Infectious Random Walks*

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Abstract

We study the dynamics of information (or virus) dissemination by \( m \) mobile agents performing independent random walks on an \( n \)-node grid. We formulate our results in terms of two scenarios: broadcasting and gossiping. In the broadcasting scenario, the mobile agents are initially placed uniformly at random among the grid nodes. At time 0, one agent is informed of a rumor and starts a random walk. When an informed agent meets an uninformed agent, the latter becomes informed and starts a new random walk. We study the broadcasting time of the system, that is, the time it takes for all agents to know the rumor. In the gossiping scenario, each agent is given a distinct rumor at time 0 and all agents start random walks. When two agents meet, they share all rumors they are aware of. We study the gossiping time of the system, that is, the time it takes for all agents to know all rumors. We prove that both the broadcasting and the gossiping times are \( \Theta(n/\sqrt{m}) \) w.h.p., thus achieving a tight characterization up to logarithmic factors. Previous results for the grid provided bounds which were weaker and only concerned average times. In the context of virus infection, a corollary of our results is that static and dynamically moving agents are infected at about the same speed.

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1 Introduction

The dynamics of multiple random walks moving in a common domain provides an attractive combinatorial framework for studying diffusion processes such as rumor spreading and virus infection. In this work we consider two related scenarios on an $n$-node square grid (or torus) $G_n$.

**Broadcasting scenario:** $m$ agents are initially placed uniformly and independently at random in the nodes of $G_n$. At time 0 one agent is informed of a rumor and starts a random walk. When an informed agent meets an uninformed agent at a node, the latter becomes informed of the rumor and starts its own random walk. How long does it take until all agents are informed (broadcasting time)?

**Gossiping scenario:** $m$ mobile agents start independent random walks from $m$ nodes of $G_n$ chosen uniformly and independently at random. At time 0 each agent has a distinct rumor. When two agents meet, each receives a copy of all rumors carried by the other agent. How long does it take until all of the $m$ agents have received all of the $m$ rumors (gossiping time)?

We prove that both the broadcasting and the gossiping times are $\tilde{\Theta}(n/\sqrt{m})$ with high probability, thus achieving a tight characterization (up to polylogarithmic factors in $n$) of the complexity of rumor spreading in both scenarios. In the context of virus infection, a corollary of our results is that static agents placed at random locations and dynamic agents moving as independent random walks are infected at about the same speed.

Although at a first glance the two models look similar, we do not have a simple reduction between them, hence we develop separate upper and lower bound proofs for each scenario. While the general approach employed in both cases is similar, there are some important differences in addressing the time dependencies, since in the first scenario random walks hit static agents, while in the second, random walks of informed agents need to collide with the random walks of the uninformed ones.

It is not hard to verify a $\tilde{\Theta}(n^2/m)$ tight bound for the broadcasting and gossiping times on an $n$-node line or ring. Combined with our results, this yields a $\tilde{\Theta}\left((n^2/m)^{1/d}\right)$ bound for $n$ node $d$-dimensional grids, with $d = 1, 2$. It remains open whether this relation generalizes to finite grids of higher dimensions.
1.1 Related work

Information dissemination has been extensively studied in the literature under a variety of scenarios and objectives. Due to space limitations, we restrict our attention to the results more directly related to our work.

A prolific line of research has addressed broadcasting and gossiping in static graphs, where the nodes of the graph represent active entities which exchange messages along incident edges according to specific protocols (e.g., push, pull, push-pull). The most recent results in this area relate the performance of the protocols to expansion properties of the underlying topology, with particular attention to the case of social networks, where broadcasting is often referred to as rumor spreading [6]. (For a relatively recent, comprehensive survey on this subject, see [12].)

With the advent of mobile ad-hoc networks there has been growing interest in studying information dissemination in dynamic scenarios, where a number of agents move either in a continuous space or along the nodes of some underlying graph and exchange information when their positions satisfy a specified proximity constraint. In [7, 8] the authors study the time it takes to broadcast information from one of \( m \) mobile agents to all others. The agents move on a square grid of \( n \) nodes and in each time step, an agent can (a) exchange information with all agents at distance at most \( R \) from it, and (b) move to any random node at distance at most \( \rho \) from its current position. The results in these papers only apply to a very dense scenario where the number of agents is linear in the number of grid nodes (i.e., \( m = \Theta(n) \)). They show that the broadcasting time is \( \Theta(\sqrt{n}/R) \) w.h.p., when \( \rho = O(R) \) and \( R = \Omega(\log n) \) [7], and it is \( O((\sqrt{n}/\rho) + \log n) \) w.h.p., when \( \rho = \Omega(\max\{R,\sqrt{\log n}\}) \) [8]. These results crucially rely on \( R + \rho = \Omega(\sqrt{\log n}) \), which implies that the range of agents' communications or movements at each step defines a connected graph.

