Planning under Uncertainty to Goal Distributions
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Abstract—Goals for planning problems are typically conceived of as subsets of the state space. However, for many practical planning problems in robotics, we expect the robot to predict goals, e.g. from noisy sensors or by generalizing learned models to novel contexts. In these cases, sets with uncertainty naturally extend to probability distributions. While a few works have used probability distributions as goals for planning, surprisingly no systematic treatment of planning to goal distributions exists in the literature. This article serves to fill that gap. We argue that goal distributions are a more appropriate goal representation than deterministic sets for many robotics applications. We present a novel approach to planning under uncertainty to goal distributions, which we use to highlight several advantages of the goal distribution formulation. We build on previous results in the literature by formally framing our approach as an instance of planning as inference. We additionally derive reductions of several common planning objectives as special cases of our probabilistic planning framework. Our experiments demonstrate the flexibility of probability distributions as a goal representation on a variety of problems including planar navigation among obstacles, intercepting a moving target, rolling a ball to a target location, and a 7-DOF robot arm reaching to grasp an object.

I. INTRODUCTION

Goals enable a robot to act with intention in its environment and provide an interface for humans to specify the desired behavior of the robot. Defining and representing goals is therefore a fundamental step in formalizing robotics problems [48, 78]. Goals are most commonly represented as elements of a (possibly infinite) subset $G \subseteq \mathcal{X}$ of the robot’s state space $\mathcal{X}$. In practice, it is common to select a particular goal state $g \in G$ to pursue, or to simply define the goal set as the singleton $G = \{g\}$.

The goal state $g$ is typically incorporated into a goal-parameterized cost function $C_g : \mathcal{X} \to \mathbb{R}$ over the robot’s state space $\mathcal{X}$. For example, $C_g$ may be defined as a goal-parameterized distance function $d_g : \mathcal{X} \to \mathbb{R}$ (e.g. Euclidean distance) between the goal state $g$ and the current state $x_t \in \mathcal{X}$. The cost function $C_g$ is often used to monitor progress to a point-based goal [6], bias graph creation in sampling-based motion planners [48], or used directly as an objective for trajectory optimization [46]. In the latter case, an action cost is often further incorporated so that $C_g : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ can encode, for example, distance to goal while enforcing smoothness constraints on the actions $u_t \in \mathcal{U}$ from the robot’s action space $\mathcal{U}$.

The point-based goal formulation just described provides formal elegance but overlooks many common challenges of robotic systems, including stochastic dynamics, numerical imprecision, continuous state and action spaces, and uncertainty in the robot’s current state and goal. Each of these phenomena make it difficult for a robot to arrive in a desired state in a verifiable manner. As a result, it is common to utilize a catch-all fuzzy notion of goal achievement: the robot achieves its goal $g$ if it arrives in a terminal state $x_T \in \mathcal{X}$ such that $d_g(x_T) < \varepsilon$ where $\varepsilon \in \mathbb{R}^+$ is a small tolerance governing how much error is permissible. This common relaxation fails to differentiate the various sources of uncertainty and imprecision a robot faces in pursuing a goal.

We advocate for probability distributions as a more suitable goal representation. Goals as probability distributions extend the traditional notion of set-based goals to include a measure of belief (i.e. uncertainty) that a point in the state space satisfies the goal condition. Goal uncertainty arises frequently...
in robotics applications whenever the robot must predict its
goal, e.g. estimating a goal from noisy sensors [44], forecasting a
dynamic goal [39, 55], or in generalizing a learned
goal representation [3, 16, 42, 63]. See Fig. 1 for a couple of
examples we examine experimentally in this article. Goal
distributions are also a natural representation in many domains,
e.g. mixture models represent targets for distributed multi-
agent systems [23, 74] and encode heuristic targets for grasping
objects [16, 52, 57]. Goal distributions formally subsume
traditional point-based and set-based goals as Dirac-delta and
uniform distributions, respectively. In spite of the myriad use
cases and benefits of goal distributions in various domains,
planning and reinforcement learning frameworks continue to
primarily rely on point-based goal representations, and no
formal treatment of planning to goal distributions exists.

To address this gap, we formalize the use of probability
distributions as a goal representation for planning (Sec. III)
and discuss encoding common goal representations as distri-
butions (Sec. IV). As our main theoretical contribution in
this article, we derive a framework (Sec. V) for planning to
goal distributions under state uncertainty as an instance of planning
as inference [50, 71, 82]. Planning as inference seeks to infer
an optimal sequence of actions by conditioning on achieving
optimality and inferring the posterior distribution of optimal
trajectories. We derive a planning as inference objective that
enables the robot to plan under state uncertainty to a goal state
distribution by minimizing an information-theoretic loss be-
 tween its predicted state distribution and the goal distribution.
We show that several common planning objectives are special
cases of our probabilistic planning framework with particular
choices of goal distributions and loss function (Sec. VI). We
provide a practical and novel algorithm (Sec. VII) that utilizes
the unscented transform [83] for state uncertainty propagation
to efficiently compute the robot’s terminal state distribution
and expected running cost.

We present experimental results (Sec. VIII) from applying
our approach to a variety of different problems, goal distri-
butions, and planning objectives. We showcase the flexibility
of probability distributions as a goal representation on the
problem of planar navigation among obstacles (Sec. VIII-A).
These results also exhibit the ease with which our planning
framework accommodates different models of goal uncer-
tainty simply by swapping in different goal distributions and
information-theoretic losses. We provide an example of how
our planning approach can leverage sources of uncertainty in
the environment to achieve a target state distribution (Sec.
VIII-B) in a ball-rolling task. We also apply our approach to
the problem of intercepting a moving target in which the
agent updates its belief of the goal as it acquires noisy
observations of the target (Sec. VIII-C). Finally, we apply our
approach to the higher-dimensional problem of a 7-DOF robot
arm reaching to grasp an object from a table (Sec. VIII-D),
where we model target end-effector poses as a mixture of pose
distributions about an object. We show that we are able to
plan to reachable poses using this distribution directly as our
goal representation without checking for reachability, while
a point-based goal requires computing inverse kinematics to
check reachability prior to planning.

We conclude the article in Sec. IX where we discuss the
opportunities to expand the use of goal distributions in other
existing planning as inference frameworks (e.g. reinforcement
and imitation learning).

II. RELATED WORK

We first review works on planning as inference, since we
formally situate the problem of planning to goal distributions
as an instance of planning as inference. Planning as inference
has proven to be a powerful framework to formalize problems
in robotics [5, 46, 50, 58, 71, 82, 87]. It constitutes a Bayesian
view of stochastic optimal control [46, 71, 82] from which
popular decision-making algorithms can be derived such as
max-entropy reinforcement learning [50] and sampling-based
solvers like model-predictive path integral control (MPPC) [9, 46, 88] and the cross-entropy method (CEM) [40, 46]. The
probabilistic perspective of planning as inference enables
elegant problem definitions as factor graphs that can be solved
by message-passing algorithms [82, 87] and non-linear least
squares optimizers [58]. Goal distributions have not been
explicitly considered in the planning as inference framework.
Part of our contribution in this article is formulating the
problem of planning to goal distributions as an instance of
planning as inference, thereby connecting to the rich literature
on planning as inference and enabling access to a variety of
existing algorithms to solve the problem.

As mentioned in Sec. I, goal distributions have cropped up
in various sub-domains of robotics, but the topic has not yet
received systematic attention. The closest work we are aware
of is [61] which advocates for goal-distribution-conditioned
policies as an extension to goal-conditioned policies in rein-
forcement learning. A limited class of goal distributions are
considered in [61] and the formulation is specific to reinforce-
ment learning. We view the present article as complementary
to [61], where we consider more general classes of goal
distributions and develop an approach for planning to goal
distributions in a unified way.

Target distributions have been utilized in reinforcement
learning to encourage exploration [49] and expedite policy
learning [4]. State marginal matching [49] learns policies
for which the state marginal distribution matches a target distri-
bution (typically uniform) to encourage targeted exploration
of the environment. Goal distributions have also been used as
sample generators for goal-conditioned reinforcement learning
[60, 69]. Recent improvements on hindsight experience replay (HER) [4] have sought to estimate goal distributions
that generate samples from low density areas of the replay
buffer state distribution [43, 67, 93] or that form a curriculum
of achievable goals [72, 67]. We note that HER-based methods
are typically used in multi-goal reinforcement learning where
states are interpreted as goals irrespective of their semantic
significance as a task goal.

A closely related but distinct area of research is covariance
steering [32] which optimizes feedback controller parameters
and an open-loop control sequence to move a Gaussian state
distribution to a target Gaussian distribution. Recent work
on covariance steering has focused on satisfying chance con-
straints [62, 66, 73], including generating constraint-satisfying
samples for MPPI [51]. Covariance steering typically assumes
linear dynamics and is limited to Gaussian state distributions,
where the objective is often to ensure the terminal state
covariance is fully contained within the goal state covariance.
Covariance steering also decouples control of the distribution
into mean steering and covariance steering, which assumes
homogeneous uncertainty and accuracy of control over the
state space. These assumptions do not hold for most robotics
domains. For example, visual odometry estimates degrade
when entering a dark room [81], and it is harder to maneuver
over varied terrain [19]. In contrast, our approach easily
accommodates non-linear dynamics, non-homogeneous state
uncertainty, and non-Gaussian state distributions.

We utilize information-theoretic costs in our approach
as they are fitting objectives for optimizing plans to goal
distributions. A variety of information theoretic costs have
been utilized in planning and policy optimization which we
briefly review here. Probabilistic control design [45] minimizes
Kullback-Leibler (KL) divergence between controlled and
desired state-action distributions and bears some similarity
to planning as inference previously described. KL divergence
is also commonly used to constrain optimization iterations
in policy search [15] 65 75 and planning [2]. Broader
classes of divergences including f-divergence [8] 24 37
and Tsallis divergence [86] have also been utilized for policy
improvement [8], imitation learning [24, 37], and stochastic
optimal control [86]. Stochastic optimal control and planning
techniques often seek to minimize the expected cost of the
trajectory [20] 89 or maximize the probability of reaching a
goal set [13] 51. Entropy of a stochastic policy is utilized
in maximum-entropy reinforcement learning [29] 30 27 and
inverse reinforcement learning [94] to prevent unnecessarily
biasing the policy class. We demonstrate the use of cross-
entropy and KL divergence in our formulation of planning
to goal distributions. However, our approach is general enough
to admit other information-theoretic losses between the robot’s
predicted state distribution and a goal distribution. We discuss
this point further in Sec. [IX]

III. PROBLEM STATEMENT

We focus our work on planning problems with continuous
state and action spaces in which the robot must reach a
desired goal while acting under stochastic dynamics. These
problems fall in the domain of stochastic optimal control and
planning [77] 71. We consider a robot with continuous state
space $\mathcal{X} \subseteq \mathbb{R}^{N_x}$ and continuous action space $\mathcal{U} \subseteq \mathbb{R}^{N_u}$
and focus on a discrete time setting, although extending to
to continuous time would be straightforward. The stochastic
dynamics function $f: \mathcal{X} \times \mathcal{U} \times \Omega \rightarrow \mathcal{X}$ determines the resulting
state $x_{t+1} = f(x_t, u_t, \omega_t)$ from applying action $u_t \in \mathcal{U}$ in
state $x_t \in \mathcal{X}$ at time $t$ subject to noise $\omega_t \sim \Omega$.

