On the crack inverse problem for pressure waves in half-space

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Abstract

After formulating the pressure wave equation in half-space minus a crack with a zero Neumann condition on the top plane, we introduce a related inverse problem. That inverse problem consists of identifying the crack and the unknown forcing term on that crack from overdetermined boundary data on a relatively open set of the top plane. This inverse problem is not uniquely solvable unless some additional assumption is made. However, we show that we can differentiate two cracks $\Gamma_1$ and $\Gamma_2$ under the assumption that $\mathbb{R}^3 \setminus \Gamma_1 \cup \Gamma_2$ is connected. If that is not the case we provide counterexamples that demonstrate non-uniqueness, even if $\Gamma_1$ and $\Gamma_2$ are smooth and “almost” flat. Finally, we show in the case where $\mathbb{R}^3 \setminus \Gamma_1 \cup \Gamma_2$ is not necessarily connected that after excluding a discrete set of frequencies, $\Gamma_1$ and $\Gamma_2$ can again be differentiated from overdetermined boundary data.

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1 Introduction

In this paper, we study an inverse problem consisting of identifying a crack in half-space and the unknown forcing term on that crack from overdetermined boundary data on a relatively open set of the top plane. For the forward problem, the governing equations involve the Helmholtz operator, a zero Neumann condition on the top plane, and continuity of the normal derivative across the crack. A jump across the crack constitutes the forcing term. Some decay at infinity is enforced by requiring that the solution lie in an adequately weighted Sobolev space.

Closely related inverse problems have been extensively studied in the steady state case. In fact, the steady state case has been investigated for the Laplace operator [9, 19] and for

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the linear elasticity operator \([3, 2, 21, 22, 18, 20, 16, 17, 15]\). \([9, 15]\) cover a related
eigenvalue problem derived from stability analysis. In \([3]\), the direct crack problem for half
space elasticity was analyzed under weaker regularity conditions (weaker than \(H^1\) regular).
In \([2]\), this direct problem was proved to be uniquely solvable in case of piecewise Lipschitz
elasticity coefficients and general elasticity tensors. Both \([2]\) and \([3]\) include a proof of
uniqueness for the related crack (or fault) inverse problem under appropriate assumptions.
\([14]\) and \([19]\) focus on the Lipschitz stability of the reconstruction of cracks based on the
assumption that only planar cracks are admissible. The seismic model introduced in \([21]\)
was used in \([22]\) to address a real life problem in geophysics consisting of identifying a fault
using GPS measurements of surface displacements. \([18, 20]\) feature statistical numerical
methods for the reconstruction of cracks based on a Bayesian approach. In \([16]\), a related
parallel accept/reject sampling algorithm was derived. The numerical method in \([17]\) is
entirely different, it is based on deep learning.

Here, we analyze a direct and an inverse problem that generalize the Laplace based case to
the wave case. The waves considered here are time harmonic pressure waves and can be
modeled by an inhomogeneous Helmholtz equation. We can actually model heterogeneous
media by assuming that the wavenumber \(k^2\) is an \(L^\infty\) function, as long as it is non-negative,
bounded away from zero, and equal to a constant \(k_0\) outside a bounded set. In section \([2]\) we
prove that the direct pressure wave problem in half space minus a crack is uniquely solvable
and well posed in the space of functions
\[
\{ v \in H^1_{loc}(\mathbb{R}^3 - \Gamma) : \frac{v}{\sqrt{1 + r^2}}, \frac{\nabla v}{\sqrt{1 + r^2}}, \frac{\partial v}{\partial r} - ik_0 v \in L^2(\mathbb{R}^3 - \Gamma) \},
\]
where \(\Gamma\) is the crack and \(r\) is the distance to the origin. The proof of uniqueness for the inverse
problem relies on unique continuation for elliptic equations from Cauchy data. The earliest
such continuation results relied on properties of analytic functions. Later, Nirenberg proved
that for second order PDEs whose leading term is the Laplacian, it suffices to assume that
solutions are \(C^1\) with piecewise continuous second derivatives for the unique continuation
property to hold \([13]\). This result was further improved by Aronszajn et al. where in \([1]\) it was
extended to such PDEs with only Lipschitz coefficients. Unfortunately, demanding Lipschitz
continuity is impractical in applications since it does not even cover the piecewise constant
case. More recently, Barcelo et al. \([4]\) proved a stronger result. In particular, their unique
continuation result implies that a solution to the pressure wave equation \((\Delta + k^2)u = 0\) in
an open set of \(\mathbb{R}^3\) satisfies the unique continuation property if \(k^2\) is in \(L^\infty_{loc}(\mathbb{R}^3)\). This unique
continuation property will help us show in section \([2.2]\) a uniqueness result for the pressure
wave inverse problem in the half-space \(\{ x : x_3 < 0 \} \) minus a crack: if \(\Gamma_i, i = 1, 2\) are two
cracks where the forcing terms \(g_i, i = 1, 2\) defining pressure discontinuity across \(\Gamma_i\) have full
support in \(\Gamma_i\) leading to the solution \(u_i\) of the forward problem, if \(\mathbb{R}^3 - \Gamma_1 \cup \Gamma_2\) is connected,
and the Cauchy data for \(u_1\) and \(u_2\) are the same on a relatively open set of the top boundary
\(\{ x : x_3 = 0 \} \) then \(\Gamma_1 = \Gamma_2\) and \(g_1 = g_2\).

In section \([3]\) we show counterexamples where uniqueness for the crack inverse problem fails
if \(\mathbb{R}^3 \setminus \Gamma_1 \cup \Gamma_2\) is not connected. In a first class of counterexamples, \(\Gamma_1 \cup \Gamma_2\) is a sphere
and we use the first Neumann eigenvalue for the Laplace operator inside an open ball that
is odd about the equator. Such a function is necessarily zero on the equator and thus the
values on the top half sphere can be extended by zero to the lower half sphere without losing
its $H^{3/2}$ character. One might argue that this first counterexample is unsatisfactory in the sense that it involves a geometry that is quite “round”: if $n$ is the normal vector on $\Gamma_1$ pointing up the range of $n \cdot e_3$ is $(0, 1]$. For that reason we provide a second counterexample where that range can be made arbitrarily narrow, and such that $\Gamma_1$ can be continued into a plane in a $C^1$ regular fashion. Constructing this family of counterexamples requires using arguments borrowed from the analysis of elliptic PDEs on domains with cusps. To make our argument easier to follow, this example is first constructed in dimension 2, then generalized to the three dimensional case using cylindrical coordinates. Let $\Gamma_a$ be the open curve $x_2 = a(x_1 - 1)^2(x_1 + 1)^2 - 2$, $x_2 \in (-1, 1)$, where $a > 0$ is a flattening parameter. In figure 1 we sketched $\Gamma_a$ for $a = 1, .25, .05$. We also sketched in the same figure $\Gamma'_a$, obtained from $\Gamma_a$ by symmetry about the line $x_2 = -2$. We show that for some values of $k$ and some choice of forcing terms $g$ on $\Gamma_a$ and $g'$ and $\Gamma'_a$, the half space crack PDE leads to the same Cauchy data everywhere on the top boundary $x_2 = 0$. Next, this geometry is rotated in three dimensional space, and we show how to construct a counterexample to uniqueness using rotationally invariant forcing terms $g_1, g_2$. The values of $g_1, g_2$ on a cross-section are not those from the two dimensional case: slight adjustments have to be made as the volume element in cylindrical coordinates is $rdrd\theta dx_3$ and the Laplace operator for a function which is independent of $\theta$ is $\frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \partial^2_{x_3}$.