In more realistic scenarios, like the ones adopted in this paper, the number of agents is decoupled from the number of locations (i.e., the graph nodes) and a smoother dynamics is enforced by limiting agents to move only between neighboring nodes. A reasonable setting consists of a set of multiple, simple random walks on a graph, one for each agent, with communication between two agents occurring when they meet at the same node. One variant of this setting is the so-called Frog Model (corresponding to our Broadcasting scenario), where initially one of \( m \) agents is active (i.e., is performing a random walk), while the remaining agents do not move. Whenever an active agent hits an inactive one, the latter is activated and starts its own random walk. This model was mostly studied in the infinite...
grid focusing on the asymptotic (in time) shape of the set of vertices containing all active agents [3, 13]. A model similar to our Gossiping scenario is often employed to model the spreading of computer viruses in networks and the spreading time is also referred to as infection time. In [9], the authors provide a general bound on the average infection time when \( m \) agents (one of them initially affected by the virus) move in an \( n \)-node graph. For general graphs, this bound is \( O(t^* \log m) \), where \( t^* \) denotes the maximum average meeting time of two random walks on the graph, and the maximum is taken over all pairs of starting locations of the random walks. Also, in the paper tighter bounds are provided for the complete graph and for expanders. Observe that the \( O(t^* \log m) \) bound specializes to \( O(n \log n \log m) \) for the \( n \)-node grid by applying the known bound on \( t^* \) of [1]. A tight bound of \( \Theta(n \log n \log m/m) \) on the infection time on the grid is claimed in [17], based on a rather informal argument where some unwarranted independence assumptions are made. Our results show that this latter bound is incorrect.

Finally, a related line of research deals with the cover time of a random walk on a graph, that is, the expected time when all of the graph nodes are touched by the random walk. (See [1] for a comprehensive account of the relevant literature.) The cover time is strictly related to the hitting time [4], namely the average time required of a random walk to reach a specified node. For \( n \)-node meshes, it is known that the hitting time is \( O(n \log n) \), while the cover time is \( O(n \log^2 n) \) [18, 5]. Bounds on the speed-up achieved on the cover time by multiple random walks as opposed to a single one are proved in [2, 10].

### 1.2 Organization of the paper

The rest of the paper is organized as follows. In Section 2, we define the problem of interest in the two scenarios and establish some technical facts which are used in the analysis. Section 3 and Section 4 contain our results for the Broadcasting and Gossiping scenario, respectively.

### 2 Preliminaries

We study the dynamics of multiple independent random walks moving on an \( n \)-node 2-dimensional square grid \( G_n = (V_n, E_n) \), where \( V_n = \{(i,j) | 1 \leq i, j \leq \sqrt{n} \} \), and \( E_n = \{((i,j), (i,j+1)) | 1 \leq i \leq \sqrt{n}, 1 \leq j < \sqrt{n} \} \cup \{((i,j), (i+1,j)) | 1 \leq i < \sqrt{n}, 1 \leq j \leq \sqrt{n} \} \cup \{(1,j), (\sqrt{n}, j), (i,1), (i, \sqrt{n}) | 1 \leq i, j \leq \sqrt{n} \} \). We also add self loops to the boundary nodes \( \{(1,j), (\sqrt{n}, j), (i,1), (i, \sqrt{n}) | 1 \leq i, j \leq \sqrt{n} \} \) so to avoid bipartiteness and equalize the steady state distribution. We remark that all
results in the paper can be immediately ported to the torus graph, where wrap-around edges substitute the self-loops.

Although the two scenarios are defined with respect to a fixed number $m$ of agents distributed uniformly and independently at random among the nodes of $G_n$ (the exact model), technically, when deriving the upper bounds to broadcasting and gossiping times, it is easier to work with a slightly modified model in which each node $v$ originally holds $m_v$ agents, where $m_v$ is distributed as a binomial variable $B(m, 1/n)$ independently of the other nodes (the binomial model). We denote by $p(n) = m/n$ the density of the binomial model, and by $\tilde{m}$ the random variable denoting the number of agents in a given instance of the model. For a sufficiently large $p(n)$, we have $\tilde{m} = \Theta(m)$ with high probability:

**Lemma 1.** Let $p(n) = m/n \geq (17 \log n)/n$, then, with probability $1 - 1/n^2$, the number of agents in the system, $\tilde{m}$, satisfies $\frac{1}{2}np(n) \leq \tilde{m} \leq \frac{3}{2}np(n)$.

