Bulk scalar field in warped extra dimensional models

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This work presents a general formalism to analyze a generic bulk scalar field in a multiple warped extra dimensional model with arbitrary number of extra dimensions. The Kaluza-Klein mass modes along with the self interaction couplings are determined and the possibility of having lowest lying KK mode masses near TeV scale are discussed. It is argued that the appearance of large number of closely spaced KK modes with enhanced coupling may prompt possible new signatures in collider physics.

I. INTRODUCTION

Theories with extra spacetime dimensions have drawn considerable attention ever since the original proposal by Kaluza and Klein. There have been renewed interest in such theories after the emergence of string theory. Several new ideas in this context has been proposed and have interesting consequences for particle phenomenology and cosmology (see [1], [2], [3], [4], [5]). In these higher dimensional models, spacetime is usually taken to be product of a four dimensional spacetime and a compact manifold of dimension ‘n’. While gravity can propagate freely through the extra dimensions, standard model particles are confined to the four dimensional spacetime. Observers in this three dimensional wall (a “3-brane”) will measure an effective Planck scale $M_{pl}^2 = M_n^{n+2}V_n$, where $V_n$ is the volume of the compact space. If $V_n$ is large enough it could make Planck scale of the order of TeV, thus removing the hierarchy between the Planck and the weak scale.

Subsequently, Randall et. al (see [6], [7]) proposed a higher dimensional scenario that is based on non-factorizable geometry and accounts for the hierarchy without introducing large extra dimensions. However, the brane-world model itself is not stable and it was shown by Goldberger and Wise(GW) (see [8]) that by introducing a scalar field in the bulk, the modulus-namely the brane separation in RS model can be stabilized without any fine tuning. Assumption of negligibly small scalar back-reaction on the metric in the GW approach prompted further work in this direction where the modification of the RS metric due to back-reaction of the bulk fields have been derived (see [9]). The stability issues in such cases have been re-examined for time-dependent cases (see [10], [11]), also the effect of gauge fields or higher form fields have been studied in several works (see [12]).

In an effort to search for the signatures of extra dimensions, the roles of the Kaluza-Klein modes of different bulk fields on the phenomenology at the standard model brane are of crucial importance.

For the 5-dimensional RS model, Goldberger and Wise determined the bulk scalar KK modes and their self-interactions. It is found that due to the exponential redshift factor in the RS model, KK scalar modes in this spacetime has TeV scale mass splitting and inverse Tev couplings (see [7]). This is in sharp contrast to the KK decomposition in product spacetimes, which for a large compactified dimensions, gives rise to a large number of light KK modes (see [13]) with a very small coupling with brane fields. Due to this very distinct feature the RS model has interesting consequences (see [14] and [15]).

Motivated by string theory and other extra dimensional models where one can have several extra dimensions, in this work we extend the results of the bulk scalar model in 5-dimensional RS space-time to arbitrary number of warped dimensions and have obtained the KK decomposition of the scalar KK masses. We have shown that in these multiply warped models we have much larger number of KK modes

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than the 5-dimensional RS counterpart with effective couplings in the inverse TeV range. Our results establish general formulae for determining these KK masses and couplings in presence of any arbitrary number of extra dimensions.

The paper is organized as follows: We give a brief explanation for six dimensional doubly warped spacetime in section II. The calculation for bulk scalar field has been done in this six dimensional spacetime in section III. The same calculations have been finally extended to higher dimensional spacetime with arbitrary number of extra dimensions in sections IV and V. The paper ends with a short discussion on our results.

II. SIX DIMENSIONAL DOUBLY WARPED SPACETIME AND EINSTEIN’S EQUATION

In this section we shall discuss doubly compactified six dimensional spacetime with $Z_2$ orbifolding along each compactified direction. For a detailed discussion we refer the reader to [15]. The manifold under consideration is given by, $M^{1,5} = [M^{1,3} \times S^1/Z_2] \times S^1/Z_2$ [12].

We let the compactified dimensions to be $x^\mu$ ($\mu = 0, 1, 2, 3$). The moduli along the compact dimensions are given by $R_y$ and $r_z$ respectively. The corresponding metric ansatz is taken as,

$$ds^2 = b^2(z) \left[ a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu + R_y^2 dy^2 \right] + r_z^2 dz^2$$

(1)

with $\eta_{\mu\nu} = diag(-1,1,1,1)$. Thus we have four orbifold fixed points which are given by $(y,z) = (0,0), (0,\pi), (\pi,0), (\pi,\pi)$ respectively.

