Formation of singularities for the relativistic Euler equations

Nikolaos Athanasiou∗
Nikolaos.Athanasiou@maths.ox.ac.uk

Shengguo Zhu†
Shengguo.Zhu@maths.ox.ac.uk

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Abstract
This paper contributes to the study of large data problems for $C^1$ solutions of the relativistic Euler equations. In the $(1+1)$-dimensional spacetime setting, if the initial data are away from vacuum, a key difficulty in proving the global well-posedness or finite time blow-up is coming up with a way to obtain sharp enough control on the lower bound of the mass-energy density function $\rho$. First, for $C^1$ solutions of the 1-dimensional classical isentropic compressible Euler equations in the Eulerian setting, we show a novel idea of obtaining a mass density time-dependent lower bound by studying the difference of the two Riemann invariants, along with certain weighted gradients of them. Furthermore, using an elaborate argument on a certain ODE inequality and introducing some key artificial (new) quantities, we apply this idea to obtain the lower bound estimate for the mass-energy density of the $(1+1)$-dimensional relativistic Euler equations. Ultimately, for $C^1$ solutions with uniformly positive initial mass-energy density of the $(1+1)$-dimensional relativistic Euler equations, we give a necessary and sufficient condition for the formation of singularity in finite time, which gives a complete picture for the $(C^1)$ large data problem in dimension $(1+1)$. Moreover, for the $(3+1)$-dimensional relativistic fluids, under the assumption that the initial mass-energy density vanishes in some open domain, we give two sufficient conditions for $C^1$ solutions to blow up in finite time, no matter how small the initial data are. We also do some interesting studies on the asymptotic behavior of the relativistic velocity, which tells us that one can not obtain any global regular solution whose $L^\infty$ norm of $u$ decays to zero as time $t$ goes to infinity.

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1 Introduction

This paper is devoted to the Cauchy problem of the relativistic Euler equations (henceforth denoted by $RE$) with large data. In fluid mechanics and astrophysics, the relativistic Euler equations are a generalization of the Euler equations that account for the effects of special relativity. On a fixed $(d+1)$-dimensional Minkowski background, they are given by

$$\begin{cases}
\left(\frac{\rho + P|u|^2/c^4}{1 - |u|^2/c^2}\right)_t + \text{div}\left(\frac{\rho + P/c^2}{1 - |u|^2/c^2}u\right) = 0, \\
\left(\frac{\rho + P/c^2}{1 - |u|^2/c^2}\right)_t + \text{div}\left(\frac{\rho + P/c^2}{1 - |u|^2/c^2} u \otimes u\right) + \nabla P = 0.
\end{cases}
$$

(1.1)

Here and throughout, $\rho \geq 0$ denotes the mass-energy density, $u = (u^{(1)},...,u^{(d)})^\top \in \mathbb{R}^d$ denotes the relativistic velocity, $d \geq 1$ the dimension of the space, $c > 0$ a large constant corresponding to the speed of light, $P$ the pressure of the fluid, $x = (x^{(1)},...,x^{(d)})^\top \in \mathbb{R}^d$ the Eulerian spatial coordinate and finally $t \in \mathbb{R}_{\geq 0}$ denotes the time coordinate. The constitutive relation $P = P(\rho)$ considered throughout most of this paper is

$$P(\rho) = k^2 \rho^\gamma,$$

(1.2)

for fixed constants $k > 0$ and $\gamma \geq 1$.

∗Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX26GG, Oxford, United Kingdom. E-mail: Nikolaos.Athanasiou@maths.ox.ac.uk

†Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX26GG, Oxford, United Kingdom. E-mail: Shengguo.Zhu@maths.ox.ac.uk
Abiding by the fact that the theory of special relativity, in a regime of low velocities, should reduce to the classical Newtonian theory, the system (1.1) formally reduces to the classical $d$-dimensional isentropic compressible Euler equations (CE) when $c$ approaches infinity:

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = 0.
\end{cases}
\]  

(1.3)

It is well-known that, for nonlinear hyperbolic conservation laws, a singularity can form in finite time from initial compression no matter how small or smooth the data are. Classical results including Liu [24], Li-Zhou-Kong [21] et al confirm that when the initial data are small smooth perturbations of constant states, a blowup in the gradient of the solutions occurs in finite time if and only if the initial data contain any compression in a genuinely nonlinear characteristic field.

A natural follow-up question is whether this dichotomy persists, at least for archetypal systems of conservation laws such as the $(1+1)$-dimensional relativistic Euler equations (1.1) (resp. the classical compressible Euler equations (1.3)), when one passes to the framework of large data problems. It turns out, as we shall promptly explain in this work, that one of the key issues towards establishing a similar dichotomy for large data is finding an effective way to obtain sharp enough control on the lower bound of the mass-energy density (resp. the mass density).

For the $1$-dimensional classical compressible Euler equations (1.3), some important progress has been achieved for large data problems. When $\gamma \geq 3$ in the pressure law (1.2), the argument in Lax [17] for general $2 \times 2$ symmetric hyperbolic systems can be applied as is to the large data problem for the isentropic Euler equations (see Section 4 of the current paper for CE)\footnote{See also a generalization to full Euler equations by Chen-Young-Zhang [4].}. What renders this possible is that, when $\gamma \geq 3$, a lower bound control for the density is not needed. Therefore, the essence of the problem is to establish the finite time singularity formation for the compressible Euler equations in the most physically relevant case $1 < \gamma < 3$. For piecewise Lipschitz continuous solutions, in an interesting paper, Lin [23] argues that the density has a (sharp) $O(1 + t)^{-1}$ lower bound and proceeds to infer the corresponding global well-posedness. However satisfying, Lin’s result comes with an important caveat: It only applies to initial data that are purely rarefactive, i.e. devoid of any compression. For general $C^1$ solutions including compressions in the solution, further novelties were required. In a recent paper [1], Chen-Pan-Zhu find an $O(1 + t)^{-\sigma}$ lower bound when $1 < \gamma < 3$ and when the data is uniformly away from vacuum. This result helps them to prove that gradient blowup of $\rho$ and/or $u$ happens in finite time if and only if the initial data are forward or backward compressive somewhere, thus establishing the same dichotomy observed for small initial data [21, 24]. Some further developments were achieved in [2] where, for general Lipschitz continuous solutions of (1.3) with $1 < \gamma < 3$, the authors improve the lower bound on the density from $O((1 + t)^{-(\gamma-1)/2})$ in [1] to the optimal order of $O((1 + t)^{-1})$. Finally, in Chen-Chen-Zhu [3] the authors provide a new method to extend the theory to more general initial density profiles including possible far field vacuum, such as when $\rho(0, x) \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$. Also, the authors provide the first global continuous non-isentropic solution including weak compression, using a new method involving solving an inverse Goursat problem. The solution they obtain is almost classical, except on two characteristics, along which the solution has a weak discontinuity (continuous but non-smooth). There is a large literature of works on sufficient conditions for formation of singularities in solutions to the compressible Euler equations and systems of hyperbolic conservation laws in multiple space dimensions. See a brief list in [7, 8, 9, 22, 25, 27, 34, 36].

In contrast, for the $(1+1)$-dimensional relativistic Euler equations, studies on the singularity formation and global well-posedness of solutions with large data are very few. Under the assumption that $P = \sigma^2 \rho$ for some positive constant $\sigma < c$, Smoller-Temple establish in their seminal work [37] the existence of a global BV weak solution to the Cauchy problem, by crucially noticing that the shock curves for (1.1) in one dimension satisfy very strong geometric properties. For a general $\gamma$-law $P(\rho)$, Chen [5] studies the corresponding Riemann problem. Later, Hsu-Lin-Markino [11] establish the existence of global $L^\infty$ weak solutions with initial data containing the vacuum state. For smooth solutions, Ruan-Zhu [35] first prove the global well-posedness of $C^1$ solutions with large data to the Cauchy problem for the $(1+1)$-dimensional
relativistic Euler equations if the initial data do not have compression (see Definition 2). For the (d+1)-
dimensional (d ≥ 1) relativistic fluids, Pan-Smoller [33] introduce two sufficient conditions for the formation
of singularities of smooth solutions: the initial data are compactly supported, or the radial component of
the initial generalized momentum is sufficiently large. However, for the sufficient and necessary condition
for singularity formation in $C^1$ solutions with large data, according to our discussion in Section 5, due to
lack of the lower bound estimate for the mass-energy density, the problem has remained hitherto unexplored.
The current paper addresses this problem for the system (1.1) in the (1 + 1)-dimensional spacetime setting,
which is a solid step in the study of relativistic Euler equations. Moreover, for the (3 + 1)-dimensional
relativistic fluids, under the assumption that the initial mass-energy density vanishes in some open domain,
we give two sufficient conditions for the regular solution to blow up in finite time, no matter how small the
initial data are. Compared with the ones in [33], we have removed the crucial assumptions that the initial
data are compactly supported via an interesting observation for the multi-dimensional relativistic fluids: the
invariance of the mass-energy density’s centroid. We also do some interesting studies on the asymptotic
behavior of the relativistic velocity, which tells us that one can not obtain a global regular solution whose
$L^\infty$ norm of $u$ decays to zero as time $t$ goes to infinity. Some interesting works on the study of the smooth
solutions with vacuum for multi-dimensional relativistic fluids can be found in [12, 13, 14, 28, 29, 30, 31, 32].

Our paper is divided into 8 sections. In Section 2, we introduce some basic notations and equations.
In Section 3, we state the main results. In Section 4, for the smooth solutions with large data of the
1-dimensional (classical) isentropic Euler equations in Eulerian coordinates, we present our novel idea for
obtaining a time-dependent mass density lower bound. The idea we present involves a careful study of the
difference of the two Riemann invariants (and the study of certain weighted gradients of them). Naturally,
why a new idea is needed in the first place is something that requires explanation. The reason is that in this
problem, unlike the classical Euler equations, the introduction of Lagrangian coordinates cannot demystify
the mathematical structure of the system under study in the same, efficient way it did in [1, 2, 3]. In other
words, upon a thorough read of [1, 2, 3], one can see that the relation

$$
\frac{(1/\rho)_t}{2} = \frac{r_x + s_x}{2},
$$

where $r$ and $s$ are the Riemann invariants of the so-called $p$-system in the Lagrange coordinate setting, is
the cornerstone of the argument for obtaining the desired lower bound estimate. In the Eulerian setting this
relation, or others of similar simplicity, are unavailable. Indeed, here one has

$$(1/\rho)_t = -u(1/\rho)_x + (1/\rho)u_x,$$

and therefore in order to get the mass density lower bound, i.e., $(1/\rho)'s$ upper bound, we should first have
the upper bound of $-u(1/\rho)_x + (1/\rho)u_x$, which seems hard to obtain. In Section 5, combining an elaborate
argument on a particular ODE inequality and introducing the crucial artificial quantity

$$
\left( \frac{k \rho^{(\gamma-1)/2}/c}{\sqrt{1 + k^2 \rho^{\gamma-1}/c^2}} \right) \frac{\gamma^2 - 2}{\gamma^2 - 4} \left( 1 + \frac{k^2 \rho^{\gamma-1}/c^2}{\gamma^2 - 4} \right)^{\gamma-1} := Y,
$$

we apply our idea to get a lower bound estimate for the mass-energy density of the (1+1)-dimensional
relativistic Euler equations. Ultimately, for $C^1$ solutions with large data and uniformly positive initial
mass-energy density of the (1+1)-dimensional relativistic Euler equations, we give a necessary and sufficient
condition for the formation of singularities in finite time. In Section 6, we shift attention to (3+1)-dimensional
relativistic fluids. Via introducing the particle number $n(\rho)$, we give a clear description on the time evolution
of the vacuum boundary. Then we give the proof for the corresponding two sufficient conditions of singularity
formation. In Section 7, we provide an extension of the 1-dimensional singularity formation results to more
general pressure laws. Finally, we include an appendix to show the related local-in-time well-posedness in
multi-dimensional spacetime that is used in our paper.
2 Basic setup

Before introducing the main results of this paper, we provide some equations and estimates for $C^1$ solutions of (1.1)-(1.2) or (1.3) with (1.2), together with initial data

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{for} \quad x \in \mathbb{R}^d,$$  

(2.1)

for future reference, where $d = 1$ or 3.

2.1 (1+1)-dimensional Relativistic Euler equations

Let $d = 1$ in (1.1) and (2.1). We first define the $C^1$ solutions as follows:

**Definition 1** Let $T > 0$ be some time. The pair $(\rho(t,x), u(t,x))$ is called a $C^1$ solution to the relativistic Euler equations (1.1)-(1.2) on $(0, T) \times \mathbb{R}$ if

$\rho > 0, \quad \rho \in C^1((0, T) \times \mathbb{R}), \quad u \in C^1((0, T) \times \mathbb{R}),$

and the equations (1.1)-(1.2) are satisfied in the pointwise sense on $(0, T) \times \mathbb{R}$. It is called a $C^1$ solution to the Cauchy problem (1.1)-(1.2) with (2.1) if it is a $C^1$ solution to the equations (1.1)-(1.2) on $(0, T) \times \mathbb{R}$ and admits the initial data (2.1) continuously.

