ON THE LOCAL BEHAVIOR OF THE MAPPINGS WITH NON–BOUNDED CHARACTERISTICS

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Abstract

The present paper is devoted to the study of space mappings, which are more general than quasiregular mappings. The questions of the behavior of differentiable mappings having the so-called $N, N^{-1}, ACP$ and $ACP^{-1}$ – properties are studied in the work. Under some additional conditions, it is showed that the modulus of such mappings $f$ can be more than each degree of logarithmic function at every neighborhood of the isolated essential singularity of $f$.

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1 Introduction

Here are some definitions. Everywhere below, $D$ is a domain in $\mathbb{R}^n$, $n \geq 2$, $m$ be a measure of Lebesgue in $\mathbb{R}^n$, and $\text{dist}(A, B)$ is the Euclidean distance between the sets $A$ and $B$ in $\mathbb{R}^n$. A mapping $f : D \to \mathbb{R}^n$ is said to be a discrete if the pre-image $f^{-1}(y)$ of any point $y \in \mathbb{R}^n$ consists of isolated points, and an open if the image of any open set $U \subset D$ is open in $\mathbb{R}^n$. The notation $f : D \to \mathbb{R}^n$ assumes that $f$ is continuous on its domain. In what follows, a mapping $f : D \to \mathbb{R}^n$ is supposed to be orientation preserving, i.e. the topological index $\mu(y, f, G)$ is greater than zero for an arbitrary domain $G \subset D, \overline{G} \subset D$ and an arbitrary $y \in f(G) \setminus f(\partial G)$, (see, for example, §2 of the Ch. II in [Re]). Let $f : D \to \mathbb{R}^n$ be an arbitrary mapping and suppose that there is a domain $G \subset D, \overline{G} \subset D$, for which $f^{-1}(f(x)) = \{x\}$. Then the quantity $\mu(f(x), f, G)$, which is referred to as the local topological index, does not depend on the choice of the domain $G$ and is denoted by $i(x, f)$. In what follows $(x, y)$ denotes the standard scalar multiplication of the vectors $x, y \in \mathbb{R}^n$, $\text{diam} A$ is Euclidean diameter of the set $A \subset \mathbb{R}^n$,

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}, \quad \mathbb{B}^n := B(0, 1),$$

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad \mathbb{S}^{n-1} := S(0, 1),$$

$\omega_{n-1}$ denotes the quare of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, $\Omega_n$ is a volume of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$. Given a mapping $f : D \to \mathbb{R}^n$, a set $E \subset D$, and a point $y \in \mathbb{R}^n$ we define the
multiplicity function \( N(y, f, E) \) as the number of pre-images of \( y \) in \( E \), that is,

\[
N(y, f, E) = \text{card} \{ x \in E : f(x) = y \}.
\]

Recall that a mapping \( f : D \to \mathbb{R}^n \) is said to have the \( N \) – property (of Luzin) if \( m(f(S)) = 0 \) whenever \( m(S) = 0 \) for all such sets \( S \subset \mathbb{R}^n \). Similarly, \( f \) has the \( N^{-1} \) – property if \( m(f^{-1}(S)) = 0 \) whenever \( m(S) = 0 \).

We write \( f \in W^{1,n}_{\text{loc}}(D) \), iff all of the coordinate functions \( f_j, f = (f_1, \ldots, f_n) \), have the partitional derivatives which are locally integrable in the degree \( n \) in \( D \).

Recall that a mapping \( f : D \to \mathbb{R}^n \) is said to be a mapping with bounded distortion, if the following conditions hold:

1) \( f \in W^{1,n}_{\text{loc}} \),
2) a Jacobian \( J(x, f) := \det f'(x) \) of the mapping \( f \) at the point \( x \in D \) preserves the sign almost everywhere in \( D \),
3) \( \|f'(x)\|^n \leq K \cdot |J(x, f)| \) at a.e. \( x \in D \) and some constant \( K < \infty \), where

\[
\|f'(x)\| := \sup_{h \in \mathbb{R}^n : |h| = 1} |f'(x)h|,
\]

see., e.g., § 3 Ch. I in [Re2], or definition 2.1 of the section 2 Ch. I in [Ri].

Active investigations of the mappings with bounded distortion were started by Yu.G. Reshetnyak. In particular, he has proved that the mappings \( f \) with bounded distortion are open and discrete, see Theorems 6.3 and 6.4, § 6, Ch. II in [Re2], are differentiable a.e., see Theorem 4 in [Re1], in and have \( N \) – property, see Theorem 6.2 Ch. II in [Re3]. From other hand, the \( N^{-1} \) – property of the mappings with bounded distortion was proved by B. Bojarski and T. Iwaniec, see Theorem 8.1 in [BI].

We recall that an isolated point \( x_0 \) of the boundary \( \partial D \) of a domain \( D \) in \( \mathbb{R}^n \) is said to be a removable singularity if there is a finite limit \( \lim_{x \to x_0} f(x) \). If \( f(x) \to \infty \) as \( x \to x_0 \), then \( x_0 \) is referred to as a pole. An isolated point \( x_0 \) of \( \partial D \) is called an essential singularity of a mapping \( f : D \to \mathbb{R}^n \) if the limit \( \lim_{x \to x_0} f(x) \) does not exist.

In 1972, in the work of J. Väisälä was proved the following, see e.g. Theorem 4.2 in [Va3].

**Statement 1.** Let \( b \in D \) and \( f : D \setminus \{b\} \to \mathbb{R}^n \) be a mapping with bounded distortion. Suppose that there exists a number \( \delta > 0 \) such that

\[
|f(x)| \leq C|x - b|^{-p},
\]

at every \( x \in B(b, \delta) \setminus \{b\} \) and some positive constants \( p > 0 \) and \( C > 0 \). Then \( b \) is a removable singularity or a pole of the mapping \( f \).

