HYDRODYNAMIC LIMIT AND PROPAGATION OF CHAOS FOR BROWNIAN PARTICLES REFLECTING FROM A NEWTONIAN BARRIER

CLAYTON BARNES

Abstract. In 2001, Knight constructed a stochastic process modeling the one dimensional interaction of two particles, one being Newtonian in the sense that it obeys Newton’s laws of motion, and the other particle being Brownian. We construct a multi-particle analog, using Skorohod map estimates in proving a propagation of chaos and characterizing the hydrodynamic limit as the solution to a PDE with free boundary condition. Stochastic methods are used to show existence and uniqueness for the free boundary problem, and also present an algorithm of approximating the solution.

1. Introduction

1.1. Introduction. This paper characterizes the hydrodynamic limit for a multiparticle generalization of a process originally introduced by Knight [21]. A hydrodynamic limit result characterizes the continuous dynamics of interacting particles when the number of these particles tends to infinity. The main novelty of this article is in our technique of proof, which utilizes continuity properties of Skorohod maps between function spaces; see section 2. By applying this method with a stochastic representation (Corollary 3.11) we prove existence and uniqueness of the free boundary problem. This is the first existence and uniqueness result for this free boundary problem we study, as it seems not to be subsumed by known results in the analysis literature. This paper is organized as follows. A short

![Figure 1](image_url)  

**Figure 1.** Simulations of 20, 40, and 200 Brownian particles reflecting from the Newtonian barrier.

historical background for the origin of our model, and a brief on related hydrodynamic limits, is in section 1.2 below. In section 1.3 we informally describe the multiparticle, then introduce

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its formal mathematical model, discussing the free boundary problem in section 1.4, followed by our main results in section 1.5. The strong existence of our stochastic model, along with the construction and results of the Skorohod maps, are developed in section 2, while section 3 utilizes these to prove the hydrodynamic limit and propagation of chaos. Finally, section 4 concludes with the proof of uniqueness for the free boundary problem.

1.2. Background. The case of one Brownian particle interacting with a barrier was originally introduced and constructed by Knight in [21] where he studied density of the final velocity. Later, White [27] generalized Knight’s construction and studied several related models. This inspired a higher dimensional version of a reflected process whose velocity vector is proportional to the boundary local time, and the stationary distribution of its position and velocity was studied by Bass, Burdzy, Chen, and Hairer [3]. For a history of hydrodynamic limits see [16] and [7]. The methods used to establish a hydrodynamic limit are varied. See [25], where Varadhan uses entropy methods to examine a spin system on a lattice when the mesh goes to zero. Entropy and relative entropy methods are general methods. However, these do not always seem feasible. See [7], for instance, where Chen and Fan study a system of particles reflecting from a separating interface. For an introductory reading on hydrodynamic limits, see the book [20] where Kipnis and Landim present a self contained treatment of hydrodynamic limits via the study of the generalized exclusion process and the zero-range process. Other hydrodynamic limit results have biological motivations in neuron modeling. See [23], [8], and [17, Chapter 4.3].

Hydrodynamic limits are related to the theory of partial differential equations since the empirical measure of the particles converge to a solution of a PDE or free boundary problem. In [6] Chayes and Swindle study the one dimensional model of hot random walkers which are emitted by a source and which annihilate cold particles which remain stationary. When a Brownian scaling is introduced, the density of the hot particles together with the cold region converge to a solution of the Stefan problem. The Stefan problem is a free boundary problem modeling the melting of ice next to a heat source. The heat particles are killed upon reaching boundary of ice, i.e. a Dirichlet boundary condition is imposed at the ice barrier, while the melting of this ice barrier is proportional to the flux of heat across it. In this way the density of heat and the ice barrier interact, producing the free boundary effect. The hydrodynamic limit we study in this paper resembles that of the Stefan problem but with some distinctive features which we point out in the next subsection; see (1.2).

Existence of a hydrodynamic limit to a free boundary problem is typically shown by tightness of the induced measures. The proof of uniqueness for the limit often appeals to known uniqueness results for the corresponding PDE from the analysis literature.

1.3. Description of the model. We start with an informal description of the model. Consider \( n \) Brownian particles \( X_1^{(n)}(t), \ldots, X_n^{(n)}(t) \) on the real line, reflecting from the same side of a moving barrier \( Y^{(n)}(t) \). The moving barrier is “massive” in the sense that it is not Brownian but obeys Newton’s laws of motion. By this we mean the barrier is modeled to have momentum, and that it experiences an impulse upon colliding with one of the Brownian particles. Impulse is equivalent to the change in momentum, and in Newtonian physics is proportional to the change in velocity. In this way the Brownian particles drive the massive barrier by increasing its velocity. We assume the Brownian particles have an equal “mass” of \( n^{-1} \) so the total mass of the system is 1, and we fix a constant \( K \geq 0 \), the impulse constant which determines the strength of the Brownian particles’ interaction with the massive
barrier. Increasing $K$ will give the Brownian particles more ability to increase the massive barrier’s velocity. If $K = 0$ the Brownian particles have no influence on the barrier, the Brownian particles become independent reflecting Brownian motions while the barrier will travel with constant speed. If $K > 0$, however, the Brownian particles are dependent. This can be seen intuitively, for in the event that one Brownian particle happens to impart a large velocity to the massive barrier, it influences the barrier’s trajectory and alters the region where the other Brownian particles are allowed to disperse themselves.

We now present a formal mathematical model describing the above scenario, and begin by assuming the standard setting: a filtered probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t)_{t \geq 0})$ with the filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying the usual conditions. Take i.i.d. $(\mathcal{F}_t)_{t \geq 0}$-Brownian motions $B^{(1)}, \ldots, B^{(n)}$, a coefficient $K \geq 0$ and an initial velocity $v \in \mathbb{R}$ for the massive particle. Consider continuous $\mathcal{F}_t$ adapted processes which satisfy the system of stochastic differential equations for $t \in [0, T]$ and $i = 1, \ldots, n$:

\[
\begin{align*}
    dX_i^{(n)} &= dB^{(i)} + dL_i^{(n)}, \\
    dY^{(n)} &= V^{(n)}(t)dt := \left( v - \frac{K}{n} \sum_{i=1}^{n} L_i^{(n)}(t) \right)dt,
\end{align*}
\]

\[X_i^{(n)}(t) \geq Y^{(n)}(t), \text{ for all } t, \text{ almost surely},
\]

$L_i^{(n)}$ is nondecreasing, and is flat away from the set $\{s : X_i^{(n)}(s) = Y^{(n)}(s)\}$.

By flat we mean

\[
\int_{\mathbb{R}} 1(X_i^{(n)}(s) > Y^{(n)}(s))dL_i^{(n)}(s) = 0.
\]

In other words, $L_i^{(n)}$ increases only when $X_i^{(n)}$ makes contact with $Y^{(n)}$. These conditions imposed on $L_i^{(n)}$ imply that $L_i^{(n)}$ is the local time of $X_i^{(n)}$ on $Y^{(n)}$, which we will see in the proof of Proposition 2.11. We can in fact let

\[
L_i^{(n)}(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{[0,\epsilon]}(X_i^{(n)}(s) - Y^{(n)}(s))ds \text{ for all } t \in [0, T], \text{ almost surely}.
\]

We assume the initial conditions $Y^{(n)}(0) = 0$, $V^{(n)}(0) = v$ and that $X_i^{(n)}(0), i = 1, \ldots, n$ are drawn from i.i.d. samples of an $L^1([0, \infty))$ random variable. In this case we require $\mathcal{F}_0$ to be large enough to contain $\sigma\{X_i^{(n)}(0) : 1 \leq i \leq n\}$. See the figure above for sample path realizations. Existence of a strong solution to this system is proved in Proposition 2.11.

A system $(X_1^{(n)}, \ldots, X_n^{(n)}, Y^{(n)}, V^{(n)})$ satisfying (1.1) above will be called a system of Brownian particles reflecting from a massive barrier with impulse coefficient $K$. The processes $X_1^{(n)}, \ldots, X_n^{(n)}$ are the Brownian particles, $Y^{(n)}$ is the reflecting barrier with $V^{(n)}$ its velocity.

### 1.4. Free boundary problem.

In Theorem 1.2 we characterize the hydrodynamic limit of the empirical process together with the random barrier. The hydrodynamic limit is the solution to a free boundary problem given as a pair $(p(t, x), y(t))$, both of which interact according to the PDE below. We think of $p(t, x)$ as the temperature at time-space location $(t, x)$ and $y(t)$ as an insulating barrier. In our convention the heat is concentrated above the insulating barrier. That is, $p(t, \cdot)$ is supported in $[y(t), \infty)$. Our initial temperature distribution is given by $\pi_0(dx)$ which need not have a continuous density.
\[
\begin{align*}
\frac{\partial p(t, x)}{\partial t} &= \frac{1}{2} \Delta_x p, \quad x > y(t), \\
\frac{\partial^+ p(t, x)}{\partial x^+} &= -2y'(t)p(t, x), \quad x = y(t), \\
y''(t) &= -(K/2)p(s, y(s)), \quad y(0) = 0, \quad y'(0) = v \in \mathbb{R}, \\
\lim_{t \downarrow 0} p(t, x) dx &= \pi_0(dx).
\end{align*}
\] (1.2)

The second condition is a one sided derivative on the positive side, and is mathematically equivalent to conservation of heat. The function \(y(t)\) acts as an insulating barrier. The third condition says the insulating barrier has an acceleration proportional to its temperature. This is contrasted with the Stefan problem mentioned in Section 1.1 in that the barrier reflects the heat back into the domain rather than absorbing it, and its acceleration is proportional to its temperature as opposed to the velocity being proportional to the heat flux. The unique solution will be one in which the equalities above hold in the classical sense. That is, \(p(t, x)\) is a differentiable function in its domain \(\{(t, x) : 0 \leq t \leq T, x \geq y(t)\}\), which is \(C^2\) in time, \(C^1\) in space, and where \(y \in C^2([0, T], \mathbb{R})\).

**Notation.** For ease of reference we introduce notation which will be used along the paper. First let \((E, d)\) be a metric space.

1.1. \(C(E_1, E_2)\) is the space of continuous functions from \((E_1, d_1)\) to \((E_2, d_2)\).

1.2. \(P(E)\) is the space of probability measures on \(E\). We may abbreviate \(P(\mathbb{R})\) as \(\mathcal{P}\).

1.3. The **Prohorov metric** is the metrization of distributional convergence for the space \(P(E)\). This is also a metric on the space of \(E\)-valued random variables through their induced measure on \(E\).

1.4. For \(f \in C([0, T], \mathbb{R})\) and \([a, b] \subset [0, T]\)

\[
\|f\|_{[a,b]} := \max_{x \in [a,b]} |f(x)|.
\]

1.5. For \(f = (f_1, \ldots, f_n) \in C([0, T], \mathbb{R}^n)\) and \([a, b] \subset [0, T]\)

\[
\|f\|_{[a,b]} := \sum_{i=1}^{n} \|f_i\|_{[a,b]}.
\]

1.6. For \(p > 0\), \((\mathcal{P}(E), W_p)\) is the space of probability measures on \(\mathbb{R}\) together with the **Wasserstein-p distance**

\[
W_p(\mu, \nu) := \left( \inf_{(X,Y)} \mathbb{E}d(X, Y)^p \right)^{1/p}
\]

where the infimum is taken over pairs \((X, Y)\) defined on the same probability space, with \(X \overset{d}{=} \mu\) and \(Y \overset{d}{=} \nu\). If \((E, d)\) is complete then so is \((\mathcal{P}(E), W_p)\). We consider \(p \geq 1\). See [26].
1.7. For $f \in C([0,T], (E, d))$ and $\delta > 0$ we define the modulus of continuity for $f$ by
\[
\omega_{(E,d)}(f, \delta) := \sup_{0<s<t<T \atop |t-s|<\delta} d(f(t), f(s)).
\]

1.8. When $\nu_t \in C([0,T], (\mathcal{P}, W_p))$ we let $\omega'(\nu, \delta) := \omega_{(\mathcal{P}, W_p)}(\nu, \delta)$.

1.5. Main results.

**Theorem 1.1.** There exists a unique classical solution to the free boundary problem (1.2).

For the hydrodynamic limit we consider the empirical measure
\[
\pi_t^{(n)}(a) = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i^{(n)}(t)}(a).
\]

For fixed $t \geq 0$, $\pi_t^{(n)}$ is a random variable with values in the space $\mathcal{P}(\mathbb{R})$. For a time horizon $T > 0$, $\{\pi_t^{(n)}: t \in [0,T]\}$ is a process with paths in the space $C([0,T], (\mathcal{P}, W_p))$ with metric
\[
\|\nu' - \nu''\|_{[0,T]} := \max_{t \in [0,T]} \sup_{x \in \mathbb{R}} W_p(\nu'(t), \nu''(t)).
\]

That this process indeed has a.s. continuous paths is proved in Lemma 3.13. In other words, $\{\pi_t^{(n)}: t \in [0,T]\}$ is a continuous measure-valued process. As such, $\pi^{(n)}$ induces a probability measure on $C([0,T], (\mathcal{P}, W_p))$. The hydrodynamic limit characterizes this distribution for large $n$.

