New graceful diameter-6 trees by transfers

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Abstract

It is proved that five classes of diameter-6 and higher-diameter trees are graceful, including the class of depth-3 trees such that each internal vertex has an odd number of children. To do this, the transfer technique of Hrnčiar & Haviar [Discrete Math. 233 (2001)] is modified, by subsuming type-2 transfers under type-1 transfers, rearranging small subtrees of a tree, and using new sequences of transfers like the backwards double-8 transfer.

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1. Introduction

Given a graph $G$, a labeling of $G$ is an injective function $f : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. Under the labeling $f$, the label of a vertex $v$ is $f(v)$, and the induced label of an edge $uv$ is $|f(u) - f(v)|$. The labeling $f$ is graceful if:

- The labels of the vertices are $\{0, 1, \ldots, |V(G)| - 1\}$, and
- The induced labels of the edges are distinct.

Note that if $G$ is a tree, the induced labels must be $\{1, \ldots, |V(G)| - 1\}$. The graph $G$ is graceful if it has a graceful labeling. The Graceful Tree Conjecture posits that all trees are graceful (Kotzig, see Bermond [1]).

Since the late 1960’s, the Graceful Tree Conjecture has captivated many by its remarkable simplicity and surprising difficulty. Originally, the conjecture was posed as a reformulation of a problem about graph decompositions, but as time passed, the conjecture gained a life of its own, inspiring the study of countless other graph labeling problems. Though the conjecture remains open, many special classes of trees are known to be graceful, as documented by Gallian’s survey [2].
We now trace an important sequence of ideas toward the conjecture, culminating in the result that all diameter-5 trees are graceful (Hrnčiar & Haviar [3]). The modifications to these ideas that produce new best results on diameter-6 trees are highlighted in boxes throughout the introduction.

1.1. Transfers from Graceful Stars

In 1978, Kotzig suggested the idea of transforming one graceful tree into another by deleting an edge and adding another edge with the same induced label (see Bermond [1]). The main challenge in applying this idea is the difficulty of organizing a sequence of such transformations.

It is easiest to handle this difficulty when working with complete trees (rooted trees with all leaves at the same level) or trees with similar structure. In 1990, Cahit [4] used his technique of canonic spiral labeling to prove the following theorem without transfers:

**Theorem 1.** (Cahit [4]) Let $T$ be a complete tree with root $v$, such that each internal vertex has an odd number of children. Then $T$ has a graceful labeling $f$ with $f(v) = 0$.

In 2001, Hrnčiar & Haviar [3] introduced the idea of a transfer, extending Kotzig’s original idea by manipulating multiple edges at once. This idea is best understood by example; see Fig. 1. Each step of Fig. 1 is a transfer of leaves, though Hrnčiar & Haviar [3] also introduce transfers of branches. They established the following now-standard conventions:

- We identify vertices with labels; “$k$” can refer to the vertex labeled $k$.
- We use the notation $i \rightarrow j$ to represent a transfer from (vertex) $i$ to $j$.

For example, we describe the transfers in Fig. 1 as follows:

$$0 \rightarrow 1 \rightarrow 12 \rightarrow 1 \rightarrow 11 \rightarrow 2$$

Our example reflects the standard strategy of Hrnčiar & Haviar [3], of beginning with a gracefully labeled star and using the following transfers:

$$0 \rightarrow n \rightarrow 1 \rightarrow n - 1 \rightarrow \cdots$$

As in Fig. 1 this sequence of transfers builds trees in a breadth-first way (see Lemma [14]). This strategy easily reproduces the graceful complete trees of Theorem 1 and extends to many trees with similar structure.
Figure 1: Performing a sequence of transfers on a gracefully labeled star.
1.2. Type-1 and Type-2 Transfers

Hrnčiar & Haviar [3] introduced two basic types of transfers of leaves. Their definitions and important properties (assuming the standard sequence of transfers $0 \rightarrow n \rightarrow 1 \rightarrow \cdots$) are as follows:

- A **type-1 transfer** is a transfer $u \rightarrow v$ of leaves $k, k + 1, \ldots, k + m$, possible if $f(u) + f(v) = k + (k + m)$. A type-1 transfer:
  - leaves behind an odd number of leaves.
  - can be followed by a type-1 or type-2 transfer.

- A **type-2 transfer** is a transfer $u \rightarrow v$ of leaves $k, k + 1, \ldots, k + m, l, l + 1, \ldots, l + m$, possible if $f(u) + f(v) = k + (l + m)$. A type-2 transfer:
  - leaves behind an odd or even number of leaves.
  - (must leave behind an even number if it follows a type-2 transfer)
  - can only be followed by a type-2 transfer.

In Fig. [4] all four transfers are type-1, leaving behind odd numbers of leaves. From the final tree, we could, e.g., perform a type-2 transfer $2 \rightarrow 10$, transferring the leaves 5, 7, and leaving the vertices 2, 10 each with two children.

In the same way, the standard approach is to perform type-1 transfers to obtain some vertices with odd numbers of children, and then switch to type-2 transfers to obtain some vertices with even numbers of children.

**Modification #1: Subsuming type-2 transfers under type-1 transfers.**

1.3. Rearranging Subtrees

Using the standard approach often requires rearranging the subtrees of a tree, as in the cases of **banana trees** and **generalized banana trees**.

A **banana tree** is a tree obtained from a collection of stars $K_{1,m_i}$ with $m_i \geq 1$ by joining a leaf of each star by an edge to a new vertex $v$, called the **apex**. A **generalized banana tree** is obtained from a banana tree by replacing the edges incident with $v$ with paths of fixed length $h \geq 0$.

