],[BL, Corollary 1.6]) :

\[
\tau(G) \equiv h_\infty(G) \cdot \tau(G(K) \backslash G(\mathbb{A}_\infty)) = h_\infty(G) \cdot \sum_{y \in Y} \tau(yU)
\]
(13)

\[
= h_\infty(G) \cdot \sum_{y \in Y} \frac{\tau(U)}{|yU y^{-1} \cap \Gamma^0|}
= q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot \prod_p \omega_p(\mathcal{O}_p) \cdot \sum_{y \in Y} \frac{1}{|yU y^{-1} \cap \Gamma^0|}
= q^{-(g-1)\dim(G)} \cdot h_\infty(G) \cdot i_\infty(G) \cdot \prod_p \omega_p(\mathcal{O}_p)
\]

where \( i_\infty(G) = \sum_{y \in Y} \frac{1}{|yU y^{-1} \cap \Gamma^0|} \).

**Lemma 5.3.** With the previously introduced notations one has

\[
\frac{i_\infty(G)}{i_\infty(G^\text{sc})} = \frac{t_\infty(G)}{j_\infty(G)}.
\]

**Proof.** Let us regard the double cosets groups in formulas[11] represented by \( Y^\text{sc} \) and \( Y \) respectively. The representatives of \( G^\text{sc}(\mathbb{A}_\infty)/U^\text{sc} \cong G^\text{sc}_0(\hat{K}_\infty)/G^\text{sc}_0(\hat{O}_\infty) \) and of \( G(\mathbb{A}_\infty)/U \cong G_\infty(\hat{K}_\infty)/G_\infty(\hat{O}_\infty) \) correspond to vertices in the orbits of \( x \) in \( B_\infty \) obtained by the actions of \( G^\text{sc}_0(\hat{K}_\infty) \) and \( G(\hat{K}_\infty) \) respectively (the Iwahori subgroup is the kernel of this action in each case). Since these actions are transitive on the alcoves in \( B_\infty \), it is sufficient to compare the orbits inside each alcove. Recall that \( t_\infty(G) \) is the number of (special) points in the orbit of \( x \) in one alcove. For each \( y \in Y^\text{sc} \), let \( x_1, ..., x_{t_\infty(G)} \) be the types representatives in the \( G_\infty(\hat{K}_\infty) \)-orbit of \( x_1 = \pi(y) \). Since the \( G^\text{sc}_\infty(\hat{K}_p) \)-action on \( B_\infty \) is type preserving, this correspondence of types in the \( x \)-orbit is one to one.

To accomplish the comparison between \( Y^\text{sc} \) and \( Y \), the above right quotients, taken modulo the discrete subgroups \( \Gamma^\text{sc} \) and \( \Gamma^0 \) (from the left) respectively, correspond to vertices in some fundamental domains of the aforementioned orbits of \( x \). In Lemma 5.2 we compared between these subgroups and got that \( \Gamma^0 \) is bijective to \( j_\infty(G) \) times \( \Gamma^\text{sc} \). Moreover, along any orbit the Bruhat-Tits schemes are isomorphic (see [Tit, § 2.5., p. 47]) and have isomorphic reductions. Thus:
∀i : |x_i U x_i^{-1} \cap \Gamma^0| = |\pi(y) U \pi(y)^{-1} \cap \Gamma^0|. We get:

\[ i_{\infty}(G) = \sum_{y \in Y} \frac{1}{|x_i U x_i^{-1} \cap \Gamma^0|} = \sum_{y \in Y} \frac{1}{|\pi(y) U \pi(y)^{-1} \cap \Gamma^0|} \]

[12]

\[ \frac{t_{\infty}(G)}{j_{\infty}(G)} \sum_{y \in Y^{G_c}} \frac{1}{|y U^{G_c} y^{-1} \cap \Gamma^{G_c}|} = \frac{t_{\infty}(G)}{j_{\infty}(G)} \cdot i_{\infty}(G^{G_c}). \]

Recall that

\[ \tau(G) = 13 \cdot q^{-(g-1) \dim G} \cdot h_{\infty}(G) \cdot i_{\infty}(G) \cdot \prod_p \omega_p(L^0_p(\hat{O}_p)). \]

Clearly the invariant \( q^{-(g-1) \dim G} \) is the same for both \( G \) and \( G^{G_c} \), as well as the volume of the compact subgroups (see Lemma 3.5 and Remark 4.1). We conclude that:

\[ \frac{\tau(G)}{\tau(G^{G_c})} = \frac{h_{\infty}(G)}{h_{\infty}(G^{G_c})} \cdot \frac{i_{\infty}(G)}{i_{\infty}(G^{G_c})}. \]

Now assuming the validity of the Weil Conjecture: \( \tau(G^{G_c}) = 1 \) and due to the strong approximation related to \( G^{G_c} \) for which \( h_{\infty}(G^{G_c}) = 1 \) (see Remark 4.7), plus Lemma 5.3 we finally deduce that:

**Corollary 5.4.**

\[ \tau(G) = h_{\infty}(G) \cdot \frac{t_{\infty}(G)}{j_{\infty}(G)}. \]

If \( G \) is quasi-split and \( C \) is of genus 0, according to Lemma 5.2 this formula simplifies to:

\[ \tau(G) = h_{\infty}(G) \cdot t_{\infty}(G). \]

### 6. Number of types in the orbit of a special point

We retain the notation and terminology introduced in the preceding sections.

**Lemma 6.1.** For any prime \( p \) one has \( H^1(\langle \sigma_p \rangle, \pi(G^{G_c}_p(\hat{K}_p^{un}))) = 1. \)

**Proof.** At any prime \( p \) we may consider the following exact sequence of \( \hat{K}_p \)-groups:

\[ 1 \to F_p \to G^{G_c}_p(\hat{K}_p^{un}) \to \pi(G^{G_c}) \to 1. \]

Due to Harder [Har2, Satz A] we know that \( H^1(\langle \sigma_p \rangle, G^{G_c}_p(\hat{K}_p^{un})) = 1 \), hence \( \langle \sigma_p \rangle \)-cohomology gives rise to the exact sequence:

\[ 1 \to H^1(\langle \sigma_p \rangle, \pi(G^{G_c}_p(\hat{K}_p^{un}))) \to H^2(\langle \sigma_p \rangle, F_p(\hat{K}_p^{un})) \]

on which the right term is trivial as \( F_p(\hat{K}_p^{un}) \) is finite. This gives the required result. \( \square \)
Lemma 6.2. The number $t_\infty(G)$ of (special) types in the $G_\infty(\hat{K}_\infty)$-orbit of the fundamental special vertex $x$ in $B_\infty$ is given by

$$t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty}| = |\overline{F}_\infty^{G_\infty}|.$$ 

Proof. Galois $I_\infty$ and $g_\infty$-cohomology yield the exact diagram

$$
1 \longrightarrow F_\infty(\hat{K}_\infty^{un}) \quad \longrightarrow \quad G_\infty^{sc}(\hat{K}_\infty^{un}) \quad \longrightarrow \quad G_\infty(\hat{K}_\infty^{un}) \quad \longrightarrow \quad H^1(I_\infty, F_\infty(\hat{K}_\infty^s)) \quad \longrightarrow \quad 1
$$

$$
1 \longrightarrow F_\infty(\hat{K}_\infty) \quad \longrightarrow \quad G_\infty^{sc}(\hat{K}_\infty) \quad \longrightarrow \quad G_\infty(\hat{K}_\infty) \quad \longrightarrow \quad H^1(g_\infty, F_\infty(\hat{K}_\infty^s)) \quad \longrightarrow \quad 1
$$

