Anderson Metal-to-Critical Transition in QCD

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A picture of thermal QCD phase change based on the analogy with metal-to-insulator transition of Anderson type was proposed in the past. In this picture, a low-\(T\) thermal state is akin to a metal with deeply infrared (IR) Dirac modes abundant and extended, while a high-\(T\) state is akin to an insulator with IR modes deplete and localized below a mobility edge \(\lambda_{\text{IR}} > 0\). Here we argue that, while \(\lambda_{\text{IR}}\) exists in QCD, a high-\(T\) state is not an insulator in such an analogy. Rather, it is a critical state arising due to a new singular mobility edge at \(\lambda_{\text{IR}} = 0\). This new mobility edge appears upon the transition into the recently proposed IR phase. As a key part of such a \textit{metal-to-critical} scenario, we present evidence using pure-glue QCD that deeply infrared Dirac modes in the IR phase extend to arbitrarily long distances. This is consistent with our previous suggestion that the IR phase supports scale invariance in the infrared. We discuss the role of Anderson-like aspects in this thermal regime and emphasize that the combination of gauge field topology and disorder plays a key role in shaping its IR physics. Our conclusions are conveyed by the structure of Dirac spectral non-analyticities.

Keywords: QCD phase transition, quark-gluon plasma, IR phase, Anderson localization, scale invariance

1. Introduction. A fruitful aspect of progress in physics is the interplay of ideas between particle/nuclear and condensed matter areas. The development of the renormalization group \cite{1, 2} or the complicated history of Anderson-Higgs mechanism \cite{3, 4} illustrate this point. Among the ideas that originated in the realm of condensed matter, the notion of disorder-induced (Anderson) localization \cite{5} had a strong impact on other parts of physics. Yet, its influence on the basic framework of elementary particle physics (Standard Model) has not been substantial.

One context where Anderson localization may prove relevant is in thermal phase changes of strongly interacting matter. Here the electron eigenstates in random potentials of Anderson type could be analogous to Dirac quark modes in backgrounds of non-Abelian gauge fields at finite temperature. If the influence of pure disorder on electron dynamics turns out to be similar to that of thermal agitation on quarks, this could be important for understanding the mechanism of QCD transition.

A decade after it was first mentioned that an Anderson-like feature could appear in finite temperature Dirac spectra \cite{6}, it was proposed in Refs. \cite{7, 8} that the QCD (chiral) transition may be a disorder-driven phenomenon analogous to Anderson metal-to-insulator transition. Here the “insulating” nature of a hot thermal state expresses the absence of long-distance (\(> 1/T\)) physics. It is specific to this scenario that, in addition to a depletion of the Dirac spectrum in this range, such physics is inhibited by any remaining infrared (IR) modes being exponentially localized at shorter scales.

The existence of Anderson-like features in hot QCD has been investigated and the critical spectral point \(\lambda_{\text{IR}} > 0\) has been found \cite{9–11}. However, describing a hot thermal state as an Anderson insulator is at odds with its chiral polarization properties \cite{12} which suggest the presence of long-distance physics. This inconsistency became sharp in light of our recent suggestion \cite{13} that, above the crossover temperatures (\(\approx 150–180\) MeV), there is a thermal phase transition at temperature \(200 < T_{\text{IR}} < 250\) MeV, involving a proliferation of deep IR modes (\(0 \leq \lambda \ll T\)) and the appearance of long-range scale-invariant IR physics. This new dynamical regime is referred to as the IR phase.

Although metal-to-insulator scenario (no IR physics) is incompatible with IR phase (long-range physics), a fruitful Anderson-like interpretation can be given to Dirac modes of IR phase, while keeping the long-range feature intact. Clarifying this is the main focus of the present paper.

