CAN TIME-HOMOGENEOUS DIFFUSIONS PRODUCE ANY DISTRIBUTION?

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Abstract. Given a centred distribution, can one find a time-homogeneous martingale diffusion starting at zero which has the given law at time 1? We answer the question affirmatively if generalized diffusions are allowed.

1. Introduction

Consider a distribution with finite mean on the real line. Can this distribution be recovered as the distribution of a diffusion at time 1? Clearly not if the support of the distribution is disconnected since diffusions are continuous, but the answer is yes, and in many ways, if the target distribution is sufficiently regular. What if we also require our process to be a martingale or to be time-homogeneous? Again, easy constructions exist (see Remark 2.4 below) to show that a suitable process exists. But, what if we require our process to be both time-homogeneous and a martingale? The original motivation for us to study this problem comes from a calibration problem in mathematical finance.

Our approach involves the speed measure of a diffusion and time-changes of Brownian motion, and makes no assumptions on the regularity of the target law. Indeed, to allow for target distributions with arbitrary support the natural class of processes to consider is the class of generalized diffusions (sometimes referred to as gap diffusions) discussed below. Our main result is to show that within this class of processes there is a time-homogeneous martingale with the given distribution at time 1.

In Section 2 we introduce the class of generalised diffusions as time-changes of a Brownian motion. We also formulate our main result, Theorem 2.3 which states that given a distribution on $\mathbb{R}$ with finite mean, there exists a generalised diffusion that is a martingale and that has this distribution at time 1. In Section 3 we collect some general results about time-changes and generalised diffusions. In Section 4 we study a discrete version of the inverse problem. For a given distribution with mass only in a finite number of points, we show that there exists a time homogeneous martingale Markov chain with the given distribution at time 1. In Section 5 we consider the case of a general distribution on the real axis. By approximating the distribution with finitely supported distributions, the existence of a
solution to the inverse problem is obtained, thereby proving Theorem 2.3. In Section 6, we apply our results to solve a calibration problem in mathematical finance. In Section 7, we characterize the speed measures for which the corresponding generalised diffusion is a (local) martingale. Finally, Section 8 concludes with some open problems.

2. Construction of generalised diffusions

In this section we construct, following [6] (see also [11] and [12]), time-homogeneous generalised diffusion processes as time-changes of Brownian motion. The time-change is specified in terms of the so-called speed measure \( \nu \).

Let \( \nu \) be a nonnegative Borel measure on the real line; \( \nu \) may be finite or infinite. Let \( B_t \) be a Brownian motion starting at \( x_0 \), and let \( L^x_u \) be its local time at the point \( x \) up to time \( u \). Recall that (a.s.) \( L^x_u \) is continuous in \((u, x) \in [0, \infty) \times \mathbb{R}\) and increasing in \( u \). Define the increasing process

\[
\Gamma_u := \int_{\mathbb{R}} L^x_u \nu(dx),
\]

noting that \( \Gamma_u \in [0, \infty] \), and let

\[
A_t := \inf\{u : \Gamma_u > t\}
\]

be its right-continuous inverse. The process

\[
X_t := B_{A_t}
\]

will be called a generalised diffusion with speed measure \( \nu \).

**Example 2.1.** If \( \nu(dx) = \frac{\sigma^2}{\sigma^2(x)} dx \) for some continuous non-vanishing function \( \sigma \), then \( X_t \) is a weak solution of

\[
dX_t = \sigma(X_t) dW_t.
\]

In this case, \( X_t \) is a diffusion. This example is the motivation for calling the measure \( \nu \) ‘speed measure’, but note that \( \nu \) rather measures the inverse of the speed.

**Remarks 2.2.**

(i) Almost surely, \( L^x_u \to \infty \) for every \( x \) as \( u \to \infty \); hence if \( \nu \) is non-zero, then \( \Gamma_u \to \infty \) a.s. as \( u \to \infty \), and thus \( A_t < \infty \) and \( X_t \) is well-defined for every \( t \in [0, \infty) \) a.s. However, we have to exclude the exceptional case \( \nu = 0 \), when \( \Gamma_u = 0 \) for every \( u \) and \( A_t = \infty \) for every \( t \geq 0 \), so \( X_t \) is not defined. (For technical reasons, we allow \( \nu = 0 \) when discussing \( A_t \), but we always assume \( \nu \neq 0 \) when considering \( X_t \), sometimes without explicitly saying so.)

(ii) \( \Gamma_u \) is left-continuous (by monotone convergence) and continuous at every \( u \) such that \( \Gamma_{u+} < \infty \) (by dominated convergence); hence \( u \mapsto \Gamma_u \) is continuous everywhere except that there might exist a single (random) finite \( u_0 \) where \( \Gamma_u \) jumps to \( +\infty \) (see Example 3.6 below.)
(iii) By (2.2) and the left-continuity of $\Gamma_u$, for all $t,u \geq 0$,

$$A_t < u \iff t \geq \Gamma_u > t.$$  

Equivalently, $u \leq A_t \iff t \geq \Gamma_u$.

It follows that $\Gamma_{A_t} \leq t \leq \Gamma_{A_t^+}$ for all $t$, and thus, by (ii), $\Gamma_{A_t^+} = t$ for all $t$ such that $\Gamma_{A_t^+} < \infty$; see further Lemma 7.4.

(iv) $\Gamma_u$ is finite for all $u$ a.s. if and only if $\nu$ is a locally finite measure (also called a Radon measure), i.e., $\nu(K) < \infty$ for every compact $K$. In this case, $\Gamma_u$ is continuous by (ii) and $\Gamma_u$ is a continuous additive functional (CAF) of $B$; conversely, every CAF of Brownian motion is of this type, see [10, Theorem 22.25] and [14, Chapter X].

(v) Although $B_0 = x_0$, $A_0$ may be strictly positive, and in general $X_0 \neq x_0$, see Lemma 3.3 below for a precise result. In [6], $X_0$ is defined to be $x_0$, but this has the disadvantage of making $X_t$ possibly not right-continuous at $t = 0$. We follow here instead the standard practise of considering right-continuous processes. We define $A_{0-} = 0$ and $X_{0-} = B_{A_{0-}} = x_0$, and thus allow the possibility that $X_{0-} \neq x_0$.

(vi) We let $(\mathcal{F}_t)_{t \geq 0}$ denote the standard completed Brownian filtration. Then each $A_t$ is a $(\mathcal{F}_t)$-stopping time, and $X_t$ is adapted to the filtration $(\mathcal{G}_t) = (\mathcal{F}_{A_t})$, $t \geq 0$. (In particular, $\mathcal{G}_{0-} = \mathcal{F}_0$ is trivial.)

In the sequel, we let “stopping time” mean $(\mathcal{F}_t)$-stopping time unless we say otherwise.

(vii) Even though the process $X_t$ is constructed as a time change of Brownian motion, it is in general not necessarily a martingale or even a local martingale; see Section 7 for a detailed discussion. However, we are mainly interested in cases in which $X_t$ is a local martingale, and preferably an integrable martingale. Recall the convention just made that $X_{0-} = x_0$ while $X_0$ may be different. We say that $(X_t)_{t \geq 0}$ is a (local) martingale if $(X_t)_{t \geq 0}$ is a (local) martingale (for the filtration $(\mathcal{G}_t)$) and, further, $\mathbb{E} X_0 = X_{0-}$. (This is equivalent to the standard definition interpreted for the index set $\{0-\} \cup [0, \infty)$.)

Our main result is that any given distribution with finite mean can be obtained as the distribution of $X_1$ for some such generalised diffusion with a suitable choice of speed measure $\nu$.

**Theorem 2.3.** Let $\mu$ be a distribution on the real axis with finite mean $\overline{\mu} = \int_{\mathbb{R}} x \mu(dx)$. Then there exists a generalised diffusion $X$ such that $X_{0-} = \overline{\mu}$, $(X_t)_{0- \leq t \leq 1}$ is a martingale, and the distribution of $X_1$ is $\mu$. Furthermore:

(i) $X_0 = \overline{\mu}$ if and only if $\overline{\mu} \in \text{supp } \mu$.

(ii) $\mathbb{E} A_1 = \text{Var}(\mu) := \int_{\mathbb{R}} (x - \overline{\mu})^2 \mu(dx) \leq \infty$. In particular, $\mathbb{E} A_1 < \infty$ if and only if $\text{Var}(\mu) < \infty$.

It follows from Theorem 7.9 that, actually, $(X_t)_{0- \leq t < \infty}$ is a martingale.

**Remark 2.4.** Theorem 2.3 guarantees the existence of a time-homogeneous (generalised) diffusion which is a martingale and has a certain distribution at
time 1. Note that the problem is much easier if some of these requirements are dropped. For example, for a given distribution $\mu$, one can find a non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f(B_t)$ has law $\mu$ (in particular, $f = F^{-1}_\mu \circ \Phi$, where $F_\mu$ is the cumulative distribution function associated with $\mu$ and $\Phi$ is the normal CDF) and then the process $X_t = f(B_t)$ is time-homogeneous with distribution $\mu$ at time 1. Further, if $\mu$ has a density which is strictly positive and differentiable, then $X$ is a time-homogeneous diffusion with $dX_t = a(X_t)dB_t + b(X_t)dt$ where $a := f' \circ f^{-1}$ and $2b := f'' \circ f^{-1}$. Typically, however, it is not a martingale.

Similarly, with $f$ as above, the process $M_t = \mathbb{E}(f(B_1) \mid \mathcal{F}_t)$ is a martingale with distribution $\mu$ at time 1, but typically it is not time-homogeneous. Again, in the regular case, we have $M_t = g(B_t, t)$ for some function $g$ solving the heat equation, and $dM_t = g'(g^{-1}(M_t, t), t)dB_t$ is a time-inhomogeneous martingale diffusion, where $g^{-1}(\cdot, t)$ is the space-inverse of $g(\cdot, t)$, and $g'(\cdot, t)$ denotes the spatial derivative.

**Remark 2.5.** In [9], a similar inverse problem is studied. Instead of allowing for generalised diffusions, the authors consider the case of diffusions with regular diffusion coefficients obtaining an approximate solution to the inverse problem using variational techniques. In [3] the authors solve a related inverse problem in which the goal is to construct a generalised diffusion $Y$ such that $Y$ is a martingale and such that the distribution of $Y_{\tau_E}$ is $\mu$, where $\tau_E$ is an independent random exponential time. Stopping at an independent exponential time is more tractable than stopping at a fixed time.

**Remark 2.6.** One interpretation of our results is that we construct a stopping time $\tau \equiv A_1$ such that $B_\tau \equiv B_{A_1} \equiv X_1$ has law $\mu$. The general problem of finding a stopping time $T$ such that $B_T$ has a given distribution is known as the Skorokhod stopping problem, see e.g. [13, Section 5.3], where other constructions of such stopping times are given.

**Example 2.7.** As a very simple case, if $\mu$ is the normal distribution $N(x_0, \sigma^2)$, we may take $d\nu = \sigma^{-2}dx$, a constant multiple of the Lebesgue measure. Then $\Gamma_u = \sigma^{-2}u$ and $A_t = \sigma^2t$ (both non-random), so $X_t \equiv B_{\sigma^2t}$, cf. Example 2.1. In general, however, it seems difficult to find $\nu$ explicitly.

### 3. Preliminaries

Recall that the *support* of a measure $\nu$ on $\mathbb{R}$ is

$$\text{supp } \nu := \mathbb{R} \setminus \bigcup \{U \subseteq \mathbb{R} : U \text{ is open and } \nu(U) = 0\}.$$ 

In other words, $x \in \text{supp } \nu$ if and only if every neighbourhood of $x$ has positive measure.

Similarly, we define the *infinity set* $\text{supp}_\infty \nu$ by

$$\text{supp}_\infty \nu := \mathbb{R} \setminus \bigcup \{U \subseteq \mathbb{R} : U \text{ is open and } \nu(U) < \infty\}.$$
In other words, \( x \in \text{supp}_\infty \nu \) if and only if every neighbourhood of \( x \) has infinite measure. Thus, \( \text{supp}_\infty \nu = \emptyset \) if and only if \( \nu \) is locally finite. By definition, \( \text{supp} \nu \) and \( \text{supp}_\infty \nu \) are closed subsets of \( \mathbb{R} \).

If \( S \) is any Borel set in \( \mathbb{R} \), we let \( H_S := \inf \{ t \geq 0 : B_t \in S \} \) denote the hitting time of \( S \) (for the Brownian motion \( B_t \)). Note that if \( S \neq \emptyset \), then \( H_S < \infty \) a.s. We will only consider cases when \( S \) is closed, and then \( B_{H_S} \in S \). For \( x \in \mathbb{R} \), we write \( H_x \) for \( H_{\{x\}} \).

