ON THE MODULUS OF CONTINUITY FOR SPECTRAL MEASURES IN
SUBSTITUTION DYNAMICS

ALEXANDER I. BUFETOV AND BORIS SOLOMYAK

Abstract. The paper gives first quantitative estimates on the modulus of continuity of the
spectral measure for weakly mixing suspension flows over substitution automorphisms. The main
results are, first, a Hölder estimate for the spectral measure of almost all suspension flows with a
piecewise constant roof function; second, a log-Hölder estimate for self-similar suspension flows;
and, third, a Hölder asymptotic expansion of the spectral measure at zero for such flows. The
second result implies log-Hölder estimates for the spectral measures of translation flows along
stable foliations of pseudo-Anosov automorphisms. The Appendix explains the connection of
these results with the theory of Bernoulli convolutions.

1. Introduction

Substitution dynamical systems and their spectral properties have been studied for a long time,
see [34, 20] and references therein. These systems are of intrinsic interest, but also have many
links to other areas, both in dynamics and beyond. An incomplete list of these links includes, first
of all, Bratteli-Vershik (adic) transformations [40, 41], especially in the stationary case; interval
exchange transformations, which are periodic for the Rauzy-Veech induction, and translation
flows along stable/unstable flows for pseudo-Anosov automorphisms, see [5, 6]. Substitutions and
associated dynamical systems are also widely used in mathematical physics, in particular, in the
study of quasicrystals, see e.g. [2, 24, 21, 32].

The aim of this paper is to estimate the modulus of continuity for the spectral measures of
suspension flows over substitution dynamical systems. Our main assumption is that the substitu-
tion matrix have at least two eigenvalues outside the unit circle, which implies that almost every
suspension flow, in particular, the self-similar suspension flow, is weakly mixing [9, 39].

Our first main result is Theorem 4.1 that gives a uniform Hölder bound away from zero for
the spectral measure of almost all suspension flows with piecewise constant roof functions. This
result does not, however, give specific examples of flows with Hölder spectrum. In the special case
of self-similar suspension flows we are only able to obtain log-Hölder estimates on the spectrum;
these are contained in our second main result, Theorem 5.1. Our third main result, Theorem 6.1
gives a Hölder asymptotic expansion for the spectral measure of our suspension flows at zero; the
Hölder exponent is explicitly computed.
The paper is organized as follows. In Section 2 we represent spectral measures of substitution automorphisms by matrix Riesz products; the construction is the natural generalization of the well-known Riesz product representation of the spectrum for substitutions of constant length, see [34]. In Section 3 we analyze our matrix Riesz products; the main result of this section is Proposition 3.2 an upper estimate on the entries of our Riesz-type matrices. In the following sections, Proposition 3.2 is used to reduce problems about spectral measures of substitutions to problems in Diophantine approximation. In Section 4, we formulate and prove Theorem 4.1 on the Hölder property for the spectrum for almost all suspension flows with piecewise constant roof function; the Hausdorff dimension of the set of exceptional roof functions is estimated in Theorem 4.3 whose proof relies on a generalization of the “Erdős-Kahane argument” in the theory of Bernoulli convolutions. In Section 5 we formulate and prove Theorem 5.1 on the log-Hölder property for self-similar suspension flows. As a corollary, we obtain log-Hölder estimates for the spectral measures of translation flows along stable foliations of pseudo-Anosov automorphisms (Corollary 5.4). The main step in the argument is Proposition 5.6. As an aside, Proposition 5.6 allows us to obtain new upper bounds on Fourier transforms of Bernoulli convolutions (see Corollary 7.5). In Section 6, we give an asymptotics of spectral measure at zero using the asymptotics of ergodic integrals of substitution dynamical systems in terms of finitely-additive invariant measures from [5, 6, 7].

Finally, in the Appendix, we first collect some well-known facts about spectral measures of dynamical systems and then discuss the connection of our results with the theory of Bernoulli convolutions; in particular, from Proposition 5.6 we derive Corollary 7.5 that gives new upper bounds for Fourier transforms of Bernoulli convolutions corresponding to inverses of algebraic integers.

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2. Matrix Riesz products for substitutions

Our aim in this section is to represent the spectral measure of a substitution dynamical system via a matrix analog of Riesz products. The reader is referred to \[34, 20\] for the background on substitutions and substitution dynamical systems. The main result of this section is Lemma 2.2.

2.1. Substitutions. For \(m \geq 2\) consider the finite alphabet \(A = \{1, \ldots, m\}\) and the set \(A^+\) of nonempty words with letters in \(A\). A substitution is a map \(\zeta : A \to A^+\), extended to \(A^+\) and \(A^\mathbb{N}\) by concatenation. The substitution space is defined as the set of bi-infinite sequences \(x \in A^\mathbb{Z}\) such that any word in \(x\) appears as a subword of \(\zeta^n(a)\) for some \(a \in A\) and \(n \in \mathbb{N}\). The substitution dynamical system is the left shift on \(A^\mathbb{Z}\) restricted to \(X_\zeta\), which we denote by \(T_\zeta\).

The substitution matrix \(S = S_\zeta = (S_{i,j})\) is the \(m \times m\) matrix, where \(m = \#A\), such that \(S_{i,j}\) is the number of symbols \(i\) in \(\zeta(j)\). The substitution is primitive if \(S^n\) is strictly positive (entrywise) for some \(n \in \mathbb{N}\). It is well-known that primitive substitution \(\mathbb{Z}\)-actions are minimal and uniquely ergodic. We assume that the substitution is primitive and non-periodic, which in the primitive case is equivalent to the space \(X_\zeta\) being infinite.

The length of a word \(u\) is denoted by \(|u|\). The substitution \(\zeta\) is said to be of constant length \(q\) if \(|\zeta(a)| = q\) for all \(a \in A\), otherwise, it is of non-constant length. Spectral properties of substitution dynamical systems have been studied extensively, see \[34, 20\] and references therein. These systems are never strongly mixing \[12\], hence there is always a singular spectral component. The discrete part of the spectrum is understood completely. For substitutions of constant length \(q\) the group of eigenvalues is non-trivial and contains the group of \(q\)-adic rationals. Non-constant length substitutions may be weak mixing; this depends on algebraic and number-theoretic properties of the substitution matrix, see \[27, 18\]. Many papers have been devoted to the question of pure discrete spectrum, but this is no our focus here. It is an interesting problem to decide when the spectrum is purely singular, as well as to determine finer properties of the spectral measures, such as their modulus of continuity. To our knowledge, results in this direction have only been available in the constant length. For instance, pure singular spectrum is known for the Thue-Morse substitution \[28\] and its “abelian” generalizations \[34, 3\]. On the other hand, there are substitutions of constant length with a Lebesgue spectral component, such as the Rudin-Shapiro substitution, see \[34\], and its generalizations, due to N. P. Frank \[19\].

2.2. Spectral measures. Let \((X, T, \mu)\) be a measure-preserving transformation and let \(U = U_T : f \mapsto f \circ T\) be the Koopman operator on \(L^2(X, \mu)\). Recall that for \(f, g \in L^2(X, \mu)\) the (complex) spectral measure \(\sigma_{f,g}\) is determined by the equations

\[
\hat{\sigma}_{f,g}(-k) = \int_0^1 e^{2\pi ik\omega} d\sigma_{f,g}(\omega) = \langle U^k f, g \rangle, \quad k \in \mathbb{Z}.
\]

We write \(\sigma_f = \sigma_{f,f}\). The following is standard; see Appendix for the proof.
Lemma 2.1. For any \( f, g \in L^2(X, \mu) \) we have
\[
\sigma_{f,g} = \text{weak*}\lim_{N \to \infty} \frac{1}{N} \left( \sum_{n=0}^{N-1} e^{-2\pi i n \omega} U^n f, \sum_{n=0}^{N-1} e^{-2\pi i n \omega} U^n g \right) \, d\omega,
\]
where in the right-hand side we consider the weak*-limit of absolutely continuous measures with the given density.

If \( X \) is a metric space and \( (X, T, \mu) \) is uniquely ergodic, then for all \( f, g \in C(X) \) and all \( k \in \mathbb{Z} \):
\[
\hat{\sigma}_{f,g}(-k) = \langle (U^k f) g, \rangle = \int_X f(T^k x) g(x) \, d\mu(x)
\]
(2.1)
\[
= \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{n+k} x) g(T^n x),
\]
where \( x \in X \) is arbitrary and the limit is uniform in \( x \).

2.3. Correlation measures. For substitutions the most important spectral measures are
\[
\sigma_a := \sigma_{1_{[a]}} \quad \text{and} \quad \sigma_{a,b} := \sigma_{1_{[a]}}, 1_{[b]}.
\]
They are also known as correlation measures. In view of (2.1), since primitive substitution dynamical systems are uniquely ergodic,
\[
\hat{\sigma}_{a,b}(-k) = \lim_{N \to \infty} \frac{1}{|\zeta_N^N(\gamma)|} \text{card} \{ 0 \leq n + k < |\zeta_N^N(\gamma)| : \zeta_N(\gamma)_{n+k} = a, \zeta_N(\gamma)_n = b \}
\]
(2.2)
for any \( \gamma \in \mathcal{A} \). For a word \( v = v_0 v_1 \ldots \in A^+ \) let
\[
\Phi_a(v, \omega) = \sum_{j=0}^{|v|-1} \delta_{v_j,a} e^{2\pi i \omega j},
\]
(2.3)
where \( \delta_{v_j,a} \) is Kronecker \( \delta \) (one if \( v_j = a \) and zero otherwise). Then it is immediate from (2.2), as in Lemma 2.1, that
\[
\sigma_{a,b} = \text{weak*}\lim_{N \to \infty} \frac{1}{|\zeta_N^N(\gamma)|} \Phi_a(\zeta_N^N(\gamma), \omega) \cdot \Phi_b(\zeta_N^N(\gamma), \omega) \, d\omega
\]
(2.4)
for any \( \gamma \in \mathcal{A} \).

Observe that for any two words \( u, v \) and the concatenated word \( uv \) we have
\[
\Phi_a(uv, \omega) = \Phi_a(u, \omega) + e^{2\pi i \omega |u|} \Phi_a(v, \omega).
\]
(2.5)
Suppose \( \zeta(b) = u^{(b)}_1 \ldots u^{(b)}_k \) for \( b \in \mathcal{A} \). Then \( \zeta^n(b) = \zeta^{n-1}(u^{(b)}_1) \ldots \zeta^{n-1}(u^{(b)}_k) \) for \( n \geq 1 \), hence
(2.5) implies for all \( b \in \mathcal{A} \):
\[
\Phi_a(\zeta^n(b), \omega) = \sum_{j=1}^k \exp \left[ 2\pi i \omega \left( |\zeta^{n-1}(u^{(b)}_1)| + \cdots + |\zeta^{n-1}(u^{(b)}_{j-1})| \right) \right] \Phi_a(\zeta^{n-1}(u^{(b)}_j), \omega).
\]
Let

\[ (2.6) \quad \bar{\Psi}_n^{(a)}(\omega) := \begin{pmatrix} \Phi_a(\zeta^n(1), \omega) \\ \vdots \\ \Phi_a(\zeta^n(m), \omega) \end{pmatrix} \quad \text{and} \quad \Pi_n(\omega) = [\bar{\Psi}_n^{(1)}(\omega), \ldots, \bar{\Psi}_n^{(m)}(\omega)] \]

where \( \Pi_n(\omega) \) is the \( m \times m \) matrix-function specified by its column vectors. It follows that

\[ (2.7) \quad \Pi_n(\omega) = M_{n-1}(\omega)\Pi_{n-1}(\omega), \quad n \geq 1, \]

where \( M_{n-1}(\omega) \) is an \( m \times m \) matrix-function, whose matrix elements are trigonometric polynomials given by

\[ (2.8) \quad (M_{n-1}(\omega))(b, c) = \sum_{j \leq k : u_j^{(b)} = c} \exp \left[ 2\pi i \omega \left( |\zeta^{n-1}(u_1^{(b)})| + \cdots + |\zeta^{n-1}(u_j^{(b)})| \right) \right] \]

(if \( j = 1 \), the expression reduces to \( \exp(0) = 1 \) by definition). Note that \( M_n(0) = S^t = S_\zeta^t \), the transpose of the substitution matrix, for all \( n \in \mathbb{N} \).

**Example.** Let \( \zeta \) be a substitution on \( \{1, 2\} \) given by \( \zeta(1) = 1222, \ \zeta(2) = 1 \). Then \( S = \begin{pmatrix} 1 & 1 \\ 3 & 0 \end{pmatrix} \).

\[ M_n(\omega) = \begin{pmatrix} 1 & e^{2\pi i \omega|\zeta^n(1)|} + e^{2\pi i \omega(|\zeta^n(1)| + |\zeta^n(2)|)} + e^{2\pi i \omega(|\zeta^n(1)| + 2|\zeta^n(2)|)} \\ 1 & 0 \end{pmatrix}, \quad n \geq 0. \]

Since \( \bar{\Psi}_0^{(a)}(\omega) = \vec{e}_a \) (the basis vector corresponding to \( a \in \mathcal{A} \)), it follows from (2.7) that

\[ (2.9) \quad \bar{\Psi}_n^{(a)}(\omega) = M_{n-1}(\omega)M_{n-2}(\omega) \cdots M_0(\omega)\vec{e}_a, \]

hence

\[ (2.10) \quad \Pi_n(\omega) = M_{n-1}(\omega)M_{n-2}(\omega) \cdots M_0(\omega). \]

Denote by \( \Sigma_\zeta \) the \( m \times m \) matrix of correlation measures for \( \zeta \); that is, \( \Sigma_\zeta(a, b) = \sigma_{a, b} \). Below \( \vec{I} \) denotes the vector \( [1, \ldots, 1]^t \) of all 1’s and \( \langle \vec{x}, \vec{y} \rangle \) stands for the scalar product in \( \mathbb{R}^m \). We write \( x_n \sim y_n \) when \( \lim_{n \to \infty} x_n/y_n = 1 \).

