DAEHEE NUMBERS AND POLYNOMIALS

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Abstract. We consider the Witt-type formula for Daehee numbers and polynomials and investigate some properties of those numbers and polynomials. In particular, Daehee numbers are closely related to higher-order Bernoulli numbers and Bernoulli numbers of the second kind.

1. Introduction

As is known, the $n$-th Daehee polynomials are defined by the generating function to be

$$\left(\log \left(1 + \frac{t}{x}\right)\right)\left(1 + t\right)^x = \sum_{n=0}^{\infty} D_n(x) \frac{t^n}{n!}, \text{ (see [5,6,8,9,10,11])}.$$  

In the special case, $x = 0$, $D_n = D_n(0)$ are called the Daehee numbers.

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the rings of $p$-adic integers, the fields of $p$-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $|\cdot|_p$ is normalized by $|p|_p = 1/p$. Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu_0(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n-1} f(x), \text{ (see [6]).}$$

Let $f_1$ be the translation of $f$ with $f_1(x) = f(x + 1)$. Then, by (1.2), we get

$$I(f_1) = I(f) + f'(0), \text{ where } f'(0) = \left. \frac{df(x)}{dx} \right|_{x=0}.$$  

As is known, the Stirling number of the first kind is defined by

$$(x)_n = x(x-1) \cdots (x-n+1) = \sum_{l=0}^{n} S_1(n, l) x^l,$$

and the Stirling number of the second kind is given by the generating function to be

$$(e^t - 1)^m = m! \sum_{l=m}^{\infty} S_2(l, m) \frac{t^l}{l!}, \text{ (see [2,3,4]).}$$

For $\alpha \in \mathbb{Z}$, the Bernoulli polynomials of order $\alpha$ are defined by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}, \text{ (see [1,2,8]).}$$

When $x = 0$, $B_n^{(\alpha)} = B_n^{(\alpha)}(0)$ are called the Bernoulli numbers of order $\alpha$.

In this paper, we give a $p$-adic integral representation of Daehee numbers and polynomials, which are called the Witt-type formula for Daehee numbers and polynomials. From our integral representation, we can derive some interesting properties related to Daehee numbers and polynomials.
2. Witt-type formula for Daehee numbers and polynomials

First, we consider the following integral representation associated with falling factorial sequences:

\[(2.1) \int_{\mathbb{Z}_p} (x)_n \, d\mu_0 (x), \quad \text{where } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.\]

By (2.1), we get

\[(2.2) \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n \, d\mu_0 (x) \frac{t^n}{n!} = \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \binom{x}{n} t^n \, d\mu_0 (x) = \int_{\mathbb{Z}_p} (1 + t)^x \, d\mu_0 (x),\]

where \( t \in \mathbb{C}_p \) with \(|t|_p < p^{-\frac{1}{p-1}}\).

For \( t \in \mathbb{C}_p \) with \(|t|_p < p^{-\frac{1}{p-1}}\), let us take \( f(x) = (1 + t)^x \). Then, from (1.3), we have

\[(2.3) \int_{\mathbb{Z}_p} (1 + t)^x \, d\mu_0 (x) = \frac{\log (1 + t)}{t}.

By (1.1) and (2.3), we see that

\[(2.4) \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} = \frac{\log (1 + t)}{t} = \int_{\mathbb{Z}_p} (1 + t)^x \, d\mu_0 (x) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (x)_n \, d\mu_0 (x) \frac{t^n}{n!}.

Therefore, by (2.4), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), we have

\[\int_{\mathbb{Z}_p} (x)_n \, d\mu_0 (x) = D_n.\]

For \( n \in \mathbb{Z} \), it is known that

\[(2.5) \left( \frac{t}{\log (1 + t)} \right)^n (1 + t)^{x-1} = \sum_{k=0}^{\infty} B_k^{(k-n+1)} (x) \frac{t^k}{k!}, \quad \text{(see [2,3,4])}.\]

Thus, by (2.5), we get

\[(2.6) D_k = \int_{\mathbb{Z}_p} (x)_k \, d\mu_0 (x) = B_k^{(k+2)} (1), \quad (k \geq 0),\]

where \( B_k^{(n)} (x) \) are the Bernoulli polynomials of order \( n \).

In the special case, \( x = 0 \), \( B_k^{(n)} = B_k^{(n)} (0) \) are called the \( n \)-th Bernoulli numbers of order \( n \).

