Nonequilibrium evolution and symmetry structure of the large-$N$ $\Phi^4$ model at finite temperature

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Abstract

We consider the large-$N$ $\Phi^4$ theory with spontaneously broken symmetry at finite temperature. We study, in the large-$N$ limit, quantum states which are characterized by a time dependent, spatially homogeneous expectation value of one of the field components, $\phi_N(t)$, and by quantum fluctuations of the other $N - 1$ components, that evolve in the background of the classical field. Investigating such systems out of equilibrium has recently been shown to display several interesting features. We extend here this type of investigations to finite temperature systems. Essentially the novel features observed at $T = 0$ carry over to finite temperature. This is not unexpected, as the main mechanisms that determine the late-time behavior remain the same. We extend two empirical - presumably exact - relations for the late-time behavior to finite temperature and use them to define the boundaries between the region of different asymptotic regimes. This results in a phase diagram with the temperature and the initial value of the classical field as parameters, the phases being characterized by spontaneous symmetry breaking resp. symmetry restoration. The time evolution is computed numerically and agrees very well with the expectations.

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1 Introduction

The investigation of the $O(N)$ vector model at large $N$ has a long-standing history in quantum field theory [1, 2, 3]. One of the main aspects was the question of symmetry restoration at high temperature that for some time was controversial. The dynamical exploration of a special class of nonequilibrium properties has been developed only recently [4, 5, 6, 7].

The out-of-equilibrium configuration that has been studied mainly is characterized by an initial state in which one of the components has a spatially homogeneous classical expectation value $\phi(t)$. This implies that the other $N-1$ components $\psi_i(x, t), i = 1 \ldots N-1$ have a mass that is different from the mass in the ground state. This means that their initial state is related to the Fock space vacuum state by a Bogoliubov transformation. The evolution of the system is governed by the classical equation of motion for the field $\phi(t)$ and by the mode equations for the quantum fields $\psi(x, t)$. The expectation value $\langle \psi(x, t)\psi(x, t) \rangle$ appears in both equations of motion, this constitutes the quantum back reaction. In the one-loop approximation, in contrast to the large-$N$ approximation, this quantum back reaction only appears in the classical equation of motion. This leads to decisive differences in the late time behavior.

We have previously [8] carried out such dynamical computations for the $O(N)$ vector model in the limit of large $N$ at finite temperature for the case of unbroken symmetry, i.e., with a positive mass term. Here we will consider the case of spontaneously broken symmetry. In this case, at low temperatures the fields $\psi_i(x, t)$ will be the Goldstone modes. This is the case for the ground state at $T = 0$ and at finite temperature; for nonequilibrium initial states these modes become massless when the system settles to a stationary state at late times. Symmetry restoration happens at high temperature and at large values of the initial field $\phi(0)$; then at late times these modes stay massive while the classical field vanishes, and thereby the spontaneous symmetry breaking disappears.

Our investigation, as well as the analogous ones at $T = 0$, are limited to fields, masses (as solutions of the gap equation) and temperatures much smaller than the scale of the Landau ghost $m_x = m_1 \exp(8\pi^2/\lambda)$, where $m_1$ is a renormalization scale, taken of order $\sqrt{\lambda}v$. So the question of symmetry non-restoration at “really” high temperatures [3] will not be addressed here.

The plan of the paper is as follows: in section 2 we introduce the model and set up the equations governing the nonequilibrium evolution. In section 3 we discuss the renormalization of the equations of motion and of the energy-momentum tensor, some details are referred to Appendix A. In section 4 we discuss the phase structure of the system as a function of temperature and initial conditions. In section 5 we present the results of the numerical computations. Some conclusions are drawn in section 6.
2 Formulation of the model

We consider the $O(N)$ vector model with the Lagrangian

$$L = \frac{1}{2} \partial_\mu \phi_i \partial^\mu \phi_i - \frac{\lambda}{4N} (\phi_i \phi_i - Nv^2)^2$$

where $\phi_i, i = 1, \ldots, N$ are $N$ real scalar fields. The nonequilibrium state of the system is characterized by a classical expectation value which we take in the direction of $\phi_N$. We split the field into its expectation value $\phi$ and the quantum fluctuations $\psi$ via

$$\phi_i(x, t) = \delta_i N \sqrt{N} \phi(t) + \psi_i(x, t).$$

In the large-$N$ limit one neglects, in the Lagrangian, all terms which are not of order $N$. In particular terms containing the fluctuation $\psi_N$ of the component $\phi_N$ are at most of order $\sqrt{N}$ and are dropped, therefore. The fluctuations of the other components are identical, their summation produces factors $N - 1 = N(1 + O(1/N))$. In the broken symmetry case these are the Goldstone modes. Identifying all the fields $\psi_1, \ldots, \psi_{N-1}$ as $\psi$ the leading order term in the Lagrangian then takes the form

$$L = N (L_\phi + L_\psi + L_1),$$

with

$$L_\phi = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\lambda}{4} (\phi^2 - v^2)^2,$$

$$L_\psi = \frac{1}{2} \partial_\mu \psi \partial^\mu \psi + \frac{\lambda}{2} v^2 \psi^2 + \frac{\lambda}{4} (\psi^2)^2,$$

$$L_1 = -\frac{\lambda}{2} \psi^2 \phi^2,$$

where $\psi^2$ is to be identified with $\sum \psi_i^2 / N$.

We decompose the fluctuating field into momentum eigenfunctions via

$$\psi(x, t) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[ a_k U_k(t) e^{ikx} + a_k^\dagger U_k^*(t) e^{-ikx} \right],$$

with $\omega_k = \sqrt{m_0^2 + k^2}$. The mass $m_0$ will be specified below. This field decomposition defines a vacuum state as being annihilated by the operators $a_k$.

The equations of motion for the field $\phi(t)$ and of the fluctuations $U_k(t)$ have been derived in this formalism by various authors [11, 12, 13].

We include in the following the counterterms that we will need later in order to write the renormalized equations. The equation of motion for the field $\phi$ becomes

$$\ddot{\phi}(t) + \delta m^2 \phi(t) - \lambda v^2 \phi(t) + (\lambda + \delta \lambda) \phi(t) \left[ \phi^2(t) + \mathcal{F}(t, T) \right] = 0.$$
Here $\mathcal{F}(t, T)$ is the divergent fluctuation integral; it is given by the average of the fluctuation fields defined by the initial density matrix. For a thermal initial state of quanta with energy $\omega_{k0} = \sqrt{k^2 + m_0^2}$ it is given by

$$\mathcal{F}(t, T) = \langle \psi^2(x, t) \rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \coth \frac{\beta \omega_{k0}}{2} |U_k(t)|^2 . \quad (2.9)$$

The mode functions satisfy the equation:

$$\left[ \frac{d^2}{dt^2} + \omega_k^2(t) \right] U_k(t) = 0 , \quad (2.10)$$

and the initial conditions

$$U_k(0) = 1 ; \quad \dot{U}_k(0) = -i \omega_{k0} . \quad (2.11)$$

The time dependent frequency $\omega_k(t)$ is given by

$$\omega_k^2(t) = k^2 + \mathcal{M}^2(t) \quad (2.12)$$

with the time dependent mass

$$\mathcal{M}^2(t) = -\lambda v^2 + \delta m^2 + (\lambda + \delta \lambda) \left[ \phi^2(t) + \mathcal{F}(t) \right] . \quad (2.13)$$

Using this definition the classical equation of motion can be rewritten as

$$\ddot{\phi}(t) + \mathcal{M}^2(t) \phi(t) = 0 \quad (2.14)$$

which is the same equation as the one for $U_k(t)$ with $k = 0$ (zero mode). Of course the initial conditions are different and $\phi(t)$ is real.

