Nonlinear effects of stabilization in ship models with non-smooth nonlinearities using P-control

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Abstract

Nonlinear terms in standard models of ship maneuvering entail continuous non-smooth terms so that bifurcations of straight motion are not amenable to standard center manifold reduction. In this paper, we determine the criticality of Hopf bifurcations that arise in stabilizing the straight motion based on a recently developed analytical approach. For such a 3 DOF model of ship motion with yaw damping and yaw restoring control, we present a detailed analysis of the possibilities to stabilize the straight motion, and the resulting nonlinear effects. To facilitate the analysis, we consider a combination of rudder and propeller forces into an effective thruster force. We identify existence, location and geometry of the stability boundary in terms of the controls, including the dependence on the propeller diameter and the thruster position. We find that ‘safe’ supercritical Andronov-Hopf bifurcations are predominant and by means of numerical continuation we provide a global bifurcation analysis, which identifies the arrangement and relative location of stable and unstable equilibria and periodic orbits. We illustrate the resulting stable quasi-periodic ship motions in Earth-fixed coordinates and present some direct numerical simulations.

Key words: Stability, Non-Smoothness, Hopf Bifurcation, Lyapunov Coefficient

AMS subject classifications: 34H15, 34H20, 37N35, 93D20

1 Introduction

In this paper we analyze the effectiveness of proportional control (P-control) to stabilize the ship motion on linear and nonlinear levels in a standard class of models for ship maneuvering with continuous non-smooth nonlinearities. Such a theoretical study is not possible by standard center manifold reduction, but is relevant for predicting properties and guiding design choices, also with certification standards in mind [1]. Indeed, linear stability can be misleading if nonlinear effects produce an ‘unsafe’ so-called subcritical bifurcation, where the stabilized state has

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such a small basin of attraction that its linear stability may be practically irrelevant. In addition, already a detailed understanding of the specific possibilities for linear stabilization in parameter space is a challenge for a given model.

For smooth models of marine vehicles the common reduction techniques are a standard tool, e.g., [9]. A general review in the context of smooth nonlinear ship models and motion can already be found in [10]. Our main goal in this manuscript is to present a method that can also handle the widely used continuous non-smooth models. In a selected model class we furthermore aim for a rather complete mathematical analysis of controllability and its failure, and of the local and global nonlinear effects associated with such a stabilization. We are specifically concerned with the standard problem of stabilising straight motion [1, 5], which, however, has so far not been analysed in this form to our knowledge.

Models of ship maneuvering entail a selection of displacements and rotations of a marine craft, which are on the one hand caused (principally) by sea waves: roll (rotation about the longitudinal axis), pitch (rotation about the transverse axis) and heave (vertical -up/down- motion); and on the other hand surge (longitudinal motion), sway (sideways motion) and yaw (rotation about the vertical axis), caused (mainly) by internal forces, such as rudders and propellers, and external forces, such as maritime conditions, cf. [5]. Reduced-order models are frequently considered and a common simplification for relatively weak sea waves is the 3 DOF horizontal plane model for the velocities surge, \( u \), sway, \( v \), and yaw, \( r \) [5, 7]; see Fig. 1. We consider such a 3 DOF model from [1, 2, 13] that explicitly includes the yaw angle as a fourth variable, and is augmented by a P-controller for yaw damping and yaw restoring, thus resulting in a 4-dimensional system of nonlinear ordinary differential equations (ODE). Parameters of the ship model are taken from the so-called Hamburg Test Case (HTC). Following [2], we combine the rudder and the propeller forces into a single thruster force. This acts on the hull with an angle \( \eta \) that is taken as the controlled variable: \( \eta = \varepsilon_r (r - r_0) + \varepsilon_\psi (\psi - \psi_0) \) with non-negative control strengths \( \varepsilon_r, \varepsilon_\psi \). This realises a P-control for \( r \), \( \psi \) that combines yaw damping and yaw restoring control (which can be viewed as a PD-control for \( \psi \) or a PI-control for \( r \)), cf. [5].

Our overall results confirm the linear control paradigm in the sense that – in the parameter ranges we consider – the nonlinear effects do not counteract linear stabilization. However, this is not at all clear a priori, and requires rather detailed analysis as well as new methods that we have developed in [11] motivated by the current problem. Specifically, we focus on stabilizing the straight motion \( \psi_0 = r_0 = 0 \), which appears as an unstable fixed point in the 4-dimensional ODE system.

The linear stability problem entails, as usual, controlling the sign of the real part of the roots of a characteristic polynomial, which here takes the form \( Q_p(\lambda) = \lambda^3 + c_2 \lambda^2 + c_1 \lambda + c_0 \), and

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Ship-fixed coordinates with 3 DOF: surge \( u \), sway \( v \) and yaw \( r \) velocities as well as yaw angle \( \psi \) and steering angle \( \eta \).}
\end{figure}
whose coefficients depend, in particular, on the control parameters $\varepsilon_r$ and $\varepsilon_\psi$. We determine the existence, location and geometry of the resulting region in the positive quadrant of the $(\varepsilon_r, \varepsilon_\psi)$-plane, for which the straight motion is linearly stable. In fact, we provide this in terms of an explicit formula. We show that the stability region may be bounded or even empty (within the positive quadrant). For this we investigate exhaustively the influence on the stability region of the propeller diameter and the longitudinal thruster location $x_T$ in percent of length from midship. In Fig. 2 we illustrate the resulting different forms of the stability region depending on the value of $x_T$ with respect to certain thresholds $x_{T_0}, x_{T_1}, x_{T_s}$; note that the natural domain of $x_T$ is $[-0.5, 0.5]$.

As might be expected, the further away from the aft, the more constrained the stability region is, and beyond the threshold $x_{T_s}$ the straight motion cannot be stabilized at all, at least not by the fixed P-control design. In a nutshell, either stabilization is possible by $\varepsilon_r$ alone (for $\varepsilon_\psi = 0$) or by no means, and – except in case 4 – the stable region is unbounded and can be reached by increasing $\varepsilon_r$ for any $\varepsilon_\psi \geq 0$. In fact, we prove that this scenario occurs not only for the HTC but much more broadly in the space of model parameters.

Concerning nonlinear effects of the stabilizing control, we study the character of the associated bifurcations, which is complicated by the fact that the nonlinear terms in the model equations are non-smooth, featuring second order modulus terms such as $v |r|$ due to the hydrodynamic forces, cf. §2. Such terms prevent the use of standard bifurcation theory, in particular for the predominant Andronov-Hopf bifurcations to periodic solutions. For vanishing yaw restoring control, $\varepsilon_\psi = 0$, steady state bifurcations occur and we prove this is a non-smooth pitchfork bifurcation. For $\varepsilon_\psi > 0$, the mentioned Hopf bifurcations occur, which we handle by the recently developed method of [11]. To leading order, this transforms the problem to a
Figure 3: (a) Non-smooth Hopf-bifurcation diagram for \( \varepsilon_r = 0 \) computed by numerical continuation; the horizontal line are straight motion equilibrium points and the bifurcating branch corresponds to periodic orbits (only maximum value shown). (b) Plot of the ship track in Earth coordinates corresponding to the periodic solution in (a) at \( \varepsilon_\psi \approx 43 \).

system of the form

\[
\begin{align*}
\dot{r} &= \mu r + \chi(\phi)r^2 + \mathcal{O}(r^3), \\
\dot{\phi} &= \omega + \Omega(\phi)r + \mathcal{O}(r^2),
\end{align*}
\]

for certain polar coordinates \((r, \phi)\), bifurcation parameter \(\mu\) and \(2\pi\)-periodic functions \(\chi, \Omega\). As detailed in [11], it follows that a Hopf-type bifurcation occurs, where the amplitudes of the emerging periodic solutions are, to leading order, given by

\[
r = -\frac{2\pi}{\Sigma} \mu + \mathcal{O}(\mu^2), \quad \Sigma := \int_0^{2\pi} \chi(\phi) d\phi.
\]

This bifurcation comes in two generic types, analogous to the usual smooth case: supercritical (unstable fixed point coexists with stable periodic orbits, \(\Sigma < 0\)) or subcritical (stable fixed point coexists with unstable periodic orbits, \(\Sigma > 0\)). In the subcritical case, upon moving the control parameter through the stability threshold, the corresponding solution will typically not track the stabilized equilibrium, but will be repelled from the unstable periodic orbit. In contrast, in the ‘safe’ supercritical case, the trajectories will track the stabilized state. Hence, from a mathematical as well as an engineering point of view, it is relevant to a priori determine this criticality from the equations, i.e., to obtain a formula for \(\Sigma\). In the smooth setting this corresponds to the so-called first Lyapunov coefficient, which is found from a center manifold reduction and normal form computation based on Taylor expansion. However, for the non-smooth ship model case this approach is not possible and we instead employ the results of [11] in order to determine \(\Sigma\). We emphasize that in the smooth case the amplitudes of the bifurcating periodic orbits, and thus the basin of attraction, scale as \(|\mu|^{1/2}\) rather than the linear scaling of the present case. As mentioned, this analysis reveals that for the ship model the bifurcations at the stability boundary are supercritical in the HTC and for different thruster positions. While this means all bifurcations are safe for the parameters we considered, it is not at all clear for which other modification of the HTC parameters this remains the case. We plot an example in Fig. 3.
However, the local analysis near a bifurcation does not reveal the overall organization of nonlinear states. In order to gain insight into this, we have performed a detailed numerical continuation analysis. This corroborates our analytical predictions and exposes how the stable and unstable equilibria as well as periodic orbits are organized globally in the positive quadrant of control strengths $\varepsilon_r, \varepsilon_\psi$. The periodic orbits limit, on the one hand, to the stability boundary curve and to the steady states at $\varepsilon_\psi = 0$, and on the other hand, to certain heteroclinic bifurcations. Such a global analysis requires the vector field to be $2\pi$-periodic with respect to the yaw angle $\psi$ and, for definiteness, we modify the control law to $\eta = \varepsilon_r r + \varepsilon_\psi \sin(\psi)$, which is nonlinear and has no impact on the local bifurcations to leading order. This global cylinder geometry also induces a winding number for periodic orbits, which is a topological invariant and enters into the global bifurcation scenario.

For further illustration, we present several simulations of solutions from ship motions also on Earth-fixed position coordinates.

This paper is structured as follows. We present the model equations and some background in §2. In the main section §3 we theoretically analyze the straight motion in terms of linear stability and bifurcations. The numerically bifurcation analysis is presented in §4 and we conclude with a brief discussion in §5.

2 Model Equations and Background

Detailed derivations of the equations of motion for a marine craft can be found in, e.g., [5]. Here we briefly discuss ingredients that are most relevant for our subsequent analysis. The kinetics is based on Newton’s second law and Euler’s axiom, where the rigid-body equations of motion take the form $M \ddot{v} = F(v)$, with $v$ the vector of ship-fixed velocities, $\dot{v}$ its time derivative, $M$ the matrix of added mass coefficients and moments of inertia, and $F$ containing forces from hull, rudder, propeller and hydrodynamics, as well as the Coriolis term.

In the modeling of the hydrodynamic forces most relevant for us is the nature of the arising nonlinear terms since their non-smooth character is decisive in the analysis of bifurcations, and requires new theory as developed in [11], motivated by the present problem. One approach to the nonlinear terms follows the drag equation for high Reynolds number given by

$$F_D = -\frac{1}{2} \rho C_D A u |u|,$$

where $u$ is the velocity of the body, $\rho$ the density of the water, $C_D$ the drag coefficient, and $A$ the effective drag area, [5]. This equation is a consequence of the experimental observation that the drag force $F_D = F_D(\rho, A, u)$ is a function of $\rho, A, u$; together with the fact that as an opposing force, this must be odd with respect to $u$, and that dimensional analysis in a power law ansatz $F_D = \rho^\alpha A^\beta u^\gamma C$ for $u > 0$ with exponents $\alpha, \beta, \gamma \in \mathbb{R}$ and a constant $C$, implies $\alpha = \beta = 1, \gamma = 2$. Indeed, polynomial regression studies investigating the representation of hydrodynamic forces confirm that $u|u|, v|v|, r|r|$ and the mixed terms $v|r|, r|v|$, are the relevant higher order terms [14, 15]. These second-order modulus terms can be regarded as square law damping in this context and may also be heuristically motivated by a Taylor expansion with correction for
the signs \textsuperscript{[1]}. We remark that ship models with third-order Taylor approximations have been analyzed, e.g., in \textsuperscript{[8]}.

Concerning the choice of model, we quote from \textsuperscript{[1, p. 75]} that the ‘choice of the form of non-linear terms is a matter of accuracy and convenience [...] also dependent on in which form the hydrodynamic derivatives were obtained in PMM [planar motion mechanism] test.’ However, when it comes to bifurcation analysis, there is a significant difference between the second order modulus and cubic terms as discussed in detail in \textsuperscript{[11]; see also §3.3.2.}

The specific model equations that we will investigate are a variation of the 3 DOF model from \textsuperscript{[1, 13]}, for which some basic analysis was conducted in \textsuperscript{[2]}. The model parameters stem from the ‘Hamburg Test Case’ (HTC) characteristics that we collect in \textsuperscript{[A.1]} as needed. We adopt these values throughout, except when analyzing the impact of selected parameter changes. The general 3 DOF model takes the form

\[
\begin{pmatrix}
m + m_{uu} & 0 & 0 \\
0 & m + m_{vv} & m_{vr} \\
0 & m_{rv} & I_z + m_{rr}
\end{pmatrix}
\begin{pmatrix}
\ddot{u} \\
\ddot{v} \\
\ddot{r}
\end{pmatrix}
=
\begin{pmatrix}
mvr + X \\
-mur + Y \\
N
\end{pmatrix},
\] (2.1)

where \(u, v\) and \(r\) are the surge, sway and yaw velocities, respectively. The external forces \(X, Y, N\) for the ‘rudder model’ of \textsuperscript{[13]} are of the form \(X = X_H + X_R + X_P, Y = Y_H + Y_R, N = N_H + N_R\) with the contributions from the hull (H), the rudder (R) and the propeller (P) of the vessel.