From (2.4), we note that

\[(2.7) (1 + t)^x \int_{\mathbb{Z}_p} (1 + t)^y \, d\mu_0 (y) = \frac{\log (1 + t)}{t} (1 + t)^x = \sum_{n=0}^{\infty} D_n (x) \frac{t^n}{n!}.\]
Thus, by (2.7), we get
\begin{equation}
(2.8) \quad \int_{\mathbb{Z}_p} (x + y)_n \, d\mu_0(y) = D_n(x), \quad (n \geq 0),
\end{equation}
and, from (2.5), we have
\begin{equation}
(2.9) \quad D_n(x) = B_n^{(n+2)}(x+1).
\end{equation}
Therefore, by (2.8) and (2.9), we obtain the following theorem.

**Theorem 2.** For $n \geq 0$, we have
\begin{equation}
D_n(x) = \int_{\mathbb{Z}_p} (x + y)_n \, d\mu_0(y),
\end{equation}
and
\begin{equation}
D_n(x) = B_n^{(n+2)}(x+1).
\end{equation}
By Theorem 1, we easily see that
\begin{equation}
(2.10) \quad D_n = \sum_{l=0}^{n} S_1(n, l) B_l,
\end{equation}
where $B_l$ are the ordinary Bernoulli numbers.

From Theorem 2, we have
\begin{equation}
(2.11) \quad D_n(x) = \int_{\mathbb{Z}_p} (x + y)_n \, d\mu_0(y) = \sum_{l=0}^{n} S_1(n, l) B_l(x),
\end{equation}
where $B_l(x)$ are the Bernoulli polynomials defined by generating function to be
\begin{equation}
\frac{t}{e^t - 1} = \sum_{n=0}^\infty B_n(x) \frac{t^n}{n!}.
\end{equation}
Therefore, by (2.10) and (2.11), we obtain the following corollary.

**Corollary 3.** For $n \geq 0$, we have
\begin{equation}
D_n(x) = \sum_{l=0}^{n} S_1(n, l) B_l(x).
\end{equation}
In (2.1), we have
\begin{equation}
(2.12) \quad \frac{t}{e^t - 1} = \sum_{n=0}^\infty D_n \frac{1}{n!} (e^t - 1)^n
= \sum_{n=0}^\infty D_n \frac{1}{n!} \sum_{m=n}^\infty S_2(m, n) \frac{t^m}{m!}
= \sum_{m=0}^\infty \left( \sum_{n=0}^{m} D_n S_2(m, n) \right) \frac{t^m}{m!}
\end{equation}
and
\begin{equation}
(2.13) \quad \frac{t}{e^t - 1} = \sum_{m=0}^\infty B_m \frac{t^m}{m!}.
\end{equation}
Therefore, by (2.12) and (2.13), we obtain the following theorem.
Theorem 4. For \( m \geq 0 \), we have
\[
B_m = \sum_{n=0}^{m} D_n S_2 (m, n).
\]

In particular,
\[
\int_{\mathbb{Z}_p} x^m d\mu_0 (x) = \sum_{n=0}^{m} D_n S_2 (m, n).
\]

Remark. For \( m \geq 0 \), by (2.11), we have
\[
\hat{\mathcal{Z}}_p x^m d\mu_0 (x) = \sum_{n=0}^{m} D_n S_2 (m, n).
\]

For \( n \in \mathbb{Z}_\geq 0 \), the rising factorial sequence is defined by
\[
(x^n) = x (x + 1) \cdots (x + n - 1).
\]

Let us define the Daehee numbers of the second kind as follows :
\[
(2.15) \quad \hat{\mathcal{D}}_n = \hat{\mathcal{Z}}_p (-x)^n d\mu_0 (x), \quad (n \in \mathbb{Z}_\geq 0).
\]

By (2.15), we get
\[
(2.16) \quad x^{(n)} = (-1)^n (-x)_n = \sum_{l=0}^{n} S_1 (n, l) (-1)^{n-l} x^l.
\]

From (2.15) and (2.16), we have
\[
(2.17) \quad \hat{\mathcal{D}}_n = \int_{\mathbb{Z}_p} (-x)^n d\mu_0 (x) = \int_{\mathbb{Z}_p} x^{(n)} (-1)^n d\mu_0 (x)
= \sum_{l=0}^{n} S_1 (n, l) (-1)^l B_l.
\]

Therefore, by (2.17), we obtain the following theorem.