**Theorem 2.** (Hrnčiar & Monoszova [5]) Let $T$ be a generalized banana tree, with apex $v$ of odd degree. Then $T$ has a graceful labeling $f$ with $f(v) = 0$.

**Proof.** Rearrange subtrees so that the stars occur in the following order:
1. Stars $K_{1,m_i}$ with $m_i$ even.
2. Stars $K_{1,m_i}$ with $m_i$ odd and $m_i > 1$.
3. Stars $K_{1,m_i}$ with $m_i = 1$.

Consider a gracefully labeled star with central vertex labeled 0, and apply type-1 transfers $0 \rightarrow n \rightarrow 1 \rightarrow \cdots$, switching to type-2 at the last level. ■

By attaching a caterpillar (see Lemma 2 of Hrnčiar & Haviar [3]), we get:

**Theorem 3.** (Hrnčiar & Monoszova [5]) All generalized banana trees are graceful.

It is now common to rearrange subtrees, but only at the apex.

**Modification #2: Rearranging small subtrees of a tree.**

1.4. Alternative Sequences of Transfers

In order to overcome the limitations of type-1 and type-2 transfers, Hrnčiar & Haviar [3] introduced the backwards double 8 transfer, an alternative sequence of type-1 transfers

$$0 \rightarrow n \rightarrow 1 \rightarrow n-1 \rightarrow 0 \rightarrow n \rightarrow 1 \rightarrow n-1 \rightarrow 2$$

This sequence of transfers leaves an even number of leaves at each of 0, $n$, 1, $n-1$, and can be followed by returning to the standard sequence of transfers $2 \rightarrow n-2 \rightarrow 3 \rightarrow \cdots$ with type-1 or type-2 transfers. Using this idea, Hrnčiar & Haviar [3] prove their main result:

**Theorem 4.** (Hrnčiar & Haviar [3]) All diameter-5 trees are graceful.

**Modification #3: Using new sequences of transfers.**

1.5. New Results

Using the three modifications above, we prove the following results:

**Theorem 5.** Let $T$ be a tree with central vertex and root $v$, such that each vertex not in the last (farthest from the root) two levels has an odd number of children. Also, suppose $T$ satisfies one of the following sets of conditions:

(a) $T$ is a diameter-6 complete tree.
(b) $T$ is a diameter-6 tree, such that
- No two leaves of distance 2 from $v$ have the same parent
- Each leaf of distance 2 from $v$ has a sibling with an even number of children.

(c) $T$ is a diameter-$2r$ complete tree, such that the number of vertices of distance $r - 1$ from $v$, with an even number of children, is not 3 (mod 4).

(d) $T$ is a diameter-$2r$ tree, such that the number of vertices of distance $r - 1$ from $v$, with an even number of children, is not 3 (mod 4), and
- No two leaves of distance $r - 1$ from $v$ have the same parent.
- Each leaf of distance $r - 1$ from $v$ has a sibling with an even number of children.

(e) $T$ is a diameter-6 tree, such that each internal vertex has an odd number of children.

Then $T$ has a graceful labeling $f$ with $f(v) = 0$.

**Corollary 6.** Let $T$ be a diameter-6 tree with central vertex and root $v$, such that each internal vertex has an odd number of children. Then $T$ has a graceful labeling $f$ with $f(v) = 0$.

**Proof.** Starting with a tree from Thm. (e), attach leaves to the root. ■

2. Preliminary Results

2.1. Definitions

In this section, we work toward defining *attainable* and *nicely attainable* sequences. Essentially,

- An *attainable* sequence is a list of numbers of leaves that can be left behind by a *well-behaved* sequence of transfers.

- A *nicely attainable* sequence is a list of numbers of leaves that can be left behind by a *well-behaved* sequence of transfers, such that the sequence of transfers can continue onward.

**Definition 7.** Let $T$ be a tree, let $f$ be a graceful labeling of $T$, let $v_1, \ldots, v_m$ be a sequence of vertices of $T$, and let $a, b, c, d$ be integers. We say that $(T, f, v_1, \ldots, v_m, a, b, c, d)$ is a transfer context if the following conditions hold:
• The labels of \(v_1, \ldots, v_m\), in order, are
\[
a, b - 1, a + 1, b - 2, \ldots \quad \text{or} \quad a, b + 1, a - 1, b + 2, \ldots
\]

• \(v_1\) is adjacent to leaves with labels \(c, c + 1, \ldots, d\).
\((v_1\) may also be adjacent to other leaves.\)

• \(a + b = c + d.\)
\((\text{This ensures that the labels of the leaves at } v_1 \text{ are the same as if the leaves were just transferred from a previous vertex } v_0 \text{ with label } b.\))

The two possible sequences of labels share the essential characteristic of the standard sequence \(0 \rightarrow n \rightarrow 1 \rightarrow \cdots\), that the sum of adjacent labels alternates between constants \(k, k + 1\) at each step.

**Definition 8.** Given a transfer context \((T, f, v_1, \ldots, v_m, a, b, c, d)\), a **well-behaved** sequence of transfers is a finite sequence of transfers
\[
v_{i_1} \rightarrow v_{i_2} \rightarrow \cdots \rightarrow v_{i_k}
\]
with \(i_1 = 1\), such that

• All transfers are type-1 transfers.

