Semigroups in 3-graded Lie groups and endomorphisms of standard subspaces

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March 18, 2022

Abstract

Let $V$ be a standard subspace in the complex Hilbert space $\mathcal{H}$ and $U : G \to U(\mathcal{H})$ be a unitary representation of a finite dimensional Lie group. We assume the existence of an element $h \in g$ such that $U(\exp th) = \Delta^V_t$ is the modular group of $V$ and that the modular involution $J_V$ normalizes $U(G)$. We want to determine the semigroup $S_V = \{ g \in G : U(g)V \subseteq V \}$. In previous work we have seen that its infinitesimal generators span a Lie algebra on which $\text{ad} h$ defines a 3-grading, and here we completely determine the semigroup $S_V$ under the assumption that $\text{ad} h$ defines a 3-grading on $g$. Concretely, we show that the $\text{ad} h$-eigenspaces $g^{\pm 1}$ contain closed convex cones $C^\pm$, such that $S_V = \exp(C^+)G_V\exp(C^-)$, where $G_V = \{ g \in G : U(g)V = V \}$ is the stabilizer of $V$. To obtain this result we compare several subsemigroups of $G$ specified by the grading and the positive cone $C_U$ of $U$. In particular, we show that the orbit $O_V = U(G)V$ with the inclusion order is an ordered symmetric space covering the adjoint orbit $O_h = \text{Ad}(G)h$, endowed with the partial order defined by $C_U$.

MSC 2010: Primary 22E45; Secondary 81R05, 81T05.

Keywords: Standard subspace, Quantum Field Theory, graded Lie group, endomorphism semigroup, ordered symmetric space

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1 Introduction

A real subspace $V$ of a complex Hilbert space $H$ is called standard if it is closed and
\[ V \cap iV = \{0\} \quad \text{and} \quad H = V + iV \quad (1) \]
(cf. [Lo08] for the basic theory of standard subspaces). If $V \subseteq H$ is a standard subspace, then $TV: D(T_V) := V + iV \rightarrow H, \ x + iy \mapsto x - iy \quad (2)$ defines a closed operator with $V = \text{Fix}(T_V)$, called the Tomita operator of $V$. Its polar decomposition can be written as $T_V = J_V \Delta_V^{1/2}$, where $J_V$ is a conjugation (an antiunitary involution) and $\Delta_V$ is a positive selfadjoint operator such that the modular relation
\[ J_V \Delta_V J_V = \Delta_V^{-1} \quad (3) \]
holds. We call $(\Delta_V, J_V)$ the pair of modular objects associated to $V$.

Standard subspaces arise naturally in the modular theory of von Neumann algebras. If $M \subseteq B(H)$ is a von Neumann algebra and $\Omega \in H$ is cyclic (for $M\Omega$ is dense in $H$) and separating (the map $M \rightarrow H, M \mapsto M\Omega$ is injective), then the Tomita–Takesaki Theorem ([BR87, Thm. 2.5.14]) implies that $V := \{M\Omega: M = M^* \in M\}$ is standard, and that $J_V \Delta_V J_V = M'$ and $\Delta_V^{it} M \Delta_V^{-it} = M$ for $t \in \mathbb{R}$.

So we obtain a one-parameter group of automorphisms of $M$ (the modular group) and a symmetry between $M$ and its commutant $M'$, implemented by $J_V$.

Building on the Haag–Kastler theory of local observables in Quantum Field Theory (QFT) ([Ha96], [BS93], [BDFS00], [FR19]), the current interest in standard subspaces arose in the 1990s from the work of Borchers and Wiesbrock ([Bo92, Wi93]). This in turn led to modular localization in Quantum Field Theory introduced by Brunetti, Guido and Longo in [BGL02, BGL94, BGL93]; see also [Le15, LL15] for important applications of this technique.

The order on the set $\text{Stand}(H)$ of standard subspaces of $H$ is of particular importance because it reflects inclusions of corresponding von Neumann algebras (see [NO17] §4.2, [Lo08] and [Ta10] for more details on the translation process). As the order on $\text{Stand}(H)$ is hard to understand, it is natural to probe the ordered space $\text{Stand}(H)$ by finite dimensional homogeneous submanifolds arising as orbits under unitary representations of finite dimensional Lie groups. To link such a representation as closely as possible to standard subspaces, we consider the following setting. Let $V \subseteq H$ be a standard subspace and $U: G \rightarrow U(H)$ be a unitary representation of the connected Lie group $G$. We further assume that there exists an involutive automorphism $\tau_G$ of $G$ and $h \in \mathfrak{g}$ fixed by $\tau = L(\tau_G)$ such that $U$ extends to an antiunitary representation of $G \rtimes \{\text{id}_G, \tau_G\}$ such that
\[ U(\exp th) = \Delta_V^{-it/2\pi} \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad J_V U(g) J_V = U(\tau_G(g)) \quad \text{for} \quad g \in G. \quad (4) \]

Then the order on the orbit $O_V := U(G)V$ is determined by the subsemigroup
\[ S_V := \{g \in G: U(g)V \subseteq V\}. \quad (5) \]
In [Ne19] we managed to calculate its Lie wedge

\[ L(S_t) = \{ x \in \mathfrak{g} : \exp(\mathbb{R}_+ t) \subseteq S_t \}, \]

i.e., the set of generators of its one-parameter subsemigroups in the Lie algebra \( \mathfrak{g} \) of \( G \). To formulate this result, for \( x \in \mathfrak{g} \), we write \( \partial U(x) \) for the (possibly unbounded) skew-adjoint operator on \( \mathcal{H} \) with \( U(\exp t x) = e^{i \partial U(x) t} \) for \( t \in \mathbb{R} \), and

\[ C_U := \{ x \in \mathfrak{g} : -i \partial U(x) \geq 0 \} \]

for the positive cone of \( U \). The Structure Theorem ([Ne19 Thm. 4.4]) asserts that, under the natural assumption that \( \ker(U) \) is discrete,

\[ L(S_t) = C_- \oplus \mathfrak{g}_\tau \oplus C_+, \quad \text{where} \quad \mathfrak{g}_\tau = L(G_\tau), \quad G_\tau = \{ g \in G : U(g) \mathcal{V} = \mathcal{V} \}, \quad (6) \]

and

\[ C_\pm := \{ x \in C_U : \tau(x) = -x, [h, x] = \pm x \}. \]

We also show that the cone \( L(S_t) \) spans a Lie subalgebra which is 3-graded by \( \text{ad} h \) and on which \( \tau \) coincides with \( e^{\pi i \text{ad} h} \).

In the present paper we therefore focus on the situation where \( g \) is 3-graded by \( \text{ad} h \) in the sense that the \( \text{ad} h \)-eigenspaces \( \mathfrak{g}^\lambda := \mathfrak{g}^\lambda(h) := \ker(\text{ad}(h - \lambda 1)) \) satisfy

\[ g = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1, \quad \text{and} \quad \tau = e^{\pi i \text{ad} h}. \quad (7) \]

The \( \tau \)-eigenspaces are then \( h = \mathfrak{g}^0 \) and \( q = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1} \). We also assume that \( C \subseteq \mathfrak{g} \) is a closed pointed \( \text{Ad}(G) \)-invariant convex cone satisfying \( \tau(C) = -C \) and that \( G \) is a 1-connected Lie group with Lie algebra \( \mathfrak{g} \). The involution \( \tau \) then integrates to an involution \( \tau_G \) on \( G \) and

\[ C_\pm := \pm C \cap \mathfrak{g}^{\pm 1} \]

are pointed convex cones invariant under the action of the group \( G^0 := \{ g \in G : \text{Ad}(g) h = h \} \).

These structures lead to three subsemigroups of \( G \):

- The Olshanski semigroup \( S(C_\ell) := G^0 \exp(C_\ell) \) for \( C_\ell := C_+ \oplus C_- \subseteq q \),
- the semigroup \( S(h, C) := \{ g \in G : h - \text{Ad}(g) h \in C \} \), and
- using the complex Olshanski semigroup \( S(iC) := G \exp(iC) \) (see Subsection 2.2.4 for details), we define the subsemigroup \( G_{e i}(C) \) of those elements \( g \in G \) for which the orbit map \( \beta^g : \mathbb{R} \to G, \beta^g(t) = \exp(th)g \exp(-th) \) extends analytically to a map \( \beta^g : \mathbb{C} \to G \).

The main results of Section 2 are the equalities

\[ S(C_\ell) = S(h, C) = \exp(C_+ G^0 \exp(C_-) = G_{e i}(C). \quad (8) \]

The first two equalities constitute the Decomposition Theorem 2.16, and the last equality is Theorem 2.21. The key point of the identity (8) is that it provides three rather different perspectives on the same subsemigroup of \( G \), and this contains important information on the semigroups \( S_t \).

To see this connection, let us first consider an antiunitary representation \( (U, \mathcal{H}) \) with discrete kernel for a semidirect product \( G \rtimes \{ \text{id}_G, \tau_G \} \), where \( G \) is a connected Lie group, and \( \tau_G \) is an involutive automorphism of \( G \). We consider the standard subspace \( \mathcal{V} \subseteq \mathcal{H} \) specified by the modular objects

\[ J_\mathcal{V} = U(\tau_G) \quad \text{and} \quad \Delta_\mathcal{V} = e^{2 \pi i \theta U(h)} \quad \text{for some} \quad h \in \mathfrak{g}^\tau \quad (9) \]

In the theory of Lie semigroups ([HHL90] [HN93]) Lie wedges are the semigroup analogs of the Lie algebras of closed subgroups. A Lie wedge is a closed convex cone \( W \) in a Lie algebra \( \mathfrak{g} \) such that \( e^{\text{ad} x} W = W \) for \( x \in W \cap -W \). In particular, linear subspaces are Lie wedges if and only if they are Lie subalgebras.
but make no further assumptions on $h$. Building on some observations by Borchers and Wiesbrock (Theorem 3.1), we show in the Monotonicity Theorem (Theorem 3.3) that

$$S_V \subseteq S(h, C_U).$$

Here the main point is that, for two standard subspaces $V_1 \subseteq V_2$, we have $\log \Delta_{V_2} \leq \log \Delta_{V_1}$ in the sense of quadratic forms. Since these logarithms are typically not semibounded, the order relation requires some explanation that we provide in Appendix A. Put differently, the Monotonicity Theorem asserts that the well-defined $G$-equivariant map

$$O_V = \mathcal{U}(G)\mathcal{V} \cong G/G_V \to O_h \cong G/G_0, \quad \mathcal{U}(g)\mathcal{V} \mapsto \mathcal{U}(g)h$$

is monotone, hence the name.

Combining the Monotonicity Theorem with the identities (3), it is now easy to determine the semigroup $S_V$ for the case where $g$ is 3-graded by $ad\,h$, and $\tau = e^{\pi i} ad\,h$. It is given by

$$S_V = G_V \exp(C_q) = \exp(C_+)G_V \exp(C_-) \subseteq S(h, C) \quad \text{for} \quad C = C_U.$$

As a consequence for the general case, the infinitesimally generated subsemigroup $\langle \exp(L(S_V)) \rangle$ of $S_V$ coincides with $\exp(C_+)G_V \exp(C_-)$.

We conclude this paper with a short section on perspectives, where we explain how the present results can be used to explore covariant nets of standard subspaces on the abstract level (MN20) and how to find them in Hilbert spaces of holomorphic functions on tubes (NOO20) and in Hilbert spaces of distributions on Lie groups (NO20). Another important issue is the classification of all tuples $(g, h, C)$, where $g$ is 3-graded by $ad\,h$ and generated by $C_\pm$ and $h$, and, more generally, the subalgebras generated by $C_\pm$ and $3(h)$ for tuples $(g, h, \tau, C)$, where $\tau(h) = h, C$ is a pointed invariant convex cone satisfying $\tau(C) = -C$ and $C_\pm = \pm C \cap q^\pm(h)$.

For first steps in this classification program we refer to (Oeh20).

Finally, we note that Longo and Witten obtain in [LW11] some results on semigroups $S_V$ that can be interpreted as infinite dimensional versions, where $G \cong E \times \alpha \mathbb{R}$, and $E$ is a topological vector space. Their results show that, extending our results to infinite dimensional groups requires completely new techniques that are different from what we use in the finite dimensional case, here and in [Ne19].

**Notation**

- For a Lie group $G$, we write $\mathfrak{g}$ for its Lie algebra, $\text{Ad}: G \to \text{Aut}(\mathfrak{g})$ for the adjoint action of $G$ on $\mathfrak{g}$, induced by the conjugation action of $G$ on itself, and $\text{ad}\,x(y) = [x, y]$ for the adjoint action of $\mathfrak{g}$ on itself.
- $\text{AU}(\mathcal{H})$ is the group of unitary or antiunitary operators on a complex Hilbert space $\mathcal{H}$.
- If $\tau_G \in \text{Aut}(G)$ is an order two automorphism, then $G_\tau := G \times \{\text{id}_G, \tau_G\}$ becomes a Lie group and an antiunitary representation of $G_\tau$ is a homomorphism $\mathcal{U}: G \to \text{AU}(\mathcal{H})$ for which $J := \mathcal{U}(\tau_G)$ is antiunitary and $\mathcal{U}(G) \subseteq \text{U}(\mathcal{H})$. Antiunitary representations are assumed to be continuous with respect to the strong operator topology on $\text{AU}(\mathcal{H})$.
- The automorphism of $\mathfrak{g}$ induced by $\tau_G$ is denoted $\tau$ and we write

$$\mathfrak{h} := \{x \in \mathfrak{g} : \tau(x) = x\} \quad \text{and} \quad \mathfrak{q} := \{x \in \mathfrak{g} : \tau(x) = -x\}$$

for its eigenspaces. We put $g^\perp := \tau_G(g)^{-1}$.
- For a real standard subspace $V \subseteq \mathcal{H}$, we write $(\Delta_V, J_\delta)$ for the corresponding pair of modular objects specified by $V = \text{Fix}(J_\delta \Delta_V^{1/2})$.
- Horizontal strips in the complex plane are denoted $S_\beta := \{z \in \mathbb{C} : 0 < \text{Im} \, z < \beta\}$ for $\beta > 0$. 


For a unitary representation \( U : G \to \mathcal{U}(\mathcal{H}) \) of a finite dimensional Lie group \( G \), we write \( \mathcal{H}^\infty \) for the dense subspace of smooth vectors \( \xi \), i.e., the orbit map \( U^\xi : G \to \mathcal{H}, g \mapsto U_g \xi \) is smooth. The infinitesimal generator of the unitary one-parameter group \( (U(\exp(tx)))_{t \in \mathbb{R}} \) is denoted \( \partial U(x) \). The closed convex \( \text{Ad}(G) \)-invariant cone

\[
C_U := \{ x \in \mathfrak{g} : -i \partial U(x) \geq 0 \}
\]

is called the positive cone of the representation \( U \).

