On Generalized Theories of Varying Fine Structure Constant

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ARTICLE

ABSTRACT

We work with a class of scalar extended theory of gravity that can drive the present cosmic acceleration as well as accommodate a mild cosmic variation of the fine structure constant $\alpha$. The motivation comes from a vintage theory developed by Bekenstein, Sandvik, Barrow and Magueijo. The $\alpha$ variation is introduced by a real scalar field interacting with charged matter. We execute a cosmological reconstruction based on a parametrization of the present matter density of the Universe. Observational consistency is ensured by comparing the theoretical estimates with JLA + OHD + BAO data sets, using a Markov chain Monte Carlo simulation. An analysis of molecular absorption lines from HIRES and UVES spectrographs is considered as a reference for the variation of $\alpha$ at different redshifts. Two examples are discussed. The first explores a field-dependent kinetic coupling of the scalar field interacting with charged matter. The second example is a generalized Brans-Dicke formalism where the varying $\alpha$ is fitted in as an effective matter field. This generates a simultaneous variation of the Newtonian constant $G$ and $\alpha$. The pattern of this variation can have a crucial role in cosmic expansion history.

Key words: cosmology: theory; dark energy; variation of fundamental constants

1 INTRODUCTION

In physics, ‘Nature’ is usually defined as a coalescence of phenomenology on different energy scales. For this notion to uphold, one requires simple and mathematically consistent theories. More often than not, these theories need to introduce new structures, as in fields or symmetries or more importantly, fundamental constants. The title ‘fundamental constant’ is allotted only to a few pre-assigned parameters which can not be derived. They can only be measured and other parameters of the theory can be expressed in terms of them. These constants define patches of the physical world in an axiomatic manner while their origin remains a riddle. An idea that accommodates a possible variation of these constants is a trial of the standard theories and a motivation to think beyond our usual understanding of physical reality.

The ‘Large Numbers hypothesis’ of Dirac (1937, 1938) can be thought of as a precursor to the idea of a varying fundamental constant. The hypothesis tips that the universal constants can be different for different phases of the evolving universe and should be treated as varying entities. Most of the early attempts (see for instance the review of Unzicker (2009)) to implement this notion into a theory of gravity went astray, until a successful field-theoretic approach was considered by Jordan (1937), allowing variations of gravitational coupling as well as the fine-structure constant. The observational viability of this theory was subsequently discussed by Fierz (1956). A special example was considered soon after by Brans and Dicke (1961) with only the gravitational coupling being varied. This particular case is now popularly known as the Brans-Dicke theory and regarded as the pioneer of a larger class of theories called the Scalar-Tensor theories (Damour and Esposito-Farese 1992). The possibility of a cosmic variation of fine structure constant $\alpha$ at Hubble rate was first proposed by Gamow (1967). This idea boosted interest in a cosmological theory with varying $\alpha$ immediately, leading to a series of works in succession challenging Gamow’s original claim. For instance, nuclear mass systematics (Peres 1967; Dyson 1967, 1972) and the laboratory analysis of the fine-structure splittings in radiogalaxy emission lines (Bahcall and Schmidt 1967) did predict a much more mildly varying $\alpha$ than the Hubble rate. Comparison of cesium atomic clocks and superconducting cavity clocks (Turneaure and Stein 1976), observation of active galactic nucleus such as a BL-Lacertae object (Wolfle, Brown and Roberts 1976) and the analysis of fission product isotopes in natural reactors also require particular attention (Shlyakhter 1976) in this regard. More advanced constraints on this variation can be found from the $SZ$ to X-ray flux ratio analysis of galaxy clusters Bora and Desai (2021), and from

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the relativistic transitions in molecular absorption lines of Quasar spectra at different redshifts (Savedoff 1956; Bahcall, Sargent and Schmidt 1967; Nunes and Lidsey 2004; Parkinson, Bassett and Barrow 2004; Doran 2005; Webb et al. 2001; Murphy, Webb and Flambaum 2003; Uzan 2003; Chand, Srianand, Petitjean and Aracil 2004). The latter will be of particular interest in this manuscript. More inclusive and motivating discussions on theories allowing a variation of natural constants are available in the literature (Uzan 2003, 2011; Chiba 2011) and may provide further insights to the readers.

We focus on a theory of gravity that allows a mild variation of $\alpha$ in the cosmological past and an asymptotic approach to the desired value at present. Theories of unification (e.g. string theory) inspire such a cosmic variation by arguing that a coupling in 4-dimension is just a projection of fundamental constants defined in higher dimensions (Antoniadis 1999). A General Relativistic (GR) description of this variation primarily requires a dynamical framework and an evolution equation for $\alpha$. A rather radical approach of constructing this is through a variation of the speed of light $c$ where $\alpha = c^2/\hbar c$ (Moffat 1993; Albrecht and Magueijo 1999). Although theories with varying $c$ have received some interest in possible resolutions of cosmological problems (Barrow and Magueijo 1999; Barrow and Magueijo 1999), they come along with a general breakdown of Lorentz invariance. We work with a class of theory where the electron charge is written as an evolving real scalar field (the $\alpha$-field) which drives a variation of $\alpha$. The philosophy goes in parallel with standard GR where the Hubble function is driven by the total energy density of the universe through the cosmological equations; and here the evolving density of an electromagnetic entity does the trick. We receive motivation from the theory of gravity proposed by Bekenstein (1982). The original theory is simple and consistent about standard requirements such as general covariance, Lorentz invariance, causality, and scale invariance of the $\alpha$-field. There is a general breakdown of charge conservation but the theory remains a nice platform to combine GR with Maxwell’s theory of electromagnetism (Landau, Sisterna and Vucetich 2001). The theory needs to be generalized to satisfy cosmological requirements (Barrow and Magueijo 1998; Magueijo, Sandvik and Kibble 2001). In particular, the class of generalized theory developed by Sandvik, Barrow and Magueijo (2002) is now popular as the Bekenstein-Sandvik-Barrow-Magueijo or the BSBM theory. We aim to work with extensions of a standard BSBM theory and the resulting non-trivial variations in space-time geometry, especially from a cosmological purview.

In many ways, GR has left behind some riddles in cosmology that are difficult to decode. For example, the standard theory and most of its existing modifications fail to allow a smooth transition of the universe into the present acceleration from a preceding era of deceleration. This is now confirmed by advanced astrophysical observations, such as the Luminosity Distance measurement of Supernova (Riess et al. 2001; Betoule et al. 2014). Theoretically, the effective fluid distribution of Dark Energy (DE) driving this phenomenon should have an evolving Equation of State (EOS) (Maor and Brunstein 2003; Maor, Brunstein and Steinhardt 2001; Upadhye, Ishak and Steinhardt 2005). It is easier to illustrate this using the Deceleration parameter $q$ which should evolve into negative values somewhere in the recent past (Padmanabhan and Roychoudhury 2003; Roychoudhury and Padmanabhan 2005). This is exactly where approaches like energy corrections of the order of a cosmological constant goes awry; in reference with the contradictions with observations (Riess et al. 2004; Eisenstein et al. 2005) and the ‘coincidence problem’ (Schutzhold 2002; Velten, Marttens and Zimdahl 2014). Simple scalar fields are used to develop toy models which can drive different phases of the cosmic expansion depending on whether the scalar self-interaction dominates over the kinetic part or not (Zlatev, Wang and Steinhardt 1999; Sahni and Starobinski 2000; Copeland, Sami and Tsujikawa 2006). Any scalar extended theory of gravity carries some motivations from theories of unification but all are not equally acceptable. For example, models that introduce a dark energy through slow-rolling scalar fields can not account for a variation in the EOS (Sen, Sen and Sami 2010; Slepean, Gott and Zinn 2014) and are effectively ruled out. Similarly, the Quintessence models become superfluous in view of the constraints on scalar-baryon interaction, derived from the ‘fifth-force experiments’ (Adelberger, Heckel and Nelson 2003). A trick perhaps lies in the choice of the self-interaction, for example, a quantum field theory inspired pseudo-Nambu-Goldstone-Boson (pNGB) case (Frieman, Hill, Stebbins and Waga 1995) works remarkably well in cosmological aspects. An interesting alternative is to allow the scalar field to interact with ordinary matter such that it decouples in high density regions of the cosmos (e.g. around the earth) and avoids detection from local experiments (Khoury and Weltman 2004; Hinterbichler and Khoury 2010). These generalizations give a set of constraints such that the theory does not allow a violation of the Equivalence Principle (EP) on the solar system scales (Jain and Khoury 2010; Will 2001, 2005; Gubser and Khoury 2004; Upadhye, Gubser and Khoury 2005; Brax et al. 2004; Damour and Polyakov 1994). However, recent research on the screening of Milky Way galaxy predicts that a scalar field satisfying the solar system constraints can not, on their own, drive the present acceleration of the universe (Wang, Hui and Khoury 2012). Therefore, it is natural to keep looking for a better theoretical framework that can support the cosmic acceleration without violating basic observational requirements.