The total bulk-brane action could be given by,

$$S = S_6 + S_5 + S_4$$

(2)

$$S_6 = \int d^4xdydz \sqrt{-g_6} \left( R_6 - \Lambda \right)$$

(3)

$$S_5 = \int d^4xdydz \left[ V_1(y) \delta(y) + V_2(y - \pi) \right] + \int d^4xdydz \left[ V_3(z) + V_4(z - \pi) \right]$$

(4)

$$S_4 = \sum_{i=1}^2 \int d^4xdydz \sqrt{-g_{vis}} \left( L - V \right) \delta(y - y_i) \delta(z - z_i)$$

(5)

Here the brane potentials in general have the particular functional dependence $V_{1,2} = V_{1,2}(z)$ and $V_{3,4} = V_{3,4}(y)$. Finally the full six dimensional Einstein’s equation is given by,

$$- M^4 \sqrt{-g_6} \left( R_{MN} - \frac{\Lambda}{2} g_{MN} \right) = \Lambda_6 \sqrt{-g_6 g_{MN}}$$

$$+ \sqrt{-g_5 V_1(z)} g_{\alpha\beta} \delta^\alpha_M \delta^\beta_N \delta(y)$$

$$+ \sqrt{-g_5 V_2(z)} g_{\alpha\beta} \delta^\alpha_M \delta^\beta_N \delta(y - \pi)$$

$$+ \sqrt{-g_5 V_3(y)} g_{\alpha\beta} \delta^\alpha_M \delta^\beta_N \delta(y)$$

$$+ \sqrt{-g_5 V_4(z - \pi)} g_{\alpha\beta} \delta^\alpha_M \delta^\beta_N \delta(z - \pi)$$

(6)

In the above expression $M$, $N$ are bulk indices, $\alpha, \beta$ run over the usual four spacetime coordinates given by $x^\mu$. The quantities $g$ and $\tilde{g}$ are the metrics in $(4 + 1)$ dimensional spaces respectively.

The line element derived from the above Einstein’s equation turns out to be [15],

$$ds^2 = \frac{\cosh^2(kz)}{\cosh^2(k\pi)} \left[ cexp(-2c|y|)\eta_{\mu\nu}dx^\mu dx^\nu + R_y^2 dy^2 \right] + r_z^2 dz^2$$

(7)

In the above line element we have the following identification for the constants $k$ and $c$ given by,

$$c \equiv \frac{R_y k}{r_z \cosh(kz)}$$

$$k \equiv r_z \sqrt{\frac{\Lambda_6}{4\Lambda}}$$

(8)
The boundary terms lead to the brane tensions and using the Einstein’s equation across the two boundaries at \( y = 0, y = \pi \) respectively, we readily obtain,

\[
V_1(z) = -V_2(z) = 8M^2 \sqrt{-\frac{\Lambda}{10}} \text{sech}(kz)
\]  

(9)

Thus the two 4-branes situated at \( y = 0 \) and \( y = \pi \) would have a \( z \)-dependent brane tension. The fact that the two tensions are equal and opposite is reminiscent of the original RS-form. Similarly we get the brane tensions for other two 4-branes as,

\[
V_3(y) = 0; V_4(y) = -\frac{8M^4k}{r_z} \tanh(k\pi)
\]  

(10)

Here \( V_{3,4} \) were introduced to account orbifolding along \( z \)-direction and with \( g_{zz} \) being a constant, the resulting hypersurface should have only a constant energy density. The fact that \( g_{yy} \) is dependent on the co-ordinate \( z \) makes the two hypersurfaces for \( y \)-orbifolding to have a \( z \)-dependent energy density.

The 3-brane located at \( (y = 0, z = \pi) \) suffers no warping and can be identified with the Planck brane. The other three can be valid choices for Standard Model (visible) brane. However if we assume that there is no brane having lower energy scale than ours, we are forced to identify the SM brane to be located at \( (y = \pi, z = 0) \).

III. BULK FIELD IN SIX DIMENSIONAL DOUBLY WARPED SPACETIME

In this section we carry out the Kaluza-Klein decomposition of a non-gravitational bulk scalar field propagating in the spacetime described by equation (7) in the spirit of the work of Goldberger and Wise. We find that in these multiply warped spacetime the SM brane contains larger number of TeV scale scalar KK modes than 5D RS model. This has significant phenomenological consequences [16].

