ON A STRONG LAW OF LARGE NUMBERS RELATED
TO MULTIPLE TESTING NORMAL MEANS*

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Very recently the strong law of large numbers (SLLN) for the normalized number of conditional rejections has been derived by [4] through a technique called “principal factor approximation (PFA)” for testing which components of a Normal random vector (rv) have zero means when each component has unit variance. Further, this law has been claimed to hold under arbitrary covariance dependence among the components. However, the original proof of this result in [4] does not seem be sufficient to establish its validity. Due to the importance of such a result for multiple testing (MT) under dependence, we provide in this article a complete justification for it under some conditions that exclude certain types of covariance dependence. Our reformulation reveals the intricate harmony to be satisfied by the speed of PFA to the original Normal rv, the dependency among the components of this rv, and the magnitudes of the conditional component means to induce such a law.

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1. Introduction. Multiple testing (MT) and control of the false discovery rate (FDR, [2]) has been conducted in a variety of scientific endeavours including the microarray study in [3] and the analysis of spatial data in [1]. It is very challenging to develop MT procedures with a desired FDR when the test statistics are strongly dependent. Recently, [4] considered MT in the following setting. Let $Z \sim N_m(\mu, \Sigma)$ be an $m$-dimensional Normal random vector (rv) with mean $\mu = (\mu_1, ..., \mu_m)^T \in \mathbb{R}^m$ and a known correlation matrix $\Sigma \geq 0$. Given an observation $z = (z_1, ..., z_m)^T$ of $Z$, the $i$th null $H_i : \mu_i = 0$ versus its alternative $H_i^* : \mu_i \neq 0$ is tested using $p_i = 2\Phi(-|z_i|)$ and some $t \in [0, 1]$ such that $H_i$ is rejected if and only if $p_i \leq t$, where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard Normal rv. To tackle this MT problem, [4] proposed the technique of principal factor approximation (PFA) to decompose the dependence embedded in $\Sigma$ and used Theorem 6 in [6] together with the PFA to obtain the strong law of large numbers (SLLN) for the normalized number of conditional rejections

$$\tilde{R}(t) = m^{-1}R(t|\tilde{w}_k) = m^{-1}\sum_{i=1}^m 1\{p_i \leq t|\tilde{w}_k\}$$

for arbitrary $\mu \in \mathbb{R}^m$ and $\Sigma \geq 0$, where $1_A$ is the indicator of a set $A$ and $\tilde{w}_k \sim N_k(0, I)$ are called by [4] the “principal factors” (see Section 2 for more details). The SLLN for $\tilde{R}(\cdot)$ (and that for another related quantity) enables [4] to derive an almost sure (a.s.) approximation to the false discovery proportion (FDP, [5]) and it is considered a theoretical breakthrough in FDR control of MT under dependence.

Unfortunately, we found that the arguments given by the authors of [4] do not seem to be fully sufficient to yield the SLLN for $\tilde{R}(\cdot)$ for arbitrary $\mu \in \mathbb{R}^m$ and $\Sigma \geq 0$, mainly due to their delicate handle of the involved
functional remainders in the Taylor expansions and their use of some implicit assumptions to derive certain auxiliary asymptotic assertions. In this article, we will provide a complete justification of this law with clarifications on all needed assumptions. In Section 2 we will revisit the technique of PFA and provide examples on different speeds of the PFA to components of the original rv. In Section 3, we will briefly review the strategy on the proof used in [4] and point out the extra arguments needed to justify this law. In Section 4, we present our detailed proof. Our reformulation provides an integrated view on how the speed of the PFA to $Z$, the dependence among $z_i$’s, and the magnitudes of $\mu_i$’s should interact with each other in order to validate this law. We end the article with a short discussion in Section 5.

2. PFA with different component speeds. Let the triple $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space on which all involved rv’s are defined. We restate the technique of PFA developed in [4] as follows. Let $w = (w_1, \ldots, w_m)^T \sim \mathcal{N}_m(0, I)$. The spectral decomposition of $\Sigma$ with eigenvalues (counting multiplicity) $\lambda_{1,m} \geq \lambda_{2,m} \geq \cdots \geq \lambda_{m,m} \geq 0$ and corresponding eigenvectors $\gamma_i = (\gamma_{i1}, \ldots, \gamma_{im})^T$ for $i = 1, \ldots, m$ implies

$$Z = \mu + \eta + v,$$

where $\eta = (\eta_1, \ldots, \eta_m)^T$ with $\eta_i = \sum_{j=1}^{k} \sqrt{\lambda_{j,m}} \gamma_{ij} w_j$ and $v = (v_1, \ldots, v_m)^T$ with $v_i = \sum_{j=k+1}^{m} \sqrt{\lambda_{j,m}} \gamma_{ij} w_j$ for any $0 \leq k \leq m$. In (2.1), $\eta = 0$ is set when $k = 0$. From (2.1), we see that $\omega_{i,m} = \mathbb{V}[\eta_i] = \sum_{j=1}^{k} \lambda_{j,m} \gamma_{ij}^2$ and $\sigma_{i,m} = \mathbb{V}[v_i] = 1 - \omega_{i,m}$, where $\mathbb{V}[\cdot]$ denotes the variance operator. Note that $(w_1, \ldots, w_k)^T$ is exactly $\tilde{w}_k$.

For (2.1) with $k \geq 1$, [4] pointed out that for a fixed $\delta > 0$ there always
exists some \(1 \leq k \leq m\) (\(k\) can depend on \(m\)) such that

\[
(2.2) \quad \vartheta_m = m - 1 \sqrt{\lambda_{k+1,m}^2 + \ldots + \lambda_{m,m}^2} \leq C m^{-\delta}
\]

for some finite constant \(C > 0\). It should be noted that (2.2) never holds for any \(\delta > 0\) when \(k = 0\) and \(\Sigma > 0\). Further, \([4]\) defined

\[a_{i,m} = (1 - \omega_{i,m})^{-1/2}\]

with the implicit assumption that \(\omega_{i,m} < 1\) for \(1 \leq i \leq m\). Note that \(a_{i,m} \geq 1\) for any \(1 \leq i \leq m\) and finite \(m\).

Let \(\text{cov}(\cdot, \cdot)\) be the covariance operator and \(A = \text{cov}(v, v) = (q_{ij})_{m \times m}\). The magnitudes of \(\{a_{i,m}\}_{i=1}^m\) play a crucial role in the asymptotic analysis on \(\tilde{R}(\cdot)\) since they control the speed of PFA via \(\eta\) to \(Z\) and affect the dependence structure, i.e., entries of \(A\), among components of \(v\). Most importantly, \(\{a_{i,m}\}_{i=1}^m\) affect the remainders of the Taylor expansion used in [4]. Let \(T_m = (\gamma_{ij})_{m \times m}, D_m = \text{diag}\{\lambda_{1,m}, \ldots, \lambda_{m,m}\}\) and \(\Sigma = \Sigma_m\), where the subscript \(m\) denotes the dimension of matrices. We give an example where \(\omega_{i,m} = 1\), i.e., \(a_{i,m} = \infty\) for \(1 \leq i \leq k\) for some \(k\).

**Lemma 2.1.** For all even \(m \geq 4\) and any \(\mu \in \mathbb{R}^m\), there exists \(\Sigma_m > 0\), a block diagonal (not diagonal) matrix, such that \(Z \sim N_m(\mu, \Sigma_m)\) obeys (2.1) and that (2.2) holds with \(k = 2^{-1}m, C = 1\) and \(\delta \in (0, 2^{-1})\) but \(a_{i,m} = \infty\) for \(1 \leq i \leq k\) and \(a_{i,m} = 1\) for \(k + 1 \leq i \leq m\).