The group $\pi_\infty(G_\infty^{sc}(\hat{K}_\infty^{un})) \cap G_\infty(\hat{K}_\infty)$ is the largest type-preserving subgroup of $G_\infty(\hat{K}_\infty)$. By the classification of affine Dynkin diagrams an automorphism of $B_\infty$ preserves the types of special vertices in $B_\infty$ if and only if it preserves types of arbitrary vertices. Therefore the cosets of $\pi_\infty(G_\infty^{sc}(\hat{K}_\infty^{un})) \cap G_\infty(\hat{K}_\infty)$ in $G_\infty(\hat{K}_\infty)$ are in 1-to-1 correspondence with the types of special vertices in the $G_\infty(\hat{K}_\infty)$-orbit. We conclude that

$$t_\infty(G) = \left| G_\infty(\hat{K}_\infty)/\left(\pi(G_\infty^{sc}(\hat{K}_\infty^{un})) \cap G_\infty(\hat{K}_\infty)\right) \right|.$$ 

The exact sequence

$$1 \rightarrow F_\infty(\hat{K}_\infty^{un}) \rightarrow G_\infty^{sc}(\hat{K}_\infty^{un}) \rightarrow G_\infty(\hat{K}_\infty^{un}) \rightarrow H^1(I_\infty, F_\infty(\hat{K}_\infty^s)) \rightarrow 1$$

can be shortened to

$$1 \rightarrow \pi(G_\infty^{sc}(\hat{K}_\infty^{un})) \rightarrow G_\infty(\hat{K}_\infty^{un}) \rightarrow H^1(I_\infty, F_\infty(\hat{K}_\infty^s)) \rightarrow 1.$$ 

Applying $(\sigma_\infty)$-cohomology on this exact sequence gives the exact sequence

$$1 \rightarrow \pi(G_\infty^{sc}(\hat{K}_\infty^{un})) \cap G_\infty(\hat{K}_\infty) \rightarrow G_\infty(\hat{K}_\infty) \rightarrow H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty} \rightarrow H^1(\langle \sigma_\infty \rangle, \pi(G_\infty^{sc}(\hat{K}_\infty^{un})))$$

on which the right-hand group is trivial by Lemma 6.1. Hence $t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty}|$.

More explicitly, the Kottwitz epimorphism together with Galois descent, yields an epimorphism $T_\infty(\hat{K}_\infty) \rightarrow X_\sigma(T_\infty)_{I_\infty}^{\sigma_\infty}$ whose kernel is the Iwahori subgroup $T_\infty^{\sigma_\infty} \cap \hat{O}_\infty$ (see [Bli, Corollary 3.2]). We get the following exact diagram

$$
1 \longrightarrow F_\infty(\hat{O}_\infty) \quad \longrightarrow \quad T_\infty^{sc}(\hat{O}_\infty) \quad \longrightarrow \quad T_\infty^{\sigma_\infty}(\hat{O}_\infty) \quad \longrightarrow \quad H^1(\langle \sigma_\infty \rangle, F_\infty(\hat{O}_\infty^{sh})) \quad \longrightarrow \quad 0
$$

$$
1 \longrightarrow F_\infty(\hat{K}_\infty) \quad \longrightarrow \quad T_\infty^{sc}(\hat{K}_\infty) \quad \longrightarrow \quad T_\infty^{\sigma_\infty}(\hat{K}_\infty) \quad \longrightarrow \quad H^1(\langle \sigma_\infty \rangle, F_\infty(\hat{K}_\infty^s)) \quad \longrightarrow \quad 0
$$

$$
0 \longrightarrow X_\sigma(T_\infty)_{I_\infty}^{\sigma_\infty} \quad \longrightarrow \quad X_\sigma(T_\infty)_{I_\infty}^{\sigma_\infty} \quad \longrightarrow \quad H^1(I_\infty, F_\infty(\hat{K}_\infty^s))^{\sigma_\infty} \quad \longrightarrow \quad 0
$$
on which the lower row can be also obtained by the following steps: applying the contravariant
left-exact functor \( \text{Hom}(-, \mathbb{Z}) \) on the exact sequence of character \( g_\infty \)-modules
\[
0 \to X^*(T_\infty) \to X^*(T_\infty^\text{sc}) \to \widehat{F_\infty} \to 0,
\]
on which \( \widehat{F_\infty} = \text{Hom}(F_\infty \otimes \hat{K}_\infty^*, \mathbb{G}_m, \hat{K}_\infty^*) \), gives the exact sequence
\[
0 \to 0 = \text{Hom}(\widehat{F_\infty}, \mathbb{Z}) \to X_s(T_\infty^\text{sc}) \overset{\pi^\vee}{\longrightarrow} X_s(T_\infty) \to \text{Ext}^1(\widehat{F_\infty}, \mathbb{Z}) \to \text{Ext}^1(X^*(T_\infty^\text{sc}), \mathbb{Z}) = 0. \tag{14}
\]
Applying the functor \( \text{Hom}(\widehat{F_\infty}, -) \) on the resolution
\[
0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0
\]
gives rise to a long exact sequence on which as \( \widehat{F_\infty} \) is finite, \( \text{Hom}(\widehat{F_\infty}, \mathbb{Q}) = 0 \) and \( \text{Ext}^1(\widehat{F_\infty}, \mathbb{Q}) = 0 \), showing the existence of an isomorphism
\[
\text{Ext}^1(\widehat{F_\infty}, \mathbb{Z}) \cong \text{Hom}(\widehat{F_\infty}, \mathbb{Q}/\mathbb{Z}) = \widehat{F_\infty}^*\]
where \( \widehat{F_\infty}^* \) is the Pontryagin dual of \( \widehat{F_\infty} \), i.e., the group of finite order characters of \( \widehat{F_\infty} \), see also [Mi2 p. 23]. Being finite, these duals are isomorphic. So sequence \( \text{(14)} \) can be rewritten as
\[
0 \to X_s(T_\infty^\text{sc}) \overset{\pi^\vee}{\longrightarrow} X_s(T_\infty) \to \widehat{F_\infty}^* \to 0. \tag{15}
\]
The \( I_\infty \)-coinvariants functor is in general only right exact, but here as \( T_\infty^\text{sc} \) is connected, \( X_s(T_\infty^\text{sc})_{I_\infty} \) is free (see [Bit Formula (3.1)]) and embedded into \( X_s(T_\infty)_{I_\infty} \). Thus applying this functor on
\[
0 \to X_s(T_\infty^\text{sc}) \overset{\pi^\vee}{\longrightarrow} X_s(T_\infty) \to \widehat{F_\infty}^* \cong \widehat{F_\infty} \to 0
\]
also leaves the left hand side exact
\[
0 \to X_s(T_\infty^\text{sc})_{I_\infty} \overset{\pi^\vee}{\longrightarrow} X_s(T_\infty)_{I_\infty} \to \widehat{F_\infty}^*_{I_\infty} \to 0.
\]
Now applying the Galois \( \langle \sigma_\infty \rangle \)-cohomology gives the exact lower row on the above diagram
\[
0 \to X_s(T_\infty^\text{sc})_{I_\infty} \overset{\pi^\vee_{I_\infty}}{\longrightarrow} X_s(T_\infty)_{I_\infty} \overset{\pi^\vee}{\longrightarrow} (\widehat{F_\infty}^*)_{I_\infty} \sigma_\infty \to H^1((\sigma_\infty), X_s(T_\infty^\text{sc})) = 0. \tag{16}
\]
Returning to the diagram, as \( \widehat{F_\infty}^* \) being finite is isomorphic as a \( g_\infty \)-module to \( \widehat{F_\infty} \), we finally get
\[
t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{K}_\infty^*))_{\sigma_\infty}| = |\text{coker}(\pi^\vee_{I_\infty})| = |\widehat{F_\infty}^g_{I_\infty}|. \tag*{□}
\]

**Remark 6.3.**

(a) Sequence \( \text{(16)} \) illustrates the fact that the number \( t_\infty(G) \) of types in the
orbit of \( x \) depends only on the embedding of \( X_s(T_\infty^\text{sc}) \) in \( X_s(T_\infty) \).

