Regularity Theorems in the Nonsymmetric Gravitational Theory

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Abstract

Regularity theorems are presented for cosmology and gravitational collapse in non-Riemannian gravitational theories. These theorems establish conditions necessary to allow the existence of timelike and null path complete spacetimes for matter that satisfies the positive energy condition. Non-Riemannian theories of gravity can have solutions that have a non-singular beginning of the universe, and the gravitational collapse of a star does not lead to a black hole event horizon and a singularity as a final stage of collapse. A perturbatively consistent version of nonsymmetric gravitational theory is studied that, in the long-range approximation, has a nonsingular static spherically symmetric solution which is path complete, does not have black hole event horizons and has finite curvature invariants. The theory satisfies the regularity theorems for cosmology and gravitational collapse. The elimination of black holes resolves the information loss puzzle.
I. INTRODUCTION

Einstein’s gravitational theory (EGT) [1] is an aesthetically pleasing theory, which agrees with all observational tests to date. However, all these tests are for weak gravitational fields; there is no convincing evidence that the theory will be valid for strong gravitational fields. The occurrence of singularities in the theory, when physical quantities such as the density and pressure of matter become infinite, produces an unphysical feature of the theory. The theorems of Hawking and Penrose [2–7] prove that for physical matter in EGT, these singularities cannot be avoided.

A new version of the nonsymmetric gravitational theory (NGT) has recently been proposed which is perturbatively consistent [8–10]. It is free of ghost poles and tachyons in the linear approximation and in a linear expansion of $g_{\mu\nu}$ about a curved Einstein background. The static spherically symmetric Wyman solution of the NGT vacuum field equations was found to be nonsingular with no event horizons and with finite curvature invariants [11,12]. These nonsingular results are valid in the long-range approximation of the new version of NGT. There are two basic parameters in the solution that enter as constants of integration: the conserved gravitational charge $m$ and a dimensionless constant $s$. The limit to EGT, as $s \to 0$, is not analytic at points $r < 2m$. A final collapsed object (FCO), which is stable for an arbitrarily large mass, replaces the black hole [11–13].

The problem of black hole information loss, first considered in detail by Hawking, has received wide attention recently [14]. Since spacetime is divided by an event horizon into two causally disconnected manifolds, and since Hawking radiation is approximately thermal, all the information associated with a collapsing star which passes through the event horizon is lost to an outside observer. After the black hole has radiated away to nothing by the process of Hawking radiation, all information about the star will have disappeared. Since an initial pure quantum state before collapse becomes a mixed state in the final stage of collapse, the unitarity of the scattering S-matrix is violated. A possible way out of this breakdown of the predictability of physics is to modify EGT, so that the event horizon and the singularity which occur as final features of gravitational collapse are removed. This would solve the information loss puzzle in a simple way at the classical level.

It would seem that we are faced with an almost insurmountable problem when attempting to discover a classical modification of EGT, that removes all its singular features. The reason for this is that such a modification might be expected to lead to a solution of the field equations, which can be mapped into one that lies near the Schwarzschild solution of EGT, and shares its singular properties. Given that the extra degrees of freedom associated with the modified gravitational theory are related to certain fields and coupling constants, if the limit to EGT is smooth and analytic in the coupling constants, then it would follow that a set of initial data on a Cauchy surface can always be found that leads, on solving the field equations of the theory, to a black hole as the final stage of collapse. Therefore, the modified theory of gravity should have a global exact solution for collapse which is non-perturbative, and does not have a smooth limit in the strong gravitational field regime at the event horizon radius, $r = 2m$, and at $r = 0$. This would mean that there is no smooth limit to the Schwarzschild solution for $r \leq 2m$. Even for a very large black hole with weak curvature at $r = 2m$, the modified theory must not display a null surface (event horizon), so we must require that the solution of the modified field equations be non-perturbative.
and non-analytic in the new degrees of freedom at $r \leq 2m$. Of course, for large values of $r$, corresponding to weak gravitational fields, the solution should be smoothly analytic in the coupling constants in the EGT limit, in order to retain the agreement of EGT with observations. This is the case for the nonsingular Wyman solution of NGT \cite{11,12,15}.

We shall study the necessary conditions that must be satisfied by NGT to guarantee the existence of regular solutions in cosmology and gravitational collapse. Because NGT is based on non-Riemannian geometry, it is possible to satisfy a generalized positive energy condition and permit the existence of timelike and null path complete spacetimes. Thus, in a non-Riemannian geometry, the singularity theorems of Hawking and Penrose can be circumvented for physical matter in strongly causal spacetimes. In contrast to Einstein’s theory of gravity, the gravitational field can be repulsive for physical matter fields. NGT is among a class of non-Riemannian theories that possess this feature. The non-singular static Wyman solution of the NGT field equations does not contain any trapped surfaces for $r < 2m$, and it is shown to satisfy the regularity theorem for gravitational collapse.

Another class of theories that can circumvent the Hawking-Penrose theorems for positive density and non-negative pressure is higher-derivative theories based on Riemannian geometry \cite{16–18}. These theories share the feature with NGT that gravity is not always attractive, so that the focussing effect of geodesics typical of gravity theories based on Riemannian geometry can be removed, thus allowing singularity-free solutions for gravitational collapse. However, these theories generally have ghost poles, tachyons or second-order poles in the linear approximation when they are expanded about Minkowski spacetime. Thus, these theories lead to unstable solutions and negative energy modes.

Another alternative theory of gravitation has been proposed by Yilmaz \cite{19}. This theory contains an energy-momentum tensor density defined in terms of a scalar field $\phi$, which acts as a source for the gravitational field. The static spherically symmetric solution is free of event horizons, but the singularity at $r = 0$ remains as an essential singularity in the metric tensor. This is an undesirable feature of a gravitational theory, because the absence of an event horizon in Yilmaz’s theory produces a naked singularity which destroys the Cauchy solution and any predictable power of the theory. This feature is shared by Einstein’s gravitational theory for solutions which do not have black hole event horizons and by any theory which does not remove the essential singularity at $r = 0$. In NGT, there is no singularity at $r = 0$ and the removal of the black hole event horizon does not destroy the predictability of the theory.

In Section 2, we present a perturbatively consistent NGT based on non-Riemannian geometry, while in Section 3 we give useful formulas connected with affine connections and curvature tensors. We determine, in Section 4, the properties of the geodesic and path equations of motion in NGT, and discuss the geodesic and path deviation equations. The static spherically symmetric solution of the NGT field equations is treated in Section 5, while Section 6 is devoted to the properties of geodesic and path congruences and the consequences of the generalized Raychaudhuri equation. In Sections 7 and 8, regularity theorems in cosmology and gravitational collapse are proved, and we present concluding remarks in Section 9.
II. NONSYMMETRIC GRAVITATIONAL THEORY

The non-Riemannian geometry is based on the nonsymmetric field structure with a nonsymmetric $g_{\mu\nu}$ \[^{[20,21]}\]

\[ g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}, \quad (2.1) \]

where

\[ g_{(\mu\nu)} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu}), \quad g_{[\mu\nu]} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu}). \quad (2.2) \]

Moreover, the affine connection is nonsymmetric:

\[ \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)} + \Gamma^\lambda_{[\mu\nu]}. \quad (2.3) \]

The contravariant tensor $g^{\mu\nu}$ is defined in terms of the equation:

\[ g^{\mu\nu} g_{\sigma\nu} = g^{\nu\mu} g_{\nu\sigma} = \delta^\mu_\sigma. \quad (2.4) \]

