Dk GRAVITATIONAL INSTANTONS AS SUPERPOSITIONS OF ATIYAH–HITCHIN AND TAUB–NUT GEOMETRIES

B.J. SCHROERS AND M.A. SINGER

Abstract. We obtain Dk ALF gravitational instantons by a gluing construction which captures, in a precise and explicit fashion, their interpretation as non-linear superpositions of the moduli space of centred charge two monopoles, equipped with the Atiyah–Hitchin metric, and k copies of the Taub–NUT manifold. The construction proceeds from a finite set of points in euclidean space, reflection symmetric about the origin, and depends on an adiabatic parameter which is incorporated into the geometry as a fifth dimension. Using a formulation in terms of hyperKähler triples on manifolds with boundaries, we show that the constituent Atiyah–Hitchin and Taub–NUT geometries arise as boundary components of the 5-dimensional geometry as the adiabatic parameter is taken to zero.

1. INTRODUCTION AND CONCLUSION

1.1. First statement of the main result. A 4-dimensional gravitational instanton is a complete hyperKähler 4-manifold (M, g), possibly with a decay condition on the curvature at infinity. Michael Atiyah was fascinated by gravitational instantons from the early 1980s onwards, and much progress was made by the Oxford group, led by Sir Michael, until he left for the mastership of Trinity College Cambridge in 1990. In particular, his student Peter Kronheimer, building on the work of Nigel Hitchin (e.g. [20]) and others, gave a complete classification of the asymptotically locally euclidean (ALE) gravitational instantons, using the hyperKähler quotient construction [22, 23]. At about the same time, Atiyah and Hitchin computed the metric on the moduli space $\mathcal{M}_2^0$ of centred SU(2) monopoles, which is an example of an asymptotically locally flat (ALF) gravitational instanton [3, 4].

The classification of ALF gravitational instantons has proved to be more difficult, but, following substantial progress [11, 12, 10, 33, 34, 7, 9] which we review below, is now much better understood. In particular, there are two infinite families, the $A_k$ and $D_k$ ALF gravitational instantons, labelled by a non-negative integer k and distinguished by the fundamental group of the asymptotic region of M. The $A_k$ gravitational instantons can all be constructed by the Gibbons–Hawking Ansatz [16, 34]. In particular, the $A_0$ graviton is the euclidean (positive mass) Taub-NUT space which we denote TN in the following.

Constructions of $D_k$ gravitational instantons are not so explicit and either use twistor theory [11, 12, 10] or rely on gluing or desingularization constructions [8, 7]. In this paper we shall present a construction in which $D_k$ ALF gravitational instantons appear as (nonlinear) superpositions of $\mathcal{M}_2^0$ and k copies of TN. A first version of the theorem to be proved is as follows:

Theorem 1.1. Given a configuration of $k > 0$ distinct points in $\mathbb{R}^3$, there exists $\varepsilon_0 > 0$, and a 1-parameter family of $D_k$ ALF gravitational instantons $(M, g_\varepsilon)$, for $\varepsilon \in (0, \varepsilon_0)$, with the following properties:

(a) There is a compact subset $K_0$ of M which is diffeomorphic to a compact subset $K'_0$ of $\mathcal{M}_2^0$, and such that $g_\varepsilon$ approaches the Atiyah–Hitchin metric as $\varepsilon \to 0$, under the identification of $K_0$ with $K'_0$;

(b) For $j=1,\ldots,k$, there is a compact subset $K_j$ of M which is diffeomorphic to a closed ball $K'_j$ in TN, and such that $g_\varepsilon$ approaches the Taub–NUT metric as $\varepsilon \to 0$, under the identification of $K_j$ with $K'_j$;

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(c) There is an asymptotic region $U \subset M$ such that $\tilde{g}_\varepsilon$ on $U$ is identifiable with the asymptotic Gibbons–Hawking metric

$$\left(1 + \frac{(k-2)\varepsilon}{|x|}\right) \frac{|dx|^2}{\varepsilon^2} + \left(1 + \frac{(k-2)\varepsilon}{|x|}\right)^{-1} \alpha^2, \quad (|x| \gg 1), \quad (1.1)$$

factored out by an involution $\iota$ covering $x \mapsto -x$ on $\mathbb{R}^3$.

In the remainder of this extended introduction we explain the relation of this result to Michael Atiyah’s interest in geometrical models of matter, provide some technical background and use it to state a more detailed version of the theorem.

1.2. Motivation. Theorem (1.1) has its origin in a speculative proposal for purely geometric models of physical particles made in [6] by Michael Atiyah, Nick Manton and the first named author of the current paper. While our main concern here is geometry, we briefly recall the physical motivation.

The idea developed in [6] is to use non-compact hyperKähler 4-manifolds to model electrically charged particles like the electron or the proton. Outside a compact core region, or at least asymptotically, the 4-manifolds are required to be circle fibrations over physical 3-dimensional space. In this asymptotic region, the model is interpreted as a dual Kaluza-Klein picture: the Chern class of the asymptotic circle bundle, which would be the magnetic charge in Kaluza-Klein theory, is taken to represent the negative of the electric charge. The further requirement that the 4-manifold has cubic volume growth means that the allowed geometric models are in effect ALF gravitational instantons [9].

In [6], these ideas were illustrated with two main examples, namely the Taub–NUT and Atiyah–Hitchin manifolds as potential models of, respectively, the electron and the proton. Following the convention of [6] we write $\text{AH}$ for the Atiyah–Hitchin manifold by which we mean the simply-connected double cover of the moduli space $\mathcal{M}_0^2$ of centred 2-monopoles in critically coupled $SU(2)$ Yang-Mills-Higgs theory [4].

Geometries which are obtained by gluing together copies of TN and of $\text{AH}$ or $\mathcal{M}_0^2$ are potential geometric models for electrons interacting with each other and a proton, and therefore interesting arenas for exploring if and how geometrical models can make contact with physics beyond basic quantum numbers like electric charge and baryon number. In particular, the model for a single electron interacting with the proton would need to account for the formation of the hydrogen atom and its excited states.

The gluing process is well-understood when dealing only with copies of TN, where it leads to the multi-center Taub–NUT spaces which make up the $A_k$ series of ALF gravitational instantons, with the positive integer $k + 1$ counting the number of centres or ‘NUTs’ [16, 34].

However, the interpretation of $D_k$ ALF gravitational instantons, even in some asymptotic region, as a composite of more elementary geometries is less clear. Here we address this issue by constructing $D_k$ gravitational instantons via a desingularization procedure first outlined in a paper of Sen [35]. While Sen’s proposal was made in the context of $M$-theory, it is similar in spirit to the motivation coming from geometric models of matter. In both cases one aims to obtain a $D_k$ space as a non-linear superposition of $\mathcal{M}_0^2$ and $k$ copies of TN, thus interpreting it as a composite object or bound state.

Our construction has two main ingredients, namely a singular and suitably symmetric gravitational instanton of the Gibbons–Hawking form, and a further manifold, obtained as a $\mathbb{Z}_2$-quotient of a branched cover $\overline{\text{AH}}$ of the Atiyah–Hitchin space $\text{AH}$, which we call $\text{HA}$. We now discuss these ingredients in turn, but should alert the reader that, while $\text{AH}$ and $\mathcal{M}_0^2$ have smooth hyperKähler metrics, the lifts of these metrics to the branched covers $\overline{\text{AH}}$ and $\text{HA}$ are singular on the branching locus.

1.3. The adiabatic Gibbons–Hawking Ansatz. The definition of ALF gravitational instantons allows for the complement of all sufficiently large compact subsets to have a non-trivial fundamental group $\Gamma$. Apart from a few exceptional cases, $\Gamma$ must be a finite subgroup of $SU(2)$, more specifically a cyclic group $\mathbb{Z}_\ell$ or the binary dihedral group $D_{2\ell}$ of order $4\ell$, for a suitable
non-negative integer \( \ell \). The corresponding ALF gravitational instantons are called \( A_{\ell-1} \) and \( D_{\ell+2} \) ALF gravitational instantons. In fact, it is natural to extend this correspondence to \( D_k \) instantons for non-negative integers \( k \) as follows.

To fix notation, our presentation of \( D_k \) as a subgroup of \( SU(2) \) is as the group generated by

\[
R_\ell = \begin{pmatrix}
  e^{-i\frac{\ell\pi}{2}} & 0 \\
  0 & e^{i\frac{\ell\pi}{2}}
\end{pmatrix}, \quad S = \begin{pmatrix}
  0 & -1 \\
  1 & 0
\end{pmatrix}, \quad \ell \geq 1,
\]

so that \( D_1 \cong \mathbb{Z}_4 \) and \( D_2 \) is the lift of the Vierergruppe, viewed as the group of rotations by \( \pi \) around orthogonal axes in \( \mathbb{R}^3 \), to \( SU(2) \). For our purposes it is convenient to define also \( D_0 \) as the infinite group with generators \( R_0, S \) and relations \( SR_0S^{-1} = R_0^{-1} \) and \( S^4 = \text{id} \). With these conventions, the fundamental group of the asymptotic region of \( D_k \) ALF gravitational instantons is \( D_{k^*} \), where

\[
k^* = |k - 2|.
\]

Sen’s proposal amounts to constructing a \( D_k \)-gravitational instanton by dividing a suitably \( S \)-invariant but singular \( A_{2k^*-1} \) gravitational instanton (defined below) by \( S \) and then resolving singularities. We construct the required gravitational instanton by a procedure we call the \textit{adiabatic Gibbons–Hawking Ansatz}, and which we now explain.

Let \( P \subset \mathbb{R}^3 \) be a finite set of points and consider the harmonic function on \( \mathbb{R}^3 \) given by

\[
h_\varepsilon(x) = 1 + \sum_{p \in P} \frac{\varepsilon m(p)}{2|x - p|},
\]

where \( m(p) \) is an integer for each \( p \). It is well known that if \( m(p) = 1 \) for each \( p \), then the Gibbons–Hawking metric

\[
g_\varepsilon = h_\varepsilon \frac{|dx|^2}{\varepsilon^2} + h_\varepsilon^{-1} \alpha^2
\]

defines, for fixed \( \varepsilon > 0 \), a hyperKähler metric which lives on a 4-manifold \( M \) carrying a circle-action with quotient \( M/S^1 = \mathbb{R}^3 \). Denoting the quotient map by \( \phi \), the action is free away from \( \phi^{-1}(P) \) where the circles collapse to points (the NUTs). So we have a \( S^1 \)-bundle with projection

\[
\phi : M \setminus \phi^{-1}(P) \to \mathbb{R}^3 \setminus P,
\]

and a connection 1-form \( \alpha \) (on \( M \setminus \phi^{-1}(P) \)) satisfying the monopole equation

\[
d\alpha = \ast_x dh_\varepsilon.
\]

If the integers \( m(p) \) are all positive but not all equal to 1, then the above construction yields an orbifold \( M \), and \( h_\varepsilon \) is a smooth orbifold metric on \( M \). If some \( m(p) < 0 \), the construction of \( M \) as an orbifold still goes through, but we no longer have \( h_\varepsilon > 0 \) everywhere. Nonetheless, \( (1.5) \) still defines a smooth (orbifold) metric on the subset \( \{ h_\varepsilon > 0 \} \) of \( M \). (In writing this of course we really mean the subset of \( M \) on which the pull-back of \( h_\varepsilon \) is positive. There seems little danger of confusion in this abuse of notation, and we shall continue to use it.)

To obtain the correct asymptotic topology for a \( D_k \) gravitational instanton, Sen’s proposal is to choose \( P \) to be symmetric \((-P = P)\), to include 0 as an element of \( P \), take \( m(0) = -4 \) and all other \( m(p) = 1 \). Then if \( 2k \) is the number of non-zero elements of \( P \), near \( \infty \),

\[
h_\varepsilon(x) = 1 + \frac{\varepsilon(2k - 4)}{2|x|} + O(\varepsilon|x|^{-3}).
\]

This means that, in terms of \( k^* \) defined in \( 1.3 \), the asymptotic topology of \( M \) is \( \mathbb{R}^3 \times S^1 \) if \( k^* = 0 \), and \( \mathbb{R}^4/\mathbb{Z}_{2k^*} \) otherwise. Because \( P \) is symmetric, there is an orientation-preserving involution, \( \iota \), say, of \( M \), covering \( x \mapsto -x \) on \( \mathbb{R}^3 \). This involution corresponds to the generator \( S \) above acting on \( \mathbb{R}^4/\langle R_{k^*} \rangle \), where \( R_{k^*} \) is as in equation \( 1.2 \) with \( \ell = k^* \). Thus \( M/\iota \) is an orbifold with a single singularity over \( x = 0 \), and the correct topology at \( \infty \) for a \( D_k \) instanton.

Moreover, \( \alpha \) can be chosen so that \( \iota^* \alpha = -\alpha \), and then \( 1.5 \) defines a smooth metric on the subset \( \{ h_\varepsilon > 0 \} \) of \( M/\iota \). From the asymptotic form of the metric, moreover, this will be an ALF metric on this \( D_k \) space. Thus we have a \( D_k \) orbifold \( M/\iota \) and an asymptotic hyperKähler ALF metric on \( M/\iota \), that is to say a hyperKähler metric on the complement of a compact subset.
With the above choices, the Gibbons–Hawking geometry on $M$ obtained from

$$\hat{h}_ε(x) = 1 - \frac{2ε}{|x|} + \epsilon \sum_{p \in \partial C} \frac{ε}{2|x-p|},$$

(1.9)

is the required singular $A_{2k+1}$ gravitational instanton. The singularity near 0 of $M/\iota$ can be resolved, metrically and topologically, by gluing into $M$ a copy of the manifold $HA$, introduced at the end of [122] and discussed carefully below, which is such that $HA/\iota$, suitably defined, is the moduli space $\mathcal{M}_0^2$ of centred 2-monopoles.

More concretely, in terms of the new variable $x' = x/ε$,

$$\hat{h}_ε(x') = 1 - \frac{2}{|x'|} + O(ε).$$

(1.10)

To leading order for $|x'| \to \infty$, this expansion, inserted into (1.3), gives the ‘negative-mass Taub–NUT’ asymptotics of the Atiyah–Hitchin metric [18], reviewed below (1.22). Using a suitable cut-off function $χ$ we can therefore construct a smooth metric, $g_{ε}$, say, on a manifold obtained by gluing $HA$ to $M$ and dividing by $\iota$. The manifold obtained in this way (unlike the metric) is independent of $χ$ and $ε$ for $ε > 0$, and we call it the Sen space $Se_ε$.

The problem then is to deform $g_{ε}$ (for small $ε > 0$) to yield a hyperKähler metric on $Se_ε$ without spoiling the ALF asymptotics of $g_{ε}$ (which are the same as those of $g_{ε}$). This is the problem we address in this paper, bearing in mind that the parameter $ε$ has the effect of scaling lengths in the base by a factor $1/ε$, while the asymptotic length of the circle fibre of tends to $2π$. Correspondingly, the metric $ε^2 g_{ε}$ collapses, as $ε \to 0$, away from $P$, to the flat metric on $\mathbb{R}^3$.

1.4. The Atiyah–Hitchin manifold and related spaces. For a precise definition of the space $HA$ we need to review the definition of the Atiyah–Hitchin manifold $AH$, its branched cover $\tilde{AH}$, and the relation of both to the moduli space $\mathcal{M}_0^2$ of centred 2-monopoles.

Michael Atiyah revisited the geometry of the Atiyah-Hitchin manifold on several occasions. Even though it arose in the specific physical context of magnetic monopoles, he hoped for an application to real and fundamental physics, and pursued this in the Skyrme model of nuclear particles [5] and in geometric models of matter [6]. In all these studies, he stressed and used the interpretation of $AH$ and its branched cover as parameter spaces of oriented ellipses, up to scale, in euclidean space.

We have also found this picture helpful, and develop it further in this section and Appendix[A] in order to clarify the discrete symmetries and their action on the core and asymptotic regions. We begin by noting that, as manifolds,

$$\tilde{AH} = TS^2 \cong \mathbb{C}P_1 \times \mathbb{C}P_1 \backslash \mathbb{C}P_1^{\text{diag}},$$

(1.11)

where $\mathbb{C}P_1^{\text{diag}}$ is the anti-diagonal $\mathbb{C}P_1$ in $\mathbb{C}P_1 \times \mathbb{C}P_1$. This manifold is a branched cover of the Atiyah–Hitchin manifold $AH$, which, as already explained, is the double cover of the moduli space $\mathcal{M}_0^2$ of centred 2-monopoles. We would like to make this explicit, and to define the manifold $HA$ in terms of $\tilde{AH}$.

In Appendix[A] we derive the concrete realisation of $\tilde{AH}$ as

$$\tilde{AH} = \{Y \in \mathbb{C}^3|Y_1^2 + Y_2^2 + Y_3^2 = 1\},$$

(1.12)

where we wrote $Y$ for the vector in $\mathbb{C}^3$ with coordinates $Y_1, Y_2$ and $Y_3$. The real and imaginary parts of $Y = y + iη$ are orthogonal, with magnitudes related via $|y|^2 = 1 + |η|^2$. We can picture this description in terms of an oriented ellipse, called the $Y$-ellipse in the following, with major axis $y$ and minor axis $η$. When $|η| = 0$ the $Y$-ellipse degenerates to an oriented line. The set of these lines is a two-sphere to which $\tilde{AH}$ retracts and which we call the core in the following. It is the diagonal submanifold of $\mathbb{C}P_1 \times \mathbb{C}P_1$, and we denote it by $\mathbb{C}P_1^{\text{diag}}$.

This description of $\tilde{AH}$ is useful for understanding its symmetries and the structure near the core, but less useful when studying the asymptotic region away from the core, which for us means simply $|η| \neq 0$. In this region it is convenient to switch to a dual description, derived in
the appendix, in terms of a complex vector \( X \) whose components also satisfy \( X^2_1 + X^2_2 + X^2_3 = 1 \), but whose real and imaginary parts are

\[
X = \tilde{x} + i\eta, \quad \tilde{x} = \frac{y \times \eta}{|\eta|^2}, \quad \xi = -\frac{\eta}{|\eta|^2}.
\]  

(1.13)

One checks that \( |\tilde{x}|^2 = 1 + |\xi|^2 \), and in Appendix \( A \) we explain that \( \tilde{x} \) and \( \xi \) are the major and minor axes of a family of ellipses which we call \( X \)-ellipses and which are dual to the \( Y \)-ellipses.

The \( X \)-ellipses degenerate into oriented lines in the direction of \( \tilde{x} \) when \( |\xi| = 0 \). The directions of these lines make up the sphere at spatial infinity in the asymptotic region of \( \overline{\mathbb{A}H} \), which is \( \mathbb{CP}^1_{\text{diag}} \) in the description (1.11). The core \( \mathbb{CP}^1_{\text{diag}} \) of \( \overline{\mathbb{A}H} \) is obtained in another degenerate limit of the \( X \)-ellipses, namely in the limit \( |\xi| \to \infty \), where they become circles of infinite radius.

We are interested in the quotients of \( \overline{\mathbb{A}H} \) by discrete symmetries which arise naturally from its description as \( \mathbb{CP}^1 \times \mathbb{CP}^1 \backslash \mathbb{CP}^1_{\text{diag}} \), namely the factor switching map \( s \), the antipodal maps on both factors \( a \) and the composition \( r = as \). It follows from the description of these maps in Appendix \( A \) that they act in the following way on the ellipse parameters, where the formulation in terms of \( (\tilde{x}, \xi) \) assumes that \( |\eta| \neq 0 \):

\[
s : (y, \eta) \mapsto (y, -\eta), \quad (\tilde{x}, \xi) \mapsto (-\tilde{x}, -\xi),
\]

\[
r : (y, \eta) \mapsto (-y, -\eta), \quad (\tilde{x}, \xi) \mapsto (\tilde{x}, -\xi),
\]

\[
a : (y, \eta) \mapsto (-y, \eta), \quad (\tilde{x}, \xi) \mapsto (-\tilde{x}, \xi).
\]  

(1.14)

In particular we see that \( s \) fixes the core \( \mathbb{CP}^1_{\text{diag}} \) but acts as the antipodal map on the sphere at spatial infinity \( \mathbb{CP}^1_{\text{diag}} \), while \( r \) fixes the sphere at spatial infinity and acts as the antipodal map on the core. Writing \( 1 \) for the identity map and defining the Vierergruppe

\[
\text{Vier} = \{1, s, r, a\},
\]  

(1.15)

we can characterise the Atiyah–Hitchin manifold \( \overline{\mathbb{A}H} \) and the moduli space \( \mathcal{M}_2^0 \) of centred 2-monopoles as the quotients

\[
\overline{\mathbb{A}H} = \overline{\mathbb{A}H}/s, \quad \mathcal{M}_2^0 = \overline{\mathbb{A}H}/r = \overline{\mathbb{A}H}/\text{Vier}.
\]  

(1.16)

It follows from our discussion of the generators, that in \( \mathbb{A}H \) the core is still a two-sphere, but the space of directions at spatial infinity is now \( \mathbb{CP}^1_{\text{diag}}/\mathbb{Z}_2 \cong \mathbb{RP}^2 \). Finally, in \( \mathcal{M}_2^0 \) both the core and the space of directions at spatial infinity are isomorphic to \( \mathbb{RP}^2 \).

The manifold obtained by quotienting \( \overline{\mathbb{A}H} \) by the free action of \( r \) is, literally, central to the construction of the Sen space. We therefore define

\[
\overline{\mathbb{A}H}/r = \mathbb{A}H.
\]  

(1.17)

This manifold still has a two-sphere of directions at spatial infinity, but its core is isomorphic to \( \mathbb{RP}^2 \). It allows us to write the moduli space \( \mathcal{M}_2^0 \) of centred 2-monopoles also as the quotient

\[
\mathcal{M}_2^0 = \overline{\mathbb{A}H}/s,
\]  

(1.18)

and this is precisely what we require for our construction. The situation is summed up in Fig. 1.

**Figure 1.** Coverings and quotients of the Atiyah–Hitchin manifold

Having defined the manifolds, we turn to their symmetries and metric structure. The rotation group \( SO(3) \) acts on all four manifolds in Fig. 1 by the obvious action of \( G \in SO(3) \) on \( X \in \mathbb{C}^3 \).
This action commutes with the action of the Vierergruppe $[14]$, so that the generic $SO(3)$ orbit is $SO(3)$ for $\widetilde{AH}$, $SO(3)/\mathbb{Z}_2$ for both $AH$ and $HA$ and $SO(3)/Vier$ for $\mathcal{M}_2^0$, with the generators $s, r$ and $\alpha$ realised as rotations by $\pi$ around three orthogonal axes. Away from the core $\mathbb{CP}_1^{diag}$, we have a $U(1)$ action which commutes with the $SO(3)$ action and which fixes the asymptotic direction $m = \tilde{x}/|\tilde{x}|$:

$$(e^\theta, (\tilde{x}, \xi)) \mapsto (\tilde{x}, R_m(\theta)\xi),$$

(1.19)

where $R_m(\theta)$ is the rotation about $m$ by an angle $\theta \in [0, 2\pi]$.

