Sequential edge-coloring on the subset of vertices of almost regular graphs

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Abstract

Let $G$ be a graph and $R \subseteq V(G)$. A proper edge-coloring of a graph $G$ with colors $1, \ldots, t$ is called an $R$-sequential $t$-coloring if the edges incident to each vertex $v \in R$ are colored by the colors $1, \ldots, d_G(v)$, where $d_G(v)$ is the degree of the vertex $v$ in $G$. In this note, we show that if $G$ is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $G$ has an $R$-sequential $r$-coloring with $|R| \geq \left\lceil \frac{(r-1)n_r+n}{2} \right\rceil$, where $n = |V(G)|$ and $n_r = |\{v \in V(G) : d_G(v) = r\}|$. As a corollary, we obtain the following result: if $G$ is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $\Sigma'(G) \leq \left\lfloor \frac{2n_r(2r-1)+n(r-1)(r^2+2r-2)}{4r} \right\rfloor$, where $\Sigma'(G)$ is the edge-chromatic sum of $G$.

1 Introduction

In this note we consider graphs which are finite, undirected, and have no loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. The degree of a vertex $v \in V(G)$ is denoted by $d_G(v)$ and the chromatic index of $G$ by $\chi'(G)$. For a graph $G$, let $\Delta(G)$ and $\delta(G)$ denote the maximum and minimum degrees of vertices in $G$, respectively. An $(a, b)$-biregular bipartite graph $G$ is a bipartite graph $G$ with the vertices in one part all having degree $a$ and the vertices in the other part all having degree $b$. The terms and concepts that we do not define can be found in [5].

A proper edge-coloring of a graph $G$ is a mapping $\alpha : E(G) \to \mathbb{N}$ such that $\alpha(e) \neq \alpha(e')$ for every pair of adjacent edges $e, e' \in E(G)$. If $\alpha$ is a proper
edge-coloring of a graph $G$, then $\Sigma'(G, \alpha)$ denotes the sum of the colors of the edges of $G$. For a graph $G$, define the edge-chromatic sum $\Sigma'(G)$ as follows: $\Sigma'(G) = \min_\alpha \Sigma'(G, \alpha)$, where minimum is taken among all possible proper edge-colorings of $G$. A proper $t$-coloring is a proper edge-coloring which makes use of $t$ different colors. If $\alpha$ is a proper $t$-coloring of $G$ and $v \in V(G)$, then $S(v, \alpha)$ denotes set of colors appearing on edges incident to $v$. Let $G$ be a graph and $R \subseteq V(G)$. A proper edge-coloring of a graph $G$ with colors $1, \ldots, t$ is called an $R$-sequential $t$-coloring if the edges incident to each vertex $v \in R$ are colored by the colors $1, \ldots, d_G(v)$.

The concept of sequential edge-coloring of graphs was introduced by Asratian [1]. In [1, 2], he proved the following result.

**Theorem 1.** If $G = (X \cup Y, E)$ is a bipartite graph with $d_G(x) \geq d_G(y)$ for every $xy \in E(G)$, where $x \in X$ and $y \in Y$, then $G$ has an $X$-sequential $\Delta(G)$-coloring.

On the other hand, in [2] Asratian and Kamalian showed that the problem of deciding whether a bipartite graph $G = (X \cup Y, E)$ with $\Delta(G) = 3$ has an $X$-sequential 3-coloring is NP-complete. Some other results on sequential edge-colorings of graphs were obtained in [3, 4]. In particular, in [4] Kamalian proved the following result.

**Theorem 2.** If $G$ is a $(r-1, r)$-biregular ($r \geq 3$) bipartite graph with $n$ vertices, then $G$ has an $R$-sequential $r$-coloring with $|R| \geq \left\lceil \frac{rn}{r-1} \right\rceil$.

In this note we generalize last theorem. As a corollary, we also obtain the following result: if $G$ is a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $\Sigma'(G) \leq \left\lfloor \frac{2n(2r-1)+n(r-1)(r^2+2r-2)}{4r} \right\rfloor$.

