ON TUNNEL NUMBER ONE KNOTS
THAT ARE NOT (1, n)

JESSE JOHNSON AND ABIGAIL THOMPSON

Abstract. We show that the bridge number of a t bridge knot in
$S^3$ with respect to an unknotted genus t surface is bounded below
by a function of the distance of the Heegaard splitting induced by
the t bridges. It follows that for any natural number n, there is a
tunnel number one knot in $S^3$ that is not (1, n).

1. Introduction

A compact, connected, closed, orientable surface $S$ embedded in $S^3$ is
standardly embedded if the closure of each component of its complement
is a handlebody. Equivalently, $S$ is a Heegaard surface for $S^3$. A knot
$K$ is in n-bridge position with respect to $S$ if the intersection of $K$ with
each handlebody is a collection of $n$ boundary parallel arcs.

For $n \geq 1$, we will say that $K$ is $(t, n)$ if $K$ can be put in n-bridge
position with respect to a standardly embedded, genus $t$ surface $S$. We
will say that $K$ is $(t, 0)$ if $K$ can be isotoped into $S$. If $K$ is $(t, n)$ for
some $n$ then $K$ is $(t, m)$ for every $m \geq n$. Thus the important number
is the smallest $n$ such that $K$ is $(t, n)$.

A set of arcs properly embedded in the complement of a knot $K$
is an unknotting system if the complement of a regular neighborhood
of $K$ and the arcs is a handlebody. The tunnel number of $K$ is the
minimum number of arcs in an unknotting system for $K$.

Let $K$ be a knot in $S^3$ and $\Sigma$ the Heegaard splitting of the knot com-
plement induced by a t-tunnel decomposition for $K$. Hempel defined
a distance $d(\Sigma)$ for Heegaard splittings using the curve complex. We
will prove the following:

1. Theorem. If $K$ is $(t, n)$ then $K$ is $(t, 0)$ or $d(\Sigma) \leq 2n + 2t$.

Every tunnel number $t$ knot is $(t + 1, 0)$. The question is for what
values of $n$ can a tunnel number $t$ knot be $(t, n)$. Moriah and Ru-
binstein [7] showed that there exist tunnel number one knots that are

1991 Mathematics Subject Classification. Primary 57M.
Key words and phrases. Tunnel number one, bridge number, curve complex.
This research was supported by NSF VIGRE grant 0135345.
(1, 2), but not (1, 1). Morimoto, Sakuma and Yokota [8] and Eudave-Muñoz [1] constructed further examples of knots that are not (1, 1). Eudave-Muñoz has recently announced the existence of tunnel number one knots that are not (1, 2). The first author of this paper [4] showed that for tunnel number one knots, \(d(\Sigma)\) can be arbitrarily large. Thus Theorem 1 implies the following:

2. Corollary. For every \(n \in \mathbb{N}\), there is a tunnel number one knot \(K\) such that \(K\) is not \((1, n)\).

The proof in [4] is non-constructive and therefore does not provide actual examples of knots with high toroidal bridge number. Since this note first appeared as a preprint, Minsky, Moriah and Schleimer [6] have given a constructive proof that there are \(t\)-tunnel knots in \(S^3\) with arbitrarily high distance splittings. They conclude, using Theorem 1, that for every \(t\) and \(k\), there is a \(t\) tunnel knot that is not \((t, k)\).

We describe weakly incompressible surfaces in Section 2 and the curve complex in Section 3. Theorem 1 and Corollary 2 are proved in Section 4.

2. Weakly Compressible Surfaces

A properly embedded, two sided surface \(S\) in a 3-manifold \(M\) is compressible if there is a disk \(D\) in \(M\) such that \(\partial D\) is an essential simple closed curve in \(S\) and the interior of \(D\) is disjoint from \(S\). If \(S\) is not compressible then \(S\) is incompressible.

