JORDAN PROPERTY FOR ALGEBRAIC GROUPS AND AUTOMORPHISM GROUPS OF PROJECTIVE VARIETIES IN ARBITRARY CHARACTERISTIC

FEI HU

ABSTRACT. We show an analogue of Jordan’s theorem for algebraic groups defined over a field \( \mathbb{K} \) of arbitrary characteristic. As a consequence, a Jordan-type property holds for the automorphism group of any projective variety over \( \mathbb{K} \).

1. INTRODUCTION

In 1878, Camille Jordan [Jor78] proved the following remarkable theorem.

Theorem 1.1 (cf. [Jor78]). For any positive integer \( n \), there exists a constant \( J(n) \) such that any finite subgroup \( \Gamma \) of \( \text{GL}_n \) over a field of characteristic zero contains a normal abelian subgroup \( A \) of index \( \leq J(n) \).

However, the above theorem is false for fields of characteristic \( p > 0 \) due to the existence of unipotent elements of finite order. For instance, the group \( \text{GL}_n(\mathbb{F}_p) \) contains arbitrarily large subgroups of the form \( \text{SL}_n(\mathbb{F}_{p^r}) \) which are simple modulo their centers. Nevertheless, for any finite subgroup \( \Gamma \) of \( \text{GL}_n(\mathbb{K}) \) of order not divisible by \( \text{char}(\mathbb{K}) \), there still exists a normal abelian subgroup \( A \) of \( \Gamma \) with \( [\Gamma : A] \leq J(n) \) for the same \( J(n) \) as in Theorem 1.1 (see e.g. [BF66, 2.9]). Later, Serre showed that the Cremona group \( \text{Cr}_2(\mathbb{K}) \) of rank 2 over a field \( \mathbb{K} \) also has this property (cf. [Ser09, Theorem 5.3]). This motivates us to make the following definition.

Definition 1.2. Let \( p \) be a prime number or zero. A group \( G \) is called a \( p \)-Jordan group, if there exists a constant \( J(G) \), depending only on \( G \), such that every finite subgroup \( \Gamma \) of \( G \) whose order is not divisible by \( p \) contains a normal abelian subgroup \( A \) of index \( \leq J(G) \).

Note that when \( p = 0 \), this notion coincides with Popov’s [Pop11, Definition 2.1]. The above mentioned results can be reformulated as follows. Both general linear groups \( \text{GL}_n(\mathbb{K}) \) and the Cremona group \( \text{Cr}_2(\mathbb{K}) \) of rank 2 are \( p \)-Jordan, where \( p = \text{char}(\mathbb{K}) \). Our first result below shows that in addition to above, any algebraic group over \( \mathbb{K} \) is \( p \)-Jordan.

Theorem 1.3. Any algebraic group \( G \) defined over a field \( \mathbb{K} \) of characteristic \( p \geq 0 \) is \( p \)-Jordan. Namely, there exists a constant \( J(G) \), depending only on \( G \), such that every finite subgroup \( \Gamma \) of \( G(\mathbb{K}) \) whose order is not divisible by \( p \) contains a normal abelian subgroup of index \( \leq J(G) \).

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Remark 1.4. Even in characteristic zero, Theorem 1.3 is nontrivial and has just been proved by Meng and Zhang recently (cf. [MZ18, Theorem 1.3]). The main obstruction is that the Jordan property may not be preserved under group extensions (see e.g. [Pop11, Remark 2.12] or [Pop14, Example 7]). See also [Zar14] for a counterexample. Note, however, that the argument in [MZ18] also depends on the Levi decomposition of connected algebraic groups (cf. [Mos56]), which is not available in prime characteristic (see e.g. [Bri15, Remark 4.9] and references therein). Our approach is based on the investigation of the Jordan property of quotient groups (see e.g. Lemma 2.2 and Step 2 in the proof of Theorem 1.3).

Another generalization of Jordan’s theorem to prime characteristic was due to Brauer and Feit [BF66] by allowing arbitrary finite subgroup $\Gamma$ of $GL_n(\mathbb{F}_p)$ whose order may be divisible by $p > 0$. They showed that $\Gamma$ contains a normal abelian subgroup whose index is bounded by a constant depending on $n$ as well as the order of the $p$-Sylow subgroup $\Gamma_p$ of $\Gamma$. Larsen and Pink [LP11] has subsequently extended Brauer–Feit [BF66] as follows.

