Majorana Zero Modes in Superconducting Proximity-coupled Magnetic Domain Wall

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We propose a simple model consisting of a magnetic domain wall proximity-coupled to an s-wave superconductor for realization of Majorana zero-energy modes. A spin-dependent gauge transformation translates the rotating magnetic profile through the domain wall to effective spin-orbit and Zeeman terms. The Hamiltonian breaks time reversal and chiral symmetries, while preserving particle-hole symmetry, placing itself into topological D class characterized by the $\mathbb{Z}_2$ topological invariant for quasi one-dimensional system. The low-energy sector of the model maps to the one isomorphic with Kitaev’s model, known to support Majorana modes at its end.2 There have been several interesting proposals to realize MFs in one-dimensional systems with an effective rather than intrinsic spin-orbit coupling.15–18 The search for MFs in these systems is both rewarding and difficult as a definite experimental observation of the predicted MFs has so far remained formidable.

Here, we describe a very simple model to realize MFs using a ferromagnetic domain wall (DW) on top of an s-wave superconductor (as Fig. 1), without requiring intrinsic spin-orbit coupling and an external magnetic field. Domain wall is a structure with spatially-varying magnetization configuration which separates regions with uniform magnetization. It is widely studied and also easily fabricated in magnets of submicrometer size.19–27 Electronic transport through ferromagnetic domain walls has been currently a subject of extensive investigations, both theoretically and experimentally.20,22,24,28–36 Now, such a domain wall on an s-wave superconductor provides the necessary ingredient to generate MFs. It will be shown here that by putting a DW on top of an s-wave superconductor, MFs will be present at the ends of the DW.

Model.— We propose a setup for realizing Majorana fermions in one-dimensional magnetic domain wall, with the length of $L$, proximity coupled to a conventional $s$-wave superconductor.

\begin{equation}
H_0 = \frac{p^2}{2m^*} - \mu - V_{\text{ex}}(r) \cdot \sigma
\end{equation}

\text{Figure 1. (Color online) A schematic configuration of proposed device for realizing Majorana fermions in one-dimensional magnetic domain wall, with the length of $L$, proximity coupled to a conventional $s$-wave superconductor.}
where $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli spin matrices and $m^*$ and $\mu$ are the effective mass and chemical potential, respectively. The last term represents the exchange coupling between the $s$ conduction electron spin and the $d$ electron spin of the local magnetization $V_{\text{ex}} = V_{\text{ex}}n(r)$, where $V_{\text{ex}}$ is the spin splitting strength and $n(r)$ the direction of the local magnetization.

The functional form of the $V_{\text{ex}}(r)$ describes the shape of magnetic domain wall. We consider a "Néel wall" with magnetization vector parallel to the $x$ axis in the leads far from the domain wall center and turns by $180^\circ$ in the $xz$ plane within the wall. We assume a trigonometric magnetization profile in the DW, $V_{\text{ex}} = (\cos \theta(x), 0, \sin \theta(x))$ where $\theta(x) = \nu \pi x/L$ with $\nu$ being the winding number, the total angle (phase) change along the system, and then do a spin-dependent gauge transformation in spin space from the fixed reference frame to the rotated frame, which is in the direction of local magnetization vector $V_{\text{ex}}(r)$. In our model, it is given by a rotation about the $y$ axis $R = e^{i \theta_y \sigma_y / 2}$. The DW Hamiltonian in the rotating frame $H_r$ takes the form

$$H_r = R_y^{-1}(\theta)H_0R_y(\theta) = e^{-i \theta_y \sigma_y / 2}H_0 e^{i \theta_y \sigma_y / 2}$$

$$= \frac{\hbar^2}{2m^*} \tau_z \otimes \sigma_0 + \frac{\hbar^2 \omega^2}{2m^*} - \frac{\hbar}{m^*} \omega p \tau_y + V_{\text{ex}} \tau_x$$

where $\omega = \partial_\theta \theta(x)/2$. Due to the gauge transformation the electrons experience an effective spin-orbit interaction ($\hbar \omega/m^* p \tau_y$, Zeeman field $V_{\text{ex}} \sigma_y$ and modified chemical potential $\mu - \hbar^2 \omega^2 / 2m^*$ in the rotated frame. The wave function in the fixed reference frame (along the $x$ axis) can be obtained from the relation $\Psi_s(x) = \mathcal{R} \Phi_s(x)$.

