ON THE GEOMETRY OF $W_n$ GRAVITY

SURESH GOVINDARAJAN

Theoretical Physics Group, Tata Institute of Fundamental Research
Bombay, 400 005, India

ABSTRACT

We report work done with T. Jayaraman in this talk. In a recent paper, Hitchin introduced generalisations of the Teichmuller space of Riemann surfaces. We relate these spaces to the Teichmuller spaces of $W$-gravity. We show how this provides a covariant description of $W$-gravity and naturally leads to a Polyakov path integral prescription for $W$-strings.

1. Introduction

In an interesting paper, Hitchin obtained a class of Teichmuller spaces as the space of solutions of self-duality equations. These spaces have the following features:

(i) They are natural generalisations of the Teichmuller space of compact Riemann surfaces with genus $g > 1$ (which we shall call $T_2$).
(ii) $T_2$ embeds in these generalised Teichmuller spaces.
(iii) These spaces have the dimension one would expect for Teichmuller spaces for $W$-gravity (as given by the zero modes of the antighosts). For example, for the case of $W A_{n-1}$-gravity, the ghost system consists of $(b^{(j)}, c^{(1-j)})$ systems for $j = 2, \ldots, n$, where the superscript refers to the spin of the field. The number of antighost zero modes (for genus $g > 1$) is given by

$$\sum_{j=2}^{n} (\text{# of } b^{(j)} \text{ zero modes}) = (2g - 2) \sum_{j=2}^{n} (2j - 1) = (2g - 2) \dim[SL(n, \mathbb{R})]$$

In this talk, I will demonstrate the relation of the Teichmuller spaces of Hitchin to $W$-gravity. As we shall see this will enable us to give a gauge independent description of $W$-gravity using generalisations of vielbeins and spin-connection of usual gravity.

Before that let me discuss some open questions in $W$-gravity. First, from the viewpoint of string theory, one would like to know if there are any non-trivial examples of $W$-strings. The known ones are generalisations of “Liouville theory” which suggest that the $c=1$ barrier exist even in the case of $W$-strings. Another question which

---

*Talk presented at the International Colloquium on Modern Quantum Field Theory II, Tata Institute of Fundamental Research, Bombay during Jan. 5-11, 1994.
†E-mail: suresh@theory.tifr.res.in
arises is whether there is any geometric structure underlying W-symmetry. This geometric structure could be of the form of extra data on the Riemann surface or some immersion in a higher dimensional manifold. Also, we would like to have a gauge independent description of W-symmetry. An important feature of W-symmetry is its non-linear nature. Can we provide a setting where these non-linear transformations become linear? Can we introduce a \(w\)-coordinate like the Grassmann coordinates in supergravity? Is there a Polyakov path-integral prescription for W-strings? In this talk I will address some of these issues and hope to address them all in the future. Of course, I must point out that some of these questions have already been addressed in some form or the other in existing literature. However, no one has provided a complete answer to all these questions.

The plan of the talk is as follows: First, I will briefly discuss various definitions of the Teichmüller space of Riemann surfaces. Then, I will introduce self-duality equations and illustrate it using the bosonic string. This will provide another definition for Teichmüller space. We will then generalise to the case of arbitrary W-gravity but will use \(W_3 = WA_2\) to give more details. Finally, I will discuss how W-diffeomorphisms arise in this formulation.

2. Various definitions of Teichmüller space

Consider a compact Riemann surface \(\Sigma\) of genus \(g > 1\). Teichmüller space can be defined as follows

\[
T_2(\Sigma) = \frac{\text{(space of all metrics on } \Sigma)}{\{\text{diff}\}_0 \times \{\text{Weyl}\}} \quad (1)
\]

\[
= \frac{\text{(space of all metrics with constant negative curvature on } \Sigma)}{\{\text{diff}\}_0} \quad (2)
\]

\[
= \text{a component of the space } \text{Hom}(\pi_1(\Sigma); PSL(2, \mathbb{R}))/PSL(2, \mathbb{R}) \ , \quad (3)
\]

