A generalization of Taub-NUT deformations

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Abstract

We introduce a generalization of Taub-NUT deformations for large families of hyper-Kähler quotients including toric hyper-Kähler manifolds and quiver varieties. It is well-known that Taub-NUT deformations are defined for toric hyper-Kähler manifolds, and the similar deformations were introduced for ALE hyper-Kähler manifolds of type $D_k$ by Dancer, using the complete hyper-Kähler metric on the cotangent bundle of complexification of compact Lie group. We generalize them and study the Taub-NUT deformations for the Hilbert schemes of $k$ points on $\mathbb{C}^2$.

1 Introduction

1.1 Taub-NUT spaces

A hyper-Kähler manifold is a Riemannian manifold $(M, g)$ equipped with orthogonal integrable complex structures $I_1, I_2, I_3$ with quaternionic relations

\[ I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1 \]

so that each $(M, g, I_i)$ is Kählerian. Then $M$ admits three symplectic forms $\omega_i := g(I_i \cdot , \cdot )$, each of which is Ricci-flat Kähler metric with respect to $I_i$. Throughout of this article, we regard $(M, I_1)$ as a complex manifold with a Ricci-flat Kähler metric $\omega_1$ and a non-degenerate closed $(2,0)$-form $\omega_2 + \sqrt{-1} \omega_3$, so-called a holomorphic symplectic structure.

The Euclidean space $\mathbb{C}^2 = \mathbb{R}^4$ is the trivial example of complete hyper-Kähler manifold, whose Ricci-flat Kähler metric is Euclidean and the holomorphic symplectic structure is given by $dz \wedge dw$, where $(z, w) \in \mathbb{C}^2$ is the standard holomorphic coordinate.

In [7], Hawking constructed a complete hyper-Kähler metric on $\mathbb{R}^4$ with cubic volume growth which is called a Taub-NUT space.
LeBrun [13] showed that the Taub-NUT space and the Euclidean space $\mathbb{C}^2$ are isomorphic as holomorphic symplectic manifolds, consequently biholomorphic. It means that the complex manifold $\mathbb{C}^2$ admits at least 2 complete Ricci-flat Kähler metrics which are not isometric. Such a phenomenon should never occur on compact complex manifolds, due to the uniqueness of the Ricci-flat Kähler metrics in each Kähler class. The similar relation also holds between multi Eguchi-Hanson spaces and multi Taub-NUT spaces.

A generalization to the higher dimensional case are obtained by Gibbons, Rychenkova, Goto [6] and Bielawski [1]. They construct Taub-NUT like hyper-Kähler metrics by deforming the toric hyper-Kähler manifolds, using tri-Hamiltonian torus actions and hyper-Kähler quotient method.

In [2], Dancer has defined the analogy of Taub-NUT deformations for some of the ALE spaces of type $D_k$ using $U(2)$-actions. His results are based on the existence of hyper-Kähler metrics on $T^*G^\mathbb{C}$ for any compact Lie group $G$ constructed by Kronheimer [12]. Another generalization to noncommutative case is considered in Section 5 of [3]. They considered hyper-Kähler modifications for hyper-Kähler manifolds with a tri-Hamiltonian $H$-action, where $H$ is a compact Lie group which is possibly noncommutative. Note that the case of [3] does not contains the results in [2], since the ALE spaces of type $D_k$ have no nontrivial tri-Hamiltonian actions.

In this paper, we generalize Taub-NUT deformations for some kinds of hyper-Kähler quotients, which enable us to treat the above three cases [6, 1], [2] and Section 5 of [3] uniformly. As a consequence, we apply the Taub-NUT deformations for the Hilbert schemes of $k$-points on $\mathbb{C}^2$.

### 1.2 Notation and a main result

Here, we describe the main result in this paper more precisely. Let a compact connected Lie group $H$ act on a hyper-Kähler manifold $(M, g, I_1, I_2, I_3)$ preserving the hyper-Kähler structure and there exists a hyper-Kähler moment map $\hat{\mu} : M \to \text{Im} \mathbb{H} \otimes \mathfrak{h}^*$ with respect to $H$-action, where $\text{Im} \mathbb{H} \cong \mathbb{R}^3$ be the pure imaginary part of quaternion $\mathbb{H}$ and $\mathfrak{h}^*$ is the dual space of the Lie algebra $\mathfrak{h} = \text{Lie}(H)$. Moreover, suppose the $H$-action extends to holomorphic $H^\mathbb{C}$ action on $(M, I_1)$, where $H^\mathbb{C}$ is the complexification of $H$. Let $\rho : H \to G \times G$ is a homomorphism of Lie groups, where $G$ is compact connected Lie group. Then $H_\rho := \rho^{-1}(\Delta_G) \subset H$ acts on $M$, where $\Delta_G \subset G \times G$ is the diagonal subgroup, and the inclusion $\iota : H_\rho \to H$ induces a hyper-Kähler moment map $\mu := \iota^* \circ \hat{\mu}$. If we denote by $Z_H \subset \mathfrak{h}^*$ the subspace of fixed points by coadjoint action of $H$ on $\mathfrak{h}^*$, then we have the hyper-Kähler quotient $\mu^{-1}(\iota^* \zeta)/H_\rho$ for each $\zeta \in \text{Im} \mathbb{H} \otimes Z_H$. In this paper we define Taub-NUT deformations for $\mu^{-1}(\iota^* \zeta)/H_\rho$ by the following way.
Let $N_G = T^*G^c$ be the hyper-Kähler manifolds with a $G \times G$-action constructed by Kronheimer [12], and $\nu : N_G \to \text{Im}H \otimes (g \oplus g)^*$ be its hyper-Kähler moment map described by Dancer and Swann [3]. Then $H$ acts on $M \times N_G$ by $\rho$, and for $(x, p) \in M \times N_G$, $\sigma(x, p) := \mu(x) + \rho^*(\nu(p))$ becomes the hyper-Kähler moment map, accordingly we obtain a hyper-Kähler quotient $\sigma^{-1}(\zeta)/H$ for each $\zeta \in \text{Im}H \otimes Z_H$. Now we have two hyper-Kähler quotients $\mu^{-1}(t^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$. If they are smooth, then there are Ricci-flat Kähler metrics $\omega_1^{i\zeta}, \omega_2^{i\zeta}$ and holomorphic symplectic structures $\omega_2^{i\zeta} + \sqrt{-1}\omega_3^{i\zeta}, \omega_2^{i\zeta} + \sqrt{-1}\omega_3^{i\zeta}$, respectively. Next we extend $\rho$ to the holomorphic homomorphism $H^c \to G^c \times G^c$ and obtain a holomorphic map

$$\bar{\rho} : H^c/H^c_\rho \to (G^c \times G^c)/\Delta G^c.$$  \hfill (1)

Then the main result is described as follows.

**Theorem 1.1.** Let $M = \mathbb{H}^N$, and $H \subset \text{Sp}(N)$ acts on $M$ naturally. Assume $\bar{\rho}$ is surjective. Then there exists a biholomorphism as complex analytic spaces

$$\psi : \mu^{-1}(t^*\zeta)/H_\rho \to \sigma^{-1}(\zeta)/H$$

for each $\zeta \in Z_H$. Moreover, if $H_\rho$ acts on $\mu^{-1}(t^*\zeta)$ freely, then $\mu^{-1}(t^*\zeta)/H_\rho$ and $\sigma^{-1}(\zeta)/H$ are smooth complete hyper-Kähler manifolds and $\psi$ satisfies

$$[\psi^*\omega_1^{i\zeta}]_{\text{DR}} = [\omega_1^{i\zeta}]_{\text{DR}}, \quad \psi^*(\omega_2^{i\zeta} + \sqrt{-1}\omega_3^{i\zeta}) = \omega_2^{i\zeta} + \sqrt{-1}\omega_3^{i\zeta},$$

where $[,]_{\text{DR}}$ is a de Rham cohomology class.

### 1.3 Hilbert schemes of $k$ points on $\mathbb{C}^2$

We apply the above theorem to $M = \text{End}(\mathbb{C}^k) \otimes \mathbb{H}$, $H = U(k) \times U(k)$, $G = U(k)$ and $\rho = \text{id}$. Then $Z_H = \mathbb{R}$, and $\mu^{-1}(t^*\zeta)/H_\rho$ becomes a quiver varieties constructed in [13], called the Hilbert scheme of $k$-points of $\mathbb{C}^2$. In particular, $\mu^{-1}(0)/H_\rho$ is isomorphic to $(\mathbb{C}^2)^k/S_k$ with Euclidean metric as hyper-Kähler orbifolds. In this case, $\sigma^{-1}(\zeta)/H$ becomes a smooth hyper-Kähler manifolds diffeomorphic to $\mu^{-1}(t^*\zeta)/H_\rho$ by Theorem 1.1 and the hyper-Kähler metric on $\sigma^{-1}(0)/H$ can be described concretely.

**Theorem 1.2.** In the above situation, we have an isomorphism

$$\sigma^{-1}(0)/H \cong (\mathbb{C}^2_{\text{Taub-NUT}})^k/S_k$$

as hyper-Kähler orbifolds, where $\mathbb{C}^2_{\text{Taub-NUT}}$ is the Taub-NUT space.
1.4 Outline of the proof

Theorem 1.1 is proven in the following way. The hyper-Kähler moment map \( \mu \) is decomposed into \( \mu = (\mu_1, \mu_2 := \mu_2 + \sqrt{-1} \mu_3) \) along the decomposition \( \text{Im} \mathbb{H} = \mathbb{R} \oplus \mathbb{C} \), where \( \mu_1 : M \to \mathfrak{h}_c^* \) and \( \mu_2 : M \to (\mathfrak{h}^C_c)^* \), and the other hyper-Kähler moment maps and the parameter \( \zeta \in \text{Im} \mathbb{H} \otimes Z_H \) are also decomposed in the same manner. Define sets of “stable points” by

\[
\mu_c^{-1}(\iota^* \zeta)_{\iota \zeta}, \quad \sigma_c^{-1}(\zeta)_{\iota \zeta}
\]

induce

\[
\mu^{-1}(\iota^* \zeta)/H_\rho \to \mu_c^{-1}(\iota^* \zeta), \quad \sigma^{-1}(\zeta)/H \to \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C,
\]

which are isomorphisms of complex analytic spaces by [8]. Here, to regard \( \mu_c^{-1}(\iota^* \zeta), \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C \) as complex analytic spaces, we consider the sets of “semistable points” \( \mu_c^{-1}(\iota^* \zeta), \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C \) in Section 1.3.

