Cosmic Problems for Condensed Matter Experiment

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Condensed matter analogs of the cosmological environment have raised the hope that laboratory experiments can be done to test theoretical ideas in cosmology. I will describe Unruh’s sonic analog of a black hole (“dumbhole”) that can be used to test Hawking radiation, and some recent proposals on how one might be able to create a dumbhole in the lab. In this context, I also discuss an experiment already done on the Helium-3 AB system by the Lancaster group.

Cosmology is a unique science where observations of the current or recent universe are used to infer about the very early universe. Furthermore, the physical ideas used to extrapolate back to the early universe are based on ideas that were developed in a non-cosmological setting. Surprisingly most of these ideas work extremely well. Yet there are also ideas that have no experimental verification but are used on the basis of theoretical extrapolation. “Cosmology in the lab” is an emerging research area to search for analogs of the cosmological environment to enable experimental tests of ideas in cosmology.

Cosmological problems are of two types. The first type contains problems that are based on physics in which the cosmological environment is not crucial. It is just that the particular problem is more relevant to cosmology than to some terrestrial system. The outcomes of phase transitions fall into this class of problems. In cosmology, remnants of a phase transition, such as topological defects, are crucial since they could be observed today and lead to important clues about an earlier epoch.

Problems of the second type are strongly based on the gravitational environment. For example, these could involve quantum effects in the pres-
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ence of a “horizon”. In models of cosmic inflation, quantum field theoretic effects on scales larger than the cosmic horizon are supposed to have led to density fluctuations that then grew into galaxies. However, these quantum field theoretic effects are built upon the success quantum field theory (QFT) has had on very small (atomic) scales. Systems with horizons have not been constructed in the lab so far, and there are no experimental tests of QFT on superhorizon scales. Another system with a horizon is a black hole. Particles within the black hole cannot classically escape to the region outside the horizon. However, quantum effects are supposed to modify this picture, leading to Hawking radiation from the black hole. It would be very desirable to find a way to experimentally test these ideas.

Research on “cosmology in the lab” should be understood strictly as an attempt to simulate the cosmological environment in laboratory systems. It should not become an attempt to equate cosmological (or gravitational) phenomenon to some feature of a particular laboratory system. For example, cosmology in the lab cannot say if there is an underlying atomistic structure to spacetime. However, if there was an atomistic theory of spacetime, condensed matter experiments might prove to be useful analogs. From my perspective, cosmology in the lab is similar to numerical studies of systems carried out on a computer. If the simulation is accurate, experiments can tell us things about cosmology that we would otherwise have no hope of finding out.

In 1981 Unruh proposed an experimental analog of a black hole. He considered a fluid that is flowing at subsonic speed in the upstream region and at supersonic speed in the downstream region (see Fig. 1). Any fish in the downstream region cannot send sound signals to the fish in the upstream region. Therefore there is a sonic event horizon. Classically, the fish upstream will receive no sound from the sonic hole i.e. the hole is “dumb”. With quantum effects taken into account, however, there will be Hawking radiation of sound from the horizon of the dumbhole just as there is Hawking radiation of light from a black hole.

To put Unruh’s dumbhole on a more quantitative footing, the fluid is assumed to obey the irrotational, inviscid, Navier-Stokes equations:

$$\nabla \times \mathbf{v} = 0, \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \Phi$$

(1)

where $\Phi$ denotes an external potential (e.g. gravitational potential). The pressure ($p$) is assumed to be a function of the density ($\rho$) alone. Fluctuations
Fig. 1. Unruh’s vision of a waterfall as a sonic black hole or “dumbhole”. The fish in the subsonic flow region cannot hear the screams of the fish in the supersonic region because the emitted sound travels too slowly to propagate upstream. The fish upstream sees a sonic horizon at the location where the fluid velocity becomes supersonic.

around some background flow will correspond to sound. If \( \rho_0 \) and \( v_0 \) denote the background flow,

\[
\rho = \rho_0 + \delta \rho \\
v = v_0 + \nabla \phi
\]

Then it can be shown that

\[
\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu \nu} \partial_\nu \phi) = 0
\]

where the metric, \( g_{\mu \nu} \), experienced by the sound is given by:

\[
ds^2 = (c_s^2 - v_0^2) dt^2 + 2v_0 dt dr - dr^2 - r^2 d\Omega^2
\]

This is the Painlevé-Gullstrand-Lemaître form of a black hole metric if we assume a spherically symmetric, stationary, convergent, background flow. The horizon is at \( v_0 = c_s \), that is, at the location where the fluid velocity equals the sound speed.

