Around spin Hurwitz numbers

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Abstract
We present a review of the spin Hurwitz numbers, which count the ramified coverings with spin structures. It is known that they are related to the characters of the Sergeev group and to the $Q$ Schur functions. This allows one to put the whole story into the context of matrix models and integrable hierarchies. The generating functions of the spin Hurwitz numbers $\tau_{\pm}$ are hypergeometric $\tau$-functions of the BKP integrable hierarchy; we present their fermionic realization. The cut-and-join equation in the form of a heat equation is written down. We explain, how a special $d$-soliton $\tau$-functions of KdV and Veselov–Novikov hierarchies generate the spin Hurwitz numbers $H_{\pm}^b(\Gamma_d^b)$ and $H_{\pm}^b(\Gamma_d^b, \Delta)$. We present the well-known Kontsevich matrix integral as the BKP $\tau$-function in the form of special neutral fermion vacuum expectation values (few different ones). We also explain how to rewrite certain BKP $\tau$-functions (including the Kontsevich one) as the hypergeometric BKP $\tau$-functions using certain relations between the projective Schur functions.

Keywords Spin Hurwitz numbers · Integrable hierarchies · $Q$ Schur functions

To the memory of Boris Dubrovin.

Sergey Natanzon passed away on December, 7, 2020.

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1 Introduction

Boris Dubrovin was one of the brightest minds in modern mathematics; in fact, his well-known achievements cover only a small part of all what he thought about and planned to do. Long ago, as young students, we were inspired by the brilliant volume [16,17], to which Boris also contributed a big part of his knowledge and vision. During his entire life in science, he tried to make new fields as clear and transparent, as happened to the classical subjects in that unique textbook. This paper was supposed to have a single author, Sergey Natanzon, and was planned as his personal tribute to his close friend, Boris. Unfortunately, Sergey was not given a chance to fulfill this duty. The present note is a brief review of what he taught us about the increasingly important subject of spin Hurwitz numbers, and it certainly lacks the vision which Sergey had on it and which we, survivors, will still need to rediscover in our future work.

Hurwitz numbers [9,15,21,26] count ramified coverings, and their significance in physics is dictated by the role that complex curves play in string theory. Applications range from the free field calculus on ramified Riemann surfaces [33] to genus expansion in matrix models and topological recursion [3,4,20]. The spin Hurwitz numbers [19,24] do just the same things to Riemann surfaces with spin structures labeled by theta-characteristics. The matrix model counterparts in this case appear to be the cubic Kontsevich [29,30,35] and Brezin-Gross-Witten (BGW) [4,43] models, which are non-obvious generalizations of the Hermitian matrix model [51–53] equivalent to a quadratic counterpart of Kontsevich theory [10,11,31]. As usual, the most efficient technique to develop spin Hurwitz calculus relies on algebraic approach, which finally puts the problem into the framework of integrable systems. The basic of this approach includes the following ingredients [44]:

- A relevant substitute of the Schur functions, which in the case of cubic Kontsevich and BGW models, is provided by the $Q$ Schur functions
- Their relation to the Hall–Littlewood polynomials [39]
- Their relation to characters of the Sergeev algebra [64], which provides a relevant generalization of symmetric group characters
- The Fröbenius formula, expressing the spin Hurwitz numbers through the Sergeev characters and the $Q$ Schur functions
- Commuting system of cut-and-join $\hat{W}$ operators [44,45], which have $Q$ Schur functions as their common eigenfunctions and the Sergeev characters as eigenvalues
- Free fermion representations [14,27,58,67], which allow one to represent $Q$ Schur functions as Pfaffians
- Integrability properties of the spin Hurwitz numbers [38,44]. As in ordinary case [45–47], spin Hurwitz $\tau$-functions form a broader and still comprehensible variety, but, in special cases, we get soliton and other solutions to the BKP hierarchy [14,27]
• Hypergeometric $\tau$-functions of the 2BKP hierarchy [56,58]. By definition, they are bilinear in the $Q$ Schur functions with coefficients of a very special product form (DP denotes the whole set of strict partitions)

$$\tau_{2BKP}\{\mathbf{p}, \mathbf{p}^*\} = \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha\{\mathbf{p}\} Q_\alpha\{\mathbf{p}^*\} \cdot \prod_{i=1}^{\ell(\alpha)} f(\alpha_i) \quad (1.1)$$

Important examples are provided by the ratios

$$\frac{Q_{N\alpha}\{\delta_{k, r}\}}{Q_\alpha\{\delta_{k, r}\}} = \prod_{i=1}^{\ell(\alpha)} f(\alpha_i) \quad \forall \text{ coprime } N, r \quad (1.2)$$

where $\ell(\alpha)$ denotes the number of lines in the Young diagram $\alpha$, and $N\alpha$ denotes the Young diagram with lengths $N\alpha_i$. In particular, for the cubic Kontsevich model [42],

$$\tau_{K_3}\{\mathbf{p}\} = \sum_{\alpha \in \text{DP}} Q_\alpha\{\mathbf{p}\} Q_\alpha\{\delta_{k, 1}\} \cdot \frac{Q_{2\alpha}\{\delta_{k, 1}\}}{Q_{2\alpha}\{\delta_{k, 3}\}} \cdot \frac{Q_\alpha\{\delta_{k, 3}\}}{Q_\alpha\{\delta_{k, 3}\}} \cdot \frac{1}{2^{\ell(\alpha)}} \quad (1.3)$$

while, for the BGW model [1],

$$\tau_{BGW}\{\mathbf{p}\} = \sum_{\alpha \in \text{DP}} Q_\alpha\{\mathbf{p}\} Q_\alpha\{\delta_{k, 1}\} \cdot \left(\frac{Q_\alpha\{\delta_{k, 1}\}}{Q_{2\alpha}\{\delta_{k, 1}\}}\right)^2 \cdot \frac{1}{2^{\ell(\alpha)}} \quad (1.4)$$

We refer to the very recent paper [1] for some complimentary details.

As to the ordinary integrability, its intimate relation to characters is well known. The Schur functions are themselves solutions to the Hirota bilinear equations, and general KP/Toda $\tau$-functions are their linear combinations with the coefficients satisfying the Plücker relations, which have determinants and their free fermion realizations as natural solutions. The $Q$ Schur functions play the same role for the BKP hierarchies; only solutions are now Pfaffians, which can be described in terms of “neutral” fermions. In this context, it is interesting that the KdV hierarchy can be considered as a reduction of the both KP and BKP hierarchies [2,12–14,27].

In the present paper, we review some auxiliary aspects, related to the free-fermion description of the $Q$ Schur functions, the BKP and KdV hierarchies. In particular, as in the case of ordinary Hurwitz numbers, the lowest “cut-and-join” $\hat{W}$ operators are compatible with the BKP Hirota equations and generate an especially simple hypergeometric “tau”-function; as a result of a restriction of one set of times this $\tau$-function can be also considered as an infinite-soliton $\tau$-function of the KdV hierarchy. Technically, the BKP $\tau$-function can be defined as a direct counterpart of the Toda lattice $\tau$-function, but for the matrix elements of $SL(\infty)$ generated by a restricted set of “neutral” fermions and depending only on odd sets of times. Proper weights made of exponentials of power sums of Young diagram lengths in (1.1) (“completed cycles”) in this formalism provides the hypergeometric $\tau$-functions, which are easy
and very straightforward to work with. The full (spin) Hurwitz τ-functions involve far more complicated weights made out of all (Sergeev) symmetric characters, and they provide an important generalization beyond (B)KP theory, which still awaits an efficient language and deep investigation.

**Notation.** Throughout the paper, we denote through \([x]\) the integer part of a number, through \(\{x\}\) its fractional part. For an integer \(k\), \((k)_r = r \{k/r\}\) denotes the value mod \(r\). For the strict partition \(\alpha\), \(\ell(\alpha)\) is the number of parts, and \(\tilde{\ell}(\alpha) := 2 \cdot \left[ \frac{\ell(\alpha)+1}{2} \right] \).

## 2 Hurwitz numbers

### 2.1 Classical Hurwitz numbers

Consider a compact Riemann surface \(S\) of genus \(g\) with a finite number of points \(x_1, \ldots, x_n \in S\). Consider a set of Young diagrams \((\Delta^1, \ldots, \Delta^n)\) of the same degree \(d = |\Delta_i|\). The lengths of the rows \(\Delta^i_1, \ldots, \Delta^i_{\ell_i}\) of the Young diagram \(\Delta^i\) give the partition of the number \(d\).

Denote by \(\tilde{M}(\Delta^1, \ldots, \Delta^n)\) the set of holomorphic mappings of compact Riemann surfaces \(\varphi : P \rightarrow S\), whose critical values lie in \(\{x_1, \ldots, x_n\}\), and the pre-images \(\varphi^{-1}(x_i)\) consist of points, where \(\varphi\) has degrees \(\Delta^i_1, \ldots, \Delta^i_{\ell_i}\). We call the mappings \(\varphi : P \rightarrow S\) and \(\varphi' : P' \rightarrow S\) as equivalent if there exists a biholomorphic mapping \(\phi\) such that \(\varphi = \varphi' \phi\). Let \(M(\Delta^1, \ldots, \Delta^n)\) denote the set of equivalence classes in the set \(\tilde{M}(\Delta^1, \ldots, \Delta^n)\).

The classical Hurwitz number [15] is the number

\[
H_d(g | \Delta^1, \ldots, \Delta^n) = \sum_{\varphi \in M(\Delta^1, \ldots, \Delta^n)} \frac{1}{|\text{Aut}(\varphi)|}
\]

There is the Fröbenius formula that gives a combinatorial expression for the Hurwitz numbers [9],

\[
H_d(g | \Delta^1, \ldots, \Delta^n) = \frac{[\Delta_1]\ldots[\Delta_k]}{(d!)^2} \sum_R \frac{\psi_R(\Delta_1)\ldots\psi_R(\Delta_k)}{\psi_R(1)^{k-2}}
\]

where \([\Delta]\) is the number of permutations of the cyclic type \(\Delta\), i.e., the number of elements in the conjugacy class of the symmetric group \(\Sigma_d\) given by the Young diagram \(\Delta\), \(|\Delta| = d\); \(\psi_R(\Delta)\) is value of the character \(\psi_R\) of the representation \(R\) of the symmetric group \(\Sigma_d\) on the permutation of cyclic type \(\Delta\), \(\psi_R(1)\) is the value on the permutation with all unit cycles, \(\Delta = [1, \ldots, 1]\), and the sum is taken over all characters of irreducible representations of \(\Sigma_d\).

Among the classical Hurwitz numbers, we will be interested only in the so-called double Hurwitz numbers

\[
H_d(\Delta, \Delta^*, b) = H_d(0|\Delta, \Delta^*, \Gamma^b_2)
\]
Here $\Gamma_2$ is a Young diagram with one row of length 2 and the rest of rows of length 1. The $\Gamma_2^b$ means a set from $b$ diagrams $\Gamma_2$.

Consider two infinite sets of variables $p = (p_1, p_2, \ldots)$ and $p^* = (p_1^*, p_2^*, \ldots)$. Associate monomial $p_\Delta = p_{\Delta_1} \cdots p_{\Delta_\ell}$ to the Young diagram $\Delta = [\Delta_1, \ldots, \Delta_\ell]$ and monomial $p_\Delta^* = p_{\Delta_1^*} \cdots p_{\Delta_\ell^*}$ to Young diagram $\Delta^* = [\Delta_1^*, \ldots, \Delta_\ell^*]$. To the double Hurwitz number $H_d(\Delta, \Delta^*, b)$ we associate the monomial $H_d(\Delta, \Delta^*, b) p_\Delta p_\Delta^*$. 

As a generating function for the double Hurwitz numbers, one usually considers the function proposed in [57]

$$
\tau(\Delta, \Delta^*, \beta, q) = \sum_{d>0} \sum_{|\Delta|=|\Delta^*=0} \sum b, b, \Delta, \Delta^* q^d \frac{p^b}{b!} H_d(\Delta, \Delta^*, b) p_\Delta p_\Delta^* \tag{2.4}
$$

According to [5,32,57,59,60], this function is a $\tau$-function of the 2D Toda lattice (it was called hypergeometric in [59,60]).

Moreover, according to [23,45], it satisfies the cut-and-join equation

$$
\frac{\partial \tau(\Delta, \Delta^*, \beta, q)}{\partial \beta} = W(\Delta, \Delta^*, \beta, q) \tag{2.5}
$$

where

$$
W = \frac{1}{2} \sum_{a,b>0} \left( (a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + abp_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right) \tag{2.6}
$$

### 2.2 Spin Hurwitz numbers

The notion of spin Hurwitz numbers was introduces in [19] (see also [36,40]) and later worked out in a set of papers, see [24,37,38,44].

A line bundle $L$ on a Riemann surface is called a spin bundle if the tensor square of $L$ is isomorphic to the cotangent bundle. The parity of the space of holomorphic sections of the bundle $L$ is called the parity of the bundle (see [8,54]) and is denoted by $Arf(L) \in \{0, 1\}$. The surface of genus 0 has exactly one spin bundle, and it is even.

Consider a holomorphic mapping $\varphi : P \to S$ whose critical point orders are odd. Such a mapping associates the spin bundle $L$ onto $S$ with the spin bundle $\varphi^* (L)$ onto $P$. Its parity $Arf(\varphi) = Arf(\varphi^*(L))$ depends only on $\varphi$ and $Arf(L)$.