It is well-known\(^2\) that there exists a local-in-time $C^1$ solution $(\rho, u)$ in $[0, T] \times \mathbb{R}$ for some $T > 0$, when

$$\inf_{x \in \mathbb{R}} \rho_0 > 0, \quad (\rho_0, u_0) \in C^1(\mathbb{R}).$$

(2.2)

We proceed with a rudimentary analysis of the Cauchy problem (1.1)-(1.2) with (2.1). First, the two eigenvalues $\lambda_1$ and $\lambda_2$ of equations (1.1)-(1.2) can be given by

$$\lambda_1 = \frac{u - \sqrt{P'}}{1 - \frac{u}{c^2} \sqrt{P'}} \quad \text{and} \quad \lambda_2 = \frac{u + \sqrt{P'}}{1 + \frac{u}{c^2} \sqrt{P'}}.$$  

(2.3)

We denote the directional derivatives as

$$t = \partial_t + \lambda_1 \partial_x, \quad \nu = \partial_t + \lambda_2 \partial_x$$

(2.4)

along two characteristic directions

$$\frac{dx^1}{dt} = \lambda_1 \quad \text{and} \quad \frac{dx^2}{dt} = \lambda_2,$$  

(2.5)

respectively and introduce the corresponding Riemann variables

$$w = \frac{c}{2} \ln \left( \frac{c + u}{c - u} \right) + \int_0^{\rho} \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} d\sigma,$$

(2.6)

$$z = \frac{c}{2} \ln \left( \frac{c + u}{c - u} \right) - \int_0^{\rho} \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} d\sigma.$$  

(2.7)

Then, it is easy to see that $w$ and $z$ satisfy

$$w' = 0 \quad \text{and} \quad z' = 0.$$  

(2.8)

Let $h_1$ and $h_2$ be functions satisfying

$$h_{1w} = \frac{\lambda_{1w}}{\lambda_1 - \lambda_2}, \quad h_{2z} = \frac{\lambda_{2z}}{\lambda_2 - \lambda_1}.$$  

(2.9)

\(^2\)See [35] and the references cited therein.
Define $\alpha = z_x, \beta = w_x$ and introduce
\[ \xi = e^{b_1 \alpha}, \quad \zeta = e^{b_2 \beta}. \]

We continue with a simplification for $s(\rho)$, as can be found for example in [5],
\[ s(\rho) = 2c\sqrt{\gamma} \arctan\left( \frac{k\rho^{(\gamma-1)/2}}{c} \right). \]

Another important calculation is the following expression for $\sqrt{P'(\rho)}$ in terms of the Riemann invariants:
\[ \sqrt{P'(\rho)} = k\sqrt{\gamma\rho^{(\gamma-1)/2}} = c\sqrt{\gamma} \tan\left( \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} \right). \]

Finally, we define the compression and rarefaction characters.

**Definition 2** The local R/C character for a classical solution of (1.1)-(1.2) with (2.1) is

- Forward $R$ iff $w_x > 0$;  Forward $C$ iff $w_x < 0$;
- Backward $R$ iff $z_x > 0$;  Backward $C$ iff $z_x < 0$.

Although this definition was not clearly provided in Lax [17], his result on some cases of $2 \times 2$ hyperbolic conservation laws can be explained as follows: a singularity forms in finite time if and only if there exists some backward or forward compression under Definition 2. According to the results obtained in this paper, we see that this definition of compression and rarefaction gives a clean cut on the singularity formation.

### 2.2 1-dimensional classical compressible Euler equations

Let $d = 1$ in (1.3) and (2.1). To make the corresponding statement precise, we first define the $C^1$ solutions as follows:

**Definition 3** Let $T > 0$ be some time. The pair $(\rho(t, x), u(t, x))$ is called a $C^1$ solution to the non-relativistic Euler equations (1.3) with (1.2) on $(0, T) \times \mathbb{R}$ if

- $\rho > 0, \quad \rho \in C^1([0, T) \times \mathbb{R}), \quad u \in C^1([0, T) \times \mathbb{R}),$

and the equations (1.3) with (1.2) are satisfied pointwise on $(0, T) \times \mathbb{R}$. It is called a $C^1$ solution to the Cauchy problem (1.3) with (1.2) and (2.1) if it is a $C^1$ solution to the equations (1.3) with (1.2) on $(0, T) \times \mathbb{R}$ and admits the initial data (2.1) continuously.

As in the relativistic case, it is well-known that there exists a local-in-time $C^1$ solution $(\rho, u)$ in $[0, T] \times \mathbb{R}$ for some $T > 0$, when (2.2) is satisfied. We proceed with a rudimentary analysis of the Cauchy problem (1.3) with (1.2) and (2.1). First, the two eigenvalues $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ of equations (1.3) with (1.2) can be given by
\[ \tilde{\lambda}_1 = u - \sqrt{P'}, \quad \tilde{\lambda}_2 = u + \sqrt{P'}. \]

We denote the directional derivatives as
\[ \partial_- = \partial_t + \tilde{\lambda}_1 \partial_x, \quad \partial_+ = \partial_t + \tilde{\lambda}_2 \partial_x \]

along two characteristic directions
\[ \frac{dy^1}{dt} = \tilde{\lambda}_1 \quad \text{and} \quad \frac{dy^2}{dt} = \tilde{\lambda}_2, \]

respectively, and introduce the corresponding Riemann variables
\[ \tilde{\omega} = u + \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma} \, d\sigma, \quad \tilde{z} = u - \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma} \, d\sigma. \]
Then, it is easy to see that $\tilde{w}$ and $\tilde{z}$ satisfy

$$\partial_+ \tilde{w} = 0 \quad \text{and} \quad \partial_- \tilde{z} = 0.$$  \hfill (2.17)

Let $\tilde{h}_1, \tilde{h}_2$ be functions satisfying

$$\tilde{h}_1 \tilde{w} = \tilde{\lambda}_1 \tilde{w}, \quad \tilde{h}_2 \tilde{z} = \tilde{\lambda}_2 \tilde{z}.$$  \hfill (2.18)

Define $\tilde{\alpha} = \tilde{z}_x$, $\tilde{\beta} = \tilde{w}_x$ and introduce

$$\phi = e^{\tilde{h}_1 \tilde{\alpha}}, \quad \psi = e^{\tilde{h}_2 \tilde{\beta}}.$$  \hfill (2.19)

Finally, we define the compression and rarefaction characters.

**Definition 4** The local R/C character for a classical solution of $(1.3)$ with $(1.2)$ and $(2.1)$ is

- Forward R iff $\tilde{w}_x > 0$;
- Forward C iff $\tilde{w}_x < 0$;
- Backward R iff $\tilde{z}_x > 0$;
- Backward C iff $\tilde{z}_x < 0$.

According to the results obtained [1, 2, 3], we know that this definition on the compression and rarefaction gives a clean cut on the singularity formation.

### 2.3 (3+1)-dimensional Relativistic Euler equations with vacuum

Let $d = 3$ in $(1.1)$ and $(2.1)$. To make the corresponding statement precise, we give the definition of regular solutions considered in this paper:

**Definition 5** Let $T > 0$ be some time. The pair $(\rho(t,x), u(t,x))$ is called a regular solution in $[0, T] \times \mathbb{R}^3$ to the Cauchy problem $(1.1)$-$(1.2)$ with $(2.1)$ if $(\rho, u)$ satisfies this problem in the sense of distributions and:

- $(A)$ $\rho \geq 0$, $(\rho, u) \in C^1([0, T] \times \mathbb{R}^3)$;
- $(B)$ $|u| < c$, $\sqrt{P'(\rho)} < c$;
- $(C)$ $u_t + u \cdot \nabla u = 0$ whenever $\rho(t,x) = 0$.

Let $x(t;x_0)$ be the particle path starting from $x_0$ at $t = 0$, i.e.,

$$\frac{d}{dt} x(t;x_0) = u(t, x(t;x_0)), \quad x(0;x_0) = x_0.$$  \hfill (2.20)

In the rest of this section, we will use the following useful physical quantities in some domain $\Omega \subset \mathbb{R}^3$:

- $m(t) = \int_\Omega \hat{\rho}(t,x) dx$ \hspace{1cm} (total energy),
- $X^*(t) = \frac{\int_\Omega \hat{x} \rho dx}{m(t)}$ \hspace{1cm} (centroid of $\Omega$),
- $M(t) = \int_\Omega \hat{\rho}(t,x)|x|^2 dx$ \hspace{1cm} (second moment),
- $F(t) = \int_\Omega \hat{\rho}(t,x) u(t,x) \cdot x dx$ \hspace{1cm} (radial component of momentum),
- $P(t) = \int_\Omega \hat{\rho}(t,x) u(t,x) dx$ \hspace{1cm} (momentum),

where $\hat{\rho}$ and $\hat{\rho}$ are given by

$$\hat{\rho} = \frac{\rho + P|u|^2/c^4}{1 - |u|^2/c^2} = \frac{1}{c^2} \hat{\rho}|u|^2 + \rho \quad \text{and} \quad \hat{\rho} = \frac{(\rho + P/c^2)}{1 - |u|^2/c^2} = \hat{\rho} + \frac{P}{c^2}.$$

Based on the physical quantities introduced in this subsection, we define one solution class as follows:
Definition 6 Let \( T > 0 \) be some time and \( \Omega = \mathbb{R}^3 \) in the definitions of \( m(t) \) and \( P(t) \). The pair \((\rho(t,x), u(t,x))\) is said to be in the solution class \( D(T) \) of the Cauchy problem (1.1)-(1.2) with (2.1) if \((\rho, u)\) satisfies this problem in the sense of distributions and

\[
(A) \quad \rho \geq 0, \quad (\rho, u) \in C^1([0,T] \times \mathbb{R}^3);
\]

\[
(B) \quad |u| < c, \quad \sqrt{P'(\rho)} < c;
\]

\[
(C) \quad \text{Conservation of total energy:} \quad 0 < m(0) = m(t) < \infty \text{ for any } t \in [0,T];
\]

\[
(D) \quad \text{Conservation of momentum:} \quad 0 < |P(0)| = |P(t)| < +\infty \text{ for any } t \in [0,T].
\]

The corresponding local-in-time well-posedness of smooth solutions defined in Definitions 5 and 6 has been established by Lefloch-Ukai [19] (see also our Appendix).

3 Statement of main results

In this section, we will state our main results in the following two subsections.

3.1 (1+1)-dimensional case

Let \( d = 1 \) in (1.1) and (2.1). From now on, we make the following assumption throughout the rest of this paper:

Assumption 1 Assume that\(^3\)

\[
\inf_{x \in \mathbb{R}} (w_0 - z_0) > 0, \quad \sqrt{P'(\rho)} \left( J_{rel}^{-1} \left( \frac{w_{\text{max}} - z_{\text{min}}}{2} \right) \right) \leq c,
\]

\[(w_0, z_0) \in C^1(\mathbb{R}), \quad \| (w_0, z_0) \|_{C^1(\mathbb{R})} \leq M_0,\]

for some constant \( M_0 > 0 \), where

\[
(w_0, z_0)(x) = (w(0, x), z(0, x)), \quad J_{rel}(x) = \int_0^x \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c}} ds,
\]

\[w_{\text{max}} = \sup\{ w_0(x) \mid x \in \mathbb{R} \}, \quad z_{\text{min}} = \inf\{ z_0(x) \mid x \in \mathbb{R} \},\]

and \( J_{rel}^{-1} \) denotes the inverse function of \( J_{rel}(x) \).

Theorem 3.1 For polytropic gas \( \gamma > 1 \) in (1.2), if \((w_0(x), z_0(x))\) satisfy the conditions in Assumption 1, then the Cauchy problem (1.1)-(1.2) with (2.1) has a unique global-in-time \( C^1 \) solution if and only if

\[w_z(x, 0) \geq 0 \quad \text{and} \quad z_x(x, 0) \geq 0, \quad \text{for all} \ x \in \mathbb{R}. \tag{3.1}\]

For more general pressure laws \( P(\rho) \), first we give the following two assumptions:

Assumption 2

\[P(\rho) > 0, \quad P'(\rho) > 0, \quad P''(\rho) > 0, \]

\[P(0) = 0, \quad \lim_{\rho \to +\infty} \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c}} d\sigma < \infty. \tag{3.2}\]

\(^3\)For the polytropic gas, the condition \( \sqrt{P'(\rho)} \left( J_{rel}^{-1} \left( \frac{w_{\text{max}} - z_{\text{min}}}{2} \right) \right) \leq c \) can be read as

\[w_{\text{max}} - z_{\text{min}} < \frac{4c}{\gamma - 1} \arctan \left( \frac{1}{\sqrt{\gamma}} \right).\]
Assumption 3 There exists a positive constant $A$ such that, for all $\rho > 0$, there holds
\[
\rho^8 \left( (5 + A)P''(\rho)^2 - 4P'(\rho)P'''(\rho) \right) + (4A - 4) \left( \rho^8 P'(\rho)^2 + \rho^7 P'(\rho)P''(\rho) \right) \geq 0. \tag{3.3}
\]

Now we can state the following theorem:

Theorem 3.2 For a general pressure law $P(\rho)$ satisfying Assumptions 2-3, if $(w_0(x), z_0(x))$ satisfy the conditions in Assumption 1, then the Cauchy problem $(1.1)$-$(1.2)$ with $(2.1)$ has a unique global-in-time $C^1$ solution if and only if
\[
w_x(x,0) \geq 0 \quad \text{and} \quad z_x(x,0) \geq 0, \quad \forall \quad x \in \mathbb{R}. \tag{3.4}
\]
Here, the corresponding definitions of $(w, z)$ under a general pressure law can be found in Section 7.2.

The proof of Theorems 3.1-3.2 can be found in Sections 5 and 7 respectively. We make some necessary remarks on our conclusions at this point:

Remark 1 The assumption $\sqrt{P'(\rho)} \left( \frac{\rho}{\rho_{\text{rel}}} \right) < c$ is imposed for one to be able to show that the local sound speed $\sqrt{P'(\rho)}$ is bounded away from the light speed $c$ for $C^1$ solutions, which is natural in the sense of physics.

Remark 2 Our conclusion establishes a necessary and sufficient condition for the formation of singularities for the $(1 + 1)$-dimensional relativistic Euler equations, namely the existence of compression in the initial data, which gives a complete picture on the formation of singularities for the Cauchy problem of the system $(1.1)$ in $(1 + 1)$-dimensional Minkowski spacetime.

Remark 3 A question of physical significance is to determine the type of singularity that forms. It is generally expected that the discontinuities developed, among others, in the works of Chen-Pan-Zhu [1], Lax [17] and the current work are indeed discontinuity singularities, i.e. shock waves. The proof of disproof of such a fact, in full generality, remains an open problem. The best partial results known to the authors, however can be found in Kong [15]. There, for a general class of strictly hyperbolic $2 \times 2$ systems with two genuinely nonlinear characteristic fields, it is shown that if a singularity forms then it develops as a shock wave if either

- one of the two Riemann invariants, $w$ or $z$, is initially a constant\(^4\), or
- certain a priori conditions, essentially quantitative bounds on the size of the derivatives, hold at the blow-up point; conditions which are, however, difficult to verify.

