Mesoscopic sensitivity of speckles in disordered nonlinear media
to changes of disordered potential

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Abstract

We show that the sensitivity of wave speckle patterns in disordered nonlinear media to changes of scattering potential increases with sample size. For large enough sample size this quantity diverges, which implies that at given coherent wave incident on a sample there are multiple solutions for the spatial distribution of the wave’s density. The number of solutions increases exponentially with the sample size.

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If a coherent wave described by a field $\phi(r, \epsilon)$ propagates in an elastically scattering medium, the spatial dependence of its "density" $n(r, \epsilon) = |\phi(r, \epsilon)|^2$ exhibits speckle: $n(r, \epsilon)$ is a random, sample specific function of coordinate $r$. Here $\epsilon$ is the wave’s energy. In the cases of noninteracting electrons and electromagnetic waves propagating in linear media the theory of sensitivity of speckle patterns to a change in scattering potential was developed long ago [1–5]. It was shown that the sensitivity is very large, but finite.

In this article we consider the same question in the case where a wave propagates in nonlinear media. For the sake of concreteness we consider the situation where the propagation of the wave is described by a nonlinear Schrodinger equation

$$(-\frac{1}{2m} \frac{\partial^2}{\partial r^2} - \epsilon + u(r) + \tilde{u}(r))\phi(r, \epsilon) = 0$$

(1)

Here $m$ is the wave mass, $\tilde{u}(r) = \beta n(r)$ is the effective nonlinear potential and $u(r)$ is a scattering potential which is a random function of the coordinates. Similar equations appear in the theory of electromagnetic waves propagating in nonlinear media [6], the theory of hydrodynamic turbulence [7], and the theory of turbulent plasma [8]. We will assume white noise statistics in $u(r)$: $\langle u(r) \rangle = 0$, $\langle u(r)u(r_1) \rangle = \frac{\pi l m^2}{\epsilon m^2} \delta(r-r_1)$. Here brackets $\langle \rangle$ correspond to averaging over realizations of $u(r)$ and $l$ is the elastic mean free path ($l \gg k^{-1} = (2\epsilon m)^{-1/2}$).

Let us consider the case where a coherent wave $\phi_0(r) = \sqrt{n_0} \exp(ikr)$ with momentum $k$ is incident on a disordered sample of the thickness $L \gg l$ (See the insert in Fig.1). We will show that the sensitivity of the nonlinear speckle pattern $n(r)$ to a small change in $u(r)$ increases with sample size $L$. At arbitrarily small $n_0$ and for arbitrary sign of $\beta$ the sensitivity become infinite provided $L$ is large enough. This implies that at given coherent wave incident on a sample Eq.1 has many solutions. This is very different from the case of uniform nonlinear media, where types of instabilities depend on the sign of $\beta$. (See, for example, [6].)

The $r$ dependence of the average density $\langle n(r, \epsilon) \rangle$ can be described by the diffusion equation, which is equivalent to calculation of the diagrams shown in Fig.2.a. We use the usual diagram technique for averaging over realizations of random potential [9]. If $|\beta n_0| \ll
one can neglect the nonlinear corrections to the diffusion coefficient $D = \frac{k_B}{3m}$. We obtained this criterion by calculating the transport scattering cross section on the effective potential $\beta n(r)$. To do so we used calculated in [11,4] spatial correlation functions of the density $[10]$. In the case of the sample geometry shown in the insert of Fig.1, we have $\langle n(r) \rangle = n_0$.