As in our previous work we rewrite the mode equation in the form

$$\left[ \frac{d^2}{dt^2} + \omega_k^2(t) \right] U_k(t) = -\mathcal{V}(t) U_k(t) , \quad (2.15)$$

whereby we have defined the time-dependent potential $\mathcal{V}(t) = \mathcal{M}^2(t) - \mathcal{M}^2(0)$; we further identify $m_0 = \mathcal{M}(0)$ as the “initial mass”.

The average of energy with respect to the initial density matrix is given by

$$\mathcal{E} = \frac{1}{2} \dot{\phi}^2(t) + \frac{1}{2} (-\lambda v^2 + \delta m^2) \phi^2(t) + \frac{\lambda + \delta \lambda}{4} \phi^4(t) + \delta \Lambda$$

$$+ \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \coth \frac{\beta \omega_{k0}}{2} \left\{ \frac{1}{2} |\dot{U}_k(t)|^2 + \frac{1}{2} \omega_k^2(t) |U_k(t)|^2 \right\} \quad (2.16)$$

$$- \frac{\lambda + \delta \lambda}{4} \mathcal{F}^2(t, T) .$$

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$^3$Note that twice the last term, with positive sign, is included in the fluctuation energy, since $\omega_k^2(t)$ contains $\mathcal{F}(t, T)$. 

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It is easy to check, using the equations of motion (2.14) and (2.10), that the energy is conserved. The energy density is the 00 component of the energy-momentum tensor. The average of the energy momentum tensor for our system is diagonal, its space-space components define the pressure which is given by

\begin{align}
p &= \phi^2(t) - E + \delta \xi \frac{d^2}{dt^2} \left[ \phi^2(t) + \mathcal{F}(t, T) \right] \tag{2.17} \\
&\quad + \int \frac{d^3 k}{(2\pi)^3 2\omega_{k0}} \coth \frac{\beta \omega_{k0}}{2} \left( \omega_{k0}^2 + \frac{k^2}{3} \right) |U_k(t)|^2.
\end{align}

\( \delta \xi \) is the renormalization of the conformal coupling term \( \xi (g_{\mu\nu} \partial^2 - \partial_{\mu} \partial_{\nu}) \phi^2 \), which has been used for the improved energy momentum tensor [12].

3 The renormalized equation of motion

The expressions for the time-dependent mass \( \mathcal{M}^2(t) \), the energy density \( \mathcal{E}(t) \) and the pressure are still undefined as they involve divergent integrals over the fluctuations. Our approach to regularization and renormalization has been presented previously [13, 8]. It is based on expanding the fluctuations \( U_k(t) \) and subsequently the various integrals involving these fluctuations with respect to the time-dependent potential \( V(t) \). As this procedure has been presented elsewhere in detail we just give the outline, here.

The expansion of the fluctuations with respect to \( V(t) \) is given in Appendix A. We use this perturbative expansion in order to single out the divergent contributions in the fluctuation integral. One finds

\begin{align}
\mathcal{F}(t) &= I_{-1}(m_0, T) - I_{-3}(m_0, T) \left[ \mathcal{M}^2(t) - \mathcal{M}^2(0) \right] + \mathcal{F}_{\text{fin}}(t, T), \tag{3.1}
\end{align}

where the finite part of \( \mathcal{F}(t, T) \) can be written as

\begin{align}
\mathcal{F}_{\text{fin}}(t, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{k0}^3} \int_0^t dt' \cos [2\omega_{k0} (t - t')] \dot{V}(t') \coth \frac{\beta \omega_{k0}}{2} \\
&\quad + \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left[ 2 \text{Re} f^2_k(t) + |f^{(1)}_k(t)|^2 \right] \coth \frac{\beta \omega_{k0}}{2}, \tag{3.2}
\end{align}

and where the divergent integrals are defined as

\begin{align}
I_{-1}(m_0, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{k0}} \left( 1 + \frac{2}{e^{\beta \omega_{k0}} - 1} \right) = I_{-1}(m_0) + \Sigma_{-1}(m_0, T) \tag{3.3} \\
I_{-3}(m_0, T) &= \int \frac{d^3 k}{(2\pi)^3} \frac{1}{4\omega_{k0}^3} \left( 1 + \frac{2}{e^{\beta \omega_{k0}} - 1} \right) = I_{-3}(m_0) + \Sigma_{-3}(m_0, T). \tag{3.4}
\end{align}
The integrals $I_{-k}(m_0)$ are the genuine divergences which appear in the renormalization at $T = 0$. Their dimensionally regularized form is given by

$$I_{-3}(m_0) = \left\{ \frac{d^3k}{(2\pi)^3} \frac{1}{4\omega^3_{k0}} \right\}_{\text{reg}} = \frac{1}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma \right\},$$

(3.5)

$$I_{-1}(m_0) = \left\{ \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega^3_{k0}} \right\}_{\text{reg}} = -\frac{m_0^2}{16\pi^2} \left\{ \frac{2}{\epsilon} + \ln \frac{4\pi\mu^2}{m_0^2} - \gamma + 1 \right\}$$

$$= -m_0^2 I_{-3}(m_0) - \frac{m_0^2}{16\pi^2}.$$  

(3.6)

The additional temperature dependent terms $\Sigma_{-k}(m_0, T)$ are finite. They are defined as

$$\Sigma_{-1}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega_{k0} \left( e^{\beta\omega_{k0}} - 1 \right)},$$

(3.7)

$$\Sigma_{-3}(m_0, T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega^3_{k0} \left( e^{\beta\omega_{k0}} - 1 \right)}.$$  

(3.8)

It is convenient to include these finite terms into the definition of $F_{\text{fin}}(t, T)$. Then the time dependent mass takes the form

$$M^2(t) = \lambda(\phi^2 - v^2) + \delta\lambda \phi^2 + \delta m^2 + (\lambda + \delta\lambda) \left[ I_{-1}(m_0) - I_{-3}(m_0)\mathcal{V}(t) + \tilde{F}_{\text{fin}}(t, T) \right],$$

(3.9)

with

$$\tilde{F}_{\text{fin}}(t, T) = \Sigma_{-1}(m_0, T) - \mathcal{V}(t) \Sigma_{-3}(m_0, T) + F_{\text{fin}}(t, T).$$

(3.10)

The time dependent mass (3.9) contains both renormalization constants $\delta m$ and $\delta\lambda$. Furthermore, its definition by this equation is implicit, $M^2(t)$ also appears on the right hand side of (3.9) in $\mathcal{V}(t)$.

