Expansion of infinite series containing modified Bessel functions of the second kind

Guglielmo Fucci\(^1,3\) and Klaus Kirsten\(^2\)

\(^1\)Department of Mathematics, East Carolina University, Greenville, NC 27858, USA
\(^2\)GCAP-CASPER, Department of Mathematics, Baylor University, Waco, TX 76798, USA

E-mail: fuccig@ecu.edu and Klaus_Kirsten@Baylor.edu

Received 8 June 2015, revised 2 September 2015
Accepted for publication 16 September 2015
Published 7 October 2015

Abstract

The aim of this work is to analyze general infinite sums containing modified Bessel functions of the second kind. In particular we present a method for the construction of a proper asymptotic expansion for such series valid when one of the parameters in the argument of the modified Bessel function of the second kind is small compared to the others. We apply the results obtained for the asymptotic expansion to specific problems that arise in the ambit of quantum field theory.

Keywords: asymptotic expansion, Bessel functions, infinite sums

1. Introduction

In quantum field theory the one-loop effective action is expressed in terms of the functional determinant of the (elliptic and self-adjoint) operator of small disturbances [6, 9]. Since the real eigenvalues of an elliptic self-adjoint operator \(L\) grow without bound, a naive computation of its functional determinant would give, as a result, a meaningless infinite quantity. A regularized expression for the functional determinant of the operator \(L\) can be obtained by using the spectral zeta function [6, 11, 19, 22, 29]. By denoting with \(\lambda_n\), \(n \in \mathbb{N}\), the real spectrum of \(L\), which acts on suitable functions defined on a smooth compact Riemannian manifold \(M\), the spectral zeta function is defined as the following sum

\[
\zeta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s},
\]

where \(s \in \mathbb{C}\). The above series is well defined in the half-plane \(\Re(s) > D/2\), where \(D\) represents the dimension of the manifold \(M\), and can be analytically continued in the entire...
complex plane to a meromorphic function possessing only simple poles [25]. The zeta regularized functional determinant of the operator $L$ is then expressed in terms of the derivative at $s = 0$ of the associated (analytically continued) spectral zeta function.

Different types of methods have been developed with the purpose of finding the analytic continuation of the spectral zeta function (see e.g. [12, 22]). The choice of which specific technique to use, however, depends heavily on whether or not the spectrum of the operator under consideration is explicitly known. The analytically continued expression of a certain class of spectral zeta functions constructed from explicitly known spectra often contains either single or double infinite series whose terms include modified Bessel functions of the second kind $K_\nu(z)$ [11, 12]. The double Bessel series that appears in most applications has the general form

$$f(s, \beta, B) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{m\beta}{\alpha_n} \right)^{s} \cos(2\pi mB)K_{\nu-s}(2\alpha_n m\beta),$$

(1.2)

where $\beta > 0$ is a dimensionless parameter, $B \in \mathbb{R}$ or $B \in i\mathbb{R}$, and the term $\alpha_n$ represents an increasing sequence of real numbers. The most commonly encountered Bessel series are

$$h(s, \beta, B) = \sum_{m=1}^{\infty} (m\beta)^{s} \cos(2\pi mB)K_{\nu-s}(2m\beta),$$

(1.3)

which can be obtained from (1.2) by setting $\alpha_n = 1$, and

$$g(s, \beta) = \sum_{n \in \mathbb{Z}^d} \left( \frac{\beta}{|n|} \right)^{s} K_{\nu-s}(2|n|\beta),$$

(1.4)

where $|n| = (n_1^2 + \cdots + n_d^2)^{1/2}$ and the prime indicates the omission of the origin of $\mathbb{Z}^d$. We would like to point out that infinite series of the form displayed above do not arise only in the analytically continued expressions of spectral zeta functions but they can also be found, for instance, in problems leading to lattice sums [31].

From the expressions (1.2)–(1.4) it is not very difficult to realize that the series converge quickly when $\beta \gg 1$ by virtue of the following asymptotic behavior of the modified Bessel function of the second kind

$$K_\nu(z) \sim \frac{\nu!}{\sqrt{2\pi z}} e^{-z},$$

(1.5)

for $z \to \infty$ with $\nu$ fixed [17]. This implies that $f(s, \beta, B)$, $h(s, \beta, B)$, and $g(s, \beta)$ become very useful for numerical evaluations when $\beta$ is large. For $\beta \ll 1$, instead, the expressions (1.2)–(1.4) lose their accuracy. This is quite undesirable since it is often of particular interest to obtain an asymptotic expansion of the series (1.2)–(1.4) valid when the parameter $\beta$ is small. For instance, in the study of finite temperature effects in quantum field theory on manifolds, the parameter $\beta$ is proportional to the inverse of the temperature $T$ [8, 10]. Therefore, the small $\beta$ expansion describes the high temperature behavior of the system under consideration [12]. In addition, series of the form (1.2) are obtained when studying the zeta regularized one-loop effective action of fields propagating on Kaluza–Klein manifolds of the type $M \times S^1$ [11]. In this case the parameter $\beta$ is related to the radius of the compactified dimension.

Due to its importance, the small-$\beta$ expansion of series of the form (1.2)–(1.4) has been analyzed by several authors. Unfortunately, however, this expansion has not always been performed correctly. In fact, in many occasions the small-$\beta$ expansion of (1.2)–(1.4) has been obtained first by substituting the small argument expansion of the modified Bessel functions of the second kind in the series (1.2)–(1.4) and then by erroneously regularizing the ensuing divergent expression with a suitable zeta function. The problem with this approach lies in the
fact that the argument of the modified Bessel functions of the second kind that appears in the series (1.2)–(1.4) cannot be assumed to be small for all \( n \). In fact, from the expressions (1.2)–(1.4) it is easy to realize that the argument of the modified Bessel functions of the second kind has the general form \( a_n \beta \) where \( a_n, n \in \mathbb{N}^+ \), is an increasing sequence of positive real numbers. Now, for any small \( \beta \) there always exists an integer \( m \in \mathbb{N}^+ \) such that \( a_n \beta > 1 \) for any \( n > m \). This implies that the argument of the modified Bessel function fails to be small uniformly in \( n \). This fact seems to have been overlooked also in the study of problems leading to spectral zeta functions of the form

\[
\sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left( n + c \right)^2 + a_n^2 \beta^{-\gamma},
\]

where \( c \in \mathbb{R}^+ \) and \( a_n \) denotes, for instance, the eigenvalues of an elliptic self-adjoint differential operator. In this framework the small-\( \beta \) expansion of (1.6) has been obtained by erroneously assuming that the product \( a_n^2 \beta \) is small for all values of \( n \). It is important to mention, at this point, that even though the small-\( \beta \) expansion of series of the form (1.2)–(1.4) has been often a source of confusion, examples of properly performed expansions can be found, for instance, in [2, 3, 11, 18, 21, 33].

The aim of this work is to present a method for obtaining the correct small-\( \beta \) expansion of series of the form (1.2)–(1.4) which is valid for arbitrary values of \( B \) and \( s \). Although, as mentioned above, correct small-\( \beta \) expansions of (1.2)–(1.4) have already been obtained, they are scattered in the literature and, most importantly, they have been found mostly for special cases. With this work we want to provide a quite general method for the small-\( \beta \) expansion which can be used to find expansions performed for special cases and can serve as a useful tool for future research in this area.

The outline of the paper is as follows. In the next two sections we utilize a complex integral representation of the modified Bessel function of the second kind to obtain, through Cauchy’s residue theorem, the small-\( \beta \) expansion of the series (1.2)–(1.4). In section 4 we apply our general results to special cases that are of wide interest in physical applications. In the last section we summarize and discuss the main results of this work.

2. Small parameter expansion of the double Bessel series

In this section we develop the small-\( \beta \) expansion of the double Bessel series (1.2). Due to the exponential decay of the modified Bessel function of the second kind, the double-series (1.2) converges for all values of \( s \) and, hence, defines an entire function in the finite complex \( s \)-plane. Clearly, the same remark also applies to the series (1.3) and (1.4) which allows us to conclude that also (1.3) and (1.4) are entire functions in the finite complex \( s \)-plane. In order to obtain an expression for (1.2) that is suitable for a small-\( \beta \) expansion we use the following complex integral representation

\[
K_\nu(z) = \left( \frac{z}{2} \right)^\nu \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(t) \Gamma(t-s) \left( \frac{z}{2} \right)^{-2t} dt,
\]

with \( c > \max \{0, \Re(s)\} \) [27 equation (10.32.13)], in the function \( f(s, \beta, B) \) to obtain

\[
f(s, \beta, B) = \frac{1}{4\pi^2} \sum_{\nu=1}^{\infty} \sum_{m=1}^{\infty} \cos(2\pi mB) \int_{c-i\infty}^{c+i\infty} \Gamma(t) \Gamma(t+s)(\beta m)^{-2\nu} a_n^{2t-2s} dt.
\]

By assuming that the increasing sequence of positive real numbers \( \{a_n\}_{n \in \mathbb{N}^+} \) represents the eigenvalues of a Laplace-type operator on a smooth compact Riemannian manifold \( M \) of
dimension $D$, we define the associated spectral zeta function as follows
\[
\zeta_M(s) = \sum_{n=1}^{\infty} n^{-s},
\] (2.3)
which is valid for $\mathcal{R}(s) > D/2$. The function (2.3) can be analytically continued to a meromorphic function with simple poles at $s = (D - k)/2$ with $k = \{0, \ldots, D - 1\}$ and $s = -(2l + 1)/2$ with $l \in \mathbb{N}_0$ [30]. In addition, for $\mathcal{R}(s) > 1/2$ it is not very difficult to obtain
\[
\sum_{m=1}^{\infty} m^{-2s} \cos(2\pi mB) = \frac{1}{2} \left[ \text{Li}_2\left(e^{2\pi iB}\right) + \text{Li}_2\left(e^{-2\pi iB}\right) \right],
\] (2.4)
where $\text{Li}_i(z)$ represents the polylogarithmic function. For $\nu < 1$ the polylogarithm has a branch point at $z = 1$ [15]. Therefore, it is convenient to distinguish between the case $B = 0$ and $B = 0$. For $\mathcal{R}(s) > 1/2$ and $B = 0$ we use the relation (2.4), for $\mathcal{R}(s) > 1/2$ and $B = 0$ the series on the left-hand-side of (2.4) becomes instead the Riemann zeta function $\zeta(s)$.

For $c > \max\{1/2, D/2 - \mathcal{R}(s)\}$ we can interchange the sum and the integral in (2.2) and then use the definition (2.3) and the result (2.4) to write, for $B = 0$,
\[
f(s, \beta, B) = \frac{1}{8\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\Gamma(t + s)\beta^{-2t}\zeta_M(t + s)\left[ \text{Li}_2\left(e^{2\pi iB}\right) + \text{Li}_2\left(e^{-2\pi iB}\right) \right] dt.
\] (2.5)
For $B = 0$ we have, instead, the following expression
\[
f(s, \beta, 0) = \frac{1}{4\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\Gamma(t + s)\beta^{-2t}\zeta_M(t + s)\zeta(2t) dt.
\] (2.6)
The integral representations (2.5) and (2.6) are particularly suitable for performing an expansion as $\beta \to 0$. This expansion is obtained by closing the integration contour to the left and by computing the resulting integral by using Cauchy’s residue theorem.

