Transport in Hamiltonian systems with slowly changing phase space structure

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Abstract

Transport in Hamiltonian systems with weak chaotic perturbations has been much studied in the past. In this paper, we introduce a new class of problems: transport in Hamiltonian systems with slowly changing phase space structure that are not order one perturbations of a given Hamiltonian. This class of problems is very important for many applications, for instance in celestial mechanics. As an example, we study a class of one-dimensional Hamiltonians that depend explicitly on time and on stochastic external parameters. The variations of the external parameters are responsible for a distortion of the phase space structures: chaotic, weakly chaotic and regular sets change with time. We show that theoretical predictions of transport rates can be made in the limit where the variations of the stochastic parameters are very slow compared to the Hamiltonian dynamics. Exact asymptotic results can be obtained in the classical case where the Hamiltonian dynamics is integrable for fixed values of the parameters. For the more interesting chaotic Hamiltonian dynamics case, we show that two mechanisms contribute to the transport. For some range of the parameter variations, one mechanism -called "transport by migration together with the mixing regions"- is dominant. We are then able to model transport in phase space by a Markov model, the local diffusion model, and to give reasonably good transport estimates.

1 Introduction

Transport in Hamiltonian systems and is a classical field of dynamical system theory [1, 2, 3, 4], with a huge number of applications [5, 6]. Beyond deterministic dynamical systems, a lot of work has been devoted in the past to study the effect of random perturbations [7], more specifically on Hamiltonian systems and on area preserving maps, for instance in the context of plasma physics [8, 9]. Noise is always present in real natural systems and in experiments because of the effect of hidden chaotic degrees of freedom. Even of small amplitude, noise plays a very important role in the long term behavior of the dynamics, and on transport properties. One usually models the hidden degrees of freedom by an additional stochastic process of small amplitude acting on the system. For example, the effect of noise on the standard map or on other classical area preserving maps has been early studied [10, 11, 2], motivated by the dynamics of charged particles in accelerators. More recently, with the development of stochastic calculus, [12, 13] have studied the generic effect of small stochastic perturbations of Hamiltonian flows, and [14, 15] have derived a diffusion equation for the slow action variable in Hamiltonian systems. In particular, [12, 13] have rigorously justified the averaging principle in Hamiltonian systems and studied the slow diffusive motion of action variables. The important point is that all those works fall in the dynamical framework

\[ \dot{x} = F(x) + \epsilon \beta(x,t), \]

(1)

where \( F \) is a Hamiltonian vector field, \( \beta \) is a deterministic or a stochastic perturbation of the Hamiltonian vector field, and \( \epsilon \) is a small parameter. Qualitatively, we can describe the dynamics of (1) saying that it follows the regular or chaotic orbits of the unperturbed dynamics \( \dot{x} = F(x) \), and deviates slowly from those orbits because of the effect of the small perturbation \( \beta \).

In this paper, we introduce a new framework, which cannot be reduced to the much studied model (1), and still is essential for many applications. We study a one-dimensional Hamiltonian dynamics which Hamiltonian depends on a slow parameter

\[ \dot{x} = J \nabla H(x, \nu(t), t), \]

(2)

where \( x := (p, q) \) represents the vector of canonical variables, and \( J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).
Let us consider a simple example to illustrate such a change in phase space structure. We take the one degree of freedom Hamiltonian

\[ H(p,q,\nu,t) := \frac{p^2}{2} + \cos(q) + \cos(q-\nu t), \]

where \( \nu \) is a frequency and plays the role of the external parameter. In the Hamiltonian (3), a resonance is defined as the value of \( p \) for which one of the two angles \( q \) or \( q+\nu t \) has zero beating frequency. We have plotted in figure 1 a snapshot of the phase space structure for the Hamiltonian (3) for three different values of \( \nu \). The frequency \( \nu \) is decreasing from the left picture to the right picture. The Poincaré section described by (3) displays two major libration regions, one centered at \( p = 0 \), and one centered at \( p = \nu \). A libration region is defined as a region of phase space where the canonical angle \( q \) has bounded oscillations between two extremal values in \([0,2\pi]\). The libration regions can be very easily recognized on the Poincaré section of figure 1 as they look like "cat’s eyes" surrounding the fix points of resonances. The Hamiltonian flow is only very weakly chaotic in the initial state because the two main resonances are far away from each other, but it becomes more and more chaotic as the two main resonances become closer (for a precise description of such systems, see [2]). The red curves in figure 1 are particular trajectories that start inside the upper libration region. For large enough values of \( \nu \), the trajectories are trapped inside the upper libration region (first and second panel of figure 1). On the contrary, when \( \nu \) is lower than some critical value, the trajectory can freely transit from one cat’s eye to the other (third panel of figure 1). Imagine now that the frequency \( \nu \) in Hamiltonian (3) is no longer fixed, but slowly depends on time. Consider then the Hamiltonian

\[ H(p,q,\nu t), t := \frac{p^2}{2} + \cos(q) + \cos\left(q - \int_0^t \nu(s)ds\right), \]

where \( \nu t \) slowly evolves with time, the geometry and topology of those regions are slowly distorted, and can be dramatically changed for large variations of \( \nu \) that occur on long times. As a consequence of the distortion of the phase space structure, some regions that might not have been accessible for the system at \( \nu \) tori. Right panel: \( \nu \approx 5 \) and the two main cat’s eyes are now connected through a strongly chaotic region. KAM tori have been destroyed and the red trajectory can freely do the transition from one libration region to the other.

Generally the dynamics (1) can be very different from the dynamics (2) because the variations of \( \nu \) in Hamiltonian (2) are slow, but are not supposed to be small. For any fixed value of the parameter \( \nu \), for the Hamiltonian \( H(x,\nu,t) \), the dynamics in its phase space is characterized by strongly chaotic regions, weakly chaotic regions, and in some cases, KAM tori. We call phase space structure the geometry and topology of these chaotic, weakly chaotic, and regular areas. When the parameter \( \nu \) slowly evolves with time, the geometry and topology of those regions are slowly distorted, and can be dramatically changed for large variations of \( \nu \) that occur on long times. As a consequence of the slow changes in phase space structure, create a particular transport mechanism that is very different from the transport created by small perturbation as in system (1).
the present paper, we investigate the case \( \nu \) where the parameter \( \nu \) is stochastic. We show that transport in phase space is the result of two mechanisms. We call the first mechanism noise driven transport in regular regions. This transport mechanism comes from the strong irregularity of the stochastic trajectories \( \nu(t) \). We call the second mechanism transport by slow migration of chaotic regions. This second transport mechanism occurs when trajectories are trapped in chaotic regions and follow their displacement to access to other parts of phase space. For generic Hamiltonians \( \nu \), both mechanisms are present and contribute to transport. Our aim is to separately describe and quantify transport due to the one and the other mechanisms.

It is natural to describe long-term transport in systems following the dynamics \( \nu \) on the timescale \( \frac{1}{\epsilon} \). All over the paper, we thus consider the Hamilton’s equations

\[
\dot{x} = \frac{1}{\epsilon} J \nabla H(x, \nu(t), t, \epsilon)
\]

in the limit \( \epsilon \to 0 \). We first consider in section 2 the case of a one-degree of freedom Hamiltonian of the form \( H(p, q, \nu(t)) \), which has an integrable dynamics for any fixed value of \( \nu \). Would the parameter \( \nu \) be a regular function of time, transport would be described by the classical theory of adiabatic invariants \cite{16, 17}. For a stochastic parameter \( \nu \), we show that the transport is completely due to the mechanism of noise driven transport in regular regions, and we give a stochastic differential equation to describe the diffusion of adiabatic invariants in the limit \( \epsilon \to 0 \). This result is obtained using standard averaging techniques. We also illustrate our results in the two concrete examples of the harmonic oscillator and the pendulum with slow stochastic frequencies.

The second part 3 of the paper is probably the most original and interesting one. We deal with the much more difficult case where the Hamiltonian dynamics \( \nu \) has an explicit time dependance and is thus chaotic even for fixed values of \( \nu \). We show that both transport mechanisms, "noise driven transport in regular regions" and "transport by slow migration of chaotic regions", take place. We explain that the second mechanism is dominant for some range of parameters. The transport can then be reduced to a fully Markovian process in the limit \( \epsilon \to 0 \). We call this process the instantaneous local diffusion model, because it consists in modeling the chaotic regions of phase space by diffusive regions with infinite diffusion coefficient. We check numerically that the Markov model gives reasonably good estimation of transport rates when the mechanism of transport by slow migration of chaotic regions is dominant.

Our results are relevant in celestial mechanics to study the long term dynamics of the spin axis of planets, and the secular dynamics of Mercury. Some times ago \cite{18} already pointed out that the dynamics of the obliquity of Mars could be reduced to a model of a pendulum with slowly varying parameters. He used the theory of adiabatic invariants to estimate the probability of Mars to enter into libration. However, this model still considered that the parameters are regular functions of time. With the work of \cite{19, 20}, it has been shown that the solar system is chaotic on a timescale of few million years. The set of fundamental frequencies of the Laplace–Lagrange solution, that plays the role of parameters in the Hamiltonian describing the dynamics of spin axes of planets, has thus a stochastic long-term evolution \cite{21}. This justifies the relevance of a model such as \( \nu \). This paper was thought of as a theoretical work to put in a general framework the ideas developed in \cite{22} to study the long term dynamics of the Earth obliquity.

2 The integrable case

2.1 Formulation of the model and theoretical results

In this section, we consider a one-degree of freedom Hamiltonian \( H(p, q, \nu) \). The set \( (p, q) \) is the set of canonical variables, which dynamics follows the Hamilton’s equations. \( \nu \) is an external parameter in the Hamiltonian. For any fixed value of the parameter \( \nu \), the Hamiltonian has one degree of freedom, and its dynamics is thus integrable. When the parameter \( \nu \) is a regular, slowly varying function of time, the slow action dynamics is described by the old theory of adiabatic invariants. Even when transport occurs through separatrix crossing, the theory of adiabatic invariants can be extended to account for the discontinuity in the action definition (see \cite{23} for a review). We consider here a case where the parameter \( \nu \) is stochastic, and slowly varying with time, and we assume that no separatrix crossing occurs during transport. Let \( \epsilon \ll 1 \) be a small parameter that sets the time scale separation between the Hamiltonian dynamics and the stochastic dynamics of \( \nu \). The set of Hamilton’s equations that describes the dynamics of the canonical variables \( (p, q) \) and the parameter \( \nu \) on the timescale \( \frac{1}{\epsilon} \) is

\[
\begin{align*}
\dot{p} &= -\frac{1}{\epsilon} \frac{\partial H}{\partial q}(p, q, \nu(t)) \\
\dot{q} &= \frac{1}{\epsilon} \frac{\partial H}{\partial p}(p, q, \nu(t))
\end{align*}
\]
and
\[ d\nu = a(\nu)dt + b(\nu)dW. \] (7)

In the stochastic differential equation (7), the stochastic product \( b(\nu)dW \) is defined with the It\(\hat{\text{A}}\)Z convention of stochastic calculus (see [21] chapter 4).

