Periodic Sudden Freezing and Thawing of Entanglement in Multiparty Pure-State Quantum Systems

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Within the one-excitation context of two identical two-level atoms interacting with a common cavity, we are concerned with the dynamics of all bipartite one-to-other entanglements between each qubit and the remaining part of the whole system, from the perspective of resource sharing. We find a new non-analytic “sudden” dynamical behavior of entanglement. Specifically, the sum of the three one-to-other entanglements of the system can be suddenly frozen at its maximal value or can be suddenly thawed from this value in a periodic manner. We calculate the onset timing of sudden freezing and sudden thawing under several different initial conditions. The phenomenon of the permanent freezing of entanglement is also found. Further analyses about freezing and thawing processes reveal quantitative and qualitative laws of resource sharing.

Introduction. Entanglement as a special quantum correlation plays a crucial role in quantum computation and quantum information [1]. Recently, a new measure of correlation, called quantum discord [2, 3], has been regarded as a generalization of entanglement. There exist separable mixed states that have nonzero discord, previously considered as classical states in that their entanglements are zero, which can be utilized for computational speed-up [1]. Specifically, for certain systems, even though entanglement has already decayed to zero, quantum discord can persist and then decay suddenly [4]. This phenomenon, labeled as the freezing of quantum discord, together with the sudden transition on its dynamics, has been well studied (see [5–8]). Experimental observations have confirmed the existence of these sudden transitions [9, 10]. Recently, the freezing of entanglement itself has also been reported [11, 12]. All of these studies have been concerned with freezing under the influence of one or another decoherence mechanism. Many questions still remain. For example, is decoherence necessary for freezing? Another question is whether freezing is a phenomenon reserved for mixed states. In this Letter we provide definitely negative answers to both of these open questions. We will demonstrate sudden freezing of entanglement in a lossless dynamical context. The demonstration also introduces entanglement sudden thawing in the same pure-state lossless context. We show that it is sufficient to focus on bipartite entanglements that arise in three-qubit interactions that are accessible via familiar cavity QED experimental arrangements. Our results demonstrate that the presence of decoherence is not needed to obtain entanglement freezing.

Analysis of the dynamical phases of freezing and thawing reveals quantitative and qualitative relations of the resource sharing implied by them. Furthermore, as we show, permanent freezing of three-party entanglement is also possible. This can happen when the entanglement resource favors two of the three participants, while the third member has no chance to share the resource during the three-party activity. Finally we identify a non-trivial upper limit for the sum of three individual entanglement addends.

Physical model. To make our model and results concrete, we begin by confining our attention to a system in which two identical atoms labelled by 1 and 2 with transition frequency \( \omega_0 \), are located in a common high-Q cavity with frequency \( \omega \), and are coupled to the cavity with the same coupling constant \( g \). The ground and excited states for the atoms are denoted by \( |g\rangle \) and \( |e\rangle \). We write the bare Hamiltonian in the usual way as a sum of atoms and cavity contributions (\( \hbar = 1 \))

\[
H_{\text{at}} = \omega_0 T_{ee}^{(1)} + \omega_0 T_{ee}^{(2)}, \quad \text{and} \quad H_{\text{cav}} = \omega a^\dagger a, \tag{1}
\]

where \( T_{ee}^{(i)} \) is the usual excited state occupation number operator for atom \( i \), defined as \( T_{ee}^{(i)} = |e\rangle_i \langle e|_i \), while \( a \) and \( a^\dagger \) are the annihilation and creation operators of the cavity mode. The usual interaction Hamiltonian between the atoms and the cavity (under the rotating wave approximation) is

\[
H_{\text{int}} = g \left( a^\dagger T_{ge}^{(1)} + T_{eg}^{(1)} a \right) + g \left( a^\dagger T_{ge}^{(2)} + T_{eg}^{(2)} a \right), \tag{2}
\]

where \( T_{ge}^{(i)} \) and \( T_{eg}^{(i)} \) are the atomic transition operators, given by \( T_{ge}^{(i)} = |g\rangle_i \langle e|_i \) and \( T_{eg}^{(i)} = |e\rangle_i \langle g|_i \).