The spin Hurwitz number was defined in [19] as

$$
H_d^{Arf(L)}(g|\Delta_1, \ldots, \Delta_n) = \sum_{\varphi \in M(\Delta_1, \ldots, \Delta_n)} (-1)^{Arf(\varphi)} \frac{|\text{Aut}(\varphi)|}{|\text{Aut}(\varphi)|} \tag{2.7}
$$

Depending on whether the parity of the bundle is even or odd, later on, we use the superscripts $+$ and $-$, respectively.
We are only interested in the double spin Hurwitz numbers, which, in this case, have the form

\[ H^P_d(\Delta, \Delta^*, b) = H^0_d(0|\Delta, \Delta^*, \Gamma_3^b) \]  

(2.8)

where \( \Gamma_3 \) is a Young diagram with one row of length 3 and other rows of length 1, and \( \Gamma_3^b \) means \( b \) of such diagrams. The generating functions for such numbers can be written in different ways. The simplest generating functions are proposed and investigated in [38,44]. They satisfy both the 2BKP hierarchy and modified cut-and-join equations. We discuss these issues in the present paper.

3 Q Schur functions

In this paper, we define the \( Q \) Schur functions in a peculiar normalization, which is conventional for fermionic representations, but different from the natural one for the Cauchy identities and matrix models used in [1,42].

3.1 Projective Schur functions

To define the \( Q \) Schur functions \( Q_\alpha \), we begin following [39] by defining an infinite skew-symmetric matrix \( (Q_{ij})_{i,j \in \mathbb{N}} \), whose entries are symmetric functions of the infinite sequence of indeterminates \( x = (x_1, x_2, \ldots) \), via the following formula:

\[
Q_{ij}(x) := \begin{cases} 
q_i(x)q_j(x) + 2 \sum_{k=1}^{\min(i,j)} (-1)^k q_{i+k}(x)q_{j-k}(x) & \text{if } (i, j) \neq (0, 0), \\
0 & \text{if } (i, j) = (0, 0), 
\end{cases}
\]  

(3.1)

where the \( q_i(x) \)’s are defined by the generating function:

\[
\prod_{i=1}^{\infty} \frac{1 + zx_i}{1 - zx_i} = \sum_{i=0}^{\infty} z^i q_i(x) 
\]  

(3.2)

For instance, \( q_1(x) = 2 \sum_i x_i \). In particular,

\[
Q_{(j,0)}(x) = -Q_{(0,j)}(x) = q_j(x) \quad \text{for } j \geq 1 
\]  

(3.3)

For a strict partition \( \alpha \) of even cardinality \( 2n \) (including a possible zero part \( \alpha_{2n} = 0 \)), let \( M_\alpha(x) \) denote the \( 2n \times 2n \) skew-symmetric matrix with entries

\[
(M_\alpha(x))_{ij} := Q_{\alpha_i\alpha_j}(x), \quad 1 \leq i, j \leq 2n. 
\]  

(3.4)

The \( Q \) Schur function is defined as its Pfaffian [39]

\[
Q_\alpha(x) := Pf(M_\alpha(x)) 
\]  

(3.5)
and, for completeness,

$$Q_{\emptyset} := 1.$$  \hspace{1cm} (3.6)

Equivalently, these may be viewed as functions $q_j(p)$, $Q_{ij}(p)$ of the odd-indexed power sum symmetric functions $p = (p_1, p_3, \ldots)$

$$p_{2i-1} = p_{2i-1}(x) = \sum_{a=1}^{\infty} x_a^{2i-1}, \quad a = 1, 2, \ldots.$$  \hspace{1cm} (3.7)

Following [39], we use the agreement

$$Q_\alpha \{ p(x) \} := Q_\alpha (x).$$  \hspace{1cm} (3.8)

In particular, $Q_{[1]} \{ p \} = 2p_1 = 2 \sum_i x_i$, $Q_{[2]} \{ p \} = 2p_1^2 = 2 (\sum_i x_i)^2$.

The following relation is known

$$Q_\alpha \{ \delta_{k,1} \} = \frac{2^{[\alpha]}}{\prod_{i=1}^{\ell(\alpha)} \alpha_i!} \Delta^*(\alpha), \quad \Delta^*(\alpha) := \prod_{i<j \leq \ell(\alpha)} \frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}.$$  \hspace{1cm} (3.9)

Remark 3.1 The role of the projective Schur functions in the BKP theory was recognized in [55,67]. In the literature on integrable systems, the variables often used and called times of the BKP hierarchy (see [12,14,25,56,58,67]) are $\frac{2p_m}{m}$ with $m$ odd.

In the present paper, we use the BKP hierarchy; however, we rewrite known BKP formulas in the power sum variables.

### 3.2 Neutral fermions and projective Schur functions

In this section, we very briefly recall the known facts (details can be found either in the original papers [14,27,67], or in [58], the results of which we will use). We need to fix the notation.

#### 3.2.1 From KP to BKP

A natural way to construct the BKP hierarchy is to start with the KP hierarchy. A standard way to describe this latter is to realize the $\tau$-function of the hierarchy as a fermionic average

$$\tau(p_f, p_f^\dagger) = \langle 0 | \gamma(p_f) g \gamma^\dagger(p_f^\dagger) | 0 \rangle$$  \hspace{1cm} (3.10)

where $p_f := (p_1, p_2, p_3, \ldots)$ and analogously $p_f^\dagger$ are sets of KP time variables,

$$\gamma(p_f) := e^{\sum_{m>0} \frac{1}{m} \mathcal{J}_m p_m}, \quad \gamma^\dagger(p_f) := e^{\sum_{m>0} \frac{1}{m} \mathcal{J}_m p_m}$$
\[ J_m := \sum_{i \in \mathbb{Z}} \psi_i \psi_{i+m}^*, \quad g = \exp \left( \sum_{i,j} A_{ij} \psi_i \psi_j^* \right) \]  

(3.11)

and \( \psi_i, \psi_i^* \) are charged fermions,

\[ [\psi_i, \psi_j]_+ = 0, \quad [\psi_i, \psi_j^*]_+ = \delta_{ij}, \quad [\psi_i^*, \psi_j^*]_+ = 0. \]  

(3.12)

Now one can notice that an embedding into the KP hierarchy of the \( \tau \)-function that depends on only odd time variables can be naturally achieved by introducing the two sets of neutral fermions \( \{ \phi_i, \ i \in \mathbb{Z} \} \)

\[ \phi_j = \frac{\psi_j + (-1)^j \psi_j^*}{\sqrt{2}}, \quad \hat{\phi}_j = i \frac{\psi_j - (-1)^j \psi_j^*}{\sqrt{2}} \]  

(3.13)

with the canonical anticommutation relations:

\[ [\phi_j, \phi_k]_+ = (-1)^j \delta_{j+k,0}, \quad [\hat{\phi}_j, \hat{\phi}_k]_+ = (-1)^j \delta_{j+k,0}, \quad [\phi_j, \hat{\phi}_k]_+ = 0 \]  

(3.14)

In particular, \((\phi_0)^2 = \frac{1}{2}\). Acting on the left and right vacua \(|0\rangle, \langle 0|\), one obtains

\[ \phi_{-j}|0\rangle = 0 = \langle 0|\phi_j, \quad \forall j > 0. \]  

(3.15)

and similarly for the second set of fermions.

The pairwise expectation values are:

\[ \langle 0|\phi_j \phi_k|0\rangle = \begin{cases} (-1)^k \delta_{j,-k} & \text{if } k > 0, \\ \frac{1}{2} \delta_{j,0} & \text{if } k = 0, \\ 0 & \text{if } k < 0, \end{cases} \]  

(3.16)

For (Euclidean) Fermi fields \( \phi(\zeta) := \sum_{n \in \mathbb{Z}} \zeta^n \phi_n \), where \( \zeta \in \mathbb{CP}^1 \), the variables \( T = -\log |\zeta| \) and \( \varphi = \arg \zeta \) are treated as time and space variables, respectively. The multipoint correlation functions use the chronological ordering convention that states that the field assigned to a larger time variable is put to the right of the field assigned to a smaller one, and the permutation sign should be taken into account. For example, if \(|\zeta_1| > |\zeta_2|\), one writes \(|0\rangle \phi(\zeta_2) \phi(\zeta_1) 0\rangle\) as \(-(|0\rangle \phi(\zeta_1) \phi(\zeta_2) 0\rangle\), which by (3.16) is equal to \(-\frac{1}{2} + \zeta_2 \zeta_1^{-1} - \cdots = -\frac{1}{2} + \zeta_2 \zeta_1^{-1} \cdots\). For a given strict partition \( \alpha = (\alpha_1, \ldots, \alpha_{2r}) \) where \( \alpha_{2r} \geq 0 \), we use the notation \( \Delta^*(\zeta_\alpha) \) defined by the relation

\[ \langle 0\rangle \phi(\zeta_{\alpha_1}) \cdots \phi(\zeta_{\alpha_{2r}})|0\rangle = 2^{-r} \prod_{i < j \leq 2r} \frac{\zeta_{\alpha_i} - \zeta_{\alpha_j}}{\zeta_{\alpha_i} + \zeta_{\alpha_j}} =: 2^{-r} \Delta^*(\zeta_\alpha) \]  

(3.17)
The first equality is obtained by the Wick theorem and by a simple analytical reasoning about poles and zeroes.

An important fact is that the factor \( \gamma(p_f) \) becomes a product of two factors at all even times vanishing:

\[
\gamma(p_f) \bigg|_{p_{2k}=0} = e^{\sum_{m \in \mathbb{Z}^+_{\text{odd}}} \left( \frac{2}{m} p_m J_m + \frac{2}{m} p_m \hat{J}_m \right)}
\]

(3.18)

where

\[
J_m = \frac{1}{2} \sum_{i \in \mathbb{Z}} (-1)^i : \phi_{-i-m} \phi_i :
\]

(3.19)

and similarly for \( \gamma(p^*) \). Here, \( : X : \) denotes the normal ordering (which is \( : X := X - \langle 0 | X | 0 \rangle \) for \( X \) quadratic in fermions).

Now we can consider only “half” of this system leaving only one set of the neutral fermions. In this system, there are two mutually commuting Abelian groups of the BKP flows

\[
\gamma(p) := e^{\sum_{m \in \mathbb{Z}^+_{\text{odd}}} \frac{2}{m} J_m p_m}, \quad \gamma(p^*) := e^{\sum_{m \in \mathbb{Z}^+_{\text{odd}}} \frac{2}{m} J_{-m} p_m}
\]

(3.20)

One has

\[
J_n J_m - J_m J_n = \frac{n}{2} \delta_{n+m,0}
\]

(3.21)

and

\[
J_m |0\rangle = 0 = \langle 0 | J_{-m}, \quad \forall m > 0
\]

(3.22)

which results in

\[
\gamma(p) |0\rangle = |0\rangle, \quad \langle 0 | \gamma^\dagger(p) = \langle 0 |
\]

(3.23)

and in

\[
\gamma(p) \gamma^\dagger(p^*) = e^{\sum_{m \in \mathbb{Z}^+_{\text{odd}}} \frac{2}{m} p_m p_m^* \gamma^\dagger(p^*) \gamma(p)}
\]

(3.24)

### 3.2.2 Q Schur function as fermionic average

One can construct the \( Q \) Schur functions as fermionic averages much similar to how the ordinary Schur functions are realized as fermionic averages of charged fermions [27].

Any nonzero partition with distinct parts (also known as strict partition), say \( \alpha \), can be written as \( \alpha = (\alpha_1, \ldots, \alpha_r) \), where \( r \) is an even number, and \( \alpha_1 > \cdots > \alpha_r \geq 0 \).

As usual, the length of a partition is the number of nonvanishing parts of \( \alpha \), and it is...
denoted $\ell(\alpha)$; thus, $\ell(\alpha)$ is an odd number if and only if $\alpha_r = 0$. The length of the zero partition is 0. Following [39], we denote by DP the set of all partitions with distinct parts (or the same: the set of all strict partitions). The weight of a partition $\lambda$ is denoted $|\lambda| := \sum_{i=1}^{\infty} \lambda_i$.

Let us introduce the notation

$$
\Phi_{\alpha} = 2^n \phi_{\alpha_1} \cdots \phi_{\alpha_{2n}}, \quad \Phi_\alpha^\dagger = (-1)^{\sum_{i=1}^{2n} \alpha_i} 2^n \phi_{\alpha_{2n-1}} \cdots \phi_{\alpha_1},
$$

$$
J_\Delta^\dagger = J_{\Delta_1} \cdots J_{\Delta_m}, \quad J_\Delta = J_{-\Delta_m} \cdots J_{-\Delta_1}.
$$

(3.25)

where $\Delta = (\Delta_1, \ldots, \Delta_m)$ is a partition with odd parts (the total set of such partitions we denote OP). We have

$$
\langle 0 | \Phi_\beta^\dagger \Phi_\alpha | 0 \rangle = 2^\ell(\alpha) \delta_{\alpha,\beta} = < Q_\alpha, Q_\beta >,
$$

(3.26)

$$
\langle 0 | J_\Delta^\dagger J_\Delta | 0 \rangle = 2^{-\ell(\Delta)} z_\Delta \delta_{\Delta,\Delta} = < p_\Delta, p_\Delta >,
$$

(3.27)

where $<, >$ denotes the scalar product in the space of symmetric functions, see (8.12) in [39]. Here, $z_\Delta$ is the standard symmetric factor of the Young diagram (order of the automorphism), and $p_\alpha := \prod_i \phi_{\alpha_i}$. To get the first equality in (3.26), we use

$$
\phi_0^2 = \frac{1}{2} \text{ for odd } \ell(\alpha).
$$

The key relation we need was found in [67] and, in our notations, is

$$
Q_\alpha \{ p \} = \langle 0 | \gamma(p) \Phi_\alpha | 0 \rangle = \langle 0 | \Phi_\alpha^\dagger \gamma^\dagger(p) | 0 \rangle
$$

(3.28)

which results in (3.5) according to the Pfaffian form of the Wick theorem with the choice $p_m = \sum_i x_i^m$.