A goal of the present paper is a proof of the analogue of the statement 1 for more general classes of mappings of finite length distortion, including the classes of mappings with bounded distortion. Mappings with finite length distortion were introduced by O. Martio, V. Ryazanov, U. Srebro and E. Yakubov in 2002, see e.g. in the work [MRSY1], or Chapter 8 in [MRSY2]. The considering of it is actually in the connection with the study of the so-called mappings with finite distortion, which are actively investigated at the last time, see e.g. Chapter 20 in [AIM] or Chapter 6 in [IM]. In this connection, see also the works [BGMV], [Mikl], [Sal] and [UV].
A curve $\gamma$ in $\mathbb{R}^n$ is a continuous mapping $\gamma : \Delta \to \mathbb{R}^n$ where $\Delta$ is an interval in $\mathbb{R}$. Its locus $\gamma(\Delta)$ is denoted by $|\gamma|$. Given a family of curves $\Gamma$ in $\mathbb{R}^n$, a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called admissible for $\Gamma$, abbr. $\rho \in \text{adm } \Gamma$, if curvilinear integral of the first type $\int_\gamma \rho(x)dx$ satisfies the condition

$$\int_\gamma \rho(x)dx \geq 1$$

for each $\gamma \in \Gamma$. The modulus $M(\Gamma)$ of $\Gamma$ is defined as

$$M(\Gamma) = \inf_{\rho \in \text{adm } \Gamma} \int_{\mathbb{R}^n} \rho^n(x)dm(x)$$

interpreted as $+\infty$ if $\text{adm } \Gamma = \emptyset$. The properties of the above modulus are analogous to the properties of the measure of Lebesgue $m$ in $\mathbb{R}^n$. Namely, a modulus of the empty family equals to zero, $M(\emptyset) = 0$, a modulus has a property of monotonicity by the relation to families of curves $\Gamma_1$ and $\Gamma_2 : \Gamma_1 \subset \Gamma_2 \Rightarrow M(\Gamma_1) \leq M(\Gamma_2)$, and has a property of subadditivity, $M\left(\bigcup_{i=1}^\infty \Gamma_i\right) \leq \sum_{i=1}^\infty M(\Gamma_i)$, see Theorem 6.2 in [Va].

We say that a property $P$ holds for almost every (a.e.) curves $\gamma$ in a family $\Gamma$ if the subfamily of all curves in $\Gamma$ for which $P$ fails has modulus zero.

If $\gamma : \Delta \to \mathbb{R}^n$ is a locally rectifiable curve, then there is the unique increasing length function $l_\gamma$ of $\Delta$ onto a length interval $\Delta_\gamma \subset \mathbb{R}$ with a prescribed normalization $l_\gamma(t_0) = 0 \in \Delta_\gamma$, $t_0 \in \Delta$, such that $l_\gamma(t)$ is equal to the length of the subcurve $\gamma|_{[t_0,t]}$ of $\gamma$ if $t > t_0$, $t \in \Delta$, and $l_\gamma(t)$ is equal to $-l(\gamma|_{[t_0,t]})$ if $t < t_0$, $t \in \Delta$. Let $g : |\gamma| \to \mathbb{R}^n$ be a continuous mapping, and suppose that the curve $\tilde{\gamma} = g \circ \gamma$ is also locally rectifiable. Then there is a unique increasing function $L_{\gamma,g} : \Delta_\gamma \to \Delta_{\tilde{\gamma}}$ such that $L_{\gamma,g}(l_\gamma(t)) = l_{\tilde{\gamma}}(t) \ \forall \ t \in \Delta$. A curve $\gamma$ in $D$ is called here a lifting of a curve $\tilde{\gamma}$ in $\mathbb{R}^n$ under $f : D \to \mathbb{R}^n$ if $\tilde{\gamma} = f \circ \gamma$. Recall that if $f \in ACP$ and only if a curve $\tilde{\gamma} = f \circ \gamma$ is locally rectifiable for a.e. curves $\gamma$ in $D$, and $L_{\gamma,f}$ is absolutely continuous on closed subintervals of $\Delta_\gamma$ for a.e. curves $\gamma$ in $D$. We say that a discrete mapping $f$ is absolute continuous on curves in the inverse direction, abbr. $ACP^{-1}$, if for a.e. curves $\tilde{\gamma}$ a lifting $\gamma$ of $\tilde{\gamma}$, $\gamma = f \circ \gamma$, is locally rectifiable, and $L_{\gamma,f}^{-1}$ is absolutely continuous on closed subintervals of $\Delta_{\tilde{\gamma}}$ for a.e. curves $\gamma$ in $f(D)$ and for each lifting $\gamma$ of $\tilde{\gamma}$. A mapping $f : D \to \mathbb{R}^n$ is said to be of finite length distortion, abbr. $f \in \text{FLD}$, if $f$ is differentiable a.e. in $D$, has $N$ – and $N^{-1}$ – properties, and $f \in ACP \cap ACP^{-1}$.

**Remark 1.1.** The notion of the mappings with finite length distortion can be given in more general case, when $f$ does not supposed to be a discrete, see e.g. in [MRSY1], see also section 8.1 in [MRSY2]. Of course, the above definition is equivalent to the correspondent general case, see, for instance, section 8.1 and corollary 8.1 in [MRSY2], or Corollary 3.14 in [MRSY1]. In this connection, the word ”discrete” will be present in the text, if it is necessary.

For the classes $W_{1,loc}^{1,n}$, and, in particular, for the mappings with bounded distortion, the $ACP$ property is well-known as B. Fuglede’s lemma, see, for instance, Theorem 28.2 in [Va]. Besides that, the $ACP^{-1}$ property was proved by E.A. Poletskii for it, see e.g. Lemma 6 in [Pol]. Taking into account all of the comments given above, we conclude that every mapping with bounded distortion is a mapping with finite length distortion, see also Theorem 4.7 in [MRSY1], or Theorem 8.2 in [MRSY2] in this connection.
We say that a function $\varphi : D \to \mathbb{R}$ has a \textit{finite mean oscillation} at the point $x_0 \in D$, write $\varphi \in \text{FMO}(x_0)$, if

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \cdot \varepsilon^n} \int_{B(x_0, \varepsilon)} |\varphi(x) - \varphi_\varepsilon| \, dm(x) < \infty,$$

where $\varphi_\varepsilon = \frac{1}{\Omega_n \cdot \varepsilon^n} \int_{B(x_0, \varepsilon)} \varphi(x) \, dm(x)$. Functions of finite mean oscillation were introduced by A. Ignat’ev and V. Ryazanov in the work [IR], see also section 11.2 in [MRSY]. There are the generalization and localization of the space BMO, that is bounded mean oscillation functions by F. John and L. Nirenberg, see for instance [JN].

