KRULL–GABRIEL DIMENSION OF COHEN–MACAULAY MODULES
OVER HYPERSURFACES OF COUNTABLE COHEN–MACAULAY
REPRESENTATION TYPE

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Abstract. We calculate the Krull–Gabriel dimension of the functor category of the (sta-
ble) category of maximal Cohen–Macaulay modules over hypersurfaces of countable Cohen–
Macaulay representation type. We show that the Krull–Gabriel dimension is 0 if the hyper-
surface is of finite Cohen–Macaulay representation type and that is 2 if the hypersurface is of
countable but not finite Cohen–Macaulay representation type.

1. Introduction

The notion of Krull–Gabriel dimension has been considered under a functorial approach view-
point of representation theory of finite dimensional algebras. It was introduced by Gabriel [6] and
has been studied by many authors including Geigle [7] and Schröer [21]. The notion is considered
for an abelian category and defined by a length of a certain filtration of Serre subcategories.
See Section 2 for the precise definition. Let $A$ be a finite dimensional algebra and $\text{mod}(A)$ a
category of finitely generated $A$-modules. The functor category $\text{mod}(\text{mod}(A))$ of $\text{mod}(A)$ is an
abelian category, so that the Krull-Gabriel dimension of $\text{mod}(\text{mod}(A))$, which is denoted by
$\text{KGdim mod(mod}(A))$, can be investigated. The Krull–Gabriel dimension is closely related to
representation types of algebras. It was proved by Auslander [2] that $A$ is of finite representation
type if and only if $\text{KGdim mod(mod}(A)) = 0$. Krause [13] shows that there are no algebras
such that $\text{KGdim mod(mod}(A)) \neq 1$ and Geigle [7] shows that every tame hereditary algebra is of
Krull–Gabriel dimension 2. Geigle [7] also shows that an algebra which is of wild representation
type has Krull–Gabriel dimension $\infty$.

Let $R$ be a commutative Cohen–Macaulay local ring and $C(R)$ the category of maximal
Cohen–Macaulay $R$-modules. In this paper, we study the Krull–Gabriel dimension of the functor
category of $C(R)$. More precisely we calculate the Krull–Gabriel dimension of $\text{mod}(C(R))$; the
category of finitely presented contravariant additive functors $F$ with $F(R) = 0$ from $C(R)$ to a
category of abelian groups. First, we shall show the following theorem, which gives an analogy
of a result due to Auslander.

Theorem 1.1. (Theorem 2.6) Let $R$ be a complete Cohen–Macaulay local ring. Then $R$ is of
finite Cohen–Macaulay representation type if and only if $\text{KGdim mod}(C(R)) = 0$.

Let $k$ be an algebraically closed uncountable field of characteristic not two. Next, we inves-
tigate the case when $R$ is a hypersurface that is of countable but not finite Cohen–Macaulay
representation type. Namely $R$ is isomorphic to the ring $k[[x_0, x_1, x_2, \ldots, x_n]]/(f)$, where $f$ is of
the following:

$$f = \begin{cases} x_0^2 + x_1^2 + \cdots + x_n^2 & (A_\infty), \\ x_0^2x_1 + x_2^2 + \cdots + x_n^2 & (D_\infty). \end{cases}$$
Theorem 1.2. [Corollary 5.10] Let $k$ be an algebraically closed uncountable field of characteristic not two. Let $R$ be a hypersurface of countable but not finite Cohen–Macaulay representation type. Then $\text{KGD} \dim \mod(\mathcal{C}(R)) = 2$.

The studies of Krull–Gabriel dimension of maximal Cohen–Macaulay modules over 1-dimensional hypersurfaces of type $(A_\infty)$ and $(D_\infty)$ are given by Puninski [18] and by Los and Puninski [15]. Their studies investigate the Krull–Gabriel dimension of the definable category of maximal Cohen–Macaulay modules in the category of all $R$-modules so that our studies are different from theirs.

The organization of this paper is as follows. In Section 2, we review the notion of Krull–Gabriel dimension and consider the case when $R$ is of finite Cohen–Macaulay representation type (Theorem 2.6). Section 3 is devoted to compute the Krull–Gabriel dimension of mod($R$) where $R$ is a 1-dimensional hypersurface singularities of type $(A_\infty)$. In Section 4 we calculate the Krull–Gabriel dimension over a 2-dimensional hypersurface of type $(D_\infty)$. In Section 5 we investigate how Krull–Gabriel dimension changes with Knörrer’s periodicity (Theorem 5.9). Using it we attempt to compute the Krull–Gabriel dimension for higher (or lower) dimensional cases (Corollary 5.10).

2. Preliminaries

Let us recall the definition of Krull–Gabriel dimension for an abelian category. Let $A$ be an abelian category. We say that a full subcategory $S$ of $A$ is a Serre subcategory if $S$ is closed under taking subobjects, quotients, and extensions.

Definition 2.1. [7, Definition 2.1] Let $A$ be an abelian category. Define $A_{-1} = 0$. For each $n \geq 0$, let $A_n$ be the category of all objects which are finite length in $A/A_{n-1}$. We define $\text{KGD} \dim A = \min\{n \mid A = A_n\}$ if such a minimum exists, and $\text{KGD} \dim A = \infty$ else. For an object $X$ in $A$, we define $\text{KGD} \dim X$ by a minimum number $n$ such that $X$ is in $A_n$.

To show a simpleness of an object in a quotient category, the following lemma is useful.

Lemma 2.2. [8, Lemma 1.1] Let $A$ be an abelian category and $S$ the Serre subcategory. For an exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ in $A/S$, there is an exact sequence $0 \rightarrow N' \rightarrow M \rightarrow L' \rightarrow 0$ in $A$ such that $N \cong N'$ and $L \cong L'$ in $A/S$. Therefore the object $X$ of $A$ becomes simple in $A/S$ if $X$ is not an object of $S$ and if for each subobject $V$ of $X$ either $V$ or $X/V$ belongs to $S$.

Let $R$ be a commutative Noetherian ring with a finite Krull dimension. We denote by $\mod(R)$ a category of finitely generated $R$-modules with $R$-homomorphisms. We compute the Krull–Gabriel dimension of $\mod(R)$.

Lemma 2.3. Let $A$ be an abelian category and $S$, $S'$ the Serre subcategories with $S' \subseteq S$. Suppose that $M \cong N$ in $A/S'$. Then $M \in S$ if and only if $N \in S$.

Proof. Since $M \cong N$ in $A/S'$, there is a morphism $f \in \text{Hom}_A(M', N/N')$ where $M/M'$, $N' \in S'$. Then $f$ is a pseudo-isomorphism in $A$, that is $\text{Ker} f$ and $\text{Coker} f$ belong to $S'$ (cf. [8, Lemma 4, p. 367]). Let $N$ belong to $S$. The quotient module $N/N'$ belongs to $S$. One can also show that $\text{Ker} f$ and $\text{Coker} f$ belong to $S$ since $S' \subseteq S$. Thus $M'$ is contained in $S$. Since $M/M'$ belongs to $S$ we have $M \in S$. The converse holds by the same argument. \qed

Proposition 2.4. Let $R$ be a commutative Noetherian ring with a finite Krull dimension. Then $\text{KGD} \dim(\mod(R)) = \dim R$.

Proof. We denote by $S_i$ the subcategory of $\mod(R)$ consisting of all finitely generated $R$-modules $M$ with $\dim M \leq i$. Notice that $S_i$ is a Serre subcategory (see [22, Example 4.5.(9)]). We shall show $R/p$ is a simple object in $\mod(R)/\mod(R)_{i-1}$ for a prime ideal $p$ with $\dim R/p = i$ and $\mod(R)_i = S_i$. We prove it by induction on $i$. First, for a maximal ideal $m$, $R/m$ is a field, so
that it is simple in mod($R$). We also remark that a finitely generated $R$-module $M$ has finite length iff $\dim M = 0$. Thus mod($R$)$_0 = S_0$. Let $i > 0$ and $p$ be a prime ideal $p$ with $\dim R/p = i$. We consider an exact sequence $0 \rightarrow V \rightarrow R/p \rightarrow C \rightarrow 0$ in mod($R$). Localizing by $p$, we have $0 \rightarrow V_p \rightarrow (R/p)_p \rightarrow C_p \rightarrow 0$. Since $(R/p)_p$ is a field, $V_p$ or $C_p = 0$. One has $\text{Ass}(V) \subseteq \text{Ass}(R/p) = \{p\}$, so that $V_p \neq 0$. Thus we have $C_p = 0$. Let $q \in \text{Supp}(C)$. Then $(R/p)_q \neq 0$, thus $p \subseteq q$. Since $C_p = 0$, $\text{ann}(C) \not\subseteq p$. Hence $p \not\subseteq q$. It yields that $\dim C < \dim R/p = i$, so that $C \in S_{i-1}$. By the induction hypothesis, one has $S_{i-1} = \text{mod}(R)_{i-1}$. Consequently, $R/p$ is simple in mod($R$)/mod($R$)$_{i-1}$ by Lemma $2.2$. Suppose that $S$ is a simple object in mod($R$)/mod($R$)$_{i-1}$. According to [22] Theorem 4.1, Example 4.5.(9)], a finitely generated $R$-module $M$ belongs to $S_i$ iff $\text{Ass}(M)$ is contained in $P_i = \{p \in \text{Spec}(R) \mid \dim R/p \leq i\}$. Since $S$ is not in mod($R$)$_{i-1}$, $\text{Ass}(S)$ is not contained in $P_{i-1}$. For a prime ideal $p$ in $\text{Ass}(S)$, $S$ has a submodule that is isomorphic to $R/p$. If the prime ideal $p$ satisfies $\dim R/p = i$, $S$ is isomorphic to $R/p$ in mod($R$)/mod($R$)$_{i-1}$ for the simplicity of $S$. Assume that a prime ideal $p$ with $\dim R/p > i$ belong to $\text{Ass}(S)$. Then $R/p$ is also isomorphic to $S$ in mod($R$)/mod($R$)$_{i-1}$ since $R/p \not\subseteq S_{i-1} = \text{mod}(R)_{i-1}$. However, since $\dim R/p > i$, there exists a prime ideal $q$ such that $p \subseteq q$ and $\dim R/q = i$. Moreover one has the exact sequence in mod($R$): $0 \rightarrow q/p \rightarrow R/p \rightarrow R/q \rightarrow 0$. Notice that $\text{Ass}(q/p) \subseteq \text{Ass}(R/p)$. Particularly $q/p$ is not in mod($R$)$_{i-1}$. This concludes that $R/p$, hence $S$, is not simple in mod($R$)/mod($R$)$_{i-1}$, and it is a contradiction. The observation shows that every simple object in mod($R$)/mod($R$)$_{i-1}$ is isomorphic to $R/p$ with $\dim R/p = i$.