Few contextual details are important before we start. (1) A fixed 3d Anderson model can describe varied electronic behaviors (conductor, insulator, criticality) simply by suitably choosing the Fermi level. This freedom does not exist in QCD, whose symmetries require such reference to be at zero. Hence, the Anderson analog of the QCD thermal state, if any, is determined by properties of modes near \(\lambda_{\text{IR}} \approx 0\). (2) Despite its IR mode proliferation, a thermal state in the IR phase could be an insulator if these modes were exponentially localized in IR spectral regime \(\lambda < T\) with bounded range. A key ingredient in reconciling IR phase with Anderson-like aspects will be the evidence that this does not occur. Rather, we conclude that the length scale associated with modes at eigenvalue \(\lambda\) diverges for \(\lambda \rightarrow 0\). (3) The available evidence is consistent with the mobility edge \(\lambda_{\text{IR}}\) appearing upon crossing into the IR phase. This point is solid in pure-glue QCD \cite{14} and, although existing full QCD estimates are somewhat below the \(T_{\text{IR}}\) window \cite{8, 15}, a reliable determination is lacking. [We thank T. Kovács for clarifying this.] In either case, thermal state at \(T < T_{\text{IR}}\) would not act as an insulator since deep IR modes here are at least as mobile as in IR phase (see above).
Item (2) above points to the presence of singular mobility edge at \( \lambda_{\text{IR}} = 0 \). Indeed, the length scale \( \ell \) we will use to characterize modes is a proxy to exponential localization range: if modes exponentially decay in space, their \( \ell \) is related to localization range by a finite transformation with their ratio approaching unity at large \( \ell \). Our approach to show \( \langle 2 \rangle \) will follow the one we used to study spatial dimensions of modes in the vicinity of \( \lambda_{\text{IR}} \) [14]. To recapitulate, exact zeromodes were found to be \( d_{\text{IR}} = 3 \) whereas the deep IR modes have lower dimensions, with \( d_{\text{IR}} \to 2 \) for \( \lambda \to 0 \). This makes the isolated point \( \lambda = \lambda_{\text{IR}} \) similar to the band of extended modes \( \lambda \geq \lambda_{\text{IR}} \), and deep IR modes \( (\lambda \geq \lambda_{\text{IR}}) \) akin to those near \( \lambda_{\text{IR}} \) \( (\lambda < \lambda_{\text{IR}}) \). Here we find similar structure in metric properties which, among other things, leads to \( \langle 2 \rangle \). Hence, the deep-IR Dirac dynamics of QCD in the IR phase is similar to Anderson-like critical dynamics: rather than metal-to-insulator, the relevant analogy for QCD phase transition occurring at \( T_{\text{IR}} \) is that of a metal-to-critical transition.

2. QCD Perspective. This work is part of our ongoing program to infer the properties of QCD vacuum and thermal states from spectral features of the Dirac operator. At the lattice-regularized level, the probing Dirac operator can be chosen to best reflect the properties of interest, with universality expected in the continuum limit. Given the important role of quark chirality and glue topology in low-energy QCD, we use the overlap operator [16] for these purposes. Refs. [12–14, 17, 18] are most relevant to the topic studied here.

The overlap spectral feature from which the notion of IR phase eventually arose, the infrared mode density peak, was first observed in Ref. [19]. Renewed interest appeared when its peculiar chiral polarization properties were noted [12]. Further studies showed that these IR modes persist both in the thermodynamic and the continuum limit [18]. This led to a new perspective in which the peak reflects a dynamical IR property of glue that drives the phase changes in SU(3) gauge theories with fundamental quarks [18]. We later identified this property with IR scale invariance of strongly coupled glue fields [13].

The relevance of Anderson-Dirac feature at \( \lambda_{\text{IR}} \) in this context was pointed out in Ref. [14]. Remarkably, such spectral non-analyticity is exactly what is needed to realize the IR-UV decoupling important for the proposed mechanism of IR scale invariance [13]. This non-analyticity is manifest in dimension \( d_{\text{IR}}(\lambda) \) of the modes, whose structure near \( \lambda_{\text{IR}} = 0 \) was found to be similar to that near \( \lambda_{\text{IR}} \). Here we study the spectral function \( \gamma(\lambda) \) expressing the “sizes” (distances) of the modes and find the same qualitative scenario. We thus propose that an Anderson-like aspect is present not only at \( \lambda_{\text{IR}} \) but also at \( \lambda_{\text{IR}} \). Hence, although the origin of IR phase is likely topological and complex [14], some of its IR properties, including certain features of scale invariance, may be understood in terms of Anderson-like criticality.