Set $l(f'(x)) := \inf_{h \in \mathbb{R}^n:|h|=1} |f'(x)h|$. Recall that \textit{inner dilatation} of the mapping $f$ at a point $x$ is defined as

$$K_I(x, f) = \begin{cases} \frac{|J(x, f)|}{|f'(x)|}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}.$$

\textit{Outer dilatation} of the mapping $f$ at the point $x$ can be defined as

$$K_O(x, f) = \begin{cases} \frac{|f'(x)|}{|J(x, f)|}, & J(x, f) \neq 0, \\ 1, & f'(x) = 0, \\ \infty, & \text{otherwise} \end{cases}.$$

It is well–known that $K_I(x, f) \leq K^n_{I}(x, f)$ everywhere at the points, where there are well–defined, see for instance formulae (2.7) and (2.8) of the section 2.1 of Ch. I in [Re]. In particular, for the mappings with bounded distortion we have $K_I(x, f) \leq K^n_{I}$ at a.e. $x$, that follows from it’s definition. The main result of the paper is the following.

**Statement 1'.** Let $b \in D$ and $f : D \setminus \{b\} \to \mathbb{R}^n$ be an open and discrete mapping with finite length distortion. Suppose that there exists $\delta > 0$ such that

$$|f(x)| \leq C \left( \log \frac{1}{|x - b|} \right)^p,$$

at every $x \in B(b, \delta) \setminus \{b\}$ and some constants $p > 0$ and $C > 0$. Let there exists a function $Q : D \to [1, \infty]$, such that $K_I(x, f) \leq Q(x)$ a.e. $x \in D$ and $Q(x) \in \text{FMO}(b)$. Then a point $b$ is a removable singularity, or a pole of the mapping $f$.

**Remark 1.2.** Note that the condition (1.2) is stronger than the requirement (1.1), and that from the Statement 1' it follows the Statement 1.

## 2 Preliminaries. The main Lemma

The following definitions can be found in the monograph [Rii], Ch. II, Section. 3, see also section 3.11 in [Vak]. Let $f : D \to \mathbb{R}^n$ be an arbitrary mapping, $\beta : [a, b) \to \mathbb{R}^n$ is a path and $x \in f^{-1}(\beta(a))$. A path $\alpha : [a, c) \to D$ is called a \textit{maximal f – lifting} of $\beta$ starting at $x$, if (1) $\alpha(a) = x$; (2) $f \circ \alpha = \beta|_{[a, c)}$; (3) if $c < c' \leq b$, then there does not exist a path $\alpha' : [a, c') \to D$ such that $\alpha = \alpha'|_{[a, c)}$ and $f \circ \alpha' = \beta|_{[a, c')}$. 

Let $x_1, \ldots, x_k$ be $k$ different points of $f^{-1}(\beta(a))$ and let

$$m = \sum_{i=1}^{k} i(x_i, f).$$

We say that the sequence $\alpha_1, \ldots, \alpha_m$ is a maximal sequence of $f$-lifting of $\beta$ starting at points $x_1, \ldots, x_k$, if

1. each $\alpha_j$ is a maximal $f$-lifting of $\beta$,
2. $\text{card} \{ j : a_j(a) = x_i \} = i(x_i, f), \ 1 \leq i \leq k$,
3. $\text{card} \{ j : a_j(t) = x \} \leq i(x, f)$ for all $x \in D$ and for all $t$.

Let $f$ be a discrete open mapping and $x_1, \ldots, x_k$ are distinct points in $f^{-1}(\beta(a))$. Then $\beta$ has a maximal sequence of $f$-liftings starting at points $x_1, \ldots, x_k$, see Theorem 3.2 Ch. II in [Ri]. The following statement was proved by author, see for instance Theorem 3.1 in [Sev3].

**Proposition 2.1.** Let $f : D \to \mathbb{R}^n$ be a discrete and an open mapping with finite length distortion, $\Gamma$ a path family in $D$, $\Gamma'$ a path family in $\mathbb{R}^n$ and $m$ a positive integer such that the following is true. Suppose that for every path $\beta$ in $\Gamma'$ there are paths $\alpha_1, \ldots, \alpha_m$ in $\Gamma$ such that $f \circ \alpha_j \subset \beta$ for all $j = 1, \ldots, m$ and such that for every $x \in D$ and all $t$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices $j$. Then

$$M(\Gamma') \leq \frac{1}{m} \int_D K_1(x, f) \cdot \rho^n(x) \ dm(x) \quad (2.1)$$

for every $\rho \in \text{adm} \Gamma$.

In particular, the Proposition 2.1 generalizes the corresponding result of O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, that is the above statement under $m = 1$, see for instance Theorem 6.10 in [MRSY1], of Theorem 8.6 in [MRSY2]. By $\Gamma(E, F, D)$ we denote the family of all curves $\gamma : [a, b] \to \mathbb{R}^n$ connecting $E$ and $F$ in $D$, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ as $t \in (a, b)$. A compact set $G \subset \mathbb{R}^n$ is said to be a set of capacity zero, write $\text{cap} G = 0$, if there exists $T \subset \mathbb{R}^n$, such that $M(\Gamma(T, G, \mathbb{R}^n)) = 0$, see, for instance, Section 2 of Ch. III and Proposition 10.2 of Ch. II in [Ri]. By definition, an arbitrary set has a zero capacity if and only if every it’s compact subset has a zero capacity. The sets of capacity zero are totally disconnected, i.e., the condition $\text{cap} G = 0$ implies that $\text{Int} G = 0$, see e.g. Corollary 2.5 of Ch. III in [Ri]. Open set $U \subset D$, $\overline{U} \subset D$, is said to be a normal neighborhood of the point $x \in D$ under the mapping $f : D \to \mathbb{R}^n$, iff $U \cap f^{-1}(f(x)) = \{x\}$ and $\partial f(U) = f(\partial U)$, see e.g. Section 4 of Ch. I in [Ri].

**Proposition 2.2.** Let $f : D \to \mathbb{R}^n$ be an open discrete mapping, then for every $x \in D$ there exists $s_x$ such that, for every $s \in (0, s_x)$, the $x$-component of $f^{-1}(B(f(x), s))$, denoted by $U(x, f, s)$, is a normal neighborhood of $x$ under $f$, $f(U(x, f, s)) = B(f(x), s)$ and $\text{diam} U(x, f, s) \to 0$ as $s \to 0$, see, for instance, Lemma 4.9 of Ch. I in [Ri].