**Proof.** First, we construct the needed positive eigenvalues \(\{\lambda_{i,m}\}_{i=1}^m\) (with \(\lambda_{i,m} \geq \lambda_{i+1,m}\)). Pick \(\delta \in (0, 2^{-1})\), \(k = 2^{-1}m\) and \(\{\varepsilon_j\}_{j=1}^k\) such that \(0 < \varepsilon_j < \varepsilon_{j+1} < 1\) for all \(1 \leq j \leq k-1\). Let \(\lambda_{k+j,m} = 1 - \varepsilon_j\) and \(\lambda_{j,m} = 1 + \varepsilon_j\)
for $1 \leq j \leq k$. Then $\sum_{i=1}^{m} \lambda_{i,m} = m$. Since $\min\{\sqrt{2}m^{1/2-\delta}, 1\} = 1$ whenever $m \geq 1$ and

$$\lambda_{k+1,m} \leq \sqrt{2}m^{1/2-\delta}$$

implies

$$m^{-1}\sqrt{\sum_{j=k+1}^{m} \lambda_{j,m}^2} \leq 2^{-1/2}m^{-1/2}\lambda_{k+1,m} \leq m^{-\delta},$$

it follows that the choice of $\{\lambda_{i,m}\}_{i=1}^{m}$, $C$, $\delta$ and $k$ validates (2.2).

Next, we construct the desired $\Sigma_{m}$ and $Z$. Keep $k = 2^{-1}m$. Let $Q_1 \in O_k$ and $Q_2 \in O_{m-k}$, where $O_n$ denotes the set of $n \times n$ orthogonal matrices. Define $T_m = \text{diag} \{Q_1, Q_2\}$. Then, $T_m = (\gamma_{ij})_{m \times m}$ is orthogonal such that $\max_{1 \leq i \leq k} \max_{k+1 \leq j \leq m} \gamma_{ij} = 0$ but $\sum_{j=k+1}^{m} \gamma_{ij}^2 = 1$ for all $k+1 \leq i \leq m$.

Now setting $Z = \mu + T_m\sqrt{\Sigma_m}w$ for any $\mu \in \mathbb{R}^m$ gives $Z \sim N_m(\mu, \Sigma_m)$ with $\Sigma_m = T_m\Sigma_m T_m^T$. Obviously, $Z$ admits decomposition (2.1) with this $k = 2^{-1}m$ and $\Sigma_m > 0$ is a block diagonal matrix. However, $\omega_{i,m} = 1$ for $1 \leq i \leq k$ and $\omega_{i,m} = 0$ for $k+1 \leq i \leq m$, meaning that $a_1 = \ldots = a_k = \infty$ and $a_{k+1} = \ldots = a_m = 1$. This completes the proof.

It can be easily seen that $\Sigma_m > 0$ is a block diagonal matrix if and only $T_m$ is so. The Normal rv $Z \sim N_m(\mu, \Sigma_m)$ provided in Lemma 2.1 has a block diagonal correlation matrix $\Sigma_m$ and has a decomposition $Z = \mu + \eta + v$ where $\eta_i = 0$ a.s. for $1 \leq i \leq k$ and $v_i = 0$ a.s. for $k+1 \leq i \leq m$. Such Normal rv’s $Z$ present a simpler case for multiple testing which $\mu_i$’s are zero since $(z_1, \ldots, z_k)^T$ are independent of $(z_{k+1}, \ldots, z_m)^T$, the latter of which has weakly dependent components as defined by [4].

We have the following example for which each $a_{i,m}$ for $1 \leq i \leq m$ is finite for any finite $m$: 
Lemma 2.2. For any $m \geq 2$, there exists an orthogonal matrix $T_m = (\gamma_{ij})_{m \times m}$ such that $\gamma_{ij} \neq 0$ for all $1 \leq i \leq j \leq m$. Thus, for any $\mu \in \mathbb{R}^m$ there exists $\Sigma_m > 0$ for which $Z \sim N_m (\mu, \Sigma_m)$ admits (2.1) for any $1 \leq k \leq m$ such that $1 < a_{i,m} < \infty$ for each $1 \leq i \leq m$ and finite $m$. In particular, for $m \geq 4$ even, $\Sigma_m > 0$ can be chosen to induce (2.2) with $k = 2^{-1}m$, $C = 1$ and $\delta \in (0, 2^{-1})$.

Proof. First, we show the existence of such a $T_m$. Let $S_{m-1} = \{ x \in \mathbb{R}^m : \| x \| = 1 \}$.

Denote by $\langle \cdot, \cdot \rangle$ the inner product in Euclidean space and by $\perp$ the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$. Pick $u = (u_1, \ldots, u_m)^T \in S_{m-1}$ such that $0 < \min_{1 \leq i \leq m} |u_i| < \max_{1 \leq i \leq m} |u_i| < 1$ and $2u_i^2 \neq 1$ for all $1 \leq i \leq m$. Define $\Pi = \{ x \in \mathbb{R}^m : \langle x, u \rangle = 0 \}$. Then $\Pi$ is a hyperplane in $\mathbb{R}^m$ with normal $u$. Let $L = \{ xu : x \in \mathbb{R} \}$. Then $\Pi = L \perp$. Let $\tilde{T}_m$ be the reflection with respect to $\Pi$ (that keeps $\Pi$ invariant but flips $u$). Then $\tilde{T}_m x = x - 2 \langle x, u \rangle u$ for all $x \in \mathbb{R}^m$. In particular, $\tilde{T}_m e_i = e_i - 2 \langle e_i, u \rangle u = e_i - 2u_i u$, where $e_i \in \mathbb{R}^m$ has the only non-zero entry, 1, at its $i$th entry. By the construction of $u$, for each $1 \leq i \leq m$ each entry of $\tilde{T}_m e_i$ is non-zero. Consequently, the matrix $T_m$ with the $m$ columns $\gamma_i = \tilde{T}_m e_i = (\gamma_{i1}, \ldots, \gamma_{im})^T$ is orthogonal and none of the $\gamma_{ij}$'s is zero.

Now we construct the needed $\Sigma_m > 0$. Take any $m$ positive numbers $\{ \lambda_{i,m} \}_{i=1}^m$ and set $D_m = \text{diag} \{ \lambda_{1,m}, \ldots, \lambda_{m,m} \}$. Then $Z = \mu + T_m \sqrt{D_m} w$ for any $\mu \in \mathbb{R}^m$ satisfies $Z \sim N_m (\mu, \Sigma_m)$ with $\Sigma_m = T_m D_m T_m^T$. Further, (2.1) holds and $1 < a_{i,m} < \infty$ for each each $1 \leq i \leq m$ and all finite $m$. Specifically, for $m \geq 4$ even, if $\{ \lambda_{i,m} \}_{i=1}^m$ is chosen to be those given in the
proof of Lemma 2.1, then (2.2) holds with the desired $k$, $C$ and $\delta$. The proof is completed.

Let $O_n$ denote the set of $n \times n$ orthogonal matrices and $S_n$ the set of $n \times n$ positive-definite matrices. We provide the third example where $\limsup_{m \to \infty} a_{i,m} < \infty$ for some $1 \leq i \leq m$.

**Corollary 2.1.** For $m \geq 4$ even and $\mu_m \in \mathbb{R}^m$, there exists $\{\Sigma_m\}_{m \geq 4}$ with $\Sigma_m \in S_n$ such that the following hold:

1. $\liminf_{m \to \infty} \lambda_{m,m} = \lambda_0$ for some $\lambda_0 > 0$.
2. For each $Z_m \sim N_m (\mu_m, \Sigma_m)$, (2.1) holds for each $1 \leq k \leq m$ and (2.2) holds with $k = 2^{-1}m$, $C = 1$ and $\delta \in (0, 2^{-1})$.
3. $1 < a_{i,m} < \infty$ for each $1 \leq i \leq m$ and finite $m$ but $\limsup_{m \to \infty} a_{m,m} < \infty$.