By inducing superconductivity into the DW via proximity to a conventional $s$-wave superconductor described within the effective BCS mean-field approximation, the low-energy effective single-particle BdG Hamiltonian in the basis $\Phi(x) = (u_\uparrow(x), u_\downarrow(x), v_\uparrow(x), -v_\downarrow(x))$ can be written as

$$H_{\text{BdG}} = \begin{pmatrix} \xi_p & V_{\text{ex}} \tau_0 & \Delta_0 \tau_x & \sigma_0 \\ V_{\text{ex}} \sigma_0 & \xi_p & \Delta_0 \tau_x & \tau_y \\ -\sigma_0 & \Delta_0 \tau_x & \xi_p & V_{\text{ex}} \tau_0 \\ \tau_y & -\sigma_0 & -\tau_y & \xi_p \end{pmatrix}$$

where $\mu' = \mu - \hbar^2 \omega^2 / 2m^*$ and the $\tau_i$ Pauli matrices act in the particle-hole (Nambu) space. It can easily be verified that the BdG Hamiltonian has a particle-hole symmetry (PHS) $\Xi H_{\text{BdG}} \Xi^{-1} = -H_{\text{BdG}}$ with respect to the particle-hole anti-unitary operator $\Xi = \tau_y \sigma_y \mathbf{K}$, where $\mathbf{K}$ is the complex conjugate operator, satisfying $\Xi^2(x) = \mathcal{A} \Phi(x)$. Both the time-reversal symmetry (TRS), represented by the operator $\Theta = i \sigma_y \mathbf{K}$, and the chiral symmetry, represented by the operator $\mathcal{C} = -i \tau_y \mathbf{K}$ are broken. The absence of TRS and chiral symmetry in the presence of PHS ensures that the Hamiltonian is in the D symmetry class characterized by the $\mathbb{Z}_2$ topological invariant for the quasi one-dimensional system. This implies that, under appropriate conditions, the system can support localized Majorana modes that remain topologically protected against local perturbations.

The spectrum of the bulk states is given by

$$\varepsilon_\pm^2(p) = \xi_p^2 + V_{\text{ex}}^2 + \Delta_0^2 + (\hbar \omega p / m^*)^2$$

$$\pm 2 \sqrt{V_{\text{ex}}^2 (\Delta_0^2 + \xi_p^2) + (\hbar \omega p \xi_p / m^*)^2}$$

where $\xi_p = p^2 / 2m^* - \mu'$ and the two $\pm$ branches are due to the spin splitting as shown in Fig. (2). The minimum gap is between the two excitation branches $\varepsilon_{\pm}^<(p)$ at $p = 0$ (measured from Fermi level) given by

$$\varepsilon_{\text{gap}} = \frac{1}{2} \left| \varepsilon_{\pm}^<(0) - \varepsilon_{\pm}^>(0) \right| = |V_{\text{ex}} - \sqrt{\Delta_0^2 + \mu'^2}|$$

which can be closed at the critical point $V_{\text{ex}}^2 = \Delta_0^2 + \mu'^2$, where the superscript $> (<)$ refers to the energy band above (below) the Fermi level ($E = 0$). The spectrum of the magnetic domain wall comprises two spin-splitted bands, along with two non-gapped degenerate Kramer’s pairs and the energy gap at the $p = 0$ point due to the broken time-reversal symmetry, for the electrons and their particle-hole related partners with the negative energy as indicated in Fig.2(a). Inducing the isotropic superconducting pair potential $\Delta_0$ via the proximity with an $s$-wave conventional superconductor opens a gap at the outer wings of the dispersion, which eliminates the possibility of high-momentum gapless excitations and leaving only the chiral states near $p = 0$ as the low energy excitations and modifies the gap forming near $p = 0$, as demonstrated in Fig.2(b). The topological quantum phase transition, specified by the closing and reopening of the gap $\varepsilon_{\text{gap}}$, with opposite sign is shown in the bottom panels of Fig.(2), where the system goes from the superconducting-dominated trivial gap Fig.2(c), to the exchange field-dominated nontrivial gap Fig.2(e) while pass the critical point Fig.2(d).16