Theorem 1.5 (cf. [LP11, Theorem 0.4]). For any positive integer $n$, there exists a constant $J'(n)$ such that any finite subgroup $\Gamma$ of $GL_n$ over a field $\mathbb{F}_p$ of characteristic $p > 0$ contains a normal abelian $p'$-subgroup $A$ of index $\leq J'(n) \cdot |\Gamma_p|^3$.

Here a finite group is called a $p$-group (resp. $p'$-group) if its order is some power of $p$ (resp. relatively prime to $p$). In an analogous way, we introduce the following notion.

Definition 1.6. Let $p$ be a prime number. A group $G$ is called a strongly $p$-Jordan group, if there exist constants $J'(G)$ and $e(G)$, depending only on $G$, such that every finite subgroup $\Gamma$ of $G$ contains a normal abelian $p'$-subgroup $A$ of index $\leq J'(G) \cdot |\Gamma_p|^{e(G)}$.

Below is our second main result extending Theorem 1.5 to arbitrary algebraic groups.

Theorem 1.7. Any algebraic group $G$ defined over a field $\mathbb{F}$ of characteristic $p > 0$ is strongly $p$-Jordan. That is, there are constants $J'(G)$ and $e_G$, depending only on $G$, such that any finite subgroup $\Gamma$ of $G(\mathbb{F})$ contains a normal abelian $p'$-subgroup $A$ of index $\leq J'(G) \cdot |\Gamma_p|^{e(G)}$.

Remark 1.8. We may think of Theorem 1.7 as a stronger version of Theorem 1.3 in positive characteristic, if we only care about the existence of those constants (i.e., $J(G)$ and $J'(G)$). Actually, we will see in Section 3 that our constant $J(G) = c_G J(n)^{\mu G}$, where $c_G$ is the number of connected components of $G$ and $n$ is the least dimension of a faithful representation of $(G^\circ)_{aff}$ over $\mathbb{F}$; see §2.2 for the meaning of $(G^\circ)_{aff}$. But $J'(G)$ is much more complicated and involved. Also, it will be shown that $e(G) = 3(r_G + 1)c_G$, where $r_G$ is bounded by the rank of $(G^\circ)_{aff}$.

In characteristic zero, it has also been proved by Meng and Zhang that the automorphism group $Aut(X)$ is Jordan for any projective variety $X$ (cf. [MZ18, Theorem 1.6]). In this note, as a byproduct of our main theorems, we also deduce two Jordan-type properties for automorphism groups of projective varieties in arbitrary characteristic.
Theorem 1.9. Let $X$ be a projective variety defined over a field $\mathbb{k}$ of characteristic $p \geq 0$. Then there exists a constant $J_X$, depending only on $X$, such that every finite $p'$-subgroup $\Gamma$ of $\text{Aut}(X)$ contains a normal abelian subgroup $A$ of index $\leq J_X$.

Theorem 1.10. Let $X$ be a projective variety defined over a field $\mathbb{k}$ of characteristic $p > 0$. Then there exist constants $J'_X$ and $e_X$, depending only on $X$, such that every finite subgroup $\Gamma$ of $\text{Aut}(X)$ contains a normal abelian $p'$-subgroup $A$ of index $\leq J'_X \cdot |\Gamma(p)|^{e_X}$.

We also note that in characteristic zero, Prokhorov and Shramov proved that the group $\text{Bir}(X)$ of birational self-maps of any non-uniruled variety $X$ is Jordan (cf. [PS14, Theorem 1.8(ii)]). Assuming the Borisov–Alexeev–Borisov conjecture, which was recently proved in Birkar’s pioneering work [Bir16], they even showed in [PS16] that $\text{Bir}(X)$ is (uniformly) Jordan for any rationally connected variety $X$, generalizing Serre’s [Ser09, Theorem 5.3] (the characteristic zero side). Quite recently, as a consequence of the aforementioned Jordan property, Reichstein obtained new low bounds on the essential dimension of a series of finite groups which was not previously known even for special cases (cf. [Rei18, Theorem 3]). At the end of our introduction, we raise the following natural question (see also [Ser09, Question 6.1] for a related question).

Question 1.11. Let $\text{Cr}_n(\mathbb{k})$ be the Cremona group of rank $n \geq 2$ defined over a field $\mathbb{k}$ of characteristic $p > 0$. Then is $\text{Cr}_n(\mathbb{k})$ strongly $p$-Jordan?

2. Preliminaries

2.1. Two group-theoretic lemmas. We need the following two group-theoretic lemmas which are quite useful in dealing with (strongly) $p$-Jordan groups (see Definitions 1.2 and 1.6). See [Pop11, Lemmas 2.6 and 2.8] and [Pop14, Theorem 3] for related results.