An extended BSBM-type theory of gravity can potentially provide this framework, fitting in beautifully with a number of requirements. The scalar field in this theory is, in fact, a prototype of the so-called chameleon fields and readily deals with questions regarding possible equivalence principle violations. A few examples of extended BSBM have already received attention in cosmological context (Barrow, Sandvik and Magueijo 2002; Barrow, Magueijo and Sandvik 2002; Barrow and Lip 2012). We work with two different generalizations of BSBM. The first one allows the coupling constant in the kinetic term governing the $\alpha$-field dynamics to be a function of the field itself. The second example is a unification of an extended BSBM and a generalized Brans-Dicke (BD) theory where the fine structure constant $\alpha$ and the Newtonian constant $G$ can vary simultaneously. We take a field-dependent
kinetic coupling of the e-field and assume the BD parameter \( \omega_{BD} \) to be a function of the BD scalar. It is well known that the standard BD setup, while widely acknowledged, could not quite deliver a ‘better theory of gravity’. This is primarily due to the observational constraints on \( \omega_{BD} \) (Bertotti, Iess and Tortora 2003) and the lack of viable cosmological solution for all epochs (Banerjee and Sen 1997; Faraoni 1999). A generalization like field-dependent \( \omega_{BD} \) can potentially resolve these issues and provide a more complete formulation of the theory. We refrain ourselves from choosing any functional form of the field profiles or their interactions. We formulate a simple way to reverse engineer the structure of the theory, a cosmological reconstruction from \( Om(z) \) parametrization. This parameter has received quite rigorous attention in the analysis of observational data and subsequent comparisons of modified dark energy models (Sahni, Saini, Starobinski and Alam 2003; Sahni et. al. 2008; Lu et. al. 2009; Tong and Zhang 2009). \( Om(z) \) is a constant \( \sim \Omega_m \) for standard \( \Lambda \)CDM model, denoting the present matter density of the universe. The reconstruction gives an observationally viable Hubble function as a function of redshift which is used to solve the field equations of the generalized theories. We discuss the role of different components of the theory in different epochs of the cosmic evolution and try to identify which components can play a probable key role in driving the transition between successive epochs. The evolution of the e-field with redshift tells us how the fine structure constant might have gone through a mild evolution alongside the cosmic expansion. In addition, the second generalization gives us a scope to study the mild evolution alongside the cosmic expansion. We also try to provide an idea how these fundamental couplings can be directly related to one another, plotting \( \alpha \) as a function of \( G \). This inter-relation is the hint of a more general background formalism (perhaps geometric) relating all the fundamental couplings, which is at this moment beyond our scope.

In Section 2, we briefly review the standard BSBM formalism. In Section 3, we discuss a bit about the observational constraints to be considered throughout the manuscript. This includes an execution outline of the reconstruction based on \( Om(z) \) and the analysis of data from Molecular Absorption Spectroscopy. Sections 4 and 5 include discussions on the structure of generalized BSBM theories. We make some concluding remarks and finish in Section 6.

2 THE BEKENSTEIN-SANDVIK-BARROW-MAGUEIJO (BSBM) THEORY : A BRIEF REVIEW

We have already mentioned that in the standard BSBM theory, the electron charge \( e \) is assumed to be a function of coordinates. It is written as a dimensionless scalar field \( \epsilon \) called the e-field.

\[
e = e_0 \epsilon(x^\nu).
\]

(1)

\( e_0 \) has the dimension of \( e \). All particle charges vary uniformly through this universal scalar.

\[
\alpha = e^2/c\hbar,
\]

where \( c \) and \( \hbar \) are constants.

A constant \( \alpha \) is a signature of Maxwell’s electrodynamics. Here, the vector potential interacts minimally with matter which works as a ‘dictum’ to help us decide which adjustments in the standard laws of physics are rational and which are not. A variation of \( \alpha \) requires some careful tweaking of the standard Maxwellian construct, for example, by allowing a general breakdown of charge conservation. Moreover, the modified theory must allow a model-independent framework and abide by generally acknowledged principles of physics. For instance, we should be able to derive the dynamical equations for \( \alpha \) from an invariant action using a corresponding action principle. We must also have second-order, hyperbolic evolution equations to ward off non-causality or runaway solutions.

The Lorentz invariant Lagrangian for a charged particle in flat spacetime is written as

\[
L = -mc(-u^\mu u_\mu)^2 + \frac{e_0 \epsilon}{c} u^\mu A_\mu,
\]

(3)

where \( m \) is the rest mass and \( e_0 \epsilon \) is the charge of the particle. The proper time is written as \( \tau \) and the four-velocity as \( u^\mu = \frac{dx^\mu}{d\tau} \). We note that the Lagrangian has a minimally interacting vector potential term and is invariant under the gauge transformation

\[
e A_\mu = \epsilon A_\mu + \chi \mu,
\]

(4)

\( \chi \) being an arbitrary function. The electromagnetic field is identified from Eq. (3) by writing the Lagrange equation

\[
d \left[ m u_\mu + \frac{e_0 \epsilon}{c} \epsilon A_\mu \right] = -m \epsilon c^2 + \frac{e_0}{c} (\epsilon A_\mu)\mu u^\nu.
\]

(5)

\( u_\mu u^\mu = -c^2 \) is the normalization used above and the rest mass is written as a function of coordinates. Eq. (5) can be simplified into

\[
\frac{d(mu_\mu)}{d\tau} = -m \epsilon c^2 + \frac{e_0}{c} [(\epsilon A_\mu)\mu - A_\mu,\nu] u^\nu.
\]

(6)

Eq. (6) is important for two reasons. Primarily, it introduces the concept of anomalous force through the term \( m \epsilon c^2 \) (Dicke 1965). More importantly, one can identify a gauge-invariant electromagnetic field from the Lorentz force term (second term on the RHS) and write, following Barrow, Sandvik and Magueijo (2002)

\[
F_{\mu\nu} = \frac{\{(e A_\nu)\mu - (e A_\mu)\nu\}}{\epsilon}.
\]

(7)

\( F_{\mu\nu} \) is also invariant under a constant rescaling of \( \epsilon \). The corresponding lagrangian contribution is written as

\[
\mathcal{L}_{em} = -F_{\mu\nu} F^{\mu\nu}/4.
\]

(8)

However, an evolution equation for \( \epsilon \) cannot be written from this simple construct. A separate lagrangian which can govern the dynamics of \( \epsilon \) and satisfy the dimensional requirement, was given by Bekenstein (Bekenstein 1982) as

\[
\mathcal{L}_\epsilon = -\frac{1}{2\epsilon} \frac{(\epsilon u^\nu u_\nu)}{c^2}.
\]

(9)