We can characterize the speckle pattern $n(r)$ and its sensitivity to a small change of scattering potential $\Delta u(r) = u'(r) - u(r)$ by correlation functions $\langle \delta n(r) \delta n(r_1) \rangle$ and $K(r, r_1) = \langle \Delta n(r) \Delta n(r_1) \rangle$. Here $\delta n(r) = n(r) - \langle n(r) \rangle$; $\Delta n(r) = n'(r, \{u'(r)\}) - n(r, \{u(r)\})$, $n(\epsilon, r, \{u(r)\})$ and $n'(\epsilon, r, \{u'(r)\})$ are solutions of Eq.1 with scattering potentials $u(r)$ and $u'(r)$ respectively, and the brackets $\langle \rangle$ correspond to averaging over both realizations of $u(r)$ and realizations of $\Delta u(r)$. We will assume that $\langle \langle u(r)u'(r_1) \rangle \rangle = U^2 \exp\left(\frac{|r-r_1|}{\tau_0}\right)$. To get the value of $K(r, r_1)$ we will use $|r - r_1| \gg l$ and in the first in $\Delta u(r)$ approximation one can generalize the Langevin approach for calculations of mesoscopic fluctuations $[4,11]$ to include nonlinear effects

$$\frac{d}{dr} \delta J(r) = 0; \quad \delta J(r) = -D \frac{d}{dr} \delta n(r) + J_{ext}(r); \quad (2)$$

$$J_{ext}(r, \{u'(r)\}) = J_{ext}(r, \{u(r)\}) + \int dr' \frac{\delta J_{ext}(r)}{\delta u(r')} (\Delta u(r') + \Delta \tilde{u}(r')) \quad (3)$$

$$\langle J^i_{ext}(r) J^j_{ext}(r_1) \rangle = \frac{2\pi l}{\delta m^2} (\langle n(r) \rangle)^2 \delta(r - r_1) \delta_{ij} \quad (4)$$

$$\langle \frac{\delta J^i_{ext}(r) \delta J^j_{ext}(r_1)}{\delta u(r')} \delta u(r'_1) \rangle = \frac{6\pi}{lk^2} \delta_{ij} \delta(r - r_1) \{G(r', r_1) \langle n(r) \rangle \times (\langle n(r'_1) \rangle G(r', r) + \langle n(r') \rangle G(r'_1, r) - \langle n(r') \rangle \langle n(r'_1) \rangle G(r', r) \} G(r'_1, r) \} \quad (5)$$

$$\Delta \tilde{u}(r) = \tilde{u}'(r) - \tilde{u}(r) = \beta \Delta n(r) \quad (6)$$

where $G(r, r_1)$ is the Green function of the equation

$$- \frac{d^2}{d^2 r} G(r, r_1) = \delta(r - r_1); \quad (7)$$
$J(r) = \frac{1}{2m} Im\phi^*(r) \frac{d}{dr} \phi(r)$ is the current density, \(\delta J(r) = J(r) - \langle J(r) \rangle\), \(J_{\text{ext}}(r, \{u(r)\})\) is a random external current source, \(\langle J_{\text{ext}}(r) \rangle = \langle \delta J_{\text{ext}}(r) \delta u(r') \rangle = \langle J_i^{\text{ext}}(r) \delta J_j^{\text{ext}}(r') \rangle = 0\), and \(i, j\) are the coordinate’s indices. Eqs.2,3,7 require the usual diffusion boundary conditions:

\[ G(r, r') = 0, \quad \langle n(r) \rangle = n_0 \text{ at } x = 0 \quad \text{and} \quad n \cdot \partial_r G(r, r') = n \cdot \partial_r \langle n(r) \rangle = n \cdot J_{\text{ext}}(r) = 0 \text{ at the closed sample’s boundaries.} \]

Here \(n\) is a the unit vector normal to the boundary.

Eqs.2-7 are a closed system which differ from \([4,11]\) by the term in Eq.3 proportional to \(\beta\). They are equivalent to the summation of diagrams shown in Fig.2b-g. Diagrams, shown in Fig.2h, are responcible for the small nongaussian part of the distribution function of \(\delta J_{\text{ext}}(r) \delta u(r')\). They are proportional to a small parameter \(\frac{1}{k^2L} \ll 1\) in the three dimentional case \((d=3)\) and can be neglected. All diagrams responcible for localisation effects can be neglected as well.

