COHERENT STATES, PHASES AND SYMPLECTIC AREA OF GEODESIC TRIANGLES

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Abstract

On certain manifolds, the phase which appears in the scalar product of two coherent state vectors is twice the symplectic area of the geodesic triangle determined by the corresponding points on the manifold and the origin of the system of coordinates. This result is proved for compact Hermitian symmetric spaces using the generalization via coherent states of the shape invariant for geodesic triangles and re-obtained on the complex Grassmannian by brute-force calculation.

1. INTRODUCTION

The aim of this paper is to find a geometric interpretation of the phase which appears in the scalar product of two coherent vectors. In 1994, in Białowieża, Askold Perelomov added this question on my list [1] of other 6 questions referring to coherent states and geometry. An explicit answer to this question for the Riemann sphere is given by Perelomov himself (cf. Ref. [2], p. 63). Earlier, S. Pancharatnam [3, 4] showed that the phase difference between the initial and final state is $\langle A|A' \rangle = \exp(-i\Omega_{ABC}/2)$, where $\Omega_{ABC}$ is the solid angle subtended by the geodesic triangle $ABC$ on the Poincaré sphere. The holonomy of a loop in the projective Hilbert space is twice the symplectic area of any two-dimensional submanifold whose boundary is the given loop (see Prop. 5.1 in [5], where this result is attributed to Aharonov and Anandan).

The main result communicated at this conference is the following: on certain manifolds, the phase $\Phi$ which appears in the scalar product of two coherent state vectors is twice the symplectic area of the geodesic triangle determined by the corresponding points on the manifold and the origin of the system of coordinates. This result was proved on a restricted class of manifolds: the compact, homogeneous, simply connected Hodge manifolds, which are in the same time naturally reductive. I mention also that
During the Workshop Martin Bordemann pointed out that the class of manifolds considered by me consists in fact only of the Hermitian symmetric spaces [1]. Indeed, any naturally reductive space with an invariant Kähler structure is locally Hermitian symmetric [2] and simply connectedness implies Hermitian symmetry. On the other side, the results of the present paper are still true for other manifolds than those considered here. For example, the results are true for the Heisenberg-Weyl group [2] as well as for the noncompact dual of the complex Grassmann manifold [3].

One idea of the present paper is to use in connection with geodesic triangles the generalization via coherent states of the shape invariant [9]. A more precise formulation of this generalization via coherent states of the shape invariant [9]. A more precise formulation of this generalization is fixed.

§ 4 presents briefly the calculation of the symplectic area of geodesic triangles on \( G_n(C^{m+n}) \) using the technique and notation from Ref. [12].

The paper is laid out as follows. In § 2 the notation on holomorphic line bundles and coherent states is fixed. § 3 deals with the shape invariant of Blaschke and Terheggen and its generalization to coherent states. The results on phases of coherent states are proved in § 4. § 5 presents briefly the calculation of the symplectic area of geodesic triangles on \( G_n(C^{m+n}) \) using the technique and notation from Ref. [12].

2. HOLOMORPHIC LINE BUNDLES AND COHERENT STATES

Let \( \tau : L \rightarrow M \) be a holomorphic line bundle over the Kähler manifold \((M, \omega)\), with the connection \( \nabla_L \) compatible with the hermitian metric \( h_L \). With respect to a holomorphic frame and a holomorphic coordinate system, \( \nabla_L = \partial + \partial_L + \overline{\partial_L} \). \( \tau_L = \partial \log h_L \), and \( \Theta_L = \overline{\theta}_L \), where \( \theta_L (\Theta_L) \) is the connection (respectively, curvature) matrix.

Assume that \( \tau \) is a prequantum line bundle, i.e. \( \omega = \frac{i}{2} \omega_{FS} \). Above \( N = \dim \mathcal{H} - 1 \), where \( \mathcal{H} = H^0(M,L) = \Gamma_{hol}(M,L) \). If \( S_i \) is a basis of global sections, orthonormal with respect to the scalar product on \( \mathcal{H} \), then the embedding \( \iota \) is given by

\[
\iota(z) = (s_1(z) : s_2(z) : \ldots : s_{N+1}(z)).
\]  

Rawnsley's [13] coherent states are defined as usual: if \( q \in L \setminus \{0\} = L_0 \), is fixed, then the evaluation of the section \( s \in \mathcal{H} \) determines uniquely the coherent vector \( e_q \in \mathcal{H} \), \( s(\tau(q)) = (e_q, s)q \).