As the Hamiltonian \( H(p,q,\nu) \) is integrable for fixed \( \nu \), we can find a set of action-angle canonical variables \((P,Q)\). With this change of variables, the new Hamiltonian \( \tilde{H}(P,\nu) \) does not depend on \( Q \). If the value of \( \nu \) is fixed, the action variable \( P(p,q,\nu) \) is a constant of motion under the dynamics of the Hamiltonian \( \tilde{H} \). In the model defined by equations (10), the parameter \( \nu \) is slowly varying with time, and follows a stochastic differential equation. This implies that the action variable \( P \) also follows a stochastic differential equation, on the timescale \( \frac{1}{\epsilon} \).

With the principles of It\(\hat{\text{A}}\)Z stochastic calculus, the stochastic differential equations on \( P \) and \( Q \) write
\[
\begin{align*}
dP & = -\frac{1}{\epsilon} \frac{\partial P}{\partial p} \frac{\partial H}{\partial q} dt + \frac{1}{\epsilon} \frac{\partial P}{\partial q} \frac{\partial H}{\partial p} dt + \frac{1}{2} \frac{\partial^2 P}{\partial p \partial q} b^2(\nu) dt, \\
dQ & = -\frac{1}{\epsilon} \frac{\partial Q}{\partial p} \frac{\partial H}{\partial q} dt + \frac{1}{\epsilon} \frac{\partial Q}{\partial q} \frac{\partial H}{\partial p} dt + \frac{1}{2} \frac{\partial^2 Q}{\partial p \partial q} b^2(\nu) dt.
\end{align*}
\] (8)

Because the set \((P,Q)\) is a set of action-angle variables, the term \(-\frac{\partial P}{\partial p} \frac{\partial H}{\partial q} dt + \frac{1}{\epsilon} \frac{\partial P}{\partial q} \frac{\partial H}{\partial p} dt \) vanishes. This comes as a consequence of the fact that \( P \) is constant if \( \nu \) is constant. The term \(-\frac{\partial Q}{\partial p} \frac{\partial H}{\partial q} dt + \frac{1}{\epsilon} \frac{\partial Q}{\partial q} \frac{\partial H}{\partial p} dt \) defines the dynamics of the angle variable \( Q \) when \( \nu \) is fixed, and thus
\[ -\frac{\partial Q}{\partial p} \frac{\partial H}{\partial q} dt + \frac{\partial Q}{\partial q} \frac{\partial H}{\partial p} dt = \omega(P,\nu) dt, \]
where we have defined the pulsation \( \omega(P,\nu) := \frac{\partial \tilde{H}}{\partial \nu}(P,\nu) \). Equations (8) become
\[
\begin{align*}
dP & = \left[ \frac{\partial P}{\partial \nu} a(\nu) + \frac{1}{2} \frac{\partial^2 P}{\partial p \partial q} b^2(\nu) \right] dt + \frac{1}{\epsilon} \frac{\partial^2 P}{\partial p \partial q} b(\nu) dW, \\
dQ & = \frac{1}{\epsilon} \omega(P,\nu) dt + \left[ \frac{\partial Q}{\partial \nu} a(\nu) + \frac{1}{2} \frac{\partial^2 Q}{\partial p \partial q} b^2(\nu) \right] dt + \frac{1}{\epsilon} \frac{\partial^2 Q}{\partial p \partial q} b(\nu) dW,
\end{align*}
\] (9)
and
\[ d\nu = a(\nu) dt + b(\nu) dW. \]

The set \((P,Q,\nu)\) evolves according to a slow-fast dynamics: the angle variable \( Q \) evolves on a timescale of order one, whereas the variables \( P \) and \( \nu \) are slow variables, evolving on a timescale of order \( \frac{1}{\epsilon} \). Our aim is to average the system (9) over the dynamics of \( Q \) to obtain a close system of equations describing the dynamics of \((P,\nu)\) over the timescale \( \frac{1}{\epsilon} \).

Before doing the averaging procedure, we first recall a very classical result of Hamiltonian systems with slow time dependance (see e.g. [17] or [2] section 2.3). There exists a smooth function \( H_1(P,Q,\nu) \) such that the differential of \((P,Q)\) with respect to \( \nu \) can be expressed as
\[
\begin{pmatrix}
\frac{\partial P}{\partial \nu} \\
\frac{\partial Q}{\partial \nu}
\end{pmatrix} = -\frac{\partial H_1}{\partial Q}, \\
\frac{\partial \nu}{\partial \nu} = \frac{\partial H_1}{\partial \nu}. \] (10)

We propose a simple proof of this result in Appendix A using differential two-forms. With the important result (10), we can compute the second order differential of \( P \) using the differentials of \( H_1 \). We define the canonical Poisson brackets of any functions \( f(p,q) \) and \( g(p,q) \) by
\[ \{f,g\}_{p,q} := \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}. \]

We have
\[ \frac{\partial^2 P}{\partial \nu^2} = \frac{\partial}{\partial \nu} \left( -\frac{\partial H_1}{\partial Q} \right) \]
\[ = -\frac{\partial^2 H_1}{\partial P \partial Q} \frac{\partial P}{\partial \nu} - \frac{\partial^2 H_1}{\partial Q^2} \frac{\partial Q}{\partial \nu} + \frac{\partial^2 H_1}{\partial \nu \partial Q}, \]
and using once more (10) we obtain
\[ \frac{\partial^2 P}{\partial \nu^2} = -\left\{ H_1, \frac{\partial H_1}{\partial Q} \right\}_{p,q} - \frac{\partial^2 H_1}{\partial \nu \partial Q}. \]
with \( \left\{ H_1, \frac{\partial H_1}{\partial \xi} \right\}_{P,Q} \) the canonical Poisson bracket with the set of variables \((P,Q)\). A similar computation leads to

\[
\frac{\partial^2 Q}{\partial \nu^2} = \left\{ H_1, \frac{\partial H_1}{\partial P} \right\}_{P,Q} + \frac{\partial^2 H_1}{\partial \nu \partial P}. \]

We write the slow-fast system of equations (9) as

\[
\begin{align*}
\frac{dQ}{d\nu} &= \frac{1}{\epsilon}(P, \nu)dt + \text{terms of order 0 in } \frac{1}{\epsilon}, \\
\frac{dP}{d\nu} &= -\frac{\partial H_1}{\partial Q} \sigma(\nu) - \frac{1}{\epsilon} \left( \left\{ H_1, \frac{\partial H_1}{\partial Q} \right\}_{P,Q} + \frac{\partial^2 H_1}{\partial \nu \partial Q} \right) b^2(\nu) dt - \frac{\partial H_1}{\partial \nu} b(\nu) dW, \\
\frac{d\nu}{d\nu} &= a(\nu)dt + b(\nu)dW.
\end{align*}
\]  

We average now the dynamics of \( P \) over the fast dynamics of the angle \( Q \). To leading order in \( \frac{1}{\epsilon} \), the dynamics of \( Q \) is simply

\[
\dot{Q} = \frac{1}{\epsilon}(P, \nu).
\]

Therefore, the invariant measure of this dynamics, for any fixed values of \( P \) and \( \nu \), is just the constant measure over the range \([0, 2\pi]\). To find the limit stochastic process for \( P \) when \( \epsilon \) goes to zero, we have to average equation (11) using the invariant measure of \( Q \). Some terms are very easy to compute. Let \( \langle \cdot \rangle_Q := \frac{1}{2\pi} \int_0^{2\pi} \cdot \, dQ \) be the averaging operator over the fast dynamics of \( Q \), we have

\[
\begin{align*}
\langle \frac{\partial H_1}{\partial Q} \rangle_Q &= 0 \quad \text{and} \quad \langle \frac{\partial^2 H_1}{\partial \nu \partial Q} \rangle_Q = 0.
\end{align*}
\]

In the deterministic part on the equation for \( P \), the only nonzero dependence comes from the average of the Poisson bracket \( \langle \left\{ H_1, \frac{\partial H_1}{\partial Q} \right\} \rangle_Q \).

The average of the stochastic term \( \frac{\partial H_1}{\partial Q} b(\nu) dW \) should be done cautiously, because it is coupled to the stochastic term \( b(\nu) dW \) in the equation for \( \nu \). To average over \( Q \), we introduce first the matrix

\[
\sigma := \left( \begin{array}{c} b(\nu) \\ b(\nu) \frac{\partial H_1}{\partial Q} \end{array} \right).
\]

The noise term acting on the set of slow variables \((\nu, P)\) can be written as \( \sigma dW \), where \( W \) is a Wiener process. We then use the standard result that the noise term

\[
\sigma \left( Q \left( \frac{t}{\epsilon} \right) \right) dW
\]

is equivalent (for weak convergence) when \( \epsilon \to 0 \) to a Gaussian white noise process, the variance of which is given by the averaged correlation matrix \( \langle \sigma \sigma^T \rangle_Q \) (see e.g. [12] chapter 8). The correlation matrix of the noise is

\[
\sigma \sigma^T = \begin{pmatrix} b^2(\nu) & b^2(\nu) \frac{\partial H_1}{\partial Q} \\ b^2(\nu) \frac{\partial H_1}{\partial Q} & b^2(\nu) \left( \frac{\partial H_1}{\partial Q} \right)^2 \end{pmatrix},
\]

and averaging over \( Q \) gives

\[
\begin{pmatrix} \langle \sigma \sigma^T \rangle_Q = \begin{pmatrix} b^2(\nu) & 0 \\ 0 & b^2(\nu) \left( \langle \frac{\partial H_1}{\partial Q} \rangle_Q \right)^2 \end{pmatrix} \end{pmatrix}.
\]

The important consequence of (12) is that the average over the fast dynamics eliminates the correlations between the noise terms in the equations of \( P \) and \( \nu \).

