On nonlinear Miyadera-Voigt perturbations

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Abstract. Let $A, C, P : D(A) \subset X \to X$ be linear operators on a Banach space $X$ such that $-A$ generates a strongly continuous semigroup on $X$, and $F : X \to X$ be a globally Lipschitz function. We study the well-posedness of semilinear equations of the form $\dot{u}(t) = G(u(t))$, where $G : D(A) \to X$ is a nonlinear map defined by $G = -A + C + F \circ P$. In fact, using the concept of maximal $L^p$-regularity and a fixed point theorem, we establish the existence and uniqueness of a strong solution for the above-mentioned semilinear equation. We illustrate our results by applications to nonlinear heat equations with respect to Dirichlet and Neumann boundary conditions, and a nonlocal unbounded nonlinear perturbation.

Mathematics Subject Classification. Primary 60H15, 35F20, 47H14; Secondary 35R60, 35B25, 30K40.

Keywords. Unbounded nonlinear perturbations, Maximal $L^p$-regularity, Semilinear equations.

1. Introduction. Semilinear parabolic equations of the type

$$\dot{u}(t) + Au(t) = f(u(t)), \quad u(0) = x, \quad t \in [0, \tau],$$

where $-A : D(A) \subset X \to X$ is a generator of an analytic semigroup on a Banach space $X$ and $f : X_\alpha \subset X \to X$ is a locally Lipschitz function where $X_\alpha, \alpha \in (0, 1)$, can be an interpolation space between $D(A)$ and $X$ or $X_\alpha = D((-A)^\alpha)$, are well-studied, see e.g. Lunardi [14]. As already discussed in [14, page 254], we may have situations where $f$ is set to a domain that is neither an interpolation space between $X$ and $D(A)$ nor $X_\alpha$.

The first author was supported by the National Center for Scientific and Technical Research (CNRST), Morocco.
In the present work, we study the well-posedness of the following abstract parabolic equation
\[
\dot{u}(t) + Au(t) = Pu(t) + F(Cu(t)), \quad u(0) = x, \quad t \in [0, \tau],
\]
where \( F : X \to X \) is a globally Lipschitz function and \( P, C : D(A) \to X \) are linear operators not generally closed or closeable. If we select
\[
f : D(A) \to X, \quad f(x) = Px + F(Cx),
\]
then we may find a constant \( \gamma > 0 \) such that
\[
\|f(x) - f(y)\| \leq \gamma\|x - y\|_{D(A)} \quad (x, y \in D(A)).
\]
This estimate shows that \( f \) is not a Lipschitz function with respect to the norm of \( X \). This fact offers many difficulties in applying the usual fixed point theorems and thus makes the study of the existence and uniqueness of the solutions of the equation (1.1) interesting. It is worth noting that the key references [5, Chap. 11] and [14, Chap. 3] do not cover semilinear parabolic equations of the type (1.1). We also mention that nonlinear perturbations of the kind (1.2) appeared for the first time in [15], where \( A \) is a Hille-Yosida operator. On the other hand, the author of [20] studied operators of the form \(-A + B\) with \( B : X \to X \) a continuous nonlinear accretive operator. Some applications of unbounded nonlinear perturbations can also be found in [19]. Recently the work [9] treated the well-posedness of the stochastic version of (1.1), where the approach is based on the concept of Yosida extensions and the approximation theory.

In the present work, we propose another approach based on the concept of maximal \( L^p \)-regularity to prove the existence and uniqueness of strong solutions of the semilinear equation (1.1). To this end, we assume that \( A \) has the maximal \( L^p \)-regularity \( (p \in (1, \infty)) \), and for some \( \alpha > 0 \), there exists \( \gamma > 0 \) such that
\[
\|Pe^{-\alpha A}x\|_{L^p([0, \alpha], X)} + \|Ce^{-\alpha A}x\|_{L^p([0, \alpha], X)} \leq \gamma\|x\| \quad (x \in D(A)). (1.3)
\]
Further, we assume that the nonlinear function \( F : X \to X \) satisfies
\[
\|F(x) - F(y)\| \leq \kappa\|x - y\| \quad (x, y \in X) \quad (1.4)
\]
for a constant \( \kappa > 0 \). We mention that operators satisfying the condition (1.3) are called admissible observation operators, see e.g. [16, Chap. 3] for more properties on such operators. Furthermore, using the conditions (1.3)-(1.4) and Hölder’s inequality, we can prove that there exist \( \alpha_0 > 0 \) and \( \tilde{\gamma} \in (0, 1) \) such that
\[
\int_0^{\alpha_0} \|f(T(t)x) - f(T(t)y)\|\,dt \leq \tilde{\gamma}\|x - y\| \quad (1.5)
\]
for any \( x, y \in D(A) \). A map \( f \) satisfying the estimate (1.5) will be called a nonlinear Miyadera-Voigt perturbation for \( A \) (see [15, 19]).

