Singular locus of $q$-logarithmic foliations

Ariel Molinuevo$^*$ and Federico Quallbrunn$^†$

$^*$Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brazil.
$^†$Departamento de Matemática, Universidad CAECE, Argentina.

Abstract

We determine the structure of the singular locus of generic codimension-$q$ logarithmic foliations and its relation with the unfoldings of said foliations. In the case where the ambient variety is the projective space $\mathbb{P}^n$ we calculate the graded ideal defining the scheme of persistent singularities.

1 Introduction

Unfoldings of singular foliations, and particularly first order infinitesimal unfoldings (cf. Definition 2.4 below) where introduced independently by Suwa and Mattei (see [Suw95] and references therein). Although the original definition of Suwa was stated in full generality, the focus of these earlier works was in germs of codimension-1 foliations around a singularity. In latter years, the work of Suwa have been expanded in several articles (see [MMQ21] and references within).

In some of these articles, unfoldings of global foliations in projective space were explicitly computed. Explicit computation of unfoldings of codimension-1 foliations in projective space was possible mainly because in this case there is an isomorphism between first order infinitesimal unfoldings of a foliation and the space of global sections of the defining ideal of the scheme of persistent singularities (cf.: Definition 2.8 below).

Other results obtained in these latter articles relate the scheme of persistent singularities with that of Kupka singularities (cf.: Definition 2.9). These type of results where obtained for codimension-1 as well as for codimension-$q$ foliations.

In this article we use the relation between Kupka and persistent singularities to establish in Theorem 4.10 the dimension of the irreducible components of a generic logarithmic foliation on a non-singular complete intersection $X$, as well as determining the equality between the schemes of Kupka and persistent singularities. Then in Corollary 4.12 we use the result stated above to compute the graded ideal of persistent singularities in the case where $X = \mathbb{P}^n$. These results are of particular interest as not much is known in general about the singular set of a higher codimension foliation, so logarithmic foliations represent an interesting first example where the dimension of the irreducible components of the singular set as well as the scheme of persistent singularities are defined.

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2 Codimension $q$ Foliations

Along this section we will give the definition of codimension-$q$ foliation on a smooth variety $X$. Then we recall the definition of unfolding of a codimension-$q$ foliation on a variety $X$. Finally we review the definitions of persistent singularities and of Kupka singularities for codimension-$q$ foliations, see Definition 2.8 and Definition 2.9 respectively, and we state some general facts regarding the ideal of persistent singularities and the Kupka scheme. We refer the reader to [MMQ21] for a complete treatment of the subject.

Let us fix a smooth variety as $X$. If $\Xi \in \Gamma(U, \bigwedge^q T X)$ is a multivector and $\varpi \in \Gamma(U, \Omega^q_X)$ a $q$-form we will denote by $i_{\Xi} \varpi \in \Gamma(U, \bigwedge^{q-1} T X)$ the contraction. Recall that the Plücker relations for $\varpi$ are given by

$$i_{\Xi} \varpi \wedge \varpi = 0$$

for any $\Xi \in \bigwedge^{q-1} T X$.

When $\varpi(p) \neq 0$ for some closed point $p \in X$ then $\varpi$ is locally decomposable as a product $\varpi = \varpi_1 \wedge \cdots \wedge \varpi_q$ of $q$ 1-forms.

**Definition 2.1.** Let $\mathcal{L}$ be a line bundle and $\omega : \mathcal{L} \to \Omega^q_X$, with $1 \leq q \leq \dim(X) - 1$, be a (non trivial) morphism of sheaves, we will say that the morphism is integrable if

- $\Omega^q_X/\mathcal{L}$ is torsion free.
- The map $i_{\Xi} \omega : \mathcal{L} \to \Omega^{q+1}_X \otimes \mathcal{L}^{-1}$ is zero for every local section $\Xi$ of $\bigwedge^{q-1} T X$.
- For every local section $s$ of $\mathcal{L}$ and $\Xi$ of $\bigwedge^{q-1} T X$, $\omega(s)$ verifies

$$d(i_{\Xi} \omega(s)) \wedge \omega(s) = 0. \quad (1)$$

We also say that $\omega$ determines a codimension-$q$ foliation.

Let $\omega : \mathcal{L} \to \Omega^q_X$ be an integrable $q$-form. Then, we can consider two maps,

$$\bigwedge^{q-1} T X \otimes \mathcal{L} \xrightarrow{i_{\Xi} \omega} \Omega^q_X \xrightarrow{\omega \wedge -} \Omega^{q+1}_X \otimes \mathcal{L}^{-1}$$

The integrability condition on $\omega$ implies that this diagram is a complex and it is easy to check that its homology is supported over the points where $\omega$ is not decomposable.

**Definition 2.2.** We define the sheaf associated to $\omega$, denoted $\mathcal{E} = \mathcal{E}(\omega)$, as the kernel of $\omega \wedge -$ as a subsheaf of $\Omega^q_X$.

Notice that by [Har80, Proposition 1.1, pp. 124] we get that $\mathcal{E}$ is a reflexive sheaf.

Composing a morphism $\omega : \mathcal{L} \to \Omega^q_X$ with the contraction of forms with vector fields give us a morphism

$$\bigwedge^q T X \otimes \mathcal{L} \to \mathcal{O}_X.$$
Definition 2.3. The ideal sheaf \( \mathcal{I}(\omega) \) is defined to be the sheaf-theoretic image of the morphism \( \bigwedge^q TX \otimes \mathcal{L} \to \mathcal{O}_X \). The subscheme it defines is called the singular scheme of \( \omega \) and denoted \( \text{Sing}(\omega) \subseteq X \). We will denote it just as \( \mathcal{I} \) if no confusion arises.

