Totally real algebraic integers in short intervals, Jacobi polynomials, and unicritical families in arithmetic dynamics

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Abstract
We classify all postcritically finite unicritical polynomials defined over the maximal totally real algebraic extension of \( \mathbb{Q} \). Two auxiliary results used in the proof of this result may be of some independent interest. The first is a recursion formula for the \( n \)th diameter of an interval, which uses properties of Jacobi polynomials. The second is a numerical criterion that allows one to give a bound on the degree of any algebraic integer having all of its complex embeddings in a real interval of length less than 4.

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1  |  INTRODUCTION

Let \( d \geq 2 \) be an integer, let \( c \in \mathbb{Q} \), and consider the polynomial map \( f : \mathbb{Q} \to \mathbb{Q} \) defined by \( f(x) = x^d + c \). Define \( f^n : \mathbb{Q} \to \mathbb{Q} \) to be the \( n \)-fold composition

\[
f^n = f \circ f \circ \ldots \circ f
\]

of \( f \) with itself. The polynomial \( f(x) \) is said to be postcritically finite (PCF), if its unique critical point 0 is preperiodic with respect to iteration; in other words, if the critical orbit 0, \( f(0) \), \( f^2(0) \), \ldots takes only finitely many distinct values. It has been well established in complex and arithmetic dynamics that PCF maps are a class well worthy of study, as they often possess dynamical...
properties that are highly distinguishable from arbitrary maps. For convenience, we define

\[ \text{PCF}_d = \{ c \in \overline{\mathbb{Q}} \mid f(x) = x^d + c \text{ is PCF} \} \]

to be the set of all PCF parameters \( c \) with respect to the degree \( d \) unicritical family \( x^d + c \).

Given an algebraic number \( c \in \overline{\mathbb{Q}} \) of degree \( n = [\mathbb{Q}(c) : \mathbb{Q}] \), let \( F_c(X) \in \mathbb{Q}[X] \) be its minimal polynomial over \( \mathbb{Q} \). Recall that \( c \) is said to be totally real if \( F_c(X) \) has \( n \) real roots. For each \( d \geq 2 \), one might ask how often a unicritical polynomial \( x^d + c \) can be both PCF and defined over \( \mathbb{Q}^{tr} \). In other words, what can we say about the intersection \( \text{PCF}_d \cap \mathbb{Q}^{tr} \), and, in particular, is this intersection finite? Although \( \text{PCF}_d \) is a set of bounded height, the field \( \mathbb{Q}^{tr} \) is an infinite degree extension of \( \mathbb{Q} \), and therefore, such a finiteness result does not follow directly from the Northcott property of heights.

In fact, the finiteness result

\[ |\text{PCF}_d \cap \mathbb{Q}^{tr}| < +\infty \]  

for each \( d \geq 2 \) is not difficult to obtain using a theorem of Fekete in arithmetic capacity theory. First, one observes that the generalized Mandelbrot set

\[ \mathcal{M}_d = \{ c \in \mathbb{C} \mid \text{the forward orbit of 0 under } x^d + c \text{ is bounded} \} \]

meets the real line at an interval of length \(< 4 \); in the case \( d = 2 \), this follows from the well-known fact that the ordinary Mandelbrot set \( \mathcal{M}_2 \) meets the real line at the interval \([-2, 1/4] \), and in the case \( d \geq 3 \), the comparable calculation has been done by Parisé–Rochon [3] and Parisé–Ransford–Rochon [2]. The second relevant observation is that the set \( \text{PCF}_d \) consists entirely of algebraic integers. Consequently, \( \text{PCF}_d \cap \mathbb{Q}^{tr} \) is confined to an adelic set of arithmetic capacity \(< 1 \), and hence, must be finite. (A standard source for arithmetic capacity theory is Rumely [5].) We give more of the details in the proof of Theorem 10.

With more care, we can prove the following explicit result.

**Theorem 1.** It holds that

\[ \text{PCF}_2 \cap \mathbb{Q}^{tr} = \{-2, -1, 0\}, \]

\[ \text{PCF}_d \cap \mathbb{Q}^{tr} = \{-1, 0\} \text{ if } d \geq 4 \text{ is even}, \]

\[ \text{PCF}_d \cap \mathbb{Q}^{tr} = \{0\} \text{ if } d \geq 3 \text{ is odd}. \]

Thus, the only quadratic PCF maps defined over \( \mathbb{Q}^{tr} \) in the unicritical family \( x^2 + c \) are the squaring map \( x^2 \), the basilica map \( x^2 - 1 \), and the Chebyshev map \( x^2 - 2 \). In the families \( x^d + c \) for \( d \geq 3 \), we find only the powering maps \( x^d \), as well as \( x^d - 1 \) when \( d \geq 4 \) is even.

Our approach is to turn the capacity-theoretic finiteness result \( |\text{PCF}_d \cap \mathbb{Q}^{tr}| < +\infty \) into a quantitative upper bound on the degree of any totally real PCF parameter. The primary tool is the \( n \)th diameter

\[ d_n(E) := \sup_{x_1, x_2, \ldots, x_n \in E} \prod_{i<j} |x_i - x_j|^{2/n(n-1)} \]  

(2)
associated to a compact subset $E$ of $\mathbb{C}$ and each integer $n \geq 2$. Thus, $d_n(E)$ is the maximal possible value of the geometric mean of the pairwise differences among any choice of $n$ points in $E$. A collection $x_1, x_2, \ldots, x_n$ of $n$ points in $E$ on which the supremum is achieved are called Fekete points for $E$; such points are guaranteed to exist by the compactness of $E$. Elementary arguments show that the $n$th diameter has the transformation property $d_n(\alpha E + \beta) = |\alpha|d_n(E)$ for $\alpha, \beta \in \mathbb{C}$, and the monotonicity property $d_n(E_1) \leq d_n(E_2)$ whenever $E_1 \subseteq E_2$.

The sequence $\{d_n(E)\}$ is monotone decreasing, and therefore, the limit

$$d_\infty(E) := \lim_{n \to +\infty} d_n(E)$$

exists. The quantity $d_\infty(E)$ is known as the transfinite diameter of $E$, and it also coincides with the capacity of $E$; see [4, §5.5].

We recall the well-known calculation that the transfinite diameter of a real interval $[a, b]$ is a quarter of its length, that is, $d_\infty([a, b]) = \frac{b-a}{4}$; see [4, Corollary 5.2.4]. In §2, we prove the following explicit recursion formula for the $n$th diameter of an interval; this result may be of some independent interest.

