Absolute Bound On the Number of Solutions of Certain Diophantine Equations of Thue and Thue-Mahler Type

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Abstract

Let $F \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $d \geq 7$ and content one. Let $\alpha$ be a root of $F(x, 1)$ and assume that the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. We prove that, for every sufficiently large prime power $p^k$, the number of solutions to the Diophantine equation of Thue type

$$|F(x, y)| = tp^k$$

in integers $(x, y, t)$ such that $\gcd(x, y) = 1$ and $1 \leq t \leq (p^k)\lambda$ does not exceed 24. Here $\lambda = \lambda(d)$ is a certain positive, monotonously increasing function that approaches one as $d$ tends to infinity. We also prove that, for every sufficiently large prime number $p$, the number of solutions to the Diophantine equation of Thue-Mahler type

$$|F(x, y)| = tp^z$$

in integers $(x, y, z, t)$ such that $\gcd(x, y) = 1$, $z \geq 1$ and $1 \leq t \leq (p^z)^{10d-61\over 20d+40}$ does not exceed 1992. Our proofs follow from the combination of two principles of Diophantine approximation, namely the generalized non-Archimedean gap principle and the Thue-Siegel principle.

1 Introduction

In this article we analyze certain Diophantine equations of Thue and Thue-Mahler type. A Thue equation is an equation of the form

$$F(x, y) = m,$$  \hspace{1cm} (1)

where $F \in \mathbb{Z}[x, y]$ is a homogeneous polynomial of degree $d \geq 3$ with nonzero discriminant $D(F)$, $m$ is a fixed positive integer, and $x, y$ are integer variables. In 1909 Thue \cite{13} established that there is a finite upper bound on the number of solutions of (1), provided that $F$ is irreducible. Since Thue’s time, the estimates
on the number of solutions of (1) have been improved significantly. In 1933, assuming that $F$ is irreducible, Mahler established the existence of a number $C$, dependent only on $F$, such that the number of primitive solutions to (1), — that is, solutions with $x$ and $y$ coprime, — does not exceed $C^{1+\omega(m)}$, where $\omega(m)$ denotes the number of distinct prime divisors of $m$ [8]. In fact, his result was even stronger: if instead of (1) we consider the equation

$$F(x, y) = p_1^{k_1} \cdots p_t^{k_t},$$

(2)

where $p_1, p_2, \ldots, p_t$ are distinct fixed prime numbers, then it follows from Mahler’s argument that the number of integer solutions $(x, y, k_1, k_2, \ldots, k_t)$ to (2), with $x, y$ coprime and $k_i$ non-negative, does not exceed $C^{1+t}$. The equation (2) is called a Thue-Mahler equation. Further improvements to this estimate have been made by Erdős and Mahler [4], and Lewis and Mahler [7].

It was conjectured by Siegel that the number of primitive solutions to (1) should not depend on the coefficients of $F$. Siegel’s conjecture was established in 1984 by Evertse [6], who proved that the number of primitive solutions to (2) does not exceed

$$2 \cdot 7^{d^3(2t+3)},$$

(3)

where a binary form $F$ of degree $d$ was assumed to be divisible by at least three pairwise linearly independent linear forms in some algebraic number field.

An estimate on the number of solutions to (1) thus follows by replacing the number $t$ in (3) with $\omega(m)$.

When integers $x$ and $y$ are arbitrary, the number of solutions to (1) can be large. For example, in 2008 Stewart [14] proved that when $F$ is of degree 3 and $D(F) \neq 0$ then there is a positive number $c = c(F)$ such that the number of solutions to (1) is at least $c(\log m)^{1/2}$. However, if we restrict our attention only to primitive solutions, then their number does not seem to increase with the growth of $m$. In 1987 it was conjectured by Erdős, Stewart and Tijdeman [5] that the number of primitive solutions to (1) does not exceed some constant, which depends only on $d$. In the same year Bombieri and Schmidt [2] proved that the number of primitive solutions to (1) does not exceed

$$C d^{1+\omega(m)},$$

where the constant $C$ is absolute. In 1991 Stewart [13] replaced $\omega(m)$ in the above estimate with $\omega(g)$, where $g$ is a divisor of $m$ satisfying $g \gg_F m^{(4+d)/3d}$ (this is the statement of [13, Theorem 1] with $\varepsilon = 1/2$). In the same paper, Stewart conjectured the following.