The Lagrangian density is given by

\[ \mathcal{L}_{NGT} = \mathcal{L}_R + \mathcal{L}_M, \quad (2.5) \]

where

\[ \mathcal{L}_R = g^{\mu\nu} R_{\mu\nu}(W) - 2\Lambda \sqrt{-g} - \frac{1}{4}[g_{\mu\nu}^2 g_{[\mu\nu]} - \frac{1}{6} g^{\mu
u} W_{\mu\nu}], \quad (2.6) \]

where $\Lambda$ is the cosmological constant and $\mu^2$ is the square of a mass associated with $g_{[\mu\nu]}$. Moreover, $\mathcal{L}_M$ is the matter Lagrangian density ($G = c = 1$):

\[ \mathcal{L}_M = -8\pi g^{\mu\nu} T_{\mu\nu}. \quad (2.7) \]

Here, $g^{\mu\nu} = \sqrt{-g} g^{\mu\nu}$ and $R_{\mu\nu}(W)$ is the NGT contracted curvature tensor:

\[ R_{\mu\nu}(W) = W^\beta_{\mu\nu,\beta} - \frac{1}{2}(W^\beta_{\mu\beta,\nu} + W^\beta_{\nu\beta,\mu}) - W_{\alpha\beta} W^{\alpha\beta}_{\mu\nu}, \quad (2.8) \]

defined in terms of the unconstrained nonsymmetric connection:

\[ W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{2}{3} \delta^\lambda_\mu W_{\nu}, \quad (2.9) \]

where

\[ W_{\mu} \equiv W^\lambda_{[\mu\lambda]} = \frac{1}{2}(W^\lambda_{\mu\lambda} - W^\lambda_{\lambda\mu}). \quad (2.10) \]

(2.9) leads to the result:

\[ \Gamma_\mu = \Gamma^\lambda_{[\mu\lambda]} = 0. \quad (2.11) \]
The NGT contracted curvature tensor can be written as

$$R_{\mu\nu}(W) = R_{\mu\nu}(\Gamma) + \frac{2}{3} W_{[\mu,\nu]},$$  \hspace{1cm} (2.12)

where \( R_{\mu\nu}(\Gamma) \) is defined by

$$R_{\mu\nu}(\Gamma) = \Gamma^\beta_{\mu\nu,\beta} - \frac{1}{2} \left( \Gamma^\beta_{(\mu,\beta),\nu} + \Gamma^\beta_{(\nu,\beta),\mu} \right) - \Gamma^\beta_{\alpha\nu} \Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{(\alpha\beta)} \Gamma^\alpha_{\mu\nu}. $$  \hspace{1cm} (2.13)

The field equations in the presence of matter sources are given by:

$$G_{\mu\nu}(W) + \Lambda g_{\mu\nu} + \frac{1}{4} \mu^2 C_{\mu\nu} - \frac{1}{6} \left( P_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P \right) = 8\pi T_{\mu\nu}, \hspace{1cm} (2.14)$$

$$g^{\mu\nu} + g^{\nu\sigma} W_{\rho\sigma} + g^{\mu\rho} W_{\sigma\rho} - g^{\mu\rho} W_{\sigma\rho} + \frac{2}{3} \delta^\rho_{\sigma} g^{\mu\rho} W_{[\rho\beta]} + \frac{1}{6} (g^{(\mu\beta)} W_{\delta\sigma} - g^{(\nu\beta)} W_{\delta\sigma}) = 0.$$ \hspace{1cm} (2.16)

Here, we have

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \hspace{1cm} (2.17)$$

$$C_{\mu\nu} = g_{[\mu\nu]} + \frac{1}{2} g_{\mu\nu} g^{[\sigma\rho]} g_{[\rho\sigma]} + g^{[\sigma\rho]} g_{\mu\sigma} g_{\rho\nu}, \hspace{1cm} (2.18)$$

$$P_{\mu\nu} = W_{\mu} W_{\nu}, \hspace{1cm} (2.19)$$

and \( P = g^{\mu\nu} P_{\mu\nu} = g^{(\mu\nu)} W_{\mu} W_{\nu} \).

With the help of (2.9) and the relation obtained from (2.15):

$$W_{\mu} = -\frac{2}{\sqrt{-g}} s_{\mu\rho} g^{[\rho\sigma]}_{\sigma}, \hspace{1cm} (2.20)$$

where \( s_{\mu\rho} \) is defined by

$$s_{\mu\rho} g^{(\rho\nu)} = s_{\mu\rho} s^{\rho\nu} = \delta^\nu_{\mu}, \hspace{1cm} (2.21)$$

we can recast (2.16) in the form:

$$g_{\mu\nu,\sigma} - g_{\rho\nu} \Gamma^\rho_{\mu\sigma} - g_{\rho\mu} \Gamma^\rho_{\sigma\nu} = Y^\rho \Lambda_{\mu\nu\rho}, \hspace{1cm} (2.22)$$

where

$$Y^\rho = \frac{1}{3} \sqrt{-g} g^{[\rho\sigma]}_{\sigma}, \hspace{1cm} (2.23)$$
and

\[ \Lambda_{\mu\nu\rho} = g_{\mu\rho} g_{\sigma\nu} - g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{[\sigma\rho]} . \]  

(2.24)

The field equations (2.14) can be written:

\[
S_{\mu\nu} \equiv R_{\mu\nu}(\Gamma) - \frac{4}{3\sqrt{-g}} (s_{\mu\rho} g^{[\rho\sigma],\sigma},\nu) + \Lambda g_{\mu\nu} \\
+ \frac{1}{2} \mu^2 (C_{\mu\nu} - \frac{1}{2} g_{\mu\nu} C) - \frac{1}{6} P_{\mu\nu} = 8\pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T),
\]

where \( C = g^{\mu\nu} C_{\mu\nu} \) and we now have

\[ P_{\mu\nu} = -\frac{4}{9} s_{\mu\rho} s_{\nu\sigma} g^{[\rho\sigma],\sigma} g^{[\tau\alpha],\alpha}. \]

(2.25)

The generalized Bianchi identities:

\[
[g_{\alpha\nu} G_{\rho\nu}(\Gamma) + g_{\nu\alpha} G_{\nu\rho}(\Gamma)]_{,\alpha} + g^{\mu\rho} G_{\mu\nu} = 0,
\]

(2.27)

give rise to the matter response equations:

\[
g_{\mu\rho} T_{\mu\nu,\nu} + g_{\rho\mu} T_{\nu\mu,\nu} + (g_{\mu\rho,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}) T^{\mu\nu} = 0.
\]

(2.28)

The Christoffel symbols, in NGT, are defined by

\[
\left\{ \lambda_{\mu\nu} \right\} = \frac{1}{2} s^{\lambda\rho} (s_{\mu\rho,\nu} + s_{\nu\rho,\mu} - s_{\mu\nu,\rho}).
\]

(2.29)

**III. AFFINE CONNECTIONS AND CURVATURE TENSORS**

We shall adopt the covariant derivative notation:

\[
D_{\mu} v^\lambda = v^\lambda_{,\mu} + v^\sigma \Gamma^\lambda_{\rho\mu}, \quad D_{\mu} v_\lambda = v_{\lambda,\mu} - v_\rho \Gamma^\rho_{\lambda\mu},
\]

and

\[
\nabla_{\mu} v^\lambda = v^\lambda_{,\mu} + v^\rho \left\{ \lambda \atop \rho \mu \right\}, \quad \nabla_{\mu} v_\lambda = v_{\lambda,\mu} - v_\rho \left\{ \rho \atop \lambda \mu \right\},
\]

(3.1)

(3.2)

where \( v^\mu \) is an arbitrary real vector.