The Atiyah–Hitchin metric is most easily expressed in terms of the $SO(3)$ matrix $G$ and one transversal coordinate $\rho$ (bijectively related to ellipse parameter $|\tilde{x}|$). We define left-invariant 1-forms on $SO(3)$ via $G^{-1}dG = \sigma_1 t_1 + \sigma_2 t_2 + \sigma_3 t_3$ for generators $t_1, t_2, t_3 \in \mathfrak{so}(3)$ of the rotations around three orthogonal axes, satisfying $[t_i, t_j] = \varepsilon_{ijk} t_k$. Then the Atiyah–Hitchin metric is

$$g_{AH} = f^2 d\rho^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2,$$

(1.20)

where the choice of $f$ amounts to fixing the transversal coordinate $\rho$, and the radial functions $a, b$ and $c$ obey coupled differential equations which follow from the hyperKähler property of the metric $[4]$. As explained in $[18]$, the choice $f = -b/\rho$ results in a radial coordinate in the range $\rho \in [\pi, \infty)$, with $\rho = \pi$ corresponding to the core $\mathbb{CP}_1^{diag}$, and coefficient functions $a, b$ and $c$ with the asymptotic form

$$a \sim b \sim \rho \sqrt{1 - \frac{2}{\rho}}, \quad c \sim -\frac{2}{\sqrt{1 - \frac{2}{\rho}}},$$

(1.21)

and exponentially small corrections. Substituting the asymptotic form into (1.20) yields the negative-mass Taut-NUT metric as the leading term

$$g_{AH} = \left(1 - \frac{2}{|x'|}\right) |dx'|^2 + \left(1 - \frac{2}{|x'|}\right)^{-1} |\alpha|^2 + O(e^{-|x'|}),$$

(1.22)

where we made the identifications

$$\frac{x'}{|x'|} = G \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad |x'| = \rho, \quad \alpha = 2\sigma_3.$$

(1.23)

In the following we will refer to (1.20) as the Atiyah–Hitchin metric and to (1.22) as its asymptotic form regardless of whether the underlying manifold is $\widetilde{AH}, AH, HA$ or $\mathcal{M}_2^0$, even though the metric is singular at the core on $\widetilde{AH}$ and HA.

We can tie together the asymptotic Taub–NUT geometry with the description of $\widetilde{AH}$ (and its quotients) in terms of the $X$-ellipses by noting that both are $U(1)$ bundles over $\mathbb{R}^3$, with $x$ and the major axis $\tilde{x}$ being coordinates on the base. The directions of both $x$ and $\tilde{x}$ parametrise the two-sphere at spatial infinity and can be identified. The magnitudes of $\tilde{x}$ and $x$ are bijectively related, but not in any obvious way.

To end this review of the Atiyah–Hitchin geometry we note that the moduli space $\mathcal{M}_2^0$ equipped with the hyperKähler metric (1.20) is the, up to scaling, unique $D_0$ ALF gravitational instanton $[4]$, and, suitably interpreted, fits into the general construction outlined in $[13]$ with $k = 0$. In this case, no gluing is required since the manifold $HA$ has the required asymptotic structure, both topologically and metrically. The quotient (1.18) realises the division by an involution $\iota = s$ which is covered by the generator $S$ in (1.22), see our Appendix $A$ for details.

1.5. Further background. ALF gravitational instantons can be interpreted and realised in a number of different ways. A gauge-theoretical model was proposed by Cherkis and Kapustin in $[12]$, where they showed that the moduli space of a smooth and unit-charge $SU(2)$ monopole moving in the background of $k$ singular $U(2)$ monopoles is an $A_{k-1}$ ALF gravitational instanton of the Gibbons–Hawking form, and argued that the moduli space of a smooth and strongly central charge-two $SU(2)$ monopole moving in the background of $k$ singular $U(2)$ monopoles is a $D_k$ ALF gravitational instanton. Subsequently, Cherkis and Hitchin $[10]$ used twistor methods and a generalised Legendre transform developed in $[24]$ and $[21]$ for a rigorous construction of
$D_k$ ALF instantons. However, it was only shown in [34] that all $A_k$ ALF gravitational instantons are of the Gibbons–Hawking form and, even more recently in [9], that all $D_k$ ALF gravitational instantons are described by the Cherkis–Hitchin–Ivanov–Kapustín–Lindström–Roček metric which emerged from the papers [24, 21, 12, 10].

The interpretation of the $A_k$ and $D_k$ ALF spaces as moduli space of monopoles with prescribed singularities provides useful intuition about the geometry of these spaces. In particular, it suggests that, at least for singularities which are well-separated from the origin, one should be able to isolate a core region of the $D_k$ ALF space where the smooth and strongly centred monopoles do not ‘see’ the $k$ singular monopoles and which should therefore be well-approximated by the Atiyah–Hitchin geometry. The picture also suggests that there should be an asymptotic region of the moduli space where the two smooth monopoles, with their centre fixed at the origin, are well-separated and move between (and over) the $k$ singularities. Orienting the line joining the smooth monopoles amounts to double-covering this part of the moduli space, and so this double cover should be well-approximated by the moduli space of a single monopole moving in background of $2k$ symmetrically spaced singularities, i.e. by the $A_{2k-1}$ ALF geometry.

It is this intuitive picture, also captured in Sen’s proposal, which our theorem makes precise. It differs from that underlying the gluing construction using ALE instantons and the Eguchi-Hanson geometry, carried out in [27] and [8], even though the mathematical techniques are related. It also clarifies that our method will not produce all $D_k$ ALF gravitational instantons, but only those where the $k$ singularities are well-separated from the origin.

A compact version of the construction outlined in §1.3 was recently carried out in [15]: on K3, sequences of hyperKähler metrics were obtained which, away from a finite set of points, collapse to the flat metric on $T^3/Z_2$. While our construction is very close to this, our approach to the analysis is different and, as we shall now explain, in a certain sense more precise.

1.6. Statement of the main result. The initial version of our main theorem 1.1 describes our construction in classical terms. In order to formulate the final version, we shall write down a framework in which the parameter $\varepsilon$ is incorporated into the geometry of the problem. We will then carry out the construction in such a way that our family $g_\varepsilon$ is smooth in all variables (including $\varepsilon$) down to and including $\varepsilon = 0$. In that limit, the space $\mathcal{M}_3$ with the Atiyah–Hitchin metric and $k$ copies of the Taub–NUT geometry emerge as building blocks of the geometry, thereby justifying the interpretation, announced in our title, of the $D_k$ ALF geometry as a superposition of these spaces.

To explain this, start from the cartesian product $W_0 = \text{Se}_k \times I$, where $I = [0, \varepsilon_0)$. Then, at least for $\varepsilon > 0$, we can clearly think of our family $g_\varepsilon^k$ as a metric on the vertical tangent bundle $T(W_0/I)$ of $W_0$, vertical with respect to the projection

$$\pi_0 : W_0 = \text{Se}_k \times I \to I.$$ (1.24)

To emphasise this change of viewpoint, we shall denote by $g^k$ this metric on $T(W_0/I)$: then $g^k|\pi^{-1}(\varepsilon) = g_\varepsilon^k$. Now $g^k$ is not smooth at the boundary $\varepsilon = 0$ of $W_0$, but there is a modification, to be denoted by $W$, essentially obtained from $W_0$ by blowing up at $k+1$ points in the boundary $\varepsilon = 0$, on which, with suitable interpretation, it becomes smooth. The details appear in [5].

The projection $\pi_0$ is replaced by a smooth projection

$$\pi : W \to I,$$ (1.25)

$W \setminus \pi^{-1}(0)$ is canonically diffeomorphic to $W_0 \setminus \pi_0^{-1}(0)$, so in particular $\pi^{-1}(\varepsilon)$ is still the Sen manifold $\text{Se}_k$ for $\varepsilon > 0$.

However, $\pi^{-1}(0)$ is singular, a union of 4-manifolds

$$X_0 \cup X_1 \cup \ldots \cup X_k \cup X_{\text{ad}}.$$ (1.26)

Here $X_0 = \mathcal{M}_3^0$ and each of $X_1, \ldots, X_k$ is a copy of the Taub–NUT space. The subscript ad on the final boundary hypersurface stands for ‘adiabatic’. The interior of this hypersurface is the total space of the $S^1$-bundle $M^{\phi^{-1}}(P)$ which arises from the Gibbons–Hawking construction with $\tilde{h}_0$ as in (1.0), factored out by $\phi$. Denoting by $g_{\text{TN}}$ the Taub–NUT metric, our main theorem is stated as follows (see also Fig. 2).
Figure 2. Schematic picture of the spaces $W$ (left) and $W_0$ (right).

**Theorem 1.2.** There is a 5-manifold-with-corners $W$ which has the following properties:

(a) There is a smooth map $\pi: W \to I$ so that for $\varepsilon > 0$, $\pi^{-1}(\varepsilon)$ has a $D_k$ ALF hyperKähler metric $\tilde{g}_\varepsilon$ which is asymptotically the metric $g_{\varepsilon}$ [(1.5)] with potential [(1.9)];

(b) The fibre $\pi^{-1}(0)$ is a singular union of 4-manifolds [(1.26)], where for each $\nu$, the boundary of $X_\nu$ is joined to a boundary component of $X_{\text{ad}}$;

(c) There is a smooth vector bundle $T_\phi(W/I)$ on $W$ (a rescaled version of the vertical tangent bundle of $W$) such that

$$T_\phi(W/I)|\pi^{-1}(\varepsilon) = T\pi^{-1}(\varepsilon) \text{ for all } \varepsilon > 0,$$

and

$$T_\phi(W/I)|X_\nu = TX_\nu \text{ for } \nu = 0, \ldots, k.$$

for all $\nu$.

(d) The one-parameter family of metrics $\tilde{g}_\varepsilon$ is smooth on $W$ in the sense that there is a metric $\tilde{g}$ on $T_\phi(W/I)$, smooth on $W$ (up to and including $\pi^{-1}(0)$), such that

$$\tilde{g}|\pi^{-1}(\varepsilon) = \tilde{g}_\varepsilon, \tilde{g}|X_0 = g_{\text{AH}} \text{ and } \tilde{g}|X_\nu = g_{T_\nu} \text{ for } \nu = 1, \ldots, k.$$ (1.27)

We defer the definition of $T_\phi(W/I)$ to [43] essentially, it is spanned, locally, by the vector fields $\varepsilon \partial_\varepsilon$, and $\partial_\theta$, and so absorbs the factor $1/\varepsilon^2$ in the adiabatic family [(1.5)]. In particular it has a well-defined restriction to $X_{\text{ad}}$ and, the limit of this metric

$$\lim_{\varepsilon \to 0} \left( \frac{|dx|^2}{\varepsilon^2} + \alpha^2 \right)$$ (1.28)

makes sense as a smooth metric on $T_\phi(W/I)|X_{\text{ad}}$.

The use of manifolds with corners in the analysis of partial differential equations in non-compact and singular settings was pioneered by Richard Melrose. Of particular relevance to the underlying analytical techniques are [25, 29, 26, 27]. We note also references such as [1, 31, 32, 13] in which techniques including real blow-up and rescaling the tangent bundle are used in a variety of geometric contexts.

The present construction, in which gluing is combined with an adiabatic limit, seems to present some new challenges that have not been addressed before. On the other hand, there are also some special geometric features of this problem which simplify the analysis at a number of points, and we take full advantage of this, rather than developing the machinery in generality.
1.7. **Plan.** The proof of Theorem (1.2) proceeds via a reformulation of the problem it addresses in a number of ways. Instead of solving for hyperKähler metrics we use the formalism of hyperKähler triples [14] to obtain an elliptic formulation of the gluing problem. This method is described in §2. Then, instead of working on non-compact manifolds we formulate the problem on compact spaces with fibred boundaries. In §3 we explain why these compactified spaces are natural domains for the family of hyperKähler metrics on the Se spaces $S_{2k}$, and how the structure of the compact manifolds is such that the asymptotic behaviour of the metric can be encoded in smoothness and decay at all boundary hypersurfaces. After a short discussion of hyperKähler triples for the Gibbons–Hawking metrics (including the asymptotic form of the Atiyah–Hitchin metric) and their primitives in §4, we take the final step in the reformulation by incorporating the scaling parameter $\varepsilon$ in the geometry of the problem. This is dealt with in §5, where we also write down the initial approximation for the $D_k$ ALF gravitational instantons by gluing together the Atiyah–Hitchin and the multi-centre Gibbons–Hawking metric. In §6, we gather some preliminaries about the linearised problem and proceed to construct a formal solution, that is a family $g_{\varepsilon}(\varepsilon)$ of metrics that is hyperKähler to all orders in $\varepsilon$. The proof is completed in final §7 by using the inverse function theorem to perturb this family to be exactly hyperKähler for all sufficiently small positive $\varepsilon$.

1.8. **Outlook.** Our construction and main result can be extended in a number of ways by replacing the space $M^0$ with other dihedral ALF gravitational instantons and adapting the adiabatic Gibbons–Hawking Ansatz correspondingly.

The simplest generalisation in this spirit is to replace $M^0$ by AH, which is an example of a $D_2$ ALF gravitational instanton (and one which we cannot obtain by the construction of this paper). The branched cover of AH is the manifold $\tilde{AH}$ whose asymptotics, when written in the standard Gibbons–Hawking form, is that of a singular $A_2$ space with a single pole of weight $-2$. Thus one could construct $D_k$ ALF gravitational instantons for $k \geq 2$ by taking the adiabatic Gibbons–Hawking Ansatz [12] with $m(0) = -2$ and $2(k - 1)$ symmetrically placed NUTs, resolving the singularity at the origin with a copy of $\tilde{AH}$ and dividing by the involution $\iota$. In this way one would obtain $D_k$ instantons for $k \geq 2$ which are not included in the family constructed here, but which arise in the limit as one pair of NUTs approaches zero.

More generally, one could also iterate the construction by gluing the branched cover of a previously obtained $D_{k'}$ ALF gravitational instanton into a singular Gibbons–Hawking space with symmetrically placed NUTs. If the adiabatic Gibbons–Hawking Ansatz has weight $m(0) = 2k' - 4$, chosen to match the asymptotics of the given $D_{k'}$ ALF space, and a further $2k''$ symmetrically placed NUTs, one should obtain a $D_{k'+k''}$ ALF gravitational instanton in this way.

Finally, one may wonder if the geometrical interpretation of the 5-manifold $W$ in Theorem 1.2 can be extended from one where each fibre has a metric to a fully geometrical picture, with a 5-dimensional metric on $W$. This is possible for the (single NUT) Taub–NUT geometry, which can be extended to a warped product with a Lorentzian geometry satisfying the (4+1)-dimensional Einstein equations [17]. The coordinate $t = 1/\varepsilon$ is a natural time coordinate in this geometry, which fascinated Michael Atiyah as a possible time-dependent model of the electron [2]. It would clearly be interesting if this solution could be generalised to natural 5-dimensional metrics on the manifold $W$ for general ALF instantons, but we have not pursued this here.

2. **HyperKähler triples**

Let $M$ be an oriented riemannian 4-manifold.

**Definition 2.1.** A *symplectic triple* on $M$ is a triple $(\omega_1, \omega_2, \omega_3)$ of symplectic forms on $M$ such that the matrix $q$ with

$$q_{ij} = \omega_i \wedge \omega_j$$

is positive-definite at every point. A *hyperKähler triple* is a symplectic triple for which $q$ is a multiple of the identity at every point.
Remark 2.2. To clarify the definition, the matrix \( q \) is a symmetric \( 3 \times 3 \) matrix with values in \( \Lambda^4 T^* \). If \( \nu \in C^\infty(M, \Lambda^4 T^*) \) is any smooth positive section, then \( \nu^{-1} q \) is a genuine symmetric \( 3 \times 3 \) matrix at each point. To say that \( q \) is positive-definite is to say that \( \nu^{-1} q \) is positive-definite for one or equivalently any positive section \( \nu \) of \( \Lambda^4 T^* \).

If \( \omega \) is a hyperKähler triple, taking the trace of \( q_{ij} \omega \delta_{ij} \), we obtain
\[
\omega_i \wedge \omega_j = \frac{1}{3} (\omega^2_1 + \omega^2_2 + \omega^2_3) \delta_{ij}.
\]
Thus this equation holds if and only if \( \omega \) is a hyperKähler triple.

The reason for the introduction of these triples, and the notation, is explained by the following

**Theorem 2.3.** Let \( M \) be an oriented 4-manifold. If \( (\omega_1, \omega_2, \omega_3) \) is a symplectic triple on \( M \), then there is a unique metric \( g \) on \( M \) with
\[
\Lambda^2_+(g) = \langle \omega_1, \omega_2, \omega_3 \rangle
\]
and \( d\mu_g = \text{tr}(g) \). If \( \omega \) is a hyperKähler triple, then \( g \) is hyperKähler and the \( \omega_j \) are the Kähler forms associated with the three complex structures,
\[
\omega_j(\xi, \eta) = g(\xi, I_j \eta).
\]

In fact, the complex structures are defined by
\[
I_j \xi = *_g (\omega_j \wedge \xi).
\]

A relatively recent reference for the formalism of triples is [13], though the formulation was surely previously known to experts. The same formalism was used by [15, 9] in their recent work on 4-dimensional hyperKähler metrics. The advantage of working with triples is that (after taking account of gauge freedom) the hyperKähler condition is reformulated as a nonlinear PDE whose linearization is a Dirac operator, hence elliptic.

The proof of Theorem 2.3 rests on an old observation about the correspondence between metrics on an oriented 4-manifold \( M \) and maximal, positive subbundles of \( \Lambda^2 T^* M \). Of course a metric \( g \) on \( M \) determines the subbundle \( \Lambda^2_+(g) \) of self-dual 2-forms. In fact, this subbundle depends only upon the conformal class of \( g \), but it is always a maximal positive subbundle for the (conformal) quadratic form \( \alpha \mapsto \alpha \wedge \alpha \) on \( \Lambda^2 T^* M \). The correspondence is bijective: to any maximal positive subbundle \( P \subset \Lambda^2 T^* M \) there corresponds a unique conformal class \([g]\) with \( \Lambda^2_+ [g] = P \). Thus a metric representative \( g \) of the conformal class is determined by the choice of a volume form on \( M \), and in the context of Theorem 2.3 this is furnished by \( \text{tr}(g) \). The hyperKähler part of the theorem comes from the fact that the connection induced on \( \Lambda^2_+ \) by the metric is also characterized by metric-compatibility and a torsion-free condition. It turns out that if \( \omega \) is a hyperKähler triple, then this connection is flat, the \( \omega_i \) are parallel, and it follows (since they are pointwise orthogonal), that they are the Kähler forms of a hyperKähler metric.

### 2.1. Perturbative formulation.

For any triple of 2-forms \( \omega \), set
\[
Q(\omega) = \omega_i \wedge \omega_j - \frac{1}{3} (\omega^2_1 + \omega^2_2 + \omega^2_3) \delta_{ij}.
\]
Then \( Q \) is a symmetric, trace-free \( 3 \times 3 \) matrix, with values in \( \Lambda^2 T^* M \) and by (2.1), \( Q(\omega) = 0 \) if and only if \( \omega \) is a hyperKähler triple.

We shall study the perturbative version of this equation. That is, we fix a symplectic triple \( \omega \) with \( Q(\omega) \) small and seek a \((C^1\text{-small})\) triple of 1-forms \( a \) so that
\[
Q(\omega + da) = 0.
\]
More formally, \( a \mapsto Q(\omega + da) \) is a nonlinear differential operator
\[
\Omega^1(M) \otimes \mathbb{R}^3 \longrightarrow C^\infty(M, S_0^2 \mathbb{R}^3 \otimes \Lambda^4).
\]
This equation cannot be elliptic as the rank of the bundle on the left here is 12, while the rank on the right is 5. The difference in ranks, 7, is accounted for by the gauge-freedom of
the problem. Indeed, (2.5) is left invariant by the action of the orientation-preserving diffeomorphisms \( \text{Diff}^+(M) \); it is clearly also unchanged if \( a \) is replaced by \( a + df \), for any triple of functions \( f = (f_1, f_2, f_3) \). Thus there are 7 gauge degrees of freedom, and this count matches the difference in the ranks of the bundles in (2.6).

By fixing the gauge, we shall obtain an elliptic equation for the triple \( a \). To state the theorem, write \( D \) for the coupled Dirac operator

\[
d^* + d_+ : \Omega^1(M) \otimes \mathbb{R}^3 \to \Omega^0(M) \oplus \Omega^2_+ (M).
\]

This is determined by the metric \( g(\omega) \) of Theorem 2.3. To avoid excessive notation, we shall not distinguish between \( D \) and its tripled version, in which every bundle in (2.7) is tensored with \( \mathbb{R}^3 \).

**Lemma 2.4.** Let \( \omega \) be a symplectic triple as above and let \( a \) be a triple of 1-forms. Let \( \tilde{\omega} = \omega + da \) and let \( q = (q_{ij}) = \omega_i \wedge \omega_j \). Write \( p_{ij} \) for the inverse matrix \( q^{-1} \) and \( R = R_{ij} \) for the matrix

\[
R = \frac{1}{2}Q(\omega) + \frac{1}{2}Q(da), \tag{2.8}
\]

Then the equation

\[
d_+ a_j = -R_{js} p_{sk} \omega_k, \tag{2.9}
\]

implies that \( \omega + da \) is a hyperKähler triple.

Here the summation convention holds for all repeated indices.

**Proof.** From the definitions, \( Q(\omega + da) = 0 \) is equivalent to the equation

\[
Q(\omega, \omega) + 2Q(\omega, da) + Q(da, da) = 0 \tag{2.10}
\]

where we have committed an abuse of notation by writing \( Q(\omega, \eta) \) for the polarized version of the quadratic form \( Q \) (i.e. \( Q(\omega, \omega) = Q(\omega) \)). Thus for triples of 2-forms \( \omega \) and \( \eta \), \( Q(\omega, \eta) \) is by definition the projection onto the trace-free symmetric part of the matrix \( (\omega \wedge \eta) \). Thus (2.10) is implied by

\[
Q(\omega)_{js} + 2da_j \wedge \omega_s + Q(da)_{js} = 0,
\]

which we rewrite as

\[
da_j \wedge \omega_s = -R_{js}, \tag{2.11}
\]

using the definition of \( R \), and we claim this is equivalent to (2.9).