The operator for the total number of excitations of the system, defined by the operator \( M = a^\dagger a + T_{ee}^{(1)} + T_{ee}^{(2)} \) commutes with the total Hamiltonian and is thus an integral of motion. The interaction Hamiltonian only causes effective transitions among the eigenstates of \( M \) with eigenvalue \( m \), namely, \( |m−2, e, e\rangle, |m−1, e, g\rangle, |m−1, g, e\rangle, |m, g, g\rangle \), where the three available slots for the ket state represent the cavity, atom 1 and atom 2 respectively. In the basis of these eigenstates, the total Hamiltonian of the system takes on a block structure along the main diagonal:
valid for an arbitrary initial state is given by \( H_{\text{tot}} \), defined as the first \( 1 \times 1 \) block on the diagonal with the single number 0 implies no interaction between the state \( |0, g, g \rangle \) and the others. The second block is a \( 3 \times 3 \) matrix that represents the case of one excitation in the system. The other blocks along the main diagonal are \( 4 \times 4 \) matrices for all \( m \geq 2 \), as is shown in Eqn. (3). Our interest is for the case \( m = 1 \), so it is clear that the cavity can only occupy \( |0 \rangle \) and \( |1 \rangle \) states, and the cavity can then be considered as a third qubit itself.

Clearly, we can add arbitrarily many more atoms of the same transition frequency and coupling strength. If all atoms are initially in their ground state and if we still allow for only one single photon initially in the cavity, the behavior of dynamics for the entire system will not change greatly.

To solve for the 3-qubit state dynamics we denote the general form of the wavefunction for the system as

\[
|\psi(t)\rangle = a_0(t) |1, g, g\rangle + a_1(t) |0, e, g\rangle + a_2(t) |0, g, e\rangle ,
\]

where \( a_0(t) \), \( a_1(t) \) and \( a_2(t) \) in turn represent the probability amplitudes that the cavity, atom 1 and atom 2 are in the excited state. When substituting these into the Schrödinger’s equation, we obtain three first order differential equations,

\[
\begin{align*}
\dot{a}_1(t) &= -ig\tilde{a}_0(t)e^{it\Delta t}, \\
\dot{a}_2(t) &= -ig\tilde{a}_0(t)e^{it\Delta t}, \\
\dot{a}_0(t) &= -ig[\tilde{a}_1(t) + \tilde{a}_2(t)]e^{-it\Delta t},
\end{align*}
\]

where \( \tilde{a}_1(t) \), \( \tilde{a}_2(t) \) and \( \tilde{a}_0(t) \) are the slowly varying amplitudes, defined as \( \tilde{a}_1(t) = a_1(t)e^{i\omega_0 t} \), \( \tilde{a}_2(t) = a_2(t)e^{i\omega_0 t} \), \( \tilde{a}_0(t) = a_0(t)e^{i\gamma t} \), and \( \Delta \) is the detuning parameter defined as \( \Delta \equiv \omega_0 - \omega \). A simple solution of these equations, valid for an arbitrary initial state is given by

\[
\begin{align*}
\tilde{a}_1(t) &= \tilde{a}_1(0) - \alpha + \left\{ \alpha \cos\left(\frac{\Omega}{2} t\right) - i\beta \sin\left(\frac{\Omega}{2} t\right) \right\} e^{i\phi t}, \\
\tilde{a}_2(t) &= \tilde{a}_2(0) - \alpha + \left\{ \alpha \cos\left(\frac{\Omega}{2} t\right) - i\beta \sin\left(\frac{\Omega}{2} t\right) \right\} e^{i\phi t}, \\
\tilde{a}_0(t) &= \tilde{a}_0(0) \cos\left(\frac{\Omega}{2} t\right) - i\gamma \sin\left(\frac{\Omega}{2} t\right) e^{-i\phi t}.
\end{align*}
\]

where \( \Omega = \sqrt{\Delta^2 + G^2} \) is a detuned Rabi oscillation frequency with \( G = \sqrt{8g} \). Also, \( a_0(0), a_1(0) \) and \( a_2(0) \) are the initial values of the probability amplitudes, with