## 2 Groups and semigroups in 3-graded Lie groups

In this section we take a closer look at the groups and semigroups that arise naturally in our setting. Throughout, \( G \) is a 1-connected (connected and simply connected) Lie group with Lie algebra \( \mathfrak{g} \), and \( h \in \mathfrak{g} \) defines a 3-grading \( \mathfrak{g} = \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \) (see (7)). In Subsection 2.1 we study basic properties of the subgroups

\[
G^{\pm 1} := \exp(\mathfrak{g}^{\pm 1}), \quad G^0 = \{ g \in G : \text{Ad}(g)h = h \} \quad \text{and} \quad P_q := G^0 G^{q+1}.
\]

In Subsection 2.2 we then also take an \( \text{Ad}(G) \)-invariant pointed closed convex cone \( C \) into account which also satisfies \( \tau(C) = -C \) for \( \tau = e^{x \text{ad} h} \). First, we show that we have an Olshanski semigroup

\[
S(C_q) := G^0 \exp(C_q) \quad \text{for} \quad C_q := C_+ \oplus C_-
\]

In Subsection 2.3 we use the pointed invariant cone \( C \) to define an \( \text{Ad}(G) \)-invariant partial order on \( \mathfrak{g} \) by \( x \leq_C y \) if \( y - x \in C \). Clearly, each adjoint orbit thus inherits an invariant order structure, and the orbit \( O_h := \text{Ad}(G)h \) is of particular interest. It is an ordered symmetric space, which leads us to the semigroup

\[
S(h, C) := \{ g \in G : \text{Ad}(g)h \leq_C h \}.
\]

One of our main results is Theorem 2.16 asserting that \( S(C_q) = S(C, h) \) and that this semigroup has a triangular decomposition \( S(h, C) = \exp(C_-) G^0 \exp(C_+) \) (Subsection 2.3). In Theorem 2.21 we further show that this subsemigroup also coincides with \( G_{\leq}(C) \). So we obtain three rather different perspectives on the same subsemigroups of \( G \) (Subsection 2.4).

### 2.1 Subgroups associated with the 3-grading

**Lemma 2.1.** If \( G \) is a 1-connected Lie group and \( \mathfrak{g} \) is 3-graded by the element \( h \in \mathfrak{g} \), then there exists a Levi complement \( l \subseteq \mathfrak{g} \) invariant under \( \text{ad} h \). We write \( L \) for the corresponding integral subgroup and \( R \) for the solvable radical of \( G \). Then the following assertions hold:

(i) \( G \cong R \rtimes L \) and \( G^j = R^j \rtimes L^j \) for \( j = -1, 0, 1 \).

(ii) \( R = R^{-1} R^0 R^1 \).

(iii) The projection \( p_l : G \to L \) satisfies \( G^{-1} G^0 G^1 = p_L^{-1} (L^{-1} L^0 L^1) \).

**Proof.** Since the derivation \( \text{ad} h \) is semisimple, [Ne19 Thm. B.2] implies the existence of a Levi complement \( l \) in \( \mathfrak{g} \), invariant under \( \text{ad} h \).

(i) As \( G \) is simply connected, we have \( G \cong R \rtimes L \), where \( R \) and \( L \) are both simply connected.

Since both factors are invariant under conjugation with \( \exp(th) \) for \( t \in \mathbb{R} \), and \( g \in G^0 \) is equivalent to \( g \exp(th)^{-1} \exp(th) \) for all \( t \in \mathbb{R} \), which in turn is equivalent to \( g = \exp(th) g \exp(th)^{-1} \) for \( t \in \mathbb{R} \), we obtain \( G^0 = R^0 \rtimes L^0 \). The relation \( G^j = R^j \rtimes L^j \) for \( j = \pm 1 \) follows from \( g^{\pm 1} = r^{\pm 1} \rtimes l^{\pm 1} \), which follows from the invariance of \( r \) and \( l \) under \( \text{ad} h \).
(ii) As \( t^{±1} \subseteq [h, t] \subseteq [g, t] \) is contained in the maximal nilpotent ideal of \( g \) \cite[Ch. 1, §5.3, Thm. 1]{Bon90}, we have \( t = u + t^i(h) \). Hence it suffices to show that the multiplication map \( U^{-1} \times U^0 \times U^1 \to U \) of the integral subgroup \( U \) corresponding to \( u \) is a diffeomorphism. As \( u^{±1} \) and \( u^{±1} + u^0 \) are Lie subalgebras of the nilpotent Lie algebra \( u \), this follows by applying \[ \text{Lemma 11.2.13} \] twice.

(iii) As \( p_l : g \to l \) is a morphism of 3-graded Lie algebras, we have \( p_l(g^j) = l^j \) for \( j = 0, ±1 \), and therefore \( p_L(G^{±1}) = L^{±1} \). To see that \( p_L(G^0) \subseteq L^0 \), we note that \( g \in G^0 \) means that \( \text{Ad}(g)h = h \), and applying \( p_l \) leads to

\[ h_t := p_l(h) = p_l(\text{Ad}(g)h) = \text{Ad}(p_L(g))h_t. \]

If, conversely, \( g_L \in L \) satisfies \( \text{Ad}(g_L)h_t = h_t \), then \( h_t := h - h_t \in t \) satisfies \( [h_t, t] = \{0\} \).

As \( L \) is connected, it follows that \( \text{Ad}(g_L)h_t = h_t \). We conclude that \( \text{Ad}(g_L)h = h \), and thus \( p_L(G^0) = L^0 \). This shows that

\[ p_L(G^{−1}G^0G^1) = L^{−1}L^0L^1. \tag{12} \]

Further, \( R = R^{−1}R^0R^1 \) by (ii), so that the subgroup property of \( G^0G^1 \) and \[ \text{Lemma 2.1(ii)} \] lead to

\[ p_L^{−1}(L^{−1}L^0L^1) = RG^{−1}G^0G^1 = G^{−1}RG^0G^1 = G^{−1}R^{−1}R^0R^1G^0G^1 = G^{−1}G^0G^1. \]

The following lemma is useful for reductions from \( G \) to its adjoint group.

**Lemma 2.2.** The subgroup \( G^0 = \{ g \in G : \text{Ad}(g)h = h \} \) coincides with

\[ G_{ad,h} := \{ g \in G : \text{Ad}(g) \text{ad} h \text{Ad}(g)^{-1} = \text{ad} h \} = \{ g \in G : \text{Ad}(g)h - h \in z(g) \}. \]

**Proof.** As \( G^0 \subseteq G_{ad,h} \), we have to show that, if \( \text{Ad}(g) \) commutes with \( \text{ad} h \), i.e., if it preserves the 3-grading, then \( \text{Ad}(g)h = h \). Let \( g = t \times l \) be an \( h \)-invariant Levi decomposition (Lemma 2.1) and write \( h = h_t + h_l \), accordingly. Then \( [h_t, t] = \{0\} \) because \( \text{ad} h \) and \( \text{ad} h_l \) have the same restriction on \( l \). We write \( G = R \times L \) for the corresponding decomposition of \( G \) and, accordingly, \( g \in G \) as \( g = g_Rg_L \) with \( g_R \in R \), \( g_L \in L \).

Assume that \( \text{Ad}(g) \) commutes with \( \text{ad} h \). Then \( \text{Ad}(g_L) \) commutes with \( \text{ad} h_t \), and this implies that \( \text{Ad}(g_L)h_t = h_t \) because \( z(l) = \{0\} \). For \( g = g_Rg_L \) we also have \( \text{Ad}(g_L)h_t = h_t \) because \( h_t \) commutes with \( l \). We thus obtain

\[ \text{Ad}(g)h = \text{Ad}(g_R)h_R \in h + z(g). \]

Next we write \( g_R = \exp(x_1)\exp(x_{−1})g_0 \) with \( g_0 \in R^0 \) and \( x_{±1} \in t^{±1} \) (Lemma 2.1(ii)) and obtain

\[ \text{Ad}(g_R)h = e^{\text{ad}x_1}e^{\text{ad}x_{−1}}h = e^{\text{ad}x_1}(h + [x_{−1}, h]) = e^{\text{ad}x_1}(h + x_{−1}) \in h + x_{−1} + t^0 + t^1. \]

Therefore \( \text{Ad}(g)h \in h + z(g) \subseteq h + t^0 \) implies \( x_{−1} = 0 \). We likewise obtain \( x_1 = 0 \), so that \( \text{Ad}(g)h = \text{Ad}(g_R)h = h \). \( \square \)

**Remark 2.3.** (a) The subgroup \( G^0 \) contains the center of \( G \) but \( \tau_G \) may act non-trivially on the center. A typical example arises for \( G = \text{SL}_2(\mathbb{R}) \) with \( Z(G) \cong \mathbb{Z} \cong \pi_1(\text{PSL}_2(\mathbb{R})) \). Here the fundamental group of \( \text{PSL}_2(\mathbb{R}) \) is generated by the loop obtained from the inclusion \( \text{PSO}_2(\mathbb{R}) \to \text{PSL}_2(\mathbb{R}) \). Let \( \tau_{\mathbb{C}} \in \text{Aut}(G) \) be the involution given on the Lie algebra level by

\[ \tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}, \quad \text{which is} \quad e^{π_1\text{ad}h} \quad \text{for} \quad h = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{13} \]

Then \( \tau_G \) induces the inversion on \( Z(G) \). Here \( \exp(\mathbb{R}h) \) is the identity component of \( G^0 \), \( Z(G) \subseteq G^0 \), and \( Z(G) \cap \exp(\mathbb{R}h) = \{e\} \) so that \( Z(G) \cong \pi_0(G^0) \). In particular, \( G^0 \) has infinitely many connected components.
Proposition 2.5.\( P \) Let \( R \subseteq G \) denote the solvable radical, i.e., the maximal connected normal solvable subgroup. Then Levi’s Theorem, and the 1-connectedness of \( G \) imply that \( G \cong R \times L \) for a 1-connected semisimple Lie group \( L \). Then [Ne19 Thm. B.2(ii)] implies that \( R^0 = G^0 \cap R \) is connected, but \( G^0 \) need not be connected, as we have seen in (a) for \( G = \text{SL}_2(\mathbb{R}) \).

(c) Let
\[
G^2 := \{ g \in G \setminus \text{Ad}(g) \tau \text{Ad}(g)^{-1} = \tau \} = \text{Ad}^{-1}(\text{Ad}(G)^{\tau}) \supseteq G^0.
\]
Then \( g \in G^2 \) is equivalent to \( g \tau(g)^{-1} = gg^2 \in \ker \text{Ad} = \text{Z}(G) \). The map
\[
\gamma : G^2 \to \text{Z}(G), \quad \gamma(g) = gg^2
\]
is a group homomorphism because \( \gamma(g)\tau(h)h = g\gamma(g)h^2 = \gamma(gh) \) for \( g, h \in G^2 \) follows from \( hh^2 \in \text{Z}(G) \). As \( G \) is 1-connected, \( G^\tau = \ker \gamma \) is connected ([Ne19 Thm. B.2]), so that \( \pi_0(G^2) \cong \gamma(G^2) \) can be identified with a subgroup of \( Z(G) \).

For \( g \in G^2 \) we have \( \gamma(g)^2 = \gamma(g) \), i.e., \( \tau(\gamma(g)) = (\gamma(g))^{-1} \). Therefore \( \gamma(g) \) is contained in the discrete subgroup
\[
Z_1 := \{ z \in Z(G) \setminus G^2 : \gamma(z) = \gamma(z) \}.
\]
The discreteness of this subgroup follows from \( L(Z_1) = \{ 1 \} \cap \gamma = \{ 0 \} \). We also note, since \( G \) is 1-connected, \( \text{exp} \mid_{\gamma(\gamma)} : \gamma(\gamma) \to Z(G)^0 \) is bijective, so that \( Z(G)^0 \cap Z_1 \) is a closed subgroup of \( G^2 \).

Lemma 2.4. For the subgroups \( P_\pm := G^0G^\pm_1 \), the following assertions hold:

(i) \( P_\pm = \{ g \in G : \text{Ad}(g)h \in h + \mathfrak{g}^\pm_1 \} \), and both subgroups are closed.

(ii) \( P_+ \cap P_- = G^0 \).

(iii) The subalgebras \( \mathfrak{p}_\pm = \mathfrak{g}^0 \oplus G^\pm_1 \) are self-normalizing.

(iv) The exponential function \( \exp : G^\pm_1 \to G^\pm_1 \) is a global diffeomorphism.

(v) The multiplication map \( G^1 \times G^0 \times G^1 \to G, (g_1, g_0, g_{-1}) \mapsto g_1g_0g_{-1} \) is a diffeomorphism onto the open subset \( G^1G^0G^1 \) of \( G \).

Proof. (i) As \( \text{Ad}(P_\pm)h = \text{Ad}(G^\pm_1)h = h + \mathfrak{g}^\pm_1 \), we have \( \text{Ad}(g)h - h \in \mathfrak{g}^\pm_1 \) for \( g \in P_\pm \). If, conversely, \( g \in G \) satisfies \( x_{\pm 1} := \text{Ad}(g)h - h \in \mathfrak{g}^\pm_1 \), then
\[
\text{Ad}(\exp(\pm x_{\pm 1}))g = e^{\pm x_{\pm 1}} h = h + x_{\pm 1} \pm [x_{\pm 1}, h] = h + x_{\pm 1} - x_{\pm 1} = h
\]
implies that \( \exp(\pm x_{\pm 1})g \in G^0 \), so that \( g \in G^\pm_1 G^0 = P_\pm \).

(ii) Clearly, \( G^1 \subseteq P_+ \cap P_- \). If, conversely, \( g \in P_+ \cap P_- \), then \( \text{Ad}(g)h - h \in \mathfrak{g}^1 \cap \mathfrak{g}^{-1} = \{ 0 \} \) by (i), so that \( g \in G^1 \).

(iii) In view of \( h \in \mathfrak{g}^0 \), for \( x = x_{-1} + x_0 + x_1 \) with \( x_j \in \mathfrak{g}^j \), the relation \( x_1 - x_{-1} = [h, x] \in \mathfrak{p}_\pm \) implies \( x_{\pm 1} = 0 \), so that \( x \in \mathfrak{p}_\pm \). This shows that \( \mathfrak{p}_\pm \) is self-normalizing.

(iv) As \( \mathfrak{g}^\pm_1 \) is abelian, the exponential function \( \exp : \mathfrak{g}^\pm_1 \to G^\pm_1 \) is a surjective group homomorphism. That it actually is a diffeomorphism follows from \( \text{Ad}(\exp h) = h + [h, h] = h + x \) for \( x \in \mathfrak{g}^\pm_1 \).

(v) The direct product group \( \mathbb{G}^1 \times P_- \) acts smoothly on \( G \) by \( (p, x) := gxp^{-1} \). Now the orbit map \( F : \mathbb{G}^1 \times P_- \to G, (g, p, e) := gxp^{-1} \) has in \( (e, e) \) the surjective differential
\[
g^1 \times P_- \to G, \quad (x, y) \mapsto x - y.
\]
This implies that \( F(G^1 \times P_-) = G^1G^0G^{-1} \) is an open subset of \( G \) and that \( F \) is a local diffeomorphism. By (ii), the stabilizer group of \( e \) is isomorphic to \( G^1 \cap P_- = G^1 \cap G^0 = \{ e \} \).

This follows from (ii) and the fact that, for \( x \in \mathfrak{g}^1 \), we have \( e^{x_{\pm 1}}h = h + [x, h] = h - x \). This implies that \( F \) is injective, hence a diffeomorphism onto an open subset of \( G \).

Proposition 2.5. \( P_+ \) coincides with the flag stabilizer
\[
G_{(P_+, \mathfrak{g}^1)} := \{ g \in G : \text{Ad}(g)p_+ = p_+, \text{Ad}(g)\mathfrak{g}^1 = \mathfrak{g}^1 \}.\]
Proof. Clearly, \( P_+ \subseteq G_{(p_+ + 1)} \). If, conversely, \( g \in G_{(p_+ + 1)} \), then \( h' := \text{Ad}(g)h \) defines a 3-grading with \( g^1(h') = g^2 \) and \( g^0(h') + g^1 = p_+ \). Therefore [HN03, Cor. 1.7] implies the existence of \( x_1 \in g^1 \) with \( \text{ad}(h' - h) = \text{ad}x_1 \), i.e., \( h' = h + x_1 + \mathfrak{z}(g) \). Then \( e^{\text{ad}x_1}h' = h' + [x_1, h'] = h' - x_1 \in h + \mathfrak{z}(g) \) shows that \( \exp(x_1)g \) fixes \( ad(h) \), hence preserves the 3-grading. This shows that \( G_{(p_+ + 1)} = G^3G_{ad}h \), so that the assertion follows from Lemma 27. \( \Box \)

2.2 The Olshanski semigroup \( S(C_q) \)

In this subsection we turn to the subsemigroups of \( G \) determined by the invariant cone \( C \), resp., its intersections \( C_{\pm} = \pm C \cap g^{\pm 1} \). As before, \( G \) is a 1-connected Lie group with Lie algebra \( \mathfrak{g} \), which is 3-graded by \( ad(h) \).