Lemma 2.1. Let $G_1$ and $G_2$ be two groups and $G$ their direct product $G_1 \times G_2$.

1. If $G_1$ and $G_2$ are $p$-Jordan, then so is $G_1 \times G_2$ and one can take $J(G_1 \times G_2) = J(G_1)J(G_2)$.
2. If $G_1$ and $G_2$ are strongly $p$-Jordan, then so is $G_1 \times G_2$ and one can take $J'(G_1 \times G_2) = J'(G_1)J'(G_2)$, $e(G_1 \times G_2) = e(G_1) + e(G_2) - 1$.

Proof. (1) It follows directly from [Pop14, Theorem 3(2)]. For the sake of completeness, we present the proof here. Let $G$ denote the direct product $G_1 \times G_2$ and $\pi_i : G \to G_i$ the projection homomorphism. Let $\Gamma$ be a finite $p'$-subgroup of $G$. Then $\Gamma_i := \pi_i(\Gamma) \leq G_i$ contains a normal abelian subgroup $A_i$ such that

$$[\Gamma_i : A_i] \leq J(G_i).$$

The subgroup $\widetilde{A}_i := \pi_i^{-1}(A_i) \cap \Gamma$ is normal in $\Gamma$ and $\Gamma/\widetilde{A}_i$ is isomorphic to $\Gamma_i/A_i$. We thus have

$$[\Gamma : \widetilde{A}_i] = [\Gamma_i : A_i] \leq J(G_i).$$

Since $A := \widetilde{A}_1 \cap \widetilde{A}_2$ is the kernel of the diagonal homomorphism

$$\Gamma \longrightarrow \Gamma/\widetilde{A}_1 \times \Gamma/\widetilde{A}_2$$
defined by the canonical projections $\Gamma \to \Gamma/\tilde{A}_i$, we conclude that

$$[\Gamma : A] \leq [\Gamma : \tilde{A}_1] \cdot [\Gamma : \tilde{A}_2] \leq J(G_1)J(G_2).$$

By the construction, $A$ is a subgroup of the abelian group $A_1 \times A_2$, so is abelian. Hence $A$ is a normal abelian subgroup of $\Gamma$ of index $\leq J(G_1)J(G_2)$ as claimed.

(2) We need to modify the above proof appropriately. More precisely, using the notation there, let $\Gamma$ be a finite subgroup of $G = G_1 \times G_2$. Then $\Gamma_i := \pi_i(\Gamma) \leq G_i$ contains a normal abelian $p'$-subgroup $A_i$ such that

$$[\Gamma_i : A_i] \leq J'(G_i) \cdot |(\Gamma_i)_{(p)}|^{e(G_i)}.$$  

Note that $|\Gamma_{(p)}| = |(\Gamma_1)_{(p)}| \cdot |(\tilde{A}_1)_{(p)}|$ because $\Gamma/\tilde{A}_i \cong \Gamma_i/A_i$ and $A_i$ is a $p'$-subgroup of $\Gamma_i$. Let $A := \tilde{A}_1 \cap \tilde{A}_2 \leq A_1 \times A_2$ as above, which is a normal abelian $p'$-subgroup of $\Gamma$. It follows that

$$[\Gamma : A] = [\Gamma/A] = \frac{|\Gamma/\tilde{A}_1| \cdot |\Gamma/\tilde{A}_2|}{|\Gamma/\tilde{A}_1\tilde{A}_2|} \leq J'(G_1) \cdot |(\Gamma_1)_{(p)}|^{e(G_1)} \cdot J'(G_2) \cdot |(\Gamma_2)_{(p)}|^{e(G_2)}$$

$$= J'(G_1)J'(G_2) \cdot \frac{|(\Gamma_1)_{(p)}|^{e(G_1)} \cdot |(\Gamma_2)_{(p)}|^{e(G_2)} \cdot |(\tilde{A}_1\tilde{A}_2)_{(p)}|}{|\Gamma_{(p)}|}$$

$$= J'(G_1)J'(G_2) \cdot \frac{|(\Gamma_1)_{(p)}|^{e(G_1)} \cdot |(\Gamma_2)_{(p)}|^{e(G_2)} \cdot |(\tilde{A}_1)_{(p)}| \cdot |(\tilde{A}_2)_{(p)}|}{|\Gamma_{(p)}|}$$

$$\leq J'(G_1)J'(G_2) \cdot |(\Gamma_1)_{(p)}|^{e(G_1)e(G_2)} - 1,$$

which proves Lemma 2.1. \hfill \Box

**Lemma 2.2.** Let $G$ be a group and $K$ a finite normal subgroup of $G$.

1. Suppose that $G$ is $p$-Jordan and one of the following conditions holds:
   1. the order of $K$ is not divisible by $p$,
   2. $p > 0$ and $K$ has a normal Sylow $p$-subgroup $K_{(p)}$.

