Hardy spaces and divergence operators on strongly Lipschitz domains of $\mathbb{R}^n$

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Abstract

Let $\Omega$ be a strongly Lipschitz domain of $\mathbb{R}^n$. Consider an elliptic second order divergence operator $L$ (including a boundary condition on $\partial \Omega$) and define a Hardy space by imposing the non-tangential maximal function of the extension of a function $f$ via the Poisson semigroup for $L$ to be in $L^1$. Under suitable assumptions on $L$, we identify this maximal Hardy space with atomic Hardy spaces, namely with $H^1(\mathbb{R}^n)$ if $\Omega = \mathbb{R}^n$, $H^1_\mathbb{D}(\Omega)$ under the Dirichlet boundary condition, and $H^1_\Sigma(\Omega)$ under the Neumann boundary condition. In particular, we obtain a new proof of the atomic decomposition for $H^1_\Sigma(\Omega)$. A version for local Hardy spaces is also given. We also present an overview of the theory of Hardy spaces and BMO spaces on Lipschitz domains with proofs.

Keywords: strongly Lipschitz domain, elliptic second order operator, boundary condition, Hardy spaces, maximal functions, atomic decomposition.

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1 Introduction

Hardy spaces on $\mathbb{R}^n$, and especially $H^1(\mathbb{R}^n)$, were studied in great detail in the 60’s and 70’s. A nice review on this is in [27].

Originally defined by means of Riesz transforms (see the seminal paper of Stein and Weiss [31]), the usefulness of this space in analysis as a substitute for $L^1(\mathbb{R}^n)$ comes from its many characterizations, beginning from the work of Fefferman-Stein (see [14]). Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a function such that $\int_{\mathbb{R}^n} \phi(x) \, dx = 1$. For all $t > 0$, define $\phi_t(x) = t^{-n} \phi(x/t)$. A locally integrable function $f$ on $\mathbb{R}^n$ is said to be in $H^1(\mathbb{R}^n)$ if the vertical maximal function

$$Mf(x) = \sup_{t > 0} |\phi_t * f(x)|$$

belongs to $L^1(\mathbb{R}^n)$. If it is the case, define

$$\|f\|_{H^1(\mathbb{R}^n)} = \|Mf\|_1.$$

Recall that a function $f \in H^1(\mathbb{R}^n)$ satisfies $\int_{\mathbb{R}^n} f(x) \, dx = 0$.

Another equivalent definition of $H^1(\mathbb{R}^n)$ involves the non-tangential maximal function associated with the Poisson semigroup (or the heat semigroup) generated by $\Delta$, the Laplace operator on $\mathbb{R}^n$. If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, the following are equivalent:

$$f \in H^1(\mathbb{R}^n),$$

$$\sup_{|y-x| \leq t} \left| e^{-t(-\Delta)^{1/2}} f(y) \right| \in L^1(\mathbb{R}^n).$$

See [14], Theorem 11, p. 183.

The atomic decomposition obtained by Coifman and Latter was a key step in the theory (see [11] when $n = 1$, [21] when $n \geq 2$). A function $a$ on $\mathbb{R}^n$ is an $H^1(\mathbb{R}^n)$-atom if it is supported in a cube $Q$, has mean-value zero and satisfies $\|a\|_2 \leq |Q|^{-1/2}$. Then, $f \in H^1(\mathbb{R}^n)$ if and only if $f = \sum_Q \lambda_Q a_Q$ where the $a_Q$’s are $H^1(\mathbb{R}^n)$-atoms and the sequence of complex numbers $(\lambda_Q)_{Q}$ is in $l^1$. The norm $\|f\|_{H^1(\mathbb{R}^n)}$ is comparable with the infimum of $\sum_Q |\lambda_Q|$ taken over all such decompositions.

In recent years, a quite complete theory of Hardy spaces on domains has been developed ([20], [24], [1], [8], [22]). The Hardy spaces are defined in terms of restrictions or support conditions from $H^1(\mathbb{R}^n)$ or in terms of some “grand’ maximal function. For these spaces, atomic decomposition have been obtained in particular on special Lipschitz domains and bounded Lipschitz domains of $\mathbb{R}^n$. However, is there a maximal characterization using the Poisson semigroup? More precisely, replace in (1), $\mathbb{R}^n$ by $\Omega$ and take for $\Delta$ the Laplacian with Dirichlet or Neumann boundary condition. This defines two maximal Hardy spaces on $\Omega$. One of the aims of the present paper is to identify each one with one of the “geometrical” Hardy spaces mentioned above. It turns out that the choice of boundary condition is meaningful in the answer. Roughly, the maximal space corresponding to the Dirichlet Laplacian is $H^1_{r}(\Omega)$ and to the Neumann Laplacian $H^1_{z}(\Omega)$. In the Dirichlet case, we shall use the existing atomic decomposition of $H^1_{r}(\Omega)$. On the other hand, in the Neumann case, we obtain in passing the atomic decomposition of $H^1_{z}(\Omega)$. We also
make the statements valid for general strongly Lipschitz domains (which include also exterior domains).

Another question we ask here is: does the Laplacian play a specific role? In other words, can it be replaced by an other second order elliptic operator? In [2], it was shown that $H^1(\mathbb{R})$ has a maximal characterization using the Poisson semigroup of elliptic operators. We give here an affirmative answer in higher dimensions and on domains, provided the elliptic operator satisfies a technical condition. For example any real elliptic operator will do. This also emphasizes the prominent role of the boundary condition in these questions.

The similar questions for local Hardy spaces have comparable answers.

Using the recent work of Dafni et al. [8], one can certainly extend our results to $H^p$ and $h^p$ spaces for a range of $p$’s smaller than 1. We have not done so to keep the length of the paper reasonable.

The plan of this paper is the following. First, we treat the case of global Hardy spaces on strongly Lipschitz domains: we review their definitions and recall their atomic decompositions (and clarify some points in the literature). We then introduce our maximal Hardy spaces and state the main theorem. Next, we recall a few facts about $BMO$ and duality. Then we turn to proving some auxiliary results involving square functions, Carleson measures and tent spaces before proving the main theorem. We also give proofs (sometimes new) of classical atomic decomposition and of duality. In a second part, we study the corresponding theory for local Hardy spaces. We also present different maximal functions characterizing our maximal Hardy space. We conclude with two appendices, one about kernel estimates and the other about the elementary geometry of Lipschitz domains.

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2 Global Hardy spaces on strongly Lipschitz domains

In what follows, it is understood without mention that $\Omega$ belongs to the class of strongly Lipschitz domains of $\mathbb{R}^n$, that is $\Omega$ is a proper open connected set in $\mathbb{R}^n$ whose boundary is a finite union of parts of rotated graphs of Lipschitz maps, at most one of these parts possibly infinite.

This class includes special Lipschitz domains, bounded Lipschitz domains and exterior domains. Some facts about such domains are presented in Appendix B.

Some statements may be valid for a restricted class and we shall indicate this when it is the case.

2.1 Hardy spaces: definitions

Let us begin with defining various Hardy spaces on a domain. Some definitions will differ from ([4, 8]). For the atomic spaces, we have privileged $L^2$ normalized atoms. We will not address the equivalent definitions obtained by taking $L^p$ normalized atoms with $p > 1$. 

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The first category is made up of restrictions to \( \Omega \) of certain functions in \( H^1(\mathbb{R}^n) \).

**Definition of** \( H^1_r(\Omega) \): A function \( f \) on \( \Omega \) is said to be in \( H^1_r(\Omega) \) if it is the restriction to \( \Omega \) of a function \( F \in H^1(\mathbb{R}^n) \). If \( f \in H^1_r(\Omega) \), define \( \| f \|_{H^1_r(\Omega)} \) by
\[
\| f \|_{H^1_r(\Omega)} = \inf \| F \|_{H^1(\mathbb{R}^n)},
\]
the infimum being taken over all the functions \( F \in H^1(\mathbb{R}^n) \) such that \( F|_\Omega = f \).

**Definition of** \( H^1_z(\Omega) \): A function \( f \) on \( \Omega \) belongs to \( H^1_z(\Omega) \) if the function \( F \) defined by
\[
F(x) = \begin{cases} 
  f(x) & \text{if } x \in \overline{\Omega}, \\
  0 & \text{if } x \notin \Omega 
\end{cases}
\]
belongs to \( H^1(\mathbb{R}^n) \). When \( f \in H^1_z(\Omega) \), its norm \( \| f \|_{H^1_z(\Omega)} \) is \( \| F \|_{H^1(\mathbb{R}^n)} \). Note that it is a strict subspace of \( H^1_r(\Omega) \) (in particular, a function \( f \in H^1_z(\Omega) \) satisfies \( \int_\Omega f(x)dx = 0 \), whereas this may not happen for \( f \in H^1_r(\Omega) \)). This space is nothing but the subspace of \( H^1(\mathbb{R}^n) \) of all functions supported in \( \Omega \).

The second category of Hardy spaces on \( \Omega \) consists of atomic spaces. We list three such spaces.

**Definition of type (a) and (b) cubes:** A cube \( Q \) is said to be a type (a) cube [with respect to \( \Omega \)] if \( 4Q \subset \Omega \), a type (b) cube if \( 2Q \subset \Omega \) and \( 4Q \cap \partial \Omega \neq \emptyset \).

**Definition of type (a) and (b) atoms:** A measurable function \( a \) on \( \Omega \) is called a type (a) atom if it is supported in a type (a) cube \( Q \) with
\[
\int_Q a(x)dx = 0 \quad \text{and} \quad \| a \|_2 \leq |Q|^{-1/2}.
\]
A measurable function \( a \) on \( \Omega \) is called a type (b) atom if it is supported in a type (b) cube \( Q \) with
\[
\| a \|_2 \leq |Q|^{-1/2}.
\]
Note that a type (b) atom is not supposed to have mean value zero.

A more speaking terminology would be interior atoms for type (a) atoms and boundary atoms for type (b) atoms. We have kept the terminology in the literature.

**Definition of** \( H^1_{r,a}(\Omega) \): A function \( f \) defined on \( \Omega \) belongs to \( H^1_{r,a}(\Omega) \) if
\[
f = \sum_{(a)} \lambda_Q a_Q + \sum_{(b)} \mu_Q b_Q
\]
where the \( a_Q \)'s are type (a) atoms, the \( b_Q \)'s are type (b) atoms and \( \sum_{(a)} |\lambda_Q| + \sum_{(b)} |\mu_Q| < +\infty \). Define \( \| f \|_{H^1_{r,a}} \) as the infimum of \( \sum_{(a)} |\lambda_Q| + \sum_{(b)} |\mu_Q| \) over all such decompositions.
Definition of $H^1_{z,a}(\Omega)$: A function $f$ defined on $\Omega$ belongs to $H^1_{z,a}(\Omega)$ if

$$ f = \sum_{(a)} \lambda_Q a_Q $$

where the $a_Q$’s are type (a) atoms and $\sum_{(a)} |\lambda_Q| < +\infty$. Define $\|f\|_{H^1_{z,a}}$ as the infimum of $\sum_{(a)} |\lambda_Q|$ over all such decompositions.

Note that this definition gives a smaller space than the atomic space considered in [9]: there, the $a_Q$’s are taken as $H^1_z(\Omega)$-atoms, that is $H^1(\mathbb{R}^n)$-atoms supported in cubes contained in $\Omega$. We shall show that our definition coincides with theirs (this fact is implicit in [9] when $\Omega$ is special Lipschitz or bounded as pointed out in [9], p. 1612). An immediate advantage of our definition of $H^1_{z,a}(\Omega)$ is the strict containment of $H^1_{z,a}(\Omega)$ in $H^1_{r,a}(\Omega)$ and the evident role of the boundary of $\Omega$. If $\Omega$ were arbitrary, that definition could be vacuous for application as $H^1_{z,a}(\Omega)$ could be too small.

Definition of $H^1_{CW}(\Omega)$: Finally, since $\Omega$ is strongly Lipschitz, it is a space of homogeneous type and one may also consider on $\Omega$ the Hardy space of Coifman and Weiss as defined in [12], which will be denoted in the sequel by $H^1_{CW}(\Omega)$. An $H^1_{CW}(\Omega)$-atom is a function $a$ supported in $Q \cap \bar{\Omega}$, where $Q$ is a cube centered in $\Omega$ (but not necessarily included in $\Omega$) and satisfying

$$ \int a(x)dx = 0 \quad \text{and} \quad \|a\|_2 \leq |Q \cap \Omega|^{-1/2}. $$

If $\Omega$ has finite measure, the constant function $\frac{1}{|\Omega|}$ is not an atom with our definition in opposition with that of [12].

A function $f$ is in $H^1_{CW}(\Omega)$ if $f$ can be written as

$$ f = \sum_Q \lambda_Q a_Q. $$

where the $a_Q$’s are $H^1_{CW}(\Omega)$-atoms and $\sum_Q |\lambda_Q| < \infty$. The norm is defined as usual. This space is also a special case of the Hardy space $H^1(F)$ considered in [20] on closed sets $F$ of $\mathbb{R}^n$ with the Markov property (here, $F = \bar{\Omega}$).

Theorem 1. (a1) $H^1_r(\Omega) \subset H^1_{r,a}(\Omega)$.

(a2) $H^1_{r,a}(\Omega) = H^1_r(\Omega)$ provided $\Omega$ is unbounded.

(b1) $H^1_{z,a}(\Omega) = H^1_{CW}(\Omega)$.

(b2) $H^1_{z,a}(\Omega) = H^1_z(\Omega)$.

Each inclusion is here a continuous embedding between Banach spaces. In this paper, one finds a self-contained proof. But, as this is not the main object of our paper, let us comment on this result now and postpone proofs till later.
Assertions (a1), (a2) are known results when $\Omega$ is a special Lipschitz domain or is bounded \[24, 9\] and are simple to prove. We note that the restriction on $\Omega$ in (a2) is necessary as a counterexample will show (See Section 2.8).

Concerning (b1), the embedding $H^1_{z,a}(\Omega) \subset H^1_{CW}(\Omega)$ is straightforward. The converse can be obtained (but this is not straightforward) by typical arguments in harmonic analysis on abstract homogeneous spaces combined with the geometry of the boundary: maximal functions, Calderon-Zygmund decomposition and Whitney coverings. However, we shall present a quite interesting argument due to Lou and McIntosh \[22\], which uses more the differential structure of $\mathbb{R}^n$ (See Section 2.8).

That $H^1_{z,a}(\Omega) \subset H^1_z(\Omega)$ in (b2) is a triviality. The remaining embedding $H^1_z(\Omega) \subset H^1_{z,a}(\Omega)$ is the deepest of all. It is proved by a constructive method in \[3\] on special Lipschitz domains and on bounded Lipschitz domains for the local Hardy spaces (see Section 3). In \[7\], it is derived from an extension theorem by Jones for BMO \[19\] and duality. Another argument is to use the result that $H^1_z(\Omega) = H^1(\Omega)$ in \[20\], combined with $H^1_{CW}(\Omega) = H^1(\Omega)$ and (b1).

A byproduct of our maximal spaces defined below (Section 2.2) is another proof of the embedding $H^1_z(\Omega) \subset H^1_{CW}(\Omega)$.

Up until Section 2.8, we assume knowledge of Theorem 1 but $H^1_z(\Omega) \subset H^1_{z,a}(\Omega)$ which is proved in Section 2.7.

### 2.2 Maximal Hardy spaces and statement of the main result

We introduce a third category of Hardy spaces on $\Omega$ defined via maximal functions associated with second order elliptic operators in divergence form. We briefly describe these operators, the most typical being the Laplacian with appropriate boundary condition. If $\Omega = \mathbb{R}^n$ or if $\Omega$ is a strongly Lipschitz domain of $\mathbb{R}^n$, we will denote by $W^{1,2}(\Omega)$ the usual Sobolev space on $\Omega$ equipped with the norm $(\|f\|_2^2 + \|
abla f\|_2^2)^{1/2}$, whereas $W^{1,2}_0(\Omega)$ stands for the closure of $C_0^\infty(\Omega)$ in $W^{1,2}(\Omega)$.

If $A : \mathbb{R}^n \to M_n(\mathbb{C})$ is a measurable function, define
\[
\|A\|_\infty = \sup_{x \in \mathbb{R}^n, \|\xi\| = \|\eta\| = 1} |\langle A(x)\xi, \eta \rangle|.
\]

Here and subsequently in the paper, the notation sup is used for esssup. For all $\delta > 0$, denote by $A(\delta)$ the class of all measurable functions $A : \mathbb{R}^n \to M_n(\mathbb{C})$ satisfying, for all $x, \xi \in \mathbb{R}^n$:
\[
\|A\|_\infty \leq \delta^{-1} \text{ and } \Re \langle A(x)\xi, \xi \rangle \geq \delta |\xi|^2.
\]

Denote by $A$ the union of all $A(\delta)$ for $\delta > 0$.

When $A \in A$ and $V$ is a closed subspace of $W^{1,2}(\Omega)$ containing $W^{1,2}_0(\Omega)$, denote by $L$ the maximal-accretive operator on $L^2(\Omega)$ with largest domain $\mathcal{D}(L) \subset V$ such that
\[
\langle Lf, g \rangle = \int_\Omega A \nabla f \cdot \overline{\nabla g}, \forall f \in \mathcal{D}(L), \forall g \in V.
\] (2)

We will write $L = (A, \Omega, V)$. Say that $L$ satisfies the Dirichlet boundary condition (DBC) when $V = W^{1,2}_0(\Omega)$, the Neumann boundary condition (NBC) when $V = W^{1,2}(\Omega)$. 

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We turn to the definition of maximal Hardy spaces associated with such operators. Let 
$L = (A, \Omega, V)$ be as above. This operator has a unique maximal accretive square root $L^{1/2}$ so 
that $-L^{1/2}$ is the generator of an $L^2(\Omega)$-contracting semigroup $P_t = e^{-tL^{1/2}}, t > 0$, the Poisson 
semigroup for $L$. We will need that $P_t$ also acts on $L^1(\Omega)$. Let us then introduce a technical 
condition on $L$.