3. The \( \gamma \)-index. We study thermal QCD represented by a field system in 4-volume \( L^4 / T \) and regularized via hypercubic lattice with shortest distance \( a \). Scales \( L / a \) and \( 1 / a \) represent the IR and UV cutoffs. Our analysis will focus on eigenmodes \( D\psi_\lambda(x) = i\gamma_5 \psi_\lambda(x) \) of Dirac operator \( D \), represented on the lattice by the overlap operator.

Unlike the effective volume whose definition requires careful consideration [20, 21], an acceptable effective distance spanned by a mode can be defined as a simple probability mean. Indeed, let \( \psi = \psi(x_i) \) be a generic eigenmode with \( x_i, i = 1, 2, \ldots, N = (L / a)^3 / (T a) \), denoting the positions of lattice sites. The probability vector \( P \) associated with \( \psi \) is specified by \( P = (p_1, p_2, \ldots, p_N) \), \( p_i = \psi^\dagger \psi(x_i) \). The effective distance can be quantified by the average separation from the location of the probability maximum, namely

\[
\ell[\psi] = \ell[P] = \sum_{i=1}^{N} p_i |x_m - x_i| , \quad p_m \geq p_i , \quad \forall i \tag{1}
\]

where \(| \ldots | \) is the Euclidean norm on a periodic lattice. Distance \( \ell[\psi] \) can be viewed as a version of “mode radius”. Its response to the removal of IR cutoff

\[
\langle \ell[\psi] \rangle_{a,L,\lambda} \propto L^{\gamma(a,\lambda)} \text{ for } L \to \infty \tag{2}
\]

defines the index \( 0 \leq \gamma \leq 1 \). Note that \( \gamma = 0 \) for exponentially localized states and \( \gamma = 1 \) for plane waves.

4. The Results. We studied the above characteristics in pure-glue lattice QCD at \( T = 1.12 T_{\text{IR}} \). Wilson action with \( \beta = 6.054 (a = 0.085 \text{ fm}, r_0 = 0.5 \text{ fm}) \) was used, and the eigenmodes of the overlap operator \( (\rho = 26/19) \) were analyzed on systems with \( L / a = 24, 28, 32, 40, 48, 56, 64 \) and \( 1/(T a) = 7 \). The implementation of overlap is described in Refs. [22–24]. Additional technical details can be found in Refs. [13, 14]. Overlap discretization is crucial to distinguish zero modes from the nearby modes. On our largest volumes the low-lying modes have unusually small eigenvalues and we numerically diagonalize the overlap operator \( D \) rather than more conventional \( D^\dagger D \). Using
Fig. 2. The spectral dependence of index $\gamma$ in pure-glue QCD.

during this approach, we find that low-lying modes are separated from zero modes by several orders of magnitude.

Our basic finding is already conveyed by coarse spectral data for $\ell$ shown in Fig. 1. We observe a central plateau, insensitive to the IR cutoff, and the “rises” near $\lambda_{IR} = 0$ and $\lambda_{A} \approx 750 \text{MeV}$, where $\ell$ increases with $L$. Point $\lambda_{A}$, marking the Anderson-like critical point, was determined for this setup in Ref. [14]. The two IR-responsive regimes act as “finite-volume critical regions” in Anderson-like language. The novelty is the existence of one near $\lambda_{IR}$.

Asymptotic tendency to extend into distance is expressed by index $\gamma$, Eq. (2). Numerically, we compute this index by fitting $\ell$, obtained on different ensembles, to a power-law in $L$. The errors on $\gamma$ are calculated using standard $\chi^2$-fitting methods. Its spectral behavior, shown in Fig. 2, characterizes the above features of raw data in a robust manner. Thus, we again observe a plateau, approximately centered around the scale of temperature, with $\gamma \approx 0$. The latter is consistent with the presumed exponentially localized nature of these modes. At the same time, the left and right rises appear below $\lambda \approx 50 \text{MeV}$ and above $\lambda \approx 550 \text{MeV}$ respectively. The examples illustrating the scaling behavior of $\ell/L$ in these three spectral regimes are shown in Fig. 3. The red point in Fig. 2 is the finite volume result for zero modes. Extrapolated value $\gamma = 1$ is predicted by their $d_{IR} = 3$ [14], and has indeed materialized in the present $\ell[\psi]$ data. The black horizontal line above $\lambda_{A}$ marks the extended regime of the bulk.