• The set of leaves transferred in the first step is a subset of the set of leaves \(\{c, \ldots, d\}\). The set of leaves transferred in each subsequent step is a subset of the leaves transferred in the previous step.

• Each transfer \(v_{i_j} \rightarrow v_{i_{j+1}}\) has indices \(i_j, i_{j+1}\) of different parity.

The result of a well-behaved sequence of transfers is the sequence \(n_1, \ldots, n_m\) of non-negative integers, such that after performing the transfers, each \(v_k\) is adjacent to \(n_k\) of the leaves \(c, \ldots, d\) originally adjacent to \(v_1\).

**Definition 9.** A sequence \(n_1, \ldots, n_m\) of non-negative integers is **attainable** if, for any transfer context \((T, f, v_1, \ldots, v_m, a, b, c, d)\) with
\[
n_1 + \cdots + n_m = |\{c, \ldots, d\}|,
\]
there exists a well-behaved sequence of transfers with result \(n_1, \ldots, n_m\).
Definition 10. A sequence $n_1, \ldots, n_m$ of non-negative integers is nicely attainable if, for any positive integer $n_{m+1}$, and for any transfer context $(T, f, v_1, \ldots, v_{m+1}, a, b, c, d)$ with

$$n_1 + \cdots + n_{m+1} = |\{c, \ldots, d\}|,$$

there exists a well-behaved sequence of transfers with result $n_1, \ldots, n_{m+1}$, such that the last transfer is $v_m \rightarrow v_{m+1}$, and $v_{m+1}$ occurs in no other transfer.

Lemma 11. The following results hold:

(a) If $n_1, \ldots, n_m$ is nicely attainable, and $n'_1, \ldots, n'_{m'}$ is attainable, then $n_1, \ldots, n_m, n'_1, \ldots, n'_{m'}$ is attainable.

(b) If $n_1, \ldots, n_m$ is nicely attainable, and $n'_1, \ldots, n'_{m'}$ is nicely attainable, then $n_1, \ldots, n_m, n'_1, \ldots, n'_{m'}$ is nicely attainable.

Proof. Concatenate the two corresponding sequences of transfers. ■

2.2. New Sequences of Transfers

For the standard sequence of transfers $0 \rightarrow n \rightarrow 1 \rightarrow \cdots$, we can leave behind any odd number of leaves at each step (see Fig. 1), essentially because the sum of adjacent labels changes by one at each step. In general, if the sum changes by $k$, then we can leave behind $l$ leaves, where $l$ is any integer with $l \geq k$ and $l \equiv k \pmod{2}$. The following lemma makes this idea precise:

Lemma 12. Let $(T, f, v_1, \ldots, v_m, a, b, c, d)$ be a transfer context. Consider performing a well-behaved sequence of transfers, beginning with

$$v_{i_1} \rightarrow \cdots \rightarrow v_{i_j}, \quad i_1 = 1, \quad j > 1.$$

Let $v_{i_{j+1}}$ be a vertex with $i_{j+1} \neq i_j \pmod{2}$, and let $l$ be an integer such that the previous transfer $v_{i_{j-1}} \rightarrow v_i$ transfers at least $l$ leaves, and such that $l \geq |i_{j+1} - i_{j-1}|/2$ and $l \equiv |i_{j+1} - i_{j-1}|/2 \pmod{2}$. Then there exists a transfer $v_{i_j} \rightarrow v_{i_{j+1}}$ of the first type, leaving behind $l$ leaves at $v_{i_j}$.

Proof. $(T, f, v_1, \ldots, v_m, a, b, c, d)$ is a transfer context, so $|f(v_{i_{j+1}}) - f(v_{i_{j-1}})| = |i_{j+1} - i_{j-1}|/2$. Suppose the previous transfer $v_{i_{j-1}} \rightarrow v_{i_j}$ transfers leaves with consecutive labels $c', \ldots, d'$, so that $f(v_{i_{j-1}}) + f(v_{i_j}) = c' + d'$. Then

$$f(v_{i_j}) + f(v_{i_{j+1}}) = c' + d' \pm |i_{j+1} - i_{j-1}|/2$$

8
Therefore, a transfer \(v_{i_j} \to v_{i_{j+1}}\) of the first type transfers at most the leaves with labels (depending on the sign in the expression above)

\[
\{c' + |i_{j+1} - i_{j-1}|/2, \ldots, d'\} \quad \text{or} \quad \{c', \ldots, d' - |i_{j+1} - i_{j-1}|/2\}.
\]

Therefore, we can leave behind \(|i_{j+1} - i_{j-1}|/2\) leaves at \(v_{i_j}\). By removing \(k\) elements from each end of the relevant set above, we can more generally leave behind \(l = |i_{j+1} - i_{j-1}|/2 + 2k\) leaves for any \(k \geq 0\), as long as there are at least \(l\) leaves at \(v_{i_j}\) to leave behind. ■

Taking \(i_0 = 0\) (see end of Def. 7) gives the analogous result for \(j = 1\):

**Lemma 13.** Let \((T, f, v_1, \ldots, v_m, a, b, c, d)\) be a transfer context. Let \(v_{i_2}\) be a vertex with \(i_2\) even, and let \(l\) be an integer such that \(\{c, \ldots, d\}\) is a set of at least \(l\) leaves, and such that \(l \geq |i_2|/2\) and \(l \equiv |i_2|/2\) (mod 2). Then there exists a transfer \(v_1 \to v_{i_2}\) of the first type, leaving behind \(l\) leaves at \(v_1\).

