STEADY FLOWS OF AN OLDROYD FLUID WITH THRESHOLD SLIP

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ABSTRACT. We consider a mathematical model that describes 3D steady flows of an incompressible viscoelastic fluid of Oldroyd type in a bounded domain under mixed boundary conditions, including a threshold-slip boundary condition. Using the concept of weak solutions, we reduce the original slip problem to a coupled system of variational inequalities and equations for the velocity field and stresses. For arbitrary large data (forcing and boundary data) and suitable material constants, we prove the existence of weak solutions and establish some of their properties.

1. Introduction. It is well known that, under certain circumstances, many non-Newtonian fluids do not satisfy the classical no-slip boundary condition. For instance, polymeric flows can slip over solid surfaces when the shear stress exceeds a critical value. As the departure from the no-slip condition occurs in various ways, numerous mathematical models have been proposed to describe slip effects (see, e.g., the short survey [34]).

In this paper, we investigate a model that describes internal steady flows of an incompressible viscoelastic fluid of Oldroyd type [32, 33] in a bounded domain \( \Omega \subset \mathbb{R}^3 \) subject to mixed boundary conditions, including a threshold-slip boundary condition [19]. Namely, the problem under consideration takes the form

\[
\begin{align*}
(v \cdot \nabla)v - \text{div} S + \nabla p &= f \quad \text{in } \Omega, \\
S &= E + (1 - \alpha)\eta D(v) \quad \text{in } \Omega, \\
\nabla \cdot v &= 0 \quad \text{in } \Omega, \\
E + \lambda \frac{D}{Dt} E &= \alpha \eta D(v) \quad \text{in } \Omega, \\
v \cdot n &= 0 \quad \text{on } \Gamma, \\
|\langle S_n \rangle_{\tan} | &\leq q \quad \text{on } \Gamma_0, \\
|\langle S_n \rangle_{\tan} | &< q \implies v_{\tan} = 0 \quad \text{on } \Gamma_0, \\
|\langle S_n \rangle_{\tan} | &= q \implies v_{\tan} \uparrow \downarrow \langle S_n \rangle_{\tan} \quad \text{on } \Gamma_0, \\
v &= 0 \quad \text{on } \Gamma \setminus \Gamma_0.
\end{align*}
\]

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Here $\Omega$ is the domain occupied by the fluid, $\Gamma$ denotes the boundary of $\Omega$, $\mathbf{v}$ is the velocity, $\mathbf{S}$ is the extra-stress tensor, $p$ is the pressure, $\mathbf{f}$ denotes the body force, $\mathbf{E}$ is the elastic part of the extra-stress tensor, $\mathbf{D}(\mathbf{v})$ is the strain-rate tensor,

$$\mathbf{D}(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} + (\nabla \mathbf{v})^T),$$

the operator $\mathcal{D}_\rho / \mathcal{D}t$ is the regularized Jaumann derivative [42]. In the stationary case, this operator is defined by the formula

$$\mathcal{D}_\rho \mathbf{E} = (\mathbf{v} \cdot \nabla) \mathbf{E} + \mathbf{E} \mathbf{W}_\rho(\mathbf{v}) - \mathbf{W}_\rho(\mathbf{v}) \mathbf{E},$$

$\mathbf{W}(\mathbf{v})$ and $\mathbf{W}_\rho(\mathbf{v})$ denote the vorticity tensor and the regularized vorticity tensor, respectively,

$$\mathbf{W}(\mathbf{v}) = \frac{1}{2} (\nabla \mathbf{v} - (\nabla \mathbf{v})^T),$$

$$\mathbf{W}_\rho(\mathbf{v})(x) = \int_{\mathbb{R}^3} \rho(x - y) \mathbf{W}(\mathbf{v})(y) \, dy,$$  \hspace{1cm} (10)

where $\rho: \mathbb{R}^3 \to \mathbb{R}$ is a smooth function with compact support such that

$$\int_{\mathbb{R}^3} \rho(x) \, dx = 1$$

and $\rho(x) = \rho(y)$ whenever $|x| = |y|$. In formula (10), we set $\mathbf{W}(\mathbf{v})(\mathbf{y}) = 0$ if $\mathbf{y} \in \mathbb{R}^3 \setminus \Omega$.

The material constants $\eta$ (the viscosity coefficient) and $\lambda$ (the stress relaxation time) are assumed to be positive, and $\alpha \in (0, 1)$ is a dimensionless parameter.

Finally, $\mathbf{n}$ denotes the unit outer normal to $\Gamma$, $(\cdot)_{\text{tan}}$ stands for the tangential component, i.e., $\mathbf{v}_{\text{tan}} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$, and $q$ is a non-negative function defined on the subset $\Gamma_0 \subset \Gamma$.

The symbol $\mathbf{a} \parallel\mathbf{b}$ is used to denote oppositely directed vectors. In other words, for any vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$

$$\mathbf{a} \parallel\mathbf{b} \iff \mathbf{a} \cdot \mathbf{b} + |\mathbf{a}||\mathbf{b}| = 0.$$

In system (1)–(9), the unknowns are $\mathbf{v}$, $\mathbf{S}$, $\mathbf{E}$, and $p$, while all other quantities are assumed to be given.

**Remark 1.** Starting with the pioneering works of Renardy [35] and Guillopè & Saut [20], mathematical models of viscoelastic fluids of Oldroyd type have been studied by many authors. We mention here only the papers [2, 3, 5, 9, 10, 12, 13, 14, 15, 16, 17, 18, 21, 22, 26, 28, 30, 39, 40, 41]. A detailed analysis of different problems and results related to the Oldroyd model and other similar non-Newtonian models can be found in the review article [37]. The present work extends the results of the conference paper [8], in which a slip problem for Oldroyd fluids is studied using the substantial derivative

$$\frac{d\mathbf{E}}{dt} = (\mathbf{v} \cdot \nabla) \mathbf{E}$$

in the constitutive law instead of the Jaumann derivative. Namely, in [8], we deal with the following rheological equation of state, which is a simplified version of (4):

$$\mathbf{E} + \lambda \frac{d\mathbf{E}}{dt} = \alpha \eta \mathbf{D}(\mathbf{v}).$$  \hspace{1cm} (11)
In contrast to (11), the constitutive law (4) is frame-indifferent (see [42]), i.e., the form of (4) does not change after a change of spatial variables. This means that the model considered here does not violate the principle of material objectivity [38].

Note also that, for the case of flows of viscoelastic fluids subject to Navier’s slip boundary condition, which does not take into account the effect of threshold slip, the corresponding boundary and initial-boundary value problems are examined in the recent papers [4, 6, 7, 25].