**Lemma 2.2.** Let \( \theta \) be the Perron-Frobenius eigenvalue of the substitution matrix \( S \) and let \( \vec{r}, \vec{\ell} \) be respectively the (right) eigenvectors of \( S, S^t \) corresponding to \( \theta \), normalized by the condition \( \langle \vec{r}, \vec{\ell} \rangle = 1 \). Then

\[ (2.11) \quad \Sigma_\zeta = \frac{1}{\langle \vec{r}, \vec{I} \rangle \langle \vec{I}, \vec{\ell} \rangle} \cdot \lim_{n \to \infty} \theta^{-n} \Pi_n(\omega) \Pi_n(\omega) d\omega. \]
Proof. In view of (2.6), the \((a, b)\) entry of \(\Pi_n^*(\omega)\Pi_n(\omega)\) is \(\sum_{j=1}^{m} \Phi_a(\zeta^n(j), \omega) \cdot \Phi_b(\zeta^n(j), \omega)\). We have

\[|\zeta^n(j)| = \langle S^n\vec{e}_j, \vec{1} \rangle \sim \theta^n \langle \vec{e}_j, \vec{r} \rangle \langle \vec{r}, \vec{1} \rangle\]

by the Perron-Frobenius Theorem, hence

\[\theta^{-n} \sum_{j=1}^{m} \Phi_a(\zeta^n(j), \omega) \cdot \Phi_b(\zeta^n(j), \omega) \sim \sum_{j=1}^{m} \frac{\langle \vec{e}_j, \vec{r} \rangle \langle \vec{r}, \vec{1} \rangle}{|\zeta^n(j)|} \Phi_a(\zeta^n(j), \omega) \cdot \Phi_b(\zeta^n(j), \omega),\]

and the desired claim follows from (2.4), in view of the fact that \(\vec{1} = \sum_{j=1}^{m} \vec{e}_j\).

\[\Box\]

Remark 2.3. Representing spectral measures as Riesz products for substitutions of constant length has been used for a long time, especially for the Thue-Morse substitution and its generalizations \[31\]. Queffelec \[34, Theorem 8.1\] proved a variant of Lemma 2.2 for substitutions of constant length, but this, to our knowledge, has not been done in the non-constant length case.

2.4. Other spectral measures and maximal spectral type. Substitution \(\zeta\) can also be extended to \(A^Z\), so that \(\zeta(x_0)\) starts from 0-th position. Consider the sets \(\zeta^k[a]\), \(a \in A\), \(k \geq 0\). It is proved in \[31, 5.6.3\] that

\[\mathcal{P}_k = \{T_\zeta^i(\zeta^k[a]), a \in A, 0 \leq i < |\zeta^k(a)|\}\]

is a Kakutani-Rokhlin partition for the substitution dynamical system, and \(\{\mathcal{P}_k\}_{k \geq 0}\) generate the Borel \(\sigma\)-algebra. It follows that the maximal spectral type of \((X_\zeta, T_\zeta, \mu)\) is equivalent to

\[\sum_{k \geq 0, a \in A} 2^{-k} \sigma_{1 \cdot \zeta^k[a]}\]

The spectral measures \(\sigma_{1 \cdot \zeta^k[a]}\) can be analyzed similarly to the correlation measures \(\sigma_a\). In fact, it is not hard to see that the matrix of measures

\[\Sigma_\zeta^{(k)} := [\sigma_{1 \cdot \zeta^k[a]} : \zeta^k[0]]_{a,b \in A}\]

can be expressed, analogously to (2.11), as

\[\Sigma_\zeta^{(k)} = \frac{1}{\langle \vec{r}, \vec{1} \rangle \langle \vec{1}, \vec{r} \rangle} \cdot \text{weak}^* \lim_{n \to \infty} \theta^{-n}(\Pi_n^{(k)})^*(\omega) \Pi_n^{(k)}(\omega) d\omega,\]

where

\[\Pi_n^{(k)}(\omega) = M_{n+k-1}(\omega) \cdots M_k(\omega).\]
3. Estimating spectral measures via matrix Riesz products

In this section we study the local quantitative behavior of spectral measures; more precisely, upper bounds for the measures of small balls. Using matrix products from Section 2, this problem is essentially reduced to some questions of Diophantine approximation.

It should be pointed out that matrix products, analogous to those that we consider, have been studied in the physics literature, in particular, in the papers by Aubry-Godrèche-Luck [2] and Gähler-Klitzing [21]. These papers study the diffraction spectrum of structures associated to substitutions, mostly in the continuous setting, as our self-similar suspension flows in Section 5, and their higher-dimensional generalizations (self-similar tilings). As was shown by S. Dworkin [15] (see also [23]), the diffraction spectrum is a “part” of the dynamical spectrum that we consider. The papers [2, 21] analyzed the growth of expressions like our $|\Phi_a(\zeta^n(b), \omega)|$ (called “structure factor”), with the help of finite matrix products, and used them to make conclusions (sometimes heuristically) about the diffraction spectrum. The paper [21] focused on the discrete part of the spectrum and showed that the “maximal growth” of $|\Phi_a(\zeta^n(b), \omega)|$ occurs in the Pisot case on a dense set of spectral parameters, which, in dynamical terms means eigenvalues. This can be compared to a theorem from [39] asserting that the self-similar substitution tiling system is weakly mixing if and only if the expansion factor is non-Pisot. The paper [2] examined in detail some non-Pisot examples and obtained “intermediate growth” of the structure factor for a dense set of spectral parameters, from which singular-continuous diffraction spectrum was deduced heuristically. A rigorous argument in this direction was made by A. Hof [26]; we outline it in Lemma 3.1 below. See also the papers [24, 8, 30, 43] for related work in the physics literature.

The next lemma essentially appears in the paper by A. Hof [26]. First we need to introduce some notation. Let $(X, T, \mu)$ be a measure-preserving system. For $f \in L^2(X, \mu)$ let

$$(3.1) \quad G_N(f, \omega) = N^{-1} \left\| \sum_{n=0}^{N-1} e^{-2\pi i n \omega} f \circ T^n \right\|_{L^2}^2.$$ 

Note that

$$\sigma_f = \text{weak}^* \lim_{N \to \infty} G_N(f, \omega) d\omega$$

by Lemma 2.1. For $x \in X$ and $f \in L^2(X, \mu)$ let

$$(3.2) \quad S_N^x(f, \omega) = \sum_{n=0}^{N-1} e^{-2\pi i n \omega} f(T^n x),$$

so that

$$(3.3) \quad G_N(f, \omega) = N^{-1} \int_X |S_N^x(f, \omega)|^2 d\mu(x).$$
Lemma 3.1. Let $\Omega(r)$ be a continuous increasing function on $[0,1)$, such that $\Omega(0) = 0$, and suppose that for some fixed $\omega \in [0,1)$ there exists $N_0 \in \mathbb{N}$ such that

\[(3.4) \quad G_N(f,\omega) \leq CN \Omega(1/N) \quad \text{for } N \geq N_0.\]

Then

\[(3.5) \quad \sigma_f(B(\omega,r)) \leq \frac{\pi^2C}{4} \Omega(3r) \quad \text{for all } r \leq (2N_0)^{-1}.\]

In particular, (3.5) holds provided

\[(3.6) \quad \sup_{x \in X} |S^x_N(f,\omega)| \leq N\sqrt{C\Omega(1/N)} \quad \text{for } N \geq N_0.\]

The proof is deferred to the Appendix; it is a minor modification of [26, Theorem 2.1].

Now we go back to the substitution dynamical system $(X_\zeta, T_\zeta, \mu)$ and let $f = \mathbb{1}_{[a]}$, that is, $f(x) = \delta_{x_0,a}$. Then

\[(3.7) \quad S^x_N(\mathbb{1}_{[a]},\omega) = \Phi_a(x[0,N-1],\omega), \quad \text{where } x[0,N-1] = x_0 \ldots x_{N-1},\]

see (2.3). We will estimate this expression in absolute value from above, as $N \to \infty$, uniformly in $x$, yielding estimates of $G_N(\mathbb{1}_{[a]},\omega)$ via (3.3) and then estimates of $\sigma_f(B(\omega,r))$ via Lemma 3.1.

First we need some terminology. A word $v$ is called a return word for the substitution $\zeta$ if $v$ starts with some letter $c$, doesn’t contain other $c$’s, and $vc$ appears as a word in (any of) the sequences in $X_\zeta$ (by the minimality of the substitution dynamical system, the set of words appearing in $x \in X_\zeta$ is the same for all $x$). That is, the return word separates two successive occurrences of a given letter (with the letter included at the beginning).

Fix a return word $v$ starting with $c$. We can replace $\zeta$ by $\zeta^\ell$ without loss of generality since this does not change the space $X_\zeta$ and hence the substitution system. Since $\zeta$ is primitive, we can thus assume without loss of generality that $vc$ occurs as a subword in every $\zeta^b$, $b \in A$. Let

$$\|x\| := \text{dist}(x,\mathbb{Z}) \quad \text{for } x \in \mathbb{R}. $$

This is standard notation in the theory of Diophantine approximation; when we use $\| \cdot \|$ for a norm, this is always indicated by a subscript, as in (3.1).

The following proposition is a key result: it shows (i) that it suffices to estimate (3.7) for $x$ starting with $\zeta^b(b)$ for some $b \in A$, and this in turn reduces to estimating matrix Riesz products; (ii) how to estimate matrix Riesz products in terms of ordinary products. See also Lemma 7.1 in the Appendix for additional information about local dimension of spectral measures.

Proposition 3.2. Let $\zeta$ be a primitive aperiodic substitution on $A$ and $v$ a return word starting with $c \in A$ such that $vc$ occurs as a subword in $\zeta(b)$ for every $b \in A$. Let $a \in A$, $\omega \in \mathbb{T}$, and $S^x_N(\mathbb{1}_{[a]},\omega)$ be defined by (3.2) with $T = T_\zeta$. Then there exist $C,C',c_1,c_2 > 0$, depending only on the substitution $\zeta$, such that
(i) for all $a, b \in A$, $n \in \mathbb{N}$, and $\omega \in [0, 1)$,

$$\Phi_a(\zeta^n(b), \omega) \leq C|\zeta^n(b)| \prod_{k=0}^{n-1} \left(1 - c_1 \|\omega|\zeta^k(v)||^2\right);$$

(ii) for all $N \in \mathbb{N}$, $\omega \in [0, 1)$, and $a \in A$,

$$S_N^a(\omega) \leq C' N \prod_{k=0}^{\lfloor c_2 \log N \rfloor} (1 - c_1 \|\omega|\zeta^k(v)||^2)$$

for all $x \in X_\zeta$.

Remark 3.3. This result should be compared with the criterion for $e^{2\pi i \omega}$ to be an eigenvalue of the substitution dynamical system [27, 18], which, under some technical assumptions, says that

$$e^{2\pi i \omega} \text{ is an eigenvalue } \iff \sum_{k=0}^{\infty} \|\omega|\zeta^k(v)||^2 < \infty \text{ for every return word } v.$$

Proposition 3.2 and Lemma 3.1 imply that on certain subsets of $T$ defined in terms of Diophantine properties, the spectral measures satisfy a Hölder condition with a uniform exponent.

Corollary 3.4. Suppose that the assumptions of Proposition 3.2 are satisfied. Fix small $\delta > 0$, $\varepsilon > 0$, and consider

$$\Omega_{\delta, \varepsilon} := \left\{ \omega \in T : \liminf_{n \to \infty} \frac{\# \{k \leq n : \|\omega|\zeta^k(v)|| \geq \delta \}}{n} > \varepsilon \right\}.$$

Then for all $a, b \in A$ and $\omega \in \Omega_{\delta, \varepsilon}$ there exists $C > 0$ such that

$$\sigma_a(B(\omega, r)) \leq Cr^\beta,$$

where $\beta = \beta(\zeta, \delta, \varepsilon)$.

Proof. It follows from Proposition 3.2 that (3.4) holds with $\Omega(r) = r^{2c_1c_2\delta^2}$ for $f = 1_{[a]}$, so Lemma 3.1 yields (3.12), since $c_1, c_2$ depend only on the substitution $\zeta$. \qed

Remark 3.5. 1. One can show that for sufficiently small $\delta > 0$ we have $\dim_H(T \setminus \bigcup_{\varepsilon > 0} \Omega_{\delta, \varepsilon}) = 0$. This is similar in spirit to the well-known fact that the set of reals with lower frequency of some decimal digit, say, 7, equal to zero, has Hausdorff dimension zero.

2. It follows from [42] that for Lebesgue-a.e. $\omega$ the sequence $\{\omega|\zeta^k(v)||\}_{k \geq 0}$ is uniformly distributed modulo 1, which yields an explicit Hölder exponent for spectral measures $\sigma_a$ almost everywhere. This is, however, not very interesting, since any finite positive measure on $[0, 1]$ has an a.e. differentiable cumulative distribution function, hence that measure satisfies a Lipschitz condition almost everywhere.

Proof of Proposition 3.2. We write $C$ to denote positive constants depending only on $\zeta$, which may differ from line to line, and use the symbol $\asymp$ when two quantities are comparable up to such multiplicative constants.
Step 1: reduction (3.8) $\Rightarrow$ (3.9). We emphasize that the reduction is not automatic and that our argument below is specific to substitution dynamical systems; see also Remark 3.4 in Hof [26]. We shall need the following well-known construction of prefix-suffix decomposition.

Lemma 3.6. Let $x \in X_\xi$ and $N \geq 1$. Then

\[
x[0, N - 1] = u_0 \zeta(u_1) \zeta^2(u_2) \ldots \zeta^n(u_n) \zeta^{n-1}(v_{n-1}) \ldots \zeta(v_1)v_0,
\]

where $n \geq 1$ and $u_i, v_i, i = 1, \ldots, n$, are respectively proper prefixes and suffixes of the words $\zeta(b), b \in A$. The words $u_i, v_i$ may be empty, except that at least one of $u_n, v_n$ is nonempty.

Now we prove the validity of the reduction. By (3.13) and (2.5) we have

\[
|\Phi_a(x[0, N - 1], \omega)| \leq \sum_{j=0}^{n} (|\Phi_a(\zeta^j(u_j), \omega)| + |\Phi_a(\zeta^j(v_j), \omega)|)
\]

(3.14)

\[
\leq 2L \sum_{j=0}^{n} \max_{b \in A} |\Phi_a(\zeta^j(b), \omega)|,
\]

where $L = \max_{b \in A} |\zeta(b)|$, since $u_j, v_j$ are subwords of substituted symbols. Therefore, assuming (3.8), we have

\[
|\Phi_a(x[0, N - 1], \omega)| \leq C \sum_{j=0}^{n} \max_{b \in A} |\zeta^j(b)| \prod_{k=0}^{j-1} \left(1 - c_1 \|\zeta^k(\omega)\|^2\right)
\]

(3.15)

\[
\leq C \sum_{j=0}^{n} \theta^j \prod_{k=0}^{j-1} \left(1 - c_1 \|\zeta^k(\omega)\|^2\right),
\]

using Perron-Frobenius Theorem in the last line (recall that $\theta$ is the Perron-Frobenius eigenvalue of $S$, hence $|\zeta^j(b)| = (S^j \tilde{e}_b, \bar{1}) \asymp \theta^j$ for all $b$). Now, assuming

\[
c_1 \leq 2(1 - \theta^{-1}),
\]

and using that $\|x\| \leq 1/2$ for any $x \in \mathbb{R}$, we see that the the ratio of consecutive terms in the sum (3.15) is at least $\theta(1 - c_1/4) \geq \theta + 1/2 > 1$. Thus the sum in (3.15) can be estimated from above by a constant times the last term, whence

\[
|\Phi_a(x[0, N - 1], \omega)| \leq C \theta^n \prod_{k=0}^{n-1} \left(1 - c_1 \|\zeta^k(\omega)\|^2\right).
\]

It follows from (3.13) that

\[
\min_{b \in A} |\zeta^n(b)| \leq N \leq 2 \max_{b \in A} |\zeta^{n+1}(b)|,
\]

hence $C^{-1} N \leq \theta^n \leq CN$ for some $C > 1$ depending only on $\zeta$, and we obtain

\[
|\Phi_a(x[0, N - 1], \omega)| \leq C N \prod_{k=0}^{\lfloor c_2 \log N \rfloor} \left(1 - c_1 \|\zeta^k(\omega)\|^2\right).
\]
This proves (3.9), in view of (3.7), and completes the reduction step.

