ENumeration of singular tropical surfaces in $\mathbb{R}^3$

HANNAH MARKWIG, THOMAS MARKWIG, AND EUGENII SHUSTIN

Abstract. We enumerate uninodal surfaces in a given divisor class in an arbitrary toric surface satisfying point conditions or, in other words, we compute the degree of the discriminant of the dual polytope. For this purpose, we solve the analogous tropical surface counting problem and provide a correspondence theorem suitable to deduce the algebraic count from the tropical count. To solve the tropical counting problem, we present an algorithm which can be viewed as a three-dimensional version of Mikhalkin’s lattice path algorithm. The latter was originally invented to count plane tropical curves satisfying point conditions. Our result relies on the classification of singular tropical surfaces [15].

1. Introduction

The tropical approach to enumerative geometry, initiated by Mikhalkin’s correspondence theorem [6], has led to remarkable success involving different kinds of enumerative numbers. An important example is the study of Welschinger invariants. These are weighted numbers of rational real plane curves satisfying point conditions which can be viewed as real analogues of plane Gromov-Witten invariants [14].

Mikhalkin originally used tropical methods to count curves in toric surfaces satisfying point conditions. Nowadays there are also tropical approaches for curves satisfying tangency conditions in addition [13, 9], for covers satisfying ramification conditions [11] and for curves in higher dimensional varieties satisfying point conditions [16]. Little is known about the enumerative geometry of surfaces in toric threefolds and the tropical counterparts. With this paper, we contribute a first step towards the establishment of tropical methods in such higher-dimensional enumerative problems.

More concretely, we focus on extending a combinatorial tool which has played an important role for the study of plane curves: Mikhalkin’s lattice path algorithm. Roughly, this algorithm can be viewed as the ”dual formulation” of the task to find all plane tropical curves of a given degree and genus satisfying given point conditions. By dual we mean the combinatorics of the subdivision of the Newton polytope which is dual to a tropical hypersurface. For a certain choice of point conditions, the translation to the dual setting is particularly nice. In the planar case, it turned out that the lattice path algorithm can be viewed as a variation of the Caporaso-Harris algorithm [13]. In the tropical world, one can use the analogue of the Caporaso-Harris degeneration technique in a non-recursive, global way leading to the so-called floor diagrams [10]. The latter can be viewed as the combinatorial essence of the lattice path algorithm. The lattice path algorithm and the Caporaso-Harris degeneration in the tropical setting have been used to derive new results and recursive formulas for Welschinger invariants [14]. The floor diagram count of plane tropical curves has led to new results concerning node polynomials [12].

In this paper, we establish a higher-dimensional version of the lattice path algorithm that counts singular surfaces in toric threefolds satisfying point conditions by means of tropical geometry. More precisely, we fix a divisor class in a compact toric threefold by fixing a lattice polytope. We count surfaces in this divisor class which are singular and satisfy the right number of generically chosen point conditions, or equivalently, we compute the degree of the discriminant of the lattice points in our polytope. The genericity of the point conditions implies that the surfaces in question have

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precisely one singular point, namely a node. Our tropical lattice path algorithm relies on the
classification of singular tropical surfaces we considered in an earlier paper [15].

Compared to Mikhalkin’s lattice path count for plane tropical curves, we encounter several
difficulties and new phenomena in our algorithm. First, the edges dual to faces passing through
one of our fixed points in the dual subdivision to a singular tropical surface do not necessarily form
a lattice path but may consist of two components. We call these situations “lattice paths with a
gap”. Second, the combinatorics of circuits dual to the singular point gets much more complicated.
Notably, even in the count of singular surfaces in $\mathbb{P}^3$ of degree $d$, that we tropically construct
in terms of floor decomposed tropical singular surfaces, we can observe different types of circuits
including ones with involved combinatorial structure. More precisely, we can have pentatopes,
squares and double edges as circuits in the count of degree $d$ surfaces.

The paper is organized as follows. In Section 2, we collect preliminaries, introduce notation
and present the problem (see Problem 2.2). In Section 3, we present a lattice path algorithm
listing all lattice paths that are used to enumerate singular tropical surfaces. In Section 4, we
compute multiplicities for each case that can appear, i.e. the number of singular algebraic surfaces
tropicalizing to a given tropical singular surface. Section 3 and 4 should be viewed as our main
results and give a complete answer to problem 2.2 stated in Section 2. In Section 5, we apply
our algorithm to surfaces in $\mathbb{P}^3$ of degree $d$, giving exact answers for $d = 2, 3$ and recovering the
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2. Preliminaries

We work over the field of complex Puiseux series $\mathbb{K} = \bigcup_{m \geq 1} \mathbb{C}\{t^{1/m}\}$ possessing the non-
Archimedean valuation $\text{Val} (\sum_r a_r t^r) = -\min \{ r \in \mathbb{Q} : a_r \neq 0 \}.$

Let $\Delta \subset \mathbb{R}^3$ be a non-defective convex lattice polytope such that the set \{ $u - u' : u, u' \in \Delta \cap \mathbb{Z}^3$\}
generates the lattice $\mathbb{Z}^3$. Let $N = |\Delta \cap \mathbb{Z}^3| - 2 > 0$. Denote by $\text{Tor}_\mathbb{K}(\Delta)$ the toric variety over $\mathbb{K}$
associated to the polytope $\Delta$. Let $\mathcal{L}_\Delta$ be the tautological line bundle on $\text{Tor}_\mathbb{K}(\Delta)$. Sections of $\mathcal{L}_\Delta$
are (Laurent) polynomials with support inside $\Delta$. Denote by $|\mathcal{L}_\Delta|$ the linear system of divisors of $0 \neq \varphi \in H^0(\mathcal{L}_\Delta)$. Clearly, $\dim |\mathcal{L}_\Delta| = |\Delta \cap \mathbb{Z}^3| - 1 = N + 1$. Define the discriminant $\text{Sing}(\Delta) \subset |\mathcal{L}_\Delta|$ to
be the family parameterizing divisors with a singularity in $(\mathbb{K}^*)^3$. The discriminant $\text{Sing}(\Delta)$ is a
hypersurface, and it is natural to ask for its degree $\text{deg } \text{Sing}(\Delta)$.

Example 2.1. In several cases the answer is known, for instance, if $\Delta$ is the simplex with vertices
$(0, 0, 0), (d, 0, 0), (0, d, 0)$ and $(0, 0, d)$, then $\text{deg } \text{Sing}(\Delta) = 4(d - 1)^3$.

Geometrically, the degree can be seen as $\#(\text{Sing}(\Delta) \cap \mathcal{P})$, where $\mathcal{P} \subset |\mathcal{L}_\Delta|$ is a generic pencil.
For example, we can take the pencil $\{ S \in |\mathcal{L}_\Delta| : S \supset \mathcal{P} \}$, where $\mathcal{P} = (p_1, ..., p_N)$ is a configuration
of $N$ points in $(\mathbb{K}^*)^3$ in general position. Such a geometric view of the degree of $\text{Sing}(\Delta)$ can be
translated to the tropical setting.

Denote by $\text{Sing}^t(\Delta)$ the tropical discriminant parameterizing singular tropical surfaces with
Newton polytope $\Delta$, i.e. tropicalizations of algebraic surfaces $S \in \text{Sing}(\Delta)$ (background on singular
tropical hypersurfaces can be found in [11] [12] [15]). Suppose that $\mathcal{F} = (x_1, ..., x_N) = \text{Val}(\mathcal{P}) \subset \mathbb{Q}^3$ is
a configuration of $N$ distinct points, which are in general position. Then the set $\text{Sing}^t(\Delta, \mathcal{P}) :=$
{ $S \in \text{Sing}^t(\Delta) : S \supset \mathcal{F}$} is finite, all tropical surfaces in this set are of maximal-dimensional
geometric type, and the points $x_1, ..., x_N$ are interior points of 2-faces in each of these surfaces
(see [15] Theorem 1 and Section 2.3]). We fix a general configuration $\mathcal{F} \subset \mathbb{Q}^3$ and suppose that
$\mathcal{F} \subset (\mathbb{K}^*)^3$ is generic among the configurations tropicalizing to $\mathcal{F}$.
Problem 2.2. (1) Describe the combinatorics of tropical surfaces $S \in \Sing^\tr(\Delta)$ passing through a given configuration $\vec{x} \subset \mathbb{Q}^3$ of $N$ points in general position. We denote this set by $\Sing^\tr(\Delta, \vec{x})$.
(2) Given a tropical surface $S \in \Sing^\tr(\Delta, \vec{x})$, calculate $\mt(S, \vec{x})$, the cardinality of the set $\Sing(\Delta, \vec{p}, S)$ of surfaces $S \in \Sing(\Delta)$ that tropicalize to $S$ and pass through a fixed generic configuration $\vec{p} \subset (\mathbb{K}^*)^3$ of $N$ points such that $\Val(\vec{p}) = \vec{x}$.

We present a solution to the first part of the problem in Section 3, where we develop a three-dimensional version of Mikhalkin’s lattice path algorithm for plane tropical curves [4, 6]. A solution to the second part is presented in Section 4. The following corollary is an immediate consequence.

Corollary 2.3. The degree of $\Sing(\Delta)$ can be computed by means of a tropical lattice path algorithm as

$$\deg \Sing(\Delta) = \sum_{S \in \Sing^\tr(\Delta, \vec{x})} \mt(S, \vec{x}).$$

Moreover, the patchworking procedure of Section 4 allows us to explicitly exhibit all algebraic surfaces $S \in \Sing(\Delta)$ passing through any fixed configuration $\vec{p} \subset (\mathbb{K}^*)^3$ such that $\Val(\vec{p})$ is in general position in $\mathbb{R}^3$. In Section 5 we analyze the formula of Example 2.1: we prove it for $d = 2, 3$ and compute the top asymptotic term in the general case.

Our result relies on the classification of singular tropical surfaces in $\mathbb{R}^3$ of maximal-dimensional geometric type [15]. The dual subdivision of such a singular tropical surface contains a unique circuit whose dual cell in the surface contains the singular points. The possible circuits of affine lattice point configurations in three-space are classified (up to integer unimodular transformations), see Figure 1. We have to consider different cases for the lattice path algorithm according to the

$$\begin{align*}
\text{(A)} & \quad \text{(B)} & \quad \text{(C)} & \quad \text{(D)} & \quad \text{(E)}
\end{align*}$$

Figure 1. The possible circuits in the dual subdivision of a singular tropical surface.

type of the circuit in the dual subdivision of a singular tropical surface passing through the points.

3. The lattice path algorithm in dimension 3

In this section, we present a solution for Problem 2.2 (1).

3.1. Tropical point constraints in Mikhalkin’s position. To apply a lattice path algorithm similar to the one for tropical curves [4, 6 Section 7.2], we place the points in the following special position. Choose a line $L \subset \mathbb{R}^3$ passing through the origin and directed by a vector $\vec{v} \in \mathbb{Q}^3$, which is not parallel or orthogonal to any proper affine subspace of $\mathbb{R}^3$ spanned by a non-empty subset $A \subset \Delta \cap \mathbb{Z}^3$; then pick the following (ordered) configuration $\vec{x} = (x_1, \ldots, x_N)$ of marked points

$$\begin{align*}
x_i = M_i \vec{v} \in L, \quad i = 1, \ldots, N, \quad \text{where} \quad 0 \ll M_1 \ll \ldots \ll M_N \text{ are positive rationals.}
\end{align*}$$

Remark 3.1. The configuration (2) is generic. The set $\Sing^\tr(\Delta, \vec{x})$ is finite, and all its elements are singular tropical surfaces of maximal-dimensional geometric type as described in [15] Theorem 2]. Moreover, for any $S \in \Sing^\tr(\Delta, \vec{x})$, each marked point $x_i, 1 \leq i \leq N$ is in the interior of a 2-face $F_i$ of $S$, and $F_i \neq F_j$ as $i \neq j$.

We will solve Problem 2.2 (1) for point configurations satisfying (2).
3.2. The dual reformulation. Introduce the partial order in $\mathbb{R}^3$: $\mathbf{u} \succ \mathbf{u}' \iff \langle \mathbf{u} - \mathbf{u}', \mathbf{v} \rangle > 0$ to obtain a linear order on $\Delta \cap \mathbb{Z}^3$:

$$\Delta \cap \mathbb{Z}^3 = \{ \mathbf{w}_0, ..., \mathbf{w}_{N+1} \}, \quad \mathbf{w}_i < \mathbf{w}_{i+1} \text{ for all } i = 0, ..., N.$$ 

Given a subset $A \subset \Delta \cap \mathbb{Z}^3$, consisting of $m \geq 2$ points $a_1 < a_2 < ... < a_m$, the complete lattice path supported on $A$ is the union $P(A)$ of segments $[a_i, a_{i+1}]$, $1 \leq i < m$. A (partial) lattice path supported on $A$ is a union of a non-empty subset of $\{ [a_i, a_{i+1}] : i = 1, ..., m-1 \}$ that contains the whole set $A$.

Let $F_S : \mathbb{R}^3 \to \mathbb{R}$ be a tropical polynomial defining a singular tropical surface $S \in \text{Sing}^\text{tr}(\Delta, \overline{\mathbf{x}})$, $\nu_S : \Delta \to \mathbb{R}$ the Legendre dual piecewise linear function, whose linearity domains determine the subdivision $\Sigma_S$ of $\Delta$ dual to $S$. Denote by $e_i$, $i = 1, ..., N$ the edge of $\Sigma_S$ dual to the 2-face $F_i$ of $S$ containing the point $x_i$ in its interior. We denote by $P(S, \overline{\mathbf{x}}) = \bigcup_{i=1}^N e_i \subset \Delta$ the lattice path corresponding to the pair $(S, \overline{\mathbf{x}})$.

**Lemma 3.2.** For a singular tropical surface $S$ passing through $\overline{\mathbf{x}}$, the lattice path $P(S, \overline{\mathbf{x}})$ defined above satisfies:

(i) Either $P(S, \overline{\mathbf{x}}) = P(A') \cup P(A'')$, where $A' = \{ \mathbf{w}_0, ..., \mathbf{w}_k \}$, $A'' = \{ \mathbf{w}_{k+1}, ..., \mathbf{w}_{N+1} \}$ for some $1 \leq k \leq N$; we call this path $\Gamma_{k,k+1}$;

(ii) or $P(S, \overline{\mathbf{x}}) = P(A)$, where $A = \Delta \cap \mathbb{Z}^3 \setminus \{ \mathbf{w}_k \}$ for some $0 \leq k \leq N+1$; we call this path $\Gamma_k$.

We call the lattice paths $\Gamma_k$, $k = 0, ..., N+1$ and $\Gamma_{k,k+1}$, $k = 1, ..., N$ the marked lattice paths for $\Delta$.

**Proof.** By the duality of $S$ and the subdivision $\Sigma_S$ (see [3, Section 2.1]), the components of $\mathbb{R}^3 \setminus S$ are in one-to-one correspondence with a subset of $\Delta \cap \mathbb{Z}^3$ (including all the vertices of $\Delta$). Due to the convexity of these components, different connected components of $L \setminus \overline{\mathbf{x}}$ cannot intersect the same component of $\mathbb{R}^3 \setminus S$. Since $L \setminus \overline{\mathbf{x}}$ has $|\overline{\mathbf{x}}| + 1 = N + 1 = |\Delta \cap \mathbb{Z}^3| - 1$ components, we encounter the following situations:

(a) both $L \setminus \overline{\mathbf{x}}$ and $\mathbb{R}^3 \setminus S$ consist of $N + 1$ components;

(b) $L \setminus \overline{\mathbf{x}}$ consists of $N + 1$ components, and $\mathbb{R}^3 \setminus S$ consists of $N + 2$ components.

Now note that if $\mathbf{w}_i$ and $\mathbf{w}_j$ are dual to the components $\mathbf{w}_i^*$, $\mathbf{w}_j^*$ intersecting $L$ along neighboring intervals, and the vector $\mathbf{v}$ points from $\mathbf{w}_i^*$ to $\mathbf{w}_j^*$, then $\mathbf{w}_j \succ \mathbf{w}_i$.