From [Suw93](4.6) Definition, p. 192] we get the following definition for a codimension-\( q \) foliation:

Definition 2.4. Let \( T \) be a scheme, \( p \in T \) a closed point, and \( \mathcal{L} \to \Omega^q_X \) a codimension-\( q \) foliation on \( X \). An unfolding of \( \omega \) is a codimension-\( q \) foliation \( \tilde{\omega} : \tilde{\mathcal{L}} \to \Omega^q_X \otimes T \) on \( X \times T \) such that \( \tilde{\omega}|_{X \times \{p\}} \equiv \omega \). In the case \( T = \text{Spec}(k[x]/(x^n)) \) we will call \( \tilde{\omega} \) a first order infinitesimal unfolding.

Let \( \mathcal{D} = \text{Spec}(k[\epsilon]/(\epsilon^2)) \) be the scheme of dual numbers over \( k \), \( 0 \in \mathcal{D} \) be its closed point, \( p : X \times \mathcal{D} \to \mathcal{D} \) be the projection and \( \iota : X \cong X \times \{0\} \hookrightarrow X \times \mathcal{D} \) be the inclusion. Then the sheaf \( \Omega^q_{X \times \mathcal{D}} \) can be decomposed as direct sum of \( \iota_* (\mathcal{O}_X) \)-modules as

\[
\Omega^q_{X \times \mathcal{D}} \cong \iota_* \Omega^q_X \oplus \epsilon \cdot (\iota_* \Omega^q_X) \oplus \iota_* \Omega^{q-1}_X \wedge \epsilon d\epsilon.
\]

Given a codimension-\( q \) foliation determined by a morphism \( \mathcal{L} \to \Omega^q_X \), and a first order infinitesimal unfolding \( \tilde{\omega} : \tilde{\mathcal{L}} \to \Omega^q_X \otimes T \) of \( \omega \), we take local generators \( \omega \) of \( \mathcal{L}(U) \) and \( \tilde{\omega} \) of \( \tilde{\mathcal{L}}(U \times \mathcal{D}) \). Suppose \( \omega \) and \( \tilde{\omega} \) are locally decomposable, then we may take \( U \) small enough such that \( \omega \) and \( \tilde{\omega} \) decompose as products

\[
\omega = \omega_1 \wedge \cdots \wedge \omega_q, \quad \tilde{\omega} = \tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_q.
\]

Then we can write \( \tilde{\omega}_i = \omega_i + \epsilon \eta_i + h_i d\epsilon \) and the equations \( d \tilde{\omega}_i \wedge \tilde{\omega} = 0 \) for \( i = 1, \ldots, q \) are equivalent to the equations

\[
\begin{align*}
\begin{cases}
\epsilon \eta_i + d\omega_i \wedge \left( \sum_{j=1}^q (-1)^j \eta_j \omega_j \right) = 0, & (i = 1, \ldots, q), \\
\epsilon (h_i - \eta_i) + d\omega_i \wedge \left( \sum_{j=1}^q (-1)^j h_j \omega_j \right) = 0, & (i = 1, \ldots, q),
\end{cases}
\end{align*}
\]

where \( \omega_j = \omega_1 \wedge \cdots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \cdots \wedge \omega_q \in \Omega^{q-1}_X(U) \).

As is shown in [Suw93] proof of 6.1 Theorem, p. 199] the second equation implies the first. So we finally get that the equations \( d \tilde{\omega}_i \wedge \tilde{\omega} = 0 \) for \( i = 1, \ldots, q \) are equivalent to

\[
\begin{align*}
\begin{cases}
\epsilon (h_i - \eta_i) + d\omega_i \wedge \left( \sum_{j=1}^q (-1)^j h_j \omega_j \right) = 0, & (i = 1, \ldots, q),
\end{cases}
\end{align*}
\]

(2)

For the case \( q = 1 \), we recall the following proposition from [MMQ21] Proposition 3.6, pp. 8] which gives the name to the ideal of persistent singularities:
Proposition 2.5. Let \( p \in X \) be a point in \( \operatorname{Sing}(\omega) \), \( \mathcal{O}_X, p \) the local ring around \( p \), and \( X_p = \operatorname{Spec}(\mathcal{O}_X, p) \). Then \( p \) is in the subscheme of persistent singularities if and only if for any infinitesimal first order unfolding \( \tilde{\omega} \) of \( \omega \) in \( X_p \), the point \( (p, 0) \in X_p \times \mathcal{D} \) is a singular point of \( \tilde{\omega} \).

Let \( \mathcal{L} \xrightarrow{\sim} \Omega_X^p \) be an integrable morphism inducing a morphism \( \mathcal{E} \to \Omega_X^p \). Composing \( \omega \) with wedge product gives a morphism \( \mathcal{L} \otimes \Omega_X^2 \xrightarrow{\omega \wedge -} \Omega_X^{q+2} \). As \( \omega \) is integrable the sheaf \( \mathcal{E} \otimes \Omega_X^p \) is in the kernel of \( \omega \wedge - \).

**Definition 2.6.** Following the notation above, we define the sheaf \( H^2(\omega) \) as

\[
H^2(\omega) := \ker(\omega \wedge -)/\mathcal{E} \otimes \Omega_X^p.
\]

**Remark 2.7.** The restriction of the de Rham differential to \( \mathcal{E} \) gives a sheaf map \( \mathcal{E} \to \Omega_X^1 \) which is not \( \mathcal{O}_X \)-linear but whose image is in \( \ker(\omega \wedge -) \) as \( \omega \) is integrable. The projection of this map to \( H^2(\omega) \) is however \( \mathcal{O}_X \)-linear as \( dg\varpi \equiv gd\varpi \mod \mathcal{E} \otimes \Omega_X^1 \) for every local section \( \varpi \) of \( \mathcal{E} \).