We can now present our main theoretical result. The dynamics of the slow process \((P, \nu)\) follows when \( \epsilon \) goes to zero the averaged equations

\[
\begin{align*}
\frac{dP}{d\nu} &= -\frac{1}{2} b^2(\nu) \left\langle \left\{ H_1, \frac{\partial H_1}{\partial Q} \right\} \right\rangle_Q dt + b(\nu) \sqrt{\left\langle \left( \frac{\partial H_1}{\partial Q} \right)^2 \right\rangle_Q} dW_1, \\
\frac{d\nu}{d\nu} &= a(\nu)dt + b(\nu) dW_2.
\end{align*}
\]
There are some interesting comments to do on equations (13). First, we note that the Wiener processes \( W_1 \) and \( W_2 \) involved in the two equations on \( P \) and \( \nu \) are independent. This beautiful property is a consequence of the relation \( \frac{\partial P}{\partial \nu} = -\frac{\partial H_1}{\partial Q} \). It would not be the case if we had chosen to study the dynamics of another slow variable of the system, for example the Hamiltonian \( H \). Also, we observe that the diffusion of the action \( P \) only comes from the stochastic part of the equation for \( \nu \), the coefficient \( b(\nu) \). For a smooth function \( \nu(t) \), the action cannot diffuse. This is in accordance with the theory of adiabatic invariants, that states that for a smooth time dependance of \( \nu(\epsilon t) \), there exists an adiabatic invariant conserved to any order in \( \epsilon \) (This result was found by [16] for the harmonic oscillator, and [17] in the general case). In the case where such an invariant exists, the action cannot diffuse.

An other interesting remark is that the stochastic equation (13) for \( P \) can be written in a more compact way. From the relation

\[
\frac{\partial}{\partial Q} \left( \frac{\partial H_1}{\partial P}, \frac{\partial^2 H_1}{\partial Q^2} \right) = \frac{\partial H_1}{\partial P} \frac{\partial^2 H_1}{\partial Q^2} + \frac{\partial^2 H_1}{\partial Q \partial P} \frac{\partial H_1}{\partial Q} \]

averaged over \( Q \), we get

\[
\left\langle \frac{\partial H_1}{\partial P}, \frac{\partial^2 H_1}{\partial Q^2} \right\rangle_Q = -\left\langle \frac{\partial^2 H_1}{\partial Q \partial P} \frac{\partial H_1}{\partial Q} \right\rangle_Q. \tag{14}
\]

Using (14), equation (13) for \( P \) can be equivalently written

\[
dP = b^2(\nu) \frac{\partial}{\partial P} \left\langle \left( \frac{\partial H_1}{\partial Q} \right)^2 \right\rangle_Q \, dt + b(\nu) \sqrt{\left\langle \left( \frac{\partial H_1}{\partial Q} \right)^2 \right\rangle_Q} \, dW_1.
\]

The last equation corresponds exactly to the stochastic equation

\[
dP = b(\nu) \sqrt{\left\langle \left( \frac{\partial H_1}{\partial Q} \right)^2 \right\rangle_Q} \times dW_1, \tag{15}
\]

where the stochastic product \( \times dW \) should be understood with the anti-Itô convention (also called sometimes the Hänggi convention).

In the following sections 2.2-2.3, we present two simple applications of the theoretical result (13). The first one is the well-known harmonic oscillator with random frequency. For this simple example, everything can be computed analytically. The second application, a bit more technical, is the pendulum with random frequency.

### 2.2 Application to the harmonic oscillator

The Hamiltonian of the harmonic oscillator is

\[
H(p, q, \nu) = \frac{1}{2} p^2 + \frac{1}{2} \nu^2 q^2. \tag{16}
\]

We consider that the frequency of the oscillator \( \nu \) is a parameter in the Hamiltonian (16), which time evolution is given by the Itô stochastic differential equation

\[
d\nu = a(\nu)dt + b(\nu)dW. \tag{17}
\]

The small parameter \( \epsilon \ll 1 \) sets the timescale separation between the Hamiltonian dynamics, and the dynamics of the random frequency. Equations (16-17) define our model.

We introduce the classical action-angle variables \((P, Q)\) defined by the relations

\[
p = \sqrt{2\nu P} \cos Q, \quad q = \sqrt{\frac{2P}{\nu}} \sin Q, \tag{18}
\]

and the new Hamiltonian \( \tilde{H}(P, \omega) \) simply writes

\[
\tilde{H}(P, \nu) = \nu P.
\]
To use the result (13), we have to find the expression of $H_1$ as a function of $P, Q$ and $\nu$. From the expression $P = \frac{\nu^2}{2} + \frac{\nu q^2}{2}$ we compute

$$\frac{\partial P}{\partial \nu} = -\frac{1}{2\nu^2} \nu^2 + \frac{1}{2} q^2$$
$$= -\frac{1}{\nu} P \cos^2 Q + \frac{1}{\nu} P \sin^2 Q$$
$$= -\frac{\partial}{\partial Q} \left[ \frac{1}{2\nu} P \sin (2Q) \right].$$

A straightforward calculation also shows that

$$\frac{\partial Q}{\partial \nu} = \frac{1}{2\nu} \sin (2Q)$$
$$= \frac{\partial}{\partial P} \left[ \frac{1}{2\nu} P \sin (2Q) \right].$$

This gives the function $H_1(P, Q, \nu)$ (section 2.3)

$$H_1 = \frac{1}{2\nu} P \sin (2Q).$$

In the case of the harmonic oscillator, equations (13) writes

$$\begin{cases}
\mathrm{d}P = b(\nu) \frac{P}{\nu^2} \mathrm{d}t + b(\nu) \frac{P}{\sqrt{2\nu}} \mathrm{d}W_1, \\
\mathrm{d}\nu = a(\nu) \mathrm{d}t + b(\nu) \mathrm{d}W_2.
\end{cases}$$

In order to illustrate the result (19), we perform a numerical simulation. We have to choose a stochastic process. As an example, we choose a process such that the frequency $\nu$ is always strictly positive. We thus set $b(\nu) = \sqrt{2\sigma^2}$, where $\sigma$ is a constant parameter, and $a(\nu) = -\nabla V(\nu)$, where the potential $V(\nu) = \frac{1}{\nu} + \frac{1}{\nu_{max} - \nu}$ is chosen such that the frequency $\nu$ is trapped in the range $[0, \nu_{max}]$.

We integrate Hamilton’s equations of motion

$$\begin{cases}
\dot{q} = \frac{1}{\epsilon} p, \\
\dot{p} = -\frac{1}{\epsilon} q,
\end{cases}$$

using a symplectic integrator of order 2. And we integrate simultaneously the stochastic equation for $\nu$

$$\mathrm{d}\nu = \left[ \frac{1}{\nu^2} - \frac{1}{(\nu_{max} - \nu)^2} \right] \mathrm{d}t + \sqrt{2\sigma^2} \mathrm{d}W,$$

using a stochastic Euler algorithm. The integration is done over 10,000 realizations of the stochastic frequency, and the same initial conditions $(p_0, q_0) = (1, 0)$, $\nu_0 = \frac{\nu_{max}}{2}$ for each trajectory. The parameters are $\nu_{max} = 2.0$, $\sigma = 0.3$ and $\epsilon = 0.01$. The histogram of the action $P = \frac{\nu^2}{2} + \frac{\nu q^2}{2}$ is represented at different times on figure (2) by the histograms.

Secondly, we perform 10,000 integrations of the averaged equations

$$\begin{cases}
\mathrm{d}P = \sigma^2 \frac{P}{\nu^2} \mathrm{d}t + \sigma \frac{P}{\nu} \mathrm{d}W_1, \\
\mathrm{d}\nu = \left[ \frac{1}{\nu^2} - \frac{1}{(\nu_{max} - \nu)^2} \right] \mathrm{d}t + \sqrt{2\sigma^2} \mathrm{d}W_2,
\end{cases}$$

with the same parameters and initial conditions. The histogram of the action is represented by the curve on figure (2). Both agree up to sampling errors, as expected. The main interest of using the averaged equations to compute a PDF is of course the drastic reduction of the time of integration. Moreover, one can then recover the PDF of the initial variables $(p, q)$ using the change of variable and the fact that the PDF of $Q$ is uniform over $[0, 2\pi]$. 

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2.3 Application to the pendulum

The pendulum with random frequency is much more involved than the harmonic oscillator, because the canonical action \( P \) cannot be expressed with usual functions. The pendulum is defined by the Hamiltonian

\[
H(p, q, \nu) = \frac{1}{2}(p - \nu)^2 + \cos(q),
\]

and the frequency \( \nu \) evolves with the Itô stochastic equation

\[
d\nu = a(\nu)dt + b(\nu)dW.
\]

The phase space of the pendulum is displayed in figure (3). The eye of the libration domain is located at \( p = \nu \).

With the stochastic evolution of \( \nu \), the eye goes up and down, and the orbits in the circulation domain are slowly distorted. The distortion can be very large, because we do not constrain the fluctuations of \( \nu \) to be small. Because of the stochastic distortion of the orbits, the trajectories are no longer constrained on a close orbit in phase space, and diffusion of the action occurs.

The action of the pendulum has two different definitions depending if the system is located in the circulation domain or in the libration domain. For simplicity, we will only consider trajectories in the circulation domain, and we will not treat the problem of crossing the separatrix where a singularity in the diffusion coefficient of the action occurs. The specific problem of separatrix crossing has already been investigated in \cite{2} section 5.7.

In the circulation domain, the canonical action is defined as \cite{25}

\[
P(p, q, \nu) := \frac{1}{2\pi} \int_{H(p', q', \nu) = H(p, q, \nu)} p' dq'.
\]

(21)

In the integral of (21), the integration of the variables \((p', q')\) is performed over the curve of equation \( H(p', q', \nu) = H(p, q, \nu) \). \( P \) is thus the area of phase space below one level curve of the Hamiltonian. For convenience, let us introduce the function

\[
f(E) = \frac{\sqrt{2}}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{E - \cos(q')}} dq'.
\]

Then with the expression of the Hamiltonian \cite{20}, it is easy to see that the action (21) writes

\[
P(p, q, \nu) = \nu + f(H(p, q, \nu)).
\]

(22)

The function \( f \) is very useful because its derivative is related to the period \( T \) of the pendulum by

\[
f'(E) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{\sqrt{2\sqrt{E - \cos(q')}}} dq' = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{q'} dq' = \frac{T}{2\pi}.
\]
Figure 3: Representation of the orbits in phase space for the pendulum. The separatrix is displayed in red. When the libration domain centered at $p = \nu$ moves up and down, the orbits are distorted by the displacement of the cat’s eye.

