On kinetic theory viscosity in a rotating gas

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Clarke and Pringle (2004) derived a proper viscosity formula in a rotating gas by applying mean free path theory. We study their argument in detail and show that their result can be derived with a much simpler calculational procedure and a physically clearer picture.

§1. Introduction

The subject of angular momentum transport is important in theories concerning accretion disks. The $\alpha$ model of Shakura and Sunyaev\(^1\) assumes outward transportation of angular momentum due to some viscous force. This viscosity is believed to originate from turbulence, because the molecular viscosity is far too small in this case.

Because the viscosity formula for turbulence is not well known, generally accepted practices have relied on the formula for molecular viscosity, with the viscosity coefficient assumed to be far larger than that for actual molecular viscosity.\(^1\) In the present paper we investigate the molecular viscosity formula, which can be unambiguously defined, applicable to a rotating gas. The viscosity formula for shear flow is well known. The shear stress is proportional to the rate of strain. For a shear flow represented by $u = (0, U(x))$, the $x$-$y$ component of the shear stress is given by

$$\sigma_{xy} = -\eta \left( \frac{dU}{dx} \right),$$

where $\eta$ is the viscosity coefficient.

We next consider a rotating flow represented by $u = (0, R\Omega(R))$ in cylindrical coordinates, where $R$ and $\Omega$ are the radial distance and angular velocity, respectively. In this case the $R$-$\phi$ component of the shear stress is

$$\sigma_{R\phi} = -\eta R \left( \frac{d\Omega}{dR} \right).$$

Microscopically, the formula (1) can be derived with mathematical rigor from the Boltzmann equation using the Chapman-Enskog expansion.\(^2\) It can also be derived more readily by applying the mean free path theory of the kinetic theory of gases heuristically.\(^2,3\) A simple application of the mean free path theory to a rotating gas, however, leads to the incorrect result $\sigma_{R\phi} \propto -d(R^2 \Omega)/dR$.\(^4\) The correct derivation starting from the Boltzmann equation naturally leads to the formula (2).\(^5,6,7\) It is a puzzle why the application of the mean free path theory, which seems physically plausible, does not yield the correct result.
Recently, Clarke and Pringle\textsuperscript{8}) have shown that application of the mean free path theory to a rotating gas can indeed lead to the correct formula, \[ (2) \]. They consider the case in which the shear velocity of the gas is sufficiently smaller than the thermal velocity of the molecules. Their argument employs the inertial frame and approximates molecular orbits by straight lines.

We have been inspired by Ref. 8) and, focusing on that paper, show here that their result can be derived with a simpler calculation procedure and a physically clearer picture.

§2. Non-rotating, linear shear flow

We consider, for simplicity, an inertial system in two dimensions, in which there exists a parallel flow in the $y$ direction. We consider a point $S$ in the flow and define its coordinates as $(x_0, 0)$ (see Fig. 1 with $U' < 0$, where $U'$ is the velocity gradient of the flow). An observer moves with $S$, and our argument employs his rest frame, which is also an inertial frame. As seen by the observer, the velocity of the flow can be written $\Delta u = (0, (x - x_0)U')$. Let us consider a point $E$ that is at a distance $\lambda/2$ from $S$, where $\lambda$ is the mean free path of gas molecules.\textsuperscript{*}) Hereafter, we assume that the flight lengths between successive collisions for all molecules are the same, i.e. $\lambda$, for simplicity.

The velocity of the flow at the point $E$ as seen from $S$ is directed in the $y$ direction, and its magnitude can be written as

$$\Delta u = -\frac{1}{2} \lambda U' \cos \alpha,$$

where $\alpha$ denotes the angle between the lines $SE$ and $SA$. The flow velocity at $A$ is also $\Delta u$, as is clear from Fig. 1. The situation here is similar to that in Ref. 8).