Planning to a goal state $g \in \mathcal{X}$ in this setting requires the
agent to utilize the dynamics function $f$ to find a sequence of
actions $u_0, \ldots, u_{T-1} \in \mathcal{U}^T$ from its initial state $x_0 \in \mathcal{X}$ to

1We use $x_0$ to indicate the state from which the agent is planning, but note
the timestep is arbitrary and replanning from any timestep is permissible, as
is common in model-predictive control schemes [40] 88.

the goal state $g$. Due to sensor noise and partial observability,
the robot rarely knows its current state precisely and must
therefore plan from an initial estimated distribution of states
$p(x_0)$, e.g. as the output of a state estimator like a Kalman
filter [81]. Typically the robot must minimize the expected cost
under state uncertainty and stochastic dynamics defined by a
cost function $C: \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}$. As noted in Sec. [I] the cost
function typically includes a distance function parameterized
by $g$ to induce goal-seeking behavior.

Instead of planning to particular states, we consider the
more general problem of planning to a goal state distribution.
A goal distribution $p(x|g = 1)$ encodes the robot’s belief (i.e. uncertainty)
that a particular state $x$ belongs to the goal set $\mathcal{G}$. Using Bayes rule, we see for any particular $x$ the goal
 density is proportional to the goal likelihood $p(x|g = 1) \propto
p(g = 1|x)$. We abbreviate the goal distribution as $p_g(x)$. In
this article, we assume the goal distribution is given in order
to focus our efforts on formalizing the problem of planning
to goal distributions. We note that goal distributions can be
set as desired if known in parametric form, which can be as
simple as adding uncertainty bounds to a point-based goal,
e.g. adding a Gaussian covariance or setting uniform bounds
on a region centered about a target point. Goal distributions
can also be estimated from data [3] 16 42 63. We discuss
this point further in Sec. [IX]

Given a goal distribution, we require two main ingredients
to generate a plan to it. First, we need a means of predicting
the robot’s terminal state distribution after following a planned
sequence of actions. This is a form of state uncertainty
propagation which we discuss further in Sec. [VII-A] Second,
we require a loss function $L: Q \times \mathcal{P} \rightarrow \mathbb{R}$ that quantifies
the difference between distributions $q \in Q$ and $p \in \mathcal{P}$,
where $Q$ and $\mathcal{P}$ are arbitrary families of distributions. In
particular, we are interested in quantifying the difference
between the robot’s terminal state distribution and the
goal state distribution. Typically $L$ will take the form of a statistical
divergence (e.g. KL divergence, total variation, etc.), but we
leave open the possibility for other losses which may not meet
the formal definition of a divergence (e.g. cross-entropy).

We can now formally state our problem as minimizing
the information-theoretic loss $L$ between the terminal state
distribution $q(x_T | X_{T-1}, U_{T-1})$ and $p_g(x_T)$, where we
abbreviate $U_t \equiv (u_0, \ldots, u_t)$, and $X_t \equiv (x_0, \ldots, x_t)$.
We formulate this as the following constrained optimization
problem

$$\arg\min_{\pi} L(q(x_T | X_{T-1}, U_{T-1}), p_g(x_T)) \quad (1a)$$

$$+ E_{q(x_{T-1}, U_{T-1})} \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] \quad (1b)$$

s.t. $X_T \in \mathcal{X}^T, U_{T-1} \in \mathcal{U}^T$ \quad (1c)

$q(x_0) = p(x_0)$ \quad (1d)$

where $\pi$ defines the policy being optimized. In its simplest
form, the policy can just be a sequence of actions $\pi = U_{T-1}$,
but we note that more general policy parameterizations can
also be utilized. The sequence of states $X_t$ is induced by the
dynamics function $f$ together with the sequence of actions $U_t$. 
Eq. [12] is the loss between the terminal state distribution under the policy and the goal distribution. Eq. [13] is the expected cost accumulated over the trajectory, where $c_t : X \times U \rightarrow \mathbb{R}$ encapsulates arbitrary running costs. Eq. [14] ensures states and actions are from the robot's state-action space. Eq. [15] is a constraint that ensures planning initiates from the robot's belief about its initial state.

We present a more concrete instantiation of this optimization problem in Sec. VII-A, where we present a tractable method for computing the terminal state distribution utilized in the information-theoretic loss in Eq. [12]. Before diving into the details of our planning formulation in Sec. V, we first define some useful goal distributions in Sec. IV.

### IV. Goal Distributions

We present a selection of goal distributions we explore in this article. Our list is by no means exhaustive and we emphasize there are likely domains that benefit from less standard distributions [1].

#### A. Dirac-delta

A Dirac-delta distribution has a density function with infinite density at its origin point $p$ and zero at all other points:

$$\delta_p(x) = \begin{cases} \infty & \text{if } x = p \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

such that $\int_X \delta_p(x) dx = 1$ over its domain $X$. The Dirac-delta is a probabilistic representation of a point-based goal [14]. Point-based goals are the most common goal representation in both planning [48] and reinforcement learning [78].

#### B. Uniform

A uniform distribution is a probabilistic representation of a set-based goal and has the density function

$$U_A(x) = \begin{cases} \frac{1}{\text{vol}(A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

where vol$(A)$ defines the volume of the set $A$. Uniform distributions are useful for encoding a bounded region of acceptable goal states without any preference to any particular state within that region. Examples include navigating to be in a particular room [69], or placing an object on a desired region of a table surface [18]. Uniform goal distributions have also proven useful as goal sample generators, particularly when learned to bias goal-conditioned policies to reach desirable states [69].

#### C. Gaussian

A Gaussian distribution has the density function

$$\mathcal{N}(x | \mu, \Sigma) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp \left( -\frac{1}{2} d_M(x; \mu, \Sigma) \right) \quad (4)$$

where $\mu$ and $\Sigma$ define the mean and covariance of the distribution, respectively, and $d_M(x; \mu, \Sigma) = \sqrt{(x - \mu)^T \Sigma^{-1} (x - \mu)}$ is the Mahalanobis distance [56]. Gaussian distributions have been utilized in learning from demonstration to encode goals learned in a data-driven fashion from sub-optimal experts [8, 16, 42, 63]. Gaussians also naturally encode uncertainty the agent might have about its goal, e.g. in dynamic tasks like object handovers [55] and catching moving objects [49], or estimating a goal online from noisy observations, e.g. footstep planning [44].

We also consider a truncated Gaussian [79] distribution with density function

$$\mathcal{N}(x | \mu, \Sigma, A) = \begin{cases} \lambda_i (v_i^T x)^2 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

which is a common model for bounded Gaussian uncertainty [17].

#### D. Bingham

The Bingham distribution [11] is an antipodally symmetric distribution on the unit hypersphere $S^d$ with density function

$$B(x | \Lambda, V) = \frac{1}{F(\Lambda)} \exp \left( \sum_{i=1}^{d} \lambda_i (v_i^T x)^2 \right) \quad (6)$$

where $x \in S^d \subset \mathbb{R}^{d+1}$ is constrained to the unit hypersphere $S^d$, $\Lambda$ is a diagonal matrix of concentration parameters, the columns of $V$ are orthogonal unit vectors, and $F(\Lambda)$ is a normalization constant. The Bingham density function bears resemblance to the multivariate Gaussian density function because it is derived from a zero-mean Gaussian conditioned to lie on the unit hypersphere $S^d$.

The Bingham distribution on $S^3$ is of particular interest to robotics as it encodes a Gaussian distribution over orientations in 3D [27]. The key difficulty in utilizing the Bingham distribution is computing the normalization constant $F(\Lambda)$ since no closed-form solution exists. In spite of this, efficient and accurate approximations have enabled use of the Bingham distributions for 6-DOF object pose estimation [28], tracking of moving objects [27], and pose uncertainty quantification in deep neural networks [64].

#### E. Mixture Models

Mixture models comprise a weighted combination of multiple probability distributions. A Gaussian mixture model (GMM) is the most commonly used mixture model with density function

$$\mathcal{M} \left( x | \alpha_i, \mu_i, \Sigma_i \right) = \sum_{i=1}^{M} \alpha_i \mathcal{N}(x | \mu_i, \Sigma_i) \quad (7)$$

where $\alpha_i$ are mixture weights associated with each component such that $\sum_{i=1}^{M} \alpha_i = 1$. GMMs have been used to encode goals learned in a data-driven fashion where a single mode does not suffice, such as the desired pre-grasp pose to pick up an object [16, 52, 57]. Mixture models also provide a natural goal representation for distributed multi-agent systems [22, 74]. We highlight that mixture models are
F. Goal Likelihood Classifier

We can model the likelihood that a state achieves goal \( p(g = 1|x) = f(x; \theta) \) using a discriminative classifier, such as logistic regression, parameterized by \( \theta \). This model offers great flexibility \cite{23} and can be used to model complex goal relations such as planning to achieve grasps for multi-fingered hands \cite{53} and deformable object manipulation from image observations \cite{76}.

V. PLANNING TO GOAL DISTRIBUTIONS

Our proposed use of goal distributions discussed in Sec. \[ III \] fits naturally within the methods of planning and control as probabilistic inference \cite{50, 71}. In this section, we first present a background on traditional planning as inference frameworks in Sec. \[ V-A \] We then provide our novel derivation and theoretical analysis of planning to goal distributions within the planning as inference framework Sec. \[ V-B \]

A. Planning as Inference

Planning as inference leverages a duality between optimization and probabilistic inference for motion control and planning problems \cite{50, 71, 82}. In the planning as inference framework, we consider distributions of state-action trajectories \( p(\tau) \) where \( \tau = (X_T, U_{T-1}) \). We introduce a binary random variable \( O_\tau \in \{0, 1\} \) where values of \( O_\tau = 1 \) and \( O_\tau = 0 \) denote whether a trajectory \( \tau \) is optimal or not, respectively. We use the notation \( O_\tau \) to represent \( O_\tau = 1 \) for brevity, and similarly we use \( O_t \) to represent \( O_t = 1 \) to denote optimality at a particular timestep in the trajectory \( \tau \). We treat optimality as an observed quantity and seek to infer the posterior distribution of optimal trajectories \( p(\tau | O_\tau) \). Algorithms based on variational inference are commonly used to optimize a proposal distribution \( q(\tau) \) from a known family \( Q \) (e.g. exponential) by solving the following minimization:

\[
q^* = \arg \min_{q \in Q} D_{\text{KL}} \left( q(\tau) \left| \left| p(\tau | O_\tau) \right) \right. \right)
\]

This minimization is equivalently solved by (details in Appendix \[ A \]

\[
q^* = \arg \min_{q \in Q} -E_q \left[ \log p(O_\tau | \tau) \right] + D_{\text{KL}} \left( q(\tau) \left| \left| p_0(\tau) \right) \right. \right)
\]

(9)

where we have labeled the first and second terms in Eq. \[ 9 \] as \( T_1 \) and \( T_2 \), respectively, for ease of reference in Sec. \[ V-B \]

The objective in Eq. \[ 9 \] seeks to maximize the log-likelihood of being optimal in expectation under the trajectory while being regularized by a state-action trajectory prior \( p_0(\tau) \). A salient example of a trajectory prior from the literature is a Gaussian process prior to ensure trajectory smoothness \cite{58}. The state-action trajectory distribution \( p_0(\tau) \) is induced by the stochastic policy prior \( \pi_0(u_t|x_t) \). We elaborate further on different prior policies in Sec. \[ V-B \]

The likelihood \( p(O_\tau | \tau) \) in Eq. \[ 9 \] is key to planning as inference, as it connects the optimization to task-specific objectives. The likelihood may be set to any density function, but it is most commonly set as the exponentiated negative cost \cite{71}:

\[
p(O_\tau | \tau) = \exp(-\alpha C(\tau))
\]

(10)

for \( C(\tau) = c_{\text{term}}(x_T) + \sum_{t=0}^{T-1} c_t(x_t, u_t) \) where \( c_{\text{term}}(\cdot) \) is a terminal cost function defined for the final timestep in the planning horizon and \( c_t(\cdot, \cdot) \) is the cost function for all other timesteps.