Perelomov’s [2] coherent states are defined by the triplet \((G, \pi, \mathcal{H})\), where \( G \) is a Lie group, \( \pi \) a unitary irreducible representation on the complex separable Hilbert space \( \mathcal{H} \). Let \( e_0 \in \mathcal{H} \) be fixed and \( e_g = \pi(g)e_0 \). With the notation \( \tilde{\psi} = \{ e^{i\alpha} \psi | \alpha \in \mathbb{R} \}, \psi \in \mathcal{H} \), \( \{ e_g \}_{g \in G} \) is a family of coherent vectors, while \( \{ \tilde{e}_g \}_{g \in G} \) is a family of coherent states. If \( K = \{ k \in G | \pi(k)e_0 = e^{i\alpha(k)}e_0 \} \), then \( M = \{ \tilde{\pi}(g)e_0 | g \in G \} \) and \( M \approx G/K \). Let \( \chi \) be a character of \( K \). Then, in Perelomov’s construction, \( L = M \times \chi \mathbb{C} \) is a \( G \)-homogeneous line bundle associated by the character \( \chi \) to the principal \( K \)-bundle \( K \rightarrow G \rightarrow M \).

In fact, Perelomov’s [2] coherent vectors are

\[
\mathbf{e}_{Z,j} = \exp \sum_{\varphi \in \Delta^+} (Z_{\varphi}F^+_{\varphi})j, \quad \mathbf{e}_{Z,j} = (\mathbf{e}_{Z,j}^* ; \mathbf{e}_{Z,j})^{-1/2} \mathbf{e}_{Z,j},
\]  

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Let us consider the projection $\xi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{CP}^n$, $\xi(x) = [x]$.

Let us consider the function: $\Psi_{\mathbb{CP}^n} : \mathbb{CP}^n \times \mathbb{CP}^n \times \mathbb{CP}^n \rightarrow \mathbb{C}$

$$\Psi_{\mathbb{CP}^n}([x],[y],[z]) = \frac{(x,y)(y,z)(z,x)}{||x||^2||y||^2||z||^2}, \quad x, y, z \in \mathbb{C}^{n+1} \setminus \{0\},$$

where the scalar product $(x,y)$ in $\mathbb{C}^{n+1}$ is linear in the second entry. Let us use the notation $d_C([x],[y]) = \arccos \frac{|(x,y)|}{||x||||y||}$ for the Cayley distance.

The phase $\Phi$ on a closed loop passing through three states in the projective space was considered by Bargmann [15]. Here I correlate this phase with the shape invariant used by Blaschke and Terheggen for $\mathbb{CP}^2$ [9] and by Brehm for $\mathbb{CP}^n$ [16]. They have proved that:

$$\Psi_{\mathbb{CP}^n}([x],[y],[z]) = \cos a \cos b \cos c \exp(i\Phi_{\mathbb{CP}^n}),$$

where $0 \leq \Phi_{\mathbb{CP}^n} < 2\pi$, and $a, b, c < \pi/2$ (in order to assure the uniqueness of the geodesic arcs) are the sides of the triangle $[x],[y],[z]$: $a = d_C([y],[z]), \quad b = d_C([z],[x]), \quad c = d_C([x],[y])$.

**Theorem 1 (Hangan, Masala [17])** Given a geodesic triangle with vertices $[x],[y],[z]$ in the projective space $\mathbb{CP}^n$, let $S$ be the surface generated by the geodesic arcs issued from $[x]$ with end-points on the geodesic arc between $[y]$ and $[z]$. Let $\mathcal{J}$ be the integral of the two-form $\omega$ on $S$. Then

$$\Phi_{\mathbb{CP}^n} = -2\mathcal{J} + 2k\pi, \quad k \in \mathbb{Z}.$$  

**Remark 1 (Hangan, Masala)** As $\omega$ is closed, we have as a consequence of Stokes’ theorem that $\mathcal{J}$ does not vary when $S$ is continuously deformed such that its boundary is fixed.