In section 4 our last result states that the pressure wave crack inverse problem is uniquely solvable within the class $\mathcal{P}$ of Lipschitz open surfaces that are finite unions of polygons, except possibly for a discrete set of frequencies. A change of frequency amounts to changing the wavenumber $k^2$ to $t^2k^2$, for some $t > 0$. If $\Gamma_1$ and $\Gamma_2$ are in $\mathcal{P}$, and if $t$ is not in some discrete set, we show that if the values of the corresponding solutions $u_1, u_2$ to the pressure wave crack inverse problem in half space for the wavenumber $t^2k^2$, are equal on a relatively open set of the top boundary then $\overline{\Gamma}_1 = \overline{\Gamma}_2$, as long as the jump of $u_i$ has full support across $\Gamma_i$. 

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Figure 1: The shapes $\Gamma_a$ (solid curves) and $\Gamma'_a$ (dashed curves) for three values of $a$. We prove that it is impossible to distinguish $\Gamma_a$ from $\Gamma'_a$ at some wavenumbers and some forcing terms on $\Gamma_a$ and $\Gamma'_a$ for the half space crack PDE from Cauchy data, even if the Cauchy data is given everywhere on the top boundary.

2 The direct and inverse crack problems for pressure waves in half space

2.1 Problem formulation and physical interpretation

Let $\mathbb{R}^3^-$ be the open half space $\{x = (x_1, x_2, x_3) : x_3 < 0\}$. Let $\Gamma$ be a Lipschitz open surface in $\mathbb{R}^3^-$. Assume that $\Gamma$ is strictly included in $\mathbb{R}^3^-$ so that the distance from $\Gamma$ to the plane $\{x_3 = 0\}$ is positive and let $D$ be a domain in $\mathbb{R}^3^-$ with Lipschitz boundary such that $\Gamma \subset \partial D$. The trace theorem (which is also valid in Lispchitz domains, [7, 8]), allows us to define an inner and outer trace in $H^{\frac{1}{2}}(\partial D)$ of functions defined in $\mathbb{R}^3^- \setminus \partial D$ with local $H^1$ regularity. Let $\tilde{H}^{\frac{1}{2}}(\Gamma)$ be the space of functions in $H^{\frac{1}{2}}(\partial D)$ supported in $\Gamma$. Let $k$ be in $L^\infty(\mathbb{R}^3^-)$ such that,

$(H1)$ $k$ is real-valued,

$(H2)$ there is a positive constant $k_{min}$ such that $k \geq k_{min}$ almost everywhere in $\mathbb{R}^3^-$,

$(H3)$ there is an $R_0 > 0$ and a $k_0 > 0$ such that if $|x| \geq R_0$, and $x \in \mathbb{R}^3^- \setminus \overline{\Gamma}$, $k(x) = k_0$. 

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We define the direct pressure wave crack problem in half-space to be the boundary value problem,

\((\Delta + k^2)u = 0 \text{ in } \mathbb{R}^3 \setminus \Gamma,\)

\(\partial_{x_3} u = 0 \text{ on the surface } x_3 = 0,\)

\(\frac{\partial u}{\partial n} = 0 \text{ across } \Gamma,\)

\([u] = g \text{ across } \Gamma,\)

\(u \in V,\)

where \(n\) is a unit normal vector on \(\Gamma\), \([v]\) denotes the jump of a function \(v\) across \(\Gamma\) in the normal direction, \(g\) is in \(\tilde{H}^{1/2}(\Gamma)\), and \(V\) is a functional space ensuring that this PDE is well posed and that the solution \(u\) depends continuously on \(g\). Following [12], section 2.6, we introduce

\[ V := \{ v \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \Gamma) : \frac{v}{\sqrt{1+r^2}}, \frac{\nabla v}{\sqrt{1+r^2}}, \frac{\partial v}{\partial r}, -ik_0v \in L^2(\mathbb{R}^3 \setminus \Gamma) \}, \]

where as previously \(r = |x|\) for \(x\) in \(\mathbb{R}^3\).

**Proposition 2.1.** Problem (1-5) is uniquely solvable. The solution \(u\) in \(V\) depends continuously on the forcing term \(g\) in \(\tilde{H}^{1/2}(\Gamma)\).

**Proof:** We first show uniqueness. Assume that \(g = 0\). It then follows from (1-5) that \((\Delta + k^2)u = 0\), weakly in \(\mathbb{R}^3\). Next we extend \(u\) to \(\mathbb{R}^3\) by setting \(u(x_1, x_2, x_3) = u(x_1, x_2, -x_3)\) if \(x_3 > 0\) and \(k\) is extended in a similar fashion. Since \(u \in V\), using (2) we have that \((\Delta + k^2)u = 0\), weakly in \(\mathbb{R}^3\). Let \(S_R\) be the sphere centered at the origin with radius \(R\). Applying Green’s theorem shows that \(\text{Im} \int_{S_R} u \frac{\partial v}{\partial r} = 0\). Next, since \(u \in V\), there is a sequence \(R_n \to \infty\) such that,

\[ \lim_{n \to \infty} \int_{S_{R_n}} \left| \frac{\partial u}{\partial r} - ik_0u \right|^2 = 0, \]

so altogether we have that,

\[ \lim_{n \to \infty} \int_{S_{R_n}} \left| \frac{\partial u}{\partial r} \right|^2 + |u|^2 = 0. \]

Due to Rellich’s lemma for far field patterns, it follows that \(u(x) = 0\), if \(|x| > R_0\). Since the only regularity assumption on \(k\) is that it is in \(L^\infty\) there is no elementary argument for showing that \(u(x) = 0\) if \(|x| \leq R_0\). However, we can use results from the unique continuation literature. As in our case \(k^2\) is in \(L^\infty\), \(u\) is in \(H^2_{\text{loc}}\) so the corollary of theorem 1 in [4] can be used to claim that \(u\) is zero throughout \(\mathbb{R}^3\) since \(2 > \frac{6n-4}{3n+2}, n = 3\).

Next, we prove existence of a solution to (1-5). Fix \(R' > R_0\). Let \(B_{R'}\) the open ball centered at the origin of \(\mathbb{R}^3\) with radius \(R'\) and define the bilinear functional,

\[ B(v, w) = \int_{B_{R'}} \nabla v \cdot \nabla w - k^2 vw - \int_{S_{R'}} T_{R', k_0} vw, \]
for \( v, w \in H^1(B_{R'}) \) and where \( T_{R',k_0} \) is the Dirichlet to Neumann map for radiating solutions to the Helmholtz equation in the exterior of \( B_{R'} \) with wavenumber \( k_0 \). \( T_{R',k_0} \) is known to be a continuous mapping from \( H^\frac{1}{2}(S'_{R'}) \) to \( H^{-\frac{1}{2}}(S'_{R'}) \), while \(-T_{R',0}\) is strictly coercive, and \( T_{R',k_0} - T_{R',0} \) is compact from \( H^\frac{1}{2}(S'_{R'}) \) to \( H^{-\frac{1}{2}}(S'_{R'}) \), see \cite{5}, section 5.3, or \cite{12}, section 2.6.5. According to the uniqueness property covered above, we have that if \( v \in H^1(B_{R'}) \) and \( B(v, w) = 0 \) for all \( w \in H^1(B_{R'}) \), then \( v = 0 \).