\[
\begin{align*}
\alpha &= \frac{4g^2[a_1(0) + \tilde{a}_2(0)]}{\Omega^2 - \Delta^2}, \\
\beta &= \frac{4g^2\Delta[a_1(0) + \tilde{a}_2(0)] + 2g(\Omega^2 - \Delta^2)\tilde{a}_0(0)}{\Omega(\Omega^2 - \Delta^2)}, \\
\gamma &= \frac{2g[a_1(0) + \tilde{a}_2(0)] - \Delta \tilde{a}_0(0)}{\Omega}.
\end{align*}
\]

We notice that the dependence of the slowly varying probability amplitudes \( \tilde{a}_1(t) \) and \( \tilde{a}_2(t) \) on the coupling strength \( g \) is closely related to their initial conditions. For example, in the initial case of \( \tilde{a}_0(0) = 1 \) and \( \tilde{a}_1(0) = \tilde{a}_2(0) = 0 \), their time evolution only depends on \( g \) to the first power since the exchanges of the excitation from the cavity to the atoms are completed through their individual atom-cavity interactions. However, this is not true for the initial case of \( \tilde{a}_1(0) = 1 \) and \( \tilde{a}_2(0) = \tilde{a}_0(0) = 0 \) or \( \tilde{a}_2(0) = 1 \) and \( \tilde{a}_1(0) = \tilde{a}_0(0) = 0 \). We can see from our interaction Hamiltonian (2) that it doesn’t allow a direct interaction, between the two atoms, but only interaction between the cavity and the atoms, such that the exchange of excitation between the atoms must be mediated by the cavity. Hence, the time evolution of \( \tilde{a}_1(t) \) and \( \tilde{a}_2(t) \) necessarily depend on \( g \) to the second power.

Throughout this paper, we consider only the resonant situation \( \Delta = 0 \) for simplicity. This is enough to demonstrate the main results. In this way, the three original amplitudes \( a_0(t), a_1(t) \) and \( a_2(t) \) differ from their slowly varying forms \( \tilde{a}_0(t), \tilde{a}_1(t) \) and \( \tilde{a}_2(t) \) only by a global phase. Thus, the slowly varying solutions can be exchanged just as the rapidly varying ones.

**Entanglement Measure.** In the three-qubit scenario, there exist various types of entanglement concerning different compositions of entangled parties, including tripar-
tite entanglement, one-to-other bipartite entanglement and one-to-one bipartite entanglement within any two-qubit subsystems. We are here concerned with the dynamics of one-to-other entanglement, from the perspective of resource sharing. In what follows, we will adopt as our measure of entanglement the Schmidt weight $K(t)$ [13], but in the normalized form, labelled as $Y(t)$, which is defined in earlier work [14]

$$Y = 1 - \sqrt{\frac{2}{K}} - 1,$$

(8)

where

$$K = \frac{1}{\mu_1^2 + \mu_2^2},$$

(9)

with $\mu_1$ and $\mu_2$ being the corresponding eigenvalues of a Schmidt-bipartitioned reduced density matrix. $Y$ is monotonic with the concurrence $C$ [15] via $Y = 1 - \sqrt{1 - C^2}$, and is a good entanglement measure.

The time dependent state [3] is not in the Schmidt basis, but our coefficient matrix procedure [16] easily allows us to compute the Schmidt weight $K$. We will explicitly calculate the bipartite entanglement between the cavity and the remaining two atoms. The other two $Y$ values follow by symmetry. If we bi-partition $|\psi(t)\rangle$ in Eqn. [11] to obtain the Schmidt coefficient matrix $V$ with the cavity part in the basis $|0\rangle, |1\rangle$, and its two-atom partner in the basis $|g, g\rangle, |g, e\rangle, |e, g\rangle, |e, e\rangle$, we obtain

$$V = \begin{bmatrix} 0 & a_2(t) & a_1(t) & 0 \\ a_0(t) & 0 & 0 & 0 \end{bmatrix},$$