**Definition 2.** For $0 < \tau \leq +\infty$, we call $(G_\tau)$ the conjunction of (3) and (4) below: The kernel 
of $e^{-tL}$, denoted by $K_t(x, y)$, is a measurable function on $\Omega \times \Omega$ and there exist $C, \alpha > 0$ such 
that, for all $0 < t < \tau$ and almost every $x, y \in \Omega$,

$$|K_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-\alpha \|x-y\|^2}. \quad (3)$$

For all $x \in \Omega$ and all $0 < t < \tau$, the function $y \mapsto K_t(x, y)$ is Hölder continuous in $\Omega$ and 
there exist $C, \mu \in [0, 1]$ such that, for all $0 < t < \tau$ and all $x, y, y' \in \Omega$,

$$|K_t(x, y) - K_t(x, y')| \leq \frac{C}{t^{n/2}} \frac{|y - y'|^\mu}{t^{\mu/2}}. \quad (4)$$

When $\tau$ is finite, we set $\tau = 1$ without loss of generality.

For those readers only interested in the Laplacian or real symmetric operators (under BDC 
or NBC), this condition is always satisfied on $\mathbb{R}^n$ or on Lipschitz domains with $\tau = \infty$ except 
under NBC with $\Omega$ bounded for which we have $\tau$ finite.

**Lemma 3.** When $(G_\infty)$ holds, the Poisson kernel of $L$, i.e. the kernel $p_t(x, y)$ of $P_t$ satisfies

$$|p_t(x, y)| \leq \frac{Ct}{(t + |x-y|)^{n+1}} \quad (5)$$

and

$$|p_t(x, y) - p_t(x, y')| \leq \frac{C}{t^n} \frac{|y - y'|^\mu}{t^{\mu}} \quad (6)$$

for all $t \in [0, \infty[$, for some $C > 0$ and $\mu \in [0, 1[$.

This follows from the subordination formula (see [29]):

$$p_t(x, y) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} K_{\frac{u}{4t}}(x, y) e^{-u^{-1/2}} du. \quad (7)$$

If $(G_\infty)$ holds and $f \in L^1_{\text{loc}}(\Omega)$ so that $y \mapsto |y|^{n-1} f(y) \in L^1(\Omega)$, define, for all $x \in \Omega$,

$$f^*_L(x) = \sup_{y \in \Omega, t > 0, |y-x| < t} |P_t f(y)|.$$

Say that $f \in H^1_{\text{max}, L}(\Omega)$ if $f^*_L \in L^1(\Omega)$ and define

$$\|f\|_{H^1_{\text{max}, L}(\Omega)} = \|f^*_L\|_{L^1(\Omega)}. \quad (8)$$

Note that $H^1_{\text{max}, L}(\Omega)$ depends, in particular, on the boundary condition. Since $P_t$ tends to 
the identity strongly in $L^1(\Omega)$ we see that $H^1_{\text{max}, L}(\Omega) \subset L^1(\Omega)$.

One of the aims of this paper is to identify this maximal space. Our result is the following:
Theorem 4. Let \( \Omega = \mathbb{R}^n \) or \( \Omega \) be a strongly Lipschitz domain of \( \mathbb{R}^n \), and \( L = (A, \Omega, V) \) satisfying \((G_\infty)\).

(a) If \( \Omega = \mathbb{R}^n \), one has \( H^1(\mathbb{R}^n) = H^1_{\text{max}, L}(\mathbb{R}^n) \).

(b) If \( \Omega \) is unbounded and \( L \) satisfies the Dirichlet boundary condition, then one has \( H^1_{r,a}(\Omega) = H^1_{\text{max}, L}(\Omega) = H^1(\Omega) \).

(c) If \( \Omega \) is unbounded and \( L \) satisfies the Neumann boundary condition, then one has \( H^1_{z,a}(\Omega) = H^1_{\text{max}, L}(\Omega) = H^1_z(\Omega) \).

If we assume \( \Omega \) is bounded in (b) then \( H^1_{r,a}(\Omega) \subset H^1_{\text{max}, L}(\Omega) \). We have not succeeded in proving the converse. Assuming \( \Omega \) unbounded in (c) is no restriction as the condition \((G_\infty)\) in never satisfied under NBC and \( \Omega \) bounded. Note that (c) contains the equality between Hardy spaces alluded to in Section 2.1 in the case where \( \Omega \) is unbounded. The case where \( \Omega \) is bounded will be addressed via local spaces in Section 2.1. We turn to some intermediate results and begin with discussing about \textit{BMO} spaces.

2.3 \textit{BMO} spaces

Definition of \( \textit{BMO}(\mathbb{R}^n) \): A locally square-integrable function \( f \) on \( \mathbb{R}^n \) is said to be in \( \textit{BMO}(\mathbb{R}^n) \) if

\[
\|f\|_{\textit{BMO}(\mathbb{R}^n)}^2 = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q|^2 \, dx < +\infty
\]

where the supremum is taken over all the cubes \( Q \in \mathbb{R}^n \) with sides parallel to the axes. Here, \( f_Q = \frac{1}{|E|} \int_E f(x) \, dx \) is the mean of \( f \) over \( E \) and \( |E| \) is the Lebesgue measure of \( E \).

Definition of \( \textit{VMO}(\mathbb{R}^n) \): Define \( \textit{VMO}(\mathbb{R}^n) \) as the closure of \( C_c(\mathbb{R}^n) \) (the space of continuous functions on \( \mathbb{R}^n \) with compact support) in \( \textit{BMO}(\mathbb{R}^n) \). This \( \textit{VMO} \) space is the one in the sense of Coifman and Weiss [12] and is different from the one considered by Sarason in [20], which is the closure of the space of all uniformly continuous \( \textit{BMO} \)-functions on \( \mathbb{R}^n \). See the recent work of G. Bourdaud for clarifications [6]. It is well-known that \( \textit{BMO}(\mathbb{R}^n) \) is the dual of \( H^1(\mathbb{R}^n) \), the latter being the dual of \( \textit{VMO}(\mathbb{R}^n) \) [14, 12].

We next introduce the first category of \( \textit{BMO} \)-spaces on \( \Omega \).

Definition of \( \textit{BMO}_z(\Omega) \): The space \( \textit{BMO}_z(\Omega) \) is defined as being the space of all functions in \( \textit{BMO}(\mathbb{R}^n) \) supported in \( \overline{\Omega} \), equipped with the norm \( \|f\|_{\textit{BMO}_z(\Omega)} = \|f\|_{\textit{BMO}(\mathbb{R}^n)} \).

Definition of \( \textit{VMO}_z(\Omega) \): We define \( \textit{VMO}_z(\Omega) \) is the closure of \( C_c(\Omega) \), the space of continuous functions with support in \( \Omega \), in \( \textit{BMO}_z(\Omega) \).
**Definition of $BMO_r(\Omega)$:** The space $BMO_r(\Omega)$ is defined as being the space of all restrictions to $\Omega$ of functions in $BMO(\mathbb{R}^n)$. If $f \in BMO_r(\Omega)$ define $\|f\|_{BMO_r(\Omega)}$ by

$$
\|f\|_{BMO_r(\Omega)} = \inf \|F\|_{BMO(\mathbb{R}^n)},
$$

the infimum being taken over all the functions $F \in BMO(\mathbb{R}^n)$ such that $F|_\Omega = f$.

Next, we turn to the second category of $BMO$-spaces, defined in terms of mean square oscillation.

**Definition of $BMO_{z,a}(\Omega)$:** A locally square-integrable function $f$ on $\Omega$ is in $BMO_{z,a}(\Omega)$ if

$$
\|\phi\|_{BMO_{z,a}(\Omega)}^2 = \sup \left( \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^2 dx, \sup_{Q} \frac{1}{|Q|} \int_Q |\phi|^2 dx \right) < +\infty,
$$

where $\sup_{(a)}$ (resp. $\sup_{(b)}$) means that the supremum is taken over all type (a) cubes (resp. all type (b) cubes).

**Definition of $BMO_{r,a}(\Omega)$:** A locally square-integrable function $f$ on $\Omega$ is in $BMO_{r,a}(\Omega)$ if

$$
\|\phi\|_{BMO_{r,a}(\Omega)}^2 = \sup_{(a)} \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^2 dx < +\infty.
$$

**Definition of $BMO_{CW}(\Omega)$:** A locally square-integrable function $\phi$ on $\Omega$ is in $BMO_{CW}(\Omega)$ if

$$
\|\phi\|_{BMO_{CW}(\Omega)}^2 = \sup \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} |\phi(x) - \phi_{Q \cap \Omega}|^2 dx < +\infty,
$$

where the supremum is taken over all cubes centered in $\Omega$. This is the space defined in [12]. A slight variation is that the indicator function of $\Omega$ is not in $BMO_{CW}(\Omega)$ when $\Omega$ has finite measure.

**Definition of $VMO_{CW}(\Omega)$:** The space $VMO_{CW}(\Omega)$ is the closure of $C_c(\Omega)$ in $BMO_{CW}(\Omega)$.

Note that $BMO_{r,a}(\Omega)$, $BMO_r(\Omega)$ and $BMO_{CW}(\Omega)$ are defined modulo constants. We ignore this well-understood issue.

Let us mention the duality results.

**Theorem 5.**

(a) The dual of $H^1_{r,a}(\Omega)$ is $BMO_{z,a}(\Omega)$.

(b) The dual of $H^1_r(\Omega)$ is $BMO_z(\Omega)$, the dual of $VMO_z(\Omega)$ is $H^1_r(\Omega)$.

(c) The dual of $H^1_{CW}(\Omega)$ is $BMO_{CW}(\Omega)$, the dual of $VMO_{CW}(\Omega)$ is $H^1_{CW}(\Omega)$. 

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(d) The dual of $H^1_\varepsilon(\Omega)$ is $BMO_r(\Omega)$.

(e) The dual of $H^1_{\varepsilon,a}(\Omega)$ is $BMO_{r,a}(\Omega)$.

The result corresponding to Theorem 1 for $BMO$-spaces is the following

**Theorem 6.** (a1) $BMO_{\varepsilon,a}(\Omega) \subset BMO_{\varepsilon}(\Omega)$.

(a2) $BMO_{\varepsilon,a}(\Omega) = BMO_{\varepsilon}(\Omega)$ provided $\varepsilon\Omega$ is unbounded.

(b1) $BMO_{CW}(\Omega) = BMO_{r,a}(\Omega)$.

(b2) $BMO_r(\Omega) = BMO_{r,a}(\Omega)$.

Again, let us just comment on these results of which we shall only need (b) and (c) of Theorem 6, which will be proved in Section 2.10.

Concerning Theorem 6 (c) is already known [12] (our change in the definition does not induce any modification in the proof) and all the other statements but (e) are not so deep. It is easy to show that $(H^1_{\varepsilon,a}(\Omega))' \subset BMO_{r,a}(\Omega)$ but the converse is harder (in particular, we think that the argument proposed in [7], Theorem 2.1, for bounded domains and local spaces has a gap).

Assume for the moment all the above is proved and let us argue for Theorem 6. First, (a1) and (a2) follow by duality (easy direct proofs are also possible). Next and we have $BMO_r(\Omega) \subset BMO_{CW}(\Omega) \subset BMO_{r,a}(\Omega)$ (duality or direct proof). Using Theorem 6, (b1), and the already observed inclusion in Theorem 5, (e), we have

$$BMO_{CW}(\Omega) \subset BMO_{r,a}(\Omega) \subset (H^1_{\varepsilon,a}(\Omega))' = (H^1_{CW}(\Omega))' = BMO_{CW}(\Omega).$$

Hence, $BMO_{CW}(\Omega) = BMO_{r,a}(\Omega)$ and $BMO_{r,a}(\Omega)$ is the dual of $H^1_{\varepsilon,a}(\Omega)$.

This completes the proof of Theorem 6, (e), and of Theorem 6, (b1).

The remaining embedding in Theorem 6, (b2), namely, $BMO_{r,a}(\Omega) \subset BMO_r(\Omega)$, given duality and the existing embeddings, is equivalent to the embedding $H^1_{\varepsilon}(\Omega) \subset H^1_{\varepsilon,a}(\Omega)$.

A result by Jones [19] characterizes the domains having an extension property for $BMO$. Lipschitz domains fall in that class. The embedding $BMO_{r,a}(\Omega) \subset BMO_r(\Omega)$ is a slightly stronger extension property since the definition of $BMO_{r,a}(\Omega)$ requires bounded mean oscillation only on cubes of type (a) while Jones assumes bounded mean oscillation on all cubes inside $\Omega$.

### 2.4 Area integrals and maximal functions

Let $\Omega = \mathbb{R}^n$ or $\Omega$ be a strongly Lipschitz domain of $\mathbb{R}^n$. Consider $L = (A, \Omega, V)$ with Dirichlet or Neumann boundary condition. Define for $x \in \Omega$,

$$S_\alpha f(x) = \left( \int_{\Gamma_\alpha(x)} t^{1-n} |\nabla P_t f(y)|^2 dy dt \right)^{1/2},$$
and

\[ S^\varepsilon_{\alpha}^R f(x) = \left( \int_{\Gamma^\varepsilon_{\alpha}^R(x)} t^{1-n} |\nabla P_t f(y)|^2 \, dy \right)^{1/2}, \]

with \( \nabla u = (\nabla u, \partial_x u) \), \( |\nabla u|^2 = |\nabla u|^2 + |\partial_x u|^2 \), \( P_t = e^{-tL^{1/2}} \) and where \( \Gamma_{\alpha}(x) \) and \( \Gamma^\varepsilon_{\alpha}^R(x) \) are the respectively the cones and the truncated cones defined by

\[ \Gamma_{\alpha}(x) = \{(y, t) \in \Omega \times ]0, +\infty[; |y - x| < \alpha t\} \]

and

\[ \Gamma^\varepsilon_{\alpha}^R(x) = \{(y, t) \in \Omega \times ]\varepsilon, R[; |y - x| < \alpha t\}, \]

for \( \alpha > 0 \), \( 0 < \varepsilon < R < +\infty \). We can also write \( S_{\alpha} = S_{\alpha}^{0, \infty} \). Here, \( | \cdot | \) is the sup norm on \( \mathbb{R}^n \) (for which the balls are cubes with sides parallel to the axes).

**Lemma 7.** Assume \( \alpha < 1 \). Then, one has for \( f \in L^2(\Omega) \),

\[ S^\varepsilon_{\alpha}^R f(x) \leq C(1 + |\ln(R/\varepsilon)|)f^*_L(x) \]

for some constant depending on \( \alpha \).

The truncated square function is well-defined for \( f \in L^2(\Omega) \) since \( \nabla P_t \) is bounded on \( L^2(\Omega) \). Let us also recall that \( u_t(y) = P_t f(y) \) satisfies the elliptic equation \( \nabla \cdot B \nabla u_t(y) = 0 \) (in the weak sense on \( \Omega \times ]0, \infty[ \)) where \( B \) is the \((n + 1) \times (n + 1)\) block diagonal matrix with components \( A \) and \( 1 \). Moreover, we have prescribed Dirichlet or Neumann data on the lateral boundary \( \partial \Omega \times ]0, \infty[ \). Hence, we have interior and boundary Caccioppoli inequalities (see [8]): for some \( \rho > 0 \) and \( C \) depending on \( \Omega \) and ellipticity,

\[ \int_E |\nabla u_t(y)|^2 \, dy \leq C r^{-2} \int_E |u_t(y)|^2 \, dy \]

for all sets \( E = B((z, \tau), r) \cap (\Omega \times ]0, \infty[) \) with \( \tilde{E} = B((z, \tau), 2r) \cap (\Omega \times ]0, \infty[) \) provided \( x \in \Omega \), \( \tau > 0 \) and \( r \leq \inf(\rho, \tau) / 4 \). Here \( \tilde{B}((z, \tau), r) \) is the open ball defined by \( \sup(|z - y|, |\tau - t|) < r \).

For \( (z, \tau) \in \Gamma^\varepsilon_{\alpha}^R(x) \), let \( E(z, \tau) = \tilde{B}((z, \tau), r) \cap (\Omega \times ]0, \infty[) \) with \( r = \delta \inf(\tau, \rho) \). Here \( \delta \) is some small number. By Besicovitch covering argument, pick a subcollection \( E_j = E_j(z_j, \tau_j) \) covering \( \Gamma^\varepsilon_{\alpha}^R(x) \) and having bounded overlap. Remark that \( (y, t) \in E_j \) implies \( t \sim d_j \), the distance from \( E_j \) to the bottom boundary \( \Omega \times \{0\} \). Remark also that if \( \delta \) is small enough, \( (y, t) \in \tilde{E}_j \) implies \( (y, t) \in \Gamma_1(x) \), hence \( |u_t(y)| \leq f^*_L(x) \). Thus we obtain from the bounded overlap and Caccioppoli’s inequality,

\[ S^\varepsilon_{\alpha}^R f(x)^2 \leq C \sum_j d_j^{1-n} r_j^{-2} |\tilde{E}_j| f^*_L(x)^2. \]

Observe that \( |\tilde{E}_j| \leq C |E_j| \) and so that the bounded overlap of the \( E_j \)'s again easily yields by inspection,

\[ \sum_j d_j^{1-n} r_j^{-2} |\tilde{E}_j| \leq C(1 + |\ln(R/\varepsilon)|). \]
**Proposition 8.** Assume that \((G_\infty)\) holds. There exists \(C > 0\) such that, for all \(f \in H^1_{\text{max}, L}(\Omega)\), 
\[
\|S_1 f\|_1 \leq C \|f\|_{H^1_{\text{max}, L}}.
\]

The proof follows ideas from [14], Theorem 8, p. 161 and [11], Section 6, see also [3], Lemme II.10. It relies on a “good \(\lambda\)” inequality. We need though variants of the truncated square functions in order to compensate the lack of pointwise regularity. Set

\[
\tilde{S}_\alpha^{\varepsilon, R} f(x) = \left( \int_1^2 \int_{\Gamma_{\infty}^{\alpha, s}(x)} t^{1-n} \left| \nabla P_t f(y) \right|^2 dy dt da \right)^{1/2}, \quad x \in \Omega.
\]

Fairly elementary arguments show that

\[
S_\alpha^{2\varepsilon, R} f \leq \tilde{S}_\alpha^{\varepsilon, R} f \leq S_{2\alpha}^{\varepsilon, R} f.
\]

We shall prove

**Lemma 9.** There exists \(c > 0\) such that, for all \(0 < \gamma \leq 1\), all \(\lambda > 0\), all \(0 < \varepsilon < R < \infty\) and all \(f \in H^1_{\text{max}, L} \cap L^2(\Omega)\),

\[
\left| \left\{ x \in \Omega; \tilde{S}_1^{\varepsilon, R} f(x) > 2\lambda, f^+(x) \leq \gamma \lambda \right\} \right| \leq c\gamma^2 \left| \left\{ x \in \Omega; \tilde{S}_1^{\varepsilon, R} f(x) > \lambda \right\} \right|.
\]

We will also use the comparability of the square functions. See [11], Proposition 4, p. 309.