Thus the proof of Theorem 1.1 is reduced to construct an isomorphism between \( \mu_c^{-1}(\iota^* \zeta), \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C \). First of all, we define an \( H^C_\rho \) equivariant holomorphic map \( \psi : \mu_c^{-1}(\iota^* \zeta) \to \sigma_c^{-1}(\zeta)_{\iota \zeta} \) so that an induced map \( \hat{\psi} : \mu_c^{-1}(\iota^* \zeta) / H^C_\rho \to \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C \) is bijective.

Then it suffices to see that \( \psi \) gives a one-to-one correspondence between \( \mu_c^{-1}(\iota^* \zeta), \sigma_c^{-1}(\zeta)_{\iota \zeta} / H^C \). To show it, we describe some equivalent conditions for \( x \) and \( \psi(x) \) to be \( x \in \mu_c^{-1}(\iota^* \zeta)_{\iota \zeta} \) and \( \psi(x) \in \sigma_c^{-1}(\zeta)_{\iota \zeta} \) in Section 3 using some convex functions on \( G \backslash G_c \). We also need the description of the Kähler potential of \( N_c \), which is discussed in Section 2.

This paper is organized as follows. We review the construction of hyper-Kähler structures on \( N_G = T^* G_c \) along [12], and describe hyper-Kähler moment map by [2] in Section 2. Moreover, we describe the Kähler potentials of the hyper-Kähler metrics using the method of [9] and [10].

In Section 3, we obtain other description of \( \mu_c^{-1}(\iota^* \zeta)_{\iota \zeta} \) and \( \sigma_c^{-1}(\zeta)_{\iota \zeta} \), we study the relation between a Kähler moment map and some geodesically convex functions on Riemannian symmetric spaces.

In Section 4, we prove Theorem 1.1 by using the description of Kähler potentials obtained in Section 2 and the methods in Section 3.

In Section 5, we apply Theorem 1.1 to the Hilbert schemes of \( k \) points on \( \mathbb{C}^2 \) and show Theorem 1.2. Moreover, we see that Theorem 1.1 can be applied to quiver varieties and toric hyper-Kähler varieties.
2 Hyper-Kähler structures on $T^*G^\mathbb{C}$

2.1 Riemannian description

Here we review briefly the hyper-Kähler quotient construction of $N_G$ along [12], and describe hyper-Kähler moment map $\nu$ along [2] [3].

Let $G$ be a compact connected Lie group, and $\|\cdot\|$ is a norm on $\mathfrak{g}$ induced by an $\text{Ad}_G$-invariant inner product. Consider the following equations

$$\frac{dT_i}{ds} + [T_0, T_i] + [T_j, T_k] = 0 \quad \text{for } (i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2), (2)$$

for $T := (T_0, T_1, T_2, T_3) \in C^1([0, 1], \mathfrak{g}) \otimes \mathbb{H}$. Put

$$N_G := \{ T \in C^1([0, 1], \mathfrak{g}) \otimes \mathbb{H}; \text{ equations (2) holds} \}.$$

Then a gauge group $\mathcal{G} := C^2([0, 1], G)$ acts on $N_G$ by

$$g \cdot T := (\text{Ad}_g T_0 + g\frac{d}{ds}g^{-1}, \text{Ad}_g T_1, \text{Ad}_g T_2, \text{Ad}_g T_3),$$

and we obtain

$$N_G := N_G / G_0,$$

where $G_0 := \{ g \in \mathcal{G}; g(0) = g(1) = 1 \}$. It is shown in [12] that $N_G$ becomes a $C^\infty$ manifold of dimension $4 \dim G$, and the standard hyper-Kähler structure on $C^1([0, 1], \mathfrak{g}) \otimes \mathbb{H}$ induces a hyper-Kähler structure $g_G, I_{G,1}, I_{G,2}, I_{G,3}$ on $N_G$. Here, $g_G$ is induced from the $L^2$-inner product on $C^1([0, 1], \mathfrak{g}) \otimes \mathbb{H}$ using Ad$_G$-invariant inner product on $\mathfrak{g}$.

Now we have a Lie group isomorphism $\mathcal{G}/G_0 = G \times G$ defined by $g \mapsto (g(0), g(1))$. Since $\mathcal{G}$ acts on $N_G$, there exists a $G \times G$-action on $N_G$ preserving the hyper-Kähler structure.

**Theorem 2.1** (2 [3]). The hyper-Kähler moment map $\nu = (\nu^0, \nu^1) : N_G \to \text{Im}\mathbb{H} \otimes (\mathfrak{g}^* \oplus \mathfrak{g}^*)$ with respect to the action of $G \times G$ on $N_G$ is given by

$$\nu^0([T]) = (T_1(0), T_2(0), T_3(0)), \quad \nu^1([T]) = -(T_1(1), T_2(1), T_3(1)),$$

under the identification $\mathfrak{g}^* \cong \mathfrak{g}$ using Ad$_G$-invariant inner product. Here we denote by $[T] \in N_G$ the equivalence class represented by $T \in N_G$. 
2.2 Holomorphic description

In this subsection, we review that \((N_G, I_G, 1)\) is identified with a holomorphic cotangent bundle \(T^*G^C \cong G^C \times g^C\) as complex manifolds along \([5][12]\).

For each \(T \in N_G\), there exists a solution \(u : [0, 1] \to G^C\) for an ordinary differential equation

\[
\frac{du}{ds}u^{-1} = -(T_0 + \sqrt{-1}T_1),
\]

then \(T_2 + \sqrt{-1}T_3 = \text{Ad}_{u(s)u(0)^{-1}}\eta\) for some \(\eta \in g^C\) \([12]\). Then a holomorphic map \(\Phi : (N_G, I_G, 1) \to G^C \times g^C\) is obtained by \([T] \mapsto (u(0)u(1)^{-1}, \eta)\).

**Theorem 2.2 \([5][12]\).** The map \(\Phi\) is biholomorphic and preserves holomorphic symplectic structures.

The moment map \(\nu\) is decomposed into \(\nu = (\nu_R = \nu_1, \nu_C = \nu_2 + \sqrt{-1}\nu_3)\) along the decomposition \(\text{Im}H \cong \mathbb{R} \oplus \mathbb{C}\). Then

\[
\nu_C(Q, \eta) = (\eta, \text{Ad}_{Q^{-1}}(\eta))
\]

under the identification \(N_G = G^C \times g^C\). \(\nu_C\) is a holomorphic moment map with respect to \(G^C \times g^C\) action on \(T^*G^C\). This action is given by

\[
(g_0, g_1)(Q, \eta) = (g_0Qg_1^{-1}, \text{Ad}_{g_0}\eta)
\]

for \((g_0, g_1) \in G^C \times G^C\) and \((Q, \eta) \in G^C \times g^C\).

2.3 Kähler potentials

Next we describe the Kähler potential of the Kähler manifold \((N_G, g_G, I_{G,1})\).

**Lemma 2.3 \([9]\).** Let \((M, g, I_1, I_2, I_3)\) be a hyper-Kähler manifold with isometric \(S^1\) action generated by a Killing field \(X\), which satisfies

\[
L_X\omega_1 = \omega_2, \quad L_X\omega_2 = -\omega_1, \quad L_X\omega_3 = 0,
\]

where \(\omega_i\) is the Kähler form of \((M, g, I_i)\). If \(\mu\) is the moment map with respect to the \(S^1\)-action on \((M, \omega_3)\), then \(\omega_1 = 2\sqrt{-1}\partial_1\bar{\partial}_1\mu, \omega_2 = 2\sqrt{-1}\partial_2\bar{\partial}_2\mu\).

We apply this lemma as follows. Let \(\omega_G := g_G(I_{G,i}, \cdot, \cdot), \) and define \(e^{i\theta} \cdot [T] := [T_0, \cos \theta T_1 + \sin \theta T_2, -\sin \theta T_1 + \cos \theta T_2, T_3]\) for \([T] \in N_G\), which is an \(S^1\)-action on \(N_G\) preserving \(\omega_G\) and satisfies the assumption of Lemma
and a $G$-action defined by $e^{\theta}$: $[T] := [T_0, -\sin \theta T_3 + \cos \theta T_1, T_2, \cos \theta T_3 + \sin \theta T_1]$, which preserves $\omega_{G,2}$. In this case the moment map is given by $\|T_1\|^2_{L^2} + \|T_3\|^2_{L^2}$. Thus we obtain the followings from Lemma 2.3.

**Proposition 2.4.** Put $\mathcal{E} : N_G \to \mathbb{R}$ to be

$$\mathcal{E}([T]) := \|T_1\|^2_{L^2} + \frac{1}{2}(\|T_2\|^2_{L^2} + \|T_3\|^2_{L^2}).$$

Then $\omega_{G,1} = 2\sqrt{-1}\partial\bar{\partial}\mathcal{E}$.

Next we describe the Kähler potential $\mathcal{E}$ as a function on $T^*G^C = G^C \times g^C$. The $\text{Ad}_G$-invariant inner product on $g$ induces a homogeneous Riemannian metric on $G \backslash G^C$. Define an antiholomorphic involution of $G^C$ by $(g e^{\sqrt{-1}T})^* := e^{\sqrt{-1}T} g^{-1}$ for $g \in G$ and $\xi \in g$, by using the polar decomposition $G^C \cong G \cdot \exp(\sqrt{-1}g)$. Then $G \backslash G^C$ is identified with $\exp(\sqrt{-1}g)$ by $G \cdot g \mapsto g \cdot g$ for $g \in G^C$. This metric on $G \backslash G^C$ is naturally extended to a hermitian metric on $T_h(G \backslash G^C) \otimes \mathbb{C}$, which is also denoted by $\| \cdot \|_h$.

**Proposition 2.5.** For each $(Q, \eta) \in G^C \times g^C = N_G$,

$$\mathcal{E}(Q, \eta) = \frac{1}{2} \min_{h \in P(a^*a, b^*b)} \int_0^1 \left( \left\| \frac{dh}{ds} \right\|^2_h + \| h \text{Ad}_a^{-1}(\eta) \|_h^2 \right) ds,$$

where $ab^{-1} = Q$ and $P(h_0, h_1) := \{ h \in C^\infty([0, 1], G \backslash G^C); h(0) = h_0, h(1) = h_1 \}$.

