Property of various correlation measures of open Dirac system with Hawking effect in Schwarzschild space–time

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A B S T R A C T
We explore the performance of various correlation measures for open Dirac system with Hawking effect in Schwarzschild space–time. Our results indicate that the impact of Hawking effect on physical accessible entanglement is weaker than that of decoherence. For generalized amplitude damping (GAD) channel, the entanglement sudden death (ESD) is analyzed in detail, and the inequivalence of quantization for Dirac particles in the black hole and Kruskal space–time is verified via quantum discord measure. In addition, as an example for interpreting Bell non-locality, we study the GAD channel with Hawking effect. It can be noticed that there is a boundary line of Bell violation for physically accessible states. That is, quantum non-locality would disappear when Hawking temperature exceeds a certain value. This critical temperature increases as a decoherence parameter decreases. In the case of phase damping (PD) channel, the interaction between the particle and noise environment does not produce bipartite system–environment entanglement. Then we discuss entanglement distributions, and find that the reduced physically accessible entanglement can be redistributed to physical inaccessible region. At last, we extend our investigation to an N-qubit system, and obtain a universal expression of the physical accessible entanglement.

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1. Introduction

Recently, researches about combination of quantum information science and relativity theory have drawn a lot of attention [1–6], since it not only promises a deeper comprehension about quantum mechanics [7] but also offers a new method of understanding the information paradox existing in black hole [8–10]. For instance, in the background of a black hole, Pan et al. [1] discussed the quantum entanglement for scalar field, and Deng et al. [4] studied how the Hawking effect and prepared states could affect entanglement distillability of Dirac fields. More recently, Xu et al. [6] expanded the investigation of the effect of Hawking radiation on multipartite entanglement in Schwarzschild spacetime. However, the above investigations are confined to an isolated system for the studies of quantum information. As the real quantum system inevitably suffers from quantum decoherence, this reciprocal interaction between the system and its external noise environment would lead to the degradation of quantum coherence and, in certain cases, produce entanglement sudden death (ESD). It therefore raises the question of how to understand the behaviors of quantum information for open system in Schwarzschild space–time.

To solve the problem, we probe the effect of Hawking radiation [11] on various quantum correlation measures for Dirac particles involved in dissipative environment in Schwarzschild space–time. Our aim is to unveil some interesting phenomena about the characteristic of correlation measurements with Hawking effect and decoherence channel, which may lead to a much better understanding of the Hawking–Unruh effects in quantum information processing. To illustrate this problem properly, we propose the scheme in a situation where two observers, Alice and Bob, share a generically entangled state at the same initial point in the flat Minkowski space–time. During the same time interval, Alice remains at the asymptotically flat region undergoing a decoherence channel, while Bob freely falls in toward a Schwarzschild black hole and eventually locates near the event horizon. Our work here is to provide a better insight into entanglement redistribution and information paradox of the black holes from the perspective of quantum mechanics.

This paper is organized as follows. In Section 2, a brief review of the vacuum structure and Hawking radiation for Dirac fields in Schwarzschild space–time is given. In Section 3, we explore
the properties of various correlation measures under two different decoherence environments with Hawking effect in the background of a Schwarzschild black hole. Finally, Section 4 summarizes our conclusions.

2. Vacuum structure of Dirac particles in the background of a Schwarzschild black hole

We first introduce a concise review of vacuum structure for Dirac particles in Schwarzschild space–time. The Dirac equation [12] in a curve space–time can be read as

\[ \gamma^{a}e_{a}^{\mu}(\partial_{\mu} + \Gamma_{\mu})\psi = 0 \]  

while the metric for the Schwarzschild space–time is given by

\[ ds^{2} = -(1 - \frac{2M}{r})dt^{2} + (1 - \frac{2M}{r})^{-1}dr^{2} + r^{2}(d\theta^{2} + \sin^{2}\theta d\phi^{2}). \]  

Combining Eq. (1) with Eq. (2) then leads to the Dirac equation in the Schwarzschild space–time in the form of

\[ -\frac{\gamma_{0}}{\sqrt{1 - \frac{2M}{r}}} \frac{\partial \psi}{\partial t} + \gamma_{1} \sqrt{1 - \frac{2M}{r}} \left[ \frac{\partial}{\partial r} + \frac{1}{2r} - \frac{M}{2(r - 2M)} \right] \psi + \frac{\gamma_{2}}{2} \frac{\partial \psi}{\partial \phi} + \frac{\gamma_{3}}{r \sin \theta} \frac{\partial \psi}{\partial \theta} = 0. \]  