(2.22) is equivalent to the set of equations:

\[
D_\sigma g_{\mu\nu} = 2\Gamma^\rho_{[\sigma\mu]} g_{\rho\nu} + L_{\mu\nu},
\]

(3.3)

where

\[
L_{\mu\nu} = Y^\rho \Lambda_{\mu\nu\rho}.
\]

(3.4)

The system of equations (3.3) admits a solution of the form [22].
\[ \Gamma^\lambda_{\mu\nu} = \left\{ \frac{\lambda}{\mu\nu} \right\} + K_{(\mu\nu)}^\lambda + \Gamma^\lambda_{[\mu\nu]}, \] (3.5)

where

\[ K_{(\mu\nu)}^\lambda = s^\lambda\gamma \Gamma^\beta_{[\alpha(\nu]^a\beta\mu]} + \Omega^\lambda_{(\mu\nu)}, \] (3.6)

\[ \Omega^\lambda_{(\mu\nu)} = \frac{1}{2} s^\lambda\alpha (L_{(\alpha\nu)\mu} + L_{(\alpha\mu)\nu} - L_{(\mu\nu)\alpha}), \] (3.7)

and we have used the notation \( a_{\mu\nu} = g_{[\mu\nu]} \).

The following identities contain the curvature tensor:

\[ 2D[\nu D \rho]v^\lambda = -R^\lambda_{\mu\nu\rho}(\Gamma) v^\mu + 2\Gamma^\alpha_{[\nu\rho]} D_{\alpha} v^\lambda, \] (3.8)

\[ 2D[\nu D \rho]v_{\mu} = R^\lambda_{\mu\nu\rho}(\Gamma) v_{\lambda} + 2\Gamma^\alpha_{[\nu\rho]} D_{\alpha} v_{\mu}, \] (3.9)

where

\[ R^\lambda_{\mu\nu\rho}(\Gamma) = \Gamma^\lambda_{\mu\nu,\rho} - \Gamma^\lambda_{\mu\rho,\nu} - \Gamma^\lambda_{\alpha\nu} \Gamma^\alpha_{\mu\rho} + \Gamma^\lambda_{\alpha\rho} \Gamma^\alpha_{\mu\nu}. \] (3.10)

The curvature tensor satisfies the identities:

\[ R^\lambda_{\rho\mu\nu}(\Gamma) = \Gamma^\lambda_{\rho|\mu\nu}(\Gamma), \] (3.11)

\[ R^\lambda_{[\mu\nu\rho]}(\Gamma) = 4 \Gamma^\alpha_{[[\nu\rho]} \Gamma^\lambda_{\rho|\mu]\alpha] + 2D[\nu \Gamma^\lambda_{\rho|\mu}], \] (3.12)

\[ D[\sigma R^\lambda_{\rho|\mu\nu}(\Gamma)] = -2\Gamma^\alpha_{[[\sigma\nu]} R^\lambda_{\rho|\mu]\alpha(\Gamma). \] (3.13)

Let us define the two contracted curvature tensors:

\[ W_{\mu\nu} = R^\beta_{\mu\nu\beta}(\Gamma), \quad V_{\nu\rho} = R^\beta_{\beta\nu\rho}(\Gamma), \] (3.14)

where

\[ W_{\mu\nu} = \Gamma^\beta_{\mu\nu,\beta} - \Gamma^\beta_{\mu\beta,\nu} - \Gamma^\beta_{\alpha\nu} \Gamma^\alpha_{\mu\beta} + \Gamma^\beta_{\alpha\beta} \Gamma^\alpha_{\mu\nu}, \] (3.15)

and

\[ V_{\nu\rho} = \Gamma^\beta_{\beta\nu,\rho} - \Gamma^\beta_{\beta\rho,\nu}. \] (3.16)

Then, from (2.11) and (2.13), we have

\[ R_{\mu\nu}(\Gamma) = W_{\mu\nu} + \frac{1}{2} (\Gamma^\beta_{(\mu\beta),\nu} - \Gamma^\beta_{(\nu\beta),\mu}), \] (3.17)

and
\[ V_{\nu \rho} = \Gamma^\beta_{(\beta \nu),\rho} - \Gamma^\beta_{(\beta \rho),\nu}. \]  
(3.18)

The Riemann-Christoffel curvature tensor \( B^\lambda_{\mu \nu \rho} \) is defined by
\[
B^\lambda_{\mu \nu \rho} = \left\{ \frac{\lambda}{\mu \nu} \right\}_{,\rho} - \left\{ \frac{\lambda}{\mu \rho} \right\}_{,\nu} - \left\{ \frac{\lambda}{\alpha \nu} \right\}_{,\mu \rho} + \left\{ \frac{\lambda}{\alpha \rho} \right\}_{,\mu \nu} - \left\{ \frac{\beta}{\alpha \nu} \right\}_{,\mu \rho} + \left\{ \frac{\beta}{\alpha \rho} \right\}_{,\mu \nu} + \left\{ \frac{\alpha}{\mu \nu} \right\}_{,\lambda}. \]  
(3.19)

By performing the contraction, \( B^\beta_{\nu \rho \beta} = B_{\nu \rho} \), we get
\[
B_{\mu \nu} = \left\{ \frac{\alpha}{\mu \nu} \right\}_{,\alpha} - \left\{ \frac{\alpha}{\mu \alpha} \right\}_{,\nu} - \left\{ \frac{\beta}{\alpha \nu} \right\}_{,\mu \beta} + \left\{ \frac{\beta}{\alpha \beta} \right\}_{,\mu \nu} + \left\{ \frac{\alpha}{\mu \nu} \right\}_{,\lambda}. \]  
(3.20)

The identities (3.11, 3.12, 3.13) reduce for the tensor \( B^\lambda_{\mu \nu \rho} \) to:
\[
B^\lambda_{\rho \mu \nu} = B^\lambda_{\rho \nu \mu}, \]  
(3.21)
\[
B^\lambda_{[\mu \nu]} = 0, \]  
(3.22)
\[
\nabla_{[\sigma} B^\lambda_{\rho \mu \nu]} = 0. \]  
(3.23)

The latter is the well-known Bianchi identity of Riemannian geometry.

Let us put
\[
X^\lambda_{\mu \nu} = K^\lambda_{(\mu \nu)} + \Gamma^\lambda_{[\mu \nu]}, \]  
(3.24)

Then, we have from (3.5) and (3.10):
\[
\Gamma^\lambda_{\mu \nu} = \left\{ \frac{\lambda}{\mu \nu} \right\} + X^\lambda_{\mu \nu}, \]  
(3.25)

and
\[
R^\lambda_{\mu \nu \rho}(\Gamma) = B^\lambda_{\mu \nu \rho} + X^\lambda_{\mu \nu, \rho} - X^\lambda_{\mu \rho, \nu} - X^\lambda_{\alpha \mu} X^\alpha_{\nu \rho} + X^\lambda_{\alpha \rho} X^\alpha_{\mu \nu}
- \left\{ \frac{\lambda}{\alpha \nu} \right\} X^\alpha_{\mu \rho} - \left\{ \frac{\alpha}{\mu \rho} \right\} X^\lambda_{\alpha \nu} + \left\{ \frac{\lambda}{\alpha \rho} \right\} X^\alpha_{\mu \nu} + \left\{ \frac{\alpha}{\mu \nu} \right\} X^\lambda_{\alpha \rho}. \]  
(3.26)

By contracting on \( \lambda \) and \( \rho \), we obtain from (2.13) and (3.26):
\[
R_{\mu \nu}(\Gamma) = B_{\mu \nu} + X^\alpha_{\mu \nu, \alpha} - \frac{1}{2}(X^\alpha_{\mu \alpha \nu} + X^\alpha_{\nu \alpha \mu}) - X^\beta_{\alpha \nu} X^\alpha_{\mu \beta} + X^\beta_{\alpha \beta} X^\alpha_{\mu \nu}
- \left\{ \frac{\beta}{\alpha \nu} \right\} X^\alpha_{\mu \beta} - \left\{ \frac{\alpha}{\mu \beta} \right\} X^\beta_{\alpha \nu} + \left\{ \frac{\beta}{\alpha \beta} \right\} X^\alpha_{\mu \nu} + \left\{ \frac{\alpha}{\mu \nu} \right\} X^\beta_{\alpha \beta}. \]  
(3.27)
The line element is defined by
\[ d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu. \] (4.1)

Let us adopt the normalization condition:
\[ g_{\mu\nu}u^\mu u^\nu = 1, \] (4.2)
where \( u^\mu = dx^\mu/d\tau \) and \( \tau \) is the proper time along the world line of a particle.

