Area and Perimeter of the Convex Hull of Stochastic Points

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Abstract

Given a set $P$ of $n$ points in the plane, we study the computation of the probability distribution function of both the area and perimeter of the convex hull of a random subset $S$ of $P$. The random subset $S$ is formed by drawing each point $p$ of $P$ independently with a given rational probability $\pi_p$. For both measures of the convex hull, we show that it is NP-hard to compute the probability that the measure is at least a given bound $w$. For $\varepsilon \in (0, 1)$, we provide an algorithm that runs in $O(n^{9/\varepsilon})$ time and returns a value that is between the probability that the area is at least $w$, and the probability that the area is at least $(1 - \varepsilon)w$. For the perimeter, we show a similar algorithm running in $O(n^{9/\varepsilon})$ time.

Finally, given $\varepsilon, \delta \in (0, 1)$, and for any measure, we show an $O(n \log n + (n/\varepsilon^2) \log(1/\delta))$-time Monte Carlo algorithm that returns a value that with probability at least $1 - \delta$ differs at most $\varepsilon$ from the probability that the measure is at least $w$.

1 Introduction

Let $P$ be a given set of $n$ points in the plane, where each point $p$ of $P$ is assigned a probability $\pi_p$. Given any subset $X \subseteq P$, let $\text{area}(X)$ and $\text{perim}(X)$ denote the area and perimeter, respectively, of the convex hull of $X$. In this paper, we study the random variables $\text{area}(S)$ and $\text{perim}(S)$, where $S$ is a random subset of $P$, formed by drawing each point $p$ of $P$ independently with probability $\pi_p$. We assume the model in which the probability $\pi_p$ of every point $p$ of $P$ is a rational number, and where deciding whether $p$ is present in a random sample of $P$ can be done in constant time. Then, any random sample of $P$ can be generated in $O(n)$ time. For the area, we show the following results:

1. Given $w \geq 0$, computing $\Pr[\text{area}(S) \geq w]$ is NP-hard.

2. Given $w \geq 0$ and $\varepsilon \in (0, 1)$, a value $\sigma$ so that $\Pr[\text{area}(S) \geq w] \leq \sigma \leq \Pr[\text{area}(S) \geq (1 - \varepsilon)w]$ can be computed in $O(n^{9/\varepsilon})$ time.

3. Given $\varepsilon, \delta \in (0, 1)$, a value $\sigma'$ satisfying $\Pr[\text{area}(S) \geq w] - \varepsilon < \sigma' < \Pr[\text{area}(S) \geq w] + \varepsilon$ with probability at least $1 - \delta$, can be computed in $O(n \log n + (n/\varepsilon^2) \log(1/\delta))$ time.

We show similar results for the perimeter. Particularly for the area, we further show that if $P \subset [0, U]^2$ for some $U > 0$, given $\varepsilon \in (0, 1)$ and $w \geq 0$, a value $\tilde{\sigma}$ satisfying $\Pr[\text{area}(S) \geq w + \varepsilon] \leq \tilde{\sigma} \leq \Pr[\text{area}(S) \geq w - \varepsilon]$ can be computed in $O(n^5 \cdot U^4/\varepsilon^2)$ time.

For the ease of explanation, we assume that the point set $P$ does not hold each of the next properties: there exist three collinear points in $P$, there exist two points of $P$ in the same vertical or horizontal line, and there exist two different parallel lines each passing through two points of $P$. All our results can be extended to consider point sets satisfying these properties.

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Notation

Given three different points \( p, q, r \) of the plane, let \( \Delta(p, q, r) \) denote the triangle with vertex set \( \{p, q, r\} \), \( \ell(p, q) \) denote the directed line through \( p \) in direction to \( q \), \( h(p) \) denote the horizontal line through \( p \), \( pq \) denote the segment with endpoints \( p \) and \( q \), and \( \|pq\| \) denote the length of \( pq \). We say that a triangle defined by three vertices of the convex hull of a random sample \( S \subseteq P \) is canonical if the triangle contains the topmost point of \( S \).

Outline

In Section 2, we show that computing the probability that the area is at least a given bound is NP-hard, and provide the algorithms to approximate this probability. In Section 3, we show the results for the perimeter.

1.1 Related work

Stochastic finite point sets in the plane, as the one considered in this paper, appear in a natural manner in many database scenarios in which the gathered data has many false positives \([2, 6, 10]\). In the last years, algorithmic problems and solutions considering stochastic points have emerged. In 2011, Chan et al. \([3]\) studied the computation of the expectation \( E[MST(S)] \), where \( S \) is a random sample drawn on the point set \( P \) and \( MST(S) \) is the total length of the minimum Euclidean spanning tree of \( S \). Each point is included in the sample \( S \) independently with a given rational probability. They motivate this problem from the following three situations: the point set \( P \) may denote all possible customer locations, each with a known probability of being present at an instant, or it may denote sensors that trigger and upload data at unpredictable times, or it may be a set of multi-dimensional observations, each with a confidence value. Among other results, they proved that computing \( E[MST(S)] \) is \( \#P \)-hard and provided a random sampling based algorithm running in \( O\left(\frac{n^5}{\varepsilon^2} \log(n/\delta)\right) \) time, that returns a \((1+\varepsilon)\)-approximation with probability at least \( 1-\delta \). In 2014, Chan et al. \([4]\) studied the probability that the distance of the closest pair of points is at most a given parameter, among \( n \) stochastic points. Computing the closest pair of points among a set of precise points is a classic and well-known problem with an efficient solution in \( O(n \log n) \) time. When introducing the stochastic imprecision, computing the above probability becomes \( \#P \)-hard \([4]\).