The main tool under the proof of the basic results of the present work is the following

**Lemma 2.1.** Let $b \in D$ and $f : D \setminus \{b\} \to \mathbb{R}^n$ be an open and a discrete mapping with finite length distortion. Suppose that there exists $\delta > 0$ such that

$$|f(x)| \leq C \left( \log \frac{1}{|x - b|} \right)^\nu,$$

(2.2)
at every $x \in B(b, \delta) \setminus \{b\}$ and some constants $p > 0$ and $C > 0$. Follow, suppose that there exist a measurable function $Q : D \to [1, \infty]$, numbers $\varepsilon_0 > 0$, $\varepsilon_0 < \text{dist} (b, \partial D \setminus \{b\})$, $A > 0$ and a Borel function $\psi(t) : [0, \varepsilon_0] \to (0, \infty)$ such that $K_I(x, f) \leq Q(x)$ a.e., and

$$\int_{|x - b| < \varepsilon_0} Q(x) \cdot \psi^{n}(|x - x_0|) \, dm(x) \leq \frac{A \cdot I^{n}(\varepsilon, \varepsilon_0)}{(\log \log \frac{1}{\varepsilon})^{n-\rho \varepsilon}} \quad \forall \ \varepsilon \in (0, \varepsilon_0/2),$$  

(2.3)

where

$$0 < I(\varepsilon, \varepsilon_0) = \int_{\varepsilon}^{\varepsilon_0} \psi(t) dt < \infty \quad \forall \ \varepsilon \in (0, \varepsilon_0).$$  

(2.4)

Then a point $b$ is a removable singularity, or a pole of the mapping $f$.

Proof. Suppose the contrary, i.e., a point $b$ is an essential singularity of $f$. Without loss of generality, we can consider that $b = 0$ and $C = 1$. In this case, there exists $R > 0$, such that

$$f(S(0, \delta)) \subset B(0, R).$$  

(2.5)

Since $b = 0$ is an essential singularity of $f$, from the conditions (2.1), (2.3) and another author’s result, see Lemma 3.1, Lemma 5.1 and Theorem 6.5 in [Sev], we have

$$N(y, f, B(0, \delta) \setminus \{0\}) = \infty$$

for all $y \in \mathbb{R}^n \setminus E$, where $\text{cap} E = 0$. Since $E$ of zero capacity, $\mathbb{R}^n \setminus E$ is unbounded. Thus, there exists $y_0 \in \mathbb{R}^n \setminus (E \cup B(0, R))$.

Let $k_0 > \frac{A \rho^{n-1}}{4a^{n-1}}$, $k_0 \in \mathbb{N}$. Since $N(y_0, f, B(0, \delta) \setminus \{0\}) = \infty$, there exist the points $x_1, \ldots, x_{k_0} \in f^{-1}(y_0)$, $x_1, \ldots, x_{k_0} \in B(0, \delta) \setminus \{0\}$. By Proposition 2.2, there exists $r > 0$, such that every point $x_j$, $j = 1, \ldots, k_0$, has a normal neighborhood $U_j := U(x_j, f, r)$ with $\cup_{j} \cap \cup_{m} = \emptyset$ at all $l \neq m$, $l, m \in \mathbb{N}$, $1 \leq l \leq k_0$ and $1 \leq m \leq k_0$.

Set $d := \min \{\varepsilon_0, \text{dist} (0, \cup_{1} \cup \ldots \cup_{k_0})\}$. Let $a \in (0, d)$ and $V := B(0, \delta) \setminus B(0, a)$. By (2.2), taking into account that $\partial f(V) \subset f(\partial V)$ and $C = 1$, we have

$$f(V) \subset B \left(0, \left(\log \frac{1}{a}\right)^{p}\right).$$  

(2.6)

Since $z_0 := y_0 + re \in B(y_0, r) = f \left(\cup_{j} \cup_{f, r}\right)$, $j = 1, \ldots, k_0$, we have $z_0 \in f(V)$. Thus there exists a sequence of the points $\tilde{x}_1, \ldots, \tilde{x}_{k_0}$, $\tilde{x}_j \in \overline{U}_j$, $1 \leq j \leq k_0$, such that $f(\tilde{x}_j) = z_0$.

Note that $k_0 \leq \sum_{j=1}^{k_0} i(\tilde{x}_j, f) = m'$. Let $H$ be a hemisphere $H = \{e \in S^{n-1} : (e, y_0) > 0\}$, $\Gamma'$ be a curve’s family $\beta : [r, (\log \frac{1}{a})^{p}] \to \mathbb{R}^n$ of the type $\beta(t) = y_0 + te$, $e \in H$, and $\Gamma$ be a sequence of maximal $f$ – liftings of $\beta$ under the mapping $f$ in $V$, starting at the points $\tilde{x}_1, \ldots, \tilde{x}_{k_0}$, $\tilde{x}_j \in \overline{U}_j$, $1 \leq j \leq k_0$, consisting from $m'$ curves, where $m' = \sum_{j=1}^{k_0} i(\tilde{x}_j, f)$. Such a sequence exists by Theorem 3.2 of Ch. II in [Ri]. By Proposition 2.1

$$M(\Gamma') \leq \frac{1}{m'} \int_{D} K_I(x, f) \cdot \rho^{n}(x) dm(x) \leq \frac{1}{k_0} \int_{D} K_I(x, f) \cdot \rho^{n}(x) dm(x)$$  

(2.7)
for every $\rho \in \text{adm} \Gamma$.