In case (a), there exists a unique point $\mathbf{w}_k$, $0 \leq k \leq N+1$, that is not a vertex of the subdivision $\Sigma_S$. Then $P(S, \overline{\mathbf{x}}) = \Gamma_k$.

In case (b), if there is a component $\mathbf{w}_k^*$ of $\mathbb{R}^3 \setminus S$ disjoint from $L$, then $P(S, \overline{\mathbf{x}})$ again is $\Gamma_k$ for some $0 \leq k \leq N + 1$. Otherwise, we have an extra intersection point $\mathbf{y} \in L \setminus \overline{\mathbf{x}}$ of $L \cap S$, and then: if $\mathbf{y} < \mathbf{x}_1$ we get the path $\Gamma_0$, if $\mathbf{x}_k < \mathbf{y} < \mathbf{x}_{k+1}$ for some $k = 1, ..., N$, we get the path $\Gamma_{k,k+1}$, and at last, if $\mathbf{x}_N < \mathbf{y}$ we get the path $\Gamma_{N+1}$.

We can now refine problem 2.2(1) as follows:

**Problem 3.3.** Given a marked lattice path $P$, find all subdivisions $\Sigma$ of $\Delta$ that contain the path $P$ (i.e., each edge of $P$ is an edge of the subdivision $\Sigma$) and are dual to singular tropical surfaces $S$ passing through $\overline{\mathbf{x}}$ (such that the edge dual to the 2-face $F_i$ containing $x_i$ is in $P$).

We suggest a solution to Problem 3.3 which can be regarded as a three-dimensional version of Mikhalkin’s lattice path algorithm [10]. By [15, Theorem 2], the desired subdivision $\Sigma$ has one circuit of type $A$, $B$, $C$, $D$, or $E$ as depicted in Figure 1 and all its three-dimensional cells that do not contain the circuit are simplices, i.e. tetrahedra whose only integral points are their vertices. In the next Section 3.3, we present an auxiliary construction that completes the subdivision outside the circuit. In Section 3.4 we explain how to fit a circuit in a subdivision for a given lattice path.
3.3. The smooth subdivision algorithm. We will first show in general terms, how to extend a given subdivision when the underlying polytope is enlarged.

Lemma 3.4. Let us be given the following data:

- a convex lattice polytope $\delta' \subset \mathbb{R}^n$ and a convex piecewise linear function $\nu' : \delta' \to \mathbb{R}$, whose linearity domains define a subdivision $\sigma'$ of $\delta'$ into convex lattice subpolytopes;
- a convex lattice polytope $\delta'' \subset \mathbb{R}^n$ such that $\delta_0 = \delta' \cap \delta''$ is a cell of the subdivision $\sigma'$ and a face of $\delta''$ of codimension 1.

Pick a point $w \in \delta'' \cap \mathbb{Z}^n \setminus \delta'$. Then there exists a unique extension of $\sigma'$ to a convex subdivision $\sigma$ of $\delta = \text{Conv}(\delta' \cup \delta'')$ such that

- the vertices of $\sigma$ are the vertices of $\sigma'$ and of $\delta''$,
- $\delta''$ is a cell of $\sigma$,
- the cells of $\sigma$ are linearity domains of a convex piecewise linear function $\nu : \delta \to \mathbb{R}$ such that $\nu|_{\delta'} = \nu'$ and $\nu(w) \geq \max \nu'$.

Proof. Clearly, $w$ does not lie in the affine subspace of $\mathbb{R}^n$ spanned by $\delta_0$. Hence the (linear) function $\nu'|_{\delta_0}$ and the value $\nu(w)$ induce a unique linear function $\nu''$ on $\delta''$. Furthermore, the condition $\nu(w) \geq \max \nu'$ ensures that any segment in $\mathbb{R}^{n+1}$ joining an interior point of the graph of $\nu'$ and an interior point of the graph of $\nu''$ lies above these graphs. Hence the lower facets of $\text{Conv}(\text{Graph}(\nu') \cup \text{Graph}(\nu''))$ (i.e. the facets whose outer normal vector has a negative last coordinate) defines a graph of a convex piecewise linear function $\nu : \delta \to \mathbb{R}$ as required. Finally, we note that there is a $\mu \gg \max \nu'$ such that the subdivision of $\delta$ defined by the linearity domains of $\nu$ does not depend on the choice of the value $\nu(w) > \mu$. \qed

Example 3.5. Let $\delta' \subset \mathbb{R}^n$ and $\nu' : \delta' \to \mathbb{R}$ be as in Lemma 3.4, $w \in \mathbb{Z}^n \setminus \delta'$, $\delta = \text{Conv}(\delta' \cup \{w\})$. Let $v \in \mathbb{Q}^n$ be a vector which is not parallel or orthogonal to any segment joining any two distinct points of $\delta$. Suppose that $w >_v w'$ for any $w' \in \delta'$. Then the construction of Lemma 3.4 works as follows. Note that there exists a point $\tilde{w} \in \delta'$ which satisfies $\tilde{w} >_v w'$ for all $w' \in \delta' \setminus \{\tilde{w}\}$ and that the segment $[\tilde{w}, w]$ intersects with $\delta'$ only at $\tilde{w}$. Then we can put $\delta'' = [\tilde{w}, w]$ and extend the subdivision $\sigma'$ of $\delta'$ to a convex subdivision of $\delta$. We call the subdivision $\sigma$ of $\delta$ the smooth extension of $\sigma'$.

An important particular case is the following construction.

Lemma 3.6. Let $\Delta = \text{Conv}(A)$, where $A \subset \Delta \cap \mathbb{Z}^3$, $|A| = N + 1$, and $A = \{a_0, \ldots, a_N\}$, $a_0 < a_1 < \ldots < a_N$ (order defined by $v$). Let $\overline{\mathcal{X}}$ be a sequence of points of $\mathbb{R}^3$ given by (2). Then:

(i) In the space of tropical surfaces defined by tropical polynomials of the form

$$F : \mathbb{R}^3 \to \mathbb{R}, \quad F(X) = \max_{\omega \in A} (c_i + \langle a_i, X \rangle), \quad c_i \in \mathbb{R}, \quad i = 0, \ldots, N,$$

there exists a unique surface $S = S(A, \overline{\mathcal{X}})$, that passes through $\overline{\mathcal{X}}$.

(ii) Each point of $\overline{\mathcal{X}}$ belongs to the interior of some 2-face of $S$, and distinct points belong to distinct faces.

(iii) The dual subdivision $\Sigma_S$ consists of only tetrahedra, and it is constructed by a sequence of smooth extensions, when starting with the point $a_0$ and subsequently adding the points $a_1, \ldots, a_N$. The edges dual to the faces of $S$, that intersect $\overline{\mathcal{X}}$, form the lattice path $P(A)$ subsequently going through the points $a_0, \ldots, a_N$.

Notice that we view the space of tropical surfaces defined by tropical polynomials as above as $\mathbb{R}^{|A|}/\{1, \ldots, 1\}$. In particular, we can always assume that the first coefficient of the tropical polynomial satisfies $c_0 = 0$.

Proof. Statements (ii) immediately follows from the general position of $\overline{\mathcal{X}}$. Thus, we explain only parts (i) and (iii). The polynomial $F_S(X)$ defining $S$ can be computed from the formulas:

$$c_0 = 0, \quad c_{i-1} + \langle a_{i-1}, x_i \rangle = c_i + \langle a_i, x_i \rangle, \quad i = 1, \ldots, N,$$
or, equivalently,
\[ c_0 = 0, \quad c_i - c_{i-1} = -M_i(a_i - a_{i-1}, v), \quad i = 1, \ldots, N. \] (3)
The function \( \nu_S : \Delta \to \mathbb{R} \) takes value \(-c_i\) at the point \( a_i, \quad i = 0, \ldots, N \). Since \( 0 \ll M_1 \ll \cdots \ll M_N \), we have \( \nu_S(a_i) \gg \nu_S(a_{i-1}) \) for all \( i = 1, \ldots, N \), which is required in Lemma 3.4 and Example 3.5.

\[ \square \]

3.4. **Subdivisions with prescribed type of circuit.** In this section, we study how the types of circuits depicted in Figure 1 fit into subdivisions for a given lattice path.

3.4.1. **Subdivisions with circuit of type B, C, or E (see Figure 1).**

**Lemma 3.7.** (1) A marked lattice path \( P \) admits an extension to a subdivision \( \Sigma \) of \( \Delta \), dual to a surface \( S \in \text{Sing}^\text{tr}(\Delta, \mathcal{F}) \) and having a circuit of type B, C, or E, only if \( P = \Gamma_k \) (see Lemma 3.2), where \( 1 \leq k \leq N \), and \( w_k \) is not a vertex of \( \Delta \). Moreover, this subdivision is unique and it can be constructed by the smooth triangulation algorithm of Lemma 3.6(iii) supported on the set \( A = \Delta \cap \mathbb{Z}^3 \setminus \{w_k\} \).

(2) Let \( P = \Gamma_k \), where \( 1 \leq k \leq N \) and \( w_k \) is not a vertex of \( \Delta \). Then the subdivision \( \Sigma \) of \( \Delta \), constructed as in item (1), is dual to a surface \( S \in \text{Sing}^\text{tr}(\Delta, \mathcal{F}) \) if and only if one of the following conditions holds true:

- the point \( w_k \) belongs to the interior of a three-dimensional cell of \( \Sigma \) (i.e. \( w_k \) is the interior point of a circuit of type B);
- the point \( w_k \) belongs to the interior of a two-dimensional cell of \( \Sigma \), and, if \( w_k \in \partial \Delta \), the subdivision \( \Sigma \) additionally satisfies the third condition in [15, Theorem 4] (i.e. \( w_k \) is the interior point of a circuit of type C);
- the point \( w_k \) is the midpoint of an edge of \( \Sigma \), and, if \( w_k \in \partial \Delta \), the subdivision \( \Sigma \) additionally satisfies the fourth condition in [15, Theorem 4] (i.e. \( w_k \) is the interior point of a circuit of type E).

**Proof.** Statement (1) is straightforward. Statement (2) follows from [15, Theorem 4].

**Remark 3.8.** It follows from the smooth triangulation algorithm of Lemma 3.6(iii) that the coefficients \( c_i, i \neq k \), of the tropical polynomial defining the unique surface \( S \in \text{Sing}^\text{tr}(\Delta, \mathcal{F}) \) dual to a subdivision extending \( \Gamma_k \) and containing a circuit of type B, C, or E according to Lemma 3.7(2) are determined by the point conditions \( \mathcal{F} \). Furthermore, the lattice points \( w_l \) forming the circuit satisfy a unique up to nonzero multiple relation \( \sum \lambda_l w_l = 0 \) with \( \sum \lambda_l = 0 \). Since the circuit is part of the subdivision, it follows that \( \sum \lambda_l c_l = 0 \), which allows us to deduce the value of \( c_k \) from the others. We call the equation \( \sum \lambda_l c_l = 0 \) defining \( c_k \) the circuit relation for the coefficients of the tropical polynomial.

3.4.2. **Subdivisions with circuit of type D (see Figure 1).** For circuits of type D, we have to treat the case of a connected path \( \Gamma_k \) or a disconnected path \( \Gamma_{k,k+1} \) (see Lemma 3.2) separately.

(1) **The case of a connected path \( P \).**

**Lemma 3.9.** Let \( P = \Gamma_k \) for some \( k = 0, \ldots, N + 1 \), and let \( P \) extend to a subdivision \( \Sigma \) of \( \Delta \) with a circuit \( C \) of type D, that is dual to a surface \( S \in \text{Sing}^\text{tr}(\Delta, \mathcal{F}) \). Then

- (i) the circuit \( C \) contains \( w_k \) and three more vertices \( w_i, w_j, w_l \), \( i < j < l \);
- (ii) the subdivision \( \Sigma \) is uniquely determined by the pair \((k, C)\), in particular,
  - it contains a smooth triangulation of \( \text{Conv}(P(l^*)) \) as in Lemma 3.6, where \( P(l^*) \) is the part of \( P \) bounded from above by the vertex \( w_i \) preceding \( w_l \) in \( P \);
  - the parallelogram \( \text{Conv}(C) \) intersects \( \text{Conv}(P(l^*)) \) along the edge \([w_i, w_j] \);
  - \( \Sigma \) is obtained from the triangulation of \( \text{Conv}(P(l^*)) \) by the extension to \( \text{Conv}(P(l^*)) \cup C \) as in Lemma 3.4 and by a sequence of smooth extensions as in Example 3.5 when subsequently adding the points of \( P \) following \( w_l \).
Proof. We explain only the first claim in statement (ii), since the rest is straightforward. As in Lemma 3.6 and Remark 3.8, we obtain the coefficients of a tropical polynomial defining the surface $S$ from the point conditions and the circuit relation. Let $\nu_S$ be the piece-wise linear function defined by this polynomial.

Suppose that $w_s \in P$, $\Delta$, $\nu = \text{Conv}(P(s))$ (where $P(s)$ is the part of $P$ bounded from above by the vertex $w_s$) is smoothly triangulated, and $s$ is maximal like that. Assume that $s < l^\ast$. The fact that the triangulation of $\Delta$ does not extend to a smooth triangulation of $\text{Conv}(\Delta \cup \{w_{s+1}\})$ means that in the graph of $\nu$, there exists a line segment $\sigma_1$ joining $(w_{s+1}, \nu_S(w_{s+1}))$ with a point $(z_1, \nu_S(z_1)) \in \Delta \times \mathbb{R}$ and a line segment $\sigma_2$ joining a point $(w_m, \nu_S(w_m))$, $m > s + 1$, or the point $(w_k, \nu_S(w_k))$ with a point $(z_2, \nu_S(z_2)) \in \Delta \times \mathbb{R}$, such that $\sigma_1 \cap (\Delta \times \mathbb{R}) = (z_1, \nu_S(z_1))$, $\sigma_2 \cap (\Delta \times \mathbb{R}) = (z_2, \nu_S(z_2))$, $\sigma_2$ lies in a lower face of the graph of $\nu_S$, and the projections of $\sigma_1, \sigma_2$ onto $\mathbb{R}^3$ intersect in the interior of the projection of this face. This, however, contradicts the convexity of the function $\nu_S : \Delta \rightarrow \mathbb{R}$, since the values $\nu_S(w_m)$, where $m > s + 1$ or $m = k$, are much larger than $\nu_S(w_{s+1})$ (for $m \neq k$ this follows from the smooth triangulation algorithm Lemma 3.6, for $m = k$ from the circuit relation as in Remark 3.8).

Lemma 3.9 provides only necessary conditions for a subdivision with circuit with type D dual to a singular tropical surface passing through the given point configuration. To formulate sufficient conditions, consider the univariate tropical polynomial

$$F_S|_L(\tau) = \max_{0 \leq s \leq N+1} \left( c_s + \tau \langle w_s, v \rangle \right).$$

(4)

Its coefficients $c_0, ..., c_{N+1}$ are determined by the following relations (point conditions and circuit relation, see Remark 3.8):

- for $k = 0$
  $$c_1 = 0, \quad c_{s+1} + M_s(w_{s+1}, v) = c_s + M_s(w_s, v), \quad 1 \leq s \leq N, \quad c_0 + c_l = c_i + c_j,$$

- for $k = N + 1$
  $$c_0 = 0, \quad c_s + M_s(w_s, v) = c_{s-1} + M_s(w_{s-1}, v), \quad 1 \leq s \leq N, \quad c_i + c_{N+1} = c_j + c_l,$$

- for $1 \leq k \leq N$
  $$c_0 = 0, \quad \begin{cases} c_s + M_s(w_s, v) = c_{s-1} + M_s(w_{s-1}, v), & \text{as } 1 \leq s < k, \\ c_{k+1} + M_k(w_{k+1}, v) = c_{k-1} + M_k(w_{k-1}, v), \\ c_{s+1} + M_s(w_{s+1}, v) = c_s + M_s(w_s, v), & \text{as } k < s \leq N, \end{cases}$$

$$\begin{cases} c_k + c_l = c_i + c_j, & \text{if } k < i, \\ c_i + c_l = c_k + c_j, & \text{if } i < k < l, \\ c_i + c_k = c_j + c_l, & \text{if } k > l. \end{cases}$$

Lemma 3.10. The subdivision $\Sigma$ constructed in Lemma 3.9 is dual to a tropical surface $S \in \text{Sing}^n(\Delta, \mathbf{F})$ if and only if the following conditions hold:

(i) the face of $\Sigma$ given by the circuit does not lie on $\partial \Delta$;

(ii) $i < k < l$.