**Theorem 2.** For $n \geq 2$, the $n$th diameter of a real interval $[\alpha, \beta]$ is given by

$$d_n([\alpha, \beta]) = (\beta - \alpha)D_1^{1/(n-1)},$$

where $\{D_n\}_{n=2}^\infty$ is the sequence defined recursively by $D_2 = 1$ and

$$D_n = \frac{n^n(n-2)^{n-2}}{2^{2n-2}(2n-3)^{2n-3}} D_{n-1} \quad (n \geq 3).$$

This result can be easily converted into a closed-form expression for $d_n([\alpha, \beta])$, but such a formula would not as useful for our purposes as the recursion formula described in Theorem 2. For later use, we give the first few values of $\{D_n\}$ here:

$$D_2 = 1 \quad D_3 = \frac{1}{16} \quad D_4 = \frac{1}{3125} \quad D_5 = \frac{27}{210827008}.$$  

It is immediately evident from the definition (2) that

$$d_n(E)^{(n-1)} = \sup_f |\text{disc}(f)|,$$

where the supremum is taken over all monic polynomials $f(X) \in \mathbb{C}[X]$ of degree $n$ having all $n$ roots in the set $E$. This fact, together with the “electrostatic interpretation” of the zeros of Jacobi polynomials (an idea going back at least to Szegő [6]), is the main idea behind the proof of Theorem 2. The $n$-Fekete points associated to the interval $[-1, 1]$ are the endpoints together with the roots of the Jacobi polynomial of degree $n-2$ and weight $(1,1)$. Like all orthogonal polynomials, the Jacobi polynomials satisfy a recursion, and there is a method (due to Schur) to calculate their discriminant; we provide the details in §2.

In order to apply Theorem 2 to the proof of Theorem 1, we consider more generally the following problem.
Given a real interval \([\alpha, \beta]\) of length \(L = \beta - \alpha < 4\), find a bound \(n_0\), depending on the length \(L\), such that if \(\theta\) is an algebraic integer, all of whose embeddings into \(\mathbb{C}\) lie in the real interval \([\alpha, \beta]\), then \([\mathbb{Q}(\theta) : \mathbb{Q}] < n_0\).

The requirement that \(L < 4\) is necessary because if \(\zeta\) is a root of unity, then all complex embeddings of \(\zeta + \zeta^{-1}\) lie in the real interval \([-2, 2]\); thus, no such \(n_0\) exists for this interval. On the other hand, it follows from general principles in arithmetic capacity theory that such a bound \(n_0\) does exist when \(L < 4\), but finding a bound written as an explicit expression in \(L\) seems to be difficult. Instead, in Theorem 8, we give a numerical criterion that has practically the same effect when applied in specific examples. This result allows one to fairly easily calculate such a bound \(n_0\) for any interval of some given particular length \(L < 4\).

The main idea behind Theorem 8 is the following. Suppose that \(\theta\) is an algebraic integer of degree \(n = [\mathbb{Q}(\theta) : \mathbb{Q}]\), such that all \(n\) embeddings of \(\theta\) into \(\mathbb{C}\) lie in the real interval \([\alpha, \beta]\). Then Theorem 2 provides an explicit upper bound on \(|\text{disc}(F_\theta)|\), where \(F_\theta(X) \in \mathbb{Z}[X]\) is the minimal polynomial of \(\theta\) over \(\mathbb{Q}\). A lower bound on \(|\text{disc}(F_\theta)|\) is provided by Minkowski’s theorem, and combining these inequalities leads to a contradiction for large enough \(n\). The statement of this result is slightly complicated, so we delay it until § 3.

Using Minkowski’s bound is not strictly necessary, as using the trivial lower bound \(|\text{disc}(F_\theta)| \geq 1\) would also lead to a contradiction for large enough \(n\). But, in practice, using Minkowski’s bound leads to a smaller value of \(n_0\), which can make a significant difference in any intended application. To illustrate, in our proof of Theorem 1 in the \(d = 2\) case, we are able to use Theorem 8 to show that if \(c\) is totally real and \(x^2 + c\) is PCF, then \([\mathbb{Q}(c) : \mathbb{Q}] < 3\). We then dispose of the cases \([\mathbb{Q}(c) : \mathbb{Q}] = 1, 2\) with elementary arguments. If instead we had used only the trivial lower bound \(|\text{disc}(F_c)| \geq 1\), we could only have deduced that \([\mathbb{Q}(c) : \mathbb{Q}] < 6\), leading to a much more difficult computational challenge to finish the proof that \(\text{PCF}_2 \cap \mathbb{Q}^{\text{tr}} = \{-2, -1, 0\}\).

Theorem 8 should be of some independent interest and certainly has other uses beyond the proof of Theorem 1. In a paper under preparation [1], the first author considers the problem of classifying precisely which parameters \(c \in \mathbb{Q}\) have the property that the map \(x^2 + c\) has only finitely many totally real preperiodic points. A fully definitive solution to this problem is given by the first author [1] using methods in arithmetic capacity theory, and for several of the parameters \(c\), a complete calculation of the set of all totally real preperiodic points for \(x^2 + c\) can be made using Theorem 8.

The plan of this paper is as follows. In § 2, we prove Theorem 2, the recursion formula for the \(n\)th diameter of an interval. In § 3, we use this recursion and Minkowski’s bound to prove our main result on totally real algebraic integers in short intervals. Finally, in § 4, we apply Theorems 2 and 8 to the proof of Theorem 1, the dynamical application of classifying totally real PCF parameters in unicritical families.

### 2 The \(n\)th Diameter of an Interval

In this section, we use properties of Jacobi polynomials to prove Theorem 2, the recursion formula for the \(n\)th diameter of an interval.

Let \(\{P_m(x)\}_{m=0}^{\infty}\) be the family of Jacobi polynomials of weight \((\alpha, \beta) = (1, 1)\). Thus, each \(P_m(x)\) has real coefficients, \(\deg P_m = m\), and the family \(\{P_m(x)\}_{m=0}^{\infty}\) is orthogonal with respect to the inner product \(\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)(1 - x^2)dx\). These assumptions determine each \(P_m(x)\) up to
a real multiplicative constant, and we choose the normalization in which each polynomial $P_m(x)$ is monic. Szegő [6] is a standard source for orthogonal polynomials.