For the metric \( g = g(\omega) \) of Theorem 2.3 \( \Lambda_+^2 (g) \) is spanned by the \( \omega_j \), and so we have

\[
d_+ a_j = u_{js} \omega_s \tag{2.12}
\]

for some collection of functions \( u_{js} \). Then (2.11) is equivalent to

\[
u_{jk} \omega_k \wedge \omega_s = u_{jk} q_{ks} = -R_{js}
\]

and multiplying by the inverse of \( q \), we obtain

\[
u_{jk} = -R_{js} p_{sk}
\]

and hence, using (2.12),

\[
d_+ a_j = -R_{js} p_{sk} \omega_k,
\]

as required. \( \square \)

**Remark 2.5.** If \( \omega \) is itself a hyperKähler triple then \( Q(\omega) = 0 \) and (2.9) takes the simpler form

\[
d_+ a_j = -\frac{1}{2} \nu^{-1} Q(da)_{jk} \omega_k
\]

where \( \nu = q_{jj}/3 = (\omega^2_1 + \omega^2_2 + \omega^2_3)/3 \).

The following gives our elliptic formulation of the perturbative hyperKähler problem.
Theorem 2.6. Let the notation be as in Lemma 2.4. Define a nonlinear mapping
\[ \mathcal{F} : \Omega^1(M) \otimes \mathbb{R}^3 \to (\Omega^0(M) \oplus \Omega^2_+(M)) \otimes \mathbb{R}^3 \]
by
\[ a_j \mapsto (d^*a_j, d_+a_j + R_{js}p_{sk}\omega_k). \]  
Then if \( \omega + da \) is a symplectic triple and \( \mathcal{F}(a) = 0 \), it follows that \( \omega + da \) is a hyperKähler triple. Furthermore, the linearization of \( \mathcal{F} \) at \( a = 0 \) is the (tripled version of the) Dirac operator \( D_g(\omega) \).

Proof. Immediate from Lemma 2.4. \( \square \)

Let us write \( \mathcal{F} \) in the form
\[ \mathcal{F}(a) = Da + e + \hat{r}(da) \]  
where \( e = \mathcal{F}(0) \) and \( \hat{r}(da) \) is the part of \( R_{js}p_{sk}\omega_k \) which is quadratic in \( da \). Then \( e \) will be small if \( \omega \) is approximately hyperKähler. To find a small \( a \) solving (2.14), we shall seek
\[ a = D^*u, \quad u \in (\Omega^0(M) \oplus \Omega^2_+(M)) \otimes \mathbb{R}^3. \]  
Substituting into (2.14),
\[ \mathcal{F}(a) = 0 \iff DD^*u = -e - \hat{r}(dD^*u). \]  
A standard Weitzenböck formula relates \( DD^* \) to the rough Laplacian \( \nabla^*\nabla \) of the metric \( g = g(\omega) \), the curvature terms being the self-dual part \( W_+ \) of the Weyl curvature and the scalar curvature \( s \). These both vanish if \( g \) is hyperKähler and will be small if \( e \) is small. This suggests that if \( e \) is small enough, then \( DD^* \) should be invertible, given a suitable Fredholm framework for \( DD^* \), and then (2.16) should be solvable for \( u \) by the implicit function theorem. Since we want to solve \( \mathcal{F}(a) = 0 \) on an ALF space, finding the right Fredholm framework is one of the technical issues that we deal with this in this paper: then we shall be able to apply the implicit function theorem to (2.16) to construct hyperKähler triples on \( S_\kappa \), thereby proving Theorem 1.2.

3. ALF spaces and manifolds with fibred boundary

The purpose of this short section is to explain how to pass from an ALF space, which is normally regarded as a complete riemannian manifold with a metric with certain asymptotic behaviour, to its compactification as a manifold with fibred boundary and smooth \( \phi \)-metric. This latter point of view informs and motivates the analytical construction in subsequent sections.

Definition 3.1. Let \( X \) be a compact manifold with boundary. We say that \( X \) has fibred boundary, or \( \phi \)-structure, if its boundary is equipped with a smooth fibration \( \phi : \partial X \to Y \), where \( Y \) is a compact manifold.

Given such a manifold with fibred boundary, one can always choose local coordinates \( (\rho, y, z) \) near a boundary point, where \( \rho \geq 0 \) is (the restriction of) a boundary defining function (bdf), \( y \) are local coordinates in the base, and \( z \) are coordinates along the fibres. Then in these coordinates, \( \phi(y, z) = y \).

Definition 3.2. Given a manifold \( X \) with fibred boundary, the \( \phi \)-tangent bundle, \( T_\phi X \), is locally spanned over \( C^\infty(X) \), by the vector fields
\[ \rho^2 \frac{\partial}{\partial \rho}, \rho \frac{\partial}{\partial y}, \frac{\partial}{\partial z}. \]  
A \( \phi \)-metric on \( X \) is then a smooth (up to and including \( \partial X \)) metric on \( T_\phi X \).

These definitions appear in [20]. The point is that a \( \phi \)-metric on \( X \) defines, by restriction, a complete metric on the interior \( X^\circ \) of \( X \), with what may be called ‘generalized ALF’ asymptotics.
Example 3.3. Let $M = \mathbb{R}^n \times Z$, where $Z$ is a compact manifold without boundary. We may compactify $M$ as a manifold $X$ with fibred boundary by taking $X = \mathbb{R}^n \times Z$, where \( \mathbb{R}^n \) is the radial compactification of \( \mathbb{R}^n \), cf. [3.3]. Then $\partial X = \partial \mathbb{R}^n \times Z = S^{n-1} \times Z$ and the $\phi$-structure is just the projection on the first factor $S^{n-1}$. Then $M$ is identified with the interior $X^\circ$ of $X$.

Consider a product metric $(g_0, g_Z)$ on $M$, where $g_0$ is the euclidean metric on $\mathbb{R}^n$ and $g_Z$ is any riemannian metric on $Z$. This is the restriction to $X^\circ$ of a smooth $\phi$-metric on $X$. This follows from the fact that near the boundary $\partial \mathbb{R}^n$ of the radial compactification, the euclidean metric is quadratic in $\rho^2 \partial_\rho$ and the $\rho \partial y_j$, where $\rho = 1/|x|$ and the $y_j$ are local coordinates on $S^{n-1}$.

Example 3.4. If $h$ is a positive harmonic function on a subset $B = \{|x| > R\} \subset \mathbb{R}^3$ with $h(x) \to 1$ for $|x| \to \infty$, then the Gibbons–Hawking Ansatz gives an ALF metric on the total space of a circle-bundle $\phi : U_0 \to B_0$. One checks that $h$ extends smoothly to the closure $B$ of $B_0$ in $\mathbb{R}^3$. There is also an extension $U \to B$ of the circle-bundle, and one checks that the connection 1-form $\alpha$ on $U_0$ also extends smoothly to $U$. The associated Gibbons–Hawking metric then extends smoothly to define a $\phi$-metric on $U$, with boundary fibration equal to the restriction of $\phi$ to $U$.

Observe that in this example, we have a neighbourhood of $\partial U$ which carries a circle-action which acts isometrically on our $\phi$-metric.

Remark 3.5. In the previous example, if the Gibbons–Hawking metric is also invariant by a finite group $\Gamma$ acting freely on $U$ and respecting $\phi$, then the compactified quotient will again be an example of a $\phi$-manifold with $\phi$-metric.

The next definition captures the notion of a $\phi$-metric on a four-manifold begin asymptotically modelled by a Gibbons–Hawking metric:

Definition 3.6. Let $(X, g)$ be a $\phi$-manifold of dimension 4, with smooth $\phi$-metric $g$. Let $\rho$ be the bdf of $\partial X$ in $X$.

We say that $g$ is strongly ALF if $\phi : \partial X \to Y$ is the total space of a principal $S^1$-bundle and with respect to some extension $\phi : U \to B$, where $U$ is a collar neighbourhood of $\partial X$ in $X$, the circle-action preserves $g$ to infinite order in $\rho$, $\mathcal{L}_{\partial_\phi} g = O(\rho^2)$.

Suppose that $(X, g)$ is strongly ALF, hyperKähler, and that the $S^1$-action also preserves the hyperKähler triple to infinite order in $\rho$. Then in $U$, $g$ must arise from the Gibbons–Hawking Ansatz (Example 3.4) up to error terms that are $O(\rho^2)$.

Notation 3.7. In order to avoid constant changes of variables from euclidean variables $x$ to coordinates $(\rho, y_1, y_2)$ adapted to the boundary, it is often convenient to work with local coordinates $(x_1, x_2, x_3, \theta)$ in an asymptotic chart. When doing so it has to be remembered that the $\partial/\partial x_j$ and $\partial/\partial \theta$ define a local basis of $T_\rho X$ all the way up to the boundary (which is at $|x| = \infty$), and that ‘smooth in $(x, \theta)$’ also means smooth all the way to the boundary: in other words smooth after the change of variables $(\rho, y_1, y_2, \theta)$, up to and including $\rho = 0$.

4. HyperKähler triples for Gibbons–Hawking metrics

In this section we shall see how the formalism of hyperKähler triples, introduced in [12] works for Gibbons–Hawking metrics and (asymptotically) for the Atiyah–Hitchin metric.

For the adiabatic Gibbons–Hawking metric $g_\varepsilon$ in (4.1), it is easy to verify that

$$
\omega_1 = \alpha \wedge \frac{dx_1}{\varepsilon} + h_\varepsilon \frac{dx_2 \wedge dx_3}{\varepsilon^2} \\
\omega_2 = \alpha \wedge \frac{dx_2}{\varepsilon} + h_\varepsilon \frac{dx_3 \wedge dx_1}{\varepsilon^2} \\
\omega_3 = \alpha \wedge \frac{dx_3}{\varepsilon} + h_\varepsilon \frac{dx_1 \wedge dx_2}{\varepsilon^2}
$$

form a hyperKähler triple. It is also straightforward to check that for fixed $\varepsilon > 0$, the $\omega_j$ extend to define smooth sections of $\Lambda^2 T_\phi X$, where $X$ is the compactification of $M$ described in the previous section.
4.1. Primitives of Gibbons–Hawking triples. When we patch our hyperKähler triples together, we want a simple construction which at least yields a symplectic triple. This means working at the level of primitives of the $\omega_j$. For the harmonic function

$$h_\varepsilon = 1 - \frac{2\varepsilon}{|x|} + \sum_{p \in P\setminus\{0\}} \frac{\varepsilon}{2|x-p|}$$

(4.2)

introduced in (1.9), decompose $h_\varepsilon$ near 0 as

$$h_\varepsilon = H_\varepsilon + u_\varepsilon$$

(4.3)

where

$$H_\varepsilon = 1 + \mu\varepsilon - \frac{2\varepsilon}{|x|}.$$  

(4.4)

Then

$$u_\varepsilon$$

is harmonic and $O(\varepsilon|x|^2)$ near $x = 0$.  

(4.5)

This estimate arises from expanding the formula for $h_\varepsilon$ in powers of $x$ and observing that there can be no linear term because of the reflection-invariance. The constant term in the expansion of $h_\varepsilon$ is absorbed by $\mu$ in $H_\varepsilon$,

$$\mu = \sum_{p \in P\setminus\{0\}} \frac{1}{2|p|}. $$

(4.6)

Working in a fixed small ball $B(0, r)$, let $\omega_Z$ be the hyperKähler triple (4.1) of the Gibbons–Hawking metric with with potential $H_\varepsilon$ and let $\eta$ be the triple of 2-forms obtained by replacing $h_\varepsilon$ by $u_\varepsilon$ in (4.1). Observe that $\eta$ is a triple of 2-forms on $\mathbb{R}^3$, so that $\eta \land \eta = 0$ (as 4-forms in 3 dimensions).

Then we have

$$\omega_\varepsilon = \omega_Z + \eta.$$  

(4.7)

The next result shows how to write $\eta = d b$ in $B(0, r)$ with an estimate on the size of the coefficients of $b$. We give a statement that is slightly more general than the one we need:

**Proposition 4.1.** Let $u = u_\varepsilon = O(\varepsilon|x|^n)$ be smooth and harmonic in $B(0, r)$ and let $\eta$ be as above. There exists a primitive $b_i$ for $\eta_i$, $d b_i = \eta_i$ such that the coefficients of $b_i$, when expanded in the rescaled basis $d x_j/\varepsilon$, are smooth and $O(|x|^{n+1})$ in $B(0, r)$.

**Proof.** It is enough to consider

$$\eta_1 = \gamma \land \frac{d x_1}{\varepsilon} + u \frac{d x_2 \land d x_3}{\varepsilon^2}$$

(4.8)

where the 1-form $\gamma$ satisfies $d\gamma = \ast\varepsilon d u$. We first need to estimate the size of $\gamma$.

For this and the subsequent estimate of the size of $b$, we use a simple quantitative form of the Poincaré Lemma. It is convenient to work with the rescaled basis

$$e_i = d x_i/\varepsilon$$

(4.9)

Suppose that $f$ is a $p$-form in the ball in $\mathbb{R}^3$, whose coefficients are $O(|x|^m)$ with respect to the basis (4.9). Then the proof of the Poincaré Lemma using the retraction of the ball to the origin gives a $(p-1)$-form $v$ with $d v = f$ and all of whose coefficients are $O(\varepsilon^{-1}|x|^{m+1})$ in the same basis.

Applying this to the equation $d\gamma = \ast\varepsilon d u$, we see that $\gamma$ can be found with $O(\varepsilon|x|^n)$ coefficients. This follows because $\ast\varepsilon d u$, expanded in the basis $e_i$, has $O(\varepsilon^2|x|^{n-1})$ coefficients.

Substituting in (4.8), $\eta$ has $O(\varepsilon|x|^n)$ coefficients. Using the Poincaré Lemma again, we find that $d b = \eta$ can be solved with coefficients that are $O(|x|^{n+1})$ in the basis (4.9). $\square$

Applying this in the case of interest:

**Proposition 4.2.** Let $h_\varepsilon$, $H_\varepsilon$ and $u_\varepsilon$ be as in equations (4.2, 4.3), and let the hyperKähler triples of $g_\varepsilon$ and $g_Z$ be denoted $\omega_\varepsilon$ and $\omega_Z$. Then in a small neighbourhood of 0, there is a triple of 1-forms $B$, whose coefficients in the basis $d x_j/\varepsilon$ of (4.9) are $O(|x|^3)$, such that

$$\omega_\varepsilon = \omega_Z + d B.$$  

(4.10)
4.2. **HyperKähler triples for AH.** A hyperKähler triple for the Atiyah–Hitchin metric was written down in [21]. We shall not use this directly, because the important thing for us is to compare the AH triple with the triple of the asymptotic Taub–NUT model. This is straightforward because these metrics differ by exponentially small terms (1.22).

In the metric on $\mathcal{M}_2^0$, the size of the circle at $\infty$ can be varied. In particular, there is a 1-parameter family of AH metrics approximated in an asymptotic Gibbons–Hawking chart by the Gibbons–Hawking metric determined by the harmonic function

$$H'(x') = 1 + \mu \varepsilon - \frac{2}{|x'|}$$

(4.11)

where $\mu > 0$ will be defined by (4.6). Denote by $g_{Z,\varepsilon}$ the TN metric with this potential and by $g_{AH,\varepsilon}$ the AH metric on $\mathcal{M}_2^0$ asymptotic to $g_{Z,\varepsilon}$. Since $g_{AH,\varepsilon}$ is exponentially close to $g_{Z,\varepsilon}$ for large $|x'|$, the following result, comparing the hyperKähler triples for these two metrics, is straightforward:

**Proposition 4.3.** Denote by $\omega_Z$ the hyperKähler triple of $g_{Z,\varepsilon}$ and by $\omega_{AH,\varepsilon}$ the hyperKähler triple of $g_{AH,\varepsilon}$. Then there exists a triple of 1-forms $A$ such that for $\varepsilon \geq 0$,

$$\omega_{AH,\varepsilon} = \omega_Z + dA,$$

with $A$ (and all derivatives) exponentially decaying as $|x'| \to \infty$.

5. **Gluing space and initial approximation**

The goal of this section is to give a systematic discussion of the space $W$ and the bundle $T_\phi(W/I)$ that appear in Theorem 1.2. One should view $W$ as a space on which our family of metrics $g^x$ are ‘resolved’, i.e. become smooth. As a warm-up, we start with the resolution of the adiabatic family of Gibbons–Hawking metrics $g_\varepsilon$, and for this we start by resolving the family of harmonic functions $h_\varepsilon$ defined in (1.4) (with all $m(p) = 1$).

**Figure 3.** Schematic of the spaces $B$ (left) and $\mathbb{R}^3 \times [0, \varepsilon_0]$ (right). The horizontal arrow is the blow-up map $\beta$ and the horizontal lines are the singularity sets respectively of $h$ and $h_\varepsilon$.

5.1. **Resolution of $h_\varepsilon$.** Recall the definition (1.4)

$$h_\varepsilon = 1 + \sum_{p \in P} \frac{\varepsilon}{2|x - p|},$$

(5.1)
where $P$ is a finite subset of $\mathbb{R}^3$. Of course for fixed $\varepsilon > 0$, $h_\varepsilon$ is singular for $|x - p| \to 0$, but we are concerned here with the indeterminacy in $\varepsilon/|x - p|$ for $\varepsilon \to 0$, $|x - p| \to 0$. This is resolved by passing to the real blow-up of $\mathbb{R}^3 \times [0, \varepsilon_0)$ in the finite set $P \times \{0\}$,

$$B = [\mathbb{R}^3 \times [0, \varepsilon_0); P \times \{0\}]$$

(5.2)

The reader is referred to [1,4] for a brief description of this blow-up: a more thorough introduction can be found, for example, in [1].

Denote by $\Sigma$ the lift to $B$ of $P \times [0, \varepsilon_0)$. Denote by $Y$ the lift of $\varepsilon = 0$ to $B$, and for $p \in P$, denote by $F_p$ the boundary hypersurface of $B$ which arises from the blow-up of $\{p\} \times \{0\}$ (the front face or exceptional divisor)—Figure 3. Then $F_p$ is a radially compactified $\mathbb{R}^3$ and $x'_p = (x - p)/\varepsilon$ are euclidean coordinates on its interior (cf. [1,4]).

**Proposition 5.1.** Denote by $h$ the pull-back of $h_\varepsilon$ to $B$. Then $h$ is smooth on $B \setminus \Sigma$, and for each $p$, $h - 1/2|x'_p|$ is smooth near $F_p$.

**Proof.** The assertion is essentially the definition of blow-up. Writing $h_\varepsilon$ in terms of the new variable $x'_p$, for some given $p$, we have

$$h = 1 + \frac{1}{2|x'_p|} + \sum_{q \neq p} \frac{\varepsilon}{2|p - q + \varepsilon x'_p|}$$

(5.3)

We may use $(x'_p, \varepsilon)$ as coordinates on any set of the form $\delta|x'_p| < 1$ inside $B$, in other words in a collar neighbourhood of ball in $F_p$. Then it is clear that $h - 1/2|x'_p|$ is smooth on such a set. Near the corner $F_p \cap Y_{ad}$, $\rho = 1/|x'_p|$ and $\sigma = \varepsilon|x'_p|$ are local boundary defining functions, and these are completed with local coordinates $(y_1, y_2) \in S^2$. In such coordinates,

$$h = 1 + \frac{1}{2\rho} + \sum_{q \neq p} \frac{\rho\sigma}{2|p - q + \sigma y|}$$

and for small $\sigma$ this is smooth in $(\rho, \sigma, y_1, y_2)$.

Finally, consider $h$ near the ‘adiabatic’ face $Y_{ad}$. Away from the corners, $|x - p| > \delta$ for all $p \in P$ and some $\delta$ and $(x, \varepsilon)$ are valid coordinates in such regions of $B$. Clearly $h$ is smooth on such a set. □

As indicated in Figure 3 there is a natural map $\pi : B \to [0, \varepsilon_0)$, the composite of the blow-up map and the projection $\pi_0$ of $\mathbb{R}^3 \times [0, \varepsilon_0)$ on its second factor. In terms of this map, $h|\pi^{-1}(\varepsilon) = h_\varepsilon$ for $\varepsilon > 0$.

5.2. **Rescaled tangent bundles and resolution of the adiabatic Gibbons–Hawking family.** For each $\varepsilon > 0$, the Gibbons–Hawking metric $g_\varepsilon$ lives on a manifold $M$ independent of $\varepsilon$ (see [1,4,7]), and equipped with a map $\phi : M \to \mathbb{R}^3$. The family $g_\varepsilon$ will be resolved on the space $W$ in the following definition:

**Definition 5.2.** (Figure 4) Let $W \to B$ be the pull-back by $\text{pr}_1 \circ \beta : B \to \mathbb{R}^3$ of $\phi : M \to \mathbb{R}^3$. Abuse notation by writing $\phi : W \to B$ for the pull-back of $\phi$, and write $\pi$ for the projection $W \to I$. Label the boundary hypersurfaces of $W$ as follows: $X_p$ is the pre-image by $\phi$ of $F_p$, for each $p \in P$; $X_{ad}$, the ‘adiabatic boundary hypersurface’ is the pre-image of $Y_{ad} \subset B$; and ‘spatial infinity’ $I_\varepsilon$ is the lift of the radial boundary $\partial \mathbb{R}^3 \times I$ of $B$ to $W$.

The most important points about $W$ are summarized in the following Proposition:

**Proposition 5.3.** For each $p \in P$ the restriction $\phi : X_p \to F_p$ is (the radial compactification of) the standard Hopf map from $TN$ to $\mathbb{R}^3$. (In particular, each $X_p$ is a radially compactified $\mathbb{R}^4$, equipped with a boundary fibration.) The adiabatic boundary hypersurface $X_{ad}$ is the total space of a circle-bundle over $Y_{ad} \subset B$; the restriction to the interior of $X_{ad}$ is canonically identifiable with the restriction of $\phi : M \setminus \phi^{-1}(P) \to \mathbb{R}^3 \setminus P$.

For $\varepsilon > 0$, $\pi^{-1}(\varepsilon)$ is equal to $M$, but $\pi^{-1}(0)$ is the union of $X_{ad}$ and the $X_p$, with $\partial X_p$ attached to the corresponding component of the boundary of $X_{ad}$.

**Proof.** This can be proved by calculations similar to those in the proof of Proposition 5.1. □
introduce a modification of the tangent bundle of $W$ in which the base directions are being stretched relative to the fibres as

Strictly speaking, the above definition only makes sense where $\phi$ is a fibration. Now $\phi$ fails to be a fibration only near the centres of the faces $\Sigma \subset B$ (not shown) and for $\varepsilon > 0$, $\pi^{-1}(\varepsilon)$ is just a copy of $M$, the 4-manifold on which $g_\varepsilon$ is defined.