A path is a curve in the spacetime, \((M, g_{\mu\nu})\), whose tangent \( u^\mu \) is parallel transported along itself:
\[ u^\sigma D_\sigma u^\mu = du^\mu/d\tau + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = \phi(\tau)u^\mu. \] (4.3)

From (4.2), we have
\[ \frac{1}{2}D_\sigma(s_{\mu\nu}u^\mu u^\nu) = \frac{1}{2}D_\sigma s_{\mu\nu}u^\mu u^\nu + s_{\mu\nu}u^\nu D_\sigma u^\mu = 0. \] (4.4)

Multiplying (4.4) by \( u^\nu \) leads to the result:
\[ u^\nu \left( \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta \right) s_{\mu\nu} = u^\nu \Sigma_{\nu}, \] (4.5)
where
\[ \Sigma_{\nu} = -\frac{1}{2}u^\sigma D_\sigma s_{\mu\nu}u^\mu. \] (4.6)

From (4.3) it follows that
\[ \Sigma_{\nu} = -\left( \Gamma^\rho_{[\sigma(\nu)\mu]} + \frac{1}{2}L_{(\mu\nu)\sigma} \right) u^\rho u^\mu. \] (4.7)

When \( u^\nu \Sigma_{\nu} = 0 \), we get \( \phi(\tau) = 0 \) and the path equation of motion for test particles:
\[ \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta}u^\alpha u^\beta = 0. \] (4.8)

From the action:
\[ I = \int d\tau^2, \] (4.9)
where \( d\tau^2 \) is given by (4.1), we can derive from Fermat’s principle the geodesic equation of motion:
\[ \frac{du^\mu}{d\tau} + \left\{ \begin{array}{c} \mu \\ \alpha\beta \end{array} \right\} u^\alpha u^\beta = 0, \] (4.10)
which is equivalent to the equation:

\[ u^\sigma \nabla_\sigma u^\mu = 0. \]  \hspace{1cm} (4.11)

The equation \( u^\nu \Sigma_\nu = 0 \) will not in general be satisfied in NGT. However, we are interested in studying the path completeness of the NGT spacetime, so in the following we shall adopt (4.8) as an equation of motion of particles that probe the regularity properties of the spacetime geometry.

The geodesic equation of motion is not equivalent to the path equation, because in general:

\[ \left( \Gamma^\mu_{\alpha\beta} - \left\{ \begin{array}{c} \mu \\ \alpha \beta \end{array} \right\} \right) u^\alpha u^\beta \neq 0, \]  \hspace{1cm} (4.12)

although \( g_{\mu\nu} u^\mu u^\nu \) is a constant of the motion for both curves.

Let us define the torsion by

\[ T(A, B) = D_A B - D_B A - [A, B], \]  \hspace{1cm} (4.13)

which is equivalent to

\[ T(A, B) = 2A^\mu B^\nu \Gamma^\alpha_{[\mu\nu]} e_\alpha, \]  \hspace{1cm} (4.14)

where \( A^\mu \) and \( B^\mu \) are two arbitrary real vectors and \( e_\alpha \) is a basis vector.

Consider a congruence of curves with tangent vectors \( V^\mu \). The separation vector \( \eta^\mu \) of two nearby curves is a vector which connects them at equal values of their curve parameter \( \lambda \). The rate of change of \( \eta^\mu \) along the congruence is zero:

\[ [V, \eta] = (V^\alpha \partial_\alpha \eta^\beta - \eta^\alpha \partial_\alpha V^\beta)e_\beta = 0. \]  \hspace{1cm} (4.15)

The geodesic deviation equation is defined by

\[ \nabla_V \nabla_V \eta = B(V, \eta)V + \nabla_\eta(\nabla_V V), \]  \hspace{1cm} (4.16)

where

\[ B(V, \eta)V = \nabla_V \nabla_\eta V - \nabla_\eta \nabla_V V - \nabla_{[V, \eta]} V. \]  \hspace{1cm} (4.17)

The path deviation equation is defined by

\[ D_V D_V \eta = R(V, \eta)V + D_V [T(V, \eta)] + D_\eta (D_V V), \]  \hspace{1cm} (4.18)

where

\[ R(V, \eta)V = D_V D_\eta V - D_\eta D_V V - D_{[V, \eta]} V. \]  \hspace{1cm} (4.19)

In general the geodesic and path deviation equations are not equivalent in the spacetime \((M, g_{\mu\nu})\).
In the case of a static spherically symmetric field, the canonical form of $g_{\mu \nu}$ is given by:

$$g_{\mu \nu} = \begin{pmatrix} -\alpha & 0 & 0 & w \\ 0 & -\beta & f \sin \theta & 0 \\ 0 & -f \sin \theta & -\beta \sin^2 \theta & 0 \\ -w & 0 & 0 & \gamma \end{pmatrix}, \quad (5.1)$$

where $\alpha, \beta, \gamma$ and $w$ are functions of $r$. The tensor $g_{\mu \nu}$ has the components:

$$g^{\mu \nu} = \begin{pmatrix} \frac{\gamma}{w^2 - \alpha \gamma} & 0 & 0 & \frac{w}{w^2 - \alpha \gamma} \\ 0 & -\frac{\beta}{\beta^2 + f^2} & \frac{f \csc \theta}{\beta^2 + f^2} & 0 \\ 0 & -\frac{\beta \csc^2 \theta}{\beta^2 + f^2} & 0 & 0 \\ -\frac{w}{w^2 - \alpha \gamma} & 0 & 0 & \frac{-\alpha}{w^2 - \alpha \gamma} \end{pmatrix}. \quad (5.2)$$

For the theory in which there is no NGT magnetic monopole charge, we have $w = 0$ and only the $g^{[23]}$ component of $g_{\mu \nu}$ survives. The line element for a spherically symmetric body is given by

$$ds^2 = \gamma(r)dt^2 - \alpha(r)dr^2 - \beta(r)(d\theta^2 + \sin^2 \theta d\phi^2). \quad (5.3)$$

We have

$$\sqrt{-g} = \sin \theta [\alpha \gamma (\beta^2 + f^2)]^{1/2}. \quad (5.4)$$

For the static spherically symmetric field with $w = 0$, it follows from (2.20) that $W_\mu = 0$.

Let us assume the long-range approximation for which the $\mu^2$ contributions in the vacuum field equations can be neglected and we assume that $\Lambda = 0$. We then obtain the static, spherically symmetric Wyman solution [11,12]:

$$\gamma = \exp(\nu), \quad (5.5)$$

$$\alpha = m^2 (\nu')^2 \exp(-\nu)(1 + s^2)(\cosh(a \nu) - \cos(b \nu))^{-2}, \quad (5.6)$$

$$f = [2m^2 \exp(-\nu)(\sinh(a \nu) \sin(b \nu) + s(1 - \cosh(a \nu) \cos(b \nu))](\cosh(a \nu) - \cos(b \nu))^{-2}, \quad (5.7)$$

where

$$a = \left(\frac{\sqrt{1 + s^2} + 1}{2}\right)^{1/2}, \quad b = \left(\frac{\sqrt{1 + s^2} - 1}{2}\right)^{1/2}, \quad (5.8)$$

prime denotes differentiation with respect to $r$, and $\nu$ is implicitly determined by the equation:

$$\exp(\nu)(\cosh(a \nu) - \cos(b \nu))^2 \frac{r^2}{2m^2} = \cosh(a \nu) \cos(b \nu) - 1 + s \sinh(a \nu) \sin(b \nu). \quad (5.9)$$
Moreover, $s$ is a dimensionless constant of integration.