By assuming conditions (1.3)-(1.4) and that \( A \) has the maximal \( L^p \)-regularity, we will prove that the semilinear equation (1.1) admits a unique
strong solution $u \in W^{1,p}([0, \tau], X) \cap L^p([0, \tau], D(A))$ whenever the initial state $x$ belongs to the trace space.

In Sect. 2, we first give a concise background on maximal regularity and prove Theorem 2.2, the main result of the paper. In the last section, we illustrate our results by applications to the nonlinear heat equation with respect to Dirichlet and Neumann boundary conditions.

2. Existence, uniqueness, and regularity of the solution of a class of semilinear parabolic equations. Let $A : D(A) \subset X \to X$ be a linear closed and densely defined operator on a Banach space $X$. Let $p > 1$ and $\tau > 0$ be real numbers and take $g \in L^p([0, \tau], X)$. A strong solution of the equation

$$\dot{u}(t) + Au(t) = g(t), \quad u(0) = 0, \quad t \in [0, \tau],$$

is a function $u \in W^{1,p}([0, \tau], X) \cap L^p([0, \tau], D(A))$ such that $u(0) = 0$ and $u$ satisfies (2.1) for almost every $t \in [0, \tau]$.

**Definition 2.1.** We say that $A$ has the maximal $L^p$-regularity on $[0, \tau]$ if for any $g \in L^p([0, \tau], X)$, there exists a unique strong solution $u \in W^{1,p}([0, \tau], X) \cap L^p([0, \tau], D(A))$ of the problem (2.1).

It is to be noted that the concept of maximal $L^p$-regularity is independent of $\tau > 0$ and $p \in (0, \infty)$. We then only talk about maximal $L^p$-regularity. Moreover, if $A$ has maximal $L^p$-regularity, then the operator $(−A, D(A))$ generates an analytic semigroup on $X$. The converse is also true if $X$ is a Hilbert space. For more details and justifications of these facts, we refer to e.g. [7].

Motivated by the Definition 2.1, we define the space of maximal regularity

$$MR_p(0, \tau) := W^{1,p}([0, \tau], X) \cap L^p([0, \tau], D(A)).$$

On this space, we define the following norm

$$\|u\|_{MR_p} := \|u\|_{W^{1,p}([0,\tau],X)} + \|u\|_{L^p([0,\tau],D(A))}, \quad u \in MR_p(0,\tau).$$

Then $(MR_p(0,\tau), \|\cdot\|_{MR_p})$ is a Banach space. We also define the trace space

$$Tr_p := \{u(0) : u \in MR_p(0,1)\}.$$ 

Endowed with the following norm

$$\|x\|_{Tr_p} := \inf \{\|u\|_{MR_p(0,1)} : u \in MR_p(0,1), \quad \text{and} \ \ u(0) = x\},$$

$Tr_p$ is a Banach space satisfying the following dense and continuous embedding

$$D(A) \hookrightarrow Tr_p \hookrightarrow X.$$

We define the following linear operators

$$\mathcal{A} u := Au(\cdot), \quad D(\mathcal{A}) := L^p([0, \tau], D(A)),$$

$$\partial u := \dot{u}, \quad D(\partial) := \{u \in W^{1,p}([0, \tau]) : u(0) = 0\} =: W_0^{1,p}([0, \tau], X),$$

$$\mathcal{L}_A := \partial + \mathcal{A}, \quad D(\mathcal{L}_A) := D(\mathcal{A}) \cap D(\partial).$$

