Global Gaussian Estimates for the Heat Kernel of Homogeneous Sums of Squares

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Received: 22 March 2020 / Accepted: 20 October 2021 / Published online: 8 November 2021
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Abstract
Let $\mathcal{H} = \sum_{j=1}^{m} X_j^2 - \partial_t$ be a heat-type operator in $\mathbb{R}^{n+1}$, where $X = \{X_1, \ldots, X_m\}$ is a system of smooth Hörmander’s vector fields in $\mathbb{R}^n$, and every $X_j$ is homogeneous of degree 1 with respect to a family of non-isotropic dilations in $\mathbb{R}^n$, while no underlying group structure is assumed. In this paper we prove global (in space and time) upper and lower Gaussian estimates for the heat kernel $\Gamma(t, x; s, y)$ of $\mathcal{H}$, in terms of the Carnot-Carathéodory distance induced by $X$ on $\mathbb{R}^n$, as well as global upper Gaussian estimates for the $t$- or $X$-derivatives of any order of $\Gamma$. From the Gaussian bounds we derive the unique solvability of the Cauchy problem for a possibly unbounded continuous initial datum satisfying exponential growth at infinity. Also, we study the solvability of the $\mathcal{H}$-Dirichlet problem on an arbitrary bounded domain. Finally, we establish a global scale-invariant Harnack inequality for non-negative solutions of $\mathcal{H}u = 0$.

Keywords
Heat kernel · Gaussian estimates · Homogeneous Hörmander vector fields · Carnot–Carathéodory spaces · Cauchy problem · Harnack inequality

Mathematics Subject Classification (2010) 35K65 · 35K08 · 35H10 (primary); 35C15 · 35R03 · 35K15

1 Introduction

Let us consider a family $X = \{X_1, \ldots, X_m\}$ of smooth Hörmander’s vector fields in $\mathbb{R}^n$ (precise definitions will be given later). The study of the corresponding heat-type operator

$\mathcal{H} := \sum_{j=1}^{m} X_j^2 - \partial_t$ on $\mathbb{R}^{n+1}$
and its fundamental solution (heat kernel) has a long history and, by now, a vast literature. The study of operators of the kind ‘sum of squares of Hörmander’s vector fields’, \( \mathcal{L} = \sum_{j=1}^{m} X_j^2 \), as well as their evolutive counterpart, \( \mathcal{H} = \mathcal{L} - \partial_t \), is usually characterized by the following dichotomy:

- \textbf{local} properties of Hörmander operators of the kind \( \mathcal{L} \) or \( \mathcal{H} \) have been established for \textit{general} families of Hörmander’s vector fields \( X_1, \ldots, X_m \) (some cornerstones in this context are [17–19, 28, 30, 32]), while

- \textbf{global} properties of \( \mathcal{L} \) or \( \mathcal{H} \) have been established almost exclusively when the vector fields \( X_1, \ldots, X_m \) are left invariant on some Lie group.

In particular, starting with the famous paper [14] by Folland, a rich theory exists under the assumption that \( X_1, \ldots, X_m \) be both left invariant with respect to some group of translations, and homogeneous with respect to some family of dilations (hence, \( X_1, \ldots, X_m \) are the generators of a Carnot group \( \mathbb{G} \) in \( \mathbb{R}^n \)). In that context, the heat kernel has the form

\[
\Gamma(t, x; s, y) = \gamma(y^{-1} * x, t - s)
\]  

with \( \gamma \) satisfying a two-sided Gaussian bound:

\[
\frac{1}{Ct^{Q/2}} \exp \left( -\frac{C \|x\|^2}{t} \right) \leq \gamma(x, t) \leq \frac{C}{t^{Q/2}} \cdot \exp \left( -\frac{\|x\|^2}{Ct} \right)
\]

for every \( x \in \mathbb{G}, t > 0 \). Here \( Q \) is the homogeneous dimension of the group, and \( \| \cdot \| \) is a homogeneous norm in \( \mathbb{G} \). Analogous upper bounds hold for the derivatives of every order:

\[
|\partial_t^m X_I \gamma(x, t)| \leq \frac{C}{t^{(Q+|I|+2m)/2}} \cdot \exp \left( -\frac{\|x\|^2}{Ct} \right)
\]

where \( X_I = X_{i_1} X_{i_2} \ldots X_{i_k} \) with \( i_1, \ldots, i_k \in \{1, 2, \ldots, m\} \), and \( |I| = k \). The above Gaussian bounds on Carnot groups are a special case of the more general results proved for heat kernels corresponding to left invariant, but not necessarily homogeneous, Hörmander’s vector fields, by Varopoulos, Saloff-Coste, Coulhon in [36]. They proved that for heat kernels on \textit{nilpotent} Lie groups, a context where one still has Eq. 1.1, the function \( \gamma \) satisfies a two-sided bound

\[
\frac{1}{c|B_X(0, \sqrt{t})|} \exp \left( -\frac{cd_X^2(x, 0)}{t} \right) \leq \gamma(x, t) \leq \frac{c}{|B_X(0, \sqrt{t})|} \exp \left( -\frac{d_X^2(x, 0)}{ct} \right),
\]  

(1.2)

and an upper bound on derivatives of every order:

\[
|\partial_t^m X_I \gamma(x, t)| \leq \frac{C}{|B_X(0, \sqrt{t})|} \cdot \exp \left( -\frac{d_X^2(x, 0)}{ct} \right),
\]  

(1.3)

where \( d_X \) is the control distance induced by \( X_1, \ldots, X_m \) and \( B_X(0, r) \) the corresponding balls (see [36, Thm. IV.4.2, Thm. IV.4.3]). Also, they proved that on \textit{unimodular Lie groups with polynomial volume growth}, that is satisfying

\[
c_1 t^D \leq |B_X(0, \sqrt{t})| \leq c_2 t^D \quad \text{for } t \geq 1
\]

and some \( D > 0 \), the above results Eqs. 1.2 and 1.3 still hold (see [36], Thm. VIII.2.7, Thm. 8.2.9). For a different approach to Gaussian estimates in the context of Lie groups with polynomial growth, see also the monograph [13] by Dungey, ter Elst, Robinson. For the special case of Gaussian estimates on Carnot groups, that we will explicitly exploit in this paper, we refer to the more recent paper [10] by Bonfiglioli, Lanconelli, Uguzzoni.
For a general system of Hörmander’s vector fields, i.e., with no underlying group structure, Gaussian bounds for the heat kernel
\[ \Gamma(t, x; s, y) = \gamma(t - s, x, y) \]
have been proved by Jerison-Sanchez Calle [19, Thms. 2, 3, 4] in the form:
\[
\frac{1}{C|B_X(x, \sqrt{t})|} \exp\left( - \frac{C d^2_X(x, y)}{t} \right) \leq \gamma(t, x, y) \leq \frac{C}{|B_X(x, \sqrt{t})|} \exp\left( - \frac{d^2_X(x, y)}{C t} \right) \tag{1.4}
\]
for every multiindices \( I, J \), with \( x, y \) ranging in a compact set and \( t \in (0, T) \). Using probabilistic techniques, Kusuoka-Stroock have extended the above results to \( x, y \) in \( \mathbb{R}^N \) and \( t \in (0, T) \), in [21], and later to \( x, y \) in \( \mathbb{R}^N \) and \( t > 0 \) in [22]. However, Kusuoka-Stroock require that the coefficients of the vector fields belong to \( C^\infty_b(\mathbb{R}^N) \). For instance, vector fields with polynomial coefficients are not covered by their theory (at least as far as global results are concerned). Related results (under the same \( C^\infty_b(\mathbb{R}^N) \) assumptions on the vector fields) have been proved by Léandre in [25, 26]. Davies in [12] has improved the constant in the exponent of the upper bound in Eq. 1.4, for a system of Hörmander’s vector fields on a compact manifold.

On the other hand, a general setting which allows to develop an interesting global theory, without assuming the existence of a group of translations, and allowing unboundedness of the coefficients of \( X_1, \ldots, X_m \) and their derivatives, is that of Hörmander vector fields which are only assumed to be 1-homogeneous with respect to a family of non-isotropic dilations of the form
\[ \delta_\lambda(x) := (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n), \]
where \( 1 = \sigma_1 \leq \ldots \leq \sigma_n \) are positive integers. In other words,
\[ X_j(u \circ \delta_\lambda) = (X_j u) \circ \delta_\lambda \]
for every \( j = 1, \ldots, m \), every \( u \in C^\infty(\mathbb{R}^n) \) and every \( \lambda > 0 \). Under this assumption (without any underlying group structure), Biagi-Bonfiglioli in [2] have built a global homogeneous fundamental solution for \( \mathcal{L} = \sum_{j=1}^m X_j^2 \) and have studied some of its properties. The idea of this construction is that, according to a procedure originally devised by Folland in [15] and adapted in [2], a system of 1-homogeneous Hörmander’s vector fields can always be lifted to a higher dimensional Carnot group where the corresponding sum of squares is known to possess a global, left invariant, homogeneous fundamental solution. Saturating this fundamental solution with respect to the added variables, in [2] a homogeneous fundamental solution for the original operator is produced. More explicit estimates for this kernel have been established in [5], in terms of the distance induced by the vector fields. The general strategy of [2] has been later implemented in [4] for heat operators corresponding to 1-homogeneous vector fields, showing the existence of a global, homogeneous, heat kernel, obtained by saturating the heat kernel of a higher dimensional operator living on a Carnot group.

The aim of this paper is to prove sharp global explicit Gaussian estimates for this heat kernel, in terms of the intrinsic distance induced by the vector fields. More precisely, we will prove Gaussian estimates Eqs. 1.4–1.5 for every \( x, y \in \mathbb{R}^n \) and \( t > 0 \), for heat
operators corresponding to 1-homogeneous (but not left invariant) Hörmander’s vector fields (see Theorem 2.4).

Our global Gaussian bounds in particular allow to improve known results about the Cauchy problem for this heat operator. In [4, Thm. 4.1] it is proved that for every bounded continuous initial datum \( f \) there exists one and only one bounded solution to the Cauchy problem. We will prove that a solution to the Cauchy problem actually exists, at least for small times, as soon as the initial datum \( f \) satisfies a growth condition of the kind

\[
\int_{\mathbb{R}^n} |f(y)| \exp \left( -\mu d_X^2(y, 0) \right) \, dy < +\infty
\]

for some constant \( \mu > 0 \). The solution is unique in the class of functions satisfying a condition

\[
\int_0^T \int_{\mathbb{R}^n} \exp \left( -\delta d_X^2(x, 0) \right) |u(t, x)| \, dt \, dx < +\infty
\]

for some \( \delta > 0 \). Moreover, if \( f \) satisfies a stronger bound of the kind

\[
\int_{\mathbb{R}^n} |f(y)| \exp \left( -\mu d_X^{\alpha}(y, 0) \right) \, dy < +\infty \quad \text{for some } \alpha \in (0, 2),
\]

then the solution exists for all \( t > 0 \) (see Theorem 6.2 and Proposition 6.5). In Section 7 we shall present an application of our global Gaussian estimates to the study of the \( \mathcal{H} \)-Dirichlet problem. In fact, by crucially exploiting these estimates, we shall show that it is possible to apply to our operators \( \mathcal{H} \) the axiomatic approach developed in the series of papers [20, 23, 24, 35]; this will lead to some necessary and sufficient conditions for the regularity of boundary points of any bounded open set \( \Omega \). Finally, in the last part of the paper we will prove a scale-invariant parabolic Harnack inequality for non-negative solutions of \( \mathcal{H}u = 0 \) (see Theorem 8.1 in Section 8).

We close this introduction with a few remarks about some related fields of research. Gaussian bounds for heat kernels have been studied, besides the Euclidean setting, in the context of Riemannian manifolds. We can quote under this respect the well-known paper [27] by Li-Yau where Gaussian bounds are proved on manifolds with nonnegative curvature (see also the monograph [16] by Grigor’yan and the references therein). Some extensions of these geometric techniques to sub-Riemannian manifolds have been done, see e.g. the paper [7] by Baudoin, Bonnefont, Garofalo. Gaussian bounds have been studied also in the abstract context of Dirichlet forms, see e.g. the papers [33, 34] by Sturm. These researches have made apparent a general relation existing between the validity of Gaussian bounds for the heat kernel, the validity of global forms of Poincaré’s inequality and doubling condition, and the validity of a parabolic Harnack inequality. For a discussion of these general relations see also the monograph [31] by Saloff-Coste. In the context of homogeneous Hörmander vector fields studied in the present paper, global forms of Poincaré’s inequality and doubling condition are known, after [5]. Therefore, our results about Gaussian bounds and Harnack inequality are not unexpected. Nevertheless, we have not been able to find in the literature a precise theorem, directly applicable to our context, implying our results. As far as we know, this is the first case of global (in space and time) Gaussian estimates explicitly proved, for both the heat kernel and its derivatives of every order, in the context of Hörmander’s vector fields (with possibly unbounded coefficients) in absence of an underlying group structure.
2 Assumptions and Statements of Gaussian Bounds

We denote by $\mathfrak{X}(\mathbb{R}^n)$ the Lie algebra of the smooth vector fields on $\mathbb{R}^n$ (with $n \geq 2$). Given a set $X \subseteq \mathfrak{X}(\mathbb{R}^n)$, we indicate by $\text{Lie}(X)$ the smallest Lie sub-algebra of $\mathfrak{X}(\mathbb{R}^n)$ containing $X$. Finally, if $Z \in \mathfrak{X}(\mathbb{R}^n)$ is a smooth vector field of the form

$$Z = \sum_{j=1}^{n} a_j(x) \frac{\partial}{\partial x_j}$$

for some $a_1, \ldots, a_n \in C^\infty(\mathbb{R}^n)$ and if $x \in \mathbb{R}^n$, we denote by $Z(x)$ the vector $(a_1(x), \ldots, a_n(x)) \in \mathbb{R}^n$.

**Assumptions 2.1** Let $X = \{X_1, \ldots, X_m\}$ (with $m \geq 2$) be a fixed family of linearly independent smooth vector fields in Euclidean space $\mathbb{R}^n$ satisfying the following structural assumptions:

(H1) there exists a family $\{\delta_\lambda\}_{\lambda > 0}$ of non-isotropic dilations of the form

$$\delta_\lambda(x) := (\lambda^{\sigma_1}x_1, \ldots, \lambda^{\sigma_n}x_n), \quad (2.1)$$

where $1 = \sigma_1 \leq \ldots \leq \sigma_n$ are positive integers, with respect to which $X_1, \ldots, X_m$ are homogeneous of degree 1. This means that

$$X_j(u \circ \delta_\lambda) = (X_j u) \circ \delta_\lambda \quad (2.2)$$

for every $j = 1, \ldots, m$, every $u \in C^\infty(\mathbb{R}^n)$ and every $\lambda > 0$. We define the $\delta_\lambda$-homogeneous dimension of $\mathbb{R}^n$ as

$$q := \sum_{j=1}^{m} \sigma_j. \quad (2.3)$$

Note that $q \geq n$.

(H2) $X_1, \ldots, X_m$ satisfy Hörmander’s rank condition at $x = 0$, that is,

$$\dim \{Y(0) : Y \in \text{Lie}(X)\} = n. \quad (2.4)$$

**Remark 2.2** By combining assumptions (H1) and (H2), it is not difficult to recognize that Hörmander’s rank condition is actually satisfied at every point $x \in \mathbb{R}^n$, that is,

$$\dim \{Y(x) : Y \in \text{Lie}(X)\} = n \quad \text{for all } x \in \mathbb{R}^n$$

(this is proved in [5, Remark 3.2]). Thus, by Hörmander’s Hypoellipticity Theorem (see [17]), both the operators $L$ and $\mathcal{F}$ are $C^\infty$-hypoelliptic in every open subset of $\mathbb{R}^n$.

In order to state our result, we first recall the following standard

**Definition 2.3** (Carnot-Carathéodory distance) Let $Y = \{Y_1, \ldots, Y_h\}$ be a family of smooth vector fields defined on some space $\mathbb{R}^k$. We assume that the $Y_j$’s satisfy Hörmander’s rank condition at every point of $\mathbb{R}^k$. The Carnot-Carathéodory (CC, shortly) distance associated with $Y$ is

$$d_Y(x, y) = \inf \{r > 0 : \text{there exists } \gamma \in C(r) \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\},$$

where $C(r)$ is the set of the absolutely continuous curves $\gamma : [0, 1] \to \mathbb{R}^k$ satisfying (a.e. on $[0, 1]$)

$$\gamma'(t) = \sum_{j=1}^{h} a_j(t) Y_j(\gamma(t)), \quad \text{with } |a_j(t)| \leq r \text{ for all } j = 1, \ldots, h.$$
We will denote by $B_Y(x, \rho)$ the metric ball $\{y \in \mathbb{R}^k : d_Y(x, y) < \rho\}$.

Well-known results assure that under the above assumptions $d_Y(x, y)$ is finite for every couple of points in $\mathbb{R}^k$ and that $(\mathbb{R}^k, d_Y)$ is a metric space; moreover, $d_Y$ is topologically, but not metrically, equivalent to the Euclidean distance.

