TROPICAL RATIONAL EQUIVALENCE ON $\mathbb{R}^r$

LARS ALLERMANN AND JOHANNES RAU

Abstract. We introduce an improved version of rational equivalence in tropical intersection theory which can be seen as a replacement of [AR07, chapter 8]. Using this new definition, rational equivalence is compatible with push-forwards of cycles. Moreover, we prove that every tropical cycle in $\mathbb{R}^r$ is equivalent to a uniquely determined affine cycle, called its degree.

This article can be seen as an additional chapter of our paper [AR07]. We stick to the definitions and notations introduced there.

As discussed in [AR07, remark 8.6] our previous definition of rational equivalence was not compatible with push-forwards of cycles. In this work we give a stronger definition of rational equivalence that is able to resolve this problem without losing other useful features. Moreover, using this new definition we are able to prove that the $k$-th Chow group $A_k(\mathbb{R}^r)$ of $\mathbb{R}^r$ is isomorphic to the group $Z^k_{\text{aff}}(\mathbb{R}^r)$ of affine $k$-cycles in $\mathbb{R}^r$.

Definition 1. Let $C$ be a cycle and let $D$ be a subcycle. We call $D$ rationally equivalent to zero on $C$, denoted by $D \sim 0$, if there exists a morphism $f : C' \to C$ and a bounded rational function $\phi$ on $C'$ such that $f^*(\phi \cdot C') = D$.

Let $D'$ be another subcycle of $C$. Then we call $D$ and $D'$ rationally equivalent if $D - D'$ is rationally equivalent to zero.

Lemma 2. Let $D$ be a cycle in $C$ rationally equivalent to zero. Then the following holds:

(a) Let $E$ be another cycle. Then $D \times E$ is also rationally equivalent to zero.
(b) Let $\varphi$ be a rational function on $C$. Then $\varphi \cdot D$ is also rationally equivalent to zero.
(c) Let $g : C \to \tilde{C}$ be a morphism. Then $g_*(D)$ is also rationally equivalent to zero.
(d) Assume $C = \mathbb{R}^r$ and let $E$ be another cycle in $\mathbb{R}^r$. Then $D \cdot E$ is also rationally equivalent to zero (where "\cdot" denotes the intersection product of cycles in $\mathbb{R}^r$ introduced in [AR07, definition 9.3]).
(e) Assume that $D$ is zero-dimensional. Then $\deg(D) = 0$.

Proof. Let $f : C' \to C$ be a morphism and $\phi$ a bounded function on $C'$ such that $f^*(\phi \cdot C') = D$. Then $f \times \text{id} : C' \times E \to C \times E$ provides (a), restricting $f$ to $f : f^*(\varphi) \cdot C' \to C$ provides (b) and composing $f$ with $g$ provides (c).
(d): The intersection product $D \cdot E$ is defined to be

$$\pi_*\left(\max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (D \times E)\right),$$

where the $x_i$ (resp. $y_i$) are the coordinates of the first (resp. second) factor of $\mathbb{R}^r \times \mathbb{R}^r$ and $\pi : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ is the projection onto the first factor. Thus we can apply (a) – (c).
(e): In this case $C'$ must be one-dimensional and we can apply [AR07, lemma 8.3], which shows that the degree of $\phi \cdot C'$ is zero. But pushing forward preserves degree. □

An easy example of rationally equivalent cycles are translations.
Lemma 3. Let $C$ be a cycle in $\mathbb{R}^r$ and let $C(v)$ denote the translation of $C$ by an arbitrary vector $v \in \mathbb{R}^r$. Then the equation

$$C(v) \sim C$$

holds.