Let us fix \( \mathcal{L} \xrightarrow{\sim} \Omega_X^p \) be an integrable morphism determining a subsheaf \( \mathcal{E} \to \Omega_X^p \). Then we have the following definitions:

**Definition 2.8.** The subscheme of persistent singularities of \( \omega \) is the one defined by the ideal sheaf \( \mathcal{I}(\omega) \) to be the annihilator of \( d(\mathcal{E}) \) in \( H^2(\omega) \). In other words the local sections of \( \mathcal{I}(\omega) \) in an open set \( U \subseteq X \) are given by

\[
\mathcal{I}(\omega)(U) = \{ h \in \mathcal{O}_X(U) : \forall \varpi \in \mathcal{E}(U), \text{ } hd\varpi = \sum_j \alpha_j \wedge \omega_j \},
\]

for some local 1-forms \( \alpha_j \) and forms \( \omega_j \) in \( \mathcal{E}(U) \). We will denote it just as \( \mathcal{I} \) if no confusion arises.

We also define as \( \mathcal{Ker}(\omega) \) the subvariety of \( X \) associated to the variety defined by \( \mathcal{I}(\omega) \), i.e., \( \mathcal{Ker}(\omega) = \mathcal{O}_X/\mathcal{I}(\omega) \).

**Definition 2.9.** The subscheme of Kupka singularities of \( \omega \) is the one defined by the ideal sheaf \( \mathcal{K}(\omega) = \text{ann}(\{d\omega\}) \in \Omega_X^{q+1} \otimes \mathcal{O}_{\text{Sing}(\omega)} \otimes \mathcal{L}^{-1} \). We will denote it just as \( \mathcal{K} \) if no confusion arises.

We also define as \( \mathcal{Kup}(\omega) \) the subvariety of \( X \) associated to the variety defined by \( \mathcal{K}(\omega) \), i.e., \( \mathcal{Kup}(\omega) = \mathcal{O}_X/\mathcal{K}(\omega) \).

Let \( p \in \mathcal{Kup}(\omega) \). We will call \( p \) a strictly Kupka point if \( d\omega|_p \neq 0 \), meaning that the evaluation of \( d\omega \) at the point \( p \) is different from zero.

We will need the following proposition, also from [MMQ21] Proposition 4.9, p. 14, for later use.

**Proposition 2.10.** Given an integrable morphism \( \mathcal{L} \xrightarrow{\sim} \Omega_X^p \) we have the inclusions \( \mathcal{I}(\omega) \subseteq \mathcal{I}(\omega) \) and \( \mathcal{I}(\omega) \subseteq \mathcal{K}(\omega) \). If moreover \( \omega \) is locally decomposable (i.e. if \( \mathcal{E} \) is locally free) then we have \( \mathcal{I}(\omega) \subseteq \mathcal{I}(\omega) \subseteq \mathcal{K}(\omega) \).

From now on we will denote \( \bigoplus_{\ell \in \mathbb{Z}} H^1(\mathbb{P}^n, \mathcal{F}(\ell)) \) by \( H^1_*(\mathbb{P}^n, \mathcal{F}) \) for a sheaf \( \mathcal{F} \) on \( \mathbb{P}^n \), the ring of homogeneous coordinates of \( \mathbb{P}^n \) by \( S \), i.e., \( S = \mathbb{C}[x_0, \ldots, x_n] = H^0_*(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \) and the homogeneous component of degree \( \ell \) of a graded \( S \)-module \( M \) by \( M(\ell) \).
3 Differential forms with logarithmics poles

Along this section recall the definitions of $q$-logarithmic differential forms and of residue of a differential form due to Deligne. For that we will follow [Del06, 3. Pôles logarithmiques, p. 72].

We will refer to the set $\{1, \ldots, s\}$ by $[s]$ and by $|I|$ to the cardinality of the (sub) set $I \subset [s]$.

**Definition 3.1.** Let us consider $D$ a normal crossing divisor and let $j : X \setminus D \to X$ the inclusion. We define the *sheaf of logarithmic differential forms along $D$* as the smallest sheaf $\Omega_{X}^{\ast}(\log D)$ of $j_{\ast}\Omega_{X}^{\ast}\setminus D$, stable by wedge products and such that $\frac{df}{f}$ is a local section of $\Omega_{X}^{1}(\log D)$ when $f$ is a local section of $j_{\ast}\mathcal{O}_{X}$, meromorphic along $D$.

By [Del06, Proposition 3.2, p. 72] we know that $\Omega_{X}^{1}(\log D)$ is locally free and

$$\Omega_{X}^{q}(\log D) = \bigwedge^{q} \Omega_{X}^{1}(\log D).$$

Let us consider $D = (\bigcup_{i=1}^{s} D_{i})$, where each $D_{i}$ is an irreducible component of the divisor $D$ and $\bar{D}' = \bigcup_{I \subseteq [s], |I| = r} \bigcap_{i \in I} D_{i}$, for $r \geq 1$. In the case $r = 1$ we will skip the superindex in $\bar{D}'$ and write just $\bar{D}$ instead.