**Proof.** We will employ the techniques used in the previous proofs. Take the eigenvalues $\{\lambda_{i,m}\}_{i=1}^m$ constructed in the proof of Lemma 2.1 but restrict $\varepsilon_{2^{-1}m}$ to be such that $\liminf_{m \to \infty} \varepsilon_{2^{-1}m} = \varepsilon_0$ for some $\varepsilon_0 > 0$. Take the $\mathbf{u}$ and $\tilde{T}_m$ constructed in the proof of Lemma 2.2 but let $u_m = u_0$ for a fixed, small positive constant $u_0$ (e.g., $u_0 = 10^{-5}$ can be used); take the $T_m = (\gamma_{ij})_{m \times m}$ induced by $\tilde{T}_m$ under the canonical orthonormal basis $\{e_i\}_{i=1}^m$ such that the $i$th column of $T_m$ is $\tilde{T}_m e_i$. Then none of the entries $\gamma_{ij}$ of $T_m$ is zero, $\gamma_{im} = -2u_i u_0$ for $1 \leq i \leq m - 1$ but $\gamma_{mm} = 1 - 2u_0^2$. So,

$$\sigma_{m,m} = \sum_{j=k+1}^m \lambda_{j,m} \gamma_{mj}^2 \geq \lambda_{m,m} \gamma_{mm}^2 = \lambda_{m,m} (1 - 2u_0^2)^2$$

and

$$\liminf_{m \to \infty} \sigma_{m,m} \geq (1 - \varepsilon_0) (1 - 2u_0^2)^2.$$
Therefore,

\[ \limsup_{m \to \infty} a_{m,m} = \left( \liminf_{m \to \infty} \sigma_{m,m}^{1/2} \right)^{-1} \leq \frac{1}{(1 - 2u_0^2) \sqrt{1 - \varepsilon_0}} < \infty. \]

The proof is completed. \qed

In fact, we can further construct more elaborate sequence of \( \{Z_m\}_m \) with \( Z_m \sim N_m (\mu_m, \Sigma_m) \) such that among \( \{a_{i,m}\}_i \) all the following three types of behaviour occur for some \( 1 \leq i, i', i'' \leq m \):

1. \( 1 < a_{i,m} < \infty \) for each \( m \) but \( \limsup_{m \to \infty} a_{i,m} < \infty \).
2. \( 1 < a_{i',m} < \infty \) for each \( m \) and \( \lim_{m \to \infty} a_{i',m} = \infty \).
3. \( a_{i'',m} = \infty \) for some finite \( m \).

**Corollary 2.2.** For large and even \( m \geq 8 \) and \( \mu_m \in \mathbb{R}^m \), there exists \( \{\Sigma_m\}_{m \geq 8} \) with \( \Sigma_m \in \mathcal{S}_n \) such that the following hold:

1. \( \liminf_{m \to \infty} \lambda_{m,m} = \lambda_0 \) for some \( \lambda_0 > 0 \).
2. For each \( Z_m \sim N_m (\mu_m, \Sigma_m) \), (2.1) holds for each \( 1 \leq k \leq m \) and (2.2) holds with \( k = 2^{-1}m, C = 1 \) and \( \delta \in (0, 2^{-1}) \).
3. \( 1 < a_{i,m} < \infty \) for each \( 1 \leq i \leq m - 1 \) and finite \( m \) and but \( a_{m,m} = \infty \).
4. \( \lim_{m \to \infty} a_{1,m} = \infty \) and \( \limsup_{m \to \infty} a_{k+1,m} < \infty \).

**Proof.** Take the eigenvalues \( \{\lambda_{i,m}\}_{i=1}^m \) constructed in the proof of Corollary 2.1. Take \( u = (u_1, ..., u_m)^T \in S^{m-1} \) such that \( u_{k+1} = \tilde{u}_0 \) for some fixed, small constant \( 0 < \tilde{u}_0 < 8^{-1} \), \( u_m = 2^{-1}\sqrt{2} \), \( u_i = 0 \) for \( i = k + 2, ..., m - 1 \), and \( u_i > 0 \) for \( 1 \leq i \leq k \) but \( \lim_{m \to \infty} u_1 = 0 \). Define \( \Pi \) and \( L \) as in Lemma 2.2 and let \( \tilde{T}_m \) be the reflection with respect to \( \Pi \). Then \( \tilde{T}_m e_i = e_i - 2u_i u \). Let
the matrix $T_m$ have its $i$th column $\gamma_i = \tilde{T}_i e_i$. We see that

$$\sigma_{k+1,m} = \lambda_{k+1,m} \left( 1 - 2u_{k+1}^2 \right)^2 + \lambda_{m,m} \left( 2u_{k+1}u_m \right)^2 \geq (1 - \varepsilon_0) \left[ (1 - 2\tilde{u}_0^2)^2 + 2\tilde{u}_0^2 \right]$$

and $\sigma_{k+1,m} < 1$ for any such $m$. However, $\sigma_{1,m} = 4\lambda_{k+1,m}u_1^2\tilde{u}_0^2 + 2^{-1}\lambda_{m,m}u_1^2$, $\sigma_{m,m} = 0$ and $0 < \sigma_{i,m} < \infty$ for $i \notin \{ k + 1, m \}$ for any such $m$. Therefore,

$$\limsup_{m \to \infty} a_{k+1,m} = \left( \liminf_{m \to \infty} \sigma_{k+1,m}^{1/2} \right)^{-1} < \infty$$

and $\lim_{m \to \infty} a_{1,m} = \left( \liminf_{m \to \infty} \sigma_{1,m}^{1/2} \right)^{-1} = \infty$. The proof is completed.

The examples we constructed demonstrate the different behaviour of $\{a_{i,m}\}_{i=1}^m$ in the PFA to $Z_m \sim N_m(\mu_m, \Sigma_m)$ as $m$ changes and hence the need for a careful analysis of the terms involving $a_{i,m}$’s and ratios between the $a_{i,m}$’s. However, such an analysis has not been explicitly and carefully conducted in [4].

3. Review of strategy in [4]. Let $\mathbb{E}$ be the expectation operator. Before we start reviewing the strategy in [4] to show the SLLN for $\tilde{R}(\cdot)$, we quote as Lemma 3.1 Theorem 6 of [6] as follows:

**Lemma 3.1 ([6]).** Let $\{X_n\}_{n=1}^\infty$ be a sequence of real-valued random variables such that $\mathbb{E} \left[ |X_n|^2 \right] \leq 1$. If $|X_n| \leq 1$ a.s. and

$$\sum_{N=1}^\infty N^{-1} \mathbb{E} \left[ \left| N^{-1} \sum_{n=1}^N X_n \right|^2 \right] < \infty,$$

then $\lim_{m \to \infty} N^{-1} \sum_{n=1}^N X_n = 0$ a.s.
By (2.2) and the Cauchy-Schwarz inequality,

\[ m^{-2} \sum_{1 \leq i \leq j \leq m} |q_{ij}| \leq \vartheta m \leq Cm^{-\delta}, \]

where \( q_{ij} = \text{cov}(v_i, v_j) \) and \( \rho_{ij} = \frac{q_{ij}\sigma_{i,m}^{-1/2}\sigma_{j,m}^{-1/2}}{\sigma_{i,m}\sigma_{j,m}} \) if \( \sigma_{i,m}\sigma_{j,m} \neq 0 \) and \( \rho_{ij} = 0 \) otherwise. Namely, \( \{v_i\}_{i=1}^m \) are “weakly dependent” as termed in [4]. This implies that \( Z \) conditional on \( \eta \) (or equivalently \( \tilde{w}_k \)) has weakly dependent components. Hence it may be possible to apply Lemma 3.1 to \( \{X_i\}_{i=1}^m \) with \( X_i = 1_{\{p_i \leq t\tilde{w}_k\}} - \mathbb{E} \left[ 1_{\{p_i \leq t\tilde{w}_k\}} \right] \) as \( m \to \infty \) to yield the SLLN for \( \tilde{R}(t) = m^{-1} \sum_{i=1}^m X_i \).