where \(\{\text{diff}\}_0\) refers to diffeomorphisms connected to the identity. The definition given in the third line follows from the uniformisation theorem. We know that \(\Sigma\) can be described by quotient of the upper half plane (with the standard Poincare metric) by a discrete subgroup (Fuchsian) of \(PSL(2, \mathbb{R})\) which is isomorphic to \(\pi_1(\Sigma)\). This provides a homomorphism of \(\pi_1(\Sigma)\) to \(PSL(2, \mathbb{R})\) which is defined up to conjugation in \(PSL(2, \mathbb{R})\) (since two Fuchsian subgroups give the same surface if they belong to the same conjugacy class). However the space \(\text{Hom}(\pi_1(\Sigma); PSL(2, \mathbb{R}))/PSL(2, \mathbb{R})\) has many disconnected components. The component of interest is specified by requiring that the first Chern class of the \(U(1)\) part of the flat \(PSL(2, \mathbb{R})\) bundle (associated with every homomorphism) is equal to \((2g - 2)\).

3. Self-duality equations

The self-duality equations (SDE) are

\[
F_A + [\Phi, \Phi^\dagger] = 0 \ , \quad (4)
\]
where \( A \) is a unitary connection on a holomorphic vector bundle \( V \), \( F_A \) is its field strength. \( \Phi \) is a holomorphic section of \( \text{End}(V) \otimes K \) (\( K \) is the canonical line bundle on \( \Sigma \)). \((V, \Phi)\) form a Higgs bundle. If a certain stability condition is satisfied and \( c_1(V) = 0 \), then the connection \( A \) (which is compatible with the holomorphic structure) satisfying (4) is unique. The holomorphicity condition on \( \Phi \) is
\[
\bar{\partial}_A \Phi = 0.
\]
Equations (4) and (5) imply that \( A \equiv A + \Phi + \Phi^\dagger \) is generically a flat \( GL(n, \mathbb{C}) \) connection\(^\dagger\). We shall now illustrate how SDE’s occur in the bosonic as well as W-strings.

### 3.1. Bosonic String

Choose \( V = K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}} \) and
\[
\Phi = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \ dz,
\]
where the metric on \( \Sigma \) is given by \( g = h^2 \ dz d\bar{z} \). Here \( A \) is a \( SU(2) \) connection and \( \mathcal{A} = A + \Phi + \Phi^\dagger \) is a flat \( PSL(2, \mathbb{C}) \) connection. The self-duality equations (4) imply
\[
F_A = \begin{pmatrix} \frac{1}{2} F^0 & 0 \\ 0 & -\frac{1}{2} F^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} h^2 dz \wedge d\bar{z}.
\]
From the above equation, we can see that the \( SU(2) \) connection is reducible to a \( U(1) \) connection which we will identify with the spin-connection on \( \Sigma \). Then the condition on \( F^0 \) is nothing but the constant curvature condition on the metric. Hitchin has shown that the connection \( \mathcal{A} \) has holonomy contained in a real form of \( PSL(2, \mathbb{C}) \) which is \( PSL(2, \mathbb{R}) \). In all the examples considered here, the holonomy will be contained in a real form. Hence, we shall treat \( \mathcal{A} \) as a flat \( PSL(2, \mathbb{R}) \) connection.

Deform \( \Phi \) to
\[
\Phi = \begin{pmatrix} 0 & h \\ \frac{a}{h} & 0 \end{pmatrix} \ dz,
\]
where \( a \in \text{Hom}(K^{-\frac{1}{2}}; K^{\frac{1}{2}}) \otimes K = K^2 \) is a holomorphic (follows from equation (5)). Again, it can be shown that the self-duality equations imply the constant negative curvature condition. This leads us to the following conclusion. The space of solutions of SDE connected to \( \Phi = \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \) is the same as the space of constant negative curvature metrics. As already discussed this space is \( T_2 \). So we now have an alternate definition for \( T_2 \).

We shall now make a direct connection with 2d gravity. We have already identified the \( U(1) \) connection \( A \) with the spin-connection on \( \Sigma \). We make another identification
\[
\Phi + \Phi^\dagger = \text{vielbeins} \ e^+, \ e^- \ on \ \Sigma.
\]
\(^\dagger\)However, in the examples we will discuss here, one obtains a flat \( SL(n, \mathbb{C}) \) connection.
This implies that \((A,e^+,e^-)\) form a flat \(PSL(2,R)\) connection. This has been observed earlier in the context of topological gravity\(^6\) and later by H. Verlinde\(^7\). With this identification, one can see that the holomorphicity condition on the Higgs field \(\Phi\) correspond to the torsion constraints imposed on the vielbein (in the conformal gauge). The metric \(g_2\) is given by the quadratic \(SO(2)\) invariant \(-g_2 = tr(E^2)\), where \(E \equiv \Phi + \Phi^\dagger\).