For simplicity, G, c, h and k are regarded as unity throughout this paper. Then, by solving Eq. (3), we obtain the positive (fermions) frequency outgoing solutions for the outside and inside region near the event horizon [13]

\[ \psi_{I}^{+} = e^{-\mu \omega} (r > r_{+}), \]

\[ \psi_{II}^{+} = e^{\mu \omega} (r < r_{+}) \]  

where \( \omega \) is a 4-component Dirac spinor [14], \( \omega \) is a monochromatic frequency of the Dirac field, \( u = t - r^{*} \) and \( r^{*} = r + 2M \ln(r - 2M)/(2M) \) represent the tortoise coordinate. By using the above complete orthogonal basis (Eqs. (4)), we then quantize the Dirac fields in Schwarzschild space–time

\[ \psi_{out} = \sum_{k} \int dk \left( c_{k}^{+} \psi_{k}^{+} + b_{k}^{+} \psi_{k}^{-} \right) \]  

where \( k = (l, l), c_{k}^{+} \) and \( b_{k}^{+} \) denote the fermion annihilation and antifermion creation operators, while \( I \) and \( II \) correspond to the state of exterior and interior region, respectively.

On the other hand, the generalized light-like Kruskal coordinates for the Schwarzschild black hole is introduced as

\[ u = -4M \ln \left[ U/(4M) \right], \quad v = 4M \ln \left[ V/(4M) \right], \quad \text{if} \ r < r_{+}, \]

\[ u = -4M \ln \left[ -U/(4M) \right], \quad v = 4M \ln \left[ V/(4M) \right], \quad \text{if} \ r > r_{+}. \]  

And based on the Damour–Ruffini’s suggestion [15] we find another complete basis for positive energy modes by making an analytic continuation for Eqs. (4):

\[ \gamma_{I}^{l+} = e^{2\pi \hbar \omega_{k}} \psi_{k}^{l+} + e^{-2\pi \hbar \omega_{k}} \psi_{k}^{l-}, \]  

\[ \gamma_{II}^{l+} = e^{2\pi \hbar \omega_{k}} \psi_{k}^{II} + e^{-2\pi \hbar \omega_{k}} \psi_{k}^{II}, \]  

Then the decomposition of the Dirac fields in the Kruskal space–time can be given in terms of these bases:

\[ \psi_{out} = \sum_{k} \int dk \left[ 2 \cosh(4\pi M \omega_{k}) \right]^{-1/2} \left( c_{k}^{+} \gamma_{I}^{l+} + b_{k}^{+} \gamma_{II}^{l+} \right) \]  

where the \( c_{k}^{+} \) and \( b_{k}^{+} \) represent the annihilation and creation operators in the Kruskal vacuum.

Obviously, these two quantization processes are unequal, and their relationship can be reflected by the Bogoliubov transformations [16] between the creation and annihilation operators in the Schwarzschild and Kruskal coordinates. Given the Bogoliubov relationships being diagonal, each annihilation operator \( c_{k}^{+} \) can be represented as

\[ c_{k}^{+} \rightarrow \left( e^{-8\pi M \omega_{k}} + 1 \right)^{1/2} a_{k}^{+} - \left( e^{8\pi M \omega_{k}} + 1 \right)^{-1/2} b_{k}^{+}. \]  

Through a series of calculation, the vacuum and excited states of the Kruskal particle for mode \( k \) can be expressed in the basis of Schwarzschild Fock space

\[ |0\rangle_{k}^{+} \rightarrow \left( e^{-\omega_{k}/T} + 1 \right)^{-1/2} \exp \left[ -\omega_{k}/2T \right] b_{k}^{+} |0\rangle_{II}^{+} \]

\[ = \left( e^{-\omega_{k}/T} + 1 \right)^{-1/2} |0\rangle_{II}^{+} + \left( e^{-\omega_{k}/T} + 1 \right)^{-1/2} |1\rangle_{II}^{+} |1\rangle_{II}^{+}, \]

where modes \( k \) are the spherical harmonics with fixed values of the orbital angular momentum \( l \) and the total angular momentum \( J \). \( |n_{k}\rangle_{I} \) and \( |n_{k}\rangle_{II} \) are the orthonormal bases for the outside and inside regions of the event horizon respectively, the superscript on the kets \(+, -\) denotes the particle and antiparticle vacua, and \( T = \frac{1}{2\pi \hbar} \) is the Hawking temperature. Note that states \( |0\rangle \) and \( |1\rangle \) refer to mode population numbers. For brevity, \( |n_{k}\rangle_{I} \) and \( |n_{k}\rangle_{II} \) is replaced by \( |n_{I}\rangle \) and \( |n_{II}\rangle \), respectively.