As a result, one has

$$
\gamma^\dagger(p) | 0 \rangle = \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} | \Phi_\alpha | 0 \rangle Q_\alpha \{ p \} = \sum_{\Delta \in \text{OP}} 2^\ell(\Delta) J_\Delta | 0 \rangle \frac{p_\Delta}{z_\Delta},
$$

$$
\langle 0 | \gamma(p) = \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} | Q_\alpha \{ p \} | 0 \rangle | \Phi_\alpha^\dagger = \sum_{\Delta \in \text{OP}} 2^\ell(\Delta) \frac{p_\Delta}{z_\Delta} \langle 0 | J_\Delta^\dagger
$$

(3.29)

thus, one gets

$$
\sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_\alpha \{ p^* \} = \langle 0 | \gamma(p) \gamma^\dagger(p) | 0 \rangle = \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_\alpha \{ p^* \}
$$

(3.30)

where the first equality follows from (3.22 and (3.23)), see also (8.13) in [39].
Let us write down the unity operator which acts in the Fock space spanned by \( \Phi_\alpha |0\rangle \):

\[
1 = \sum_{\alpha \in \text{DP}} \Phi_\alpha |0\rangle 2^{-\ell(\alpha)} \langle 0| \Phi_\alpha^\dagger = \sum_{\Delta \in \text{OP}} J_\Delta |0\rangle \frac{2^{\ell(\Delta)}}{z_\Delta} (0| J_\Delta^\dagger)
\] (3.31)

The sets \( \{ \Phi_\alpha |0\rangle, \alpha \in \text{DP} \} \) and \( \{ J_\Delta |0\rangle, \Delta \in \text{OP} \} \) form two different bases in the fermionic Fock space.

### 3.2.3 Sergeev characters

Now note that the quantity

\[
\chi_\alpha(\Delta) := 2^{-\ell(\alpha)} \langle 0| J_\Delta^\dagger \Phi_\alpha |0\rangle = 2^{-\ell(\alpha)} \langle 0| \Phi_\alpha^\dagger J_\Delta |0\rangle
\] (3.32)

is nothing but the character of the Sergeev group [64]. To see this, we notice that

\[
\Phi_\alpha |0\rangle = \sum_{\alpha} \chi_\alpha(\Delta) \Phi_\alpha |0\rangle, \quad \langle 0| J_\Delta^\dagger = \sum_{\alpha} \chi_\alpha(\Delta) \langle 0| \Phi_\alpha
\]

\[
\Phi_\alpha |0\rangle = \sum_{\Delta} \frac{2^{\ell(\alpha) + \ell(\Delta)}}{z_\Delta} \chi_\alpha(\Delta) J_\Delta |0\rangle, \quad \langle 0| \Phi_\alpha^\dagger = \sum_{\Delta} \frac{2^{\ell(\alpha) + \ell(\Delta)}}{z_\Delta} \chi_\alpha(\Delta) \langle 0| J_\Delta^\dagger
\] (3.33)

By (3.29), we obtain the bosonization of these formulae:

\[
p_\Delta = \sum_{\alpha} \chi_\alpha(\Delta) Q_{\alpha} \{ p \}, \quad Q_{\alpha} \{ p \} = \sum_{\Delta} \frac{2^{\ell(\alpha) + \ell(\Delta)}}{z_\Delta} \chi_\alpha(\Delta) p_\Delta
\] (3.34)

This is a counterpart of the Fröbenius formula for the \( Q \) Schur functions (see [39, Sec.I.7]), and, hence, \( \chi_\alpha(\Delta) \) are, indeed, characters of the Sergeev group, [24,37].

Introduce

\[
f_\alpha(\Delta) := 2^{-\ell(\Delta)} \langle 0| J_\Delta^\dagger \Phi_\alpha |0\rangle \frac{1}{z_\Delta} \frac{1}{Q_{\alpha} \{ \delta_{k,1} \}} = \frac{2^{\ell(\alpha) + \ell(\Delta)}}{z_\Delta Q_{\alpha} \{ \delta_{k,1} \}} \chi_\alpha(\Delta)
\] (3.35)

(In [44], we used the notation \( \Phi_\alpha(\Delta) \) for this quantity, it is a basic building block for a spin counterpart of the Fröbenius formula for the Hurwitz numbers, see formula (5.4).)

With this quantity,

\[
Q_{\alpha} \{ p \} = Q_{\alpha} \{ \delta_{k,1} \} \sum_{\Delta \in \text{OP}} f_\alpha(\Delta) p_\Delta
\] (3.36)
4 BKP $\tau$-functions

Any vacuum expectation value of the form

$$\tau(p, p^*) = \langle 0 | \gamma(p) g \gamma^\dagger(p^*) | 0 \rangle$$

(4.1)

where $p = (p_1, p_3, \ldots)$ and $p^* = (p_1^*, p_3^*, \ldots)$ are two independent sets of parameters, and

$$g = e^{\sum_{i,k \in \mathbb{Z}} A_{ik} \phi_i \phi_k}$$

(4.2)

is a BKP $\tau$-function with respect to the set of times $p$, and, at the same time, it is a BKP $\tau$-function with respect to the set of times $p^*$, hence, it can be called a two-sided BKP, we call it either BKP $\tau$-function or 2BKP $\tau$-function. Here, $\{A_{ik}\}_{i,k \in \mathbb{Z}}$ are the elements of a doubly infinite skew-symmetric matrix $A$. The choice of matrix $A$ or, what is the same, the choice of $g$ defines a common solution to all equations of the 2BKP integrable hierarchy.

4.1 Hypergeometric BKP vs KP $\tau$-functions

The term hypergeometric $\tau$-function was introduced in [59,60] where it was emphasized that the series

$$\sum_R \prod_{l=1}^p \frac{1}{(a_l)_R} \cdot \text{Schur}_R(x) \cdot \text{Schur}_R(y), \quad (z)_R := \prod_{(i,j) \in R} (z + j - i)$$

(4.3)

This series can be presented in the form

$$\sum_R \text{Schur}_R(p) \text{Schur}_R(p^*) e^{\sum_k iz_k C_k(R)}, \quad C_k(R) := \sum_i (R_i - i + 1/2)^k - (1/2 - i)^k$$

(4.4)

(see, e.g., [6]), where $C_k(R)$ are the eigenvalues of the peculiarly chosen Casimir operators of $GL(\infty)$ in representation $R$ of $SL(N)$ at large enough $N > \ell(R)$. This is a KP $\tau$-function with $g$ of the form [32,57,59,60]

$$g = e^{\sum_{n \in \mathbb{Z}} \psi_n \psi_n^\dagger : \sum_k \frac{1}{k} n^k t_k}$$

(4.5)

Quite similarly, the special choice of $g$,

$$g^\pm = g^\pm (t) = e^{\sum_{n>0} (-1)^n \phi_n \phi_{-n} \left( \sum_{k \in \mathbb{Z}_{\text{odd}}} \frac{1}{k} n^k t_k - \delta^\pm \right)}, \quad \delta^\pm = \frac{i \pi (\pm 1 - 1)}{2},$$

(4.6)
where \( t = (t_1, t_3, \ldots) \) is a set of parameters, describes the family of \( \tau \)-functions called hypergeometric BKP \( \tau \)-functions [56,58]. We will denote the related \( \tau \)-function through \( \tau^{\pm}(p, p^*|t) \).

One can verify that

\[
g^{\pm} \phi_i \left( g^{\pm} \right)^{-1} = \pm e^{\sum_{k \in \mathbb{Z}_{odd}} \frac{1}{k} k_i \phi_i}, \quad i \neq 0
\]

(4.7)

Let us note that \( g^{\pm}|0\rangle = |0\rangle. \) From (3.14), (3.29), it follows that

\[
\tau^{\pm}(p, p^*|t) := \langle 0 | \gamma(p) g^{\pm}(t) \gamma(p^*) | 0 \rangle
\]

\[
= \sum_{\alpha \in \text{DP}} e^{\sum_{k \in \mathbb{Z}_{odd}} \frac{1}{k} k_i \omega_k(\alpha)} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_\alpha \{ p^* \}
\]

\[
\pm \sum_{\alpha \in \text{DP}} e^{\sum_{k \in \mathbb{Z}_{odd}} \frac{1}{k} k_i \omega_k(\alpha)} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_\alpha \{ p^* \}:
\]

(4.8)

i.e.,

\[
\tau^{\pm}(p, p^*|t) = R_{\pm} \cdot \sum_{\alpha \in \text{DP}} e^{\sum_{k \in \mathbb{Z}_{odd}} \frac{1}{k} k_i \omega_k(\alpha)} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_\alpha \{ p^* \}
\]

(4.9)

where \( R_{\pm} \) is the projection operator, and

\[
\omega_k(\alpha) = \sum_{i=1}^{\ell(\alpha)} \alpha_i^k
\]

(4.10)

are spin counterparts of the completed cycles in the case of ordinary Hurwitz numbers. Notice the difference between these quantities and (4.4): in the spin case, the main quantities involve the integers \( \alpha_i \), while the same quantities in the non-spin case, \( \alpha_i - i \). This is related to the fact that, in the former case, all the formulas involve the strict partitions, and, in the latter one, the formulas involve all partitions.

**Remark 4.1** For every hypergeometric BKP \( \tau \)-function (4.9), its square is a hypergeometric KP \( \tau \)-function, with all even times put to zero, details see in [58, sec.3,Prop.2].

The issue of the hypergeometric \( \tau \)-functions can be explained in a different way: the KP \( \tau \)-function can be generally expanded into the Schur functions,

\[
\tau(p) = \sum_{\alpha} c_\alpha \text{Schur}_\alpha \{ p \}
\]

(4.11)

This linear combination solves the KP hierarchy iff the coefficients satisfy the Plücker relations for the infinite-dimensional Grassmannian [27]

\[
c_{(\vec{\alpha} | \vec{\beta})_{[x_i, x_j; y_i, y_j]}} \cdot c_{(\vec{\alpha} | \vec{\gamma})_{[x_i; y_i]}} - c_{(\vec{\alpha} | \vec{\gamma})_{[x_i; y_i]}} \cdot c_{(\vec{\alpha} | \vec{\gamma})_{[x_j; y_j]}} + c_{(\vec{\alpha} | \vec{\gamma})_{[x_i; y_j]}} \cdot c_{(\vec{\alpha} | \vec{\gamma})_{[x_j; y_i]}} = 0
\]

(4.12)
where we used the Fröbenius (hook) parametrization for the Young diagrams: \( \alpha = (x_1, \ldots, x_h | y_1, \ldots, y_h) = (\vec{x} | \vec{y}) \), where \( x_i, y_j \) are the hook arms and legs correspondingly [39], and denoted through \([x_i]; [y_j]\) removing a subset \([x_i]; [y_j]\) from the set of hook legs and arms.

Now we note that if \( c_\alpha \)'s solve these relations, so do \( c_\alpha \prod_{i,j \in \alpha} f(i-j) \) with an arbitrary function \( f(x) \). Since a particular solution to the Plücker relations is given by the Schur functions of arbitrary set of times (parameters), and since the exponential in (4.4) is of the type \( \prod_{i,j \in \alpha} f(i-j) \) [6], we finally come to the hypergeometric \( \tau \)-function (4.4).

Similarly, a linear combination of the \( Q \) Schur functions, \( \tau(p) = \sum_{\alpha \in \mathbb{D}P} c_{\alpha}^{BK} \mathcal{Q}_\alpha[p] \) (4.13)
solves the BKP hierarchy iff the coefficients satisfy the Plücker relations for the isotropic Grassmannian [27]

\[
\begin{align*}
& c_{[\alpha_1, \ldots, \alpha_k]}^{BK} c_{[\alpha_1, \ldots, \alpha_k, \beta_1, \beta_2, \beta_3, \beta_4]}^{BK} - c_{[\alpha_1, \ldots, \alpha_k, \beta_1, \beta_2]}^{BK} c_{[\alpha_1, \ldots, \alpha_k, \beta_3, \beta_4]}^{BK} + \\
& + c_{[\alpha_1, \ldots, \alpha_k, \beta_1, \beta_3]}^{BK} c_{[\alpha_1, \ldots, \alpha_k, \beta_2, \beta_4]}^{BK} - c_{[\alpha_1, \ldots, \alpha_k, \beta_1, \beta_3]}^{BK} c_{[\alpha_1, \ldots, \alpha_k, \beta_2, \beta_3]}^{BK} = 0
\end{align*}
\] (4.14)

Now we again note that if \( c_\alpha^{BK} \)’s solve these relation, so do \( (\pm 1)^{\ell(\alpha)} e^{\sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} t_m \omega_m(\alpha)} c_\alpha^{BK} \). Since a particular solution to the Plücker relations is given by the \( Q \) Schur functions of arbitrary set of times (parameters), we finally come to the hypergeometric \( \tau \)-function (4.9).

### 4.2 Bosonization

Now we need to bosonize the operators that act on the Fock vectors\(^1\). The BKP hierarchy and related objects that we will review in this section were introduced in a series of papers by Kyoto school [12,14,27]. However, we shall use here the approach due to [28,65].

Let \( z \in S^1 \)

\[
V(z, \hat{p}) = \frac{1}{\sqrt{2}} D_\eta e^{\sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} \frac{2}{m} z^m p_m} e^{-\sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} \frac{1}{m} \frac{\partial}{\partial p_m}} \] (4.15)

be the vertex operator as it was introduced in [28]. Here \( D_\eta = \eta + \frac{\partial}{\partial \eta} \), where \( \eta \) is an auxiliary odd Grassmannian variable: \( \eta^2 = 0, D^2_\eta = 1 \). The symbol \( \hat{p} \) denotes the set of two collection \( p_1, p_3, p_5, \ldots \) and \( \frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_3}, \ldots \).