Proposition 2.6. If \( C \) is pointed, then the following assertions hold:

(i) The cone \( C_q := C_{+} \cap C_{-} \subseteq q = g^1 \oplus g^{-1} \) is weakly hyperbolic, i.e., Spec(ad\( x \)) \( \subseteq \mathbb{R} \) for \( x \in C_q \).

(ii) \( S(C_q) := G^0\exp(C_q) \) is a closed subsemigroup of \( G \) invariant under \( s \mapsto s^2 = \tau(s)^{-1} \).

(iii) The polar map \( \Phi : G^0 \times C_q \rightarrow S(C_q), (g,x) \mapsto g\exp x \) is a homeomorphism.

Proof. (i) From [Ne09] Prop. VII.3.4] it follows that \( C \) is weakly elliptic in the ideal \( \mathfrak{g}_C := C - C \subseteq \mathfrak{g} \), i.e., Spec(ad\( x \)) \( \subseteq \mathbb{R} \) for \( x \in C \). For \( x \in \mathfrak{g}_C \), we have ad\( x (g) \subseteq \mathfrak{g}_C \), so that Spec(ad\( x q \)) \( \subseteq \) Spec(ad\( \mathfrak{g}_C \)) \( \cup \{ 0 \} \), and therefore \( C \) is also weakly elliptic in \( \mathfrak{g} \).

Consider the isomorphism

\[ \zeta := e^{\frac{2\pi i}{3} \text{ad}h} : \mathfrak{g} \rightarrow \mathfrak{g}^\circ := \mathfrak{h} + i\mathfrak{q}, \quad x_1 + x_0 + x_{-1} \mapsto ix_1 + x_0 - ix_{-1}. \tag{14} \]

Then \( \zeta(C_q) = i(C \cap q) \) is weakly hyperbolic because \( C \cap q \) is weakly elliptic. As \( \zeta \) is an isomorphism of real Lie algebras, the cone \( C_q \) is also weakly hyperbolic.

(ii), (iii): We consider the quotient group \( G_{ad} := \text{Ad}(G) \) with Lie algebra \( \mathfrak{g}_{ad} := \text{ad}g \cong g/\mathfrak{z}(g) \) and the subgroup

\[ G_{ad}^* := (G_{ad})^\circ = \{ g \in G_{ad} : g \circ \tau = \tau \circ g \}. \]

As \( Z(G) \subseteq G^0 \), the Lie algebra \( \mathfrak{g}_{ad} \) inherits a natural 3-grading induced by \( adh \in \mathfrak{g}_{ad} \). For every \( g \in G^0 \), the automorphism \( \text{Ad}(g) \) commutes with \( adh \), hence also with \( \tau \), so that \( \text{Ad}(G^0) \subseteq (G_{ad})^\circ \).

For the weakly hyperbolic cone \( \text{ad}(C_q) \subseteq \text{ad}q \) the relation \( \mathfrak{z}(\mathfrak{g}_{ad}) \subseteq \mathfrak{g}_{ad}^0 \) implies that

\[ \text{ad}(C_q - C_q) \cap \mathfrak{z}(\mathfrak{g}_{ad}) = \{ 0 \}. \]

Therefore Lawson’s Theorem ([HN93 Thm. 7.34]) implies that the map

\[ \Psi : (G_{ad})^\circ \times C_q \rightarrow (G_{ad})^\circ \exp(\text{ad}C_q), \quad (g,x) \mapsto g\exp(x) \]

is a homeomorphism onto a closed subset of \( G_{ad} \). Restricting to the open, hence closed subgroup \( \text{Ad}(G^0) \) of \( G_{ad}^* \), it follows that

\[ \Psi : \text{Ad}(G^0) \times C_q \rightarrow \text{Ad}(G^0) \exp(\text{ad}C_q) = \text{Ad}(S(C_q)), \quad (g,x) \mapsto g\exp(x) \]

is a homeomorphism onto a closed subset of \( G_{ad} \). As \( Z(G) \subseteq G^0 \), the set \( S(C_q) = G^0\exp(C_q) \) satisfies \( S(C_q) = S(C_q)Z(G) \), so that \( S(C_q) = \text{Ad}^{-1}(\text{Ad}(S(C_q))) \) is a closed subset of \( G \).

Clearly, the map \( \Phi \) is continuous and surjective. Further, the map

\[ \sigma : S(C_q) \rightarrow C_q, \quad g\exp x \mapsto x \]

is well-defined and continuous because \( \Psi \) is a homeomorphism and \( \text{ad}|_q \) is injective since \( \ker(\text{ad}) = \mathfrak{z}(g) \subseteq \mathfrak{g}^0 \). Now \( \Phi(g_1, x_1) = \Phi(g_2, x_2) \) implies \( x_1 = \sigma(g_1, \exp(x_1)) = \sigma(g_2, \exp(x_2)) = x_2 \) which further entails that \( g_1 = g_2 \). Therefore \( \Phi \) is injective, and its inverse map

\[ \Phi^{-1} : S(C_q) \rightarrow G^0 \times C_q, \quad s \mapsto (s \cdot \exp(-\sigma(x)), x) \]
is also continuous. This shows that \( \Phi \) is a homeomorphism onto \( S(C_q) \).

Now we show that \( S(C_q) \) is a subsemigroup of \( G \). As \( G \) is 1-connected, [HN93 Cor. 7.35] implies that \( S(C_q)_0 := G_q \exp(C_q) \) is a subsemigroup of \( G \). As \( \Ad(G^0)C_q = C_q \), the subgroup \( G^0 \) normalizes the subsemigroup \( S(C_q)_0 \), and thus \( S(C_q) = G^0 \cdot S(C_q)_0 \) also is a subsemigroup of \( G \).

For \( s \in \exp(C_q) \), we have \( s^2 = s \), and for \( s \in G^0 \) we have \( s^2 = \tau(s)^{-1} \in G^0 \), so that \( S(C_q) \) is \( \tau \)-invariant. This completes the proof of (ii) and (iii).

\[ \square \]

### 2.3 The ordered symmetric space \( O_h \) and the semigroup \( S(h, C) \)

The pointed closed convex cone \( C \subseteq g \) defines a partial order on \( g \) by

\[ x \leq_C y \quad \text{if} \quad y - x \in C. \]

The invariance of \( C \) under the adjoint group \( \Ad(G) \) implies that \( x \leq_C y \) implies \( \Ad(g)x \leq_C \Ad(g)y \) for every \( g \in G \), so that \( G \) acts on \( g \) by order isomorphisms.

For the formulation of the following proposition, we recall the concept of a symmetric space in the sense of O. Loos:

**Definition 2.7.** Let \( M \) be a smooth manifold and \( M \times M \to M, (x, y) \mapsto x \bullet y := s_x(y) \) be a smooth map with the following properties: each \( s_x \) is an involution for which \( x \) is an isolated fixed point and

\[ s_x(y \bullet z) = s_x(y) \bullet s_x(z) \quad \text{for all} \quad x, y \in M, \quad \text{i.e.,} \quad s_x \in \Aut(M, \bullet). \]

Then we call \( (M, \bullet) \) a symmetric space (in the sense of Loos; see [Lo69]).

**Proposition 2.8.** (The ordered symmetric space \( O_h \)) We consider the adjoint orbit \( O_h := \Ad(G)h \cong G/G^0 \). Then the following assertions hold:

(i) \( O_h \) carries the structure of a Loos symmetric space (Definition 2.7), defined by

\[ x \bullet y = e^{\tau_x \Ad y} x, \quad \text{resp.} \quad (\Ad(g)h) \bullet y = \Ad(g)\tau \Ad(g)^{-1} y. \]

(ii) \( O_h \) carries an \( \Ad(G) \)-invariant partial order defined by restriction of \( \leq_C \).

(iii) The order intervals \([x, y]\) in \( (O_h, \leq) \) are compact.

**Proof.** (i) follows from the fact that, for every \( x \in O_h \), the automorphism \( \tau_x := e^{\tau_x \Ad} x \) is an involutive automorphism of \( g \) for which the fixed point \( x \) is isolated in \( O_h = O_x \).

(ii) is trivial.

(iii) First we observe that the centralizer \( g^0 \) of the ad-diagonalizable element \( h \) contains a Cartan subalgebra of \( g \) because every ad-semisimple element is contained in a Cartan subalgebra ([Bon99 Ch. VII, §2, no. 3, Prop. 10]). Now [Ne99 Thm. L13] implies that the adjoint orbit \( O_h = \Ad(G)h \subseteq g \) is closed. As \( O_h \) is closed in \( g \), the compactness of the order interval

\[ \uparrow x \cap \downarrow y = O_h \cap (x + C) \cap (y - C), \quad \text{where} \quad \uparrow x = \{ z \in O_h : x \leq z \}, \downarrow y = \{ z \in O_h : z \leq y \}, \]

follows from the compactness of \( (x + C) \cap (y - C) \), a consequence of the pointedness of \( C \). Therefore the order intervals in \( (O_h, \leq) \) are compact.

\[ \square \]

**Remark 2.9.** (a) Note that \( \Ad(G^{\pm 1})h = h + g^{\pm 1} \subseteq O_h \) are affine subspaces of \( O_h \) intersecting in \( h \). As these subspaces are invariant under the stabilizer group \( G^0 \), they define two \( G \)-invariant families of affine subspaces of \( O_h \). For \( x = \Ad(g)h \in O_h \), the corresponding two affine subspaces through \( x \) are given by

\[ x + g^{\pm 1}(x) = \Ad(g)(h + g^{\pm 1}(h)). \]

(b) The affine subspace \( h + g^{\pm 1} \subseteq O_h \) are symmetric subspaces with respect to the canonical symmetric space structure defined by \( h' \bullet (h' + x) = h' - x \) for \( h' \in h + g^{\pm 1} \).
Now we turn to the analysis of the semigroup \( S(h, C) := \{ g \in G : \text{Ad}(g)h \leq_C h \} \). We first take a closer look at an important example.

**Example 2.10.** A typical example arises for
\[
g = \mathfrak{sl}_2(\mathbb{R}) = \{ x \in \mathfrak{gl}_2(\mathbb{R}) : \text{tr}x = 0 \} \quad \text{and} \quad G = \widehat{\text{SL}}_2(\mathbb{R}).
\]
The determinant defines an invariant Lorentzian form on the 3-dimensional Lie algebra \( \mathfrak{sl}_2(\mathbb{R}) \) and \( \text{Ad}(G) \) can be identified with the connected group \( \text{SO}_{1,2}(\mathbb{R})_0 \) acting on 3-dimensional Minkowski space. Accordingly,
\[
C := \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{R}) : b \geq 0, c \leq 0, a^2 \leq -bc \right\}
\]
and this leads to the two one-dimensional cones \( C_+ = \mathbb{R}E_{12} \) and \( C_- = \mathbb{R}E_{21} \). For \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \) we now have
\[
\text{Ad}(g)h = \begin{pmatrix} \frac{ad+bc}{cd} & -\frac{ab}{cd} \\ \frac{bc}{cd} & -\frac{a}{cd} - \frac{b}{cd} \end{pmatrix} = \left( \begin{array}{cc} \frac{1}{2} + bc & -ab \\ bc & -\frac{1}{2} - bc \end{array} \right)
\]
and therefore \( h - \text{Ad}(g)h = \begin{pmatrix} -bc & ab \\ -cd & bc \end{pmatrix} \in C \) if and only if \( ab \geq 0, cd \geq 0, (bc)^2 \leq abcd = bc(1+bc) \), where the latter condition is equivalent to \( bc \geq 0 \). We conclude that
\[
S(h, C) = \{ g \in \text{SL}_2(\mathbb{R}) : ad \geq 0, cd \geq 0, bc \geq 0 \}.
\]
This semigroup contains
\[
\text{SL}_2(\mathbb{R})_+ := \{ g \in \text{SL}_2(\mathbb{R}) : (\forall j, k) g_{jk} \geq 0 \} = \exp(C_+) \exp(Rh) \exp(C_-)
\]
as its identify component, and \( S(h, C) = \text{SL}_2(\mathbb{R})_+ \cup - \text{SL}_2(\mathbb{R})_+ \) (see Theorem 2.16 and Corollary 2.17 below).

The following lemma records some trivial relations between the invariant cone \( C \) and the 3-grading.

**Lemma 2.11.** The following assertions hold for the subalgebras \( \mathfrak{p}_\pm = \mathfrak{g}^{\pm 1} \times \mathfrak{g}^0 \):
(i) \( C \subseteq C_+ \oplus \mathfrak{g}^0 \oplus -C_- \).
(ii) \( C_+ \oplus \mathfrak{g}^0 \oplus C_- = \{ x \in \mathfrak{g} : [h, x] \in C \} = (\text{ad} h)^{-1}(C) \).
(iii) \( \downarrow h = \mathcal{O}_h \cap (h - C) \subseteq C_+ \oplus \mathfrak{g}^0 \oplus C_- \).
(iv) \( (C + \mathfrak{p}_\pm)/\mathfrak{p}_\pm = \mp C_\mp \text{ in } \mathfrak{g}^{\pm 1} \cong \mathfrak{g}/\mathfrak{p}_\mp \).

**Proof.** (i) As \( C \) is invariant under \( e^{t \text{ad} h} \), for \( x = x_1 + x_0 + x_{-1} \in C \), we have
\[
x_{\pm 1} = \lim_{t \to \infty} e^{-t} e^{\pm t \text{ad} h} x \in C \cap \mathfrak{g}^{\pm 1} = \pm C_\pm.
\]
(ii), (iii) follow from (i).
(iv) follows from \( \mp C_\mp \subseteq C \) and (i).
Remark 2.12. The invariant cone $C$ is in general not uniquely determined by $C_+$ and $C_-$. However, given $C_\pm$, the closed convex invariant cone

$$C^\sharp := \bigcap_{g \in G} \text{Ad}(g)(C_+ \oplus g^0 \oplus -C_-)$$

contains $C$ (Lemma 2.11(i)) and is contained in the product set $C_+ \oplus g^0 \oplus -C_-$. It is the maximal invariant cone with this property. In particular, we have $C^\sharp \cap g^{\pm 1} = \pm C_\pm$. As $C_\pm$ are pointed, the subspace

$$C^\flat \cap -C^\sharp = \bigcap_{g \in G} \text{Ad}(g)g^0$$

is the largest ideal of $g$ contained in $g^0$.

On the other hand, the closed convex cone $C^\flat$ generated by $\text{Ad}(G)(C_+ - C_-)$ is the minimal invariant cone with $C_\pm^\flat = C_\pm$.

We are now ready to take a closer look at the semigroup $S(h, C)$ defined by the order structure on $O_1$. This will later be complemented by the result that $S(h, C) = S(C)$ (Theorem 2.16).

Proposition 2.13. The set

$$S(h, C) = \{ g \in G : \text{Ad}(g)h \leq_C h \}$$

is a closed subsemigroup of $G$ with the following properties:

(i) $G^0$ is the unit group $S(h, C) \cap S(h, C)^{-1}$ of $S(h, C)$.

(ii) $L(S(h, C)) := \{ x \in g : \exp(\mathbb{R} x) \subseteq S(h, C) \}$ equals $C_+ \oplus g^0 \oplus C_-$.

(iii) $S(h, C)$ is $\leq_C$-invariant.