(b) By the geometric version of Čebotarev’s density theorem (see in [Ja2]), one may choose the
point \( \infty \) such that \( G_\infty \) is split. In this case \( t_\infty(G) = |F_\infty| \).

Now Corollary 5.4 together with Lemma 6.2 show the Main Theorem.
Main Theorem. Assuming the Weil conjecture validity one has:

\[ \tau(G) = h_\infty(G) \cdot \frac{t_\infty(G)}{j_\infty(G)}. \]

The number \( t_\infty(G) \) satisfies

\[ t_\infty(G) = |H^1(I_\infty, F_\infty(\hat{\mathcal{K}}_\infty))^{\sigma_\infty}| = |\hat{F}_\infty^{g_\infty}| \]

and:

\[ j_\infty(G) = \frac{|H^1(\mathcal{O}_\infty(\hat{\mathcal{K}}_\infty), \mathcal{F})|}{|\mathcal{F}(\mathcal{O}_\infty(\hat{\mathcal{K}}_\infty))|}. \]

If in particular \( G \) is quasi-split and the genus \( g \) of the curve \( C \) is 0 then \( j_\infty(G) = 1 \) and so

\[ \tau(G) = h_\infty(G) \cdot t_\infty(G) = h_\infty(G) \cdot |\hat{F}_\infty^{g_\infty}|. \]

7. Application and examples

In this section we describe an application of our Main Theorem in case \( G \) is quasi-split and \( g = 0 \). We combine our result with [Ono1, Formula (3.9.1')] and the techniques from [PR § 8.2] in order to relate the cokernels of Bourqui’s degree maps \( \text{deg}_{T^{sc}} \) and \( \text{deg}_T \) from [Bou, Section 2.2], where \( T^{sc} \) and \( T \) denote suitable Cartan subgroups of \( G^{sc} \) and \( G \) respectively; cf. Proposition 7.8 below. These concrete computations will allow us to also provide a wealth of examples for which we compute the relative Tamagawa numbers. Ono’s formula was originally designed for groups over number fields and was generalized to the function field case in [BD, Theorem 6.1]. We will use freely the notation concerning algebraic tori introduced in [Ono1]. In this section we will usually assume that Weil’s conjecture \( \tau(G^{sc}) = 1 \) holds.

Remark 7.1. According to the Bruhat–Tits construction \( G_p(\hat{\mathcal{O}}_p) = T_p(\hat{\mathcal{O}}_p) \mathcal{X}(\hat{\mathcal{O}}_p) \). As \( G_p \) is quasi-split one has (see [BT2, Corollary 4.6.7]) \( G_p^0(\hat{\mathcal{O}}_p) = T_p^0(\hat{\mathcal{O}}_p) \mathcal{X}(\hat{\mathcal{O}}_p) \) and so

\[ [G_p(\hat{\mathcal{O}}_p) : G_p^0(\hat{\mathcal{O}}_p)] = [T_p(\hat{\mathcal{O}}_p) : T_p^0(\hat{\mathcal{O}}_p)]. \]

Definition 7.2. The finite group \( W(T) = T(K) \cap T^{c}(\mathbb{A}) = T(\mathbb{F}_q) \) is the group of units of \( T \) and its cardinality is denoted by \( w(T) \).

Lemma 7.3.

\[ \frac{w(T)}{w(T^{sc})} = \frac{|T(\mathbb{F}_q)|}{|T^{sc}(\mathbb{F}_q)|} = \frac{|T(\mathbb{F}_q)|}{|T^0(\mathbb{F}_q)|} = [T(\mathbb{F}_q) : T^0(\mathbb{F}_q)]. \]

Proof. Under the assumptions of \( G \) being quasi-split and \( g = 0 \) the finite groups \( T^{sc}(\mathbb{F}_q) \) and \( T^0(\mathbb{F}_q) \) are of the same cardinality (see in the proof of Lemma 5.2). The assertion follows. \( \square \)
For an algebraic $K$-torus $T$ we set the following subgroup of the adelic group $T(\mathbb{A})$

$$T^1(\mathbb{A}) := \{ x \in T(\mathbb{A}) : ||\chi(x)|| = 1 \ \forall \chi \in X^*(T)_K \}. \quad (17)$$

Let $\mathfrak{g} = \text{Gal}(K^s/K)$. Following J. Oesterlé in [Oes 1.5.5], D. Bourqui defines in [Bou §2.2.1] the morphism

$$\text{deg}_T : T(\mathbb{A}) \to \text{Hom}(X^*(T)^0, \mathbb{Z})$$

with $\text{ker}(\text{deg}_T) = T^1(\mathbb{A})$ and a finite cokernel (see [Bou Proposition 2.21]). The maximal compact subgroup of $T(\mathbb{A})$ is denoted by

$$T^c(\mathbb{A}) := \prod_p T_p(\hat{O}_p).$$

**Definition 7.4.** The class number of $T$ is $h(T) := [T^1(\mathbb{A}) : T^c(\mathbb{A}) T(K)].$

By [Ono Formula (3.9.1)] for a $K$-isogeny $\pi : T \to T'$ of tori $T$, $T'$ defined over $K$ one has

$$\tau(\pi) := \frac{\tau(T')}{\tau(T)} = \frac{w(T) h(T')}{w(T') h(T)} \prod_p \frac{L_p(1, \chi_{T'}) \cdot \omega_p(T_{\hat{O}_p})}{L_p(1, \chi_{T} \cdot \omega_p(T_{\hat{O}_p}))}. \quad (18)$$

We shall need the following

**Lemma 7.5.** Let $H_p$ be an affine, smooth and connected group scheme defined over $O_p$. Then $H^1((\sigma_p), H_p(O^h_p)) = 1$.

**Proof.** As $O_p$ is Henselian, we have $H^1((\sigma_p), H_p(O^h_p)) \cong H^1((\sigma_p), H_p(k^s_p))$ (see Remark 3.11(a) in [Mii Chapter III, §3]). The group on the right hand side is trivial by Lang’s Theorem (see [Lan] and [Ser Chapter VI, Proposition 5]). □

**Remark 6.** As $G^sc_p$ is quasi-split and simply connected, its Cartan subgroup $T^sc_p$ is a quasi-trivial torus (i.e. a product of Weil tori). Thus not only $H^1(\mathfrak{g}_p, G^sc_p(K^s_p)) = 1$ (which is due to Harder as aforementioned), but also $H^1(\mathfrak{g}_p, T^sc_p(K^s_p)) = 1$ as well as $H^1(\mathfrak{g}, G^sc(K^s)) = 1$ and $H^1(\mathfrak{g}, T^sc(K^s)) = 1$.