**Lemma 10.** For \(\alpha, \beta > 0\), \(0 \leq \varepsilon < R \leq +\infty\), one has

\[
\|S_\alpha^{\varepsilon, R} f\|_1 \sim \|S_\beta^{\varepsilon, R} f\|_1,
\]

where the implicit constants do not depend on \(f, \varepsilon, R\).

Let us deduce Proposition 8. Assume first that \(f \in H^1_{\text{max}, L} \cap L^2(\Omega)\). As a consequence of Lemma 9, by integrating both sides with respect to \(\lambda\), one obtains

\[
\|\tilde{S}_1^{\varepsilon, R} f\|_1 \leq \gamma^{-1} \|f_1^+\|_1 + c\gamma^2 \|\tilde{S}_1^{\varepsilon, R} f\|_1.
\]

Thanks to Lemma 10 and the comparisons between the square functions and their variants, one has

\[
\|S_1^{\varepsilon, R} f\|_1 \leq C \|\tilde{S}_1^{\varepsilon, R} f\|_1
\]

and by Lemma 9

\[
\|\tilde{S}_1^{\varepsilon/2, R} f\|_1 \leq \|S_1^{\varepsilon/2, R} f\|_1 \leq \|S_1^{\varepsilon/2, \varepsilon} f\|_1 + \|S_1^{\varepsilon, R} f\|_1 + \|S_1^{R, 2R} f\|_1 \leq \|S_1^{\varepsilon, R} f\|_1 + C \|f_1^+\|_1.
\]

Hence, by choosing \(\gamma\) appropriately and using the a priori knowledge that \(\|S_1^{\varepsilon, R} f\|_1 < +\infty\) one obtains

\[
\|S_1^{\varepsilon, R} f\|_1 \leq C \|f_1^+\|_1.
\]
By letting $\varepsilon \downarrow 0$ and $R \uparrow +\infty$, the conclusion of Proposition \textcolor{red}{8} in the case $f \in L^2$ follows.

To complete the proof of Proposition \textcolor{red}{8}, we have to relax the assumption $f \in L^2(\Omega)$. But, if $f_L^* \in L^1$, then $f \in L^1(\Omega)$ and together with the kernel estimates on the kernel of $P_t$, one has $f_\varepsilon \in L^2(\Omega)$ for all $\varepsilon > 0$, where $f_\varepsilon(x) = P_\varepsilon f(x)$. It follows that $\|Sf_\varepsilon\|_1 \leq C \|(f_\varepsilon)_L\|_1 \leq \|f_L^*\|_1$. Letting $\varepsilon \downarrow 0$, one obtains $\|Sf\|_1 \leq C \|f_L^*\|_1$ by monotone convergence. \hfill \blacksquare

We turn to the proof of Lemma \textcolor{red}{9}. In the next argument, $\varepsilon, R, \lambda$ are fixed. Also $f \in H_{\text{max}, L} \cap L^2(\Omega)$. Define $O = \left\{ x \in \Omega; \tilde{S}_{1/2} f(x) > \lambda \right\}$. We may assume that $O \neq \emptyset$. Let $O = \bigcup_k Q_k$ be a Whitney decomposition of $O$ (with respect to $\Omega$) by dyadic cubes (of $\mathbb{R}^n$), so that, for all $k$, $20Q_k \subset O \subset \Omega$, but $4Q_k$ intersects $\Omega \setminus O$. Since $\left\{ \tilde{S}_{1/20} f > 2\lambda \right\} \subset \left\{ \tilde{S}_{1/2} f > \lambda \right\}$, it is enough to show that

$$\left| \left\{ x \in Q_k; \tilde{S}_{1/20} f(x) > 2\lambda, f_L^*(x) \leq \gamma \lambda \right\} \right| \leq c \gamma^2 |Q_k|.$$

From now on, fix $k$ and denote by $t$ the side length of $Q_k$.

If $x \in Q_k$,

$$\tilde{S}_{1/20}^{\sup(10t, \varepsilon), R} f(x) \leq \lambda.$$

Indeed, pick $x_k \in 4Q_k$ with $x_k \notin O$. If $|y - x| < \frac{t}{20}$ and with $t \geq \sup(10t, \varepsilon)$, then one has $|x_k - y| < \frac{t}{20} + 4t \leq \frac{t}{2}$. Hence $\tilde{S}_{1/20}^{\sup(10t, \varepsilon), R} f(x) \leq \tilde{S}_{1/2}^{\sup(10t, \varepsilon), R} f(x_k) \leq \lambda$.

If $\varepsilon \geq 10t$, we are done. Otherwise, using, $\tilde{S}_{1/20}^{\varepsilon, R} f(x) \leq \tilde{S}_{1/20}^{\varepsilon, 10t} f(x) + \tilde{S}_{1/20}^{10t, R} f(x)$, it remains to show that

$$\left| \left\{ x \in Q_k \cap F; g(x) > \lambda \right\} \right| \leq c \gamma^2 |Q_k|$$

where

$$g(x) = \tilde{S}_{1/20}^{\varepsilon, 10t} f(x)$$

and

$$F = \left\{ x \in \Omega; f_L^*(x) \leq \gamma \lambda \right\}.$$

By Tchebytchev’s inequality, this follows from

$$\int_{Q_k \cap F} g^2 \leq c \gamma^2 \lambda^2 |Q_k|.$$

We note that the condition $(G_\infty)$ implies that $F$ is a closed set of $\Omega$.

If $5l \leq \varepsilon$, then the argument using Caccioppoli’s inequality shows that

$$\int_{Q_k \cap F} g^2 \leq c \int_{Q_k \cap F} (f_L^*)^2 \leq c \gamma^2 \lambda^2 |Q_k \cap F|.$$

Assume from now on that $\varepsilon < 5l$. By geometric considerations,

$$\int_{Q_k \cap F} g(x)^2 dx \leq c \int_1^2 \int_{E_\varepsilon} t |\nabla u_t(y)|^2 dy dtda,$$
where
\[ E_a = \{(y, t) \in \Omega \times ]0, \infty[; a\psi(y) < t\} \]
with \( \psi(y) \) the Lipschitz function equal to 20 dist \((y, Q_k \cap F)\). Recall also that \( u_t(y) = P_t f(y) \).

Observe that \( E_a = \{(y, at); (y, t) \in E_1\} \). Define \( E = \{y; (y, t) \in E_1\} \): this is an open set in \( \Omega \). For a connected component \( C \) of \( E \), let \( \mathcal{C}_a = \{(y, t) \in E_a; y \in C\} \). It suffices to show that
\[
\int_1^2 t \int_{\mathcal{C}_a} |\nabla u_t(y)|^2 \, dy dt da \leq c\gamma^2 \lambda^2 |\mathcal{C}|.
\]
Indeed, summing over all connected components of \( E \), we get
\[
\int_1^2 t \int_{E_a} |\nabla u_t(y)|^2 \, dy dt da \leq c\gamma^2 \lambda^2 |E|,
\]
and it remains to observe that \( E \subset 2Q_k \). Indeed, if \( y \in E \), there is a point \((y, t)\) above contained \( E_1 \), hence there exists \( x \in Q_k \cap F \) such that \(|y - x| < \frac{t}{20}\). Since \( t < 10l \), we have \(|y - x| < \frac{t}{2} \) and the desired inclusion follows.

We next fix a connected component \( C \) of \( E \). Consider \( a \in ]1, 2[ \) and note that \( \mathcal{C}_a \) is connected and has Lipschitz boundary. The ellipticity condition for \( A \) shows that
\[
\int_{\mathcal{C}_a} t |\nabla u_t(y)|^2 \, dy dt \leq C \text{ Re } \int_{\mathcal{C}_a} tB \nabla u_t(y) \overline{\nabla u_t(y)} \, dy dt = C \text{ Re } I_a,
\]
where \( B \) is the \((n + 1) \times (n + 1)\) block diagonal matrix with components \( A \) and \( 1 \). The function \( u_t(y) \) satisfies the equation \( \nabla \cdot B \nabla u_t(y) = 0 \) (in the weak sense on \( \Omega \times ]0, \infty[ \)) so that we wish to integrate by parts.

To do so let us make some observations. We claim that for \((y, t) \in \overline{\mathcal{C}_a}\), then \( y \in 2Q_k \subset \Omega \) and \((y, t) \in E_1 \). Indeed, since \( F \) is closed, there exists \( x \in Q_k \cap F \) such that \(|y - x| \leq \frac{t}{20}\). Since \( t < 10l \), we have \(|y - x| \leq \frac{t}{2} \) and the first claim is true. Moreover, \(|y - x| \leq \frac{t}{20l} < t \), hence the second claim.

It follows in particular that \( \overline{\mathcal{C}_a} \) remains far from the boundary of \( \Omega \times ]0, \infty[ \), so that we do not care about the boundary values of \( u_t(y) \), and that \(|u_t(y)| \leq \gamma \lambda \) on \( \overline{\mathcal{C}_a} \).

The Green-Riemann formula shows that \( I_a \) is equal to
\[
-\int_{\mathcal{C}_a} \partial_t u_t(y) \overline{u_t(y)} \, dy dt + \int_{\partial \mathcal{C}_a} tB \nabla u_t(y) \cdot N_a(y, t) \overline{u_t(y)} \, d\sigma_a(y, t).
\]
In this computation, \( N_a(y, t) \) is the unit normal vector outward \( \mathcal{C}_a \) whereas \( d\sigma_a \) is the surface measure over \( \partial \mathcal{C}_a \). Moreover, the Green-Riemann formula again yields
\[
2 \text{ Re } \int_{\mathcal{C}_a} \partial_t u_t(y) \overline{u_t(y)} \, dy dt = \int_{\partial \mathcal{C}_a} |u_t(y)|^2 N_a(y, t) \cdot (0, \ldots, 0, 1) \, d\sigma_a(y, t).
\]
Finally,
\[
\int_{\mathcal{C}_a} t |\nabla u_t(y)|^2 \, dy dt \leq C \int_{\partial \mathcal{C}_a} |u_t(y)|^2 \, d\sigma_a(y, t)
+ C \int_{\partial \mathcal{C}_a} t |u_t(y)| |\nabla u_t(y)| \, d\sigma_a(y, t).
\]
Since $|u_t(y)| \leq \gamma \lambda$ on $\partial C_a$, we obtain that
\[
\int_1^2 \int_{\partial C_a} |u_t(y)|^2 \, d\sigma_a(y, t) \, da \leq \gamma^2 \lambda^2 \int_1^2 \int_{\partial C_a} d\sigma_a(y, t) \, da.
\]
We claim that
\[
\int_1^2 \int_{\partial C_a} d\sigma_a(y, t) \, da \leq c |C|.
\]
Indeed, this integral is bounded by $c \int_{\mathcal{G}} \frac{dz ds}{s}$ where $\mathcal{G}$ is the union of the sets $\partial C_a$ for $1 < a < 2$. This is the set of points $(z, s)$ with $z \in C$ and $\varepsilon < s < 2\varepsilon$ or $\psi(z) < s < 2\psi(z)$ or $10l < s < 20l$. The claim follows readily.

It remains to establish
\[
\int_1^2 \int_{\partial C_a} t |u_t(y)| |\nabla u_t(y)| \, d\sigma_a(y, t) \, da \leq c \gamma^2 \lambda^2 |C|.
\]
Using the previous notation and a change of variables, this integral is bounded by
\[
\gamma \lambda \int_{\mathcal{G}} |\nabla u_t(y)| \, dy dt.
\]
Pick a covering of $\mathcal{G}$ with bounded overlap by balls $B_j = B\left((x_j, t_j), \frac{\varepsilon t_j}{20}\right)$. Remark that $(x, t) \in B_j$ implies $t \sim t_j \sim r(B_j)$, the radius of $B_j$. Then using Hölder’s inequality and again Caccioppoli’s inequality
\[
\int_{\mathcal{G}} |\nabla u_t(y)| \, dy dt \leq c \sum_{B_j} \int_{\mathcal{G}} |\nabla u_t(y)| \, dy dt
\]
\[
\leq c \sum_{B_j} |B_j|^{1/2} r(B_j)^{-1} \left( \int_{2B_j} |u_t(y)|^2 \, dy dt \right)^{1/2}
\]
\[
\leq c \gamma \lambda \sum_{B_j} |B_j| r(B_j)^{-1}
\]
\[
\leq c \gamma \lambda \int_{\mathcal{G}} \frac{dz ds}{s}
\]
\[
\leq c \gamma \lambda |C|.
\]
Here, $\mathcal{G}$ is a set like $\mathcal{G}$ but slightly enlarged: it is contained set of points $(z, s)$ with $z \in C$ and $\varepsilon/2 < s < 4\varepsilon$ or $\psi(z)/2 < s < 4\psi(z)$ or $5l < s < 40l$. 

**Remark 11.** When $L$ is the Laplacian, then $(y, t) \mapsto u_t(y)$ is harmonic so that the Caccioppoli inequality can be replaced by the mean value property and one can proceed directly using the square functions (and not their variants).

When $\Omega \neq \mathbb{R}^n$, the Whitney cubes $Q_k$ are designed to stay away from the boundary of $\Omega$ so that interior estimates suffice. When $\Omega = \mathbb{R}^n$, one can proceed directly and prove the good lambda inequality on $\mathbb{R}^n$. Details are left to the reader.
2.5 Carleson measure estimates and BMO

In the present section, let \( \Omega = \mathbb{R}^n \) or \( \Omega \) be a strongly Lipschitz domain of \( \mathbb{R}^n \). Consider an operator \( L = (A, \Omega, V) \) satisfying (3) and the restrictions of Theorem 4 on \( \Omega \). Set \( P_t = e^{-tL/2} \).

We intend to show that, when a function \( \phi \) belongs to \( BMO_{z,a}(\Omega) \) (resp. \( BMO_{CW}(\Omega) \)), \( t|\partial_t P_t \phi(x)|^2 \, dx \, dt \) is a Carleson measure when \( V = W_{0}^{1,2}(\Omega) \) (resp. \( V = W^{1,2}(\Omega) \)), with the obvious modifications when \( \Omega = \mathbb{R}^n \). When \( Q \) is a cube with center \( x_Q \in \Omega \) and radius \( r = \ell(Q)/2 \), define the tent over \( Q \) by

\[
T(Q) = \{(y,t) \in \Omega \times [0, +\infty[; |y - x_Q| < r - t\}. 
\]

We recall that \( |\cdot| \) is the sup norm on \( \mathbb{R}^n \). When \( \phi \) is locally integrable on \( \Omega \), set

\[
T\phi(x) = \left( \sup_{Q \ni x} \frac{1}{|Q \cap \Omega|} \int_{T(Q)} |\partial_t P_t \phi(\eta)|^2 \, d\eta \right)^{1/2},
\]

where the supremum is taken over all the cubes \( Q \) centered in \( \Omega \) and containing \( x \).

We will need a few facts from functional calculus for the operator \( L = (A, \Omega, V) \) (see [23] and [33]).

First note that \( L \) is one-to-one (except if \( \Omega \) is bounded and \( V = W^{1,2}(\Omega) \), which is excluded from our discussion), and if \( \omega = \sup_{x \in \Omega, \xi \in \mathbb{C}^n} |\arg A(x)\xi| \), one has \( \omega < \pi/2 \) and \( L \) is \( \omega \)-accretive on \( V \) (see [3]). If \( \mu \in ]\omega, \pi[ \) and \( \Gamma_\mu = \{ z \in \mathbb{C}; |z| < \mu \} \), for all function \( f \in H^\infty(\Gamma_\mu) \), one can define a bounded operator \( f(L) \) on \( L^2(\Omega) \) such that \( \|f(L)\| \leq c_\mu \|f\|_{H^\infty(\Gamma_\mu)} \).

If \( \psi \in H^\infty(\Gamma_\mu) \) and if there exist \( c, s > 0 \) such that, for all \( \zeta \in \Gamma_\mu \), \( |\psi(\zeta)| \leq c |\zeta|^s (1 + |\zeta|)^{-2s} \), then \( \psi(L) \) may be computed thanks to the Cauchy formula:

\[
\psi(L) = \frac{1}{2i\pi} \int_{\gamma} (\zeta - L)^{-1} \psi(\zeta) d\zeta
\]

where \( \gamma \) is made of two rays \( re^{\pm i\nu}, r > 0, \omega < \nu < \mu \) and is described counterclockwise. If \( \psi \) is such a function and is not identically zero, and if one defines \( \psi_t(\zeta) = \psi(t\zeta) \), there exists \( c_\psi > 0 \) such that, for all \( f \in L^2(\Omega) \),

\[
c_\psi \|f\|_2 \leq \left( \int_0^{+\infty} \|\psi_t(L)f\|_2^2 \frac{dt}{t} \right)^{1/2} \leq c_\psi^{-1} \|f\|_2.
\]

See [23]. This remark applies to \( \psi(z) = -z^{1/2}e^{-z^{1/2}} \) and \( \psi_t(L) = (t\partial_t P_t) \).

The result of this section is a follows (see [4], Theorem 3, (iii), p. 145):

**Proposition 12.** Assume that \( L = (A, \Omega, V) \) satisfies (3).

(a) If \( \Omega = \mathbb{R}^n \), then there exists \( C > 0 \) such that, for all \( \phi \in BMO(\mathbb{R}^n) \), \( \|T\phi\|_{L^\infty(\mathbb{R}^n)} \leq C \|\phi\|_{BMO(\mathbb{R}^n)} \).

(b) If \( L \) satisfies DBC, then there exists \( C > 0 \) such that, for all \( \phi \in BMO_{z,a}(\Omega) \), \( \|T\phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{BMO_{z,a}(\Omega)} \).
(c) If $\Omega$ is unbounded and $L$ satisfies NBC, then there exists $C > 0$ such that, for all $\phi \in BMO_{CW}(\Omega)$, $\|T\phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{BMO_{CW}(\Omega)}$.