Remark 2. Let us explain the reasons for using the regularized vorticity tensor $W_{\rho}$ instead of $W$. The goal of this work is to construct weak solutions of boundary value problem (1)–(9) with arbitrary large data. The main difficulty arises in the passage-to-limit procedure in the terms $E_n W(v_n)$ and $W(v_n) E_n$ as $n \to \infty$. To overcome this difficulty, it is convenient to use the following property: if $v_n \to v_0$ weakly in the Sobolev space $H^1(\Omega, \mathbb{R}^3)$, then $W_{\rho}(v_n) \to W_{\rho}(v_0)$ strongly in $L^\infty(\Omega, \mathbb{R}^{3 \times 3})$ as $n \to \infty$. This property is one of the key points in the proof of the existence theorem (see Sect. 4).

Remark 3. The set of relations (6)–(8) represents a special case of the following slip boundary condition [34]:

$$| (\mathbf{T} n)_{\text{tan}} | \leq \psi(\mathbf{x}, | (\mathbf{T} n)_{\text{tan}} |),$$

$$\mathbf{T} n_{\text{tan}} \cdot \mathbf{v}_{\text{tan}} = -\psi(\mathbf{x}, | (\mathbf{T} n)_{\text{tan}} |) | \mathbf{v}_{\text{tan}} |,$$

where the tensor $\mathbf{T} = -\rho \mathbf{I} + \mathbf{S}$ is the Cauchy stress tensor, $\mathbf{I}$ is the identity tensor, $(\mathbf{T} n)_{\text{tan}} = ((\mathbf{T} n) \cdot \mathbf{n}) \mathbf{n}$, and $\psi: \Gamma_0 \times [0, +\infty) \to [0, +\infty)$ is a given function. Indeed, if $\psi(\mathbf{x}, \xi) = q(\mathbf{x})$ for any $(\mathbf{x}, \xi) \in \Gamma_0 \times [0, +\infty)$, then it can easily be checked that system (12), (13) is equivalent to (6)–(8).

Remark 4. If we take $\alpha = \lambda = 0$ formally, then (1)–(9) reduces to the well-known Navier–Stokes equations with threshold-slip boundary conditions. For the corresponding slip problems, well-posedness results were obtained by H. Fujita [19]; in this regard, see also [23, 24].

2. Weak formulation of the problem. We are interested in the existence of weak (generalized) solutions to problem (1)–(9). Before performing our study, let us introduce certain function spaces and notations.

We denote by $| \mathbf{x} |$ the Euclidean norm of a vector $\mathbf{x}$ and by $| \mathbf{G} |$ the Frobenius norm of a tensor $\mathbf{G}$:

$$| \mathbf{x} |^2 = \mathbf{x} \cdot \mathbf{x}, \quad | \mathbf{G} |^2 = \text{trace}(\mathbf{G G}^T).$$

By $\mathbf{G} : \mathbf{F}$ denote the scalar product of two tensors $\mathbf{G}$ and $\mathbf{F}$:

$$\mathbf{G} : \mathbf{F} = \text{trace}(\mathbf{G F}^T).$$

We shall employ the following notations

$$L^p(\Omega, \mathbb{R}^d), \quad H^m(\Omega, \mathbb{R}^d) = W^{m,2}(\Omega, \mathbb{R}^d)$$

for the Lebesgue and Sobolev spaces of functions defined on $\Omega$ and with values in $\mathbb{R}^d$.

Let $\mathbb{R}^{3 \times 3}_{\text{sym}}$ be the space symmetric matrices of size $3 \times 3$. Denote by $C^\infty_0(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}})$ the space of infinitely differentiable functions with compact support contained in $\Omega$ and with values in $\mathbb{R}^{3 \times 3}_{\text{sym}}$.

By definition, put

$$H^2_0(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}}) = \text{the closure of the set } C^\infty_0(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}}) \text{ in } H^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}}).$$
Assuming that $\Omega$ is a bounded domain in $\mathbb{R}^3$ and $\Gamma = \partial \Omega$ is of class $C^2$, we equip this space with the scalar product

$$
(\Phi, \Psi)_{H^2_0(\Omega, \mathbb{R}^{3 \times 3})} = \int_{\Omega} \Delta \Phi : \Delta \Psi \, dx.
$$

It follows from properties of the Laplace operator $\Delta$ (see, e.g., [36, Chapter 9]) that this scalar product is well defined and the norm

$$
\|\Phi\|_{H^2_0(\Omega, \mathbb{R}^{3 \times 3})} = (\Phi, \Phi)^{1/2}_{H^2_0(\Omega, \mathbb{R}^{3 \times 3})}
$$

is equivalent to the norm induced from the Sobolev space $H^2(\Omega, \mathbb{R}^{3 \times 3})$.

Introduce the spaces:

$$
L^2_+(\Gamma_0, \mathbb{R}) = \{ a \in L^2(\Gamma_0, \mathbb{R}) : a(x) \geq 0 \text{ for a.e. } x \in \Gamma_0 \},
$$

$$
Q(\Omega, \mathbb{R}^3) = \{ v \in C^\infty(\overline{\Omega}, \mathbb{R}^3) : \nabla \cdot v = 0, \ v|_\Gamma \cdot n = 0, \ v|_{\Gamma \setminus \Gamma_0} = 0 \},
$$

$$
X(\Omega, \mathbb{R}^3) = \text{the closure of the set } Q(\Omega, \mathbb{R}^3) \text{ in the space } H^1(\Omega, \mathbb{R}^3),
$$

and define the scalar product in $X(\Omega, \mathbb{R}^3)$ by the formula

$$
(v, u)_{X(\Omega, \mathbb{R}^3)} = \int_{\Omega} \nabla(v) : \nabla(u) \, dx,
$$

assuming that at least one of the following two conditions holds:

(A1) the domain $\Omega \subset \mathbb{R}^3$ is not a body of revolution, i.e., $\Omega$ is not a body obtained by rotating a plane curve around some straight line that lies on the same plane;

(A2) the 2-dimensional Lebesgue measure of the set $\Gamma \setminus \Gamma_0$ is positive.

As the next proposition shows, the scalar product (14) is well defined and the norm

$$
\|v\|_{X(\Omega, \mathbb{R}^3)} = (v, v)^{1/2}_{X(\Omega, \mathbb{R}^3)}
$$

is equivalent to the norm induced from the space $H^1(\Omega, \mathbb{R}^3)$.

**Proposition 1.** (Korn’s inequalities, see [29, Chapter I, Theorems 2.2 and 2.3] and [31, Lemma 6.2]).

Let $\Omega$ be a bounded locally Lipschitz domain in $\mathbb{R}^3$. The following statements hold.

(i) If $U$ is not a body of revolution, then

$$
\| \nabla(v) \|_{L^2(U, \mathbb{R}^{3 \times 3})}^2 \geq C_1(U) \|v\|_{H^1(U, \mathbb{R}^3)}^2, \quad C_1(U) = \text{const} > 0,
$$

for all $v \in H^1(U, \mathbb{R}^3)$ such that $v_{|_U} \cdot n = 0$.