Thus, our theorem above along with Kong’s result readily implies that, if one of the two initial data variables $w_0, z_0$ is a constant, a shock forms and if and only if there exists initial compression in the non-constant Riemann invariant variable. In any case, further study on the type of singularities obtained in this work and several others promises to be a meaningful and interesting direction for research.

3.2 $(3+1)$-dimensional case

Let $d = 3$ in $(1.1)$ and $(2.1)$. We present two scenarios for singularity development in finite time from initial data with vacuum in some local domain. The first one is the isolated mass group:

Definition 7 (Isolated mass group) The initial data pair $(\rho_0(x), u_0(x))$ is said to have an isolated mass group $(A_0, B_0)$ if there exist two smooth, bounded and connected open sets $A_0 \subset \mathbb{R}^3$ and $B_0 \subset \mathbb{R}^3$ satisfying
\[
\begin{cases}
A_0 \subset B_0 \subset B_{B_0} \subset \mathbb{R}^3, \\
\rho_0(x) = 0, \quad \forall \quad x \in B_0 \setminus A_0, \\
u_0(x)|_{\partial A_0} = \pi_0, \\
\int_{A_0} \rho_0(x) dx > 0, \\
\int_{A_0} \bar{\rho}(0,x) u_0 dx = 0.
\end{cases} \tag{3.5}
\]

\(^4\)Thus the study of the system reduces to the study of a scalar conservation law, which in general has a complete theory. See also Lebaud [18] for the 1-D classical Euler equations.
for some positive constant $R_0$ and constant vector $\overline{r}_0 \in \mathbb{R}^3$, where $B_{R_0}$ is the ball centered at the origin with radius $R_0$.

Our first blowup result shows that the existence of an isolated mass group in the initial data guarantees the finite time singularity formation of regular solutions.

**Theorem 3.3 (Blow-up by isolated mass group)** If the initial data $(\rho_0, u_0)(x)$ have an isolated mass group $(\mathcal{A}_0, \mathcal{B}_0)$, then the regular solution $(\rho, u)(t, x)$ in $[0, T_m) \times \mathbb{R}^3$ defined in Definition 5 with maximal existence time $T_m$ to the Cauchy problem (1.1)-(1.2) and (2.1) blows up in finite time, i.e., $T_m < +\infty$.

For the second scenario, we explore the hyperbolic structure for the system in a vacuum region. For this purpose, we introduce the following concept:

**Definition 8 (Hyperbolic singularity set)** We define the smooth, open set $V \subset \Omega$ as a hyperbolic singularity set, if $V$ and $(\rho_0, u_0)$ satisfy

\[
\begin{align}
\rho_0(x) &= 0, \quad \forall \ x \in V; \\
Sp(\nabla u_0) \cap \mathbb{R}^- &\neq \emptyset, \quad \forall \ x \in V,
\end{align}
\]

where we denote by $Sp(\nabla u_0(x))$ the spectrum of the Jacobian matrix of $u_0$.

Then we show the following:

**Theorem 3.4 (Blow-up by the hyperbolic singularity set)** If the initial data $(\rho_0, u_0)(x)$ have a hyperbolic singularity set $V$, then the regular solution $(\rho, u)(t, x)$ in $[0, T_m) \times \mathbb{R}^3$ defined in Definition 5 with maximal existence time $T_m$ to the Cauchy problem (1.1)-(1.2) and (2.1) blows up in finite time, i.e., $T_m < +\infty$.

A natural question to ask is whether the local solution in [19] can be extended globally in time if we can identify some initial data which avoid the above two blow-up mechanisms and what the large time behavior is. In the following theorem, we give a very interesting observation for the solution’s asymptotic behavior.

**Theorem 3.5** Let $\gamma > 1$ in (1.2). For the Cauchy problem (1.1)-(1.2) with (2.1), there is no solution $(\rho, u) \in D(\infty)$ satisfying

\[
\limsup_{t \to +\infty} ||u(t, x)||_{L^\infty(\mathbb{R}^3)} = 0.
\]

## 4 The mass density lower bound of 1-dimensional CE

Let $d = 1$ in (1.3) and (2.1). Recall at this point the notation introduced in Subsection 2.2. We dedicate this section to the presentation of the new approach for obtaining the crucial lower bound estimate on the mass density for the classical compressible Euler equations (1.3). The main idea is that, instead of obtaining a transport equation for $\rho$ and using it to obtain the estimate, we focus instead on the difference of the two Riemann invariants, in the classical case given by $\tilde{w} - \tilde{z}$. The function $\tilde{w} - \tilde{z}$ is an increasing function of $\rho$ and therefore control on $\tilde{w} - \tilde{z}$ translates to control on $\rho$. To best exhibit our approach, we apply it in the first subsection to the classical Euler equations. In the final subsection, we lay down the main argument for singularity formation in finite time, which is essentially that of [17]. We explain then why a lower bound estimate is of such importance; and how we may use to conclude our argument.

### 4.1 The mass density lower bound estimate in the Eulerian setting

We begin by noticing that the weighted gradients satisfy certain Riccati equations.

**Lemma 4.1** For the $C^1$ solution of the system (1.3), the following Riccati ODEs hold:

\[
\partial_- \phi = -\left(e^{-\frac{1}{2} \lambda_{17}}\right) \phi^2, \quad \partial_+ \psi = -\left(e^{-\frac{1}{2} \lambda_{26}}\right) \psi^2.
\]
Moreover, let $y^i(t, y^i_0)$ $(i = 1, 2)$ be two characteristic curves (defined in Section 2.2) starting from $(0, y^i_0)$. One has
\[
\frac{1}{\phi(t, y^i(t, y^i_0))} = \frac{1}{\phi(0, y^i_0)} + \int_0^t \left( e^{-\tilde{h}_1 \tilde{\lambda}_1} \right) (\sigma, y^i(\sigma, y^i_0)) \, d\sigma.
\]
\[
\frac{1}{\psi(t, y^2(t, y^2_0))} = \frac{1}{\psi(0, y^2_0)} + \int_0^t \left( e^{-\tilde{h}_2 \tilde{\lambda}_2} \right) (\sigma, y^2(\sigma, y^2_0)) \, d\sigma.
\]

Proof. Differentiating the last equation of (2.17) with respect to $x$, and recalling the definition of $\tilde{h}_1$ from (2.18), we arrive at
\[
\partial_\lambda \tilde{\alpha} + (\partial_\lambda \tilde{h}_1) \tilde{\alpha} + \tilde{\lambda}_1 \tilde{\alpha}^2 = 0,
\]
which, along with $\phi = e^{\tilde{h}_1 \tilde{\alpha}}$, implies the desired ODE on $\tilde{\phi}$. The formula of $\phi$ along the backward characteristic curve can be obtained by solving the Riccati ODE that we obtained. The proof for $\psi$ is similar, and here we omit the details.

Our strategy will be to work towards obtaining a time-dependent lower bound on $\bar{w} - \bar{z}$. To achieve this, we must first rewrite $\tilde{\lambda}_1, \tilde{\lambda}_2$ in terms of $\bar{w}, \bar{z}$:
\[
\begin{cases}
\tilde{\lambda}_1 = \frac{\bar{w} + \bar{z}}{2} - (\bar{w} - \bar{z})(\gamma - 1), \\
\tilde{\lambda}_2 = \frac{\bar{w} + \bar{z}}{2} + (\bar{w} - \bar{z})(\gamma - 1).
\end{cases}
\]
(4.2)

One can get that $\tilde{\lambda}_1 = \tilde{\lambda}_2 = \tilde{h} = \frac{\gamma - 3}{2\gamma - 2} \ln(\bar{w} - \bar{z})$, (4.3)

which implies that
\[
e^{-\tilde{h}_1 \tilde{\lambda}_1} = e^{-\tilde{h}_2 \tilde{\lambda}_2} = \frac{\gamma + 1}{4} (\bar{w} - \bar{z})^\frac{\gamma - 3}{2\gamma - 2}.
\]
(4.4)

Based on these observations and Lemma 4.1, standard ODE theory then leads us to the following result:

**Proposition 4.1** Denote
\[
\tilde{Q}_1 = \max \left\{ 0, \sup_x \phi(0, x) \right\}, \quad \tilde{Q}_2 = \max \left\{ 0, \sup_x \psi(0, x) \right\}.
\]

For the $C^1$ solution of the system (1.3), there holds $\phi \leq \tilde{Q}_1$, $\psi \leq \tilde{Q}_2$.

Finally, we can get the desired lower bound estimates of the mass density.

**Lemma 4.2** Let $y^i(t, y^i_0)$ $(i = 1, 2)$ be two characteristic curves starting from $(0, y^i_0)$. There holds
\[
\left( e^{-\tilde{h}_1 \tilde{\lambda}_1} \right) (t, y^1(t, y^1_0)) \geq \frac{1}{C_1 + C_2 t}, \quad \left( e^{-\tilde{h}_2 \tilde{\lambda}_2} \right) (t, y^2(t, y^2_0)) \geq \frac{1}{C_1 + C_2 t},
\]
for positive constants $C_i$ $(i = 1, 2)$ independent of the time.

**Proof.** According to (2.17), and the definition of $\psi$, one can obtain that
\[
\partial_\lambda (\bar{w} - \bar{z}) = (\bar{w} - \bar{z})_t + \tilde{\lambda}_1 (\bar{w} - \bar{z})_x = (\tilde{\lambda}_1 - \tilde{\lambda}_2) \bar{w}_x = -\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{e^{\tilde{h}_2}} \psi.
\]
(4.6)

which, along with the Proposition 4.1 and the relation (4.4), implies that
\[
\partial_\lambda (\bar{w} - \bar{z}) \geq -\frac{\tilde{\lambda}_2 - \tilde{\lambda}_1}{e^{\tilde{h}_2}} \tilde{Q}_2 = -\frac{(\gamma - 1)\tilde{Q}_2}{2} (\bar{w} - \bar{z})^\frac{\gamma - 3}{2\gamma - 2}.
\]
(4.7)
Denoting $C' = \frac{(\gamma - 1)\tilde{Q}_2}{2}$, and integrating (4.7) along $y^1(t, y_0^1)$ over $[0, t]$, yields

\[(\tilde{w} - \tilde{z})(t, y^1(t, y_0^1)) \geq \left(\left((\tilde{w} - \tilde{z})(0, y_0^1)\right)^{\frac{\gamma - 2}{4\gamma} + \frac{3 - \gamma}{2\gamma}} + \frac{3 - \gamma}{2\gamma - 2}C't\right)^{\frac{2\gamma - 2}{4\gamma - 3}}. \tag{4.8}\]

In particular, taking (4.4) into account, one can obtain the time-dependent lower bound

\[\left(e^{-\tilde{h}_1\tilde{\lambda}_1}\right)(t, y^1(t, y_0^1)) \geq \frac{1}{C_1 + C_2t}, \tag{4.9}\]

for positive constants $C_i \ (i = 1, 2)$ independent of $t$. Similarly, we can obtain the second estimate. □

**Remark 4** Taking into account that $w - z = C\rho^{\frac{\gamma - 1}{2}}$ for some universal constant $C > 0$ and Lemma 4.2, we recover precisely the result of [1]:

\[
\rho(t, x) \geq \left(\frac{1}{C_1 + C_2t}\right)^{\frac{1}{1 - \gamma}}.
\]

### 4.2 Formation of singularity

It is important at this stage to highlight the main mechanism that shall be used throughout the paper to obtain the formation of singularities. The argument within this subsection can be traced back to P.D. Lax in his 1964 paper on $2 \times 2$ systems. This is best described in the context of system (1.3). The Riccati type equations in Lemma 4.1 are precisely what gives us a clear passage to study the singularity formation and/or global existence of classical solutions for hyperbolic systems with two unknowns.

Without loss of generality, we assume that there exists a point $(0, y_0^1)$ on the initial data line $t = 0$ such that $\phi(0, y_0^1) < 0$. Then we see that a sufficient condition for the breakdown of the classical solution is

\[
\int_{0}^{\infty} \left(e^{-\tilde{h}_1\tilde{\lambda}_1}\right)(t, y^1(t, y_0^1))dt = \infty, \tag{4.10}
\]

which, actually can be verified by the conclusions obtained in Lemma 4.2. The proof of our main theorem in the next section essentially comes down to establishing a statement of the form (4.10) for the system of $(1 + 1)$-dimensional relativistic Euler equations.

### 5 Formation of singularities for the (1+1)-dimensional RE

In this section we shall lay down the proof of Theorem 3.1. Let $d = 1$ in (1.1) and (2.1). As we mentioned in Subsection 2.1, it is a well-known result that given initial data as in Theorem 3.1, there exists a $T \in (0, \infty)$ such that there exists a local-in-time $C^1$ solution to the Cauchy problem (1.1) – (1.2) with (2.1). We proceed by obtaining estimates on the solution in the slab $[0, T] \times \mathbb{R}$.

#### 5.1 Preliminaries

Before giving the detailed proof, we first give several fundamental lemmata for the RE equations. First, we show that the relativistic fluid velocity $u$ is less than the light speed $c$.

**Lemma 5.1** For the $C^1$ solution of the Cauchy problem (1.1)-(1.2) with (2.1), under the Assumption 1, the absolute value $|u|$ of the velocity function is uniformly bounded away from the light speed $c$.

**Proof.** According to (2.17), one can obtain that

\[
|\ln \left(\frac{c + u}{c - u}\right)| = \left|\frac{w + z}{c}\right| \leq \frac{2M_0}{c}. \tag{5.1}
\]

That is to say,

\[e^{-\frac{2M_0}{c}} < \frac{c + u}{c - u} < e^{\frac{2M_0}{c}},\]

which implies that $|u|$ is uniformly bounded away from $c$. □
Second, we confirm that the mass-energy density will keep the positivity property.