We also denote by $\{P^*_m(x)\}_{m=0}^{\infty}$ the family of Jacobi polynomials of weight $(\alpha, \beta) = (1,1)$, but normalized as in [6] so that $P^*_m(1) = m + 1$. It is shown in [6] § IV.4.21 that the leading coefficient of $P^*_m(x)$ is given by

$$L_m = \frac{(2m + 2)!}{2^m m!(m + 2)!}.$$

Thus, $P^*_m(x) = L_m P_m(x)$, which implies that

$$|P^*_m(\pm 1)| = P^*_m(1) = \frac{m + 1}{L_m} = \frac{2^m(m + 1)!(m + 2)!}{(2m + 2)!}.$$ (5)

It is useful for recursion purposes to calculate the ratio

$$\frac{P_m(1)}{P_{m-1}(1)} = \frac{m + 2}{2m + 1},$$ (6)

which follows from (5).

By the general theory of orthogonal polynomials, there is an alternate characterization of the family $\{P_m(x)\}_{m=0}^{\infty}$. For each $m \geq 0$, $y = P_m(x)$ is the unique monic polynomial of degree $m$ satisfying the differential equation

$$(1 - x^2)y'' - 4xy' + m(m + 3)y = 0;$$ (7)

this is proved in [6, Theorem 4.2.2].

**Lemma 3.** The monic Jacobi polynomials $\{P_m(x)\}_{m=0}^{\infty}$ satisfy the recursion $P_0(x) = 1$, $P_1(x) = x$, and

$$P_m(x) = xP_{m-1}(x) - C_m P_{m-2}(x) \quad (m \geq 2),$$

where

$$C_m = \frac{m^2 - 1}{4m^2 - 1}. \quad (8)$$

Moreover, the discriminant of Jacobi polynomials satisfy the recursion $|\text{disc}(P_1)| = 1$ and

$$|\text{disc}(P_m)| = \frac{m^m(m + 2)^{m-2}}{(2m + 1)^{2m-3}|\text{disc}(P_{m-1})|} \quad (m \geq 2). \quad (9)$$

**Proof.** The calculations of $P_0(x)$ and $P_1(x)$ follow from the monic assumption together with the orthogonality $\langle P_0, P_1 \rangle = 0$. All orthogonal polynomials satisfy a recursion of the type

$$P_m(x) = (A_m x + B_m)P_{m-1}(x) - C_m P_{m-2}(x) \quad (m \geq 2),$$
see [6, §3.1]. That $A_m = 1$ follows from the monic assumption. That $B_m = 0$ follows from the fact that $P_m(x)$ is either even or odd according to the parity of $m$ (see [6, §4.1]), and therefore, each $P_m(x)$ has vanishing $x^{m-1}$ term. To calculate the $C_m$, we evaluate the recursion at $x = 1$, we find $P_m(1) = P_{m-1}(1) - C_m P_{m-2}(1)$, and thus,

$$ C_m = \frac{P_{m-1}(1) - P_m(1)}{P_{m-2}(1)} = \frac{m^2 - 1}{4m^2 - 1}, $$

by (5).

To prove the recursion (9) for the discriminant, we use a method of Schur (see [6, §6.7]). For each $m \geq 2$, set

$$ \Delta_m = |\text{Res}(P_m, P_{m-1})| = \prod_{P_m(\alpha) = 0} |P_{m-1}(\alpha)| = \prod_{P_{m-1}(\beta) = 0} |P_m(\beta)|. $$

The recursion $P_m(x) = xP_{m-1}(x) - C_m P_{m-2}(x)$ shows that

$$ \Delta_m = C_m^{m-1} \Delta_{m-1}. $$

(10)

To apply this recursion to the discriminant of the Jacobi polynomials $P_m(x)$, we also note that

$$ (1 - \alpha^2)P'_m(\alpha) = N_m P_{m-1}(\alpha) \quad \text{whenever} \quad P_m(\alpha) = 0, $$

where

$$ N_m = \frac{mP_m(1)}{P_{m-1}(1)} = \frac{m(m+2)}{2m+1}; $$

this is derived in [6, §4.5]. Therefore,

$$ |\text{disc}(P_m)| = |\text{Res}(P_m, P'_m)| $$

$$ = \prod_{P_m(\alpha) = 0} |P'_m(\alpha)| $$

$$ = N_m^m \prod_{P_m(\alpha) = 0} |1 - \alpha^2|^{-1} |P_{m-1}(\alpha)| $$

$$ = N_m^m |P_m(1)|^{-1} |P_m(-1)|^{-1} \prod_{P_m(\alpha) = 0} |P_{m-1}(\alpha)| $$

$$ = N_m^m P_m(1)^{-2} \Delta_m. $$

We conclude that

$$ \frac{|\text{disc}(P_m)|}{|\text{disc}(P_{m-1})|} = \frac{N_m^m}{N_{m-1}^{m-1}} \cdot \frac{P_{m-1}(1)^2}{P_m(1)^2} \cdot \frac{\Delta_m}{\Delta_{m-1}} $$

$$ = \frac{m^m(m+2)^{m-2}}{(2m+1)^{2m-3}}. $$
by substituting the definition of $N_m$ together with (6) and (10) and simplifying; this completes the proof of the recursion (9). □

The following lemma may be viewed as a consequence of the “electrostatic interpretation” of the zeros of Jacobi polynomials, an idea that goes back at least to Szegő [6]. For completeness, we sketch the proof given in [6, §VI.6.7].

**Lemma 4.** For each $n \geq 2$, the supremum

$$\sup_{x_1, x_2, \ldots, x_n \in [-1, 1]} \prod_{i<j} |x_i - x_j|^2$$

is achieved when the points $x_1, x_2, \ldots, x_n$ are the roots of the polynomial

$$Q_n(x) = (x^2 - 1)P_{n-2}(x).$$

In particular, $d_n([-1, 1])^{n(n-1)} = |\text{disc}(Q_n)|$.

**Proof.** Let $-1 \leq x_1 < x_2 < \cdots < x_{n-1} < x_n \leq 1$ be an ordered list of points at which the desired supremum is achieved; such points exist by compactness. Since the expression inside the supremum increases as $x_n$ increases, we must have $x_n = 1$, and similarly, $x_1 = -1$.