(ii) If $\Sigma \subset \partial U$ and the 2-dimensional Lebesgue measure of $\Sigma$ is positive, then

$$
\| \nabla(v) \|_{L^2(U, \mathbb{R}^{3 \times 3})}^2 \geq C_2(U) \|v\|_{H^1(U, \mathbb{R}^3)}^2, \quad C_2(U) = \text{const} > 0,
$$

for all $v \in H^1(U, \mathbb{R}^3)$ such that $v_{|_\Sigma} = 0$.

Employing the variational inequalities approach [19], we introduce the concept of weak solutions to problem (1)–(9).

Let $f$ and $q$ be given such that $f \in L^2(\Omega, \mathbb{R}^3)$, $q \in L^2_+(\Gamma_0, \mathbb{R})$. 

Definition 2.1. We shall say that a triplet 
\[(v, S, E) \in X(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{3\times3}) \times L^2(\Omega, \mathbb{R}^{3\times3})\]
is a weak solution of problem (1)–(9) if condition (2) is valid and

\[- \sum_{i=1}^{3} \int_{\Omega} v_i \frac{\partial \varphi}{\partial x_i} \, dx + \int_{\Omega} E : D(\varphi) \, dx - \frac{1}{\alpha \eta} \int_{\Omega} |E|^2 \, dx \]
\[+ (1 - \alpha) \eta \int_{\Omega} D(v) : D(\varphi - v) \, dx + \int_{\Gamma_0} q|\varphi| \, d\Gamma_0 - \int_{\Gamma_0} q|v| \, d\Gamma_0 \geq \int_{\Omega} f \cdot (\varphi - v) \, dx \quad \forall \varphi \in X(\Omega, \mathbb{R}^3),\]

\[\int_{\Omega} E : F \, dx - \lambda \sum_{i=1}^{3} \int_{\Omega} v_i E : \frac{\partial F}{\partial x_i} \, dx + \lambda \int_{\Omega} (EW_\rho(v) - W_\rho(v)E) : F \, dx \]
\[= \alpha \eta \int_{\Omega} D(v) : F \, dx \quad \forall F \in H^2(\Omega, \mathbb{R}^{3\times3}).\]

Remark 5. Let us compare weak and classical solutions to problem (1)–(9). Suppose that \((\tilde{v}, \tilde{S}, \tilde{E})\) is a classical solution to problem (1)–(9). Following exactly the same steps as in [8, Remark 3], we obtain relations (15) and (16). On the other hand, it can be proved that if a weak solution \((\tilde{v}, \tilde{S}, \tilde{E})\) of problem (1)–(9) is sufficiently smooth, then there exists a function \(\tilde{p}\) such that \((\tilde{v}, \tilde{S}, \tilde{E}, \tilde{p})\) is a classical solution to (1)–(9).

3. Main results. The following theorem gives our main results.

Theorem 3.1. Assume that \(\Omega\) is a bounded domain in \(\mathbb{R}^3\) with boundary \(\Gamma \in C^2\) and at least one of conditions (A1), (A2) holds. Furthermore, let \(f \in L^2(\Omega, \mathbb{R}^3)\) and \(q \in L^2_+(\Gamma_0, \mathbb{R})\). Then

(i) boundary value problem (1)–(9) has at least one weak solution \((v, S, E)\) such that

\[(1 - \alpha) \eta \int_{\Omega} |D(v)|^2 \, dx + \frac{1}{\alpha \eta} \int_{\Omega} |E|^2 \, dx + \int_{\Gamma_0} q|v| \, d\Gamma_0 \leq \int_{\Omega} f \cdot v \, dx; \quad (17)\]

(ii) if \((v_0, S_0, E_0)\) is a weak solution of problem (1)–(9) such that

\[(v_0, S_0, E_0) \in X(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{3\times3}) \times H^1(\Omega, \mathbb{R}^{3\times3}),\]

then

\[(1 - \alpha) \eta \int_{\Omega} |D(v_0)|^2 \, dx + \frac{1}{\alpha \eta} \int_{\Omega} |E_0|^2 \, dx + \int_{\Gamma_0} q|v_0| \, d\Gamma_0 = \int_{\Omega} f \cdot v_0 \, dx; \quad (18)\]

(iii) the set of weak solutions to problem (1)–(9) is sequentially weakly closed in the space \(X(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{3\times3}) \times L^2(\Omega, \mathbb{R}^{3\times3}).\)

The proof of this theorem uses methods for solving variational inequalities with pseudo-monotone operators and convex functionals [11], the method of introduction of auxiliary viscosity [27], and a passage-to-limit procedure based on energy estimates of approximate solutions and compactness arguments.
4. Proof of the theorem 3.1. First we establish the existence result (i). The proof of this statement is derived in six steps.

**Step 1.** Let us fix \( u \in X(\Omega, \mathbb{R}^3) \) and consider the following linear problem depending on a positive parameter \( \varepsilon \):

Find \( E \in H^3_0(\Omega, \mathbb{R}^{3 \times 3}) \) such that

\[
\varepsilon \int_\Omega \Delta E : \Delta F \, dx + \int_\Omega E : F \, dx - \lambda \sum_{i=1}^3 \int_\Omega u_i E : \frac{\partial F}{\partial x_i} \, dx \\
+ \lambda \int_\Omega (EW_\rho(u) - W_\rho(u)E) : F \, dx = \alpha \eta \int_\Omega D(u) : F \, dx \tag{19}
\]

for any \( F \in H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \).

Using the Lax–Milgram theorem (see, e.g., [36, Theorem 9.14]) and the following relations

\[
\| G \|_{H^2(\Omega, \mathbb{R}^{3 \times 3})} \leq C \| \Delta G \|_{L^2(\Omega, \mathbb{R}^{3 \times 3})}, \quad C = \text{const},
\]

\[
\sum_{i=1}^3 \int_\Omega u_i G : \frac{\partial G}{\partial x_i} \, dx = 0, \quad \int_\Omega (GW_\rho(u) - W_\rho(u)G) : G \, dx = 0,
\]

which hold for any \( G \in H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \), we deduce that problem (19) admits a unique solution, for any \( \varepsilon > 0 \) and \( u \in X(\Omega, \mathbb{R}^3) \). Denote by \( R_\varepsilon(u) \) the solution of this problem.