Let $M$ be a finitely generated $R$-module. According to [17] Theorem 6.4, we have a filtration of $M$ in mod($R$)

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

such that $M_k/M_{k-1} \cong R/p_k$ with prime ideals $p_k$. Suppose that $\dim M = i$. Then $\dim R/p_k \leq i$ for the prime ideals $p_k$ since $\text{Ass}(M) \subseteq \{p_1, \ldots, p_n\}$. Since $R/p$ with $\dim R/p = i$ is a simple object in mod($R$)/mod($R$)$_{i-1}$, $M$ belongs to mod($R$)$_i$. This shows that $S_i \subseteq \text{mod}(R)_i$. Conversely, for each $M \in \text{mod}(R)_i$, we have a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ in mod($R$)/mod($R$)$_{i-1}$ such that $M_k/M_{k-1} \cong R/p_k$ in mod($R$)/mod($R$)$_{i-1}$ with $\dim R/p_k = i$. By Lemma $2.3$, $M_k/M_{k-1}$ belongs to $S_i$ since $R/p_k$ is in $S_i$. Hence $M$ belongs to $S_i$. Consequently mod($R$)$_i = S_i$.

For each $M \in \text{mod}(R)$, one has $\text{KDim} M \leq \max\{\dim R/p_k \mid k = 1, \ldots, n\}$ by the filtration (2.1). Thus $\text{KDim} \text{mod}(R) \leq \sup\{\dim R/p \mid p \in \text{Spec}R\} \leq \dim R$. On the other hand, take a minimal associated prime ideal $p$ of $R$, then $\dim R/p = \dim R$, so that $\dim R \leq \text{KDim} \text{mod}(R)$. Therefore we obtain $\text{KDim} \text{mod}(R) = \dim R$. \qed

From now we focus on a category of maximal Cohen–Macaulay (abbr. MCM) modules. In the rest of the paper we always assume that $(R, \mathfrak{m})$ is a complete CM local ring. We denote by $C(R)$ the full subcategory of mod($R$) consisting of all MCM $R$-modules and by $C_0(R)$ the full subcategory of $C(R)$ consisting of all modules that are locally free on the punctured spectrum of $R$.

Now let us recall the full subcategory of the functor category of $C(R)$ which is called the Auslander category. We give a brief review of the Auslander category. See [23] Chapter 4 and 13) for the details. The Auslander category mod($C(R)$) is the category whose objects are finitely presented contravariant additive functors from $C(R)$ to a category of abelian groups and whose morphisms are natural transformations between functors. We denote by mod($C(R)$) the full subcategory mod($C(R)$) consisting of functors $F$ with $F(R) = 0$. Note that every object $F \in \text{mod}(C(R))$ is obtained from a short exact sequence in $C(R)$. Namely, we have the short exact sequence $0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0$ such that

$$0 \rightarrow \text{Hom}_R(, N) \rightarrow \text{Hom}_R(, M) \rightarrow \text{Hom}_R(, L) \rightarrow F \rightarrow 0$$

is exact in mod($C(R)$).
We denote by $\mathcal{C}(R)$ the stable category of $\mathcal{C}(R)$. The objects of $\mathcal{C}(R)$ are the same as those of $\mathcal{C}(R)$, and the morphisms of $\mathcal{C}(R)$ are elements of $\text{Hom}_R(M, N) = \text{Hom}_R(M, N)/P(M, N)$ for $M, N \in \mathcal{C}(R)$, where $P(M, N)$ denote the set of morphisms from $M$ to $N$ factoring through free $R$-modules. For a finitely generated $R$-module $M$, we denote by $\text{syz}_1^R(M)$ the reduced first syzygy of $M$.

**Remark 2.5.** (1) Since $R$ is complete, $\mathcal{C}(R)$, thus $\mathcal{C}(R)$, is a Krull-Schmidt category.
(2) The categories $\text{mod}(\mathcal{C}(R))$ and $\text{mod}(\mathcal{C}(R))$ are abelian categories (cf. [24 (4.17), (4.19)]).
(3) The category $\text{mod}(\mathcal{C}(R))$ is equivalent to the Auslander category $\text{mod}(\mathcal{C}(R))$ of $\mathcal{C}(R)$. See [24 Remark 2.6]. Moreover, according to [24 Remark 4.16], for $F \in \text{mod}(\mathcal{C}(R))$ with $0 \to \text{Hom}_R(\ , N) \to \text{Hom}_R(\ , M) \to \text{Hom}_R(\ , L) \to F \to 0$, we have an exact sequence $\text{Hom}_R(\ , N) \to \text{Hom}_R(\ , M) \to \text{Hom}_R(\ , L) \to F \to 0$.
(4) If $R$ is Gorenstein the stable category $\mathcal{C}(R)$ has a structure of a triangulated category with the suspension (shift) functor defined by $(-)[-1] = \text{syz}_R^1(-)$ (cf. [9]).

In the paper, we use a theory of Auslander-Reiten (abbr. AR) sequences. For the detail, we recommend the reader to refer to [23]. Let $0 \to Z \to Y \to X \to 0$ be an AR sequence in $\mathcal{C}(R)$. Then the functor $S_X$ defined by an exact sequence

$$0 \to \text{Hom}_R(\ , Z) \to \text{Hom}_R(\ , Y) \to \text{Hom}_R(\ , X) \to S_X \to 0$$

is a simple object in $\text{mod}(\mathcal{C}(R))$ and all the simple objects in $\text{mod}(\mathcal{C}(R))$ are obtained in this way from AR sequences (cf. (4.12)).

For a functor $F \in \text{mod}(\mathcal{C}(R))$, we denote by $\text{Supp}(F)$ a set of isomorphism classes of indecomposable $\text{MCM}$ $R$-modules $M$ with $F(M) \neq 0$:

$$\text{Supp}(F) = \{M \mid F(M) \neq 0\} \cong \varnothing.$$

Let us show the first result of the paper, which is an analogical result due to Auslander. We say that $R$ is of finite CM representation type if there are only a finite number of isomorphism classes of indecomposable $\text{MCM}$ $R$-modules.

**Theorem 2.6.** Let $R$ be a complete CM local ring. Then $R$ is of finite CM representation type if and only if KGdim $\text{mod}(\mathcal{C}(R)) = 0$.

**Proof.** Suppose that $R$ is of finite representation type. As mentioned in [23 Chapter 13], every functor $F \in \text{mod}(\mathcal{C}(R))$ has finite length. Hence KGdim $\text{mod}(\mathcal{C}(R)) = 0$.

Conversely suppose that KGdim $\text{mod}(\mathcal{C}(R)) = 0$. According to [10 Lemma 2.1], there exists $X \in \mathcal{C}(R)$ such that $\text{Hom}_R(M, X) \neq 0$ for all non free $\text{MCM}$ $R$-modules $M$. That is, $\text{Supp}(\text{Hom}_R(\ , X)) \cup \{R\} = \text{Ind}(\mathcal{C}(R))$. Since $\text{Hom}_R(\ , X) \in \text{mod}(\mathcal{C}(R))$, $\text{Hom}_R(\ , X)$ belongs to $\text{mod}(\mathcal{C}(R))_0$. Thus there are only finitely many indecomposable $\text{MCM}$ $R$-modules $M$ such that $\text{Hom}_R(M, X) \neq 0$, so that $R$ is of finite CM representation type.

**Remark 2.7.** We note that the Krull–Gabriel dimension of $\text{mod}(\mathcal{C}(R))$ is not always 0 even if $R$ is of finite CM representation type. Actually let $R = k[x]$. Then $\mathcal{C}(R) = \text{add}\{R\}$. Thus $R$ is of finite CM representation type. Since $\text{mod}(\mathcal{C}(R)) = \text{mod}(R)$ (cf. [16 Lemma 6.4]), we have the equality KGdim $\text{mod}(R) = \dim R = 1$ by Proposition 2.4.

Suppose that $R$ is a Gorenstein local ring. Since $\mathcal{C}(R)$ is a triangulated category, we can define

$$\text{Hom}_R(\ , M[-1]) \to \text{Hom}_R(\ , L[-1]) \to F[-1] \to 0$$

for every $F \in \text{mod}(\mathcal{C}(R))$ with $\text{Hom}_R(\ , M) \to \text{Hom}_R(\ , L) \to F \to 0$. For the later use, we state a lemma.