Since

\[ \mathbb{V} \left[ m^{-1} \sum_{i=1}^m X_i \right] = m^{-2} \sum_{i=1}^m \mathbb{V}(X_i) + 2m^{-2} \sum_{1 \leq i < j \leq m} \text{cov}(X_i, X_j) \]

\[ \leq 4m^{-1} + 2m^{-2} \sum_{1 \leq i < j \leq m} \text{cov}(X_i, X_j), \]

we see that if

\[ 2m^{-2} \sum_{1 \leq i < j \leq m} \text{cov}(X_i, X_j) \leq M m^{-\delta_1} \]

for some \( \delta_1 > 0 \), then

\[ \sum_{m=1}^{\infty} m^{-1} \mathbb{V} \left[ m^{-1} \sum_{i=1}^m X_i \right] \leq 4 \sum_{m=1}^{\infty} m^{-2} + 2 \sum_{m=1}^{\infty} m^{-1-\delta_1} < \infty, \]

where \( M > 0 \) is a generic constant that can assume different (and appropriate) values at different occurrences. Once (3.4) is established, Lemma 3.1 immediately implies

\[ \lim_{m \to \infty} \left| \tilde{R}(t) - \mathbb{E} \left[ \tilde{R}(t) \right] \right| = 0 \text{ a.s.} \]

To establish (3.3), [4] applied Taylor expansion with respect to \( q_{ij}^{1/2} \) (when \( q_{ij} > 0 \)) to each term

\[ \zeta_{ij} = \mathbb{P}(p_i \leq t, p_j \leq t|\tilde{w}_k), i \neq j \]
and this produces a total of $2^{-1}m (m - 1)$ functional remainders whose various orders are mainly affected by the magnitudes of $\{a_{i,m}\}_{i=1}^m$, the ratios between certain pairs among $\{a_{i,m}\}_{i=1}^m$, and the corresponding functional mean values. However, it seems that [4] failed to fully (and correctly) handle all these remainders for which $|\rho_{ij}|$ are close 1 and $a_{i,m}^{-1}a_{j,m}^{-1}$ (or $a_{i,m}^{-1}a_{j,m}^{-1}$) grows exponentially fast as $m$ increases.

4. SLLN for normalized number of conditional rejections. We are ready to provide the extra arguments for the complete proof of the SLLN for $\tilde{R}(\cdot)$. We state the key result of [4] in our notations as follows:

**Proposition 4.1 ([4]).** Suppose $Z_m \sim N_m(\mu_m, \Sigma_m)$ with $\Sigma_m \geq 0$. Choose an appropriate $k$ such that (2.2) holds for some $\delta > 0$. Then

$$\forall \left[ m^{-1}R(t|\tilde{w}_k) \right] = O_P\left( m^{-\delta} \right)$$

and

$$\mathbb{P}\left\{ \lim_{m \to \infty} \left| m^{-1}R(t|\tilde{w}_k) - \mathbb{E} \left[ m^{-1}R(t|\tilde{w}_k) \right] \right| = 0 \right\} = 1.$$

To present our proof, we introduce some notations and sets. For any two positive sequences $\{x_m, y_m : m \geq 1\}$, we write $x_m \asymp y_m$ if and only if $0 < \liminf_{m \to \infty} x_my_m^{-1} \leq \limsup_{m \to \infty} x_my_m^{-1} < \infty$. Define three sets for the magnitudes of $\{a_{i,m}\}_{i=1}^m$ as

$$E_1 = \{ i \in \mathbb{N} : a_{i,m} < \infty \text{ for any } m \text{ and } \lim_{m \to \infty} a_{i,m} = \infty \},$$

$$E_2 = \{ i \in \mathbb{N} : a_{i,m} = \infty \text{ for some } m \},$$

$$E_3 = \{ i \in \mathbb{N} : \limsup_{m \to \infty} a_{i,m} < \infty \},$$

where it is understood that $a_{i,m}$ is undefined when $i > m$ for each finite $m$. Additionally, define the set counting how many pairs of $v_i$ and $v_j$ can be
highly dependent but not necessarily linearly dependent a.s. as

$$S_{\varepsilon,m} = \{(i,j) : 1 \leq i < j \leq m, |\rho_{ij}| > 1 - \varepsilon\}$$

for $\varepsilon \in (0,1)$ since $m^{-2} \sum_{1 \leq i \leq j \leq m} |q_{ij}| = O\left(m^{-\delta}\right)$ does not necessarily imply

$$m^{-2} \sum_{1 \leq i \leq j \leq m} |\rho_{ij}| = O\left(m^{-\delta}\right).$$

Proposition 4.1 is reformulated as follows:

**Theorem 4.1.** Let $Z_m \sim N_m(\mu_m, \Sigma_m)$ with $\Sigma_m \geq 0$ and choose $k = k_m$ such that (2.2) holds for some $\delta > 0$. Suppose the following hold:

1. When $E_1 \neq \emptyset$, there exists some $q \in \mathbb{N}$ independent of $m$ such that

   $$a_{(m)} = a_{(1)}^q,$$

   where $a_{(1)} = \min_{i \in E_1, 1 \leq i \leq m} a_{i,m}$ and $a_{(m)} = \max_{i \in E_1, 1 \leq i \leq m} a_{i,m}$.

2. For some $\tilde{\varepsilon} \in (0,1)$,

   $$\limsup_{m \to \infty} m^{-2+\delta} |S_{\tilde{\varepsilon}}| < \infty.$$

Then,

$$\mathbb{V} \left[ \tilde{R}(t) \right] = O\left(m^{-\delta}\right)$$

holds except on the event

$$G_{t,\varepsilon} = \bigcup_{m \geq 1} \bigcup_{i \in E_1, 1 \leq i \leq m} \{\omega \in \Omega : |\pm z_{t/2} - \eta_i - \mu_i| < \varepsilon\}$$

for any $\varepsilon > 0$, where $z_{t/2} = \Phi^{-1}(t/2)$ and $\Phi^{-1}$ is the inverse of $\Phi$. 
Proof. Let \( Y_i = 1 - X_i \). Then \( \text{cov}(Y_i, Y_j) = \text{cov}(X_i, X_j) \) and (3.2) is equivalent to

\[
\forall \left[ m^{-1} \sum_{i=1}^{m} X_i \right] \leq 4m^{-1} + 2m^{-2} \sum_{(i,j) \in S_{\varepsilon,m}} \text{cov}(Y_i, Y_j) \\
+ 2m^{-2} \sum_{(i,j) \in S_{\varepsilon,m}^C} \text{cov}(Y_i, Y_j) \\
\leq 4m^{-1} + Mm^{-\delta} + 2m^{-2} \sum_{(i,j) \in S_{\varepsilon,m}^C} \text{cov}(Y_i, Y_j),
\]

since (4.4) implies \( m^{-2+\delta}|S_{\varepsilon,m}| < M \) for all \( m \in \mathbb{N} \), where \( A^C \) denotes the complement of a set \( A \). This means we only need to show

\[
2m^{-2} \sum_{(i,j) \in S_{\varepsilon,m}^C} \text{cov}(Y_i, Y_j) = O \left( m^{-\delta_1} \right)
\]

for some \( \delta_1 > 0 \). Since \( \text{cov}(X_i, X_j) = 0 \) for all \( 1 \leq j \leq m \) when \( i \in E_2 \), the inequality in (4.7) is not affected by such summands (asymptotically).