Theorem 2 below gives in the particular case of the complex projective space a new proof of the theorem of Hangan and Masala. For the complex Grassmann manifold, an explicit calculation, independent of Theorem 3 is briefly presented in §5.

Now we shall consider a generalization of the definition (3.1) for the line bundle $\tau$ in the context of Rawnsley’s coherent states. So, let us take $x,y,z \in M$ and $q,q',q'' \in L$ such that $\tau(q) = x, \tau(q') = y, \tau(q'') = z$. A generalization of (3.1) is given by the three-point function $\Psi_M : M \times M \times M \rightarrow \mathbb{C}$ (see also [11]):

$$\Psi_M(x,y,z) = \frac{(e_q,e_{q'}) (e_{q'}, e_{q''}) (e_{q''}, e_q)}{||e_q||^2 ||e_{q'}||^2 ||e_{q''}||^2},$$

which is globally defined and does not depend of the representatives in the fibre.
Theorem 2 a). Let $M$ be a compact, Hodge manifold, admitting the kählerian embedding $\cite{[2,4]}$. Then we have the Cauchy formula:

\[ \Psi_M(x, y, z) = \Psi_{\mathbb{C}P^n}(\iota(x), \iota(y), \iota(z)). \] (3.4)

Let $0 \leq \Phi_M < 2\pi$ be the phase

\[ \Psi_M(x, y, z) = |\Psi_M(x, y, z)| \times e^{i\Phi_M(x, y, z)}. \] (3.5)

Then we have

\[ \Phi_M(x, y, z) = \Phi_{\mathbb{C}P^n}(\iota(x), \iota(y), \iota(z)) \mod 2k\pi, k \in \mathbb{Z}, \] (3.6)

\[ |\Psi_M(x, y, z)| = \cos a \cos b \cos c, \] (3.7)

where

\[ a = d_C(\iota(y), \iota(z)), b = d_C(\iota(z), \iota(x)), c = d_C(\iota(x), \iota(y)). \] (3.8)

b). Let us suppose that $M$ is a compact Hermitian symmetric space. Let us consider that the points $x, y, z \in M$ are such that any pair of them can be joined by a unique geodesic arc, which determine the loop $\gamma(x, y, z)$. Then the angle $\Phi_M$ (3.3) can be given by an equation of the type (3.2), but on the manifold $M$:

\[ \Phi_M(x, y, z) = -2 \int_{\sigma(x, y, z)} \omega_M. \] (3.9)

where $\sigma(x, y, z)$ is the surface of the geodesic triangle determined by the points $x, y, z$ or a deformation surface of $\gamma(x, y, z)$.

Proof. a). Eq. (3.4) is proved using the Cauchy formula $\cite{[1]}$ for the complex two-point functions in a local representation of sections. See also Prop. 4.7 in Ref. $\cite{[10]}$.

Let us also consider the real two-point function

\[ \psi_M(x, y) = \frac{|(e_q, e_{q'})|}{||e_q||^{1/2}||e_{q'}||^{1/2}}, \tau(q) = x, \tau(q') = y. \] (3.10)

The two-point function verifies locally the Cauchy relation

\[ \psi_M(x, y) = \psi_{\mathbb{C}P^n}(\iota(x), \iota(y)) \] (3.11)