Observe that

$$D(\mathcal{L}_A) = \{u \in MR_p(0, \tau) : u(0) = 0\} =: M^0_p(0, \tau).$$
The operator $\mathcal{L}_A$ generates an evolution semigroup on $L^p([0, \tau], X)$ given by

$$(e^{-s\mathcal{L}_A}g)(t) = \begin{cases} e^{-sA}g(t-s), & t \in [s, \tau], \\ 0, & t \in [0, s], \end{cases}$$

see [3, Chap. 2]. We also mention that the operator $\mathcal{L}_A$ is used in [2,4], and [6] to study regularities of evolution equations. If $A$ has the maximal $L^p$-regularity, then the operator $\mathcal{L}_A$ is invertible and the solution of (2.1) $u \in M^0_p(0, \tau)$ is given by

$$u(t) = (\mathcal{L}_A^{-1}g)(t) = \int_0^t e^{-(t-s)A}g(s)ds, \quad t \in [0, \tau].$$

The main result of the paper is the following:

**Theorem 2.2.** Assume that $A$ has the maximal $L^p$-regularity and the operators $C, P,$ and $F$ satisfy the conditions (1.3)–(1.4). Then for any initial condition $u(0) = x \in Tr_p$, the parabolic problem (1.1) has a unique strong solution $u \in MR^0_p(0, \tau)$, satisfying

$$u(t) = e^{-tA}x + \int_0^t e^{-(t-s)A}Pu(s)ds + \int_0^t e^{-(t-s)A}F(Cu(s))ds, \quad t \in [0, \tau].$$

(2.2)

**Proof.** Let $x \in Tr_p$ and $\alpha \in (0, \tau)$ and select

$$\Phi(v) = P(\mathcal{L}_A^{-1}v + e^{-A}x) + F(C(\mathcal{L}_A^{-1}v + e^{-A}x)), \quad v \in L^p([0, \alpha], X).$$

According to [10, Proposition 3.3], the condition (1.3) implies

$$\|P(\mathcal{L}_A^{-1}v + e^{-A}x)\|_{L^p([0, \alpha], X)} \leq c_p \left( \alpha^{\frac{1}{q}} \gamma \|v\|_{L^p([0, \tau], X)} + \gamma \|x\| \right),$$

$$\|C(\mathcal{L}_A^{-1}v + e^{-A}x)\|_{L^p([0, \alpha], X)} \leq c_p \left( \alpha^{\frac{1}{q}} \gamma \|v\|_{L^p([0, \tau], X)} + \gamma \|x\| \right)$$

for a constant $c_p > 0$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. This shows that $\Phi : L^p([0, \alpha], X) \to L^p([0, \alpha], X)$. On the other hand, for $v_1, v_2 \in L^p([0, \alpha], X)$, we have

$$\|\Phi(v_1) - \Phi(v_2)\|_{L^p([0, \alpha], X)}$$

$$\leq \|P(\mathcal{L}_A^{-1}(v_1 - v_2))\|_{L^p([0, \alpha], X)} + \kappa \|C(\mathcal{L}_A^{-1}(v_1 - v_2))\|_{L^p([0, \alpha], X)}$$

$$\leq (1 + \kappa)\gamma \alpha^{\frac{1}{q}} \|v_1 - v_2\|_{L^p([0, \tau], X)}.$$

Choose $\alpha_0 > 0$ such that $(1 + \kappa)\gamma \alpha_0^{\frac{1}{q}} < 1$. Then $\Phi$ is a contraction on $L^p([0, \alpha_0], X)$. Thus, by using Banach’s fixed point theorem, there exists a unique $v \in L^p([0, \alpha_0], X)$ such that $v = \Phi(v)$. Furthermore, there exists a unique $w \in MR^0_p(0, \alpha_0)$ such that $\mathcal{L}_Aw = v$. This shows that $w(0) = 0$ and

$$\dot{w} + Aw = \mathcal{L}_Aw = P(w + e^{-A}x) + F(C(w + e^{-A}x)) \quad \text{on } [0, \alpha_0].$$
As \( x \in Tr_p \), then \( t \mapsto e^{-tA}x \) is differentiable and \( \frac{d}{dt}e^{-tA}x = -Ae^{-tA}x. \) Now we put \( u(t) = w(t) + e^{-tA}x \) for \( t \in [0, \alpha_0] \). Then \( u(0) = x \) and \( u \in MR_p(0, \alpha_0) \). Moreover, by a simple computation, we have \( \dot{u}(t) + Au(t) = Pu(t) + F(Cu(t)) \) for a.e. \( t \in [0, \alpha_0] \). This shows that \( u \in MR_p(0, \alpha_0) \) is a strong solution of (1.1). Using standard arguments in nonlinear analysis [17], one can extend this solution to \([0, \tau]\).