We can now state our main result:

**Theorem 2.4** Let $X = \{X_1, \ldots, X_m\}$ be a family of smooth vector fields in $\mathbb{R}^n$ satisfying Assumptions 2.1, and let $\mathcal{H}$ the heat-type operator

$$\mathcal{H} := \mathcal{L} - \partial_t = \sum_{j=1}^m X_j^2 - \partial_t \quad \text{on } \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n.$$ (2.5)

Moreover, let $\gamma(t, x, y) := \gamma(t - s, x, y)$ be the global heat kernel of $\mathcal{H}$, that will be precisely defined in Eq. 3.10. Then, the following facts hold.

(i) There exists a constant $\varrho > 1$ such that

$$\frac{1}{e |B_x(x, \sqrt{t})|} \exp \left( - \frac{\varrho d_Y^2(x, y)}{t} \right) \leq \gamma(t, x, y) \leq \frac{1}{e |B_x(x, \sqrt{t})|} \exp \left( - \frac{d_Y^2(x, y)}{\varrho t} \right),$$ (2.6)

for every $x, y \in \mathbb{R}^n$ and every $t > 0$.

(ii) For any nonnegative integers $k, r$ there exists $C = C_{k, r} > 0$ such that

$$\left| \left( \frac{\partial}{\partial t} \right)^k \cdots Y_r \gamma(t, x, y) \right| \leq C \frac{t^{-(k+r)/2}}{|B_x(x, \sqrt{t})|} \exp \left( - \frac{d_Y^2(x, y)}{Ct} \right),$$ (2.7)

for every choice of vector fields $Y_1, \ldots, Y_r \in \{X_1^x, \ldots, X_m^x, X_1^y, \ldots, X_m^y\}$, and every choice of $x, y \in \mathbb{R}^n$, $t > 0$.

The results about the Cauchy problem for $\mathcal{H}$ will be stated and proved in Section 6, while our scale-invariant Harnack inequality will be stated and proved in Section 8.

**Example** We conclude this section by presenting some concrete examples of smooth vector fields $X_1, \ldots, X_m \in \mathcal{X}(...)$ which satisfy our structural assumptions (H1)-(H2) (see also [5, 6]).

1. Let $p \in \mathbb{N}$ be fixed. In Euclidean space $\mathbb{R}^2$, we consider the vector fields

$$X_1 := \partial_{x_1}, \quad X_2 := x_1^p \partial_{x_2}.$$ 

Then, the family $X := \{X_1, X_2\}$ is linearly independent in $\mathcal{X}(\mathbb{R}^2)$ and satisfies assumption (H1); moreover, since $X_1, X_2$ are homogeneous of degree 1 with respect to the dilations

$$\delta_{\lambda}(x) := (\lambda x_1, \lambda^{p+1} x_2),$$

also assumption (H2) is fulfilled. Here,

$$\mathcal{H} = \mathcal{L} - \partial_t = \partial_{x_1}^2 + x_1^{2p} \partial_{x_2}^2 - \partial_t.$$ 

The operator $\mathcal{L} = X_1^2 + X_2^2$ is the so-called Bauendi-Grushin operator (of step $p$).
2. Let \( k, n_1, n_2 \in \mathbb{N} \) be arbitrarily fixed, and let \( n := n_1 + n_2 \geq 2 \). We denote a generic point \( x \in \mathbb{R}^n \equiv \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) by \( x = (y, z) \), where \( y \in \mathbb{R}^{n_1} \) and \( z \in \mathbb{R}^{n_2} \), and we define
\[
Y_i := \partial y_i \quad (i = 1, \ldots, n_1);
\]
\[
Z_{i,j} := y^k_i \partial z_j \quad (i = 1, \ldots, n_1 \text{ and } j = 1, \ldots, n_2).
\]
Then, it is not difficult to recognize that the family
\[
X := \{ Y_i, Z_{i,j} : i = 1, \ldots, n_1 \text{ and } j = 1, \ldots, n_2 \}
\]
is linearly independent in \( \mathcal{X}(\mathbb{R}^n) \) and satisfies assumption (H1); moreover, since the vector fields in \( X \) are homogeneous of degree 1 with respect to the dilations
\[
\delta_\lambda(x) = (\lambda y, \lambda^{k+1} z),
\]
also assumption (H2) is fulfilled. In this case, we have
\[
\mathcal{H} = L - \partial_t = \Delta_y + \left(y_{k1}^2 + \cdots + y_{kn_1}^2\right) \Delta_z - \partial_t.
\]

3. In Euclidean space \( \mathbb{R}^n \), with \( n \geq 2 \), we consider the vector fields
\[
X_1 := \partial x_1, \quad X_2 := x_1 \partial x_2 + \frac{x_1^2}{2!} \partial x_3 + \cdots + \frac{x_1^{n-1}}{(n-1)!} \partial x_n.
\]
Then, the family \( X := \{ X_1, X_2 \} \) is linearly independent in \( \mathcal{X}(\mathbb{R}^n) \) and satisfies assumption (H1); moreover, since \( X_1, X_2 \) are homogeneous of degree 1 with respect to the dilations
\[
\delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^n x_n),
\]
also assumption (H2) is fulfilled. Here, we have
\[
\mathcal{H} = L - \partial_t = \partial_{x_1}^2 + \left(x_1 \partial x_2 + \cdots + \frac{x_1^{n-1}}{(n-1)!} \partial x_n\right)^2 - \partial_t.
\]

3 Preliminaries and Known Results

3.1 Carnot Groups, Lifting and Construction of the Heat Kernel for \( \mathcal{H} \)

We begin by recalling the definition of homogeneous Carnot group and some related notions (see, e.g., [9] for an exhaustive treatment of this topic).

We say that \( \mathbb{G} = (\mathbb{R}^N, *, D_\lambda) \) is a homogeneous group if \( (\mathbb{R}^N, *) \) is a Lie group (with group identity \( e = 0 \)) and if there exists a one-parameter family of group automorphisms \( \{ D_\lambda \}_{\lambda > 0} \) acting as in (H1). We shall call the Lie group operation \( * \) ‘translation’ and the automorphisms \( D_\lambda \) ‘dilations’.

We say that a smooth vector field \( X \) is left invariant if, for every \( f \in C^\infty(\mathbb{R}^N) \), we have
\[
X \left( x \mapsto f(y * x) \right) = (Xf)(y * x) \quad \text{for all } x, y \in \mathbb{G}.
\]
For \( i = 1, 2, \ldots, n \), let \( X_i \) be the only left invariant vector field which agrees at the origin with \( \partial x_i \). Assume that for some positive integer \( m < N \) we have that \( X_1, \ldots, X_m \) are \( 1 \)-homogeneous (in the sense of Eq. 2.2) and that \( X_1, \ldots, X_m \) satisfy Hörmander’s condition as in (H2) (at the origin and then, by left invariance, at every point). Then we say that
\[
\mathbb{G} \text{ is a Carnot group and } X_1, \ldots, X_m \text{ are its generators.}
\]
A continuous function $\| \cdot \| : \mathbb{G} \to [0, +\infty)$ is called a homogeneous norm on $\mathbb{G}$ if there exists $c > 0$ such that, for every $u, v \in \mathbb{G}$, the following hold:

(i) $\|u\| = 0$ if and only if $u = 0$;
(ii) $\|D_\lambda(u)\| = \lambda \|u\|$ for every $\lambda > 0$;
(iii) $\|u * v\| \leq c(\|u\| + \|v\|)$;
(iv) $\|u^{-1}\| \leq c \|u\|$.

If $X = \{X_1, \ldots, X_m\}$ are the generators of a Carnot group $\mathbb{G}$ and $d_X$ the Carnot-Carathéodory distance associated with $X$, then $\|u\| = d_X(u, 0)$ is a homogeneous norm on $\mathbb{G}$, further satisfying properties (iii)-(iv) with $c = 1$.

A key information for the study of the operator $H$ (and of its associated heat kernel) is the dimension of the Lie algebra $\mathfrak{a} := \text{Lie}(X).$ Under our assumptions (H1)-(H2), it is easy to see that $\mathfrak{a}$ has finite dimension: in fact, using [3, Theorem A.11] and [9, Proposition 1.3.10]), one has

$$\mathfrak{a} = \bigoplus_{k=1}^{\sigma_n} \mathfrak{a}_k$$

where $\mathfrak{a}_1 := \text{span}(X) = \text{span}\{X_1, \ldots, X_m\}$ and

$$\mathfrak{a}_k := \text{span}\{[Y, Z] : Y \in \mathfrak{a}_1, Z \in \mathfrak{a}_{k-1}\} \quad (\text{for } k \geq 2).$$

In particular, we obtain

$$N = \dim(\mathfrak{a}) \geq \dim\{Y(0) : Y \in \mathfrak{a}\} = n.$$  \hfill (3.1)

As a consequence of Eq. 3.1, only the following two cases can occur.

(i) $N = n$. In this case, by taking into account the $\delta_\lambda$-homogeneity of $X_1, \ldots, X_m$, we can apply some results in [8], ensuring the existence of an operation $\ast$ on $\mathbb{R}^n$ such that

$$\mathbb{F} = (\mathbb{R}^n, \ast, \delta_\lambda)$$

is a homogeneous Carnot group with Lie($\mathbb{F}$) = $\mathfrak{a}$.

Hence, the vector fields $X_1, \ldots, X_m$ are left invariant on $\mathbb{F}$, and the operator $\mathcal{H}$ becomes the canonical heat operator on $\mathbb{R} \times \mathbb{F}$. This is a well-studied scenario, in which all the results of this paper are well-known (see, for example, [10]).

(ii) $N > n$. In this case, instead, we derive from [1, Theorem 1.4] that there cannot exist any Lie-group structure in $\mathbb{R}^n$ with respect to which $X_1, \ldots, X_m$ are left invariant. In particular, the operator $\mathcal{H}$ is not a canonical heat operator on some Carnot group.

In view of the above discussion, throughout the sequel, in the proof of our results, we also make the following ‘dimensional’ assumption.

(H3) Using the notation $\mathfrak{a} = \text{Lie}(X)$ and $N = \dim(\mathfrak{a})$, we assume that

$$p := N - n \geq 1.$$  \hfill (3.2)

**Remark 3.1** Note that condition (H3) is not a further assumption that we require in order for our results to be true. It is a further condition that is not restrictive to assume within the proofs, because if our Assumptions 2.1 hold and (H3) is not true, then our Theorem 2.4 is already known. Note that all the families $X$ considered in the examples at the end of Section 2 also satisfy (H3).

Even if assumption (H3) implies that $X_1, \ldots, X_m$ cannot be left invariant with respect to any Lie-group structure in $\mathbb{R}^n$, it is proved in [2] that the $X_j$’s can be lifted (in a suitable
sense) to vector fields $Z_1, \ldots, Z_m$ which are left invariant on a higher-dimensional Carnot group:

**Theorem 3.2** (Lifting, see [2, Theorem 3.1]) Let us suppose that assumptions (H1)-to-(H3) are satisfied. Then, it is possible to construct a homogeneous Carnot group $G = (\mathbb{R}^N, \ast, D_\lambda)$ satisfying the following properties:

1. $G$ has $m$ generators;
2. denoting the points of $\mathbb{R}^N$ as $u = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^p$, the family of dilations $\{D_\lambda\}_{\lambda > 0}$ takes the following 'lifted' form:
   \[ D_\lambda(u) = D_\lambda(x, \xi) = (\delta_\lambda(x), \delta_\lambda^*(\xi)), \]
   where $\delta_\lambda^*(\xi) = (\lambda^{\tau_1} \xi_1, \ldots, \lambda^{\tau_p} \xi_p)$ for some integers $1 \leq \tau_1 \leq \ldots \leq \tau_p$;
3. there exists a system of Lie-generators $Z = \{Z_1, \ldots, Z_m\}$ of $\text{Lie}(G)$ s.t.
   \[ Z_j(x, \xi) = X_j(x) + R_j(x, \xi), \]
   where the $R_j$’s are smooth vector fields operating only in the variables $\xi \in \mathbb{R}^p$, but with coefficient possibly depending on $(x, \xi)$. In particular, $R_1, \ldots, R_m$ are $D_\lambda$-homogeneous of degree 1.

**Notation 3.3** Throughout the paper, we will handle points in the ‘original’ space $\mathbb{R}^n$, and points in the ‘lifted’ space $\mathbb{R}^N$, according to Theorem 3.2. To this end, we shall use the notation

- $x, y, z, \ldots$ for points in $\mathbb{R}^n$;
- $u = (x, \xi), v = (y, \eta), \ldots$ for points in $\mathbb{R}^N \equiv \mathbb{R}^n \times \mathbb{R}^p$,

denoting by Greek letters the added variables in the lifting procedure. The scalar time variables will be denoted by letters $t, s, \tau$. Moreover, we shall indicate by $d_X$ and $d_Z$ the Carnot-Carathéodory distances associated with $X$ and $Z$, respectively, and with $B_X(x, \rho), B_Z(u, \rho)$ the $d_X$-ball, $d_Z$-ball, respectively, with centre $x \in \mathbb{R}^n, u \in \mathbb{R}^N$, and radius $\rho > 0$.

Since the lifted vector fields $Z_1, \ldots, Z_m$ in Theorem 3.2 are left invariant on $G$, many properties of $\mathcal{H}_G$ and its associated heat kernel are well-known. In fact, the following theorem holds.

**Theorem 3.4** ([10, Theorems 2.1, 2.5]) There exists a function

\[ \gamma_G : \mathbb{R}^{1+N} \to \mathbb{R}, \]

smooth away from the origin, such that

\[ \Gamma_G(t, u; s, v) := \gamma_G(t - s, v^{-1} * u) \]

is the global heat kernel of $\mathcal{H}_G = \mathcal{L}_G - \partial_t$; this means, precisely, that

- for every fixed $(t, z) \in \mathbb{R}^{1+N}$, one has $\Gamma_G(t, z; \cdot) \in L^1_{\text{loc}}(\mathbb{R}^{1+N})$;
- for every $\varphi \in C_0^\infty(\mathbb{R}^{1+N})$ and every $(t, u) \in \mathbb{R}^{1+N}$, one has

\[ \mathcal{H}_G \left( \int_{\mathbb{R}^{1+N}} \Gamma_G(t, u; s, v) \varphi(s, v) \, ds \, dv \right) = \int_{\mathbb{R}^{1+N}} \Gamma_G(t, u; s, v) \mathcal{H}_G \varphi(s, v) \, ds \, dv = -\varphi(t, u). \]
Furthermore, \( \gamma_G \) satisfies the following properties:

(i) \( \gamma_G \geq 0 \) and \( \gamma_G(t, u) = 0 \) if and only if \( t \leq 0 \);

(ii) \( \gamma_G(t, u) = \gamma_G(t, u^{-1}) \) for every \( (t, u) \in \mathbb{R}^{1+N} \);

(iii) for every \( \lambda > 0 \) and every \( (t, u) \), we have

\[
\gamma_G(\lambda^2 t, D_x(u)) = \lambda^{-Q} \gamma_G(t, u),
\]

where \( Q \) is the homogeneous dimension of the group \( G \), that is,
\[
Q := q + q^*, \quad \text{with } q \text{ as in Eq. 2.3 and } q^* := \sum_{k=1}^p \tau_k; \tag{3.6}
\]

(iv) \( \gamma_G \) vanishes at infinity, that is, \( \gamma_G(t, u) \to 0 \) as \( |(t, u)| \to +\infty \);

(v) for every \( t > 0 \), we have

\[
\int_{\mathbb{R}^N} \gamma_G(t, u) \, du = 1.
\]

Finally, the following Gaussian estimates for \( \gamma_G \) hold:

(a) there exists a constant \( c \geq 1 \), only depending on \( G \) and \( Z \), s.t.

\[
c^{-1} t^{-Q/2} \exp \left( -\frac{c \|u\|_2^2}{t} \right) \leq \gamma_G(t, u) \leq c t^{-Q/2} \exp \left( -\frac{\|u\|_2^2}{c t} \right), \tag{3.7}
\]

for every \( u \in \mathbb{R}^N \) and every \( t > 0 \).

(b) for every nonnegative integers \( h, k \) there exists a constant \( \tilde{c} > 0 \) s.t.

\[
\left| \partial^{i_1} \cdots \partial^{i_h} \left( \frac{\partial}{\partial t} \right)^k \gamma_G(t, u) \right| \leq \tilde{c} t^{-(Q+h+2k)/2} \exp \left( -\frac{\|u\|_2^2}{\tilde{c} t} \right) \tag{3.8}
\]

for any \( u \in \mathbb{R}^N \), any \( t > 0 \) and every choice of \( i_1, \ldots, i_h \in \{1, \ldots, m\} \).

Now, the ‘lifting property’ Eq. 3.4 contained in Theorem 3.2 easily implies that

\[
\mathcal{H} \left( t, x \mapsto u(t, \pi(x)) \right) = (\mathcal{H} u)(t, \pi(x)), \quad \text{for all } u \in C^2(\mathbb{R}^n) \tag{3.9}
\]

where \( \pi \) is the projection of \( \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p \) on \( \mathbb{R}^n \). By combining Eq. 3.9 with Theorem 3.4, it is proved in [4] the following result.

