Distinct Distances: Open Problems and Current Bounds

Adam Sheffer*

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Abstract

This document surveys the variants of the Erdős distinct distances problem, and lists the best known bounds for each of them.

1 Introduction

Given a set $P$ of $n$ points in $\mathbb{R}^2$, let $D(P)$ denote the number of distinct distances that are determined by pairs of points from $P$. Let $D(n) = \min_{|P|=n} D(P)$; that is, $D(n)$ is the minimum number of distinct distances that a set of $n$ points in $\mathbb{R}^2$ can determine. In his celebrated 1946 paper [18], Erdős derived the bound $D(n) = O(n/\sqrt{\log n})$. More specifically, Erdős proved that a $\sqrt{n} \times \sqrt{n}$ integer lattice determines $\Theta(n/\sqrt{\log n})$ distinct distances (for more details, see Section 2). Though almost 70 years have passed since Erdős considered this lattice structure, no configuration that determines an asymptotically smaller number of distinct distances was discovered.

For the celebrations of his 80th birthday, Erdős compiled a survey of his favorite contributions to mathematics [23], in which he wrote

"My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems."

After 64 years and a series of increasingly larger lower bounds, Guth and Katz [30] derived the bound $D(n) = \Omega(n/\log n)$, almost matching the best known upper bound (a comprehensive list of the previous bounds can be found in [29]). To derive this bound, Guth and Katz developed several novel techniques, relying on tools from algebraic geometry. Notice that a small gap of $O(\sqrt{\log n})$ remains between the best known lower and upper bounds.

**Problem 1** Find the exact asymptotic value of $D(n)$.

Since the problem is almost completely solved, one might wonder what is the purpose of this document. The answer is that there are many challenging variants of the distinct distances problem that are still wide open (many of these also posed by Erdős). For some of these variants, such as the ones presented in Section 2, even after several decades of work hardly anything non-trivial is known. This document is meant to be dynamic, keeping

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*School of Computer Science, Tel Aviv University, Tel Aviv 69978, Israel. sheffera@tau.ac.il

1See also William Gasarch’s webpage [http://www.cs.umd.edu/~gasarch/erdos_dist/erdos_dist.html](http://www.cs.umd.edu/~gasarch/erdos_dist/erdos_dist.html)
track of the best known bounds and conjectures concerning the various distinct distances variants.

The main part of this survey is partitioned into six sections, each discussing a different family of problems. Section 2 discusses the structure of planar point sets that determine a small number of distinct distances. Section 3 surveys problems in which a planar point set is constrained in some manner. Section 4 considers distinct distances in higher dimensions. Section 5 discusses subsets of points that determine every distance at most once. Section 6 surveys another family of restricted point sets, in which every subset of \( k \) points determines at least \( l \) distinct distances. Finally, Section 7 presents several additional problems that do not fit any of the other sections.

For those who are only interested in the “main” problems, it is the personal view of the author (and likely of others) that currently the most challenging/interesting distinct distances problems are Problem 3 of Section 2 and Problem 16 of Section 4. Another great collection of open problems related to distinct distances can be found in [6, Sections 5.3 – 5.6].

A more general version of Problem 1 asks whether every planar point set \( P \) contains a point that determines many distinct distances with the other points of \( P \) (e.g., see Erdős [22]). More formally, what is the minimum value \( \hat{D}(n) \), such that for every set \( P \) of \( n \) points in the plane there exists a point \( p \in P \) that determines at least \( \hat{D}(n) \) distinct distances with the other points of \( P \)? An immediate upper bound is \( \hat{D}(n) \leq D(n) = O(n^{1/2} \log n) \). However, Guth and Katz’s bound does not immediately imply a matching lower bound for \( \hat{D}(n) \). The best known lower bound, obtained by Katz and Tardos [31], is \( D(n) = \Omega(n^{(58 - 14e)/(55 - 16e)}) \approx \Omega(n^{0.864}) \).

**Problem 2** Find the asymptotic value of \( \hat{D}(n) \).

## 2 The structure of point sets with few distinct distances

In this section we discuss attempts to characterize point sets that determine a small number of distinct distances. This seems to be the family of problems for which we know the least: even after decades of studying them, hardly anything is known. We say that a set \( P \) of \( n \) points in the plane is optimal if \( D(P) = D(n) \). Since even the asymptotic value of \( D(n) \) is still unknown, we simply consider sets \( P \) of \( n \) points such that \( D(P) = O(n/\sqrt{\log n}) \), and refer to such sets as near-optimal. All the point sets in this section are planar.

Figure 1: (a) An integer lattice. (b) A lattice that can be obtained from the lattice in (a) either by rotation and uniform scaling, or by removing from (a) every point whose coordinates sum to an odd number. (c) A triangular lattice (of equilateral triangles).

Erdős asked whether every optimal (or near-optimal) set “has lattice structure” [21]. To make this question clearer, let us first consider some of the known near-optimal sets. In

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2The arXiv version is updated only after significant changes. A more up to date version might be found in [http://adamsheffer.wordpress.com/pdf-files/](http://adamsheffer.wordpress.com/pdf-files/)
the introduction we already mentioned that the $\sqrt{n} \times \sqrt{n}$ integer lattice (e.g., see Figure 1(a)) determines $O(n/\sqrt{\log n})$ distinct distances. This was observed by Erdős, who noticed that this is an immediate corollary of the following theorem from number theory.

**Theorem 2.1 (Landau-Ramanujan [5, 7, 32])** The number of positive integers smaller than $n$ that are the sum of two squares is $\Theta(n/\sqrt{\log n})$.

Every distance in the $\sqrt{n} \times \sqrt{n}$ integer lattice is the square root of a sum of two squares between 0 and $n$. Thus, Theorem 2.1 implies that the number of distinct distances in this case is $\Theta(n/\sqrt{\log n})$.

The above implies that the $\sqrt{n} \times \sqrt{n}$ integer lattice is a near-optimal set. More generally, for any integer $c \geq 1$, every $n$ point subset of the $c\sqrt{n} \times c\sqrt{n}$ integer lattice is a near-optimal set (since we can still apply Theorem 2.1 in such cases), and so are translations, rotations, and uniform scalings of such lattices. For example, the lattice in Figure 1(b) can be obtained either by rotating and scaling the $\sqrt{n} \times \sqrt{n}$ integer lattice, or by removing from the $2\sqrt{n} \times 2\sqrt{n}$ integer lattice every point whose coordinates sum to an odd number.

Another interesting set of points is the triangular lattice, which is depicted in Figure 1(c). This lattice corresponds to the vertices in a tiling of equilateral triangles (unlike the set in Figure 1(b), which can be seen as the vertices of a tiling of isosceles right triangles). The points of the triangular lattice can be written as

$$\left\{a \cdot (1,0) + b \cdot (1/2, \sqrt{3}/2) \mid a, b \in \mathbb{Z}\right\}. \quad (1)$$

Due to the irrational $y$-coordinates, it can be easily noticed that such a lattice cannot be obtained from the integer lattice without applying a non-uniform scaling. However, the number of distinct distances determined by an $\sqrt{n} \times \sqrt{n}$ triangular lattice is still $\Theta(n/\sqrt{\log n})$. To see this, we first perform a uniform scaling of (1), obtaining

$$\left\{a \cdot (2,0) + b \cdot (1, \sqrt{3}) \mid 1 \leq a, b \leq \sqrt{n}\right\}.$$

Any distance between two points of this lattice can be written as $\sqrt{i^2 + 3j^2}$, where $i, j \in \mathbb{Z}$. A variant of Theorem 2.1 [4, pages 59 and 115–116] (a brief description in English can be found, e.g., in the introduction of [7]) implies that the number of of positive integers smaller than $n$ that can be expressed as $i^2 + 3j^2$ is $\Theta(n/\sqrt{\log n})$, immediately implying the above assertion. After obtaining optimal sets for a few small values of $n$, Erdős and Fishburn [24] conjectured that, for infinitely many values of $n$, there is an $n$-point subset of the triangular lattice that is an optimal set.