Let us first consider the linear case \(\beta = 0, \tilde{u}(r) = 0\). Index \((0)\) will indicate quantities calculated at \(\beta = 0\). Solving Eqs.2-5,7 at \(|r - r_1| > l\) in \(d = 3\) case we get \([4]\)

\[ K^{(0)}(r, r_1) = \langle \langle \Delta n^{(0)}(r) \Delta n^{(0)}(r_1) \rangle \rangle \sim \left( \frac{\tau_D}{\tau_f} \right)^2 < \delta n(r) \delta n(r_1) > \sim \frac{n_0^2}{k^2l|r - r_1|} \left( \frac{\tau_D}{\tau_f} \right)^2 \]

(8)

where \(\tau_D = L^2/D\), and \(\frac{1}{\tau_f} = \frac{mU}{L}\) characterizes the change in scattering potential. This can also be obtained by calculating the diagrams shown in Fig.2b,c. The characteristic time \(\tau_f^{(0)} \sim L^2/D\) corresponds to a complete change in the speckle pattern due to the change of the scattering potential \(\Delta u(r)\). One can get the same estimate from the requirement that an additional phase \(\chi^{(0)} \sim \sqrt{\frac{L^2}{Dr_f^{(0)}}}\), which the traveling wave aquires due to the change in the potential \(\Delta u(r)\), is of order \(\pi\). If impurities have a cross-section of order \(\frac{1}{k^2}\) and they are shifted from their initial position by distances of order \(\frac{1}{k}\) then in the \(d = 3\) case one has to change the positions of the \(N^{(0)} = Llk^2\) impurities to in order change the speckle pattern significantly \([2]\). Characteristic changes of energy \(\Delta \epsilon^{(0)*} = \frac{L}{k^2}\) and of the angle of incidence \(\Delta \theta^{(0)*} = \frac{1}{kL}\) (see the insert in Fig.1), which change the speckle pattern significantly, can be obtained in a similar way \([1,4]\).

Let us now turn to the case \(\beta \neq 0\). Expanding Eqs.2-6 up to second order in \(\beta\), and performing the average over realizations of \(u(r)\) and \(\Delta u(r)\) in the \(d=3\) case we get the
correction to \(K^{(0)}(r, r_1)\).

\[
K^{(1)}(r, r_1) \sim \gamma K^{(0)}(r, r_1)
\]

(9)

where

\[
\gamma = \left(\frac{3 \eta_0 \beta}{\epsilon}\right)^2 \left(\frac{L}{l}\right)^3
\]

(10)

The index (1) indicates quantities proportional to \(\beta^2\). It can also be obtained by calculating the diagrams shown in Fig.2d-g or by estimating the additional phase which the wave traveling along a typical diffusion path will pick up due to the change in the effective potential \(\Delta \tilde{u}^{(1)}(r) = \beta \Delta n^{(0)}(r)\)

\[
\langle\langle (\Delta \chi^{(1)})^2 \rangle\rangle = \left(\frac{k \beta}{2 \epsilon}\right)^2 \langle\langle \int ds ds_1 \Delta n^{(0)}(s) \Delta n^{(0)}(s_1) \rangle\rangle \sim \gamma \langle\langle (\chi^{(0)})^2 \rangle\rangle
\]

(11)

Here integration is taken along typical diffusion paths of length \(\frac{L^2}{l}\).

Eqs.9,11 imply that \(\langle\langle (\chi^{(1)})^2 \rangle\rangle \gg \langle\langle (\chi^{(0)})^2 \rangle\rangle\) and that Eq.1 has many solutions. Let us estimate the number of the solutions in the \(D=3\) case. It is convenient to expand

\[
\tilde{u}(r) = \frac{D}{\sqrt{L}} \sum_m m \hat{\bar{u}}_m n_m(r)
\]

(12)

over a complete set of eigenstates \(n_m(r)\) of diffusion equation Eq.7

\[
- \frac{D}{d^2r} n_m(r) = E_m n_m(r)
\]