**Theorem 3.5** ([4, Theorem 1.4]) Let \( X = \{X_1, \ldots, X_m\} \) be a set of smooth vector fields on \( \mathbb{R}^n \) satisfying axioms (H1)-to-(H3), and let \( \mathcal{H} \) be the heat-type operator defined in Eq. 2.5. Moreover, let \( G = (\mathbb{R}^N, \ast, D_x) \) and \( Z = \{Z_1, \ldots, Z_m\} \) be as in Theorem 3.2.

Then, if \( \gamma_G \) is as in Theorem 3.4, the following facts hold.

(i) The function \( \Gamma \) defined by

\[
\Gamma(t, x; s, y) := \gamma(t - s, x, y) := \int_{\mathbb{R}^p} \gamma_G(t - s, (y, 0)^{-1} \ast (x, \eta)) \, d\eta,
\]

is the global heat kernel of \( \mathcal{H} \). This means, precisely, that

(i)_1 for any fixed \( (t, x) \in \mathbb{R}^{1+n} \), we have \( \Gamma(t, x; \cdot) \in L^1_{\text{loc}}(\mathbb{R}^{1+n}) \);

(i)_2 for every \( \varphi \in C_0^\infty(\mathbb{R}^{1+n}) \) and every \( (t, x) \in \mathbb{R}^{1+n} \), we have

\[
\mathcal{H} \left( \int_{\mathbb{R}^{1+n}} \gamma(t - s, x, y) \varphi(s, y) \, ds \, dy \right) = \int_{\mathbb{R}^{1+n}} \gamma(t - s, x, y) \mathcal{H} \varphi(s, y) \, ds \, dy = -\varphi(t, x).
\]
There exists a constant $c \geq 1$ such that
\[
e^{-1} t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{c \| (y, 0)^{-1} * (x, \eta) \|^2}{t} \right) d\eta \leq \gamma(t, x, y)
\]
\[
\leq c t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\| (y, 0)^{-1} * (x, \eta) \|^2}{c t} \right) d\eta,
\]
for every $x, y \in \mathbb{R}^n$ and every $t > 0$. \hfill (3.11)

- $\gamma \geq 0$ and $\gamma(t, x, y) = 0$ if and only if $t \leq 0$. \hfill (iii)

- $\gamma$ is symmetric in the space variables, i.e. $\gamma(t, x, y) = \gamma(t, y, x)$ for every $x, y \in \mathbb{R}^n$ and every $t > 0$. \hfill (iv)

- $\Gamma$ is smooth out of the diagonal of $\mathbb{R}^{1+n} \times \mathbb{R}^{1+n}$. \hfill (v)

- For every fixed $(t, x) \in \mathbb{R}^{1+n}$, with $t > 0$, we have
\[
\int_{\mathbb{R}^n} \gamma(t, x, y) dy = 1.
\]

- If $\varphi \in C^0_b(\mathbb{R}^n)$, then the function
\[
u(t, x) := \int_{\mathbb{R}^n} \gamma(t, x, y) \varphi(y) dy
\]
defined for $(t, x) \in \Omega = (0, +\infty) \times \mathbb{R}^n$ is the unique bounded classical solution of the homogeneous Cauchy problem for $\mathcal{H}$, that is,
\[
\begin{cases}
\mathcal{H} u = 0 & \text{in } \Omega \\
u(0, x) = \varphi(x) & \text{for } x \in \mathbb{R}^n.
\end{cases}
\]

- The function $\Gamma^\ast(t, u; s, v) = \Gamma(s, v; t, u)$ is the global heat kernel of the (formal) adjoint operator $\mathcal{H}^\ast := \mathcal{L} + \partial_t$, and satisfies dual statements with respect to (i). \hfill (viii)

In the above theorem, $\| \cdot \|$ is any homogeneous norm on $\mathcal{G}$.

Remark 3.6 Points (ii) and (iv) in the above theorem also imply that, with the same constant $c \geq 1$ as in (ii), for all $x, y \in \mathbb{R}^n$ and any $t > 0$ one has
\[
e^{-1} t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{c \| (x, 0)^{-1} * (y, \eta) \|^2}{t} \right) d\eta \leq \gamma(t, x, y)
\]
\[
\leq c t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\| (x, 0)^{-1} * (y, \eta) \|^2}{c t} \right) d\eta,
\]
(with the switched roles of $x, y$ in the Gaussians). It will be sometimes convenient to use Eq. 3.11 in this alternative form.

### 3.2 Review of Known Results on the CC Distance

Throughout the sequel, we will handle two distinct families of Hörmander’s vector fields, each one inducing a Carnot-Carathéodory distance:

- the original family of vector fields $X_1, \ldots, X_m$, defined in $\mathbb{R}^n$, and satisfying (H1)-to-(H3);
the lifted vector fields $Z_1, \ldots, Z_m$, defined on the higher dimensional Carnot group $\mathbb{G}$ in $\mathbb{R}^N$.

Both the $X_i$’s and the $Z_i$’s are 1-homogeneous with respect to suitable dilations, which implies some properties of the distances and the corresponding balls. The $Z_i$’s are also left invariant, which implies more properties for the corresponding distance. Finally, the $Z_i$’s are a lifting of the $X_i$’s. The next proposition collects the basic properties which follow from these facts.

**Proposition 3.7** With the previous notation and assumptions about the systems of vector fields $X$ and $Z$, the following properties hold.

(i) Homogeneity:
\[
d_X(\delta_\lambda(x), \delta_\lambda(y)) = \lambda d_X(x, y) \quad \text{for all } x, y \in \mathbb{R}^n \text{ and } \lambda > 0
\]
\[
d_Z(D_\lambda(u), D_\lambda(v)) = \lambda d_Z(u, v) \quad \text{for all } u, v \in \mathbb{R}^N \text{ and } \lambda > 0
\]
\[
\delta_\lambda(B_X(x, \rho)) = B_X(\delta_\lambda(x), \lambda \rho) \quad \text{for all } x \in \mathbb{R}^n \text{ and } \lambda, \rho > 0
\]
\[
D_\lambda(B_Z(u, \rho)) = B_Z(D_\lambda(u), \lambda \rho) \quad \text{for all } u \in \mathbb{R}^N \text{ and } \lambda, \rho > 0
\]

(ii) Left invariance:
\[
d_Z(u, v) = d_Z(u \ast w, v \ast w) \quad \text{for all } u, v, w \in \mathbb{R}^N
\]
\[
u \ast B_Z(v, \rho) = B_Z(u \ast v, \rho) \quad \text{for all } u, v \in \mathbb{R}^N \text{ and } \rho > 0
\]

(iii) Projection:
\[
d_X(x, y) \leq d_Z((x, \xi), (y, \eta)) \quad \text{for all } (x, \xi), (y, \eta) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p
\]
\[
\pi(B_Z((x, \xi), \rho)) = B_X(x, \rho) \quad \text{for all } (x, \xi) \in \mathbb{R}^N \text{ and } \rho > 0
\]

where $\pi$ is the projection from $\mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^p$ into $\mathbb{R}^n$. In particular, since $\pi$ is surjective, the last equality in (iii) means that
\[
\forall \ y \in B_X(x, \rho), \ \xi \in \mathbb{R}^p \ \exists \ \eta \in \mathbb{R}^p \ s.t. \ (y, \eta) \in B_Z(x, \xi, \rho). \quad (3.13)
\]

(iv) Volume of $Z$-balls: setting $\omega_Q = |B_Z(0, 1)|$, we have
\[
|B_Z(u, \rho)| = |B_Z(0, \rho)| = \omega_Q \rho^Q \quad \text{for all } u \in \mathbb{R}^N \text{ and } \rho > 0. \quad (3.14)
\]

(v) Homogeneous norm: if we let
\[
\|u\| = d_Z(u, 0) \quad \text{for every } u \in \mathbb{R}^N,
\]
then $\| \cdot \|$ is a homogeneous norm, and we also have
\[
d_Z(u, v) = \|v^{-1} \ast u\| = \|u^{-1} \ast v\| \quad \text{for every } u, v \in \mathbb{R}^N.
\]

Throughout the following, the symbol $\| \cdot \|$ in $\mathbb{R}^N$ will always denote this special norm.

The proof of Proposition 3.7 can be found in [2], or is immediate.

A much deeper result describes the volume of $X$-balls. The following theorem specializes a celebrated result by Nagel, Stein and Wainger [28] to the case of our 1-homogeneous vector fields $X$ (for a proof see [5, Theorem B]):
Theorem 3.8 Let \( X = \{X_1, \ldots, X_m\} \), \( n \) and \( q \) be as before. Then, there exist constants \( \gamma_1, \gamma_2 > 0 \) such that, for every \( x \in \mathbb{R}^n \) and every \( \rho > 0 \), one has the estimates
\[
\gamma_1 \sum_{j=1}^{q} f_j(x) \rho^j \leq |B_X(x, \rho)| \leq \gamma_2 \sum_{j=1}^{q} f_j(x) \rho^j.
\] (3.15)

Here, the functions \( f_k, \ldots, f_q : \mathbb{R}^n \to \mathbb{R} \) satisfy the following properties:

1. \( f_k, \ldots, f_q \) are continuous and non-negative on \( \mathbb{R}^n \);
2. for every \( j \in \{n, \ldots, q\} \), the function \( f_j \) is \( \delta \lambda \)-homogeneous of degree \( q - j \).

Remark 3.9 From estimate Eq. 3.15 it can be easily derived the following notable fact: for any \( x \in \mathbb{R}^n \) and any \( 0 < r < \rho \), one has
\[
\gamma_1 \left( \frac{\rho}{r} \right)^n \leq \frac{|B_X(x, \rho)|}{|B_X(x, r)|} \leq \gamma_2 \left( \frac{\rho}{r} \right)^q.
\] (3.16)

In particular, the following global doubling property holds:
\[
|B_X(x, 2\rho)| \leq 2^q \gamma_2 |B_X(x, \rho)| \quad \text{for all } x \in \mathbb{R}^n \text{ and } \rho > 0. \quad (3.17)
\]

Together with the global doubling property Eq. 3.17, it is possible to prove (see [5]) also the following global Poincaré inequality on \( d_X \)-balls (which, however, we will not use in the following): there exists a constant \( C_p > 0 \) such that, for every \( x \in \mathbb{R}^n \), \( r > 0 \) and every \( u \in C^1(\overline{B}(x, 2r)) \), one has
\[
\int_{B_X(x, r)} \left| u(y) - u_{B_X(x, r)} \right| \, dy \leq C_p r \int_{B_X(x, 2r)} \sqrt{\sum_{i=1}^{m} |X_i u|^2} \, dy,
\]
where \( u_{B_X(x, r)} \) is the integral mean of \( u \) on \( B_X(x, r) \).

Inequalities Eqs. 3.16–3.17 easily imply that the function
\[
\frac{1}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{t} \right)
\]
(which plays a key role in our estimates) is not so asymmetric in \( x, y \) as could seem. More precisely, we have the following proposition.

Proposition 3.10 For every \( \theta > 0 \) there exists a constant \( C_1 > 0 \) such that
\[
\frac{1}{|B_X(y, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{\theta t} \right) \leq C_1 \frac{1}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{C_1 \theta t} \right),
\] (3.18)
for every \( x, y \in \mathbb{R}^n \) and every \( t > 0 \).

Proof To prove this, let us distinguish two cases.

• If \( d_X(x, y) \leq \sqrt{t} \), we infer from Eq. 3.17 that \( |B_X(y, \sqrt{t})| \) and \( |B_X(x, \sqrt{t})| \) are equivalent, and thus the above inequality holds.
If \( d_X(x, y) > \sqrt{t} \), then by Eqs. 3.16 and 3.17 we have

\[
\frac{1}{|B_X(y, \sqrt{t})|} \leq \frac{\gamma_2}{|B_X(y, d_X(x, y))|} \cdot \left( \frac{d_X(x, y)}{\sqrt{t}} \right)^q
\]

\[
\leq \frac{2^q (\gamma_2)^2}{|B_X(x, d_X(x, y))|} \cdot \left( \frac{d_X(x, y)}{\sqrt{t}} \right)^q
\]

\[
\leq \frac{2^q (\gamma_2)^2}{\gamma_1} \cdot \frac{1}{|B_X(x, \sqrt{t})|} \cdot \left( \frac{d_X(x, y)}{\sqrt{t}} \right)^{q - n}.
\]

From this, we readily obtain Eq. 3.18 (see, e.g., Eq. 4.2).

This ends the proof.

Another deep known result that will play a key role in our estimates is the following ‘global’ version of a well-known result by Sanchéz-Calle [32] (see also [28, Lemma 3.2] and [18]), which compares the volumes of \( B_X(x, \rho) \) and \( B_Z((x, \xi), \rho) \). For a proof of this result see [5, Theorem C].

**Theorem 3.11** Under the previous assumptions and notation, there exist constants \( \kappa \in (0, 1) \) and \( c_1, c_2 > 0 \) such that, for every \( x \in \mathbb{R}^n \), every \( \xi \in \mathbb{R}^p \) and every \( \rho > 0 \) one has the estimates:

\[
\left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((x, \xi), \rho) \} \right| \leq c_1 \frac{|B_Z((x, \xi), \rho)|}{|B_X(x, \rho)|}, \quad \forall y \in \mathbb{R}^n, \quad (3.19)
\]

\[
\left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((x, \xi), \rho) \} \right| \geq c_2 \frac{|B_Z((x, \xi), \rho)|}{|B_X(x, \rho)|}, \quad \forall y \in B_X(x, \kappa \rho). \quad (3.20)
\]

We wish to stress that Theorems 3.8 and 3.11 contain global results, adapted to our context of homogeneous vector fields. In contrast with this, the original versions of these results, contained in [28, 32], and related to general systems of Hörmander’s vector fields, express local results.

### 4 Gaussian Estimates for \( \Gamma \)

The aim of this section is to prove upper/lower Gaussian estimates for the global heat kernel \( \Gamma(t, x; s, y) \) of \( \mathcal{H} \) (or, equivalently, for \( \gamma(t, x, y) \) as defined in Eq. 3.10). Broadly put, our approach is the following: on account of Eq. 3.12, we already know that \( \Gamma \) satisfies the ‘quasi-Gaussian’ estimates

\[
\gamma(t, x, y) \approx t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\| (x, 0)^{-1} * (y, \eta) \|^2}{t} \right) d\eta,
\]

where \( Q \) is as in Eq. 3.6; we then derive ‘pure’ Gaussian estimates for \( \Gamma \) by showing that, for any \( x, y \in \mathbb{R}^n \) and any \( t > 0 \), one has

\[
t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\| (x, 0)^{-1} * (y, \eta) \|^2}{t} \right) d\eta \approx \frac{1}{|B_X(x, \sqrt{t})|} \cdot \exp \left( - \frac{d_X(x, y)^2}{t} \right). \quad (4.1)
\]

To begin with, for a future reference, we state the following lemma.
Lemma 4.1 The following estimates hold true:

(i) for every $\nu > 0$ and $\delta \in (0, 1)$ there exists $c > 0$ such that
$$\tau^\nu e^{-\tau^2} \leq c e^{-\delta \tau^2} \text{ for every } \tau \geq 0; \quad (4.2)$$

(ii) for every positive $\nu, \theta$ there exists $c > 0$ such that
$$\tau^{-\nu} \geq c e^{-\theta \tau^2} \text{ for every } \tau > 0. \quad (4.3)$$

We then proceed by proving Eq. 4.1, and we start with the upper estimate.

Proposition 4.2 There exists a constant $\kappa > 1$ such that
$$t^{-Q/2} \int_{\mathbb{R}^p} \exp \left(-\frac{\| (x, 0) - 1 \ast (y, \eta) \|^2}{t} \right) d\eta \leq \frac{\kappa}{|B_X(x, \sqrt{t})|} \exp \left(-\frac{d_X^2(x, y)}{2t} \right), \quad (4.4)$$
for every $x, y \in \mathbb{R}^n$ and every $t > 0$.

Proof Let $x, y \in \mathbb{R}^n$ be arbitrarily fixed, and let $t > 0$.