We consider a free scalar field in the bulk for which the action is given by,

\[
S = \frac{1}{2} \int d^4x \int dy \int dz \sqrt{-G} \left( G^{AB} \partial_A \Phi \partial_B \Phi + m^2 \Phi^2 \right)
\]  

(11)

where \( G^{AB} \) with \( A, B = \mu, y, z \) is given by equation (7), and \( m \) is of order of \( M_{pl} \). After an integration by parts, this can be written as,

\[
S = \frac{1}{2} \int d^4x \int dy \int dz \left[ R_y r_z e^{-2\sigma} \frac{\cosh^3(kz)}{\cosh^3(k\pi)} \eta_{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi 
+ R_y r_z e^{-4\sigma} \frac{\cosh^5(kz)}{\cosh^5(k\pi)} m^2 \Phi^2 
- \frac{r_z}{R_y} \Phi \partial_y \left( e^{-4\sigma} \frac{\cosh^3(kz)}{\cosh^3(k\pi)} \partial_y \Phi \right) 
- \frac{R_y}{r_z} \Phi \partial_z \left( e^{-4\sigma} \frac{\cosh^3(kz)}{\cosh^3(k\pi)} \partial_z \Phi \right) \right]
\]  

(12)

where \( \sigma = c|y| \). Now we make the following substitution for KK decomposition,

\[
\Phi(x, y, z) = \sum_{n,m} \phi_{nm}(x) \frac{\alpha_n(y) \beta_m(z)}{\sqrt{R_y} \sqrt{r_z}}
\]  

(13)

The following normalization conditions are imposed on the fields \( \alpha \) and \( \beta \),

\[
\int dy e^{-2\sigma} \alpha_n \alpha_m = \delta_{nm}
\]  

(14)

\[
\int dz \frac{\cosh^3(kz)}{\cosh^3(k\pi)} \beta_p \beta_q = \delta_{pq}
\]  

(15)
The differential equation satisfied by the function $\alpha_n(y)$ is,

$$- \frac{1}{R_y^2} \frac{d}{dy} \left( e^{-4\sigma} \frac{d\alpha_m}{dy} \right) = A_m^2 e^{-2\sigma} \alpha_m$$

(16)

which further simplifies to the following form,

$$\frac{d^2\alpha_m}{dy^2} - 4kR_y \frac{d\alpha_m}{dy} + A_m^2 R_y^2 e^{2\sigma} \alpha_y = 0$$

(17)

The above equation can be solved in terms of Bessel functions of first and second order, as,

$$\alpha_m = \frac{e^{2\sigma}}{N_m} \left[ J_2 \left( \frac{A_m e^\sigma}{k} \right) + b_m Y_2 \left( \frac{A_m e^\sigma}{k} \right) \right]$$

(18)

Similar analysis for $\beta$ yields,

$$\frac{d^2\beta_m}{dz^2} + 5ktanh(kz) \frac{d\beta_m}{dz} + \frac{1}{z^2} B_m^2 \cosh^2(k\pi) \beta_m = 0$$

(19)

The solution, apart from an overall normalization, can be expressed as,

$$\beta_m(z) = e^{\frac{5}{2} k^2 z^2} \left[ H_{\frac{5}{2} k^2 z^2} \left( \frac{-10k^2 + B_m^2 r^2 (1 + \cosh(2k\pi))}{10k^2} \right) + E_m e^{\frac{5}{2} k^2 z^2} \right] F_1 \left( \frac{-10k^2 + B_m^2 r^2 (1 + \cosh(2k\pi))}{10k^2}, \frac{1}{2}, \frac{5k^2 z^2}{2} \right)$$

(20)

where $F_1$ is the Kummer confluent hypergeometric function. Using the above equations we readily obtain the following action for the field $\phi(x)$ as,

$$S = \frac{1}{2} \int d^4x \left( \sum_{n,m} \eta^{\mu\nu} \partial_\mu \phi_{nm} \partial_\nu \phi_{nm} + \sum_{n,m,p,q} M_{nmpq} \phi_{nm} \phi_{pq} \right)$$

(21)

$$M_{nmpq} = \{ A_n^2 \delta_{np} \delta_{mq} + B_n^2 \delta_{np} P_{mq} + m^2 P_{np} Q_{mq} \}$$

(22)

where we have the following expression for the element $Q_{nm}$,

$$Q_{nm} = \int dz \frac{\cosh^5(kz)}{\cosh^5(k\pi)} \beta_n \beta_m$$

(23)

and $P_{nm}$ as,

$$P_{nm} = \int dy e^{-4\sigma} \alpha_n \alpha_m$$

(24)

Now from the previous discussion we have the solution for these two sets of functions $\alpha_n(y)$ and $\beta_n(z)$, which can be used to determine $Q_{nm}$ and $P_{nm}$ in order to obtain the masses of the KK modes by evaluating the quantity $M_{nmpq}$. In contrast to the five dimensional situation (see [14]) where the masses of the bulk fields appear as a diagonalized mass matrix, in this case the bulk field $\Phi(x, y, z)$ manifests itself to some four dimensional observer as an infinite KK tower with mass being determined by the the quantity $M_{nmpq}$ such that a scalar $\phi_{nm}$ has a mass $M_{nmnm}$ after an appropriate diagonalization procedure.