**Conjecture 1.1.** (Stewart, [13, Section 6]) There exists an absolute constant $c_0$ such that for any binary form $F \in \mathbb{Z}[x, y]$ with nonzero discriminant and degree at least three there exists a number $C = C(F)$, such that if $m$ is an integer larger than $C$, then the Thue equation (1) has at most $c_0$ solutions in coprime integers $x$ and $y$.

The most notable step forward towards Conjecture 1.1 can be found in the work of Thunder [16]. Based on [13] he gives a heuristic that supports the
conjecture of Stewart when the degree of the form $F$ is at least five. By using a generalization of the non-Archmiedean gap principle established in [10], we develop new methods for estimating the number of primitive solutions of (1) and (2) in the case $t = 1$, thus providing theoretical evidence in support of Stewart’s conjecture. Instead of looking at (1) and (2) though, we study equations of the form

$$|F(x, y)| = tp^z,$$

with $p^z$ a prime power and $t$ an integer variable, which is “small” in comparison to $p^z$. We demonstrate that it is possible to provide an absolute bound on the number of primitive solutions, provided that $F$ is irreducible of degree $d \geq 7$ and the order of the Galois group of $F(x, 1)$ over $\mathbb{Q}$ is equal to $d$.

In order to state the main results given in Theorems 1.2 and 1.3, we need to introduce the notion of an enhanced automorphism group of a binary form. For a $2 \times 2$ matrix $M = (s \ u; t \ v)$, with complex entries, define the binary form $F_M$ by

$$F_M(x, y) = F(sx + uy, tx + vy).$$

Let $\overline{\mathbb{Q}}$ denote the algebraic closure of the rationals and let $K$ be a field containing $\mathbb{Q}$. We say that a matrix $M = (s \ u; t \ v) \in M_2(K)$ is a $K$-automorphism of $F$ (resp., $|F|$) if $F_M = F$ (resp., $F_M = \pm F$). The set of all $K$-automorphisms of $F$ (resp., $|F|$) is denoted by $\text{Aut}_K F$ (resp., $\text{Aut}_K |F|$). We define

$$\text{Aut}' F = \left\{ \frac{1}{\sqrt{|sv - tu|}} \begin{pmatrix} s & u \\ t & v \end{pmatrix} : s, t, u, v \in \mathbb{Z} \right\} \cap \text{Aut}_{\overline{\mathbb{Q}}} |F|$$

and refer to it as the enhanced automorphism group of $F$. See [10, Lemma 7.2] for a proof that $\text{Aut}' |F|$ contains at most 24 elements, provided that $d \geq 3$.

For a nonzero polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$, we define the content of $P$ to be the greatest common divisor of its coefficients. For an arbitrary finite set $X$, let $\#X$ denote its cardinality. Let

$$f(d) = \frac{20d - 41}{80} \left( \sqrt{d^2 + 16d} - 1 \right) - 1$$

and notice that $f(d)$ is a positive monotonously increasing function on the interval $[7, \infty)$, which approaches one as $d$ tends to infinity. We prove the following.

**Theorem 1.2.** Let $F \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $d \geq 7$ and content one. Let $\alpha$ be a root of $F(x, 1)$ and assume that the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. Let $\lambda$ be such that $0 \leq \lambda < f(d)$, where $f(d)$ is defined in (5). Let $p$ be prime, $k$ a positive integer, and consider the Diophantine equation

$$|F(x, y)| = tp^k.$$  

Provided that $p^k$ is sufficiently large, the number of solutions to (6) in integers $(x, y, t)$ such that

$$\gcd(x, y) = 1 \quad \text{and} \quad 1 \leq t \leq (p^k)^{\lambda}$$

is bounded. The precise upper bound is given by

$$N(F, p^k, \lambda) = \frac{20d - 41}{80} \left( \sqrt{d^2 + 16d} - 1 \right) - 1.$$
is at most $\# \text{Aut}'|F|$. In particular, it does not exceed 24. More precisely, for any two solutions $(x_1, y_1, t_1), (x_2, y_2, t_2)$ there exists a matrix $M = (sv-tu)^{-1/2}$. In Aut$'|F|$ such that

$$\frac{x_2}{y_2} = \frac{sx_1 + uy_1}{tx_1 + vy_1}.$$

**Theorem 1.3.** Let $F \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $d \geq 7$ and content one. Let $\alpha$ be a root of $F(x, 1)$ and assume that the field extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. Let $\lambda$ be such that

$$0 \leq \lambda < 1 - 8.1/(d + 2).$$

Let $p$ be prime and consider the Diophantine equation

$$|F(x, y)| = tp^z. \quad (7)$$

Provided that $p$ is sufficiently large, the number of solutions to (7) in integers $(x, y, z, t)$ such that $\gcd(x, y) = 1$, $z \geq 1$ and $1 \leq t \leq (p^z)^{\lambda}$ is at most

$$\# \text{Aut}'|F| \cdot \left[1 + \frac{11.51 + 1.5 \log d + \log ((d - 2.05)/(1 + \lambda))}{\log((d - 2.05)/(1 + \lambda) - 0.5d)} \right].$$