The validity of this procedure can be justified as follows. Instead of the integral in (2.6) we consider an integral $f_c$ which is the same as the one in (2.6) but with a contour of integration $C$ being a rectangle containing $N - 1$ poles of the integrand. The contour $C$ has vertices at $c \pm i\eta'$ and $-c_N \pm i\eta'$ with $c > D/2$ and $c_N > 0$ where $c_N$ is a point on the real axis between the $N$th and $(N - 1)$th pole of the integrand. By using Cauchy’s residue theorem on the integral $f_c$ one can prove that the sum of the first $(N - 1)$ residues of the integrand is equal to the sum of the integral over the vertical segment at $c$, the integral over the vertical segment at $-c_N$ and the integrals over the remaining two horizontal segments. To give an estimate for the integrals over the different parts of the contour $C$ we need the following remarks. It is well-known (see e.g. [16]) that the trace of the heat kernel of a Laplace operator $K(t)$ defined over a smooth compact Riemannian manifold and its associated spectral zeta function $\zeta(s)$ are related via the Mellin transform
\[
\Gamma(s)\zeta(s) = \int_0^{\infty} t^{s-1}K(t)dt,
\] (2.7)
valid for $\mathcal{R}(s) > D/2$. Since it can be proved that the trace of the heat kernel has the following expansion $K(t) = \sum_{k=0}^{N} A_k t^{k-D/2} + O(t^{N+1-D/2})$ as $t \to 0^+$ [16, 25, 30] and since $K(t)$ is exponentially small as $t \to \infty$ we can conclude (see [5]) that for any $x \leq D/2$, $\lim_{t \to \infty} ||\Gamma(x + iy)\zeta(x + iy)|| = 0$. By using the inverse Mellin transform in (2.7) one gets
\[ K(\beta) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\beta^{-t} \zeta(t) \, dt. \] (2.8)

By integrating (2.8) over a rectangular contour having vertices \( c \pm iT \) and \(-c \pm iT\), with \( c' > 0 \) a point between the \( n \)th and the \((N-1)\)th pole of the integrand in (2.8) and by then considering the limit \( T \to \infty \) we find

\[ K(\beta) = \sum_{k=0}^{N} A_k \beta^{-(k-D)/2} + \frac{1}{2\pi i} \int_{-c_N-i\infty}^{-c_N+i\infty} \Gamma(t)\beta^{-t} \zeta(t) \, dt, \] (2.9)

since the horizontal segments give no contributions due to the behavior \( \lim_{y \to \infty} \Gamma(x + iy)\zeta(x + iy) = 0 \). By comparing (2.9) with the asymptotic expansion of the trace of the heat kernel one can prove that

\[ \frac{1}{2\pi i} \int_{-c_N-i\infty}^{-c_N+i\infty} \Gamma(t)\beta^{-t} \zeta(t) \, dt = O(\beta^{N+1-D/2}). \] (2.10)

From the above comments and from the fact that \( \lim_{|y| \to \infty} \left| \Gamma(x + iy)\zeta_R(x + iy) \right| = 0 \) one can prove by using Sterling’s formula for \( \Gamma(z) \), \( z \in \mathbb{C} \), and the behavior \( \zeta_R(x + iy) \approx O(\mu(x) \ln^\nu t) \) with \( \mu(x) > 0 \) and \( \nu > 0 \) [28, 32], one can conclude that the integral over the horizontal segments in \( f_c \) does not contribute when \( T \to \infty \). This means that the integral in (2.6) is given by the sum of the first \((N-1)\) residues of the integrand and the integral

\[ \frac{1}{4\pi i} \int_{-c_N-i\infty}^{-c_N+i\infty} \Gamma(t)(t + s)\beta^{-2t} \zeta_M(t + s) \zeta_R(2t) \, dt. \] (2.11)

By using the estimate (2.10) and the inequality [28]

\[ \left| \Gamma(t)\beta^{-t} \zeta_R(2t) \right| = \left| \pi^{-2t-1/2} \xi_R(1 + 2t) \Gamma(t + \frac{1}{2}) \right| \leq \pi^{-2c_N-1} \xi_R(1 + 2c_N) \Gamma(c_N + \frac{1}{2}). \] (2.12)

which can be obtained from the functional equation of the Riemann zeta function and the inequalities \( \zeta_R(x + iy) \leq \zeta_R(x) \) for \( x > 1 \) and \( \left| \Gamma(x + iy) \right| \leq \Gamma(x) \) for \( x > 0 \) [28] one can prove that (2.11) is \( O(\beta^{N+1-D/2}) \). We are, hence, justified in shifting the integration contour to the left in (2.6) in order to obtain an asymptotic expansion of \( f(s, \beta, 0) \) for small \( \beta \).

An argument similar to the one outlined above can be applied to the integral in (2.5). We utilize the Mellin transform to obtain, for \( |\varepsilon| < 1 \), [15]

\[ \Gamma(s)\zeta_E(z) = \int_0^\infty t^{s-1} \frac{z}{e^t - z} \, dt, \] (2.13)

valid when \( \Re(s) > 0 \). Since \( z/(e^t - z) = O(1) \) as \( t \to 0 \) and is exponentially small when \( t \to \infty \) we have that for \( x < 0 \) \( \lim_{|y| \to \infty} \left| \Gamma(x + iy)\zeta_E(z) \right| = 0 \) [5]. Now, we consider the inverse transform of (2.13) with \( \varepsilon > 0 \)

\[ \frac{z}{e^t - z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(t)\beta^{-t} \zeta_E(z) \, dt, \] (2.14)

and evaluate the integral over the rectangle with vertices \( c \pm iT \) and \(-c' \pm iT\), \( c' > 0 \). By Cauchy’s residue theorem the sum of the residues at \( t = -n, n \in \mathbb{N}_0 \), equals the integral in (2.14) and the integral over the vertical line of fixed abscissa \(-c' \) since the integrals over the horizontal segments give no contribution due to the limit \( \lim_{|y| \to \infty} \left| \Gamma(x + iy)\zeta_E(z) \right| = 0 \). It is easy to realize that the sum of the residues at \( t = -n \) of the integrand coincides with the small-\( t \) asymptotic expansion, up to the order \( N \), of \( z/(e^t - z) \). This allows us to conclude that
the integral over the vertical line at \(-c'\) is of order \(O(t^{N+1})\). By utilizing these estimates and an argument similar to the one described above for the integral (2.6), one can justify the shifting the contour of integration to the left in (2.5) in order to get the small-\(\beta\) asymptotic expansion of \(f(s, \beta, B)\).

We perform the shift of the contours of integration in (2.5) and (2.6). For the case \(B \neq 0\) the terms in the integrand of (2.5) have poles at the points

\[
t = -s + \frac{D - k}{2}, \quad t = -s - \frac{2l + 1}{2}, \quad t = -s - l, \quad t = -l,
\]

with \(k \in \{0, \ldots, D - 1\}\) and \(l \in \mathbb{N}_0\). From (2.15) it is clear that depending on the value of \(s\), the integrand of (2.5) might have either simple or double poles. In fact, it is not difficult to realize that for \(s \neq k\), with \(k \in \mathbb{Z}\), and for \(s = \pm(2n + 1)/2\), \(n \in \mathbb{N}_0\), none of the points in (2.15) coincide and, therefore, the integrand in (2.5) possesses only simple poles. When either \(s = k\) or \(s = \pm(2n + 1)/2\) the integrand in (2.5) will develop, in addition to simple poles, also double poles.

For \(s \neq k\) and \(s \neq \pm(2n + 1)/2\), we close the contour of integration in (2.5) to the left as to include the simple poles of the integrand. From the poles at \(t = -s + (D - k)/2\) we obtain the following contribution

\[
f_1(s, \beta, B) = \frac{\beta^2}{4} \sum_{k=0}^{D-1} \Gamma\left(-s + \frac{D - k}{2}\right) \Gamma\left(\frac{D - k}{2}\right) \beta^{-D} \text{Res} \zeta_M \left(\frac{D - k}{2}\right)
\]

\[
\times \left[\text{Li}_{-2s + D - k} \left(e^{2\pi i B}\right) + \text{Li}_{-2s - D - k} \left(e^{-2\pi i B}\right)\right].
\]

Since the residues of the spectral zeta function are proportional to the coefficients \(A^M_{n/2}\) of the asymptotic expansion of the trace of the heat kernel associated with the Laplace-type operator on a smooth compact Riemannian manifold \(M\) as follows [30]

\[
A^M_{\frac{n}{2}} = \Gamma\left(-\frac{2l + 1}{2}\right) \text{Res} \zeta_M \left(-\frac{2l + 1}{2}\right),
\]

and

\[
A^M_{\frac{n}{2} + 1} = \Gamma\left(-\frac{2l + 1}{2}\right) \text{Res} \zeta_M \left(-\frac{2l + 1}{2}\right),
\]

the expression in (2.16) becomes

\[
f_1(s, \beta, B) = \frac{\beta^2}{4} \sum_{k=0}^{D-1} \Gamma\left(-s + \frac{D - k}{2}\right) \beta^{-D} A^M_{\frac{n}{2}} \left[\text{Li}_{-2s + D - k} \left(e^{2\pi i B}\right) + \text{Li}_{-2s - D - k} \left(e^{-2\pi i B}\right)\right].
\]

(2.19)

From the poles at \(t = -s - (2l + 1)/2\) we have the following contribution

\[
f_2(s, \beta, B) = \frac{\beta^{2l+1}}{4} \sum_{l=0}^{\infty} \Gamma\left(-s + \frac{2l + 1}{2}\right) \beta^{2l} A^M_{\frac{n}{2} + 1} \times \left[\text{Li}_{-2s - 2l - 1} \left(e^{2\pi i B}\right) + \text{Li}_{-2s - 2l - 1} \left(e^{-2\pi i B}\right)\right].
\]

(2.20)

which has been obtained by using the relation (2.18).
The residues associated with the poles located at $t = -s - l$ contribute

$$f_3(s, \beta, B) = \frac{\beta^{2s}}{4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \Gamma(-s - l) \beta^{2l} \zeta_M(-l) \left[ \text{Li}_{-2s-2l}(e^{2\pi i l}) + \text{Li}_{-2s-2l}(e^{-2\pi i l}) \right].$$

(2.21)

By using the relation [30]

$$\zeta_M(-l) = (-1)^l l! A_{\frac{M}{2}+l}^M,$$

(2.22)

in (2.21) we obtain

$$f_3(s, \beta, B) = \frac{\beta^{2s}}{4} \sum_{l=0}^{\infty} \Gamma(-s - l) \beta^{2l} A_{\frac{M}{2}+l}^M \left[ \text{Li}_{-2s-2l}(e^{2\pi i l}) + \text{Li}_{-2s-2l}(e^{-2\pi i l}) \right].$$

(2.23)

Finally, from the poles at $t = -1$ we have

$$f_4(s, \beta, B) = \frac{\beta^{2s}}{4} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \Gamma(-s - l) \beta^{2l} \zeta_M(-l) \left[ \text{Li}_{-2s-2l}(e^{2\pi i l}) + \text{Li}_{-2s-2l}(e^{-2\pi i l}) \right].$$

(2.24)

By noticing that for $n \in \mathbb{N}_0$ the polylogarithmic function satisfies the following relations [15]

$$\text{Li}_n(e^{2\pi i}) + \text{Li}_n(e^{-2\pi i}) = -\frac{(2\pi i)^n}{n!} B_n(s),$$

(2.25)

we can conclude that the sum of polylogarithmic functions in (2.24) vanishes identically except when $l = 0$. The last remarks allow us to write that

$$f_4(s, \beta, B) = -\frac{1}{4} \Gamma(s) \zeta_M(s).$$

(2.27)

At this point, by adding the results obtained in (2.19), (2.20), (2.23), and (2.27) we have

$$f(s, \beta, B) \sim \frac{\beta^{2s}}{4} \sum_{l=-D}^{\infty} \Gamma\left(-s - \frac{l}{2}\right) \beta^{2l} A_{\frac{M}{2}+l}^M \left[ \text{Li}_{-2s-2l}(e^{2\pi i l}) + \text{Li}_{-2s-2l}(e^{-2\pi i l}) \right] - \frac{1}{4} \Gamma(s) \zeta_M(s),$$

(2.28)

where, here and in the rest of this paper, we use $\sim$ to represent the small-$\beta$ asymptotic expansion.