**Step 2.** Now let us estimate the \( H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \)-norm of the vector function \( R_\varepsilon(u) \). Setting \( E = R_\varepsilon(u), F = R_\varepsilon(u) \) in (19), we find

\[
\varepsilon \int_\Omega |\Delta R_\varepsilon(u)|^2 \, dx + \int_\Omega |R_\varepsilon(u)|^2 \, dx = \alpha \eta \int_\Omega D(u) : R_\varepsilon(u) \, dx. \tag{20}
\]

In particular, we have

\[
\int_\Omega |R_\varepsilon(u)|^2 \, dx \leq \alpha \eta \int_\Omega D(u) : R_\varepsilon(u) \, dx \\
\leq \alpha \eta \left( \int_\Omega |D(u)|^2 \, dx \right)^{1/2} \left( \int_\Omega |R_\varepsilon(u)|^2 \, dx \right)^{1/2}
\]

and hence

\[
\left( \int_\Omega |R_\varepsilon(u)|^2 \, dx \right)^{1/2} \leq \alpha \eta \left( \int_\Omega |D(u)|^2 \, dx \right)^{1/2}.
\]

Using this inequality, we deduce from (20) that

\[
\varepsilon \int_\Omega |\Delta R_\varepsilon(u)|^2 \, dx \leq \alpha \eta \left( \int_\Omega |D(u)|^2 \, dx \right)^{1/2} \left( \int_\Omega |R_\varepsilon(u)|^2 \, dx \right)^{1/2}
\]

or equivalently,

\[
\| R_\varepsilon(u) \|_{H^2_0(\Omega, \mathbb{R}^{3 \times 3})} \leq \alpha \eta \varepsilon^{-1/2} \| u \|_{X(\Omega, \mathbb{R}^3)}. \tag{21}
\]

**Step 3.** We shall show that the operator \( R_\varepsilon : X(\Omega, \mathbb{R}^3) \to H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \) is completely continuous, i.e., if \( u_n \to u_0 \) weakly in \( X(\Omega, \mathbb{R}^3) \) as \( n \to \infty \), then

\[
R_\varepsilon(u_n) \to R_\varepsilon(u_0) \text{ strongly in } H^2_0(\Omega, \mathbb{R}^{3 \times 3}), \tag{22}
\]

as \( n \to \infty \).
Since the embedding $H^1(\Omega, \mathbb{R}^3) \hookrightarrow L^4(\Omega, \mathbb{R}^3)$ is compact (see [1, Theorem 6.3]), the embedding $X(\Omega, \mathbb{R}^3) \hookrightarrow L^4(\Omega, \mathbb{R}^3)$ is compact too. Consequently, we have

$$u_n \to u_0 \text{ strongly in } L^4(\Omega, \mathbb{R}^3)$$  \hfill (23)

as $n \to \infty$. Moreover, since the trace operator $\gamma_0: H^1(\Omega, \mathbb{R}^3) \to L^2(\Gamma, \mathbb{R}^3)$ is compact, we see that

$$u_n|\Gamma \to u_0|\Gamma \text{ strongly in } L^2(\Gamma, \mathbb{R}^3)$$  \hfill (24)

as $n \to \infty$.

In equation (19), we replace $u$ by $u_n$, $E$ by $R_\varepsilon(u_n)$, and $F$ by $R_\varepsilon(u_n) - R_\varepsilon(u_0)$; this gives

$$\varepsilon \int_\Omega \Delta R_\varepsilon(u_n) : \Delta (R_\varepsilon(u_n) - R_\varepsilon(u_0)) \, dx$$

$$+ \int_\Omega R_\varepsilon(u_n) : (R_\varepsilon(u_n) - R_\varepsilon(u_0)) \, dx$$

$$- \lambda \sum_{i=1}^3 \int_\Omega u_n iR_\varepsilon(u_n) : \frac{\partial (R_\varepsilon(u_n) - R_\varepsilon(u_0))}{\partial x_i} \, dx$$

$$+ \lambda \int_\Omega (R_\varepsilon(u_n) W_\rho(u_n)) : (R_\varepsilon(u_n) - R_\varepsilon(u_0)) \, dx$$

$$- \lambda \int_\Omega (W_\rho(u_n) R_\varepsilon(u_n)) : (R_\varepsilon(u_n) - R_\varepsilon(u_0)) \, dx$$

$$= \alpha \eta \int_\Omega D(u_n) : (R_\varepsilon(u_n) - R_\varepsilon(u_0)) \, dx.$$  \hfill (25)

Further, if we take $u = u_0$, $E = R_\varepsilon(u_0)$, and $F = -[R_\varepsilon(u_n) - R_\varepsilon(u_0)]$ in (19), we obtain

$$- \varepsilon \int_\Omega \Delta R_\varepsilon(u_0) : \Delta (R_\varepsilon(u_0) - R_\varepsilon(u_0)) \, dx$$

$$- \int_\Omega R_\varepsilon(u_0) : (R_\varepsilon(u_0) - R_\varepsilon(u_0)) \, dx$$

$$+ \lambda \sum_{i=1}^3 \int_\Omega u_0 iR_\varepsilon(u_0) : \frac{\partial (R_\varepsilon(u_0) - R_\varepsilon(u_0))}{\partial x_i} \, dx$$

$$- \lambda \int_\Omega (R_\varepsilon(u_0) W_\rho(u_0)) : (R_\varepsilon(u_0) - R_\varepsilon(u_0)) \, dx$$

$$+ \lambda \int_\Omega (W_\rho(u_0) R_\varepsilon(u_0)) : (R_\varepsilon(u_0) - R_\varepsilon(u_0)) \, dx$$

$$= -\alpha \eta \int_\Omega D(u_0) : (R_\varepsilon(u_0) - R_\varepsilon(u_0)) \, dx.$$  \hfill (26)
Now, summing equalities (25) and (26), we get
\[
\varepsilon \int_{\Omega} |\Delta (R_c(u_n) - R_c(u_0))|^2 \, dx + \int_{\Omega} |R_c(u_n) - R_c(u_0)|^2 \, dx
\]
\[= \lambda \sum_{i=1}^{3} \int_{\Omega} (u_{ni} - u_{0i}) R_c(u_n) : \frac{\partial (R_c(u_n) - R_c(u_0))}{\partial x_i} \, dx + \int_{\Omega} (R_c(u_n) - R_c(u_0)) : (R_c(u_n) - R_c(u_0)) \, dx
\]
\[= \lambda \int_{\Omega} (W_\rho(u_n - u_0)R_c(u_n)) : (R_c(u_n) - R_c(u_0)) \, dx
\]
\[= \alpha \eta \int_{\Omega} D(u_n - u_0) : (R_c(u_n) - R_c(u_0)) \, dx.
\]
Moreover, in view of (21), we easily derive that
\[
\|R_c(u_n)\|_{H^2_0(\Omega, \mathbb{R}^{3\times 3})} \leq \alpha \eta \varepsilon^{-1/2} \|u_n\|_{X(\Omega, \mathbb{R}^3)} \leq C,
\]
where $C$ denotes positive constant, which is independent of $n$.