The coefficients of the metric $g_\varepsilon$ are resolved on $W$, by Proposition 5.1, but we have to introduce a modification of the tangent bundle of $W$ to deal with the adiabatic behaviour of $g_\varepsilon$, in which the base directions are being stretched relative to the fibres as $\varepsilon \to 0$. This is quite analogous to the introduction of the $\phi$-tangent bundle in the discussion of ALF metrics in §3.

**Definition 5.4.** Define $T_\phi(W/I) \to W$ to be the smooth vector bundle locally spanned over $C^\infty(W)$ by the lifts of $\varepsilon \partial / \partial x_j$ and $\partial / \partial \theta$ on $M \times I$. The notation $W/I$ indicates that the sections of $T_\phi(W/I)$ are tangent to the fibres of $\pi : W \to I$, at least where these are smooth; the subscript $\phi$ indicates that for fixed $\varepsilon > 0$, the restriction of this bundle to $\pi^{-1}(\varepsilon)$—which is a compact manifold with fibred boundary—is canonically isomorphic to $T_\phi \pi^{-1}(\varepsilon)$.

**Remark 5.5.** Strictly speaking, the above definition only makes sense where $\phi$ is a fibration. Now $\phi$ fails to be a fibration only near the centres of the faces $X_p$, and here we define $T_\phi(W/I)$ simply to be the $\pi$-vertical sub-bundle of $TW$. This slightly subtle point shows that in the definition of $T_\phi(W/I)$, all that is really needed is a fibration $\phi$ defined near the adiabatic face $X_{ad}$.

Recall that $\phi$ in particular equips $X_p$ with a boundary fibration structure. Therefore $T_\phi X_p$ makes sense, and we shall see that the restriction of $T_\phi(W/I)$ to any of the $X_p$ is canonically isomorphic to $T_\phi X_p$. We shall denote the restriction of $T_\phi(W/I)$ to $X_{ad}$ by $T_{\phi-ad}(X_{ad})$. This is spanned formally by the $\varepsilon \partial / \partial x_j$ and $\partial / \partial \theta$ at $\varepsilon = 0$, and it is not hard to see that (at least over the interior of $X_{ad}$) this is the direct sum of $TY$ and the vertical tangent bundle of the fibration $X_{ad} \to Y$.

**Proposition 5.6.** The lift of $g_\varepsilon$ to $W$ defines a smooth metric $g$ on $T_\phi(W/I)$. The restriction of $g$ to $X_p$ is a copy of the Taub–NUT metric, and its restriction to $X_{ad}$ is

$$g_{ad} := e_1^2 + e_2^2 + e_3^2 + \alpha_{ad}^2,$$

where $e_j = dx_j / \varepsilon$ and $\alpha_{ad}$ is the restriction of $\alpha$ from (1.7) to $X_{ad}$.

**Proof.** These are straightforward computations, given a bit of familiarity with the spaces $B$ and $W$. If we look at a hypersurface $X_p$, then staying away from its boundary, we may use the local coordinates $x'_p = (x - p) / \varepsilon$ introduced in Proposition 5.1, $\varepsilon$ and $\theta$. Then $dx'_p = dx / \varepsilon$ and so the metric near the interior of $X_p$ has the form

$$\left(1 + \frac{1}{2|x'_p|}\right)|dx'_p|^2 + \left(1 + \frac{1}{2|x'_p|}\right)^{-1}(\alpha'_p)^2 + O(\varepsilon),$$

(5.4)
where the \( O(\varepsilon) \) error term is smooth in \( x' \), and \( \alpha_p' \) is the standard \( SO(3) \)-invariant connection 1-form on the Hopf bundle over \( \mathbb{R}^3 \). This shows that restriction of our metric is smooth in neighbourhoods of the interior of \( X_p \), for any \( p \).

Near the corner, the main point is to see the behaviour of the basis 1-forms, since we have already verified that \( h \) is smooth there. If, as in Proposition 5.1 we use the adapted coordinates \((\rho, \sigma, y_1, y_2, \theta)\), \( T^*_{\phi}(W/I) \) is spanned by

\[
\frac{d\sigma}{\rho\sigma}, \frac{d\rho}{\rho^2}, \frac{dy}{\rho}, d\theta,
\]

with

\[
\frac{d\sigma}{\rho\sigma} + \frac{d\rho}{\rho^2} = 0
\]

in \( T^*_{\phi}(W/I) \). (Remember that \( T^*_{\phi}(W/I) \) is the quotient of \( T^*W \) by the conormal to the fibres, which is spanned by \( \frac{d\varepsilon}{\varepsilon} = \frac{d\rho}{\rho} + \frac{d\sigma}{\sigma} \).) The forms

\[
\frac{d\sigma}{\rho\sigma} = \frac{d\sigma}{\varepsilon}, \frac{dy}{\rho} = \frac{\sigma dy}{\varepsilon}, d\theta
\]

are more convenient for the discussion of the metric near \( X_{\text{ad}} \). Indeed, \((\sigma, y_1, y_2)\) are local polar coordinates around \( x = p \), so this basis is smoothly identifiable with the basis \((\varepsilon_1, \varepsilon_2, \varepsilon_3, \alpha_{\text{ad}})\) appearing in the statement of the Proposition. These arguments identify the claimed restrictions of \( T^*_{\phi}(W/I) \) to \( X_p \) and \( X_{\text{ad}} \) and also the assertions about the smoothness of \( g \) on \( W \) as well as its restrictions to the various boundary hypersurfaces. \(\square\)

5.3. Sen space. We now modify the above construction to define a variant of our space \( W \) in which the smooth fibre \( \pi^{-1}(\varepsilon) \) is no longer an \( ALF \) \( A_k \) gravitational instanton but instead the Sen model \( S_{\varepsilon k} \) of a \( D_k \) ALF gravitational instanton.

The function \( h_\varepsilon \) is replaced by \( \hat{h}_\varepsilon \) from (1.9):

\[
\hat{h}_\varepsilon = 1 - \frac{2\varepsilon}{|x|} + \sum_{p \in P \setminus \{0\}} \frac{\varepsilon}{2|x - p|}
\]

and the main point is to see that in the construction of \( W \), it is possible to resolve the singularity in \( M \) over \( x = 0 \) by gluing in the branched cover \( \text{HA} \) of \( \mathcal{M}_g^2 \). Here are the steps we need to take—we put hats on everything to avoid confusion with what we did in the previous section:

- Choose \( P \) to be symmetric, \( P = -P \) and to contain 0.
- Let \( \hat{B} \) be constructed by blowing up \( P \times \{0\} \) inside \( \mathbb{R}^3 \times [0, \varepsilon_0] \), cf. (5.2). Let \( \hat{Y}_{\text{ad}} \) be the lift of \( \varepsilon = 0 \) as before.
- Let \( \hat{W}_0 \) and \( \hat{W} \) be defined as in (5.2). The only difference is that now these spaces have orbifold singularities over \( \{0\} \times [0, \varepsilon_0] \) because the degree of the circle-bundle around 0 is 4 rather than 1. Denote by \( \hat{X}_{\text{ad}} \) the inverse image by \( \phi \) of \( \hat{Y}_{\text{ad}} \).
- Define \( T^*_{\phi}(\hat{W}/I) \) as above. Observe that in the construction of \( \hat{g}_\varepsilon \) from (5.3), symmetry of \( P \) means that the circle-bundle \( \hat{M}_\varepsilon \) can be chosen to carry an involution \( \iota \) covering \( x \rightarrow -x \) on \( \mathbb{R}^3 \), and such that \( \iota^* \hat{\alpha} = -\alpha \).
- The analogue of Proposition 5.6 holds as stated apart from at the boundary face \( \hat{X}_0 \), where the metric is not defined because \( \hat{h}_\varepsilon \) is not everywhere positive.

Notation 5.7. Because the non-zero elements \( p \) and \( -p \) are identified when we divide by the involution, it is now convenient to pick \( p_1, \ldots, p_k \in P \setminus \{0\} \) so that

\[
P \setminus \{0\} = \{\pm p_1, \pm p_2, \ldots, \pm p_k\}.
\]

Then \( \pi^{-1}(0) = X_{\text{ad}} \cup X_0 \cup X_1 \cup \ldots \cup X_k \), as in the statement of Theorem 1.2 in the Introduction.

To avoid excessive notation, we shall now use \( W \) for the quotient \( \hat{W}/\iota \), hoping that this does not cause undue confusion the previous section (Definition 5.2).

The properties of our newly defined \( W \) are summarised in the following result.
Proposition 5.8. There exists a smooth 5-manifold (with corners) \( W \) with boundary hypersurfaces \( X_{ad}, X_{p} (\nu = 0, 1, \ldots, k) \) and \( I_{c} \). The boundary hypersurface \( X_{ad} \) is the total space of a circle-bundle \( \phi : X_{ad} \to Y_{ad} \) and \( W \) is equipped with a smooth projection \( \pi : W \to [0, \varepsilon] \).

Moreover:

(i) for \( \varepsilon > 0 \), \( \pi^{-1}(\varepsilon) \) is a copy of the \( Se_{k} \);

(ii) there is a smooth metric \( g^{X} \) on \( T_{\phi}(W/I) \) whose restriction to \( X_{0} \) is the \( AH \) metric on \( \mathcal{M}_{2}^{0} \), and whose restriction to \( X_{\nu} \) for \( \nu \neq 0 \) is the Taub-NUT metric;

(iii) \( X_{ad} \) and \( Y_{ad} \) are respectively \( \mathbb{Z}_{2} \) quotients of \( \tilde{X}_{ad} \) and \( \tilde{Y}_{ad} \) from the above construction of \( \tilde{W} \), and the restriction of \( g^{X} \) to \( X_{ad} \) is the quotient by \( \iota \) of \( \tilde{g}_{ad} \) (cf. Proposition 5.6).

Proof. In \( \tilde{W} \), consider a neighbourhood of the boundary \( \tilde{X}_{0} \). As before local coordinates can be taken to be \( \rho = 1/|x'|, \sigma = \varepsilon|x'| = |x|, \) along with local coordinates \( y \in \partial \tilde{X}_{0} \) and \( \theta \) along the fibre of \( \phi \). In terms of \( x' \) and \( \varepsilon \) (which are coordinates valid over \( \tilde{X}_{0} \)),

\[
\hat{h}_{\varepsilon} = 1 - \frac{2}{|x'|} + \mu \varepsilon + O(\varepsilon|x'|^{2}) \tag{5.6}
\]

where

\[
\mu = \sum_{\nu = 1}^{k} \frac{1}{|p_{\nu}|} \tag{5.7}
\]

and the error term can also be written \( O(\rho \varepsilon^{3}) \). By design, the metric \( \hat{g}_{\varepsilon} \), near the boundary of \( \tilde{X}_{0} \), matches the known asymptotics of the \( AH \) manifold and metric. It is convenient to include the next term in the expansion (5.6), so we take the produce \( HA \times [0, \varepsilon] \) equipped with the one-parameter family \( g_{AH, \varepsilon} \) of \( AH \) metrics where the coefficient of \( d\theta^{2} \) is \( 1 + \mu \varepsilon \) at \( \infty \) (cf. (4.2)).

Then we identify a region of the form

\[
\{(x', \theta, \varepsilon) \in \tilde{X}_{0} : 1 < \delta|x'| < 2 \} \subset HA
\]

with a subset \( \delta/2 < \rho < \delta \) near the corner, mapping \( (x', \theta, \varepsilon) \) to \( \rho = 1/|x'|, \quad y = x'/|x| \) and \( \sigma = \varepsilon|x'|. \) In this way we replace the orbifold \( \tilde{X}_{0} \) in \( \tilde{W} \) by a copy of \( HA \), getting a new version of \( W \), which we shall briefly denote by \( \tilde{W}' \). The construction of \( W \) is completed by dividing \( \tilde{W}' \) by the involution \( \iota \). This has the effect of replacing \( HA \) by \( \mathcal{M}_{2}^{0} \) with the \( AH \) metric, and identifying the other faces \( X_{p_{\nu}} \) and \( X_{-p_{\nu}} \) in pairs. This gives the description of the boundary hypersurfaces of \( W \) as claimed in the Proposition.

To construct a smooth metric on \( T_{\phi}(W/I) \) with the claimed restriction properties, let

\[
H_{\varepsilon}(x') = 1 + \mu \varepsilon - \frac{2}{|x'|} \tag{5.8}
\]

(cf. (4.11)) and let \( g_{Z} \) be the associated family of Gibbons–Hawking metrics. Then we have

\[
g_{AH, \varepsilon} - g_{Z} = u = O(\rho^{\infty}) \tag{5.9}
\]

and

\[
\hat{g}_{\varepsilon} - g_{Z} = v = O(\rho \varepsilon^{3}) \tag{5.10}
\]

Let \( \chi(t) \) be a standard smooth cut-off function equal to 1 for \( t < \delta/2 \) and equal to 0 for \( t \geq \delta \). Let

\[
g^{X} = g_{Z} + \chi(\sigma)u + \chi(\rho)v \tag{5.11}
\]

This is defined initially in the collar neighbourhood \{0 < \rho, \sigma < \delta\}, but for \( \rho < \delta/2 \) is equal to

\[
g_{Z} + v + \chi(\sigma)u = \hat{g}_{\varepsilon} + \chi(\sigma)u
\]

which therefore extends smoothly to the whole of \( W \). For \( \sigma > \delta \), this is identically equal to \( \hat{g}_{\varepsilon} \) but because \( u = 0 \) on \( X_{ad} \) by (4.9), it is also the case that \( g_{Z}|X_{ad} = \hat{g}_{\varepsilon}|X_{ad} \). Similarly, for \( \sigma < \delta/2 \), (5.10) may be rewritten

\[
g^{X} = g_{Z} + u + \chi(\rho)v = g_{AH, \varepsilon} + \chi(\rho)v. \tag{5.12}
\]

In this form it is clear that \( g^{X} \) extends smoothly over \( X_{0} \) (after factoring out by the involution to remove the singularity of \( g_{AH, \varepsilon} \) at the core, when viewed as a metric on \( HA \)), is identically
equal to \( g_{AH,\varepsilon} \) over the region \( \delta |x'| < 1 \) and is equal to \( g_{AH,\varepsilon} \) on \( X_0 \) itself because \( v \) vanishes on \( X_0 \) (which is defined by \( \sigma = 0 \)).

**Remark 5.9.** Our construction started from the adiabatic Gibbons–Hawking family: by making it smooth, we were led to the space \( W \), with the AH and TN geometries appearing naturally at the boundary.

It is also possible to start with \( \mathcal{M}_2^0 \) and construct the same space \( W \) as follows. Let \( X_0 \) be the compactification of \( \mathcal{M}_2^0 \) as a \( \phi \)-manifold, with boundary defining function \( \rho = 1/|x'| \), where \( x' \) are euclidean coordinates in the basic fibre of the fibration near \( \partial X_0 \). Take \( X_0 \times [0, \varepsilon_0) \) and let \( Z \) be the blow-up of the corner \( \partial X_0 \times \{0\} \). The front face of \( Z \) is a ‘stretched’ version of the asymptotic region of \( X_0 \). In particular it is (up to the action of \( i \)) the total space of a non-trivial circle-bundle over \( \mathbb{R}^4 \times [0, \varepsilon_0] \). The natural euclidean coordinate in the base is \( x = \varepsilon x' \), and so the negative-mass Taub–NUT asymptotic form of AH metric lifts to

\[
\left( 1 - \frac{2\varepsilon}{|x|} \right) \frac{|dx|^2}{\varepsilon^2} + \left( 1 - \frac{2\varepsilon}{|x|} \right)^{-1} \alpha^2
\]

(5.13)

near \( Z \). We can now pick \( k \) points \( p_1, \ldots, p_k \) on \( Z \) and ‘insert’ copies of Taub–NUT by adding terms \( \varepsilon/2|x - p_\nu| \) to \( 1 - 2\varepsilon/|x| \) and modifying \( \alpha \) accordingly (the circle-bundle over \( Z \) will be changed to a circle-bundle over \( Z \setminus \{p_1, \ldots, p_k\} \). Our space \( W \) is recovered from this point of view by passing to the blow-up of \( Z \) in the points \( (p_\nu, 0) \).

6. Formal solution

In this section, \( W \) will be as in Proposition 5.8 and the notation will be as there. The goal of this section is the construction of smooth families of approximate hyperKähler triples on the fibres of \( \pi \) inside \( W \), whose limits, at \( \varepsilon = 0 \), correspond to the given hyperKähler metrics, in other words \( g_{AH} \) on \( X_0 \), \( g_{TN} \) on the \( X_\nu \), and \( g_{ad} \), the limiting adiabatic metric on \( X_\nu \).

The steps in this construction are an initial approximation (a modification of the metric gluing construction given in Proposition 5.8), followed by an iterative argument that improves this approximation order by order in \( \varepsilon \).

We begin with some necessary technical preliminaries about fibrewise symplectic and hyperKähler triples on \( W \), and then proceed to the statement of the main result of this section.

6.1. (Fibrewise) symplectic and hyperKähler triples on \( W \).

**Notation 6.1.** (Boundary defining functions for the boundary hypersurfaces of \( W \).) Recall the boundary hypersurfaces of \( W \) were denoted by \( X_\nu \), for \( \nu = 0, \ldots, k \), \( X_{ad} \) and \( I_\varepsilon \). Boundary defining functions for these hypersurfaces will be denoted \( \sigma_\nu \) for \( X_\nu \), \( \rho \) for \( X_{ad} \) and \( \sigma_I \) for \( I_\varepsilon \). We may and shall assume that \( \rho \sigma_\nu = \varepsilon \) in a neighbourhood of the corner \( X_{ad} \cap X_\nu \) for all \( \nu \). It is convenient to define \( \sigma = \sigma_0 \sigma_1 \ldots \sigma_k \sigma_I \).

**Notation 6.2.** If \( U \subset W \) is an open set, denote by \( \Omega^k_\phi(U) \) the space of smooth sections over \( U \) of \( \Lambda^k T^*_\phi(W/I) \). Further, write \( \Omega^k_{\phi, ei} \) for the subspace of \( \text{essentially invariant} \) forms. This is the subspace of forms \( \alpha \) such that \( \mathcal{L}_{\zeta_\phi} \alpha = O((\rho \sigma_I)^2) \). Write \( \Omega^k_{\phi, eb} \) for the subspace of \( \text{essentially basic} \) forms. These are the essentially invariant forms \( \alpha \) which also satisfy \( \iota_{\zeta_\phi} \alpha = O((\rho \sigma_I)^2) \).

Write \( d_\sigma \) for the relative differential

\[
d_\sigma : \Omega^k_\phi \rightarrow \Omega^{k+1}_\phi.
\]

(6.1)

**Remark 6.3.** Observe that if \( \alpha \) is essentially invariant (resp. essentially basic) then \( d_\sigma \alpha \) is essentially invariant (resp. essentially basic).

**Definition 6.4.** By a symplectic triple \( \omega \) on an open subset \( U \) of \( W \) we shall always mean a triple \((\omega_1, \omega_2, \omega_3)\) with \( \omega_j \in \Omega^2_\phi(U) \), such that \( d_\sigma \omega_j = 0 \) and such that the \( 3 \times 3 \) matrix

\[
(\omega_j \wedge \omega_k)
\]

is positive-definite at every point of \( U \).

(6.2)

A symplectic triple on \( U \subset W \) is called hyperKähler if \( Q(\omega) = 0 \), where

\[
Q(\omega)_{jk} = \omega_j \wedge \omega_k - \frac{1}{3}(\omega_1^2 + \omega_2^2 + \omega_3^2)\delta_{jk}.
\]

(6.3)
Remark 6.5. While it would be more accurate to call the triples appearing in this definition relative symplectic or hyperKähler triples, we believe that no serious confusion will result from this definition. However, the reader should bear in mind that symplectic triples on $W$ are to be thought of informally as $\varepsilon$-dependent smooth families of symplectic triples on the ‘Sen space’ $\mathcal{S}_g$ with some rather strong control on their behaviour in the limit as $\varepsilon \to 0$. As previously in this paper, we shall try to be consistent in our use of bold symbols for $\varepsilon$-dependent families, viewed as data on the 5-dimensional space $W$.

The $3 \times 3$ matrices appearing in (6.2) and (6.3) take values in the trivial real line-bundle $\lambda := \Lambda^4 T^*\rho_0(W/I)$. The condition that (6.2) be positive-definite makes sense for one and hence any trivialization of this bundle.

Notation 6.6. For each $\nu$, denote by $U_\nu$ a neighbourhood of the form $\{\sigma_\nu < \delta\}$ of $X_\nu$ in $W$. Denote by $V$ a collar neighbourhood of the form $\{\rho < \delta\}$ of $X_{ad}$ in $W$. For each $\nu$ there is a natural map $\kappa_\nu : U_\nu \to X_\nu$. For $\nu > 0$ this follows from the definition of blow-up: $\kappa_\nu$ sends the point $(x, \varepsilon)$ with $\varepsilon > 0$ near $p_0$ to point $(x - p_0)/\varepsilon \in X_\nu$. This extends smoothly to give $\kappa_\nu : U_\nu \to X_\nu$, equal to the identity on $X_\nu$. For $X_0$ it follows from the explicit construction of $W$. It is easily checked in local coordinates near $X_0 \cap X_{ad}$ that $\kappa_0$ is smooth.

For $\nu = 1, \ldots, k$, we shall denote by $\omega_\nu$, the hyperKähler triple of $X_\nu$ and define $\omega_\nu = \chi_\nu \kappa^* \omega_\nu$, where $\chi_\nu = \chi(\sigma_\nu)$ is cut-off function equal to 1 in a neighbourhood of $X_\nu$ and with compact support in $U_\nu$. Define $\omega_0$ in $U_0$ to be the hyperKähler triple of the 1-parameter family of metrics $g_{AH,\varepsilon}$ used in the proof of Proposition 5.3.

Similarly, we shall denote by $\omega_{ad}$ the lift to $V$ of the hyperKähler triple of the Gibbons–Hawking family $\tilde{g}_e$ (factored out by the involution) and by $\omega_{ad}$ its restriction to $X_{ad}$.

In Proposition 6.9 we shall construct a symplectic triple $\omega^X$ on $W$ such that

$$\omega^X|X_\nu = \omega_\nu, \quad \omega^X|X_{ad} = \omega_{ad}. \quad (6.4)$$

This is a crude patching construction analogous to the construction of $g^X$.

The main theorem to be proved in this section is the following:

Theorem 6.7. On $W$, there is a smooth symplectic triple $\zeta$ satisfying

$$Q(\zeta) = O(\varepsilon^X \sigma^X). \quad (6.5)$$

Moreover, $\zeta|X_0$ is the AH–hyperKähler triple, $\zeta|X_\nu$ is the TN symplectic triple, and $\zeta|X_{ad}$ is the adiabatic symplectic triple on $X_{ad}$. More precisely, $\zeta - \omega_\lambda$ is smooth, essentially basic and $O(\varepsilon \rho^2 \sigma^2)$ on $W$.