A notable feature of our results is conveyed by the inset of Fig. 2, showing $\gamma$ for near-zero modes from variable interval $(0, \lambda_{max})$. Data from available volumes suggest that $\lim_{\lambda \to \lambda_{IR}} \gamma(\lambda) \lesssim 0.8$. The left vicinity of $\lambda_{A}$ (vertical line in Fig. 2) also suggests $\gamma < 1$. We thus observe a discrepancy of these limiting values relative to both fully extended ($\gamma = 1$) and exponentially localized ($\gamma = 0$) regimes. This suggests that $\gamma(\lambda)$ has the same kind of critical non-analyticity structure as mode dimension $d_{IR}(\lambda)$. (See Fig. 5 in Ref. [14].)

Such scenario is supported by volume tendencies in $\gamma$. Fig. 4 shows the dependence of $\gamma$ on $1/L_{max}$ within the two rises, with $L_{max}$ being the maximal lattice size used in the analysis. Four systems of consecutive available sizes were used to extract each value shown from a power fit. The top plots suggest the extrapolation toward $\gamma = 0$ for interiors of the rises, while in the immediate vicinity of $\lambda_{IR}$ and $\lambda_{A}$ (bottom plots) the values remain stable. In case of $\lambda_{IR}$, we split the outer critical region into three parts with approximately equal statistics of modes to show that $\gamma \to 0$ in each. This indeed supports the singularity structure shown in the inset of the bottom right plot.

5. Discussion. Important consequence of our findings is that the length scale $\ell$ of non-zero modes in IR regime $0 < \lambda < T$ is unbounded in infinite volume. Indeed, $\ell(\lambda)$ increases toward infrared ($\lambda \to 0^+$, left rise). Its rate of growth with $L$, namely $\gamma(\lambda)$ increases toward infrared as well (Fig. 2), with deepest near-zero modes governed by $\gamma(0^+) \lesssim 0.8$ (inset of Fig. 2). Hence, $\ell(\lambda)$ in infinite volume diverges, at least for $\lambda \to 0^+$. Since $\gamma(\lambda, L_{max})$ tends to zero with $L_{max}$, at least for $\lambda > 1 \text{MeV}$ (Fig. 4, top left), we conclude that the most likely scenario is that
the two-component nature of IR phase and its IR scale remains finite at any fixed $0 < \lambda < T$, and $\lambda_{ir} = 0$ is a new mobility edge. Given the point (1) of Sec. 1, this in turn means that crossing into IR phase can be described as metal-to-critical transition.

In Ref. [14] we proposed a new feature of QCD Dirac modes in IR phase: their tendency to spread into available volume, expressed by IR dimension function $d_{ir}(\lambda)$, is non-analytic at $\lambda_{ir} = 0$ and $\lambda > 0$. Both non-analyticities feature a characteristic double discontinuity. Results of the present work suggest a similar structure in mode tendency to extend into available distances $\gamma(\lambda)$ (inset of Fig. 4). Since $d_{ir}$ and $\gamma$ reflect distinct geometric features of Dirac modes (volume and distance), the consistency of these findings reinforce the picture of $\lambda_{ir}$ and $\lambda_{A}$ as two points of Dirac non-analyticity present in IR phase.

The chief ramification of the above is that the vicinity of $\lambda_{ir}$ can support long-range physics. Indeed, $\ell(\lambda) \geq 1/T$ for all $\lambda < T$, and diverges for $\lambda \to 0^+$. This is a prerequisite for IR phase to support IR scale-invariant glue [13]. Moreover, the added aspect of Anderson-like criticality will likely play a relevant role in the eventual full understanding of the scale invariant dynamics involved. More concretely, we expect the modes in IR “critical region” of size $1/L$ to encode the zero glueball screening masses conjectured to characterize the IR phase [13].

The presented evidence is in the context of pure-glue QCD and the analysis should be extended to QCD with physical quarks. We point out, however, that the overlap-inferred glue response to IR transition has been previously found to be very similar in these theories [13].