The proofs of the three assertions are similar, and we give the one of (b). Consider $\phi \in BMO_{z,a}(\Omega)$ with $\|\phi\|_{BMO_{z,a}(\Omega)} \leq 1$, $x \in \Omega$ and a cube $Q$ centered in $\Omega$ and containing $x$. Write

$$\phi = \phi_{2Q \cap \Omega} + (\phi - \phi_{2Q \cap \Omega}).$$

It is classical using the square function estimate for $\partial_t P_t$, the decay of the kernel of $\partial_t P_t$ and $BMO$ inequalities that

$$\frac{1}{|Q \cap \Omega|} \int_{T(Q)} |\partial_t P_t(\phi - \phi_{2Q \cap \Omega})(y)|^2 t dy dt \leq c \|\phi\|_{BMO_{CW}(\Omega)}^2$$

and since $BMO_{z,a}(\Omega) \subset BMO_{r,a}(\Omega) = BMO_{CW}(\Omega)$, $\|\phi\|_{BMO_{CW}(\Omega)} \leq c$.

It remains to control $I_Q = \frac{1}{|Q \cap \Omega|} \int_{T(Q)} |\partial_t P_t(\phi_{2Q \cap \Omega})(y)|^2 t dy dt$ by a constant which does not depend on $Q$.

We use the following lemma.

**Lemma 13.** Let $\phi \in BMO_{z,a}(\Omega)$ with $\|\phi\|_{BMO_{z,a}(\Omega)} \leq 1$ and $Q$ be a cube centered in $\Omega$. Then

$$|\phi_{Q \cap \Omega}| \leq C \sup(\ln \left(\frac{\delta}{r}\right), 1),$$

where $r$ is the radius of $Q$ and $\delta \geq 0$ its distance to the boundary of $\Omega$.

**Proof:** Notice first that, for any cube $Q$ of type (a), one has

$$|\phi_Q - \phi_{2Q}| \leq C.$$

Moreover, if $Q$ is of type (b), then $|\phi_Q| \leq 1$ by definition of $BMO_{z,a}(\Omega)$ and Hölder’s inequality.

Assume first that $Q$ is of type (a), and let $k$ be the smallest integer such that $2^k Q$ is of type (b). Then, one has

$$|\phi_Q| \leq \sum_{i=0}^{k-1} |\phi_{2^i Q} - \phi_{2^{i+1} Q}| + |\phi_{2^k Q}| \leq C(k + 1) \leq C' \ln \left(\frac{\delta}{r}\right).$$

Assume now that $4Q \cap \partial \Omega \neq \emptyset$. Take a Whitney decomposition of $Q \cap \Omega$ with respect to $\partial \Omega$,

$$Q \cap \Omega = \bigcup_k Q_k,$$

where, for each $k$, $Q_k$ is a type (b) cube. Therefore, one has

$$|\phi_{Q \cap \Omega}| \leq \sum_k \frac{|Q_k|}{|Q \cap \Omega|} |\phi_{Q_k}| \leq 1.$$
Let us come back to the proof of Proposition 12. Lemma 34 (see Appendix A) shows that, for all $y \in \mathbb{Q} \cap \Omega$,
\[
\left| \int_{\Omega} \partial_t p_t(z, y) dz \right| \leq C \left( 1 + \frac{\delta(y)}{t} \right)^{-1}
\]
where $\delta(y)$ is the distance from $y$ to the boundary of $\Omega$. It is then fairly easy using that $\Omega$ is strongly Lipschitz to show that
\[
\frac{1}{|Q \cap \Omega|} \int_{T(Q)} C \left( 1 + \frac{\delta(y)}{t} \right)^{-2} tdydt \leq C \inf \left( \frac{t^2}{\delta^2}, 1 \right).
\]
See [5], Lemma 29, for details in a related situation. This and Lemma 13 prove that $I_Q \leq C$. Assertion $(b)$ is complete.

For assertions $(a)$ and $(c)$, decompose $\phi$ as above and since $\partial_t P_t$ annihilates constants, only the first term arises. Proposition 12 is proved. ■

2.6 A weakly dense class

We shall need the following lemma (see [3], Lemme II.11):

**Lemma 14.** Assume that $L = (A, \Omega, V)$ satisfies $(G_\infty)$. For all function $f \in H^1_{max,L}(\Omega)$, there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of functions in $H^1_{max,L}(\Omega) \cap L^2(\Omega)$, such that, for all $\phi \in C_0(\Omega)$,
\[
\lim_{k \to +\infty} \langle f_k, \phi \rangle = \langle f, \phi \rangle
\]
and, for all $k \in \mathbb{N}$,
\[
\|f_k\|_{H^1_{max,L}(\Omega)} \leq C \|f\|_{H^1_{max,L}(\Omega)}.
\]

We briefly sketch the proof. Let $f \in H^1_{max,L}(\Omega)$. Define $f_k = P_{2^{-k}} f$. Then, the decay of the kernel of $P_{2^{-k}}$ and $f \in L^1(\Omega)$ imply $f_k \in L^2(\Omega)$. It is obvious that $\|f_k\|_{H^1_{max,L}(\Omega)} \leq \|f\|_{H^1_{max,L}(\Omega)}$. Using arguments analogous to [2], p. 776, which rely on the decay of the kernel of $P_{2^{-k}}$, one obtains the weak convergence. ■

2.7 Proof of Theorem 4

We begin this section with the proof of assertion $(a)$ in Theorem 4.

$H^1(\mathbb{R}^n) \subset H^1_{max,L}(\mathbb{R}^n)$: If $a$ is an $H^1(\mathbb{R}^n)$-atom, it is plain to see, using $(G_\infty)$, that $\|a^*_L\|_1 \leq C$ (see, for instance, [13]).
$H_{\max,L}^1(\mathbb{R}^n) \subset H^1(\mathbb{R}^n)$: Consider first $f \in H_{\max,L}^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Observe that, for all $z$ in a sector $\Gamma_\mu$ with $\mu \in ]\omega, \pi[$,

$$
\int_0^{+\infty} \left(-t z_1^{1/2} e^{-t z_1^{1/2}}\right) \left(-t z_1^{1/2} e^{-t z_1^{1/2}}\right) \frac{dt}{t} = \frac{1}{4}.
$$

As a consequence, one has

$$
\text{Id} = 4 \int_0^{+\infty} (t L^{1/2} P_t)(t L^{1/2} P_t) \frac{dt}{t},
$$

where the integral converges strongly in $L^2(\mathbb{R}^n)$. Note that $t L^{1/2} P_t = -t \partial_t P_t$. Thus, if $f \in L^2(\mathbb{R}^n)$ and $\phi$ is continuous with compact support in $\mathbb{R}^n$, one has

$$
\int_{\mathbb{R}^n} f(y) \overline{\phi(y)} dy = 4 \int_{\mathbb{R}^n} \int_0^{+\infty} (t \partial_t P_t)(f)(y)(t \partial_t P_t^*)(\phi)(y) \frac{dydt}{t}.
$$

We want to show that

$$
\left| \int_{\mathbb{R}^n} f(y) \overline{\phi(y)} dy \right| \leq C \|f\|_{H_{\max,L}^1(\mathbb{R}^n)} \|\phi\|_{BMO(\mathbb{R}^n)}.
$$

Well-known arguments from the theory of tent spaces (see [11, 30]) show that the right-hand side of (9) is bounded by

$$
C \|sf\|_{L^1(\mathbb{R}^n)} \|T\phi\|_{L^\infty(\mathbb{R}^n)},
$$

where $T\phi$ is defined in Section 2.5 with $L$ replaced by $L^*$ and

$$
sf(x) = \left( \int_{\Gamma_1(x)} t^{1-n} |\partial_t P_t f(y)|^2 dydt \right)^{1/2}.
$$

On the one hand, it is clear that $sf \leq S_1 f$ where $S_1$ is defined in Section 2.5 and Proposition 8 shows that

$$
\|S_1 f\|_{L^1(\mathbb{R}^n)} \leq C \|f\|_{H_{\max,L}^1(\mathbb{R}^n)}.
$$

On the other hand, assertion $(a)$ in Proposition 12 yields

$$
\|T\phi\|_{L^\infty(\mathbb{R}^n)} \leq C \|\phi\|_{BMO(\mathbb{R}^n)}.
$$

This ends the proof of (10).

Up to now, we have proved that, when $f \in L^2(\mathbb{R}^n) \cap H_{\max,L}^1(\mathbb{R}^n)$,

$$
\|f\|_{H^1(\mathbb{R}^n)} \leq C \|f\|_{H_{\max,L}^1(\mathbb{R}^n)}.
$$

Consider $f \in H_{\max,L}^1(\mathbb{R}^n)$. Let $(f_k)_{k \in \mathbb{N}}$ be a sequence given by Lemma 4. Then, for all $k \in \mathbb{N}$, since $f_k \in H_{\max,L}^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, one has

$$
\|f_k\|_{H^1(\mathbb{R}^n)} \leq C \|f_k\|_{H_{\max,L}^1(\mathbb{R}^n)} \leq C \|f\|_{H_{\max,L}^1(\mathbb{R}^n)}.
$$
Since \( H^1(\mathbb{R}^n) \) is the dual of \( VMO(\mathbb{R}^n) \), there exists a subsequence \( (f_{\phi(k)})_{k\in\mathbb{N}} \) and a function \( g \in H^1(\mathbb{R}^n) \) such that, for all \( \phi \in C_c(\mathbb{R}^n), \langle f_{\phi(k)}, \phi \rangle \to \langle g, \phi \rangle \). Since Lemma 14 shows that \( \langle f_{\phi(k)}, \phi \rangle \to \langle f, \phi \rangle \), one has \( f = g \). Therefore, \( f \in H^1(\mathbb{R}^n) \) and

\[
\|f\|_{H^1(\mathbb{R}^n)} \leq \lim \|f_{\phi(k)}\|_{H^1(\mathbb{R}^n)} \leq C \|f\|_{H^1_{max,L}(\mathbb{R}^n)}.
\]

We now turn to the proof of assertion \((b)\) in Theorem 1.

\( H^1_{r,a}(\Omega) \subset H^1_{max,L}(\Omega) \): Let \( a \) be an atom of type \((a)\) supported in a cube \( Q \). Then, using \((G_\infty)\), one sees that \( \|a\|_{H^1_{max,L}(\Omega)} \leq C \) (see [13]). If \( a \) is of type \((b)\), write that \( P_a(x) = \int_Q [p_t(x,y) - p_t(x,y_0)] a(y) \, dy \) where \( y_0 \) is a point on \( \partial \Omega \) such that \( |y - y_0| \sim d(y, \partial \Omega) \) whenever \( y \in \text{Supp} \, a \) (remember that \( p_t(x,y_0) = 0 \) since \( y_0 \in \partial \Omega \)) and use this representation and \((G_\infty)\) to show that \( \|a\|_{H^1_{max,L}(\Omega)} \leq C \).

\( H^1_{max,L}(\Omega) \subset H^1_{r,a}(\Omega) \): As in the case of \( \mathbb{R}^n \), consider first \( f \in H^1_{max,L}(\Omega) \cap L^2(\Omega) \). Arguing as before, one obtains that, if \( \phi \) is continuous with compact support, one has

\[
\int_\Omega f(y)\overline{\phi(y)} \, dy = 4 \int_\Omega \int_0^{+\infty} (t\partial_t P_t)(f)(y)(t\partial_t P_t^*)(\phi)(y) \, \frac{dydt}{t}.
\]

We want to show that

\[
\left| \int_\Omega f(y)\overline{\phi(y)} \, dy \right| \leq C \|f\|_{H^1_{max,L}(\Omega)} \|\phi\|_{BMO_{z,a}(\Omega)}.
\]

Use the theory of tent spaces again (see [13]) to obtain

\[
\left| \int_\Omega f(y)\overline{\phi(y)} \, dy \right| \leq C \|S_1 f\|_{L^1(\Omega)} \|T\phi\|_{L^\infty(\Omega)}.
\]

Proposition 8 shows that

\[
\|S_1 f\|_{L^1(\Omega)} \leq C \|f\|_{H^1_{max,L}(\Omega)},
\]

whereas assertion \((b)\) in Proposition 12 yields

\[
\|T\phi\|_{L^\infty(\Omega)} \leq C \|\phi\|_{BMO_{z,a}(\Omega)},
\]

which ends the proof of [12]. This inequality implies that there exists \( C > 0 \) such that, for all \( f \in H^1_{max,L}(\Omega) \cap L^2(\Omega) \),

\[
\|f\|_{H^1_{r,a}(\Omega)} \leq C \|f\|_{H^1_{max,L}(\Omega)}.
\]

Indeed, since \( BMO_z(\Omega) = BMO_{z,a}(\Omega) \) and \( H^1_{r,a}(\Omega) = H^1_{r}(\Omega) = (VMO_z(\Omega))^\prime \), we have that \( \|f\|_{H^1_{r,a}(\Omega)} \sim \sup \{ |\langle f, \phi \rangle|; \phi \in C_c(\Omega), \|\phi\|_{BMO_{z,a}(\Omega)} = 1 \} \) (see Section 2.3).
One gets rid of the condition $f \in L^2(\Omega)$ using Lemma 14. Indeed, if $f \in H^1_{\text{max}, L}(\Omega)$, there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of functions in $H^1_{\text{max}, L}(\Omega) \cap L^2(\Omega)$, such that, for all $\phi \in C_c(\Omega)$,
\[
\lim_{k \to +\infty} \langle f_k, \phi \rangle = \langle f, \phi \rangle
\] (14)
and, for all $k \in \mathbb{N}$,
\[
\| f_k \|_{H^1_{\text{max}, L}(\Omega)} \leq C \| f \|_{H^1_{\text{max}, L}(\Omega)}.
\] (15)
By (13) and (15) and $H^1_r(\Omega) = H^1_{r,a}(\Omega)$, the $f_k$’s are bounded in $H^1_r(\Omega)$. Since $H^1_r(\Omega)$ is the dual of $\text{VMO}_z(\Omega)$, there exists a subsequence $(f_{\phi(k)})$ which converges $\ast$-weakly to $g \in H^1_r(\Omega)$. Then, (14) implies that $f = g$. Moreover,
\[
\| f \|_{H^1_{r,a}(\Omega)} \sim \| f \|_{H^1_r(\Omega)} \leq C \| f \|_{H^1_{\text{max}, L}(\Omega)}.
\]

We are now left with the task of proving assertion $(c)$ in Theorem 4. We shall prove that $H^1_z(\Omega) \subset H^1_{\text{max}, L}(\Omega) \subset H^1_{CW}(\Omega)$ since $H^1_{CW}(\Omega) = H^1_{z,a}(\Omega) \subset H^1_z(\Omega)$ by Theorem 1.

$H^1_z(\Omega) \subset H^1_{\text{max}, L}(\Omega)$: Recall that $p_t(x, y)$ is the Poisson kernel for $L$. Recall that $(G_\infty)$ and the subordination formula imply that
\[
|p_t(x, y)| \leq Ct^{-n} \left(1 + \frac{|x - y|}{t}\right)^{-n-1}
\]
and
\[
|p_t(x, y) - p_t(x, y')| \leq Ct^{-n} \left(\frac{|y - y'|}{t}\right)^{\nu} \left(1 + \frac{|x - y|}{t}\right)^{-n-1-\nu} + \left(1 + \frac{|x - y'|}{t}\right)^{-n-1-\nu}
\]
for some $\nu \in (0, 1]$. For all $t > 0$ and $x \in \Omega$, define
\[
F_{x,t}(y) = t^n \left(1 + \frac{|x - y|}{t}\right)^{n+1} p_t(x, y).
\]
It is easy to show that
\[
|F_{x,t}(y)| \leq C
\]
and
\[
|F_{x,t}(y) - F_{x,t}(y')| \leq C \left(\frac{|y - y'|}{t}\right)^{\nu}
\]
for all $y, y' \in \Omega$. Thus, the function $F_{x,t}$ may be extended to a bounded Hölder continuous function on $\overline{\Omega}$, then on $\mathbb{R}^n$ (see [23], Chapter 6, p. 174, Theorem 3). If this extension is denoted by $\tilde{F}_{x,t}$, one has
\[
|\tilde{F}_{x,t}(y)| \leq C_0 C
\]
and
\[ |\widetilde{F}_{x,t}(y) - \widetilde{F}_{x,t}(y')| \leq C_0 C \left( \frac{|y - y'|}{t} \right)^\nu, \]
for all \( y, y' \in \mathbb{R}^n \), where \( C_0 \) only depends on \( \Omega \). Define now
\[ \tilde{p}_t(x,y) = t^{-n} \left(1 + \frac{|x - y|}{t}\right)^{-n-1} \tilde{F}_{x,t}(y). \]
Then, one has
\[ |\tilde{p}_t(x,y)| \leq C t^{-n} \left(1 + \frac{|x - y|}{t}\right)^{-n-1} \quad (16) \]
and
\[ |\tilde{p}_t(x,y) - \tilde{p}_t(x,y')| \leq C t^{-n} \left( \frac{|y - y'|}{t} \right)^\nu \quad (17) \]
for all \( x \in \Omega, y, y' \in \mathbb{R}^n \) and all \( t > 0 \). Moreover, for all \( t > 0 \) and all \( x, y \in \Omega \), \( p_t(x,y) = \tilde{p}_t(x,y) \).

Consider now a function \( f \in H^1_x(\Omega) \), extended by 0 outside \( \Omega \), so that \( f \in H^1(\mathbb{R}^n) \) and \( \|f\|_{H^1_x(\Omega)} = \|f\|_{H^1(\mathbb{R}^n)}. \) For all \( x \in \Omega \), one has
\[ \int_{\Omega} p_t(x,y) f(y) dy = \int_{\mathbb{R}^n} \tilde{p}_t(x,y) f(y) dy. \quad (18) \]
Using the atomic decomposition of \( f \) into \( H^1(\mathbb{R}^n) \)-atoms and the estimates (16) and (17) for \( \tilde{p}_t \), one easily deduces from (18) that
\[ \|f^\natural\|_{L^1(\Omega)} \leq C \|f\|_{H^1(\mathbb{R}^n)}. \]

This ends the proof of the inclusion \( H^1_x(\Omega) \subset H^1_{\text{max},L}(\Omega) \).