3. Performance of various correlation measures under noise environments for Dirac particles in Schwarzschild space–time

Assume that Alice stays static with a detector which only detects mode \( |n_{I}\rangle \) in the asymptotically flat region undergoing the influence of decoherence environments, while Bob, with a detector, is sensitive only to mode \( |n_{II}\rangle \), freely falls toward a Schwarzschild black hole and hovers at a fixed finite nearest distance away from the event horizon. They share a generically entangled state

\[ |\phi\rangle = \alpha |00\rangle + \sqrt{1 - \alpha^{2}} |11\rangle, \]  

where \( \alpha \) is a state parameter that runs from 0 to 1. We then utilize Eq. (10) to rewrite Eq. (11) in terms of Minkowski mode for Alice and black hole mode for Bob. Since the exterior region is causally disconnected from the interior region of the black hole, the physical accessible density matrix \( \rho_{AB} \) by tracing over the state of the interior region \( B_{II} \) can be obtained in the form of

\[ \rho_{AB} = \alpha^{2} \mu^{2} |00\rangle \langle 00| + \alpha^{2} \nu^{2} |01\rangle \langle 01| + (1 - \alpha^{2}) |11\rangle \langle 11| + \alpha \mu \sqrt{1 - \alpha^{2}} |00\rangle \langle 11| + |11\rangle \langle 00|, \]

where \( \mu = (e^{-\omega_{k}/T} + 1)^{-1/2} \) and \( \nu = (e^{\omega_{k}/T} + 1)^{-1/2} \).
calculate relevant results. The dissipation course under GAD channel can be described by a quantum operation as [18]

\[ \varepsilon_{\text{GAD}}(\rho) = \sum_{i=0}^{3} E_i \rho E_i^\dagger. \]  

(13)

The Kraus operators \( E_i \) [18] on a single qubit are given by

\[ E_0 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ \sqrt{1-r} & 0 \end{pmatrix}, \quad E_1 = \sqrt{p} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{r} \end{pmatrix}, \]
\[ E_2 = \sqrt{1-p} \begin{pmatrix} \sqrt{1-r} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \sqrt{1-p} \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{r} \end{pmatrix}. \]

(14)

Here \( \{ p, r \} \) is usually a function of environment temperature \( T' \).

\[ r = 1 - e^{-\gamma T'}, \quad \gamma = \left[ \frac{2}{\exp\left(-\frac{\hbar \omega}{k_B T'}\right) - 1} + 1 \right] \gamma_0, \]
\[ p = \frac{1}{1 + \exp\left(-\frac{\hbar \omega}{k_B T'}\right)}. \]

\( \gamma_0 \) is the energy relaxation rate, \( r \) is the storage period, \( \hbar \omega \) is the transition energy of quantum system and \( k_B \) is the Boltzmann constant. Note that setting \( p = 1 \) would reduce GAD channel to the well-informed amplitude damping (AD) channel.

Then, in this case, the ultimate physical accessible density matrix can be written as

\[ \rho_{ABI} = (1 - r + pr) \alpha^2 \mu^2 |00\rangle\langle 00| \]
\[ + \left[ (1 - r + pr) \alpha^2 \mu^2 + (1 - \alpha^2) pr \right] |01\rangle\langle 01| \]
\[ + (1 - p) \alpha^2 \mu^2 |10\rangle\langle 10| \]
\[ + \left[ (1 - pr) (1 - \alpha^2) + (1 - p) \alpha^2 \mu^2 \right] |11\rangle\langle 11| \]
\[ + \alpha \mu \sqrt{(1 - r) (1 - \alpha^2)} (|00\rangle\langle 11| + |11\rangle\langle 00|). \]

(15)

3.1.1. Concurrency

Concurrency [19] is used to capture quantum entanglement in this work. The concurrency of bipartite state is defined as \( C = \max_{i} \{ \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0 \} \), where \( \lambda_i \) (\( i = 1, 2, 3, 4 \)) are the square roots of the eigenvalues in descending order of the operator \( R = \rho \sigma_i^x \otimes \sigma_j^x + \rho \sigma_i^y \otimes \sigma_j^y + \rho \sigma_i^z \otimes \sigma_j^z \). Therefore, the expression of physical accessible quantum entanglement of subsystem \( AB_1 \) is

\[ C'_{AB_1} = 2 \alpha \mu \sqrt{(1 - r) (1 - \alpha^2)} \]
\[ - 2 \sqrt{(1 - r + pr) \alpha^2 \mu^2 + (1 - \alpha^2) pr} (1 - p) \alpha^2 \mu^2. \]

(16)