The computation of the coefficients in the stochastic equation for the action is a bit tricky, because we do not have the explicit expression of the function $H_1(P,Q,\nu)$. Therefore the averaging should be done directly on the expressions $\frac{\partial P}{\partial \nu}$ and $\frac{\partial^2 P}{\partial \nu^2}$ of the derivatives of $P$. To spare the reader lengthy calculations, we give directly the result of the averaging procedure. All details can be found in Appendix B. The stochastic equations for the slow variables $(P,\nu)$ are

$$
\begin{align*}
\text{d}P &= \frac{\sigma^2(\nu)}{2} \left[ (P - \nu) \frac{T(P)}{2\pi} \right] \text{d}t + b(\nu) \sqrt{(P - \nu) \frac{T(P)}{2\pi} - 1} \text{d}W_1, \\
\text{d}\nu &= a(\nu) \text{d}t + b(\nu) \text{d}W_2.
\end{align*}
$$

One should be careful that in equation (23), the period of the pendulum $T(P)$ should be considered as a function of the action $P$ and not of the Hamiltonian $H$. Another remark is that the expression of the diffusion coefficient $(P - \nu) \frac{T(P)}{2\pi} - 1$ in front of the Wiener process is physically very consistent. For large values of $P$, the pulsation of the pendulum is asymptotically $P - \nu$, and thus $(P - \nu) \frac{T(P)}{2\pi} - 1$ goes to zero. Away from the libration domain, there is no diffusion of the action, because the orbits of the pendulum are just straight lines, and are not distorted by any variation of $\nu$. On the other hand, the diffusion coefficient has a singularity at the separatrix, because the period $T(P)$ goes to infinity. This is a manifestation of the break down of equation (23) at the separatrix, and that it cannot describe properly the separatrix crossing.

To illustrate equations (23), we perform again a numerical simulation. We choose for the equation on $\nu$ the coefficients $b(\nu) = \sqrt{2\sigma^2}$ and $a(\nu) = -\nabla V(\nu)$ with a symmetric potential around zero $V(\nu) := \frac{1}{\nu + \nu_{\text{max}}} + \frac{1}{\nu_{\text{max}} - \nu}$. The frequency $\nu$ has thus random variations in the range $[-\nu_{\text{max}}, \nu_{\text{max}}]$.

We integrate Hamilton’s equations of motion

$$
\begin{align*}
\dot{q} &= \frac{(p - \nu)}{\epsilon}, \\
\dot{p} &= \frac{1}{\epsilon} \sin(q),
\end{align*}
$$

using a symplectic integrator of order 2. And we integrate simultaneously the stochastic equation for $\nu$

$$
\text{d}\nu = \left[ \frac{1}{(\nu + \nu_{\text{max}})^2} - \frac{1}{(\nu_{\text{max}} - \nu)^2} \right] \text{d}t + \sqrt{2\sigma^2} \text{d}W;
$$

using a stochastic Euler algorithm. The integration is done over 10,000 realizations of the stochastic frequency, and the same initial conditions $(p_0, q_0) = (8.0, \pi)$, $\nu_0 = 0$ for each trajectory. The initial condition is taken far enough
from the separatrix such that no trajectory crosses the separatrix during the time of integration. The parameters are $\nu_{\text{max}} = 2.0$, $\sigma = 1.0$ and $\epsilon = 0.01$. The PDF of the action $P = \nu + f(H)$ is represented at different times on figure (4) by the histograms.

Secondly, we perform 10,000 integrations of the averaged equations

$$
\begin{align*}
\frac{dP}{dt} &= \sigma \frac{d}{dP} \left[ \frac{T(P)}{2\pi} \right] dt + \sqrt{2\sigma^2} \sqrt{\frac{T(P)}{2\pi}} - 1dW_1, \\
\frac{d\nu}{dt} &= \frac{1}{(\nu + \nu_{\text{max}})} - \frac{1}{(\nu_{\text{max}} - \nu)} \frac{\epsilon}{(\nu_{\text{max}} - \nu)} dt + \sqrt{2\sigma^2} dW_2,
\end{align*}
$$

with the same parameters and initial conditions. The action PDF is represented by the curve on figure (4).

3 The chaotic case

3.1 Formulation of the model

We now turn to a more difficult situation. We study the dynamics (5) where the Hamiltonian $H(p, q, t, \nu)$ has an explicit time dependance. We assume that the explicit time dependance in $H$ creates many resonances overlap, and thus the Hamiltonian dynamics for fixed $\nu$ has both regular and chaotic trajectories. The phase space is divided into regions with different mixing properties. In the regions in $(p, q)$-space where at least two resonances overlap, the dynamics is strongly chaotic. In regions of phase space that are far away from the main resonances, the dynamics is almost regular. KAM-tori may subsist or not, depending on the amplitude and localization of the resonant frequencies. The parameter $\nu$ is assumed to be stochastic, and evolving on a time scale of order one, whereas the canonical variables evolve on the small timescale $\epsilon \ll 1$. We want to characterize transport in phase space on the slow timescale.

By contrast with the simplest integrable case of section 2, there is now no global action variable. The transport cannot be described by a diffusive equation of some kind of action variable. What we will explain in the present section is that the transport is very different in the present case where the dynamics of $H$ is chaotic. In the chaotic case, there are two competitive transport mechanisms, that can be qualitatively described as follows:

1. The chaotic regions are moving in phase space. If the system is in one of those chaotic regions, it is carried together with the region upwards or downwards. At any time, it can leave the region and enter in the regular part of phase space. But depending on when the system leaves the region, it may be carried up or down far away from its initial position. We refer to this mechanism as “transport by migration with chaotic regions”.

2. The regular regions far away from the resonances are very similar to the integrable Hamiltonian dynamics of section 2. The slow stochastic variations of $\nu$ distort the orbits, and enhances the chaotic diffusion of these regions. As a result, no KAM tori can subsist and the system diffuses through phase space on a time scale of order one. We refer to this mechanism as “noise driven transport in regular regions”.

Depending on the parameters of the Hamiltonian, one of the two mechanisms described above may overcome the other. This section provides a concrete example of a Hamiltonian where the two mechanisms of transport are present. We also determine in which parameter regime the transport by migration with chaotic regions is dominant.
To complete our Hamiltonian model (24), we need to specify the stochastic process for the set of frequencies. To this end, we choose to prescribe the stochastic process for the set of initial angles \(\varphi_k\) and timescale separation \(\epsilon\). The Hamiltonian (24) together with the stochastic equation (25) completely defines our model. In the simulations of the dynamics (24) with Hamiltonian (24), the parameters \(\varphi_k\), the timescale separation \(\epsilon\) and the mean frequencies \(\nu_k^*\) were fixed as given in Table 1, whereas the amplitude \(\sigma\) is a control parameter that we changed in the different simulations.

In Figure 5, we have represented two pictures to help the reader understand the chaotic structure of phase space. On the left, we have represented the cat’s eyes of the main resonances in red. All the regions of phase space covered by the eyes are strongly chaotic. The two first order resonances appear with amplitude 1 in the Hamiltonian (24), and thus create cat’s eyes with extension 4. The second order resonances create a chaotic region around \(p = 5\), and its extension can be found using Lie transformations of the Hamiltonian (24). We found that the second order resonances come with typical amplitude \(\sqrt{2}/\nu_i^*\) and have thus an extension \(4\sqrt{2}/\nu_i^* \approx 0.56\). Those three chaotic regions are separated by two regular bands, in which transport is only possible either through the weak chaos created by higher order resonances, that are of much smaller amplitudes, or through noise driven transport in regular regions. On the right of Figure 5, we have represented the bands associated with the chaotic regions of first and second orders. For convenience, we have labelled the regions. Regions 1 and 2 are the chaotic regions of first order resonances, and region 3 is the chaotic region of second order resonances. The regions of weak chaos, which we also call the “regular” regions, are labelled as regions 4 and 5. Within the bands 1, 2 and 3, chaos is strongly developed, and the system is thus rapidly carried throughout the band on the timescale \(\epsilon\). The bands are separated by regions 4 and 5 of much weaker chaos: for fixed values of the frequencies \(\nu\), the system can hardly cross those regular regions, and therefore the migration between one band to the other is very slow.

The important point to emphasize is that Figure 5 is not static: the set of four frequencies \(\nu\) is slowly varying around its mean values \(\nu^*\), with stochastic variations defined by equation (25). This slow variation of the frequencies imply that the transport in phase space does not only happen through the well known "chaotic diffusion" observed in chaotic maps (e.g. the standard map, see [2] chapter 5). It comes from two other mechanisms that we have previously referred to as "transport by migration with chaotic regions" and "noise driven transport in regular regions". We look in section 3.4 at the problem of first passage time at \(p = 0\) starting at \(p = 10\). We will see that the first mechanism, that is, transport by migration with the chaotic regions, is dominant for large noise amplitude.

Let us assume for example that we are in a range of parameters for which the transport is mainly due to migration with the chaotic regions. The system starts at \(p = 10\). It can reach the value \(p = 0\) through successive

| Frequencies \(\nu_k^*(s^{-1})\) | Initial angles \(\varphi_k\) | Timescale separation \(\epsilon\) |
|-------------------------------|-----------------------------|-------------------------------|
| 10.0                          | 0.0                         | \(10^{-2}\)                   |
| 9.9                           | \(\pi\)                     |                               |
| 0.1                           | \(\pi\)                     |                               |
| 0.0                           | 0.0                         |                               |

Table 1: Fixed parameters of the model (24-25)

We propose to study the Hamiltonian dynamics (5), where the Hamiltonian depends on a set of four frequencies \(\nu := (\nu_1, \nu_2, \nu_3, \nu_4)\) plays the role of the external parameters. The frequencies are divided in two groups of two frequencies which create resonance overlap, and create two main chaotic regions around \(p = 0\) and \(p = 10\) respectively. The parameters \(\{\varphi_k\}_{k=1,4}\) are some initial phases, and \(\epsilon\) is a small parameter to model the timescale separation between the fast dynamics of the canonical variables and the stochastic dynamics of the frequencies. To complete our Hamiltonian model (24), we need to specify the stochastic process for the set of frequencies \(\nu\). We choose for the variations of \(\nu\) an Ornstein-Uhlenbeck process defined by

\[
d\nu = -(\nu - \nu^*)dt + \sqrt{2\sigma^2}dW. \tag{25}
\]

In Equation (25), \(W(t)\) is a 4-dimensional Wiener process. For simplicity, in order to keep the size of the stochastic regions constant, we choose to prescribe the same noise for two frequencies of the same set, that is, \(W_1 = W_2\) and \(W_3 = W_4\). The correlation function of the noise is \(\langle dW_1(t)dW_3(t') \rangle = 0\) and \(\langle dW_1(t)dW_1(t') \rangle = \langle dW_2(t)dW_2(t') \rangle = \delta(t-t')dt\). The parameter \(\sigma\) quantifies the noise amplitude and is the same for all frequencies. The term \(-(\nu - \nu^*)\) keeps the frequencies close to their averaged values defined by the set \(\nu^*\).