**Step 2: proof of (3.8).** We fix $\omega$ and omit it from notation, so that $M_n = M_n(\omega)$. We are going to use (2.6) and (2.9). By assumption, for any $b \in A$, we can write

$$\zeta(b) = p^{(b)}vqc^{(b)},$$

where $p^{(b)}$ and $q^{(b)}$ are words, possibly empty, and $v$ starts with $c$. We are going to estimate the absolute value of the trigonometric polynomial $M_n(b,c)$ given by (2.8). Note that $M_n(b,c)$ is a trigonometric polynomial with $S_t(b,c)$ exponential terms and all coefficients equal to one. By (2.8) and (3.16), the expression for $M_n(b,c)$ includes the terms $e^{2\pi i\omega|\zeta^n(p^{(b)})|} + e^{2\pi i\omega|\zeta^n(p^{(b)}v)|}$, hence

$$|M_n(b,c)| \leq S_t(b,c) - 2 + |1 + e^{2\pi i\omega|\zeta^n(v)|}|.$$

From the inequality

$$|1 + e^{2\pi i\tau}| \leq 2 - \frac{1}{2} \|\tau\|^2, \quad \tau \in \mathbb{R},$$

we have

$$|M_n(b,c)| \leq S_t(b,c) - \frac{1}{2} \|\omega|\zeta^n(v)|\|^2.$$

We will use the following notation:

- for vectors $\vec{x}, \vec{y} \in \mathbb{R}^m$, the inequality $\vec{x} \leq \vec{y}$ means componentwise inequality;
- the operation of taking absolute values of all entries for a vector $\vec{x}$ and a matrix $A$ will be denoted $|\vec{x}|_1$ and $|A|_1$.

It is clear that for any, generally speaking, rectangular matrices $A, B$ such that the product $AB$ is well-defined, we have

$$|AB|_1 \leq |A|_1 |B|_1.$$

For an arbitrary $\vec{x} = [x_1, \ldots, x_m]^t > \vec{0}$ and $k \geq 0$, using (3.18) we can estimate

$$(M_k^t \vec{x})_b = \sum_{j=1}^m |M_k(b,j)|x_j$$

$$\leq \sum_{j=1}^m S_t(b,j)x_j - \frac{1}{2} \|\omega|\zeta^k(v)|\|^2 x_c$$

$$\leq (1 - c_3(\vec{x})\|\omega|\zeta^k(v)|\|^2) \cdot \sum_{j=1}^m S_t(b,j)x_j$$

$$= (1 - c_3(\vec{x})\|\omega|\zeta^k(v)|\|^2) \cdot (S_t \vec{x})_b,$$

where

$$c_3(\vec{x}) = \frac{x_c}{2m \max_j S_t(b,j) \cdot \max_j x_j}.$$
Observe that (3.19) implies
\[
(\vec{\Psi}_n^{(a)}(\omega))|_\cdot| = (M_{n-1} \cdots M_0 \vec{e}_a)|_\cdot| \leq M_{n-1}^{|_\cdot|} \cdots M_0^{|_\cdot|} \vec{e}_a \leq M_{n-1}^{|_\cdot|} \cdots M_0^{|_\cdot|} \vec{1}.
\]
Thus, using (3.20) inductively we obtain
\[
(\vec{\Psi}_n^{(a)}(\omega))|_\cdot| \leq n-1 \prod_{k=0}^{n-1} \left( 1 - c_3((S^k(\vec{1}))_n(\omega|\zeta^k(v)|^2) \cdot (S^k)^n \vec{1}.
\]
It remains to note that
\[
c_1 := \inf_{k \geq 0} c_3((S^k(\vec{1}))_n(\omega|\zeta^k(v)|^2) > 0
\]
and \((S^k)^n \vec{1} \sim \theta^a \vec{1} \sim |\zeta^n(b)| \vec{1}\), all by Perron-Frobenius, and the desired inequality (3.8) follows, in view of (2.6). Proposition 3.2 is proved completely.

4. Hölder continuity of spectral measures for almost every substitution suspension flow

Let \(\zeta\) be a primitive substitution on \(A = \{1, \ldots, m\}\), and let \((X_\zeta, T_\zeta)\) be the corresponding uniquely ergodic \(\mathbb{Z}\)-action. For a strictly positive vector \(\vec{s} = (s_1, \ldots, s_m)\) we consider the suspension flow over \(T_\zeta\), with the piecewise-constant roof function, equal to \(s_j\) on the cylinder set \([j]\). The resulting space will be denoted \(X^{\vec{s}}_\zeta\) and the flow by \((X^{\vec{s}}_\zeta, h_t)\). This flow can also be viewed as the translation action on a tiling space, with interval prototiles of length \(s_j\). Such dynamical systems have been studied e.g. in [4, 9]. Denote by \(\tilde{\mu}\) the unique invariant probability measure for the suspension flow \((X^{\vec{s}}_\zeta, h_t)\). We have, by definition,
\[
X^{\vec{s}}_\zeta = \bigcup_{a \in A} X_a, \quad \text{where} \quad X_a = \{(x, t) : x \in X_\zeta, x_0 = a, 0 \leq t \leq s_a\}.
\]
and this union is disjoint in measure. Let \(f \in L^2(X^{\vec{s}}_\zeta, \tilde{\mu})\). By the Spectral Theorem for measure-preserving flows, there is a finite positive Borel measure \(\sigma_f\) on \(\mathbb{R}\) such that
\[
\int_{-\infty}^{\infty} e^{2\pi i \omega t} d\sigma_f(\omega) = \langle f \circ h_t, f \rangle \quad \text{for} \ t \in \mathbb{R}.
\]
We will focus on the spectral measures of characteristic functions of \(X_a\):
\[
\sigma_a := \sigma_f \quad \text{for} \ f = \mathbb{1}_{X_a}, \ a \in A.
\]
For a word \(v\) in the alphabet \(A\) denote by \(\vec{\ell}(v) \in \mathbb{Z}^m\) its “population vector” whose \(j\)-th entry is the number of \(j\)'s in \(v\), for \(j \leq m\). We will need the “tiling length” of \(v\) defined by
\[
|v|_{\vec{s}} := \vec{\ell}(v) \cdot \vec{s}, \quad (4.1)
\]
It is not hard to show, similarly to the case of substitution \(\mathbb{Z}\)-actions, as discussed in Section 2.4, that the maximal spectral type of the flow \((X_{\vec{s}}^\mathcal{Z}, h_t)\) is equivalent to
\[
\sum_{k \geq 0, \ a \in A} 2^{-k} \sigma_{\mathcal{Z}^k[a]},
\]
where \(\sigma_{\mathcal{Z}^k[a]}\) is the spectral measure of the characteristic function of the set
\[
\mathcal{X}_{\mathcal{Z}^k[a]} := \{ h_t(x, 0) : x \in \mathcal{Z}^k[a], \ 0 \leq t \leq |\mathcal{Z}^k(a)|\},
\]
and these spectral measures can be analyzed similarly to \(\sigma_a\).

Suspension flows for \(\vec{s}\), which differ by a constant multiple, are related by a “time-scale change”. It follows that if \(\vec{s}' = c \vec{s}\) for \(c > 0\), then
\[
\sigma_{\mathcal{Z}'} \big|_{\mathcal{E}}(E) = \sigma_{\mathcal{Z}}(c^{-1} E) \quad \text{for Borel } E,
\]
where the superscript indicates which suspension flow is considered. Thus it makes sense to normalize suspension flows, for instance, assuming that \(\vec{s} \in \Delta^{m-1} := \{ \vec{y} \in \mathbb{R}^m_+ : \sum_{j=1}^{m} y_j = 1 \}\).

**Theorem 4.1.** Let \(\mathcal{Z}\) be a primitive aperiodic substitution on \(A = \{1, \ldots, m\}\), with substitution matrix \(S\). Suppose that the characteristic polynomial of \(S\) is irreducible and the second eigenvalue satisfies \(|\theta_2| > 1\). Then for Lebesgue-almost every suspension flow the spectral measures of the dynamical system \((X_{\vec{s}}^\mathcal{Z}, h_t)\) are Hölder continuous away from zero.

More precisely, there exists a constant \(\gamma > 0\), depending only on the substitution \(\mathcal{Z}\), such that for Lebesgue-almost every \(\vec{s} \in \Delta^{m-1}\) and \(B > 1\), there exist \(r_0 = r_0(\vec{s}, B) > 0\) and \(C = C(\vec{s}, B) > 0\) such that
\[
\sigma_{\mathcal{Z}}([\omega - r, \omega + r]) \leq Cr^\gamma, \quad \text{for all } |\omega| \in [B^{-1}B], \ 0 < r \leq r_0, \ \text{and } a \in A.
\]

**Remark 4.2.** When the characteristic polynomial of \(S\) is irreducible and \(|\theta_2| \geq 1\), then almost every suspension flow is weak mixing by \([9]\) (see, in particular, \([9]\) Theorem 2.7). On the other hand, if all eigenvalues of \(S\), other than \(\theta\), are inside the unit circle (the “PV case”), then all suspension flows have dense point spectrum and \((4.3)\) cannot hold \([9]\). The “Salem case” (see Definition 7.3) is more complicated, but we expect that \((4.3)\) fails in the Salem case as well.

In fact, we get a more precise estimate of the exceptional set in Theorem 4.1.

**Theorem 4.3.** Let \(\mathcal{Z}\) be a primitive aperiodic substitution on \(A = \{1, \ldots, m\}\), with substitution matrix \(S\). Suppose that the characteristic polynomial of \(S\) is irreducible, and there are exactly \(q\) eigenvalues of absolute value \(\leq 1\), for some \(0 \leq q < m - 1\). Then for every \(\eta > 0\) there exists \(\gamma = \gamma(\eta) > 0\) and an exceptional set \(\mathcal{E}_\eta \subset \Delta^{m-1}\) of Hausdorff dimension at most \(q + \eta\) such that for every \(\vec{s} \in \Delta^{m-1} \setminus \mathcal{E}_\eta\) and any \(B > 1\) there exist \(r_0 = r_0(\vec{s}, B) > 0\) and \(C = C(\vec{s}, B) > 0\) for which we have the estimate \((4.3)\).
Thus, in particular, we get some Hölder estimate for all suspension flows outside a zero-dimensional set, assuming $S$ has all eigenvalues outside the unit circle. Note that Theorem 4.1 is an immediate corollary of Theorem 4.3, just choose $\eta < 1$ and note that $q + \eta < m - 1 = \dim(\Delta^{m-1})$.

For the proof of the theorem, we need an analog of Lemma 3.1 for flows. Let

$$S^R_R(f, \omega) = \int_0^R e^{-2\pi i \omega t} f \circ h_t(x,t) \, dt.$$

**Lemma 4.4.** Suppose that for some fixed $\omega \in \mathbb{R}$, $R_0 > 0$, and $\alpha \in (0,1)$ we have

$$\sup_{(x,t) \in X^\omega} |S^R_R(f, \omega)| \leq C_1 R^\alpha$$

for all $R \geq R_0$. Then

$$\sigma_f([\omega - r, \omega + r]) \leq \pi^2 C_1^2 r^2 (1 - \alpha)$$

for all $r \leq (2R_0)^{-1}$.

See the Appendix for the proof.

Next, consider an analog of (2.3): for $v = v_0 \ldots v_{N-1} \in A^+$, let

$$\Phi^\zeta_a(v, \omega) = \sum_{j=0}^{N-1} \delta_{v_j,a} \exp(2\pi i \omega|v_0 \ldots v_j|_\zeta).$$

Then

$$S^R_R(1_{X^a}, \omega) = 1 - e^{-2\pi i \omega s_a} \cdot \Phi^\zeta_a(x[0,N-1], \omega)$$

for $R = |x[0,N-1]|_\zeta$.

We also have an analog of Proposition 3.2 for suspension flows:

**Proposition 4.5.** Let $\zeta$ be a primitive substitution on $A$ and $v$ a return word starting with $c \in A$ such that $vc$ occurs as a subword in $\zeta(b)$ for every $b \in A$. Let $\bar{s} \in \Delta^{m-1}$. Then there exist $C, C', c_1, c_2 > 0$, depending only on the substitution $\zeta$ and $\min_j s_j$, such that

(i) for all $a, b \in A$, $n \in \mathbb{N}$, and $\omega \in \mathbb{R}$,

$$|\Phi^\zeta_a(\zeta^n(b), \omega)| \leq C|\zeta^n(b)|_\zeta \cdot \prod_{k=0}^{n-1} \left(1 - c_1 |\omega| \zeta^k(v)|_\bar{s}\right)^2;$$

(ii) for all $R > 1$, $\omega \in \mathbb{R}$, and $a \in A$,

$$|S^R_R(1_{X^a}, \omega)| \leq C'R \prod_{k=0}^{\lfloor c_2 \log R \rfloor} (1 - c_1 |\omega| \zeta^k(v)|_\bar{s})^2$$

for all $(x,t) \in X^\omega$.