We now display how the decoherence parameter and Hawking temperature affect the evolution of concurrence \( C'_{AB_1} \) for an initially maximal entangled state. As depicted in Fig. 1(a), the accessible entanglement \( C'_{AB_1} \) always decreases as the Hawking temperature increases, and the existence of decoherence apparently accelerates its attenuation, probably even produces the ESD in certain circumstance. Moreover, when Hawking radiation is infinite, which corresponds to the black hole tends to evaporate completely, the residual entanglement exists only for the smaller decoherence parameter \( r \). That is, the ESD occurs at the critical Hawking temperature if the decoherence is strong enough. For example, setting the decoherence parameter \( r = 0.75 \) and \( p = 0.5 \), the ESD appears with the critical value \( T = 7.4889 \). In addition, due to the Pauli exclusion principle in Fermi–Dirac statistics, the quantum states cannot be excited infinitely with Hawking effect for Fermi particles. This explains why the accessible entanglement \( C'_{AB_1} \) does not decrease to totally vanish, when decoherence is negligible, i.e., \( r = p = 0 \), even if Hawking temperature increases to infinity. Nevertheless, from Fig. 1(b), we note that the decoherence can directly leads to ESD alone (the case of \( T = 0 \)). It indicates that the influence of Hawking effect on physical accessible entanglement is weaker than decoherence. Meanwhile, we also find that the critical values of \( r \) which produce ESD decrease with the increase of the Hawking temperature.

3.1.2. Quantum discord

To understand better the performance of quantum correlation, we also investigate a studier measurement, quantum discord [20], which has been introduced for characterizing all the non-classical correlations. For a bipartite state \( \rho^{AB} \), quantum discord \( D(\rho^{AB}) \) [21] is given by subtracting classical correlation \( \text{CC}(\rho^{AB}) \) from the total amount of correlation \( I(\rho^{AB}) \), namely,

\[ D(\rho^{AB}) = I(\rho^{AB}) - \text{CC}(\rho^{AB}), \]

(17)

where \( I(\rho^{AB}) = S(\rho^{A}) + S(\rho^{B}) - S(\rho^{AB}) \), \( \rho^{AB} \) is the density operator of a composite bipartite quantum system \( AB \), \( \rho^{A(B)} = \text{tr}_{B(A)}(\rho^{AB}) \) denotes the reduced density operator of the partition \( A(B) \) and \( S(\rho) = -\text{tr}(\rho \log_2 \rho) \) is the Von Neumann entropy. The classical correlation depends on the maximal information obtained with measurement on one of the subsystems and can be expressed as

\[ \text{CC}(\rho^{AB}) = \max_{\{\Pi_k^B\}} \left[ S(\rho^A) - S(\rho^A|\{\Pi_k^B\}) \right] \]

(18)

where \( \{\Pi_k^B\} \) indicate a complete set of positive-operator-valued measurements (POVM) preformed on subsystem \( B \). The quantum conditional entropy is defined as \( S(\rho^A|\{\Pi_k^B\}) = \sum_k p_k S(\rho_k) \).
through the definition of conditional density operator $\rho_{AB}^{\mathcal{R}} = \frac{1}{p_k} \text{Tr}_B((I^A \otimes \Pi_k^B))\rho_{AB}(I^A \otimes \Pi_k^B)$ with $p_k = \text{Tr}(I^A \otimes \Pi_k^B)\rho_{AB}(I^A \otimes \Pi_k^B)$ as the probability of obtaining the outcome $k$.

For the physically accessible state $|\Psi\rangle_{AB}$, to acquire the minimum conditional entropy of subsystem $A$, we perform a complete set of projective measurements $|\psi_1\rangle = \cos(\theta/2)|1\rangle + e^{i\phi}\sin(\theta/2)|0\rangle$ and $|\psi_2\rangle = \sin(\theta/2)|1\rangle - e^{i\phi}\cos(\theta/2)|0\rangle$ $(0 \leq \theta < \pi/2$ and $0 \leq \phi < 2\pi)$ on subsystem $B_i$. One can easily get the probability $p_k = \frac{1}{2}(1 + (-1)^k(1 - 2\alpha^2\mu^2)\cos\theta)$ and the two eigenvalues of the corresponding $\rho_k$ as $\lambda_\pm(\rho_k) = \frac{1 \pm \sqrt{1 - 2(1 - \alpha^2 - pr + r\alpha)^2 + 4\alpha^2\mu^2(1-r)(1-\alpha^2)}}{2}$, where $\Omega_k = \frac{1}{4}(1 - 2(1 - \alpha^2 - pr + r\alpha)^2 + (-1)^k\cos\theta(1 - 2\mu))$ and $\mu = (1 - r + pr\alpha^2 \mu^2 + (1 - pr)(1 - \alpha^2) - (1 - p)r\alpha^2/2)$. Meanwhile, the entropy of $\rho_k$ can be written in terms of its eigenvalues as $S(\rho_k) = f(\lambda_+(\rho_k))$. Note that here we define the function $f(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$. Then the quantum conditional entropy can be represented as $S(\rho_{AB}^{\mathcal{R}}|\Pi_k^B) = p_1 S(\rho_1) + p_2 S(\rho_2)$, and its minimum can be obtained by setting its partial derivatives with respect to $\theta$ and $\delta$ to zero. After some tedious calculations, we reach the formula for minimum conditional entropy