CASE I: $d_X(x, y) > \sqrt{t}$. In this case, for every $n = 0, 1, 2, \ldots$, we define
$$A_n := \{ \eta \in \mathbb{R}^p : 2^n d_X(x, y) \leq \| (x, 0) - 1 \ast (y, \eta) \| < 2^{n+1} d_X(x, y) \}, \quad (4.5)$$
and we observe that, by Proposition 3.7-(iii), it holds $\mathbb{R}^p = \bigcup_{n \geq 0} A_n$. Hence,

$$\int_{\mathbb{R}^p} \exp \left(-\frac{\| (x, 0) - 1 \ast (y, \eta) \|^2}{t} \right) d\eta = \sum_{n=0}^{+\infty} \int_{A_n} \exp \left(-\frac{\| (x, 0) - 1 \ast (y, \eta) \|^2}{t} \right) d\eta$$

$$\leq \sum_{n=0}^{+\infty} \exp \left(-\frac{2^n d_X^2(x, y)}{t} \right) \cdot |A_n|$$

$$\leq \sum_{n=0}^{+\infty} \exp \left(-\frac{2^n d_X^2(x, y)}{t} \right) \cdot \left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_{B_X((x, 0), 2^{n+1} d_X(x, y))} \} \right|$$

$$=:(\star).$$

Next, by combining Theorem 3.11 and Eq. 3.14, for every $n \geq 0$ we have

$$\left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_{B_X((x, 0), 2^{n+1} d_X(x, y))} \} \right| \leq c_1 \frac{|B_{B_X((x, 0), 2^{n+1} d_X(x, y))}|}{|B_X(x, 2^{n+1} d_X(x, y))|}$$

$$= c_1 \omega_Q \frac{2^{Q(n+1)} d_X^Q(x, y)}{|B_X(x, 2^{n+1} d_X(x, y))|} \leq c_1 \omega_Q \frac{2^{Q(n+1)} d_X^Q(x, y)}{|B_X(x, d_X(x, y))|}$$

$$\leq c_1 \omega_Q \frac{2^{Q(n+1)} d_X^Q(x, y)}{|B_X(x, \sqrt{t})|}.$$
since $d_X(x, y) > \sqrt{t}$. As a consequence, we obtain

$$
(\star) \leq c_1 \omega_Q \sum_{n=0}^{+\infty} \exp \left( - \frac{2^n d_X^2(x, y)}{t} \right) \frac{2^Q(n+1) d_X^Q(x, y)}{|B_X(x, \sqrt{t})|}
$$

$$
= 2^Q c_1 \omega_Q \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} \sum_{n=0}^{+\infty} \left( \frac{2^n d_X(x, y)}{\sqrt{t}} \right)^Q \exp \left( - \frac{2^n d_X^2(x, y)}{t} \right)
$$

(by estimate Eq. 4.2, with $\nu = Q$ and, e.g., $\delta = 1/2$)

$$
\leq \frac{\alpha_Q t^{Q/2}}{|B_X(x, \sqrt{t})|} \sum_{n=0}^{+\infty} \exp \left( - \frac{2^n d_X^2(x, y)}{2t} \right) =: (\star\star),
$$

for some constant $\alpha_Q$ depending on $Q$. On the other hand, since we are assuming that $d_X(x, y) > \sqrt{t}$, for any $n \geq 0$ we have

$$
\exp \left( - \frac{2^n d_X^2(x, y)}{2t} \right) = \exp \left( - \frac{d_X^2(x, y)}{2t} \right) \cdot \exp \left( - \frac{d_X^2(x, y)}{2} \cdot (2^n - 1) \right)
$$

$$
\leq \exp \left( - \frac{d_X^2(x, y)}{2t} \right) \cdot \exp \left( - \frac{2^n - 1}{2} \right),
$$

from which we derive that

$$
(\star\star) \leq \frac{\alpha_Q t^{Q/2}}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{2t} \right) \cdot \sum_{n=0}^{+\infty} \exp \left( - \frac{2^n - 1}{2} \right)
$$

$$
= \frac{\alpha_Q' t^{Q/2}}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{2t} \right).
$$

Finally, using this last estimate, we obtain

$$
t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\|x, 0\|^{-1} \ast (y, \eta)^2}{t} \right) d\eta \leq \frac{\alpha_Q'}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{2t} \right) \quad (4.6)
$$

which is precisely Eq. 4.4 (with $\kappa = \alpha_Q'$).

**CASE II:** $d_X(x, y) \leq \sqrt{t}$. First of all, for every non-negative integer $n$ we consider the set

$$
B_n := \{ \eta \in \mathbb{R}^p : 2^n \sqrt{t} \leq \|x, 0\|^{-1} \ast (y, \eta) \| < 2^{n+1} \sqrt{t} \}; \quad (4.7)
$$

moreover, we define

$$
B := \{ \eta \in \mathbb{R}^p : \|x, 0\|^{-1} \ast (y, \eta) \| < \sqrt{t} \}. \quad (4.8)
$$
Then we have:

\[
\int_{\mathbb{R}^p} \exp \left( -\frac{\| (x, 0)^{-1} \ast (y, \eta) \|^2}{t} \right) \, d\eta = \int_B \{ \ldots \} \, d\eta + \sum_{n=0}^{+\infty} \int_{B_n} \{ \ldots \} \, d\eta \\
\leq |B| + \sum_{n=0}^{+\infty} \exp \left( -2^{2n} \right) \cdot |B_n| \\
\leq \left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((x, 0), \sqrt{t}) \} \right| \\
+ \sum_{n=0}^{+\infty} \exp \left( -2^{2n} \right) \cdot \left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((x, 0), 2^{n+1} \sqrt{t}) \} \right| =: (\star). 
\]

Now, again by Theorem 3.11 and Eq. 3.14, for every \( n \geq 0 \) we have

\[
\left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((x, 0), 2^n \sqrt{t}) \} \right| \leq c_1 \frac{|B_Z((x, 0), 2^n \sqrt{t})|}{|B_X((x, 0), 2^n \sqrt{t})|}. 
\]

As a consequence, we obtain

\[
(\star) \leq c_1 \omega_Q \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} + c_1 \omega_Q \cdot \sum_{n=0}^{+\infty} \exp \left( -2^{2n} \right) \frac{2^{(n+1)Q/2}}{|B_X((x, 0), 2^n \sqrt{t})|} \\
= c_1 \omega_Q \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} \cdot \left( 1 + \sum_{n=0}^{+\infty} \exp \left( -2^{2n} \right) \frac{2^{Q(n+1)}}{|B_X((x, 0), 2^n \sqrt{t})|} \right) \\
= \beta_Q \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} = (\star\star). 
\]

On the other hand, since we are assuming that \( d_X(x, y) \leq \sqrt{t} \), we have

\[
\exp \left( -\frac{d_X^2(x, y)}{2t} \right) \geq e^{-1/2}, 
\]

from which we derive that

\[
(\star\star) \leq \beta_Q \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} \cdot \exp \left( -\frac{d_X^2(x, y)}{2t} \right). 
\]

Finally, using this last estimate, we obtain

\[
t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( -\frac{\| (x, 0)^{-1} \ast (y, \eta) \|^2}{t} \right) \, d\eta \leq \frac{\beta_Q}{|B_X(x, \sqrt{t})|} \cdot \exp \left( -\frac{d_X^2(x, y)}{2t} \right), \quad (4.9) 
\]

and this is again Eq. 4.4. Gathering Eqs. 4.6 and 4.9, we conclude that estimate Eq. 4.4 holds for every \( x, y \in \mathbb{R}^n \) and every \( t > 0 \) by choosing

\[
\kappa := \max\{\alpha'_1, \beta_Q\} > 1. 
\]

This ends the proof. 

\[ \square \]
In order to prove lower estimate of \( \Gamma \), we need the following property.

**Lemma 4.3** With the above notation and assumption, let \( b > a > 0 \) be fixed real numbers, and let \( x, y \in \mathbb{R}^n \) satisfying
\[
d_X(x, y) < a. \tag{4.10}
\]
Then, for every \( \xi \in \mathbb{R}^p \) there exists \( \eta = \eta_{x, y, \xi} \in \mathbb{R}^p \setminus \{0\} \) such that
\[
\{ \eta \in \mathbb{R}^p : a \leq d_Z((x, \xi), (y, \eta)) < b \} \supseteq \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((y, \eta), \frac{1}{2}(b - a)) \}. \tag{4.11}
\]

**Proof** Since \( y \in B_X(x, a) \), if \( \xi \in \mathbb{R}^p \) is arbitrarily fixed, by Eq. 3.13 there exists \( \eta_0 \in \mathbb{R}^p \) such that
\[
g(\lambda) = d_Z((x, \xi), (y, \delta^\lambda_{\xi}(\eta_0))) < a. \tag{4.12}
\]
In particular, since the set in the right-hand side of Eq. 4.12 is open, we can assume that \( \eta_0 = 0 \). We then consider the function
\[
g(\lambda) = d_Z((x, \xi), (y, \delta^\lambda_{\xi}(\eta_0))) \quad \text{where} \quad \delta^\lambda_{\xi}(\eta) = (\lambda \xi_1, \ldots, \lambda \xi_p \eta_p).
\]
Clearly, we have that \( g \) is continuous on the whole of \([1, +\infty)\); moreover, from Eq. 4.12 we infer that
\[
g(1) < a. \tag{4.13}
\]
We now claim that
\[
\lim_{\lambda \to +\infty} g(\lambda) = +\infty. \tag{4.14}
\]
To prove Eq. 4.14 we first notice that, by triangle’s inequality, we have
\[
g(\lambda) \geq d_Z((0, 0), (y, \delta^\lambda_{\xi}(\eta_0))) - d_Z((0, 0), (x, \xi)) \quad \text{(for all } \lambda \geq 1) ; \tag{4.15}
\]
moreover, since the vector fields \( Z_1, \ldots, Z_m \) are \( D_{\lambda} \)-homogeneous of degree 1, by Proposition 3.7-(i) we deduce that
\[
d_Z((0, 0), (y, \delta^\lambda_{\xi}(\eta_0))) = d_Z(0, 0), (x, \xi) \tag{4.16}
\]
Since \( y_{\lambda, \xi} = \delta_{1/\lambda}(y) \to 0 \in \mathbb{R}^n \) as \( \lambda \to +\infty \), and since \( \eta_0 \neq 0 \), we have
\[
\lim_{\lambda \to +\infty} d_Z((0, 0), (y_{\lambda, \xi}, \eta_0)) = d_Z((0, 0), (0, \eta_0)) > 0 ;
\]
as a consequence, taking the limit as \( \lambda \to +\infty \) in Eq. 4.16 we obtain
\[
\lim_{\lambda \to +\infty} d_Z((0, 0), (y, \delta^\lambda_{\xi}(\eta_0))) = +\infty. \tag{4.17}
\]
Gathering Eqs. 4.17 and 4.15, we obtain the claimed Eq. 4.14.

Next, using the continuity of \( g \), together with Eqs. 4.13 and 4.14, we infer the existence of a suitable \( \bar{\lambda} \in (1, +\infty) \) such that
\[
g(\bar{\lambda}) = d_Z((x, \xi), (y, \delta^\bar{\lambda}_{\xi}(\eta_0))) = \frac{b + a}{2}. \tag{4.18}
\]
Setting \( \bar{\eta} := \delta^\bar{\lambda}_{\xi}(\eta_0) \), we prove Eq. 4.11 by showing the stronger inclusion
\[
\{ z \in \mathbb{R}^N : a \leq d_Z((x, \xi), z) < b \} \supseteq B_Z((y, \bar{\eta}), \frac{1}{2}(b - a)). \tag{4.19}
\]
To this end, let $u \in B_Z((y, \eta), \frac{1}{2}(b-a))$ be fixed. On the one hand, we have
\[
d_Z((x, \xi), u) \leq d_Z((x, \xi), (y, \eta)) + d_Z((y, \eta), u) < \frac{b+a}{2} + \frac{b-a}{2} = b;
\]
on the other hand, since we also have
\[
d_Z((x, \xi), u) \geq d_Z((x, \xi), (y, \eta)) - d_Z(u, (y, \eta)) > \frac{b+a}{2} - \frac{b-a}{2} = a,
\]
we conclude that Eq. 4.19 holds. This ends the proof.

We can now prove the estimate from below in Eq. 4.1.

**Proposition 4.4** There exists a constant $\vartheta > 1$ such that
\[
t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( -\frac{\| (x,0)^{-1} * (y,\eta) \|^2}{t} \right) \, d\eta \geq \frac{1}{\vartheta |B_X(x, \sqrt{t})|} \exp \left( -\frac{\partial d_X^2(x, y)}{t} \right).
\]
for every $x, y \in \mathbb{R}^n$ and every $t > 0$.

**Proof** Let $x, y \in \mathbb{R}^n$ be arbitrarily fixed, and let $t > 0$.

**CASE I:** $d_X(x, y) > \sqrt{t}$. In this case, we consider the set
\[
A := \{ \eta \in \mathbb{R}^p : 2d_X(x, y) \leq \| (x,0)^{-1} * (y,\eta) \| < 4d_X(x, y) \}.
\]
By applying Lemma 4.3 (with $a := 2d_X(x, y)$ and $b := 2a$), one has
\[
A \supseteq \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((y, \eta), d_X(x, y)) \}
\]
(for a suitable $\eta = \eta_{x,y} \in \mathbb{R}^p \setminus \{0\}$); as a consequence, we obtain
\[
\int_{\mathbb{R}^p} \exp \left( -\frac{\| (x,0)^{-1} * (y,\eta) \|^2}{t} \right) \, d\eta \geq \int_A \exp \left( -\frac{\| (x,0)^{-1} * (y,\eta) \|^2}{t} \right) \, d\eta
\]
(since $\| (x,0)^{-1} * (y,\eta) \|^2 \leq 16 d_X^2(x, y)$ for $\eta \in A$)
\[
\geq \exp \left( -\frac{16 d_X^2(x, y)}{t} \right) \cdot |A|
\]
\[
\geq \exp \left( -\frac{16 d_X^2(x, y)}{t} \right) \cdot \left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((y, \eta), d_X(x, y)) \} \right| =: (\bigstar).
\]
On the other hand, by using Theorem 3.11 (with the choice $(x, \xi) = (y, \eta)$) and Eq. 3.14, we get
\[
\left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_Z((y, \eta), d_X(x, y)) \} \right| \geq c_2 \frac{B_Z((y, \eta), d_X(x, y))}{B_X(y, d_X(x, y))}
\]
\[
= c_2 \omega_Q \frac{d_X^Q(x, y)}{B_X(y, d_X(x, y))}
\]
(since we are assuming that $d_X(x, y) > \sqrt{t}$)
\[
> c_2 \omega_Q \frac{t^{Q/2}}{B_X(y, d_X(x, y))},
\]
from which we derive the estimate

\[(\star) \geq c_2 \omega Q \frac{t^{Q/2}}{B_X(y, d_X(x, y))} \cdot \exp \left( - \frac{16 d_X^2(x, y)}{t} \right) \]

(since \(B_X(y, d_X(x, y)) \subseteq B_X(x, 2d_X(x, y))\))

\[\geq c_2 \omega Q \frac{t^{Q/2}}{B_X(x, 2d_X(x, y))} \cdot \exp \left( - \frac{16 d_X^2(x, y)}{t} \right) =: (\star \star)\].

We now observe that, since \(X_1, \ldots, X_m\) are \(\delta\)-homogeneous of degree 1, and since we are assuming that \(d_X(x, y) > \sqrt{t}\), we can apply Eq. 3.16, getting

\[|B_X(x, 2d_X(x, y))| \leq \gamma_2 |B_X(x, \sqrt{t})| \cdot \left( \frac{2d_X(x, y)}{\sqrt{t}} \right)^q,\]

where \(q\) is as in Eq. 2.3. As a consequence, we deduce that

\[(\star \star) \geq c_2 \omega Q \frac{t^{Q/2}}{2^q \gamma_2 |B_X(x, \sqrt{t})|} \cdot \left( \frac{d_X(x, y)}{\sqrt{t}} \right)^{-q} \exp \left( - \frac{16 d_X^2(x, y)}{t} \right) \]

(by estimate Eq. 4.3, with \(v = q\) and, e.g., \(\theta = 4\))

\[\geq \frac{t^{Q/2}}{\alpha_{q, Q} |B_X(x, \sqrt{t})|} \exp \left( - \frac{20 d_X^2(x, y)}{t} \right),\]

for some constant \(\alpha_{q, Q}\) depending on \(q, Q\). Finally, by exploiting this last estimate, we obtain

\[t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( - \frac{\|(x, 0)^{-1} \ast (y, \eta)\|^2}{t} \right) d\eta \geq t^{-Q/2} \cdot \left[ \frac{t^{Q/2}}{\alpha_{q, Q} |B_X(x, \sqrt{t})|} \exp \left( - \frac{20 d_X^2(x, y)}{t} \right) \right] \]

(setting \(\vartheta_1 = \max(\alpha_{q, Q}, 20)\))

\[\geq \frac{1}{\vartheta_1 |B_X(x, \sqrt{t})|} \exp \left( - \frac{\vartheta_1 d_X^2(x, y)}{t} \right), \quad (4.22)\]

which exactly the desired Eq. 4.20 (with \(\vartheta = \vartheta_1 > 1\)).

