Analytic Investigation for Synchronous Firing Patterns Propagation in Spiking Neural Networks

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Abstract
Based on the moment closure method and mean field theory, a Gaussian random field is constructed to quantitatively and analytically characterize the dynamics of a random point field. The approach provides a theoretical tool to investigate synchronized spike propagation in a feedforward or recurrent spiking neural network. We show that the balance between the excitation and inhibition postsynaptic potentials is required for the occurrence of synfire chains. In particular, with a balanced network, the critical packet size of invasion and annihilation is observed. We also derive a sufficient analytic condition for the synchronization propagation in an asynchronous environment, which further allows us to disclose the possibility of spatial synaptic structure to sustain a stable synfire chain. Our findings are in good agreement with simulations and help us understand the propagation of spatio-temporal patterns in a random point field.

Keywords Spiking neural networks · Synfire chain · Gaussian random field · Mean field theory

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1 Introduction

Towards fully understanding an evolutionary random point field, the (joint) probability distribution density is often very hard, if it is possible, to be calculated and estimated. However, in many cases, the first few moments are sufficient to present a holistic picture of the evolution of this random point field [1–4]. To this end, the moment closure approach is proposed to approximate random fields by employing their first few moments [5–8], which transforms the spatio-temporal random point field into a few non-random dynamical systems of the moments respectively. Then, this idea enables us to investigate these moment dynamical systems for the depiction of the asymptotical properties of the random point field. For instance, the spatio-temporal pattern of random point field can be regarded as attracting dynamics of the corresponding moment dynamical system.

Mathematically, neural activities can be formulated as random point processes (spike trains). A large ensemble of neural spike trains with an underlying geometric structure (manifold) comprises a random point field [9]. In particular, the propagation of spatio-temporal patterns of neural spiking activities, as the essence of cortical function, has been widely studied in recent decades [10–12]. A typical example of spatio-temporal pattern, the so-called pulse-packet, is a synfire chain. It encodes a piece of information and reliably transmits it from one layer of the nervous system to the other. At each layer, spikes are synchronized inside a packet of neurons but asynchronous outside the packet [13–15] (see Fig. 1 for illustration). There are many studies showing that synfire chain can transmit information between neuronal populations experimentally or theoretically [16–18]. These coexistence of synchronous and asynchronous states of neural activities can also be categorized as “chimera” phenomenon [19–21].

The synfire chain can be naturally described in the framework of random point fields. Traditionally, dynamical models of mean firing rate are used to depict the evolution of spike spatio-temporal patterns in cortical circuits [22–24]. However, such models are only true...
under the condition of independent or weak correlation of spiking activities between neurons. And they have limitations to fully account for the evolution of the spiking patterns [25–28]. On the other hand, temporal dispersion in terms of pulse-packets and correlation mapping represents the degree of synchrony well. Still, it fails to realistically describe the pattern in a random point field [29–31]. How the survival of a pulse packet in a multilayer neural network depends on the structural and physiological characteristics of the network has not been answered analytically in the literature.

Our contributions can be briefly summarized as follows. In this paper, we develop a novel and general theoretical framework to investigate the synchrony dynamics of the random point field of multilayer feedforward neural network (FNN) with leaky integrate-and-fire (LIF) neurons. Additionally, we also find similar dynamical features in discrete-time recurrent spiking neural network (RNN) using our model. By constructing the moment dynamical system that includes the first (mean firing rate) and second-order statistics (variance and correlation) of spiking random point fields, the synfire chain dynamics can be regarded as a certain sort of attracting set of the correlation mappings. Combining with the mean field approach, we discover the necessity of the balance of network for stable synchronization propagation. In addition, we analytically derive a sufficient condition for the existence of this attracting set that enables us to obtain the appropriate size of this synfire chain and the proper synaptic density of the neural network. These results are in good agreement with the numerical results and help understand the evolution of synchronization pattern in spiking neural network (SNN).

The remaining sections are organized as follows. In Sect. 2, firstly, we introduce a general way to approximate a discrete random point field using a Gaussian random field. Then, considering the first-order (mean) and the second-order (variance, correlation) moments, we adopt the moment closure method to analyze the evolution of the Gaussian random field. We verify that a SNN can be naturally treated as a group of Gaussian random fields. The firing rate and the spiking correlation between two neurons can be described by the moment closure. Since the moment function of correlation has been established, we propose an analytical approach to study the synchronization pattern of FNN (synfire chain) based on mean field theory. In Sect. 3, we conduct experiments to find suitable conditions for sustaining a stable synfire chain and prove them using our theoretical framework. We find our theories are in good agreement with experiments. Furthermore, we demonstrate that the propagation of synchronization in RNN can be treated similarly so that those findings in FNN are also applied for RNN. Finally, in Sect. 4, we draw our conclusions and introduce our future works.

2 Methods

2.1 Gaussian Approximation of Random Point field

We start with an evolutionary random point field, \( \xi(x, \tau) \), standing for the number of events occurring at \( x \) for the duration \([0, \tau]\), where \( x \) is the spatial variable and \( t \) is time. We discretize the continuous time into time bins \([0, \Delta, 2\Delta, \ldots, k\Delta, \ldots]\) with a length \( \Delta \). In each time bin, \([k\Delta, (k + 1)\Delta]\) with \( k \geq 0 \), we define a random field \( \xi^k(x, t) = \xi(x, t + k\Delta) - \xi(x, k\Delta) \), which counts the number of events occurring in \([k\Delta, k\Delta + t]\) for each \( t \in [0, \Delta] \).

For each \( \xi^k(x, t) \), the number of events occurring in an infinitesimal interval of time, \([t, t + dt]\), denoted by \( d\xi^k(x, t) \), follows a probability distribution, \( P^k(\xi; x, dt) \). Inspired by the moment closure approach, with assuming that \( \xi(x, t) \) is a renewal process, we are in the
stage to approximate $P^k(\xi; x, dt)$ by the first moment, mean, the second statistics, variation, and correlation. Based on the central limit theorem, we approximate $d\xi^k(x, t)$ by a Gaussian field as follows:

$$d\xi^k(x, t) \approx \mu^k(x, t)dt + \sigma^k(x, t)d\eta^k(x, t)$$ (1)

where $\mu^k(x, t)$ and $\sigma^k(x, t)$ are the functions of mean and standard variance, $\eta^k(x, t)$ is a Brownian motion with zero mean and unit variance: $E(\eta^k(x, t)) = 0$, $\text{var}(d\eta^k(x, t)) = dt$, but correlated. Here, $E(\cdot)$ stands for the expectation and $\text{var}(\cdot)$ stands for the variance.