Remark 6.8. The meaning of (6.3) is that for every $N$, there is a constant $C_N$ such that

$$|Q(\zeta)| \leq C_N \varepsilon^N \sigma^N. \quad (6.6)$$

Since $Q$ is smooth, all derivatives also vanish faster than any power of $\varepsilon$.

6.2. Initial approximation. Our initial approximation to $\zeta$ is furnished by the following

Proposition 6.9. There exists a smooth symplectic triple $\omega^X$ on $W$ satisfying (6.4) and such that

$$Q(\omega^X) \in \varepsilon^3 \rho^X \sigma^X \mathbb{R}^3 \otimes (W, S^2_\varepsilon \mathbb{R}^3 \otimes \lambda). \quad (6.7)$$

(Recall that we have defined $\lambda$ to be the ‘relative density bundle’ $\lambda = \Lambda^4 T^*\rho_0(W/I)$.)

Proof. Refer to the notational conventions set up in paragraph 6.6. Let $Z = X_0 \cap X_{ad}$. Then we have the metric family $g_Z$ and its associated triple $\omega_Z$ associated to the family of Gibbons–Hawking metrics determined by the family of harmonic functions $H_e$ from (5.3). Then $\omega_Z \in \Omega^2_\varepsilon(U_0 \cap V)$. Recall again that local coordinates in $U_0 \cap V$ are the two defining function $\rho$ and $\sigma_0$, where $\sigma_0 = \varepsilon|x|$ and $\rho = |x|^{-1}$ in terms of the ‘original’ $x$ variables on $\mathbb{R}^3$ and the rescaled asymptotic base variable $x'$ in $\mathcal{M}_e^g$.

By Propositions 4.2 and 4.3 we have

$$A \in \rho^2 \Omega^1_\varepsilon(U \cap V) \otimes \mathbb{R}^3 \text{ such that } \omega_0 = \omega_Z + d_x A \quad (6.8)$$
and
\[ B \in \sigma_0^3 \Omega^1_{\phi,eb}(U \cap V) \otimes \mathbb{R}^3 \] such that \( \omega_{ad} = \omega_Z + d_\pi B \), \hfill (6.9)
both (6.8) and (6.9) being valid in \( U \cap V \).

Let \( \chi(t) \) be as in Proposition 5.8 equal to 1 for \( t \leq \delta/2 \) and vanishing for \( t \geq \delta \). Then we claim that
\[ \omega^X = \omega_Z + d_\pi (\chi(\rho)A + \chi(\sigma_0(B))) \] \hfill (6.10)
fits the bill.

To verify this, it is useful to record the following

**Lemma 6.10.** The relative differential \( d_\pi \) is a differential operator in
\[ \text{Diff}^1(\mathbb{W}/I; \Lambda^k T^*_\phi(W/I), \Lambda^{k+1} T^*_\phi(W/I)). \]
In particular, if \( \alpha \in \rho^m \sigma^m \Omega^k(\mathbb{W} \cap V) \), then \( d_\pi \alpha \in \rho^{m+1} \sigma^m \Omega^{k+1}(U \cap V) \).

**Proof.** See Definition 7.1 for the definition of this space of differential operators. Once this is understood, the verification of the result is straightforward. \( \square \)

If we restrict to the neighbourhood \( \sigma_0 < \delta/2 \) of \( X_0 \), then \( \chi(\sigma_0) = 1 \) and so
\[ \omega^X = \omega_Z + dA + d_\pi (\chi(\rho)B) = \kappa^* \omega_0 + d_\pi (\chi(\rho)B) \] \hfill (6.11)
in this subset of \( U \cap V \). Thus \( \omega^X \) can be extended smoothly to \( \{ \sigma_0 < \delta/2 \} \) by defining it to be equal to \( \omega_0 \) away from \( Z \). By Lemma 6.10
\[ d_\pi (\chi(\rho)B) = O(\rho \sigma_0^3) \] \hfill (6.12)
and is supported in \( U_0 \cap V \), so the restriction of \( \omega^X \) to \( X_0 \) is \( \omega_{AH} \).

Similarly, the restriction of \( \omega^X \) to \( \{ \rho < \delta/2 \} \) can be extended by \( \tilde{\omega} \) to a collar neighbourhood of \( X_{ad} \cup X_\phi \), we have
\[ \omega^X = \omega_{ad} + d_\pi (\chi(\sigma_0)A) \] \hfill (6.13)
in this set,
\[ d_\pi (\chi(\sigma_0)A) = O(\rho^2) \] \hfill (6.14)
and is supported in \( U_0 \cap V \), so the restriction of \( \omega^X \) to \( X_{ad} \) is \( \omega_{ad} \).

It is clear that \( \omega^X \) is smooth and that \( Q(\omega^X) \) is also smooth. To prove (6.7), it is sufficient to compute \( Q(\omega^X) \) away from the corner. In \( U_0 \), away from the corner, we may use (6.12). Then
\[ Q(\omega^X) = Q(\omega_0) + 2Q(\omega_0, d_\pi (\chi(\rho)B) + Q(d_\pi (\chi(\rho)B), d_\pi (\chi(\rho)B))). \] \hfill (6.15)
Since \( B \) is essentially basic, the third term is automatically \( O(\rho^2) \) for degree reasons. The first term is zero because \( \omega_0 \) is hyperKähler, and so the second is \( O(\rho \sigma_0^3) \) for \( \sigma_0 \to 0 \) away from \( \rho = 0 \).

Using (6.10) similarly we see that \( Q(\omega^X) = O(\rho^2) \) for \( \rho \to 0 \) away from \( \sigma_0 = 0 \). Combining these two calculations gives (6.7). \( \square \)

**Notation 6.11.** In the interest of readability we shall write \( d \) for \( d_\pi \) in the rest of this section.

Theorem 6.7 is proved by induction. The inductive assumption is that we have found
\[ c \in \sigma_1^2 \Omega^1_{\phi,eb}(W) \otimes \mathbb{R}^3 \] \hfill (6.16)
such that
\[ Q(\omega^X + \varepsilon dc) \in \varepsilon^N F + \varepsilon^{N+3} G \] \hfill (6.17)
where
\[ F \in \rho^\varepsilon \sigma^\varepsilon \Gamma (W, S_{\phi,3}^2 \mathbb{R}^3 \otimes \lambda), \quad G \in \sigma^\varepsilon \Gamma(S_{\phi,3}^2 \mathbb{R}^3 \otimes \lambda). \] \hfill (6.18)
The decomposition (6.9) is well-defined up to smooth sections which are \( O(\varepsilon^\varepsilon \sigma^\varepsilon) \), in other words very small at all boundary hypersurfaces. These very small terms will be unimportant in this section. Equations (6.17) and (6.18) imply that the restriction of \( \omega^X + \varepsilon dc \) to \( \pi^{-1}(\varepsilon) \) is a symplectic triple that is ‘approximately hyperKähler to order \( \varepsilon^N \).

We need to record the fine structure of the error term as in (6.18) to be sure of the smoothness of the triple \( \zeta \) are aiming for in Theorem 6.7.
We shall construct \( a \) defined near \( \bigcup X_\nu \) and \( b \) defined near \( X_{ad} \), essentially basic and decaying near spatial infinity, so that with
\[
c' = c + \varepsilon^{N-1}da + \varepsilon^{N+1}db, \tag{6.19}
\]
we have
\[
Q(\omega^X + \varepsilon dc') = \varepsilon^{N+1}F' + \varepsilon^{N+4}G'
\tag{6.20}
\]
where \( F' \) and \( G' \) are in the same spaces as \( F \) and \( G \) in \([6.18]\). Thus we have improved the error term in \([6.17]\) by one order in \( \varepsilon \).

The induction starts because of Proposition \([6.9]\) which is the case \( N = 3 \) of \([6.17] \). Given \([6.17] \), the required \( a \) and \( b \) are obtained by solving a Poisson equation respectively over \( \bigcup X_\nu \) and on the base \( Y_{ad} \) of the \( S^1 \)-bundle \( X_{ad} \to Y_{ad} \).

6.3. **Construction of \( a \) — linear theory.** We gather in this subsection the linear theory of the equation \( Q(\omega, da) = f \), on an ALF gravitational instanton \( X \), where the RHS is rapidly decreasing near \( \partial X \).

The following is an explicit version of the infinitesimal diffeomorphism gauge invariance of the linearized equations:

**Lemma 6.12.** Let \( X \) be a hyperKähler 4-manifold with hyperKähler triple \( \omega \). For any vector field \( v \), we have \( Q(\omega, d(t_v \omega)) = 0 \).

**Proof.** The equation is gauge-invariant, so we have \( Q(\phi_t^* \omega) = \phi_t^* Q(\omega) = 0 \) where \( \phi_t \) is the one-parameter family of diffeomorphisms generated by \( v \). Then the derivative at \( t = 0 \) of \( \phi_t^* \omega \) is just \( d(t_v \omega) \) by Cartan’s formula, and the Lemma follows at once. \( \square \)

**Theorem 6.13.** Let \((X, g)\) be an ALF gravitational instanton with hyperKähler triple \( \omega \). Suppose that \( f \in \rho^0 C^\infty(X, S^0_+ \mathbb{R}^3 \otimes \lambda) \) is smooth and rapidly decreasing with all derivatives at the boundary. Then there exists \( a \in \rho^0 \Omega^4_{\phi, ch}(X) \otimes \mathbb{R}^3 \) such that \( Q(\omega, da) = f \). \( \tag{6.21} \)

(Here \( \rho \) is the boundary defining function of \( \partial X \).)

**Proof.** As in \([22]\) regard \( f \) as a section of \((\Lambda^0 \oplus \Lambda^2) \otimes \mathbb{R}^3 \) (trivializing \( \Lambda^2_+ \) using the triple. By Theorem \([C,3]\) there exists
\[
\phi \in \rho C^\infty(X, (\Lambda^0 \oplus \Lambda^2) \otimes \mathbb{R}^3), \quad DD^* \phi = f. \tag{6.22}
\]
Then \( u = D^* \phi \) is \( O(\rho^0) \), essentially invariant, and \( Da_0 = f \), which also implies \( Q(\omega, du) = f \).

In order to get an essentially basic solution \( a \), we shall find a vector field \( v \), supported near \( \partial X \), such that
\[
a = u + \iota_v \omega \tag{6.23}
\]
is essentially basic. By Lemma \([6.12]\) we shall still have \( Q(\omega, da) = f \). In an asymptotic Gibbons–Hawking chart with local coordinates \((x, \theta)\), write
\[
u_j = u_{0j} \alpha + \sum u_{ij} dx_i
\]
we then have three essentially invariant functions \((u_{01}, u_{02}, u_{03})\). Let
\[
v = u_0 \delta x_j.
\]
Then
\[
\iota_v \omega_1 = \iota_v (\alpha \wedge dx_1 + h dx_2 \wedge dx_3) = -u_{01} \alpha + h(u_{02} dx_3 - u_{03} dx_2),
\]
with similar formulae for the \( \iota_v \omega_2 \) and \( \iota_v \omega_3 \). Defining \( a = u + \iota_v \omega \)
(\( v \) is cut off to zero away from the boundary of \( X \)) gives the required essentially basic solution of \([6.21]\). \( \square \)
6.4. Construction of \( a \).

**Proposition 6.14.** Given \( c \) satisfying (6.16) and (6.17), there exists \( a \in \rho^2 \sigma^2 \Omega_{\phi, eb}^1(W) \) such that

\[
Q(\omega^\tau + \varepsilon dc + \varepsilon N da) \in \varepsilon^{N+1} F' + \varepsilon^{N+3} G'
\]

where \( F' \) and \( G' \) are in the spaces shown in (6.18).

Moreover \( a \) can be chosen to be supported arbitrarily close to \( \bigcup X_\nu \) (and in particular away from spatial infinity \( \mathcal{I}_\infty \)).

**Proof.** Let us write \( \omega' = \omega^\tau + \varepsilon dc \). If

\[
a \in \rho^2 \sigma^2 \Omega_{\phi, eb}^1(W)
\]

then we calculate

\[
Q(\omega' + \varepsilon N da) = Q(\omega') + \varepsilon^N Q(\omega', da) + \varepsilon^{2N} Q(da, da)
\]

Since \( a \) is essentially basic, the third term on the RHS is \( O(\varepsilon^{2N} \rho^2) \). Since the correction term \( c \) is also essentially basic, the second term on the RHS differs from \( \varepsilon^N Q(\omega', da) \) by \( O(\varepsilon^{N+1} \rho^2) \) in each collar neighbourhood \( U_\nu \). Thus, using \( F' \) and \( G' \) to denote elements of the spaces (6.18) that are allowed to vary from line to line, (6.20) can be rewritten

\[
Q(\omega' + \varepsilon N da) = \varepsilon^N F + \varepsilon^{N} Q(\omega', da) + \varepsilon^{N+3} G
\]

in each \( U_\nu \).

This equation has a well-defined leading term at \( X_\nu \) obtained by dividing by \( \varepsilon^N \) and taking the limit \( \sigma_\nu \to 0 \). The leading coefficient, \( f_\nu \), say, of \( F \) at \( X_\nu \) does not depend upon \( a \) and so the leading term in the RHS of (6.27) is

\[
f_\nu + Q(\omega', da|X_\nu).
\]

Now \( f_\nu \) is \( O(\rho^2) \) on \( X_\nu \) so by Theorem 6.13 there exists a solution \( a_\nu \in \rho^2 \Omega^1_{\phi, eb}(X_\nu) \otimes \mathbb{R}^3 \) so that

\[
Q(\omega', da_\nu) = -f_\nu
\]

on \( X_\nu \). Define \( a \) in \( U_\nu \) to be \( \chi(\sigma_\nu) \kappa^*_\nu(a_\nu) \). As the neighbourhoods \( U_\nu \) are pairwise disjoint, we may regard this as defining \( a \) over the whole of \( W \), and so defined, \( a \) is supported in the union of the \( U_\nu \).

Finally we claim that \( a \), so defined, satisfies (6.24). In \( U_\nu \), we have, from (6.27),

\[
Q(\omega' + \varepsilon N da) = \varepsilon^N (f_\nu + Q(\omega', da(\chi(\kappa_\nu^*(a_\nu))) + \varepsilon^{N+1} F' + \varepsilon^{N+3} G'
\]

Because \( \chi \) is identically 1 near \( X_\nu \), \( d\chi \) is supported away from \( X_\nu \). Furthermore, \( d\chi = O(\rho) \) and \( a_\nu = O(\rho^2) \), so the first term on the RHS is \( O(\sigma^2 \varepsilon^N \rho^3) = O(\varepsilon^{N+3} \sigma^2 x) \). Hence this term can be absorbed by the \( \varepsilon^{N+3} G' \) term, and (6.24) is proved. \( \square \)

6.5. **Construction of \( b \) – linear theory.** In this section we summarize the linear theory for the Laplacian of the adiabatic family of metrics \( g_\epsilon \). By a straightforward calculation, for

\[
g = h_\epsilon \frac{|dx|^2}{\varepsilon^2} + h_\epsilon^{-1} \alpha^2,
\]

we have

\[
\Delta_{g_\epsilon} = \varepsilon^2 h^{-1} \tilde{\Delta}_0 - h \tilde{c}_\theta^2
\]

where \( \Delta_0 \) is the laplacian of the (unrescaled) euclidean metric, \( \tilde{\Delta}_0 \) is its horizontal lift and \( \tilde{c}_\theta \) denotes the generator of the circle action. As an aside, if in local coordinates,

\[
\alpha = d\theta + \sum a_j dx_j
\]

then

\[
\tilde{\Delta}_0 = -\sum \nabla_j^2, \text{ where } \nabla_j = \frac{\partial}{\partial x_j} - a_j \frac{\partial}{\partial \theta}.
\]
In particular, if \( u \) is invariant, then regarding it without change of notation as a function on \( \mathbb{R}^3 \), we have
\[
\Delta_{g_s} u = \varepsilon^2 h^{-1} \Delta_0 u. \tag{6.34}
\]
Recall that \( D = d^* + d_+ \),
\[
D : \Omega^1 \longrightarrow \Omega^0 \oplus \Omega_+^2.
\]
As in §2 on a hyperKähler 4-manifold \( M \), \( \Lambda^2 \) has a flat orthonormal trivialization by a hyper-Kähler triple \((\omega_j)\). Using this trivialization, if
\[
\phi = (\phi_0, \phi_j \omega_j) \in \Omega^0 \oplus \Omega_+^2
\]
then \( DD^* \) acts as the scalar Laplacian on the coefficients \((\phi_0, \ldots, \phi_3)\).

We shall need the formula for \( D^* \) for the metric \([6.32]\), acting on invariant functions. A simple calculation gives that if
\[
D^* \phi = w_0 e_0 + w_j e_j,
\]
where
\[
e_0 = \frac{\alpha}{\sqrt{h}}, \quad e_j = \sqrt{h} \frac{dx_j}{\varepsilon},
\]
then
\[
\begin{bmatrix}
w_0 \\
w_1 \\
w_2 \\
w_3
\end{bmatrix} = \frac{\varepsilon}{\sqrt{h}} \begin{bmatrix}
0 & \hat{\omega}_1 & \hat{\omega}_2 & \hat{\omega}_3 \\
-\hat{\omega}_1 & 0 & -\hat{\omega}_2 & \hat{\omega}_2 \\
-\hat{\omega}_2 & \hat{\omega}_3 & 0 & -\hat{\omega}_1 \\
-\hat{\omega}_3 & -\hat{\omega}_2 & \hat{\omega}_1 & 0
\end{bmatrix} \begin{bmatrix}
\phi_0 \\
\phi_1 \\
\phi_2 \\
\phi_3
\end{bmatrix}.
\tag{6.35}
\]
(One can verify \( DD^* = \Delta_g \) directly from this formula and its adjoint.)

In the next theorem, we write \( \omega_{ad} \) for the lift of the hyperKähler triple of \((\phi_0, \ldots, \phi_3)\) to a collar neighbourhood \( V \) of \( X_{ad} \) in \( W \). The notation for boundary defining functions is as in paragraph 6.1.

**Theorem 6.15.** Let
\[
G \in \sigma^x C^\infty_c (V, S_0^2 \mathbb{R}^3 \otimes \lambda)
\]
be an essentially invariant section. Then there exists
\[
b \in \sigma^1_3 \Omega^1_{\phi, \omega} (V) \otimes \mathbb{R}^3
\]
such that
\[
Q(\omega_{ad}, db) = \varepsilon G + O(\varepsilon^x \sigma^x)
\]
in \( V \).

**Proof.** Suppose that \( G \) is exactly \( S^1 \)-invariant. Then we may regard \( G \) as a function \( V/S^1 \). Because \( G \) is rapidly vanishes to all orders in \( \sigma \) near \( X_p \), we may regard \( G \) as a smooth function on \( \mathbb{R}^3 \times [0, \varepsilon_0] \), which vanishes to all orders in \( |x-p| \) at every point \( p \) of \( P \).

We solve \((6.37)\) in two stages. First, the formula
\[
\mathbf{u}(x, \varepsilon) = \frac{1}{4\pi^2 \varepsilon} \int |x-y| h_\varepsilon(y) G(y, \varepsilon) \, dy
\]
gives a function on \( \mathbb{R}^3 \) such that
\[
\Delta_{g_s} \mathbf{u}(x, \varepsilon) = \varepsilon G. \tag{6.39}
\]
Moreover, \( \varepsilon \mathbf{u}(x, \varepsilon) \) is smooth in all variables because the singularities in \( h \) at the points of \( P \) are cancelled by the vanishing of \( G \) to all orders at these points. It is also \( O(|x|^{-1}) \) for \( |x| \to \infty \).

As before, regard \( G \) and \( \mathbf{u} \) as sections of \((\Lambda^0 \oplus \Lambda^2_+)^* \otimes \mathbb{R}^3 \). Then if \( b_0 = D^*_g \mathbf{u} \), we have \( D_g b_0 = \varepsilon G \). From the formula \((6.35)\), \( b_0 \), is initially defined on \( \mathbb{R}^3 \setminus P \) and lifts to a smooth section of \( \Omega^1_{\phi, \omega}(V) \otimes \mathbb{R}^3 \). Indeed, near each of the \( X_p \), \( h_\varepsilon = 1 + O(\sigma) \), where the ‘\( O \)’ is smooth for small \( \sigma \).

We can now correct \( b_0 \) exactly as we did in Theorem 6.13 to obtain \( b \) satisfying \((6.37)\) and \((6.38)\).

We started the proof by assuming that \( G \) was exactly invariant. If \( G \) is only essentially invariant, we can write \( G = G_0 + G_1 \) where \( G_0 \) is exactly invariant and \( G_1 \) is \( O(\varepsilon^x \sigma^x) \). Then we apply the previous argument with \( G \) replaced by \( G_0 \) to obtain the result. \( \square \)
6.6. Construction of $b$. The second half of the inductive step is contained in the following result:

**Proposition 6.16.** Let $c$ and $a$ be as in Proposition 6.14. Then there exists $b$, supported near $\rho = 0$, essentially basic and $O(\sigma^2)$, so that

$$Q(\omega' + \varepsilon^N da + \varepsilon^{N+1}db) = \varepsilon^{N+1} F'' + \varepsilon^{N+2} G''$$

where $F''$ and $G''$ are in the spaces in (6.18).

**Proof.** Let us write $\omega'' = \omega' + \varepsilon^N da$. If we calculate the LHS of (6.41) in $V$, assuming that $b$ is essentially basic in $V$, our collar neighbourhood of $X_{ad}$, we obtain

$$Q(\omega'' + \varepsilon^{N+2}db) = Q(\omega'') + \varepsilon^{N+2}Q(\omega'', db) + O(\varepsilon^{2N+2}\rho^\infty)$$

where as before the last term vanishes to all orders in $\rho$ because $b$ is essentially basic. Using $F'$ and $G'$ for generic functions in the spaces (6.18) which may vary from line to line, simplifications analogous to those in the proof of Proposition 6.14 yield

$$Q(\omega'' + \varepsilon^{N+2}db) = \varepsilon^{N+1} F' + \varepsilon^{N+3} G' \varepsilon^{N+2} Q(\omega_{ad}, db).$$

Now $G'$ satisfies the hypotheses of $G$ in Theorem 6.15 so there exists $b$ as in (6.37) and satisfying (6.38) (with $G$ replaced by $-G'$). In order to extend $b$ to $W$, we must replace it by $\chi(\rho)b$. Then

$$Q(\omega'' + \varepsilon^{N+2}d(\chi(\rho)b)) = \varepsilon^{N+1} F' + \varepsilon^{N+4} G' + \varepsilon^{N+2} Q(\omega_{ad}, d\chi \wedge b).$$

The last term is $O(\rho^\infty)$ because $\chi$ is identically 1 near $X_{ad}$ and $b$ is smooth near each $X_\nu$. Thus this last term can be absorbed into $\varepsilon^{N+1} F'$, yielding (6.41). \qed

6.7. Completion of proof of Theorem 6.7. The inductive argument given in §6.3–6.6 shows that there is a solution $\bar{\zeta}$ of (6.5) in formal power series in $\varepsilon$. However, it is well known that given such a formal power series, there exists smooth $\zeta$ whose derivatives at all boundary hypersurfaces agree with those of $\bar{\zeta}$ (Borel’s Lemma). This observation completes the proof of Theorem 6.7.