Foschini at al. \([8]\) studied in 2011 the expected Klee’s measure of \( n \) stochastic axis-aligned hyper-rectangles, where each hyper-rectangle is present with a given probability. They showed that the expected Klee’s measure can be computed in polynomial time (assuming the dimension is a constant), provided a data structure for maintaining the expected Klee’s measure over a dynamic family of such probabilistic hyper-rectangles, and proved that it is NP-hard to compute the probability that the Klee’s measure exceeds a given value even in one dimension, using a reduction from the Subset-Sum problem \([9]\).

With respect to the convex hull of stochastic points, in the same model that we consider (called unipoint model \([11]\)), Suri et al. \([12]\) investigated the most likely convex hull of stochastic points, which is the convex hull that appears with the most probability. They proved that such a convex hull can be computed in \( O(n^3) \) time in the plane, and its computation is NP-hard in higher dimensions.

In a more general model of discrete probabilistic points (called multipoint model \([11]\)), each of the \( n \) points either does not occur or occurs at one of finitely many locations, following its own discrete probability distribution. In this model that generalizes the one considered in this paper, Agarwal et al. \([1]\) gave exact computations and approximations of the probability that a query point lies in the convex hull, and Feldman et al. \([7]\) considered the minimum enclosing
ball problem and gave a $(1 + \varepsilon)$-approximation. In this more general model and other ones, Jørgensen et al. [10] studied approximations of the distribution functions of the solutions of geometric shape-fitting problems, and described the variation of the solutions to these problems with respect to the uncertainty of the points. They noted that in the multipoint model the distribution of area or perimeter of the convex hull may have exponential complexity if all the points lie on or near a circle.

More recently, in 2014, Li et al. [11] considered a set of $n$ points in the plane colored with $k$ colors, and studied, among other computation problems, the computation of the expected area or perimeter of the convex hull of a random sample of the points. Such random samples are obtained by picking for each color a point of that color uniformly at random. They proved that both expectations can be computed in $O(n^2)$ time. We note that their arguments can be used to compute both $E[\text{area}(S)]$ and $E[\text{perim}(S)]$, each one in $O(n^2)$ time. In the case of the expected perimeter, similar arguments were discussed by Chan et al. [3].

2 Probability distribution function of area

2.1 NP-hardness

Theorem 1. Given a stochastic point set $P$ at rational coordinates, and an integer $w > 0$, it is NP-hard to compute the probability $\Pr[\text{area}(S) \geq w]$ that the area of the convex hull of a random sample $S \subseteq P$ is at least $w$.

Proof. We show a Turing reduction from the SUBSET-SUM problem that is NP-complete [9].

Our Turing reduction assumes an unknown algorithm $A(P, w)$ computing $\Pr[\text{area}(S) \geq w]$, that will be called twice. The SUBSET-SUM problem receives as input a set $\{a_1, \ldots, a_n\} \subseteq \mathbb{N}$ of $n$ numbers and a target $t \in \mathbb{N}$, and asks whether it exists a subset $J \subseteq [1..n]$ such that $\sum_{i \in J} a_i = t$. Let $([a_1, \ldots, a_n], t)$ be an instance of the SUBSET-SUM problem. We assume w.l.o.g. that each of the numbers $a_1, \ldots, a_n$ can be represented in a polynomial number of bits (refer to the NP-completeness proof of the SUBSET-SUM problem [9]), then the base-2 logarithm of each of them is polynomially bounded. To show that computing $\Pr[\text{area}(S) \geq w]$ is NP-hard, we construct in polynomial time the $2n + 1$ stochastic points $p_1, p_2, \ldots, p_{n+1}$ and $q_1, q_2, \ldots, q_n$ at rational coordinates such that:

(a) the points $p_1, p_2, \ldots, p_{n+1}$, together with the points $q_1, q_2, \ldots, q_n$, are in convex position and appear in the order $p_1, q_1, p_2, q_2, \ldots, p_n, q_n, p_{n+1}$ clockwise;
(b) $\pi_{p_i} = 1$ for every $i \in [1..n+1]$ and $\pi_{q_j} < 1$ for every $j \in [1..n]$;
(c) for some positive $b \in \mathbb{N}$, $\text{area}([p_j, q_j, p_{j+1}]) = b \cdot a_j \in \mathbb{N}$ for all $j \in [1..n]$; and
(d) $\text{area}([p_1, \ldots, p_{n+1}]) \in \mathbb{N}$.