(13)

where \(E_m \sim \tau_D^{-1} m^{\frac{2}{3}}\) are eigenvalues of Eq.13 and \(m = 1, 2...\) labels the eigenmodes. Let us first regard \(\bar{u}_m\) as independent variables. The solution of Eq.1 can be written as \(n(r) = n(r, \bar{u}_1, .. \bar{u}_k...)\). Then using the selfconsistency equation \(\tilde{u} = \beta n\) we get

\[
\gamma^{-\frac{1}{2}} m^{\frac{2}{3}} \bar{u}_m = F_m(\bar{u}_1, .. \bar{u}_k...)
\]

(14)

where \(F_m(\bar{u}_1, ...) = k L^{-1} n_1^{-1} m^{\frac{1}{2}} l^{\frac{1}{2}} \int d^n r n(\bar{r}, \bar{u}_1...)n_m(r)\) are random sample specific functions.

The problem of the investigation of properties of \(F_m(\bar{u}_1, ...\) as a function of \(\bar{u}_n\) is equivalent to the linear problem considered in [1-5]. To characterize the dimensionless functions
we calculate the following correlation functions with the help of Eqs.2-5,7: (a) mesoscopic fluctuations of modes with \( m \neq n \) are uncorrelated \( \langle \delta F_m \delta F_n \rangle = 0 \), where \( \delta F_m = F_m - \langle F_m \rangle \);

(b) \( \langle (\delta F_m)^2 \rangle \sim 1 \); and (c)

\[
\frac{\langle [F_m(\bar{u}_1,..\bar{u}_n + \Delta \bar{u}_n,..) - F_m(\bar{u}_1,..\bar{u}_n,..)]^2 \rangle}{\langle (\delta F_m)^2 \rangle} \sim (\Delta \bar{u}_n)^2.
\]

Eq.15 means that the characteristic period of random oscillations of \( F_m \) as a function of \( \bar{u}_n \) is of order unity, \( \Delta \bar{u}_n \sim 1 \). In anticipation of these results we have introduced the factor \( m^{\frac{3}{4}} \) in Eq.12.

It is important that Eqs.14 with large enough \( m > M = \gamma^{\frac{2}{3}} \) have unique solutions \( \bar{u}_m \sim m^{-\frac{2}{3}} \ll 1 \), provided the values \( \bar{u}_{m<M} \) are given. We get the estimate for \( M \) from the requirement \( \gamma^{-\frac{1}{2}} m^{\frac{2}{3}} \ll 1 \). Therefore at \( \gamma \sim 1 \) modes with \( m \simeq 1 \) are the most important for determination of the number of solutions of Eq.1. This also follows, for example, from the long range \( r - r' \) dependence of \( K_0(r, r') \) and from the the fact that the main contribution to Eq.9 and the diagrams shown in Fig.2 d-g is from integration over intermediate coordinates with \( |r - r'| \sim L \). Therefore we would like to introduce a model which captures the main features of the problem at \( \gamma \sim 1 \): we put \( \bar{u}_{m>1} = 0 \) in the first of Eqs.14 for \( m = 1 \) and get the equation

\[
\gamma^{-\frac{1}{2}} \bar{u}_1 = F_1(\bar{u}_1, 0, 0, ..)
\]

It is equivalent to substitution in Eq.1 \( \bar{u}(r) \rightarrow \frac{D}{\sqrt{L}} \bar{u}_1 n_1(r) \). (Then expanding Eq.1 with respect to powers of \( \beta \) one can reproduce the values of the diagrams Fig.2b-g with the precision of the factor of order of unity.) In Fig.1 we show a qualitative ”graphical” solution of Eq.16 which corresponds to intersection of two functions:\( F_1(\bar{u}_1, 0, 0, ..) \) and \( \gamma^{-\frac{1}{2}} \bar{u}_1 \). It follows from Fig.1 that at \( \gamma > 1 \) both Eq.16 with a typical realization of the potential \( u(r) \), and consequently Eq.1, have many solutions. In this case the sensitivity defined as

\[
(\frac{\beta}{\tau_D})^2 K(r, r')
\]