**Proof.** The essential part of the proof is obtained in [5], and we explain the outline. Define $\mathcal{L} : C^1([0, 1], g) \otimes \mathbb{H} \to \mathbb{R}$ by $\mathcal{L}(T) := \|T_1\|^2_{L^2} + \frac{1}{2}(\|T_2\|^2_{L^2} + \|T_3\|^2_{L^2})$. Fix $T \in N_G$, then $\mathcal{L}_{|G_0^C \cdot T}$ attains its minimum value at $T$ by Lemma 2.3 of [5], hence

$$\mathcal{E}([T]) = \min_{g \in G_0^C} \mathcal{L}(g \cdot T).$$

Here $G_0^C$ is the complexified gauge group defined by

$$G_0^C := \{ g \in G^C = C^2([0, 1], G^C); g(0) = g(1) = 1 \},$$

and a $G_0^C$ action on $C^1([0, 1], g) \otimes \mathbb{H}$ is defined by $g \cdot (T_0 + \sqrt{-1}T_1, T_2 + \sqrt{-1}T_3) := (\text{Ad}_g(T_0 + \sqrt{-1}T_1), g \cdot (T_2 + \sqrt{-1}T_3))$. Take $u : [0, 1] \to G^C$ as in Section 2.2 and let $(Q, \eta) \in G^C \times g^C$ be corresponding to $[T] \in N_G$.
under the identification given in Section 2.2. Then we have
\[ Q = u(0)u(1) - 1, \]
\[ T_2(s) + \sqrt{-1} T_3(s) = \text{Ad}_{u(s)u(0)^{-1}}\eta. \]
For \( g \in G_C \),
\[ g \cdot T = \left( g(s)u(s) \frac{d}{ds} (gu)^{-1}(s), \text{Ad}_{g(s)u(s)} \text{Ad}_{u(0)^{-1}}\eta \right). \]

Now we extend the \( \text{Ad}_G \)-invariant inner product on \( g \) to the \( C \) bilinear form on \( g_C \). From the calculation in the proof of Lemma 2.3 of [5], we obtain
\[ L(g \cdot T) = \int_0^1 \left( \frac{1}{4} \| h' \|_h^2 + \frac{1}{2} \| h \text{Ad}_{u(0)^{-1}}(\eta) \|_{h_0}^2 \right) ds, \]
where \( h = (gu)^*(gu) \). Thus we obtain
\[ \mathcal{E}([T]) = \min_{h_0 \in P(h_0, h_1)} \int_0^1 \left( \frac{1}{4} \| h' \|_h^2 + \frac{1}{2} \| h \text{Ad}_{u(0)^{-1}}(\eta) \|_{h_0}^2 \right) ds, \]
where \( h_0 = u(0)^*u(0) \), \( h_1 = u(1)^*u(1) \) and \( u(0)u(1)^{-1} = Q \), hence we have the assertion.

3 Kähler manifolds with Hamiltonian actions

Let \( L^C \) be a complexification of connected compact Lie group \( L \), and \( L^C \) acts on a complex manifold \( (X, I) \) holomorphically. Let \( \omega \) be a Kähler form on \( X \), and \( L \) acts on \( (X, I, \omega) \) isometrically. Suppose that there is a symplectic moment map \( m : X \to \mathfrak{l}^* \) with respect to \( L \) action, where \( \mathfrak{l} \) is the Lie algebra of \( L \).

Fix an \( \text{Ad}_L \)-invariant inner product on \( \mathfrak{l} \), then the Riemannian metric on the homogeneous space \( L/L^C \) is induced. We denote by \( Lg \in L/L^C \) the equivalence class represented by \( g \in L^C \), and put
\[ \xi_g := \frac{d}{dt} \bigg|_{t=0} L e^{\sqrt{-1} t \xi} g \in T_{Lg}(L/L^C) \]
for each \( \xi \in \mathfrak{l} \), where \( \{L e^{\sqrt{-1} t \xi} g\}_{t \in \mathbb{R}} \) is a geodesic through \( Lg \).

Now define a 1-form \( \alpha_{x, \zeta} \in \Omega^1(L/L^C) \) by \( (\alpha_{x, \zeta})_{Lg}(\xi_g) := \langle m(gx) - \zeta, \xi \rangle \) for \( \zeta \in Z_L \), which is easily checked to be closed. Since \( L/L^C \cong 1 \) is simply-connected, \( \alpha_{x, \zeta} \) is \( d \)-exact and there is a unique primitive function up to constant. Accordingly, there is a unique function \( \Phi_{x, \zeta} : L/L^C \to \mathbb{R} \) satisfying \( d\Phi_{x, \zeta} = \alpha_{x, \zeta} \) and \( \Phi_{x, \zeta}(L \cdot 1) = 0 \). Here, the latter equality is a normalization for removing the ambiguity, and it is not essential. Then it is easy to check that \( \Phi_{x, \zeta} \) is a geodesically convex function on the Riemannian symmetric space \( L/L^C \).
From the argument in [10][16], the naturally induced map
\[ m^{-1}(\zeta)/L \to X_\zeta/L^C \]
becomes a homeomorphism, where
\[ X_\zeta := \{ x \in X ; \Phi_{x,\zeta} \text{ has a critical point} \}. \]

In this subsection we show some equivalent conditions for the existence of the critical point of \( \Phi_{x,\zeta} \).

For \( g \in L^C \), we denote by \( R_g \) the isometry on \( L \setminus L^C \) given by the right action of \( L^C \) on \( L \setminus L^C \). Then we can check that \( R_g^*\alpha_{x,\zeta} = \alpha_{gx,\zeta} \), and \( R_g^*\Phi_{x,\zeta} - \Phi_{gx,\zeta} \) becomes a constant function, hence \( X_\zeta \) is \( L^C \)-closed.

Let \( \text{Stab}(x)^C := \{ g \in L^C ; gx = x \} \), and \( \text{stab}(x)^C \) be the Lie algebra of \( \text{Stab}(x)^C \). We put
\[ \text{Stab}(x) := \{ g \in L ; gx = x \} = \text{Stab}(x)^C \cap L, \]
\[ \text{stab}(x) := \text{Lie}(\text{Stab}(x)) = \text{stab}(x)^C \cap l. \]

Note that \( \text{Stab}(x)^C \) contains the complexification of \( \text{Stab}(x) \) as a subgroup, though it is not necessary to be equal. Let \( \pi_{\text{im}} : 1^C \to 1 \) be defined by \( \pi_{\text{im}}(a + \sqrt{-1}b) = b \) for \( a, b \in L \), and put \( \overline{\text{stab}}(x) := \pi_{\text{im}}(\text{stab}(x)^C) \). Then there is the orthogonal decomposition \( 1 = \overline{\text{stab}}(x) \oplus V_x \) with respect to \( \text{Ad}_L \)-invariant inner product on \( 1 \).

**Lemma 3.1.** For each \( g \in L^C \), there are \( \gamma \in \text{Stab}(x)^C \) and \( \xi \in V_x \) such that \( Lg\gamma = Lg\sqrt{-1}\xi \).

**Proof.** Consider the smooth function \( f : Lg \cdot \text{Stab}(x)^C \to \mathbb{R} \) defined by
\[ f(Lg) := \text{dist}_{L \setminus L^C}(L \cdot 1, Lg\gamma)^2, \]
where \( Lg \cdot \text{Stab}(x)^C \subset L \setminus L^C \) is the \( \text{Stab}(x)^C \)-orbit through \( Lg \). Now \( \text{Stab}(x)^C \) is closed in \( L^C \), then \( Lg \cdot \text{Stab}(x)^C \) is the closed orbit, hence \( f \) is proper. Since \( f \) is bounded from below, there is a minimum point \( Lg\gamma_0 \in Lg \cdot \text{Stab}(x)^C \).

By the polar decomposition \( L^C \cong L \times 1 \), we can take \( h \in L \) and \( \xi \in 1 \) such that \( hg\gamma_0 = e^{\sqrt{-1}\xi} \). Under the identification \( T_{Lg\gamma_0}(L \setminus L^C) \cong 1 \), the subspace \( T_{Lg\gamma_0}(Lg \cdot \text{Stab}(x)^C) \) is identified with \( \overline{\text{stab}}(hg\gamma_0 x) = \text{stab}(hx) \). Since the derivative of \( f \) at \( Lg\gamma_0 \) vanishes, we have \( \xi \in V_{hx\gamma_0 x} \).
Take \( \hat{b} \in \text{stab}(x) \) arbitrarily, and fix \( \hat{a} \in \mathcal{I} \) such that \( \hat{a} + \sqrt{-1} \hat{b} \in \text{stab}(x)^C \). Since \( \text{stab}(hg\gamma_0x)^C = \text{Ad}_{hg\gamma_0}(\text{stab}(x)^C) \), there is \( a + \sqrt{-1}b \in \text{stab}(hg\gamma_0x)^C \) and \( a + \sqrt{-1}b = \text{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b}) \). Since \( b \in \text{stab}(hg\gamma_0x) \), we have

\[
0 = 2\sqrt{-1}\langle \xi, b \rangle = \langle \xi, \text{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b}) \rangle + \langle \text{Ad}_{hg\gamma_0}(\hat{a} + \sqrt{-1}\hat{b})^*, (\hat{a} + \sqrt{-1}\hat{b}) \rangle
\]

\[
= \langle \text{Ad}_{hg\gamma_0}^{-1}\xi, \hat{a} + \sqrt{-1}\hat{b} \rangle + \langle \text{Ad}_{hg\gamma_0}\xi, (\hat{a} + \sqrt{-1}\hat{b})^* \rangle
\]

\[
= \langle \text{Ad}_{\sqrt{-1}\gamma_0}\xi, \hat{a} + \sqrt{-1}\hat{b} \rangle + \langle \text{Ad}_{\sqrt{-1}\gamma_0}\xi, (\hat{a} + \sqrt{-1}\hat{b})^* \rangle
\]

\[
= \langle \xi, \hat{a} + \sqrt{-1}\hat{b} \rangle + \langle \xi, -\hat{a} + \sqrt{-1}\hat{b} \rangle
\]

\[
= 2\sqrt{-1}\langle \xi, \hat{b} \rangle.
\]

Thus we obtain \( \xi \in V_x \).

\[
\square
\]

**Proposition 3.2.** \( \Phi_{x,\zeta} \) has a critical point if and only if all of the following conditions are satisfied for some \( g \in L^C \); (i) \( \Phi_{gx,\zeta} \) is Stab(\( gx \))^C invariant, (ii) \( \lim_{t \to \infty} \Phi_{gx,\zeta}(Le^{\sqrt{-1}\zeta}) = \infty \) for any \( \zeta \in \mathcal{I} \).