Therefore, it suffices to show

\[
2m^{-2} \sum_{(i,j) \in I_{\varepsilon,m}} \text{cov}(Y_i, Y_j) = O \left( m^{-\delta_1} \right),
\]

where

\[
I_{\varepsilon,m} = \{(i, j) : (i, j) \in S_{\varepsilon,m}^C \cap ((E_1 \otimes E_1) \cup (E_3 \otimes E_3))\}
\]

and \( \otimes \) denotes the Cartesian product. We can break \( I_{\varepsilon,m} = I_{\varepsilon,m}^+ \cup I_{\varepsilon,m}^- \) and show

\[
2m^{-2} \sum_{(i,j) \in I_{\varepsilon,m}^+} \text{cov}(Y_i, Y_j) = O \left( m^{-\delta_1} \right)
\]

and

\[
2m^{-2} \sum_{(i,j) \in I_{\varepsilon,m}^-} \text{cov}(Y_i, Y_j) = O \left( m^{-\delta_1} \right),
\]
where \( I^+_{\xi,m} = \{(i,j) \in I_{\xi,m} : \rho_{ij} \geq 0\} \) and \( I^-_{\xi,m} = \{(i,j) \in I_{\xi,m} : \rho_{ij} < 0\} \).

Since the techniques of proving (4.9) and (4.10) are the same, in the sequel we will just show (4.9).

Let \( h_1(x) = (b^2 - x^2)^{-1/2} \) for \(|b| \neq |x|\). Then

\[
(4.11) \quad h_1(x) = h(0) + h_1'(\theta)(x - 0) = b^{-1} + \frac{\theta}{(b^2 - \theta^2)^{3/2}} x
\]

for some \( \theta \) strictly between 0 and \( x \), where \( h_1'(\theta) = \theta (b^2 - \theta^2)^{-3/2} \). Set \( o(x) = h_1'(\theta) x \), which does not necessarily satisfy \( \lim_{x \to 0} |x^{-1} o(x)| = 0 \) if \( \lim_{x \to 0} b = 0 \). By Taylor expansion, we have, for \(|b| \neq |x|\),

\[
\Phi \left( \frac{xz + bc}{(b^2 - x^2)^{1/2}} \right) = \Phi (c + r_1(c,x)) = \Phi (c) + \phi(c) xzb^{-1} + \phi(c) (xz + bc) o(x) + \frac{1}{2} [-\theta_* \phi(\theta_*)] r_1^2(c,x) = \Phi (c) + \phi(c) xzb^{-1} + \tilde{o}(c,x),
\]

for some \( \theta_* \) strictly between \( c \) and \( c + r_1(c,x) \), where \( \phi(t) = \frac{d}{dt} \Phi(t) \), \( \tilde{o}(c,x) = r_2(c,x) + r_3(c,x) \) and

\[
(4.12) \quad \begin{cases} 
  r_1(c,x) = xzb^{-1} + (xz + bc) o(x), \\
  r_2(c,x) = -\frac{1}{2\sqrt{2\pi}} \theta_* e^{-\theta_*^2/2} r_1^2(c,x), \\
  r_3(c,x) = \phi(c) (xz + bc) o(x).
\end{cases}
\]

We will now omit the subscript \( m \) in \( a_{i,m} \) and write \( a_{i,m} \) as \( a_i \). Set \( b_{ij} = a_i^{-1} a_j^{-1}, r_{1,i} = -z_{t/2} - \eta_i - \mu_i, r_{2,i} = z_{t/2} - \eta_i - \mu_i \) and \( c_{i,l} = a_i r_{l,i} \) with \( l = 1, 2 \).

Since \( z_{t/2} \leq 0 \) for \( t \in [0,1] \), then \( p_i \geq t \) if and only if \( c_{2,i}/a_i \leq v_i \leq c_{1,i}/a_i \).
Using the formula from [7] or page 192 of [8] for $0 \leq \rho_{ij} < 1$, we have

\[(4.13) \quad \mathbb{E}[Y_i Y_j] = \mathbb{P}(\{p_i > t, p_j > t | \tilde{w}_k\})\]

\[= \mathbb{P}(\{c_{2,i}/a_i < v_i < c_{1,i}/a_i, c_{2,j}/a_j < v_j < c_{1,j}/a_j\})\]

\[= \int_{-\infty}^{+\infty} [\varphi(\rho_{ij}, c_{1,i}, z) - \varphi(\rho_{ij}, c_{2,i}, z)] \times [\varphi(\rho_{ij}, c_{1,j}, z) - \varphi(\rho_{ij}, c_{2,j}, z)] \phi(z) dz,\]

where $\varphi(x_1, x_2, z) = \Phi\left(\frac{\sqrt{x_1}z + x_2}{\left(1 - x_1\right)^{1/2}}\right)$ for $0 \leq x_1 < 1$. Set $b = b_{ij}^{1/2}$ and $x = q_{ij}^{1/2}$, for which a summation involving $x$ and/or $b$ is summing $q_{ij}^{1/2}$ and/or $b_{ij}^{1/2}$ over certain pairs of $(i, j)$. Applying Taylor expansion to $\varphi(\rho_{ij}, c_{l,i}, z)$ but only to the second order with respect to $x$ (in [4] this expansion is to the third order) gives

\[\varphi(\rho_{ij}, c_{l,i}, z) = \Phi\left(\frac{xz + bc_{l,i}}{b^2 - x^2}^{1/2}\right) = \Phi(c_{l,i}) + \phi(c_{l,i}) b^{-1}xz + \tilde{o}(c_{l,i}, x)\]

and

\[d_i = \varphi(\rho_{ij}, c_{1,i}, z) - \varphi(\rho_{ij}, c_{2,i}, z)\]

\[= \Phi(c_{1,i}) + \phi(c_{1,i}) b^{-1}xz + [-\Phi(c_{2,i}) - \phi(c_{2,i}) b^{-1}xz] + \Delta\tilde{o}_i(c, x)\]

\[= [\Phi(c_{1,i}) - \Phi(c_{2,i})] + [\phi(c_{1,i}) - \phi(c_{2,i})] b^{-1}xz + \Delta\tilde{o}_i(x)\]

\[= \Delta\Phi(c_i) + \Delta\phi(c_i) b^{-1}xz + \Delta\tilde{o}_i(x),\]

where the corresponding functional mean value $\theta$ in (4.12) is denoted by $\theta_{l,i}$, the mean value $\theta$ in (4.11) by $\theta_i$, and