### 3.2. W String

The generalisation to the W-string case is now easy. We shall restrict our discussion to the \(WA_n\) string for simplicity. Following Hitchin\(^2\), we consider the vector bundle \(V_n\) given by

\[
V_n \equiv S^{n-1}(K^{-\frac{1}{2}} \oplus K^\frac{1}{2}) = K^{-\frac{n-1}{2}} \oplus K^{-\frac{n-3}{2}} \oplus \ldots \oplus K^\frac{n-1}{2}
\]

and choose the Higgs field

\[
\Phi_n = \begin{pmatrix}
0 & h & 0 & \ldots & 0 \\
\frac{a_1}{h} & \ddots & \ddots & \ddots & \vdots \\
\frac{a_2}{h} & \ddots & \ddots & h & 0 \\
\vdots & \ddots & \frac{a_3}{h} & 0 & h \\
\frac{a_{n-1}}{h^{(n-1)}} & \ldots & \frac{a_2}{h} & \frac{a_1}{h} & 0
\end{pmatrix} \, dz ,
\]

where \(a_i \in K^i\). Again \(A \equiv A + \Phi + \Phi^\dagger\) is a flat \(PSL(n,C)\) connection provided the self-duality equations are satisfied. However, the flat connection \(A\) has its holonomy contained in a split real form of \(PSL(n,C)\) which is isomorphic to \(PSL(n,R)\).

The bosonic string example suggests that we make the following identifications

\[-A \quad generalised\ spin-connection\ for\ W-gravity.\]

\[-\Phi + \Phi^\dagger \quad generalised\ vielbein\ for\ W-gravity.\]

Does this make sense? First, consider the space

\(T_n \equiv \{\text{a component of the space of solutions of SDE for } (V_n, \Phi_n)\}\)\(^\S\)

This component has dimension \((2g-2)\text{dim}[SL(n,R)]\), which as we have seen earlier is the dimension we expect for the dimension for the Teichmuller space for \(WA_n\) gravity. Further, local deformations are given by quadratic, cubic, \ldots differentials. Also, the presence of higher order invariants lets us define higher order symmetric tensors. For example, for \(n = 3\), the quadratic \(SO(3)\) invariant gives the metric \((-g_2 \equiv tr(E^2))\) and the cubic one gives a symmetric third rank tensor \((-g_3 \equiv tr(E^3))\). So this seems to suggest that the identifications are sensible. In the So this seems to suggest that these identifications are sensible. In the latter part of the talk, we shall demonstrate how the “usual” \(W\)-diffeomorphisms are recovered in the conformal gauge. The self-duality equations \((\mathbb{E}_4)\) and \((\mathbb{E}_5)\) can now be interpreted as generalisations of the constant curvature condition and torsion constraints respectively.

\(^\S\text{This component corresponds to the Teichmuller component of Hitchin.}\)
In the bosonic case, the constant curvature condition corresponded to gauge fixing the Weyl degree of freedom. So restoring the Weyl degree corresponds to relaxing the constant curvature condition. In a similar fashion, in the general case, we can restore the the W-Weyl degrees of freedom by relaxing the generalised constant curvature condition (which is given by equation (4)). Further, the W-Weyl transformations can be obtained using a method due to Howe. Here Weyl transformations are seen as transformations which keep the torsion constraints invariant.

We shall discuss the $W_3$ case in more detail now. The $PSL(3, \mathbb{R})$ connection can be parametrised as follows

$$
\mathcal{A} = \begin{pmatrix}
\omega - \frac{e^+}{\sqrt{3}} & (\omega^- + e^-) & \sqrt{2}e^-\\
(\omega^+ - e^+) & \frac{2e^+}{\sqrt{3}} & (-\omega^- + e^-) \\
\sqrt{2}e^{++} & (-\omega^+ - e^+) & -\omega - \frac{e^+}{\sqrt{3}}
\end{pmatrix}
$$

(12)

where $\omega, \omega^\pm$ form the $SO(3)$ connection and $e^*$ are the generalised vielbein. We would like to make the following observations.