Equivalently, we can use the following coefficient of variation (CV) to stand for the second moment:

$$\text{CV}^k(x, t) = \frac{\sigma^k(x, t)}{\mu^k(x, t)}.$$ (2)

The Pearson correlation coefficient (CC) between locations $x$ and $y$ at $t$ is written as:

$$\rho^k(x, y, t) = \frac{E[\eta^k(x, t), \eta^k(y, t)]}{\sigma^k(x)\sigma^k(y)}.$$ (3)

Assume that $d\xi^k(x, t)$ is asymptotically stationary. Then, with a sufficiently large $\Delta$, the long-term duration behaviours of $\mu^k(x, t)$, $\sigma^k(x, t)$ and $\rho^k(x, y, t)$ can be represented by the asymptotics, namely, $\mu^k(x)$, $\sigma^k(x)$ and $\rho^k(x, y)$. Thus, (1) can be rewritten as:

$$d\xi^k(x, t) \approx \mu^k(x)dt + \sigma^k(x)d\eta^k(x, t).$$ (4)

Here, the correlation between $d\eta^k(x, t)$ and $d\eta^k(y, t)$ is $\rho^k(x, y)dt$. In this stationary case, CC can be equivalently defined in the following way. The shifting correlation between $\xi^k(x, t)$ and $\xi^k(y, t)$ in a sliding window with length $T$ is defined as:

$$\rho^k_T(x, y) = \frac{\text{cov}(n^k(x, \tau), n^k(y, \tau))}{\sqrt{\text{var}(n^k(x, \tau))\text{var}(n^k(y, \tau))}},$$ (5)

where $n^k(x, \tau)$ is the number of events occurring at location $x$ in the time interval $[\tau, \tau + T]$, and $\text{cov}(\cdot, \cdot)$ stands for the covariance. The covariance and variance are calculated with respect to $\tau$. Thus, the CC between this spike train pair is defined as its limit $\rho^k_T(x, y) = \lim_{T \to \infty} \rho^k_T(x, y)$, which equals formula (3) in stationary state [31].

### 2.2 Construction of Moment Closure

The above analysis approximates an intractable discrete point process (which is assumed as a renewal process) by a stationary Gaussian random field system (4)–(5). Describing spiking activities with probability distribution function directly is often difficult. It is more efficient to focus on their moments of distribution. Specifically, suppose $\Lambda$ is a finite index set and $\{m_i : i \in \Lambda\}$ represents the moments we need. Then, we can transfer a stochastic system into a moment system:

$$\{m_i : i \in \Lambda\} = F(\{m_i : i \in \Lambda\}),$$ (6)

using moment closure methods [8], where $F(\cdot)$ represents corresponding moment mapping. Different from the famous model proposed by Wilson and Cowan [32] which only includes the first order moment, we hope to find a more complex model based on higher order, such as the second order moments (variance, correlation). Gaussian part in the right-hand side of
(4) can naturally provide the information of the second order moments and yield a moment closure.

We regard the long-term dynamics in the time bin \([ (k - 1) \Delta, k \Delta] \), \( \xi^{k-1}(x, t) \), as the input to the \( \xi^k(x, t) \) in the successive time bin, \([ k \Delta, (k + 1) \Delta] \). We aim to formulate the iteration of these first and second moments in the form of

\[
\begin{align*}
\mu^k(x) &= \mathcal{H}^x_{1} \left[ \mu^{k-1}(\cdot), \sigma^{k-1}(\cdot), \rho^{k-1}(\cdot, \cdot) \right], \\
\sigma^k(x) &= \mathcal{H}^x_{2} \left[ \mu^{k-1}(\cdot), \sigma^{k-1}(\cdot), \rho^{k-1}(\cdot, \cdot) \right], \\
\rho^k(x, y) &= \Psi^{x,y} \left[ \mu^{k-1}(\cdot), \sigma^{k-1}(\cdot), \rho^{k-1}(\cdot, \cdot) \right].
\end{align*}
\]

(7)

Here, \( \mathcal{H}^x_{1,2} \) stand for the functions of mean and variance with respect to the last moment mean and covariance, named mean and variance mapping respectively. \( \Psi^{x,y} \) stands for the correlation function, named correlation mapping. Thus, the set of mean, variance and correlation mappings compose of the moment dynamic system (7) that represents the random point field \( \xi(x, \tau) \) in the moment closure fashion. The problem of spatio-temporal pattern of \( \xi(x, \tau) \), which can be described by the first and second moments/statistics, is transformed into the dynamics of (7). For instance, synchronization pattern of a random point process can be described by the CC (3). \( \rho^k(x, y) = 1 \) implies the processes at \( x \) and \( y \) completely synchronise; \( \rho^k(x, y) = 0 \) means uncorrelated and \( \rho^k(x, y) = -1 \) means complete desynchronization. The asymptotic dynamics, namely, asymptotic attractor, can depict the pattern related to synchronization. If \( \xi(x, t) \) is asymptotically steady, we can equivalently study the equilibrium functions: \( \mu^{k-1}(x) = \mu^k(x) \), \( \sigma^{k-1}(x) = \sigma^k(x) \) and \( \rho^{k-1}(x, y) = \rho^k(x, y) \), towards understanding the asymptotic pattern of the random point process. We highlight that this approach can be extended to include the high-order moments and correlations of the random point field.

We emphasize that constructing the explicit expressions of \( \mathcal{H}^x_{1,2} \) and \( \Psi^{x,y} \) is the core of the moment closure method. Those three mappings highly depend on the corresponding situation. We depict how to formulate them for SNNs in the following section.