**Proof.** Analogous to proof of Lemma 12. ■

By these lemmas, we can establish the following attainable and nicely attainable sequences, where \(o\), \(e\), and \(e/0\) represent any positive odd integer, positive even integer, and non-negative even integer, respectively:

| Nicely attain. seq. | Corresponding well-behaved sequence of transfers |
|---------------------|-----------------------------------------------|
| \(o\)               | \(v_1 \to v_2\)                             |
| \(e, o, o, o, e\)   | \(v_1 \to v_4 \to v_3 \to v_2 \to v_5 \to v_6\) |
| \(e, o, e, o, e\)   | \(v_1 \to v_4 \to v_5 \to v_2 \to v_3 \to v_6 \to v_7\) |
| \(e, e/0, o, \ldots, o, e/0, e\) | \(v_1 \to [v_2 \to v_1] \to [v_4 \to v_3] \to \cdots \) |
|                       | \(\to [v_{2k} \to v_{2k-1}] \to v_{2k} \to v_{2k+1}\) |

| Attainable sequence  | Corresponding well-behaved sequence of transfers |
|----------------------|-----------------------------------------------|
| \(e, \ldots, e\)    | \(v_1 \to v_2 \to \cdots \to v_{k-1} \to v_k \to v_{k-1} \to \cdots \to v_1\) |
| \(e, e/0, e, o\)    | \(v_1 \to v_2 \to v_1 \to v_4 \to v_3\) |
| \(e, e/0, \ldots, o\) | \(v_1 \to [v_2 \to v_1] \to [v_4 \to v_3] \to \cdots\) |
|                       | \(\to [v_{2k} \to v_{2k-1}]\) |

Each nicely attainable sequence above is also an attainable sequence, by removing the last transfer. Also, note that the backwards double-8 transfer of Hrnčiar & Havíar [3] corresponds to the nicely attainable sequence \(e, e, e, e\); we leave it out because \(e, e, e, e\) is a special case of the last nicely attainable sequence above.
2.3. Order of Leaves

For the standard transfers $0 \rightarrow n \rightarrow 1 \rightarrow \cdots$, leaves are left behind in the same order as vertices appear in the sequence of transfers. The following lemma makes this idea precise, and extends it to certain other transfers. Note that we allow the list $v_1, \ldots, v_m$ to be long enough to include some of $c, \ldots, d$.

Lemma 14. Let $(T, f, v_1, \ldots, v_m, a, b, c, d)$ be a transfer context, where the list $v_1, \ldots, v_m$ includes at least $l$ of the leaves $c, \ldots, d$, and let $v_0$ be a vertex with $f(v_0) = b$. If a type-1 transfer $v_1 \rightarrow v_0$ or $v_1 \rightarrow v_2$ leaves behind $l$ of the leaves $c, \ldots, d$, then it leaves behind the $l$ leaves of $c, \ldots, d$ that occur first in the list $v_1, \ldots, v_m$.

Proof. Suppose $a < b$. Since $v_1, \ldots, v_m$ includes some of $c, \ldots, d$, the vertices $v_1, \ldots, v_m$ must have labels $a, b-1, a+1, b-2, \ldots$. Suppose the leaves transferred by the transfer $v_1 \rightarrow v_0$ or $v_1 \rightarrow v_2$ have labels $c', \ldots, d'$.

- If we transfer $v_1 \rightarrow v_0$, then $c' + d' = a + b$, so $c' - a = b - d'$.
- If we transfer $v_1 \rightarrow v_2$, then $c' + d' = a + b - 1$, so $c' - a + 1 = b - d'$.

Therefore, $c', d'$ occur consecutively in $v_1, \ldots, v_m$, so the leaves left behind are exactly the leaves that occur before $c', d'$ in the list $v_1, \ldots, v_m$.

If $a > b$, consider the labeling $g(v) = n - f(v)$, and apply the above. ■

This lemma allows for the standard breadth-first construction (see Fig. 1).

3. Main Results

We now prove our main result, Thm. 5 (see p. 5). We can easily obtain any of these trees by transfers, up to the last level (see Fig. 1). The key idea is to rearrange subtrees to make the transfers at the last level possible.

3.1. Proof of Thm. 5(a): A Class of Diameter-6 Complete Trees

Proof. Let $v$ have children $v_1, \ldots, v_k$, let each $v_i$ have children $v_{i,1}, \ldots, v_{i,m_k}$, and let each $v_{i,j}$ have $n_{i,j}$ children. Then we can represent $T$ as

$$( (n_{1,1}, \ldots, n_{1,m_1}), \ldots, (n_{k,1}, \ldots, n_{k,m_k}) ).$$

By defining the $v_{i,j}$ differently, we can permute each $m_i$-tuple and also the list of $m_i$-tuples, giving a different sequence $n_{1,1}, \ldots, n_{k,m_k}$; we say that the list
of $m_i$-tuples corresponds to any such sequence of integers. It suffices to prove that each possible list of $m_i$-tuples corresponds to an attainable sequence.

We denote by $a/b$ any permutation of the $b$-tuple $\langle e, \ldots, e, o, \ldots, o \rangle$. By the conditions, each $b$ is odd.

**Claim:** If each $m_i$-tuple is one of 0/1, 1/3, 2/3, 3/5, then the list of $m_i$-tuples corresponds to some sequence of integers of the form $n_1, \ldots, n_m, e, \ldots, e$, where $n_1, \ldots, n_m$ is nicely attainable.