When either $s = k$, $k \in \mathbb{Z}$, or $s = \pm(2n + 1)/2$, $n \in \mathbb{N}_0$, the integrand in (2.5) will develop both simple and double poles. The contributions coming from the double poles can be easily computed by using the following argument. The integrand appearing in the integral representations utilized in this work can be viewed as a product of three functions, $I(t) = p(t)q(t)r(t)$. One of the functions, which we assume to be $r(t)$, is analytic while $p(t)$ and $q(t)$ possess $N$, finitely or infinitely many, coinciding simple poles at $t = c_i$. The residue of each resulting double pole at $t = c_i$ can then be computed to be

$$\text{Res} I(c_i) = \left(\text{Res} p(c_i)\right)\left(\text{FP} q(c_i)\right)r(c_i) + \left(\text{Res} q(c_i)\right)\left(\text{FP} p(c_i)\right)r(c_i) + \left(\text{Res} p(c_i)\right)\left(\text{Res} q(c_i)\right)r'(c_i).$$

(2.29)

We start by analyzing the case $s = n$, with $n \in \mathbb{N}^+$, and an even-dimensional smooth compact Riemannian manifold $M$, namely $D = 2d$. Under this assumption the integrand in (2.5) has simple poles for $t = -n - (2m + 1)/2$, $m \in \mathbb{N}_0$, and for $t = -n + d - p - 1/2,$
\( \{0, \ldots, d-1\} \), and double poles for \( t = -q \), where \( q \in \mathbb{N}^+ \) such that \( q \geq n \). Moreover, if \( 0 < n \leq d, \ t = -n + d - j \) represents a simple pole for \( j = \{0, \ldots, d - n - 1\} \) and a double pole when \( j = \{d - n, \ldots, d - 1\} \). If \( n \geq d + 1, \ t = -n + d - j \) is a double pole for \( j = \{0, \ldots, d - n - 1\} \), and \( t = -q \) with \( q = \{0, \ldots, n - d - 1\} \), represents instead a simple pole. The integral in (2.5) is computed by closing the integration contour to the left. The contributions coming from the simple poles can be computed as before while the contributions from the double poles are found by using the argument leading to (2.29). This gives the result

\[
\begin{align*}
f_{D=2d}(n, \beta, B) & \sim \frac{1}{4} \left[ -(n-1)! \text{FP } \zeta_M(n) + A_{d-n}^M(\gamma - \Psi(n) + 2 \ln \beta) \right] \Theta(d-n) \\
& + \Theta(d-n-1) \frac{\beta^{2n-2d} d^{-n-1}}{4} \sum_{j=0}^{d-n-1} \Gamma(-n+d-j) \beta^2 A_j^M \left[ \text{Li}_{-2n+2d-2j}(e^{2\pi iB}) \right] \\
& + \text{Li}_{-2n-2d-2j}(e^{-2\pi iB}) \\
& + \frac{\beta^{2n-2d}}{4} \sum_{m=-d}^{\infty} \Gamma\left(-n - \frac{2m+1}{2}\right) \beta^2 A_{d-m}^M \left[ \text{Li}_{-2n-2m-1}(e^{2\pi iB}) \right] \\
& + \text{Li}_{-2n-2m-1}(e^{-2\pi iB}) \\
& + \frac{\beta^{2n-2d}}{2} \sum_{j=\max\{d-n,0\}}^{d-1} \frac{(-1)^{n-d+j}}{(n-d+j)!} \beta^2 A_j^M \left[ \text{Li}'_{-2n+2d-2j}(e^{2\pi iB}) \right] \\
& + \text{Li}'_{-2n+2d-2j}(e^{-2\pi iB}) \\
& + \frac{1}{2} \sum_{m=n}^{\infty} \frac{(-1)^m}{m!} \beta^2 A_{d+m-n}^M \left[ \text{Li}'_{-2n}(e^{2\pi iB}) + \text{Li}'_{-2n}(e^{-2\pi iB}) \right] \\
& - \frac{1}{4} (n-1)! \zeta_M(n) \Theta(n-d-1),
\end{align*}
\]

where \( \Theta(x) \) is the unit step-function, \( \gamma \) is the Euler–Mascheroni constant, \( \Psi(x) \) represents the logarithmic derivative of the gamma function, and \( \text{Li}_s'(z) = \partial_s \text{Li}_s(z) \). For \( s = n \), with \( n \in \mathbb{N}^+ \), and \( D = 2d + 1 \), the integrand in (2.5) presents simple poles for \( t = -n - (2m+1)/2 \), \( m \in \mathbb{N}_0 \), and for \( t = -n + d - j - 1/2, j = \{0, \ldots, d\} \), and double poles for \( t = -q \), where \( q \in \mathbb{N}^+ \) with \( q \geq n \). In addition, if \( 0 < n \leq d, t = -n + d - p \) is a simple pole for \( p = \{0, \ldots, d - n\} \) and a double pole when \( j = \{d - n, \ldots, d - 1\} \). If \( n \geq d + 1, t = -n + d - p \) is a double pole for \( j = \{0, \ldots, d - 1\} \) and \( t = -q \), with \( q = \{0, \ldots, n - d - 1\} \), is a simple pole. The application of Cauchy’s residue theorem then gives

\[
\begin{align*}
f_{D=2d+1}(n, \beta, B) & \sim \frac{1}{4} \left[ -(n-1)! \text{FP } \zeta_M(n) + A_{d-n+1/2}^M(\gamma - \Psi(n) + 2 \ln \beta) \right] \Theta(d-n) \\
& + \Theta(d-n-1) \frac{\beta^{2n-2d} d^{-n-1}}{4} \sum_{p=0}^{d-n-1} \Gamma(-n+d-p) \beta^2 A_p^M \left[ \text{Li}_{-2n+2d-2p}(e^{2\pi iB}) \right] \\
& + \text{Li}_{-2n+2d-2p}(e^{-2\pi iB}) \\
& + \frac{\beta^{2n+1}}{4} \sum_{m=-d+1}^{\infty} \Gamma\left(-n - \frac{2m+1}{2}\right) \beta^2 A_{d+m-1/2}^M \left[ \text{Li}_{-2n-2m-1}(e^{2\pi iB}) \right]
\end{align*}
\]
\[ + \text{Li}_{-2n-2m-1}(e^{-2\pi i\beta}) \]
\[ + \frac{\beta^{2n+2d-2p}}{2} \sum_{p=\max\{d-n,0\}}^{d-1} \frac{(-1)^p}{(n-d+p)!} \beta^{2p} A^M \frac{1}{p+\frac{1}{2}} \text{Li}_{-2n-2d-2p}(e^{2\pi i\beta}) \]
\[ + \text{Li}_0'(\frac{e^{2\pi i\beta}}{2} \sum_{m=n}^{\infty} \frac{(-1)^m}{m!} \beta^{2m} A^M \frac{1}{m+n+\frac{1}{2}} \left[ \text{Li}_{2n}(e^{2\pi i\beta}) + \text{Li}_{2m}(e^{-2\pi i\beta}) \right] \]
\[ - \frac{1}{4} (n-1)! \zeta_M(n) \Theta(n-d-1). \] (2.31)

When \( s = -n \) with \( n \in \mathbb{N}_0 \) for both even and odd-dimensional smooth compact Riemannian manifolds \( M \), the integrand in (2.5) contains simple poles at \( t = n + (D-k)/2 \), with \( k = \{0, \ldots, D-1\} \), at \( t = n - (2m+1)/2, m \in \mathbb{N}_0 \), and at \( t = q \) with \( q = \{1, \ldots, n\} \). Double poles appear, instead, at the points \( t = -q \) with \( q \in \mathbb{N}_0 \). Taking into account the above poles, the integrand in (2.5) leads to the result
\[ f(-n, \beta, B) \]
\[ \sim \frac{1}{4} A^M_{\frac{d-n+1}{2}} \left[ (\gamma - \Psi(1+n) + 2 \ln \beta) - \frac{(-1)^{p}}{n!} \zeta_M'(-n) \right] \]
\[ + \frac{\beta^{2n+2d-2p}}{4} \sum_{k=0}^{D-1} \Gamma \left( n + \frac{D-k}{2} \right) \beta^{2k} A^M \frac{1}{k+\frac{1}{2}} \left[ \text{Li}_{2n+D-k}(e^{2\pi i\beta}) + \text{Li}_{2n+D-k}(e^{-2\pi i\beta}) \right] \]
\[ + \frac{\beta^{2n+1}}{4} \sum_{m=0}^{\infty} \Gamma \left( n - m + \frac{D}{2} \right) \beta^{2m} A^M \frac{1}{m+n+\frac{1}{2}} \left[ \text{Li}_{2n-2m+1}(e^{2\pi i\beta}) + \text{Li}_{2n-2m+1}(e^{-2\pi i\beta}) \right] \]
\[ + \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \beta^{2m} A^M \frac{1}{m+n+\frac{1}{2}} \left[ \text{Li}_{2n}(e^{2\pi i\beta}) + \text{Li}_{2m}(e^{-2\pi i\beta}) \right] \]
\[ + \frac{\Theta(n-1)}{4} \sum_{m=1}^{n} (m-1)! \beta^{2m} A^M \frac{1}{m+n-1} \left[ \text{Li}_{2n}(e^{2\pi i\beta}) + \text{Li}_{2m}(e^{-2\pi i\beta}) \right]. \] (2.32)

We consider next the values \( s = (2n+1)/2, n \in \mathbb{N}_0 \), and \( D = 2d \). In this case the integrand in (2.5) has simple poles at \( t = -n - (2m+1)/2, m \in \mathbb{N}_0 \), at \( t = -n + d-j - 1/2, j = \{0, \ldots, d-1\} \), and double poles at \( t = -q \), with \( q \in \mathbb{N}_0^+ \) with \( q \geq n+1 \). In addition, if \( 0 \leq n \leq d-1 \), then \( t = -n + d - p - 1 \) is a simple pole for \( p = \{0, \ldots, d-n-2\} \) and a double pole for \( p = \{d-n-1, \ldots, d-1\} \). If instead \( n \geq d \), then \( t = -n + d - p - 1 \) represents a double pole for \( p = \{0, \ldots, d-1\} \) and \( t = -q \), with \( q = \{0, \ldots, n-d\} \), is a simple pole. By computing (2.5) in this case we obtain
\[ f_{D=2d} \left( \frac{2n+1}{2}, \beta, B \right) \sim \frac{1}{4} \left[ -\Gamma \left( n + \frac{1}{2} \right) \text{FP} \zeta_M \left( n + \frac{1}{2} \right) \right] \]
\[ + \frac{A^M_{\frac{d-n+1}{2}}}{2} \left[ (\gamma - \Psi \left( n + \frac{1}{2} \right) + 2 \ln \beta) \right] \Theta(d-n-1) \]
\[ + \frac{\Theta(d-n-2)}{4} \sum_{j=0}^{d-n-2} \Gamma(-n + d - j - 1) \beta^{2j} A^M_{\frac{j+1}{2}} \]
\[
\begin{align*}
&\times \left[ \text{Li}_{-2n+2d-2j-2}(e^{2\pi iB}) + \text{Li}_{-2n+2d-2j-2}(e^{-2\pi iB}) \right] \\
&+ \frac{\beta^{2n+1}}{4} \sum_{m=-d}^{\infty} \Gamma \left( -n - \frac{2m+1}{2} \right) \beta_{2m}A_{d+m}\left[ \text{Li}_{-2n-2m-1}(e^{2\pi iB}) \right] \\
&+ \text{Li}_{-2n-2m-1}(e^{-2\pi iB}) \\
&+ \frac{\beta^{2n-2d+2}}{2} \sum_{j=\max(d-n-1,0)}^{d-1} \frac{(-1)^{n-d+j+1}}{(n-d+j+1)!} \beta_{2j}A_{M}^{d} \frac{1}{j+\frac{1}{2}} \\
&\times \left[ \text{Li}_{-2n+2d-2j-2}(e^{2\pi iB}) + \text{Li}_{-2n+2d-2j-2}(e^{-2\pi iB}) \right] \\
&+ \frac{1}{2} \sum_{m=n+1}^{\infty} \frac{(-1)^{m}}{m!} \beta_{2m}A_{M}^{d+m-n-\frac{1}{2}} \left[ \text{Li}_{-2m}(e^{2\pi iB}) + \text{Li}_{-2m}(e^{-2\pi iB}) \right] \\
&- \frac{1}{4} \Gamma \left( n + \frac{1}{2} \right) \zeta_{M} \left( n + \frac{1}{2} \right) \Theta(n-d),
\end{align*}
\] (2.33)