**Lemma 3.1.** If \( \nu \neq 0 \), then a.s. \( X_t = B_{A_t} \in \text{supp} \nu \) for all \( t \geq 0 \).

**Proof.** If \( s > 0 \) and \( B_s \notin \text{supp} \nu \), then there exists \( \varepsilon > 0 \) such that \( B_u \notin \text{supp} \nu \) for \( u \in [s - \varepsilon, s + \varepsilon] \), and then \( L^x_u \) is constant for \( u \in [s - \varepsilon, s + \varepsilon] \) for each \( x \in \text{supp} \nu \); hence \( \Gamma_u \) is constant for \( u \in [s - \varepsilon, s + \varepsilon] \), and thus \( A_t \neq s \) for all \( t \geq 0 \).

Similarly, if \( s = 0 \) and \( B_s \notin \text{supp} \nu \), then there exists \( \varepsilon > 0 \) such that \( \Gamma_u = 0 \) for \( u \in [0, \varepsilon] \). Therefore \( A_t > 0 = s \) for all \( t \geq 0 \). \( \square \)

**Lemma 3.2.** If \( T \) is any finite stopping time for \( B_t \), then a.s., for all \( \varepsilon > 0 \),

\[
L^x_{T+\varepsilon} > L^x_T \geq 0.
\]

Consequently, there is for every \( \varepsilon > 0 \) a.s. a (random) \( \delta > 0 \) and an open set \( U \) containing \( B_T \) such that \( L^x_{T+\varepsilon} - L^x_T \geq \delta \) for all \( x \in U \).

**Proof.** The first claim is an immediate consequence of the strong Markov property and the fact that \( L^x_T > 0 \) a.s. for a Brownian motion started at \( x \) and any \( \varepsilon > 0 \).

The second claim follows since \( L \) is continuous. \( \square \)

**Lemma 3.3.** \( A_0 = H_{\text{supp} \nu} \), the hitting time of the support of \( \nu \), a.s. In particular, if \( x_0 \in \text{supp} \nu \), then \( A_0 = 0 \) and \( X_0 = x_0 \) a.s., but if \( x_0 \notin \text{supp} \nu \), then \( A_0 > 0 \) and \( X_0 \neq x_0 \) a.s.

**Proof.** If \( u < H = H_{\text{supp} \nu} \), then \( \Gamma_u = 0 \). On the other hand, by Lemma 3.2, for any \( \varepsilon > 0 \), \( L^x_{H+\varepsilon} > 0 \) a.s. and, moreover, \( L^x_{H+\varepsilon} > 0 \) for \( x \) in a neighbourhood \( U \) of \( B_H \). Since \( B_H \in \text{supp} \nu \), we have \( \nu(U) > 0 \) and thus \( \Gamma_{H+\varepsilon} > 0 \). In other words, a.s. \( \Gamma_u > 0 \) for all \( u > H \).

The definition (2.2) of \( A_t \) now shows that \( A_0 = H \) a.s., and the result follows, recalling Lemma 3.1. \( \square \)

**Lemma 3.4.** We have \( \mathbb{E} X_0 = x_0 = X_{0-} \) if and only if neither \( \text{supp} \nu \subset (-\infty, x_0) \) nor \( \text{supp} \nu \subset (x_0, \infty) \).

**Proof.** If \( \text{supp} \nu \) intersects both \([x_0, \infty)\) and \((-\infty, x_0] \), then \( B_{u \wedge H_{\text{supp} \nu}} \) is a bounded martingale and thus \( \mathbb{E} X_0 = \mathbb{E} B_{H_{\text{supp} \nu}} = B_0 = x_0 \).

On the other hand, if \( \text{supp} \nu \subset (-\infty, x_0) \), then Lemma 3.1 implies \( x_0 < X_0 \) a.s., so \( \mathbb{E} X_0 < x_0 \). Similarly, if \( \text{supp} \nu \subset (x_0, \infty) \), then \( \mathbb{E} X_0 > x_0 \). \( \square \)
We define, for given $\nu$ and $x_0$,
\begin{align}
x_+ := \inf \{ x \geq x_0 : x \in \text{supp}\nu \}, \\
x_- := \sup \{ x \leq x_0 : x \in \text{supp}\nu \}, \\
x_+^\infty := \inf \{ x \geq x_0 : x \in \text{supp}_\infty\nu \}, \\
x_-^\infty := \sup \{ x \leq x_0 : x \in \text{supp}_\infty\nu \}.
\end{align}

Note that these may be $\pm \infty$ (when the corresponding sets are empty). In general,
\[-\infty \leq x_-^\infty \leq x_- \leq x_0 \leq x_+ \leq x_+^\infty \leq \infty.\]

It follows from Lemmas 3.3 and 3.4 that we have the following cases:

(i) If $x_0 \in \text{supp}\nu$ (i.e., $x_- = x_+ = x_0$), then $X_0 = x_0$.
(ii) If $\text{supp}\nu \subset (x_0, \infty)$ (i.e., $x_- = -\infty$ and $x_0 < x_+ < \infty$), then $X_0 = x_+$.
(iii) If $\text{supp}\nu \subset (-\infty, x_0)$ (i.e., $x_+ = +\infty$ and $-\infty < x_- < x_0$), then $X_0 = x_-$.
(iv) Otherwise (i.e., if $-\infty < x_- < x_0$ and $x_0 < x_+ < \infty$), then $X_0 \in \{ x_- , x_+ \}$, with the unique distribution satisfying $\mathbb{E}X_0 = x_0$.

**Lemma 3.5.** Let $H = H_{\text{supp}_\infty\nu}$ be the hitting time for $B_t$ of the infinity set of $\nu$. Then a.s. $\Gamma_u = \infty$ for all $u > H$, and thus $A_t \leq H$ for all $t \geq 0$.

On the other hand, $\Gamma_u < \infty$ a.s. for all $u < H$.

**Proof.** If $H < \infty$, then $B_H \in \text{supp}_\infty\nu$. For any $\varepsilon > 0$, by Lemma 3.2, $L_{H+\varepsilon}^x \geq \delta > 0$ for $x$ in a neighbourhood $U$ of $B_H$ and some $\delta > 0$. Since $B_H \in \text{supp}_\infty\nu$, we have $\nu(U) = \infty$ and thus $\Gamma_{H+\varepsilon} \geq \int_U L_{H+\varepsilon}^x \nu(dx) = \infty$.

The definition (2.2) of $A_t$ now shows that $A_t \leq H$ a.s.

If $u < H$, then $K := \{ B_s : 0 \leq s \leq u \}$ is a compact interval disjoint from $\text{supp}_\infty\nu$, and thus $\nu(K) < \infty$. Since $x \mapsto L_0^x$ is continuous and vanishes outside $K$, we have $\Gamma_u = \int_K L_0^x \nu(dx) < \infty$. \hfill $\square$

**Example 3.6.** Let $\nu$ be an infinite point mass at $x_1 \in \mathbb{R}$. Then $\text{supp}\nu = \text{supp}_\infty\nu = \{ x_1 \}$. If $H_{x_1}$ is the hitting time of $x_1$, then $\Gamma_u = 0$ for $u < H_{x_1}$, cf. Lemma 3.3 and its proof, but a.s. $\Gamma_u = \infty$ for $u > H_{x_1}$ by Lemma 3.5.

Hence, a.s., $A_t = H_{x_1}$ and $X_t = x_1$ for every $t \geq 0$.

In particular, if $x_1 = x_0$, then $H_{x_1} = 0$ a.s., and thus $A_t = 0$ a.s. for all $t \geq 0$.

More generally, if $\nu$ is a measure such that $\nu(S) = 0$ or $\nu(S) = \infty$ for every Borel set $S$, then $\text{supp}_\infty\nu = \text{supp}\nu$, and Lemmas 3.5 and 3.3 imply that a.s. $A_t = H_{\text{supp}\nu}$ for all $t \geq 0$, so $X_t = X_0$ is the first point of $\text{supp}\nu$ hit by $B_u$.

**Lemma 3.7.** Suppose that $\text{supp}_\infty\nu \cap (-\infty, x_0]$ and $\text{supp}_\infty\nu \cap [x_0, \infty)$ are both non-empty. Then $B_{A_t \wedge u} \in [x_-^\infty, x_+^\infty]$ for all $t, u \geq 0$, and thus $(B_{A_t \wedge u})_{u \geq 0}$ is a bounded martingale for each fixed $t$. 

Proof. Recall $x^\infty_t$ from (3.3)–(3.4). By assumption, $-\infty < x^\infty_0 \leq x^\infty_t < \infty$. Let $H = H_{supp_\nu} = H_{[x^\infty_0,x^\infty_t]}$. Then $x^\infty_u \leq B_u \leq x^\infty_\pm$ for all $u \leq H$; thus Lemma 5.5 implies that $x^\infty_u \leq B_u \wedge A_t \leq x^\infty_\pm$ for any $u \geq 0$ and $t \geq 0$. Finally, $(B_{A_t \wedge u})_{u \geq 0}$ is a martingale since $A_t$ is a stopping time. \qed

Lemma 3.8. If $E A_t < \infty$, then $E X_t = x_0$ and $\text{Var } X_t = E(X_t - x_0)^2 = E A_t < \infty$.

Proof. This is an instance of Wald’s lemmas, see e.g. [13, Theorems 2.44 and 2.48]. \qed

Lemma 3.9. If $E A_t < \infty$ for some $t_0 < \infty$, then $(X_t)_{0- \leq t \leq t_0}$ is a square integrable martingale.

Proof. Since each $A_t \wedge n$ is a bounded stopping time, if $0- \leq s < t$, then

$$
E(B_{A_t \wedge n} | F_{A_s}) = B_{A_t \wedge n \wedge A_s} = B_{A_s \wedge n}
$$

a.s. (see e.g. [10] Theorem 7.29), so $B_{A_t \wedge n}$, $t \geq 0-$, is a martingale. Furthermore, by Wald’s lemma, i.e., since $(B_t - x_0)^2 - t$ is a martingale,

$$
E(B_{A_t \wedge n} - x_0)^2 = E(A_t \wedge n).
$$

It follows that for any fixed $t \leq t_0$, $E(B_{A_t \wedge n} - x_0)^2 \leq E A_t < \infty$; hence the variables $B_{A_t \wedge n}$, $n \geq 1$, are uniformly integrable, and thus $B_{A_t \wedge n} \rightarrow B_{A_t}$ in $L^1$. If $0- \leq s \leq t \leq t_0$, we thus obtain, by letting $n \rightarrow \infty$ in (3.5),

$$
E(B_t - x_0)^2 = E(B_t | F_{A_s}) = B_{A_s} \text{ a.s.}
$$

Thus $X_t = B_{A_t}$, $0- \leq t \leq t_0$, is an integrable martingale; it is square integrable by Lemma 3.8. \qed

Note that the converse to Lemma 3.8 does not always hold: we may have $\text{Var } X_t < \infty$ also when $E A_t = \infty$. For example, this happens in Example 3.6 if $x_1 \neq 0$. We give a simple sufficient condition for $E A_t < \infty$.

Lemma 3.10. Suppose that $\text{supp}_\nu \cap (-\infty, x_0]$ and $\text{supp}_\nu \cap [x_0, \infty)$ both are non-empty. Then $(X_t)_{t \geq 0-}$, is a bounded martingale with $E X_t = x_0$ and $E A_t = E(X_t - x_0)^2 < \infty$ for every $t \geq 0$.

Proof. Lemma 3.7 shows that $x^\infty_t \leq X_t = B_{A_t} \leq x^\infty_t$, for any $t \geq 0$, so $(X_t)$ is uniformly bounded.

For each $n$, $A_t \wedge n$ is a bounded stopping time and (3.6) holds. Letting $n \rightarrow \infty$, we find $E(X_t - x_0)^2 = E A_t$ by dominated and monotone convergence, and thus $E A_t \leq \max\{(x_0 - x^\infty_0)^2, (x^\infty_0 - x_0)^2\} < \infty$. Finally, Lemma 3.9 shows that $X_t$ is a martingale. \qed

We have defined $A_t$ in (2.22) so that it is right-continuous. The corresponding left-continuous process is

$$
A_t^- := \inf\{u \geq 0 : \Gamma_u \geq t\};
$$

note that for $t > 0$, $A_t^- = \lim_{s \nearrow t} A_s$, while $A_0^- = 0$ (as defined in Remark 2.24(v) above), and $A_t^- \leq t$ is a stopping time. It is possible that $A_t^- < A_t$, i.e. that $A_t$ jumps; this corresponds to time intervals where $\Gamma_u$ is constant,
because $B_u$ moves in the complement of supp $\nu$, so unless supp $\nu = \mathbb{R}$, it will a.s. happen for some $t$. However, the next lemma shows that there is a.s. no jump for a fixed $t > 0$. (Equivalently, for a fixed $t > 0$, there is a.s. at most one $u > 0$ such that $\Gamma_u = t$.)