\(^1\) Note that, for the first time, such a bosonization relation was obtained in an unpublished preprint preceding the article [63], the article was not accepted for publication because the result was rather unusual.

\( \square \) Springer
Introducing

\[ 2\theta(z, p) := \sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} \frac{2}{m} z^m p_m - \sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} z^{-m} \frac{\partial}{\partial p_m} = \sum_{m \in \mathbb{Z}^{+}_{\text{odd}}} \frac{2}{m} J^b_m z^m \]  

(4.16)

where

\[ J^b_m(\hat{p}) = \begin{cases} 
  p_m & \text{if } m > 0 \text{ odd} \\
  -\frac{m}{2} \frac{\partial}{\partial p_m} & \text{if } m < 0 \text{ odd} \\
  0 & \text{if } m \text{ even},
\end{cases} \]  

(4.17)

one can rewrite (4.15) as

\[ V(z, p) = \frac{D_n}{\sqrt{2}} \cdot e^{2\theta(z, p)} \]  

(4.18)

where \( \cdot \) denotes the so-called bosonic normal ordering which means that all derivatives are moved to the right.

Using

\[ [J^b_m, J^b_n] = -\frac{m}{2} \delta_{m+n,0}, \]  

(4.19)

one can verify that

\[ V(z_1, p) V(z_2, p) = \frac{1 - z_2 z_1^{-1}}{1 + z_2 z_1^{-1}} :V(z_1, p) V(z_2, p): \]  

(4.20)

which results in the commutation relations

\[ V(z_1, p) V(z_2, p) + V(z_2, p) V(z_1, p) = \sum_{n \in \mathbb{Z}} \left( -\frac{z_1}{z_2} \right)^n = \delta(\varphi_1 - \varphi_2 - \pi) \]  

(4.21)

where the right hand side is the Dirac delta function, \( z_i = e^{\varphi_i}, i = 1, 2 \). For the Fourier modes \( V(z, p) = \sum_{i \in \mathbb{Z}} V_i(p) z_i^i \), one obtains

\[ V_n(p) V_m(p) + V_m(p) V_n(p) = (-1)^n \delta_{n+m} \]  

(4.22)

Now one can note that up to the sign factor the fermionic and bosonic currents have the same commutation relations

\[ J^b_n J^b_m - J^b_m J^b_n = J_m J_n - J_n J_m = \frac{m}{2} \delta_{n+m,0} \]  

(4.23)
and that the operator

\[ \phi(z) := \phi_0 \gamma([z]) \gamma^\dagger(-[z^{-1}]) \]  

(4.24)

where \([z] := (z, z^3, z^5, \ldots)\), has the same commutation relations as the vertex operator (4.15). This describes a correspondence between bosonic and fermionic operators.

4.3 \(W_{1+\infty}^B\) algebra

The general construction of the algebra \(W_{1+\infty}^B\) is presented in [65,66]. In this section, we derive all the relations important for our work. Following the standard procedure, one can expand the product of the two vertex operators in the generators of \(W_{1+\infty}^B\) algebra:

\[ \frac{1}{2} V(z e^\frac{y}{2}, p) V(-z e^{-\frac{y}{2}}, p) = \frac{1}{2} 1 + e^{-y} = \frac{1}{4} e^y - 1 \left( \theta(z e^\frac{y}{2}) + \theta(-z e^{-\frac{y}{2}}) \right) \]

(4.25)

As follows from the left hand side of this formula, \(\Omega_{mn}\) vanish when \(n\) and \(m\) have the same parity.

In particular, the operators \(\Omega_{m,0} = J_m^b\) with odd \(m\) form the bosonic current algebra, \(\Omega_{m,1}\) with even \(m\) form the Virasoro algebra, etc. Our main interest is the commutative algebra of operators \(\Omega_n := \Omega_{0,n}\) with odd \(n\). In particular,

\[ \Omega_1 = \sum_{n>0} n p_n \partial_n \]

\[ \Omega_3 = \frac{1}{2} \sum_{n>0} n^3 p_n \partial_n + \frac{1}{2} \sum_{n>0} n p_n \partial_n + 4 \sum_{n_1,n_2,n_3 \text{ odd}} p_{n_1} p_{n_2} p_{n_3} (n_1 + n_2 + n_3) \partial_{n_1+n_2+n_3} \]

\[ + 3 \sum_{n_1+n_2+n_3 \text{ odd}} p_{n_1} p_{n_2} n_4 \partial_{n_4} + \sum_{n_1,n_2,n_3 \text{ odd}} p_{n_1+n_2+n_3} \partial_{n_1} \partial_{n_2} \partial_{n_3} \]  

(4.26)
The fermionic counterpart of (4.25) is much simpler:

\[
\frac{1}{2} : \phi(ze^{\frac{\chi}{2}})\phi(-ze^{-\frac{\chi}{2}}) := \frac{1}{2} \sum_{m,j \in \mathbb{Z}} z^m e^{\frac{\chi}{2} (m+2j)} (-1)^j : \phi_{m+j}\phi_{-j} :
\]

\[
:= \sum_{m \in \mathbb{Z}} \frac{1}{n!} \gamma^n m \Omega^F_{mn}
\]

(4.27)

Again, as follows from the left hand side of this formula, \( \Omega_{mn} = 0 \) when \( n \) and \( m \) have the same parity. One gets

\[
\Omega^F_{mn} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \left( \frac{m}{2} + j \right)^n (-1)^j : \phi_{m+j}\phi_{-j} : \quad (4.28)
\]

\[
= \text{res}_z \left( z^{-\frac{m}{2}} \cdot \left( z \frac{\partial}{\partial z} \right)^n \cdot z^{-\frac{m}{2}} \cdot \phi(z) \right) \phi(-z) \frac{dz}{z}
\]

(4.29)

In particular, the current algebra:

\[
J_m := \Omega^F_{-m,0} = \begin{cases} \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^j \phi_{j-m}\phi_{j} & \text{if } m \text{ odd} \\ 0 & \text{if } m \text{ even} \end{cases}
\]

(4.30)

the Virasoro algebra

\[
L^F_m := \Omega^F_{-2m,1} = \frac{1}{2} \sum_{j \in \mathbb{Z}} (j - m) (-1)^j (-1)^j \phi_{j-2m}\phi_{j} \quad (4.31)
\]

and

\[
\Omega^F_n := \Omega^F_{0,n} = \frac{1}{2} \sum_{j \in \mathbb{Z}} (-1)^j j^n : \phi_j\phi_{-j} := \sum_{j=1,3,...} (-1)^j j^n \phi_j\phi_{-j}, \quad n \text{ odd}.
\]

(4.32)

One can easily see that

\[
[\Omega^F_n, \Omega^F_m] = 0
\]

for each pair of \( n, m \).

The boson-fermion correspondence gives rise to the following relation

\[
e^{l^m}_m \Omega_{mn, m^*n^*} \Omega_{m^*, n^*} (\mathbf{p}) \cdot \tau (\mathbf{p}, \mathbf{p}^*) = \langle 0 | \gamma (\mathbf{p}) e^{l^m}_m \Omega_{mn, n^*} g e^{l^m}_n \Omega_{m^*, n^*} \gamma^\dagger (\mathbf{p}^*) | 0 \rangle
\]

\[
= \tau (\mathbf{p}, \mathbf{p}^*) | l^m_m, l^m_{n^*} \rangle
\]
The flows with respect to the parameters $t_{mn}$ and $t_{m'n'}$ are called additional symmetries, see [61, 65, 66].

**Lemma 4.1** For $r \in \mathbb{Z}_{\geq 0}$ introduce

\[
d_{2k,n} := z^{-k} \cdot \left( z \frac{\partial}{\partial z} \right)^n \cdot z^{-k}
\]

\[
\Omega_{2k,r}^{(r)} := \text{res}_{z} \left( d_{2k,n}^{r} \cdot \phi(z) \right) \phi(-z) \frac{dz}{z}
\]

\[
= \sum_j \phi_{2(r+j)j} (-1)^j (j + k)^n (j + 3k)^n \cdots (j + (2r + 1)k)^n \tag{4.33}
\]

Then:

- $\Omega_{2k,1}(1) = \Omega_{2k,1}$ and $\Omega_{2k,1}(r) = 0$ if $r$ is even.
- $[\Omega_{2k,n}(r), \Omega_{2k,n}(r')] = 0$.

We define

\[
\gamma_{m,n}(p) := e^{\sum_{r>0, \text{odd}} p_r \Omega_{2k,r}^{(r)}} \tag{4.34}
\]

where $p = (p_1, p_3, \ldots)$ is a set of parameters.

It can be shown that the vacuum expectation value

\[
\langle 0 | \gamma_{m,n}(p) \gamma_{m',n'}(p^*) | 0 \rangle
\]

where $m < 0$ and $m' > 0$ is a BKP $\tau$-function with respect to $p$ variables and a BKP $\tau$-function with respect to $p^*$ variables.

Consider the Virasoro element

\[
\Omega_{2,1} = L_{-1} = \frac{1}{2} \sum_{j \in \mathbb{Z}} \phi_{2r+j - j} (-1)^j (1 + j)
\]

and the set of related commuting operators

\[
\Omega_{2,1}(r) := L_{-1}(r) = \frac{1}{2} \text{res}_{z} \left( \frac{\partial}{\partial z} \cdot \frac{1}{z} \right)^r \phi(z) \phi(-z) \frac{dz}{z} \tag{4.35}
\]

\[
= \frac{1}{2} \sum_{j \in \mathbb{Z}} \phi_{2(r+j)j} (-1)^j (1 + j)(3 + j) \cdots (2r + j - 1), \quad r = 1, 3, 5, \ldots
\tag{4.36}
\]
We have the following counterpart of (3.28)

**Proposition 4.2**

\[ \langle 0 | \Phi_\alpha^\dagger \gamma_{2,1}(p) | 0 \rangle = \begin{cases} Q_{2 \alpha}(p) \prod \frac{1}{(\alpha_i - 1)!} & \text{each } \alpha_i \text{ is even} \\ 0 & \text{otherwise} \end{cases} \]  

(4.37)

The proof is achieved with the observation that \( \phi_{-j}(p) := (\gamma_{2,1}(p))^{-1} \phi_{-j} \gamma_{2,1}(p) \) is a finite linear combination of the modes \( \phi_{-i} \) where \( 1 \leq i \leq j \) and where \( j \) is odd in case \( j \) is an odd positive number. Then, it follows that the presence of the odd parts leads to zero vacuum expectation value. We take into account that \( \langle 0 | \gamma_{2,1} = \langle 0 | \) and apply the Wick theorem for \( \phi_{\alpha_i}(p) \) and use the definition (3.5) of the projective Schur functions to get the right hand side of (4.37).

**Corollary 4.3** Let \( g = \exp \sum_{i>0} \phi_i \phi_{-i} \log \frac{F(\frac{1}{3})}{(2i-1)!} \) where \( F \) is a function on the lattice \( \mathbb{Z}_{>0} \). We have

\[ \langle 0 | \gamma(p) g \gamma_{2,1}(p^*) | 0 \rangle = \tilde{\gamma}_{2,1}^{\text{Bos}}(p^*; \hat{p}) \cdot 1 = \sum_{\alpha \in \text{DP}} Q_{2 \alpha}(p) Q_{\alpha}(p^*) \prod_{i=1}^{\ell(\alpha)} F(\alpha_i) \]  

(4.38)

where \( \tilde{\gamma}_{2,1}^{\text{Bos}}(p^*; \hat{p}) \) is the bosonic version of \( g \gamma_{2,1}(p^*) g^{-1} \).

Let us note that the right hand side of (4.38) has the form of the right hand side of (1.3) and (1.4).

**Remark 4.2** The perturbation series for the partition functions of the well-known \( N \times N \) two-matrix model (and therefore also one-matrix model) are obtained from the similar construction [59,60]. Let us consider the set of operators commuting with the KP Virasoro generator \( L_{-1} \):

\[ L_{-1}^{\text{KP}}(r) := \text{res}_z \left( \frac{d^r \psi(z)}{d z^n} \right) \psi^\dagger(z), \quad r = 1, 2, 3, \ldots \]  

(4.39)

Then,

\[ e^{\sum_{r>0} \frac{1}{r} p^* L_{-1}^{\text{Bos}}(r)} \cdot 1 = \sum_{\lambda} (N)_{\lambda} \mathbf{s}_\lambda(p) \mathbf{s}_\lambda(p^*) \]  

(4.40)

which is a counterpart of (4.34) for \( \gamma_{2,1}(p) \), (4.35), and (4.38).

At the end of this section, we note that the action of the bosonic version of \( \gamma_{m,n} \) with even \( n \) (and odd \( m \)) on \( 1 \) always gives a version of the BKP hypergeometric tau function.
5 Properties of spin Hurwitz $\tau$-functions

5.1 Integrable properties

Due to the bosonization relations, the $\tau$-function (4.9) can be presented as the result of action of the commuting operators $\{\Omega_n, n \in \mathbb{Z}_{odd}^+\}$ on the simplest $\tau$-functions

**Proposition 5.1** Consider $\tau$-functions (4.9). We have

$$\tau^\pm(p, p^*|t) = e^{\sum_{n \in \mathbb{Z}_{odd}^+} \frac{1}{n} \Omega_n(p)} \cdot \tau^\pm(p, p^*|0)$$

(5.1)

where

$$\tau^\pm(p, p^*|0) := \sum_{\alpha \in DP} 2^{-\ell(\alpha)} Q_\alpha(p) Q_\alpha(p^*) \pm \sum_{\alpha \in DP} 2^{-\ell(\alpha)} Q_\alpha(p) Q_\alpha(p^*)$$

(5.2)

Indeed, by bosonization formulas the action of $\Omega_n$-flows on

$$\tau^\pm(p, p^*|0) = \langle 0|\gamma(p)g^\pm(0)\gamma^+(p^*)|0\rangle$$

is equal to

$$\langle 0|\gamma(p)e^{\sum_{n \in \mathbb{Z}_{odd}^+} \frac{1}{n} \Omega_n^F} g^\pm(0)\gamma^+(p^*)|0\rangle$$

and in addition $\Omega_n^F|\alpha\rangle = |\alpha\rangle \omega_n(\alpha)$ which follows from (3.14).