We find for $2m/r \ll 1$ and $0 < sm^2/r^2 < 1$ that the metric takes the Schwarzschild form:

$$
\gamma \sim \alpha^{-1} \sim 1 - \frac{2m}{r},
$$

(5.10)

and

$$
f \sim \frac{sm^2}{3}.
$$

(5.11)

Near $r = 0$ we can develop expansions where $r/m < 1$ and $0 < |s| < 1$. The leading terms are [11,12]:

$$
\gamma = \gamma_0 + \frac{\gamma_0(1 + \mathcal{O}(s^2))}{2|s|} \left( \frac{r}{m} \right)^2 + \mathcal{O}\left( \left( \frac{r}{m} \right)^4 \right),
$$

(5.12)

$$
\alpha = \frac{4\gamma_0(1 + \mathcal{O}(s^2))}{s^2} \left( \frac{r}{m} \right)^2 + \mathcal{O}\left( \left( \frac{r}{m} \right)^4 \right),
$$

(5.13)

$$
f = m^2 \left( 4 - \frac{|s|\pi}{2} + s|s| + \mathcal{O}(s^3) \right) + \frac{|s| + s^2\pi/8 + \mathcal{O}(s^3)}{4} r^2 + \mathcal{O}(r^4),
$$

(5.14)

$$
\gamma_0 = \exp \left( - \frac{\pi + 2s}{|s|} + \mathcal{O}(s) \right) \ldots.
$$

(5.15)

These solutions clearly illustrate the non-analytic nature of the limit $s \to 0$ in the strong gravitational field regime for $r < 2m$.

The singularity caused by the vanishing of $\alpha(r)$ at $r = 0$ is a coordinate singularity, which can be removed by transforming to another coordinate frame of reference. The curvature invariants do not, of course, contain any coordinate singularities. We can transform to a coordinate frame in which the spacetime $(M, g_{\mu\nu})$ is completely free of singularities [11,12].

The NGT curvature invariants such as the generalized Kretschmann scalar:

$$
K = R^{\lambda\mu\nu\rho} R_{\lambda\mu\nu\rho}
$$

(5.16)

are finite. However, the curvature invariants formed from the Riemann-Christoffel tensor are singular at $r = 0$. For example, the Kretschmann scalar:

$$
S = B^{\lambda\mu\nu\rho} B_{\lambda\mu\nu\rho},
$$

(5.17)

is singular like $S \sim m^4/r^8$ near $r = 0$ [13]. The Christoffel symbol defined by (2.23) is singular at $r = 0$, and shares the same analytic spacetime properties as the Christoffel symbol in EGT. Thus, only the fully non-Riemannian geometry of NGT describes a nonsingular spacetime.
The static spherically symmetric solution is everywhere non-singular and there is no event horizon at $r = 2m$. A black hole is replaced in the theory by a FCO which can be stable for an arbitrarily large mass $[11–13]$. It has been shown by Cornish [23], that the static spherically symmetric solution of the field equations for $\mu \neq 0$ only satisfies the flat space asymptotic conditions at $r \to \infty$ if $w = 0$. This demonstrates that the Wyman solution is the unique static spherically symmetric solution of the field equations and that only one degree of freedom is permitted globally for this solution.

Because the Christoffel symbol $\{^{\mu}_{\alpha\beta}\}$ is singular at $r = 0$ in the spacetime, $(M, g_{\mu\nu})$, it follows that the Wyman solution is not geodesically complete. On the other hand, the connection $\Gamma^\lambda_{\mu\nu}$ is non-singular everywhere in the spacetime, $(M, g_{\mu\nu})$, including at $r = 0$, so the path equation (4.8) can be complete for the Wyman solution. Because $Y^\mu = 0$ and therefore $u^\nu \Sigma_\nu = 0$ for the Wyman solution, it follows that the path equation (4.8) holds for this solution.

VI. PATH CONGRUENCES

Let a smooth congruence of timelike paths be parameterized by proper time $\tau$, so that the vector field, $V^\mu$, of tangents is normalized to unit length:

$$V^\mu V_\mu = s_{\lambda\mu} V^\lambda V^\mu = 1.$$  \hspace{1cm} (6.1)

We define a tensor field $Q_{\mu\nu}$ by

$$Q_{\mu\nu} = D_\nu V_\mu.$$  \hspace{1cm} (6.2)

It is purely spatial in character:

$$Q_{\mu\nu} V^\mu = Q_{\mu\nu} V^\nu = 0.$$  \hspace{1cm} (6.3)

Let $\gamma$ denote a smooth one-parameter subfamily of paths in the congruence, and let $\eta^\mu$ describe an infinitesimal displacement from one such path $\gamma_0$ to a nearby path in the subfamily. Then, we have

$$V^\nu D_\nu \eta^\mu = \eta^\nu D_\nu V^\mu = Q^\mu_{\nu\eta} \eta^\nu.$$  \hspace{1cm} (6.4)

Let us define a “spatial” metric tensor $h_{\mu\nu}$:

$$h_{\mu\nu} = s_{\mu\nu} - V_\mu V_\nu,$$  \hspace{1cm} (6.5)

which satisfies $h_{\mu\nu} V^\mu = h_{\mu\nu} V^\nu = 0$ and $h_{\mu\nu} = s^{\mu\sigma} h_{\mu\sigma}$. Then, the expansion $\theta$, shear $\sigma_{\mu\nu}$, and twist $\omega_{\mu\nu}$ of the congruence are given, respectively, by

$$\theta = Q^\mu_{\nu\eta} h_{\mu\nu},$$  \hspace{1cm} (6.6)

$$\sigma_{\mu\nu} = Q_{\mu\nu} - \frac{1}{3} \theta h_{\mu\nu},$$  \hspace{1cm} (6.7)
\[ \omega_{\mu\nu} = Q_{(\mu\nu)}. \]

We have
\[ Q_{\mu\nu} = \frac{1}{3} \theta \hat{h}_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}, \]
and \( \sigma_{\mu\nu} V^\mu = \omega_{\mu\nu} V^\nu = 0. \)

It can be shown from (3.8) that
\[ V^\sigma D_\sigma Q^\lambda_{\ \nu} = -Q^\lambda_{\ \nu} Q^\lambda_{\ \sigma} - R^\lambda_{\ \mu\nu\sigma}(\Gamma) V^\mu V^\sigma + 2\Gamma^\alpha_{[\sigma\nu]} Q^\lambda_{\ \alpha} V^\sigma, \]
(6.10)
Contracting on the suffixes \( \lambda \) and \( \nu \), we get the generalized Raychaudhuri equation \[24]:
\[ \frac{d\theta}{d\tau} = -\frac{1}{3} \theta^2 - \sigma^2 + \omega^2 - R_{\mu\nu}(\Gamma) V^\mu V^\nu - 2\Gamma^\alpha_{[\mu\nu]} Q^\alpha_{\ \nu}, \]
(6.11)
where we have used \( W_{\mu\nu} V^\mu V^\nu = R_{\mu\nu}(\Gamma) V^\mu V^\nu \), which follows from (3.17). Moreover,
\[ \sigma^2 = h_{\mu\alpha} h_{\nu\beta} \sigma^{\alpha\beta} \sigma_{\mu\nu}, \quad \omega^2 = h_{\mu\alpha} h_{\nu\beta} \omega^{\alpha\beta} \omega_{\mu\nu}. \]
(6.12)
Next we consider a congruence of null paths with tangent field \( Z^\mu \). We associate the metric \( \hat{h}_{\mu\nu} = \hat{h}_{\nu\mu} \) with the space of null vectors \( Z^\mu \) and we define the tensor field:
\[ \hat{Q}_{\mu\nu} = D_\nu Z_\mu, \]
(6.13)
which can be decomposed as:
\[ \hat{Q}_{\mu\nu} = \frac{1}{2} \hat{\theta} \hat{h}_{\mu\nu} + \hat{\sigma}_{\mu\nu} + \hat{\omega}_{\mu\nu}, \]
(6.14)
where, as before, \( \hat{\theta}, \hat{\sigma}_{\mu\nu} \) and \( \hat{\omega}_{\mu\nu} \) denote the expansion, shear and twist, respectively, given by
\[ \hat{\theta} = \hat{Q}^\mu_{\ \nu} \hat{h}_{\mu\nu}, \]
(6.15)
\[ \hat{\sigma}_{\mu\nu} = \hat{Q}_{(\mu\nu)} - \frac{1}{2} \hat{\theta} \hat{h}_{\mu\nu}, \]
(6.16)
\[ \hat{\omega}_{\mu\nu} = \hat{Q}_{[\mu\nu]}. \]
(6.17)
Then, for an affine parameter \( \lambda \), the generalized Raychaudhuri equation takes the form:
\[ \frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2} \hat{\theta}^2 - \hat{\sigma}^2 + \hat{\omega}^2 - R_{\mu\nu}(\Gamma) Z^\mu Z^\nu - 2\Gamma^\alpha_{[\mu\nu]} \hat{Q}^\alpha_{\ \nu} Z^\nu. \]
(6.18)
In Einstein’s theory of gravity, we have
\[ a_{\mu\nu} = 0, \quad \Gamma^\lambda_{[\mu\nu]} = 0, \]
(6.19)
and (3.3) becomes
\[ \nabla_\sigma s_{\mu\nu} = 0. \] (6.20)