We have so far presented planning as inference in its standard formulation akin to \cite{71} and \cite{46}. We now turn to our novel contributions to incorporate goal distributions in planning as inference.

B. Goal Distributions in Planning as Inference

We examine incorporating goal distributions into the planning as inference framework just described and derive its relation to the optimization problem from Eq. \[ 11 \]

We first address the optimality likelihood \( p(O_\tau | \tau) \). We define optimality at the terminal state to mean reaching the goal, i.e. \( p(O_T) = p_g(x_T) \). We then define the trajectory optimality likelihood as

\[
p(O_\tau | \tau) = p_g(x_T) \exp \left( -\alpha \sum_{t=0}^{T-1} c_t(x_t, u_t) \right)
\]

(11)

which captures our proposed notion of optimality, namely satisfying high probability under the goal distribution density function \( p_g(x_T) \) while accounting for arbitrary running costs over the rest of the trajectory. We note this is equivalent to defining \( c_{\text{term}} = -\frac{1}{\alpha} \ln p_g(x_T) \) in Eq. \[ 11 \]. However, since \( p_g(x_T) \) is a density function, we find it more appropriate to directly incorporate it as a factor in the optimality likelihood.

We now consider the implications of using the optimality likelihood from Eq. \[ 11 \] by plugging it into Eq. \[ 9 \]. Plugging Eq. \[ 11 \] into \( T_1 \) we get

\[
T_1 = -E_q \left[ \log \left( p_g(x_T) \exp \left( -\alpha \sum_{t=0}^{T-1} c_t(x_t, u_t) \right) \right) \right]
\]

(12)

\[
= -E_q \left[ \log p_g(x_T) + \logexp \left( -\alpha \sum_{t=0}^{T-1} c_t(x_t, u_t) \right) \right]
\]

(13)

\[
E_q \left[ \log p_g(x_T) \right] + E_q \left[ \alpha \sum_{t=0}^{T-1} c_t(x_t, u_t) \right]
\]

(14)

where \( T_4 \) is simply the expected running cost accumulated over non-terminal timesteps of the trajectory, which we inherit from the standard planning as inference formulation. Our formulation differs for the final timestep with term \( T_3 \) which we further expand as \( T_3 = E_q \left[ -\log p_g(x_T) \right] = \)
\[ \mathbb{E}_q(x_T | \tau) \left[ - \log p_g(x_T) \right] \] This quantity is the cross-entropy of \( p_g(x_T) \) with respect to the terminal state distribution \( q(x_T | \tau) \) which we denote by \( \mathcal{H}(q(x_T | \tau), p_g(x_T)) \).

Recombinining the terms above, we restate the planning as inference objective from Eq. 8 for the case of planning to goal distributions as

\[ q^* = \arg \min_{q \in \mathcal{Q}} \mathcal{H}(q(x_T | \tau), p_g(x_T)) + \mathbb{E}_q \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] + \text{KL}(q(\tau) \parallel p_0(\tau)) \]  

\[ + \frac{\text{KL}(q(\tau) \parallel p_0(\tau))}{D_{KL}(q(\tau) \parallel p_0(\tau))} \]  

**Remark 1.** Planning as inference for planning to goal distributions is equivalent to minimizing the cross-entropy between the terminal state distribution and the goal distribution while minimizing expected running costs and regularizing to a prior state-action distribution.

We now discuss some special cases of this result and relate the objective from Eq. [15] to the optimization problem we defined in Eq. [1].

**1) Deterministic Policy:** If we assume a deterministic policy \( \pi(u_t | x_t) = \delta_{u_t=\phi(x_t)} \), then the KL regularizer in Eq. [15] reduces to \( D_{KL}(q(\tau) \parallel p_0(\tau)) = \mathbb{E}_q \left[ - \sum_{t=0}^{T-1} \log \pi_0(u_t | x_t) \right] \) (see Appendix B for details) and we simplify Eq. [15] to

\[ q^* = \arg \min_{q \in \mathcal{Q}} \mathcal{H}(q(x_T | \tau), p_g(x_T)) + \mathbb{E}_q \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] + \mathbb{E}_q \left[ - \sum_{t=0}^{T-1} \log \pi_0(u_t | x_t) \right] \]  

We assume a deterministic policy in the rest of this article. We discuss extensions for reinforcement learning later in Sec. IX.

**2) Uniform Prior Policy:** If we specify a uniform distribution for the policy prior \( \pi_0(u_t | x_t) = \delta(u_t) \), its value becomes constant with respect to the optimization [71], reducing the problem to exactly that defined in Eq. [1]a with the cross entropy as loss \( L(q(x_T | \tau), p_g(x_T)) = \mathcal{H}(q(x_T | \tau), p_g(x_T)) \).

**Remark 2.** Planning as inference for planning to goal distributions with a uniform prior policy is equivalent to solving the planning to goal distribution problems with a cross entropy loss.

**3) Maximum Entropy Terminal State:** We consider setting the loss in Eq. [1]a to be the KL-divergence, \( D_{KL}(q(x_T | \tau) \parallel p_g(x_T)) \). Then we have the following objective for the planning to goal distribution problem

\[ q^* = \arg \min_{q \in \mathcal{Q}} D_{KL}(q(x_T | \tau) \parallel p_g(x_T)) + \mathbb{E}_q \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] \]  

\[ + \mathbb{E}_q \left[ - \sum_{t=0}^{T-1} \log \pi_0(u_t | x_t) \right] \]  

\[ = \arg \min_{q \in \mathcal{Q}} \mathcal{H}(q(x_T | \tau), p_g(x_T)) + \mathbb{E}_q \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] - \mathcal{H}(p_g(x_T)) \]  

following from the relation \( \mathcal{H}(p_1, p_2) = D_{KL}(p_1 \parallel p_2) + \mathcal{H}(p_1) \). If we set Eq. [18] equal to Eq. [16] we see the first two terms cancel and we are left with the equality

\[ \mathcal{H}(q(x_T | \tau)) = \mathbb{E}_q \left[ \sum_{t=0}^{T-1} \log \pi_0(u_t | x_t) \right] \]  

**Remark 3.** Planning as inference for planning to goal distributions with a prior policy that maximizes entropy of the terminal state is equivalent to solving the planning to goal distributions problem with a KL divergence loss.

**4) M-Projections for Planning to Finite Support Goals:** The KL divergence in the variational inference objective for planning as inference in Eq. [8] is formulated as an information projection. Since KL divergence is an asymmetric loss between distributions, there are two possible projections an optimizer can solve to minimize KL divergence:

\[ q^* = \arg \min_{q} D_{KL}(q(x) \parallel p(x)) \]  

(\text{I-projection})

\[ q^* = \arg \min_{q} D_{KL}(p(x) \parallel q(x)) \]  

(\text{M-projection})

The information projection (I-projection) exhibits mode-seeking behavior while the moment projection (M-projection) seeks coverage of all regions where \( p(x) > 0 \) and thus exhibits moment-matching behavior [59]. Importantly, when \( p(x) \) has finite support (e.g. uniform, Dirac-delta, truncated Gaussian), it is necessary to use an M-projection to avoid the division by zero that would occur in the I-projection over regions outside the support of \( p(x) \).

Our formulation has so far only considered an I-projection. In general, the M-projection is intractable to solve for arbitrary planning as inference problems. This is due to the fact that one would require access already to the full distribution of optimal trajectories \( p(\tau | O_\tau) \) in order to compute the optimization. However, since we do assume access to the goal distribution, we can compute either the I-projection or M-projection for the KL divergence at the terminal timestep. Thus as a final objective for investigation, we examine setting the distributional loss in Eq. [14] to be the M-projection KL divergence, instead of the I-projection as previously examined. We get the following objective:

\[ \pi^* = \arg \min_{\pi} D_{KL}(p_g(x_T) \parallel q(x_T | \tau)) \]  

\[ + \mathbb{E}_q \left[ \sum_{t=0}^{T-1} c_t(x_t, u_t) \right] \]  

This is nearly equivalent to the planning as inference with a maximum-entropy prior result from Sec. VII-B3, however the KL divergence term is now an M-projection. A key advantage this affords us is our method naturally accommodates goal distributions with finite support, which we explore in our experiments in Sec. VIII-A. We thus have a unified framework for planning to arbitrary goal distributions under uncertain dynamics utilizing information-theoretic loss functions.

Note that for a fixed goal distribution, the M-projection of cross-entropy is equivalent to the objective in Eq. [20] since the entropy of the goal distribution is constant with respect to the decision variables. However, this is not necessarily the case for goal distributions that may change over time based on the agent’s observations. We discuss this point further in Sec. IX.
VI. COST REDUCTIONS

We present an additional theoretical contribution to illustrate how our probabilistic planning framework encompasses common planning objectives in the literature. We examine the cross-entropy cost term (term $T_3$ in Eq. 14) between the predicted terminal state distribution $q(x_T | \tau)$ and the goal distribution $p_g(x_T)$ for several common distribution choices. Looking at both the I-projection $D_{EC}(q(x_T | \tau) \parallel p_g(x_T))$ and the M-projection $D_{CE}(p_g(x_T) \parallel q(x_T | \tau))$, we reduce the costs to commonly used cost functions from the literature.

A. (Weighted) Euclidean Distance

For a Gaussian goal distribution $p_g(x_T) = \mathcal{N}(x_T | \mu_g, \Sigma_g)$ and minimizing the I-projection of cross-entropy we have:

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{q(\tau)} [-\log \mathcal{N}(x_T | \mu_g, \Sigma_g)]$$

(21)

$$= \arg\min_{\pi} \mathbb{E}_{q(\tau)} [-\log \text{exp}\{-(x_T - \mu_g)^T \Sigma_g^{-1}(x_T - \mu_g)\}]$$

(22)

$$= \arg\min_{\pi} \mathbb{E}_{q(\tau)} [(x_T - \mu_g)^T \Sigma_g^{-1}(x_T - \mu_g)]$$

(23)

$$= \arg\min_{\pi} \mathbb{E}_{q(\tau)} \|x_T - \mu_g\|^2_{\Sigma_g^{-1}}$$

(24)

$$= \arg\min_{\pi} \mathbb{E}_{q(\tau)} \|x_T - \mu_g\|^2_{\Lambda_g}$$

(25)

For the case of deterministic dynamics, that is $q(x_T | \tau) = \delta_{x_T | T}(x)$, the expectation simplifies to a single point evaluation and we recover the common weighted Euclidean distance. Since it is weighted by the precision (i.e. inverse covariance) of the goal distribution, it is equivalent to the Mahalanobis distance [56]. The same cost arises for the case of a goal point (i.e. Dirac delta distribution) and Gaussian state uncertainty if we consider the M-projection of cross-entropy. In this case, the distance is weighted by the precision of the terminal state distribution instead of the goal precision.

