A Poset View of the Major Index

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Abstract

We introduce the Major MacMahon map from $\mathbb{Z}(a, b)$ to $\mathbb{Z}[q]$, and show how this map commutes with the pyramid and bipyramid operators. When the Major MacMahon map is applied to the ab-index of a simplicial poset, it yields the $q$-analogue of $n!$ times the $h$-polynomial of the polytope. Applying the map to the Boolean algebra gives the distribution of the major index on the symmetric group, a seminal result due to MacMahon. Similarly, when applied to the cross-polytope we obtain the distribution of one of the major indexes on the signed permutations, due to Reiner.

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1 Introduction

One hundred and one years ago in 1913 Major Percy Alexander MacMahon [9] (see also his collected works [11]) introduced the major index of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of the multiset $M = \{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\}$ of size $n$ to be the sum of the elements of its descent set, that is,

$$\text{maj}(\pi) = \sum_{\pi_i > \pi_{i+1}} i.$$ 

He showed that the distribution of this permutation statistic is given by the $q$-analogue of the multinomial Gaussian coefficient, that is, the following identity holds:

$$\sum_{\pi} q^{\text{maj}(\pi)} = \frac{[n]!}{[\alpha_1]! \cdot [\alpha_2]! \cdots [\alpha_k]!} = \frac{n!}{\alpha!},$$

where $\pi$ ranges over all permutations of the multiset $M$. Here $[n]! = [n] \cdot [n-1] \cdots [1]$ denotes the $q$-analogue of $n!$, where $[n] = 1 + q + \cdots + q^{n-1}$.

Many properties of the descent set of a permutation $\pi$, that is, $\text{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$, have been studied by encoding the set by its ab-word; see for instance [6, 12]. For a multiset

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permutation $\pi \in \mathfrak{S}_M$ the \textit{ab}-word is given by $u(\pi) = u_1 u_2 \cdots u_{n-1}$, where $u_i = b$ if $\pi_i > \pi_{i+1}$ and $u_i = a$ otherwise.

Inspired by this definition, we introduce the \textit{Major MacMahon map} $\Theta$ on the ring $\mathbb{Z}(a, b)$ of non-commutative polynomials in the variables $a$ and $b$ to $\mathbb{Z}[q]$, polynomials in the variable $q$, by

$$\Theta(w) = \prod_{i : u_i = b} q^i,$$

for a monomial $w = u_1 u_2 \cdots u_n$ and extend $\Theta$ to all of $\mathbb{Z}(a, b)$ by linearity. In short, the map $\Theta$ sends each variable $a$ to $1$ and the variables $b$ to $q$ to the power of its position, read from left to right. A Swedish example is $\Theta(abba) = q^5$.

## 2 Chain enumeration and products of posets

Let $P$ be a graded poset of rank $n+1$ with minimal element $\hat{0}$, maximal element $\hat{1}$ and rank function $\rho$. Let the rank difference be defined by $\rho(x, y) = \rho(y) - \rho(x)$. The \textit{flag} $f$-vector entry $f_S$, for $S = \{s_1 < s_2 < \cdots < s_k\}$ a subset $\{1, 2, \ldots, n\}$, is the number of chains $c = \{0 = x_0 < x_1 < x_2 < \cdots < x_{k+1} = 1\}$ such that the rank of the element $x_i$ is $s_i$, that is, $\rho(x_i) = s_i$ for $1 \leq i \leq k$. The \textit{flag} $h$-vector is defined by the invertible relation

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T.$$

For a subset $S$ of $\{1, 2, \ldots, n\}$ define two \textit{ab}-polynomials of degree $n$ by $u_S = u_1 u_2 \cdots u_n$ and $v_S = v_1 v_2 \cdots v_n$ by

$$u_i = \begin{cases} a & \text{if } i \notin S, \\ b & \text{if } i \in S, \end{cases} \quad \text{and} \quad v_i = \begin{cases} a - b & \text{if } i \notin S, \\ b & \text{if } i \in S. \end{cases}$$

The \textit{ab-index} of the poset $P$ is defined by the two equivalent expressions:

$$\Psi(P) = \sum_S f_S \cdot v_S = \sum_S h_S \cdot u_S,$$

where the two sums range over all subsets $S$ of $\{1, 2, \ldots, n\}$. For more details on the \textit{ab-index} see [7] or the book [16 Section 3.17].

