Universality and Leading Corrections in Few-Body Systems

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Abstract. A large two-body scattering length leads to universal behavior in few-body systems. In particular, the three-body system displays interesting features such as exact discrete scale invariance in the bound state spectrum in the limit of infinite scattering length. Here, I will discuss how an effective field theory (EFT) can be used to study these features and how the finite range of the underlying interaction impacts the bound state spectrum at first order in the EFT expansion.

1 Introduction

Few-body systems with a large two-body scattering length \( a \) display interesting universal features. In the two-body system a large positive two-body scattering length will lead to a bound state with binding energy proportional to \( 1/(Ma^2) \) (where \( M \) denotes the mass). Vitaly Efimov showed that the situation in the three-body system is more complicated. For example, for infinite scattering length the binding energies of different states labeled with \( n \) and \( n^* \) are related by

\[
B_3^{(n)} = (e^{-2\pi/s_0})^{n-n^*} B_3^{(n^*)},
\]

where \( s_0 \approx 1.00624 \). The geometric spectrum is a signature of discrete scaling symmetry with scaling factor \( e^{-2\pi/s_0} \). Efimov pointed out furthermore that these results are also relevant for finite scattering length \( a \) as long as \( a \gg l \), where \( l \) denotes the range of the underlying interaction \([1,2]\).

Over the last years an effective field theory (EFT) has been developed which is tailored to calculate the low-energy properties of few-body systems with a large two-body scattering length \([3]\). This short-range EFT is the appropriate description of ultracold atoms close to a Feshbach resonance and nucleons at very low energies. At leading order, the short-range EFT provides a powerful framework to calculate observables in the zero-range limit and reproduces therefore the results derived by Efimov for the three-body sector exactly. It allows furthermore to calculate the effects of the finite range of the underlying interaction systematically and to compute electroweak reactions relevant to nuclear astrophysics.
2 One-Parameter Correlations and Universality

A particular feature of the short-range EFT is the appearance of a three-body force at leading order. Once this three-body counterterm is adjusted such that a known three-body datum is reproduced all remaining observables can be predicted. Three-body observables will therefore depend not only on the scattering length $a$ but also on one additional three-body parameter.

The necessity of this counterterm is more than just an artefact of the field-theoretic formulation of the problem but is instead strongly tied to the aforementioned discrete scale invariance in the three-body system. Its appearance at leading order implies furthermore that two types of one-parameter correlations can be generated within this framework. Either the three-body counterterm is kept constant while the two-body scattering length is varied or $a$ is kept constant while the three-body counterterm is varied.

One example for each of these types of one-parameter correlations is shown in Fig. 1. The correlation between the charge radius of the triton and its binding energy is shown in the left panel of Fig. 1. The two different solid lines correspond to two different choices of fixing the two-body counterterms, the circles and triangles denote Faddeev calculations with different potentials, the square denotes the experimental value. The difference between the experimental result and the short-range EFT result is due to the importance of finite range corrections.

The right panel shows an Efimov plot which includes results for the four-body system obtained with the short-range EFT. The circles and triangles denote the binding momentum of two different four-body bound states which lie between two...
successive three-body states. The lower solid line denotes the shallower of these two three-bound states. The upper solid line denotes the next three-body state in the Efimov spectrum. The dashed and dot-dashed lines denote the thresholds for decay into atom plus dimer and two dimers, respectively.

3 Finite Range Corrections

A systematic calculation of higher order corrections is required for an appropriate description of observables if the range of the underlying interaction leads to a sizeable expansion parameter. This is the case in nuclear physics where the ratio of effective range over scattering length is roughly $\sim 1/3$.

Higher order corrections in the EFT expansion have been studied extensively over the last years [6, 7, 8], however, analytical information on the form of these range corrections is very limited. In [9] we used the fact that the wave functions of the Efimov trimers are known in the unitary limit. This allowed us to calculate the shift in the binding energies linear in the effective range in perturbation theory. It is therefore necessary to calculate first the perturbing hyperspherical potential [10, 11, 9]

$$V_{NLO} = -\frac{s_0^2 \xi_0 r_s}{R^3}, \quad (2)$$

which is done by implementing a next-to-leading order Bethe-Peierls condition into the hyperangular equation. The shift in the binding energy of the $n$th bound state can then be found by calculating the integral

$$\frac{2M}{\hbar^2} \Delta B_n^{(1)} = s_0^2 r_s \xi_0 \left[ \int_0^\infty dR f_n^2 (R) \frac{1}{R^3} - \frac{2H_1 M}{\hbar^2 s_0^2 r_s \xi_0} A^2 f_n^2 \left( \frac{1}{A} \right) \right] , \quad (3)$$

where $f_n(R)$ is the leading-order wave function of the $n$th three-body bound state. The second term on the right hand side arises from a three-body force

$$V_{SR}^{(1)} (R) = H_1 (A) A^2 \delta \left( R - \frac{1}{A} \right), \quad (4)$$

which has been included to regularize the divergent first term. The expression is renormalized by demanding that the shift in the binding energy of the state with index $n_*$ is 0. It turns out that this condition leads to the surprising result that the complete three-body spectrum remains unperturbed, i.e.

$$\Delta B_n^{(1)} = 0 , \quad (5)$$

for all $n$. This result which was also found numerically by Thøgersen et al. [12] shows that the discrete scaling symmetry in the three-body system constrains the form of higher order corrections strongly.

4 Summary

Effective field theories can be applied to any system in which a separation of scales is present. They are not only perfectly suited to calculate observables in a
systematic low-energy expansion, but also provide a reliable error estimate and a well-defined domain of applicability. An EFT appropriate for short-range interactions has been applied to a large variety of physical systems. I discussed how this short-range EFT can be used to study universal relations in the three-body sector and how range corrections affect the three-body bound state spectrum.

It is a surprising result that the Efimov spectrum in the unitary limit remains unchanged at next-to-leading order. It will be interesting to see how a finite range affects the universal relations between different three-body observables such as the relation between the minima in the three-body recombination rate and the binding energy of Efimov trimers in the unitary limit. It might furthermore be possible to obtain analytic results at next-to-next-to-leading order in the unitary limit.

The consistent inclusion of finite range corrections is required for future calculations of electroweak reactions in few-body systems relevant to nuclear astrophysics and will also be useful in applications of the short-range EFT to Halo-nuclei [13] or α-clusters [14].

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