As mentioned in the introduction, in order to facilitate the presentation of the mathematical method and analysis, we combine the rudder and propeller forces as in \textsuperscript{[2]} into a simpler ‘thruster force’, which gives

\[
X = X_H + X_P \cos \eta,
\]

\[
Y = Y_H + X_P \sin \eta,
\]

\[
N = N_H + x_T X_P \sin \eta.
\] (2.2)

Here the thruster acts on the hull at longitudinal position \(x_T\) relative to the length \(L_{pp}\) and exerts a force in direction \(\eta\) of amplitude given by the propeller force \(X_P\). As the default position we take that of the rudder in the HTC, namely \(x_T = -(0.5 + \frac{0.571}{100})L_{pp} = -0.49429L_{pp}\), with \(x_B = -0.571\) being the longitudinal center of gravity with respect to midship in percent of length; note that \(x_T\) is measured from the midship towards the front, so its natural domain is \([-0.5L_{pp}, 0.5L_{pp}]\).
The hull forces have the following general expressions, cf. [1, 13]:

$$X_H = \frac{1}{2} \rho L_{pp} T \left( X_{u|u|u} |u| + X_{\beta \gamma} L_{pp} v r \right),$$

$$Y_H = \frac{1}{2} \rho L_{pp} T \left( Y_{\beta} |u| v + Y_{\gamma} L_{pp} u r + Y_{\beta \beta} |v| |v| + Y_{\gamma \gamma} L_{pp}^2 r |r| + Y_{\beta \gamma} L_{pp} v r \right)$$

$$+ Y_{\beta} |\beta| L_{pp} v |v| + Y_{ab} |u^a y^b| \text{sgn}(v) V^{2-a_n-b_n},$$

$$N_H = \frac{1}{2} \rho L_{pp}^2 T \left( N_{\beta} u v + N_{\gamma} L_{pp} r |u| + N_{\beta \beta} L_{pp} r |v| + N_{\gamma \gamma} |v| |v| \right)$$

$$+ N_{\beta \gamma} r^2 L_{pp} V^{-1} + N_{\beta \gamma} r^2 L_{pp}^2 V^{-1} \text{sgn}(u) + N_{\gamma \gamma} L_{pp} c_n |u| r c_n |V| r c_n \text{sgn}(r)$$

$$+ N_{ab} |u^a y^b| |V|^{-a_n-b_n+2} \text{sgn}(uv),$$

where $V = \sqrt{u^2 + v^2}$ and all coefficients are by default those of the HTC as listed in [A.1]. Note that these forces implement the aforementioned second order modulus form of the drag.

The propeller force taken from [13] reads

$$X_P = (1-t) T_P(u),$$

with propeller thrust $T_P(u) = \rho n_p^2 D_p^4 K_T$, where $K_T = \sum_{i=0}^5 J_i$, and $J_p = \frac{u(1-w)}{n_p \beta_p}$. Later, it turns out to be relevant that the propeller thrust is monotone decreasing with respect to the surge $u$, i.e.,

$$\partial_u X_P < 0,$$  \hspace{1cm} (2.3)

which can readily be verified numerically for the HTC.

The model is completed by the control law for the steering angle $\eta = \eta_1 + \eta_2$, which is a combination of the yaw damping P-control $\eta_1$ and yaw restoring P-control $\eta_2$; we refer to [5] for a general background. The first consists of adding a proportional compensation to the yaw velocity $r$ by setting $\eta_1 = \eta_0 + \epsilon_r (r - r_0)$, with target yaw velocity $r_0$, initial thruster angle $\eta_0$ and control parameter $\epsilon_r \geq 0$. Analogously, for the yaw angle $\psi$ we have $\eta_2 = \eta_0 + \epsilon_\psi (\psi - \psi_0)$ with target yaw angle $\psi_0$ and second control parameter $\epsilon_\psi \geq 0$. Notably, for $\epsilon_\psi > 0$ this requires to add $\psi = r$ as a fourth equation to (2.1). Since we are focusing on controlling a straight line trajectory, without loss of generality, we set $r_0 = \psi_0 = \eta_0 = 0$, which gives

$$\eta = \epsilon_r r + \epsilon_\psi \psi.$$  \hspace{1cm} (2.4)

We remark that any P-control involving $\psi$ is consistent only for sufficiently small values of $\psi$ since it is an angular variable, which means that in general, all model equations must be $2\pi$-periodic with respect to $\psi$. This is not relevant in practice (usually) and not for the existence and linear stability analysis of the straight motion, and not even for weakly nonlinear bifurcation analysis. However, for the global bifurcation computations and analysis in [4], we need to modify (2.4) to be periodic.

Regarding the propeller frequency $n_p$, we reflect the following: all forces scale quadratically with respect to the velocities so that $u = n_p \ddot{u}$, gives, e.g., $X_P = n_p^2 \ddot{X}_P(\ddot{u})$, where $\ddot{X}_P$ is independent
of \( n_p \). This implies the natural relation that all velocities are proportional to the propeller speed: rescaling all velocities proportional to \( n_p \) gives a factor \( n_p^2 \) on the right hand side of (2.1) and a factor \( n_p \) on the left hand side. Time rescaling with \( n_p \) gives a second factor on the left so that \( n_p \) is removed upon division. Hence, any motion for one value of \( n_p > 0 \) can directly be rescaled to the motion corresponding to another positive value of \( n_p \). Therefore, without loss of information, we can fix \( n_p > 0 \) arbitrarily; in the HTC this is \( n_p = 2 \). Concerning the propeller diameter \( D_p \), we also observe that scaling \( u = D_p\bar{u} \) gives \( X_p = D_p\bar{X}_p(\bar{u}) \), where \( \bar{X}_p \) is independent of \( D_p \). However, not all forces scale in the same way and, as we will explain below, the value of \( D_p \) enters into the analysis of the model.

In order to facilitate the implementation and analysis, we rescale (2.1) in terms of the length \( L_{pp} \) between perpendiculars of the vessel, but do not pursue a full non-dimensionalization. As the starting point, we note that the surge and sway velocities have dimensions \([u] = [v] = m/s\), while the dimension of the angular yaw velocity is \([\psi] = 1/s\). We equate these by introducing \( \bar{r} = L_{pp}r \) and then rescale (2.1), together with (2.2), accordingly. Denoting the rescaled quantities with overbar, except using \( \tau \) for the rescaled thruster force amplitude, we thus obtain

\[
\begin{pmatrix}
\bar{m} + \bar{m}_{uu} & 0 & 0 \\
0 & \bar{m} + \bar{m}_{uv} & \bar{m}_{vr} \\
0 & \bar{m}_{rv} & \bar{I}_z + \bar{m}_{rr}
\end{pmatrix}
\begin{pmatrix}
\bar{u} \\
\bar{v} \\
\bar{r}
\end{pmatrix}
= \begin{pmatrix}
L_{pp}^{-1}\bar{m}\bar{v}\bar{r} + \bar{X}_H + \tau \cos \eta \\
-L_{pp}^{-1}\bar{m}\bar{u}\bar{r} + \bar{Y}_H + \tau \sin \eta \\
\bar{N}_H + \bar{x}_T \tau \sin \eta
\end{pmatrix},
\]  

(2.5)

where we have scaled

\[
\bar{m} = B_2^{-1}m, \quad \bar{m}_{uu} = B_2^{-1}m_{uu}, \quad \bar{m}_{uv} = B_2^{-1}m_{uv}, \quad \bar{I}_z = B_2^{-1}I_z, \\
\bar{m}_{vr} = B_3^{-1}m_{vr}, \quad \bar{m}_{rv} = B_3^{-1}m_{rv}, \quad \bar{m}_{rr} = B_3^{-1}m_{rr}, \quad \tau = B_2^{-1}X_p, \\
\bar{X}_H = B_2^{-1}X_H, \quad \bar{Y}_H = B_2^{-1}Y_H, \quad \bar{N}_H = B_3^{-1}N_H, \quad \bar{x}_T = L_{pp}^{-1}x_T,
\]

with \( B_i = \frac{1}{2}\rho L_{pp}^i T \). More specifically, this gives

\[
\bar{X}_H = L_{pp}^{-1}\left(X_{u|u|u^2} + X_{\beta\gamma uv}\right), \\
\bar{Y}_H = L_{pp}^{-1}\left(Y_{\beta uv} + Y_{\gamma u\bar{r}} + Y_{\beta|\gamma|\bar{r}}|\bar{v}| + Y_{\beta|\gamma|\bar{r}}|\bar{r}| + Y_{\beta|\gamma|v}|\bar{r}| + Y_{\beta|\gamma|v}|\bar{v}| + \\
+ Y_{abu|d|v}(|v| \text{sgn}(v))V^{2-a_s-b_y}\right), \\
\bar{N}_H = L_{pp}^{-1}\left(N_{\beta uv} + N_{\gamma u\bar{r}} + N_{\beta\gamma u|v|}|\bar{r}|^2V^{-c_n+1}\text{sgn}(\bar{r}) + N_{\beta\gamma|v|\bar{r}}|\bar{r}| + N_{\beta\gamma|v|v} |\bar{r}| + \\
+ N_{\beta\gamma\gamma\bar{r}}V^{-1} + N_{\beta\gamma\gamma\bar{r}}V^{-1} + N_{abu|d|v}(|v| \text{sgn}(v))V^{2-a_n-b_n+2}\text{sgn}(v)\right).
\]

To ease exposition, in the following we refer to the rescaled hydrodynamic bare hull coefficients \( \bar{X}_j = L_{pp}^{-1}X_j, \bar{Y}_j = L_{pp}^{-1}Y_j, \bar{N}_j = L_{pp}^{-1}N_j \), where \( j \) denotes the corresponding subindex; and likewise introduce \( m_L = L_{pp}^{-1}\bar{m} \).

Concerning the control law (2.4), it is natural to scale \( \bar{\psi} = L_{pp} \psi \) so that the added equation \( \dot{\psi} = r \) becomes \( \dot{\bar{\psi}} = \bar{r} \); note that \( \bar{\psi} \) is then \( 2\pi/L_{pp} \)-periodic. Therefore, in (2.4) it is natural to scale \( \bar{\varepsilon}_r = L_{pp}^{-1}\varepsilon_r \) and \( \bar{\varepsilon}_\psi = L_{pp}^{-1}\varepsilon_\psi \), which yields

\[
\eta = \bar{\varepsilon}_r \bar{r} + \bar{\varepsilon}_\psi \bar{\psi}.
\]  

(2.6)
In the following we omit all overbars and study the rescaled 4D ‘thruster model’

\[
\begin{pmatrix}
m + m_{u_0} & 0 & 0 & 0 \\
0 & m + m_{v_0} & m_{v_0} & 0 \\
0 & m_{r_0} & I_z + m_{r_0} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\dot{u} \\
\dot{v} \\
\dot{r} \\
\dot{\psi} \\
\end{pmatrix} =
\begin{pmatrix}
mL \dot{v} + X_H + \tau(u) \cos \eta \\
-mL \dot{u} + Y_H + \tau(u) \sin \eta \\
N_H + x_T \tau(u) \sin \eta \\
r \\
\end{pmatrix},
\]

written more compactly as \( M \dot{\nu} = F(\nu) \), with \( M \) the invertible matrix on the left hand side and \( F(\nu) \) the right hand side. The equivalent explicit compact form of (2.7) thus reads

\[
\dot{\nu} = M^{-1} F(\nu).
\]

3 Theoretical Analysis

In this section we analyze the impact of the yaw-damping yaw-restoring control (2.6) on the stability and bifurcation of the straight motion with constant speed. We include variations of the selected design parameters \( D_p \), the propeller diameter, and \( x_T \), the thruster position, in order to illustrate the methodology. We start discussing the existence of the straight motion as an equilibrium in the ship-fixed coordinates, then turn to the linear stability and finally analyze the resulting bifurcations.

3.1 Equilibrium Straight Motion

The straight motion of the vessel with constant speed corresponds to an equilibrium point \((u_0, v_0, r_0, \psi_0) = (u_0, 0, 0, 0)\), with \( u_0 > 0 \) and the reference direction \( \psi_0 = 0 \) of system (2.7). Equilibria are those \( \nu = (u, v, r, \psi)^\top \) for which \( F(\nu) \) in (2.8) vanishes. Setting \( \nu = r = \psi = 0 \), the last three components of \( F(\nu) \) vanish, which in fact holds for any \( \psi_0 \) if \( \epsilon_{\psi} = 0 \), so that in this case we obtain a line of equilibrium motion in any direction. For \( \epsilon_{\psi} \neq 0 \), this is constrained to the reference direction \( \psi_0 = 0 \). The remaining first component of \( F(\nu) \) now reads \( X_H + \tau(u) \cos(0) \), which gives the condition for the equilibrium straight velocity \( u_0 \) as

\[
X_{u|u} u_0^2 + \tau(u_0) = 0,
\]

independent of the control parameters \( \epsilon_r, \epsilon_{\psi} \). This equation possesses a unique positive solution if (2.3) holds, i.e., \( \partial_u \tau(u) < 0 \), since \( \tau(0) > 0 \) and \( X_{u|u} \) \(< 0 \), and therefore the left hand side is strictly decreasing for \( u > 0 \). These conditions hold for the HTC values (see A.1), and then \( u_0 = u_0^{HTC} \approx 8.71 \).

Regarding the parameters \( D_p, x_T \), the equilibrium location is independent of \( x_T \) since this does not appear in (3.1). For the propeller diameter \( D_p \), we scale \( \bar{u} = D_p \tilde{u} \) and \( \tilde{\tau}(\tilde{u}) := \tau(u)/D_p^2 \) so that (3.1) becomes \( X_{u|u} \bar{u}_0^2 + D_p^2 \tilde{\tau}(\bar{u}_0) = 0 \), where the first addend is independent of \( D_p \). Hence, the equilibrium depends on \( D_p \), but for large values its location is approximately proportional to \( D_p \), cf. Fig. 4 (a,b): indeed, the rescaled (3.1), upon multiplication by \( (D_p^2 X_{u|u})^{-1} \), takes the form

\[
D_p^{-2} \bar{u}_0^2 + C_0 + P(\bar{u}_0) = 0,
\]

where \( C_0 \) and \( P(\bar{u}_0) \) vanish for large values of \( D_p \).
where all \( \tilde{C}_i \) are positive constants. Therefore, for \( u_0 \) exactly proportional to \( D_p \) we have \( \partial_u \tau(u_0)(D_p) = \tilde{C}D_p^3 \), with \( \tilde{C} < 0 \) and the approximate proportionality implies \( \partial_u \tau(u_0) < 0 \) for \( D_p \gg 1 \).