diverges. The main contribution to this divergency comes from realizations of \( u(r) \), when \( F_1(\bar{u}_1, 0, 0, ..) \) and \( \gamma^{-\frac{1}{2}} \bar{u}_1 \) are tangent to each other. The criterion \( \gamma > 1 \) is equivalent to the inequality \( \langle (\frac{\beta}{\tau_D})^2 \int n(r)d\mathbf{r} \rangle > 1 \). In such a form this criterion is similar to the criterion of Stoner ferromagnetic instability in metals [13].
We would like to mention that even at $\gamma < 1$ there are rear realizations of $u(r)$ which correspond to several solutions of Eq.1. Therefore, formally speaking, the sensitivity diverges at any $\gamma$. Obviously the conventional diagram technique is unable to describe the existence of many solutions of Eq.1.

At $\gamma \gg 1$ the number of solutions of Eq.16 shown in Fig.1 is of order $\gamma^{\frac{3}{2}}$. However, if $\gamma \gg 1$ not only $\bar{u}_1$, but also higher modes with $1 < m < M$, are relevant. In this case Eqs.14 have multiple solutions in the intervals $|\bar{u}_m| < \gamma^{\frac{3}{4}}m^{-\frac{3}{4}}$. Since both the amplitude of fluctuations and the periods in $m$-th direction of randomly rippled hypersurfaces $F_m(\bar{u}_1...)$ are of order unity, the number of solutions $N$ of Eqs.14,1 is proportional to the volume of the manifold $|\bar{u}_m| < \gamma^{\frac{3}{4}}m^{-\frac{3}{4}}$, $m < M$. As a result we have

$$N \sim \gamma^{\frac{3M}{2}} \prod_{m=1}^{M} m^{-\frac{3}{4}} = \exp(a\gamma^{\frac{3}{4}}) \tag{17}$$

where $a \sim 1$.

Similar phenomenon may occur in disordered metals with interacting electrons. The system can be unstable with respect to creation of random magnetic moments. In this case $n(r)$ would play the role of magnetization density. This would correspond to Finkelshtain’s scenario \cite{12}. However, in this case to get a self consistency equation for $n(r)$ we have to integrate over electron energies up to the Fermi energy, which decreases the amplitude of mesoscopic fluctuations of $n(r)$. As a result, at small electron-electron interaction constant the situation with many solutions occurs only in the D=2 case and the characteristic spatial scale of integration over $r$ will be of the order of the localization length in the linear problem. Thus the problem of interacting electrons in disordered metals remains unsolved.

Above we considered the case when $\phi(r,t) = \phi(r,\epsilon) \exp(i\epsilon t)$ is a complex quantity and $n(r) = |\phi(r,t)|^2$ is time independent. Therefore the third harmonic, proportional to $\exp(3i\epsilon t)$ is not generated. In the case of propagation of electromagnetic waves in nonlinear media $\phi(r)$ should be considered as a real quantity, which leads to generation of third harmonics. In this case the presented above consideration is valid only as long as the amplitude of the third harmonic is smaller than the amplitude of the first harmonic. It is
the case provided that $\frac{\kappa l^2}{L} \ll 1$. The letter criterion does not contradict to the requirement $\gamma \gg 1$.

Finally, we would like to mention that the problem considered above is similar to the problem of classical chaos, where the sensitivity of trajectories of motion to changes in boundary conditions exponentially increases with the sample size (See for example [14]).

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FIG. 1. Graphical solution of Eq.16. The wavy line corresponds to $F_1(\bar{u}_1)$ while straight lines 1 and 2 correspond to $\gamma - \frac{1}{2} \bar{u}_1$ in the cases $\gamma \sim 1$ and $\gamma \gg 1$ respectively.
FIG. 2. Solid lines correspond to Green functions of Eq.1 with $\beta = 0$, dashed lines correspond to $\frac{\pi}{m^2} \delta(r-r_1)$, the four solid lines vertices correspond to the factor $\beta$, thick wavy line correspond to $\Delta u(r)$ and thin wavy lines correspond to density vertexes.