For now, hardly anything is known regarding Erdős’s conjecture that every near-optimal set has a lattice structure. As a first step, Erdős [21] suggested to determine whether every near-optimal point set contains $\Omega(\sqrt{n})$ points on a line, and can thus be covered by a small number of lines. Since this also appears to be quite difficult, Erdős asked whether there exists a line with $\Omega(n^\varepsilon)$ points of the set. Embarrassingly, even this easier variant remains open.

**Problem 3 (Erdős [21])** Prove or disprove: For a sufficiently small $\varepsilon > 0$, every near-optimal point set contains $\Omega(n^\varepsilon)$ points on a common line.

A simple extension of Lemma 3.1 (see Section 3) implies that for every near-optimal set $P$ of $n$ points, there exists a line $\ell$ such that $|\ell \cap P| = \Omega(\sqrt{\log n})$. No better lower bound is known.
Sheffer, Zahl, and de Zeeuw [43] considered the complement problem — proving that no line can contain many points of a near-optimal set. Specifically, they proved that for every near-optimal set \( \mathcal{P} \) of \( n \) points, no line contains \( \Omega(n^{7/8}) \) points of \( \mathcal{P} \).

**Problem 4** Prove or disprove: For a sufficiently small \( \varepsilon > 0 \), no near-optimal point set contains \( \Omega(n^{0.5+\varepsilon}) \) points on a common line.

In Section 3 we mention a result by Pach and de Zeeuw [36], implying that if a set of \( n \) points is contained in a constant-degree curve, then these points span \( \Omega(n^{4/5}) \) distinct distances, unless the curve contains a line or a circle. That is, for every near-optimal set \( \mathcal{P} \) of \( n \) points, any constant degree algebraic curve that does not contain lines and circles cannot contain \( \Omega(n^{3/4}) \) points of \( \mathcal{P} \). In [43], it is also proved that for every near-optimal set \( \mathcal{P} \) of \( n \) points, no circle contains \( \Omega(n^{5/6}) \) points of \( \mathcal{P} \). Combining these three results, we obtain that for every near-optimal set \( \mathcal{P} \) of \( n \) points, no constant-degree algebraic curve contains \( \Omega(n^{7/8}) \) points of \( \mathcal{P} \).

**Problem 5** Prove or disprove: For every near-optimal set \( \mathcal{P} \) of \( n \) points and sufficiently small \( \varepsilon > 0 \), no constant-degree curve contains \( \Omega(n^{0.5+\varepsilon}) \) points of \( \mathcal{P} \).

It is known that the points of the integer grid (depicted in Figure 1(a)) do not determine any equilateral triangles. Thus, there are near-optimal sets that determine no equilateral triangles. On the other hand, it can easily be noticed that the points of any near-optimal set must determine many isosceles triangles. Erdős [21] conjectured that any near-optimal set must contain either an equilateral triangle or a square.

**Problem 6** (Erdős [21]) Prove or disprove: The vertices of every near optimal set span either a square or an equilateral triangle.

We conclude this section with a related conjecture:

**Problem 7** (Guth and Katz [30]) Prove or disprove: For any near-optimal point set \( \mathcal{P} \), there are many rotations of the plane that map many points of \( \mathcal{P} \) to other points of \( \mathcal{P} \).

(The quantity “many” in this last problem is a bit vague; see [30] for more details.)

3 Constrained sets of points

In this section we consider variants of the distinct distances problem where the point sets are constrained in some way. The best known bounds for these problems are listed in Table 1; see Figure 2 and the text below for an explanation of the notation used in the table. Unless stated otherwise, the point sets in this section are planar.

General and convex position. We begin with a set of problems that ask for exact bounds, instead of asymptotic ones; that is, where the “fight” is for the best constant of proportionality. The hierarchy between the different types of constraints that are presented in this part is depicted in Figure 2(a). Denote by \( D_{\text{no3}}(n) \) the minimum number of distinct distances that can be determined by a set of \( n \) points, no three of which are collinear (that is, \( D_{\text{no3}}(n) = \min_{|\mathcal{P}|=n} D(\mathcal{P}) \), where the minimum is taken over all sets of \( n \) points containing no three collinear points). Notice that the vertices of a regular \( n \)-gon, such as
Table 1: The best known bounds for constrained sets of points.

| Variant            | Lower bound                                      | Upper bound                                      |
|--------------------|--------------------------------------------------|--------------------------------------------------|
| \( D_{\text{no3}}(n) \) | \((n-1)/3\) (Szemerédi)                          | \(n/2\) [18]                                    |
| \( D_{\text{no3}}(n) \) | \([n-1]/3\) (Szemerédi)                         | \(n/2\) [18]                                    |
| \( D_{\text{conv}}(n) \) | \( \left( \frac{14}{30} + \frac{1}{2700} \right) n + O(1) \) [11, 34] | \(n/2\) [18]                                    |
| \( D_{\text{gen}}(n) \) | \( \Omega(n) \) (trivial)                       | \(n^{2}\sqrt{\log n}\) [25]                    |
| \( D_{\text{para}}(n) \) | \([n-1]/3\) (trivial)                          | \(n - O(1)\) (trivial)                         |
| \( D_{\text{lines}}(n) \) | \( \Omega(n^{4/3}) \) [40]                     | \(O(n^{2}/\sqrt{\log n})\) [14]               |
| \( D_{\text{curves}}(n) \) | \( \Omega(n^{4/3}) \) [36]                     | \(O(n^{2}/\sqrt{\log n})\) [14]               |

Figure 2: (a) The hierarchy of the constrained point sets in Section 3. Every arrow goes from a constrained problem to its generalization. (b) The vertices of a regular 10-gon.

the set depicted in Figure 2(b), satisfy this property and determine \( \left\lfloor \frac{n}{2} \right\rfloor \) distinct distances, implying \( D_{\text{no3}}(n) \leq \left\lfloor \frac{n}{2} \right\rfloor \). The best known lower bound, due to Szemerédi (communicated by Erdős in [20]; the proof can also be found in [35, Chapter 13]), is \( D_{\text{no3}}(n) \geq \left\lceil \frac{n-1}{3} \right\rceil \). Szemerédi also conjectured that \( D_{\text{no3}}(n) = \left\lfloor \frac{n}{2} \right\rfloor \); see [22, 27].

**Problem 8** Find the exact value of \( D_{\text{no3}}(n) \).

As before, let \( \hat{D}_{\text{no3}}(n) \) denote the maximum number satisfying that for any set \( \mathcal{P} \) of \( n \) points, no three of which are collinear, there exists a point \( p \in \mathcal{P} \) such that the pairs of \( \{p\} \times \mathcal{P} \) determine at least \( \hat{D}_{\text{no3}}(n) \) distinct distances.

**Problem 9** Find the exact value of \( \hat{D}_{\text{no3}}(n) \).

It can be easily noticed that the regular \( n \)-gon configuration implies \( \hat{D}_{\text{no3}}(n) \leq \left\lfloor \frac{n}{2} \right\rfloor \), and Szemerédi’s bound for \( D_{\text{no3}}(n) \) also remains valid for \( \hat{D}_{\text{no3}}(n) \). Thus, the best known bounds for Problems 8 and 9 are currently identical (as opposed to the case of Problems 1 and 2).