Let \( u_g \) be in \( H^1(B_{R'} \setminus \overline{T}) \) such that \([u_g] = g\) across \( \Gamma \) and \( u_g \) is zero in a neighborhood of \( S'_{R'} \). For example, we can set

\[
 u_g(x) = \phi(x) \frac{1}{4\pi} \int_{\Gamma} \nabla_y \left( \frac{1}{|x-y|} \right) \cdot n(y) g(y) d\sigma(y),
\]

where \( \phi \in C^\infty_c(B_{R'}) \) is constantly equal to 1 in a neighborhood of \( \Gamma \). It is known that the \( H^1 \) norm of \( u_g \) is bounded by a constant times the \( H^\frac{1}{2} \) norm of \( g \), see theorem 1 in \cite{6}. Consider the variational problem,

\[
\text{find } v \in H^1(B_{R'}) \text{ such that } \forall w \in H^1(B_{R'}), \\
B(v, w) = -B(u_g, w). 
\]

(7)

We already know that this problem has at most one solution. Existence follows by arguing that this problem is in the form strictly coercive plus compact, which is the case thanks to the properties of the operator \( T_{R',k_0} \) recalled above. Next, integrating by parts will show that \((\Delta + k^2)(v + u_g) = 0\), in \( B_{R'} \setminus \overline{T} \) and that \([\frac{\partial}{\partial n}(v + u_g)] = 0\) across \( \Gamma \). \([v + u_g] = [u_g] = g\) across \( \Gamma \) is clear by construction. Thanks to the operator \( T_{R',k_0} \), \( v \) can be extended to a function in \( H^1_{loc}(\mathbb{R}^3 \setminus \overline{T}) \) such that \( \frac{v}{\sqrt{1+\xi^2}}, \frac{\nabla v}{\sqrt{1+\xi^2}}, \frac{\partial v}{\partial r} - i k_0 v \in L^2(\mathbb{R}^3 \setminus \overline{T}) \) and \((\Delta + k^2)v = 0\) in \( \mathbb{R}^3 \setminus \overline{B_{R'}} \) as well as in a neighborhood of \( S'_{R'} \). \( u_g \), which is zero in a neighborhood of \( S'_{R'} \) is just extended by zero. Finally, we set for \( x_3 < 0 \)

\[
 u(x_1,x_2,x_3) = (v + u_g)(x_1,x_2,x_3) + (v + u_g)(x_1,x_2,-x_3)
\]

to find a solution to (1-5). \( \square \)

2.2 The crack inverse problem for pressure waves in half space: formulation and uniqueness of solutions

We now prove a theorem stating that the crack inverse problem related to problem (1-5) has at most one solution. The data for the inverse problem is Cauchy data over a portion of the top plane \( \{x_3 = 0\} \). The forcing term \( g \) and the crack \( \Gamma \) are both unknown in the inverse problem.

**Theorem 2.1.** Let \( i = 1, 2, \) let \( \Gamma_i \) be a Lipschitz open surface such that its closure is in \( \mathbb{R}^{3-} \), let \( u^i \) be the unique solution to (1-5) with \( \Gamma_i \) in place of \( \Gamma \) and the jumps \( g^i \) in \( H^{1/2}(\Gamma_i) \) in place of \( g \). Let \( V \) be a non-empty relatively open subset of the top plane \( \{x : x_3 = 0\} \). Assume that \( \mathbb{R}^{3-} \setminus \overline{\Gamma_1 \cup \Gamma_2} \) is connected and that \( g^i \) has full support in \( \overline{\Gamma_i}, i = 1, 2 \). If \( u^1 = u^2 \) on \( V \), then \( \overline{\Gamma_1} = \overline{\Gamma_2} \) and \( g^1 = g^2 \) almost everywhere.
Proof: Let $U = \mathbb{R}^3^- \setminus \overline{\Gamma_1 \cup \Gamma_2}$ and set $u = u^1 - u^2$ in $U$. Since $(\Delta + k^2)u = 0$ in $U$ and $u = \partial_{x_3} u = 0$ on $V$, $u$ can be extended by zero to an open set $U'$ of $\mathbb{R}^3$ such that $U \subset U'$, $U'$ is connected, $U' \cap \{x : x_3 > 0\}$ is non-empty, $u$ is in $H^2_{\text{loc}}(U')$ and satisfies $(\Delta + k^2)u = 0$ in $U'$, where we can set $k = 0$ in $U' \cap \{x : x_3 > 0\}$. As $u$ is in $H^2_{\text{loc}}(U')$ and $k^2$ is in $L^\infty(U')$, the unique continuation property (corollary of theorem 1 in [3]) can be applied, and $u$ is zero in $U'$.

Arguing by contradiction, suppose that there is an $x$ in $\Gamma_1$ such that $x \notin \Gamma_2$. Then there is an open ball $B(x, r)$ centered at $x$ with radius $r > 0$ such that $B(x, r) \cap \overline{\Gamma_2} = \emptyset$. Since $[u] = [u^2] = 0$ across $B(x, r) \cap \overline{\Gamma_1}$, it follows that $[u^1] = 0$ across $B(x, r) \cap \overline{\Gamma_1}$: this contradicts that $g^1$ has full support in $\overline{\Gamma_1}$. We conclude that $\Gamma_1 \subset \Gamma_2$. Reversing the roles of $\Gamma_1$ and $\Gamma_2$ we then find that $\overline{\Gamma_1} = \overline{\Gamma_2}$. Using one more time that $u$ is zero in $U$, since $[u] = 0$ across $\Gamma_1 = \Gamma_2$, it follows that $g_1 - g_2 = 0$ almost everywhere in $\Gamma_1$. □.

3 Examples where $\mathbb{R}^3^- \setminus \overline{\Gamma_1 \cup \Gamma_2}$ is not connected and uniqueness for the crack inverse problem for pressure waves fails

3.1 A counter example involving half-spheres

We start from an eigenvalue for the Neumann problem in the ball $B(0, 1), k_1 > 0$, and $\psi$ an associated eigenfunction

$$(\Delta + k_1^2)\psi = 0, \text{ in } B(0, 1), \partial_\nu \psi = 0, \text{ on } B(0, 1).$$

We may choose $\psi$ to be odd in $x_3$ so that $\psi(x_1, x_2, 0) = 0$. More specifically, denoting $j_1(r) = \frac{\sin r}{r} - \cos r$ the spherical harmonic of order 1, it is known that $s(r, \theta, \phi) = j_1(r) \cos \theta$ (where in the spherical coordinates $(r, \theta, \phi)$, $\theta$ is the co-latitude ) satisfies $(\Delta + 1)s = 0$. Now let $k_1$ be the first positive zero of $j'_1$. Let $\psi(r, \theta, \phi) = j_1(k_1 r) \cos \theta$, and let $S(0, 1)$ be the unit sphere centered at the origin. Define the half spheres $S^+(0, 1) = \{x \in S(0, 1) : x_3 > 0\}$, $S^-(0, 1) = \{x \in S(0, 1) : x_3 < 0\}$.

Now we set

$$\Gamma_1 = \{x \in \mathbb{R}^3^- : x + (0, 0, 2) \in S^+(0, 1)\},$$

$$g_1(x_1, x_2, x_3) = \psi(x_1, x_2, x_3 + 2), (x_1, x_2, x_3) \in \Gamma_1,$$

and likewise,

$$\Gamma_2 = \{x \in \mathbb{R}^3^- : x + (0, 0, 2) \in S^-(0, 1)\},$$

$$g_2(x_1, x_2, x_3) = \psi(x_1, x_2, x_3 + 2), (x_1, x_2, x_3) \in \Gamma_2.$$

We decide that on $\Gamma_1$ and on $\Gamma_2$, the normal vector will have a positive third coordinate (pointing “up”). Note that by construction $g_1 \in H^2(\Gamma_1)$ and $g_2 \in H^2(\Gamma_2)$ since $\psi(x_1, x_2, 0) = 0$. Let $u'$ be the unique solution to (1.5) with $\Gamma_i$ in place of $\Gamma$ and the jumps $g^i$ in $H^{1/2}(\Gamma_i)$.
in place of \( g \). Let \( u = u^1 - u^2 \).
We now show that \( u \) is zero outside \( B(0, 1) - (0, 0, 2) \). Set
\[
v(x_1, x_2, x_3) = \begin{cases} 
\psi(x_1, x_2, x_3 + 2), & \text{if } x \in B(0, 1) - (0, 0, 2), \\
0, & \text{if } x \in \mathbb{R}^3 - \setminus B(0, 1) - (0, 0, 2).
\end{cases}
\]
By construction, \( \frac{\partial v}{\partial n} = 0 \), and \( [v] = \psi = [u] \). Since problem (1-5) is uniquely solvable, \( u = v \) in \( \mathbb{R}^3 \). We conclude that \( u^1 = u^2 \) everywhere on the top plane \( \{ x : x_3 = 0 \} \). Note that \( g_i \) has full support on \( \Gamma_i, i = 1, 2 \).