**Lemma 7.7.** The map $\pi_K^*: \text{Hom}(X^*(T^sc)^0, \mathbb{Z}) \to \text{Hom}(X^*(T)^0, \mathbb{Z})$ is injective. One has

$$h_\infty(G) \cdot \frac{h(T^sc)}{h(T)} = \frac{|\text{coker}(\pi^*_K)|}{l_\infty(G) \cdot |D|} \cdot \frac{\prod_p |T_p(\hat{O}_p) : T^0_p(\hat{O}_p)|}{[T(F_q) : T^0(F_q)]}.$$ 

**Proof.** Since $G$ is of non-compact type, the exact sequence of $K$-groups

$$1 \to F \to G^sc \xrightarrow{\pi_K} G \to 1$$

induces the exactness over the adelic ring $\mathbb{A}$

$$1 \to F(\mathbb{A}) \to G^sc(\mathbb{A}) \xrightarrow{\pi_K^A} G(\mathbb{A}) \xrightarrow{\text{coker}(\pi^*_A)} \prod_p H^1(\mathfrak{g}_p, F^\times_p(K^s_p)).$$
where \( g_p := \text{Gal}(\hat{K}_p^*/K_p) \) – see [PR § 8.2] and 3) in the proof of Thm. 3.2. in [Tha] for the function field case. According to [PR Proposition 8.8] one has

\[
h_{\infty}(G) = [\psi_h(G(A)) : \psi_h(G(\hat{A}_\infty)G(K))].
\]

Denote \( G^0(\hat{A}_\infty) = G_{\infty}(\hat{K}_\infty) \times \prod_{p \neq \infty} G^0_p(\hat{O}_p) \). Define the finite set \( S := \{p \mid p \text{ ramified}\} \). If \( S = \emptyset \) then \( G^0(\hat{A}_\infty) = G^0(\hat{A}_\infty) \) (see Remark 3.2). Otherwise, by the Borel density theorem (e.g. in the guise of [CM Thm. 2.4, Prop. 2.8]) \( G(\hat{O}(1_{\infty\cup S})) \) is Zariski-dense in \( \prod_{p \in S \setminus \{\infty\}} G_p \). This implies the equality \( G(\hat{A}_\infty)G(K) = G^0(\hat{A}_\infty)G(K) \), and so

\[
h_{\infty}(G) = [\psi_h(G(A)) : \psi_h(G^0(\hat{A}_\infty)G(K))].
\]

(19)

Since \( F \) is central in \( G^{sc} \), it is embedded in \( T^{sc} \). The corresponding exact sequence of \( K \)-groups of multiplicative type

\[
1 \to F \to T^{sc} \xrightarrow{\pi} T \to 1
\]

induces by \( g \)-cohomology the exact sequences over \( K \) (see Remark 7.6):

\[
1 \to F(K) \to G^{sc}(K) \xrightarrow{\pi} G(K) \xrightarrow{\delta_K} H^1(\langle \sigma_p \rangle, F(K^s)) \to 1
\]

\[
1 \to F(K) \to T^{sc}(K) \xrightarrow{\pi} T(K) \xrightarrow{\delta_K} H^1(\langle \sigma_p \rangle, F(K^s)) \to 1
\]

showing that \( \delta_K(G(K)) = \delta_K(T(K)) \). At any \( p \), as \( G^{sc}_p \) is connected, by Lemma 7.5 and Remark 7.6 one has

\[
\text{coker}[G^{sc}_p(\hat{O}_p) : G^0_p(\hat{O}_p)] = \text{coker}[T^{sc}_p(\hat{O}_p) : T^0_p(\hat{O}_p)] = H^1(\langle \sigma_p \rangle, F_p(\hat{O}_p)),
\]

\[
\text{coker}[G^{sc}_p(\hat{K}_p) \to G_p(\hat{K}_p)] = \text{coker}[T^{sc}_p(\hat{K}_p) \to T_p(\hat{K}_p)] = H^1(\langle \sigma_p \rangle, F_p(\hat{K}_p)).
\]

Thus together with \( [G_p(\hat{O}_p) : G^0_p(\hat{O}_p)] = [T_p(\hat{O}_p) : T^0_p(\hat{O}_p)] \) (see Remark 7.1), we may infer that

\[
\psi_h(G(\hat{A})) = \text{coker}[G^{sc}(\hat{A}) \to G(\hat{A})] = \text{coker}[T^{sc}(\hat{A}) \to T(\hat{A})] = \psi_h(T(\hat{A})).
\]

In particular, over \( \hat{A}_\infty \), due to Corollary 3.4 Galois cohomology yields an exact sequence

\[
1 \to F(\hat{A}_\infty) \to G^{sc}(\hat{A}_\infty) \xrightarrow{\pi} G^0(\hat{A}_\infty) \xrightarrow{\psi_h} H^1(\langle \sigma_p \rangle, F(\hat{K}_\infty^s)) \times \prod_{p \neq \infty} H^1(\langle \sigma_p \rangle, F_p(\hat{O}_p)),
\]

and similarly for the tori, showing that \( \psi_h(G^0(\hat{A}_\infty) = \psi_h(T(\hat{A}_\infty)) \). These cokernel equalities enable us to express \( h_{\infty}(G) \) as given in (19) via \( T \), namely

\[
h_{\infty}(G) = [\psi_h(T(\hat{A})) : \psi_h(T^0(\hat{A}_\infty)T(K))].
\]

(20)
Applying the Snake Lemma on its two middle rows, we get the exactness of the diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & F(A) & \rightarrow & (T^{sc})^1(A) & \rightarrow & T^1(A) & \rightarrow & \psi_A & \rightarrow & \psi_A(T^1(A)) & \rightarrow & 1 \\
1 & \rightarrow & F(A) & \rightarrow & T^{sc}(A) & \rightarrow & T(A) & \rightarrow & \psi_A & \rightarrow & \psi_A(T(A)) & \rightarrow & 1 \\
& & & \downarrow & \text{deg}_{T^{sc}} & \downarrow & \text{deg}_T & & & & & \\
0 & \rightarrow & \text{Hom}(X^*(T^{sc})^\theta, \mathbb{Z}) & \rightarrow & \text{Hom}(X^*(T)^\theta, \mathbb{Z}) & \rightarrow & \text{coker}(\pi_K^\vee) & \rightarrow & 0 \\
& & & & & & & & & & & \\
& & & \rightarrow & \text{coker}(\text{deg}_{T^{sc}}) & \rightarrow & \text{coker}(\text{deg}_T) & \rightarrow & D & \rightarrow & 0 \\
\end{array}
\]

(note that the elements in ker($\pi_A$) are units, and so belong to $T^1(A)$) from which we see that:

\[
[\psi_A(T(A)) : \psi_A(T^1(A))] = |\text{coker}(\pi_K^\vee)|/|D|. \quad (21)
\]