### 7. Completion of Proof

To complete the proof of our main theorem, we need to modify $\zeta$ in Theorem 6.7 by a triple $a \in \Omega^1(\phi(W) \otimes \mathbb{R}^n)$, say, so that

$$Q(\zeta + da) = 0$$

on (the fibres of) $W$.

This is an application of the implicit function theorem, uniformly on the fibre $\pi^{-1}(\varepsilon)$, for $\varepsilon > 0$. The key step is the uniform invertibility result for the linearization, Theorem 7.6 below. To have this invertibility, we shall need to use the freedom to choose $\varepsilon_0$ to be very small.

Let us agree to denote by $g_\zeta$ the metric on $T_\phi(W/I)$ determined by the symplectic triple $\zeta$, and by $\Delta_\zeta$ the associated Laplacian.

We shall use the reformulation $a = D_\zeta^b u$ discussed in (2.15)–(2.16) to reduce our work to the study of the Laplacian $\Delta_\zeta$ of $g_\zeta$.

#### 7.1. (Fibrewise) differential operators on $W$

The $\phi$-vertical tangent bundle $T_\phi(W/I)$ was introduced in Definition 5.4. Denote by $\gamma_\phi(W/I)$ the space of smooth sections of $T_\phi(W/I)$. This space of vector fields is used to define the relevant space of differential operators on $W$:

**Definition 7.1.** The space $\text{Diff}^m(W/I)$ is the space of differential operators which are polynomial (of degree $\leq m$) in the vector fields from $T_\phi(W/I)$ with smooth coefficients. The space $\text{Diff}^0(W/I)$ is the set of differential operators on $W$ which are polynomial in $\gamma_\phi(W/I)$, the $b$-vector fields on $W$ that are tangent to the fibres of $\pi$.

These definitions also make sense for differential operators acting between sections of vector bundles over $W$. From the definitions, we see that $(\rho\sigma_1)^m \text{Diff}^m(W/I) \subset \text{Diff}^m(W/I)$.
Example 7.2. We have already seen the relative exterior derivative $d_\pi$ as an example of an operator in $\text{Diff}^1_\pi(W/I)$.

If $g$ is a smooth metric on $T_\phi(W/I)$, then the Laplacian $\Delta_\zeta$ is an operator in $\text{Diff}^2_\phi(W/I)$.

**Definition 7.3.** A function $f$ on $W$ is said to be essentially invariant if
\[
\frac{\partial f}{\partial \theta} = O((\rho \sigma_I)^\zeta) \in V.
\] (7.2)

The definition makes sense for a given choice of $\alpha$, $\beta$, $m$. Let $D$ be the space just defined which will serve as a domain for the Laplacian $\Delta_\zeta$.

**Example 7.2.** We have already seen the relative exterior derivative $d_\pi$ as an example of an operator in $\text{Diff}^1_\pi(W/I)$.

**Definition 7.4.** Let $\mathcal{B}$ be the space of functions $u$ such that $Pu \in H^m_b(W)$ for all $P \in \text{Diff}_\phi^m(W/I)$. Then by definition, $P$ extends to define a bounded map $H^{n,m} \rightarrow H^m_b$ for any $m$. Define also $H^{n,m}_{b,b}(W)$ to be the space of functions $u$ such that $Qu \in H^m_b(W)$ for all $Q \in \text{Diff}_\phi^m(W/I)$.

In order to deal with the Laplacian, we shall need to split off the $S^1$-invariant component of these elements in these Sobolev spaces. This only makes sense near $X_{\text{ad}} \cup I_\infty$, which means that our definitions are a little complicated. Recall that $V$ is a fixed tubular neighbourhood of $X_{\text{ad}}$.

**Definition 7.5.** Let $\beta > \alpha + 2$, $\alpha > 0$, and fix a bump function $\chi(t) = 1$ for $t \leq 1/2$ and equal to zero for $t \geq 1$. Define
\[
\mathcal{R}_{\alpha,\beta,m}(W) = \{ \chi(\rho/\delta)f_0 + f_1 \} \text{ where } f_0 \in (\rho \sigma_I)^{\alpha+2}H^m_b(W) \text{ and } \frac{\partial f_0}{\partial \theta} = 0, f_1 \in (\rho \sigma_I)^{\beta}H^m_b(W).
\] (7.4)

Since $\alpha + 2 < \beta$, this allows for the zero-Fourier mode $f_0$ to be larger than the non-zero Fourier modes. In practice, we shall take $\alpha \in (0,1)$ and $\beta$ can be as large as we like. The space just defined will serve as a range space for the Laplacian. The definition of the domain is similar:

**Definition 7.6.** With $\alpha$ and $\beta$ as in Definition 7.3 let
\[
\mathcal{D}_{\alpha,\beta,m+2}(W) = \{ \chi(\rho/\delta)u_0 + u_1 \} \text{ where } u_0 \in (\rho \sigma_I)^{\alpha}H^{2,m}_{b,b}(W) \text{ and } \frac{\partial u_0}{\partial \theta} = 0, u_1 \in (\rho \sigma_I)^{\beta}H^{2,m}_{b,b}(W).
\] (7.5)

The Laplacian of a smooth, essentially invariant metric on $T_\phi(W/I)$ extends to define a bounded linear map
\[
\mathcal{D}_{\alpha,\beta,m+2}(W) \rightarrow \mathcal{R}_{\alpha,\beta,m}(W)
\]
for every $m$ and $\beta > \alpha + 2$.

The main linear result to be given in this section is the invertibility of the Laplacian between these spaces.

**Theorem 7.6.** Let $g_\zeta$ be the metric on $T_\phi(W/I)$ determined by the symplectic triple $\zeta$ on $W$, and let $\Delta_\zeta$ be the associated Laplacian. Fix $\alpha \in (0,1)$ and $\beta > \alpha + 2$ and $m$. Then there exists $\varepsilon_0 > 0$ so that with $W = \pi^{-1}(0,\varepsilon_0)$,
\[
\Delta : \mathcal{D}_{\alpha,\beta,m+2}(W) \rightarrow \mathcal{R}_{\alpha,\beta,m}(W)
\] (7.6)
is invertible.

**Proof.** This follows by patching inverses on the $X_{\nu}$ to an ‘adiabatic’ inverse on $V$. For this we first need to localize functions on $W$ near the different boundary hypersurfaces.

To simplify notation, having fixed $\alpha, \beta$ and $m$, write
\[
\mathcal{R} = \mathcal{R}_{\alpha,\beta,m}, \mathcal{D} = \mathcal{D}_{\alpha,\beta,m+2}
\] (7.7)
and write $\mathcal{R}(W)$, $\mathcal{R}(X_\nu)$ etc. to distinguish between function spaces on $W$ and on $X_\nu$ (cf. (C.3)–(C.4) below). As before, let $\chi(t)$ be a standard cut-off function, $0 \leq \chi(t) \leq 1$, $\chi(t) = 1$ for $t \leq \frac{1}{2}$ and vanishing for $t \geq 1$. Let $\delta > 0$ be small. Let $\chi_\nu = \chi(\sigma_\nu/\delta)$ and let $\chi_{ad} = 1 - \sum_\nu \chi_\nu$. If $f \in C^\infty(W)$ then $\chi_\nu f$ is smooth and supported in $U_\nu$ and $\chi_{ad} f$ is smooth and supported in $V$.

Now we identify $U_\nu$ with a subset of the product $X_\nu \times [0, \varepsilon_0)$ by the map $w \mapsto (\kappa_\nu(w), \pi(w))$, where $\kappa_\nu$ was introduced in (6.6). Under this identification, a set of the form $\sigma_\nu < a$ maps to the subset

\[
\{(x_\nu', \theta_\nu, \varepsilon) : |x_\nu'| < a\varepsilon^{-1}\} \subset X_\nu \times [0, \varepsilon_0)
\]

and in particular $\chi_\nu f$, when transferred to the product, will be compactly supported in each slice $X_\nu \times \{\varepsilon\}$ for $\varepsilon > 0$ (though these compact sets grow as $\varepsilon \to 0$).

From Theorem C.2, we have an inverse $G_\nu : \mathcal{R}(X_\nu) \to \mathcal{D}(X_\nu)$ of the Laplacian $\Delta_{g_\nu}$. Using the identification of $U_\nu$ with the product (6.8), now define a lift $G_\nu$ of $G_\nu$ to act on functions on $W$ by the formula:

\[
G_\nu(f) = \eta_\nu G_\nu \chi_\nu f
\]

where

\[
\eta_\nu = \chi(\log \sigma_\nu/\log \delta).
\]

Then $\eta_\nu$ goes from 1 to 0 as $\sigma_\nu$ goes from $\delta$ to $\sqrt{\delta}$ and in particular $\eta_\nu$ is identically 1 on $\text{supp}(\chi_\nu)$, so

\[
\eta_\nu \chi_\nu = \chi_\nu.
\]

From the boundedness of $G_\nu : \mathcal{R}(X_\nu) \to \mathcal{D}(X_\nu)$, it follows that

\[
G_\nu : \mathcal{R}(W) \to \mathcal{D}(W)
\]

is bounded, and that

\[
\Delta_{g_\nu} G_\nu = \chi_\nu - \iota_\nu.
\]

The key point is that we can choose $\delta$ so that

the operator norm of $\iota_\nu : \mathcal{R}(W) \to \mathcal{R}(W)$ is bounded by

\[
\frac{1}{10(k + 1)} + C_\nu(\delta)\varepsilon_0.
\]

To prove this, let $f \in \mathcal{R}(W)$ and observe that $G_\nu f$ is supported in $U_\nu$, and in this set, $\Delta_\zeta = \Delta_{g_\nu} + O(\varepsilon_0)$, where $g_\nu$ is the original ALF hyperKähler metric on the hypersurface $X_\nu$. Thus

\[
\Delta_\zeta G_\nu = \Delta_{g_\nu} G_\nu + O(\varepsilon_0) = \Delta_{g_\nu} \eta_\nu G_\nu \chi_\nu + O(\varepsilon_0) = \eta_\nu \Delta_{g_\nu} G_\nu \chi_\nu + [\Delta_{g_\nu}, \eta_\nu] G_\nu \chi_\nu + O(\varepsilon_0) = \chi_\nu + [\Delta_{g_\nu}, \eta_\nu] G_\nu \chi_\nu + O(\varepsilon_0)
\]

(7.15) using (7.11) to obtain the first term in (7.15). The commutator is an operator in $\text{Diff}_\nu^1(W/I)$ of the form

\[
[\Delta_{g_\nu}, \eta_\nu] = c_0 + c_1 \nabla
\]

(7.16)

where $c_0 = \Delta_{g_\nu} \eta_\nu$ and $c_1 = \nabla \eta_\nu$. Now for any smooth function $\beta$ with compact support in $U_\nu$, $\beta G_\nu \chi_\nu$ defines a bounded linear map $\mathcal{R} \to \mathcal{D}$. Noting that functions in $\mathcal{D}$ decay like $\rho^a$ while functions in $\mathcal{R}$ decay at the faster rate $\rho^{a+2}$, in order that (7.16) have small norm, we require that the coefficient of $\nabla$ be bounded by $o(\delta)\rho$ and the order-0 term be bounded by $o(\delta)\rho^2$. This is where the specific form $\eta$ comes in. Indeed, we have

\[
d\eta_\nu = \chi' \left( \frac{\log \sigma_\nu}{\log \delta} \right) \frac{d\sigma_\nu}{\sigma_\nu \log \delta}
\]

(7.17)

and since the norm $\sigma_\nu^{-1} d\sigma_\nu$ is $O(\rho)$, this term is bounded by a multiple of $\rho/|\log \delta|$. Similarly $|\nabla^2 \eta| = O(\rho^2/|\log \delta|)$. To obtain (7.16), take $\delta$ so small that the operator norm of the commutator in (7.15) is $< \frac{1}{10(k + 1)}$. Then (7.14) is obtained.

\[\text{1}\text{We have here suppressed explicit mention of the map identifying $U_\nu$ with the product}\]
We now need to complete the definition of $G$ by finding an approximate inverse localized near $V$. For this, set
\[ \eta_{ad}(x) = 1 - \sum_{n} \chi \left( \log \frac{2\sigma_{\nu}}{2 \log \delta} \right) \]
so that $\eta_{ad}$ is identically 1 on the support of $\chi_{ad}$ and goes from 1 to 0 as any of the $\sigma_{\nu}$ goes from $\delta/2$ to $\delta^2/2$. We shall construct an operator $G_{ad}$ as a sum $\sum \eta_{ad} G_n \chi_{ad}$ where $G_n$ acts on the $n$-th Fourier coefficient of $\chi_{ad} f \in \mathcal{H}(W)$. Our operator will have properties analogous to those of $G_{\nu}$,
\[ G_{ad} : \mathcal{H}(W) \to \mathcal{H}(W) \text{ is bounded and } \Delta G_{ad} = \chi_{ad} - e_{ad} \]
and we can choose $\delta$ so that
\[ \text{the operator norm of } e_{ad} : \mathcal{H}(W) \to \mathcal{H}(W) \text{ is bounded by } \frac{1}{10} + C_{ad}(\delta) \varepsilon_0. \]
For the construction of $G_{ad}$ we are localized to $V$, we have the $S^1$-action and for $f \in \mathcal{H}(W)$ we may split $\chi_{ad} f$ into its Fourier modes. For the zero Fourier mode, define $u_0 \in \mathcal{D}(W)$ by the formula
\[ u_0 = \varepsilon^{-2} \eta_{ad} G_0(\chi_{ad} f_0) \]
where $G_0$ is the Green’s operator of the euclidean Laplacian $\mathbb{R}^3$. It is not hard to check that this is bounded between the given spaces, and
\[ (\varepsilon^2 \Delta_0) u_0 = [\Delta_0, \eta_{ad}] G_0 \chi_{ad} f_0 + \chi_{ad} f_0. \]
As discussed above, the operator norm of the first term can be made as small as we please by choosing $\delta$ sufficiently small: the derivatives of $\eta_{ad}$ give factors of $1/|\log \delta|$ in the coefficients of the commutator.

On the $n$-th Fourier mode we invert the model operator
\[ \varepsilon^2 \Delta_0 + n^2 \]
using the explicit Green’s operator
\[ G_n(x - x') = \frac{e^{-|n||x-x'|/\varepsilon}}{4\pi|x-x'|} \]
on $\mathbb{R}^3$. Then the formula
\[ \hat{u}_n(x) = \eta_{ad}(x) \int G_n(x - x') \chi_{ad}(x') \hat{f}_n(x') \, dx', \quad n \neq 0 \]
gives the $n$-th Fourier coefficient of a function $u$ in $\mathcal{D}(W)$ if $\chi_{ad} \hat{f}_n$ is the $n$-th Fourier coefficient of $\chi_{ad} f$, with $f \in (\rho \sigma_{\nu})^3 \mathcal{H}_0^{m} \subset \mathcal{H}_{a, \beta, m}(W)$.

Combining the definitions (7.21) and (7.25), we obtain an operator $G_{ad}$, which is a bounded linear map from $\mathcal{H}(W) \to \mathcal{D}(W)$.

Now, in $V$, $\Delta_\zeta$ differs from $\Delta_{ad}$ by an operator $\rho A$ where $A \in \text{Diff}_0^2(W/I)$,
\[ \Delta_\zeta = \Delta_{ad} + \rho A \text{ in } V. \]
Then
\[ \Delta_\zeta G_{ad} = \chi_{ad} + O \left( \frac{1}{|\log \delta|} \right) + \rho A G_{ad}. \]
Choose $\delta$ so small that the operator norm of the second term on the RHS is less than $\frac{1}{10}$. With $\delta$ fixed in this way, the support of $\rho A G_{ad}$ is bounded away from the $X_{\nu}$ and so $\rho$ can be bounded here by $C_{ad}(\delta) \varepsilon_0$.

Choosing $\delta$ to satisfy (7.14) and (7.20), and defining
\[ G = \sum_{\nu} G_{\nu} + G_{ad}, \]
we have a bounded operator $\mathcal{H}(W) \to \mathcal{D}(W)$ with the property
\[ \Delta_\zeta G = 1 - e_\zeta \]
where the operator norm of $\epsilon_\zeta$ has the form $\frac{1}{\delta} + C\epsilon_0$. Thus, picking $\epsilon_0$ sufficiently small, $1 - \epsilon_\zeta$ is invertible and $G^{-1} = (1 - \epsilon_\zeta)^{-1}$ is the required inverse.

**Remark 7.7.** The glued inverse operator $G$ appears to depend upon $m$, in that in general if $m$ is increased, we shall need to take $\delta$ smaller. However, the argument shows that if

$$f \in \bigcap_{m,n \geq 0} \epsilon^n \mathcal{R}_{\alpha,\beta,m}$$

then the solution $u$ of $\Delta_\zeta u = f$ given by the Theorem will lie in the intersection

$$u \in \bigcap_{m,n \geq 0} \epsilon^n \mathcal{R}_{\alpha,\beta,m+2}$$

**Theorem 7.8.** Let $\zeta$ be as in Theorem 6.7. Then there exists $\epsilon_0 > 0$ and

$$a \in \epsilon^X \sigma_1^2 \Omega^1_{\phi,e}(W) \otimes \mathbb{R}^3$$

such that $\zeta + da$ is a hyperKähler triple on $W$.

**Proof.** We obtain a finite-regularity solution by the implicit function theorem, and then iterate to obtain smoothness. Seek $a = D^* G\phi$, where $G_\zeta$ is the inverse of $\Delta_\zeta$ from Theorem 7.6.

By (2.16) we require $\phi$ to solve the nonlinear equation

$$\phi = -e - \hat{r}(dD^* G\phi)$$

where $e = Q(\zeta)$ and $\hat{r}$ is quadratic.

We find a solution $\phi \in \mathcal{R}_{\alpha,\beta,m}$. Since $dD^*$ maps $\mathcal{R}_{\alpha,\beta,m+2}$ into $\mathcal{R}_{\alpha,\beta,m}$, we need to know that $u \mapsto u \otimes u$ is bounded from $\mathcal{R}$ to $\mathcal{R}$. If we choose $m > 5/2$ (remember that $W$ is 5-dimensional) then since the weights force decay, this is indeed satisfied. Because $Q(\zeta)$ is smooth and rapidly decreasing in $\varepsilon$ and $\sigma_f$, by taking $\epsilon_0$ small, $Q(\zeta)$ can be arranged to have very small norm in any fixed $\mathcal{R}_{\alpha,\beta,m}(W)$. Because $\hat{r}$ is quadratic, $\phi \mapsto -e - \hat{r}(dD^* G\phi)$ is a contraction for $\epsilon_0$ small enough, giving a solution to (7.33) for any given $m$.

For the regularity statement, note first that the solution will be smooth in any bounded open subset of the interior of $W$ by elliptic regularity, because $Q(\zeta)$ is itself smooth. To see boundary regularity consider first the boundary components $X_\nu$ and $X_{ad}$, staying away from spatial infinity $I_\infty$. The solution for a given $m$ is defined in a subset $\varepsilon < \epsilon_0$, and by construction this solution has $b$-regularity of order $m$ at $\pi^{-1}(0)$. The solution is also $O(\epsilon^n)$ for every $n$, because this is true of $Q(\zeta)$. If we pass to a larger value $m_1 > m$, then we obtain a different solution $u_1$, defined in $\varepsilon < \epsilon_1$, where in general, $\epsilon_1 < \epsilon_0$. However, the solution is unique, so $u_0|_{\{\varepsilon < \epsilon_1\}} = u_1$, and it follows that $u_0$ also has $b$-regularity of order $m_1$. Hence indeed in any neighbourhood $O$ of any boundary point of $\pi^{-1}(0)$, $u \in \epsilon^X H^0_0(\hat{\omega})$ and so is smooth and vanishes to all orders at the boundary of $O$.

For the regularity at $I_\infty$, we need to take a closer look at the asymptotic form of $g_\zeta$. We claim that mod $O(\varepsilon \sigma_1)^\infty$, $g_\zeta$ is given by the Gibbons–Hawking Ansatz near $I_\infty$.

Recall that a hyperKähler metric in 4 dimensions can be expressed using the Gibbons–Hawking Ansatz if it admits an isometric triholomorphic $S^1$-action. The Euclidean coordinates $x_j$ emerge as the three components of the hyperKähler moment map for this action, and the Gibbons–Hawking form of the metric follows by using these coordinates.

If we write $\zeta = \zeta_0 + \zeta_1$ and correspondingly $g_\zeta = g_0 + g_1$, where $\zeta_0$ and $g_0$ are exactly $S^1$-invariant, then the error terms $\zeta_1$ and $g_1$ will be $O((\varepsilon \sigma_1)^\infty)$. Now $\zeta$ is a modification of $\omega_{ad}$ by essentially basic forms, which means that

$$I_{\zeta_0}(\zeta - \omega_{ad}), I_{\zeta_1}(\zeta_0 - \omega_{ad}) \text{ are } O((\varepsilon \sigma_1)^\infty) \text{ near } I_\infty.$$ (7.34)

Thus the $x_j$ are approximate moment maps for $g_0$ and following through the Gibbons–Hawking Ansatz we obtain a harmonic function $h$, say, defined near $I_\infty$, such that

$$g_0 = h \frac{|dx|^2}{\varepsilon^2} + h^{-1} \alpha_\zeta^2$$ (7.35)

The smoothness of decaying solutions of the Laplace equation now follows as it did for $X_\nu$ in Theorem C.3. \qed
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Appendix A. The manifold $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1}$ and dual ellipses

A.1. Homogeneous coordinates and projection operators. The non-compact manifolds obtained from $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1}$ by removing the diagonal and anti-diagonal $\mathbb{CP}^1$ can be interpreted in terms of oriented ellipses in two dual ways. The goal of this appendix is to derive this picture, which was used in §1.4 to explain the geometry and mutual relationships of the spaces $\mathcal{M}_0^2, \mathcal{AH}, \overline{\mathcal{AH}}$ and $\mathcal{HA}$.