   Then $G/K$ is $p$-Jordan and one can take $J(G/K) = J(G)$.

2. Suppose that $G$ is strongly $p$-Jordan. Then $G/K$ is strongly $p$-Jordan and one can take

   $$J'(G/K) = J'(G) \cdot |K_{(p)}|^{e(G)} \cdot e(G/K) = e(G).$$

**Proof.** (1) The first case is easy; see e.g. [Pop11, Lemma 2.6]. We now consider the case that $p > 0$ and $K_{(p)}$ is nontrivial. Let $\Gamma$ be a finite $p'$-subgroup of $G/K$. Let $H$ be the inverse of $\Gamma$ in $G$. Since $p \nmid |\Gamma|$ by the assumption, $K_{(p)}$ is also a Sylow $p$-subgroup of $H$. It follows from $K \leq H$ that $K_{(p)}$ is also normal in $H$. Hence $K_{(p)} = H_{(p)}$ is the normal Sylow $p$-subgroup of $H$. Namely, we have the following exact sequence:

$$1 \rightarrow K/K_{(p)} \rightarrow H/H_{(p)} \rightarrow \Gamma \rightarrow 1.$$
By the Schur–Zassenhaus theorem (cf. [Rob96, Theorem 9.1.2]), there is a complement $K_C$ (resp. $H_C$) of $K_{(p)} = H_{(p)}$ in $K$ (resp. $H$), which satisfies that $K = K_{(p)} \rtimes K_C$ (resp. $H = H_{(p)} \rtimes H_C$). (Their theorem also states that all complements are conjugate to each other, here we do not need this conjugation result though). Then we rewrite the above exact sequence as follows:

$$1 \rightarrow K_C \rightarrow H_C \rightarrow \Gamma \rightarrow 1.$$ 

Note that our $H_C$ now is a finite $p'$-subgroup of $G$. It follows that there is a normal abelian subgroup $A_{H_C}$ of $H_C$ such that $[H_C : A_{H_C}] \leq J(G)$. Let $A$ be the image of $A_{H_C}$ in $\Gamma$. Then $[\Gamma : A] \leq [H_C : A_{H_C}] \leq J(G)$.

(2) Let $\Gamma$ be a finite subgroup of $G/K$ and $H$ the inverse of $\Gamma$ in $G$. Then $H$ contains a normal abelian $p'$-subgroup $A_H$ of index $\leq J'(G) \cdot |H_{(p)}|^{e(G)}$. Let $A$ be the image of $A_H$ in $\Gamma$. Noting that $|H_{(p)}| = |K_{(p)}| \cdot |\Gamma_{(p)}|$, we thus have

$$[\Gamma : A] \leq [H : A_H] \leq J'(G) \cdot |H_{(p)}|^{e(G)} = J'(G) \cdot |K_{(p)}|^{e(G)} \cdot |\Gamma_{(p)}|^{e(G)}.$$ 

This yields the assertion (2) and hence Lemma 2.2 follows. \qed

2.2. Two algebraic group-theoretic theorems. Let $G$ be a connected algebraic group defined over a perfect field $\mathbb{K}$. We record the following classical decomposition theorem of algebraic groups. By the Chevalley’s structure theorem, there is a smallest connected normal affine subgroup scheme $G_{\text{aff}}$ of $G$ such that the quotient $G/G_{\text{aff}}$ is an abelian variety (cf. [Bri17, Theorem 2]). On the other hand, $G$ has a smallest connected normal subgroup scheme $G_{\text{ant}}$ such that the quotient $G/G_{\text{ant}}$ is affine; moreover, $G_{\text{ant}}$ is smooth and contained in the center $Z(G)$ of $G$ (cf. [Ros56, §5]). We have the following Rosenlicht’s decomposition theorem (see e.g. [Bri17, §5.1]).

