The tensionless path from closed to open strings

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We reconsider the tensionless limit on bosonic closed string theory, where the 3d Bondi-Metzner-Sachs (BMS) algebra appears as symmetries on the worldsheet, as opposed to two copies of the Virasoro algebra in the case of the usual tensile theory. This is an ultra-relativistic limit on the worldsheet. We consider the induced representations of the BMS algebra in the oscillator basis and show that the limit takes the tensile closed string vacuum to the “induced” vacuum which is identified as a Neumann boundary state. Hence, rather remarkably, an open string emerges from closed strings in the tensionless limit. We also follow the perturbative states in the tensile theory in the limit and show that there is a Bose-Einstein like condensation of all perturbative states on this induced vacuum. This ties up nicely with the picture of the formation of a long string from a gas of strings in the Hagedorn temperature, where the effective string tension goes to zero.

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Introduction. The very recent first visual evidence of the existence of black holes has reignited interest in the field of gravity and what lies beyond Einstein’s theory even in the non-scientific world. Quantum gravity has been the Holy Grail of modern theoretical physics for several decades now. Of the explored avenues, string theory remains the most viable framework to construct a quantum theory of gravity. String theory is endowed with an intrinsic length-scale, the tension of the fundamental strings, as in the point particle case, become null [1]. In this limit, it has been long speculated (see e.g. [2]) that the distinction between closed and open strings become blurred. In this note, we put forward concrete calculations based on the underlying worldsheet symmetries of the tensionless string, to show how, contrary to conventional wisdom, open strings emerge from closed strings in the tensionless limit. We also find a surprising condensation of all perturbative closed string degrees of freedom on the emergent open string, leading us to speculate that this is the indication of a phase transition.

Classical Tensionless Closed Strings. We start with a quick recap of the important features of the classical tensionless closed string theory. The Polyakov action for bosonic tensile string theory is

\[
S = -\frac{T}{2} \int d^2 \xi \sqrt{-g} g^{\alpha \beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu \nu}. \tag{1}
\]

The action is invariant under worldsheet diffeomorphisms and gauge fixing is required. It is convenient to fix the conformal gauge \( g_{\alpha \beta} = e^{2\phi} \eta_{\alpha \beta} \), but there is still some gauge symmetry left over. This residual symmetry is given by two copies of the Virasoro algebra with generators \( \mathcal{L}_n, \bar{\mathcal{L}}_n \) following:

\[
[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}. \tag{2}
\]

The signature of the tensionless limit is that the world-sheet metric \( g^{\alpha \beta} \) degenerates as we take tension to zero. This can be explicitly shown by looking at the Hamiltonian formulation when one equates the phase space action to the Polyakov form [3]. We implement this by replacing \( T \sqrt{-g} g^{\alpha \beta} \) by \( V^\alpha V^\beta \) where \( V^\alpha \) is a vector density. The action in the \( T \to 0 \) limit then becomes [3]

\[
S = \int d^2 \xi \ V^\alpha V^\beta \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu \nu}. \tag{3}
\]

This action is again invariant under world-sheet diffeomorphisms and one needs to fix gauge. In the tensionless analogue of the conformal gauge \( V^\alpha = (1,0) \), there is again a residual symmetry, which in this case turns out to be the 3d Bondi-Metzner-Sachs algebra (or equivalently the 2d Galilean Conformal Algebra) [3–5]:

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c_L}{12} m(m^2 - 1) \delta_{m+n,0},
\]

\[
[L_m, M_n] = (m - n)M_{m+n} + \frac{c_M}{12} m(m^2 - 1) \delta_{m+n,0},
\]

\[
[M_m, M_n] = 0. \tag{4}
\]

This action also appears as the asymptotic symmetries of 3d flat spacetimes at null infinity [6] (thus has been used in studies of holography in flat spacetimes, see e.g. [7, 8]) and also in non-relativistic conformal systems [9, 10]. In eq (4) above, \( c_L, c_M \) are central charges consistent with Jacobi identities.
In the tensionless limit, the fundamental string grows long and floppy, and the length of the string becomes infinite. For the co-ordinates on the world-sheet, this limit is best viewed as \[ \tau \rightarrow \epsilon \tau, \sigma \rightarrow \sigma, \text{ with } \epsilon \rightarrow 0. \] (5)