**Proof of Claim.** We call a list of $m_i$-tuples nicely attainable if it corresponds to some nicely attainable sequence of integers. We list all minimal nicely attainable lists of $m_i$-tuples below, where brackets indicate nicely attainable subsequences (see Lemma 11, tables on p. 9):

| Minimal nicely attainable lists of $m_i$-tuples |
|-----------------------------------------------|
| (0/1) | $\langle (o) \rangle$ |
| (1/3, 1/3) | $\langle (e, o, o), (o, e, o) \rangle$ |
| (1/3, 3/5) | $\langle (o, o, e), (e, e, e, o, o) \rangle$ |
| (2/3, 2/3) | $\langle (o, e, e), (e, e) \rangle$ |
| (2/3, 3/5, 3/5) | $\langle (o, e, e, e), (e, e, e, o, o, e), (e, e, e, o, o, o) \rangle$ |
| (3/5, 3/5, 3/5, 3/5) | $\langle (o, o, o, o, e, e, e, e, e, o, o, o), (e, e, e, o, o, o, e, e, e, e, e) \rangle$ |

We call a list of $m_i$-tuples 0/1, 1/3, 2/3, 3/5 irreducible if no subset is nicely attainable. We list all irreducible lists of $m_i$-tuples below; the right column shows that each corresponds to a sequence of the desired form:

| Irreducible lists of $m_i$-tuples |
|-----------------------------------|
| $\emptyset$ | $\emptyset$ |
| (1/3) | $\langle (o, o, e) \rangle$ |
| (2/3) | $\langle (o, e, e) \rangle$ |
| (3/5) | $\langle (o, o, e, e, e) \rangle$ |
| (1/3, 2/3) | $\langle (e, o, o), (o, e, e) \rangle$ |
| (2/3, 3/5) | $\langle (o, e, e), (e, e, o, o, e) \rangle$ |
| (3/5, 3/5) | $\langle (o, o, e, e, e), (e, o, o, e, e) \rangle$ |
| (3/5, 3/5, 3/5) | $\langle (o, o, e, e, e), (e, o, o, e, e), (e, e, o, o, e) \rangle$ |
By the definition of an irreducible list of \( m_i \)-tuples, we can permute any list of \( m_i \)-tuples \( 0/1, 1/3, 2/3, 3/5 \) into a list of the following form:

\[
\begin{array}{ccc}
\frac{a_1}{b_1}, \ldots, \frac{a_{i_1}}{b_{i_1}}, \ldots, \frac{a_{k-1}}{b_{k-1}+1}, \ldots, \frac{a_{i_k}}{b_{i_k}}, \frac{a_{i_k+1}}{b_{i_k+1}}, \ldots, \frac{a_{i_k+1}}{b_{i_k+1}} \\
\text{nearly attainable} & \text{nearly attainable} & \text{irreducible}
\end{array}
\]

Therefore, any such list corresponds to a sequence of the desired form. 

**Claim:** If each \( m_i \)-tuple is one of \( 0/1, 1/3, 2/3, 3/5, 1/1 \), then the list of \( m_i \)-tuples corresponds to some attainable sequence of integers.

**Proof of Claim.** Permute the \( m_i \)-tuples so that all \( 1/1 \)'s are at the end. 

We now prove the desired result. We associate each general \( a/b \) with one of \( 0/1, 1/3, 2/3, 3/5, 1/1 \), giving five classes of \( m_i \)-tuples \( a/b \):

| If...                      | Then associate \( a/b \) with...
|---------------------------|-------------------------------|
| \( a < b, a \equiv 0 \pmod{4} \), | \( 0/1 \).                      |
| \( a < b, a \equiv 1 \pmod{4} \), | \( 1/3 \).                      |
| \( a < b, a \equiv 2 \pmod{4} \), | \( 2/3 \).                      |
| \( a < b, a \equiv 3 \pmod{4} \), | \( 3/5 \).                      |
| \( a = b \)                  | \( 1/1 \).                      |

Consider repeating the proofs of the claims above, but replacing each of \( 0/1, 1/3, 2/3, 3/5, 1/1 \) with a general \( a/b \) in its class. To show that the proofs still hold, it suffices to prove the following three statements:

1. The nearly attainable lists of \( m_i \)-tuples are still nearly attainable.
2. The irreducible lists of \( m_i \)-tuples still correspond to sequences of the form \( n_1, \ldots, n_m, e, \ldots, e \), where \( n_1, \ldots, n_m \) is nearly attainable.
3. The lists associated with \( 1/1 \) correspond to \( e, \ldots, e \).

Since (3) is obvious, consider (1) and (2), which only involve \( m_i \)-tuples associated with \( 0/1, 1/3, 2/3, 3/5 \). We can obtain these general \( m_i \)-tuples from \( 0/1, 1/3, 2/3, 3/5 \) by repeatedly inserting \( o \) or \( e, e, e, e \). Consider the following modified versions of the tables above:
Minimal nicely attainable lists of $m_i$-tuples

| $(0/1)$ | $((\ldots, o))$ |
| $(1/3, 1/3)$ | $((\ldots, e, o, o), (o, e, \ldots, o))$ |
| $(1/3, 3/5)$ | $((\ldots, o, o, e), (e, e, \ldots, o, o))$ |
| $(2/3, 2/3)$ | $((\ldots, o, e), (e, e, \ldots, o))$ |
| $(2/3, 3/5, 3/5)$ | $((\ldots, o, e, e), (e, e, \ldots, o, o), (e, e, \ldots, o, o))$ |
| $(3/5, 3/5, 3/5, 3/5)$ | $((\ldots, o, o, e, e), (e, e, \ldots, o, o, e), (e, e, \ldots, o, o))$ |