**Lemma 5.2** For the $C^1$ solution of the Cauchy problem (1.1)-(1.2) with (2.1), under the Assumption 1, $\rho > 0$.

**Proof.** According to (2.12), one has

$$\rho = \left(\frac{c}{k} \tan \left(\frac{(w-z)(\gamma - 1)}{4c^\gamma}\right)\right) := F(w-z).$$

Notice that $F(0) = 0$. Denote $\theta = w - z$. We can then rewrite

$$\lambda_1 - \lambda_2 = \frac{2\sqrt{P'(\rho)(1-u^2/c^2)}}{1-u^2P'(\rho)/c^2} - \frac{2\sqrt{P'(\theta)(1-u^2/c^2)}}{1-u^2P'(\theta)/c^2} = g(\theta, u).$$

Notice then that $g(0, u) = 0$ and

$$-\frac{2\sqrt{P'(\theta)/c^2}}{1-u^2P'(\theta)/c^2} = \frac{\partial g(\theta, u)}{\partial u},$$

where $\theta_1$ is between 0 and $\theta$. Thus,

$$(w-z)_t + \lambda_2(w-z)_x = \frac{\partial g(\theta, u)}{\partial u} (w-z)_x.$$  

Let $x^2 = x^2(t, x^2)$ denote the the forward characteristic curve starting from the point $(0, x^2_0)$. Integrating along this forward characteristic over $[0, t]$, one can get

$$(w-z)(t, x) = (w_0 - z_0)(x^2_0) \exp \left( \int_0^t \frac{\partial g(\theta, u)}{\partial \theta}(\sigma, x^2(\sigma, x^2_0)) d\sigma \right),$$

which, along with Assumption 1, implies the desired conclusion. \hfill \Box

Next, we show that the local sound speed $\sqrt{P'(\rho)}$ is also less than the light speed $c$.

**Lemma 5.3** For the $C^1$ solution of the Cauchy problem (1.1)-(1.2) with (2.1), there holds $\sqrt{P'(\rho)} < c$.

**Proof.** According to (2.12), Assumption 1 and $w - z \leq w_{max} - z_{min}$, one gets

$$\sqrt{P'(\rho)} \leq c \sqrt{\gamma} \tan \left(\frac{(w_{max} - z_{min})(\gamma - 1)}{4c^\gamma}\right) < c,$$

which can be directly translated to an upper bound on the density $\rho$,

$$\rho < c^{\frac{2}{\gamma - 1}} k^{-\frac{2}{\gamma - 1}} 2^{-\frac{2}{\gamma - 1}}.$$

\hfill \Box

Lemmas 5.1-5.3 show us that both $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1 < \lambda_2$, which implies that the system (1.1)-(1.2) is strictly hyperbolic. Now we note down the Riccati equations satisfied by $\xi$ and $\zeta$ defined in Section 2.1.

**Lemma 5.4** For the $C^1$ solution of the Cauchy problem (1.1) – (1.2) with (2.1), under the Assumption 1, the following Riccati ODEs hold:

$$\xi' = - \left( e^{-h_1}\lambda_{1z} \right) \xi^2, \quad \zeta' = - \left( e^{-h_2}\lambda_{2z} \right) \zeta^2.$$  

Moreover, let $x^i(t, x^2_0)$ ($i = 1, 2$) be two characteristic curves starting from $(0, x^2_0)$. One has

$$\frac{1}{\xi(t, x^i(t, x^2_0))} = \frac{1}{\xi(0, x^2_0)} + \int_0^t \left( e^{-h_1}\lambda_{1z} \right)(\sigma, x^1(\sigma, x^2_0)) d\sigma.$$

$$\frac{1}{\zeta(t, x^2(t, x^2_0))} = \frac{1}{\zeta(0, x^2_0)} + \int_0^t \left( e^{-h_2}\lambda_{2z} \right)(\sigma, x^2(\sigma, x^2_0)) d\sigma.$$
The proof is identical to that of Lemma 4.1. We omit the details. In what follows, it turns out that there is a clear distinction in the proof between the cases $\gamma \geq 3$ and the physical range $1 \leq \gamma < 3$. Because of that, we will lay down the proof for each of those two cases in separate subsections.

5.2 Proof for the case $\gamma \geq 3$ of Theorem 3.1

Instead of working with (1.1)-(1.2), here and throughout we will rephrase the problem entirely in the language of Riemann invariants $w$ and $z$, i.e. system (2.8). To that end, we must first focus our attention on rewriting $\lambda_1$ and $\lambda_2$ as functions of $w$ and $z$ instead of $\rho$ and $u$.

Lemma 5.5 Define the following functions

$$f(w, z) := \frac{w + z}{c} + \ln \left( \frac{1 - \sqrt{\gamma} \tan \left( \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} \right)}{1 + \sqrt{\gamma} \tan \left( \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} \right)} \right), \quad (5.5)$$

$$g(w, z) := \frac{w + z}{c} + \ln \left( \frac{1 + \sqrt{\gamma} \tan \left( \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} \right)}{1 - \sqrt{\gamma} \tan \left( \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}} \right)} \right). \quad (5.6)$$

For the $C^1$ solution of the Cauchy problem (1.1) – (1.2) with (2.1), under the Assumption 1, there holds

$$\lambda_1(w, z) = c \frac{e^f - 1}{e^f + 1}, \quad \lambda_2(w, z) = c \frac{e^g - 1}{e^g + 1}. \quad (5.7)$$

Proof. We notice that, the functions $\lambda_1, \lambda_2$, written in terms of $\rho$ and $u$, are reminiscent of (in fact identical to) the relativistic addition formulae for $u$ and $\sqrt{\gamma}(\rho)$. In particular,

$$\ln \left( \frac{c + \lambda_1}{c - \lambda_1} \right) = \ln \left( \frac{c + u}{c - u} \right) + \ln \left( \frac{c - \sqrt{\gamma} \rho}{c + \sqrt{\gamma} \rho} \right) = \tilde{f}(\rho, u) = f(w, z), \quad (5.8)$$

$$\ln \left( \frac{c + \lambda_2}{c - \lambda_2} \right) = \ln \left( \frac{c + u}{c - u} \right) + \ln \left( \frac{c + \sqrt{\gamma} \rho}{c - \sqrt{\gamma} \rho} \right) = \tilde{g}(\rho, u) = g(w, z), \quad (5.9)$$

which, along with (2.12) and solving for $\lambda_1, \lambda_2$, yields the desired relations. \hfill \Box

Now we are ready to give the proof for the case $\gamma \geq 3$ of Theorem 3.1.

Proof. Assume now, without loss of generality, that there exists $x \in \mathbb{R}$ such that $z_0'(x) < 0$. According to Lemma 5.4, what we need to show is just the divergence of the integral

$$\int_0^\infty (e^{-h_1 \lambda_1 z})(\sigma, x^1(\sigma, x_0^1)) \, d\sigma.$$ 

For this purpose, we divide the rest of the proof into two steps.

Step 1: The detailed formula of $e^{-h_1 \lambda_1 z}$. Introduce, for convenience, the notation

$$Y = \frac{(w-z)(\gamma-1)}{4c\sqrt{\gamma}}.$$

It follows from direct calculations that

$$\lambda_{1w} = \frac{e^{\frac{\pi}{\sqrt{2}} \frac{2 - 2\gamma + (1 + \gamma)\cos(2Y)\sec(Y)^2}{\left(1 + e^{\frac{\pi}{\sqrt{2}}} - (e^{\frac{\pi}{\sqrt{2}}} - 1) \sqrt{\gamma} \tan(Y)\right)^2}}}{\left(1 + e^{\frac{\pi}{\sqrt{2}}} - (e^{\frac{\pi}{\sqrt{2}}} - 1) \sqrt{\gamma} \tan(Y)\right)^2}. \quad (5.10)$$

\*\*If instead $w_0'(x) < 0$ the proof is precisely the same after relabelling the corresponding variables.\*\*
and
\[
\lambda_1 - \lambda_2 = \frac{8 c e^{-\frac{c}{\gamma^2}}}{-(1 + e^{-\frac{c}{\gamma^2}})^2 + (e^{-\frac{c}{\gamma^2}} - 1)^2 \gamma \tan(Y)^2}. \tag{5.11}
\]

Upon simplification, we have
\[
\frac{\lambda_{1w}}{\lambda_1 - \lambda_2} = \frac{(2 - 2\gamma + (1 + \gamma) \cos(2Y)) \left( (e^{\frac{c}{\gamma^2}} - 1) \sqrt{\gamma} + (1 + e^{\frac{c}{\gamma^2}}) \cot(Y) \right) \csc(Y) \sec(Y)}{8 c \gamma (e^{\frac{c}{\gamma^2}} - 1) - 8 c \sqrt{\gamma} (1 + e^{\frac{c}{\gamma^2}}) \cot(Y)}. \tag{5.12}
\]

It should be pointed out that the complicated expression (5.12) can be explicitly integrated with respect to \( w \), which provides us with an explicit form for the function \( h \) satisfying (2.9):
\[
\begin{aligned}
h &= \frac{3\gamma - 1}{2\gamma - 2} \ln(\cos(Y)) + \frac{\gamma - 3}{2\gamma - 2} \ln(\sin(Y)) + \frac{w - z}{2c} \\
&\quad - \ln\left( (1 + e^{\frac{c}{\gamma^2}}) \cos(Y) - \left( 1 + e^{\frac{c}{\gamma^2}} \right) \sin(Y) \right).
\end{aligned} \tag{5.13}
\]

Before we notice that the term
\[
(1 + e^{\frac{c}{\gamma^2}}) \cos(Y) - \left( 1 + e^{\frac{c}{\gamma^2}} \right) \sin(Y)
\]
contained inside the ln-function is positive, as \( \tan(Y) < \frac{1}{\sqrt{\gamma^2}} < 1 \) because of (2.12) and Assumption 1.

It follows from the direct calculation that
\[
\lambda_{1z} = \frac{e^{\frac{c}{\gamma^2}} (1 + \gamma) \cos(2Y) \sec(Y)^2}{(1 + e^{\frac{c}{\gamma^2}} - (e^{\frac{c}{\gamma^2}} - 1) \sqrt{\gamma} \tan(Y))^2}. \tag{5.14}
\]

It is, at this point, more useful to actually rewrite \( e^{-h_1 \lambda_{1z}} \) in terms of the original \((\rho, u)\)-coordinates. Define for convenience
\[
y = \frac{k \rho^{(\gamma - 1)/2}}{c},
\]
then the following formula holds:
\[
e^{-h_1 \lambda_{1z}} = \frac{ce^{\frac{2\gamma \rho}{\gamma - 1} \tan(Y) - c + u}(u + 1) \gamma + 1}{2c^2 - 2u \sqrt{P'(\rho)}} (1 + y^2)^{\frac{\gamma + 1}{\gamma - 1}} (1 - y^2) > 0. \tag{5.15}
\]

**Step 2:** the uniform lower bound of \( e^{-h_1 \lambda_{1z}} \). We analyze the above formulae of \( e^{-h_1 \lambda_{1z}} \) term by term:

- the term \( c e^{\frac{2\gamma \rho}{\gamma - 1} \tan(Y)} \) can be bounded below by \( ce^{-\frac{c}{\gamma^2}} \);
- the term \( \frac{c + u}{2c^2 - 2u \sqrt{P'(\rho)}} \) can be uniformly bounded below by a positive constant \( C_0 \) depending only on the initial data and \( c \), because of Lemmas 5.1 and 5.3;
- the term \( \left( \frac{u}{\sqrt{y^2 + 1}} \right)^{\frac{\gamma + 1}{\gamma - 1}} (1 + y^2)^{\frac{\gamma + 1}{\gamma - 1}} \) is bounded below by 1 for all \( \gamma \geq 3 \);
- the term \( 1 - y^2 \) can be bounded below by \( 1 - \frac{1}{\gamma} \), as \( \sqrt{P'(\rho)} < c \) implies
\[
y < \gamma - \frac{1}{\gamma}, \tag{5.16}
\]
which will play an important role in dealing with the physical case \( 1 \leq \gamma \leq 3 \) later.

Then, according to the solution’s formula shown in Lemma 5.4 and the uniform lower bound of \( e^{-h_1 \lambda_{1z}} \) obtained above, our theorem thus follows for the case \( \gamma \geq 3 \).

□

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Proof. The key to the proof lies in showing that $e^{-h_1 \lambda_{1z}}$ and $e^{-h_2 \lambda_{2w}}$ are non-negative. For this purpose, according to (5.14) and the formula

$$
\lambda_{2w} = \frac{e^{\omega_{2w}}}{(1 + \gamma) \cos(2Y) \sec(Y)} \left[ 1 + e^{\omega_{2w}} (e^{\omega_{2w}} - 1)^2 \sqrt{\gamma} \tan(Y) \right],
$$

Remark 5 It is easy to see by the above discussion that the term which will ultimately dictate the divergence, or the (hopefully not to be encountered!) convergence of the integral of $e^{-h \lambda_{1z}}$ in time is

$$
\left( \frac{k p^{(\gamma-1)/4}/c}{\sqrt{1 + k^2 \rho^{\gamma-1}/c^2}} \right)^{\frac{2}{\gamma+1}} (1 + \frac{k^2 \rho^{\gamma-1}}{c^2})^{\frac{2}{\gamma+1}} := \mathcal{Y}.
$$

5.3 Proof for the case $1 \leq \gamma < 3$ of Theorem 3.1

Our starting point is Remark 5 from Subsection 5.2. We observe that the behavior of $\mathcal{Y}$ ultimately dictates whether or not the integral $\int_0^\infty (e^{-h_1 \lambda_{1z}})(\sigma,x^1(\sigma,x_0^1)) d\sigma$ diverges. In particular, it becomes clear that in order to prove the Theorem 3.1 for $1 \leq \gamma < 3$, we are required to give a proper time-dependent lower bound on $\mathcal{Y}$, a function itself of the density $\rho$, strong enough such that $\int_0^\infty (\sigma,x^1(\sigma)) d\sigma = \infty$.