\( H^1_{\text{max},L}(\Omega) \subset H^1_{\text{CW}}(\Omega) \): Arguing as in the proof of \( H^1_{\text{max},L}(\Omega) \subset H^1_{r,a}(\Omega) \) under DBC, one proves that, for all \( f \in H^1_{\text{max},L}(\Omega) \cap L^2(\Omega) \) and all function \( \phi \) continuous and compactly supported in \( \Omega \),
\[ \left| \int_{\Omega} f(x) \overline{\phi(x)} dx \right| \leq C \|f\|_{H^1_{\text{max},L}(\Omega)} \|\phi\|_{BMO_{\text{CW}}(\Omega)}. \]
The proof uses the theory of tent spaces, Proposition 8 and assertion (c) of Proposition 12.

Then, one gets rid of the assumption \( f \in L^2(\Omega) \) as before, using Lemma 14 and the fact that \( H^1_{\text{CW}}(\Omega) \) is the dual space of \( VMO_{\text{CW}}(\Omega) \).
2.8 Some consequences

Assume the hypotheses of Theorem 4. We list some consequences of the proofs.

Each maximal Hardy space is characterized by the square functions $S_1 f$ and $sf$ being in $L^1$. Indeed, we have already seen that $\|sf\|_1 \leq \|S_1 f\|_1 \leq C \|f^*_L\|_1$ and the argument via tent spaces and Carleson measures of Section 2.7 shows in fact that $sf \in L^1$ implies $f$ is in an atomic space. As the atomic space is contained in a maximal space, we have a full circle of implications. See the forthcoming paper [1] where a general theory for Hardy spaces defined through square functions of type $sf$ associated to abstract operators $L$ is developed.

Each of the atomic BMO space, has a characterization in terms of Carleson measures. In other words, the converse to the inequalities of Proposition 12 between BMO norms and $\|T\phi\|_\infty$ hold (provided $\phi$ satisfies an integrability condition as in [14]). We leave to the reader the care of checking this.

2.9 “Easy” embeddings between Hardy spaces

We prove part of Theorem 1 for the global Hardy spaces.

Assertion (a1): It is proved in [1] (p. 305, proof of Theorem 2.7, (1) $\Rightarrow$ (2)) for local spaces and when $\Omega$ is bounded. We briefly recall the argument for completeness. Let $f \in H^1_*(\Omega)$ and $F \in H^1(\mathbb{R}^n)$ be an extension of $f$ satisfying $\|F\|_{H^1(\mathbb{R}^n)} \leq 2 \|f\|_{H^1_*(\Omega)}$. The function $F$ may be decomposed into

$$ F = \sum_Q \lambda_Q a_Q. $$

In this sum, we are only interested in the cubes $Q$ which intersect $\Omega$. If $4Q \subset \Omega$, consider $a_Q$ as a type (a) atom. If $2Q \subset \Omega$ and $4Q \cap \partial \Omega \neq \emptyset$, consider $a_Q$ as a type (b) atom. Finally, consider the case when $2Q \cap \partial \Omega \neq \emptyset$ and perform a Whitney decomposition of $Q \cap \Omega$ with respect to $\partial \Omega$:

$$ Q \cap \Omega = \bigcup_k Q_k $$

where each $Q_k$ is a type (b) cube and decompose

$$ a_Q 1_\Omega = \sum_k |Q_k|^{1/2} \|a_Q 1_{Q_k}\|_2 \frac{a_Q 1_{Q_k}}{|Q_k|^{1/2} \|a_Q 1_{Q_k}\|_2}. $$

Since each $Q_k$ is a type (b) cube, $\frac{a_Q 1_{Q_k}}{|Q_k|^{1/2} \|a_Q 1_{Q_k}\|_2}$ is a type (b) atom. Moreover,

$$ \sum_k |Q_k|^{1/2} \|a_Q 1_{Q_k}\|_2 \leq \left( \sum_k |Q_k| \right)^{1/2} \left( \sum_k \|a_Q 1_{Q_k}\|_2^2 \right)^{1/2} \leq |Q|^{1/2} \|a_Q\|_2 \leq 1. $$
Thus, we have obtained a decomposition of $f = F|_{\Omega}$ into $H^1_{r,a}(\Omega)$-atoms.

**Assertion (a2):** Let $a$ be an $H^1_{r,a}(\Omega)$-atom. If $a$ is of type (a), then its extension by 0 outside $\Omega$ is an $H^1(\mathbb{R}^n)$-atom. Hence $a \in H^1_{r}(\Omega)$.

If $a$ is of type (b), let $Q$ be a type (b) cube on which $a$ is supported. We use the following claim, whose proof is deferred to Appendix B:

**Claim:** There exists $\rho \in [0, +\infty[\,$, such that if $Q$ is a type (b) cube and $\ell(Q) < \rho$, there exists a cube $\tilde{Q} \subset ^c \Omega$ such that $|\tilde{Q}| \sim |Q|$ and the distance from $\tilde{Q}$ to $Q$ is comparable to $\ell(Q)$. Furthermore, $\rho = \infty$ if $\Omega$ is unbounded.

Define an extension of $a$ as follows: Let $A(x) = a(x)$ if $x \in Q$, $A(x) = -\frac{1}{|Q|} \int_Q a$ if $x \in \tilde{Q}$. Let $Q_0$ be the smallest cube in $\mathbb{R}^n$ containing $Q$ and $\tilde{Q}$. It is clear that $\text{Supp} A \subset Q_0$, $\|A\|_2 \leq C|Q_0|^{-1/2}$, that $\int A = 0$ and that $a$ is the restriction of $A$ to $\Omega$. Hence, $a \in H^1_{r}(\Omega)$

**Assertion (b1):** The inclusion $H^1_{z,a}(\Omega) \subset H^1_{CW}(\Omega)$ is obvious. We give the converse using an argument due to Lou and McIntosh.

Let $a$ be an $H^1_{CW}(\Omega)$-atom associated to a cube $Q$. We want to show that $a$ belongs to $H^1_{z,a}(\Omega)$. In fact, we are going to show that $a$ can be written as a sum of type (a) atoms.

If $a$ is supported on a type (a) cube, we do nothing. If not, $a$ is supported by $Q \cap \overline{\Omega}$ where $Q$ is a cube centered in $\Omega$ which is not of type (a). Since $a$ is square integrable with mean value 0 on the Lipschitz domain $Q \cap \Omega$, we can invoke the following corollary of a result by Nečas [25], Chapter 3, Lemma 7.1.

**Lemma 15.** Let $D$ be a bounded Lipschitz domain. The divergence operator is a (continuous) map from $H^1_0(D)^n$ onto $L^2(D) = \{ f \in L^2(D); \int_D f = 0 \}$: there exists $C > 0$ depending only on the Lipschitz constant of $D$ such that for all $f \in L^2_0(D)$, there exists $g \in H^1_0(D)^n$ such that $\text{div} g = f$ and $\int_D |\nabla g|^2 \leq C \int_D |f|^2$.

Indeed, Nečas proves that the gradient operator is one-one with closed range from $L^2_0(D)$ into $H^{-1}(D)^n$ with $\|\nabla f\|_{H^{-1}(D)^n} \geq C\|f\|_2$ and controls the constant.

Hence, pick $b \in H^1_0(Q \cap \Omega)^n$ with $a = \text{div} b$ and $\int |\nabla b|^2 \leq C \int |a|^2$, the constant $C$ depending only on the Lipschitz constant of $Q \cap \Omega$, henceforth only on $\Omega$. Extend $b$ by 0 outside of $Q \cap \Omega$.

Pick a Whitney decomposition $(Q_k)$ of $\Omega$ by cubes from a dyadic grid (of $\mathbb{R}^n$) containing $Q$ and so that $8Q_k \subset \Omega$. Again

$$Q \cap \Omega = \bigcup_{k \in K} Q_k$$

and

$$\sum_{k \in K} |Q_k| = |Q \cap \Omega|.$$

for some index set $K$. Let $(\eta_k)$ be a smooth partition of unity associated to this covering with $\eta_k$ supported in $2Q_k$ and $\|\eta_k\|_\infty \leq 1$ and $\|\nabla \eta_k\|_\infty \leq C(\ell(Q_k))^{-1}$. Write $a = \sum_{k \in K} \text{div}(\eta_k b)$ and set

$$a_k = \frac{\text{div}(\eta_k b)}{\|\text{div}(\eta_k b)\|_2 |2Q_k|^{1/2}} \quad \text{and} \quad \lambda_k = \|\text{div}(\eta_k b)\|_2 |2Q_k|^{1/2}$$

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whenever this number is not zero. Otherwise set $\lambda_k = 0$ and $a_k = 0$. It is clear from its construction that $a_k$ is a type (a) atom. It remains to show that $\sum \lambda_k \leq C$ independent of $Q$. By Cauchy-Schwarz inequality this sum does not exceed
\[
\left( \sum_{k \in K} 2^n |Q_k| \right)^{1/2} \left( \sum_{k \in K} \int_{2Q_k} |\text{div}(\eta_k b)|^2 \right)^{1/2}
\]
so it suffices to establish that the second term $\Sigma$ is controlled by $\|a\|_2 + \|\nabla b\|_2$.

To this end, write
\[
\Sigma \leq \left( \sum_{k \in K} \int_{2Q_k} |\eta_k \text{div}b|^2 \right)^{1/2} + \left( \sum_{k \in K} \int_{2Q_k} |\nabla \eta_k \cdot b|^2 \right)^{1/2}
\]
For the term containing $\text{div}b = a$ using the finite covering property of the cubes $2Q_k$ which leads to the bound $C\|a\|_2$. For the other term, observe that when $x \in 2Q_k$ then $d(x, \partial \Omega) \sim \ell(Q_k)$.

Hence, this and the finite overlap property of the cubes $2Q_k$ lead to
\[
\left( \sum_{k \in K} \int_{2Q_k} |\nabla \eta_k \cdot b|^2 \right)^{1/2} \leq C \left( \sum_{k \in K} \int_{2Q_k} \left| \frac{b(x)}{d(x, \partial \Omega)} \right|^2 \, dx \right)^{1/2} \leq C \left( \int_{\Omega} \left| \frac{b(x)}{d(x, \partial \Omega)} \right|^2 \, dx \right)^{1/2}
\]
Now use Hardy’s inequality (see, e.g., [13], Chapter 1, Section 5)
\[
\left( \int_{\Omega} \left| \frac{b(x)}{d(x, \partial \Omega)} \right|^2 \, dx \right)^{1/2} \leq \left( \int_{Q \cap \Omega} \left| \frac{b(x)}{d(x, \partial (Q \cap \Omega))} \right|^2 \, dx \right)^{1/2} \leq C \|\nabla b\|_2
\]
since $b \in H^1_0(Q \cap \Omega)$ and $Q \cap \Omega$ is strongly Lipschitz and bounded. Note that the constant $C$ in this inequality depends only on the domain $\Omega$ and not, in particular, on the size of $Q$ (by a scaling argument). This ends the proof of assertion (b1).

**Assertion (b2):** It is obvious that $H^1_{z,a}(\Omega) \subset H^1_z(\Omega)$.

**Remark 16.** Let us see that Theorem 4 (a2), is sharp. Let us take $\Omega = \mathbb{R} \setminus [0,1]$ (although $\Omega$ is not connected: the calculations are the same in any dimension). Let $a = -(2N)^{-1} X_{[N,3N]}$ where $N$ is a positive integer. Then $a$ is a type (b) atom. Suppose that the continuous embedding $H^1_{r,a}(\Omega) \subset H^1_r(\Omega)$ holds. Then $\|a\|_{H^1_r(\Omega)} \leq C$ uniformly with respect to $N$. Pick $A \in H^1(\mathbb{R})$ with norm not exceeding $C + 1$ and whose restriction to $\Omega$ is $a$. Since $\int_{\mathbb{R}} A = 0$ it must be that $\int_{[0,1]} A = 1$. Observe that
\[
\|A\|_{H^1(\mathbb{R})} \geq c \int_2^{N/4} \sup_{2 < t < N/2} |\phi_t * A(x)| \, dx
\]
because the maximal definition of $H^1(\mathbb{R})$ with $\phi$ smooth, supported in $[0,1]$ and $\phi(x) > 0$ for $x \in [0,1]$. Write for $x \in [2, N/4]$, and $t \in [2, N/2]$
\[
\phi_t * A(x) = \int_0^1 (\phi_t(x-y) - \phi_t(x)) A(y) \, dy + \phi_t(x)
\]
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Then, for all $x$ in that range, the support condition of $\phi$ implies that $\sup_{2^{t} < N/2} |\phi_t(x)| \geq C/x$ while $\sup_{2^{t} < N/2} \int_0^1 (\phi_t(x - y) - \phi_t(x)) A(y) \, dy \leq C/x^2$. We obtain therefore, $\|A\|_{H^1(\mathbb{R})} \geq C \ln N$, which is a contradiction.

### 2.10 Duality results and $BMO$ embeddings

We prove the parts of theorem 3 left aside in Section 2.3.

**Assertion (a):** It is plain to see that, if $\phi \in BMO_{z,a}(\Omega)$, $\phi$ defines a bounded linear functional $L$ on $H^1_{r,a}(\Omega)$ with $\|L\| \leq \|\phi\|_{BMO_{z,a}(\Omega)}$.

Consider now a bounded linear functional $L$ on $H^1_{r,a}(\Omega)$ and assume that $\|L\| = 1$. For all cube $Q \subset \Omega$ of type (a), define $L_0^2(Q) = \left\{ f \in L^2(Q) : \int_Q f(x) \, dx = 0 \right\}$. Then, for all $f \in L_0^2(Q)$, $\frac{f}{\|f\|_{L^2(Q)}^{1/2}}$ is a type (a) atom, so that $L$ defines a bounded linear functional on $L_0^2(Q)$. As a consequence, there exists $b_Q \in L_0^2(Q)$ such that, for all $f \in L_0^2(Q)$,

$$L f = \int_Q f(x) b_Q(x) \, dx.$$  

Moreover, $\|b_Q\|_2 \leq |Q|^{1/2}$. Similarly, if $Q$ is a type (b) cube, $L$ defines a bounded linear functional on $L^2(Q)$ and there exists $b_Q \in L^2(Q)$ with $\|b_Q\|_2 \leq |Q|^{1/2}$ such that, for all $f \in L^2(Q)$,

$$L f = \int_Q f(x) b_Q(x) \, dx.$$ 

Observe that, whenever $Q_1$ and $Q_2$ are type (b) cubes, $b_{Q_1}$ and $b_{Q_2}$ coincide on $Q_1 \cap Q_2$. Indeed, whenever $f \in L^2(Q_1) \cap L^2(Q_2)$, one has

$$L f = \int f(x) b_{Q_1}(x) \, dx = \int f(x) b_{Q_2}(x) \, dx.$$ 

Similarly, whenever $Q_1$ is a type (a) cube and $Q_2$ is a type (b) cube, $b_{Q_1} - b_{Q_2}$ is constant on $Q_1 \cap Q_2$. Indeed, whenever $f$ is supported in $Q_1 \cap Q_2$, $f \in L^2(Q_1) \cap L^2(Q_2)$ and has zero integral, one has

$$L f = \int f(x) b_{Q_1}(x) \, dx = \int f(x) b_{Q_2}(x) \, dx.$$ 

The key observation at this point is that for any $x \in \Omega$, there exists a type (b) cube that contains $x$ and one defines $b(x) = b_Q(x)$ where $Q$ is any such cube. This definition is consistent because of the previous remark.

Consider now a type (a) cube $Q$. Then, $Q$ is contained in a type (b) cube, hence there exists $c_Q \in \mathbb{C}$ such that $b = b_Q + c_Q$ on $Q$. One has

$$\frac{1}{|Q|} \int_Q |b(x) - c_Q|^2 \, dx = \frac{1}{|Q|} \int_Q |b_Q(x)|^2 \, dx \leq 1.$$
If $Q$ is a type $(b)$ cube,
\[
\frac{1}{|Q|} \int_Q |b(x)|^2 \, dx = \frac{1}{|Q|} \int_Q |b_Q(x)|^2 \, dx \leq 1.
\]

Hence, $\|b\|_{BMO_z,\omega}(\Omega) \leq 1$. One easily checks that
\[
\mathcal{L} f = \int f(x) b(x) \, dx
\]
whenever $f$ is a finite linear combination of atoms of type $(a)$ or $(b)$ in $H^1_{r,a}(\Omega)$.

**Assertion (b):** Let $\phi$ be a function in $BMO_{z}(\Omega)$. Denote by $D(\mathbb{R}^n)$ the vector space generated by $H^1(\mathbb{R}^n)$-atoms and by $D_r(\Omega)$ the space of restrictions to $\Omega$ of functions in $D(\mathbb{R}^n)$. By definition and density of $D(\mathbb{R}^n)$ in $H^1(\mathbb{R}^n)$, one has that $D_r(\Omega)$ is dense in $H^1_r(\Omega)$ and for $f \in D_r(\Omega)$, $\|f\|_{H^1_r(\Omega)} = \inf \|F\|_{H^1(\mathbb{R}^n)}$ where the infimum is taken over all $F \in D(\mathbb{R}^n)$ which coincide with $f$ on $\Omega$. For $f \in D_r(\Omega)$, define
\[
\mathcal{L} f = \int \Omega f(x) \phi(x) \, dx.
\]

Then, for any function $F \in D(\mathbb{R}^n)$ which coincides with $f$ on $\Omega$, one has
\[
|\mathcal{L} f| = \left| \int \mathbb{R}^n F(x) \phi(x) \, dx \right| \leq \|F\|_{H^1(\mathbb{R}^n)} \|\phi\|_{BMO_{z}(\Omega)},
\]
which shows that
\[
|\mathcal{L} f| \leq \|f\|_{H^1_r(\Omega)} \|\phi\|_{BMO_{z}(\Omega)}.
\]

Hence the dual of $H^1_r(\Omega)$ contains $BMO_{z}(\Omega)$.