**CASE II:** \(d_X(x, y) \leq \sqrt{t}\). The proof is similar to that of CASE I, letting now

\[A = \{ \eta \in \mathbb{R}^p : 2\sqrt{t} \leq \|(x, 0)^{-1} \ast (y, \eta)\| < 4\sqrt{t} \}.\]

Applying Lemma 4.3 (with \(a := 2\sqrt{t} > d_X(x, y)\) and \(b := 2a\), we get

\[A \supset \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_{\mathbb{Z}}((y, \eta), \sqrt{t}) \} \neq \emptyset\]

(for a suitable \(\overline{\eta} = \overline{\eta}_{x,y} \in \mathbb{R}^p \setminus \{0\}\)); as a consequence, we obtain

\[\int_{\mathbb{R}^p} \exp \left( - \frac{\|(x, 0)^{-1} \ast (y, \eta)\|^2}{t} \right) d\eta \geq \int_{A} \exp \left( - \frac{\|(x, 0)^{-1} \ast (y, \eta)\|^2}{t} \right) d\eta \geq e^{-16} \cdot |A| \geq e^{-16} \cdot \left| \{ \eta \in \mathbb{R}^p : (y, \eta) \in B_{\mathbb{Z}}((y, \overline{\eta}), \sqrt{t}) \} \right| =: (\star).\]
On the other hand, by using Theorem 3.11 (with the choice \((x, \xi) = (y, \eta)\)) and Eq. 3.14, we get
\[
\left\{ \eta \in \mathbb{R}^p : (y, \eta) \in B_{\mathbb{Z}}((y, \sqrt{t}), \sqrt{t}) \right\} \geq c_2 \frac{|B_{\mathbb{Z}}((y, \sqrt{t}), \sqrt{t})|}{|B_X(y, \sqrt{t})|}
\]
\[
= c_2 \omega_Q \frac{t^{Q/2}}{|B_X(y, \sqrt{t})|},
\]
from which we derive the estimate (remind that we are assuming \(d_X(x, y) \leq \sqrt{t}\))
\[
(\star) \geq c_2 \frac{t^{Q/2}}{e^{16}} \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|}
\]
\[
(\text{since } B_X(y, \sqrt{t}) \subseteq B_X(x, d_X(x, y) + \sqrt{t}) \subseteq B_X(x, 2\sqrt{t}) )
\]
\[
\geq c_2 \frac{t^{Q/2}}{e^{16}} \frac{t^{Q/2}}{|B_X(x, 2\sqrt{t})|} =: (\star\star).
\]
By Eq. 3.17, we have
\[
|B_X(x, 2\sqrt{t})| \leq \gamma_2 2^q |B_X(x, \sqrt{t})| \quad \text{(where } q \text{ is as in Eq. 2.3)};
\]
as a consequence, we deduce that
\[
(\star\star) \geq c_2 \frac{t^{Q/2}}{2^q e^{16}} \frac{t^{Q/2}}{|B_X(x, \sqrt{t})|} \geq \frac{t^{Q/2}}{\beta_{q, Q} |B_X(x, \sqrt{t})|} \cdot \exp \left( -\frac{d_X^2(x, y)}{t} \right),
\]
for some constant \(\beta_{q, Q}\) depending on \(q, Q\). Using this last estimate, we get
\[
t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( -\frac{\| (x, 0)^{-1} \ast (y, \eta) \|^2}{t} \right) d\eta
\]
\[
\geq t^{-Q/2} \cdot \left[ \frac{t^{Q/2}}{\beta_{q, Q} |B_X(x, \sqrt{t})|} \cdot \exp \left( -\frac{d_X^2(x, y)}{t} \right) \right]
\]
\[
(\text{setting } \vartheta_2 := \max\{\beta_{q, Q}, 1\})
\]
\[
\geq \frac{1}{\vartheta_2 |B_X(x, \sqrt{t})|} \exp \left( -\frac{\vartheta_2 d_X^2(x, y)}{t} \right),
\]
(4.23)
and this is again the desired Eq. 4.20 (this time with \(\vartheta = \vartheta_2 \geq 1\)). Gathering Eqs. 4.22 and 4.23, we conclude that estimate Eq. 4.20 holds for every \(x, y \in \mathbb{R}^n\) and every \(t > 0\) by choosing
\[
\vartheta := \max\{\vartheta_1, \vartheta_2\} > 1.
\]
This ends the proof.

Thanks to Propositions 4.2 and 4.4, we can now prove Eq. 2.6 in Theorem 2.4.

**Proof of Theorem 2.4-(i)** For every \(x, y \in \mathbb{R}^n\) and every \(t > 0\), we set
\[
H(x, y, t) := t^{-Q/2} \int_{\mathbb{R}^p} \exp \left( -\frac{\| (x, 0)^{-1} \ast (y, \eta) \|^2}{t} \right) d\eta.
\]
On account of Eq. 3.12, we know that there exists a constant $c \geq 1$, only depending on $G$ and on $Z$ (which, in their turn, only depend on the set $X$), such that
\[
e^{-Q/2} H (x, y, c t) \leq \gamma(t, x, y) \leq e^{-Q/2} H (x, y, ct)
\] (4.24)
for every $x, y \in \mathbb{R}^n$ and every $t > 0$. These bounds, together with the preceding Propositions 4.2 and 4.4, immediately give Eq. 2.6.

5 Estimates for the Derivatives of $\Gamma$

The aim of this section is to establish (upper) Gaussian estimates for the space derivatives along $X_1, \ldots, X_m$ and for the ‘time derivatives’ of arbitrary order of $\gamma$, that is Theorem 2.4-(ii). To begin with, we state the following theorem proved in [4], which provides integral representations (analogous to formula Eq. 3.10) for any space/time derivative of $\gamma$.

**Theorem 5.1** (See [4, Theorem 3]) Under the previous assumption, and keeping the notation of Theorem 3.5, for any nonnegative integers $\alpha, h, k$ and any choice of indexes $i_1, \ldots, i_h, j_1, \ldots, j_k$ in $\{1, \ldots, m\}$, we have the following representation formulas
\[
\frac{\partial}{\partial t}^{\alpha} X_{i_1}^1 \cdots X_{i_h}^h \chi(t, x, y) = \int_{\mathbb{R}^\rho} \left( \frac{\partial}{\partial t}^{\alpha} Z_{i_1} \cdots Z_{i_h} \chi \circ G(t, y, 0) \right) (t, (y, 0)^{-1} (x, \eta)) \, d\eta;
\]
\[
\frac{\partial}{\partial t}^{\alpha} X_{j_1}^1 \cdots X_{j_k}^k \chi(t, x, y) = \int_{\mathbb{R}^\rho} \left( \frac{\partial}{\partial t}^{\alpha} Z_{j_1} \cdots Z_{j_k} \chi \circ G(t, x, 0) \right) (t, (x, 0)^{-1} (y, \eta)) \, d\eta;
\]
\[
\frac{\partial}{\partial t}^{\alpha} X_{j_1}^1 \cdots X_{j_k}^k X_{i_1}^1 \cdots X_{i_h}^h \chi(t, x, y) = \int_{\mathbb{R}^\rho} \left( \frac{\partial}{\partial t}^{\alpha} Z_{j_1} \cdots Z_{j_k} \left( (Z_{i_1} \cdots Z_{i_h} \chi) \circ \tau \right) \right) (t, (x, 0)^{-1} (y, \eta)) \, d\eta,
\]
holding true for every $(t, x) \neq (0, y) \in \mathbb{R}^{1+n}$. Here $\tau : \mathbb{R}^{1+N} \to \mathbb{R}^{1+N}$ is the map defined by
\[
\tau(t, u) = (t, u^{-1})
\]
and $u^{-1}$ is the inverse of $u$ in $G = (\mathbb{R}^N, \ast)$.

While the proof of our Gaussian estimates for the derivatives appearing in Eq. 5.2 and Eq. 5.1 is, by now, quite straightforward, for the mixed case in Eq. 5.3 it will require some extra work. We start establishing the following proposition, which will be useful for the case of mixed derivatives.

**Proposition 5.2** With the above notation, for any nonnegative integers $\alpha, h, k$ and any choice of indexes $i_1, \ldots, i_h, j_1, \ldots, j_k \in \{1, \ldots, m\}$, there exists $c_1, c_2 > 0$ such that
\[
\left| \left( \frac{\partial}{\partial t} \right)^\alpha Z_{j_1} \cdots Z_{j_k} \left( (Z_{i_1} \cdots Z_{i_h} \chi) \circ \tau \right) (t, u) \right| \leq c_1 t^{-(Q+2\alpha+h+k)/2} \exp \left( - \frac{\|u\|^2}{c_2 t} \right).
\]
for every $u \in \mathbb{G}$ and every $t > 0$.

In turn, Proposition 5.2 follows from two facts which are stated separately in the next two lemmas, since they may be of independent interest.

**Lemma 5.3** Let $Y$ be a 1-homogeneous (but not necessarily left invariant) smooth vector field on $\mathbb{G}$. Then, it is possible to find another 1-homogeneous smooth vector field $\tilde{Y}$ such that

$$Y(f \circ \iota) = (\tilde{Y} f) \circ \iota$$

for every $f \in C^\infty(\mathbb{R}^N)$, where $\iota(u) = u^{-1}$ is the inversion map on $\mathbb{G}$.

**Proof** First of all, let us write the dilations on $\mathbb{G}$ as follows:

$$D_\lambda(u_1, \ldots, u_N) = (\lambda^{\alpha_1} u_1, \ldots, \lambda^{\alpha_N} u_N) \quad \text{(for any } \lambda > 0 \text{ and } u \in \mathbb{G}).$$

Moreover, since $Y$ is 1-homogeneous, we write

$$Y = \sum_{j=1}^N b_j(u) \frac{\partial}{\partial u_j}$$

where $b_j(u)$ is a $(\alpha_j - 1)$-homogeneous polynomial function. Using the structure of the inversion map on homogeneous groups (see, e.g., [9, Corollary 1.3.16]), we know that the $k$-th component $\iota_k$ of the map $\iota$ is a $\alpha_k$-homogeneous polynomial function; as a consequence, for every $j = 1, \ldots, N$ we have that $\partial_{u_j} \iota_k$ is a $(\alpha_k - \alpha_j)$-homogeneous polynomial function. Therefore, we obtain

$$Y(f \circ \iota)(u) = \sum_{j=1}^N b_j(u) \sum_{k=1}^N \frac{\partial f}{\partial u_k}(\iota(u)) \frac{\partial \iota_k}{\partial u_j}(u)$$

$$= \sum_{k=1}^N \left( \sum_{j=1}^N b_j(u) \frac{\partial \iota_k}{\partial u_j}(u) \right) \frac{\partial f}{\partial u_k}(\iota(u)) = \sum_{k=1}^N c_k(u) \frac{\partial f}{\partial u_k}(\iota(u)).$$

We now claim that $c_k$ is a homogeneous polynomial function of degree $\alpha_k - 1$: in fact, taking into account the homogeneity of $b_j$ and of $\partial_{u_j} \iota_k$, we see that $b_j \partial_{u_j} \iota_k$ is homogeneous of degree $$(\alpha_j - 1) + (\alpha_k - \alpha_j) = \alpha_k - 1;$$

as a consequence, the function $c_k = \sum_{j=1}^N b_j \partial_{u_j} \iota_k$ is a homogeneous polynomial of degree $\alpha_k - 1$, as claimed. Bearing in mind this last information, we then define

$$\tilde{c}_k = c_k \circ \iota.$$

Since the dilations $D_\lambda$ are group automorphisms, we have $D_\lambda \circ \iota = \iota \circ D_\lambda$; from this, we get

$$\tilde{c}_k(D_\lambda(u)) = c_k(D_\lambda(\iota(u))) = \lambda^{\alpha_k - 1} c_k(\iota(u)) = \lambda^{\alpha_k - 1} \tilde{c}_k(u). \quad (5.4)$$

Thus, $\tilde{c}_k$ is $(\alpha_k - 1)$-homogeneous as well, and we can write

$$Y(f \circ \iota)(u) = \sum_{k=1}^N \tilde{c}_k(\iota(u)) \frac{\partial f}{\partial u_k}(\iota(u)) \equiv (\tilde{Y} f)(\iota(u)),$$

where $\tilde{Y} := \sum_{k=1}^N \tilde{c}_k(u) \partial_{u_k}$ is a 1-homogeneous vector field (in view of Eq. 5.4). \qed
Next, let us prove the following:

**Proposition 5.4** Let \( \alpha, r \) be nonnegative integers, and let \( Y_1, ..., Y_r \) be 1-homogeneous (but not necessarily left invariant) smooth vector fields on \( \mathbb{G} \). Then, there exist constants \( c_1, c_2 > 0 \) such that, for every \( u \in \mathbb{G} \) and every \( t > 0 \), the following Gaussian bound holds

\[
\left| \left( \frac{\partial}{\partial t} \right)^\alpha Y_1 \cdots Y_r \gamma_{\mathbb{G}}(t, u) \right| \leq c_1 t^{-\left( Q/2 + \alpha + r/2 \right)} \exp \left( -\frac{\|u\|^2}{c_2 t} \right).
\]

**Proof** If \( Y_1, ..., Y_r \) are 1-homogeneous and left invariant vector fields on \( \mathbb{G} \), this result is proved by [10, Theorem. 2.5] (see also Eq. 3.8 in Theorem 3.4). We are going to show that the result for left invariant 1-homogeneous vector fields easily implies our more general statement.

In fact, let \( X_1, ..., X_N \) be the canonical basis of \( \mathbb{G} \), i.e., \( X_i \) is the unique left invariant vector field on \( \mathbb{G} \) such that \( X_i(0) = \partial u_i \). Up to possibly reordering the \( X_i \)'s, we can assume that \( X_i \) is \( \alpha_i \)-homogeneous, with \( 1 = \alpha_1 = \ldots = \alpha_m < \alpha_{m+1} \leq \alpha_{m+2} \ldots \leq \alpha_N = s \), and \( s \) is the step of \( \mathbb{G} \) (that is, \( \text{Lie}(\mathbb{G}) \) is nilpotent of step \( s \)). Then, for homogeneity reasons, we have

\[
X_i = \partial u_i + \sum_{k=1}^{N \atop \alpha_k > \alpha_i} b_{ik}(u) \partial u_k \quad \text{(for } i = 1, 2, ..., N),
\]

where \( b_{ik}(u) \) is a \((\alpha_k - \alpha_i)\)-homogeneous polynomial function. In particular, since \( X_N = \partial u_N \), we can solve the above system in \( \partial u_1, ..., \partial u_N \) using backward substitution, thus writing

\[
\partial u_i = X_i + \sum_{k=1}^{N \atop \alpha_k > \alpha_i} c_{ik}(u) X_k \quad \text{(for } i = 1, 2, ..., N), \tag{5.5}
\]

where \( c_{ik}(u) \) is a \((\alpha_k - \alpha_i)\)-homogeneous polynomial function.

Let now \( Y \) be a 1-homogeneous vector field. Owing to Eq. 5.5, we have

\[
Y = \sum_{i=1}^{N} \beta_i(u) \partial u_i = \sum_{i=1}^{N} \beta_i(u) \left( X_i + \sum_{k=1}^{N \atop \alpha_k > \alpha_i} c_{ik}(u) X_k \right) = \sum_{i=1}^{N} \gamma_i(u) X_i,
\]

where \( \gamma_i(u) \) is a \((\alpha_i - 1)\)-homogeneous polynomial function. Notice that, since \( X_1, ..., X_m \) are generators of \( \text{Lie}(\mathbb{G}) \), every \( X_i \) with \( i > m \) can be written as a linear combination (with constant coefficients) of commutators of \( X_1, ..., X_m \), of length \( \alpha_i \). Thus, since the Gaussian bound holds for left invariant vector fields (see Eq. 3.8 in Theorem 3.4), we obtain

\[
|Y \gamma_{\mathbb{G}}(t, u)| \leq \sum_{i=1}^{N} |\gamma_i(u)| \cdot |X_i \gamma_{\mathbb{G}}(t, u)| \leq \hat{c} \sum_{i=1}^{N} |\gamma_i(u)| \cdot t^{-\left( Q + \alpha_i \right)/2} \exp \left( -\frac{\|u\|^2}{\hat{c} t} \right)
\]

\[
\leq \kappa \sum_{i=1}^{N} \|u\|^{|\alpha_i|-1} \cdot t^{-\left( Q + \alpha_i \right)/2} \exp \left( -\frac{\|u\|^2}{\hat{c} t} \right)
\]

\[
= \kappa t^{-\left( Q + 1 \right)/2} \sum_{i=1}^{N} \left( \frac{\|u\|}{\sqrt{t}} \right)^{|\alpha_i|-1} \cdot \exp \left( -\frac{\|u\|^2}{\hat{c} t} \right)
\]

\[
\leq c_1 t^{-\left( Q + 1 \right)/2} \exp \left( -\frac{\|u\|^2}{c_2 t} \right).
\]
where the last inequality follows from Eq. 4.2. The general case then follows by iteration.

We are now ready to prove Proposition 5.2.

**Proof of Proposition 5.2.** By repeatedly applying Lemma 5.3, we can rewrite

\[
\left( \frac{\partial}{\partial t} \right)^{\alpha} Z_{j_1} \cdots Z_{j_k} \left( (Z_{i_1} \cdots Z_{i_h} \gamma) \circ \mathcal{T} \right) = \left\{ \left( \frac{\partial}{\partial t} \right)^{\alpha} \tilde{Z}_{j_1} \cdots \tilde{Z}_{j_k} (Z_{i_1} \cdots Z_{i_h} \gamma) \right\} \circ \mathcal{T}
\]

with \( \mathcal{T}(t, u) = (t, u^{-1}) \). Here the \( Z_i \)'s are 1-homogeneous and left invariant, whereas the \( \tilde{Z}_i \)'s are just 1-homogeneous. Anyhow, we can apply Proposition 5.4 and get the desired result.