This $\tau$-function is a $\tau$-function of the BKP hierarchy with respect to two sets of time variables $p$ and $p^*$. However, one can look at its dependence not only on the time variables $p$ and $p^*$, but also on $t$ variables, and, as we demonstrate in sec.5.3, it is related to the solitonic solutions to the KdV hierarchy.

5.2 Cut-and-join equation

A specific form of the exponential means that the BKP $\tau$-function (4.9) is a generating function of the Hurwitz numbers corresponding to the completed cycles. In the case of the ordinary Hurwitz numbers, one could consider the generating function of the simplest double Hurwitz numbers with two branching profiles fixed and all other ramifications being just double ramification points, $H_d(0|\Delta, \Delta^*, \Gamma_1^b)$. This generating function was a (KP) $\tau$-function [57]. On the contrary, in the spin case, even the simplest generating function of the spin Hurwitz numbers $H^\pm(\Gamma_d, \Delta^1, \Delta^2)$ where

$$\Gamma_d = (3, 1^{d-3})$$

(5.3)

is not a $\tau$-function [38,44].

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Indeed, the Hurwitz numbers can be represented as \[ H^\pm(\Delta^1, \ldots, \Delta^k) = R_\pm \cdot \sum_{\alpha \in \text{DP}} (Q_\alpha(\delta_{k,1}))^2 f_\alpha(\Delta^1) \cdots f_\alpha(\Delta^k) \] (5.4)

and (see [38] and [44, Eq.(102) and derivation in sec.6])

\[ f_\alpha(\Gamma_d) = \frac{1}{3} \omega_3(\alpha) - (\omega_1(\alpha))^2 + \frac{2}{3} \omega_1(\alpha), \quad |\alpha| = d \geq 3 \] (5.5)

One can see that this expression is not a linear combination of the completed cycles \(\omega_k(\alpha)\). However, we can still derive an equation for \(\tau^\pm(\mathbf{p}, \mathbf{p}^*|\mathbf{t})\), which is a counterpart of the celebrated cut-and-join equation [23].

To this end, with the help of (5.5), we rewrite (4.9) as follows

\[ \tau^\pm(\mathbf{p}, \mathbf{p}^*|\mathbf{t}) = R_\pm \cdot \sum_{\alpha \in \text{DP}} 2^{-\ell(\alpha)} e^{t_1 d + t_3 (f_\alpha(\Gamma) + d^2 - \frac{2}{3} d)} \]

\[ e^{\sum_{n>3, \text{odd}} \frac{1}{n} t_n \omega_n(\alpha)} Q_\alpha(\mathbf{p}) Q_\alpha(\mathbf{p}^*) \]

\[ = c \sum_{d \geq 0} e^{t_1(d-\frac{1}{2}) + t_3(d-\frac{1}{3})^2} \Phi^\pm_d(\mathbf{p}, \mathbf{p}^*|t_3, t_5, \ldots) \] (5.6)

where \(c = e^{\frac{t_1}{2} - \frac{t_3}{3}}\) and where

\[ \Phi^\pm_d(\mathbf{p}, \mathbf{p}^*|t_3, t_5, \ldots) \]

\[ = R_\pm \cdot \sum_{\alpha \in \text{DP} \atop |\alpha| = d} e^{t_1 f_\alpha(\Gamma) + \sum_{n>3, \text{odd}} \frac{1}{n} t_n \omega_n(\alpha)} Q_\alpha(\mathbf{p}) Q_\alpha(\mathbf{p}^*) \] (5.7)

Let us put \(t_i = 0, \quad i > 3\). Then, we get

\[ \Phi^\pm_d(\mathbf{p}, \mathbf{p}^*|t_3) = \sum_{\Delta^1, \Delta^2 \atop |\Delta^1| = |\Delta^2|} \sum_{b \geq 0} \frac{t_3^b}{b!} H^\pm\left( \Gamma_d^b, \Delta^1, \Delta^2 \right) p_{\Delta^1} p^*_{\Delta^2} \] (5.8)

where

\[ H^\pm\left( \Gamma_d^b, \Delta^1, \Delta^2 \right) = R_\pm \cdot \sum_{\alpha \in \text{DP} \atop |\alpha| = d} (Q_\alpha(\delta_{k,1}))^b (f_\alpha(\Gamma))^b f_\alpha(\Delta^1) f_\alpha(\Delta^2) \] (5.9)

Since the BKP \(\tau\)-function is defined up to a constant factor \(c\), we obtain
Theorem 5.2 Multiply the BKP $\tau$-function (4.9) with the factor $c^{-1} = e^{-\frac{1}{3}t_1 + \frac{1}{3}t_3}$. Then, one gets the cut-and-join equation in the form

\[
\left( \frac{\partial}{\partial t_3} - \left( \frac{\partial}{\partial t_1} \right)^2 \right) \cdot \tau^\pm(\mathbf{p}, \mathbf{p}^*|\mathbf{t}) = \mathcal{W} \cdot \tau^\pm(\mathbf{p}, \mathbf{p}^*|\mathbf{t})
\]  

(5.10)

or, which is the same

\[
\frac{\partial \Phi_3^\pm(\mathbf{p}, \mathbf{p}^*|t_3)}{\partial t_3} = \mathcal{W} \cdot \Phi_3^\pm(\mathbf{p}, \mathbf{p}^*|t_3)
\]

(5.11)

where the cut-and-join operator

\[
\mathcal{W} = \frac{1}{3} \Omega_3(\mathbf{p}) - \Omega_1(\mathbf{p})\Omega_1(\mathbf{p}) + \frac{2}{3} \Omega_1(\mathbf{p}) = -\left( \sum_{n > 0} np_n \partial_n \right)^2 + \left( \frac{2}{3} + \frac{1}{6} \right) \sum_{n > 0} np_n \partial_n \\
+ \frac{1}{6} \sum_{n > 0} n^3 n \partial_n \partial_n + \frac{4}{3} \sum_{n_1, n_2, n_3 \text{ odd}} p_{n_1} p_{n_2} p_{n_3} (n_1 + n_2 + n_3) \partial_{n_1 + n_2 + n_3} \\
+ \sum_{n_1 + n_2 = n_3 + n_4 \text{ odd}} p_{n_1} p_{n_2} p_{n_3} p_{n_4} \partial_{n_1} \partial_{n_2} \partial_{n_3} + \frac{1}{3} \sum_{n_1, n_2, n_3 \text{ odd}} p_{n_1 + n_2 + n_3} \partial_{n_1} \partial_{n_2} \partial_{n_3}
\]

(5.12)

The cut-and-join equation has a form of the quantum heat equation with the potential $\mathcal{W}$.

By analogy with the case of the ordinary Hurwitz numbers analyzed by Boris Dubrovin in [18], we have

Remark 5.1 Each $\Omega_n(\mathbf{p})$ can be treated as a Hamiltonian of the quantum dispersionless modified KdV equation on the circle with the eigenstates given by $Q_\alpha(\mathbf{p})$ and the eigenvalues given by $\omega_n(\alpha)$. If one introduces the Plank constant $\hbar$ and puts $\nu = \hbar^{-\frac{1}{2}} \sum_n e^{i n \phi} J_n^b$, then the first non-trivial Hamiltonian $\hbar^2 \Omega_3$ is

\[
\mathcal{H}_3 = \int_0^{2\pi} d\phi \left( \nu^4 + \hbar (\nu \phi)^2 \right) d\phi
\]

(5.13)

5.3 Solitons of the KdV and BKP hierarchies which generate Hurwitz numbers

So far, we discussed integrable properties with respect to time variables $\mathbf{p}$ that are the variables of the $Q$ Schur functions. In this subsection, we discuss a relation to the soliton solution in $t_n$. That is, we want to show that if we fix both sets $\mathbf{p}, \mathbf{p}^*$ so that they are equal to $\mathbf{p}_1 := (1, 0, 0, \ldots)$, then $\tau^\pm(\mathbf{p}_1, \mathbf{p}_1|\mathbf{t})$ turns into a recognizable soliton $\tau$-function of the KdV equation, where the role of times is played by the set $\mathbf{t} = (t_1, t_3, \ldots)$. 
Theorem 5.3 Consider the $\tau$-function (5.6) where we restrict the times to be $p = p^* = p_1$ and introduce

$$u^\pm(t) = 2 \frac{\partial^2}{\partial t_1^2} \log \tau^\pm(p_1, p_1|t)$$

(5.14)

Then each $u^\pm$ is the ($\infty$)-soliton solution of the KdV hierarchy with respect to the times $t = (t_1, t_3, \ldots)$. In particular,

$$12u^\pm_{t_3} = u^\pm_{t_1 t_3} + 6u^\pm_{t_1}$$

(5.15)

If one puts all $t_i = 0$ for $i > 3$ and denote $(t_1, t_3) = (x, t)$, so that $x, t$ are the space-time coordinates in the standard KdV theory [41], then

$$\tau^\pm(p_1, p_1|x + \log q, t, 0, 0, \ldots) = c(x, t) \sum_{d \geq 3} q^d e^{x(d - \frac{1}{3}) + t\left(d - \frac{1}{3}\right)^2} \sum_{b \geq 0} t^b b! H^\pm \left(\Gamma^b_{d}\right)$$

(5.16)

Proof This solution is well known: it is a multi-soliton $\tau$-function. Indeed, as it follows from (4.9),(3.9) we have

$$\tau^\pm(p_1, p_1|t) = 1 + \sum_{k=1}^\infty \sum_{1 \leq \alpha_1 < \ldots < \alpha_k} 2^{2|\alpha| - k} \prod_{1 < j \leq k} \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}\right) \prod_{i=1}^k e^{\eta(\alpha_i, t) + \delta^\pm} \frac{1}{(\alpha_i!)^2}$$

(5.17)

where $\delta^\pm$ is defined in (4.6) and where

$$\eta(x, t) := \sum_{m=1,3,5,\ldots} \frac{1}{m} t_m x^m$$

(5.18)

which a special case of the general soliton solution of KdV hierarchy given by

$$\tau^\text{Sol}^\pm_{\text{KdV}}(t) = 1 + \sum_{k>0} \sum_{\alpha \in \Delta_P} \left(\Delta^\alpha(\xi_\alpha)\right)^2 \prod_{i=1}^k e^{\eta(\alpha_i, t) + \delta^\pm}$$

(5.19)

$$= 1 + \sum_i \eta^+_i + \sum_{i<j} \frac{(\xi_i - \xi_j)^2}{(\xi_i + \xi_j)^2} e^{\eta^+_i + \eta^+_j} + \sum_{i<j<k} \frac{(\xi_i - \xi_j)^2(\xi_i - \xi_k)^2(\xi_j - \xi_k)^2}{(\xi_i + \xi_j)^2(\xi_i + \xi_k)^2(\xi_j + \xi_k)^2} e^{2\eta^+_i + \eta^+_j + \eta^+_k} + \ldots$$

(5.20)
where $\eta_j^\pm := \eta(\xi_j, t) + a_j - \delta^\pm$ and

$$\Delta^\pm(\zeta_\alpha) := \prod_{i < j \leq k} \frac{\zeta_{a_i} - \zeta_{a_j}}{\zeta_{a_i} + \zeta_{a_j}}$$  (5.21)

We recall that the parameters $\xi_j$ play the role of the soliton momenta, the parameters $a_j$ define initial positions of solitons, these are solitons $u^+ \sim \frac{1}{2} \xi_j^2 \cosh^{-2}\left(\frac{1}{2} \xi_j x + \frac{1}{6} \xi_j^3 t + \frac{1}{2} a_j\right)$ for $\delta^+$ case and solitons $u^- \sim \frac{1}{2} \xi_j^2 \sinh^{-2}\left(\frac{1}{2} \xi_j x + \frac{1}{6} \xi_j^3 t + \frac{1}{2} a_j\right)$ for $\delta^-$ case. Factors (5.21) describe interactions of solitons.

Choosing $\xi_j = j$, $j = 1, 2, 3, \ldots$ and $a_j = -\log 2^{1-2j}(j!)^2$, and taking into account (3.9), (4.9) one obtains (5.17). Formula (5.16) translates the sum over the number of sheets in the covering problem related to $H(\Gamma^b_d)$.

Remark 5.2 To generate the Hurwitz numbers for $d$-sheeted coverings, it is enough to consider the $N$-soliton KdV $\tau$-function with the parameters $\xi_i$ which fill the string $0, 1, \ldots, d$ where $d$ does not exceed the number of solitons $N$. Such a $\tau$-function is holomorphic in $x, t$ variables, and one can write

$$H^\pm(\Gamma^b_d) = (2\pi)^{-1} \text{res}_{i=0} t^{b-1} e^{-t\left(d^2 + \frac{2}{3} d\right)} \int_0^{2\pi} dx e^{-ixd} \tau^\pm_N(1, x, t)$$  (5.22)

A similar solitonic $\tau$-function can be obtained at $p^* = \delta_{k,1} := p_1$: $\tau(p, p_1|t)$ is the $\tau$-function of the two-component BKP hierarchy [28] (or of the Veselov–Novikov hierarchy) with respect to the times $p = (p_1, p_3, \ldots)$ and $t = (t_1, t_3, \ldots)$.