By a simple rescaling argument, for any positive constant \( k^2 \), we can likewise construct two half-spheres \( \Gamma_1, \Gamma_2 \) and forcing terms \( g_1, g_2 \) with full support in \( \Gamma_1, \Gamma_2 \) such that corresponding \( u^1, u^2 \) solving (1-5) satisfy \( u^1 = u^2 \) everywhere on the top plane \( \{ x : x_3 = 0 \} \).

3.2 A counterexample in case of cracks that are nearly flat, the two-dimensional case

In the previous counterexample the range of \( n \cdot e_3 \), where \( n \) is the normal vector to \( \Gamma_1 \) and \( e_3 = (0, 0, 1) \), was wide, namely this range was \( (0, 1] \). By contrast, in this section we construct a counterexample where this range can be made arbitrarily small. The price to pay is that constructing this counterexample requires a thorough regularity analysis on domains with cusps. We will construct this counterexample in dimension 2 to make the argument clearer and the notations easier to follow. Let \( f : [-1, 1] \rightarrow \mathbb{R}, f(t) = (t - 1)^2(t + 1)^2 \), and consider the domain with two cusps
\[
\Omega := \{(x_1, x_2) : -1 < x_1 < 1, -f(x_1) < x_2 < f(x_1)\}.
\]
\( \Omega \) is sketched in figure 2.

Figure 2: The domain \( \Omega \) bounded by \( \Gamma_1 \) and \( \Gamma_2 \) and the subdomain \( \Omega_R \).
Lemma 3.1. \( H^1(\Omega) \) is compactly embedded in \( L^2(\Omega) \).

Proof: \( \Omega \) presents two power-type cusps. This lemma holds thanks to the theory of Sobolev spaces on domains with cusps. More precisely, we refer to Maz’ya and Poborchi’s textbook, p 430, \[11\]. □

Thanks to lemma 3.1, classical arguments can be used to show that the eigenvalues for the Neumann problem for the Laplace operator in \( \Omega \) form an increasing sequence diverging to infinity. We now claim that we can find an eigenfunction which is odd in \( x_2 \):

Lemma 3.2. There exists a function \( \varphi \) in \( H^1(\Omega) \) and \( \mu > 0 \) such that

\[
\int_{\Omega} \varphi^2 = 1, \quad (\Delta + \mu^2)\varphi = 0, \quad \frac{\partial \varphi}{\partial n} = 0, \quad \text{almost everywhere on } \partial \Omega \quad \text{and} \quad \varphi(x_1, -x_2) = -\varphi(x_1, x_2), \quad \text{for } (x_1, x_2) \text{ in } \Omega.
\]

Proof: Let

\[
H^{1,\pm}(\Omega) := \{ v \in H^1(\Omega) : v(x_1, -x_2) = \pm v(x_1, x_2), (x_1, x_2) \in \Omega \}.
\]

\( H^{1,+}(\Omega) \) and \( H^{1,-}(\Omega) \) are orthogonal complements of each other with regard to the inner product in \( H^1(\Omega) \). Let

\[
\mu^2 = \inf_{v \in H^{1,-}(\Omega), \int_\Omega v^2 = 1} \int_\Omega |\nabla v|^2.
\]

Standard arguments show that a minimizing sequence for this inf converges to some \( \varphi \) in \( H^{1,-}(\Omega) \) such that for all \( \phi \in H^{1,-}(\Omega) \) \( \int_\Omega \nabla \varphi \cdot \nabla \phi - \mu^2 \varphi \phi = 0 \). Now since \( \varphi \in H^{1,-}(\Omega) \), for all \( \phi \in H^{1,+}(\Omega) \) \( \int_\Omega \nabla \varphi \cdot \nabla \phi - \mu^2 \varphi \phi = 0 \). We conclude that \( (\Delta + \mu^2)\varphi = 0 \) and \( \frac{\partial \varphi}{\partial n} = 0 \), almost everywhere on \( \partial \Omega \). □

We would now like to use \( \varphi \) to construct a relevant counterexample just in the way that \( \psi \) was used in section 3.1. However, one important point remains. Since \( \Omega \) presents cusp singularities, not all functions in \( H^1(\Omega) \) can be extended to functions in \( H^1(\mathbb{R}^2) \). The following analysis will show that such an extension is possible for \( \varphi \). The argument borrows from the theory of elliptic PDEs on domains with cusps and takes advantage of the fact that \( \varphi(x_1, 0) = 0, -1 < x_1 < 1 \), in the sense of traces.

**Lemma 3.3.** Let \( \varphi \) be as in lemma 3.2. Let \( R \in (0, 1) \) and

\[
\Omega_R := \{(x_1, x_2) \in \Omega : -1 < x_1 < -1 + R\},
\]

(see figure 3). Let \( \sigma \) be the weight \( \sigma(x_1, x_2) = \sqrt{(x_1 + 1)^2 + x_2^2} \). Then

\[
\int_{\Omega_R} \varphi^2 = O(R^4), \quad (9)
\]

and \( \sigma^{-\alpha} \varphi \) is in \( L^2(\Omega) \) for any \( \alpha < 2 \).

Proof: Let \( \eta \in C^1(\Omega) \cap H^1(\Omega) \) be such that \( \eta(x_1, 0) = 0 \). For \( 0 < x_2 < f(x_1) \),

\[
|\eta(x_1, x_2)| = \left| \int_0^{x_2} \partial_{x_2} \eta(x_1, t) dt \right| \leq |x_2|^{\frac{1}{2}} \left( \int_0^{x_2} |\partial_{x_2} \eta(x_1, t)|^2 dt \right)^{\frac{1}{2}},
\]
thus we have for $R \in (0, 1)$,
\[
\int_{-1+R}^{-1} \int_{0}^{f(x_1)} |\eta(x_1,x_2)|^2 dx_2 dx_1 \leq \int_{-1+R}^{-1} \int_{0}^{f(x_1)} (|x_2| \int_{0}^{x_2} |\partial_{x_2} \eta(x_1,t)|^2 dt) dx_2 dx_1 \\
= \int_{-1+R}^{-1} \int_{0}^{f(x_1)} \int_{0}^{f(x_1)} |\partial_{x_2} \eta(x_1,t)|^2 dt dx_2 dx_1 \\
\leq \int_{-1+R}^{-1} \int_{0}^{f(x_1)} \frac{f(x_1)}{2} dx_2 dx_1 \\
\leq 8R^4 \int_{-1+R}^{-1} \int_{0}^{f(x_1)} |\partial_{x_2} \eta(x_1,t)|^2 dt dx_1,
\]
that is, $\int_{\Omega_R} |\eta|^2 \leq 16R^4 \int_{\Omega_R} |\partial_{x_2} \eta|^2$. Estimate (9) follows by density since $\varphi(x_1,0) = 0$, $-1 < x_1 < 1$, in the sense of traces. Next,
\[
\int_{\Omega} |\varphi(x_1,x_2)|^2 ((x_1 + 1)^2 + x^2)^{-\alpha} dx_1 dx_2 = \sum_{n=0}^{\infty} \int_{-1+\frac{1}{2^n}}^{-1} \int_{-f(x_1)}^{f(x_1)} |\varphi(x_1,x_2)|^2 ((x_1 + 1)^2 + x^2)^{-\alpha} dx_1 dx_2 \\
\leq C \sum_{n=0}^{\infty} \frac{2^{(2n+2)\alpha}}{2^{4n}},
\]
is finite if $\alpha < 2$, thus $\sigma^{-\alpha} \varphi$ is in $L^2(\Omega)$, if $\alpha < 2$. \(\square\)