Given $e \in H$, we show that, for every curve $\beta = y_0 + te$ and it’s maximal lifting $\alpha(t) : [r, c) \to V$ starting at the point $\tilde{x}_{j_0}$, $\alpha \in \Gamma$, $1 \leq j_0 \leq k_0$, there exists a sequence $r_k \in [r, c]$ with $r_k \to c - 0$ as $k \to \infty$ such that $\text{dist} (\alpha(r_k), \partial V) \to 0$ as $k \to \infty$. Suppose the contrary, i.e., there exists $e_0 \in H$, such that $\beta(t), t \in [r, c)$, is a maximal lifting of $\beta = y_0 + te_0$, and $\alpha(t)$ lies inside of $V$ with it’s closure. Let $C(c, \alpha(t))$ denotes a cluster set of $\alpha$ as $t \to c - 0$. For every $x \in C(c, \alpha(t))$ there exists a sequence $t_k \to \infty$ such that $x = \lim_{k \to \infty} \alpha(t_k)$. Since $f$ is continuous and $C(c, \alpha(t)) \subset V$ by the assumption, we have $f(x) = f(\lim_{k \to \infty} \alpha(t_k)) = \lim_{k \to \infty} \beta(t_k) = \beta(c)$, from what it follows that $f$ is a constant on $C(c, \alpha(t))$. Since $f$ is a discrete and a set $C(c, \alpha(t))$ is connected, we have $C(c, \alpha(t)) = p_1 \in V$. Let $c \neq b_0 : = (\log \frac{1}{a})^p$. In this case, we can construct a lifting $\alpha'$ of $\beta$ started at $p_1$. Unit the liftings $\alpha$ and $\alpha'$, we obtain another maximal lifting $\alpha''$ of $\beta$ starting at the point $\tilde{x}_{j_0}$, that contradicts to the maximal property of the first lifting $\alpha$. Thus, $c = b_0$ and hence $\alpha$ can be extend to closed curve defined on the segment $[r, (\log \frac{1}{a})^p]$ (we don’t change the notion of the extended curve). Then, at every $t \in [r, (\log \frac{1}{a})^p]$, we have $\beta(t) = f(\alpha(t)) \subset f(V)$. In particular, by (2.6)

$$z_1 := y_0 + \left( \log \frac{1}{a} \right)^p e_0 \in f(V) \subset B \left( 0, \left( \log \frac{1}{a} \right)^p \right). \quad (2.8)$$

However, since $e_0 \in H$, we have

$$|z_1| = \left| y_0 + \left( \log \frac{1}{a} \right)^p e_0 \right| = \sqrt{|y_0|^2 + 2 \left( y_0, \left( \log \frac{1}{a} \right)^p e_0 \right) + \left( \log \frac{1}{a} \right)^2} \geq \sqrt{|y_0|^2 + \left( \log \frac{1}{a} \right)^2} \geq \left( \log \frac{1}{a} \right)^p. \quad (2.9)$$

However, the relation (2.9) contradicts to (2.8), which disproves the assumption that $\alpha(t)$ consists in the set $V$ with it’s closure. Consequently, dist $(\alpha(r_k), \partial V) \to 0$ as $k \to c - 0$ and some sequence $r_k \in [r, c]$ such that $r_k \to c - 0$ as $k \to \infty$.

Note that the situation when dist $(\alpha(r_k), S(0, \delta)) \to 0$ as $k \to c - 0$ is excluded. In fact, suppose that there exist $p_2 \in S(0, \delta)$ and a sequence $k_l, l \in \mathbb{N}$, such that $\alpha(r_k) \to p_2$ as $l \to \infty$. By the continuously of $f$ we have that $\beta(r_k) \to f(p_2)$ as $l \to \infty$, that is impossible by (2.7), because for every $e \in H$ and $t \in [r, (\log \frac{1}{a})^p]$ we have $|\beta(t)| = |y_0 + te| = \sqrt{|y_0|^2 + 2t(y_0, e) + t^2} \geq |y_0| > R$ by the choosing of $y_0$.

It follows from above that there exists a sequence $r_k \in [r, c]$ such that $r_k \to c - 0$ as $k \to \infty$ and $\alpha(r_k) \to p_3 \in S(0, a)$. Besides that, every such a curve $\alpha \in \Gamma$ intersects the sphere $S(0, d)$ because $\alpha$ has a start outside of $B(0, d)$. Consider the function

$$\rho_a(x) = \begin{cases} 
\psi(|x|)/I(a, d), & x \in B(0, d) \setminus B(0, a), \\
0, & x \in \mathbb{R^n} \setminus (B(0, d) \setminus B(0, a))
\end{cases},$$

where $I(a, d)$ is defined as in (2.4) and $\psi$ be a function from the condition of Lemma. Since $\psi(t) > 0$, $I(a, d) > 0$ for every $0 < a < d$. Thus, a function $\rho_a(x)$ which is given above is well–defined. Note that a function $\rho_a(x)$ is Borel, moreover, since $\rho_a(x)$ is a radial function,
by the above properties of curves of $\Gamma$ and by Theorem 5.7 in [Va1], for every curve $\alpha \in \Gamma$ we have

$$\int_\alpha \rho_a(x)dx \geq \frac{1}{I(a,d)} \int_a^d \psi(t)dt = 1,$$

i.e., $\rho_a(x) \in \text{adm} \Gamma$. Thus, from (2.3) and (2.7) we have

$$M(\Gamma') \leq \frac{1}{k_0 \cdot I^n(a,d)} \int_{a < |x| < d} K_I(x, f) \cdot \psi^n(|x|)dm(x) \leq$$

$$\leq \frac{I^n(a, \varepsilon_0)}{k_0 \cdot I^n(a,d) \cdot I^n(a,\varepsilon_0)} \int_{a < |x| < \varepsilon_0} Q(x) \cdot \psi^n(|x|)dm(x) =$$

$$= \left(1 + \frac{I(d, \varepsilon_0)}{I(a,d)} \right)^n \frac{1}{k_0 \cdot I^n(a,\varepsilon_0)} \int_{a < |x| < \varepsilon_0} Q(x) \cdot \psi^n(|x|)dm(x) \leq$$

$$\leq \frac{2}{k_0 \cdot I^n(a,\varepsilon_0)} \int_{a < |x| < \varepsilon_0} Q(x) \cdot \psi^n(|x|)dm(x)$$

at every $a \in (0, d_1)$ and some $d_1, d_1 \leq d$, because by (2.3), $I^n(a, d) \to \infty$ as $a \to \infty$. Now, from (2.3), we conclude that

$$M(\Gamma') \leq \frac{2A}{k_0 \left(\log \log \frac{1}{a}\right)^{n-1}} \quad (2.10)$$

for $a \in (0, d_1)$. From other hand, by section 7.7 in [Va1],

$$M(\Gamma') = \frac{1}{2} \frac{\omega_{n-1}}{\left(\log \left(\frac{\log ^\frac{1}{p} r}{a}\right)\right)^{n-1}} \cdot \quad (2.11)$$