\[
\begin{cases}
\Delta\Phi(c_i) = \Phi(c_{1,i}) - \Phi(c_{2,i}), \\
\Delta\phi(c_i) = \phi(c_{1,i}) - \phi(c_{2,i}), \\
\Delta\tilde{o}_i(x) = \tilde{o}(c_{1,i}, x) - \tilde{o}(c_{2,i}, x).
\end{cases}
\]
Therefore
\[
d_{i}d_{j} = \Delta \Phi (c_{i}) \Delta \Phi (c_{j}) + \Delta \Phi (c_{i}) \Delta \phi (c_{j}) b^{-1} x z + \Delta \Phi (c_{i}) \Delta \tilde{o}_{j} (x)
\]
\[
+ \Delta \phi (c_{i}) b^{-1} x z \Delta \Phi (c_{j}) + \Delta \phi (c_{i}) \Delta \phi (c_{j}) b^{-2} x^{2} z^{2} + \Delta \phi (c_{i}) b^{-1} x z \Delta \tilde{o}_{j} (x)
\]
\[
+ \Delta \tilde{o}_{i} (x) \Delta \Phi (c_{j}) + \Delta \tilde{o}_{i} (x) \Delta \phi (c_{j}) b^{-1} x z + \Delta \tilde{o}_{i} (x) \Delta \tilde{o}_{j} (x)
\]
and
\[
\int d_{i}d_{j} \phi (z) dz = \Delta \Phi (c_{i}) \Delta \Phi (c_{j}) + \Delta \phi (c_{i}) \Delta \phi (c_{j}) b^{-2} x^{2} + \int g_{ij} (\cdot) \phi (z) dz,
\]
where
\[
g_{ij} (\cdot) = \Delta \Phi (c_{i}) \Delta \tilde{o}_{j} (x) + \Delta \tilde{o}_{i} (x) \Delta \Phi (c_{j}) + \Delta \tilde{o}_{i} (x) \Delta \tilde{o}_{j} (x).
\]
Consequently
\[
\int d_{i}d_{j} \phi (z) dz = \Delta \Phi (c_{i}) \Delta \Phi (c_{j}) + \Delta \phi (c_{i}) \Delta \phi (c_{j}) b^{-2} x^{2} + \int g_{ij} (\cdot) \phi (z) dz.
\]
It is easy to see that the set \( \left\{ \tilde{w}_{k} : \eta = z_{t/2} + \mu_{m} \text{ or } \eta = -z_{t/2} + \mu_{m} \right\} \) is \( \mathbb{P} \)-null. Therefore,
\[
P (D^{*}) \leq \lim_{m \to \infty} \sum_{i=1}^{m} P \left( \bigcup_{l=1}^{2} \{ r_{l,i} = 0 \} \right) = 0,
\]
where \( D^{*} = \lim_{m \to \infty} \bigcup_{l=1}^{m} \bigcup_{l=1}^{2} \{ r_{l,i} = 0 \} \). Further, on the event \( \tilde{D} = (D^{*} \cup G_{t,\varepsilon})^{C} \) with \( G_{t,\varepsilon} \) defined in (4.6), we have
\[
\lim_{m \to \infty} \min_{i \in E_{1}, 1 \leq i \leq m} \min \{ |r_{1,i}|, |r_{2,i}| \} \geq \varepsilon > 0.
\]
In what follows, we will just work on the event \( \tilde{D} \), where (4.17) will help us obtain uniform boundedness of the functional coefficients in (4.15).

From (4.17), we see
\[
\sup_{m \geq 1} \max_{(i,j) \in T_{t,m}^{+}} \sup_{(r_{1,i}, r_{2,i}, a_{i})} \{ |\phi (c_{1,i}) - \phi (c_{2,i})| a_{i} \} \leq M.
\]
Thus

\[(4.18) \quad m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} |\Delta \phi (c_i) \Delta \phi (c_j)| b^{-2} x^2 \leq M m^{-2\delta}.\]

Clearly,

\[|g_{ij}(\cdot)| = |\Delta \Phi (c_i) \Delta \tilde{o}_j (x) + \Delta \tilde{o}_i (x) \Delta \Phi (c_j) + \Delta \tilde{o}_i (x) \Delta \tilde{o}_j (x)| \leq 2 |\Delta \tilde{o}_j (x)| + 2 |\Delta \tilde{o}_i (x)| + |\Delta \tilde{o}_i (x) \Delta \tilde{o}_j (x)|.\]

If \(\Delta \tilde{o}_j (x)\) for \(j = 1, \ldots, m\) are dominated by a polynomial \(g^* (|x|, |z|)\) with uniformly bounded functional coefficients without the constant term, then

\[(4.19) \quad m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \int |\Delta \tilde{o}_j (x)| \phi (z) \, dz \leq m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \tilde{g} (|x|) \leq M m^{-\beta_0\delta},\]

for \(0 < \beta_0 \leq 2\), where \(\tilde{g} (\cdot)\) is a polynomial in \(|x|\) and \(\beta_0 > 0\) is the smallest degree of \(|x|\) in \(g^* (|x|, |z|)\) and we have used \((3.1)\) and

\[(4.20) \quad m^{-2} \sum_{1 \leq i \leq j \leq m} |q_{ij}|^\beta = O \left( m^{-\delta \min (\beta, 2)} \right)\]

for all \(\beta \in (0, 2]\) derived from \((2.2)\) by Hölder’s inequality. Obviously, \((4.19)\) implies

\[(4.21) \quad m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \int |\Delta \tilde{o}_i (x)| |\Delta \tilde{o}_j (x)| \phi (z) \, dz \]

\[\leq m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \left( \int |\Delta \tilde{o}_i (x)|^2 \phi (z) \, dz \right)^{1/2}
\times \left( \int |\Delta \tilde{o}_j (x)|^2 \phi (z) \, dz \right)^{1/2}
\leq M m^{-2\beta_0\delta},\]

from which \((4.9)\) holds with \(\delta_1 = 2\beta_0\delta\). Therefore, it suffices to justify \((4.19)\).
By their definitions,

\[ \begin{align*}
\bar{o} (c_{l,i}, x) &= \phi (c_{l,i}) x^2 z h'_1 (\theta_i) + \phi (c_{l,i}) b c_{l,i} h'_1 (\theta_i) x - \frac{1}{2} \theta_{l,i} \phi (\theta_{l,i}) r^2_1 (c_{l,i}, x), \\
\Delta \bar{o}_i (x) &= r_{3,1} (c_{i,x}) x^2 z + r_{3,2} (c_{i,x}) x + r_4 (c_{i,x}),
\end{align*} \]

where

\[ \begin{align*}
r_{3,1} (c_{i,x}) &= \phi (c_{1,i}) h'_1 (\theta_i) - \phi (c_{2,i}) h'_1 (\theta_i), \\
r_{3,2} (c_{i,x}) &= [c_{1,i} \phi (c_{1,i}) h'_1 (\theta_i) - c_{2,i} \phi (c_{2,i}) h'_1 (\theta_i)] b, \\
r_4 (c_{i,x}) &= -\frac{1}{2} \theta_{1,i} \phi (\theta_{1,i}) r^2_1 (c_{1,i}, x) + \frac{1}{2} \theta_{2,i} \phi (\theta_{2,i}) r^2_1 (c_{2,i}, x). \\
\end{align*} \]

From

\[ \sup_{m \geq 1} \max_{(i,j) \in I_{\xi,m}^+} |\rho_{i,j}| \leq 1 - \bar{\varepsilon}, \]

it follows that

\[ b^{-2} \tilde{\rho} \geq b^{-2} \frac{|\theta/b|}{1 - (x/b)^2} \left[ 1 - (1 - \bar{\varepsilon})^2 \right]^{-3/2} \geq |h'_1 (\theta)| \geq \frac{1}{b^3}, \]

where

\[ (4.23) \quad \tilde{\rho} = (1 - \bar{\varepsilon}) \left[ 1 - (1 - \bar{\varepsilon})^2 \right]^{-3/2}. \]

Using (4.17), (4.3) and the property of \( \phi (\cdot) \), we have

\[ (4.24) \quad \sup_{m \geq 1} \max_{(i,j) \in I_{\xi,m}^+} \left| \phi (c_{l,i}) h'_1 (\theta_i) \right| \leq M \]

and

\[ (4.25) \quad \sup_{m \geq 1} \max_{(i,j) \in I_{\xi,m}^+} \left| c_{l,i} \phi (c_{l,i}) h'_1 (\theta_i) a_i^{-1/2} a_j^{-1/2} \right| \leq M. \]