**Lemma 3.4.** Let $\varphi$ be as in lemma 3.2. $\sigma^{-\beta} \nabla \varphi$ is in $L^2(\Omega)$, if $\beta < 1$.

**Proof:** We know that $\int_{\Omega} \nabla \varphi \cdot \nabla \phi - \mu^2 \varphi \phi = 0$, for all $\phi$ in $H^1(\Omega)$. Let $\beta < 1$, $0 < \epsilon < 1$, and
\[
\phi_\epsilon = \begin{cases} 
\varphi \sigma^{-2\beta}, & \text{if } \sigma > \epsilon \\
\varphi \sigma^{-2\beta}, & \text{if } \sigma \leq \epsilon 
\end{cases}
\]
$\phi_\epsilon$ is in $H^1(\Omega)$. From $\int_{\Omega} \nabla \varphi \cdot \nabla \phi_\epsilon = \mu^2 \varphi \phi_\epsilon$,
\[
\int_{\Omega} (|\nabla \varphi|^2 \sigma^{-2\beta} - 2\beta \varphi \sigma^{-2\beta-1} \nabla \varphi \cdot \nabla \sigma) 1_{\sigma > \epsilon} + \int_{\Omega} |\nabla \varphi|^2 \epsilon^{-2\beta} 1_{\sigma \leq \epsilon} \\
= \mu^2 (\int_{\Omega} \varphi^2 \sigma^{-2\beta} 1_{\sigma > \epsilon} + \int_{\Omega} \varphi^2 \epsilon^{-2\beta} 1_{\sigma \leq \epsilon}),
\]
we infer,
\[
\int_{\Omega} |\nabla \varphi|^2 \sigma^{-2\beta} 1_{\sigma > \epsilon} \leq 2\beta \int_{\Omega} \varphi \sigma^{-2\beta-1} \nabla \varphi \cdot \nabla \sigma 1_{\sigma > \epsilon} + \mu^2 \int_{\Omega} \varphi^2 \sigma^{-2\beta}.
\]
Noting that $|\nabla \sigma| = 1$, applying Cauchy-Schwarz, and rearranging terms, we obtain
\[
\frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 \sigma^{-2\beta} 1_{\sigma > \epsilon} \leq 2\beta^2 \int_{\Omega} \varphi^2 \sigma^{-2\beta-2} + \mu^2 \int_{\Omega} \varphi^2 \sigma^{-2\beta}. \tag{10}
\]
Now using lemma 3.3 as $-2\beta - 2 > -4$, letting $\epsilon \to 0$, the result is proved. □

Interestingly, now that we know that $\sigma^{-\beta}\nabla \varphi$ is in $L^2(\Omega)$ for $\beta < 1$, the estimate in the proof of lemma 3.3 can be improved leading to stronger decay estimates near the cusp.

**Proposition 3.1.** Let $\varphi$ be as in lemma 3.2. $\sigma^{-\alpha} \varphi$ and $\sigma^{1-\alpha} \nabla \varphi$ are in $L^2(\Omega)$ for any $\alpha < 3$.

**Proof:** The proof of lemma 3.3 indicates that $\int_{\Omega_R} |\varphi|^2 \leq 16R^4 \int_{\Omega_R} |\partial_{x_2} \varphi|^2$. Thus for $\beta < 1$,

$$
\int_{\Omega_R} |\varphi|^2 \leq 16R^4 \int_{\Omega_R} \sigma^{2\beta} |\sigma^{-\beta} \partial_{x_2} \varphi|^2 \leq 32R^{4+2\beta} \int_{\Omega_R} |\sigma^{-\beta} \partial_{x_2} \varphi|^2,
$$

for all $R$ small enough. Following the calculation in the proof of lemma 3.3 we now have,

$$
\int_{\Omega_1} |\varphi(x_1, x_2)|^2 ((x_1 + 1)^2 + x^2)^{-\alpha} dx_1 dx_2 \leq C \sum_{\alpha\in\mathbb{N}} 2^{(2n+2)\alpha} (2n)^{4+2\beta},
$$

which is finite if $\alpha < 2 + \beta$, thus $\sigma^{-\alpha} \varphi$ is in $L^2(\Omega)$ if $\alpha < 3$. In other words, $\sigma^{-\alpha} \varphi$ is in $L^2(\Omega)$ if $\alpha < 3$. We now rewrite estimate (10) with a parameter $\gamma$,

$$
\frac{1}{2} \int_{\Omega} |\nabla \varphi|^2 \sigma^{-2\gamma} 1_{\sigma > \epsilon} \leq 2\gamma^2 \int_{\Omega} \varphi^2 \sigma^{-2\gamma - 2} + \mu^2 \int_{\Omega} \varphi^2 \sigma^{-2\gamma}.
$$

For $\gamma < 2$, the right hand side is finite and we let $\epsilon$ tend to zero. □

**Lemma 3.5.** Let $\varphi$ be as in lemma 3.2. $\varphi$ can be extended to a function in $H^1(\mathbb{R}^2)$ which is odd in $x_2$.

**Proof:** Let $\eta$ be a function in $C^\infty_c(\mathbb{R}^2)$ equal to 1 in a neighborhood of $(-1, 0)$ and zero if $x_1 > 0$. Set $\varphi' = \varphi \eta$. Let

$$
H^1_{\sigma^2}(\mathbb{R}^2) := \{ \sigma^{\frac{1}{2}} v, \sigma^{\frac{1}{2}} \nabla v \in L^2(\mathbb{R}^2) \},
$$

endowed with its natural norm,

$$
\| \sigma^{\frac{1}{2}} v \|_{L^2(\mathbb{R}^2)} + \| \sigma^{\frac{1}{2}} \nabla v \|_{L^2(\mathbb{R}^2)}.
$$

Given that the cusp of $\Omega$ at $(-1, 0)$ is of order 2, there is a continuous extension operator from $H^1(\Omega)$ to $H^1_{\sigma^2}(\mathbb{R}^2)$: see [10], section 1.5.4. Proposition 3.1 implies that $\sigma^{-\frac{1}{2}} \varphi'$ is in $H^1(\Omega)$. Let $E$ be its extension to $H^1_{\sigma^2}(\mathbb{R}^2)$. We may assume that the support of $E$ is compact. By definition of $H^1_{\sigma^2}(\mathbb{R}^2)$, $\sigma^{\frac{1}{2}} E$ and $\sigma^{\frac{1}{2}} \nabla E$ are in $L^2(\mathbb{R}^2)$. Thus $\sigma^2 E$ is in $L^2(\mathbb{R}^2)$. But $\nabla (\sigma^2 E) = \sigma^{\frac{1}{2}} \nabla E + \frac{3}{2} \sigma^{\frac{1}{2}} E \nabla \sigma$ is also in $L^2(\mathbb{R}^2)$, as $\nabla \sigma$ is bounded. Thus $\sigma^2 E$ is an extension of $\varphi'$ which is in $H^1(\mathbb{R}^2)$. The cusp at $(1, 0)$ can be handled likewise: we extend $\varphi(1 - \eta)$ to a function in $H^1(\mathbb{R}^2)$. Adding the two extensions, we obtain a function $\tilde{E}$ in $H^1(\mathbb{R}^2)$ equal to $\varphi$ in $\Omega$. Finally, $\frac{1}{2} (\tilde{E}(x_1, x_2) - \tilde{E}(x_1, -x_2))$ is in $H^1(\mathbb{R}^2)$, is odd in $x_2$, and equals $\varphi$ in $\Omega$, since $\varphi(x_1, -x_2) = -\varphi(x_1, x_2)$. □
Construction of the related counterexample for the unique solvability of the crack inverse problem.