### 3.2 Linear Stability of Straight Motion

We first recall from [2] that the straight motion without control is unstable. Linearizing the right hand side of (2.5) without control at \( u = u_0, v = r = 0 \) and multiplying by the inverse of the mass matrix gives

\[
\tilde{S} := \frac{1}{D} \begin{pmatrix}
\frac{D}{m+m_{uu}} & 0 & 0 \\
0 & I_z + m_{rr} & -m_{rv} \\
0 & -m_{rv} & m + m_{vv}
\end{pmatrix}
\begin{pmatrix}
2X_{u|u}|u_0 + \partial_u \tau(u_0) & 0 & 0 \\
0 & Y_{\beta}u_0 & (Y_\gamma - m_L)u_0 \\
0 & N_\beta u_0 & N_\gamma u_0
\end{pmatrix},
\]

with \( D := (m + m_{vv})(I_z + m_{rr}) - m_{rv}m_{rr} \). Due to the block structure, the upper left entry is an eigenvalue \( \lambda_1 \) of \( \tilde{S} \), which is always negative since \( X_{u|u}|u_0 < 0 \) and \( \partial_u \tau(u_0) < 0 \) as just discussed. More specifically, for the HTC, see [\( \Lambda \)] \( X_{u|u} \approx -0.00009, \ m = 0.2328, \ m_{uu} = 0.0247 \) and \( u_0 = u_0^{\text{HTC}} \approx 8.71, \ \partial_u \tau(u_0) \approx -0.0008, \) yielding \( \lambda_1 \approx -0.0094 \). Since \( \lambda_1 < 0 \), the stability of the straight motion equilibrium point \((u_0, 0, 0)\) is determined by the lower right \( 2 \times 2 \) submatrix.
of \( \tilde{S} \), which we denote by \( S \). Its trace and determinant are, respectively,

\[
\text{tr}(S) = D^{-1}((I_z + m_{rr})Y_\beta + (m + m_{rv})N_\gamma - m_{v\gamma}N_\beta - m_{r\gamma}(Y_\gamma - m_L))u_0,
\]

\[
\text{det}(S) = D^{-1}(Y_\beta N_\gamma - N_\beta(Y_\gamma - m_L))u_0^2.
\]

For the HTC, see again \([1] \) both trace and determinant are negative, which implies that the eigenvalues \( \lambda_2, \lambda_3 \) are non-zero with opposite signs. Specifically, \( \text{tr}(S) \approx -0.0657, \text{det}(S) \approx -0.0059, \lambda_2 \approx 0.0506 \) and \( \lambda_3 \approx -0.1163 \).

Hence, for the system without control, the straight motion equilibrium is unstable and we next discuss the impact of the P-control \((2.6) \) on the eigenvalues, but first note that the free parameters \( D_p, x_T \) do not change the instability. Clearly, \( x_T \) does not influence the stability analysis since the matrix and equilibrium do not depend on it. The propeller diameter \( D_p \) enters only in \( \partial_a \tau(u_0) \), which modifies the first eigenvalue \( \lambda_1 \) of \( \tilde{S} \), but – as noted above – it is negative for any value of \( D_p \).

In the remainder of this section we analyze the eigenvalues for the rescaled thruster model \((2.7) \) with the P-control \((2.6) \). We denote by \( J \) the linearization of the right-hand side matrix of \((2.7) \) at the equilibrium point \((u, v, r, \psi) = (u_0, 0, 0, 0) \), which reads

\[
J = \begin{pmatrix}
2X_{u|u|}u_0 + \partial_u \tau(u_0) & 0 & 0 & 0 \\
0 & Y_\beta |u_0| & (Y_\gamma - m_L)u_0 + \tau(u_0)e_r & \tau(u_0)e_\psi' \\
0 & N_\beta u_0 & N_\gamma u_0 + x_T \tau(u_0)e_r & x_T \tau(u_0)e_\psi' \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then, \( F(v) = Jv + h(v) \), where \( h(v) = O(||v||^2) \) contains the quadratic and higher order terms; recall that due to the second order modulus terms, \( h \) is Lipschitz-continuous but not differentiable. The resulting linear part of \((2.8) \) is \( M^{-1}J \) (with upper left 3-by-3 block \( \tilde{S} \) from above) and has the form

\[
A := M^{-1}J = \begin{pmatrix}
p_{11} & 0 & 0 & 0 \\
0 & p_{22} & p_{23} & p_{24} \\
0 & p_{32} & p_{33} & p_{34} \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
(3.3)
\]

where

\[
p_{11} = (m + m_{uu})^{-1}(2X_{u|u|}u_0 + \partial_u \tau(u_0)),
\]

and we define the other matrix entries as follows, noting the dependencies on \( u_0, e_\psi', e_r \):

\[
p_{22} = p_{22a}u_0, \quad p_{23} = p_{23a}u_0 + q_{23} \tau(u_0)e_r, \quad p_{24} = q_{24} \tau(u_0)e_\psi',
\]

\[
p_{32} = p_{32a}u_0, \quad p_{33} = p_{33a}u_0 + q_{33} \tau(u_0)e_r, \quad p_{34} = q_{34} \tau(u_0)e_\psi',
\]
which we denote by \( P \) above, both cases, the real part of the traversing eigenvalues has non-zero derivative with respect to \( s \).

We analyze the eigenvalues of the linear part of (2.7) given by

\[
\begin{align*}
p_{22u} &= D^{-1}((I_z + m_{rr})Y_\beta - m_{rr}N_\beta), \\
p_{23u} &= D^{-1}((I_z + m_{rr})Y_\gamma - m_{rr}N_\gamma), \\
q_{23} &= D^{-1}(I_z + m_{rr} - m_{rr}x_T), \\
p_{32u} &= D^{-1}(-m_{rL}Y_\beta + (m + m_{rr})N_\beta), \\
p_{33u} &= D^{-1}(-m_{rL}Y_\gamma - m_{rr} + (m + m_{rr})N_\gamma), \\
q_{33} &= D^{-1}(-m_{rL} + (m + m_{rr})x_T).
\end{align*}
\]

We remark that the coefficient \( q_{23} = \) the same for \( p_{23} \) and \( p_{24} \) as well as \( q_{33} \) for \( p_{33} \) and \( p_{34} \). Further we define the following, which enter in the stability result:

\[
\begin{align*}
K_{11} &= q_{33}^2 \tau(u_0)^2, \\
K_{02} &= q_{33}(p_{32u}q_{23} - p_{22u}q_{33})u_0 \tau(u_0)^2, \\
K_{01} &= [(p_{22u} + p_{33u})(p_{32u}q_{23} - p_{22u}q_{33}) + q_{33}(p_{23u}p_{32u} - p_{22u}p_{33u})]u_0^2 \tau(u_0), \\
K_{10} &= [(p_{22u} + p_{33u})q_{33} + p_{32u}q_{23} - p_{22u}q_{33}]u_0 \tau(u_0), \\
K_{00} &= (p_{22u} + p_{33u})(p_{23u}p_{32u} - p_{22u}p_{33u})u_0^3.
\end{align*}
\]

With these preparations we can formulate our main result concerning the change of stability of the unstable straight motion equilibrium for the HTC values. This is a refinement of the result in [12], and implies ‘global controllability’ of the straight motion in the sense that stabilization by the P-control is possible along any direction in the control parameter space, i.e., the positive quadrant of the \((\epsilon_r, \epsilon_\psi)\)-plane.

**Theorem 3.1.** Consider the rescaled thruster model (2.7) with the HTC values and define

\[
\epsilon_\psi(\epsilon_r) := -\frac{K_{02}\epsilon_r^2 + K_{01}\epsilon_r + K_{00}}{K_{11}\epsilon_r + K_{10}},
\]

with \( K_{ij} \) from (3.4) and \( \epsilon_r, \epsilon_\psi \) the P-control parameters from (2.6). Fix any \( \epsilon_r, \epsilon_\psi \geq 0 \) and for \( s \geq 0 \) consider control parameters on the ray \((\epsilon_r, \epsilon_\psi)(s) := s \cdot (\epsilon_r, \epsilon_\psi)\). Then, as \( s \) increases, the equilibrium \((u_0, 0, 0, 0)\) of (2.7) is stabilized when \((\epsilon_r, \epsilon_\psi)(s)\) crosses the curve defined by (3.5). This crossing point lies at a unique \( s^* > 0 \) and for the eigenvalues of \( J \) the following holds: for \( \epsilon_\psi^* > 0 \), a complex pair of eigenvalues traverses the imaginary axis as \( s \) crosses \( s^* \), while for \( \epsilon_\psi = 0 \) one eigenvalue is fixed at zero and a simple real eigenvalue traverses zero at \( s = s^* \). In both cases, the real part of the traversing eigenvalues has non-zero derivative with respect to \( s \) at \( s = s^* \).

**Proof.** We analyze the eigenvalues of the linear part of (2.7) given by \( A \) in (3.3). As noted above, \( p_{11} < 0 \) for the HTC values, so that it suffices to consider the lower right \( 3 \times 3 \)-matrix, which we denote by \( P \). Its characteristic polynomial reads

\[
Q_P(\lambda) = \det(\lambda I - P) = \lambda^3 + c_2\lambda^2 + c_1\lambda + c_0,
\]

where

\[
\begin{align*}
p_{22u} &= D^{-1}((I_z + m_{rr})Y_\beta - m_{rr}N_\beta), \\
p_{23u} &= D^{-1}((I_z + m_{rr})Y_\gamma - m_{rr}N_\gamma), \\
q_{23} &= D^{-1}(I_z + m_{rr} - m_{rr}x_T), \\
p_{32u} &= D^{-1}(-m_{rL}Y_\beta + (m + m_{rr})N_\beta), \\
p_{33u} &= D^{-1}(-m_{rL}Y_\gamma - m_{rr} + (m + m_{rr})N_\gamma), \\
q_{33} &= D^{-1}(-m_{rL} + (m + m_{rr})x_T).
\end{align*}
\]
with $I$ the 3-by-3 identity matrix and

\[ c_0 = p_{34}p_{22} - p_{24}p_{32} = (p_{22a}q_{33} - p_{32a}q_{23})u_0 \tau(u_0)\epsilon_\psi, \]
\[ c_1 = p_{22}p_{33} - p_{23}p_{32} - p_{34} \]
\[ = (p_{22a}p_{33u} - p_{33a}p_{22u})u_0^2 + (p_{22a}q_{33} - p_{32a}q_{23})u_0 \tau(u_0)\epsilon_r - q_{33} \tau(u_0)\epsilon_\psi, \]
\[ c_2 = -p_{22} - p_{33} = -(p_{22a} + p_{33a})u_0 - q_{33} \tau(u_0)\epsilon_r. \]

The Routh-Hurwitz criterion, see [6], states that all eigenvalues of $P$ have negative real part if and only if $c_2, c_0 > 0$ and $c_2c_1 - c_0 > 0$. The first condition is always satisfied for the HTC values and $\epsilon_\psi > 0$ since for these $p_{22}, p_{33} < 0$, which implies $c_2 > 0$; and $p_{24} > 0$, $p_{32}, p_{34}, p_{22} < 0$ yield $c_0 > 0$. Concerning the second condition, using $K_{ij}$ from (3.4) we have

\[ c_2c_1 - c_0 = q_{33}^2 \tau(u_0)^2 \epsilon_\psi \epsilon_r + q_{33}(p_{32u}q_{23} - p_{22u}q_{33})u_0 \tau(u_0)^2 \epsilon_r^2 \]
\[ + \left[ (p_{22u} + p_{33u})(p_{32u}q_{23} - p_{22u}q_{33}) + q_{33}(p_{23u}p_{32u} - p_{22u}p_{33u}) \right] u_0^2 \tau(u_0)\epsilon_r \]
\[ + \left[ (p_{22u} + p_{33u})q_{33} + p_{32u}q_{23} - p_{22u}q_{33} \right] u_0 \tau(u_0)\epsilon_\psi \]
\[ + (p_{22u} + p_{33u})(p_{23u}p_{32u} - p_{22u}p_{33u})u_0^3 \]
\[ = K_{11} \epsilon_\psi \epsilon_r + K_{02} \epsilon_r^2 + K_{01} \epsilon_\psi + K_{10} \epsilon_\psi + K_{00}. \]

Hence, as claimed, precisely those $(\epsilon_r, \epsilon_\psi)$ ‘above’ the convex curve defined by $c_2c_1 - c_0 = 0$ or equivalently (3.5), provide eigenvalues with negative real part.

In addition, for the control values satisfying (3.5) and $\epsilon_\psi(\epsilon_r) \neq 0$, it follows that there exist a pair of complex conjugates with vanishing real part. Indeed, for the curve $c_2c_1 - c_0 = 0$ the characteristic polynomial can be factorized as $Q_p(\lambda) = (c_2\lambda^2 + c_0) \left( \frac{1}{c_2} \lambda + 1 \right)$, and since $c_2, c_0 > 0$, the eigenvalues of the first factor correspond to a pair of purely complex conjugates, $\lambda_{\pm} = \pm \sqrt{-\frac{c_0}{c_2}}$.

Finally, if $\epsilon_\psi = 0$, then the last column of the matrix (3.3) vanishes, and $c_0 = 0$, so it has a fixed zero eigenvalue and the conditions for the other eigenvalues to have positive real parts are $c_2 > 0$, which holds for the HTC, and $c_1 > 0$. The latter occurs for $\epsilon_r$ larger than the root $\epsilon_{r_1} > 0$ of $c_1$, where the characteristic polynomial reads $Q_p(\lambda) = \lambda^2(\lambda + c_2)$, with a double zero eigenvalue, as we wanted to prove.