Gas molecules are assumed to be ejected from the point $E$ isotropically when observed from the frame moving with $E$, and at constant speed $c$.\textsuperscript{8}) The flow velocity and molecular velocity added give $v$. Application of cosine theorem to the triangle formed by $\Delta u$, $c$ and $v$ gives

$$c^2 = v^2 + (\Delta u)^2 - 2v \Delta u \cos \left( \frac{\pi}{2} - \alpha \right).$$

Following Ref. 8), we assume $c^2 \gg (\Delta u)^2$, ignore higher-order terms, and obtain

$$v = c \left( 1 - \frac{1}{2} \frac{\lambda U'}{c} \cos \alpha \sin \alpha \right) = c \left( 1 - \frac{1}{4} \frac{\lambda U'}{c} \sin 2 \alpha \right).$$

Gas molecules ejected from the point $E$ pass through the point $S$ without a change in their velocities or directions of motion. Equation (4) shows that the velocity distribution of the gas molecules at $S$ is no longer isotropic but exhibits an oval shape whose major axis is inclined $45^\circ$ toward the $x$-axis.

\textsuperscript{*}) Although Clarke and Pringle chose this distance as $\lambda$, we consider $S$ to be the midpoint of the trajectory of a molecule and take this distance to be $\lambda/2$. 
The \( x \)-\( y \) component of the viscous stress tensor \( \sigma_{xy} \) is the net \( y \) momentum, carried by the gas molecules through a line of unit length along the \( y \)-axis per unit time. The \( x \) and \( y \) components of the velocity are \( v_x = v \cos \alpha \) and \( v_y = v \sin \alpha \), respectively. The \( y \) momentum is \( mv_y \), and the mass flux is \( nv_x \), where \( m \) and \( n \) are mass of a molecule and the number density of molecules, respectively. Thus, 
\[
\sigma_{xy} = mn \langle v_x v_y \rangle,
\]
where \( \langle \cdots \rangle \) represents an average taken over velocity space.

Because we assume that the molecule velocity at \( E \) has a constant value \( c \) in all directions, the distribution function is non-zero only on the circle with radius \( c \), and therefore it must have the form 
\[
f(v) \sim \delta(v - c),
\]
where \( \delta \) is the delta function and \( v \) is the radial coordinate in velocity space. Integration of \( \delta(v - c) \) over \( v \)-\( \alpha \) space yields \( 2\pi c \). Then, using \( v \) in Eq. \ref{eq:n}, our distribution function is found to be

\[
f(v, \alpha) = \frac{1}{2\pi c} \delta \left( v - c \left( 1 - \frac{1}{4} \frac{\lambda U'}{c} \sin 2\alpha \right) \right).
\] (5)
Substituting \( v_x \) and \( v_y \) into \( \sigma_{xy} = mn\langle v_x v_y \rangle \), we obtain

\[
\sigma_{xy} = mn \langle v^2 \sin \alpha \cos \alpha \rangle = mn \int_0^{2\pi} \int_0^\infty f(v, \alpha) v^3 \sin \alpha \cos \alpha dv d\alpha.
\]

Fixing \( \alpha \) and integrating the above over \( v \) from 0 to \( \infty \) gives

\[
\sigma_{xy} = \frac{mnc^2}{4\pi} \int_0^{2\pi} \left( 1 - \frac{1}{4} \frac{\lambda U'}{c} \sin 2\alpha \right)^3 \sin 2\alpha d\alpha
\approx \frac{\rho c^2}{4\pi} \int_0^{2\pi} \left( 1 - \frac{3}{4} \frac{\lambda U'}{c} \sin 2\alpha \right) \sin 2\alpha d\alpha = -\frac{3}{16} \rho c \lambda U',
\]

where \( \rho = nm \) is the density of the gas. This equation, together with formula (1), gives the viscosity coefficient \( \eta = 3\rho c \lambda / 16 \). The kinematic viscosity \( \nu = \eta / \rho \) is, from the above formula, \( \nu = 3c \lambda / 16 \). The usual heuristic procedure based on mean free path theory gives the coefficient 1/3 (for a three-dimensional treatment) or 1/2 for a two-dimensional treatment, instead of 3/16.