Before showing the detailed proof, we first give a clear outline of our strategy:

- (1) We rewrite the equations (2.8) in the form of the difference $(w - z)$ of the Riemann invariants:

$$
(w - z)' = (w - z) + \lambda_1(w - z)_z = (\lambda_1 - \lambda_2)w_z,
$$

which implies that

$$
(w - z)' = \frac{\lambda_1 - \lambda_2}{\rho h_2} \zeta \geq -\frac{\lambda_1 - \lambda_2}{\rho h_2} Q_2,
$$

under the assumption that $\zeta = e^{h_2 w_z}$ has a uniform upper bound $Q_2$ independent of the time. Actually, such kind a bound is established in Subsection 5.3.1;

- (2) In order to introduce a suitable ODEs inequality for $\mathcal{Y}$, we first obtain an ODEs inequality for $y$ from (5.19), which requires us to rewrite the quantities $w - z$ and $\frac{\lambda_1 - \lambda_2}{\rho h_2}$ explicitly in terms of $\rho, u$. To this end, there is a crucial observation that should be pointed out, that in $\frac{\lambda_1 - \lambda_2}{\rho h_2}$, all the terms involving $u$ have a uniform upper and lower bound. This can be seen in Section 5.3.2;

- (3) Based on the analysis on $y$ and the relation between $y$ and $\mathcal{Y}$, we successfully introduce a proper ODEs inequality for $\mathcal{Y}$, which could effectively control the behaviour of the density with respect to time. This can be seen in Section 5.3.3;

- (4) Finally, we show that the estimates obtained above are indeed sufficiently strong so that the desired quantity (5.17) has a divergent integral over time along the characteristic curve. Then we can obtain the desired conclusion stated in Theorem 3.1.

With that in mind, let us begin to show the detailed proof step by step.

5.3.1 Upper bound of the weighted gradients of the Riemann invariants

Now we need to give the upper bound of the weighted gradients of the Riemann invariants.

Lemma 5.6 Define the non-negative constants

$$
Q_1 := \max \left\{ 0, \sup_x \xi(x,0) \right\}, \quad Q_2 := \max \left\{ 0, \sup_x \zeta(x,0) \right\}.
$$

For the $C^1$ solution of the system (1.1)-(1.2), under Assumption 1, one has

$$
\xi(x,t) \leq Q_1, \quad \zeta(x,t) \leq Q_2.
$$

Proof. The key to the proof lies in showing that $e^{-h_1 \lambda_{1z}}$ and $e^{-h_2 \lambda_{2w}}$ are non-negative. For this purpose, according to (5.14) and the formula

$$
\lambda_{2w} = \frac{e^{\omega_{2w}}}{(1 + \gamma) \cos(2Y) \sec(Y)} \left[ 1 + e^{\omega_{2w}} (e^{\omega_{2w}} - 1)^2 \sqrt{\gamma} \tan(Y) \right],
$$
one gets that what we need is \( \cos(2Y) > 0 \), which, actually, can be obtained quickly from (5.16), and the following formulas

\[
Y = \frac{(w - z)(\gamma - 1)}{4c\sqrt{\gamma}} = \arctan\left(\frac{kp^{(\gamma - 1)/2}}{c}\right) \quad \text{and} \quad \cos(2\arctan(x)) = \frac{1 - x^2}{1 + x^2}.
\]

Thus, the conclusion of this lemma follows from the Riccati ODEs established in Lemma 5.4.

□

It should be pointed out that, so far, (5.19) has been proved.

### 5.3.2 Establishing ODE inequality of \( y \)

As mentioned before, now we need to rewrite the quantities \( w - z \) and \( \lambda_2 - \lambda_1 \) explicitly in terms of \( \rho, u \). First, according to (2.12) and (5.5), one has

\[
f = \ln\left(\frac{c + u}{c - u}\right) + \ln\left(\frac{1 - \sqrt{\gamma}k\rho/(\gamma - 1)/2}{1 + \sqrt{\gamma}k\rho/(\gamma - 1)/2}\right),
\]

\[
g = \ln\left(\frac{c + u}{c - u}\right) + \ln\left(\frac{1 + \sqrt{\gamma}k\rho/(\gamma - 1)/2}{1 - \sqrt{\gamma}k\rho/(\gamma - 1)/2}\right).
\]

Second, from (1.2) and (2.3), one can get

\[
\lambda_1 = \frac{c^2(\rho - k\sqrt{\gamma}\rho^{(\gamma - 1)/2})}{c^2 - k\rho^{(\gamma - 1)/2}},
\]

\[
\lambda_2 = \frac{c^2(\rho + k\sqrt{\gamma}\rho^{(\gamma - 1)/2})}{c^2 + k\rho^{(\gamma - 1)/2}},
\]

whence the following simplified expression for \( \lambda_2 - \lambda_1 \) follows:

\[
\lambda_2 - \lambda_1 = \frac{2e^2 k (c^2 - u^2) \sqrt{\gamma} \rho^{(\gamma - 1)/2}}{c^4 - k^2 u^4 \rho^{\gamma - 1}}.
\]

Moreover, the following explicit forms for \( h_1 \) and \( h_2 \) can be obtained:

\[
h_1 = \frac{3\gamma - 1}{2\gamma - 2} \ln(\cos(Y)) + \frac{\gamma - 3}{2\gamma - 2} \ln(\sin(Y)) + \frac{w - z}{2c}
- \ln\left(\left(1 + e^{\frac{w - z}{c}}\right)\cos(Y) - \left(-1 + e^{\frac{w - z}{c}}\right)\sqrt{\gamma}\sin(Y)\right),
\]

\[
h_2 = \frac{3\gamma - 1}{2\gamma - 2} \ln(\cos(Y)) + \frac{\gamma - 3}{2\gamma - 2} \ln(\sin(Y)) + \frac{w - z}{2c}
- \ln\left(\left(e^{\frac{w - z}{c}} + e^{\frac{w - z}{c}}\right)\cos(Y) + \left(e^{\frac{w - z}{c}} - e^{\frac{w - z}{c}}\right)\sqrt{\gamma}\sin(Y)\right).
\]

Here, as always, we denote \( Y = \arctan\left(\frac{kp^{(\gamma - 1)/2}}{c}\right) = \frac{(w - z)(\gamma - 1)}{4c\sqrt{\gamma}} \).

Now we are ready to develop one ODEs inequality for \( y \) from (5.19).

**Lemma 5.7** For the \( C^1 \) solution of the Cauchy problem (1.1) – (1.2) with (2.1), under Assumption 1, there holds

\[
y' \geq -C_g y^{5/2} (1 + y^2)^{3/2},
\]

for some positive constant \( C_g \) independent of the time.
Proof. First, it follows from the direct computations that the following simplified form for $\lambda_2 - \lambda_1$ holds:

$$\lambda_2 - \lambda_1 = \frac{4c\sqrt{\pi} \text{Arctan}(y)}{\sqrt{y^2+1}} (c + u) c \sqrt{7} y \left( \frac{w}{\sqrt{y^2+1}} \right) \frac{\gamma}{\pi + 1} (1 + y^2)^{\frac{\gamma + 1}{2}}$$

where $y = \frac{b_{\gamma}^{(\gamma-1)/2}}{c}$. 

Second, according to Lemmas 5.1-5.3, one can obtain that

$$C_1 - 1 \leq -C_g y \left( \frac{y}{\sqrt{y^2+1}} \right)^{\frac{\gamma}{\gamma-1}} (1 + y^2)^{\frac{\gamma + 1}{2}} = -C_g \left( \frac{y}{\sqrt{y^2+1}} \right)^{\gamma + 1} (1 + y^2)^{\frac{\gamma}{2}}.$$ 

Notice that $w - z = \frac{4c\sqrt{\gamma}}{\gamma-1} \text{Arctan}(y)$, and $(w - z)' = C_g \frac{y'}{y^2+1}$, so that we can rewrite (5.30) as (5.27). □

5.3.3 Establishing one ODE inequality of $Y$

We now recall that what determines the convergence/divergence of the integral of $e^{-h_1 \lambda_1 z}$ in time is the variable $Y$ from (5.17), which can be rewritten as

$$Y = \left( \frac{y}{\sqrt{y^2+1}} \right)^{\frac{3-\gamma}{\gamma-1}} (1 + y^2)^{\frac{\gamma+1}{2}}.$$ 

At this point, without loss of generality, that there exists $x \in \mathbb{R}$ such that $z'(x) < 0$. According to Lemma 5.4, what we need to show is just the divergence of the integral

$$\int_0^\infty (e^{-h_1 \lambda_1 z}(\sigma, x^1(\sigma, x_0^1))) d\sigma.$$ 

Then, according to fact on $Y$ mentioned above, our task therefore reduces to showing that

$$\int_0^\infty \left( \frac{y}{\sqrt{y^2+1}} \right)^{\frac{3-\gamma}{\gamma-1}} (1 + y^2)^{\frac{\gamma+1}{2}} (t, x^1(t, x_0^1)) dt$$

diverges, based on Lemma 5.7.

However, the explicit solution of the differential equation

$$y' = -C_g \frac{y}{\sqrt{y^2+1}} \left( 1 + y^2 \right)^{3/2}$$

is very hard to handle, as it involves hypergeometric functions. We instead adopt an indirect approach and look at a transport equation for the variable $Y$ itself:

$$Y = \left( \frac{y}{\sqrt{y^2+1}} \right)^{\frac{3-\gamma}{\gamma-1}} (1 + y^2)^{\frac{\gamma+1}{2}} = \frac{y}{\sqrt{y^2+1}} (1 + y^2)^{\frac{\gamma}{2}}.$$ 

It turns out that, one can obtain the following Riccati-type inequality for $Y$:

\text{If instead } w_0'(x) < 0 \text{ the proof is precisely the same after relabelling the corresponding variables.}
Lemma 5.8 For the $C^1$ solution of the Cauchy problem (1.1) – (1.2) with (2.1), under the Assumption 1, there holds

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2$$

for some universal constant $C_g$ independent of the time.

Proof. First, it follows from the direct calculation that

$$\mathcal{Y}' = \frac{y^{\gamma-2}}{2\gamma-2} \frac{((\gamma+1)g^2 + (3-\gamma))}{\gamma-\gamma^2+1}.$$

which, along with (5.27), implies that

$$\mathcal{Y}' \geq -C_g y^{\frac{\gamma-2}{\gamma-1}} (1+g^2) (\gamma+1)g^2 + (3-\gamma).$$

Actually, the above ODE inequality can equivalently be rewritten as

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2 (\gamma+1)g^2 + (3-\gamma).$$

At this point it is important to observe the following:

Second, it follows from the fact $\sqrt{P'(\rho)} < c$ that

$$y = \frac{k\rho^{(\gamma-1)/2}}{c} < \gamma^{-\frac{1}{2}}.$$

Therefore

$$(\gamma+1)g^2 + (3-\gamma) \leq \frac{\gamma+1}{\gamma} + (3-\gamma),$$

which, along with (5.35), implies that

$$\mathcal{Y}' \geq -C_g \mathcal{Y}^2,$$

for some universal constant $C_g$ independent of the time.

\[\square\]

5.3.4 Mass-density’s lower bound estimates and formation of singularity

Now, based on the conclusions obtained in Sections 5.3.1-5.3.3, we are ready to finish the proof of Theorem 3.1.

Actually, from Lemma 5.8, one can obtain that

$$\mathcal{Y}(t) \geq \frac{1}{C_1 + C_2 t},$$

for some universal constants $C_i$ ($i = 1, 2$) independent of the time.

From (5.15), one can get

$$e^{-h} \Lambda_2 = \frac{ce^{-2\sqrt{\tan(y)}}}{2c^2 - 2u \sqrt{P'(\rho)}}$$

From the analysis shown in Step 2 of the proof for the case $\gamma \geq 3$ of Theorem 3.1 in Section 5.2, one obtains

$$C_g^{-1} \leq \mathcal{H} \leq C_g,$$

for some universal constant $C_g$ independent of the time. Also, we know that $\mathcal{Y}(0)$ is positive.

Then, finally, according to the solution’s formula shown in Lemma 5.4, the desired conclusion stated in Theorem 3.1 for $1 \leq \gamma < 3$ has been proved.

Remark 6 Notice that $\mathcal{Y} = C\rho^{\frac{3-\gamma}{\gamma}} \sqrt{\frac{k^2\rho^{\cdot\cdot2\gamma-1}}{c^2}} < 2C\rho^{\frac{3-\gamma}{\gamma}}$, because of the upper bound on $\rho$. Together with (5.37), we obtain the same bound for $\rho$ as in the classical case:

$$\rho(t, x) = O(1 + t)^{\frac{1}{3-\gamma}}.$$
6 Formation of singularities for the (3+1)-dimensional RE

Let \( d = 3 \) in (1.1) and (2.1). In this section we will do some studies on the formation of singularities for the (3+1)-dimensional relativistic fluids.

6.1 Proof of Theorem 3.3

In this subsection, we always assume that \((\rho, u)(t, x)\) is the regular solution in \([0, T_m] \times \mathbb{R}^3\) defined in Definition 5 to the Cauchy problem (1.1)-(1.2) and (2.1). Before showing the singularity formation caused by the so-called isolated mass group, defined in Definition 7, we first consider the time evolution of the vacuum domain. For this purpose, next we introduce one physical quantity, namely the particle number.

Definition 9 Define the particle number \( n \) by

\[
 n(\rho) = n(1) \exp \left( \int_{1}^{\rho} \frac{d\sigma}{\sigma + \frac{P(\sigma)}{c^2}} \right). \tag{6.1}
\]

In the following lemma, we will show that the evolution of the vacuum still can be tracked by the particle path defined by (2.20).

Lemma 6.1

\[
 n(\rho) = 0 \quad \text{if and only if} \quad \rho = 0. \tag{6.2}
\]

Moreover, one has

\[
\frac{n}{\sqrt{1 - u^2/c^2}}(t, x(t; x_0)) = \frac{n}{\sqrt{1 - u^2/c^2}}(0, x_0) \exp \left( \int_{0}^{t} - \text{div}(u, x(x; x_0)) d\sigma \right), \tag{6.3}
\]

where \( x(t; x_0) \) is the particle path starting from \((0, x_0)\). Also, if \( \rho(0, x_0) = 0 \), then one gets

\[
\rho(0, x_0) = n(0, x_0) = n(t, x(t; x_0)) = \rho(t, x(t; x_0)) = 0 \quad \text{for} \quad t \in [0, T_m].
\]

Proof. First, the equivalence relation (6.2) follows from the facts \( n(0) = 0 \) and \( \frac{dn}{d\rho} > 0 \).