### 2.3 Multilayer Feedforward Spiking Network

As an application, let us specify a random point field described by multilayer FNN of spiking neurons with a sparse random coupling structure [14, 33]. At each layer, there are exactly \( N = 5n \) neurons. Among them, there are \( N_E = 4n \) excitatory (E-) neurons and \( N_I = n \) inhibitory (I-) neurons. For the E-neuron group at the \( k \)-th layer, a synfire chain is carried by a packet of E-neurons, denoted by \( \mathcal{W}^k \) with the identical size \( \#[\mathcal{W}^k] = W \) for all \( k \), as the packet size, where \( \#[\cdot] \) is the number of elements in a finite set. Between any two successive layers, for instance, from the \((k - 1)\)-th to the \( k \)-th layer, each neuron in \( \mathcal{W}^k \) receives inputs from all neurons in \( \mathcal{W}^{k-1} \). This constructs a fully connected FNN in the packet, which is believed as the basic model for synfire chain [33]. This fully connected FNN is embedded into a sparse network [14] and the couplings for other neurons outside \( \mathcal{W}^k \) are randomly picked with an equal probability so that the total number of excitation and inhibition synaptic links are \( K_E = \lambda N_E \) and \( K_I = \lambda N_I \) respectively (see Fig. 1). Here, \( \lambda \in (0, 1) \) represents the sparsity of the synaptic density. To ensure that every neuron receives an equal number of synapses, \( W \leq K_E \) is necessary.

We use a discretized random point field \( N_i^k(t) \) to stand for the spike counts from the neuron \( i \) at the \( k \)-th layer, i.e., \( N_i^k(t) = \sum_l H(t - t_i^{l,k}) \), where \( H(\cdot) \) is the Heaviside step function and \( \{ t_i^{l,k} : l = 1, 2, \ldots \} \) are the time points of the presynaptic spikes from neuron.
\( i \). Here, the neuron label \( i \) stands for the spatial variable, \( t \) for the continuous-time variable at the \( k \)-th layer.

The evolution of the random point field follows the LIF model. The potential activity of each neuron \( i \) at layer \( k \) is described as:

\[
\tau_m dV^k_i(t) = -V^k_i(t) dt + I^k_{ext,i} + I^k_{syn,i}, \tag{8}
\]

for \( i = 1, \ldots, N \). Here, \( \tau_m \) is the capacitance constant, \( I^k_{ext,i} \) is the external current stimulus at layer \( k \), and \( I^k_{syn,i} \) is the synaptic stimulus from the neurons at layer \( k - 1 \):

\[
I^k_{syn,i} = \sum_j w^k_{ij} dN^{k-1}_j \tag{9}
\]

with \( w^k_{ij} \) standing for the strength of the excitatory postsynaptic potential (EPSP) or inhibitory postsynaptic potential (IPSP) from neuron \( j \) at the \((k - 1)\)-th layer to neuron \( i \) at the \( k \)-th layer. In this paper, we take values of EPSP/IPSP as follows. If there is a synaptic link from \( j \) to \( i \), \( w^k_{ij} = w_0 \) if \( j \) is an E-neuron and \( w^k_{ij} = -rgw_0 \) if \( j \) is an I-neuron, for some constant \( w_0 > 0 \), where \( g = 4 \) equalizes the ratio between the numbers of E-synapses over I-synapses and thus \( r \) serves as the ratio of ISPs over ESPS. If there is no link from \( j \) to \( i \), \( w_{ij} = 0 \).

Once \( V^k_i(t) \) reaches a threshold \( (V_{ih}) \), neuron \( i \) is depolarized and emits a spike. Then, \( V^k_i(t) \) is reset to \( V_\tau \) after a period of refractory time \( T_{ref} \). By this way, neuron \( i \) at layer \( k \) emits a spike train, which is the input to the neurons at the \((k + 1)\)-th layer that are linked with neuron \( i \). Therefore, the random point field of spike trains, \( N^k_i(t) \), can be generated.

Since \( N^{k-1}_j(t) \) can be regarded as a renewal process naturally (e.g., a Poisson process [34, 35]), using the approach mentioned in Sects. 2.1–2.2, we are to approximate the spike train of neuron \( j \) (a random point process) by a Gaussian process:

\[
dN^{k-1}_j(t) \sim \mu^{k-1}_j dt + \sqrt{\tau_m \sigma^{k-1}_j} dB^{k-1}_j, \tag{10}
\]

where \( B^{k-1}_j, j = 1, \ldots, N \) are correlated Brownian motions [36, 37]. \( \mu^{k-1}_j \) and \( \sigma^{k-1}_j \) are the mean and variance of neuron \( j \) (at the \((k - 1)\)-th layer), which can be derived using the renew process theory [38]. Here, \( \tau_m \) stands for the time-scale constant. According to (9)–(10), (8) becomes correlated Ornstein-Uhlenbeck (OU) processes:

\[
\tau_m dV^k_i(t) = -V^k_i(t) dt + \hat{\mu}^k_i dt + \sqrt{\tau_m \hat{\sigma}^k_i} dB^{k-1}_i(t), \tag{11}
\]

\( i = 1, \ldots, N \). Here \( \hat{\mu}^k_i = \sum_j w^k_{ij} \mu^{k-1}_j \) and \( \hat{\sigma}^k_i = \sqrt{\sum_{j,l} w^k_{ij} \sigma^{k-1}_j \rho^{k-1}_{jl} w^k_{il} \sigma^{k-1}_l} \) are the mean and standard variance of the sum of the post synapses received by neuron \( i \) respectively, where \( \rho^{k-1}_{jl} \) is the CC between the \( j \)-th and \( l \)-th synapses of neuron \( i \).