We now have to fix the renormalization counterterms in such a way that the relation between the time dependent mass and $\phi(t)$ becomes finite. An additional constraint derives from the requirement that the renormalization counterterms should not depend on the initial condition, but only on the parameters appearing in the Lagrangian, i.e., $\lambda$ and $v$ and renormalization conventions.

We first determine $\delta\lambda$ by considering the difference

$$\mathcal{V}(t) = M^2(t) - M^2(0)$$

$$= (\lambda + \delta\lambda) \left[ \phi^2(t) - \phi^2(0) - I_{-3}(m_0)\mathcal{V}(t) + \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right].$$

(3.11)

The divergent parts depend on the initial mass $m_0$. We have to replace this by a renormalization scale independent of the initial conditions. In Ref. [8] we had chosen the scale $m$, where $m$ was the mass parameter appearing in the Lagrangian. Here the analogous mass squared would be $m^2 = -\lambda v^2$ and so $m$ would be imaginary. We therefore choose another scale $m_1$ which we do not
specify here. In the numerical computations we have used the physical Higgs mass \(m_1^2 = m_H^2 = 2\lambda v^2\).

We rewrite the implicit equation for \(V(t)\) as

\[
V(t) \left[ 1 + (\lambda + \delta \lambda) I_{-3}(m_1) \right] = (\lambda + \delta \lambda) \left\{ \phi^2(t) - \phi^2(0) - [I_{-3}(m_0) - I_{-3}(m_1)] V(t) \right. \\
+ \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \left. \right\}
\]

and require

\[
\frac{\lambda + \delta \lambda}{1 + (\lambda + \delta \lambda) I_{-3}(m_1)} = \lambda.
\]

Solving with respect to \(\delta \lambda\) we find

\[
\delta \lambda = \frac{\lambda^2 I_{-3}(m_1)}{1 + \lambda I_{-3}(m_1)}.
\]

Inserting this relation into (3.12) we find

\[
V(t) = \lambda C \left[ \phi^2(t) - \phi^2(0) - \tilde{F}_{\text{fin}}(t, T) - \tilde{F}_{\text{fin}}(0, T) \right].
\]

with

\[
C = \frac{1}{1 + \lambda [I_{-3}(m_0) - I_{-3}(m_1)]} = \frac{1}{1 + \frac{\lambda}{16\pi^2} \ln \left( \frac{m_1^2}{m_0^2} \right)}.
\]

Eq. (3.15) is a finite relation for the potential \(V(t)\) since the difference \([I_{-3}(m_0) - I_{-3}(m_1)]\) is finite. Going back to Eq. (3.10) we realize that \(\tilde{F}_{\text{fin}}\) on the right hand side contains itself a term proportional to \(V(t)\). Taking account of this term we rewrite \(V(t)\) in terms of \(F_{\text{fin}}\) as

\[
V(t) = \lambda C_T \left[ \phi^2(t) - \phi^2(0) + F_{\text{fin}}(t, T) \right]
\]

with

\[
C_T = \frac{1}{1 + \frac{\lambda}{16\pi^2} \ln \left( \frac{m_1^2}{m_0^2} \right) + \lambda \Sigma_{-3}(m_0, T)}.
\]

Recall that \(F_{\text{fin}}(t)\) is the mode integral of second order in \(V(t)\) and vanishes at \(t = 0\).

We now go back to equation (3.9) which we take at the initial time \(t = 0\):

\[
m_0^2 \equiv M^2(0) = \lambda [\phi^2(0) - v^2] + \delta \lambda \phi^2(0) + \delta m^2 + (\lambda + \delta \lambda) \left[ I_{-1}(m_0) + \tilde{F}_{\text{fin}}(0, T) \right].
\]

This is an implicit relation between \(m_0\) and \(\phi(0)\) which, however, contains still the infinite quantities \(\delta \lambda\), \(\delta m\) and \(I_{-1}(m_0)\). Using Eq. (3.6) we can rewrite Eq. (3.19) as

\[
m_0^2 = \left( -\lambda v^2 + \delta m^2 \right) + (\lambda + \delta \lambda) \left[ \phi^2(0) - m_0^2 I_{-3}(m_0) - \frac{m_0^2}{16\pi^2} + \tilde{F}_{\text{fin}}(0, T) \right].
\]
As renormalization condition we require $m_0$ to vanish, for temperature $T = 0$, at the minimum of the potential $\phi = v$, as it is the case on the tree level. We note that $m_0^2 = 0$ is not the curvature of the tree level potential at $\phi = v$ which is $m_H^2 = 2\lambda v^2$. It is the mass of the fluctuations at $\phi = v$ in the large-$N$ approximation. For $T = 0$ we have $\mathcal{F}_{\text{fin}}(t = 0, T = 0) = \Sigma_{-1}(m_0, T = 0) = 0$. Setting $m_0 = 0$, $\phi(0) = v$ in the gap equation (3.20) we get immediately

$$\delta m^2 = -\delta \lambda v^2 = -\frac{\lambda^2 v^2 I_{-3}(m_1)}{1 - \lambda I_{-3}(m_1)}.$$  \hspace{1cm} (3.21)

Inserting this into Eq. (3.20) we obtain the renormalized gap equation

$$m_0^2 = \lambda C \left[ \phi^2(0) - v^2 - \frac{m_0^2}{16\pi^2} + \frac{\Sigma_{-1}(m_0, T)}{1} \right].$$  \hspace{1cm} (3.22)

For the numerical computation it is easier to choose some $m_0^2 \geq 0$ and to use the gap equation solved for $\phi^2(0)$:

$$\phi^2(0) = \frac{m_0^2}{\lambda} + v^2 + \frac{m_0^2}{16\pi^2} \left( 1 + \ln \frac{m_1}{m_0^2} \right) - \Sigma_{-1}(m_0, T).$$  \hspace{1cm} (3.23)

For $t > 0$ the renormalized relation for the mass squared $\mathcal{M}^2(t)$ we find, using Eqns. (3.15) and (3.22), is

$$\mathcal{M}^2(t) = m_0^2 + \mathcal{V}(t) = \lambda C \left[ \phi^2(t) - v^2 - \frac{m_0^2}{16\pi^2} + \mathcal{F}_{\text{fin}}(t, T) \right].$$  \hspace{1cm} (3.24)

Having thus obtained a finite relation between $\phi(t)$ and $\mathcal{M}(t)$ the equations of motion for the classical field $\phi(t)$ and for the modes $U_k(t)$ are well-defined and finite.