Einstein’s gravitational field equations give:
\[ B_{\mu\nu} V^\mu V^\nu = 8\pi (T_{(\mu\nu)} - \frac{1}{2} s_{\mu\nu} T) V^\mu V^\nu = 8\pi (T_{(\mu\nu)} V^\mu V^\nu - \frac{1}{2} T). \] (6.21)

For physical matter we have
\[ T_{(\mu\nu)} V^\mu V^\nu \geq 0, \] (6.22)
for all timelike vectors \( V^\mu \), which is called the weak energy condition. Moreover,
\[ T_{(\mu\nu)} V^\mu V^\nu \geq \frac{1}{2} T, \] (6.23)
for timelike \( V^\mu \) is the strong energy condition.

In the case of null geodesics, we obtain from Einstein’s field equations:
\[ B_{\mu\nu} = 8\pi T_{(\mu\nu)} Z^\mu Z^\nu, \] (6.24)
and we postulate that
\[ T_{(\mu\nu)} Z^\mu Z^\nu \geq 0, \] (6.25)
which is a weaker requirement than for the timelike vectors \( V^\mu \).

For Einstein’s theory, (6.11) and (6.18) become
\[ \frac{d\theta}{d\tau} = -\frac{1}{3} \sigma^2 - \dot{\sigma}^2 + \omega^2 - B_{\mu\nu} V^\mu V^\nu, \] (6.26)
and
\[ \frac{d\hat{\theta}}{d\lambda} = -\frac{1}{2} \dot{\hat{\sigma}}^2 - \hat{\sigma}^2 + \hat{\omega}^2 - B_{\mu\nu} Z^\mu Z^\nu. \] (6.27)

From (6.22) and (6.25), it follows that the last terms of (6.26) and (6.27) are negative. The terms \( \sigma^2 \) and \( \dot{\sigma}^2 \), in (6.26) and (6.27), are nonpositive and if the congruence of geodesics is hypersurface orthogonal, we have \( \omega_{\mu\nu} = \dot{\omega}_{\mu\nu} = 0 \), so that the third terms in (6.26) and (6.27) vanish. If we set \( z = \theta^{-1} \) and \( \dot{z} = dz/d\tau \), then we get from (6.26):
\[ \dot{z} \geq \frac{1}{3}, \] (6.28)
or,
\[ z(\tau) \geq z_0^{-1} + \frac{1}{3} \tau, \] (6.29)
where \( z_0 \) is the initial value of \( z \). If \( \theta_0 \) is negative, corresponding to an initially converging geodesic congruence, then (6.29) requires that \( z \) must pass through zero, which means that
\[ \theta \to -\infty \text{ within a proper time } \tau \leq 3/|\theta_0|. \] Unboundedness of \( \theta \) implies that the volume of a tube of matter shrinks to zero and for normal matter, \( \rho \to \infty \) and \( p \to \infty \), generating a singularity.

In NGT, the symmetric part of the field equations (2.25) is given by

\[ S_{(\mu\nu)}V^\mu V^\nu = 8\pi(T_{(\mu\nu)}V^\mu V^\nu - \frac{1}{2}T). \quad (6.30) \]

As in EGT, we assume for physical matter that (6.22) and (6.23) hold.

Let us write

\[ X(\tau) = R_{\mu\nu}(\Gamma)V^\mu V^\nu + \sigma^2 + 2\Gamma^\alpha_{[\mu\nu]}Q^\mu_{\alpha} V^\nu. \quad (6.31) \]

For a hypersurface orthogonal congruence of paths, \( \omega_{\mu\nu} = 0 \), and (6.11) can be written as

\[ \dot{z} - X(\tau)z^2 - \frac{1}{3} = 0, \quad (6.32) \]

which has the form of a Riccati equation. We have

\[ X(\tau) = \dot{z} - \frac{1}{3}z^2. \quad (6.33) \]

If \( X(\tau) \) is a smooth analytic function of \( \tau \), then \( z \neq 0 \) for all \( \tau \). We exclude the exceptional case when \( \dot{z} \to \frac{1}{3} \) as \( z^2 \) for some \( \tau = \tau_0 \).

We have proven the following Lemma:

**Lemma 1** Let \( V^\mu \) be the tangent field of a hypersurface orthogonal timelike path congruence. Suppose that \( X(\tau) \) is a smooth analytic function of \( \tau \). Then, \( \theta < \infty \) along a path in that congruence for any \( \tau \) and the path can be extended indefinitely.

For a congruence of null paths with tangent field \( Z^\mu \), we write

\[ \dot{X}(\tau) = R_{\mu\nu}(\Gamma)Z^\mu Z^\nu + \dot{\sigma}^2 + 2\Gamma^\alpha_{[\mu\nu]}Q^\mu_{\alpha} Z^\nu, \quad (6.34) \]

and we obtain from (6.18) for a hypersurface orthogonal congruence of paths:

\[ \dot{X}(\tau) = \frac{dz/d\lambda - \frac{1}{2}}{z^2}. \quad (6.35) \]

We can state the following Lemma:

**Lemma 2** Let \( Z^\mu \) be the tangent field of a hypersurface orthogonal null path congruence. Let us suppose that \( \dot{X}(\tau) \) is a smooth analytic function of \( \tau \). Then, for a path in the congruence, \( \theta < \infty \) along that path for any affine length \( \lambda \), and the null path can be extended indefinitely.

A Jacobi field on a path \( \gamma \) with tangent \( V^\mu \) is a solution of the path deviation equation (4.18). Points \( a, b \in \gamma \) are conjugate if there exists a field \( \eta^\mu \) which vanishes at both \( a \) and \( b \). Conjugate points in spacetime signal the moment when a curve fails to be a local maximum of proper time between two points and a null path fails to remain on the boundary of the future of a point. In Riemannian geometry, they signal locally the moment when a curve fails to take on its minimal length. We can now state the following ([5], [7]):
Proposition 1 Let \((M, g_{\mu\nu})\) be a strongly causal spacetime, and let \(a, b \in M\) with \(b \in J^+(b)\), then the length function \(\tau\), defined on a Cauchy surface \(\Sigma(p)\), attains a finite value at \(\gamma \in \Sigma(p)\) when \(\gamma\) is a path orthogonal to \(\Sigma\) with no point conjugate to \(\sigma\) between \(\Sigma\) and \(a\).