The next step is to include quantum field theoretic effects. For this we have to go back to Eq. (2), find the mode functions, and compare their behavior near the horizon to that in the asymptotic region. This is the usual Hawking calculation and we will not show it here. We simply state Unruh’s result for the Hawking temperature:

\[
T_H = \frac{1}{2 \pi} \left. \frac{\partial v_0}{\partial r} \right|_{hor} = (3 \times 10^{-7} \text{ K}) \left[ \frac{c_s}{300 \text{ m/s}} \right] \left[ \frac{1 \text{ mm}}{R} \right]
\]

where \( R \) is the distance over which \( c_s \) changes.
Unruh’s estimate highlights the difficulty of the problem. We need to accelerate the fluid by 300 m/s over a distance of 1 mm to get a temperature of a mere $\sim 10^{-7}$ K. In units of the acceleration due to gravity, the required fluid acceleration is $\sim 10^7 g$! Achieving such an enormous acceleration is impractical given that we also want the fluid to remain very cold ($\mu$K), so that we are able to detect the Hawking radiation. There are all sorts of instabilities that set in under such extreme conditions for all known systems. What is the alternative? A ray of hope emerges when one considers Visser’s generalization of Unruh’s result. Unruh in his pioneering contribution only considered a fluid in which the sound speed is fixed. Visser generalized the setting to include the possibility that the sound speed may vary within the fluid. Then he found:

$$T_{sH} = \left( \frac{\hbar}{2\pi k_B} \right) \left( \frac{\partial}{\partial r} (c_s - v) \right) \bigg|_{\text{hor}}$$  \hspace{1cm} (5)

where $c_s = c_s(t, x)$ is the sound speed and the derivative is evaluated at the location of the horizon ($v = c_s$).

The ray of hope is that we need not have huge gradients in $v$. Instead we might be able to arrange $c_s$ to vary very rapidly. An extreme case would be where $v \approx 0$ everywhere but the properties of the fluid change at a boundary that is moving. The fluid is stationary with respect to the container but is moving with respect to the boundary.

This was also the underlying idea in a proposal made by Jacobson and Volovik where they considered a moving texture or domain wall in $^3$He. The sound speed within the wall is different from that outside. And the wall needs to be moved relatively slowly with respect to the container for there to be Hawking radiation. One complication there is that the order parameter varies within the wall in such a way that there is also radiation due to other effects.

The basic experimental set-up seems very simple (see Fig. 2). There is a tube with the fluid in one phase on one side and a second phase on the other side. We assume that the sound speed in phase 2 is less than the sound speed in phase 1: $c_1 > c_2$. If the phase boundary propagates from phase 2 into phase 1 at a speed $v$ such that $c_1 > v > c_2$, sound from within the phase 2 region cannot enter the phase 1 region. Alternately, in the rest frame of the boundary, the fluid is flowing with velocity $v$ toward the phase 2 side, and the Unruh set-up is exactly duplicated. The phase boundary is the location of the sonic horizon. Since the fluid is not moving with respect to the container, there won’t be any instabilities due to surface effects. Although, the phase boundary does move with respect to the container and one needs to worry if this will lead to instabilities.
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Fig. 2. A supercooled or superheated system with two phases can be a black hole analog if the phase boundary propagates with subsonic speed with respect to Phase 1 but supersonic speed with respect to Phase 2 ($c_1 > v > c_2$). In the rest frame of the boundary, the fluid is moving to the right at a speed $v$ that is subsonic to the left of the boundary and supersonic to its right.

The Hawking temperature for the propagating phase boundary can be calculated in the rest frame of the boundary. Then the metric is of the Painlevé-Gullstrand-Lemaître form (Eq. 3) and one can directly use the calculations in Refs. 7,8 in which Hawking radiation is viewed as the tunneling of ingoing particles from inside the black hole to outgoing particles on the outside. The advantage of this approach is that it is largely system independent. It does require, however, that the particles be massless both inside and outside the black hole. Let us now outline this calculation following Ref. 8.