The way in which we have renormalized has made the cutoff disappear. This was possible only to the extent that we could safely neglect corrections of order $\epsilon$ in the evaluation of the divergent integrals. One way of achieving this is to take the limit $\epsilon \to 0$. This implies for the bare coupling $\lambda_0$

$$\lambda_0 = \lim_{\epsilon \to 0} \frac{\lambda}{1 - \frac{\lambda/2}{16\pi^2 \epsilon}} = 0^-,$$  \hspace{1cm} (3.25)

so this is the case of “negative bare coupling” as discussed in [3]. One can leave the cutoff finite, however, as long as the masses and momenta are much smaller than the scale of the Landau ghost, $m_x = m_0^2 \exp(8\pi^2/\lambda)$. This will be case here. This is not related to a pragmatic momentum cutoff that we apply to the convergent integrals of the finite part.
While we have found here the gap equation as a self-consistency condition, it can also be derived \[3, 7\] from a potential (free energy) which here takes the form

\[
V(m_0^2, \Phi^2, T) = \frac{m_0^2}{2} \left\{ \phi^2 - v^2 - \frac{m_0^2}{2\lambda} + \frac{m_0^2}{32\pi^2} \left[ \ln \left( \frac{m_0^2}{m_1^2} \right) - \frac{3}{2} \right] \right\} + \int \frac{d^3k}{(2\pi)^3} \frac{1}{\beta} \ln \left[ \frac{1}{\exp(-\beta \omega_k)} \right].
\] (3.26)

The gap equation then follows from the condition

\[
\frac{\partial V(m_0^2, \phi^2, T)}{\partial m_0^2} = 0.
\] (3.27)

It should be mentioned here that the gap equation has two solutions, one of which lies above the scale of the Landau ghost, \(m_m = m_1 \exp(8\pi^2/\lambda^2)\). In the sense that we consider here the model as giving rise to a low energy effective theory we discard this high mass solution, and its discussion. The solution we consider is the low energy one which is of order \(\sqrt{\lambda}v\) or \(m_1\).

The energy density is given by

\[
\mathcal{E} = \frac{1}{2} \dot{\phi}^2(t) + \frac{1}{4} \left( \lambda + \delta \lambda \right) \left( \phi^2 - v^2 \right)^2 + \delta \Lambda
\]

\[
+ \mathcal{E}_\text{fl}(t, T) - \frac{\lambda + \delta \lambda}{4} \mathcal{F}^2(t, T).
\] (3.28)

Here we have used already that \(\delta m^2 = -\delta \lambda v^2\), and part of the “cosmological constant” counterterm \(\delta \Lambda\) is included in \(\delta \lambda v^4/4\). The fluctuation energy is given by

\[
\mathcal{E}_\text{fl}(t, T) = \int \frac{d^3k}{(2\pi)^3 \omega_{\omega_k}} \coth \frac{\beta \omega_k}{2} \left\{ \frac{1}{2} |\dot{U}_k(t)|^2 + \frac{1}{2} \omega_k |U_k(t)|^2 \right\}.
\] (3.29)

We again split off the temperature-dependent contribution via

\[
\mathcal{E}_\text{fl}(t, T) = \mathcal{E}_\text{fl}(t, 0) + \Delta \mathcal{E}_\text{fl}(t, T),
\] (3.30)

where the second term on the right hand side

\[
\Delta \mathcal{E}_\text{fl}(t, T) = \int \frac{d^3k}{(2\pi)^3} \frac{2}{\omega_{\omega_k} - 1} \left\{ \frac{1}{2} |\dot{U}_k(t)|^2 + \frac{1}{2} \omega_k^2 |U_k(t)|^2 \right\},
\] (3.31)

is finite. The divergences of the first term are given \[13\] by the decomposition

\[
\mathcal{E}_\text{fl}(t, 0) = I_1(m_0) + \frac{1}{2} \mathcal{V}(t) I_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) I_{-3}(m_0) + \mathcal{E}_{\text{fl,fin}}(t, 0)
\] (3.32)
with
\[ E_{\text{fl,fin}}(t,0) = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_{k0}} \left\{ \frac{1}{2} |\tilde{f}_k^{(1)}|^2 + \frac{\mathcal{V}(t)}{2} \left[ 2\text{Re}\tilde{f}_k^{(1)} + |\hat{f}_k^{(1)}|^2 \right] + \frac{\mathcal{V}^2(t)}{8\omega_{k0}^2} \right\}. \] (3.33)

We denote the sum of \( E_{\text{fl,fin}}(t,0) \) and \( \Delta E_{\text{fl}}(t,T) \) finite contributions as \( E_{\text{fl,fin}}(t,T) \).

The expression for the energy then takes the form
\[ \mathcal{E} = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda + \delta \lambda}{4} \left( \phi^2 - v^2 \right)^2 + E_{\text{fl,fin}}(t,T) + I_1(m_0) + \frac{1}{2} \mathcal{V}(t) I_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) I_{-3}(m_0) \]
\[ -\frac{\lambda + \delta \lambda}{4} \mathcal{F}^2(t,T) + \delta \Lambda. \] (3.34)

In addition to the divergences arising from \( E_{\text{fl}}(t,T) \) we have to take into consideration those of \( \mathcal{F}^2(t,T) \) which we have analyzed above. If all divergences and the renormalization constant \( \delta \lambda \) are inserted, the expression turns out to be finite, i.e., the remaining counterterm \( \delta \Lambda \) is needed only for a finite renormalization.

We require the energy to vanish at \( T = 0 \) for \( \phi(t) \equiv v \), which implies \( m_0 = 0 \). Then \( \delta \Lambda = 0 \). There remains a finite constant dependent on the initial condition
\[ \Delta \Lambda = \frac{m_0^4}{128\pi^2} \left( 1 + \frac{2\lambda C}{16\pi^2} \right). \] (3.35)

and the energy is given by
\[ \mathcal{E} = \frac{1}{2} \dot{\phi}^2 + \frac{\lambda}{4} C (\phi^2 - v^2)^2 + \frac{1}{2} \Delta m^2 (\phi^2 - v^2) \]
\[ + E_{\text{fl,fin}}(t,T) - \frac{\lambda}{4} C \mathcal{F}^2_{\text{fin}}(t,T) + \Delta \Lambda. \] (3.36)

Here \( \Delta m^2 \) is given by
\[ \Delta m^2 = -\lambda C \frac{m_0^2}{16\pi^2}. \] (3.37)

We write the pressure in the form
\[ p = \dot{\phi}^2(t) - \mathcal{E} + p_{\text{fl,fin}}(t,T) + \delta \xi \frac{d^2}{dt^2} \left[ \phi^2(t) + \mathcal{F}(t,T) \right]. \] (3.38)

The renormalization does not differ from the case of unbroken symmetry discussed in Ref. [8] and is not presented again. We find
\[ \delta \xi = \frac{\lambda x}{6(1 - \lambda x)} = \frac{\lambda I_{-3}(m)}{6(1 - \lambda I_{-3}(m))}. \] (3.39)