Applying integration by parts, we also obtain
\[
\|(W_\rho)_{ij}(u)\|_{L^\infty(\Omega, \mathbb{R})}
\]
\[= \frac{1}{2} \sup_{x \in \Omega} \left| \int_{\Omega} \rho(x - y) \left( \frac{\partial u_i(y, t)}{\partial y_j} - \frac{\partial u_j(y, t)}{\partial y_i} \right) \, dy \right|
\]
\[\leq \frac{1}{2} \sup_{x \in \Omega} \left| - \int_{\Omega} \frac{\partial \rho(x - y)}{\partial y_j} u_i(y, t) \, dy + \int_{\Omega} \frac{\partial \rho(x - y)}{\partial y_i} u_j(y, t) \, dy + \int_{\Gamma} \rho(x - y)n_j u_i(y, t) \, d\Gamma - \int_{\Gamma} \rho(x - y)n_i u_j(y, t) \, d\Gamma \right|
\]
\[\leq C_p(\|u\|_{L^2(\Omega, \mathbb{R}^3)} + \|u\|_{L^2(\Gamma, \mathbb{R}^3)}), \quad C_p = \text{const},
\]
and
\[
\left| \int_{\Omega} D(u) : G \, dx \right|
\]
\[= \frac{1}{2} \sum_{i,j=1}^{3} \int_{\Omega} \left( \frac{\partial u_i}{\partial y_j} + \frac{\partial u_j}{\partial y_i} \right) G_{ij} \, dx
\]
\[= \frac{1}{2} \sum_{i,j=1}^{3} \int_{\Omega} \left( u_i \frac{\partial G_{ij}}{\partial y_j} + u_j \frac{\partial G_{ij}}{\partial y_i} \right) \, dx
\]
\[\leq C_0 \|u\|_{L^2(\Omega, \mathbb{R}^3)} \|G\|_{H^1_0(\Omega, \mathbb{R}^{3\times 3})}, \quad C_0 = \text{const},
\]
for any $u \in X(\Omega, \mathbb{R}^3)$ and $G \in H^1_0(\Omega, \mathbb{R}^{3\times 3})$. 
Using the Hölder inequalities and the estimates (29) and (30), we deduce from equality (27) that

\[
\int_\Omega |\Delta (R_\varepsilon (u_n) - R_\varepsilon (u_0))|^2 \, dx \\
\leq \frac{\lambda}{\varepsilon} \left( \sum_{i=1}^3 \int_\Omega (u_{ni} - u_{0i})R_\varepsilon (u_n) : \frac{\partial (R_\varepsilon (u_n) - R_\varepsilon (u_0))}{\partial x_i} \, dx \right) \\
+ \frac{\lambda}{\varepsilon} \int_\Omega (R_\varepsilon (u_n)W_\varepsilon (u_n - u_0)) : (R_\varepsilon (u_n) - R_\varepsilon (u_0)) \, dx \\
+ \frac{\lambda}{\varepsilon} \int_\Omega (W_\varepsilon (u_n - u_0)R_\varepsilon (u_n)) : (R_\varepsilon (u_n) - R_\varepsilon (u_0)) \, dx \\
+ \frac{\alpha \eta}{\varepsilon} \left( \int_\Omega D(u_n - u_0) : (R_\varepsilon (u_n) - R_\varepsilon (u_0)) \, dx \right) \\
\leq \frac{\lambda}{\varepsilon} \sum_{i=1}^3 \left( \int_\Omega |(u_{ni} - u_{0i})|^4 \, dx \right)^{1/4} \left( \int_\Omega |R_\varepsilon (u_n)|^4 \, dx \right)^{1/4} \\
\times \left( \int_\Omega \left| \frac{\partial (R_\varepsilon (u_n) - R_\varepsilon (u_0))}{\partial x_i} \right|^2 \, dx \right)^{1/2} \\
+ \frac{2\lambda}{\varepsilon} \sup_{x \in \Omega} |W_\varepsilon (u_n - u_0)| \left( \int_\Omega |R_\varepsilon (u_n)|^2 \, dx \right)^{1/2} \\
\times \left( \int_\Omega |R_\varepsilon (u_n) - R_\varepsilon (u_0)|^2 \, dx \right)^{1/2} \\
+ \frac{\alpha \eta}{\varepsilon} \left( \int_\Omega D(u_n - u_0) : (R_\varepsilon (u_n) - R_\varepsilon (u_0)) \, dx \right) \\
\leq C_1 \|u_n - u_0\|_{L^4 (\Omega, \mathbb{R}^3)} \|R_\varepsilon (u_n)\|_{L^4 (\Omega, \mathbb{R}^{3x3})} \|R_\varepsilon (u_n) - R_\varepsilon (u_0)\|_{H_0^1 (\Omega, \mathbb{R}^{3x3})} \\
+ C_2 \left\{ \|u_n - u_0\|_{L^2 (\Omega, \mathbb{R}^3)} + \|u_n - u_0\|_{L^2 (\Gamma, \mathbb{R}^3)} \right\} \\
\times \|R_\varepsilon (u_n)\|_{L^2 (\Omega, \mathbb{R}^{3x3})} \|R_\varepsilon (u_n) - R_\varepsilon (u_0)\|_{L^2 (\Omega, \mathbb{R}^{3x3})} \\
+ C_3 \|u_n - u_0\|_{L^2 (\Omega, \mathbb{R}^3)} \|R_\varepsilon (u_n) - R_\varepsilon (u_0)\|_{H_0^1 (\Omega, \mathbb{R}^{3x3})}.
\]  

(31)

Here and in the succeeding discussion, the symbols $C_1, C_2, \ldots$ denote positive constants that are independent of $n$.

Further, taking into account the estimate (28) and the strong convergence (23) as well as (24), we derive from (31) that

\[
\int_\Omega |\Delta (R_\varepsilon (u_n) - R_\varepsilon (u_0))|^2 \, dx \leq C_4 \|u_n - u_0\|_{L^4 (\Omega, \mathbb{R}^3)} \\
+ C_5 \|u_n - u_0\|_{L^2 (\Omega, \mathbb{R}^3)} + C_6 \|u_n - u_0\|_{L^2 (\Gamma, \mathbb{R}^3)} \to 0
\]

as $n \to \infty$. This means that (22) holds. Thus, the operator $R_\varepsilon$ is completely continuous.