Proof. The first condition is necessary by [15] Theorem 4. Having it fulfilled, we have to ensure that the roots of the tropical polynomial $F_S|_L(\tau)$ (that is the restriction of the tropical polynomial defining $S$ to the line $L$) are $M_s$, $1 \leq s \leq N$, and maybe one more root outside the range $[M_1, M_N]$. Since the tropical polynomial

$$\bar{F}(\tau) = \max_{0 \leq s \leq N+1, \ s \neq k} \left( c_s + \tau \langle w_s, v \rangle \right)$$
has precisely the roots $M_1, \ldots, M_N$, we end up with inequalities
\[
\begin{cases}
    c_0 + M_1 \langle w_0, v \rangle \leq M_1 \langle w_1, v \rangle = c_2 + M_1 \langle w_2, v \rangle, & \text{if } k = 0, \\
    c_k + M_k \langle w_k, v \rangle \leq c_{k-1} + M_k \langle w_{k-1}, v \rangle = c_{k+1} + M_k \langle w_{k+1}, v \rangle, & \text{if } 1 \leq k \leq N, \\
    c_{N+1} + M_N \langle w_{N+1}, v \rangle \leq c_N + M_N \langle w_N, v \rangle = c_{N-1} + M_N \langle w_{N-1}, v \rangle, & \text{if } k = N + 1.
\end{cases}
\]  

Condition (5) is necessary as well, cf. the proof of Lemma 3.2. The sufficiency of conditions (i) and (5) comes again from [15, Theorem 4] (i.e., $S$ is singular), and from the fact that $F_S|_L(\tau)$ defines the intersection points of $L$ and $S$:

We show the equivalence of (ii) and (5). Suppose that $i < j < l < k \leq N$. Then from (5) and the circuit relation we derive
\[
c_i = o(M_i), \quad c_j = o(M_j), \quad c_l = o(M_l), \quad c_k = c_i + c_j - c_l = O(M_l),
\]
and hence
\[
c_k + M_k \langle w_k, v \rangle \leq c_{k-1} + M_k \langle w_{k-1}, v \rangle \iff c_k \leq c_{k-1} - M_k \langle w_k - w_{k-1}, v \rangle \iff O(M_l) \leq -M_k \langle w_k - w_{k-1}, v \rangle + o(M_k),
\]
a contradiction.

Suppose that $i < j < l < N < k = N + 1$. Again (5) and the circuit relations yield
\[
c_{N+1} + M_N \langle w_{N+1}, v \rangle \leq c_N + M_N \langle w_N, v \rangle \iff c_{N+1} \leq c_N - M_N \langle w_{N+1} - w_N, v \rangle \iff O(M_l) \leq -M_N \langle w_{N+1} - w_N, v \rangle + o(M_N),
\]
a contradiction, since $l < N$, and hence $c_l = O(M_l) = o(M_N)$.

Suppose that $i < j < l = N < k = N + 1$. Then similarly we get
\[
c_{N+1} + M_N \langle w_{N+1}, v \rangle \leq c_N + M_N \langle w_N, v \rangle \iff c_{N+1} \leq c_N - M_N \langle w_{N+1} - w_N, v \rangle \iff c_j - c_i \leq -M_N \langle w_{N+1} - w_N, v \rangle,
\]
which is a contradiction.

Suppose that $1 \leq k < i < j < l$. Then we have
\[
\begin{cases}
    c_i = o(M_{l-1}), \quad c_j = o(M_{l-1}), \quad c_l = -M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}), \\
    c_{k+1} = o(M_{l-1}), \quad c_k = c_i + c_j - c_l = -M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}),
\end{cases}
\]
and hence
\[
c_k + M_k \langle w_k, v \rangle \leq c_{k+1} + M_k \langle w_{k+1}, v \rangle \iff c_k \leq c_{k+1} + M_k \langle w_{k+1} - w_k, v \rangle \iff M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}) \leq o(M_{l-1}),
\]
a contradiction.

In the case $k = 0 < i < j < l$, we again have relations (7), and hence
\[
c_0 + M_1 \langle w_0, v \rangle \leq M_1 \langle w_1, v \rangle \iff c_0 \leq M_1 \langle w_1 - w_0, v \rangle \iff M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}) \leq o(M_{l-1}),
\]
a contradiction.

In the case $k = 0$, $i = 1 < j < l$, we similarly obtain
\[
c_0 \leq M_1 \langle w_1 - w_0, v \rangle = O(M_1) = o(M_{l-1}),
\]
since $l - 1 \geq 2$. However, from the circuit relation, we get
\[
c_0 = c_j - c_i = M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}),
\]
which contradicts the former conclusion.
Suppose that \( i < k < l \). Then the equations for \( c_s, 0 \leq s \leq N + 1 \), yield
\[
c_t = \begin{cases} -M_{l-1} \langle w_{l-1}, v \rangle + o(M_{l-1}), & \text{if } k < l - 1, \\
-M_{l-1} \langle w_{l-2}, v \rangle + o(M_{l-1}), & \text{if } k = l - 1.
\end{cases}
\]

If \( k < l - 1 \), the required relation reads
\[
c_k + M_k \langle w_k, v \rangle \leq c_{k+1} + M_k \langle w_{k+1}, v \rangle \implies c_k \leq c_{k+1} + M_k \langle w_{k+1} - w_k, v \rangle = O(M_k) = o(M_{l-1})
\]
\[
\iff c_k + c_i - c_j \leq o(M_{l-1}) \iff -M_{l-1} \langle w_l - w_{l-1}, v \rangle + o(M_{l-1}) \leq o(M_{l-1}),
\]
which holds true.

If \( k = l - 1 \), the required relation reads
\[
c_i + c_j - M_{l-1} \langle w_{l-1}, v \rangle \leq c_i + M_{l-1} \langle w_l, v \rangle \iff c_i - c_j = o(M_{l-1}) \leq M_{l-1} \langle w_l - w_{l-1}, v \rangle,
\]
which again holds true.

(2) The case of a disconnected path \( P \).

Lemma 3.11. Let \( P = \Gamma_{k,k+1} \) for some \( 1 \leq k < N \). Then it cannot be extended to a subdivision of \( \Delta \) with a circuit of type \( D \) that is dual to a surface \( S \in \text{Sing}^D(\Delta, \mathcal{F}) \).

Proof. Observe that \( L \cap S \) contains the marked points \( x_s = M_s v, 1 \leq s \leq N \), and one more point \( x_0 = M_0 v \) such that \( M_k < M_0 < M_{k+1} \), which separates the intervals \( w_k^* \cap L \) and \( w_{k+1}^* \cap L \) (here, \( w_k^* \) denotes the connected component of \( \mathbb{R}^3 \setminus S \) dual to \( w_k \), cf. the proof of Lemma 3.2). If \( P \) extends to a subdivision of \( \Delta \) dual to a surface \( S \in \text{Sing}^D(\Delta, \mathcal{F}) \), then the coefficients of the tropical polynomial \( F_S|_L(\tau) \) (see (8)) can be computed from
\[
c_0 = 0, \quad c_{s+1} = \begin{cases} c_s - M_{s+1} \langle w_{s+1} - w_s, v \rangle, & \text{if } 0 \leq s < k, \\
c_k - M_0 \langle w_{k+1} - w_k, v \rangle, & \text{if } s = k, \\
c_s - M_s \langle w_{s+1} - w_s, v \rangle, & \text{if } k < s \leq N.
\end{cases}
\]

Assume the circuit of type \( D \) consists of the points \( w_i, w_j, w_l \) and \( w_m \), with \( i < j < l < m \). Joining relations (8) and \( 0 \ll M_1 \ll \ldots \ll M_N \), we can write the circuit relation \( c_i + c_m = c_j + c_k \) as \( c_m + O(c_m) = O(c_j) \), which yields \( c_m = O(c_j) \), and hence it can hold only if \( l = k, m = k + 1, \) and \( M_0 = O(M_k) \). However, under these conditions, the circuit relation \( c_i + c_{k+1} = c_j + c_k \) converts to
\[
c_{k+1} - c_k = c_j - c_i \implies -M_0 \langle w_{k+1} - w_k, v \rangle = O(M_j),
\]
which is a contradiction, since \( M_0 > M_k \gg M_j \) as \( j < k \). \( \square \)

3.4.3. Subdivisions with circuit of type \( A \) (see Figure 1). Recall (cf. [15], Theorem 2]) that a circuit of type \( A \) is formed by the vertices of a pentatope, which up to \( \mathbb{Z} \)-affine transformation can be identified with
\[
\Pi_{p,q} = \text{Conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,p,q)\}, \quad p, q > 0, \gcd(p, q) = 1.
\]

The circuit relation (see Remark 3.3) means that the points \( (\omega, -c_\omega) \in \mathbb{R}^4, \omega \in \Pi_{p,q}, c_\omega \in \mathbb{R} \), lie in one 3-plane, and it can be written as
\[
c_{100} + pc_{010} + qc_{001} = (p+q)c_{000} + c_{1pq}.
\]

(1) The case of a connected path \( P \).

Lemma 3.12. Let the lattice path \( P = \Gamma_k \) (see Lemma 3.2), \( 0 \leq k < N + 1 \), admit an extension to a subdivision \( \Sigma \) of \( \Delta \) with a circuit \( C = \{w_i, w_j, w_l, w_m, w_n\} \), \( i < j < l < m < n \), of type \( A \) dual to a surface \( S \in \text{Sing}^D(\Delta, \mathcal{F}) \). Then
\[
(i) \ k \in \{i, j, l, m, n\};
\]
\[
(ii) \ the \ cases \ k = n \leq N \ and \ k = n = N + 1 > m + 1 \ are \ not \ possible;
\]
\[
(iii) \ the \ subdivision \ \Sigma \ is \ uniquely \ determined \ by \ the \ pair \ (k, C) \ and \ satisfies \ the \ following:
\]

it contains a smooth triangulation of $\Delta_{m-1} = \text{Conv}\{w_s : 0 \leq s < m, \ s \neq k\}$;
• the pentatope $\text{Conv}(C)$ intersects $\Delta_{m-1}$ along their common 2-face spanned by the first three points of $C \setminus \{w_k\}$;
• $\Sigma$ is obtained from the triangulation of $\Delta_{m-1}$ by the extension to $\text{Conv}(\Delta_{m-1} \cup C)$ as in Lemma 3.12 and by a sequence of smooth extensions as in Example 3.5 when subsequently adding the points of $P$ following $w_n$.

Proof. Claim (i) immediately follows from formulas (3), since in case $k \notin \{i, j, l, m, n\}$, we would have $|c_n| \gg \max\{|c_i|, |c_j|, |c_l|, |c_m|\}$ contrary to the circuit relation (10) (combined with a proper $\mathbb{Z}$-affine transformation).

Suppose now that $k = n \leq N$. The necessary condition in this case is (see (5))

$$c_n + M_n\langle w_n, v \rangle \leq c_{n-1} + M_n\langle w_{n-1}, v \rangle$$

which again contradicts the circuit relation (10) (combined with a proper $\mathbb{Z}$-affine transformation).

Similarly to the proof of Lemma 3.10, we have to check conditions (5), which read

$$c_n = O(|c_i| + |c_j| + |c_l| + |c_m|) = O(M_m) = o(M_n),$$

whereas from the circuit relation (10) we get

$$c_n = O(|c_i| + |c_j| + |c_l| + |c_m|) = O(M_m) = o(M_N),$$

a contradiction.

Suppose that $k = n = N + 1 > m + 1$. Then the necessary condition (5) yields

$$c_{N+1} + M_N\langle w_{N+1}, v \rangle \leq c_{N-1} + M_N\langle w_{N-1}, v \rangle$$

which again contradicts the circuit relation

$$c_n = O(|c_i| + |c_j| + |c_l| + |c_m|) = O(M_m) = o(M_N).$$

Claim (iii) is proved analogously to Lemma 3.9(ii). \(\square\)

Lemma 3.13. In the notation of Lemma 3.12 let the data $k$ and $C$ satisfy conditions (i) and (ii), and let a subdivision $\Sigma$ of $\Delta$ be constructed as in item (iii). Write the circuit relation (10) in the form

$$c_k = \sum_{s \in \{i, j, l, m, n\} \setminus \{k\}} \lambda_s c_s. \quad (11)$$

Then $\Sigma$ is dual to a tropical surface $S \in \text{Sing}^\text{tr}(\Delta, \mathcal{F})$ if and only if the following holds:

• for $k = n = N + 1$, $m = N$, either

$$\langle (\lambda_N - 1)(w_N - w_{N-1}) - (w_{N+1} - w_N), v \rangle > 0, \quad (12)$$

or

$$\langle (\lambda_N - 1)(w_N - w_{N-1}) - (w_{N+1} - w_N), v \rangle = 0 \quad \text{and} \quad \begin{cases} \text{either } l < N - 1, \\
\text{or } l = N - 1, \lambda_N - 1 + \lambda_{N-1} > 0, \\
\text{or } l = N - 1, \lambda_N - 1 + \lambda_{N-1} = 0, \lambda_j > 0, \quad (13)\end{cases}$$

• for $0 \leq k < n$, we have $\lambda_s > 0$.

Proof. Similarly to the proof of Lemma 3.10 we have to check conditions (5), which read

$$\begin{align*}
c_{N+1} + M_N\langle w_{N+1}, v \rangle \leq c_N + M_N\langle w_N, v \rangle, & \quad \text{if } k = n = N + 1, \ m = N, \\
c_k + M_k\langle w_k, v \rangle \leq c_{k-1} + M_k\langle w_{k-1}, v \rangle, & \quad \text{if } 0 < k < n, \\
c_0 + M_1\langle w_0, v \rangle \leq M_1\langle w_1, v \rangle, & \quad \text{if } k = 0. \quad (14)
\end{align*}$$

In the first case, we plug the circuit relation (11) and the relation $c_N = c_{N-1} - M_N\langle w_N - w_{N-1}, v \rangle$ into (14) and obtain

$$(\lambda_N - 1)c_{N-1} + \lambda_ic_i + \lambda_jc_j + \lambda_kc_k \leq M_N((\lambda_N - 1)(w_N - w_{N-1}) - (w_{N+1} - w_N), v). \quad (15)$$
Since the left-hand side is of order $o(M_N)$, we immediately see that $O(12)$ is sufficient for $O(14)$, and that the opposite strict inequality contradicts $O(13)$. If the right-hand side of $O(15)$ vanishes, we get $\lambda_N - 1 > 0$, and hence conditions $O(13)$ in view of

$$c_{N-1} = -M_{N-1}(w_{N-1} - w_{N-2}, v) + o(M_{N-1}), \quad c_i = -M_i(w_i - w_{i-1}, v) + o(M_i), \quad c_j, c_i = o(M_i).$$

In the second case, we again plug the circuit relation into $O(14)$ and obtain

$$\begin{cases} \lambda_n c_n + \sum_{s \in \{i,j,i,m\}\{k\}} \lambda_s c_s - c_{k-1} \leq -M_k(w_k - w_{k-1}, v), & \text{if } k \neq 0, \\
\lambda_n c_n + \lambda_m c_m + \lambda_i c_i + \lambda_j c_j \leq M_1(w_1 - w_0, v), & \text{if } k = i = 0,
\end{cases}$$

which holds if and only if $\lambda_n > 0$ in view of

$$c_n = -M_n(w_n - w_{n-1}, v) + o(M_n), \quad c_i, c_j, c_m = o(M_n)$$

(recall that $\lambda_n \neq 0$).

(2) The case of a disconnected path $P$.