The Hamiltonian (24) together with the stochastic equation (25) completely defines our model. In the simulations of the dynamics (5) with Hamiltonian (24), the parameters \(\varphi_k\), the timescale separation \(\epsilon\) and the mean frequencies \(\nu_k^*\) were fixed as given in Table 1, whereas the amplitude \(\sigma\) is a control parameter that we changed in the different simulations.
Figure 5: Schematic representation of the chaotic structure of phase space for the Hamiltonian (24). The left panel represents the eyes of the resonances of first and second order in the system. Resonance overlap creates strongly chaotic regions in phase space. The chaotic regions are represented by the bands on the right panel. The arrows indicate that the chaotic regions are slowly moving with time.

Jumps from one region to the other, until it eventually reaches the chaotic region around the frequency \( \nu_1^* = 0 \). To illustrate the transport mechanism, we have represented in figure (6) the different steps of the transport. We have labelled the chaotic and regular regions the same way as in figure (5). First, one downward fluctuation of the couple of frequencies \((\nu_1, \nu_2)\) brings together the chaotic region 1 and the system around \( p = 7 \). Then, region 1 moves upward again, but the system leaves the chaotic region and is thus trapped in region 4. A simultaneous displacement upwards of the frequencies \( \nu_3 \) and \( \nu_4 \) brings the region 3 of second order resonances upwards, and it captures the system. The system has thus passed from region 1 to region 3 thanks to stochastic migration with the chaotic regions. It then passes from region 3 to region 2 by a similar trajectory mechanism: it is transported downwards and left in region 5 and an eventual upward displacement of region 2 captures it and brings it to \( p = 0 \).

Figure (7) displays an example of a trajectory starting at \( p = 10 \) and reaching \( p = 0 \). The example of figure (7) shows that transport is not straightforward to \( p = 0 \). The system can be captured and released many times by a chaotic region before being captured by another chaotic region. In the example of figure (7), the system spends most of the time in the vicinity of region 1, and the transition to regions 3 and 2 only occurs at the very end of the trajectory.

The mechanism of transport we just described, composed of successive jumps between regions of different types, is typical to go from \( p = 10 \) to \( p = 0 \) when chaotic diffusion in the regular regions is negligible. As the reader would surely have noticed, it is not important where the system is exactly located when it is inside a chaotic region of type 1, 2 or 3. The mixing in those strongly chaotic regions is so fast compared to the timescale of frequency variations, that the system has time to explore the whole region before any significant displacement of the region. To say it another way, the underlying Hamiltonian dynamics is not important to determine the transport characteristics. In fact, there are only two properties of the dynamics that matter. The first one is the conservation of area which is characteristic of Hamiltonian dynamics (the Hamiltonian flow is a symplectic transformation). The second one is that the phase space is partitioned into several strongly mixing regions (where chaos is strongly developed), separated by regular regions. Those two properties prompted us to perform a kind of “averaging” of the Hamiltonian dynamics and build an even simpler, fully stochastic model, that we call the local diffusion model.

### 3.2 Averaging of the dynamics: the local diffusion model

The local diffusion model is a purely stochastic model built from the Hamiltonian model (24) with stochastic frequencies (25). The idea is to average the dynamics over an intermediate timescale which is much longer than the timescale of the Hamiltonian dynamics, but much smaller than the timescale of stochastic variations of the frequencies. We use the hypothesis that the regions of strong chaos are also mixing. As the timescale of the Hamiltonian dynamics is typically of order \( \frac{1}{\nu_1^*} \), the averaging procedure is done over a time \( \tau_{av} \) satisfying

\[
\epsilon \frac{1}{\nu_1^*} \ll \tau_{av} \ll \frac{1}{\nu_1^*}. \tag{26}
\]

Over the timescale \( \tau_{av} \), the Hamiltonian dynamics in the chaotic regions is assumed to be mixing. Strong chaos separates the neighboring trajectories exponentially fast, on a timescale of order \( \frac{1}{\nu_1^*} \), and the system has thus
Figure 6: Schematic representation of a trajectory in the Hamiltonian stochastic model \[24\]. The system is represented by the red triangle. The picture displays the chaotic regions 1, 2 and 3 at 8 different times. The system is initially located in region 1 and is carried to region 2 through the mechanism of transport by migration of the chaotic regions.

Figure 7: Example of a trajectory in the stochastic Hamiltonian model \[24\] transported from \( p = 10 \) to \( p = 0 \) by the stochastic migration of the chaotic regions. The blue curve is the action \( p(t) \). The red and orange curves represent the frequencies \( \nu_1(t) \) and \( \nu_4(t) \). When the system is located in the chaotic region 1, the action value has fast and large fluctuations around \( \nu_1(t) \). When the system leaves region 1 and enters in the weakly chaotic region 4, the action fluctuations are much smaller. The left panel is an enlargement of the end of the trajectory, when the system is captured by the chaotic region 2 and exits through the boundary at \( p = 0 \).
completely “forgotten” its initial condition over the timescale $\tau_{av}$. This means that if the system has an initial condition inside the chaotic region 1 of figure (5) (for example), it can be anywhere inside the region 1 after a time $\tau_{av}$. On the other hand, we assume that chaos is weak enough in the regular regions 4 and 5 of figure (6) to keep the system to an average value very close to $\nu$ within the time $\tau_{av}$.

The local diffusion model of second order is represented in figure (8). It consists of three patches of infinite diffusion coefficient $D$ in $p$-space. Two of them have an extension $\delta_1$, and correspond in figure (5) to the chaotic regions 1 and 2 of main resonances. The third one has a smaller extension $\delta_2$ and corresponds to region 3 of second order resonances in figure (6). We assume that chaotic transport is negligible away from the resonances of first and second orders, that’s why we set the diffusion coefficient to zero out of the diffusion patches. The diffusion patches are then moved according to the stochastic dynamics (25).

The solution cannot be represented any more by a trajectory, but only through the probability distribution $\rho_\nu(p, t)$ to find the system at impulsion $p$ at time $t$, given a realization of the stochastic process $\nu(t)$. Let $\rho_i$ be the extension of the $i$-th diffusion region that we call $R_i$. The diffusion region $R_i$ then covers the interval $[\nu_i - \delta_i/2, \nu_i + \delta_i/2]$. If the impulsion $p$ is out of all the diffusion regions, the function $\rho_\nu(p, t)$ remains the same at step $t + dt$. If $p$ is inside the diffusion region $R_i$, the probability distribution at step $t + dt$ is the average of the probability distribution over the whole region. The dynamics of the distribution $\rho_\nu(p, t)$ can be implemented following the equations

$$
\rho_\nu(p, t + dt) = \begin{cases} 
\frac{1}{\delta_i} \int_{\nu(t) - \delta_i/2}^{\nu(t) + \delta_i/2} \rho_\nu(p', t) dp' & \text{if } p \text{ is in region } R_i \\
\rho_\nu(p, t) & \text{otherwise} 
\end{cases}
$$

(27)

and

$$
\nu(t + dt) = \nu(t) - (\nu - \nu^*) dt + \sqrt{2\sigma^2} dW(t)
$$

(28)

The second equation (28) is just another way to write Equation (25). We have $dW_1 = dW_2$, $dW_3 = dW_4$, and $\langle dW_1(t) dW_1(t') \rangle = \delta(t - t') dt$ and $\langle dW_1(t) dW_3(t') \rangle = 0$. The consequence of equation (27) is that at each step, the probability distribution is constant over each region $R_i$. But as the reader can see on equation (28), the region $R_i$ is moving because of the variations of the frequencies $\{\nu_i\}_{i=1..4}$. Therefore, at the next step, the average is performed over a region which has slightly moved during the time step $dt$.

### 3.3 The Markov process that corresponds to the local diffusion model

In this section, we give a rigorous mathematical definition of the process described by equations (27,28). We explain that there is no proper Markov process for the variable $(p, \nu)$ corresponding exactly to equations (27,28), but that an equivalent Markov process exists on an extended space. For simplicity, we define the Markov process for a single diffusion region $R_i$, but the following discussion can be straightforwardly generalized to many diffusion regions.

Let us consider the stochastic process $(p_t, \nu_t)$ defined by the following rules. First, $d\nu_t = -(\nu_t - \nu^*) dt + \sqrt{2\sigma^2} dW(t)$, and then $p_t$ is a jump process such that if $p_t \in R$ then $p_t$ jumps to any other point of $R$ at a rate $1/(\epsilon\delta)$ (where $\delta$ is the width of the diffusion region), and that if $p_t$ does not belong to $R$ then it stays constant.
This defines a Markov process. One can think of \( \epsilon \) as being the same as in section 3.1 or being another unrelated parameter. One can write the infinitesimal generator of this process. If \( \phi \) is a test function on the space \((p, \nu)\), then the infinitesimal generator writes

\[
G_{(p, \nu)}[\phi] = \frac{1}{\epsilon} \left[ \frac{1}{\epsilon} \int_{\nu-\delta/2}^{\nu+\delta/2} \phi(p, \nu') \, dp_1 - \phi(p, \nu) \right] I_{R}(p) + L_{\nu} [\phi],
\]

(29)

where \( I_{R}(p) \) is the step function equal to 1 if \( p \in R \) and zero otherwise, and \( L_{\nu} \) is the infinitesimal generator of the diffusion \( \nu \).

The process described by (27-28) corresponds to the infinite rate limit \((\epsilon \to 0 \text{ or } 1/\epsilon \to \infty)\) of the process with infinitesimal generator (29). We could equivalently say that (27-28) is the infinite rate limit of the finite rate jump process (29). As is clearly seen from (29), the infinitesimal generator (29) does not have a simple limit when \( \epsilon \to 0 \). In this limit, when \( p_{t} \) is inside \( R \), it oscillates faster and faster, uniformly over the set \( R \). In order to define properly the limit for such a fast oscillating variable, we would need the formalism of Young measures and weak convergence.