As is clear from the above calculation, if the velocity distribution at S is isotropic, the viscous stress there will become zero. Accordingly, the presence of viscosity requires anisotropy in the velocity distribution. In our calculation, the anisotropy in the velocity distribution of molecules at the point S has been derived assuming isotropy of the velocity distribution of molecules in the frame of the gas at the point E. This procedure resembles that in which, using the Chapman-Enskog expansion of the Boltzmann equation, one first assumes an isotropic Maxwellian distribution as the zeroth-order approximation in order to derive an anisotropic velocity distribution as the first-order approximation.

We emphasize that this isotropy is only a calculational means for obtaining the anisotropic velocity distribution. If necessary, the velocity distribution obtained using the formula (4) can be taken as that at the point E to calculate a second-order correction for that at the point S.

§3. Circular flow

3.1. Calculations in a rotating frame

We next consider gas rotating about an axis with the velocity \( \mathbf{u} = (0, R \Omega(R)) \), as seen from the inertial frame. For the moment, we ignore the effects of the Coriolis force and gravity and continue to assume that the molecule orbit is a straight line. In this case, the situation is again similar to that depicted in Fig. 1. The center of the rotation, O, is located far to the left (not shown in the figure). Let us denote the distance OE by \( R_E \) and the angle SOE by \( \phi \). If \( \lambda \ll R \), then \( \phi \approx \lambda / R \), and so \( \cos \phi \approx 1 \). We also have \( OE = OA \), i.e. \( R_E \approx R - \frac{1}{2} \lambda \cos \alpha \). In the above derivation, we have ignored terms of second and higher orders of \( \lambda / R \).

The \( y \) component of the flow velocity at the point E as seen from the observer located at the point S differs from that given in the formula (4). If the observer at
S moves with the inertial frame adopted in Ref. 8), it becomes

\[ \Delta_i u = R_E \Omega(R_E) \cos \phi - R\Omega \approx R_E \Omega(R_E) - R\Omega \approx -\frac{1}{2}(\Omega + R\Omega')\lambda \cos \alpha. \]  \tag{6} 

The \( x \) component of the flow velocity at E in the inertial frame is

\[ \Delta_x u = R_E \Omega(R_E) \sin \phi \approx \mathcal{O}(\lambda/R), \]

which cannot be ignored in the present approximation.

The \( y \) component of the velocity, however, when the observer at the point S moves with the rotating frame having an angular velocity of \( \Omega(R) \), becomes

\[ \Delta_r u = R_E (\Omega(R_E) - \Omega(R)) \cos \phi \approx R_E (\Omega(R_E) - \Omega(R)) \approx -\frac{1}{2}R\Omega'\lambda \cos \alpha. \]  \tag{7} 

This can also be expressed as

\[ \Delta_r u = \Delta_i u + \Omega\lambda \cos \alpha/2. \]  \tag{4} 

Indeed, for rigid body rotation, where \( \Omega \) is constant, this shows, as expected, that \( \Delta_r u \) becomes zero, while \( \Delta_i u \) is clearly non-zero. In the rotating frame, the \( x \) component of mean flow velocity can be ignored, because it is \( R_E (\Omega(R_E) - \Omega(R)) \sin \phi = \mathcal{O}(\lambda/R)^2 \). This is the advantage of employing the rotating frame rather than the inertial frame. However, the rotating frame has the disadvantage of the appearance of the Coriolis force. The Coriolis force, however, does not introduce a problem, as discussed below.

Both the inertial and rotating frames can, with correct calculations, yield the same result. The important point is which provides a simpler calculation. The rotating frame obviously insures a simpler calculation, and for this reason we adopt this frame here.

Comparison of the formula (7) with (3) clearly shows that the replacement of \( U' \) (for a plane shear flow in the inertial frame) by \( R\Omega' \) (for a circular flow in the rotating frame). It is then clear that the viscosity formula for the rotating flow has the same form as the formula (2).