**Proof.** Let \( \Phi_{x,\zeta} \) has a critical point \( Lg \in L \setminus L^C \) for \( g \in L^C \). Since \( \Phi_{gx,\zeta} - R_g^*\Phi_{x,\zeta} \) is a constant function, we may suppose \( g = 1 \) by the homogeneity. Then \( \frac{d}{dt} |_{t=0} \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) = 0 \) for all \( \zeta \in \mathcal{I} \). If \( \zeta \not\in \text{stab}(x) \), especially \( \zeta \not\in \text{stab}(x) \),

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) = ||\zeta||_\omega^2 > 0
\]

and there exists sufficiently small \( \delta > 0 \) and \( \frac{d}{dt} \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) \geq \delta \) for all \( t \geq 1 \). Thus we obtain \( \lim_{t \to \infty} \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) = \infty \). For any \( \zeta \in \mathcal{I} \), we have \( \frac{d}{dt} \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) \geq 0 \) for any \( t > 0 \) and obtain \( \Phi_{x,\zeta}(Le^{\sqrt{-1}\zeta}) \geq \Phi_{x,\zeta}(L \cdot 1) \), hence \( \Phi_{x,\zeta}(L \cdot 1) \) is the minimum value of \( \Phi_{x,\zeta} \). Especially \( \Phi_{x,\zeta} \) is bounded from below. Next we show that \( \Phi_{x,\zeta} \) is Stab(\( x \))^C invariant. For any \( \gamma \in \text{Stab}(x)^C \), \( R_\gamma^*\Phi_{x,\zeta} - \Phi_{x,\zeta} \) is a constant function. If we put \( R_\gamma^*\Phi_{x,\zeta} - \Phi_{x,\zeta} = c_\gamma \in \mathbb{R} \), then we have \( \Phi_{x,\zeta}(L \gamma) = \Phi_{x,\zeta}(L \cdot 1) + c_\gamma \) and \( \Phi_{x,\zeta}(L^{-1}) = \Phi_{x,\zeta}(L \cdot 1) - c_\gamma \). Since \( \Phi_{x,\zeta}(L \cdot 1) \) is a minimum value, then \( c_\gamma \) should be zero, which implies \( \Phi_{x,\zeta} \) is Stab(\( x \))^C invariant.

Conversely, assume that the conditions (i)-(ii) hold. It suffices to show that \( \Phi_{gx,\zeta} \) has a minimum point in \( L \setminus L^C \). To see it, it suffices to see that \( \Phi_{gx,\zeta}|_{\exp(V_{gx})} \) has a minimum point by applying Lemma 3.1, where

\[
\exp(V_{gx}) := \{ L \cdot e^{\sqrt{-1}\zeta} \in L \setminus L^C ; \zeta \in V_{gx} \}
\]

and \( L = \text{stab}(gx) \oplus V_{gx} \) is the orthogonal decomposition with respect to the Ad_L-inverse inner product. Define a smooth function \( F : S(V_{gx}) \times \mathbb{R} \to \mathbb{R} \).
by

\[ F(\xi, t) := \frac{d}{dt}\Phi_{g_x,\xi}(e^{\sqrt{-1}\xi}), \]

where \( S(V_{gx}) := \{ \xi \in V_{gx}; \|\xi\| = 1 \} \). Now, recall that \( t \mapsto \Phi_{g_x,\xi}(e^{\sqrt{-1}\xi}) \)

is convex and we have assumed that \( \lim_{|t| \to \infty} \Phi_{g_x,\xi}(e^{\sqrt{-1}\xi}) = \infty \) for all \( \xi \in S(V_{gx}) \). It implies that there exists a unique \( \bar{t}(\xi) \in \mathbb{R} \) for each \( \xi \in S(V_{gx}) \) such that \( F(\xi, \bar{t}(\xi)) = 0 \). Since \( \frac{\partial}{\partial t} F(\xi, t) = \|\xi_x\|_\omega^2 > 0, \bar{t} : S(V_{gx}) \to \mathbb{R} \)

is smooth by Implicit Function Theorem. In particular, \( \xi \mapsto \Phi_{g_x,\xi}(e^{\sqrt{-1}\xi}) \)

becomes a smooth function on the compact manifold \( S(V_{gx}) \), hence it has a minimum point \( \xi_{min} \in S(V_{gx}) \). It is easy to see \( \exp(\bar{t}(\xi_{min}))\xi_{min} \in \exp(V_{gx}) \)

is a minimum point of \( \Phi_{g_x,\xi}\exp(V_{gx}) \).

Since each \( \text{Stab}(x)^C \)-orbit is closed in \( L \setminus L^C \), the distance on \( L \setminus L^C \)

induces a structure of a metric space on \( L \setminus L^C / \text{Stab}(x)^C \). If \( \Phi_{x,\xi} \)

is \( \text{Stab}(x)^C \)-invariant, then it induces a function \( \tilde{\Phi}_{x,\xi} : L \setminus L^C / \text{Stab}(x)^C \to \mathbb{R} \).

**Proposition 3.3.** \( \Phi_{x,\xi} \) has a critical point if and only if \( \Phi_{x,\xi} \) is \( \text{Stab}(x)^C \)-invariant, and \( \tilde{\Phi}_{x,\xi} \) is proper and bounded from below.

**Proof.** Assume that \( \Phi_{x,\xi} \) has a critical point. Then the conditions (i)-(ii) in Proposition 3.2 are satisfied for some \( g \in L^C \), accordingly it suffices to show the properness of \( \tilde{\Phi}_{x,\xi} \). Define an equivalence relation in \( \exp(V_{gx}) \)

by

\( Lg_1 \sim Lg_2 \) if \( Lg_1 \) and \( Lg_2 \) lie on the same \( \text{Stab}(x)^C \) orbit. Then the homeomorphism \( \tilde{\Phi}_{x,\xi} : \exp(V_{gx})/ \sim \to L \setminus L^C / \text{Stab}(x)^C \)

is naturally induced. Since \( \Phi_{x,\xi} \exp(V_{gx}) \) is proper, \( \tilde{\Phi}_{x,\xi} \) is also proper.

Conversely, assume that \( \Phi_{x,\xi} \) is \( \text{Stab}(x)^C \)-invariant, and that \( \tilde{\Phi}_{x,\xi} \) is proper and bounded from below. Then the minimizing sequence of \( \Phi_{x,\xi} \) always converges, therefore \( \Phi_{x,\xi} \) has a minimum point.

**Proposition 3.4.** Assume that \( (d \Phi_{x,\xi})_{L^1} = 0 \). Let a diffeomorphism \( \Psi_L : L \times 1 \to L^C \)

be defined by \( \Psi_L(g, \xi) := ge^{\sqrt{-1}\xi} \). Then the restriction

\[ \Psi_{L|\text{Stab}(x) \times \text{stab}(x)} : \text{Stab}(x) \times \text{stab}(x) \to \text{Stab}(x)^C \]

is a diffeomorphism.

**Proof.** Let \( (d \Phi_{x,\xi})_{L^1} = 0 \). Take \( \gamma \in \text{Stab}(x)^C \), and put \( \gamma = ge^{\sqrt{-1}\xi} \) for some \( g \in L \) and \( \xi \in 1 \). It suffices to show \( g \in \text{Stab}(x) \) and \( \xi \in \text{stab}(x) \). By the proof of Proposition 3.2 we have \( \Phi_{x,\xi}(L \cdot 1) = \Phi_{x,\xi}(L \cdot \gamma) = \Phi_{x,\xi}(L \cdot e^{\sqrt{-1}\xi}) \).

Since \( \Phi_{x,\xi} \) is geodesically convex and \( \{ L \cdot e^{\sqrt{-1}\xi} \}_{t \in \mathbb{R}} \) is a geodesic, \( \Phi_{x,\xi} \) have to be constant on this geodesic. Thus we have

\[ 0 = \frac{d^2}{dt^2} \bigg|_{t=0} \Phi_{x,\xi}(e^{\sqrt{-1}\xi}) = \|\xi_x^*\|^2, \]
which implies $\xi \in \text{stab}(x)$. Then we obtain
\[ g = \gamma e^{-\sqrt{-1}t\xi} \in \text{Stab}(x)^C \cap L = \text{Stab}(x). \]

\[ \square \]

**Remark 3.1.** It is shown by Corollary 2.15 in [17] that $\text{stab}(x)^C$ are reductive for all $x$ such that $(d\Phi_{x,\zeta})_{L^{-1}} = 0$.

Next we assume that there is an $L$-invariant function $\varphi \in C^\infty(L \setminus L^C)$ and satisfies $\omega = \sqrt{-1}\partial\bar{\partial}\varphi$. Then we have $\Phi_{x,\zeta}(L \cdot e^{\sqrt{-1}t\xi}x) = \varphi(e^{\sqrt{-1}t\xi}x) - \langle \zeta, \xi \rangle + c$ for some constant $c$ by the discussion of (2.6) in [8]. Here we may assume $c = 0$, since the existence of the critical point does not depend on the value of $c$.

**Proposition 3.5.** Assume that $\lim_{t \to \infty} \varphi(e^{\sqrt{-1}t\xi}x)/t = \infty$ holds for all $\xi \in \mathfrak{l}$ which satisfy $\lim_{t \to \infty} \varphi(e^{\sqrt{-1}t\xi}x) = \infty$. If $\Phi_{x,\zeta}$ has a critical point, then $\Phi_{x,s\zeta}$ also has a critical point for each $s > 0$.

**Proof.** We may assume the conditions (i)(ii) of Proposition 3.2 are satisfied for $\Phi_{x,\zeta}$. It suffices to show that $\Phi_{x,s\zeta}$ also satisfies (i)(ii). Since $\Phi_{x,\zeta}$ is $\text{Stab}(x)^C$ invariant, we have
\[ \Phi_{x,s\zeta}(Lg\gamma) = s\Phi_{x,\zeta}(Lg\gamma) + (1-s)\varphi(g\gamma x) \]
\[ = s\Phi_{x,\zeta}(Lg) + (1-s)\varphi(gx) = \Phi_{x,s\zeta}(Lg) \]
for all $Lg \in L \setminus L^C$ and $\gamma \in \text{Stab}(x)^C$, thus $\Phi_{x,s\zeta}$ is $\text{Stab}(x)^C$ invariant. Next we take $\xi \in \mathfrak{l} - \text{stab}(x)$ and consider the behavior of $\Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi}x)$ for $t \to \infty$. Since $\lim_{t \to \infty} \Phi_{x,\zeta}(e^{\sqrt{-1}t\xi}x) = \infty$, we have $\lim_{t \to \infty} \varphi(e^{\sqrt{-1}t\xi}x) = \infty$ or $-\langle \zeta, \xi \rangle > 0$. If the latter case occurs, then we obtain $\lim_{t \to \infty} \Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi}) = \infty$. Let the former case occur. From the assumption we have $\varphi(e^{\sqrt{-1}t\xi}x)/t \to \infty$ for $t \to \infty$, thus $\lim_{t \to \infty} \Phi_{x,s\zeta}(e^{\sqrt{-1}t\xi}) = \infty$ for all $s > 0$. \[ \square \]

## 4 Main results

### 4.1 Correspondence of orbits

In this section we prove Theorem 1.1. Let $(M,g,I_1,I_2,I_3)$, $H,G,\rho$ and $H_\rho$ be as in Section 1.2. First of all, we show the following lemma.