The output spike trains derived from (11) (with potential threshold \( V_{ih} \)) are also approximated as Gaussian processes. This establishes mappings of the first (mean mapping), second-order statistics (variance and correlation mappings) for the random point field \( N^{k}_i(t) \) of the successive layer, which is a specific case of the moment system (7). Using the Siegert’s expression [39], the first two mappings can be formulated as:

\[
\mu^k_i = S_1(\hat{\mu}^k_i, \hat{\sigma}^k_i), \]

\[
\sigma^k_i = S_2(\hat{\mu}^k_i, \hat{\sigma}^k_i)\sqrt{S_1(\hat{\mu}^k_i, \hat{\sigma}^k_i)}, \quad i = 1, \ldots, N, \tag{12}
\]

where \( S_1 \) and \( S_2 \) are the mean and variance mappings (see Appendix A for details of the formulations).
As pointed out in Ref. [31], the evolution of the CC from layer \( k - 1 \) to layer \( k \) is formulated as:

\[
\rho_{i,j}^k = \Phi(\hat{\rho}_{i,j}^k), \quad i, j = 1, \ldots, N,
\]

where \( \hat{\rho}_{i,j}^k \) is the correlation coefficient between the collection of synaptic inputs of neuron \( i \) and \( j \) at layer \( k \):

\[
\hat{\rho}_{i,j}^k = \frac{\sum_{p,q} w_{ip}^k \sigma_p^{k-1} w_{jq}^k \sigma_q^{k-1} \rho_{pq}^{k-1}}{\hat{\sigma}_i^k \hat{\sigma}_j^k},
\]

\[
p, q = 1, 2, \ldots, K_E + K_I.
\]

\( \Phi \) is the correlation mapping, which is assumed to be the identity mapping, namely, \( \Phi(\hat{\rho}) = \hat{\rho} \) in this paper, according to the arguments in Ref. [36]. However, the following results still hold when \( \Phi(\hat{\rho}) \) is monotonous increasing with respect to \( \hat{\rho} \) as shown in Ref. [31, 37] with some minor modifications.

To sum up, the spike trains of neurons in the FNN are modeled by the discrete random point field \( N_k^i(t) \). With Gaussian random field assumption and moment closure method, their evolution through layers is formulated by the iteration of the first and second moments/statistics, which is completely described by the mean, variance, correlation mappings and called the moment neural network (MNN) [36] if we adopt the explicit expressions (12)–(13).

In particular, the evolution equation of the correlation mapping through a multilayer FNN can be utilized to analytically and quantitatively study the synchronization propagation.

### 2.4 Spatial Mean Field Approximation for Synfire Chain

The (stable) synfire chain is defined by an attractor of the correlation mapping (13) together with the mean and variance mappings (12) with two properties: \((P_1)\) the CCs between the neurons in the packet of synchrony are large; \((P_2)\) the CCs between the other pairs of neurons are low.

Hence to characterize the synfire chain, we focus on the correlation mapping, which turns out to be a useful way to understand the complex dynamics. Let us use the subscript \( + \) to denote all variables inside the packet \( W_k^+ \) and \( - \) outside the packet at each layer. Then, given two small constants \( 0 < \epsilon < 1 \) and \( 0 < \delta < 1 \), we define \( \Lambda_{\epsilon,\delta} := \{(\rho_+,\rho_-) : \rho_+ > 1 - \epsilon, |\rho_-| < \delta\} \). If \( \Lambda_{\epsilon,\delta} \) is an attracting set of the FNN of LIF neurons or the theoretical model (13), we call it synfire attractor and \( \rho_+ - \rho_- \) the synfire gap. Thus, a synfire chain can be defined as:

**Definition 1** The multilayer FNN is said to possess a synfire chain (with respect to \( \epsilon \) and \( \delta \)) if \( \Lambda_{\epsilon,\delta} \) is an attracting set of the iteration mapping (13) accompanied with (12).

With the mean field approximation, the correlation mapping allows to provide analytic inference and insights on the stability of synfire chain by some algebras, and so greatly simplifies the analysis of the transmission of spiking point process in multilayer FNN. According to whether the link or neuron belongs to the packet or not, we substitute the specific correlation between neurons \( p \) and \( q \), \( \rho_{pq}^k \), and the variance (or mean firing rate) of neuron \( p \), \( \sigma_p^k \) \((\mu_p^k)\), by \( \rho_{\pm}^k \) and \( \sigma_{\pm}^k \). In the steady-state case, we omit the superscript \( k \). In detail, the mean in-packet value can be calculated through (13)–(14) as:

\[
\rho_+^{k+1} = \left\langle \Phi(\hat{\rho}_{i,j}^{k+1}) \right\rangle_{(i,j) \in W_{k+1}^+ \times W_{k+1}^+}
\]
\[ \langle \cdot \rangle_S \] represents the average calculation over the set \( S \), noting that the correlation is assumed to be the identity mapping in this paper. Similarly, the out-of-packet CCs are approximately written as:

\[
\rho_{k+1}^- = \langle \Phi(\hat{\rho}_{k+1}^+ \cdot \hat{\rho}_{k+1}^-) \rangle_{(i,j) \notin W_{k+1} \times W_{k+1}} \\
\approx a_1 \left( \frac{A_{k+1}^+}{B_{k+1}^+} \right) + a_2 \left( \frac{A_{k+1}^-}{\sqrt{B_{k+1}^+ B_{k+1}^-}} \right) \approx \frac{A_{k+1}^-}{B_{k+1}^+},
\]

(16)

where \( a_1 = \frac{(N-W)^2}{N^2} \approx 1 \) and \( a_2 = \frac{2W(N-W)}{N^2} \approx 0 \), when \( N > W \), are two weights corresponding to the proportions of neuron pairs with all out-of-packet neuron pairs and those with one in-packet and the other out-of-packet respectively. Here,

\[
\begin{align*}
A_{k+1}^+ &= \sum_{p,q} w_{ip}^{k+1} \sigma_p^k w_{jq}^{k+1} \sigma_q^k \rho_{pq}^k \left|_{i,j \in W_{k+1}} \right., \\
B_{k+1}^+ &= \sum_{p,q} w_{ip}^{k+1} \sigma_p^k w_{iq}^{k+1} \sigma_q^k \rho_{pq}^k \left|_{i \in W_{k+1}} \right., \\
A_{k+1}^- &= \sum_{p,q} w_{ip}^{k+1} \sigma_p^k w_{jq}^{k+1} \sigma_q^k \rho_{pq}^k \left|_{i,j \notin W_{k+1}} \right., \\
B_{k+1}^- &= \sum_{p,q} w_{ip}^{k+1} \sigma_p^k w_{jq}^{k+1} \sigma_q^k \rho_{pq}^k \left|_{i \notin W_{k+1}} \right.,
\end{align*}
\]

(17)

To analyze the synchronization of neurons theoretically, we need to find a way to calculate \((17)\). Considering the significant impact of neuron’s position (inside the packet or not) and type (excitatory or inhibitory), we propose a squared decomposition in Fig. 2 to help identify different values of variance and CC among different positions and types of neurons. For brevity, we demonstrate the specific mean-field expressions of \( A_{k+1}^\pm \) and \( B_{k+1}^\pm \) and their detailed derivations in Appendix B.