Let $G = \text{area}([p_1, \ldots, p_{n+1}])$, and let $S$ be any random sample of the point set $P = \{p_1, \ldots, p_{n+1}, q_1, \ldots, q_n\}$. Let $J = \{j \in [1..n] \mid q_j \in S\}$. Observe that

$$\text{area}(S) = G + \sum_{j \in J} \text{area}([p_j, q_j, p_{j+1}]) = G + b \sum_{j \in J} a_j.$$

Calling twice the algorithm $A(P, w)$, we can compute $\Pr[\text{area}(S) \geq G + bt]$ and $\Pr[\text{area}(S) \geq G + bt + 1]$. Hence, there exists such a subset $J \subseteq [1..n]$ with $\sum_{i \in J} a_i = t$ if and only if

$$\Pr[\text{area}(S) \geq G + bt] > \Pr[\text{area}(S) \geq G + bt + 1].$$

3
We show now how the above stochastic point set $P$ can be built in polynomial time. Let $p_i = ((2i-1)^2, 2i-1)$ for every $i \in [1..n+1]$, and $s_j = ((2j)^2, 2j)$ for every $j \in [1..n]$. Observe that the points $p_1, \ldots, p_{n+1}, s_1, \ldots, s_n$ belong to $\mathbb{N}^2$, are in convex position, and they appear in the order $p_1, s_1, p_2, s_2, \ldots, p_n, s_n, p_{n+1}$ clockwise (see Figure 1). Let $c_i = \text{area}([p_i, s_i, p_{i+1}])/a_i \in \mathbb{Q}$ for all $i \in [1..n]$, and $\hat{c} = \min\{c_1, \ldots, c_n\}$. For every $i \in [1..n]$, we build the point $q_i$ on the segment $s_i m_i$, where $m_i = (p_i + p_{i+1})/2$ is the midpoint of the segment $p_ip_{i+1}$ (see Figure 2). The point $q_i$ is such that

$$\frac{q_i m_i}{s_i m_i} = \frac{\hat{c}}{c_i} \leq 1.$$ 

Observe then that $q_i \in \mathbb{Q}^2$, and $\text{area}([p_i, q_i, p_{i+1}]) = a_i \cdot \hat{c}$ for all $i \in [1..n]$. Let $b, d \in \mathbb{N}$ be the smallest natural numbers such that $\hat{c} = b/(2d)$. Observe that $b$ and $d$ can be computed in polynomial time. Indeed, let $k \in [1..n]$ such that $\hat{c} = \text{area}([p_k, s_k, p_{k+1}])/a_k$. Note that each of $2 \cdot \text{area}([p_k, s_k, p_{k+1}]) \in \mathbb{N}$ and $a_k$ can be represented in a polynomial number of bits, then the maximum common divisor of $2 \cdot \text{area}([p_k, s_k, p_{k+1}])$ and $a_k$ can be computed in $O(\log \min\{\text{area}([p_k, s_k, p_{k+1}]), a_k\})$ polynomial-time operations between natural numbers $[5]$. Finally, we scale the point set $P = \{p_1, \ldots, p_{n+1}, q_1, \ldots, q_n\}$ by $2d$. We have now that $\text{area}([p_i, q_i, p_{i+1}]) = a_i \cdot b \in \mathbb{N}$ and that $G = \text{area}([p_1, \ldots, p_{n+1}]) \in \mathbb{N}$ since every new $p_i$ has even integer coordinates (see Figure 3). By considering $\pi_{p_i} = 1$ for every $i \in [1..n+1]$ and $\pi_{q_j} < 1$ for every $j \in [1..n]$, the point set $P$ satisfies the properties (a)-(d). \qed
2.2 Approximations

**Lemma 2.** Assuming that the area of each triangle defined by points of $P$ is a natural number, given an integer $w \geq 0$, the probability $\Pr[\text{area}(S) \geq w]$ can be computed in $O(n^5 \cdot w)$ time.

*Proof.* Given two different points $a, b \in P$, let $E_{a,b}$ denote the event for the random sample $S \subseteq P$ in which $a$ is the topmost point of $S$, and $b$ is the vertex following $a$ in the counterclockwise order of the vertices of the convex hull of $S$ (see Figure 4). Observe that we have

$$
\Pr[\text{area}(S) \geq w] = \sum_{a,b \in P} \Pr[\text{area}(S) \geq w \mid E_{a,b}] \cdot \Pr[E_{a,b}],
$$

and that $\Pr[E_{a,b}]$ can be computed in $O(n)$ time. We show now how to compute the probability $\Pr[\text{area}(S) \geq w \mid E_{a,b}]$ using dynamic programming. Let $B_a \subset P$ denote the points below the line $h(a)$. Let $\mathbf{P}_{a,b} \subset ((a) \cup B_a)^2$ denote the set of pairs of points $(u, v)$ such that either $v = a$ and $u = b$, or $v \neq a$ and $u$ is to the left of the directed line $\ell(a, v)$. For every $(u, v) \in \mathbf{P}_{a,b}$, let $Z_{u,v} \subset \mathbb{R}^2$ denote the region of the points below the line $h(a)$, to the left of the line $\ell(a, u)$, and to the left of the line $\ell(v, u)$ (see Figure 5). Now, for every $z \in [0..w]$, consider the entry

$$
T[u, v, z] \text{ of the table } T, \text{ defined as }
$$

$$
T[u, v, z] = \Pr[\text{area}((S \cap Z_{u,v}) \cup \{a, u\}) \geq z],
$$

which stands for the event that the convex hull of the random sample restricted to $Z_{u,v}$, together with the points $a$ and $u$, is at least $z$. Then, note that

$$
T[b, a, w] = \Pr[\text{area}(S) \geq w \mid E_{a,b}].
$$

Figure 4: The event $E_{a,b}$ for points $a, b \in P$.