**Lemma 2.8.** Let $R$ be a Gorenstein local ring and $S$ a simple object in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_{n-1}$. Then $S[-1]$ is also simple in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_{n-1}$. 

Then the following equality holds for each indecomposable \( k \)-dimensional \( R \)-module. Here \( k \) is as in Proposition 3.2.

Lemma 3.3. Let \( R \) be an algebraically closed uncountable field of characteristic not two and \( R = k[[x, y]]/(x^2) \). Then \( \text{KDim} \mod(C(R)) = 2 \).

Proof. We prove by induction on \( n \). Suppose that \( S \) is simple in \( \mod(C(R)) \), that is \( S \) is an algebraically closed uncountable field of characteristic not two and \( R = k[[x, y]]/(x^2) \). This section is devoted to calculate the Krull-Gabriel dimension of \( \mod(C(R)) \). It is known that \( R \) is of countable representation type, namely there exist infinitely but only countably many isomorphism classes of indecomposable MCM \( R \)-modules. The non free indecomposable MCM \( R \)-modules are as follows:

\[ \begin{align*}
I &= \text{Coker}(x) : R \rightarrow R, \\
I_n &= \text{Coker}(\begin{pmatrix} x^n & y^n \\ 0 & x \end{pmatrix}) : R^{\oplus 2} \rightarrow R^{\oplus 2} \quad (n \geq 1).
\end{align*} \]

Next we attempt to compute \( \text{dim}_k \text{Hom}_R(I, I_n) \). Let the multiplicity of \( U \) as a direct summand of \( X \) in \( C(R) \).

3. Krull–Gabriel dimension of \( \mod(C(k[[x, y]]/(x^2))) \)

Let \( k \) be an algebraically closed uncountable field of characteristic not two and \( R = k[[x, y]]/(x^2) \). Then \( \text{KDim} \mod(C(R)) = 2 \).

To prove the theorem, we shall do some preparation.

Proposition 3.2. [11] Proposition 2.14 (1)] Let \( 0 \rightarrow Z \rightarrow Y \rightarrow X \rightarrow 0 \) be an AR sequence. Then the following equality holds for each indecomposable \( U \in C(R) \):

\[ \text{dim}_k \text{Hom}_R(U, X) + \text{dim}_k \text{Hom}_R(U, Z) - \text{dim}_k \text{Hom}_R(U, Y) = \mu(U, X) + \mu(U, X[-1]). \]

Here \( \mu(U, X) \) is the multiplicity of \( U \) as a direct summand of \( X \) in \( C(R) \).

Lemma 3.3. Let \( R \), \( I \), and \( I_n \) be as above. The following statements hold.

1. \( \text{dim}_k \text{Hom}_R(I, I_n) = \begin{cases} 2n & m \geq n, \\
2m & m \leq n. \end{cases} \)
2. \( \text{dim}_k \text{Hom}_R(I, I_n) = \text{dim}_k \text{Hom}_R(I_n, I) = n \) for \( n \geq 1 \).
3. \( \text{dim}_k \text{Hom}_R(I, I) = \infty \).

Proof. First we shall compute \( \text{dim}_k \text{Hom}_R(-, I_1) \). Since \( R \) is Gorenstein \( \text{Hom}_R(M, N) \cong \text{Ext}_R^1(M, \text{syz}_R^2(N)) \) (cf. [23] (12.10)). One has \( I_n \cong \text{syz}_R^2(I_n) \). Thus it is enough to compute the dimension of \( \text{Ext}_R^1(-, I_1) \). Notice that that \( I_1 = (x, y)R \). We have the complex:

\[ \begin{array}{c}
\cdots \longrightarrow I_1 \oplus \beta \alpha \gamma \delta I_1 \oplus \cdots \longrightarrow \end{array} \]

and \( \text{Ext}_R^1(I_1, I_1) \cong \text{ker} \alpha/\text{Im} \beta \). Let \( \alpha(b) \in \text{Hom}_R(I_1, I_n) \). Assume that \( \alpha(b) = \begin{pmatrix} x^n & y^n \\ ax & bx \end{pmatrix} \). Since \( ax \in (x^2) \), \( a \in (x) \cap I_1 = (x) \). Moreover, since \( ay^n - bx \in (x^2) \), one has \( axy^n - bx = cx^2 \) for some \( c \in k[[x, y]] \). It implies that \( b = ay^n - cx \in (x, y^n) \) with \( a = ax \). Thus, \( \text{Ext}_R^1(I_n, I_1) \cong \{ \begin{pmatrix} a & a' y^n \\ -ax & -ax + a' y^n \end{pmatrix} \mid a' x \in (x)/(x^2, yx), -ax + a' y^n \in (x, y^n)/(x^2, xy, y^{n+1}) \} \cong k^{\oplus 2} \). Hence \( \text{dim}_k \text{Hom}_R(I_n, I_1) = \text{dim}_k \text{Ext}_R^1(I_n, I_1) = 1 \). Similarly, we have \( \text{Ext}_R^1(I, I_1) \cong (x)/(x^2, yx) \cong k \), so that \( \text{dim}_k \text{Hom}_R(I, I_1) = 1 \).

Next we attempt to compute \( \text{dim}_k \text{Hom}_R(-, I_n) \) by using Proposition 3.4. For each \( n \geq 1 \), there exists an AR-sequence: \( 0 \rightarrow I_n \rightarrow I_{n+1} \oplus I_{n-1} \rightarrow I_n \rightarrow 0 \) where \( I_0 \cong R \). We note again
that every MCM $R$-module $M$ is isomorphic to $\text{syz}_R^1(M)$, namely $M \cong M[-1]$ in $\mathcal{C}(R)$. Then we claim that the following equation holds for $n \geq 1$:

**Claim.** We have the following equation.

(3.1) $\dim_k \text{Hom}_R(-, I_n) = n \dim_k \text{Hom}_R(-, I_1) - \sum_{i=1}^{n-1} 2(n-i)\mu(-, I_i)$.

We prove it by induction on $n$. The case $n = 1$ is clear. For $n > 1$, by Proposition 3.4, we have

$$\dim_k \text{Hom}_R(-, I_{n+1}) = 2 \dim_k \text{Hom}_R(-, I_n) - \dim_k \text{Hom}_R(-, I_{n-1}) - 2\mu(-, I_n).$$

By the induction hypothesis,

$$\dim_k \text{Hom}_R(-, I_{n+1}) = 2 \{n \dim_k \text{Hom}_R(-, I_1) - \sum_{i=1}^{n-1} 2(n-i)\mu(-, I_i)\} - \{(n-1) \dim_k \text{Hom}_R(-, I_1) - \sum_{i=1}^{n-2} 2(n-1-i)\mu(-, I_i)\} - 2\mu(-, I_n)$$

$$= \{2n - (n-1)\} \dim_k \text{Hom}_R(-, I_1) + \sum_{i=1}^{n-1} 2(n+1-i)\mu(-, I_i) - 4\mu(-, I_{n-1}) - 2\mu(-, I_n)$$

$$= (n+1) \dim_k \text{Hom}_R(-, I_1) + \sum_{i=1}^{n} 2(n+1-i)\mu(-, I_i).$$

Hence the equation (3.1) holds. \hfill \Box

Now we calculate the dimension of $\text{Hom}_R(I_m, I_n)$. Suppose that $m \geq n$. By virtue of the equation (3.1), $\dim_k \text{Hom}_R(I_m, I_n) = n \dim_k \text{Hom}_R(I_m, I_1) - \sum_{i=1}^{n-1} 2(n-i)\mu(I_m, I_i) = n \dim_k \text{Hom}_R(I_m, I_1) = 2n$. If $m \leq n$ then we have the equality

$$\dim_k \text{Hom}_R(I_m, I_n) = n \dim_k \text{Hom}_R(I_m, I_1) - \sum_{i=1}^{n-1} 2(n-i)\mu(I_m, I_1) = 2n - 2(n-m)\mu(I_m, I_m) = 2m.$$

Moreover, the equation (3.1) also shows that $\dim_k \text{Hom}_R(I_n, I_n) = n \dim_k \text{Hom}_R(I_n, I_1) - \sum_{i=1}^{n-1} 2(n-i)\mu(I_n, I_1) = n$. By AR duality (see Remark 3.4), we also have $\dim_k \text{Hom}_R(I_n, I) = n$. Therefore the assertions (1) and (2) hold. (3) follows from the isomorphism $\text{Ext}_R^1(I, I) \cong (x)/(x^2) \cong k[y]$. \hfill \Box

**Remark 3.4.** It is known as AR duality that $\text{Hom}_R(\text{Hom}_R(M, N), E(k)) \cong \text{Ext}_R^1(N, \tau M)$ for $N \in \mathcal{C}(R)_0$. Here $E(k)$ is the injective hull of $k$ and $\tau M$ is the AR translation of $M$. See [23 (3.10)]. It follows from the Matlis duality that $\dim_k \text{Hom}_R(M, N) = \dim_k \text{Hom}_R(N, \tau M)$. Hence, in Lemma 3.3 (1)(2), one can show from either equation the other equation. For instance, one has $\dim_k \text{Hom}_R(-, I_n) = \dim_k \text{Hom}_R(I_n, -)$ since $\tau M \cong M$ in this case.