**Proof.** The proof is similar to that of Proposition 3.2. One just has to replace the usual length of words by their “tiling length,” defined in (4.1). Of course, the constants in the proof will depend on the ratio

$$\frac{\max_j s_j}{\min_j s_j} \leq (\min_j s_j)^{-1}.$$
The implication (i) \(\Rightarrow\) (ii) is obtained as in Step 1 of the proof of Proposition 3.2, using the prefix-suffix decomposition. For the proof of (i) we use the inequality, obtained by a straightforward extension of the arguments in Section 2:

\[ |\Phi_{\vec{s}}(\zeta^n(b), \omega)| \leq \|M_{n-1}^\vec{s}(\omega) \cdots M_0^\vec{s}(\omega)\|, \]

where

\[ (M_\ell^\vec{s}(\omega))(b, c) = \sum_{j \leq k_\ell: \ u_j^{(b)} = c} \exp \left[ 2\pi i \omega |\zeta^\ell(u_1^{(b)} \cdots u_{j-1}^{(b)})|_\vec{s} \right], \ \ell \in \mathbb{N}, \]

by analogy with (2.8). The matrix product is then estimated as in Step 2 of the proof of Proposition 3.2.

**Proof of Theorem 4.3.** Recall that, passing to a power \(\zeta^\ell\) if necessary, we can always obtain a return word \(v\) as in the statement of Proposition 4.5, and the existence of such a word (for \(\zeta\) itself) will be the standing assumption until the end of the section.

Let \(\theta_1 = \theta, \theta_2, \ldots, \theta_m\) be the eigenvalues of the substitution matrix \(S\), ordered by magnitude, and let \(\vec{e}_j^\vec{s}\) be the corresponding eigenvectors of its transpose \(S^t\) (real and complex). By the assumptions on the matrix \(S\), it is diagonalizable over \(\mathbb{C}\) and

\[ |\theta_{m-q}| > 1, \quad |\theta_{m-q+1}| \leq 1 \]

(we do not exclude the possibility of \(q = 0\); in that case the second inequality is vacuous). Let \(\{\vec{e}_j^\vec{s}\}_{j=1}^m\) be the dual basis, i.e. \(\vec{e}_j\) is the eigenvector of \(S\) corresponding to \(\theta_j\) and \(\langle \vec{e}_i, \vec{e}_j^\vec{s} \rangle = \delta_{ij}\). Then \(\vec{s} = \sum_{j=1}^m \langle \vec{e}_j, \vec{s} \rangle \vec{e}_j^\vec{s}\), hence

\[ |\zeta^n(v)|_{\vec{s}} = \langle \vec{\ell}(\zeta^n(v)), \vec{s} \rangle = \langle S^n \vec{\ell}(v), \vec{s} \rangle = \sum_{j=1}^m \langle \vec{e}_j, \vec{s} \rangle \langle \vec{\ell}(v), \vec{e}_j^\vec{s} \rangle \theta_j^n, \quad j \leq m, \ n \geq 0. \]

Let

\[ b_j = \langle \vec{e}_j, \vec{s} \rangle \langle \vec{\ell}(v), \vec{e}_j^\vec{s} \rangle, \quad j = 1, \ldots, m, \]

so that

\[ |\zeta^n(v)|_{\vec{s}} = \sum_{j=1}^m b_j \theta_j^n. \]

We always have \(b_1 > 0\), since \(\theta_1\) is the Perron-Frobenius eigenvalue, both eigenvectors \(\vec{e}_1\) and \(\vec{e}_1^\vec{s}\) are strictly positive, \(\vec{s}\) is strictly positive, and \(\vec{\ell}(v) \neq \vec{0}\) is non-negative. Further, since \(\vec{\ell}(v)\) is an integer vector and the characteristic polynomial of \(S\) is irreducible, we have \(\langle \vec{\ell}(v), \vec{e}_j^\vec{s} \rangle \neq 0\) for all \(j \leq m\). Indeed, otherwise \(S\) would have a rational invariant subspace, spanned by \(S^n \vec{\ell}(v), \ n \geq 0, \)
of dimension less than \( m \), contradicting the fact that its eigenvalues are algebraic integers of degree \( m \). Note also that \( b_j = \overline{b_{-j}} \) for \( \theta_j = \overline{\theta_j} \). Let

\[
\mathcal{H}^{m-1} = \left\{ (a_1, \ldots, a_m) \in \mathbb{C}^m : a_1 = 1, \ a_j = \overline{a_{-j}} \right\},
\]

let \( P_{m-q} \) be the projection from \( \mathcal{H}^{m-1} \) to the subspace spanned by the first \( m - q \) coordinates, and let \( \mathcal{H}^{m-q-1} = P_{m-q}\mathcal{H}^{m-1} \). It is clear that \( \mathcal{H}^{m-1} \) is a real affine-linear space of dimension \( m - 1 \) and \( \mathcal{H}^{m-q-1} \) is a real affine-linear space of dimension \( m - q - 1 \). It is convenient to pass from \( \Delta^{m-1} \) to a subset of \( \mathcal{H}^{m-1} \) when parametrizing the suspension flows. To this end, consider the map \( F : \mathbb{C}^{m-1} \to \mathbb{C}^{m-1} \) given by

\[
F(\tilde{s}) = \left( \frac{\langle \tilde{e}_j, \tilde{s} \rangle \langle \tilde{e}(v), \tilde{e}_j \rangle}{\langle \tilde{e}_1, \tilde{s} \rangle \langle \tilde{e}(v), \tilde{e}_1 \rangle} \right)_{1 \leq j \leq m}.
\]

The map \( F \) is by definition a change of basis transformation, followed by an invertible diagonal map. Consequently, the map \( F \) is linear and invertible. Let \( \tilde{\Delta}^{m-1} := \{ \tilde{y} \in \mathbb{R}^m : \sum_{j=1}^m y_j = 1 \} \) be the affine-linear space spanned by \( \Delta^{m-1} \). Note that \( F(\tilde{\Delta}^{m-1}) \subset \mathcal{H}^{m-1} \), and since both \( \tilde{\Delta}^{m-1} \) and \( \mathcal{H}^{m-1} \) are affine-linear spaces of real dimension \( m - 1 \), we have that \( F|_{\tilde{\Delta}^{m-1}} \) is a real affine-linear invertible map onto \( \mathcal{H}^{m-1} \), which preserves Hausdorff dimension.

The following proposition contains the core of the proof of Theorem 4.3. We will need the Vandermonde matrix

\[
\Theta = \begin{pmatrix}
1 & \cdots & 1 \\
\vdots & \ddots & \vdots \\
\theta_1^{m-1} & \cdots & \theta_m^{m-1}
\end{pmatrix}
\]

and its \( \ell^\infty \) operator norm \( \|\Theta\|_\infty \); note that \( \Theta \) is invertible, since all \( \theta_j \) are distinct.

**Proposition 4.6.** Let \( B > 1 \) and \( k \in \mathbb{N} \). Consider two constants, depending only on substitution, defined as follows:

\[
\rho := \frac{1}{2} (1 + \theta_1 \|\Theta\|_\infty \|\Theta^{-1}\|_\infty)^{-1} \quad \text{and} \quad L := 2 + \theta_1 \|\Theta\|_\infty \|\Theta^{-1}\|_\infty.
\]

Let \( E_k^N(B) \) be the set of \( (a_1, \ldots, a_{m-q}) \in \mathcal{H}^{m-q-1} \) such that there exist \( \omega \in [B^{-1}, B] \) and \( a_{m-q+1}, \ldots, a_m \), with \( (a_1, \ldots, a_m) \in F(\Delta^{m-1}) \), for which

\[
\text{card} \left\{ n \in [1, N] : \left\| \omega \sum_{j=1}^m a_j \theta_j^n \right\| \geq \rho \right\} < \frac{N}{k}.
\]

Further, let \( E_k(B) := \bigcap_{N_0=1}^\infty \bigcup_{N=N_0}^\infty E_k^N(B) \). Then

\[
\text{dim}_H(E_k(B)) \leq \frac{\log[2L^{m+1}k]}{k \log |\theta_{m-q}|}.
\]
We first derive Theorem 4.3 from Proposition 4.6.

Set

\[ E_k = \bigcup_{B > 1} E_k(B). \]

Choose \( k \in \mathbb{N} \) in such a way that \( \dim_H(E_k(B)) < \eta \) for all \( B > 0 \). Then we also have

\[ \dim_H(E_k) < \eta. \]

Let

\[ E_\eta := (F|_{\Delta^m})^{-1}(F)_{m-q}(E_k). \]

Note that \( P_{m-q}(E_k) \) is the direct product of \( E_k \) with a real \( q \)-dimensional linear space, hence \( \dim_H(E_\eta) = \dim_H(E_k) + q < \eta + q \). We want to show that \( E_\eta \) is the desired exceptional set in Theorem 4.3. To this end, let \( \bar{s} \in \Delta^{m-1} \setminus E_\eta \) and \( B > 1 \). Consider the coefficients \( b_j \) defined by (4.8), so that (4.9) holds; then

\[ F(\bar{s}) = (1, b_2/b_1, \ldots, b_m/b_1) =: (a_1, \ldots, a_m). \]

Observe that

\[ b_1 = \langle \bar{e}_1, \bar{s} \rangle \langle \tilde{\ell}(v), e_1^* \rangle \in [C_2^{-1}, C_2], \]

where \( C_2 > 1 \) depends only on \( \zeta \) and \( v \), since

\[ \min_j (\bar{e}_1)_j \leq \langle \bar{e}_1, \bar{s} \rangle \leq \max_j (\bar{e}_1)_j \quad \text{for all} \quad \bar{s} \in \Delta^{m-1}. \]

By assumption,

\[ (a_1, \ldots, a_{m-q}) \not\in E_k(C_2B), \]

hence there exists \( N_0 = N_0(\bar{s}, C_2B) \in \mathbb{N} \) such that

\[ (a_1, \ldots, a_{m-q}) \not\in E_k^N(C_2B) \]

for all \( N \geq N_0 \). By the definition of \( E_k^N(C_2B) \) and (4.9), rescaling by \( b_1 \), we obtain that for all \( \omega \in [B^{-1}, B] \) there are at least \( [N/k] \) integers \( n \in [1, N] \) for which

\[ \| \omega | \zeta^n(v) \bar{s} \| \geq \rho, \]

hence

\[ \max_{|\omega| \in [B^{-1}, B]} \prod_{n=1}^N (1 - c_1 \| \omega | \zeta^n(v) \bar{s} \|^2) \leq (1 - c_1 \rho^2)^{[N/k]} \quad \text{for all} \quad N \geq N_0. \]

Combined with Proposition 4.5(ii), this estimate implies

\[ \sup \left\{ |S_{R}^{(x,t)}(\mathbb{1}_{xa}, \omega)| : (x, t) \in \mathbb{F}_x, |\omega| \in [B^{-1}, B] \right\} \leq C' R^\alpha \quad \text{for} \quad R \geq R_0(\bar{s}, B), \]

where

\[ \alpha = 1 + \frac{c_2 \log(1 - c_1 \rho^2)}{k}. \]
Now the claim of Theorem 4.3 follows from Lemma 4.4, and it remains to prove Proposition 4.6. □

Proof of Proposition 4.6. This is proved by a variant of the “Erdős-Kahane argument,” which was invented to prove power decay of the Fourier transform of almost every Bernoulli convolution (see [17], [29]). Here we need a generalization which differs in many details, but the basic scheme is the same. See Appendix (Section 7.1) for a discussion of Bernoulli convolutions.

We can assume that $S$ has no zero eigenvalue, that is, $\theta_m \neq 0$. Indeed, if $\theta_m = 0$, we can just ignore the last coordinate and work with the vectors $(a_1, \ldots, a_{m-1})$ in (4.13).

A technical complication is that some of the coordinates of a point in $H^{m-q-1}$ might be zero. In order to address this issue, we let $\Upsilon$ be the collection of subsets $J \subset \{2, \ldots, q\}$ which satisfy

$$j \in J, \quad \theta_j = \theta_j' \Rightarrow j' \in J.$$  

Then we can write

$$H^{m-q-1} = \bigcup_{\mathcal{Y} \in \Upsilon} \bigcup_{\beta > 1} H^{m-q-1}(\mathcal{J}, \beta),$$

where

$$H^{m-q-1}(\mathcal{J}, \beta) := \{(a_1, \ldots, a_{m-q}) \in H^{m-q-1} : |a_j| \in [\beta^{-1}, \beta], \text{ for all } j \in \mathcal{J}; a_j = 0, \text{ for all } j \in \{2, \ldots, q\} \setminus \mathcal{J}\}$$

(recall that $a_1 = 1$ for all points in $H^{m-q-1}$). Then we define

$$E_k^N(B, \mathcal{J}, \beta) := E_k^N(B) \cap H^{m-q-1}(\mathcal{J}, \beta) \quad \text{and} \quad E_k(B, \mathcal{J}, \beta) := \bigcap_{N_0=1}^{\infty} \bigcup_{N=N_0}^{\infty} E_k^N(B, \mathcal{J}, \beta).$$

Clearly,

$$E_k(B) = \bigcup_{\mathcal{J} \in \Upsilon} \bigcup_{n=1}^{\infty} E_k(B, \mathcal{J}, 1/n),$$

hence it is enough to get the desired upper bound (4.14) for $\dim_H(E_k(B, \mathcal{J}, \beta))$ for some fixed $\mathcal{J} \neq \emptyset$ and $\beta > 0$ (if $\mathcal{J} = \emptyset$, the set $E_k(B, \mathcal{J}, \beta)$ is trivially a singleton).