Furthermore, from the following exact diagram

\[
\begin{array}{ccccccc}
1 & \rightarrow & F(A) & \rightarrow & (T^{sc})^c(A)T^{sc}(K) & \rightarrow & (T^{c})^0(A)T(K) & \rightarrow & \psi_A & \rightarrow & \psi_A((T^{c})^0(A)T(K)) & \rightarrow & 1 \\
1 & \rightarrow & F(A) & \rightarrow & (T^{sc})^1(A) & \rightarrow & T^1(A) & \rightarrow & \psi_A & \rightarrow & \psi_A(T^1(A)) & \rightarrow & 1 \\
& & & & & & & & & & & \\
1 & \rightarrow & \text{Cl}(T^{sc}) & \rightarrow & \text{Cl}(T^0) & \rightarrow & \psi_A(T^1(A))/\psi_A((T^{c})^0(A)T(K)) & \rightarrow & 1 \\
\end{array}
\]

with $(T^{c})^0(A) := \prod_p T^0_p(\hat{O}_p)$ one can see that

\[
\frac{h(T)}{h(T^{sc})} = \frac{h(T^0)/h(T^{sc})}{[T^{c}(A)T(K) : (T^{c})^0(A)T(K)]} = \frac{[\psi_A(T^1(A)) : \psi_A((T^{c})^0(A)T(K))]}{[T^{c}(A)T(K) : (T^{c})^0(A)T(K)]}. \quad (22)
\]

Using the third and second isomorphism Theorems one has

\[
T^{c}(A)T(K)/((T^{c})^0(A)T(K)) \cong T^{c}(A)T(K)/(T^{c})^0(A)T(K)/T(K) \cong T^{c}(A)T(K)/T(F_q)(T^{c})^0(A)/T^0(F_q)
\]

whence

\[
[T^{c}(A)T(K) : (T^{c})^0(A)T(K)] = \frac{\prod_p [T^0_p(\hat{O}_p) : T^0_p(\hat{O}_p)]}{[T(F_q) : T^0(F_q)]}. \quad (23)
\]

Similarly,

\[
T^0(A_\infty)T(K)/((T^{c})^0(A)T(K)) \cong T^0(A_\infty)T(K)/(T^{c})^0(A)T(K)/T(K) \cong T^0(A_\infty)/T^0(F_q)((T^{c})^0(A)/T^0(F_q) \cong T^0(A_\infty)/(T^{c})^0(A).
\]

In order to compute the cardinality of the latter ratio image under $\psi$, we may use cohomology again. Fix a separable closure $\hat{K}_\infty^s$ of $\hat{K}_\infty$ containing the maximal unramified extension $\hat{K}_\infty^{un}$ of $\hat{K}_\infty$ with absolute Galois group $g_\infty$ and inertia subgroup $I_\infty = \text{Gal}(\hat{K}_\infty^s/\hat{K}_\infty^{un})$. The spectral sequence
then induces the exact sequence (see [Ser3 1.2.6(b)])

\[ 0 \to H^1(\langle \sigma_\infty \rangle, F_\infty(\hat{K}_\infty^{\text{un}})) \xrightarrow{\text{inf}} H^1(g_\infty, F_\infty(\hat{K}_\infty^{\text{a}})) \xrightarrow{\text{reg}} H^1(I_\infty, F_\infty(\hat{K}_\infty^{\text{a}}))^{\sigma_\infty} \to H^2(\langle \sigma_\infty \rangle, F_\infty(\hat{K}_\infty^{\text{un}})) = 0 \]

which shows that

\[
[\psi_A(T^0(\mathbb{A}_\infty)) : \psi_A((T^e)^0(\mathbb{A}_e))] = \frac{|H^1(g_\infty, F_\infty(\hat{K}_\infty^{\text{a}}))|}{|H^1(\langle \sigma_\infty \rangle, \mathcal{E}_\infty(\hat{G}^{\text{sh}}))|} = |H^1(I_\infty, F_\infty(\hat{K}_\infty^{\text{a}}))^{\sigma_\infty}| \overset{(\ref{25})}{=} t_\infty(G) \tag{25}
\]

All together we finally get

\[
h_\infty(G) \cdot \frac{h^{(T^{sc})}}{h(T)} \overset{(21), (22), (23)}{=} \frac{[\psi_A(T(\mathbb{A}_e)) : \psi_A((T^e)^0(\mathbb{A}_e)T(\mathbb{K}))]}{[\psi_A(T^1(\mathbb{A}_e)) : \psi_A((T^e)^0(\mathbb{A}_e)T(\mathbb{K}))]} \cdot \prod_p \left[ \mathcal{I}_p(\hat{\mathcal{O}}_p) : \mathcal{I}_p^0(\hat{\mathcal{O}}_p) \right] \cdot \prod_p \left[ \mathcal{I}_p(\hat{\mathcal{O}}_p) : \mathcal{I}_p^0(\hat{\mathcal{O}}_p) \right] \overset{(24)}{=} \frac{[\psi_A(T(\mathbb{A}_e)) : \psi_A((T^e)^0(\mathbb{A}_e)T(\mathbb{K}))]}{[\psi_A(T^0(\mathbb{A}_\infty)) : \psi_A((T^e)^0(\mathbb{A}_e)T(\mathbb{K}))]} \cdot \prod_p \left[ \mathcal{I}_p(\hat{\mathcal{O}}_p) : \mathcal{I}_p^0(\hat{\mathcal{O}}_p) \right] \overset{(21), (25)}{=} \frac{\text{coker}(\pi_\hat{\mathcal{K}})}{|D| \cdot t_\infty(G)} \cdot \prod_p \left[ \mathcal{I}_p(\hat{\mathcal{O}}_p) : \mathcal{I}_p^0(\hat{\mathcal{O}}_p) \right] = \frac{\text{coker}(\pi_\hat{\mathcal{K}})}{|D| \cdot t_\infty(G)} \cdot \prod_p \left[ \mathcal{I}_p(\hat{\mathcal{O}}_p) : \mathcal{I}_p^0(\hat{\mathcal{O}}_p) \right] \]

The following proposition now is an immediate consequence of the Main Theorem, Lemma \ref{7.7}

**Proposition 7.8.** \(|D| = |\text{coker}(\pi_\hat{\mathcal{K}})| = 1.\)

**Proof.** Following [Ono3] by the proof of Theorem 6.1 in [BD] one has

\[
\tau(G) = \tau(G^{\text{sc}}) \cdot \frac{\tau(T)}{\tau(T^{\text{sc}})} \cdot |\text{coker}(\pi_\mathbb{K})|. \tag{26}
\]