**Lemma 3.5.** For $n > 0$, we have the exact sequence:

(3.2) $0 \to I_n \xrightarrow{(\alpha \tau)} I \oplus R \xrightarrow{(y^n - x)} I \to 0$.

**Proof.** It is straightforward. \hfill \Box

Thanks to Lemma 3.5 we obtain the finitely presented functor:

(3.3) $0 \to \text{Hom}_R(-, I_n) \to \text{Hom}_R(-, I) \oplus \text{Hom}_R(-, R) \to \text{Hom}_R(-, I) \to H_n \to 0$.

First, we shall show the functor $H_1$ is a simple object in $\text{mod}(\mathcal{C}(R))/_0\text{mod}(\mathcal{C}(R))$.

**Proposition 3.6.** The functor $H_1$ is simple in $\text{mod}(\mathcal{C}(R))/_0\text{mod}(\mathcal{C}(R))$. 
Proof. By [24, Proposition 3.3], the exact sequence (3.3) induces the long exact sequence:
\[
\cdots \rightarrow \text{Hom}_R(-, I_1) \rightarrow \text{Hom}_R(-, I) \rightarrow \text{Hom}_R(-, I) \rightarrow 0
\]
(3.4) \[
\rightarrow \text{Hom}_R(-, I_1[-1]) \rightarrow \text{Hom}_R(-, I[-1]) \rightarrow \text{Hom}_R(-, I[-1]).
\]
For each indecomposable \( X \in \mathcal{C}_0(R) \), since \( \dim_k \text{Hom}_R(X, I_1) \) and \( \dim_k \text{Hom}_R(X, I) \) are finite, we have \( \dim_k H_1(X) = \frac{1}{2} \dim_k \text{Hom}_R(X, I_1) = 1 \). Since \( \text{Hom}_R(I, I) \cong k[y] \), one has \( H_1(I) \cong k[y]/yk[y] \). Consequently \( \dim_k H_1(X) = 1 \) for all indecomposable \( X \in \mathcal{C}(R) \).

Let \( 0 \rightarrow V \rightarrow H_1 \rightarrow C \rightarrow 0 \) be an admissible exact sequence in \( \text{mod}(\mathcal{C}(R)) \). Since \( V \in \text{mod}(\mathcal{C}(R)) \), we have the exact sequence \( 0 \rightarrow \text{Hom}_R(-, Z) \rightarrow \text{Hom}_R(-, Y) \rightarrow \text{Hom}_R(-, X) \rightarrow V \rightarrow 0 \). Then, for all \( M \in \mathcal{C}_0(R) \),
\[
\dim_k V(M) = \frac{1}{2} \left\{ \dim_k \text{Hom}_R(M, X) + \dim_k \text{Hom}_R(M, Z) - \dim_k \text{Hom}_R(M, Y) \right\}.
\]
Let \( X = I^{\oplus a_0} \oplus I_1^{\oplus a_1} \oplus \cdots \oplus I_\ell^{\oplus a_\ell}, \ Y = I^{\oplus b_0} \oplus I_1^{\oplus b_1} \oplus \cdots \oplus I_m^{\oplus b_m}, \) and \( Z = I^{\oplus c_0} \oplus I_1^{\oplus c_1} \oplus \cdots \oplus I_n^{\oplus c_n} \).

We put \( m = \max \{ l_1, \ldots, l_\ell, m_1, \ldots, m_m, n_1, \ldots, n_n \} \). For \( m \leq n < \infty \),
\[
\dim_k V(I_n) = \frac{1}{2} \left( \sum_i m \cdot a_i + \sum_i n \cdot c_i - \sum_i m \cdot b_i \right).
\]
This equation yields that \( \dim_k V(I_n) = 0 \) for \( m \leq n < \infty \) since \( V \) is a subfunctor of \( H_1 \).

Assume that \( \dim_k V(I_n) = 0 \) for \( m \leq n \). Then \( V(I_n) = 0 \) except for a finite number of \( I_n \). Namely \( \text{Supp}(V) \) is a finite set, and we shall show \( I \notin \text{Supp}(V) \). If it holds, \( V \) is in \( \text{mod}(\mathcal{C}(R))_0 \). Assume that \( I \in \text{Supp}(V) \). For \( I' \in \text{Supp}(V) \setminus \mathcal{C}_0(R) \), there is an epimorphism from \( V \rightarrow S_{I'} \). (See the proof of [23, (4.12)],.) Put the kernel of the epimorphism as \( V' \). Then \( V' \in \text{mod}(\mathcal{C}(R)) \) and \( \text{Supp}(V') = \text{Supp}(V) \setminus \{ I' \} \). Repeating the procedure, we obtain the functor \( \tilde{V} \in \text{mod}(\mathcal{C}(R)) \) such that \( \text{Supp}(\tilde{V}) = \{ I \} \) and \( \dim_k \tilde{V}(I) = 1 \). It yields that \( \tilde{V} \) is a simple object with \( \tilde{V}(I) \neq 0 \), so that the AR sequence ending in \( I \) exists ([23, (4.13)]). Namely \( I \in \mathcal{C}_0(R) \) ([23, (3.4)]). This is a contradiction. Hence \( I \notin \text{Supp}(V) \).

Assume that \( \dim_k V(I_n) = 1 \) for \( m \leq n \). Then \( \dim_k C(I_n) = 0 \) for \( m \leq n \). Apply the same argument to \( C \) and we also conclude that \( C \) is contained in \( \text{mod}(\mathcal{C}(R))_0 \). Consequently we get the assertion. \( \square \)

Remark 3.7. Since \( H_1 \) is a subfunctor of \( \text{Hom}_R(-, I_1) \), we have an exact sequence \( 0 \rightarrow H_1 \rightarrow \text{Hom}_R(-, I_1) \rightarrow H'_1 \rightarrow 0 \) in \( \text{mod}(\mathcal{C}(R)) \). By the calculation in the proof of Proposition 3.6, \( \dim_k H'_1(I_n) = 1 \) for all \( n \) and \( \dim_k H_1(I) = 0 \). By using the same argument of Proposition 3.6 one can also prove that \( H'_1 \) is a simple object in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \). Therefore, \( \ell(\text{Hom}_R(-, I_1)) = 2 \) in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \).

Proposition 3.8. The length of \( \text{Hom}_R(-, I_n) \) is finite in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \) for \( n \geq 1 \).

Proof. First we claim that \( \ell(\text{Hom}_R(-, I_1)) < \infty \) in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \), and it holds by Proposition 3.6 and Remark 3.7. Suppose that \( n > 1 \). Since there is an AR sequence \( 0 \rightarrow I_n \rightarrow I_{n+1} \oplus I_{n-1} \rightarrow I_n \rightarrow 0 \) ([20, 6.1]), we obtain the sequence: \( \text{Hom}_R(-, I_n) \rightarrow \text{Hom}_R(-, I_{n+1}) \oplus \text{Hom}_R(-, I_{n-1}) \rightarrow \text{Hom}_R(-, I_n) \). Since \( \text{mod}(\mathcal{C}(R))_1 \) is a Serre subcategory, \( \text{Hom}_R(-, I_n) \) belongs to \( \text{mod}(\mathcal{C}(R))_1 \). That is \( \ell(\text{Hom}_R(-, I_n)) = 2 \) in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \). \( \square \)

Remark 3.9. In the Grothendieck group of \( \text{mod}(\mathcal{C}(R)) \), an AR sequence gives \( \text{Hom}_R(-, I_{n+1}) + \text{Hom}_R(-, I_{n-1}) = 2[\text{Hom}_R(-, I_n)] - 2[S_{I_n}] \). Combining the equality with Remark 3.7 one has \( \ell(\text{Hom}_R(-, I_n)) = 2n \) in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \) for \( n \geq 1 \). By [1, Proposition 2.1 (1)], there is an exact sequence \( 0 \rightarrow I \rightarrow I_n \rightarrow I \rightarrow 0 \) for \( n \geq 1 \). Then \( 2\ell(\text{Hom}_R(-, I)) \geq \ell(\text{Hom}_R(-, I_n)) \) in \( \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_0 \). This yields that \( \text{Hom}_R(-, I) \) does not belong to \( \text{mod}(\mathcal{C}(R))_1 \).
Lemma 3.10. Let $F$ be a finitely presented functor with the exact sequence
$$0 \to \text{Hom}_R(-, Z) \to \text{Hom}_R(-, Y) \to \text{Hom}_R(-, X \oplus C) \to F \to 0.$$ 
Then there is an exact sequence of functors
$$F' \xrightarrow{\rho} F \to F/\text{Im}\rho \to 0$$ such that $F'$ and $F/\text{Im}\rho$ are images of $\text{Hom}_R(-, X)$ and $\text{Hom}_R(-, C)$ respectively.

Proof. The following diagram is obtained by the taking pullback:

Thus we get

Thus we get

Proposition 3.11. The functor $\text{Hom}_R(-, I)$ is simple in $\text{mod}(C(R))/\text{mod}(C(R))_1$.