For \( s = (2n+1)/2, n \in \mathbb{N}_{0} \), and \( D = 2d + 1 \) one finds simple poles at \( t = -n - (2m+1)/2 \), with \( m \in \mathbb{N}_{0} \), and double poles at \( t = -q \), where \( q \in \mathbb{N}^{+} \) with \( q \geq n + 1 \). Furthermore, if \( 0 \leq n \leq d \), then \( t = -n + d - j \) is a simple pole for \( j = \{0, \ldots, d-n-1\} \) and a double pole for \( j = \{d-n, \ldots, d\} \). If \( n \geq d+1 \), then \( t = -n + d - j \) is a double pole for \( j = \{0, \ldots, d-1\} \) and \( t = -q \), with \( q = \{0, \ldots, n-d-1\} \), is a simple pole. In this case the integral in (2.5) gives

\[
\begin{align*}
&f_{D=2d+1} \left( \frac{2n+1}{2}, \beta, B \right) \\
&\sim \frac{1}{4} \left[ -\Gamma \left( n + \frac{1}{2} \right) \text{FP} \zeta_{M} \left( n + \frac{1}{2} \right) + A_{M}^{d+m} \left( \gamma - \Psi \left( n + \frac{1}{2} \right) + 2 \ln \beta \right) \right] \Theta(d-n) \\
&+ \Theta(d-n-1) \frac{\beta^{2n-2d} d^{-n-1}}{4} \sum_{j=0}^{d-1} \Gamma(-n-d+j) \beta_{2j}A_{M}^{d} \\
&\times \left[ \text{Li}_{-2n+2d-2j}(e^{2\pi iB}) + \text{Li}_{-2n+2d-2j}(e^{-2\pi iB}) \right] \\
&+ \frac{\beta^{2n+1}}{4} \sum_{m=-d}^{\infty} \Gamma \left( -n - \frac{2m+1}{2} \right) \beta_{2m}A_{d+m+1} \left[ \text{Li}_{-2n-2m-1}(e^{2\pi iB}) \right. \\
&\left. + \text{Li}_{-2n-2m-1}(e^{-2\pi iB}) \right] \\
&+ \frac{\beta^{2n-2d}}{2} \sum_{j=\max(d-n-1,0)}^{d-1} \frac{(-1)^{n-d+j}}{(n-d+j)!} \beta_{2j}A_{M}^{d} \\
&\times \left[ \text{Li}_{-2n+2d-2j}(e^{2\pi iB}) + \text{Li}_{-2n+2d-2j}(e^{-2\pi iB}) \right] \\
&+ \frac{1}{2} \sum_{m=n+1}^{\infty} \frac{(-1)^{m}}{m!} \beta_{2m}A_{M}^{d+m-n-\frac{1}{2}} \left[ \text{Li}_{-2m}(e^{2\pi iB}) + \text{Li}_{-2m}(e^{-2\pi iB}) \right] \\
&\left. - \frac{1}{4} \Gamma \left( n + \frac{1}{2} \right) \zeta_{M} \left( n + \frac{1}{2} \right) \Theta(n-d-1), \right]
\end{align*}
\] (2.34)

Lastly, for \( s = -(2n+1)/2, n \in \mathbb{N}_{0} \), and for both even and odd-dimensional smooth compact Riemannian manifold \( M \), the integrand in (2.5) has simple poles at \( t = n + (D-k+1)/2 \), with \( k = \{0, \ldots, D-1\} \), at \( t = n - m + 1/2 \), for \( m \in \mathbb{N}_{0} \), and at \( t = -q \), with
$q = \{1, \ldots, n\}$. Double poles appear, instead, for $t = -q$ with $q \in \mathbb{N}_0$. In this case (2.5) becomes

$$f\left(\frac{2n+1}{2}, \beta, B\right)$$

$$\sim -\frac{1}{4} A_{n+1}^{M} \left[ \psi\left(\frac{2n+1}{2}\right) - \gamma - 2 \ln \beta \right] - \frac{1}{4} \Gamma\left(\frac{2n+1}{2}\right) \text{FP} \zeta_M\left(\frac{2n+1}{2}\right)$$

$$+ \frac{\beta^{-2n-D-1}}{4} \sum_{k=0}^{D-1} \Gamma\left(n + \frac{D-k+1}{2}\right) \beta^k A_n^{M} \left[ \text{Li}_{2n+D-k+1}\left(e^{2\pi i B}\right) + \text{Li}_{2n+D-k+1}\left(e^{-2\pi i B}\right) \right]$$

$$+ \frac{1}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(n - m + \frac{1}{2}\right) \beta^{2m} A_{n+m+1}^{M} \left[ \text{Li}_{2n-2m+1}\left(e^{2\pi i B}\right) + \text{Li}_{2n-2m+1}\left(e^{-2\pi i B}\right) \right]$$

$$+ \frac{\Theta(n-1)}{4} \sum_{m=0}^{n} \frac{(m-1)!}{2} \beta^{-2m} A_{n+m-n-m}^{M} \left[ \text{Li}_{2m}\left(e^{2\pi i B}\right) + \text{Li}_{2m}\left(e^{-2\pi i B}\right) \right].$$

(2.35)

Let us now assume that $B = 0$. In this case the relevant integral is $f(s, \beta, 0)$ in (2.6). Once again, when $s = k$, with $k \in \mathbb{Z}$, and when $s = \pm (2n+1)/2, n \in \mathbb{N}_0$ it is not difficult to verify that the integrand in (2.6) contains only simple poles which are located at the points given in (2.15) and at the point $t = 1/2$. The integral $f(s, \beta, 0)$ is then evaluated by closing the contour to the left and by using Cauchy’s residue theorem. From the poles positioned at $t = -s + (D - k)/2$ we obtain the following contribution

$$f_1(s, \beta, 0) = \frac{\beta^{2s}}{2} \sum_{k=0}^{D-1} \Gamma\left(-s + \frac{D-k}{2}\right) \beta^k A_n^{M} \zeta_R\left(-2s + D - k\right).$$

(2.36)

From the poles at $t = -s - (2l + 1)/2$ we get

$$f_2(s, \beta, 0) = \frac{\beta^{2s+1}}{2} \sum_{l=0}^{\infty} \Gamma\left(-s - \frac{2l+1}{2}\right) \beta^{2l} A_{n+l}^{M} \zeta_R\left(-2s - 2l - 1\right),$$

(2.37)

while from the ones at $t = -s - l$ we have

$$f_3(s, \beta, 0) = \frac{\beta^{2s}}{2} \sum_{l=0}^{\infty} \Gamma\left(-s - l\right) \beta^{2l} A_{n+l}^{M} \zeta_R\left(-2s - 2l\right).$$

(2.38)

Since $\zeta(-2k) = 0$ for $k \in \mathbb{N}^+$, the only contribution coming from the poles at $t = -l$ can be found to be

$$f_3(s, \beta, 0) = -\frac{1}{4} \Gamma(s) \zeta_M(s).$$

(2.39)

Lastly, from the pole at $t = 1/2$ we obtain

$$f_4(s, \beta, 0) = \frac{\sqrt{\pi}}{4\beta} \Gamma\left(s + \frac{1}{2}\right) \zeta_M\left(s + \frac{1}{2}\right).$$

(2.40)
By adding the terms obtained in (2.36) through (2.40) we obtain the result
\[ f(s, \beta, 0) \sim \frac{\beta^{2s}}{2} \sum_{l=-D}^{\infty} \Gamma\left(-s - \frac{1}{2}\right) \beta^l A_{n+l}^M \zeta_R(-2s - l) - \frac{1}{4} \Gamma(s) \zeta_M(s) \]
\[ + \frac{\sqrt{\pi}}{4\beta} \left( s + \frac{1}{2} \right) \zeta_A \left( s + \frac{1}{2} \right). \tag{2.41} \]

Similarly to the previous case, when either \( s = k \), \( k \in \mathbb{Z} \), or \( s = \pm(2n + 1)/2 \), \( n \in \mathbb{N}_0 \), the integrand in (2.6) will have simple as well as double poles. We start by considering the values \( s = n \), with \( n \in \mathbb{N}^+ \), and \( D = 2d \). The simple and double poles of the integrand in (2.6) coincide with the ones listed just before (2.30). In addition to those poles, we have that the point \( t = 1/2 \) is a simple pole when \( n \geq d \) and a double pole when \( 1 \leq n \leq d - 1 \). By closing the contour of integration to the left, the integral in (2.6) can be computed by using Cauchy’s residue theorem to obtain
\[ f_{D=2d}(n, \beta, 0) \]
\[ \sim \frac{1}{4} \left[ -(n - 1)! \text{FP} \zeta_M(n) + A_{d-n}^M (\gamma - \Psi(n) + 2 \ln \beta) \right] \Theta(d - n) \]
\[ + \Theta(d - n - 1) \left[ \frac{\beta^{2n-2d}}{2} \sum_{j=0}^{d-n-1} \Gamma(-n - d - j) \beta^{2j} A_{d-n+j}^M \zeta_R(-2n + 2d - 2j) \right] \]
\[ + \frac{\sqrt{\pi}}{4\beta} \left( n + \frac{1}{2} \right) \text{FP} \zeta_M \left( n + \frac{1}{2} \right) - \frac{\sqrt{\pi}}{2\beta} A_M^{d-n-\frac{1}{2}} \left( \ln 4\beta - \sum_{k=1}^{n} \frac{1}{2k - 1} \right) \]
\[ + \frac{\beta^{2n+1}}{2} \sum_{m=n-d}^{\infty} \Gamma \left( n - \frac{2m + 1}{2} \right) \beta^{2m} A_{d+m+1}^M \zeta_R(-2n - 2m - 1) \]
\[ + \beta^{2n-2d} \sum_{j=\max(d-n,0)}^{d-1} \frac{(-1)^{n-d+j}}{(n - d + j)!} \beta^{2j} A_j^M \zeta_R'(-2n + 2d - 2j) \]
\[ + \frac{1}{4}(n - 1)! \zeta_M(n) \Theta(n - d - 1) + \frac{\sqrt{\pi}}{4\beta} \left( n + \frac{1}{2} \right) \zeta_M \left( n + \frac{1}{2} \right) \Theta(n - d). \tag{2.42} \]

