Ancillary Gaussian modes activate the potential to witness non-Markovianity

Dario De Santis1,∗, Donato Farina1, Mohammad Mehboudi2 and Antonio Acín1,3
1 IFCO-Institut de Ciencies Fotoniques, The Barcelona Institute of Science and Technology, 08860 Castelldefels (Barcelona), Spain
2 Département de Physique Appliquée, Université de Genève, 1211 Genève, Switzerland
3 ICREA—Institució Catalana de Recerca i Estudis Avançats, 08010 Barcelona, Spain
∗ Author to whom any correspondence should be addressed.
E-mail: Dario.DeSantis@alumni.icfo.eu
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Abstract
We study how the number of employed modes impacts the ability to witness non-Markovian evolutions via correlation backflows in continuous-variable quantum dynamics. We first prove the existence of non-Markovian Gaussian evolutions that do not show any revivals in the correlations between the mode evolving through the dynamics and a single ancillary mode. We then demonstrate how this scenario radically changes when two ancillary modes are considered. Indeed, we show that the same evolutions can show correlation backflows along a specific bipartition when three-mode states are employed, and where only one mode is subjected to the evolution. These results can be interpreted as a form of activation phenomenon in non-Markovianity detection and are proven for two types of correlations, entanglement and steering, and two classes of Gaussian evolutions, a classical noise model and the quantum Brownian motion model.

1. Introduction

The interaction between any given quantum system and the surrounding environment can never be completely avoided; the theory of open quantum systems is an indispensable framework to describe the realistic dynamics of quantum systems [1, 2]. The interaction with the environment is usually detrimental for quantum resources, like quantum coherence or entanglement, and in general makes the resulting map on the system no longer unitary, but by a quantum channel, described by a completely-positive trace-preserving (CPTP) map. The evolution in time is then described by a continuous family of quantum channels, which can be classified as Markovian or non-Markovian. While the former class is characterized by the continuous degradation of any type of information encoded in the system, in the latter the decoherence process is not monotonic in time—these recoherences are often called backflows of information. Non-Markovian evolutions have attracted much interest, not only because of their fundamental interest, but also because the associated backflows can have a positive effect in various quantum information tasks, such as metrology [3], quantum key distribution [4], quantum teleportation [5], entanglement generation [6], quantum communication [7], information screening [8] and quantum thermodynamics [9–12].

The mathematical property used to define Markovianity is called completely positive divisibility (CP-divisibility); the dynamics is Markovian if and only if it is possible to describe the evolution between any two times through the action of a physical quantum channel, that is, a CPTP map (for reviews on this topic see [13–15]). Various strategies have been adopted to connect this mathematical definition to more physically motivated ones. This has lead to the development of witnesses of non-Markovianity through backflows of different quantities, such as the error probability in state discrimination [16–18], channel capacity [7], Fisher information [19, 20], the volume of accessible states [21] and correlations [22–28]. In the case of correlations, the standard method for witnessing non-Markovianity works as follows: (a) prepare an initial state of two particles, which we name system and ancilla; (b) apply the considered evolution to one of the two particles, the system, while the ancilla remains untouched and (c) monitor how the correlations
between the two particles change during the evolution. If the correlations do not decrease monotonically with time, the dynamics gives rise to a correlation backflow and therefore is non-Markovian. Beyond this recipe, in general it is not known whether and how to construct the initial two-particle state for a given non-Markovian dynamics or, even simpler, what is the minimal dimension of the ancillary system that is needed for this task. Even less is known for continuous-variable systems and, in particular, for Gaussian dynamics, despite their prominent role in many physically relevant scenarios. While for finite-dimensional, several works have studied correlation backflows in quantum evolutions [25, 26, 29], continuous-variable settings have not been explored beyond the use of a single ancillary mode [27, 28, 30].

In this work, we firstly ask ourselves whether considering a single ancillary mode is sufficient to witness quantum correlation backflows for arbitrary non-Markovian dynamics. Particularly, we use both entanglement and Gaussian steerability as correlations. Our results show that indeed using a single ancillary mode is not always sufficient to witness backflows. Secondly, and motivated by this shortcoming, we ask ourselves whether deploying a secondary ancillary system would be advantageous. We show through two examples, namely the dynamics of a single mode under (a) a classical noise model and (b) the quantum Brownian motion model, that the secondary ancillary mode allows witnessing non-Markovian evolutions that are impossible to detect with any possible single ancillary mode initialization. Finally, we show that, while for some open dynamics two ancillary modes are sufficient for witnessing non-Markovianity, for some other dynamics one may need even a higher number of ancillary modes.

The article is structured as follows. In section 2 we briefly introduce Gaussian states and the entangled initializations of interest. Such initializations are assumed to undergo local Gaussian dynamics, characterized in section 3, and their quantum correlations are quantified through Gaussian steerability and entanglement, section 4. The advantage stemming from the use of more than one ancillary mode is presented in section 5, through paradigmatic examples. Finally, we summarize our results in section 6.

2. Preliminaries

In this section we set the notation and introduce the adopted formalism to describe quantum Gaussian systems. An $n$-mode continuous variable quantum system is defined through states over the Hilbert space $\mathcal{H}^{(n)} = \otimes_{i=1}^{n} \mathcal{H}_i$, where $\mathcal{H}_i$ is the Hilbert space of a bosonic harmonic oscillator corresponding to the $i$th mode of the system. We call $S(\mathcal{H}^{(n)})$ the state space of density operators $\rho$ associated to $\mathcal{H}^{(n)}$. The quadrature operators of the $i$th mode are $\hat{q}_i = (\hat{a}_i + \hat{a}_i^\dagger)$ and $\hat{p}_i = -i(\hat{a}_i - \hat{a}_i^\dagger)$, where $\hat{a}_i$ and $\hat{a}_i^\dagger$ are, respectively, the annihilation and creation operator for the $i$th mode. By grouping these operators in the vector $\hat{X} = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2, \ldots)$, we can write the canonical commutation relations as $[\hat{X}_i, \hat{X}_j] = 2i\Omega^{(i)}(\delta_{ij})$, where

$$\Omega^{(n)} = \bigoplus_{i=1}^{n} \Omega^{(1)}, \quad \Omega^{(1)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(1)

the $2n \times 2n$ matrix $\Omega^{(n)}$ being the $n$-mode symplectic form and $\Omega^{(1)}$ the corresponding single-mode form.

A quantum state $\rho \in S(\mathcal{H}^{(n)})$ is called Gaussian when the first and second moment of the quadrature vector $\hat{X}$, namely

$$d_i = \langle \hat{X}_i \rangle_\rho \quad \text{and} \quad \sigma_{ij} = \frac{1}{2} \langle \{\hat{X}_i, \hat{X}_j\} \rangle_\rho = \langle \hat{X}_i \rangle_\rho \langle \hat{X}_j \rangle_\rho,$$

(2)

are sufficient to fully describe $\rho$, where $\langle \hat{O} \rangle_\rho = \text{Tr}[\rho \hat{O}]$ is the expectation value of the operator $\hat{O}$ on the state $\rho$. The $2n \times 2n$ real symmetric matrix $\sigma$ is called the covariance matrix of the system. Two Gaussian states with different first moments and same covariance matrix can be mapped one into the other by a displacement unitary transformation. We underline that, since in the following we are interested only on the information contained in the covariance matrix, we ignore $d_i$.

In case of a bipartite scenario, where Alice owns the first $n_A$ modes and Bob owns the last $n_B$, the covariance matrix $\sigma_{AB}$ of a shared Gaussian state can be written as follows:

$$\sigma_{AB} = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix},$$

(3)

where the $2n_A \times 2n_A$ matrix $A$ ($2n_B \times 2n_B$ matrix $B$) is the covariance matrix of Alice’s (Bob’s) system and the correlation matrix $C$ is $2n_A \times 2n_B$. In order for $\sigma_{AB}$ to correspond to a physical quantum state, namely to satisfy the uncertainty principle, the following condition has to be satisfied:

$$\sigma_{AB} + i\Omega^{(n)} \succeq 0,$$

(4)

where $n = n_A + n_B$ and the inequality means that the matrix in the l.h.s. is positive semi-definite.
2.1. Two-mode entangled states

In the following we consider two main classes of Gaussian states: the two-mode squeezed states [31] and the three-mode Greenberger–Horne–Zeilinger/W. Dür (GHZ/W) states [32], the corresponding covariance matrices being indicated, respectively, as \( \sigma_{2,r} \) and \( \sigma_{3,r} \). First, in order to define \( \sigma_{2,r} \), consider a two-mode scenario, where Alice and Bob own each a single mode \( (n_A = n_B = 1) \). The covariance matrix \( \sigma_{2,r} \) corresponding to a two-mode squeezed state is given by equation (3), with \( A = B = \cosh(2r)I \) and \( C = \sinh(2r)Z \), where I is the identity matrix and \( Z = \text{diag}(1, -1) \), namely:

\[
\sigma_{2,r} = \begin{pmatrix}
\cosh(2r) & \sinh(2r)
\sinh(2r) & \cosh(2r)
\end{pmatrix},
\tag{5}
\]

As the squeezing parameter \( r \geq 0 \) increases, the two modes become more and more correlated (entangled) [31], where \( r = 0 \) corresponds to a separable state, namely the two-mode vacuum state. The maximally entangled Einstein–Podolsky–Rosen (EPR) state corresponds to \( \lim_{r \to \infty} \sigma_{2,r} \). It must be noticed that \( \sigma_{2,\infty} \) corresponds to an infinite energy state [31] and therefore it cannot be realized experimentally.