Proof. Let

be an admissible exact sequence in $\text{mod}(C(R))$. Since $V \in \text{mod}(C(R))$, we may assume that $V$ has the sequence $0 \to \text{Hom}_R(-, Z) \to \text{Hom}_R(-, Y) \to \text{Hom}_R(-, X) \to V \to 0$ for some $X,Y,Z \in C(R)$. If $X \in C_0(R)$, $V \in \text{mod}(C(R))_1$ because $V$ is an image of $\text{Hom}_R(-, X)$ ([23, (4.16)]). Thus the proof is completed. Therefore we assume that $X$ contains $I$ as a direct summand. Let $X \cong I^\oplus \oplus M$ where $M \in C_0(R)$. By Lemma 3.10 there exits the sequence $V' \xrightarrow{\rho} V \to V/\text{Im}\rho \to 0$ such that $V'$ and $V/\text{Im}\rho$ are images of $\text{Hom}_R(-, I^\oplus)$ and $\text{Hom}_R(-, M)$.
respectively. Then the commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & V & \rightarrow & \text{Hom}_R(I, I) & \rightarrow & C & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \rightarrow & \text{Im}\rho & \rightarrow & \text{Hom}_R(I, I) & \rightarrow & C' & \rightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 & \rightarrow & V/\text{Im}\rho & \rightarrow & 0 & & 0 & & 0 & \\
\end{array}
\]

is obtained. Since \( V/\text{Im}\rho \in \text{mod}(C(R))_0 \), \( V \cong \text{Im}\rho \) and \( C \cong C' \) in \( \text{mod}(C(R))/\text{mod}(C(R))_0 \). Hence we may consider \( 0 \rightarrow \text{Im}\rho \rightarrow \text{Hom}_R(-, I) \rightarrow C' \rightarrow 0 \) instead of the sequence (3.5). Now we remark that \( \text{Im}\rho \) is also the image of \( \text{Hom}_R(-, I^{\oplus l}) \), so that we have

\[
\text{Hom}_R(-, I^{\oplus l}) \xrightarrow{\psi} \text{Hom}_R(-, I) \rightarrow C' \rightarrow 0.
\]

By Yoneda’s lemma, \( \varphi \in \text{Hom}_R(I^{\oplus l}, I) \). Since \( \text{Hom}_R(I, I) \cong k[y] \), \( \varphi \) is of the form \( (y^{n_1} \ y^{n_2} \ \cdots \ y^{n_l}) \), where \( n_1 \leq n_2 \leq \cdots \leq n_l \). The mapping cone of \( C(\varphi) \) is \( I_{n_1} \oplus I^{\oplus l-1} \). Actually,

\[
C(\varphi) = \text{Coker} \left( \begin{array}{c}
x \\
\vdots \\
x y^{n_l} \\
\end{array} \right) : R^{\oplus l+1} \rightarrow R^{\oplus l+1}.
\]

By considering basic matrix transformations, one can obtain the isomorphism \( C(\varphi) \cong I_{n_1} \oplus I^{\oplus l-1} \). Since \( C(\varphi)[-1] \cong (I_{n_1} \oplus I^{\oplus l-1})[-1] \cong I_{n_1} \oplus I^{\oplus l-1} \), we can make the triangle:

\[
(I_{n_1} \oplus I^{\oplus l-1}) \rightarrow I^{\oplus l} \xrightarrow{\varphi} I \xrightarrow{\psi} (I_{n_1} \oplus I^{\oplus l-1})[1].
\]

Now we claim that \( I^{\oplus l-1} \) is split out. Set \( \psi = \left( \begin{array}{c}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_l \\
\end{array} \right) \), and we shall show \( \psi_i = 0 \) for \( i \geq 2 \). Since \( \psi \circ \varphi = 0 \),

\[
\left( \begin{array}{cccc}
\psi_1 y^{n_1} & \psi_2 y^{n_2} & \cdots & \psi_l y^{n_l} \\
\psi_2 y^{n_1} & \psi_2 y^{n_2} & \cdots & \psi_2 y^{n_l} \\
\vdots & \vdots & \ddots & \vdots \\
\psi_l y^{n_1} & \psi_l y^{n_2} & \cdots & \psi_l y^{n_l} \\
\end{array} \right) = 0.
\]

Then, for \( i \geq 2 \), \( \psi_i y^{n_i} = y^{n_i} \psi_i = 0 \) in \( \text{Hom}_R(I, I) \). Since \( \text{Hom}_R(I, I) \cong k[y] \), \( y^{n_i} \) is a non zero divizor on \( \text{Hom}_R(I, I) \), so that \( \psi_i = 0 \). Consequently, \( \psi_i = 0 \) for all \( i \geq 2 \). Thus we have the
Since $\beta$ may assume that $C$ is a retraction. Thus $\gamma$ is a section. Moreover $C(\psi_1) \cong I$. Therefore we get

$$I \xrightarrow{(\psi_1, 0)} I_n \oplus I^{\oplus l-1} \xrightarrow{\alpha} I^{\oplus l} \longrightarrow I$$

One can see that $\beta$ is a retraction. Thus $\gamma$ is a section. Moreover $C(\psi_1) \cong I$. Therefore we get

$$I \xrightarrow{(\psi_1, 0)} I_n \oplus I^{\oplus l-1} \xrightarrow{(\alpha', 0)} I^{\oplus l} \longrightarrow I$$

Since $\beta$ (resp. $\gamma$) is a retraction (resp. a section), $(\gamma, \beta)$ gives an isomorphism. Therefore we may assume that $C'$ has the presentation:

$$\text{Hom}_R(-, I_n) \rightarrow \text{Hom}_R(-, I) \rightarrow \text{Hom}_R(-, I) \rightarrow C' \rightarrow 0.$$
We compute them by the same methods used in Lemma 3.3. Since the are AR triangles and \( \dim_k \text{Hom}_R(k) \neq 0 \), instead of \( \text{Hom}_R(k) \) we have the following equality.

\[
\text{dim}_{k} \text{Hom}_R(L_n, I) = m \quad \text{for} \quad n \geq 1.
\]

Proof. We compute them by the same methods used in Lemma 3.3. Since there are AR triangles \( L_n \rightarrow L_{n+1} \oplus L_{n-1} \rightarrow L_n \rightarrow L_n[1] \) for \( n \geq 1 \), here \( L_0 = 0 \), the equation

\[
\dim_k \text{Hom}_R(-, L_n) = n \dim_k \text{Hom}_R(-, L_1) - \sum_{i=1}^{n-1} 2(n - i)\mu(-, L_i)
\]

holds. Hence it is enough to compute the dimension of \( \text{Hom}_R(-, L_1) \). To do this, we use the results due to Knörrer (see Section 4 for the detail). Let \( R = k[x, y]/(x^2 y) \). All non free indecomposable MCM \( R \)-modules are

\[
R/[x, y], R/[x^2], R/[y],
\]

and

\[
M_n^+ = \text{Coker} \left( \begin{array}{ccc} x & y^n \\ 0 & -x \end{array} \right) : R^\oplus 2 \rightarrow R^\oplus 2, \quad M_n^- = \text{Coker} \left( \begin{array}{ccc} x y & y^{n+1} \\ 0 & -xy \end{array} \right) : R^\oplus 2 \rightarrow R^\oplus 2,
\]

\[
N_n^+ = \text{Coker} \left( \begin{array}{ccc} x & y^n \\ 0 & -y \end{array} \right) : R^\oplus 2 \rightarrow R^\oplus 2, \quad N_n^- = \text{Coker} \left( \begin{array}{ccc} x & y^n \\ 0 & -y \end{array} \right) : R^\oplus 2 \rightarrow R^\oplus 2, \quad (n \geq 1).
\]

As mentioned in Corollary 5.5, there is an adjoint pair of functors \((A, B)\) between \( \mathcal{L}(R) \) and \( \mathcal{L}(R^2) \). Particularly we have the isomorphism \( \text{Hom}_R(A(-, L_1) \cong \text{Hom}_R(-, B(L_1)) \). Notice that \( A(R/[x]) = A(R/[x^2]) = I, A(R/[x^3]) = A(R/[y]) = L_1, A(M_n^+) = A(M_n^-) = L_{2n+1} \) and \( A(N_n^+) = A(N_n^-) = L_{2n} \). Hence we may compute the dimension of \( \text{Hom}_R(-, B(L_1)) \) instead of \( \text{Hom}_R(-, L_1) \). Since \( B(L_1) = R/(x^2) \oplus R/(y) \), we have \( \dim_k \text{Hom}_R(-, B(L_1)) = \dim_k \text{Hom}_R(-, R/(x^2) \oplus R/(y)) \).

Claim. We have the following equality.

(i) \( \dim_k \text{Hom}_R(R/(y), R/(x^2) \oplus R/(y)) = 2 \).
(ii) \( \dim_k \text{Hom}_R(M_n^+, R/[x^2] \oplus R/(y)) = 2 \).
(iii) \( \dim_k \text{Hom}_R(N_n^+, R/[x^2] \oplus R/(y)) = 2 \).
(iv) \( \dim_k \text{Hom}_R(R/(x), R/[x^2] \oplus R/(y)) = 1 \).
(v) \( \dim_k \text{Hom}_R(R/(x), R/[x] \oplus R/(xy)) = \infty \).

Proof. (i) First we note that \( R/[x^2] \cong (y) \) and \( R/(y) \cong (x^2) \). It follows from the complexes

\[
\cdots \rightarrow (y) \rightarrow (y) \rightarrow \cdots
\]
Remark 4.2. Also gives (3).

(i) We have the complex:

\[ \cdots \longrightarrow (x^2) \longrightarrow (x^2) \longrightarrow (x^2) \longrightarrow \cdots \]

that \( \text{Ext}^1_R(R/(y), (y)) \cong (y)/(y^2) \cong k \oplus k \) and \( \text{Ext}^1_R(R/(y), (x^2)) = 0 \). Thus \( \dim_k \text{Hom}_R(R/(y), R/(x^2) \oplus R/(y)) = \dim_k \text{Ext}^1_R(R/(y), (y)) = 2 \).