In [Del06, p. 77] there is the definition of *residue of a $q$-differential form* which consists of the extending by linearity the following application:

$$\begin{array}{ccc}
\Omega_{X}^{q}(\log D) & \xrightarrow{\text{Res}^{q}} & \mathcal{O}_{\bar{D}'}, \\
\frac{df_{1}}{f_{1}} \wedge \cdots \wedge \frac{df_{q}}{f_{q}} & \xrightarrow{\text{Res}^{q}} & 1 \in \mathcal{O}_{\bar{D}'}, \\
\omega & \xrightarrow{\text{Res}^{q}} & 0, \text{ if } \omega \in \Omega_{X}^{q}.
\end{array}$$

Then if $\omega \in H^{0}(X, \Omega_{X}^{q}(\log D))$ we have

$$\text{Res}^{q}(\omega) \in H^{0}(X, \mathcal{O}_{\bar{D}'}) = \bigoplus_{I \subseteq [s], |I| = q, 1 \in I} H^{0}(X, \mathcal{O}_{\bigcap_{i \in I} D_{i}}).$$

We can also define a residue application $R : \Omega_{X}^{q}(\log D) \to \Omega_{D}^{q-1}(\log D')$ in the following way: first consider $D'$ a divisor of $\bar{D}$ defined by

$$D' = \bigcup_{i=1}^{s} \bar{D}_{i} = \bigcup_{j \neq i}^{s} \left(D_{i} \cap D_{j}\right),$$

then we define $R$ as the map

$$\begin{array}{ccc}
\Omega_{X}^{q}(\log D) & \xrightarrow{R} & \Omega_{D}^{q-1}(\log D') \\
\frac{df_{i}}{f_{i}} \wedge \eta & \xrightarrow{R} & \eta|_{D_{i}},
\end{array} \quad (3)$$
and we extend it by linearity.

Notice that $\eta|_{D_i}$ will have logarithmic poles in the intersections of $D_i$ with the other components of $D$.

**Remark 3.2.** Consider $D_1$ as an irreducible component of $D$ and $\eta$ a $q$-logarithmic form in $\log D$. The $q-1$ logarithmic form $\eta|_{D_1}$ has as a residue

$$\text{Res}^{q-1}(\eta|_{D_1}) \in \bigoplus_{\mathclap{\left| I \right| = q, \ 1 \in I}} H^0(X, \mathcal{O}_{\bigcap_{i \in I} D_i}).$$

Meaning that the residue of $\eta|_{D_1}$ consists of those $\lambda_i$ that are residues of $\eta$ and such that they correspond to the intersection of $q$-components in which $D_1$ is one of them. In particular, if all the $\lambda_i$ of $\eta$ are $\neq 0$, then the same is true for the residues of the $q-1$-differential form $R(\eta)$.

**Proposition 3.3.** Keeping the notation of the residue maps as above, let us consider $\omega \in H^0(X, \Omega^1_X(\log D))$ and define $\eta$ as $R(\omega) \in H^0(X, \Omega^{q-1}_D(\log D'))$. Then we have that

$$\text{Sing}(\omega) \cap D \subset \text{Sing}(\eta).$$

**Proof.** Let us take $p \in \text{Sing}(\omega) \cap D$, such that $p \in D_1$. Locally around $p$ we can write

$$\omega = \frac{df}{f^1} \wedge \tilde{\eta} + \tilde{\omega},$$

where $\tilde{\eta}$ is a $q-1$-differential form in $X$ such that $\tilde{\eta}|_{D_1} = \eta$ and $\tilde{\omega}$ is an holomorphic $q$-differential form. Being $p$ a singular point then

$$\frac{df}{f^1} \wedge \tilde{\eta}|_p + \tilde{\omega}|_p = 0,$$

since $\tilde{\omega}$ is holomorphic it is linearly independent with $\frac{df}{f^1} \wedge \tilde{\eta}$ near $p$ implying that

$$\frac{df}{f^1} \wedge \tilde{\eta}|_p = \tilde{\omega}|_p = 0.$$

Then $\tilde{\eta}|_p = 0$ because $\frac{df}{f^1}$ is a local generator of $\Omega^1_X(\log D)$. Then we conclude that $\eta|_p = 0$ as we wanted to see.

4 Main results

Now we will define our main object of study, which are $q$-logarithmic foliations on a smooth variety $X$. For that, we first define a 1-logarithmic foliation. Then we will state and prove our main results, which are Theorem 4.10, Corollary 4.12.

**Definition 4.1.** Let $\varpi \in H^0(X, \Omega^1_X(\log D))$ be a 1-logarithmic form, where $D$ is a normal crossing divisor locally defined as $D = (\prod_{i=1}^s f_i)$, written as

$$\varpi = \sum_{i=1}^s \lambda_i \frac{df_i}{f_i}.$$
Then we define the 1-logarithmic foliation associated to $\varpi$ as

$$\omega = \left( \prod_{i=1}^{s} f_i \right) \sum_{i=1}^{s} \lambda_i \frac{df_i}{f_i} = \sum_{i=1}^{s} \lambda_i \hat{F}_i df_i ,$$

where $\hat{F}_i = \prod_{j \neq i} f_j$. This defines a global section

$$F\varpi \in H^0(X, \Omega_X^1 \otimes L) ,$$

for some line bundle $L$, and $F$ locally given by $\prod_{i=1}^{s} f_i$.

**Definition 4.2.** Let $\varpi \in H^0(X, \Omega_X^q (\log D))$ be a $q$-logarithmic differential form, where $D$ is a normal crossing divisor locally defined as $D = (\prod_{i=1}^{s} f_i)$, written as

$$\varpi = \sum_{I \subset [s], |I| = q} \lambda_I \frac{df_I}{f_I} .$$

Then we define the $q$-logarithmic foliation associated to $\varpi$ as

$$\omega = \left( \prod_{i=1}^{s} f_i \right) \left( \sum_{I \subset [s], |I| = q} \lambda_I \frac{df_I}{f_I} \right) = \sum_{I \subset [s], |I| = q} \lambda_I \hat{F}_I df_I ,$$

where $\hat{F}_I$ denotes the product $\prod_{I \subset [s], j \neq I} f_j$ and $df_I = \wedge_{i \in I} df_i$. This defines a global section

$$F\varpi \in H^0(X, \Omega_X^q \otimes L) ,$$

for some line bundle $L$ and $F$ locally given by $\prod_{i=1}^{s} f_i$.