The present results combined with those of Refs. [13, 14] suggest a Dirac spectral “phase diagram” of thermal QCD similar to one shown in Fig. 5. The thick red lines are phase boundaries separating the regular $d_{ir} = 3$, $\gamma = 1$ modes outside of the closed region, from the rest. Solid parts of the boundary feature the double discontinuity structure. In the absence of evidence to the contrary, the dashed parts are presumed to be simple discontinuities. Critical lines $\pm \lambda_{A}(T)$ coincide [14] with mobility edges studied previously [9–11]. The critical line $\lambda_{ir}(T) = 0$ is also the line of singularities in mode density $\rho(\lambda, T) \propto 1/\lambda$ for $\lambda \to 0$, defining the IR phase [13]. Note that the non-zero value of $\lambda_{A}(T_{in})$ (see also [25]) allows for the possibility that the two types of critical lines have distinct critical properties.

Finally, we wish to make several comments. (i) The range $(-\lambda_{A}, \lambda_{A})$ widens with increasing $T$ due to growing disorder. However, increasing $T$ also weakens the interaction which biases modes toward free-like mobility and hence softens the effect of disorder alone. In fact, it is unknown how these competing tendencies resolve near the expected onset $T_{uv}$ of perturbative plasma, where critical lines end. Indeed, it would also be consistent with asymptotic freedom if lines $\pm \lambda_{A}(T)$ turned back toward the $T$-axis at sufficiently high temperatures, but there is currently no evidence of that. We also note that the possibility that $T_{uv} \to \infty$ cannot be excluded at this point. (ii) It would be valuable to understand the full scope and limitations of similarities between 3d Anderson transitions and QCD points $\lambda_{A}$, $\lambda_{ir}$. This is non-trivial since, although the former are simple idealizations describing the effects of pure disorder on free particle propagation, the latter express the influence of thermal disorder on structured and complicated dynamics. Nevertheless, the existence of two distinct critical lines in the QCD spectral diagram offers a schematic understanding of this added complexity. Indeed, $\lambda_{A}(T)$ can be understood at high temperatures $T \ll T_{uv}$ as resulting from the above competition of disorder and asymptotic freedom. The appearance of $\lambda_{ir}$, on the other hand, is natural when $T$ approaches $T_{ir}$ from below. Here the increased thermal fluctuations erode IR non-perturbative fields, until only those essential for supporting glue topology remain, being the most impervious to disorder. This is the nature of the transformation occurring at $T_{ir}$. The resulting prominent role of glue topology in the IR phase is supported by integer effective dimensions [14] and other results [26, 27]. (iii) Incorporating the above into the original IR phase proposal [13] suggests the following scenario. As the temperature of the system increases, the Dirac spectral crossover at $T_{A} \approx 150$ MeV signals the onset of condensate melting. By the time the temperature $T_{ir}$ ($T_{ir} \in (200, 250)$ MeV) is reached, thermal fluctuations have eroded away most non-perturbative IR fields, including those responsible for the scale anomaly. At this point a phase transformation occurs in which the system separates into the scale invariant IR component, characterized by the power-law mode density $\rho(\lambda, T) \propto 1/\lambda$, and the bulk. This is accompanied by the appearance of Anderson-like critical lines $\lambda_{ir}(T) = 0$ and $\pm \lambda_{A}(T)$ (Fig. 5). The critical region associated with $\lambda_{ir}$ gives the physics of the IR phase its long-range character and the attribute of competition between glue topology and disorder. The criticality of $\lambda_{A}$, on the other hand, provides a localization barrier that makes the IR component independent of the bulk, effectively allowing for non-analytic running and scale invariance below some energy scale $\Lambda_{ir} < T$. The two-component nature of IR phase and its IR scale

![FIG. 5. Conjectured Dirac spectral phase diagram of thermal pure glue QCD.](image-url)
invariance are expected to be important in generating the 
near-perfect fluidity of hot strongly interacting matter. (iv) The features described in this work are also relevant for chiral limit considerations. Here the problem of $U_A(1)$ anomaly survival at high temperatures [28] is an open question. For recent complementary views on this see, e.g., Refs. [29–31]. One should also mention here that the validity of the original metal-to-insulator scenario in the chiral limit remains an open question in a similar sense.

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