(iv) $S(h, C) \cap (G^{-1} G^0 G^1) = \exp(C_-)G^0 \exp(C_+) \subseteq S(C) \subseteq S(h, C)$.

Proof. (i) That $S(h, C)$ is a subsemigroup follows immediately from the $\text{Ad}(G)$-invariance of the order $\leq_C$, and its closedness follows from the closedness of $h - C$. Clearly $G^0$ is a subgroup of the monoid $S(h, C)$. Conversely, $g \in S(h, C) \cap S(h, C)^{-1}$ implies that

$$\text{Ad}(g^{-1})h - h = \text{Ad}(g^{-1})(h - \text{Ad}(g)h) \in \text{Ad}(g^{-1})C = C$$

(15)

and $\text{Ad}(g^{-1})h - h \in -C$, so that $C \cap -C = \{ 0 \}$ yields $\text{Ad}(g)h = h$, i.e., $g \in G^0$.

(ii) For $x \in g^{\pm 1}(h)$ we have $\text{Ad}(\exp x)h - h = e^{\text{ad} x}h - h = [x, h] = \mp x$, so that $\exp(C_+), \exp(C_-) \subseteq S(h, C)$, and we thus obtain

$$\exp(C_+)G^0 \exp(C_-) \subseteq S(h, C).$$

(16)

This shows that $C_+ \oplus g^0 \oplus C_- \subseteq L(S(h, C))$. Conversely, for any $x \in L(S(h, C))$ and $t \in \mathbb{R}$, we have $e^{t \text{ad} x}h - h \in -C$ for $t \geq 0$, so that

$$[h, x] = \lim_{t \to 0^+} \frac{1}{t} (h - e^{t \text{ad} x}h) \in C.$$

Now Lemma 2.11(ii) implies that $x \in C_+ \oplus g^0 \oplus C_-$. (iii) For $g \in S(h, C)$ we have $\text{Ad}(g^{-1})h - h \in C$ by (15), and thus

$$\text{Ad}(g^\beta)h = \text{Ad}(\tau_G(g)^{-1})h = \tau \text{Ad}(g)^{-1}h \in \tau(h + C) = h + \tau(C) = h - C.$$

(17)

(iv) As $S(h, C)$ contains $G^0$ and $C_\pm \subseteq L(S(h, C))$ by (ii), we have

$$\exp(C_-)G^0 \exp(C_+) \subseteq S(h, C) \cap (G^{-1} G^0 G^1).$$
As $S(C_q)$ is a subsemigroup, we also have $\exp(C_-)G^0\exp(C_+) \subseteq S(C_q)$. Further, $G^0 \subseteq S(h, C)$, and $C_q = C_+ + C_- \subseteq L(S(h, C))$ yield $S(C_q) = G^0\exp(C_q) \subseteq S(h, C)$. It remains to verify that $S(h, C) \cap (G^{-1}G^0G^1) \subseteq \exp(C_-)G^0\exp(C_+)$. So let $g = \exp(x_1)g_0\exp(x_1)$ with $g_0 \in G^0$ and $x_{\pm 1} \in \mathfrak{g}^{\pm 1}$ and assume that $g \in S(h, C)$. Then
\[ \text{Ad}(g)h \in h - C \subseteq -C_+ + \mathfrak{p}_- \]
by Lemma [2.11](i). From $[x_-, \mathfrak{g}] \subseteq \mathfrak{p}_-$ we derive that $\text{Ad}(\exp(x_-))$ acts trivially on the quotient space $\mathfrak{g}/\mathfrak{p}_-$. This leads to
\[ -C_+ + \mathfrak{p}_- \ni \text{Ad}(g_0)e^{\text{ad}x_1}h = \text{Ad}(g_0)(h - x_1) = h - \text{Ad}(g_0)x_1, \]
which implies that $x_1 \in \text{Ad}(g_0)^{-1}C_+ = C_+$. We likewise obtain from
\[ \text{Ad}(g)^{-1}h \in h + C \subseteq -C_- + \mathfrak{p}_+ \]
that
\[ -C_- + \mathfrak{p}_+ \ni \text{Ad}(g_0)^{-1}e^{-\text{ad}x_-}h = \text{Ad}(g_0)^{-1}(h - x_-) = h - \text{Ad}(g_0)^{-1}x_- \]
which implies that $x_- \in \text{Ad}(g_0)C_- = C_-$. Putting everything together, we see that
\[ S(h, C) \cap (G^{-1}G^0G^1) \subseteq \exp(C_-)G^0\exp(C_+). \]

To obtain finer information on $S(h, C)$, we shall use the Levi decomposition of $G$ (Lemma 2.2) to reduce matters to the case of simple Lie algebras which we consider next. If $\mathfrak{g}$ is simple, then $\mathfrak{g}^1$ carries the structure of a simple euclidean Jordan algebra, which provides an important unifying perspective. For more on euclidean Jordan algebras we refer to [FK94].

**Lemma 2.14.** If $\mathfrak{g}$ is simple and 3-graded by $\text{ad}h$, then $\mathfrak{g}$ is hermitian, $\mathfrak{g}^1$ carries the structure of a euclidean Jordan algebra $E$, and $\mathfrak{g}$ is isomorphic to the Lie algebra $\text{conf}(E)$ of conformal vector fields on $E$. For any connected Lie group $G$ with Lie algebra $\mathfrak{g}$ and a maximal proper invariant cone $C \subseteq \mathfrak{g}$, we have
\[ S(h, C) = S(C_q) \subseteq G^{-1}G^0G^1. \]  

**Proof.** The first assertion $\mathfrak{g} \cong \text{conf}(E)$ follows from ([HÖ97 Thms. 1.3.11, 3.2.8]). Here $E \cong \mathfrak{g}^1$ corresponds to the constant vector fields on $E$, $\mathfrak{g}^0$ consists of linear vector fields, and $\mathfrak{g}^{-1}$ of homogeneous quadratic ones. The flows of these vector fields generate the group $\text{Conf}(E)$ (the identity component of the conformal group of $E$) which acts on $E$ by birational maps. Choosing a Jordan identity in $C \cap \mathfrak{g}^1$, it follows from ([HNO94 Rem. V.4]) that $C_+ = E \cap C = \mathfrak{g}^1 \cap C$ coincides with the positive cone $E_+$ of squares in the Jordan algebra $E$.

We first consider the 1-connected Lie group $G$ with Lie algebra $\mathfrak{g}$. Then $\text{Ad}(G) \cong \text{Conf}(E_0)$, so that we may consider the adjoint representation as a homomorphism $\text{Ad} : G \to \text{Conf}(E_0)$. In [Ne18 Thm. A.1] we have shown that the subsemigroup
\[ \text{Comp}(E_+) := \{ g \in \text{Conf}(E_0) : gE_+ \subseteq E_+ \} \]
is maximal. We now show that $\text{Comp}(E_+) = \text{Ad}(S(C_q))$. In view of the polar decomposition
\[ \text{Comp}(E_+) = (\text{Aut}(E_+) \cap \text{Conf}(E_0))\exp(C_q), \]
it suffices to show that $\text{Ad}(G^0) = \text{Aut}(E_+) \cap \text{Conf}(E_0)$.

Clearly, $\text{Ad}(G^0)$ acts on $E \cong \mathfrak{g}^1$ by linear maps preserving the positive cone $E_+ = C_+$ in the Jordan algebra $E$. Suppose, conversely, that a linear automorphism $\varphi$ of the convex cone $E_+$ is contained in the connected conformal group $\text{Conf}(E_0) \cong \text{Ad}(G)$. Then $\varphi$ defines a linear automorphism of $E$, hence fixes the linear vector field corresponding to $\text{ad}h|_E = Id|_E$ (the Euler vector field on $E$). This means that $g \in \text{Ad}(G)_h = \text{Ad}(G^0)$ (Lemma 2.2).

As explained above, we conclude that $\text{Ad}(S(C_q)) = \text{Comp}(E_+)$. This implies that $\text{Ad}(S(C_q))$ is a
maximal subsemigroup of $\text{Ad}(G)$. As its inverse image $S(C_q)$ in $G$ contains $Z(G) = \ker(\text{Ad})$, it is maximal as well.

For the corresponding grading element $h$ and the maximal invariant cone $C \subseteq \mathfrak{g}$ containing $C_+$, this implies that the semigroup $S(h, C)$, which contains $S(C_q)$ by Proposition 2.13(iv), actually coincides with $S(C_q)$. Further, $S(C_q) \subseteq G^{-1}G^0G^1$ follows from Koufany’s Theorem (Ko95) and also Ne18, Thm. 3.8).

Any connected Lie group with Lie algebra $\mathfrak{g}$ is of the form $G_\Gamma := G/\Gamma$, where $\Gamma \subseteq Z(G)$ is a discrete subgroup. Since all three sets in (18) are $\Gamma$-saturated, we obtain

$$S(h, C)/\Gamma = \{g \in G_\Gamma: \text{Ad}(g)h - h \in -C\} \cong S(C_q)/\Gamma = G^0_\Gamma \exp(C_q) \subseteq G^{-1}_\Gamma G^0_\Gamma G^1,$$

and this proves (18) for the general case.

\[\square\]

**Remark 2.15.** In general the subgroup

$$\text{Aut}(E_+) \cap \text{Ad}(G) = \text{Aut}(E_+) \cap \text{Conf}(E)_0$$

is not connected.

For $E = \text{Sym}_n(\mathbb{R})$ and $\text{conf}(E) \cong \text{sp}_{2n}(\mathbb{R})$, we have for $G = \text{Sp}_{2n}(\mathbb{R})$ (not simply connected), $G^0 = \text{GL}_n(\mathbb{R})$, acting on $E$ by $g.A = gAg^t$. Therefore $g = -1$ acts trivially, so that

$$\text{Conf}(E)_0 = \text{Ad}(G) \cong \text{Sp}_{2n}(\mathbb{R})/\{\pm 1\}.$$

If $n$ is even, then $\det(-1) = 1$, so that $\text{GL}_n(\mathbb{R})/\{\pm 1\} \subseteq \text{Aut}(E_+)$ has two connected components.

The following theorem is the main result of this section. It shows that the two semigroups $S(C_q)$ and $S(h, C)$ actually coincide and decompose according to the 3-grading.

**Theorem 2.16.** (Decomposition Theorem)

$$S(h, C) = \exp(C_-)G^0\exp(C_+) = \exp(C_+)G^0\exp(C_-) = S(C_q).$$

**Proof.** Claim 1: $S(h, C) \subseteq G^{-1}G^0G^1$. In view of Proposition 2.13 the first equality follows from Claim 1. This further implies that $\exp(C_-)G^0\exp(C_+)$ is a closed subsemigroup, hence coincides with the subsemigroup generated by $G^0$ and $\exp(C_\pm)$, and this in turn coincides with $S(C_q) = G^0\exp(C_q)$. We also obtain from Proposition 2.6 that

$$S(h, C) = S(h, C)^t = (\exp(C_-)G^0\exp(C_+))^t = \exp(C_+)G^0\exp(C_-).$$

So it remains to verify Claim 1.

In view Lemma 2.14, $\mathfrak{g}$ contains an $\text{ad } h$-invariant Levi complement $\mathfrak{l}$. Let $\mathfrak{l} = \mathfrak{l}_1 \oplus \cdots \oplus \mathfrak{l}_m$ denote the decomposition into simple ideals. Then each ideal is invariant under $\text{ad } h$ because all derivations of $\mathfrak{l}$ are inner. If $\mathfrak{l}_i$ is compact, then $\mathfrak{l}_i \not\subseteq \mathfrak{l}^0$ because all derivations of $\mathfrak{l}_i$ are elliptic. If the grading of $\mathfrak{l}_i$ is non-trivial, then $\mathfrak{l}_i$ is contained in the ideal $\mathfrak{g}_C := C - C$ of $\mathfrak{g}$ which contains $\mathfrak{g}^{\pm 1} = C_\pm - C_\pm$. By Lemma 2.13, $\mathfrak{l}_i$ is hermitian. It follows in particular that $\mathfrak{l}_i^i \oplus \mathfrak{h}^{-1}$ is contained in the sum $\mathfrak{l}_i$ of all simple hermitian ideals of $\mathfrak{l}$.

**Claim 2:** $\mathfrak{p}_{\mathfrak{l}_i}(C) \subseteq \mathfrak{l}_i$ is a pointed generating invariant cone. In $\mathfrak{g}_C$ the cone $C$ is pointed and generating, so that $\mathfrak{g}_C$ contains a compactly embedded Cartan subalgebra $\mathfrak{t}$, compatible with the Levi decomposition $\mathfrak{g}_C = \mathfrak{t} \times \mathfrak{l}$ (Ne99 Prop. VII.1.9). By Ne99 Thm. VII.3.8 there exists an adapted positive system $\Delta^+ \subseteq \Delta(\mathfrak{g}_C, \mathfrak{t})$ (cf. Appendix B) such that

$$C \cap \mathfrak{t} \subseteq C_{\text{max}}(\Delta^+) = (\Delta^+)^*.$$

Here $\Delta^+ \subseteq \Delta$ is the subset of non-compact roots which contains the subset $\Delta_{p,s}$ of non-compact simple roots corresponding to $\mathfrak{l}_i$. All these roots vanish on $\mathfrak{t} \cap \mathfrak{r}$, so that

$$(\Delta^+)^* \subseteq (\Delta_{p,s})^* = (\Delta^+)^* + \mathfrak{t} \cap \mathfrak{r}.$$
This implies that
\[ p_{\mathfrak{h}}(C \cap t) \subseteq (i\Delta_{\mathfrak{h},s}^+)^*, \]
and since \( \text{Ad}(G)(C \cap t) \) is dense in \( C \), it follows that
\[ p_{\mathfrak{h}}(C) \subseteq \text{Ad}(L_0)C_{\text{max}}(\Delta_{\mathfrak{h},s}^+) \subseteq W_{\text{max},s} := \text{Ad}(L)(C_{\text{max}}(\Delta_{\mathfrak{h},s}^+)). \] (20)

Here \( W_{\text{max},s} \) is a pointed closed convex invariant cone in \( \mathfrak{h} \). As \( C \) generates \( \mathfrak{g}_C \), the cone \( p_{\mathfrak{h}}(C) \) generates \( \mathfrak{h}_0 \). This proves Claim 2.

If the grading on a hermitian ideal \( \mathfrak{i}_j \) is non-trivial, the projection of \( C^+ \) into \( \mathfrak{i}_j^1 \) is contained in a pointed invariant cone, and this in turn implies that \( \mathfrak{i}_j^1 \) can be identified with a euclidean Jordan algebra \( E_j \) for which \( \mathfrak{i}_j \cong \mathfrak{c}n(1.E_j) \) is the Lie algebra of the conformal group. This follows from Lemma 2.14 which further entails that, for the invariant cone \( C_j := W_{\text{max},s} \cap \mathfrak{i}_j \) in \( \mathfrak{i}_j \), we have
\[ S(h_j, C_j) = S(C_j, q) \subseteq L_j^{-1}L_j^0L_j^1. \]

We conclude that, for the grading element \( h_1 = \sum_{j=1}^m h_j \) of the 3-graded semisimple Lie algebra \( \mathfrak{i} \), we have
\[ S(h_1, \sum_j C_j) = \prod_{j=1}^m S(h_j, C_j) \subseteq L^{-1}L^0L^1, \] (21)
where we use that \( L_j = L_j^0 \) if \( h_j = 0 \).