**Lemma 3.11.** Let $t$ be fixed with $0 < t < \infty$. Then a.s. $A_{t^-} = A_t$.

Note that the result fails for $t = 0$, see Lemma 3.3.

**Proof.** Let $\varepsilon > 0$. Since $A_{t^-}$ is a stopping time, Lemma 3.2 shows that a.s. there exists a neighbourhood $U$ of $B_{A_{t^-}}$ and some $\delta > 0$ such that $L_{A_{t^-}+\varepsilon}^x \geq L_{A_{t^-}}^x + \delta$ for $x \in U$. Further, since $A_{t-1/n} \to A_{t^-}$ as $n \to \infty$, Lemma 3.1 implies that a.s. $B_{A_{t^-}} \in \text{supp} \nu$ and thus $\nu(U) > 0$; hence either $\Gamma_{A_{t^-}} = \infty$ or

$$\Gamma_{A_{t^-}+\varepsilon} \geq \Gamma_{A_{t^-}} + \delta \nu(U) > \Gamma_{A_{t^-}}. \quad (3.8)$$

If $\Gamma_{A_{t^-}+\varepsilon} < \infty$, then $\Gamma_u$ is continuous at $u = A_{t^-}$, see Remarks 2.2(ii), and it follows from (3.7) that $\Gamma_{A_{t^-}} \geq t$ (and actually $\Gamma_{A_{t^-}} = t$); thus (3.8) yields $\Gamma_{A_{t^-}+\varepsilon} > t$. This is trivially true also when $\Gamma_{A_{t^-}+\varepsilon} = \infty$.

Thus, a.s., $\Gamma_{A_{t^-}+\varepsilon} > t$, which implies that that $A_t \leq A_{t^-} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, and $A_{t^-} \leq A_t$, the result follows.

When considering several speed measures $\nu_n$, we use $n$ as a superscript to denote the corresponding $\Gamma_{u_n}$, $A_{t_n}$ and $X_{t_n}$; we use always the same $B_t$.

If $S$ is a topological space (in our case $\mathbb{R}$ or an interval in $\mathbb{R}$), we let $C_c(S)$ denote the space of continuous functions $S \to \mathbb{R}$ with compact support, and $C_c^+(S)$ the subset of such functions $S \to [0, \infty)$.

**Lemma 3.12.** Let $\nu, \nu_1, \nu_2, \ldots$ be a sequence of measures on $\mathbb{R}$. Assume either

(i) $\int_{\mathbb{R}} \varphi \, d\nu_n \to \int_{\mathbb{R}} \varphi \, d\nu$ as $n \to \infty$ for every $\varphi \in C_c^+(\mathbb{R})$,

or, more generally,

(ii) there exists an interval $(a, b)$ with $-\infty \leq a < x_0 < b \leq \infty$ such that $\int_{\mathbb{R}} \varphi \, d\nu_n \to \int_{\mathbb{R}} \varphi \, d\nu$ as $n \to \infty$ for every $\varphi \in C_c^+(a, b)$, and also for every $\varphi \in C_c^+(\mathbb{R})$ such that $\varphi(a) > 0$ or $\varphi(b) > 0$.

Then, for each $t > 0$, $A_{t_n} \to A_t$ a.s., and thus, if $\nu, \nu_1, \nu_2, \ldots$ are non-zero, $X_{t_n} \to X_t$ a.s., where $X, X^1, X^2, \ldots$ are the corresponding generalised diffusions constructed from the same Brownian motion.

**Proof.** The local time $L_u^x \in C_c^+(\mathbb{R})$, as a function of $x$, for every $u \geq 0$. In (i) we thus have, for every $u \geq 0$,

$$\Gamma_u^n = \int_{\mathbb{R}} L_u^x \nu_n(dx) \to \int_{\mathbb{R}} L_u^x \nu(dx) = \Gamma_u. \quad (3.9)$$

In (ii), let $H = H_{\{a, b\}}$ be the hitting time of $\{a, b\}$. If $u < H$, then the support of $L_u^x$ is contained in $(a, b)$, so $L_u^x \in C_c^+(a, b)$ and $\Gamma_u^n \to \Gamma_u$ as in (3.9).
If \( u > H \), then a.s. \( L_H^u > 0 \) by Lemma 3.2. Since \( \varphi(x) := L_H^x \) is continuous and \( B_H = a \) or \( B_H = b \), the assumption shows that (3.9) holds in this case too.

Hence, in both (i) and (ii), \( \Gamma^n_u \to \Gamma_u \) for all \( u \geq 0 \) except possibly when \( u = H \).

Let \( s = A_t \). By (2.2), \( \Gamma_u > t \) for \( u > s \) and \( \Gamma_u \leq t \) for \( u < s \). Further, if \( \Gamma_u = t \) for some \( u < s \), then \( A_{t-} \leq u < s = A_t \), which has probability 0 by Lemma 3.11. Consequently, a.s. \( \Gamma_u < t \) if \( u < s \).

Assume for simplicity that \( s = A_t < \infty \). (The case \( A_t = \infty \) is similar.) If \( \varepsilon > 0 \) and \( s + \varepsilon \neq H \), then thus \( \Gamma^n_{s+\varepsilon} \to \Gamma_{s+\varepsilon} > t \). Hence, for sufficiently large \( n \), \( \Gamma^n_{s+\varepsilon} > t \) and thus \( A^n_t \leq s + \varepsilon \). Since \( \varepsilon > 0 \) is almost arbitrary, it follows that \( \limsup_{n \to \infty} A^n_t \leq s \) a.s. Similarly, considering \( s - \varepsilon \neq H \), it follows that \( \liminf_{n \to \infty} A^n_t \geq s \) a.s. Consequently, \( A^n_t \to A_t \) a.s. as \( n \to \infty \).

Finally, if \( A_t < \infty \), then \( X^n_t = B_{A^n_t} \to B_{A_t} = X_t \) a.s. follows, since \( B_u \) is continuous.

Lemma 3.13. Let \( a > b > x_0 \) and \( \delta > 0 \). Then there exists a constant \( C = C(a,b,x_0,\delta) \) such that if \( \mathbb{P}(X_1 \geq a) \geq \delta \), then \( \nu[x_0,b] \leq C \).

Proof. Assume for convenience \( x_0 = 0 \). By replacing \( a \) by \((a+b)/2\), we may also assume that \( \mathbb{P}(X_1 > a) \geq \delta \).

Let \( H = H_a = \inf\{u : B_u = a\} \). By definition, \( X_1 = B_{A_1} \), so if \( X_1 > a \), then \( H < A_1 \) and thus \( \Gamma_H \leq 1 \). Consequently, \( \mathbb{P}(\Gamma_H \leq 1) \geq \delta \).

The local time \( L_H^x \) is a continuous function of \( x \in \mathbb{R} \), and it is a.s. strictly positive on \([0,a)\) by Ray’s theorem [13 Thm 6.38] (a consequence of the Ray–Knight theorem which gives its distribution). Hence, \( Y = \inf_{x \in [0,b]} L_H^x \geq 0 \) a.s., and thus there is a constant \( c > 0 \) such that \( \mathbb{P}(Y < c) < \delta \).

Hence, with positive probability \( \Gamma_H \leq 1 \) and \( Y \geq c \). However, then

\[
1 \geq \Gamma_H = \int_{\mathbb{R}} L_H^x \nu(dx) \geq \int_0^b L_H^x \nu(dx) \geq Y \nu[0,b] \geq c \nu[0,b],
\]

so \( \nu[0,b] \leq 1/c \).

Lemma 3.14. For every \( K > 0 \) there exists \( \kappa = \kappa(K) > 0 \) such that if \( \nu \) is a speed measure such that \( \mathbb{E}[X_1] \leq K \), and further \( \text{supp}_{\infty} \nu \cap (-\infty,x_0] \) and \( \text{supp}_{\infty} \nu \cap [x_0,\infty) \) both are non-empty, then \( \nu[x_0 - 2K,x_0 + 2K] \geq \kappa \).

Proof. We may assume that \( x_0 = 0 \). By Lemma 3.7, \( (B_{u \land A_1})_{u \geq 0} \) is a bounded, and thus uniformly integrable, martingale, closed by \( B_{A_1} = X_1 \).

Let \( \tilde{H} = H_{\{\pm 2K\}} \) be the hitting time of \( \pm 2K \). Then

\[
\mathbb{P}(\Gamma_{\tilde{H}} \leq 1) = \mathbb{P}(\tilde{H} \leq A_1) \leq \mathbb{P}(\sup_u |B_{u \land A_1}| \geq 2K) \leq \frac{\mathbb{E}[B_{A_1}]}{2K} \leq \frac{K}{2K} = \frac{1}{2}.
\]

(3.10)

Let \( Y = \max_x L_{\tilde{H}}^x \); this is a finite random variable so there exists \( c > 0 \) such that \( \mathbb{P}(Y > c) < 1/2 \). (Note that \( Y \) and \( c \) depend on \( K \) but not on \( \nu \).) With positive probability we thus have both \( \Gamma_{\tilde{H}} > 1 \) and \( Y \leq c \). This
implies, since \( L^x_H = 0 \) when \( |x| > 2K \),
\[
1 < \Gamma_H = \int_{-2K}^{2K} L^x_H \nu(dx) \leq \int_{-2K}^{2K} Y \nu(dx) \leq c\nu[-2K,2K],
\]
and the result follows with \( \kappa = c^{-1} \). \( \square \)

4. THE DISCRETE CASE

In this section we treat the inverse problem in a discrete setting. We fix points \( y_0 < y_1 < y_2 < \cdots < y_{n+1} \) and consider discrete speed measures \( \nu = \sum_{i=0}^{n+1} b_i \delta_{y_i} \), where \( \delta_a \) is a unit point mass at the point \( a \) and \( b_i \) takes values in \([0, \infty)\). We assume that \( b_0 = b_{n+1} = \infty \). We also fix a starting point \( x_0 \in (y_0, y_{n+1}) \). (We could for simplicity assume that \( x_0 = y_0 \) for some \( i_0 \in \{1, 2, \ldots, n\} \), but that is not necessary.)

Given such a speed measure \( \nu \) and \( x_0 \), we construct a generalised diffusion \( X \) as described in Section 2 above. By Lemma 3.1, the process \( (X_t)_{t \geq 0} \) only takes values in the set \( \{y_i\}_{i=0}^{n+1} \). Moreover, since \( b_0 = b_{n+1} = \infty \), the states \( y_0 \) and \( y_{n+1} \) are absorbing, so \( X \) is bounded, and it follows from Lemma 3.2 (or Theorem 7.3) that \( X \) is a martingale; in particular \( \mathbb{E} X_t = x_0 \). Let \( p_i = \mathbb{P}(X_1 = y_i) \) be the probability that \( X \) at time 1 is in state \( y_i \). Then \( 0 \leq p_i \leq 1 \), \( \sum_{i=0}^{n+1} p_i = 1 \), and we also have \( \sum_{i=0}^{n+1} y_i p_i = \mathbb{E} X_1 = x_0 \).

This defines a mapping \( G \) from the set of speed measures above to the set of distributions with mean \( x_0 \). More precisely, we write \( G(b_1, \ldots, b_n) : = (p_0, \ldots, p_{n+1}) \), and note that \( G : B^n \to \Pi^n \), where \( B^n : = [0, \infty]^n \) and
\[
\Pi^n : = \{ \pi = (\pi_0, \ldots, \pi_{n+1}) \in [0,1]^{n+1} : \sum_{i=0}^{n+1} \pi_i = 1 \text{ and } \sum_{i=0}^{n+1} y_i \pi_i = x_0 \}.
\]

Lemma 4.1. The function \( G : B^n \to \Pi^n \) is continuous.

Proof. This is a direct consequence of Lemma 3.12. (Note that the possibility that one or several \( b_i = \infty \) is no problem when we verify condition (i).) \( \square \)

We will use algebraic topology to show that \( G \) is surjective; see e.g. Chapter IV for standard definitions and results used below. We begin by studying the sets \( B^n \) and \( \Pi^n \). The set \( B^n \) is homeomorphic to the unit cube \([0,1]^n\) with the boundary \( \partial B^n : = [0,\infty]^n \setminus (0,\infty)^n \) corresponding to the boundary \( \partial[0,1]^n : = [0,1]^n \setminus (0,1)^n \). We write
\[
\partial B^n = \bigcup_{j=1}^n (\partial_{j0} B^n \cup \partial_{j\infty} B^n)
\]
where \( \partial_{js} B^n : = \{(b_i)_{i=0}^n \in B^n : b_s = s\} \).