Recall that a graded poset $P$ is \textit{Eulerian} if every non-trivial interval has the same number of elements of even as odd rank. Equivalently, a poset is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in $P$. When the graded poset $P$ is Eulerian then the \textit{ab-index} $\Psi(P)$ can be written in terms of the non-commuting variables $c = a + b$ and $d = ab + ba$ and it is called the \textit{cd-index}; see [2]. For an $n$-dimensional convex polytope $P$ its face lattice $\mathcal{L}(P)$ is an Eulerian poset of rank $n+1$. In this case we write $\Psi(P)$ for the \textit{ab-index} (\textit{cd-index}) instead of the cumbersome $\Psi(\mathcal{L}(P))$.

There are also two products on posets that we will study. The first is the \textit{Cartesian product}, defined by $P \times Q = \{(x, y) : x \in P, y \in Q\}$ with the order relation $(x, y) \leq_{P \times Q} (z, w)$ if $x \leq_P z$
and \( y \leq Q w \). Note that the rank of the Cartesian product of two graded posets of ranks \( m \) and \( n \) is \( m + n \). As a special case we define \( \text{Pyr}(P) = P \times B_1 \), where \( B_1 \) is the Boolean algebra of rank 1. The geometric reason is that this operation corresponds to the geometric operation of taking the pyramid of a polytope, that is, \( \mathcal{L}(\text{Pyr}(P)) = \text{Pyr}(\mathcal{L}(P)) \) for a polytope \( P \).

The second product is the dual diamond product, defined by
\[
P \diamond^* Q = (P - \{ \hat{1}_P \}) \times (Q - \{ \hat{1}_Q \}) \cup \{ \hat{1} \}.
\]
The rank of the product \( P \diamond^* Q \) is the sum of the ranks of \( P \) and \( Q \) minus one. This is the dual to the diamond product \( \diamond \) defined by removing the minimal elements of the posets, taking the Cartesian product and adjoining a new minimal element. The product \( \diamond \) behaves well with the quasi-symmetric functions of type \( B \). (See Sections 5 and 6.) However, we will dualize our presentation and keep working with the product \( \diamond^* \).

Yet again, we have an important special case. We define \( \text{Bipyr}(P) = P \diamond^* B_2 \). The geometric motivation is the connection to the bipyramid of a polytope, that is, \( \mathcal{L}(\text{Bipyr}(P)) = \text{Bipyr}(\mathcal{L}(P)) \) for a polytope \( P \).

### 3 Pyramids and bipyramids

Define on the ring \( \mathbb{Z}(a, b) \) of non-commutative polynomials in the variables \( a \) and \( b \) the two derivations \( G \) and \( D \) by
\[
G(1) = 0, \quad G(a) = ba, \quad G(b) = ab,
D(1) = 0, \quad D(a) = D(b) = ab + ba.
\]
Extend these two derivations to all of \( \mathbb{Z}(a, b) \) by linearity. The pyramid and the bipyramid operators are given by
\[
\text{Pyr}(w) = G(w) + w \cdot c \quad \text{and} \quad \text{Bipyr}(w) = D(w) + c \cdot w.
\]
These two operators are suitably named, since for a poset \( P \) we have
\[
\Psi(\text{Pyr}(P)) = \text{Pyr}(\Psi(P)) \quad \text{and} \quad \Psi(\text{Bipyr}(P)) = \text{Bipyr}(\Psi(P)).
\]
For further details, see [7].

**Theorem 3.1.** The Major MacMahon map \( \Theta \) commutes with right multiplication by \( c \), the derivation \( G \), the pyramid and the bipyramid operators as follows:
\[
\Theta(w \cdot c) = (1 + q^{n+1}) \cdot \Theta(w), \quad (3.1)
\]
\[
\Theta(G(w)) = q \cdot [n] \cdot \Theta(w), \quad (3.2)
\]
\[
\Theta(\text{Pyr}(w)) = [n + 2] \cdot \Theta(w), \quad (3.3)
\]
\[
\Theta(\text{Bipyr}(w)) = [2] \cdot [n + 1] \cdot \Theta(w), \quad (3.4)
\]
where \( w \) is a homogeneous \( ab \)-polynomial of degree \( n \).

**Proof.** It is enough to prove the four identities for an \( ab \)-monomial \( w \) of degree \( n \). Directly we have that \( \Theta(w \cdot a) = \Theta(w) \) and \( \Theta(w \cdot b) = q^{n+1} \cdot \Theta(w) \). Adding these two identities yields equation (3.1).
Assume that \( w \) consists of \( k \) b's. We will label the \( n \) letters of \( w \) as follows: The \( k \) b's are labeled 1 through \( k \) reading from right to left, whereas the \( n - k \) a's are labeled \( k + 1 \) through \( n \) reading left to right. As an example, the word \( w = \text{aababba} \) is written as \( w_4w_5w_3w_6w_2w_1w_7 \).