If \( C_1 \subseteq C_2 \) are invariant cones in \( \mathfrak{i} \), then we clearly have \( S(h_1, C_1) \subseteq S(h_1, C_2) \), so that \( 20 \) and \( 21 \) show that
\[ p_L(S(h, C)) \subseteq S(h_1, W_{\text{max},s}) \subseteq L^{-1}L^0L^1. \] (22)

With Lemma 2.14(iii), we now obtain \( S(h, C) \subseteq p_L^{-1}(L^{-1}L^0L^1) = G^{-1}G^0G^1. \)

The subgroup \( G^r \) is connected because \( G \) is 1-connected ([Ne99, Thm. B.2]), so it coincides with \( (G^0)_0 \), and the preceding theorem implies that:

**Corollary 2.17.** The identity component of \( S(C_q) \) is
\[ S(C_q)_0 = G^r \exp(C_{q}^0) = \exp(C_+^0)G^r \exp(C_-^0). \]

**Remark 2.18.** The Decomposition Theorem shows in particular that the semigroup \( S(h, C) \) only depends on the cone \( C_q = C_+ \oplus C_- \), i.e., that \( S(h, C) = S(h, C') \) if \( C_\pm = C_\pm' \). As we have seen in the proof of Lemma 2.14 this only leads to two different semigroups \( S(h, C) \) and \( S(h, -C) = S(h, C)^{-1} \).

**Example 2.19.** (a) Suppose that \( \mathfrak{g} \) is solvable and that \( C_\pm \) span \( \mathfrak{g}^{\pm 1} \). Then \( [\mathfrak{g}, \mathfrak{g}] \) is a nilpotent ideal containing \( \mathfrak{g}^{\pm 1} \). For any pointed invariant cone \( C \), the cone \( C \cap [\mathfrak{g}, \mathfrak{g}] \) is a pointed invariant cone in the nilpotent Lie algebra \([\mathfrak{g}, \mathfrak{g}]\), so that its span is abelian by [Ne99, Ex. VII.3.21]. Then \([\mathfrak{g}, \mathfrak{g}^{-1}] = \{0\}\) and
\[ \mathfrak{g} \cong \mathfrak{g}^1 \oplus \mathfrak{g}^{-1} \times \mathfrak{g}^0. \]

Conversely, any involution \( D: E \to E \) of a finite dimensional vector space \( E \) defines a solvable Lie algebra
\[ \mathfrak{g} := E \rtimes_D \mathbb{R} \quad \text{with the bracket} \quad [(v, t), (v', t')] := (tDv' - t'Dv, 0). \]
For \( h := (0, 1) \) we then obtain the \( \text{ad} \ h \)-eigenspaces
\[ \mathfrak{g}^0 = Rh \quad \text{and} \quad \mathfrak{g}^{\pm 1} = E^{\pm}(D). \]

(b) Let \( (V, \omega) \) be a symplectic vector space and \( \mathfrak{h} = \mathfrak{h}(V, \omega) = \mathbb{R} \oplus V \) be the corresponding Heisenberg algebra with the bracket \([z, v], (z', v') = (\omega(v, v'), 0)\). Then any pointed invariant cone \( C \) is contained in the center, hence (up to sign) of the form \( C = \mathbb{R}_+ (1, 0) \). If \( \tau \) is an
involutive automorphism of $\mathfrak{his}(V, \omega)$ with $\tau(C) = -C$, then (up to equivalence) it has the form
\[ \tau(z, v) = (-z, \tau_V(v)), \]
where $\tau_V : V \to V$ is antisymplectic, i.e., $\tau_V^* \omega = -\omega$.

Extending $\mathfrak{his}(V, \omega)$ by a diagonalizable derivation $D$ to $g := \mathfrak{his}(V, \omega) \rtimes_D \mathbb{R}$, we may also consider the corresponding element $h := (0, 0, 1)$ for which $\text{ad} h$ coincides with $D$ on the Heisenberg algebra and extend $\tau$ by $\tau(h) = h$ to $g$ (which works if $D$ commutes with $\tau_V$). Suppose that $\text{ad} h$ defines a 3-grading with $\tau = e^{\text{ad} h}$ and, w.l.o.g., that $Dz = z$ for the central element $z = (1, 0) \in \mathfrak{his}(V, \omega)$. Then $V_h := \text{Fix}(\tau_V)$ and $V_q := \text{Fix}(-\tau_V)$ are Lagrangian subspaces with $V = V_h \oplus V_q$ and $g^0 = V_h \oplus \mathbb{R}h$. From $[V_q, V_h] \subseteq \mathbb{R}z \subseteq g^1$, it follows that $V_q \subseteq g^1$.

We therefore have
\[ g^{-1} = \{0\} \quad \text{and} \quad g^1 = V_q = \text{Fix}(-\tau_V). \]

For more complicated examples we refer to Subsection 3.3.

### 2.4 The semigroup $G_{\pi i}(C)$

Let $\eta_C : G \to G_C$ denote the universal complexification of $G$, i.e., $G_C$ is the 1-connected Lie group with Lie algebra $g_C$, and $L(\eta_C) : g \to g_C$ is the canonical inclusion. For the pointed generating invariant cone $C$, Lawson’s Theorem ([Ne99 Thm. IX.1.10]) implies the existence of a semigroup $S(iC)$ which is a covering of the subsemigroup $\eta_C(G) \exp(iC)$ of the universal complexification $G_C$ (the simply connected group with Lie algebra $g_C$). Then the exponential function $\exp : g + iC \to G_C$ lifts to an exponential function $\text{Exp} : g + iC \to S(iC)$ and the polar map
\[ G \times C \to S(iC) = G \exp(iC), \quad (g, x) \mapsto g \exp(ix) \]
is a homeomorphism, and, if $C$ has non-empty interior\footnote{Note that we do not assume that $C$ has interior points. A typical example where this is not the case arises from the Poincaré group; see Example 3.3.} a diffeomorphism of $G \times C^0$ onto the complex manifold $S(iC)^0$.

For $z \in C$ with $\text{Im} z > 0$, we write $G_z = G_z(C) \subseteq G$ for the closed subsemigroup of all elements $g \in G$ for which the orbit map $\beta^z(t) := \exp(th)g \exp(-th)$ extends analytically to a continuous map
\[ \beta^z : S_{\text{Im} z} = \{w \in C : 0 \leq \text{Im} w \leq \text{Im} z\} \to S(iC) \]
(see [Ne99 Lemma 3.9] for details). Here “analytic” means that, on the open strip the composed map $\beta^z : S_{\text{Im} z} \to S(iC) \to G_C$ is holomorphic.

**Lemma 2.20.** $(G^1)_{\pi i} = \exp(C_\pi)$ and $(G^{-1})_{\pi i} = \exp(C_{-\pi})$.

**Proof.** For the abelian subgroup $G^1 \cong g^1$ and $G^0_1 \cong g^1_1$, we have
\[ G^1_{\pi i} = \{\exp(x) : x \in g^1_1, (\forall y \in [0, \pi]) \exp(ey x) \in G^1 \exp(iC_\pi) = \exp(g^1_1 + iC_\pi)\}. \]
As the exponential function of $(G^1)_C$ is bijective\footnote{This follows from the same argument as Lemma 2.4(iv).} $\exp(x) \in G^1_{\pi i}$ is equivalent to
\[ e^{iy}x = \cos(y)x + i\sin(y) x \in g^1_1 + iC_\pi \quad \text{for} \quad y \in [0, \pi], \]
and this is equivalent to $x \in C_\pi$. This proves the first assertion, and the second follows similarly.

**Theorem 2.21.** $G_{\pi i} = \exp(C_{-\pi})G^0 \exp(C_\pi)$. 

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Step 2. \( G_{\pi_i}(C) \subseteq G^0G \):

By Theorem 2.16 we have \( S(h, C) = \exp(C^+)G^0 \exp(C^-) \). Since \( G_{\pi_i}(C) \) is a subsemigroup of \( G \) which obviously contains \( G^0 \) (the elements with constant orbit maps), it suffices to see that \( \exp(C_{\pm}) \subseteq G_{\pi_i} \). This follows from Lemma 2.20.

**Proof.** Step 1.

By Theorem 2.16 we only have to determine which elements in \( G_{\pi_i}(C) \) are \( G^0 \)-maximal. This follows from Lemma 2.20.

Step 2. \( G_{\pi_i}(C) \subseteq G^0G \):

Let \( G \cong R \times L \) be a Levi decomposition with \( [h, l] \subseteq l \) (Lemma 2.1) and write \( p_L : G \to L \) for the corresponding morphism of 3-graded Lie groups. In view of Lemma 2.1(iii), it suffices to show that \( p_L(G_{\pi_i}) \subseteq L^{-1}L^0L^1 \).

We have already seen in the proof of Theorem 2.16 that \( C_1 := q_l(C) \subseteq l \) is a pointed invariant cone whose span \( l_C \) is a direct sum of hermitian ideals. All other simple ideals of \( l \) are contained in \( l_1 \). For \( L_{\pi_i} = L_{\pi_i}(C_1) \), it follows that \( p_L(G_{\pi_i}) \subseteq L_{\pi_i} \). Enlarging the cone \( C_1 \) to a maximal pointed invariant cone \( C_{\max} \) in \( l_C \), we have \( L_{\pi_i}(C_1) \subseteq L_{\pi_i}(C_{\max}) \). As \( C_{\max} \) is adapted to the decomposition \( l_C = l_1 \oplus \ldots \oplus l_m \) into simple ideals in the sense that

\[
C_{\max} = \bigoplus_{j=1}^m (C_{\max} \cap l_j), \quad \text{it follows that} \quad L_{\pi_i}(C_{\max}) = \bigoplus_{j=1}^m L_{j, \pi_i}(C_{\max} \cap l_j).
\]

Therefore it suffices to show that \( L_{\pi_i} \subseteq L^{-1}L^0L^1 \) if \( L \) is simple hermitian and \( C = C_{\max} \). Then \( L_{\pi_i} \) is a closed subgroup of \( L \) containing the maximal subsemigroup

\[
\exp(C^+)L^0 \exp(C^-) = S(C_0)
\]

(Lemma 2.23). As \( (L^0)_{\pi_i} = L^1 \cap L_{\pi_i} = \exp(C_1) \) follows from Lemma 2.20 \( L_{\pi_i} \neq L \), so that the maximality of \( S(C_0) \) implies that \( L_{\pi_i} = S(C_0) \subseteq L^{-1}L^0L^1 \).

Step 3. \( G_{\pi_i}(C) \subseteq \exp(C^+)G^0 \exp(C^-) = S(h, C) \):

In view of Step 2, it remains to show that \( g = g_lg_0g_{-1} \in G_{\pi_i} \) with \( g_j \in G^j \) implies that \( g_{\pm 1} \in \exp(C^\pm) \). To this end, we consider the projections

\[
\pi_\pm : G \to M_\pm := G/P_\pm.
\]

Clearly, \( \pi_+ \) maps \( G_{\pi_i} \) into the subset of all elements \( m \in M_+ \) for which the orbit map \( \tau_m^o(t) := \exp(th).m \) extends analytically to a map from \( S_\pi \) with values in

\[
\pi_-(\exp(g_1 + ic_+)) \subseteq M_{+, \mathbb{C}} := G_{\mathbb{C}}/P_{-, \mathbb{C}}.
\]

Writing \( g_1 = \exp(x) \), we have \( \pi_+(g) = g_1P_- = \exp(x)P_- \) and

\[
\gamma_{x_1}(z) = \exp((a + ib)x_1) \exp(c_x) = \exp(e^{a_+}(b) + i\sin(b)x)P_{-, \mathbb{C}}.
\]

As \( \eta_+ : G_+ \to M_{+, \mathbb{C}}, z \mapsto \exp(x)P_- \) also is an open embeddings (Lemma 2.21(v)), we see that \( g \in G_{\pi_i} \) implies \( x \in C_+ \). A similar argument shows that \( g_{-1} \in \exp(C^-) \).

Combining Steps 1-3, the assertion follows.

**Corollary 2.22.** We have

\[
G_{\pi_i}(C, \tau_G) := \{ g \in G_{\pi_i}(C) : \beta^0(\pi_i) = \tau_G(g) \} = \exp(C^-)(G^0)^T \exp(C^+).
\]

**Proof.** By Theorem 2.21 we only have to determine which elements in \( G_{\pi_i} = \exp(C^-)G^0 \exp(C^+) \) satisfy \( \beta^0(\pi_i) = \tau_G(g) \). Writing \( g = g_{-1}g_0g_{-1} \) with \( g_{\pm 1} \in \exp(C_{\pm}) \), we have \( \beta_{\pm 1}(\pi_i) = g_{\pm 1}^{-1} = \tau_G(g_{\pm 1}) \). We thus obtain

\[ \beta^0(\pi_i) = g_{-1}g_0g_{-1} \quad \text{and} \quad \tau_G(g) = g_{-1}^{-1}\tau_G(g_0)g_{-1}^{-1}. \]

Equality of these elements is equivalent to \( \tau_G(g_0) = g_0 \).
Corollary 2.23. If $(U, \mathcal{H})$ is an antunitary representation of $G \times \{1, \tau_G\}$ with discrete kernel, $J_\xi = U(\tau_G)$, $\Delta_\xi = e^{2\pi i U(b)}$, and $C = C_U$, then

$$\{g \in G; U(\beta^n(\pi_i)) = U(\tau_G(g))\} = \exp(C_-)G_\xi \exp(C_+) \subseteq S_\xi,$$

where $G_\xi = \{g \in G; U(g) = \mathbb{V}\}$.

Proof. As in the proof of Corollary 2.22, we see that $g = g_{-1}g_0g_1$ is contained in the set on the left hand side if and only if $U(g_0) = U(\tau_G(g_0)) = J_\xi U(g_0) J_\xi$. Since $\ker(U)$ is discrete, the assertion now follows from

$$G_\xi = \{g \in G; U(g) = J_\xi U(g), U(g) \Delta U(g)^{-1} = \Delta_\xi\} = \{g \in G^0; U(g) J_\xi = J_\xi U(g)\}.$$

The inclusion $\exp(C_-)G_\xi \exp(C_+) \subseteq S_\xi$ now follows from the Inclusion Theorem, which is [Ne19, Thm. 3.11].

3 The semigroup $S_\mathbb{V}$ in the 3-graded case

In this section we eventually turn to the compression semigroup $S_\mathbb{V}$ of a standard subspace $\mathbb{V}$. So we consider an antunitary representation $(U, \mathcal{H})$ of a semidirect product $G \times \{1, \tau_G\}$ with discrete kernel, where $G$ is a connected Lie group, $\tau_G$ is an involutive automorphism of $G$ and $h \in g^\prime$. We consider the standard subspace $\mathbb{V} \subseteq \mathcal{H}$ specified by the modular objects $J_\xi = U(\tau_G)$ and $\Delta_\xi = e^{2\pi i U(b)}$.

3.1 The general monotonicity theorem

The following result is essentially contained in the work of Borchers and Wiesbrock, see for instance [Be03] §II.1. For the formulation we refer to the discussion of the order on the space of selfadjoint operators in Appendix A.