Szemerédi’s proof is so simple and elegant that it is hard to resist stating it here.

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Lemma 3.1 \( \hat{D}_{\text{no3}}(n) \geq \left\lceil \frac{n-1}{3} \right\rceil \)

Proof. Consider a set \( \mathcal{P} \) of \( n \) points, no three of which are collinear. Let \( x \) denote the minimum number satisfying that every point \( p \in \mathcal{P} \) determines at most \( x \) distinct distances with the points of \( \mathcal{P} \setminus \{p\} \).

Let \( T = \{(a,p,q) \in \mathcal{P}^3 \mid |ap| = |aq|\} \), where \( a,p,q \) are three distinct points and where \((a,p,q)\) and \((a,q,p)\) are counted as the same triple. The proof is based on double counting \(|T|\), and we begin by deriving an upper bound for it. Given a pair of points \( p,q \in \mathcal{P} \), the triplet \((a,p,q)\) is in \( T \) if and only if \( a \) is on the perpendicular bisector of the segment \( pq \). By the assumption, each such perpendicular bisector contains at most two points of \( \mathcal{P} \), which implies

\[
|T| \leq 2 \binom{n}{2} = n(n-1). \tag{2}
\]

For the lower bound, notice that for every point \( p \in \mathcal{P} \), the points of \( \mathcal{P} \setminus \{p\} \) are contained in at most \( x \) concentric circles around \( p \). We denote these circles as \( C_{p,1}, \ldots, C_{p,x} \) and set \( n_{p,i} = |C_{p,i} \cap \mathcal{P}| \). Notice that \( \sum_{i=1}^{x} n_{p,i} = n - 1 \) for every \( p \in \mathcal{P} \). By the Cauchy-Schwarz inequality, we have \( \sum_{i=1}^{x} n_{p,i}^2 \geq \frac{1}{2} (n - 1)^2 \). This in turn implies

\[
|T| = \sum_{p \in \mathcal{P}} \sum_{i=1}^{x} \left( \frac{n_{p,i}}{2} \right) = \frac{1}{2} \sum_{p \in \mathcal{P}} \sum_{i=1}^{x} (n_{p,i}^2 - n_{p,i}) \geq \frac{1}{2} \sum_{p \in \mathcal{P}} \left( \frac{1}{x} (n-1)^2 - (n-1) \right) = \frac{n(n-1)(n-1-x)}{2x}. \tag{3}
\]

Combining (2) and (3) immediately implies the assertion of the lemma.

Though Problems 8 and 9 have been stuck for several decades, more recent advances have been obtained for more restricted variants of them. Similarly to \( D_{\text{no3}}(n) \), let \( D_{\text{conv}}(n) \) denote the minimum number of distinct distances that can be determined by a set of \( n \) points in (strict) convex position. Let \( \hat{D}_{\text{conv}}(n) \) denote the maximum number satisfying that for any set \( \mathcal{P} \) of \( n \) points in convex position, there exists a point \( p \in \mathcal{P} \) such that \( \{p\} \times \mathcal{P} \) determines at least \( D_{\text{conv}}(n) \) distinct distances.

By considering the regular \( n \)-gon once again, we notice that \( D_{\text{conv}}(n) \leq \left\lceil \frac{n}{2} \right\rceil \) and that \( \hat{D}_{\text{conv}}(n) \leq \left\lceil \frac{n}{2} \right\rceil \). Already in his 1946 paper, Erdős [18] conjectured that \( D_{\text{conv}}(n) = \left\lceil \frac{n}{2} \right\rceil \). This was proven by Altman [1, 2], which led Erdős to suggest the stronger conjecture \( \hat{D}_{\text{conv}}(n) = \left\lceil \frac{n}{2} \right\rceil \). Since for any set of points in convex position no three points can be collinear, we have \( D_{\text{conv}}(n) \geq \hat{D}_{\text{no3}}(n) \geq \left\lceil \frac{n-1}{3} \right\rceil \). In 2006, Dumitrescu [11] derived the improved bound \( \hat{D}_{\text{conv}}(n) \geq \left\lceil \frac{13n-6}{30} \right\rceil \). Recently, the slightly improved bound \( \hat{D}_{\text{conv}}(n) \geq \left( \frac{13}{30} + \frac{1}{27n} \right) n + O(1) \) was obtained by Nivasch, Pach, Pinchasi, and Zerbib [34].

Problem 10 Find the exact value of \( \hat{D}_{\text{conv}}(n) \).

We say that a set of points is in general position if no three points are collinear and no four points are cocircular. Denote by \( D_{\text{gen}}(n) \) the minimum number of distinct distances that can be determined by a set of \( n \) points in general position. The convex \( n \)-gon configuration is not in general position, and it is in fact unknown whether \( D_{\text{gen}}(n) = \Theta(n) \) or not. The best known upper bound \( D_{\text{gen}}(n) = n^2 O(\sqrt{\log n}) \) was derived by Erdős, Füredi, Pach, and Ruzsa [25]. This bound is obtained by considering a very different construction: taking an integer grid \( G \) in a \( d \)-dimensional space (where \( d \) is roughly \( \sqrt{\log n} \)), considering a subset \( G' \)
of the points of $G$ that lie on a common sphere, and projecting $G'$ on a generic plane. The
generic projection guarantees that the resulting planar set is in general position, while the
integer-grid-on-a-sphere structure implies a relatively small number of distinct distances.
For an easy lower bound, notice that $D_{\text{gen}}(n) \geq D_{\text{no3}}(n) = \Omega(n)$.

**Problem 11** Find the asymptotic value of $D_{\text{gen}}(n)$.

Erdős [22] also suggested to study the maximum number $\hat{D}_{\text{gen}}(n)$ satisfying that for any
set $P$ of $n$ points in general position, there exists a point $p \in P$ such that $\{p\} \times P$ determines
at least $\hat{D}_{\text{gen}}(n)$ distinct distances. Consider a point set $P$ in general position and a point
$p \in P$. If $p$ has a distance of $d$ from four other points of $P$, then each of these four points
is incident to the circle whose center is $p$ and radius is $d$, contradicting the general position
assumption. Thus, a trivial lower bound is $\hat{D}_{\text{gen}}(n) \geq \lceil (n - 1)/3 \rceil$. No non-trivial bound is
known for $\hat{D}_{\text{gen}}(n)$ (neither a lower nor an upper bound), and when discussing the problem,
Erdős [22] wrote “It is rather frustrating that I got nowhere with…”.

**Problem 12** Find the exact value of $\hat{D}_{\text{gen}}(n)$.

We conclude the discussion about this family of problems with an even more con-
strained type of point sets. The point configuration that implies $D_{\text{gen}}(n) = n2^O(\sqrt{\log n})$
spans many duplicate vectors, which leads to the following notation. Let $D_{\text{para}}(n)$ denote
the minimum number of distinct distances determined by a set of $n$ points in general po-
sition that do not determine any parallelograms. Erdős, Hickerson, and Pach [27] asked
whether $D_{\text{para}}(n) = o(n^2)$. This was recently confirmed by Dumitrescu [12], who proved
$D_{\text{para}}(n) = O(n^2/\sqrt{\log n})$ by considering, for prime $n$, the point set
\[
\{ (i, j) \mid i = 0, 1, \ldots, (n - 1)/4, \ j = i^2 \mod n \}.
\]
The trivial $D_{\text{para}}(n) = \Omega(n)$ is currently the best known lower bound.