## 2 The Result

**Theorem 3.** If $G$ a graph with $\Delta(G) - \delta(G) \leq 1$ and $\chi'(G) = \Delta(G) = r$ ($r \geq 3$), then $G$ has an $R$-sequential $r$-coloring with $|R| \geq \left\lceil \frac{(r-1)n_r+n}{r} \right\rceil$, where $n = |V(G)|$ and $n_r = \{v \in V(G) : d_G(v) = r\}$.

**Proof.** Since $\chi'(G) = \Delta(G) = r$, there exists a proper $r$-coloring $\alpha$ of the graph $G$ with colors $1, 2, \ldots, r$. For $i = 1, 2, \ldots, r$, define the set $V_\alpha(i)$ as follows:

$$V_\alpha(i) = \{v \in V(G) : i \notin S(v, \alpha)\}.$$

Clearly, for any $i', i''$, $1 \leq i' < i'' \leq r$, we have

$$V_\alpha(i') \cap V_\alpha(i'') = \emptyset \quad \text{and} \quad \bigcup_{i=1}^{r} V_\alpha(i) = V(G) \setminus V_r,$$
where \( V_r = \{ v \in V(G) : d_G(v) = r \} \).

Hence,

\[
|V(G)| - |V_r| = \sum_{i=1}^{r} |V_\alpha(i)| = \sum_{i=1}^{r} |V_\alpha(i)|.
\]

This implies that there exists \( i_0 \), 1 \( \leq \) \( i_0 \) \( \leq \) \( r \), for which \(|V_\alpha(i_0)| \geq \left\lceil \frac{n-n_r}{r} \right\rceil\). Let \( R = V_r \cup V_\alpha(i_0) \). Clearly, \(|R| \geq n_r + \left\lceil \frac{n-n_r}{r} \right\rceil\).

If \( i_0 = r \), then \( \alpha \) is an \( R \)-sequential \( r \)-coloring of \( G \); otherwise define an edge-coloring \( \beta \) as follows: for any \( e \in E(G) \), let

\[
\beta(e) = \begin{cases} 
\alpha(e), & \text{if } \alpha(e) \neq i_0, r, \\
i_0, & \text{if } \alpha(e) = r, \\
r, & \text{if } \alpha(e) = i_0.
\end{cases}
\]

It is easy to see that \( \beta \) is an \( R \)-sequential \( r \)-coloring of \( G \) with \(|R| \geq \left\lceil \frac{(r-1)n_r+n}{r} \right\rceil\). □

**Corollary 1.** If \( G \) is an \((r-1, r)\)-biregular \((r \geq 3)\) bipartite graph with \( n \) vertices, then \( G \) has an \( R \)-sequential \( r \)-coloring with \(|R| \geq \left\lceil \frac{r n}{2(r-1)} \right\rceil\).

**Corollary 2.** If \( G \) is a graph with \( \Delta(G) - \delta(G) \leq 1 \) and \( \chi'(G) = \Delta(G) = r \) \((r \geq 3)\), then

\[
\Sigma'(G) \leq \left\lfloor \frac{2n_r(2r-1)+n(r-1)(r^2+2r-2)}{4r} \right\rfloor
\]

**Proof.** Let \( \alpha \) be an \( R \)-sequential \( r \)-coloring of \( G \) with \(|R| \geq \left\lceil \frac{(r-1)n_r+n}{r} \right\rceil\) described in the proof of Theorem 3. Now, we have

\[
\Sigma'(G) \leq \Sigma'(G, \alpha) \leq \sum_{i=1}^{r} \left( \frac{n_r - \frac{r(r+1)}{2}}{2} + \left\lfloor \frac{n-n_r}{r} \right\rfloor \left( \frac{r(r-1)}{2} \right) \right) + \frac{(n-n_r)(r-1)(r^2+2r-2)}{4r}
\]

□
References

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