3. Dirichlet and Neumann boundary problem for semilinear parabolic equations.

3.1. Dirichlet boundary problem. Let \( \Omega \subset \mathbb{R}^n \) be an open bounded set and put \( X = L^2(\Omega) \). We consider the following semilinear initial value problem

\[
\begin{align*}
\partial_t u &= (\Delta + (-\Delta)^\alpha)u + f((-\Delta)^\beta u) \quad \text{on } [0, T] \times \Omega, \\
u_{|t=0} &= \xi, \quad \text{on } \Omega, \\
u(t, \cdot) &= \mathcal{M} u(t), \quad \text{on } \partial \Omega,
\end{align*}
\]

where \( \alpha, \beta \in (0, \frac{1}{4}) \), \( \xi \in L^2(\Omega) \),

\[ (\mathcal{M} \varphi)(x) = \int \frac{K(y, x)\varphi(y)dy}{\varphi(x)} \in L^2(\Omega), \quad \varphi \in L^2(\Omega), \quad (3.2) \]

for a kernel \( K \in L^\infty(\partial \Omega \times \Omega) \). We assume that \( f \) is a real valued function, globally Lipschitz. Define the following operators

\[ A := -\Delta, \quad D(A) = \left\{ \varphi \in H^2(\Omega) : \varphi(x) = \int \frac{K(x, y)\varphi(y)dy}{\varphi(x)} , \quad x \in \partial \Omega \right\}, \]

\[ P := (-\Delta)^\alpha, \quad D(P) := D(A), \]

\[ C := (-\Delta)^\beta, \quad D(C) := D(A), \]

\[ (F\phi)(x) = f(\phi(x)), \quad x \in \Omega. \]

With the use of the above operators, the equation (3.1) can be reformulated as the abstract semilinear equation (1.1). Now according to Theorem 2.2, to prove that the equation (3.1) admits a unique strong solution, it suffices to show that \( A \) has the maximal \( L^2 \)-regularity, \( P \) and \( C \) satisfy the condition (1.3).

**Lemma 3.1.** The operator \( A \) has the maximal \( L^p \)-regularity for any \( p \in (1, \infty) \).

**Proof.** As \( X \) is a Hilbert space, it suffices to prove that \((-A, D(A))\) generates an analytic semigroup on \( X \). We first remark that the operator defined by \( -A_0 = \Delta \) with domain \( D(A_0) = H^2(\Omega) \cap H_0^1(\Omega) \) is a generator of an analytic semigroup on \( L^2(\Omega) \). Now let \( \mathcal{D} : L^2(\partial \Omega) \rightarrow L^2(\Omega) \) be the Dirichlet map, defined by \( y = \mathcal{D} v \), where \( \Delta y = 0 \) on \( \Omega \) and \( y_{|\partial \Omega} = v \). If we put \( \theta_\varepsilon := \frac{1}{4} - \varepsilon \) for \( \varepsilon \in (0, \frac{1}{4}) \), then \( \mathcal{D} \in \mathcal{L}(L^2(\partial \Omega), D(A_{0, \varepsilon}^\theta)) \), due to [13,18]. Now, by the closed graph theorem, it follows that \( A_{0, \varepsilon}^\theta \mathcal{D} \in \mathcal{L}(L^2(\partial \Omega), L^2(\Omega)) \). Then there exists a
constant $c > 0$ such that
\[
\|A_0 e^{-tA_0} \mathcal{D}\| = \|A_0^{1-\theta_\varepsilon} e^{-tA_0} A_0^{\theta_\varepsilon} \mathcal{D}\| \\
\leq ct^\theta_\varepsilon - 1, \quad t > 0.
\] (3.3)