For each index $2 \leq k \leq n - 1$, by the maximality assumption, the point $x_k$ must be a critical point for the function

$$g(x) = \log |x + 1| + \log |x - 1| + \sum_{2 \leq j \leq n-1 \atop j \neq k} \log |x - x_j|,$$

and therefore,

$$\frac{1}{x_k + 1} + \frac{1}{x_k - 1} + \sum_{2 \leq j \leq n-1 \atop j \neq k} \frac{1}{x_k - x_j} = 0. \tag{11}$$

Define $f(x) = \prod_{j=2}^{n-1} (x - x_j)$, so the proof will be complete if we show that $f(x) = P_{n-2}(x)$. For each $2 \leq k \leq n - 1$, let $f_k(x) = f(x) / (x - x_k)$. Thus,

$$\frac{f'_k(x)}{f_k(x)} = \sum_{2 \leq j \leq n-1 \atop j \neq k} \frac{1}{x - x_j}$$

and

$$f'(x) = (x - x_k)f'_k(x) + f_k(x),$$

$$f''(x) = (x - x_k)f''_k(x) + 2f'_k(x).$$

For each $2 \leq k \leq n - 1$, we obtain $f'(x_k) = f_k(x_k)$ and $f''(x_k) = 2f'_k(x_k)$, and therefore,

$$\frac{f'''(x_k)}{2f''(x_k)} = \frac{f'_k(x_k)}{f_k(x_k)} = \sum_{2 \leq j \leq n-1 \atop j \neq k} \frac{1}{x_k - x_j}.$$
Combining this with (11), we obtain

\[
\frac{f''(x_k)}{2f'(x_k)} + \frac{1}{x_k - 1} + \frac{1}{x_k + 1} = 0,
\]

which simplifies to \((x_k^2 - 1)f''(x_k) + 4x_k f'(x_k) = 0\). This tells us that the polynomials \(f(x)\) and \((x^2 - 1)f''(x) + 4x f'(x)\) share the same roots. Since the former is monic and the latter has leading coefficient \((n - 2)(n - 3) + 4(n - 2) = (n + 1)(n - 2)\), we obtain

\[
(x^2 - 1)f''(x) + 4x f'(x) = (n + 1)(n - 2)f(x).
\]

Since \(f(x)\) is monic and satisfies the differential equation (7) with \(m = n - 2\), we conclude that \(f(x) = P_{n-2}(x)\), completing the proof. \(\square\)

**Lemma 5.** The polynomials \(\{Q_n(x)\}_{n=2}^{\infty}\) defined by \(Q_n(x) = (x^2 - 1)P_{n-2}(x)\) satisfy the recursion \(\|\text{disc}(Q_2)\| = 4\) and

\[
\|\text{disc}(Q_n)\| = \frac{n^n(n-2)^{n-2}}{(2n-3)^{2n-3}}\|\text{disc}(Q_{n-1})\| \quad (n \geq 3).
\]

**Proof.** We have \(\|\text{disc}(Q_2)\| = \|\text{disc}(x^2 - 1)\| = 4\). If \(f(x), g(x) \in \mathbb{C}[x]\) are monic polynomials, recall the well-known identity

\[
\|\text{disc}(fg)\| = \|\text{disc}(f)\| |\text{Res}(f, g)|^2 \|\text{disc}(g)\|
\]

for the discriminant of their product, where \(|\text{Res}(f, g)| = \prod_{f(\alpha) = 0} |g(\alpha)|\). Using this and the fact that \(|P_m(-x)| = |P_m(x)|\), we have

\[
\|\text{disc}(Q_n)\| = \|\text{disc}(x^2 - 1)| |\text{Res}(x^2 - 1, P_{n-2})|^2 \|\text{disc}(P_{n-2})\|
\]

\[
= 4|P_{n-2}(1)|^4 \|\text{disc}(P_{n-2})\|,
\]

and therefore, when \(n \geq 3\), we use (6) and Lemma 3 to obtain

\[
\frac{|\text{disc}(Q_n)|}{|\text{disc}(Q_{n-1})|} = \frac{|P_{n-2}(1)|^4}{|P_{n-3}(1)|^4} \frac{|\text{disc}(P_{n-2})|}{|\text{disc}(P_{n-3})|}
\]

\[
= \frac{n^n(n-2)^{n-2}}{(2n-3)^{2n-3}},
\]

which is the desired recursion. \(\square\)

**Proof of Theorem 2.** For \(n \geq 2\), we define \(D_n = 2^{-n(n-1)}d_n([-1, 1])^n(n-1)\). Thus,

\[
d_n([-1, 1]) = 2D_n^{1/n(n-1)},
\]

and more generally, in view of the transformation property \(d_n(rs + t) = |r|d_n(s)\) for \(r, t \in \mathbb{R}\), for \(L = (\beta - \alpha)/2\), we have

\[
d_n([\alpha, \beta]) = d_n([\beta - L, L]) = L d_n([-1, 1]) = L 2 D_n^{1/n(n-1)} = (\beta - \alpha) D_n^{1/n(n-1)}.
\]
We now just have to show that the sequence \({D_n}_{n=2}^\infty\) satisfies the initial condition and recurrence relation given in the statement of the theorem. Clearly, \(D_2 = 2^{-2}d_2([-1,1])^2 = 1\). Using Lemmas 4 and 5, for \(n \geq 3\), we have

\[
\frac{D_n}{D_{n-1}} = \frac{2^{-n(n-1)}d_n([-1,1])^{n(n-1)}}{2^{-(n-1)(n-2)}d_{n-1}([-1,1])^{(n-1)(n-2)}}
\]

\[
= 2^{-2n+2} \left| \text{disc}(Q_n) \right| \left| \text{disc}(Q_{n-1}) \right|
\]

\[
= \frac{n^n(n-2)^{n-2}}{2^{2n-2}(2n-3)^{2n-3}},
\]

which is the stated recurrence relation for the sequence \({D_n}_{n=2}^\infty\).

\[\square\]

3 \hspace{1cm} ALGEBRAIC INTEGERS WITH CONJUGATES IN A SHORT INTERVAL

In this section, we prove our main result on totally real algebraic integers with all conjugates in a short interval. We first need two preliminary lemmas.

**Lemma 6.** Let \(n_0\) be an integer and let \({a_n}_{n=n_0}^\infty\) and \({b_n}_{n=n_0}^\infty\) be sequences of positive real numbers. If \(a_{n_0} < b_{n_0}\), and \(a_{n+1}/a_n < b_{n+1}/b_n\) for all \(n \geq n_0\), then \(a_n < b_n\) for all \(n \geq n_0\).