**Lemma 4.1.** Let $\bar{\rho}$ be defined as (7), and assume it is surjective. Then a linear map
\[ \rho^*|_{\text{Ann} \Delta g} : \text{Ann} \Delta g \to \text{Ann} h_\rho \]
is bijective, where

\[
\text{Ann.} \Delta_g := \{ \varphi \in g^* \oplus g^*; \varphi|_{\Delta_x} = 0 \}, \\
\text{Ann.} h_\rho := \{ \varphi \in h^*; \varphi|_{h_\rho} = 0 \}.
\]

**Proof.** The assertion is obvious since \(\rho^*|_{\text{Ann.} \Delta_g}\) is the adjoint map of

\[
\hat{\rho}_* : h/h_\rho \to (g \oplus g)/\Delta_g,
\]

under the identification \(\{h/h_\rho\}^* \cong \text{Ann.} h_\rho\) and \(\{(g \oplus g)/\Delta_g\}^* \cong \text{Ann.} \Delta_g\).

Any \(x \in \mu_{c^{-1}}(\iota^* \zeta_C)\) satisfies \(\hat{\mu}_C(x) - \zeta_C \in \ker \iota^* = \text{Ann.} h^g_C\) by the definition. Consequently, there exists a unique \(\eta(x) \in (g^c)^*\) such that

\[
\hat{\mu}_C(x) - \zeta_C = \rho^*(-\eta(x), \eta(x))
\]

by Lemma [L.1], which implies \((x, 1, \eta(x)) \in \sigma_{c^{-1}}(\zeta_C)\). Here we identify \(N_G = G^c \times g^c\) and \((g^c)^* = g^c\) by the Ad-G-invariant \(C^\infty\) bilinear form. Thus we obtain a map \(\psi : \mu_{c^{-1}}(\iota^* \zeta_C) \to \sigma_{c^{-1}}(\zeta_C)\) defined by \(\psi(x) := (x, 1, \eta(x))\), which induces a map \(\psi : \mu_{c^{-1}}(\iota^* \zeta_C)/H^c_\rho \to \sigma_{c^{-1}}(\zeta_C)/H^c\).

**Proposition 4.2.** \(\psi\) is well-defined and a homeomorphism.

**Proof.** First of all we check the well-definedness. Let \(x \in \mu_{c^{-1}}(\iota^* \zeta_C)\) and \(h \in H^c_\rho\). Now we may write \(\rho(h) = (h_0, h_1) \in G^c \times G^c\), and suppose \(h_0 = h_1\). Then \(hx \in \mu_{c^{-1}}(\iota^* \zeta_C)\) and \(\hat{\psi}(hx) = (hx, 1, \eta(hx))\), where \(\eta(hx) \in g^c\) is uniquely defined by \(\hat{\mu}_C(hx) - \zeta_C = \rho^*(-\eta(hx), \eta(hx))\). Now we have

\[
\hat{\mu}_C(hx) - \zeta_C = \text{Ad}_{h^{-1}}^* (\hat{\mu}_C(x) - \zeta_C) = \text{Ad}_{h_0^{-1}}^* \rho^*(-\eta(x), \eta(x)) = \rho^*(-\text{Ad}_{h_0^{-1}}^* \eta(x), \text{Ad}_{h_0^{-1}}^* \eta(x)),
\]

where \(\text{Ad}_g^* \in GL((g^c)^*)\) is defined by

\[
\langle \text{Ad}_g^* y, \xi \rangle := \langle y, \text{Ad}_g \xi \rangle
\]

for \(y \in (g^c)^*, \xi \in g^c, g \in G^c\), hence \(\eta(hx) = \text{Ad}_{h_0^{-1}}^* \eta(x)\) holds by the uniqueness. Since \(\text{Ad}_{h_0^{-1}}^*\) corresponds to \(\text{Ad}_{h_0}\) under the identification \((g^c)^* \cong g^c\), we obtain \(\hat{\psi}(hx) = h \hat{\psi}(x)\).
Next we show the injectivity. Take \(x, x' \in \mu^{-1}_{\mathcal{C}}(\iota^*\zeta_{\mathcal{C}})\) to be \(\hat{\psi}(x) = h\hat{\psi}(x')\) for some \(h \in H^\mathcal{C}\). Then \((x, 1, \eta(x)) = (hx', h_0h_i^{-1}, \text{Ad}_{h_0}\eta(x'))\), accordingly we obtain \(h_0h_i^{-1} = 1\) which implies \(h \in H^\rho_{\mathcal{C}}\).

The surjectivity is shown by constructing the inverse map of \(\psi\) as follows. Take \((x, Q, \eta) \in \sigma^{-1}_{\mathcal{C}}(\zeta_{\mathcal{C}})\) arbitrarily. From Section 2.2, we have \(\sigma_{\mathcal{C}}(x, Q, \eta) = \hat{\mu}_{\mathcal{C}}(x) + \rho^*(\nu(Q, \eta)) = \hat{\mu}_{\mathcal{C}}(x) + \rho^*(\eta, -\text{Ad}_{Q^{-1}}\eta) = \zeta_{\mathcal{C}}\) (4)

From the surjectivity of the map (1), there exist some \(h \in H^\mathcal{C}\) such that \(h_0h_i^{-1} = Q\). Then \(\rho(h)^{-1} = (h_0^{-1}, h_0^{-1}Q) \in H^\mathcal{C}\), and we have

\[\rho^*(\eta, -\text{Ad}_{Q^{-1}}\eta) = \rho^*\text{Ad}_{\rho(h)}(\text{Ad}_{h_0^{-1}}\eta, -\text{Ad}_{h_0^{-1}}\eta) = \text{Ad}_{\rho(h)}(\text{Ad}_{h_0^{-1}}\eta, -\text{Ad}_{h_0^{-1}}\eta).\] (5)

By combining (1)(5), we obtain

\[\hat{\mu}_{\mathcal{C}}(h^{-1}x) + \rho^*(\text{Ad}_{h_0^{-1}}\eta, -\text{Ad}_{h_0^{-1}}\eta) = \zeta_{\mathcal{C}},\]

which means \(h^{-1}x \in \mu_{\mathcal{C}}^{-1}(\iota^*\zeta_{\mathcal{C}})\) and

\[\hat{\psi}(h^{-1}x) = (h^{-1}x, 1, \text{Ad}_{h_0^{-1}}\eta) = h^{-1}(x, Q, \eta).\]

Thus we have the surjectivity of \(\psi\). Here, we can take \(h\) depending on \(Q\) continuously in local, therefore the inverse of \(\psi\) becomes continuous. \(\square\)

We can give group isomorphisms between the stabilizers as follows. Let

\[
\begin{align*}
\text{Stab}(x)^{\mathcal{C}} := & \{g \in H^\rho_{\mathcal{C}}; gx = x\}, \\
\text{Stab}(x, Q, \eta)^{\mathcal{C}} := & \{g \in H^\mathcal{C}; g(x, Q, \eta) = (x, Q, \eta)\}.
\end{align*}
\]

Then it is easy to check that the inclusion \(\text{Stab}(x)^{\mathcal{C}} \hookrightarrow \text{Stab}(\hat{\psi}(x))^{\mathcal{C}}\) is surjective, hence we obtain a Lie group isomorphism

\[
\text{Stab}(x)^{\mathcal{C}} \cong \text{Stab}(\hat{\psi}(x))^{\mathcal{C}}.
\] (6)

### 4.2 Correspondence of stability

Put

\[
\begin{align*}
\mu_{\mathcal{C}}^{-1}(\iota^*\zeta_{\mathcal{C}})_{\iota_{\mathcal{C}}} := & \{x \in \mu_{\mathcal{C}}^{-1}(\iota^*\zeta_{\mathcal{C}}); \Phi_x, \iota^*\zeta_{\mathcal{C}} \text{ has a critical point}\}, \\
\sigma_{\mathcal{C}}^{-1}(\zeta_{\mathcal{C}})_{\zeta_{\mathcal{C}}} := & \{(x, Q, \eta) \in \sigma_{\mathcal{C}}^{-1}(\zeta_{\mathcal{C}}); \Phi_{(x, Q, \eta), \zeta_{\mathcal{C}}} \text{ has a critical point}\}.
\end{align*}
\]
In this subsection we prove that \( \psi \) is a bijection from \( \mu_{C}^{-1}(\tau^{*}\zeta\chi)_{C}\cap H^{C}_{\rho} \) to \( \sigma_{C}^{-1}(\zeta\chi)_{C}\cap H^{C} \) by using the results in Section 3.

Along Section 3 we define geodesically convex functions

\[
\Phi_{x,\zeta}: H\setminus H^{C} \to \mathbb{R}, \quad \Phi_{x,\tau^{*}\zeta}: H_{\rho}\setminus H^{C}_{\rho} \to \mathbb{R}, \quad \Phi_{(x,\eta_{1},\eta_{2})}: H\setminus H^{C} \to \mathbb{R},
\]

for \( x \in M \) and \( (x, Q, \eta) \in M \times N_{G} \), corresponding to the moment maps \( m = \mu_{1}, \mu_{2}, \sigma_{1} \), respectively. Since \( H_{\rho} \) is a closed subgroup of \( \rho \), \( H_{\rho}\setminus H^{C}_{\rho} \) is naturally embedded in \( H\setminus H^{C} \). Then we have \( \Phi_{x,\tau^{*}\zeta}(H_{\rho}h) = \Phi_{x,\zeta}(Hh) \) for all \( h \in H^{C}_{\rho} \). Moreover we may write \( \Phi_{(x,\eta_{1},\eta_{2})}(H\hat{h}) = \Phi_{x,\zeta}(H\hat{h}) + \mathcal{E}(\hat{h}_{0}Q_{h}^{-1}, A_{h_{0}}\eta) \) for all \( \hat{h} \in H^{C} \) from Proposition 2.4 and (2.6) in [6].

**Proposition 4.3.** Let \( x \in \mu_{C}^{-1}(\tau^{*}\zeta\chi)_{C}\cap \zeta \). Then \( \hat{\psi}(x) \in \sigma_{C}^{-1}(\zeta\chi)_{C} \).

**Proof.** It suffices to show that \( \Phi_{\hat{\psi}(x),\zeta} \) has a critical point if \( \Phi_{x,\tau^{*}\zeta} \) has a critical point.

First of all, it is easy to check that \( \Phi_{\hat{\psi}(x),\zeta} \) is \( \text{Stab}(\hat{\psi}(x))^{C} \) invariant, since \( \text{Stab}(x)^{C} = \text{Stab}(\hat{\psi}(x))^{C} \) and \( \Phi_{x,\tau^{*}\zeta} \) is \( \text{Stab}(x)^{C} \) invariant.