For a standard BSBM theory \( \omega = \frac{\omega_{BD}}{c^2} \), \( l \) comes in as a dimensional correction and works as a length scale of the theory. The theory allows the electric field to be Coulombic for a point charge only above this length scale. This puts additional constraints on the corresponding energy scale \( \frac{\hbar \epsilon}{m c} \). A generalization of this setup was given by
Sandvik, Barrow and Magueijo (2002) using a transformed gauge
\[ a_\mu = e A_\mu, \]  
and a modified field tensor
\[ f_{\mu\nu} = e F_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu. \]  
Using \( \psi = \ln e \) as a variable, the action is written as
\[ S = \int d^4x \sqrt{-g} \left( L_\psi + L_{\text{mat}} + L_\phi + L_{\text{em}} e^{-2\psi} \right). \]  
The setup has a similarity with dilaton-type theories (Forgacs and Horvath 1979; Marciano 1984; Barrow 1987; Damour and Polyakov 1994). While a standard dilaton field interacts only with the electromagnetic sector. In Eq. (12) the \( \psi \)-contribution is
\[ L_\psi = -\frac{\omega}{2} \partial_\mu \psi \partial^\mu \psi, \]  
and the electromagnetic contribution is
\[ L_{\text{em}} = -\frac{1}{4} f_{\mu\nu} f^{\mu\nu}. \]  
\( \omega \) is a coupling constant and \( L_\phi = \frac{1}{8\pi G} R \) is the standard GR part. A usual metric variation of the action produces the modified field equations
\[ G_{\mu\nu} = 8\pi G \left( T_{\mu\nu}^{\text{mat}} + T_{\mu\nu}^\psi + T_{\mu\nu}^{\text{em}} e^{-2\psi} \right). \]  
Similarly, a \( \psi \) variation produces the scalar field evolution equation responsible for \( \alpha \)-dynamics
\[ \Box \psi = \frac{2}{\omega} e^{-2\psi} L_{\text{em}}. \]  
Before moving forward to the next section, let us write the cosmological equations for a standard BSBM setup. The independent field equations for a spatially-flat FRW geometry are
\[ \left( \frac{\dot a}{a} \right)^2 = \frac{8\pi G}{3} \left( \rho_m \left( 1 + \zeta_m e^{-2\psi} \right) + \rho_r e^{-2\psi} + \frac{\omega}{2} \psi^2 \right), \] 
and the \( \psi \) evolution equation
\[ \ddot \psi + 3H \dot \psi = -\frac{2}{\omega} e^{-2\psi} \zeta_m \rho_m. \]  
All the equations are in \( G = c = 1 \) unit. One needs to specify the nature of \( L_{\text{em}} \) to write these equations. This is done from a parametrization \( \zeta = \frac{\delta_m}{\delta_r} \). \( \rho \) is the total baryon energy density. This parameter defines the non-relativistic matter contribution in \( L_{\text{em}} \), \( \zeta_m \) is the cosmological value of \( \zeta \) which depends on the non-baryonic matter content of the universe, including Dark matter. The interplay between electric and magnetic fields in the cold dark matter distribution is tipped to be a crucial factor that restricts the cosmological value of \( \zeta \) to be between \(-1\) and \(+1\) (Barrow, Sandvik and Magueijo 2002). Based on comparison with spectroscopic analysis of molecular absorption lines from Quasars, a model with negative \( \zeta_m \) is thought to be slightly disfavored (Barrow and Lip 2012). However, it is more rational to call these arguments speculative since there exists, to date, no clear knowledge of a Dark matter distribution. The fluid contents of the effective energy-momentum distribution, i.e., radiation and the matter, generate their respective conservation equations
\[ \dot \rho_m + 3H \rho_m = 0, \]  
\[ \dot \rho_r + 4H \rho_r = 2\psi \rho_r. \]  
A solution of this set of Eqs. (17), (18), (20) and (20) dictates the cosmic evolution of fine structure constant, written as
\[ \alpha = \exp (2\psi) c_0^2 / hc. \]  
However, an exact solution in analytical form is never guaranteed. It is tactically better to work with some ansatz over one of the components of the equations, preferably supported by phenomenological evidences. This is where a scheme of reconstruction can prove to be much more effective. In the next section we discuss about the particular scheme we are interested in before moving on to generalized BSBM setups.

3 COMPARISON WITH OBSERVATIONAL DATA

3.1 Cosmological Reconstruction from \( Om(z) \)
Reconstruction is a way to develop a theory in reverse order. One usually starts from one or more widely accepted phenomenological facts, directly or indirectly inspired by astrophysical observations. For example, in cosmology, a natural intuition would be to start from an optimum behavior any parameter that governs the evolution of the universe. The field equations of the theory can then be solved to write the best possible structure. Reconstruction schemes based on dimensionless kinematic quantities are particularly popular (Bernstein and Jain 2004; Visser 2005; Cattoen and Visser 2007; Dunajski 2008). These quantities are written as parameters involving higher-order derivatives of the scale factor, such as deceleration, jerk or statefinder (Alam, Sahni, Saini and Starobinsky 2003; Mukherjee and Banerjee 2016; Chakrabarti 2021). In this manuscript, we work with the \( Om(z) \) parameter, which has gained much popularity for its utility in categorical comparison of cosmological models with observational data. The parameter is a measure of the present matter density contrast of the universe, written as a constant value \( \Omega_m0 \) for standard cosmology. Quite a number of schemes are popular based on this parameter (Sahni, Saini, Starobinski and Alam 2003; Sahni et. al. 2008; Lu et. al. 2009; Tong and Zhang 2009). We illustrate a rather simple method to parametrize \( \Omega_m0 \), introducing two new parameters at the outset. This allows a mild variation of the parameter with redshift. The aim is to compare this parametric form with astrophysical observations and to estimate the new parameters for a best possible behavior for Hubble and deceleration. The standard parameter is written as
\[ \Omega_{m0} = \frac{\left( h(z) \right)^2 - 1}{(1 + z)^3 - 1}. \]  
\[ h(z) = \frac{H(z)}{H_0}. \]  
The present value of Hubble is given by \( H_0 \). The parametrization is introduced by replacing \( \Omega_{m0} \) with
\[ Om(z) = \lambda_0 (1 + z) \] 
Alternatively, we can write Hubble as a function of redshift
\[ h(z) = \left[ 1 + \lambda_0 (1 + z)^{\delta} \right] \left( (1 + z)^3 - 1 \right)^{\frac{1}{2}}, \]
in closed analytical form. We estimate the model parameters using a statistical analysis, in comparison with data-sets provided by observations from: (i) the Joint Light Curve Analysis of Supernova distance modulus (SDSS – II and SNLS collaborations) ([Betoule et al. 2014], (ii) the Hubble parameter measurements (OHD) (Simon, Verde and Jimenez 2005; Stern et. al. 2010; Blake et. al. 2012; Moresco et. al. 2012; Chuang and Wang 2013; Planck collaboration 2014; Delubac et. al. 2015) and (iii) the Baryon Acoustic Oscillation (BAO) data (6df Galaxy Survey, BOSS LOWZ and BOSS CMASS) ([Beutler et al. 2011; BOSS collaboration 2012]. The argument of differentiation in Eq. (25) is changed into redshift $z$ and the Hubble is written in its dimensionless form by scaling $H_0$ by 100 km Mpc$^{-1}$ sec$^{-1}$

$$h(z) = \frac{H(z)}{H_0} = \frac{H(z)}{100 \times h_0}. \quad (26)$$

We estimate the present value of dimensionless Hubble ($h_0 = H_0/100\text{ km Mpc}^{-1}\text{ sec}^{-1}$) and the deceleration parameter directly using a Markov Chain Monte Carlo simulation (MCMC) written in python (Foreman-Mackey, Hogg, Lang and Goodman 2013). The parameter space confidence contours reveal the best fit values along with associated uncertainty in the estimation, as shown in Fig. 1. We also write the best-fit parameter values and 1σ error estimation in Table 1 for convenience.

The present value of Hubble parameter is well consistent with recent observations ([Planck collaboration 2014]. A departure from standard cosmology can be noted, particularly from the estimated best fit value of $\delta$. While for a $\Lambda$CDM model $\delta$ would be exactly 0, for the extended cosmology the parameter is found to be in the range $0.052^{+0.038}_{-0.037}$. There is also a slight difference in the estimate of present matter density contrast, given by $\lambda_0$. These departures can be used to identify modified models of Dark Energy and categorize their behavior as Quintessence-like or Phantom-like ([Shafieloo, Alam, Sahni and Starobinsky 2006; Wang and Tegmark 2005]). However, an advanced cosmological analysis is not within the scope of this manuscript and we are happy with an overall optimum behavior of the evolving DE. The Hubble $H(z)$ is plotted in Fig. 2 along with the data points from observation. This evolution has sufficient observational validity for a reasonable range of redshift. To demonstrate the transition into present acceleration from a deceleration we also plot the numerical solution of scale factor with cosmic time, for a convenient range of reference.

This transition can be better understood from the acceleration term $\ddot{a}$. This is explained in Fig. 3 where we plot $q(z)$ and the jerk parameter $j(z)$, both for the best fit parameters (bold blue) and for the regions of uncertainty (faded blue shades). The present value of $q(z)$ is found to be $\sim -0.62$. The deceleration moves from a positive zone into negative at a redshift $z_t < 1$, which marks the transition of the
we plot whose evolution confirms that higher-order late-time acceleration. For larger redshifts, firms an effective negative pressure during the DE dominated distribution of the universe has an effective EOS written as of gravity under consideration. The total energy-momentum rate and dynamical evolutions are independent of the theory requirements. We also plot the jerk parameter in the bottom panel of Fig. 3 whose evolution confirms that higher-order parameters show a clear departure compared to a ΛCDM structure formation in the universe. We look into this by first is necessary to avoid a large departure from the observed density to follow a corresponding ΛCDM pattern closely. This transition redshift as well as the present value of deceleration provide a good agreement with observational assumptions about the background density is homogeneous, written as ρ_m. Small deviations from ρ_m are written as δρ_m. The ‘matter density contrast’ is then defined as

\[ \delta_m = \frac{\delta \rho_m}{\rho_m}. \]

\( \delta_m \) can not accelerate away with Hubble or the scale factor. Its dynamics is governed by a non-linear evolution equation in locally over-dense distributions such as around a star or a collapsing distribution. However, for a spatially homogeneous late-time cosmology a linearized evolution equation is efficient enough.

\[ \delta_m + 2H\dot{\delta}_m = 4\pi G\rho_m \delta_m. \]

We change the time derivatives of Eq. (31) into derivatives with respect to scale factor and solve for \( \delta_m \) numerically. We take the initial conditions for scale factor as \( a_i = 0.01 \) and for overdensity as \( \delta_m(a_i) = 0.01 \) and \( \delta_m(a_i) = 0 \). The solution for the best fit parameter values is plotted in Fig. 5 and a behavior closely following ΛCDM model can be seen.