The solution for $\alpha_n(y)$ is presented in equation (15). A similar solution was obtained by Wise et. al (see [14]) except for the fact that we have Bessel functions of second order. Following their discussion we can argue in a similar manner that the lightest KK modes have mass parameter $A_m$ suppressed exponentially with respect to the the scale $m$ appearing in equation (11). Since we have taken $m$ to be order of Planck scale and $c$ to be around 12, by stabilization these mass modes $A_m$ are in the TeV range. Also from the solution we could observe that the modes $\alpha_n(y)$ are larger in the region $y = \pi$.

The solution for $\beta_n(z)$ has been presented in equation (20). The solution has an overall factor of $\exp \left[ -\frac{5}{2} k^2 z^2 \right]$ and we see that the solution has maximum value around $z = 0$. Hence the bulk field being
a product of these two functions $\alpha_m(y)$ and $\beta_n(z)$ as shown in equation [12] has mass parameter in the TeV range and has maximum value around $(y = \pi, z = 0)$. Now from section [11] this is precisely the SM brane. Hence bulk field has a maximum in the SM brane i.e. the KK modes are most likely to be found in that region where $A_m$ and $B_m$ are in the TeV range. This sets the stage for KK excitations to have TeV scale mass splitting on the SM brane.

We now present the self interactions of the bulk scalar field. From the four dimensional point of view we start with a seven dimensiona l spacetime where the space-time like dimensions are successively warped. In other words the manifold of interest could be given by, $[\{ f^{1,3} \times [S^1/Z_2] \} \times S^1/Z_2] \times S^1/Z_2$. Thus the effective four dimensional coupling constants are,

$$\lambda_{eff} = \frac{4\alpha_p}{(M_{R_y})^{m-1}(M_{R_z})^{m-1}M^{2m-4}} \int_0^\pi dy \int_0^\pi dz \int_0^{\pi} d\rho \rho \frac{\sin^5(\rho^2)}{\sin^5(\pi \rho^2)} \left( \frac{\alpha_p}{\sqrt{R_y}} \right)^{2m}$$

which reduces to,

$$\lambda_{eff} \approx 4\alpha \left( \frac{c}{M_{R_y}} \right)^{m-1} \left( \frac{1}{M_{R_z}} \right)^{m-1} \left( Me^{-c \pi} \right)^{4-2m} \int_0^{\pi} r^{4m-5} dr \left[ \frac{J_2(A_p \frac{c}{k \pi} r)}{A_p} \right]^{2m}$$

in large $kR_y$ and $kR_z$ limit. Hence we observe that the relevant scale for four dimensional physics is not the scale set by Planck scale i.e. $M$, but this is $Me^{-c \pi} \frac{1}{\cosh k \pi}$. Hence the KK reduction has lead the couplings from Planck scale to the TeV scale by the warp factor on the SM brane located at $(y = \pi, z = 0)$.

IV. SEVEN AND HIGHER DIMENSIONAL SPACETIME WITH MULTIPLE WARPING

With an aim to arrive at a generic result we shall now try to extend our analysis with one more extra dimension. For that purpose we start with a seven dimensional spacetime where the space-like dimensions are successively warped. In other words the manifold of interest could be given by, $[\{ f^{1,3} \times [S^1/Z_2] \} \times S^1/Z_2] \times S^1/Z_2$. Then the total brane bulk action can be given by,

$$S = S_7 + S_6 + S_5 + S_4$$

$$S_7 = \int d^4x dy dz dw \sqrt{g_7} (R_7 - \Lambda_7)$$

$$S_6 = \int d^4x dy dz dw \left[ V_1 \delta(w) + V_2 \delta(w - \pi) \right] + \int d^4x dy dz dw \left[ V_3 \delta(z) + V_4 \delta(z - \pi) \right] + \int d^4x dy dz dw \left[ V_5 \delta(y) + V_6 \delta(y - \pi) \right]$$

with appropriate actions $(S_5)$ for twelve possible 4-branes at the edges $(z, w) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$, $(z, y) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$ and $(y, w) = (0, 0), (0, \pi), (\pi, 0), (\pi, \pi)$. We also have eight possible 3-branes at the corners $(y, z, w) = (0, 0, 0), (0, 0, \pi), (\pi, 0, 0), (\pi, \pi, 0), (\pi, 0, \pi), (\pi, \pi, \pi)$. (\pi, \pi, \pi)$.
By natural extension of the method as illustrated in previous section we get the line element and other parameters such that 