In both the exterior and interior regions of the sonic hole (Phases 1 and 2 in Fig. 2), the dispersion relation is assumed to be that for a massless fermionic particle. In the fluid rest frame:

\[ E(p) = \pm c_1 p, \quad \text{exterior} \quad (6) \]

\[ E(p) = \pm c_2 p, \quad \text{interior} \quad (7) \]

For convenience we will write these equations as:

\[ E(p) = \pm cp \quad (8) \]

where $c(x)$ is $c_1$ in the exterior region and $c_2$ in the interior region. The transition in $c(x)$ is assumed to be sharp compared to any other length scale of interest but $c(x)$ is still assumed to be continuous.

As for any fermion, the dispersion relation has states with both negative and positive energy. The negative energy levels are all occupied and form the Dirac sea.

If we work in the rest frame of the phase boundary, the fluid moves to the right with speed $v$ in Fig. 2. Now the velocity of a quasiparticle, defined as $\nabla_p E$, is shifted by the fluid velocity. So the dispersion relation becomes:

\[ E(p) = (\pm c + v)p \quad (9) \]
Fig. 3. The dispersion relation outside (left diagram) and inside (right diagram) the dumbhole. Due to the supersonic flow inside the dumbhole the occupied levels in the Dirac sea (dashed lines) emerge and get positive energy. Now particles in the emerged states can tunnel to an unoccupied state in the exterior region.

The usual situation is when $v < c_2 < c_1$. Then the filled levels (Dirac sea) of both branches of the dispersion relation have negative energy. What happens if $v > c_2$? Then one branch of the Dirac sea in the interior region has positive energy as illustrated in Fig. 3. This is what makes it possible for particles of energy $E$ in the interior vacuum to emerge out of the dumbhole. As shown in Fig. 3 it is the particles that have velocity directed to the right (further into the interior of the dumbhole), that can tunnel out onto the branch for the left-moving particles in the exterior region and then escape out to the asymptotic region.

The energy is related to the momentum of the particle via $E = (\pm c + v)p$. We are interested in the particle that is outgoing in the exterior region, therefore we choose the $-$ sign.

The Hawking temperature is calculated by finding the tunneling rate, which can be obtained from the imaginary part of the action:

$$\text{Im}(S) = \text{Im} \int_{-\infty}^{+\infty} p(x)dx = \text{Im} \int_{-\infty}^{+\infty} \frac{E}{v - c(x)}dx$$

The function $v - c(x)$ has a zero at the location of the phase boundary ($x = 0$). So we need to go over to the complex plane, deform the contour of integration around the pole at the origin, and then evaluate the integral using the residue theorem. This gives:

$$\text{Im}(S) = \frac{\pi E}{|c|}$$
where \( |c'| \) is the magnitude of the derivative of the sound speed at the location of the horizon. The rate at which particles can tunnel to the exterior region is given by \( \exp(2 \text{ Im} S) = \exp(E/T_H) \) where \( T_H = |c'|/2\pi \) is the Hawking temperature.

Then the formula for the Hawking temperature is the same as in Eq. (4). The Hawking temperature is estimated to be:

\[
T_{sH} = 0.04 \, \text{K} \left( \frac{\delta c}{300\,\text{m/s}} \right) \left( \frac{100\text{Å}}{\xi} \right)
\]  

(12)

where, \( \xi \) is the thickness of the phase boundary.

Note that \( v \) enters the estimate since it determines the location of the horizon at which gradient of \( c \) is evaluated. However, we will assume that \( T_{sH} \) is (roughly) independent of \( v \) as long as it satisfies \( c_1 < v < c_2 \).

The hardest problem, of course, is to find an experimental realization of a dumbhole. Since the Hawking temperature is very low, the system must either be a fluid or a solid. Only a few fluids are known at this temperature. Solids may be used too if there is a suitable melting transition. Another possibility is to use Bose-Einstein condensates that have been discussed in other related contexts in Refs. 9, 10, 11.