Before concluding this section we discuss the thermodynamic equilibrium of the cosmological system in brief. The notion is that a thermodynamic system consisting of the universe surrounded by a cosmological horizon is not too different from a black hole in thermodynamic equilibrium (Gibbons and Hawking 1977; Jacobson 1995; Padmanabhan 2000). To recover the first law of blackhole thermodynamics

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**Figure 3.** Top Panel: Deceleration parameter as a function of z. Bottom Panel: Jerk parameter as a function of z. The plots are for the best fit parameter values of \( H_0, \delta \) and \( \lambda_0 \) (bold blue) as well as for the regions of uncertainty (faded blue shades).

**Figure 4.** The effective equation of state as a function of z. The plot is for the best fit parameter values of \( H_0, \delta \) and \( \lambda_0 \) (bold blue) as well as for the regions of uncertainty (faded blue shades).

**Figure 5.** Plot of \( \delta_m \) vs \( a \) for the best fit parameter values of \( H_0, \delta \) and \( \lambda_0 \).
for a spatially flat cosmological system the horizon is given by $r_h = 1/H$ and is known as the Hubble horizon (Bak and Rey 2000). The total entropy of this system surrounded by the Hubble horizon must not decrease with cosmic expansion. We write the total entropy $S$ as a sum of the boundary entropy and the entropy of the constituent fluid

$$S = S_b + S_f.$$  

(32)

The constraints on the total entropy are written as

$$\frac{dS}{dn} \geq 0,$$  

(33)

$$\frac{d^2S}{dn^2} < 0.$$  

(34)

The horizon entropy is proportional to the horizon area $A = 4\pi r_h^2$. In a $h = k_B = c = 8\pi G = 1$ unit it is written as,

$$S_h = 8\pi^2 r_h^2.$$  

(35)

The temperature of the horizon depends on its radius (Jacobson 1995; Bak and Rey 2000; Frolov and Kofman 2003)

$$T_h = 1/2\pi r_h.$$  

(36)

Assuming that a late time cosmology is dominated by a coexistence of dark energy and dark matter, the total fluid entropy is

$$S_f = S_{cdm} + S_{de}.$$  

(37)

Then the first law of thermodynamics gives

$$TdS_{cdm} = dE_{cdm} + p_m dV = dE_{cdm},$$  

(38)

$$TdS_{de} = dE_{de} + p_d dV.$$  

(39)

$p_d$ and $p_m$ are the pressures of two fluid components and the fluid temperature $T$ is uniform. If $V = 4\pi r_h^3/3$, the energy contributions are written as

$$E_{cdm} = \frac{4\pi^2 \rho_{cdm} r_h^3}{3},$$  

(40)

$$E_{de} = \frac{4\pi^2 \rho_{de} r_h^3}{3}.$$  

(41)

If we assume that $T = T_h$, i.e., the horizon has the same temperature as the fluid, the rate of change of entropy with time can be written as

$$\dot{S} = \dot{S}_{cdm} + \dot{S}_{de} + \dot{S}_h = 4\pi H r_h^2 [\rho_m + (1 + \omega_{de})\rho_{de}] = 4\pi H r_h^2 [\rho_m + (1 + \omega_{de})\rho_{de}].$$  

(42)

Eq. (33) ensures a thermodynamic equilibrium depending on how the total entropy $S$ changes with $n = \ln a$ in the first order and in the second order. Using Eq. (42), we can write

$$S_{cdm} = \frac{16\pi^2}{H^2} (H \dot{n})^2,$$  

(43)

$$S_{mn} = 2S_{n} \left( H \dot{n} - \frac{2Hn}{H} \right) = 2S_{n} \Psi.$$  

(44)

We note that $\Psi < 0$ ensures the thermodynamic equilibrium. For the present work, all we need to do is to bring in the exact form of Hubble in Eq. (43) from Eq. (25) and plot $\Psi$ as a function of the scale factor $a$. Fig. 6 shows the plot and one can find $\Psi$ to be negative during late-times. There is an interesting evolution from positive into the negative domain during the cosmic expansion which hints that perhaps the universe tends to move towards a thermodynamic equilibrium. We should mention here that $\dot{S}$ may have some additional role in the estimation of cosmological quantities for different types of dark energy models, as discussed quite recently by Jamil, Saridakis and Setare (2010).

In a nutshell, this simple formulation is used to write a desired late-time cosmological behavior based on a list of observations. It can not be denied that this is restricted only in the late-time era. For a unified picture of $\alpha$ variation along with cosmic expansion, one requires a different analysis altogether. Moreover, this is not the best possible structure, but it can provide an overall qualitative idea going by usual conventions. However, with the closed form of Hubble we will be able to avoid any speculative assumptions to simplify the field equations. Now, in subsequent sections, we will demonstrate the application of this scheme for two extended versions of BSBM theory. In particular, we need to solve for the $e$-field and determine the evolution of the fine structure coupling. For that, we also need to fit in with measurements of Quasar absorption spectra which is briefly discussed in the next subsection.

### 3.2 Observational Constraints of $\alpha$

In the introductory notes, it is already mentioned that mild cosmic variation of different fundamental couplings are observed and reported in the literature (Uzan 2011; Martins 2015). They receive serious attention particularly in view of the standard model of particle physics. Most of these observations come from relativistic transitions in molecular absorption lines of Quasar spectra at different redshifts. These provide enough motivation for one to keep looking beyond standard cosmology. Even if the considerations are somewhat unorthodox, they can potentially provide practical solutions to avoid violations of the Equivalence Principle (Martins et. al. 2015; Leite et. al. 2014; Webb et. al. 2011).

In this work we refer to a combined analysis of constraints from molecular absorption spectroscopy (Webb et. al. 2011; Ferreira et. al. 2014; Ferreira and Martins 2015; Whitmore and Murphy 2015; Pinho and Martins 2016; Martins and Pinho 2017) and the late-time cosmological observations.

**Figure 6.** Evolution of $\Psi = \left( \frac{H_{mn}}{H_{n}} - \frac{2H_{n}}{H} \right)$ with respect to scale factor for the best fit parameter values of $H_0$, $\delta$ and $\lambda_0$. 

MNRAS 000, 000–000 (2022)
| Source          | Redshift | $\Delta \alpha / \alpha$ (ppm) | Spectrograph. |
|-----------------|----------|------------------------------|---------------|
| J0026$-$2857    | 1.02     | $3.5 \pm 8.9$               | UVES          |
| J0058$+$0041    | 1.07     | $-1.4 \pm 7.2$              | HIRES         |
| 3 sources       |          |                              |               |
| HS1549$+$1919   | 1.14     | $-7.5 \pm 5.5$              | UVES/HIRES/HDS|
| HE0515$-$4414   | 1.15     | $-1.4 \pm 0.9$              | UVES          |
| J1237$+$0106    | 1.31     | $-4.5 \pm 8.7$              | HIRES         |
| HS1549$+$1919   | 1.34     | $-0.7 \pm 6.6$              | UVES/HIRES/HDS|
| J0841$+$0312    | 1.34     | $3.0 \pm 4.0$               | HIRES         |
| J0841$+$0312    | 1.34     | $5.7 \pm 4.7$               | UVES          |
| J0108$-$0037    | 1.37     | $-8.4 \pm 7.3$              | UVES          |
| HE0001$-$2340   | 1.58     | $-1.5 \pm 2.6$              | UVES          |
| J1029$+$1039    | 1.62     | $-1.7 \pm 10.1$             | HIRES         |
| HE1104$-$1805   | 1.66     | $-4.7 \pm 5.3$              | HIRES         |
| HE2217$-$2818   | 1.69     | $1.3 \pm 2.6$               | UVES          |
| HS1946$+$7658   | 1.74     | $-7.9 \pm 6.2$              | HIRES         |
| HS1549$+$1919   | 1.80     | $-6.4 \pm 7.2$              | UVES/HIRES/HDS|
| Q1103$-$2645    | 1.84     | $3.5 \pm 2.5$               | UVES          |
| Q2206$-$1958    | 1.92     | $-4.6 \pm 6.4$              | UVES          |
| Q1755$+$57      | 1.97     | $4.7 \pm 4.7$               | HIRES         |
| PHL957          | 2.31     | $-0.7 \pm 6.8$              | HIRES         |
| PHL957          | 2.31     | $-0.2 \pm 12.9$             | UVES          |

Table 2. The dataset for $\alpha$ variations, written as $\Delta \alpha / \alpha$ in a unit of parts per million. The absorption spectra are for different Qusar sources, measured by different spectrographs at different redshifts. Some values are written as weighted average of separate independent measurements (Songaila and Cowie 2014).