Second, according to the system (1.1), one can obtain that \( n \) satisfies the following equation

\[
 \left( \frac{n}{\sqrt{1 - u^2/c^2}} \right)_t + \text{div} \left( \frac{nu}{\sqrt{1 - u^2/c^2}} \right) = 0, \tag{6.4}
\]

which implies the formula (6.3). \( \square \)

For simplicity, we define the following images of \( A_0, B_0, \) and \( B_0 \setminus A_0 \), respectively, under the flow map of (2.20).

Definition 10 (Particle path and flow map) Let \( A(t), B(t), B(t) \setminus A(t) \) be the images of \( A_0, B_0, \) and \( B_0 \setminus A_0 \), respectively, under the flow map of (2.20), i.e.,

\[
 A(t) = \left\{ x(t; x_0) | x_0 \in A_0 \right\},
 B(t) = \left\{ x(t; x_0) | x_0 \in B_0 \right\},
 B(t) \setminus A(t) = \left\{ x(t; x_0) | x_0 \in B_0 \setminus A_0 \right\}.
\]

The following lemma establishes the invariance of the volume \(|A(t)|\) for regular solutions.

Lemma 6.2 Suppose that the initial data \((\rho_0, u_0)(x)\) have an isolated mass group \((A_0, B_0)\). Then, for the regular solution \((\rho, u)(t, x)\) on \( \mathbb{R}^3 \times [0, T_m) \) to the Cauchy problem (1.1)-(1.2) and (2.1), we have

\[
 |A(t)| = |A_0|, \quad t \in [0, T_m).
\]
Proof. From Lemma 6.1 and the definition of regular solutions, one has

$$u_t + u \cdot \nabla u = 0, \quad \text{in} \quad B(t) \setminus A(t).$$

Therefore, \( u \) is invariant along the particle path \( x(t; x_0) \) with \( x_0 \in B_0 \setminus A_0 \).

For any \( x_0^1, x_0^2 \in \partial A_0 \), we define

$$\frac{d}{dt} x^i(t; x_0^i) = u(x^i(t; x_0^i), t), \quad x^i(0; x_0^i) = x_0^i, \quad \text{for} \quad i = 1, 2,$$

(6.6)

Then we have

$$\frac{d}{dt} (x^1(t; x_0^1) - x^2(t; x_0^2)) = u(x^1(t; x_0^1), t) - u(x^2(t; x_0^2), t) = \dot{u}_0 - \dot{u}_0 = 0,$$

(6.7)

which implies that

$$|A(t)| = |A_0|, \quad t \in [0, T_m].$$

We point out that, although the volume of \( A(t) \) is invariant, the vacuum boundary \( \partial A(t) \) travels with constant velocity \( \overline{u} \). The following well-known Reynolds transport theorem (c.f. [10]) is useful.

**Lemma 6.3** For any \( G(t, x) \in C^1(\mathbb{R}^3 \times \mathbb{R}^+), \) one has

$$\frac{d}{dt} \int_{A(t)} G(t, x) dx = \int_{A(t)} \dot{G}_t(x) dx + \int_{\partial A(t)} G_t(x)(u(t, x) \cdot n) dS,$$

where \( n \) is the outward unit normal vector to \( \partial A(t) \) and \( u \) is the velocity of the fluid.

Based the observations in Lemmas 6.1-6.3, now we can obtain the following conservation laws of total energy and total momentum, and the invariance of the centroid.

**Lemma 6.4** Suppose that the initial data \( (\rho_0, u_0)(x) \) have an isolated mass group \( (A_0, B_0) \), then for the regular solution \( (\rho, u)(t, x) \) on \( \mathbb{R}^3 \times [0, T_m) \) to the Cauchy problem (1.1)-(1.2) with (2.1), we have

$$m(t) = m(0), \quad \overline{P}(t) = P(0), \quad X^*(t) = X^*(0), \quad A(t) \subseteq B_{2R_0 + |X_0^*|}, \quad \text{for} \quad t \in [0, T_m).$$

Proof. From (1.1) and Lemma 6.3, direct computation shows

$$\frac{d}{dt} m(t) = \int_{A(t)} \dot{\rho} dx + \int_{\partial A(t)} \dot{\rho} u \cdot n dS$$

$$= \int_{A(t)} - \text{div}(\dot{\rho} u) dx = \int_{\partial A(t)} - \dot{\rho} u \cdot n dS = 0,$$

which implies that \( m(t) = m(0) \).

Similarly, one has

$$\frac{d}{dt} \overline{P}(t) = \int_{A(t)} (\dot{\rho} u) dx + \int_{\partial B(t)} \dot{\rho} u (u \cdot n) dS$$

$$= \int_{A(t)} \left( - \text{div}(\dot{\rho} u \otimes u) - \nabla P \right) dx = \int_{\partial A(t)} \left( - \dot{\rho} u \otimes u - PI_3 \right) \cdot n dS = 0,$$

which implies that \( \overline{P}(t) = P(0) = 0 \).

Finally, from the definition of \( X^*(t) \), \( m(t) = m_0 \) and \( \overline{P}(t) = P(0) = 0 \), one has

$$\frac{d}{dt} \left( \int_{A(t)} x \dot{\rho} dx \right) = \int_{A(t)} x \dot{\rho} dx + \int_{\partial A(t)} x \dot{\rho} (u \cdot n) dS$$

$$= - \int_{A(t)} x \text{div}(\dot{\rho} u) dx = \int_{A(t)} \dot{\rho} u dx = \overline{P}(0) = 0,$$

which means that \( X^*(t) = X^*(0) \). Moreover, since the total energy on \( A(t) \) is conserved, thus \( X^*(0) \) is contained in the closed convex hull of \( A(t) \) from \( X^*(t) = X^*(0) \), we easily know that \( A(t) \subseteq B_{2R_0 + |X^*(0)|}. \)
We are now ready to give the proof of Theorem 3.3:

**Proof.** First, it follows from the equation (1.1) and the integration by parts that

\[
\frac{d}{dt} M(t) = \int_{A(t)} \hat{\rho} |x|^2 dx + \int_{\partial A(t)} \hat{\rho} |x|^2 (u \cdot n) dS = 2F(t). \tag{6.8}
\]

Similarly, according to Lemma 6.3, the equation (1.1) and integration by parts, one can obtain that

\[
\frac{d}{dt} F(t) = \int_{A(t)} (\hat{\rho}u)_t \cdot x dx + \int_{\partial A(t)} \hat{\rho}u \cdot x (u \cdot n) dS
\]

\[
= \int_{A(t)} \hat{\rho}u^2 dx + 3 \int_{A(t)} Pdx = c^2 \int_{A(t)} \left( \frac{1}{c^2} \hat{\rho}u^2 + \frac{3P}{c^2} \right) dx. \tag{6.9}
\]

By Jensen’s inequality, one has

\[
\int_{A(t)} Pdx \geq |A(0)|P(\bar{m}(t)), \tag{6.10}
\]

where \( \bar{m}(t) = \frac{I_{A(t)} \rho dx}{|A(0)|} \). Then from (6.9)-(6.10), one obtains

\[
\frac{d^2}{dt^2} M(t) \geq 2c^2 \int_{A(t)} \left( \frac{1}{c^2} \hat{\rho}u^2 \right) dx + 6|A(0)|P(\bar{m}) \equiv 2c^2 N(t). \tag{6.11}
\]

Now we consider the following two cases:

- If \( \int_{A(t)} \frac{1}{c^2} \hat{\rho}u^2 dx \geq \frac{1}{2} m(0) \), one gets

  \[
  N(t) \geq \frac{1}{2} m(0); \tag{6.12}
  \]

- If \( \int_{A(t)} \frac{1}{c^2} \hat{\rho}u^2 dx \leq \frac{1}{2} m(0) \), then according to

  \[
  m(t) = m(0) = \int_{A(t)} \hat{\rho} dx \leq \frac{1}{2} m(0) + \int_{A(t)} \rho dx,
  \]

  one has

  \[
  \int_{A(t)} \rho dx \geq \frac{1}{2} m(0), \tag{6.14}
  \]

  which implies that

  \[
  N(t) \geq \frac{3}{c^2} |A(0)|P(\bar{m}(0)) \equiv D_0 m(0) > 0, \tag{6.15}
  \]

  where \( D_0 = \frac{3}{c^2} |A(0)|P(\bar{m}(0)) \).

Denote \( D = 2c^2 \min\{\frac{1}{2}, D_0\} \), one has

\[
\frac{d^2}{dt^2} M(t) \geq Dm(0) > 0. \tag{6.16}
\]

Integrating (6.8) and (6.16) over \([0, t]\), respectively, one can obtain that

\[
M(t) \geq M_0 + 2F_0 t + \frac{1}{2} Dm(0)t^2. \tag{6.17}
\]
According to Lemma 6.4, it yields

\[ M(t) = \int_{A(t)} \hat{\rho}|x|^2 \, dx \leq R_1^2 m_0, \]  
(6.18)

where \( R_1 = |X^*(0)| + 2R_0 \).

Combining (6.17) with (6.18), one has

\[ R_1^2 m_0 \geq M_0 + 2F_0 t + \frac{1}{2} Dm(0) t^2, \]  
(6.19)

which means that \( T_m < +\infty \). \( \square \)

### 6.2 Proof of Theorem 3.4

**Proof.** We denote by \( V(t) \) the evolved domain that is the image of \( V \) under the flow map, i.e.,

\[ V(t) = \{ x | x = x(t; x_0), \quad \forall \ x_0 \in V \}, \]  
(6.20)

where \( x(t; x_0) \) is the particle path starting from \((0, x_0)\). It follows from Lemma 6.1 that the mass-energy density is simply supported along the particle paths, so

\[ \rho(t, x) = 0, \quad \text{when} \quad x \in V(t). \]

Thus, via the Definition 5 for regular solutions, we deduce that

\[ u_t + u \cdot \nabla u = 0, \quad \text{when} \quad x \in V(t), \]  
(6.21)

which means that \( u \) is a constant along the particle path \( x(t; x_0) \). Then for any \( x \in V(t) \), we obtain that

\[ u(t, x) = u_0(x - tu(t, x)), \]

which immediately implies that

\[ \nabla u(t, x) = \left( I_3 + t \nabla u_0(x - tu(t, x)) \right)^{-1} \nabla u_0, \quad \text{for} \quad x \in V(t). \]  
(6.22)

If there is any \( \lambda \in \text{Sp}(\nabla u_0) \) satisfying \( \lambda < 0 \), then from (6.22), it is obvious that the quantity \( \nabla u \) will blow up in finite time, i.e.,

\[ T_m < +\infty. \]  
\( \square \)

### 6.3 Proof of Theorem 3.5

In this subsection, we simply denote

\[ \int_{\mathbb{R}^3} f \, dx = \int f \, dx. \]

Now we are ready to prove Theorem 3.5. **Proof.** Let \( T > 0 \) be any constant, and \((\rho, u) \in D(T)\). It follows from the definitions of \( m(t), \mathbb{P}(t) \hat{\rho} \) and \( \hat{\rho} \) that

\[ |\mathbb{P}(0)| = |\mathbb{P}(t)| \leq \| u(t) \|_{L^\infty(\mathbb{R}^3)} \int \hat{\rho} \, dx \leq \| u(t) \|_{L^\infty(\mathbb{R}^3)} \left( m(0) + \int_{\mathbb{R}^3} \frac{P}{c^2} \, dx \right). \]  
(6.23)
Notice that
\[
\int P \, dx = \left| \int_0^\rho P'(\sigma) \, d\sigma \right| \leq \left\| P'(\rho) \right\|_{L^\infty(\mathbb{R}^3)} \rho \, dx \\
\leq c^2 \int \rho \, dx \leq c^2 m(t) = c^2 m_0.
\] (6.24)

Then one obtains that there exists a positive constant \( C_u = \frac{|P(0)|}{2m(0)} \) such that
\[
\|u(t)\|_{L^\infty(\mathbb{R}^3)} \geq C_u \text{ for } t \in [0,T].
\]
Thus one obtains the desired conclusion as shown in Theorem 3.5.
\[\square\]

7 Remarks on the general pressure law of the 1-dimensional case

Let \( d = 1 \) in (1.1)-(1.3) and (2.1). We revisit in this section the study of the relativistic Euler equations in \( 1 + 1 \) dimensions. Its main aim is to enlarge the set of pressure laws \( P = P(\rho) \) from those in (1.2) to more general ones which satisfy Assumptions 2-3. As mentioned before, we first show our basic idea for the classical compressible Euler equations, and then give detailed proof for the relativistic flow. Throughout this section, we always assume that \( P(\rho) \) satisfies the Assumptions 2-3.

7.1 The classical Euler equations

7.1.1 Notations and relations.

First, the eigenvalues \( \tilde{\lambda}_i \) \((i = 1, 2)\), directional derivatives \( \partial_- \) and \( \partial_+ \), the characteristic directions \( y^1 \) and \( y^2 \), Riemann variables \( \tilde{w} \) and \( \tilde{z} \), \( \tilde{h}_1 \) and \( \tilde{h}_2 \), \( \tilde{\alpha} \) and \( \tilde{\beta} \), \( \phi \) and \( \psi \) are still given by or satisfy (2.13)-(2.19).

Second, we define the function \( J_{\text{clas}}(x) = \int_0^x \frac{\sqrt{P'(\sigma)}}{\sigma} \, d\sigma \), and then \( \tilde{w} - \tilde{z} = 2 J_{\text{clas}}(\rho) \). It is easy to see that Assumption 2 implies \( J_{\text{clas}} \) is strictly increasing. Thus, we can write
\[
\rho = J_{\text{clas}}^{-1} \left( \frac{\tilde{w} - \tilde{z}}{2} \right),
\]
and the eigenvalues \( \tilde{\lambda}_i \) \((i = 1, 2)\) can be rewritten as functions of the Riemann invariants:
\[
\tilde{\lambda}_1 = \frac{\tilde{w} + \tilde{z}}{2} - \sqrt{\Sigma}, \quad \tilde{\lambda}_2 = \frac{\tilde{w} + \tilde{z}}{2} + \sqrt{\Sigma},
\]
where \( \Sigma = P'' \left( J_{\text{clas}}^{-1} \left( \frac{\tilde{w} - \tilde{z}}{2} \right) \right) \).