(see §4.2 in Ref. $\cite{[10]}$ for a more precise formulation). The functions (3.10) are introduced in eq. (3.3), the Cauchy relation (3.11) is taken into account and eq. (3.6) follows. Eq. (3.7) follows if in the Cauchy relation (3.11) it is observed that

\[ \psi_{\mathbb{C}P^n}(\iota(x), \iota(y)) = \cos d_C(\iota(x), \iota(y)). \] (3.12)

b). Now we consider the manifold $M$ to be to be compact Hermitian symmetric. So, $M$ is a compact, homogeneous, simply connected, naturally reductive, Hodge manifold, which admits a holomorphic and isometric embedding in a projective space.
Let us consider a closed piece-wise smooth curve \( \gamma \) in \( M \). Because the manifold is Hodge and simply connected, we are under the conditions of Thm. 2.2.1 in \cite{18}. From \cite{18} we need only the expression (1.8.3) of the parallel transport function. However, in order to put in accord the notation from \cite{18} with our notation, we repeat some parts of the proof. The same notation is used also in Theorem 4 below. Let \( P_\gamma : L_p \to L_p \) the parallel transport along \( \gamma \). Then \( P_\gamma(s) = Q_\gamma s \). Let the notation \( A_L = i\theta_L \). The scalar parallel transport function \( Q(\gamma) = \exp i\beta \) is calculated with the Stokes’ formula, where the phase \( \beta \) is

\[
\beta = \oint_\gamma A_L = \oint_\gamma i\theta_L = \int_\sigma dA_L.
\]  

(3.13)

Here \( \sigma \) is a surface of deformation of \( \gamma \). We recall that \( \gamma : I \to M \) is homotopic to a point if there is a rectangle \( R = [a, b] \times [c, d] \) in the plane and a piece-wise smooth parametrization \( \rho : I \to \hat{R} \) of the boundary \( \hat{R} \) of \( R \) oriented counter-clockwise such that \( \sigma \circ \rho = \gamma \). Such a map defines an oriented surface with \( \hat{R} \) oriented counter-clockwise as boundary and is called surface of deformation of \( \gamma \).

Considering the (positive) line bundle \( L \) over \( M \), we have (also cf. eq. (2.2.1) in \cite{18})

\[
\beta = i\int_\sigma d\theta_L = i\int_\sigma \Theta_L = 2\int_\sigma \omega_M,
\]  

(3.14)

\[
Q(\gamma) = \exp(i\beta) = \exp(2i\int_\sigma \omega_M).
\]  

(3.15)

Now a closed path \( \gamma : x \to y \to z \to x \) in \( M \) is considered. Because the manifold \( M \) is naturally reductive, the coherent states realise parallel transport on geodesics (cf. Remark 3 in the second Ref. \cite{14} and Remark 1 in the third Ref. \cite{1}). Taking as auto-parallel section \( s \) along the piece-wise smooth curve \( \gamma \) in the formula of the parallel transport \( P_\gamma(s) = Q_\gamma s \) the normalized coherent state vector \( ||e_q||^{-1}e_q \), the holonomy \( \beta = \Phi(x, z, y) \) on the geodesic path \( \gamma = \gamma(x, y, z, x) \) is \( \beta = -\Phi(x, y, z) \). So, eq. (3.9) is proved. More details will be given elsewhere \cite{11}.

\section*{4. PHASES AND COHERENT STATES}

\textbf{Theorem 3} Let \( M \) be a compact Hermitian symmetric manifold. Let \((L, h_L, \nabla_L)\) be a homogeneous line bundle supposed to be very ample. Let us consider on the manifold of coherent states \( M \) the Perelomov’s coherent vectors (2.2) in a local chart, corresponding to the fundamental representation \( \pi \). Let us consider the points \( Z, Z' \in V_0 \subset M \) such that \( 0, Z, Z' \) is a geodesic triangle. Then the phase \( \Phi_M \) defined by the relation

\[
(e_{Z'}, e_Z) = |(e_{Z'}, e_Z)| \exp(i\Phi_M(Z', Z))
\]  

(4.1)

is given by twice the integral of the symplectic two-form on the surface \( \sigma(0, z, z') \) of the geodesic triangle \( \gamma(0, Z, Z') \subset M \)

\[
\Phi_M(Z', Z) = 2\int_{\sigma(0, Z', Z')} \omega_M.
\]  

(4.2)
Also
\[ |(e_{Z'}, e_Z)| = |(e_{Z}, e_{Z'})| = \cos d_C(\iota(Z'), \iota(Z)). \] (4.3)

**Remark 2** If the line bundle is not very ample, then an integer \( m \) appears in front of the integral in the eq. (4.2), corresponding to the power \( L^m \) for which the ample line bundle \( L \) becomes very ample.