4 GENERALIZED BSBSM : THEORY AND RECONSTRUCTION

We first apply this analysis to a simple extension of BSBSM theory that was introduced by Barrow and Lip (2012). This extension allows $\omega$, the coupling of the kinetic term to be a function of the $\epsilon$-field, $\psi$. We briefly discuss the action and the field equations of the theory without making any particular choice of $\omega(\psi)$. The reconstruction allows us more freedom to solve the field equations numerically and see how $\psi$ might have evolved in the recent past. From this, we can determine an optimum dynamics for $\alpha$ that does not alter the course of a viable late-time cosmology. More importantly, we can compare the theoretical evolution of $\Delta \alpha$ with the data from molecular absorption spectra as in Table. 2.

We work with the Lagrangian

$$\mathcal{L} = \sqrt{-g}(\mathcal{L}_g + \mathcal{L}_{\text{mat}} + \mathcal{L}_\psi + \mathcal{L}_\epsilon e^{-2\psi}).$$

Apart from the usual gravitational part

$$\mathcal{L}_g = \frac{R}{16\pi G},$$

there is a contribution of the scalar field

$$\mathcal{L}_\psi = -\frac{\omega(\psi)}{2} \partial_\mu \psi \partial^\mu \psi,$$

and the electromagnetic part

$$\mathcal{L}_\epsilon = -\frac{1}{4} f_{\mu \nu} f^{\mu \nu},$$

in the Lagrangian. The condition $\omega(\psi) \geq 0$ is known as the no-ghost condition and is enforced at the outset for a positive energy density of the $\epsilon$-field. $\alpha$ evolves as

$$\alpha = \alpha_0 e^{2\psi},$$

similar to a standard BSBSM setup. In natural units, the spatially-flat FRW equation with a $-,+,+,+$ signature are written from a metric variation of Eq. (45)

$$\ddot{\psi} + 3H \dot{\psi} + \frac{\omega'(\psi)}{2\omega} \dot{\psi}^2 = -\frac{2\kappa \rho_m}{\omega} e^{-2\psi}.$$
The noninteracting radiation density \( \rho_r \) is covariantly conserved, leading to the condition

\[ \rho_{\nu r} = \rho_r e^{-2\psi} \propto a^4. \]  

(53)

As discussed by Barrow and Lip (2012), a GR limit of the theory can be defined by the conditions \( |\zeta| e^{-2\psi} \ll 1 \) and \( \psi^2 \zeta \ll \rho_r \) under which Eq. (50) gives back the standard Friedmann equation. \( \rho_r \) is the primary matter content of the epoch one is considering, i.e., different for a cold dark matter or a \( \Lambda \) epoch. We do not enforce these conditions at the outset as our aim is a bit different, to formulate a scalar dominated theory exhibiting enough departure from GR. However, we recall that the interaction of electric and magnetic fields in a cold dark matter distribution is expected to constrain the cosmological value of \( \zeta \) heavily (Barrow, Sandvik and Magueijo 2002) and therefore, we do take \( \zeta \) to be between \(-1\) and \(+1\).

To solve Eqs. (50), (52) and (53) we call in the reconstruction of Hubble from \( Om(z) \) parametrization, given by Eqs. (24) and (25). We take the best fit parameter values of \( \lambda_0, \delta \) and \( H_0 \) from Table 1, and use

\[ H(z) = H_0 \left[ 1 + \lambda_0(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right]^{1/2}. \]  

(54)

First, we change the arguments of the equations from cosmic time into redshift and change the derivatives. To demonstrate in brief,

\[ \dot{\psi} = \frac{d\psi}{dz} \frac{da}{dt}, \]  

(55)

\[ = -H_0 \psi'(1+z) \left[ 1 + \lambda_0(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right]^{1/2}, \]  

(56)

where we have used \( \psi' = d\psi/dz \) and \( a = 1/(1+z) \). Using this the second derivative of \( \psi \) can be written as

\[ \ddot{\psi} = H_0^2 (1+z)^2 \psi'' \psi^\zeta \left[ 1 + \lambda_0(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right] + H_0^2 (1+z)^2 \psi' \left[ 1 + \lambda_0(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right] + \frac{H_0^2}{2} (1+z)^2 \psi^\zeta \left[ \delta \lambda_0(1+z)^\delta - 1 \right] \left\{ (1+z)^3 - 1 \right\} + 3\lambda_0(1+z)^{\delta+2}. \]  

(57)

We also use Eq. (50) to define \( \omega(\psi) \) as

\[ \omega(\psi) = \frac{2}{H_0^2 \psi^2 (1+z)^2 \left[ 1 + \lambda(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right]} \left[ 3H_0^2 \left[ 1 + \lambda(1+z)^\delta \left\{ (1+z)^3 - 1 \right\} \right] - \rho_0 (1 + \zeta e^{-2\psi}) (1+z)^3 - \rho_1 (1+z)^4 - \rho_2 (1+z)^5 \right]. \]  

(58)

\( \rho_0, \rho_1 \) and \( \rho_2 \) are proportionality constants. Using Eq. (58), we solve the \( c \)-field evolution Eq. (52) numerically and plot \( \psi \) as a function of redshift \( z \) in the top panel of Fig. 7. It can be seen that just before the redshift of transition, \( z_t \sim 1 \), the scalar field crosses into a positive domain.

For most of the earlier epochs, the scalar evolves through the negative sector. The overall variation of the scalar is mild \(~ [0.000005, -0.30] \). This can be understood from the bottom panel of Fig. 7 where \( \psi \) evolution is shown for larger redshifts as well.

This already indicates a mild variation of the fine structure constant which is exponentially related to \( \psi \). In Fig. 8 we plot \( \alpha \) with \( z \). Moreover, taking \( \alpha_0 \) and \( \alpha_z \) to be the values of the coupling at the present epoch and at some redshift \( z \), we derive the evolution of the quantity \( (\alpha_z - \alpha_0)/\alpha_0 = \Delta \alpha/\alpha \). The theoretically estimated \( \Delta \alpha/\alpha \) is illustrated in Fig. 9 while the observational data points as in Table 2 are fitted in. The graph on the top panel shows that at low redshift, the theory gives a good fit with the observations of molecular absorption spectra. The bottom panel shows \( \Delta \alpha/\alpha \) for a larger range of redshift, until \( z \sim 400 \). The variation is
The second scalar field is the \( \sigma \phi \) as in a standard BSBM setup. We also introduce a dimensionless BD coupling \( \omega_{BD} \). It seems rational too, given the fact that we have a notion that all the fundamental couplings of physics are somehow related to one another provides the primary motivation. It seems rational too, given the fact that we have clear interpretation of their origin or their constant nature. The formulation is inspired by a similar approach of Barrow, Magueijo and Sandvik (2002). We consider a two-scalar extended theory of gravity. One of the scalar fields, \( \phi \), is of Brans-Dicke (BD) nature, i.e., geometric and accounts for the variation of the effective gravitational coupling (Brans and Dicke 1961). The second scalar field is the \( \phi \)-field as in a standard BSBM setup. We also introduce a dimensionless BD coupling \( \omega_{BD} \). However, in mind the stringent constraints on the value of BD parameter from local astronomical tests, we deliberately take \( \omega_{BD} \) to be a function of \( \phi \). Such extensions were first considered by Nordtvedt Jr. (1970) and since then have received a few revisits in the context of cosmological analysis of modified BD theories (Barker 1978; Schwinger 1978; Van den Bergh 1982). The varying \( \alpha \) in this generalized Brans-Dicke setup behaves as an effective matter field and therefore we call the theory a generalized Brans-Dicke-BSBM: THEORY AND RECONSTRUCTION

In this section, we work with a generalized Brans-Dicke-BSBM setup which can support a variation of the Newtonian coupling \( G \) as well as the fine structure constant \( \alpha \).