**Theorem 2.3** (cf. [Bri17, Theorem 5.1.1 and Remark 5.1.2]). *Keep the above notation and assumptions. The following statements hold.*

1. $G = G_{\text{aff}} \cdot G_{\text{ant}} \cong G_{\text{aff}} \times G_{\text{ant}}/(G_{\text{aff}} \cap G_{\text{ant}})$.
2. $G_{\text{aff}} \cap G_{\text{ant}}$ contains $(G_{\text{ant}})_{\text{aff}}$.
3. The quotient $(G_{\text{aff}} \cap G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}}$ is finite.
4. The multiplication map of $G$ induces an isogeny

$$m: (G_{\text{aff}} \times G_{\text{ant}})/(G_{\text{ant}})_{\text{aff}} \rightarrow G,$$

where $(G_{\text{ant}})_{\text{aff}}$ is viewed as a subgroup scheme of $G_{\text{aff}} \times G_{\text{ant}}$ via $x \mapsto (x, x^{-1})$.

The theorem below is a special case of a theorem due to Lucchini Arteche [LA17] which plays an important role in the proof of our main theorems. It could be regarded as an effective version of Brion’s theorem on the existence of the quasi-splitness of an extension of algebraic groups with finite quotient (cf. [Bri15, Theorem 1.1]).
Theorem 2.4 (cf. [LA17, Theorem 3.2]). Let \( \mathbb{k} \) be an algebraically closed field of characteristic \( p \geq 0 \). Let \( \Gamma \) be a smooth finite \( \mathbb{k} \)-group of order \( n \), and \( G \) an arbitrary smooth \( \mathbb{k} \)-group. Then for any group extension
\[
1 \rightarrow G \rightarrow H \rightarrow \Gamma \rightarrow 1,
\]
there exist a finite smooth \( \mathbb{k} \)-subgroup \( S \) of \( G \) and a commutative diagram with exact rows
\[
\begin{array}{c}
1 \\ \downarrow \downarrow \\ S \rightarrow F \\ \downarrow \downarrow \\ G \rightarrow H \rightarrow \Gamma \rightarrow 1
\end{array}
\]
Moreover, if \( G \) is an algebraic torus with rank \( r \), then \( S \) can be taken as a subgroup of the \( n \)-torsion subgroup \( G[\mathbb{n}] \) of \( G \). In particular, the order of \( S \) divides \( n^r \).

3. Proofs of theorems

We are eventually interested only in questions concerning algebraic groups or varieties defined over algebraically closed fields. So from now on, we will assume that \( \mathbb{k} \) is algebraically closed.

Proof of Theorem 1.3. We may assume that \( G \) is smooth (or equivalently, reduced), since we only consider finite subgroups of the group \( G(\mathbb{k}) \) of \( \mathbb{k} \)-rational points. We first consider the case that \( G \) is connected. In the following Steps 1-3, we will show the theorem under this case.

Step 1. Let \( G_{\text{aff}} \) and \( G_{\text{ant}} \) denote the affine part and the anti-affine part of \( G \), respectively (see §2.2). We claim that \( G_{\text{aff}} \times G_{\text{ant}} \) is \( p \)-Jordan. Indeed, since both \( G_{\text{aff}} \) and \( G_{\text{ant}} \) are \( p \)-Jordan, the claim follows from Lemma 2.1(1). More precisely, let \( n \) be the least dimension of a faithful representation of \( G_{\text{aff}} \) over \( \mathbb{k} \). Then there is a constant \( J(n) \) which is essentially from Theorem 1.1 such that every finite \( p' \)-subgroup \( \Gamma \) of \( G_{\text{aff}} \times G_{\text{ant}} \) contains a normal abelian subgroup of index \( \leq J(n) \).

Step 2. Let \( N \) denote \( (G_{\text{ant}})_{\text{aff}} \). Consider the following exact sequence of algebraic groups:
\[
1 \rightarrow N \rightarrow G_{\text{aff}} \times G_{\text{ant}} \rightarrow (G_{\text{aff}} \times G_{\text{ant}})/N \rightarrow 1.
\]
We claim that the quotient group \( (G_{\text{aff}} \times G_{\text{ant}})/N \) is \( p \)-Jordan. Let \( \Gamma \) be a finite \( p' \)-subgroup of \( (G_{\text{aff}} \times G_{\text{ant}})/N \). Let \( H \) be the inverse (or pullback) of the finite group \( \Gamma \) in \( G_{\text{aff}} \times G_{\text{ant}} \) (cf. [Bri17, Proposition 2.8.3]). Then according to Theorem 2.4, there exist a finite smooth \( \mathbb{k} \)-subgroup \( S \) of \( N \) and a commutative diagram of algebraic groups:
\[
\begin{array}{c}
1 \\ \downarrow \downarrow \\ S \rightarrow F \\ \downarrow \downarrow \\ G \rightarrow H \rightarrow \Gamma \rightarrow 1
\end{array}
\]