This is an ultra-relativistic (UR) or Carrollian limit on the worldsheet \[5, 11\], where the (worldsheet) speed of light goes to zero. It can also be thought of as an infinite boost that makes an ordinary string into a null string. In terms of the generators, this takes the form:

\[ L_n = \mathcal{L}_n - \mathcal{L}_{-n}, \quad M_n = \epsilon (\mathcal{L}_n + \mathcal{L}_{-n}). \] (6)

We now turn to mode expansions of the bosonic tensionless string. In the V\( \alpha \) = (1, 0) gauge, the action can be used to derive equations of motion and constraints:

\[ \partial_{\sigma}^2 X^\mu = 0; \quad (\partial_{\tau} X)^2 = \partial_{\sigma} \cdot \partial_{\sigma} X = 0. \] (7)

The EOM can be solved to yield the mode expansion \[12\]:

\[ X^\mu(\sigma, \tau) = x^\mu + \sqrt{2} \epsilon \mathcal{B}_0^\mu \tau + \sqrt{2} \epsilon \sum_{n \neq 0} \frac{i}{n} (A_n^\mu - i n \tau B_n^\mu) e^{-i n \sigma}. \] (8)

Here we have already put in closed string boundary conditions \[ X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau). \] Defining

\[ L_n = \sum_m A_{-m} \cdot B_{m+n}, \quad M_n = \sum_m B_{-m} \cdot B_{m+n} \] (9)

the constraint equations in (7) become

\[ T(1) = \sum_n (L_n - i n \tau M_n) e^{-i n \sigma} = 0, \] (10a)

\[ T(2) = \sum_n M_n e^{-i n \sigma} = 0. \] (10b)

These combinations are the energy-momentum tensors of a 2d BMS invariant field theory and can be derived just from the symmetry algebra. The classical algebra of the oscillator modes \((A, B)\) is:

\[ \{A_m^\mu, A_n^\nu\} = \{B_m^\mu, B_n^\nu\} = 0, \quad \{A_m^\mu, B_n^\nu\} = -2i m \delta_{m+n,0} \eta^{\mu\nu}. \] (11)

This is not the algebra of harmonic oscillator modes. This will be an important point as we go forward. One can obtain the mode expansion (8) by looking at the tensile mode expansion and using the limit (5). This relates the tensile oscillators \((\alpha, \tilde{\alpha})\) to \((A, B)\):

\[ A_n^\mu = \frac{1}{\sqrt{\epsilon}} (\alpha_n^\mu - \tilde{\alpha}_n^\mu), \quad B_n^\mu = \sqrt{\epsilon} (\alpha_n^\mu + \tilde{\alpha}_n^\mu). \] (12)

It can be easily seen that using (9) and the above relation (12), we get back (6).
where

\[ \beta_\pm = \frac{1}{2} \left( \sqrt{\epsilon} \pm \frac{1}{\sqrt{\epsilon}} \right) \]

(21)

It is clear that since there is a mixing of tensile raising and lowering operators in \( C, \tilde{C} \), the \( C \) vacuum \( |0\rangle_c \) defined by

\[ |0\rangle_c : C^\mu_n |0\rangle_c = 0 = \tilde{C}^\mu_n |0\rangle_c \quad \forall n > 0. \]

(22)

is different from tensile vacuum \( |0\rangle_\alpha \) which in turn is defined by

\[ |0\rangle_\alpha : \alpha^\mu_n |0\rangle_\alpha = 0 = \tilde{\alpha}^\mu_n |0\rangle_\alpha \quad \forall n > 0. \]

(23)

Let us now turn our attention to the vacuum in the induced representation, which we denote by \( |I\rangle \). We have

\[ B^\nu_c |I\rangle = 0, \quad \forall n. \]

(24)

Since there is no ordering ambiguity in \( M_0 \) when acting on this vacuum, the mass of the induced vacuum has to be zero. In terms of \( C \) oscillators, the induced vacuum conditions are:

\[ (C^\mu_n + \tilde{C}^\mu_{-n}) |I\rangle = 0, \quad \forall n. \]

(25)

This is the condition of a Neumann boundary state and the solution is given by

\[ |I\rangle = \mathcal{N} \prod_{n=1}^{\infty} \exp \left( -\frac{1}{n} C_{-n} \cdot \tilde{C}_{-n} \right) |0\rangle_c \]

(26)

where \( \mathcal{N} \) is a (infinite) normalisation constant.