**Step 4.** Consider one more auxiliary problem:
Find \( v \in X(\Omega, \mathbb{R}^3) \) such that
\[
-\sum_{i=1}^{3} \int_{\Omega} u_i v \cdot \frac{\partial \varphi}{\partial x_i} \, dx + \int_{\Omega} R_{\varepsilon}(v) : D(\varphi - v) \, dx \\
+ (1 - \alpha)\eta \int_{\Omega} D(v) : D(\varphi - v) \, dx + \int_{\Gamma_0} q|\varphi| \, d\Gamma_0 - \int_{\Gamma_0} q|v| \, d\Gamma_0 
\]
(32)

By \([X(\Omega, \mathbb{R}^3)]^*\) denote the dual space of \( X(\Omega, \mathbb{R}^3) \) and introduce the operators:

\( A : X(\Omega, \mathbb{R}^3) \to [X(\Omega, \mathbb{R}^3)]^* \), \( A(u, \varphi) = (1 - \alpha)\eta \int_{\Omega} D(u) : D(\varphi) \, dx \)

\( K_{\varepsilon} : X(\Omega, \mathbb{R}^3) \to [X(\Omega, \mathbb{R}^3)]^* \),

\( \langle K_{\varepsilon}(u), \varphi \rangle = -\sum_{i=1}^{3} \int_{\Omega} u_i u_i \cdot \frac{\partial \varphi}{\partial x_i} \, dx + \int_{\Omega} R_{\varepsilon}(u) : D(\varphi) \, dx \)

\( J : X(\Omega, \mathbb{R}^3) \to \mathbb{R} \), \( J(u) = \int_{\Gamma_0} q|u| \, d\Gamma_0. \)

Then problem (32) can be written as the following variational inequality
\[
\langle A(v) + K_{\varepsilon}(v) - f, \varphi - v \rangle + J(\varphi) - J(v) \geq 0 \quad \forall \varphi \in X(\Omega, \mathbb{R}^3).
\]

Next, we observe that the operator \( A \) is monotone and
\[
\langle A(u), u \rangle = (1 - \alpha)\eta \|u\|_{X(\Omega, \mathbb{R}^3)}^2, \quad \langle K_{\varepsilon}(u), u \rangle \geq 0
\]
for any \( u \in X(\Omega, \mathbb{R}^3) \).

Moreover, since the operator \( R_{\varepsilon} \) is completely continuous, it can easily be checked that \( K_{\varepsilon} \) is completely continuous too, i.e., if \( u_n \rightharpoonup u_0 \) weakly in \( X(\Omega, \mathbb{R}^3) \) as \( n \to \infty \), then
\[
K_{\varepsilon}(u_n) \rightharpoonup K_{\varepsilon}(u_0) \text{ strongly in } [X(\Omega, \mathbb{R}^3)]^* \text{ as } n \to \infty.
\]
(33)

Indeed, we have
\[
|\langle K_{\varepsilon}(u_n) - K_{\varepsilon}(u_0), \varphi \rangle| \leq C_\varepsilon \left\{ \|u_n - u_0\|_{L^4(\Omega, \mathbb{R}^3)} + \|R_{\varepsilon}(u_n) - R_{\varepsilon}(u_0)\|_{L^2(\Omega, \mathbb{R}^{3x3})} \right\} \|\varphi\|_{X(\Omega, \mathbb{R}^3)},
\]
for any \( \varphi \in X(\Omega, \mathbb{R}^3) \). This means that
\[
\|K_{\varepsilon}(u_n) - K_{\varepsilon}(u_0)\|_{[X(\Omega, \mathbb{R}^3)]^*} \leq C_\varepsilon \left( \|u_n - u_0\|_{L^4(\Omega, \mathbb{R}^3)} + \|R_{\varepsilon}(u_n) - R_{\varepsilon}(u_0)\|_{L^2(\Omega, \mathbb{R}^{3x3})} \right).
\]
Since (22) and (23) hold, we arrive at the convergence (33).
Using the above-mentioned properties of the operators $A$ and $K_\varepsilon$, we deduce that the sum $A + K_\varepsilon$ is a pseudo-monotone operator and
\[
\frac{(A(u) + K_\varepsilon(u), u) + J(u)}{\|u\|_{X(\Omega, \mathbb{R}^3)}} \to +\infty
\]
as $\|u\|_{X(\Omega, \mathbb{R}^3)} \to +\infty$. Taking into account the existence results for variational inequalities with pseudo-monotone operators and convex functionals (see [11, Corollary 30, p. 138]), we see that, for any $\varepsilon > 0$, problem (32) possesses a solution $v_\varepsilon \in X(\Omega, \mathbb{R}^3)$.

**Step 5.** We want to obtain estimates that are independent of $\varepsilon$. Let $\varepsilon_n > 0$ be a sequence such that $\varepsilon_n \to 0$ as $n \to \infty$. Denote by $E_{\varepsilon_n}$ the vector function $R_{\varepsilon_n}(v_{\varepsilon_n})$.

We clearly have
\[
\varepsilon_n \int_\Omega \Delta E_{\varepsilon_n} : \Delta F \, dx + \int_\Omega E_{\varepsilon_n} : F \, dx - \lambda \sum_{i=1}^3 \int_\Omega v_{\varepsilon_n} E_{\varepsilon_n} : \frac{\partial F}{\partial x_i} \, dx
\]
\[
+ \lambda \int_\Omega (E_{\varepsilon_n} W_\rho(v_{\varepsilon_n}) - W_\rho(v_{\varepsilon_n}) E_{\varepsilon_n}) : F \, dx
\]
\[
= \alpha \eta \int_\Omega D(v_{\varepsilon_n}) : F \, dx \quad \forall F \in H_0^2(\Omega, \mathbb{R}^{3 \times 3}),
\]
\[
- \sum_{i=1}^3 \int_\Omega v_{\varepsilon_n} v_{\varepsilon_n} : \frac{\partial \phi}{\partial x_i} \, dx + \int_\Omega E_{\varepsilon_n} : D(\phi - v_{\varepsilon_n}) \, dx
\]
\[
+ (1 - \alpha) \eta \int_\Omega D(v_{\varepsilon_n}) : D(\phi - v_{\varepsilon_n}) \, dx + \int_{\Gamma_0} q|\phi| d\Gamma_0 - \int_{\Gamma_0} q|v_{\varepsilon_n}| d\Gamma_0
\]
\[
\geq \int_\Omega f \cdot (\phi - v_{\varepsilon_n}) \, dx \quad \forall \phi \in X(\Omega, \mathbb{R}^3).
\]

Putting $F = E_{\varepsilon_n}$ in (34) and using the following equalities:
\[
\sum_{i=1}^3 \int_\Omega v_{\varepsilon_n} E_{\varepsilon_n} : \frac{\partial E_{\varepsilon_n}}{\partial x_i} \, dx = 0,
\]
\[
\int_\Omega (E_{\varepsilon_n} W_\rho(v_{\varepsilon_n}) - W_\rho(v_{\varepsilon_n}) E_{\varepsilon_n}) : E_{\varepsilon_n} \, dx = 0,
\]
we find
\[
\varepsilon_n \int_\Omega |\Delta E_{\varepsilon_n}|^2 \, dx + \int_\Omega |E_{\varepsilon_n}|^2 \, dx = \alpha \eta \int_\Omega D(v_{\varepsilon_n}) : E_{\varepsilon_n} \, dx.
\]