**Lemma 3.14.** Let the lattice path $P = \Gamma_{k,k+1}, 1 \leq k \leq N$, (see Lemma 3.3, 3.4) admit an extension to a subdivision $\Sigma$ of $\Delta$ with a circuit $C = \{w_i, w_j, w_l, w_m, w_n\}$, $i < j < l < m < n$, of type $A$ dual to a surface $S \in \text{Sing}^{\text{tr}}(\Delta, \pi)$. Then

(i) either $m = k, n = k + 1$, or $m = k + 1, n = k + 2$;

(ii) the subdivision $\Sigma$ is uniquely determined by the pair $(k, C)$ and satisfies the following:

- it contains a smooth triangulation of $\Delta_{m-1} = \text{Conv}\{w_s : 0 \leq s < m, s \neq k\}$;
- the pentatope $\text{Conv}(C)$ intersects $\Delta_{m-1}$ along their common 2-face $\text{Conv}\{w_i, w_j, w_l\}$;
- $\Sigma$ is obtained from the triangulation of $\Delta_{m-1}$ by the extension to $\text{Conv}(\Delta_{m-1} \cup C)$ as in Lemma 3.4, and by a sequence of smooth extensions as in Example 3.5 when subsequently adding the points of $P$ following $w_n$.

**Proof.** From equations (8), we get that $c_i, c_j, c_m = o(|c_n|)$, and hence the circuit relation (10) yields that $c_m$ and $c_n$ must be of the same order. This is only possible if either $m = k, n = k + 1$, and $M_0$ is comparable with $M_k$, or $m = k + 1, n = k + 2$, and $M_0$ is comparable with $M_{k+1}$. Claim (ii) can be proved as Lemma 3.9(ii).

**Lemma 3.15.** In the notation of Lemma 3.14, let the data $k$ and $C$ satisfy condition (i), and let a subdivision $\Sigma$ of $\Delta$ be constructed as in item (ii). Write the circuit relation (10) in the form

$$c_n = \lambda_m c_m + \lambda_i c_i + \lambda_j c_j + \lambda_i c_i.$$  

(16)

Then $\Sigma$ is dual to a tropical surface $S \in \text{Sing}^{\text{tr}}(\Delta, \pi)$ if and only if the following holds:

- for $m = k, n = k + 1$, we have $\lambda_k - 1 > 0$ and either
  
  $$(\lambda_k - 1)(w_k - w_{k-1}, v) < 0,$$

  or
  
  $$(\lambda_k - 1)(w_k - w_{k-1}, v) = 0, \text{ and } \begin{cases} l < k, \\
  \text{or } l = k - 1, \lambda_k + \lambda_{k-1} < 0, \\
  \text{or } l = k - 1, \lambda_k + \lambda_{k-1} = 0, \lambda_j > 0,
\end{cases}$$

- for $m = k + 1, n = k + 2$, we have $\lambda_{k+1} - 1 > 0$ and either
  
  $$(\lambda_{k+1} - 1)(w_{k+1} - w_k, v) < 0,$$

  or
  
  $$(\lambda_{k+1} - 1)(w_{k+1} - w_k, v) = 0, \text{ and } \begin{cases} l < k, \\
  \text{or } l = k, \lambda_{k+1} + \lambda_k < 0, \\
  \text{or } l = k, \lambda_{k+1} + \lambda_k = 0, \lambda_j > 0.
\end{cases}$$


HANNAH MARKWIG, THOMAS MARKWIG, AND EUGENII SHUSTIN

4. Multiplicities of singular tropical surfaces

In Section 3 we formulated a lattice path algorithm to construct dual subdivisions for all tropical surfaces, and with that, all tropical surfaces $S \in \text{Sing}^d(\Delta, \mathfrak{F})$ for a point configuration $\mathfrak{F}$ satisfying condition (19) on page 6. Now we compute the multiplicity $\text{mt}(S, \mathfrak{F})$, which is the number of singular algebraic surfaces over $\mathbb{K}$ with Newton polytope $\Delta$, passing through the a generic configuration $\mathfrak{p} \subset ((\mathbb{K}^*)^3)^N$, $\text{Val}(\mathfrak{p}) = \mathfrak{F}$, and tropicalizing to $S$, thus solving Problem 2.2 (2). This number is finite due to the general position of the configuration $\mathfrak{p}$, but it may vanish as we see below, since the singular lifts of a tropical surface $S \in \text{Sing}^d(\Delta, \mathfrak{F})$ may avoid the configuration $\mathfrak{p}$.

If $x_i = (x_{i1}, x_{i2}, x_{i3})$ then $p_i = (p_{i1}, p_{i2}, p_{i3})$, where $p_{ij} = (\xi_{ij} + O(t^{\geq 0}))t^{-x_{ij}}, \xi_{ij} \neq 0$ for all $1 \leq i \leq N, j = 1, 2, 3$. We denote $\text{Ini}(p_i) = \xi_i := (\xi_{i1}, \xi_{i2}, \xi_{i3})$ and $\text{Ini}(\mathfrak{p}) = \xi := (\xi_1, ..., \xi_N)$.

Introduce also the following auxiliary notation. If the circuit $C_S$ in the dual subdivision of $S$ is of type A, we fix an affine automorphism $\Phi_S : \mathbb{Z}^3 \rightarrow \mathbb{Z}^3$ taking $C_S$ to a canonical pentatope $\Pi_{p,q}$.
The discriminant equation of a polynomial \( \sum_{\omega \in \Pi_{p,q}} a_\omega Z^\omega \) can be written in the form
\[
(-1)^{1+p+q} a_{000}^{-1} a_{001}^{-q} a_{010}^{-p} a_{1pq} = 1.
\] (23)

Denote the exponent of a coefficient \( a_\omega \) in this equation by \( d(\omega) \).

4.1. Enhanced singular tropical surfaces. Let \( x \in (\mathbb{R}^3)^N \) be given a point configuration \( x \), a generic point configuration \( \overline{p} \in ((\mathbb{K}^*)^3)^N \) such that \( \text{Val}(\overline{p}) = x \). A tropical surface \( S \in \text{Sing}^{\text{tr}}(\Delta, x) \) and its defining tropical polynomial
\[
F_S(X) = \max_{\omega \in \Delta \cap \mathbb{Z}^3} (c_\omega + \langle X, \omega \rangle).
\] (24)

Denote by \( \nu_S : \Delta \rightarrow \mathbb{R} \) the convex piecewise linear function Legendre dual to \( F_S \), by \( \Sigma_S \) the subdivision of \( \Delta \) dual to \( S \), by \( C_S \) the circuit, and by \( P_S \) the corresponding lattice path (formed by the edges dual to the 2-faces of \( S \) containing the points of \( x \)). Observe that \( \nu_S(\omega) = -c_\omega \) for all points \( \omega \in \Delta \cap \mathbb{Z}^3 \).

**Lemma 4.1.** Any surface \( S \in \text{Sing}(\Delta) \) that tropicalizes to \( S \) is defined by a polynomial
\[
\varphi_S(Z) = \sum_{\omega \in \Delta \cap \mathbb{Z}^3} (\alpha_\omega + O(t^{>0})) t^{\nu_S(\omega)} Z^\omega,
\] (25)

where \( Z^\omega = z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3} \), and \( O(t^{>0}) \) accumulates the terms containing \( t \) to a positive power, and \( \alpha_\omega \neq 0 \) for all \( \omega \in \Delta \cap \mathbb{Z}^3 \). Furthermore, the (complex) polynomial
\[
\text{Ini}^{C_S}(\varphi_S)(Z) := \sum_{\omega \in C_S} \alpha_\omega Z^\omega
\]
has a singularity in \((\mathbb{C}^*)^3\).

**Proof.** We have to explain only the last claim. Viewing the surface \( S \) as an analytic equisingular family of singular complex surfaces (cf. [8, Section 2.3]), we obtain an induced family of singular points with the limit belonging to the big torus of \( \text{Tor}(\Delta) \) for some cell \( \Delta \) of the subdivision \( \Sigma_S \) (the cell dual to the face of \( S \) containing the tropical singular point). It is easy to see that, for any cell \( \Delta \neq \text{Conv}(C_S) \) of \( \Sigma_S \), any (nonzero) polynomial \( \sum_{\omega \in \Delta \cap \mathbb{Z}^3} \beta_\omega Z^\omega \) has no singularities in \((\mathbb{C}^*)^3\). Hence \( \text{Ini}^{C_S}(\varphi_S) \) must have singularity in \((\mathbb{C}^*)^3\).

**Lemma 4.2.** If a polynomial \( \varphi(Z) \) of the form \( (24) \) defines a surface in \((\mathbb{K}^*)^3\) passing through the configuration \( \overline{p} \), and if the polynomial \( \text{Ini}^{C_S}(\varphi) \) has a singularity in \((\mathbb{C}^*)^3\), then the point \( x := (\alpha_\omega)_{\omega \in \Delta \cap \mathbb{Z}^3} \in \mathbb{P}^{N+1} \) belongs to a finite set denoted by \( A(S, \overline{p}) \). Furthermore,

(i) If \( C_S \) is of type A, then

- for \( P_S = \Gamma_k \), we have \( |A(S, \overline{p})| = |d(\Phi_S(w_k))| \);  
- for \( P_S = \Gamma_{k,k+1} \) and \( w_{k+1} = \max C_S \), we have \( |A(S, \overline{p})| = |d(\Phi_S(w_{k+1}))| \);  
- for \( P_S = \Gamma_{k,k+1} \) and \( w_{k+2} = \max C_S \), we have \( |A(S, \overline{p})| = |d(\Phi_S(w_{k+2})) + d(\Phi_S(w_{k+1}))| \).

(ii) If \( C_S \) is of type B, then \( |A(S, \overline{p})| = \text{Vol}(\text{Conv}(C_S)) \) (the lattice volume of \( \text{Conv}(C_S) \)) when the tetrahedron \( \text{Conv}(C_S) \) cannot be taken to \( \text{Conv}\{(0,0,0),(1,0,0),(0,1,0),(3,7,20)\} \) by an automorphism of \( \mathbb{Z}^3 \) (cf. [15, Theorem 2]), and \( |A(S, \overline{p})| = \frac{1}{4}\text{Vol}(\text{Conv}(C_S)) = 4 \) when the tetrahedron \( \text{Conv}(C_S) \) can be transformed to \( \text{Conv}\{(0,0,0),(1,0,0),(0,1,0),(3,7,20)\} \) by an automorphism of \( \mathbb{Z}^3 \).

(iii) If \( C_S \) is of type C, then \( |A(S, \overline{p})| = 3 \) (the lattice area of \( \text{Conv}(C_S) \)).

(iv) If \( C_S \) is of type D, then \( |A(S, \overline{p})| = 1 \).

(v) If \( C_S \) is of type E, then \( |A(S, \overline{p})| = 1 \) or 2 according as \( \text{Conv}(C_S) \) is an edge of the lattice path or not.
Proof.\ We start by investigating the effect of the conditions imposed by the marked points \( p_i \).

Tropically, the marked point \( x_i, \ 1 \leq i \leq N \) lies on a 2-face \( F_i \) of \( S \) dual to an edge \( E = [\omega^0, \omega^1] \subset P_S \).

In particular,
\[
b := c_{\omega^0} + \langle x_i, \omega^0 \rangle = c_{\omega^1} + \langle x_i, \omega^1 \rangle > c_{\omega} + \langle x_i, \omega \rangle \quad \text{for all } \omega \in \Delta \setminus E ,
\]
and then the condition imposed by the marked point \( p_i \) is
\[
0 = \varphi(p_i) = t^{-b} (\text{In}(\varphi)(\xi_i) + O(t^{>0})) , \quad \text{In}(\varphi)(Z) = \sum_{\omega \in E} \alpha_{\omega} Z^\omega .
\]
The lattice length \(|E|\) of \( E \) is either 1 or 2. If \(|E| = 1\), we obtain
\[
\alpha_{\omega^1} = -\alpha_{\omega^0} c_{\omega^1} \omega^1 . \quad (26)
\]
If \(|E| = 2\), then \( \text{In}(\varphi)(Z) \) has a singularity in \((\mathbb{C}^*)^3\); hence it is a monomial multiplied by the square of a binomial, which then implies
\[
\alpha_{\omega^1} = \alpha_{\omega^0} c_{\omega^1} \omega^1 , \quad \alpha_{\omega} = -2\alpha_{\omega^0} c_{\omega^1} \omega^1 / 2 , \quad \omega = \omega^0 + \omega^1 . \quad (27)
\]

It follows, in particular, that \( \overline{\alpha} \) is uniquely defined if \( C_S \) is of type \( E \) and \( \text{Conv}(C_S) \) is an edge of the lattice path \( P_S \). If \( C_S \) is of type \( E \) and \( \text{Conv}(C_S) \not\subset P_S \), then we uniquely determine \( \alpha_{\omega^0} \) and \( \alpha_{\omega^1} \) for the end points \( \omega^0, \omega^1 \) of \( C_S \), and by Lemma 1.1 obtain two values \( \alpha_{\omega} = \pm 2\sqrt{\alpha_{\omega^0} \alpha_{\omega^1}} \) for the midpoint \( \omega \) of \( C_S \), and hence two singular points of \( \text{In}(\varphi)(Z) \). Thus statement (v) is proved.

Now consider other types of circuits.

Suppose that \( P_S = \Gamma_{k,k+1}, 1 \leq k \leq N \). As shown in Section 3.4, \( C_S \) must be of type \( A \). Equations (26) yield \( \overline{\alpha} \in \mathbb{P}^{N+1} \) in the form
\[
(a_{\omega^0}, ..., a_{\omega_k}, \lambda a'_{\omega_{k+1}}, ..., \lambda a'_{\omega_{N+1}}) ,
\]
where \( (a_{\omega^0}, ..., a_{\omega_k}, a'_{\omega_{k+1}}, ..., a'_{\omega_{N+1}}) \) is a uniquely defined generic point of \( \mathbb{P}^{N+1} \), and \( \lambda \neq 0 \) is an unknown parameter, one which can compute from the discriminant equation (28) of the pentatope \( \Phi_S(\text{Conv}(C_S)) \), obtaining \(|d(\Phi_S(w_k))|\) many solutions if \( w_{k+1} = \max C_S \) and \(|d(\Phi_S(w_{k+2})) + d(\Phi_S(w_{k+1}))|\) many solutions if \( w_{k+2} = \max C_S \).

Suppose that \( P_S = \Gamma_k, 0 \leq k \leq N + 1 \). Then equations (26) determine the (nonzero) values \( \alpha_{\omega}, \omega \neq w_k \), up to proportionality.

If \( C_S \) is of type \( A \), we obtain \(|d(\Phi_S(w_k))|\) values for the coefficient \( a_{w_k} \) from the discriminant equation (23) of the pentatope \( \Phi_S(\text{Conv}(C_S)) \). Thus (i) is proved.

If \( C_S \) is of type \( B \), then \( w_k \) is the interior point of the tetrahedron \( \text{Conv}(C_S) \). After a suitable transformation of the lattice \( \mathbb{Z}^3 \) and a coordinate change, we obtain the equivalent question: How many values of \( a \in \mathbb{C}^* \) are there such that the polynomial
\[
\psi(x, y, z) = 1 + x + y + x^i y^j z^l + a x^{i'} y^{j'} z^{l'}
\]
has a singularity in \((\mathbb{C}^*)^3\), where
\[
(i, j, l) = (3, 3, 4), (2, 2, 5), (2, 4, 7), (2, 6, 11), (2, 7, 13), (2, 9, 17), (2, 13, 19), \text{ or } (3, 7, 20)
\]
and \((i', j', l')\) is the unique interior integral point of the tetrahedron
\[
\text{Conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (i, j, l)\} .
\]
The system of equations \( \psi = \psi_x = \psi_y = \psi_z = 0 \) reduces to
\[
x = \lambda , \quad y = \mu , \quad z^l = \nu , \quad a = \rho z^{-l'} \quad (28)
\]
with some nonzero constants \( \lambda, \mu, \nu, \rho \). In all cases except for \((i, j, l) = (3, 7, 20)\), we have \( \gcd(l', l) = 1 \), and hence \( l = \text{Vol}(\text{Conv}(C_S)) \) solutions for \( a \). In the remaining case \((i, j, k) = (3, 7, 20), (i', j', l') = (1, 2, 5)\), and we obtain 4 = Vol(\text{Conv}(C_S))/5 values for \( a \). We obtain (ii).