(ii) We have the complex:

\[ \cdots \longrightarrow (x^2) \oplus 2 \beta = (x^{y^1 - x}) \longrightarrow (x^2) \oplus 2 \alpha = (x^{y^1 - x}) \longrightarrow (x^2) \oplus 2 \longrightarrow \cdots . \]

Suppose that \( \alpha((\alpha))^2 = \left(\begin{array}{c} a \\ b \end{array}\right) \) is an element such that \( \alpha((\alpha))^2 = 0 \). Then \( a \in (x^2 \cap (x) = (x^2) \).

Since \( a y^{n+1} - b x y \in (x^2 y) \) and \( a \in (x^2) \), \( b \in (x^2) \cap (x) = (x^2) \). Thus \( \ker \alpha \cong (x^2) \oplus (x^2) \).

Since \( \beta((\alpha))^2 = \left(\begin{array}{c} a \\ b \end{array}\right) \), \( \text{Im} \beta \cong (x^3) \oplus (x^3) \). We have that \( \text{Ext}^1_R(M_n^+, (x^2)) = \ker \alpha / \text{Im} \beta \cong (x^2)/(x^3) \oplus (x^2)/(x^3) \cong k \oplus k \). Also, we have the complex:

\[ \cdots \longrightarrow (y^2) \oplus 2 \beta = (x^{y^1 - x}) \longrightarrow (y^2) \oplus 2 \alpha = (x^{y^1 - x}) \longrightarrow (y^2) \oplus 2 \longrightarrow \cdots . \]

Let \( (\alpha)^2 \) be a square matrices such that \( \alpha((\alpha)^2) = 1 \). Then \( a \in (y^2 \cap (x) = (y^2) \).

Since \( \text{Im} \beta \cong (y^2) \oplus (y^2,y^3+1) \), \( \text{Ext}^1_R(M_n^+, (y)) \cong (y^2)/(y^2) \oplus (y^2,y^3+1)/(y^2,y^3+1) = (0) \). Therefore \( \dim_k \text{Hom}_R(M_n^+, R/(x^2) \oplus R/(y)) = \dim_k \text{Ext}^1_R(M_n^+, R/(x^2) \oplus R/(y)) = 2 \).

(iii) Similarly, the complexes

\[ \cdots \longrightarrow (x^2) \oplus 2 \beta = (x^{y^1 - x}) \longrightarrow (x^2) \oplus 2 \alpha = (x^{y^1 - x}) \longrightarrow (x^2) \oplus 2 \longrightarrow \cdots . \]

and

\[ \cdots \longrightarrow (y^2) \oplus 2 \beta = (x^{y^1 - x}) \longrightarrow (y^2) \oplus 2 \alpha = (x^{y^1 - x}) \longrightarrow (y^2) \oplus 2 \longrightarrow \cdots . \]

say that \( \text{Ext}^1_R(N_n^+, (x^2)) \cong (x^2)/(x^3) \cong k \) and \( \text{Ext}^1_R(N_n^+, (y)) \cong (y^2)/(y^2) \cong k \), so that \( \dim_k \text{Hom}_R(N_n^+, R/(x^2) \oplus R/(y)) = \dim_k \text{Ext}^1_R(N_n^+, (x^2) \oplus (y)) = 2 \).

(iv) From the complexes

\[ \cdots \longrightarrow (y) \longrightarrow (y) \longrightarrow (y) \longrightarrow \cdots \]

and

\[ \cdots \longrightarrow (x^2) \longrightarrow (x^2) \longrightarrow (x^2) \longrightarrow \cdots . \]

one has \( \text{Ext}^1_R(R/(x), (y)) = 0 \) and \( \text{Ext}^1_R(R/(x), (x^2)) \cong (x^2)/(x^3) \cong k \). Therefore \( \dim_k \text{Hom}_R(R/(x), R/(x^2) \oplus R/(y)) = \dim_k \text{Ext}^1_R(R/(x), (x^2)) = 1 \).

(v) Notice that \( R/(x) \cong (x) \), \( R/(yx) \cong (x) \). One can show from the complexes

\[ \cdots \longrightarrow (xy) \longrightarrow (xy) \longrightarrow (xy) \longrightarrow \cdots \]

and

\[ \cdots \longrightarrow (x) \longrightarrow (x) \longrightarrow (x) \longrightarrow \cdots \]

that \( \text{Ext}^1_R(R/(x), (y)) \cong (x) \) and \( \text{Ext}^1_R(R/(x), (x)) \cong (x)/(x^2) \). This implies (v).

The claim (i), (ii), (iii), (iv) and the equation [1.1] give the assertion (1), (2). The claim (v) also gives (3).

Remark 4.2. As mentioned in the proof of Lemma [1.1.3], \( \text{Hom}_R(I, I) \) is isomorphic to \( k[y] \oplus k[y] \). Let \( T = S[z] / (f + z^2) \) with \( f \in m_S \). Then there is a one-to-one correspondence between the isomorphism classes of MCM \( T \)-modules and the equivalence classes of square matrices \( \varphi \) with entries in \( S \) such that \( \varphi^2 = -f \). We also remark that giving a \( T \)-homomorphism \( g : M \to N \) is equivalent to giving an \( S \)-homomorphism \( \alpha : S^\oplus m \to S^\oplus n \) such that \( \alpha \circ \varphi_M = \varphi_N \circ \alpha \). By using this, one has \( \text{Hom}_R(I, I) \cong \{ (a, b) \mid a, c \in k[x, y] \} \). Note that \( I \) corresponds to
the matrix \( \begin{pmatrix} 0 & -x y \\ x & 0 \end{pmatrix} \). Moreover, since \( R^2 \) corresponds to \( \begin{pmatrix} 0 & -x^2 y \\ 1 & 0 \end{pmatrix} \), \( P(I, I) \) is isomorphic to \( \{(\frac{x^c}{a}, -x^c y) | a, c \in k[x, y]\} \), so that we have \( \text{Hom}_{R^2}(I, I) \cong \{(\frac{x^c}{a}, -x^c y) | a, c \in k[y]\} \). And we see that \( \frac{y}{x}^n \) (resp. \( y^n \)) in \( \text{Hom}_{R^2}(I, I) \) corresponds to \( \begin{pmatrix} 0 & -y^{n+1} \\ y & 0 \end{pmatrix} \) (resp. \( \begin{pmatrix} y & 0 \\ 0 & y^n \end{pmatrix} \)). Therefore \( \text{Hom}_{R^2}(I, I) \) is generated by \( \frac{y}{x} \) and 1 as a \( k[y] \)-module. Now we calculate the mapping cone \( C(\varphi) : I \to I \). One has

\[
C\left(\frac{y}{x}^n\right) = \text{Coker} \begin{pmatrix} z & -xy & 0 \\ x & z & -y^n & 0 \\ 0 & 0 & z & xy \\ 0 & 0 & -x & z \end{pmatrix} : R^2 \oplus 4 \to R^2 \oplus 4 \cong M = L_{2n+1},
\]

\[
C(y^n) = \text{Coker} \begin{pmatrix} z & -xy & -y^n & 0 \\ x & z & 0 & -y^n \\ 0 & 0 & z & xy \\ 0 & 0 & -x & z \end{pmatrix} : R^2 \oplus 4 \to R^2 \oplus 4 \cong N = L_{2n}
\]

respectively.

**Theorem 4.3.** Let \( R^2, I, L_n \) be as above. Then the following statements hold.

1. \( \text{Kgdim}_{R^2}(-, L_n) = 1 \) for \( n \geq 1 \).
2. \( \text{Kgdim}_{R^2}(-, I) = 2 \).

Consequently \( \text{Kgdim}_{\text{mod}(C(R^2))} = 2 \).

**Proof.** The arguments in Section 3 are valid. We have the exact sequence:

\[
0 \to L_1 \begin{array}{c} \left( \frac{x^2}{1} \right) \end{array} I \oplus R \begin{array}{c} \left( \frac{z}{-x} \right) \end{array} I \to 0.
\]

We consider the functor induced by the sequence (4.2), that is, \( \text{Hom}_{R^2}(-, L_1) \to \text{Hom}_{R^2}(-, I) \to \text{Hom}_{R^2}(-, I) \to G_1 \to 0 \). The functor \( G_1 \) satisfies the equation \( \dim_k G_1(L) = 1 \) for each indecomposable \( L \in \text{C}(R^2) \). Thus by using the same arguments in Proposition 3.3 one has \( G_1 \) is a simple object in \( \text{mod}(C(R^2))/\text{mod}(C(R^2))_0 \). Moreover \( \ell(\text{Hom}_{R^2}(-, L_1)) = 2 \) (see Remark 3.7). Since we have an AR-triangle \( L_n \to L_{n+1} \oplus L_{n-1} \to L_n \to L_n[1] \), we have \( \ell(\text{Hom}_{R^2}(-, L_n)) = 2n \) in \( \text{mod}(C(R^2))/\text{mod}(C(R^2))_0 \), so that the assertion (1) holds. See also Proposition 3.3.

According to [1] Proposition 2.2. (2)], there is a triangle \( I \to L_n \to I \to I[1] \) for any \( n \geq 1 \). Thus we have \( 2 \ell(\text{Hom}_{R^2}(-, I)) \geq \ell(\text{Hom}_{R^2}(-, L_n)) \) in \( \text{mod}(C(R^2))/\text{mod}(C(R^2))_0 \). Hence \( \text{Hom}_{R^2}(-, I) \) does not belong to \( \text{mod}(C(R^2))_1 \). Then we can also show Kgdim \( \text{Hom}_{R^2}(-, I) = 2 \) by using the arguments in Proposition 3.3.