Remark 5.3 Two-component $\tau$-functions are constructed with the two-component fermions. We recall that any two-component BKP (2-BKP) $\tau$-function is a BKP $\tau$-function in $p = (p_1, p_3, \ldots)$ variables and it is also a BKP $\tau$-function in $t = (t_1, t_3, \ldots)$ variables, see [27, 28].

Theorem 5.4 The $\tau$-function (5.6) $\tau^{\pm}(p, p_1|t)$ that generates the Hurwitz numbers is a two-component BKP multisoliton $\tau$-function with respect to the time sets $t$ and $p$.

For the proof, it suffices to represent $\tau^{\pm}(p, p_1|t)$ in the form of an appropriate vacuum expectation value. Indeed, this 2-BKP (Veselov–Novikov) $\tau$-function $\tilde{\tau}^{\text{VN}\pm}(p, t)$ is of the following solitonic type

$$\tilde{\tau}^{\text{VN}\pm}(p, t) = (0)\gamma^{(1)}(t)\gamma^{(2)}(p) g^{\pm}(0) e^{\sqrt{-1} \sum_{i \geq 0} C_{ij}(\xi_i) \phi_i^{(2)}(i)} |0\rangle$$  (5.23)

$$= \sum_{\alpha, \beta \in \text{DP}} 2^{-\frac{1}{2} \tilde{\ell}(\alpha) - \frac{1}{2} \tilde{\ell}(\beta)} C_{\alpha, \beta} e^{\eta(\xi_\alpha, t) - \delta^\pm \Delta^\pm(\xi_\alpha) Q_{\beta}(p)}$$  (5.24)

where the symbol $g^{\pm}(0)$ is given by (4.6) where each $\phi_i$ is replaced by $\phi_i^{(2)}$ and where $\eta(\xi_\alpha, t) \text{ and } \Delta^\pm(\xi_\alpha)$ are given, respectively, by (5.18) and by (5.21), and the set $\{\xi_i\}$ is a set of free parameters (soliton momenta), and a free chosen matrix $C$. 

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defines the interaction between solitons (the net of lines in \((t_1, t_3)\) plane in the tropical approximation of the \(\tau\)-function). The series (5.24) where \(C_{\alpha,\beta} = \det (C)_{\alpha_i,\beta_j}\) is obtained by direct calculation of the vacuum expectation value in (5.23) taking into account (3.17) and (3.28). Then, if we choose \(C_{j,0} = 2^j \frac{1}{j!} \delta_{j,i}, i > 0, \) \(C_{j,0} = 2 \delta_{j,0}\) and put \(\xi_j = j\) we get \(\tilde{\tau}^{\text{VN}}(\mathbf{p}, t) = \tau^\pm(\mathbf{p}, \mathbf{p}_1|t)\) as it follows from (4.9) and (3.9).

When all \(t_i = 0\) for \(i > 3\), this \(\tau\)-function generates the Hurwitz numbers \(H^\pm(\Gamma_d, \Delta)\):

\[
\tau^\pm(\mathbf{p}, \mathbf{p}_1|t) = \sum_{d \geq 0} e^{t_1 d + t_3 (d^2 - \frac{2}{3} d)} \sum_{|\Delta| = d} \sum_{b \geq 0} \frac{t_3^b}{b!} H^\pm \left( \Gamma_d^b, \Delta \right) \mathbf{p}_\Delta
\]

Such a \(\tau\)-function describes “the net of resonant solitons” (it is either the net of regular solitons or the net of singular solitons).\(^2\)

Similarly to the previous case, one gets

**Remark 5.4** To generate the Hurwitz numbers \(H^\pm(\Gamma_d^b, \Delta)\), it is enough to consider the \(N\)-soliton \(\tau\)-function with the parameters \(\xi_i\) which fill the string \(0, 1, \ldots, d\) where \(d\) does not exceed the number of solitons \(N\). Such a \(\tau\)-function is holomorphic in \(x, t\) variables, and one can write

\[
\sum_{|\Delta| = d} H^\pm \left( \Gamma_d^b, \Delta \right) \mathbf{p}_\Delta = (2\pi)^{-1} \sum_{i=0}^{\text{res}} t_i^{-b-1} e^{-t_i \left(d^2 + \frac{2}{3} d\right)} \int_0^{2\pi} dx e^{-ixd} \tilde{\tau}^{\text{VN}}(ix, t)
\]

The superscript “\(-\)” denotes the singular soliton solution, while “\(+\)” is related to the regular one.

### 5.4 Different fermionic expressions generating Hurwitz numbers

Besides the formula (4.8) and the example discussed in (5.23), there are many others representations of the series (4.9) in terms of neutral fermions. These are different embeddings of a given function in many variables into various families of \(\tau\)-functions. Here are some more examples. In these examples, \(\eta(\alpha_i, t)\) and \(\Delta^*(\xi_{\alpha})\) are given, respectively, by (5.18) and by (5.21).

**Lemma 5.5** The multisoliton BKP \(\tau\)-function can be written as the following vacuum expectation value:

\[
\tau_1(t) = \langle 0 | \gamma(t) e^{\sum_{i>j \geq 0} A_{ij} \phi(\xi_i) \phi(\xi_j)} | 0 \rangle = \sum_{\alpha \in DP} 2^{-\frac{1}{2}} \ell(\alpha) A_{\alpha} \Delta^*(\xi_{\alpha}) \prod_{i=1}^{\ell(\alpha)} e^{\eta(\xi_{\alpha_i}, t)}
\]

\(^2\) The best presentation of such nets of regular solitons in the KP case is given in [34]. Singular resonant solitons were considered in [62]. Networks of singular solitons will be described elsewhere.
where \( A_\alpha = \text{Pf} \bar{A} \), where \( \bar{A} \) is the skew symmetric \( \bar{\ell}(\alpha) \times \bar{\ell}(\alpha) \) submatrix of the matrix \( A \) whose entries are selected by the parts of \( \alpha = (\alpha_1, \ldots, \alpha_k) \) as follows:

\[
\bar{A}_{i,j} := A_{\alpha_i,\alpha_j}, \quad i, j \leq k
\]

in case \( k \) is even. And it is

\[
\bar{A}_{i,j} := \begin{cases} A_{\alpha_i,\alpha_j}, & i, j \leq k \\ A_{\alpha_i,0}, & i \leq k, \ j = k + 1 \end{cases}
\]

in case \( k \) is odd. For any choice of \( A \), we get the multisiton BKP \( \tau \)-function (which describes the net of resonant solitons with the momentums given by the set \( \{ \xi_i \} \)). For \( \xi_i = i \), we obtain the BKP multisiton \( \tau \)-function with integer momenta.

As an example, one can choose \( A_{ij} = \frac{1}{2} Q_{Ni,Nj} \{ p \} F(i) F(j) \), \( j > 0 \) and \( A_{i,0} = Q_{Ni,0} \{ p \} F(i) \). Then \( \tau \)-function (5.27) takes the form

\[
\tau_1(t,p) = \sum_{\alpha \in \Delta P} \sum_{i=1}^{\ell(\alpha)} e^{\eta(\xi_\alpha,t)} F(\alpha_i)
\]

The right hand side of (5.30) where \( F(j) = \pm \frac{1}{i!} \) can be equated to \( \tau \)-function (5.23) if we choose \( C_{i,j} \sim \delta_{Ni,j} \) in (5.24).

If we put \( p_k = \delta_{k,1} \) and \( F(i) = \pm \frac{(Ni)!}{i!} \) we get KdV \( \tau \) function (5.17).

**Lemma 5.6** Another example of the BKP \( \tau \)-function (\( A_\alpha \) is the same as in the previous Lemma):

\[
\tau_2(p, p^*) = \langle 0 | \gamma(p) e^{\sum_{i>j \geq 0} A_{ij} \phi_i \phi_j} | 0 \rangle = \sum_{\alpha} \sum_{i=1}^{\ell(\alpha)} 2^{-\ell(\alpha)} A_\alpha \{ p \} \prod_{\alpha} F(\alpha_i)
\]

If we choose \( A \) the same as in example (5.30) then \( \tau \)-function (5.31) takes the form

\[
\tau_2(p, p^*) = \sum_{\alpha \in \Delta P} 2^{-\ell(\alpha)} Q_\alpha \{ p \} Q_{\alpha^*} \{ p^* \} \prod_{i=1}^{\ell(\alpha)} F(\alpha_i)
\]

which is of form considered in Sect. 6 below. For \( N = 1 \), (5.32) is the hypergeometric \( \tau \)-function given by (4.9) if we choose \( F(j) = \pm \exp \sum_m j^m t_m \).

**Lemma 5.7** An example of the 2-component BKP (2-BKP) \( \tau \)-function is

\[
\tau_3(p^{(1)}, p^{(2)}) = \langle 0 | \gamma^{(1)}(p^{(1)}) | 0 \rangle = \sum_{\alpha,\beta \in \Delta P} 2^{-\frac{1}{2} \ell(\alpha) - \frac{1}{2} \ell(\beta)} C_{\alpha,\beta} \{ p^{(1)} \} \{ p^{(2)} \}
\]

where \( C_{\alpha,\beta} = \text{det} \{ \gamma(\alpha) \} \).
For instance, one can choose $C_{ij} = F(j)\delta_{i,N_j} + \sum_{j > 0} C_{i,j}$ and once again get the series in the right hand side in (5.32).

**Lemma 5.8**  Another example of the 2-BKP $\tau$-function is

$$
\tau_4(p^1, p^2) = \langle 0 | \gamma^{(1)}(p^{1})\gamma^{(2)}(p^{2})e^{\sqrt{-1}\sum_{i,j>0} C_{ij}p_i^{1}p_j^{2}}|0 \rangle
$$

(5.34)

$$
= \sum_{\alpha,\beta \in DP} 2^{-\frac{1}{2}\ell(\alpha)-\frac{1}{2}\ell(\beta)} C_{\alpha,\beta} \prod_{i=1}^{\ell(\alpha)} e^{\eta(\zeta_i^{(1)}, p^1)} \prod_{i=1}^{\ell(\alpha)} e^{\eta(\zeta_i^{(2)}, p^2)} \Delta^\alpha_{\alpha} \Delta^\beta_{\beta}
$$

(5.35)

To get KdV multisoliton $\tau$-function it is enough to take $\zeta_i^{(1)} = \zeta_i^{(2)}$, $i \geq 0$. To get $\tau$-function (5.17), we take $\zeta_i = i$ and $C_{ij} = 2^{2j} (j!)^{-2}\delta_{i,j}$, $j > 0$, $C_{i,0} = C_{0,i} = 2\delta_{i,0}$.

**Remark 5.5**  Note that the reduction BKP $\rightarrow$ KdV equation has the simplest form when one uses the two-component BKP approach and imposes the condition

$$
\left( \frac{\partial}{\partial p_i^{(1)}} - \frac{\partial}{\partial p_i^{(2)}} \right) \tau(p^1, p^2) = 0
$$

(5.36)

and the KdV higher times can be identified with $p^1 + p^2$. It can be also formulated in terms of the Lax operator for the Veselov–Novikov equation, which is a 2D Schrödinger operator (with a reduction 2D Schrödinger $\rightarrow$ 1D Schrödinger operator the latter being the Lax operator for the KdV equation). Another way is to present the KdV hierarchy as the so-called rational reduction of the BKP hierarchy.

For some other aspects of relation between the KdV and BKP hierarchies, see [2,12–14,27].

The example of the KdV and of Veselov–Novikov solitons considered above can be presented as a specification of $\tau_1, \ldots, \tau_4$.

**Remark 5.6**  In all cases, the sums in the exponents can be replaced by integrals with the appropriate measures

Any particular fermionic representation is convenient for writing down string equations, which are stationary points for special combinations of $W_B^{\infty}$ symmetries, discussed in Sect. 4.3.

**5.5 Matrix model $\tau$-functions as fermion averages**

The partition functions of the matrix models in external field considered in the Introduction, (1.3) and (1.4), which are hypergeometric BKP $\tau$-functions can be considered as particular cases of described fermion averages.

The $\tau$-function of the BGW model in (1.4) can be presented as examples discussed in the previous subsection by a proper specification of parameters. That is,
Proposition 5.9  The series (1.4) has the following fermionic representations:

\[
\sum_{\alpha \in \Delta \{P\}} 2^{-\ell(\alpha)} Q_\alpha \{p\} Q_\alpha \{\delta_{k,1}\} \prod_{i=1}^{\ell(\alpha)} \left( \frac{(2\alpha_i)!}{\alpha_i!} \right)^2
= (0|\gamma(p) e_{\sum_{n>p} \log \left( \frac{(2n)!}{n!} \right)^2 (-1)^n \phi_n \phi_n \{\delta_{k,1}\} |0)
= (0|\gamma(p) e_{\sum_{i\geq j \geq 0} \frac{1}{2} \left( \frac{(2i)!}{i!i!} \right)^2 \tilde{Q}_{i,j} \phi_i \phi_j |0)
= (0|\gamma(1(p)) \gamma(2)\{\delta_{k,1}\} e^{\sqrt{-1} \sum_{i\geq 0} C_i \left( \frac{(2i)!}{i!i!} \right)^2 \phi_i(1) \phi_i(2) |0)
\]  \tag{5.37}

Here, \( \tilde{Q}_{i,j} = Q_{i,j} \{\delta_{k,1}\} = 2^{i+j} \frac{1}{i!j!} \frac{i-j}{i+j} \); \( \tilde{Q}_{i,0} = 2 Q_{i,0} \{\delta_{k,1}\} = 2^{i+1} \frac{1}{i!} \) and \( C_i = 1, i > 0; C_0 = 2 \), and we used (3.9).