Applying the functor \(\text{Hom}(\cdot, \mathbb{Z})\) on the sequence:

\[
0 \to X^*(T)^{\theta} \xrightarrow{\pi_\mathbb{K}} X^*(T^{\text{sc}})^{\theta} \to M := \text{coker}(\pi_\mathbb{K}) \to 0 \tag{27}
\]

gives rise to the exact sequence

\[
0 \to 0 = \text{Hom}(M, \mathbb{Z}) \to \text{Hom}(X^*(T^{\text{sc}})^{\theta}, \mathbb{Z}) \xrightarrow{\pi_\mathbb{K}} \text{Hom}(X^*(T)^{\theta}, \mathbb{Z}) \to \text{Ext}^1(M, \mathbb{Z}) \cong \text{Hom}(M, \mathbb{Q}/\mathbb{Z}) \to 0
\]
which shows that \( \text{coker}(\pi_K) \) is the Pontryagin dual of \( \text{coker}(\hat{\pi}_K) \). As both groups are finite, they therefore have the same cardinality. Hence from formula (26) we get

\[
\tau(G) = \tau(G^{sc}) \cdot |\text{coker}(\pi_K)| \cdot \frac{\tau(T)}{\tau(T^{sc})}
\]

\[
\text{LS} \quad \tau(G^{sc}) \cdot |\text{coker}(\pi_K)| \cdot \frac{h(T)}{h(T^{sc})} \cdot \frac{w(T^{sc})}{w(T)} \prod_p L_p(1, \chi_T) \cdot \omega_p(T_p(\hat{\mathcal{O}_p}))
\]

\[
\text{Bit} \quad \text{sc} \quad \tau(G^{sc}) \cdot |\text{coker}(\pi_K)| \cdot \frac{h(T)}{h(T^{sc})} \cdot \frac{w(T^{sc})}{w(T)} \prod_p [T_p(\hat{\mathcal{O}_p}) : T^{0}_p(\hat{\mathcal{O}_p})]
\]

\[
\text{L3} \quad \tau(G^{sc}) \cdot |\text{coker}(\pi_K)| \cdot \frac{h(T)}{h(T^{sc})} \cdot \frac{\prod_p[T_p(\hat{\mathcal{O}_p}) : T^{0}_p(\hat{\mathcal{O}_p})]}{|T(\mathbb{F}_q) : T^{0}(\mathbb{F}_q)|}
\]

\[
\tau(G^{sc}) \cdot h_\infty(G) \cdot t_\infty(G) \cdot |D| \quad \text{LL}
\]

\[
\tau(G^{sc}) \cdot \tau(G) \cdot |D|.
\]

This implies \( |D| = \frac{1}{\tau(G^{sc})} = 1 \) due to the Weil conjecture. \( \square \)

**Remark 7.9.** Any isogenous \( K \)-tori \( T^{sc} \) and \( T \) with \( T^{sc} \) quasi-trivial can be realized as Cartan subgroups of semisimple and quasi-split groups \( G^{sc} \) and \( G \) respectively, with \( G^{sc} \) simply connected. E.g., given the isogeny \( \pi : T^{sc} \to T \), then each factor \( R_{L/K}(G_{m}^{d}) \) in \( T^{sc} \), is a Cartan subgroup of the quasi-split and simply connected group \( G^{sc} = R_{L/K}(\text{SL}_{d+1}) \), and \( T \) is a Cartan subgroup of \( G = G^{sc}/\ker(\pi) \), cf. Examples 3.2 below. Hence we may generalize Proposition 7.8 to the statement that for any isogeny \( T^{sc} \to T \), the induced map \( \text{coker}(\deg_{T^{sc}}) \to \text{coker}(\deg_{T}) \) is surjective.

Quite naturally, our Main Theorem reproduces the following well-known facts. Recall that in this section we assume the validity of the Weil Conjecture.

**Corollary 7.10.** If \( G \) is \( K \)-split and \( g = 0 \) then \( h_\infty(G) = 1 \) and \( \tau(G) = t_\infty(G) = |F| \).

**Proof.** If \( G \) is \( K \)-split, then \( T^{sc} \) and \( T \) are \( K \)-split thus having connected reduction everywhere and \( h(T) = h(T^{sc}) \). Furthermore, \( |\text{coker}(\pi_K)| = |F| = |F_\infty| = t_\infty(G) \) whence by Lemma 7.11 \( h_\infty(G) = 1 \). Hence according to the Main Theorem 1 we get \( \tau(G) = t_\infty(G) = |F_\infty| = |F| \). \( \square \)

**Remark 7.11.** We have assumed in this section that the genus \( g \) of \( C \) is 0. Otherwise, if \( g > 0 \), \( h_\infty(G) \) does not need to be 1, though \( G \) splits over \( K \). For example, let \( G = \text{PGL}_n \) defined over \( K = \mathbb{F}_q(C) \) where \( C \) is an elliptic curve (\( g = 1 \)). Let \( \infty \) be a \( K \)-rational point and let \( G \) be an affine, smooth, flat, connected and of finite type model of \( G \) defined over \( \text{Spec} \mathcal{O}_{\{\infty\}} \) as been constructed above. Let \( \mathcal{G}_L \) be a similar construction for \( \text{GL}_n \) and \( \mathcal{G}_m \) for \( \mathcal{G}_m \). According to
Nisnevich exact sequence (see [Nis, 3.5.2] and also [Gon, Thm. 3.4]), since the Shafarevich-Tate group w.r.t. \( S = \{ \infty \} \) is trivial in this split case, we have:

\[
\text{Cl}_\infty(G) \cong H^1_{\text{ét}}(O_{(\infty)}, \mathcal{G}).
\]

The exact sequence of smooth \( O_{(\infty)} \)-groups

\[
1 \to \mathcal{G}_m \to \mathcal{GL}_n \to \mathcal{G} \to 1,
\]
gives rise by flat cohomology to the following exact sequence

\[
\text{Pic} \ (C_{\text{aff}}) \xrightarrow{\partial} H^1_{\text{ét}}(O_{(\infty)}, \mathcal{GL}_n) \xrightarrow{\delta} H^1_{\text{ét}}(O_{(\infty)}, \mathcal{G}) \to H^2_{\text{ét}}(O_{(\infty)}, \mathcal{G}_m)
\]
on which \( H^1_{\text{ét}}(O_{(\infty)}, \mathcal{GL}_n) \) classifies the rank \( n \) vector bundles defined over \( C_{\text{aff}} := C - \{ \infty \} \). Every rank-\( n \) vector bundle over a Dedekind domain is a direct sum \( O_{C_{\text{aff}}}^{n-1} \oplus \mathcal{L} \), where \( [\mathcal{L}] \in \text{Pic} \ (C_{\text{aff}}) \) and \( O_{C_{\text{aff}}} \) is the trivial line bundle. As \( \partial : [\mathcal{L}] \mapsto n\mathcal{L} \) we have:

\[
\text{im}(\delta) \cong H^1_{\text{ét}}(O_{(\infty)}, \mathcal{GL}_n) / \ker(\delta) = H^1_{\text{ét}}(O_{(\infty)}, \mathcal{GL}_n) / \text{im}(\partial) \cong \text{Pic} \ (C_{\text{aff}})[n] := \text{Pic} \ (C_{\text{aff}})/n\text{Pic} \ (C_{\text{aff}}).
\]

Moreover, as \( C_{\text{aff}} \) is smooth, one has (see [Mil1, Prop. 2.15]): \( H^2_{\text{ét}}(O_{(\infty)}, \mathcal{G}_m) = \text{Br}(O_{(\infty)}) \), classifying Azumaya \( O_{(\infty)} \)-algebras (see [Mil1, § 2]). At each prime \( p \): \( \text{Br}(O_{(\infty)}) \subset \text{Br}((O_{(\infty)})_p) = \text{Br}(O_p) \).