The final result for the renormalized pressure reads
\[ p = \dot{\phi}^2(t) - \mathcal{E} + \mathcal{P}_{\text{fin}}(t,T) - \frac{m_0^4}{96\pi^2} - \frac{m_0^2}{48\pi^2} \mathcal{V}(t) - \frac{1}{96\pi^2} \left[ \ln \left( \frac{m_1^2}{m_0^2} \right) + 2 \right] \mathcal{J}(t). \] (3.40)
with

\[
p_{\text{fl,fin}}(t, 0) = \int \frac{d^3 k}{(2\pi)^3 2\omega_{k0}} \left\{ \left( \omega_{k0}^2 + \frac{k^2}{3} \right) \left[ 2\text{Re} f_\text{f}^{(2)}(t) + |f_\text{f}^{(1)}(t)|^2 \right] \right.
\]
\[
+ \left( \frac{1}{6\omega_{k0}^2} - \frac{m_0^2}{24\omega_{k0}^4} \right) \int_0^t dt' \cos 2\omega_{k0}(t - t') \tilde{\bar{\nu}}(t') \right.
\]
\[
+ \left( \frac{1}{12\omega_{k0}^2} + \frac{m_0^2}{24\omega_{k0}^4} \right) \cos(2\omega_{k0}t) \tilde{\bar{\nu}}(0)
\]
\[
+ |f_\text{f}^{(1)}(t)|^2 - 2\text{Re} \left[ i\omega_{k0} f_\text{f}^{(1)}(t) + i\omega_{k0} f_\text{f}^{(1)(t)} f_\text{f}^{(1)}(t) \right]. \quad (3.41)
\]

\[
4 \quad \text{Analysis of the gap equation and of the phase structure}
\]

The dynamical evolution of the nonequilibrium system depends on two parameters, the temperature \(T\) and the initial amplitude of the classical field \(\phi(0) = \phi_0\) which in analogy with thermal equilibrium systems can be considered as an external parameter. There are two regions from which we can start the system, which we will call regions II and III. There is, in addition, one region into which the system can evolve when one considers \(\phi(t = \infty) = \phi_\infty\) and not \(\phi_0\) as the external parameter. We call it region I. In this section we will characterize these regions and describe the dynamical evolution as to be expected from the analysis at \(T = 0\). This analysis is based on certain empirical results [6, 7] that, though unproven, seem to be at least almost exact. We will generalize these results to finite temperature in a plausible heuristic way, to be confirmed by the numerical computations. We think that the way in which we generalize these results will give a further clue to understanding them.

\[4.1 \quad \text{Region I, } m_0^2 < 0\]

The gap equation requires \(m_0^2\) to be positive. The point where \(m_0^2 = 0\) marks an initial condition that leads to a solution \(\phi = \text{const.}, \phi_0 = 0\) as we will assume in the following. For \(T = 0\) this stationary amplitude is \(\phi = v\). For \(T > 0\) we can easily find this amplitude as well. Indeed for \(m_0 = 0\) the integral \(\Sigma_{-1}(m_0, T)\) is given by its value for massless quanta, i.e.,

\[
\Sigma_{-1}(0, T) = \frac{T^2}{12}, \quad (4.1)
\]

therefore

\[
\phi_1^2(T) = \phi_1^2|_{m_0=0} = v^2 - \frac{T^2}{12}. \quad (4.2)
\]
For $\phi_0 < \phi_0(T)$ the gap equation has no real solution $m_0$. The region below the boundary (1.2) is region I.

If nevertheless one wants to start the system with $\phi_0$ in region I one faces the problem that in this region the gap equation requires $m_0^2$ to be negative. Then the low-momentum modes with $k^2 < -m_0^2$ have imaginary frequencies. So from an orthodox point of view (to which we adhere here) the system cannot be quantized properly. One may avoid this problem by redefining the dispersion relation for the initial frequencies via $\omega_{k_0}^2 = k^2 + |m_0^2|$ in this region. Of course at $t > 0$ $\mathcal{M}^2(t)$ will be negative so the “mass squared” changes sign at $T = 0$, a situation called “quench” in analogy by a similar transition form a stable to an unstable state by a sudden drop of temperature or inversion of a magnet field. On the other hand the amplitude $\phi(t)$ can reach this region at late times, but then it is in a quantum state different form the ones we use as initial states.

4.2 Region II: $m_0^2 > 0, m_\infty^2 = 0$

We now assume $\phi_0$ is started above the boundary value (1.2). If $\phi_0$ is not too large the system may, at $t > 0$, enter a region where $\mathcal{M}^2(t) < 0$, i.e., region I. Then the quantum fluctuations with momenta $k^2 < -\mathcal{M}(t)$ will increase exponentially, signalling instability. This causes $\mathcal{M}(t)$ to increase so that it is driven back to a value $\mathcal{M}(t) > 0$. If the initial amplitude $\phi_0$ is sufficiently small this forth-and-back reaction will lead $\mathcal{M}(t)$ to stabilize at $\mathcal{M}\infty = 0$. So at late times $\tilde{\mathcal{F}}_{\text{fin}}(t, T)$ is determined by quantum modes $U_k(t)$ that oscillate with time-independent frequencies $\omega_\infty = k$, it becomes stationary as well and will be positive. Therefore $\phi(t)$ stabilizes at some value

$$\phi_\infty^2 = v^2 - \tilde{\mathcal{F}}_{\text{fin}}(\infty, T) < v^2 - \tilde{\mathcal{F}}_{\text{fin}}(0, T). \quad (4.3)$$

This is entirely analogous to the behavior found at $T = 0$ [7]. We call the region of initial values $\phi_0$ leading to this late-time behavior region II.

The stabilization by back-reaction onto the fluctuations obtained in the large-$N$ approximation is not present in the one-loop approximation. In this approximation, once $\phi(t)$ dips into the unstable region $\phi(t) < v/\sqrt{3}$, the mass squared of the fluctuations becomes negative and the low momentum modes evolve exponentially. The effective mass of the classical field increases exponentially as well, and continues to do so, but the mass squared of the fluctuations stays negative. The amplitude $\phi(t)$ is driven towards zero. Nevertheless the classical energy continues to increase, as the field oscillates faster and faster, this energy being extracted from the energy of the quantum fluctuations. Obviously this signals the instability of the quantum vacuum, as already apparent from the fact that the effective potential is complex in this region. We will illustrate this by a numerical example, to be presented in the next section.
At \( T = 0 \) the final value \( \phi_\infty \) was found to be related to the initial value \( \phi_0 \) by an empirical relation

\[
\phi_\infty^2 = \sqrt{\phi_0^2(2v^2 - \phi_0^2)} \quad T = 0.
\]  

(4.4)

It is not obvious how to generalize this relation to finite temperature. It was remarked in Ref. [7] that the relation only depends on the initial, purely classical, energy, which is given by \( E = \lambda(\phi^2 - v^2)^2/4 \). Obviously it satisfies the constraints that \( \phi_\infty^2 = v^2 \) if \( \phi_0 = v^2 \), and that \( \phi_\infty = 0 \) if classically the system can reach the maximum of the potential; this happens at \( \phi_0^2 = \phi_0^2(T = 0) = 2v^2 \). So Eq. (4.4) seems to be related to energy considerations. We further observe that the classical turning point is at \( \bar{\phi}_0^2 = 2v^2 - \phi_0^2 \) so that one may write Eq. (4.4) as the geometric mean

\[
\phi_\infty^2 = \sqrt{\phi_0^2 \bar{\phi}_0^2}.
\]  

(4.5)

This form turns indeed out to lead to the correct generalization for finite temperature.