The theorem is a consequence of the following claim. Applying the derivation \( G \) only to the letter \( w_i \) and then applying the Major MacMahon map yields \( q^i \cdot \Theta(w) \), that is,

\[
\Theta(u \cdot G(w_i) \cdot v) = q^i \cdot \Theta(u \cdot w_i \cdot v), \tag{3.5}
\]

where \( w \) is factored as \( u \cdot w_i \cdot v \). To see this, first consider when \( 1 \leq i \leq k \). There are \( i \) b's to the right of \( w_i \) including \( w_i \) itself. They each are shifted one step to the right when replacing \( w_i = b \) with \( G(b) = ab \) and hence we gain a factor of \( q^i \). The second case is when \( k + 1 \leq i \leq n \). Then \( w_i \) is an a and is replaced by \( ba \) under the derivation \( G \). Assume that there are \( j \) b's to the right of \( w_i \). When these \( j \) b's are shifted one step to the right they contribute a factor of \( q^j \). We also create a new \( b \). It has \( i - k - 1 \) a's to the left and \( k - j \) b's to the left. Hence the position of the new \( b \) is \( (i - k - 1) + (k - j) + 1 = i - j \) and thus its contribution is \( q^{i - j} \). Again the factor is given by \( q^i \cdot q^{i - j} = q^i \), proving the claim. Now by summing over these \( n \) cases, identity (3.2) follows. Identity (3.3) is the sum of identities (3.1) and (3.2).

To prove identity (3.4), we use a different labeling of the monomial \( w \). This time label the \( k \) b's with the subscripts 0 through \( k - 1 \), rather than 1 through \( k \). That is, in our example \( w = \text{aababba} \) is now labeled as \( w_4w_5w_2w_6w_1w_7 \). We claim that for \( w = u \cdot w_i \cdot v \) we have that

\[
\Theta(u \cdot D(w_i) \cdot v) = q^i \cdot [2] \cdot \Theta(w).
\]

The first case is \( 0 \leq i \leq k - 1 \). Then \( w_i = b \) has \( i \) b's to its right. Thus when replacing \( b \) with \( ba \) there are \( i \) b's that are shifted one step, giving the factor \( q^i \). Similarly, when replacing \( w_i \) with \( ab \), there are \( i + 1 \) b's that are shifted one step, giving the factor \( q^{i+1} \). The sum of the two factors is \( q^i \cdot [2] \). The second case is \( k + 1 \leq i \leq n \). It is as the second case above when replacing \( w_i \) with \( ba \), yielding the factor \( q^i \). When replacing \( w_i \) with \( ab \) there is one more shift, giving \( q^{i+1} \). Adding these two subcases completes the proof of the claim.

It is straightforward to observe that

\[
\Theta(c \cdot w) = q^k \cdot [2] \cdot \Theta(w). \tag{3.4}
\]

Calling this the case \( i = k \), the identity (3.4) follows by summing the \( n + 1 \) cases \( 0 \leq i \leq n \).

Iterating equations (3.3) and (3.4) we obtain that the Major MacMahon map of the \( ab \)-index of the \( n \)-dimensional simplex \( \Delta_n \) and the \( n \)-dimensional cross-polytope \( C_n^* \).

**Corollary 3.2.** The \( n \)-dimensional simplex \( \Delta_n \) and the \( n \)-dimensional cross-polytope \( C_n^* \) satisfy

\[
\Theta(\Psi(\Delta_n)) = [n + 1]!,
\]

\[
\Theta(\Psi(C_n^*)) = [2]^n \cdot [n]!.
\]

4 Simplicial posets

A graded poset \( P \) is **simplicial** if all of its lower order intervals are Boolean, that is, for all elements \( x \leq \hat{1} \) the interval \([0, x] \) is isomorphic to the Boolean algebra \( B_{\rho(x)} \). It is well-known that all the
flag information of a simplicial poset of rank \( n + 1 \) is contained in the \( f \)-vector \((f_0, f_1, \ldots, f_n)\), where \( f_0 = 1 \) and \( f_i = f(i) \) for \( 1 \leq i \leq n \). The \( h \)-vector, equivalently, the \( h \)-polynomial \( h(P) = h_0 + h_1 \cdot q + \cdots + h_n \cdot q^n \), is defined by the polynomial relation
\[
h(q) = \sum_{i=0}^{n} f_i \cdot (q - 1)^{n-i}.
\]
See for instance [19, Section 8.3]. The \( h \)-polynomial and the bipyramid operation commutes as follows
\[
h(\text{Bipyr}(P)) = (1 + q) \cdot h(P).
\]

We can now evaluate the Major MacMahon map on the \( ab \)-index of a simplicial poset.