### 3 Results

To verify that our theories are in good agreement with experiments, we carry out several experiments using our theoretical model (MNN) and the FNN with LIF neurons in this section. For the LIF network, we simulate 2000 LIF neurons in each layer with the initial Poisson input (20 Hz) to the first layer for a sufficiently long period. We record the moments of each spike for each neuron. Based on these recorded data, we can calculate the average firing rate \( (\mu_\pm) \) inside and outside the packet in each layer, and the CC \( (\rho_\pm) \). For the MNN, the key to simulating it is to keep the connection weights \( (w_{ij}^k) \) of each layer consistent with the corresponding LIF network, and the initial values of the MNN consistent with the Poisson input. Therefore, the initial values of the expectation and the standard deviation are 20 and \( \sqrt{20} \) according to the nature of Poisson distribution. The CC is initialized by a matrix in which all the diagonal elements are equal to 1 and all the other elements are 0.2. Then the MNN is simulated according to \((12)\)–\((14)\) and \((A.1)\)–\((A.2)\). We adopt the above settings unless stated otherwise.
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Fig. 2 The graphical computation of $\sum_{p,q} u_{ip}^{k+1} \sigma_p^k u_{jq}^{k+1} \sigma_q^k \rho_{pq}^k$. We lined up the subscripts $p = 1, 2, \ldots, W, \ldots, K_E$, $q = 1, 2, \ldots, W, \ldots, K_E$, in two mutually perpendicular directions to form two squares for $A(i,j) \in W^{k+1} \times W^{k+1}$ and $B(i,j) \notin W^{k+1} \times W^{k+1}$. Both of the squares can be decomposed into 3 diagonal intervals and 7 blocks according to the value of $\rho_{pq}^k$, the location of $p$ and $q$ (inside the packet or not) and the attribute of $p$ and $q$ (excitatory or inhibitory). The value of the number of neuron pairs, variance weight, CC on each region and the expressions of $A_{\pm}^{k+1}$ and $B_{\pm}^{k+1}$ are listed in the right tables.

### 3.1 Non-Constant Coefficient of Variation

To our best knowledge, in most existing literature, the spike trains of LIF neurons are simulated by Poisson or sub-/supra- Poisson processes through FNNs. The CV of the spike trains are assumed constant [14, 40]. Thus, the first order moment (mean) can be utilized to depict the second-order dynamics of synfire propagation. However, this assumption could be away from the facts. To justify it, we simulate an FNN with a initial Poisson input to see...
the evolution of the CV. As shown in Fig. 3A, CV is non-stationary for layers. Moreover, even if the CV goes steady after many layers, it varies for parameters. For instance, there is a moderate rise of CV with increasing packet size $W$ or decreasing ratio $r$, as shown in Fig. 3B. This motivates us to employ Gaussian process to approximate the spike trains, instead of assuming fixed CV.

3.2 Balanced Network

The parameter $r$ measures the ratio of ISPS over ESPS. $r = 1$ means that the excitatory and inhibitory are exactly balanced. Consider the correlation mapping in the mean field sense, (15)–(16) and their detailed expressions (B.1)–(B.4). Noting $K_E = \lambda N_E$ with $N_E = 4n = \frac{4}{N}$, the terms of the highest order in the denominator, $B_{k+1}^{\pm}$, and the numerator, $A_{k+1}^{\pm}$, are both $(r-1)^2 K_E^2 (\sigma_{\pm}^k)^2 \rho_{\pm}^k$ as $N \to \infty$. So, if $r \neq 1$, the identical term of the highest order in both denominator and numerator implies that $\rho_{\pm}^{k+1} = A_{k+1}^{\pm}/B_{k+1}^{\pm}$ approaches to 1 as $k$ goes to infinity when $N$ is sufficiently large. Therefore, with a large size, $r = 1$ is the critical value of the I-ESPS ratio for the existence of synfire chain in terms of the property $P_2$. As illustrated in Fig. 4A, one can disclose that $A_{\epsilon,\delta}$ exists with some small values of $\epsilon$ and $\delta$ only when $r \approx 1$, which implies the largest synfire gap ($\rho_+ - \rho_- > 0.7$). As shown in the left part of Fig. 4A, the CC outside the packet ($\rho_-$) will gradually approach 1 when $r < 1$, so that the synfire chain disappears. When $r > 1$, the out-of-packet mean firing rate decreases quickly so that the out-of-packet spike frequency almost disappears (lower than 1 Hz) when $r > 1.2$. Therefore, to maintain a stable pattern of synchronization, a balanced network, namely, $r = 1$, is necessary.

Figure 5 clearly illustrates the spatio-temporal message passing through a balanced LIF network. We build an FNN with 20 layers and 2500 neurons per layer. We arrange 2500 neurons of each layer into a large $50 \times 50$ square and select the specific neurons that form the shape of a small $8 \times 8$ square. We fully connect those neurons between two neighboring layers while the other neurons are connected randomly and sparsely. With the initial Poisson
Fig. 4 Critical window of $r$ for the existence of synfire chain from the MNN model (mean, variance and correlation mappings) and numerical LIF simulations. A Comparison between the theoretical results of the correlation mapping and simulation results of LIF neurons. $\rho_\pm$ (solid and ◻) and $\rho_-$ (dash and ◦) vary with respect to the I-ESPS ratio $r$ with $W = 32$ under the theoretical model (blue) or LIF simulation (red). $\rho_\pm$ are calculated at the 20-th layer of the FNN. The mean firing rate (corresponds to the red bar and the right vertical axis) of the out-of-packet neurons $\mu_-$, vary with respect to $r$ when $r > 1$. B Simulation results of the propagation of $\mu_\pm$, $\sigma_\pm$ and $\rho_\pm$ (the embedded subfigure) through the LIF network (8). At the initial layer, $N$ Poisson spike trains are generated in a time interval of 20 s with the firing rate of 20Hz. $W$ trains in the packet are strongly correlated ($\rho_+ \approx 1$) and the rest out-of-packet $N-W$ trains are weakly correlated ($\rho_- < 0.2$).

input, we record all the spikes of each neuron in 20 layers. The phenomenon of synfire chain starts from the 7-th layer via our observations. We sample and show the spiking activities of neurons in the 7-th, 8-th, 10-th, ..., and 20-th layers. It is obvious that the synfire chain propagates through layers stably.