The extrinsic curvature, \(K_{\mu\nu}\), on a spacelike hypersurface \(\Sigma\) is defined by

\[
K_{\mu\nu} = \nabla_{\mu} V_{\nu} = Q_{\mu\nu}.
\] (6.36)

\(K_{\mu\nu}\) is purely spatial i.e., \(K_{\mu\nu} V^\mu = K_{\mu\nu} V^\nu = 0\). The trace of \(K_{\mu\nu}\) is given by

\[
K = K^{\mu\nu} h_{\mu\nu} = \theta,
\] (6.37)

where \(\theta\) is the expansion of the path congruence orthogonal to \(\Sigma\).

VII. REGULARITY THEOREMS FOR COSMOLOGY

When a spacetime manifests timelike or null geodesic incompleteness, then this has the immediate physical significance that there are freely falling observers whose histories did not exist after, or before, a finite interval of proper time. Thus, timelike and null geodesic completeness are minimum conditions for a singularity-free spacetime. A spacetime which is null or timelike geodesically incomplete contains a singularity. The notion of geodesic completeness can be extended to bundle-completeness (b-completeness) i.e., that a spacetime \((M, g_{\mu\nu})\) is b-complete if there is an endpoint for every \(C^1\) curve of finite length as measured by a generalized affine parameter. Thus, a spacetime is singularity-free if it is b-complete \[25\]. For non-Riemannian gravitational theories these statements can be extended to autoparallel paths of the connection \(\Gamma^\lambda_{\mu\nu}\).

Let us consider proving two theorems which establish the necessary conditions for the existence of nonsingular solutions in NGT cosmology. We shall use the properties of the non-Riemannian geometry of NGT, whose associated timelike and null paths reveal under what conditions these paths can be complete.

We shall first assume that the universe is globally hyperbolic and that at one instant of time it is expanding everywhere at a nonzero rate. Then the universe will have begun in a nonsingular state a finite time ago if:

Theorem 1 Let \((M, g_{\mu\nu})\) be a globally hyperbolic spacetime in which the strong energy condition, \(S_{(\mu\nu)} V^\mu V^\nu \geq 0\), holds and \(X(\tau)\) is a smooth analytic function of \(\tau\) for all timelike \(V^\mu\). Assume that there exists a smooth spacelike Cauchy surface \(\Sigma\) for which the trace of the extrinsic curvature for the past directed normal path congruence satisfies \(K \leq C < 0\) everywhere (where \(C\) is a constant). Then all timelike paths from \(\Sigma\) are complete.

Proof. Let there exist a past timelike directed curve, \(\gamma\), from \(\Sigma\). Let \(a\) be a point on \(\gamma\) lying beyond length \(3/C\) from \(\Sigma\). By Lemma 1 and Proposition 1, there exists a curve \(\gamma\) from \(a\) to \(\Sigma\), which also must have length greater than \(3/C\). Since \(|\theta| < \infty\) at all points on \(\gamma\), then \(\gamma\) is a path with no conjugate point between \(\Sigma\) and \(a\) and therefore the path is complete. ■

We can eliminate the assumption of global hyperbolicity in the same way that is done in the proof of the Hawking-Penrose theorem \[4\], which establishes the initial singularity in EGT cosmology, by assuming that \(\Sigma\) is compact (the universe is closed).
Theorem 2 Suppose that $S_{(\mu\nu)}N^\mu N^\nu \geq 0$, which follows from the strong energy condition for timelike and null vectors $N^\mu$. Let $(M, g_{\mu\nu})$ be a strongly causal spacetime such that $X(\tau)$ and $\tilde{X}(\tau)$ are smooth analytic functions of $\tau$. Let there exist a compact, edgeless, achronal, smooth spacelike hypersurface $\Gamma$ such that the past directed normal path congruence from $\Gamma$ has $K < 0$ everywhere on $\Gamma$. Then all the past directed timelike and null paths can be extended beyond the value $3/|C|$ (where $C$ denotes the maximum value of $K$, so that $K \leq C < 0$) and therefore the paths are complete.

Proof. According to Lemmas 1 and 2 all timelike and null paths past directed from $\Gamma$ will have $|\theta| < \infty$ at all points on the paths, which means that there will be no conjugate points on the paths beyond the value $3/|C|$. Then these paths can be extended indefinitely. ☐

Let us define

$$Y(\tau) = B_{\mu\nu}V^\mu V^\nu + \sigma^2,$$ (7.1)

and

$$\dot{Y}(\tau) = B_{\mu\nu}Z^\mu Z^\nu + \dot{\sigma}^2,$$ (7.2)

where $B_{\mu\nu}$ is given by (3.21).

In Einstein’s gravitational theory, the Hawking-Penrose theorems establish that if the positive energy condition, $B_{\mu\nu}N^\mu N^\nu \geq 0$, holds for timelike and null vectors $N^\mu$, then at least one past directed timelike geodesic has no length greater than $3/|C|$, and is therefore inextendible. This means that $z$ must vanish at some point on the geodesic and, therefore, $Y(\tau)$ and $\dot{Y}(\tau)$ cannot be smooth analytic functions for all values of the proper time or affine parameter. The physical interpretation of the positive energy condition is that gravity is always attractive i.e., neighboring geodesics near any one point accelerate, on the average, toward each other.

In NGT, we can prove that all paths are complete provided that for hypersurface normal congruences, $X(\tau)$ and $\tilde{X}(\tau)$ are smooth analytic functions of $\tau$. This is now possible because the positive energy condition, $S_{(\mu\nu)}N^\mu N^\nu \geq 0$, does not conflict with the analytic property of $X(\tau)$ or $\tilde{X}(\tau)$. The functions $X(\tau)$ and $\tilde{X}(\tau)$ can be positive or negative in the spacetime manifold without violating the positive energy condition. Thus, NGT belongs to a class of non-Riemannian theories in which gravity is not always attractive, although physical matter satisfies the required positive energy conditions. This is why singular curvature can be avoided in gravitationally unstable situations. It is clear that any gravity theory based on a purely Riemannian geometry (excluding $R^2$ curvature terms in the Lagrangian density) must have incomplete geodesics and that all cosmological solutions begin their expansion in a singular state. Any external fields in such a theory of gravity will be incorporated in the positive energy conditions and inevitably lead to the Hawking-Penrose theorems. An example of this situation is provided by the de Sitter solution in EGT cosmology for the equation of state: $\rho = -p$. The solution is nonsingular, but violates the positive energy condition. On the other hand, NGT is based on a non-Riemannian geometry which allows nonsingular solutions in cosmology to exist.
VIII. REGULARITY THEOREM FOR STELLAR COLLAPSE

The solution of the NGT field equations outside a star is necessarily that part of the asymptotically flat region of the Wyman solution for which $r$ is greater than some value $R$ corresponding to the radius of the star. For a given pressure and density in the star a solution will exist, which is joined, for $r > R$, onto the exterior solution. Since time-like Killing vectors remain time-like for all $r$ between $r = 0$ and $r = \infty$, the star can have $r < 2m$. This is in contrast to the Schwarzschild solution in EGT, for which the star must have $r > 2m$, because only in this case is there a time-like Killing vector. Let us now suppose that the nuclear fuel of the star is exhausted, causing the star to contract under the influence of its gravity. If the contraction cannot be halted by the pressure before $r \leq 2m$, due to the star having a mass greater than some critical mass $m_c$, and if the collapse ends in a final static spherically symmetric configuration, then the solution outside the star is the nonsingular Wyman solution, and we can anticipate that the final collapsed star has no event horizon and no singularity at $r = 0$.