For finite \( \epsilon \), we first define a Markov process over the space of measures as follows: we first regularize the process \( \nu_{t} \) by introducing a correlation time \( \tau_{c} \), and \( \nu_{t} \) is the diffusion process

\[
\dot{\nu}_{t} = -(\nu_{t} - \nu^{*}) + \sqrt{2\sigma^{2}} \eta(t),
\]

(30)

where \( \eta \) is a continuous random process such that \( \langle \eta(t), \eta(t') \rangle = \frac{1}{\epsilon} e^{-t/\tau_{c}} \). With the above definition, the velocity \( \dot{\nu}_{t} \) is a well-defined variable. The mixing region is still defined as \( R_{\tau} := [\nu_{t} - \delta/2, \nu_{t} + \delta/2] \). Then we introduce the process \( \mu_{t} \) in the space of measures and the jump process \( p_{t}^{0} \in R \) with the following rules:

1. The state \((\mu_{t} = \delta(p - p_{0}^{0}), p_{t}^{0})\) is invariant if \( p_{t}^{0} \not\in R \). This law translates in measure space the fact that the jump process \( p_{t} \) does not move outside \( R_{\tau} \).
2. The process \((\mu_{t}, p_{t}^{0})\) jumps from \((\mu_{t} = \delta(p - p_{0}^{0}), p_{t}^{0})\) to \((\mu_{t} = I_{R}(p), p_{t}^{0})\) with rate \( \frac{1}{\epsilon} \) when \( p_{t}^{0} \in R_{\tau} \). This law translates in measure space the fact that the jump process \( p_{t} \) can jump to any point in \( R_{\tau} \) with rate \( 1/(\epsilon \delta) \).
3. We define the variable \( B_{R_{\tau}} := \nu_{t} - \text{sgn}(\dot{\nu}_{t}) \frac{\delta}{2} \), which corresponds to the boundary of \( R_{\tau} \) opposite to the moving direction of \( \nu_{t} \). The process \((\mu_{t}, p_{t}^{0})\) jumps from \((\mu_{t} = I_{R}(p), p_{t}^{0})\) to \((\mu_{t} = \delta(p - (B_{R_{\tau}})), B_{R_{\tau}})\) with rate \( \frac{2}{\epsilon} \). This corresponds to the exit from region \( R_{\tau} \). With the regularization (30) of the process \( \nu_{t} \), the jumping rate is perfectly defined.

The Markov process \((\nu_{t}, \mu_{t}, p_{t}^{0})\) described by the rules 1.2 and 3 is equivalent to the Markov jump process \( p_{t} \) which infinitesimal generator is given by (29). With the formalism of Young measures, the \( \epsilon \to 0 \) limit becomes straightforward: the process converges to the jump process defined by rules 1 and 3, and with the infinite rate limit for the second rule. This means that the process \((\mu_{t}, p_{t}^{0})\) jumps instantaneously from \((\mu_{t} = \delta(p - p_{0}^{0}), p_{t}^{0})\) to \((\mu_{t} = I_{R}(p), p_{t}^{0})\) as soon as \( p_{t}^{0} \in R_{\tau} \). This gives a precise definition of the local diffusion model (27-28). We note however that the limit \( \tau_{c} \to 0 \) in this model leads to some troubles as the jump rate \( \frac{2}{\epsilon} \) becomes infinite. The question of the limit \( \tau_{c} \to 0 \) is a very interesting mathematical one, but it is out of the scope of this paper.

In this subsection, we have set up the mathematical framework. We have built two models: the first model is a Hamiltonian model (24) that depends on time and on frequencies with a slow stochastic evolution (25). The second model is completely stochastic, and can be thought of as the “averaging” of the first model. It is called the local diffusion model, and defined by equations (27-28). The great interest of the local diffusion model is that it is completely stochastic, and thus much simpler to study than the first model, which still keeps the complexity inherent to a chaotic dynamics.

We still did not explain on which conditions the local diffusion model should give relevant predictions for the stochastic Hamiltonian model, this is one crucial question that is answered in section 3.4. Section 3.4 presents the numerical simulations performed with the first model (24-25), with comparisons with the local diffusion model. We study in section 3.4 first exit times \( \tau \) from a domain, both for the model (24-25) and for the local diffusion model. More precisely, we define the variable \( \tau \) as the first time to reach \( p = 0 \) starting from \( p = 10 \). Our aim is then to compute numerically the probability distribution \( \rho(\tau) \). In particular, we focus on the typical time \( \tau^{*} \) where the probability distribution \( \rho(\tau) \) reaches its maximal value, and on the dependance of the distribution on the noise amplitude \( \sigma \).
Table 2: Noise amplitudes and integration times of the five simulations in figure [10].

| Simulation | σ  | $T_{max} \times \frac{\nu_1}{2\pi} \times 10^{-1}$ |
|------------|----|-----------------------------------------------|
| 1          | 3.0| 1.0                                           |
| 2          | 2.2| 2.0                                           |
| 3          | 1.84| 3.0                                          |
| 4          | 1.1| 10.0                                         |
| 5          | 0.7| 50.0                                         |

3.4 Comparison of the dynamics of the stochastic Hamiltonian model with the local diffusion model

3.4.1 Simulations of the Hamiltonian dynamics with stochastic frequencies

We now present the numerical results obtained with the stochastic Hamiltonian model defined by (24, 25). The Hamiltonian [24] has the form $A(p) + B(q, t)$ with

$$A(p) := \frac{p^2}{2},$$
$$B(q, t) := \sum_{k=1}^{4} \cos \left( q - \frac{1}{\epsilon} \int_{0}^{t} \nu_k(s) ds - \varphi_k \right).$$

We have thus used the symplectic integrator of order 4 $SBAB_3$ [26]. At each time step, we integrate the frequencies from equation [25] with a stochastic Euler algorithm. The parameter $\epsilon$, the mean frequencies $\nu_k^*$ and the initial phases $\varphi_k$ were fixed to their nominal values given in Table 1.

As we have explained at the end of Section 3.2, we are mainly interested in the first exit time $\tau$ defined as the first time to leave the region $p > 0$ starting from $p = 10$. We want to compute numerically its probability distribution function $\rho(\tau)$, and determine how it depends on the noise amplitude $\sigma$. To achieve this aim, we have performed a set of five numerical simulations using the values of $\sigma$ given in Table 2. For each simulation, we ran 5000 trajectories all starting at the same point $(p, q) = (10, 0)$. Each trajectory is run with a different realization of the noise $W(t)$. The results of the simulations are the histograms displayed in figure [10]. The histograms represent the distributions $\rho(\tau)$ for each simulation.

3.4.2 Simulations of the local diffusion model

The local diffusion model is given by equations (27, 28). So far, we did not prescribe the values of the parameters $\delta_1, \delta_2$ corresponding to the extensions of the diffusion patches $R_1, R_2, R_3$. $\delta_1$ and $\delta_2$ should correspond to the effective extension of the strongly chaotic regions of the stochastic Hamiltonian model. The parameters $\delta_1$ and $\delta_2$ could be estimated from the Chirikov criterion of resonance overlap. However, the direct numerical simulations show that the size of the chaotic regions is smaller than the theoretical predictions of the Chirikov criterion. To obtain a better agreement with the simulations, we prescribed the size of the diffusion patches with the following method.

To estimate the size of the chaotic region 1 (see figure 9), we ran a numerical simulation of Hamilton equations with the Hamiltonian [24], except that we kept the frequencies fixed to their reference values $\nu_1^*$. We simulated 2000 trajectories with initial conditions $p = \nu_1^*$ and $q$ equally distributed over the range $[0, 2\pi]$, over a time $T = 300 \times \frac{2\pi}{\nu_1^*}$. The final coordinates are then distributed over the chaotic region 1, and only very few of them have exited region 1. Figure 9 shows a typical histogram of final momenta. We then define the boundaries of the chaotic region as the symmetric interval $[p_1, p_2]$ centered at $p = \frac{\nu_1^* + \nu_2^*}{2}$ in which 90% of the probability distribution is concentrated. Then, the empirical estimate of $\delta_1$ is $\delta_1 \approx p_2 - p_1$.

Using the same method, we have estimated the extension $\delta_2, \delta_3, \delta_4$ of the chaotic regions corresponding to the resonances of second, third and fourth orders respectively. The regions are located around the values $p = 3.3, p = 6.6$ for the resonances of third order, and $p = 2.5$ and $p = 7.5$ for the resonances of fourth order. The values of $\{\delta_i\}_{i=1,4}$ are reported in table 3.

We ran $M = 1000$ simulations using for each a new realization of the stochastic process $\nu(t)$. For each of the realization $\nu(t)$, we could compute with (27) the density $g_\nu(p, t)$ of the probability to find the system at $p$ at time $t$. At the beginning, the system is in $p = \nu_1^*$, which corresponds to the initial condition $g_\nu(p, 0) = \delta(p - \nu_1^*)$. We
Figure 9: Histogram of the density in the chaotic region 1 after about 300 turnover times. The two red lines at $p_1$ and $p_2$ show the boundaries of the strongly chaotic region. The mixing region used in the local diffusion model is defined as the symmetric interval $[p_1, p_2]$ centered at $p = \nu_1^* + \nu_2^*$ in which 90% of the probability distribution is concentrated. Its extension is given by $\delta_1 = p_2 - p_1$.

want to compute the probability of first hitting time at $p = 0$. This means that we have to prescribe the boundary condition $g_\nu(p < 0, t) = 0$. In practice, this condition amounts to set $g_\nu(p, t) = 0$ for $p \in [\nu_4(t) - \delta_1/2, \nu_4(t) + \delta_1/2]$ because if the system enters in the diffusion patch $R_2$, it is immediately transported across the patch and reaches $p = 0$.

The complete probability distribution $g(p, t)$ is simply the average of $g_\nu(p, t)$ over the realizations of the stochastic process $\nu(t)$. Let $\{\nu^k(t)\}_{k=1}^M$ be the $M$ realizations of $\nu(t)$, we have

$$g(p, t) = \frac{1}{M} \sum_{k=1}^{M} g_{\nu^k}(p, t).$$

Then the probability of first hitting times $g(\tau)$ is given by

$$\rho(\tau) = -\frac{d}{d\tau} \int_0^{+\infty} g(p, \tau) dp.$$

The simulations are performed with the set of parameters given in table 2. The results are displayed on the different graphs of figure 10 together with the histograms obtained by the direct Hamiltonian simulations. The curves show the results of $\rho(\tau)$ obtained with the simulations of the local diffusion model, whereas the histograms show the results for $\rho(\tau)$ obtained with the simulations of the stochastic Hamiltonian model. On the simulations 1, 2 and 3, we have only used the local diffusion model including the resonances up to second order. But on the simulations 4 and 5, there are two curves for $\rho(\tau)$: the lower one is the distribution $\rho(\tau)$ computed with the local diffusion model with resonances up to second order, but on simulation 4 and 5, we included the resonances up to fourth order in the local diffusion model. The results are displayed by the upper curves in simulations 4 and 5.