Assume that \(S\) separates \(M\) into components \(X\) and \(Y\). Then \(S\) is strongly compressible if there are disks \(D_1\) and \(D_2\) such that \(\partial D_1\) and \(\partial D_2\) are disjoint, essential simple closed curves in \(S\), the interior of \(D_1\) is contained in \(X\) (disjoint from \(S\)) and the interior of \(D_2\) is contained in \(Y\). If \(S\) is not strongly compressible then \(S\) is weakly incompressible.

A properly embedded surface \(S\) is boundary compressible if there is a disk \(D \subset M\) such that \(\partial D\) consists of an essential arc in \(S\) and an arc in \(\partial M\). A separating surface \(S\) is strongly boundary compressible if there are boundary compressing disks on opposite sides of \(S\) with disjoint boundaries, or a boundary compressing disk and a compressing disk on opposite sides of \(S\) with disjoint boundaries. A surface is weakly boundary incompressible if \(S\) is not strongly boundary compressible and \(S\) is not strongly compressible.

3. Lemma. Let \(M\) be a compact 3-manifold and \(F\) a closed, separating, incompressible torus embedded in \(M\). Let \(A, B\) be the closures of the components of the complement of \(F\). Let \(S\) be a second surface which separates \(M\). If \(S \cap A\) is weakly boundary incompressible in \(A\) and
ON TUNNEL NUMBER ONE KNOTS

$S \cap B$ is empty or incompressible and boundary incompressible in $B$ then $S$ is weakly incompressible in $M$. If $S \cap A$ and $S \cap B$ are both incompressible and boundary incompressible, then $S$ is incompressible in $M$.

Proof. Assume for contradiction $S$ is strongly compressible. Then there are disks $D_1, D_2$ properly embedded on opposite sides of $S$ such that $\partial D_1 \cap \partial D_2$ is empty.

Assume $D_1$ and $D_2$ have been chosen transverse to $F$ and with a minimal number of components in $(D_1 \cup D_2) \cap F$. If $D_1$ and $D_2$ are disjoint from $F$ then both disks must be in $A$ because $S \cap B$ is incompressible. This contradicts the assumption that $S \cap A$ is weakly boundary incompressible. Without loss of generality, assume $F \cap D_1$ is not empty.

Because $F$ is incompressible and any loop in $D_1$ is trivial in $D_1$, any loop component of $D_1 \cap F$ must be trivial in $F$. Compressing $D_1$ along an innermost such loop will reduce the number of components of intersection without changing its boundary. Thus minimality implies $D_1 \cap F$ is a collection of arcs. Similarly, if $D_2 \cap F$ is not empty then $D_2 \cap F$ is a collection of arcs.

An outermost arc $\beta$ in $D_1$ cuts off a disk whose boundary consists of an arc $\alpha$ in $F$ and an arc $\beta$ in $S \cap A$ or $S \cap B$. If the arc $\beta$ is trivial in $S \cap B$ or $S \cap A$ then it can be pushed across $F$ (taking any other arcs with it) and reducing $(D_1 \cup D_2) \cap F$. Thus we can assume that $\beta$ is essential in $S \cap A$ or $S \cap B$.

If $\beta$ is in $S \cap B$ then the outermost disk is a boundary compression disk for $S \cap B$. Because $S \cap B$ is boundary incompressible, this is not possible so $\beta$ must be in $S \cap A$ and $D_1$ contains a boundary compression disk $D$ for $S \cap A$.

If $D_2$ is disjoint from $F$ then $D_2$ is a compression disk for $S \cap A$. This compression disk is on the opposite side from $D$ and $\partial D$ is disjoint from $\partial D_2$. This contradicts the assumption that $S \cap A$ is weakly boundary incompressible. If $D_2$ intersects $F$ then, as with $D_1$, an outermost disk argument implies that $D_2$ contains a boundary compressing disk $D'$ for $S \cap A$. The disks $D$ and $D'$ are disjoint and on opposite sides of $S \cap A$, again contradicting weak boundary incompressibility.