Theorem 3.1. (Borchers–Wiesbrock Monotonicity) If $\mathbb{V}_1 \subseteq \mathbb{V}_2$ are standard subspaces of $\mathcal{H}$, then

$$\Delta_{\mathbb{V}_1} \leq \Delta_{\mathbb{V}_2},$$

and we also have $\log(\Delta_{\mathbb{V}_2}) \leq \log(\Delta_{\mathbb{V}_1})$ in the sense that

$$\|\theta_{\log}(\Delta_{\mathbb{V}_2})(\xi, \xi) \leq \theta_{\log}(\Delta_{\mathbb{V}_1})(\xi, \xi) \quad \text{for} \quad \xi \in \mathcal{D}[\log(\Delta_{\mathbb{V}_2})] \cap \mathcal{D}[\log(\Delta_{\mathbb{V}_1})].$$ (23)

Proof. Let $T_{\mathbb{V}_1} \subseteq T_{\mathbb{V}_2}$ be the Tomita operators of $\mathbb{V}_1$ and $\mathbb{V}_2$, respectively. Their graphs

$$\Gamma_j := \Gamma(T_j) = \{\xi; T_j \xi; \xi \in \mathcal{D}(T_j)\} \subseteq \mathcal{H} \oplus \mathcal{H}$$

are closed subspaces of $\mathcal{H} \oplus \mathcal{H}$ with $\Gamma_1 \subseteq \Gamma_2$. Hence the orthogonal projections $P_j$ on $\mathcal{H} \oplus \mathcal{H}$ with range $\Gamma_j$ satisfy $P_1 \leq P_2$. Identifying $B(\mathcal{H} \oplus \mathcal{H})$ with the algebra $M_2(B(H))$ of $(2 \times 2)$-matrices with entries in $B(H)$, we write $P_j$ as a $(2 \times 2)$-matrix. We obtain from $\mathbb{V}_1 \subseteq \mathbb{V}_2$ that $\Gamma(T_{\mathbb{V}_1}) \subseteq \Gamma(T_{\mathbb{V}_2})$, i.e., $P_1 \leq P_2$, and hence that $(P_1)_{11} \leq (P_2)_{11}$. Therefore Lemma [A1] leads to

$$(1 + \Delta_{\mathbb{V}_1})^{-1} = (1 + T_{\mathbb{V}_1}^* T_{\mathbb{V}_1})^{-1} \leq (1 + T_{\mathbb{V}_2}^* T_{\mathbb{V}_2})^{-1} = (1 + \Delta_{\mathbb{V}_2})^{-1}.$$

As the function $x \mapsto -\frac{1}{x}$ is operator monotone on $(0, \infty)$ (cf. [Sch12] Cor. 10.13), we obtain $\Delta_{\mathbb{V}_2} \leq \Delta_{\mathbb{V}_1}$, and (23) follows from Theorem [A4].

Remark 3.2. Note that the relation $\Delta_{\mathbb{V}_2} \leq \Delta_{\mathbb{V}_1}$ conversely implies that

$$\mathbb{V}_1 + i\mathbb{V}_1 = \mathcal{D}(\Delta_{\mathbb{V}_1}^{1/2}) = \mathcal{D}(\Delta_{\mathbb{V}_1}) \subseteq \mathcal{D}[\Delta_{\mathbb{V}_2}] = \mathcal{D}(\Delta_{\mathbb{V}_2}^{1/2}) = \mathbb{V}_2 + i\mathbb{V}_2$$ (24)

(Definition [A2]). In general, the inclusion (24) is weaker than $\Delta_{\mathbb{V}_2} \subseteq \Delta_{\mathbb{V}_1}$. If, for instance, $\Delta_{\mathbb{V}_1}$ and $\Delta_{\mathbb{V}_2}$ are bounded, then $\mathbb{V}_1 + i\mathbb{V}_1 = \mathbb{V}_2 + i\mathbb{V}_2 = \mathcal{H}$, but $\Delta_{\mathbb{V}_2} \leq \Delta_{\mathbb{V}_1}$ does not always hold.
We now apply Theorem 3.1 to obtain information on $S_v$.

**Theorem 3.3.** (The Monotonicity Theorem) Let $(U, H)$ be an antiunitary representation of $G \times \{id_G, \tau_G\}$, $h \in \mathfrak{g}^\tau$, and let $V$ be an invariant subspace. Then

$$S_v = \{g \in G: U(g)V \subseteq V\} \subseteq S(h, C_U) = \{g \in G: \text{Ad}(g)h \in h - C_U\}.$$

**Proof.** For $g \in S_v$, we have $U(g)V \subseteq V$. As $\Delta_{U(g)V} = e^{2\pi i \partial U(g)h}$ and $H^\infty \subseteq D(i \partial U(h)) \cap D(i \partial U(\text{Ad}(g)h)) \subseteq D[i \partial U(h)] \cap D[i \partial U(\text{Ad}(g)h)]$, Theorem 3.1 implies that

$$\langle \xi, i \partial U(\text{Ad}(g)h)\xi \rangle \geq \langle \xi, i \partial U(h)\xi \rangle \quad \text{for} \quad \xi \in H^\infty. \quad (25)$$

As the operators $i \partial U(x)$, $x \in \mathfrak{g}$, are the closures of their restriction to the $U(G)$-invariant subspace $H^\infty$ of smooth vectors, we conclude from (25) that $\text{Ad}(g)h - h \in -C_U$, so that $\text{Ad}(g)h \in h - C_U$, i.e., $g \in S(h, C_U)$. This proves the theorem. \qed

### 3.2 The semigroup $S_v$ in the 3-graded case

In the context that we studied throughout this paper, where $\mathfrak{g}$ is 3-graded by $\text{ad} \ h$, we have the following stronger result:

**Theorem 3.4.** Suppose, in addition to the setting of Theorem 3.3 that $\mathfrak{g}$ is 3-graded by $\text{ad} \ h$, $\tau = e^{\pi i \text{ad} \ h}$ and $C = C_U$, where $\ker(U)$ is discrete. Then

$$S_v = \exp(C_+)G_v \exp(C_-). \quad (26)$$

**Proof.** First we recall from Theorem 2.16 that, under the stated assumptions, $S(h, C) = \exp(C_+)G^0 \exp(C_-)$. Next Theorem 3.3 shows that $S_v \subseteq S(h, C)$. With Corollary 2.26, we thus obtain

$$\exp(C_+)G_v \exp(C_-) \subseteq S_v \subseteq S(h, C) = \exp(C_+)G^0 \exp(C_-). \quad (27)$$

Let $g = g_+g_0g_-$ with $g_+ \in \exp(C_+)$ and $g_0 \in G^0$ be an element of $S(h, C)$. If $g \in S_v$, then $U(g)V \subseteq V$ implies that the orbit map $\alpha^{U(g)}(t) := \Delta_{U(g)}^{-\pi t/2\pi}U(g)\Delta_{U(g)}^t$ extends to $S_v$ with

$$\alpha^{U(g)}(\pi i) = U(\tau_G(g)) \quad (28)$$

(see the Araki–Szidó Theorem; [AZ01, Ne19 Thm. 2.3]). We know already that $\alpha^{U(g_\pm)}(\pi i) = U(g_\pm)^{-1}$ exists, and, since $\alpha^{U(g_0)}(\pi i) = U(g_0)$, we obtain from (28)

$$\alpha^{U(g)}(\pi i) = \alpha^{U(g_+)}(\pi i)\alpha^{U(g_0)}(\pi i)\alpha^{U(g_-)}(\pi i) = U(g_+)^{-1}U(g_0)U(g_-)^{-1}$$

and

$$U(\tau_G(g)) = U(g_+)^{-1}U(\tau_G(g_0))U(g_-)^{-1}$$

that $U(g_0) = U(\tau_G(g_0))$, so that $g_0 \in G_v$. This shows that

$$S_v = S_v \cap S(h, C) \subseteq \exp(C_+)G_v \exp(C_-),$$

and with (27) we obtain (26). \qed
3.3 Examples

Example 3.5. (Poincaré group) In Quantum Field Theory on Minkowski space, the natural symmetry group is the proper Poincaré group \( P(d) \cong \mathbb{R}^{1,d-1} \rtimes O_{1,d-1}(\mathbb{R}) \) acting by orientation preserving isometries on \( d \)-dimensional Minkowski space \( \mathbb{R}^{1,d-1} \). Its Lie algebra is \( \mathfrak{g} := \mathfrak{p}(d) \cong \mathbb{R}^{1,d-1} \rtimes \mathfrak{so}_{1,d-1}(\mathbb{R}) \) and the closed forward light cone

\[
C := \{ (x_0, x) \in \mathbb{R}^{1,d-1} : x_0 \geq 0, x_0^2 \geq x^2 \} \tag{29}
\]

is a pointed invariant cone in \( \mathfrak{p}(d) \). The generator \( h \in \mathfrak{so}_{1,d-1}(\mathbb{R}) \) of the Lorentz boost on the \((x_0, x_1)\)-plane

\[
h(x_0, x_1, \ldots, x_{d-1}) = (x_1, x_0, 0, \ldots, 0)
\]

defines a 3-grading on \( \mathfrak{g} \) because \( \text{ad} h \) is diagonalizable with spectrum \( \{-1, 0, 1\} \), and \( \tau := e^{\pi i \text{ad} h} \) defines an involution on \( \mathfrak{g} \), acting on the ideal \( \mathbb{R}^{1,d-1} \) (Minkowski space) by

\[
\tau_M(x_0, x_1, \ldots, x_{d-1}) = (-x_0, -x_1, x_2, \ldots, x_{d-1}).
\]

To connect with the results above, we have to apply them to the universal cover \( \tilde{G} \) of the group \( G := P(d)_{\text{discrete}} \cong \mathbb{R}^{1,d-1} \rtimes \mathbb{R}^{1,d-1} \).

A unitary representation \((U, \mathcal{H})\) of \( G \) is called a positive energy representation if \( C \subset C \mathcal{U} \).

If \( \text{ker}(U) \) is discrete, then \( C \mathcal{U} \) is pointed, and \( C = C \mathcal{U} \) follows from the fact that this is, up to sign, the only non-zero pointed invariant cone in the Lie algebra \( \mathfrak{g} = \mathfrak{p}(d) \) for \( d > 2 \); for \( d = 2 \) there are four pointed invariant cones which are quarter planes.

The Lie algebra \( \mathfrak{g} \) is 3-graded by \( \text{ad} h \), but \( \mathfrak{g}^0 \) and the two cones \( C_{\pm} \) generate a proper Lie subalgebra. Here \( \mathfrak{g}^0 = \ker(\text{ad} h) = \mathfrak{h} \) is the centralizer of the Lorentz boost:

\[
\mathfrak{g}^0 = \{ (0, 0) \times \mathbb{R}^{d-2} \times (\mathfrak{so}_{1,1}(\mathbb{R}) \oplus \mathfrak{so}_{d-2}(\mathbb{R})) \cong (\mathbb{R}^{d-2} \times \mathfrak{so}_{d-2}(\mathbb{R})) \oplus \mathbb{R} h,
\]

and,

\[
C_+ = C \cap \mathfrak{g}^1 = \mathbb{R}_+(e_1 + e_0) \quad \text{and} \quad C_- = -C \cap \mathfrak{g}^{-1} = \mathbb{R}_+(e_1 - e_0).
\]

The subsemigroup \( S(h, C) := \{ g \in G : h - \text{Ad}(h) g \in C \} \) is easy to determine. The relation \( \text{Ad}(g) h - h \in \mathbb{R}^d \) implies that \( g = (v, I) \) with \( \text{Ad}(I) h = h \), and then \( \text{Ad}(g) h = \text{Ad}(v, I) h = -hv \in -C \) is equivalent to \( hv \in C \), which specifies the closure \( \overline{W_R} \) of the standard right wedge

\[
W_R := \{ x \in \mathbb{R}^{1,d-1} : x_1 > |x_0| \}.
\]

We therefore obtain

\[
S(h, C) = \overline{W_R} \rtimes (\mathfrak{so}_{1,1}(\mathbb{R})^\dagger \times \mathfrak{so}_{d-2}(\mathbb{R})) = \{ g \in G : g W_R \subseteq W_R \}
\]

(see [NO17] Lemma 4.12 for the last equality). As the subgroup \( \mathfrak{so}_{1,1}(\mathbb{R})^\dagger \times \mathfrak{so}_{d-2}(\mathbb{R}) \subseteq \mathfrak{so}_{1,d-1}(\mathbb{R}) \) commutes with \( h \) and \( \tau \). For any antiunitary positive energy representation of

\[
G \rtimes \{ 1, \tau_M \} = \mathbb{R}^{1,d-1} \rtimes O_{1,d-1}(\mathbb{R})^\dagger,
\]

the semigroup \( S_t \) corresponding to the standard subspace specified by \( U(\tau_M) = J_t \) and \( \Delta_t = e^{\pi t \text{ad} U(h)} \) satisfies

\[
S_t = S(h, C) = \overline{W_R} \rtimes (\mathfrak{so}_{1,1}(\mathbb{R})^\dagger \times \mathfrak{so}_{d-2}(\mathbb{R})), \quad \text{where} \quad \mathfrak{so}_{1,1}(\mathbb{R})^\dagger = \text{exp}(R h).
\]

For the covering group \( \tilde{G} \) we obtain the same picture because the involution acts trivially on the covering \( (\tilde{G})^0 \) of \( G^0 \).
Example 3.6. (Conformal groups $SO_{2,d}(\mathbb{R})$) The Lie algebra of the conformal group $G := SO_{2,d}(\mathbb{R})^1$ of Minkowski space is $\mathfrak{g} = \mathfrak{so}_{2,d}(\mathbb{R})$, which contains the Poincaré algebra as those elements corresponding to affine vector fields on $E := \mathbb{R}^{1,d-1}$. For $d \geq 3$ it is a simple hermitian Lie algebra. It contains many elements $h$ defining a 3-grading on $\mathfrak{g}$, but all these elements are conjugate. One arises from the element $h_0 = \text{id}$ corresponding to the Euler vector field on $E$. Then $\mathfrak{g}'(h_0)$, $j = -1, 0, 1$, are spaces of vector fields on $E$ which are linear (for $j = 0$), constant (for $j = 1$) and quadratic (for $j = -1$)\footnote{We encountered similar structures in the proof of Lemma 2.14 for more general euclidean Jordan algebras.}. Another important example is the element $h_1 \in \mathfrak{so}_{1,1}(\mathbb{R}) \subseteq \mathfrak{so}_{2,d-1}(\mathbb{R})$ corresponding to a Lorentz boost in the Poincaré algebra (see Example 3.5).

We consider the minimal invariant cone $C \subseteq \mathfrak{g}$ which intersects $E$ in the positive light cone $C_+(h_0) \subseteq E$. For all these elements $h$ we obtain a complete description of the corresponding semigroups $S_t$ as $\exp(C_+ \mathcal{G}_t) \exp(C_-)$, and here these semigroups have interior points because $C_\pm$ generate the subspaces $\mathfrak{g}_\pm$.