A notion that all the fundamental couplings of physics are somehow related to one another provides the primary motivation. It seems rational too, given the fact that we have clear interpretation of their origin or their constant nature. The formulation is inspired by a similar approach of Barrow, Magueijo and Sandvik (2002). We consider a two-scalar extended theory of gravity. One of the scalar fields, \( \phi \), is of Brans-Dicke (BD) nature, i.e., geometric and accounts for the variation of the effective gravitational coupling (Brans and Dicke 1961). The second scalar field is the \( \phi \)-field as in a standard BSBM setup. We also introduce a dimensionless BD coupling \( \omega_{BD} \). However, in mind the stringent constraints on the value of BD parameter from local astronomical tests, we deliberately take \( \omega_{BD} \) to be a function of \( \phi \). Such extensions were first considered by Nordtvedt Jr. (1970) and since then have received a few revisits in the context of cosmological analysis of modified BD theories (Barker 1978; Schwinger 1978; Van den Bergh 1982). The varying \( \alpha \) in this generalized Brans-Dicke setup behaves as an effective matter field and therefore we call the theory a generalized...
**Brans-Dicke-BSBM (BDBSBM)** setup. The combined action is written as

\[
S = \int d^4 x \sqrt{-g} \left( R \phi + \frac{16\pi}{c^4} \mathcal{L} - \omega_{BD}(\phi) \frac{\phi_{,\mu}\phi^{,\mu}}{\phi} \right). \tag{59}
\]

The energy momentum Lagrangian consists of three parts,

\[
\mathcal{L} = \mathcal{L}_m + \mathcal{L}_{em} \exp(-2\psi) + \mathcal{L}_\psi. \tag{60}
\]

The e-field contribution to the energy-momentum tensor is written as

\[
\mathcal{L}_\psi = -\frac{\omega(\psi)}{2} \psi_{,\mu} \psi^{,\mu}. \tag{61}
\]

As usual, we are interested in a spatially homogeneous cosmology for which the independent equations are written as

\[
3H^2 = \frac{8\pi}{\phi} \left( \rho_m(1 + |\zeta| \exp(-2\psi)) + \rho_r \exp(-2\psi) + \rho_\psi \right)
\]

\[
-3 \frac{\dot{a}}{a} \frac{\dot{\phi}}{\phi} + \frac{\omega_{BD}(\phi)}{2} \frac{\phi'^2}{\phi^2} - \frac{k}{a^2}, \tag{62}
\]

\[
\ddot{\psi} + 3H \dot{\psi} = \frac{8\pi}{3} \frac{\rho_m - 2\rho_\psi}{2\omega_{BD}(\phi)} - \frac{\dot{\phi}^2}{3 + 2\omega_{BD}(\phi)}, \tag{63}
\]

\[
\frac{d\omega_{BD}(\phi)}{d\phi} + \frac{1}{2\omega(\psi)} \frac{d\omega(\psi)}{d\psi} \dot{\psi}^2 = -\frac{2\exp(-2\psi)}{\omega(\psi)} \zeta \rho_m. \tag{64}
\]

Apart from the BD field and its functions, other terms are quite similar to the case considered in the last section. \(\omega(\psi)\) signifies the energy scale of \(\psi\) and should not be confused with the Brans-Dicke counterpart \(\omega_{BD}(\phi)\). \(\rho_\psi\) is the kinetic energy density contribution of \(\psi\) and is given by

\[
\rho_\psi = \frac{\omega(\psi)}{2} \dot{\psi}^2. \tag{65}
\]

\(\rho_m\) is proportional to \(a^{-3}\). Conservation of the noninteracting radiation density \(\rho_r\) leads to the condition

\[
\rho_r = \rho_r e^{-2\psi} \propto a^{-4}. \tag{66}
\]

In a unit \(\hbar = c = 1\), the fine structure coupling has a correlation with \(\psi\) as

\[
\alpha = \alpha_0 e^{\frac{2\psi}{\psi}}, \tag{67}
\]

\(\alpha_0\) being the value of fine structure constant as we know today. We solve the above set of field equations, primarily to derive \(\psi\). Since the unknown functions in this case outnumber the independent equations, we choose \(\omega(\psi)\) at the outset as

\[
\omega(\psi) = \omega_0 e^{\psi}, \tag{68}
\]

with the constraint that \(\omega_0 > 0\) for all time. We also set the present value of Newtonian coupling \(G\) to be unity such that while solving for \(\phi\) we have a fixed initial condition (\(\phi \propto 1/G\)). \(\zeta = \mathcal{L}_{em}/\rho_m\) is the parameter defining non-relativistic matter contribution to \(\mathcal{L}_{em}\). It demands specific attention for this particular example of extended BSBM as it can establish a correlation between the scalar dynamics and the cold dark matter constituents of our universe. For instance, \(\zeta < 0\) corresponds to models where the magnetic energy of the interacting scalar-charged matter system dominates over the electric field energy.

Example of Dark matter being dominated by magnetic coupling can be found in superconducting cosmic strings for which \(\zeta = -1\). These cases have received some attention in literature and they support a mildly varying \(\alpha\) during the dust-dominated era followed by an almost constant \(\alpha\) during late times (Sandvik, Barrow and Magueijo 2002; Barrow, Sandvik and Magueijo 2002). In comparison, a positive \(\zeta\) case is expected contradict with observations of molecular absorption spectroscopy (Dvali and Zaldarriaga 2002). Since there is no clear knowledge of the Dark matter distribution in our universe, we take a diplomatic approach and look into the evolution of \(\alpha\), \(\Delta \alpha/\alpha\) and the scalar fields for both \(\zeta < 0\) and \(\zeta > 0\), while always restricting \(\zeta\) to be between \(-1\) and \(+1\). Since we rely on a cosmological reconstruction supported by a set of widely accepted observations, the expansion scale factor is never affected by the \(\alpha\) variations. Therefore the choices of \(\zeta\) can only modify the e-field evolution and do not come at odds with the primary cosmological requirements.

The differential Eqs. (62), (63), and (64) are solved using the reconstructed Hubble,

\[
H(z) = H_0 \left[ 1 + \lambda_0 (1 + z)^{\delta} \{(1 + z)^3 - 1\} \right]^\frac{1}{2}. \tag{69}
\]

We take the best fit parameter values of \(\lambda_0\), \(\delta\) and \(H_0\) while doing this. Once again, the primary trick is to change the arguments of the equations from cosmic time into redshift and to solve the equations numerically (See Eqs. (55) and (57) for reference). First, we solve the set of equations for \(\zeta < 0\). The plots in Fig. 11 show the numerical solution for \(\psi(z)\). The top panel of the Figure shows the \(\psi\) evolution for low \(z\). Around the redshift of transition, the scalar field crosses into a positive domain. For most of the
Figure 12. Generalized Brans-Dicke-BSBM : The evolution of $\alpha$ with $z$ for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta > 0$. Top Panel shows the evolution for low redshift and the Bottom Panel shows the evolution for high redshift.

Figure 13. Generalized Brans-Dicke-BSBM : The evolution of $\Delta\alpha/\alpha$ with $z$ for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta > 0$. The Top Panel shows a plot for low redshifts alongwith the molecular spectroscopy data. The Bottom panel shows the plot for a larger range of redshifts.

Figure 14. Generalized Brans-Dicke-BSBM : The evolution of the Brans-Dicke scalar $\phi$ (Top Panel) and the coupling $\omega_{BD}$ (Bottom Panel) with $z$ for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta > 0$. The initial condition for the numerical solution is taken to be $\dot{\phi} > 0$.

Figure 15. Generalized Brans-Dicke-BSBM : The evolution of the Brans-Dicke scalar $\phi$ (Top Panel) and the coupling $\omega_{BD}$ (Bottom Panel) with $z$ for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta > 0$. The initial condition for the numerical solution is taken to be $\dot{\phi} < 0$. 
We plot the fine structure constant in Fig. 16. The top panels of both the Figures suggest that \( \psi(z) \) evolves in a different manner for \( \zeta < 0 \). Nevertheless, the fact that \( \psi \) evolves in a different manner for \( \zeta < 0 \), does not contribute much to the low redshift variation of \( \Delta \alpha/\alpha \). This is quite clear from the top panel of Fig. 18 where a good fit with the molecular absorption spectroscopic data is once again found. However, for larger ranges of redshift, the evolution is clearly different as compared to a \( \zeta > 0 \) dynamics, as in the bottom panel of Fig. 18). \( \Delta \alpha/\alpha \) evolves in and out of the negative domain within a redshift range of \( z \in (5, 30) \). The formation of this minima coincides with the end of matter domination and it may have a role in setting up the system for the onset of late-time acceleration. We also solve for the BD field \( \phi \) and the coupling \( \omega_{BD} \) for a \( \zeta < 0 \) system. The numerical solution is, once again, sensitive on the initial conditions \( \phi > 0 \) and \( \phi < 0 \). These are shown in Fig. 19 and Fig. 20. The evolution is quite similar to the \( \zeta > 0 \) cases. Therefore,
$\omega_{BD}(\phi) \sim \delta_1 e^{-\delta_2 - \delta_3 \phi}$ remains a good possible choice for the generalized BD coupling for all initial conditions under consideration.