Next we need to find suitable \( \tilde{h}_1 \) and \( \tilde{h}_2 \) satisfying (2.18). It follows from direct calculations that
\[
\tilde{\lambda}_1 \tilde{w} = \tilde{\lambda}_2 \tilde{z} = \frac{1}{2} - \frac{1}{4 \sqrt{P'(\rho)}} P''(\rho) \left( J_{\text{clas}}^{-1} \right)' \left( \frac{\tilde{w} - \tilde{z}}{2} \right).
\]

Notice that, via denoting inf\(_{x \in \mathbb{R}}(w_0 - z_0) = \epsilon > 0 \) for some constant \( \epsilon \), we can choose\(^7\)
\[
\tilde{h}_1 = \tilde{h}_2 = \frac{\epsilon}{4} \ln \Sigma - \int_\epsilon^{\frac{\sqrt{\Sigma}}{2}} \frac{d\sigma}{\sqrt{\Sigma(\sigma)}} := \tilde{h},
\]
\(^7\)By a slight abuse of notation, whenever we write \( \Sigma \) alone, we mean the expression \( P'' \left( J_{\text{clas}}^{-1} \left( \frac{\tilde{w} - \tilde{z}}{2} \right) \right) \) but when we write \( \Sigma(\sigma) \) we mean the function \( \Sigma(x) = P'' \left( J_{\text{clas}}^{-1} \left( \frac{x}{2} \right) \right) \).
which, along with the same argument used in the proof of Lemma 4.1, implies that

\[
\partial_-\phi = -e^{-h}\lambda \phi^2, \quad \partial_+\psi = -e^{-h}\lambda \psi^2.
\]

Since \(\lambda_1 = \lambda_2 > 0\), according to Assumptions 2-3, there exist positive constants \(\overline{Q}_1, \overline{Q}_2\) such that

\[
\phi \leq \overline{Q}_1, \quad \psi \leq \overline{Q}_2,
\]

within the lifespan of the \(C^1\) solution.

### 7.1.2 Derivation of the desired ODE inequality

According to the same argument used in the proof of Lemma 4.2, we therefore have,

\[
\partial_-\left(\tilde{w} - \tilde{z}\right) \geq -\frac{2\sqrt{\Sigma}}{\sqrt{\Sigma}} \exp\left(\int_{\tilde{z}}^{\tilde{w}} \frac{d\sigma}{2\sqrt{\Sigma(\sigma)}}\right) \overline{Q}_2,
\]

or equivalently due to \(\partial_-(\tilde{w} - \tilde{z}) = 2\sqrt{P''(\rho)}\partial_+\rho\),

\[
\partial_-\rho \geq -2\overline{Q}_2 \frac{\rho}{\sqrt{P''(\rho)}} \exp\left(\int_{\tilde{z}}^{\tilde{w}} \frac{d\sigma}{2\sqrt{\Sigma(\sigma)}}\right).
\]

### 7.1.3 Lower bound estimates of the mass density.

Now we hope that, based on the above ODE inequality, we can obtain the divergence of the integral of

\[
e^{-h}\lambda_1 = \exp\left(\int_{\tilde{z}}^{\tilde{w}} \frac{d\sigma}{2\sqrt{\Sigma(\sigma)}}\right) (G_1(\rho) + G_2(\rho)),
\]

with respect to the time over \([0, +\infty)\), where

\[
G_1(\rho) = \frac{1}{2} P'(\rho)^{-\frac{5}{4}}, \quad G_2(\rho) = \frac{1}{7} \rho P''(\rho) P'(\rho)^{-\frac{5}{4}}.
\]

Then similarly to the proof shown in Section 4, we can obtain the if and only if condition on the singularity formation of the classical Euler equations with general pressure law. For simplicity, we also denote \(G(\rho) = G_1(\rho) + G_2(\rho)\).

Next we give a proper ODEs inequality for \(G(\rho)\). First, it follows from direct calculations that

\[
G'_1(\rho) = -\frac{1}{8} P'(\rho)^{-\frac{5}{4}} P''(\rho),
\]

\[
G'_2(\rho) = \frac{1}{4} \left( P'(\rho)^{-\frac{5}{4}} P''(\rho)^{\frac{5}{4}} - \frac{5}{16} \rho P''(\rho)^2 P'(\rho)^{-\frac{5}{4}}.\right.
\]

which implies that

\[
G'(\rho) = \exp\left(\int_{\tilde{z}}^{\tilde{w}} \frac{d\sigma}{2\sqrt{\Sigma(\sigma)}}\right) \rho^2 P'(\rho) P''(\rho) - \frac{5}{16} P''(\rho)^2 + \rho P'(\rho) P''(\rho) + P'(\rho)^2.
\]

Notice that Assumptions 2-3 implies that

\[
G'(\rho) \leq \exp\left(\int_{\tilde{z}}^{\tilde{w}} \frac{d\sigma}{2\sqrt{\Sigma(\sigma)}}\right) \frac{4}{4P'(\rho)^{\frac{5}{4}}}.\]

Now we consider the following two cases:
7.2 The general pressure law for the Relativistic Euler equations

In this subsection, we will give the proof for Theorem 3.2.

7.2.1 Notations and relations.

We note that the directional derivatives $\tau$ and $\lambda$, the characteristic directions $x_1$ and $x_2$, Riemann variables $w$ and $z$, $h_1$ and $h_2$, $\alpha$ and $\beta$, $\xi$ and $\zeta$ are still given by or satisfy (2.3)-(2.10). Also, $f(w, z)$, $g(w, z)$, $f(\rho, u)$ and $g(\rho, u)$ are given by Lemma 5.5. Define the helpful function

$$J_{\text{rel}}(x) = \int_0^x \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{\rho}{c^2}} d\sigma.$$  

Rewriting $f(w, z)$ and $g(w, z)$ in terms of Riemann invariants, we denote

$$\ln \left(\frac{c + \lambda_1}{c - \lambda_1}\right) = \frac{w + z}{c} + \ln \left(\frac{c - \sqrt{\Lambda}}{c + \sqrt{\Lambda}}\right) := F(w, z),$$  

$$\ln \left(\frac{c + \lambda_2}{c - \lambda_2}\right) = \frac{w + z}{c} + \ln \left(\frac{c + \sqrt{\Lambda}}{c - \sqrt{\Lambda}}\right) := G(w, z),$$  

where $\Lambda = P'(\frac{J_{\text{rel}}^{-1}(\frac{w+z}{2})}{2})^8$. It follows from the direct calculation that

$$\Lambda_1(w, z) = c \left(1 - \frac{2}{e^{F(w, z)} + 1}\right), \quad \Lambda_2(w, z) = c \left(1 - \frac{2}{e^{G(w, z)} + 1}\right),$$  

$$\Lambda_1 = 2e \left(\frac{e^G - e^F}{e^F + 1}\right) \left(\frac{4\sqrt{\Lambda}}{e^G - e^F}\right),$$  

$$\Lambda_1 w = \frac{2 e^{F(w, z)}}{e^{F(w, z)} + 1} F_w(w, z), \quad e^{F(w, z)} = \frac{e^{\lambda_1}(c \pm \sqrt{\Lambda})}{c \pm \sqrt{\Lambda}}.$$  

*By a slight abuse of notation, whenever we write $\Lambda$ alone, we mean the expression $P'\left(J_{\text{rel}}^{-1}\left(\frac{w+z}{2}\right)\right)$ but when we write $\Lambda(\sigma)$ we mean the function $\Sigma(x) = P'\left(J_{\text{rel}}^{-1}\left(\frac{x}{2}\right)\right)$. 

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and
\[ F_w(w, z) = \frac{1}{c} - \frac{2c}{c^2 - \Lambda} \cdot P^\prime \left( J^{-1}_\text{rel} \left( \frac{w-z}{2} \right) \right) \left( J^{-1}_\text{rel} \left( \frac{w-z}{2} \right) \right), \] (7.13)

which imply that
\[ \frac{\lambda_{1w}}{\lambda_1 - \lambda_2} = -\frac{2c e^F}{(e^{F+1})} \frac{F_w}{e^{G-G}_w} = -\frac{e^G + 1}{(e^F + 1)(e^G - e^F)} (e^G)_w = -\left( \frac{1}{e^F + 1} + \frac{1}{e^G - e^F} \right) (e^F)_w. \] (7.14)

Notice that
\[ \ln \left( e^{G+F} - e^{2F} \right)_w = \frac{1}{e^{G+F} - e^{2F}} \left( \frac{2c e^Z}{e^G + 1} - 2e^F (e^F)_w \right) = -2(e^F)_w \frac{e^G - e^F}{c(e^{G+F} - e^{2F})}. \]

Then one can obtain that
\[ -\frac{1}{e^G - e^F} (e^F)_w = \frac{1}{2} \ln (e^{G+F} - e^{2F}) - \frac{e^{2(F+1)}}{c(e^{G+F} - e^{2F})}. \]

Then, together with (7.14), one gets
\[ \frac{\lambda_{1w}}{\lambda_1 - \lambda_2} = -\left( \ln \left( e^F + 1 \right) \right)_w + \frac{1}{2} \left( \ln \left( e^{G+F} - e^{2F} \right) \right)_w - \frac{1}{c(1 - e^{F-G})}. \] (7.15)

Therefore, we can choose
\[ h_1 = -\ln \left( e^F + 1 \right) + \frac{1}{2} \ln \left( e^{G+F} - e^{2F} \right) - \int_{\epsilon/2}^{\frac{\Lambda}{\sqrt{\Lambda}}} \left( c + \sqrt{\Lambda(\sigma)} \right)^2 2c^2 \sqrt{\Lambda(\sigma)} d\sigma. \] (7.16)

Similarly, one can obtain
\[ \lambda_{2z} = \frac{2c e^G}{(e^G + 1)} G_z. \] (7.17)

Then one has
\[ \frac{\lambda_{2z}}{\lambda_2 - \lambda_1} = \frac{2c e^G}{(e^{F+1})} \frac{G_z}{e^{G-G}_z} = \frac{e^F + 1}{(e^G + 1)(e^G - e^F)} (e^G)_z \]
\[ = -\left( \frac{1}{e^G + 1} + \frac{1}{e^G - e^F} \right) (e^G)_z \]
\[ = -\left( \ln \left( e^G + 1 \right) \right)_z + \frac{1}{e^G - e^F} (e^G)_z. \] (7.18)

It follows from the direct calculation that
\[ \frac{1}{2} \ln \left( e^{2G} - e^{G+F} \right)_z = \frac{(e^G)_z}{e^G - e^F} - \frac{1}{c} \frac{e^{G+F}}{e^{2G} - e^{G+F}} \] (7.19)

Then
\[ \frac{1}{e^G - e^F} (e^G)_z = \frac{1}{2} \ln \left( e^{2G} - e^{G+F} \right)_z + \frac{1}{c(e^{G-F} - 1)}. \] (7.20)

We can therefore choose
\[ h_2 = -\ln \left( e^G + 1 \right) + \frac{1}{2} \ln \left( e^{2G} - e^{G+F} \right) - \int_{\epsilon}^{\frac{1}{2}} \frac{c + \sqrt{\Lambda(\sigma)}}{2c^2 \sqrt{\Lambda(\sigma)}} d\sigma. \] (7.21)
Set, for convenience,

\[ I_1 = \int \frac{c + \sqrt{\Lambda(\sigma)}}{2 c^2 \sqrt{\Lambda(\sigma)}} \, d\sigma, \quad I_2 = \int \frac{c - \sqrt{\Lambda(\sigma)}}{2 c^2 \sqrt{\Lambda(\sigma)}} \, d\sigma. \]  

(7.22)

According to (7.12) and (7.21), one can obtain

\[ \frac{\lambda_2 - \lambda_1}{e^{\hat{b}_2}} = C_g \left( \frac{e^G - e^F}{e^G + 1} \right) \cdot \frac{e^G + 1}{\sqrt{e^{2G} - e^{G+F}}} \cdot \exp(I_2) \]

\[ = C_g \left( \frac{e^G + 1}{\sqrt{e^{2G} - e^{G+F}}} \right) \cdot \frac{e^G + 1}{\sqrt{e^{2G} - e^{G+F}}} \cdot \exp(I_2) \]

\[ = C_g \left( \frac{1}{\sqrt{P'(\rho)}} \right) \left( 1 + e^{\frac{4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}} \right) \cdot \exp(I_2) \]

(7.23)

for some positive constant \( C_g \) independent of the time.

Furthermore,

\[ e^{-h_1 \lambda_{1z}} = \frac{e^F + 1}{\sqrt{e^{2G} - e^{G+F}}} \cdot \exp(I_1) \left( \frac{1}{c} + \frac{2 c P''(\rho) \left( \rho + \frac{P'(\rho)}{c} \right)}{4 P'(\rho)(c^2 - P'(\rho))} \right) \]

\[ = \frac{1 + e^{\frac{-4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}}}{\sqrt{e^{2G} - e^{G+F}}} \cdot \exp(I_1) \left( \frac{1}{c} + \frac{2 c P''(\rho) \left( \rho + \frac{P'(\rho)}{c} \right)}{4 P'(\rho)(c^2 - P'(\rho))} \right) \]

\[ = C_g \left( \frac{c + \sqrt{P'(\rho)}}{e^{\frac{4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}}} \right) \cdot \exp(I_1) \left( \frac{1}{c} + \frac{2 c P''(\rho) \left( \rho + \frac{P'(\rho)}{c} \right)}{4 P'(\rho)(c^2 - P'(\rho))} \right) \]

\[ := H(\rho). \]

Define

\[ H_1 = \frac{1}{\sqrt{P'(\rho)}}, \quad H_2 = \frac{P''(\rho) \left( \rho + \frac{P'(\rho)}{c} \right)}{2 \left( 1 - \frac{P'(\rho)}{c} \right) P'(\rho) \frac{c}{2^2}}. \]  

(7.25)

then one has

\[ e^{-h_1 \lambda_{1z}} = C_g \left( \frac{1 + \sqrt{P'(\rho)}}{c} \right) \cdot \frac{1 + e^{\frac{-4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}}}{e^{\frac{4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}}} \cdot \exp(I_1) \left( H_1(\rho) + H_2(\rho) \right). \]

It is easy to check that the conclusions of Lemmas 5.1-5.4 and 5.6 still hold for the current pressure law.