**Theorem 1.2.** Assume that
\[
W_p(\pi_0^{(n)} , \pi_0) \to 0, \ n \to \infty,
\]
where $\pi_0$ has support in $[0, \infty)$. Then
\[
(\pi^{(n)}, Y^{(n)}) \to (p(t, w) dw, y(t)), \ n \to \infty,
\]
in the Prohorov metric on $C([0,T], (\mathcal{P}, W_p) \times \mathbb{R})$ where $y \in C^2([0,T], \mathbb{R})$, $p(t, x)$ is a probability density supported in $[y(t), \infty)$, and with $(p(t, x), y(t))$ solving (1.2).

The proof is at the end of Section 3.

The third result is about propagation of chaos—the dependence of any finite collection of tagged particles disappears as the number of particles $n$ tends to infinity.

**Theorem 1.3.** Assume $X_i^{(n)}(0) = \xi_i$, $i = 1, \ldots, n$, where $\xi_i, i \in \mathbb{N}$ are i.i.d. samples of nonnegative integrable random variable. Fix positive integers $i_1, \ldots, i_k$. Then
\[
(X_{i_1}^{(n)}, \ldots, X_{i_k}^{(n)}) \to (X_{i_1}^{(\infty)}, \ldots, X_{i_k}^{(\infty)}) \text{ a.s. as } n \to \infty,
\]
where the limit consists of independent processes $X_{i_1}^{(\infty)}, \ldots, X_{i_k}^{(\infty)}$.

The $\xi_i$ are given so the processes have an initial condition which does not depend on $n$ in the triangular array. This ensures that after $n \geq \max i_k$ the initial conditions for the $X_{i_1}^{(n)}, \ldots, X_{i_k}^{(n)}$ are all defined and unchanging with $n$. The proof is in Section 3.

The paper is organized as follows. In Section 2, we construct the processes $X_i^{(n)}$ path-by-path on any probability space supporting an infinite sequence of i.i.d. Brownian motions.
as well as the initial random variables $X_i^{(n)}(0)$ for all $1 \leq i \leq n, n \in \mathbb{N}$. We do this by constructing a functional to which we apply pathwise to the $n$ Brownian motions $B^{(1)}, \ldots, B^{(n)}$. In Proposition 2.11 we show this pathwise construction gives a system of processes satisfying (1.1). Such a method for reflected processes is called a Skorohod map, since Skorohod used the method to construct a reflected Brownian motion on the positive half-line $\mathbb{R}_+ := [0, \infty)$. For instance, if $B(t)$ is a standard Brownian motion and $m(t) = \sup_{0 < s < t} [-B(s) \vee 0]$, then $B(t) + m(t)$ has the same distribution as $X$, where $dX = dB + dL$ and $L$ is the semimartingale local time of $X$ at zero; see [19, Section 3.6C]. Here $m(t)$ would be the Skorohod map which corresponds to reflected Brownian motion.

In Section 3 Theorems 1.2 and 1.3. We use the estimates derived in the second section to demonstrate almost sure convergence of the barrier $Y^{(n)}$ to a unique deterministic function $y$ in the form of a functional strong law of large numbers; see Propositions 3.5 and 3.7. In this section we introduce properties of the measure-valued process $\pi^{(n)}$ mentioned above. In Proposition 3.19, we prove a uniform stochastic equicontinuity, which is stronger than the typical stochastic equicontinuity necessary for tightness of processes in some metric space. This is needed because we used the Wasserstein metric on the space of probability measures. Note that the Wasserstein metric is much stronger than the Prohorov metric which is often used in hydrodynamic limits. See Remark 3.18.

We conclude the paper with Section 4, where we use our stochastic tools to prove uniqueness of the free boundary problem in the statement of Theorem 1.2.

2. SKOROHOD MAP: CONSTRUCTION AND ESTIMATES

In this section we construct the system given in (1.1) by applying a Skorohod map to the collection of Brownian paths. First we recall the classical Skorohod equation.

**Lemma 2.1** (Skorohod, see [19]). Let $f \in C([0, T], \mathbb{R})$ with $f(0) \geq 0$. There is a unique, continuous, nondecreasing function $m(t)$ such that

\[ x_f(t) = f(t) + m_f(t) \geq 0, \]

\[ m_f(0) = 0, \quad m_f(t) \text{ is flat off } \{ s : x_f(s) = 0 \}, \]

and is given by

\[ m_f(t) = \sup_{0 < s < t} [-f(s)] \vee 0. \]

**Remark 2.2.** As stated in the introduction, flatness off $\{ z : x_f(z) = 0 \}$ for $m_f$ means

\[ \int_0^t 1(x_f(s) > 0)(s)dm_f(s) = 0. \]

The classical Lévy’s theorem says when $f$ is replaced by a Brownian motion, the corresponding process $x_f$ is distributed as $|B|$.

**Remark 2.3.** The solution of the Skorohod equation has a time shift property: For any $0 \leq s \leq t \leq T$,

\[ x_f(t) = x_f(t-s), \]

where $g(t) = x_f(s) + f(t) - f(s)$.

The following lemmas will be useful later when proving tightness of our processes; see Lemma 3.2.
Lemma 2.4. Let \( f, g \in C([0,T], \mathbb{R}) \) and assume that \( f \geq g \). Then \( m_f(t) \leq m_g(t) \), for all \( t \in [0,T] \).

Proof. From Lemma 2.1,

\[
m_f(t) = \sup_{0<u<t} [-f(u)] \vee 0 \leq \sup_{0<u<t} [-g(u)] \vee 0 = m_g(t).
\]

\( \square \)

Lemma 2.5. Let \( f, y_1, y_2 \in C([0,T], \mathbb{R}) \) and assume that \( y_1(0) = y_2(0), f(0) + y_1(0) \geq 0 \), and

\[
y_1(t) - y_1(s) > y_2(t) - y_2(s) \text{ for all } 0 \leq s < t \leq T.
\]

Then

\[
m_2(t) - m_2(s) \geq m_1(t) - m_1(s), \text{ for all } 0 \leq s < t \leq T,
\]

where \( m_{f+y_i} \), \( i = 1, 2 \) correspond to the solution of the Skorohod problem provided by Lemma 2.4.

Proof. We first show that \( x_{f+y_1}(t) \geq x_{f+y_2}(t) \) for all \( t \in [0,T] \). That this holds for \( t = 0 \) is guaranteed by the assumption on the initial conditions, which imply \( x_{f+y_1}(0) = x_{f+y_2}(0) \). Assume for the sake of contradiction that there is some \( t^* \in [0,T] \) such that \( x_{f+y_2}(t^*) > x_{f+y_1}(t^*) \geq 0 \). Let

\[
\tau = \sup \{ t < t^* : x_{f+y_2}(t) = 0 \}
\]

be the last zero of \( x_{f+y_2} \) before time \( t^* \). It follows that \( m_{f+y_2} \) is flat on the interval \([\tau, t^*] \). In other words,

\[
0 = m_{f+y_2}(t^*) - m_{f+y_2}(\tau) \leq m_{f+y_1}(t^*) - m_{f+y_1}(\tau).
\]

By shifting the Skorohod solution by time \( \tau \) as in Remark 2.3, using (2.2) and the fact that \( x_{f+y_1}(\tau) \geq 0 = x_{f+y_2}(\tau) \) together with assumption (2.1),

\[
x_{f+y_1}(t^*) = x_{f+y_1}(\tau) + f(t^*) - f(\tau) + y_1(t^*) - y_1(\tau) + m_{f+y_1}(t^*) - m_{f+y_1}(\tau)
\]

\[
\geq x_{f+y_2}(\tau) + f(t^*) - f(\tau) + y_2(t^*) - y_2(\tau) + m_{f+y_2}(t^*) - m_{f+y_2}(\tau)
\]

\[
= x_{f+y_2}(t^*)
\]

which contradicts the definition of \( t^* \). Therefore \( x_{f+y_1}(t) \geq x_{f+y_2}(t) \) for all \( t \in [0,T] \).

For a fixed \( s \in [0,T] \) let

\[
g_i(t) = x_{f+y_i}(s) + f(t) - f(s) + y_i(t) - y_i(s) \text{ for } s \leq t \leq T,
\]

and \( i = 1, 2 \). The assumption (2.1) on \( y_i \) together with the fact that \( x_{f+y_1} \geq x_{f+y_2} \) imply \( g_1(t) \geq g_2(t) \). Apply Lemma 2.4 to \( g_1, g_2 \) and shift time by \( s \) as in Remark 2.3 to see

\[
m_{f+y_1}(t) - m_{f+y_1}(s) = m_{g_1}(t-s) \leq m_{g_2}(t-s) = m_{f+y_2}(t) - m_{f+y_2}(s),
\]

proving the result.

\( \square \)

We construct a generalization of the Skorohod map for \( f = (f_1, \ldots, f_n) \in C([0,T], \mathbb{R}^n) \), \( v \in \mathbb{R} \) and \( K \geq 0 \). If any \( f_i(0) < 0 \), the velocity of our corresponding inert particle will immediately receive a negative jump of \( \frac{1}{n} \sum_{i=1}^n (f_i(0) \wedge 0) \). Therefore by allowing any initial real velocity we may assume without loss of generality that \( f_i(0) \) are nonnegative.
Theorem 2.6. Corresponding to each \( f = (f_1, \cdots, f_n) \in C([0, T], \mathbb{R}^n), v \in \mathbb{R}, K \geq 0 \) is a pair of continuous functions

\[
(I^{(n)}_f(t), V^{(n)}_f(t)) =: \Gamma_n f(t) \in C([0, T], \mathbb{R}^2)
\]

satisfying

\[
x_i(t) := f_i(t) + I^{(n)}_f(t) + m_i(t) \geq 0,
\]

\[
m_i(t) \text{ is flat off } \{ t : x_i(t) = 0 \},
\]

\[
V^{(n)}_f(t) = -v + \frac{K}{n} \sum_{i=1}^{n} m_i(t), \quad v \in \mathbb{R},
\]

\[
I^{(n)}_f(t) = \int_0^t V^{(n)}_f(s)ds,
\]

for all \( t \in [0, T] \).

Remark 2.7. It follows from the classical Skorohod equation that

\[
m_i(t) = \sup_{0 < s < t} \left[ -(f_i(s) + I^{(n)}_f(s)) \right] \geq 0.
\]

This is used in the proof of Proposition 2.11 below.

Proof. Uniqueness: We prove continuity estimates which holds for any solutions of (2.3) - (2.6). That is, assume that (2.3) - (2.6) holds for two collections of functions \( f = (f_1, \cdots, f_n), g = (g_1, \cdots, g_n) \in C([0, T], \mathbb{R}^n) \). Let \((I^{(n)}_f, V^{(n)}_f), (I^{(n)}_g, V^{(n)}_g)\) correspond to solutions of the Skorohod problem. Since we are proving uniqueness we are assuming such solutions exist for \( f, g \). By Remark 2.7, \( m^f_i(t) \) is the running minimum of \( f_i + I^{(n)}_f \) below zero until time \( t \), and the same holds for \( m^g_i(t) \). Hence

\[
\|m^f_i - m^g_i\|_{[0,t]} \leq \| (f_i + I^{(n)}_f) - (g_i + I^{(n)}_g) \|_{[0,t]}.
\]

By the triangle inequality, (2.5), (2.6), and (2)

\[
\alpha(t) := \sum_{i=1}^{n} \| (f_i + I^{(n)}_f) - (g_i + I^{(n)}_g) \|_{[0,t]}
\]

\[
\leq \sum_{i=1}^{n} \left( \| f_i - g_i \|_{[0,t]} \right) + n \| I^{(n)}_f - I^{(n)}_g \|_{[0,t]}
\]

\[
\leq \| f - g \|_{[0,t]} + K \int_0^t \sum_{i=1}^{n} |m^f_i(s) - m^g_i(s)|ds
\]

\[
\leq \| f - g \|_{[0,t]} + K \int_0^t \sum_{i=1}^{n} |m^f_i(s) - m^g_i(s)|ds
\]

Now apply Grönwall’s inequality to attain

\[
\alpha(t) \leq \| f - g \|_{[0,t]} \exp(Kt).
\]
Consequently,

\[(2.8) \quad \|V_f^{(n)} - V_g^{(n)}\|_{[0,t]} \leq \frac{K}{n} \sum_{i=1}^{n} |m_f^i(t) - m_g^i(t)| \leq \frac{(K/n)\alpha(t)}{\alpha(t)} \leq (K\|f - g\|_{[0,t]}/n) \exp(Kt).\]

This holds for any \(f, g\) and any two pairs \((I_f^{(n)}, V_f^{(n)}), (I_g^{(n)}, V_g^{(n)})\) solving the equations \((2.3) - (2.6)\). Taking \(g = f\) in \((2.8)\) shows \((I_f^{(n)}, V_f^{(n)})\) is unique, and \(\Gamma_n\) is well defined granted existence.