Let $H^{-1}(\Omega)$ be the topological dual of $H_0^1(\Omega)$ with respect to the pivot space $L^2(\Omega)$. It is known (see e.g. [8, page 126]) that the extension of the semigroup $(e^{tA_0})_{t \geq 0}$ to $H^{-1}(\Omega)$ is a strongly continuous semigroup $(e^{t\tilde{A}})_{t \geq 0}$ on $H^{-1}(\Omega)$ whose generator $\tilde{A} : L^2(\Omega) \to H^{-1}(\Omega)$ is the extension of $-A_0$ to $L^2(\Omega)$. As $\mathcal{D} \varphi \in \ker(\Delta)$, for any $\varphi \in L^2(\partial \Omega)$,
\[
(\Delta - \tilde{A}) \mathcal{D} \varphi = (-\tilde{A}) \mathcal{D} \varphi := B \varphi.
\] (3.4)

If we consider the operator $\mathcal{D} \varphi = \varphi|_{\partial \Omega}$, then $\mathcal{D} = (\mathcal{D}|_{\ker(\Delta)})^{-1}$. Then the equation (3.4) becomes
\[
\Delta = \tilde{A} + B \mathcal{D} \quad \text{on } H^2(\Omega).
\]

Using this relation and the proof of [11, Theorem 4.1], we can write
\[
-A = \tilde{A} + \mathcal{B}, \quad D(A) = \{ \varphi \in L^2(\Omega) : (\tilde{A} + \mathcal{B}) \varphi \in L^2(\Omega) \},
\] (3.5)

where $\mathcal{B} := B.M \in L(L^2(\Omega), H^{-1}(\Omega))$, due to $M \in L(L^2(\Omega), L^2(\partial \Omega))$. On the other hand, for $t > 0$, $p > \frac{1}{\theta_\varepsilon}$, and $\psi \in L^p([0, t], L^2(\Omega))$,
\[
\int_0^t e^{(t-s)} \tilde{A} \mathcal{B} \psi(s) ds = \int_0^t A_0 e^{-(t-s)A_0} \mathcal{B} \mathcal{M} \psi(s) ds
\]
and, by (3.3) and Hölder’s inequality,
\[
\left\| \int_0^t e^{(t-s)} \tilde{A} \mathcal{B} \psi(s) ds \right\|_{L^2(\Omega)} \\
\leq c \left( \int_0^t s^{q(\theta_\varepsilon - 1)} ds \right)^{\frac{1}{q}} \left( \int_0^t \| \mathcal{M} \psi(s) \|^p ds \right)^{\frac{1}{p}}
\] (3.6)

where $\frac{1}{p} + \frac{1}{q} = 1$, $0 < q(1 - \theta_\varepsilon) < 1$, and a constant $\tilde{c} := \tilde{c}(t, q, \varepsilon, c) > 0$. According to [8, page 188, Corollary 3.4], $\mathcal{B}$ is a Desch-Schappacher perturbation for $-A$. Now the relation (3.3) shows that the operator $-A$ generates a strongly continuous semigroup on $X$ given by
\[
e^{-tA} \phi = e^{-tA_0} \phi + \int_0^t e^{-(t-s)A_0} \mathcal{B} e^{-sA} \phi ds
\] (3.7)
for any $t \geq 0$, and $\phi \in X$, see e.g. [8, page 186]. From (3.5), $\lambda + A = (\lambda + A_0)(I - R(\lambda, -A_0)\mathcal{B})$ for $\lambda \in \rho(-A_0)$. Thus for any $\lambda \in \rho(-A_0)$, $\lambda \in \rho(-A)$ if and only if $1 \in \rho(R(\lambda, -A_0)\mathcal{B})$, and in this case, we have
\[
R(\lambda, -A) = (I - R(\lambda, -A_0)\mathcal{B})^{-1} R(\lambda, -A_0).
\]
To be more self-contained, let us recall from the proof of [1, Theorem 8] how to prove that the semigroup generated by $-A$ is analytic. As $-A_0$ generates an analytic semigroup, there exist $\omega > \omega_0(-A_0)$ and a constant $c > 0$ such that $\|R(\lambda, -A_0)\| \leq c|\lambda - \omega|^{-1}$ for any $\lambda \in \mathbb{C}$ with $\text{Re}\lambda > \omega$. On the other hand, from (3.6) and [16, Chap. 3], we also have $\|R(\lambda, -A_0)\mathcal{B}\| \leq \eta\|\mathcal{M}\|(\text{Re}\lambda - \omega)^{\frac{1}{p}}$ for any $\text{Re}\lambda > \omega$ and for a certain constant $\eta > 0$. Now for any $\lambda \in \mathbb{C}$ such that $\text{Re}\lambda > \omega_1 := \omega + (2\eta\|\mathcal{M}\|)^q$, we have $\|(I - R(\lambda, -A_0)\mathcal{B})^{-1}\| \leq 2$, and then $\|R(\lambda, -A)\| \leq 2c|\lambda - \omega_1|^{-1}$. This ends the proof. □