Figure 5: The region $Z_{u,v}$. Left: general case. Right: particular case $u = b$ and $v = a$. 
We show now how to compute \( T[u, v, z] \) recursively for every \( u, v, z \). For every point \( u' \in P \cap Z_{u,v} \), let \( E_{u'} \) stand for the event in which \( u' \) satisfies the following properties: \( u' \in S \) and \( u' \) is the vertex of the convex hull of \( (S \cap Z_{u,v}) \cup \{a, u\} \) that follows the vertex \( u \) in counter-clockwise order, that is, \( uu' \) is an edge of the convex hull of \( (S \cap Z_{u,v}) \cup \{a, u\} \) and the elements of \( (S \cap Z_{u,v}) \setminus \{u'\} \) are to the left of the line \( \ell(u, u') \) (see Figure 6(left)). Note that \( u' \) is also the first point of \( S \cap Z_{u,v} \) hit by the line \( \ell(v, u) \) when rotated counter-clockwise centered at \( u \). Then, we have that

\[
T[u, v, 0] = 1 \quad \text{for all } (u, v) \in P_{a,b}
\]

and

\[
T[u, v, z] = \sum_{u' \in Z_{u,v}} \Pr[E_{u'}] \cdot F(u, v, z, u') \quad \text{for all } (u, v) \in P_{a,b} \text{ and } z \in [1..w],
\]

where

\[
F(u, v, z, u') = \begin{cases} 
T[u', u, z - \text{area}(\{u, u', a\})] & \text{if } \text{area}(\{u, u', a\}) < z \\
1 & \text{if } \text{area}(\{u, u', a\}) \geq z,
\end{cases}
\]

(see Figure 6(right)). Since the points in \( P \cap Z_{a,v} \) can be sorted radially around \( u \) in \( O(n) \) time, by computing the dual arrangement of \( P \) in \( O(n^2) \) time as a unique preprocessing, the probabilities \( \Pr[E_{u'}] \), \( u' \in P \cap Z_{a,v} \), can be computed in overall \( O(n) \) time by following such radial sorting of \( P \cap Z_{a,v} \). Then, all entries \( T[u, v, z] \), including \( T[b, a, w] = \Pr[\text{area}(S) \geq w \mid E_{a,b}] \), can be computed in \( O(n^3 \cdot w) \) time, and then \( \Pr[\text{area}(S) \geq w] \) can be computed in \( O(n^5 \cdot w) \) time.

The result thus follows.

**Theorem 3.** Given \( \varepsilon \in (0, 1) \) and \( w \geq 0 \), a value \( \sigma \) satisfying

\[
\Pr[\text{area}(S) \geq w] \leq \sigma \leq \Pr[\text{area}(S) \geq (1 - \varepsilon)w]
\]

can be computed in \( O(n^9 / \varepsilon) \) time.

**Proof.** The idea is to use a conditioning of the samples on subsets of \( P \) of bounded area of the convex hull, to apply on such conditionings a rounding strategy to the areas of each triangle, and to use Lemma 2. Given three points \( p, q, r \in P \), let \( E_{p,q,r} \) denote the event in which the random sample \( S \subseteq P \) satisfies the following conditions (see Figure 7):

- \( p \) is the leftmost point of \( S \);
- \( q \) is the rightmost point of \( S \);
- \( r \) is the point of \( S \) that maximizes the vertical distance to the line \( \ell(p, q) \). Furthermore, to make all the events \( E_{p,q,r} \) independent, if \( r \) is below \( \ell(p, q) \), then there is no point of \( S \) above \( \ell(p, q) \) at such a vertical distance to \( \ell(p, q) \).
Figure 7: The random sample $S \subseteq P$ conditioned on the event $E_{p,q,r}$ in which the point $r$ is above the line $t(p, q)$.

Consider from this point forward that the random samples $S \subseteq P$ are conditioned on the event $E_{p,q,r}$ for given points $p, q, r \in P$. Let $\lambda$ denote the area of the triangle with vertex set $\{p, q, r\}$ (see Figure 7). Observe that for every random sample $S$ (conditioned on $E_{p,q,r}$) we have that