**Existence:** The case \(n = 1\) is in [27]. To demonstrate existence, we use a limiting procedure to construct the processes \(I_f^{(n)}, V_f^{(n)}\) which in turn produce the map \(\Gamma_n\). For a fixed \(\epsilon > 0\), define the functions \(I_{M\epsilon}^{\epsilon}, V_{M\epsilon}^{\epsilon}\) as recursively in the intervals \([0, \epsilon], [\epsilon, 2\epsilon], \ldots, [(M - 1)\epsilon, M\epsilon]\) as follows.

1. **On the interval** \([0,\epsilon]\), simply let \(I_0^{\epsilon}(t) = 0\) and \(V_0^{\epsilon} = v\).
2. Assume we are given \(I_{M\epsilon}^{\epsilon}, V_{M\epsilon}^{\epsilon}\). Let
   \[I_{(M+1)\epsilon}^{\epsilon}\big|_{[0,M\epsilon]} = I_{M\epsilon}^{\epsilon}\] and \(V_{(M+1)\epsilon}^{\epsilon}\big|_{[0,M\epsilon]} = V_{M\epsilon}^{\epsilon}\).

For \(t \in [M\epsilon, (M+1)\epsilon]\) let
\[V_{(M+1)\epsilon}^{\epsilon}(t) = \frac{K}{n} \sum_{i=1}^{n} \max_{0 \leq u \leq M\epsilon} (\lceil f_i(u) + I_{M\epsilon}^{\epsilon}(u) \rceil \vee 0)\]
be the average of the running minimum below zero of \(f_i + I_{M\epsilon}^{\epsilon}\) until time \(M\epsilon\).

3. **Extend** \(I_{(M+1)\epsilon}^{\epsilon}\) to \([M\epsilon, (M+1)\epsilon]\) linearly by giving it slope \(V_{(M+1)\epsilon}^{\epsilon}\).
4. **Set** \(I_f^{(n,\epsilon)}, V_f^{(n,\epsilon)}\) as the functions produced once the recursion covers the interval \([0, T]\).

A couple observations follow easily from this construction. First,
\[I_f^{(n,\epsilon)}(t) = \int_0^t V_f^{(n,\epsilon)}(s)ds.\]

Second, \(V_f^{(n,\epsilon)}\) is increasing, so \(I_f^{(n,\epsilon)}\) is differentiable and convex. By construction
\[\|V_f^{(n,\epsilon)}\|_{[0,T]} \leq |v|T + \frac{K}{n} \sum_{i=1}^{n} \max_{0 \leq u \leq T} (\lceil f_i(u) \rceil \vee 0) < \infty,\]
for every \(\epsilon > 0\), and therefore \(\{\|V_f^{(n,\epsilon)}\|_{[0,T]} : \epsilon > 0\}\) is a bounded set. Consequently the collection \(\{I_f^{(n,\epsilon)} : \epsilon > 0\}\) is uniformly Lipschitz, and since \((I_f^{(n,\epsilon)}(0) = 0\) for all \(\epsilon > 0\) it is pointwise bounded as well. Hence the family \(\{I_f^{(n,\epsilon)} : \epsilon > 0\}\) satisfies the Arzel`a-Ascoli criterion. By taking a subsequence \(\epsilon_k \to 0\) there is a continuous function \(I_f^{(n)}\) such that
\[I_f^{(n,\epsilon_k)}(t) = \int_0^t V_f^{(n,\epsilon_k)}(s)ds \longrightarrow I_f^{(n)}(t)\]
uniformly for \(t \in [0,T]\). By the construction of \(V_f^{(n,\epsilon_k)}\), this implies
\[V_f^{(n,\epsilon_k)}(t) = \frac{K}{n} \sum_{i=1}^{n} \max_{0 \leq u \leq \lceil t/\epsilon_k \rceil \epsilon_k} (\lceil f_i(u) + I_f^{(n,\epsilon_k)}(u) \rceil \vee 0) \longrightarrow \frac{K}{n} \sum_{i=1}^{n} \max_{0 \leq u \leq t} (\lceil f_i(u) + I_f^{(n)}(u) \rceil \vee 0)\]
uniformly for \( t \) in \([0, T]\), as \( \epsilon_k \to 0 \). Set \( m_i(t) = \max_{0 \leq u \leq t} [-f_i(u) + I^{(n)}(u)] \vee 0 \) so that

\[
V^{(n)}(t) = \frac{K}{n} \sum_{i=1}^{n} m_i(t).
\]

Then \( m_i \) is flat off \( \{ s : f_i(s) + I^{(n)}(s) + m_i(s) = 0 \} \) by Skorohod’s lemma [2.1] By the dominated convergence theorem,

\[
I^{(n)}(t) = \int_0^t V^{(n)}(s) ds,
\]

and clearly \( f_i(t) + I^{(n)} + m_i(t) \geq 0 \). Therefore \((I^{(n)}, V^{(n)})\) satisfy the equations (2.3)–(2.6).

We state the bounds attained in (2.8) as we have shown that \( V^{(n)} \), as a map between function spaces \( C([0, T], \mathbb{R}^n) \to C([0, T], \mathbb{R}) \), is Lipschitz with Lipschitz constant \((K/n) \exp(KT)\).

**Proposition 2.8.** (Lipschitz property of \( V^{(n)} \)) For any \( v \in \mathbb{R}, K \geq 0 \), take \( f, g \in C([0, T], \mathbb{R}^n) \) such that \( \|f - g\|_{[0, T]} < \eta \). We have

\[
\|V_f^{(n)} - V_g^{(n)}\|_{[0, T]} \leq (K\eta/n) \exp(KT),
\]

and

\[
\|I_f^{(n)} - I_g^{(n)}\|_{[0, T]} \leq (K\eta/n) T \exp(KT).
\]

**Remark 2.9.** Clearly

\[
\frac{1}{n} \|f - g\|_{[0, T]} = \frac{1}{n} \sum_{i=1}^{n} \|f_i - g_i\|_{[0, T]}
\]

is the average distance between the \( f_i, g_i \). Proposition 2.8 says that if this average distance is small, the difference in the drifts \( V_f^{(n)}, V_g^{(n)} \) is small as well.

**Proof.** The first bound (2.9) is shown on (2.8), while (2.10) follows as

\[
\|I_f^{(n)} - I_g^{(n)}\|_{[0, T]} = \sup_{0 \leq u \leq T} \int_0^u |V_f^{(n)}(s) - V_g^{(n)}(s)| ds
\]

\[
\leq \sup_{0 < u < T} \int_0^u |V_f^{(n)}(s) - V_f^{(n)}(s)| ds
\]

\[
\leq \int_0^T \|V_f^{(n)} - V_g^{(n)}\|_{[0, T]} ds
\]

\[
\leq T(K\eta/n) \exp(KT).
\]

□

Consider the above sequence \( I_f^{(n, \epsilon)} \) defined above for a given \( f = (f_1, \ldots, f_n) \in \mathbb{R}^n \). By Proposition 2.8, \( I_f^{(n, \epsilon)} \) converges in the uniform norm on \( C([0, T], \mathbb{R}) \) to a unique continuous function. The Proposition below says this rate of convergence only depends on \( \|f\|_{[0, T]} \).
Proposition 2.10. Consider the sequence $I_f^{(n,e)}$ defined above for a given $f = (f_1, \ldots, f_n) \in C([0, T], \mathbb{R}^n)$. If $l < m$, then
\[
\|I_f^{(n,2^{-l})} - I_f^{(n,2^{-m})}\|_{[0,T]} \leq ((2 + K)\|f\|_{[0,T]}/n)2^{-l} \exp(KT).
\]

Proof. The proof is in the similar vein as that of Proposition 2.8. We make a couple notational
of conveniences. For $j = l, m$ we will write $I^j$ in place of $I_f^{(n,2^{-j})}$, and $I^j_{k2^{-j}}$ in place of $I_f^{(n,2^{-j})}_{k2^{-j}}$. Recall $I^l$ is piecewise linear with definition.
\[
D(k) := \sup_{0 < t < k2^{-l}} |I^l(t) - I^m(t)| = \frac{1}{n} \sum_{i=1}^{n} \sup_{0 < t < k2^{-l}} |f_i(t) + I^l(t) - (f_i(t) + I^m(t))|
\]
\[
= \|(f + I^l) - (f + I^m)\|_{[0,k2^{-l}]/n}.
\]

By construction $I^l \equiv 0$ on $[0, 2^{-l}]$. Due to nonegativity of $I^m$, for any $t \in [0, T]$
\[
|V^{(n,2^{-m})}(t)| \leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0 < u < T} [-f_i(u) - I^m] \vee 0
\]
\[
\leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0 < u < T} [-f_i(u)] \vee 0 \leq K\|f\|/n.
\]

Therefore $I^m$ is piecewise linear with a slope not exceeding $K\|f\|/n$, and so
\[
D(1) \leq (K\|f\|/n)2^{-l}.
\]

Assume we are given $D(k)$. We wish to bound the difference between $I^l$ and $I^m$ on the
interval $[0, (k + 1)2^{-l}]$. We know $I^l_{[0,k2^{-l}]} = I^l_{k2^{-l}}$ and $I^m_{[0,k2^{-l}]} = I^m_{k2^{-l}}$. Similarly the
function $I^l$ has constant slope on $[k2^{-l}, (k + 1)2^{-l}]$, with its slope adjustment being at the
end of this interval at $(k + 1)2^{-l}$; to this end $I^l_{k2^{-l}} = I^l_{(k+1)2^{-l}}$ on $[k2^{-l}, (k + 1)2^{-l}]$. On the
other hand, $I^m$ has a slope adjustment at each time $k2^{-l} + 2^{-m}, k2^{-l} + 2^{-m+1}, \ldots, (k + 1)2^{-l}$. Note that the difference in the slope between $I^l, I^m$ at time $k2^{-l}$ is not more than $D(k)$. By the triangle inequality
\[
D(k + 1) = \|I^l_{(k+1)2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]}
\]
\[
\leq \|I^l_{k2^{-l}} - I^m_{k2^{-l}}\|_{[0,(k+1)2^{-l}]} + \|I^m_{k2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]}
\]
\[
\leq \|I^l_{k2^{-l}} - I^m_{k2^{-l}}\|_{[0,k2^{-l}]} + \|I^m_{k2^{-l}} - I^m_{[k2^{-l},(k+1)2^{-l}]}\| + \|I^m_{k2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]}
\]
\[
\leq D(k) + KD(k)2^{-l} + \|I^m_{k2^{-l}} - I^m_{(k+1)2^{-l}}\|_{[0,(k+1)2^{-l}]}
\]

Let
\[
\beta_k = \frac{1}{n} \sum_{i=1}^{n} \sup_{0 < t < (k+1)2^{-l}} [-((f(t) + I^m(t))] - \frac{1}{n} \sum_{i=1}^{n} \sup_{0 < t < k2^{-l}} [-((f(t) + I^m(t))],
\]
so that $K\beta_k$ is the amount the velocity $I^m$ increases in the interval $[k2^{-l}, (k + 1)2^{-l}]$. From
this telescoping definition of $\beta_k$ we see that
\[
\sum_{k=1}^{[T2^l]} \beta_k = \frac{1}{n} \sum_{i=1}^{n} \sup_{0 < s < T} [-((f(s) + I^m(s))] \vee 0 \leq \frac{1}{n} \|f\|_{[0,T]}.
\]
Combine with (2.12) above to see that
\[(2.14) \quad D(k + 1) \leq D(k) + KD(k)2^{-l} + K\beta_k 2^{-l} = D(k) + K(D(k) + \beta_k)2^{-l}.\]
Set \(A(k)\) to be recursively defined with the above inequality taken as the equality, i.e.,
\[A(k + 1) = A(k) + KA(k)2^{-l} + K\beta_k 2^{-l}.\]
We have \(D(k) \leq A(k)\) for all \(k\). Since the total sum of the \(\beta_k\) does not exceed \(\|f\|_{[0,T]}/n\), it follows that \(A(k)\) is maximized when all the mass of \(\sum_{i=1}^{2^l} \beta_k \leq \|f\|_{[0,T]}/n\) is placed at \(\beta_1\). Intuitively this allows the entire mass of \(\sum A\) it follows that
\[12\] CLAYTON BARNES
\[\begin{aligned}
\text{then (2.15)}
\end{aligned}\]
producing the pair of processes \(D\) we have
\[\begin{aligned}
\text{Set (2.14)}
\end{aligned}\]
which concludes the result.
\[\square\]