**Theorem 4.1.** For a simplicial poset \( P \) of rank \( n + 1 \) the following identity holds:
\[
\Theta(\Psi(P)) = [n]! \cdot h(P).
\]

**Proof.** Let \( B_n \cup \{\hat{1}\} \) denote the Boolean algebra \( B_n \) with a new maximal element added. Note that \( B_n \cup \{\hat{1}\} \) is indeed a simplicial poset and its \( h \)-polynomial is 1. Furthermore, equation (4.1) holds for \( B_n \cup \{\hat{1}\} \) since
\[
\Theta(\Psi(B_n \cup \{\hat{1}\})) = \Theta(\Psi(B_n)) = [n]! = [n]! \cdot h(B_n \cup \{\hat{1}\}).
\]
Also, if (4.1) holds for a poset \( P \) then it also holds for \( \text{Bipyr}(P) \), since we have
\[
\Theta(\Psi(\text{Bipyr}(P))) = [2] \cdot [n + 1] \cdot \Theta(\Psi(P)) = [2] \cdot [n + 1] \cdot [n]! \cdot h(P) = [n + 1]! \cdot h(\text{Bipyr}(P)).
\]

Observe that both sides of (4.1) are linear in the \( h \)-polynomial. Hence to prove it for any simplicial poset \( P \) it is enough to prove it for a basis of the span of all simplicial posets of rank \( n + 1 \). Such a basis is given by the posets
\[
B_n = \left\{ \text{Bipyr}^i(B_{n-i} \cup \{\hat{1}\}) \right\}_{0 \leq i \leq n}.
\]
This is a basis since the polynomials \( h(\text{Bipyr}^i(B_{n-i} \cup \{\hat{1}\})) = (1 + q)^i \), for \( 0 \leq i \leq n \), are a basis for polynomials of degree at most \( n \).

Finally, since every element in the basis is built up by iterating bipyramids of the posets \( B_n \cup \{\hat{1}\} \), the theorem holds for all simplicial posets.

Observe that the poset \( \text{Bipyr}^i(B_{n-i} \cup \{\hat{1}\}) \) is the face lattice of the simplicial complex consisting of the \( 2^i \) facets of the \( n \)-dimensional cross-polytope in the cone \( x_1, \ldots, x_{n-i} \geq 0 \).

For an Eulerian simplicial poset \( P \), the \( h \)-vector is symmetric, that is, \( h_i = h_{n-i} \). In other words, the \( h \)-polynomial is palindromic. Stanley [15] introduced the simplicial shelling components, that is, the \( cd \)-polynomials \( \Phi_{n,i} \) such that the \( cd \)-index of an Eulerian simplicial poset \( P \) of rank \( n + 1 \) is given by
\[
\Psi(P) = \sum_{i=0}^{n} h_i \cdot \Phi_{n,i}.
\]
These \( cd \)-polynomials satisfies the recursion \( \Phi_{n,0} = \Psi(B_n) \cdot c \) and \( \Phi_{n,i} = G(\Phi_{n-1,i-1}) \); see [7, Section 8]. The Major MacMahon map of these polynomials is described by the next result.
Corollary 4.2. The Major MacMahon map of the simplicial shelling components is given by

$$\Theta(\Phi_{n,i}) = q^i \cdot [2(n-i)] \cdot [n-1]!.$$  

Proof. When \(i = 0\) we have \(\Theta(\Phi_{n,0}) = \Theta(\Psi(B_n) \cdot c) = (1 + q^n) \cdot [n]! = [2n] \cdot [n-1]!\). Also when \(i \geq 1\) we obtain \(\Theta(\Phi_{n,i}) = \Theta(G(\Phi_{n-1,i-1})) = q^i \cdot [2(n-i)] \cdot [n-1]!\).  

We end with the following observation.

Theorem 4.3. For an Eulerian poset \(P\) of rank \(n + 1\), the polynomial \([2]^{\lceil n/2 \rceil}\) divides \(\Theta(\Psi(P))\).