$$\min_{\Pi_k^B} S(\rho_{AB}^{\mathcal{R}}|\Pi_k^B) = \min(\xi_1, \xi_2),$$  \hspace{1cm} (19)$$

where

$$\xi_1 = f(1 + \sqrt{1 - 2(1 - \alpha^2 - pr + r\alpha)^2 + 4\alpha^2\mu^2(1-r)(1-\alpha^2)})/2$$

and

$$\xi_2 = -(1 - r - pr)\alpha^2 \mu^2 \log_2[(1 - r - pr)\alpha^2 \mu^2]$$

$$- (1 - pr)\alpha^2 \mu^2 \log_2[(1 - pr)\alpha^2 \mu^2]$$

$$- [(1 - pr)(1 - \alpha^2) + (1 - p)\alpha^2 r^2 \mu^2]$$

$$\times \log_2[(1 - pr)(1 - \alpha^2) + (1 - p)\alpha^2 r^2 \mu^2]$$

$$- [(1 - r + pr)\alpha^2 \mu^2 (1 - \alpha^2)]$$

$$\times \log_2[(1 - r + pr)\alpha^2 \mu^2 (1 - \alpha^2)] - f(\alpha^2 \mu^2).$$

Finally, it is straightforward to inserting the above relevant contents into Eq. (17), resulting in the analytical expression of QD.

From Fig. 2(a), we observe that the value of QD is always nonzero except for the point where the decoherence parameter $r = 1$. It manifests that QD is more robust than entanglement and it can reveal more properties of the given system, namely, QD can be regarded as a better resource for implementing quantum information processing in the practical application. Moreover, QD as a function of the state parameter $\alpha^2$ with fixed decoherence parameters and $t_{ij}$. For more details, please refer to the original paper [23,24].
and

\[
B_2 = 2(4\alpha^2 \mu^2 (1-r)(1-\alpha^2) + [\alpha^2 (1-p-r+2pr) \mu^2 - \nu^2] + (1-2pr)(1-\alpha^2))^2)^{1/2}.
\]

From discussions above and contour plot Fig. 3, we are able to draw the conclusions below: (i) When the Hawking effect is close to extinct, namely, in the case of an almost extreme black hole, the physically accessible states will violate the CHSH inequality if and only if \(0 \leq r < 0.5\); (ii) The non-local region shrinks as the intensity of Hawking effect grows; (iii) When \(0 \leq r < 0.5\), there exists a boundary line for quantum non-locality, which means the quantum non-locality will disappear when the Hawking temperature exceeds a fixed value which increases with the decrease of parameter \(r\). Especially, when \(r\) approaches to 0, the quantum non-locality always exists whatever the Hawking temperature is. Further analysis informs us that the strength of non-locality decreases with the intensity of Hawking effect enhancing, so the non-locality can be regarded as one of the performances of quantum correlation.

### 3.2. Phase damping channel

In this subsection, we discuss another widespread noise environment, the phase damping (PD) channel, which depicts the loss of quantum coherence without losing energy [18]. The PD environment is especially representative, since it provides a revealing view of decoherence in realistic physical situations. It can be regarded as an elastic scattering, that is, the particle’s state does not change and the state of environment experiences a transition without any system–environment energy exchange. Similarly, here we consider Alice stays in an asymptotically flat region undergoing the PD channel, while Bob hovers at a fixed finite nearest distance away from the event horizon.