### 3.3 Synfire Chain Condition and Packet Size

We discuss the packet size in the synfire chain: $\Lambda_{\epsilon, \delta}$ with $\epsilon = \delta = 0.3$ as an attracting set of the correlation mapping (14) in a balanced network [41, 42]. As illustrated in Fig. 6A, to maintain $\Lambda_{\epsilon, \delta}$, the packet of synfire chain should have an appropriate size. A large $W$ will enhance synchronization between neurons not only inside the packet but also outside the packet, and thus the in-packet synchrony will invade over the whole network, which leads the synfire gap to disappear due to the large packet size. That is, property $P_2$ fails to hold (invasion). There exists a corresponding threshold value of $W$ when synchronization invasion occurs, named invasion packet size and denoted by $W_i$. In comparison, a very small $W$ may depress synchronization propagation in the packet and at the same time destruct the maintaining of the mean firing rates of the whole network. Thus, the in-packet synfire chain will annihilate, which also leads the synfire gap disappears due to the small packet size. That is, property $P_1$ cannot hold (annihilation). There also exists a corresponding threshold value of $W$ when synchronization annihilation occurs, named annihilation packet size and denoted by $W_a$. Therefore, an ideal $W$ to maintain a stable synfire chain is medium, belonging to the interval $(W_a, W_i)$, which leads both the synchronous synfire packet and an asynchronous background with regular firing rates. As illustrated by Fig. 6B, the simulation results of LIF neurons in the FNN have good agreement with the theoretical results. The slight difference
Fig. 5 Illustration of spiking records during the simulation (a black dot represents the neuron is spiking at that time). We arrange 2500 neurons of each layer into a large $50 \times 50$ square and select the specific neurons which form the shape of a small $8 \times 8$ square. It shows that the small square pulse-packet is transmitted temporally and spatially primarily results from the absence of higher-order moments in the MNN and the introduce of the steady-state assumption and the central limit theorem.

Using the methods introduced in Sect. 2, we can derive an analytic result (a sufficient condition) with respect to $\epsilon, \delta$ and $\lambda$ for the existence of attractor $\Lambda_{\epsilon, \delta}$ in a balanced network ($r = 1$). We consider the asymptotic stationary state of system (15)–(16), in which all the mean-field variables such as $\sigma^t_{\pm}, \rho^t_{\pm}, A^t_{\pm}$ and $B^t_{\pm}$ have reached their steady states as shown in Fig. 4B, whose values are denoted by $\sigma_{\pm}, \rho_{\pm}, A_{\pm}$ and $B_{\pm}$ respectively. For simplicity, let $P = 5K_E \sigma^2(1-\rho_\pm)$, $Q = [\sigma^2_+ (\rho_+-\rho_-)+(\sigma_+-\sigma_-)^2 \rho_-]$ and $R = 2K_E \sigma_- (\sigma_+-\sigma_-) \rho_- + (\sigma_+)^2 (1-\rho_+)-(\sigma_-)^2 (1-\rho_-)$. With $r = 1$, according to (15)–(16) and (B.1)–(B.4), by some algebras, we have

\[
\rho_+ \geq \frac{\lambda P + Q W^2 + R W}{P + Q W^2 + R W} \geq \frac{\lambda P + \lambda Q W^2 + R W}{P + \lambda Q W^2 + R W} \tag{18}
\]

and

\[
|\rho_-| \leq \frac{\lambda P + \lambda^2 Q W^2 + \lambda R W}{P + \lambda^2 Q W^2 + \lambda R W}. \tag{19}
\]

Thus, it can be seen that if

\[
\frac{\lambda P + \lambda Q W^2 + R W}{P + \lambda Q W^2 + R W} > 1 - \epsilon, \tag{20}
\]

and

\[
\frac{\lambda P + \lambda^2 Q W^2 + \lambda R W}{P + \lambda^2 Q W^2 + \lambda R W} < \delta. \tag{21}
\]
Fig. 6 The mean in-packet and out-of-packet CCs: $\rho_+$ (solid lines) and $\rho_-$ (dash lines) associated with the left vertical axis, and the mean firing rates $\mu$ (bars) associated with the right vertical axis vary with respect to packet size $W$, for different values of $r$. The synfire region (between two grey lines and corresponds to $r = 1$) is defined by $\Lambda_{\epsilon, \delta}$ with $\epsilon = \delta = 0.3$. The curves and bars are plotted for the theoretical model (A) and the LIF network (B). The solid vertical grey line is for the critical annihilation packet size $W_a$ and the dotted grey line is for the invasion size $W_i$.

hold for some $W$, then the existence of $\Lambda_{\epsilon, \delta}$ can be guaranteed in the mean-field sense. By some algebras, one can derive that if

$$\epsilon > \frac{\lambda}{\delta} \frac{1 - \lambda}{1 - \delta}$$

holds, named synfire condition, then (20)–(21) hold.