Proof. It is enough to show this result for a cd-monomial \(w\) of degree \(n\). A c in an odd position \(i\) of \(w\) yields a factor of \(1 + q^i\). A d that covers an odd position \(i\) of \(w\) yields either \(q^i - 1 + q^i\) or \(q^i + q^{i+1}\). Each of these polynomials contributes a factor of \(1 + q\). The result follows since there are \(\lceil n/2 \rceil\) odd positions.  

5 The Cartesian product of posets

We now study how the Major MacMahon map behaves under the Cartesian product. Recall that for a poset \(P\) the ab-index \(\Psi(P)\) encodes the flag f-vector information of the poset \(P\). There is another encoding of this information as a quasi-symmetric function. For further information about quasi-symmetric functions, see [17, Section 7.19].

A composition \(\alpha\) of \(n\) is a list of positive integers \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) such that \(\alpha_1 + \alpha_2 + \cdots + \alpha_k = n\). Let \(\text{Comp}(n)\) denote the set of compositions of \(n\). There are three natural bijections between ab-monomials \(u\) of degree \(n\), subsets \(S\) of the set \(\{1, 2, \ldots, n\}\) and compositions of \(n + 1\). Given a composition \(\alpha \in \text{Comp}_{n+1}\) we have the subset \(S_\alpha\), the ab-monomial \(u_\alpha\) and the ab-polynomial \(v_\alpha\) defined by

\[
S_\alpha = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\},\\
u_\alpha = a^{\alpha_1-1} \cdot b \cdot a^{\alpha_2-1} \cdot b \cdots b \cdot a^{\alpha_k-1},\\v_\alpha = (a-b)^{\alpha_1-1} \cdot b \cdot (a-b)^{\alpha_2-1} \cdot b \cdots b \cdot (a-b)^{\alpha_k-1}.
\]

For \(S\) a subset of \(\{1, 2, \ldots, n\}\) let \(\text{co}(S)\) denote associated composition.

The monomial quasi-symmetric function \(M_\alpha\) is defined as the sum

\[
M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} t_{i_1}^{\alpha_1} \cdot t_{i_2}^{\alpha_2} \cdots t_{i_k}^{\alpha_k}.
\]

A second basis is given by the fundamental quasi-symmetric function \(L_\alpha\) defined as

\[
L_\alpha = \sum_{S_\alpha \subseteq T \subseteq \{1, 2, \ldots, n\}} M_{\text{co}(T)}.
\]

Following [8] define an injective linear map \(\gamma : \mathbb{Z}(a, b) \rightarrow \text{QSym}\) by

\[
\gamma(v_\alpha) = M_\alpha,
\]
for a composition $\alpha$ of $n \geq 1$. The image of $\gamma$ is all quasi-symmetric functions without constant term. Moreover, the image of the $ab$-monomial $u_\alpha$ under $\gamma$ is the fundamental quasi-symmetric function $L_\alpha$, that is,
\[ \gamma(u_\alpha) = L_\alpha. \]

Another way to encode the flag vectors of a poset $P$ is by the quasi-symmetric function of the poset. It is quickly defined as $F(P) = \gamma(\Psi(P))$. A more poset-oriented definition is the following limit of sums over multichains
\[ F(P) = \lim_{k \to \infty} \sum_{\delta = x_0 \leq x_1 \leq \cdots \leq x_k = \hat{1}} t_1^{\rho(x_0, x_1)} t_2^{\rho(x_1, x_2)} \cdots t_k^{\rho(x_{k-1}, x_k)}. \]

For more on the quasi-symmetric function of a poset, see [5].

The stable principal specialization of a quasi-symmetric function is the substitution $\text{ps}(f) = f(1, q, q^2, \ldots)$. Note that this is a homeomorphism, that is, $\text{ps}(f \cdot g) = \text{ps}(f) \cdot \text{ps}(g)$.

For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ let $\alpha^*$ denote the reverse composition, that is, $\alpha^* = (\alpha_k, \ldots, \alpha_2, \alpha_1)$. This involution extends to an anti-automorphism on $\text{QSym}$ by $M^*_\alpha \mapsto M_{\alpha^*}$. Define $\text{ps}^*$ by the relation $\text{ps}^*(f) = \text{ps}(f^*)$. Informally speaking, this corresponds to the substitution $\text{ps}^*(f) = f(\ldots, q^2, q, 1)$.