The set \( \Pi^n \) is the intersection \( \Delta_{n+1} \cap M_{x_0} \) of the simplex
\[
\Delta_{n+1} : = \{ (\pi_i)_{i=0}^{n+1} : \pi_i \geq 0 \text{ and } \sum_{i=0}^{n+1} \pi_i = 1 \}
\]
and the hyperplane

\[ M_{x_0} := \{ (\pi_i)_{i=0}^{n+1} : \sum_{i=0}^{n+1} y_i \pi_i = x_0 \} \]

in \( \mathbb{R}^{n+2} \). Further, \( \Delta_{n+1} \) lies in the hyperplane \( L := \{ (\pi_i)_{i=0}^{n+1} : \sum_{i=0}^{n+1} \pi_i = 1 \} \). Thus \( \Pi^n \) is a compact convex set in the \( n \)-dimensional plane \( L \cap M_{x_0} \), which can be identified with \( \mathbb{R}^n \). Since \( y_0 < x_0 < y_{n+1} \), \( M_{x_0} \) contains an interior point of \( \Delta_{n+1} \), so \( \Pi^n = \Delta_{n+1} \cap M_{x_0} \) has a non-empty relative interior in \( L \cap M_{x_0} \). Thus \( \Pi^n \) is an \( n \)-dimensional compact convex set in \( L \cap M_{x_0} \) and its boundary is

\[ \partial \Pi^n = \Pi^n \cap \partial \Delta_{n+1} = \bigcup_{j=0}^{n+1} \partial_j \Pi^n, \]

where \( \partial_j \Pi^n := \{ (\pi_i)_{i=0}^{n+1} : \pi_j = 0 \} \).

Consequently, both \( B^n \) and \( \Pi^n \) are homeomorphic to compact convex sets in \( \mathbb{R}^n \) with non-empty interiors. Every such set is homeomorphic to the unit ball \( D^n := \{ x \in \mathbb{R}^n : |x| \leq 1 \} \) via a homeomorphism mapping the boundary onto the boundary \( \partial D^n = S^{n-1} \). Thus there are homeomorphisms \( (B^n, \partial B^n) \approx (D^n, S^{n-1}) \) and \( (\Pi^n, \partial \Pi^n) \approx (D^n, S^{n-1}) \).

**Lemma 4.2.** The function \( G \) maps \( \partial B^n \) into \( \partial \Pi^n \), and thus \( G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n) \).

**Proof.** If \( (b_i)_{i=1}^n \in \partial B^n \), then \( y_j \notin \text{supp} \nu \), so \( X_1 \neq y_j \) a.s. by Lemma 3.1 and thus \( \pi_j = 0 \), so \( (\pi_i)_{i=0}^{n+1} \in \partial_j \Pi^n \).

If \( (b_i)_{i=1}^n \in \partial B^n \), then \( y_j \in \text{supp} \nu \). If further \( y_j \leq x_0 \), then Lemma 3.5 implies that \( X_1 \geq y_j \) a.s., and thus \( \pi_i = 0 \) for \( 0 \leq i < j \) and, e.g., \( (\pi_i)_{i=0}^{n+1} \in \partial_{j-1} \Pi^n \). Similarly, if \( y_j \geq x_0 \), then \( \pi_i = 0 \) for \( j < i \leq n + 1 \) and \( (\pi_i)_{i=0}^{n+1} \in \partial_{j+1} \Pi^n \).

The homeomorphisms above induce isomorphisms of the relative homology groups \( H_n(B^n, \partial B^n) \approx H_n(\Pi^n, \partial \Pi^n) \approx H_n(D^n, S^{n-1}) \approx \mathbb{Z} \). The mapping degree of the function \( G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n) \) can thus be defined as the integer \( \deg(G) \) such that the homomorphism \( G_* : H_n(B^n, \partial B^n) \to H_n(\Pi^n, \partial \Pi^n) \) corresponds to multiplication by \( \deg(G) \) on \( \mathbb{Z} \). More precisely, this defines the mapping degree up to sign; the sign depends on the orientation of the spaces, but we have no reason to care about the orientations so we ignore them and the sign of \( \deg(G) \).

**Lemma 4.3.** For any \( n \geq 1 \), any \( y_0 < y_1 < \cdots < y_{n+1} \), and \( x_0 \in (y_0, y_{n+1}) \), \( \deg(G) = \pm 1 \).

**Proof.** We use induction on the dimension \( n \). We sometimes write \( G = G_n \) for clarity. For the induction step, we assume \( n \geq 2 \). The long exact
homology sequence yields the commutative diagram

\[
\begin{array}{c}
0 = H_n(B^n) \longrightarrow H_n(B^n, \partial B^n) \xrightarrow{\partial} H_{n-1}(\partial B^n) \longrightarrow H_{n-1}(B^n) = 0 \\
\downarrow G_* \quad \quad \quad \quad \downarrow G_* \\
0 = H_n(\Pi^n) \longrightarrow H_n(\Pi^n, \partial \Pi^n) \xrightarrow{\partial} H_{n-1}(\partial \Pi^n) \longrightarrow H_{n-1}(\Pi^n) = 0
\end{array}
\]

where the rows are exact; thus the connecting homomorphisms \( \partial \) are isomorphisms, and the degree of \( G : (B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n) \) equals the degree of the restriction \( G : \partial B^n \to \partial \Pi^n \).

Assume \( x_0 < y_n \). (Otherwise \( x_0 \geq y_n > y_1 \), and we may argue similarly using \( \partial \_\infty B^n \) and \( \partial_0 \Pi^n \).) We single out the faces \( \partial_\infty B^n \) and \( \partial_{n+1} \Pi^n \) of the boundaries and define \( \partial_* B^n := \bigcup_{i=0}^n \partial_i B^n \cup \bigcup_{i=1}^{n-1} \partial_i \_\infty B^n \) and \( \partial_* \Pi^n := \bigcup_{i=0}^n \partial_i \Pi^n \). By the proof of Lemma 4.2 \( G : \partial_\infty B^n \to \partial_{n+1} \Pi^n \) and \( G : \partial_* B^n \to \partial_* \Pi^n \).

We claim that the degree of \( G : \partial B^n \to \partial \Pi^n \) equals the degree of \( G : (\partial_\infty B^n, \partial_\infty B^n \cap \partial_* B^n) \to (\partial_{n+1} \Pi^n, \partial_{n+1} \Pi^n \cap \partial_* \Pi^n) \). Using homeomorphisms \( \partial B^n \approx S^{n-1} \) and \( \partial \Pi^n \approx S^{n-1} \) that map the faces \( \partial_\infty B^n \) and \( \partial_{n+1} \Pi^n \) onto the upper hemisphere \( S^{n-1}_+ \), this is an instance of the general fact that if \( F : S^{n-1} \to S^{n-1} \) is continuous and maps \( S^{n-1}_- \to S^{n-1}_- \), then the degree of \( F : (S^{n-1}_+, S^{n-1}_+ \cap S^{n-1}_-) \to (S^{n-1}_-, S^{n-1}_- \cap S^{n-1}_-) \) equals the degree of \( F : S^{n-1} \to S^{n-1} \).

If \( (b_i)_1^n \in \partial_\infty B^n \), then \( \nu \) has infinite point masses at both \( y_n \) and \( y_{n+1} \), with \( x_0 < y_n < y_{n+1} \). By Lemma 3.5 we can ignore \( y_{n+1} \) and we obtain the same generalized diffusion \( X \) as with the speed measure \( \sum_{i=0}^n b_i \delta y_i \). We can thus identify \( \partial_\infty B^n \) with \( B^{n-1} \) (based on the points \( (y_i)_1^n \)). Furthermore, there is an obvious identification \( \partial_{n+1} \Pi^n = \Pi^{n-1} \), and with these identifications, \( G : \partial_\infty B^n \to \partial_{n+1} \Pi^n \) corresponds to \( G_{n-1} : B^{n-1} \to \Pi^{n-1} \). Moreover, the various boundaries correspond so that we have the commutative diagram

\[
\begin{array}{c}
H_{n-1}(\partial_\infty B^n, \partial_\infty B^n \cap \partial_* B^n) \xrightarrow{G_*} H_{n-1}(B^{n-1}, \partial B^{n-1}) \\
\downarrow \quad \quad \quad \quad \downarrow \\
H_{n-1}(\partial_{n+1} \Pi^n, \partial_{n+1} \Pi^n \cap \partial_* \Pi^n) \xrightarrow{G_{n-1}*} H_{n-1}(\Pi^{n-1}, \partial \Pi^{n-1})
\end{array}
\]

where the rows are the isomorphisms given by these identifications. Hence the degree of \( G_* : (\partial_\infty B^n, \partial_\infty B^n \cap \partial_* B^n) \to (\partial_{n+1} \Pi^n, \partial_{n+1} \Pi^n \cap \partial_* \Pi^n) \) equals \( \deg(G_{n-1}) \).

Combining this with the equalities above, we see that \( \deg(G_n) = \deg(G_{n-1}) \), which completes the induction step.

It remains to treat the initial case \( n = 1 \). In this case \( B^1 \) and \( \Pi^1 \) are intervals, and can be parametrized by \( b_1 \) and \( p_1 \). It is easy to see that the mapping \( G : b_1 \mapsto p_1 \) is strictly increasing, and thus a homeomorphism \( B^1 \to \Pi^1 \); hence \( G_* : H_1(B^1, \partial B^1) \to H_1(\Pi^1, \partial \Pi^1) \) is an isomorphism, so \( \deg(G) = \pm 1 \). Alternatively, we may use the first commutative diagram
above also in the case \( n = 1 \), replacing the homology groups \( H_{n-1} = H_0 \) by the reduced homology groups \( \tilde{H}_0 \). The sets \( \partial B^1 \) and \( \partial \Pi^1 \) contain exactly two elements each, and it is easy to see that \( G : \partial B^1 \to \partial \Pi^1 \) is a bijection and thus \( G_* : H_0(\partial B^1) \to H_0(\partial \Pi^1) \) is an isomorphism. \( \square \)

**Theorem 4.4.** The function \( G \) is surjective, for any \( n \geq 1 \) and any \( y_0 < y_1 < \cdots < y_{n+1} \) and \( x_0 \in (y_0, y_{n+1}) \). Consequently, the discrete inverse problem has a solution.

**Proof.** An immediate consequence of Lemma 4.3 since a function \((B^n, \partial B^n) \to (\Pi^n, \partial \Pi^n)\) (or, equivalently, \((D^n, S^{n-1}) \to (D^n, S^{n-1})\)) that is not surjective has mapping degree 0. \( \square \)

5. **The general case**

In this section we study the inverse problem for arbitrary distributions on the real axis. To do this, we approximate the given distribution with a sequence of discrete distributions. For each discrete distribution we can find a discrete speed measure that solves the inverse problem according to Section 4. We then show that the sequence of discrete speed measures has a convergent subsequence, and that the limit solves the inverse problem. We begin with a lemma giving the approximation that we shall use.

**Lemma 5.1.** Let \( \mu \) be a probability measure on \( \mathbb{R} \) with finite mean \( \bar{x} = \int_{\mathbb{R}} x \mu(dx) \). Then there exists a sequence \( \mu_n, n \geq 1 \), of probability measures with finite supports such that, as \( n \to \infty \),

(i) \( \mu_n \to \mu \) weakly;
(ii) \( \text{inf supp } \mu_n \to \text{inf supp } \mu \);
(iii) \( \text{sup supp } \mu_n \to \text{sup supp } \mu \);
(iv) each \( \mu_n \) has the same mean \( \bar{x} \) as \( \mu \).
(v) \( \int_{\mathbb{R}} |x| d\mu_n(x) \to \int_{\mathbb{R}} |x| d\mu(x) \).

If further \( \mu \) has finite variance \( \text{Var}(\mu) = \int_{\mathbb{R}} (x - \bar{x})^2 \mu(dx) \), then \( \mu_n \) can be chosen such that

(vi) \( \text{Var}(\mu_n) \to \text{Var}(\mu) \).

**Proof.** Let \( Y \) be a random variable with distribution \( \mu \). First, truncate \( Y \) at \( \pm n \) by defining

\[ Y_n' := (Y \wedge n) \vee (-n). \]

Then, letting all limits in this proof be for \( n \to \infty \),

\[ \mathbb{E} |Y_n' - Y| \leq \mathbb{E}(|Y|; |Y| > n) \to 0. \]

Next, discretize by defining

\[ Y_n'' := \frac{1}{n} \lfloor n Y_n' \rfloor. \]

Clearly, \( |Y_n'' - Y_n'| < 1/n \). It follows that

\[ | \mathbb{E} Y_n'' - \mathbb{E} Y | \leq \mathbb{E} |Y_n'' - Y| \leq \frac{1}{n} + \mathbb{E} |Y_n' - Y| \to 0. \]
Finally, we adjust the mean by defining
\[ Y_n := Y_{n”} - \mathbb{E}(Y_{n”} - Y). \] (5.1)
Thus \( \mathbb{E} Y_n = \mathbb{E} Y = \overline{\nu} \).