Conversely, let $\mathcal{L}$ be a bounded linear functional in $H^1_r(\Omega)$. For all $f \in H^1(\mathbb{R}^n)$, define
\[
\tilde{\mathcal{L}}(f) = \mathcal{L}(f|_{\Omega}).
\]

The definition of the norm in $H^1_r(\Omega)$ shows that $\tilde{\mathcal{L}}$ is a bounded linear functional on $H^1(\mathbb{R}^n)$: for all $f \in H^1(\mathbb{R}^n)$,
\[
|\tilde{\mathcal{L}}(f)| = |\mathcal{L}(f|_{\Omega})| \leq \|\mathcal{L}\| \|f|_{\Omega}\|_{H^1_r(\Omega)} \leq \|\mathcal{L}\| \|f\|_{H^1(\mathbb{R}^n)}.
\]

Therefore, $\|\tilde{\mathcal{L}}\| \leq \|\mathcal{L}\|$. Since $BMO(\mathbb{R}^n)$ is the dual of $H^1(\mathbb{R}^n)$, there exists $\phi \in BMO(\mathbb{R}^n)$ such that, for any finite linear combination of $H^1(\mathbb{R}^n)$-atoms,
\[
\tilde{\mathcal{L}}(f) = \int f(x) \phi(x) \, dx.
\]
As a consequence,
\[
\|\phi\|_{BMO(\mathbb{R}^n)} \leq \|\mathcal{L}\|.
\]
Observe that, if \(f\) is any \(H^1(\mathbb{R}^n)\)-atom supported outside \(\Omega\), \(\tilde{L}(f) = 0\), which shows that \(\phi\) is constant on each connected component of \(\partial\Omega\). Fix two such components \(C \neq C'\). We let \(c, c'\) be the value of \(\phi\) on \(C, C'\) respectively. Let \(Q\) and \(Q'\) be two cubes of same size respectively contained in \(C\) and \(C'\). Define \(a(x) = 1, x \in Q\) and \(a(x) = -1, x \in Q'\). Then, \(a\) is a multiple of an \(H^1(\mathbb{R}^n)\)-atom with support contained in the smallest cube of \(\mathbb{R}^n\) containing \(Q\) and \(Q'\). Since its support is contained outside of \(\Omega\), we have \(\tilde{L}(a) = 0\). By construction of \(a\), we have
\[
\tilde{L}(a) = c|Q| - c'|Q'|.
\]
Hence \(c = c'\) and \(\phi = c\) outside \(\Omega\).

Let \(\tilde{\phi} = \phi - c\). Clearly, \(\tilde{\phi}|_{\Omega} \in BMO_{z}(\Omega)\). If \(f = F|_{\Omega}\) with \(F \in D(\mathbb{R}^n)\), one has
\[
L(f) = \tilde{L}(F) = \int_{\mathbb{R}^n} F(x)\tilde{\phi}(x)dx = \int_{\Omega} f(x)\tilde{\phi}(x)dx.
\]
This proves \((H^1_{r}(\Omega))' \subset BMO_{z}(\Omega)\).

We now show that the dual of \(VMO_{z}(\Omega)\) is \(H^1_{f}(\Omega)\). This is a consequence of the following Banach space principle. If \(X\) is a Banach space and \(Y\) is a closed subspace of \(X\), then \(Y'\) is isometric to \(X'/Y^\perp\), where \(Y^\perp = \{L \in X' : L(y) = 0 \forall y \in Y\}\). Here, we have \(X = VMO(\mathbb{R}^n)\) and \(Y = VMO_{z}(\Omega)\).

Assertion \((c)\): is in [12] with minor changes due to our modification of definition.

Assertion \((d)\): We apply the above abstract principle with \(X = H^1(\mathbb{R}^n)\) and \(Y = H^1_{f}(\Omega)\).

Assertion \((e)\): That \((H^1_{r,a}(\Omega))' \subset BMO_{z,a}(\Omega)\) is straightforward. The converse is already observed in Section 2.3.

3 Local Hardy and \(BMO\) spaces on strongly Lipschitz domains

We now give localized versions of the previous results.

3.1 Local Hardy spaces

We first recall the definition of \(h^1(\mathbb{R}^n)\) and its atomic decomposition from [17].

Definition of \(h^1(\mathbb{R}^n)\): Let \(\phi \in S(\mathbb{R}^n)\) be a function such that \(\int_{\mathbb{R}^n} \phi(x)dx = 1\). For all \(t > 0\), define \(\phi_t(x) = t^{-n}\phi(x/t)\). A locally integrable function \(f\) on \(\mathbb{R}^n\) is said to be in \(h^1(\mathbb{R}^n)\) if the maximal function
\[
m_f(x) = \sup_{0 < t < 1} |\phi_t * f(x)|
\]
belongs to $L^1(\mathbb{R}^n)$. If it is the case, define
\[ \|f\|_{h^1(\mathbb{R}^n)} = \|mf\|_1. \]
One has $H^1(\mathbb{R}^n) \subset h^1(\mathbb{R}^n)$. It should be noted that a function in $h^1(\mathbb{R}^n)$ does not necessarily have zero integral. We note that other maximal functions $\sup_{0 < t < \delta} |\phi_t \ast f(x)|$ with $\delta > 0$ would lead to an equivalent norm.

Replacing $t > 0$ by $0 < t < 1$ in (1), one obtains a characterization of $h^1(\mathbb{R}^n)$ in terms of a non tangential maximal function associated with the heat or the Poisson semigroup generated by $\Delta$ (see [17]).

**Atomic decomposition of $h^1(\mathbb{R}^n)$:** A function $a$ is an $h^1(\mathbb{R}^n)$-atom if it is supported in a cube $Q$, satisfies $\|a\|_2 \leq |Q|^{-1/2}$ and has mean-value zero if $\ell(Q) < 1$. Then, $f \in h^1(\mathbb{R}^n)$ if and only if $f = \sum Q \lambda_Q a_Q$, where the $a_Q$’s are $h^1(\mathbb{R}^n)$-atoms and $\sum_Q |\lambda_Q| < \infty$. Moreover, $\|f\|_{h^1(\mathbb{R}^n)}$ is comparable with the infimum of $\sum_Q |\lambda_Q|$ taken over all such decompositions.

We now turn to local Hardy spaces on $\Omega$. As for global spaces, there are three categories of local Hardy spaces on $\Omega$ may be considered. The first category are restriction spaces.

**Definition of $h^1_r(\Omega)$ and $h^1_z(\Omega)$:** The spaces $h^1_r(\Omega)$ and $h^1_z(\Omega)$ are defined in the same way as $H^1_r(\Omega)$ and $H^1_z(\Omega)$, replacing $H^1(\mathbb{R}^n)$ by $h^1(\mathbb{R}^n)$. Observe that $h^1_z(\Omega)$ is a strict subspace of $h^1_r(\Omega)$ (see [8], Proposition 6.4).

The second category is made up of atomic spaces. Here, we adopt definitions different from [2] and [8]. We feel they are more natural ones.

**Definition of type (a) and type (b) local cubes:** Let $\delta > 0$. A cube $Q$ is a type (a) local cube if $\ell(Q) < \delta$ and $4Q \subset \Omega$, a type (b$_{far}$) local cube if $\ell(Q) \geq \delta$ and $4Q \subset \Omega$, and a type (b$_{close}$) local cube if $2Q \subset \Omega$ and $4Q \cap \partial \Omega \neq \emptyset$. The type (b) local cubes are those of type (b$_{far}$) or (b$_{close}$). We always arrange $\delta$ so that the class of type (b$_{far}$) local cubes is not empty. To simplify the exposition, we fix $\delta = 1$.

**Definition of type (a) and type (b) local atoms:** A measurable function $a$ on $\Omega$ is called a type (a) local atom if it is supported in a type (a) local cube $Q$ with
\[ \|a\|_2 \leq |Q|^{-1/2} \text{ and } \int a(x) dx = 0. \]
A measurable function $a$ on $\Omega$ is called a type (b$_{far}$) (resp. (b$_{close}$)) local atom if it is supported in a type (b$_{far}$) (resp. (b$_{close}$)) local cube $Q$ with
\[ \|a\|_2 \leq |Q|^{-1/2}. \]
Note that type (b) local atoms do not have mean value zero.
Definition of $h^1_{r,a}(\Omega)$: A function $f$ defined on $\Omega$ belongs to $h^1_{r,a}(\Omega)$ if

$$f = \sum_{(a)} \lambda_Q a_Q + \sum_{(b)} \mu_Q b_Q$$

where the $a_Q$'s are type $(a)$ local atoms, the $b_Q$'s are type $(b)$ local atoms and $\sum_{(a)} |\lambda_Q| + \sum_{(b)} |\mu_Q| < +\infty$. Define $\|f\|_{h^1_{r,a}}$ as the infimum of $\sum_{(a)} |\lambda_Q| + \sum_{(b)} |\mu_Q|$ over all such decompositions.

Definition of $h^1_{z,a}(\Omega)$: A function $f$ defined on $\Omega$ belongs to $h^1_{z,a}(\Omega)$ if

$$f = \sum_{(a)} \lambda_Q a_Q + \sum_{(b_{far})} \mu_Q b_Q$$

where the $a_Q$'s are type $(a)$ local atoms, the $b_Q$'s are type $(b_{far})$ local atoms and $\sum_{(a)} |\lambda_Q| + \sum_{(b_{far})} |\mu_Q| < +\infty$. Define $\|f\|_{h^1_{z,a}}$ as the infimum of $\sum_{(a)} |\lambda_Q| + \sum_{(b_{far})} |\mu_Q|$ over all such decompositions.

Definition of $h^1_{CW}(\Omega)$: An $h^1_{CW}(\Omega)$-atom is a function $a$ supported in $Q \cap \Omega$, where $Q$ is a cube centered in $\Omega$ (but not necessarily included in $\Omega$) with

$$\|a\|_2 \leq \frac{1}{|Q \cap \Omega|^{1/2}}, \quad \text{and} \quad \ell(Q) < 1, \int a(x)dx = 0.$$  

A function $f$ is in $h^1_{CW}(\Omega)$ if it can be written

$$f = \sum_{Q} \lambda_Q a_Q,$$

where the $a_Q$'s are $h^1_{CW}(\Omega)$-atoms and $\sum_{Q} |\lambda_Q| < \infty$. The norm is defined as usual.

Remark 17. Each global Hardy space is contained in the corresponding local space. Also, $h^1_{z,a}(\Omega)$ and $h^1_{z}(\Omega)$ are respectively strict subspaces of $h^1_{r,a}(\Omega)$ and $h^1_{r}(\Omega)$. If $\Omega$ is bounded, one can see that $h^1_{r,a}(\Omega) = H^1_{r,a}(\Omega)$, $h^1_{r}(\Omega) = H^1_{r}(\Omega)$ and that $h^1_{CW}(\Omega) = H^1_{CW}(\Omega) + \mathbb{C}\chi_{\Omega}$, $h^1_{z,a}(\Omega) = H^1_{z,a}(\Omega) + \mathbb{C}\chi_{\Omega}$, and $h^1_{z}(\Omega) = H^1_{z}(\Omega) + \mathbb{C}\chi_{\Omega}$. Here $\chi_{\Omega}$ is the indicator function of $\Omega$. All these facts but the inclusion $h^1_{z}(\Omega) \subset H^1_{z}(\Omega) + \mathbb{C}\chi_{\Omega}$ are easy to prove. For the latter one uses the following observation using maximal characterizations: if $f \in h^1(\mathbb{R}^n)$ has compact support and vanishing mean, then $f \in H^1(\mathbb{R}^n)$. Details are left to the reader.

These local Hardy spaces compare as follows.

Theorem 18. 

(a) $h^1_{r}(\Omega) = h^1_{r,a}(\Omega)$.

(b) $h^1_{z,a}(\Omega) = h^1_{CW}(\Omega)$.  

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Note that (a) holds with no restriction on $\Omega$ while it is not true for global Hardy spaces. We admit this result for the moment but the inclusion $h^1_\omega(\Omega) \subset h^1_{z,a}(\Omega)$, which will be seen in the course of proving the next theorem.

Finally, we consider the third category of local Hardy spaces.

**Definition of $h^1_{\max,L}(\Omega)$:** If $L = (A, \Omega, V)$ is a second order elliptic operator in divergence form and if $f \in L^1_{\text{loc}}(\Omega)$ with slow growth, we say that $f \in h^1_{\max,L}(\Omega)$ if

$$f^*_{\text{loc},L}(x) = \sup_{|y-x|<t \leq 1} |e^{-tL^{1/2}} f(y)| \in L^1(\Omega).$$

Define

$$\|f\|_{h^1_{\max,L}} = \left\| f^*_{\text{loc},L} \right\|_1.$$

It is evident that $H^1_{\max,L}(\Omega) \subset h^1_{\max,L}(\Omega)$.

The local version of Theorem 4 is as follows.

**Theorem 19.** Let $\Omega = \mathbb{R}^n$ or $\Omega$ be a strongly Lipschitz domain, and $L = (A, \Omega, V)$ satisfying $(G_\infty)$.

(a) One has $h^1(\mathbb{R}^n) = h^1_{\max,L}(\mathbb{R}^n)$.

(b) Assume that $L$ satisfies the DBC. Then, one has $h^1_{r,a}(\Omega) = h^1_{\max,L}(\Omega) = h^1_{r}(\Omega)$.

(c) Assume that $L$ satisfies the NBC. Then, one has $h^1_{z,a}(\Omega) = h^1_{\max,L}(\Omega) = h^1_{z}(\Omega)$.

**Proposition 20.** For a bounded Lipschitz domain, statements (b) and (c) in Theorem 19 hold for $L = (A, \Omega, V)$ satisfying $(G_1)$.

This result applies when the coefficients of $A$ are complex-valued BUC functions or in the closure of BUC in bmo (See [4]).

**Remark 21.** When $\Omega$ is bounded and $L$ satisfies $(G_1)$, $h^1_{\max,L}(\Omega) = H^1_{\max,L}(\Omega)$. Indeed, one inclusion holds. For the converse, consider $x \in \Omega$, $t > 1$ and $y \in \Omega$ satisfying $|y-x| < t$. Lemma 21 in Appendix A yields

$$|P_t f(y)| \leq \int_{\Omega} \frac{Ct}{t+|y-z|} |f(z)| \, dz \leq C \|f\|_1.$$

As a consequence, for all $x \in \Omega$,

$$|f^*_{L}(x)| \leq C \left( \| f^*_{\text{loc},L}(x) \| + \| f \|_1 \right)$$

and

$$\| f^*_{L} \|_1 \leq C' \left( \| f^*_{\text{loc},L} \|_1 + \| f \|_1 \right).$$
The strategy to prove Theorem 19 and Proposition 20 is essentially the same as for the global spaces: we need a few local bmo-spaces and some duality results, comparison between maximal functions and area functionals, and the theory of tent spaces.

**Remark 22.** Assertion (c) in Proposition 21 applies to the Neumann Laplacian on a bounded $\Omega$. Together with Remark 17 this completes the proof of Theorem 1, (b2).

### 3.2 bmo spaces

A locally square-integrable function $f$ on $\mathbb{R}^n$ is said to be in $bmo(\mathbb{R}^n)$ if

$$\|\phi\|_{bmo(\mathbb{R}^n)}^2 = \sup \left( \sup_{\ell(Q) \leq 1} \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^2 \, dx, \sup_{\ell(Q) > 1} \frac{1}{|Q|} \int_Q |\phi(x)|^2 \, dx \right) < +\infty.$$ 

Define $vmo(\mathbb{R}^n)$ as the closure of $C_c(\mathbb{R}^n)$ in $bmo(\mathbb{R}^n)$. It is well-known that $bmo(\mathbb{R}^n)$ is the dual of $h^1(\mathbb{R}^n)$, which is the dual of $vmo(\mathbb{R}^n)$ [17].

Define $bmo_z(\Omega)$, $vmo_z(\Omega)$ and $bmo_r(\Omega)$ analogously to the corresponding global $BMO$ or $VMO$ spaces, replacing $BMO(\mathbb{R}^n)$ by $bmo(\mathbb{R}^n)$ and $VMO(\mathbb{R}^n)$ by $vmo(\mathbb{R}^n)$.

A locally square-integrable function $f$ on $\Omega$ is in $bmo_{z,a}(\Omega)$ if

$$\|\phi\|_{bmo_{z,a}(\Omega)}^2 = \sup \left( \left( \sup_{(a)} \left( \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^2 \, dx, \sup_{(b)} \frac{1}{|Q|} \int_Q |\phi(x)|^2 \, dx \right) \right) < +\infty,$$

where $sup$ (resp. $sup$) means that the supremum is taken over all type $(a)$ (resp. $(b)$) local cubes.

A locally square-integrable function $f$ on $\Omega$ is in $bmo_{r,a}(\Omega)$ if

$$\|\phi\|_{bmo_{r,a}(\Omega)}^2 = \sup \left( \left( \sup_{(a)} \left( \frac{1}{|Q|} \int_Q |\phi(x) - \phi_Q|^2 \, dx, \sup_{(b)} \frac{1}{|Q|} \int_Q |\phi(x)|^2 \, dx \right) \right) < +\infty.$$

A locally square-integrable function $\phi$ defined on $\Omega$ is in $bmo_{CW}(\Omega)$ if

$$\|\phi\|_{bmo_{CW}(\Omega)}^2 = \sup \left( \left( \sup_{(a)} \left( \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} |\phi(x) - \phi_{Q \cap \Omega}|^2 \, dx, \sup_{(b)} \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} |\phi(x)|^2 \, dx \right) \right) < +\infty,$$

where the cubes have center in $\Omega$. The space $vmo_{CW}(\Omega)$ is defined as the closure of $C_c(\Omega)$ in $bmo_{CW}(\Omega)$.

The duality results for local spaces and the comparisons between $bmo$ spaces are the same as for the global spaces. Let us state them for completeness.

**Theorem 23.** (a) The dual of $h^1_{r,a}(\Omega)$ is $bmo_{z,a}(\Omega)$.

(b) The dual of $h^1_r(\Omega)$ is $bmo_z(\Omega)$, the dual of $vmo_z(\Omega)$ is $h^1_r(\Omega)$. 

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(c) The dual of $h^1_{CW}(\Omega)$ is $bmo_{CW}(\Omega)$, the dual of $vmo_{CW}(\Omega)$ is $h^1_{CW}(\Omega)$.

(d) The dual of $h^1_2(\Omega)$ is $bmo_r(\Omega)$.

(e) The dual of $h^1_{z,a}(\Omega)$ is $bmo_{r,a}(\Omega)$.

Theorem 24. (a) $bmo_{z,a}(\Omega) = bmo_z(\Omega)$.

(b1) $bmo_{CW}(\Omega) = bmo_{r,a}(\Omega)$.

(b2) $bmo_r(\Omega) = bmo_{r,a}(\Omega)$.