**Theorem 5.1.** For a homogeneous $ab$-polynomial $w$ of degree $n - 1$ the Major MacMahon map is given by
\[ \Theta(w) = (1 - q)^n \cdot [n!] \cdot \text{ps}^*(\gamma(w)). \tag{5.1} \]
For a poset $P$ of rank $n$ this identity is
\[ \Theta(\Psi(P)) = (1 - q)^n \cdot [n!] \cdot \text{ps}^*(F(P)). \tag{5.2} \]

**Proof.** It is enough to prove identity (5.1) for an $ab$-monomial $w$ of degree $n - 1$. Let $\alpha$ be the composition of $n$ corresponding to the reverse monomial $w^*$. Furthermore, let $e(\alpha)$ be the sum $\sum_{i \in S_\alpha} (n - i)$. Note that $e(\alpha)$ is in fact the sum $\sum_{i \in S} i$, where $S$ is the subset associated with the $ab$-monomial $w$. That is, we have $q^{e(\alpha)} = \Theta(w)$. Equation (5.1) follows from Lemma 7.19.10 in [17]. By applying the first identity to $\Psi(P)$, we obtain identity (5.2). \qed

Since the quasi-symmetric function is multiplicative under the Cartesian product, we have the next result.

**Theorem 5.2.** For two posets $P$ and $Q$ of ranks $m$, respectively $n$, the following identity holds:
\[ \Theta(\Psi(P \times Q)) = \left[\begin{array}{c} m + n \\ n \end{array}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)). \tag{5.3} \]
Proof. The proof is a direct verification as follows:

\[
\Theta(\Psi(P \times Q)) = (1 - q)^{m+n} \cdot [m+n]! \cdot \text{ps}(F(P^* \times Q^*)) \\
= \binom{m+n}{m} \cdot (1 - q)^{m+n} \cdot [m]! \cdot [n]! \cdot \text{ps}(F(P^*)) \cdot \text{ps}(F(Q^*)) \\
= \binom{m+n}{m} \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).
\]

\[\square\]

6 The dual diamond product

Define the quasi-symmetric function of type $B^*$ of a poset $P$ to be the expression

\[
F_{B^*}(P) = \sum_{\hat{0} \leq x < \hat{1}} F([\hat{0}, x]) \cdot s^{\rho(x, \hat{1}) - 1}.
\]

This is an element of the algebra $\text{QSym} \otimes \mathbb{Z}[s]$ which we view as the quasi-symmetric functions of type $B^*$. We view $\text{QSym}_{B^*}$ as an subalgebra of $\mathbb{Z}[t_1, t_2, \ldots; s]$, which is quasi-symmetric in the variables $t_1, t_2, \ldots$. For instance, a basis for $\text{QSym}_{B^*}$ is given by a $M_{\alpha} \cdot s^i$ where $\alpha$ ranges over all compositions and $i$ over all non-negative integers.

Furthermore, the type $B^*$ quasi-symmetric function $F_{B^*}$ is multiplicative respect to the product $\circ^*$, that is, $F_{B^*}(P \circ^* Q) = F_{B^*}(P) \cdot F_{B^*}(Q)$; see [8, Theorem 13.3].

Let $f$ be a homogeneous quasi-symmetric function such that $f \cdot s^j$ is a quasi-symmetric function of type $B^*$. We define the stable principal specialization of the quasi-symmetric function $f \cdot s^j$ of type $B^*$ to be $\text{ps}_{B^*}(f \cdot s^j) = q^{\text{deg}(f)} \cdot \text{ps}^*(f)$. This is the substitution $s = 1$, $t_k = q$, $t_{k-1} = q^2$, \ldots as $k$ tends to infinity, since $f(\ldots, q^3, q^2, q) = q^{\text{deg}(f)} \cdot f(\ldots, q^2, q, 1)$. Especially, for a poset $P$ we have

\[
\text{ps}_{B^*}(F_{B^*}(P)) = \sum_{\hat{0} \leq x < \hat{1}} q^{\rho(x)} \cdot \text{ps}^*(F([\hat{0}, x])).
\]

Theorem 6.1. For a poset $P$ of rank $n+1$ the relationship between the Major MacMahon map and the stable principal specialization of type $B^*$ is given by

\[
\Theta(\Psi(P)) = (1 - q)^n \cdot [n]! \cdot \text{ps}_{B^*}(F_{B^*}(P^*)).
\]