Let \( \mu_n = \mathcal{L}(Y_n) \), the distribution of \( Y_n \). Then \( \mu_n \) has finite support and (iv) holds by the construction. Furthermore, by (5.1),
\[ \mathbb{E}|Y_n - Y| \leq 2\mathbb{E}|Y_{n”} - Y| \to 0, \] (5.2)
which implies \( Y_n \overset{d}{\to} Y \) and thus (i). From (5.2) we also have \( \mathbb{E}|Y_n| \to \mathbb{E}|Y| \), which is
\[ (v). \]

If \( \text{inf supp } \mu = -\infty \), then (i) implies that \( \text{inf supp } \mu_n \to -\infty \), so (ii) holds.

Suppose now that \( \text{inf supp } \mu = a > -\infty \). If \( n > |a| \), then \( \text{inf supp } \mathcal{L}(Y_n) = a \), and it follows that
\[ |\text{inf supp } \mu_n - a| \leq \frac{1}{n} + |\mathbb{E} Y_{n”} - \mathbb{E} Y| \to 0, \]
which shows that (ii) hold in this case too.

The proof of (iii) is similar, mutatis mutandis.

If \( \mu \) has finite variance, then \( \mathbb{E} Y^2 < \infty \), and \( \mathbb{E}|Y_n|^2 = \mathbb{E}(|Y| \wedge n)^2 \to \mathbb{E} Y^2 \).

Taking square roots we find \( \|Y_n\|_2 \to \|Y\|_2 \). Minkowski’s inequality yields
\[ \|Y_n - Y\|_2 \leq \|Y_n - Y_{n”}\|_2 \leq \frac{1}{n} + \mathbb{E}|Y_{n”} - Y| \to 0. \]

Consequently, \( \|Y_n\|_2 \to \|Y\|_2 \), and thus \( \mathbb{E} Y_n^2 \to \mathbb{E} Y^2 \). Since \( \mathbb{E} Y_n = \mathbb{E} Y \), this implies \( \text{Var}(Y_n) \to \text{Var}(Y) \), which shows (v).

**Proof of Theorem 2.3.** We may for simplicity assume that \( x_0 = \overline{\nu} = 0 \).

Let \( a_- = \text{inf supp } \mu \geq -\infty \) and \( a_+ = \text{sup supp } \mu \leq +\infty \). Since \( \overline{\nu} = 0 \), we have \( a_- \leq 0 \leq a_+ \). Moreover, if \( a_- = 0 \) or \( a_+ = 0 \), then necessarily \( \mu = \delta_0 \). In this case we may simply take \( \nu \) as an infinite point mass at 0; then \( A_t = 0 \) and \( X_t = 0 \) for all \( t \geq 0 \) a.s., see Example 3.6. In the sequel we thus assume \( -\infty < a_- < 0 < a_+ \leq \infty \).

Let \( \mu_n \) be a sequence of distributions satisfying (i) in Lemma 5.1. The distributions \( \mu_n \) have finite supports, and thus Theorem 1.4 shows that there exist speed measures \( \nu_n \) so that the corresponding generalised diffusion \( X_n \) has distribution \( \mu_n \) at time 1.

If \( 0 < b < a_+ \), choose \( a \in (b,a_+) \). Then \( a > b > 0 \) and \( a < \text{sup supp } \mu \), so \( \mu(a,\infty) > 0 \). Since \( \mu_n \to \mu \), \( \liminf_{n \to \infty} \mu_n(a,\infty) \geq \mu(a,\infty) \), so for all large \( n \), \( \mu_n(a,\infty) > \frac{1}{2} \mu(a,\infty) > 0 \). Lemma 3.14 applies and implies that \( \nu_n[0,b] \leq C = C(b) \) for all large \( n \), i.e., \( \limsup_{n \to \infty} \nu_n[0,b] < \infty \) for every \( b < a_+ \).

By a symmetric argument, we also have \( \limsup_{n \to \infty} \nu_n[0,b] < \infty \) for every \( b > a_- \).

Choose sequences \( b^m_- \searrow a_- \) and \( b^m_+ \nearrow a_+ \). On each interval \( [b^m_-, b^m_+] \), the measures \( \nu_n \) are, as we just have shown, uniformly bounded if we exclude a finite number of small \( n \). Since \( [b^m_-, b^m_+] \) is compact, we may thus choose a subsequence of \( \nu_n \) such that the restrictions to \( [b^m_-, b^m_+] \) converge to some
measure \( \nu_m \). By a diagonal procedure, we can do this simultaneously for all \( m \), which provides a subsequence of \( \nu_n \) such that (along the subsequence) \( \nu_n \to \nu_m \) on \([b_n^-, b_n^+]\) for every \( m \). In the sequel we consider only this subsequence.

It follows that \( \Lambda(f) = \lim_{n \to \infty} \int f \, d\nu_n \) exists and is finite for every \( f \in C^+ C_c(a_-, a_+) \). Clearly, \( \Lambda \) is a positive linear functional on \( C^+ C_c(a_-, a_+) \), so by the Riesz representation theorem there exists a Borel measure \( \nu \) on \( (a_-, a_+) \)

\[ \Lambda f = \int_{a_-}^{a_+} f \, d\nu. \]

Similarly, \( \nu \) is a measure on \( (a_-, a_+) \) with \( \nu((a_-, a_+)) = 1 \) on \( f \in C^+ C_c(a_-, a_+) \).

We define \( \nu \) by adding infinite point masses at \( a_- \) and \( a_+ \), if these are finite: \( \nu = \nu + \infty \cdot \delta_{a_-} + \infty \cdot \delta_{a_+} \) (where \( \delta_{a_\pm} \) is the Dirac delta function at \( a_\pm \)).

If \( a_- \) and \( a_+ \) are finite and \( f \in C^+ C_c(\mathbb{R}) \) with \( f(a_-) > 0 \), then \( \int f \, d\nu \geq f(a_-) \nu\{a_-\} = \infty \). Furthermore, \( a_\pm := \inf \supp \mu_n \to a_\pm \), so \( f(a_\pm) > 0 \) for all large \( n \). The construction of \( \nu_n \) in Theorem 3.1(a) gives \( \nu_n \) an infinite point mass at \( a_\pm \), so \( \int f \, d\nu_n = \infty \) for all large \( n \). Thus \( \int f \, d\nu_n \to \int f \, d\nu = \infty \) as \( n \to \infty \). Similarly, \( \int f \, d\nu_n \to \int f \, d\nu = \infty \) as \( n \to \infty \) if \( f(a_+) > 0 \).

The assumptions of Lemma 3.12(ii) are satisfied, and thus a.s. \( A_1^n \to A_1 \) and, if \( \nu \neq 0 \), \( X_1^n \to X_1 \) as \( n \to \infty \). In particular, then \( X_1^n \to X_1 \) in distribution, and since \( X_1^n \) has distribution \( \mu_n \) and \( \mu_n \to \mu \), it follows that the distribution of \( X_1 \) is \( \mu \).

It remains to verify that the measure \( \nu \) is non-zero. If \( a_- \) or \( a_+ \) is finite, this is clear since \( \nu \) by construction has a point mass there.

If \( a_\pm = \pm \infty \), we use Lemma 5.4(v) which yields \( \mathbb{E}[|X_1^n|] = \int_{\mathbb{R}} |x| \, d\mu_n(x) - \int_{\mathbb{R}} |x| \, d\mu(x) < \infty \). Let \( K := \sup_n \mathbb{E}[|X_1^n|] < \infty \). By Lemma 3.13 there exists \( \kappa > 0 \) such that \( \nu_n[-2K, 2K] \geq \kappa \) for every \( n \). Let \( f \in C^+ \mathbb{R}(\mathbb{R}) \) with \( f = 1 \) on \([-2K, 2K] \). As shown above, \( \int_{\mathbb{R}} f \, d\nu_n \to \int_{\mathbb{R}} f \, d\nu = \int_{\mathbb{R}} f \, d\nu \) as \( n \to \infty \). Since \( \int_{\mathbb{R}} f \, d\nu_n \geq \nu_n[-2K, 2K] \geq \kappa \), this implies \( \int_{\mathbb{R}} f \, d\nu \geq \kappa > 0 \), and thus \( \nu \neq 0 \).

This proves the existence of a non-zero speed measure \( \nu \) such that \( X_1 \) has the desired distribution \( \mu \). We next prove that \( X_1 \) is a martingale.

We have shown that \( X_1^n \to X_1 \) a.s. and, by Lemma 5.4(v), \( \mathbb{E}[|X_1^n|] \to \mathbb{E}[|X_1|] \). This implies \( \mathbb{E}[X_1^n - X_1] \to 0 \), i.e. \( X_1^n \to X_1 \) in \( L^1 \) (see e.g. [10, Proposition 4.12]). For each \( n \), \( (B_{A_1^n \wedge u})_{u \geq 0} \) is by Lemma 3.7 a bounded martingale with limit \( B_{A_1^n} = X_1^n \), and thus \( B_{A_1^n \wedge u} = \mathbb{E}(X_1^n \mid F_u) \) for every \( u \). As \( n \to \infty \), a.s. \( A_1^n \to A_1 \) and thus \( B_{A_1^n \wedge u} \to B_{A_1 \wedge u} \). Further, we have just shown \( X_1^n \to X_1 \) in \( L^1 \), and this implies \( B_{A_1^n \wedge u} = \mathbb{E}(X_1^n \mid F_u) \to \mathbb{E}(X_1 \mid F_u) \) in \( L^1 \). The two limit results both hold in probability, so the limits must coincide: \( B_{A_1 \wedge u} = \mathbb{E}(X_1 \mid F_u) \).

This proves that \( (B_{A_1 \wedge u})_{u \geq 0} \) is a uniformly integrable martingale. Consequently, for any \( (\tau_t) \)-stopping time \( \tau \), \( B_{A_1 \wedge \tau} = \mathbb{E}(X_1 \mid F_\tau) \). In particular, for \( 0 \leq t \leq 1 \), \( X_1 = B_{A_1} = \mathbb{E}(X_1 \mid F_{A_1}) \), which proves that \( (X_t)_{t \leq 1} \) is a martingale. This completes the main part of the proof, and we turn to (ii) and (iii).

By Lemma 3.3 \( X_0 = \mathbb{P} \) if and only if \( \mathbb{P} \in \text{supp } \nu \). If \( \mathbb{P} \notin \text{supp } \nu \), then there exists an open set \( U \) with \( \mathbb{P} \in U \) and \( \nu(U) = 0 \). By Lemma 3.3 \( X_1 \notin U \)
a.s., so $\pi \notin \text{supp} \mu$. On the other hand, if $\pi \in \text{supp} \nu$ and $U$ is any open set containing $\pi$, then $\nu(U) > 0$. It is easy to see that there is a positive probability that $B_t$ will remain in $U$ until $\Gamma_t > 1$, and thus $X_1 \in U$. Hence $\mu(U) = P(X_1 \in U) > 0$ for any such $U$, so $\pi \in \text{supp} \mu$.

If $\mu$ has finite variance, we may by Lemma 5.1(v) assume that $\text{Var} \mu_n \to \text{Var} \mu$, and thus $\text{sup} \text{Var} \mu_n < \infty$. Lemma 3.10 applies to every $\nu_n$ and yields

$$E A_1^n = E (X_1^n)^2 = \text{Var}(\mu_n).$$

Consequently, by Fatou’s lemma,

$$E A_1 = E \lim_{n \to \infty} A_1^n \leq \liminf_{n \to \infty} E A_1^n = \liminf_{n \to \infty} \text{Var}(\mu_n) = \text{Var}(\mu) < \infty.$$  \hspace{1cm} (5.3)

Lemma 3.8 shows that $E A_1 = \text{Var}(\mu)$ in this case (so there is equality in (5.3)), and also if $\text{Var}(\mu) = \infty$. \hspace{1cm} \square

Example 5.2. As an illustration of Theorem 2.3, let $Y_t = |B_t^{(3)}|^{-1}$, where $B^{(3)}$ is a 3-dimensional Brownian motion with $|B_0^{(3)}| = y_0^{-1}$ for some $y_0 > 0$. It is well-known that $Y$ is a local martingale bounded from below, hence a supermartingale, but it is not a true martingale, compare Theorem 7.9 below. Moreover, $Y$ can be represented as the solution of

$$\begin{cases}
  dY_t = Y_t^2 dW_t, & t > 0, \\
  Y_0 = y_0
\end{cases} \hspace{1cm} (5.4)$$

for some standard Brownian motion $W$. Note that $Y$ is a diffusion by (5.4), and by Example 2.1 (together with a stopping argument), $Y$ is the generalized diffusion with speed measure $x^{-4} dx$, $x > 0$. (Note that $Y_t$ tends to 0 as $t \to \infty$, but never reaches 0.) Being the reciprocal of a Brownian motion, $Y$ has an explicit density, compare [2, Example 2.2.2]. The density of $Y_1$ decays like $C x^{-4}$ for large $x$, so only moments of order strictly less than three exist finitely. Moreover, the expected value of $Y_1$ is strictly smaller than the starting value $y_0$.