#### 3.2.1. Concurrence

To get a better understanding of the degrees of freedom of decoherence environment, we choose another way to discuss this case. The dissipative interaction between the qubit and its environment can be expressed as

\[
\begin{align*}
|0\rangle_S|0\rangle_E & \rightarrow |0\rangle_S|0\rangle_E, \\
|1\rangle_S|0\rangle_E & \rightarrow \sqrt{1-r_p}|1\rangle_S|0\rangle_E + \sqrt{r_p}|1\rangle_S|1\rangle_E.
\end{align*}
\]

Utilizing Eq. (24) to rewrite Eq. (12), the final physical accessible density matrix can be given by

\[
\rho^{\mu}_{\text{AE/EB/AB}_1} = \alpha^2 \mu^2 |000\rangle \langle 000| + \alpha^2 \nu^2 |001\rangle \langle 001| + (1-\alpha^2)(1-r_p)|101\rangle \langle 101| + r_p(1-\alpha^2)|111\rangle \langle 111|
\]

\[
+ \alpha \mu \sqrt{(1-\alpha^2)(1-r_p)}(|000\rangle \langle 101| + |101\rangle \langle 000|) + (1-\alpha^2)\sqrt{r_p(1-r_p)}(|101\rangle \langle 111| + |111\rangle \langle 101|)
\]

\[
+ \alpha \mu \sqrt{r_p(1-\alpha^2)}(|000\rangle \langle 111| + |111\rangle \langle 000|).
\]

The reduced density matrix of partition \(AB_1\), which can be obtained by tracing over the degree of freedom about environment, is expressed as

\[
\rho^{\mu}_{\text{AB}_1} = \alpha^2 \mu^2 |00\rangle \langle 00| + \alpha^2 \nu^2 |01\rangle \langle 01| + (1-\alpha^2)|11\rangle \langle 11| + r_p(1-\alpha^2)|10\rangle \langle 10|
\]

\[
+ \alpha \mu \sqrt{(1-\alpha^2)(1-r_p)}(|00\rangle \langle 11| + |11\rangle \langle 00|).
\]

Similarly, taking the trace over the degree of freedom of \(B_1\) and \(A\), respectively, gives us the reduced density matrix of \(AE\) and \(EB_1\):

\[
\rho^{\mu}_{\text{AE}} = \alpha^2 |00\rangle \langle 00| + (1-\alpha^2)(1-r_p)|10\rangle \langle 10|
\]

\[
+ r_p(1-\alpha^2)|11\rangle \langle 11|
\]

\[
+ \alpha \sqrt{(1-\alpha^2)(1-r_p)}(|10\rangle \langle 11| + |11\rangle \langle 10|).
\]

\[
\rho^{\mu}_{\text{EB}_1} = \alpha^2 \mu^2 |00\rangle \langle 00| + \alpha^2 \nu^2 + (1-\alpha^2)(1-r_p)|01\rangle \langle 01|
\]

\[
+ r_p(1-\alpha^2)|11\rangle \langle 11|
\]

\[
+ \alpha \sqrt{r_p(1-r_p)}(|01\rangle \langle 11| + |11\rangle \langle 01|).
\]

On the other hand, through mathematical analysis, the concurrence of \(\rho^{\mu}_{\text{AB}_1}\) is

\[
C^{\mu}_{\text{AB}_1} = 2\alpha \mu \sqrt{(1-\alpha^2)(1-r_p)}
\]

The properties of quantum entanglement of subsystem \(AB_1\) are plotted in Fig. 4 with \(\alpha = \frac{\nu}{\sqrt{\nu^2 + \mu^2}}\). We find that \(C_{AB_1}\) decreases with the increase of decoherence parameter or Hawking temperature. However, the ESD can’t appear all the time, which is the most apparent difference from GAD case. The intrinsic physical reason is that for GAD channel, Alice not only can switch from the excited state to the ground state by undergoing spontaneous emission, but also can jump from the ground state to the excited energy state by absorbing energy from the external nonzero temperature environment. Besides, the concurrence of original state vanishes if decoherence parameter satisfies \(r_p = 1\). It indicates that the influence of decoherence on \(C_{AB_1}\) is more ruthless than Hawking effect, which is the same to the GAD case. In particular, according to the inchoate argument of Hawking [8], which states that the smaller black holes have a higher temperature and so the radiation of them is more violent than that of more massive black holes, namely, the mass of the black hole affects the radiation temperature directly. Our result demonstrates that when the decoherence parameter is fixed, quantum entanglement decreases as the mass of black hole decreases, which verifies the Hawking’s inference.

Furthermore, we generalize our investigation to the case of an \(N\)-qubit system

\[
|\Phi\rangle_{1,2,3, \ldots, N} = \left(\eta|0\rangle^\otimes N + \sqrt{1-\eta^2}|1\rangle^\otimes N\right)_{1,2,3, \ldots, N}.
\]

If \(n\) (\(n < N\) particles freely fall in toward a Schwarzschild black hole and eventually locate near the event horizon, and the
other particles remain in the asymptotically flat region including \( n' \) \((n' < (N - n))\) particles under the PD decoherence environment.