Let \( 0 \to V \to \text{Hom}_{R^2}(-, I) \to C \to 0 \) be an admissible sequence in \( \text{mod}(C(R^2))/\text{mod}(C(R^2))_0 \). By virtue of Lemma 3.10 we may assume that \( C \) has the presentation \( \text{Hom}_{R^2}(-, I^{\oplus l}) \xrightarrow{\varphi} \text{Hom}_{R^2}(-, I) \to C \to 0 \). According to Remark 4.2 \( \varphi \) is of the form \( \left(a_1 y^{n_1} + b_1 \frac{z}{x} y^{m_1} a_2 y^{n_2} + b_2 \frac{z}{x} y^{m_2} \cdots a_l y^{n_l} + b_l \frac{z}{x} y^{m_l}\right) \) where \( a_i, b_i \) are units in \( k[[y]] \). Set \( n = \min\{n_1, \ldots, n_l\} \) and \( m = \min\{m_1, \ldots, m_l\} \). Suppose that \( n \geq m \), the mapping cone \( C(\varphi) \cong M = L_{2n+1} \oplus I^{\oplus l-1} \). Suppose that \( n < m \), then \( C(\varphi) \cong N = L_{2n} \oplus I^{\oplus l-1} \). Since \( a_i y^{n_i} + b_i \frac{z}{x} y^{m_i} \) is a non zero divisor on \( \text{Hom}_{R^2}(I, I) \), \( I^{\oplus l-1} \) is split out as shown in Proposition 3.3.

Hence we also have \( C \) is a subfunctor of \( \text{Hom}_{R^2}(-, L_{2n+1}) \) or \( \text{Hom}_{R^2}(-, L_{2n}) \), so that \( C \) is in \( \text{mod}(C(R^2))_1 \). Consequently, \( \text{Hom}_{R^2}(-, I) \) is simple in \( \text{mod}(C(R^2))/\text{mod}(C(R^2))_1 \), which means Kgdim \( \text{Hom}_{R^2}(-, I) = 2 \). \( \square \)

5. **Knörrer’s periodicity**

In this section we investigate how the Krull–Gabriel dimension behaves concerning Knörrer’s periodicity. We recall some observations given in [19, 12].

Let \( \mathcal{C} \) and \( \mathcal{D} \) be additive categories with a functor \( \mathcal{A} : \mathcal{C} \to \mathcal{D} \). Then \( \mathcal{A} \) induces the functor \( \mathcal{A} : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{D}) \) by \( \mathcal{A}(\text{Hom}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})) = \text{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{A}(\mathcal{C})) \).
Lemma 5.1. Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories with functors $\mathcal{A} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{B} : \mathcal{D} \to \mathcal{C}$. Suppose that $(\mathcal{B}, \mathcal{A})$ is an adjoint pair of functors. Then the induced functor $\mathcal{A} : \text{mod}(\mathcal{C}) \to \text{mod}(\mathcal{D})$ is an exact functor.

Proof. By the adjointness of $(\mathcal{B}, \mathcal{A})$, one can show that $\mathcal{A}(\mathcal{F})(-) \cong \mathcal{F}(\mathcal{B}(-))$ for $\mathcal{F} \in \text{mod}(\mathcal{C})$. The assertion follows from the isomorphism.

Let $R$ be a hypersurface, that is, $R = S/(f)$ where $S = \mathbb{k}[x_0, x_1, \ldots, x_n]$ is a formal power series ring with a maximal ideal $m_S = (x_0, x_1, \ldots, x_n)$ and $f \in m_S$. For the ring $R$, we denote $R^2 = S[z]/(f + z^2)$. Then the group $G = \mathbb{Z}/2\mathbb{Z}$ acts on $R^2$ by $\sigma : z \to -z$. Denote the skew group ring by $R^2 * G$. We say that a finitely generated $R^2 * G$-module is MCM if it is MCM as an $R^2$-module. For an $R^2$-module $M$ and the involution $\sigma$ in $G$, we define an $R^2$-module $\sigma^* M$ by $M = \sigma^* M$ as a set and $r \circ m = \sigma(r)m$. For the detail, refer to [12, Section 2].

Remark 5.2. For an $R^2$-module $M$, $M \oplus \sigma^* M$ has an $R^2 * G$-module structure. For $(a, b) \in M \oplus \sigma^* M$, we define the action of $\sigma$ by $\sigma(a, b) = (b, a)$. Moreover, for an $R^2$-homomorphism $f : M \to N$, we see that $\begin{pmatrix} f & 0 \\ f & 0 \end{pmatrix} : M \oplus \sigma^* M \to N \oplus \sigma^* N$ is an $R^2 * G$-homomorphism.

The following theorem is due to Knörrer [12].

Theorem 5.3. [12] Let $R$, $R^2 *, G$, $R^2$ be as above. We have the functors

$$
\mathcal{C}(R) \xrightarrow{\Omega} \mathcal{C}(R^2 * G) \xrightarrow{\mathcal{F}} \mathcal{C}(R^2),
$$

where the functor $\Omega(-)$ is defined by $\text{syz}^1_{R^2}(\mathcal{F})$, $\mathcal{F}$ is a forget-functor and $\mathcal{F}(\mathcal{A}(-)) = - \otimes_{R^2} R^2 * G$ is its adjoint. Then, for $X \in \mathcal{C}(R)$ and $Y \in \mathcal{C}(R^2)$, the following statements hold.

1. The functor $\Omega$ gives the categorical equivalence.
2. $\Omega^{-1} \circ \mathcal{F} \circ \mathcal{A} \circ \Omega$ is equivalent to the functor $X \to X \oplus \text{syz}^1_{R}(X)$.
3. $\mathcal{F} \circ \mathcal{A} \circ \mathcal{F}^{-1} \circ \mathcal{A}$ is equivalent to the functor $Y \to Y \oplus \sigma^* Y$.

Proof. (1) [12, Proposition 2.1, Remark 2.2(ii)]. (2), (3) [12, Proposition 2.4, Lemma 2.5].

Lemma 5.4. Let $\Omega$, $\mathcal{F}$ and $\mathcal{A}$ be as above. Set $\mathcal{A} = \mathcal{F} \circ \Omega$ and $\mathcal{B} = \Omega^{-1} \circ \mathcal{A}$. Then $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ are adjoint pairs.

Proof. Since $\Omega$ gives the equivalence it is enough to show that $(\mathcal{A}, \mathcal{F})$ and $(\mathcal{F}, \mathcal{A})$ are adjoint pairs. It is well-known that $(\mathcal{A}, \mathcal{F})$ is an adjoint pair, and we show $(\mathcal{F}, \mathcal{A})$ is an adjoint pair. Note that, for $X \in \mathcal{C}(R^2)$, $\mathcal{A}(-) = \mathcal{F}(\mathcal{B}(-)) = \mathcal{F}(\mathcal{B}(\mathcal{A}(-))) = Y \oplus \sigma^* Y$ as $R^2 * G$-module ([12, Lemma 2.5.(i)]). The isomorphism gives the correspondence:

$$
\text{Hom}_{R^2 * G}(X, \mathcal{A}(-)) \cong \text{Hom}_{R^2 * G}(X, Y \oplus \sigma^* Y) \quad f \mapsto [x \mapsto (f_1(x), f_2(x))].
$$

Since $f(\sigma(x)) = \sigma(f(x)) = (f_1(\sigma(x)), f_2(\sigma(x))) = (f_1(x), f_2(x)) = f_2(f_1(x)) = f_2(f_1)$ (see Remark 5.2). Thus $f_2 = f_1 \circ \sigma$. We determine the morphism $\Phi : \text{Hom}_{R^2 * G}(X, \mathcal{A}(-)) \to \text{Hom}_{R^2 * G}(\mathcal{F}(X), Y)$ by $\phi(f) = f_1$. Conversely, by the observation above, we determine the morphism $\Psi : \text{Hom}_{R^2 * G}(\mathcal{F}(X), Y) \to \text{Hom}_{R^2 * G}(X, Y)$ by $\Psi(g) = (g, g \circ \sigma)$. By using the isomorphism $\mathcal{A}(-) \cong Y \oplus \sigma^* Y$, we see that $\Phi$ and $\Psi$ give natural isomorphisms between $\text{Hom}_{R^2 * G}(X, \mathcal{A}(-))$ and $\text{Hom}_{R^2 * G}(\mathcal{F}(X), Y)$. Compare with the proof of [19, Theorem 3.2].

One can also show that $(\mathcal{A}, \mathcal{B})$ and $(\mathcal{B}, \mathcal{A})$ are adjoint pairs between $\mathcal{C}(R)$ and $\mathcal{C}(R^2)$ since the functors $\Omega$, $\mathcal{F}$ and $\mathcal{A}$ preserve projectivity. Actually, $\Omega(R) = \text{syz}^1_{R^2}(R) \cong R^2$ and $\Omega^{-1}(R^2 * G) = (R^2 * G)^G / (R^2 * G)^0 \cong R$ where $(R^2 * G)^G$ (resp. $(R^2 * G)^0$) is the set of $\sigma$-invariant (resp. $\sigma$-antiinvariant) elements of $R^2 * G$ ([12, Proposition 2.1]). The indecomposable projective $R^2 * G$-modules are $R^2$ and $\sigma^* R^2$ ([12, Section 2]). This implies that $\mathcal{F}$ sends projective $R^2 * G$-modules to projective (free) $R^2$-modules since $\sigma^* R^2 \cong R^2$ as an $R^2$-module. It also implies that the functor $\mathcal{A}$ sends projective (free) $R^2$-modules to projective $R^2 * G$-modules. The fact induce
that $\langle A, B \rangle$ and $\langle B, A \rangle$ are also adjoint pairs between $\mathcal{C}(R)$ and $\mathcal{C}(R^2)$. Hence we get the following consequence.