2.2. Three-mode entangled states

The GHZ/W state is a three-mode Gaussian state which is entangled among each mode. It is realized by three squeezed beams mixed in a tritter [32]. In case of equal squeezings, the corresponding covariance matrix is given by

\[
\sigma_{3,r} = \begin{pmatrix}
\sigma(r) & \epsilon(r) & \epsilon(r)
\epsilon(r) & \sigma(r) & \epsilon(r)
\epsilon(r) & \epsilon(r) & \sigma(r)
\end{pmatrix},
\tag{6}
\]

where \( \sigma(r) = \text{diag}(\sqrt{r^2 + 2e^{-2r}}, (e^{-2r} + 2e^{2r})/3) \) and \( \epsilon(r) = (2/3)\sinh(2r)Z \). The parameter \( r \geq 0 \) is the global squeezing parameter of the state, where \( r = 0 \) corresponds to the separable case and \( r \to \infty \) provides maximal entanglement.

Consider the bipartite scenario where Alice owns the first two modes of \( \sigma_{3,r} \), while Bob owns the last mode. The covariance matrix \( \sigma_{3,r} \) can be divided into blocks as in equation (3), where

\[
A = \begin{pmatrix}
\sigma(r) & \epsilon(r) \\
\epsilon(r) & \sigma(r)
\end{pmatrix}, \quad B = \sigma(r) \quad \text{and} \quad C = \begin{pmatrix}
\epsilon(r) \\
\epsilon(r)
\end{pmatrix}.
\]

3. Gaussian channels and evolutions

Gaussian channels are those that preserve Gaussianity of quantum states, that is, they map Gaussian states into Gaussian states. They can be fully characterized by their application on the displacement vector and the covariance matrix. Nonetheless, since we are interested only in the information contained in the covariance matrix of Gaussian states, we represent the action of a generic Gaussian transformation as [31]

\[
\Lambda : \sigma \mapsto \sigma' = T\sigma T^\dagger + N,
\tag{7}
\]

where \( T \) and \( N \) are \( 2n \times 2n \) matrices of reals. Moreover, \( N \) must be symmetric to preserve the symmetry of the covariance matrices. Thus, any Gaussian channel can be represented by the pair \( \Lambda \equiv (T, N) \), such that \( \sigma' = (N, T)\sigma \). It is clear from equation (7) that the identity map corresponds to \((T, N) = (I, 0)\), where, again, \( I \) is the identity matrix and \( 0 \) is the null matrix.

The channel \((T, N)\) CPTP if and only if [33]

\[
N - iT\Omega T^\dagger + i\Omega \succeq 0.
\tag{8}
\]

For single-mode channels, condition (8) reduces to the following two conditions

\[
N = N^\top \succeq 0,
\tag{9}
\]

\[
\det N \geq (\det T - 1)^2.
\tag{10}
\]

A Gaussian dynamical evolution can be denoted by the time-parametrized family \( \{\Lambda_t\}_{t \geq 0} = \{T_t, N_t\}_{t \geq 0} \), where the channel \((T_t, N_t)\) that represents the evolution at time \( t \) is called dynamical map. One expects that,
4. Non-Markovianity witnesses

Given a functional $I : S(\mathcal{H}) \rightarrow \mathbb{R}^+$ that maps quantum states into non-negative real numbers, we call it an information quantifier if it is non-increasing under CPTP maps, namely if $I(\rho) \geq I(\Lambda(\rho))$ for all CPTP maps $\Lambda : S(\mathcal{H}) \rightarrow S(\mathcal{H})$ and states $\rho \in S(\mathcal{H})$. The minimum value $I = 0$ is interpreted as the absence of the considered information in the state. It follows that all evolutions $\{\Lambda_t\}_{t \geq 0}$ cannot increase the amount of information contained in the initial state $\rho(0)$, namely $I(\hat{\rho}(0)) \geq I(\hat{\rho}(t))$ for all $t \geq 0$ and $\rho(0) \in S(\mathcal{H})$, where $\hat{\rho}(t) := \Lambda_t(\rho(0))$. Nonetheless, it could be the case that an intermediate map between two states $s < t$ is not CPTP and that we obtain the increase $I(\hat{\rho}(s)) < I(\hat{\rho}(t))$. Hence, since Markovian evolutions are characterized by having CPTP intermediate maps, any increase, or backflow, of $I$ witnesses non-Markovianity.

Notice that, in general, it does not suffice to consider the evolution of a single initial state $\rho(0)$ in order to state that an evolution is Markovian due to the monotonicity of $I(\hat{\rho}(t))$. Indeed, even if a state $\hat{\rho}(0)$ does not allow observing backflows of $I$, there may be a different state $\hat{\rho}'(0)$ for which an increase of $I(\hat{\rho}'(t))$ can be observed. The same is true for $I$: some quantifiers are not able to witness certain types of non-Markovian evolutions.

In this context, a key ingredient is the use of ancillary systems. Indeed, we can use initial states $\hat{\rho}(0) \in S(\mathcal{H} \otimes \mathcal{H}')$ such that $\hat{\rho}(t) = \Lambda_t \otimes I'(\hat{\rho}(0))$, where the non-evolving ancillary system is defined over the Hilbert space $\mathcal{H}'$ and $I'$ is the identity map on $S(\mathcal{H}')$. In general, initializations that make use of ancillas allow witnessing non-Markovianity with higher precision. Indeed, for some evolutions and information quantifiers, we can obtain backflows if and only if particular system-ancilla initializations are considered [17, 29, 34]. Moreover, the dimension of the ancilla is also important: depending on the case, a minimal ancillary size could be required to obtain backflows.

In the following, we exploit two information quantifiers as non-Markovianity witnesses: Gaussian steerability and entanglement. Our results reveal that there exist some non-Markovian Gaussian evolutions that (a) cannot be witnessed by means of the aforementioned correlations with any two-mode Gaussian initializaiton, but (b) can be witnessed by using three-mode initial Gaussian states, where in (a) and (b) we respectively consider one and two ancillary modes. Thus, we highlight the crucial role that ancillary modes can play.

More in details, we consider the scenario where Alice and Bob share a Gaussian correlated system $A_1A_2B$, where $A_1$ is Alice’s evolving system, $A_2$ is Alice’s ancillary system (in case there is one) and $B$ is Bob’s ancillary system. Hence, we compare the potentials of the settings $A_1B$ and $A_1A_2B$ to provide correlation backflows, where $A_1$, $A_2$ and $B$ are one-mode systems. As described before, the aim of this work is to describe the advantages of using the three-mode setup $A_1A_2B$.

Finally, we propose the following analogy between finite and infinite dimensional evolutions to discuss the minimal ancillary sizes of $A_2$ and $B$ needed to observe correlation backflows when $A_1$ is a generic $n$-mode Gaussian evolving system. There exists a hierarchy for the degree of non-Markovianity of $d$-dimensional evolutions called $k$-divisibility [35], which is based on the minimal ancillary dimension needed to obtain information backflows [34]. The backflows considered here correspond to increases in distinguishability of...
two states given with a-priori probabilities \( p \) and \( 1 - p \), which are defined over the evolving system and an ancilla. If an invertible evolution is \( k \)-divisible but not \( k + 1 \)-divisible, we can obtain backflows if and only if \( k + 1 \)-dimensional (or larger) ancillas are considered. In this framework, Markovianity corresponds to \( d \)-divisibility. Hence, \( d \) is the largest ancillary dimension needed to witness non-Markovianity, which is required for \( d - 1 \)-divisible evolutions. In terms of correlation backflows, by considering the setting \( A_1A_2|B \) explained above, \( d + 1 \) and 2 are, respectively, the minimum dimensions of \( A_2 \) and \( B \) that have been proven to be sufficient to witness any invertible non-Markovian evolution [24].