(10)

from which the reduced density matrix $\rho_0$ for the cavity is then given by

$$\rho_0 = V V^\dagger = \begin{bmatrix} |a_1(t)|^2 + |a_2(t)|^2 & 0 & 0 \\ 0 & |a_0(t)|^2 \end{bmatrix}.$$  

(11)

It is clear that the two time-dependent eigenvalues for the reduced density matrix $\rho_0$ are

$$\mu_1^{(0)}(t) = |a_1(t)|^2 + |a_2(t)|^2, \quad \mu_2^{(0)}(t) = |a_0(t)|^2.$$  

(12)

On substituting Eqn. [12] first into Eqn. [9] then into Eqn. [3], and using the fact that $\mu_1^{(0)}(t) + \mu_2^{(0)}(t) = 1$, after some straightforward algebra, we can express the bipartite entanglement between cavity and its remaining part $Y_{0(12)}(t)$ in a compact form

$$Y_{0(12)}(t) = 2 \min \left( |a_1(t)|^2 + |a_2(t)|^2, |a_0(t)|^2 \right).$$  

(13)

Now we can obtain the other two entanglement members corresponding to atom 1 and atom 2 by the permutation of the position between $|a_1(t)|^2$, $|a_2(t)|^2$ and $|a_0(t)|^2$ in Eqn. [13] respectively, viz.

$$Y_{1(02)}(t) = 2 \min \left( |a_0(t)|^2 + |a_2(t)|^2, |a_1(t)|^2 \right),$$  

(14)

with the two eigenvalues for the reduced density matrix of atom 1 being $\mu_1^{(1)} = |a_0(t)|^2 + |a_2(t)|^2$, $\mu_2^{(1)} = |a_1(t)|^2$.

Similarly,

$$Y_{2(01)}(t) = 2 \min \left( |a_1(t)|^2 + |a_0(t)|^2, |a_2(t)|^2 \right),$$  

(15)

with the two eigenvalues for the reduced density matrix of atom 2 being $\mu_1^{(2)} = |a_0(t)|^2 + |a_1(t)|^2$, $\mu_2^{(2)} = |a_2(t)|^2$.

To have an effective and different understanding of the entanglement of the system, we now move to the perspective of resource sharing. That is, we will examine the dynamics of $Y_S(t)$, the sum of the three individual one-to-other entanglements above:

$$Y_S(t) := Y_{0(12)}(t) + Y_{1(02)}(t) + Y_{2(01)}(t).$$  

(16)

By substitution of [13], [14] and [15] into [16], we find that the sum of the three one-to-other entanglements of this system can be expressed as

$$Y_S(t) = 2 \left[ \min \left( |a_1(t)|^2 + |a_2(t)|^2, |a_0(t)|^2 \right) + \min \left( |a_2(t)|^2 + |a_0(t)|^2, |a_1(t)|^2 \right) + \min \left( |a_1(t)|^2 + |a_0(t)|^2, |a_2(t)|^2 \right) \right].$$

(17)

Eqn. [17] shows that the sum of entanglements is fixed up to the modulus of probability amplitudes. This means that three participants in different states can share the same amount of entanglement.

Since the three individual $Y_{i(jk)}$ entanglements are all normalized, ranging between 0 and 1, a natural guess for the range of their sum would be 0 to 3. However, it can be easily checked from the expression [17] that the normalization restriction $|a_0|^2 + |a_1|^2 + |a_2|^2 = 1$ effectively prevents $Y_S$ from exceeding the maximal value 2, namely

$$Y_{0(12)} + Y_{1(02)} + Y_{2(01)} \leq 2.$$  

(18)