To construct our system (1.1) in the introduction from Proposition 2.6 we apply the map \(\Gamma_n\) pathwise with
\[\begin{aligned}
(f_1, \ldots, f_n) = (B^{(1)} + X_1^{(n)}(0), \ldots, B^{(n)} + X_n^{(n)}(0)) =: B + X(0),
\end{aligned}\]
producing the pair of processes
\[\begin{aligned}
\Gamma_n(B + X(0)) = \left(\frac{f^{(n)}_{B + X(0)}}{\text{B} + X(0)}, \frac{\tilde{V}^{(n)}_{B + X(0)}}{\text{B} + X(0)}\right).
\end{aligned}\]
Set
\[\begin{aligned}
X^{(n)} = X^{(n)}(0) + B^{(i)} + m^{(n)}_i, \quad V^{(n)} = -\tilde{V}^{(n)}_{B + X(0)},
\end{aligned}\]
Then
\[\begin{aligned}
\text{Proposition 2.11.} \quad (X_1^{(n)}, \ldots, X_n^{(n)}, V^{(n)}, Y^{(n)}) \text{ satisfies (1.1), therefore giving a strong solution to that system of SDE's.}
\end{aligned}\]
\[\begin{aligned}
\text{Proof.} \quad \text{We begin from (2.3) - (2.6) with the } f_i(t) \text{ replaced with } B^{(i)}(t) + X_i^{(n)}(0). \text{ The following holds almost surely:}
\end{aligned}\]
\[\begin{aligned}
(2.16) \quad X_t^{(n)} = X_t^{(n)}(0) + B^{(i)}(t) + m^{(n)}_i(t) \geq Y^{(n)}(t), \text{ for all } 0 < t < T,
\end{aligned}\]
\[\begin{aligned}
(2.17) \quad V^{(n)}(t) = \nu - \frac{K}{n} \sum_{i=1}^{n} m^{(n)}_i(t),
\end{aligned}\]
\[\begin{aligned}
(2.18) \quad Y^{(n)}(t) = \int_0^t V^{(n)}(s)ds,
\end{aligned}\]
\[\begin{aligned}
(2.19) \quad m^{(n)}_i \text{ is flat off of } \{t : X_t^{(n)}(t) = -Y^{(n)}(t)\}.
\end{aligned}\]
We take \( v = 0 \) for convenience. The fact that we have a strong solution of the system follows from the path-by-path construction. We apply a transformation of measure argument. As mentioned in Remark 2.7 for a fixed time \( t \in [0,T] \),

\[
V^{(n)}(t) = -\frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<t} (-B^{(i)}(u) + X^{(n)}_i(0) - Y^{(n)}(u)) \lor 0,
\]

which, due to nonnegativity of \( X^{(n)}_i(0) \) and the fact that \( Y^{(n)} \leq 0 \),

\[
\sup_{u \in [0,T]} |V^{(n)}(u)| \leq \frac{K}{n} \sum_{i=1}^{n} \sup_{0<u<T} (-B^{(i)}(u) \lor 0).
\]

This is equivalent to saying Brownian motion with a nonnegative drift has a running minimum below zero less than the running minimum of a simple Brownian motion. It follows from continuity of the processes on \([0,T] \) that \( \sup_{0<u<T} |V^{(n)}(u)| \leq |V^{(n)}(T)| < \infty \) almost surely. Therefore

\[
Z(t) = \exp \left( \frac{K}{n} \sum_{i=1}^{n} \int_{0}^{t} V^{(n)}(s) dB^{(i)}(s) - nY^{(n)}(t) \right)
\]

is a local martingale, and therefore there exists a collection of exhaustive stopping times \( \tau_k \xrightarrow{k \to \infty} \infty \) such that \( Z(t \wedge \tau_k) \) is a true martingale for each \( k \). We will apply a Girsanov transformation of measure, see [19, Ch. 3.5]. Let \( \mathbb{Q} \) be defined by \( d\mathbb{Q}/d\mathbb{P} = Z(t \wedge \tau_k) \). Under \( \mathbb{Q} \) each \( \tilde{B}^{(i)}(t \wedge \tau_k) := B^{(i)}(t \wedge \tau_k) - Y^{(n)}(t \wedge \tau_k) \) has the law of a Brownian motion, and the joint law of \((X^{(n)}_1, \ldots, X^{(n)}_n)\), when stopped at \( \tau_k \), has the same law as \( \tilde{X}_k(t \wedge \tau_k) := X^{(n)}_1(0) + \tilde{B}^{(i)}(t \wedge \tau_k) + \tilde{m}_i(t \wedge \tau_k) \geq 0 \), where \( \tilde{m}_i(t) = \sup_{0 < u < t} [-\tilde{B}^{(i)}(u) + X^{(n)}_i(0)] \lor 0 \). The \( \tilde{m}_i \) are incidentally equal to \( m^{(n)}_i := \sup_{0 < u < t} [-B^{(i)}(u) + X^{(n)}_i(0) - Y^{(n)}(u)] \lor 0 \). Because \( \tilde{m}_i \) is flat off \( \{ t : \tilde{X}_i(t) = 0 \} \), the classical Lévy’s theorem [19, Chapter 3] shows that this system \((\tilde{X}_1, \ldots, \tilde{X}_n)\) is equivalent in law to processes solving

\[
d\tilde{X}_i = dB^{(i)} + d\tilde{L}_i, \quad \tilde{X}_i(0) = X^{(n)}_i(0),
\]

when stopped at \( \tau_k \), and where \( \tilde{L}_i \) is the local time at zero of \( \tilde{X}_i \). That is,

\[
\tilde{L}_i(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{[0,\epsilon]}(\tilde{X}_i(s)) ds = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{0}^{t} 1_{[0,\epsilon]}(X^{(n)}_i(s) - Y^{(n)}(s)) ds =: L^{(n)}_i(t),
\]

for all \( t \), almost surely. Additionally, \( \tilde{m}_i \) is the local time of \( \tilde{X}_i \) at zero, which by definition is the local time of contact between \( X^{(n)}_i \) and \( Y^{(n)} \). Since \( m^{(n)}_i = m^{(n)}_i \) this shows that \( m^{(n)}_i(t) = L^{(n)}_i(t) \) for all \( t \), almost surely. This means that under \( \mathbb{P} \), as processes stopped at \( \tau_k \), solutions to (2.10)–(2.18) are solutions of

\[
dX^{(n)}_i = dB^{(i)} + dL^{(n)}_i, \quad dY^{(n)} = -\frac{K}{n} \sum_{i=1}^{n} L^{(n)}_i(t) dt,
\]

with the given initial conditions and where \( L^{(n)}_i \) is the local time of \( X^{(n)}_i - Y^{(n)} \) at zero. This latter process is the definition of (1.1). Since \( \tau_k \to \infty \) almost surely, the equivalence in law holds as processes defined on \([0,T] \).
Lemma 2.12. Let \( (Y^{(n)}, V^{(n)}) \) be defined as in equation (2.15), then
\[
\|V^{(n)}\|_{[0,T]} \leq v + K \left( vT + \frac{1}{n} \sum_{i=1}^{n} m_i(T) \right),
\]
where \( m_i(t) = \sup_{0 < s < t} [-B^{(i)}(s)] \vee 0. \)

Proof. Clearly \( \sup_{0 < s < T} V^{(n)}(s) \leq v \), which implies that \( Y^{(n)}(t) \leq vt \) for \( t \in [0, T] \). From Remark 2.7 and Lemma 2.5, we have
\[
\|V^{(n)}\|_{[0,T]} = \sup_{0 < s < T} \left( v - \frac{K}{n} \sum_{i=1}^{n} \left( \sup_{0 < u < s} \left[ -(B^{(i)}(u) - Y^{(n)}(u)) \right] \vee 0 \right) \right)
\]
\[
\leq v + \frac{K}{n} \sum_{i=1}^{n} \left( \sup_{0 < s < T} \left[ -(B^{(i)}(s) - vs) \right] \vee 0 \right)
\]
\[
\leq v + K \left( vT + \frac{1}{n} \sum_{i=1}^{n} \left( \sup_{0 < s < T} [-B^{(i)}(s)] \vee 0 \right) \right).
\]

\[\square\]

3. Hydrodynamic Limit and Propagation of Chaos

Lemma 3.1. The collection \( \{(Y^{(n)}(t), V^{(n)}(t)) : t \in [0, T]\}_{n \geq 1} \) is tight in the space of continuous functions.

Proof. It suffices to show \( V^{(n)} \) is tight, since \( Y^{(n)}(t) = \int_{0}^{t} V^{(n)}(s)ds \). By our representation of \( V^{(n)} \) as (2.5), together with Remark 2.7, we apply Lemma 2.5 with \( y_1 = -Y^{(n)} \) and \( y_2 = 0 \) to show the increment \( V^{(n)}(t + \delta) - V^{(n)}(t) \) is not more than the change of the running maximum of the Brownian paths, averaged over \( n \). That is, letting \( m_1(t), \ldots, m_n(t) \) denote the respective running minimum below zero of the \( B^{(1)}, \ldots, B^{(n)} \)
\[
0 \leq |V^{(n)}(t + \delta) - V^{(n)}(t)| \leq K \sum_{i=1}^{n} (m_i(t + \delta) - m_i(t)) \text{ for all } t \in [0, T - \delta]
\]
for any positive delta, almost surely. Since the \( m_i \) are nondecreasing, we have the same inequality but for the modulus of continuity:
\[
\omega(V^{(n)}, T, \delta) \leq \frac{1}{n} \sum_{i=1}^{n} \omega(m_i, T, \delta)
\]
almost surely. By independence of the \( m_i \) and the strong law of large numbers, for each rational \( q \in [0, T], S_n(q) \to \mathbb{E}m_1(q) = \sqrt{2q/\pi} \) as \( n \to \infty \). Note that \( S_n \) is monotone for each \( n \), almost surely. It is known that if a sequence of continuous monotone functions converge to a continuous function pointwise on a dense subset of a compact set, the entire sequence converges uniformly. Hence \( S_n(t) \to \sqrt{2t/\pi} \) uniformly in \( t \in [0, T] \), almost surely. If \( f_n \to f \) uniformly in \( C([0, T], \mathbb{R}) \), then sup\( \omega(f, T, \delta) \to 0 \) as \( \delta \to 0 \). Therefore sup\( \omega(S_n, T, \delta) \to 0 \) as \( \delta \to 0 \), almost surely. By (3.1) we know sup\( \omega(V^{(n)}(T, \delta) \to 0 \) almost surely as well. Since \( V^{(n)}(0) = v \), almost surely, this is sufficient for tightness of the \( V^{(n)} \). \[\square\]
Lemma 3.2. The collection \( \{ (Y^{(n)}(t), V^{(n)}(t)) : t \in [0, T] \} \) is tight in the space of continuous functions.

Remark 3.3. We use Proposition 3.5 in the proof.

Proof. (Tightness of \( Y^{(n)} \)) For simplicity we take the initial velocity \( v = 0 \). We first show that \( Y^{(n)} \) is tight. By our representation of \( V^{(n)} \) as (2.5), together with Remark 2.7, the maximum velocity \( \|V^{(n)}\|_{[0,T]} \) is bounded almost surely by the scaled running minimum of the Brownian paths below zero, averaged over \( n \). In other words, letting \( m_1(t), \ldots, m_n(t) \) denote the running minimum below zero of \( B^{(1)}, \ldots, B^{(n)} \), Lemma 2.12 gives

\[
\|V^{(n)}\|_{[0,T]} \leq \frac{K}{n} \sum_{i=1}^{n} m_i(T).
\]

Consequently for any \( 0 < \delta < T \),

\[
\omega(Y^{(n)}, \delta, T) := \sup_{\delta \leq t < T - \delta} \sup_{t - \delta < s < t + \delta} |Y^{(n)}(t) - Y^{(n)}(s)| \leq \delta \|V^{(n)}\|_{[0,T]} \leq \delta \frac{K}{n} \sum_{i=1}^{n} m_i(T)
\]

almost surely. Taking expectations, we have

(3.2) \[
\mathbb{E} \omega(Y^{(n)}, \delta, T) \leq \delta K \mathbb{E} m_1(T) = \delta K \sqrt{2T/\pi}.
\]

Fix \( \epsilon > 0 \) and apply Markov’s inequality, we see

\[
\sup_n \mathbb{P}(\omega(Y^{(n)}, \delta, T) > \epsilon) \leq \epsilon^{-1} \sup_n \mathbb{E} \omega(Y^{(n)}, \delta, T) \leq \epsilon^{-1} \delta K \sqrt{2T/\pi},
\]

which implies

\[
\lim_{\delta \to 0} \sup_n \mathbb{P}(\omega(Y^{(n)}, \delta, T) > \epsilon) = 0.
\]

Together with \( \mathbb{P}(Y^{(n)}(0) = 0) = 1 \), this is sufficient for tightness of the sequence of continuous processes \( \{ Y^{(n)}(t) : t \in [0, T] \} \).