Irreducible lists of $m_i$-tuples

| $\emptyset$ | $((\ldots))$ |
| $(1/3)$ | $((\ldots, o, o, e))$ |
| $(2/3)$ | $((\ldots, o, e, e))$ |
| $(3/5)$ | $((\ldots, o, o, e, e))$ |
| $(1/3, 2/3)$ | $((\ldots, o, o), (o, e, \ldots, e))$ |
| $(2/3, 3/5)$ | $((\ldots, o, e, e), (e, e, \ldots, o, o))$ |
| $(3/5, 3/5)$ | $((\ldots, o, o, e, e), (e, e, \ldots, o, o, e))$ |
| $(3/5, 3/5, 3/5)$ | $((\ldots, o, o, e, e, e), (e, e, \ldots, o, o, o, o))$ |

By inserting each $o$ or $e, e, e, e$ at the dots, (1) and (2) also hold.

### 3.2. Proof of Thm. 5(b): A Class of Diameter-6 Non-Complete Trees

**Proof.** We adapt the proof above. We again represent trees by lists of $m_t$-tuples, denoting by $a/b$ any permutation of the $b$-tuple of the form

- $(e/0, e, \ldots, e, o, \ldots, o)$ if $a > 1$
- $(e, o, \ldots, o)$ if $a = 1$
- $(o, \ldots, o)$ if $a = 0$

Since our $m_t$-tuples now contain $e/0$’s, we no longer put all 1/1’s at the end, and we classify our $m_t$-tuples differently. We now associate each general $a/b$ with one of 0/1, 1/3, 2/3, 3/3, 1/1:
If...

| Condition                      | Then associate a/b with... |
|--------------------------------|----------------------------|
| $a \equiv 0 \pmod{4}$          | 0/1                        |
| $a \equiv 1 \pmod{4}, a < b,$  | 1/3                        |
| $a \equiv 1 \pmod{4}, a = b,$  | 1/1                        |
| $a \equiv 2 \pmod{4}$          | 2/3                        |
| $a \equiv 3 \pmod{4}$          | 3/3                        |

We list our minimal nicely attainable lists of $m_i$-tuples below:

| Minimal nicely attainable lists of $m_i$-tuples |
|-----------------------------------------------|
| (0/1)                                          |
| ((...))                                       |
| (1/3, 1/3)                                     |
| ((...e, o, o), (o, e, ..., o))                 |
| (1/3, 3/3)                                     |
| ((...o, o, e), (e/0, e, e, ...))               |
| (2/3, 2/3)                                     |
| ((...o, e/0), (e/0, e, e, ..., o))             |
| (3/3, 1/1)                                     |
| ((...e, e/0, e), (e, ...))                    |
| (1/3, 2/3, 1/3)                                |
| ((...o, o, e), (...e), (e/0, e, ..., o))       |
| (2/3, 3/3, 1/3)                                |
| ((...o, e, e/0), (e/0, e, e, ..., e), (e/0, e, e, ...)) |
| (2/3, 1/1, 1/1)                                |
| ((...o, e/0), (...e), (..., e))                |
| (3/3, 3/3, 3/3, 3/3)                          |
| ((...e, e/0, e), (...e, e/0, e, e/0, e, ..., e), (e/0, e, e, ...)) |
| (1/3, 1/1, 1/1, 1/1)                          |
| ((...o, o, e), (...e), (...e), (...e))        |
| (1/1, 1/1, 1/1, 1/1)                          |
| ((..., e), (...e), (...e), (...e))             |

As before, we call a list of $m_i$-tuples **irreducible** if no subset is nicely attainable. We list all irreducible lists of $m_i$-tuples below; the right column shows that each corresponds to a sequence of the desired form:

| Irreducible lists of $m_i$-tuples |
|-----------------------------------|
| φ                                 |
| ((...))                           |
| (1/3)                             |
| ((...o, o, e))                    |
| (2/3)                             |
| ((...o, e, e/0))                  |
| (3/3)                             |
| ((...e, e/0))                     |
| (1/1)                             |
| ((...e))                          |
Therefore, any list of $m_i$-tuples $0/1, 1/3, 2/3, 3/3, 1/1$ corresponds to some attainable sequence of integers. Consider replacing each of $0/1, 1/3, 2/3, 3/3, 1/1$ with a general $a/b$ in its class. It suffices to prove the following:

1. The nicely attainable lists of $m_i$-tuples are still nicely attainable.
2. The irreducible lists of $m_i$-tuples still correspond to attainable sequences.