B. Goal Set Indicator

For a uniform goal distribution $p_g(x_T) = \mathcal{U}_G(x_T)$ and deterministic dynamics $q(x_T | \tau) = \delta_{x_T | \tau}(x)$, minimizing the I-projection of cross-entropy amounts to

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{q(\tau)} [-\log \mathcal{U}_G(x_T)]$$

(26)

$$= \arg\min_{\pi} \mathbb{E}_{q(\tau)} \left\{-\int_{x_T} -\delta_{x_T | \tau}(x) \log \mathcal{U}_G(x_T) dx_T \right\}$$

(27)

$$= \arg\min_{\pi} \left\{-\log u_G \text{ if } x_T \in \mathcal{G} \right\}$$

(28)

where $u_G = 1/\sqrt{\text{vol}(\mathcal{G})}$ as described in Sec. [IV-B]. Hence the minimum is obtained with a constant cost if the terminal state $x_T$ from executing trajectory $\tau$ reaches any point in the goal set, while any state outside the goal set receives infinite cost. Note the function is non-differentiable as expected from the set-based goal definition. We can treat a single goal state naturally as a special case of this function. The non-differentiable nature of this purely set-based formulation motivates using search-based and sampling-based planners over optimization-based approaches in these deterministic settings.

C. Chance-Constrained Goal Set

For a uniform goal distribution $p_g(x_T) = \mathcal{U}_G(x_T)$, minimizing the M-projection of cross-entropy amounts to

$$\pi^* = \arg\min_{\pi} \mathbb{E}_{U_G(x_T)} [-\log q(x_T | \tau)]$$

(29)

$$= \arg\min_{\pi} \int_{x_T \in \mathcal{G}} -U_G(x_T) \log q(x_T | \tau) dx_T$$

(30)

$$= \arg\min_{\pi} -u_G \int_{x_T \in \mathcal{G}} \log q(x_T | \tau) dx_T$$

(31)

$$= \arg\max_{\pi} \int_{x_T \in \mathcal{G}} q(x_T | \tau) dx_T$$

(32)

which defines the probability of reaching any state in the goal set $\mathcal{G}$, a commonly used term for reaching a goal set in chance-constrained control (e.g. Equation (6) in [13]).

D. Maximize Probability of Reaching Goal Point

A special case of the previous result in Sec. [VII-C] is a Dirac-delta goal distribution $p_g(x_T) = \delta_{g_T}(x)$ instead of a uniform goal distribution. We get

$$\pi^* = \arg\max_{\pi} q(x_T = g | \tau)$$

(33)

which maximizes the probability of reaching a point-based goal $g$ following trajectory $\tau$.

VII. PRACTICAL ALGORITHM

In this section, we formulate planning to goal distributions as a practical instantiation of our method described in Sec. [VI]. We first describe how we compute the agent’s predicted terminal state distribution with the unscented transform in Sec. [VII-A]. In Sec. [VII-B] we incorporate the unscented transform uncertainty propagation into a more concrete formulation of the constrained optimization problem defined in Eq. 1 from Sec. [III]. We then discuss tractable methods for computing our information-theoretic losses in Sec. [VII-C]. We use this formulation in our experiments in Sec. [VIII].

A. State Uncertainty Propagation

As noted in Sec. [III], a robot typically maintains a probabilistic estimate of its state $\hat{p}(x_0)$ to cope with aleatoric uncertainty from its sensors and the environment [38, 81]. In order to compute the terminal state distribution $q(x_T | \tau)$ and in turn compute the information-theoretic losses for Eq. 1a, we require a means of propagating the state uncertainty over the planning horizon given the initial state distribution $\hat{p}(x_0)$, a sequence of actions $U_{T-1}$, and the robot’s stochastic (possibly nonlinear) dynamics function $f : \mathcal{X} \times \mathcal{U} \times \Omega \rightarrow \mathcal{X}$. Our choice of uncertainty propagation method is informed by the evolution of nonlinear Bayesian filters [81, 57]. Perhaps the simplest method is Monte Carlo sampling, i.e. sampling initial states $x_0 \sim \hat{p}(x_0)$ and sequentially applying the dynamics function to acquire a collection of samples from which the terminal state distribution can be approximated. However, this approach can require thousands of samples to estimate the distribution well, which imposes a computational burden when estimating auxiliary costs and constraints (e.g.
Taking the sample mean and covariance of the points in $P$ under Gaussian state uncertainty propagated by the unscented transform provides an accurate estimate of the state distribution propagated under nonlinear dynamics (akin to the unscented Kalman filter). We leverage the unscented transform in our approach since we can achieve an accurate estimate of the propagated state marginal using only a small set of deterministically computed query points without having to compute Jacobians or Hessians of the dynamics function.

However, we emphasize that other uncertainty propagation state estimation techniques assume a Gaussian state distribution in Sec. VIII to illustrate our approach. Most popular techniques are possible in our framework. We discuss this in Sec IX.

Given a state distribution $\mathcal{N}(x_t \mid \mu_t, \Sigma_t)$, the unscented transform computes a small set of sigma points $P_t = \{\mu_t \pm \beta L_t[i]\}_{i=1}^N$ for state size $N$ where $L_t[i]$ is the $i$-th row of the Cholesky decomposition of the covariance $\Sigma_t$ and $\beta$ is a hyperparameter governing how spread out the sigma points are from the mean. Given an action $u_t \in \mathcal{U}$, we estimate how the distribution will transform under the robot’s dynamics by evaluating the noise-free dynamics function, $f(x_t, u_t, 0)$ at each of the sigma points $p_i \in P_t$ together with the action $u_t$ to get a new set of points $P'_t = \{f(p_i, u_t, 0) \mid p_i \in P_t\}$.

Taking the sample mean and covariance of the points in $P'_t$ provides an estimate of the distribution at the next timestep $\mathcal{N}(x_{t+1} \mid \mu_{t+1}, \Sigma_{t+1})$. This procedure is similar to the unscented dynamics described in [33] as well as its use in unscented model predictive control [21, 85].

We assume Gaussian state distributions and use the unscented transform for uncertainty propagation in our experiments in Sec. VII to illustrate our approach. Most popular state estimation techniques assume a Gaussian state distribution [31, 83], so this is not a particularly limiting assumption. However, we emphasize that other uncertainty propagation techniques are possible in our framework. We discuss this point further in Sec IX.

### B. Planning to Goal Distributions with Unscented Transform

We now present a more concrete instantiation of the abstract optimization problem from Sec. III which we will subsequently use in our experiments in Sec. VIII. We formulate the optimization problem as a direct transcription of a trajectory optimization for planning to a goal distribution $p_g(x_T)$ under Gaussian state uncertainty propagated by the unscented transform as described in Sec. VII-A.

$$\min_{\Theta} \mathcal{L}(\mathcal{N}(x_T \mid \mu_T, \Sigma_T), p_g(x_T))$$  \hspace{1cm} (34a)$$

$$+ \sum_{t=0}^{T-1} \sum_{p_i \in P_t} \mathcal{N}(p_t \mid \mu_t, \Sigma_t) \cdot c_t(p_t', u_t)$$  \hspace{1cm} (34b)$$

s.t. $x_{\text{min}} \leq \mu_t \leq x_{\text{max}}, \forall t \in [0, T]$ \hspace{1cm} (34c)$$

$u_{\text{min}} \leq u_t \leq u_{\text{max}}, \forall t \in [0, T - 1]$ \hspace{1cm} (34d)$$

$\mu_0 = x_0$, $\Sigma_0 = \Sigma_{x_0}$ \hspace{1cm} (34e)$$

$\mu_t = \frac{1}{|P'_t|} \sum_{p'_i \in P'_t} p'_i$, $\forall t \in [1, T]$ \hspace{1cm} (34f)$$

$\Sigma_t = \frac{1}{|P'_t|} \sum_{p'_i \in P'_t} (p'_i - \mu_t)(p'_i - \mu_t)^T + R_t$, $\forall t \in [1, T]$ \hspace{1cm} (34g)$$

where the decision variables $\Theta$ include the sequence of actions $U_{T-1}$ and, due to the direct transcription formulation, the sequence of means $\mu_0, \ldots, \mu_T$ and covariances $\Sigma_0, \ldots, \Sigma_T$ parameterizing the state distributions over the trajectory.

Eq. 34a is an information-theoretic loss between the terminal state and goal distributions. In our experiments in Sec. VIII we focus on the cross-entropy and KL divergence losses that resulted from our derivations in Sec. VII-B. Eq. 34b encapsulates the arbitrary running costs from Eq. 1b adapted here to be applied to the sigma points resulting from uncertainty propagation with the unscented transform. Note the costs are weighted by the probability of the sigma points under the predicted state distribution for the associated timestep.

Regarding constraints, Eqs. 34c and 34d are bound constraints on the decision variables. Eq. 34e ensures the initial state distribution matches the observed initial state distribution. Eqs. 34f and 34g constrain the distribution parameters at each timestep to match the empirical mean and covariance, respectively, estimated from the transformed sigma points at each timestep $P'_t$ as described in Sec. VII-A. As a reminder, $P'_t = \{f(p_i, u_t, 0) \mid p_i \in P_t\}$ is the set of transformed sigma points under the noise-free dynamics function. The stochasticity of the dynamics is accounted for by the term $R_t$ in Eq. 34g.

Eq. 34h ensures the covariance matrices remain symmetric positive semi-definite. We note that in practice we use $L_t = \text{choi}(\Sigma_t)$ for the covariance optimization such that $L_t L_t^T = \Sigma_t$, which makes it easier for optimizers to satisfy this constraint. Finally, Eqs. 34i and 34j enable us to incorporate arbitrary inequality and equality constraints on the decision variables. For example, we add collision-avoidance constraints on the sigma points computed from the distribution parameters at each timestep in our experiments in Sec. VIII similar to [33].
C. Approximate Information-Theoretic Losses

Cross-entropy and KL divergence are two salient information-theoretic losses that came out of our planning as inference derivation in Sec. VII-B and we primarily utilize these losses in our experiments in Sec. VIII. There are often closed form solutions for computing these losses (e.g. between two Gaussian distributions \[^{34}\]). However, there are instances where no closed form solutions exist and approximations are necessary (e.g. between two Gaussian mixture models \[^{31}\], truncated Gaussians \[^{17}\]). Since we assume Gaussian state uncertainty in this article, we utilize a simple and accurate approximation where needed based on the sigma point computation discussed in Sec. VII-A. For a Gaussian distribution \(p_1(x) = \mathcal{N}(x | \mu_1, \Sigma_1)\) and an arbitrary distribution \(p_2(x)\), we compute approximate cross-entropy as

\[
D_{\text{CE-app}}(p_1(x) \parallel p_2(x)) = \frac{1}{|\mathcal{P}|} \sum_{p \in \mathcal{P}} -p_1(p) \log p_2(p) \tag{35}
\]

where \(\mathcal{P}\) is the set of sigma points as discussed in Sec. VII-A. A similar approximation follows for KL divergence. These approximations are based on the unscented approximation described in \[^{31}\] which demonstrated its accuracy and efficiency over Monte Carlo estimates. We utilize this approximation in our experiments when closed form solutions do not exist.

VIII. Experiments

We present experiments performed on four different environments. In Sec. VIII-A we demonstrate some of the basic behaviors and flexibility of our approach on a simple planar navigation among obstacles problem. In Sec. VIII-B we further explore the behavior of different information-theoretic losses on an underactuated planar ball-rolling environment. We explore a more realistic scenario in Sec. VIII-C in which the agent must intercept a moving target and update its belief of the target’s location with noisy sensor readings. Finally, in Sec. VIII-D we apply our approach to a higher dimensional problem of a 7-DOF robot arm reaching to grasp an object and uncover some intriguing benefits of a distribution-based goal in a manipulation setting.

Additional details about the different solvers and environment parameters we use in our experiments can be found in Appendix D. All code\[^{4}\] and data associated with these experiments will be released upon acceptance.

A. Flexibility of Goal Distribution Representation

We first apply our approach to a simple 2D navigation problem to highlight the flexibility of modeling task goals as goal distributions in our probabilistic planning framework. The task objective is to navigate from a start configuration to a goal while avoiding obstacles.