We plot the curve defined by (3.5) in Fig. 5 In agreement with the theorem statement, we numerically find that the linearization in the equilibrium, $A$, on this curve possesses two real negative eigenvalues and a pair of complex conjugates with zero real part. Furthermore, in the shaded region below the curve, two eigenvalues have positive real part while in the white region beyond this curve, the real part of all eigenvalues is negative. Thus, one expects a Hopf bifurcation occurs at $(\epsilon_r, \epsilon_\psi)$ satisfying (3.5). However, as mention before, for this non-smooth system, the analysis is more delicate than usual.

Next we discuss the impact of varying the parameters $n_p$, $D_p$ and $x_T$ on the stabilization and thus linear controllability of the straight motion.
Equation (3.5) thus becomes but from (3.2) we have \( \hat{\varepsilon} \) proportional to \( n \) 

\[ K \]

where \( \hat{\varepsilon} \) can be removed from the uncontrolled system by rescaling. Here we consider the dependence 

\[ \frac{D_{\text{stability}}}{\text{plane, in terms of the unscaled control parameters}}, \text{i.e., corresponding to (2.4) rather than (2.6).} \]

The straight motion is unstable in the colored area below the curve, and it is stable in the white region. The transition to stability for \( \varepsilon_{\psi} > 0 \) goes via a complex conjugate pair of eigenvalues, and the transition for \( \varepsilon_{\psi} = 0 \) by a real eigenvalue.

### 3.2.1 Impact of changing the parameters \( n_p \) and \( D_p \)

First, we briefly discuss the influence of the propeller frequency \( n_p \). From §2 we recall that \( n_p \) can be removed from the uncontrolled system by rescaling. Here we consider the dependence on \( n_p \) of \( \varepsilon_{\psi}(n_p) \), i.e., of \( K_{ij} \) in (3.4). As in §2 we scale \( u = n_p \hat{u} \) and \( \tau(u) = n_p^2 \hat{\tau}(\hat{u}) \). Note that all \( g_{ij} \) are multiplied by \( \tau \) and thus \( K_{11} = \tau'(\tau^2), K_{02} = \tau'(u^2 \tau^2), K_{01} = \tau'(u^2 \tau), K_{10} = \tau'(u \tau), K_{00} = \tau'(u^3) \). This yields

\[ \varepsilon_{\psi}(n_p; \hat{u}) = -\frac{\hat{K}_{02} n_p^2 \hat{u}^2 \overline{\varepsilon_r} + \hat{K}_{01} n_p \hat{u}^2 \tau_{\psi} + \hat{K}_{00} \hat{u}^3}{\hat{K}_{11} n_p \tau^2 \overline{\varepsilon_r} + \hat{K}_{10} \hat{u} \hat{\tau}}, \]

where \( \hat{K}_{ij} \) are real constants. Hence, \( \varepsilon_{\tau} \) should be scaled as \( \hat{\varepsilon}_{\tau} = n_p \varepsilon_{\tau} \) so that \( \varepsilon_{\tau} \) is inversely proportional to \( n_p \), while \( \varepsilon_{\psi}(0; \hat{u}_0; n_p) \) is independent of \( n_p \).

Concerning \( D_p \), we scale as in §3.1 \( u = D_p \hat{u} \) and \( \tau(u) = D_p^4 \hat{\tau}(\hat{u}) \), so \( \hat{\tau} \) is independent of \( D_p \) but from (3.2) we have \( \hat{u}_0 \neq 0 \) as \( D_p \rightarrow \infty \), where \( \hat{u}_0 \) is independent of \( D_p \). Equation (3.5) thus becomes

\[ \varepsilon_{\psi}(\varepsilon_{\tau}, \hat{u}; D_p) = -\frac{\hat{K}_{02} D_p^6 \hat{u}^2 \overline{\varepsilon_r} + \hat{K}_{01} D_p^3 \hat{u}^2 \tau_{\psi} + \hat{K}_{00} \hat{u}^3}{\hat{K}_{11} D_p^5 \tau^2 \overline{\varepsilon_r} + \hat{K}_{10} D_p^2 \hat{u} \hat{\tau}}, \] (3.6)

where \( \hat{K}_{ij} \) are real constants. Setting \( E := \hat{\tau} D_p^3 \varepsilon_{\tau}, \gamma := \hat{\tau} D_p^2 \) gives

\[ \varepsilon_{\psi}(E, \gamma) = -\frac{\hat{K}_{02} \hat{u}^2 E + \hat{K}_{01} \hat{u}^2 E + \hat{K}_{00} \hat{u}^3}{(\hat{K}_{11} E + \hat{K}_{10} \hat{u}) \gamma}, \]

which has the same functional form with same signs of coefficients as (3.5). Therefore, the stability boundary is qualitatively the same for different \( D_p \). Specifically, on the one hand, for \( \varepsilon_{\tau} = 0 \) we get from (3.6) that

\[ \varepsilon_{\psi}(0; \hat{u}; D_p) = -\frac{\hat{K}_{00} \hat{u}^2}{\hat{K}_{10} D_p^2 \hat{\tau}} = \tau^2(D_p \hat{\tau}). \]
On the other hand, \( \varepsilon_\psi = 0 \) in (3.6) gives a function

\[
\varepsilon_r(u) = \varepsilon_r(\tilde{u}; D_p) = \frac{\tilde{u}}{D_p^3} \left( -\tilde{K}_{01} \pm \sqrt{\tilde{K}_{01}^2 - 4 \tilde{K}_{02} \tilde{K}_{00}} \right),
\]

and \( \tilde{u}_0(D_p) \rightarrow \tilde{u}_0^* \neq 0 \) implies \( \varepsilon_r(\tilde{u}_0; D_p) = \mathcal{O}(D_p^{-3}) \). Hence,

\[
u_0 \varepsilon_r(u_0) = D_p \tilde{u}_0 \varepsilon_r(\tilde{u}_0; D_p) = \mathcal{O}(D_p^{-2})
\]

and particularly, if \( D_p = D_f \cdot D_p^{\text{HTC}} \), where \( D_p^{\text{HTC}} = 6.105 \) is the value of the propeller diameter in the HTC, then for \( D_f \gg 1 \) we have \( u_0 > u_0^{\text{HTC}} \) and \( \varepsilon_r(u_0) < \varepsilon_r(u_0^{\text{HTC}}) \), where \( u_0^{\text{HTC}} \approx 8.71 \) and \( \varepsilon_r(u_0^{\text{HTC}}) \approx 130.13 \), which means that for larger propeller diameter \( D_p \), it is ‘easier’ to stabilize the straight motion by \( \varepsilon_r \). In fact, numerically the latter holds for all \( D_p \). Indeed, for small \( D_p \) we have \( u_0 = \mathcal{O}(D_p^2) \), see Fig. 4(a), and arguing similarly as above, this gives the relation

\[
\frac{u_0}{u_0^{\text{HTC}}} \approx \frac{\varepsilon_r(u_0^{\text{HTC}})}{\varepsilon_r(u_0)},
\]

which holds in a relatively broad range. For example, if \( D_p = D_f \cdot D_p^{\text{HTC}} \), then taking \( D_p = 1.5D_p^{\text{HTC}} = 9.1575 \), we get \( u_0 \approx 16.57 \) and thus \( \varepsilon_r(u_0) \approx 68.40 \), yielding

\[
\frac{u_0}{u_0^{\text{HTC}}} \approx \frac{\varepsilon_r(u_0^{\text{HTC}})}{\varepsilon_r(u_0)} \approx 1.9025;
\]

while for \( D_p = 0.5D_p^{\text{HTC}} = 3.0525 \), we obtain \( u_0 \approx 2.70 \) and \( \varepsilon_r(u_0) \approx 419.14 \), which gives

\[
\frac{u_0}{u_0^{\text{HTC}}} \approx \frac{\varepsilon_r(u_0^{\text{HTC}})}{\varepsilon_r(u_0)} \approx 0.31046.
\]

3.2.2 Impact of changing the thruster position parameter \( x_T \)

As shown next, understanding the impact of the thruster position is more involved. It turns out that for the HTC, when increasing \( x_T \) from the default value near \(-0.49\), i.e., placing the thruster more forwards, beyond \( x_T \approx -0.38 \) we lose the ability to stabilize the straight motion by \( \varepsilon_\psi \) alone: there is \( \varepsilon_\psi^* > 0 \) such that for \( \varepsilon_r \in [0, \varepsilon_r^*] \) there is no \( \varepsilon_\psi \) that stabilizes. Moreover, beyond \( x_T \approx 0.17 \) the P-control cannot stabilize the straight motion at all, thus creating an ‘uncontrollable’ situation.

Before preparing the precise statement and proof, we first note that while (3.1) does not depend on \( x_T \), the function (3.5) does. In fact, all \( q_{ij} \) in (3.4) depend on \( x_T \), which results in \( K_{02}, K_{11} \) being quadratic in \( x_T \), and \( K_{01}, K_{10} \) linear; only \( K_{00} \) does not depend on \( x_T \). However, it is not clear that the graph of \( \varepsilon_\psi(\varepsilon_r; x_T) \) remains a stability boundary since this only accounts for one of the criteria for stable eigenvalues. Indeed, the first condition for the Routh-Hurwitz criterion in the proof of Theorem 3.1 requires \( c_0 \geq 0 \) and \( c_2 > 0 \), but for instance at \( x_T = 0.3 \) we have \( \varepsilon_\psi(\varepsilon_r; x_T) < 0 \) for \( \varepsilon_r < \varepsilon_r^2 \approx 144.47 \), and \( c_2 < 0 \) for \( \varepsilon_r > \varepsilon_r^2 \), where \( \varepsilon_r^2 \) is the unique root of \( c_2 \); see Fig. 8(a). Therefore, in this case the curve \( \varepsilon_\psi(\varepsilon_r; x_T) \) is not a stability boundary, and the straight motion cannot be stabilized by the P-control (2.6).

In the following we provide a detailed, and somewhat tedious, analysis that explains all possibilities to stabilize the straight motion depending on \( x_T \). Recall that \( u_0 \) does not depend on \( x_T \),
but for the coefficients $c_0, c_1, c_2$ of the eigenvalue problem we have

\[
\begin{align*}
c_0 &= p_{34}p_{22} - p_{24}p_{32} = (\tilde{\alpha} + \tilde{\beta}x_T)\epsilon_y, \\
c_1 &= p_{22}p_{33} - p_{23}p_{32} - p_{34} = \delta + (\tilde{\alpha} + \tilde{\beta}x_T)\epsilon_r + (\alpha + \beta x_T)\epsilon_y, \\
c_2 &= -p_{22} - p_{33} = \gamma + (\alpha + \beta x_T)\epsilon_r,
\end{align*}
\]

where

\[
\begin{align*}
\alpha &:= D^{-1}m_{rv}\tau(u_0) > 0, \\
\beta &:= -D^{-1}(m + m_{rv})\tau(u_0) < 0, \\
\tilde{\alpha} &:= -D^{-1}[m_{rv}p_{22u} + (I_z + m_{rr})p_{32u}]u_0\tau(u_0) > 0, \\
\tilde{\beta} &:= D^{-1}[(m + m_{rv})p_{22u} + m_{rr}p_{32u}]u_0\tau(u_0) < 0, \\
\gamma &:= D^{-1}[m_{rr}N_{r} - (I_z + m_{rr})Y_{r} - (m + m_{rv})N_{r} + m_{rr}(Y_{r} - m_{L})]u_0 > 0, \\
\delta &:= D^{-1}[(m + m_{rv})N_{r} - m_{rv}(Y_{r} - m_{L})]p_{22u} + (m_{rr}N_{r} - (I_z + m_{rr})(Y_{r} - m_{L}))p_{32u}]u_0^2 < 0.
\end{align*}
\]

The signs of all these coefficients are given for the values of the HTC and remain the same for any $D_p > 0$ due to (2.3). Furthermore, the singularity of $\epsilon_y(\epsilon_r; x_T)$ can be written as

\[
\epsilon_r^* = \frac{\tilde{\alpha} - \gamma \alpha + (\tilde{\beta} - \gamma \beta)x_T}{(\alpha + \beta x_T)^2};
\]

see Fig. [7]. For the HTC values, large negative values of $x_T$ give negative $\epsilon_r^* < 0$, a sign change occurs at $x_T \approx -0.38$, and a singularity near $x_T = 0.016$ takes place. In particular, $\epsilon_r^* < 0$ for the HTC default value $x_T \approx -0.49$. As long as $\epsilon_r^* < 0$, the singularity lies outside the positive range of the control parameters and is thus disregarded. However, for values of $x_T$ where $\epsilon_r^* > 0$, stabilization by $\epsilon_y$ alone is not possible and fixing $\epsilon_r < \epsilon_r^*$ creates an ‘uncontrollable’ situation. We plot an example in Fig. [6](a). Furthermore, for $\epsilon_r^* > 0$, there may be two positive values of $\epsilon_r$ for which $\epsilon_y(\epsilon_r; x_T) = 0$, i.e., potentially a stabilization and subsequent destabilization when increasing $\epsilon_r$ from zero. We plot an example in Fig. [8](a). In addition, comparing Figs. [6](a) and [8](a), a switching from convexity to concavity of $\epsilon_y(\epsilon_r; x_T)$ has occurred, which in particular takes place at $x_T \approx -0.17$, cf. Fig. [7](b,c).