With Proposition 5.2 in hand, we can prove the Gaussian estimates on the derivatives.

**Proof of Theorem 2.4-(ii)** We distinguish three different cases.

**CASE 1.** \( Y_1, \ldots, Y_r = X_{i_1}^x \cdots X_{i_r}^x \). Then, by Eqs. 5.1, 3.8 and Proposition 4.2 we have

\[
\left| \left( \frac{\partial}{\partial t} \right)^{\alpha} X_{i_1}^x \cdots X_{i_r}^x \gamma(t, x, y) \right| \leq \hat{c} t^{-(Q/2 + \alpha + r/2)} \int_{\mathbb{R}^p} \exp \left( - \frac{\|(y, 0)^{-1} * (x, \eta)\|}{\hat{c} t} \right) \, d\eta
\]

\[
\leq c t^{-(\alpha + r/2)} \frac{1}{|B_X(y, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{C t} \right).
\]

The assertion then follows by Remark 3.10.

**CASE 2.** \( Y_1, \ldots, Y_r = X_{j_1}^y \cdots X_{j_r}^y \). Then, by Eqs. 5.2, 3.8 and Proposition 4.2, we have

\[
\left| \left( \frac{\partial}{\partial t} \right)^{\alpha} X_{j_1}^y \cdots X_{j_r}^y \gamma(t, x, y) \right| \leq \hat{c} t^{-(Q/2 + \alpha + r/2)} \int_{\mathbb{R}^p} \exp \left( - \frac{\|(x, 0)^{-1} * (y, \eta)\|}{\hat{c} t} \right) \, d\eta
\]

\[
\leq c t^{-(\alpha + r/2)} \frac{1}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{C t} \right).
\]

**CASE 3.** \( Y_1, \ldots, Y_r = X_{j_1}^y \cdots X_{j_k}^y X_{i_1}^x \cdots X_{i_h}^x \) (with \( k + h = r \)). In this last case, by exploiting Eq. 5.3, Proposition 5.2 and again Proposition 4.2, we obtain

\[
\left| \left( \frac{\partial}{\partial t} \right)^{\alpha} X_{j_1}^y \cdots X_{j_k}^y X_{i_1}^x \cdots X_{i_h}^x \gamma(t, x, y) \right| \leq c_1 t^{-(Q/2 + \alpha + r/2)} \int_{\mathbb{R}^p} \exp \left( - \frac{\|(x, 0)^{-1} * (y, \eta)\|^2}{c_2 t} \right) \, d\eta
\]

\[
\leq c t^{-(\alpha + r/2)} \frac{1}{|B_X(x, \sqrt{t})|} \exp \left( - \frac{d_X^2(x, y)}{C t} \right).
\]

This ends the proof.
As anticipated in the Introduction, in this section we exploit the global Gaussian bounds of $\Gamma$ to study the unique solvability of the Cauchy problem for $\mathcal{H}$. More precisely, we extend the result proved in [4, Thm. 4.1], where the Cauchy problem is studied for bounded continuous initial data, to possibly unbounded continuous initial data, fastly growing at infinity.

As for the proof of Gaussian estimates, we will make Assumptions 2.1 and will also assume (H3) (see Section 3). As noted before, condition (H3) amounts to assuming that we are not in a Carnot group (see also Remark 6.4 after the proof of our result for some explanation on this point).

We start with the following

**Definition 6.1** Let $S_\tau := (0, \tau) \times \mathbb{R}^n$, for some fixed $\tau \in (0, +\infty]$. Given any function $f \in C(\mathbb{R}^n)$, we say that $u : S_\tau \rightarrow \mathbb{R}$ is a classical solution of the Cauchy problem

$$\begin{cases}
\mathcal{H}u = 0 & \text{in } S_\tau, \\
u(0, x) = f(x) & \text{for } x \in \mathbb{R}^n
\end{cases}$$

if it satisfies the following properties:

1. $u \in C^2(S_\tau)$ and $\mathcal{H}u = 0$ pointwise on $S_\tau$;
2. $\lim_{t \rightarrow 0^+} u(t, x) = f(x)$ for every fixed $x \in \mathbb{R}^n$.

Using the upper Gaussian estimates of $\Gamma$, we are able to prove that Eq. 6.1 admits (at least) one classical solution when the initial datum $f$ grows at most exponentially. In what follows, we set

$$\rho_X(x) := d_X(0, x) \quad (x \in \mathbb{R}^n).$$

**Theorem 6.2** There exists $T > 0$ such that, if $f \in C(\mathbb{R}^n)$ satisfies the growth condition

$$\int_{\mathbb{R}^n} |f(y)| \exp \left( -\mu \rho_X^2(y) \right) dy < +\infty \quad (6.2)$$

for some constant $\mu > 0$, then the function

$$u(t, x) := \int_{\mathbb{R}^n} \Gamma(t, x; 0, y) f(y) dy = \int_{\mathbb{R}^n} \gamma(t, x, y) f(y) dy \quad (6.3)$$

is a classical solution of Eq. 6.1 on the strip $S_T/\mu$.

Furthermore, it is possible to find constants $\tau, \delta > 0$ (depending on $\mu$) such that

$$\int_{S_\tau} \exp \left( -\delta \rho_X^2(x) \right) |u(t, x)| dt dx < +\infty. \quad (6.4)$$

Finally, if $u_1, u_2$ are two classical solutions of Eq. 6.1 with the same continuous initial datum $f$, and if $u_1, u_2$ satisfy condition Eq. 6.4 in two strips $S_{\tau_1}, S_{\tau_2}$, respectively, then

$$u_1 \equiv u_2 \text{ in } S_{\tau}, \quad \text{for } \tau = \min\{\tau_1, \tau_2\}.$$
Proof Since $\phi$ is bounded, it suffices to show that $\phi$ is integrable at infinity. To this end, if $\sigma_1, \ldots, \sigma_n$ are as in Eq. 2.1, we consider the homogeneous norm
\[ N(y) := \sum_{j=1}^{n} |y_j|^{1/\sigma_j} \quad (y \in \mathbb{R}^n), \]
and we prove that $\phi$ is integrable on the set $\mathcal{O} := \{ N \geq 1 \}$. Now, using Lemma 4.1, and taking into account that both $N$ and $\rho_X$ are $\delta_\lambda$-homogeneous of degree 1, we have
\[ \int_{\mathcal{O}} \phi(y) \, dy \leq c \int_{\rho_X^2 < 2R} \frac{1}{\rho_X^{2q}(y)} \, dy \cdot \sum_{k=0}^{+\infty} 1 < +\infty. \]
(performing the change of variable $y = \delta_{2k}(u)$)
This ends the proof. \qed

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2 Assume that $f \in C(\mathbb{R}^n)$ satisfies Eq. 6.2, and let $u$ be as in Eq. 6.3.

STEP I. Let us show that $u$ is well defined and solves Eq. 6.1 in some $\mathcal{S}_T$. To this end, let $R > 0$ be arbitrarily fixed, and let $\phi_R \in C^0(\mathbb{R}^n)$ satisfy the following properties
\begin{itemize}
  \item $\phi_R \equiv 1$ on $\{ \rho_X < R \}$;
  \item $\phi_R \equiv 0$ on $\{ \rho_X > 2R \}$;
  \item $0 \leq \phi_R \leq 1$ on $\mathbb{R}^n$.
\end{itemize}
Then, we can write
\[ u(t, x) = \int_{\mathbb{R}^n} \gamma(t, x, y) f(y) \phi_R(y) \, dy + \int_{\mathbb{R}^n} \gamma(t, x, y) f(y)(1 - \phi_R(y)) \, dy \equiv u_1(t, x) + u_2(t, x). \]
Since $f \phi_R$ is bounded continuous, by [4, Theorem 4.1] we know that $u_1$ is well defined for every $t > 0$, and it solves Eq. 6.1 with initial datum $f \phi_R$ on the whole of on $(0, +\infty) \times \mathbb{R}^n$. In particular, since $\phi_R \equiv 1$ on the set $\{ \rho_X < R \}$, for every $x \in \mathbb{R}^n$ with $\rho_X(x) < R$ we have
\[ \lim_{t \to 0^+} u_1(t, x) = (f \phi_R)(x) = f(x). \]
We now prove that there exists a suitable $T > 0$, independent of the chosen $R$, such that the following facts hold on the bounded stripe $\mathcal{S}_{T/\mu, R} := (0, T/\mu) \times \{ \rho_X < R \}$:

(i) $u_2$ is well defined; (ii) $u_2$ it solves the equation $\mathcal{F}(u) = 0$; (iii) $u_2(t, x) \to 0$ as $t \to 0^+$. As for (i)-(ii) we observe that, by the Gaussian estimate Eq. 2.6, we have
\[ |u_2(t, x)| \leq \frac{\Theta}{|B(x, \sqrt{t})|} \int_{\{\rho_X(y) > 2R\}} \exp \left(-\frac{d_X^2(x, y)}{\mu^2 t} + \mu \rho_X^2(y)\right) |f(y)| \, dy \]
\[ = \frac{\Theta}{|B(x, \sqrt{t})|} \int_{\{\rho_X(y) > 2R\}} \exp \left(-\frac{d_X^2(x, y)}{\mu^2 t} + \mu \rho_X^2(y)\right) |f(y)| \exp \left(-\mu \rho_X^2(y)\right) \, dy. \]
On the other hand, for every \( x, y \in \mathbb{R}^n \) satisfying \( \rho_X(x) < R \) and \( \rho_X(y) > 2R \), one has

\[
d_X(x, y) \geq \rho_X(y) - \rho_X(x) \geq \frac{\rho_X(y)}{2};
\]
as a consequence, we get

\[
\exp \left( -\frac{d^2_X(x, y)}{q t} + \mu \rho_X^2(y) \right) \leq \exp \left( -\rho_X^2(y) \left( \frac{1}{4qt} - \mu \right) \right) \leq 1, \quad (6.7)
\]
as soon as \( x \in \{ \rho_X < R \} \) and \( \frac{1}{4qt} - \mu > 0 \), that is (setting \( T_1 := 1/(4q) \))

\[
t < \frac{T_1}{\mu}.
\]

Gathering together all these facts, for fixed \((t, x) \in ST/\mu, R\) we obtain

\[
|u_2(t, x)| \leq c t, x \int_{\mathbb{R}^n} |f(y)| \exp \left( -\mu \rho_X^2(y) \right) dy < +\infty.
\]

Now, using the Gaussian estimates Eq. 2.7 for the derivatives of \( \gamma \), and arguing exactly as above, one can easily prove that \( u_2 \in C^2(ST/\mu, R) \) and \( \mathcal{H} u_2 = 0 \) on \( ST/\mu, R \), where

\[
T := \min \left\{ T_1, \frac{1}{4C} \right\} \quad \text{and} \quad C \text{ is as in Eq. 2.7}. \quad (6.8)
\]

Next, we show that for \( t \to 0^+ \) we have \( u_2(t, x) \to 0 \) if \( x \in \mathbb{R}^n \) satisfies \( \rho_X(x) < R \). To this end we first observe that, by Eq. 3.15, for every \( t > 0 \) and \( x \in \mathbb{R}^n \) we have

\[
|B_X(x, \sqrt{t})| \geq \gamma_1 \sum_{h=n}^{q} f_h(y) t^{h/2} \geq \gamma_1 f_q t^{q/2} = \kappa_q t^{q/2}, \quad (6.9)
\]
with \( \kappa_q := \gamma_1 f_q \) (remind that \( f_1, \ldots, f_q \geq 0 \) and \( f_q \) is a positive constant). As a consequence, by combining Eqs. 6.9, 6.6 and 6.7, for every \((t, x) \in ST/\mu, R\) we obtain the estimate

\[
|u_2(t, x)| \leq c \int_{\{ \rho_X(y) > 2R \}} e^{-\rho_X^2(y) \left( \frac{1}{4qt} - \mu \right)} \cdot |f(y)| \exp \left( -\mu \rho_X^2(y) \right) dy.
\]

We are going to show that, by Lebesgue’s theorem, the last integral goes to zero as \( t \to 0^+ \). On the one hand, for every fixed \( y \in \mathbb{R}^n \) with \( \rho_X(y) > 2R \), we have

\[
\lim_{t \to 0^+} \left( e^{-\rho_X^2(y) \left( \frac{1}{4qt} - \mu \right)} \cdot |f(y)| e^{-\mu \rho_X^2(y)} \right) \leq c_y \cdot \lim_{t \to 0^+} \frac{e^{-4R^2 \left( \frac{1}{4qt} - \mu \right)}}{t^{q/2}} = 0.
\]

On the other hand, for every \( t > 0 \) and every \( y \in \mathbb{R}^n \) satisfying \( \rho_X(y) > 2R \), one has

\[
e^{-\rho_X^2(y) \left( \frac{1}{4qt} - \mu \right)} \cdot |f(y)| e^{-\mu \rho_X^2(y)} \leq \frac{e^{-4R^2 \left( \frac{1}{4qt} - \mu \right)}}{t^{q/2}} \cdot |f(y)| e^{-\mu \rho_X^2(y)} = \sup_{t > 0} \left( \frac{e^{-4R^2 \left( \frac{1}{4qt} - \mu \right)}}{t^{q/2}} \right) \cdot |f(y)| e^{-\mu \rho_X^2(y)} \equiv c |f(y)| e^{-\mu \rho_X^2(y)} \in L^1(\mathbb{R}^n),
\]
and thus, by Lebesgue’s theorem, we conclude that

\[
\lim_{t \to 0^+} u_2(t, x) = 0 \quad \text{for every} \ x \in \mathbb{R}^n \text{ with} \ \rho_X(x) < R.
\]

Summing up, we have proved that \( u_2 \) satisfies (i)-to-(iii) on \( ST/\mu, R \), as desired.
Finally, due to the arbitrariness of \( R > 0 \), we then conclude that \( u \) is a classical solution of problem Eq. 6.1 on the stripe \( ST/\mu \) (with \( T \) as in Eq. 6.8).

**STEP II.** Let us show that \( u \) satisfies a bound as in Eq. 6.4 for some \( \delta, \tau > 0 \). Since \( u \) is a continuous function on the stripe \( ST/\mu \), the integral

\[
\int_0^\tau \int_{\{|\rho_X(x)\leq R\}} |u(t, x)| \exp\left(-\delta \rho_X^2(x)\right) dt \, dx
\]

is finite for every choice of \( \delta, R > 0 \) and every \( 0 < \tau < T/\mu \). So, it is enough to show that there exist suitable \( \delta \in (0, +\infty) \) and \( 0 < \tau < T/\mu \) such that

\[
\int_0^\tau \int_{\{|\rho_X(x)\geq R\}} |u(t, x)| \exp\left(-\delta \rho_X^2(x)\right) dt \, dx < +\infty.
\]

By the very definition of \( u \) in Eq. 6.3, we have

\[
\int_0^\tau \int_{\{|\rho_X(x)\geq R\}} |u(t, x)| \exp\left(-\delta \rho_X^2(x)\right) dt \, dx 
\leq \int_0^\tau \int_{\{|\rho_X(x)\geq R\}} \left( \int_{\mathbb{R}^n} \gamma(t, x, y) |f(y)| \, dy \right) \exp\left(-\delta \rho_X^2(x)\right) dt \, dx.
\]

We then split the space integral as follows

\[
\int_{\{|\rho_X(x)\geq R\}} \left( \int_{\mathbb{R}^n} \gamma(t, x, y) |f(y)| \, dy \right) \exp\left(-\delta \rho_X^2(x)\right) dx 
= \int_{\{|\rho_X(x)\geq R\}} \left( \int_{\{|\rho_X(y)\geq 2\rho_X(x)\}} \gamma(t, x, y) |f(y)| \, dy \right) \exp\left(-\delta \rho_X^2(x)\right) dx
\]

\[
+ \int_{\{|\rho_X(x)\geq R\}} \left( \int_{\{|\rho_X(y)\leq 2\rho_X(x)\}} \gamma(t, x, y) |f(y)| \, dy \right) \exp\left(-\delta \rho_X^2(x)\right) dx
\equiv A(t) + B(t).
\]

As for \( A(t) \), by combining the Gaussian estimate Eq. 2.6 with Eq. 6.9, we get

\[
A(t) \leq \frac{c_\rho}{t^{q/2}} \int_{\{|\rho_X(x)\geq R\}} \left( \int_{\{|\rho_X(y)\geq 2\rho_X(x)\}} e^{-\frac{d_X^2(x, y)}{q t} + \mu \rho_X^2(y)} \cdot e^{-\mu \rho_X^2(y) |f(y)| \, dy} \cdot e^{-\delta \rho_X^2(x)} \, dx; \right.
\]

moreover, using the fact that \( d_X(x, y) \geq \rho_X(y) - \rho_X(x) \) for every \( x, y \in \mathbb{R}^n \), one has

\[
\exp\left(-\frac{d_X^2(x, y)}{q t} + \mu \rho_X^2(y)\right) \leq \exp\left(\frac{\rho_X^2(x)}{q t}\right) \cdot \exp\left(-\rho_X^2(y)\left(\frac{1}{2 q t} - \mu\right)\right) = (\star).
\]