If \( C_S \) is of type \( C \), then \( w_k \) is the interior point of the triangle \( \text{Conv}(C_S) \). For given \( \omega \neq 0 \) at the vertices \( \omega \) of \( \text{Conv}(C_S) \), there are exactly \( \text{Area}(\text{Conv}(C_S)) = 3 \) values \( a_{w_k} \), corresponding to singular polynomials \( \text{In}(\varphi_S) \) (cf. [8, Lemma 3.5]). This yields (iii).
If \( C_S \) is of type D, then \( w_k \) is a vertex of the parallelogram \( \text{Conv}(C_S) \). If \( w_i, w_j, w_l \) are the other vertices of \( \text{Conv}(C_S) \), and \( w_j \) is opposite to \( w_k \), then the fact that \( \text{Im}^{C_S}(\varphi_S) \) has a singularity in \((C^*)^3\) yields \( \alpha_{w_k} = \alpha_{w_l} \alpha_{w_j}^{-1} \), which defines \( \alpha_{w_k} \) uniquely. Thus statement (iv) is proved. \( \square \)

**Remark 4.3.** Observe that, in the case of the lattice path \( \Gamma_{k,k+1} \) and a circuit of type A containing the points \( w_{k+1}, w_{k+2} \), one may obtain an empty set \( A(S,\overline{p}) \).

We call the points \( \overline{p} \in A(S,\overline{p}) \) enhancements of \( S \), and the pairs \( (S,\overline{p}) \) enhanced singular tropical surfaces.

### 4.2. Singular points of tropical surfaces

By \([15\text{ Theorem 2}]\), the position of a tropical singular point \( y \in S \) is defined uniquely whenever the circuit \( C_S \) is of type A, B, or D. For circuit types C and E there may be several possible positions for \( y \). We will describe these possibilities via the geometry of \( \text{Graph}(\nu_S) \). Namely, to determine the position of \( y \), it is enough to determine the translation of \( S \) which moves \( y \) to the origin. In turn, translations of \( S \) are in one-to-one correspondence with changes \( \nu_S \mapsto \nu_S + \Lambda \), where \( \Lambda \) is any affine linear function. To move the singularity to the origin, we use \([15\text{ Lemma 10}]\).

Without loss of generality, we assume that (cf. \([15\text{ Theorem 2}]\))

\[
C_S = \begin{cases} 
\{(1,0,0), (2,1,0), (0,2,0), (1,1,0)\}, & \text{if of type C}, \\
\{(0,0,0), (0,0,1), (0,0,2)\}, & \text{if of type E}.
\end{cases}
\]

**Lemma 4.4.** Let \( C_S \) be of type \( C, \Lambda' : \Delta \to \mathbb{R} \) the unique affine linear function, depending only on \( x \) and \( y \), which coincides with \( \nu_S \) along \( \text{Conv}(C_S) \). Set \( \nu' = \nu_S - \Lambda' \) and introduce the following convex piecewise linear function on the projection \( \text{pr}_z(\Delta) \) of \( \Delta \) to the \( z \)-axis: Set

\[
\begin{align*}
-c'_m &= \min\{\nu'(\omega) : \omega \in \Delta \cap \mathbb{Z}^3, \text{pr}_z(\omega) = m\}, \quad m \in \text{pr}_z(\Delta) \cap \mathbb{Z} \setminus \{0\}, \\
-c'_0 &\gg \max\{-c'_m, m \neq 0\},
\end{align*}
\]

and then define a function \( \nu_z : \text{pr}_z(\Delta) \to \mathbb{R} \), whose graph is the lower convex hull of

\[
\text{Conv}\{(m, -c'_m) : m \in \text{pr}_z(\Delta) \cap \mathbb{Z}\}.
\]

Then the possible singular points \( y \in S \) are in one-to-one correspondence with linear functions \( \Lambda'' : \Delta \to \mathbb{R} \) that vanish at the origin, are strictly less than \( \nu_z \), and whose graph is parallel to an edge of \( \text{Graph}(\nu_S) \) which projects to one of the following segments:

\[
[-3,-1], \ [-3,1], \ [-1,3], \ [-1,1], \ [-1,-3], \ [1,3].
\]

**Proof.** The statement follows from \([15\text{ Theorem 2 and Section 4.3}]\): According to the type of the weight class (see \([15\text{ Lemma 10}]\)), we have to pick two points \( \omega^1 \) and \( \omega^2 \) in \( \Delta \cap \mathbb{Z}^3 \) whose coefficients \( c_{\omega^1} \) and \( c_{\omega^2} \) become equal and maximal among the \( \omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S \) after subtracting \( \Lambda' \) and \( \Lambda'' \). Lemma 18 in \([15]\) yields the restriction that these points have to be picked with lattice distance one or three to the circuit \( C_S \). \( \square \)

**Lemma 4.5.** Let \( C_S \) be of type \( E, \Lambda' : \Delta \to \mathbb{R} \) the unique affine linear function, depending only on \( z \), which coincides with \( \nu_S \) along \( \text{Conv}(C_S) \). Set \( \nu' = \nu_S - \Lambda' \) and introduce the following convex piecewise linear function on the projection \( \text{pr}_{x,y}(\Delta) \) of \( \Delta \) to the \( (x,y) \)-plane: Set

\[
\begin{align*}
-c'_m &= \min\{\nu'(\omega) : \omega \in \Delta \cap \mathbb{Z}^3, \text{pr}_{x,y}(\omega) = m\}, \quad m \in \text{pr}_{x,y}(\Delta) \cap \mathbb{Z}^2 \setminus \{0\}, \\
-c'_0 &\gg \max\{-c'_m, m \neq 0\},
\end{align*}
\]

and then define a function \( \nu_{x,y} : \text{pr}_z(\Delta) \to \mathbb{R} \), whose graph is the lower convex hull of

\[
\text{Conv}\{(m, -c'_m) : m \in \text{pr}_{x,y}(\Delta) \cap \mathbb{Z}^2\}.
\]
4.3. Patchworking singular algebraic surfaces. We will now see how given enhancements \(\bar{\tau}\) can be lifted to equations of algebraic surfaces in \(\text{Sing}(\Delta, \bar{\nu}_S, S)\) using patchworking techniques. We start with the most difficult case, namely circuits of type E.

Lemma 4.6. Let \(S \in \text{Sing}^{\text{II}}(\Delta, \bar{\nu})\) have a circuit \(C_S\) of type E, and \(C_S = \{(0, 0, 0), (0, 0, 1), (0, 0, 2)\}\).

(1) Suppose that a tropical singular point \(y \in S\) is associated with a triangle \(\delta \subset \mathbb{R}^2 \hookrightarrow \mathbb{R}^3\) from the list \((30)\), as specified in Lemma 4.5(i). Then there exist precisely \(2 \cdot \text{Area}(\delta)\) surfaces \(S \in \text{Sing}(\Delta, \bar{\nu}_S, S)\) that have a singular point tropicalizing to \(y\).
(2) Suppose that a tropical singular point \( y \in S \) is associated with a pair of edges \( E_1, E_2 \) as specified in Lemma 4.3(ii). Then there exist precisely 8 surfaces \( S \in \text{Sing}(\Delta, \mathcal{P}, S) \) that have a singular point tropicalizing to \( y \).

**Proof.** In both cases the lattice path is \( P_S = \Gamma_k \) for some \( 1 \leq k \leq N \). Furthermore, we have the following options:

(i) either the segment \( \text{Conv}(C_S) \) is a part of the lattice path \( \Gamma_k \), and its dual 2-face of \( S \) contains a marked point \( x_{k_0} = (\lambda, \mu, 0) \), where we can suppose that \( \lambda, \mu \) are generic in the sense of (33);

(ii) or \( \text{Conv}(C_S) \) is not an edge of \( \Gamma_k \).

**Step 1.** Consider the possibility (i). Then the enhancement \( \overline{\tau} \) is uniquely restored from formulas (26) and (27), when we set \( \omega^0 = (0, 0, 2) \). We have

\[
\varphi_S(Z) = z_3^2 - 2a_{001}z_3 + a_{000} + \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_\omega t^{-c_\omega} Z^\omega
\]

with \(-c_\omega > 0\) for all \( \omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S \) and \( \text{Val}(a_\omega) = 0 \) for all \( \omega \), and we have

\[
\Delta \cap \mathbb{Z}^3 = \{w_0, \ldots, w_{N+1}\}
\]

with

\[
C_S = \{w_{k-1} = (0, 0, 2), w_k = (0, 0, 1), w_{k+1} = (0, 0, 0)\}.
\]

We intend to solve the system of equations

\[
\varphi_S(p_i) = 0, \ i = 1, \ldots, N, \ \varphi_S(q) = \frac{\partial \varphi_S}{\partial x}(q) = \frac{\partial \varphi_S}{\partial y}(q) = \frac{\partial \varphi_S}{\partial z}(q) = 0
\]

with respect to the variables \( a_{001}, a_{000}, a_\omega, \omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S \), and the coordinates \( z_1, z_2, z_3 \) of the singular point \( q \) with the aid of the implicit function theorem.

Recall that, in the framework of Lemma 4.3, \( y = \text{Val}(q) \) is the origin, i.e. \( \text{Val}(z_i) = 0 \), and let \( \text{Ini}(q) = (z_{10}, z_{20}, z_{30}) \). Indeed, \( z_{30} = 1 \), which follows from the equation \( \varphi_S(q) = 0 \) and (31).

(1) In the first case, let \( \delta = \text{Conv}\{(i_1, j_1), (i_2, j_2), (i_3, j_3)\} \). There exist uniquely defined \( l_1, l_2, l_3 \in \mathbb{Z} \) such that, in the notation of Lemma 4.3,

\[
(i_r, j_r, l_r) \in \Delta \quad \text{and} \quad \nu_{x,y}(i_r, j_r) = \nu'(i_r, j_r, l_r), \ r = 1, 2, 3.
\]

Then by formula (31) we have

\[
-c_{i_r, j_r, l_r} = s < -c_\omega
\]

for \( r = 1, 2, 3 \) and all other \( \omega \not\in \{\omega_1 = \omega_2 = 0\} \). The equations

\[
\left(t^{-s} \frac{\partial \varphi_S}{\partial z_1}\right)_{t=0, z_3 = z_{30}} = \left(t^{-s} \frac{\partial \varphi_S}{\partial z_2}\right)_{t=0, z_3 = z_{30}} = 0
\]

yield that the coordinates \( z_{10}, z_{20} \) of \( \text{Ini}(q) = (z_{10}, z_{20}, z_{30}) \) correspond to critical points in \((\mathbb{C}^*)^2\) of the polynomial

\[
Q(z_1, z_2) = \sum_{r=1}^3 \alpha_{i_r, j_r, l_r} z_1^{i_r} z_2^{j_r},
\]

which gives us \( \text{Area}(\delta) \) solutions \((z_{10}, z_{20})\) as possible initial values for \( q \) in total.

In order to apply the implicit function theorem we have to replace the equation \( \varphi_S(q) = 0 \) by two possible other equations, since it is unsuitable itself being of degree two in \( z_3 \). We now first want to derive these new equations. For that we consider the equation \( \varphi_S(p_k) = 0 \),

\[
1 - 2a_{001} + a_{000} + \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_\omega t^{-c_\omega + \lambda \omega_1 + \mu \omega_2} = 0,
\]
together with $\frac{\partial \varphi}{\partial z_3}(q) = 0$,

$$2z_3 - 2a_{001} + \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} t^{-c_\omega} \varphi_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1} = 0.$$  

The equations lead to

$$a_{001} = z_3 + \frac{1}{2} \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} t^{-c_\omega} \varphi_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1}$$

and

$$a_{000} = -1 + 2z_3 + \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} \cdot (t^{-c_\omega} \varphi_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1} - t^{-c_\omega + \lambda \omega_1 + \mu \omega_2}).$$

Plugging these equations into (31) and reorganizing the terms we get

$$(z_3 - 1)^2 = \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} t^{-c_\omega} \cdot (Z^\omega - t^{\lambda \omega_1 + \mu \omega_2} + (z_3 - 1) \cdot \omega_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1}),$$

and taking square roots we get two equations

$$\psi_\pm = z_3 - 1 \pm \sqrt{\sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} t^{-c_\omega} \cdot (Z^\omega - t^{\lambda \omega_1 + \mu \omega_2} + (z_3 - 1) \cdot \omega_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1})} = 0$$  

(32)

to replace $\varphi_S(q) = 0$ with.

We now consider the polynomial map $\Psi$, that maps

$$\zeta = (t, a_{w_1}, \ldots, a_{w_{k-2}}, z_1, z_2, z_3, a_{w_k}, \ldots, a_{w_{N+1}})$$
to

$$\left( t^{-s_1} \varphi_S(p_1), \ldots, t^{-s_{k-1}} \varphi_S(p_{k-1}), t^{-s} \frac{\partial \varphi_S}{\partial z_1}, t^{-s} \frac{\partial \varphi_S}{\partial z_2}, \varphi_\pm, \frac{\partial \varphi_S}{\partial z_3}, t^{-s_k} \varphi_S(p_k), \ldots, t^{-s_{N-1}} \varphi_S(p_{N-1}) \right),$$

where $s_i = \text{Val}(\varphi_S(p_i))$. Note, that the initial values give a zero

$$\zeta_0 = (0, \alpha_{w_1}, \ldots, \alpha_{w_{k-2}}, z_{10}, z_{20}, z_{30}, \alpha_{w_k}, \ldots, \alpha_{w_{N+1}})$$
of $\Psi$. We assume that the values $\lambda$ and $\mu$ are generic in the sense that

$$-c_\omega + \lambda \omega_1' + \mu \omega_2' \neq -c_\omega + \lambda \omega_1 + \mu \omega_2 \neq -c_\omega'$$  

(33)

whenever $\omega \neq \omega'$. This ensures that the term under the square root in $\psi_\pm$ is non-zero, if we evaluate it at $(\alpha_{w_1}, \ldots, \alpha_{w_{k-2}}, z_{10}, z_{20}, z_{30}, \alpha_{w_k}, \ldots, \alpha_{w_n})$, so that $\psi_\pm$ is analytic locally in $\zeta_0$. Moreover, computing derivatives in [32] we get

$$\frac{\partial \psi_\pm}{\partial z_3}(\zeta_0) = 1 \pm \sum_{\omega \in \Delta \cap \mathbb{Z}^3 \setminus C_S} a_{\omega} t^{-c_\omega} \cdot (\omega_3 z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 1} + (z_3 - 1) \omega_3 (\omega_3 - 1) z_1^{\omega_1} z_2^{\omega_2} z_3^{\omega_3 - 2}) = 1$$

and all other derivatives of $\psi_\pm$ vanish at $\zeta_0$, since due to the genericity assumption on $\lambda$ and $\mu$ the valuation of the denominator in the fraction is at most half the valuation of the numerator. Similar computations for the other component functions of $\Psi$ lead to the following Jacobian of $\Psi$.
with respect to all variables but \( t \) evaluated at \( \zeta_0 \),

\[
\left( \begin{array}{cccc}
\xi_1^{w_0} & \xi_1^{w_1} & \cdots & \xi_1^{w_{k-2}} \\
\cdots & \cdots & \cdots & \cdots \\
\xi_k^{w_{k-1}} & \xi_k^{w_{k-2}} & \cdots & \xi_k^{w_0} \\
\end{array} \right) 
\left( \begin{array}{cccc}
\frac{\partial^2 \varphi_S}{\partial z_1^2} (\zeta_0) & \frac{\partial^2 \varphi_S}{\partial z_2^2} (\zeta_0) & \cdots & \frac{\partial^2 \varphi_S}{\partial z_k^2} (\zeta_0) \\
\frac{\partial^2 \varphi_S}{\partial z_1 \partial z_2} (\zeta_0) & \frac{\partial^2 \varphi_S}{\partial z_1 \partial z_3} (\zeta_0) & \cdots & \frac{\partial^2 \varphi_S}{\partial z_k \partial z_1} (\zeta_0) \\
\frac{\partial^2 \varphi_S}{\partial z_1 \partial z_k} (\zeta_0) & \frac{\partial^2 \varphi_S}{\partial z_2 \partial z_k} (\zeta_0) & \cdots & \frac{\partial^2 \varphi_S}{\partial z_k \partial z_k} (\zeta_0) \\
\end{array} \right) 
\left( \begin{array}{cccc}
\xi_1^{w_0} & \xi_1^{w_1} & \cdots & \xi_1^{w_{k-2}} \\
\cdots & \cdots & \cdots & \cdots \\
\xi_k^{w_{k-1}} & \xi_k^{w_{k-2}} & \cdots & \xi_k^{w_0} \\
\end{array} \right) ^{-1}
\]

where all missing entries are zero and the stars denote possibly non-zero entries. Since the critical point \((z_{10}, z_{20})\) of \( Q \) is non-degenerate the Hessian in the middle block has a non-vanishing determinant and thus the determinant of the Jacobian does not vanish. Applying the implicit function theorem we get in each of the two cases \( \psi_- \) and \( \psi_+ \) a unique solution, and since we have \( \text{Area}(\delta) \) choices for \((z_{10}, z_{20})\) we end up with \( 2 \cdot \text{Area}(\delta) \) algebraic surfaces \( S \in \text{Sing}(\Delta, \overline{p}, S) \) having a singular point \( q \) with \( \text{Trop}(q) = y \).