**Proof.** The proof is an induction with base case \(n = n_0\). If \(a_n < b_n\) for some \(n \geq n_0\), then

\[
\frac{a_{n+1}}{b_{n+1}} = \frac{a_{n+1}/a_n}{b_{n+1}/b_n} \cdot \frac{a_n}{b_n} < 1,
\]

and therefore, \(a_{n+1} < b_{n+1}\).

\[\square\]

**Lemma 7.** Let \(F(X) \in \mathbb{Z}[X]\) be a monic, irreducible polynomial of degree \(n \geq 2\) with \(n\) real roots. Then \(|\text{disc}(F)| \geq n^{2n}/n!^2\).

**Proof.** Let \(\theta \in \mathbb{R}\) be a root of \(F(X)\), let \(K = \mathbb{Q}(\theta)\), and let \(\Delta_K \in \mathbb{Z}\) be the discriminant of \(K\). Let \(\sigma_i : K \hookrightarrow \mathbb{R}\) \((i = 1, 2, \ldots, n)\) denote the \(n\) distinct embeddings of \(K\) into \(\mathbb{R}\). By a standard calculation, \(|\text{disc}(F)| = |\det V|^2\), where \(V = (\sigma_i(\theta^j))\), and thus,

\[
|\text{disc}(F)| = |\det V|^2 = |O_K / \mathbb{Z}[[\theta]]|^2 \cdot |\Delta_K| \geq |\Delta_K| \geq n^{2n}/n!^2.
\]

Here, the final inequality is Minkowski’s lower bound on the discriminant in the special case of a totally real number field \(K\).

\[\square\]

We are now ready to state and prove the main result of this section.
Theorem 8. Let $[\alpha, \beta]$ be a real interval of length $0 < \beta - \alpha < 4$. Define sequences $\{a_n\}_{n=2}^{\infty}$ and $\{b_n\}_{n=2}^{\infty}$ by

$$a_n = d_n([\alpha, \beta])^{n(n-1)} = (\beta - \alpha)^{n(n-1)}D_n$$

$$b_n = \frac{n^{2n}}{n!^2},$$

where the sequence $\{D_n\}$ is defined in Theorem 2. Suppose that there exists an integer $n_0 \geq 2$ with the properties that

$$a_{n_0} < b_{n_0} \quad \text{and} \quad \frac{a_{n_0+1}}{a_{n_0}} < \frac{b_{n_0+1}}{b_{n_0}}. \quad \text{(12)}$$

If $\theta$ is an algebraic integer such that the minimal polynomial $F_\theta(X) \in \mathbb{Z}[X]$ of $\theta$ over $\mathbb{Q}$ has all $n$ roots in the interval $[\alpha, \beta]$, then $[\mathbb{Q}(\theta) : \mathbb{Q}] < n_0$.

Proof. We are going to show that $a_n < b_n$ for all $n \geq n_0$. This will be sufficient to prove the theorem, because if this holds, and if $\theta$ is an algebraic integer of degree $n = [\mathbb{Q}(\theta) : \mathbb{Q}]$, with minimal polynomial $F_\theta(X) \in \mathbb{Z}[X]$ having all $n$ roots in the interval $[\alpha, \beta]$, then Lemma 7 and the definition of the $n$th diameter of the interval $[\alpha, \beta]$, respectively, imply the lower and upper bounds

$$b_n \leq |\text{disc}(F_\theta)| \leq a_n.$$

But as $a_n < b_n$ for all $n \geq n_0$, it follows that $n < n_0$, which is the desired result of the theorem.

To show that $a_n < b_n$ for all $n \geq n_0$, note that since it holds for $n = n_0$ by assumption, according to Lemma 6, we just have to show that $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$ for all $n \geq n_0$, or equivalently that

$$\frac{a_n}{a_{n-1}} < \frac{b_n}{b_{n-1}}. \quad \text{(13)}$$

for all $n \geq n_0 + 1$. We are also given that (13) holds for $n = n_0 + 1$ by assumption. So, we just have to show that (13) holds more generally for all $n \geq n_0 + 1$.

By an elementary calculation,

$$\frac{b_n}{b_{n-1}} = \frac{n^{2n}}{n!^2} \cdot \frac{(n-1)!^2}{(n-1)^2(n-1)!^2} = \frac{n^{2n-2}}{(n-1)^{2n-2}}.$$

Abbreviating $L = \beta - \alpha$ and using Theorem 2, we have

$$\frac{a_n}{a_{n-1}} = \frac{L^{n(n-1)}D_n}{L^{(n-1)(n-2)}D_{n-1}} = \frac{L^{2n-2}n^n(n-2)^{n-2}}{2^{2n-2}(2n-3)^{2n-3}}.$$

Thus, (13) is equivalent to

$$\frac{L^{2n-2}n^n(n-2)^{n-2}}{2^{2n-2}(2n-3)^{2n-3}} < \frac{n^{2n-2}}{(n-1)^{2n-2}} \quad \text{(14)}$$
and an elementary manipulation shows that this is, in turn, equivalent to the inequality

\[ 2 \log(L/2) < \frac{1}{n-1} \log \left( \frac{n^{n-2}(2n-3)^{2n-3}}{(n-1)^{2n-2}(n-2)^{n-2}} \right). \]  \hspace{1cm} (15) 

By assumption, (15) holds for \( n = n_0 + 1 \), and in order to conclude that it holds for all \( n \geq n_0 + 1 \), we just have to show that the function defined by

\[ h(x) = \frac{1}{x-1} \log \left( \frac{x^{x-2}(2x-3)^{2x-3}}{(x-1)^{2x-2}(x-2)^{x-2}} \right) \]

is increasing for \( x \geq 3 \). An elementary calculation shows that

\[ h'(x) = \frac{1}{(x-1)^2} \left( \frac{2}{x} + \log(2x-3) + \log x - \log(x-2) - 2 \right). \]

A simple calculus argument shows that \( h'(x) > 0 \) for all \( x \geq 3 \). Thus, \( h(x) \) is increasing and we conclude that (15) and hence (13) holds for all \( n \geq n_0 + 1 \), completing the proof of the theorem. \( \square \)

The following sample application of Theorem 8 shows that it can be combined with elementary arguments to give sharp results. This corollary will also immediately imply the degree \( d \geq 3 \) case of Theorem 1. Given a real interval \([\alpha, \beta]\), denote by \( T_{[\alpha, \beta]} \) the set of all algebraic integers \( \theta \) with the property that all \( \mathbb{Q}(\theta) : \mathbb{Q} \) embeddings of \( \mathbb{Q}(\theta) \) into \( \mathbb{C} \) map \( \theta \) into the real interval \([\alpha, \beta]\).