Since these models are special cases of generalized Brans-Dicke theory, the evolution of $G_{eff} \propto 1/\phi$ is an important factor that determines the nature of the theory. Generally, no pre-assigned expectation regarding this evolution can be found in literature, except, the novel bound given by Weinberg (1972)

$$\frac{\dot{G}}{G} = -\frac{\dot{\phi}}{\phi} = -\frac{k}{H}. \quad (71)$$

$k \leq 1$. For a scalar-tensor theory this evolution is solution-dependent and we have two different $\phi$ profiles, as in Fig. 14 and Fig. 15. We plot $G_{eff}$ against Hubble, in Fig. 21. The top panel shows the evolution for the initial condition $\dot{\phi} > 0$. In this case, $G_{eff}$ decays smoothly with Hubble, indicating that gravitational interaction was weaker in the past, when Hubble or the natural scale of energy was higher. The maximum allowed value of $G_{eff}$ is equal to the present value 1. This particular version of the theory can be called an ‘asymptotically free theory’. However, if $G_{eff}$ decays with Hubble, i.e., increases with cosmic expansion, we can not rule out a probable conflict with the $H_0$ tension on some scale. (Banerjee, Cai, Heisenberg, Colgain, Sheikh-Jabbari and Yang 2021; Heisenberg, Villarrubia-Rojo and Zosso 2022; Lee, Lee, Colgain, Sheikh-Jabbari and Thakur 2022). The second scenario with $\dot{\phi} < 0$ might help us avoid this issue. In this case, due to a monotonically decreasing profile of $\phi$, $G_{eff}$ increases with Hubble. This indicates a decay of the gravitational coupling with cosmic time.

Figure 18. Generalized Brans-Dicke-BSBM : The evolution of $\Delta \alpha/\alpha$ with $z$ or the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta < 0$. Top Panel shows a plot for low redshifts alongwith molecular spectroscopy data fitted in. The Bottom panel is for larger span of redshifts.

Figure 19. Generalized Brans-Dicke-BSBM : The evolution of the Brans-Dicke scalar $\phi$ (Top Panel) and the coupling $\omega_{BD}$ with $z$ (Bottom Panel) for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta < 0$. The initial condition for the numerical solution is taken to be $\dot{\phi} > 0$.

Figure 20. Generalized Brans-Dicke-BSBM : The evolution of the Brans-Dicke scalar $\phi$ (Top Panel) and the coupling $\omega_{BD}$ with $z$ (Bottom Panel) for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$ and $\zeta < 0$. The initial condition for the numerical solution is taken to be $\dot{\phi} < 0$. 

MNRRAS 000, 000–000 (2022)
until the minimum allowed value 1 is reached. This avoids a further complicated $H_0$ tension at the expense of giving away the asymptotically free nature, which may compromise some of the standard model phenomenology (Sola, Karimkhani and Khodam-Mohammadi 2017). We curiously note that, (i) the present epoch enjoys an extremum (either the maxima or the minima) of $G_{\text{eff}}$ and (ii) $\zeta$ leaves no contribution in these evolutions.

The models given in this section do have some similarities in structure with theories where a 'dynamical vacuum' can drive a variation in proton-to-electron mass ratio (Cruz Perez and Sola 2018; Sola et. al. 2019). Both of these models support a similar pattern in the mild variation of natural couplings. The connection is potentially intriguing as the dynamical vacuum models are closely related to a cosmic variation of Higgs vacuum expectation value (Sola et. al. 2020; Sola, Karimkhani and Khodam-Mohammadi 2017; Chakrabarti 2021). This, again indicates a direct feedback of the theory on the Quark masses and overall, on particle physics phenomenology. We also mention here that a variation of $\Delta \alpha/\alpha$ already allows a $\mu$-variation through

$$\frac{\Delta \mu}{\mu} \sim \frac{\Delta \Lambda_{\text{QCD}}}{\Lambda_{\text{QCD}}} - \frac{\Delta \nu}{\nu} \sim R \frac{\Delta \alpha}{\alpha}. \quad (72)$$

Here $R < 0$ is a model-dependent parameter and can be estimated from high energy experiments of unified theories (Avelino, Martins, Nunes and Olive 2006). Overall, these analogies motivate a requirement to merge extended theories of gravity with phenomenologies of particle physics.

Figure 21. Generalized Brans-Dicke-BSBM: Variation of Newtonian Coupling, scaled as $G/G_0$ with Hubble for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$. $G_0$ is the present value of $G$ (scaled to 1 in natural units). Top Panel: Plot for $\dot{\phi} > 0$. Bottom Panel: Plot for $\dot{\phi} < 0$.

Figure 22. Generalized Brans-Dicke-BSBM: A comparison between variations of the fine structure coupling, $\Delta \alpha/\alpha$, and the gravitational coupling $\Delta G/G_0$. $G_0$ is the present value of $G$, scaled to 1 in natural units. The plots are for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$.

Figure 23. Generalized Brans-Dicke-BSBM: The evolution of $\alpha$ vs $G/G_0$ for the best fit parameter values of $\lambda_0$, $\delta$ and $H_0$. $G$ is scaled by $G_0$, the present value of $G$ (scaled to 1 in natural units). The top panel is for $\zeta > 0$ and the bottom panel is for $\zeta < 0$. The initial condition $\dot{\phi} > 0$ is enforced.
Before concluding the section, we briefly talk about an interesting possibility. It is quite reasonable to imagine that more than one fundamental couplings of physics should vary simultaneously and constraining any of them can, inadvertently, constrain the others. The generalized Brans-Dicke-BSBM is an example of the same, where a dimensionless $\alpha$ and a dimensionful $G$ vary alongside each other. The comparative scale of these variations can be realized from Fig. 22, where, $\Delta \alpha/\alpha$ and $\Delta G/G_0$ are drawn on the top and the bottom panel respectively, as functions of redshift. Note that, the gravitational coupling $G$ is scaled by $G_0$ and written as a dimensionless quantity. $G_0$ is the present value of $G$ (1 in natural units) which contains all the dimensional informations. This scaling is indeed necessary, in reference to the notion that a concept of varying natural constant is rational if and only if it is dimensionless (Dirac 1937, 1938; Duff 2014, 2016). Similar variations of more than one fundamental couplings have been addressed to some extent in different scenarios, for instance, in primordial nucleosynthesis (Campbell and Olive 1995; Coelho, Nunes, Olive, Uzan and Vangioni 2007). Some constraints on a coupled variation can also be determined using Optical atomic clocks (Luo, Olive and Uzan 2011; Ferreira, Juliao, Martins and Monteiro 2012). However, the possibility that one coupling may simply vary as a function of another due to an independent background mechanism, has never been considered. In a recent research on the concurrent variation of $G$, the speed of light $c$, the Planck constant $h$, and the Boltzmann constant $k_B$, the need for an alternative cosmological setup was discussed (Gupta 2022). If such a setup exists, it should generate from a fundamental, unified theory. At this moment no such concrete theory is known and we shall have to be content with intuitions, based on whatever evidences we can find. We compare the numerical solutions of fine structure constant $\alpha$ and gravitational coupling $G$ and plot one of them as a function of the other. In Fig. 23 we plot $\alpha$ vs $G/G_0$ for the two signs of $\zeta$. The condition $\dot{\phi} > 0$ is assumed which means that a $G_{eff}$ decaying with Hubble is chosen for this plot. Therefore the allowed range of $G_{eff}$ is $[0, 1]$. We take particular note of the formation of a minima in the evolution of $\alpha$. An early universe is signified by higher energy scales, or higher Hubble, implying lower values of $G$ (See Fig. 21). $G$ increases with Hubble to the present value, which is scaled to 1. The two end-points of the plot give two maximum allowed values of $\alpha$ during the cosmic expansion. We speculate that the universe evolves maintaining a correlation with this pattern. It starts evolving with rapid early acceleration where one maxima of $\alpha$ is found. The evolution gradually leads the universe towards the minima of $\alpha$ which coincides with an extended epoch of deceleration. Finally, the universe moves back into the recent acceleration with $\alpha$ approaching the second maxima. The two panels in the Figure are for different signs of $\zeta$.

In Fig. 24 we plot $\alpha$ vs $G/G_0$ for the two signs of $\zeta$, taking $\dot{\phi} < 0$. This case is for the version of the theory where gravitational interaction was stronger in past and decays with cosmic time. $G_{eff}$ can in principle vary in the range $[1, \infty)$ in this case. Quite similar to the previous version of the theory, a clear formation of minima is noted: $\alpha$ evolves negligibly in the cosmological past (i.e., for $G_{eff} > 1$) and rolls along the variation of $G_{eff}$. The two end-points of the graph, i.e., two maximum allowed values of $\alpha$ can mark the epochs of cosmic acceleration, while the minima signals an epoch of deceleration. The overall scale of the variation is quite mild. However, around the present epoch where $\alpha$ starts growing in a rather dominant manner, this is a bit contrary to physical expectations.

Depending on a combination of calculative guess and basic fitting we give a rough functional form that describes how $\alpha$ might have evolved with $G$,

$$
\alpha = \alpha_0 \left[ 1 - \gamma_0 \left\{ (-1)^{\frac{G_0}{G}} - 1 \right\} \right]^{\gamma_2} \epsilon^{\gamma_3 \left\{ (-1)^{\frac{G_0}{G}} - 1 \right\}^{\gamma_4}} \quad \text{(73)}
$$

$\alpha_0$ is the present value of the fine structure coupling, which is close to $\simeq 0.0073$. The ratio $\frac{G_0}{G}$ is dimensionless, where $G_0$ is the present value of the coupling.