With these preparations we can formulate the following main result concerning controllability in terms of $x_T$. This involves the threshold for $x_T$ given by

\[
x_{T_s} = \frac{\delta \alpha - \gamma \tilde{\alpha}}{\gamma \tilde{\beta} - \delta \beta} > 0,
\]

which is independent of $u_0$ and therefore of $D_p$. Specifically, it is the unique value of $x_T$ for which the roots $\epsilon_r = \epsilon_{r_{j}}$ of $c_{j}$, $j = 1,2$, at fixed $\epsilon_y = 0$ coincide. For the HTC values we have $x_{T_s} \approx 0.17$. As before, the stability region and its boundary refer to the positive quadrant in the $(\epsilon_r, \epsilon_y)$-plane.
Theorem 3.2. Consider $x_T$ as a free parameter and assume for all other parameters the same signs of coefficients within $c_0, c_1, c_2$ as above for the HTC values. Then two scenarios occur. In case $-\alpha/\beta < -\tilde{\alpha}/\tilde{\beta}$, for any $x_T \leq -\alpha/\beta$ the statement of Theorem 3.1 holds true for all $\varepsilon_r > 0$, and the stability boundary is a strictly monotone curve intersecting the $\varepsilon_r$-axis, but possibly not the $\varepsilon_\psi$-axis with a vertical asymptote at $\varepsilon_r^\ast$. Moreover, if $x_T \geq x_{T^\ast}$, the straight motion cannot be stabilized by any $\varepsilon_r, \varepsilon_\psi > 0$, while for $-\alpha/\beta < x_T < x_{T^\ast}$, the stability region is bounded with boundary of parabolic shape, intersecting twice the $\varepsilon_\psi$-axis. In case $-\tilde{\alpha}/\tilde{\beta} < -\alpha/\beta$, the stability boundary is strictly monotone if $x_T < -\tilde{\alpha}/\tilde{\beta}$ and otherwise stabilization is impossible.

Before presenting the proof we proceed with some remarks. The theorem in particular implies the following alternative: either the straight motion can be stabilized by increasing $\varepsilon_r$ for all $\varepsilon_\psi > 0$ or none, except in the case of a bounded stability region. In all cases, the possibility to stabilize is determined by the case $\varepsilon_\psi = 0$ alone. Specifically, in the case $-\alpha/\beta < x_T < -\tilde{\alpha}/\tilde{\beta}$, which occurs in the HTC, if the longitudinal location of the thruster of the vessel, $x_T$, is too far from the aft, i.e., near the fore, then the straight motion cannot be stabilized by the P-control (2.6). On the other hand, in the case $-\tilde{\alpha}/\tilde{\beta} < x_T < -\alpha/\beta$, stabilizing is easier the closer the thruster is to the aft, though we do not know whether this case can be realized in an actual ship.

Proof of Theorem 3.2 We recall that the Routh-Hurwitz criterion from the proof of Theorem 3.1 consists of $c_0, c_2 > 0$ and $c_2 c_1 > c_0$, which must be attainable for stabilization in case $\varepsilon_\psi > 0$; for $\varepsilon_\psi = 0$ we have $c_0 = 0$ and the criterion becomes $c_1, c_2 > 0$. The common condition $c_2 > 0$ is independent of the control parameter $\varepsilon_\psi$ and equivalent to either $x_T \leq -\alpha/\beta$ (and any $\varepsilon_r \geq 0$), or $\varepsilon_r < \varepsilon_r^2 = -\gamma/(-\alpha/\beta x_T)$.

Concerning $\varepsilon_\psi = 0$, the sign conditions $\tilde{\alpha} > 0$ and $\tilde{\beta} < 0$ yield $c_1 > 0$ if and only if $x_T < -\tilde{\alpha}/\tilde{\beta}$ and $\varepsilon_r > \varepsilon_r^\ast = -\tilde{\gamma}/(-\tilde{\alpha} + \tilde{\beta} x_T)$. In particular, only $\varepsilon_r^\ast > 0$ is relevant. In case $x_T > -\alpha/\beta$, the two bounds on $\varepsilon_r$ are compatible if and only if $\varepsilon_r < \varepsilon_r^\ast$, which is equivalent to $x_T < x_{T^\ast}$. Hence, we find that controllability for $\varepsilon_\psi = 0$ requires

$$x_T < \min\left\{-\tilde{\alpha}/\tilde{\beta}, \max\{-\alpha/\beta, x_{T^\ast}\}\right\},$$

which covers the claimed statements in this case. (In particular, for the HTC values we have $x_{T^\ast} = (\delta \alpha - \gamma \tilde{\alpha})/(\gamma \beta - \delta \tilde{\beta}) \approx 0.17$, and $-\tilde{\alpha}/\tilde{\beta} \approx 0.83$, so the threshold would be $x_T = x_{T^\ast}$.)
We plot relevant quantities as functions of $x_T$, we only consider $\epsilon_1 < 0$ for brevity. In case $\alpha + \beta x_T = 0$, we have $\epsilon_\psi(\epsilon_r) = \gamma \delta / (\tilde{\alpha} + \beta x_T) + \gamma \epsilon_r$ with positive slope and root at $\epsilon_{r_1}$. This is the stability boundary since in this case $c_2 > 0$ and $c_0 > 0$ for $\epsilon_\psi > 0$. For $\alpha + \beta x_T \neq 0$ the value $\epsilon_{r_2} = -\gamma / (\alpha + \beta x_T)$ is finite and we can write

$$\epsilon_\psi(\epsilon_r) = \frac{(\epsilon_r - \epsilon_{r_2})(\epsilon_r - \epsilon_{r_1})}{\epsilon_r - \epsilon^*_r}, \quad a = -\frac{\tilde{\alpha} + \beta x_T}{\alpha + \beta x_T},$$

with $\epsilon^*_r = \epsilon_{r_2} + (\tilde{\alpha} + \beta x_T)/(\alpha + \beta x_T)^2$, so that $\epsilon^*_r > \epsilon_{r_2}$. The sign of $a$ is that of $\alpha + \beta x_T$ since we only consider $x_T < -\tilde{\alpha}/\tilde{\beta}$. Notably, the graph consists of one convex and one concave branch and at least one intersects $\epsilon_\psi = 0$ at $\epsilon_r = \epsilon_{r_1} > 0$. On the one hand, we directly compute

$$\partial_{\epsilon_r} \epsilon_\psi(\epsilon_{r_1}) = \frac{(\epsilon_{r_1} - \epsilon_{r_2})(\epsilon_{r_1} - \epsilon^*_r)}{(\epsilon_{r_1} - \epsilon^*_r)^2},$$

which has the sign of $-(\alpha + \beta x_T)(\epsilon_{r_1} - \epsilon_{r_2})(\epsilon_{r_1} - \epsilon^*_r)$. On the other hand, via $F := c_1 c_2 - c_0 = 0$, that defines $\epsilon_\psi(\epsilon_r)$, we have $\partial_{\epsilon_r} \epsilon_\psi(\epsilon_r) = -\partial_{\epsilon_r} F / \partial_{\epsilon_\psi} F$ and $\epsilon_r = \epsilon_{r_2}$ gives $\partial_{\epsilon_r} F = (\partial_{\epsilon_r} c_2)c_1$, with $\partial_{\epsilon_r} c_2 = \alpha + \beta x_T$, and $\partial_{\epsilon_\psi} F = -\partial_{\epsilon_\psi} c_0 = -(\tilde{\alpha} + \beta x_T)$. Since $\tilde{\alpha} + \beta x_T > 0$, we have

$$\text{sgn}(\partial_{\epsilon_r} \epsilon_\psi(\epsilon_{r_2})) = \text{sgn}((\alpha + \beta x_T)c_1).$$

Due to the conditions $\epsilon_{r_1} > 0$ and $\epsilon_{r_2} < \epsilon^*_r$ from above, we have six possible orderings of $\epsilon_{r_1}, \epsilon_{r_2}, \epsilon^*_r$ and $\epsilon_r = 0$ which we present below. In addition, we discuss the relation of these cases to the value of $x_T$. For that, we define $x_{T_0} = (\gamma \alpha - \tilde{\alpha})/(\tilde{\beta} - \gamma \beta)$, which solves $\epsilon^*_r(x_{T_0}) = 0$, and $x_{T_-}, x_{T_+}$ as the roots of $\epsilon^*_r - \epsilon_{r_1}$, where $x_{T_-} < x_{T_+}$. From the sign conditions, note that $\gamma \alpha - \gamma \tilde{\alpha} < 0$ and $\gamma \tilde{\beta} - \delta \beta < 0$. Furthermore, it turns out that relevant is the sign of $\alpha / \beta - \tilde{\alpha} / \tilde{\beta}$. 

**Figure 7:** We plot relevant quantities as functions of $x_T$, and mark the HTC value $x_T = -0.49429$ (red vertical line) as well as the region $x_T > -\tilde{\alpha}/\tilde{\beta}$ (shaded area). Orange curves: $\epsilon^*_r(x_T)$ from (3.7) for the singularity of $\epsilon_\psi(\epsilon_r;x_T)$; blue curves: $\epsilon_{r_1}(x_T)$, the root of $c_1$ at $\epsilon_\psi = 0$; green curves: $\epsilon_{r_2}(x_T)$, the root of $c_2$. 

| $x_T$ | $\epsilon_{r_1}$ | $\epsilon_{r_2}$ | $\epsilon^*_r$ |
|------|-----------------|-----------------|--------------|
| -2   | 0               | -4              | -4           |
| 0    | -2              | -2              | -2           |
| 1    | 1               | 1               | 1            |
| 2    | 2               | 2               | 2            |
We first consider $-\alpha/\beta < -\bar{\alpha}/\bar{\beta}$, in which $\varepsilon_{r_1}(x_{T_r}) = \varepsilon_{r_1}(x_{T_l}) > 0$; see Fig. 7 for the HTC values. Here $x_{T_0} < x_{T_-} < x_{T_0} < x_{T_+}$ applies. Indeed, since $\varepsilon^*_r$ vanishes for a unique $x_{T_0}$, left and right of this point the function is negative and positive, respectively, and $\varepsilon_{r_1} > 0$ is monotone increasing, so that $x_{T_0} < x_{T_-}$ holds. Note that $\forall x_T < -\bar{\alpha}/\bar{\beta}$, we have $\varepsilon^*_r > \varepsilon_{r_2}$, and therefore $x_{T_0} < x_{T_-}$. Finally, again since $\varepsilon_{r_1} > 0$ is monotone increasing, $x_{T_-} < x_{T_0}$ holds.

Case (1). $\varepsilon_{r_2} < \varepsilon^*_r < 0 < \varepsilon_{r_1}$ (which occurs in the HTC). Here $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) < 0$ and the graph of $\varepsilon_\psi(\varepsilon_r) \geq 0$ connects the $\varepsilon_r$- and $\varepsilon_\psi$-axes, is strictly decreasing and forms the stability boundary as in Fig. 5. This case is equivalent to $x_T < x_{T_0}$ if $\bar{\beta} > \gamma\beta$, or $x_T > x_{T_0}$ if $\bar{\beta} < \gamma\beta$, since then $\varepsilon^*_r < 0$. However, as for the HTC values, we assume $\bar{\beta} > \gamma\beta$ and thus this case corresponds to $x_T < x_{T_0}$.

Case (2). $0 < \varepsilon_{r_2} < \varepsilon^*_r < \varepsilon_{r_1}$. Here $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) < 0$ and again the graph of $\varepsilon_\psi(\varepsilon_r) \geq 0$ is strictly decreasing and forms the stability boundary, but it no longer connects to the $\varepsilon_\psi$-axis. More precisely, $\varepsilon^*_r$ is the lower bound for the stability threshold $\varepsilon_r$ at any given $\varepsilon_\psi > 0$. The situation is as in Fig. 6(a). Regarding $x_T$ we consider, as before, $\bar{\beta} > \gamma\beta$, and we then require $x_T < -\alpha/\beta$ such that $\varepsilon_{r_2} > 0$. Moreover, $x_T \in (x_{T_-}, x_{T_+})$ for $\varepsilon_{r_1} < \varepsilon^*_r$. Thus, $x_T \in (x_{T_-}, -\alpha/\beta)$. Notice that at $\varepsilon_r = \varepsilon_{r_1} = \varepsilon^*_r$, i.e., at $x_T = x_{T_-}$, the graph of the function $\varepsilon_\psi(\varepsilon_r)$ degenerates to the union of a vertical and a non-vertical line; cf. Figs. 6(b,c).

Case (3). $0 < \varepsilon_{r_1} < \varepsilon_{r_2} < \varepsilon^*_r$. Here $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) > 0 > \partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_2})$, the graph of $\varepsilon_\psi(\varepsilon_r) \geq 0$ is strictly increasing near $\varepsilon_{r_1}$ with a local maximum below $\varepsilon_{r_2}$, and this part forms the stability boundary to a bounded stable region since $c_2 > 0$ between both roots; see Fig. 8(c). In terms of $x_T$, this case is translated into $x_T \in (-\alpha/\beta, x_{T_+})$. The transition between case 3 and 4 occurs at $x_T = -\alpha/\beta$, where we have confirmed controllability in this situation above.

The following last two cases have $x_T \geq x_{T_0}$ and correspond to fully uncontrollable situations.

Case (5). $0 < \varepsilon_{r_2} < \varepsilon_{r_1} < \varepsilon^*_r$. Here $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) < 0 < \partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_2})$, the graph of $\varepsilon_\psi(\varepsilon_r) \geq 0$ is strictly increasing near $\varepsilon_{r_2}$ with a local maximum below $\varepsilon_{r_1}$, but it does not form a stability boundary since $c_2 < 0$ for $\varepsilon_r > \varepsilon_{r_2}$. This situation happens for $x_T \in (x_{T_-}, x_{T_+})$. The transition between case 4 and 5 takes place at $x_T = x_{T_r}$, which yields $\varepsilon_{r_1} = \varepsilon_{r_2}$ and thus $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) = 0$. The graph is negative or concave and therefore no control is possible; see Fig. 8(b).

Case (6). $0 < \varepsilon_{r_2} < \varepsilon^*_r < \varepsilon_{r_1}$. Here $\partial_{\varepsilon_r}\varepsilon_\psi(\varepsilon_{r_1}) > 0$ and the graph of $\varepsilon_\psi(\varepsilon_r) \geq 0$ is strictly increasing, but no longer forms a stability boundary since $c_2 < 0$ in the region where $\varepsilon_\psi(\varepsilon_r) > 0$; see Fig. 8(a). It is easy to see that this case is equivalent to $x_T \in (x_{T_-}, -\bar{\alpha}/\bar{\beta})$. For $\varepsilon_r = \varepsilon_{r_1} = \varepsilon^*_r$, i.e., at $x_T = x_{T_0}$, again the graph of the function $\varepsilon_\psi(\varepsilon_r)$ degenerates to the union of a vertical line in the region where $c_2 < 0$, and a line with positive slope and root at $c_2 = 0$.