**Lemma 3.2.** Let $p \in (1, \infty)$ and $\sigma \in (0, \frac{1}{p})$. Then the operator $A^\sigma$ satisfies the condition (1.3) with respect to the exponent $p$.

**Proof.** According to the proof of Lemma 3.1, we know that $-A$ generates an analytic semigroup on $L^2(\Omega)$. Thus for any $\sigma$, we have

$$\|A^\sigma e^{-tA}\| \leq \frac{M}{t^\sigma} \quad (t > 0)$$

for a constant $M > 0$. Now for $\sigma \in (0, \frac{1}{p})$, the function $t \mapsto \frac{M}{t^\sigma}$ is $p$-integrable on any interval $[0, r]$ with $r > 0$, and

$$\int_0^r \|A^\sigma e^{-tA}\phi\|^p \leq \gamma^p\|\phi\|^p_{L^2(\Omega)}$$

for any $\phi \in D(A)$ and a constant $\gamma := \gamma(r, p, \sigma, M) > 0$. □

The following result follows immediately from Lemma 3.1, Lemma 3.2, and Theorem 2.2.

**Theorem 3.3.** Let $p \in (1, \infty)$ and $\alpha, \beta \in (0, \frac{1}{p})$. Then the semilinear equation (3.1) has a unique strong solution

$$u \in W^{1,p}([0, T], L^2(\Omega)) \cap L^p([0, T], D(A)).$$

### 3.2. Neumann boundary problem with nonlocal perturbation term.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with a $C^2$ boundary $\partial\Omega$ and outer unit normal $\nu$ and put $X = L^2(\Omega)$. We consider the following semilinear initial value problem

$$\begin{cases}
\dot{u}(t, x) = \Delta u(t, x) + f \left( \int_{\partial\Omega} \Upsilon(x, y)c(y)u(t, y)dy \right), & x \in \Omega, t \in [0, T], \\
u u(t, x)|_{\nu(x)} = \int_{\Omega} K(x, y)u(t, y)dy, & x \in \partial\Omega, t \in [0, T],
\end{cases} \tag{3.8}
$$

where $\Upsilon, K \in L^\infty(\partial\Omega \times \Omega)$, $c \in C_c(\partial\Omega)$, and $f$ is a globally Lipschitz real valued function. We define the following operators:

$$\mathcal{R} : L^2(\partial\Omega) \rightarrow L^2(\Omega), \quad (\mathcal{R}\varphi)(x) = \int_{\partial\Omega} \Upsilon(x, y)\varphi(y)dy, \quad x \in \Omega,$$

$$(\Theta\varphi)(y) = c(y)\varphi(y), \quad y \in \partial\Omega,$$

$$C = \mathcal{R}\Theta : H^2(\Omega) \rightarrow L^2(\Omega).$$
On the other hand, let $F : L^2(\Omega) \to L^2(\Omega)$ and $\mathcal{M} \in L(L^2(\Omega), L^2(\partial\Omega))$ be as in Sect. 3.1. We now define the operator

$$A := -\Delta, \quad D(A) = \{ \varphi \in H^2(\Omega) : \nabla \varphi |_{\partial \Omega} = \mathcal{M} \varphi \}.$$ 