(Tightness of \( V^{(n)} \)) Take any subsequence \( n' \) for which \( Y^{(n')} \) converges to some process \( Y \) in distribution. By Proposition 3.3, \( Y \) is deterministic. Recall that the initial conditions \( X_i^{(n)}(0), 1 \leq i \leq n \), are i.i.d. samples with distribution \( \pi_0^{(n)} \) and that \( W_p(\pi_0^{(n)}, \pi_0) \to 0 \) by assumption. Let \( X_i^{(\infty)}(0) \), \( i \in \mathbb{N} \), be independent samples with distribution \( \pi_0 \). By definition of the \( W_p \) metric there is a probability space supporting all the processes \( \{ X_i^{(n)}(t) : t \in [0, T], 1 \leq n, n \in \mathbb{N} \} \) such that

\[
\sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \to 0, \text{ as } n \to \infty.
\]

Hence,

\[
\sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1 \leq i \leq n} \left( \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \right)^{1/p} \to 0, \text{ as } n \to \infty.
\]

This enlarged probability space can be constructed by taking \( \mathcal{F}_t \times \sigma \{ X_i^{(n)}(0) : 1 \leq i \leq n, n \in \mathbb{N} \} \) as our new filtration. Then,

\[
\frac{1}{n} \mathbb{E} \sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)| \leq \sup_{1 \leq i \leq n} \mathbb{E}|X_i^{(n)}(0) - X_i^{(\infty)}(0)| \to 0,
\]
and so
\[
\frac{1}{n} \sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)| \xrightarrow{P} 0.
\]
Hence every sequence \( n' \) has a further subsequence \( n'_k \) with
\[
(3.3) \quad \frac{1}{n'_k} \sum_{i=1}^{n'_k} |X_i^{(n'_k)}(0) - X_i^{(\infty)}(0)| \to 0
\]
almost surely. Without loss of generality we relabel such a sequence \( n'_k \) as \( n' \). For \( i = 1, \ldots, n' \), define
\[
m_i^{(n)}(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(n)}(0)) - Y^{(n)}(u)] \vee 0,
\]
\[
\tilde{m}_i(t) = \sup_{0<u<t} -[(B^{(i)}(u) + X_i^{(\infty)}(0)) - Y(u)] \vee 0.
\]
Since
\[
V^{(n')}(t) = -\frac{K}{n'} \sum_{i=1}^{n'} m_i^{(n')}(t),
\]
we compute
\[
\frac{K}{n'} \sum_{i=1}^{n'} \tilde{m}_i - V^{(n')} \bigg\|_{[0,T]} 
\leq \frac{K}{n'} \sum_{i=1}^{n'} \|\tilde{m}_i - m_i^{(n')}\|
\leq \frac{K}{n'} \sum_{i=1}^{n'} (|X_i^{(\infty)}(0) - X_i^{(n')}(0)| + \|Y^{(n')} - Y\|_{[0,T]}) \to 0
\]
almost surely. In words, \( V^{(n')} \) and the average of the running minimum of the i.i.d. Brownian paths below the curve \( Y \) become arbitrarily close in the uniform distance. By the strong law of large numbers \( \frac{1}{n'} \sum_{i=1}^{n'} \tilde{m}_i(t) \to E\tilde{m}_i(t) \) almost surely for each \( t \). That is,
\[
(3.5) \quad \lim_{n' \to \infty} V^{(n')}(t) = -\lim_{n' \to \infty} \frac{K}{n'} \sum_{i=1}^{n'} \tilde{m}_i(t) = -KE\tilde{m}_i(t),
\]
and by \( (3.4) \), \( V^{(n')} \) converges in the uniform norm to \( -KE\tilde{m}_i(t) \). This implies tightness for \( \{V^{(n)}(t) : t \in [0, T]\} \).

We will show that by combining Proposition \( 3.5 \) with tightness will give subsequential limits of \( V^{(n)} \) to deterministic functions. Proposition \( 2.10 \) will be used to show there is a unique limit.

**Remark 3.4.** Tightness of \( \{V^{(n)}(T) : t \in [0, T]\} \) implies there exists a subsequence \( V^{(n')} \) which converges in distribution to some process \( V \) in \( C([0, T], \mathbb{R}) \). By the Skorohod representation theorem one can exhibit a probability space supporting an entire sequence of processes, \( U^{(n')} \), and \( U \) such that \( U^{(n')} \to U \) almost surely in the uniform norm on \( C([0, T], \mathbb{R}) \) and where \( U^{(n')} \) (resp. \( U \)) has the same distribution as \( V^{(n')} \) (resp. \( V \)). Consequently if \( U \) is deterministic \( V \) is also deterministic, and therefore the conclusion of the next proposition also holds when \( V^{(n')} \) converges to \( V \) in distribution.
Proposition 3.5. Let \( n' \) be some sequence such that \( V^{(n')} = \frac{K}{n'} \sum_{i=1}^{n'} L_i^{(n)} \) converges uniformly on \([0, T]\) to \( V \), almost surely, on some probability space supporting underlying Brownian motions \( \{B^{(i)} : i \in \mathbb{N}\} \) and \( \{X_i^{(n)}(0) : 1 \leq i \leq n, n \in \mathbb{N}\} \). Then \( V \) is deterministic.

Remark 3.6. Since

\[
Y^{(n')}(t) = \int_0^t V^{(n')}(s)ds
\]

for all \( 0 < t < T \), almost surely, we see that \( Y^{(n')} \) converges uniformly to some deterministic \( Y \) if the sequence \( n' \) is as in Proposition 3.5.

Proof. We assume (3.3) as demonstrated in the proof of tightness for \( V^{(n)} \) in Lemma 3.2. That is, assuming \( n' \) is as in the statement of Proposition 3.5, we may assume without loss of generality that

\[
\frac{1}{n'} \mathbb{E} \sum_{i=1}^{n'} |X_i^{(n')}(0) - X_i^{(\infty)}(0)| \to 0
\]

almost surely. From this and Proposition 2.8, we have:

\[
\|V^{(n')}_{(B^{(1)}+X_1^{(n')}(0),...,B^{(n')}+X_{n'}^{(n')}(0))} - V^{(n')}_{(B^{(1)}+X_1^{(\infty)}(0),...,B^{(n')}+X_{n'}^{(\infty)}(0))}\|_{[0,T]} \leq K \frac{1}{n'} \sum_{i=1}^{n'} |X_i^{(n')}(0) - X_i^{(\infty)}(0)| \exp(\mathcal{K}T) \to 0
\]

almost surely. Therefore it suffices to show \( V^{(n')}_{(B^{(1)}+X_1^{(\infty)}(0),...,B^{(n')}+X_{n'}^{(\infty)}(0))} \) converges to a deterministic limit. For almost all \( \omega \) in our probability space and each \( k \geq 1 \), there is a constant \( C(\omega, k) \) such that \( \|B^{(1)}(x) = (B^{(1)}(0), \ldots, B^{(k)}(x) + X_k(x))\|_{[0,T]} < C(\omega, k) < \infty \). This follows from continuity of the \( B^{(i)} \) and the assumption that the initial samples \( X_i^{(\infty)}(0) \) come from an almost surely finite random variable. Apply Proposition 2.8 to \( f = (B^{(1)}(0), \ldots, B^{(n')}(0)) \), \( g = (B^{(1)}(0), \ldots, B^{(n')} + X_{n'}^{(\infty)}(0)) \) and \( \eta = \|f - g\|_{[0,T]} < C(\omega, k) \) to get

\[
\|V^{(n')}_{(B^{(1)}+X_1^{(\infty)}(0),...,B^{(n')}+X_{n'}^{(\infty)}(0))} - \mathcal{F}_T^{k+1,\infty}\|_{[0,T]} \leq (K C(\omega, k)/n') \exp(\mathcal{K}T) \to 0 \text{ as } n' \to \infty.
\]

Therefore

\[
V = \lim_{n' \to \infty} V^{(n')}_{(0, \ldots, 0, B^{(k+1)}(0), \ldots, B^{(n')} + X_{n'}^{(\infty)}(0))} \in \mathcal{F}_T^{k+1,\infty}
\]

where \( \mathcal{F}_T^{k+1,\infty} \) is the sigma-field generated by \( \{B^{(i)}(t) + X_i^{(\infty)}(0) : 0 < t < T, k \leq i\} \). By definition this means the continuous function \( V \) is adapted to the tail sigma-field of the infinite sequence of i.i.d. processes. Hence \( \{V(t) : t \in [0,T]\} \) is adapted to a trivial sigma-field, implying that \( V \) is deterministic.

Proposition 3.7 (Uniqueness of Limit). All subsequential limits given Proposition 3.5 are in fact the same.
It follows from the strong law of large numbers that
\( (3.6) \)
\[ Y^i = \lim_{n_k \to \infty} \lim_{2^{l-i} \to 0} I^{(n_k,2^{-l})} \]
\[ (B^{(1)} + X_1^{(\infty)}(0), \ldots, B^{(n_k)} + X_m^{(\infty)}(0)) \].

It follows from the strong law of large numbers that \( \| (B^{(1)} + X_1^{(\infty)}(0), \ldots, B^{(n_k)} + X_m^{(\infty)}(0)) \|_{[0,T]} / n < C(\omega) < \infty \) for almost each \( \omega \). Applying Proposition 2.10, we see that
\[ \| I^{(n_k,2^{-l})} - I^{(n_k,2^{-m})} \|_{[0,T]} \leq (2 + K)C(\omega)2^{-l} \exp(KT). \]

Let \( m \to \infty \) and we have
\[ \| I^{(n_k,2^{-l})} - I^{(n_k)} \|_{[0,T]} \leq (2 + K)C(\omega)2^{-l} \exp(KT). \]

In other words,
\[ \sup_{n_k \geq 1, i=1,2} \| I^{(n_k,2^{-l})} - I^{(n_k)} \|_{[0,T]} \leq (2 + K)C(\omega)2^{-l} \exp(KT), \]

and as \( 2^{-l} \to 0 \) the convergence of \( I^{(n_k,2^{-l})} \) to \( I^{(n_k)} \) is uniform over \( (n_k)_{k \geq 1} \), almost surely. By the Moore-Osgood theorem this guarantees an interchange of the limiting operations in \( (3.6) \). Hence,
\[ Y^i = \lim_{2^{-l} \to 0} \lim_{n_k \to \infty} I^{(n_k,2^{-l})}. \]

We will use the strong law of large numbers to show \( \lim_{n_k \to \infty} I^{(n_k,2^{-l})} = \lim_{n_k \to \infty} I^{(n_k,2^{-l})} \).

This can be seen by induction on \( [0,N2^{-l}] \): By construction of the \( I^{(n_k,2^{-l})} \) the two limits are identically zero on \( [0,2^{-l}] \). Assume the two limits agree on \( [0,N2^{-l}] \). This induction hypothesis implies the slope of \( I^{(n_k,2^{-l})} \) and the slope of \( I^{(n_k,2^{-l})} \) become arbitrarily close as \( k \to \infty \). Since the slope of \( I^{(n_k,2^{-l})} \) on \( [N2^{-l}, (N+1)2^{-l}] \) is the average of the positive part of the running minimum of the \( B^{(1)} + I^{(n_k,2^{-l})}, \ldots, B^{(n_k)} + I^{(n_k,2^{-l})} \), and because the limit in the strong law of large numbers is independent on the subsequence chosen, the slopes of \( I^{(n_k,2^{-l})}, I^{(n_k,2^{-l})} \) become arbitrarily close on \( [0, (N+1)2^{-l}] \) as \( k \to \infty \). This completes proves the induction step. \( \square \)

The previous two propositions imply the following corollary.

**Corollary 3.8.** There are deterministic functions \( \{(Y(t), V(t)) : 0 < t < T\} \) with \( dY/dt = V \), such that for any \( q \geq 1 \),
\[ (Y^{(n)}, V^{(n)}) \overset{W_q}{\to} (Y, V). \]

Furthermore, for \( t \in [0,T] \)
\[ V(t) = v - K\mathbb{E} m(t), \]
where \( m(t) = \sup_{0 \leq u < t} [B(u) + X_1^{(\infty)}(0) - Y(u)] \vee 0 \) is the running minimum of the Brownian motion under the curve \( Y \).