Similarly, in order to observe correlation backflows from \( n \)-mode non-Markovian Gaussian evolutions, we may expect to need \( n + 1 \)-mode \( A_2 \) ancillas. Nonetheless, Gaussian non-Markovianity follows a simpler hierarchy [36]: intermediate maps are either CP, positive or non-positive, where Markovianity corresponds to 1-divisibility. Therefore, we expect that, given a generic invertible non-Markovian Gaussian evolution, a minimal requirement for a bipartite system \( A_1A_2|B \) to provide correlation backflows is that \( A_2 \) and \( B \) are respectively (at most) two-mode and one-mode Gaussian systems, no matter the number of evolving modes of \( A_1 \).

### 4.1. Gaussian steerability

Gaussian steerability is a form of quantum correlations and, similarly to other correlation measures, is non-increasing under the action of CPTP local maps. For instance, it means that if one is interested in a single mode dynamical channel \( (T_t, N_t) \), one can construct a local evolution as in (12) and Gaussian steerability can be considered as an information quantifier and used to witness non-Markovianity through backflows, as suggested in [28]. There, the authors show that \( \sigma_{AB}(0) = \sigma_2 \), can be deployed as initial state to witness non-Markovianity for the quantum Brownian motion model—see the description in section 5.2. In what follows, we first present the mathematical description of Gaussian steerability. Then, we provide examples of dynamics where this correlation cannot witness non-Markovianity when using any two-mode Gaussian system. We then proceed by showing that using three-modes one can witness non-Markovianity in many cases where two modes fail. We furthermore show that even with three entangled modes it can happen that some non-Markovian evolutions cannot be witnessed using Gaussian steerability.

Consider a bipartite scenario where Alice and Bob share an \( n_A + n_B \)-mode Gaussian state with covariance matrix \( \sigma_{AB} \), where Alice holds the first \( n_A \) modes and Bob holds the last \( n_B \) modes—see equation (3). A quantifier for the potential of Alice to steer Bob’s share through Gaussian measurements has been introduced in [37]. It turns out that \( \sigma_{AB} \) is Gaussian steerable from Alice to Bob, or \( A \rightarrow B \) steerable with Gaussian measurements, if and only if the following condition is violated [38]:

\[
\sigma_{AB} + i(0_A \oplus \Omega_B) \not\succeq 0,
\]

where \( 0_A \) is the \( 2n_A \times 2n_A \) null matrix and \( \Omega_B = \Omega^{(n_B)} \). This condition is equivalent to the Schur complement of \( B \)

\[
M_B^G = B - C^T A^{-1} C,
\]

not being a physical covariance matrix; that is a violation of the following inequality

\[
M_B^G + i\Omega_B \succeq 0.
\]

Therefore, \( A \rightarrow B \) Gaussian steerability can be verified by studying the set \( \{ \nu_i \}_{i=1}^{n_B} \) of symplectic eigenvalues of \( M_B^G \). Recall that these are associated to the absolute value of the eigenvalues \( \{ \pm \nu_i \}_{i=1}^{n_B} \) of the matrix \( i\Omega_B M_B^G \) [39]. It can be shown that equation (15) is violated if and only if \( \nu_i < 1 \) for one or more \( i \) [31]. Hence, following [37], one can quantify \( A \rightarrow B \) Gaussian steerability as:

\[
\mathcal{G}_{A \rightarrow B}(\sigma_{AB}) = \max \left\{ 0, -\sum_{\nu_i < 1} \log \nu_i \right\}.
\]

In case we want to evaluate \( B \rightarrow A \) Gaussian steerability, we replace \( M_B^G \) with the Schur complement of \( A \), namely \( M_A^G = A - C B^{-1} C^T \), and evaluate its symplectic eigenvalues. Notice that in general steering is not symmetric, i.e. \( \mathcal{G}_{A \rightarrow B}(\sigma_{AB}) \neq \mathcal{G}_{B \rightarrow A}(\sigma_{AB}) \).

#### 4.1.1. Measurement incompatibility and steering

A necessary condition for Gaussian steerability is given by Gaussian measurement incompatibility. Imagine an \( A \rightarrow B \) Gaussian steering scenario, where Alice owns \( n_A \) Gaussian modes which are transformed by the Gaussian channel \( (T, N) \). In case the action of the (dual) channel \( (T, N) \) makes the set of Alice’s Gaussian
measurements compatible, no \( A \to B \) Gaussian steering can be performed—that is
\[
G_{A\to B}((T \otimes I_B, N \otimes 0_B)\sigma_{AB}) = 0 \text{ for all initializations } \sigma_{AB}. \]
In turn, a Gaussian channel breaks incompatibility of all Gaussian measurements if and only if \([40–42]\)
\[
N - iTYT^\dagger \geq 0. \tag{17}
\]

We call a channel \((T, N)\) Gaussian incompatibility breaking (GIB) in case it satisfies equation (17). This immediately leads to the following observation:

**Observation 1.**—Consider a dynamics that breaks incompatibility of all Gaussian measurements on Alice’s side within some time interval \( t \in (t_1, t_2) \). Any non-Markovian behavior, namely the violation of the CP-divisibility condition, of the dynamics within this interval cannot be witnessed by Gaussian steerability from Alice to Bob. Indeed, steering is equal to zero in the time interval \((t_1, t_2)\).

On the other hand, Alice can always extend her system to include one or more new Gaussian modes which do not undergo the Gaussian channel. Such an extension leads to our second observation:

**Observation 2.**—If one or more of Alice’s modes do not undergo the Gaussian dynamics, namely if Alice extends her modes by at least one such that her total share undergoes the Gaussian dynamics \((T_t \otimes I, N_t \otimes 0)\), the criterion (17) is always violated, i.e. the dynamics on Alice is never GIB.

Note that Observation 2 does not imply that Gaussian steerability from Alice’s extended system to Bob can witness non-Markovianity. Indeed, on the one hand measurement incompatibility is a necessary but not sufficient condition for Gaussian steerability. On the other hand, provided that one has non-zero steering, it is not guaranteed that steering backflows are always observed when non-Markovianity is at play. Nonetheless, we can increase the chance to witness a bigger class of non-Markovian dynamics by simply extending the number of modes. We showcase this through some examples in sections 5.1.1 and 5.2.1.

### 4.2. Entanglement

A necessary condition for the separability of a bipartite \( n_A + n_B \)-mode Gaussian state \( \tilde{\rho}_{AB} \) with covariance matrix \( \sigma_{AB} \), is given by the positivity of the partial transposition of the density matrix, namely the positive partial transpose (PPT) condition \([43, 44]\), which states that separable states satisfy the condition:

\[
\sigma_{AB} + i \Omega_A \otimes \Omega_B^\dagger \geq 0, \tag{18}
\]
where \( \Omega_A = \Omega^{(n_A)} \) and \( \Omega_B = \Omega^{(n_B)} \). Hence, a quantifier for the entanglement in \( \sigma_{AB} \) can be defined as:

\[
\mathcal{E}_{\text{PPT}}(\sigma_{AB}) = \max \left\{ 0, -\sum_{\mu_i < 0} \mu_i \right\}, \tag{19}
\]
where \( \mu_i \) is the \( i \)th eigenvalue of \( \sigma_{AB} + i \Omega_A \otimes \Omega_B^\dagger \). The PPT condition (18) is a necessary separability condition in general, but turns out to be sufficient for any \( 1 + n_B \)-mode and \( n_A + 1 \)-mode Gaussian state, namely when at least one of the two parties is single mode (the case we will be considering in section 5).

A Gaussian channel \((T, N)\) applied to Alice’s share is entanglement breaking (EB), i.e. nullifies the entanglement content of any bipartite input state, if and only if the matrix \( N \) admits \([45]\)
\[
N = N_1 + N_2, \text{ where } N_1 \geq i \Omega^{(n_A)} \text{ and } N_2 \geq iTY(\Omega^{(n_A)}) T^\dagger. \tag{20}
\]

**Remark.** Any EB channel is also GIB. To see this for Gaussian channels considered here, note that (20) implies that a necessary condition for EB is to have \( N_1 \geq 0 \). When we add this to the condition for \( N_2 \), we revive (17). Also, notice that the reverse is not necessarily true.

We can now make the following two observations analogous to Observations 1 and 2.