**From closed to open strings.** The relation between the \( C \)-oscillators and the \( \alpha \)-oscillators is a Bogoliubov transformation on the worldsheet:

\[ \alpha^\mu_n = e^{iG} C_n e^{-iG} = \cosh \theta \, C^\mu_n - \sinh \theta \, \tilde{C}^\mu_n, \quad (27) \]

\[ \tilde{\alpha}^\mu_n = e^{iG} \tilde{C}_n e^{-iG} = -\sinh \theta \, C^\mu_n + \cosh \theta \, \tilde{C}^\mu_n, \]

where

\[ G = i \sum_{n=1}^{\infty} \theta \left[ C_{-n} \tilde{C}_{-n} - C_n \tilde{C}_n \right], \quad \tanh \theta = \frac{\epsilon - 1}{\epsilon + 1}. \]

(28)

We can use this to relate the two vacua:

\[ |0\rangle_\alpha = \exp[iG] |0\rangle_c \]

\[ = \left( \frac{1}{\cosh \theta} \right)^{1+1+\ldots} \prod_{n=1}^{\infty} \exp \left[ \frac{\tanh \theta}{n} C_{-n} \cdot \tilde{C}_{-n} \right] |0\rangle_c \]

(29)

Using the regularisation: \( 1 + 1 + 1 + \ldots \infty = \zeta(0) = -\frac{1}{2} \), we finally get

\[ |0\rangle_\alpha = \sqrt{\cosh \theta} \prod_{n=1}^{\infty} \exp \left[ \frac{\tanh \theta}{n} C_{-n} \cdot \tilde{C}_{-n} \right] |0\rangle_c \]

(30)

This is exactly the induced vacuum \( |I\rangle \) that we introduced in (24)–(26). As we mentioned there, this is a Neumann boundary state. This is thus an open string free to move in all dimensions [35]. We have thus obtained an open string by taking a tensionless limit on a closed string theory. Physically, how this is happening can be visualised as in Figure (1).

An open string with Neumann boundary conditions in all directions can also be interpreted as a space-filling D-25 brane. An intuitive picture of how a closed string becomes a spacefilling brane is shown below in Fig. (2).

![FIG. 1: Formation of open strings from closed strings.](image1)

![FIG. 2: Formation of spacefilling D-brane from closed strings.](image2)
there is a truncation of the algebra to its Virasoro subalgebra [10]. Thus in this case, when we dial $\epsilon$ away from 1, the symmetry algebra stays two copies of the Virasoro algebra, until it reaches $\epsilon = 0$, where it becomes BMS$_3$, which in turn reduces to a single copy of the Virasoro due to the absence of the central term $c_M$. So even from this perspective, there is a clear hint of an open string appearing from the closed string worldsheet as the tension is dialled down to zero.

**Bose-Einstein Condensation on the Worldsheet.** We now describe a novel process by which this emergent open string is formed from the states of the tensile closed string theory. Consider any perturbative state in the original tensile theory $|\Psi\rangle = \xi_{\mu\nu} \alpha_{-n}^\mu \alpha_{-n}^\nu |0\rangle_\alpha$ where $\xi_{\mu\nu}$ is a polarisation tensor. Let us attempt to understand the evolution of the state as $\epsilon \to 0$. Close to $\epsilon = 0$, the alpha vacuum can be approximated as follows:

$$|0\rangle_\alpha = |I\rangle + \epsilon |I_1\rangle + \epsilon^2 |I_2\rangle + \ldots.$$  

In this limit, the conditions on the alpha vacuum $(\alpha_n |0\rangle_\alpha = \tilde{\alpha}_n |0\rangle_\alpha = 0, \ n > 0)$ translate to:

$$B_n |I\rangle = 0, \ \forall n \neq 0; \quad (32)$$

$$A_n |I\rangle + B_n |I_1\rangle = 0, \ A_{-n} |I\rangle - B_{-n} |I_1\rangle = 0, \ n > 0.$$  

One can now take this limit on the state $|\Psi\rangle = \alpha_{-n} \tilde{\alpha}_{-n} |0\rangle_\alpha$ (where now we have suppressed the space-time indices) which is now rewritten as

$$|\Psi\rangle = \frac{1}{\epsilon} (B_{-n} + \epsilon A_{-n}) (B_n - \epsilon A_n) (|I\rangle + \epsilon |I_1\rangle + \ldots).$$