Next we take \( \hat{\xi} \in \mathfrak{h} \), put \( \rho_{\hat{\xi}}(\hat{\xi}) = (\hat{\xi}_{0}, \hat{\xi}_{1}) \) and consider the behavior of \( \Phi_{\hat{\psi}(x),\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \) for \( t \to \infty \). Since \( \Phi_{x,\zeta} \) is geodesically convex, there is a constant \( c_{\hat{\xi}}\mathbb{R} \) and \( \lim_{t \to +\infty} \frac{d}{dt} \Phi_{x,\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \geq c_{\hat{\xi}} \). Then we have an inequality \( \Phi_{x,\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \geq c_{\hat{\xi}}t - N_{1} \) for all \( t \in \mathbb{R} \), for some sufficiently large \( N_{1} \). If \( \hat{\xi}_{0} \neq \hat{\xi}_{1} \), then

\[
\Phi_{\hat{\psi}(x),\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \geq \mathcal{E}(e^{\sqrt{-1}t\hat{\xi}_{0}}, e^{-\sqrt{-1}t\hat{\xi}_{1}}, \text{Ad}_{e^{\sqrt{-1}t\hat{\xi}_{0}}\eta}(x)) + c_{\hat{\xi}}t - N_{1}
\]

\[
\geq \min_{\hat{h} \in P(e^{\sqrt{-1}t\hat{\xi}_{0}}, e^{-2\sqrt{-1}t\hat{\xi}_{1}})} \int_{0}^{1} \frac{1}{4} \|h'\|_{h}^{2} + c_{\hat{\xi}}t - N_{1}
\]

\[
\geq \text{dist}_{G\setminus G^{C}}(e^{2\sqrt{-1}t\hat{\xi}_{0}}, e^{2\sqrt{-1}t\hat{\xi}_{1}}) + c_{\hat{\xi}}t - N_{1}
\]

Now \( G\setminus G^{C} \) is a Hadamard manifold, therefore the function

\[
t \mapsto \text{dist}_{G\setminus G^{C}}(e^{2\sqrt{-1}t\hat{\xi}_{0}}, e^{2\sqrt{-1}t\hat{\xi}_{1}})
\]

is convex. Since \( \hat{\xi}_{0} \neq \hat{\xi}_{1} \), there exists a positive constant \( N_{2} > 0 \) and

\[
\text{dist}_{G\setminus G^{C}}(e^{2\sqrt{-1}t\hat{\xi}_{0}}, e^{2\sqrt{-1}t\hat{\xi}_{1}})^{2} \geq N_{2}t^{2}
\]

for \( t \geq 1 \), and we obtain \( \Phi_{\hat{\psi}(x),\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \to \infty \) for \( t \to \infty \).

If \( \hat{\xi}_{0} = \hat{\xi}_{1} \), then \( \hat{\xi} \in \mathfrak{h}_{\rho} \). In this case we have

\[
\Phi_{\hat{\psi}(x),\zeta}(He^{\sqrt{-1}t\hat{\xi}}) \geq \Phi_{x,\tau^{*}\zeta}(H_{\rho}e^{\sqrt{-1}t\hat{\xi}}) \to \infty
\]
for $t \to \infty$, if we take $\hat{\xi} \notin \text{stab}(\hat{\psi}(x)) = \text{stab}(x)$, where the $\text{stab}$ is defined in the next section. Thus $\Phi_{\hat{\psi}(x),\hat{\xi}}$ has a critical value by Proposition\textsuperscript{3.2}.

Next we show the converse correspondence. From now on, we assume that there is an $H$-invariant global Kähler potential $\varphi : M \to \mathbb{R}$ of $(M, I_1, \omega_1)$, then we have

$$\Phi_{x, \iota^* \xi} (H_\rho e^{\sqrt{-1} \xi t}) = \varphi(e^{\sqrt{-1} \xi t}) - \langle \iota^* \xi, \xi \rangle + \text{const.},$$
$$\Phi_{(x, Q, \eta), \xi} (H e^{\sqrt{-1} \xi t}) = \varphi(e^{\sqrt{-1} \xi t}) + \mathcal{E}(Q, \eta) - \langle \xi, \hat{\xi} \rangle + \text{const.},$$

where $\xi \in H_\rho$ and $\hat{\xi} \in h$. Here we may assume the constant in the right hand sides of equalities are equal to 0.

**Proposition 4.4.** Assume that there exists a smooth function $q : \mathbb{R} \to \mathbb{R}$ such that $\|\eta(x)\|^2 \leq q(\varphi(x))$ and $q'(\varphi(x)) \geq 0$ for any $x \in M$. Suppose that if $\varphi(e^{\sqrt{-1} \xi t} : x) \to \infty$ for $t \to \infty$ then $\lim_{t \to \infty} \varphi(e^{\sqrt{-1} \xi t} : x)/t = \infty$ for any $\xi \in h_\rho$. If $\hat{\psi}(x) \in \sigma_{\Xi}^{-1}(\xi_{\hat{\xi}})_{\hat{\xi}}$, then $x \in \mu_1^{-1}(\iota^* \xi)_{\iota^* \xi}$.

**Proof.** Assume that $\Phi_{\hat{\psi}(x), \hat{\xi}}$ has a critical point. From Proposition\textsuperscript{3.3}, $\Phi_{\hat{\psi}(x), \hat{\xi}}$ is $\text{Stab}(\hat{\psi}(x))^C$ invariant and the induced map

$$\Phi_{\hat{\psi}(x), \hat{\xi}} : H \backslash H^C/\text{Stab}(\hat{\psi}(x))^C \to \mathbb{R}$$

is proper, and bounded from below. Since $H_\rho \backslash H^C/\text{Stab}(x)^C$ is a closed subset of $H \backslash H^C/\text{Stab}(\hat{\psi}(x))^C$, $F := \Phi_{\hat{\psi}(x), \hat{\xi}}|_{H_\rho \backslash H^C/\text{Stab}(x)^C}$ is also proper and bounded from below.

If $(Q, \eta) := (1, \eta(gx))$ for $g \in H^C_\rho$, we have an upper estimate

$$\mathcal{E}(1, \eta(gx)) \leq \|\eta(x)\|^2 \leq q(\varphi(gx)),$$

by taking a path $h \in P(1,1)$ to be $h(s) = 1$. Hence if we take $\xi \in h_\rho$, then

$$\Phi_{\hat{\psi}(x), \hat{\xi}} (H e^{\sqrt{-1} \xi t}) \leq \Phi_{x, \iota^* \xi} (H_\rho e^{\sqrt{-1} \xi t} + q(\varphi(e^{\sqrt{-1} \xi t}))) =: \hat{F}_+ (H_\rho e^{\sqrt{-1} \xi t}).$$

Now $\hat{F}_+$ induces a function $F_+ : H_\rho \backslash H^C_\rho/\text{Stab}(x)^C \to \mathbb{R}$, which satisfies $F_+ \geq F$, therefore $F_+$ is also proper and bounded from below. Thus $\hat{F}_+$ has a minimum point $e^{\sqrt{-1} \xi t} \in H_\rho \backslash H^C_\rho$, and have

$$0 = (d\hat{F}_+)_{e^{\sqrt{-1} \xi t}} = (1 + q'(\varphi(e^{\sqrt{-1} \xi t}))) \mu(e^{\sqrt{-1} \xi t} - \iota^* \xi).$$

Now we have shown that $\Phi_{x, \iota^* \xi}$ has a critical point if we put $s = 1 + q'(\varphi(e^{\sqrt{-1} \xi t}))$, hence $\Phi_{x, \iota^* \xi}$ also has a critical point by Proposition\textsuperscript{3.3}.

**Remark 4.1.** The assumption of Proposition\textsuperscript{4.4} is always satisfied if $M$ is the quaternionic vector space $\mathbb{H}^N$ with Euclidean metric, and $H \subset \text{Sp}(N)$ acts on $M$ linearly.
4.3 Proof of the main theorem

**Proposition 4.5.** \(\text{Stab}(x) \subset H_\rho\) and \(\text{Stab}(y) \subset H\) are isomorphic as Lie groups for any \(x \in \mu^{-1}(\iota^*\zeta)\) and \(y \in \sigma^{-1}(\zeta)\) satisfying \(yH = \psi(xH_\rho)\).

**Proof.** The assertion follows directly from Proposition 3.3 and the isomorphism [1].

**Proof of Theorem 1.1.** Define an open subset \(\mu^{-1}(\iota^*\zeta)_{\iota^*\zeta_1} \subset \mu^{-1}(\iota^*\zeta_1)\) and \(\sigma^{-1}(\zeta_{\iota^*\zeta_1})_{\sigma^{-1}(\zeta_1)}\) by

\[
\mu^{-1}(\iota^*\zeta_{\iota^*\zeta_1}) := \{x \in \mu^{-1}(\iota^*\zeta_1); \overline{H_\rho \cdot x} \cap \mu^{-1}(\iota^*\zeta_1) \neq \emptyset\},
\sigma^{-1}(\zeta_{\iota^*\zeta_1}) := \{y \in \mu^{-1}(\iota^*\zeta_1); \overline{H_\rho \cdot y} \cap \sigma^{-1}(\zeta_1) \neq \emptyset\}.
\]

Then the naturally induced maps

\[
\mu^{-1}(\iota^*\zeta)/H_\rho \to \mu^{-1}(\iota^*\zeta)_{\iota^*\zeta_1}/H_\rho, \quad \sigma^{-1}(\zeta)/H \to \sigma^{-1}(\zeta)_{\iota^*\zeta_1}/H
\]

gives an isomorphisms as complex analytic spaces by the main theorem in [8], where // is the categorical quotient. Moreover \(\mu^{-1}(\iota^*\zeta)_{\iota^*\zeta_1}\) and \(\sigma^{-1}(\zeta_{\iota^*\zeta_1})\) are the minimal open subsets of \(\mu^{-1}(\iota^*\zeta_1)\) and \(\sigma^{-1}(\zeta_1)\) containing \(\mu^{-1}(\iota^*\zeta_{\iota^*\zeta_1})\) and \(\sigma^{-1}(\zeta\iota^*\zeta_{\iota^*\zeta_1})\), respectively. Then \(\psi\) gives a bijective map \(\mu^{-1}(\iota^*\zeta)_{\iota^*\zeta_1}/H_\rho \to \sigma^{-1}(\zeta_{\iota^*\zeta_1})_{\iota^*\zeta_1}/H\) by Propositions 1, 2, 3, and 4. Moreover it is biholomorphic since \(\psi\) is obviously holomorphic and the inverse of \(\psi\) is also holomorphically defined in the proof of Proposition 12.