**Corollary 9.** If \([\alpha, \beta]\) is a real interval of length \( \beta - \alpha < \sqrt{5} \), then \( T_{[\alpha, \beta]} \subseteq \mathbb{Z} \).

This corollary is sharp in the sense that a counterexample exists if \( \beta - \alpha = \sqrt{5} \). The irrational algebraic integer \( \theta = \frac{1 + \sqrt{5}}{2} \) and its algebraic conjugate \( \theta' = \frac{1 - \sqrt{5}}{2} \) both lie in the interval \([\theta', \theta]\), which has length \( \sqrt{5} \).

**Proof of Corollary 9.** We wish to prove that if \( \theta \in T_{[\alpha, \beta]} \), then \( \theta \in \mathbb{Q} \). That, in fact, \( \theta \in \mathbb{Z} \) follows at once since \( \theta \) is assumed to be an algebraic integer. The strategy of proof is to use Theorems 2 and 8 to prove that \( \mathbb{Q}(\theta) : \mathbb{Q} \neq 2 \), and then to use elementary arguments to show that \( \mathbb{Q}(\theta) : \mathbb{Q} \neq 2 \).

We wish to apply Theorem 8, and so for each \( n \geq 2 \), we set

\[ a_n = d_n([\alpha, \beta])^{n(n-1)} = (\beta - \alpha)^{n(n-1)} D_n \]
\[ b_n = \frac{n^{2n}}{n!^2} \]

using Theorem 2 to calculate the \( n \)th diameters of the interval. In particular,

\[ a_2 = d_2([\alpha, \beta])^{2(2-1)} = (\beta - \alpha)^2 D_2 = (\beta - \alpha)^2, \]
\[ a_3 = d_3([\alpha, \beta])^{3(3-1)} = (\beta - \alpha)^6 D_3 = (\beta - \alpha)^6 \frac{1}{16}, \]
\[ a_4 = d_4([\alpha, \beta])^{4(4-1)} = (\beta - \alpha)^{12} D_4 = (\beta - \alpha)^{12} \frac{1}{3125}. \]
and

\[ b_2 = \frac{2^4}{2!^2} = 4, \]
\[ b_3 = \frac{3^6}{3!^2} = \frac{81}{4}, \]
\[ b_4 = \frac{4^8}{4!^2} = \frac{1024}{9}, \]

using \( D_2 = 1, D_3 = \frac{1}{16}, \) and \( D_4 = \frac{1}{3125} \) as described after the statement of Theorem 2.

We have \( a_3 < b_3 \), as this inequality simplifies to \((\beta - \alpha)^3 < 18\), which holds because \((\beta - \alpha)^3 < (\sqrt{5})^3 \approx 11.18 \ldots\). We also have \( \frac{a_3}{b_3} < \frac{b_4}{b_3} \) because this inequality simplifies to

\[ (\beta - \alpha)^6 < \frac{800000}{729} \approx 1097.393 \ldots, \]

which holds because \((\beta - \alpha)^6 < \sqrt{5}^6 = 125\). We conclude using Theorem 8 that if \( \theta \in T_{[\alpha, \beta]} \), then \([\mathbb{Q}(\theta) : \mathbb{Q}] < 3\).

It remains only to prove that \([\mathbb{Q}(\theta) : \mathbb{Q}] \neq 2\) for \( \theta \in T_{[\alpha, \beta]} \). If, in fact, \([\mathbb{Q}(\theta) : \mathbb{Q}] = 2\) and \( \theta \) has minimal polynomial \( F(X) = X^2 + aX + b \in \mathbb{Z}[X] \), then \( \text{disc}(F) = a^2 - 4b \) must be a positive nonsquare, as \( F(X) \) is irreducible over \( \mathbb{Q} \) and has two real roots. But \( \text{disc}(F) = a^2 - 4b = 2 \) and \( \text{disc}(F) = a^2 - 4b = 3 \) are both impossible as 2 and 3 are not squares modulo 4. So, \( \text{disc}(F) \geq 5 \). If \( \theta' \) denotes the algebraic conjugate of \( \theta \), then \( \text{disc}(F) = (\theta - \theta')^2 \) and hence \(|\theta - \theta'| \geq \sqrt{5} \). But this contradicts the assumption that both \( \theta \) and \( \theta' \) are elements of the interval \([\alpha, \beta]\) that has length \(< \sqrt{5} \).

We have proved that \( T_{[\alpha, \beta]} \subseteq \mathbb{Q} \) and hence \( T_{[\alpha, \beta]} \subseteq \mathbb{Z} \) as \( T_{[\alpha, \beta]} \) contains only algebraic integers. \( \square \)

4 | APPLICATION TO POSTCRITICALLY FINITE POLYNOMIALS IN UNICRITICAL FAMILIES

Our goal in this section is to apply Theorems 2 and 8 to the proof of Theorem 1, the dynamical application of classifying totally real PCF parameters in unicritical families.

Although superseded by Theorem 1, we first include the following qualitative finiteness result, because it has a fairly elementary proof and illustrates the main ideas behind the proof of Theorem 1 in a simple way. Recall that \( \mathbb{Q}^{\text{tr}} \) denotes the maximal totally real subfield of \( \mathbb{Q} \), and that for each integer \( d \geq 2 \), we denote

\[ \text{PCF}_d = \{ c \in \overline{\mathbb{Q}} \mid f(x) = x^d + c \text{ is PCF} \}. \]

**Theorem 10.** For each \( d \geq 2 \), the set \( \text{PCF}_d \cap \mathbb{Q}^{\text{tr}} \) is finite.

**Proof.** We begin with the \( d = 2 \) case. For a parameter \( c \), define \( f_c(x) = x^2 + c \), and note that if \( f_c \) is PCF, then 0 is preperiodic and hence \( f_i'(0) = f_j'(0) \) for some integers \( i < j \). In other words, \( c \) is a root of the polynomial \( G(X) = f_i'(X)(X) - f_j'(X)(0) \), which is monic with integral coefficients, and so,
c must be an algebraic integer. Let $F_c(X) \in \mathbb{Z}[X]$ be the monic minimal polynomial of c. Since the parameter c is a root of the monic integral polynomial $f^i(X(0)) - f^j(X(0))$ in X for some $i < j$, it follows that $F_c(X)$ is a divisor of $f^i(X(0)) - f^j(X(0))$, and hence, all of the roots of $F_c(X)$ have the property that $f_c$ is PCF. It follows from this and the totally real hypothesis that all of the roots of $F_c(X)$ are real and also in the Mandelbrot set

$$\mathcal{M}_2 = \{c \in \mathbb{C} \mid x^2 + c \text{ has bounded critical orbit}\}.$$  

It is well known that $\mathbb{R} \cap \mathcal{M}_2 = [-2, 1/4]$. Since the transfinite diameter of an interval is one quarter of its length ([4, Corollary 5.2.4]), we have $d_{\infty}(\mathbb{R} \cap \mathcal{M}_2) < 1$, and hence, $\mathbb{R} \cap \mathcal{M}_2$ contains only finitely many complete sets of conjugates of algebraic integers, completing the proof that $\text{PCF}_d \cap \mathbb{Q}^{tr}$ is finite.