One particularly intriguing system is superfluid \( ^3\text{He} \) since the AB phase boundary has been studied quite extensively both experimentally and theoretically. Indeed the AB interface has been made to oscillate by applying suitable magnetic fields, and the Lancaster group has also measured the radiation from the oscillating phase boundary\(^{12} \). The experimental setup is essentially that shown in Fig. 2 except that the container is vertical. The fluid is kept at 150 \( \mu\)K and 0 bar pressure. At such low temperatures there are few excitations present and the system is essentially in its vacuum state. A non-uniform magnetic field is applied along the vertical such that it is stronger in the lower part of the container than in the upper part. This causes the lower part to be in the A-phase and the upper part to be in the B-phase. The AB phase boundary is at the location where the magnetic field attains a critical value, \( B_{AB} \). Next, a small oscillating magnetic field is applied along the vertical so that the total magnetic field is:

\[
B(t, z) = B_0(z) + B_{AC} \sin(\omega t)
\]  

(13)

The location of the AB phase boundary in the absence of dissipation can be be found by setting \( B(t, z) = B_{AB} \). Due to the oscillating component of the magnetic field, the equilibrium location of the interface is time-dependent: \( z_0 = a \sin(\omega t) \). The amplitude of oscillation is given by: \( a = B_{AC}/\nabla B_0 \). (The gradient of \( B_0 \) is roughly constant in the region where the phase boundary oscillates.) The frequency of oscillation of the phase boundary can be
changed by changing the frequency, $\nu = \omega / 2\pi$, of the applied oscillating field. The quasiparticle radiation is detected by a vibrating wire resonator placed at the upper end of the container. The data for the radiated power versus oscillation frequency is shown in Fig. 4.

The behavior of the experimental curves in Fig. 4 are explained phenomenologically as follows. Assuming a linear restoring force and dissipative motion, the equation of motion for $z(t)$, the position of the interface, is:

$$-k(z - z_0(t)) - \gamma \frac{dz}{dt} = 0$$  \hspace{1cm} (14)

where $z_0(t) = a \sin(\omega t)$ ($a = B_{AC}/\nabla B$) is the equilibrium position of the interface. Then,

$$v(t) = \frac{dz}{dt} = v_0 \cos(\omega t - \phi) + O(e^{-\kappa t})$$  \hspace{1cm} (15)

where

$$\kappa = \frac{k}{\gamma}, \quad v_0 = \frac{a \kappa \omega}{\sqrt{\kappa^2 + \omega^2}}, \quad \tan \phi = \frac{\omega}{\kappa}$$  \hspace{1cm} (16)

Therefore $v_0 \propto \omega$ for $\omega \ll \kappa$ and $v_0$ is independent of $\omega$ for $\omega \gg \kappa$. The average power dissipated over an oscillation time period is

$$P = \gamma \langle v^2 \rangle$$  \hspace{1cm} (17)

and this grows quadratically with $\omega$ for $\omega < \kappa$ and is constant for $\omega > \kappa$. This is consistent with the log-log plot of the data for $\nu < 1$ Hz in Fig. 4 for
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the larger value of $B_{AC}$, and over the whole range of $\nu$ for the smaller value of $B_{AC}$ provided we treat $\gamma$ as a fitting parameter that depends on $B_{AC}$.

There are three unexplained features of the data as discussed in Ref. [12]:

1. Theoretical estimates of $\gamma$ are small by several orders of magnitude.

2. For the smaller value of $B_{AC}$, the required value of $\gamma$ is about 5 times smaller than what is needed to fit the data for larger $B_{AC}$. If $\gamma$ is calculated from the $B_{AC} = 0.643$ mT data, and used to predict the $B_{AC} = 0.214$ mT curve, one gets the lower solid curve shown in Fig. [1] and this does not even come close to matching the data.