**Proof.** Convergence in \( W_q \) for any \( q \geq 1 \) is shown once we can establish that \( Y^{(n)} \) and \( V^{(n)} \) converge almost surely and in \( L_q \) to \( Y \) and \( V \), respectively, in some probability space supporting a sequence of i.i.d. Brownian motions and the initial conditions \( \{X_i^{(n)}(0) : 1 \leq i \leq m(0)\} \).
Convergence in distribution follows from Propositions 3.5 and 3.7. By Skorohod's representation there is a probability space where convergence holds almost surely. The convergence in $L_q$ comes from the bound indicated in the proof of Proposition 2.11 that

$$|V^{(n)}(t)| \leq \frac{K}{n} \sum_{i=1}^{n} L'_i(t)$$

where the $L'_i$ are i.i.d. local times at zero of Brownian motion. Now use the fact that $\frac{1}{n} \sum_{i=1}^{n} L'_i$ converges almost surely and in $L_q$ to its mean function, see [15], and apply the (generalized) dominated convergence theorem [14, Chapter 2.3]. The fact that $V(t) = v - K_E m(t)$ is given as (3.5) in Lemma 3.2 under the case $v = 0$. □

We are now in a position to prove the propagation of chaos result.

**Proof of Theorem 1.3.** It suffices to prove the theorem for two particles $X_1^{(n)}, X_2^{(n)}$. The initial conditions $\xi_1$ and $\xi_2$ are independent by assumption. From the Skorohod representation theorem there is a probability space supporting all our processes such that $Y^{(n)}$ converges almost surely to $Y$ in $C([0, T], \mathbb{R})$. For $l = 1, 2$ set

$$m_l(t) = \sup_{0 < u < t} -[B^{(l)}(u) + \xi_l - Y(u)] \vee 0,$$

$$m_l^{(n)}(t) = \sup_{0 < u < t} -[B^{(l)}(u) + \xi_l - Y^{(n)}(u)] \vee 0.$$  

By an argument similar to the one in Proposition 2.11, $\xi_l + B^{(l)} + m_l^{(n)}$ has the same distribution as $X_l^{(n)}$. Since $Y^{(n)} \to Y$ almost surely, $m_l^{(n)} \to m_l$ almost surely as well. Hence $\xi_l + B^{(l)} + m_l^{(n)} \to \xi_l + B^{(l)} + m_l$, almost surely. Clearly $\xi_1 + B^{(1)} + m_1$ and $\xi_2 + B^{(2)} + m_2$ are independent as each is a Brownian motion reflected from $Y$, driven by different independent Brownian motions with independent initial positions. □

In [4], Burdzy, Chen and Sylvester study the density of Brownian motion reflected inside a time dependent domain. They assume the boundary is $C^3$ in both time and space, see [4, Section 2]. In our case $n = 1$, and their results hold under the weaker assumption that the space-time boundary is $C^2$. Let $g(t) \in C^2([0, T], \mathbb{R})$ be a twice differentiable function with $g(0) = 0$. Given a Brownian motion $B(t)$ and $x \geq 0$, let $p(t, y)$ be the transition density of the reflected Brownian motion solving $dX(t) = dB(t) + dL(t)$, and the initial condition $X(0) = x$, where $L$ is the local time of $X$ on $g$. That is, for a given Borel set $A \subset [g(t), \infty),$

$$\mathbb{P}_x(X(t) \in A) = \int_A p(t, y)dy.$$  

**Proposition 3.9** [4, Theorem 2.9]. The transition density $p(t, y)$ defined above solves the following heat equation in a time-dependent domain:

$$\frac{\partial p(t, y)}{\partial t} = \frac{1}{2} \Delta_y p(t, y), \quad y > g(t),$$

$$\frac{\partial^+ p(t, y)}{\partial^+ y} = -2g'(t)p(t, y), \quad y = g(t),$$

$$\lim_{t \downarrow 0} p(t, y)dy = \delta_x(dy).$$
Remark 3.10. Here
\[ \frac{\partial^+ p(t, y)}{\partial^+ y} = \lim_{h \downarrow 0} \frac{p(t, y + h) - p(t, y)}{h} \]
is the one sided derivative on the positive side.

Corollary 3.11. Let \( \xi \) be a random variable with law \( \pi_0(dx) \), independent from the Brownian motion \( B \), both supported on \( (\Omega, \mathbb{P}, \mathcal{F}_t) \). Let \( g \in C^2([0, T], \mathbb{R}) \) and
\[ X(t) = \xi + B(t) + m(t), \quad m(t) = \sup_{0 < s < t} (-\xi + B(s) - g(s)) \vee 0. \]
Then \( p(t, x) := \mathbb{P}(X(t) = dx) \) solves the PDE
\begin{align*}
\frac{\partial p}{\partial t} &= \frac{1}{2} \Delta_y p, \quad y > g(t), \\
\frac{\partial p}{\partial y} &= -2g'(t)p, \quad y = g(t), \\
\lim_{t \downarrow 0} p(t, y) dy &= \pi_0(dy). 
\end{align*}

Proof. As in the proof of Proposition 2.11, it follows from Lévy’s theorem applied after a Girsanov change of measure that \( X \) is distributed as a Brownian motion reflected from the curve \( g \). Now apply Proposition 3.9 after conditioning on \( \xi \). \( \square \)

For a given time \( 0 < t < T \) and fixed value of \( n \), the definition of our interacting diffusions gives us \( n \) particles \( X_1^{(n)}(t), \ldots, X_n^{(n)}(t) \) which all lie in \( [Y^{(n)}(t), \infty) \). Recall that
\[ \pi_t^{(n)} = \frac{1}{n} \sum_{i=1}^n \delta_{\{X_i^{(n)}(t)\}} \]
denotes the empirical process of the arrangement of these particles. Similarly recall the definition of \( W_p \) in 1.6. The main property of \( W_p \) we will need is that \( \mathcal{P} \) is separable and complete under \( W_p \). Clearly \( \pi_t^{(n)} \) is a random variable with state space \( \mathcal{P} \). In this way \( \{\pi_t^{(n)} : t \in [0, T]\} \) is a \( (\mathcal{P}, W_p) \)—valued stochastic process. It follows from Proposition 3.13 below that \( \pi_t^{(n)} \) is continuous, and \( (\pi_t^{(n)}, Y^{(n)}(\cdot), V^{(n)}(\cdot)) \) has the strong Markov property.

Lemma 3.12. For any collection \( x_i, y_i \in \mathbb{R}, \ i = 1, \ldots, n \) we have
\[ W_p\left( \frac{1}{n} \sum_{i=1}^n \delta_{\{x_i\}}, \frac{1}{n} \sum_{i=1}^n \delta_{\{y_i\}} \right) \leq \left( \frac{1}{n} \sum_{i=1}^n |x_i - y_i|^p \right)^{1/p} \]

Proof. This follows from coupling \( (X, Y) \) with
\[ X \overset{d}{=} \frac{1}{n} \sum_{i=1}^n \delta_{\{x_i\}}, \quad Y \overset{d}{=} \frac{1}{n} \sum_{i=1}^n \delta_{\{y_i\}} \]
so that \( X \) has mass on \( \{x_i\} \) exactly when \( Y \) has mass on \( \{y_i\} \). \( \square \)

Proposition 3.13. The pair \( \{(\pi_t^{(n)}, Y^{(n)}(t), V^{(n)}(t)) : 0 < t < T\} \) is a continuous strong Markov process on \( \mathcal{P} \times \mathbb{R}^2 \) under the product metric \( W_p \times | \cdot | \).
Proof. The strong Markov property follows from the strong Markov property of \((X_1^{(n)}, \ldots, X_n^{(n)}, Y^{(n)}, V^{(n)})\). We need only show continuity of \(\pi^{(n)}\) since \((Y^{(n)}, V^{(n)})\) is continuous. By Lemma 3.12

\[
W_p(\pi_t^{(n)}, \pi_s^{(n)}) \leq \left( \frac{1}{n} \sum_{i=1}^{n} |X_i^{(n)}(t) - X_i^{(n)}(s)|^p \right)^{1/p},
\]

and continuity follows from the continuity of the \(X_i^{(n)}\). □

As \(\pi^{(n)}\) is a continuous \(\mathcal{P}\)–valued process, it induces a probability measure on \(C([0, T], (\mathcal{P}, W_p))\).

We will abuse notation, which should be clear from context, by letting \(\pi^{(n)}\) denote the measure on \(C([0, T], \mathcal{P})\), and \(\pi_t^{(n)}\) to denote either the stochastic process or the element in \(\mathcal{P}\) for fixed \(t\). Let

\[
\tilde{\pi}_t^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \delta_{\{\tilde{X}(i)(t)\}},
\]

where

\[
d\tilde{X}(i) = dB(i) + d\tilde{L}(i), \quad X_i^{(\infty)}(0) \overset{d}{=} \pi_0 \text{ for } i = 1, \ldots, n,
\]

the \(X_i^{(\infty)}(0)\) are i.i.d. and \(\tilde{L}(i)\) is the local time of \(\tilde{X}_i^{(n)}\) on the function \(Y\) given in Corollary 3.8.

**Proposition 3.14.** There is a probability space supporting \(\pi^{(n)}, \tilde{\pi}_t^{(n)}\) for all \(n\) such that

\[
\sup_{0 < t < T} W_p(\pi_t^{(n)}, \tilde{\pi}_t^{(n)}) \rightarrow 0
\]

almost surely.

**Remark 3.15.** This shows that distributional convergence of \(\tilde{\pi}_t^{(n)}\) to a probability measure on \(C([0, T], \mathcal{P})\) implies convergence of \(\pi_t^{(n)}\). In fact, distributional convergence of \(\pi_t^{(n)}\) and convergence of \(\tilde{\pi}_t^{(n)}\) are equivalent. They will approach the same limiting measure should one (hence both) of them converge.

**Proof.** We work on the probability space supporting all the \(\{B(i)(t) : 0 < t < T\}\) together with the initial conditions \(\{X_i^{(n)}(0) : 1 \leq i \leq n, n \in \mathbb{N}\}\). This space will then support \(Y^{(n)}, Y\) as well. By Corollary 3.8 we may also assume \(Y^{(n)} \rightarrow Y\) almost surely. As in the proof of Lemma 3.2, \(\{X_i^{(\infty)}(0) : i \in \mathbb{N}\}\) are i.i.d. samples with distribution \(\pi_0\). By our assumption that \(\pi_0^{(n)} \rightarrow \pi_0\) in \((\mathcal{P}, W_p)\), we may further choose our probability space so that

\[
\frac{1}{n} \sum_{i=1}^{n} |X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p \rightarrow 0
\]

almost surely. We use the same representation of our processes as in the proof of the propagation of chaos. That is,

\[
X_i^{(n)}(t) = X_i^{(n)}(0) + B(i)(t) + m_i^{(n)}(t),
\]

\[
\tilde{X}(i)(t) = X_i^{(\infty)}(0) + B(i)(t) + \tilde{m}_i(t),
\]
for $i = 1, \ldots, n$, and $t \in [0, T]$, where

\begin{align}
(3.14) & \quad m_i^{(n)}(t) = \sup_{0 < u < t} -[(B^{(i)}(u) + X_i^{(n)}(0)) - Y^{(n)}(u)] \lor 0, \\
(3.15) & \quad \tilde{m}_i(t) = \sup_{0 < u < t} -[(B^{(i)}(u) + X_i^{(\infty)}(0)) - Y(u)] \lor 0.
\end{align}

By the triangle inequality

\begin{equation}
(3.16) \quad \|m_i^{(n)} - \tilde{m}_i\|_{[0,t]} \leq |X_i^{(n)}(0) - \bar{X}_i^{(\infty)}(0)| + \|Y^{(n)} - Y\|_{[0,t]}
\end{equation}

for any $t \in [0, T]$. For any nonnegative numbers $a$ and $b$, $(a + b)^p \leq (2(a \lor b))^p \leq 2^p(a^p + b^p)$. Using (3.16), Lemma 3.12, (3.11) and the fact that $\|Y^{(n)} - Y\|_{[0,T]} \rightarrow 0$ almost surely,

\[
\sup_{0 < t < T} W_p(\pi_i^{(n)}, \tilde{\pi}_i^{(n)}) \leq \sup_{0 < t < T} \left( \frac{1}{n} \sum_{i=1}^{n} \|X_i^{(n)}(t) - \bar{X}_i^{(\infty)}(t)||^p \right)^{1/p} \\
\leq \sup_{0 < t < T} \left( \frac{1}{n} \sum_{i=1}^{n} \left( |X_i^{(n)}(0) - X_i^{(\infty)}(0)| + \|m_i^{(n)} - \tilde{m}_i\|_{[0,t]} \right)^p \right)^{1/p} \\
= \left( \frac{1}{n} \sum_{i=1}^{n} 2^{p+1} |X_i^{(n)}(0) - X_i^{(\infty)}(0)|^p + 2^p \|Y^{(n)} - Y\|_{[0,T]}^p \right)^{1/p} \rightarrow 0,
\]

almost surely. \quad \square

Recall the following notions of modulus of continuity. For $\gamma \in C([0, T], (\mathcal{P}, W_p))$,

\[ \omega'(\gamma, T, \delta) = \sup_{0 < t < T \atop |t - s| < \delta} W_p(\gamma_t, \gamma_s), \]

and similarly for $f \in C([0, T], \mathbb{R})$,

\[ \omega(f, T, \delta) = \sup_{0 < t < T \atop |t - s| < \delta} |f(t) - f(s)|. \]

In our method of showing tightness of the collection $\pi^{(n)}$ we utilize $p$-th moment bounds of $\omega(B, T, \delta)$ for a Brownian motion $B$. This is to be compared to Lévy’s theorem on the modulus of continuity for Brownian motion which deals with the almost sure behavior of the modulus of continuity for small values of $\delta$. We cite the article [13], where the authors Fischer and Nappo give a more general statement concerning moment bounds of $\omega(X, T, \delta)$, when $X$ is an Itô process.