\[
 ds^2 = \frac{\cosh^2(\ell w)}{\cosh^2(\ell \pi)} \left\{ \frac{\cosh^2(kz)}{\cosh^2 k\pi} \left[ \exp(-2cy)\eta_{\mu\nu}dx^\mu dx^\nu + R_y^2 dy^2 \right] + r_z^2 dz^2 \right\} + \eta_{w}^2 dw^2 \\
 \ell^2 = \frac{\Lambda_7 R_w^2}{15} \\
k = \frac{\ell r_z}{R_w \cosh(\ell \pi)} \\
c = \frac{\ell R_y}{R_w \cosh(k\pi) \cosh(\ell \pi)} = \frac{kR_y}{r_z \cosh(k\pi)} \\
\tag{32}
\]

It may be of interest that the 5-brane at \( w = \pi \) does not represent a flat metric (y and z dependences). In order to obtain substantial warping along the \( w \) direction (from \( w = \pi \) to \( w = 0 \)), one need to make \( \ell \pi \) substantial (same order of magnitude as RS scenario). The seven dimensional or triply warped model has a structure analogous to that of six dimensional one, not only in the, form of functional dependence but also on the nature of warping. This method can easily be extended to even higher dimensions. Also note that the orbifolding requires branes situated at edges of n-dimensional hypercube with 3-branes at the corners. If one of the direction suffers a large warping any other direction should have small warping so that there is no large hierarchy coming from the moduli. In this case also we have several candidates for our SM brane. However applying the fact that no brane should have less energy than ours, leads to (\( y = \pi, z = 0, w = 0 \)) to be SM brane.

V. BULK FIELDS IN SEVEN AND HIGHER DIMENSIONAL SPACETIME

Following the methods of previous sections we shall carry out the Kaluza-Klein decomposition of a bulk scalar field propagating in the spacetime given by equation \ref{eq:32}. As in the previous section in this case as well we can write the bulk scalar field in terms of product of four functions. By making KK decomposition we again end up with KK mass modes having TeV scale masses and splittings. The action for the bulk scalar field in this seven dimensional spacetime can be given as,

\[
 S = \frac{1}{2} \int d^4x \int dy \int dz \int dw \sqrt{-G} \left[ G_{AB} \partial^A \Phi \partial^B \Phi + m^2 \Phi^2 \right] \\
\tag{33}
\]

From the line element as given by equation \ref{eq:29} we readily obtain the following form for the action,

\[
 S = \frac{1}{2} \int d^4x \int dy \int dz \int dw \left[ R_y r_z R_w e^{-2\sigma} \cosh^4(\ell w) \cosh^3(kz) \eta_{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \\
+ \frac{1}{2} R_y R_w r_z \cosh^4(\ell \pi) \cosh^3(k\pi) e^{-4\sigma} (\partial_y \Phi)^2 \\
+ \frac{1}{2} R_y r_z R_w \cosh^4(\ell w) \cosh^3(kz) e^{-4\sigma} (\partial_z \Phi)^2 \\
+ \frac{1}{2} R_w r_z R_y \cosh^4(\ell \pi) \cosh^3(k\pi) e^{-4\sigma} (\partial_w \Phi)^2 \\
+ \frac{1}{2} m^2 R_w r_z R_y \cosh^6(\ell w) \cosh^5(kz) \cosh^5(\ell \pi) \cosh^3(k\pi) e^{-4\sigma} \Phi^2 \right] \\
\tag{34}
\]

where \( \sigma = c|y| \). We make the following substitution for the bulk field,

\[
\Phi = \sum_{pqrs} \phi_{pqrs}(x) \frac{\alpha_p}{\sqrt{R_y}} \frac{\beta_q}{\sqrt{r_z}} \frac{\gamma_r}{\sqrt{R_w}} \\
\tag{35}
\]

We also impose the following normalization for the functions \( \alpha_p, \beta_q \) and \( \gamma_r \),

\[
\int e^{-2\alpha_m} \alpha_n dy = \delta_m^n \\
\tag{36}
\]
The third equation (42) has the following solution along with an overall normalization,

\[ \int \frac{\cosh^3(kz)}{\cosh^3(k\pi)} \beta_m \beta_n dz = \delta_{mn} \]  

(37)

\[ \int \frac{\cosh^4(\ell w)}{\cosh^4(\ell \pi)} \gamma_m \gamma_n dw = \delta_{mn} \]  