The case in which $S \cap A$ and $S \cap B$ are both incompressible and boundary incompressible proceeds similarly, but more easily. □

To apply Lemma 3 to knots, we need a result regarding thin position for a knot in the 3-sphere with respect to a standard genus $g$ Heegaard splitting. The result follows from unpublished work of C. Feist [2]. His Theorem 5.5 implies:
4. **Lemma.** If a knot $K$ is $(t, n)$ and not $(t, 0)$ then either (case 1) there is a bicompressible, weakly boundary incompressible meridinal genus $t$ surface with at most $2n$ boundary components in the complement of $K$ or (case 2) there is an incompressible, boundary incompressible meridinal surface with genus at most $t$ and at most $2n$ boundary components in the complement of $K$.

3. **The Curve Complex**

Let $H$ be a 3-manifold with boundary and let $\Sigma$ be a component of $\partial H$.

5. **Definition.** The curve complex $C(\Sigma)$ is the graph whose vertices are isotopy classes of simple closed curves in $\Sigma$ and edges connect vertices corresponding to disjoint curves.

For more detailed descriptions of the curve complex, see [3] and [5].

6. **Definition.** The boundary set $H \subset C(\Sigma)$ corresponding to $H$ is the set of vertices $\{ l \in C(\Sigma) : l$ bounds a disk in $H \}$.

Given vertices $l_1, l_2$ in $C(\Sigma)$, the distance $d(l_1, l_2)$ is the geodesic distance: the number of edges in the shortest path from $l_1$ to $l_2$. This definition extends to a definition of distances between subsets $X, Y$ of $C(\Sigma)$ by defining $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$ and for distances between a point and a set similarly.

Given a compact, connected, orientable 3-manifold $M$ and a compact, connected, closed, separating surface $\Sigma$, let $A$ and $B$ be the closures of the complement in $M$ of $\Sigma$. Then $\Sigma$ is a component of $\partial A$ and a component of $\partial B$. Let $X, Y$ be the boundary sets in $C(\Sigma)$ of $A$ and $B$, respectively. If $X$ and $Y$ are non-empty, we will define $d(\Sigma) = d(X, Y)$.

This situation arises in a knot complement as follows: Let $M$ be the complement of a regular neighborhood of a knot $K$ in $S^3$ and let $\tau_1, \ldots, \tau_t$ be a collection of properly embedded arcs in $M$. The arcs $\tau_1, \ldots, \tau_t$ are called a collection of unknotting tunnels for $K$ if the complement in $M$ of a regular neighborhood $N$ of $\bigcup \tau_i \cup \partial M$ is a handlebody. Let $\Sigma$ be the boundary component of the closure of $N$ that is disjoint from $\partial M$. The surface $\Sigma$ separates $M$ and allows us to define $d(\Sigma)$ as above. For $t = 1$, Lemma 4 and Lemma 11 of [4] imply the following Lemma:

7. **Lemma.** For every $N$, there is a knot $K$ in $S^3$ and an unknotting tunnel $\tau$ such that for $\Sigma$ constructed as above $d(\Sigma) > N$. 
In [4], it is shown that $d(\Sigma)$ bounds below both the bridge number of $K$ and the Seifert genus of $K$. Theorem 1 provides a similar bound for the toroidal bridge number.

4. Bounding Distance

A compact, separating surface $\Sigma$ properly embedded in a manifold $M$ is called \textit{bicompressible} if there are compressing disks for $\Sigma$ in both components of $M \setminus \Sigma$.

Given a bicompressible, weakly incompressible surface $\Sigma$, let $A$, $B$ be the closures of the complements of $M \setminus \Sigma$. If we compress $\Sigma$ into $A$, the resulting surface, $\Sigma'$, separates $A$. It may be possible to compress $\Sigma'$ still further into the component of $A \setminus \Sigma'$ which does not contain $\Sigma$, creating a new surface which again separates $A$.