We emphasize that in all examples in this section, we use the same planning algorithm that we formalized in Sec. VII. Merely by changing the family and/or parameterization of the goal distribution in each example (and selecting an appropriate information-theoretic loss), we are able to plan to point-based and set-based goals as well as goals with varied models of uncertainty associated with them.

1) Environment Description: We use the Dubins car model from \[^{40}\], a simple vehicle model with non-holonomic constraints in the state space \(X = SE(2)\). The state \(x = (p_x, p_y, \phi)\) denotes the car’s planar position \((p_x, p_y)\) and orientation \(\phi\). The dynamics obey

\[
\dot{p}_x = v \cos \phi, \quad \dot{p}_y = v \sin \phi, \quad \dot{\phi} = r \tag{36}
\]

where \(v \in [0,v_{\text{max}}]\) is a linear speed and \(r \in [-\tan \psi_{\text{max}}, \tan \psi_{\text{max}}]\) is the turn rate for \(\psi_{\text{max}} \in (0, \frac{\pi}{2})\). We use an arclength parameterization similar to \[^{40}\] where actions \(\mathbf{u} = (v, r)\) are applied at each timestep for duration \(\Delta t\) such that the robot moves in a straight line with velocity \(v\) if \(r = 0\) and arcs with radius \(\frac{v}{\max}\) otherwise. An action sequence has the form \(U_{t-1} = (v_1, r_1, \ldots, v_{T-1}, r_{T-1})\). We extend the model in \[^{40}\] to have stochastic dynamics by adding Gaussian noise \(\omega_t \sim \mathcal{N}(x | 0, \alpha I)\) to the state updates at each timestep.

We manually define a Gaussian estimate \(\mathcal{N}(x_0 | \mu_0, \Sigma_0)\) for the robot’s initial state. We compute the marginal terminal state distribution \(\mathcal{N}(x_T | \mu_T, \Sigma_T)\) using the unscented transform as described in Sec. VII-A. We use spherical obstacles and utilize the negative signed distance function (SDF) in an inequality constraint \(-d_{\text{SDF}}(x_t, o_i, r_i) \leq 0\) for all timesteps \(t \in [0, T]\) where \(T = 45\) is the planning horizon and \((o_i, r_i)\) are the origin point \(o_i \in \mathbb{R}^2\) and radius \(r_i \in \mathbb{R}\) of a spherical obstacle. These constraints apply to the mean trajectory as well as the computed sigma points at each timestep as described in Sec. VII-B.

2) Navigating to a goal region: We define the goal region \(G\) as the blue, hatched rectangle shown in Fig. 2a. We are able to plan directly to this region by modeling it as a uniform goal distribution \(p_g(x_T) = U_G(x_T)\). Here we utilize the M-projection of the cross-entropy loss \(D_{\text{CE}}(U_G(x_T) \parallel \mathcal{N}(x_T | \mu_T, \Sigma_T))\) since the uniform distribution has finite support as discussed in Sec. VII-B and Sec. VII-C. As a reminder, this objective is equivalent to the chance-constrained goal set objective \[^{13}\] as discussed in Sec. VII-C. The resulting plan is shown in Fig. 2a where the orange dotted line shows the planned mean path of the robot and the orange ellipses show two standard deviations of the robot’s predicted Gaussian state uncertainty over the trajectory. This result demonstrates our probabilistic planning framework accommodates traditional set-based goal representations.

In contrast to traditional planning approaches that sample a particular point \(g \in G\) from the set \(G\) to plan to \[^{48}\], we are able to plan directly to the region.

While we are able to plan directly to the goal region \(G\) as just described, we can also plan to any particular point \(g \in G\) (blue dot in Fig. 2b) by modeling the goal point as a Dirac-delta distribution \(p_g(x_T) = \delta_g(x_T)\) as discussed in Sec. IV-A. We again use the M-projection cross-entropy loss \(D_{\text{CE}}(\delta_g(x_T) \parallel \mathcal{N}(x_T | \mu_T, \Sigma_T))\) due to the finite support of the Dirac-delta distribution. As a reminder, this objective amounts to maximizing the probability of reaching a goal point (Sec. VI-D). We show the resulting plan for reaching this arbitrarily, sampled point in Fig 2b. This results shows

\[^{2}\] A preliminary release of our code can be found here: https://bitbucket.org/robot-learning/distribution_planning

\[^{3}\] A preliminary release of our code can be found here: https://bitbucket.org/robot-learning/distribution_planning
that we retain point-based and set-based goal representations in our probabilistic framework.

We depart from traditional set-based and point-based goal representations in Fig. 2 with a Gaussian goal distribution $p_g(x_T) = \mathcal{N}(x_T \mid \mu_g, \Sigma_g)$ as described in Sec. IV-C. The Gaussian distribution has infinite support and therefore we use the I-projection of cross-entropy $D_{\text{CE}}(\mathcal{N}(x_T \mid \mu_T, \Sigma_T) \parallel \mathcal{N}(x_T \mid \mu_g, \Sigma_g))$. The resulting plan is shown in Fig. 2c. We set the mean of the Gaussian to the same point sampled from the previous example for comparison. We note that in this particular unimodal example, the M-projection plan looks similar. This result demonstrates we are able to generate plans to a goal location while explicitly accounting for the robot’s Gaussian belief about which goal point it should navigate to.

We additionally consider a truncated Gaussian goal distribution $p_g(x_T) = \mathcal{N}(x_T \mid \mu_g, \Sigma_g, \mathcal{G})$ shown in Fig. 2d. We choose the same Gaussian distribution as in the previous example, but bound the PDF to the goal region $\mathcal{G}$ as illustrated by the green rectangle in Fig. 2d such that the PDF is equal to zero outside the goal region. As discussed in Sec. IV-C, this serves as a model of bounded uncertainty in which the agent has Gaussian uncertainty about a point-based goal in the goal region, but the well-defined goal region limits the agent’s belief to be contained within the goal region. Note we again use the M-projection of cross-entropy $D_{\text{CE}}(\mathcal{N}(x_T \mid \mu_g, \Sigma_g, \mathcal{G}) \parallel \mathcal{N}(x_T \mid \mu_T, \Sigma_T))$ since the truncated Gaussian has finite support. We also note we use the approximate cross-entropy discussed in Sec. VII-C since there is no closed form solution for truncated Gaussians.

We now consider a goal classifier learned from data as shown in Fig. 2e. The blue dots indicate samples that are considered goals. We acquire negative samples by sampling uniformly from the environment region outside the goal region.
We now look at a Gaussian mixture model (GMM) goal distribution. Even when planning to the same goal, differently can result in paths from different homotopy classes. We show two I-projection paths (purple paths, one to each component) and one M-projection path (orange path). Purple and orange ellipses show the associated state uncertainty for the respective paths.

We train a simple neural network model $f(x; \theta)$ consisting of fully connected layers with ReLU activations. We utilized two hidden layers of width 32 and trained the model with binary cross-entropy loss. We again utilize the approximate cross-entropy loss from Sec. VII-C. As discussed in Sec. IV-F, this is a very general goal representation that can model complex goals, and we can use it directly as a goal in our planning framework.

For our last example in this section, we show several plans to different point-based goals (i.e. Dirac-delta distributions) with varying values of the $\beta$ parameter discussed in Sec. VII-A. The $\beta$ parameter governs how spread out the sigma points get in propagating state uncertainty with the unscented transform. Since we apply collision costs to the sigma points, this is not a well-defined measure of spread. Instead, it provides a way to adjust the mass of the terminal state distribution to cover all modes of the goal distribution. The M-projection results in a plan that terminates in between the two modes of the goal distribution (orange path in Fig. 3). This is a well-known feature of the information and moment projections for multimodal distributions [59]. However, we find it insightful to demonstrate this behavior in our probabilistic planning framework. In most planning problems it is desirable to optimize to a particular mode, and thus the I-projection will be preferred. However, there are instances where the M-projection may be desirable for a multimodal distribution, e.g. surveillance or allocation problems that require an agent to be in proximity to several targets simultaneously [7].

## B. Leveraging sources of uncertainty for planning

In Sec. VII-A we showed that both cross-entropy and KL divergence are meaningful information theoretic losses when planning to goal distributions from the perspective of planning as inference. The examples in Sec. VII-A did not have notable differences between cross-entropy and KL divergence. This is because in the M-projection cases, the entropy of the goal distribution is constant in the optimization and thus minimizing cross-entropy and KL divergence are equivalent. For the I-projections, the terminal covariance is largely determined by the horizon of the plan, since we only considered homogeneous dynamics noise over the environment. We now consider an example with heteroscedastic noise that shows a clear difference between these two objectives, and compare also to point-based planners.

We consider a 2D ball-rolling environment (pictured in Fig. 1) in which the agent must select the initial position and velocity of a ball such that the ball will end up as close to a target location as possible. Note that in contrast to the previous environment, the agent only applies control input at the initial timestep and must leverage the passive dynamics of the environment to get the ball to the desired target location.

We use the state and dynamics model of [84]. The state space $X \subseteq \mathbb{R}^6$ consists of the planar position $p$, velocity $v = \dot{p}$, and acceleration $a = \ddot{p}$ of the ball. The acceleration of the ball is computed by $a = F/m$ where $m = 0.045$ kg is the mass of the ball in kilograms and the force

$$F = -\mu_f ||N|| \frac{v}{||v||}$$

is the frictional force applied to the ball from the ground where $\mu_f$ is the coefficient of friction and $N$ is the normal force applied to the ball from the ground. We assume a flat surface so that the normal force is simply counteracting gravity, i.e. $N = (0, 0, mg)$ for $g = 9.8 \, \text{m/s}^2$. We use a friction coefficient in Sec. IV-E. In contrast to the unimodal example from Fig. 2c the I-projection and M-projection of cross-entropy for a multimodal GMM exhibit notably different behavior as shown in Fig. 3. As discussed in Sec. VII-A, the I-projection is mode-seeking and thus is capable of generating plans to either mode depending on the parameter initialization provided to the solver. We show one instance of a plan to each GMM mode in Fig. 3 (purple paths). The M-projection exhibits moment-matching behavior and will strive to concentrate the mass of the terminal state distribution to cover all modes of the goal distribution.
of $\mu_f = 0.04$. We compute velocity and position using Euler integration from the computed acceleration with a timestep of $dt = 0.3$. We constrain the position to the blue line shown in Fig. 4.

We apply a small additive isotropic Gaussian noise $\omega \sim \mathcal{N}(\alpha \mid 0, \alpha I)$ to the acceleration for $\alpha = 0.0001$. However, this environment has heterogeneous noise. The black, circular gradient shown in Fig. 4a is a noise amplifier defined by a Gaussian distribution that adds an additional $\omega' \sim \mathcal{N}(\alpha' \mid 0, \alpha'I)$ where $\alpha'$ is proportional to the Mahalanobis distance between the agent and the center of the noise amplifier. Intuitively, the closer the ball gets to the center of the noise amplifier, the more dynamics noise is added to the acceleration. This behavior mimics exacerbated state uncertainty induced by the environment, e.g. a patch of ice.

We consider a Gaussian goal distribution $p_g(x) = \mathcal{N}(x \mid \mu_g, \Sigma_g)$ illustrated by the blue ellipses in Fig. 4. We again use the unscented transform to compute the marginal terminal state distribution $\mathcal{N}(x_T \mid \mu_T, \Sigma_T)$. In order to compare the terminal distributions resulting from each plan, we run a total of 500 rollouts per trial and fit a Gaussian distribution to the points at the last timestep using maximum likelihood estimation. Terminal distributions are visualized by grey ellipses in Fig. 4 showing standard deviations. Note we estimate the terminal distributions using rollouts instead of the unscented transform prediction in order to put the comparison on an even footing with the deterministic point-based planner we will discuss shortly, which has no associated method of uncertainty propagation.