Especially, for a homogeneous $ab$-polynomial $w$ of degree $n$ the Major MacMahon map is given by

\[
\Theta(w) = (1 - q)^n \cdot [n]! \cdot \text{ps}_{B^*}(\gamma_{B^*}(w^*)).
\]
Rearranging terms yields
\[
\lim_{k \to \infty} \sum_{\emptyset = x_0 \leq x_1 \leq \ldots \leq x_k = \hat{1}} (q^{k-1})^{\rho(x_0,x_1)} \cdots (q^{2})^{\rho(x_{k-3},x_{k-2})} \cdot q^{\rho(x_{k-2},x_{k-1})} \cdot 1^{\rho(x_{k-1},x_k)}
\]
= \lim_{k \to \infty} \sum_{\emptyset = x_0 \leq x_1 \leq \ldots \leq x_k = \hat{1}} (q^{k-1})^{\rho(x_0,x_1)} \cdots (q^{2})^{\rho(x_{k-3},x_{k-2})} \cdot q^{\rho(x_{k-2},x_{k-1})}
= \lim_{k \to \infty} \sum_{\emptyset = x_0 \leq x_1 \leq \ldots \leq x_k = \hat{1}} q^{\rho(x_{k-1})} \cdot (q^{k-2})^{\rho(x_0,x_1)} \cdots q^{\rho(x_{k-3},x_{k-2})} \cdot 1^{\rho(x_{k-2},x_{k-1})}
= \sum_{\emptyset \leq x \leq \hat{1}} q^{\rho(x)} \cdot ps^*(F([\emptyset, x]))
= \sum_{\emptyset \leq x < \hat{1}} q^{\rho(x)} \cdot ps^*(F([\emptyset, x])) + q^{n+1} \cdot ps^*(F(P)).
\]
Rearranging terms yields
\[
\sum_{\emptyset \leq x < \hat{1}} q^{\rho(x)} \cdot ps^*(F([\emptyset, x])) = (1 - q^{n+1}) \cdot ps^*(F(P))
= (1 - q^{n+1}) \cdot ps(F(P^*))
= (1 - q^{n+1}) \cdot \frac{\Theta(\Psi(P))}{(1 - q)^{n+1} \cdot [n + 1]!}
= \frac{\Theta(\Psi(P))}{(1 - q)^n \cdot [n]!}.
\]
Combining the last identity with (6.1) yields the desired result.

**Theorem 6.2.** For two posets \( P \) and \( Q \) of ranks \( m+1 \), respectively \( n+1 \), the identity holds:
\[
\Theta(\Psi(P \circ^* Q)) = \left[\begin{array}{c} m+n \\ n \end{array}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).
\] (6.4)

**Proof.** The proof is a direct verification as follows:
\[
\Theta(\Psi(P \circ^* Q)) = (1 - q)^{m+n} \cdot [m + n]! \cdot ps_{B^*}(F_{B^*}(P^* \circ^* Q^*))
= \left[\begin{array}{c} m+n \\ m \end{array}\right] \cdot (1 - q)^{m+n} \cdot [m]! \cdot [n]! \cdot ps_{B^*}(F_{B^*}(P^*)) \cdot ps_{B^*}(F_{B^*}(Q^*))
= \left[\begin{array}{c} m+n \\ m \end{array}\right] \cdot \Theta(\Psi(P)) \cdot \Theta(\Psi(Q)).
\]

7 Permutations

One connection between permutations and posets is via the concept of \( R \)-labelings. For more details, see [16 Section 3.14]. Let \( \mathcal{E}(P) \) be the set of all cover relations of \( P \), that is, \( \mathcal{E}(P) = \{(x, y) \in P^2 : x < y\} \). A graded poset \( P \) has an \( R \)-labeling if there is a map \( \lambda : \mathcal{E}(P) \to \Lambda \), where
Iterating Theorem 5.2 evaluates the Major MacMahon map on $L_{\lambda} < \lambda < JH_{\lambda}$. Theorem 7.1. For an R-labeling $\lambda$ of a graded poset $P$ we have that

$$\Psi(P) = \sum_{c} u_{\lambda(c)},$$

where the sum is over the Jordan–Hölder set $JH(P)$.

This a reformulation of a result of Björner and Stanley [3, Theorem 2.7]. The reformulation can be found in [5] Lemma 3.1.

As a corollary we obtain MacMahon’s classical result on the major index on a multiset; see [5]. For a composition $\alpha$ of $n$ let $\mathfrak{S}_\alpha$ denote all the permutations of the multiset $\{1^{\alpha_1}, 2^{\alpha_2}, \ldots, k^{\alpha_k}\}$.