### 7.2.2 Derivation of the desired ODE inequality

First, it is obvious that

\[ C_g^{-1} \leq 1 + \frac{\sqrt{P'(\rho)}}{c} \leq C_g, \quad C_g^{-1} \leq 1 + e^{\frac{-4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}} \leq C_g, \quad C_g^{-1} \leq e^{\frac{-4 c - \sqrt{P'(\rho)}}{c + \sqrt{P'(\rho)}}} \leq C_g, \]

(7.26)

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for some positive constant $C_g$ independent of the time.

Given two functions $f_1, f_2$, we denote $f_1 \approx f_2$ if and only if there are positive constants $C_1$ and $C_2$ such that

$$C_1 f_1 \leq f_2 \leq C_2 f_1,$$

pointwisely.

With this in mind, the above three remarks allow us to conclude that

$$e^{-h_1 \lambda_{1z}} \approx \exp(I_1) \left( H_1(\rho) + H_2(\rho) \right). \quad (7.27)$$

Second, we observe that

**Lemma 7.1**

$$H_1(\rho) + H_2(\rho) \approx G_1(\rho) + G_2(\rho),$$

where $G_1$ and $G_2$ are given by (7.4).

**Proof.** First, $H_1 \equiv 2G_1 \geq 0$. Second,

$$\frac{2(1 + \gamma^2)}{1 - \gamma^2} G_2(\rho) \geq H_2(\rho) = 2G_2(\rho) \left( 1 + \frac{P(\rho)}{P'(\rho)} \right) \geq 2G_2(\rho),$$

where one has used the facts that $P(0) = 0$ and $P'(\rho) \leq c^2\gamma^2$. Then, one can obtain that

$$\frac{1 + \gamma^2}{1 - \gamma^2}(G_1 + G_2) \geq H_1 + H_2 \geq 2(G_1 + G_2),$$

pointwisely. \hfill \Box

Next, we will show that

**Proposition 7.1** \(\exp(I_1) \geq C_g \exp\left( \int_\frac{J_{\text{rel}}(\rho)}{J_{\text{clas}}(\rho)} \frac{d\sigma}{2\sqrt{2}(\sigma)} \right)\).

**Proof.** First, notice that, from Assumption 2, one has

$$P(\rho) \leq \rho P'(\rho). \quad (7.28)$$

Second, set $x = J_{\text{rel}}^{-1}(s)$ and substitute the integral variable, then one can rewrite $\exp(I_1)$ as

$$\exp(I_1) = \exp\left( \int_\rho^0 \Psi(x) dx \right) \quad \text{with} \quad \Psi(x) = \left( 1 + \frac{\sqrt{P'(x)}}{c} \right)^2 \frac{1}{2 \left( x + \frac{P(x)}{c^2} \right)}. \quad (7.29)$$

Similarly, one has

$$\exp\left( \int_\frac{J_{\text{rel}}(\rho)}{J_{\text{clas}}(\rho)} \frac{d\sigma}{2\sqrt{2}(\sigma)} \right) = \exp\left( \int_\frac{J_{\text{rel}}^{-1}(\rho)}{J_{\text{clas}}^{-1}(\rho/2)} \frac{1}{2x} dx \right).$$

It follows from (7.28) that

$$\left( 1 + \frac{\sqrt{P'(x)}}{c} \right)^2 \geq \frac{1}{x} \left( x + \frac{P(x)}{c^2} \right).$$

Then, combining the above relations, one can obtain that

$$\exp(I_1) = \exp\left( \int_\rho^0 \Psi(x) dx \right) = C_g \exp\left( \int_\rho^0 \Psi(x) dx \right) \geq C_g \exp\left( \int_\rho^0 \frac{1}{2x} dx \right). \quad (7.29)$$

\hfill \Box
This brings us back to the classical expression of Subsection 7.1.

7.2.3 Lower bound estimate on the mass-energy density

According to the conclusions obtained in the above two steps, now we give the desired lower bound estimates. First, according to the definitions of \( w \) and \( z \), (5.19) and (7.23), one can obtain that

\[
\rho' \geq -C_g \exp(\mathcal{I}_2) \frac{\rho + \frac{P(\rho)}{c^2}}{P'(\rho)^{1/4}} \geq -C_g \exp(\mathcal{I}_2) \frac{\rho}{P'(\rho)^{1/4}},
\]

(7.30)

where one has used the fact that \( P(\rho) \leq c^2 \rho \).

It follows from (7.30) that the integral of the classical expression

\[
\exp\left( \int_{\mathcal{I}_1}^{\rho} \frac{1}{2} \frac{dx}{P(\rho)} \right) (G_1(\rho) + G_2(\rho))
\]

diverges according to the results obtained in Subsection 7.1. Since

\[
\exp(\mathcal{I}_1)(H_1(\rho) + H_2(\rho)) \geq C_g \exp\left( \int_{\mathcal{I}_1}^{\rho} \frac{1}{2} \frac{dx}{P(\rho)} \right) (G_1(\rho) + G_2(\rho)),
\]

the result follows from the arguments used in Section 7.1.3.

8 Appendix

In order to support our theory on the singularity formation shown in Section 6, in this appendix, we show the local-in-time well-posedness of the smooth solution for the Relativistic Euler equations (1.1) with initial vacuum in multi-dimensional spacetime. The detailed proof can be found in Lefloch-Ukai [19].

8.1 Symmetric formulation of (1.1) allowing vacuum

First for simplicity, in the following we denote \( c_s = \sqrt{P'(\rho)} \) as the sound speed in the fluid. We refer to

\[
z_{\pm} := S(\rho) \pm R(u)
\]

as the generalized Riemann invariant variables, where

\[
R(u) := \frac{c}{2} \ln \left( \frac{c + |u|}{c - |u|} \right), \quad S(\rho) := \int_0^\rho \frac{\sqrt{P'(\sigma)}}{\sigma + \frac{P(\sigma)}{c^2}} d\sigma.
\]

(8.2)

We also introduce the projection operator and the normalized velocity

\[
E(u) := I_3 - \tilde{u} \otimes \tilde{u}, \quad \tilde{u} = \frac{u}{|u|}
\]

(8.3)

where \( I_3 \) represents a \( 3 \times 3 \) unit matrix. Then one has

Lemma 8.1 [19] In terms of the generalized Riemann invariant variables \( (z_+, z_-) \) and the normalized velocity \( \tilde{u} \) defined in (8.1)-(8.3), the relativistic Euler equations (1.1) take the following symmetric form

\[
A_0(W)W_t + \sum_{j=1}^3 A_j W_{z_j} = 0,
\]

(8.4)

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with $W = (z_+, z_-, \bar{u})$, in which $A^0(W)$ and $A^j(W)$ are

$$A^0(W) = \begin{pmatrix} a_0 & 0 & 0 \\ 0 & b_0 & 0 \\ 0 & 0 & c_0|u|^2I_3 \end{pmatrix}, \quad A^j(W) = \begin{pmatrix} a_1\bar{u}_j & 0 & a_2|u|e^j \\ 0 & b_1\bar{u}_j & a_2|u|e^j \\ a_2ve^j & -a_2|u|e^j & c_0|u|^2I_3 \end{pmatrix}, \quad (8.5)$$

and

$$a_0 = 1 + |u|c_s/c^2, \quad b_0 = 1 - |u|c_s/c^2, \quad c_0 = \frac{2}{1 - |u|^2/c^2}, \quad (8.6)$$

Using the Lorentz invariance property of the Euler equations, (8.4) can be also expressed in the transformed coordinates $(\bar{t}, \bar{x})$ defined by (8.8), that is

$$A^0(\bar{W})(\bar{W}) + \sum_{j=1}^{3} A^j(\bar{W})\bar{x}_j = 0, \quad (8.9)$$

where $\bar{W} = (\bar{z}_+, \bar{z}_-, \bar{u})$ is defined from the transformed unknowns $(\bar{\rho}, \bar{u})$. The expression of (8.7) becomes

$$< A^0(\bar{W})\eta, \eta > = \bar{a}_0|\eta_1|^2 + \bar{b}_0|\eta_2|^2 + \bar{c}_0|\bar{u}|^2|\bar{\eta}|^2, \quad (8.10)$$

where $\bar{a}_0$, $\bar{b}_0$ and $\bar{c}_0$ are defined by (8.6) with $\rho, u$ replaced by $\bar{\rho}, \bar{u}$.

We consider the Cauchy problem associated with (8.4) where initial data given on the initial hyperplane

$$H_0: \quad t = 0.$$ 

First, we observe that

$$< A^0(W)\eta, \eta > = a_0|\eta_1|^2 + b_0|\eta_2|^2 + c_0|u|^2|\eta|^2, \quad (8.7)$$

where $<, >$ denotes the Euclidian inner product in $\mathbb{R}^5$ and

$$\eta = (\eta_1, \eta_2, \ldots, \eta_5) = (\bar{z}_1, \bar{z}_2, \bar{u}), \quad \bar{\eta} = (\eta_3, \eta_4, \eta_5) \in \mathbb{R}^3.$$ 

From (8.6), the matrix $A^0(W)$ can be positive definite only if the velocity $u$ never vanishes. According to Friedlichs-Lax-Kato theory [16, 26], a local in time solution exists and is unique in the Sobolev space $H^\sigma$ for $\sigma > \frac{2}{7}$. However, this lower bound on velocity is not physically realistic.

In order to deal with this difficulty, Lefloch and Ukai [19] apply a well-chosen Lorentz transformation, which allows for the fact that the Lorentz-transformed velocity does not exceed some threshold and remains bounded away from the light speed.

### 8.2 Lorentz transformation

The Lorentz transformation $(t, x) \rightarrow (\bar{t}, \bar{x})$ associated with the vector $U \neq 0$ is defined by

$$\begin{cases}
\bar{t} = \varpi \left( t - \frac{U \cdot x}{c^2} \right), \\
\bar{x} = -\varpi Ut + \left( I_3 + \left( \varpi - 1 \right) \frac{U \otimes U}{U^2} \right) x, \\
\bar{\rho}(\bar{t}, \bar{x}) = \rho(t, x), \\
\bar{u} = \frac{dx}{dt} = \frac{1}{1 - |u|^2/c^2} \left( -U + \left( \varpi I_3 + \left( 1 - \varpi^{-1} \right) \frac{U \otimes U}{U^2} \right) u \right),
\end{cases} \quad (8.8)$$

where $\varpi = \frac{1}{\sqrt{1 - |u|^2/c^2}}$ represents the Lorentz factor.

Using the Lorentz invariance property of the Euler equations, (8.4) can be also expressed in the transformed coordinates $(\bar{t}, \bar{x})$ defined by (8.8), that is

$$A^0(\bar{W})(\bar{W}) + \sum_{j=1}^{3} A^j(\bar{W})\bar{x}_j = 0,$$
In view the upper and lower bounds (8.10), we conclude the transformed matrix \( A^0(\tilde{W}) \) is positive definite in the coordinate system \((\tilde{t}, \tilde{x})\). Hence Friedrichs-Lax-Kato theory \([16, 26]\) applies to the initial value problem for (8.9), provided initial data are imposed on the initial hypersurface \( \tilde{t} = 0 \). In the relativistic setting, the initial plane \( H : t = 0 \) is not preserved by the transformation (8.8). However, in the new coordinate system \((\bar{t}, \bar{x})\) the initial plane becomes

\[
\bar{H}_0 : \quad \bar{t} = -U \cdot \bar{x}.
\]

In order to prove local well-posedness for the oblique initial-value problem (8.9) with data in \( \bar{H}_0 \), it is convenient to introduce a further change of coordinates

\[
t' = \bar{t} + \frac{U \cdot \bar{x}}{c^2}, \quad x' = \bar{x},
\]

which maps the hyperplane \( \bar{H}_0 \) to the hyperplane

\[
\bar{H}'_0 : \quad t' = 0.
\]

This transformation puts the system (8.10) into the form

\[
B^0(W')(W')_{t'} + \sum_{j=1}^{3} B^j(W') W'_{x_j} = 0,
\]

where \( W' = (z'_+, z'_-, u')^T \) is defined from the transformed unknowns \((\rho'(t', x') = \bar{\rho}(\bar{t}, \bar{x}), u'(t', x') = \bar{u}(\bar{t}, \bar{x}))\),

\[
B^0(W') = A^0(\tilde{W}) + \frac{1}{c^2} \sum_{j=1}^{d} U_j A_j(\tilde{W}), \quad B_j(W') = A_j(\tilde{W}).
\]

It has been proved in [19] that \( B^0(W') \) is positive definite via choosing proper \( U \). Here we omit its proof.

### 8.3 Desired local-in-time well-posedness

In turn, Friedrichs-Lax-Kato theory \([16, 26]\) guarantees the existence of a solution defined in a small neighborhood of this hyperplane \( \bar{H}_0 \). Making the transformation back to the original variables, we obtain a solution in a small neighborhood of the initial line \( t = 0 \). This finally gives the following theorem:

**Theorem 8.1** [19] If the initial data \((\rho_0, u_0)\) satisfy the following regularity conditions:

\[
0 \leq \rho_0 \leq M_1, \quad |u_0| < c, \quad \sqrt{P'(\rho_0)} < c, \quad (\rho_0, u_0) \in H^3_{ul}(\mathbb{R}^3),
\]

for some constants \( M_1 > 0 \), then there exists a positive time \( T_* \) and a unique classical solution \((\rho, u)(t, x)\) in \([0, T_*] \times \mathbb{R}^3\) to the Cauchy problem (1.1)-(1.2) with (2.1) satisfying

\[
\begin{align*}
\rho & \geq 0, \quad |u| < c, \quad \sqrt{P'(\rho)} < c, \\
(\rho, u) & \in C([0, T_*]; H^3_{ul}(\mathbb{R}^3)) \cap C^1([0, T_*]; H^2_{ul}(\mathbb{R}^3)), \\
u_t + u \cdot \nabla u & = 0 \quad \text{whenever} \quad \rho(t, x) = 0.
\end{align*}
\]

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