Example 3.7. Another interesting example which is neither semisimple nor an affine group is given by the Lie algebra

$$\mathfrak{g} = \mathfrak{hcsp}(V,\omega) := \mathfrak{heis}(V,\omega) \times \mathfrak{csp}(V,\omega),$$

where $(V,\omega)$ is a symplectic vector space, $\mathfrak{heis}(V,\omega) = \mathbb{R} \oplus V$ is the corresponding Heisenberg algebra with the bracket $[[z,v],(z',v')] = \omega(v,v'),0$, and

$$\mathfrak{csp}(V,\omega) := \mathfrak{sp}(V,\omega) \oplus \mathbb{R} \text{id}_V$$

is the conformal symplectic Lie algebra of $(V,\omega)$. The hyperplane ideal $j := \mathfrak{heis}(V,\omega) \times \mathfrak{sp}(V,\omega)$ (the Jacobi algebra) can be identified by the linear isomorphism

$$\varphi : j \to \text{Pol}_{\leq 2}(V), \quad \varphi(z,v,x)(\xi) := z + \omega(v,\xi) + \frac{1}{2}\omega(x,\xi), \quad \xi \in V$$

with the Lie algebra of polynomials $\text{Pol}_{\leq 2}(V)$ of degree $\leq 2$ on $V$, endowed with the Poisson bracket \cite[Prop. A.IV.15]{Ne99}. The set

$$C := \{f \in \text{Pol}_{\leq 2}(V) : f \geq 0\}$$

is a pointed generating invariant cone in $j$. The element $h_0 := \text{id}_V$ defines a derivation on $j$ by $(\text{ad } h_0)(z,v,x) = (2z,v,0)$ for $z \in \mathbb{R}, v \in V, x \in \mathfrak{sp}(V,\omega)$. Any involution $\tau_V$ on $V$ satisfying $\tau_V^2 = -\omega$ defines by

$$\tau_V(z,v,x) := (-z,\tau_V(v),\tau_V x \tau_V)$$

an involution on $\mathfrak{g}$ with $\tau_V(h_0) = h_0$, and $-\tau_V(C) = C$ follows from

$$\varphi(\tau_V(z,v,x)) = -\varphi(z,v,x) \circ \tau_V.$$

Considering $\tau_V$ as an element of $\mathfrak{sp}(V,\omega)$, the element $h := \frac{1}{2}(\text{id}_V + \tau_V) \in \mathfrak{csp}(V,\omega)$ defines a 3-grading of $\mathfrak{g}$ because $\text{ad } h$ is diagonalizable with eigenvalues $\pm 1, 0$. Writing $V = V_1 \oplus V_{-1}$ for the $\tau_V$-eigenspace decomposition, we have

$$\mathfrak{g}^{-1} = 0 \oplus 0 \oplus \mathfrak{sp}(V,\omega)^{-1}, \quad \mathfrak{g}^0 = 0 \oplus V_{-1} \oplus \mathfrak{sp}(V,\omega)^0 \cong V_{-1} \times \mathfrak{gl}(V_{-1}), \quad \mathfrak{g}^1 = \mathbb{R} \oplus V_1 \oplus \mathfrak{sp}(V,\omega)^1.$$

Note that

$$e^{\pi i \text{ad } h} = (-\tau_V)^2.$$  

Here $\mathfrak{g}^1$ can be identified with the space $\text{Pol}_{\leq 2}(V_{-1})$ of polynomials of degree $\leq 2$ on $V_{-1}$ and

$$C_+ = C \cap \mathfrak{g}^{-1} = \{f \in \text{Pol}_{\leq 2}(V_{-1}) : f \geq 0\}.$$
This cone is invariant under the natural action of the affine group $G^0 \cong \text{Aff}(V_{-1})_0 \cong V_{-1} \times \text{GL}(V_{-1})_0$ whose Lie algebra is $\mathfrak{g}^0$. We also note that
\[ \mathfrak{g}^{-1} = \mathfrak{sp}(V, \omega)^{-1} \cong \text{Pol}_2(V_1) \quad \text{and} \quad C_- = -C \cap \mathfrak{g}^{-1} = \{ f \in \text{Pol}_2(V_1) : f \leq 0 \}.
\]

Now we turn to the corresponding group and one of its irreducible unitary representations. Choosing a symplectic basis, we obtain an isomorphism with $V \cong V_{-1} \oplus V_1 \cong \mathbb{R}^n \oplus \mathbb{R}^n$ with the canonical symplectic form specified by $\omega((q,0), (0,p)) = (q,p)$ and $\tau_V (q,p) = (-q,p)$. Let $\text{Mp}_2n(\mathbb{R})$ denote the metaplectic group, which is the unique non-trivial double cover of $\text{Sp}_{2n}(\mathbb{R})$. We consider the group
\[ G := \text{Heis}(\mathbb{R}^{2n}) \rtimes_{\alpha} (\mathbb{R}_+^\times \times \text{Mp}_{2n}(\mathbb{R})), \]
where $\mathbb{R}_+^\times$ acts on $\text{Heis}(\mathbb{R}^{2n}) = \mathbb{R} \times \mathbb{R}^{2n}$ by $z, r = (r^{-2} z, rv)$. Its Lie algebra is $\mathfrak{g} = \mathfrak{h} \mathfrak{sp}(V, \omega)$. Then
\[ \mathcal{H} := L^2 \left( \mathbb{R}_+^\times \cdot \frac{d\lambda}{\lambda}, L^2(\mathbb{R}^n) \right) \cong L^2 \left( \mathbb{R}_+^\times \times \mathbb{R}^n, \frac{d\lambda}{\lambda} \otimes dx \right), \]
carries an irreducible representation of $G$, where $L^2(\mathbb{R}^n) \cong L^2(V_{-1})$ carries the oscillator representation $U_0$ of $\text{Heis}(\mathbb{R}^{2n}) \times \text{Mp}_{2n}(\mathbb{R})$. The Heisenberg group $\text{Heis}(\mathbb{R}^{2n})$ is represented on $\mathcal{H}$ by
\begin{align*}
(U(z, 0, 0)f)(\lambda, x) &= e^{i\lambda^2 \omega} f(\lambda, x), \\
(U(0, q, 0)f)(\lambda, x) &= e^{i\lambda(q, x)} f(\lambda, x), \\
(U(0, 0, p)f)(\lambda, x) &= f(\lambda, x - \lambda p).
\end{align*}

The group $\text{Mp}_{2n}(\mathbb{R})$ acts by the metaplectic representation on $L^2(\mathbb{R}^n)$ via
\[ (U(g)f)(\lambda, \cdot) := U_0(g) f(\lambda, \cdot), \]

independently of $\lambda$. The one-parameter group $\mathbb{R}_+^\times = \exp(\mathbb{R} h_0)$ acts by
\[ (U(r)f)(\lambda, x) := f(r\lambda, x) \quad \text{for} \quad r > 0. \]

We also note that we have a conjugation $J$ on $\mathcal{H}$ defined by
\[ (J f)(\lambda, x) := \overline{f(\lambda, -x)} \quad \text{satisfying} \quad J U(g) J = U(\tau_G(g)), \]
where $\tau_G$ induces on $\mathfrak{g}$ the involution $e^{i \pi h} = (-\tau_V)^*$. The positive cone $C_U \subset \mathfrak{g}$ is the same as the one of the metaplectic representation. It intersects $\mathfrak{sp}(V, \omega)$ in its unique invariant cone of non-negative polynomials of degree 2 on $V$. This implies that $(C_U)_+ = C_+$. To determine $(C_U)_+ = C_U \cap \mathfrak{g}^1$, we observe that $\mathfrak{g}^1$ acts on $L^2(\mathbb{R}^n) \cong L^2(V_-)$ by multiplication operators. This shows that we also have $(C_U)_+ = C_+$, so that we can determine the semigroup $S_V$ for the standard subspace $\mathcal{V} \subseteq \mathcal{H}$ with $\Delta_V = e^{2\pi i \delta_U(h)}$ and $J_V = J$. It takes the form
\[ S_V = \exp(C_+) G_V \exp(C_-), \]
where $G_V = G^0$ is a double cover of $\text{Aff}(\mathbb{R}^n)_0$, its inverse image in $\text{Mp}_{2n}(\mathbb{R})$.

### 4 Perspectives

For an antiunitary representation $(U, \mathcal{H})$ of the Lie group $G \times \{1, \tau_G\}$, any element $h \in \mathfrak{g}^*$ specifies a standard subspaces of $\mathcal{H}$ by the relations
\[ J_V = U(\tau_G) \quad \text{and} \quad \Delta_V = e^{2\pi i \delta_U(h)}. \]
4.1 The spaces \( \mathcal{O}_T \) and \( \mathcal{O}_h \)

As we mentioned already in the introduction, the \( G \)-orbit \( \mathcal{O}_T := U(G)V \cong G/G_T \) is a homogeneous space on which the inclusion order is invariant, and the order is encoded in the semigroup \( S_T \) by

\[
U(g_1)V \subseteq U(g_2)V \iff g_2^{-1}g_1 \in S_T.
\]

The semigroup \( S(C_U, h) \) likewise encodes the order on the homogeneous space \( (\mathcal{O}_h, \leq C_U) \) and the Monotonicity Theorem (Theorem 3.3) asserts that \( S_T \subseteq S(h, C_U) \), so that the natural map \( \pi : \mathcal{O}_T \to \mathcal{O}_h \) is monotone. If \( g \) is 3-graded by \( ad\ h \) and \( \tau = e^{\varepsilon_d ad\ h} \), then \( G_T \) is an open subgroup of \( G_h \) and \( \pi \) is a covering with \( S(h, C_U) = G_h S_T \), containing \( S_T \) as an open subsemigroup, so that the concrete ordered space \( \mathcal{O}_h \) is a very good model for \( (\mathcal{O}_T, \subseteq) \).

**Example 4.1.** In general, the connection between \( \mathcal{O}_h \) and \( \mathcal{O}_T \) is much less intimate, as the example of the 2-dimensional non-abelian Lie group shows. Consider \( g = Rh \oplus \mathbb{R}x \) with \( [h, x] = \lambda x \) for some \( \lambda > 0 \). Then \( C = \mathbb{R}_+ x \) is an invariant cone in \( g \) and the adjoint orbit of \( h \)

\[
\mathcal{O}_h = e^{R ad\ h} h = h + \mathbb{R}x,
\]

edowed with its natural order \( \leq_C \) and

\[
S(h, C) = \exp(Rh) \exp(\mathbb{R}_+ \lambda x).
\]

If \( \lambda \neq 1 \), then \( C_\pm = \{ 0 \} \) leads for representations with \( C = C_U \) to \( S_T = G_T \), so that the order on \( \mathcal{O}_T \) is trivial. Only for \( \lambda = \pm 1 \) we have \( S_T = S(h, C) \) This follows from our result above, but it also can be derived directly from the Borchers–Wiesbrock Theorem (\cite{NO17 §3.4}).

**Problem 4.2.** For a pointed closed convex invariant cone \( C \subsetneq g \) and \( h \in g \), determine the tangent wedge \( L(S(h, C)) \) of the semigroup \( S(h, C) \) in concrete terms.

Clearly, \( h - e^{\varepsilon_{ad\ h}} h \in C \) for \( t \geq 0 \) implies \([h, x] \in C \), so that

\[
L(S(h, C)) \subset (ad\ h)^{-1}(C).
\]

We have seen above that, in the 3-graded case we have equality because

\[
(ad\ h)^{-1}(C) = C_+ \oplus g^0 \oplus C_-.
\]

The most important case is when \( ad\ h \) is real diagonalizable, so that \( g = \oplus_{\lambda \neq 0} g^\lambda h \). Then \( T_h(\mathcal{O}_h) \cong [g, h] = \sum_{\lambda \neq 0} g^\lambda(h) \) and \( g^\lambda h = g_\lambda \subseteq L(S(h, C)) \). In general it seem rather complicated to determine

\[
(ad\ h)^{-1}(C) \cap [h, g] = \left\{ x = \sum_{\lambda \neq 0} x_\lambda : [h, x] = \sum_{\lambda \neq 0} \lambda x_\lambda \in C \right\}.
\]

Only the maximal and minimal eigenvalues \( \lambda_{\text{min}} \) and \( \lambda_{\text{max}} \) have the property that \( x \in C \) implies \( x_{\lambda_{\text{max}}} = \lim_{t \to \infty} e^{-t\lambda_{\text{max}}} e^{t ad\ h} x \in C \), and likewise \( x_{\lambda_{\text{min}}} \in C \).

4.2 Covariant nets of standard subspaces

As we have seen in Example 3.3 for the Poincaré group \( G = P(d) \),

\[
S(h, C) = \{ g \in G : gW_T \subseteq W_R \},
\]

so that the ordered space \( (\mathcal{O}_h, \leq C) \) is isomorphic to the wedge space \( W = G.W_T \) of wedge domains in \( R^{1+d-1} \). As such, it provides a natural index set whose elements may be interpreted as “special-space-time domains”.

If \( \tau \) does not coincide with \( e^{\varepsilon_1 ad\ h} \), it is more natural to consider pairs \( (h, \tau) \in g \times \text{Aut}(g) \), where \( \tau \) is an involution fixing \( h \) and to consider \( G \)-orbits \( \mathcal{O}_{(h, \tau)} \subseteq g \times \text{Aut}(g) \) of such pairs. For more on the rich geometric structures of such pairs as dilation spaces, we refer to \cite{No18}.
Any pointed convex invariant cone $C \subseteq \mathfrak{g}$ now specifies a natural order on the homogeneous space $O_{(h, \tau)}$ corresponding to the semigroup

$$S = \exp(C_+)G_{(h, \tau)}\exp(C_-), \quad C_\pm := \pm C \cap \mathfrak{g}^- \cap \ker(\text{ad} \, h \mp 1).$$

Considering the pairs $(h, \tau)$ as abstractions of wedge domains in spacetimes, it is now natural to try to classify $G$-covariant maps $O_{(h, \tau)} \to \text{Stand} (\mathcal{H})$ and to study the Bisognano–Wichmann property, and their causality and duality properties. This project is pursued in [MN20].

### 4.3 Standard subspaces in Hilbert spaces of distributions

From the perspective of Quantum Field Theory, it is also interesting to see how standard subspaces arise as concrete subspaces of Hilbert spaces of distributions. Here one considers a smooth manifold $M$ and a positive definite distribution $D$ on $M \times M$, so that

$$\langle \xi, \eta \rangle_D := D(\xi \otimes \eta)$$

defines a positive semidefinite form on the space $C_c^\infty(M, \mathbb{C})$ of test functions on $M$, hence a Hilbert space of distributions $\mathcal{H}_D \subseteq C_c^\infty(M)$ (cf. [NO18, Ex. 2.4.4]). For every open subset $\Omega \subseteq M$, we thus obtain a closed real subspace $V(\Omega)$ as the closure of the image of $C_c^\infty(\Omega, \mathbb{R})$.

We also assume that $\alpha: \mathbb{R}^\times \to \text{Diff}(M)$ defines an action, such that $\alpha(\mathbb{R}_+^\times)$ leaves $D$ invariant and that the involution $\tau_M := \alpha(-1)$ satisfies

$$\langle (\tau_M)\xi, (\tau_M)\eta \rangle_D = \langle \eta, \xi \rangle_D.$$

Then we obtain an antiunitary representation $U$ of $\mathbb{R}^\times$, and this specifies a standard subspace $V \subseteq \mathcal{H}_D$ by

$$U(e^t) = \Delta^{-it/2\pi} \quad \text{for} \quad t \in \mathbb{R} \quad \text{and} \quad J_V = U(-1).$$

**Problem 4.3.** Find necessary and sufficient conditions on pairs $(\alpha, \Omega)$ such that $V = V(\Omega)$.

This question is studied in [NO20] for the case where $M = G$ is a Lie group, $D$ is left invariant (hence defined by a positive definite distribution on $G$), and $\alpha(e^t)(g) = \exp(th)g\exp(-th)$ for $t \in \mathbb{R}$. In this case the semigroups constructed in this article provide natural domains on which the real test functions generate a standard subspace.

In [NOO20] we study the same problem for groups of the form $G = (E, +) \times \mathbb{R}^\times$, where the Hilbert space $\mathcal{H}_D$ consists of boundary values of holomorphic functions on a tube domain. If $E$ is Minkowski space, then our findings show that wedge domains $\Omega \subseteq E$ and the corresponding boosts provide pairs $(\alpha, \Omega)$ with $V_\alpha = V(\Omega)$.

### Acknowledgments

We thank Gandalf Lechner for an invitation to the Simons Center workshop “Operator Algebras and Applications” in June 2019, where some of the results of this paper have been obtained. In particular, we are most grateful to Roberto Longo and Gandalf Lechner for pointing out that a proof of Theorem 3.1 (Borchers–Wiesbrock Monotonicity) is essentially contained in [Bo00]. We also thank Konrad Schm"udgen for illuminating discussions on the subtleties of the order on the space of unbounded selfadjoint operators.

Last, but not least, we also thank Daniel Oeh for reading earlier versions of this manuscript.