As shown in Fig. 4a, minimizing the I-projection of KL divergence $D_{KL}(\mathcal{N}(x_T \mid \mu_T, \Sigma_T) \parallel \mathcal{N}(x_T \mid \mu_g, \Sigma_g))$ results in a plan that has the ball pass close by the noise amplifier. The terminal distribution matches the goal distribution with a KL divergence value of 0.796. However, minimizing the I-projection of cross-entropy $D_{CE}(\mathcal{N}(x_T \mid \mu_T, \Sigma_T) \parallel \mathcal{N}(x_T \mid \mu_g, \Sigma_g))$ results in a plan that avoids the noise amplifier and keeps a tight covariance for the terminal state distribution as shown in Fig. 4b. The resulting distribution has a KL divergence value of 2.272 with respect to the goal distribution. This higher KL divergence value is due to cross-entropy seeking to maximize the expected (log) probability of reaching the goal, which incentivizes a lower entropy terminal state distribution. This is evident when we consider that $D_{CE}(p1 \parallel p2) = H(p1) + D_{KL}(p1 \parallel p2)$ as discussed previously in Sec. V.B.2.

We further consider two point-based planners. First, we use the negative log probability of the computed terminal state distribution $\mathcal{N}(x_T \mid \mu_T, \Sigma_T)$ as the cost. We sample 50 goal points from the Gaussian goal distribution (blue dots in Fig. 4c) and generate a plan for each goal point. We perform 10 rollouts for each of the 50 plans for a total of 500 rollouts and fit the Gaussian distribution displayed with grey ellipses in Fig. 4c. We see that the terminal distribution matches very closely to the goal distribution with a KL divergence of 0.770. This value is only slightly lower than the distribution we achieved with 500 rollouts of the single plan resulting from optimizing KL divergence. Second, we consider a deterministic planner that plans to each of the goal samples.

![Figure 4](image-url)

**Fig. 4**: Comparison of different planning objectives for a 2D ball-rolling environment. The initial ball position (red dot) is constrained to the dark blue line. Blue ellipses denote the Gaussian goal distribution. Grey ellipses illustrate the terminal state distribution fitted from rollouts. The black gradient is a noise amplifier that increases dynamics noise the closer the ball gets to darker-colored regions. (a) KL divergence plans leverage the additional noise uncertainty from the amplifier to more closely match the terminal covariance of the goal distribution. Purple lines show paths from 20 rollouts of the plan. (b) Cross-entropy plans avoid the noise amplifier and achieve a tight covariance about the goal mean. (c) Using the negative log PDF of the terminal state distribution as the cost, we plan to 50 samples from the goal distribution (blue dots) and avoid the noise amplifier. Lines indicate the planned path for the 50 different goal samples. (d) A deterministic planner ignores the noise amplifier in planning to the goal samples, resulting in a terminal distribution that does not match the goal distribution.
Fig. 5: A selection of frames from an MPC execution of a planar navigation agent intercepting a moving target. The actual goal path is denoted by the blue line, the blue ellipses display the agent’s Gaussian belief about where the goal is at each timestep, and the green ellipses display the agent’s projected Gaussian belief of where the goal will be at the end of the planning horizon. The agent’s path is denoted by the red line in each frame, and the gray dotted line displays its current plan. Dark spheres are obstacles to avoid. See Sec. VIII-C for details.

and assumes there is no noise in the dynamics. As shown in Fig. 4d, we get solutions where the initial position of the ball is spread out over the range of the start region. Some of the plans then have the ball pass directly over the noise amplifier, resulting in rollouts that may disperse the ball far away from its target location. We again perform 10 rollouts for each of the 50 plans and find the terminal Gaussian distribution to have a KL divergence of 1.388, nearly double the value of the KL divergence plan and log probability plans. It is therefore not sufficient to deterministically plan to sample points from the goal distribution, and we achieve better results utilizing the predicted terminal state uncertainty. Note that with the point-based planners, we must compute a separate plan for every point. In contrast, we computed a single plan optimizing KL divergence that resulted in a terminal state distribution closely matching the goal distribution.

These examples illustrate that our framework is able to advantageously leverage sources of uncertainty in the environment to achieve a target state distribution. We note similar results can also be achieved in environments with homogeneous noise if the agent optimizes stochastic policies, e.g. with reinforcement learning.

C. Intercepting a moving target

We have so far only considered examples with static goal distributions. We now consider a more realistic scenario in which the goal distribution changes over time and is updated based on the agent’s observations. The robot’s task is to intercept a moving target while maintaining a Gaussian belief of the target’s location from noisy observations.

We use the widely utilized double integrator system \[0, 46\] for both the agent and the moving target in a planar environment. The agent’s state space \(\mathcal{X} \subseteq \mathbb{R}^4\) consists of a planar position \(p \in \mathbb{R}^2\) and velocity \(v \in \mathbb{R}^2\) and the action space \(u \subseteq \mathbb{R}^2\) is the agent’s acceleration \(a \in \mathbb{R}^2\). The dynamics obey \(v = \dot{p}\) and \(a = \dot{v}\) and we compute the position and velocity through double Euler integration of the applied acceleration (hence the name “double integrator”).

The agent maintains a Gaussian belief of the target’s location (blue ellipses in Fig. 5) that is updated at every timestep using a Kalman filter \[36\]. The agent also uses the Kalman filter to project its Gaussian belief of the target’s location over the planning horizon to represent the agent’s Gaussian belief of the targets future location (green ellipses in Fig. 5). Note that the target moves deterministically in a straight line with constant velocity from left to right. The agent uses this motion model in the Kalman updates, but assumes the agent moves stochastically. The agent receives noisy observations of the target’s location at every timestep, where the observations get more accurate the closer the agent gets to the target.

We use the I-projection of cross-entropy as our planning objective. We interleave planning and control with a model predictive control (MPC) scheme. The agent creates a plan for a small horizon, executes the first action of the plan in the environment, acquires an observation from the environment, and then re-plans. This procedure continues until a termination condition is achieved. We use model predictive path integral control (MPPI) as our plan update procedure. MPPI is a sampling-based solver that is widely used due to its computation efficiency and efficacy on complex domains. See Appendix D-C for details on the solver and MPC scheme
we use.

We see in Fig. 5 that initially (timesteps \( t = 0 \) and \( t = 15 \)) the agent plans a path between the two obstacles to the projected goal distribution. However, as the target advances in its trajectory, the agent gets new observations that push the projected goal distribution far enough along that the agent starts planning paths that go to the right of all obstacles (starting at \( t = 20 \)). The agent thus switched the homotopy class of its planned path to better intercept the target based on its prediction.

By timestep \( t = 45 \), the agent has intercepted the target. Note that the agent’s uncertainty at this point is much lower than earlier in the execution (denoted by the smaller ellipses for the Gaussian belief in Fig. 5). This is due to the observation model we define that enables the agent to incorporate more accurate observations into the Kalman filter state estimation as the agent gets closer to the target. Once the agent intercepts the target (\( t = 45 \)), it continues to move along in sync with the target, as shown in the final frame of Fig. 5 for \( t = 65 \). Note that the agent’s path does not precisely follow the target but deviates slightly. This is due to the agent executing with stochastic dynamics, and there is some irreducible uncertainty about where the target is located due to the aleatoric uncertainty of the agent’s sensors.

This example demonstrates an application of our approach to a more realistic setting in which the agent must update its belief of its goal online using noisy observations. Importantly, we were able to use the output of a Kalman filter directly as our goal representation for planning, as opposed to only using the mean or a sample from the belief distribution. We only explored the I-projection of cross-entropy in this simple example. However, we believe other objectives (e.g. the M-projection) discussed in our planning framework deserve more detailed attention, a point we discuss further in Sec. IX.

D. Goal distributions are a better goal representation for robot arm reaching

We now turn to a more complex domain than the previous experiments to demonstrate an advantage of a distribution-based goal over a point-based goal. We address the problem of a 7-DOF robot arm reaching to grasp an object shown in Fig. 6. The objective for the robot is to reach its end-effector to a pre-grasp pose near an object such that closing the fingers of the hand would result in the robot grasping the object.

A common heuristic for generating pre-grasp pose candidates is to select poses that align the palm of the hand to a face of an axis-aligned bounding box of the object. Zero-mean Gaussian noise may also be added to each candidate to induce further variation in the pose candidates. We model this explicitly as a mixture of distributions (Sec. IV-E) where each component is a distribution over SE(3) poses. We represent each SE(3) pose distribution as a Gaussian distribution (Sec. IV-C) over the 3D position together with a Bingham distribution (Sec. IV-D) over the SO(3) orientations of the end-effector. We refer to this distribution as a pose mixture model. The distribution is defined relative to the object, and therefore as the object is placed in different locations on the table shown in Fig. 6 the goal distribution is transformed to the object’s reference frame. This distribution is similar to the Gaussian mixture model used to encode goals fit from data for object-reaching in a learning from demonstration setting in [16]. However, our pose mixture model is more correct in the sense we properly model the distribution of orientations as a Gaussian distribution in SO(3).

We use the simulated 7-DOF KUKA iiwa arm shown in Fig. 6 with state space \( \mathcal{X} \subset \mathbb{R}^7 \times \text{SE}(3) \) consisting of joint positions \( q \in \mathbb{R}^7 \) in radians together with the pose of the end-effector \( p \in \text{SE}(3) \). We use quaternions to represent the end-effector orientation which naturally pairs with our use of the Bingham distribution as described in Sec. IV-D. The robot’s action space \( \mathcal{U} \subset \mathbb{R}^7 \) consists of changes in joint angles \( \Delta q \in \mathbb{R}^7 \). The state transition dynamics are governed by

\[
q_{t+1} = q_t + \Delta q_t \tag{38}
\]

\[
p_{t+1} = FK(q_{t+1}) \tag{39}
\]

where \( FK(\cdot) \) is the robot’s forward kinematics. An Allegro hand is mounted on the robot arm. The robot’s objective is to orient the palm link of the hand to a pre-grasp pose. We use a fixed grasp pre-shape joint configuration for the fingers. We use the “sugar box” object from the YCB dataset in our experiments, shown in Fig. 6. We defined a goal pose mixture model with 6 components following the heuristic from [52].

This problem is traditionally solved by selecting a particular point-based goal to create a motion plan for [52]. For example, we can generate a sample from the pose mixture model we have defined. However, many of the samples will be unreachable based on the kinematics of the robot arm. A target pose relative to the object may be reachable when the object is placed in some poses on the table, and unreachable in others. We quantify this for our environment by spawning the object in 100 random locations on the table. We then generate 100 samples from the pose mixture model distribution and use the robot’s inverse kinematics (IK) to compute joint configurations that would enable the robot to reach to each sample. A sample is considered reachable if an IK solution can be found for that sample, and unreachable otherwise. We use Drake [80] to compute IK solutions, where we make up to 10 attempts to find a solution, each time seeding the solver with a different joint configuration sampled uniformly within the robot’s joint limits.