By (2.10) and (2.11) we have

$$\frac{1}{2} \frac{\omega_{n-1}}{\left(\log \left(\frac{\log ^\frac{1}{p} r}{a}\right)\right)^{n-1}} \leq \frac{2A}{k_0 \left(\log \log \frac{1}{a}\right)^{n-1}},$$

from what

$$\left(\log \left(\frac{\log ^\frac{1}{p} r}{a}\right)^{\frac{2}{\omega_{n-1}}}\right)^{n-1} \geq \left(\log \left(\frac{\log ^\frac{1}{p} r}{a}\right)^{\frac{k_0}{2A}}\right)^{\frac{n-1}{n}}$$

$$\left(\frac{\log ^\frac{1}{p} r}{a}\right)^{\frac{2}{\omega_{n-1}}} \geq \left(\frac{\log ^\frac{1}{p} r}{a}\right)^{\frac{k_0}{2A}}$$
the composition of the mappings \( h \circ f \). It cannot be an essential singularity of \( f \).

In the right-hand part of it we obtain that \( f \) has no singularity of \( f \) at \( m \).

Note that a mapping with finite length distortion.

Letting to the limit as \( a \to 0 \) in the right-hand part of it we obtain that
\[
\frac{1}{r} \geq \left( \log \frac{1}{a} \right)^{ \frac{1}{n-1} } .
\]

Since by the choosing \( k_0 > \frac{4A \omega_{n-1}}{\omega_{n-1}} \), in the right-hand part of the above relation the logarithmic function presents in some positive degree. Letting to the limit as \( a \to 0 \) in the right-hand part of it we obtain that
\[
\frac{1}{r} \geq \infty ,
\]

that is impossible. The contradiction obtained above disproves the assumption that \( b = 0 \) is an essential singularity of \( f \). \( \square \)

The next statement follows directly from Lemma 5 in \[Sev\] as \( \psi(t) = \frac{1}{t \log t} \) and from the estimate \((2.1)\) at \( m = 1 \).

**Proposition 2.3.** Let \( b \in D \) and \( f : D \to \mathbb{R}^n \) be an open and a discrete mapping with finite length distortion. Suppose that there exist a measurable function \( Q : D \to [1, \infty] \), the numbers \( \varepsilon_0 > 0, \varepsilon_0 < \text{dist} (b, \partial D) \), and \( A > 0 \) such that \( K_I(x, f) \leq Q(x) \) a.e., such that the relations \((2.3)\) and \((2.4)\) hold as \( \psi(t) = \frac{1}{t \log t} \), i.e.,

\[
\int_{\varepsilon < |x-b| < \varepsilon_0} \frac{Q(x)}{|x-b| \log^n \frac{1}{|x-b|}} \, dm(x) \leq A \cdot \log \frac{1}{\log \frac{1}{\varepsilon_0}} \quad \forall \ \varepsilon \in (0, \varepsilon_0) . \tag{2.12}
\]

Then for every \( x \in B(b, \varepsilon_0) \)

\[
|f(x) - f(b)| \leq \frac{\alpha_n (1 + R^2)}{\delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x-b|}} \right\}^{\beta_n} , \tag{2.13}
\]

where \( \alpha_n \) and \( \beta_n = \left( \frac{\omega_{n-1}}{A} \right)^{1/(n-1)} \) depend only on \( n \), and \( \delta \) depends only on \( R \).

**Corollary 2.1.** Under the conditions of Lemma 2.1, suppose that the condition \((2.12)\) take a place instead of \((2.3)\) and \((2.4)\), and the condition

\[
\lim_{x \to b} \frac{|f(x)|}{(\log \frac{1}{|x-b|})^{\beta_n}} = 0 , \tag{2.14}
\]

take a place instead of \((2.2)\), where \( \beta_n = \left( \frac{\omega_{n-1}}{A} \right)^{1/(n-1)} \). Then a point \( x = b \) is a removable singularity of \( f \).

**Proof.** Without loss of generality, we can consider that \( b = 0 \). By Lemma 2.1 a point \( b \) can not to be an essential singularity of \( f \). Suppose that \( b = 0 \) is a pole of \( f \). Consider the composition of the mappings \( h = g \circ f \), where \( g(x) = \frac{x}{|x|^2} \) is inversion under the sphere \( \mathbb{S}^{n-1} \).

Note that a mapping \( h \) to be a mapping with finite length distortion, \( K_I(x, f) = K_I(x, h) \) and \( h(0) = 0 \). Moreover, \( h \) is bounded in some neighborhood of zero. Thus there exist \( \varepsilon_1 > 0 \) and \( R > 0 \) such that \( |h(x)| \leq R \) at every \( |x| < \varepsilon_1 \). Now we apply a Proposition 2.3 By \((2.13)\),

\[
|h(x)| = \frac{1}{|f(x)|} \leq \frac{\alpha_n (1 + R^2)}{\delta} \left\{ \frac{\log \frac{1}{\varepsilon_0}}{\log \frac{1}{|x|}} \right\}^{\beta_n} .
\]
Consequently,
\[
\frac{|f(x)|}{\left\{ \log \frac{1}{|x|} \right\}^{\beta_n}} \geq \frac{\delta}{\alpha_n(1 + R^2)\left\{ \log \frac{1}{\varepsilon_0} \right\}^{\beta_n}}.
\]
However, the last relation contradicts to the (2.14). The contradiction obtained above prove
that \( b = 0 \) is a removable singularity of \( f \). □

3 The proof of the main results

Proof of the statement 1’ follows from (2.12) which holds for every function \( Q \in FMO(b) \),
see, for instance, Corollary 2.3 in [IR], or Lemma 6.1 of Ch. VI in [MRSY], and from the
Lemma 2.1. □

Corollary 3.1. Let \( f : D \setminus \{ b \} \to \mathbb{R}^n \) be an open and a discrete mapping with finite
length distortion. Suppose that there exists a measurable function such that \( Q : D \to [1, \infty] \),
such that \( K_1(x, f) \leq Q(x) \) at a.e. \( x \in D \) and \( Q(x) \in FMO(b) \).
There exists \( p_0 > 0 \) such that
\[
\lim_{x \to b} \frac{|f(x)|}{\left\{ \log \frac{1}{|x-b|} \right\}^{p_0}} = 0
\]
implies that \( b = 0 \) is a removable singularity of \( f \).