Proof. Consider the cycle $C \times \mathbb{R}$ in $\mathbb{R}^r \times \mathbb{R}$ with morphism

$$f : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R}^r, \quad (x,t) \mapsto x + t \cdot e_i,$$

where $e_i$ is the $i$-th unit vector in $\mathbb{R}^r$. For $\mu \in \mathbb{R}$ let $\phi_\mu$ be the bounded function

$$\phi_\mu(x,t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq \mu \\ \mu & t \geq \mu. \end{cases}$$

Then $f_* (\phi_\mu \cdot C \times \mathbb{R}) = C - C(\mu \cdot e_i)$, which proves the claim. \qed

Definition 4. Let $C$ be a cycle in $\mathbb{R}^r$ of codimension $k$. Then we define $d_C$ to be the map

$$Z_k(\mathbb{R}^r) \to \mathbb{Z}, \quad D \mapsto \deg(C \cdot D).$$

Lemma 5. Let $C = \{(X, \omega_X)\}$ be a $d$-dimensional affine cycle in $\mathbb{R}^r$. Then there always exists a representative $(X', \omega_{X'})$ of $C$ and a complete simplicial fan $\Theta$ such that $X' \subseteq \Theta$.

Proof. Let $X_0 := X = \{\sigma_1, \ldots, \sigma_N\}$ and let $\sigma_i = \{x \in \mathbb{R}^r | f^r_i(x) \geq 0, \ldots, f^r_{k_\sigma}(x) \geq 0\}$. Moreover, let $Y_0 := \{\mathbb{R}^r\}$ and for $f \in \Lambda^r$ let

$$H_f := \{\{x | f(x) \geq 0\}, \{x | f(x) = 0\}, \{x | f(x) \leq 0\}\}.$$

For all $i = 1, \ldots, N$ we construct refinements

$$X_i := X_{i-1} \cap H_{f^r_{1_i}} \cap \ldots \cap H_{f^r_{k_\sigma}}$$

and

$$Y_i := Y_{i-1} \cap H_{f^r_{1_i}} \cap \ldots \cap H_{f^r_{k_\sigma}}$$

as described in [GKM07, 2.5(e)]. This construction yields fans $X_N$ and $Y_N$ with $X_N^{(k)} \subseteq Y_N^{(k)}$ for all $k$ and $|X| = |X_N|$. Moreover, $Y_N$ is a complete fan in $\mathbb{R}^r$. We can make $Y_N$ into a simplicial fan by further subdividing its cones: Let $\Theta := Y_N$. If $\sigma \in \Theta(p)$ is generated by vectors $v_1, \ldots, v_q$ then remove $\sigma$ and add all cones $\mathbb{R}_{\geq v_{i_1}} + \ldots + \mathbb{R}_{\geq v_{i_k}}$ for $1 \leq k \leq p$ and $1 \leq i_1 < \ldots < i_k \leq q$ to $\Theta$. Finally, we take $(X', \omega_{X'}) := (X, \omega_X) \cap \Theta$ as described in [GKM07, 2.11(b)]. \qed

Lemma 6. Let $C_1$ and $C_2$ be affine cycles in $\mathbb{R}^r$ with $C_1 \sim C_2$. Then $C_1 = C_2$.

Proof. Note that $C_1 \sim C_2$ implies $d_{C_1} = d_{C_2}$ by lemma \ref{lem:lem5} (d) and (e). Hence it suffices to show the following: If $C = [(X, \omega_X)]$ is a tropical cycle with $d_C = 0$ then $C = 0$.

We prove this by induction on $d := \text{dim}(X)$. For $d = 0$ the situation is trivial, as we have $X = d_x(\mathbb{R}^r) \cdot \{0\}$. Therefore $d_X = 0$ implies $C = [(X, \omega_X)] = 0$.