If we let $\lambda(d) = 0.5 - 4.05/(d + 2)$, then the function

$$g(d) = 1 + \frac{11.51 + 1.5 \log d + \log ((d - 2.05)/(1 + \lambda(d)))}{\log((d - 2.05)/(1 + \lambda(d)) - 0.5d)}$$

is monotonously decreasing on the interval $[7, \infty)$. Since $g(7) \approx 83.3$, we can use the upper bound $\# \text{Aut}'|F| \leq 24$ as well as Theorem 1.3 to conclude that the number of solutions $(x, y, z, t)$ to (7) satisfying the aforementioned conditions does not exceed $24 \cdot g(7) = 1992$ when $d \geq 7$. Furthermore, since $g(10^{15}) < 4$ and $\lim_{d \to \infty} g(d) = 3.5$, we can also conclude that it does not exceed $24 \cdot g(10^{15}) = 72$ when $d \geq 10^{15}$. The proof of Theorem 1.2 follows from the generalized non-Archimedean gap principle, whose statement is given in Section 2, Lemma 2.3. The proof of Theorem 1.3 follows from the combination of the non-Archimedean gap principle and the Thue-Siegel principle, as formulated by Bombieri and Mueller [1]. Both of these principles have been utilized in [10] so to establish the result stated in Lemma 2.5. Unfortunately, due to the application of Roth’s Theorem [11] it is not yet possible to determine how large a prime power $p^k$ in Theorem 1.2 or a prime $p$ in Theorem 1.3 should be in order for the respective absolute bound to hold. The author expects that it is possible to overcome this problem if one is able to generalize the non-Archimedean gap principle even further and
extend the range of $\mu$ from $(d/2) + 1 < \mu < d$ to, say, $\sqrt{2d} < \mu < d$, as it was done by Siegel \cite{Siegel12} and Dyson \cite{Dyson3} in the context of (what was later called) the Thue-Siegel principle.

Let us see an application of Theorems \ref{thm:1.2} and \ref{thm:1.3}. For an integer $n \geq 3$, let
\[
\Psi_n(x, y) = \prod_{1 \leq k < \frac{n}{2}, \gcd(k, n) = 1} \left( x - 2 \cos \left( \frac{2\pi k}{n} \right) y \right)
\]
denote the homogenization of the minimal polynomial of $2 \cos \left( \frac{2\pi n}{1} \right)$. Then $\Psi_n$ has degree $d = \frac{\varphi(n)}{2}$, where $\varphi(n)$ is the Euler’s totient function. Further, the Galois group of $\Psi_n(x, 1)$ has order $d$ \cite[Lemma 3.1]{Huang09}. Assume that $d \geq 5$ and let $M = |sv - tu|^{-1/2}, \begin{pmatrix} s & u \\ t & v \end{pmatrix}$ be an element of $\# \text{Aut}^1 |F|$. Since $d \geq 5$, it follows from Lemma \ref{lem:2.3} (see Section \ref{sec:2}) that there exists a positive integer $j$ coprime to $n$, with $1 \leq j < n/2$, such that
\[
2 \cos \left( \frac{2\pi j}{n} \right) = 2 \cos \left( \frac{2\pi}{n} \right) v - u - 2 \cos \left( \frac{2\pi}{n} \right) t + s.
\]
By \cite[Lemma 3.5]{Huang09}, it must be the case that $s \neq 0$, $s = v$ and $t = u = 0$. Thus, $M = |sv - tu|^{-1/2}, \begin{pmatrix} s & u \\ t & v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Hence $\text{Aut}^1 |F| = \{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \}$, and so the following results hold.

**Corollary 1.4.** Let $n$ be a positive integer such that $\varphi(n) \geq 14$. Let $\lambda$ be such that $0 \leq \lambda < f(d)$, where $f(d)$ is defined in \cite{Huang09}. Let $p$ be prime, $k$ a positive integer, and consider the Diophantine equation
\[
|\Psi_n(x, y)| = tp^k.
\]
Provided that $p^k$ is sufficiently large, the equation \ref{eq:8} has either no solutions in integers $(x, y, t)$ such that
\[
\gcd(x, y) = 1 \quad \text{and} \quad 1 \leq t \leq (p^k)^\lambda,
\]
or exactly two solutions, namely $(x, y, t)$ and $(-x, -y, t)$.