Now we consider the second situation, $-\bar{\alpha}/\bar{\beta} < -\alpha/\beta$, where $\varepsilon_{r_1}(x_{T_r}) = \varepsilon_{r_2}(x_{T_l}) < 0$. The scenario is analogous to the previous cases 1 to 3 when accounting for the reversed ordering $x_{T_-} < x_{T_0} < x_{T_0} < x_{T_+}$. Indeed, case 1 happens for $x_T \in (x_{T_0}, -\bar{\alpha}/\bar{\beta})$, case 2 for $x_T \in (x_{T_-}, x_{T_0})$,
\[ \dot{\bar{u}} = -\frac{m_L v r + X_H(\bar{u}) + \bar{\tau}(\bar{u}) \cos \eta}{-m_L (u_0 + \bar{u}) r + Y_H(\bar{u}) + \bar{\tau}(\bar{u}) \sin \eta} \]

where \( \bar{\tau}(\bar{u}) := \tau(u_0 + \bar{u}) \). In the following we omit the tilde from \( \bar{u} \) to simplify the notation. We rewrite, expand in \( u \) and, based on the results in [11], already omit all cubic order terms, which gives

\[ \begin{pmatrix} \dot{u} \\ \dot{\psi} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \tau_{11} (\cos \eta - 1) + k_1 u + k_2 u^2 + k_3 v r + [\tau_{12} u + \tau_{13} u^2] \cos \eta \\ p_{22} v + k_5 r + k_6 u v + k_7 u r + f_1 (v, r) + [\tau_{21} + \tau_{22} u + \tau_{23} u^2] \sin \eta \\ p_{32} v + k_9 r + k_{10} u v + k_{11} u r + f_2 (v, r) + [\tau_{31} + \tau_{32} u + \tau_{33} u^2] \sin \eta \end{pmatrix} r \]

where \( \tau_{ij} \) are the coefficients resulting from the expansion and

\[ f_1 (v, r) = a_{11} v |v| + a_{12} v |r| + a_{21} r |v| + a_{22} r |r|, \]
\[ f_2 (v, r) = b_{11} v |v| + b_{12} v |r| + b_{21} r |v| + b_{22} r |r|, \]
with coefficients
\[
\begin{align*}
a_{11} &= D^{-1}((I_z + m_{rr})Y_{\beta}\beta - m_{vr}N_{\beta}\beta), \\
a_{12} &= D^{-1}(I_z + m_{rr})Y_{\beta}\gamma', \\
a_{21} &= D^{-1}((I_z + m_{rr})Y_{\beta}\gamma), \\
a_{22} &= D^{-1}((I_z + m_{rr})Y_{\gamma}\gamma' - m_{vr}N_{\gamma}\gamma'), \\
b_{11} &= D^{-1}((m + m_{vv})N_{\beta}|\beta| - m_{rv}Y_{\beta}|\beta|), \\
b_{12} &= -D^{-1}m_{rv}Y_{\beta}|\beta|, \\
b_{21} &= -D^{-1}m_{rr}Y_{\beta}|\beta|, \\
b_{22} &= D^{-1}((m + m_{vv})N_{\gamma}|\gamma| - m_{rv}Y_{\gamma}|\gamma|).
\end{align*}
\]

Using again that our analysis focuses on the vicinity of the origin, we expand the functions \(\cos \eta = 1 + O(\eta^2)\), \(\sin \eta = \eta + O(\eta^3)\) and omit higher order terms. Using also the control form \(\eta = \epsilon_r r + \epsilon_\psi \psi\) from (2.6) this reduces (3.9) to
\[
\begin{pmatrix}
    \dot{u} \\
    \dot{v} \\
    \dot{\psi}
\end{pmatrix} = 
\begin{pmatrix}
    p_{11}u + U(u, v, r, \psi) \\
    p_{22}v + p_{23}r + p_{24} \psi + k_6 uv + (k_7 + \tau_{22} \epsilon_r)ur + \tau_{22} \epsilon_\psi u \psi + f_1(v, r) \\
    p_{32}v + p_{33}r + p_{34} \psi + k_{10} uv + (k_{11} + \tau_{32} \epsilon_r)ur + \tau_{32} \epsilon_\psi u \psi + f_2(v, r)
\end{pmatrix},
\]
where \(U(u, v, r, \psi)\) is a second order nonlinear function and the coefficients correspond to \(p_{11} = k_1 + \tau_{12}, p_{23} = k_2 + \tau_{21} \epsilon_r, p_{24} = \tau_{21} \epsilon_\psi, p_{33} = k_6 + \tau_{31} \epsilon_r, p_{34} = \tau_{31} \epsilon_\psi\). We recall the entries of the matrix \(A\) from (3.3) and notice that these \(p_{ij}\) are exactly the same as before since we have moved the fixed point to the origin and expanded the function \(r\).

As a first step towards the nonlinear analysis, we discuss the simpler case of the pitchfork bifurcation, and then turn to the more involved Hopf bifurcation analysis.

### 3.3.1 Pitchfork Bifurcation for \(\epsilon_\psi = 0\)

We keep \(\epsilon_\psi = 0\) fixed so that the last equation in (3.10) can be dropped. The linearization in the steady forward motion gives the matrix \(A\) from (3.3) reduced to the 3-by-3 upper left matrix, with block diagonal structure and hence an eigenvalue \(p_{11} \neq 0\) for the HTC values. The remaining 2-by-2 block, which we call \(B\), has determinant that is linear in \(\epsilon_r\) and vanishes at the unique positive value \(\epsilon_{r_1}\) from (3.3), as in the proof of Theorem 3.1. We readily find numerically that for the HTC values \(B_0 := B|_{\epsilon_r=\epsilon_{r_1}}\) has eigenvectors \(e_0\) for the eigenvalue zero, and \(e_1\) for the non-zero eigenvalue \(\lambda\). This means that the bifurcation upon changing \(\epsilon_r\) will be purely of steady states and we thus seek solutions of the reduced (3.10) with zero left-hand side. Since \(f_1, f_2\) are quadratic of second order modulus type, we expect a non-smooth pitchfork bifurcation as in the truncated normal form \(\sigma x|x| + \tilde{\epsilon}_r x = 0\), where the sign of \(\sigma \neq 0\) determines the super- or subcritical character of the bifurcation. Here, \(\tilde{\epsilon}_r = g(\epsilon_r)\) with a local bijection \(g\).

The first steady state equation of the reduced (3.10) can be solved for \(u\) by the implicit function theorem since \(p_{11} \neq 0\) and \(U\) is nonlinear; the resulting solution satisfies \(u = O(v^2 + r^2)\) so that substitution into the second and third equations contributes a term of cubic order.

We write \(\epsilon_r = \epsilon_{r_1} + \tilde{\epsilon}_r\) and \(B = B_0 + \tilde{\epsilon}_r B_1\); note that \(B_1\) has vanishing first column. Next we choose the eigenvectors \(e_j^*\) of the adjoint \(B_0^\dagger\) so that \(\langle e_j, e_j^* \rangle = 0\) and \(\langle e_j, e_j^* \rangle = 1\), \(j = 1, 2\). Changing coordinates \((v, r) = xe_0 + ye_1\) we project the second and third equations onto \(\text{span}(e_1)\), which results in
\[
0 = \langle (A_0 + \tilde{\epsilon}_r A_1)(xe_0 + ye_1), e_1^* \rangle + \langle f(v, r), e_1^* \rangle = \lambda y + \tilde{\epsilon}_r (x\langle A_1 e_0, e_1^* \rangle + y\langle A_1 e_1, e_1^* \rangle) + \langle f(v, r), e_1^* \rangle.
\]
Figure 9: Different views of the non-smooth pitchfork bifurcation diagram of equilibrium points in the 3D reduced system for $\varepsilon_\psi = 0$ computed by numerical continuation.

Once more, we may solve by the implicit function theorem since $\lambda \neq 0$, which gives $y = y(\tilde{e}_r,x) = \mathcal{O}(|\tilde{e}_r x|^2)$. It remains to solve the projection onto $\text{span}(e_0)$, which is given by

$$0 = \langle (A_0 + \tilde{e}_r A_1)(xe_0 + ye_1), e_0^* \rangle + \langle f(v,r), e_0^* \rangle$$
$$= \tilde{e}_r x \langle A_1 e_0, e_0^* \rangle + y(\tilde{e}_r, x) \langle A_1 e_1, e_0^* \rangle + \langle f(v,r), e_0^* \rangle$$
$$= \tilde{e}_r x \langle A_1 e_0, e_0^* \rangle + x|x| \langle f(e_0), e_0^* \rangle + \mathcal{O}(3),$$

where $\mathcal{O}(3)$ is of cubic order in $\tilde{e}_r, x$, and where we used that $f$ is of second order modulus form. The truncated bifurcation equation thus reads

$$\tilde{e}_r x \langle A_1 e_0, e_0^* \rangle + x|x| \langle f(e_0), e_0^* \rangle = 0,$$

and numerical evaluation for the HTC values gives negative coefficients

$$\langle A_1 e_0, e_0^* \rangle \approx -4.04 \cdot 10^{-2},$$
$$\langle f(e_0), e_0^* \rangle \approx -1.32 \cdot 10^{-3}.$$

Therefore, a non-smooth pitchfork bifurcation occurs and it is supercritical since the steady state is stable for $\tilde{e}_r > 0$. This means that a branch of stable steady motions emerges when decreasing $\varepsilon_r$ from $\varepsilon_{r_1}$. Indeed, we numerically find this as plotted in Fig. 9.

For the full 4D system (3.10), the $\psi$-component resulting from such an equilibrium with $r$-component $r_0$ is $\psi(t) = r_0 t + \psi_0(t)$. Since the equilibria are stable, the corresponding trajectory in the 4D system is normally hyperbolic, and also compact. In fact, it is a hyperbolic periodic orbit due to the periodicity of the angle variable $\psi$. Hence, for the control law modified to be periodic in $\psi$, the robustness of hyperbolic periodic orbits implies that these persist for $0 < \varepsilon_\psi \ll 1$ as nearby periodic solutions.

We remark that in the 4D system, straight motions appear for any $\varepsilon_r, \varepsilon_\psi$ as the equilibrium point with $r = v = \psi = 0, u = u_0$ solving (3.1), but for $\varepsilon_\psi = 0$ as the line of equilibria with arbitrary $\psi$. Therefore, at $\varepsilon_\psi = 0$ the 4-by-4 linearization $A$ from (3.3) possesses a zero eigenvalue for any $\varepsilon_r$, which is double at $\varepsilon_{r_1}$ with a 2-by-2 Jordan block, as well as two nonzero real eigenvalues. We thus expect the unfolding to contain elements of a Bogdanov-Takens bifurcation with symmetry, e.g., [3], including various heteroclinic orbits. However, the rigorous analysis is beyond the scope of this paper since the situation is degenerated by the occurrence of a line of equilibria and the non-smooth nonlinearity, which makes already the Hopf-bifurcation analysis more involved as discussed next.

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3.3.2 Hopf Bifurcations

We have shown in §3.2 that, for any pair of control parameters \((\varepsilon_r, \varepsilon_\psi)\) with \(\varepsilon_r \geq 0, \varepsilon_\psi > 0\) on the stability boundary, the eigenvalues of the linearization at the equilibrium point \((u, v, r, \psi) = (u_0, 0, 0, 0)\) possess one complex conjugate pair with non-zero imaginary parts, and the two other eigenvalues are real and negative. Hence, for any such \((\varepsilon_r, \varepsilon_\psi)\), the linear part of (3.10) can be transformed into

\[
\begin{pmatrix}
\lambda_1 & 0 & 0 & 0 \\
0 & \mu & -\omega & 0 \\
0 & \omega & \mu & 0 \\
0 & 0 & 0 & \lambda_4
\end{pmatrix},
\]

where \(\mu \in \mathbb{R}, \omega > 0\) and the eigenvalues are \(\lambda_1, \lambda_4 < 0\) and \(\lambda_\pm = \mu \pm i\omega\). Note from (3.10) that \(\lambda_1 = p_{11}\).

In the following, we perform the coordinate changes and identification of terms that allow to apply the theory from [11], which will then justify to neglect the terms we drop in the coming steps. In particular, since \(\lambda_1 < 0\) the results from [11] imply that we can neglect the first equation of (3.10) for our bifurcation analysis. Hence, we next analyze the lower right 3 \(\times\) 3-matrix \((v, r, \psi)\), which contains the linearly oscillating part and the coupled equation \(\psi = r\). Moreover, we define the matrix \(T = (a \mid b \mid s)\) with columns \(a, b, s \in \mathbb{R}^3\) from the eigenvectors \(\zeta_+ = a + ib, \quad \zeta_- = a - ib, \quad s\), of the eigenvalues \(\mu + i\omega, \mu - i\omega, \lambda_4\), respectively. Notably, \(\mu = 0\) along the curve \(\varepsilon_\psi(\varepsilon_r)\).