To prove the induction step, we first use lemma \ref{lem:lem5} which shows that we can assume that $X$ is the $d$-skeleton of a complete simplicial fan $\Theta$ with certain (possibly zero) weights on the $d$-dimensional cones. We have to show that, if we assume $d_X = 0$, all these weights are actually zero. So let $\sigma$ be a facet of $X$ and let $v_1, \ldots, v_d$ denote primitive vectors that generate $\sigma$. As $\Theta$ is simplicial, for each $i = 1, \ldots, d$ there exists a unique rational function $\varphi_i$ on $\Theta$ which fulfills $\varphi_i(v_i) = 1$ and is identically zero on all other rays of $\Theta$. We now want to compute the weight of $\tau := \langle v_1, \ldots, v_{d-1} \rangle \mathbb{R}_{\geq 0}$ in the intersection product $\varphi_d \cdot X$. As a representative of the primitive vector $v_{\sigma/\tau}$ we can use $\frac{1}{|\Lambda_{\sigma/\tau} + \tau|} v_d$ (it might not be an integer vector, but modulo $V_{\tau}$, it is a primitive generator of $\sigma$). Now, as $\varphi_d$ is identically zero on
all facets containing \(\tau\) but \(\sigma\) (in particular, \(\varphi_d\) is identically zero on \(\tau\)), the weight of \(\tau\) can be computed to be

\[
\omega_{\varphi_d \cdot X}(\tau) = \omega_X(\sigma) \frac{1}{|\Lambda_\sigma / \Lambda_r + \mathbb{Z}v_d|} \varphi_d(v_d) = \omega_X(\sigma) \frac{1}{|\Lambda_\sigma / \Lambda_r + \mathbb{Z}v_d|}.
\]

Our assumption \(d_C = d_X = 0\) implies \(d_X = 0\) as \((\varphi_d \cdot X) \cdot Y = X \cdot (\varphi_d \cdot Y)\) for all cycles \(Y\) of complementary dimension. We apply the induction hypothesis to \(\varphi_d \cdot X\) and conclude that \(\varphi_d \cdot X = 0\) and thus \(\omega_{\varphi_d \cdot X}(\tau) = 0\). But our above computation shows that this implies \(\omega_X(\sigma) = 0\) and therefore \(C = [(X, \omega_X)] = 0\).

**Theorem 7.** Let \(C\) be a cycle in \(\mathbb{R}^r\). Then there exists an affine cycle \(\delta(X)\) in \(\mathbb{R}^r\) with \(X \sim \delta(X)\).

**Proof.** Let \((X_1, \omega_{X_1})\) be a representative of \(C_1 := C\). Refining \((X_1, \omega_{X_1})\) we may assume that every polyhedron \(\sigma \in X_1\) is the convex hull of its 1-skeleton (see for example [Z95, 1.2 and 2.2]) and that every polyhedron \(\sigma \in X_1\) contains at least one vertex \(\sigma \supseteq P_0 \in X_1^{(0)}\).

The 1-skeleton of \(X_1\) is a finite graph \(\Gamma\) with edges \(X_1^{(1)} = \{e_1, \ldots, e_N\}\) and vertices \(X_1^{(0)} = \{P_0, \ldots, P_M\}\). By lemma 3 we may assume that \(P_0\) is the origin. On every edge \(e_i\) of this graph we choose an orientation and a primitive direction vector \(v_i \in \Lambda_{e_i}\) respecting this orientation (see figure 1(a)). Then for \(i = 1, \ldots, N\) let \(l_i \cdot ||v_i||\) be the length of the edge \(e_i\) (we set \(l_i = \infty\) if \(e_i\) is unbounded).

Adjacency of the edges in the graph \(\Gamma\) yields a system of linear equations in the variables \(l_i\) having the entries of the vectors \(v_i\) as coefficients (see figure 1(b)). As the system is solved by the given lengths \(l_i \in \mathbb{R}_{>0}\) and all vectors \(v_i\) are integral there exists a positive and integral solution \(l'_1, \ldots, l'_N\). Using these numbers \(l'_i\) we construct a polyhedral complex \(X'_1\), \(t \in \mathbb{R}\) as follows: We keep the position of the point \(P_0\) fixed and for all \(i = 1, \ldots, N\) we change the length of the edge \(e_i\) to \(l'_i + t \cdot l'_i\). For a given polyhedron \(\sigma \in X_1\) this process yields a deformation \(\sigma^t\) of \(\sigma\) which is not necessarily a polyhedron, but that can be decomposed into polyhedra \(\sigma^t_1, \ldots, \sigma^t_N\) (see figure 1(c)). If such a polyhedron \(\sigma^t_i\) is of dimension \(\dim(C)\), then we define its weight to be