(2) The second case works along the same lines. With the notation of Lemma 4.5 the relations

\[
\begin{cases}
(-1,0,l_1), (1,0,l_2), (i,1,l_3), (j,-1,l_4) \in \Delta \cap \mathbb{Z}^3, \\
\nu_{x,y}(-1,0) = \nu'(-1,0,l_1), \quad \nu_{x,y}(1,0) = \nu'(1,0,l_2), \\
\nu_{x,y}(i,1) = \nu'(i,1,l_3), \quad \nu_{x,y}(j,-1) = \nu'(j,-1,l_4)
\end{cases}
\]

uniquely determine integers \( l_1, l_2, l_3, l_4 \) and valuations \( s_2 > s_1 > 0 \), such that

\[
s_1 = -c_{-1,0,l_1} = -c_{1,0,l_2} < -c_\omega
\]

for all other \( \omega \in \Delta \cap \mathbb{Z}^3 \) of the form \( \omega = (i,0,l) \) and such that

\[
s_2 = -c_{i,1,l_3} = -c_{j,1,l_4} < -c_\omega
\]

for all remaining \( \omega \not\in \{ \omega_2 = 0 \} \). Defining

\[
Q_1(z_1, z_3) = \alpha_{-1,0,l_1} z_1^{-1} z_3 + \alpha_{1,0,l_2} z_1 z_3^2, \quad Q_2(z_1, z_2, z_3) = \alpha_{i,1,l_3} z_1^i z_2 z_3^j + \alpha_{j,1,l_4} z_1^j z_2 z_3^i,
\]

the critical points of \( Q_1 \) and \( Q_2 \) respectively determine the possible pairs \((z_{10}, z_{20})\) for \( \text{Init}(q) \) \((z_{10}, z_{20}, 1)\) via the equations

\[
\left( t^{-s_1} \frac{\partial^{\varphi_S}}{\partial z_1} \right)_{t=0} (z_{10}, z_{20}, 1) = \left( t^{-s_2} \frac{\partial^{\varphi_S}}{\partial z_2} \right)_{t=0} (z_{10}, z_{20}, 1) = 0.
\]

They are thus the solutions of the system

\[
- \alpha_{-1,0,l_1} z_{10}^{-2} + \alpha_{1,0,l_2} = \alpha_{i,1,l_3} z_{10}^i z_{20}^-2 + \alpha_{j,1,l_4} z_{10}^j z_{20}^-2 = 0,
\]

from which we get \( 4 \) solutions \((z_{10}, z_{20}) \in (\mathbb{C}^*)^2 \). Replacing the equations \( t^{-s} \frac{\partial^{\varphi_S}}{\partial z_1} \) and \( t^{-s} \frac{\partial^{\varphi_S}}{\partial z_2} \) in case (1) by \( t^{-s} \frac{\partial^{\varphi_S}}{\partial z_1} \) and \( t^{-s} \frac{\partial^{\varphi_S}}{\partial z_2} \), we can continue as in case (1) and find \( 8 \) surfaces \( S \in \text{Sing}(\Delta, \overline{p}, S) \) having a singular point \( q \) tropicalizing to \( y \).

Step 2. In the situation (ii), the above argument appears to be rather simpler. Note that the equations \( \varphi_S(p_i) = 0, i = 1, ..., N \), and the condition \( \omega^0 = (0,0,2) \) uniquely determine the values \( \alpha_\omega \) for all \( \omega \neq w_k \). For \( \omega = w_k \), we obtain two values \( \alpha_{w_k} = \alpha_{001} = \pm \sqrt{\alpha_{000}} \) (cf. Lemma 4.1), and respectively \( z_{30} = -\alpha_{001} \). Independently of the choice of \( \alpha_{001} \), we obtain \( \text{Area}(\delta) \) pairs \((z_{10}, z_{20})\) in
the case (1), or 4 pairs \((z_0, z_{20})\) in the case (2). The application of the implicit function theorem reduces to the computation of the Jacobian at \(t = 0\) for the system

\[
\frac{\partial \varphi_S}{\partial z_3}(q) = t^{-s_1} \frac{\partial \varphi_S}{\partial z_1}(q) = t^{-s_2} \frac{\partial \varphi_S}{\partial z_2}(q) = 0
\]

in the case (1), or the system

\[
\frac{\partial \varphi_S}{\partial z_3}(q) = t^{-s_1} \frac{\partial \varphi_S}{\partial z_1}(q) = t^{-s_2} \frac{\partial \varphi_S}{\partial z_2}(q) = 0
\]

in the case (2). The nondegeneracy of these Jacobians is straightforward. \(\square\)

**Lemma 4.7.** Let \(S \in \text{Sing}^{tr}(\Delta, \mathcal{F})\), and let the circuit \(C_S\) in the subdivision dual to \(S\) be of type \(A\) or \(B\) as in Figure [1]. Then

(i) if \(C_S\) is not \(\mathbb{Z}\)-affine equivalent to \(\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (3, 7, 20)\}\), then, for any point \(\mathbf{p} \in A(S, \mathcal{P})\), (see Lemma 4.2) there exists a unique algebraic surface \(\mathcal{S} \in \text{Sing}(\Delta, \mathcal{P}, S)\);

(ii) if \(C_S\) is \(\mathbb{Z}\)-affine equivalent to \(\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (3, 7, 20)\}\), then, for any point \(\mathbf{p} \in A(S, \mathcal{P})\), there exist 5 algebraic surfaces \(\mathcal{S} \in \text{Sing}(\Delta, \mathcal{P}, S)\) matching the enhancement \(\mathbf{p}\).

**Proof.** The required statement can again be viewed as a patchworking theorem, and it follows from a suitable version of the implicit function theorem. Namely, we look for polynomials

\[
\varphi_S(Z) = \sum_{\omega \in \Delta \cap \mathbb{Z}^3} a_\omega t^{\nu_S(\omega)} Z^\omega
\]

(cf. formula (25)), where \(a_{\omega^0} \equiv 1\) for some vertex \(\omega^0\) of the subdivision \(\Sigma_S\), and the remaining coefficients \(a_\omega = \alpha_\omega + O(t^{>0})\) are obtained from the conditions to pass through \(\mathcal{P}\) and to have a singular point \(q\) with \(\text{Ini}(q) = z\), a singular point of \(\text{Ini}^C(S)(\varphi_S)(Z) = \sum_{\omega \in \Delta \cap \mathbb{Z}^3} a_\omega Z^\omega\) in \((\mathbb{C}^*)^3\). At \(t = 0\) these conditions turn into the system of equations (20) in the coefficients \(a_\omega, \omega \neq \omega^0\), and the discriminant equation for the circuit \(C_S\).

In the case (i), if the lattice path is \(\Gamma_k\) for some \(k\), the Jacobian of the above system at \(t = 0\) is a (suitably arranged) lower triangular matrix with the nonzero entries from (26) and the discriminant equation for the circuit. If the lattice path is \(\Gamma_{k,k+1}\), we obtain a matrix whose column corresponding to \(a_{\nu_{k+1}}\) has only one nonzero entry, and is (suitably arranged) lower triangular after erasing this column and the corresponding row.

In the case (ii), without loss of generality suppose that

\[
C_S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (3, 7, 20), (1, 2, 5)\}.
\]

Any singular complex polynomial \(\text{Ini}^C(S)(\varphi_S)\) supported at \(C_S\) has 5 singular points in \((\mathbb{C}^*)^3\), obtained from each other by the \(\mathbb{Z}_5\)-action \(z \mapsto z \varepsilon, \varepsilon^5 = 1\). Each singular point \(z \in (\mathbb{C}^*)^3\) of \(\text{Ini}^C(S)(\varphi_S)\) is an ordinary node, in particular,

\[
\det(\text{Hessian}(\text{Ini}^C(S)(\varphi_S))(z)) \neq 0.
\] (35)

Then we consider the system of equations in the coefficients \(a_\omega, \omega \neq \omega^0\), of the sought polynomial \(\varphi_S\) and the coordinates \(z_i, i = 1, 2, 3\), of the singular point, where \(z_i = z_i(0) + O(t^{>0})\). The equations induced by the conditions \(S \supset \mathcal{P}\) and

\[
\varphi_S(z_1, z_2, z_3) = \frac{\partial \varphi_S}{\partial z_i}(z_1, z_2, z_3) = 0, \ i = 1, 2, 3,
\] (36)

and this system has a unique solution, since its Jacobian at \(t = 0\) does not vanish:

- the block coming from the conditions \(\varphi_S(p_i) = 0, \ i = 1, ..., N\), is the Jacobian of the nondegenerate linear system (26),
- for the block coming from the system (36), the nondegeneracy follows from (35). \(\square\)
Lemma 4.8. Let \( S \in \text{Sing}^r(\Delta, \mathfrak{F}) \), \( C_S \) be of type \( C \). Let us be given an enhancement \( \overline{\alpha} \in A(S, \mathfrak{F}) \) and a tropical singular point \( y \in S \) associated with a segment \( \sigma = [m, n] \) as specified in Lemma 4.4. Then there are \((n - m)\) algebraic surfaces \( S \in \text{Sing}(\Delta, \overline{\mathfrak{F}}, S) \), matching the given data \( \overline{\alpha} \) and \( y \).

**Proof.** Without loss of generality we can suppose that the lattice path \( P_S = \Gamma_k \), the left out point \( w_k \) is \((1,1,0)\), the circuit is \( C_S = \{(1,0,0), (2,1,0), (0,2,0), (1,1,0)\} \), the tropical singular point is \( y = (0,0,0) \), and the sought polynomial takes form (cf. [15, Theorem 2(b.1)])

\[
\varphi_S = \sum_{(i,j,0) \in C_S} a_{ij0} z_1^i z_2^j + \sum_{(i,j,0) \in \Delta \setminus C_S} O(t>0) \cdot z_1^i z_2^j + t^s \left( a_{i_1j_1m} z_1^{i_1} z_2^{j_1} z_3^m + a_{i_2j_2n} z_1^{i_2} z_2^{j_2} z_3^n \right) + O(t>^s),
\]

where \( s > 0 \), and

\[
a_{100} = 1, a_{210} = \alpha_{210} + O(t>0), \ a_{020} = \alpha_{020} + O(t>0), \ a_{110} = \alpha_{110} + O(t>0), \ a_{i_1j_1m} = \alpha_{i_1j_1m} + O(t>0), \ a_{i_2j_2n} = \alpha_{i_2j_2n} + O(t>0).
\]

The equations

\[
(\varphi_S)_{t=0}(z_{10}, z_{20}, z_{30}) = \left( \frac{\partial \varphi_S}{\partial z_1} \right)_{t=0}(z_{10}, z_{20}, z_{30}) = \left( \frac{\partial \varphi_S}{\partial z_2} \right)_{t=0}(z_{10}, z_{20}, z_{30}) = t^{-s} \left( \frac{\partial \varphi_S}{\partial z_3} \right)_{t=0}(z_{10}, z_{20}, z_{30}) = 0
\]

for \((z_{10}, z_{20}, z_{30}) = \text{Ini}(q)\), \( q \) being a singular point of the sought surface \( S \in \text{Sing}(\Delta, \overline{\mathfrak{F}}, S) \), give a unique solution \((z_{10}, z_{20})\) for the singularity of \( \text{Ini}^C_S(\varphi_S) \) in \((\mathbb{C}^*)^2\), and the last equation,

\[
m\alpha_{i_1j_1m} z_{10}^{i_1} z_{20}^{j_1} z_{30}^m + n\alpha_{i_2j_2n} z_{10}^{i_2} z_{20}^{j_2} z_{30}^n = 0,
\]

yields \((n - m)\) nonzero solutions for \( z_{30} \).

We claim that each solution induces a unique surface \( S \in \text{Sing}(\Delta, \overline{\mathfrak{F}}, S) \) matching the requirements of Lemma. Indeed, the implicit function theorem applies: the system

\[
\varphi_S(p_i) = 0, \quad i = 1, ..., N,
\]

linearizes into the nondegenerate linear system [20] with respect to the variables \( a_\omega \), \( \omega \in \Delta \cap \mathbb{Z}^3 \setminus \{(1,0,0), (1,1,0)\} \), and the Jacobian evaluated at \( t = 0 \) for the system

\[
\varphi_S(q) = \frac{\partial \varphi_S}{\partial z_1}(q) = \frac{\partial \varphi_S}{\partial z_2}(q) = t^{-s} \frac{\partial \varphi_S}{\partial z_3}(q) = 0
\]

with respect to \( a_{110} \) and the coordinates of \( q \) takes form of a lower block-triangular matrix

\[
\begin{pmatrix}
\begin{bmatrix}
z_{10} z_{20} & 0 & 0 \\
* & \text{Hessian}(\text{Ini}^C_S(\varphi))(z_{10}, z_{20}) & 0 \\
* & * & Q_{zz}(z_{10}, z_{20}, z_{30})
\end{bmatrix}
\end{pmatrix}
\]

where \( Q = \alpha_{i_1j_1m} z_{10}^{i_1} z_{20}^{j_1} z_{30}^m + \alpha_{i_2j_2n} z_{10}^{i_2} z_{20}^{j_2} z_{30}^n \). The nondegeneracy of this matrix (coming particularly from the fact that \((z_{10}, z_{20})\) is an ordinary node of \( \text{Ini}^C_S(\varphi) \) and that the nonzero roots \( z_{30} \) of (37) are simple) completes the proof. \( \square \)

Lemma 4.9. Let \( S \in \text{Sing}^r(\Delta, \mathfrak{F}) \), \( C_S \) be of type \( D \). Then there are two surfaces \( S \in \text{Sing}(\Delta, \overline{\mathfrak{F}}, S) \).
Proof. Without loss of generality we can suppose that the lattice path $P_S = \Gamma_k$, the left out point $w_k$ is the origin, the circuit is $C_S = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\}$, the unique tropical singular point is $y = (0, 0, 0)$, and the sought polynomial takes the form (cf. \cite[Theorem 2(b.2)]{15})

$$
\varphi(z_1, z_2, z_3) = z_1z_2 + a_{100}(t)z_1 + a_{010}(t)z_2 + a_{000}(t) + \sum_{(u,v,0) \in \Delta \setminus C_S} O(t^{>0}) \cdot z_1^u z_2^v + t^s(a_{ij1}(t)z_1^j z_2^j + a_{m,n,-1}(t)z_1^m y^n z_3^{-1}) + O(t^{>s}),
$$

where $s > 0$, and

$$
a_{100}(t) = a_{100} + O(t^{>0}), \quad a_{010}(t) = a_{010} + O(t^{>0}), \quad a_{000}(t) = a_{000} + O(t^{>0}),
$$

$$
a_{ij1}(t) = a_{ij1} + O(t^{>0}), \quad a_{m,n,-1}(t) = a_{m,n,-1} + O(t^{>0}).
$$

A possible singular point of a sought surface $S \in \text{Sing}(\Delta, \mathcal{P}, S)$ should be $q = (z_{10} + O(t^{>0}), z_{20} + O(t^{>0}), z_{30} + O(t^{>0}))$, where $(z_{10}, z_{20}, z_{30})$ are found from the system $\varphi|_{t=0} = \frac{\partial \varphi}{\partial z_1}|_{t=0} = \frac{\partial \varphi}{\partial z_2}|_{t=0} = (t^{-s} \frac{\partial \varphi}{\partial z_3})|_{t=0} = 0$, which reduces to

$$
a_{000} = a_{100}a_{010}, \quad z_{10} = -a_{010}, \quad z_{20} = -a_{100}, \quad a_{ij1}z_{10}^j z_{20} - a_{m,n,-1}z_{10}^m z_{20} z_{30}^2 = 0.
$$

Thus, we get two solutions $(z_{10}, z_{20}, z_{30})$, and we claim that each of them induces a unique algebraic surface $S \in \text{Sing}(\Delta, \mathcal{P}, S)$. Again we apply the implicit function theorem to the system of equations

$$
\varphi(p_i) = 0, \quad i = 1, ..., N, \quad \varphi(q) = \frac{\partial \varphi}{\partial z_1}(q) = \frac{\partial \varphi}{\partial z_2}(q) = t^{-s} \frac{\partial \varphi}{\partial z_3}(q) = 0 \tag{38}
$$

in the coordinates of $q$ and the coefficients $a_\omega, \omega \in \Delta \cap \mathbb{Z}^3 \setminus \{(1, 1, 0)\}$. Similarly to the proof of Lemma \ref{4.7} the Jacobian, evaluated at $t = 0$, splits into a block coming from the nondegenerate linear system \ref{20} and a block coming from the last four equations in \ref{38}:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \alpha_{100} & 0 & 0 \\
0 & 0 & \alpha_{010} & 0 \\
0 & * & * & 2\alpha_{m,n,-1}z_{10}^m z_{20} z_{30}^{-3}
\end{pmatrix}
$$

that is nondegenerate too.