While the full BdG Hamiltonian Eq. (3) is not that of a spinless $p$-wave superconductor, if the Fermi energy lies between the two bands $\varepsilon_{\pm}^<(p)$, the system is effectively an one-dimensional spinless system. The effective low energy physics of the BdG Hamiltonian Eq. (3) then exactly maps to the one-dimensional spinless $p$-wave superconductor. To this end, we consider the effective Hamiltonian Eq. (2) without superconducting pair potential $\Delta_0 = 0$ and diagonalize it using a unitary transformation, $H_{\text{eff}} = U \mathcal{U}^\dagger$ where $\Lambda = \text{diag}(\varepsilon_+, \varepsilon_-, -\varepsilon_-, -\varepsilon_+)$ with the eigenvalues given by

$$\varepsilon_\pm = p^2 / 2m^* - \mu' \pm \sqrt{V_{\text{ex}}^2 + (\hbar \omega p / m^*)^2}$$

We then project the BdG Hamiltonian onto the Hilbert space of the eigenstates of the Hamiltonian $H_0$ without superconductivity using the same unitary transformation, $H_{\text{eff}} \rightarrow U^\dagger H_{\text{BdG}} U$, which yields

$$H_{\text{eff}} = \Lambda + \Delta_0 \cos \vartheta \tau_x - \Delta_0 \sin \vartheta \tau_y \sigma_x$$

where $\Delta_0$ is given by the relation $\sqrt{V_{\text{ex}}^2 + (\hbar \omega p / m^*)^2} \sin \vartheta = \hbar \omega p / m^*$. To study the Majorana zero modes of interest, it suffices to retain the
Majorana Zero Modes Solution.— The full BdG Hamiltonian Eq. (3) can be diagonalized by using a Bogoliubov transformation with the quasiparticle operators \( \alpha \) and \( \alpha^\dagger \) satisfying the anti-commutation relations for complex fermions. The corresponding self-adjoint Majorana operators \( \gamma_+ = (\alpha^\dagger + \alpha)/2 \) and \( \gamma_- = (\alpha^\dagger - \alpha)/2 \) are given by \( \gamma_\pm = \int dx [u_{\gamma\pm}(x) \Phi^\dagger_k(x) + u_{\gamma\mp}(x) \Phi_k(x)] \), where \( u_{\gamma\pm}(x) \) are the solutions to the BdG equations subject to boundary conditions and \( \pm \) denotes the solutions for the positive and negative energy bands, respectively. It can be easily demonstrated from the explicit expressions for the Majorana operators that \( \gamma_\pm \) are the Majorana wave function envelope decays exponentially with distance into the the bulk from the boundaries with localization length determined by the effective superconducting gap \( \Delta_0 \) and the effective superconducting coherence length \( \xi \sim \hbar v_F/\Delta \). The effective pairing gap \( \Delta \) can be obtained by linearizing \( \Delta(p) \) in Eq. (9) around \( p = 0 \) which gives \( \Delta = \hbar v_F (\omega \Delta_0)/V_{ex} \) and hence \( \xi \sim V_{ex}/(\omega \Delta_0) \) where \( \omega = \pi v_F/(2L) \). The overlap of the two Majorana states at both ends is proportional to \( e^{-L/\xi} \), hence increasing the winding number \( \nu \) makes the end states to be more independent as indicated in Figs. (b)-(d).
To gain deeper insight into the Majorana zero-energy modes, we solve the BdG equation analytically with higher modes ((c) and (d)). Decoupling this system of coupled second order differential equations yields a fourth order homogeneous differential equation for each \( u_\nu(\hat{x}) \), which in general has solutions of the form \( u_\nu(\hat{x}) \sim e^{m_{\nu} \hat{x}} \) and the corresponding characteristic equation for \( m \) reads,

\[
m^4 + 4(\mu' + \nu^2)m^2 + 8\nu\lambda\Delta_0 m + 4\mathcal{G} = 0
\]

(10)

where \( \mathcal{G} = \Delta_0^2 + \mu'^2 - \tilde{V}_{ex}^2 \). The roots of the polynomial should satisfy the following constraints,

\[
\prod_{n=1}^{4} m_n = 4\mathcal{G}, \quad \sum_{n=1}^{4} m_n = 0.
\]

(12)

It can be verified using Eq. (12) that for \( \mathcal{G} > 0 \) (\( \mathcal{G} < 0 \)), there are always an even (odd) number of solutions with positive real part \( \text{Re}(m) > 0 \), implying an even (odd) number of band crossings at the Fermi level. The boundary and normalizability conditions for a localized Majorana wave function solution can only be satisfied when \( \mathcal{G} < 0 \), permitting the zero-energy modes to exist. We can thus conclude that \( \mathcal{G} = 0 \) determines the topological quantum phase transition indicated by the closing and reopening the bulk gap at the topological critical point.