In characteristic zero, there is a normal abelian subgroup \( A_F \) of \( F \) such that \( [F : A_F] \leq J(n) \) by Step 1. It follows that the image \( A \) of \( A_F \) in \( \Gamma \) is a normal abelian subgroup of \( \Gamma \) of index \( [\Gamma : A] \leq [F : A_F] \leq J(n) \).
Now we assume that \( p = \text{char}(\mathbb{k}) > 0 \). By [Bri17, Proposition 5.5.1], any anti-affine group over \( \mathbb{k} \) is a semi-abelian variety. Thus \( N = (G_{\text{ant}})_{\text{aff}} \) is an algebraic torus over \( \mathbb{k} \). In particular, \( N \) is commutative and so is \( S \). So \( S(p) \) is the unique normal Sylow \( p \)-subgroup of \( S \). By the assumption on \( \Gamma \) that \( p \nmid |\Gamma| \), \( S(p) \) is also a Sylow \( p \)-subgroup of \( F \). It follows from \( S \leq F \) that \( S(p) \) is also normal in \( F \). Then the Schur–Zassenhaus theorem asserts that there are complements \( S_C \) and \( F_C \) of \( S(p) = F(p) \) in \( S \) and \( F \) respectively; see also the proof of Lemma 2.2(1). It follows that \( F_C \) is a \( p' \)-subgroup of \( G_{\text{aff}} \times G_{\text{ant}} \) and hence contains a normal abelian subgroup \( A_{F_C} \) of index \( \leq J(n) \) by Step 1 as in the previous case; the rest is the same.

**Step 3.** By Rosenlicht’s decomposition Theorem 2.3, we have the following exact sequence of algebraic groups:

\[
1 \rightarrow K \rightarrow (G_{\text{aff}} \times G_{\text{ant}})/N \xrightarrow{m} G \rightarrow 1,
\]

where \( N = (G_{\text{ant}})_{\text{aff}} \) as in Step 2 and \( K := (G_{\text{aff}} \cap G_{\text{ant}})/N \) is a finite group so that \( m \), induced from the multiplication map of \( G \), is an isogeny. Note that \( G_{\text{ant}} \) is commutative (cf. [Bri17, Proposition 3.3.5]). Then \( K \) is abelian and hence \( K(p) \) is the normal Sylow \( p' \)-subgroup of \( K \). Let \( \Gamma \) be a finite \( p' \)-subgroup of \( G \). It follows from Lemma 2.2(1) and Step 2 that \( \Gamma \) contains a normal abelian subgroup \( A \) of index \( \leq J(n) \).

**Step 4.** Finally, with the aid of [Pop11, Lemma 2.11], we are able to deal with non-connected algebraic groups as well. Indeed, let \( G^o \) be the neutral component of \( G \). Denote by \( c_G \) the order of the group \( \pi_0(G) := G/G^o \) of connected components of \( G \). Then any finite \( p' \)-subgroup \( \Gamma \) of \( G \) contains a normal abelian subgroup \( A \) of index \( \leq c_G J(n)^{c_G} \), where \( n \) is the least dimension of a faithful representation of \( (G^o)_{\text{aff}} \) over \( \mathbb{k} \).

We finally conclude the proof of Theorem 1.3 by letting \( J(G) = c_G J(n)^{c_G} \). \( \square \)

**Proof of Theorem 1.7.** The proof of Theorem 1.7 basically follows the strategy of the proof of Theorem 1.3. Note, however, that the group \( \Gamma \) may be of order divisible by \( p \) now.

We first consider the case that \( G \) is a connected algebraic group. We may assume that \( \mathbb{k} \) is algebraically closed. In Step 1, we have that \( J'(G_{\text{aff}}) = J'(n) \), \( e(G_{\text{aff}}) = 3 \) by Theorem 1.5, and \( J'(G_{\text{ant}}) = 1 \), \( e(G_{\text{ant}}) = 1 \) because \( G_{\text{ant}} \) is commutative (cf. [Bri17, Proposition 3.3.5]). Here \( n \) is the least dimension of a faithful representation of \( G_{\text{aff}} \) over \( \mathbb{k} \) as usual. Thus by Lemma 2.1(2), \( J'(G_{\text{aff}} \times G_{\text{ant}}) = J'(G_{\text{aff}})J'(G_{\text{ant}}) = J'(n) \) and \( e(G_{\text{aff}} \times G_{\text{ant}}) = e(G_{\text{aff}}) + e(G_{\text{ant}}) - 1 = 3 \). In other words, any finite subgroup \( \Gamma \) of \( G_{\text{aff}} \times G_{\text{ant}} \) contains a normal abelian \( p' \)-subgroup of index \( \leq J'(n) \cdot |\Gamma(p)|^3 \).