**Corollary 1.5.** Let $n$ be a positive integer such that $\varphi(n) \geq 14$. Let $\lambda$ be such that
\[
0 \leq \lambda < 1 - 8/(d + 2).
\]
Let $p$ be prime and consider the Diophantine equation
\[
|\Psi_n(x, y)| = tp^z.
\]
Provided that $p$ is sufficiently large, the number of solutions to \ref{eq:9} in integers $(x, y, z, t)$ such that
\[
\gcd(x, y) = 1, \quad z \geq 1 \quad \text{and} \quad 1 \leq t \leq (p^z)^\lambda
\]
is at most
\[
2 \left( 1 + \frac{11.51 + 1.5 \log d + \log ((d - 2.05)/(1 + \lambda))}{\log((d - 2.05)/(1 + \lambda) - 0.5d)} \right).
\]
If we let $d = \varphi(n)/2$ and $\lambda = 1 - 8.5/(d + 2)$, then it is a consequence of Corollary 1.4 that the number of solutions in integers $(x, y, z, t)$ to (9) does not exceed 166 for all $d \geq 7$ and it does not exceed 6 for all $d \geq 10^{15}$.

The article is structured as follows. In Section 2 we outline a number of auxiliary results, which are used in later sections. We recommend the reader to skip this section and use it as a reference when reading proofs of Theorems 1.2 and 1.3, which are outlined in Sections 3 and 4, respectively.

2 Auxiliary Results

This section contains several definitions and results, which we utilize in the remaining part of the article. We recommend the reader to skip this section and refer to it when reading the proofs outlined in Sections 3 and 4.

We begin with a number of definitions. For an arbitrary polynomial $P \in \mathbb{Z}[x_1, x_2, \ldots, x_n]$, we let $H(P)$ denote the maximum of Archimedean absolute values of its coefficients, and refer to this quantity as the height of $P$. For a point $(x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n$, we define $H(x_1, x_2, \ldots, x_n) = \max_{i=1,2,\ldots,n} |x_i|$ and refer to this quantity as the height of $(x_1, x_2, \ldots, x_n)$.

Let $P \in \mathbb{C}[x]$ be a polynomial that is not identically equal to zero, with leading coefficient $c_P$. The Mahler measure of $P$, denoted $M(P)$, is defined to be $M(P) = |c_P|$ if $P(x)$ is the constant polynomial and

$$M(P) = |c_P| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

otherwise, where $\alpha_1, \ldots, \alpha_d \in \mathbb{C}$ are the roots of $P$. For a binary form $Q \in \mathbb{C}[x, y]$, we define the Mahler measure of $Q$ as $M(Q) = M(Q(x, 1))$. The following lemma is a consequence of a well-known result of Lewis and Mahler [7]. See [10] for its proof.

**Lemma 2.1** (see [10] Lemma 2.6). Let

$$F(x, y) = c_dx^d + c_{d-1}x^{d-1}y + \cdots + c_0y^d$$

be a binary form of degree $d \geq 2$ with integer coefficients such that $c_0c_d \neq 0$. Let $x_0, y_0$ be nonzero integers. There exists a root $\alpha$ of $F(x, 1)$ such that

$$\min \left\{ \left| \alpha - \frac{x_0}{y_0} \right|, \left| \alpha^{-1} - \frac{y_0}{x_0} \right| \right\} \leq \frac{C |F(x_0, y_0)|}{H(x_0, y_0)^{d-1}},$$

where

$$C = \frac{2^{d-1}d^{(d-1)/2}}{M(F)^{d-2}} \left( \frac{1}{|D(F)|^{1/2}} \right).$$

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Lemma 2.2 (Thunder, [16 Lemma 2]). Let $p$ be a rational prime and let $\overline{\mathbb{Q}}_p$ denote the algebraic closure of the field of $p$-adic numbers $\mathbb{Q}_p$. Let $F \in \mathbb{Z}[x, y]$ be an irreducible homogeneous polynomial of degree $d \geq 2$ and content one, and denote the roots of $F(x, 1)$ by $\alpha_1, \ldots, \alpha_d \in \mathbb{Q}_p$. Let $x$ and $y$ be coprime integers.

