EXPLICIT APPROXIMATION OF THE SUM OF THE RECIPROCAL OF THE IMAGINARY PARTS OF THE ZETA ZEROS

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Abstract. In this note, we give some explicit upper and lower bounds for the summation \( \sum_{0 < \gamma \leq T} \frac{1}{\gamma} \), where \( \gamma \) is the imaginary part of nontrivial zeros \( \rho = \beta + i\gamma \) of \( \zeta(s) \).

1. Introduction

The Riemann zeta function is defined for \( \Re(s) > 1 \) by \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) and extended by analytic continuation to the complex plane with one singularity at \( s = 1 \); in fact a simple pole with residues 1. The functional equation for this function in symmetric form, is
\[
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s),
\]
where \( \Gamma(s) = \int_{0}^{\infty} e^{-t} t^{s-1} dt \) is a meromorphic function of the complex variable \( s \), with simple poles at \( s = 0, -1, -2, \ldots \) (see [3]). By this equation, trivial zeros of \( \zeta(s) \) are \( s = -2, -4, -6, \ldots \). Also, it implies symmetry of nontrivial zeros (other zeros \( \rho = \beta + i\gamma \) which have the property \( 0 \leq \beta \leq 1 \)) according to the line \( \Re(s) = \frac{1}{2} \). The summation
\[
\mathcal{A}(T) = \sum_{0 < \gamma \leq T} \frac{1}{\gamma},
\]
where \( \gamma \) is the imaginary part of nontrivial zeros appears in some explicit approximation of primes, and having some explicit approximations of it can be useful for careful computations. This is a summation over imaginary part of zeta zeros, and for approximating such summations we use Stieltjes integral and integrating by parts; let \( N(T) \) be the number of zeros \( \rho \) of \( \zeta(s) \) with \( 0 < \Im(\rho) \leq T \) and \( 0 \leq \Re(\rho) \leq 1 \). Then, supposing \( 1 < U \leq V \) and \( \Phi(t) \in C^1(U, V) \) to be non-negative, we have
\[
\Phi(\gamma) = \int_{U}^{V} \Phi(t) dN(t) = - \int_{U}^{V} N(t) \Phi'(t) dt + N(V) \Phi(V) - N(U) \Phi(U).
\]

About \( N(T) \), Riemann [5] guessed that
\[
N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).
\]
This conjecture of Riemann proved by H. von Mangoldt more than 30 years later [1] [2]. An immediate corollary of above approximate formula, which is known as Riemann-van Mangoldt formula is \( \mathcal{A}(T) = O(\log^2 T) \), which follows by partial summation from Riemann-van Mangoldt formula [2]. In 1941, Rosser [6] introduced the following approximation of \( N(T) \):
\[
|N(T) - F(T)| \leq R(T) \quad (T \geq 2),
\]

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where
\[ F(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}, \]
and
\[ R(T) = \frac{137}{1000} \log T + \frac{433}{1000} \log \log T + \frac{397}{250}. \]
This approximation allows us to make some explicit approximation of \( A(T) \).

2. Approximation of \( A(T) \)

2.1. Approximate Estimation of \( A(T) \). As we set above, \( \gamma_1 \) is the imaginary part of first nontrivial zero of the Riemann zeta function in the upper half plane and computations \([4]\) give us \( \gamma_1 = 14.13472514 \cdots \). On using (1.1) with \( \Phi(\gamma) = \frac{1}{\gamma}, 0 < U < \gamma_1 \) and \( V = T \), we obtain
\[
(2.1) \quad A(T) = \int_U^T \frac{dN(t)}{t} = \int_U^T \frac{N(t)}{t^2} dt + \frac{N(T)}{T}.
\]
Substituting \( N(T) \) from (1.2), we obtain
\[
A(T) = \frac{1}{2\pi} \int_U^T \frac{\log \left( \frac{T}{2\pi} \right)}{t} dt - \frac{1}{2\pi} \int_U^T \frac{\log T}{t} dt + \frac{1}{2\pi} \log T - \frac{1}{2\pi} + O \left( \int_U^T \frac{\log(t)}{t^2} dt \right) + O \left( \frac{\log T}{T} \right).
\]
Computing integrals and error terms, and then letting \( U \to \gamma_1^- \), we get the following approximation
\[
A(T) = \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + O(1).
\]