(38)

Now applying integration by parts to the integral as presented in equation [34] we readily obtain,

\[
S = \frac{1}{2} \int d^4x \int dy \int dz \int dw \left\{ \sum_{pqrabc} \frac{e^{-2\sigma}}{\cosh^4(\ell \pi)cosh^3(k\pi)} (\eta^{\mu\nu} \partial_\mu \phi_{pqr} \partial_\nu \phi_{abc}) \alpha_p \alpha_q \beta_r \gamma_c \right\} 
\]

\[
- \frac{1}{2 R_y^2} \sum_{pqrabc} \frac{cosh^4(\ell w) \cosh^3(k\pi)}{cosh^4(\ell \pi) \cosh^3(k\pi)} \phi_{pqr} \phi_{abc} \beta_q \gamma_c \alpha_p \partial_y (e^{-4\sigma} \partial_y \alpha_a) 
\]

\[
- \frac{1}{2 r_z^2} \sum_{pqrabc} \frac{cosh^4(\ell w) \cosh^3(k\pi)}{cosh^4(\ell \pi) \cosh^3(k\pi)} \phi_{pqr} \phi_{abc} \alpha_p \alpha_q \gamma_c \partial_z (\cosh^5(kz) \partial_z \beta_n) 
\]

\[
- \frac{1}{2 M_w^2} \sum_{pqrabc} \frac{cosh^5(kz) \cosh^3(k\pi)}{cosh^5(k\pi) \cosh^3(k\pi)} e^{-4\sigma} \phi_{pqr} \phi_{abc} \alpha_p \alpha_q \beta_r \gamma_c 
\]

(39)

Then we make the following choice for the differential equations satisfied by the functions \( \alpha_n, \beta_n \) and \( \gamma_n \),

\[
- \frac{1}{R_y^2} \partial_y (e^{-4\sigma} \partial_y \alpha) = A_n^2 e^{-2\sigma} \alpha_n 
\]

(40)

\[
- \frac{1}{r_z^2} \partial_z (\cosh^5(kz) \partial_z \beta_n) = B_n^2 \cosh^3(k\pi) \beta_n 
\]

(41)

\[
- \frac{1}{M_w^2} \partial_w (\cosh^6(\ell w) \partial_w \gamma_n) = C_n^2 \cosh^4(\ell \pi) \gamma_n 
\]

(42)

The first equation as presented in [44] can be solved and has identical solution as obtained in the previous section. However for convenience we rewrite the solution,

\[
\alpha_p = \frac{e^{2\sigma}}{N_p} \left[ J_2 \left( \frac{A_p e^{\sigma}}{k} \right) + b_p Y_2 \left( \frac{A_p e^{\sigma}}{k} \right) \right] 
\]

(43)

The second equation as given by equation [45] has the same solution as presented in equation [20] but we rewrite it here,

\[
\beta_q(z) = \exp \left[ -\frac{5}{2} k^2 z^2 \right] H_{\sqrt{5/2}kz} \left( \frac{-10k^2 + B_q^2 r^2 (1 + Cosh(2k\pi))}{10k^2} \right) 
\]

\[
+ E_q \exp \left[ -\frac{5}{2} k^2 z^2 \right] \text{1F1} \left( \frac{-10k^2 + B_q^2 r^2 (1 + Cosh(2k\pi))}{10k^2}, \frac{1}{2}, \frac{5k^2 z^2}{2} \right) 
\]

(44)

The third equation [46] has the following solution along with an overall normalization,

\[
\gamma_r(w) = \exp \left[ -3\ell^2 w^2 \right] H_{\sqrt{3}\ell w} \left( \frac{-12\ell^2 + C_\ell^2 r^2 (1 + Cosh(2\ell\pi))}{12\ell^2} \right) 
\]

\[
+ F_r \exp \left[ -3\ell^2 w^2 \right] \text{1F1} \left( \frac{-12\ell^2 + C_\ell^2 r^2 (1 + Cosh(2\ell\pi))}{24\ell^2}, \frac{1}{2}, \frac{3\ell^2 w^2}{2} \right) 
\]

(45)

Hence our final expression for the action is given by,

\[
S = \frac{1}{2} \int d^4x \left\{ \sum_{pqr} \eta^{\mu\nu} \partial_\mu \phi_{pqr} \partial_\nu \phi_{abc} + \sum_{abcq} M_{pqrabc} \phi_{pqr} \phi_{abc} \right\} 
\]