**Observation 3.**—Consider a dynamics on Alice’s side that is EB within some time interval \( t \in (t_1, t_2) \). Any non-Markovian behavior, namely the violation of the CP-divisibility condition, of the dynamics within this interval cannot be witnessed by entanglement between Alice’s and Bob’s system. Indeed, entanglement is always zero in \((t_1, t_2)\).

**Observation 4.**—If one or more of Alice’s modes do not undergo the Gaussian dynamics—i.e. if Alice extends her modes by at least one, such that her total share undergoes the Gaussian channel \((T_t \otimes I, N_t \otimes 0)\)—the criterion (20) is always violated, i.e. the dynamics on Alice is never EB.

It turns out that for a generic one-mode Gaussian channel \((T, N)\), the EB character can be tested by applying the channel locally over the maximally entangled two-mode squeezed state \( \sigma_{2,\infty} = \lim_{r \to \infty} \sigma_{2,r} \), i.e. the separability condition \( \mathcal{E}_{\text{PPT}}((T \otimes I, N \otimes 0)\sigma_{2,\infty}) = 0 \) is a necessary and sufficient condition for the EB character of the channel \([39]\).
Figure 1. Schematic setup for the detection of non-Markovianity with (a) two-mode squeezed input state and with (b) GHZ/W three-mode input state. Notice the selected bipartition concerning Alice’s and Bob’s shares (black horizontal line) where only Alice is allowed to own more than one mode. We consider Gaussian evolutions \((T_t, N_t)\) that are applied only to the first mode of Alice’s share. Non-monotonic behaviors of Gaussian steering/entanglement quantifiers as function of time (backflows) are used as non-Markovianity witnesses. The GHZ/W three-mode configuration can activate the potential to witness non-Markovian behaviors which in the two-mode scenario do not imply any revivals of quantum correlations.

5. Paradigmatic examples

Here, we demonstrate how by using an extra auxiliary mode on Alice’s share, one can witness non-Markovianity through correlation backflows within a bigger class of Gaussian dynamics, if compared to the one-ancillary mode scenario. The potential of the method was anticipated in Observations 2 and 4: since some of the modes owned by Alice do not undergo the channel and can maintain their correlations with Bob’s side the correlation quantifier does not nullify implying more chances of observing its backflow. A schematic representation of our settings is reported in figure 1.

5.1. Classical noise channel

We start by probably the simplest example, i.e. a classical noise channel applied to a single mode. It is given by \cite{31}

\[
(T_t, N_t) = (I, \eta(t)I),
\]

where \(\eta(t) \geq 0\) is a time-dependent continuous function such that \(\eta(0) = 0\). The form for the intermediate map of this evolution can be derived using equation (11), obtaining, for any \(s, t\) such that \(0 < s < t\),

\[
(T_{ts}, N_{ts}) = (I, (\eta(t) - \eta(s))I).
\]

It follows that this Gaussian evolution is Markovian if and only if \(\eta(t)\) is monotonically increasing. Indeed, from equation (9), as soon as \(\eta(t) - \eta(s) < 0\) the corresponding intermediate channel is non-CPTP.

5.1.1. Gaussian steerability

Consider a two-mode Gaussian state shared between Alice and Bob described by the covariance matrix \(\sigma_{AB}\). When acting on Alice’s mode, the dynamical map \((I, \eta(t)I)\) makes the set of all Gaussian measurements compatible if and only if \(\eta(t) \geq 1\) (see equation (17) or \cite{42}). Therefore, according to Observation 1, it is not possible to obtain information backflows through Gaussian steerability \(\mathcal{G}_{A\rightarrow B}\) at times \(\{t \mid \eta(t) > 1\}\). However, inspired by Observation 2, Alice can extend her system to contain an auxiliary mode, such that her share undergoes a local evolution on the first mode, see equation (12). For instance, if we take \(\sigma_{AB} = \sigma_{3,r}\), where Alice holds the first two modes, there will be a backflow in \(\mathcal{G}_{A\rightarrow B}\) at anytime that \(\eta(t)\) decreases, i.e. whenever the dynamics is not CP-divisible. Interestingly, this is true for any squeezing \(r > 0\). We provide the proof of this result in appendix B, where we obtain the analytical forms of \(\mathcal{G}_{A\rightarrow B}(\sigma_{2,r}(t))\) and \(\mathcal{G}_{A\rightarrow B}(\sigma_{3,r}(t))\) for any \(r\) and \(\eta(t)\).

As an example, let the noise assume the following time dependence

\[
\eta(t) = \frac{t^2}{t^2 - 2t + 2},
\]

Since \(\eta(t)\) is monotonically decreasing for \(t \geq 2\), it follows that the evolution is non-Markovian and the intermediate map \((T_{ts}, N_{ts})\) is not CPTP for any \(2 \leq s < t\). Nonetheless, \((T_t, N_t)\) breaks the incompatibility of all Gaussian measurements for \(t \geq 1\) and therefore no backflow of \(\mathcal{G}_{N_{ts}}\) can be observed when Alice holds
only one mode. This is showcased in figure 2(a) where we take $\sigma_{AB} = \sigma_{2,r}$. This is contrary to when we take $\sigma_{AB} = \sigma_{3,r}$, where Alice holds the first two modes. As seen from figure 2(a), in this case $G_{A\rightarrow B}$ increases whenever the dynamics is not CP-divisible.

5.1.2. Entanglement
The dynamical map $(I,\eta(t)f)$ breaks the entanglement of all states if and only if $\eta(t) \geq 2$, see equation (20). Accordingly, we can choose $\eta(t)$ to be the following time-dependent function,

$$\eta(t) = \frac{2t^2}{t^2 - 2t + 2},$$

namely the function (23) rescaled by a factor 2, implying that the non-Markovian interval is the same as before, i.e. $[2,\infty]$ where the function is decreasing. For $t \geq t_{EB} = 1$ we have the EB property ($\eta(1) = 2$ and $\eta(t) \geq 2$ if and only if $t \geq 1$). Analogously to what we observed for steering, in figure 2(b) we show that whether we cannot observe any backflow of $E_{PPPT}(\sigma_{2,r}(t))$ (this quantity nullifies for $t \geq t_{EB}$), we do observe, instead, a backflow of $E_{PPPT}(\sigma_{3,r}(t))$ (red line).

5.2. Lossy channel and the quantum Brownian motion
The second example we consider is the wider class of lossy channels, which are the evolutions with dynamical maps

$$(T_i,s,N_i) = (\tau(t)i,I,\eta(t)i),$$

including the classical noise channel (21) in the particular cases $\tau(t) = 1$. The intermediate maps of these evolutions assume the form

$$(T_i,t,s,N_i) = (\tau(t,i),s,\eta(t,i)),$$

where $\tau(t,s) = \tau(t)/\tau(s)$ and $\eta(t,s) = \eta(t) - \eta(s)(\tau(t)/\tau(s))^2$. As we show in appendix C, the lossy channel is not CP-divisible, namely the intermediate map $(T_{t+\epsilon,t},N_{t+\epsilon})$ is not CPTP, at times $t$ if and only if either one or both of the following inequalities are violated

$$\dot{\eta}(t) - 2(\eta(t) \pm 1) \frac{\dot{\tau}(t)}{\tau(t)} \geq 0.$$

The class of lossy channels is relevant in the description of quantum Brownian motion, where a harmonic oscillator with frequency $\omega_0$ undergoes a dissipative dynamics by interacting with a bosonic bath at temperature $T$. The total Hamiltonian is quadratic, which guarantees that the system undergoes a Gaussian dynamics—details of the interaction and the derivation of the dynamics are given in the
appendix D. The two dynamical parameters $\eta(t)$ and $\tau(t)$ are connected to the physical parameters describing the system and the bath as follows

$$\eta(t) = \tau(t)^2 \int_0^t ds \Delta(s)/\tau(s)^2, \quad \tau(t) = \exp \left[-\int_0^t ds \frac{\gamma(s)}{2}\right].$$

Here, $\Delta(t)$ is the so-called diffusion coefficient and $\gamma(t)$ is the damping coefficient. The parameter $\alpha$ quantifies the system-bath interaction strength.

For our simulations, we follow [28] and choose a spectral density $J(\omega)$ of the bath with a Lorentz–Drude cutoff

$$J(\omega) = \frac{2 \omega'^2 - \omega^2}{\pi} \frac{\omega^2 - \omega'^2}{\omega^2 + \omega'^2},$$

where $\omega_c$ is the cutoff frequency. The parameter $s$ defines the ohmicity of the spectral density: $s < 1$ corresponds to the sub-Ohmic regime, $s = 1$ to the Ohmic regime, and $s > 1$ the super-Ohmic regime.