The physical accessible concurrence of this system without the degrees of freedom of the environment can be given by

\[
C(\varphi_{1,2,\ldots,n}) = 2\eta \sqrt{1 - \eta^2 (1 - r_p)^{n/2}} \left( e^{-\alpha \eta} + 1 \right)^{-n/2}.
\]

The corresponding detailed proof is given in Appendix A. Surprisingly, the result shows that N-qubit accessible concurrence is independent of N. This is due to that the initial system just makes the physical accessible density matrix be a particular X matrix. Essentially, in this case, the value of N only controls quantum mutual information besides entanglement.

### 3.2.2. Entanglement distribution

In addition, using PD case as an example, we now study the reason why the physical accessible entanglement is reduced with the growing Hawking temperature. As the existence of Hawking effect makes the mode of Bob divide into mode \(|n\rangle_{B1}\) and \(|n\rangle_{B2}\), we guess that the disappeared entanglement is distributed to the physical inaccessible region. Utilizing Eqs. (10) and (24) to regroup Eq. (11), one can obtain

\[
|\phi\rangle_{AB1B2} = \alpha |00\rangle + \beta |11\rangle + \sqrt{1 - |\alpha|^2 |(1 - r_p)|1010\rangle + |\alpha|^2 |1010\rangle + |\alpha|^2 |1100\rangle}.
\]

It is easy to get the reduced density matrices of partition \( AB_1 \) and \( AB_2 \), by tracing over \( EB_{II} \) and \( EB_{I} \), respectively.

\[
\rho_{AB_1} = \alpha^2 |00\rangle\langle 00| + \alpha^2 |01\rangle\langle 01| + (1 - |\alpha|^2) |10\rangle\langle 10|
\]

\[
+ \alpha |\alpha|^2 |10\rangle\langle 10| + |\alpha|^2 |11\rangle\langle 11|,
\]

\[
\rho_{AB_2} = \alpha^2 |00\rangle\langle 00| + (1 - |\alpha|^2) |10\rangle\langle 10| + \alpha^2 |11\rangle\langle 11|.
\]

Notably, the quantum properties of \( \rho_{AB_1}^{''} \) and \( \rho_{AB_2}^{''} \) have nothing to do with decoherence parameter \( r_p \), because only the Alice’s mode undergoes the PD channel. Our calculation indicates that \( C_{EB_{II}} = C_{AE} = C_{EB_{I}} = 0 \), that is, the interaction between system and noise environment does not produce bipartite particle–environment entanglement in the PD case. In Fig. 5, the redistributions of the quantum entanglement show how the Hawking temperature changes all the nonzero bipartite concurrences. When the Hawking effect is extinct, there is no entanglement between modes \( B_1 \) and \( B_2 \), or between modes \( A \) and \( B_2 \). As the intensity of Hawking effect grows, the physical accessible entanglement decreases while the inaccessible entanglements increases rapidly. Consequently, the vanishing accessible entanglement is distributed to physical inaccessible entanglement. It is also worth mentioning that when the black hole evaporates completely, the quantum entanglements of \( \rho_{AB_1} \) and \( \rho_{AB_2} \) are tantamount.

### 4. Conclusions

By employing various correlation measures, such as concurrence, quantum discord, and Bell violation, we have fully investigated the properties of open Dirac system with Hawking effect in the background of a Schwarzschild black hole, as well as two different realistic decoherence environments, GAD and PD channels. Our main conclusions are that quantum entanglement always decreases when the Hawking temperature increases, and that the existence of decoherence can accelerate its attenuation. The latter suggests that the influence of Hawking effect on physical accessible entanglement is weaker than that of decoherence. The reason is that the quantum states cannot be infinitely excited due to the principle of Pauli exclusion in Fermi–Dirac statistics. Moreover, the most significant difference between GAD case and PD or AD case is that the ESD can appear in GAD channel with certain circumstances. We also notice that the Hawking temperature affects the critical values of the decoherence parameter, which is related to the production of ESD. Meanwhile, the inequivalence of quantization for Dirac particles in the black hole and Kruskal space–time is verified via quantum discord measure. The quantum non–locality is also assessed, resulting in the following conclusions: (i) When the Hawking effect is close to extinct, the physically accessible states will violate the CHSH inequality if and only if the decoherence parameter \( r \) satisfies \( 0 \leq r < 0.5 \), and there exists a boundary line for Bell violation; (ii) The non-local region shrinks...
when the intensity of Hawking effect grows; (iii) The strength of non-locality decreases with the increasing intensity of Hawking effect because non-locality is one of the performances of quantum correlation. For the case of PD channel, we discover that \( C_{\phi_{B}}'' = C_{\phi_{A}}'' = 0 \), namely, the interaction between system and noise environment does not produce bipartite system–environment entanglement. We also find that the reduced physically accessible entanglement is redistributed to physical inaccessible region. Additionally, we extend our investigation to the case of an N-qubit system and obtain a universal expression of the physical accessible entanglement. Surprisingly, our results show that this expression is actually independent of the total number \( N \).