For \( s = n \), \( n \in \mathbb{N}^+ \), and \( D = 2d + 1 \), the simple and double poles of the integrand in (2.6) are the same as the ones given before the expression (2.31). Moreover, \( t = 1/2 \) is a simple pole for \( n \geq d + 1 \) and a double pole for \( 1 \leq n \leq d \). In this case we obtain
\[ f_{D=2d+1}(n, \beta, 0) \]
\[ \sim \frac{1}{4} \left[ -(n - 1)! \text{FP} \zeta_M(n) + A_{d-n+\frac{1}{2}}^M (\gamma - \Psi(n) + 2 \ln \beta) \right] \Theta(d - n) \]
\[ + \frac{\sqrt{\pi}}{\beta} \left( n + \frac{1}{2} \right) \text{FP} \zeta_M \left( n + \frac{1}{2} \right) - \frac{2\sqrt{\pi}}{\beta} A_{d-n}^M \left( \ln 4\beta - \sum_{k=1}^{n} \frac{1}{2k - 1} \right) \Theta(d - n) \]
\[ + \Theta(d - n - 1) \frac{\beta^{2n-2d}}{2} \sum_{p=0}^{d-n-1} \Gamma(-n - d - p) \beta^{2p} A_{d-n+\frac{1}{2}}^M \zeta_R(-2n + 2d - 2p) \]
When \( s = -n \), with \( n \in \mathbb{N}_0 \), and \( M \) is either an even or odd-dimensional smooth compact Riemannian manifold, the poles of the integrand of (2.6) are located at the points indicated before equation (2.32) with the addition of a double pole at \( t = 1/2 \). In this case the integral (2.6) gives

\[
f(n, \beta, 0) \sim \frac{1}{4} A^M_{d+n} \left[ \gamma - \Psi(1 + n) + 2 \ln \beta \right] - \frac{1}{4} \frac{(-1)^n}{n!} \zeta'_{\nu_M}(-n)
\]

\[
+ \frac{\beta^{-2-2D} D-2}{2} \sum_{k=0}^{D-2} \Gamma \left( n + \frac{D-k}{2} \right)
\]

\[
\times \beta^2 A^M_\frac{d+n}{2} \zeta_R(2n + D - k) + \Theta(n - 1) \beta^{-2n-1} \frac{1}{2} \Gamma \left( n + \frac{1}{2} \right) \beta^2 A^M_\frac{d+n}{2} \zeta_R(2n + 1)
\]

\[
+ \frac{\beta^{-2n+1}}{2} \sum_{m=0}^{\infty} \Gamma \left( n - \frac{2m+1}{2} \right) \beta^2 A^M_{d+n-m} \zeta_R(2n - 2m - 1)
\]

\[
+ \frac{\Theta(n-1)}{2} \sum_{m=1}^{n} \frac{(-1)^m}{m+1} \beta^2 A^M_{d+n} \zeta_R(-2m)
\]

\[
+ \frac{\sqrt{\pi}}{4\beta} \Gamma \left( \frac{1}{2} - n \right) \left[ \ln \frac{4\beta}{A^M_{d+n-1}} - \sum_{k=1}^{n} \frac{1}{2k-1} \right]
\]

\[
(2.43)
\]

We next focus on the values \( s = (2n + 1)/2 \), \( n \in \mathbb{N}_0 \), and \( D = 2d \). In this case, the integrand in (2.6) has poles at the points listed before (2.33). In addition, \( t = 1/2 \) is a simple pole if \( n \geq d \) and a double pole if \( 0 \leq n \leq d - 1 \). Taking into account the above poles, the integral in (2.6) leads to the result

\[
f_{D=2d} \left( \frac{2n+1}{2}, \beta, 0 \right) \sim \frac{1}{4} \left[ -\Gamma \left( n + \frac{1}{2} \right) \zeta_M \left( n + \frac{1}{2} \right) + A^M_{d-n-\frac{1}{2}} \left( \gamma - \Psi \left( n + \frac{1}{2} \right) + 2 \ln \beta \right) \right]
\]
$$\frac{1}{\beta} n! \text{FP} \zeta_{M}(n+1) - \frac{2}{\beta} A_{d-n}^{M} \left( \ln 2\beta - 2H_{n} \right) \Theta(d - n - 1)$$

$$+ \Theta(d - n - 2) \frac{\beta 2n - 2d - 2 - d - n - 2}{2} \sum_{j=0}^{d-n-2} \Gamma(-n - d - j - 1)$$

$$\times \beta^{2n+1} \zeta_{R}(-2n + 2d - 2j - 2)$$

$$+ \frac{\beta^{2n+1}}{2} \sum_{m=-d}^{d} \sum_{m=n+1}^{d-n-1} \Gamma\left(-n - \frac{2m + 1}{2}\right) \beta^{2m} A_{d+m}^{M} \zeta_{R}(-2n - 2m - 1)$$

$$+ \frac{\beta^{2n-2d+2}}{2} \sum_{j=\max(d-n,1)}^{d-1} \frac{(-1)^{n-d+j+1}}{(n - d + j + 1)!} \beta^{2j} A_{d+m}^{M} \zeta^{j}_{R}(-2n - 2d - 2j - 2)$$

$$+ \sum_{m=n+1}^{\infty} \frac{(-1)^{m}}{m!} \beta^{2m} A_{d+m}^{M} \zeta^{j}_{R}(-2m)$$

$$- \frac{1}{4} \Theta(n - d) \left[ \Gamma\left(n + \frac{1}{2}\right) \zeta_{M}(n + \frac{1}{2}) - \frac{2}{\beta} n! \zeta_{M}(n + 1) \right] \Theta(d - n).$$

$$\text{(2.45)}$$

where $H_{n}$ denotes the $n$th harmonic number. For $s = (2n + 1)/2, n \in \mathbb{N}_{0}$, and $D = 2d + 1$, the integrand in (2.6) develops poles at the points indicated above (2.34). Furthermore, the point $t = 1/2$ becomes a simple pole when $n \geq d + 1$ and a double pole when $0 \leq n \leq d$. In this case the integrand (2.6) gives

$$f_{D=2d+1}\left(\frac{2n + 1}{2}, \beta, 0\right)$$

$$\sim \frac{1}{4} \left[ -\Gamma\left(n + \frac{1}{2}\right) \text{FP} \zeta_{M}(n + \frac{1}{2}) + A_{d-n}^{M} \left( \gamma - \Psi\left(n + \frac{1}{2}\right) + 2 \ln \beta \right) \right] \Theta(d - n)$$

$$+ \Theta(d - n - 1) \frac{\beta^{2n-2d} d - n - 1}{2} \sum_{j=0}^{d-n-1} \Gamma(-n - d - j) \beta^{2j} A_{d-j}^{M} \zeta_{R}(-2n + 2d - 2j)$$

$$+ \frac{\sqrt{\pi}}{4\beta^{n!}} \text{FP} \zeta_{M}(n + 1) - \frac{2}{\beta} A_{d-n}^{M} \left( \ln 2\beta - 2H_{n} \right)$$

$$+ \frac{\beta^{2n+1}}{2} \sum_{m=-d}^{d} \sum_{m=n+1}^{d-n-1} \Gamma\left(-n - \frac{2m + 1}{2}\right) \beta^{2m} A_{d+m}^{M} \zeta_{R}(-2n - 2m - 1)$$

$$+ \frac{\beta^{2n-2d}}{2} \sum_{j=\max(d-n,1)}^{d} \frac{(-1)^{n-d+j+1}}{(n - d + j + 1)!} \beta^{2j} A_{d-j}^{M} \zeta^{j}_{R}(-2n - 2d - 2j - 2)$$

$$+ \sum_{m=n+1}^{\infty} \frac{(-1)^{m}}{m!} \beta^{2m} A_{d+m}^{M} \zeta^{j}_{R}(-2m)$$

$$- \frac{1}{4} \Gamma\left(n + \frac{1}{2}\right) \zeta_{M}(n + \frac{1}{2}) \Theta(n - d - 1),$$

$$\text{(2.46)}$$

Finally, when $s = -(2n + 1)/2$, with $n \in \mathbb{N}_{0}$, and $M$ is either an even or odd-dimensional smooth compact Riemannian manifold, the integrand in (2.6) has poles located at the points indicated before equation (2.35) and a double pole at $t = 1/2$. In this case the integrand (2.6)
gives the result
\[ f \left( \frac{-2n + 1}{2}, \beta, 0 \right) \]
\[ \sim -\frac{1}{4} A_{\frac{1}{2}+n+1}^M \left[ \psi \left( \frac{-2n + 1}{2} \right) - \gamma - 2 \ln \beta \right] - \frac{1}{4} \Gamma \left( \frac{-2n + 1}{2} \right) \text{FP} \zeta_M \left( \frac{-2n + 1}{2} \right) \]
\[ + \frac{\beta^{-2n-D-1}}{2} \sum_{k=0}^{D-1} \Gamma \left( n + \frac{D - k + 1}{2} \right) \beta^{2n} A_{\frac{1}{2}+n+1}^M \zeta_R \left( 2n + D - k + 1 \right) \]
\[ + \frac{\beta^{-2n-1}}{2} \sum_{m=0}^{\infty} \Gamma \left( n - m + \frac{1}{2} \right) \beta^{2m} A_{\frac{1}{2}+n}^M \zeta_R \left( 2n - 2m + 1 \right) \]
\[ + \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \beta^{2n} A_{\frac{1}{2}+n}^M \zeta_R \left( -2m \right) \]
\[ + \frac{\Theta(n-1)}{2} \sum_{m=1}^{n} \frac{n}{(m-1)!} \beta^{-2m} A_{\frac{1}{2}+n}^M \zeta_R \left( 2m \right) \]
\[ + \sqrt{\pi} \frac{(-1)^n}{n!} \zeta'_{\frac{1}{2}} \left( -n \right) + \frac{\sqrt{\pi}}{4\beta} \frac{A_{\frac{1}{2}+n}^M}{n!} \left( -2 \ln 2\beta + H_n \right) \]
\[ (2.47) \]

3. Small parameter expansion of the Bessel series

Let us now turn our attention to the series (1.3) and (1.4) defined in the section 1. By utilizing the complex integral representation (2.1) for the modified Bessel function of the second kind in the expression for \( h(s, \beta, B) \) in (1.3), we obtain

\[ h(s, \beta, B) = \frac{1}{8\pi i} \int_{-\infty}^{\infty} \Gamma(t) \Gamma(t + s)^{-2} \left[ \text{Li}_{2s-2} \left( e^{2\pi i \beta} \right) + \text{Li}_{2s-2} \left( e^{-2\pi i \beta} \right) \right] \text{d}t, \]
\[ (3.1) \]

with \( c = \max \{0, -\Re(s)\} \). The integrand can be shown to have only simple poles when \( s = k \), with \( k \in \mathbb{Z} \), positioned at the points \( t = -s - n \) and \( t = -n \) where \( n \in \mathbb{N}_0 \). For \( s = k \), instead, the integrand develops both simple and double poles. By assuming that \( s \neq k \) we close the integration contour to the left in (3.1) and apply Cauchy’s residue theorem to obtain the following contribution

\[ h_1(s, \beta, B) = \frac{\beta^{2s}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-s-n) \beta^{2n} \left[ \text{Li}_{-2s-2} \left( e^{2\pi i \beta} \right) + \text{Li}_{-2s-2} \left( e^{-2\pi i \beta} \right) \right] \]
\[ (3.2) \]

from the poles at \( t = -s-n \). From the poles at \( t = -n \) we have, instead,

\[ h_2(s, \beta, B) = -\frac{1}{4} \Gamma(s) \]
\[ (3.3) \]

By adding the results in (3.2) and (3.3) we have

\[ h(s, \beta, B) \sim \frac{\beta^{2s}}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-s-n) \beta^{2n} \left[ \text{Li}_{-2s-2} \left( e^{2\pi i \beta} \right) + \text{Li}_{-2s-2} \left( e^{-2\pi i \beta} \right) \right] - \frac{1}{4} \Gamma(s) \]
\[ (3.4) \]