We find that for the 100 random poses of the object, a mean value of 39.53 of the 100 generated samples are unreachable with a standard deviation of 16.97. We additionally quantify the number of unreachable components per object pose by generating 100 samples from each component. We determine a component to be unreachable if at least 50 of the 100 component samples are unreachable. We find that on average 2.38 of the 6 components are unreachable with a standard deviation of 1.08 (see Appendix C for more visualizations of reachability). These numbers suggest that simply generating goal samples for a point-based planner will frequently result in goal points that are not feasible (i.e. unreachable). The point-based planner will therefore need to first validate the point is reachable by computing inverse kinematics prior to planning.
Fig. 6: Examples of a 7-DOF arm reaching its end-effector to a goal pose mixture model distribution about an object to be grasped. Close-up views of the object are shown in the first frame in each row, where green and red spheres indicate whether a pose is reachable or not, respectively. Each row shows the reaching motion where time increases from left to right.

Using our probabilistic planning approach and the pose mixture model as the goal distribution, we are able to generate reaching plans without having to check for reachability with inverse kinematics. A selection of executions are shown in Fig. 6. We define a pose distribution for the end-effector with a fixed covariance which we transform over the planning horizon to mimic uncertainty propagation. We make this simplification since the dynamics noise on industrial robot arms like the KUKA iiwa we utilize is typically negligible. We note that proper uncertainty propagation can be performed using unscented orientation propagation [25]. We use the approximate cross-entropy loss described in Sec. VII-C since there is no closed form solution between a pose distribution and pose mixture model.

We ran our approach 10 times on all 100 object poses, each time with a different seed for random number generators, to quantify how often the plans reach to a reachable component, where component reachability is determined as described above. Our approach reaches to a reachable component with a mean percentage of 96.4% and standard deviation of 2.059% over the 10 trials. The instances where our approach does not plan to a reachable component are due to the solver getting stuck in local optima. This effect could be mitigated by a more advanced solver, e.g. using Stein variational methods [46].

In summary, a pose mixture model is a common heuristic representation for target pose candidates in reaching to grasp an object. Instead of planning to point-based samples, our approach is capable of using the pose mixture model distribution directly as the goal representation. In contrast to the point-based goal representation, we do not have to compute inverse kinematics to first determine if the target pose is reachable or not.

IX. DISCUSSION AND CONCLUSION

In this paper, we have argued that goal distributions are a more suitable goal representation than point-based goals for many problems in robotics. Goal distributions not only subsume traditional point-based and set-based goals but enable varied models of uncertainty the agent might have about its goal. We derived planning to goal distributions as an instance of planning as inference, thereby connecting our approach to the rich literature on planning as inference and enabling the use of a variety of solvers for planning to goal distributions. We additionally derived several cost reductions of our probabilistic planning formulation to common planning objectives in the literature. Our experiments showcased the flexibility of probability distributions as a goal representation, and the ease with which we can accommodate different models of goal uncertainty in our framework.

We believe there are many exciting avenues for future
research in planning to goal distributions. One interesting direction is incorporating learning into our probabilistic planning framework. We see two key areas that will benefit from learning. First, learning conditional generative models of goal distributions will provide a powerful and flexible goal representation for more complex environments and behaviors. This is also advocated for in [61], albeit for goal-conditioned policies in reinforcement learning. For example, learning mixture density networks [12] conditioned on a representation of the current environment configuration would enable learning scene-dependent multi-modal goal distributions. This could be paired with a multi-modal distribution in the planner as in Stein-variational methods [46] to enable parallelized planning to multiple goal regions simultaneously.

A second area we believe learning will benefit our probabilistic planning framework is in propagating the robot’s state uncertainty. The state spaces and dynamics functions utilized in our experiments were simple for demonstrative purposes, but more complex environments may require more advanced uncertainty propagation techniques to estimate the terminal state distribution well. For example, if the state space itself is a learned latent representation space, normalizing flows [41] may be an interesting technique to estimate how the latent state distribution transforms over the planning horizon.

In this paper, we only investigated cross-entropy and KL divergence as information-theoretic losses in our optimization. As we discussed in Sec. III other losses are possible such as $f$-divergence [8] [24] [37] and Tsallis divergence [86]. Given the use of these broader classes of divergence in imitation learning [24] [37], reinforcement learning [8], and stochastic optimal control [86], we believe there are further opportunities in these related disciplines to utilize goal distributions beyond the planning framework we presented in this paper. We also foresee the utility of goal distributions in deterministic planning with uncertain goals, e.g. having Gaussian goal uncertainty and using Mahalanobis as a heuristic bias in sampling-based planning algorithms like RRT [47].

Our experiments in Sec. VIII-A demonstrated that minimizing the M-projection of cross-entropy is often necessary when we wish to model goals as distributions with finite support. We discussed in Sec. VIII-B how minimizing the M-projection of KL divergence and cross-entropy are equivalent for our optimizations since the entropy of the goal distribution is constant with respect to the decision variables. However, this is not always the case. Our experiments in Sec. VIII-C required the agent to maintain a belief of a moving target’s location which was updated at every timestep based on observations from a noisy sensor. The goal distribution was therefore dependent on the agents observations, which the agent has some control over. That is, the agent’s action determines in part what it will observe next, and the observation impacts what the goal distribution will be at the next timestep. Optimizing the M-projection in this case could therefore encourage the agent to reduce uncertainty about its goal and lead to exploratory behavior. However, the agent requires a model to predict what it will observe over its planning horizon, e.g. a maximum likelihood estimate in a POMDP setting [68]. This scenario is beyond the scope of the current paper but we believe this is an exciting application of our framework that deserves further investigation.

We showed in our results in Sec. VII-B that our planning approach could leverage a source of state uncertainty in the environment to more accurately match a goal state distribution. By embracing uncertainty both in the goal representation and how the agent predicts its state to transform over the planning horizon, our approach opens a wider array of behaviors that would typically be avoided by deterministic plans to point-based goals. We believe leveraging sources of uncertainty in a controlled manner will open up more diverse behaviors in various robotics domains. For example, predicting the distribution of poses an object might settle in after a robot tosses it [92] would enable planning to target locations outside the robot’s reachable workspace. As another example, driving on mixed terrain [19] could be made more robust by modeling how the robot will behave driving over treacherous terrain like ice and mud.

We believe goal distributions are a versatile and expressive goal representation for robots operating under uncertainty. The probabilistic planning framework we have presented in this article easily accommodates different goal distributions and probabilistic planning objectives. We foresee these techniques being particularly applicable to real-world robotics problems where state and goal uncertainty are inherent. We are excited by the prospect of embracing goal uncertainty explicitly in planning and control, especially when considering state or observation uncertainty. We anticipate many fruitful avenues of further research beyond what we have suggested here.

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APPENDIX A

VIARIATIONAL INFERENCE FOR PLANNING AS INFERENCE

We described at a high-level in Sec. V-A how variational inference techniques are often used to solve planning as inference problems. We now provide a more detailed derivation for the optimization objective in Eq. 9. This derivation is similar to that provided in [46] (cf. Appendix D in [46]).

Claim 1. Consider the distribution of optimal trajectories \( p(\tau \mid O_\tau) \) and a proposal distribution \( q(\tau) \in \mathcal{Q} \) from a family of distributions \( \mathcal{Q} \). The objective

\[
q^* = \arg\min_{q \in \mathcal{Q}} D_{KL}(q(\tau) \parallel p(\tau \mid O_\tau))
\]

is equivalently solved by

\[
q^* = \arg\min_{q \in \mathcal{Q}} -E_q[\log p(O_\tau \mid \tau)] + D_{KL}(q(\tau) \parallel p(\tau))
\]

Proof: We have

\[
q^* = \arg\min_{q \in \mathcal{Q}} D_{KL}(q(\tau) \parallel p(\tau \mid O_\tau)) = \arg\min_{q \in \mathcal{Q}} \int q(\tau) \log \frac{q(\tau)}{p(\tau \mid O_\tau)} d\tau = \arg\min_{q \in \mathcal{Q}} \int q(\tau) \log \frac{q(\tau)}{p(O_\tau \mid \tau)p(\tau)} d\tau = \arg\min_{q \in \mathcal{Q}} \int q(\tau) \left[ \log q(\tau) - \log p(O_\tau \mid \tau) - \log p(\tau) \right] d\tau = \arg\min_{q \in \mathcal{Q}} -\int q(\tau) \log p(O_\tau \mid \tau) d\tau + \int q(\tau) \log \frac{q(\tau)}{p(\tau)} d\tau = \arg\min_{q \in \mathcal{Q}} -\int q(\tau) \log p(O_\tau \mid \tau) d\tau + \int q(\tau) \log \frac{q(\tau)}{p(\tau)} d\tau = \arg\min_{q \in \mathcal{Q}} -\mathbb{E}_q[\log p(O_\tau \mid \tau)] + D_{KL}(q(\tau) \parallel p(\tau))
\]

This establishes Eq. 9 in V-A. Note the KL term in Eq. 48 regularizes the trajectory distribution \( q(\tau) \) to a prior distribution \( p(\tau) \). As such, we denote the prior distribution by \( p_0(\tau) \) in Eq. 9. This objective is equivalent to maximizing the evidence lower bound (ELBO) expressed by

\[
q^* = \arg\max_{q \in \mathcal{Q}} \mathbb{E}_q[\log p(O_\tau \mid \tau)] - D_{KL}(q(\tau) \parallel p(\tau))
\]

which is a common variational objective in machine learning more generally.

APPENDIX B

ALTERNATIVE DERIVATION FOR GOAL DISTRIBUTIONS IN PLANNING AS INFERENCE

We now present an alternative derivation for our results in Sec. V-B. We re-derive the planning as inference and stochastic optimality duality from [71] taking special care to introduce the optimal goal distribution for the terminal cost to fit our needs.

In planning as inference we wish to minimize the following objective

\[
q^* = \arg\min_q D_{KL}(q(\tau) \parallel p(\tau \mid O = 1))
\]

where the variational \( q \) has the form:

\[
q(\tau) = q(x_0) \prod_{t=0}^{T-1} q(x_{t+1} \mid x_t, u_t) \pi_q(u_t \mid x_t)
\]

and the optimal trajectory distribution has the form

\[
p(\tau \mid O = 1) \propto p(\tau) p(O_\tau = 1 \mid x_T) \prod_{t=0}^{T-1} p(x_t \mid x_{t+1} \mid x_t, u_t) p(x_{t+1} \mid x_t, u_t) \pi_0(u_t \mid x_t)
\]
We now plug the trajectory distribution definitions into the KL objective and expanding terms.

\[
D_{\text{KL}}\left(q(\tau) \parallel p(\tau \mid \mathcal{O} = 1)\right) = \mathbb{E}_{q(\tau)} \left[ \ln \frac{q(x_0)p(O = 1 \mid x_T) \prod_{t=0}^{T-1} q(x_{t+1} \mid x_t, u_t) \pi_q(u_t \mid x_t)}{p(x_0)p(O = 1 \mid x_T) \prod_{t=0}^{T-1} p(O = 1 \mid x_t, u_t)p(x_{t+1} \mid x_t, u_t)\pi_0(u_t \mid x_t)} \right]
\]

\[
= \mathbb{E}_{q(\tau)} \left[ \ln \left( \prod_{t=0}^{T-1} q(x_{t+1} \mid x_t, u_t)\pi_q(u_t \mid x_t) \right) - \ln \left( \prod_{t=0}^{T-1} p(O = 1 \mid x_t, u_t)p(x_{t+1} \mid x_t, u_t)\pi_0(u_t \mid x_t) \right) \right]
\]

\[
= \mathbb{E}_{q(\tau)} \ln q(x_0) - \ln p(x_0) + \sum_{t=0}^{T-1} \ln q(x_{t+1} \mid x_t, u_t) - \sum_{t=0}^{T-1} \ln p(x_{t+1} \mid x_t, u_t)
\]

\[
+ \sum_{t=0}^{T-1} \ln \pi_q(u_t \mid x_t) - \ln p(O = 1 \mid x_T) - \sum_{t=0}^{T-1} \ln p(O = 1 \mid x_t, u_t) - \sum_{t=0}^{T-1} \ln \pi_0(u_t \mid x_t)
\]