We begin by fixing notation for the vector space $\mathbb{C}^2$ and its projective space $\mathbb{CP}^1$. We write

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix},$$

(A.1)

for elements $z, w \in \mathbb{C}^2$ which also serve as homogeneous coordinates for $\mathbb{CP}^1$. We will require both the anti-symmetric bilinear form

$$z \wedge w = z_1 w_2 - w_1 z_2 \quad \text{(A.2)}$$

and the sesquilinear form $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^2$, linear in the second argument, given by

$$\langle w, z \rangle = \bar{w}_1 z_1 + \bar{w}_2 z_2. \quad \text{(A.3)}$$

Defining

$$w^\perp = \begin{pmatrix} -\bar{w}_2 \\ \bar{w}_1 \end{pmatrix}, \quad \text{(A.4)}$$

we note that $z \wedge w^\perp = \langle w, z \rangle$.

We are interested in the non-compact manifolds obtained from $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1}$ by removing the diagonal or anti-diagonal, i.e.,

$$\mathbb{CP}^1 \times \overline{\mathbb{CP}^1} \setminus \mathbb{CP}^1_{\text{diag}} = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2 | z \wedge w \neq 0 \} / \sim, \quad \text{(A.5)}$$

$$\mathbb{CP}^1 \times \overline{\mathbb{CP}^1} \setminus \mathbb{CP}^1_{\text{adiag}} = \{(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2 | z \wedge w^\perp \neq 0 \} / \sim,$$

where $\sim$ is division by the scaling action of $\mathbb{C}^* \times \mathbb{C}^*$ on $(z, w)$. A convenient description of these quotient spaces is in terms of the projection operators

$$P(z, w) = \frac{1}{w \wedge z} z \langle w^\perp, \cdot \rangle, \quad Q(z, w) = \frac{1}{\langle w, z \rangle} z \langle w, \cdot \rangle, \quad \text{(A.6)}$$

naturally representing points in, respectively, $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1} \setminus \mathbb{CP}^1_{\text{diag}}$ and $\mathbb{CP}^1 \times \overline{\mathbb{CP}^1} \setminus \mathbb{CP}^1_{\text{adiag}}$. Clearly $P^2 = P$ and $Q^2 = Q$. With

$$Q^\dagger(z, w) = \frac{1}{\langle z, w \rangle} w \langle z, \cdot \rangle, \quad \text{(A.7)}$$

we also note the identities

$$PQ = Q, \quad PQ^\dagger = 0, \quad \text{(A.8)}$$

and

$$QP = P, \quad QP^\dagger = 0. \quad \text{(A.9)}$$

If we now define traceless $2 \times 2$ matrices $M$ and $N$ via

$$P = \frac{1}{2} (\text{id} + M), \quad Q = \frac{1}{2} (\text{id} + N), \quad \text{(A.10)}$$

then
\[ M^2 = N^2 = \text{id}, \]

as well as \( Q^\dagger = \frac{1}{2}(\text{id} + N^\dagger). \) The identities \([A.8]\) are equivalent to
\[ MN = \text{id} + N - M, \quad MN^\dagger + N^\dagger + M + \text{id} = 0. \]

Before we leave the discussion of the projectors \( P \) and \( Q, \) we note that
\[ P^\dagger(z, w) = P(z, w) \iff w^* = z, \quad Q^\dagger(z, w) = Q(z, w) \iff w = z, \]

so that \( P \) is Hermitian precisely on the anti-diagonal inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \) and \( Q \) is Hermitian precisely on the diagonal inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}}. \)

### A.2. Ellipses in Euclidean space.

To obtain the description of \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \) and \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \) in \( \mathbb{C}^3 \) we use the Pauli matrices
\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

(A.14)
to expand
\[
M = X_1 \sigma_1 + X_2 \sigma_2 + X_3 \sigma_3, \quad N = Y_1 \sigma_1 + Y_2 \sigma_2 + Y_3 \sigma_3,
\]

(A.15)
and then assemble the Cartesian components into vectors \( X = (X_1, X_2, X_3)^t, Y = (Y_1, Y_2, Y_3)^t \)
in \( \mathbb{C}^3, \) with real and imaginary parts
\[
X = \tilde{x} + i\xi, \quad Y = y + i\eta.
\]

(A.16)
Then the constraint \([A.11]\) implies
\[
|\tilde{x}|^2 = |\xi|^2 + 1, \quad \tilde{x} \cdot \xi = 0 \quad |y|^2 = |\eta|^2 + 1, \quad y \cdot \eta = 0.
\]

(A.17)
We thus arrive at pairs of vectors \( (\tilde{x}, \xi) \) and \( (y, \eta) \) satisfying the constraints \([A.17]\) as natural coordinates on, respectively \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \) and \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}}. \) They make explicit the isomorphisms
\[
\mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \cong T^* S^2, \quad \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}} \cong TS^2,
\]

(A.18)
with \( m = \tilde{x}/|\tilde{x}|, \) \( n = y/|y| \) taking values on the round sphere in Euclidean space, and \( \xi \) and \( \eta \) being co-tangent and tangent vectors at \( m \) and \( n. \) It follows from \([A.13]\) that \( X = m \) on the anti-diagonal \( \mathbb{CP}_1^{\text{diag}} \) inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}}, \) so that \( m \) is a natural coordinate there. Similarly, \( Y = n \) on the diagonal \( \mathbb{CP}_1^{\text{diag}} \) inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus \mathbb{CP}_1^{\text{diag}}, \) so that \( n \) is a natural coordinate there. This picture is consistent with the self-intersection numbers of the zero-section of \( TS^2 \) and the diagonal inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \) both being \( +2, \) and the self-intersection numbers of the zero-section of \( T^* S^2 \) and the anti-diagonal inside \( \mathbb{CP}_1 \times \mathbb{CP}_1 \) both being \( -2. \)

The coordinates \( (\tilde{x}, \xi) \) and \( (y, \eta) \) naturally parametrise oriented ellipses up to scale in Euclidean space, which we call the \( X- \) and \( Y- \)ellipse. In this interpretation, \( \tilde{x} \) and \( \xi \) are major and minor axes of the \( X- \)ellipse, while \( y \) and \( \eta \) are the major and minor axes of the \( Y- \)ellipse. When \( \xi = 0 \) the \( X- \)ellipse degenerates into a line along \( \tilde{x} \) and when \( \eta = 0, \) the \( Y- \)ellipse degenerates into a line along \( y. \) The special case of the ellipse becoming a circle is only obtained in the limit of \( |\xi| \to \infty \) for the \( X- \)ellipse and \( |\eta| \to \infty \) for the \( Y- \)ellipse.

We can now state and prove the main result of this appendix.

**Lemma A.1.** The \( X- \) and \( Y- \)ellipses are dual to each other in the sense that
\[
\tilde{x} = \frac{y \times \eta}{|\eta|^2}, \quad \xi = -\frac{\eta}{|\eta|^2},
\]

(A.19)
where \( |\eta| \neq 0. \) This map is an involution of \( \mathbb{CP}_1 \times \mathbb{CP}_1 \setminus (\mathbb{CP}_1^{\text{diag}} \cup \mathbb{CP}_1^{\text{diag}}) \), where we also have
\[
y = \frac{\tilde{x} \times \xi}{|\xi|^2}, \quad \eta = -\frac{\xi}{|\xi|^2}.
\]

(A.20)
In particular, the degeneration of the \( X- \)ellipse into a line corresponds to the degeneration of the \( Y- \)ellipse into a circle at right angles to that line, and conversely.
Proof: We deduce from (A.12) that the Hermitian matrices $X$ and $Y$ characterising the $X$- and $Y$-ellipses satisfy

$$M(N - N^\dagger) = 2 \text{id} + N + N^\dagger. \quad (A.21)$$

Writing $\sigma$ for the vector with cartesian component $\sigma_1, \sigma_2, \sigma_3$, we have

$$N - N^\dagger = 2i\eta \cdot \sigma, \quad (N - N^\dagger)^2 = -4|\eta|^2 \quad (A.22)$$

showing that $N - N^\dagger$ is invertible when $|\eta| \neq 0$, with inverse

$$(N - N^\dagger)^{-1} = -\frac{N - N^\dagger}{4|\eta|^2}. \quad (A.23)$$

Multiplying (A.21) from the right by this inverse, and using and

$$NN^\dagger - N^\dagger N = 4y \times \eta \cdot \sigma \quad (A.24)$$

we deduce

$$M = -\frac{(N + N^\dagger + 2)(N - N^\dagger)}{4\eta^2} \quad (A.25)$$

which is the claimed relation (A.20). The proof of (A.19) is elementary, but also follows from the dual relation of projectors (A.9). One checks that the degeneration of the $X$-ellipse into lines along $\tilde{x}$ as $|\xi| \to 0$ makes the $Y$-ellipse degenerate into a circle of infinite radius in the plane orthogonal to $\tilde{x}$. Conversely, in the limit $|\eta| \to 0$ the $Y$-ellipses degenerate into lines along $y$ while the $X$-ellipses become circles of infinite radius in the orthogonal plane.

A.3. Symmetries. In §1.4 of the main text we also make use of discrete symmetries of $\mathbb{C}P_1 \times \mathbb{C}P_1 \setminus \mathbb{C}P_1^\text{diag}$. The factor switching map

$$s : (z, w) \mapsto (w, z) \quad (A.26)$$

induces the map $Q \mapsto Q^\dagger$ at the level of projectors and hence

$$s : TS^2 \to TS^2, \quad Y \mapsto \tilde{Y}. \quad (A.27)$$

When $|\eta| \neq 0$, it also induces the map $X \mapsto -X$.

The factor switching map composed with the antipodal map on both factors

$$r : (z, w) \mapsto (w^\perp, z^\perp) \quad (A.28)$$

induces the map $Q \mapsto \text{id} - Q$ at the level of projectors and hence

$$r : TS^2 \to TS^2, \quad Y \mapsto -Y. \quad (A.29)$$

When $|\eta| \neq 0$, it also induces the map $X \mapsto \tilde{X}$. The maps $s$ and $r$ commute, and generate the Vierergruppe. The product $a = rs = sr$ is the antipodal map on both factors, so

$$a : (z, w) \mapsto (z^\perp, w^\perp). \quad (A.30)$$

It maps

$$a : TS^2 \to TS^2, \quad Y \mapsto -\tilde{Y}. \quad (A.31)$$

When $|\eta| \neq 0$, it also induces the map $X \mapsto -\tilde{X}$.

The $SU(2)$ action on the homogeneous coordinates induces the adjoint $SO(3)$ action on both $X$ and $Y$, so a rotation

$$X \mapsto GX, \quad Y \mapsto GY, \quad (A.32)$$

of the complex vectors $X, Y \in \mathbb{C}^3$ by $G \in SO(3)$. This action commutes with the action of the Vierergruppe given above.
To make contact with the discussion in the main text we also require a lift of the Vierergruppe to the subgroup $D_2$ of $SU(2)$. This can be achieved by noting that the traceless complex matrix $M$ can be expressed in terms of the magnitudes of $\tilde{x}$ and $\xi$ as

$$M = g(|\tilde{x}| \sigma_3 + i|\xi| \sigma_1) g^\dagger, \quad g \in SU(2) \quad (A.33)$$

and that, for given $M$, this fixes $g$ up to sign when $\xi \neq 0$. Then one checks that, with the matrices $R_\ell = \exp(-i \frac{2\pi}{\ell} \sigma_3)$ and $S = -i \sigma_2$ defined in [12], the right-multiplication by $S$,

$$g \mapsto gS, \quad (A.34)$$

induces the map $s : (\tilde{x}, \xi) \mapsto (-\tilde{x}, -\xi)$ given in (1.14), and the right-multiplication by $R$,

$$g \mapsto gR_\ell, \quad (A.35)$$

induces the map $(\tilde{x}, \xi) \mapsto (x, R_m(2\pi/\ell)\xi)$, where we used the notation defined after (1.19). In particular, the right-multiplication by $R_2$ induces the map $r$ given in (1.2). Identifying $(g, |\xi|)$ for $\xi \neq 0$ with $g(|\xi|, 0)^t \in \mathbb{C}^2$, this is the lift of the Vierergruppe to the binary dihedral group $D_2$ acting on $\mathbb{C}^2$ which is used in the main text.

**APPENDIX B. MANIFOLDS WITH CORNERS**

**B.1 Definitions.** In this paper, we have used manifolds with corners (MWCs) systematically to resolve singularities (for example the indeterminacy in the adiabatic Gibbons–Hawking family in [5]) and to obtain a smooth family of $D_k$ ALF gravitational instantons on the Sen space. We gather here the most important definitions of the theory, the aim being to make the rest of the paper more self-contained, rather than to give a systematic development. For more details, the reader is referred to [30] or the short summary in [28]. Another good introduction is contained in [1].

We give start with an extrinsic definition of MWC, referring to the above references for the intrinsic approach.

Let $M$ be a real manifold of dimension $n$. A subset $X \subset M$ is an $n$-dimensional manifold with corners (MWC) if $X$ is a finite non-empty intersection of ‘half-spaces’ $H_j = \{ \rho_j \geq 0 \}$, where the $\rho_j \in C^\infty(M)$ and $d\rho_j \neq 0$ on the zero-set of $\rho_j$. (Here $j$ lies in some finite index set $J$.) This condition guarantees that $Z_j = \{ \rho_j = 0 \}$ is a smooth embedded submanifold of $M$ of codimension 1.

We assume that the interior $X^\circ$ of $X$ (in $M$) is non-empty, so that $X^\circ$ is an $n$-manifold. We assume also that there is no redundancy in the set $\{ H_j \}$, so that the intersection of any proper subset of the $H_j$ is strictly larger than $X$. In particular $Y_j = X \cap Z_j$ is non-empty, and more importantly its interior in $Z_j$ is a manifold of dimension of $n - 1$. It is customary to suppose that the $Y_j$ are connected. (This can be achieved by renumbering, and possibly shrinking the ambient manifold $M$.) The $Y_j$ are called the boundary hypersurfaces of $X$ and $\rho_j$ is the boundary defining function (bdf) of $Y_j$.

The final technical point is that all non-empty intersections of the boundary hypersurfaces should be cut out transversely by the $\rho_j$—we do not want any pair of boundary hypersurfaces to meet tangentially, for example. Thus we insist that if

$$\rho_j(p) = 0 \text{ for } j = 1, \ldots, k \quad (B.1)$$

then

$$d\rho_1(p) \wedge \cdots \wedge d\rho_k(p) \neq 0. \quad (B.2)$$

(It is to be understood that this holds for all subsets of $J$.) It follows in particular that at most $n$ boundary hypersurfaces can meet in $X$.

**Example B.1.** If we have a finite collection of generically chosen half-spaces in $\mathbb{R}^n$, then there intersection (if non-empty) will be a manifold with corners. One may think of a general manifold with corners as a ‘curvilinear version’ of this, though of course there is no reason for a general MWC to be homeomorphic to a ball.

**Example B.2.** A closed (solid) octahedron in $\mathbb{R}^3$ is not an example of a MWC because four faces come together at each vertex, which is not allowed in a MWC of dimension 3.
In this paper, all our MWCs have corners only up to codimension 2: in other words, there are non-empty intersections of certain pairs of boundary hypersurfaces, but any intersection of three boundary hypersurfaces is empty. We now explain what is meant by adapted coordinates in this setting.

**Example B.3.** First of all, suppose that \( p \) lies on some boundary hypersurface \( Y \) but is not in any intersection \( Y \cap Y' \) of boundary hypersurfaces. Then adapted coordinates in a neighbourhood \( \Omega \) of \( p \) in \( X \) are \( (\rho, y_1, \ldots, y_{n-1}) \), where the \( y_j \)'s are local coordinates on \( Y \cap \Omega \) centred at \( p \); thus \( p \) is identified with the origin of this coordinate system.

**Example B.4.** Similarly, if \( p \in Y \cap Y' \) and the bdfs of \( Y \) and \( Y' \) are respectively \( \rho \) and \( \sigma \), adapted coordinates in a neighbourhood \( \Omega \) of \( p \) in \( X \) are \( (\rho, \sigma, y_1, \ldots, y_{n-2}) \), where now the \( y_j \)'s are local coordinates on \( Y \cap Y' \cap \Omega \) (which is an ordinary \( (n-2) \)-manifold because there are no corners of codimension 3 or more), centred at \( p \). Again, \( p \) is identified with the origin of this coordinate system.

**B.2. The b-tangent bundle.** The references mentioned above develop a suitable category of MWCs and smooth maps. In this development, the so-called b-tangent bundle of MWCs is the ‘correct’ replacement for the tangent bundle in ordinary differential analysis.

Let \( X \) be a compact MWC, of dimension \( n \). The set of all smooth vector fields which are tangent to all boundary faces of \( X \) is denoted \( \mathcal{V}_b(X) \). There is a smooth vector bundle the b-tangent bundle \( T_bX \) with the property that \( C^\infty(X, T_bX) = \mathcal{V}_b(X) \), where on the LHS we have unrestricted smooth sections over \( X \).

Let us give a local description of \( T_bX \) in the case of the two examples above.

**Example B.5.** With the notation of Example B.3 a local basis for \( T_b\Omega \) is given by the vector fields

\[
\frac{\partial}{\partial \rho}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-1}} \tag{B.3}
\]

and \( \mathcal{V}_b(\Omega) \) is the space of all linear combinations of these vector fields with coefficients in \( C^\infty(\Omega) \).

**Example B.6.** With the notation of Example B.4 a local basis for \( T_b\Omega \) is given by the vector fields

\[
\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \sigma}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n-2}} \tag{B.4}
\]

and \( \mathcal{V}_b(\Omega) \) is the space of all linear combinations of these vector fields with coefficients in \( C^\infty(\Omega) \).

If we change adapted local coordinates in either of these examples, we get new local bases \( [B.3] \) or \( [B.4] \) and it is easy to see that these are related by transition functions in \( C^\infty(\Omega) \). This is one way to verify the existence of the bundle \( T_bX \).

For every point \( p \) of \( X \) there is an ‘evaluation map’ \( T_{b,p}X \rightarrow T_pX \), but this is not an isomorphism if \( p \in \partial X \). On the other hand the restriction of \( T_bX \) to the interior \( X^0 \) of \( X \) is canonically isomorphic to \( TX^0 \). However, if \( v \in \mathcal{V}_b(X) \), the smoothness of the coefficients up to and including the boundary of \( X \) means that \( v|X^0 \) will have some controlled vanishing near each of the boundary hypersurfaces.

It is convenient to introduce the following notation.

**Notation B.7.** Let \( X \) be a MWC as above. Let \( \Omega \) be an open set of \( X \). Then \( C^\infty(\Omega) \) is the space of smooth functions on \( \Omega \) (up to including the boundary of \( X \), if \( \Omega \cap \partial X \neq \emptyset \)). The space \( C^\infty_0(\Omega) \) is the subspace of functions with compact support of \( \Omega \). The support of such a function meets \( \partial X \) in a compact subset of \( \partial X \cap \Omega \) but need not be empty. By contrast, the subspace \( \hat{C}^\infty(\Omega) \) consists of those functions which vanish to all orders at the boundary hypersurfaces with non-empty intersection with \( \Omega \). If \( f \in \hat{C}^\infty(\Omega) \) we say that \( f \) is rapidly decreasing at \( \partial \Omega \) and this has to be understood in the precise sense of the previous sentence. In the setting of Example B.4 we say that \( f \in C^\infty(\Omega) \) is rapidly decreasing at \( Y \) if \( f \) vanishes to all orders in the bdf \( \rho \) of \( Y \).
B.3. Radial compactification and the scattering tangent bundle. Let $E$ be a euclidean vector space of dimension $n$. The radial compactification $\overline{E}$ is a compact manifold with boundary, obtained by adjoining the ‘sphere at $\infty$’ to $E$. Concretely, $\overline{E}$ is obtained from the disjoint union

$$E \cup S^{n-1} \times \{0, 1\}$$

by identifying $x \in \{|x| > 1\}$ with the point $(x/|x|, 1/|x|)$ in $S^{n-1} \times (0, 1)$. For the smooth structure defined on $\overline{E}$ in this way, $\rho = 1/|x|$ is a bdf for $\partial \overline{E}$ in $\overline{E}$, smooth in a neighbourhood of the boundary. (By cutting off $\rho$ to be equal to 1 in a $B(0, 1/2)$, say, we can define a bdf for $\partial \overline{E}$ which is in $C^\infty(\overline{E})$.)

If we adjoin local coordinates $(y_1, \ldots, y_{n-1})$ in $S^{n-1}$ to $\rho$, we then have adapted local coordinates near a point of $\partial \overline{E}$. It is not hard to see that the standard euclidean vector fields $\partial/\partial x_j$ become smooth linear combinations of the vector fields

$$\rho^2 \frac{\partial}{\partial \rho}, \rho \frac{\partial}{\partial y_1}, \ldots, \rho \frac{\partial}{\partial y_{n-1}}. \quad (B.5)$$

This observation motivates the definition of the scattering tangent bundle $T_{sc} \overline{E}$ which is locally spanned over $C^\infty(\overline{E})$ by the vector fields $(B.5)$. Correspondingly, the euclidean metric extends as a smooth metric on the bundle $T_{sc} \overline{E}$ over $\overline{E}$—another example of a rescaled tangent bundle on the compactification of a non-compact manifold.

One of the advantages of the radial compactification $\overline{E}$ of $E$ is that important subspaces of $C^\infty(E)$ have simple descriptions [27, Appendix A] using $\overline{E}$. We mention some of these:

- The Schwarz space $\mathcal{S}(E) \subset C^\infty(E)$ of all whose derivatives vanish faster than any power of $|x|$ for $|x| \to \infty$ corresponds exactly $C^\infty(\overline{E})$.

- The space $S^0(E)$ of Kohn-Nirenberg symbols of order 0 on $E$, that is, the functions $f$ such that

$$\sup_{\overline{E}} |\partial_\alpha f| \leq C_\alpha (1 + |x|)^{-|\alpha|} \quad (B.6)$$

for all multi-indices $\alpha$ corresponds to the space $\mathcal{S}^0(\overline{E})$ of conormal functions (of order 0): this space can be defined as the set of functions $f$ for which

$$\sup_{\overline{E}} |v_1 \ldots v_N f| < \infty \quad (B.7)$$

for any collection of b-vector fields $v_j \in \mathcal{V}_b(\overline{E})$.

- The subspace $C^\infty(\overline{E}) \subset \mathcal{S}^0(\overline{E})$ then corresponds to the space $S^0_E$ of classical symbols, meaning those that have classical asymptotic expansions as sums of homogeneous functions for large $x$.

- More generally, the space of symbols of order $m$ on $E$ corresponds to $\rho^{-m} \mathcal{S}^0(\overline{E})$, with $\rho^{-m}C^\infty(\overline{E})$ corresponding to the subspace of classical symbols of order $m$.