Thus the equation (3.8) is reformulated in $L^2(\Omega)$ as

$$\dot{u}(t) + Au(t) = F(Cu(t)), \quad u(0) = g, \quad t \in [0, \tau].$$

**Theorem 3.4.** The equation (3.8) has a unique strong solution

$$u \in W^{1,2}([0, \tau], L^2(\Omega)) \cap L^2([0, \tau], D(A)).$$

**Proof.** Step 1: We will prove that the operator $A$ has the maximal $L^2$-regularity. In fact, let us first define the operator $A_0 := -\Delta$ with domain $D(A_0) = \{ \varphi \in H^2(\Omega) : \nabla \varphi |_{\partial \Omega} = 0, \ x \in \partial \Omega \}$. It is well known that $A_0$ generates an analytic semigroup on $L^2(\Omega)$. Second, denote by $\varphi = \mathcal{N} \psi \in H^2(\Omega)$ the solution of the elliptic boundary value problem $\varphi = 0$ on $\Omega$ and $\nabla \varphi |_{\partial \Omega} = \psi$ on $\partial \Omega$ for $\psi \in L^2(\partial \Omega)$. Then $\mathcal{N}$ is continuous from $L^2(\partial \Omega)$ to $D(A_0^2)$ for any $\beta \in (0, \frac{3}{4})$. Moreover, for any $t > 0, \ alpha \geq 0,$ and $0 < \beta < 3/4,$ we have

$$A_0^\alpha e^{-tA_0} A_0 \mathcal{N} \in L(L^2(\partial \Omega), X) \ and \ \| A_0^\alpha e^{-tA_0} A_0 \mathcal{N} \| \leq \kappa_\beta t^{\beta - \alpha - 1}$$

(3.9)

for a constant $\kappa_\beta > 0$, due to [12]. We put $\mathcal{P} := A_0 \mathcal{N} \mathcal{M}$ and choose $\beta \in (\frac{1}{2}, \frac{3}{4})$ and $\alpha = 0$. Then by using (3.9) and the fact that $2(1 - \beta) < 1$, we obtain

$$\left\| \int_0^t e^{-(t-s)A_0} \mathcal{P} v(s) ds \right\|_{L^2(\Omega)} \leq \delta_\beta \| \mathcal{M} \| \| v \|_{L^2([0, t], L^2(\Omega))}$$

for all $t > 0$ and $v \in L^2([0, t], L^2(\Omega))$, where $\delta_\beta > 0$ is a constant. By the same argument as in Sect. 3.1, $\mathcal{P}$ is a Desch-Schappacher perturbation operator for $A_0$ which implies that the operator $\mathfrak{A} := -A_0 + \mathcal{P}$ with domain $D(\mathfrak{A}) = \{ \varphi \in L^2(\Omega) : (A_0 + \mathcal{P}) \varphi \in L^2(\Omega) \}$ generates an analytic semigroup on $L^2(\Omega)$. Furthermore, we have $-A = \mathfrak{A}$, see e.g. [11]. Thus $-A$ has the maximal $L^2$-regularity. In addition,

$$e^{-tA} \varphi = e^{-tA_0} \varphi + \int_0^t e^{-(t-s)A_0} \mathcal{P} e^{-sA} \varphi ds$$

(3.10)

for any $t > 0$ and $\varphi \in L^2(\Omega)$.

Step 2: We will prove that the operator $C$ satisfies (1.3) with respect to $A$. In fact, the operator $C : D(A_0^\alpha) \to L^2(\Omega)$ is uniformly bounded for any $\alpha > \frac{1}{4}$, so that $C A_0^{-\alpha} \in L(L^2(\Omega))$ and $\| C A_0^{-\alpha} \| \leq \eta$ for a constant $\eta > 0$. Let us now choose $\alpha \in (\frac{1}{4}, \frac{1}{2})$. We have

$$\| C e^{-tA_0} \| \leq \eta \frac{M}{t^\alpha} \quad (t > 0).$$

(3.11)
On the other hand, by using (3.9) and the fact that then $\frac{1}{2} < 1 - (\beta - \alpha) < 1$, we obtain

$$
\left\| C \int_0^t e^{-(t-s)\mathcal{A}_0} \mathcal{P} e^{-s\mathcal{A}} \varphi ds \right\| \leq \eta \kappa \beta \int_0^t \frac{1}{(t-s)^{1-(\beta - \alpha)}} \| \mathcal{M} e^{-s\mathcal{A}} \varphi \| ds
$$

$$
\leq \eta \kappa \beta \| e^{\omega |t| t^{\beta - \alpha}} \| \varphi \|
$$

(3.12)

for any $\varphi \in D(\mathcal{A})$, and $\omega > \omega_0(-\mathcal{A})$, the type of the semigroup generated by $-\mathcal{A}$. Now the fact that $C$ satisfies (1.3) for $\mathcal{A}$ follows by combining (3.10), (3.11), and (3.12). Finally, we can use Theorem 2.2 to conclude.