This model is non-Markovian, i.e. the infinitesimal intermediate map $(T_{t+\epsilon,t}, N_{t+\epsilon,t})$ is not CPTP, for those times $t$ such that one or both of the following inequalities are violated

$$\Delta(t) \pm \gamma(t) \geq 0,$$

a situation that is typically encountered when the cutoff frequency is smaller than the characteristic frequency of the oscillator, i.e. $\omega_c < \omega_0$ (see, e.g. [28]). The criterion (30) above can be also found directly by substituting (28) in (27).

5.2.1. Gaussian steerability

Analogously to the classical noise channel, we first consider a two-mode state $\sigma_{AB}$, where Alice’s share undergoes the dynamical map $(\tau(t) I, \eta(t) I)$. [28] uses such a setting to witness non-Markovianity of the channel in the context of quantum Brownian motion—where the initial state is $\sigma_{AB} = \sigma_{2,r}$. In several scenarios this technique allows to witness non-Markovianity via Gaussian steerability backflows. However, under such a channel, all Gaussian measurements on Alice’s mode become compatible if and only if $\eta(t) \geq \tau^2(t)$ (see equation (17) or [42]). Therefore, it is not possible to witness non-Markovianity through Gaussian steerability at times $\{t \mid \eta(t) \geq \tau^2(t)\}$. In appendix C we derive the analytical form of $\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t))$ for any $r$, $\eta(t)$ and $\tau(t)$. Moreover, we show that a Gaussian steerability backflow is obtained, more precisely $\partial_t \mathcal{G}_{A \rightarrow B} > 0$, if and only if the inequalities

$$\dot{\eta}(t) - 2\eta(t) \frac{\tau(t)}{\tau(t)} < 0, \quad \eta(t) - \tau(t)^2 < 0,$$

are satisfied simultaneously (the latter inequality being the aforementioned non-GIB condition). Notice that the first inequality is the arithmetic average of the two possible violations of (27), and hence more restrictive than (27). This implies that some non-Markovian evolution cannot be witnessed.

Nonetheless, when Alice’s share is extended to include an auxiliary mode that does not undergo the channel, i.e. the dynamical map is $(\tau(t) I \oplus I, \eta(t) I \oplus 0)$—the set of Gaussian measurements will remain incompatible at all times. In particular, let Alice and Bob share the three-mode squeezed state $\sigma_{AB} = \sigma_{3,r}$, where Alice holds the first two modes. In appendix C we derive the analytical form of $\mathcal{G}_{A \rightarrow B}(\sigma_{3,r}(t))$ for any $r$, $\eta(t)$ and $\tau(t)$. Moreover, we show that we have a backflow $\partial_t \mathcal{G}_{A \rightarrow B} > 0$ if and only if

$$\dot{\eta}(t) - 2\eta(t) \frac{\tau(t)}{\tau(t)} < 0,$$

which for the quantum Brownian motion is equivalent to $\Delta(t) < 0$. This criterion is clearly less restrictive than (31)—in that it does not require $\eta(t) < \tau^2(t)$. Thus, using the extra mode one can witness information backflow for a bigger class of dynamics. Notice that, the criterion (32) is still more restrictive than
in which breakdown of inequality equation (temperatures. As a benchmark, we also depict the breakdown of CP-divisibility (shaded regions), i.e. the steerability. Y et, it is not guaranteed that by doing so one can witness all non-Markovian dynamics with Gaussian evolutions one might consider a different three-mode initialization or increase the number of ancillary modes. Y et, it is not guaranteed that by doing so one can witness all non-Markovian dynamics with Gaussian evolutions one might consider a different three-mode initialization or increase the number of ancillary modes. Yet, it is not guaranteed that by doing so one can witness all non-Markovian dynamics with Gaussian steerability.

In figure 3(a), we plot Gaussian steerability $G_{A\rightarrow B}$ for the quantum Brownian motion model at high temperatures. As a benchmark, we also depict the breakdown of CP-divisibility (shaded regions), i.e. the breakdown of inequality equation (30). W e compare two cases with $\sigma_{AB}=\sigma_{2,r}$ and $\sigma_{AB}=\sigma_{3,r}$. As the inset shows, using the two-mode state is not successful for some of the parameter regimes that we consider. In particular, if the interaction strength $\alpha$ is too large, the evolution of two-mode initializations cannot show any backflow of $A \rightarrow B$ Gaussian steerability because the evolution breaks the incompatibility of Alice’s measurements. F urthermore, as one increases $\alpha$, the time intervals during which one cannot witness backflow of Gaussian steerability in the two-mode configuration increases. On the contrary, in the three-mode scenario, if Alice owns the first two modes of $\sigma_{3,r}$, the incompatibility of Alice’s measurements is not broken and we can witness non-Markovianity via backflows for all $\alpha$.

5.2.2. Entanglement

F irstly, notice that the dynamical map $(\tau(t)I, \eta(t)I)$ is EB iff $\eta(t) \geq \tau^2(t) + 1$ (see equation (20)). W hen we use the two-mode squeezed state for small $r$, one can show that $\partial_t \mathcal{E}_{PPT} > 0$ if the following two inequalities are satisfied simultaneously

$$\eta(t) - \tau^2(t) - 1 < 0, \quad (33)$$

$$\eta(t) - 2(\eta(t) - 1) \frac{\dot{\tau}(t)}{\tau(t)} < 0. \quad (34)$$

The first inequality expresses the fact that steerable states are entangled but in general not vice-versa, i.e. entanglement is more robust to noise. I nterestingly, provided steerability is non-zero, the second inequality, being one of the two possible violations of (27), when compared with (31) suggests that, steerability (entanglement) could be preferable for observing backflows if $\frac{\dot{\tau}(t)}{\tau(t)} > 0$, the two figures of merit yielding the same performance in the case $\dot{\tau}(t) = 0$.

W hile we are not able to retrieve analytical conclusions about entanglement backflows in the case of the three-mode initialization $\sigma_{3,r}$, the potential of this configuration can be shown numerically for the quantum Brownian motion model at high temperatures. In figure 3(b), we plot $\mathcal{E}_{PPT}$ as function of time comparing the two cases $\sigma_{2,r}(t)$ (inset) and $\sigma_{3,r}(t)$. Again, using the two-mode state is not successful for large values of $\alpha$.

On the contrary, in the three-mode scenario, if Alice owns the first two modes of $\sigma_{3,r}$ we can witness non-Markovianity via backflows of entanglement for all $\alpha$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{(a) Gaussian steerability $G_{A\rightarrow B}$ as function of time for the three-mode squeezed state initialization $\sigma_{3,r}(0)$ and Alice’s first mode subjected to the quantum Brownian motion, for different values of the coupling parameter $\alpha$: 0.25 (purple), 0.35 (blue), 0.42 (red) and 0.7 (orange). Inset: $G_{A\rightarrow B}$ for the two-mode squeezed state initialization $\sigma_{2,r}(0)$ with the same parameters. Here for $\alpha$ greater than $\approx 0.42$ (red), we cannot witness non-Markovianity as the dynamical maps of the evolution become GIB before any non-Markovian behavior. (b) Same as in (a) but for entanglement $\mathcal{E}_{PPT}$ and for the following values of $\alpha$: 0.25 (purple), 0.35 (blue), 0.395(red), threshold value for the $\mathcal{E}_{PPT}(\sigma_{3,r}(t))$ sensitivity; 0.7 (orange). Both $G_{A\rightarrow B}(\sigma_{2,r}(t))$ and $\mathcal{E}_{PPT}(\sigma_{3,r}(t))$ show backflow whenever non-Markovianity is present (shadow blue regions, according to condition (30)). All the plots are made in the Ohmic regime ($s = 1$), setting $r = 2, \omega_0 = 7, \omega_r = 1, T = 100$ (high temperatures).}
\end{figure}
6. Discussion

We considered Gaussian steerability and entanglement as quantifiers of the correlations contained in a two-party Gaussian system and at the same time their backflows as non-Markovianity witnesses. We were interested in those non-Markovian evolutions that cannot be witnessed by backflows of these correlations when two-mode Gaussian initializations are considered. This happens on time intervals where the considered one-mode evolutions are incompatibility and EB, nullifying Gaussian steerability and entanglement, respectively. Therefore, we considered a strategy that makes use of three-mode Gaussian states, namely the GHZ/W three-mode squeezed states, where Alice owns the first two modes which allows to overcome the problem of GIB and EB. For classical noise evolutions, we also showed that our use of the GHZ/W three-mode squeezed states allows to witness any non-Markovian evolution of this kind. However, in the more general case of evolutions characterized by lossy channels, our results show that, while the GHZ/W three-mode squeezed states allows witnessing non-Markovian evolutions that cannot be witnessed with two-mode initializations, this three-mode state could not be enough for detecting all non-Markovian dynamics. In order to increase the potential to witness non-Markovianity via correlation backflows, it would be interesting to consider three-mode initializations different from the GHZ/W states or to increase the size of the ancillas. Nonetheless, for the quantum Brownian motion, a particular instance of the lossy channels of experimental interest (but still more sophisticated than the classical noise channels), the GHZ/W three-mode squeezed state strategy accomplishes the task of detecting non-Markovianity through correlation backflows for values of the parameters where any two-mode initialization fails.