When \( s = n \), with \( n \in \mathbb{N}_+ \), the integrand in (3.1) has poles at \( t = -j \), \( j \in \mathbb{N}_0 \), which are simple when \( j = \{0, \ldots, n-1\} \) and double when \( j \geq n \). In this case the integral (3.1) gives
For $s = -n$, with $n \in \mathbb{N}_0$, the integrand in (3.1) has simple poles at $t = j$, with $j = \{1, \ldots, n\}$, and double poles for $t = -j$, with $j \geq 0$. The integral (3.1) then provides the result

$$h(-n, \beta, B) \sim \frac{1}{2} \frac{(-1)^n}{n!} \Gamma(\gamma + \ln \beta - 2H_0)$$

$$+ \frac{(-1)^n}{2} \sum_{j=0}^{\infty} \frac{\beta^{2j}}{j!(j + n)!} \left[ \text{Li}'_{-2j}(e^{2\pi i B}) + \text{Li}'_{-2j}(e^{-2\pi i B}) \right]$$

$$+ \frac{1}{4} \sum_{j=1}^{n} \frac{(-1)^n-j(j-1)!}{(n-j)!} \beta^{-2j} \left[ \text{Li}_{j}(e^{2\pi i B}) + \text{Li}_{j}(e^{-2\pi i B}) \right].$$

(3.6)

For $B = 0$ the series $h(s, \beta, B)$ in (1.3) becomes

$$h(s, \beta, 0) = \sum_{m=1}^{\infty} (m\beta)^s \mathcal{K}_s(2m\beta).$$

(3.7)

The integral representation (2.1) allows us to rewrite (3.7) as follows

$$h(s, \beta, 0) = \frac{1}{4\pi i} \int_{c-\infty}^{c+\infty} \Gamma(t) \Gamma(t + s) \beta^{-2s} \zeta_R(2t) \, dt,$$

(3.8)

where $c > \max\{1/2, -\Re(s)\}$. The integrand in (3.8) develops poles at the points $t = -s - n$, $t = -n$, and at $t = 1/2$ with $n \in \mathbb{N}_0$. These poles are simple for values of $s = k$, with $k \in \mathbb{Z}$, and $s = -(2n + 1)/2$, with $n \in \mathbb{N}_0$. For all $s \in \mathbb{C}$ except for the integers and negative half-integers we close the contour of integration to the left and we obtain, from the first set of poles, the contribution

$$h_1(s, \beta, 0) = \frac{\beta^{2s}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-s - n) \beta^{2n} \zeta_R(-2s - 2n).$$

(3.9)

From the poles at $t = -n$ we have that $h_2(s, \beta, 0) = h_2(s, \beta, B)$, while the pole at $t = 1/2$ leads to the result

$$h_3(s, \beta, 0) = \frac{\sqrt{\pi}}{4\beta} \Gamma\left(s + \frac{1}{2}\right).$$

(3.10)

By adding the contributions from all the poles we obtain the expansion

$$h(s, \beta, 0) \sim \frac{\beta^{2s}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-s - n) \beta^{2n} \zeta_R(-2s - 2n) - \frac{1}{4} \Gamma(s) + \frac{\sqrt{\pi}}{4\beta} \Gamma\left(s + \frac{1}{2}\right).$$

(3.11)

Now, for $s = m, m \in \mathbb{N}_0^+$, the integrand in (3.8) has simple poles for $t = -j$, negative integers, with $j = \{0, \ldots, n - 1\}$ and at $t = 1/2$. Double poles are present, instead, for $t = -j$ where $j \geq n$. In this case the integral (3.8) becomes

$$h(n, \beta, 0) \sim (-1)^n \sum_{j=0}^{\infty} \frac{\beta^{2j}}{j!(j - n)!} \zeta_R(-2j) - \frac{1}{4} \Gamma(n) + \frac{\sqrt{\pi}}{4\beta} \Gamma\left(n + \frac{1}{2}\right).$$

(3.12)

For $s = -m$ with $m \in \mathbb{N}_0$, the integrand in (3.8) has simple poles for $t = j$, $j = \{1, \ldots, n\}$ and $t = 1/2$, and double poles for $t = -j$ with $j \in \mathbb{N}_0$. The integral (3.8) then gives
Lastly, when \( s = -(2n + 1)/2 \), the integrand in (3.8) has simple poles for \( t = -m \), with \( m \in \mathbb{N}_0 \), and \( t = -m + n + 1/2 \) with \( m \neq n \). In addition, a double pole develops at \( t = 1/2 \).

In this case we have

\[
h\left(\frac{-2n + 1}{2}, \beta, 0\right) \sim \frac{\beta^{-2n-1}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(-m + n + \frac{1}{2}\right) \beta^{2m} \zeta_R(-2m + 2n + 1) \]

\[
- \frac{1}{4} \Gamma\left(\frac{1}{2} - n\right) - \frac{\sqrt{\pi} \, (-1)^n}{4\beta^{-1} \, n!} \left(2 \ln 2\beta - H_n\right).
\]

The last Bessel series considered in this work is \( g(s, \beta) \) defined in (1.4). By making use of the integral representation (2.1) we write the series (1.4) as

\[
g(s, \beta) = \frac{1}{4\pi i} \int_{c - i\infty}^{c + i\infty} \Gamma(t)\Gamma(t + s)\beta^{-2t} \zeta_E(s + t)\, dt,
\]

where \( c > \max\{0, d/2 - \Re(s)\} \) and \( \zeta_E(u) \) denotes the Epstein zeta function \([13, 14]\)

\[
\zeta_E(u) = \sum_{n \in \mathbb{Z}^d} \left(\frac{1}{n_1^2 + \cdots + n_d^2}\right)^u.
\]

which is valid for \( \Re(u) > d/2 \). The following reflection formula for the Epstein zeta function (3.16)

\[
\zeta_E(u) = \frac{\pi^{2u - d/2}}{\Gamma(u)} \Gamma\left(\frac{d}{2} - u\right) \zeta_E\left(\frac{d}{2} - u\right)
\]

provides the analytic continuation of \( \zeta_E(s) \) to the entire complex plane with a single simple pole at the point \( u = d/2 \) \([13, 14]\). When \( s \neq k, k \in \mathbb{Z} \), and \( s = (2n + 1)/2 \), with the latter holding only for the case of odd \( d \), the integrand in (3.15) possesses simple poles at the points \( t = -s - n \), \( t = -n \), and \( t = d/2 - s \) where \( n \in \mathbb{N}_0 \). Under the above assumptions, by closing the integration contour to the left we obtain from the first set of poles

\[
g_1(s, \beta) = \frac{\beta^{2s}}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(-s - n) \beta^{2n} \zeta_E(-n).
\]

Since one can prove from (3.17) that \( \zeta_E(-p) = 0 \) for \( p \in \mathbb{N} \) and \( \zeta_E(0) = -1 \), we can write (3.18) as

\[
g_1(s, \beta) = -\frac{\beta^{2s}}{2} \Gamma(-s).
\]

From the poles at \( t = -n \) we obtain the contribution

\[
g_2(s, \beta) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(s - n) \beta^{2n} \zeta_E(s - n),
\]

\[
\text{J. Phys. A: Math. Theor. 48 (2015) 435203}
\]
and from the ones at \( t = d/2 - s \) we have

\[
g_s(s, \beta) = \frac{\beta^{2s-d/2}}{2} \Gamma\left(\frac{d}{2} - s\right),
\]

which can be proved by noticing that

\[
\text{Res} \ \zeta_E \left( \frac{d}{2} \right) = \frac{\pi^d}{\Gamma\left( \frac{d}{2} \right)}.
\]

Adding the above results allows us to obtain

\[
g(s, \beta) \sim \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Gamma(s - n) \beta^{2n} \zeta_E(s - n) - \frac{\beta^{2s}}{2} \Gamma(-s) + \frac{\beta^{2s-d/2}}{2} \Gamma\left(\frac{d}{2} - s\right),
\]

To compute the small-\( \beta \) expansion of \( g(s, \beta) \) when \( s \) is an integer or a positive half-integer, it is convenient to distinguish between even and odd values of \( d \).

We consider, first, the case of even \( d \), namely \( d = 2l \) with \( l \in \mathbb{N}^+ \). For \( s = n, \, n \in \mathbb{N}_0 \), and the integrand in (3.15) has simple poles for \( t = -j \), \( j = 0, \ldots, n - 1 \) with \( j \neq n - l \), and double poles at \( t = -j \), with \( j \geq n \). In addition, \( t = l - n \) is a simple pole when \( l \geq n + 1 \), while it becomes a double pole when \( l \in \{1, \ldots, n\} \). In this case the integral (3.15) gives

\[
g_{d=2l}(n, \beta) \sim \frac{1}{2} \sum_{j=0}^{n-1} \frac{(-1)^j}{j!} \Gamma(n - j) \beta^{2j} \zeta_E(n - j) + \frac{(-1)^n}{2} \sum_{j=n, j \neq n-l}^{\infty} \frac{\beta^{2j}}{j!} \zeta_E(n - j) + \Theta(n - l) \left[ \frac{\pi^{-l}(l - 1)! \text{FP} \zeta_E(l) + \Psi(n - l + 1) + \Psi(l) - 2 \ln \beta}{2(n - l)!} \right].
\]

For \( s = -n \), with \( n \in \mathbb{N}^+ \), the integrand in (3.15) has simple poles at \( t = j \), with \( j = \{1, \ldots, n\} \) and at \( t = l + n \). Double poles, instead, are present at the points \( t = -j \), \( j \in \mathbb{N}_0 \). The integral (3.15) then becomes

\[
g_{d=2l}(-n, \beta) \sim -\frac{(n - 1)!}{2 \beta^{2n}} + \frac{\pi^l}{2 \beta^{2l+2n}} \Gamma(n + l) + \frac{(-1)^n}{2} \sum_{j=0}^{\infty} \frac{\beta^{2j}}{j!} \zeta_E(-j - n).
\]

We now analyze the case \( d = 2l + 1 \), with \( l \in \mathbb{N}_0 \). For \( s = n, \, n \in \mathbb{N}_0 \), the integrand in (3.15) has simple poles at \( t = -j \), with \( j = \{0, \ldots, n - 1\} \), and at \( t = l - n + 1/2 \). Double poles appear at the points \( t = -j \), with \( j \geq n \). In this case we obtain
When \( s = -n \), with \( n \in \mathbb{N}^+ \), the integrand in (3.15) develops simple poles at \( t = j \), with \( j = \{1, \ldots, n\} \), and at \( t = l + n + 1/2 \), and double poles at \( t = -j \), where \( j \in \mathbb{N}_0 \). The integral (3.15) can then be computed to give

\[
g_{d=2l+1}(-n, \beta) \sim -\frac{(n - 1)!}{2\beta^{2n}} + \frac{\pi^{l+1/2}}{2\beta^{2l+2n+1}} \Gamma\left(n + 1 + \frac{1}{2}\right) \\
+ \frac{(-1)^n \beta^{2j}}{2} \sum_{j=0}^{\infty} \frac{j!}{(n + j)!} \zeta_E\left(-j - n\right).
\]  

(3.27)