Because it gives a simple illustration of the ideas used in the proof of Theorem 1, we can quickly describe the proof of Fekete’s theorem in this special case. Suppose on the contrary that \{c_m\} is an infinite sequence of distinct points in $\mathbb{Q}^{tr}$ such that $f_{c_m}$ is PCF for all $m \geq 1$. Since the $c_m$ are algebraic integers and all of the complex embeddings of each $c_m$ are in $[-2, 1/4]$, the sequence \{c_m\} has bounded height, and so, by Northcott’s property, we know that $n_m : = [\mathbb{Q}(c_m) : \mathbb{Q}] \to \infty$ as $m \to +\infty$. Since $c_m$ is an algebraic integer, $\text{disc}(F_m)$ is a nonzero rational integer. Therefore,

$$1 \leq |\text{disc}(F_m)|^{1/n_m(1/m-1)} \leq d_{n_m}([-2, 1/4]) \to d_{\infty}([-2, 1/4]) = 9/16,$$

a contradiction as $m \to +\infty$.

For $d \geq 3$, the same proof works, using properties of the degree $d$ analogue

$$\mathcal{M}_d = \{c \in \mathbb{C} \mid x^d + c \text{ has bounded critical orbit}\}$$

of the Mandelbrot set. It follows from work of Parisé–Rochon [3] and Parisé–Ransford–Rochon [2] that for all $d \geq 3$, the set $\mathcal{M}_d \cap \mathbb{R}$ is an interval of length less than 4. Thus, $d_{\infty}(\mathcal{M}_d \cap \mathbb{R}) < 1$ for $d \geq 3$ and the same contradiction is obtained.

Proof of Theorem 1 in the case $d=2$. We seek to prove that

$$\text{PCF}_2 \cap \mathbb{Q}^{tr} = \{-2, -1, 0\}$$

that is, that the only totally real $c$ for which $x^2 + c$ is PCF are $c = -2, -1, 0$.

Suppose that $c \in \overline{\mathbb{Q}}$ is totally real and that $x^2 + c$ is PCF. In particular, $c$ must be an algebraic integer (as explained in the proof of Theorem 10). Since all of the $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugates of $c$ in $\overline{\mathbb{Q}}$ are also PCF and are real when embedded into $\mathbb{C}$, the minimal polynomial of $c$ over $\mathbb{Q}$ has all roots in $\mathcal{M}_2 \cap \mathbb{R}$, where $\mathcal{M}_2$ is the ordinary Mandelbrot set. Recall that $\mathcal{M}_2 \cap \mathbb{R} = [-2, 1/4]$.

We now proceed in an argument similar the proof of Corollary 9, following these steps:

Step 1: We use Theorems 2 and 8 to prove that $[\mathbb{Q}(c) : \mathbb{Q}] < 3$.
Step 2: We use elementary arguments to show that $[\mathbb{Q}(c) : \mathbb{Q}] \neq 2$.
Step 3: We recall a well-known argument to conclude that $c \in \{-2, -1, 0\}$. 

\[\text{PCF}_2 \cap \mathbb{Q}^{tr} = \{-2, -1, 0\}; \quad (16)\]
Step 1: We apply Theorem 8 with $L = 9/4$, the length of the interval $[-2, 1/4]$. For each $n \geq 2$, set

$$a_n = d_n([-2, 1/4])^{n(n-1)} = (9/4)^{n(n-1)}D_n$$

$$b_n = \frac{n^{2n}}{n!^2},$$

where the sequence $\{D_n\}$ is defined in Theorem 2. We calculate

$$a_2 = d_2([-2, 1/4])^{2(2-1)} = (9/4)^2D_2 = \frac{81}{16} = 5.0625,$$

$$a_3 = d_3([-2, 1/4])^{3(3-1)} = (9/4)^6D_3 = \frac{531441}{65536} \approx 8.109 \ldots,$$

$$a_4 = d_4([-2, 1/4])^{4(4-1)} = (9/4)^{12}D_4 = \frac{282429536481}{52428800000} \approx 5.386 \ldots,$$

and

$$b_2 = \frac{2^4}{2!^2} = 4,$$

$$b_3 = \frac{3^6}{3!^2} = \frac{81}{4} = 20.25,$$

$$b_4 = \frac{4^8}{4!^2} = \frac{1024}{9} \approx 113.777 \ldots,$$

using $D_2 = 1$, $D_3 = \frac{1}{16}$, and $D_4 = \frac{1}{3125}$ as described after the statement of Theorem 2. Moreover, using these calculations, we have

$$\frac{a_4}{a_3} = \frac{282429536481/52428800000}{531441/65536} = \frac{531441}{800000} \approx 0.664 \ldots,$$

$$\frac{b_4}{b_3} = \frac{1024/9}{81/4} = \frac{4096}{729} \approx 5.618 \ldots,$$

and thus $\frac{a_4}{a_3} < \frac{b_4}{b_3}$. We conclude using Theorem 8 with $n_0 = 3$ that $[\mathbb{Q}(c) : \mathbb{Q}] < 3$.