(46)

\[
M_{pqrabc} = \left\{ A_p^2 \delta_{pa} \delta_{qb} \delta_{rc} + B_p^2 P_{rc} \delta_{pa} \delta_{qb} + C_p^2 P_{qb} Q_{rc} \delta_{pa} + m^2 P_{pa} Q_{qb} R_{rc} \right\} 
\]

(47)
Thus these mass modes can be seen to include exponential factors such as $\exp \beta_n(z)$ with respect to the scalar mass $m$. The effective four dimensional coupling constants therefore are being given by equation (43) as well. From the solution of $\gamma_n(y)$ which in the large $\mathcal{R}_y$ range, solutions have maximum value around $y = 0$ as obtained earlier in section III as well. From the solution of $\gamma_n(w)$ it is evident that the solution has maximum value around $w = 0$. Hence the bulk field being a product of these three functions $\alpha_n(y)$, $\beta_n(z)$ and $\gamma_n(w)$ as shown in equation (25), has mass parameter $B_M$ is of the order of TeV, solutions have maximum value to find the modes around $(y = \pi, z = 0, w = 0)$ which is the location of the SM brane. Also the bulk field is maximum in the SM brane i.e. the KK modes are most likely to be found in the TeV region as the $A_m$, $B_m$ and $C_m$ are in the TeV range.

For completeness we present the self interactions of the bulk scalar field in this seven dimensional spacetime. From the four dimensional point of view these self interactions induce couplings between the KK modes. In this case also the effective self-couplings are suppressed by the warp factor and if the Planck scale sets the six dimensional couplings and the low lying KK modes have TeV range self-interactions. We present the interaction term in the action with coupling parameter $\lambda$ such that,

$$ S_{\text{int}} = \int d^4x \int^\pi_{-\pi} dy \int^\pi_{-\pi} dz \int^\pi_{-\pi} dw \sqrt{G} \frac{\lambda}{M^{5m-7}} \phi^{2m} $$

where the coupling $\lambda$ is of the order of unity. Then we could expand in modes and hence the self-interactions of light KK states are given by,

$$ S_{\text{int}} = \int d^4x \int^\pi_{-\pi} dy \int^\pi_{-\pi} dz \int^\pi_{-\pi} dw R_y r_z R_w e^{-4\sigma} \frac{\cos^5(kz) \cos^6(\ell w)}{\cos^5(k\pi) \cos^6(\ell\pi)} \frac{\lambda}{M^{5m-7}} \phi^{2m} $$

The effective four dimensional coupling constants therefore are being given by,

$$ \lambda_{eff} = \frac{8\lambda}{(MR_y)^{m-1}(MR_z)^{m-1}(MR_w)^{m-1}M^{2m-1}} \int^\pi_0 dy e^{-4\sigma} a_p^2 \int^\pi_0 dz \frac{\cos^5(kz)}{\cos^5(k\pi)} \beta_q^2 \int^\pi_0 dw \frac{\cos^5(\ell w)}{\cos^5(\ell\pi)} \gamma_r^2 $$

which in the large $k R_y$, $k r_z$ and $\ell R_w$ limit reduces to,

$$ \lambda_{eff} \simeq 8\lambda \left( \frac{c}{MR_y} \right)^{m-1} \left( \frac{1}{MR_z} \right)^{m-1} \left( \frac{1}{MR_w} \right)^{m-1} \left( Me^{-c\pi} \frac{1}{\cos^2 k\pi \cos(\ell\pi)} \right)^{4-2m} $$

$$ \int^1_0 r^{4m-5} dr \left[ J_2 \left( \frac{A e^{-c\pi} r}{k} \right) \right]^{2m} \int^\pi_0 dz (\beta_q)^2 \int^\pi_0 dw (\gamma_r)^2 $$
Hence we observe that the relevant scale for four dimensional physics is not the scale set by Planck scale but $M e^{-c_\pi \frac{1}{\cosh k \pi x} \frac{1}{\cosh (\ell \pi)}}$. Hence the KK reduction lead to the TeV scale couplings by the warp factor on the SM brane located at $(y = \pi, z = 0, w = 0)$.