In Corollary 4.12 we will see that under generic conditions the ideal of persistent singularities $\mathcal{I}(\varpi)$ is defined by

$$\mathcal{I}(\varpi) = \bigcap_{K \subset [s], |K| = q+1} \sum_{i \in K} \mathcal{I}(f_i) ,$$

when $X = \mathbb{P}^n$ and where by $\mathcal{I}(f)$ we denote the ideal generated by $f$. For that, we would like to state now the following technical lemma, which is a generalization of [Suw83, Corollary 1.6, p. 9].

**Lemma 4.3.** Let us consider $f_1, \ldots, f_s \in S$ homogeneous polynomials, for $s \in \mathbb{N}$ such that $s > q$. Suppose that $\text{ht}(f_{i_1}, \ldots, f_{i_{s-q}}) = s - q$ if $i_1, \ldots, i_{s-q} \in \{1, \ldots, s\}$ are distinct. Assume that $J$ runs through every subset of cardinality $q$ of $\{1, \ldots, s\}$, $q \geq 1$, then we have

$$\sum_{J \subset [s], |J| = q} \mathcal{I}(\hat{F}_J) = \bigcap_{K \subset [s], |K| = q+1} \sum_{i \in K} \mathcal{I}(f_i) .$$

**Proof.** To see the first inclusion

$$\sum_{J \subset [s], |J| = q} \left( \hat{F}_J \right) \subseteq \bigcap_{K \subset [s], |K| = q+1} \sum_{i \in K} \mathcal{I}(f_i)$$
we just notice that $\hat{F}_J$ has $s - q$ factors. Now, since $s - q + 1 = s + 1 > s$
then for every subset $K \subset \{1, \ldots, s\}$ such that $|K| = q + 1$ there exists a $k_0 \in K$
such that $k_0 \notin J$. This implies that $f_{k_0}|\hat{F}_J$ and $f_{k_0} \in \sum_{i \in K} I(f_i)$.

Since this happens for every $J \subset \{1, \ldots, s\}$, $|J| = q$, and $K \subset \{1, \ldots, s\}$, $|K| = q + 1$, we have that $\bar{F}_J \in \sum_{i \in K} I(f_i)$. Then

$$\sum_{J \subset \{s\}, |J| = q} \bar{F}_J \subset \bigcap_{K \subset \{s\}} \sum_{|K| = q + 1} I(f_i)$$

as we wanted to see.

For the other inclusion we will use the transversality of the $\{f_1, \ldots, f_s\}$ in the following sense: we can compute the intersection of $I(f_i)$ in the sense that we would get the least amount of polynomials in the product. Let us suppose that we choose $f_1$ as many times as possible, then we take $f_2$ as many times as possible and so on. For $f_1$ we have $(\binom{s}{q})$ possible choices, meaning that there are $(\binom{s}{q})$ ideals with $f_1$ in the intersection; for $f_2$ we have $(\binom{s-1}{q})$ possible choices, and it keeps going on like this. By the following formula

$$\sum_{i=1}^{s-q} \binom{s-i}{q} = \binom{s}{q + 1}$$

we are seeing that we can take, at least, $s - q$ polynomials to fill the $(\binom{s}{q+1})$
possible choices of each one of the ideals in the intersection. Then, if $|J| = q$, we have that $\bar{F}_J \in \bigcap_{K \subset \{s\}, |K| = q + 1} \sum_{i \in K} I(f_i)$ concluding our proof.

From now on let us fix $X$ to be a smooth complete intersection, $D = \sum_{i=1}^t D_i = (\prod_{i=1}^t f_i)$ a normal crossing divisor such that its components $D_i = (f_i)$ are irreducible, smooth, and ample.

We will also consider $\varpi \in H^0(X, \Omega_X^q (\log D))$ a $q$-logarithmic differential form and $\omega = F \cdot \varpi \in H^0(X, \Omega_X^q \otimes L)$ the associated logarithmic $q$-differential foliation, where $F = \prod_{i=1}^t f_i$, as in Definition 4.2.

**Proposition 4.4.** Let $\varpi \in H^0(X, \Omega_X^q (\log D))$ be a $q$-logarithmic differential form such that every residue is not zero (i.e. such that every $\lambda_i$ is $\neq 0$). Then $\text{Sing}(\varpi)$, the zero locus of $\varpi$, has dimension $\dim(\text{Sing}(\varpi)) \leq q - 1$.

**Proof.** The case $q = 1$ is the main result of [CSV06]. Let $q \geq 2$ and suppose $\dim(\text{Sing}(\varpi)) \geq q$. The ampleness of the components $D_i$ implies that the intersection of the zeroes of $\varpi$, with $D$ has dimension greater or equal to $q - 1$. By Proposition 4.3, the intersection of $\text{Sing}(\varpi)$ with $D$ is inside the singular locus of the form $R(\varpi)$. $R(\varpi)$ is a $q - 1$-logarithmic form on a complete intersection, the residues of $R(\varpi)$ are a subset of the residues of $\varpi$, so they are all non-null. By induction this cannot be, therefore $\dim(\text{Sing}(\varpi)) \leq q - 1$. 


Theorem 1.11] is proven that a universal unfolding for any singularity \( p \) is of the form \( \eta \). Take where \( \omega \) is a (strictly) Kupka point. The case \( q = 1 \) follows from the fact that, if \( \text{Sing}(\omega) \) is reduced, then \( \mathcal{K}(\omega) \) is the closure of the set of strictly Kupka points i.e., points \( p \in \text{Sing}(\omega) \) such that \( d\omega|_p \neq 0 \).