To this end, we fix a non-empty $\mathcal{J} \in \Upsilon$ and $\beta > 0$, and suppose that $(a_1, \ldots, a_{m-q}) \in E_k(B, \mathcal{J}, \beta)$. Then $(a_1, \ldots, a_{m-q})$ belongs to $E_k^N(B, \mathcal{J}, \beta)$ for infinitely many $N$’s. Fix such an $N$. By definition, there exist $\omega \in [B^{-1}, B]$, and $a_{m-q+1}, \ldots, a_m$ for which (4.13) holds (recall that we do not exclude the case $q = 0$; then no additional $a_j$’s are fixed). Write

$$\omega \sum_{j=1}^{m} a_j \theta_j^N = K_n + \varepsilon_n, \quad K_n \in \mathbb{N}, \quad |\varepsilon_n| \leq 1/2, \quad n \geq 1,$$
so that $\|\omega \sum_{j=1}^{m} a_j \theta^n_j\| = |\varepsilon_n|$. Denote

$$\vec{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}, \quad \vec{K}_n = \begin{pmatrix} K_n \\ \vdots \\ K_{n+m-1} \end{pmatrix}, \quad \vec{\varepsilon}_n = \begin{pmatrix} \varepsilon_n \\ \vdots \\ \varepsilon_{n+m-1} \end{pmatrix};$$

then equations (4.17) for $n, n + 1, \ldots, n + m - 1$ combine into

$$\omega \begin{pmatrix} \theta_1^n & \cdots & \theta_m^n \\ \vdots & \ddots & \vdots \\ \theta_1^{n+m-1} & \cdots & \theta_m^{n+m-1} \end{pmatrix} \vec{a} = \vec{K}_n + \vec{\varepsilon}_n. \quad (4.18)$$

Let $\text{Diag}[\theta^n_j]$ be the diagonal matrix with the diagonal entries $\theta^n_1, \ldots, \theta^n_m$, then (4.18) becomes

$$\omega \Theta \cdot \text{Diag}[\theta^n_j] \vec{a} = \vec{K}_n + \vec{\varepsilon}_n, \quad n \geq 1,$$

where $\Theta$ is the Vandermonde matrix (4.11). The Vandermonde matrix is invertible, since $\theta_j$ are all distinct, and all $\theta_j$ were assumed nonzero, hence

$$\vec{a} = \omega^{-1} \text{Diag}[\theta^{-n}_j] \Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n), \quad n \geq 1. \quad (4.19)$$

It follows that

$$a_j = \omega^{-1} \theta^{-n}_j [\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_j, \quad j = 1, \ldots, m, \quad n \geq 1, \quad (4.20)$$

where $[\cdot]_j$ denotes the $j$-th component of a vector. Since $\omega \in [B^{-1}, B], a_1 = 1$, and $|a_j| \in [\beta^{-1}, \beta]$ for $j \in J$, we have

$$B^{-1} \theta_1^n \leq |[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_1| \leq B \theta_1^n, \quad n \geq 1, \quad (4.21)$$

and

$$(B \beta)^{-1} |\theta_j|^n \leq |[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_j| \leq B \beta |\theta_j|^n, \quad j \in J, \quad n \geq 1. \quad (4.22)$$

From (4.20), recalling that $a_1 = 1$, we obtain

$$a_j = \frac{\theta^{-n}_j [\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_j}{\theta^{-n}_1 [\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_1}, \quad (4.23)$$

and we want to show that $a_j$ is approximately equal to

$$\frac{\theta^{-n}_j [\Theta^{-1}\vec{K}_n]_j}{\theta^{-n}_1 [\Theta^{-1}\vec{K}_n]_1}$$

for $j \in J$ and $n$ sufficiently large.

In fact,

$$\frac{|[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_j|}{|[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_1|} - \frac{|[\Theta^{-1}\vec{K}_n]_j|}{|[\Theta^{-1}\vec{K}_n]_1|} \leq \frac{|[\Theta^{-1}\varepsilon_n]_1|}{|[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_1|} + \frac{|[\Theta^{-1}\varepsilon_n]_1 [\Theta^{-1}\vec{K}_n]_1|}{|[\Theta^{-1}(\vec{K}_n + \vec{\varepsilon}_n)]_1 [\Theta^{-1}\vec{K}_n]_1|}, \quad (4.24)$$
Observe that
\[ \|\Theta^{-1}\varepsilon_n\|_\infty \leq \|\Theta^{-1}\|_\infty \|\varepsilon_n\|_\infty \leq (1/2)\|\Theta^{-1}\|_\infty =: C_\Theta, \]
hence (4.21) and (4.22) yield two-sided estimates of \(\|[\Theta^{-1}K_n]_j\|\) for \(j \in \{1\} \cup \mathcal{J}, \ n \geq 1\). Thus we can continue (4.24) to obtain for \(j \in \mathcal{J}, \ n \geq 1\):
\[ (4.24) \leq \frac{C_\Theta}{B^{-1}q_1}(1 + \frac{B\beta|\theta_j|^n + C_\Theta}{B^{-1}q_1 - C_\Theta}) \leq 2BC_\Theta\theta_1^{-n} \]
for \(n\) sufficiently large, depending on \(\beta, B, \) and \(\Theta\), since \(|\theta_j| < \theta_1\). Therefore, by (4.23),
\[ (4.25) \quad \left| a_j - \frac{\theta_j^{-n}[\Theta^{-1}K_n]_j}{\theta_1^{-n}[\Theta^{-1}K_n]_1} \right| \leq 2BC_\Theta \cdot |\theta_j|^{-n}, \quad j \in \mathcal{J}, \ n \geq n_0(\Theta, B, \beta). \]
It is crucial, of course, that \(|\theta_j| > 1\) for \(j \in \mathcal{J} \subset \{2, \ldots, m - q\}\).

The last inequality will be useful a bit later. Now, comparing (4.19) with the same equality for \(n + 1\), we obtain
\[ (4.26) \quad K_{n+1} + \varepsilon_{n+1} = \Theta \text{Diag}[\theta_j] \Theta^{-1}(K_n + \varepsilon_n). \]

**Lemma 4.7.** Let \(\rho\) and \(L\) be the constants given by (4.12). Consider arbitrary \(\omega > 0\) and \(\vec{a} = (a_1, \ldots, a_m) \in \mathcal{H}^{m-1}\), and define \(K_n, \varepsilon_n, n \geq 1\), by the formula (4.17).

(i) if \(\max\{|\varepsilon_n|, \ldots, |\varepsilon_{n+m}|\} < \rho\), then \(K_{n+m}\) is uniquely determined by \(K_n, K_{n+1}, \ldots, K_{n+m-1}\), independent of \(\omega\) and \(\vec{s}\);

(ii) given \(K_n, K_{n+1}, \ldots, K_{n+m-1}\), there are at most \(L\) possibilities for \(K_{n+m}\).

**Proof.** It follows from (4.26) that
\[ \|K_{n+1} - \Theta \text{Diag}[\theta_j] \Theta^{-1}K_n\|_\infty \leq \|\varepsilon_{n+1}\|_\infty + \theta_1\|\Theta\|_\infty \|\Theta^{-1}\|_\infty \|\varepsilon_n\|_\infty \]
we used \(\|\text{Diag}[\theta_j]\|_\infty = \theta_1\) here). Note that \(K_{n+m} = (K_{n+1})_m\). In part (i) we have \(\max\{\|\varepsilon_n\|, \|\varepsilon_{n+1}\|\} < \rho\) here, hence
\[ |K_{n+m} - (\Theta \text{Diag}[\theta_j] \Theta^{-1}K_n)_m| < \rho(1 + \theta_1\|\Theta\|_\infty \|\Theta^{-1}\|_\infty) \leq 1/2 \]
by (4.12), hence \(K_{n+m}\) is determined uniquely, being an integer. In part (ii) we have
\[ |K_{n+m} - (\Theta \text{Diag}[\theta_j] \Theta^{-1}K_n)_m| \leq (1 + \theta_1\|\Theta\|_\infty \|\Theta^{-1}\|_\infty)/2, \]
and the number of possible integers \(K_{n+m}\) is at most \(1 + \theta_1\|\Theta\|_\infty \|\Theta^{-1}\|_\infty + 1 = L\), as desired. \(\square\)

Now we conclude the proof of Proposition 4.6. Recall that we are now working with the set \(E_k(B, \mathcal{J}, \beta)\), see (4.10), and we estimate its Hausdorff dimension from above by producing efficient covers of the sets \(E^N_k(B, \mathcal{J}, \beta)\) for sufficiently large \(N\). We take an arbitrary point
\[ (a_1, \ldots, a_{m-q}) \in E^N_k(B, \mathcal{J}, \beta) \]
and find the numbers $K_n, \varepsilon_n$ from (4.17). The inequality (4.25) was proved for $n \geq n_0(\theta, B, \beta)$, and we apply it for $n = N - m + 1$. Using that

$$|\theta_{m-q}| = \min_{j \leq m-q} |\theta_j| > 1,$$

we obtain that $(a_1, \ldots, a_{m-q})$ is contained in the closed $\ell^{\infty}$ ball of radius $2BC_\Theta \cdot |\theta_{m-q}|^{-N+m-1}$, centred at the point

$$(x_1, \ldots, x_{m-q}), \text{ where } x_1 = 1; \ x_j = 0, \ j \notin J; \text{ and } x_j = \frac{\theta_j^{-N+m-1}[1 - 1]}{\theta_1^{-N+m-1}[1 - 1]}, \ j \in J.$$

The number of such balls does not exceed the number of possible vectors $\tilde{K}_{N-m+1}$. This, in turn, is bounded above by the number of possible sequences $K_1, \ldots, K_N$. Now we use the crucial assumption (4.13) in the definition of the set $E^N_k(B) \supset E^N_k(B, J, \beta)$. The set

$$\{n \in [1, N] : |\varepsilon_n| \geq \rho\}$$

has cardinality less than $N/k$, and we can enlarge it arbitrarily to get a set $\Gamma \subset [1, N] \cap \mathbb{N}$ with $	ext{card}(\Gamma) = \lceil \frac{N}{k} \rceil$. There are $\lceil \frac{N}{N/k} \rceil$ such subsets $\Gamma$, and it remains to estimate the number of possible sequences $K_1, \ldots, K_N$ for a fixed $\Gamma$.

Since $(a_1, \ldots, a_m) \in \mathcal{F}(\Delta^{m-1})$, it follows from (4.10) that there exists a constant $C_3 > 0$, depending only on the substitution matrix and $v$, such that $|a_j| \leq C_3$, $j = 1, \ldots, m$. Further, $|\omega| \in [B^{-1}, B]$, hence (4.17) implies an upper bound

$$|K_n| \leq BC_3 m \theta_1^n + 1, \ n \geq 1.$$

Thus, there are at fewer than $C_4$ possibilities for the number of initial parts of the sequence $K_1, \ldots, K_m$, where $C_4 = (BC_3 m \theta_1^n + 1)^m$.

Now we fix $\Gamma \subset [1, N] \cap \mathbb{N}$ and consider those $(a_1, \ldots, a_{m-q})$ for which $|\varepsilon_n| < \rho$ for $n \in [1, N] \setminus \Gamma$. Once $K_1, \ldots, K_N$ are determined, for $m \leq n \leq N - 1$, we check whether $\{n - m + 1, \ldots, n + 1\}$ intersects $\Gamma$. If it does, there are at most $L$ possibilities for $K_{n+1}$ by Lemma (4.7) ii). If it does not, then there is only one choice of $K_{n+1}$. It follows that the number of sequences $K_1, \ldots, K_N$ for the given $\Gamma$ does not exceed $C_4 \cdot L^{(m+1)|\text{card}(\Gamma)|}$.

Thus, the total number sequences, hence the balls of radius $2BC_\Theta \cdot |\theta_{m-q}|^{-N+m-1}$ needed to cover $E^N_k(B, J, \beta)$ is at most

$$C_4 \binom{N}{[N/k]} \cdot L^{(m+1)|N/k|}.$$

Therefore,

$$\dim_H(E_k(B, J, \beta)) \leq \lim_{N \to \infty} -\log \left( C_4 \binom{N}{[N/k]} \cdot L^{(m+1)|N/k|} \right) \leq \frac{\log [2L^{m+1}k]}{k \log |\theta_{m-q}|}.$$

The proof of Proposition 4.6 is complete. Theorem 4.3 is proved completely. \qed
5. Self-similar suspension flows

We continue to study suspension flows \((X^\xi, h_t)\) over substitution \(\mathbb{Z}\)-actions, but now we focus on the special choice of the roof function, which makes the system “geometrically self-similar”: namely we assume that \(\vec{s}\) is the Perron-Frobenius eigenvector of the transpose substitution matrix \(S^t: S^t \vec{s} = \theta \vec{s}\). In this case

\[
|\zeta^n(v)|_\xi = \langle S^n \tilde{\ell}(v), \vec{s} \rangle = \langle \tilde{\ell}(v), (S^t)^n \vec{s} \rangle = \theta^n \langle \tilde{\ell}(v), \vec{s} \rangle = \theta^n |v|_\xi.
\]

Self-similar suspension flows over substitutions are a special case of self-similar tiling dynamical systems, studied in [39, 7] and many other papers. Much of what we do in this paper can be extended to the tiling setting. Below we omit the superscript \(\vec{s}\) from the notation, and let \((X^\xi, h_t)\) be the self-similar substitution suspension flow.

The connection between self-similar substitutions and self-similar substitution suspension flows can be expressed using the formalism of Vershik’s automorphisms. Recall that a general construction of Vershik [40] endows an arbitrary ergodic automorphism of a Lebesgue space with a sequence of Rokhlin towers that intersect in Markovian way; the initial automorphism is thus represented as a Vershik automorphism of a Markov compactum. Orbits of our automorphism are identified with leaves of the tail equivalence relation in the Markov compactum.

In the particular case of self-similar substitutions, the construction of Vershik and Livshits [41] yields a representation of our initial substitution automorphism as an “adic transformation” in a special Markov compactum, the space of one-sided infinite paths in a fixed finite graph. Considering now the space of bi-infinite paths, we arrive at a natural symbolic coding of the self-similar translation flow, whose orbits are again identified with leaves of the tail equivalence relation in the two-sided Markov compactum, see [5]. Self-similar renormalization is effectuated by the Markov shift; note that in the two-sided case, the invariant measure of the flow is precisely the measure of maximal entropy for the Markov shift.

In the last section we proved Hölder continuity for almost every suspension flow. However, this says nothing about specific examples. Our next theorem is a quantitative result in the self-similar case, which is weaker (logarithmic modulus of continuity instead of the Hölder property), but has the advantage of being specific.

**Theorem 5.1.** Let \(\zeta\) be a primitive aperiodic substitution on \(A = \{1, \ldots, m\}\), and let \(\theta = \theta_1\) be the Perron-Frobenius eigenvalue of the substitution matrix. Suppose that \(\theta\) admits a Galois conjugate \(\theta_2\) outside the unit circle. Let \((X^\xi, h_t)\) be the self-similar suspension flow over the substitution dynamical system \((X^\xi, T^\xi)\), and let \(\sigma_a\) be the spectral measure corresponding to the cylinder set \([a]\) for \(a \in A\). Then there exists \(\gamma > 0\) such that for any \(B > 1\) there exist \(C_B\) and \(r_0(B)\) such that

\[
\sigma_a([\omega - r, \omega + r]) \leq C_B (\log(1/r))^{-\gamma}, \quad \text{for all } 0 < r \leq r_0(B), \quad a \in A, \quad |\omega| \in [B^{-1}, B].
\]
Remark 5.2. The assumption that $\theta$ admit a Galois conjugate $\theta_2$ outside the unit circle is equivalent to saying that $\theta$ is not a PV or Salem number. The definition of PV and Salem numbers is recalled in the Appendix (Section 7.1).

Remark 5.3. If $\theta$ is a PV number, then the substitution suspension flow has a dense point spectrum, so (5.1) cannot hold [39, 9]. We do not know whether (5.1) holds for Salem numbers $\theta$.