3. The data clearly shows an unexpected increase in dissipation at frequencies $\nu > 1$ Hz when $B_{AC} = 0.643$ mT.

Can Hawking radiation play any role in the increased dissipation seen for $\nu > 1$ Hz when $B_{AC} = 0.643$ mT? The data does seem to indicate a new source of dissipation at high frequencies. The velocity of the interface at $\nu \sim 1$ Hz is about 1 cm/s. This also happens to be close to the quasiparticle velocity in the A phase in the direction orthogonal to the interface ($c_A = 3$ cm/s for certain excitations as explained below). It is also less than the quasiparticle velocity in the B phase ($c_B = 55$ m/s). Furthermore, the power emitted in Hawking radiation using the temperature in Eq. (12) is of order 1 pW and is in the range seen. Hawking radiation is only expected during the part of the oscillation when the velocity of the interface lies in the suitable range: $c_B > v > c_A$. As this duration increases, we expect the amount of Hawking radiation to grow proportionally.

At $\nu \sim 1$ Hz in the experiment, the simple model for the motion of the interface must break down. This could happen in two ways. The restoring force might get non-linear corrections, or the dissipation parameter $\gamma$ could get some velocity dependent corrections. Purely for illustrative purposes, let us assume that the form of the velocity (Eq. (15)) continues to hold even at large $\omega$ but with a different amplitude:

$$v = \alpha \omega \cos(\omega t - \tilde{\phi})$$ (18)

The amplitude $\alpha$ is assumed to be an $\omega$ independent parameter in the frequency range of interest. The exact form of $v$ is not crucial for us. For example, the power of $\omega$ in the amplitude could be different from 1.

Let us now assume that Hawking radiation at temperature $T_{sH}$ (Eq. (12)) is emitted when $v = \dot{z} > c_A$. Then, for $j$ “light” species of radiation, this gives:

$$P_H = j\sigma_s T_{sH}^4 A \left( \frac{\delta t}{\tau} \right) = j\sigma_s T_{sH}^4 A \frac{1}{\pi} \cos^{-1} \left( \frac{c_A}{2\pi \alpha \nu} \right)$$ (19)
where $\delta t/\tau$ is the fraction of time for which $v > c_A$. Note that the formula holds only for radiation for which the mass is less than the Hawking temperature. Since $T_{sH} \approx 3\text{mK} > \Delta_B \approx 1.7\text{mK}$ this is self-consistent. (If the Hawking temperature is less than the mass of the particles, the radiation is exponentially suppressed.) Let us write

$$P_H = \frac{P_0}{\pi} \cos^{-1} \left( \frac{\nu_s}{\nu} \right)$$

(20)

where

$$P_0 \approx 116 \left( \frac{j}{2} \right) \left( \frac{d}{4.3\text{mm}} \right)^2 \left( \frac{\delta c_s}{60\text{m/s}} \right)^2 \left( \frac{100\text{Å}}{\xi} \right)^4 \text{pW}$$

(21)

and $d$ is the cell diameter (4.3 mm in the experiment). The value of $P_0$ is in the observed range if $\xi \sim 300\text{ Å}$.

Can we estimate the critical frequency $\nu_s$? In our illustrative model

$$\nu_s = \frac{c_A}{2\pi\alpha}$$

(22)

If $\alpha = B_{AC}/\nabla B_0|_{\text{interface}}$ then the values of $\nu_s$ are shown in the following Table.

| $\nabla B_0$ T/m | $B_{AC}$ mT | $\nu_s$ Hz |
|-----------------|-------------|-----------|
| 2.00            | 0.643       | 14.8      |
| 1.00            | 0.643       | 7.4       |
| 0.53            | 0.643       | 3.9       |
| 1.00            | 0.214       | 22.2      |

Clearly the values of $\nu_s$, the critical value of the frequency at which Hawking radiation starts, do not agree with data. The experiment shows anomalous radiation starting at some critical frequency, whereas Hawking radiation would start at some critical velocity. On the other hand, there is no reason to adopt $\alpha = B_{AC}/\nabla B_0$ except that this holds at low frequencies. The Hawking radiation explanation can only work if $\alpha$ is independent of $\nabla B_0$ at high frequencies. However, as we will now see, the interpretation in terms of Hawking radiation has additional theoretical issues.