If \( q > 1 \) then from the local expression of \( \omega \) follows that \( \bigcap_{i=1}^{q+1} D_i \subseteq \text{Sing}(\omega) \) for any intersection of \( q+1 \) components of \( D \). Let \( p \) be a regular point of \( \bigcup_{K \subseteq \{s\} \colon |K|=q+1} \bigcap_{i \in K} D_i \) and suppose \( d\omega|_p = 0 \). The logarithmic \( q - 1 \)-form \( \eta := R(\omega)|_{D_j} \) defines a logarithmic foliation on \( D_j \) given by the twisted form \( \tilde{F}_j \eta \) whose Kupka scheme consists of the \( q + 1 \)-wise intersections of components of \( D \) containing \( D_j \). From the definition of \( R(\omega) \), Eq. (3), and Proposition 3.3 it follows that if \( d\omega|_p = 0 \) then \( d(\tilde{F}_j \eta)|_p = 0 \) which contradicts the inductive hypothesis on \( q \). So \( p \) is a Kupka point of \( \omega \) and the result follows.

\[ \mathcal{K}(\omega) = \bigcup_{K \subseteq \{s\} \colon |K|=q+1} \left( \bigcap_{i \in K} D_i \right). \]

Proposition 4.5. Following the notation above, the Kupka scheme of \( \omega \) is given by

\[ \mathcal{K}(\omega) = \bigcup_{K \subseteq \{s\} \colon |K|=q+1} \left( \bigcap_{i \in K} D_i \right). \]

Proof. We will proceed by induction on \( q \). Suppose \( q = 1 \), then \[ \text{CSV06} \] implies \( \text{Sing}(\omega) \) consists of isolated points and a codimension-2 subvariety given by the intersections of the components of \( D \). If \( p \in D_i \cap D_j \) is a regular point of \( D_i \cap D_j \) then locally around \( p \) we have \( d\omega|_p = g df_i \wedge df_j \) where \( g \) is a unit of \( \mathcal{O}_{D_i \cap D_j \cap p} \). Therefore \( p \) is a (strictly) Kupka point. The case \( q = 1 \) follows form the fact that, if \( \text{Sing}(\omega) \) is reduced, then \( \mathcal{K}(\omega) \) is the closure of the set of strictly Kupka points i.e., points \( p \in \text{Sing}(\omega) \) such that \( d\omega|_p \neq 0 \).

 Proposition 4.6. Following the notation above, if \( p \notin \mathcal{K}(\omega) \) then there is a local infinitesimal unfolding \( \tilde{\omega} \) of \( \omega \) around \( p \) such that \( \tilde{\omega}|_p \neq 0 \). Moreover \( \tilde{\omega} = F \tilde{\omega} \) where \( \tilde{\omega} \) is a logarithmic form on a neighbourhood of \( p \) in \( X \times D \) with poles on a divisor \( \tilde{D} = (\tilde{F} = 0) \), where \( \tilde{F}|_X = F \).

Proof. The case \( q = 1 \) is essentially the main result of \[ \text{Siw83}, \text{Siw83}, \text{Theorem 1.11} \] is proven that a universal unfolding for \( \omega = f_1 \cdots f_s \sum_{j=1}^s \lambda_j \frac{df_j}{f_j} \) is of the form

\[ \tilde{\omega} = \tilde{f}_1 \cdots \tilde{f}_s \sum_{j=1}^s \lambda_j \frac{d\tilde{f}_j}{f_j}, \]

where \( \tilde{f}_j(x, t) \) is a function germ such that \( \tilde{f}_j(x, 0) = f_j(x) \). Outside \( \mathcal{K}(\omega) \), any singularity \( p \) verifies that there is a local form \( \eta \) around \( p \) such that \( d\omega = \omega \wedge \eta \), which, by \[ \text{MMQ21}, \text{Proposition 3.6} \] is equivalent to the existence of an unfolding \( \tilde{\omega} \) such that \( \tilde{\omega}|_p \neq 0 \). Therefore the universal unfolding \( \tilde{\omega} \) also verifies \( \tilde{\omega}|_p \neq 0 \), and is logarithmic.

If \( q > 1 \) and \( p \in (f_1 = 0) \), write

\[ \omega = f_1 \cdots f_s \sum_{J \subseteq \{s\} \colon |J|=q} \lambda_J \frac{df_J}{f_J}. \]

Take \( \eta = R(\tilde{\omega}) \). Then by inductive hypothesis there is an unfolding of \( f_2 \cdots f_p \eta \) of the form

\[ \tilde{\eta} = \tilde{f}_2 \cdots \tilde{f}_s \sum_{J \subseteq \{2, \ldots, s\} \colon |J|=q-1} \lambda_J \frac{d\tilde{f}_J}{f_J}. \]
Now $\tilde{\eta}|_{p} \neq 0$. We can take
\[
\tilde{\omega} := f_{1}\tilde{f}_{2} \cdots \tilde{f}_{s} \sum_{J \subset [s], |J| = q} \lambda_{J} \frac{d\tilde{f}_{J}}{\tilde{f}_{J}},
\]
in such a way that $R(\tilde{\omega}) = \tilde{\eta}$, and so $\tilde{\omega}|_{p} \neq 0$.

**Corollary 4.7.** We have the inclusion of closed subvarieties $(\mathcal{Per}(\omega))_{\text{red}} \subseteq (\mathcal{Kup}(\omega))_{\text{red}}$.

**Proof.** Indeed, for a logarithmic foliation given by an $\omega$, a point $p \notin D$ is never in $\mathcal{Per}(\omega)$. If $p \in D$ then the above proposition together with [MMQ21 Proposition 4.1] is showing that $p \notin \mathcal{Kup}(\omega)$ implies $p \notin \mathcal{Per}(\omega)$, from which the corollary follows.

**Proposition 4.8.** Let $p \in \mathcal{Kup}(\omega)$ be a strictly Kupka point. Suppose that, when representing $\text{Res}^{q}(\omega)$ as a vector in
\[
\text{Res}^{q}(\omega) \in \bigoplus_{I \subset [s], |I| = q} H^{0}(X, \mathcal{O}_{\bigcap I D_{i}}) \simeq \mathbb{C}^{(I \subset [s], |I| = q)},
\]
there are two different coordinates $\lambda_{I} \neq \lambda_{J}$. Then locally around $p$ we have $\mathcal{F}_{p} \subset \mathcal{K}_{p}$.