$$\lambda \leq \text{area}(S) \leq 4\lambda.$$

We compute the value $\sigma_{p,q,r} \in [0, 1]$ as an approximation to the probability $\Pr[\text{area}(S) \geq w \mid E_{p,q,r}]$. If $w \leq \lambda$, then $\Pr[\text{area}(S) \geq w \mid E_{p,q,r}] = 1$ and we set $\sigma_{p,q,r} = 1$, and if $4\lambda < w$, then $\Pr[\text{area}(S) \geq w \mid E_{p,q,r}] = 0$ and we set $\sigma_{p,q,r} = 0$. Otherwise, if $w \in (\lambda, 4\lambda]$, we compute $\sigma_{p,q,r}$ as follows: Let $\theta = \varepsilon/n$. We round the area $a$ of each triangle defined by three points of $P$ by $\hat{a} = \lceil a \cdot \theta \cdot \lambda \rceil$, and round the target $w$ by $\hat{w} = \lfloor w \cdot \theta \cdot \lambda \rfloor$. Let $\hat{\text{area}}(S)$ denote the sum of the rounded areas of the canonical triangles of the convex hull of $S$. Given that the algorithm of Lemma 2 sums areas of canonical triangles, we compute the probability $\Pr[\hat{\text{area}}(S) \geq \hat{w}]$ in $O(n^5 \cdot \hat{w}) = O(n^5 \cdot \left\lfloor \frac{w}{\theta \cdot \lambda} \right\rfloor) \subseteq O(n^5 / \varepsilon)$ time, and set $\sigma_{p,q,r} = \Pr[\hat{\text{area}}(S) \geq \hat{w}]$. We now analyse how close $\sigma_{p,q,r}$ is from $\Pr[\text{area}(S) \geq w]$. Let $S$ be a random sample conditioned on $E_{p,q,r}$ and so that the convex hull of $S$ is triangulated into $k$ canonical triangles of areas $a_1, a_2, \ldots, a_k$, respectively. Observe that

$$w \geq \theta \lambda \left\lfloor \frac{w}{\theta \cdot \lambda} \right\rfloor = \theta \lambda \cdot \hat{w}$$

and

$$\theta \lambda (\hat{a}_1 + \cdots + \hat{a}_k) = \theta \lambda \left( \frac{a_1}{\theta \lambda} + \cdots + \frac{a_k}{\theta \lambda} \right) \geq a_1 + \cdots + a_k.$$

Then, $a_1 + \cdots + a_k \geq w$ implies $\hat{a}_1 + \cdots + \hat{a}_k \geq \hat{w}$. Hence,

$$\Pr[\text{area}(S) \geq w] \leq \Pr[\hat{\text{area}}(S) \geq \hat{w}] = \sigma_{p,q,r}. \quad (1)$$

Assume now that $\hat{a}_1 + \cdots + \hat{a}_k \geq \hat{w}$. Then, given that

$$\hat{w} = \left\lfloor \frac{w}{\theta \cdot \lambda} \right\rfloor \geq \frac{w}{\theta \cdot \lambda} - 1$$

and

$$\hat{a}_1 + \cdots + \hat{a}_k = \left\lceil \frac{a_1}{\theta \cdot \lambda} \right\rceil + \cdots + \left\lceil \frac{a_k}{\theta \cdot \lambda} \right\rceil \leq \frac{a_1}{\theta \cdot \lambda} + \cdots + \frac{a_k}{\theta \cdot \lambda} + k,$$
we have
\[ \frac{a_1}{\theta \cdot \lambda} + \cdots + \frac{a_k}{\theta \cdot \lambda} + k \geq \frac{w}{\theta \cdot \lambda} - 1 \]
which implies
\[ a_1 + \cdots + a_k \geq w - (k + 1) \cdot \theta \lambda \]
\[ \geq w - n \cdot \theta \lambda \]
\[ \geq (1 - n \cdot \theta)w \quad (\lambda \leq w) \]
\[ = (1 - \varepsilon)w \]
Then, \( \hat{a}_1 + \cdots + \hat{a}_k \geq \hat{w} \) implies \( a_1 + \cdots + a_k \geq (1 - \varepsilon)w \). Therefore,
\[ \sigma_{p,q,r} = \Pr[\text{area}(S) \geq \hat{w}] \leq \Pr[\text{area}(S) \geq (1 - \varepsilon)w]. \quad (2) \]
Finally, consider that the random samples \( S \subseteq P \) are not conditioned in any event \( E_{p,q,r} \). We can then use the fact
\[ \Pr[\text{area}(S) \geq w] = \sum_{p,q,r \in P} \Pr[\text{area}(S) \geq w | E_{p,q,r}] \cdot \Pr[E_{p,q,r}] \]
to compute the value
\[ \sigma = \sum_{p,q,r \in P} \sigma_{p,q,r} \cdot \Pr[E_{p,q,r}] \]
in \( O(n^3 \cdot n^6/\varepsilon) = O(n^9/\varepsilon) \) time, which by combining equations (1) and (2) verifies
\[ \Pr[\text{area}(S) \geq w] \leq \sigma \leq \Pr[\text{area}(S) \geq (1 - \varepsilon)w]. \]
The result thus follows. \( \square \)

Given the high running time of the algorithm in Theorem 3 and that it may happen that \( \Pr[\text{area}(S) \geq (1 - \varepsilon)w] - \Pr[\text{area}(S) \geq w] \) is close to 1, we give the following simple Monte Carlo algorithm to approximate \( \Pr[\text{area}(S) \geq w] \) with absolute error and a probability of success. A similar algorithm was given by Agarwal et al. [1] to approximate the probability that a given point is contained in the convex hull of the probabilistic points.