An interesting open question is to identify a mode geometry, and corresponding state, able to display a correlation backflow for all non-Markovian evolutions. Our results have shown that two modes are not enough for Gaussian steerability and entanglement, while three modes do provide an improvement. Are three modes enough? If not, is a finite number of modes enough? Note that in the finite dimensional case, solutions to this question were derived in \[24-26\]. In the first work, a correlation measure \(C\) was introduced based on state distinguishability and it was shown, for an evolution acting on a system \(A_1\) of dimension \(d\), how to construct an initial state defined on systems \(A_1A_2B\) with respective dimensions \(d, d+1, 2\), that displays a backflow in the correlation measure \(C\) along the bipartition \(A_1A_2\). Again for an evolution acting on a system \(S\) of dimension \(d\), it was provided an initial state consisting of systems \(A_1A_2A_3B\) with respective dimensions \(d, d+1, 2\) and such that an entanglement negativity backflow along the bipartition \(A_1A_2A_3\) could be detected for all invertible non-Markovian evolutions.

Whether a similar arrangement is possible in the Gaussian case remains an open question. Nonetheless, as discussed in section 4, we expect that any invertible \(n\)-mode non-Markovian Gaussian evolution on \(A_1\) can be witnessed with a correlation backflow along the bipartition \(A_1A_2\), where \(A_2\) and \(B\) are respectively (at most) two and one mode ancillary Gaussian systems, no matter the number \(n\) of evolving modes. It would be interesting to start this study by checking whether it is always possible to obtain correlation backflows along the bipartition \(A_1A_2\) when a generic one-mode non-Markovian Gaussian evolution is applied on \(A_1\) and \(A_2\) (or \(B\)) is a two-mode (one-mode) ancilla.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. CP-divisibility criterion for smooth Gaussian dynamics

In the main text we introduced non-Markovianity as violation of CP-divisibility of the intermediate map. For a smooth dynamics, CP-divisibility is equivalent to having an infinitesimally CP-divisible dynamics. In other words, at any time \(t\) and for infinitesimally small \(\epsilon > 0\) we should have
where the intermediate map \((T_{t+\epsilon},N_{t+\epsilon})\) is CP. On the one hand, since \(\epsilon\) is small we expect the map to take the form

\[
(T_{t+\epsilon},N_{t+\epsilon}) = (T_t,N_t) \circ (T_{\epsilon},N_{\epsilon}),
\]

(A.1)

which means any covariance matrix \(\sigma_0\) evolves as follows

\[
\sigma_{t+\epsilon} = (T_{t+\epsilon},N_{t+\epsilon})\sigma_0
= T_t\sigma_0 T_t^T + N_t + \epsilon \left( \dot{T}_t \sigma_0 T_t^T + \dot{N}_t \right) + \mathcal{O}(\epsilon^2).
\]

(A.2)

On the other hand, we should have

\[
\sigma_{t+\epsilon} = (T_{t+\epsilon},N_{t+\epsilon})\sigma_t = (T_{t+\epsilon},N_{t+\epsilon})\circ (T_t,N_t) \sigma_0
= T_{t+\epsilon} T_t \sigma_0 T_t^T T_{t+\epsilon}^T + T_{t+\epsilon} N_t T_{t+\epsilon}^T + N_{t+\epsilon}.
\]

(A.3)

By comparing equations (A.3) and (A.4) we should have the following two equivalences

\[
T_{t+\epsilon} T_t \sigma_0 T_t^T T_{t+\epsilon}^T \equiv T_t \sigma_0 T_t^T + \epsilon \left( \dot{T}_t \sigma_0 T_t^T + T_t \sigma_0 \dot{T}_t^T \right) + \mathcal{O}(\epsilon^2)
\]

\[
\Rightarrow T_{t+\epsilon} = I + \epsilon \dot{T}_t T_t^{-1} + \mathcal{O}(\epsilon^2)
\]

(A.5)

\[
T_{t+\epsilon} N_t T_{t+\epsilon}^T + N_{t+\epsilon} \equiv N_t + \epsilon \dot{N}_t + \mathcal{O}(\epsilon^2)
\]

\[
\Rightarrow N_{t+\epsilon} = \epsilon \left( \dot{N}_t - \dot{T}_t T_t^{-1} N_t - N_t T_t^{-T} T_t^T \right) + \mathcal{O}(\epsilon^2),
\]

(A.6)

where we use the convention \(A^{-T} = (A^T)^{-1} = (A^{-1})^T\) for any matrix \(A\). Furthermore, the positivity of the intermediate map implies

\[
N_{t+\epsilon} + i\Omega - i T_{t+\epsilon} \Omega T_{t+\epsilon}^T \geq 0,
\]

(A.7)

which by using equations (A.5) and (A.6)—and keeping the leading order in \(\epsilon\)—is equivalent to the following criterion

\[
N_{t+\epsilon} + i\Omega - \left( i \Omega + \epsilon \left( \dot{T}_t T_t^{-1} \Omega + i \Omega T_t^{-T} T_t^T \right) \right) \geq 0
\]

\[
\Rightarrow \dot{N}_t - \left( \dot{T}_t T_t^{-1} (i\Omega + N_t) + (i\Omega + N_t) T_t^{-T} T_t^T \right) \geq 0.
\]

(A.8)

The above condition is a necessary and sufficient condition for the intermediate map being non-Markovian, i.e. if it holds at all times, the dynamics is Markovian, otherwise, it is not. Furthermore, it can be deduced directly from equation (8) of [46], substituting in that equation at first order in \(\epsilon\), \(X(t+\epsilon,0) \rightarrow X(t,0)\epsilon + X(t,0)\) and \(Y(t+\epsilon,0) \rightarrow Y(t,0)\epsilon + Y(t,0)\), where \(X(t,0)\) and \(Y(t,0)\) of [46] are the matrices \(T_t\) and \(N_t\), respectively.

**Appendix B. Classical noise channel: potential of the GHZ/W three-mode squeezed state setup**

Here we show that using the initialization \(\sigma_{3,t} (6)\) one can witness any non-Markovianity of the classical noise evolution (21) by means of steering backflow. Interestingly, this happens for any value of the squeezing parameter \(r\). The impossibility of achieving the same result through the two-mode initialization \(\sigma_{2,t} (5)\) is also shown analytically. Finally, we conclude the section with an explanatory example of oscillating noise.

**B.1. Three-mode scenario**

Consider the state given by \(\sigma_{3,t}\) with only the first mode undergoing the classical noise channel. The total map is given by the pair \((I^{(3)}, \eta_1I^{(1)} \otimes 0^{(2)})\), where we keep using the notation where \(I^{(n)} (0^{(n)})\) is the \(2n \times 2n\) identity (null) matrix. The covariance matrix after the channel reads

\[
\sigma_{3,t}(t) = \begin{pmatrix}
\sigma(r) + \eta_1 & \epsilon(r) & \epsilon(r) \\
\epsilon(r) & \sigma(r) & \epsilon(r) \\
\epsilon(r) & \epsilon(r) & \sigma(r)
\end{pmatrix}.
\]

(B.1)
Whenever \( \partial t \eta_t \leq 0 \) the dynamics is non-Markovian (see equation (22) and related discussion). We want to see it via steering, in particular we use the first two modes (including the noisy one) in order to steer the third mode. According to (13) the state is steerable from the first two modes to the third one if and only if the following is violated

\[
\sigma_{3,t}(t) + \hat{A}(2) \otimes \Omega^{(1)} \geq 0, \tag{B.2}
\]

which is equivalent to the non-physicality of the covariance matrix

\[
M_B = B - C^T A^{-1} C, \tag{B.3}
\]

with \( A \equiv \sigma_{3,t}(t)(1:4,1:4), C \equiv \sigma_{3,t}(t)(1:4,5:6) \), and \( B \equiv \sigma_{3,t}(t)(5:6,5:6) \)—note that equation (14) corresponds to equation (B.3) for \( \sigma = \sigma_{3,t} \). This gives the matrix

\[
M_B = \begin{pmatrix}
\eta_t + 3 e^{r} + 2 r e^{4r} & 0 \\
0 & \frac{e^{r} (2 \eta_t + 3 e^{r} + r e^{4r})}{\eta_t + 2 e^{r} + 2 r e^{4r}}
\end{pmatrix}. \tag{B.4}
\]