**Proof.** Being $p$ a Kupka point, there are, by [DM00 Proposition 1.3.1, pp. 457], local coordinates $(x_{1}, \ldots, x_{n})$ in which $\omega$ is written completely in terms of $(x_{1}, \ldots, x_{q+1})$. In other words there is a local submersion $\pi : U \to \mathbb{C}^{q+1}$ and a local form $\alpha \in \Omega_{\mathbb{C}^{q+1}, 0}^{1}$ such that $\omega|_{U} = \pi^{*}\alpha$. We can take $U$ small enough in such a way that $0 \in \mathbb{C}^{q+1}$ is the only singularity of $\alpha$. Moreover, $\alpha$ also defines a logarithmic foliation in $\mathbb{C}^{q+1}$ and its isolated singularity is the normal crossing intersection of $q + 1$ of its invariant divisors. So we have local coordinates around $p$ such that the logarithmic form $\lambda$ is written as
\[
\lambda = \sum_{i=1}^{q+1} x_{i} f_{i} \left( \sum_{j \neq i} dx_{j} / x_{j} \right) + \xi_{1}.
\]

Where in this coordinates $F = x_{1} \cdots x_{q+1}$ and where $\xi_{1} \in W^{q-1} \Omega_{X}^{q}(\log D)$, the space of germs of logarithmic $q$-forms of weight at most $q - 1$ as defined in [Del00 3.5, p. 75]. Therefore we have in this coordinates the expression
\[
\omega = \sum_{i=1}^{q+1} x_{i} f_{i} \left( \sum_{j \neq i} dx_{j} \right) + \xi_{2}.
\]

In this case $\xi_{2} = F \cdot \xi_{1}$ is the germ of a holomorphic $q$-form which is actually in $(x_{1}, \ldots, x_{q+1})^{2} \Omega_{X, p}^{q} = K^{2} \cdot \Omega_{X, p}^{q}$. Now lets take $\Xi_{i,k} = \bigwedge_{j \neq i,k} dx_{j} \in \bigwedge^{q-1} T_{X, p}$ and $\varpi_{i,k} = i_{\Xi_{i,k}} \omega$, then
\[
\varpi_{i,k} = x_{j} f_{j} dx_{k} - x_{k} f_{k} dx_{j} + i_{\Xi_{i,k}} \xi_{2}.
\]
Notice that the $\varpi_{i,k}$ with $1 \leq i < k \leq q+1$ generate the sheaf $\mathcal{E}(\omega)$, as $i_{\Xi_{i,k}} \omega = 0$ for $q + 2 \leq j \leq n$. In particular every 1-form in $\mathcal{E}$ is in $K \cdot \Omega_{X, 0}^{1}$. We also have
\[
d\varpi_{i,k} = (f_{j} - f_{k}) dx_{i} \wedge dx_{k} + \xi_{3},
\]
where \( \xi_3 = d\xi_2 \) is the germ of a 2-form in \( K \cdot \Omega_{X,0}^2 \).

Now suppose \( h \in \mathcal{I}_p \), we want to show \( h = 0 \) in \( \mathcal{O}_{X,p}/K_p \). As \( h \in \mathcal{I}_p \) we have that, for every \( \varpi \in \mathcal{E} \),

\[
hd\varpi \in K \cdot \Omega_{X,p}^2.
\]

In particular we may take \( \varpi_{i,k} \) and get

\[
0 \equiv hd\varpi_{i,k} \equiv h(f_i(0) - f_k(0))dx_i \wedge dx_k \mod K.
\]

If we have two different residues in \( \omega \) then we may take \( i, k \) such that \( f_i(0) - f_k(0) \neq 0 \) (because the residues are exactly \( f_i(0) \) for \( i = 1, \ldots, q + 1 \)). Hence we must have \( h \equiv 0 \mod K \).

By using the two propositions above we can conclude the following.

**Theorem 4.9.** Let \( \omega \) be a codimension-\( q \)-foliation on \( X \) such that \( Kup(\omega) \) is reduced and the singular locus be the disjoint union of \( Kup(\omega) \) and a variety \( \mathcal{L} \), i.e., \( \text{Sing}(\omega) = Kup(\omega) \sqcup \mathcal{L} \). Then we have

\[
Kup(\omega) = \text{Per}(\omega).
\]

**Proof.** By Corollary 4.7 we have that \( (\text{Per}(\omega))_{\text{red}} \subset (Kup(\omega))_{\text{red}} \), then by our hypothesis we have that \( (\text{Per}(\omega))_{\text{red}} \subset Kup(\omega) \). And by Proposition 4.8 we know that for every \( p \in Kup(\omega) \) a strictly Kupka point we have \( \mathcal{I}_p \subset K_p \), meaning that \( Kup(\omega)_p \subset \text{Per}(\omega)_p \).

Since \( Kup(\omega) \) is reduced then the set of strictly Kupka points is dense in \( Kup(\omega) \), implying that

\[
Kup(\omega) \subset \text{Per}(\omega).
\]

By Proposition 2.10 we have that \( \text{Per}(\omega) \subset \text{Sing}(\omega) \) then \( Kup(\omega) \subset \text{Per}(\omega) \subset \text{Sing}(\omega) \). And since \( \text{Sing}(\omega) = Kup(\omega) \sqcup \mathcal{L} \) we have that

\[
Kup(\omega)_p \subset \text{Per}(\omega)_p \subset \text{Sing}(\omega)_p = Kup(\omega)_p , \forall p \in \text{Supp}(Kup(\omega))
\]

implying that \( \mathcal{K}(\omega) = \mathcal{I} \).