As a consequence, since in \( A(t) \) we have \( \rho_X(y) \geq 2 \rho_X(x) \) and \( \rho_X(x) > 1 \), we obtain

\[
(\star) \leq \exp\left(\frac{\rho_X^2(x)}{q t} - 4 \rho_X^2(x)\left(\frac{1}{2 q t} - \mu\right)\right) = \exp\left(-\rho_X^2(x)\left(\frac{1}{q t} - 4 \mu\right)\right)
\]

provided that \( t \in (0, T/\mu) \), see Eq. 6.8. Using this last estimate, we get

\[
A(t) \leq \frac{c_\rho}{t^{q/2}} \exp\left(-\frac{1}{q t} - 4 \mu\right) \left( \int_{\mathbb{R}^n} |f(y)| \exp\left(-\mu \rho_X^2(y)\right) \, dy \right) \left( \int_{\mathbb{R}^n} \exp\left(-\delta \rho_X^2(x)\right) \, dx \right)
\]

\[
= \frac{c'}{t^{q/2}} \exp\left(-\frac{1}{q t} - 4 \mu\right)
\]
where we have exploited Lemma 6.3. From this, we finally obtain
\[
\int_0^\tau A(t) \, dt \leq \int_0^\tau \frac{c_1}{\sqrt{q}} e^{-\left(\frac{1}{\sqrt{q}} - 4\mu\right) t} \, dt < +\infty, \quad \text{for any } \tau \in (0, T/\mu) \text{ and any } \delta > 0.
\]
As for \( B(t) \), since \( \rho_X(y) < 2\rho_X(x) \), we have
\[
\begin{align*}
B(t) &= \int_{\{\rho_X(x) > 1\}} \left( \int_{\rho_X(y) < 2\rho_X(x)} \gamma(t, x, y) e^{\mu \rho_X^2(y)} \cdot |f(y)| e^{-\mu \rho_X^2(y)} \, dy \right) e^{-\delta \rho_X^2(x)} \, dx \\
&\leq \int_{\mathbb{R}^n} |f(y)| \exp\left(-\mu \rho_X^2(y)\right) \left( \int_{\mathbb{R}^n} \gamma(t, x, y) e^{(4\mu - \delta) \rho_X^2(x)} \, dx \right) \, dy.
\end{align*}
\]
Thus, if we choose \( \delta \geq 4\mu \), from Theorem 3.4-(iv) and (vi) we obtain
\[
\int_{\mathbb{R}^n} \gamma(t, x, y) e^{-\left(\delta - 4\mu\right) \rho_X^2(x)} \, dx \leq \int_{\mathbb{R}^n} \gamma(t, x, y) \, dx = 1.
\]
As a consequence, we get
\[
B(t) \leq \int_{\mathbb{R}^n} |f(y)| e^{-\mu \rho_X^2(y)} \, dy =: c < +\infty,
\]
from which we derive that
\[
\int_0^\tau B(t) \, dt < +\infty \quad \text{for any } \tau \in (0, T/\mu) \text{ and any } \delta \geq 4\mu.
\]
Summing up, we conclude that \( u \) satisfies Eq. 6.4 for every \( \delta \geq 4\mu \) and every \( \tau \in (0, T/\mu) \).

**STEP III.** Let us prove the uniqueness result. By linearity, it is enough to show that if for some \( \tau > 0 \) the function \( u \in C^2(S_{\tau}) \) is a classical solution of
\[
\mathcal{H} u = 0 \quad \text{in } S_{\tau},
\]
\[
u(0, x) = 0 \quad \text{for } x \in \mathbb{R}^n, \quad (6.10)
\]
and satisfies Eq. 6.4, then \( u \equiv 0 \) on \( S_{\tau} \). Denoting again by \( \pi_n \) the projection of \( \mathbb{R}^N \) onto \( \mathbb{R}^n \), we set
\[
\widehat{u} : \widehat{S}_{\tau} := (0, \tau) \times \mathbb{R}^N \to \mathbb{R}, \quad \widehat{u}(t, z) := u(t, \pi_n(z)).
\]
Obviously, \( \widehat{u} \in C^2(\widehat{S}_{\tau}) \); moreover, since \( u \) solves Eq. 6.10 and \( \mathcal{H}_G = \sum_{j=1}^m Z_j^2 - \partial_t \) is a lifting of \( \mathcal{H} \) (see Eq. 3.9), it is easy to check that \( \widehat{u} \) is a classical solution of
\[
\begin{align*}
\mathcal{H}_G \widehat{u} &= 0 \quad \text{in } \widehat{S}_{\tau}, \\
\widehat{u}(0, z) &= 0 \quad \text{for } z \in \mathbb{R}^N. \quad (6.11)
\end{align*}
\]
We claim that there exists \( \widehat{\delta} > 0 \) such that
\[
\int_{\widehat{S}_{\tau}} \exp\left(-\widehat{\delta} \|z\|^2\right) |\widehat{u}(t, z)| \, dt \, dz < +\infty. \quad (6.12)
\]
Once this is proved, by [10, Theorem 6.5] we derive that \( \widehat{u} \equiv 0 \) on \( \widehat{S}_{\tau} \), and thus \( u \equiv 0 \) on \( S_{\tau} \).

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To prove Eq. 6.12, let $\hat{\nu} > 0$ to be fixed in a moment. By using Proposition 4.2 (with $x = 0$ and $t = \delta^{-1} > 0$), we obtain the following computation

$$
\int_{\mathcal{S}_\tau} \exp \left( -\hat{\nu} \|z\|^2 \right) |\hat{u}(t, z)| \, dt \, dz = \int_{\mathcal{S}_\tau} \exp \left( -\hat{\nu} \|(x, \xi)\|^2 \right) |\hat{u}(t, (x, \xi))| \, dx \, d\xi \\
= \int_{\mathcal{S}_\tau} \left( \int_{\mathbb{R}^n} \exp \left( -\hat{\nu} \|(x, \xi)\|^2 \right) d\xi \right) |u(t, x)| \, dx \\
\leq \frac{\kappa}{\delta^{Q/2} |B_X(0, \delta^{-1/2})|} \int_{\mathcal{S}_\tau} \exp \left( -\frac{\delta \rho_X^2(x)}{\kappa} \right) |u(t, x)| \, dx \, dx,
$$

(6.13)

for a suitable constant $\kappa > 1$. As a consequence, if we choose $\hat{\delta} := \delta \cdot \kappa$, with $\delta$ as in Eq. 6.4, from Eq. 6.13 we immediately deduce the claimed Eq. 6.12. This ends the proof.

Remark 6.4 In the special case of Carnot groups, the uniqueness part of our result was already known, after [10, Thm. 6.5], and in our proof (Step III) we have explicitly exploited that result, relying on the assumption (H3) and the lifting technique. On the other hand, in the proof of our existence result (Steps I-II) we have never exploited assumption (H3) and the lifting technique. Actually, our proof in Steps I-II works also in Carnot groups, and our existence result extends the one proved in [10, Corollary 6.2], where a stronger pointwise (instead of integral) bound was assumed on $f$.

If the initial datum satisfies a slightly stronger assumption than Eq. 6.2, we can refine the previous results getting existence and uniqueness of the solution for every $t > 0$:

**Proposition 6.5** Let $f \in C(\mathbb{R}^n)$ satisfy the growth assumption Eq. 6.2 in the following stronger form: there exist $\alpha \in (0, 2)$ and $\mu > 0$ such that

$$
\int_{\mathbb{R}^n} |f(y)| \exp \left( -\mu \rho_X^2(y) \right) dy < +\infty.
$$

(6.14)

Then, the function $u$ defined by Eq. 6.3 is a classical solution of Eq. 6.1 on $S_{\infty} := (0, +\infty) \times \mathbb{R}^n$.

**Proof** Using assumption Eq. 6.14, it is easy to see that for every fixed $\theta > 0$ one has

$$
\int_{\mathbb{R}^n} |f(y)| \exp \left( -\theta \rho_X^2(y) \right) dy < +\infty.
$$

(6.15)

As a consequence, from Theorem 6.2 we derive that the function $u$ in Eq. 6.3 is a classical solution of Eq. 6.1 on $S_{T/\theta}$ for every $\theta > 0$, hence on the whole of $S_{\infty}$.

**7 An Application to the Dirichlet Problem for $\mathcal{H}$**

The aim of this section is to show how our global Gaussian estimates for $\Gamma$ can be used to study the solvability of the $\mathcal{H}$-Dirichlet problem on an arbitrary bounded domain $\Omega \subseteq \mathbb{R}^{1+n}$. All the results we are going to present basically follow by combining the results of the previous sections with the investigations carried out (in an abstract framework) in [20, 23, 24, 35].

To begin with, we need to establish the following proposition.
Proposition 7.1 The CC distance $d_X$ associated with our system $X = \{X_1, \ldots, X_m\}$ of homogeneous Hörmander’s vector fields satisfies the so-called segment property: for every fixed $x, y \in \mathbb{R}^n$ there exists a continuous path $\gamma : [0, 1] \to \mathbb{R}^n$ such that $\gamma(0) = x$, $\gamma(1) = y$ and

$$d_X(x, y) = d_X(x, \gamma(t)) + d_X(\gamma(t), y) \quad \text{for all } 0 \leq t \leq 1.$$

Proof This fact has been proved in [9, Corollary 5.15.6] in the context of Carnot groups. Actually, the same proof can be repeated in our setting; the only nontrivial point that must be checked is that the $d_X$-balls $B_X(x, \rho)$ are bounded in the Euclidean sense (for all $x \in \mathbb{R}^n$ and $\rho > 0$).

To prove this fact, we argue as follows. First of all, since the distance $d_X$ is topologically equivalent to the Euclidean distance, there exists some $r > 0$ such that the Euclidean ball $B_E(0, 1)$ contains the $d_X$-ball $B_X(0, r)$. On the other hand, for every $R > 0$ we have $\delta_{r/R}(B_X(0, R)) = B_X(0, r) \subseteq B_E(0, 1)$; hence, $\delta_{r/R}(B_X(0, R))$ is bounded in the Euclidean sense and, by the explicit form of $\delta_{r/R}$, the same is true for $B_X(0, R)$. From this, since for any $x \in \mathbb{R}^n$ and $\rho > 0$ we have $B_X(x, \rho) \subseteq B_X(0, R)$, with $R = \rho + d_X(x, 0)$, we conclude that every $d_X$-ball is bounded in the Euclidean sense.

Using the segment property of $d_X$, jointly with the properties of $\Gamma$ listed in Theorem 3.5 and the global Gaussian estimates Eq. 2.6 in Theorem 2.4, we can apply the axiomatic approach developed in [23]; denoting by $H$ the sheaf of functions defined as

$$\Omega \mapsto H(\Omega) := \{u \in \mathcal{C}^\infty(\Omega) : \mathcal{H}u = 0 \text{ in } \Omega\},$$

we have that $(\mathbb{R}^n, H)$ is a $\beta$-harmonic space satisfying the Doob convergence property. In this context, given a fixed open set $\Omega \subseteq \mathbb{R}^{1+n}$, we say that

- a function $u : \Omega \to \mathbb{R}$ is $\mathcal{H}$-harmonic in $\Omega$ if $u \in H(\Omega)$;
- a function $u : \Omega \to (-\infty, +\infty]$ is $\mathcal{H}$-superharmonic in $\Omega$ if
  
  1. $u$ is lower semi-continuous (l.s.c., for short) in $\Omega$;
  2. the set $\{x \in \Omega : u(x) < +\infty\}$ is dense in $\Omega$;
  3. for every $v \in \mathcal{C}(\overline{\Omega})$ such that $v|_\Omega \in H(\Omega)$ and $v \leq u$ on $\partial \Omega$ one has $v \leq u$ on $\Omega$.
- a function $u : \Omega \to [-\infty, +\infty)$ is $\mathcal{H}$-subharmonic in $\Omega$ if $-u$ is $\mathcal{H}$-superharmonic in $\Omega$.

We denote by $\overline{H}(\Omega)$ (resp. $H(\Omega)$) the (convex) cone of the $\mathcal{H}$-superharmonic (resp. $\mathcal{H}$-subharmonic) functions in $\Omega$. Obviously, we have $\overline{H}(\Omega) = -H(\Omega)$ and $\overline{H}(\Omega) \cap H(\Omega) = H(\Omega)$.

Let now $\Omega \subseteq \mathbb{R}^{1+n}$ be a fixed open set, and let $\varphi \in \mathcal{C}(\overline{\Omega})$. We say that a function $u : \Omega \to \mathbb{R}$ is a classical solution of the $\mathcal{H}$-Dirichlet problem

$$\begin{cases}
\mathcal{H}u = 0 & \text{in } \Omega, \\
u|_{\partial\Omega} = \varphi
\end{cases}$$

if it satisfies the following properties:

- $u \in \mathcal{C}(\overline{\Omega})$ and $u|_\Omega \in \mathcal{C}^2(\Omega)$;
- $\mathcal{H}u = 0$ in $\Omega$ and $u|_{\partial\Omega} = \varphi$.
Since $\mathcal{H} = \mathcal{L} - \partial_t$ satisfies the Weak Maximum Principle on every open subset of $\mathbb{R}^{1+n}$ (see, e.g., [3, Example 8.20]), there exists at most one classical solution of the Dirichlet problem Eq. 7.1; however, the existence of such a solution for a general $\varphi \in C(\partial \Omega)$ is not guaranteed. For this reason, we introduce the so-called Perron–Wiener–Brelot–Bauer (PWBB, in short) solution of Eq. 7.1.

Following [23], we first consider the functions

$$H^\Omega_\varphi (x) := \inf \left\{ u(x) : u \in \mathcal{H}(\Omega) \text{ and } \liminf_{\omega \to \omega_0} u(\omega) \geq \varphi(\omega_0) \text{ for all } \omega_0 \in \partial \Omega \right\} \quad \text{and}$$

$$\overline{H}^\Omega_\varphi (x) := \sup \left\{ u(x) : u \in \mathcal{H}(\Omega) \text{ and } \limsup_{\omega \to \omega_0} u(\omega) \leq \varphi(\omega_0) \text{ for all } \omega_0 \in \partial \Omega \right\}.$$

Then, since $(\mathbb{R}^{1+n}, H)$ satisfies Doob’s convergence property, it can be proved that

$$\overline{H}^\Omega_\varphi \equiv \overline{H}^\Omega =: H^\Omega_\varphi \in H(\Omega).$$

We shall call this function the PWBB solution of Eq. 7.1. Obviously, if $u$ is the classical solution of Eq. 7.1, one has $u \equiv H^\Omega_\varphi$ on $\Omega$; on the other hand, even if $H^\Omega_\varphi$ can be constructed for an arbitrary $\varphi \in C(\partial \Omega)$ and it is always $\mathcal{H}$-harmonic in $\Omega$, one cannot expect (in general) that

$$\lim_{\omega \to \omega_0} H^\Omega_\varphi (\omega) = \varphi(\omega_0) \quad \text{for all } \omega_0 \in \partial \Omega.$$

The following definition is thus plainly justified.

**Definition 7.2** A point $\omega_0 \in \partial \Omega$ is called $\mathcal{H}$-regular if

$$\lim_{\omega \to \omega_0} H^\Omega_\varphi (\omega) = \varphi(\omega_0) \quad \text{for all } \varphi \in C(\partial \Omega). \quad (7.2)$$

Due to the validity of the segment property for $d_\mathcal{H}$, the ‘good’ behavior of $\Gamma$ in Theorem 3.5, and the validity of global Gaussian estimates for $\Gamma$, we are entitled to apply to our context all the axiomatic results established in [20, 23, 24, 35]. As a consequence, we obtain several necessary/sufficient conditions for a point $\omega_0 \in \partial \Omega$ to be $\mathcal{H}$-regular (in the sense of Definition 7.2).

**Remark 7.3** Before proceeding, we would like to spend a few words about the ‘novelty’ of the results we are going to describe. First of all, as it is clear from Definition 7.2, the notion of $\mathcal{H}$-regular point is purely local; more precisely, if $H$ is a parabolic operator in $\mathbb{R}^{1+n}$ such that

$$H \equiv \mathcal{H} \quad \text{in an open neighborhood of } \overline{\Omega},$$

then a point $\omega_0 \in \partial \Omega$ is $\mathcal{H}$-regular if and only if it is $H$-regular. On the other hand, even in the simplest case of the Heat operator $\mathcal{H} = \Delta - \partial_t$, many meaningful characterizations of the $\mathcal{H}$-regular points require global objects (such as the existence of global heat kernel of Gaussian type).