Let $\Sigma_A$ be the result of compressing $\Sigma'$ away from $\Sigma$ repeatedly, until the resulting surface has no compression disks on the side which does not contain $\Sigma$. Let $\Sigma_B$ be the result of the same operation, but compressing $\Sigma$ maximally into $B$. Define $\Sigma^*$ to be the submanifold of $M$ bounded by $\Sigma_A$ and $\Sigma_B$. Following [9] (with slightly different notation), we will say that weakly incompressible surfaces $\Sigma$ and $S$ are \textit{well separated} if $S^*$ can be isotoped disjoint from $\Sigma^*$. We will say that $\Sigma$ and $S$ are parallel if $S$ can be isotoped to be parallel to $\Sigma$. The following is Theorem 3.3 in [9].

8. Theorem (Scharlemann and Tomova [9]). If $\Sigma$ and $S$ are bicompressible, weakly incompressible, connected, closed surfaces in $M$ then either $\Sigma$ and $S$ are well separated, $\Sigma$ and $S$ are parallel, or $d(\Sigma) \leq 2 - \chi(S)$.

This theorem is the key to the following proof. Note that $2 - \chi(S)$ is precisely twice the genus of $S$.

Proof of Theorem 1. Let $M$ be the complement in $S^3$ of a neighborhood of a knot $K$ and assume $K$ is $(t, n)$. By Lemma 4, there is either an incompressible, boundary incompressible or a bicompressible, weakly boundary incompressible $2k$-punctured genus $t$ surface $T$ properly embedded in $M$ with $k \leq n$.

Let $M'$ be the complement in $S^3$ of a neighborhood of the connect sum of $k$ trefoil knots. There is a collection $T'$ of $k$ pairwise disjoint, properly embedded, essential annuli in $M'$ and there is a homeomorphism $\phi : \partial M \to \partial M'$ which sends $\partial T$ onto $\partial T'$. Let $M''$ be the result of gluing $M$ and $M'$ via the map $\phi$. The image in $M''$ of $T' \cup T$ is a closed, genus $t + k$ surface which we will call $S$. The Euler characteristic of $S$ is
2 − 2(k + t). Because \( T \) is incompressible or weakly incompressible and \( T' \) is incompressible, Lemma 3 implies that \( S \) is either incompressible or weakly incompressible.

Lemma 3 also implies that the image in \( M'' \) of \( \Sigma \) is weakly incompressible because \( \Sigma \) is weakly incompressible in \( M \) and \( \Sigma \cap M' \) is empty.

Suppose \( T' \cup T \) is compressible but weakly incompressible. Then by Theorem 8, either \( \Sigma \) and \( S \) are parallel, the surfaces are well-separated or \( d(\Sigma) \leq 2(k + t) \leq 2n + 2t \). To complete the proof of this case we will show that \( \Sigma \) and \( S \) are not parallel or well separated.

First we will show that the surfaces are not parallel. The surface \( \Sigma \) bounds a submanifold containing the closed, incompressible torus \( \partial M \). If \( \Sigma \) and \( S \) are parallel then the complement of \( S \) contains an incompressible torus \( A \), isotopic to \( \partial M \). Assume for contradiction this is the case. Any loop in the intersection \( A \cap \partial M \) must be trivial in both surfaces or essential in both, as both surfaces are incompressible. Any trivial loop of intersection can be eliminated by an isotopy of \( A \) which keeps \( A \) disjoint from \( S \), so we can assume \( A \cap S \) is empty or consists of essential loops.

If \( A \cap S \) is empty then \( A \) is contained in \( M \) or \( M' \). If \( M \) contains an essential torus then as noted in [10], \( d(\Sigma) \leq 2 \) and we are done. Thus we will assume the only incompressible surface in \( M \) is boundary parallel. Such a surface cannot be disjoint from \( T \subset S \).