Proof follows directly from the Statement 1’ and Corollary 2.1 □

In what follows \( q_{x_0}(r) \) denotes the integral average of \( Q(x) \) under the sphere \( |x - x_0| = r \),
\[
q_{x_0}(r) := \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x-x_0|=r} Q(x) dS,
\]
where \( dS \) is element of the square of the surface \( S \).

Theorem 3.1. Let \( b \in D \) and \( f : D \setminus \{ b \} \to \mathbb{R}^n \) be an open and a discrete mapping
with finite length distortion. Suppose that there exists \( \delta > 0 \) such that the relation (1.2)
holds at every \( x \in B(b, \delta) \) and some constants \( p > 0 \) and \( C > 0 \). Let there exists a function
\( Q : D \to [1, \infty] \), such that \( K_1(x, f) \leq Q(x) \) a.e. \( x \in D \) and \( q_b(r) \leq C \cdot (\log \frac{1}{r})^{n-1} \) as \( r \to 0 \).
The a point \( b \) is a pole or a removable singularity of \( f \).

Moreover, in addition, if the relation (3.1) holds as \( p_0 = \left( \frac{1}{C} \right)^{1/(n-1)} \), then a point \( b = 0 \) is
removable for \( f \).

Proof. We can consider that \( b = 0 \). Let \( \varepsilon_0 < \min \{ \text{dist} \ (0, \partial D), \ 1 \} \). Set \( \psi(t) = \frac{1}{t \log \frac{1}{r}} \).
Note that
\[
\int_{\varepsilon < |x| < \varepsilon_0} \frac{Q(x)dm(x)}{|x| \log \frac{1}{|x|}} = \int_{\varepsilon}^{\varepsilon_0} \left( \int_{|x|=r} \frac{Q(x)dm(x)}{|x| \log \frac{1}{|x|}} dS \right) dr \leq C \cdot \omega_{n-1} \cdot I(\varepsilon, \varepsilon_0),
\]
where as above \( I(\varepsilon, \varepsilon_0) := \int_{\varepsilon}^{\varepsilon_0} \psi(t) dt = \log \frac{1}{\varepsilon} \log \frac{1}{\varepsilon_0} \). Thus the conditions (2.3) and (2.4) of Lemma
2.1 hold at \( \psi \) which was given above. The second statement of the Theorem 3.1 follows from
the Corollary 2.1 □
4 Corollaries. The precision of the conditions

Recall that a point $y_0 \in D$ is said to be a branch point of the mapping $f : D \to \mathbb{R}^n$, if for every neighborhood $U$ of the point $y_0$ a restriction $f|_U$ fails to be a homeomorphism. A set of all branch sets of $f$ is denoted by $B_f$. The following statement can be found as Theorem 1 in [Sev2]. Let $f : D \to \mathbb{R}^n$ be an open discrete mapping of the class $W^{1,n}_{loc}(D)$ such that either $K_0(x,f) \in L^{n-1}_{loc}$, or $K_1(x,f) \in L^1_{loc}$, and $m(B_f) = 0$. Then $f$ to be a mapping with finite length distortion. Founded on the Statement 1' and on Theorem 4.1 we have the following.

**Theorem 4.1.** Let $b \in D$ and $f : D \setminus \{b\} \to \mathbb{R}^n$ be an open discrete mapping of the class $W^{1,n}_{loc}(D)$, for which either $K_0(x,f) \in L^{n-1}_{loc}$, or $K_1(x,f) \in L^1_{loc}$, and $m(B_f) = 0$. Suppose that there exists $\delta > 0$ such that the inequality

$$|f(x)| \leq C \left( \log \frac{1}{|x-b|} \right)^p$$

take a place for all $x \in B(b,\delta)$ and some constants $p > 0$ and $C > 0$. Besides that, suppose that there exists a measurable function $Q : D \to [1,\infty]$ such that $K_1(x,f) \leq Q(x)$ at a.e. $x \in D$ and $Q(x) \in FMO(b)$. Then a point $b$ is a removable singularity, or a pole of $f$.

Moreover, there exists a number $p_0 > 0$ such that the condition

$$\lim_{x \to b} \frac{|f(x)|}{\left( \log \frac{1}{|x-b|} \right)^{p_0}} = 0$$

implies that a point $b$ to be a removable singularity of $f$.

**Theorem 4.2.** All of the conclusions of the Theorem 4.1 take a place if the assumption $Q(x) \in FMO(b)$ to replace on the requirement $q_0(r) \leq C \cdot (\log \frac{1}{r})^{n-1}$ as $r \to 0$. In this case, we can take $p_0 = \left( \frac{1}{e} \right)^{1/(n-1)}$.

The following result shows that the conditions on $Q$ which are given above can not to be done more weaker, for instance, we can not replace it by the assumption $Q \in L^q_{loc}$, $q \geq 1$, for every sufficiently large $q$.

**Theorem 4.3.** Given $p > 0$ and $q \in [1,\infty)$, there exists a homeomorphism $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ with finite length distortion which belongs to $f \in W^{1,n}_{loc}$, and $f^{-1} \in W^{1,n}_{loc}$, such that $K_1 \in L^q_{loc}(\mathbb{B}^n)$,

$$|f(x)| \leq 2 \left( \log \frac{1}{|x|} \right)^p$$

(4.1)

at every $x \in B(0,1/e) \setminus \{0\}$, and a point $b = 0$ to be an essential singularity of $f$. Moreover, $f$ is a bounded mapping in this case.

**Proof.** The desired homeomorphism $f : \mathbb{B}^n \setminus \{0\} \to \mathbb{R}^n$ can be given as

$$f(x) = \frac{1 + |x|^\alpha}{|x|} \cdot x,$$

where $\alpha \in (0, n/q(n-1))$. We can consider that $\alpha < 1$. Note that $f$ maps $\mathbb{B}^n \setminus \{0\}$ onto the ring $\{1 < |y| < 2\}$ in $\mathbb{R}^n$, and the cluster set $C(f,0) = \{|y| = 1\}$. In particular, it follows from here that $x_0 = 0$ is an essential singularity of $f$. It is clear that $f \in C^1(\mathbb{B}^n \setminus \{0\})$ and,
consequently, \( f \in W_{\text{loc}}^{1,n} \), moreover, 
\[
K_f(x, f) = \left( \frac{1 + |x|^n}{\alpha |x|^n} \right)^{n-1} \leq \frac{C}{|x|^{n-1}},
\]
see Proposition 6.3 of Ch. VI in [MRSY_2]. Thus, 
\( K_f(x, f) \in L^q(\mathbb{B}^n) \) because \( \alpha(n-1)q < n \). Besides that, to note that \( f \) is locally quasiconformal mapping and hence \( f^{-1} \in W_{\text{loc}}^{1,n} \). Thus, \( f \) is a mapping with finite length distortion in \( \mathbb{B}^n \setminus \{0\} \) by Theorem 4.6 in [MRSY_3], see also Theorem 8.1 of Ch. VIII in [MRSY_2].