3.5 Discussion

The different simulations in figure 10 aim at showing which phenomenon governs the transport in phase space in the stochastic Hamiltonian model 24,25, and how the transport depends on the parameters of the model.

One trivial but crucial conclusion of our numerical study is that the transport in a stochastic Hamiltonian model is very different from the transport with the same Hamiltonian 24 without stochastic variations of the frequencies. If the frequencies are fixed, the trajectories starting at $p = \nu_i^*$ are just spread across the first mixing region, and
none of them reaches the value \( p = 0 \) within the time \( T_{\text{max}} \). Transport with stochastic frequencies is thus a new mechanism that completely overcomes chaotic diffusion in deterministic chaotic Hamiltonian dynamics.

The qualitative shape of the distributions of first hitting times \( \rho(\tau) \) displayed in figure (10) is typical of a distribution of first exit times from a domain in a stochastic system. The probability distribution has a maximum \( \rho^* \) reached at \( \tau^* \), that can be considered as the typical time for the exit event to occur. For times smaller than \( \tau^* \), the probability distribution goes rapidly to zero. It is thus very rare for the system to reach \( p = 0 \) in a time much smaller than the typical time \( \tau^* \), because it corresponds to exceptionally large and fast random fluctuations of the stochastic frequencies \( \{\nu_i\} \). For times larger than the typical time \( \tau^* \), the probability distribution also goes to zero because it is also a rare event, called “persistence”, that the system does not leave the domain \( p > 0 \) for a long time.

The order of magnitude of \( \tau^* \) depends on the noise amplitude \( \sigma \) acting on the frequencies. For \( \sigma = 3.0 \), it is of the order of \( 10^3 \times \frac{\pi}{\nu^*} \), whereas for \( \sigma = 0.7 \), it is two orders of magnitude larger, of the order of \( 10^5 \times \frac{\pi}{\nu^*} \). We have checked numerically that the typical exit time \( \tau^* \) does not depend on \( \epsilon \) in the limit \( \epsilon \to 0 \). This fact shows that the transport mechanism has a well defined limit when \( \epsilon \to 0 \), and confirm the relevance of the local diffusion model.

The local diffusion model presented in Section 3.2 can be seen as the averaged dynamics of the stochastic Hamiltonian model, for which transport outside of the principal chaotic regions has been neglected. This is reflected in the local diffusion model by the fact that the diffusion coefficient is zero outside the diffusion patches \( \mathcal{R}_i \). Therefore, the local diffusion model only takes into account transport by migration with the chaotic regions. If the transport of this type is dominant, it is natural to expect that the local diffusion model reproduces well the results of the stochastic Hamiltonian model. If, on contrary, transport is mainly due to the mechanism of noise driven transport in regular regions, then the exit rate at \( p = 0 \) predicted by the local diffusion model is much slower than the real transport of the Hamiltonian dynamics with stochastic parameters.

On figure (10), it can be seen that the local diffusion model is able to capture quite well the probability distribution of first exit times \( \rho(\tau) \). For the three simulations with \( \sigma = 3.0/2.2/1.84 \), the local diffusion model including resonances up to second order gives excellent results. It reproduce qualitatively and quantitatively the histogram of \( \rho(\tau) \), with the same location \( \tau^* \) of the maximal value, and reproduces the decrease of the distribution \( \rho(\tau) \) for long times. For the two simulations with \( \sigma = 1.1/0.7 \) the local diffusion model of second order predicts a transport rate which is much smaller than the real transport. In particular for \( \sigma = 0.7 \), even the qualitative shape of \( \rho(\tau) \) for the Hamiltonian model is not reproduced, the typical time \( \tau^* \) is by far overestimated. This means that for \( \sigma \) values below 1.84, transport through resonances of order higher than two can no longer be neglected. The local diffusion model including resonances up to fourth order is able to reproduce qualitatively the distribution of exit

| Resonances | Extension of the Chaotic Region | Numerical Estimation of \( p_2 - p_1 \) |
|------------|--------------------------------|---------------------------------|
| First Order | \( \delta_1 \)               | 2.25                            |
| Second Order | \( \delta_2 \)            | 0.50                            |
| Third Order | \( \delta_3 \)            | 0.28                            |
| Fourth Order | \( \delta_4 \)           | 0.09                            |

Table 3: Width of the diffusion patches of the local diffusion model.

Figure 10: First exit time distributions \( \rho(\tau) \) for three simulations with \( \sigma = 3.0 \) (left), \( \sigma = 2.2 \) (middle) and \( \sigma = 1.84 \) (right). We display the distributions in terms of the non-dimensional time \( \tau' := \tau \times \frac{\pi}{\nu^*} \). The histograms display the results of the simulations with the Hamiltonian model. The curves show the results of the corresponding local diffusion model including resonances up to second order.
the distributions in terms of the non-dimensional time $\tau' := \tau \times \frac{\sigma}{\pi}$. The histograms display the results of the simulations with the Hamiltonian $[24]$. The lower curves (red) show the results of the corresponding local diffusion model including resonances up to second order, and the upper curves (yellow) show the results of the local diffusion model including resonances up to fourth order.

Figures 11: First exit time distributions $\rho(\tau)$ for two simulations with $\sigma = 1.1$ (left) and $\sigma = 0.7$ (right). We display the distributions in terms of the non-dimensional time $\tau' := \tau \times \frac{\sigma}{\pi}$. The histograms display the results of the simulations with the Hamiltonian $[24]$. The lower curves (red) show the results of the corresponding local diffusion model including resonances up to second order, and the upper curves (yellow) show the results of the local diffusion model including resonances up to fourth order.

times $\rho(\tau)$, and gives also a good order of magnitude of the value of $\rho(\tau)$. This is illustrated by the upper curves for the simulations 4 and 5 on figure [10]. For values of $\sigma$ below 0.7, the local diffusion model is no longer able to reproduce transport in phase space, even with resonances up to fourth order. This comes from the fact that the overlap of two neighboring chaotic regions becomes very rare when $\sigma$ decreases, and thus the mechanism of noise driven transport in the regular regions overcomes the first mechanism.

The relevance of the local diffusion model to predict transport depends on the balance between the amplitude $\sigma$ of frequency fluctuations and the distance to cross between two neighboring diffusion regions. For example, in the local diffusion model of order two, the initial distance between two neighboring diffusion patches is

$$\text{Dist}_{R_1 \to R_3} := \left( \frac{\nu_1^* + \nu_2^*}{2} - \frac{\delta_1}{2} \right) - \left( \frac{\nu_1^* + \nu_2^*}{2} + \frac{\delta_2}{2} \right) \approx 3.575.$$ 

The sum of the variances of the fluctuations of the diffusion patches $R_1$ and $R_3$ gives the typical amplitude of the fluctuations of the mixing regions

$$\sqrt{V_{R_1} + V_{R_3}} = \left( 1 + \frac{1}{\sqrt{2}} \right) \sigma,$$

where $V_{R_i}$ is the variance of fluctuations of region $i$. Therefore, the efficiency of the transport through migration of the mixing regions depends on whether the parameter

$$\Delta := \frac{\text{Dist}_{R_1 \to R_3}}{\sqrt{V_{R_1} + V_{R_3}}} \approx \frac{2.1}{\sigma}$$

is large or small compared to one.

This means that for the three simulations with $\sigma = 3.0/2.2/1.84$, the initial distance to cross between two diffusion patches is larger or of same order as the amplitude of the frequency fluctuations. The jump between one diffusion patch to another is thus possible with “typical” fluctuations of the frequencies. Transport does not require an exceptionally large fluctuation. But this is no longer the case for $\sigma = 1.1/0.7$. Frequency fluctuations are two small to pass directly from the mixing region surrounding the first order resonances to the mixing region surrounding the second order resonances (or equivalently, from $R_1$ to $R_3$), and transport is due to higher order resonances. For example, in the local diffusion model of fourth order, the initial distance to cross to jump from the diffusion patch $R_1$ to the next patch $R_i$ is

$$\text{Dist}_{R_1 \to R_i} := \left( \frac{\nu_i^* + \nu_j^*}{2} - \frac{\delta_i}{2} \right) - \left( \frac{3\nu_i^* + \nu_j^*}{2} + \frac{\delta_i}{2} \right) \approx 1.28.$$ 

On the other hand, the combined fluctuations of the two patches have a variance of

$$\sqrt{V_{R_1} + V_{R_i}} \left( 1 + \frac{\sqrt{10}}{\pi} \right) \sigma.$$ 

Thus the parameter $\Delta$ is of the order of $\Delta \approx \frac{1.28}{1 + \sqrt{10}/\pi} \approx 0.71$. This argument explains why the local diffusion model up to order four is able to predict the transport for values of $\sigma$ of the order of 0.7, but fails for lower values of $\sigma$. 

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In the present section, we have shown how a system that satisfies a Hamiltonian dynamics with stochastic frequencies can be transported slowly through phase space by the slow displacement of the chaotic regions. We have shown that this kind of transport can be reproduced qualitatively and quantitatively by a Markov model, the local diffusion model. The local diffusion model gives a representation of the strongly chaotic regions created by resonance overlap by diffusion patches with infinite diffusion coefficient. The relevance of this Markov model to predict transport rates in the stochastic Hamiltonian model mainly depends on the amplitude of the frequency stochastic variations. We have shown that for decreasing amplitudes of the variations, the local diffusion model should take into account resonances of higher and higher orders. In this section, we have presented a model including resonances up to order four. But one cannot expect the local diffusion model to be valid for all range of the amplitude fluctuations $\sigma$, even if we include resonances up to higher orders. The reason for that is that transport is also due to noise driven transport in the regular regions, away from the resonances. If the amplitude of the fluctuations is too small, then transport is mainly due to noise driven transport in the regular regions. We have shown that transport is mainly due to migration with the chaotic regions if the typical fluctuations of the frequencies are similar to the distance in phase space between two neighboring chaotic regions.

We have performed an other numerical simulation where the stochastic process for the variations of $\nu$ is a jump process with exponential distribution of the jumps, and we found that the results are in agreement with the general picture we give in this section. This suggests that the transport mechanism described in this article is robust to a change in the stochastic process for the evolution of the external parameter.