Let \( \varphi \) be as in lemma 3.2. According to lemma 3.2, the function \( \varphi^+ \) defined by \( \varphi^+(x_1,x_2) = \varphi(x_1,x_2) \) if \( x_2 > 0, (x_1,x_2) \in \Omega, \varphi^+(x_1,x_2) = 0 \) if \( x_2 < 0, (x_1,x_2) \in \Omega, \) is still in \( H^1(\Omega) \) and according to lemma 3.5, \( \varphi^+ \) can be extended to a function in \( \tilde{H}^\frac{1}{2}(\partial \Omega^+) \) which is zero if \( x_2 < 0. \) This explains why the trace of \( \varphi \) restricted to \( \partial \Omega^+ = \{ (x_1,x_2) \in \partial \Omega : x_2 > 0 \} \) is in \( \tilde{H}^\frac{1}{2}(\partial \Omega^+) \).

Define \( \Gamma_1 = \{ (x_1,x_2) : (x_1,x_2 + 2) \in \partial \Omega^+ \}, \Gamma_2 = \{ (x_1,x_2) : (x_1,x_2 + 2) \in \partial \Omega, x_2 = 0 \}, \) and \( g_i(x_1,x_2) = \varphi(x_1,x_2 + 2), (x_1,x_2) \in \Gamma_i, i = 1, 2. \) We now know that \( g_i \) is in \( \tilde{H}^\frac{1}{2}(\Gamma_i) \). Let \( u^i \) be the unique solution to the two dimensional analog of system (1-5) with \( \Gamma \) in place of \( \Gamma \), where the continuous normal vector on \( \Gamma \) is such that \( n \cdot e_2 > 0, g^2 \) in \( \tilde{H}^{1/2}(\Gamma_i) \) is in place of \( g, k^2 = \mu^2 \) as in lemma 3.2 and the solutions are sought in the functional space

\[
\mathcal{V}_2 := \{ v \in H^1_{loc}(\mathbb{R}^2 - \bar{\Gamma}) : \frac{v}{\sqrt{1 + r^2}}, \frac{\nabla v}{\sqrt{1 + r^2}}, \frac{\partial v}{\partial r} - i \mu v \in L^2(\mathbb{R}^2 - \bar{\Gamma}) \}.
\]

Let \( u = u^1 - u^2 \). We can show that \( u \) is zero outside \( \Omega - (0,0,2) \) by repeating the same argument as in the end of section 3.1. Note that \( g_i \) has full support in \( \Gamma_i \) since \( \frac{\partial \varphi}{\partial n} = 0 \) on \( \partial \Omega \), this is again due to the corollary of theorem 1 in [1,4]. Finally, by using the rescaling \( (x_1,x_2) \rightarrow (sx_1,sx_2) \), we can find a similar counterexample for any \( k^2 > 0 \). Replacing \( f \) by \( f_a(x_1) = a(x_1 - 1)^2(x_1 + 1)^2 - 2, x_2 \in (-1,1), \) where \( 0 < a < 1 \) instead of \( f \), the same argument as above can be repeated using the domain with cusps \( \{ -f_a(x_1) < x_2 < f_a(x_1) : -1 < x_2 < 1 \} \). Picking \( a \) small can make the corresponding curves \( \Gamma_1 \) and \( \Gamma_2 \) arbitrarily flat. Note that \( \Gamma_1 \) and \( \Gamma_2 \) can be extended by line segments while preserving their \( C^1 \) regularity.

3.3 A counterexample in case of cracks that are nearly flat, the three-dimensional case

The previous two-dimensional counterexample can serve as an inspiration for constructing a three-dimensional counterexample with cylindrical symmetries. We have to implement relevant modifications to the two dimensional functional spaces that we are using. Let \((r,\theta,x_3)\) denote the cylindrical coordinates in \(\mathbb{R}^3\). Since we will focus on functions that are independent of \(\theta\), for any open subset \(U\) of \(\mathbb{R}^2\) where the points in \(U\) are denoted by \((r,x_3)\), define the space \(L^2_r(U)\) of measurable functions \(u\) in \(U\) such that \(\int_U u^2 r dr dx_3 < \infty\) equipped with the norm

\[
\|u\|_{L^2_r(U)} = \left( \int_U u^2 r dr dx_3 \right)^{\frac{1}{2}}.
\]

Similarly, let \(H^1_r(U)\) be the subspace of functions \(u\) in \(L^2_r(U)\) that have weak derivatives in \(L^2_r(U)\). \(H^1_r(U)\) will be equipped with the norm

\[
\|u\|_{H^1_r(U)} = \|u\|_{L^2_r(U)} + \left( \int_U |\nabla u|^2 r dr dx_3 \right)^{\frac{1}{2}},
\]
where $\nabla u = (\partial_r u, \partial_x u)$. Next, we define the open subset of $\mathbb{R}^2$,

$$
\Omega_0 := \{(r, x_3) : 0 < r < 1, -f(r) < x_3 < f(r)\},
$$

where $f$ is the same as in (8), and the open subset of $\mathbb{R}^3$,

$$
\Omega_{\mathbb{R}^3} := \{(r, \theta, x_3) : 0 < r < 1, 0 \leq \theta < 2\pi, -f(r) < x_3 < f(r)\}.
$$

The shape in figure 2 is then a cross-section of $\Omega_{\mathbb{R}^3}$ by any plane containing the $x_3$ axis.

**Lemma 3.6.** $H^1_r(\Omega_0)$ is compactly embedded in $L^2_r(\Omega_0)$.

**Proof:** We set $U_1 = \{(r, x_3) \in \Omega_0 : r < \frac{3}{4}\}$, $U_2 = \{(r, x_3) \in \Omega_0 : r > \frac{1}{4}\}$. Rotating $U_1$ about the $x_3$ axis, we obtain a bounded Lipschitz domain $U_{1,\mathbb{R}^3}$ in $\mathbb{R}^3$. As $H^1(U_{1,\mathbb{R}^3})$ is compactly embedded in $L^2(U_{1,\mathbb{R}^3})$, it follows that $H^1_r(U_1)$ is compactly embedded in $L^2_r(U_1)$. In $U_2$, $\frac{1}{4} < r < 1$, thus the spaces $L^2_r(U_2)$ are the same, as well as the spaces $H^1_r(U_2)$ and $H^1(U_2)$. It now follows from lemma 3.1 that $H^1_r(U_2)$ is compactly embedded in $L^2_r(U_2)$. Finally, fix a smooth function $p : (0, \infty) \rightarrow \mathbb{R}$ such that $0 \leq p \leq 1$, $p(r) = 1$, if $r < \frac{1}{4}$, $p(r) = 0$, if $r > \frac{3}{4}$. For $u$ in $H^1_r(\Omega_0)$, write $u = pu + (1-p)u$ and the result follows. □

**Lemma 3.7.** There exists a function $\varphi$ in $H^1(\Omega_{\mathbb{R}^3})$ and $\mu > 0$ such that $\int_\Omega \varphi^2 = 1$, $(\Delta + \mu^2)\varphi = 0$, $\frac{\partial \varphi}{\partial n} = 0$, almost everywhere on $\partial \Omega_{\mathbb{R}^3}$, $\varphi$ does not depend on $\theta$, and $\varphi(r, -x_3) = -\varphi(r, x_3)$, for $(r, x_3)$ in $\Omega$.