For \( s = (2n + 1)/2 \) with \( n \in \mathbb{N}_0 \), the integrand in (3.15) has simple poles at \( t = -j \), \( j \in \mathbb{N}_0 \) with \( j > n - l \), and at \( t = -m - n - 1/2 \), for \( m \in \mathbb{N}_0 \). Moreover, \( t = -n + l \) is a simple pole for \( l \geq n + 1 \), while it is a double pole for \( l = \{0, \ldots, n\} \). The integral in (3.15) then gives

\[
g_{d=2l+1}\left(\frac{2n + 1}{2}, \beta\right) \\
\sim \frac{1}{2} \sum_{j=0}^{n - l - 1} \frac{(-1)^j}{j!} \Gamma\left(n - j + 1 + \frac{1}{2}\right) \beta^{2j} \zeta_E\left(n - j + 1 + \frac{1}{2}\right) - \frac{\beta^{2n+1}}{2} \Gamma\left(-n - 1\right) \\
+ \frac{\Theta(n - l)}{2(n - l)!} \Gamma\left(l + 1 + \frac{1}{2}\right) \frac{\pi^{-l-1/2} \zeta_E\left(l + 1 + \frac{1}{2}\right)}{FP} \\
+ \Psi(n - l + 1) + \Psi\left(l + 1 + \frac{1}{2}\right) - 2 \ln \beta \\
+ \Theta(l - n - 1) \frac{\beta^{2n-2l+1}}{2} \Gamma(l - n).
\]  

(3.28)

4. Specific examples

The general results for the small-\( \beta \) expansion of Bessel series of the form (1.2)-(1.4) found in the previous sections will now be used to study cases that are of particular interest in applications. The double Bessel series (1.2), for instance, appears in the expression of the spectral zeta function associated with the Laplace operator \( L \) defined on quite general product manifolds. In fact, let \( M \) be a \( D \)-dimensional product manifold of the type \( M = I \times U \), where \( I = [a, b] \subset \mathbb{R} \) and \( U = \mathbb{R}^d \times N \) with \( N \) being a smooth Riemannian manifold with or without boundary for which we have \( \dim N = D - d - 1 \). In the appropriate coordinate system the eigenvalue equation \( L \varphi = \lambda^2 \varphi \) can be separated and one can prove that the spectrum has the general form

\[
\lambda^2 = \alpha^2 + \gamma^2 + \sum_{i=1}^{d} k_i^2,
\]  

(4.1)
where $\alpha^2$ are the eigenvalues of the Laplacian on $N$, $\gamma^2$ are the ones associated with the Laplacian on $I$ and $k_i^2$ represents the continuous spectrum resulting from the Laplacian on $\mathbb{R}^d$.

In this work we will restrict our attention to the class of boundary conditions at the endpoints of $I$ that produce eigenvalues of the form

$$\gamma^2 = \frac{\beta^2}{\beta^2}(m + B)^2,$$  \hspace{1cm} (4.2)

with $\beta > 0$ and $B \equiv -m$. Depending on the specific boundary conditions chosen we can have either $m \in \mathbb{Z}$ or $m \in \mathbb{N}_0$. We will continue our discussion under the assumption that $m \in \mathbb{Z}$. This is not restrictive since the spectral zeta function associated with a problem for which $m \in \mathbb{N}_0$ can be obtained from the one with $m \in \mathbb{Z}$. The eigenvalues (4.1) with the definition (4.2) define the following spectral zeta function density

$$\zeta(s) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \sum_{m \in \mathbb{Z}} \left( \alpha^2 + \beta^2 \gamma^2(m + B)^2 + \sum_{i=1}^{d} k_i^2 \right)^{-s} \, dk.$$  \hspace{1cm} (4.3)

By using the integral [17]

$$\int_{0}^{\infty} \left[ 1 + \sum_{i=1}^{d} (rk_i)^2 \right]^{-s} \, dk = 2^{-d-\frac{d}{2}} \frac{\Gamma(s - \frac{d}{2})}{\Gamma(s)} r^{-d},$$  \hspace{1cm} (4.4)

with $r > 0$, we can perform the integral over the variables $k$ in (4.3) and obtain

$$\zeta(s) = \frac{\Gamma(s - \frac{d}{2})}{(2\pi)^d \Gamma(s)} \sum_{m \in \mathbb{Z}} \left( \alpha^2 + \beta^2 \gamma^2(m + B)^2 \right)^{-s + \frac{d}{2}},$$  \hspace{1cm} (4.5)

which is valid for $\Re(s) > D/2$. To perform the analytic continuation of (4.5) to values of $s$ to the left of the abscissa of convergence $\Re(s) = 1/2$, we utilize a method based on the Poisson summation formula. For the argument of the double-series in (4.5) we use the integral representation of the Gamma function to obtain

$$\zeta(s) = \frac{1}{(2\pi)^d \Gamma(s)} \sum_{m \in \mathbb{Z}} \int_{0}^{\infty} t^{-\frac{d}{2}-1} \exp \left\{ -t \left( \alpha^2 + \beta^2 \gamma^2(m + B)^2 \right) \right\} \, dt.$$  \hspace{1cm} (4.6)

For $\Re(s) > D/2$, we first interchange the sums and the integral and then perform the sum over $m$ by using the Poisson resummation formula [20]

$$\sum_{m \in \mathbb{Z}} e^{-it^2 \gamma^2(m + B)^2} = \frac{\beta}{\sqrt{\pi t}} \left[ 1 + 2 \sum_{m=1}^{\infty} \cos(2\pi mB) e^{-\frac{\beta^2 m^2}{t}} \right],$$  \hspace{1cm} (4.7)

to finally arrive at the expression

$$\zeta(s) = \frac{\beta}{2^{d-1} \pi^{d/2} \Gamma(s)} \int_{0}^{\infty} t^{-\frac{d}{2}-1} \sum_{\alpha} e^{-\alpha^2 t} \, dt$$

$$+ \frac{2\beta}{2^{d-1} \pi^{d/2} \Gamma(s)} \sum_{m=1}^{\infty} \cos(2\pi mB) \int_{0}^{\infty} t^{-\frac{d}{2}-1} e^{-\frac{\beta^2 m^2}{t}} \, dt.$$  \hspace{1cm} (4.8)

The first term in (4.8) is the Mellin transform of the trace of the heat kernel associated with the Laplacian on $N$ and, hence, is proportional to the spectral zeta function.
while the second term can be written in terms of modified Bessel functions of the second kind by virtue of the following integral representation [17]

$$K_{s}(z) = \frac{\alpha}{2} \int_{0}^{\infty} e^{-t^2} \left(1 - \frac{t^2}{z^2}ight) dt.$$  

(4.10)

The last remarks allow us to write the expression

$$\zeta(s) = \frac{\beta}{2d} \frac{\Gamma\left(s - \frac{d+1}{2}\right)}{\Gamma(s)} \zeta_{N}(s - \frac{d+1}{2}) + \frac{\beta}{2d^2} \frac{\cos(2\pi m b)}{\pi^2} \sum_{\alpha} \sum_{n=1}^{\infty} \left( \frac{m\beta}{\alpha} \right)^{s - \frac{d+1}{2}} K_{s - \frac{d+1}{2}}(2\alpha \beta m),$$  

(4.11)

which represents the analytic continuation of \(\zeta(s)\) to a meromorphic function in the entire complex plane. Due to the exponential decay of the modified Bessel function of the second kind, the meromorphic structure of \(\zeta(s)\) is entirely encoded in the first term of (4.11). Spectral zeta functions of the form (4.11) appear frequently in the literature, especially in the ambit of Kaluza–Klein theories, Casimir energy, and in the study of finite temperature effects in quantum field theory [4, 10, 11, 23]. By utilizing the result obtained in (2.28) we find the following expression for the small-\(\beta\) expansion of the spectral zeta function in (4.11)

$$\zeta(s) \approx \frac{\beta^{2s-d}}{2d^2} \frac{\Gamma(s)}{\Gamma(s)} \sum_{l=-d}^{\infty} \zeta^{s + \frac{d+1}{2}} \frac{\alpha_{l}}{\alpha_{l+1}} \times \left[ \text{Li}_{s-l+1} \left( e^{2\pi i \beta n} \right) \right],$$  

(4.12)

valid for all values of \(s\). The expression (4.12) can be used to study the small-\(\beta\) behavior of quantities of interest. For instance, the derivative at \(s = 0\) of (4.12) with \(\mu = iB\) would provide the high temperature expansion of the finite temperature one-loop effective action associated with a quantum field propagating on \(U\) with chemical potential \(\mu\) (see e.g. [11]).

As a further example we consider the following spectral zeta function

$$\zeta(s, \beta) = -\frac{1}{2d^2} \frac{\Gamma\left(s - \frac{d-1}{2}\right)}{\Gamma(s)} \sum_{n=0}^{\alpha} \sum_{n=0}^{\alpha+1} \left[ \alpha^2 + \frac{\pi^2}{\beta^2} \left( n + \frac{1}{2} \right)^2 \right]^{-s + \frac{d-1}{2}},$$  

(4.13)

which appears when studying the Casimir energy of fermions in the setting of a higher dimensional piston geometry of the type \(M^D \times N\) with \(M^D\) a \(D\)-dimensional Euclidean space and \(N\) a compact Riemannian manifold [26]. Here, \(\beta\) represents the length of the first chamber of the piston and \(\alpha\) the eigenvalues of the Laplacian on \(N\). We will also assume that \(\dim N = Q\). By noticing that

$$\sum_{n=0}^{\alpha} \sum_{n=0}^{\alpha+1} \left[ \alpha^2 + \frac{\pi^2}{\beta^2} \left( n + \frac{1}{2} \right)^2 \right]^{-s + \frac{d-1}{2}} = \frac{1}{2} \sum_{n=0}^{\alpha} \sum_{n=0}^{\alpha+1} \left[ \alpha^2 + \frac{\pi^2}{\beta^2} \left( n + \frac{1}{2} \right)^2 \right]^{-s + \frac{d-1}{2}},$$  

(4.14)

we have

$$\zeta(s, \beta) = -2^{D-3} \zeta(s),$$  

(4.15)
with \( \zeta(s) \) given in (4.5) with the substitution \( d = D - 1 \) and \( B = 1/2 \). The last relation implies that we can use the method outlined in this section to perform the analytic continuation of (4.13) to obtain

\[
\zeta(s, \beta) = \frac{\beta}{4\pi^2} \frac{\Gamma\left(s - \frac{D}{2}\right)}{\Gamma(s)} - \frac{\beta}{\pi^2 \Gamma(s)} \sum_{n=1}^{\infty} (1 - (-1)^n) \left( \frac{n \beta}{\alpha} \right)^{-\frac{D}{2}} K_{s - \frac{D}{2} + \frac{1}{2}} (2\alpha/n). \tag{4.16}
\]

By using the result (4.12) with \( B = 1/2 \) and the fact that

\[
\text{Li}_s(e^\pi) + \text{Li}_s(e^{-\pi}) = 2 \left( 2^{1-s} - 1 \right) \zeta_R(s),
\]

one finds the following expansion of \( \zeta(s, \beta) \) valid for \( s \in \mathbb{C} \) when the size \( \beta \) of the first chamber of the piston is small

\[
\zeta(s, \beta) \sim -\frac{\beta^{2s-D+1}}{2\pi^2 \Gamma(s)} \sum_{l=0}^{\infty} \Gamma\left(-s + \frac{D-l+1}{2}\right) \beta^{l-1} \frac{A_{N_l}}{\pi} \left( 2^{2s-D-l+1} - 1 \right) \zeta_R(-2s + D - l). \tag{4.18}
\]

By setting \( s = \varepsilon - 1/2 \) in (4.18) we obtain an expression for the small-\( \beta \) expansion of the Casimir energy associated with the first chamber of the piston

\[
E_C\left(\varepsilon - \frac{1}{2}, \beta\right) \sim \frac{\beta}{16\pi^{D+1} \varepsilon} A_{N_l}^{\frac{1}{2}} + \frac{1}{8\pi^2} \sum_{l=0}^{D+1} \Gamma\left(\frac{D-l+1}{2}\right) \beta^{l-1} \frac{A_{N_l}}{\pi} \left( 2^{2l-D} - 1 \right) \zeta_R(D - l + 1), \tag{4.19}
\]

which, as \( \varepsilon \to 0 \), is divergent as expected. By adding the Casimir energy contribution coming from the second chamber, namely the expression (4.19) with the replacement \( \beta \to L - \beta \), \( L \) being the total length of the piston, and by then differentiating the resulting expression with respect to \( \beta \) [7] we obtain the following Casimir force on the piston

\[
F_C(\beta) \sim \frac{1}{8\pi^2} \sum_{l=0}^{D+1} \Gamma\left(\frac{D-l+1}{2}\right) \beta^{l-1} \frac{A_{N_l}}{\pi} \left( (L - \beta)^{D-l-1} - \beta^{D-l-1} \right) \left( 2^{2l-D} - 1 \right) \zeta_R(D - l + 1), \tag{4.20}
\]

which is the correct expansion valid when the size \( \beta \) of the first chamber is small.