We make the common assumption \cite{50} that we have the correct estimate of the initial distribution \(p(x_0) = q(x_0)\) from our state estimation process as well as the correct dynamics model \(q(x_{t+1} \mid x_t, u_t) = p(x_{t+1} \mid x_t, u_t)\). Then the objective simplifies to

\[
D_{\text{KL}}\left(q(\tau) \parallel p(\tau \mid \mathcal{O} = 1)\right) = \mathbb{E}_{q(\tau)} \left[ \sum_{t=0}^{T-1} \ln \pi_q(u_t \mid x_t) - \ln p(O = 1 \mid x_T) - \sum_{t=0}^{T-1} \ln p(O = 1 \mid x_t, u_t) - \sum_{t=0}^{T-1} \ln \pi_0(u_t \mid x_t) \right]
\]

If we further assume that the policy is deterministic \(\pi_q(u_t \mid x_t) = \delta_{u_t = \phi(x_t)}\), we are able to integrate the term out:

\[
D_{\text{KL}}\left(q(\tau) \parallel p(\tau \mid \mathcal{O} = 1)\right) = \mathbb{E}_{q(\tau)} \left[ - \ln p(O = 1 \mid x_T) - \sum_{t=0}^{T-1} \ln p(O = 1 \mid x_t, u_t) - \sum_{t=0}^{T-1} \ln \pi_0(u_t \mid x_t) \right]
\]

We now assert our optimality distributions, namely that the terminal state reaches the estimated goal, \(p(O = 1 \mid x_T) = g(x_T)\) and our non-terminal optimality conditions are exponentiated negative cost (i.e. reward), \(p(O = 1 \mid x_t, u_t) = \exp\{ - \alpha c_t(x_t, u_t) \}\). We get the objective of

\[
D_{\text{KL}}\left(q(\tau) \parallel p(\tau \mid \mathcal{O} = 1)\right) = \mathbb{E}_{q(\tau)} \left[ - \ln g(x_T) + \sum_{t=0}^{T-1} \ln \pi_0(u_t \mid x_t) \right]
\]

Which shows us that we wish to minimize the sum of the expected negative log likelihood of reaching the goal, the expected running costs (scaled by \(\alpha\)) as well as a term penalizing low entropy in the prior policy. However, if we assume a uniform prior policy \(\pi_0(u_t \mid x_t) = U(u_t \mid \mathcal{U})\) then we obtain the simpler form

\[
\arg \min_{\pi} D_{\text{KL}}\left(q(\tau) \parallel p(\tau \mid \mathcal{O} = 1)\right) = \arg \min_{\pi} \mathbb{E}_{q(\tau)} \left[ - \ln g(x_T) + \sum_{t=0}^{T-1} - \alpha c_t(x_t, u_t) \right]
\]

Which is equivalent to maximizing the expected probability of reaching the goal, times the Boltzman distribution over costs:

\[
\min \mathbb{E}_{q(\tau)} \left[ - \ln g(x_T) + \sum_{t=0}^{T-1} - \alpha c_t(x_t, u_t) \right] \equiv \max \mathbb{E}_{q(\tau)} \left[ g(x_T) \prod_{t=0}^{T-1} \exp\{ - \alpha c_t(x_t, u_t) \} \right]
\]

Let \(\lambda = \frac{1}{\alpha}\), then

\[
\pi^* = \arg \min_{\pi} \mathbb{E}_{q(\tau)} \left[ - \ln g(x_T) + \sum_{t=0}^{T-1} - \alpha c_t(x_t, u_t) \right]
\]

\[
= \arg \min_{\pi} \mathbb{E}_{q(\tau)} \left[ - \ln g(x_T) + \sum_{t=0}^{T-1} - \alpha c_t(x_t, u_t) \right] \cdot \lambda
\]

\[
= \arg \min_{\pi} \mathbb{E}_{q(\tau)} \left[ - \lambda \ln g(x_T) + \sum_{t=0}^{T-1} - c_t(x_t, u_t) \right]
\]

which we can interpret as a Lagrange multiplier on the terminal cost term, enforcing an inequality constraint that \(E_{q(\tau)}[p_g(x_T)] > (1 - \epsilon)\) and if this is not met we can update the dual parameter \(\lambda\) and re-solve the unconstrained optimization problem.
We provide some additional visualizations (Fig. 7) for our arm-reaching experiments from Sec. VIII-D to offer more intuition for the reachability of the different components in the pose mixture model we defined. As a reminder, reachability of a component is determined by computing inverse kinematics for 100 samples and counting how many samples were reachable. A component is deemed reachable if at least 50 of the samples are reachable.

We found that some components are reachable in certain object poses and not others. Additionally, for every object pose, there was typically at least one component that was unreachable. Of the 100 object poses, there were only 5 instances where all components were reachable. 88 of the 100 poses had at least one component with zero reachable samples.

We visualize the component reachability in Fig. 7. Every sub-figure shows the object mesh in the same 100 uniformly random poses we used in our experiments in Sec. VIII-D where the color of the mesh indicates the reachability of the associated component. Each sub-figure visualizes reachability for a different component. The color gradient shown in the right side of the figure determines the interpretation of the mesh colors, where yellow means the component is highly-reachable, purple means the component is highly-unreachable, and green and blue are on the spectrum in between.

Note that there are far more instances where the components are nearly completely reachable or completely unreachable in comparison to instances that fall in the middle of that spectrum. This aligns well with our intuition – if one pose is reachable, it’s likely that small perturbations of that pose will also be reachable. However, there are instances where a pose is perhaps reachable but close to the edge of the reachable workspace, or the robot may be near a joint limit. In these instances, small perturbations to a reachable pose may lead to unreachable samples.
APPENDIX D
SOLVER DETAILS

We utilized a variety of solvers in our experiments in Sec. VIII including gradient-based and sampling-based solvers. We will here describe in more details the particular solvers and hyperparameters we used. We emphasize that we did not expend undue effort optimizing hyperparameters and it is likely different solvers and hyperparameters will have better performance on our problems.

A. Dubins Planar Navigation Environment

We used a gradient-based solver for the planar navigation environment in Sec. VIII-A. We formulated our problem as a constrained optimization problem in Drake [80] and used the SNOPT [26] solver. We used the following hyperparameters and parameter settings:

| Parameter | Value | Description |
|-----------|-------|-------------|
| I         | 200   | Maximum number of iterations to run solver |
| T         | 45    | Planning horizon |
| dt        | 0.3   | Timestep |
| \( \omega_t \) | 0.002 | Variance for dynamics noise (isotropic Gaussian) |
| \( \Sigma_0 \) | 0.02  | Initial state covariance (isotropic Gaussian) |
| \( \beta \) | 2     | Parameter to unscented transform governing sigma point dispersion |

We used the default values provided by the Drake interface to SNOPT for all SNOPT hyperparameters not mentioned in the table.

B. Ball-Rolling Environment

We used the cross-entropy method (CEM) [40] for our solver for the ball-rolling problem in Sec. VIII-B. CEM is a sampling-based solver that generates solution samples from a Gaussian distribution, evaluates the cost of each sample, and re-fits the distribution for the next iteration using the samples with the lowest cost (a.k.a. the elite set). The CEM update rule has been derived from a planning as inference framework assuming an optimality likelihood with thresholded utility [46].

We note that in principle the gradient solver from Appendix D-A could be used for this problem, but in practice it was highly sensitive to the initial solution and would frequently get stuck in local optima. In contrast, CEM found solutions very quickly (within approximately 5-10 iterations) and was not as susceptible to local optima. We used the following parameter settings:

| Parameter | Value | Description |
|-----------|-------|-------------|
| I         | 50    | Maximum number of iterations to run CEM |
| N         | 500   | Number of samples generated in each CEM iteration |
| K         | 20    | Number of elite samples to re-fit CEM sampling distribution to |
| \( \sigma^2_0 \) | 0.8   | Initial covariance for the CEM sampling distribution (isotropic Gaussian) |
| H         | 100   | Planning horizon |
| dt        | 0.3   | Timestep |
| \( \omega_t \) | 0.0001 | Nominal variance for dynamics noise (isotropic Gaussian) |
| \( \omega'_t \) | 0.008 | Additional noise from amplifier (isotropic Gaussian) |
| \( \beta \) | 2     | Parameter to unscented transform governing sigma point dispersion |

We used the following default values provided by the Drake interface to SNOPT for all SNOPT hyperparameters not mentioned in the table.

C. Target Intercept Environment

We use model predictive path integral control (MPPI) [88] for our dynamic goal experiments in Sec. VIII-C. MPPI is a sampling-based solver that iteratively updates a Gaussian distribution of solutions by generating noisy perturbations of the current mean solution at each iteration. A weighted average of the samples is computed to determine the new mean solution, where higher cost samples contribute less to the update. This procedure is carried out in a model predictive control (MPC) scheme such that a plan for a small horizon is generated at every iteration, the first action of that plan is executed, and the agent re-plans at the next iteration. This procedure is widely used in robotics due to its computational simplicity, ease of parallelization [9], and applicability to complex problems even when the cost is non-differentiable. See [88] for a more detailed and formal presentation of MPPI.

We used the following parameter settings:
| Parameter | Value | Description |
|-----------|-------|-------------|
| $I$       | 70    | Number of iterations to run MPC |
| $N$       | 100   | Number of samples generated in each MPPI iteration |
| $\sigma_0^2$ | 0.02 | Initial variance for the MPPI sampling distribution (isotropic Gaussian) |
| $\sigma_N^2$ | 0.002 | Terminal variance for the MPPI sampling distribution (isotropic Gaussian) |
| $\max H$  | 25    | Maximum planning horizon |
| $\min H$  | 3     | Minimum planning horizon |

Note we employ a couple of strategies to get better results:

1) We reduce the MPPI variance from $\sigma_0^2$ to $\sigma_N^2$ in even steps over the number of MPPI iterations $I$. We found this strategy to produce better results since it can find more refined solutions as the number of MPPI iterations increases.

2) We dynamically adapt the planning horizon to start at a maximum of $\max H$ and reduce to a minimum of $\min H$ based on how far the agent is from the target. In order to adapt this value based only on information available to the agent, we use the KL divergence between the agent’s state distribution and the projected belief distribution as a measure of proximity so that as the agent gets closer to the target, it uses a shorter planning horizon.

D. Arm-Reaching Environment

We use MPPI as our solver for the arm reaching environment from our experiments in Sec. VIII-D. We note again that in principle the gradient-based solver could in principle be used for this environment. However, the solver required a good initial solution in order to find a valid plan, and the initial solutions ended up over-biasing the solver to particular components to reach to. We found the sampling scheme of MPPI to produce the least biased results. Note that we are not using MPPI in an MPC setting as we did in Appendix D-C, but instead just using the MPPI sampling and update scheme to efficiently find solutions for the full planning horizon. We used the following settings:

| Parameter | Value | Description |
|-----------|-------|-------------|
| $I$       | 30    | Maximum number of iterations to run MPPI |
| $N$       | 500   | Number of samples generated in each MPPI iteration |
| $\sigma_0^2$ | 0.002 | Initial variance for the MPPI sampling distribution (isotropic Gaussian) |
| $\sigma_N^2$ | 0.0001 | Terminal variance for the MPPI sampling distribution (isotropic Gaussian) |
| $\max H$  | 10    | Planning horizon |