**Proof of the Theorem 3.** The theorem is a direct consequence of Theorem 2 and of the remark that the Berry phase [4] is the opposite of the Bargmann phase [15]. Indeed, let us take the points \( 0, Z, Z' \) in Theorem 3 to correspond to the points \( x, y, z \) in Theorem 2. Then the three-point function \( \Psi_M(0, Z, Z') \) (3.3) becomes the complex-valued two-point function \( (e_{Z'}, e_Z) \) and \( \Phi_M(0, Z, Z') = -\Phi_M(0, Z, Z') \) in eq. (3.5) is denoted simply \( \Phi_M(Z, Z') \) in eq. (4.1). Eq. (4.3) is nothing else than eq. (3.12). \( \square \)

5. **ILLUSTRATION ON COMPLEX GRASSMANN MANIFOLD** \( G_n(\mathbb{C}^{m+n}) \)

**Theorem 4** Let \( z, z' \in V_0 \subset G_n(\mathbb{C}^{m+n}) \) be described by the Pontrjagin coordinates \( Z, Z' \). Let \( \gamma(0, z, z') \) be the geodesic triangle obtained by joining \( 0, z, z' \). Then the symplectic area of the surface \( \sigma(0, z, z') \) of the geodesic triangle \( \gamma(0, z, z') \) is given by

\[ \mathcal{J} = \int_{\sigma(0, z, z')} \omega = \frac{1}{4i} \log \frac{\det(1 + ZZ')}{\det(1 + Z'Z)} . \] (5.1)

**Proof** We apply the Stokes' formula (3.13), take into account eqs. (3.14), (3.15) and the relation \( dA_L = 2\omega \). Here \( L \) is the dual of the tautological (universal) line bundle on the Grassmann manifold. The connection is \( A_L = i\text{Tr}[dZ Z' + (1 + ZZ')^{-1}] \) [14]. In the relation \( A_L = i\theta_L \), the connection matrix \( \theta_L \) corresponds to the hermitian metric on the dual of the tautological line bundle on the Grassmann manifold \( \hat{h}_L(Z) = \det(1 + ZZ')^{-1} \). The Berry connection [4] which corresponds to \( A_L \) is

\[ A_B = \frac{i}{2} \text{Tr}[(dZ Z' - Z dZ')(1 + ZZ')^{-1}] . \] (5.2)

The corresponding two-form on \( G_n(\mathbb{C}^{m+n}) \) is

\[ \omega = \frac{i}{2} \text{Tr}[dZ(1 + Z' Z)^{-1} \wedge dZ' + (1 + ZZ')^{-1}] . \] (5.3)

The calculation is long. I indicate here only the main steps. Details will be given elsewhere [8].

a). Firstly, let \( z \in V_0 \). The explicit expression of the geodesic starting at \( 0 \in V_0 \subset G_n(\mathbb{C}^{m+n}) \) with \( \dot{Z}(0) = B \) is \( Z(t) = B \tan \frac{\sqrt{t + B^2}}{\sqrt{t + B^2}} \), where \( \dot{Z}(0) = B \) (cf. [12]).

The conditions that the points \( 0, z, z_0 \in V_0 \) to belong to the same geodesics are [8]

\[ Z_0 Z' = Z_0^+ Z' = Z^+ Z_0. \] (5.4)
b). The integral on $\gamma(0, z_1, z_2, 0)$ is calculated firstly on the geodesic arc joining $z_1, z_2$. The situation is reduced to that on calculating the integral on the geodesic joining the points $0, z$. A linear fractional transformation which sends $z_1 \to 0$ has the expression \[ Z'(Z) = (AZ + B)(CZ + D)^{-1} \] where
\[ A = (1 + Z_1 Z_1^*)^{-1/2}, B = -(1 + Z_1 Z_1^*)^{-1/2} Z_1, C = (1 + Z_1^* Z_1)^{-1/2}, D = (1 + Z_1^* Z_1)^{-1/2} \]
and makes the quasi-linear change of variables. We need the formulas:
\[ Z'(Z) = (1 + Z_1 Z_1^*)^{-1/2}(Z - Z_1'(1 + Z_1^* Z_1)^{-1}(1 + Z_1^* Z_1)^{1/2}. \] (5.5)