Setting \(\xi := (\xi_1, \xi_2, \xi_3)^T = T^{-1}(v, r, \psi)^T\), the \((v, r, \psi)\)-subsystem of (3.10) takes the form

\[
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} = \begin{pmatrix}
\mu & -\omega & 0 \\
\omega & \mu & 0 \\
0 & 0 & \lambda_4
\end{pmatrix}
\begin{pmatrix}
\xi_1 \\
\xi_2 \\
\xi_3
\end{pmatrix} + h_2(\xi) + \mathcal{R},
\]

(3.11)

where \(h_2(\xi)\) contains all relevant quadratic order terms,

\[
h_2(\xi) = T^{-1}
\begin{pmatrix}
f_1(T \cdot \xi) \\
f_2(T \cdot \xi) \\
0
\end{pmatrix} = \begin{pmatrix}
Z_{11} & Z_{12} & Z_{13} \\
Z_{21} & Z_{22} & Z_{23} \\
Z_{31} & Z_{32} & Z_{33}
\end{pmatrix}
\begin{pmatrix}
g_1(\xi) \\
g_2(\xi) \\
0
\end{pmatrix} = \begin{pmatrix}
Z_{11}g_1(\xi) + Z_{12}g_2(\xi) \\
Z_{21}g_1(\xi) + Z_{22}g_2(\xi) \\
Z_{31}g_1(\xi) + Z_{32}g_2(\xi)
\end{pmatrix},
\]

with quadratic order functions \(g_1, g_2\) discussed below,

\[
Z_{11} = \frac{b_2s_3 - b_3s_2}{\text{det}(T)}, \quad Z_{12} = \frac{-b_1s_3 + b_3s_1}{\text{det}(T)},
\]

\[
Z_{21} = \frac{-a_2s_3 + a_3s_2}{\text{det}(T)}, \quad Z_{22} = \frac{a_1s_3 - a_3s_1}{\text{det}(T)},
\]

and \(a_j, b_j, s_j, j \in \{1, 2, 3\}\), the components of the vectors \(a, b, s\), respectively.

All missing terms are collected in \(\mathcal{R}\), including the nonlinear terms involving \(u, v, ur, u\psi, uv\psi\), from (3.10), which turn out to be irrelevant at leader order due to [11] §2, §4. Furthermore, [11] Thm 2.3 implies that the third component of the transformed system (3.11), \(\xi_3\), will belong to
higher order terms (perturbation) as well. Therefore, using the shorthand $\|\cdot\| := |\cdot|$, the relevant functions $g_1, g_2$ are
\[
g_1(\xi_1, \xi_2) = a_{11}([a_1 \xi_1 + b_1 \xi_2]) + a_{12}([a_1 \xi_1 + b_1 \xi_2])|a_2 \xi_1 + b_2 \xi_2|
+ a_{21}([a_2 \xi_1 + b_2 \xi_2])|a_1 \xi_1 + b_1 \xi_2| + a_{22}([a_2 \xi_1 + b_2 \xi_2]),
\]
\[
g_2(\xi_1, \xi_2) = b_{11}([a_1 \xi_1 + b_1 \xi_2]) + b_{12}([a_1 \xi_1 + b_1 \xi_2])|a_2 \xi_1 + b_2 \xi_2|
+ b_{21}([a_2 \xi_1 + b_2 \xi_2])|a_1 \xi_1 + b_1 \xi_2| + b_{22}([a_2 \xi_1 + b_2 \xi_2]).
\]

The leading order part of (3.11) has $\xi_3$ independent of $\xi_3$, such that only the first two equations in (3.11) remain of interest. Using polar coordinates $(\xi_1, \xi_2) = (r \cos \varphi, r \sin \varphi)$, these become
\[
\begin{align*}
\dot{r} &= \mu r + \chi(\varphi) r^2 + \mathcal{O}(r^3), \\
\dot{\varphi} &= \omega + \Omega(\varphi) r + \mathcal{O}(r^2),
\end{align*}
\]
where
\[
\chi(\varphi) = (a_{11}\Lambda + b_{11}\Gamma)(|a_1 c + b_1 s|) + (a_{12}\Lambda + b_{12}\Gamma)(|a_1 c + b_1 s|)|a_2 c + b_2 s|
+ (a_{21}\Lambda + b_{21}\Gamma)(|a_2 c + b_2 s|)|a_1 c + b_1 s| + (a_{22}\Lambda + b_{22}\Gamma)|a_2 c + b_2 s|,
\]
with $c := \cos \varphi, s := \sin \varphi$ and $\Lambda := cZ_{11} + sZ_{21}, \Gamma := cZ_{12} + sZ_{22}$ to simplify the notation. More specifically, we use cylindrical coordinates and then apply [11, Thm 2.3] in order to obtain (3.12).

Before formulating the bifurcation theorem, we recall the two generic forms of a Hopf bifurcation mentioned in the introduction: in the supercritical (or safe) case the stable fixed point changes stability when the stable periodic orbit is created, while in the subcritical (or unsafe) scenario the unstable fixed point becomes stable when an unstable periodic orbit is born.

**Theorem 3.3.** For the rescaled thruster model (2.7) in the HTC case, and considering $\varepsilon_\psi > 0$, the system undergoes a Hopf bifurcation for the control values (3.5) which is supercritical if $\int_0^{2\pi} \chi(\varphi) d\varphi < 0$ and subcritical for positive sign. Furthermore, the leading order amplitude of the periodic orbit is given by
\[
r = -\frac{2\pi}{\int_0^{2\pi} \chi(\varphi) d\varphi} \mu + \mathcal{O}(\mu^2).
\]

**Proof.** We recall from Theorem [3.1] that a pair of complex conjugate eigenvalues crosses the imaginary axis for $(\varepsilon_r, \varepsilon_\psi)$ crossing the values of (3.5) so that the equivalent system (3.10) is amenable to [11, Cor. 4.7]. In particular, the non-oscillatory linear part is invertible since $\lambda_1, \lambda_4 < 0$ on the stability boundary by Theorem [3.1]. Due to [11, Cor. 4.7], the criticality of the Hopf bifurcation is that of (3.12), and determined by the sign of $\int_0^{2\pi} \chi(\varphi) d\varphi$. Finally, the directly related claimed leading order amplitude of the bifurcating periodic orbits follows from [11, Prop. 3.8]. \qed
Figure 10: Values of $\Sigma$ for (a) $\varepsilon_r \in \{1, 2, \ldots, 130\}$ in the HTC, and (b) $\varepsilon_r \in \{1, 2, \ldots, 68\}$ with $D_p = 9.1575$. See Fig. 2 case 1.

This result reduces the problem of determining criticality, i.e., safety of the stabilizing control, to the sign of

$$\Sigma := \int_0^{2\pi} \chi(\varphi) d\varphi.$$ 

We remark that $\Sigma$ from (4.29) is the same up to the natural positive factor $2\pi \omega$. However, even with the formula (3.13) and despite the fact that all terms in $\chi$ can be explicitly integrated, it appears difficult to determine the sign of $\Sigma$ analytically since the eigenvectors enter non-trivially. For illustration, two of the terms that arise in $\Sigma$ for $(\varepsilon_r, \varepsilon_\psi) = (0, \varepsilon_\psi(0))$ from (3.5) read

$$\int_0^{2\pi} a_{11} \Lambda[[a_1 c + b_1 s]] d\varphi = \frac{8}{3} \sqrt{a_1^2 + b_1^2} a_{11} (Z_{111} a_1 + Z_{12} b_1),$$

$$\int_0^{2\pi} a_{12} \Lambda(a_1 c + b_1 s) d\varphi = \frac{4}{3} \sqrt{a_1^2 + b_1^2} a_{12} (Z_{111} (a_1 (2a_2^2 + b_2^2) + a_2 b_1 b_2)$$

$$+ Z_{12} (b_1 (2b_2^2 + a_2^2) + a_1 a_2 b_2)).$$

Determining the sign of $\Sigma$ requires summing up terms of this form so that we generally need details on each of these, which requires knowing all entries of the matrix $T$, i.e., the eigenvectors.

Nevertheless, numerical evaluation of all these quantities and thus of $\Sigma$ is highly accurate and almost instantaneous on modern computers. This makes it possible to readily predict the criticality as well as the leading order expansion of the bifurcating periodic solutions. We next show various evaluations of $\Sigma$ along the stability boundary (3.5) determined in §3.2. These are all negative for the values of the propeller diameter $D_p$ that we have taken, e.g., in Fig. 10 (a) for the classical HTC. Therefore, the Hopf bifurcation appears to be supercritical for any stabilizing P-control, which means a safe control scenario.

In Fig. 10 (b) we plot $\Sigma$ for $D_p = 1.5D_p^{HTC}$, which is qualitatively the same. In addition, we note that here $\Sigma$ is much smaller for small $\varepsilon_r$, where $\varepsilon_\psi$ is relatively big, than for larger $\varepsilon_r$. In view of Theorem 3.3 the latter means that the amplitudes of the bifurcating periodic orbits
Figure 11: Values of Σ for (a) \( \varepsilon_r \in \{21,22,\cdots,152\} \) with \( x_T = -0.3 \), see Fig. 2 case 2; and (b) \( \varepsilon_r \in \{236,237,\cdots,488\} \) with \( x_T = 0.1 \), corresponding to case 4 in Fig. 2. The vertical lines mark in (a) the vertical asymptote and in (b) the intersections of the stability boundary with the \( \varepsilon_r \)-axis.

grow much quicker for large \( \varepsilon_r \). Hence, the transition from stable straight motion to the motion induced by the stable periodic orbits is much more abrupt for larger \( \varepsilon_r \) values (see §4).

Concerning the thruster position, we recall from §3.2 that there are four cases as shown in Fig. 2. At \( x_T = -0.3 \), which is case 2, control is possible for approximately \( \varepsilon_r \in (21,152.5) \); see Fig. 6 (a). We plot the resulting values of \( \Sigma \) in Fig. 11 (a). Further we compute \( \Sigma \) for \( x_T = 0.1 \), which corresponds to case 4, and display the results in Fig. 11 (b). In this last example the interval of \( \varepsilon_r \)-values for which the straight motion is stable is bounded by the zeros of the concave branch, thus approximately \( \varepsilon_r \in \{235.9,488.6\} \); compare with Fig. 8 (c), where \( x_T = 0.16 \). For \( x_T = -0.16 \), which corresponds to case 3 in Fig. 2 we numerically find a qualitatively reflected \( \Sigma \) graph. In both of these cases, the abruptness mentioned above occurs for smaller \( \varepsilon_r \), i.e., closer to the (first) intersection of the stability boundary curve with the \( \varepsilon_r \)-axis.

4 Numerical Bifurcation Analysis

In this section we present numerical results that corroborate and illustrate the analysis of the previous sections based on implementing the model in the continuation software AUTO [4]. In Fig. 12 we plot the stability boundary for the HTC that results from such a numerical continuation, which tracks the Hopf bifurcation locus. Up to numerical error, this agrees with (3.5) plotted in Fig. 5 based on the analytical prediction of Theorem 3.1. Note that all quantities in this section are dimensionally comparable.

Next, we have employed numerical bifurcation and continuation to compute the periodic orbits that bifurcate from this stability boundary along curves in the \((\varepsilon_r,\varepsilon_\psi)\)-plane for \( \varepsilon_\psi > 0 \); in Fig. 12 two such curves are plotted in violet and green. Notably, these curves lie ‘under’ the stability boundary, which corroborates the analytically predicted supercritical nature of the Hopf bifurcations. As shown in §3.3.1 along the \( \varepsilon_r \)-axis, i.e., for \( \varepsilon_\psi = 0 \), the Hopf bifurcation turns into a pitchfork bifurcation. Specifically, for \( \varepsilon_\psi = 0 \), (3.10) reduces to a 3D system for \( u,v,r \) since the vector field is then independent of \( \psi \). The numerical results plotted in Fig. 9 illustrate the analytically predicted supercritical pitchfork bifurcation, as well as the analytical
Figure 12: Stability boundary curve $\varepsilon_\psi(\varepsilon_r)$ for the HTC values in blue and unstable region shaded, stable region in white. Existence of periodic solutions at $\varepsilon_r \approx 10.6$ is marked by the violet vertical line, and on the near diagonal by the green line; both extend until $\varepsilon_\psi \approx 0.41$ just above heteroclinic bifurcations. The dot marks the point $(10.6, 30)$ for which simulations are plotted in Fig. 20.

prediction that the resulting curve of $(\varepsilon_r, u)$ is smooth while those for $(\varepsilon_r, v), (\varepsilon_r, r)$ are non-smooth, having limiting non-zero slopes at the bifurcation point.

We recall that in the 4D system (3.10), the $\psi$-component generated by such equilibria is $\psi_0(t) = r_0 t + \psi_0$, with $r_0$ the equilibrium value and $\psi_0$ an arbitrary shift in case $r_0 \neq 0$. Any straight motion with $u = u_0$ has either $v_0 = r_0 = 0$ and $\psi_0 = 0$ in case $\varepsilon_\psi \neq 0$, or otherwise the $\psi$-axis shifted by $u_0$ as a line of equilibria. The trajectories of the solutions that bifurcate from the pitchfork, denoted by $v_{\pm}(t) \in \mathbb{R}^4$, at $\varepsilon_\psi = 0$ take the form $\psi_{\pm}(t) = r_{\pm} t + \psi_{\pm}(0)$, where $r_0 = r_{\pm} \neq 0$ are the corresponding $r$-values; note that $r_- = -r_+$. We recall that for the control law modified to be periodic in $\psi$, these perturb to hyperbolic periodic orbits for $0 < \varepsilon_\psi \ll 1$.

Indeed, in order to discuss the nonlinear effects of the control it becomes relevant that $\psi$ is in fact an angular variable for the ship modeling, which means that the model equations must be $2\pi$-periodic with respect to $\psi$. This is clearly incompatible with the linear P-control scheme (2.4), but, as mentioned previously, thus far did not play a role since $\psi$ only entered as a perturbation, and the P-control is precisely intended to be applied in this regime. For the solutions $v_{\pm}(t)$, the control law is independent of $\psi$ since $\varepsilon_\psi = 0$ and is therefore trivially periodic. However, for bifurcating solutions, it is not guaranteed (and actually wrong) that solutions remain within this regime, which may also be viewed as forcefully driving a control scheme beyond its range of validity. Hence, we must modify the P-control to be periodic with respect to $\psi$, while remaining compatible with the linear nature of the P-control for small $\psi$. For definiteness, we choose to replace the linear $\psi$-term by a sinus,

$$\eta = \varepsilon_r r + \varepsilon_\psi \sin(\psi),$$

so that $\psi$ replaced by $\psi$ modulo $2\pi$ makes no difference. Being smooth and satisfying $\sin'(0) = 1$, this modification has no influence on the results so far. In particular, the stability boundary, the supercritical nature of the bifurcations and in fact the pitchfork branches of the 3D reduced system are identical to before. However, this choice has an impact on the solutions along curves in the $(\varepsilon_r, \varepsilon_\psi)$-plane for $\varepsilon_\psi > 0$ away from the stability boundary. Nevertheless, it turns out that there is even no qualitative impact for $\varepsilon_\psi \geq 1$ (at least for the numerous curves we checked); we will discuss smaller values later. Note that instead of the sinus we could choose a function that
is the identity for $|\psi|$ below some threshold, but globally smooth and periodic, which would provide results that fully coincide with the P-control as long as $|\psi|$ is below the threshold.