\[
\overline{\omega}_{X'_1}(\sigma^t_i) := (-1)^{\delta(\sigma^t_i)} \cdot \omega_{X_1}(\sigma),
\]

where \(\delta(\sigma^t_i)\) is the number of values \(t' \in \mathbb{R}\) between 0 and \(t\) such that at least one of the lengths \(l_i + t' \cdot l'_i\) occurring in the boundary of \(\sigma^t_i\) is zero. We denote by \(X'_1\) the set of all polyhedra \(\sigma^t_i\) for \(\sigma \in X_1\) and by \(\overline{\omega}_{X'_1}\) the weight function on the polyhedra of maximal dimension. Refining and possibly merging some of the \(\sigma^t_i\) (we have to add up the weights of all merged polyhedra) yields a tropical polyhedral complex \((X'_1, \omega_{X'_1})\). Note that \((X'_1, \omega_{X'_1}) = (X_1, \omega_{X_1})\). Furthermore, for \(\sigma \in X_1\) we can consider the set

\[
\overline{\sigma} := \bigcup_{t \in \mathbb{R}} \left( \bigcup_{j=1}^{P_1} \sigma^t_j \times \{t\} \right) \subseteq \mathbb{R}^r \times \mathbb{R}.
\]

This set naturally splits up into polyhedra \(\overline{\sigma}_1, \ldots, \overline{\sigma}_s\). If a polyhedron \(\overline{\sigma}_i\) is of maximal dimension we associate the weight \(\overline{\omega}_{X'_1}(\sigma^t_i)\) to it, where \(\sigma^t_i\) is a polyhedron containing a point in the relative interior of \(\overline{\sigma}_i\) (this weight is obviously well-defined). We denote by \(\overline{Z}\) the set \(\{\overline{\sigma}_1, \ldots, \overline{\sigma}_s | \sigma \in X_1\}\) and by \(\overline{\omega}_{\overline{Z}}\) the weight function on the polyhedra of maximal dimension. The choice of the weights \(\overline{\omega}_{\overline{Z}}(\overline{\sigma}_i)\) ensures that refining some of the \(\overline{\sigma}_i\) yields a tropical polyhedral complex \((Z, \omega_{\overline{Z}})\).

Now, for \(\mu \in \mathbb{R}\) let \(\varphi_{\mu}\) be the rational function defined by

\[
\varphi_{\mu} : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R} : (x, t) \mapsto \max\{0, t\} - \max\{\mu, t\}.
\]

Let \(\overline{\sigma}_i \in Z\) be a polyhedron of maximal dimension and let (possibly after a refinement of \(X'_1\)) \(\sigma^t_i \subseteq \overline{\sigma}_i\) be a polyhedron of \(X'_1\) of maximal dimension. As every polyhedron in \(X_1\) contains at least one vertex,
we associate the weight $\omega_{P_j} \subseteq \sigma_j$. Let $P_{\sigma_j}^{+1}$ be the translation of $P_{\sigma_j}$ in $X_1^{t+1}$. We have

\[ P_{\sigma_j}^{+1} - P_{\sigma_j} = \sum_{j=1}^{k} \pm l_j^t v_{i_j} \]

for some $i_j \in \{1, \ldots, N\}$. Hence

\[ \left( \sum_{j=1}^{k} \pm l_j^t v_{i_j} \right) \in (\Lambda \times \mathbb{Z})_{\sigma_j} \]

is a generator of $(\Lambda \times \mathbb{Z})_{\sigma_j}/(\Lambda \times \mathbb{Z})_{\sigma_j}$, and we can deduce that

\[ \varphi_{\mu} \cdot [(Z, \omega Z)] = [(X^t_1, \omega X^t_1)] - [(X^t_1, \omega X^t_1)]. \]

Now let $t_0 \in \mathbb{R}_{>0}$ be the largest value such that there exists an edge that has been shrunk to length 0, i.e., an edge $e_i^t \in (X^t_1)^{(1)}$ with length $l_i + t_0 \cdot l_i^t = 0$. We conclude that

\[ \varphi_{t_0} \cdot [(Z, \omega Z)] = C_1 - C_2, \]

where $C_2 := [(X^t_1, \omega X^t_1)]$ can be seen as the cycle $C = C_1$ with at least one bounded polyhedron shrunk to one dimension less.