So

\[ \sup_{m \geq 1} \max_{(i,j) \in I_{\xi,m}^+} \left| \phi (c_{1,i}) h'_1 (\theta_i) - \phi (c_{2,i}) h'_1 (\theta_i) \right| \leq M. \]
\[
\sup_{m \geq 1} \max_{(i,j) \in I^+_{\varepsilon,m} (r_1, i, r_2, i, a_i, a_j)} |\tau_{ij} (c_i, \theta_i)| \leq M. 
\]

where

\[
\tau_{ij} (c_i, \theta_i) = [c_{1,i} \phi (c_{1,i}) h'_1 (\theta_i) - c_{2,i} \phi (c_{2,i}) h'_1 (\theta_i)] a_i^{-1/2} a_j^{-1/2}. 
\]

Therefore,

\[
(4.27) \quad m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \int |r_{3,1} (c_i, x)| x^2 |z| \phi (z) dz \leq M m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} x^2 = M m^{-2 \delta} 
\]

and

\[
(4.28) \quad m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} \int |r_{3,2} (c_i, x)||x| \phi (z) dz \leq M m^{-2} \sum_{(i,j) \in I^+_{\varepsilon,m}} |x| \leq M m^{-\delta}, 
\]

where we have used the inequality (3.1) and (4.20).

The last term \( \kappa_m = m^{-2} \sum_{1 \leq i < j \leq m} \int |r_4 (c_i, x)| \phi (z) dz \) needs intricate treatment. The summands in \( \kappa_m \) that involve \( a_i \) with \( i \in E_3 \) do not inflate the order of \( \kappa_m \) to be greater than \( m^{-\delta} \) since such \( a_i \)'s are uniformly bounded from both above and below, \( \phi (\cdot) \) is rapidly decreasing and smooth, and (3.1) holds. Therefore, to assess the order of \( \kappa_m \), it suffices to deal with its summands that involve \( a_i \) and \( a_j \) with \((i,j) \in I^+_{\varepsilon,m} = I^+_{\varepsilon,m} \cap (E_1 \otimes E_1) \) and work with the set \( B (\sqrt{|c|}) = \{ z : |z| \leq \sqrt{|c|} \} \), where \( |c| \) can be either \( |c_{1,i}| \)
or \(|c_2, i|\) (which is eventually sufficiently large). Since
\[
\liminf_{|c| \to \infty} \frac{r_1(c, x)}{c} = \liminf_{|c| \to \infty} \left( \frac{xzb^{-1}}{c} + \frac{bh_1'(\theta) xc}{c} + \frac{xzh_1'(\theta) x}{c} \right)
\]
\[
\geq \liminf_{|c| \to \infty} \left( bh_1'(\theta) x + \frac{z}{c} h_1'(\theta) x^2 \right)
\]
\[
\geq \liminf_{|c| \to \infty} bh_1'(\theta) x \geq 0
\]
regardless of the sign of \(x\), we see that \(c\) and \(r_1(c, x)\) have the same sign, i.e., \(cr_1(c, x) \geq 0\). This means either \(c + r_1(c, x) < \theta_2 < c < 0\) or \(0 < c < \theta_2 < c + r_1(c, x)\), and
\[
2 \max \{|r_1(c, x)|, |c|\} \geq |\theta_2| > \max \{|r_1(c, x)|, |c|\}.
\]
Noticing further
\[
\limsup_{m \to \infty} \frac{r_1(c, x)}{c} \leq \limsup_{m \to \infty} \left( 1 + \frac{|z|}{c} \right) |b^2h_1'(\theta)|
\]
\[
\leq \limsup_{m \to \infty} \left( 1 + \frac{|z|}{c} \right) \frac{1}{\left(1 - (1 - \tilde{\varepsilon})^2\right)^{3/2}}
\]
\[
\leq (1 - \tilde{\varepsilon})^{-1}\tilde{\rho} < \infty,
\]
we see
\[
\begin{align*}
|r_1(c, x)| &\leq M (1 - \tilde{\varepsilon})^{-1}\tilde{\rho} |c| , \\
M (1 - \tilde{\varepsilon})^{-1}\tilde{\rho} |c| &\geq |\theta_2| > |c| ,
\end{align*}
\]
and
\[
|r_2(c, x)| \leq \theta_2 e^{-\theta_2^2/2} r_1^2(c, x) \leq M |c|^3 \exp \left(- |c|^2/2 \right),
\]
where we used the fact that $t\phi(t)$ is decreasing in $t > 0$. From

$$r_1^2(c, x)$$

$$= x^2b^{-2}z^2 + 2x^3b^{-1}h_1'(\theta) z^2 + 2x^2ch_1'(\theta) z + x^4h_1'^2(\theta) z^2$$

$$+ 2x^3bch_1'^2(\theta) z + b^2c^2h_1'^2(\theta)x^2$$

$$= r_{1,1}(c, z)x^2 + 2r_{1,2}(c, z)x^3 + h_1'^2(\theta)x^4z^2$$

$$\leq r_{1,1}^*(c, h_1'(\theta))x^2 + r_{1,2}^*(c, h_1'(\theta))|x|^3 + h_1'^2(\theta)|c|x^4,$$

we see

$$\int_{B(\sqrt{c})} |r_2(c, x)| \phi(z) dz \leq M|c|\exp\left(-|c|^2/2 \right) g^{**}(|x|),$$

where

$$g^{**}(|x|) = r_{1,1}^*(c, h_1'(\theta))x^2 + r_{1,2}^*(c, h_1'(\theta))|x|^3 + h_1'^2(\theta)|c|x^4$$

and

$$\begin{cases}
 r_{1,1}(c, z) = b^{-2}z^2 + 2ch_1'(\theta) z + b^2c^2h_1'^2(\theta), \\
r_{1,2}(c, z) = b^{-1}h_1'(\theta)z^2 + bch_1'^2(\theta) z, \\
r_{1,1}^*(c, h_1'(\theta)) = b^{-2}|c| + 2|c|^{3/2}|h_1'(\theta)| + b^2c^2h_1'^2(\theta), \\
r_{1,2}^*(c, h_1'(\theta)) = b^{-1}h_1'(\theta)|c| + |bh_1'^2(\theta)||c|^{3/2}.
\end{cases}$$

Since $\varepsilon a_i \leq |c_i|$ for $l = 1, 2$ and $1 \leq i \leq m$ by (4.17) and $\lim_{m \to \infty} a_i = \infty$, we get

$$\sup_{m \geq 1} \max_{(i,j) \in I_{l,m}} \sup_{c,b} \left\{ |c|^3 \exp\left(-|c|^2/2 \right) b^{-2}h_1'^2(\theta) \right\} \leq M$$

and $M|c|\exp\left(-|c|^2/2 \right) g^{**}(|x|)$ is thus dominated by a polynomial with uniformly bounded coefficients whose lowest degree in $|x|$ is 2. Hence

$$m^{-2} \sum_{(i,j) \in I_{l,m}} \int_{B(\sqrt{c})} |r_2(c, x)| \phi(z) dz \leq Mm^{-2\delta}. \quad (4.29)$$
On the other hand, on $\mathbb{R} \setminus B(\sqrt{c})$ we have