The presence of the zero-energy self-adjoint eigenstate thus relates directly to the sign of the gap \( \epsilon_{\text{gap}} \) and thus that of \( \mathcal{G} \). The sign of \( \mathcal{G} \) then characterizes the topological nature of the system and defines the associated topological invariant, which is a Z

\[
\mathcal{Z}(\mathcal{G}) = \text{sign}(\mathcal{G}) = \text{sign}(\Pi_n m_n) = \text{Pf}(\mathcal{H}_{\text{BdG}}),
\]

where \( \text{Pf} \) refers to the Pfaffian of the BdG Hamiltonian. We then deduce the topological phase diagram in the space of microscopic parameters \( V_{ex}, \Delta_0, \mu, \) and \( \nu \). The results are shown in Fig. 5. We note that the topologically nontrivial phase exists within extended portion of the phase diagram. Surprisingly, re-entrance phase transition occurs in the phase diagram, where by varying appropriate parameters, one can go from trivial to nontrivial and back to trivial phase or vice versa. To have actually Majorana modes, one should have both \( \mathcal{Q} = -1 \) and gapped bulk states. For the ferromagnetic limit \( \nu = 0 \), in the whole range where \( \mathcal{Q} = -1 \) bulk is gapless so it does not have topological phase while in the antiferromagnetic limit \( \nu = \infty \) the \( \mathcal{Q} \) never changes sign and we thus never cross the phase transition.

A stability analysis generalizing that in\(^{38}\) suggests that in order for the Majorana bound states to be stable, we have to choose the parameters to be such that \( 0 < |V_{ex} - \sqrt{\Delta_0^2 + \mu'^2}| < \Delta_0 \). This ensures that the central gap (at \( p = 0 \)) is nonzero and at the same time smaller than the two outer gaps which, in the limit of \( V_{ex} \ll \hbar \omega_F/m^* \) where \( \pm \omega_F \) are the locations of the two outer gaps, is given by \( \Delta(p_F) \approx \Delta_0 \). Considering the regime of dominant exchange field energy, which favors the nontrivial topological phase hosting MFs, the nonzero gap condition of the above criterion translates into \( |\mu'| < \sqrt{V_{ex}^2 - \Delta_0^2} \) or equivalently \( |V_{ex}| > \sqrt{\Delta_0^2 + \mu'^2} \).
Typical values of parameters in common metallic ferromagnets below its Curie temperature with thickness smaller than a critical value with easy-axis anisotropy along $x$ and very large $x\times z$ easy-plane anisotropy. The pitch of the domain wall $\lambda = L/\nu$ depends on the exchange stiffness $A$ and easy-axis anisotropy coefficient $K$ as $\lambda \sim \sqrt{A/K}$. Typical values of parameters in common metallic ferromagnets such as Co, Ni and Fe are $A \approx 100 \text{meV}/\text{Å}^2$, $K \approx 0.01 - 1 \mu \text{eV}/\text{Å}^3$, $V_{xc} \approx 1 - 10 \text{meV}$, which give $\lambda \approx 100 - 1000 \text{Å}$. The $\Delta_0 \approx 1 \text{meV}$ for elemental BCS superconductors such as Nb and Pb. This estimate readily puts our system to be within the MF-hosting nontrivial phase, requiring just few 10 mV’s gate voltage to tune the chemical potential and drive the system across the topological phase transition.

Conclusion.— We propose a very simple and easy-to-build system for realization of Majorana zero modes in a magnetic domain wall in proximity with a conventional superconductor. The Majorana fermions show themselves as localized zero-energy states at the interfaces between the domain wall and ferromagnets with uniform magnetization in opposite directions. The effective spin-orbit and Zeeman terms together with the isotropic superconducting pair potential lead to an effective anisotropic $p$-wave pairing capable of harboring Majorana fermions. Our results indicate that the nontrivial phase with its MFs is achievable in realistic situations without fine tuning while the topological quantum phase transition from the trivial phase with no Majorana zero mode to the topological phase with its localized zero mode is attainable by varying the gate voltage, making the proposal very feasible practically.

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