Let \((I_1^\zeta, I_2^\zeta, I_3^\zeta)\) be the hypercomplex structure on \(\mu^{-1}(\iota^*\zeta)/H_\rho\) induced from \((I_1, I_2, I_3)\) on \(M\). Similarly, let \((I_1^\zeta, I_2^\zeta, I_3^\zeta)\) be the hypercomplex structure on \(\sigma^{-1}(\zeta)/H\) induced from \((I_1 \times I_{G,1}, I_2 \times I_{G,2}, I_3 \times I_{G,3})\) on \(M \times N_G\). Moreover, let \(\omega_1^\zeta\) and \(\omega_2^\zeta\) be the corresponding Kähler forms.

If \(H_\rho\) acts on \(\mu^{-1}(\iota^*\zeta)\) freely, then \(H\) also acts on \(\sigma^{-1}(\zeta)\) freely from Proposition 1, 5, hence \(\mu^{-1}(\iota^*\zeta)/H_\rho\) and \(\sigma^{-1}(\zeta)/H\) become smooth hyper-Kähler manifolds by [3]. Since \(M\) and \(M \times N_G\) are complete, \(\mu^{-1}(\iota^*\zeta)/H_\rho\) and \(\sigma^{-1}(\zeta)/H\) are complete, too. See [12] for the completeness of \(N_G\).

The equality \(\psi^*(\omega_1^\zeta + \sqrt{-1}\omega_2^\zeta) = \omega_1^\zeta + \sqrt{-1}\omega_2^\zeta\) follows directly from the definition of \(\psi\) in Section 1 and the fact that any fiber of \(T^*G^\zeta\) are holomorphic Lagrangian submanifolds.

Next we show the corresponding of Kähler classes. For each \(y \in S^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3; y_1^2 + y_2^2 + y_3^2 = 1\}\), put

\[
I_y^\zeta := \sum_{i=1}^3 y_i I_i^\zeta, \quad I_y := \sum_{i=1}^3 y_i I_i.
\]
Now we take $y', y'' \in S^2$ such that \{y, y', y''\} is the orthonormal basis of $\mathbb{R}^3$ with the positive orientation. Then we can apply Theorem 1.1 for the complex structure $I_y^\zeta$ and $I_y^{\zeta'}$, and obtain a biholomorphism $\psi_y$. Thus we obtain a continuous family of diffeomorphisms $\{\psi_y\}_y$ parametrized by $y \in S^2$. Since $S^2$ is connected, the induced maps

$$
\psi_y^* : H^2(\sigma^{-1}(\zeta)/H, \mathbb{R}) \to H^2(\mu^{-1}(\nu^*\zeta)/H_\rho, \mathbb{R})
$$

does not depend on $y \in S^2$. Since each $\psi_y$ identifies the holomorphic symplectic forms with respect to $I_y^\zeta$ and $I_y^{\zeta'}$, therefore $\psi_y^*[\omega_{\zeta i}] = [\omega_{\zeta' i}]$ holds for $i = 1, 2, 3$.

Finally, we show the correspondence of the parameter spaces of two hyper-Kähler quotients.

**Proposition 4.6.** Let $\zeta, \zeta' \in \text{Im}\mathbb{H} \otimes Z_H$ satisfies $\nu^*\zeta = \nu^*\zeta'$. Then hyper-Kähler quotients $\sigma^{-1}(\zeta)/H$ and $\sigma^{-1}(\zeta')/H$ are canonically identified.

**Proof.** Take $\zeta, \zeta' \in \text{Im}\mathbb{H} \otimes Z_H$ such that $\nu^*\zeta = \nu^*\zeta'$. Then $\zeta' - \zeta \in \text{Im}\mathbb{H} \otimes \text{Ann}\mathfrak{h}_p$, and there exists a unique $A = (A_1, A_2, A_3) \in \text{Im}\mathbb{H} \otimes \mathfrak{g}^*$ such that $\rho^*(A, -A) = \zeta' - \zeta$ by Lemma 4.1. Define $\hat{A} \in N_{\zeta'}$ by $\hat{A}(t) := (0, A_1, A_2, A_3)$ for all $t \in [0, 1]$. Then a $C^\infty$ map $\sigma^{-1}(\zeta) \to \sigma^{-1}(\zeta')$ defined by $(x, [T]) \to (x, [T + \hat{A}])$ gives an isomorphism $\sigma^{-1}(\zeta)/H \to \sigma^{-1}(\zeta')/H$.

## 5 Examples

Here we raise some examples which Theorem 1.1 can be applied to.

### 5.1 Hilbert schemes of $k$ points on $\mathbb{C}^2$

Here we apply the main results obtained in the previous sections to the case of

$$
M = \text{End}(\mathbb{C}^k) \oplus \text{End}(\mathbb{C}^k) \oplus \mathbb{C} \oplus (\mathbb{C}^k)^*
$$

$G = U(k)$, $H = U(k) \times U(k)$ and $\rho = \text{id} : H \to G \times G$. Here, $H$-action on $M$ is defined by $(g_0, g_1) \cdot (A, B, p, q) := (g_0Ag_1^{-1}, g_1Bg_0^{-1}, g_0p, g_0q)$ for $g_0, g_1 \in U(k)$, $A, B \in \text{End}(\mathbb{C}^k)$, $p \in \mathbb{C}^k$ and $q \in (\mathbb{C}^k)^*$. According to [13], $Z_{H_\rho} \cong \mathbb{R}$ and $\mu^{-1}(\nu^*\zeta)/H_\rho$ is a smooth hyper-Kähler manifolds diffeomorphic to a crepant resolution of $(\mathbb{C}^2)^k/S_k$ if $\nu^*\zeta \in \text{Im}\mathbb{H}$ is given by $\nu^*\zeta = (t, 0, 0)$ for $t \neq 0$ in this situation. Here, $S_k$ is the symmetric group acting on $(\mathbb{C}^2)^k$. If $\nu^*\zeta = 0$, then $\mu^{-1}(0)/H_\rho$ is isometric to $(\mathbb{C}^2)^k/S_k$ with Euclidean metric.
Then we have a family of smooth hyper-Kähler manifolds $\sigma^{-1}(\zeta)/H$ which are biholomorphic to $\mu^{-1}(v^*\zeta)/H_\rho$. In particular, we can study $\sigma^{-1}(0)/H$ which gives a singular hyper-Kähler metric on $(\mathbb{C}^2)^k/S_k$ as follows.

**Theorem 5.1.** Let $M, H, G, \rho$ be as above. Then $\sigma^{-1}(0)/H$ is isometric to $(\mathbb{C}_\text{Taub-NUT})^k/S_k$ on their regular parts, where $\mathbb{C}_\text{Taub-NUT}$ is Taub-NUT space.

Before the proof of Theorem 5.1, we see that $N_L$ is identified with the open subset of $L \times \mathbb{I}^3$ as follows by [3], for any compact Lie group $L$. Let $T \in N_L$ and $f : [0, 1] \to L$ be the solution of the initial value problem

$$
\text{Ad}_{f(s)}T_0(s) + f(s)\frac{df(s)}{ds}f(s)^{-1} = 0, \\
f(1) = 1.
$$

Then a $C^\infty$ map $\phi : N_L \to L \times \mathbb{I}^3$ is defined by

$$
\phi([T]) := (f(0)^{-1}, T_1(1), T_2(1), T_3(1)).
$$

$\phi$ is an diffeomorphism from $N_L$ to an open subset of $L \times \mathbb{I}^3$. In particular, $\phi$ is surjective and an isomorphism of hyper-Kähler manifolds if $L$ is a torus, therefore we may assume $N_{T^k} = T^k \times \mathbb{R}^3$.

Next we begin the proof of Theorem 5.1. The inclusion $T^k \subset U(k)$ which is given by

$$
T^k = \{ \text{diag}(g_1, \ldots, g_k) \in U(k); \ g_1, \ldots, g_k \in S^1 \}
$$

induces an embedding $N_{T^k} \subset N_{U(k)}$. Now we put

$$
M_0 := \{(A, B, 0, 0) \in M; \ A = \text{diag}(a_1, \ldots, a_k), \ B = \text{diag}(b_1, \ldots, b_k)\}
$$

$$
\cong \mathbb{C}^k \oplus \mathbb{C}^k,
$$

then $\hat{M}_0 := M_0 \times N_{T^k}$ is a hyper-Kähler submanifold of $\hat{M} := M \times N_{U(k)}$.

Let a closed sub group $H_0 \subset H$ be generated by

$$
\{(g\chi, \chi) \in U(k) \times U(k); \ g \in T^k, \ \chi \in \mathcal{S}_k\},
$$

then $H_0$ is isomorphic to $\mathcal{S}_k \times T^k$. Then, $H_0$-action is closed on $\hat{M}_0$, and we obtain the hyper-Kähler moment map $\sigma_0 := \iota_0^* \circ \sigma|_{\hat{M}_0} : \hat{M}_0 \to \text{Im}\mathbb{H} \otimes h_0^*$, where $\iota_0 : h^* \to h_0^*$ is the adjoint map of the inclusion $h_0 \hookrightarrow h$. Here, $h_0 = u(k) \oplus \{0\}$ is the Lie algebra of $H_0$.

**Lemma 5.2.** We have $\sigma_0^{-1}(0) = (\sigma|_{\hat{M}_0})^{-1}(0)$, and the naturally induced map $\sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H$ is injective.
Proof. For \( x = (A, B, 0, 0) \in M \), we have
\[
\dot{\mu}(x) = (\sqrt{-1}(B^*B - AA^*), \sqrt{-1}(AB - B^*A^*), -AB + B^*A^*)
\]
\[
\oplus (\sqrt{-1}(A^*A - BB^*), \sqrt{-1}(BA + A^*B^*), BA - A^*B^*)
\]
\[
\in \text{ImH} \otimes (u(k) \oplus u(k)),
\]
where \( u(k) \) is the Lie algebra of \( U(k) \), and we identify \( u(k) \cong u(k)^* \) by the bilinear form \((u, v) \mapsto \text{tr}(uv^*)\). If \( x \in M_0 \), we can put
\[
A = \text{diag}(a_1, \ldots, a_k), \quad B = \text{diag}(b_1, \ldots, b_k),
\]
and we obtain
\[
\dot{\mu}(x) = -\sqrt{-1}\tau(x) \oplus \sqrt{-1}\tau(x) \in \text{ImH} \otimes (t^k + t^k),
\]
where \( t^k := \text{Lie}(T^k) \subset u(k) \), and \( \tau = (\tau_1, \tau_2, \tau_3) : M_0 \to \text{ImH} \otimes \mathbb{R}^k \) is the hyper-Kähler moment map with respect to the tri-Hamiltonian \( T^k \)-action on \( M_0 \) defined by
\[
\tau_1(x) = \text{diag}(|a_1|^2 - |b_1|^2, \ldots, |a_k|^2 - |b_k|^2),
\]
\[
\tau_2(x) = \text{diag}(2\text{Re}(a_1b_1), \ldots, 2\text{Re}(a_kb_k)),
\]
\[
\tau_3(x) = \text{diag}(2\text{Im}(a_1b_1), \ldots, 2\text{Im}(a_kb_k)).
\]
Under the identification \((\theta, y) \in T^k \times \mathbb{R}^{3k} = N_{T^k}\), we obtain \( \rho^*(\nu(\theta, y)) = \sqrt{-1}(y, -y) \). Thus we have \( \sigma(x, \theta, y) = \sqrt{-1}(-\tau(x) + y, \tau(x) - y) \) and \( \sigma_0(x, \theta, y) = \sqrt{-1}(-\tau(x) + y) \) for \((x, t, y) \in \hat{M}_0\), which implies \( \sigma_0^{-1}(0) = (\sigma|_{\hat{M}_0})^{-1}(0) \). Then we obtain \( \sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H \) by the inclusion \( \sigma_0^{-1}(0) \subset \sigma^{-1}(0) \).