We repeat the whole process until all bounded polyhedra are shrunk to a point, i.e., until we obtain an affine cycle $C_\mu$. By construction we have

\[ C = C_1 \sim C_2 \sim \ldots \sim C_\mu, \]

which proves the claim.

**Definition 8.** Let $C$ be a cycle in $\mathbb{R}^r$. We define the *recession cycle or degree of $C$*, denoted by $\delta(C)$, to be the affine cycle equivalent to $C$. This affine cycle exists by theorem 7 and is unique by lemma 6.

**Remark 9.** Let $\sigma$ be a polyhedron in $\mathbb{R}^r$. We define the *recession cone of $C$* to be

\[ rc(\sigma) := \{ v \in \mathbb{R}^r \mid x + \mathbb{R}_{\geq 0} v \subseteq \sigma \forall x \in \sigma \} = \{ v \in \mathbb{R}^r \mid \exists x \in \sigma \text{ s.t. } x + \mathbb{R}_{\geq 0} v \subseteq \sigma \}. \]

The two sets coincide as $\sigma$ is closed and convex.

Let $C$ be a $d$-dimensional cycle in $\mathbb{R}^r$ with representative $(X, \omega_X)$ and let

\[ \widehat{R}(C) := \{ rc(\sigma) \mid \sigma \in X \}. \]

By removing all cones of $\widehat{R}(C)$ that are not contained in a $d$-dimensional cone and by subdividing the remaining cones we can make this set into a fan $R(C)$ of pure dimension $d$. To every cone $\sigma \in R(C)^{(d)}$ we associate the weight

\[ \omega_{R(C)}(\sigma) := \sum_{\sigma' \in X \atop \sigma \subseteq rc(\sigma')} \omega_X(\sigma'). \]
The proof of theorem 7 indeed shows that
\[ \delta(C) = [(R(C), \omega_{R(C)})] \]
holds.

**Theorem 10.** Let \( C, D \) be two tropical cycles in \( \mathbb{R}^r \). Then the following are equivalent:

i) \( C \sim D \)

ii) \( d_C = d_D \)

iii) \( \delta(C) = \delta(D) \)

**Proof.** i) \( \Rightarrow \) ii) follows from lemma 2 (d) and (e). iii) \( \Rightarrow \) i) is an immediate consequence of theorem 7. ii) \( \Rightarrow \) iii) follows from theorem 7, i) \( \Rightarrow \) ii) and lemma 6. \( \square \)

**Remark 11.** In other words, the above theorem says: Rational equivalence, numerical equivalence and “having the same degree” coincide.

**Theorem 12** (General Bézout’s theorem). Let \( C, D \) be two tropical cycles in \( \mathbb{R}^r \). Then

\[ \delta(C \cdot D) = \delta(C) \cdot \delta(D). \]

**Proof.** We apply theorem 7 and get

\[ \delta(C \cdot D) \sim C \cdot D \sim \delta(C) \cdot \delta(D) \]

(the second equivalence also uses lemma 2 (d)). By lemma 6 two rationally equivalent affine cycles are equal. \( \square \)
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LARS ALLERMANN, FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT KAIERSLAUTERN, POSTFACH 3049, 67653 KAIERSLAUTERN, GERMANY

E-mail address: allerman@mathematik.uni-kl.de

JOHANNES RAU, FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT KAIERSLAUTERN, POSTFACH 3049, 67653 KAIERSLAUTERN, GERMANY

E-mail address: jrau@mathematik.uni-kl.de