Now this result can easily be extended to any higher dimension spacetime. For $n$ extra dimensions we can write the action for the bulk field as,

$$ S = \frac{1}{2} \int d^4 x \int dy \int dz \int dw \cdots \sqrt{-G} \left[ G_{AB} \partial^A \Phi \partial^B \Phi + m^2 \Phi^2 \right] $$

where $G_{AB}$ with $A, B = \mu, \nu, z, w, \cdots$ is given by a generalization of equation (32), and $m$ is of order of $M_{pl}$. Then the KK splitting for the bulk field can be expressed as the following decomposition,

$$ \Phi = \sum_{pqrs \cdots} \phi_{pqrs \cdots} (x) \frac{\alpha_p}{\sqrt{R_y}} \frac{\beta_q}{\sqrt{R_z}} \frac{\gamma_r}{\sqrt{R_w}} \cdots $$

Thus among these $n$ extra dimensions one will have the solution given by equation (43), and then the other $(n - 1)$ solutions are being given by generalization of equation (44) such that the numerical values will be different but form of the solution remains unaltered. For $n$th extra dimension ($n > 1$) the solution for the mode can therefore be expressed as,

$$ \chi_r (w) = \exp \left[ \frac{3}{2} k^2 w^2 \right] \exp \left[ -\frac{1}{2} n k^2 w^2 \right] H_{2n} \left( \frac{-6k^2 - 2nk^2 + M_r^2 (1 + Cosh (2k\pi))}{2k^2 (3 + n)} \right) $$

$$ + F_r \exp \left[ \frac{3}{2} k^2 w^2 \right] \exp \left[ -\frac{1}{2} n k^2 w^2 \right] F_1 \left( \frac{-6k^2 - 2nk^2 + M_r^2 (1 + Cosh (2k\pi))}{4k^2 (3 + n)} \right) \frac{1}{2} \frac{1}{2} (3 + n)k^2 w^2 \right) $$

Hence the bulk field as viewed by a four dimensional observer leads to a mass matrix whose components can be obtained by solving the eigenvalue problem as presented by each separable functions in the expansion given by equation (52). Also all these eigenvalues have TeV scale masses and the bulk field also has maximum value at $(y = \pi, z = 0, w = 0, \cdots)$, which is the SM brane. Hence the standard Model particles can be taken as low lying Kaluza-Klein modes of a bulk field propagating in any number of extra dimensional spacetime.

The effective self-coupling is this case turn out to be,

$$ \lambda_{eff} \simeq 2^n \lambda \left( \frac{c}{MR_y} \right)^{m-1} \left( \frac{1}{MR_z} \right)^{m-1} \left( \frac{1}{MR_w} \right)^{m-1} \cdots \left( M e^{-c_\pi \frac{1}{\cosh n-1 k \pi} \frac{1}{\cosh n-2 (\ell \pi)}} \right)^{4-2m} $$

$$ \int_0^1 r^{4m-5} dr \left[ \frac{J_2 \left( \frac{A_p c^\pi}{k} \right)}{A_p} \right]^{2m} \int_0^\pi d\beta q \int_0^\pi dw \gamma_r (2m) \cdots $$

Thus finally we have obtained the KK mass modes and their self interactions for $n$ extra dimensions. We have also observed that in all these cases the KK mass modes and self interactions are suppressed by the warp factor near the SM brane and hence all are in TeV scale. Hence this properties can be used to search for the TeV range KK mass modes and self interactions in next generation colliders.

VI. DISCUSSION

In this paper we generalize the work by Goldberger and Wise to determine the KK modes and the effective self interaction of a bulk scalar in a multiple warped space-time. For arbitrary number of extra dimensions, we have derived the expressions for the KK mode masses and their self interactions. Various components determining the masses are in the TeV range because of the warp factor suppression. Moreover the bulk scalar field has been shown to have maximum value at the SM brane. Hence the low lying KK modes for the bulk scalar fields lie in the TeV range with inverse Tev self-coupling. Thus the
appearance of KK mode masses and couplings at Tev scale are generic features of warped dimensional models with any number of extra warped dimensions as long as we want to resolve the gauge hierarchy problem without introducing any hierarchical moduli. We present a compact and generic formula to determine all the mass modes and their couplings for models with any arbitrary number of warped extra dimensions. This work now can be extended to other form of bulk fields which in turn may lead to the possibility of identifying various standard Model particles as the low lying KK excitation of various bulk fields where the small warping in multiple directions can explain mass splitting in standard model particles as discussed in [16] and [17]. The close spacing of the low lying KK modes along with enhanced coupling facilitates if likely for them to be seen as a series of closely lying resonances. In order to investigate the role of KK mass modes through collider-based experiments, we consider the interaction of various modes with itself i.e. self-interactions and we have obtained that all of them are suppressed to the TeV scale by the warp factor. The other things to be noted are that, with stronger coupling of the KK modes of the bulk scalar field, one expects the decay widths to be larger, and thus the peaks to be broader as we go to higher and higher dimensions. The nature of the line-shapes therefore will be an interesting benchmark to distinguish between higher dimensional and lower dimensional KK signals if such excitations appear during the high luminosity runs of the LHC.

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