Furthermore, putting $2v_{\varepsilon_n}$ in place of $\phi$ into (35), we get
\[
\int_\Omega E_{\varepsilon_n} : D(v_{\varepsilon_n}) \, dx + (1 - \alpha) \eta \int_\Omega |D(v_{\varepsilon_n})|^2 \, dx + \int_{\Gamma_0} q|v_{\varepsilon_n}| d\Gamma_0 \geq \int_\Omega f \cdot v_{\varepsilon_n} \, dx,
\]
where we used the equality
\[
\sum_{i=1}^3 \int_\Omega v_{\varepsilon_n} v_{\varepsilon_n} : \frac{\partial v_{\varepsilon_n}}{\partial x_i} \, dx = 0.
\]

On the other hand, by setting $\phi = 0$ in (35), we obtain
\[
\int_\Omega E_{\varepsilon_n} : D(v_{\varepsilon_n}) \, dx + (1 - \alpha) \eta \int_\Omega |D(v_{\varepsilon_n})|^2 \, dx + \int_{\Gamma_0} q|v_{\varepsilon_n}| d\Gamma_0 \leq \int_\Omega f \cdot v_{\varepsilon_n} \, dx.
\]
Combining (37) and (38), we obviously have
\[
\int_{\Omega} E_{\varepsilon_n} : D(v_{\varepsilon_n}) \, dx + (1-\alpha)\int_{\Omega} |D(v_{\varepsilon_n})|^2 \, dx + \int_{\Gamma_0} q|v_{\varepsilon_n}| \, d\Gamma_0 = \int_{\Omega} f \cdot v_{\varepsilon_n} \, dx. \tag{39}
\]
Now we multiply (36) by \( (\alpha \eta)^{-1} \) and add it to (39); this gives
\[
\frac{\varepsilon_n}{\alpha \eta} \int_{\Omega} |\Delta E_{\varepsilon_n}|^2 \, dx + \frac{1}{\alpha \eta} \int_{\Omega} |E_{\varepsilon_n}|^2 \, dx + (1-\alpha)\eta \int_{\Omega} |D(v_{\varepsilon_n})|^2 \, dx
+ \int_{\Gamma_0} q|v_{\varepsilon_n}| \, d\Gamma_0 = \int_{\Omega} f \cdot v_{\varepsilon_n} \, dx. \tag{40}
\]
In particular, we have
\[
(1-\alpha)\eta \int_{\Omega} |D(v_{\varepsilon_n})|^2 \, dx \leq \int_{\Omega} f \cdot v_{\varepsilon_n} \, dx \leq \|f\|_{X(\Omega, \mathbb{R}^3)}^* \|v_{\varepsilon_n}\|_{X(\Omega, \mathbb{R}^3)}^*
\]
and hence
\[
\|v_{\varepsilon_n}\|_{X(\Omega, \mathbb{R}^3)} \leq \frac{1}{(1-\alpha)\eta} \|f\|_{X(\Omega, \mathbb{R}^3)}^*. \tag{41}
\]
Besides, it follows from (40) and (41) that
\[
\frac{1}{\alpha \eta} \int_{\Omega} |E_{\varepsilon_n}|^2 \, dx \leq \int_{\Omega} f \cdot v_{\varepsilon_n} \, dx
\leq \|f\|_{X(\Omega, \mathbb{R}^3)}^* \|v_{\varepsilon_n}\|_{X(\Omega, \mathbb{R}^3)}^*
\leq \frac{1}{(1-\alpha)\eta} \|f\|_{X(\Omega, \mathbb{R}^3)}^*.
\]
This yields that
\[
\|E_{\varepsilon_n}\|_{L^2(\Omega, \mathbb{R}^{3\times3}_{\text{sym}})} \leq \alpha^{1/2} (1-\alpha)^{-1/2} \|f\|_{X(\Omega, \mathbb{R}^3)}^*. \tag{42}
\]

\textbf{Step 6.} We shall construct a weak solution of the original boundary value problem by passing to the limit \( n \to \infty \). In view of the estimates (41) and (42), we can assume without loss of generality that, for some \( v_* \in X(\Omega, \mathbb{R}^3) \) and \( E_* \in L^2(\Omega, \mathbb{R}^{3\times3}_{\text{sym}}) \):
\[
v_{\varepsilon_n} \rightharpoonup v_* \quad \text{weakly in } X(\Omega, \mathbb{R}^3), \tag{43}
\]
\[
E_{\varepsilon_n} \rightharpoonup E_* \quad \text{weakly in } L^2(\Omega, \mathbb{R}^{3\times3}_{\text{sym}}) \tag{44}
\]
as \( n \to \infty \). Moreover, the weak convergence (43) implies that
\[
v_{\varepsilon_n} \to v_* \quad \text{strongly in } L^4(\Omega, \mathbb{R}^3), \tag{45}
\]
\[
v_{\varepsilon_n}|_{\Gamma} \to v_*|_{\Gamma} \quad \text{strongly in } L^2(\Gamma, \mathbb{R}^3) \tag{46}
\]
as \( n \to \infty \).