Theorem 2.3 provides the existence of a time-homogeneous generalised diffusion $X$ which is a true martingale such that $X_1$ and $Y_1$ have the same distribution, and $X_0 = E Y_1 < y_0$. Consequently, there are in this case two different speed measures that give rise to the same distribution at time 1. However, only one of the corresponding processes is a martingale, and they have different starting points.

Example 5.3. A related example is when $Y$ is the solution of

$$\begin{cases}
  dY_t = (1 + Y_t^2) dW_t, & t > 0, \\
  Y_0 = 0
\end{cases}$$

By Example 2.1 the diffusion $Y$ is the generalized diffusion with speed measure $(1 + x^2)^{-2} dx$. Again, $Y$ is a local martingale, but not a martingale, see Theorems 7.3 and 7.9 below.
Define $h(x, t) = e^{Mt}g(x)$, where $g$ is a smooth positive function satisfying $g(x) = |x|$ for $|x| \geq 1$ and $g(x) \geq |x|$ everywhere, and $M$ is a positive constant so that $h_t \geq \frac{1}{2}(1 + x^2)^2h_{xx}$.

Let $u_N(x, t) = \mathbb{E}_x |Y_t| \wedge N$, where the index indicates that $Y_0 = x$. By a maximum principle argument, compare (7), $u_N(x, t) \leq h(x, t)$ independently of $N$. Consequently, by monotone convergence, $\mathbb{E}_0 |Y_t| \leq h(0, t) < \infty$.

Theorem 2.3 thus applies and provides a generalised diffusion $X_t$ which is a martingale started at 0 such that $X_1$ has the same distribution as $Y_1$. Consequently, we have in this case two different generalised diffusions with the same starting point that give rise to the same distribution at time 1. However, only one of these processes is a martingale.

6. An Application to Mathematical Finance

In this section we study an inverse problem in Mathematical Finance. Let $X$ be a non-negative martingale with $X_0 = x_0$, and consider the expected values

$$C(K, T) := \mathbb{E}(X_T - K)^+.$$  \hfill (6.1)

Here $X_t$ has the interpretation as a price process of a stock, and the expected value $C(K, T)$ is the price of a call option with strike $K$ and maturity $T$ (for the sake of simplicity, we assume that interest rates are zero). The function $C$ is non-increasing and convex as a function of $K$, and it satisfies $C(0, T) = x_0$, $C(\infty, T) = 0$ and $C(K, T) \geq (x_0 - T)^+$. In Mathematical Finance, the corresponding inverse problem is of great interest. Option prices $C(K, T)$ are observable in the market, at least for a large collection of strikes $K$ and maturities $T$, whereas the stock price model is not. Under some regularity conditions (often neglected in the literature), Dupire [5] determines a local volatility model

$$dX_t = \sigma(X_t, t) dW_t$$

such that (6.1) holds for all $K$ and $T$. To do this, naturally one needs call option prices $C(K, T)$ given for all strikes $K \geq 0$ and all maturities $T > 0$.

The assumption that call option prices are known for all maturities $T$ is often unnatural in applications - indeed, there is typically only one maturity a month. We therefore consider the situation where $C(K, T)$ is given for all strikes $K$ but for one fixed maturity $T > 0$. A natural question is then whether there exists a time-homogeneous diffusion process

$$dX_t = \sigma(X_t) dW_t$$

such that (6.1) holds for all strikes. We have the following result.

**Theorem 6.1.** Let $c : [0, \infty) \to [0, \infty)$ be a convex and non-increasing function satisfying $c(0) = x_0$, $c(\infty) = 0$ and $c(K) \geq (x_0 - K)^+$, and let $T > 0$ be fixed. Then there exists a time-homogeneous (generalised) diffusion such that $X$ is a martingale with $X_0 = x_0$, and such that $c(K) = \mathbb{E}(X_T - K)^+$. 

Proof. We construct a probability distribution \( \mu \) on \([0, \infty)\) as follows. On \((0, \infty)\), we let \( \mu \) be defined as the measure given by the second derivative of the convex function \( c \). Since \( c(\infty) = 0 \), this measure has mass \(-c'(0)\), where \( c'(0) \) is the (right) derivative of \( c \) at 0. Moreover, we let a point mass \( \epsilon = 1 + c'(0) \) be located at 0; note that the assumptions \( c(0) = x_0 \) and \( c(K) \geq (x_0 - K)^+ \geq x_0 - K \) imply that \( c'(0) \geq -1 \) so \( \epsilon \geq 0 \).

Integration by parts shows that the expected value of the distribution \( \mu \) is given by

\[
\int_0^\infty x \mu(dx) = \int_0^\infty \mu(x, \infty) dx = -\int_0^\infty c'(x) dx = c(0) = x_0.
\]

Now, by Theorem 2.3 (and an obvious scaling to general \( T \)) there exists a time-homogeneous generalised diffusion \( X \) with \( X_0 = x_0 \) such that \( X \) is a martingale and \( X_T \) has distribution \( \mu \). Finally, integration by parts gives

\[
E(X_T - K)^+ = \int_K^\infty (x - K) \mu(dx) = \int_K^\infty \mu(x, \infty) dx = c(K),
\]

thus finishing the proof. \( \Box \)

7. Martingality of generalised diffusions

The speed measure constructed in the proof of Theorem 2.3 is such that the generalized diffusion \((X_t)\) is a martingale, but as mentioned above, this is not the case for every speed measure \( \nu \). We characterize here the speed measures for which \( X_t \) is a (local) martingale.

Example 7.1. If the speed measure is given by

\[
\nu(dx) = \begin{cases} 
  dx & \text{for } x \geq 0 \\
  0 & \text{for } x < 0,
\end{cases}
\]

then the corresponding process \( X \) is a Brownian motion reflected at 0, and, in particular, not a local martingale.

This type of reflection at an extreme point of \( \text{supp } \nu \) is the only thing that can prevent \( X \) from being a local martingale. This is shown in the theorem below from [6], here somewhat extended.

We first give a lemma, essentially saying that stopping \( X \) at a point \( a \in \text{supp } \nu \) is the same as stopping \( B \) at \( a \). (Note that we cannot stop \( X \) at \( a \notin \text{supp } \nu \) since \( X_t \in \text{supp } \nu \) for all \( t \geq 0 \) by Lemma 3.1.)

Lemma 7.2. If \( a \in (x^{-\infty}, x^{+\infty}) \cap \text{supp } \nu \) and \( \tau := \inf\{t : X_t = a\} \), then \( \tau = \Gamma_{H_a} \). If \( H_a < H_{\text{supp } \nu} \), then also \( \tau < \infty \) and \( A_\tau = H_a \), while \( \tau = \infty \) if \( H_a > H_{\text{supp } \nu} \). Furthermore, \( \mathbb{P}(H_a < H_{\text{supp } \nu}) > 0 \) and \( \mathbb{P}(\tau < t_0) > 0 \) for any \( t_0 > 0 \).

Proof. If \( H_a < H_{\text{supp } \nu} \), then \( \Gamma_{H_a} < \infty \) by Lemma 3.5 and thus Lemma 3.2 and the assumption \( a \in \text{supp } \nu \) imply that \( \Gamma_{H_a + \varepsilon} > \Gamma_{H_a} \) for every \( \varepsilon > 0 \). Thus, if \( t = \Gamma_{H_a} \), then \( A_t = H_a \) and \( X_t = B_{A_t} = B_{H_a} = a \). On the other hand, if \( t < \Gamma_{H_a} \), then \( A_t < H_a \) by (2.3) so \( X_t = B_{A_t} \neq a \). Hence, \( \tau = \Gamma_{H_a} \).
If \( H_a > H_{\text{supp}\nu} \), then Lemma 3.5 yields \( \Gamma_{H_a} = \infty \). Lemma 3.5 yields further, for any \( t < \infty \), \( A_t \leq H_{\text{supp}\nu} \) so \( X_t = B_{A_t} \neq a \). Hence, \( \tau = \infty = \Gamma_{H_a} \).

For the final statement, assume that \( a \leq x_0 \), say. Note that the assumption implies \( x_0 \notin \text{supp}\nu \), so \( x_\infty^- < x_0 < x_\infty^+ \). Choose \( b \in (x_0, x_\infty^+) \). Then \( \nu[a, b] < \infty \). Let \( \epsilon := t_0/\nu[a, b] \). (The case \( \nu[a, b] = 0 \) is easy.) It is easy to see (e.g., using excursion theory) that with positive probability both \( \sup_x L_{H_a}^x < \epsilon \) and \( H_a < H_b \). Thus \( B_s \in [a, b) \) for all \( s \leq H_a \), and then \( \tau = \Gamma_{H_a} = \int_a^b L_{H_a}^x \nu(dx) < \epsilon \nu[a, b] = t_0 \).

The claim \( \mathbb{P}(H_a < H_{\text{supp}\nu}) > 0 \) follows from this, using Lemma 3.5 but it is easier to see it directly since \( H_{\text{supp}\nu} = H_{(x_\infty^- \times x_\infty^+)} \).

**Theorem 7.3.** \((X_t)_{t \geq 0}\) is a local martingale if and only if either \( \text{supp}\nu = \{x_0\} \) (then, trivially, \( X_t = x_0 \) for all \( t \)) or both the following conditions hold:

(i) \( \text{supp}\nu \cap (\neg \infty, x_0) \neq \emptyset \) or \( \text{supp}\nu \cap (\neg \infty, b) \neq \emptyset \) for all \( b < x_0 \),

(ii) \( \text{supp}\nu \cap [x_0, \infty) \neq \emptyset \) or \( \text{supp}\nu \cap [b, \infty) \neq \emptyset \) for all \( b > x_0 \).

Moreover, if \((X_t)_{0 \leq t \leq t_0}\) is a local martingale for some \( t_0 > 0 \), then \((X_t)_{t \geq 0}\) is a local martingale.

**Proof.** If \( \text{supp}\nu = \{x_0\} \), then \( X_t = x_0 \) a.s. for all \( t \) by Lemma 3.1.

Suppose now that (i) and (ii) hold. If \( \text{supp}\nu \cap (\neg \infty, x_0) \neq \emptyset \), let \( a_n^+ := x_\infty^- = \sup(x \leq x_0 : x \in \text{supp}\nu) \) for each \( n \); otherwise let \( a_n^+ \) be a sequence of points in \( \text{supp}\nu \cap (\neg \infty, x_0) \) with \( a_n^- \downarrow -\infty \) as \( n \to \infty \). Define a sequence \( a_n^+ \geq x_0 \) similarly, using (ii).

Let \( H_n := H_{(a_n^-, a_n^+)} \). For each \( n \), \((B_{t \land H_n})_{t \geq 0}\) is a bounded martingale. Thus, if \( s < t \), then \( \mathbb{E}(B_{A_t \land H_n} | \mathcal{F}_A_s) = B_{A_s \land H_n} \), so \((B_{A_t \land H_n})_{t \geq 0}\) is a martingale for the filtration \((\mathcal{G}_t) = (\mathcal{F}_A_t)\).

Set \( T_n := \Gamma_{H_n^+} = \inf(\Gamma_X : u > H_n) \). \( T_n \) is a stopping time for the filtration \((\mathcal{G}_t)\). Further, since \( H_n \leq H_{n+1} \) we have \( T_n \leq T_{n+1} \).

If \( T_n < \infty \), then \( \Gamma_{H_n} = T_n < \infty \) and it follows from Lemma 3.2 that a.s. then \( \Gamma_u > T_n \) for all \( u > H_n \), so \( A_{T_n} = H_n \); hence, \( B_{A_t \land H_n} = B_{A_{T_n} \land T_n} = B_{A_{T_n} \land T_n} = X_{t \land T_n} \). On the other hand, it \( T_n = \infty \), then \( A_t \leq H_n \) for every \( t \), and thus \( B_{A_t \land H_n} = B_{A_t} = X_t = X_{t \land T_n} \). In any case, thus \( B_{A_t \land H_n} = X_{t \land T_n} \) a.s., so \((X_{t \land T_n})_{t \geq 0}\) is a martingale.