Again, the difficult parts are $(h^1_{z,a}(\Omega))^\prime \subset bmo_{r,a}(\Omega)$ and $bmo_{r,a}(\Omega) \subset bmo_r(\Omega)$, which are proved using Theorem 23 (c) and Theorem 24 (b1).

3.3 Proofs of equalities between local Hardy spaces

Proof of Theorem 19. In each case, the most involved part is to imbed our maximal space into an atomic space. We concentrate on this.

One has the local statements corresponding to the results in Sections 2.4 and 2.5, in which the square functions and the Carleson measures are truncated at some fixed time $t < t_0$, say for example $t = 1$. Except for some technical adjustments the proofs are the same and left to the reader.

The idea is to use this in the representation formula $f = f_1 + f_2$ where

$$f_1 = 4 \int_0^1 (tL^{1/2}P_1)(tL^{1/2}P_t f) \frac{dt}{t}$$

and $f_2 = \frac{1}{4}(2L^{1/2} + I)P_1P_1 f$ for $f \in h^1_{max,L}(\Omega) \cap L^2(\Omega)$.

For $f_1$, we proceed using the tent spaces again and then eliminate the requirement that $f \in L^2(\Omega)$ to obtain $f_1 \in h^1(\mathbb{R}^n)$ or $h^1_{r,a}(\Omega)$ under DBC or $h^1_{CW}(\Omega)$ under NBC.

Let us consider $f_2$. The idea is to prove that $f_2 \in h^1_{CW}(\Omega)$ in each case. Indeed, when $\Omega = \mathbb{R}^n$ we have $h^1_{CW}(\mathbb{R}^n) = h^1(\mathbb{R}^n)$, under DBC $h^1_{CW}(\Omega) \subset h^1_{r,a}(\Omega)$ from the definitions of these spaces and easy arguments, and under NBC, $h^1_{CW}(\Omega) = h^1_{z,a}(\Omega)$ from Theorem 18 (b1).

Here is the argument. Assume first $\Omega = \mathbb{R}^n$. Observe that $P_1 f = g$ is bounded by $f_{loc,L}^*$ which is in $L^1(\mathbb{R}^n)$. Also since $L^{1/2}P_1 = -\partial_t P_t |_{t=1}$ the subordination formula yields that the kernel $K(x,y)$ of $(2L^{1/2} + I)P_1$ is bounded by $ck(x,y)$ with $k(x,y) = (1 + |x - y|)^{-n-1}$.

Take $(Q_k)$ be a covering of $\mathbb{R}^n$ by cubes with size 1 obtained by translation from the unit cube $[0,1]^n$. Let $(\eta_k)$ be a smooth partition of unity associated with this covering so that $\eta_k$ is supported in $2Q_k$. Then one has

$$f_2(x) = \sum_k b_k(x)$$

If $\Omega$ is bounded and $V = W^{1,2}(\Omega)$ then the formula holds if $\int_\Omega f = 0$. If the mean of $f$ is not zero, then it applies to $\tilde{f} = f - c\chi_\Omega$ with the constant $c$ so that the mean of $\tilde{f}$ is zero. Conclude with $\chi_\Omega \in h^1_{CW}(\Omega)$.  

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where
\[ b_k(x) = \eta_k(x) \int_{\mathbb{R}^n} K(x, y)g(y) \, dy \]

Observe that \( b_k \) is supported in \( 2Q_k \). Set \( \lambda_k = |2Q_k|^{1/2} \left( \int |b_k|^2 \right)^{1/2} \). We have
\[
\lambda_k \leq c|2Q_k|^{1/2} \left( \int_{2Q_k} \int_{\mathbb{R}^n} k(x, y)|g(y)| \, dy \, dx \right)^{1/2}.
\]

Observe that \( k(x, y) \leq c \inf_{|x-z| \leq 2} k(z, y) \) for all \( x, y \in \mathbb{R}^n \). For \( x \in 2Q_k \), since \( \ell(Q_k) \leq 2 \), \( k(x, y) \leq c \inf_{z \in 2Q_k} k(z, y) \) for all \( y \). Hence
\[
\lambda_k \leq c|2Q_k| \int_{\mathbb{R}^n} \inf_{z \in 2Q_k} k(z, y)|g(y)| \, dy \leq \int \int \chi_{2Q_k}(x)k(x, y)|g(y)| \, dy \, dx
\]

and it follows from the finite overlap property of the family \((2Q_k)\) that
\[
\sum \lambda_k \leq c \int \int k(x, y)|g(y)| \, dy \, dx \leq c\|g\|_1 \leq c\|f\|_{h^{1}_{max,L}(\mathbb{R}^n)}.
\]

If we set \( a_k = \lambda_k^{-1}b_k \) when \( \lambda_k \neq 0 \), then \( a_k \) is an \( h^1(\mathbb{R}^n) \)-atom. Thus, \( f_2 \in h^1(\mathbb{R}^n) \) with \( \|f_2\|_{h^1(\mathbb{R}^n)} \leq c\|f\|_{h^{1}_{max,L}(\mathbb{R}^n)} \).

Assume now that \( \Omega \) is strongly Lipschitz and \( L \) satisfies either boundary condition. Let \((Q_k)\) be a covering of \( \Omega \) with cubes of \( \mathbb{R}^n \) such that \( \ell(Q_k) = 1 \). We keep only those cubes which intersect \( \Omega \). If \( Q_k \) has center in \( \Omega \), we set \( \lambda_k = |2Q_k \cap \Omega|^{1/2} \left( \int |b_k|^2 \right)^{1/2} \). If \( Q_k \) has center outside of \( \Omega \), then we replace \( Q_k \) by \( \tilde{Q}_k \) with center in \( Q_k \cap \Omega \) and \( \ell(Q_k) = 2\ell(Q_k) \) and define \( \lambda_k = |2\tilde{Q}_k \cap \Omega|^{1/2} \left( \int |b_k|^2 \right)^{1/2} \). With these modifications of \( \lambda_k \), we see from the same argument that \( a_k \) is an \( h^1_{CW}(\Omega) \)-atom and that \( \sum |\lambda_k| \leq c\|f\|_{h^{1}_{max,L}(\Omega)} \) remarking that all integrals should take place on \( \Omega \).

**Proof of Proposition 20.** We want to relax the condition \((C_\infty)\) to \((G_1)\) when \( \Omega \) is bounded. The same arguments works once we make sure of small time decay estimates for the Poisson kernel. This is proved in Appendix A.

**Proof of Theorem 18.** The proof of \((a)\) is as in the global case and is skipped.

The proofs of \( h^1_{z,a}(\Omega) \subset h^1_{CW}(\Omega) \) and \( h^1_{z,a}(\Omega) \subset h^1_2(\Omega) \) are straightforward from the definitions.

It remains to prove \( h^1_{CW}(\Omega) \subset h^1_{z,a}(\Omega) \). Because of Remark 17 and the global case, this is already known if \( \Omega \) is bounded. We assume next that \( \Omega \) is unbounded. Let \( a \) be an \( h^1_{CW} \)-atom supported on a cube \( Q \) centered in \( \Omega \). Since \( \Omega \) is unbounded, we have \( |Q \cap \Omega| \sim |Q| \) (see Appendix B).

If \( \ell(Q) < 1 \), we proceed as in the global case. Either \( a \) is a type \((a)\) local atom or can be decomposed into a sum of type \((a)\) local atoms.

Assume \( \ell(Q) \geq 1 \). If \( a \) is a type \((b_{far})\) local atom, we are done. It remains to argue when \( Q \) is close to the boundary, ie \( 4Q \cap \partial \Omega \neq \emptyset \). In this case, we claim there exists a type \((b_{far})\)
local cube $Q'$ with $\ell(Q') = \ell(Q)$ and the distance between $Q$ and $Q'$ is comparable to $\ell(Q)$ (See Appendix B). Define $\tilde{a} = a - (1/|Q'|) \int_{Q'\cap \Omega} a|_{Q'\cap \Omega}$. Then $\tilde{a}$ is a multiple of $H_{CW}^1(\Omega)$-atom (supported in $cQ \cap \Omega$ for some constant $c$ that does not depend on $Q$), thus $\tilde{a} \in H_{z,a}^1 \subset h_{z,a}^1$. Now, $\tilde{a} - a$ clearly is a type $(b_{far})$ local atom, hence it belongs to $h_{z,a}^1(\Omega)$.

4 Other maximal functions

As a consequence of the atomic decomposition for the maximal space, one may use other maximal functions, such as the vertical and the non-tangential maximal functions associated with $e^{-tL}$. More precisely, the following holds:

**Theorem 25.** Let $L = (A, \Omega, V)$. Assume that $\Omega = \mathbb{R}^n$ or that $\Omega$ be a strongly Lipschitz domain of $\mathbb{R}^n$ under DBC with $\Omega$ unbounded or under NBC. Assume also that $L$ and $L^*$ satisfy $(G_\infty)$. The following are equivalent:

\begin{align}
\sup_{t>0} |e^{-tL} f(x)| &\in L^1(\Omega), \quad \text{(19)} \\
\sup_{|x-y|<\sqrt{t}} |e^{-tL} f(y)| &\in L^1(\Omega), \quad \text{(20)} \\
f &\in H_{1,1}^1(\Omega), \quad \text{(21)}
\end{align}

One also has the analogous local statement, replacing $H_{1,1}^1$ by $h_{1,1}^1$ and $t > 0$ by $0 < t < t_0$ for any $t_0 > 0$ without restriction on $\Omega$. Moreover, if $\Omega$ is bounded then $(G_1)$ suffices.

The new assumption that $L^*$ satisfies $(G_\tau)$ simply means that $\text{(4)}$ holds for $K_t(y,x)$ too. Again, when $L$ is real, this is not a supplementary hypothesis.

We write the proof for global spaces, under DBC when $\Omega$ is unbounded for example. We have already proved (Theorem 4) the implication $\text{(19)} \Rightarrow f \in H_{1,1}^1(\Omega)$ and the implication $f \in H_{1,1}^1(\Omega) \Rightarrow \text{(19)}$ is an easy consequence of the estimates for $K_t$ (as the proof of $f \in H_{r,a}^1(\Omega) \Rightarrow f \in H_{1,1}^1(\Omega))$. We therefore turn to the proofs of $\text{(19)} \Rightarrow \text{(20)}$ and $\text{(20)} \Rightarrow \text{(21)}$.

The argument for $\text{(19)} \Rightarrow \text{(20)}$ relies upon the comparison between the $L^1$ norms of two maximal functions and is inspired by [14], p.185. For all $\alpha > 0$ and $v : \Omega \times ]0, +\infty[ \to \mathbb{C}$ set

$$v_\alpha^*(x) = \sup_{|y-x|<\alpha \sqrt{t}} |v(y,t)|.$$

If $f \in L^1_{\text{loc}}$ with slow growth, set

$$u(x,t) = e^{-tL} f(x), \quad u^+(x) = \sup_{t>0} |u(x,t)|, \quad u^*(x) = u_1^*(x).$$

Recall that we assume $(G_\infty)$ so that slow growth insures that $u$ is well-defined.
Finally, for all $\varepsilon > 0$, all $N \in \mathbb{N}$ and all $x \in \Omega$, consider

$$u_{\varepsilon,N}^*(x) = \sup_{|y-x|<\sqrt{t}<\varepsilon^{-1}} |u(y,t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N (1 + \varepsilon |y|)^{-N}$$

and

$$U_{\varepsilon,N}^*(x) = \sup_{|y-x|<\sqrt{t}<\varepsilon^{-1},|y'-x|<\sqrt{t}<\varepsilon^{-1}} \left( \frac{\sqrt{t}}{|y-y'|} \right)^\mu |u(y,t) - u(y',t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N (1 + \varepsilon |y|)^{-N}.$$ for some $\mu > 0$ to be chosen later.

We intend to show the following proposition:

**Proposition 26.** There exists $C > 0$ such that, for all $f \in L^1_{loc}$, $\|u^*\|_1 \leq C \|u^+\|_1$.

Notice first that the $L^1$-norm of $u_\alpha^*$ is controlled by the $L^1$-norm of $u^*$. More precisely, the following holds (see [14], Lemma 1, p. 166):

**Lemma 27.** There exists $C$ such that, for all continuous function $v$ on $\Omega \times [0, +\infty[$ and all $\alpha > 0$,

$$\|v_\alpha^*\|_1 \leq C \alpha \|v^*\|_1.$$

Note that this inequality holds if $v$ is truncated for $t > t_0$.

The proof of Proposition 26 relies on the following observation:

**Lemma 28.** Assume that $u_{\varepsilon,N}^* \in L^1$. Then

$$\|U_{\varepsilon,N}^*\|_1 \leq C \|u_{\varepsilon,N}^*\|_1,$$

where $C$ is independent on $\varepsilon, N$ and $u$.

Fix $x \in \Omega$ and consider $y, y'$ and $t$ such that $|y - x| < \sqrt{t}$ and $|y' - x| < \sqrt{t}$. Define also

$$v(y,t) = u(y,t) \left( 1 + \varepsilon |y| \right)^{-N} \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N \chi_{[0,1]}(\varepsilon \sqrt{t})$$

so that $v_1^* = u_{\varepsilon,N}$. Start from

$$u(y,t) - u(y',t) = \int_{\Omega} \left( K_{t/2}(y,z) - K_{t/2}(y',z) \right) u(z,t/2) dz = I_0 + \sum_{k \geq 1} I_k,$$

where

$$I_0 = \int_{|z-y| \leq \sqrt{t}} |K_{t/2}(y,z) - K_{t/2}(y',z)| \ |u(z,t/2)| dz,$$

and

$$I_k = \int_{2^{k-1}\sqrt{t} < |z-y| \leq 2^k \sqrt{t}} |K_{t/2}(y,z) - K_{t/2}(y',z)| \ |u(z,t/2)| dz.$$
Using $(1 + \varepsilon |z|)^N \leq (1 + \varepsilon |y|)^N (1 + 2^k)^N$ if $|z - y| \leq 2^k \sqrt{t}$ and $\varepsilon \sqrt{t} < 1$ and

$$
\int_{2^k-1 \sqrt{t} < |z - y| \leq 2^k \sqrt{t}} \left| K_{t/2}(y, z) - K_{t/2}(y', z) \right| dz \leq c \left( \frac{|y - y'|}{\sqrt{t}} \right)^\mu e^{-\alpha 2^k} \nu
$$

for some $\mu > 0$ and $\alpha > 0$ from $(G_\infty)$ for $L^*$, we easily get

$$
\left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N \frac{|u(y, t) - u(y', t)|}{(1 + \varepsilon |y|)^N} \leq c \left( \frac{|y - y'|}{\sqrt{t}} \right)^\mu \left( v_2(x) + \sum_{k \geq 1} e^{-\alpha 2^k} (1 + 2^k)^N v_{2k+1}(x) \right).
$$

Therefore,

$$
U_{\varepsilon,N}^*(x) \leq c \left( v_2(x) + \sum_{k \geq 1} e^{-\alpha 2^k} (1 + 2^k)^N v_{2k+1}(x) \right).
$$

Lemma 28 follows at once from Lemma 27. \( \blacksquare \)

We now prove Proposition 23, following [4], p. 186. Consider $f$ such that $u^+ \in L^1$ and $N \in \mathbb{N}$ large enough, so that one easily derives that $u_{\varepsilon,N}^* \in L^1$ for all $\varepsilon > 0$. Define $G_{\varepsilon,N} = \{ x \in \Omega; U_{\varepsilon,N}^*(x) \leq Bu_{\varepsilon,N}^*(x) \}$ for some $B > 0$ to be chosen. Then, one has

$$
\int_{\Omega \setminus G_{\varepsilon,N}} u_{\varepsilon,N}^*(x) dx \leq \frac{1}{B} \int_{\Omega \setminus G_{\varepsilon,N}} U_{\varepsilon,N}^*(x) dx
$$

provided that $B$ is large enough.

Moreover, for almost all $x \in G_{\varepsilon,N}$, one has $u_{\varepsilon,N}^*(x) \leq CM(x)$, where

$$
M(x) = \sup_{Q \ni x} \left( \frac{1}{|Q \cap \Omega|} \int_{Q \cap \Omega} u^+(y) dy \right)^{1/r},
$$

with $0 < r < 1$ (in this definition, the cubes are centered in $\Omega$). Indeed, let $x \in G_{\varepsilon,N}$ for which $u_{\varepsilon,N}^*(x) < \infty$. There exist $y, t$ such that $|y - x| < \sqrt{t} < \varepsilon^{-1}$ and

$$
|u(y, t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N \left( 1 + \varepsilon |y| \right)^{-N} \geq \frac{1}{2} u_{\varepsilon,N}^*(x).
$$

Since $x \in G_{\varepsilon,N}$, if $|z - x| < \sqrt{t}$ and $|z' - x| < \sqrt{t}$, one has

$$
\left( \frac{\sqrt{t}}{|z - z'|} \right)^\mu |u(z, t) - u(z', t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N \left( 1 + \varepsilon |z| \right)^{-N} \leq 2B |u(y, t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N \left( 1 + \varepsilon |y| \right)^{-N}
$$

hence,

$$
\left( \frac{\sqrt{t}}{|z - z'|} \right)^\mu |u(z, t) - u(z', t)| \leq c |u(y, t)|.
$$
It follows that
\[ |u(z, t)| \geq \frac{1}{2} |u(y, t)| \]
when \( z \in A = \{ w; \ |w - x| < \sqrt{t} \text{ and } |w - y| < \frac{\sqrt{t}}{2C} \} \). Therefore, when \( z \in A \), one has
\[ |u(z, t)| \geq \frac{1}{2} |u(y, t)| \left( \frac{\sqrt{t}}{\sqrt{t} + \varepsilon} \right)^N (1 + \varepsilon |y|)^{-N} \geq \frac{1}{4} u_\varepsilon^*(x). \]
Hence,
\[
M(x)^r \geq \frac{c}{|B(x, 2\sqrt{t})|} \int_{B(x, 2\sqrt{t})} u^+(z)^r \, dz \\
\geq \frac{c}{|B(x, 2\sqrt{t})|} \int_{B(x, 2\sqrt{t})} |u(z, t)|^r \, dz \\
\geq c \left( \frac{1}{4} u_\varepsilon^*(x) \right)^r \frac{|A|}{|B(x, 2\sqrt{t})|} \\
\geq cu_\varepsilon^*(x)^r.
\]
Finally, using the fact that \( 1/r > 1 \), one obtains that
\[
\int_\Omega u_\varepsilon^*(x) \, dx \leq 2 \int_{G_\varepsilon,N} u_\varepsilon^*(x) \, dx \\
\leq C \int_{G_\varepsilon,N} M(x) \, dx \\
\leq C \int_{G_\varepsilon,N} M(x) \, dx \\
\leq C \int_{\Omega} u^+(x) \, dx
\]
where \( C \) does not depend on \( \varepsilon \). Letting \( \varepsilon \to 0 \) yields \( \|u^*\|_1 \leq C\|u^+\|_1 \) and (20) is proved.