\hfill \square

5. APPLICATION AND EXAMPLES

Denote

$$
\Delta_d^n = \text{Conv}\{(0, ..., 0), (d, 0, ..., 0), (0, d, 0, ..., 0), ..., (0, ..., 0, d)\} \subset \mathbb{R}^n.
$$

As we mentioned already in Example \ref{2.1} \deg \text{Sing}(\Delta_d^3) = 4(d-1)^3$. We will demonstrate a computation of the top asymptotical term in \deg \text{Sing}(\Delta_d^3)$ for the general case and a precise computation of \deg \text{Sing}(\Delta_d^3) = 32 via formula \ref{11}, based on the lattice path algorithm presented in Sections \ref{3} and \ref{4}.

5.1. Enumeration of singular algebraic surfaces of degree $d \geq 2$. Let $d \geq 2$. Fix the line $L \subset \mathbb{R}^3$ passing through the origin and directed by the vector $v = (1, \varepsilon, \varepsilon^2)$ with a sufficiently small rational $\varepsilon > 0$. It then defines the following order on $\Delta_d^3 \cap \mathbb{Z}^3 = \{w_k : k = 0, ..., N + 1\}$:

$$
w_k = (i, j, l) \prec w_{k'} = (i', j', l') \iff \begin{cases} 
either i < i', 
or i = i', j < j', 
or i = i', j = j', l < l'. \end{cases} \tag{39}
$$

We shall use also the induced order

$$
(i, j) \prec (i', j') \iff \begin{cases} 
either i < i', 
or i = i', j < j'. \end{cases} \tag{40}
$$

that is nondegenerate too.
Pick \( N = |\Delta^3_d \cap \mathbb{Z}^3| - 2 \) points \( x_1, \ldots, x_N \in L \) satisfying (2), and a generic configuration \( \widehat{p} \subset (\mathbb{K}^*)^3 \) of \( N \) points such that \( \text{Val}(p_i) = x_i, \; i = 1, \ldots, N \).

It is clear that, in the considered situation, no singular tropical surfaces with circuits of type B or C are possible. We will describe all tropical surfaces \( S \in \text{Sing}^3(\Delta^3_d, \widehat{p}) \) contributing to the top asymptotical term of \( \text{deg Sing}(\Delta^3_d) \). Using Corollaries 5.5, 5.8, and 5.11 below, we derive that

\[
\text{deg Sing}(\Delta^3_d) = \sum_{S \in \text{Sing}^3(\Delta^3_d, \widehat{p})} \text{mt}(S, \widehat{p}) = 4d^3 + O(d^2) .
\]

5.1.1. Contribution of singular tropical surfaces with a circuit of type E (see Figure 1).

Lemma 5.1. Let \( w_k = (i,j,l) \) with \( i, j > 0 \) and \( 0 < l < d - i - j \). Then there exists a unique tropical surface \( S \in \text{Sing}^3(\Delta^3_d, \widehat{p}) \) matching the lattice path \( \Gamma_k \). It has a circuit \( C_S = \{(i,j,l-1),(i,j,l),(i,j,l+1)\} \) of type E, and \( \text{mt}(S, \widehat{p}) = 8 \).

Proof. The order (39) yields that the path \( \Gamma_k \) contains the edge \([w_{k-1}, w_{k+1}] = [(i,j,l-1),(i,j,l+1)]\), and hence the circuit must be of type E. By Lemma 3.7, we get a unique surface \( S \in \text{Sing}^3(\Delta^3_d, \widehat{p}) \). Moreover, by Lemma 4.2(v), \( S \) admits a unique enhancement \( \text{tr} \). To allocate a possible singular point \( y \in S \), we follow the procedure from Lemma 4.5. Denote \( b_1 = \nu_S(i,j,l+1), b_0 = \nu_S(i,j,l-1) \). Then the function \( \Lambda' : \Delta^3_d \rightarrow \mathbb{R} \) is as follows:

\[
\Lambda'(x,y,z) = \frac{b_1 - b_0}{2} (z - l) + \frac{b_0 + b_1}{2} .
\]

In view of (41) and (39), we have

\[
0 < \nu_S(i',j',l') \leq b_0 \leq b_1 \leq \nu_S(i'',j'',l'') \text{ as long as } (i', j') \prec (i, j) \prec (i'', j'') .
\]

Hence the graph of the function \( \nu_{x,y} : \Delta^3_d \rightarrow \mathbb{R} \) is the lower convex hull of the points \( (\omega, -c'_\omega) \), \( \omega \in \Delta^3_d \cap \mathbb{Z}^3 \setminus \{(i,j)\} \), such that

\[
c'_\omega = \begin{cases} 
\nu_S(i',j',d - i' - j') - \frac{b_1 - b_0}{2} (d - i' - j' - l) - \frac{b_0 + b_1}{2} & \text{if } \omega = (i', j') \prec (i,j), \\
\nu_S(i',j',0) + \frac{b_1 - b_0}{2} l - \frac{b_0 + b_1}{2} & \text{if } (i,j) \prec \omega = (i', j') .
\end{cases}
\]

From (42) we can easily derive that all the points \( (\omega, -c'_\omega) \), \( \omega \in \Delta^3_d \cap \mathbb{Z}^3 \setminus \{(i,j)\} \), appear as vertices of the graph of \( \nu_{x,y} \), and that this function induces a smooth triangulation of \( \Delta^3_d \), built on the lattice path, which goes through the points \( \omega \in \Delta^3_d \cap \mathbb{Z}^3 \setminus \{(i,j)\} \) in the order (40) and has the unique edge \( E_1 = [(i,j-1),(i,j+1)] \) of length 2. Consider now the segment \( E_2 = [(i-1, d - i + 1), (i+1, 0)] \). The function \( \Lambda'' : \Delta^3_d \rightarrow \mathbb{R} \) defined by the conditions

\[
\Lambda''|_{E_1} = \nu_{x,y}|_{E_1} \text{ and } (\nu_{x,y} - \Lambda'')|_{E_2} = \text{const} ,
\]

can be expressed as

\[
\Lambda''(x,y) = \frac{-c_{i,j+1} + c'_{i,j-1}}{2}(y - j) + \frac{-c_{i+1,0} + c'_{i-1,d-i+1}}{2}(x - i) + \frac{-c_{i,j+1} - c'_{i,j-1}}{2} .
\]

Relations (12) and formulas (13) immediately yield that

\[
\nu_{x,y}(\omega) > \Lambda''(\omega) \text{ for all } \omega \prec (i-1,d-i+1) \text{ and } \omega \succ (i+1,0) ,
\]

and hence the hypotheses of Lemma 4.5(ii) are satisfied. By Lemma 4.6(2), we obtain 8 algebraic surfaces \( S \in \text{Sing}(\Delta^3_d, \widehat{p}, S) \).

Note that, in this construction the segment \( E_1 \) is defined uniquely, and for any segment \( E'_2 = [(i-1, j'),(i+1, j'')] \neq E_2 \), the corresponding function \( \Lambda'' = \Lambda''|_{E_1,E_2} \) will satisfy, due to (12),

\[
\begin{cases} 
\text{either } (\nu_{x,y} - \Lambda'')(i-1, d-i+1) < 0 \text{ if } j' < d - i + 1, \\
or \quad (\nu_{x,y} - \Lambda'')(i+1, 0) < 0 \text{ if } j'' > 0 ,
\end{cases}
\]

contrary to the conditions of Lemma 4.5.
Furthermore, if \( j \geq 2 \), the subdivision of \( \Delta^2_d \) contains no triangle, equivalent to that from the list \([30]\) up to the shift \((x,y) \rightarrow (x-i,y-j)\) and a \( \mathbb{Z} \)-linear transformation. In turn, if \( j = 1 \), the triangles \( T_{j'} = \text{Conv}((i-1,j'),(i-1,j'+1),(i,0)) \), \( 0 \leq j' \leq d-i \), meet the above requirement.

However, the linear function \( \Lambda'' : \mathbb{R}^2 \rightarrow \mathbb{R} \) linearly extending \( \nu_{x,y}|_{T_{j'}} \) satisfies (cf. \([42]\) and \([43]\))

\[
\Lambda''(i,1) = -c'_{i,0} - c'_{i-1,j'+1} + c'_{i-1,j'} = \nu_S(i,0,d-\nu) + \nu_S(i-1,j'+1,d-\nu-i-j'+1) - \nu_S(i-1,j',d-\nu-i-j'+1)
\]

\[
\quad - \frac{b_1 - d - i - l}{2} + \frac{b_0 - d - i - l - 2}{2} = -b_1 - d - i - l + o(b_1) < 0,
\]

contrary to the conditions of Lemma \([4.5]\).

\[ \square \]

**Lemma 5.2.** Let \( w_k = (i,0,l) \) with \( i > 0 \) and \( 0 < l < d - i \). Then there exists a unique tropical surface \( S \in \text{Sing}^{\text{tr}}(\Delta^3_d, \mathcal{X}) \) matching the lattice path \( \Gamma_k \). It has a circuit \( C_S = \{(i,0,l-1),(i,0,l),(i,0,l+1)\} \) of type \( E \), and \( \text{mt}(S, \mathcal{X}) = 2(d-i+1) \).

**Proof.** We proceed as in the proof of Lemma \([5.1]\), similarly obtaining formulas \([41]\) and \([43]\), where \( j = 0 \). The subdivision of \( \Delta^3_d \) is again a smooth triangulation, built on the lattice path, which goes through the points \( \omega \in \Delta^3_d \cap \mathbb{Z}^2 \setminus \{(i,0)\} \) in the order \([10]\). Since \( \text{pr}_{x,y}|(i,0,l) = (i,0) \), the case (ii) of Lemma \([4.5]\) is not possible. However, the triangles \( T_{j'} = \text{Conv}((i-1,j'),(i-1,j'+1),(i,1)) \), \( 0 \leq j' \leq d-i \), of the obtained subdivision of \( \Delta^3_d \) meet the conditions of Lemma \([4.5]\) (i). Moreover, the function \( \Lambda'' : \Delta^3_d \rightarrow \mathbb{R} \), linearly extending \( \nu_{x,y}|_{T_{j'}} \), satisfies

\[
\Lambda''(i,0) = -c'_{i,1} - c'_{i-1,j'} + c'_{i-1,j'+1} = \nu_S(i,1,0) + \nu_S(i-1,j',d-\nu-i-j'+1) - \nu_S(i-1,j'+1,d-\nu-i-j')
\]

\[
\quad + \frac{b_1 - b_0}{2}(l+1) - \frac{b_0 + b_1}{2} = \nu_S(i,1,0) + o(\nu_S(i,1,0)) > 0.
\]

Hence, the conditions of Lemma \([4.5]\) are satisfied, and we obtain \( \text{mt}(S, \mathcal{X}) = 2(d-i+1) \).

\[ \square \]

**Lemma 5.3.** Let \( w_k = (i,j,d-i-j) \) with \( j > 0 \) and \( i + j \leq d-1 \). Then there exists a unique tropical surface \( S \in \text{Sing}^{\text{tr}}(\Delta^3_d, \mathcal{X}) \) matching the lattice path \( \Gamma_k \) and the circuit \( C_S = \{(i,j-1,d-i-j+1), (i,j+1,d-i-j+1)\} \) of type \( E \). Furthermore, \( \text{mt}(S, \mathcal{X}) = 2(d-i-1) \).

**Proof.** The smooth triangulation of \( \Delta^3_d \) induced by the path \( \Gamma_k \) contains an edge \([i,j-1,d-i-j+1],[i,j+1,d-i-j+1]\) of length \( 2 \), which forms a circuit of type \( E \). Let \( \text{pr}_{x,y} : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be the projection onto the \((x,y)\)-plane parallel to the vector \( a = (0,1,-1) \). The point \( \text{pr}_{x,y}(C_S) = (i,d-i) \) belongs to \( \partial \Delta^3_d \), and hence the situation of Lemma \([4.5]\) (ii) is not possible. Set \( b_0 = \nu_S(i,j-1,d-i-j+1) \) and \( b_1 = \nu_S(i,j+1,d-i-j-1) \), and note that

\[
0 < \nu_S(i',j',l') \ll b_0 \ll b_1 \ll \nu_S(i'',j'',l'') \quad \text{when} \quad (i',j'+l') \prec (i,d-i) \prec (i'',j''+l''). \quad (44)
\]

The suitably modified construction of Lemma \([4.5]\) yields \( \Lambda'(y) = \frac{b_0-b_1}{2}(y-j) + \frac{b_0+b_1}{2} \), and that the graph of the function \( \nu_{x,y} \) is the lower convex hull of the set of points \((\omega, -c''_i) \), \( \omega \in \Delta^3_d \cap \mathbb{Z}^2 \setminus \{(i,d-i)\} \), where due to \([44]\) we have

\[
-c''_{i,j} = \begin{cases}
\nu_S(i',j',0) - \frac{b_0-b_1}{2}(j'-j) - \frac{b_0+b_1}{2}, & \text{if } (i',j') \leq (i,j) \\
\nu_S(i,j+1,j'-j-1)-b_1, & \text{if } i' = i, \ j < j' < d-i \\
\nu_S(i',0,j) + \frac{b_0-b_1}{2}j - \frac{b_0+b_1}{2}, & \text{if } i' > i.
\end{cases}
\]

One can see that all the points \((\omega, -c''_i) \), \( \omega \in \Delta^3_d \cap \mathbb{Z}^2 \setminus \{(i,d-i)\} \), are vertices of the graph of \( \nu_{x,y} \), and that the subdivision of \( \Delta^3_d \) induced by \( \nu_{x,y} \) is a smooth triangulation built on the lattice path, which goes through the points \( \omega \in \Delta^3_d \cap \mathbb{Z}^2 \setminus \{(i,d-i)\} \) in the order \([10]\). This subdivision contains the triangles \( T_{j'} = \text{Conv}((i,d-i-1),(i+1,j'),(i+1,j'+1)) \), \( 0 \leq j' \leq d-i-2 \), satisfying the conditions of Lemma \([4.5]\) (i). Moreover, the functions \( \Lambda'' : \Delta^3_d \rightarrow \mathbb{R} \), linearly extending \( \nu_{x,y}|_{T_{j'}} \), satisfy

\[ \square \]
Lemma 5.4. Let

\[ \Lambda''(i, d - i) = -c'_{i,d-i-1} - c'_{i+1,j'+1} + c'_{i+1,j'} \]

Then there is no surface \( S \),

Hence, by Lemma 4.5, we get \( \mu(S, x) = 2(d - i - 1) \).

\[ \nu_S(i, j + 1, d - i - j - 2) + \nu_S(i + 1, 0, j' + 1) - \nu_S(i + 1, 0, j') - b_1 \]

if \( j < d - i - 1 \); if \( j = d - i - 1 \), we obtain

\[ \Lambda''(i, d - i) = -c'_{i,d-i-1} - c'_{i+1,j'+1} + c'_{i+1,j'} \]

\[ \nu_S(i, d - i - 1, 0) + \nu_S(i + 1, 0, j' + 1) - \nu_S(i + 1, 0, j') - \frac{b_0 + b_1}{2} \]

\[ \nu_S(i + 1, 0, j' + 1) + o(\nu_S(i + 1, 0, j' + 1)) > 0. \]

Hence, by Lemma 4.5 we get \( \mu(S, x) = 2(d - i - 1) \).