(i) Here, $(e^\pm, \omega)$ form a $PSL(2, \mathbb{R})$ connection which is embedded in $PSL(3, \mathbb{R})$.

(ii) $e^{++}, e^{--}, e^{+-}$ are the new vielbein. They have been labelled by their $U(1)$ charges (w.r.t. $\omega$). Schoutens et al. introduced W-vielbein which correspond to $e^{++}$ and $e^{--}$ (but not for $e^{+-}$) in their covariant construction of an action for scalar fields coupled to $WA_2$ gravity.

(iii) The geometry is not Riemannian anymore in the sense that the torsion constraints are not sufficient to determine the connection in terms of the vielbein. However, we can always choose a gauge where we trade one of the gauge symmetries to determine the connection in terms of the vielbein.

(iv) Conformal gauge corresponds to choosing

$$
e^+_z = e^-_z = h \quad , \quad e^{++}_z = e^{--}_z = 0 \quad ,$$

which fixes the $SO(3)$ gauge freedom. Further, these gauge choices are algebraic and hence their corresponding ghosts can be ignored (since they would be non-interacting). Next, choose the following gauge choices

$$
e^+_z = e^-_z = e^{++}_z = e^{--}_z = 0 \quad ,$$

whose corresponding ghosts (anti-ghosts) are of spin $-1, -2 (2, 3)$ as expected for $W_3$ gravity. Further, the residual transformations which preserve this gauge choice correspond to holomorphic (anti - holomorphic) transformations $e^+, e^{++}$ ($e^-, e^{--}$).

4. W-diffeomorphisms
The work of Gerasimov et al.\(^\text{10}\) and subsequently that of Bilal et al.\(^\text{11}\) has provided a geometric picture of W-diffeomorphisms in the conformal gauge. They have shown that W-diffeomorphisms correspond to deformations of certain flag manifolds. In their construction, the action of \(WA_{(n-1)}\) diffeomorphisms on vector space \(V = S^{n-1}(K^{-\frac{1}{2}} \oplus K^\frac{1}{2})\) was demonstrated. This is precisely the space on which the Higgs field acts. We shall show that this is not a coincidence and show that provided a certain constraint is obeyed, the \(SL(n,\mathbb{R})\) gauge transformations are equivalent to W-diffeomorphisms. Other related works are\(^\text{12}\). The feature which is different in our formulation here is that one does not need to invoke matter fields in order to describe W-diffeomorphisms. The role of projective connections is played by combinations of the generalised spin connections.

4.1. The bosonic case

As usual, the bosonic \((PSL(2,\mathbb{R}))\) case shows us the way. Choose the Higgs field \(\Phi\) as in eqn. \(\text{(8)}\). Consider the fields \((\tilde{\psi}_1, \tilde{\psi}_2) \in (K^{-\frac{1}{2}}, K^{\frac{1}{2}})\) subject to the conditions

\[
(\bar{\partial}_z + \Phi)(\tilde{\psi}_1 \tilde{\psi}_2) = 0 ,
\]

(13)

\[
(\partial_z + \Phi)(\tilde{\psi}_1 \tilde{\psi}_2) = 0 .
\]

(14)

Eqn. \(\text{(13)}\) implies that the fields \(\psi_i\) are holomorphic while the second eqn. \(\text{(14)}\) is a constraint on \(\psi_i\). Interestingly, the self-duality equations \(\text{(4)}\) and \(\text{(5)}\) imply the consistency of the two conditions we have just imposed. The holomorphicity condition \(\text{(3)}\) on \(\Phi\) implies that

\[
\omega_z = -h^{-1}\partial_z h , \quad \omega_{\bar{z}} = h^{-1}\partial_{\bar{z}} h ,
\]

(15)

and the condition that \(a\) is holomorphic \((\partial_z a = 0)\). We shall now rescale the fields \(\psi_i\) as follows

\[
\begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix} =
\begin{pmatrix}
h^{\frac{1}{2}} \psi_1 \\
h^{-\frac{1}{2}} \psi_2
\end{pmatrix} .
\]

(16)

\((\psi_1, \psi_2)\) correspond to primary fields in the CFT sense, i.e., they have Einstein indices. This rescaling now enables us to make contact with diffeomorphisms in CFT. Further, the Christoffel connection are given by \(\Gamma_{zz} = 2\omega_z\) and \(\Gamma_{z\bar{z}} = 2\omega_{\bar{z}}\). A simple calculation shows that conditions \(\text{(13)}\) and \(\text{(14)}\) translates to the following conditions

\[
(\partial_z + \bar{\mu} \partial_z - \frac{1}{2} \partial_z \bar{\mu})\psi_1 = 0 ,
\]