Using the commutation relations and (32), we can show

$$|\Psi\rangle \to K |I\rangle \quad \text{as} \quad \epsilon \to 0, \quad (33)$$

where $K$ is a level dependent constant: $K = 2m^\mu \epsilon^\nu \xi_{\mu\nu}$. Thus, all perturbative closed string states condense on the open string induced vacuum. This condensation is like a Bose-Einstein condensation on the worldsheet and is indicative of a phase transition. A point to note here is that this is independent of the level of the state and hence very high energy perturbative states also condense to this vacuum.

**Connections with Hagedorn Physics.** Let us remind the reader that the framework of tensionless strings is useful for addressing questions of string theory near the Hagedorn temperature $T_H$. The Hagedorn temperature is the point where the partition function of the single particle states in string theory blows up and it has been long speculated that this is indicative of a phase transition to a new phase where very different degrees of freedom arise [16]. When looking at the theory of free strings, the Hagedorn phase transition can be understood as follows. This is the regime where it become thermodynamically favourable to form a long string as opposed to heating up a gas of strings [17, 18]. Strings near the Hagedorn temperature become effectively tensionless [19, 20]:

$$T_{\text{eff}} = T_0 \sqrt{1 - \frac{T^2}{T_H^2}} \quad (34)$$

Here $T_{\text{eff}}$ is the effective tension, $T_0 = 1/2\pi \alpha'$ is the usual tension of the string, and $T$ is the temperature of the system. We propose that the induced vacuum $|I\rangle$ is the emergent long string from the point of view of the worldsheet.

The Bose-Einstein condensation on the worldsheet described above is also the worldsheet manifestation of the Hagedorn phase transition. This seems to be at odds with the observation that the BE condensation is something that happens at absolute zero while the Hagedorn phase transition is a very high energy phenomenon. To clarify this, we remind the reader that the Hagedorn temperature is related to the string tension: $T_H = \frac{1}{2\pi \sqrt{2}\alpha'}$. So the tensionless limit, which is $\alpha' \to \infty$, from the point of view of the worldsheet, drives the Hagedorn temperature to zero and hence relates to the above described BE condensation.

**Summary and Future Directions.** We have shown the rather remarkable emergence of an open string from closed strings in the tensionless limit in the context of bosonic string theory and also shown that there is a condensation of all perturbative closed string modes to form this open string. It would be of interest to take this beyond bosonic string theory and generalise to the case of superstrings. From the point of view of the worldsheet, there are two different classical manifestations of the tensionless superstring, which arise from two different contractions of the fermionic generators, which have been dubbed the homogeneous [21, 22] and the inhomogeneous tensionless superstring [23]. We wish to examine both these limits and study the analogue of the induced representations of the underlying super BMS algebras and see what emerges in the quantum regime.

The classical aspects of the tensionless strings, we believe, is now well understood and the central feature is the emergence of the BMS algebra on the null worldsheet.
The Riemannian structure of the tensile worldsheet is replaced by an emergent Carrollian structure. This is similar to what happens for other null surfaces, like the null boundary of flatspace [24–26] and also black hole horizons [27]. But the quantum mechanical structure is much more intricate. We have, in this paper, shed light on one of the possible vacua, the induced vacuum. Depending on the choice of vacuum structure, the resulting quantum theory would be very different. A detailed exposition of this would be presented in a companion paper [28]. There we would also elaborate on other aspects of the induced vacuum.

We have just skimmed the surface of the representation theoretic aspects of the underlying BMS algebra for the tensionless string. There has been a recent resurgence of flat space physics relating asymptotic symmetries with soft theorems and memory effects (see e.g. [29] for a review). The story in three spacetime dimensions has not yet been properly fleshed out [36], and this is in part because of the lack of physical degrees of freedom of gravity in bulk dimensions $d = 3$. The interplay between BMS symmetries and soft theorems in $d = 3$ would have very interesting consequences for the study of tensionless strings. The analogues of soft theorems on the worldsheet may tie in with the infinite number of relations between string amplitudes in the very high energy or equivalently the tensionless limit of strings [30–32]. These remain active lines of inquiry and we hope to shed light on these aspects in the future.

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