Finally, we verify that \( T_n \to \infty \) a.s. as \( n \to \infty \). Recall that \( B_{H_n} \in \{a_n^-, a_n^+\} \); we consider for definiteness the case when \( B_{H_n} = a_n^- \), so \( H_n = H_{a_n^-} \). By construction, either \( a_n^+ \in \text{supp}\nu \), and then Lemma 3.5 implies that \( \Gamma_{H_{a_n^-}} = \infty \) a.s., or else \( a_n^- \to -\infty \) and then \( H_{a_n^-} \to \infty \) and thus \( \Gamma_{H_{a_n^-}} \to \infty \) a.s. In both cases we obtain \( T_n = \Gamma_{H_{a_n^-}} \to \infty \) a.s. This completes the proof that \((X_t)_{t \geq 0}\) is a local martingale.

For the converse, we assume that \( \text{supp}\nu \neq \{x_0\} \). Assume that (ii) fails. (The case when (ii) fails is symmetric.) Let \( \xi_- := \inf\{x : x \in \text{supp}\nu\} \). Since (ii) fails, \( \xi_- \geq -\infty \). We may assume that \( \xi_- \leq x_0 \), since otherwise Lemma 3.5 shows that \( X_t \) is not a local martingale at 0.
Lemma 7.4. If \( A_t < \text{Supp}_\infty \nu \), then \( \Gamma_{A_t} = t \). Consequently, for all \( t \geq 0 \), \( \Gamma_{A_t} = t \land \Gamma_{\text{Supp}_\infty \nu} \).

Proof. If \( A_t < \text{Supp}_\infty \nu \), then for small \( \varepsilon > 0 \) we have \( A_t + \varepsilon < \text{Supp}_\infty \nu \) and thus \( \Gamma_{A_t+\varepsilon} < \infty \) by Lemma 3.5, which by Remark 2.2(iii) yields \( \Gamma_{A_t} = t \).

Furthermore, in this case \( t = \Gamma_{A_t} \leq \Gamma_{\text{Supp}_\infty \nu} \), so \( \Gamma_{A_t} = t \land \Gamma_{\text{Supp}_\infty \nu} \).

Since \( A_t \leq \text{Supp}_\infty \nu \) by Lemma 3.5, the only remaining case is \( A_t = \text{Supp}_\infty \nu \). In this case \( \Gamma_{\text{Supp}_\infty \nu} = \Gamma_{A_t} \leq t \) so \( t \land \Gamma_{\text{Supp}_\infty \nu} = \Gamma_{\text{Supp}_\infty \nu} = \Gamma_{A_t} \). \( \square \)
Lemma 7.5. For every measurable function $f \geq 0$, a.s. for every $u \leq H_{\text{supp} \nu}$,
\begin{equation}
\int_0^u f(B_s) \, d\Gamma_s = \int_{-\infty}^\infty f(x)L_u^x \nu(dx).
\end{equation}

Proof. By monotone convergence it suffices to consider $u < H_{\text{supp} \nu}$, and then, by modifying $\nu$ outside the range of $\{B_s : s \leq u\}$, it suffices to consider locally finite $\nu$. In this case, the result is a consequence of the general theory of continuous additive functionals, see [14, Corollary X.(2.13)]. (It is also easy to make a direct proof in this case, by first assuming that $f$ is continuous, and then partitioning the interval $[0, u]$ into $N$ subintervals, and for each subinterval estimating the change of the difference between the two sides in (7.1); we omit the details.)

\[ \Box \]

Lemma 7.6. If $f \geq 0$ is a deterministic or random measurable function, then a.s. for every $T \leq H_{\text{supp} \nu}$,
\begin{equation}
\int_0^T f(s) \, d\Gamma_s = \int_0^{\Gamma_T} f(A_t) \, dt.
\end{equation}

Proof. Consider a fixed $\omega$ in our probability space. By a monotone class argument (or by seeing the two sides of (7.2) as $\int f \, d\mu_L$ and $\int f \, d\mu_R$ for two finite measures $\mu_L, \mu_R$ on $[0, \infty)$), it suffices to prove (7.2) when $f$ is the indicator of an interval $[0, u)$ for some $u > 0$. In this case $\int_0^T f(s) \, d\Gamma_s = \int_0^{\Gamma_T} f(A_t) \, dt = \Gamma_T \wedge u$ and, by Remarks 2.2(iii)
\begin{equation}
\int_0^{\Gamma_T} f(A_t) \, dt = \int_0^{\Gamma_T} 1\{t < \Gamma_u\} \, dt = \Gamma_T \wedge \Gamma_u = \Gamma_T \wedge u.
\end{equation}

\[ \Box \]

If $(X_t)_{t \geq 0}$ is a martingale, then $\mathbb{E}X_{t \wedge \tau} = x_0$ for every $X$-stopping time $\tau$ and every $t \geq 0$. This means that $\mathbb{E}B_T = x_0$ for the $B$-stopping time $T = A_t \wedge A_t$. We would like to have $\mathbb{E}B_T = x_0$ also for other $B$-stopping times $T \leq A_t$, not necessarily obtained by stopping $X$. However, this is not always possible, as is seen by the following example.

Example 7.7. If $\text{supp} \nu = \{x_0\}$, then $X_t = x_0$ for all $t \geq 0$, so $X_t$ is trivially a martingale. However, if further $\nu\{x_0\} < \infty$, for example if $\nu = \delta_{x_0}$, then for any $t > 0$ and $a \neq x_0$, $\mathbb{P}(B_{A_t \wedge H_a} = a) = \mathbb{P}(H_a < A_t) > 0$; since $B_{A_t \wedge H_a} \in \{a, x_0\}$, this implies $\mathbb{E}B_{A_t \wedge H_a} \neq x_0$.

The next lemma shows that Example 7.7 is the only counterexample. (The trivial example shows that the lemma is not as trivial as it might look.)

Lemma 7.8. Suppose that $\text{supp} \nu \neq \{x_0\}$ and that $(X_t)_{t \leq t_0}$ is a martingale for some $t_0 > 0$. Then, for every stopping time $T \leq A_{t_0}$ and every real $a$, $\mathbb{E}B_T = x_0$.

It will follow from Theorem 7.9 and its proof that, more generally, $\mathbb{E}B_T = x_0$ for any such $T$, so the $H_a$ is not really needed but it simplifies the proof.
Proof. We suppose that $a < x_0$. (The case $a > x_0$ is similar and $a = x_0$ is trivial.)

Suppose first that $a \in \text{supp } \nu$. Then Lemma 3.2 implies that a.s. $\Gamma_{H_a + \varepsilon} > \Gamma_{H_a}$ for every $\varepsilon > 0$. Hence, if $\tau := \Gamma_{H_a}$, then $A_{\tau} = H_a$ and $X_{\tau} = B_{H_a} = a$; moreover, $X_s \neq a$ for $s < H_a$. In other words, $\tau$ is the $X$-stopping time $\inf\{s : X_s = a\}$. The assumption that $(X_t)_{t \leq t_0}$ is a martingale thus implies, for $t \leq t_0$,

$$x_0 = \mathbb{E} X_{t \wedge \tau} = \mathbb{E} B_{A_{t \wedge A_{\tau}}} = \mathbb{E} B_{A_{t \wedge H_a}}.$$  \hspace{1cm} (7.3)

If $T'$ and $T''$ are two stopping times with $T' \geq T''$, then $\mathbb{E}(B_{T' \wedge u} | F_{T''}) = B_{T'' \wedge u}$ for every $u \geq 0$ (see e.g. [10, Theorem 7.29]). If $T' \leq H_a$, then $B_s - a \geq 0$ for all $s \leq T'$, and Fatou's lemma yields $\mathbb{E}(B_{T'} - a | F_{T''}) \leq B_{T''} - a$ and thus, by taking the expectation,

$$\mathbb{E} B_{T'} \leq \mathbb{E} B_{T''}.$$ \hspace{1cm} (7.4)

We apply (7.4) first to $A_{t_0} \wedge H_a$ and $T \wedge H_a$ and then to $T \wedge H_a$ and 0, obtaining

$$\mathbb{E} B_{A_{t_0} \wedge H_a} \leq \mathbb{E} B_{T \wedge H_a} \leq \mathbb{E} B_0 = x_0,$$ \hspace{1cm} (7.5)

and (7.3) shows that we have equalities. This proves the result when $a \in \text{supp } \nu$.

In general, by Theorem 3.3 either there exists some $b < a$ with $b \in \text{supp } \nu$, or there exists $b \in [a, x_0]$ with $b \in \text{supp } \nu$. In the first case $H_a < H_b$ and in the second case $H_a > H_b$ and Lemma 3.5 implies that $A_{t_0} \leq H_b$ and thus $T \leq A_{t_0} \leq H_b$. Consequently, in both cases $T \wedge H_a = T \wedge H_a \wedge H_b$, and the result follows from the case just proven applied to $T \wedge H_a$ and $b$. \hfill \Box

Theorem 7.9. The following are equivalent, for any $t_0 > 0$.

(i) $(X_t)_{t \geq t_0}$ is a martingale.
(ii) $(X_t)_{0 \leq t \leq t_0}$ is a martingale.
(iii) $\text{supp } \nu = \{x_0\}$ or $x_0 \in \text{supp } \nu$ or
\[ \int_{-\infty}^{\infty} (1 + |x|) \nu(dx) = \int_{-\infty}^{x_0} (1 + |x|) \nu(dx) = \infty. \] \hspace{1cm} (7.6)

(iv) $\text{supp } \nu = \{x_0\}$ or $x_0 \in \text{supp } \nu$ or
\[ \int_{x_0}^{\infty} |x - x_0| \nu(dx) = \int_{-\infty}^{x_0} |x - x_0| \nu(dx) = \infty. \] \hspace{1cm} (7.7)

Remark 7.10. For a related result for non-negative diffusion processes, see [4].

Proof. The result is trivial if $\text{supp } \nu = \{x_0\}$ or $x_0 \in \text{supp } \nu$. We may thus assume $\text{supp } \nu \neq \{x_0\}$ and $-\infty \leq x_- < x_0 < x_+ \leq \infty$. The equivalence of (7.6) and (7.7) then is elementary.

Let

$$\varphi(x) := \begin{cases} 2 \int_{x_0}^{x} |y| \nu(dy), & x \geq x_0, \\ -2 \int_{x_0}^{x} |y| \nu(dy), & x < x_0, \end{cases}$$  \hspace{1cm} (7.8)
and
\[
\psi(x) := \begin{cases} 
\int_{x_0}^x \varphi(y) \, dy, & x \geq x_0, \\
- \int_{x}^{x_0} \varphi(y) \, dy, & x < x_0.
\end{cases}
\] (7.9)

Then \( \psi \) is a non-negative convex function on \((x_0^-, x_0^+)\) with left derivative \( \varphi \) and thus second derivative (in distribution sense) \( \psi''(x) = 2|x|\nu(dx) \).

By the Itô–Tanaka formula \cite[Theorem VI.(1.5)]{14} and Lemma \(7.5\) for \( u \leq H_{\text{supp}, \nu} \),
\[
\psi(B_u) = \int_0^u \varphi(B_s) \, dB_s + \int_\mathbb{R} L^\varphi_s |x| \nu(dx) = \int_0^u \varphi(B_s) \, dB_s + \int_0^u |B_s| \, d\Gamma_s.
\] (7.10)

Let \( H := H_{[a,b]} \) where \( x^- < a < x_0 < b < x^+ \). Then \( H < H_{\text{supp}, \nu} \). Further, \( \varphi(B_s) \) is bounded for \( s \leq H \), and consequently \( \int_0^u \varphi(B_s) \, dB_s \), \( u \geq 0 \), is a martingale. Hence, for every bounded stopping time \( T \leq H \), \( \mathbb{E} \int_0^T \varphi(B_s) \, dB_s = 0 \), and (7.10) yields
\[
\mathbb{E}(\psi(B_T)) = \mathbb{E} \int_0^T |B_s| \, d\Gamma_s.
\] (7.11)

Moreover, for any stopping time \( T \leq H \), we can apply (7.11) to \( T \land u \) and let \( u \to \infty \). Since \( T \leq H \) we have \( B_{T \land u} \in [a, b] \), and thus \( \psi(B_{T \land u}) \) is uniformly bounded; hence dominated convergence on the left-hand side and monotone convergence on the right-hand side shows that (7.11) holds for any stopping time \( T \leq H \).