(17)

\[
(\partial_z^2 - u)\psi_1 = 0 ,
\]

(18)

where \(\bar{\mu} \equiv \frac{\bar{a}}{\bar{h}^2}\) is the Beltrami differential and \(u \equiv \frac{1}{2} \partial_z \Gamma_{zz} + \frac{1}{4} (\Gamma_{zz})^2 - a\). It is a simple exercise to check that \(u\) transforms like the Schwarzian. A similar observation was
made by Sonoda who pointed out that such a term could be added to the energy momentum tensor to make it transform like a \((2,0)\) tensor. Hence, \(u\) behaves like a projective connection. Eqn. (17) implies that \(\psi_1\) transforms as
\[
\delta \psi_1 = \xi^z \partial_z \psi_1 - \frac{1}{2} (\partial_z \xi^z) \psi_1 ,
\]
which is the standard transformation of a \((-\frac{1}{2},0)\) tensor in CFT. What we have done here is similar to what have done. However, there are some differences. The flatness condition that has been considered in is different from the flatness condition implied by the self-duality equations. The \(PSL(2,\mathbb{R})\) gauge field is a completely geometric object with no relation apriori to matter fields. However, certain special combinations of the spin-connection transforms like the Schwarzian. This combination can be related to the stress-tensor via Ward identities considered by Verlinde. It can also be seen that \(W\)-transformations have a presentation here without directly involving “matter fields”.

The compatibility of conditions (17) and (18) implies that
\[
\left[ \partial_z + \bar{\mu} \partial_z + 2(\partial_z \bar{\mu}) \right] u = \frac{1}{2} \partial_z^3 \bar{\mu} , \tag{19}
\]
which is equivalent to the standard OPE for the stress-tensor (following a procedure outlined in)
\[
T(z) T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \partial_w T(w) + \ldots , \tag{20}
\]
provided we identify \(u \rightarrow \frac{6}{c} \langle T \rangle\). We thus recover the residual diffeomorphisms which preserve conformal gauge.

4.2. The \(W_3\) case

We shall now repeat the exercise of the previous subsection for the \(WA_2\) case. Consider the multiplet \((\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3) \in (K^{-1}, K^0, K)\). The Higgs field is given by (11) to be
\[
\Phi = \begin{pmatrix} 0 & h & 0 \\ \frac{a}{h} & 0 & h \\ \frac{b}{h^2} & \frac{a}{h} & 0 \end{pmatrix} dz + \mathcal{O}(a^2, ab, b^2) .
\]
The holomorphicity condition (3) implies that
\[
\omega^+_{\mu} = \omega^-_{\mu} = 0 , \quad \omega_z = -h^{-1} \partial_z h , \quad \omega_{\bar{z}} = h^{-1} \partial_{\bar{z}} h , \tag{21}
\]
and that \((a, b)\) are holomorphic \((\partial_z a = \partial_{\bar{z}} b = 0)\). The above solution is valid to \(\mathcal{O}(a^2, ab, b^2)\). Impose the following conditions on \((\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)\).
\[
(\bar{\partial}_w + \Phi^\dagger) \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = 0 , \tag{22}
\]
\[ (\partial_\omega + \Phi) \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = 0 \quad . \tag{23} \]

Eqn. (22) implies that the fields \( \tilde{\psi}_i \) are holomorphic while the second eqn. (23) is a constraint on \( \tilde{\psi}_i \). We shall now rescale the fields \( \tilde{\psi}_i \) as follows

\[ \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \\ \tilde{\psi}_3 \end{pmatrix} = \begin{pmatrix} h\psi_1 \\ \psi_2 \\ h^{-1}\psi_3 \end{pmatrix} \quad . \tag{24} \]