From (36) it follows that
\[
\int_{\Omega} D(v_{\varepsilon_n}) : E_{\varepsilon_n} \, dx \geq \frac{1}{\alpha \eta} \int_{\Omega} |E_{\varepsilon_n}|^2 \, dx.
\]
If we combine this inequality with (35), we get
\[
- \sum_{i=1}^3 \int_{\Omega} v_{\varepsilon_n} i v_{\varepsilon_n} \cdot \frac{\partial \varphi}{\partial x_i} \, dx + \int_{\Omega} E_{\varepsilon_n} : D(\varphi) \, dx - \frac{1}{\alpha \eta} \int_{\Omega} |E_{\varepsilon_n}|^2 \, dx
+ (1-\alpha)\eta \int_{\Omega} D(v_{\varepsilon_n}) : D(\varphi - v_{\varepsilon_n}) \, dx + \int_{\Gamma_0} q|\varphi| \, d\Gamma_0
- \int_{\Gamma_0} q|v_{\varepsilon_n}| \, d\Gamma_0 \geq \int_{\Omega} f \cdot (\varphi - v_{\varepsilon_n}) \, dx \quad \forall \varphi \in X(\Omega, \mathbb{R}^3). \tag{47}
\]
Using (43), (44), (45), and (46), and the relations
\[
\int_\Omega |D(\mathbf{v}_n)|^2 \, dx \leq \liminf_{n \to \infty} \int_\Omega |D(\mathbf{v}_{\varepsilon_n})|^2 \, dx,
\]
we pass to the lower limit in (47). The result is
\[
- \sum_{i=1}^3 \int_\Omega v_i \mathbf{v}_n : \frac{\partial \Phi}{\partial x_i} \, dx + \int_\Omega \mathbf{E}_n : D(\Phi) \, dx - \frac{1}{\alpha \eta} \int_\Omega |\mathbf{E}_n|^2 \, dx
+ (1 - \alpha) \eta \int_\Omega D(\mathbf{v}_n) : D(\Phi - \mathbf{v}_n) \, dx + \int_{\Gamma_0} q|\mathbf{v}| \, d\Gamma_0 \geq \int_\Omega \mathbf{f} : (\Phi - \mathbf{v}_n) \, dx \quad \forall \Phi \in X(\Omega, \mathbb{R}^3).
\]
Let \( \Phi \) be an arbitrary vector function from the space \( C_0^\infty(\Omega, \mathbb{R}_{sym}^{3\times3}) \). Putting
\( \mathbf{F} = \Phi \) in (34) and integrating by parts the first term, we get
\[
\varepsilon_n \int_\Omega E_{\varepsilon_n} : \Delta(\Delta \Phi) \, dx + \int_\Omega E_{\varepsilon_n} : \Phi \, dx - \lambda \sum_{i=1}^3 \int_\Omega v_i E_{\varepsilon_n} : \frac{\partial \Phi}{\partial x_i} \, dx
+ \lambda \int_\Omega (E_{\varepsilon_n} W_\rho(v_{\varepsilon_n}) - W_\rho(v_{\varepsilon_n}) E_{\varepsilon_n}) : \Phi \, dx = \alpha \eta \int_\Omega D(v_{\varepsilon_n}) : \Phi \, dx.
\]
By (43), (44), (45), we pass to the limit \( n \to \infty \) in the last equality and obtain
\[
\int_\Omega \mathbf{E}_n : \Phi \, dx - \lambda \sum_{i=1}^3 \int_\Omega v_i \mathbf{E}_n : \frac{\partial \Phi}{\partial x_i} \, dx
+ \lambda \int_\Omega (E_i W_\rho(v_n) - W_\rho(v_n) E_n) : \Phi \, dx = \alpha \eta \int_\Omega D(v_n) : \Phi \, dx.
\]
Since the set \( C_0^\infty(\Omega, \mathbb{R}_{sym}^{3\times3}) \) is dense in the space \( H_0^2(\Omega, \mathbb{R}_{sym}^{3\times3}) \), the last equality remains valid if we replace \( \Phi \) with an arbitrary vector function \( \mathbf{F} \in H_0^2(\Omega, \mathbb{R}_{sym}^{3\times3}) \). Thus, we have
\[
\int_\Omega \mathbf{E}_n : \mathbf{F} \, dx - \lambda \sum_{i=1}^3 \int_\Omega v_i \mathbf{E}_n : \frac{\partial \mathbf{F}}{\partial x_i} \, dx
+ \lambda \int_\Omega (E_i W_\rho(v_n) - W_\rho(v_n) E_n) : \mathbf{F} \, dx = \alpha \eta \int_\Omega D(v_n) : \mathbf{F} \, dx
\]
for any \( \mathbf{F} \in H_0^2(\Omega, \mathbb{R}_{sym}^{3\times3}) \).
Let us define \( \mathbf{S}_n \) by the formula
\[
\mathbf{S}_n = \mathbf{E}_n + (1 - \alpha) \eta \mathbf{D}(\mathbf{v}_n).
\]
Clearly, relations (50) and (51) together with (52) mean that the triplet \((\mathbf{v}_n, \mathbf{S}_n, \mathbf{E}_n)\) is a weak solution of problem (1)–(9).
It follows from (40) that
\[
(1 - \alpha) \eta \int_\Omega |D(\mathbf{v}_{\varepsilon_n})|^2 \, dx + \frac{1}{\alpha \eta} \int_\Omega |\mathbf{E}_{\varepsilon_n}|^2 \, dx + \int_{\Gamma_0} q|\mathbf{v}_{\varepsilon_n}| \, d\Gamma_0 \leq \int_\Omega \mathbf{f} \cdot \mathbf{v}_{\varepsilon_n} \, dx.
\]
Using (45), (46), (48), and (49), we can pass to the lower limit in the last inequality (as \( n \to \infty \)) and obtain the energy estimate (17) with \( v = v_\ast \) and \( E = E_\ast \). This completes the proof of (i).

Now let us show (ii). By definition of weak solutions to (1)–(9), we have

\[
\int_\Omega E_0 : F \, dx - \lambda \sum_{i=1}^{3} \int_\Omega v_0 E_0 : \frac{\partial F}{\partial x_i} \, dx \\
+ \lambda \int_\Omega (E_0 W_\rho(v_0) - W_\rho(v_0) E_0) : F \, dx = \alpha \eta \int_\Omega D(v_0) : F \, dx
\]

for any \( F \in H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \). After integrating the second term by parts, we get

\[
\int_\Omega E_0 : F \, dx + \lambda \sum_{i=1}^{3} \int_\Omega v_0 \frac{\partial E_0}{\partial x_i} \, dx \\
+ \lambda \int_\Omega (E_0 W_\rho(v_0) - W_\rho(v_0) E_0) : F \, dx = \alpha \eta \int_\Omega D(v_0) : F \, dx.
\]  

(53)

Since \( H^2_0(\Omega, \mathbb{R}^{3 \times 3}) \) is dense in \( L^4(\Omega, \mathbb{R}^{3 \times 3}_\text{sym}) \), it follows that the last equality is valid for any vector function \( F \in L^4(\Omega, \mathbb{R}^{3 \times 3}_\text{sym}) \). Then, by setting \( F = E_0 \), we deduce from (53) that

\[
\int_\Omega |E_0|^2 \, dx = \alpha \eta \int_\Omega D(v_0) : E_0 \, dx.
\]  

(54)

Putting \( v = v_0 \), \( E = E_0 \), \( \varphi = 2v_0 \) in inequality (15) and taking into account (54), we get the following inequality

\[
(1 - \alpha) \eta \int_\Omega |D(v_0)|^2 \, dx + \frac{1}{\alpha \eta} \int_\Omega |E_0|^2 \, dx + \int_{\Gamma_0} q |v_0| \, ds \geq \int_\Omega f \cdot v_0 \, dx.
\]  

(55)

On the other hand, substituting \( v = v_0 \), \( E = E_0 \), \( \varphi = 0 \) in (15), we obtain

\[
(1 - \alpha) \eta \int_\Omega |D(v_0)|^2 \, dx + \frac{1}{\alpha \eta} \int_\Omega |E_0|^2 \, dx + \int_{\Gamma_0} q |v_0| \, ds \leq \int_\Omega f \cdot v_0 \, dx.
\]

This inequality together with (55) give precisely (18).

Finally, by using the passage-to-limit procedure as above, we derive that the set of weak solutions to problem (1)–(9) is sequentially weakly closed in the space \( X(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^{3 \times 3}_\text{sym}) \times L^2(\Omega, \mathbb{R}^{3 \times 3}_\text{sym}) \).

Theorem 3.1 is proved.

**Remark 6.** In studying flows of Oldroyd fluids subject to threshold-slip boundary conditions, a lot of important issues are still open, including the uniqueness of weak solutions under smallness of the external forces (as in the case of the Navier–Stokes equations), the existence of strong solutions, the continuous dependence of solutions on the problem data, and the well-posedness of non-steady problems.

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