The main difficulty with the interpretation is that, in our earlier estimate, we have assumed that the sound quanta are massless on either side of the interface, whereas this is not true in the AB system. The quasiparticles on the A-phase side of the interface indeed have a linear dispersion relation very close to the nodal point. This can be seen from the full dispersion relation in the rest frame of the fluid\[13\]

$$E(p) = \pm \left[ v_F^2 (|p| - p_F)^2 + \frac{\Delta_A^2}{p_F^2} (\hat{l} \times p)^2 \right]^{1/2}$$

(23)
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where \( v_F \approx 55 \text{ m/s} \) is the Fermi velocity and \( p_F = m^*v_F \) is the Fermi momentum with \( m^* \approx 3m_{He-3} \), and \( \Delta_A \approx 2.02k_BT_c \) where \( T_c \) is the transition temperature from normal to superfluid phase. At zero pressure and strong magnetic field \( T_c \approx 1 \text{ mK} \). It is easy to check that \( E(p) \) vanishes if \( p = \pm p_F \hat{l} \).

Hence the dispersion relation has a node. Now consider excitations near the node, \( p = \pm p_F \hat{l} + \delta p \), with

\[
\left| \frac{\delta p}{p_F} \right| \ll \frac{\Delta_A}{p_Fv_F} \approx 10^{-3}
\]

and with momenta perpendicular to \( \hat{l} \): \( \delta p \cdot \hat{l} = 0 \)

For these excitations the dispersion relation is linear,

\[
E = \pm c_A |\delta p|
\]

with \( c_A = \Delta_A/p_F \approx 3 \text{ cm/s} \). Therefore we can expect a dumbhole to form whenever the interface velocity exceeds 3 cm/s.

On the B-phase side, the dispersion relation in the rest frame of the fluid is:

\[
E(p) = \pm \left[ \left\{ \epsilon(p) - \mu \right\}^2 + \Delta_B^2 \frac{p^2}{p_F^2} \right]^{1/2}
\]

where \( \epsilon(p) = p^2/2m^* \), \( \mu = \epsilon(p_F) \), and the B-phase gap \( \Delta_B \approx 1.76k_BT_c \) with \( T_c \approx 0.93 \text{ mK} \) at zero pressure and magnetic field. Since \( E \) only depends on \( p^2 \) and it does not vanish at \( p^2 = p_F^2 \), there is no node in the dispersion relation. In fact, for \( p = |p| \approx p_F \) we have:

\[
E(p) = \pm \left[ v_F^2(p - p_F)^2 + \Delta_B^2 \right]^{1/2}
\]

where we also assume

\[
\left( \frac{\Delta_B}{\mu} \right)^2 \ll \left( \frac{p - p_F}{p_F} \right) \ll 1
\]

Note that \( (\Delta_B/\mu)^2 \approx 10^{-6} \).

Since the B-phase dispersion relation is not of the usual relativistic form \( E = \pm \sqrt{p^2c^2 + m^2c^4} \), it is not possible to think of the quasiparticles as propagating on a metric with just the usual second derivative kinetic term. However the black hole analogy need not break down! There is still a sonic horizon from within which quasiparticles moving at 3 cm/s in the A-phase

\[\text{1Note that the } \hat{l} \text{ vectors on the AB interface mostly lie in the plane of the interface. Hence excitations that cross the interface necessarily have non-vanishing momenta perpendicular to } \hat{l}.\]
Vacuum fluctuations in the B phase can produce an $f\bar{f}$ pair, say with $f$ having positive energy and $\bar{f}$ having negative energy. $\bar{f}$ falls into the dumbhole, and if its velocity is very low, cannot escape back out into the B-phase region. $f$ escapes and forms Hawking radiation.

This is an important ingredient for a dumbhole (see Fig. 5). The present situation is more complicated though, because not all quasiparticles in the A-phase move at 3 cm/s. Only the quasiparticles close to the nodal point have this velocity. So a negative energy particle that falls through the interface and into the A-phase region cannot escape back to the B-phase region only if its velocity in the A-phase region is of order 3 cm/s.

The dispersion relations given above are in the fluid rest frame. In the interface rest frame, the fluid is moving to the right with velocity $v$. Quasiparticle velocities in the interface rest frame will be shifted by $v$ with respect to the velocities in the fluid rest frame. Since velocity is defined by $\nabla_p E$, the dispersion relation in the interface rest frame is obtained by adding $v \cdot p$ to the dispersion relation in the fluid rest frame. As before, part of the Dirac sea on the A-phase side of the interface will emerge into the positive energy region. Furthermore, the energy of some occupied states in the A-phase side is the same as that of unoccupied states in the B-phase side. So the basic set-up of the linear dispersion relation calculation given above is available. The technical difficulty is that the equation $\epsilon = E(p, c(x))$ cannot as simply be inverted to get $p(E, c(x))$. Hence let us construct a toy model for the dispersion relation that captures some of the essential features of the full model.