Each component of the complement in \( M' \) of \( T' \) is homeomorphic to an unknot complement or a trefoil knot complement. Thus an incompressible surface in \( M' \) which does not intersect \( T' \) bounds an unknot complement or a trefoil complement. If \( \partial M \) is isotopic to one of these surfaces, then \( M \) must be an unknot or trefoil complement. In either case, \( d(\Sigma) \leq 2 \) (see [4]). Thus we will assume \( A \cap S \) must be non-empty.

Let \( A' \) be a component of \( A \cap M \). An incompressible annulus properly embedded in \( M \) is always boundary parallel, so one component of \( M \setminus A' \) is a solid torus. The surface \( S \) cannot be contained in this solid torus, so \( A' \) can be isotoped across \( \partial M \), reducing \( A \cap \partial M \). This implies \( A \) is disjoint from \( \partial M \), which we saw above is a contradiction. Hence \( A \) and \( \Sigma \) are not parallel.

To show that the surfaces are not well separated, consider the subsets \( \Sigma^* \) and \( S^* \) of \( M'' \) defined above. The surface \( \Sigma \) compresses down to a ball on one side and to a neighborhood of \( \partial M \) on the other, so we can take \( \Sigma^* \) to be the image in \( M'' \) of \( M \). If \( \Sigma \) and \( S \) are well separated then \( S \) can be isotoped out of \( M'' \). After the isotopy, \( \partial M'' \) is an incompressible surface in the complement of \( S \). Thus there is an incompressible torus, isotopic to \( \partial M \) in the complement of \( S \). We showed that no such surface exists, so \( \Sigma \) and \( S \) are not well separated.
ON TUNNEL NUMBER ONE KNOTS THAT ARE NOT $(1,n)$

Now suppose $T' \cup T$ is incompressible. The arguments of Theorem 8 apply to this case as well, although considerably simplified by the fact that $T' \cup T$ is incompressible instead of weakly incompressible. The details of this case are left to the reader. □

Proof of Corollary 2. By Lemma 7 there is a knot $K$ with unknotting tunnel $\tau$ such that for the induced Heegaard splitting $\Sigma$, $d(\Sigma) > 2n + 2$. As noted in [4], every unknotting tunnel for a torus knot has distance at most 2, so $K$ is not $(1,0)$. Thus by Theorem 1 $K$ is not $(1,n)$. □

References

[1] M. Eudave-Muñoz. (1,1)-knots and incompressible surfaces. preprint, 2002. ArXiv:math.GT/0201121.
[2] C. Feist. Results on Thin Position. Ph.D. Thesis, UC Davis, 1998.
[3] J. Hempel. 3-manifolds as viewed from the curve complex. Topology, 40:631–657, 2001. ArXiv:math.GT/9712220.
[4] J. Johnson. Bridge number and the curve complex. preprint, 2006.
[5] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves I: Hyperbolicity. Inventiones mathematicae, 138:103–149, 1999. ArXiv:math.GT/9804098.
[6] Y. N. Minsky, Y. Moriah and S. Schleimer. High distance knots. preprint, 2006. ArXiv:math.GT/0607265.
[7] Y. Moriah and H. Rubinstein. Heegaard structure of negatively curved 3-manifolds. Comm. in Ann. and Geometry, 5:375–412, 1997.
[8] K. Morimoto, M. Sakuma, and Y. Yokota. Examples of tunnel number one knots which have the property “$1 + 1 = 3$”. Math. Proc. Camb. Phil. Soc., 119:113–118, 1996.
[9] M. Scharlemann and M. Tomova. Alternate Heegaard genus bounds distance. preprint, 2004. ArXiv:math.GT/0501140.
[10] A. Thompson. The disjoint curve property and genus 2 manifolds. Topology Appl., 97(3):273–279, 1999.

Department of Mathematics, Yale University, New Haven, CT 06511, USA

E-mail address: jessee.johnson@yale.edu

Department of Mathematics, University of California, Davis, CA 95616, USA

E-mail address: thompson@math.ucdavis.edu