\( (\psi_1, \psi_2, \psi_3) \) are primary fields. Conditions (22) and (23) translate to the following conditions

\[ (\partial_z - \tilde{\mu}\partial_z + \partial_z\tilde{\mu} + \tilde{\rho}(\partial_z^2 - \frac{2}{3}\tilde{u}_2) - \frac{1}{2}\partial_z\tilde{\rho}\partial_z + \frac{1}{6}(\partial_z^2\tilde{\rho}))\psi_1 = 0 \quad (25) \]

\[ (\partial_z^3 - \tilde{u}_2\partial_z - (u_3 + \frac{1}{2}\partial_z\tilde{u}_2))\psi_1 = 0 \quad , \tag{26} \]

where \( \tilde{\mu} \equiv \frac{a}{h^2} \) and \( \tilde{\rho} \equiv \frac{b}{h^4} \) are the Beltrami differentials. Further,

\[ \tilde{u}_2 = 2\partial_z\Gamma_{zz} + (\Gamma_{zz})^2 + 2a \quad , \quad u_3 = [\partial_z - 2\Gamma_{zz}](\omega^{-z}_z h) - b \quad , \]

where we have restored dependence on \( \omega^{-z}_z \) (even though it is vanishing to \( O(a^2, ab, b^2) \)). This is to explicitly demonstrate that \( u_3 \) also behaves like a generalised projective connection in the sense of Refs. 10, 11.

Equations (25) and (26) are identical to those obtained in Refs. 10, 11. Using arguments identical to Refs. 10, 11 we can show that these two equations are equivalent to the following OPE’s.

\[ T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \ldots , \]

\[ T(z) W(w) = \frac{3W(w)}{(z-w)^2} + \frac{\partial_w W(w)}{(z-w)} + \ldots , \]

\[ W(z) W(w) = \frac{c/3}{(z-w)^6} + \left( \frac{2}{(z-w)^4} + \frac{\partial_w}{(z-w)^3} + \frac{(3/10)\partial_w^2}{(z-w)^2} + \frac{(1/15)\partial_w^3}{(z-w)} \right) T(w) \]

\[ + \frac{16}{5c} \left( \frac{2}{(z-w)^2} + \frac{\partial_w}{(z-w)} \right) \Lambda(w) + \ldots , \tag{27} \]

where \( \Lambda = T^2 \). The above OPE corresponds to the semi-classical limit \( (c \to \infty) \) of the OPE’s of the \( W_3 \)-algebra given in Ref. 14 after we make the following identifications

\[ \tilde{u}_2 \longrightarrow \frac{24}{c} \langle T \rangle \quad , \quad u_3 \longrightarrow \frac{24}{c} \langle W \rangle \quad . \]
Equation (23) implies the following transformation law for a $(-1)$-differential
\[
\delta \psi_1 = \xi^z \partial_z \psi_1 - (\partial_z \xi^z) \psi_1 + \xi^{zz}(\partial_z^2 \psi_1) - (\partial_z \xi^{zz})(\partial_z \psi_1) - \frac{2}{3}((\partial_z^2 - \ddot{u}_2)\xi^{zz}) \psi_1 ,
\]
where $\xi^z$ and $\xi^{zz}$ parametrise $W$-diffeomorphisms. These parameters are holomorphic since we are in the conformal gauge.

We have demonstrated how $W$-diffeomorphisms are obtained in our formulation. The interesting feature is that certain combinations of the generalised connection play the role of projective connections. So we do not need to introduce any new matter dependent fields to describe $W$-diffeomorphisms.

5. Conclusion and Discussions

In this talk, we have related the Teichmuller components of Hitchin to the Teichmuller spaces for $W$-gravity. This lead to the introduction of generalised vielbeins and connections. One can now write out a Polyakov path integral for the $W$-gravity sector. Of course, the hard part is to understand how to couple matter to $W$-gravity in this formulations. The work of Schoutens et al. could possibly give us a hint as to how to proceed.

In this talk, we had restricted ourselves to the genus $g > 1$ situation. What are the gauge groups for the $g = 0, 1$ cases? For the bosonic case, one knows the answer ($SO(3)$ and $ISO(2) \sim IU(1)$). This suggests the following choices: $SU(n)$ for $g = 0$ and $IU(n - 1)$ for $g = 1$.

In connection with $W_3$ gravity, we would like to mention the work of Goldman who has studied the space of convex $RP^2$ projective structures (we shall call this space $P(\Sigma)$). We would like to observe that dim $P(\Sigma) = \text{dim } T_3(\Sigma)$. Hitchin has suggested that $P(\Sigma)$ is a subspace of or possibly the whole of $T_3(\Sigma)$. We conjecture that there exists a $RP^2$ structure in the case of $W_3$-gravity. Goldman has provided Fenchel-Nielsen coordinates (pants decomposition) for $P(\Sigma)$ which implies we can discuss higher loop divergences in the context of $W$-strings just as in the bosonic string.