**Problem 13** Find the asymptotic value of $D_{\text{para}}(n)$.

![Figure 3: Two lines with $O(n)$ distinct distances.](image-url)

**Points on curves.** We next consider a different family of constrained point sets. Let $P_1$
and $P_2$ be two sets of $n$ points each, such that the points of $P_1$ (resp., $P_2$) lie on a line
$\ell_1$ (resp., $\ell_2$). Let $D(P_1, P_2)$ denote the number of distinct distances between the pairs of
$P_1 \times P_2$ (that is, only distinct distances between points on different lines are considered).
When the two lines are either parallel or orthogonal, the points can be arranged such that
$D(P_1, P_2) = \Theta(n)$; for example, see Figure 3. Purdy conjectured that if the lines are
neither parallel nor orthogonal then $D(\mathcal{P}_1, \mathcal{P}_2) = \omega(n)$ (e.g., see [6, Section 5.5]). Let us denote as $D_{\text{lines}}(n)$ the minimum number of distinct distances in such a scenario. Elekes and Rónyai [15] proved Purdy’s conjecture, though without deriving any specific superlinear lower bound. Later, Elekes [14] showed that $D_{\text{lines}}(n) = \Omega(n^{5/4})$. Recently, Sharir, Sheffer, and Solymosi [40] proved $D_{\text{lines}}(n) = \Omega(n^{4/3})$. More generally, they derived the bound $\Omega(\min\{n^{2/3}m^{2/3}, m^2, n^2\})$ for the asymmetric case where one line contains $n$ points and the other $m$ points (another recent asymmetric result, which is subsumed by the bound from [40], was derived by Schwartz, Solymosi, and de Zeeuw [39]). Elekes [14] also observed the upper bound $D_{\text{lines}}(n) = O(n^2/\sqrt{\log n})$.

**Problem 14** Find the asymptotic value of $D_{\text{lines}}(n)$.

Generalizing Problem 14, we consider the minimum number of distinct distances between two sets of $n$ points, each on a constant degree algebraic curve in $\mathbb{R}^2$ (the two curves may or may not be identical). We already know that there could be $O(n)$ distinct distances when the points are on parallel or orthogonal lines. In this more general scenario there exists a third exceptional case — the case of two concentric circles. Thus, we denote by $D_{\text{curves}}(n)$ the minimum number of distinct distances between two sets of $n$ points, each on a constant degree algebraic curve, such that the two curves do not contain orthogonal lines, parallel lines, or concentric circles (this also implies that the two curves do not contain common lines or circles). Pach and de Zeeuw [36] extended the technique of [40] to obtain $D_{\text{curves}}(n) = \Omega(n^{4/3})$. More generally, they derived the bound $\Omega(\min\{n^{2/3}m^{2/3}, m^2, n^2\})$ for the asymmetric case when one curve contains $n$ points and the other $m$ points.

**Problem 15** Find the asymptotic value of $D_{\text{curves}}(n)$.

In Section 4 we consider similar problems in higher dimensions.

## 4 Higher dimensions

In this section we consider higher-dimensional variants of the distinct distances problem. Denote by $D_d(n)$ the minimum number of distinct distances that a set of $n$ points in $\mathbb{R}^d$ can determine. Similarly to the planar case, the best known configuration for the $d$-dimensional case is an $n^{1/d} \times n^{1/d} \times \cdots \times n^{1/d}$ integer lattice. Every distance in this configuration is the square root of a sum of $d$ squares, each with a value between 0 and $n^{2/d}$. Therefore, the number of distinct distances that are determined by such a lattice is $O(n^{2/d})$, implying $D_d(n) = O(n^{2/d})$. This bound was already observed by Erdős in his 1946 paper [18], and is conjectured to be tight (for $d \geq 3$).

Solymosi and Vu [44] derived the following recursive relations on $D_d(n)$.

**Theorem 4.1** (Solymosi and Vu [44]) (i) If $D_{d_0}(n) = \Omega(n^{d_0})$, then for all $d > d_0$, we have

$$D_d(n) = \Omega\left(n^{\frac{2d}{(d+d_0+1)\lfloor(d-d_0)2d_0\sqrt{\omega}\rfloor}}\right).$$

(ii) If $D_{d_0}(n) = \Omega(n^{d_0})$, then for all $d > d_0$ where $d - d_0$ is even, we have

$$D_d(n) = \Omega\left(n^{\frac{2d+1}{(d+d_0+2)\lfloor(d-d_0)2d_0+2\sqrt{\omega}\rfloor}}\right).$$
Recall that \( D_2(n) = \Omega(n/\log n) \). Combining this with Theorem 4.1(i) implies \( D_3(n) = \Omega^*(n^{3/5}) \)\(^4\) while the above lattice example implies \( D_3(n) = O(n^{2/3}) \). Currently, these are the best known bounds for \( D_3(n) \). The best known bounds for larger values of \( d \) are obtained by combining Theorem 4.1(ii) with the bounds \( D_2(n) = \Omega^*(n) \) and \( D_3(n) = \Omega^*(n^{3/5}) \) as base cases. That is, for even \( d \geq 4 \) we have \( D_d(n) = \Omega^*(n^{2d+4d-2}) \) and for odd \( d \geq 5 \) we have \( D_d(n) = \Omega^*(n^{2d+2d-5/3}) \). Notice that as \( d \) goes to infinity \( D_d(n) \) approaches the conjectured bound \( \Theta(n^{2/d}) \).

**Problem 16** Find the asymptotic value of \( D_d(n) \).

It seems possible that the techniques that were used by Guth and Katz [30] for analyzing \( D(n) \) could also be applied to the higher dimensional variant. However, so far it is still not clear how to extend some of the steps in this analysis.

Let \( D_3^*(n) \) denote the minimum number of distinct distances that a set of \( n \) points on a hypersphere in \( \mathbb{R}^d \) can determine. Tao [45] observed that the bound from [30] remains valid when the point set is on a sphere in \( \mathbb{R}^3 \) (or on a hyperbolic plane). That is, \( D_3^*(n) = \Omega(n/\log n) \). For a lower bound, place a set of \( n \) points on a circle that is on the sphere, so that they form the vertices of a regular planar \( n \)-gon (recall figure 2(b)). This implies \( D_3^*(n) = O(n) \).

**Problem 17** Find the asymptotic value of \( D_3^*(n) \).

Erdős, Fučík, Pach, and Ruzsa [25] proved \( D_4^*(n) = O(n/\log \log n) \) and \( D_d^*(n) = O(n^{2/(d-2)}) \) for \( d > 4 \). These bounds are obtained by taking an integer lattice in \( \mathbb{R}^d \) and then choosing a sphere that contains many lattice points. No lower bound is known beyond the trivial \( D_3^*(n) = O(n) \).

**Problem 18** Find the asymptotic value of \( D_3^*(n) \) for \( d \geq 4 \).

**Restricted point sets.** Charalambides [9] considered the case where a set of \( n \) points is contained in a constant-degree curve \( C \) in \( \mathbb{R}^d \) (i.e., the \( d \)-dimensional variant of Problem 15). He showed that if \( C \) contains an algebraic helix (see [9] for a description of this family of curves. In \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) the only algebraic helices are lines and circles) the points may determine only \( O(n) \) distinct distances. On the other hand, if \( C \) does not contain any algebraic helices, the points on it determine \( \Omega(n^{5/4}) \) distinct distances. We denote by \( D_{\text{curve}}^{(d)}(n) \) the minimum number of distinct distances that are determined by \( n \) points on a \( d \)-dimensional curve which does not contain an algebraic helix.