If $i_0$ is an index with

$$\frac{|x - \alpha_{i_0}y|_p}{\max\{1, |\alpha_{i_0}|_p\}} = \min_{1 \leq i \leq d} \left\{ \frac{|x - \alpha_i y|_p}{\max\{1, |\alpha_i|_p\}} \right\},$$

then

$$\frac{|x - \alpha_{i_0}y|_p}{\max\{1, |\alpha_{i_0}|_p\}} \leq \frac{|F(x, y)|_p}{|D(F)|_p^{1/2}}.$$ 

Further, if $|F(x, y)|_p < |D(F)|_p^{1/2}$, then the index $i_0$ above is unique and $\alpha_{i_0} \in \mathbb{Q}_p$.

The following three results were established in [10]. Lemma 2.3 states the generalized non-Archimedean gap principle, which plays a crucial role in the proof of Theorem 1.2. In turn, Lemma 2.5 follows from the combination of the generalized gap principle and the Thue-Siegel principle, as formulated by Bombieri and Mueller in [1, Section II].

Lemma 2.3 (see [10 Theorem 1.2]). Let $p$ be a rational prime. Let $\alpha \in \mathbb{Q}_p$ be a $p$-adic algebraic number of degree $d \geq 3$ over $\mathbb{Q}$ and let $\beta$ be irrational and in $\mathbb{Q}(\alpha)$. Let $\mu$ be a real number such that $\frac{d}{2} + 1 < \mu < d$ and let $C_0$ be a positive real number. There exist positive real numbers $C_1$ and $C_2$, that are explicitly computable in terms of $\alpha$, $\beta$, $\mu$ and $C_0$, with the following property. If $x_1/y_1$ and $x_2/y_2$ are rational numbers in lowest terms such that $H(x_2, y_2) \geq H(x_1, y_1) \geq C_1$ and

$$|y_1 \alpha - x_1|_p < \frac{C_0}{H(x_1, y_1)^\mu}, \quad |y_2 \beta - x_2|_p < \frac{C_0}{H(x_2, y_2)^\mu},$$

then at least one of the following holds:

- $H(x_2, y_2) > C_2^{-1} H(x_1, y_1)^{\mu - d/2}$;
- There exist integers $s, t, u, v$, with $sv - tu \neq 0$, such that

$$\beta = \frac{s\alpha + t}{u \alpha + v} \quad \text{and} \quad \frac{x_2}{y_2} = \frac{s x_1 + t y_1}{u x_1 + v y_1}.$$ 

Lemma 2.4 (see [10 Proposition 7.3]). Let $F \in \mathbb{Z}[x, y]$ be an irreducible binary form of degree $d \geq 3$ and let $c_d$ denote the coefficient of $x^d$ in $F$. Let $\alpha_1, \ldots, \alpha_d$ be the roots of $F(x, 1)$. There exists an index $j \in \{1, \ldots, d\}$ such that

$$\alpha_j = \frac{v \alpha_1 - u}{-t \alpha_1 + s}. $$
for some integers \( s, t, u \) and \( v \) if and only if the matrix

\[
M = \frac{1}{\sqrt{|sv - tu|}} \begin{pmatrix} s & u \\ t & v \end{pmatrix}
\]

is in \( \text{Aut}' |F| \). Furthermore, if \( M \in \text{Aut}' |F| \), then \( |sv - tu| = \left| \frac{F(s,t)}{c_d} \right|^{2/d} \).

For an irrational number \( \alpha \), the orbit of \( \alpha \) is the set

\[
\text{orb}(\alpha) = \left\{ \frac{v\alpha - u}{-t\alpha + s} : s, t, u, v \in \mathbb{Z}, \ sv - tu \neq 0 \right\}.
\]

**Lemma 2.5.** Let \( K = \mathbb{R} \) or \( \mathbb{Q}_p \), where \( p \) is a rational prime, and denote the standard absolute value on \( K \) by \( | \cdot | \). Let \( \alpha_1 \in K \) be an algebraic number of degree \( d \geq 3 \) over \( \mathbb{Q} \) and \( \alpha_2, \alpha_3, \ldots, \alpha_n \) be distinct elements of \( \mathbb{Q}(\alpha_1) \), different from \( \alpha_1 \), each of degree \( d \). Let \( \mu \) be such that \( (d/2) + 1 < \mu < d \). Let \( C_0 \) be a real number such that

\[
A = 500^2 \left( \log \max_{i=1, \ldots, n} \{M(\alpha_i)\} + \frac{d}{2} \right).
\]

(10)