Then in Step 2, we claim that any finite subgroup \( \Gamma \) of \( (G_{\text{aff}} \times G_{\text{ant}})/N \) contains a normal abelian \( p' \)-subgroup of index \( \leq J'(n) \cdot |\Gamma(p)|^{3(r_G + 1)} \), where \( r_G \) is the rank of the algebraic torus \( N = (G_{\text{ant}})_{\text{aff}} \) which is further bounded by the rank of \( G_{\text{aff}} \). Indeed, we follow the argument as in the proof of Theorem 1.3 and get the commutative diagram (3.1). By the previous step, \( F \) contains a normal abelian \( p' \)-subgroup \( A_F \) of index \( \leq J'(n) \cdot |F(p)|^3 \). Note that Theorem 2.4 also yields that the order of \( S \) divides \( |\Gamma|^\alpha \). In particular, \( |S(p)| \) divides \( |\Gamma(p)|^\alpha \). Hence the image \( A \)
of $A_F$ in $\Gamma$ is a normal abelian $p'$-subgroup of $\Gamma$ such that

$$[\Gamma : A] \leq [F : A_F] \leq J'(n) \cdot |F(p)|^3 = J'(n) \cdot \left( |S(p)| \cdot |\Gamma(p)| \right)^3 \leq J'(n) \cdot |\Gamma(p)|^{3(r_G+1)},$$
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as claimed.

In Step 3, we consider the exact sequence (3.2). Recall that $K := (G_{aff} \cap G_{ant})/N$ is a finite group (depending on $G$ canonically). Let $\Gamma$ be a finite subgroup of $G$. Then it follows from the previous step and Lemma 2.2(2) that $\Gamma$ contains a normal abelian $p'$-subgroup $A$ of index

$$[\Gamma : A] \leq J'(n) \cdot |K(p)|^{3(r_G+1)} \cdot |\Gamma(p)|^{3(r_G+1)}.$$

Lastly, we consider the case that $G$ may be non-connected. As before, let $\pi_0(G) = G/G^o$ denote the group of connected components of $G$ and $c_G = |\pi_0(G)|$. Then any finite subgroup $\Gamma$ of $G$ contains a normal abelian $p'$-subgroup $A$ of index at most

$$c_G \left( J'(n) \cdot |K(p)|^{3(r_G+1)} \cdot |\Gamma(p)|^{3(r_G+1)} \right)^{c_G} = c_G J'(n)^{c_G} \cdot |K(p)|^{3(r_G+1)c_G} \cdot |\Gamma(p)|^{3(r_G+1)c_G},$$

where $n$ is the least dimension of a faithful representation of $(G^o)_{aff}$ over $\mathbb{F}_p$.

Let $J'(G)$ denote $c_G J'(n)^{c_G} \cdot |K(p)|^{3(r_G+1)c_G}$ and $e(G) := 3(r_G + 1)c_G$. We thus complete the proof of Theorem 1.7.

Given a projective variety $X$ defined over $\mathbb{F}_p$, the automorphism group scheme $\text{Aut}_X$ of $X$ is locally of finite type over $\mathbb{F}_p$ and the automorphism group $\text{Aut}(X)$ is just the rational $\mathbb{F}_p$-points of $\text{Aut}_X$, i.e., $\text{Aut}(X) = \text{Aut}_X(\mathbb{F}_p)$; in particular, the reduced neutral component $(\text{Aut}_X^0)^{\text{red}}$ of $\text{Aut}_X$ is a smooth algebraic group defined over $\mathbb{F}_p$ (see e.g. [Bri17, §7]). We denote $(\text{Aut}_X^0)^{\text{red}}(\mathbb{F}_p)$ by $\text{Aut}^o(X)$.