As a corollary of the theorem above we get the following decomposition of \( \text{Sing}(\omega) \) for \( \omega \) a \( q \)-logarithmic foliation.

**Theorem 4.10.** Let \( \omega \in H^0(X, \Omega^q_X \otimes \mathcal{L}) \) be a generic \( q \)-logarithmic foliation. Then we can decompose its singular locus \( \text{Sing}(\omega) \) as the disjoint union of

\[
\text{Sing}(\omega) = Kup(\omega) \sqcup \mathcal{H},
\]

where \( Kup(\omega) \) is the Kupka variety of \( \omega \) of codim \( q + 1 \) and \( \mathcal{H} \) is a variety of dimension \( q - 1 \). Even more so, we have that \( Kup(\omega) = \text{Per}(\omega) \).

**Proof.** Since \( \omega \) is a \( q \)-logarithmic foliation, we know that its singular locus is made of \( Kup(\omega) \) of codimension-\( q + 1 \) and a variety \( \mathcal{H} \) of dimension \( q - 1 \). Since \( \omega \) is generic, we can assume that \( Kup(\omega) \cap \mathcal{H} = \emptyset \).

Finally, we can apply Theorem 4.9 and conclude \( Kup(\omega) = \text{Per}(\omega) \) as we wanted to see.

In the case where \( X = \mathbb{P}^n \) we denote \( H^0(\mathbb{P}^n, \mathcal{E}) \) by \( \mathcal{E}_+ \).
Proposition 4.11. Given a codimension-$q$ foliation determined by a morphism $\mathcal{L} \to \Omega^q_{P^n}$, where $\mathcal{L} = \mathcal{O}_{P^n}(l)$, we commit an abuse of notation and denote the global section $H^0(P^n, \omega)$ by $\omega$. We compute

$$H^0(P^n, \mathcal{F}(\omega)) = \bigcap_{\omega \in \mathcal{E}_*} \left( \sum_{\eta \in \mathcal{E}_*} \eta \wedge \Omega^1_S : d\omega \right)$$

$$H^0(P^n, \mathcal{K}(\omega)) = \left( \sum_{\eta \in \mathcal{E}_*} \eta \wedge \Omega^1_S : d\omega \right)$$

where $\Omega^q_S = \Lambda^q \Omega^1_S$ with $\Omega^1_S$ is the $S$-module of Kähler differentials.

Proof. For the first equality, recalling Definition [23], it is easy to see that

$$\sum_{\eta \in \mathcal{E}_*} \eta \wedge \Omega^1_S \subset H^0(P^n, \mathcal{F}(\omega))$$

To see the other contention, we can do it locally. Let us consider a prime ideal $p \in \text{Spec}(P^n)$ and localize $\mathcal{F}(\omega)$ at $p$. We have that, for some $t \in \mathbb{Z},$

$$\mathcal{F}(\omega)(t)_p = \{ h \in \mathcal{O}_{P^n}(t)_p : \forall \omega \in \mathcal{E}(t)_p, \; hd\omega = \sum_j \alpha_j \wedge \omega_j \}$$

for some local 1-forms $\alpha_j \in \Omega^1_S(t)_p$ and forms $\omega_j$ in $\mathcal{E}_p$. Since $\mathcal{E}(t)_p$ is generated by the $\eta \in \Omega^1_S(t)_p$ such that $\omega \wedge \eta = 0$, then we get that

$$\mathcal{F}(\omega)(t)_p \subset \left( \bigcap_{\omega \in \mathcal{E}_*} \left( \sum_{\eta \in \mathcal{E}_*} \eta \wedge \Omega^1_S : d\omega \right) \right)(t)_p = \bigcap_{\omega \in \mathcal{E}_p} \left( \sum_{\eta \in \mathcal{E}_p} \eta \wedge \Omega^1_S(t)_p : d\omega \right)$$

for every $t \in \mathbb{Z}$, giving the other inclusion.

For the other equality we proceed in a similar way and the result follows. □

Using the above characterization of the graded modules associated with $\mathcal{F}(\omega)$ and $\mathcal{K}(\omega)$ we obtain the following.

Corollary 4.12. Let $X = P^n$ and $\omega$ a $q$-logarithmic foliation defined by a normal crossing divisor $D = (\prod_{i=1}^s f_i)$ as

$$\omega = F \left( \sum_{I \subseteq [s], \; |I| = q} \frac{\lambda_I \omega_I}{f_I} \right) = \sum_{I \subseteq [s], \; |I| = q} \lambda_I \omega_I \omega_I = F \omega \in H^0(X, \Omega^q_X \otimes \mathcal{L})$$

satisfying the hypotheses of Theorem [4.10], then we have

$$H^0_s(P^n, \mathcal{F}(\omega)) = H^0_s(P^n, \mathcal{K}(\omega)) = \sum_{J \subseteq [s], \; |J| = q} \mathcal{I}(\mathcal{F}_J) = \bigcap_{J \subseteq [s], \; |J| = q} \mathcal{I}(\mathcal{F}_J)$$

Proof. For $\omega$ of the form given in the statement the Kupka scheme is, by Proposition [4.23] given by the ideal $\bigcap_{J \subseteq [s], \; |J| = q+1} \sum \mathcal{I}(f_i)$. Then applying Theorem [4.10] and Lemma [4.3] we are done. □
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Ariel Molinuevo∗ arielmolinuevo@gmail.com
Federico Quallbrunn† fquallb@dm.uba.ar

∗Instituto de Matemática
Universidade Federal do Rio de Janeiro
Caixa Postal 68530
CEP. 21945-970 Rio de Janeiro - RJ
BRASIL

† Departamento de Matemática
Universidad CAECE
Av. de Mayo 866
CP C1084AAQ
Buenos Aires
Argentina