We are to identify the values of $W$ that satisfied (20)–(21) under the synfire condition. In fact, this synfire condition (22) is equivalent to $\frac{1-\lambda}{\epsilon} - 1 < \frac{\delta - \lambda}{\lambda(1-\delta)}$. Thus, for any $\tau \in \left(\frac{1-\lambda}{\epsilon} - 1, \frac{\delta - \lambda}{\lambda(1-\delta)}\right)$, picking $W$ as the solution of equation $\frac{\lambda^2 Q W^2 + \lambda R W}{\lambda P} = \tau$ with respect to $S$, one can easily verify that (20)–(21) hold. Hence, we can derive a region of synfire size:

$$\text{Reg} := \{W \text{ with } \frac{\lambda^2 Q W^2 + \lambda R W}{\lambda P} = \tau : \tau \in \left(\frac{1-\lambda}{\epsilon} - 1, \frac{\delta - \lambda}{\lambda(1-\delta)}\right)\}.$$  

(23)

For each $W \in \text{Reg}$, $\Lambda_{\epsilon, \delta}$ is an attracting set of the model (13) in the mean-field sense. In addition, $\max \text{Reg}$ and $\min \text{Reg}$ give the lower bound of the invasion packet size and the upper bound of the annihilation packet size respectively.
3.4 Sparse Synaptic Density

Furthermore, the synfire condition (22) reveals the dependence of synfire propagation on the synaptic density $\lambda$. The synfire condition (22) and the region of synfire chain packet (23) imply that a smaller $\lambda$ leads to a larger interval of synfire region in terms of the existence of $\Lambda_{\epsilon, \delta}$, which may cause a larger value of the maximum synfire gap $\max_{W}(\rho_{+} - \rho_{-})$. However, since the mean mapping in (12) can be written as $\mu_{k+1} = H(x_1 \mu_{k}, \sigma_{k}, \rho_{k})$, where $H(x_1 \mu_{k}, \sigma_{k}, \rho_{k})$ is a sigmoidal function given $\mu_{k}$ and $\sigma_{k}$, in a balanced network ($r = 1$), the mean out-of-packet firing rate will converge to 0 through layers, because $\mu_{k+1} \rightarrow 0$ as $\lambda \rightarrow 0$. That is to say, very small $\lambda$ results in the mean out-of-packet firing rate going extremely low, even near zero. As shown in Fig. 7, the maximum synfire gap $\max_{W}(\rho_{+} - \rho_{-})$ decreases with $\lambda$ but an extremely small $\lambda$ (less than 0.075) makes the spiking activities outside the packet vanish (< 0.3 Hz). Therefore, to sustain a synfire chain and the asynchronous out-of-packet spiking activities, $\lambda$ should be taken a modestly small value. For instance, $\lambda = 0.1$ is suggested in Refs. [33, 43].

3.5 Synfire Chain in RNN

Besides FNN, we claim that the above analysis still works in RNN. If we consider the dynamics of a single layer of neurons in discrete time, then (8) can be written as:

$$\tau_m dV_{s}^{i}(t) = -V_{s}^{i}(t)dt + I_{ext,i}^{s} + I_{syn,i}^{s},$$

(24)

where $s$ denotes time index. Similar to the Sect. 2.2, we iterate moment mappings and obtain the moment closure:

$$\begin{align*}
\mu^{s}(x) & = H_{1}^{s} \left[ \mu^{s-1}(\cdot), \sigma^{s-1}(\cdot), \rho^{s-1}(\cdot, \cdot) \right], \\
\sigma^{s}(x) & = H_{2}^{s} \left[ \mu^{s-1}(\cdot), \sigma^{s-1}(\cdot), \rho^{s-1}(\cdot, \cdot) \right], \\
\rho^{s}(x, y) & = \Psi^{s,y} \left[ \mu^{s-1}(\cdot), \sigma^{s-1}(\cdot), \rho^{s-1}(\cdot, \cdot) \right],
\end{align*}$$

(25)

where the mapping functions are still defined as (12) and (13) according to [37]. Therefore, FNN and RNN share the same evolution equations in our framework.

As shown in Fig. 8, to analyze RNN, firstly, we should unfold the network structure in the discrete time. Then, the RNN can be treated as an FNN with constrained weight [44], which means packets $W_s$ ($s = 1, \ldots$) share the same position and other random sparse connects are also fixed in different layers (weight sharing). If other parameters (e.g., I-ESPS proportion $r$, synaptic density $\lambda$, packet size $W$) take the same values as above, the dynamics of the RNN and FNN are almost identical during the forward propagation. The notations $A_{\pm}^{s}, B_{\pm}^{s}$ and $\rho_{\pm}^{s}$ are consistent with those used in the FNN (in Sect. 2.3). Then (17) turns to:
Fig. 8 The structure of the RNN and its equivalent form unfolded in the discrete time

Fig. 9 The mean in-packet (out-of-packet) CCs of the recurrent ($\tilde{\rho}_\pm$) and forward ($\rho_\pm$) LIF networks with the same Poisson input (20 Hz) and other parameters ($W = 50$, $r = 1$) in the simulation

\[
\begin{align*}
A_+^{s+1} &= \left\{ \sum_{p,q} w_{ip}^{s+1} \sigma_p^s w_{j_q}^{s+1} \sigma_q^s \rho_{pq}^s \right\}_{i,j \in \mathcal{W}^{s+1}}, \\
B_+^{s+1} &= \left\{ \sum_{p,q} w_{ip}^{s+1} \sigma_p^s w_{iq}^{s+1} \sigma_q^s \rho_{pq}^s \right\}_{i \in \mathcal{W}^{s+1}}, \\
A_-^{s+1} &= \left\{ \sum_{p,q} w_{ip}^{s+1} \sigma_p^s w_{j_q}^{s+1} \sigma_q^s \rho_{pq}^s \right\}_{i,j \notin \mathcal{W}^{s+1}}, \\
B_-^{s+1} &= \left\{ \sum_{p,q} w_{ip}^{s+1} \sigma_p^s w_{iq}^{s+1} \sigma_q^s \rho_{pq}^s \right\}_{i \notin \mathcal{W}^{s+1}}. \\
\end{align*}
\]  

The values of $A_\pm^{s+1}$ and $B_\pm^{s+1}$ are decided by the previous time. Therefore, if we assume that the initial input spikes in RNN and FNN are identical, using the mean field approximation, it is easy to obtain:

\[
\begin{align*}
A_\pm^{s+1} &= A_\pm^{k+1}, \\
B_\pm^{s+1} &= B_\pm^{k+1}, \text{ if } s = k, \\
\end{align*}
\]

which leads to $\rho_\pm^{s+1} = \rho_\pm^{k+1}$. From Fig. 9, we find that both networks have almost the same attractor. Furthermore, other dynamical characteristics referred in Sects. 3.1–3.4. can also be verified in the same way.
4 Conclusions

We propose a theoretical framework of Gaussian random field to study how synchronization pattern propagates in feedforward or recurrent SNNs. Using the mean field approach, we analytically prove that the balanced network is necessary for the stability of synfire chain. To derive a sufficient condition for sustaining a synfire chain, we provide an estimation of the appropriate packet size region. Our analytic results show good agreement with the simulations of the LIF network. We highlight that this approach based on the moment closure method and mean field theory is powerful and general to investigate propagation and stability of spatio-temporal patterns in a random point field. One step further, we have not included the complex interactions like brain manifold structure in the random field here and we have not considered real data such as EEG and fMRI [11, 45–47], which are our future works. Our theoretical framework should be a valuable tool for investigating the varied dynamics of the spiking patterns observed in experiments.