**Problem 19** Find the asymptotic value of \( D_{\text{curve}}^{(d)}(n) \).

Raz, Sharir, and Solymosi [38] proved the improved bound \( \Omega(n^{4/3}) \), though only for the rather special case of curves that have a polynomial parametrization. In this case, the only restriction on the curve is that it must not contain any lines (i.e., no other type of algebraic helix has such a parametrization).

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\(^4\)In the \( \Omega^*(\cdot) \) notation we neglect polylogarithmic factors; although the logarithm in the denominator of the lower bound for \( D_2(n) \) does not exactly fit the formulation of Theorem 4.1, the proof remains valid.
It also seems natural to ask what happens when the point sets are restricted to surfaces. That is, the minimum number of distances between a set $P_1$ of $n$ points on a surface $S_1$ and a set $P_2$ of $n$ points on a surface $S_2$, both in $\mathbb{R}^d$. For example, consider the case where $S_1$ and $S_2$ are non-parallel planes in $\mathbb{R}^3$. Then there are two orthogonal lines $\ell_1, \ell_2$ such that $\ell_1 \subset \Pi_1$ and $\ell_2 \subset \Pi_2$ and $\ell_1 \cap \ell_2$ is a point on $\Pi_1 \cap \Pi_2$. By placing two sets of points on $\ell_1, \ell_2$ as depicted in Figure 3, we obtain $\Theta(n)$ distinct distances. The same bound can be obtained between two spheres and between a sphere and a plane. However, this does not settle even these special cases, since an unrestricted set of points in $\mathbb{R}^3$ can determine only $\Theta(n^{2/3})$ distinct distances. No non-trivial bound is known for surfaces in $\mathbb{R}^3$.

**Problem 20** Consider two sets of points in $\mathbb{R}^3$, each contained in a surface. For which surfaces can the number of distinct distances between the two sets be sublinear? For which surfaces this number must be superlinear?

5 Subsets with no repeated distances

This section surveys problems that concern point subsets that determine every distance at most once. Table 2 lists the best known bounds for the problems that are presented in this section.

| Variant | Lower bound | Upper bound |
|---------|-------------|-------------|
| subset$(n)$ | $\Omega(n^{1/3}/\log^{1/3} n)$ [8, 30, 33] | $O(\sqrt{n}/(\log n)^{1/4})$ [26] |
| subset$(L)$ | $\Omega(n^{1/3}/\log^{1/3} n)$ [26] | $O(\sqrt{n}/(\log n)^{1/4})$ [26] |
| subset$(L_d)$ | $\Omega(n^{2/(3d)})$ [33] | (trivial) |
| subset$_d(n)$ | $\Omega\left(n^{1/(3d-3)}(\log n)^{1/3-2/(3d-3)}\right)$ [10] | $O(n^{1/d})$ (trivial) |
| subset$_p(n)$ | $\Omega(n^{4/33\frac{15}{15}})$ | $O(\sqrt{n}/(\log n)^{1/4})$ [26] |

Figure 4: A set of 25 points and a subset of 5 points that span every distance at most once.

Given a set $P$ of points in $\mathbb{R}^2$, let subset$(P)$ denote the size of the largest subset $P' \subset P$ such that every distance is spanned by the points of $P'$ at most once; that is, there are no points $a, b, c, d \in P$ such that $|ab| = |cd| > 0$ (including cases where $a = c$). Figure 4 depicts a set of 25 points and a subset of 5 points that span every distance at most once. Let subset$(n) = \min_{|P|=n} \text{subset}(P)$. In other words, subset$(n)$ is the maximum number satisfying the property that every set of $n$ points in the plane contains a subset of subset$(n)$ points that do not span any distance more than once. Erdős [19, 26, 28] posed the following problem.

**Problem 21 (Erdős)** Find the asymptotic value of subset$(n)$.
Let $\mathcal{P}$ be a point set that spans $d$ distinct distances. Notice that if all of the distances that are spanned by the points of a subset $\mathcal{P}' \subset \mathcal{P}$ are unique, then the number of distances that are spanned by $\mathcal{P}$ must be at least $\binom{|\mathcal{P}'|}{2}$, or equivalently $|\mathcal{P}'| = O\left(\sqrt{d(\mathcal{P})}\right)$. Let $\mathcal{L}$ be a $\sqrt{n} \times \sqrt{n}$ integer lattice. Erdős [18] proved that the number of distinct distances that are spanned by $\mathcal{L}$ is $\Theta(n/\sqrt{\log n})$ (e.g., see Section 2 of this document). Therefore, we have $\text{subset}(n) \leq \text{subset}(\mathcal{L}) = O\left(\sqrt{n}/(\log n)^{1/4}\right)$. Lefmann and Thiele [33] derived the bound $\text{subset}(n) = \Omega(n^{0.25})$ by using a probabilistic argument. Dumitrescu [12] improved this bound to $\text{subset}(n) = \Omega(n^{0.288})$ by combining it with an upper bound of Pach and Tardos [37] on the maximum number of isosceles triangles that can be spanned by a set of $n$ points. Recently, Charalambides [8] combined the probabilistic argument of Lefmann and Thiele with a result from Guth and Katz’s distinct distances paper [30], to obtain the following improved result and simple proof.

**Theorem 5.1 (Charalambides [8])** $\text{subset}(n) = \Omega(n^{1/3}/\log^{1/3} n)$.

**Proof.** Consider a set $\mathcal{P}$ of $n$ points in $\mathbb{R}^2$. Similarly to the Elekes-Sharir reduction, we define the set

$$Q_1 = \{(a, b, c, d) \in \mathcal{P}^4 \mid |ab| = |cd| > 0\},$$

such that every quadruple of $Q_1$ consists of four distinct points. Guth and Katz [30] proved that $|Q_1| = O(n^3 \log n)$ (also for the case where $Q_1$ is allowed to contain quadruples where not all four points are distinct). Let $Q_2$ be the set of isosceles and equilateral triangles that are spanned by points of $\mathcal{P}$. Pach and Tardos [37] proved that $|Q_2| = O(n^{2.137})$.

Let $\mathcal{P}' \subset \mathcal{P}$ be a subset that is obtained by selecting every point of $\mathcal{P}$ with a probability $0 < p < 1$ that we will fix later. We have $E[|\mathcal{P}'|] = pn$. Let $Q_1' \subset Q_1$ be the set of quadruples of $Q_1$ that contain only points of $\mathcal{P}'$. Every quadruple of $Q_1$ is in $Q_1'$ with a probability of $p^4$, so $E[|Q_1'|] \leq \alpha_1 p^4 n^3 \log n$, for a sufficiently large constant $\alpha_1$. Let $Q_2'$ be the set of triangles of $Q_2$ that contain only points of $\mathcal{P}'$. The bound of Pach and Tardos implies $E[|Q_2'|] \leq \alpha_2 p^3 n^{2.137}$, for a sufficiently large constant $\alpha_2$. Notice that the points of $\mathcal{P}'$ span every distance at most once if and only if $|Q_1'| = |Q_2'| = 0$. By linearity of expectation, we have

$$E\left[|\mathcal{P}'| - |Q_1'| - |Q_2'|\right] \geq pn - \alpha_1 p^4 n^3 \log n - \alpha_2 p^3 n^{2.137}.$$

By setting $p = 1/(2\alpha_1 n^2 \log n)^{1/3}$, for sufficiently large $n$ we obtain

$$E\left[|\mathcal{P}'| - |Q_1'| - |Q_2'|\right] \geq \frac{n^{1/3}}{3(\alpha_1 \log n)^{1/3}}.$$

Therefore, there exists a subset $\mathcal{P}' \subset \mathcal{P}$ for which $|\mathcal{P}'| - |Q_1'| - |Q_2'| \geq \frac{n^{1/3}}{3(\alpha_1 \log n)^{1/3}}$. Let $\mathcal{P}''$ be a subset of $\mathcal{P}'$ that is obtained by removing from $\mathcal{P}'$ a point from every element of $Q_1'$ and $Q_2'$. The subset $\mathcal{P}''$ does not span any repeated distances and contains $\Omega(n^{1/3}/\log^{1/3} n)$ points of $\mathcal{P}$.