There exists a positive real number \( C_3 \), which is explicitly computable in terms of \( \alpha_1, \alpha_2, \ldots, \alpha_n, \mu \) and \( C_0 \), with the following property. The total number of rationals \( x/y \) in lowest terms, which satisfy \( H(x, y) \geq C_3 \) and

\[
\left| \frac{\alpha_j - x}{y} \right| < \frac{C_0}{H(x, y)^\mu}
\]

for some \( j \in \{1, 2, \ldots, n\} \) is less than

\[
\gamma \left[ 1 + \frac{11.51 + 1.5 \log d + \log \mu}{\log(\mu - d/2)} \right],
\]

where

\[
\gamma = \max\{\gamma_1, \ldots, \gamma_n\}, \quad \gamma_i = \#\{j : \alpha_j \in \text{orb}(\alpha_i)\}.
\]

(12)

Notice that when degree \( d \) extension \( \mathbb{Q}(\alpha)/\mathbb{Q} \) is Galois and \( \alpha = \alpha_1, \ldots, \alpha_d \) are the algebraic conjugates of \( \alpha \), then for \( d \geq 3 \) it follows from Lemma 2.4 that \( \alpha_j = (v\alpha - u)/(-t\alpha + s) \) if and only if \( M = \left| \frac{F(s,t)}{c_d} \right|^{-1/2} \cdot \begin{pmatrix} s & u \\ t & v \end{pmatrix} \) is an element of \( \text{Aut}' |F| \). Thus, in this case, the quantity \( \gamma \) in (12) does not exceed \( \# \text{Aut}' |F| \). This fact plays an important role in the proof of Theorem 1.3.

### 3 Proof of Theorem 1.2

By Roth’s Theorem [11], for every complex root \( \alpha \) of \( F(x, 1) \) there exist only finitely many nonzero integers \( x, y \) such that \( \min \{ |\alpha - x/y|, |\alpha^{-1} - y/x| \} \leq H(x, y)^{-2.05} \). Since

\[
|F(x, y)| \leq (d + 1)H(F)H(x, y)^d
\]

(11)
and \( |F(x, y)| = tp^k \), we have

\[
\frac{tp^k}{(d + 1)H(F)} \leq H(x, y)^d.
\]

Hence by choosing a large enough \( p^k \) we can increase \( H(x, y) \) and make it so large that the inequality \( \min \{ |\alpha - x/y|, |\alpha^{-1} - y/x| \} \leq H(x, y)^{-2.05} \) is no longer satisfied for every complex root \( \alpha \) of \( F(x, 1) \).

Define

\[
C_0 = \frac{2^{d-1}d^{(d-1)/2}M(F)d^{-2}}{|D(F)|^{1/2}}.
\]

Assume that there exists a solution \((x, y, t)\) to (6). By Lemma 2.1,

\[
\min \left\{ \left| \frac{\alpha - x}{y} \right|, \left| \frac{\alpha^{-1} - y}{x} \right| \right\} \leq \frac{C_0tp^k}{H(x, y)^d},
\]

From our choice of \( p^k \) and the above inequality it follows that

\[
\frac{1}{H(x, y)^{2.05}} < \min \left\{ \left| \frac{\alpha - x}{y} \right|, \left| \frac{\alpha^{-1} - y}{x} \right| \right\} \leq \frac{C_0tp^k}{H(x, y)^d},
\]

which is equivalent to

\[
H(x, y) < (C_0tp^k)^{1/(d-2.05)}. \tag{13}
\]

Since \( t \leq (p^k)^\lambda \),

\[
 tp^k \leq (p^k)^{1+\lambda} \leq |F(x, y)|_p^{-(1+\lambda)}.
\]

Combining this inequality with (13), we get

\[
H(x, y)^{d-2.05} < C_0tp^k \leq C_0|F(x, y)|_p^{-(1+\lambda)}.
\]

We conclude that

\[
|F(x, y)|_p < \frac{C_0^{1/(1+\lambda)}}{H(x, y)^\mu}, \tag{14}
\]

where

\[
\mu = \frac{d - 2.05}{1 + \lambda}.
\]

Next, we take \( p^k \) sufficiently large that

\[
p^k > |D(F)|.
\]

Then

\[
|F(x, y)|_p \leq p^{-k} < |D(F)|^{-1} \leq |D(F)|_p.
\]