Step 2: Assume that $[\mathbb{Q}(c) : \mathbb{Q}] = 2$. Let $F(X) = X^2 + aX + b \in \mathbb{Z}[X]$ be the minimal polynomial of $c$ over $\mathbb{Q}$, thus $a, b \in \mathbb{Z}$ since $c$ is an algebraic integer. Since $c$ is totally real and $x^2 + c$ is PCF, we know that both complex roots $c_1, c_2$ of $F(X)$ are in the real interval $M_2 \cap \mathbb{R} = [-2, 1/4]$. Thus,

$$0 < \text{disc}(F) = a^2 - 4b = (c_1 - c_2)^2 \leq (9/4)^2 = 5.0625.$$

Moreover, disc$(F)$ is not a square since $F(X)$ is irreducible, so disc$(F)$ is either 2, 3, or 5. We cannot have disc$(F) = 2$ or disc$(F) = 3$ because disc$(F) = a^2 - 4b$ and neither 2 nor 3 is a square modulo 4.
So, we must have $\text{disc}(F) = a^2 - 4b = 5$; in particular, $a$ must be odd. Since both $c_1$ and $c_2$ are in the interval $[-2, 1/4]$, we have $a = -(c_1 + c_2) \in [-1/2, 4]$, so either $a = 1$ or $a = 3$. If $a = 1$, then $b = -1$ and $F(X) = X^2 + X - 1$; but one of the roots of this polynomial, $c_1 = \frac{-1 + \sqrt{5}}{2} \approx 0.618 \ldots$ is not contained in $[-2, 1/4]$, which gives a contradiction. If $a = 3$, then $b = 1$ and $F(X) = X^2 + 3X + 1$; but one of the roots of this polynomial, $c_1 = \frac{-3-\sqrt{5}}{2} \approx -2.618 \ldots$ is not contained in $[-2, 1/4]$, which again gives a contradiction, completing the proof that $[\mathbb{Q}(c) : \mathbb{Q}] \neq 2$.

**Step 3:** We now know that $[\mathbb{Q}(c) : \mathbb{Q}] = 1$ and hence $c \in \mathbb{Q}$. Since $c$ is an algebraic integer and hence a rational integer, and $c \in M_d \cap \mathbb{R} = [-2, 1/4]$, we conclude that $c \in \{-2, -1, 0\}$. It is elementary to check that all three of $x^2 - 2$, $x^2 - 1$, and $x^2$ are PCF, concluding the proof of (16).

To prove Theorem 1 in the case $d \geq 3$, we need to understand the intersection $M_d \cap \mathbb{R}$ of the degree $d$ generalized Mandelbrot set $M_d$ with the real line. It has been shown by Parisé–Rochon [3] when $d \geq 3$ is odd, and by Parisé–Ransford–Rochon [2] when $d \geq 4$ is even, that

$$M_d \cap \mathbb{R} = \begin{cases} [-a_d, a_d] & \text{if } d \geq 3 \text{ is odd} \\ [-b_d, a_d] & \text{if } d \geq 4 \text{ is even,} \end{cases}$$

(17)

where $a_d = (d - 1)/(d^d/(d-1))$ and $b_d = 2^{1/(d-1)}$.

**Lemma 11.** For each $d \geq 3$, the interval $M_d \cap \mathbb{R}$ has length less than $\sqrt{5}$.

**Proof.** First, we have

$$a_d = \frac{d - 1}{d^{d/(d-1)}} < 1$$

for all $d \geq 2$. Indeed, this is algebraically equivalent to the inequality

$$(d - 1) \log(d - 1) < d \log d,$$

which follows from the fact that $x \mapsto x \log x$ is increasing for $x \geq 1$. We conclude that when $d \geq 3$ is odd, the interval $M_d \cap \mathbb{R} = [-a_d, a_d]$ has length $< 2 < \sqrt{5}$.

Now consider the case $d \geq 4$ even. We want to show that $a_d + b_d < \sqrt{5}$. We can just check numerically that $a_4 + b_4 = 3/(4^{4/3}) + 2^{1/3} \approx 1.732 \ldots < \sqrt{5}$, while for $d \geq 6$ (even), we have $d - 1 \geq 5$ and so

$$a_d + b_d = a_d + 2^{1/(d-1)} \leq a_d + 2^{1/5} < 1 + 2^{1/5} \approx 2.148 < \sqrt{5}. \square$$

**Proof of Theorem 1 in the case $d \geq 3$.** We seek to prove that

$$\text{PCF}_d \cap \mathbb{Q}^{\text{tr}} = \{-1, 0\} \text{ if } d \geq 4 \text{ is even}$$

$$\text{PCF}_d \cap \mathbb{Q}^{\text{tr}} = \{0\} \text{ if } d \geq 3 \text{ is odd;}$$

(18)

that is, for $d \geq 3$, the only PCF unicritical maps $x^d + c$ for $c \in \mathbb{Q}^{\text{tr}}$ are $x^d - 1$ and $x^d$ when $d \geq 4$ is even, and $x^d$ when $d \geq 3$ is odd.
Suppose that $c \in \text{PCF}_d \cap \mathbb{Q}^{tr}$. By the same argument described in the degree $d = 2$ case, $c$ must be an algebraic integer, and all of the algebraic conjugates of $c$ in $\mathbb{C}$ lie in the real interval $\mathcal{M}_d \cap \mathbb{R}$. Since these intervals have length $< \sqrt{5}$ for all $d \geq 3$ by Lemma 11, it follows from Corollary 9 that $c \in \mathbb{Z}$. Recalling that $a_d < 1$ and $b_d = 2^{1/(d-1)}$, when $d \geq 3$ is odd the only integer in $\mathcal{M}_d \cap \mathbb{R} = [-a_d, a_d]$ is $c = 0$, and when $d \geq 4$ is even the only integers in $\mathcal{M}_d \cap \mathbb{R} = [-b_d, a_d]$ are $c = -1, 0$, completing the proof. □

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REFERENCES
1. C. Noytaptim, *Preperiodic points with local rationality conditions in the quadratic unicritical family*, arXiv:2211.10571 (2022).
2. P.-O. Parisé, T. Ransford, and D. Rochon, *Tricomplex dynamical systems generated by polynomials of even degree*, Chaotic Model. Simul. 1 (2017), 37–48.
3. P.-O. Parisé and D. Rochon, *Tricomplex dynamical systems generated by polynomials of odd degree*, Fractals 25 (2017), 1–11.
4. T. Ransford, *Potential theory in the complex plane*, London Mathematical Society Student Texts, vol. 28, Cambridge University Press, Cambridge, 1995.
5. R. S. Rumely, *Capacity theory on algebraic curves*, Lecture Notes in Mathematics, vol. 1378, Springer, Berlin, 1989.
6. G. Szegő, *Orthogonal polynomials*, 4th ed., American Mathematical Society Colloquium Publications, Vol. XXIII, American Mathematical Society, Providence, RI, 1975.