The mapping \( f \) is bounded in \( \mathbb{B}^n \setminus \{0\} \), in particular, \( f \) satisfies the inequality \( |f(x)| \leq 2 \) at \( x \in \mathbb{B}^n \setminus \{0\} \). From other hand, a function \( s(x) := \left( \log \frac{1}{|x|} \right)^q \) satisfies \( |s(x)| \geq 1 \) for all \( |x| \leq 1/e \). From here we have a relation (4.11).

Thus, we construct a mapping \( f \) which have an essential isolated singularity, and satisfying all of the conditions of the Theorem 4.3. \( \square \)

The following statement shows that the condition of the openness of the mapping \( f \) is essential.

**Theorem 4.4.** There exist a discrete mapping \( f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \) with finite length distortion such that \( K_f \equiv 1 \), satisfying to the condition

\[
|f(x)| \leq \left( \log \frac{1}{|x|} \right)^p
\]

at every \( x \in B(0, 1/e) \setminus \{0\} \) and \( p > 0 \), such that a point \( b = 0 \) is an essential singularity of \( f \).

**Proof.** Consider the division of \( \mathbb{R}^n \) by the cubes

\[
C_{k_1, \ldots, k_n} = \prod_{i=1}^n [2k_i - 1, 2k_i + 1], \quad k_i \in \mathbb{Z}.
\]

Consider a cube \( C_{k_1, \ldots, k_n} \) with \( k_1, \ldots, k_n \geq 0 \); the case of the different signs of \( k_i \) can be considered by analogy. Let \( x = (x_1, \ldots, x_n) \in C_{k_1, \ldots, k_n} \). If \( k_1 = 0 \), \( g_{m_1} := \text{id} \). Let \( k_1 > 0 \). Set \( f_{1, \ldots, 1, 1}(x) = y_{1, \ldots, 1, 1} \), where \( y_{1, \ldots, 1, 1} \) be a symmetric reflection of \( x \) under the hyperplane \( x_1 = 2k_1 - 1 \). If \( 2k_1 - 3 = -1 \), the process is finished. Let \( 2k_1 - 3 > -1 \), then \( f_{1, \ldots, 1, 2}(x) = y_{1, \ldots, 1, 2} \), where \( y_{1, \ldots, 1, 2} \) be a symmetric reflection of the point \( y_{1, \ldots, 1, 1} \) under the hyperplane \( x_1 = 2k_1 - 3 \). If \( 2k_1 - 5 = -1 \), the process is finished. In other case we continue, \( f_{1, \ldots, 1, 3}(x) = y_{1, \ldots, 1, 3} \). Etc. After a finite number of the steps \( m_1 \) we have a mapping \( g_{m_1} = f_{1, \ldots, 1, m_1} \circ \cdots \circ f_{1, \ldots, 1, 1} \), such that \( g_{m_1}(x) \in C_{0, k_2, k_3, \ldots, k_n} \).

Follow, if \( k_2 = 0 \), then \( g_{m_2} := g_{m_1} \). As \( k_2 > 0 \), we repeat the above transformations with the coordinate \( x_2 \) and the point \( x_{m_1} := g_{m_1}(x) \). Set \( f_{1, \ldots, 1, 2, m_1}(x) = y_{1, \ldots, 1, 2, m_1} \), where \( y_{1, \ldots, 1, 2, m_1} \) be a symmetric reflection of the point \( x_{m_1} \) under the hyperplane \( x_2 = 2k_2 - 1 \). If \( 2k_2 - 3 = -1 \), the process is finished. In other case we continue. Now, we have a mapping \( g_{m_2} = f_{1, \ldots, m_2, m_1} \circ \cdots \circ f_{1, \ldots, 2, m_1} \), such that \( g_{m_2}(x_{m_1}) \in C_{0, 0, k_3, \ldots, k_n} \).

Etc. After some number of the steps \( m_0 = m_1 + m_2 + \ldots + m_n \) we obtain a mapping \( G_0 = g_{m_n} \circ g_{m_{n-1}} \circ \cdots \circ g_{m_2} \circ g_{m_1} \), such that the image \( x_{m_n} \) of the point \( x \) under the mapping \( G_0 \) lies in the cube \( C_{0, 0, 0, \ldots, 0} \). The compressing \( G_1(x) = \sqrt[n]{n} \cdot x \) maps \( C_{0, 0, 0, \ldots, 0} \) into some cube \( A_0 \), which lies in \( \mathbb{B}^n \). Set \( G_2 := G_1 \circ G_0 \).

Note that a point \( z_0 = \infty \) is an essential singularity of \( G_2 \), moreover, \( C(G_2, \infty) = A_0 \subset \mathbb{B}^n \). Then a mapping

\[
g := G_2 \circ G_3,
\]

(4.3)
ON THE LOCAL BEHAVIOR ...

\[ G_3(x) = \frac{x}{|x|^2}, \] has an essential singularity \( b = 0, \) and

\[ C(g, 0) \subset \mathbb{R}^n. \]

By the construction of \( G_2, \) which is given by (4.3), \( G_2 \) is a discrete mapping, preserves the lengths of curves in \( \mathbb{R}^n, \) is differentiable a.e. and has \( N \) and \( N^{-1} \) – properties. Thus, \( g \) is a mapping with finite length distortion, moreover, it is easy to see that \( K_I(x, g) = 1. \) Finally, \( |g(x)| \leq 1 \) at every \( x \in \mathbb{R}^n \setminus \{0\}. \) Thus, (4.2) holds at every \( x \in B(0, 1/e) \setminus \{0\}. \)

The desired mapping have been constructed. □

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