**Proof:** We introduce the two subspaces of $H^1_r(\Omega_0)$

$$
H^1_r^{\pm}(\Omega_0) := \{v \in H^1_r(\Omega_0) : v(r, -x_3) = \pm v(r, x_3), (r, x_3) \in \Omega_0\}.
$$

They are orthogonal to one another for the natural inner product in $H^1_r(\Omega_0)$. Let

$$
\mu^2 = \inf_{v \in H^1_r^{\pm}(\Omega_0), \|v\|_{L^2_r(\Omega_0)} = 1} \int_{\Omega_0} |\nabla v|^2 r dr dx_3.
$$

Again, a minimizing sequence for this inf converges to some $\varphi_0$ in $H^1_r^{\pm}(\Omega_0)$ such that for all $\phi$ in $\in H^1_r^{\pm}(\Omega_0)$

$$
\int_{\Omega_0} (\nabla \varphi_0 \cdot \nabla \phi - \mu^2 \varphi_0 \phi) r dr dx_3 = 0,
$$

and by orthogonality, this holds for all $\phi$ in $H^1_r(\Omega_0)$. We then set the extension $\varphi(r, \theta, x_3) = \varphi_0(r, x_3)$ to obtain a function in $H^1(\Omega_{\mathbb{R}^3})$ which has the desired properties. □

**Proposition 3.2.** Let $\varphi_0$ be as in the proof of lemma 3.7. Let $\sigma$ be the weight $\sigma(r, x_3) = \sqrt{(r-1)^2 + x_3^2}$. Then $\sigma^{-\alpha} \varphi_0$ and $\sigma^{1-\alpha} \nabla \varphi_0$ are in $L^2_r(\Omega_0)$ for any $\alpha < 3$. □
\textbf{Proof:} Just as its analog in the purely two-dimensional case of section 3.2 this proposition is proved in two steps. In the first step we prove it for \( \alpha < 2 \). Let \( R \in (0, 1) \) and 

\[ \Omega_{0,R} := \{(r, x_3) \in \Omega_0 : 1 - R < r < 1\}. \]

Since in \( \Omega_{0,R} \) \( r \) is bounded above by 1, the proof is nearly identical to that of lemmas 3.3 and 3.4. Next, the decay estimates can be improved since

\[ \int_{\Omega_{0,R}} |\varphi_0|^2 \leq 16R^4 \int_{\Omega_{0,R}} \sigma^{2\beta} |\sigma^{-\beta} \partial_{x_3} \varphi_0|^2 r dr dx_3 \leq 32R^{4+2\beta} \int_{\Omega_{0,R}} |\sigma^{-\beta} \partial_{x_3} \varphi_0|^2 r dr dx_3, \]

for all \( R \) small enough, and the rest of the proof follows as in the proof of proposition 3.1 with minor adjustments due to the different surface are element in the present case. \( \square \)

\textbf{Lemma 3.8.} Let \( \varphi \) be as in lemma 3.7. \( \varphi \) can be extended to a function in \( H^1(\mathbb{R}^3) \) which is odd in \( x_3 \).

\textbf{Proof:} We use again the smooth function \( p : (0, \infty) \to \mathbb{R} \) such that \( 0 \leq p \leq 1, \ p(r)=1, \) if \( r < \frac{1}{4} \), \( p(r) = 0, \) if \( r > \frac{3}{4} \). It suffices to find separate extensions of \( p \varphi \) and \( (1-p) \varphi \) from \( H^1(\Omega_{R^3}) \) to \( H^1(\mathbb{R}^3) \).

It is clear that \( p \varphi \) can be extended from \( H^1(\Omega_{R^3}) \) to \( H^1(\mathbb{R}^3) \) since the bounded domain \( \Omega_{R^3} \cap \{|r| < \frac{3}{4}\} \) is Lipschitz regular.

The function \( \sigma^{\frac{\beta}{2}} (1-p) \varphi_0 \) is in \( H^1_{\sigma^\frac{\beta}{2}}(\Omega_0) \) thanks to proposition 3.2. As it is zero if \( r < \frac{1}{4} \), it is also in \( H^1(\Omega_0) \), so using the same argument as in lemma 3.5 it can be extended to a function \( E_0 \) in \( H^1_{\sigma^\frac{\beta}{2}}(\mathbb{R}^3) \). We can assume \( E_0 \) to be compactly supported. Then, just as in the proof of lemma 3.5, we claim that \( \sigma^{\frac{\beta}{2}} E_0 \) is in \( H^1(\mathbb{R}^2) \). It is also in \( H^1_{\sigma^\frac{\beta}{2}}(\mathbb{R}^3) \), since it has compact support. Let \( \psi \) be function in \( H^1(\mathbb{R}^3) \) which is independent of the polar angle \( \theta \) and whose cross-section is \( \sigma^{\frac{\beta}{2}} E_0 \). Note that \( \psi \) equals \( (1-p) \varphi \) in \( \Omega_{R^3} \). Finally, we add the extensions of \( p \varphi \) and \( (1-p) \varphi \). We can ensure that this extension is odd in \( x_3 \) by only retaining its odd part, as in the end of the proof of lemma 3.5. \( \square \)

\textbf{Construction of the related counterexample to the unique solvability of the crack inverse problem: the three-dimensional case.}

We pretty much follow the construction in the two-dimensional case demonstrated at the end of section 3.2. For \( \varphi \) be as in lemma 3.7 set \( \varphi^+ = \varphi \) if \( x_3 > 0, \ x \in \Omega_{R^3}, \ \varphi^+ = 0 \) if \( x_3 < 0, \ x \in \Omega_{R^3}, \ \varphi^+ \) is still in \( H^1(\Omega_{R^3}) \) according to lemma 3.7 and according to lemma 3.8 \( \varphi^+ \) can be extended to a function in \( H^1(\mathbb{R}^3) \) which is zero if \( x_3 < 0 \). This explains why the trace of \( \varphi \) restricted to \( \partial \Omega^+_{R^3} = \{ x \in \partial \Omega_{R^3} : x_3 > 0 \} \) is in \( \tilde{H}^\frac{1}{2}(\partial \Omega^+_{R^3}) \).

Define \( \Gamma_i = \{(r, \theta, x_3) : (r, \theta, x_3 + 2) \in \partial \Omega^+_{R^3} \}, \Gamma_i = \{(r, \theta, x_3) : (r, \theta, x_3 + 2) \in \partial \Omega_{R^3}, x_3 + 2 < 0 \}, \) and \( g_i(r, \theta, x_3) = \varphi(r, \theta, x_3 + 2), \ (r, \theta, x_3) \in \Gamma_i, \ i = 1, 2 \). We now know that \( g_i \) is in \( \tilde{H}^\frac{1}{2}(\Gamma_i) \). Let \( u^i \) be the unique solution to system (1.5) with \( \Gamma_i \) in place of \( \Gamma \), where the continuous normal vector on \( \Gamma_i \) is such that \( n \cdot e_3 > 0, \) \( g_i^* \) in \( \tilde{H}^{1/2}(\Gamma_i) \) is in place of \( g, \) \( k^2 = \mu^2 \) as in lemma 3.7. Let \( u = u^1 - u^2 \). We can show that \( u \) is zero outside \( \Omega - (0, 0, 2) \) by repeating the same argument as in the end of section 3.1. Note that \( g_i \) has full support in \( \Gamma_i \); since \( \frac{\partial \phi}{\partial n} = 0 \) on \( \partial \Omega \), this is again due to the corollary of theorem 1 in [4].
4 The crack inverse problem for pressure waves in half-space: uniqueness of solutions except for a discrete set of frequencies