We hope that the formulation of $W$-gravity presented in this talk will provide some simplifications in the description of $W$-gravity as well as will lead to more insights into the geometric structures underlying $W$-gravity and also provide new realisations of $W$-strings.

Acknowledgements: I would like to thank T. Jayaraman for a collaboration which led to the work described here. I thank the organisers of the Colloquium for the opportunity to present this work.

---

\footnote{It has been brought to our attention that Goldman and Choi have recently shown that $P(\Sigma) = T_3(\Sigma)$. We thank Pablo Ares Gastesi for this information.}
References

1. S. Govindarajan and T. Jayaraman, “A proposal for the geometry of $W_n$ gravity,” Tata Preprint TIFR/TH/94-14 = hep-th/9405146.
2. N. J. Hitchin, “Lie Groups and Teichmuller theory,” Topology 31 (1992) 451.
3. A. Bilal and J. Gervais, Nuc. Phys. B326 (1989) 222; P. Mansfield and B. Spence, Nuc. Phys. B362 (1991) 294; S. Das, A. Dhar and S. K. Rama, Mod. Phys. Lett. A6 (1991) 3055, and Int. J. of Mod. Phys. A7 (1992) 2295; C. N. Pope, L. J. Romans and K. S. Stelle, Phys. Lett. B268 (1991) 167; H. Lu, C. N. Pope, S. Schrans and K. Xu, Nuc. Phys. B385 (1992) 99.
4. N. J. Hitchin, “The self-duality equations on a Riemann surface,” Proc. London Math. Soc. 55 (1987) 59-126.
5. D. Birmingham, M. Blau, M. Rakowski and G. Thompson, “Topological field theory,” Physics Reports 209 (1991) 129-340.
6. D. Montano and J. Sonnenschein, “The topology of moduli space and quantum field theory,” Nuc. Phys. B324 (1989) 348; E. Verlinde and H. Verlinde, “A solution of two dimensional topological quantum gravity,” Nuc. Phys. B348 (1991) 435.
7. H. Verlinde, “Conformal field theory, 2-D quantum gravity and quantization of Teichmuller space,” Nuc. Phys. B337 (1990) 652.
8. P. S. Howe, “Super weyl transformations in two dimensions,” J. Phys. A12 (1979) 393.
9. K. Schoutens, A. Sevrin and van Nieuwenhuizen, “A new gauge theory for $W$–type algebras,” Phys. Lett. B243 (1990) 245-249; “Covariant formulation of classical W-gravity,” Nuc. Phys. B349 (1991) 791-814.
10. A. Gerasimov, A. Levin and A. Marshakov, “On W-gravity in two-dimensions,” Nuc. Phys. B360 (1991) 537-558.
11. A. Bilal, V. Fock and I. Kogan, “On the origin of W-algebras,” Nuc. Phys. B359 (1991) 635-672.
12. J. de Boer and J. Goeree, “W – gravity from Chern-Simons theory,” Nuc. Phys. B381 (1992) 329-359; “Covariant $W_3$ action,” Phys. Lett. B274 (1992) 289-297; K. Yoshida, “On the origin of SL$(N,C)$ current algebra in generalized 2-dimensional gravity,” Int. J. of Mod. Phys. A7 (1992) 4353-4376.
13. H. Sonoda, “The energy momentum tensor on a Riemann surface,” Nuc. Phys. B281 (1987) 546.
14. A. B. Zamolodchikov, Teor. Mat. Fiz. 65 (1985) 1205; V. A. Fateev and A. B. Zamolodchikov, Nuc. Phys. B280 (1987) 644.
15. W. M. Goldman, “Convex real projective structures on compact surfaces,” J. Diff. Geom. 31 (1990) 791-845.
16. S. Choi and W. M. Goldman, “Convex real projective structures on closed surfaces are closed,” Proc. of the AMS 118 (1993) 657.
17. E. Gava, R. Iengo, T. Jayaraman and R. Ramachandran, “Multiloop divergences in the closed bosonic string theory,” Phys. Lett. B168 (1986) 207.