Consider the one-dimensional dispersion relation:

$$E(p) = \pm \sqrt{c^2(p - P)^2 + \Delta^2} + vp$$

(29)

where $c = v_F$, $\Delta = \Delta_B$ in the B-phase region and $c = c_A$, $\Delta = 0$ in the A-phase. We will take $P = p_F \equiv P_B$ in the B-phase and $P = 0$ in the A-phase, but also keep open the possibility that $P_B = 0$. The functions $c(x)$, $\Delta(x)$ and $P(x)$ only vary appreciably within the interface. This dispersion relation goes over to Eq. (25) in the A-phase (with $v = 0$) and likewise Eq. (27) in the B-phase\(^2\). Note that the dispersion relation is written in the rest frame

\(^2\)Though the linear dispersion in the A-phase is due to the $\Delta_A^2$ term and not due to the kinetic term as in the toy model.
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of the interface in which the fluid is moving to the right with speed \( v \). Also, we are looking at the one dimensional problem where the momentum is only along the \( x \) direction.

Inverting Eq. (29) we get,

\[
q \equiv p - P = \frac{\epsilon v \pm \sqrt{\epsilon^2 v^2 - (v^2 - c^2)(c^2 - \Delta^2)}(v^2 - c^2)}{(v^2 - c^2)}
\]

(30)

where \( \epsilon \equiv E - vP \). The discriminant should be positive in the asymptotic regions for the particle to be physical (and not “under the barrier”). This imposes a constraint on the velocity:

\[
v^2 > c^2 \left( 1 - \frac{\epsilon^2}{\Delta^2} \right)
\]

(31)

In the A-phase region, \( \Delta = 0 \), and the condition is trivially satisfied as seen from Eq. (30). In the B-phase, the condition reads:

\[
v^2 > v_F^2 \left[ 1 - \frac{(E - vP_B)^2}{\Delta_B^4} \right]
\]

(32)

Hawking radiation can only occur if:

\[
E > \Delta_B \sqrt{1 - \frac{v^2}{v_F^2}} + vP_B
\]

(33)

or else if:

\[
E < -\Delta_B \sqrt{1 - \frac{v^2}{v_F^2}} + vP_B
\]

(34)

We assume that the condition in Eq. (31) is satisfied everywhere, including within the interface. If this is not the case, the discussion will be more involved.

In the experiment, \( v \sim c_A \sim 10^{-3}v_F \) and so the square root factor is essentially 1. Therefore either \( E > \Delta_B + vP_B \) or else \( E < -\Delta_B + vP_B \). Note that we are interested in \( v > \Delta_A/p_F = c_A \) and since \( \Delta_A \approx \Delta_B \), we also have \( vP_F > \Delta_B \). So the thresholds for \( E \) are both positive if \( P_B = p_F \) (but not if, for example, \( P_B = 0 \)). The first possibility is one that is expected since the minimum energy of quasiparticles in the B-phase is \( \Delta_B \) in the fluid rest frame. The second possibility corresponds to the emergence of the Dirac sea in the B-phase when viewed in the rest frame of the interface. Let us discuss both possibilities in some more detail.

At the large values of the energy required by the first possibility (Eq. (33)), our toy model starts becoming suspect and a more realistic calculation is
called for. However, if we proceed with the toy model under the assumption that it still gives the correct qualitative behavior, we can calculate the Hawking temperature. The calculation of the action follows the linear case described earlier. There is a pole at $v = c(x)$ and this gives the only contribution to the imaginary part of the action. However, the process requires a branch change. The particles on the A-phase side are on the branch with the $-$ sign in Eq. (29) while, after tunneling, the particles would end up on the branch on the B-phase side that has the $+$ sign. Further analysis is required to determine if the branch change can take place.