Actually, the sharing properties of the three individual $Y_{i(jk)}$ are quite restrictive in two aspects. First, the resource has this key limitation: the total amount of the resource to be shared is limited to 2. For example, when two of the individuals take the value 1, the third entanglement has no more free resource to share, and thus can only be 0. One example for this situation is the Bell state $(|0, e, g\rangle + |0, g, e\rangle)/\sqrt{2}$. It can be shown that for this specific state, $Y_{1(02)} = Y_{2(01)} = 1$, leaving the only possibility for $Y_{0(12)}$ to be 0. Second, the individual entanglements have their own limitations: the value of one entanglement cannot exceed the sum of the other two (or equivalently, one half of $Y_S$). Since the total resource cannot exceed 2, the maximal amount that an individual entanglement can share is only 1 in this sense. Since no single individual entanglement can exceed the sum of the others, the three one-to-other entanglements
squared amplitudes, which is given by

\[ \frac{Y}{S} \]

of the three entanglements are strictly decided by their contrary, in the freezing region, the sharing distribution

\[ Y \]

is always 2 in the freezing region, the maximal value for an individual entanglement is 1. When \( Y_{0(12)} \)
reaches the value 1 again, \( Y_S \) thaws immediately, as can be seen at \( t = 3\pi/G \).

The initial state for the second class is the partially entangled Bell state:

\[ |\psi(0)\rangle = \cos \theta |0, e, g\rangle + \sin \theta |0, g, e\rangle, \]

where the initial excitation is on the two atoms, and a parameter \( \theta \) has been introduced. After some long but straightforward calculations, we obtain the final expression of \( Y_S(t) \) in terms of the sum of three distinct minimum functions, given in Eq. (21).
The period of $Y_S(t)$ means the entanglement resource of the system belongs to the fractional evolution for the three-party system. From the antisymmetrical Bell state, for which the matrix elements are constant value 2. This is because the atoms are in the value of $Gt$ is barely any freezing. On the contrary, at the specific freezing time is about to be minimized, when there is any non-frozen. For the second class of initial condition $\theta = 0$, the period of $Gt$ doesn’t change. Fig. 2 shows the detailed behavior of $Y_S(t)$ as a function of $Gt$ and the parameter $\theta$. The analytical expression is given by Eqn. (21). The cold region of blue color takes the value 2, indicating the freezing of entanglement, while the warmer region of red and orange takes value less than 2, and thus represents thawing of entanglement.

$Y_S(t) = \begin{cases} -\frac{1}{2} \cos(Gt) - 2 \cos\left(\frac{Gt}{2}\right) + \frac{5}{2}, & \tau_0 \leq Gt \leq \tau_1 \\ 2, & \tau_1 \leq Gt \leq \tau_2 \\ -\frac{1}{2} \cos(Gt) + 2 \cos\left(\frac{Gt}{2}\right) + \frac{5}{2}, & \tau_2 \leq Gt \leq \tau_3 \end{cases}$

where $\tau_0 = 2k\pi$, $\tau_1 = 2k\pi + 2\arccos(\sqrt{2} - 1)$, $\tau_2 = 2k\pi + 2\arccos(1 - \sqrt{2})$, and $\tau_3 = 2(k + 1)\pi$ and $k$ is any non-negative integer. In fact, from Eqn. (5), it is not difficult to find that the physical process for all values of $\theta$ is that the two atoms will swap their situations at the end of the first half cycle ($t = \pi/G$), up to a global phase factor. As a result, $Y_S(t)$ must have the same evolution in these two half cycles, as shown in the case of $\theta = 0$. This is why the cycle of $Y_S(t)$ is still equal to $2\pi$ despite containing a new oscillation term $\cos(Gt/2)$ in contrast to Eqn. (20).

The values for the three individual entanglements are also plotted as functions of the dimensionless time in Fig. 3. It can be seen that the two atoms in turn occupy one half of $Y_S$ in the total resource for each thawing region. When one of them reaches the value 1, a new freezing or thawing begins. Within the freezing, the continued ratio relation still holds.

Summary. In summary, we have reported the freezing and thawing of entanglement in a lossless background for pure-state systems for the first time, and we demonstrated that dissipative environment is not a necessary condition for freezing. This study provides an analytical
understanding of dynamical effects of entanglement freezing and thawing for a quantum system consisting of three qubits, with one qubit serving as the common cavity. In addition, this study also provides an analytical understanding of entanglement sharing dynamics in the case that both the sharing resource and all sharing participants are restricted in the way that was explained in the text. The generalization of our results to an arbitrary N-qubit context is under way, with results to be reported subsequently.

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