Using symbolic coding of translation flows along stable foliations of pseudo-Anosov automorphisms by suspension flows over Vershik’s automorphisms (see Section 1.8.2 in [5] and references therein) we obtain the following

Corollary 5.4. Suppose that $g$ is a pseudo-Anosov diffeomorphism on a surface $M$ such that the induced action $g^*$ on $H^1(M, \mathbb{R})$ has dominant eigenvalue with at least one conjugate outside the unit circle. Then the conclusion of Theorem 5.1 holds for translation flows along stable/unstable foliations of $g$.

For the proof of Theorem 5.1 we need an analog of Lemma 3.1 for $\mathbb{R}$-actions, which has also been essentially worked out by Hof [26]. Let $(Y, \mu, h_t)$ be a measure-preserving $\mathbb{R}$-action. For $f \in L^2(Y, \mu)$, $R > 0$, $\omega \in \mathbb{R}$, and $y \in Y$ let

$$G_R(f, \omega) = R^{-1} \left\| \int_0^R e^{-2\pi i \omega t} f \circ h_t dt \right\|^2 \quad \text{and} \quad S^y_R(f, \omega) = \int_0^R e^{-2\pi i \omega t} f \circ h_t dt,$$

so that

$$G_R(f, \omega) = R^{-1} \int_Y |S^y_R(f, \omega)|^2 d\mu(y). \tag{5.2}$$

Lemma 5.5. Let $\Omega(r)$ be a continuous increasing function on $[0, 1)$, such that $\Omega(0) = 0$, and suppose that for some fixed $\omega \in \mathbb{R}$, $R_0 \geq 1$, $C > 0$, and $f \in L^2(Y, \mu)$ we have

$$G_R(f, \omega) \leq CR\Omega(1/R) \quad \text{for} \ R \geq R_0. \tag{5.3}$$

Then

$$\sigma_f(\omega - r, \omega + r) \leq \frac{\pi^2 C}{4} \Omega(2r) \quad \text{for all} \ r \leq (2R_0)^{-1}. \tag{5.4}$$

In particular, if

$$\sup_{y \in Y} |S^y_R(f, \omega)| \leq R\sqrt{C\Omega(1/R)} \quad \text{for} \ R \geq R_0, \tag{5.5}$$

then (5.4) holds.

The proof, which follows Hof [26], is deferred to Appendix. The key ingredient of the proof of Theorem 5.1 is the following proposition, together with Proposition [4, 5] and Lemma 5.5. Recall that the height of a polynomial is the maximum of absolute values of its coefficients.
Proposition 5.6. Suppose that $\theta = \theta_1 > 1$ is an algebraic integer which has at least one conjugate $\theta_2$ satisfying $|\theta_2| > 1$. Then there exist $C_4 > 0$ and $\alpha, \beta > 0$ such that
\[
\exp\left(-\sum_{k=0}^{N-1} \|t\theta^k\|^2\right) \leq \begin{cases} C_4(\log(1+t))^{1/\log \beta} \cdot N^{-\alpha}, & N \geq 1, \quad \text{if } t \geq 1; \\ C_4N^{-\alpha}, & N \geq 2\left\lfloor \frac{\log(1+t)}{\log \theta} \right\rfloor, \quad \text{if } t \in (0,1). \end{cases}
\]
In fact, we can take
\[
\alpha = \frac{(1+sH)^2}{\log \beta}, \quad \beta = 1 + \left\lfloor \frac{s \log \theta}{\log |\theta_2|} \right\rfloor.
\]
where $H$ is the height of the minimal polynomial of $\theta$ and $s$ is its degree.

Let us derive Theorem 5.1 first. Passing to $\zeta^k$ if necessary, we can assume that a return word $v$ as in Proposition 4.5 exists and we fix this word for the duration of the proof. Then we are going to apply (4.6) with $\omega \in [B^{-1}, B]$, keeping in mind that $|\zeta^k(v)|_{\ell^1} = \theta^k|v|_{\ell^1}$ in the self-similar case. We obtain for $(x,t) \in \mathcal{X}_\zeta$ and $R > 1$:
\[
|S_{R,t}(\mathbb{I}_{\mathcal{X}_a}, \omega)| \leq C'R \exp\left(-c_1 \sum_{k=0}^{\lfloor c_2 \log R \rfloor} \|\omega|_{\ell^\infty} \cdot \theta^k \right) \leq C_B' R (\log R)^{-c_1},
\]
using Proposition 5.6 in the last step. Then Lemma 5.5 applies with $\Omega(r) = (\log(1/r))^{-2c_1}$ since (5.5) holds with $Y = \mathcal{X}_\zeta$ and $f = \mathbb{I}_{\mathcal{X}_a}$ for $R_0 = R_0(B)$. This yields the desired estimate for the spectral measure. Now Theorem 5.1 is proved completely, modulo Proposition 5.6.

Interestingly, there is a close relationship between estimates of expressions like $\exp(-\sum_{k=0}^{N-1} \|t\theta^k\|^2)$ and decay estimates of the Fourier transforms of Bernoulli convolutions and other self-similar measures with uniform contraction rates. We discuss Bernoulli convolutions and this connection in the Appendix.

Proof of Proposition 5.6 The argument uses techniques common in the work of Pisot and Salem and the theory of numbers named after them; it also shares some common features with the proof of Proposition 4.6. We will use symbols $\preceq$ and $\succeq$ to indicate inequalities up to a multiplicative constant, depending only on $\theta$. Let $s$ be the degree of $\theta$ and let
\[
q(x) = x^s - b_1 x^{s-1} - \ldots - b_s, \quad b_j \in \mathbb{Z},
\]
be the minimal polynomial for $\theta$. First assume that $t \geq 1$ and write
\[
(5.7) \quad t\theta^k = p_k + \varepsilon_k, \quad p_k \in \mathbb{N}, \quad \varepsilon_k \in [-1/2, 1/2), \quad \text{for } k \geq 1.
\]
Thus $\|t\theta^k\| = |\varepsilon_k|$; note that $p_1 \geq 1$ by the assumption $t \geq 1$. The numbers $p_k$ and $\varepsilon_k$ depend on $t$; we suppress this dependence in the notation, but should keep it in mind. Since the sequence $\{t\theta^k\}_{k \geq 1}$ satisfies the recurrence relation with characteristic polynomial $q(x)$, we have
\[
p_{k+s} + \varepsilon_{k+s} = b_1(p_{k+s-1} + \varepsilon_{k+s-1}) + \cdots + b_s(p_k + \varepsilon_k) \quad \text{for } k \geq 1,
\]
hence

\[ p_{k+s} - b_1p_{k+s-1} - \ldots - b_sp_k = -(\varepsilon_{k+s} - b_1\varepsilon_{k+s-1} - \ldots - b_s\varepsilon_k). \]

Let \( H = \max_{j \leq s} |b_j| \) denote the height of the polynomial \( q \). Since the left-hand side of (5.8) is an integer, if

\[ \max\{|\varepsilon_k|, |\varepsilon_{k+1}|, \ldots, |\varepsilon_{k+s}|\} < (1 + sH)^{-1} =: \delta_1, \]

then the expression in (5.8) is less than one in absolute value, hence zero, and then

\[ \tilde{\varepsilon}_{k+1} := \begin{pmatrix} \varepsilon_{k+1} \\ \varepsilon_{k+2} \\ \vdots \\ \varepsilon_{k+s-1} \\ \varepsilon_{k+s} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_k \\ \varepsilon_{k+1} \\ \vdots \\ \varepsilon_{k+s-2} \\ \varepsilon_{k+s-1} \end{pmatrix} =: A\tilde{\varepsilon}_k. \]

Note that \( A \) is the companion matrix of the polynomial \( q(x) \), so its eigenvalues are precisely \( \theta \) and its conjugates, and they are simple. We see that

\[ \max\{|\tilde{\varepsilon}_k|, |\tilde{\varepsilon}_{k+1}|, \ldots, |\tilde{\varepsilon}_{k+n}|\} < \delta_1 \implies \tilde{\varepsilon}_{k+j} = A^j\tilde{\varepsilon}_k \quad \text{for } j = 1, \ldots, n. \]

Now we express \( \tilde{\varepsilon}_k \) as a linear combination of eigenvectors for \( A \). Our goal is to show that the coefficient corresponding to \( \theta_2 \) is not too small in absolute value, so we can estimate the norms of \( \tilde{\varepsilon}_{k+j} \) from below, and for \( n \sim k \) the implication (5.10) will lead to a contradiction.

Let \( \theta = \theta_1, \theta_2, \ldots, \theta_s \) be the (real and complex) zeros of \( q(x) \). Denote by \( \{\tilde{e}_{\theta_j}\}_{j \leq s} \), respectively \( \{e_{\theta_j}\}_{j \leq s} \), the eigenvectors of \( A \) and \( A^t \), which can be explicitly written as follows:

\[ \tilde{e}_{\theta_j} = \begin{pmatrix} 1 \\ \theta_j \\ \vdots \\ \theta_j^{s-1} \end{pmatrix}, \quad e_{\theta_j} = \begin{pmatrix} b_s\theta_j^{s-2} \\ b_{s-1}\theta_j^{s-2} + b_s\theta_j^{s-3} \\ \vdots \\ b_2\theta_j^{s-2} + \cdots + b_{s-1}\theta_j + b_s \end{pmatrix}. \]

The coefficient corresponding to \( \theta_2 \) in the eigenvector expansion for \( \tilde{\varepsilon}_k \) equals

\[ a_2 = \frac{\langle \tilde{e}_k, e_{\theta_2}^t \rangle}{\langle \tilde{e}_{\theta_2}, e_{\theta_2}^t \rangle}. \]

The denominator in the formula for \( a_2 \) depends only on \( \theta \). As for the numerator, it follows from (5.7) that

\[ \tilde{\varepsilon}_k = t\theta_2 k \tilde{e}_{\theta_1} - \tilde{p}_k, \quad \text{where } \tilde{p}_k = [p_k, p_{k+1}, \ldots, p_{k+s-1}]^t. \]

Thus,

\[ \langle \tilde{e}_k, e_{\theta_2} \rangle = -\langle \tilde{p}_k, e_{\theta_2} \rangle. \]
It follows from (5.11) that $(\tilde{p}_k, e^T_{β_2}) = Q(θ_2)$, where $Q$ is a nontrivial polynomial of degree $s - 1$, with integer coefficients, of height at most $p_{k+s-1}H \lesssim t^κ$. Since $Q(θ_2) \neq 0$, applying the classical estimate of Garsia [22, Lemma 1.51] we obtain

$$|Q(θ_2)| \geq \frac{\prod_{|θ_j| \neq 1, j \neq 2} |θ_j| - 1}{s^{s-2}(\prod_{|θ_j| > 1, j \neq 2} |θ_j|)^s \text{Height}(Q)^s} \gtrsim t^{-κ}s^{-κs}.$$

We have proved that $|a_2| \gtrsim t^{-κ}s^{-κs}$. Let $||| \cdot |||$ be a norm in $\mathbb{R}^s$ adapted to the eigenvector expansion for $A$ (e.g. the sup norm of the vector of coordinates with respect to the basis of eigenvectors given by (5.11)); then $|||v_k||| \geq |a_2|$, hence (5.10) yields the implication

$$\max\{|||v_k|||, |||v_{k+1}|||, \ldots, |||v_{k+n}|||\} < δ_1 \implies |||v_{k+n}||| \geq |θ_2|^n|a_2| \gtrsim |θ_2|^nt^{-κ}s^{-κs}.$$

Since all norms in $\mathbb{R}^s$ are equivalent, this will be a contradiction with $|||v_{k+n}||| \leq 1/2$ for $n \geq k \frac{s \log θ}{\log |θ_2|} + \frac{s \log t}{\log |θ_2|} + K$, where $K ∈ \mathbb{N}$ depends only on $θ$. Let

$$β := \lceil s \log θ / \log |θ_2| \rceil + 1.$$

Then

$$kβ - 1 \geq k \frac{s \log θ}{\log |θ_2|} + \frac{s \log t}{\log |θ_2|} + K \text{ for } k \geq k_0 := \lceil s \log t / \log |θ_2| \rceil + K + 1,$$

hence

$$∀ k \geq k_0, \quad \max\{|||v_k|||, |||v_{k+1}|||, \ldots, |||v_{kβ-1}|||\} \geq δ_1.$$

This implies that

$$\exp\left(-\sum_{k=k_0}^{k_0β^n} |||tθ^k|||^2\right) \leq \exp(-nδ_1^2),$$

therefore,

$$\exp\left(-\sum_{k=0}^{N} |||tθ^k|||^2\right) \lesssim k_0^{1/\log β} \exp\left(-\frac{\log N}{\log β} \cdot δ_1^2\right) \lesssim (\log(1 + t))^{1/\log β} \cdot N^{-\frac{sβ}{\log β}}\cdot \frac{1}{\log β},$$

which completes the proof of (5.6) for $t \geq 1$. Finally, for $t \in (0, 1)$ let $j = \lceil \log(t^{-1}) / \log θ \rceil = \min\{j \geq 1 : tζ^j \geq 1\}$. Then taking $τ = tζ^j$ we have from the case already proven, for $N \geq 2j$:

$$\exp\left(-\sum_{k=0}^{N} |||tθ^k|||^2\right) \leq \exp\left(-\sum_{k=0}^{N-j} |||τθ^k|||^2\right) \lesssim (N - j)^{-α} \lesssim (N/2)^{-α},$$

which yields the desired estimate. \qed
6. Spectral measure at zero for self-similar substitution flows

Here we continue the study of the spectrum for the self-similar substitution suspension flow $(X_\zeta, h_t, \bar{\mu})$ over the substitution $\mathbb{Z}$-action $(X_\zeta, T_\zeta, \mu)$, as in Section 5, but now we focus on the behavior of spectral measures $\sigma_f$ in the neighborhood of zero. Lemma 5.5 and related results in the literature indicate that this behavior is controlled by the growth of the ergodic integrals $\int_0^t f \circ h_\tau(x) \, d\tau$. (Of course, we need to take $f$ orthogonal to constants, in order to avoid the trivial zero eigenvalue.) Using available results about asymptotic behavior of such ergodic integrals, we obtain much more precise results near zero than at other points.