4 Conclusion

We have studied time dependent Hamiltonian systems with one degree of freedom, for which the Hamiltonian depends on an additional slow stochastic parameter. The slowly varying parameter introduces a timescale separation in the system, which allows us to use the theory of averaging to describe the long term evolution of the system. When the fast Hamiltonian dynamics is integrable, it has been shown that the slow evolution can be described by a diffusion process of the action variable. This first part of our work is mostly an extension of the theory of adiabatic invariants to the stochastic case. More interesting for practical applications is the case where the fast Hamiltonian dynamics is chaotic. We have shown that transport in phase space comes from the slow displacement of chaotic regions, and is equivalent, for some ranges of the parameters, to a Markov process called the local diffusion model. We have shown numerically that the local diffusion model give quantitative results in agreement with the full simulations of the Hamiltonian dynamics with stochastic parameters.

The reduction of the dynamics to a Markov process opens the possibility to use large deviation theory to compute the probability of rare events in the dynamical system, for example the probability of an exceptionally fast exit out of a domain. Simplified Hamiltonian models with few degrees of freedom and a stochastic parameter can be used as preliminary works to find qualitative behaviors and order of magnitudes before resorting to involved numerical simulations. It is hoped that our work will find interesting applications in celestial mechanics and other domains of physics.

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A Canonical change of variables and existence of a function $H_1$ such that $\frac{\partial P}{\partial \nu} = -\frac{\partial H_1}{\partial Q}$ and $\frac{\partial Q}{\partial \nu} = \frac{\partial H_1}{\partial P}$

Let $(p, q)$ be the canonical variables of the Hamiltonian $H(p, q, \nu)$ where $\nu$ an external parameter of the Hamiltonian. We assume that for a given value of $\nu$, the Hamiltonian $H$ is integrable, which means that for any $\nu$ there exist canonical variables $(P(p, q, \nu), Q(p, q, \nu))$ such that $H(p, q, \nu) = \tilde{H}(P, \nu)$, and $\tilde{H}$ does not depend on $Q$. By definition, a change of variable is canonical one whenever the differential two-form is conserved: $dP \wedge dQ = dp \wedge dq$, or equivalently the Poisson bracket $\{P, Q\}_{p,q} = 1$, where $\{f, g\}_{p,q} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q}$. The aim of this section is to prove that there exists a function $H_1(P, Q, \nu)$ such that $\frac{\partial P}{\partial \nu} = \frac{\partial H_1}{\partial Q}(p, q, \nu)$, $P(p, q, \nu)$, $\nu$) and $\frac{\partial Q}{\partial \nu} = \frac{\partial H_1}{\partial P}(p, q, \nu)$ (relation (10) in the main text).

In order to prove this result, we extend the phase space by introducing $\Lambda$ the canonical momentum associated to $\nu$. We consider an extended Hamiltonian with two degrees of freedom

$$H'(p, q, \Lambda, \nu) := H(p, q, \nu) + \Lambda.$$ 

This amounts to saying that $\nu$ has a dynamics, with $\dot{\nu} = \frac{\partial H}{\partial \nu} = 1$. We will show that we can find a new momentum $\Lambda'$ such that the change of variables $(p, q, \nu, \Lambda) \rightarrow (P, Q, \nu', \Lambda')$ is canonical, where $(P(p, q, \nu), Q(p, q, \nu))$ and the canonical variable defined above for any fixed $\nu$, and the new variable $\nu$ is such that $\nu' := \nu$. A change of variable is canonical if and only if the differential two-form is conserved, which writes

$$dP \wedge dQ + d\Lambda' \wedge d\nu' = dp \wedge dq + d\Lambda \wedge d\nu.$$ 

(31)

Then we express explicitly the differentials $dP$ and $dQ$ as

$$
\begin{align*}
    dP &= \frac{\partial P}{\partial p} dp + \frac{\partial P}{\partial q} dq + \frac{\partial P}{\partial \nu} d\nu, \\
    dQ &= \frac{\partial Q}{\partial p} dp + \frac{\partial Q}{\partial q} dq + \frac{\partial Q}{\partial \nu} d\nu,
\end{align*}
$$

and we use these relations in (31) to obtain

$$
\begin{align*}
\{P, Q\}_{p,q} dp \wedge dq + \{P, Q\}_{p,\nu} dp \wedge d\lambda + \{P, Q\}_{q,\nu} dq \wedge d\lambda + d\Lambda' \wedge d\nu' = dp \wedge dq + d\Lambda \wedge d\nu.
\end{align*}
$$

(32)

In the first wedge product, the Poisson bracket satisfies $\{P, Q\}_{p,q} = 1$, thus (32) becomes

$$
\begin{align*}
\{P, Q\}_{p,\nu} dp \wedge d\nu + \{P, Q\}_{q,\nu} dq \wedge d\nu + d\Lambda' \wedge d\nu' = d\Lambda \wedge d\nu.
\end{align*}
$$

(33)

We now use that $\nu' = \nu$. We look for a function $H_1(Q, P, \nu)$ such that the canonical momentum $\Lambda'$ has the form $\Lambda' = \Lambda - H_1$. The equality (33) writes

$$
\left( \{P, Q\}_{p,\nu} dp + \{P, Q\}_{q,\nu} dq - dH_1 \right) \wedge d\nu = 0.
$$

This means that we can find a canonical change of variables iff

$$
\{P, Q\}_{p,\nu} dp + \{P, Q\}_{q,\nu} dq = dH_1,
$$

that is, iff the differential form $\{P, Q\}_{p,\nu} dp + \{P, Q\}_{q,\nu} dq$ is exact. Following a theorem of Riemann, a form is exact on a simple connected domain if and only if it is closed. The Poisson brackets should thus satisfy

$$
\frac{\partial}{\partial q} \{P, Q\}_{p,\nu} = \frac{\partial}{\partial p} \{P, Q\}_{q,\nu}.
$$

(34)

With some simple manipulations, we can write equality (34) as

$$
\frac{\partial}{\partial \nu} \{P, Q\}_{p,q} = 0,
$$

which is satisfied thanks to the relation $\{P, Q\}_{p,q} = 0$.
We have just proven the existence of $H_1$. The new Hamiltonian writes in terms of the variables $(P, Q, \nu', \Lambda')$
\[ H'(p, q, \Lambda, \nu) = \tilde{H}(P, \nu') + H_1(P, Q, \nu') + \Lambda'. \]
Therefore, the time evolution of the canonical variables is
\[
\begin{align*}
\frac{dP}{dt} &= -\frac{\partial H_1}{\partial Q}, \\
\frac{dQ}{dt} &= \frac{\partial \tilde{H}}{\partial P} + \frac{\partial H_1}{\partial P}.
\end{align*}
\]
(35)
But on the other hand, we know that the time evolution satisfies
\[
\begin{align*}
\frac{dP}{dt} &= \frac{\partial P}{\partial \nu} \dot{\nu}, \\
\frac{dQ}{dt} &= \frac{\partial \tilde{H}}{\partial P} + \frac{\partial Q}{\partial \nu} \dot{\nu}.
\end{align*}
\]
(36)
The identification of the equalities (35) and (36), with $\dot{\nu} = 1$, gives the desired result.

B Diffusion coefficients for the pendulum

We start from the expression of the action (22)
\[ P(p, q, \nu) = \nu + f(H(p, q, \nu)), \]
with
\[ f(H) = \frac{\sqrt{2}}{2\pi} \int_0^{2\pi} \sqrt{H - \cos(q')} dq'. \]
We have to compute the derivatives of $P$ with respect to $\nu$:
\[
\frac{\partial P}{\partial \nu} = 1 + \frac{\partial H}{\partial \nu} f'(H) = 1 - (p - \nu) f'(H).
\]
(37)
For the second derivative we get
\[
\frac{\partial^2 P}{\partial \nu^2} = f'(H) + (p - \nu) f''(H).
\]
(38)
We know from the theory of section 2.1 that $\frac{\partial P}{\partial \nu}$ should be the derivative of $H_1$, but we do not have any analytic expression for $H_1$. We thus have to average $\frac{\partial P}{\partial \nu}$ over the fast motion using expression (37). The fast motion is the integrable dynamics of $H$, and $H$ is thus a constant of the fast motion. Let us check that $\left\langle \frac{\partial P}{\partial \nu} \right\rangle_Q = \left\langle \frac{\partial H_1}{\partial Q} \right\rangle_Q = 0$. The average over $Q$ gives
\[ \left\langle \frac{\partial P}{\partial \nu} \right\rangle_Q = 1 - (p - \omega) f'(H). \]
The average over $Q$ is computed as follows,
\[
\langle p - \nu \rangle_Q = \frac{1}{2\pi} \int_0^{2\pi} (p - \nu) dQ \\
= \frac{1}{T} \int_0^T (p - \nu) dt \text{ (with the change of variable } Q \leftrightarrow t) \\
= \frac{1}{T} \int_0^T \dot{q} dt = \frac{2\pi}{T}.
\]
(39)
With the relation $f'(H) = \frac{T}{2\pi}$ we obtain $\left\langle \frac{\partial P}{\partial \nu} \right\rangle_Q = 0.$
To find the coefficients of the stochastic equation (23), we have to compute $\left\langle \left( \frac{\partial P}{\partial \nu} \right)^2 \right\rangle_Q$ and $\left\langle \frac{\partial^2 P}{\partial \nu^2} \right\rangle_Q$. Both of them can be expressed in terms of $\langle (p - \nu) \rangle_Q$ and $\langle (p - \nu)^2 \rangle_Q$. We compute the same way as before

$$\langle (p - \nu)^2 \rangle_Q = \frac{1}{2\pi} \int_0^{2\pi} (p - \nu)^2 dQ$$
$$= \frac{1}{T} \int_0^{2\pi} (p - \nu) dq$$
$$= \frac{2\pi}{T} (P - \nu).$$

With the relations (39-40), it is then easy to compute

$$\left\langle \frac{\partial^2 P}{\partial \nu^2} \right\rangle_Q = f'(H) + \langle (p - \nu)^2 \rangle_Q f''(H)$$
$$= \frac{T}{2\pi} + \frac{2\pi}{T} (P - \nu) \frac{\partial P}{\partial H} \frac{T'(P)}{2\pi} \text{ with the relation } f'(H) = \frac{T}{2\pi},$$
$$= \frac{T}{2\pi} + (P - \nu) \frac{T'(P)}{2\pi} \text{ using } \frac{\partial P}{\partial H} = f'(H) = \frac{T}{2\pi},$$
$$= \frac{d}{dP} \left[ (P - \nu) \frac{T(P)}{2\pi} \right].$$

and

$$\left\langle \left( \frac{\partial P}{\partial \nu} \right)^2 \right\rangle_Q = 1 + \langle (p - \nu)^2 \rangle_Q \left( f'(H) \right)^2 - 2 \langle p - \nu \rangle f'(H)$$
$$= (P - \nu) \frac{T}{2\pi} - 1.$$

Those two expressions then give the equation (23).