Erdős and Guy [26] considered the following special case of Problem 21.

**Problem 22 (Erdős and Guy [26])** Find the asymptotic value of $\text{subset}(\mathcal{L})$, where $\mathcal{L}$ is a $\sqrt{n} \times \sqrt{n}$ integer lattice.

As mentioned above, the best known upper bound for $\text{subset}(\mathcal{L})$ is $O\left(\sqrt{n}/(\log n)^{1/4}\right)$. Erdős and Guy [26] derived the bound $\text{subset}(\mathcal{L}) = \Omega\left(n^{1/3-\varepsilon}\right)$, which was later improved by
Lefmann and Thiele [33] to $\text{subset}(L) = \Omega(n^{1/3})$. This bound is still marginally better than the bound implied by Theorem 5.1.

Erdős and Guy [26] also considered the higher dimensional variant of Problem 22. That is, they considered a $d$-dimensional lattice $L_d$ of the form $n^{1/d} \times \cdots \times n^{1/d}$. Similarly to the planar case, Erdős and Guy [26] derived the bound $\text{subset}(L_d) = \Omega((n^{2/(3d)} - \varepsilon)$, and this was later improved by Lefmann and Thiele [33] to $\text{subset}(L_d) = \Omega(n^{2/(3d)})$. It is simple to show that the points of $L_d$ span $O(n^{2/d})$ distinct distances (see Section 4 of this document), which implies $\text{subset}(L_d) = O(n^{1/d})$.

**Problem 23 (Erdős and Guy [26])** Find the asymptotic value of $\text{subset}(L_d)$, where $L_d$ is an $n^{1/d} \times \cdots \times n^{1/d}$ integer lattice.

We can also consider the higher dimensional variant of Problem 21. Let $\text{subset}_d(n)$ denote the maximum number satisfying the property that every set of $n$ points in $\mathbb{R}^d$ contains a subset of $\text{subset}_d(n)$ points that do not span any distance more than once. Thiele [46, Theorem 4.33] proved the lower bound $\text{subset}_d(n) = \Omega(n^{1/(3d-2)})$. Recently, this was improved by Conlon, Fox, Gasarch, Harris, Ulrich, and Zbarsky [10] to $\text{subset}_d(n) = \Omega(n^{1/(3d-3)}(\log n)^{1/3-2/(3d-3)})$. The best known upper bound is $\text{subset}_d(n) \leq \text{subset}(L_d) = O(n^{1/d})$.

**Problem 24** Find the asymptotic value of $\text{subset}_d(n)$ for $d \geq 3$.

In the open problems book of Brass, Moser, and Pach [6], they offer a problem of a similar flavor. Let $\text{subset}'(n)$ denote the maximum number satisfying the property that every set of $n$ points in the plane contains a subset of $\text{subset}'(n)$ points that do not span any isosceles triangles.

**Problem 25 (Brass, Moser, and Pach [6])** Find the asymptotic value of $\text{subset}'(n)$.

For a trivial upper bound, we have $\text{subset}'(n) \leq s(n) = O(\sqrt{n}/(\log n)^{1/4})$. By adapting the proof of Theorem 5.1 (i.e., removing $Q_1$ from the analysis), we obtain $s'(n) = \Omega(n^{0.4315})$.

### 6 Every $k$ points determine at least $l$ distinct distances.

Let $\phi(n, k, l)$ denote the minimum number of distinct distances that are determined by a planar $n$ point set $P$ with the property that any $k$ points of $P$ determine at least $l$ distinct distances (this notation is taken from [6, Chapter 5]). That is, by having a local property of every small subset of points, we wish to obtain a global property of the entire point set. Table 3 lists the best known bounds for most of the problems that are discussed in this section.

**The case $k=3$.** The value of $\phi(n, 3, 2)$ is the minimum number of distinct distances that are determined by a set of $n$ points that do not span any equilateral triangles. As discussed in Section 2, Erdős [18] noticed that a $\sqrt{n} \times \sqrt{n}$ integer lattice determines $\Theta(n/\sqrt{\log n})$ distinct distances. It is known that the points of the integer lattice do not determine any equilateral triangles, and thus $\phi(n, 3, 2) = O(n/\log n)$. Guth and Katz’s bound on $D(n)$ [30] implies $\phi(n, 3, 2) = \Omega(n/\log n)$. Thus, the best known bounds for $\phi(n, 3, 2)$ are identical to the ones for $D(n)$.

**Problem 26** Find the asymptotic value of $\phi(n, 3, 2)$.
The value of $\phi(n, 3, 3)$ is the minimum number of distinct distances that are determined by a set of $n$ points that do not span any isosceles triangles (here and in the following cases, we also consider degenerate polygons, such as a degenerate isosceles triangle whose three vertices are collinear). Since no isosceles triangles are allowed, every point determines $n - 1$ distinct distances with the other points of the set, and thus we have $\phi(n, 3, 3) = \Omega(n)$. Erdős [21] observed the following upper bound for $\phi(n, 3, 3)$. Behrend [3] proved that there exists a set $A$ of positive integers $a_1 < a_2 < \cdots < a_n$ such that no three elements of $A$ determine an arithmetic progression and that $a_n < n 2^{O(\sqrt{\log n})}$. Therefore, the point set $P_1 = \{(a_1,0), (a_2,0), \ldots, (a_n,0)\} \subset \mathbb{R}^2$ does not span any isosceles triangles. Since $P_1 \subset P_2 = \{(1,0), (2,0), \ldots, (a_n,0)\}$ and $D(P_2) < n 2^{O(\sqrt{\log n})}$, we have $\phi(n, 3, 3) < n 2^{O(\sqrt{\log n})}$. Erdős conjectured [21] that $\phi(n, 3, 3) = \omega(n)$.

**Problem 27** Find the asymptotic value of $\phi(n, 3, 3)$.

**The case $k=4$.** The value of $\phi(n, 4, 3)$ is the minimum number of distinct distances that are determined by a set of $n$ points that do not span any squares. This case is somewhat similar to the one of $\phi(n,3,2)$. The $\sqrt{n} \times \sqrt{n}$ triangular lattice determines $\Theta(n/\sqrt{\log n})$ distinct distances (as proved in Section 2). Since the triangular lattice does not contain any squares, we have $\phi(n, 4, 3) = O(n/\sqrt{\log n})$. The best known lower bound $\phi(n, 4, 3) = \Omega(n/\log n)$ is implied by Guth and Katz’s bound.

**Problem 28** Find the asymptotic value of $\phi(n, 4, 3)$.

![Diagram](image)

Figure 5: (a) The point $p$ is equidistant from the other three vertices of the deltoid. (b) The only possible cases of segments having the same length are $|ab| = |cd|$ and $|ac| = |bd|$.