- $\epsilon = \pm 1$. This parameter indicates the nature of the theory under consideration. For $\epsilon = +1$, $G_{eff}$ decays with Hubble, i.e., grows with cosmic expansion. For $\epsilon = -1$, $G_{eff}$ grows with Hubble, i.e., decays with cosmic expansion.
- For $\zeta > 0$, we estimate that (i) $\gamma_0 \simeq 0.0001$, (ii) $\gamma_1 \simeq 100$, (iii) $\gamma_2 \simeq 1$, (iv) $\gamma_3 \simeq 1$ and (v) $\gamma_4 \sim 0.25$.
- For $\zeta < 0$, we estimate that (i) $\gamma_0 \simeq 0.0001$, (ii) $\gamma_1 \simeq 100$, (iii) $\gamma_2 \simeq 0.5$, (iv) $\gamma_3 \simeq 0.2$ and (v) $\gamma_4 \sim 0.5$.

Since we are talking about the evolution of the universe in terms of look-back time only, an extended question in this regard would be to ask whether the universe will continue to follow this pattern in the future as well. This is a bit difficult.
to answer based solely on this analysis. We will need a dynamical system analysis of the differential equations governing the system, in order to identify the attractor fixed points. Moreover, the nature and distribution of the cold dark matter does not interfere much in this issue. Nevertheless, it can affect the slope of the curves which means it can affect the onset and span of different epochs as well as the rate of expansion, through the parameter $\zeta$. The manner in which the universe evolves is never affected.

6 CONCLUSION

Fundamental couplings are an essence of the most rudimentary postulates of physics. However, their origin remains an enigma. Many decades have passed since it was first conjectured that these couplings can characterize different states of the evolving universe and should be treated as varying entities. Today the research in this arena is quite rich, combining non-trivial theoretical formulations with advanced experimental/observational data. We contribute to this genre by considering a generalized theory of varying fine structure constant $\alpha$. The theory is inspired from a generally covariant formalism originally given by Bekenstein (1982) where $\alpha$ varies due to a real scalar field interacting with charged matter. The cosmological extension of Bekenstein’s theory, developed by Sandvik, Barrow and Magueijo (2002), is popular as a BSBM setup. We work with two examples where a standard BSBM is further generalized. The first generalization allows a field dependent-kinetic term. The second one is a unified theory of two simultaneously varying constants, $\alpha$ and the gravitational coupling $G$. This is done by including a Brans-Dicke type geometric scalar in the Lagrangian. Kinetic terms for both of the scalars are functions of the respective fields in this example. Our aim is to explore the required structure of these extended theories such that a consistent cosmological evolution can be realized alongside the mild variation of $\alpha$.

We utilize a simple methodology for this exercise, based on a parametrization of the present matter density of the universe, written as $\Omega_m(z)$. It gives us the Hubble function in closed form, simple enough to compare the theories with reasonable sets of observational data. The observations are of the present epoch and therefore this reconstruction leads to a consistent late-time cosmology, describing the smooth deceleration-to-acceleration transition very nicely. Moreover, we do not need to assume any form of a Dark energy EOS at the outset. Through a statistical analysis of the cosmological data and the comparison with theoretical calculations we establish the validity of the model. We also bring in specific measurements of $\alpha$ variations, reported as $\Delta \alpha/\alpha$ vs redshift in the spectroscopic analysis of molecular absorption lines observed at Keck and VLT telescopes. A careful comparison of these observations with the theoretically derived $\Delta \alpha/\alpha$ provides us comprehensible constraints on the allowed variation of $\alpha$ and on the extended theories as well.

The scalar field responsible for $\alpha$ variation is of chameleon nature. It interacts with matter in the Lagrangian and acquires a density-dependent mass term. This is realized from the Lagrangian Eq. (12) and the scalar Eq. (16). As a result, the scalar can decouple around massive objects due to its own interaction with matter and avoid detection. In extension, this might contribute to the emerging questions involving a violation of Equivalence Principle Constraints. With recent observations disfavoring the case of a standard chameleon field driving the cosmic expansion, the requirement for generalized formalisms such as these, increases manifold. The models in this manuscript, in a sense, provides two different examples, (i) by generalizing the kinetic part of a chameleon field and (ii) by including a second field of geometric origin. We note in passing that the scalar-matter interaction profiles ($f(\psi) \sim e^{-2\psi}$) for these two models are found to be quite different as a function of redshift, although the overall variation of their profile is mild (See Fig. 25 for reference).

Both of the models give a mild variation of $\alpha$ consistent with observations. The first model is a natural generalization with the kinetic part evolving roughly as an exponential of the $e$-field (See Fig. 10 for reference), $\omega(\psi) \sim e^{2\psi}$. The numerical solutions of the field equations suggest that $\beta$ should be positive, i.e., $\omega(\psi)$ should grow with $\psi$ for a consistent late-time cosmology. This is well consistent with the results of Barrow and Lip (2012), although here the form is derived from cosmological requirements and not assumed at the outset. For the second model, we are forced to choose a similar form of $\omega(\psi)$ to be able to solve the non-linear equations. The rate of variation of $\Delta \alpha/\alpha$ is a bit different for this model, resulting in a nicer fit with observations. Additionally, this is a special example of generalized Brans-Dicke theory with the varying $\alpha$ being fitted in as an effective matter field. The kinetic coupling $\omega_{BD}$ is a function of the
Brans-Dicke scalar $\phi$ and its evolution is also determined by solving the field equations. We have also guessed a most likely functional form of $\omega_{BD}(\phi) \sim \delta_1 e^{-\phi_0 \phi}$ by fitting in with the numerical solution. For a standard Brans-Dicke theory, $\omega_{BD}$ needs to be quite a large number to provide a viable cosmological solution. With the present generalization, we say that the function $\omega_{BD}(\phi)$ needs to behave differently for different epochs, as per requirements. It dominates since an era of late-time acceleration sets off but is subdued throughout the deceleration. This model also shows how the cold dark matter distribution in the present universe can contribute to the nature of $\alpha$ evolution, through the parameter $\zeta = \mathcal{L}_{em}/\rho_m$. The signature of $\zeta$ determines the ratio of magnetic energy to the electric field energy of the system. We show that while the low redshift behavior of $\psi, \alpha$ and $\Delta \alpha/\alpha$ do not show much of a difference between $\zeta < 0$ and $\zeta > 0$, for higher redshifts the evolutions are characteristically distinct.

The generalized Brans-Dicke-BSBM model assembles two separate natures of a scalar-tensor theory into one fold. These models allow a varying $G_{\text{eff}} \propto 1/\phi$, where $\phi$ is a geometric scalar field. Depending on the $\phi$ profile, the theory can illustrate two different characters. For a monotonically increasing $\phi$, $G_{\text{eff}}$ decays smoothly with Hubble, indicating that gravitational interaction was weaker in the past. On the other hand, for a monotonically decreasing $\phi$, $G_{\text{eff}}$ increases with Hubble, i.e., decays with cosmic time. It is, however, noted that the latter of these cases may further complicate the already non-trivial issue of $H_0$ tension. A weakening $G_{\text{eff}}$ during the present cosmic acceleration can avoid this complication, but only at the expense of giving away the asymptotically free nature of the theory. We also find that the present epoch enjoys an extremum of $G_{\text{eff}}$ irrespective of the nature of the theory, independent of the initial conditions.

We conclude with a hope that the generalized Brans-Dicke-BSBN formalism in particular, can be thought of as the special case of a more fundamental theory. This theory, if formulated in a better way, should be able to describe the variations of all the fundamental couplings within one mathematical construct. This postulation finds motivation from a comparative analysis of the variations of two natural couplings, the fine structure constant and the gravitational constant. The pattern of their mutual variation may as well shed some light on the phases of cosmic expansion. (i) The universe starts evolving from an early phase of extended deceleration where there is an extremum of $\alpha$ and $G$. (ii) The second phase sees $\alpha$ rolling down as a function of $G$, towards a minima and this coincides with an epoch of extended deceleration. (iii) Finally, $\alpha$ rolls back up to a second extremum with the universe entering the phase of recent acceleration. $G$ is either a monotonically increasing or a monotonically decreasing function of Hubble, illustrating two entirely different kind of theories within the same framework. We put forward a general prediction that any epoch of cosmic acceleration should exhibit an extremum of any natural coupling. Although the cold dark matter distribution of the universe can affect the onset and span of different epochs, it has negligible effect on the $\alpha$ vs $G$ pattern. This pattern may not be completely comprehensive, however, the analogies do motivate the requirement for a unified field theory; perhaps through a consistent assembly of extended theories of gravity, phenomenologies of particle physics and a better implementation of the Large Numbers hypothesis.

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