Corollary 7.2 (MacMahon). For a composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of $n$ the following identity holds:

$$\sum_{\pi \in \mathfrak{S}_\alpha} q^{\text{maj}(\pi)} = \frac{[n]!}{[\alpha_1]! \cdots [\alpha_k]!}.$$

Proof. Let $P_i$ denote the chain of rank $\alpha_i$ for $i = 1, \ldots, k$. Furthermore, label all the cover relations in $P_i$ with $i$. Let $L$ denote the distributive lattice $P_1 \times P_2 \times \cdots \times P_k$. Furthermore, let $L$ inherit an R-labeling from its factors, that is, if $x = (x_1, x_2, \ldots, x_k) \prec (y_1, y_2, \ldots, y_k) = y$ let the label $\lambda(x, y)$ be the unique coordinate $i$ such that $x_i < y_i$. Observe that the Jordan–Hölder set of $L$ is $\mathfrak{S}_\alpha$. Direct computation yields $\Psi(P_i) = a^{\alpha_i-1}$, so the Major MacMahon map is $\Theta(\Psi(P_i)) = 1$. Iterating Theorem 5.2 evaluates the Major MacMahon map on $L$:

$$\sum_{\pi \in \mathfrak{S}_\alpha} q^{\text{maj}(\pi)} = \Theta \left( \sum_{\pi \in \mathfrak{S}_\alpha} u(\pi) \right) = \Theta(\Psi(L)) = \left[\frac{n}{\alpha}\right].$$

For a vector $\mathbf{r} = (r_1, r_2, \ldots, r_n)$ of positive integers let an $r$-signed permutation be a list $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n+1}) = ((j_1, \pi_1), (j_2, \pi_2), \ldots, (j_n, \pi_n), 0)$ such that $\pi_1 \pi_2 \cdots \pi_n$ is a permutation in the symmetric group $\mathfrak{S}_n$ and the sign $j_i$ is from the set $S_{\pi_i} = \{-1\} \cup \{2, \ldots, r_{\pi_i}\}$. On the set of labels $\Lambda = \{(j, i) : 1 \leq i \leq n, j \in S_i\} \cup \{0\}$ we use the lexicographic order with the extra condition that $0 < (j, i)$ if and only if $0 < j$. Denote the set of $r$-signed permutations by $\mathfrak{S}_n^r$. The descent set of an $r$-signed permutation $\sigma$ is the set $\text{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and the major index is defined as $\text{maj}(\sigma) = \sum_{i \in \text{Des}(\sigma)} i$. Similar to Corollary 7.2 we have the following result.
Corollary 7.3. The distribution of the major index for $r$-signed permutations is given by

$$\sum_{\sigma \in S_n^r} q^{maj(\sigma)} = [n]! \cdot \prod_{i=1}^{n} (1 + (r_i - 1) \cdot q).$$

Proof. The proof is the same as Corollary 7.2 except we replace the chains with the posets $P_i$ in Figure 1. Note that $\Psi(P_i) = a + (r_i - 1) \cdot b$. Let $L$ be the lattice $L = P_1 \circ P_2 \circ \cdots \circ P_n$. Let $L$ inherit the labels of the cover relations from its factors with the extra condition that the cover relations attached to the maximal element receive the label 0. This is an $R$-labeling and the labels of the maximal chains are exactly the $r$-signed permutations.

For signed permutations, that is, $r = (2, 2, \ldots, 2)$, the above result follows from an identity due to Reiner [13, Equation (5)].

8 Concluding remarks

We suggest the following $q, t$-extension of the Major MacMahon map $\Theta$. Define $\Theta^{q, t} : \mathbb{Z}[a, b] \rightarrow \mathbb{Z}[q, t]$ by

$$\Theta^{q, t}(w) = \Theta(w) \cdot w_{a=1, b=q} = \prod_{i : u_i = b} q^i \cdot t,$$

for an $ab$-monomial $w = u_1 u_2 \cdots u_n$. Applying this map to the $ab$-index of the Boolean algebra yields one of the four types of $q$-Eulerian polynomials:

$$\Theta^{q, t}(\Psi(B_n)) = A^{maj, des}_n(q, t) = \sum_{\pi \in S_n} q^{maj(\pi)} t^{des(\pi)}.$$

The following identity has been attributed to Carlitz [4], but goes back to MacMahon [10] Volume 2, Chapter IV, §462,

$$\sum_{k \geq 0} [k + 1]^n \cdot t^k = \frac{A^{maj, des}_n(q, t)}{\prod_{j=0}^{n}(1 - t \cdot q^j)}.$$

For recent work on the $q$-Eulerian polynomials, see Shareshian and Wachs [14]. It is natural to ask if there is a poset approach to identity (8.2).

There are several different ways to extend the major index to signed permutations. Two of our favorites are [1, 18].
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