In the non-singular spacetime, $(M, g_{\mu \nu})$, the entire causal past of future null infinity, $J^-(\mathcal{J}^+)$, is nonsingular, and includes the entire physical spacetime. Thus, there are no black hole regions, $\mathcal{R}_B$, defined by $\mathcal{R}_B = [M - J^-(\mathcal{J}^+)]$ with an enclosing event horizon boundary: $H = J^-(\mathcal{J}^+) \cap M$. Let there be some past-directed unit timelike vector $V^\mu$ at a point $p$. A compact, two-dimensional, smooth spacelike submanifold, $\mathcal{T}$, for which the expansion, $\theta$, of both sets of ingoing and outgoing future directed null geodesics or paths orthogonal to $\mathcal{T}$ is everywhere negative is called a trapped surface. In the Schwarzschild solution, all spheres inside the black hole ($r < 2m$) are trapped surfaces. In the nonsingular Wyman solution, there will not be any trapped surfaces $\mathcal{T}$ below $r = 2m$, since the $\theta$ associated with the paths is everywhere non-negative. We can now state the following regularity theorem:

**Theorem 3** The spacetime $(M, g_{\mu \nu})$ can be path complete if:

1. Let $(M, g_{\mu \nu})$ be a spacetime which satisfies the strong energy condition for matter: $S_{(\mu\nu)}N^\mu N^\nu \geq 0$ for all timelike and null vectors $N^\mu$.

2. $X(\tau)$ and $\dot{X}(\tau)$ are smooth analytic functions of $\tau$ for every non-spacelike vector $N^\mu$.

3. There are no closed time-like curves.

4. There are no closed trapped surfaces $\mathcal{T}$.

**Proof.** Let $\theta_0 < 0$ denote the maximum value of $\theta$ for a set of path congruences in $(M, g_{\mu \nu})$. Because there are no trapped surfaces in the spacetime all paths have affine length $\geq 3/|\theta_0|$ ($\geq 2/|\theta_0|$ for null paths). The smooth analytic behavior of $X(\tau)$ and $\dot{X}(\tau)$ guarantees by Lemmas \[ and \[ that $z \neq 0$ along the paths for $\tau \geq 3/|\theta_0|$ (or $2/|\theta_0|$) and by Proposition \[, the paths will not have any conjugate points at which the paths terminate. Therefore, the paths are null and timelike complete.

In the nonsingular Wyman solution, $X(\tau)$ and $\dot{X}(\tau)$ are smooth analytic functions of $\tau$ for all timelike and null vectors $N^\mu$. It follows that this solution satisfies the conditions of Theorem \[. Gravity is not always attractive, for $R_{\mu \nu}(\Gamma) N^\mu N^\nu$ can change sign and can be negative as the gravitational field increases its strength, so that neighboring paths
will accelerate, on the average, away from each other. Even if the star is not spherically symmetric, there will still not be any trapped surface provided the departures from spherical symmetry are not too large. This will follow from the stability of the Cauchy development, since trapped surfaces cannot form in any gravitational collapse whose initial conditions are sufficiently close to initial conditions for spherical collapse. These results can be generalized to a whole manifold, and the existence and uniqueness of developments for an initial set of data can be derived. As in EGT, there exist constraint equations whose development and uniqueness for an initial set of data can be proved rigorously. The analog of the stability proof for small fluctuations of the spherically symmetric Schwarzschild solution [26] has not yet been established for the nonsingular Wyman solution. However, since the theory has a stable vacuum solution that follows from a consistent perturbative expansion scheme, we expect that the NGT static spherically symmetric solution is stable against small fluctuations.

From the static Wyman solution, we find that \( Y(\tau) \) is not an analytic smooth function everywhere in the spacetime \((M, g_{\mu \nu})\), so the Riemannian geometry associated with the NGT spacetime does not satisfy the regularity theorems established in the foregoing, and the geodesics in \((M, g_{\mu \nu})\) are not complete.

**IX. CONCLUSIONS**

When the physical matter satisfies positive and non-negative pressure conditions and \( g_{\mu \nu} = s_{\mu \nu} \), the spacetime possesses a Riemannian geometry and if the theory is of Einstein form i.e., the Lagrangian density contains no derivatives higher than the second order, then the Hawking-Penrose theorems are valid and the spacetime must contain non-coordinate singularities. Classical alternatives to this singular spacetime scenario are a gravity theory based on non-Riemannian geometry and a theory based on Riemannian geometry and a higher-derivative Lagrangian density. In particular, we have studied NGT and found that it can satisfy the regularity theorems for cosmology and gravitational collapse proved above.

We have argued that any gravity theory with a set of initial data on a Cauchy surface that has a global non-perturbative solution to the field equations, which can be mapped into solutions near the Schwarzschild solution of EGT, shares the singularity at \( r = 0 \) and the black hole event horizon of the Schwarzschild solution. For this reason, we must expect that any modified gravity theory that removes all singularities and black hole event horizons is non-perturbative and non-analytic in the new coupling constants for \( r < 2m \).

The static spherically symmetric Wyman solution of the vacuum field equations is non-singular in the long-range approximation and satisfies the conditions of the regularity theorems, namely, the spacetime is path complete and the curvature invariants are finite. The solution is not analytic in the parameter \( s \) for \( r \leq 2m \), for in the limit \( s \to 0 \) the solution does not become the Schwarzschild solution with its event horizon and singularity at \( r = 0 \). Since the solution does not have any event horizons there are no black holes. The black hole is replaced by a final collapsed object that can be made stable by the repulsive NGT forces for an arbitrarily large mass.

An observer, in EGT, falling through an event horizon of a large black hole does not experience anything unusual about the event, whereas in NGT the observer would in general detect a skew symmetric force at the event horizon, enabling him to determine the existence
of such a horizon.

Because the FCO does not have an event horizon the information loss problem is resolved at the classical level. The elimination of black holes is the simplest and possibly the least radical solution of the information loss puzzle.

It has been argued that the black hole information loss problem could be resolved by an, as yet, unknown quantum gravity theory \[27\]. However, if it is argued that quantum gravity effects could remove the black hole event horizon, then we are forced into a possibly unphysical paradox, namely, an observer falling through a large black hole event horizon with weak curvature would see entirely different physics associated with the event horizon than an observer at a large distance from the black hole. In order to overcome this paradox, long-range non-local physics has to be postulated to exist, whereby an observer deep inside a black hole can communicate to an outside observer the discrepancy detected between their observations. As yet unproved claims have to be made that what otherwise appears to be a weak curvature event horizon for a large black hole, which can be treated as a classical domain of gravity, is a region of spacetime where quantum gravity effects associated with the Planck mass describe the true physical situation. Moreover, all the information inside the black hole must be encoded in the Hawking radiation by an unknown dynamical mechanism.

The NGT that we have studied has been proved to be perturbatively consistent for weak gravitational fields \[8,9\]. It is asymptotically stable at future null infinity where the linear approximation is a stable solution of the field equations without ghost poles or tachyons. Higher-derivative theories can satisfy the regularity theorems as applied to geodesics of the Riemannian geometry. This class of theories can have repulsive gravity that retains the positive energy conditions for physical matter \[10,15\]. There are no exact solutions of the field equations in four dimensions analogous to the Wyman solution of NGT, so it is not clear whether such theories can remove event horizons and black holes. However, higher-derivative theories usually possess ghost poles, tachyons and higher-order poles of one kind or another and this feature renders the theories perturbatively unphysical. Theories with $R^2$ curvature terms in the Lagrangian exist, in which the metric tensor and the connection are treated as independent variables \[17,28\]. Ghost poles and tachyons can be absent in these theories and they could be free of singularities at $r = 0$, but it is unlikely that they are free of black hole event horizons.

The main result of this study is that classical theories of gravity can be formulated that do not have singularities and black holes. They satisfy the standard gravitational experimental tests. Whereas it is feasible that a quantum gravity program can be carried through successfully in the future, we have demonstrated that a classical modification of EGT, of the kind proposed here, can solve the singularity problem and the paradoxes associated with black holes. We then avoid having to seriously modify quantum mechanics or having to solve the difficult task of quantizing gravity to resolve the black hole information loss problem.

The question remains to be answered: Does the non-singular, NGT non-Riemannian geometry describe the true structure of spacetime, or is the geometry of spacetime described by the Riemannian geometry associated with NGT, with incomplete geodesics and black holes? As we have seen, either description exists as a possibility in NGT, and only future experiments can decide the outcome of this fundamental question.

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