**Theorem 3.16** ([13]). Let $B(t)$ be a one dimensional Brownian motion and $T > \delta > 0$. For any $q > 0$ there exists a positive constant $C_q$ independent of $T$ and $\delta$ such that

\[ \mathbb{E} \omega(B, T, \delta)^q < C_q \left( \delta \log \frac{T}{\delta} \right)^{q/2}. \]

This leads directly to the following strong law of large numbers applied to the modulus of continuity $\omega(B^{(i)}, T, \delta)$.
Corollary 3.17. Consider a sequence of independent Brownian motions \( \{B^{(i)} : i \in \mathbb{N}\} \) all defined on the same probability space. We have

\[
\frac{1}{n} \sum_{i=1}^{n} \omega(B^{(i)}, T, \delta)^q \xrightarrow{a.s.} \mathbb{E}\omega(B^{(i)}, T, \delta)^q < C_q \left( \frac{\delta \log T}{\delta} \right)^{q/2}
\]

for every \( q > 0 \), every \( \delta > 0 \), and some positive constant \( C_q \) depending on \( q \) only.

Remark 3.18. Typically when \( X_n \) are continuous stochastic process on a complete and separable metric space \((E,d)\), one demonstrates tightness of the measures induced on \( C([0,T],E) \) by showing “stochastic equicontinuity”

\[
\lim_{\delta \to 0} \sup_n \mathbb{P}(\omega(X_n, T, \delta) > \epsilon) = 0
\]

together with a compact containment condition for a countable dense set of times \([0,T]\):
given any \( \eta > 0 \) one can find a relatively compact set \( \Gamma_{t,\eta} \subset E \) such that

\[
\inf_n \mathbb{P}(X_n(t) \in \Gamma_{t,\eta}) > 1 - \eta.
\]

Consider (3.17) and the corresponding \( \delta \) for \( \epsilon = 1 \). Repeated use of the triangle inequality between time increments of size \( \delta \) can be used to bound \( X_n(t) \) with high probability uniformly in \( n \) at each time \( t \) should \( X_n \) be bounded w.h.p. uniformly in \( n \) at a fixed time \( t_0 \). Since boundedness in \( \mathbb{R}^d \) is equivalent to relative compactness, if \( E \) is Euclidean, (3.18) can be concluded from (3.17) provided there is some time \( t_0 \) such that \( X_n(t_0) \) is bounded w.h.p. uniformly in \( n \). If \( E \) is not Euclidean, finding compact sets may not be particularly easy, especially if \( E \) is not locally compact. Since our processes are \((\mathcal{P},W_p)\)—valued continuous processes, as shown in Lemma 3.13 and since \((\mathcal{P},W_p)\) is not locally compact, we face similar issues. One can use the \( p \)-th moment bounds on \( \omega(B^{(i)}, T, \delta) \) with a similar arguments in the proof of Proposition 3.19 to demonstrate (3.17). This would need to be paired with a compact containment condition as mentioned. We sidestep dealings with compact sets in \((\mathcal{P},W_p)\) by establishing almost sure pointwise convergence of subsequential limits of \( \pi_t^{(n)} \) together with a uniform stochastic equicontinuity result Proposition 3.19 below.

Proposition 3.19. For every \( \epsilon, \eta > 0 \) there corresponds a \( \delta > 0 \) such that

\[
\mathbb{P} \left( \sup_n \omega'(\pi_t^{(n)}, T, \delta) \leq \epsilon \right) > 1 - \eta.
\]

Proof. Recall the role of \( v \) in (1.1). We first prove the case when \( v \leq 0 \) so that \( Y \) is monotonically decreasing. The general case follows by applying the proof to each partition of \([0,T] = [0, t^*] \cup [t^*, T]\), where \( t^* \) is the unique zero of \( V \). From Lemma 3.12 and the
definitions of \( \omega, \omega' \) the following holds almost surely,

\[
\omega'(\bar{\pi}^{(n)}, T, \delta) := \sup_{0 \leq t < T, \quad \mid t - s \mid < \delta} W_p(\bar{\pi}_s, \bar{\pi}_t^{(n)})
\]

\[
\leq \sup_{0 < t < T, \quad \mid t - s \mid < \delta} \left( \sum_{i=1}^{n} \frac{1}{n} \left[ |B_i(s) - B_i(t)| + |\tilde{m}_i(s) - \tilde{m}_i(t)| \right] \right)^{1/p}
\]

\[
\leq \left( \sum_{i=1}^{n} \sup_{0 < t < T, \quad \mid t - s \mid < \delta} \left[ |B_i(s) - B_i(t)| + |\tilde{m}_i(s) - \tilde{m}_i(t)| \right] \right)^{1/p}
\]

\[
\leq \left( \sum_{i=1}^{n} \sup_{0 < t < T, \quad \mid t - s \mid < \delta} |B_i(t) - B_i(s)| + \sup_{0 < t < T, \quad \mid t - s \mid < \delta} |\tilde{m}_i(t) - \tilde{m}_i(s)| \right)^{1/p}
\]

\[
= \left( \sum_{i=1}^{n} \omega(B_i, T, \delta)^p + \omega(\tilde{m}_i, T, \delta)^p \right)^{1/p}.
\]

Because \( dY/dt \leq v \) is monotonically decreasing, \( \omega(\tilde{m}_i, T, \delta) \leq v\delta + \omega(B_i, T, \delta) \). That is, the maximum change the Brownian path makes below \( Y \), in the span of \( \delta \) time, is bounded by the change made by the line \( vt \) plus to the change of the Brownian path. This gives

\[
\omega'(\bar{\pi}^{(n)}, T, \delta) \leq \left( 2pv\delta + \frac{2p+1}{n} \sum_{i=1}^{n} \omega(B_i, T, \delta)^p \right)^{1/p}
\]

almost surely. For simplicity we take \( v = 0 \) in the remaining argument. Setting \( I_\epsilon = (\epsilon^p / 2p+1, \infty) \),

\[
\mathbb{P}(\sup_{n > N} \omega'(\bar{\pi}^{(n)}, T, \delta) > \epsilon) \leq \mathbb{P}\left( \sup_{n > N} \frac{1}{n} \sum_{i=1}^{n} \omega(B_i, T, \delta)^p > \frac{\epsilon^p}{2p+1} \right)
\]

\[
= \mathbb{E} 1_{I_\epsilon} \left\{ \sup_{n > N} \frac{1}{n} \sum_{i=1}^{n} \omega(B_i, T, \delta)^p \right\}.
\]

By Corollary 3.17 and the dominated convergene theorem,

\[
\lim_{N \to \infty} \mathbb{E} 1_{I_\epsilon} \left\{ \sup_{n > N} \frac{1}{n} \sum_{i=1}^{n} \omega(B_i, T, \delta)^p \right\} = \mathbb{E} 1_{I_\epsilon} \left\{ \mathbb{E} \omega(B_i, T, \delta)^p \right\}
\]

\[
\leq \mathbb{E} 1_{I_\epsilon} \left\{ C_p \left( \delta \log \frac{T}{\delta} \right)^{p/2} \right\}
\]

\[
= 1_{I_\epsilon} \left\{ C_p \left( \delta \log \frac{T}{\delta} \right)^{p/2} \right\}.
\]

In other words,

\[
(3.19) \quad \lim_{N \to \infty} \mathbb{P}(\sup_{n > N} \omega'(\bar{\pi}^{(n)}, T, \delta) > \epsilon) \leq 1_{I_\epsilon} \left\{ C_p \left( \delta \log \frac{T}{\delta} \right)^{p/2} \right\},
\]
which is 0 when $\delta$ satisfies
\[ \delta \log \frac{T}{\delta} < \frac{\epsilon^2}{4^{(p+1)/p} C_p^{2/p}}. \]
With this chosen value of $\delta$, take $N$ large enough so that
\[ \mathbb{P}(\sup_{n>N} \omega'(\bar\pi^{(n)}, T, \delta) > \epsilon) < \eta/2, \]
then appropriately shrink $\delta$ until
\[ \sum_{i=1}^{N} \mathbb{P}(\omega'((\bar\pi^{(i)}, T, \delta) > \epsilon) < \eta/2 \]
to conclude that
\[ \mathbb{P}(\sup_{n} \omega'(\bar\pi^{(n)}, T, \delta) > \epsilon) < \eta. \]

**Corollary 3.20.** The collection \{\(\bar\pi^{(n)}, n \geq 1\)\} is equicontinuous on \(C([0, T], (\mathcal{P}, W_p))\) with probability 1.

**Proof.** Apply Proposition \[3.19\] to decreasing sequences $\epsilon = 1/k$ and $\eta = 2^{-k}$ to yield a sequence $\delta_k \to 0$ with
\[ \sum_{k=1}^{\infty} \mathbb{P}(\sup_{n} \omega'(\bar\pi^{(n)}, T, \delta_k) > 1/k) < \infty. \]
By Borel-Cantelli the probability that \{\(\sup_{n} \omega'(\bar\pi^{(n)}, T, \delta_k) > 1/k\)\} occurs infinitely often is zero. Almost surely, \(A_k := \{\sup_{n} \omega'(\bar\pi^{(n)}, T, \delta_k) \leq 1/k\}\) occurs all but finitely many times. This means for an almost sure set of $\omega$ in our probability space there is a finite integer $N(\omega)$ so that $\omega \in \bigcap_{k>N(\omega)} A_k$, which in turn implies the sequence $\bar\pi^{(1)}(\omega), \bar\pi^{(2)}(\omega), \ldots$, is equicontinuous. \[\square\]

**Theorem 3.21** ([15]). For $p \geq 1$, let \{\(\xi_i : i \in \mathbb{N}\)\} be i.i.d. samples of an \(L^p\) bounded random variable $\xi$ with density $f$, all supported on the same probability space. Then
\[ \mathbb{E} W_p \left( \frac{1}{n} \sum_{i=1}^{n} \delta_{\xi_i} \right)(f) \to 0, \text{ as } n \to \infty. \]

We present one more lemma before proving the main result.

**Lemma 3.22.** Let $V$ be a continuous function, and $X$ a solution to $dX = dB + V dt + dL$ where $L$ is the local time of $X$ at zero. Then
\[ Z(t) = \exp \left( - \int_{0}^{t} V_s dB_s - \frac{1}{2} \int_{0}^{t} V_s^2 ds \right) \]
is a martingale with $Z(0) = 0$ and $\mathbb{E}[Z(t)^p] < \infty$ for any $p > 0$. 