Let $\zeta$ be a primitive aperiodic substitution on the alphabet $A = \{1, \ldots, m\}$, with substitution matrix $S$. In this section we assume, in addition, that $S$ has a positive real second eigenvalue greater than one, which dominates the remaining eigenvalues:

\[
\theta = \theta_1 > \theta_2 > |\theta_3| \geq \cdots, \quad \theta_2 > 1.
\]

In order to state our theorem, we need to recall results about the asymptotic behavior of ergodic integrals

\[
S(f, x, t) = \int_0^t f \circ h_\tau(x) \, d\tau
\]

for an appropriate class of test functions. In what follows, we use the notation and terminology from [7], specialized to $d = 1$. Recall that $X_\zeta = \{x = (y, u) : y \in X_\zeta, \ 0 \leq u \leq s_{y_0}\}$, where $\bar{s} = [s_1, \ldots, s_m]$ is the Perron-Frobenius eigenvector for $S^t$. We say that a function $f$ on $X_\zeta$ is cylindrical if it is integrable with respect to $\bar{\mu}$ and depends only on the "tile" containing the origin. More precisely, there exist functions $\psi_j \in L^1([0, s_j])$ for $j \leq m$, such that

\[
f(y, u) = \psi_{y_0}(t), \quad 0 \leq t \leq s_{y_0}.
\]

In the self-similar case we have a "geometric" substitution action

\[Z : X_\zeta \rightarrow X_\zeta,\]

which is hyperbolic (in the sense of Smale spaces) and satisfies the relation

\[Z \circ h_t = h_{\theta t} \circ Z,\]

see [7, 2.5]. Following [5, 6, 7], let $(\Phi^+_{2,x})_{x \in X_\zeta}$ be the Hölder cocycle (or finitely-additive measure) defined on the algebra generated by line segments in $\mathbb{R}$ (which may be identified with the unstable set of $x$ with respect to $Z$), corresponding to the second eigenvalue $\theta_2$. We will write

\[\Phi^+_{2,x}(t) := \Phi^+_{2,x}([0, t]).\]

Further, let $\Phi^-_2$ be the finitely-additive measure on the transversal corresponding to $\theta_2$, which induces a finitely-additive invariant measure $m_{\Phi^-_2}$ for the flow $h_t$. There is an explicit formula for
\[ m_{\Phi_2}(f) = \sum_{j=1}^{m} (\vec{e}_2)_j \int_0^{s_j} \psi_j(t) \, dt, \]

where \( \vec{e}_2 \) is the eigenvector of \( S \) corresponding to \( \theta_2 \) and \( f \) is defined by (6.3).

For a cylindrical \( f \in L^2(X_\zeta, \tilde{\mu}) \) with \( \int f \, d\tilde{\mu} = 0 \), an asymptotic formula \( S(f, x, t) \) is a very special case of results in [5]. It is also a special case (for \( d = 1 \)) of [7, Theorem 4.3] on self-similar tiling dynamical systems in \( \mathbb{R}^d \). We have

\[ S(f, x, t) = \Phi_{2,x}^{+}(t) \cdot m_{\Phi_2}(f) + R(t), \]

where

\[ |R(t)| \leq C \max(1, |t|^{\alpha - \varepsilon}), \]

and \( \Phi_{2,x}^{+}(t) = \theta_2 \Phi_{2,x}^{+}(t) \), see [7, Lemmas 3.3, 3.3]. The cocycle \( \Phi_{2,x}^{+}(t) \) is non-degenerate in the following sense:

- for a given \( x \in X_\zeta \), the function \( t \mapsto \Phi_{2,x}^{+}(t) \) is not a.e. zero [5, Prop. 2.26];
- for a given \( t \in \mathbb{R}_+ \), the function \( x \mapsto \Phi_{2,x}^{+}(t) \) is not \( \tilde{\mu} \)-a.e. constant [5, Prop. 2.28].

We should note that a similar result for substitution \( \mathbb{Z} \)-actions has been obtained by Dumont and Thomas [14], and the corresponding cocycles were further studied in [13].

**Theorem 6.1.** Let \( \zeta \) be a primitive aperiodic substitution on the alphabet \( A = \{1, \ldots, m\} \), with substitution matrix \( S \) satisfying (6.7). Let \( f \) be a cylindrical function with zero mean \( \int f \, d\tilde{\mu} = 0 \) and \( m_{\Phi_2}(f) \neq 0 \). Let \( \sigma_f \) be the corresponding spectral measure on the line. Then there exists a non-trivial positive \( \sigma \)-finite Borel measure \( \eta \) on \( \mathbb{R} \), such that

\[ \lim_{N \to \infty} \frac{\sigma_f([-c\theta^{-N}, c\theta^{-N}])}{\theta^{-N(2-2\alpha)}} = \eta([-c, c]) \quad \text{for all} \ c > 0 \ \text{such that} \ \eta(\{c\}) = 0, \]

where

\[ \alpha = \log(\theta_2/\log \theta) \in (0, 1). \]

**Remark 6.2.** 1. The conditions on the cylindrical functions can be verified directly: first, we have \( \int f \, d\tilde{\mu} = \sum_{j=1}^{m} (\vec{e}_2)_j \int_0^{s_j} \psi_j(t) \, dt \), where \( \vec{e}_1 \) is the eigenvector of \( S \) corresponding to \( \theta_1 \). Second, the value of \( m_{\Phi_2}(f) \) is computed in a similar way, see (6.4).

2. Note that if \( \alpha < 1/2 \), then our result shows, informally speaking, that the spectral measure “has a zero of order \( 1 - 2\alpha \)” at zero.

3. An estimate of the spectral measure at zero can be obtained using Hof’s argument [26], together with well-known estimates of ergodic sums for substitutions [14 4]. Note, however, that
this general argument gives much cruder estimates than those of Theorem 6.1 and, in particular, does not give an asymptotics for the spectral measure.

Proof of Theorem 6.1. Recall that the spectral measure $\sigma_f$ on $\mathbb{R}$ is defined by

$$\int_{-\infty}^{\infty} e^{2\pi i \omega t} \, d\sigma_f(\omega) = \langle f \circ h_t, f \rangle \quad \text{for } t \in \mathbb{R}.$$ 

Moreover, there is spectral isomorphism between $L^2(\mathbb{R}, \sigma_f)$ and a closed subspace of $L^2(\mathbb{R}, \tilde{\mu})$ which transforms the unitary group of multiplication by $\{e^{2\pi i \omega t}\}_{t \in \mathbb{R}}$ into the unitary group $g \mapsto g \circ h_t$. As a consequence, this spectral isomorphism maps $e^{2\pi i \omega t}$ (as a function of $\omega$) to $f \circ h_t(x)$ (a function of $x$) for all $t \in \mathbb{R}$.

Our goal is to analyse the behavior of $\sigma_f$ near zero. To this end, we fix a function $\psi \in \mathcal{S} = \mathcal{S}(\mathbb{R})$, the Schwartz class of smooth test functions, and study the integral

$$\int |\psi(T\omega)|^2 \, d\sigma_f(\omega) = T^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \circ h_t(x) \cdot \hat{\psi}(t/T) \, dt \, dx.$$

The latter is a function of $\omega$ in $L^2(d\sigma_f)$, which is mapped by the spectral isomorphism to

$$T^{-1} \int_{-\infty}^{\infty} f \circ h_t(x) \cdot \hat{\psi}(t/T) \, dt,$$

a function of $x$ in $L^2(\mathbb{R}, \tilde{\mu})$. Since the spectral isomorphism preserves the norm, we obtain

$$\int |\psi(T\omega)|^2 \, d\sigma_f(\omega) = T^{-2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f \circ h_t(x) \cdot \hat{\psi}(t/T) \, dt \, dx.$$

Integration by parts yields, in view of (6.2) and (6.5):

$$\int_{-\infty}^{\infty} f \circ h_t(x) \cdot \hat{\psi}(t/T) \, dt = -T^{-1} \int_{-\infty}^{\infty} S(f, x, t)(\hat{\psi})(t/T) \, dt$$

$$= -T^{-1} \int_{-\infty}^{\infty} \left( \Phi_2^+(t) \cdot m_{\Phi_2}(f) + \mathcal{R}(t) \right)(\hat{\psi})(t/T) \, dt.$$

First let us estimate the error term. Since $\hat{\psi}$ is in $\mathcal{S}$, we have

$$|\nabla(\hat{\psi})(t)| \leq C_{\psi, \alpha} \min(1, |t|^{-\alpha-1}).$$

Together with (6.6), this yields

$$T^{-1} \left| \int_{-\infty}^{\infty} \mathcal{R}(t)(\hat{\psi})(t/T) \, dt \right| \leq 2C_1C_{\psi, \alpha} T^{-1} \left( 1 + \int_{1}^{T} t^{\alpha-\varepsilon} \, dt + \int_{T}^{\infty} t^{\alpha-\varepsilon} (t/T)^{-\alpha-1} \, dt \right)$$

$$= O(T^{\alpha-\varepsilon}).$$

For the main term, we assume that

$$T = \theta^N, \quad N \geq 1,$$
let \( t = \theta^N \tau \), and use renormalization \( (6.7) \):

\[
\Phi_{2,x}^+(t) = \Phi_{2,x}^+(\theta^N \tau) = \theta^N \Phi_{2,z-N(x)}^+(\tau) = T^\alpha \Phi_{2,z-N(x)}^+(\tau).
\]

After the change of variable the main term in \( (6.11) \) becomes

\[
-T^\alpha m_{\Phi_2}^-(f) \int_{\mathbb{R}} \Phi_{2,z-N(x)}^+(\tau) \left( \hat{\psi} \right)'(\tau) \, d\tau,
\]

We substitute this and \( (6.12) \) into \( (6.11) \), which is then substituted into \( (6.10) \) to obtain

\[
\int |\psi(T\omega)|^2 \, d\sigma_f(\omega) = T^{2\alpha-2} (m_{\Phi_2}^-(f))^2 \int_{X_\xi} \left| \int_{\mathbb{R}} \Phi_{2,x}^+(\tau) \left( \hat{\psi} \right)'(\tau) \, d\tau \right|^2 \, d\tilde{\mu}(x) + O(T^{2\alpha-2-\epsilon}),
\]

as \( T \to \infty \), where we used that \( \tilde{\mu} \) is \( Z \)-invariant \( (7 \text{ Lemma 2.12}) \). Finally, letting \( T = \theta^N \), with \( N \to \infty \), we obtain

\[
(6.13) \quad \lim_{N \to \infty} \frac{\int |\psi(\theta^N \omega)|^2 \, d\sigma_f(\omega)}{(\theta^N)^{2\alpha-2}} = (m_{\Phi_2}^-(f))^2 \int_{X_\xi} \left| \int_{\mathbb{R}} \Phi_{2,x}^+(\tau) \left( \hat{\psi} \right)'(\tau) \, d\tau \right|^2 \, d\tilde{\mu}(x).
\]

Note that the right-hand is non-trivial (not constant zero), since \( \Phi_{2,x}^+ \) is non-degenerate and we have freedom in the choice of \( \psi \). Observe, moreover, that the entire argument can be repeated for a pair of real functions \( \psi_1, \psi_2 \in S(\mathbb{R}) \), which yields

\[
Q(\psi_1, \psi_2) := \lim_{N \to \infty} \frac{\int \psi_1(\theta^N \omega) \psi_2(\theta^N \omega) \, d\sigma_f(\omega)}{\theta^N(2\alpha-2)} =
\]

\[
(6.14) \quad (m_{\Phi_2}^-(f))^2 \int_{X_\xi} \left( \int_{\mathbb{R}} \Phi_{2,x}^+(\tau) \left( \hat{\psi}_1 \right)'(\tau) \, d\tau \cdot \int_{\mathbb{R}} \Phi_{2,x}^+(\tau) \left( \hat{\psi}_2 \right)'(\tau) \, d\tau \right) \, d\tilde{\mu}(x).
\]

We thus have a bilinear continuous functional \( Q(\psi_1, \psi_2) \) on \( S \times S \) that depends only on the product \( \psi_1 \psi_2 \), not identically zero and is nonnegative on nonnegative functions. We now recall the following well-known fact from the theory of distributions: Let \( D \) be a tempered distribution on the real line such that for any Schwartz function \( \varphi \geq 0 \) we have \( D(\varphi) \geq 0 \). Then there exists a \( \sigma \)-finite positive Radon measure \( \eta \) on \( \mathbb{R} \) such that

\[
D(\varphi) = \int_{\mathbb{R}} \varphi \, d\eta.
\]

**Claim.** There exists a \( \sigma \)-finite positive Radon measure \( \eta \) on \( \mathbb{R} \) such that for any \( \psi_1, \psi_2 \in C^0_\infty \) we have

\[
Q(\psi_1, \psi_2) = \int_{\mathbb{R}} \psi_1 \psi_2 \, d\eta.
\]

In order to prove the claim, observe that for any \( \psi_1 \in S \), \( \psi_1 \geq 0 \), the functional \( \varphi \mapsto Q(\psi_1, \varphi) \) is a tempered distribution which is non-negative on \( \varphi \geq 0 \), hence there exists a measure \( \eta_{\psi_1} \) such that

\[
Q(\psi_1, \varphi) = \int_{\mathbb{R}} \varphi \, d\eta_{\psi_1}, \quad \varphi \in S.
\]
Now note that $Q(\psi_1 \psi_2, \varphi) = Q(\psi_1, \psi_2 \varphi)$, hence

\[ d\eta_{\psi_1 \psi_2} = \psi_2 d\eta_{\psi_1}, \quad \text{for } \psi_1, \psi_2 \in S, \ \psi_1 \geq 0, \ \psi_2 \geq 0. \tag{6.15} \]

Take any $\psi > 0$ in $S$, and let

\[ d\eta := \psi^{-1} d\eta_{\psi}. \]

It follows from (6.15) that $\eta$ does not depend on the choice of $\psi$. Now let $\psi_1, \psi_2 \in C_0^\infty$. We have

\[ Q(\psi_1, \psi_2) = Q(\psi, \psi_1 \psi_2/\psi) = \int_{\mathbb{R}} \psi_1 \psi_2 \psi^{-1} d\eta_{\psi} = \int_{\mathbb{R}} \psi_1 \psi_2 d\eta, \]

and the proof of the claim is complete. In the last line we used that $\psi_1 \psi_2 \psi^{-1} \in C_0^\infty \subset S$. □

We can now conclude that the formula

\[ \lim_{N \to \infty} \int_{\mathbb{R}} \psi_1(\theta_1^N \omega) \psi_2(\theta_2^N \omega) d\sigma_f(\omega) = \int_{\mathbb{R}} \psi_1 \psi_2 d\eta, \tag{6.16} \]

holds not just for $\psi_1, \psi_2$ in $C_0^\infty$, but also if $\psi_1$ and $\psi_2$ are continuous compactly supported functions or characteristic functions of intervals whose endpoints are points of continuity of the measure $\eta$. Indeed, take $a, b \in \mathbb{R}$ with $\eta(\{a\}) = \eta(\{b\}) = 0$, let $\psi = \psi_1 = \psi_2 = \chi_{[a,b]}$, and choose sequences $\psi^{(n,+)}$, $\psi^{(n,-)}$ of compactly supported $C^\infty$ functions approximating $\chi_{[a,b]}$ from above and below, converging to $\chi_{[a,b]}$ pointwise and, for any $\delta > 0$, uniformly on the complement to the set

\[ (a - \delta, a + \delta) \cup (b - \delta, b + \delta). \]

Since $\eta$ does not have atoms at $a$ and $b$, for any $\varepsilon > 0$ there exists a sufficiently small $\delta$ such that

\[ \eta((a - \delta, a + \delta) \cup (b - \delta, b + \delta)) < \varepsilon. \]

Consequently,

\[ \lim_{n \to \infty} \int_{\mathbb{R}} (\psi^{(n,+)} \psi^{(n,+)} - \psi^{(n,-)} \psi^{(n,-)}) d\eta = 0, \]

and (6.16) holds for $\psi_1 = \psi_2 = \chi_{[a,b]}$. The case of compactly supported continuous functions is obtained even easier, by uniform approximation from above and from below. It remains to substitute the characteristic function $\chi_{[-c,c]}$ for $\psi_1 = \psi_2$ to obtain (6.8), as desired. The theorem is proved completely. □

7. Appendix

In this section we collect some additional statements, definitions and proofs, which complement the main body of the paper.