Similarly, the \( \tau \)-function of the Kontsevich model (1.3) can be presented as an example of the \( \tau \)-functions discussed in the previous subsection, and can be treated both as the BKP and as the 2-BKP \( \tau \)-function:

Proposition 5.10  The series (1.3) can be presented as fermionic averages

\[
\sum_{\alpha \in \Delta \{P\}} 2^{-\ell(\alpha)} Q_\alpha \{p\} Q_{2\alpha} \{\delta_{k,3}\} \prod_{i=1}^{\ell(\alpha)} \left( \frac{(2\alpha_i)!}{\alpha_i!} \right)^2
= (0|\gamma(p) e_{\sum_{i\geq j \geq 0} \frac{1}{2} \left( \frac{(2i)!}{i!i!} \right)^2 \tilde{Q}_{2i,2j} \{\delta_{k,3}\} \phi_i \phi_j |0)
= (0|\gamma(1(p)) \gamma(2)\{\delta_{k,3}\} e^{\sqrt{-1} \sum_{i\geq 0} C_i \left( \frac{(2i)!}{i!i!} \right)^2 \phi_i(1) \phi_i(2) |0)
\]  \tag{5.38}

Here, \( \tilde{Q}_{2i,2j} = Q_{2i,2j} \{\delta_{k,3}\} \) and \( \tilde{Q}_{2i,0} = 2 Q_{2i,0} \{\delta_{k,3}\} \), and \( C_i = 1, i > 0 \) and \( C_0 = 2 \).

6 Hypergeometric \( \tau \)-functions composed entirely of characters

In this section, we describe an important construction of particular hypergeometric \( \tau \)-functions from the ratios of characters at special loci \( p_k = \delta_{k,r} \).

6.1 Factorization properties of the \( Q \) Schur functions

The basic relation is

\[
Q_\alpha \{\delta_{k,r}\} = Q_{\alpha \gamma} \{\delta_{k,r}\} \prod_{i=1}^{\ell(\alpha)} F(\alpha_i)
\]  \tag{6.1}

with mutually prime \( N \) and \( r \) and with some function \( F(\alpha_i) \).

Define the doubled Young diagram to be \( d(\alpha) := (\alpha_1, \alpha_2, \ldots | \alpha_1 - 1, \alpha_2 - 1, \ldots) \) in the Frobenius notation [39]. Now suppose we are given a strict partition \( \alpha \) such that
the size of $\alpha$ is divisible by $r$, and the $r$-core of $d(\alpha)$ is trivial (this simultaneously implies that the $r$-core of $d(N\alpha)$ is trivial), this is the condition that $Q_d\{\delta_{k,r}\} \neq 0$. Suppose also that $N$ and $r$ are coprime ($r$ is certainly odd). Then, the following formula is correct:

$$\frac{Q_\alpha\{\frac{r}{2} \cdot \delta_{k,r}\}}{Q_N\alpha\{\frac{r}{2} \cdot \delta_{k,r}\}} = \prod_{i=1}^{\ell(\alpha)} (-1)^{\rho_{N,r}(\alpha_i)} \cdot N^{[\alpha_i/r]} \cdot [N\alpha_i/r]!$$

(6.2)

where $\{\ldots\}$ at the r.h.s. denotes the fractional part of a number, and $[\ldots]$ denotes the integer part.

At first, the formula of type (6.2) was conjectured in [1] for the case $N = 2$, $r = 3$. We derive formula (6.2) in Appendix A, here just point out that the integer-valued function $\rho_{N,r}(x)$ depends only on $(x)_r$ (the value of $x$ mod $r$), and on $(Nx)_r$ (the value of $Nx$ mod $r$). Manifestly, $\rho_{N,r}(0) = 0$, and all other $x_i$ enters the product in pairs $\rho_{N,r}(k) + \rho_{N,r}(r - k)$ so that there is no difference which sign to choose for an individual $(-1)^{\rho_{N,r}(x)}$ in the pair. For the sake of definiteness, let us choose $(-1)^{\rho_{N,r}(k)} = 1$ with $r \geq k > r/2$. Then one has

$$\rho_{N,r}(x) = \begin{cases} 
(Nx)_r - (x)_r & \text{for } 0 < (x)_r < r/2 \\
0 & \text{otherwise}
\end{cases}$$

(6.3)

In fact, there is a more fundamental factorization formula for $p_k$ nonvanishing only at $k$ divisible by $r$, (6.5), and formula (6.2) is its straightforward corollary in a particular case. We discuss this in the next subsection.

### 6.2 Factorization formula

Now we consider a more general factorization formula in the case, when an infinite set of times $t_k$ is nonvanishing. That is, we introduce a set of times $p[r] := (p_1[r], p_3[r], \ldots)$ such that

$$\frac{1}{k} p_k[r] = \frac{1}{j} p_j \delta_{k,jr}, \quad \text{both } r, j \text{ odd}$$

(6.4)

Let us consider a strict partition $\alpha$, and produce $r$ new strict partitions $\mu$, $a^c$, $b^c$ made of $[\alpha_i/r]$. Parting into these $r - 1$ partitions depends on the value $(\alpha_i)_r = x$: the parts with $x = 0$ get to partition $\mu$, the parts with $0 < x \leq (r - 1)/2$ get to partitions $a^c$, $c = x$, and those with $(r - 1)/2 < x < r$ get to partitions $b^c$, $c = r - x$. Thus, the partition $a^c$ at each color $c$ has an associated partition $b^c$.

Thus, the parts of $\alpha$ are parted into three groups:

- parts $r\mu$ that are divisible by $r$
- parts presented as $ra^c + c$ where $c = 1, \ldots, \frac{1}{2}(r - 1)$
- parts presented as $r(b^c + 1) - c$ where $c = 1, \ldots, \frac{1}{2}(r - 1)$
Suppose that \(|\alpha|\) is divisible by \(r\) and that the length of partitions \(a^c\) coincides with those of \(b^c\) (otherwise, \(Q_\alpha\{p_k[r]\} = 0\)). We denote this length through \(\kappa^c\). Then, there is a beautiful factorization formula (which we derive in Appendix B)

\[
Q_\alpha\{p_k[r]\} = (-1)^\omega \cdot 2^{-\frac{1}{2}\ell(\mu)} \cdot \frac{1}{2}(r-1) \cdot \prod_{c=1}^{\kappa^c} S_{(a^c|b^c)}(2p'_k) \cdot \prod_{i=1}^{\kappa} (-1)^{a^c_i + b^c_i + c} \tag{6.5}
\]

where \(p'_k := (p_1, 0, p_3, 0, p_5, \ldots)\), \(S_{(a^c|b^c)}\) is the ordinary Schur function in the Frobenius (hook) notation, and \(\omega\) depends on the order of embedded parts which belong to one of the three groups. Basically, it is not important for our purposes because we will be interested in rescaling of lengths of the parts \(\alpha_i \rightarrow N\alpha_i\) which keeps the order, and we get the same \(\omega\).

The factorization formula (6.5) leads to formula (6.2), see Appendix A for details.

### 6.3 Hypergeometric \(\tau\) functions written entirely by \(Q\)-functions

In this section, we describe an important construction of particular hypergeometric \(\tau\)-functions from the ratios of characters at the special loci \(p_k = \delta_{k,r}\) described in the previous section. It is actually applicable in a more general context, but we present it in the case of the \(Q\) Schur functions and the BKP hierarchy.

The relations (6.1), (6.2) imply that the bilinear combination (1.1), which is a BKP \(\tau\)-function can be specified in the form

\[
\tau_{BKP}\{p, \bar{p}\} = \sum_{\alpha \in DP} Q_\alpha\{p\} Q_{N\alpha}\{\delta_{k,r}\} \cdot \prod_{i=1}^{\ell(\alpha)} f(\alpha_i) \tag{6.6}
\]

Indeed, the restrictions for \(\alpha\) in (6.2) imply that \(Q_{N\alpha} \neq 0\) and, hence, are satisfied, since otherwise \(\alpha\) does not contribute to the sum (6.6).

In its turn, this means that not only the partition function of the Kontsevich model, [42]

\[
\tau_{K3}\{p_k\} = \sum_{\alpha \in DP} \frac{1}{2(\ell(\alpha))} \cdot \frac{Q_\alpha\{p_k\} Q_{\alpha}\{\delta_{k,1}\} Q_{2\alpha}\{\delta_{k,3}\}}{Q_{2\alpha}\{\delta_{k,1}\}} \tag{6.7}
\]

which is of form (6.6) with \(N = 2, r = 3\) [1], is a BKP \(\tau\)-function with the weight function given by

\[
\prod_{i=1}^{\ell(\alpha)} f(\alpha_i) = \frac{Q_\alpha\{\delta_{k,3}\} Q_{\alpha}\{\delta_{k,1}\}}{Q_{2\alpha}\{\delta_{k,3}\} Q_{2\alpha}\{\delta_{k,1}\}} \tag{6.8}
\]

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but a more general combination with arbitrary coprime $N$ and $r$ is still a BKP $\tau$-function.

From (6.2), it also follows that one can choose it as a product of various ratios $\frac{Q_{\alpha}(\delta_{k,r})}{Q_{N\alpha}(\delta_{k,r})}$ still preserving the BKP $\tau$-function:

$$\tau_{BKP}\{p, \bar{p}\} = \sum_{\alpha \in DP} Q_{\alpha}(p) Q_{\alpha}(\delta_{k,r}) \prod_i \left( \frac{Q_{\alpha}(\delta_{k,r})}{Q_{N\alpha}(\delta_{k,r})} \right)^{n_i}$$  (6.9)

with arbitrary sets of (coprime) integers $N_i$, $r_i$, and numbers $n_i$, however, one has to deal carefully with vanishing of $Q_{\alpha}(\delta_{k,r})$ (either choosing proper sets of $r_i$'s, or continuing formula (6.2) to all sets of $\alpha_i$ by definition). This means that one can construct non-trivial hypergeometric BKP $\tau$-functions completely in terms of characters. The cubic Kontsevich and BGW partition functions belong to this class, which once again emphasizes the intimate connection between the hypergeometric and matrix model partition functions, i.e., between hypergeometricity and string equations.

Since the Virasoro algebra acts on the $Q$ Schur functions in a very simple way [7]:

$$\hat{L}_n Q_{\alpha} \left\{ \frac{p_k}{\sqrt{2}} \right\} = \sum_{i=1}^{\ell(\alpha)} (-)^{v_i(\alpha)} (\alpha_i - n) Q_{\alpha - 2n\epsilon_i} \left\{ \frac{p_k}{\sqrt{2}} \right\},$$

$$\hat{L}_n := \sum_{k \in \mathbb{Z}_{red}^+} (k + n) p_k \frac{\partial}{\partial p_{k+n}} + \frac{1}{2} \sum_{a,b \in \mathbb{Z}_{red}^+} ab \frac{\partial^2}{\partial p_a \partial p_b}$$  (6.10)

where $\alpha - 2k\epsilon_i$ denotes the shift of $\alpha_i \rightarrow \alpha_i - 2k$. This can make it shorter than some other lines and thus imply reordering of lines in the diagram to put them back into decreasing order, then $v_i(\alpha, b)$ is the number of lines, which the $i$-th one needs to jump over. It is possible to check that (1.3) and (1.4) satisfy the Virasoro constraints (after an appropriate rescaling of time variables). This illustrates the general claim of [48,49] that superintegrability of matrix models, which underlies these expressions implies both the ordinary integrability and the Virasoro constraints: the two basic properties of matrix model partition functions [51–53]. From the very beginning, it was clear that they are intimately related [22], but a nature of this relation remained obscure. Now we understand that they are just two different corollaries of a more fundamental superintegrability feature $< character >= character$.

7 Conclusion

In this paper, we made a brief review of the details behind the formalism of $Q$ Schur functions, which can be relevant for deeper investigation of the spin Hurwitz partition functions. We put a special emphasis on the “hypergeometric” $\tau$-functions. They are made from Casimir exponentials and give rise to peculiarly-factorized coefficients in the character expansions, which is typically associated with (generalized) hypergeometric series, hence the name. There is a mounting evidence that matrix model
τ-functions (i.e., those which satisfy additional string equations) belong to this class. Until recently, this was not so obvious, because important matrix models were not brought to this form, but recently the reason has been found: the τ-functions for the Kontsevich and BGW models are expanded in the Q Schur functions rather than in the ordinary Schur functions. A posteriori, this is rather obvious because they satisfy KdV rather than the generic KP/Toda hierarchy. Moreover, now it is clear that the generalized Kontsevich model is likewise expressed through appropriately generalized Q Schur functions [50]. In this paper, we, however, concentrated on the standard Q Schur functions and perspectives of spin Hurwitz studies.

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Declaration

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A Derivation of (6.2) from (6.5)

The basic factorization formula (6.5) allows us to obtain formula (6.2) immediately with the following logic. First of all, the basic factorization formula (6.5) expresses the Q Schur functions, $Q_R$ at $p_k[r]$ through those at $p_k$, these two being related by (6.4). In particular, under the choice $p_k[r] = r \delta_k, r$, it expresses the Q Schur functions at $\delta_k, r$ through those at $\delta_k, 1$. Hence, it remains to evaluate the Q Schur functions at $p_k = \delta_k, 1$, and then insert the obtained results into (6.5) in order to obtain finally (A.9). This is what is done in this Appendix in three steps. The factorization formula itself is proved in Appendix B.

Step I. (6.5) explicitly at $p_k = r \delta_k, r/2$.