**Theorem 4.** Given \( \varepsilon, \delta \in (0, 1) \) and \( w \geq 0 \), a value \( \sigma' \) can be computed in \( O(n \log n + (n/\varepsilon^2) \log(1/\delta)) \) time so that with probability at least \( 1 - \delta \)
\[ \Pr[\text{area}(S) \geq w] - \varepsilon < \sigma' < \Pr[\text{area}(S) \geq w] + \varepsilon. \]

**Proof.** The idea is to use repeated random sampling. Let \( S_1, S_2, \ldots, S_N \subseteq P \) be \( N \) random samples of \( P \), where \( N \) is going to be specified later, and let \( X_i \ (i = 1, \ldots, N) \) be the indicator variable such that \( X_i = 1 \) if and only if \( \text{area}(S_i) \geq w \). Let \( \mu = \Pr[\text{area}(S) \geq w] \) and \( \sigma' = (1/N) \sum_{i=1}^{N} X_i \), and note that \( E[X_i] = \mu \). Using a Chernoff-Hoeffding bound, we have \( \Pr[|\sigma' - \mu| \geq \varepsilon] \leq 2 \exp(-2\varepsilon^2N) \). Then, setting \( N = \lceil (1/2\varepsilon^2) \ln(2/\delta) \rceil \), we have that \( |\sigma' - \mu| < \varepsilon \) with probability at least \( 1 - \delta \). Since after a sorting preprocessing of \( P \), the convex hull of each sample \( S_i \) can be computed in \( O(n) \) time, the running time is \( O(n \log n + N \cdot n) = O(n \log n + (n/\varepsilon^2) \log(1/\delta)) \). \( \square \)

If the coordinates of the points of \( P \) belong to some range of bounded size, then we can round the coordinates of each point of \( P \) so that in the resulting point set every triangle defined by three points has integer area. After that, we can use Lemma 2 over the resulting point set to approximate the probability \( \Pr[\text{area}(S) \geq w] \). This approach is used in the following result.
Theorem 5. If \( P \subset [0, U]^2 \) for some \( U > 0 \), then given \( \varepsilon \in (0, 1) \) and \( w \geq 0 \) a value \( \tilde{\sigma} \) satisfying

\[
\Pr[\text{area}(S) \geq w + \varepsilon] \leq \tilde{\sigma} \leq \Pr[\text{area}(S) \geq w - \varepsilon]
\]

can be computed in \( O(n^5 \cdot U^4 / \varepsilon^2) \) time.

Proof. Let \( \delta > 0 \) be a parameter to be specified later. For every random sample \( S \subseteq P \), let

\[
\tilde{S} = \left\{ \left( 2 \left\lfloor \frac{x}{\delta} \right\rfloor, 2 \left\lfloor \frac{y}{\delta} \right\rfloor \right) : (x, y) \in S \right\}.
\]

Note that the area of every triangle defined by three points of \( \tilde{S} \) is a natural number, for every \( S \subseteq P \). Furthermore, we have that

\[
\left| \text{area}(S) - \left( \frac{\delta^2}{4} \right) \text{area}(\tilde{S}) \right| < 4\delta U.
\]

Using Lemma 2, we can compute the probability

\[
\tilde{\sigma} = \Pr \left[ \text{area}(\tilde{S}) \geq \left\lceil \frac{4w}{\delta^2} \right\rceil \right]
\]

in \( O(n^5 \cdot [4w/\delta^2]) \) \( \subseteq \) \( O(n^5 \cdot U^2 / \delta^2) \) time. If \( \text{area}(\tilde{S}) \geq \left\lceil 4w/\delta^2 \right\rceil \), then

\[
w \leq \text{area}(\tilde{S}) \cdot \frac{\delta^2}{4} < \text{area}(S) + 4\delta U,
\]

which implies \( \text{area}(S) \geq w - 4\delta U \). Hence,

\[
\tilde{\sigma} = \Pr \left[ \text{area}(\tilde{S}) \geq \left\lceil 4w/\delta^2 \right\rceil \right] \leq \Pr \left[ \text{area}(S) \geq w - 4\delta U \right]. \tag{3}
\]

If \( \text{area}(S) \geq w + 4\delta U \), then

\[
w + 4\delta U \leq \text{area}(S) < \frac{\delta^2}{4} \cdot \text{area}(\tilde{S}) + 4\delta U,
\]

which implies \( \text{area}(\tilde{S}) \geq \left\lfloor 4w/\delta \right\rfloor \) since \( \text{area}(\tilde{S}) \in \mathbb{N} \). Then, we have that

\[
\Pr \left[ \text{area}(S) \geq w + 4\delta U \right] \leq \Pr \left[ \text{area}(\tilde{S}) \geq \left\lfloor 4w/\delta^2 \right\rfloor \right] = \tilde{\sigma}. \tag{4}
\]

Setting \( \delta = \frac{\varepsilon}{4\sqrt{t}} \), and combining (3) and (4), we have that \( \tilde{\sigma} \) satisfies

\[
\Pr[\text{area}(S) \geq w + \varepsilon] \leq \tilde{\sigma} \leq \Pr[\text{area}(S) \geq w - \varepsilon],
\]

and can be computed in \( O(n^5 U^4 / \varepsilon^2) \) time. \( \square \)