We would like to consider, as an additional example, the following Bessel series

\[
S(m) = \sum_{n=1}^{\infty} \left( \frac{m}{nL} \right)^{D-1} K_{2-1}(nLm), \tag{4.21}
\]

which appears in the expression for the one-loop renormalized mass in the Euclidean \( \lambda \phi^4 \) model compactified along one direction [1, 24]. Here, \( L \) denotes the size of the one-dimensional compactified subspace of a \( D \)-dimensional Euclidean space, \( m \) represents the physical mass of the field. In the study of the critical behavior of this theory one is confronted with the task of analyzing the small-\( m \) expansion of the series (4.21) [1, 24]. By setting \( m = 2\beta/L \) we can rewrite (4.21) in terms of the sum (3.7) as follows
and, hence, we can use the small-$\beta$ expansion of $h(s, \beta, 0)$ found in (3.11). When the dimension of the Euclidean space is even, $D = 2d, d \geq 1$, we have, from (3.13) the expansion

$$h(1 - d, \beta, 0) \sim \frac{1}{2} \frac{(-1)^{d-1}}{(d - 1)!} \left( \gamma + \ln \beta - 2H_{d-1} \right) + (-1)^{d-1} \frac{1}{2} \sum_{j=1}^{\infty} \beta^{2j} j! (j + d - 1)! \zeta'_R(-2j) + \frac{\sqrt{\pi}}{4\beta} \Gamma\left(\frac{3}{2} - d\right) \sum_{j=1}^{\infty} \frac{(-1)^{d-j-1}(j - 1)!}{(d - j - 1)!} \beta^{-2j} \zeta_R(2j). \tag{4.23}$$

By applying this expansion to (4.22) and by substituting back the variable $m$ we obtain for (4.21) the expansion

$$S(m) \sim \frac{m^{2d-2}(-1)^{d-1}}{2^d(d - 1)!} \left( \gamma + \ln \left(\frac{mL}{2}\right) - 2H_{d-1} \right) + \frac{(-1)^{d-1}m^{2d-2}}{2^d} \sum_{j=0}^{\infty} \frac{(mL)^{2j}}{4^j j!} \zeta'_R(-2j) + \frac{\sqrt{\pi}m^{2d-3}}{2^dL} \Gamma\left(\frac{3}{2} - d\right) + \frac{(-1)^{d-1}m^{2d-2}}{2^d} \sum_{j=1}^{d-1} \frac{(-1)^{d-j-1}(j - 1)!}{(d - j - 1)!} \zeta_R(2j) \tag{4.24}$$

valid when $D = 2d$.

Let us next consider the case of odd-dimensional Euclidean space namely $D = 2d + 1$ with $d \geq 1$. In this case we use (3.14) to obtain

$$h\left(\frac{1}{2} - d, \beta, 0\right) \sim \frac{\beta^{-2d+1}}{2} \sum_{m=d-1}^{\infty} \frac{(-1)^m}{m!} \Gamma\left(-m + d - \frac{1}{2}\right) \beta^{2m} \zeta_R(-2m + 2d - 1) - \frac{1}{4} \Gamma\left(\frac{3}{2} - d\right) - \frac{\sqrt{\pi}}{4\beta} \Gamma\left(-1\right) \left(2 \ln 2\beta - H_{d-1}\right). \tag{4.25}$$

This expansion can be used in (4.22) to obtain, when substituting $\beta = mL/2$, the following small-$m$ expansion of (4.21) valid for an odd-dimensional Euclidean space

$$S(m) \sim \frac{2^{d-1}L^{2d-1}}{2^{d-1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k k!} \left(-k + d - \frac{1}{2}\right) \left(\frac{mL}{2}\right)^{2k} \zeta_R(-2k + 2d - 1) - \frac{m^{2d-1}}{2^{d-1}L} \Gamma\left(\frac{3}{2} - d\right) - \frac{\sqrt{\pi}m^{2d-2}}{2^{d-1}L} \left(2 \ln (mL) - H_{d-1}\right). \tag{4.26}$$

The small-$m$ expansions of $S(m)$ in (4.21) obtained in (4.24) and (4.26) are valid for any dimension $D$. The expansion obtained in [1], on the other hand, needs to be somehow regularized in the case $D = 3$ which makes our results more suitable for the analysis of $S(m)$ for any given dimension $D$. 

J. Phys. A: Math. Theor. 48 (2015) 435203 G Fucci and K Kirsten
5. Concluding remarks

In this work we have analyzed the asymptotic expansion of Bessel series valid when a given parameter, which we have denoted by \( \beta \), is small. Explicit and very general asymptotic expansions have been obtained for both Bessel and double-Bessel series. The method used to obtain such general expansions is based on the representation of the modified Bessel function of the second kind in terms of a contour integral. After closing the contour of integration to the left, the integral is computed by using Cauchy’s residue theorem. This method allowed us to find in a fairly straightforward way the desired small-\( \beta \) expansion of the Bessel series under consideration valid for all values of \( s \).

In this paper we have focused our attention to the small-\( \beta \) expansion of Bessel series of the form (1.2), (1.3), and (1.4). It is important to mention, however, that the method outlined in the previous sections can easily be adapted to other types of Bessel series. For instance, if in the series (1.4) we replace \( |n| \) with \( n \cdot r \), with \( r \in \mathbb{R}_0^d \), and then sum over the lattice \( n \in \mathbb{N}_0^d \), we obtain a new Bessel series whose small-\( \beta \) expansion can be found by following the method described in section 2 for \( g(s, \beta) \). In this situation, however, the relevant zeta function appearing in the final small-\( \beta \) expansion would be the Barnes zeta function instead of the Epstein zeta function that is found in (3.23).

The type of Bessel series considered in the previous sections appear frequently in the literature and their expansions with respect to a small parameter prove to be of fundamental importance for obtaining very useful information about special limiting cases. Unfortunately, however, such expansions are not always performed correctly since the validity of the methods used is often dubious. The purpose of this paper is then to serve as a guide for properly performing the small parameter expansion of infinite series (including double series) containing modified Bessel functions of the second kind.

Acknowledgments

The research of GF is partially funded by the ORAU Ralph E Powe Junior Faculty Enhancement Award.

References

[1] Abreu L M, de Calan C, Malbouisson A P C, Malbouisson J M C and Santana A E 2005 Critical behavior of the compactified theory J. Math. Phys. 46 012304
[2] Actor A 1987 More on zeta function regularization of high-temperature expansions Fortschr. Phys. 35 793
[3] Allen B 1986 Does statistical mechanics equal one-loop quantum field theory? Phys. Rev. D 33 3640
[4] Ambjørn J and Wolfram S 1983 Properties of the vacuum: I. Mechanical and thermodynamic Ann. Phys., NY 147 1
[5] Bleistein N and Handelsman R A 1986 Asymptotic Expansions of Integrals (New York: Dover)
[6] Bytsenko A A, Cognola G, Elizalde E, Moretti V and Zerbini S 2003 Analytic Aspects of Quantum Fields (Singapore: World Scientific)
[7] Bordag M, Klimchitskaya G L, Mohideen U and Mostepanenko V M 2009 Advances in the Casimir Effect (Oxford: Oxford University Press)
[8] Camporesi R 1990 Harmonic analysis and propagators on homogeneous spaces Phys. Rep. 196 1
[9] DeWitt B S 2003 The Global Approach to Quantum Field Theory (Oxford: Oxford University Press)
[10] Dowker J S 1984 Finite temperature and vacuum effects in higher dimensions Class. Quantum Grav. 1 359
[11] Elizalde E, Odintsov S D, Romeo A, Bytsenko A and Zerbini S 1994 Zeta Regularization Techniques with Applications (Singapore: World Scientific)
[12] Elizalde E 1995 Ten Physical Applications of the Spectral Zeta Function (Berlin: Springer)
[13] Epstein P 1903 Zur Theorie allgemeiner Zetafunktionen Math. Ann. 56 615
[14] Epstein P 1907 Zur Theorie allgemeiner Zetafunktionen: II Math. Ann. 63 205
[15] Erdélyi A 1953 Higher Transcendental Functions vol 1 Bateman Project Staff (New York: McGraw-Hill)
[16] Gilkey P B 1995 Invariance Theory the Heat Equation and the Atiyah–Singer Index Theorem (Boca Raton, FL: CRC Press)
[17] Gradshtein I S and Ryzhik I M 2007 Table of Integrals Series and Products ed A Jeffrey and D Zwillinger (Oxford: Academic)
[18] Haber H E and Weldon J 1982 On the relativistic Bose–Einstein integrals J. Math. Phys. 23 1852
[19] Hawking S W 1977 Zeta function regularization of path integrals in curved spacetime Commun. Math. Phys. 55 133–48
[20] Hille E 1962 Analytic Function Theory vol 2 (Boston: Ginn)
[21] Kirsten K 1992 Connections between Kelvin functions and zeta functions with applications J. Phys. A: Math. Gen. 25 6297
[22] Kirsten K 2001 Spectral Functions in Mathematics and Physics (Boca Raton, FL: CRC Press)
[23] Fulling K and Kirsten S A 2009 Kaluza–Klein models as pistons Phys. Rev. D 79 065019
[24] Malbouisson A P C, Malbouisson J M C and Santana A E 2003 Effective potential approach to phase transitions in confined systems Phys. Lett. A 318 406
[25] Minakshisundaram S and Pleijel A 1949 Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds Can. J. Math. 1 242
[26] Oikonomou V K 2014 Casimir force of fermions coupled to monopoles in six-dimensional spacetime Int. J. Geom. Methods Mod. Phys. 11 1450011
[27] Olver F W J, Lozier D W, Boisvert R F and Clark C W (ed) 2010 NIST Handbook of Mathematical Functions (New York: Cambridge University Press)
[28] Paris R B and Kaminski D 2001 Asymptotics and Mellin–Barnes Integrals Encyclopedia of Mathematics and its Applications vol 85 (Cambridge: Cambridge University Press)
[29] Ray D B and Singer I M 1971 R-torsion and the Laplacian on Riemannian manifolds Adv. Math. 7 145
[30] Seeley R T 1968 Complex powers of an elliptic operator Singular Integrals (Proc. Symp. Pure Mathematics Chicago, 1966 vol 10) (Providence, RI: American Mathematics Society) p 288
[31] Singh S and Pathria R K 1989 Analytic evaluation of a class of lattice sums in arbitrary dimensions J. Phys. A: Math. Gen. 22 1883
[32] Titchmarsh E C 1986 The Theory of the Riemann Zeta Function ed D R Heath-Brown (Oxford: Oxford University Press)
[33] Weldon H A 1986 Proof of zeta-function regularization of high-temperature expansions Nucl. Phys. B 270 79