By Lemma 2.2 there exists a unique \( p \)-adic root \( \alpha \in \mathbb{Q}_p \) of \( F(x, 1) \) such that

\[
\frac{|y\alpha - x|_p}{\max\{1, |\alpha|_p\}} \leq \frac{|F(x, y)|_p}{|D(F)|_p^{1/2}}.
\]
Let $c_d$ denote the coefficient of $x^d$ in $F$. Since $c_d \alpha$ is an algebraic integer, we see that $|c_d\alpha|_p \leq 1$, so $\max\{1, |\alpha|_p\} \leq |c_d|_p^{-1}$. Combining this inequality with (13), we obtain

$$|y_1 \alpha - x|_p < \frac{\max\{1, |\alpha|_p\} \cdot |F(x, y)|_p}{|D(F)|^1_2 H(x, y)^\mu} \leq \frac{C_1}{H(x, y)^\mu}$$

where

$$C_1 = C_0^{1/(1+\lambda)} c_d |D(F)|^{1/2}.$$

Now, assume that there exist two solutions $(x_1, y_1, t_1)$ and $(x_2, y_2, t_2)$ to (6), ordered so that $H(x_2, y_2) \geq H(x_1, y_1)$. Then it follows from the discussion above that there exist $p$-adic roots $\alpha, \beta \in \mathbb{Q}_p$ of $F(x, 1)$ such that

$$|\alpha - x_1|_p < \frac{C_1}{H(x_1, y_1)^\mu}, \quad |\beta - x_2|_p < \frac{C_1}{H(x_2, y_2)^\mu}.$$

Since $(d/2) + 1 < \mu < d$, it follows from Lemma 2.3 that there exist positive numbers $C_2$ and $C_3$, which depend on $C_1$, $\mu$, and $F$, but not on $p$, such that if $H(x_2, y_2) \geq H(x_1, y_1) \geq C_2$, then either $H(x_2, y_2) > C_3 H(x_1, y_1)^{\mu - d/2}$, or $\alpha, \beta$ and $x_1/y_1, x_2/y_2$ are connected by means of a linear fractional transformation, or both. By choosing $p^k$ sufficiently large we can always ensure that $H(x_1, y_1) \geq C_2$. We obtain an upper bound on $H(x_2, y_2)$ by combining (13) with the inequality $|F(x_1, y_1)| \leq (d + 1) H(F) H(x_1, y_1)^d$:

$$H(x_2, y_2) \leq (C_0 t_2 p^k)^{1/(d - 2.05)} \leq (C_0 (p^k)^{1+\lambda})^{1/(d - 2.05)} \leq (C_0 (t_2 p^k)^{1+\lambda})^{1/(d - 2.05)} = (C_0 |F(x_1, y_1)|^{1+\lambda})^{1/(d - 2.05)} \leq \left( C_0 \left( (d + 1) H(F) H(x_1, y_1)^d \right)^{1+\lambda} \right)^{1/(d - 2.05)}.$$

Merging the above upper bound with the lower bound $H(x_2, y_2) > C_3 H(x_1, y_1)^{\mu - d/2}$ results in the inequality

$$C_3 H(x_1, y_1)^{\mu - d/2 - (1+\lambda)/(d - 2.05)} \leq \left( C_0 \left( (d + 1) H(F) \right)^{1+\lambda} \right)^{1/(d - 2.05)}.$$

From our choice of $\lambda$ it follows that the exponent of $H(x_1, y_1)$ is positive, and so $H(x_1, y_1)$ is bounded. Thus, by making $p^k$ (and therefore $H(x_1, y_1)$) sufficiently large we can always ensure that the inequality $H(x_2, y_2) > C_3 H(x_1, y_1)^{\mu - d/2}$ does not hold. Then $\alpha, \beta$ and $x_1/y_1, x_2/y_2$ are connected by means of a linear fractional transformation:

$$\beta = \frac{v_0 - u}{-t_0 + s} \quad \text{and} \quad \frac{x_2}{y_2} = \frac{u x_1 - u y_1}{-t x_1 + s y_1}.$$
where \( s, t, u, v \in \mathbb{Z} \) and \( sv - tu \neq 0 \). By Lemma 2.4, the matrix

\[
M = \frac{1}{\sqrt{|sv - tu|}} \begin{pmatrix} s & u \\ t & v \end{pmatrix}
\]

is an element of \( \text{Aut}'|F| \). Hence the number of solutions \((x, y, t)\) to (6) is at most \( \# \text{Aut}'|F| \).