3 Perimeter

Similar to Lemma 2 we can prove that if all the distances between the elements of \( P \) are considered integer, the probability \( \Pr[\text{perim}(S) \geq w] \) can be computed in \( O(n^5 \cdot w) \) time, for every integer \( w \geq 0 \). Then, using conditioning of the samples and a rounding strategy, we adapt the arguments of Theorem 3 to obtain the following result:
Theorem 6. Given \( \varepsilon \in (0, 1) \) and \( w \geq 0 \), a value \( \sigma' \) satisfying

\[
\Pr[\text{perim}(S) \geq w] \leq \sigma' \leq \Pr[\text{perim}(S) \geq (1 - \varepsilon)w]
\]

can be computed in \( O(n^9/\varepsilon) \) time.

**Sketch of the proof.** Let \( E_{p,q,r} \) be the event that the minimum enclosing disk of \( S \) is supported by the points \( p, q, r \in P \), and let \( \lambda \) denote the diameter of such a disk. Conditioned on \( E_{p,q,r} \), we have \( 2\lambda \leq \text{perim}(S) < \pi\lambda \). The algorithm is similar to that of Theorem 3.

We can further show that Theorem 4 also holds if perimeter is used instead of area. We complement this section by proving that, in general, computing the probability \( \Pr[\text{perim}(S) \geq w] \) is NP-hard. The arguments are similar to that of Theorem 1, but the proof requires several key details to deal with distances between points, expressed by square roots.

**Theorem 7.** Given a stochastic point set \( P \) at rational coordinates, and an integer \( w > 0 \), it is NP-hard to compute the probability \( \Pr[\text{perim}(S) \geq w] \) that the perimeter of the convex hull of a random sample \( S \subseteq P \) is at least \( w \).

**Proof.** We show a Turing reduction from the Subset-Sum problem [9]. Let \((\{a_1, \ldots, a_n\}, t)\) be an instance of the Subset-Sum problem [9]. We assume w.l.o.g. that each of the numbers \( a_1, \ldots, a_n \) can be represented in a polynomial number of bits (refer to the NP-completeness proof of the Subset-Sum problem [9]), then the base-2 logarithm of each of them is polynomially bounded. Let \( c \in \mathbb{N} \) be a big enough and polynomially bounded number that will be specified later. For every \( k \in [1..2n] \), let \( v_k \) denote de vector

\[
v_k = \left( c \cdot \frac{k^2 - 1}{k^2 + 1}, c \cdot \frac{2k}{k^2 + 1} \right).
\]

Let \( p_1 = (0, 0) \), and for \( i = 1, \ldots, n \), let \( s_i = p_i + v_{2i-1} \) and \( p_{i+1} = s_i + v_{2i} \). Let \( z_1 = p_{n+1} - v_1 \), and for \( j = 2, \ldots, 2n-1 \), let \( z_j = z_{j-1} - v_j \). Note that the \( 4n \) points \( p_1, s_1, p_2, s_2, \ldots, p_n, s_n, p_{n+1}, \ldots, z_{2n-1} \) are at rational coordinates and in convex position, and appear in this order clockwise. Further note that each edge of the convex hull of those points has length precisely \( c \), and that the perimeter is equal to \( L = 4n \cdot c \in \mathbb{N} \) (see Figure 8).

Let \( \varepsilon = 1/2n \). For every \( i \in [1..n] \), we build in polynomial time the point \( q_i \in \mathbb{Q}^2 \) in the triangle \( \Delta(p_i, s_i, p_{i+1}) \) so that

\[
c - a_i \leq p_i q_i = q_i p_{i+1} < (c - a_i) + \varepsilon.
\]
The value of \( c \) is selected so that the point \( q_i \) exists for every \( i \in \{1..n\} \). Let \( P \) denote the point set \( \{p_1, s_1, p_2, s_2, \ldots, p_n, s_n, p_{n+1}, z_1, \ldots, z_{2n-1}\} \cup \{q_1, \ldots, q_n\} \), and assume that \( \pi_u = 1 \) for all \( u \in \{p_1, p_2, \ldots, p_n, p_{n+1}, z_1, \ldots, z_{2n-1}\} \cup \{q_1, \ldots, q_n\} \), and \( \pi_v < 1 \) for all \( v \in \{s_1, \ldots, s_n\} \). Let \( S \subseteq P \) be any random sample of \( P \), \( J = \{j \in \{1..n\} \mid s_j \notin S\} \), and \( \varepsilon_j = \overline{p_jq_j} - (c - a_j) \) for every \( j \in \{1..n\} \).

Observe that

\[
\text{perim}(S) = 2n \cdot c + \sum_{j \in J} 2 \cdot \overline{p_jq_j} + \sum_{j \notin J} 2c
\]

\[
= 2n \cdot c + \sum_{j \in J} 2 \cdot ((c - a_i) + \varepsilon_j) + \sum_{j \notin J} 2c
\]

which implies that

\[
L - 2 \sum_{j \in J} a_j = |\text{perim}(S)|,
\]

given that

\[
0 \leq 2 \sum_{j \in J} \varepsilon_j < 2|J| \cdot \varepsilon \leq 2n \cdot \varepsilon = 1.
\]

Hence, there exists a subset \( J \subseteq \{1..n\} \) such that \( \sum_{j \in J} a_j = t \) if and only if

\[
\Pr[\text{perim}(S) \geq L - 2t] > \Pr[\text{perim}(S) \geq L - 2t + 1].
\]

In other words, if and only if there exists a random sample \( S \subseteq P \) such that \( |\text{perim}(S)| = L - 2t \).