The symplectic eigenvalue of the matrix above,

\[
\nu_- = \sqrt{\frac{e^{2r} [(e^{4r} + 2) \eta_t + 3 e^{2r}] (e^{2r} (2 \eta_t + 3 e^{2r}) + \eta_t)}{[(e^{4r} + 2) e^{2r} \eta_t + 2 e^{4r} + 1] [e^{r} (e^{4r} + 2 e^{2r} \eta_t + 2) + \eta_t]^2}}, \tag{B.5}
\]

is always lower than 1, since, considering the expression inside the square root, the denominator exceeds the numerator by the positive quantity \((-1 + e^{2r})^2 [\eta_t + e^{2r} (2 + e^{2r} \eta_t)]\) and both the numerator and the denominator are positive. Furthermore, the derivative with respect to \( \eta_t \),

\[
\frac{\partial \nu_-}{\partial \eta_t} = \frac{e^{2r} (e^{4r} - 1)^2 [4 e^{2r} (2 e^{4r} + 5 e^{2r} + 2) \eta_t + 9 (e^{2r} + e^{4r}) + (2 e^{12r} + 7 e^{8r} + 7 e^{4r} + 2) \eta_t^2]}{2 [(e^{4r} + 2) e^{2r} \eta_t + 2 e^{4r} + 1]^2}, \tag{B.6}
\]

is also positive. This shows that for all values of \( r \) and \( \eta_t \), one can always detect non-Markovianity. In particular, in the cases of small and large squeezing parameter \( r \), the Gaussian steerability function is less cumbersome.

The symplectic eigenvalue around \( r = 0^+ \), i.e. for epsilon-entangled states, reads

\[
\nu_- = 1 - \frac{16}{9(\eta_t + 1)} r^2 + O(r^3). \tag{B.7}
\]

which is always smaller than one. Gaussian steerability from Alice, who owns the first two modes, to Bob, who owns the third mode, reads

\[
\mathcal{G}_{A \rightarrow B}(\sigma_{3,t}(t)) = -\log(\nu_-) \approx \frac{16}{9(\eta_t + 1)} r^2 + O(r^3), \tag{B.8}
\]

which is a monotonically decreasing function of \( \eta_t \). Hence, whenever the dynamics is non-Markovian, i.e. \( \partial t \eta_t \leq 0 \), the Gaussian steerability increases and one can detect non-Markovianity. For large squeezing parameter \( r \), instead, we get

\[
\mathcal{G}_{A \rightarrow B}(\sigma_{3,t}(t)) = \frac{1}{2} \ln \left( \frac{e^{2r}}{2 \eta_t} \right) + O(e^{-2r}), \tag{B.9}
\]

leading to the same conclusion as before. These results show that the state remains always steerable from the first two modes to the third mode, no matter how large \( \eta_t \) is.
B.2. Two-mode scenario

For a scenario with two modes and input $\sigma_{2,r}$ (equation (5)) one can find that the matrix $M_B$ (see equation (14)) reads

\[
\begin{pmatrix}
\frac{\eta_r \cosh(2r)}{\eta_r + \cosh(2r)} + 1 & 0 \\
0 & \frac{\eta_r \cosh(2r)}{\eta_r + \cosh(2r)} + 1
\end{pmatrix}.
\]

(B.10)

We have

\[
\nu_- = \eta_r \cosh(2r) + 1 / (\eta_r + \cosh(2r)),
\]

(B.11)

which is smaller than 1 iff $\eta_r < 1$, and

\[
\frac{\partial \nu_-}{\partial \eta_r} = \frac{\cosh^2(2r) - 1}{(\eta_r + \cosh(2r))^2}
\]

(B.12)

is always positive. This implies that for any $r > 0$, the two-mode initialization setup cannot detect non-Markovianity if $\eta_r \geq 1$, but it does if $\eta_r < 1$ since the function $\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t)) = \max \{0, -\ln(\nu_-)\}$ is monotonically decreasing with $\eta_r$.

Again, behaviors for small and large squeezing parameter $r$ are the following. For small $r = 0^+$, we have

\[
\nu_- = 1 - \frac{1 - \eta_r}{1 + \eta_r} r^2 + \mathcal{O}(r^4),
\]

(B.13)

and therefore, at leading order, the Gaussian steerability from Alice, who owns the first mode, to Bob, who owns the second mode, reads

\[
\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t)) = \begin{cases} 
\frac{1 - \eta_r}{1 + \eta_r} r^2, & \eta_r < 1 \\
0, & \eta_r \geq 1
\end{cases}
\]

(B.14)

which, consistently, cannot detect non-Markovianity if $\eta_r \geq 1$, but it does if $\eta_r < 1$, since the function is monotonically decreasing with $\eta_r$. For large $r$ we finally obtain

\[
\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t)) = \max \{0, \ln \left( \frac{1}{\eta_r} \right) + \mathcal{O}(e^{-2r}) \}.
\]

(B.15)

B.3. Classical noise: oscillating noise

We conclude the discussion on the classical noise channel of section 5.1 by considering an oscillating noise that didactically shows when the two-mode configuration (5) fails in detecting non-Markovianity. We hence set the noise as

\[
\eta(t) = \eta_0(1 - \cos(2\pi t))/2,
\]

(B.16)

with $\eta_0$ being the constant gauging its intensity. If the constant $\eta_0 \leq 1$, an initialization in $\sigma_{2,r}(0)$ implies backflows of $\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t))$ if and only if $\eta(t)$ decreases, see figure B1(a). In this case two modes are enough to detect any non-Markovian character. On the contrary, if $\eta_0 > 1$ we observe backflows of $\mathcal{G}_{A \rightarrow B}(\sigma_{2,r}(t))$ if and only if $\eta(t)$ decreases and $\eta(t) < 1$, see figure B1(b). As analytically shown, the three-mode initialization $\sigma_{2,r}(0)$ does not suffer of this limitation providing steering backflow as soon as the map shows non-Markovian behavior and for any value of $\eta(t)$.

Figures B1(c) and (d) concern instead entanglement $\mathcal{E}_{PPT}$, again, for different values of $\eta_0$. The core message does not change by considering this other correlation quantifier. The dynamical maps of this evolution are periodically EB in finite time intervals for $\eta_0 > 2$. For this reason, whether some time intervals of non-Markovianity cannot be witnessed by backflows of entanglement in the two-mode configuration (5) (see figure B1(c)), the three-mode configuration (6) never fails in detecting non-Markovianity (see figure B1(d)).
Appendix C. Lossy channel

The lossy channel is given by $T_t = \tau_t I$ and $N_t = \eta_t I$, see equation (25). First of all, from equations (9) and (10), the complete positivity of the channel imposes

$$\eta_t \geq \tau_t^2 \left| 1 - \frac{1}{\tau_t} \right| ,$$

while $\tau_t$ can also be considered positive in full generality.

C.1. CP-divisibility

The CP-divisibility of the intermediate map breaks down if and only if equation (A.8) is violated. For the lossy channel, the matrix on the l.h.s. of (A.8) has the following eigenvalues

$$\lambda_{\pm} = \hat{\eta} - 2(\eta \pm 1) \tau_t \tau_t^*,$$

and therefore when the sign of at least one eigenvalue is negative we have non-Markovianity, formally

$$\text{sign} [\lambda_{\pm}] = \text{sign} [\tau_t^2 \hat{\eta} - 2\tau_t \tau_t^* \eta \pm 2\tau_t \tau_t^*],$$

getting the value 1 iff we are in the Markovian case. This proves inequality (27) of section 5.2.