We apply (7.11) first to \( T = A_r \land H \land u \) for some \( r, u \geq 0 \). Thus, using Lemma \(7.6\) and \( \Gamma_{A_r} \leq r \), see Remarks 2.2[iii],
\[
\mathbb{E}(\psi(B_{A_r \land H \land u})) = \mathbb{E} \int_0^{A_r \land H \land u} |B_s| \, d\Gamma_s = \mathbb{E} \int_0^{A_r \land H \land u} |B_s \land H \land u| \, d\Gamma_s
\]
\[
= \mathbb{E} \int_0^{A_r \land H \land u} |B_{A_r \land H \land u}| \, dt \leq \mathbb{E} \int_0^r |B_{A_r \land H \land u}| \, dt = \int_0^r \mathbb{E}|B_{A_r \land H \land u}| \, dt.
\] (7.12)

Suppose first that (7.7) holds. By translation we may assume that \( x_0 = 0 \). Then \( \varphi(x) \to \pm \infty \) as \( x \to \pm \infty \), and thus \( \psi(x)/|x| \to \infty \) as \( x \to \pm \infty \). In particular, \( \psi(x) \geq |x| \) for large \( |x| \), and thus \( |x| \leq \psi(x) + C \) for some constant \( C \). Consequently, (7.12) yields
\[
\mathbb{E}|B_{A_r \land H \land u}| \leq C + \mathbb{E}(\psi(B_{A_r \land H \land u})) \leq C + \int_0^r \mathbb{E}|B_{A_r \land H \land u}| \, dt.
\]
Note further that \( B_{A_r \land H \land u} \) is bounded by \( \max\{|a|, b| \}, \) so the expectations here are finite and bounded. We can thus apply Gronwall’s Lemma \cite[Appendix §1]{14} and conclude
\[
\mathbb{E}|B_{A_r \land H \land u}| \leq Ce^t.
\] (7.13)
Using this in (7.12) again yields the sharper estimate
\[ \mathbb{E}(\psi(B_{A_t \wedge H \wedge u})) \leq C\epsilon^t. \] (7.14)

Now let \( a \to x^\infty \) and \( b \to x^\infty \); then \( H \to H_{\supp \nu} = H_{\{x^\infty, x^\infty\}} \). Since \( A_t \leq H_{\supp \nu} \) by Lemma 3.5, \( B_{A_t \wedge H \wedge u} \to B_{A_t \wedge u} \), and Fatou’s lemma yields
\[ \mathbb{E}(\psi(B_{A_t \wedge u})) \leq C\epsilon^t. \] (7.15)

Since \( \psi(x)/|x| \to \infty \) as \( x \to \pm \infty \), (7.15) implies that for a fixed \( t \), the random variables \( B_{A_t \wedge u}, u \geq 0 \), are uniformly integrable [8, Theorem 5.4.3]. Moreover, \( B_{A_t \wedge u}, u \geq 0 \), is a martingale, and since it is uniformly integrable, \( B_{A_t \wedge u} = \mathbb{E}(B_{A_t} \mid \mathcal{F}_u) \), for all \( u \geq 0 \), and further, see [10, Theorem 7.29], for any \( s \leq t \),
\[ \mathbb{E}(B_{A_t} \mid \mathcal{F}_s) = B_{A_t \wedge A_s} = B_{A_s}. \] (7.16)

Conversely, suppose that \( \text{(ii)} \) holds but \( \text{(7.6)} \) fails; by symmetry we may assume that
\[ \int_{x_0}^{\infty} (1 + |x|) \nu(dx) < \infty. \] (7.17)

In particular, this shows that \( \supp \nu \cap (x_0, \infty) = \emptyset \), so \( x^\infty = \infty \). By translation, we may now also assume that \( x^\infty < 0 < x_0 \), and we now take \( a = 0 \) and any \( b > x_0 \). Thus \( H = H_{\{a,b\}} = H_0 \wedge H_b \). We apply (7.11) with \( T = A_t \wedge H \). Noting that \( H \leq H_0 \) and \( B_s \geq 0 \) for \( s \leq H \), we obtain, using also Lemma 7.6
\[ \mathbb{E}(B_{A_t \wedge H_0 \wedge H_b}) = \mathbb{E} \int_0^{A_t \wedge H_0} B_s d\Gamma_s = \mathbb{E} \int_0^{A_t \wedge H_0} B_{s \wedge H_0} d\Gamma_s \]
\[ = \mathbb{E} \int_0^{A_t \wedge H_0} B_{A_s \wedge H_0} d\Gamma_s. \] (7.18)

Since \( H < H_{\supp \nu} \), Lemma 7.4 shows that
\[ \Gamma_{A_t \wedge H} = \Gamma_{A_t} \wedge \Gamma_H = t \wedge \Gamma_{H_{\supp \nu}} \wedge \Gamma_H = t \wedge \Gamma_H = t \wedge \Gamma_{H_0} \wedge \Gamma_{H_b}. \]

Further, if \( r \geq \Gamma_{H_0} \) then \( A_r \geq H_0 \) by (2.3) and thus \( B_{A_r \wedge H_0} = B_{H_0} = 0 \). Hence, (7.18) yields
\[ \mathbb{E}(B_{A_t \wedge H_0 \wedge H_b}) = \mathbb{E} \int_0^{t \wedge \Gamma_{H_0} \wedge \Gamma_{H_b}} B_{A_r \wedge H_0} dr = \mathbb{E} \int_0^{t \wedge \Gamma_{H_b}} B_{A_r \wedge H_0} dr. \] (7.19)

Let \( C' := 2 \int_0^\infty y \nu(dy) \). Then \( |\varphi(x)| \leq C' \) for \( x \geq 0 \), and thus, for any \( x \geq 0 \),
\[ \psi(x) \leq \psi(0) + C' x \leq \psi(x_0) + C' x_0 + C' x = C' x_0 + C' x. \] (7.20)

Thus, for \( t = t_0 \), the left-hand side of (7.19) is at most, using Lemma 7.8
\[ \mathbb{E}(C' x_0 + C' B_{A_t \wedge H_0 \wedge H_b}) = 2C' x_0. \] (7.21)
Consequently (7.19) yields
\[ 2C'x_0 \geq \mathbb{E} \int_0^{t_0 \land H_b} B_{A_r \land H_0} \, dr. \tag{7.22} \]
Letting \( b \to \infty \) we have \( H_b \to \infty \) and \( \Gamma_{H_b} \to \infty \), and thus (7.22) yields by monotone convergence, using Lemma 7.8 again,
\[ 2C'x_0 \geq \mathbb{E} \int_0^{t_0} B_{A_r \land H_0} \, dr = t_0 x_0. \]
Consequently,
\[ t_0 \leq 2C' = 4 \int_0^\infty y \nu(dy). \tag{7.23} \]
This shows that (i) \( \Rightarrow \) (iii) since we then may choose \( t_0 \) arbitrarily large in (7.23) and obtain a contradiction. To extend this to (ii) with a given \( t_0 \), we argue as follows, assuming (ii) and (7.17).

We have derived (7.23) under the assumption \( x_\infty - x < 0 < x_0 \). By translation, we have in general, for any \( x_0 \) and \( x_\infty - x \), and any \( a \in (x_\infty - x, x_0) \),
\[ t_0 \leq 4 \int_a^\infty y \nu(dy). \tag{7.24} \]
Letting \( a \to x_0 \) yields
\[ t_0 \leq 4 \int_{x_0}^\infty y \nu(dy). \tag{7.25} \]
Take any \( z > x_0 \) with \( z \in \text{supp } \nu \), and let \( \tau := \Gamma_{H_z} ; \) by Lemma 7.2, \( \tau = \inf \{ t : X_t = z \} \), so \( \tau \) is an \( X \)-stopping time. Condition on the event \( \mathcal{E} := \{ \tau \leq t_0/2 \} \); we have \( \mathbb{P}(\mathcal{E}) > 0 \) by Lemma 7.2. On the event \( \mathcal{E} \), \( B'_u = B_{H_z+u} \) is a Brownian motion starting at \( z \), and the processes corresponding to \( \Gamma \), \( A \) and \( X \) defined by \( \Gamma' = \Gamma_{H_z+u} - \Gamma_{H_z} = \Gamma_{H_z+u} - \tau \), \( A'_t = A_{t+\tau} - H_z \) and \( X'_t = B'_{A'_t} = X_{t+\tau} \). Since \( \tau \) is an \( X \)-stopping time, on \( \mathcal{E} \), \( X'_t \) is a martingale for \( 0 \leq t \leq t_0/2 \), and also at \( t = 0^- \) since \( X'_0 = z = X'_0 \), cf. Lemma 3.3. Consequently, we may apply the result above to \( X'_t \), and (7.25) yields
\[ \frac{t_0}{2} \leq 4 \int_z^\infty y \nu(dy). \tag{7.26} \]
However, if (7.17) holds, then the right-hand side of (7.26) tends to 0 as \( z \to \infty \), which is a contradiction. This contradiction shows that (7.17) cannot hold, and thus (ii) \( \Rightarrow \) (iii) \( \Box \)

Note that the case \( x_0 \in \text{supp}_\infty \nu \) is not redundant in (iii). An example is given by \( x_0 = 0 \) and \( \nu(dx) = dx/x \) for \( x > 0 \); then (7.6) fails because the second integral vanishes, nevertheless \( X_t = x_0 = 0 \) for all \( t \), which trivially is a martingale.
8. Open problems

8.1. Weakening the assumptions. Theorem 2.3 shows the existence of a generalised diffusion solving the inverse problem provided the given distribution has finite expectation. Does the result hold without this assumption (allowing local martingales instead of only martingales)?

8.2. Uniqueness. Theorem 2.3 shows the existence of a generalised diffusion, defined by a speed measure $\nu$ (assuming $\mathbb{E} |X_1|$ is finite). Is the measure $\nu$ unique? Strictly speaking it is not: Lemma 3.7 shows that we will never leave the interval $[x^-_\infty, x^+\infty]$, so any change of $\nu$ outside this interval (assuming at least one of $x^-_\infty$ and $x^+\infty$ is finite) will not affect $X_t$ at all. We may normalize $\nu$ by first taking the restriction $\nu_0$ to $(x^-_\infty, x^+\infty)$; if $x^-_\infty$ is finite but $x^+\infty \notin \text{supp}_\infty \nu_0$ we also add an infinite point mass at $x^-_\infty$, and similarly for $x^+\infty$. The question is then: Given the distribution of $X_1$, is there a unique corresponding normalized speed measure such that the corresponding process $(X_t)_{t \leq 1}$ is a martingale?

The problem of uniqueness is also open in the discrete case, i.e. is the mapping $G : B^n \to \Pi^n$ studied in Section 4 an injection? (As remarked in the proof, this holds for $n = 1$.)

8.3. Relations between $\mu$ and $\nu$. Find relations between the speed measure $\nu$ (assumed normalized as above) and the distribution $\mu$ of $X_1$. For example, if $\mu$ has a point mass at $x$, does $\nu$ also have a point mass there? Does the converse hold? If $\nu$ is absolutely continuous, is $\mu$ too? Does the converse hold? For which $\nu$ does $\mu$ have a finite second moment?

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References

[1] Bredon, G. E., Topology and Geometry, Springer-Verlag, New York, 1993.
[2] Cox, A. and Hobson, D., Local martingales, bubbles and option prices. Finance Stoch. 9 (2005), 477–492.
[3] Cox, A., Hobson, D. and Obłój, J., Time homogeneous diffusions with a given marginal at a random time. [arXiv:0912.1719] To appear in ESAIM: Probability and Statistics (special issue in honour of Marc Yor).
[4] Delbaen, F. and Shirakawa, H., No arbitrage condition for positive diffusion price processes. Asia-Pacific Financial Markets 9 (2002), 159–168.
[5] Dupire, B., Pricing with a smile. Risk 7 (1994), 18–20.
[6] Ekström, E. and Hobson, D., Recovering a time-homogeneous stock price process from perpetual option prices. To appear in Ann. Appl. Probab. (2011).
[7] Ekström, E. and Tysk, J. Bubbles, convexity and the Black-Scholes equation. Ann. Appl. Probab. 19 (2009), no. 4, 1369–1384.
[8] Gut, A., Probability: A Graduate Course, Springer, New York, 2005. Corrected 2nd printing 2007.
[9] Jiang, L. and Tao Y., Identifying the volatility of underlying assets from option prices. Inverse Problems 17 (2001), 137–155.
[10] Kallenberg, O., Foundations of Modern Probability. 2nd ed., Springer, New York, 2002.
[11] Knight, F. B., Characterization of the Levy measures of inverse local times of gap diffusion. In: Seminar on Stochastic Processes, 1981 Birkhäuser, Boston, Mass., 53–78, 1981.

[12] Kotani, S. and Watanabe, S., Krein’s spectral theory of strings and generalized diffusion processes. In: Functional analysis in Markov processes, Lecture Notes in Math. 923, Springer, Berlin, 1982, 235–259.

[13] Mörters, P. and Peres, Y., Brownian Motion, Cambridge Univ. Press, Cambridge, 2010.

[14] Revuz, D. and Yor, M., Continuous Martingales and Brownian Motion, 3rd ed., Springer, Berlin, 1999.

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