We show now how to compute the value of \( c \), and how to compute the point \( q_i \) for every \( i \in \{1..n\} \).

Consider the isosceles triangle \( \Delta(p_i, s_i, p_{i+1}) \) (see Figure 9). Let \( m_i \) denote the midpoint of the segment \( p_ip_{i+1} \), and \( s = 2i - 1 \). To ensure the existence of a point \( \tilde{q}_i \in s_im_i \) such that \( \overline{p_im_i} = c - a_i \), we need to guarantee that

\[
(c - a_i)^2 > \frac{c^2}{4} \left( \frac{s^2 - 1}{s^2 + 1} + \frac{(s + 1)^2 - 1}{(s + 1)^2 + 1} \right)^2 + \left( \frac{2s}{s^2 + 1} + \frac{2(s + 1)}{(s + 1)^2 + 1} \right)^2
\]

\[
= \frac{c^2}{2} \left( 1 + \frac{s^2 - 1}{s^2 + 1} + \frac{(s + 1)^2 - 1}{(s + 1)^2 + 1} + \frac{2s}{s^2 + 1} + \frac{2(s + 1)}{(s + 1)^2 + 1} \right)
\]

\[
= \frac{c^2}{2} \left( 1 + \frac{s^4 + 2s^3 + 3s^2 + 2s}{s^4 + 2s^3 + 3s^2 + 2s + 2} \right)
\]

\[
= \frac{c^2}{2} \left( 1 - \frac{1}{s^4 + 2s^3 + 3s^2 + 2s + 2} \right),
\]

which holds if

\[
\left( 1 - \frac{a_i}{c} \right)^2 \geq \left( 1 - \frac{1}{20s^4} \right)^2 \quad (\text{i.e. } c \geq 20s^4a_i)
\]

Figure 9: Construction of the point \( q_i \).
since
\[
\left(1 - \frac{1}{20s^4}\right)^2 > 1 - \frac{1}{10s^4} \geq 1 - \frac{1}{s^4 + 2s^3 + 3s^2 + 2s + 2}.
\]
Then, we set \(c = 20 \cdot (2n)^4 \cdot \max\{a_1, \ldots, a_n\} = 320 \cdot n^4 \cdot \max\{a_1, \ldots, a_n\} \).

Let \(d = \frac{p_i}{m_i} \) and \(z = \frac{q_i}{m_i}^2 = (c - a_i)^2 - d^2 \in \mathbb{Q} \). The point \(q_i \) is a point in the segment \(s_im_i \), that is close to \(\tilde{q}_i \), such that, if \(h \) denotes the distance \(\frac{p_i}{m_i} \), then \(h \) is rational and satisfies
\[
\sqrt{z} \leq h < \sqrt{z} + \delta,
\]
where \(\delta = \frac{1}{\sqrt{k+1}} \) and \(k = \lceil \log_2((1 + 2\sqrt{z})/\varepsilon^2) \rceil \). Note that \(k \) can be computed in \(O(\log(z/\varepsilon)) \subseteq O(\log n + \log c) \subseteq O(\log c) \) time, which is a polynomial time. Further note that \(h \) can be found, by using a binary search, in polynomial \(O(\log(\sqrt{z}/\delta)) \subseteq O(\log c) \) time. Then, we have
\[
h^2 - z = (h - \sqrt{z})(h + \sqrt{z}) < \delta(\delta + 2\sqrt{z}) < \delta(1 + 2\sqrt{z}) < \varepsilon^2,
\]
which implies
\[
(c - a_i)^2 \leq d^2 + h^2 < (c - a_i)^2 + \varepsilon^2 < ((c - a_i) + \varepsilon)^2.
\]
Hence, \(c - a_i \leq \sqrt{d^2 + h^2} = \frac{p_i}{m_i} \) \(q_i \) \(p_i+1 \) \(q_i \) \(p_i+1 \) \(c - a_i + \varepsilon \). Since the slope of the line \(\ell(p_i, p_i+1) \) is rational, the slope of \(\ell(s_i, m_i) \) is also rational. Then, \(q_i \) has rational coordinates since \(\frac{p_i}{m_i} = h \in \mathbb{Q} \).

\[\square\]

4 Discussion

The results of this paper consider the unipoint model: each point has a fixed location but exists with a given probability. The arguments given for approximating the probability distribution functions of area and perimeter, respectively, seem not to work in the multipoint model, in which each point exists probabilistically at one of multiple possible sites. For the unipoint model, both the expectation and the probability distribution function of the number of vertices in the convex hull can be computed exactly in polynomial time. It suffices to consider either that the area of each triangle defined by three points is equal to one, or that the segment defined by each pair of points has length equal to one, and then use Lemma \[2\] of this paper.

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