Next we show the injectivity of \( \sigma_0^{-1}(0)/H_0 \to \sigma^{-1}(0)/H \). Let \((x, \theta, y) \in \sigma_0^{-1}(0) \) and \((g_0, g_1) \in H = U(k) \times U(k) \) satisfy \((g_0, g_1) \cdot (x, \theta, y) \in \sigma^{-1}(0) \). Since \((g_0, g_1) \cdot (x, \theta, y) = (g_0, g_1) \cdot x, g_0\theta g_1^{-1}, A_{g_1}y\), we have \( \tilde{\theta} := g_0\theta g_1^{-1} \in T^k \).

For \( x = (A, B, 0, 0) \),
\[
(g_0, g_1)x = (g_0Ag_1^{-1}, g_1Bg_0^{-1}, 0, 0)
\]
\[
= (g_0A\theta^{-1}g_0^{-1}\tilde{\theta}, \tilde{\theta}^{-1}g_0\theta Bg_0^{-1}, 0, 0) =: (\tilde{A}, \tilde{B}, 0, 0) \in M_0,
\]
then we have equalities between diagonal matrices \( g_0A\theta^{-1}g_0^{-1} = \tilde{A}\tilde{\theta}^{-1} \) and \( g_0\tilde{B}g_0^{-1} = \tilde{B}\tilde{\theta} \). By comparing the eigenvalues of both sides, we can see there exist \( \chi \in S_k \) such that \( \tilde{a}_i\theta_i^{-1} = a_i\theta_i^{-1} \) and \( \tilde{b}_i\theta_i = b_i\theta_i \) for \( i = 1, \ldots, k \), where \( \tilde{a}_i, \tilde{b}_i, \theta_i, \tilde{\theta}_i \) are the \( i \times i \) components of \( \tilde{A}, \tilde{B}, \theta, \tilde{\theta} \), respectively. This implies that \((x, t, y) \) and \((g_0, g_1) \cdot (x, \theta, y) \) lie on the same \( H_0 \)-orbit, since \( y = \tau(x) \).
\qed
Proof of Theorem 5.1. It is easy to see that $\sigma_0^{-1}(0)/H_0 \rightarrow \sigma^{-1}(0)/H$ preserves the hyper-Kähler structures. Since $\sigma_0^{-1}(0)/H_0 = (\sigma_0^{-1}(0)/T^k)/S_k$ and $\sigma_0^{-1}(0)/T^k$ is isomorphic to $(\mathbb{C}^2_{\text{Taub--NUT}})^k$, therefore $\sigma^{-1}(0)/H$ contains $(\mathbb{C}^2_{\text{Taub--NUT}})^k/S_k$ as a hyper-Kähler suborbifold. From Theorem 4.1, the quotient space $\sigma^{-1}(0)/H$ is homeomorphic to $\mu^{-1}(0)/H_p$, which is $(\mathbb{C}^2)^k/S_k$ by [14]. Since $(\mathbb{C}^2)^k/S_k$ is connected, and $(\mathbb{C}^2_{\text{Taub--NUT}})^k/S_k$ is complete, the embedding $\sigma_0^{-1}(0)/H_0 \rightarrow \sigma^{-1}(0)/H$ should be isomorphic. \hfill $\Box$

5.2 Quiver varieties

The setting considered in Section 5.1 can be generalized to quiver varieties defined by Nakajima [15], which contains ALE spaces constructed by [11]. Quiver varieties are constructed as hyper-Kähler quotient as follows.

Let $Q = (V, E, s, t)$ be a finite oriented graph, that is, $V$ and $E$ are finite sets with maps $s, t : E \rightarrow V$, where $s(h) \in V$ is a source of a quiver $h \in E$, and $t(h) \in V$ is a target. More over $E$ is decomposed into $E = \Omega \cup \overline{\Omega}$, with one to one correspondence $\Omega \rightarrow \overline{\Omega}$ denoted by $h \mapsto \tilde{h}$ satisfying $s(h) = t(\tilde{h})$ and $t(h) = s(\tilde{h})$. Next we fix a dimension vector $v = (v_k)_{k \in V}$, where each $v_k$ is a positive integer. Then the action of $\prod_{k \in V} U(v_k)$ on

$$M = \bigoplus_{h \in \Omega} \text{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}}) \oplus \bigoplus_{h \in \Omega} \text{Hom}(\mathbb{C}^{v_{s(h)}}, \mathbb{C}^{v_{t(h)}})$$

(7)

by $(g_k)_k : (A_h, B_h)_h := (g_{t(h)}A_h g^{-1}_{s(h)}; g_{s(h)}B_h g^{-1}_{t(h)}).$ Then the quiver varieties are constructed by taking hyper-Kähler quotients for this situation.

Here we explain the settings of Taub-NUT deformations for quiver varieties, which contain the case of [2]. Let $M$ be as (7). We define $H, G, \rho$ as follows so that $H_p = \prod_{k \in V} U(v_k)$. We take another finite oriented graph $\tilde{Q} = (\tilde{V}, \tilde{E}, \tilde{s}, \tilde{t})$ with a surjection $\pi : \tilde{V} \rightarrow V$ satisfying $\pi(\tilde{s}(h)) = s(h)$ and $\pi(\tilde{t}(h)) = t(h)$ for all $h \in H$. We label elements of $\pi^{-1}(k)$ numbers as $\pi^{-1}(k) = \{k_1, k_2, \ldots, k_{N_k}\}$. Note that $\tilde{Q}$ may be disconnected even if $Q$ is a connected graph. A dimension vector $v' = (v_k)_{k \in \tilde{V}}$ is determined by $v_k = v_{\pi k}$ for all $k \in \tilde{V}$. Then we define $H := \prod_{k \in \tilde{V}} U(v_k)$ and $G := \prod_{k \in V'} U(v_k)^{N_k-1}$, where $V' = \{k \in V; \pi^{-1}(k) \geq 2\}$. A homomorphism $\rho : H \rightarrow G \times G$ is defined by

$$\rho((g_{k})_{k \in \tilde{V}}) = ((g_{k_1}, g_{k_2}, \ldots, g_{k_{N_k}}), (g_{k_2}, \ldots, g_{k_{N_k-1}}, g_{k_{N_k}}))_{k \in V'}$$

Then $\mu^{-1}(\iota^*\zeta)/H_p$ becomes a quiver variety, and we obtain another hyper-Kähler quotient $\sigma^{-1}(\zeta)/H$ diffeomorphic to $\mu^{-1}(\iota^*\zeta)/H_p$. 

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5.3 Toric hyper-Kähler varieties

In the previous sections we assumed that $H$ and $G$ are compact. However, the compactness is not essential for the proof of Theorem 1.1; we need only $\text{Ad}_G$-invariant positive definite inner products on its Lie algebra and the existence of hyper-Kähler metrics on $N_G$ with tri-Hamiltonian $G \times G$-actions.

In this subsection we consider the case of noncompact abelian Lie groups $H = \mathbb{R}^N$ and $G = \mathbb{R}^N/k$, where the vector subspace $k \subset \mathbb{R}^N$ is given by $k = k_{\mathbb{Z}} \otimes \mathbb{R}$ for some submodule $k_{\mathbb{Z}} \subset \mathbb{Z}^N$. $\rho : H \to G \times G$ is defined by

$$\rho(v) := (v \mod k, 0),$$

then $\bar{\rho}$ defined by (1) is surjective. In this case we put $N_G := G \times G \times G \times G$ with the Euclidean metric, and $G \times G$-action on $N_G$ is defined by $(g_0, g_1) \cdot (h_0, h_1, h_2, h_3) := (h_0 + g_0 - g_1, h_1, h_2, h_3)$. Then Theorem 2.1, 2.2 and Proposition 2.5 hold in this case. Let $M = \mathbb{H}^N$, and define $H$-action on $M$ by

$$(t_1, \ldots, t_N) \cdot (x_1, \ldots, x_N) := (x_1 e^{-2\pi i t_1}, \ldots, x_N e^{-2\pi i t_N}).$$

The hyper-Kähler quotient $\mu^{-1}(i^*\zeta)/H_{\rho}$ becomes a toric hyper-Kähler variety, and $\sigma^{-1}(\zeta)/H$ is its Taub-NUT deformation defined in [1]. Theorem 1.1 can be also applied to this situation.

5.4 Hyper-Kähler manifolds with tri-Hamiltonian actions

Here we show that the limited case of Theorem 7 of [4] also follows from Theorem 1.1. Let $M = \mathbb{H}^N$ and $H \subset SP(N)$. Take a normal closed subgroup $H_{\rho} \subset H$ and put $G := H/H_{\rho}$. Let $\rho : H \to G \times G$ be given by

$$\rho(h) := (1, hH_{\rho}).$$

Then $X = \mu^{-1}(i^*\zeta)/H_{\rho}$ is a hyper-Kähler manifolds with tri-Hamiltonian $G$-action, and $\sigma^{-1}(\zeta)/H$ is the modification of $\mu^{-1}(i^*\zeta)/H_{\rho}$ defined in Section 5 of [1]. From Theorem 1.1, we have the following results.

**Theorem 5.3.** Let $X = \mu^{-1}(i^*\zeta)/H_{\rho}$ be a tri-Hamiltonian $G$ hyper-Kähler manifold defined as above. Then the modification of $X$ in the sense of Section 5 of [4] by the tri-Hamiltonian $G$-action is isomorphic to $X$ as holomorphic symplectic manifolds, hence diffeomorphic.

By Theorem 7 of [4], we have already known that $\sigma^{-1}(\zeta)/H$ is diffeomorphic to $\hat{\mu}^{-1}(i^!(N_G) + \zeta)/H_{\rho}$, which is an open subset of $X$. Theorem 5.3 asserts that this open subset is diffeomorphic to $X$, even if it is a proper subset of $X$.

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