The value of $\phi(n, 4, 4)$ is the minimum number of distinct distances that are determined by a set of $n$ points that do not span any rhombuses, rectangles, or deltoids with one vertex that is equidistant to the three other three (see Figure 5(a)). Dumitrescu [12] observed that $\phi(n,4,4) < n 2^{O(\sqrt{\log n})}$ by using the same point set $P_1$ from the analysis of $\phi(n,3,3)$.
Indeed, consider a subset of four points of \( P_1 \), as depicted in Figure 5(b), and notice that the only pairs of segments that are allowed to have the same length (without resulting in an arithmetic progression) are \( |ab| = |cd| \) and \( |ac| = |bd| \). Thus, every quadruple of points determines at least four distinct distances. No lower bound is known beyond \( \phi(n, 4, 4) = \Omega(n / \log n) \).

**Problem 29** Find the asymptotic value of \( \phi(n, 4, 4) \).

Not much is known about the case of \( \phi(n, 4, 5) \). Erdős [21] asked whether \( \phi(n, 4, 5) = \Theta(n^2) \), though the best known lower bound is only \( \phi(n, 4, 5) = \Omega(n) \). Indeed, when considering an \( n \) point set \( P \) with this property, we notice that any circle whose center is a point of \( P \) can be incident to at most two points of \( P \).

**Problem 30** Find the asymptotic value of \( \phi(n, 4, 5) \).

Finally, we have the trivial bound \( \phi(n, 4, 6) = n(n-1)/2 \), since every distance can occur at most once in this case.

**The case \( k=5 \).** We do not go over the various cases of \( k = 5 \), and only mention that Erdős [21] asked whether \( \phi(n, 5, 9) = \Omega(n^2) \). Again, nothing is known in this case beyond the trivial \( \phi(n, 5, 9) = \Omega(n) \).

**Problem 31** Find the asymptotic value of \( \phi(n, 5, 9) \).

**The general case.** Erdős [21] noticed that, for any \( k \geq 3 \), we have

\[
\phi \left( n, k, \binom{k}{2} - k + 3 \right) = \Omega(n).
\]

To see why, consider a set \( P \) of \( n \) points such that every \( k \)-point subset of \( P \) determines at least \( \binom{k}{2} - k + 3 \) distinct distances, and a point \( p \in P \). If \( p \) is the center of a circle that is incident to \( k - 1 \) points of \( P \), then together these \( k \) points determine at most \( \binom{k}{2} - k + 2 \) distinct distances, contradicting the assumption on \( P \). Thus, the points of \( P \setminus \{p\} \) are contained in at least \( (n-1)/(k-2) \) circles around \( p \), which in turn implies that \( p \) determines \( \Omega(n) \) distinct distances with the other points of \( P \).

Another simple observation is that, for any \( k \geq 4 \), we have

\[
\phi \left( n, k, \frac{k}{2} - \left\lfloor k/2 \right\rfloor + 2 \right) = \Omega(n^2),
\]

since in this case every distance can occur at most \( \left\lfloor k/2 \right\rfloor - 1 \) times.

Similarly to the cases of \( \phi(n, 3, 3) \) and \( \phi(n, 4, 4) \), we can use the set \( P_1 \) to obtain a bound of \( n2^{O(\sqrt{\log n})} \) for various \( \phi(n, k, l) \). For example, it is not hard to show that

\[
\phi \left( n, k, 2\left\lfloor k/2 \right\rfloor \right) = n2^{O(\sqrt{\log n})}.
\]

It seems likely that a more careful analysis would yield the same bound for larger values of \( l \).

**Problem 32** Find stronger general bounds for \( \phi(n, k, l) \).
7 Additional problems

In this section we discuss several additional “orphan” problems that did not fit into any of the previous sections.

In Problems 14, 15, and 20, we considered bipartite distinct distances problems, where we have two sets of points $\mathcal{P}_1$ and $\mathcal{P}_2$, and are interested only in distinct distances that are spanned by pairs of $\mathcal{P}_1 \times \mathcal{P}_2$ (i.e., distances between points from different sets). We now consider a somewhat different bipartite problem, where $\mathcal{P}_1$ is a set of three points and $\mathcal{P}_2$ is a set of $n$ points, both in $\mathbb{R}^2$. A simple argument shows that the number of distinct distances between two such sets is $\Omega(n^{1/2})$. Elekes [13] constructed a configuration where the three points of $\mathcal{P}_1$ are evenly spaced on a line, and the number of distinct distances between the two sets is $\Theta(n^{1/2})$, thus settling this problem.

We denote by $D(3,n)$ the minimum number of distinct distances between a set $\mathcal{P}_1$ of three non-collinear points and a set $\mathcal{P}_2$ of $n$ points. Elekes and Szabó [17] proved the bound $D(3,n) = \Omega(n^{0.502})$, showing that collinearity is necessary to obtaining a bound of $\Theta(n^{1/2})$. Recently, Sharir and Solymosi [42] obtained a more significant improvement, proving $D(3,n) = \Omega(n^{6/11})$. The best known upper bound is the trivial $D(3,n) = (n/\log n)$.

**Problem 33** Find the asymptotic value of $D(3,n)$.

Another bipartite variant was recently suggested by Sharir and Smorodinsky [41]. Denote by $L(m,n)$ the minimum number of distinct distances between a set of $m$ points and a set of $n$ lines, both in $\mathbb{R}^2$ (where the distance between a point and a line is defined in the usual way). By placing $m$ points on a line $\ell$ and then taking $n$ lines that are parallel to $\ell$, we obtain $L(m,n) = O(n)$. Sharir and Smorodinsky [41] derive the lower bound $L(m,n) = \Omega(n^{5/9} m^{2/9} / \log^{2/3} m)$.

**Problem 34** Find the asymptotic value of $L(m,n)$.

We next consider a variant of Problem 2. Given a planar set $\mathcal{P}$ of points and a point $p \in \mathcal{P}$, we denote by $\hat{D}_p(\mathcal{P})$ the number of distinct distances between $p$ and the other points of $\mathcal{P}$. We set $\hat{D}_\Sigma(\mathcal{P}) = \sum_{p \in \mathcal{P}} \hat{D}_p(\mathcal{P})$ and $\hat{D}_\Sigma(n) = \min_{|\mathcal{P}|=n} \hat{D}_\Sigma(\mathcal{P})$. It is not hard to verify that if $\mathcal{P}$ is a $\sqrt{n} \times \sqrt{n}$ integer lattice, then $\hat{D}_\Sigma(\mathcal{P}) = \Theta(n^2/\sqrt{\log n})$. Erdős conjectured [20, 22] that this should also be the value of $\hat{D}_\Sigma(n)$.

**Problem 35** Find the asymptotic value of $\hat{D}_\Sigma(n)$.

The best known lower bound $\hat{D}_\Sigma(n) = \Omega(n^{1.864})$ is immediately implied by Katz and Tardos’ [31] bound $D(n) = \Omega(n^{(48 - 14\epsilon)/(35 - 16\epsilon)}) = \Omega(n^{0.864})$.

Next, we consider a generalization from distinct distances to distinct vectors. Given a planar point set $\mathcal{P}$, we denote by $v(\mathcal{P})$ the number of distinct vectors that are spanned by pairs of points of $\mathcal{P}$ (i.e., the points $p,q$ span the vectors $p - q$ and $q - p$), and set $v(n) = \min_{|\mathcal{P}|=n} v(\mathcal{P})$. Clearly, $v(n)$ should be at least as large as $D(n)$. In fact, it is also not hard to show that $v(n) = \Theta(n)$. Indeed, the $\sqrt{n} \times \sqrt{n}$ integer lattice determines $\Theta(n)$ distinct vectors.