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Appendix A: Formulations of $S_1, S_2$ in (12)

\[
S_1(y, z) \approx \frac{1}{\left( T_{ref} + \frac{2}{L} \int I(V_{th}, y, z) \, D_-(u) du \right)},
\]

\[
S_2(y, z) \approx \left( \frac{8}{L^2} \int I(V_{th}, y, z) \, I(V_r, y, z) \, D_- \otimes D_-(u) du \right)^{1/2} \left( T_{ref} + \frac{2}{L} \int I(V_{th}, y, z) \, D_-(u) du \right).
\]

Here $T_{ref}$ represents the refractory period, $V_r$ is the rest potential and $V_{th}$ is the threshold of membrane potential to emit a spike. Additionally,

\[
I(\xi, y, z) = \frac{\xi L - y}{z},
\]

\[
D_-(u) = \exp \left( u^2 \right) \int_{-\infty}^{u} \exp \left( -v^2 \right) dv,
\]

\[
D_- \otimes D_-(u) = \exp \left( u^2 \right) \int_{-\infty}^{u} \exp \left( -v^2 \right) D_-^2(v) dv,
\]

where $D_-(u)$ is exactly the Dawson’s integral. For the details, we refer readers to Ref. [36, 37].

Appendix B: Derivations of $A_{\pm}^{k+1}$ and $B_{\pm}^{k+1}$ in (17)

As illustrated in the left squares of Fig. 2, we decompose the whole square into 10 regions according to the value of $\rho_{pq}^k$, the location of $p$ and $q$ (inside the packet or not) and the
attribute of $p$ and $q$ (excitatory or inhibitory). To calculate $\sum_{p,q} w^{k+1}_{ip} \sigma^k_p w^{k+1}_{jq} \sigma^k_q \rho^{k+1}_{pq}$, let us take region $a,b$ and $c$ for instance provided with $(i, j) \in \mathcal{V}^{k+1} \times \mathcal{V}^{k+1}$. Since region $a,b,c$ are all diagonal intervals, the correlation coefficient $\rho^{k+1}_{pq}$ equals 1 if $p$ connects to $q$. In the region $a$, note that neurons are fully connected within the clusters. $i$ and $j$ share the same $W$ neighbors from previous layers. That is to say, both $p$ and $q$ go through the same index set of length $W$. In the region $b, p$ and $q$ go through different index sets of length $K_I - W$, thus $p$ connects to $q$ for $K_E - W$ times if $i = j$ otherwise $\lambda(K_E - W)$ times. In the region $c$, $p$ and $q$ go through different index sets of length $K_I$, thus $p$ connects to $q$ for $K_I$ times if $i = j$ otherwise $\lambda K_I$ times. For the variance weight, in the region $a$, both $p$ and $q$ are inside the cluster and excitatory, thus the variance weight is $(\sigma^{k+1}_p)^2$; in the region $b$, both $p$ and $q$ are outside the cluster and excitatory, thus the variance weight is $(\sigma^{k+1}_q)^2$; and in the region $c$, both $p$ and $q$ are outside the cluster and inhibitory, thus the variance weight is $(-4r \sigma^{k+1}_q)^2$. Therefore, the contribution of these regions to $A^{k+1}_a$ and $B^{k+1}_a$ are $W(\rho^{k+1}_q)^2 \ast 1, \lambda(K_E - W)(\rho^{k+1}_q)^2 \ast 1, \lambda K_I(4r \rho^{k+1}_q)^2 \ast 1$ and $W(\rho^{k+1}_q)^2 \ast 1, (K_E - W)(\rho^{k+1}_q)^2 \ast 1, K_I(4r \rho^{k+1}_q)^2 \ast 1$ respectively. To sum up the components in each region, we have

\[
A^{k+1}_a \approx \{W(\sigma^{k+1}_p)^2 + \lambda(K_E - W)(\sigma^{k+1}_p)^2 + \lambda K_I(-4r \sigma^{k+1}_q)^2\} \ast 1 + \{(W^2 - W)(\sigma^{k+1}_q)^2\} \ast \rho^{k+1}_q
\]

\[
B^{k+1}_a \approx \{W(\sigma^{k+1}_p)^2 + \lambda(K_E - W)(\sigma^{k+1}_p)^2 + \lambda K_I(-4r \sigma^{k+1}_q)^2\} \ast 1 + \{(W^2 - W)(\sigma^{k+1}_q)^2\} \ast \rho^{k+1}_q
\]

\[
A^{k+1}_b \approx \{\lambda^2 W(\sigma^{k+1}_p)^2 + \lambda(K_E - W)(\sigma^{k+1}_p)^2 + \lambda K_I(-4r \sigma^{k+1}_q)^2\} \ast 1
\]

\[
B^{k+1}_b \approx \{\lambda^2 W(\sigma^{k+1}_p)^2 + (K_E - W)(\sigma^{k+1}_p)^2 + K_I(-4r \sigma^{k+1}_q)^2\} \ast 1
\]
\[\begin{align*}
&= \lambda W (\sigma^k_+)^2 (1 - \rho^k_+) + (K_E - \lambda W) (\sigma^k_-)^2 (1 - \rho^k_-) + 4r^2 K_E (\sigma^k_-)^2 (1 - \rho^k_-) \\
&\quad + \lambda^2 W^2 (\sigma^k_+)^2 (\rho^k_+ - \rho^k_-) + \rho^k_- (r - 1)^2 K_E (\sigma^k_-)^2 + 2\lambda K_E W (\sigma^k_- - \sigma^k_-)^2 \\
&\quad + \lambda^2 W^2 (\sigma^k_+ - \sigma^k_-)^2. \tag{B.4}
\end{align*}\]

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