Obviously the relation is characterized by the motion at early times when the quantum fluctuations have not yet evolved. When discussing renormalization we have made an expansion with respect to the “potential” \( \mathcal{V}(t) \) which vanishes at \( t = 0 \). So the same expansion can be used to study the early time behavior. In the energy the coefficients of the terms of first and second order in \( \mathcal{V} \) have been absorbed into renormalization constants. However, the thermal fluctuations are not absorbed in this way and will add to the classical terms in an early time expansion. These appear in the energy, see Eq. (3.34), via

\[
\Delta \mathcal{E}_\text{fl}(t, T) = \Sigma_1(m_0) + \frac{1}{2} \mathcal{V}(t) \Sigma_{-1}(m_0) - \frac{1}{4} \mathcal{V}^2(t) \Sigma_{-3}(m_0) + O(\mathcal{V}^3)
\]  

(4.6)

as a part of \( \mathcal{E}_{\text{fl, fin}}(t, T) \) and via Eq. (3.4) in \( \tilde{\mathcal{F}}_{\text{fin}}(t, T) \). Taking these expansions into account the energy can be written in the form

\[
E \simeq \frac{\lambda}{4} \mathcal{C} \left[ a\phi^4 + \bar{a}\phi_0^4 + b\phi^2 + \bar{b}\phi_0^2 + c\phi^2\phi_0^2 \right] + \text{const.}
\]  

(4.7)

up to terms of order \( \mathcal{V}^3 \). We need the coefficients

\[
a = 1 - \lambda\mathcal{C}_T \Sigma_{-3}(m_0, T),
\]  

(4.8)

\[
b = -2 \left[ v^2 - \Sigma_{-1}(m_0, T) \right],
\]  

(4.9)

\[
c = \lambda\mathcal{C}_T \Sigma_{-3}(m_0, T).
\]  

(4.10)

The classical turning point is given by

\[
\bar{\phi}_0^2 = -\frac{b + (a + c)\phi_0^2}{a} = \frac{1}{1 - \lambda\mathcal{C}_T \Sigma_{-3}} \left[ 2v^2 - \Sigma_{-1} - (1 + \lambda\mathcal{C}_T \Sigma_{-3})\phi_0^2 \right],
\]  

(4.11)
so that we are led to suppose

\[ \phi^2_\infty(T) = \sqrt{\frac{1}{1 - \lambda C T \Sigma_{-3}}} \sqrt{\phi_0^2 [2\nu^2 - 2\Sigma_{-1} - (1 + \lambda C T \Sigma_{-3})\phi_0^2]} . \] (4.12)

We find indeed (see below) that this relation is very well fulfilled numerically. According to this formula the region II is limited by the requirement that the expression in the square root be positive, so the boundary between region II and the new region III is given by

\[ \phi^2_0 = 2\nu^2 - \frac{\Sigma_{-1}(m_0, T)}{1 + \lambda C T \Sigma_{-3}(m_0, T)} . \] (4.13)

We note that the relation is implicit, the value of \( m_0 \) that appears on the right hand side is related to \( \phi^2_0 \) on the left hand side by the gap equation.

### 4.3 Region III, \( \phi_\infty = 0 \) and \( M^2_\infty > 0 \)

If the value \( \phi_0 \) becomes larger than \( \phi_2 \) the stationary state with constant \( \phi \) and vanishing mass \( M^2(t) \) is no longer attained, and the system reaches another asymptotic regime where \( M^2(\infty) \neq 0 \) whereas \( \phi(t) \rightarrow 0 \). This regime is similar to the one that describes the late time behavior for the unbroken symmetry case. We call the region of initial values \( \phi_0 \) that leads to such a behavior region III.

There are two phenomena that characterize the transition to this region. On the one hand the stabilization of the system is taken over by the phenomenon of parametric resonance. On the other hand the system has enough energy so that \( \phi(t) \) can move over the maximum of the potential at \( \phi = 0 \), and indeed will oscillate around \( \phi = 0 \). Accordingly the threshold value of \( \phi_0 \) at which these two phenomena set in can be characterized by two - a priori unrelated - criteria. Both rely on plausible assumptions, which at \( T = 0 \) lead to the same prediction for the critical value of \( \phi_0 \).

The criterion based on the energy consideration has been presented in the previous subsection, we now describe the criterion supplied by the phenomenon of parametric resonance. For the case of unbroken symmetry it was found at zero \[3\] and finite temperature \[8\], that the late time behavior is described by an empirical sum rule which relates \( M^2_\infty \) to the initial amplitude. For \( T = 0 \) an analogous sum rule was found to hold for the case of spontaneously broken symmetry as well \[7\]. It is given by

\[ \mu^2_\infty = -1 + \frac{\eta_0^2}{2} . \] (4.14)

Here \( \mu \) and \( \eta \) are normalized in such a way that the classical equation of motion at early times, i.e., in the parametric resonance regime without back reaction, reads

\[ \eta'' - \eta + \eta^3 = 0 , \] (4.15)
where the prime denotes a derivative with respect to $\tau = \alpha t$ and where $\eta = \beta \phi$, also $\mu = M/\alpha$. With $\eta(\tau)$ a solution of Eq. (4.15) the mode equation becomes a Lamé equation. The sum rule implies $[6]$, that the frequencies $\omega^2(t) = M^2(t) + k^2$ are shifted outside the parametric resonance band of the Lamé equation. Though there is no rigorous derivation for the sum rule, it accordingly seems related to the parametric resonance phenomenon.

As the shift of the frequencies outside the parametric resonance region must have happened at the end of the phase where the evolution of the system is described by parametric resonance, we will again consider the initial classical evolution. Again, in addition to the classical terms we have to take into account the terms due to the thermal fluctuations. In terms of the parameters introduced in the previous section the equation of motion is given by

$$\ddot{\phi} + \lambda C a \phi^3 + \frac{\lambda}{2} C (b + c \phi_0^2) \phi = 0 . \quad (4.16)$$

Comparing to the normalized equation (4.15) we determine the factors $\alpha$ and $\beta$ to be

$$\alpha = \sqrt{\frac{\lambda C}{2}} \sqrt{b + c \phi_0^2} , \quad (4.17)$$
$$\beta = \sqrt{-\frac{2a}{b + c \phi_0^2}} , \quad (4.18)$$

so that the asymptotic mass is given by

$$M^2_\infty = \alpha^2 (-1 + \frac{1}{2} \beta^2 \phi_0^2) \quad (4.19)$$

$$= \lambda C \left\{-v^2 + \Sigma_{-1}(m_0, T) + \frac{1}{2} \left[1 + \lambda C \Sigma_{-3}(m_0, T)\right] \phi_0^2\right\}.$$

Again $\phi_0$ and $m_0$ are related by the gap equation. At the transition from region II to region III the asymptotic mass vanishes. It is easily seen that this criterion leads to an identical equation for the boundary, i.e., Eq. (4.13).