Since \( O_p \) is complete, the latter group is isomorphic to \( \text{Br}(k_p) \) (see [AG, Thm. 6.5]). But \( k_p \) is a finite field thus \( \text{Br}(k_p) \) is trivial as well as \( H^2_{\text{ét}}(O_{(\infty)}, \mathcal{G}_m) \) and \( \delta \) is surjective. We get that \( h_\infty(G) = |\text{Cl}_\infty(G)| = |H^1_{\text{ét}}(O_{(\infty)}, \mathcal{G})| = |\text{Pic} \ (C_{\text{aff}})[n]|. \) In order to compute this group, as we assumed \( \infty \) is \( K \)-rational and \( (\text{char}(K), |F|) = 1 \), the restrictions of \( C \) to \( C_{\text{aff}} \) and of \( \text{Pic} \ (C) \) for \( \text{Pic}^0(C) \) inducing the exact sequences

\[
0 \to \mathbb{Z} \to \text{Pic} \ (C) \to \text{Pic} \ (C_{\text{aff}}) \to 0
\]

\[
0 \to \mathbb{Z} \to \text{Pic} \ (C) \to \text{Pic}^0(C) \to 0
\]
give an isomorphism \( \text{Pic} \ (C_{\text{aff}}) \cong \text{Pic}^0(C) \cong C(\mathbb{F}_q) \). So it is easy to find an elliptic curve \( C \) for which \( h_\infty(G) = |C(\mathbb{F}_q)|/|n| > 1 \).

Applying flat cohomology on Kummer’s exact sequence of \( O_{(\infty)} \)-schemes:

\[
1 \to \mu_n \to \mathcal{G}_m \xrightarrow{x \mapsto x^n} \mathcal{G}_m
\]
gives rise to the exact sequence of groups of \( O_{(\infty)} \)-points:

\[
(O_{(\infty)})^\times \xrightarrow{x \mapsto x^n} (O_{(\infty)})^\times \to H^1_{\text{ét}}(O_{(\infty)}, \mu_n) \to \text{Pic} \ (C_{\text{aff}}) \xrightarrow{x \mapsto x^n} \text{Pic} \ (C_{\text{aff}})
\]

which in light of the proof of Lemma 5.2 can we rewritten as

\[
1 \to \mathbb{F}_q^\times/((\mathbb{F}_q^\times))^n \to H^1_{\text{ét}}(O_{(\infty)}, \mu_n) \to \text{Pic} \ (C_{\text{aff}})[n] \to 0.
\]
We deduce that $H^1_{ct}(O_{(\infty)}, \mu_n)$ is an extension of $\mathbb{F}^\times_q / (\mathbb{F}^\times_q)^n$ by $\text{Pic} (C^{\text{af}})[n]$ and so 
\[
|H^1_{ct}(O_{(\infty)}, \mu_n)| = |\mathbb{F}^\times_q / (\mathbb{F}^\times_q)^n| \cdot |\text{Pic} (C^{\text{af}})[n]| = |H^1(\mathbb{F}_q, \mu_n)| \cdot |\text{Pic} (C^{\text{af}})[n]|.
\]
Consequently 
\[
j_\infty(G) = \frac{|H^1_{ct}(O_{(\infty)}, \mu_n)|}{|\mu_n(\mathbb{F}_q)|} = \frac{|H^1_{ct}(O_{(\infty)}, \mu_n)|}{|H^1(\mathbb{F}_q, \mu_n)|} = |\text{Pic} (C^{\text{af}})[n]| = h_\infty(G)
\]
and finally:
\[
\tau(G) = h_\infty(G) \cdot \frac{j_\infty(G)}{j_\infty(G)} = |F| = n.
\]

**Corollary 7.12.** If $G$ is adjoint (not necessarily split) and $g = 0$ then $h_\infty(G) = 1$ and $\tau(G) = \tau_\infty(G) = |\hat{F}|$, where $\hat{F} := \text{Hom}(F(K^s), \mathbb{G}_{m,K^s})$ and $g := \text{Gal}(K^s/K)$.

**Proof.** According to Ono’s formula (3.9.11’) in [Ono1], considering the isogeny of class groups of $T^{\text{sc}}$ and $T$, there exists a finite set of primes $S$ for which
\[
\frac{h(T)}{h(T^{\text{sc}})} = \left(\frac{q(\alpha_1)}{\prod_{p \in S} q(\alpha_{O_p})}\right) / \left(\frac{q(\alpha_S)}{q(\alpha_W)}\right)
\]
where for any isogeny $\alpha$, $q(\alpha)$ stands for $|\ker(\alpha)|/|\text{ker}(\alpha)|$ and (see notation in Section 4):
\[
T_S^1(\mathbb{A}) := T^1(\mathbb{A}) \cap T_S, \ T_S(K) := T(\mathbb{A}(S)) \cap T(K),
\]
\[
\alpha_1^S := (T_S^{\text{sc}})^1(\mathbb{A}) \to T_S^1(\mathbb{A}), \ \alpha_K^S := T_S^{\text{sc}}(K) \to T_S(K), \ \alpha_W := W(T^{\text{sc}}) \to W(T).
\]
As $G^{\text{sc}}$ is simply-connected and $G$ is adjoint, both quasi-split, their Cartan subgroups $T^{\text{sc}}$ and $T$ are quasi-trivial and their integral models are connected everywhere (see Remark 2.1). In this case the quantities $q(\alpha)$ related to $\alpha^{K}$ and $\alpha_K^{S}$ are equals to the ones obtained in the split case on which the class group of each $\mathbb{G}_{m,n}$ is the class group of $K$ (see Formulas (3.1.7) and (3.1.8) in [Ono1]), thus equal to 1. Also by Lemma 3.5 one may deduce that $q(\alpha_{O_p}) = 1$ at each $p$ and by Lemma 7.3 (recall $g = 0$) this can be deduced also for $\alpha_W$. Hence $T^{\text{sc}}$ and $T$ share the same class number and so by Lemma 7.7 Prop. 7.8 and our Main Theorem $\tau(G) = h_\infty(G) \cdot \tau_\infty(G) = |\text{coker}(\pi_K^{\vee})|$ (see in the proof of 7.8). But as $T$ is quasi-trivial, $X^*(T) = \bigoplus_{i=1}^{n} \text{Ind}_1^H (H_i, \mathbb{Z})$ where $H_i$ are some finite subgroups of $\mathfrak{g}$, thus by Shapiro’s lemma $H^1(\mathfrak{g}, X^*(T)) \cong \bigoplus_{i=1}^{n} H^1(H_i, \mathbb{Z}) = 0$. Consequently the exact sequence of character groups (considered as $\mathfrak{g}$-modules):
\[
0 \to X^*(T) \to X^*(T^{\text{sc}}) \to \hat{F} \to 0
\]
gives rise by $\mathfrak{g}$-cohomology to the exact sequence:
\[
0 \to X^*(T)^{\#} \xrightarrow{\tilde{\pi}_K} X^*(T^{\text{sc}})^{\#} \to \hat{F}^{\#} \to H^1(\mathfrak{g}, X^*(T)) = 0
\]
from which we can see that $\tau(G) = |\text{coker}(\tilde{\pi}_K)| = |\text{coker}(\tilde{\pi}_K)| = |\hat{F}|$. This also shows by Ono’s formula [Ono3 Main Theorem] (see Cor. 7.13 below) that $\text{III}^1(\hat{F}) = 1$. \qed
More generally, our Main Theorem leads us under this section settings: $G$ is quasi-split and $g = 0$, to the following more general result obtained by Ono at 1965 (see Main Theorem in [Ono3]). It was designed for groups over number fields and been generalized by Behrend and Dhillon at 2009 to the function field case in [BD, Theorem 6.1].

**Corollary 7.13** (Ono’s formula). One has

$$
\tau(G) = \frac{|\hat{F}^\theta|}{|\Pi^1(\hat{F})|}
$$

where the denominator is the first Shafarevitch–Tate group assigned to $\hat{F}$.