The following lemma confirms that no other surface \( S \in \text{Sing}^1(\Delta^3_d, \mathfrak{F}) \) has a circuit of type E.

Lemma 5.4. Let

(i) either \( w_k = (0, j, l), j, l \geq 0, j + l < d \),

(ii) or \( w_k = (i, j, 0), i, j > 0, j + i < d \),

(iii) or \( w_k = (i, d - i, 0), 0 < i < d \),

(iv) or \( w_k = (i, 0, 0), 0 < i < d \).

Then there is no surface \( S \in \text{Sing}^1(\Delta^3_d, \mathfrak{F}) \) matching the lattice path \( \Gamma_k \) and having a circuit of type E.

Proof. Suppose that such a surface \( S \in \text{Sing}^1(\Delta^3_d, \mathfrak{F}) \) does exist. The construction of Lemma 4.5 (as performed in the proof of Lemmas 5.1 5.3) leads

- in case (i), to a function \( \nu_{x,y} : \Delta^2_d \to \mathbb{R} \), which is defined by \( \nu_{x,y} \) and which induces a smooth triangulation of \( \Delta^2_d \), built on the lattice path going through the points \( \Delta^2_d \cap \mathbb{Z}^2 \setminus \{(0, j)\} \) in the order \( \{0\} \); if \( j > 0 \) this subdivision does not meet neither the conditions of Lemma 4.5(i), nor of Lemma 4.5(ii);

- in case (ii), to a function \( \nu_{x,z} : \Delta^2_d \to \mathbb{R} \), defined by the values

\[ -c'_{i',d} = \begin{cases} 
\nu_S(i', d - i' - l' - l'), -b_0 \frac{b_0}{2} (d - i' - l' - j) - \frac{b_0 + b_1}{2}, & \text{if } i' < i, \\
\nu_S(i, j, l') - \frac{b_0 + b_1}{2}, & \text{if } i' = i, l' \leq d - i - j, \\
\nu_S(i, d - i - l', l') + \frac{b_0 + b_1}{2} (j + i' - d) - \frac{b_0 + b_1}{2}, & \text{if } i' = i, l' > d - i - j, \\
\nu_S(i', 0, l') + \frac{b_0 + b_1}{2} j - \frac{b_0 + b_1}{2}, & \text{if } i' > i, \\
b_0 = \nu_S(i, j, 1, 0), & b_1 = \nu_S(i, j + 1, 0).
\end{cases} \]

It induces a smooth triangulation of \( \Delta^2_d \), built on the lattice path going through the points \( \Delta^2_d \cap \mathbb{Z}^2 \setminus \{(i, 0)\} \), and this subdivision contains triangles \( T_{d'} = \text{Conv}\{(i, 1), (i - 1, l'), (i - 1, l' + 1)\} \), \( 0 \leq l' \leq d - i \); however, the function \( \Lambda'' : \Delta^2_d \to \mathbb{R} \) linearly extending \( \nu_{x,z}|_{T_{d'}} \) takes the negative value

\[ \Lambda''(i, 0) = -c'_{i,1} - c'_{i-1,l'} + c'_{i-1,l'+1} \]

\[ \nu_S(i, j, 1) - \frac{b_0 + b_1}{2} \]

\[ + \left[ \frac{b_0 + b_1}{2} (d - i' - l' - j) - \frac{b_0 + b_1}{2} \right] \\
- \left[ \frac{b_0 + b_1}{2} (d - i' - l' - j) - \frac{b_0 + b_1}{2} \right] \\
= -b_1 + o(b_1) < 0, \]
Corollary 5.5. Let \( \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X}) \subset \text{Sing}_{\text{tr}}(\Delta_3^3, \mathcal{X}) \) be formed by singular tropical surfaces with a circuit of type E. Then

\[
\sum_{S \in \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X})} \text{mt}(S, \mathcal{X}) = \frac{8}{3} d^3 + O(d^2). 
\]

Proof. Straightforward from Lemmas 5.1, 5.5. \( \square \)

Remark 5.6. Note that the tropical surfaces we consider in this count may have up to \( d \) singular points (accounting for different algebraic realizations of the same tropical surface, for which the tropicalization of the singularity differs). The different singular points occur for the different triangles we pick e.g. in the case considered in Lemma 5.2.

5.1.2. Contribution of singular tropical surfaces with a circuit of type D (see Figure 7).

Lemma 5.7. (1) Let \( \mathbf{w}_k = (i, j, 0) \) with \( i > 0, 0 < j < d - i \). Then, for any quadruple

\[ Q_l = \{(i, j, 0), (i, j, 1), (i, j - 1, l), (i, j - 1, l + 1)\} \subset \Delta_3^d, \quad l = 0, \ldots, d - i - j, \]

there exists a unique tropical surface \( S \in \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X}) \) matching the lattice path \( \Gamma_k \) (see Lemma 3.2) and having the circuit \( C_S = Q_l \) of type D.

(2) Let \( \mathbf{w}_k = (i, j, d - i - j) \) with \( i > 0, 0 \leq j < d - i \). Then, for any quadruple

\[ Q_l = \{(i, j, d - i - j), (i, j, d - i - j - 1), (i, j + 1, l), (i, j + 1, l + 1)\} \subset \Delta_3^d, \quad l = 0, \ldots, d - i - j - 2, \]

there exists a unique tropical surface \( S \in \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X}) \) matching the lattice path \( \Gamma_k \) and having the circuit \( C_S = Q_l \) of type D.

(3) Let \( \mathbf{w}_k = (i, 0, 0) \) with \( 0 < i < d \). Then, for any quadruple

\[ Q_{j,l} = \{(i, 0, 0), (i, 0, 1), (i - 1, j, l), (i - 1, j, l + 1)\} \subset \Delta_3^d, \quad l = 0, \ldots, d - i - j - 1, \quad j = 1, \ldots, d - i, \]

there exists a unique tropical surface \( S \in \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X}) \) matching the lattice path \( \Gamma_k \) and having the circuit \( C_S = Q_{j,l} \) of type D.

(4) Each of the above surfaces \( S \) satisfies \( \text{mt}(S, \mathcal{X}) = 2 \).

Proof. In view of Lemmas 5.9 and 5.10 to prove claims (1)-(3) one has to only show that the quadruple \( Q_l \), resp. \( Q_{j,l} \) spans a parallelogram of lattice area 2 not contained in \( \partial \Delta_3^d \), which intersects with \( \text{Conv}\{\mathbf{w}_0, \ldots, \mathbf{w}_{k-1}\} \) along one of its edges, and that the point \( \mathbf{w}_k \) is intermediate in \( Q_l \), resp. \( Q_{j,l} \) along the order (39). The claim (4) follows from Lemma 4.9. \( \square \)

Corollary 5.8. Let \( \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X}) \subset \text{Sing}_{\text{tr}}(\Delta_3^3, \mathcal{X}) \) be formed by singular tropical surfaces with a circuit of type D. Then

\[
\sum_{S \in \text{Sing}_{\text{tr}}(\Delta_3^d, \mathcal{X})} \text{mt}(S, \mathcal{X}) = d^3 + O(d^2). 
\]

Proof. Lemma 5.7 yields that the left-hand side of (45) is at least \( d^3 + O(d^2) \). One easily checks that no other surface with a circuit of type D contributes to the leading order \( d^3 \). \( \square \)
Remark 5.9. In fact, there are other surfaces $S \in \text{Sing}^{\text{tr}}_{\mathbb{Z}}(\Delta_d^3, \mathcal{F})$, and they correspond to lattice paths $\Gamma_k$ with $w_k = (i, d - i, 0)$, $0 \leq i < d$ and quadruples 

$$Q_j = \{(i, d - i, 0), (i, d - i - 1, 1), (i + 1, j, 0), (i + 1, j - 1, 1)\}, \quad 0 < j < d - i - 1.$$ 

However, their total contribution to the left-hand side of (15) is $O(d^2)$.

5.1.3. Contribution of singular tropical surfaces with a circuit of type $A$ (see Figure 1).

Lemma 5.10. Let $w_k = (i, d - i, 0)$ with $0 \leq i < d$. Then, for any 5-tuple

$$Q'_{j,t} = \{(i, d - i, 0), (i, d - i - 1, 1), (i + 1, j, 0), (i + 1, j - 1, l), (i + 1, j - 1, l + 1)\} \subset \Delta_d^3,$$

$$j > 0, \quad l \geq 2, \quad j + l \leq d - i - 1,$$

and for any 5-tuple

$$Q''_{j,t} = \{(i, d - i, 0), (i, d - i - 1, 1), (i + 1, j, l), (i + 1, j, l + 1), (i + 1, j - 1, d - i - j)\} \subset \Delta_d^3,$$

$$j > 0, \quad l \geq 0, \quad j + l < d - i - 2,$$

there exists a unique tropical surface $S \in \text{Sing}^{\text{tr}}_{\mathbb{Z}}(\Delta_d^3, \mathcal{F})$ matching the lattice path $\Gamma_k$ (see Lemma 3.2) and having the circuit $C_S = Q'_{j,t}$, resp. $C_s = Q''_{j,t}$ of type $A$. Each of the above surfaces $S$ satisfies $\text{mt}(S, \mathcal{F}) = 1$.

Proof. Observe that each pentatope $\text{Conv}(Q'_{j,t})$ or $\text{Conv}(Q''_{j,t})$ is $\mathbb{Z}$-affine equivalent to some $\Pi_{pq}$ defined in (9). Furthermore, the last (in the sense of order (39)) point of $Q'_{j,t}$ is $w_{n'} = (i + 1, j, 0)$, and the last point of $Q''_{j,t}$ is $w_{n''} = (i + 1, j, l + 1)$, and the point $w_k$ is intermediate in both cases. Furthermore, one can see that the intersection of $\text{Conv}(Q'_{j,t})$ with $\text{Conv}\{w_s : 0 \leq s < n', s \neq k\}$ is a common 2-face $\text{Conv}\{(i, d - i - 1, 1), (i + 1, j - 1, l), (i + 1, j - 1, l + 1)\}$, and the intersection of $\text{Conv}(Q''_{j,t})$ with $\text{Conv}\{w_s : 0 \leq s < n'', s \neq k\}$ is a common 2-face $\text{Conv}\{(i, d - i - 1, 1), (i + 1, j, l), (i + 1, j - 1, d - i - j)\}$. Furthermore, for the points of $Q'_{j,t}$ we have the relation

$$w_k = (i, d - i, 0) = (i + 1, j, 0) - l \cdot (i + 1, j - 1, l) + (l - 1) \cdot (i + 1, j - 1, l + 1) + (i, d - i - 1, 1),$$

while for the point of $Q''_{j,t}$

$$w_k = (i, d - i, 0) = (d - 1 - i - j - l) \cdot (i + 1, j, l + 1) - (d - 2 - i - j - l) \cdot (i + 1, j, l) + (i + 1, j, l - i - j) + (i, d - i - 1, 1),$$

which in both cases yields, first, that $\lambda_n > 0$ (in the notation of Lemma 3.13) and, second, that $|A(S, \mathcal{F})| = 1$ for each of the considered singular tropical surfaces $S$ (in the notation of Lemma 1.2). □

It is not difficult to show that no other surfaces $S \in \text{Sing}^{\text{tr}}_{\mathbb{Z}}(\Delta_d^3, \mathcal{F})$ are possible (the use of other lattice paths necessarily leads to a pair of parallel edges in the pentatope, which is forbidden for pentatopes $\Pi_{pq}$). Hence, we obtain the following corollary:

Corollary 5.11. Let $\text{Sing}^{\text{tr}}_{\mathbb{Z}}(\Delta_d^3, \mathcal{F}) \subset \text{Sing}^{\text{tr}}(\Delta_d^3, \mathcal{F})$ be formed by singular tropical surfaces with a circuit of type $A$. Then

$$\sum_{S \in \text{Sing}^{\text{tr}}_{\mathbb{Z}}(\Delta_d^3, \mathcal{F})} \text{mt}(S, \mathcal{F}) = \frac{1}{3}d^3 + O(d^2).$$
5.2. Enumeration of singular cubic and quadric surfaces.

**Lemma 5.12.** Using our lattice path algorithm, we obtain for a point configuration \(x\) as in the beginning of subsection 5.1

\[
\sum_{S \in \text{Sing}^T(\Delta^3, x)} m_t(S, x) = 32.
\]

**Proof.** We mainly adapt the results of the preceding section to the case of \(d = 3\).

Following Lemma 5.2, we consider the lattice path \(\Gamma_k\) (see Lemma 3.2) for \(w_k = (1, 0, 1)\) and obtain one singular tropical surface with circuit of type E, which gives rise to 6 singular algebraic cubics.

Following Lemma 5.3, we consider the lattice paths \(\Gamma_k\) for \(w_k = (0, 1, 2), (0, 2, 1), (1, 1, 1)\) and obtain 3 singular tropical surfaces with circuit of type E, two of them giving rise to four singular algebraic cubics, one to two.

Following Lemma 5.7(1), we consider the lattice path \(\Gamma_k\) for \(w_k = (1, 1, 0)\) and obtain 2 singular tropical surfaces with circuit of type D, each of them giving rise to two singular algebraic cubics.

Following Lemma 5.7(3), we consider the lattice path \(\Gamma_k\) for \(w_k = (1, 0, 0)\) or \((2, 0, 0)\) and obtain 4 singular tropical surfaces with circuit of type D, each of them giving rise to 2 singular algebraic cubics.

Finally, letting \(w_k = (0, 3, 0)\), we can easily check that there exists a unique singular tropical surface matching the lattice path \(\Gamma_k\) and having circuit \(C = \{(0, 3, 0), (0, 2, 1), (1, 1, 0), (1, 0, 1)\}\) of type D. It satisfies conditions of Lemma 3.10 and hence provides 2 singular algebraic cubics. (Such extra singular tropical surfaces were mentioned in Remark 5.9 as those which do not contribute to the top asymptotics of \(\text{deg Sing}(\Delta^3_d)\).) \(\square\)

**Remark 5.13.** Using the results of the previous subsection, we can easily verify that

\[
\sum_{S \in \text{Sing}^T(\Delta^3, x)} m_t(S, x) = 4.
\]

Indeed, following Lemma 5.3 we obtain one circuit of type E for the lattice path \(\Gamma_k\) (see Lemma 8.2) for \(w_k = (0, 1, 1)\) contributing 2, and following Lemma 5.7(3), we obtain one circuit of type D for the lattice path \(\Gamma_k\) for \(w_k = (1, 0, 0)\) contributing another 2.

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Hannah Markwig, Universität des Saarlandes, Fachrichtung Mathematik, Postfach 151150, 66041 Saarbrücken, Germany

E-mail address: hannah@math.uni-sb.de

Thomas Markwig, TU Kaiserslautern, Erwin-Schrödinger-Straße, 67653 Kaisers-lautern, Germany

E-mail address: keilen@mathematik.uni-kl.de

Eugenii Shustin, School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

E-mail address: shustin@post.tau.ac.il