**Corollary 5.5.** Let $A$ and $B$ be as in Lemma 5.4. Then $\langle A, B \rangle$ and $\langle B, A \rangle$ are adjoint pairs between $\mathcal{C}(R)$ and $\mathcal{C}(R^2)$. Thus the induced functors $A : \text{mod}(\mathcal{C}(R)) \to \text{mod}(\mathcal{C}(R^2))$ and $B : \text{mod}(\mathcal{C}(R^2)) \to \text{mod}(\mathcal{C}(R))$ are exact functors.

**Remark 5.6.** According to Theorem 5.3 for each $F \in \text{mod}(\mathcal{C}(R))$ with $\text{Hom}_R(-, Y) \xrightarrow{\text{Hom}_R(-, f)} \text{Hom}_R(-, X) \rightarrow F \rightarrow 0$, one has $B \circ A(F) = F \oplus F[-1]$ defined by $\text{Hom}_R(-, Y) \oplus \text{Hom}_R(-, Y[-1]) \xrightarrow{\text{Hom}_R(-, f)} \text{Hom}_R(-, X) \oplus \text{Hom}_R(-, X[-1]) \rightarrow F \oplus F[-1] \rightarrow 0$. Similarly one also has $A \circ B(G) = G \oplus \sigma^*G$ for each $G \in \text{mod}(\mathcal{C}(R^2))$ (see also Remark 5.2).

**Proposition 5.7.** Let $R = S/(f)$ be a hypersurface and $A$, $B$ as in Lemma 5.4. Suppose that, for each $F \in \text{mod}(\mathcal{C}(R))_n$, $A(F) \in \text{mod}(\mathcal{C}(R^2))_n$. Then, for a simple object $S \in \text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_n$, $A(S)$ has finite length in $\text{mod}(\mathcal{C}(R^2))/\text{mod}(\mathcal{C}(R^2))_n$.

**Proof.** Let $S$ be a simple object in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_n$. If $A(S)$ is simple, we have nothing to prove. Thus we may assume that $A(S)$ is not simple. Then we have an exact sequence of functors $0 \to V \to A(S) \to S' \to 0$ such that $S'$ is simple in $\text{mod}(\mathcal{C}(R^2))/\text{mod}(\mathcal{C}(R^2))_n$. Apply $B$ to the sequence, one has

$$0 \to B(V) \to B \circ A(S) \to B(S') \to 0.$$

Since $B \circ A(S) \cong S \oplus S[-1]$, $\ell(B(V)) + \ell(B(S')) = 2$ in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_n$. Notice that the object $S[-1]$ is a simple object (Lemma 5.3). Suppose that $\ell(B(V)) = 0$, namely $B(V)$ belongs to $\text{mod}(\mathcal{C}(R))_n$. By the assumption $A \circ B(V) \cong V \oplus \sigma^*V$ is in $\text{mod}(\mathcal{C}(R^2))$ and so is $V$. Hence $A(S) \cong S'$ in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_n$. This is a contradiction since $A(S)$ is not simple. Suppose that $\ell(B(S')) = 0$. Similarly one shows that $S'$ is in $\text{mod}(\mathcal{C}(R^2))_n$, which is a contradiction. Consequently $\ell(B(V)) = \ell(B(S')) = 1$. Namely $B(V)$ and $B(S')$ are simple in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_n$.

Now we shall show $V$ is also simple in $\text{mod}(\mathcal{C}(R^2))/\text{mod}(\mathcal{C}(R^2))_n$. Let $0 \to V' \to V \to C \to 0$ be an admissible sequence in $\text{mod}(\mathcal{C}(R^2))$. Then we obtain the sequence $0 \to B(V') \to B(V) \to B(C) \to 0$ in $\text{mod}(\mathcal{C}(R))$. Since $B(V)$ is simple, $B(V')$ or $B(C)$ is in $\text{mod}(\mathcal{C}(R))_n$. Assume that $B(V')$ is in $\text{mod}(\mathcal{C}(R))_n$. By the assumption, $A \circ B(V')$ is in $\text{mod}(\mathcal{C}(R^2))_n$, and so is $V'$. This implies that $V$ is simple in $\text{mod}(\mathcal{C}(R^2))/\text{mod}(\mathcal{C}(R^2))_n$. By the same arguments, the case that $B(C)$ is in $\text{mod}(\mathcal{C}(R))_n$ also implies that $V$ is simple.

Consequently $A(S)$ is of length 2 in $\text{mod}(\mathcal{C}(R^2))/\text{mod}(\mathcal{C}(R^2))_n$, so that the assertion holds.

**Proposition 5.8.** Suppose that $A(S)$ belongs to $\text{mod}(\mathcal{C}(R^2))_n$ for each simple object $S$ in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_{n-1}$. Then $A(F)$ is in $\text{mod}(\mathcal{C}(R^2))_n$ for each $F$ in $\text{mod}(\mathcal{C}(R))_n$.

**Proof.** We have the filtration of $F$ in $\text{mod}(\mathcal{C}(R))$

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = F$$

such that $F_i/F_{i-1}$ are simple in $\text{mod}(\mathcal{C}(R))/\text{mod}(\mathcal{C}(R))_{n-1}$. Apply $A$ to the filtration, we obtain the filtration

$$0 = A(F_0) \subset A(F_1) \subset A(F_2) \subset \cdots \subset A(F_n) = A(F)$$

in $\text{mod}(\mathcal{C}(R^2))$. By the assumption, $A(F_i)/A(F_{i-1})$ is in $\text{mod}(\mathcal{C}(R^2))_n$. Hence $A(F)$ belongs to $\text{mod}(\mathcal{C}(R^2))_n$. \qed

**Theorem 5.9.** Let $R = S/(f)$ and $R^2 = S[z]/(f + z^2)$. Then $\text{Kdim } \text{mod}(\mathcal{C}(R)) = \text{Kdim } \text{mod}(\mathcal{C}(R^2))$.\qed
Proof. First we notice that, as mentioned in [12, Corollary 2.10], a simple object \( S \) in \( \text{mod}(C(R)) \) goes to a length-finite object in \( \text{mod}(C(R^2)) \), that is \( \mathcal{A}(S) \in \text{mod}(C(R^2))_0 \). Conversely a simple object \( S' \) in \( \text{mod}(C(R^2)) \) also goes to an object in \( \text{mod}(C(R))_0 \), that is \( \mathcal{B}(S') \in \text{mod}(C(R))_0 \). Summing up Proposition 5.7 and 5.8 one can see that \( \mathcal{A} \) gives a functor from \( \text{mod}(C(R))_n \) to \( \text{mod}(C(R^2))_n \). Suppose that \( \text{Kgdim} \ \text{mod}(C(R)) = n \). For each object \( F \in \text{mod}(C(R^2)) \), \( \mathcal{B}(F) \) belongs to \( \text{mod}(C(R))_n \). Since \( \mathcal{A} \circ \mathcal{B}(F) = F \oplus \sigma^+ F \) belongs to \( \text{mod}(C(R^2))_n \), \( F \) is contained in \( \text{mod}(C(R^2))_n \), so that \( \text{Kgdim} F \leq n \). Therefore \( \text{Kgdim} \text{mod}(C(R^2)) \leq \text{Kgdim} \text{mod}(C(R)) \).

Let \( R^2 = S[z_1, z_2]/(f + z_1^2 + z_2^2) \). Then we have an equivalence of triangulated categories \( C(R) \cong C(R^2) \) which is known as Kn"orrer’s periodicity (cf. [23, (12.10)]) Hence we also have the equivalence \( \text{mod}(C(R)) \cong \text{mod}(C(R^2)) \). Apply the above arguments to \( \text{mod}(C(R^2)) \) and \( \text{mod}(C(R^2)) \), one has \( \text{Kgdim} \text{mod}(C(R)) = \text{Kgdim} \text{mod}(C(R^2)) \leq \text{Kgdim} \text{mod}(C(R^2)) \).

Consequently \( \text{Kgdim} \text{mod}(C(R)) = \text{Kgdim} \text{mod}(C(R^2)) \).

Suppose that \( \text{Kgdim} \text{mod}(C(R)) = \infty \). The inequality \( \text{Kgdim} \text{mod}(C(R)) \leq \text{Kgdim} \text{mod}(C(R^2)) \) holds if \( \text{Kgdim} \text{mod}(C(R^2)) \) is finite. Thus \( \text{Kgdim} \text{mod}(C(R^2)) \) must be infinite.

Finally we reach the main theorem of the paper.

**Corollary 5.10.** Let \( k \) be an algebraically closed uncountable field of characteristic not two. Let \( R \) be a hypersurface of countable but not finite CM representation type. Then \( \text{Kgdim} \text{mod}(C(R)) = 2 \).

Proof. We may assume that \( R \) is a hypersurface of type \((A_\infty)\) or \((D_\infty)\). Thanks to Theorem 5.9 we have known that the Krull–Gabriel dimension is stable under Kn"orrer’s periodicity. Therefore the assertion holds from Theorem 5.1 and 1.3. \( \square \)

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