3.2. Calculations in the inertial frame

Clarke and Pringle, 8) employing the inertial frame, after complicated calculations derived the result

\[ v_y = c \sin \alpha \left( 1 - \frac{1}{2} \sin \alpha \cos \alpha \frac{\lambda R\Omega'}{c} \right), \]

where their \( \lambda \) is replaced by our \( \lambda/2 \). Similarly, \( v_x \) can be derived easily:

\[ v_x = c \cos \alpha \left( 1 - \frac{1}{2} \sin \alpha \cos \alpha \frac{\lambda R\Omega'}{c} \right). \]  \tag{8} 

No further calculations are presented here, because they are the same as those given above, except that \( U' \) has been replaced by \( R\Omega' \), as described above. The result is, naturally, the same as that obtained in the rotating frame.

3.3. Effect of curvature of molecule orbits

To this point, we have ignored the effect of the curvature of the molecule orbits. When in the rotating frame, one experiences the Coriolis force. The Coriolis force acts on a molecule, changing its trajectory from linear to circular. The radius of
this circle, which is the same as the Larmor radius of a charged particle moving in a magnetic field, is represented by $c/\Omega$. This radius is, for an accretion disk, believed to be approximately equal to the disk thickness, $H$. Under the condition $\lambda \ll H$, the trajectory of a gas molecule can therefore be regarded as linear. With the result based on the Boltzmann equation, we find that the Coriolis force does not change the form of the viscosity formula but, rather, produces anisotropy in the viscosity coefficients and suppresses the viscosity coefficient in the direction perpendicular to the rotating axis.\(^5\)–\(^7\) This effect does not appear in the present approximation, however.

The effect of gravity can be studied similarly. We consider the rotating frame moving with the point $S$, which exhibits Keplerian motion. When considered on a sufficiently small scale about the point $S$, the rotating frame can be approximated by the Hill coordinates. It is known that a test particle moves in an elliptical (not circular) orbit whose major axis to minor axis ratio is 2:1.\(^9\) The curvature radius of the orbit is of the order of $c/\Omega$, which is the same as the above result. In summary, the molecule/particle orbit can also, within the range of our approximation, be approximated as linear, even when gravity is important.

When $\lambda$ is large, the suppression of the viscosity coefficient in the rotational plane occurs for two reasons: a) shortening of the effective mean free path of gas molecules due to their curved orbits, and b) reduction of the asymmetry in the velocity distribution at $S$, because of the curvature of the particle orbit. This effect reduces $\langle v_x v_y \rangle$ and, therefore, the effect of viscosity. In the extreme case, the velocity distribution becomes symmetric about the $x$-axis as is seen from the numerical simulation of Narayan et al.\(^6\) The above effects of a) and b) are of the order of $(\lambda/H)^2$.

§4. Discussion

We now return to the original question that led to our previous paper.\(^4\) This question is set up as follows. In accretion disks, the angular momentum increases outwards, while the angular momenta of the molecules are conserved during their motion. Why, then, is angular momentum transported outward in opposition to the angular momentum gradient?

Let us consider two adjacent annuli in a Keplerian rotating gas disk. The gas in the inner annulus has a larger angular velocity and smaller angular momentum than that in the outer annulus. Note that the velocity distributions of the molecules in both annuli are much larger than the velocity difference between the two annuli. Of the molecules in the inner annulus, many have much larger specific angular momenta than the average specific angular momentum of the outer annulus (while, of course, many more have smaller ones).

We thus find that the above calculation shows that the velocity distribution at the point $S$ comes to possess an oval shape whose major axis is inclined $45^\circ$ toward the $x$-axis. This means that, of the molecules originally present in the inner annulus, those whose angular momenta are larger than the average angular momentum of the outer annulus (such molecules are ejected in the positive $y$ direction) will preferen-
tially be transported from the inner to the outer annulus. The contribution from those molecules that have smaller angular momenta and have reached the outer annulus is small. Conversely, of the molecules originally present in the outer annulus, those having angular momenta smaller than the average angular momentum of the inner annulus will preferentially be transported to the inner annulus. Accordingly, angular momentum indeed does flow against its gradient.

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