For general parabolic Hörmander operators $H$ in $\mathbb{R}^{1+n}$ (of which our $\mathcal{H} = \mathcal{L} - \partial_t$ are particular cases), the availability of a global heat kernel and of global Gaussian bounds is not guaranteed; however, it is possible to overcome this lack of ‘global objects’ as follows (see also [35]). If $\Omega \subseteq \mathbb{R}^{1+n}$ is a bounded open set, we consider a bounded stripe $S := (T_1, T_2) \times D_0 \subseteq \mathbb{R}^{1+n}$ such that

$$\Omega \subseteq S.$$

Then, we invoke the results in [11]: there exists a parabolic Hörmander operator $\tilde{H}$, defined on the whole of $\mathbb{R}^{1+n}$, which coincides with $H$ on $S \supseteq \Omega$; moreover, for this ‘extended’
operator $\tilde{H}$ there exists a global heat kernel $\Gamma$, which satisfies global Gaussian bounds. It is now possible to characterize the $H$-regular points of $\partial \Omega$ in terms of the global objects associated with $\tilde{H}$ (see [24, 35]).

The aim of this section is to point out that, since our operators $\mathcal{H}$ possess a global heat kernel which satisfies global Gaussian bounds, there is no need to exploit the ‘extension procedure’ described above: in this case, it is possible to characterize the $H$-regular points in the boundary of any bounded open set $\Omega$ by means of ‘global objects’ which only depend on $\mathcal{H}$ (and not on $\Omega$).

Summing up, the results we are going to present in this section are not new, and they hold for parabolic operators which are more general than the ones considered in this paper; what we would like to emphasize is that, when homogeneous Hörmander vector fields are involved, it is possible to state these results in a more ‘intrinsic’ and ‘global’ form.

Throughout the sequel, given any compact set $K \subseteq \mathbb{R}^{1+n}$, we define

$$ V_K(\omega) = \liminf_{z \to \omega} \left( W_K(z) \right), \quad \text{where} $$

$$ W_K(z) := \inf \{ v(z) : v \in \mathcal{H}(\mathbb{R}^n), v \geq 0 \text{ on } \mathbb{R}^{1+n} \text{ and } v \geq 1 \text{ on } K \}. \quad (7.3) $$

The function $V_K$ is usually referred to as the $H$-balayage of $u_0 \equiv 1$ on $K$.

**Theorem 7.4** [23, Thms 4.6 and 4.11] Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0, x_0) \in \partial \Omega$ be a fixed point of $\mathcal{H}$. For any $r > 0$, we define

$$ \Omega'_r(\omega_0) := \{ \omega = (t, x) \in \mathbb{R}^{1+n} \setminus \Omega : t \leq t_0, \left( d_X(x, x_0)^4 + |t - t_0|^2 \right)^{1/4} \leq r \}, $$

and we denote by $V_r$ the so-called $\mathcal{H}$-balayage of $u_0 \equiv 1$ on $\Omega'_r(\omega_0)$, that is,

$$ V_r := V_{\Omega'_r(\omega_0)}. \quad (7.4) $$

Then, following assertions are equivalent:

- $\omega_0$ is not $\mathcal{H}$-regular;
- there exists $r > 0$ such that $V_r(\omega_0) < 1$;
- $V_r(\omega) \to 0$ as $r \to 0^+$.

On the other hand, if there exist real constants $M, \rho, \theta > 0$ such that

$$ \left| \left\{ x \in \overline{B_X(x_0, M\rho)} : (t_0 - \rho^2, x) \notin \Omega \right\} \right| \geq \theta |B_X(x_0, M\rho)|, $$

then $\omega_0$ is $\mathcal{H}$-regular.

Another sufficient condition for $\mathcal{H}$-regularity is the following.

**Theorem 7.5** [20, Theorem 5.1] Let $\Omega \subseteq \mathbb{R}^{1+n}$ be an open set, and let $\omega_0 = (t_0, x_0) \in \partial \Omega$ be fixed. Moreover, let $\{ B_\lambda \}_{0 < \lambda < 1}$ be a basis of closed neighborhoods of $x_0$ in $\mathbb{R}^n$ such that

$$ B_\lambda \subseteq B_\mu \text{ if } 0 < \lambda < \mu \leq 1. $$

For every $\lambda \in (0, 1)$, we define

$$ \Omega_\lambda(\omega_0) := \left( [t_0 - \lambda, t_0] \times B_\lambda \right) \setminus \Omega \quad \text{and} \quad T_\lambda(\omega_0) := \{ x \in \mathbb{R}^n : (t_0 - \lambda, x) \in \Omega_\lambda(\omega_0) \}. $$

Then the point $\omega_0$ is $\mathcal{H}$-regular if

$$ \limsup_{\lambda \searrow 0^+} \int_{T_\lambda(\omega_0)} \gamma(\lambda, x_0, \xi) \, d\xi > 0. $$
By making use of the so-called $\mathcal{H}$-Wiener function (associated with the open set $\Omega$ and the point $\omega_0 \in \partial \Omega$), it is possible to derive a necessary and sufficient condition for $\omega_0$ to be regular.

**Theorem 7.6** [23, Theorem 5.4] Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0, x_0) \in \partial \Omega$ be fixed. Moreover, given a number $p > 0$ and a sequence $\{r_k\}_{k \in \mathbb{N}}$ converging to 0 as $k \to +\infty$, we define the $\mathcal{H}$-Wiener function (associated with $\Omega$ and $\omega_0$) as

$$W(\omega) := \sum_{k=1}^{+\infty} \frac{1 - V_k(\omega)}{p^k},$$

(7.5)

where $V_k = V_{r_k}$ and, for every $r > 0$, the function $V_r$ is as in Eq. 7.4. Then $\omega_0$ is $\mathcal{H}$-regular if and only if $W(\omega) \to 0$ as $\omega \to \omega_0$.

Finally, by making explicit use of our global Gaussian estimates for $\Gamma$, we can obtain criteria for $\mathcal{H}$-regularity which are resemblant to the classical results proved by Wiener and Landis for the heat operator $\Delta - \partial_t$. In order to clearly state these criteria, we first fix some notation.

Given a compact set $K \subseteq \mathbb{R}^{1+n}$, let $V_K$ be the $\mathcal{H}$-balayage of $u_0 \equiv 1$ on $K$ defined in Eq. 7.3. By classical results of Potential Theory, it is known that $V_K$ is $\mathcal{H}$-superharmonic on $\mathbb{R}^{1+n}$; as a consequence, there exists a unique positive Radon measure $\mu = \mu_K$ on $\mathbb{R}^{1+n}$ such that

$$\mathcal{H}V_K = -\mu_K \text{ in } \mathcal{D}'(\mathbb{R}^{1+n}) \text{ and } \text{supp}(\mu_K) = K.$$

(see, e.g., [29]). We then define the $\mathcal{H}$-capacity of $K$ as follows

$$\mathcal{C}_{\mathcal{H}}(K) := \mu_K(K).$$

Moreover, if $M^+(K)$ denotes the set of non-negative Radon measures on $\mathbb{R}^{1+n}$ with support contained in $K$, we also define the $a$-Gaussian capacity of $K$ as follows

$$\mathcal{C}_a(K) := \sup \left\{ \nu(K) : \nu \in M^+(K) \text{ and } \int_K G_a(t, x; s, y) \, d\mu(s, y) \leq 1 \text{ for all } (t, x) \in \mathbb{R}^{1+n} \right\},$$

where for every $a > 0$ we have used the notation

$$G_a(t, x; s, y) := \begin{cases} 0, & \text{if } t \leq s, \\ \frac{1}{|B_X(x, \sqrt{t-s})|} \exp \left( -a \frac{d_X^2(x, y)}{t-s} \right), & \text{if } t > s. \end{cases} \quad (7.6)$$

Notice that, using Eq. 7.6, our Gaussian estimates Eq. 2.6 reads as

$$\frac{1}{q} G_a(t, x; s, y) \geq \Gamma(t, x; s, y) \geq \varrho G_{1/q}(t, x; s, y) \quad (\text{for all } (t, x), (s, y) \in \mathbb{R}^{1+n}).$$

Here is a ‘Wiener-type’ test for $\mathcal{H}$-regularity.
Theorem 7.7 [24, Theorem 1.1] Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0, x_0) \in \partial \Omega$. For every fixed $\lambda \in (0, 1)$ and every $h, k \in \mathbb{N}$, we define

$$\Omega_k^h(\omega_0, \lambda) := \left\{ \omega = (t, x) \in \mathbb{R}^{1+n} \setminus \Omega : \lambda^{k+1} \leq t_0 - t \leq \lambda^k, \right. \left. \frac{1}{\lambda^{h-1}} \leq \exp \left( \frac{d_X^2(x_0, x)}{t_0 - t} \right) \leq \frac{1}{\lambda^h}, \left( d_X(x, x_0)^4 + |t - t_0|^2 \right)^{1/4} \leq \sqrt{\lambda} \right\}.$$ 

Then, if $\varrho > 0$ is as in Eq. 2.6, the following facts hold.

- if there exist $0 < a \leq 1/\varrho$ and $b \geq \varrho$ such that

$$\sum_{h, k=1}^{+\infty} \frac{\mathcal{E}_a(\Omega_k^h(\omega_0, \lambda))}{|B_X(x_0, \lambda^{k/2})|} \lambda^{bh} = +\infty,$$

then the point $\omega_0$ is $\mathcal{H}$-regular.

- If the point $\omega_0$ is $\mathcal{H}$-regular, then

$$\sum_{h, k=1}^{+\infty} \frac{\mathcal{E}_b(\Omega_k^h(\omega_0, \lambda))}{|B_X(x_0, \lambda^{k/2})|} \lambda^{ah} = +\infty,$$

for every $0 < a \leq 1/\varrho$ and $b \geq \varrho$.

Finally, a ‘Landis-type’ condition for $\mathcal{H}$-regularity is given by the following theorem.

Theorem 7.8 [35, Theorem 1.3] Let $\Omega \subseteq \mathbb{R}^{1+n}$ be a bounded open set, and let $\omega_0 = (t_0, x_0) \in \partial \Omega$. For every fixed $\lambda \in (0, 1/2)$ and every $k \in \mathbb{N}$, we consider the set

$$\Omega_k^s(\omega_0) := \left\{ \omega = (t, x) \in \mathbb{R}^{1+n} \setminus \Omega : \frac{1}{\lambda^{k+1}} \leq \Gamma(t_0, x_0; t, x) \leq \frac{1}{(k+1)\lambda^{k+1}} \right\} \cup \{(t_0, x_0)\},$$

where $\Gamma$ is the global heat kernel of $\mathcal{H}$. Then $\omega_0$ is $\mathcal{H}$-regular if and only if

$$\sum_{k=1}^{+\infty} V_{\Omega_k^s(\omega_0)}(\omega_0) = +\infty,$$

where $V_{\Omega_k^s(\omega_0)}$ is the $\mathcal{H}$-balayage of $u_0 \equiv 1$ on $\Omega_k^s(\omega_0)$, see Eq. 7.3.

8 Scale-Invariant Harnack Inequality for $\mathcal{H}$

In this last section we prove a scale-invariant Harnack inequality for non-negative solutions of $\mathcal{H}u = 0$. This fact easily follows, via the lifting procedure, from the analogous result proved on Carnot groups in [10, Corollary 4.5]. It is however a result which is worthwhile to be pointed out.

Given any point $\omega_0 = (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n$, any number $r > 0$, we define

$$C(\omega_0, r) := \{(t, x) \in \mathbb{R}^{1+n} : d_X(x, x_0) < r, |t - t_0| < r^2\}.$$ 

Furthermore, for every $\lambda \in (0, 1/2)$, we set

$$S_\lambda(\omega_0, r) := \{(t, x) \in \mathbb{R}^{1+n} : d_X(x, x_0) < (1 - \lambda)r, \lambda r^2 < t_0 - t < (1 - \lambda)r^2\}.$$ 

We are ready to state our result.
Theorem 8.1 For every \( h, k = 0, 1, 2, \ldots \) and every fixed \( \lambda \in (0, 1/2) \), it is possible to find a positive constant \( \nu = \nu_{h,k,\lambda} > 0 \) such that, for every \( \omega_0 = (t_0, x_0) \in (0, +\infty) \times \mathbb{R}^n \), every \( r > 0 \), and every nonnegative function \( u \in C^2(C(\omega_0, r)) \) satisfying \( \mathcal{H}u = 0 \) on \( C(\omega_0, r) \),

\[
\sup_{S_{\lambda}(\omega_0, r)} \left| \sum_{i_1, \ldots, i_h} \partial_{t_{i_1}} \cdots \partial_{t_{i_h}} u \right| \leq \nu r^{-(h+2k)} u(\omega_0),
\]

(8.1)

for every \( i_1, \ldots, i_h \in \{1, \ldots, m\} \).

Proof Letting \( v_0 := (x_0, 0) \in \mathbb{R}^N \) and \( \tilde{\omega}_0 := (t_0, v_0) \in \mathbb{R}^{1+N} \), we define

\[
\widehat{C}(\tilde{\omega}_0, r) := \{(t, v) \in \mathbb{R}^{1+N} : d_{\mathbb{Z}}(v, v_0) < r, \ |t - t_0| < r^2\}
\]

and

\[
\widehat{S}_{\lambda}(\tilde{\omega}_0, r) := \{(t, v) \in \mathbb{R}^{1+N} : d_{\mathbb{Z}}(v, v_0) < (1 - \lambda)r, \ \lambda r^2 < t_0 - t < (1 - \lambda)r^2\}.
\]

Let then \( u \in C^2(C(\omega_0, r)) \) be any non-negative function satisfying of \( \mathcal{H}u = 0 \) on \( C(\omega_0, r) \). Denoting by \( \pi_n : \mathbb{R}^N \to \mathbb{R}^n \) the canonical projection of \( \mathbb{R}^N \) onto \( \mathbb{R}^n \), we set

\[
\widehat{u}(t, v) := u(t, \pi_n(v)) \quad \quad (v \in \mathbb{R}^N).
\]

Since \( B_{\mathbb{Z}}(v_0, r) \subseteq \pi_{n}^{-1}(B_X(x_0, r)) \) (see Proposition 3.7-(iii)), we have

\[
\widehat{u} \in C^2(\widehat{C}(\tilde{\omega}_0, r)).
\]

Moreover, since \( u \geq 0 \) and \( \mathcal{H}u = 0 \) on \( C(\omega_0, r) \), from the lifting property Eq. 3.9 we derive that

\[
\widehat{u} \geq 0 \quad \text{and} \quad \mathcal{H}_{\tilde{\omega}_0} \widehat{u} = 0 \quad \text{on} \quad \widehat{C}(\tilde{\omega}_0, r).
\]

Putting together these facts, we are entitled to apply [10, Corollary 4.5], obtaining

\[
\sup_{\widehat{S}_{\lambda}(\tilde{\omega}_0, r)} \left| \sum_{i_1, \ldots, i_h} \partial_{t_{i_1}} \cdots \partial_{t_{i_h}} \widehat{u} \right| \leq \nu r^{-(h+2k)} \widehat{u}(t_0, v_0),
\]

(8.2)

where \( \nu > 0 \) is an absolute constant only depending on \( h, k \) and \( \lambda \). We now claim that the above Eq. 8.2 is precisely the desired Eq. 8.1. In fact, by the very definition of \( \widehat{u} \), we have

\[
\widehat{u}(t_0, v_0) = u(t_0, x_0) = u(\omega_0);
\]

(8.3)

moreover, by repeatedly exploiting Eq. 3.4, we get

\[
Z_{i_1} \cdots Z_{i_h} (\partial_t)^k \widehat{u}(t, v) = (\partial_t)^k \left( Z_{i_1} \cdots Z_{i_h} (v \mapsto u(t, \pi_n(v))) \right)
\]

\[
= (\partial_t)^k \left( Z_{i_1} \cdots Z_{i_{h-1}} (v \mapsto (X_{i_h} u)(t, \pi_n(v))) \right)
\]

\[
= \ldots = \left( (\partial_t)^k X_{i_1} \cdots X_{i_h} u \right)(t, \pi_n(v)) \quad \text{for all} \ (t, v) \in \widehat{C}(\tilde{\omega}_0, r).
\]

From this, taking into account that \( \pi_n (B_{\mathbb{Z}}(v_0, (1 - \lambda)r)) = B_X(x_0, (1 - \lambda)r) \), we readily obtain

\[
\sup_{\widehat{S}_{\lambda}(\tilde{\omega}_0, r)} \left| \sum_{i_1, \ldots, i_h} \partial_{t_{i_1}} \cdots \partial_{t_{i_h}} \widehat{u} \right| = \sup_{\widehat{S}_{\lambda}(\tilde{\omega}_0, r)} \left| \sum_{i_1, \ldots, i_h} \partial_{t_{i_1}} \cdots \partial_{t_{i_h}} \widehat{u} \right|.
\]

(8.4)

By combining Eqs. 8.2, 8.3 and 8.4, we finally derive Eq. 8.1, with an absolute constant \( \nu > 0 \) which depends on the chosen \( h, k \) and \( \lambda \) (but not on \( \omega_0, r \) nor \( u \)). This ends the proof. \( \square \)
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