We have covered examples where two cracks $\Gamma_1$ and $\Gamma_2$ are such that $\mathbb{R}^3 \setminus \overline{\Gamma_1 \cup \Gamma_2}$ has more than one connected component and there are some $k$ and $g_1$ and $g_2$ such that if $u^1$ and $u^2$ solve problem (1-5) with respective forcing terms $g_1$ and $g_2$, we have that $u_1 = u_2$ on $\{x : x_3 = 0\}$ even if $\Gamma_1 \neq \Gamma_2$ and $g_1, g_2$ have full support in $\Gamma_1, \Gamma_2$. By contrast, we now prove that under a general requirement on the geometry of $\Gamma$ this can only occur for a discrete set of frequencies, in other words, uniqueness for the inverse problem holds for wavenumbers $tk$, except possibly for a discrete set of scaling parameters $t$. Let $\mathcal{P}$ be the class of Lipschitz open surfaces that are finite unions of polygons. Since any polygon is a finite union of triangles, $\mathcal{P}$ is equivalently the class of Lipschitz open surfaces that can be obtained from a finite triangulation. Fix $k$ in $L^\infty(\mathbb{R}^3)$ satisfying conditions $(H1)$ through $(H3)$. For $t > 0$ consider the problem

\begin{align*}
(\Delta + i^2k^2)u &= 0 \text{ in } \mathbb{R}^3 \setminus \bar{\Gamma}, \\
\partial_{x_3}u &= 0 \text{ on the surface } x_3 = 0, \\
\frac{[\partial u]}{[\partial n]} &= 0 \text{ across } \Gamma, \\
[u] &= g \text{ across } \Gamma, \\
u &\in \mathcal{V}_i,
\end{align*}

where

\begin{equation}
\mathcal{V}_i := \{v \in H^1_{loc}(\mathbb{R}^3 \setminus \bar{\Gamma}) : \frac{v}{\sqrt{1+r^2}}, \frac{\nabla v}{\sqrt{1+r^2}}, \frac{\partial v}{\partial r} - itk_0v \in L^2(\mathbb{R}^3 \setminus \bar{\Gamma})\}.
\end{equation}

**Theorem 4.1.** For $i = 1, 2$, let $\Gamma_i$ be a Lipschitz open surface of class $\mathcal{P}$ such that its closure is in $\mathbb{R}^3$, let $u^i$ be the unique solution to (11-15) with $\Gamma_i$ in place of $\Gamma$ and the jumps $g^i$ in $\dot{H}^{1/2}(\Gamma_i)$ in place of $g$. Assume that $g_i$ has full support in $\Gamma_i$. There is a discrete set $\mathcal{D}$ in $[0, \infty)$ such that if $t \notin \mathcal{D}$, if $V$ is a non-empty relatively open subset of $\{x : x_3 = 0\}$, and if $u^1 = u^2$ on $V$, then $\Gamma_1 = \Gamma_2$ and $g^1 = g^2$ almost everywhere.

**Remark:** If $\Gamma_1$ and $\Gamma_2$ are allowed to be general Lipschitz domains, then a connected component $\Omega_i$ of $\mathbb{R}^3 \setminus \overline{\Gamma_1 \cup \Gamma_2}$ may be one of those so called bad domains discussed in chapter 2 section 5 of [11] (cusps may not be of power type). In particular, $H^1(\Omega_i)$ may not be compactly embedded in $L^2(\Omega_i)$ which significantly complicates the study of Neumann eigenvalues.

**Proof** of theorem 4.1. Since $\Gamma_1, \Gamma_2$ are Lipschitz open surfaces of class $\mathcal{P}$, $\mathbb{R}^3 \setminus \overline{\Gamma_1 \cup \Gamma_2}$ has a finite number of connected components $\Omega_i$, $i = 1, \ldots, N$. If $N = 1$, the result holds due to theorem 2.1. If $N \geq 2$, we may assume that $\Omega_1$ is unbounded, while $\Omega_i$, $i \geq 2$ is bounded. By construction, each $\Omega_i$, $i \geq 2$, is a bounded Lipschitz domain. Thanks to conditions $(H1 - H2)$, we can define on $H^1(\Omega_i)$, $i \geq 2$, the inner product,

\[ B(v, w) = \int_{\Omega_i} \nabla v \cdot \nabla w + k^2 vw, \]
which is equivalent to the natural inner product. Similarly, we define on $L^2(\Omega_i)$,

$$b(v, w) = \int_{\Omega_i} k^2 vw.$$  

By Lax-Milgram’s theorem, for any continuous linear functional $F$ on $H^1(\Omega_i)$, there is a unique $TF$ in $H^1(\Omega_i)$ such that $B(TF, w) = Fw$, for all $w$ in $H^1(\Omega_i)$, and $T$ is a continuous linear operator. Since $\Omega_i$ is a Lipschitz domain, $H^1(\Omega_i)$ is compactly embedded in $L^2(\Omega_i)$, thus we can define a compact operator $K$ from $H^1(\Omega_i)$ to its dual mapping $\phi$ to the functional $w \mapsto b(\phi, w)$. $TK$ is now a compact operator from $H^1(\Omega_i)$ to $H^1(\Omega_i)$ and satisfies $B(TKv, w) = b(v, w)$, for all $v, w$ in $H^1(\Omega_i)$. It is clear that $TK$ is symmetric, positive, and definite, thus its eigenvalues form a decreasing sequence of positive numbers $\alpha_n$ converging to zero. If $\alpha$ is not an eigenvalue, then $TK - \alpha I$ is invertible. Let $N_i$ be the set of these eigenvalues. If $\alpha \notin N_i$ and $v$ is in $H^1(\Omega_i)$, if for all $w$ in $H^1(\Omega_i)$, $B((TK - \alpha I)v, w) = 0$, then $v = 0$. As $B((TK - \alpha I)v, w) = b(v, w) - \alpha B(v, w)$, it follows that if $(\Delta + (\frac{1}{\alpha} - 1)k^2)v = 0$ in $\Omega_i$ and $\frac{\partial v}{\partial n} = 0$ on $\partial \Omega_i$, then $v = 0$. We then set

$$D_i = \left\{ \sqrt{\frac{1}{\alpha_n} - 1} : n \geq 1 \right\},$$

and $D = \bigcup_{i=2}^N D_i$. Assume that $t \notin D$. Let $U = \bigcup_{i=1}^N \Omega_i$ and set $u = u^1 - u^2$ in $U$. Since $(\Delta + t^2k^2)u = 0$ in $U$ and $u = \partial_{x_3}u = 0$ on $V$, $u$ can be extended by zero to an open set $U'$ of $\mathbb{R}^3$ such that $\Omega_i \subset U'$, $U'$ is connected, $U' \cap \{x : x_3 > 0\}$ is non-empty, $u$ is in $H^2_{\text{loc}}(U')$ and satisfies $(\Delta + t^2k^2)u = 0$ in $U'$, where we can set $k = 0$ in $U' \cap \{x : x_3 > 0\}$. By the unique continuation property (corollary of theorem 1 in [4]), $u$ is zero in $U'$. Next, using (13), we find that $\frac{\partial u}{\partial n} = 0$ almost everywhere on $\partial \Omega_i$ for $i \geq 2$: as $t \notin D$, it follows that $u$ is also zero in $\Omega_i$. To finish the proof, we just need to carry out the same argument as in the proof of theorem 2.1. □.

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