\textbf{Proof.} Since \(V\) is continuous, it is bounded, and so it follows from Novikov’s condition that \(Z\) is a martingale. In fact, if \(M(t)\) is a continuous local martingale, \(Z := \exp(M - \frac{1}{2} \langle M \rangle)\) is a local martingale from Ito’s lemma. Because it is non-negative we may apply Fatou’s lemma to an exhaustive sequence of local times \(T_n \rightarrow \infty\) to see \(\mathbb{E}(Z'(t) | \mathcal{F}_s) \leq \lim_{n \to \infty} \mathbb{E}(Z'(t \wedge T_n) | \mathcal{F}_s) = \lim_{n \to \infty} Z'(s \wedge T_n) = Z'(s)\). That is, \(Z\) is a supermartingale. Take any \(p, q, q' > 0\) with \(\frac{1}{q} + \frac{1}{q'} = 1\). Then
\[
\mathbb{E}[Z(t)^p] = \mathbb{E}\left[ \exp\left(-p \int_0^t V_s dB_s - \frac{qp^2}{2} \int_0^t V_s^2 ds\right) \exp\left(\frac{p(qp - 1)}{2} \int_0^t V_s^2 ds\right) \right].
\]

Now apply Holder’s inequality with \(q, q'\)
\[
\mathbb{E}[Z(t)^p] \leq \mathbb{E}\left[ \exp\left(-pq \int_0^t V_s dB_s - \frac{q'p^2}{2} \int_0^t V_s^2 ds\right)^{1/q}\right]
\leq 1 \cdot \mathbb{E}\left[ \exp\left(\frac{p(qp - 1)}{2} \int_0^t V_s^2 ds\right)^{1/q'} \right] = \exp\left(\frac{p(qp - 1)}{2} \int_0^t V_s^2 ds\right) < \infty.
\]

Here
\[
\mathbb{E}\left[ \exp\left(-pq \int_0^t V_s dB_s - \frac{q'p^2}{2} \int_0^t V_s^2 ds\right) \right] \leq 1
\]

since
\[
M(t) = -pq \int_0^t V_s dB_s, \quad \langle M \rangle(t) = q^2 p^2 \int_0^t V_s^2 ds
\]

and because \(\exp(M(t) - \langle M \rangle(t))\) is a supermartingale as explained above. \(\square\)

\textbf{Proof of Theorem 1.2.} We first show that \(\pi^{(n)}\) converges in distribution to the measure induced by \(p(t, \cdot)\). By Proposition 3.14 it suffices to show this for \(\tilde{\pi}^{(n)}\). Take any subsequence \(n_k\). For each rational \(0 < t < T\) we have defined \(\tilde{\pi}_t^{(n)}\) as an empirical measure of i.i.d. random variables with density \(p(t, \cdot)\) taken from Corollary 3.11 by replacing \(g\) in the Corollary statement with \(Y\). By Theorem 3.21
\[
\mathbb{E} W_p(\tilde{\pi}_t^{(n_k)}, p(t, \cdot)) \rightarrow 0, \quad \text{for each } t \in [0, T].
\]

For each rational \(t \in [0, T]\) there is a subsequence \(n'_k\) such that \(W_p(\tilde{\pi}_t^{(n'_k)}, p(t, \cdot)) \rightarrow 0\) almost surely. By a Cantor diagonalization applied to each subsequence for an enumeration of the rationals, there exists a single subsequence \(n''_k\) such that \(W_p(\tilde{\pi}_t^{(n''_k)}, p(t, \cdot)) \rightarrow 0\) for each rational \(t \in [0, T]\), almost surely. Apply the uniform equicontinuity given by Corollary 3.20 and follow the proof of Arzéla-Ascoli verbatim to see that the subsequence \(\tilde{\pi}^{(n''_k)}\) is totally bounded in the space \(C([0, T], (\mathcal{P}, W_p))\), almost surely. See [14, Chapter 4.6]. Total boundedness in a metric space is equivalent to every sequence having a Cauchy subsequence. Consequently, for almost every \(\omega\) in the probability space, every subsequence of \(\pi^{(n''_k)}(\omega)\) has a Cauchy subsequence in \(C([0, T], (\mathcal{P}, W_p))\). Because \(C([0, T], (\mathcal{P}, W_p))\) is complete, every subsequence of \(\pi^{(n''_k)}(\omega)\) has a convergent subsequence. Since \(\pi_t^{(n''_k)}(\omega)\) already converges to the continuous \(p(t, \cdot)\) along rationals, every subsequence of \(\pi^{(n''_k)}(\omega)\) has a further subsequence converging to \(p(t, \cdot)\). Therefore \(\pi^{(n_k)}\) converges to \(p(t, \cdot)\) almost surely. We have shown that every subsequence \(\pi^{(n_k)}\) has a further subsequence \(\pi^{(n'_k)}\) converging to \(p(t, \cdot)\) in law on
\( C([0, T], (\mathcal{P}, W_p)) \). This proves the claim that \( \{ \pi_t^n : t \in [0, T] \} \) converges in distribution to \( p(t, \cdot) \). Next, we show

\[
V(t) = -(K/2) \int_0^t p(s, Y(s))ds.
\]

Let

\[
m(t) = \sup_{0 < u < t} -[B^{(1)}(u) + X^{(\infty)}(0) - Y(u)] \vee 0.
\]

As in the proof of Proposition 2.11 we know \( m(t) \) is distributed as \( \tilde{L}^{(1)}(t) \), the local time of \( \tilde{X}(t) := B^{(1)}(t) + X^{(\infty)}(0) + m(t) \) on \( Y \). From Corollary 3.8 we have, almost surely,

\[
V(t) = \mathbb{E} m_1(t) = \mathbb{E} \tilde{L}^{(1)}(t)
\]

\[
= \mathbb{E} \lim_{\epsilon \to 0} -\frac{K}{2\epsilon} \int_0^t 1_{[0,\epsilon]}(\tilde{X}(s) - Y(s))ds
\]

\[
= \lim_{\epsilon \to 0} -\frac{K}{2\epsilon} \mathbb{E} \int_0^t 1_{[0,\epsilon]}(\tilde{X}(s) - Y(s))ds
\]

\[
= \frac{-K}{2} \lim_{\epsilon \to 0} \int_0^t F(s, \epsilon)ds,
\]

where \( F(s, \epsilon) = \mathbb{P}(0 \leq \tilde{X}(s) - Y(s) \leq \epsilon) \), provided we justify the passing of the limit under the expectation. In the proof of Proposition 2.11 we saw that \( \tilde{X} - Y \) solves an SDE of the form \( dW = dB + Vdt + dL \) for the continuous function \( V \), and such processes have a continuous density \( \phi(s, x) = p(s, Y(s) + x) \). That such processes have a continuous density is shown in [22]. Write

\[
\frac{1}{\epsilon} \int_0^t F(s, \epsilon)ds = \int_0^t \frac{1}{\epsilon} \int_0^\epsilon \phi(s, x)dxds = \int_0^t \phi(s, x^*)ds
\]

for some \( 0 < x^* < \epsilon \) by the mean value theorem. For all \( 0 < \epsilon < 1 \)

\[
\frac{1}{\epsilon} \int_0^t F(s, \epsilon)ds \leq \sup_{0 < x^* < 1} \int_0^t \phi(s, x^*)ds \leq \int_0^t \sup_{0 < x^* < 1} \phi(s, x^*)ds < \infty
\]

and the bounded convergence theorem justifies the passing of the limit inside the time integral,

\[
\frac{-K}{2} \lim_{\epsilon \to 0} \int_0^t \frac{F(s, \epsilon)}{\epsilon}ds = \frac{-K}{2} \int_0^t \lim_{\epsilon \to 0} \frac{F(s, \epsilon)}{\epsilon}ds = \frac{-K}{2} \int_0^t \phi(s, 0)ds.
\]

That is,

\[
V(t) = \frac{-K}{2} \int_0^t p(s, Y(s))ds.
\]

We now justify the exchange of limit in (3.20) using the definition of local time to replace the time integral with a space integral. Let \( L^{(1)}(s, a) \) denote the local time of \( \tilde{X} - Y \) at level \( a \) and time \( s \). We see

\[
\frac{1}{\epsilon} \int_0^t 1_{[0,\epsilon]}(X(s) - Y(s))ds = \int_0^t \frac{1}{\epsilon} \int_0^\epsilon \tilde{L}^{(1)}(s, z)dzds \leq \int_0^t \sup_z [\tilde{L}^{(1)}(s, z)]dz.
\]
The Lebesgue dominated convergence theorem will justify (3.20) provided we show
\[ E \int_0^t \sup_z \tilde{L}^{(1)}(s, z) ds \leq t E \sup_z \tilde{L}^{(1)}(t, z) < \infty. \]
We may apply a Girsanov change of measure as in Lemma 3.22 as \( Y \) satisfies the Novikov condition, so
\[ Z(t) = \exp \left( -\int_0^t V_s dB_s - \frac{1}{2} \int_0^t V_s^2 ds \right) \]
is an exponential martingale with \(|B|\) having the same distribution as \( \tilde{X} \) under the measure \( dQ/dP = Z(t) \) by this Girsanov transformation. Lemma 3.22 states
\[ E[|Z(T)|^2] = C < \infty. \]
From this, the change of measure formula and Cauchy-Schwarz,
\[ E \sup_z \tilde{L}^{(1)}(t, z) = E(Z(t) \sup_z L(t, z)) \leq E(Z^2(t))^{1/2} E[(\sup_z L(t, z))^2]^{1/2} \leq C^{1/2} E[(\sup_z L(t, z))^2]^{1/2} \]
where \( L(t, z) \) is the local time at level \( z \) of Brownian motion reflected from the origin. The main results in [2, Theorem 3.1] demonstrate bounds on the last term, where Barlow and Yor show the existence of a constant \( C_p \) such that
\[ E[(\sup_z L_t(z))^p] \leq C_p t^{p/2}. \]
It follows that
\[ E \sup_z \tilde{L}^{(1)}(t, z) < \infty, \]
completing the proof. \( \square \)

4. Uniqueness of the heat equation with free-boundary

In this section we give existence and uniqueness for the PDE with free boundary condition \((p(t, \cdot), y(t))\) which is the solution of our hydrodynamic limit given by (1.2). If \((p, y)\) is a solution and \( p(t, \cdot) \) represents the distribution of heat, then the equation in Theorem 1.2 is interpreted as saying the acceleration of the moving barrier \( y(t) \) is proportional to the current amount of heat on it. The hydrodynamic limit already yields existence of such a solution. In that statement of Theorem 1.2 \((\pi^{(n)}, Y^{(n)})\) converges in some sense to a solution of (1.2). Here we show this is the only solution by demonstrating uniqueness of this PDE with free boundary.

Remark 4.1. For any solution \((p, y)\) of (1.2) make a substitution \( u(t, x) = p(t, x + y(t)) \) and see \((u, y)\) solves
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \frac{1}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + y'(t)u_x(t, x), \quad \text{when } x > 0, \\
\frac{\partial u}{\partial x}(t, 0) &= -2y'(t)u(0, t), \\
y''(t) &= -\frac{K}{2} u(t, 0), \quad y(0) = 0, \quad y'(0) = v \in \mathbb{R}, \quad y'' \in C([0, T], \mathbb{R}), \\
\lim_{t \downarrow 0} u(t, x) &= f(x)dx, \quad f \in L^1(\mathbb{R}_+). 
\end{align*}
In this way the two problems are equivalent.

Theorem 4.2. The PDE problem in (4.1)-(4.3), and equivalently that in (1.2), has a unique solution for any \( K \geq 0 \).
Remark 4.3. The regularity of the boundary plays an important role because if $y''$ exists then the solution to (1.2) has a stochastic representation given from Corollary 3.11. We exploit this to show uniqueness.

Proof. Theorem 1.2 gives existence. To show uniqueness we will prove the corresponding barriers $y_1, y_2$ of any two solutions are in fact equal. Assume that $(p_1(t, \cdot), y_1(t)), (p_2(t, \cdot), y_2(t))$ are pairs solving the PDE with the given initial conditions. Following Corollary 3.11 above we know that the transition density $p_i(t, x)$ of Brownian motion reflecting from $y_i$ satisfies the PDE

\begin{align}
\frac{\partial p_i}{\partial t} &= \frac{1}{2} \Delta_y p_i, \ y > y_i(t), \\
\frac{\partial p_i}{\partial y} &= -2y_i'(t) p_i, \ y = y_i(t), \\
\lim_{t \to 0} p_i(t, y)dy &= f(y)dy \in L^1(\mathbb{R}_+).
\end{align}

Without loss of generality we assume $\int f(y)dy = 1$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space supporting a Brownian motion $B(t)$ and an independent random variable $\xi$ with density $f$. As in the proof of Theorem 1.2 we know

\[ y_i''(t) = -(K/2) \mathbb{E} m_i(t), \quad \text{where } m_i(t) = \max_{s \in [0,t]} -[B(u) + \xi - y_i(u)] \vee 0. \]

Linearity of expectation yields the following comparison between $y_1', y_2'$:

\[ |y_1'(t) - y_2'(t)| \leq \int_0^t |y_1''(s) - y_2''(s)|ds \]

\[ = \frac{K}{2} \int_0^t \mathbb{E} (m_1(s) - m_2(s))ds \leq \frac{K}{2} \int_0^t \|y_1 - y_2\|_{[0,s]}ds \leq \frac{K}{2} t \|y_1 - y_2\|_{[0,t]}.
\]

Consequently,

\[ |y_1(t) - y_2(t)| \leq \frac{K}{2} \int_0^t x\|y_1 - y_2\|_{[0,x]}dx \leq (K/2)t^2 \|y_1 - y_2\|_{[0,t]}.
\]

Because the right hand is nondecreasing this inequality holds when the left hand is maximized across time,

\[ \|y_1 - y_2\|_{[0,t]} \leq (K/2)t^2 \|y_1 - y_2\|_{[0,t]}.
\]

Therefore $\|y_1 - y_2\|_{[0,t]} \leq C \|y_1 - y_2\|_{[0,t]}$ for some $C < 1$ as long as $0 \leq t < t^* < \sqrt{2/K}$. As a result $\|y_1 - y_2\|_{[0,t^*]} = 0$ for all $t^* \in [0, \sqrt{2/K}]$. In other words, the barriers $y_1$ and $y_2$ are identical up until this fixed positive time. A renewal argument shows that $y_1$ and $y_2$ are identical across the entire interval $[0, T]$. $\square$

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