C.2. Gaussian steerability

The two-mode Gaussian steerability, where Alice and Bob own one mode, reads

$$G_{A \rightarrow B}(\sigma_{2,T}(t)) = \max \left\{ 0, \frac{2(\tau(t)^2 - \eta(t))}{\tau(t)^2 + \eta(t)} r^2 + O(r^3) \right\} \quad (C.4)$$
for small $r$ and
\[
\mathcal{G}_{A\rightarrow B}(\sigma_{2,r}(t)) = \max \left\{ 0, \ln \left( \frac{\tau^2_r}{\eta_r} \right) + \mathcal{O}(e^{-2r}) \right\},
\] (C.5)
for large $r$. As a consequence, one can check that, both for the limits of small and large $r$,
\[
\text{sign}[\mathcal{G}_{A\rightarrow B}(\sigma_{2,r}(t))] = \begin{cases} 
- \frac{\eta_r(t) + 2\eta_r(t) \tau(t)}{\tau(t)}, & \eta(t) < \tau^2(t) \\
0, & \eta(t) \geq \tau^2(t)
\end{cases}
\] (C.6)
This condition remains valid for any $r > 0$. Indeed, we have
\[
\nu_- = \frac{(\eta_r/\tau^2_r) \cosh (2r) + 1}{(\eta_r/\tau^2_r) + \cosh (2r)},
\] (C.7)
i.e. nothing but equation (B.11) with the substitution $\eta_r \rightarrow \eta_r/\tau^2_r$. Hence, (C.7) is smaller than 1 if $\eta_r < \tau^2_r$ and its time derivative is $\dot{\nu}_- = \frac{\partial \nu_-}{\partial (\eta_r/\tau^2_r)} \frac{d(\eta_r/\tau^2_r)}{dt}$. While $\frac{\partial \nu_-}{\partial (\eta_r/\tau^2_r)}$ is always positive (to see it one can directly use the results from section B.2), $\text{sign}[\mathcal{G}_{A\rightarrow B}(\sigma_{2,r}(t))] = \text{sign}[\dot{\eta}_r - 2\eta_r \dot{r}_r/\tau^2_r]$. This proves condition (C.6) and (31) for any $r > 0$.

On the other hand, the three-mode Gaussian steerability (Alice owns the first two modes and Bob owns the third mode) for small $r$ and large $r$ reads, respectively, as
\[
\mathcal{G}_{A\rightarrow B}(\sigma_{3,r}(t)) = \frac{16\tau^2_r}{9(\tau^2_r + \eta_r)} r^2 + \mathcal{O}(r^3),
\] (C.8)
\[
\mathcal{G}_{A\rightarrow B}(\sigma_{3,r}(t)) = \frac{1}{2} \ln \left( \frac{\tau^2_r e^{2r}}{2\eta_r} \right) + \mathcal{O}(e^{-2r}),
\] (C.9)
which is always positive. Therefore, Alice can (somewhat obviously) always steer the third mode. One can easily check that, both for the limits of small and large $r$,
\[
\text{sign}[\mathcal{G}_{A\rightarrow B}(\sigma_{3,r}(t))] = \text{sign}\left[ - \dot{\eta}_r(t) + 2\eta_r(t) \frac{\dot{r}_r(t)}{\tau(t)} \right],
\] (C.10)
and therefore, one can detect non-Markovianity if the sign is positive, proving condition (32). However, it is a stricter constraint than (C.3), and thus we may not detect some non-Markovian dynamics. One can finally analytically check that condition (C.10) is valid for any $r > 0$. In this case, analogously to what we observed for expression (C.7), $\nu_- \rightarrow \eta_r/\tau^2_r$ is nothing but expression (B.5) with the substitution $\eta_r \rightarrow \eta_r/\tau^2_r$. Therefore, $\nu_- < 1$ and the sign$[\dot{\nu}_-] = \text{sign}[\dot{\eta}_r - 2\eta_r \dot{r}_r/\tau^2_r]$, proving (C.10) for any $r > 0$.

Appendix D. Quantum Brownian motion

The quantum Brownian motion is a particular example of the lossy channel (25) admitting a microscopic derivation.

D.1. Microscopic derivation and master equation

We consider the following Hamiltonian $H$ for the whole system–environment compound,
\[
\hat{H} = \hat{H}_S + \hat{H}_E + \hat{H}_I ,
\] (D.1)
\[
\hat{H}_S = \omega_0 a^\dagger a, \quad \hat{H}_E = \sum_k \omega_k \hat{b}_k^\dagger \hat{b}_k, \quad \hat{H}_I = \frac{\alpha}{2} \hat{Q} \hat{Q}, \quad \hat{Q} = \sum_k \hat{Q}_k,
\] (D.2)
where $\hat{H}_S$, $\hat{H}_E$ are the local terms on the system and on the environment, respectively, $\hat{H}_I$ is the system–environment interaction term and $\hat{q} = \hat{a} + \hat{a}^\dagger$ and $\hat{Q}_k = \hat{b}_k + \hat{b}_k^\dagger$ are the position operators of the system (bosonic ladder operators $\hat{a}$, $\hat{a}^\dagger$) and environment (bosonic ladder operators $\hat{b}_k$, $\hat{b}_k^\dagger$). $\hat{H}_I$ is a dipole-like interaction having coupling constant $\alpha$ controlling its strength. Performing a second order expansion on the
exact dynamics in interaction picture and enforcing Born (weak-coupling) and first Markov approximations one arrives to the Redfield equation for the reduced system dynamics in interaction picture \[ \frac{d}{dt} \hat{\rho}(t) = -\frac{\alpha^2}{4} \int_0^t d\tau \left( \hat{\rho}(t-\tau) \hat{a}(t-\tau) - \hat{a}^\dagger(t-\tau) \hat{\rho}(t-\tau) \right) + h.c. , \] (D.3)

where \( \langle \cdot \rangle_T \) denotes the average over the bath thermal state at temperature \( T \) and we indicated operators in interaction picture as \( \hat{\rho}(t) = \exp[i(\hat{H}_S + \hat{H}_\text{B})t] \hat{\rho} \exp[-i(\hat{H}_S + \hat{H}_\text{B})t] \).

In the secular approximation, i.e., canceling the fast oscillating counter-rotating terms \( \exp(\pm i\omega_0 t) \), neglecting the Lamb shift and assuming the environment to be in the thermal state with temperature \( T \), the above equation reduces to the quantum Brownian motion master equation (see e.g. [28, 47])

\[ \frac{d}{dt} \hat{\rho}(t) = \frac{\Delta(t) + \gamma(t)}{2} \left( \hat{a} \hat{\rho}(t) \hat{a}^\dagger - \frac{1}{2} \{ \hat{a}^\dagger \hat{a}, \hat{\rho}(t) \} \right) + \frac{\Delta(t) - \gamma(t)}{2} \left( \hat{a}^\dagger \hat{\rho}(t) \hat{a} - \frac{1}{2} \{ \hat{a} \hat{a}^\dagger, \hat{\rho}(t) \} \right) , \]

with \( \Delta(t) \) and \( \gamma(t) \) being the diffusion and damping coefficients, respectively, defined in (28) and \( J(\omega) = \sum_n \gamma_n^2 \delta(\omega - \omega_n) \) is the spectral density (in the main text, assumed to be of the form (29)). This implies the following master equation for the \((1+n)\)-mode covariance matrix

\[ \frac{d}{dt} \sigma(t) = A(t) \sigma(t) + \sigma(t) A^T(t) + D(t) , \]

with

\[ A(t) = \left[ -\frac{\gamma(t)}{2} I^{(1)} \right] \otimes \theta^{(n)} \] and \( D(t) = \left[ \Delta(t) I^{(1)} \right] \otimes \theta^{(n)} , \]

where we assumed only the first mode to be affected from the channel. In order to get the dynamics of second order moments we solved numerically the integrals (28) and the master equation (D.5) for the covariance matrix. An equivalent approach would be to consider the integrated lossy channel expression (see, e.g. [28])

\[ \sigma(t) = [(\tau(t) I^{(1)} \oplus I^{(n)}) \sigma_0 (\tau(t) I^{(1)} \oplus I^{(n)})^T + \eta(t) I^{(1)} \oplus I^{(n)}] , \]

with \( \tau(t) \) and \( \eta(t) \) given in (28).

**D.2. Low temperature regime: failure of the GHZ/W three-mode squeezed state setup**

Comparing the violation of the Markovian condition (27) with the backflow conditions for Gaussian steerability (31) and (32), we infer that any non-Markovianity can be detected in the limit \( \eta(t) \gg 1 \) via steering backflow, at least for the three-mode initialization \( \sigma_{3,r}(0) \). For quantum Brownian motion this limit is achieved at high temperatures, see figure 3. Nonetheless, by lowering the temperature one expects the limitation in sensitivity to manifest. In figure D1(a), we plot \( \mathcal{G}_{A\rightarrow B} \) as a function of time for inputs \( \sigma_{3,r}(0) \).
and $\sigma_{2D}(0)$ (inset) at low temperatures. We observe that $G_{A \rightarrow B}$ does not show backflow in the non-Markovian region, for both the considered initializations. In figure D1(b) we also observe that the same insensitivity is obtained by considering entanglement $E_{\text{PTT}}$.

ORCID iD

Dario De Santis © https://orcid.org/0000-0001-5666-1770

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