Erdős, Füredi, Pach, and Ruzsa [25] studied the case of distinct vectors for point sets in general position (i.e., no three points on a line and no four points on a circle). We set $v_{\text{gen}}(n) = \min_{|\mathcal{P}|=n} v(\mathcal{P})$, where the sum is taken over every $n$ point set in general position. In [25], it is proven that $v_{\text{gen}}(n) > cn$ for every constant $c$; i.e., $v_{\text{gen}}(n) = \omega(n)$. The best
known upper bound \( v_{\text{gen}}(n) = n2^{O(\log n)} \) is immediately implied by the best known upper bound for Problem 12.

**Problem 36** Find the asymptotic value of \( v_{\text{gen}}(n) \).

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**References**

[1] E. Altman, On a problem of P. Erdős, *Amer. Math. Monthly* **70** (1963), 148–157.

[2] E. Altman, Some theorems on convex polygons, *Canad. Math. Bull.* **15** (1972), 329–340.

[3] F. A. Behrend, On sets of integers which contain no three terms in arithmetic progression, *Proc. Nat. Acad. Sci.* **32** (1946), 331–332.

[4] P. Bernays, Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante, Dissertation, Göttingen, 1912.

[5] B. C. Berndt and R. A. Rankin, *Ramanujan: Letters and Commentary*, Amer. Math. Soc., Providence, RI, 1995.

[6] P. Brass, W. Moser, and J. Pach, *Research Problems in Discrete Geometry*, Springer-Verlag, New York, 2005.

[7] D. Brink, P. Moree, and R. Osburn, Principal forms \( X^2 + nY^2 \) representing many integers, *Abh. Math. Sem. Univ. Hambg.*, **81** (2011), 129–139.

[8] M. Charalambides, A note on distinct distance subsets, *Journal of Geometry*, **104** (2013), 439–442.

[9] M. Charalambides, Exponent gaps on curves via rigidity, arXiv:1307.0870.

[10] D. Conlon, J. Fox, W. Gasarch, D. Harris, D. Ulrich, and S. Zbarsky, Distinct volume subsets, arXiv:1401.6734.

[11] A. Dumitrescu, On distinct distances from a vertex of a convex polygon, *Discrete Comput. Geom.* **36**(4) (2006), 503–509.

[12] A. Dumitrescu, On distinct distances among points in general position and other related problems, *Periodica Mathematica Hungarica* **57**(2) (2008), 165–176.

[13] G. Elekes, Circle grids and bipartite graphs of distances, *Combinatorica* **15** (1995), 167–174.

[14] G. Elekes, A note on the number of distinct distances, *Period. Math. Hung.*, **38** (1999), 173–177.

[15] G. Elekes and L. Rónyai, A combinatorial problem on polynomials and rational functions, *J. Combin. Theory Ser. A*, **89** (2000), 1–20.
[16] G. Elekes and M. Sharir, Incidences in three dimensions and distinct distances in the plane, *Combinat. Probab. Comput.* **20** (2011), 571–608.

[17] G. Elekes and E. Szabó, How to nd groups? (and how to use them in Erdos geometry?), *Combinatorica* **32** (2012), 537–571.

[18] P. Erdős, On sets of distances of $n$ points, *Amer. Math. Monthly* **53** (1946), 248–250.

[19] P. Erdős, Nehany geometriai problémáról (in Hungarian), *Matematikai Lapok*, **8** (1957), 86–92.

[20] P. Erdős, On some problems of elementary and combinatorial geometry, *Ann. Mat. Pura Appl.* **103** (1975), 99–108.

[21] P. Erdős, On some metric and combinatorial geometric problems, *Discrete Math.* **60** (1986), 147–153.

[22] P. Erdős, Some combinatorial and metric problems in geometry, *Intuitive geometry* (K. Böröczky and G. Fejes Tóth, eds.), North-Holland, Amsterdam-New York, 1987, 167–177.

[23] P. Erdős, On some of my favourite theorems, *Combinatorics, Paul Erdős is Eighty*, Vol. 2 (D. Miklós et al., eds.), Bolyai Society Mathematical Studies 2, Budapest, 1996, 97–132.

[24] P. Erdős and P. Fishburn, Maximum planar sets that determine $k$ distances, *Discrete Math.* **160** (1996), 115–125.

[25] P. Erdős, Z. Füredi, J. Pach, and I. Z. Ruzsa, The grid revisited, *Discrete Math.* **111** (1993), 189–196.

[26] P. Erdős, R. K. Guy, Distinct distances between lattice points, *Elemente Math.* **25** (1970) 121–123.

[27] P. Erdős, D. Hickerson, and J. Pach, A problem of Leo Moser about repeated distances on the sphere, *Amer. Math. Monthly* **96** (1989), 569–575.

[28] P. Erdős and G. Purdy, Extremal problems in combinatorial geometry, *Handbook of Combinatorics*, Vol. I (R. L. Graham, M. Grötschel, and L. Lovász, editors), Elsevier, Amsterdam, 1995, pp. 809– 874.

[29] J. Garibaldi, A. Iosevich, and S. Senger, *The Erdős Distance Problem*, Student Math. Library, Vol. 56, Amer. Math. Soc. Press, Providence, RI, 2011.

[30] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, arXiv:1011.4105.

[31] N. H. Katz and G. Tardos, A new entropy inequality for the Erdős distance problem, *Towards a Theory of Geometric Graphs* (*J. Pach*, ed.), Contemporary Mathematics **342**, AMS, Providence, RI, 2004, 119–126.

[32] E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindezahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate, *Arch. Math. Phys.* **13** (1908), 305–312.
[33] H. Lefmann and T. Thiele, Point sets with distinct distances, *Combinatorica* **15** (1995) 379–408.

[34] G. Nivasch, J. Pach, R. Pinchasi, and S. Zerbib, The number of distinct distances from a vertex of a convex polygon, *J. Computational Geometry*, **4** (2013), 1–12.

[35] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, Wiley-Interscience, New York, 1995.

[36] J. Pach and F. de Zeeuw Distinct Distances on Algebraic Curves in the Plane, *Proc. 30th ACM Symp. on Computational Geometry* (2014), to appear.

[37] J. Pach and G. Tardos, Isosceles triangles determined by a planar point set, *Graphs and Combinatorics* **18** (2002), 769–779.

[38] O. Raz, M. Sharir and J. Solymosi, Polynomials vanishing on grids: The Elekes-Rónyai problem revisited, *Proc. 30th ACM Symp. on Computational Geometry* (2014), to appear.

[39] R. Schwartz, J. Solymosi, and F. de Zeeuw, Extensions of a result of Elekes and Rónyai, *J. Combin. Theory Ser. A*, **120** (2013), 1695–1713.

[40] M. Sharir, A. Sheffer, and J. Solymosi, Distinct distances on two lines, *J. Combinat. Theory A*, 120 (2013), 1732–1736.

[41] M. Sharir and S. Smorodinsky, On Distinct Distances Between Points and Lines, Manuscript.

[42] M. Sharir and J. Solymosi, Distinct distances from three points, arXiv:1308.0814.

[43] A. Sheffer, J. Zahl, and F. de Zeeuw, Few distinct distances implies no heavy lines or circles, *Combinatorica*, to appear.

[44] J. Solymosi and V. H. Vu, Near optimal bounds for the Erdős distinct distances problem in high dimensions. *Combinatorica* **28**(1) (2008), 113–125.

[45] T. Tao, Lines in the Euclidean group SE(2), [http://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/](http://terrytao.wordpress.com/2011/03/05/lines-in-the-euclidean-group-se2/)

[46] T. Thiele, *Geometric selection problems and hypergraphs*, PhD thesis, Instut fur Mathematik II Freir Universität Berlin, 1995.