B.4. Blow-up of a point in a half-space. Since it is so important in this paper, we quickly describe the real blow-up of the origin in a half-space. For a more complete description of the real blow-up, two references among many are [1, 30].

Let $E$ be an $n$-dimensional euclidean space and let $X = E \times [0, \infty)$, with euclidean coordinates $x \in E$ and $\varepsilon$ in the half-line. The real blow-up $\tilde{X}$ of $X$ in $(0, 0)$, denoted by $[X; (0, 0)]$, is a MWC in which there is a new boundary hypersurface, the front face $ff$ or exceptional divisor, which parameterises the space of rays in $X$ emanating from $(0, 0)$. If such a ray does not lie in $E \times \{0\}$, then it has a unique intersection with $E \times \{1\}$, and so this part of $\text{ff}$ may be naturally identified with another copy of $E$.

The main properties of $\tilde{X}$ are the following (Figure 5). First of all it is a MWC with two boundary hypersurfaces, which we shall denote by $Y_0$ and $Y_1$. $Y_0$ is the lift of $\{\varepsilon = 0\} \subset E$ to $\tilde{X}$ and is naturally identifiable with the blow-up $[E; 0]$ of $E$ in the origin. (Thus $Y_0$ is diffeomorphic to the complement of an open ball in $E$, though not canonically.) On the other hand the front face $Y_1$ by definition is the set of rays in $X$ through 0, and is naturally identifiable with the radial compactification of another copy $E'$, say, of $E$. (It is natural to think of $E'$ as the tangent space
T_0E.) The blow-up map $\beta : \tilde{X} \to X$ maps $Y_1$ to $(0,0)$ and restricts to define a diffeomorphism $\tilde{X}\setminus Y_1 \to X\setminus(0,0)$.

The original coordinates $(x, \varepsilon)$ lift to $\tilde{X}$ and give smooth coordinates away from $Y_1$. In particular $\varepsilon$ is a local bdf for $Y_1$, in the sense that is an acceptable bdf for $Y_1$ on any subset $\Omega$ which is bounded away from $Y_1$. Similarly, $(x' = x/\varepsilon, \varepsilon)$ are local coordinates near any point of $Y_1\setminus Y_0 \cap Y_1$, and $\varepsilon$ is a local bdf for $Y_1$, away from $Y_0$. Adapted coordinates near the corner can be taken to be $(\rho, \sigma, y)$, where $y = x/|x|$, $\rho = 1/|x'| = \varepsilon/|x|$ and $\sigma = |x|$. In particular $\beta^*\varepsilon = \rho \sigma$ in these coordinates.

**Appendix C. Analysis of the Laplacian of an ALF space**

In [26] we recalled the notion of a compact manifold with fibred boundary, or $\phi$-structure. The geometrical microlocal approach to the analysis of elliptic operators in this setting was first undertaken in [26] and was further developed for the purposes of Hodge theory in [36] and [19]. In this section our focus is on the Laplacian of a strongly ALF metric (Definition 3.6), and in particular on the Poisson problem

$$\Delta_g u = f$$

(C.1)

on a strongly ALF space $(X, g)$.

The analysis of (C.1) is simpler than that of the Laplacian of a general $\phi$-metric for two reasons. First of all, $g$ is *essentially invariant* with respect to the $S^1$-action we have in the asymptotic Gibbons–Hawking chart guaranteed by Definition 3.6. It follows from $\Delta_g$ preserves the Fourier modes of $u$, up to rapidly decreasing errors near $\partial X$. In particular, modulo such very small errors, if $u$ is $S^1$-invariant in a neighbourhood of $\partial X$, then $\Delta_g u = h^{-1}\Delta_0 u$, in the Gibbons–Hawking chart, where $h$ is the harmonic function which defines the asymptotic Gibbons–Hawking metric. The very weak coupling between the different Fourier modes, together with this very simple action of $\Delta_g$ on the zero Fourier-mode yields much more regular behaviour of the solution $u$ in (C.1) if $f$, for example is smooth and supported away from $\partial X$.

In order to state our results, we have to introduce function spaces which treat the zero- and non-zero-Fourier modes of the data differently.

**Definition C.1.** Let $d\mu_{b}$ be any smooth $b$-density on $X$, and let $L^2_{b}(X)$ be the resulting space of $L^2$ functions. For positive integers $H_{\phi,b}^{n,m}(X)$ is the Sobolev space of functions on $X$ with $m$ $b$-derivatives and $n$ $\phi$-derivatives in $L^2_{b}(X)$:

$$\text{Diff}_{\phi}^{m} u \subset L^2_{b}(X).$$

As previously, the subscript $\text{ei}$ will be used to denote functions which are essentially invariant near $\partial X$. Write $H_{\phi}^{s}(X)$ for $H^{s,0}(X)$.

We now define a family of domains and ranges for $\Delta_g$ so that

$$\Delta_g : \mathcal{D}_{\alpha,\beta,m+2}(X) \to \mathcal{R}_{\alpha,\beta,m}(X)$$

is a bounded invertible linear mapping. There is some flexibility in the definition, and we choose to make the range space as close as possible to a b-Sobolev space, albeit one in which the
invariant and non-invariant components with respect to the $S^1$-action are weighted differently. For $m$ a positive integer and any numbers $\alpha > 0$ and $\beta > \alpha + 2$, define

$$\mathcal{R}_{\alpha,\beta,m}(X) = \{ \chi f_0 + f_1 \} \text{ where } f_0 \in \rho^{\alpha+2}H^0(U) \text{ and } \frac{\partial f_0}{\partial \theta} = 0, f_1 \in \rho^\beta H^0_b(X). \quad (C.3)$$

In this definition, $U$ is a collar neighbourhood of $\partial X$ and $\chi$ is cut-off function with compact support in $U$ and identically 1 in a neighbourhood of $\partial X$. We always take $\beta > \alpha + 2$, so that the invariant component decays more slowly than the non-invariant component.

For the domain, define

$$\mathcal{D}_{\alpha,\beta,m+2}(X) = \{ \chi u_0 + u_1 \} \text{ where } u_0 \in \rho^\alpha H^{m+2}_b(U) + \mathcal{H}_\alpha, \quad \frac{\partial u_0}{\partial \theta} = 0, u_1 \in \rho^\beta H^{2,m}_b(X). \quad (C.4)$$

where everything is already defined apart from $\mathcal{H}_\alpha$. This is a finite-dimensional space of finite-order harmonic multipole expansions on $\mathbb{R}^3$,

$$\mathcal{H}_\alpha = \left\{ \sum_{j=1}^{[\alpha]} h_j(x) \right\} \quad (C.5)$$

where $h_j(x)$ is homogeneous of degree $-j$ on $\mathbb{R}^3 \setminus \{0\}$. Here if $\alpha \in (0, 1)$, $\mathcal{H}_\alpha = \{0\}$ by definition.

**Theorem C.2.** Let the definitions be as above: in particular $\alpha > 0$ and $\beta > \alpha + 2$. Then there is a bounded inverse of $\Delta$,

$$G : \mathcal{R}_{\alpha,\beta,m} \to \mathcal{D}_{\alpha,\beta,m+2}, \quad \Delta G = G \Delta = 1. \quad (C.6)$$

If $f \in \dot{C}^{\infty}(X)$, then it lies in the intersection over all $(\alpha, \beta, m)$ of $\mathcal{R}_{\alpha,\beta,m}$, the solution $u = Gf$ lies in all the $\mathcal{D}_{\alpha,\beta,m}$, and so:

**Theorem C.3.** Let $f \in \dot{C}^{\infty}(X)$. Then there exists unique $u$

$$u \in |x|^{-1}C^{\infty}_{\alpha}(X) + \dot{C}^{\infty}(X) \quad (C.7)$$

solving $\Delta u = f$.

The essentially invariant part of $u$ in (C.7) is not merely smooth, but has an asymptotic expansion in homogeneous harmonic functions (multipole expansion). That is, there is a sequence of functions $h_j$ on $\mathbb{R}^3 \setminus \{0\}$ such that for any $N$,

$$u - \sum_{j=1}^{N} h_j \in |x|^{-N-1}C^{\infty}(X) \quad (C.8)$$

where $h_j$ is homogeneous of degree $-j$. Furthermore this equation can be differentiated any number of times and remains valid.

**C.1. Discussion of Proof of Theorem C.2.** In this section, we assume some familiarity with the methods and notation of [26] as presented in that paper or the others mentioned at the beginning of this Appendix.

The inverse operator $G$ is constructed in stages, using the class $\Psi^*_\phi(X)$ of $\phi$-pseudodifferential operators on $X$. These operators have kernels whose structure is clearest on the ‘stretched product’ $X_2^2$ which is a certain blow-up of the cartesian square $X^2 = X \times X$. It has two front faces, $\Pi_b$ and $\Pi_\phi$, as well as the old boundary hypersurfaces, which are the lifts of $\partial X \times X$ and $X \times \partial X$ to $X_2^2$. The space of $\phi$-smoothing operators $\Psi^{-\infty}_\phi(X)$ consists of those operators whose Schwarz kernels lift to smooth functions on $X_2^2$ which in addition rapidly decreasing at all boundary hypersurfaces apart from $\Pi_\phi$. The first step in the construction of $G$ is to find $G \in \Psi^{-2}_\phi(X)$ such that

$$\Delta P = 1 - R = P \Delta \quad (C.9)$$

where $R \in \Psi^{-\infty}_\phi(X)$. 
In an asymptotic Gibbons–Hawking chart, $R$ has kernel of the form $R(x - x', x'; \theta, \theta')$, where $R$ is smooth and rapidly decreasing in the first two variables.\(^3\) (Here $(x, \theta)$ and $(x', \theta')$ are local coordinates on two copies of the asymptotic Gibbons–Hawking chart of $X$.) The subclass $\Phi^{-\infty}(X)$ of essentially invariant smoothing operators is defined by insisting that $R$ be essentially invariant with respect to the diagonal circle action on $X^2_\phi$. Explicitly, this means that the kernel of $R$ has the form $R(x - x', x'; \theta - \theta')$ modulo a function in $\dot{C}^{\infty}(X^2_\phi)$.

The construction of $P$ depends only on the $\phi$-symbol of $\Delta_g$, and it is not hard to check that the essential invariance of $\Delta_g$ means that $P$ can be chosen to be essentially invariant so that $R \in \Psi^{-\infty}_{\phi, ei}(X)$. To summarise:

**Proposition C.4.** Let $(X, g)$ be (the compactification of) a strongly ALF space, and let $\Delta_g$ be the Laplacian of $g$. Then there exist $P \in \Psi^{-2}_{\phi, ei}(X)$ and $R \in \Psi^{-\infty}_{\phi, ei}(X)$, formally self-adjoint, such that

$$P \Delta_g = 1 - R, \quad \Delta_g P = 1 - R. \quad \text{(C.10)}$$

The error term $R$ in (C.10) is not compact as an operator $L^2(X) \to L^2(X)$, and so (in contrast to the case of a compact manifold without boundary) Proposition C.4 does not immediately imply that $\Delta_g$ is Fredholm. We shall improve $P$ by replacing it by an operator $P + Q$, so that $\Delta_g Q - R$ is still smooth but also decays ‘at $\infty$’ and in particular at $\mathbb{R}_\phi$. This condition has a precise interpretation in terms of vanishing at the various boundary hypersurfaces of $X^2_\phi$.

Just by setting $x = x' + w$, we obtain:

**Lemma C.5.** Let $(X, g)$ be a strongly ALF space with Laplacian $\Delta_g$ as above. Then the action of $\Delta_g$ on $Q(x - x', x'; \theta, \theta')$ is given by the first factor of the product) to $X^2_\phi$ is given by

$$\Delta_g Q = \frac{1}{h(x' + w)} \tilde{\Delta}_{0,w} - h(x' + w) \frac{\partial^2}{\partial \theta^2} + O(|x'|^{-\infty}) \quad \text{(C.11)}$$

where $\tilde{\Delta}_{0,w}$ is the horizontal lift of the euclidean Laplacian (in $w$) as in (6.33) (with $\varepsilon = 1$).

In general, the normal operator $N(A)$ of a $\phi$-differential operator is obtained from the lift of the operator to $X^2_{\phi}$, acting on functions defined near $\mathbb{R}_\phi$, and setting $\rho_{\phi} = 0$. For $\Delta_g$, this is achieved by taking $x'$ to $\infty$, yielding

$$N(\Delta_g) = \Delta_0 = \Delta_{\mathbb{R}^3 \times S^1}, \quad \text{(C.12)}$$

(the variables in $\mathbb{R}^3 \times S^1$ being $(w, \theta)$). Moreover, $N(R)$ is defined for any $R \in \Psi^{-\infty}_{\phi}(X)$ by restriction to $\mathbb{R}_\phi$. Thus to ‘solve away’ the leading term of $R$ at $\mathbb{R}_\phi$, it is enough to solve the model problem

$$N(\Delta_g) Q_0 = N(R) \quad \text{(C.13)}$$

for $Q_0$ defined on $\mathbb{R}_\phi$. A smooth extension of $Q_0$ to the interior will then give $Q \in \Psi^{-\infty}_{\phi}(X)$ with $N(Q) = Q_0$ and such that $\Delta_g Q - R$ vanishes at least to first order at $\mathbb{R}_\phi$.

In the language of [26] the Laplacian is not fully elliptic, meaning that one cannot solve (6.13) with a rapidly decreasing $Q_0$ even if $N(R)$ is rapidly decreasing. The problem here is the zero Fourier-mode or invariant part of $N(R)$. Thus we shall treat the zero- and non-zero Fourier modes of $N(R)$ and $Q_0$ separately.

**Definition C.6.** The space $\mathcal{C}^{\infty}_{\phi, ei}(X^2_{\phi})$ consists of those smooth functions on $X^2_\phi$ that are essentially $S^1 \times S^1$-invariant in a neighbourhood of the essential invariance $\Phi^{-\infty}(X^2_{\phi})$, and the subspace of $\Psi^{-\infty}(X)$ consisting of the essentially $S^1 \times S^1$-invariant elements.

**Remark C.7.** In other words, $(ei, ei)$ functions on $X^2_\phi$ are those that can be written in the form $f_0 + f_1$, where $f_0$ is supported near $\mathbb{R}_\phi \cup \mathbb{R}_b$ and pulled back from $U^2_{\phi, w}$, while $f_1 \in \dot{C}^{\infty}(X^2_\phi)$.

When smooth functions are regarded as Schwarz kernels of operators on functions on $X$, $P \in \Psi^{-\infty}_{ei}(X)$ will (essentially) preserve the Fourier modes: the $n$-th Fourier mode of $Pu$ near $x' \to \infty$ with $|x - x'|$ bounded corresponds to going towards $\mathbb{R}_\phi$. In fact, on any set where $|x - x'|$ is bounded, $1/|x'|$ is a local bdf for $\mathbb{R}_\phi$ in $X^2_\phi$.\(^2\)
the boundary is determined by the $n$-th Fourier mode of $u$, up to $O(\rho^n)$ errors. On the other hand, if $Q \in \Psi_{\epsilon,\epsilon}^{-\infty}$, $Qu$ is essentially invariant and depends only on the zero-Fourier mode of $u$, up to $O(\rho^n)$ errors. Thus $Q$ essentially annihilates the non-zero Fourier modes of $u$.

Similar issues were encountered and dealt with in \[36, 19\]. Our results are slightly simpler because of the essential invariance of $g$ and the simple explicit form \((C.11)\) of the lift of $\Delta_g$ to $X_\phi^2$.

**Proposition C.8.** Let $(X, g)$ be a strongly ALF space with asymptotic Gibbons–Hawking model $\gamma_{gh}$. There exists

$$G \in \Psi_{\phi,\epsilon}^{-2}(X) + \rho_0 \rho' C_{\epsilon,\epsilon}(X_\phi^2)$$

(C.14)

such that

$$\Delta G = 1, \ G\Delta = 1.$$  

(C.15)

Here $\rho_0$ is the bdf of $\gamma_{\beta}$ and $\rho$ and $\rho'$ are the bdfs of the old boundary hypersurfaces of $X_\phi^2$.

**Proof.** (Sketch) Starting from the error term $R \in \Psi_{\epsilon,\epsilon}^{-\infty}(X)$ from Proposition \((C.4)\), write $R = R_0 + R_1$, where $R_0$ is exactly invariant with respect to the $S^1 \times S^1$-action, and supported near $\gamma_{\phi}$. (This can be achieved by averaging $R$ over $S^1 \times S^1$.) Then we may think of $R_0$, at least away from $\gamma_{\phi}$, as a function $R_0(x, x')$, where $x$ and $x'$ are euclidean (asymptotic) coordinates near the base $U$ of the asymptotic Gibbons–Hawking chart of $X$. We solve

$$\Delta_{gh} Q_0 = R_0,$$  

(C.16)

near $\gamma_{\phi} \cup \gamma_{\beta}$ using the formula:

$$Q_0(x, x') = \chi(\rho)\chi(\rho') \frac{1}{4\pi} \int \frac{1}{|x - x''|} h(x'') R_0(x'', x') \, dx''.$$  

(C.17)

Here $\rho = 1/|\cdot|$, $\rho' = 1/|\cdot'|$, and $\chi(t)$ is a cut-off function identically equal to 1 for $t \leq \delta/2$ and 1 for $t \geq \delta$. So defined, $Q_0$ satisfies \((C.16)\) where $\chi(\rho) = 1$, which includes a neighbourhood of $\gamma_{\beta} \cup \gamma_{\phi}$. What has to be checked is that \((C.17)\), lifted to $X_\phi^2$, defines a function with vanishing specified in \((C.14)\), and we omit the details of this computation. In any case, we have

$$\Delta_{gh} Q_0 = \chi(\rho)\chi(\rho') R_0 + [\Delta_{gh}, \chi(\rho)]\chi(\rho') \frac{1}{4\pi} \int \frac{1}{|x - x''|} h(x'') R_0(x'', x') \, dx''.$$  

(C.18)

We may suppose that $\chi(\rho)\chi(\rho')$ is identically 1 on the support of $R_0$. Then since the commutator term is supported away from $\gamma_{\phi} \cup \gamma_{\beta}$, we have a solution to \((C.16)\), modulo an error supported near $\rho' = 0$ but away from the other faces.

For the non-invariant part, we solve

$$\tilde{\Delta}_{gh} Q_1 = R_1$$  

(C.19)

less explicitly by using the normal operator of $\tilde{\Delta}_{gh}$. From \((C.12)\), $N(\Delta_g)$ is the flat Laplacian on $\mathbb{R}^3 \times S^1$. This is easily inverted on the non-zero Fourier modes by use of the Fourier transform.

**Lemma C.9.** Let $f \in C_\infty(\mathbb{R}^3 \times S^1)$ be rapidly decreasing in $w \in \mathbb{R}^3$ and be such that the zero-Fourier mode of $f$ is 0. Then there exists smooth rapidly decreasing $u$ such that $\Delta_0 u = f$.

**Proof.** Expand $f$ in Fourier series and take the Fourier transform in $\mathbb{R}^3$. Then $f$ is replaced by a sequence $(\hat{f}_n(\eta))$ where $\hat{f}_0 = 0$, all coefficients are smooth and rapidly decreasing in $(\eta, n)$. In the dual variables, $\Delta_0$ acts on the $n$-th component as multiplication by $|\eta|^2 + n^2$. Hence we define $u$ to be the inverse Fourier transform of

$$(\hat{f}_n(\eta)) \left( |\eta|^2 + n^2 \right)$$

Because $\hat{f}_0 = 0$, each coefficient is smooth in $\eta$ and the sequence is rapidly decreasing in $(n, \eta)$. Hence $u$ is smooth and rapidly decreasing in $w$.  \(\square\)
Applying this result with $f$ replaced by $N(R)$, we obtain a first approximation $q_1$, to $Q_1$, that is
\begin{equation}
\Delta_{gh}q_1 - R_1 \in \rho_0 \Psi_{\epsilon_1}^{-\infty}(X) \tag{C.20}
\end{equation}
We can now iterate the construction successively to solve away the coefficients in the expansion of $\Delta_{gh}$ at $f\phi$. The result is $Q_1 \in \Psi_{\epsilon_1}^{-\infty}(X)$ (and with no zero Fourier-mode) such that
\begin{equation}
\Delta_{gh}Q_1 - R_1 \in \rho_0^2 \Psi_{\epsilon_1}^{-\infty}(X) \tag{C.21}
\end{equation}
Combining $Q_0$ and $Q_1$ on $X^2_\phi$, we obtain an improved parametrix $P_1 = P + Q$ which inverts $\Delta_{gh}$ mod a compact error. Standard arguments, using the formal self-adjointness and positivity of $\Delta_{gh}$ allow one to get from here to the inverse $G$ in the statement of the theorem.

To complete the proof of Theorem C.2 it has to be checked that the operator $G$ of Proposition C.3 defines a bounded linear map $\mathcal{R}_{\alpha,\beta,m}(X) \rightarrow \mathcal{R}_{\alpha,\beta,m+2}(X)$ for every $\beta > \alpha + 2$ and all $m$. This can be proved, for example by splitting $G$ as a sum of terms, one being in $\Psi_0^{-2}$ and the rest having smooth kernels on $X^2_\phi$. The boundedness is proved separately for these terms, either by hand, or by using the general push-forward theorem of [28].

As a technical remark, we observe that the asymptotic expansion of $u$ in Theorem C.3 is much better than might be expected in general. These asymptotic expansions are generally polyhomogeneous conormal expansions and arise as part of the general theory of b-differential operators, [28] Chapter 5]. The reason that our expansions are so simple is that $\Delta_{\phi} - \Delta_{gh}$ vanishes to all orders in $\rho$ near $\partial X$. This means that the asymptotic expansion of an $L^2$ solution of $\Delta_{\phi}u = f$, where $f \in C^\infty(X)$, will consist of $S^1$-invariant terms and these will be exactly as for the asymptotic solution of $\Delta_{gh}u = 0$, where $u_0$ and $f_0$ are $S^1$-invariant, and $f_0 \in C^\infty(\mathbb{R}^3)$. Once again, because $\Delta_{gh}u_0 = h\Delta_{\phi}u_0$, the asymptotic expansion for $u_0$ must be the same as for the laplacian on $\mathbb{R}^3$, which of course is the usual multipole expansion of a harmonic function defined in a region $\{|x| > R\} \subset \mathbb{R}^3$.

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DEPARTMENT OF MATHEMATICS, HERIOT-WATT UNIVERSITY AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES
E-mail address: b.j.schroers@hw.ac.uk

DEPARTMENT OF MATHEMATICS, UNIVERSITY COLLEGE LONDON
E-mail address: michael.singer@ucl.ac.uk