**Proof.** Applying Galois $g$-cohomology to the sequence of groups of characters

$$
0 \to X^*(T) \xrightarrow{\pi} X^*(T^{sc}) \to \hat{F} \to 0
$$

where $\hat{F} := \text{Hom}(F \otimes_K K^*, \mathbb{G}_{m,K^*})$ yields the relation

$$
|\text{coker}(\pi_K)| = \frac{|\hat{F}\theta|}{|H^1(g, X^*(T))|}. 
$$

(28)

The following formula for the Tamagawa number of a torus is taken from [Ono2, Main Theorem], [Oes, Corollary 3.3]

$$
\tau(T) = \frac{|H^1(g, X^*(T))|}{|\Pi^1(T)|}. 
$$

(29)

Together with $|\Pi^1(T)| = |\Pi^1(\hat{F})|$ ([Ono3, p. 102]) we conclude

$$
\tau(G) \overset{28}{=} \tau(G^{sc}) \cdot \tau(T) \cdot |\text{coker}(\pi_K)| \quad \overset{28}{=} \tau(G^{sc}) \cdot \tau(T) \cdot \frac{|\hat{F}\theta|}{|H^1(g, X^*(T))|} \quad \overset{29}{=} \tau(G^{sc}) \cdot \frac{|\hat{F}\theta|}{|\Pi^1(\hat{F})|}. 
$$

In the following examples we refer to a construction which was demonstrated by Ono over number fields, in [Ono2, Example 6.3]. Our ground field is $K = \mathbb{F}_q(t)$ with odd characteristic and $\infty$ is chosen to correspond to the pole of $t$. At each example we consider another extension $L$ of $K$. We denote $g = \text{Gal}(L/K)$. The group $G^{sc} = R_{L/K}^{\text{SL}_2}$ is the universal cover of the semisimple and quasi-split $K$-group $G = G^{sc}/F$ where $F := R_{L/K}^{(1)}(\mu_n) = \ker[R_{L/K}(\mu_n) \to \mu_n]$. Let $S$ be the diagonal $K$-split maximal torus in $G$. Then $T = \text{Cent}_G(S)$ is a maximal torus of $G$ and is isomorphic as a $g$-module to the $K$-torus $\mathbb{G}_m \times R_{L/K}^{(1)}(\mathbb{G}_m)$ where the right hand factor is the norm torus, namely the kernel of the norm map (see [San, Example 5.6])

$$
R_{L/K}^{(1)}(\mathbb{G}_m) := \ker \left[ R_{L/K}(\mathbb{G}_m) \xrightarrow{N_{L/K}} \mathbb{G}_m \right].
$$
Its preimage in $G^{sc}$ is the Weil torus $T^{sc} = R_{L/K}(G_{m,L})$, fitting into the exact sequence

$$1 \to F \to T^{sc} \xrightarrow{\pi} T \to 1.$$ 

Over any $\hat{O}_p$, the norm torus is Spec $\hat{O}_\infty[a,b]/(a^2 - pb^2 - 1)$. Its reduction provides at each place $p$, $e_p$ connected components, where $e_p$ stands for the ramification index there (see [Bil] Example 3.3]), i.e. $[\mathcal{T}_p(\hat{O}_p) : \mathcal{T}_p^0(\hat{O}_p)] = e_p$. In this construction $|\text{coker}(\hat{\pi}_K)| = 1$.

**Example 7.14.** We start by $L = \mathbb{F}_{q^2}(t)$ obtained by extending the field of constants of $K$. Since the extension is quadratic, $F_\infty = \mu_2$ is $K_\infty$-split whence $t_\infty(G) = |\hat{F}_\infty| = |F_\infty| = 2$. Moreover, as $L/K$ is imaginary and totally unramified we have $h(T)/h(T^{sc}) = 2$ (see [Mor] Example 1]). Thus by Lemma [7,14] $h_\infty(G) = 1$ whence according to our Main Theorem $\tau(G) = h_\infty(G) \cdot t_\infty(G) = 2$.

**Example 7.15.** Now let $L = K(\sqrt{d})$ where $d$ is a product of $m$ distinct finite primes $p_i$. As before, $F_\infty = \mu_2$ and $t_\infty(G) = 2$. Recall that the norm torus is the only factor in $T^{sc}$ and $T$ which might have a disconnected reduction. This time, since each $p_i$, as well as $\infty$, ramifies in $L$ with $e_p = 2$ we have

$$\prod_p [\mathcal{T}_p(\hat{O}_p) : \mathcal{T}_p^0(\hat{O}_p)] = 2^{m+1}$$

while:

$$[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}_0(\mathbb{F}_q)] = |\{x \in \mathbb{F}_q : x^2 = 1\}| = |\{\pm 1\}| = 2.$$ 

Moreover, as $h(T^{sc})/h(T) = 2^{m-1}$ (see [Mor] Example 1]) and $\text{coker}(\hat{\pi}_K) = 1$, by Lemma [7,13] we get $h_\infty(G) = 1$. Altogether, we see by the Main Theorem that $\tau(G)$ remains equal to 2, independently of $m$. Both this result and the one of the previous example agree with Ono’s formula [7,13] indeed, as $L/K$ is cyclic, $\Pi^1(\hat{F}) = 1$ and $\tau(G) = |\hat{F}^\times| = |F| = 2$.

**Example 7.16.** Let $L = K(\Lambda_f)$ be the $f$-cyclotomic extension where $f$ is an irreducible polynomial of degree $d$. Then $\mathfrak{g}$ is cyclic of order $n = q^d - 1$. We still have $h(T)/h(T^{sc}) = 1$ ([Mor] Example 2]).

The only places which ramify in $L$ are $\infty$ with $e_\infty = q-1$ and $(f)$ which is totally ramified (see [Hay] Theorem 3.2)]. Therefore $[\mathcal{T}_{(\infty)}(O_{(\infty)}) : \mathcal{T}_0(\mathbb{O}_{(\infty)})] = q - 1$ and $[\mathcal{T}_{(f)}(O_{(f)}) : \mathcal{T}_0(\mathcal{O}_{(f)})] = n$. On the units group, since $q - 1 | n$ we have

$$[\mathcal{T}(\mathbb{F}_q) : \mathcal{T}_0(\mathbb{F}_q)] = |\{x \in \mathbb{F}_q : x^n = 1\}| = |\mathbb{F}_q^\times| = q - 1.$$ 

Moreover, $t_\infty(G) = |\hat{F}_\infty^\times| = |\mu_n| = n$ and as before $\text{coker}(\hat{\pi}_K) = 1$. So by Lemma [7,7] we get

$$h_\infty(G) = \frac{\prod_p [\mathcal{T}_p(\hat{O}_p) : \mathcal{T}_p^0(\hat{O}_p)]}{t_\infty(G) \cdot [\mathcal{T}(\mathbb{F}_q) : \mathcal{T}_0(\mathbb{F}_q)]} = \frac{(q - 1) \cdot n}{n \cdot (q - 1)} = 1.$$ 

Thus by our Main Theorem we conclude that $\tau(G) = t_\infty(G) = n$. Indeed, as $L/K$ is cyclic, $\Pi^1(\hat{F}) = 1$ and $\tau(G) = |\hat{F}^\times| = |\mu_n| = n$, which agrees again with Ono’s formula [7,13].
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