In order to obtain formula (6.2), we evaluate (6.5) at $p_k = r \delta_k, r/2$. Using (3.9), we get for individual factors in (6.5) in this case:

$$Q_{\mu} \left\{ \frac{1}{2} \cdot \delta_k, 1 \right\} = \prod_{i=1}^{k^0} \frac{1}{\mu_i !} \prod_{i < j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \quad (A.1)$$

$$S_{(a^c | b^c)}(\delta_k, 1) = \frac{1}{\prod_{i=1}^{k^c} a_i^c ! b_i^c !} \frac{\prod_{i < j} (a_i^c - a_j^c)(b_i^c - b_j^c)}{\prod_{i, j} (a_i^c + b_j^c + 1)} \quad (A.2)$$

Step II. (6.5) for $N\alpha$.
Now we evaluate (6.5) for $N\alpha$. To this end, we notice that $N\alpha$ consists of the following parts:

- parts $N r \mu$ that are still divisible by $r$
- parts presented as $N r a^c + N c$ where $N c = N, \ldots, \frac{1}{2} N (r - 1)$
- parts presented as $N r (b^c + 1) - N c$ where $N c = N, \ldots, \frac{1}{2} N (r - 1)$

For each $c$, there exists $p_c$ and $c_N$

$$N c = r p_c + c_N, \quad c_N < r \quad (A.3)$$

Let

$$N < r, \quad N \text{ and } r \text{ are coprime} \quad (A.4)$$

Equation (A.3) maps each $c$ to a certain $c_N$. If $c$ and $c'$ are different, then $c_N (c)$ cannot coincide with either $r - c_N (c')$ or $c_N (c')$. Indeed, suppose it is not correct, and one has the same $c_N$ for two different $c$ ($c$ and $c'$):

$$N (c + c') = r (p_c + p'_c + r) \Rightarrow c + c' = r \frac{N}{N} (p_c + p'_c + 1)$$

However, this is impossible because of (A.4) and because both $c$ and $c'$ are less than $\frac{1}{2} r$. Similarly impossible is

$$N (c - c') = r (p_c - p'_c) \Rightarrow c - c' = r \frac{N}{N} (p_c - p'_c)$$

Notice that the partition associated by $c$ is also associated by $c_N$:

$$(a^c, b^c) = \left( r a^c_i + c, r (b^c_j + 1) - c \right) \rightarrow \left( N r a^c_i + N c, N r (b^c_j + 1) - N c \right)$$

$$= \left( r (N a^c_i + p_c) + c_N, r (N b^c_j - p_c) - c_N \right) \quad (A.5)$$

Thus, one finally obtains

$$Na = r (N a^c_i + p_i) + c_N, \quad Nb = r (N b^c_j + 1) - p_i) - c_N \quad (A.6)$$

Therefore, one gets

$$S_{(Na^c|Nb^c)} \{ \delta_{k,1} \} = \frac{N^{-c}}{\prod_{i=1}^{c} (N a^c_i + p_i)! (N b^c_i - p_i - 1)!} \prod_{i<j} (a^c_i - a^c_j) (b^c_j - b^c_j)$$

$$\prod_{i,j=1}^{c} (a^c_i + b^c_j + 1) \quad (A.7)$$

and

$$Q_{N \mu} \left\{ \frac{1}{2}, \delta_{k,1} \right\} = \frac{1}{\prod_{i=1}^{k} (N \mu_i)!} \prod_{i<j} \frac{\mu_i - \mu_j}{\mu_i + \mu_j} \quad (A.8)$$
Step III. Evaluating the ratio (6.2).
Now, using (A.1)-(A.2) and (A.7)-(A.8), we are ready to evaluate the ratio of the \( Q \) Schur functions in (6.2):

\[
\frac{Q_{\alpha} \{ \frac{r}{2} \cdot \delta_{k,r} \}}{Q_{N\alpha} \{ \frac{r}{2} \cdot \delta_{k,r} \}} = (-1)^g \left( \frac{(\ell(\mu))}{\prod_{i=1}^{\ell(\mu)} (N! / \mu_i!)} \right)
\]

\[
\frac{1}{2} (r-1)^{\ell(\alpha)} \prod_{c=1}^{k^c} N^c \prod_{i=1}^{c^c} \frac{(N\alpha_i^c + p_c)! (N\beta_i^c - p_c - 1)!}{a_i^c! b_i^c!} = (-1)^g \frac{1}{2} v \prod_{i=1}^{\ell(\alpha)} \frac{[N\alpha_i / r]!}{[\alpha_i / r]!}
\]

(A.9)

where \( v \) is the number of parts of \( \alpha \) that are not divisible by \( r \), and \( g \) originates from the sign factors at the r.h.s. of (6.5) and is equal to

\[
g = \sum_{c=1}^{\ell(\alpha)} k^c(c - c_N(c))
\]

(A.10)

where \( c_N(c) \) is given by (A.3). Since this sign factor coincides with (6.3) and \( v = 2 \sum_{i=1}^{\ell(\alpha)} [\alpha_i / r] \), we finally come to formula (6.2).

**B Factorization formula (6.5) from fermion calculus**

Here, we prove (6.5) using the fermion average representation for the \( Q \) Schur functions (3.28). To this end, we consider the fermionic representation for the \( Q \) Schur function (3.28)

\[
Q_{\alpha} \{ p_k[r] \} = 2^{\ell(\alpha)} \langle 0 | \gamma(p[r]) \phi_{\alpha_1} \cdots \phi_{\alpha_{\ell(\alpha)}} | 0 \rangle = 2^{\ell(\alpha)} \langle 0 | \phi_{\alpha_1} (p[r]) \cdots \phi_{\alpha_{\ell(\alpha)}} (p[r]) | 0 \rangle
\]

\[
\phi_k(p[r]) := \gamma(p[r]) \cdot \phi_{\alpha_1} \cdot \gamma(p[r])^{-1}
\]

(B.1)

Using the canonical anticommutation relation (3.14), we have

\[
[J_m, \phi_i] = \phi_{i-m}, \quad m \text{ odd}
\]

(B.2)

and

\[
\phi_j(p[r]) := e^{\sum_{m>0, \text{odd}} \frac{2}{nr} J_m p_m[r]} \phi_j e^{-\sum_{m>0, \text{odd}} \frac{2}{nr} J_{mr} t_m[r]} = \sum_{m \geq 0} \phi_{j-m} h_m \{ 2p_k' \}
\]

(B.3)

where \( h_i \) are complete symmetric functions restricted on the set of odd labeled times:

\[
e^{\sum_{n=0, \text{odd}} \frac{2}{nr} p_n z^n} = \sum_{n \geq 0} z^n h_n \{ 2p_k' \}, \quad h_n \{ 2p_k' \} := S_n \{ 2p_k' \}
\]

(B.4)
Let us note that the exponential (B.4) is also a generating function for the elementary projective Schur functions $Q_{n,0}$:

$$h_n\{2p'_k\} = Q_{(n,0)}\{p_k\} \quad (B.5)$$

For evaluation of the averages at the r.h.s of (B.1), we use the Wick theorem. To this end, we consider the pairwise average:

$$-\langle 0|\phi_a(p[r])\phi_b(p[r])|0 \rangle = \langle 0|\phi_b(p[r])\phi_a(p[r])|0 \rangle = \langle 0|\sum_{m \geq 0} \phi_{b-mr} h_m\{2p'_k\} \sum_{n \geq 0} \phi_{a-nr} h_n\{2p'_k\}|0 \rangle \quad (B.6)$$

where, according to the canonical pairing (3.16), contribute only the terms with

$$a - nr + b - mr = 0 \quad (B.7)$$

This implies distributing the parts of the strict partition into three groups $\mu$, $a^c$ and $b^c$ as above.

**Remark B.1** One may say that there exist a part $\alpha_x$ such that $a_i^c + p_c = [\alpha_x/r]$ and $c = (\alpha_x)_r$, and a part $\alpha_y$, with $[\alpha_y/r] = b_j^c - p_c - 1$ and $(\alpha_y)_r = -c$. In this case, $\langle 0|\phi_{\alpha_x}(p[r])\phi_{\alpha_y}(p[r])|0 \rangle \neq 0$.

Similarly, one may say that there exist a part $\alpha_x$ such that $Na_i^c + p_c = [Na_x/r]$ and $c_N = (Na_x)_r$, and a part $\alpha_y$, with $[Na_y/r] =Nb_j^c - p_c - 1$ and $(Na_y)_r = -c_N$. In this case, $\langle 0|\phi_{Na_x}(p[r])\phi_{Na_y}(p[r])|0 \rangle \neq 0$.

One obtains for a pair of $a^c$ and $b^c$ of the same color $c$ (we will denote here $a = ra^c + c$, $b = rb^c + r - c$) that (B.6) is equal to

$$\langle 0|\phi_{-c}\phi_c|0 \rangle h_{b^c+1}\{2p'_k\}h_{a^c}\{2p'_k\} + \cdots + \langle 0|\phi_{-a}\phi_a|0 \rangle h_{a^c+b^c+1}\{2p'_k\}h_{0}\{2p'_k\} = (-1)^{c}h_{b^c+1}\{2p'_k\}h_{a^c}\{2p'_k\} + \cdots + (-1)^{a+1}h_{a^c+b^c}\{2p'_k\}h_{1}\{2p'_k\} + (-1)^{a}h_{a^c+b^c+1}\{2p'_k\}h_{0}\{2p'_k\} \quad (B.8)$$

For instance, for $r = 3$ (i.e., there is the single color $c = 1 = \frac{1}{2}(r - 1)$), if taking $a^1 = a^2 = 0$, i.e., $a = 1$, $b = 2$, one obtains

$$\langle 0|\phi_2(p[3])\phi_1(p[3])|0 \rangle = \langle 0|\phi_1\phi_1|0 \rangle h_{1}\{2p'_k\}h_{0}\{2p'_k\} = (-1)^{1}h_{1}\{2p'_k\}h_{0}\{2p'_k\} = -2p_1 \quad (B.9)$$

Now notice that, since $r$, $j$ are odd, one has from (B.4)

$$h_i\{2p'_k\} = (-1)^{i}h_i\{-2p'_k\} \quad (B.10)$$

therefore (B.8) is written as

$$(-1)^{c+a^c}h_{b^c+1}\{2p'_k\}h_{a^c}\{-2p'_k\} + \cdots + (-1)^{a}h_{a^c+b^c}\{2p'_k\}h_{1}\{-2p'_k\}$$
\[ +(-1)^a h a^c + b^c + 1 \{ 2 p'_k \} h_0 \{-2 p'_k \} \]  \hspace{1cm} (B.11)

(the parity of \( a = ra^c + c \) is equal to that of \( a^c + c \) because \( r \) is odd).

We compare (B.11) with the one-hook Schur function [39]

\[ S_{(j|k)} \{ 2 p'_k \} = \sum_{i=0}^{k} h_{j+i+1} \{ 2 p'_k \} h_{k-i} \{-2 p'_k \} \]  \hspace{1cm} (B.12)

and obtain that

\[ \langle 0 | \gamma(p[r]) \phi_{r b^c + c} \phi_{r a^c + c} | 0 \rangle = (-1)^{b^c + ra^c + c} S_{(a^c|b^c)} \{ 2 p'_k \} \]
\[ = (-1)^{b^c + a^c + c} S_{(a^c|b^c)} \{ 2 p'_k \} \]  \hspace{1cm} (B.13)

The case of sub-partition \( \mu \) can be considered in a similar way. For a pairwise averages, using (B.3) in the same way as before, and using (B.5) and (3.1), one gets

\[ \langle 0 | \phi_{r \mu}(p[r]) \phi_{r \nu}(p[r]) | 0 \rangle = \sum_{m \geq 0} \phi_{\mu-mr} h_m \{ 2 p'_k \} \sum_{n \geq 0} \phi_{\nu-nr} h_n \{ 2 p'_k \} | 0 \rangle \]
\[ = 2^{-1} Q \{ \mu, \nu \} \{ p_k \} \]  \hspace{1cm} (B.14)

(here \( \mu, \nu \) are numbers). The Pfaffian of the Wick theorem yields the projective Schur function labeled by the partition \( \mu \).

As one can see, after re-numbering, the neutral fermions from complementary groups \( ac \) and \( bc \) are quite similar to the charged fermions, while the neutral fermions in the group \( \mu \) up to re-numbering the Fourier modes remain to be neutral (3.16) inside average (B.1).

Finally, applying the Wick theorem to evaluate averages with all three groups of fermions, one obtains (6.5).

### C Multicomponent KP and BKP \( \tau \)-functions

Following [27], one can define two-component charged fermions by \( \psi_j^{(i)} = \psi_{2j+i} \), \( \psi_{\bar{j}}^{(i)} = \psi_{2j+i}^{\dagger} \), where \( i = 1, 2 \) is the component number. Due to (3.12), we see that the fermions with different components anticommute. For each given component, one can construct the neutral fermions labeled with the same component number via (3.13). The neutral fermions related to different components anticommute. We construct currents labeled with the component number and the corresponding evolution operators,

\[ \gamma^{(i)}(p_j) = e \sum_{k>0, \text{odd}} \frac{1}{\bar{z}} p_k^{(i)} J_k^{(i)} \hspace{1cm} J_k^{(i)} = \sum_{n \in \mathbb{Z}} (-1)^n \phi_{-k-n}^{(i)} \phi_n^{(i)} \]  \hspace{1cm} (C.1)
Then any vacuum expectation value

\[ \tau(p^{(1)}, p^{(2)}) = \langle 0 | \gamma^{(i)}(p^{(1)}) \gamma^{(i)}(p^{(2)}) e^{-A} | 0 \rangle \]  
(C.2)

where \( A \) is quadratic in the neutral fermions is a two-component BKP \( \tau \)-function, and the sets \( p^{(1)} \) and \( p^{(2)} \) are called the sets of 2-BKP higher times. There is an important

**Remark C.1** If one fixes the set \( p^{(2)} \) (the set \( p^{(1)} \)), then \( \tau(p^{(1)}, p^{(2)}) \) can be considered as a one-component BKP \( \tau \)-function with respect to the set \( p^{(1)} \) (to the set \( p^{(2)} \)).

Details may be found in [27] or in [28].

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