The field amplitude decreases to zero at late times, in this regime. So the symmetry is restored dynamically at high excitation characterized by a high value of $\phi_0$.

At the critical temperature $T = \sqrt{12v}$ both boundaries $\phi_1(T)$ and $\phi_2(T)$ become zero. Above $T_C$ the behavior of the system is the same as for region III, for all initial values of $\phi_0$. While at the border between region I and II there was a lowest value for $\phi_0$ for obtaining real solutions of the gap equation, now there is a lowest value of $m_0$, the one for which $\phi_0 = 0$. It is obtained by solving the gap equation for $\phi_0 = 0$ and agrees with the thermodynamical equilibrium value $m_\beta$ at that temperature, as defined, e.g., in Eq. (3.38) of Ref. [2]. Of course with $\phi_0 = 0$ the system remains static.
Having defined the three regions by the two boundaries (4.2) and (4.13) we present, in Fig. 1, a phase diagram in the $\phi^2_0 - T$ plane. Fig. 2 shows the phase diagram in the $m^2_0 - T$ plane, displaying, above $T_C$, the region $m_0 < m_\beta$ which is excluded as an initial condition. We have to stress that the boundary between regions II and III relies on an empirical relation.

The symmetry restoration above a critical temperature is expected naively. However, if the temperature becomes nonperturbatively large, $T \simeq \sqrt{12m_1 \exp(8\pi^2/\lambda)}$, the gap equation does not have solutions any longer. Then the free energy attains its maximum at the boundary $m_0 = 0$ and the $O(N)$ symmetry is again broken [3]. This phenomenon of "symmetry non-restoration", as well as the existence of the second solution of the gap equation above $m_x = m_1 \exp(8\pi^2/\lambda)$ will not be discussed here, as it is not part of the low energy effective theory.

5 Numerical Results

We have discussed already in the previous section the type of nonequilibrium behavior to be expected in the different regions of phase space. The numerical results follow these expectations. We have chosen generally the parameters $v = 1$ and $\lambda = 1$. We present results for the various regions in the $T, \phi_0$ plane. The critical temperature is $2\sqrt{3} = 3.464$. We choose the temperatures between $T = 1$ and 4, the latter one being above the phase transition. The numerical method has been described in [8]. We just recall that all the integrals computed numerically are finite, so cutting off the momentum integration at some reasonable value is unrelated to cutoffs used for renormalization.

We first consider initial conditions in region II. The expectation value of $\phi$, shown in Fig. 3 becomes constant and different from zero as $t \to \infty$. This signals spontaneous breakdown of the $O(N)$ symmetry. As displayed in Fig. 4 the mass $M^2(t)$ vanishes as $t \to \infty$, as expected form the Goldstone theorem. The momentum distribution of the quantum fluctuations peaks at $k = 0$ as $|U_k(t)|^2 \propto k^{-2}$, leading to long-range correlations, a phenomenon called “dynamical Bose-Einstein condensation” in Ref. [7] and investigated further, for finite volume, in Ref. [18]. We show an example of the momentum distribution in Fig. 5, but we have not studied the phenomenon in detail.

The relation between the asymptotic value as $t \to \infty$ for $\phi(t)$ and the initial amplitude $\phi_0$ is displayed in Figs. 6 to 8, for $T = 1, 2.5$ and 3. We compare the data with our generalization (4.12) of the empirical formula (4.4) given in Ref. [7]. The data are obtained by averaging over the second half of the time interval. The agreement is excellent, except at the phase boundary where the averaging converges slowly.

As an illustration of the behavior of the system in the unstable region in the one-loop approximation we show, in Fig. 9, the evolution of the field amplitude, and, in Fig. 10, the exponential behavior of the fluctuation integral and of the
effective mass squared $\mathcal{M}^2(t)$ of the classical field.

The behavior of the system in region III is displayed in Figs. 11 and 12. The amplitude $\phi(t)$ is seen to decrease to zero. The decrease is powerlike, not exponential, a phenomenon called anomalous relaxation in Ref. [8]. Fig. 12 shows the squared mass $\mathcal{M}^2(t)$ which is seen to converge to an asymptotic value $\mathcal{M}^2_\infty$. The sum rule for this asymptotic value, Eq. (1.12), is compared to the data in Fig. 13 for $T = 1.5, 2.5$ and 4. The agreement is again excellent.

We have not presented the results for the pressure and the ratio of pressure and energy which varies between 0 for a nonrelativistic ensemble and $1/3$ for an ultrarelativistic ensemble. Here these are dominated, already at $T = 1$, by the purely thermal contributions, so that the fluctuations generated by the motion of the field $\phi(t)$ are relatively unimportant.

6 Conclusions and Outlook

The dynamical exploration of the quantum states of the $O(N) \lambda \Phi^4$ theory in the limit $N \to \infty$ has been extended here to finite temperature. We have performed numerical simulations with various initial fields $\phi_0 = \phi(0)$ and initial masses $m_0$ related by the gap equation, and for various temperatures $T$. Depending on the initial conditions we find, in analogy to computations at zero temperature [3, 4], final states with restored $O(N)$ symmetry and final states for which the symmetry is spontaneously broken. The resulting phase diagrams resemble typical phase diagrams of thermodynamical systems, with the temperature and an external variable as parameters. Instead of, e.g., the magnetic field or the pressure we have here the initial value $\phi_0$ as external parameter. While the initial states are thermal states, the final states are not.

We have generalized two empirical formulae, the relation between the initial and asymptotic field amplitudes in region II, and the formula for the asymptotic value of $\mathcal{M}^2(t)$ in region III to finite temperature, extending the plausibility arguments given in [3]. While we have not been able, either, to derive these formulae, the way of generalizing them may give some clue for such a derivation. Both relations are linked as they give the same formula for the boundary between regions II and III, though the arguments for their heuristic derivation are seemingly different. Furthermore, it is clear that both of them are based on the early time behavior. Obviously the fluctuations have to be included up to order $V^2(t)$ in a perturbative expansion. At $T = 0$ these terms are essentially absorbed into renormalization constants, so that the purely classical behavior prevails. One may also formulate the modifications at finite temperature in terms of temperature dependent masses and couplings. It is the role of the large-$N$ quantum back reaction to transmit the early time behavior into the late time one.

Unfortunately there are many interesting models for which the large-$N$ approach is not possible or not adequate. The one-loop approximation, on the other
hand, can be applied in general. However, it shows features that seem to make it obsolete for describing nonequilibrium phenomena, especially for theories with spontaneous symmetry breaking. As an illustration we have shown the typical behavior of a spontaneously broken $\lambda\Phi^4$ model in the one-loop approximation. The system does not reach a stationary state at late times: the effective mass of the classical field diverges exponentially, while the effective mass of the quantum fluctuations is and stays negative. This is due to the lack of the quantum back reaction onto the fluctuations. The fact that one finds such a pathological behavior may, however, indicate the correct physics and is not necessarily a consequence of an inadequate approximation. It is known that the system is indeed unstable for spatially constant static fields, it is an instability with respect to formation of domains [14]. For space dependent fields like minimal bubble configurations the one-loop approximation to the effective action does not display any unplausible features [15, 16, 17], though the effective potential is complex in the unstable region. So it is not clear whether the “taming” of the instability introduced by the large-$N$ approximation necessarily improves the understanding of the physics.