$$
\begin{align*}
|r_1^2(c, x)| & \leq |r_{1,1}(c, z)| x^2 + 2 |r_{1,2}(c, z)| x^3 + h_1^2(\theta) x^4 z^2 \\
& \leq [b^{-2}z^2 + 2 |ch_1'(\theta)| z] + b^2c^2 h_1^2(\theta) x^2 \\
& + \left[2b^{-1}h_1'(\theta) x^2 + 2bch_1^2(\theta) z\right] |x|^3 + h_1^2(\theta) x^4 \\
& \leq \left[b^{-2}|c| + 2 |ch_1'(\theta)||c|^{1/2} + b^2c^2 h_1^2(\theta)\right] x^2 \\
& + 2 \left(b^{-1}h_1'(\theta)|c| + 2bch_1^2(\theta)|c|^{1/2}\right) |x|^3 + h_1^2(\theta) x^4 \\
& \leq M \left(|c|^3 + |h_1'(\theta)||c|^{3/2} + c^2 h_1^2(\theta)\right) x^2 \\
& + M \left(h_1'(\theta)|c|^2 + h_1^2(\theta)|c|^{3/2}\right) |x|^3 + h_1^2(\theta) x^4 \\
& \leq M |c|^3 \left(x^2 + |x|^3 + x^4\right) \leq M |c|^3 x^2
\end{align*}
$$

and

$$
\left(\int_{\mathbb{R} \setminus B(\sqrt{c})} |r_1(c, x)| \phi(z) \, dz\right)^2 \leq \int_{\mathbb{R} \setminus B(\sqrt{c})} |r_1^2(c, x)| \phi(z) \, dz \\
\leq x^2 \int_{\mathbb{R} \setminus B(\sqrt{c})} M |c|^3 \phi(z) \, dz \leq M x^2
$$

since $\lim_{|c| \to \infty} |c|^{s'} \phi(|c|) = 0$ for any $s' > 0$. This means

$$
m^{-2} \sum_{(i,j) \in I_{c,m}} \int_{\mathbb{R} \setminus B(\sqrt{c})} |r_2(c, x)| \phi(z) \, dz \leq M m^{-2} \sum_{(i,j) \in I_{c,m}} |x| = M m^{-\delta},
$$

which, with (4.29), implies

$$
m^{-2} \sum_{(i,j) \in I_{c,m}} \int |r_2(c, x)| \phi(z) \, dz \leq M m^{-\delta}.
$$

Therefore

$$
m^{-2} \sum_{(i,j) \in I_{c,m}} \int |r_4(c_1, x)| \phi(z) \, dz \leq M m^{-\delta}.
$$
Combining (4.27), (4.28) and (4.31) gives

\[
(4.32) \quad m^{-2} \sum_{(i,j) \in I^k_{\delta,m}} \int |\Delta \tilde{\alpha}_i(x)| \phi(z) \, dz \leq Mm^{-\delta}.
\]

Thus (4.19) and (4.21) hold with \( \delta_1 = 2\delta \). Consequently,

\[
(4.33) \quad \lim_{m \to \infty} \left| m^{-1} R(t|\tilde{w}_k) - E \left[ m^{-1} R(t|\tilde{w}_k) \right] \right| = 0
\]

outside the event \( G_{t,\varepsilon} \). This completes the proof.

Using Lemma 3.1, we have the following corollary from Theorem 4.1.

**Corollary 4.1.** The same conditions of Theorem 4.1 imply

\[
(4.33) \quad \lim_{m \to \infty} \left| m^{-1} R(t|\tilde{w}_k) - E \left[ m^{-1} R(t|\tilde{w}_k) \right] \right| = 0
\]

outside the event \( G_{t,\varepsilon} \).

From the proof of Theorem 4.1, we see that among the functional remainders there are mainly two types that can diverge to infinity as \( m \to \infty \):

1. \( \phi(a_{i,m}r_i)h'(\theta_i)a_{i,m} \) for which \( |a_{i,m}r_i| = O(1), a_{i,m} \to \infty \) and \( |h'(\theta_i)| \to \infty \), where \( r_i = \pm z_{t/2} - \eta_i - u_i \).

2. \( \phi(a_{i,m}r_i)h'(\theta_i)a_{i,m}^{-1}a_{j,m} \) for which \( |a_{i,m}r_i| \to \infty, r_i = O(1), a_{i,m}, a_{j,m} \to \infty, h'(\theta_i) \to \infty \) but \( a_{j,m} \geq e^{Ma_{i,m}^2} \) for some \( M > 0 \) large when \( m \) is large enough.

Such terms will very likely inflate \( \mathbb{V} \left[ \hat{R}(t) \right] \) out of the desired order \( m^{-\delta} \) for the SLLN for \( \hat{R}(t) \) to hold. In fact, the event \( G_{t,\varepsilon} \) contains all \( \omega \in \Omega \) for which the above two cases can happen but it does not necessarily have diminishing probability.
We briefly describe the roles the two additional conditions (4.3) and (4.4).

When \( a_{i,m} \to \infty \) but \( a_{i,m} < \infty \) for finite \( m \), \( a_{i,m} \) represents the rate at which \( \tilde{\eta}_i = \eta_i + \mu_i \) stochastically approximates \( Z_i \), and condition (4.3) requires that the relative rate at which \( a_{i,m} \to \infty \) can not be exponential. Condition (4.4) controls how many pairs of \((v_i,v_j)\), \(i \neq j\) can be highly correlated and restricts the contribution of such pairs in the variance of \( m^{-1}R(t|\tilde{w}_k) \). It also controls the “speed” at which \( \Sigma_m > 0 \) can approach to singularity. In addition, it ensures the validity and accuracy of the expansion (4.13) (since (4.13) is undefined when \(|\rho_{ij}| = 1\), and prevents \( h'_1(\theta) \) in (4.22) from diverging to infinity (unexpectedly fast). Further, it induces (4.23), and together with condition (4.3) it validates (4.27), (4.28) and (4.31). These extra sufficient conditions ensure the uniform boundedness of the involved functional remainders in such Taylor expansions and induce (4.5) outside the event \( G_{t,\varepsilon} \).

5. Discussion. Under two more conditions rather than none in [4], we have reformulated the SLLN for the normalized number of conditional rejections when testing which components of a Normal rv \( Z_m \sim N_m(\mu_m, \Sigma_m) \) have zero means when its correlation matrix \( \Sigma_m \) is known. Our proof shows that the speed of the PFA to the original Normal rv, the degree of dependency between components of the Normal rv, and the magnitudes of the conditional component means should be compatible with each other in order to yield such a law. Relaxation of (4.3) and (4.4) is possible. But such a law may not hold for arbitrary \( \mu_m \) and arbitrary \( \Sigma_m \geq 0 \), since, on the event

\[
H_{t,\varepsilon,\tilde{\varepsilon}} = \left\{ \omega \in \Omega : 1 - \tilde{\varepsilon} \leq \rho_{ij} < 1, a_{i,m}^{-2} \log a_{j,m} \geq 1, i, j \in E_1, i \neq j \right\} \cap G_{t,\varepsilon}
\]
when $0 < \max \{\varepsilon, \tilde{\varepsilon}\} \ll 1$,

$$\mathbb{P} \left\{ \lim_{m \to \infty} m^{-2+\delta} \sum_{1 \leq i < j \leq m} \left| \Delta \tilde{\alpha}_j(x) \right| \phi(z) \, dz = \infty \right\} > 0$$

(see (4.14) for the definition of $\Delta \tilde{\alpha}_j(x)$) and

$$\mathbb{P} \left\{ \limsup_{m \to \infty} m^\delta \Psi \left[ m^{-1} R(t|\tilde{\omega}_k) \right] = \infty \right\} > 0$$

(5.1)

can happen for any $\delta > 0$, which then invalidates (4.1) and (4.2). Since estimating the FDP and control of the FDR under other strong but unknown types of dependency among the test statistics still remains a highly challenging problem, such an area of research requires development of new methods and clarification of the boundaries beyond which certain methods become fragile or invalid.

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