Theorem 5. For \( n \geq 0 \), we have
\[
\hat{\mathcal{D}}_n = \sum_{l=0}^{n} S_1 (n, l) (-1)^l B_l.
\]

Let us consider the generating function of the Daehee numbers of the second kind as follows :
\[
(2.18) \quad \sum_{n=0}^{\infty} \hat{\mathcal{D}}_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (-x)^n d\mu_0 (x) \frac{t^n}{n!}
= \int_{\mathbb{Z}_p} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} t^n d\mu_0 (x)
= \int_{\mathbb{Z}_p} (1 + t)^{-x} d\mu_0 (x).
\]

From (2.18), we can derive the following equation :
\[
(2.19) \quad \int_{\mathbb{Z}_p} (1 + t)^{-x} d\mu_0 (x) = \frac{(1 + t) \log (1 + t)}{t},
\]
where \( |t|_p < p^{-\frac{1}{p}} \).
By (2.18) and (2.19), we get

\[(2.20)\]
\[
\frac{1}{t} (1 + t) \log (1 + t) = \int_{\mathbb{Z}_p} (1 + t)^{-x} \, d\mu_0 (x)
\]
\[
= \sum_{n=0}^{\infty} \hat{D}_n \frac{t^n}{n!}.
\]

Let us consider the Daehee polynomials of the second kind as follows:

\[(2.21)\]
\[
\frac{1 + t}{t} \log (1 + t) = \sum_{n=0}^{\infty} \hat{D}_n (x) \frac{t^n}{n!}.
\]

Then, by (2.21), we get

\[(2.22)\]
\[
\int_{\mathbb{Z}_p} (1 + t)^{-x-y} \, d\mu_0 (y) = \sum_{n=0}^{\infty} \hat{D}_n (x) \frac{t^n}{n!}.
\]

From (2.22), we get

\[(2.23)\]
\[
\hat{D}_n (x) = \int_{\mathbb{Z}_p} (-x - y)_n \, d\mu_0 (y), \quad (n \geq 0)
\]
\[
= \sum_{l=0}^{n} (-1)^l S_1 (n, l) B_l (x).
\]

Therefore, by (2.23), we obtain the following theorem.

**Theorem 6.** For \(n \geq 0\), we have

\[
\hat{D}_n (x) = \int_{\mathbb{Z}_p} (-x - y)_n \, d\mu_0 (y) = \sum_{l=0}^{n} (-1)^l S_1 (n, l) B_l (x).
\]

From (2.21) and (2.22), we have

\[(2.24)\]
\[
\left( \frac{t}{e^t - 1} \right) e^{(1-x)t} = \sum_{n=0}^{\infty} \hat{D}_n (x) \frac{1}{n!} \left( e^t - 1 \right)^n
\]
\[
= \sum_{n=0}^{\infty} \hat{D}_n (x) \frac{1}{n!} \sum_{m=n}^{\infty} S_2 (m, n) \frac{t^m}{m!}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}_n (x) S_2 (m, n) \right) \frac{t^n}{n!},
\]

and

\[(2.25)\]
\[
\int_{\mathbb{Z}_p} e^{-(x+y)t} \, d\mu_0 (y) = \sum_{n=0}^{\infty} \hat{D}_n (x) \left( e^t - 1 \right)^n \frac{1}{n!}
\]
\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} \hat{D}_n (x) S_2 (m, n) \right) \frac{t^m}{m!}.
\]

Therefore, by (2.24) and (2.25), we obtain the following theorem.

**Theorem 7.** For \(m \geq 0\), we have

\[
B_m (1-x) = (-1)^m \int_{\mathbb{Z}_p} (x + y)^m \, d\mu_0 (y)
\]
\[
= \sum_{n=0}^{m} \hat{D}_n (x) S_2 (m, n).
\]
In particular, \[ B_m (1 - x) = (-1)^m B_m (x) = \sum_{n=0}^{m} \hat{D}_m (x) S_2 (m, n). \]

**Remark.** By (2.5), (2.20) and (2.21), we see that
\[ \hat{D}_n = B_n^{(n+2)} (2), \quad \hat{D}_n (x) = B_n^{(n+2)} (2 - x). \]

From Theorem 1 and (2.15), we have
\[ (-1)^n D_n n! = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \hat{D}_m m! = \sum_{m=0}^{n} \binom{n}{m} \hat{D}_m m! \]
and
\[ (-1)^n \frac{\hat{D}_n}{n!} = (-1)^n \sum_{m=0}^{n} \binom{n}{m} \frac{D_m}{m!} = \sum_{m=0}^{n} \binom{n}{m} \frac{D_m}{m!}. \]

Therefore, by (2.26) and (2.27), we obtain the following theorem.

**Theorem 8.** For \( n \in \mathbb{N} \), we have
\[ (-1)^n \frac{D_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{\hat{D}_m}{m!}, \]
and
\[ (-1)^n \frac{\hat{D}_n}{n!} = \sum_{m=1}^{n} \binom{n-1}{m-1} \frac{D_m}{m!}. \]

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