In this situation it is certainly very important to develop new approaches to the evolution of quantum systems for theories with spontaneously broken symmetry [19, 20]. There are indications in a large-$N$ quantum mechanical system [20] that the large-$N$ limit may be misleading, as the next-to-leading corrections become large especially at late times. It is not clear, however, what the impact of these results on quantum field theory will be. One of the problems is that, in contrast to the large-$N$ and one loop approximations, alternative wave functionals pose problems with renormalization [19]. This is not only a technical problem. It is connected (trivially) to the fact that the higher the dimension of space, the more the ultraviolet behavior of the system will be important.

We think nevertheless, that a good understanding of the leading order approximation may improve the understanding of the corrections. That these become large at late times is not too surprising, it is therefore even more important to realize (once more) that the late-time behavior is related to the early-time behavior, which will therefore set the initial conditions for other approximations as well. This should apply to the phase structure as well.

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A Perturbative expansion

The mode functions $U_k(t)$ with the initial conditions introduced in section 2 satisfy the integral equation

$$U_k(t) = e^{-i\omega k_0 t} + \int_0^\infty dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') U_k(t') , \quad (A.1)$$

with

$$\Delta_{k,\text{ret}}(t-t') = -\frac{1}{\omega k_0} \Theta(t-t') \sin (\omega k_0(t-t')) . \quad (A.2)$$

We separate $U_k(t)$ into the trivial part corresponding to the case $\mathcal{V}(t) = 0$ and a function $f_k(t)$ which represents the reaction to the potential by making the ansatz

$$U_k(t) = e^{-i\omega k_0 t}[1 + f_k(t)] . \quad (A.3)$$

$f_k(t)$ satisfies then the integral equation

$$f_k(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') [1 + f_k(t')] e^{i\omega k_0(t-t')} , \quad (A.4)$$

and an equivalent differential equation

$$\ddot{f}_k(t) - 2i\omega k_0 \dot{f}_k(t) = -\mathcal{V}(t)[1 + f_k(t)] , \quad (A.5)$$

with the initial conditions $f_k(0) = \dot{f}_k(0) = 0$. We expand now $f_k(t)$ with respect to orders in $\mathcal{V}(t)$ by writing

$$f_k(t) = f_k^{(1)}(t) + f_k^{(2)}(t) + f_k^{(3)}(t) + \cdots \quad (A.6)$$

$$= f_k^{(1)}(t) + f_k^{(2)}(t) , \quad (A.7)$$

where $f_k^{(n)}(t)$ is of n'th order in $\mathcal{V}(t)$ and $f_k^{(n)}(t)$ is the sum over all orders beginning with the n'th one:

$$f_k^{(n)}(t) = \sum_{l=n}^\infty f_k^{(l)}(t) . \quad (A.8)$$

The $f_k^{(n)}$ are obtained by iterating the integral equation (A.4) or the differential equation (A.5). The function $f_k^{(1)}(t)$ is identical to the function $f_k(t)$ itself which is obtained by solving (A.5). The function $f_k^{(2)}(t)$ can again be obtained by iteration via

$$f_k^{(2)}(t) = \int_0^t dt' \Delta_{k,\text{ret}}(t-t') \mathcal{V}(t') f_k^{(1)}(t') e^{i\omega k_0(t-t')} . \quad (A.9)$$
The integral equations can be used in order to derive the asymptotic behavior as \( \omega_{k0} \to \infty \) and to separate divergent and finite contributions. This has been described previously in extenso [13]. We illustrate the procedure by calculating the relevant leading terms for \( f_k^{(1)}(t) \). We have

\[
f_k^{(1)}(t) = \frac{i}{2\omega_{k0}} \int_0^t dt' (\exp(2i\omega_{k0}(t-t')) - 1) \mathbf{V}(t').
\]

(A.10)

Integrating by parts we obtain

\[
f_k^{(1)}(\tau) = -\frac{i}{2\omega_{k0}} \int_0^t dt' \mathbf{V}(t') - \frac{1}{4\omega_{k0}^2} \mathbf{V}(t) + \frac{1}{4\omega_{k0}^2} \int_0^t dt' \exp(2i\omega_{k0}(t-t')) \dot{\mathbf{V}}(t').
\]

(A.11)

For the expansion of the fluctuation integral \( \mathcal{F}(t) \) we need the real part of \( f_k^{(1)} \) for which we find

\[
\text{Re} \ h_k^{(1)}(t) = -\frac{1}{4\omega_{k0}^2} \mathbf{V}(t) + \frac{1}{4\omega_{k0}^2} \int_0^t dt' \cos(2\omega_{k0}(t-t')) \dot{\mathbf{V}}(t').
\]

(A.12)

The second term decreases at least as \( \omega_{k0}^{-3} \). In terms of the perturbative expansion for the functions \( f_k \) we can the mode functions appearing in the fluctuation integral as

\[
|U_k|^2 = 1 + 2\text{Re} \ f_k^{(1)} + |f_k^{(1)}|^2.
\]

(A.13)

Using Eq. (A.12) the leading behavior of this expression is

\[
1 + 2\text{Re} \ f_k^{(1)} + |f_k^{(1)}|^2 \simeq 1 - \frac{1}{2\omega_{k0}^2} \mathbf{V}(t).
\]

(A.14)

Similarly the integrand of the energy density and pressure can be expanded [13]. As these are more divergent, the calculations require more integrations by parts in order to single out the leading powers in \( \omega_{k0} \) and they become more involved.

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Figure Captions

Fig. 1: Phase diagram in the $\phi_0^2 - T$ plane.
Fig. 2: Phase diagram in the $m_0^2 - T$ plane.
Fig. 3: Evolution of classical field in region II.
Fig. 4: Evolution of $\mathcal{M}^2(t)$ in region II.
Fig. 5: The momentum spectrum for $T = 1$ at $t = 75$ displaying “dynamical Bose-Einstein condensation”, with a fit $\sin^2(kt)/k^2$.
Fig. 6: Late time amplitude $\phi(\infty)$ vs. initial amplitude $\phi_0$ for $T = 1$ (asterisks), compared with Eq. (4.12) (solid line).
Fig. 7: The same as Fig. 3 for $T = 2$.
Fig. 8: The same as Fig. 3 for $T = 3$.
Fig. 9: Evolution of the classical field in the one-loop approximation.
Fig. 10: The fluctuation integral (solid line) and $\mathcal{M}^2(t)$ (dashed line) in the one-loop approximation.
Fig. 11: Evolution of the classical field in region III.
Fig. 12: Evolution of $\mathcal{M}^2(t)$ in region III.
Fig. 13: The asymptotic sum rule for $\mathcal{M}^2(t)$. The data for $T = 1.5$ (diamonds), $T = 2.5$ (asterisks) and $T = 4$ (triangles) are compared to Eq. (4.19) (solid lines).