We can obtain these general $m_i$-tuples from $0/1, 1/3, 2/3, 3/3, 1/1$ by repeatedly inserting $o$ or a permutation of $e/0, e, e, e$. For each list of $m_i$-tuples other than $(2/3, 1/1)$, it is clear that we can insert $o$ at the dots, but for inserting a permutation of $e/0, e, e, e$ at the dots we have three cases:

**Case 1: Dots within a nicely attainable string $e, e/0, e/0, e$.**

*Proof of Case 1.* Inserting permutations of $e/0, e, e, e$ produces a string of $e$’s and $e/0$’s with length a multiple of four. We claim that we can choose the permutations so that the resulting string is of the form

$$e, e/0, e/0, e, . . . , e, e/0, e/0, e$$

Since two of every four consecutive terms of this string are $e/0$’s, our claim is correct, and we can choose the permutations appropriately. □

**Case 2: Dots within a sequence of trailing $e$’s.**

*Proof of Case 2.* Inserting permutations of $e/0, e, e, e$ produces a longer string of trailing $e$’s. We claim that we can choose the permutations so that the
resulting string is of one of the following forms:

\[ e, e/0, e/0, e, e/0, e/0, e \]
\[ e, e/0, e/0, e, e/0, e, e/0, e \]
\[ e, e/0, e, e/0, e, e/0, e, e/0, e \]
\[ e, e/0, e, e, e/0, e, e, e, e/0 \]

Since at least one of every four consecutive terms of these strings are e/0's, our claim is correct, and we can choose the permutations appropriately. □

**Case 3: All other dots.**

**Proof of Case 3.** We can insert e, e/0, e, e, which is nicely attainable. □

Finally, consider lists of \( m_t \)-tuples associated with \((2/3, 1/1)\). Each such list of \( m_t \)-tuples other than \((2/3, 1/1)\) itself can be obtained from one of the following by repeatedly inserting o or a permutation of e/0, e, e, e:

| Lists of \( m_t \)-tuples associated with \((2/3, 1/1)\) |
|------------------|
| \((2/3, 5/5)\)   | \(((\ldots, o, e, e/0), (e/0, e, 
|                  | \ldots, e, e, e))\) |
| \((2/5, 1/1)\)   | \(((\ldots, e), (o, o, o, e, 
|                  | \ldots, e/0))\) |
| \((6/7, 1/1)\)   | \(((\ldots, o, e, e/0, e, e, e), (\ldots))\) |

Therefore, (1) and (2) hold, and the theorem is proved.

**3.3. Proof of Thm. 5(c): A Class of Diameter-2r Complete Trees**

**Proof.** We generalize our representation of depth-3 trees above to all trees \( S \) as follows, where \( S \) is a complete rooted tree with root \( v \):

- If \( S \) has depth 0 or 1, then we denote \( S \) by \( n \), where \( v \) has \( n \) children.
- If \( S \) has depth greater than 1, then we denote \( S \) by \((S_1, \ldots, S_k)\), where \( S_1, \ldots, S_k \) are the connected components of \( S \backslash v \).

By the conditions, if \( S \) is a subtree of a tree satisfying the conditions of Thm. 5(c), then our representation of \( S \) has no 0's, and each set of parentheses encloses an odd number of trees.

Our intuition is that a tree \( S \) functions in certain basic ways within larger trees, depending on the number of even integers in our representation of \( S \):
To formalize this intuition, we define the ending of a sequence, so that certain sequences have ending $\emptyset$, $E_1$, $E_2$, $E_2'$, or $E_3$, as follows:

| Ending | Sequence of integers (dfs. for Thm. 5(c)) | Sequence of integers (adjusted dfs. for Thm. 5(d)) |
|--------|-------------------------------------------|--------------------------------------------------|
| $\emptyset$ | $n_1, \ldots, n_m$ | (same) |
| $E_1$ | $n_1, \ldots, n_m, e$ | (same) |
| $E_2$ | $n_1, \ldots, n_m, e, o, \ldots, o$ | $n_1, \ldots, n_m, e, e/0, o, \ldots, o$ |
| $E_2'$ | $n_1, \ldots, n_m, e, o, \ldots, o$ | $n_1, \ldots, n_m, e, e/0, o, \ldots, o$ |
| $E_3$ | $n_1, \ldots, n_m, e, o, \ldots, o, e$ | $n_1, \ldots, n_m, e, e/0, o, \ldots, o, e/0$ |

We associate a tree $S$ with a pair of endings $(E_1, E_2)$ if $S$ corresponds to a sequence $S$, such that appending $S$ to any sequence with ending $E_1$ gives a sequence with ending $E_2$. We can then combine trees; if $S_1, S_2$ are associated with $(E_1, E_2), (E_2, E_3)$, respectively, then $(S_1, S_2)$ is associated with $(E_1, E_3)$.

Our intuition suggests the following claim:

**Claim:** If our representation of $S$ has no 0’s, and each set of parentheses encloses an odd number of trees, then $S$ is associated with the following pairs, depending on the number of even integers in its representation:

| # even ints. | Pairs of endings associated with $S$ |
|--------------|-------------------------------------|
| 0 (mod 4)    | $(\emptyset, \emptyset), (E_2', E_2'), (E_2', E_2)$ |
| 1 (mod 4)    | $(\emptyset, E_1), (E_1, E_2), (E_2, E_3), (E_3, \emptyset)$ |
| 2 (mod 4)    | $(\emptyset, E_2), (\emptyset, E_2'), (E_2, \emptyset), (E_2', \emptyset)$ |
| 3 (mod 4)    | $(\emptyset, E_3), (E_1, \emptyset), (E_2, E_1), (E_3, E_2)$ |
Proof of Claim. We may assume that each set of parentheses in our representation of $S$ encloses one or three terms, by adding parentheses:

$$(S_1, \ldots, S_k) \rightarrow (\cdots((S_1, S_2, S_3), S_4, S_5), \ldots, S_k)$$

It is easy to verify that $e$ and $o$ satisfy the claim, so by induction it suffices to prove by casework that if $S_1, S_2, S_3$ satisfy the claim, then $(S_1, S_2, S_3)$ does also. We omit the details, but the following observations are helpful:

- If $S_1, S_2$ each have an odd number of $e$'s, then $(S_1, S_2)$ is associated with $(\emptyset, \emptyset)$ and $(E_2, E_2)$, or $(\emptyset, E_2)$ and $(E_2, \emptyset)$.