**Proof of Theorem 1.9.** Let $G := (\text{Aut}_X^0)^{\text{red}}$ and $G^o := \Gamma \cap \text{Aut}^o(X)$. Applying Theorem 1.3 to $G^o \leq G(\mathbb{F}_p)$, there is a normal abelian subgroup $A_G^o$ of $G^o$ of index $\leq J(G) = J(n)$, where $n$ is the least dimension of a faithful representation of $G_{aff}$ over $\mathbb{F}_p$. By [MZ18, Lemma 2.5], there is a constant $\ell_X$ depending only on $X$ such that $[\Gamma : \Gamma^o] \leq \ell_X$ (note that their proof is independent of the characteristic; see [MZ18, Remark 2.6]). It follows that

$$A := \bigcap_{g \in \Gamma} g^{-1} A_{G^o} g = \bigcap_{i=1}^{\lceil \ell_X \rceil} g_i^{-1} A_{G^o} g_i$$

is a normal abelian subgroup of $\Gamma$, where $g_i$’s are representatives of $\Gamma/G^o$. Note that $g_i^{-1} A_{G^o} g_i$ is a normal abelian group of $\Gamma^o$ of index $\leq J(n)$ for each $i$. This yields that the index of $A$ in $\Gamma^o$ is at most $J(n)^{\ell_X}$ (see also [Pop11, Lemma 2.11]). We thus have

$$[\Gamma : A] = [\Gamma : \Gamma^o] \cdot [\Gamma^o : A] \leq \ell_X J(n)^{\ell_X}.$$

To conclude the proof, we let $J_X := \ell_X J(n)^{\ell_X}$. 

**Remark 3.1.** In fact, using a theorem due to Chermak and Delgado (cf. [Isa08, Theorem 1.41]), there exists a characteristic (and hence normal) abelian subgroup $M$ of $\Gamma$ such that

$$[\Gamma : M] \leq [\Gamma : A_{G^o}]^2 = [\Gamma : \Gamma^o]^2 \cdot [\Gamma^o : A_{G^o}]^2 \leq \ell_X^2 J(n)^2.$$
Note, however, that the construction of this so-called Chermak–Delgado subgroup $M$ of $\Gamma$ is independent of $A_{\Gamma^o}$ (see [Isa08, Corollary 1.45]). Thus we do not know whether $M$ is still a subgroup of $A_{\Gamma^o}$ so that this argument breaks down in the proof of Theorem 1.10 (since $M$ may not be a $p'$-subgroup of an arbitrary finite subgroup $\Gamma$ of $G(\bar{k})$).

**Proof of Theorem 1.10.** Let $G := (\text{Aut}^*_X)_{\text{red}}$ and $\Gamma^o := \Gamma \cap \text{Aut}^o(X)$ as above. Then applying Theorem 1.7 to $\Gamma^o \leq G(\bar{k})$, there is a normal abelian $p'$-subgroup $A_{\Gamma^o}$ of $\Gamma^o$ such that

$$[\Gamma^o : A_{\Gamma^o}] \leq J'(G) \cdot |\Gamma^o_{(p)}|^{3(r_G + 1)},$$

where $\Gamma^o_{(p)}$ is the Sylow $p$-subgroup of $\Gamma^o$ and $r_G$ is the rank of $(G_{\text{ant}})_{\text{aff}}$. As in the proof of Theorem 1.9, we also have $[\Gamma : \Gamma^o] \leq \ell_X$ for some constant $\ell_X$ depending only on $X$. Similarly, there is a normal abelian $p'$-subgroup $A$ of $\Gamma$ of index

$$[\Gamma : A] = [\Gamma : \Gamma^o] \cdot [\Gamma^o : A] \leq \ell_X \left( J'(G) \cdot |\Gamma^o_{(p)}|^{3(r_G + 1)} \right)^{\ell_X} \leq \ell_X J'(G)^{\ell_X} \cdot |\Gamma_{(p)}|^{3(r_G + 1)\ell_X}.$$

The corollary follows by letting $J'_X := \ell_X J'(G)^{\ell_X}$ and $e_X = 3(r_G + 1)\ell_X$. \hfill \Box

**Remark 3.2.** It is known that if an algebraic torus $T$ acting faithfully on an algebraic variety $X$, then $T$ acts generically freely on $X$ (cf. [Dem70, §1.6, Corollaire 1]). This yields that

$$r_G = \text{rank}(G_{\text{ant}})_{\text{aff}} \leq \text{rank } G_{\text{aff}} \leq \dim X.$$

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**Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver, BC V6T 1Z2, Canada**

**Pacific Institute for the Mathematical Sciences, 2207 Main Mall, Vancouver, BC V6T 1Z4, Canada**

E-mail address: fhu@math.ubc.ca