2.2. Explicit Estimation of \( A(T) \). Considering (1.3) and using (2.1) with \( 2 \leq U < \gamma_1 \), for every \( T \geq 2 \) implies
\[
\int_U^T \frac{F(t)}{t^2} dt - \int_U^T \frac{R(t)}{t^2} dt + \frac{F(T) - R(T)}{T} \leq A(T) \leq \int_U^T \frac{F(t)}{t^2} dt + \int_U^T \frac{R(t)}{t^2} dt + \frac{F(T) + R(T)}{T}.
\]
A simple calculation, yields
\[
\frac{F(t)}{t^2} = \frac{d}{dt} \left\{ \frac{1}{4\pi} \log^2 t - \frac{1 + \log(2\pi)}{2\pi} \log t + \frac{\log^2(2\pi) - 2 \log(2\pi)}{4\pi} - \frac{7}{8t} \right\},
\]
and setting \( \mathcal{E}(t) = \int_1^t \frac{ds}{s^2} \), we also have
\[
\frac{R(t)}{t^2} = \frac{d}{dt} \left\{ -\frac{433 \log t}{1000} - \frac{137 \log t}{1000} - \frac{69}{40t} - \frac{433}{1000} \mathcal{E}(t) \right\}.
\]
The integral of \( \mathcal{E}(t) \) converges for \( t > 1 \); in fact \( \mathcal{E}(t) \sim \frac{1}{t \log t} \) when \( t \to \infty \). Using the fact \( \frac{d}{dt} \mathcal{E}(t) = -\frac{1}{t \log^2 t} \), we get
\[
\frac{1}{t \log t} - \frac{1}{t \log^2 t} < \mathcal{E}(t) < \frac{1}{t \log t} - \frac{31}{95 \log^2 t}
\]
for \( t \geq 2 \). Therefore, after letting \( U \to \gamma_1^- \), we obtain the following explicit upper bound
\[
A(T) < \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + \mathcal{E}_{au} - \frac{137 \log^2 T + 433 \log T - 433}{1000 T \log^2 T} \quad (T \geq 2),
\]
where \( \mathcal{E}_{au} = 0.43596427 \cdots < \frac{109}{250} \), and an easy computation verifies \(-\frac{137 \log^2 T + 433 \log T - 433}{1000 T \log^2 T} < 0 \) for \( T \geq 2.222 \). Thus, we obtain
\[
A(T) < \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + \frac{109}{250}.
\]
for $T \geq 2.222$. Similarly, we get

$$A(T) > \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + c_{\text{at}}$$

$$+ \frac{274 \log^3 T + 3313 \log^2 T + 866 \log T + 433 \log T - 433}{1000T \log^2 T} \quad (T \geq 2),$$

where $c_{\text{at}} = 0.00658187 \cdots > \frac{3}{50}$ and for $T \geq 2$ the last term in the above inequality is positive. So, we obtain

$$A(T) > \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T + \frac{3}{50}$$

for $T \geq 2$. Therefore we have proved the following result:

**Theorem 2.1.** Letting $A(T) = \sum_{0 < \gamma \leq T} \frac{1}{\gamma}$ with $\gamma$ is imaginary part of zeta zeros, we have

$$(2.2) \quad \frac{15}{250} < A(T) - \left\{ \frac{1}{4\pi} \log^2 T - \frac{\log(2\pi)}{2\pi} \log T \right\} < \frac{109}{250},$$

where the left hand side holds for $T \geq 2$ and the right hand side holds for $T \geq 2.222$.

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