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References

[1] Amansag, A., Bounit, H., Driouich, D., Hadd, S.: On the maximal regularity for perturbed autonomous and non-autonomous evolution equations. J. Evol. Equ. 20, 165–190 (2020)

[2] Arendt, W., Chill, R., Fornaro, S., Poupaud, C.: $L^p$-maximal regularity for non-autonomous evolution equations. J. Differential Equations 237, 1–26 (2007)

[3] Chicone, C., Latushkin, Y.: Evolution Semigroups in Dynamical Systems and Differential Equations. Mathematical Surveys and Monographs, 70. American Mathematical Society. Providence, RI (1999)

[4] Clément, Ph.: On the method of sums of operators. In: Semi-groupes d’opérateurs et calcul fonctionnel (Besançon, 1998), pp. 1–30. Publ. Math. UFR Sci. Tech. Besançon, 16. Univ. Franche-Comté, Besançon (1999)

[5] Curtain, R., Zwart, H.: Introduction to Infinite-dimensional Systems Theory. A State-space Approach. Texts in Applied Mathematics, 71. Springer, New York (2020)

[6] Da Prato, G., Grisvard, P.: Sommes d’opérateurs linéaires et équations différentielles opérationnelles. J. Math. Pures Appl. 54, 305–387 (1975)

[7] Dore, G.: Maximal regularity in $L_p$ spaces for an abstract Cauchy problem. Adv. Differential Equations 5(1–3), 293–322 (2000)

[8] Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli, and R. Schnaubelt. Graduate Texts in Mathematics, 194. Springer, New York (2000)

[9] Fkirine, M., Hadd, S.: Solving stochastic equations with unbounded nonlinear perturbations. arXiv:2102.06996 (2021)

[10] Hadd, S.: Unbounded perturbations of $C_0$-semigroups on Banach spaces and applications. Semigroup Forum 70, 451–465 (2005)

[11] Hadd, S., Manzo, R., Rhandi, A.: Unbounded perturbations of the generator domain. Discrete Contin. Dyn. Sys. A 35, 703–723 (2015)
Lasiecka, I., Triggiani, R.: Control Theory for Partial Differential Equations: Continuous and Approximation Theories. II. Abstract Hyperbolic–like Systems Over a Finite Time Horizon. Encyclopedia of Mathematics and its Applications, 75. Cambridge University Press, Cambridge (2000)

Lasiecka, I., Triggiani, R.: Dirichlet boundary control problem for parabolic equations with quadratic cost: analyticity and Riccati’s feedback synthesis. SIAM J. Control. Optim. 21, 41–67 (1983)

Lunardi, A.: Analytic Semigroups and Optimal Regularity in Parabolic Problems. Reprint of the 1995 original. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel (2013)

Thieme, H.R., Vosseler, H.: Semilinear perturbations of Hille-Yosida operators. Banach Center Publ. 63, 87–122 (2003)

Tucsnak, M., Weiss, G.: Observation and Control for Operator Semigroups. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel (2009)

Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Mathematical Sciences, 44. Springer, New York (1983)

Washburn, D.C.: A bound on the boundary input map for parabolic equations with application to time optimal control. SIAM J. Control. Optim. 17, 652–671 (1979)

Webb, G.F.: Functional differential equations and nonlinear semigroups in $L^p$-spaces. J. Differential Equations 20(1), 71–89 (1976)

Webb, G.F.: Continuous nonlinear perturbations of linear accretive operators in banach spaces. J. Funct. Anal. 10(2), 191–203 (1972)

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Received: 25 December 2021
Revised: 19 April 2022
Accepted: 5 May 2022.