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**Appendix A**

Here, we show the detailed computational process of physical accessible concurrence for the case of an N-qubit system \(|\Phi\rangle_{1,2,3,\ldots,n} = \eta |0\rangle^o N + \sqrt{1 - \eta^2} |1\rangle^o N_{1,2,3,\ldots,n} \). For \( N \)-partite pure states \(|\Psi\rangle \in H_1 \otimes H_2 \otimes H_3 \otimes \cdots \otimes H_N \), where \( H \) represents the Hilbert space, the multipartite concurrence can be defined as [27]

\[
C_{\phi_{i}} = \min_{\tau} \sqrt{2} \sqrt{1 - \text{Tr} (\rho_0^{2})},
\]

where \( \tau = \{ \tau_j \} \) represents the set of all possible bipartitions \( \{ A_i | B_i \} \) of \( \{ 1, 2, 3, \ldots, N \} \). For the given multipartite system, we assume that \( n (n < N) \) particles freely fall toward a Schwarzschild black hole and locate near the event horizon, and the other particles remain at the asymptotically flat region including \( n' (n' < (N - n)) \) particles under the PD decoherence environment. So, Eq. (29) can be rewritten as

\[
|\Phi\rangle_{1,2,3,\ldots,n} = \eta |0\rangle^o (N-n' - n) \otimes |00\rangle^o \otimes (\mu |00\rangle + u |11\rangle)^o n
+ \sqrt{1 - \eta^2} |1\rangle^o (N-n'- n)
\otimes (\sqrt{1-r}_1 |10\rangle + \sqrt{r}_1 |01\rangle)^o n' \otimes |10\rangle^o n. \tag{A.2}
\]

Tracing over the state of the physical inaccessible regions, we can get an \( N \)-qubit physical accessible state. Specially, its density matrix is a particular \( X \) matrix, whose only nonzero elements are diagonal or anti-diagonal when written in an orthonormal basis. As we known, the \( N \)-qubit \( X \) matrix can be written as

\[
\hat{X} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & z_1 \\
0 & 0 & 0 & 0 & 0 & 0 & z_2 \\
0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_n & z_n & 0 & 0 \\
0 & 0 & 0 & 0 & z_n^* & b_n & 0 \\
0 & 0 & \ddots & 0 & 0 & 0 & b_1 \\
z_1^* & 0 & 0 & 0 & 0 & 0 & b_1
\end{pmatrix}. \tag{A.3}
\]

In order to ensure \( \hat{X} \) is positive and normalized, the conditions of \(|z_1| \leq \sqrt{a_m b_m} \) and \( \sum_m (a_m + b_m) = 1 \) must be satisfied simultaneously. According to Ref. [28], the concurrence of an \( N \)-qubit state is given by the formula

\[
C = 2 \max \{ 0, |z_1| - a_m \}, \quad m = 1, 2, 3, \ldots, n, \tag{A.4}
\]

where \( a_m = \sum_{n'=m} \sqrt{a_n b_n} \). Through mathematical analysis, for the given \( N \)-qubit physical accessible density matrix without the degree of freedom of the environment, we can obtain that \( b_2 = b_3 = \cdots = b_n = 0 \). The other diagonal elements can be written as

\[
a_1 = (\eta \mu^n)^2, \quad a_2 = (\eta \mu^{n-1} \nu)^2, \\
a_3 = (\eta \mu^{n-2} \nu)^2, \quad a_4 = (\eta \mu^{n-3} \nu)^2, \quad \ldots. \tag{A.5}
\]

The value of \( b_1 \) is very intricate, but we do not need it due to the final expression has nothing to do with it. For the anti-diagonal elements,

\[
z_1 = z_1^* = \eta \sqrt{1 - \eta^2 (1 - r_p)^{n/2}} \mu^n, \tag{A.6}
\]

and the other elements are zero. Inserting these parameters into the formula (A.4), we can easily obtain the expression of the given \( N \)-qubit physical accessible state

\[
C_{\phi_{1,2,3,\ldots,n}} = 2 \eta \sqrt{1 - \eta^2 (1 - r_p)^{n/2}} \mu^n
= 2 \eta \sqrt{1 - \eta^2 (1 - r_p)^{n/2} (e^{-\alpha n/T} + 1)^{-n/2}}. \tag{A.7}
\]

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