When $Z_1 \to 0$, the point $Z_2$ becomes $Z_I = Z'(Z_2)$. So, we have to calculate
\[ I = i \int_{Z_1}^{Z_2} \text{Tr} [dZZ^+(1 + ZZ^+)^{-1}] \] (5.6)
and make the quasi-linear change of variables. We need the formulas:
\[ dZ = (A - Z'_C)^{-1} dZ'(B^+ Z' + D)^{-1}. Z^+ = (D^+ Z'^* - B^+)(A^+ - C^* Z'^*)^{-1}, \]
\[ 1 + ZZ^+ = (-Z'_C + A)^{-1}(1 + Z'Z'^*)(C^* Z'^* - A^+)^{-1}. \] (5.7)

Eq. \( (5.3) \) becomes
\[ I = i \int_{0}^{Z_1} \text{Tr} [dZ'(1 - Z_1' Z)^{-1}(Z'^* + Z_1^*)(1 + Z'Z'^*)^{-1}]. \] (5.8)

c). The condition \( (5.4) \) under the fractional transformation becomes
\[ Z_I' Z'^* = Z' Z_I^*; Z'^* Z_I = Z_I^* Z'. \] (5.9)
The last equation has the solution \( Z' = \frac{Z_{11}}{|Z_{11}|} Z_I \). Introducing the last expression in eq. \( (5.8) \) it is obtained
\[ I = i \int_{0}^{r_0} dr \text{Tr} [(1 - Ar)^{-1}(Br + A)(1 + Br^2)^{-1}], \] (5.10)
where \( A = \frac{Z_1^+ Z_1}{|Z_{11}|}, |Z_{11}| = r, B = \frac{Z_1^* Z_1}{|Z_{11}|^2}, |Z_{11}| = r_0. \) But
\[ (1 - Ar)^{-1}(Br + A)(1 + Br^2)^{-1} = Br(1 + Br^2)^{-1} + (1 - Ar)^{-1} A. \]
With the formula \( \frac{d}{dx} \log \det U = \text{Tr}(U^{-1} U' \frac{dU}{dx}) \), the integral \( (5.10) \) becomes successively
\[ I = i \log \det \frac{(1 + Br_0^2)^{1/2}}{1 - Ar_0} = i \frac{1}{2} \log \det \frac{(1 + Z_1^+ Z_I)}{(1 - Z_1^+ Z_I)^2}. \] (5.11)

But
\[ 1 - Z_1 Z_1^* = (1 + Z_1 Z_1^*)^{1/2}(1 + Z_1 Z_1^*)^{-1}(1 + Z_1 Z_1^*)^{1/2}, \]
\[ (1 + Z_1 Z_1^*)^{-1} = (1 + Z_1 Z_1^*)^{-1/2}(1 + Z_1 Z_1^*)^{-1/2} \]
\[ (1 + Z_2 Z_2^*)^{-1}(1 + Z_2 Z_2^*)^{-1/2}, \]
\[ I = i \frac{1}{2} \log \frac{\text{det}(1 + Z_2^2 Z_2^*) \text{det}(1 + Z_2 Z_2^*)}{\text{det}(1 + Z_1 Z_1^*) \text{det}(1 + Z_1 Z_1^*)}. \] (5.12)
Taking the particular values $0, Z$ and $Z'$ for $Z_1, Z_2$ in the last expression, eq. \( (5.11) \) is proved because $I = 2J$. \( \Box \)
Remark 3 Equation (5.1) contains as particular case the projective space and the sphere. The expression for the sphere can be found in [2]. Note that in the conventions of this talk, the two-form on the sphere is 
\[ \omega = \frac{1}{2} \frac{dz \wedge d\bar{z}}{1 + |z|^2} \] . This gives for the sphere of radius 1 the area \( \pi \). This also explains the difference with Pancharatnam’s formula.

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