For the sinusoidal control law, we thus find numerically the presence of a smooth surface of periodic solutions, parameterized by the control parameter values ‘under’ the stability boundary curve within the range $\varepsilon_r \geq 0, \varepsilon_\psi \geq 1$. We have confirmed this along a number of different axis-aligned curves, in particular including the $\varepsilon_\psi$-axis, for $1 \leq \varepsilon_\psi < 45.8$, which in fact shows that the surface of periodic solutions extends into negative $\varepsilon_r$. In Fig. [13] we plot different views of the bifurcation diagram along the $\varepsilon_\psi$-axis; the analytical predictions are again matched by a smooth diagram for $(\varepsilon_\psi, u)$ and non-smooth along the center eigenspace within the $(v, r, \psi)$ subspace. This is completely analogous for other continuations along different curves, in particular, along the coloured curves in Fig. [12]. Furthermore, in Fig. [14] we plot some views of the profiles of the periodic solutions along the violet line in Fig. [12] and note the growth of the $\psi$-range as $\varepsilon_\psi$ decreases.

The branch appears to terminate for $\varepsilon_\psi \approx 0.408$ in a heteroclinic bifurcation, where the periodic orbit limits on a heteroclinic cycle between a pair of nearly straight motion equilibrium points and $\psi \approx \pm 482.85$. We show one view near this cycle in Fig. [14] (c). Though incorrect for the ship modeling, it is instructive to note that the proportional control law admits numerical continuation to much smaller $\varepsilon_\psi$, e.g., 0.005. It appears that in this case, the periodic orbits limit to a different heteroclinic cycle, namely between the aforementioned ‘equilibrium’ solutions $v_\pm$ at $\varepsilon_\psi = 0$ that connect to the pitchfork bifurcation. Here the periodic solutions for small $\varepsilon_\psi$ have $(\psi, v)$-profiles that closely track one of $v_\pm$ for a period of time before switching and tracking the other. See the vertical lines plotted in Fig. [14] (b). Also for the sinusoidal law we find the solutions to switch in such a manner, though not as strongly as for the other control law. Concerning smaller values of $\varepsilon_\psi$, we recall that $v_\pm$ perturb to nearby solutions for $0 < \varepsilon_\psi \ll 1$ and we find they continue up to the heteroclinic bifurcation at $\varepsilon_\psi \approx 0.408$. Here these solutions
Figure 14: (a, b) Sample profiles of periodic orbits for $\varepsilon_r \approx 10.6$ fixed, and $\varepsilon_\psi$ between 25.9 at the Hopf bifurcation and $\varepsilon_\psi = 1$ computed by numerical continuation. In (b) we additionally plot vertical lines at the $v$-components of $v_\pm$, cf. Fig. 9. In (c) we only plot the solution near the termination point $\varepsilon_\psi \approx 0.408$ and period $T \approx 9285$.

![Figure 14](image)

Figure 15: Earth-fixed coordinate plots of periodic solutions from Fig. 14 with $z(0) = 0$, $t \in [0, 5T]$ and $T$ the corresponding period. (a) Small amplitude solution near the Hopf bifurcation with $\varepsilon_\psi \approx 25.6$ and period $T \approx 50.5$. (b) Solution with $\varepsilon_\psi \approx 23$ and period $T \approx 54$.

![Figure 15](image)

appear to terminate in one of the heteroclinic orbits involved, i.e., in Fig. 14 (c) one of the parts with fixed sign of $v$. Indeed, all the periodic solutions along the branch emerging from the Hopf bifurcations have necessarily winding number zero in the cylinder with periodic $\psi$, while $\eta_\pm$ have winding number one, which is therefore constant on the branch that emanates from these for increasing $\varepsilon_\psi$ from zero. We remark that the winding number is a topological invariant along branches, but branches with winding number zero and one can – and apparently do – meet at such a heteroclinic cycle. It appears that this is the way in which the entire region `under' the stability boundary up to the $\varepsilon_r$-axis is organized.

We turn to the resulting ship motions on the Earth-fixed position coordinates $(x, y) \in \mathbb{R}^2$. These can be conveniently expressed in complex form as $z = x + iy \in \mathbb{C}$ and the relation of $z(t)$ to the ship-fixed $(u, v, \psi)$-coordinates is given by $\dot{z} = (u + iv)\exp(i\psi)$; recall that we have chosen $\psi(0) = 0$ as the reference straight direction, but by symmetry any other angle $\psi_*$ can be implemented upon multiplication with $\exp(i\psi_*)$. Hence, for any initial $z(0)$ the positions can be readily computed by integration, using given ship-fixed coordinates. In particular, the constant $(u_0, v_0, r_0)$ corresponds to straight ($r_0 = 0$) or circular ($r_0 \neq 0$) motion of the ship with
Figure 16: (a) Earth-fixed coordinate plots with \( z(0) = 0 \) of the largest amplitude solution in Fig. 14 (b), which has period \( T \approx 311 \) for \( t \in [0, 5T] \). In (b) we plot the trajectory from (a) for a selected time interval in \([0, T]\), which shows a ‘twirl’, in- and outwinding near a circular motion that resembles the equilibrium \( \nu_+ \) at \( \epsilon_\psi = 0, \epsilon_r \approx 10.6 \) in Fig. 9.

\[
\eta = \epsilon_r r + \epsilon_\psi \sin(\psi)
\]

Figure 17: Thruster angles \( \eta = \epsilon_r r + \epsilon_\psi \sin(\psi) \) in degrees over one period \( t \in [0, T] \) of (a) solution of Fig. 15 (a); (b) solution of Fig. 15 (b); (c) solution of Fig. 16.

non-dimensional radius \( r_0^{-1} \sqrt{u_0^2 + v_0^2} \); for instance in the 3D reduced equilibrium points \( \nu_\pm \) at \( \epsilon_\psi = 0 \). We plot the results for selected solutions in Figs. 15, 16. As expected, periodic solutions near the Hopf bifurcation closely track the straight motion up to a superposed oscillation, cf. Fig. 15 (a). However, beyond the bifurcation point, the ship motion is completely altered to a figure-eight with superposed surge as plotted in Fig. 15 (b). Further decreasing \( \epsilon_\psi \), the ship track switches back- and forth between periods of following circular motions that closely resemble the equilibrium states \( \nu_\pm \) at \( \epsilon_\psi = 0 \). This can be seen in Fig. 16, where each of the ‘twirls’ in (a) is similar to the magnified one in (b). Here the aforementioned heteroclinic cycle numerically found for the original control law can provide an explanation: even though we do not numerically find this cycle for the sinusoidal control law, the ‘twirls’ may well be signatures of the continuations of the periodic solutions with winding number one that emanate from \( \nu_\pm \) into \( \epsilon_\psi > 0 \) as discussed above.

We additionally plot the thruster angle \( \eta = \epsilon_r r + \epsilon_\psi \sin(\psi) \) of selected solutions in Fig. 17. As expected, close to the Hopf bifurcation the thruster angle just mildly oscillates, while further from it, the thruster angles vary more dramatically, with various frequencies and up to angles of 35°.

Finally, we turn to time-simulations of initial value problems in the model in this regime. Recall that the analytic prediction of supercritical bifurcations implies dynamically stable so-
Figure 18: Time simulation for $\varepsilon_\psi = 0$, $\varepsilon_r \approx 10.6$ from the straight motion; (a) surge $(t,u)$, (b) sway $(t,v)$ and (c) yaw $(t,r)$ velocities.

Figure 19: Time simulation, similar to Fig. 18, for $\varepsilon_\psi = 0.2$, $\varepsilon_r \approx 10.6$ from $t = 1050$ on. The trajectories converge to the circular motion displayed in Fig. 16(b).

Solutions near bifurcation, i.e., they possess a basin of attraction so that any solution with initial condition in this basin converges exponentially towards it. Indeed, we found in the numerical continuation that all periodic solutions are stable, i.e., all Floquet exponents have negative real parts, away from the mentioned heteroclinic bifurcations, e.g., for $\varepsilon_\psi \geq 1$. Also the equilibria $\nu_{\pm}$ born from the pitchfork at $\varepsilon_\psi = 0$ are stable in this sense in the reduced 3D system, cf. Fig. 18 and as discussed in §3.3.1 in the 4D system these form exponentially stable periodic orbits, as shown for $\varepsilon_\psi = 0.2$, $\varepsilon_r \approx 10.6$ in Fig. 19. For other values we observe convergence to other solutions.

For completeness, we have also computed simulations for $(\varepsilon_r, \varepsilon_\psi)$-values above the stability boundary curve, as the violet dot, for instance, in Fig. 12. Locally, the trajectories converge, as expected, to the stable equilibrium straight motion; see Fig. 20.

Figure 20: Time simulation, similar to Fig. 18 for $\varepsilon_\psi = 30$, $\varepsilon_r \approx 10.6$ from a non-trivially perturbed initial condition.
5 Discussion

In this paper we have presented an accurate analysis of stabilizing the straight motion in a 3 DOF ship model with P-control. We have identified the location and geometry of the stability region in the control parameter space, which has been, essentially, independent of the propeller diameter, as well as its frequency, but it has strongly depended on the thruster – or rudder – position and, in the Hamburg Test Case, it has shrunk and even disappeared when moving further to the fore of the ship. On the nonlinear level, the second order modulus nonlinearities require a non-standard approach which we have applied to the current setting in detail. In this manner we have confirmed that the linearly stabilizing control produces also a ‘safe’ nonlinear system. By numerical continuation we have corroborated these analytical results and, moreover, presented the arrangement of periodic solutions and equilibria in ship-fixed coordinates globally in the control parameter space. Finally, we have illustrated the resulting ship tracks in Earth-fixed coordinates.

While the details of the analysis depend on the specific choice of some parameters, as the thruster position, the approach can be in principle adapted to any other design parameter of the ship model and even to related models.

A number of questions and further lines of research arise from this report. On the one hand, a natural next step would be to perform the same study for the full ‘rudder model’ and explore whether again only ‘safe’ bifurcations occur. An extension would include turning circle maneuvers and optimality conditions, possibly multi-objective control. On the other hand, it would be interesting to complete the analytical unfolding of the double zero eigenvalue in this cylindrical geometry and non-smooth setting, and relate it to Bogdanov-Takens points with symmetry.

Furthermore and beyond the ship-fixed equilibrium maneuvers, numerical continuation can be used as well to effectively investigate different planned movements, for instance, zig-zag maneuvers, and evaluate these, e.g., in terms of the so-called overshoots, [1].

A Appendix

A.1 HTC characteristics

| Hull forces | Coefficient | Value | Coefficient | Value |
|-------------|-------------|-------|-------------|-------|
| \( \bar{m} \) | \( \bar{m}_{uu} \) | 0.2328 | 0.0247 |
| \( \bar{I}_z \) | \( \bar{m}_{vv} \) | 0.0134 | 0.2286 |
|               | \( \bar{m}_{rr} \) |       | 0.0150 |
|               | \( \bar{m}_{rv} \) |       | 0.0074 |
|               | \( \bar{m}_{vy} \) |       | 0.0074 |

Table 1: Rescaled added mass coefficients, (2.5).
### Hull forces

| Coefficient | Value | Coefficient | Value |
|-------------|-------|-------------|-------|
| $X_{u|u|}$  | $-0.0141$ | $N_{\beta}$ | $-0.1442$ |
| $Y_{\beta}$ | $-0.1735$ | $N_{\gamma}$ | $-0.0276$ |
| $Y_{\gamma}$ | $0.0338$ | $N_{\beta|\beta|}$ | $-0.0375$ |
| $Y_{\beta|\beta|}$ | $-1.1378$ | $N_{\gamma|\gamma|}$ | $-0.0386$ |
| $Y_{\gamma|\gamma|}$ | $0.0123$ | $Y_{\hat{\beta}|\gamma}$ | $-0.0537$ |
| $Y_{\hat{\beta}|\gamma}$ | $0.1251$ | $a_{n}$ | $3$ |
| $b_{n}$ | $2$ | $c_{n}$ | $2$ |

**Table 2:** Hydrodynamic bare hull coefficients. To obtain the rescaled values for (2.5), multiply each force with $L_{pp}^{-1}$.

### Propeller characteristics

| Coefficient | Value | Coefficient | Value |
|-------------|-------|-------------|-------|
| $D_{p}$     | $6.105$ | $K_{T0}$ | $0.366897$ |
|             |       | $K_{T1}$ | $-0.345036$ |
|             |       | $K_{T2}$ | $0.068841$ |
|             |       | $K_{T3}$ | $-0.710991$ |
|             |       | $K_{T4}$ | $0.948559$ |
|             |       | $K_{T5}$ | $-0.428915$ |

**Table 3:** Propeller characteristics with propeller diameter $[D_{p}] = m$ and non-dimensional coefficients $K_{Ti}$ of the propeller thrust $T_{p}$, corresponding to the model No. 5286.

### Further parameters

| Coefficient | Value | Coefficient | Value |
|-------------|-------|-------------|-------|
| $L_{pp}$    | $153.70$ | $t$          | $0.22$ |
| $T$         | $10.30$ | $w$          | $0.38$ |
| $\rho$      | $1025$  |              |       |

**Table 4:** Further parameters: length between perpendiculars $[L_{pp}] = m$, mean draft $[T] = m$, water density $[\rho] = kg/m^3$, and non-dimensional thrust deduction fraction $t$ and wake fraction $w$.
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