**Proof of Lemma 2.7** It is enough to check that the Fourier coefficients of the a.c. measures in the right-hand side converge to $\hat{\sigma}_{f,g}(k)$ for all $k \in \mathbb{Z}$. Note that

\[ \langle e^{-2\pi i n \omega} U^n f, e^{-2\pi i \ell \omega} U^\ell g \rangle = \langle U^n f, g \rangle e^{-2\pi i (n-\ell) \omega}, \]

and $(e^{-2\pi i \omega} d\omega)(-k) = \delta_{k,n}$, and the claim follows easily. □
Proof of Lemma 3.1. It follows from (3.3) that (3.6) implies (3.4), so we only need to show that (3.4) implies (3.5). Analogously to [26] we have
\[ G_N(f, \omega) = \int_T K_{N-1}(\omega - x) d\sigma_f(x), \quad \text{where} \quad K_{N-1}(y) = \frac{1}{N} \left( \frac{\sin(N\pi y)}{\sin(\pi y)} \right)^2 \]
is the Fejér kernel. Given \( r \in (0, (2N_0)^{-1}) \), take \( N = \lceil 1/2r \rceil \). Then for \( |\omega - x| \leq r \) we have \( N\pi|\omega - x| \leq \frac{\pi}{2} \), hence
\[ |\sin(N\pi(\omega - x))| \geq \frac{2}{\pi} N\pi|\omega - x| \quad \text{and} \quad |\sin(\pi(\omega - x))| \leq \pi|\omega - x|. \]
Therefore, \( K_{N-1}(\omega - x) \geq 4N/\pi^2 \), and we obtain
\[ \sigma_f(B(\omega, r)) \leq \int_{\omega-r}^{\omega+r} \frac{\pi^2}{4N} K_{N-1}(\omega - x) d\sigma_f(x) \leq \frac{\pi^2 G_N(f, \omega)}{4N} \]
\[ \leq \frac{\pi^2 C}{4} \Omega(1/N) \leq \frac{\pi^2 C}{4} \Omega(3r), \]
since \( N^{-1} \leq 3r \), and the proof is complete. \( \square \)

Proof of Lemmas 5.5 and 4.4. We have
\[ G_R(f, \omega) = R^{-1} \left\langle \int_0^R e^{-2\pi i \omega y} f \circ h_y dy, \int_0^R e^{-2\pi i \omega z} f \circ h_z dz \right\rangle \]
\[ = R^{-1} \int_0^R \int_0^R e^{-2\pi i (y-z)\omega} (f \circ h_{y-z}, f) dy dz \]
\[ = R^{-1} \int_0^R \int_0^R e^{-2\pi i (y-z)\omega} \int_R e^{2\pi i (y-z)\tau} d\sigma_f(\tau) dy dz \]
\[ = R^{-1} \int_R \left( \int_{[0,R]^2} e^{2\pi i (y-z)(\tau-\omega)} dy dz \right) d\sigma_f(\tau) \]
\[ = \int_R K_R(\omega - \tau) d\sigma_f(\tau), \]
where
\[ K_R(y) = R^{-1} \left( \frac{\sin(\pi R y)}{\pi y} \right)^2 \]
is the Fejér kernel for \( \mathbb{R} \). Taking \( r = \frac{1}{2R} \), we have \( K_R(y) \geq \frac{4R}{\pi^2} \) on \([-r, r]\), hence \( \sigma_f([\omega - r, \omega + r]) \leq \frac{\pi^2 C}{4R} G_R(f, \omega) \), and (5.4) follows. This concludes the proof of Lemma 5.5 and Lemma 4.4 is a special case for substitution suspension flows, with \( \Omega(r) = r^{2-2\alpha}. \) \( \square \)
7.1. Local dimension of spectral measures. The lower local dimension of a measure $\nu$ at $\omega$ is defined by
\[
d(\nu, \omega) = \liminf_{r \to 0} \frac{\log \nu(B(\omega, r))}{\log r}.
\]
The upper local dimension is defined similarly, with $\limsup$; the local dimension is said to exist if there is a limit.

**Lemma 7.1.** Let $\zeta$ be a primitive aperiodic substitution on $A = \{1, \ldots, m\}$, $\omega \in [0, 1)$, and $M_n(\omega)$ are $m \times m$ matrices defined by (2.8). Suppose that $\theta$ is the Perron-Frobenius eigenvalue of the substitution matrix. Let
\[
\alpha_{\omega} = \limsup_{n \to \infty} \|M_{n-1}(\omega) \cdots M_0(\omega)\|^{1/n}.
\]
Then
\[
(7.2) \quad \limsup_{N \to \infty} \frac{\sup_{x \in X_\zeta} \log |\Phi_a(x[0, N - 1], \omega)|}{\log N} \leq \frac{\log \alpha_{\omega}}{\log \theta} \quad \text{for all } a \in A
\]
and
\[
(7.3) \quad d(\sigma_a, \omega) \geq 2 - \frac{2 \log \alpha_{\omega}}{\log \theta} \quad \text{for all } a \in A.
\]

**Proof.** To show (7.2), take an arbitrary $x \in X_\zeta$ and use the prefix-suffix decomposition (3.13) and (3.14), together with (2.9) and (2.6) — the result is then immediate.

In view of (3.7) and (3.3), the inequality (7.2) implies
\[
\limsup_{N \to \infty} \frac{\log G_N(\|a\|, \omega)}{\log N} \leq \frac{2 \log \alpha_{\omega}}{\log \theta} - 1.
\]
Now (7.1), which holds for all $r$ sufficiently small, with $N = \lfloor 1/2r \rfloor$, yields (7.3). \qed

**Remark 7.2.** There is a similar statement for suspension flows from Sections 4 and 5. In that setting,
\[
\alpha_{\omega} = \limsup_{n \to \infty} \|M_{n-1}^\zeta(\omega) \cdots M_0^\zeta(\omega)\|^{1/n},
\]
where $M_j^\zeta$ are given by (4.7). The matrix product takes a particularly pleasant form in the self-similar setting of Section 5, because then $M_j^\zeta(\omega) = M_0^\zeta(\theta^j \omega)$.

7.2. Classes of algebraic integers and Bernoulli convolutions.

**Definition 7.3.** An algebraic integer $\theta > 1$ is called a Pisot, or PV (Pisot-Vijayarghavan) number, if all its Galois conjugates, i.e. other zeros of the minimal polynomial for $\theta$, are all inside the unit circle. An algebraic integer $\theta > 1$ is called a Salem number, if it has no conjugates outside the unit circle and at least one conjugate on the unit circle. (In this case necessarily $\theta$ has even degree $\geq 4$ and all conjugates except $\theta^{-1}$ are on the unit circle).
See [36] for the basic theory of PV and Salem numbers, as well as their connection to harmonic analysis. These classes of algebraic integers play an outstanding role, not just in the theory of Diophantine approximation and uniform distribution, but also in dynamical systems and fractal geometry, in particular, in the study of Bernoulli convolutions.

**Definition 7.4.** The (infinite) Bernoulli convolution measure $\nu_\lambda$ with parameter $\lambda \in (0, 1)$ is the distribution of the random series $\sum_{n=0}^{\infty} \pm \lambda^n$, where the signs are chosen randomly and independently with probabilities $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Bernoulli convolution measures have been studied extensively since the 1930’s, but there are still many difficult open problems; among them an outstanding one is to decide for which $\lambda \in (1/2, 1)$ the Bernoulli convolution measure $\nu_\lambda$ is absolutely continuous. The following discussion is not supposed to be comprehensive; many of the statements naturally extend to more general classes of self-similar measures. We will, however, mention the case of “biased” Bernoulli convolutions $\nu_\lambda^p$, defined analogously to $\nu_\lambda$, but with the signs chosen with probabilities $(p, 1-p)$, for $p \in (\frac{1}{2}, 1)$.

The early work on Bernoulli convolutions focused on their Fourier transforms:

$$\hat{\nu}_\lambda(\xi) = \prod_{n=0}^{\infty} \cos(2\pi \lambda^n \xi).$$

Erdős [16] proved that $\hat{\nu}_\lambda(\xi) \not\to 0$ as $\xi \to \infty$ when $\lambda^{-1}$ is a PV number, hence such $\nu_\lambda$ are singular. Salem [35] proved that this characterizes PV numbers, namely, for all other $\lambda$ the Fourier transform vanishes at infinity. However, his proof does not yield any quantitative estimates of the decay of the Fourier transform. Erdős [17] proved that for almost every $\lambda$ the Fourier transform has some power decay, which he then used to show that $\nu_\lambda$ is absolutely continuous for almost every $\lambda$ sufficiently close to one. Kahane [29] observed that Erdős’ proof actually gives an estimate of the dimension of the exceptional set, and in fact the power decay of the Fourier transform (for some power depending on $\lambda$) holds for all $\lambda$ outside a set of Hausdorff dimension zero. This is what we called the “Erdős-Kahane argument” in Section 4. The survey [33] contains a detailed exposition of these results (Erdős’ Theorem with Kahane’s improvement) with quantitative estimates. Garsia [22] proved that if $\theta = \lambda^{-1}$ is an algebraic integer whose conjugates are all outside the unit circle and the constant term of the minimal polynomial equals $\pm 2$, then $\nu_\lambda$ is absolutely continuous (even with bounded density). As of this writing, this still remains the largest explicitly known class of $\lambda$ for which the Bernoulli convolution is absolutely continuous. It was proved in [11, Theorem 1.6] that the Fourier transform for these “Garsia” Bernoulli convolutions has a power decay at infinity. Using different techniques (integration over the parameter and “transversality method”), Solomyak [38] established absolute continuity of $\nu_\lambda$ for a.e. $\lambda \in (1/2, 1)$. For further advances using the transversality method, we refer the reader to [33]. Recently Hochman [25], using ideas from additive combinatorics, proved that for all $\lambda \in (1/2, 1)$ outside a set of Hausdorff dimension zero and for all algebraic $\lambda \in (1/2, 1)$, the Hausdorff dimension of $\nu_\lambda$ equals one.
Having dimension one for a measure falls just short of absolute continuity, but even more recently Shmerkin [37] combined Hochman’s Theorem with the Erdős-Kahane result mentioned above to obtain absolute continuity outside a zero-dimensional set of exceptions.

We observe that Proposition 5.6 immediately implies the following

Corollary 7.5. Let \( \theta \) be an algebraic integer which has at least one conjugate outside the unit circle, and let \( \lambda = \theta^{-1} \). Then for any \( p \in (0, 1) \) there exists \( \alpha > 0 \) such that

\[
\sup_{\xi \in \mathbb{R}} |\hat{\nu}_\lambda^p(\xi)| (\log(2 + |\xi|))^\alpha < \infty.
\]

Proof. Let \( \xi \in [\theta^N, \theta^{N+1}] \), and set \( \zeta = \theta^{-N} \xi \). Then

\[
|\hat{\nu}_\lambda^p(\xi)| = \prod_{n=0}^{\infty} \left| pe^{-2\pi i \lambda^n \xi} + (1 - p)e^{2\pi i \lambda^n \xi} \right| \\
\leq \prod_{n=0}^{N} \left| p + (1 - p)e^{4\pi i \theta^n \zeta} \right| \\
\leq \prod_{n=0}^{N} \left( 1 - \frac{1}{2} \left\|2\theta^n \zeta\right\|^2 \right),
\]

using (3.17), and Proposition 5.6 yields the result.

Remark 7.6. 1. Recently X.-R. Dai [10] obtained a similar result for rationals \( \lambda = a/b \), with \( 1 < a < b \).

2. Corollary 7.5 extends to the class of self-similar measures of the form \( \nu = \sum_{j=1}^{m} p_j (\nu \circ f_j^{-1}) \), where \( f_j(x) = \lambda x + a_j \), \( a_j \in \mathbb{Z} \), and \((p_1, \ldots, p_m)\) is a probability vector (the Bernoulli convolution \( \nu_\lambda \) is a special case with \( m = 2 \), \( a_1 = -1, a_2 = 1 \)).

3. If \( \theta > 1 \) is the Perron-Frobenius eigenvalue of a substitution matrix, and a biased Bernoulli convolution \( \nu_\lambda^p \), with \( \lambda = \theta^{-1} \) and \( p \neq \frac{1}{2} \), has a power decay of the Fourier transform with exponent \( \gamma \), then \( \sup_{t \in \mathbb{R}} \exp(N\gamma - \sum_{k=0}^{N-1} \|t\theta^k\|^2) < \infty \) and as a consequence, spectral measures of the self-similar suspension flow are Hölder continuous. (The “bias” is important here, because the factors \( pe^{-2\pi i \lambda^n \xi} + (1 - p)e^{2\pi i \lambda^n \xi} \) are bounded away from zero in absolute value.) However, power decay of the Fourier transform is not known for any biased Bernoulli convolution; the result of [11, Theorem 1.6] covers the unbiased case only.

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Alexander I. Bufetov, Laboratoire d’Analyse, Topologie, Probabilités, aix-Marseille Université, CNRS, Marseille, France; Steklov Institute, Moscow; The Institute for Information Transmission Problems, Moscow; National Research University Higher School of Economics, Moscow, Russia

Rice University, Houston TX USA

E-mail address: bufetov@mi.ras.ru

Boris Solomyak, Box 354350, Department of Mathematics, University of Washington, Seattle, WA, USA

E-mail address: solomyak@uw.edu