- If $S_1, S_2$ each have an even number of $e$'s, then $(S_1, S_2)$ is associated with $(\emptyset, \emptyset)$ and $(E_2, E_2)$, or $(\emptyset, E_2)$ and $(E_2, \emptyset)$.

□

Now we return to the original problem. If the number of even integers in our representation of $S$ is not 3 (mod 4), then $S$ is associated with one of $(\emptyset, \emptyset), (\emptyset, E1), (\emptyset, E2)$. Since the sequences associated with $\emptyset, E1, E2$ are attainable, $S$ is associated with an attainable sequence, as desired. ■

3.4. Proof of Thm. 5(d): A Class of Diameter-2r Non-Complete Trees

Proof. We use the same proof, adjusting the definitions of $E2, E2', E3$ as noted above to allow $e/0$'s, and including $(e, e, 0), (e, o, 0)$ as base cases.

3.5. Proof of Thm. 5(e): A Class of Diameter-6 Odd-Children Trees

Proof. Rearrange the subtrees of $T$ at $v$ according to the numbers of internal vertices and leaves of distance 2 from $v$ in each subtree, as follows:

1. positive odd number of internal vertices, no leaves
2. positive odd number of internal vertices, positive even number of leaves
3. positive even number of internal vertices, positive odd number of leaves
4. no internal vertices, positive odd number of leaves

Let the neighbors of $v$ be $v_1, \ldots, v_m$, such that the subtrees are as follows:

$$(v_1, \ldots, v_{i_1-1}, v_{i_1}, \ldots, v_{i_2-1}, v_{i_2}, \ldots, v_{i_3-1}, v_{i_3}, \ldots, v_m)$$

Starting with a gracefully labeled star with central vertex labeled 0, perform a 0 $\rightarrow$ $n$ transfer, leaving $m$ vertices adjacent to the root. Call them $v_1, \ldots, v_m$, in the order they would appear in a transfer context $(T, f, v_1, \ldots, v_m, n, 0, e, d)$. Then perform the transfers
Thinking of this sequence as leaving behind an odd number of leaves once at each step, and twice at the turns at \(v_{i_3-1}, v_{i_1}\) (see Lemma 12), we can leave behind the following numbers of leaves at each step (referring to the internal vertices and leaves adjacent to the \(v_j\) in each subtree):

- \(v_1, \ldots, v_{i_1-1}\) all of the internal vertices
- \(v_{i_1}, \ldots, v_{i_2-1}\) all of the internal vertices
- \(v_{i_2}, \ldots, v_{i_3-1}\) some of the internal vertices
- \(v_{i_3-1}, \ldots, v_{i_2}\) rest of the internal vertices
- \(v_{i_2-1}, \ldots, v_{i_1}\) some of the leaves
- \(v_{i_1}, \ldots, v_{i_2-1}\) rest of the leaves
- \(v_{i_2}, \ldots, v_{i_3-1}\) all of the leaves
- \(v_{i_3}, \ldots, v_m\) all of the leaves

After these transfers, the leaves left behind are in order (see Lemma 14). Therefore, we can obtain \(T\) by performing the following sequence of transfers, stopping at the last leaf left at \(v_{i_2}\) during the second pass:

\[v_m \rightarrow v_{m+1} \rightarrow v_{m+2} \rightarrow \cdots\]

Therefore, \(T\) has a graceful labeling \(f\) with \(f(v) = 0\).

We have now proved our main result, Thm. 5 (see p. 5).

4. Conclusion

Our three modifications have both simplified and expanded the transfer technique, producing new best results on diameter-6 trees. Our framework can also accommodate some important ideas of other papers:

- final \(n \rightarrow 0\) transfer of branches (Hrnčiar & Haviar [3])
  (used to prove that all diameter-5 trees are graceful)

- final transfers of branches along a path (Mishra & Panigrahi [6, 10])
  (used to prove that certain lobsters are graceful)
For example, see Fig. 11, 12 of Hrnčiar & Haviar [3]. In this final step, they use a $21 \rightarrow 0$ transfer of the branches with vertices labeled 1, 3, 18, 20. The same tree can be obtained within our framework as follows, beginning with a gracefully labeled star with central vertex labeled 0:

- Perform a type-1 transfer $0 \rightarrow 21$ of the leaves labeled 2, \ldots, 19.
- Perform a type-1 transfer $21 \rightarrow 0$ of the leaves labeled 3, \ldots, 18.
- Perform a type-1 transfer $0 \rightarrow 21$ of the leaves labeled 4, \ldots, 17.
- Continue as in Hrnčiar & Haviar [3].

An analogous remark applies to Mishra & Panigrahi [6–10].

The trees we have considered have no vertices with even numbers of children in higher levels. To address this shortcoming, it is necessary to consider different sequences of transfers in higher levels, which have a complicated effect on the order of vertices in lower levels. Understanding this effect will be essential in extending our framework.

However, to prove that all diameter-6 trees are graceful, our framework seems insufficient. Cahit’s technique of canonic spiral labeling is relevant, since it produces many of the same trees as the transfer technique, but also some trees that the transfer technique cannot produce (see slides 7-10 of Cahit [11]). Unifying the techniques may yield progress toward the conjecture.

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