### 4 Proof of Theorem 1.3

The beginning of the proof is similar to the proof of Theorem 1.2. By Roth’s Theorem [11], for every root \( \alpha \) of \( F(x, 1) \) there exist only finitely many nonzero integers \( x, y \) such that \( \min \{|\alpha - x/y|, |\alpha^{-1} - y/x|\} \leq H(x, y)^{-2.05} \). Since \( |F(x, y)| \leq (d + 1)H(F)H(x, y)^d \) and \( |F(x, y)| = tp^z \), we have

\[
\frac{p}{(d + 1)H(F)} \leq \frac{tp^z}{(d + 1)H(F)} = \frac{|F(x, y)|}{(d + 1)H(F)} \leq H(x, y)^d.
\]

Hence by choosing a large enough \( p \) we can increase \( H(x, y) \) and make it so large that the inequality \( \min \{|\alpha - x/y|, |\alpha^{-1} - y/x|\} \leq H(x, y)^{-2.05} \) is no longer satisfied for every complex root \( \alpha \) of \( F(x, 1) \).

Now, assume that there exists a solution \((x, y, z, t)\) of (7). As in the proof of Theorem 1.2, for our choice of \( p \) the inequality

\[
H(x, y) < (C_0 tp^z)^{1/(d-2.05)}
\]

holds, where

\[
C_0 = \frac{2d^{-1}d^{(d-1)/2}M(F)^{d-2}}{|D(F)|^{1/2}}.
\]

Since

\[
 tp^z \leq (p^z)^{1+\lambda} \leq |F(x, y)|_p^{-(1+\lambda)},
\]

it follows from (15) that

\[
|F(x, y)|_p < \frac{C_0^{1/(1+\lambda)}}{H(x, y)^\mu},
\]

where

\[
\mu = \frac{d - 2.05}{1 + \lambda}.
\]

We take \( p \) sufficiently large that

\[
p > |D(F)|.
\]

Then

\[
|F(x, y)|_p \leq p^{-1} < |D(F)|^{-1} \leq |D(F)|_p.
\]
Let $c_d$ denote the coefficient of $x^d$ in $F$. By Lemma 2.2 there exists a unique $p$-adic root $\alpha \in \mathbb{Q}_p$ of $F(x, 1)$ such that

\[ |y\alpha - x|_p \leq \max\{1, |\alpha|_p\} \frac{|F(x, y)|_p}{|D(F)|^{1/2}_p} < \frac{C_1}{H(x, y)\mu}, \]

where

\[ C_1 = C_0^{1/(1+\lambda)} c_d |D(F)|^{1/2}. \]

Note that $C_1$ is independent of $p$. Further, we can ensure that $p \nmid y$ by adjusting our choice of $p$ as follows:

\[ p > c_d. \]

Indeed, if $p \mid y$, then $p$ does not divide $x$, because $x$ and $y$ are coprime. Since $z \geq 1$, it is evident from equation

\[ c_d x^d + y(c_d - 1)x^{d-1} + \cdots + c_0 y^{d-1} = \pm tp^z \]

that $p$ divides $c_d$, in contradiction to our choice of $p$. Then $|y|_p = 1$, and so for any $\alpha \in \mathbb{Q}_p$ we have

\[ \left| \frac{\alpha - x}{y} \right|_p = |y\alpha - x|_p. \]

Therefore

\[ \left| \frac{\alpha - x}{y} \right|_p < \frac{C_1}{H(x, y)\mu}. \]

Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the roots of $F(x, 1)$. Since $(d/2) + 1 < \mu < d$, we apply Lemma 2.5 and conclude that there exists a positive number $C_2$, which depends on $C_1, \mu, \alpha_1, \alpha_2, \ldots, \alpha_d$, but not on $p$, such that the number of rationals $x/y$ in lowest terms satisfying $H(x, y) \geq C_2$ and

\[ \left| \frac{\alpha_j - x}{y} \right|_p < \frac{C_2}{H(x, y)\mu} \]

for some $j \in \{1, 2, \ldots, d\}$ is less than

\[ \text{# Aut'} |F| \cdot \left[ 1 + \frac{11.51 + 1.5 \log d + \log \mu}{\log(\mu - 0.5d)} \right]. \]

If we choose $p$ so that $p \geq (d + 1)H(F)C_2^d$, then

\[ C_2^d \leq \frac{p}{(d + 1)H(F)} \leq \frac{tp^z}{(d + 1)H(F)} = \frac{|F(x, y)|}{(d + 1)H(F)} \leq H(x, y)^d, \]

so the inequality $H(x, y) \geq C_2$ is satisfied. Since all solutions $(x, y, z, t)$ to (7), including those that satisfy $H(x, y) \geq C_2$, also satisfy (10), the result follows.

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