Now we discuss the second possibility (Eq. (34)). In this case, the particle from the A-phase may tunnel to a state in the Dirac sea of the B-phase. However, the tunneling cannot take place since the corresponding state in the B-phase Dirac sea is occupied. Yet the process may still be important. Suppose there is a vacuum fluctuation in the B-phase in which a particle from the Dirac sea gets excited to the upper branch. Normally the particle would fall back into the hole in the Dirac sea within a time allowed by the uncertainty principle. However, in the present situation, a particle from the A-phase can tunnel into the B-phase and fill up the hole before the original particle has had a chance to fall back. Then there is no hole left for the particle in the upper branch to return to and it must escape as a real particle (see Fig. 6). This literally corresponds to the process shown in Fig. 5 where vacuum fluctuations in the B-phase create a pair of positive and negative energy fermions and the negative energy particle falls into the dumbhole. So the second possibility given by Eq. (34) might indeed be relevant.

The calculation of the Hawking temperature proceeds as before. The tunneling rate is calculated by finding the imaginary part of the action obtained by integrating $q(x)$. Note that the $+$ sign in Eq. (30) must be chosen for there to be a pole in $q(x)$. The tunneling rate must be multiplied by the probability of having a hole of energy $E$ on the B-phase side. As long as this factor is not an exponential, the Boltzmann factor will be given by the tunneling rate alone. Since the structure of the pole is the same as in the linear calculation given above, the Hawking temperature will still be $T_{sH} = |c'|/2\pi$ in the interface rest frame. Note that since the tunneling occurs between the branches with a $-$ sign in Eq. (29), $E$ must lie in the range (for $P_B = p_F$):

$$0 < E < v_F - \Delta_B$$

where we are assuming $v \ll v_F$. Therefore, in the fluid rest frame, the radiated particle can only have energy ($E_r$) in the interval:

$$\Delta_B < E_r < v_F$$
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Fig. 6. The figure shows the schematics of the dispersion relations in the toy model for the A- and B-phases. Dashed curves denote the filled states of the Dirac sea. If there is a vacuum fluctuation in the B-phase, a hole can be created in the lower branch, which can then be re-filled by a particle tunneling out of the A-phase. The particle that had jumped onto the upper branch during the vacuum fluctuation then escapes to infinity as Hawking radiation. This process is identical to that shown in Fig. 5.

This restriction on the range of energy of radiated particles will change if branch changing processes can occur. Then one can imagine several other processes as well. These should be investigated.

The above discussion is based on a toy model of the AB system and suffers from the danger that it is inaccurate in some essential way. In particular, since the last term in Eq. (29) comes from $\mathbf{v} \cdot \mathbf{p}$, and the component of $\mathbf{p}$ along $\mathbf{v}$ is taken to be small on the A-phase side, perhaps it would have been better to replace the term by $\mathbf{v}(\mathbf{p} - P(x))$. Then, with $P_B = p_F$ and for small $p - p_F$, the B-phase Dirac sea would not emerge into the positive energy region and the only possibility for tunneling would be via branch changing processes.

Clearly it is important to do a more thorough calculation and with a realistic model of the AB interface. Such an effort would be aided by the early calculations on the dissipation from the AB interface \cite{15}, the profile of the order parameter within the interface \cite{16}, and the scattering of quasiparticles off the interface \cite{17}. To connect with experiment, one would also need a handle on the interface motion at high frequencies and on other effects that could potentially explain the anomalous radiation.

Experimentally, from the perspective of studying Hawking radiation, it would be desirable to obtain the spectrum of the emitted radiation, and to
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be able to correlate the emitted power with the dynamics (say, the velocity) of the interface.

If it turns out that Hawking radiation from the AB interface of $^3$He is highly suppressed (or absent), we will be left with the challenge of finding a quantum black hole analog. This is an exciting quest. It is probably also our only hope for experimentally testing current ideas on quantum black holes and other cosmological problems.

ACKNOWLEDGMENTS

I would like to thank the organizers of the conference on Quantum Phenomena at Low Temperatures (ULTI III Users Meeting, 2004) in Lammi, Finland, for giving me the opportunity to participate. I am grateful to Arnie Dahm, Shaun Fisher, Harsh Mathur, George Pickett, Nils Schopohl and Grisha Volovik for very informative discussions. This work was supported by the DOE (US).

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