To complete the proof of Proposition 26, it remains to see \((21) \Rightarrow (20)\). This follows easily from the subordination formula (7) and Lemma 27.

**Appendix A: Kernel estimates**

In this Appendix, we derive some consequences of the Gaussian upper bounds (3) which we assume to hold for \( 0 < t < \tau \). The first consequence is that an estimate of the form (3) holds for \( t\partial_t K_t(x, y) \) by analyticity of the semigroup (See [3], Chapter I, Lemma 19).

We first claim the following:
Lemma 29. Assume that $\tau = 1$. Then, for all $t > 1$ and all $x, y \in \Omega$, one has
\[ |K_t(x, y)| \leq C e^{-\frac{1}{2}\frac{|x-y|^2}{t}}. \]

The proof relies on the following $L^2$-maximum principle (see [18]):

Proposition 30. Assume that $A \in \mathcal{A}(c)$. Let $u(x, t)$ be a function on $\Omega \times ]0, +\infty[$ satisfying $\partial_t u(x, t) + Lu(x, t) = 0$ on $\Omega$. Then, if $\xi : \Omega \times ]0, +\infty[ \rightarrow \mathbb{R}$ is locally Lipschitz and satisfies the relation
\[ \partial_t \xi(x, t) + \alpha |\nabla \xi(x, t)|^2 \leq 0, \]
where $\alpha = \frac{1}{2c^2}$, the function
\[ I(t) = \int_{\Omega} |u(x, t)|^2 e^{\xi(x, t)}dx \]
is non increasing in $t > 0$.

Indeed, for all $t > 0$, one has
\[
I'(t) = 2\text{Re} \int_{\Omega} \partial_t u(x, t) \overline{u(x, t)} e^{\xi(x, t)}dx + \int_{\Omega} u(x, t) \overline{u(x, t)} \partial_t \xi(x, t) e^{\xi(x, t)}dx
\]
\[
= -2\text{Re} \int_{\Omega} Lu(x, t) \overline{u(x, t)} e^{\xi(x, t)}dx - \alpha \int_{\Omega} u(x, t) \overline{u(x, t)} |\nabla \xi(x, t)|^2 e^{\xi(x, t)}dx
\]
\[
= -2\text{Re} \int_{\Omega} A(x) \nabla u(x, t) \overline{\nabla u(x, t)} e^{\xi(x, t)}dx - 2\text{Re} \int_{\Omega} A(x) \nabla u(x, t) \nabla \xi(x, t) \overline{u(x, t)} e^{\xi(x, t)}dx
\]
\[
- \alpha \int_{\Omega} u(x, t) \overline{u(x, t)} |\nabla \xi(x, t)|^2 e^{\xi(x, t)}dx
\]
\[
\leq -2c \int_{\Omega} |\nabla u(x, t)|^2 e^{\xi(x, t)}dx + 2c^{-1} \int_{\Omega} |\nabla u(x, t)| |\nabla \xi(x, t)| |u(x, t)| e^{\xi(x, t)}dx
\]
\[
- \alpha \int_{\Omega} |u(x, t)|^2 |\nabla \xi(x, t)|^2 e^{\xi(x, t)}dx
\]
\[
\leq (-2c + c^{-1}/\varepsilon) \int_{\Omega} |\nabla u(x, t)|^2 e^{\xi(x, t)}dx + (\varepsilon c^{-1} - \alpha) \int_{\Omega} |u(x, t)|^2 |\nabla \xi(x, t)|^2 e^{\xi(x, t)}dx
\]
\[
= 0.
\]

In the previous computation, $\varepsilon = \alpha c = \frac{1}{2c^2}$. Proposition 30 is therefore proved.
In order to prove Lemma 31, observe that, for \( \beta > 0 \) small enough (namely, \( \beta \leq \frac{1}{10} \)), the function \( \xi(x,t) = \frac{\beta (x-y)^2}{t} \) satisfies the assumptions of Proposition 30. Then, write
\[
|K_t(x,y)| = \left| \int_{\Omega} K_{t/2}(x,z) e^{\frac{\alpha |x-z|^2}{t}} K_{t/2}(z,y) e^{\frac{\alpha |x-y|^2}{t}} e^{-\frac{\alpha |x-z|^2+|x-y|^2}{t}} \, dz \right|
\]
\[
\leq \left( \int_{\Omega} |K_{t/2}(x,z)|^2 e^{\alpha |x-z|^2} \, dz \right)^{1/2} \left( \int_{\Omega} |K_{t/2}(z,y)|^2 e^{\alpha |z-y|^2} \, dz \right)^{1/2} e^{-\frac{\alpha |x-y|^2}{t}}
\]
\[
\leq \left( \int_{\Omega} |K_{1/2}(x,z)|^2 e^{\alpha |x-z|^2} \, dz \right)^{1/2} \left( \int_{\Omega} |K_{1/2}(z,y)|^2 e^{\alpha |z-y|^2} \, dz \right)^{1/2} e^{-\frac{\alpha |x-y|^2}{t}}
\]
\[
\leq Ce^{-\frac{\alpha |x-y|^2}{t}}.
\]

As a consequence of the upper bounds for \( K_t \), we get the following estimates for the Poisson kernel:

**Lemma 31.** (a) Assume that \( \tau = +\infty \). Then, for all \( t > 0 \), all \( x, y \in \Omega \),
\[
|p_t(x,y)| \leq \frac{Ct}{(t + |x-y|)^{n+1}}.
\]

(b) Assume now that \( \tau = 1 \) and \( \Omega \) is bounded. Then, for all \( 0 < t < 1 \), all \( x, y \in \Omega \),
\[
|p_t(x,y)| \leq \frac{Ct}{(t + |x-y|)^{n+1}}.
\]

For all \( t > 1 \), all \( x, y \in \Omega \),
\[
|p_t(x,y)| \leq \frac{Ct}{t + |x-y|}.
\]

By analyticity, the same estimates hold for \( t \partial_t p_t(x,y) \).

Just use the subordination formula (3) and the upper estimates for \( |K_t(x,y)| \).

We now summarize \( L^2 \)-estimates for \( \nabla K_t(x,y) \) that follow from the assumption (3) and the Caccioppoli inequality (see [5], Proposition 15):

**Proposition 32.** (a) Assume that \( \tau = +\infty \). For all \( x \in \Omega \), all \( t > 0 \) and all \( r > 0 \),
\[
\left( \int_{r \leq |x-y| \leq 2r} |\nabla_y K_t(x,y)|^2 \, dy \right)^{1/2} \leq c C_G t^{-\frac{1}{2} - \frac{n}{4}} \left( \frac{r}{\sqrt{t}} \right)^{\frac{n-2}{2}} e^{-\frac{\beta^2}{t}}.
\]

(b) Assume that \( \tau = 1 \). Then, for all \( x \in \Omega \), all \( 0 < t < 1 \) and all \( r > 0 \),
\[
\left( \int_{r \leq |x-y| \leq 2r} |\nabla_y K_t(x,y)|^2 \, dy \right)^{1/2} \leq c C_G t^{-\frac{1}{2} - \frac{n}{4}} \left( \frac{r}{\sqrt{t}} \right)^{\frac{n-2}{2}} e^{-\frac{\beta^2}{t}}.
\]
For all $x \in \Omega$, all $t > 1$ and all $r > 0$,

$$\left( \int_{r \leq |x-y| \leq 2r} |\nabla_y K_t(y, x)|^2 \, dy \right)^{1/2} \leq c C_G t^{-\frac{1}{2}} r^{\frac{n-2}{2}} e^{-\frac{\beta r^2}{C}}.$$ 

As a consequence of Proposition 32, the following holds:

**Lemma 33.** For all $x \in \Omega$, denote by $\delta(x)$ the distance from $x$ to $\partial \Omega$.

(a) **Under NBC**, for all $x \in \Omega$,

$$\int_{\Omega} \partial_t K_t(y, x) \, dy = 0.$$ 

(b) **Under DBC**, for all $x \in \Omega$ for all $0 < t < \tau$,

$$\left| \int_{\Omega} \partial_t K_t(y, x) \, dy \right| \leq C t e^{-\frac{\beta \delta^2}{4t}}.$$ 

**Under DBC**, if $\Omega$ is bounded and $\tau = 1$, for all $x \in \Omega$ and all $t > 1$,

$$\left| \int_{\Omega} \partial_t K_t(y, x) \, dy \right| \leq C t.$$ 

Under NBC, one has $e^{-tL}1 = 1$, whence assertion (a) holds.

To prove the first part of assertion (b), choose $\psi_1 \in C^\infty_0(\Omega)$ such that $\psi_1(z) = 1$ if $d(z, y) \leq \delta/4$, $\psi_1(z) = 0$ if $d(z, y) \geq \delta/2$ and $\|\nabla \psi_1\|_\infty \leq C/\delta$. Here $\delta = \delta(x)$. Define $\psi_2 = 1 - \psi_1$. Then, one has

$$\int_{\Omega} \partial_t K_t(y, x) \, dy = \int_{\Omega} \partial_t K_t(y, x) \psi_1(y) \, dy + \int_{\Omega} \partial_t K_t(y, x) \psi_2(y) \, dy.$$ 

But Lemma 32 shows that

$$\left| \int_{\Omega} L_y K_t(y, x) \psi_1(z) \, dz \right| = \left| \int_{\Omega} A \nabla_y K_t(y, x) \nabla_y \psi_1(y) \, dy \right|$$

$$\leq C \int_{\frac{\delta}{4} \leq d(z, y) \leq \frac{\delta}{2}} |\nabla_y K_t(y, x)| |\nabla_y \psi_1(y)| \, dy$$

$$\leq C t^{-\frac{1}{4}} \left( \frac{\delta}{\sqrt{t}} \right)^{\frac{n-2}{4}} e^{-\frac{\beta}{4} \delta^2} \delta^{\frac{n-2}{2}}$$

$$= \frac{C}{t} \left( \frac{\delta}{\sqrt{t}} \right)^{n-2} e^{-\frac{\beta}{4} \delta^2}.$$ 

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Moreover,
\[
\left| \int \partial_t K_t(x,y) \psi_2(y)dy \right| \leq \int_{|y-x| \geq \delta/2} |\partial_t K_t(x,y)| dy \\
\leq \frac{C}{t} e^{-\delta^2 t}.
\]

For the second part of assertion (b), we have
\[
\int_{\Omega} |\partial_t K_t(x,y)| dy \leq \frac{C|\Omega|}{t}.
\]

From these estimates and the subordination formula, we deduce the following:

**Lemma 34.** For all \( x \in \Omega \), denote again by \( \delta(x) \) the distance from \( x \) to \( \partial \Omega \).

(a) Under NBC, for all \( x \in \Omega \),
\[
\int_{\Omega} \partial_t p_t(x,y) dy = 0.
\]

(b) Under DBC, if \( \tau = +\infty \), for all \( x \in \Omega \),
\[
\left| \int_{\Omega} \partial_t p_t(x,y) dy \right| \leq \frac{C}{t} \left( 1 + \frac{\delta(x)}{t} \right)^{-1}.
\]

Under DBC, if \( \Omega \) is bounded and \( \tau = 1 \), for all \( x \in \Omega \) and \( 0 < t < 1 \)
\[
\left| \int_{\Omega} \partial_t p_t(x,y) dy \right| \leq \frac{C}{t} \left( 1 + \frac{\delta(x)}{t} \right)^{-1}.
\]

Let us prove the second point of part (b). By differentiating the subordination formula, one has
\[
\int_{\Omega} \partial_t p_t(x,y) dy = \frac{1}{\sqrt{\pi}} \int_{\Omega} \int_{0}^{+\infty} 2u t^2 \partial_s K_s(x,y)|_{s = \frac{t}{4u}} e^{-u} u^{-1/2} du dy.
\]

Break the integral at \( u = t^2/4 \). The part for \( u \geq t^2/4 \) is controlled by \( \frac{C}{t} \left( 1 + \frac{\delta(x)}{t} \right)^{-1} \). The part for \( u \leq t^2/4 \) is bounded by \( \frac{C}{t} \int_{0}^{t^2/4} e^{-u} u^{-1/2} du \leq c \). Since \( t + \delta(x) \leq 1 + \text{diam}(\Omega) \), we obtain \( c \leq \frac{C}{t} \left( 1 + \frac{\delta(x)}{t} \right)^{-1} \). This concludes the proof.

We leave to the reader the care of studying what happens to regularity estimates for small time for \( p_t(x,y) \) when \((G_1)\) holds.
Appendix B: Elementary geometry of Lipschitz domains

A strongly Lipschitz domain is by definition a domain in $\mathbb{R}^n$ whose boundary is covered by a finite number of parts of Lipschitz graphs (up to rotations) at most one them being infinite. A special Lipschitz domain is the domain above the graph of a Lipschitz function defined on $\mathbb{R}^{n-1}$.

Let $\Omega$ be a strongly Lipschitz domain.

1. There exists a finite covering of $\mathbb{R}^n$ by open sets $U_1, U_2, \ldots, U_s$ with at most one of them being infinite such that for each $k$ either $U_k \cap \Omega = \emptyset$ or there is a special Lipschitz domain $\Omega_k$ and a rotation $R_k$ in $\mathbb{R}^n$ such that $U_k \cap \Omega = U_k \cap R_k(\Omega_k)$.

2. There exists a cube $Q_0$ such that either $\Omega \subset Q_0$ or there is a rotation $R$ and a special Lipschitz domain $\Omega_0$ such that $\omega Q_0 \cap \Omega = \omega Q_0 \cap R(\Omega_0)$.

3. There are constants $\rho \in ]0, +\infty]$ and $C > 0$ such that if $Q$ is a cube centered in $\Omega$ and $\ell(Q) \leq \rho$ then $|Q \cap \Omega| \geq C|Q|$. When $\Omega$ is an unbounded, $\rho = \infty$.

4. There exists $\rho \in ]0, +\infty]$, such that if $Q$ is a type $(b)$ cube and $\ell(Q) < \rho$, there exists a cube $\tilde{Q} \subset \omega \Omega$ such that $|\tilde{Q}| = |Q|$ and the distance from $\tilde{Q}$ to $Q$ is comparable to the side length of $Q$. Furthermore, $\rho = \infty$ is $\omega \Omega$ is unbounded.

5. Assume $\Omega$ is unbounded. Let $Q$ be a cube with $\ell(Q) \geq 1$, centered in $\Omega$ with $4Q \cap \Omega \neq \emptyset$. There exists a cube $Q'$ with $4Q' \subset \Omega$, $\ell(Q') = \ell(Q)$ and the distance between $Q$ and $Q'$ is comparable to $\ell(Q)$.

The proof of 1. is classical and skipped.

Point 2. follows easily: take $Q_0$ as the smallest cube containing the bounded $U_k$’s in point 1.

To obtain $\rho = \infty$ in the proof of 3. when $\Omega$ is a special Lipschitz domain is classical using “vertical” reflection. Localisation gives us a finite $\rho$. To obtain $\rho = \infty$ when $\Omega$ is unbounded, we argue as follows: let $Q_0$ be the cube of point 2. Let $Q$ be a cube centered in $\Omega$ with $\ell(Q) > \rho$. If $\ell(Q) \leq \lambda \ell(Q_0)$ for some $\lambda > 1$ to be chosen, then for $\tilde{Q} = \frac{\ell(Q)}{\ell(Q_0)}Q$ we have $|Q \cap \Omega| \geq |\tilde{Q} \cap \Omega| \geq C|\tilde{Q}| \geq C\left(\frac{\ell(Q)}{\ell(Q_0)}\right)^n|Q|$. If $\ell(Q) \geq \lambda \ell(Q_0)$ and the center of $Q$ belongs to $R(\Omega_0)$, then $|Q \cap \Omega| \geq |Q \cap \Omega \cap \omega Q_0| = |Q \cap R(\Omega_0) \cap \omega Q_0| \geq |Q \cap R(\Omega_0)| - |Q_0| \geq C|Q| - |Q_0|$ where $C$ is the constant obtained for the domain $R(\Omega_0)$. One chooses $\lambda$ so that $|Q_0| \leq C \rho^n/6^n$. If $\ell(Q) \geq \lambda \ell(Q_0)$ and the center of $Q$ does not belong to $R(\Omega_0)$, then this center belongs to $Q_0$ and one can find in $Q$ a point in $R(\Omega_0)$ at distance less than $\ell(Q_0)$ from the center of $Q$. It follows that $Q$ contains a cube of side length $\ell(Q)/3$ and centered in $R(\Omega_0)$. We apply the above argument to that cube.

The proof of 4. is well-known if $\Omega$ is special Lipschitz or bounded. See [3], p. 304. By the same argument, one can see it holds for some $\rho$ finite for all strongly Lipschitz domains. It remains to show that one can take drop the finiteness of $\rho$ if $\omega \Omega$ is unbounded. In that case, let $Q$ be a type $(b)$ cube contained in $\Omega$ with $\ell(Q) \geq \rho$. Pick $Q_0$, $R$ and $\Omega_0$ of point 2. In a basis $(e_1, \ldots, e_n)$, $\Omega_0$ is $x_n \geq \varphi(x_1, \ldots, x_{n-1})$. We take $\tilde{Q} = Q - c\ell(Q)R(e_n)$ for some appropriately
chosen $c$ that depends only on the domain $\Omega$. We leave to the reader the care of verifying that such a choice is possible.

To see point 5. let $Q$ is a cube of size greater than 1, centered in $\Omega$ with $4Q \cap \Omega \neq \emptyset$. Arguing as above, we take $Q' = Q + c\ell(Q)\overline{R(e_n)}$ where, since $\Omega$ is unbounded, one can pick $c$ large enough and independent of $Q$ such that $Q'$ enjoys the desired properties. Details are left to the reader.

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