### A Logarithms of positive operators

In this appendix we collect some background on the order on the space of not necessarily semibounded selfadjoint operators because it is needed in the proof of Theorem 3.1.
Definition A.1. (Quadratic form defined by a selfadjoint operator $A$) Let $P_A$ denote the spectral measure of $A$ and, for $\xi \in \mathcal{H}$, write $P^{\xi}_A := \langle \xi, P(\cdot)\xi \rangle$. Then we define

$$D[A] := \{\xi \in \mathcal{H} : \int_{\mathbb{R}} |x| \, dP^{\xi}_A(x) < \infty\} = D(|A|^{1/2})$$

and

$$q_A(\xi, \psi) := \int_{\mathbb{R}} x \, \langle \xi, dP_A(x)\psi \rangle \quad \text{for} \quad \xi, \eta \in D[A]$$

(cf. [RS80 §VIII.6], [Sch12 §10.2]). Clearly, $\mathcal{D}(A) = \mathcal{D}(|A|) \subseteq D[A] = D(|A|^{1/2})$, but if $A$ is unbounded, then this inclusion is strict.

Definition A.2. For two selfadjoint operators $A, B$, semibounded from below, we define $A \preceq B$ if

$$\mathcal{D}[B] \subseteq \mathcal{D}[A] \quad \text{and} \quad q_A(\xi, \xi) \leq q_B(\xi, \xi) \quad \text{for} \quad \xi \in \mathcal{D}[B]$$

([Sch12 Def. 10.5]). If $A$ and $B$ are not semibounded from below, we write $A \preceq B$ if

$$q_A(\xi, \xi) \leq q_B(\xi, \xi) \quad \text{for} \quad \xi \in \mathcal{D}[A] \cap \mathcal{D}[B].$$

Lemma A.3. For $\Re z > 0$, we have

$$\log z = \int_{0}^{\infty} \frac{1}{x + 1} - \frac{1}{x + z} \, dx.$$ 

Proof. For $\Re z > 0$, let $\gamma(z)(x) := \frac{1}{x + z}$, as a function on the half line $(0, \infty)$. Then

$$\gamma(z)(x) - \gamma(w)(x) = \frac{w - z}{(x + z)(x + w)}$$

is integrable over $(0, \infty)$, so that $F(z) := \int_{0}^{\infty} \frac{1}{x + 1} - \frac{1}{x + z} \, dx$ is defined. Next we observe that, for $\Re z > 0$ and $|h| < \Re z$, we have

$$\frac{\gamma(z + h)(x) - \gamma(z)(x)}{h} + \frac{1}{(x + z)^2} - \frac{1}{(x + z + h)(x + z)} = \frac{h}{(x + z)^2(x + z + h)}.$$ 

It is easy to see that this expression tends to 0 in $L^1(0, \infty)$ for $h \to 0$. This implies that $F$ is holomorphic with

$$F'(z) = \int_{0}^{\infty} \frac{1}{(x + z)^2} \, dx = -\left[ \frac{1}{x + z} \right]_{0}^{\infty} = \frac{1}{z}.$$ 

As $F(1) = 0$, it follows that $F(z) = \log z$. 

We want to use the preceding lemma to see that, for a selfadjoint operator $A > 0$ ($A \geq 0$ with $\ker A = 0$), we have

$$\log(A) = \int_{0}^{\infty} (x + 1)^{-1} - (x + A)^{-1} \, dx$$

in a suitable sense and derive a suitable version of the operator-monotonicity of log from this integral representation. Note that the integrand defines a norm-continuous function $(0, \infty) \to B(\mathcal{H})$ with a possible singularity in 0.

Theorem A.4. If $0 \leq A \leq B$ and $\ker A = 0$, then $\ker B = 0$ and

$$\log(A) \preceq \log(B),$$

i.e., $q_{\log(A)}(\xi, \xi) \leq q_{\log(B)}(\xi, \xi)$ for $\xi \in \mathcal{D}[\log(A)] \cap \mathcal{D}[\log(B)]$. 

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Proof. Let $P_A$ denote the spectral measure of $A$, so that $A = \int_0^\infty x dP_A(x)$. The condition \( \ker A = \{0\} \) means that $P_A(\{0\}) = 0$, so that the integral representing $A$ actually extends over the open interval $(0, \infty)$. Recall that

$$D(\log A) = \left\{ \xi \in \mathcal{H}: \int_0^\infty |\log(x)|^2 dP_A^x(x) < \infty \right\}$$

and

$$D[\log A] = \left\{ \xi \in \mathcal{H}: \int_0^\infty |\log(x)| dP_A^x(x) < \infty \right\} = D[\log A_{\geq 1}] \cap D[\log A_{< 1}],$$

where $A_{< 1} = P_A((0,1))A$ and $A_{\geq 1} = P_A([1,\infty))A$, so that $A = A_{< 1} \oplus A_{\geq 1}$. We write $\xi \in D[\log A]$ accordingly as $\xi = \xi_1 \oplus \xi_2$ with $\xi_1 = P_A((0,1))\xi$ and $\xi_2 = P_A([1,\infty))\xi$. Then we obtain with the Fubini–Tonelli Theorem on iterated integrals and Lemma A.3

$$q_{\log(A)}(\xi, \xi) = \int_0^\infty \log(x) dP_A^x(x)$$

$$= \int_{(0,1)} \log(x) dP_A^x(x) \int_{[1,\infty)} \log(x) dP_A^x(x)$$

$$= \int_{(0,1)} \int_0^\infty \frac{1}{t+1} - \frac{1}{t+x} dt dP_A^x(x) + \int_{[1,\infty)} \int_0^\infty \frac{1}{t+1} - \frac{1}{t+x} dt dP_A^x(x)$$

$$= \int_0^\infty \int_{(0,1)} \frac{1}{t+1} - \frac{1}{t+x} dP_A^x(x) dt + \int_0^\infty \int_{[1,\infty)} \frac{1}{t+1} - \frac{1}{t+x} dP_A^x(x) dt$$

$$= \int_0^\infty \langle \xi, (t+1)^{-1} - (t+A)^{-1} \rangle \xi dt. \quad (34)$$

Here the existence of the latter integral is a consequence of the Fubini–Tonelli Theorem. In this sense we have

$$q_{\log(A)}(\xi, \xi) = \int_0^\infty ((t+1)^{-1}\|\xi\|^2 - q_{(t+A)^{-1}}(\xi, \xi)) dt \text{ for } \xi \in D[\log(A)]. \quad (35)$$

For $0 < A \leq B$, we have

$$-(x+A)^{-1} \leq -(x+B)^{-1} \quad (36)$$

by [Sch12, Cor. 10.12], so that (36) immediately implies the theorem. \( \square \)

**Corollary A.5.** If $0 < A \leq B$ are selfadjoint operators, then

$$\langle \xi, \log(A)\xi \rangle \leq \langle \xi, \log(B)\xi \rangle \text{ for } \xi \in D(\log(A)) \cap D(\log(B)).$$

**Proof.** We only have to observe that $D(\log(A)) \subseteq D[\log(A)]$ and then use Theorem A.4 \( \square \)

**Remark A.6.** Suppose that $c > 0$ and that $A$ and $B$ are selfadjoint with $B \geq A \geq c1$. Then $\xi \in D[\log(A)]$ is equivalent to

$$\int_0^\infty \log(x) dP_A^x(x) = \int_1^c \log(x) dP_A^x(x) + \int_1^\infty \log(x) dP_A^x(x) < \infty.$$
as an equality in \( \mathbb{R} \cup \{ \infty \} \). We conclude that \( \mathcal{D}[\log(B)] \subseteq \mathcal{D}[\log(A)] \), so that we also recover from Theorem A.3 the well-known operator-monotonicity assertion \( \log(A) \leq \log(B) \).

Likewise \( \log(B) = -\log(B^{-1}) \) shows that \( 0 < A \leq B \leq C1 \) for some \( C > 0 \) implies that \( C^{-1}1 \leq B^{-1} \leq A^{-1} \), so that

\[
\mathcal{D}[\log(A)] = \mathcal{D}[\log(A^{-1})] \subseteq \mathcal{D}[\log(B^{-1})] = \mathcal{D}[\log(B)]
\]

and therefore

\[
\mathcal{D}[\log(B)] \cap \mathcal{D}[\log(A)] = \mathcal{D}[\log(A)].
\]

This shows that, if \( 0 \in \text{Spec}(A) \), i.e., \( \log(A) \) is not bounded from below, then \( \log(A) \not\leq \log(B) \) is not equivalent to \( \log(A) \leq \log(B) \) in the sense of Definition A.2, but we still have \( -\log(B) \leq -\log(A) \).

### B Root decomposition

In this appendix we recall a few concepts related to root decompositions of a finite dimensional Lie algebra \( \mathfrak{g} \) with respect to a compactly embedded Cartan subalgebra. This is used in the proofs of Theorems 2.16 and 2.21.

Let \( \mathfrak{g} \) be a finite dimensional Lie algebra and \( \mathfrak{t} \subseteq \mathfrak{g} \) be a compactly embedded Cartan subalgebra, i.e., the closure of \( e^{\mathfrak{t}t} \subseteq \text{Aut}(\mathfrak{g}) \) is compact and \( \mathfrak{t} \) coincides with its own centralizer: \( \mathfrak{t} = \text{ad}(\mathfrak{t}) \). Then we have the root decomposition

\[
\mathfrak{g}_C = \mathfrak{t}C \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_C^{\alpha}, \quad \text{where} \quad \mathfrak{g}_C^{\alpha} := \{ x \in \mathfrak{g}_C : (\forall h \in \mathfrak{t}C) [h, x] = \alpha(h)x \}
\]

and

\[
\alpha(t) \subseteq i\mathbb{R} \quad \text{for every root} \quad \alpha \in \Delta := \{ \alpha \in \mathfrak{t}^* \setminus \{0\} : \mathfrak{g}_C^{\alpha} \neq \{0\} \}.
\]

For \( x + iy \in \mathfrak{g}_C \) we put \( (x + iy)^* := -x + iy \), so that \( \mathfrak{g} = \{ x \in \mathfrak{g}_C : x^* = -x \} \). We then have \( x_\alpha \in \mathfrak{g}_C^{\alpha} \) for \( x_\alpha \in \mathfrak{g}_C^{\alpha} \). We call a root \( \alpha \in \Delta \)

- **compact**, if there exists an \( x_\alpha \in \mathfrak{g}_C^{\alpha} \) with \( \alpha([x_\alpha, x_\alpha^*]) > 0 \).
- **non-compact**, if there exists a non-zero \( x_\alpha \in \mathfrak{g}_C^{\alpha} \) with \( \alpha([x_\alpha, x_\alpha^*]) \leq 0 \).
- **non-compact simple**, if there exists a non-zero \( x_\alpha \in \mathfrak{g}_C^{\alpha} \) with \( \alpha([x_\alpha, x_\alpha^*]) < 0 \).

We write \( \Delta_k, \Delta_p, \Delta_{p,s} \subseteq \Delta \) for the subset of compact, non-compact, resp., non-compact simple roots. A subset \( \Delta^+ \subseteq \Delta \) is called a **positive system** if there exists an \( x_0 \in \mathfrak{t} \) with \( \alpha(x_0) \neq 0 \) for every \( \alpha \in \Delta \) and

\[
\Delta^+ = \{ \alpha \in \Delta : i\alpha(x_0) \geq 0 \}.
\]

A positive system \( \Delta^+ \) is said to be **adapted** if \( i\alpha(x_0) > i\beta(x_0) \) for \( \alpha \in \Delta^+_p \) and \( \beta \in \Delta_k \) (cf. [Ne99 Prop. VII.2.12]). To an adapted positive system \( \Delta^+ \), we associate the cone

\[
C_{\max} := C_{\max}(\Delta^+_p) := (i\Delta^+_p)^*.
\]

Now \( W_{\max} := \text{Ad}(G)C_{\max} \) is a closed convex invariant cone with \( W_{\max}^0 = \text{Ad}(G)C_{\max}^0 \). We also note that \( W_{\max} \cap \mathfrak{t} = C_{\max} \) ([Ne99 Lemma VIII.3.22, 27]).

### C Projections onto graphs

For the sake of completeness, we include here some arguments from [Bo00 §II.1] that are used in the proof of the Monotonicity Theorem (Theorem 3.1).
Lemma C.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be complex Hilbert spaces, let $S: \mathcal{H}_1 \supseteq \mathcal{D}(S) \to \mathcal{H}_2$ be a closed operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ denote the orthogonal projection onto the closed subspace $\Gamma(S) = \{(x, Sx) \in \mathcal{H}_1 \oplus \mathcal{H}_2 : x \in \mathcal{D}(S)\}$, written as a $(2 \times 2)$-matrix. Then

$$p_{11} = (1 + S^*)^{-1} \quad \text{and} \quad p_{12}\big|_{\mathcal{D}(S^*)} = (1 + S^*)^{-1}S^*.$$

Proof. The relation $P = P^2$ implies $p_{11} = p_{12}p_{21} + p_{11}^2$. As $P(\xi, 0) = (p_{11}\xi, Sp_{11}\xi) \in \Gamma(S)$, we have

$$(\xi, 0) - (p_{11}\xi, Sp_{11}\xi) = ((1 - p_{11})\xi, -Sp_{11}\xi) \perp \Gamma(S).$$

Further,

$$\Gamma(S)^\perp = \{(-S^*\psi, \psi) : \psi \in \mathcal{D}(S^*)\} \quad (38)$$

now shows that $1 - p_{11} = S^*Sp_{11}$, i.e., $1 = (1 + S^*)p_{11}$. As $1 + S^*$ is injective, it follows that $p_{11} = (1 + S^*)^{-1}$.

From $P(0, \xi) = (p_{12}\xi, Sp_{12}\xi) \in \Gamma(S)$, we likewise get

$$(0, \xi) - (p_{12}\xi, Sp_{12}\xi) = (-p_{12}\xi, (1 - Sp_{12})\xi) \perp \Gamma(S).$$

With $\mathcal{R}$, this leads to $p_{12} = S^*(1 - Sp_{12})$. For $\xi \in \mathcal{D}(S^*)$, we thus obtain $p_{12}\xi = S^*\xi - S^*Sp_{12}\xi$, so that $(1 + S^*)p_{12}\xi = S^*\xi$, and finally $p_{12}\xi = (1 + S^*)^{-1}S^*\xi$. \qed

Lemma C.1 can be used to characterize operators on $\mathcal{D}(S)$ which are bounded in the graph topology.

Lemma C.2. Let $\mathcal{H}_3$ be a Hilbert space and $A: \mathcal{D}(S) \to \mathcal{H}_3$ be a linear map. Then $A$ is continuous with respect to the graph topology on $\mathcal{D}(S)$ if and only if the operators

$$A(1 + S^*)^{-1}: \mathcal{H}_1 \to \mathcal{H}_3 \quad \text{and} \quad A(1 + S^*)^{-1}S^*: \mathcal{D}(S^*) \to \mathcal{H}_3$$

are bounded, where $\mathcal{D}(S^*) \subseteq \mathcal{H}_2$ carries the subspace topology.

Proof. The operator $A$ is continuous in the graph topology if and only if the operator

$$\tilde{A}: \Gamma(S) \to \mathcal{H}_3, \quad (\xi, S\xi) \mapsto A\xi$$

is bounded, and this is equivalent to the boundedness of $\tilde{A} \circ P: \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_3$. As this operator has the form $\tilde{A}P(\xi_1, \xi_2) = A(p_{11}\xi_1 + p_{12}\xi_2)$, its continuity is by Lemma C.1 equivalent to the boundedness of

$$Ap_{11} = A(1 + S^*)^{-1} \quad \text{and} \quad Ap_{12}\big|_{\mathcal{D}(S^*)} = A(1 + S^*)^{-1}S^*.$$

\[ \square \]

Remark C.3. If $S: \mathcal{H} \supseteq \mathcal{D}(S) \to \mathcal{H}$ is an antilinear operator and $\mathcal{H}^\text{op}$ denotes $\mathcal{H}$, endowed with the opposite complex structure, then Lemma C.1 applies with $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathcal{H}^\text{op}$.

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