**Proof.** The number of agents is a binomial random variable with expectation $m = 17 \log n$. Applying a Chernoff–Hoeffding bound to their number, yielding

$$\Pr\left( |\tilde{m} - m| \geq \frac{1}{2} m \right) \leq 2e^{-\frac{1}{8}m} \leq \frac{1}{n^2},$$

for a sufficiently large $n$. \hfill \qed

Observe that a high-probability result for the binomial model with $p(n) = m/n$ implies a similar high-probability result in the exact model with $m$ agents (see [15, Corollary 5.9]).

Moving agents follow independent, simple, symmetric random walks, that is, at each step an agent moves to a neighbor of its current location, chosen uniformly at random. Moreover, time is discrete, and moves are synchronized. For a given rumor $r$, we say that an agent is informed (of $r$) if it has a copy of $r$, otherwise it is uninformed (of $r$). In both the broadcasting and gossiping scenarios defined in the introduction, whenever an agent informed of $r$ meets another agent uninformed of $r$, the latter becomes informed of $r$.

**Definition 1** (broadcasting time, gossiping time). In the Broadcasting scenario, the broadcasting time $T_B$ is the first time at which all agents are informed of the single rumor. In the Gossiping scenario, the gossiping time $T_G$ is the first time at which all agents are informed of all rumors.

We denote by $||x - y||$ the Manhattan ($L_1$) distance between nodes $x$ and $y$ of $G_n$. Our analysis uses the following two technical lemmas which
characterize the set of nodes visited by a random walk within a given time interval.

**Lemma 2.** Consider a random walk on $G_n$, starting at time $t = 0$ at node $v_0$. There exists a positive constant $c_1$ such that for any $v \neq v_0$,

$$\Pr (v \text{ is visited within } (||v - v_0||)^2 \text{ steps}) \geq \frac{c_1}{\max\{1, \log(||v - v_0||)\}}.$$ 

**Proof.** The Lemma is proven in [3, Theorem 2.2] for the infinite grid $\mathbb{Z}^2$. By the “Reflection Principle” [11, Page 72], for each walk in $\mathbb{Z}^2$ that started in $G_n$, crossed a boundary and then crossed the boundary back to $G_n$, there is a walk with the same probability that does not cross the boundary and visits all the nodes in $G_n$ that were visited by the first walk. Thus, restricting the walks to $G_n$ can only change the bound by a constant factor. \hfill $\square$

**Lemma 3.** Consider the first $\ell$ steps of a random walk in $G_n$ which was at node $v_0$ at time 0.

1. The probability that at any given step $1 \leq i \leq \ell$ the random walk is at distance at least $\geq \lambda \sqrt{\ell}$ from $v_0$ is at most $2e^{-\lambda^2/2}$.

2. There is a constant $c_2$ such that, with probability greater than $1/2$, by time $\ell$ the walk has visited at least $c_2 \ell / \log \ell$ distinct nodes in $G_n$.

**Proof.** We observe that the distance from $v_0$ in each coordinate defines a martingale with bounded difference 1. Then, the first property follows from the Azuma-Hoeffding Inequality [15, Theorem 2.6].

As for the second property, let $R_\ell$ be the set of nodes reached by the walk in $\ell$ steps. By Lemma 2, $E(R_\ell) = \Omega(\ell / \log \ell)$ (even when $v_0$ is near a boundary), while $\text{Var}(R_\ell) = \Theta(\ell^2 / \log^4 \ell)$ (see [16]). The result follows by applying Chebyshev’s inequality. \hfill $\square$

3 The Broadcasting scenario

In the following two subsections we derive upper and lower bounds on $T_B$, the first time at which all agents are informed of the single rumor.

3.1 An upper bound on the broadcasting time

Since the cover time of $G_n$ is $O(n \log^2 n)$ (see [18, 5]), we can easily achieve an $\tilde{O}(n / \sqrt{m})$ upper bound on $T_B$, with high probability, when $m$ is polylogarithmic in $n$. Therefore, in what follows we focus on the case $m = \Omega(\log^3 n)$. 

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To ease the analysis, it is convenient to envision the spreading process as partitioned into three successive phases (see Figure 1; the value $\ell_1$ and the constants $q, q'$ will be set by the analysis):

**Phase I** (*Initial diffusion*) The source agent informs a set $A_1$ of at least $q \log^2 n$ agents, all with origins in a square $B_1$ of side length $8\ell_1 \log^{3/2} n$.

**Phase II** (*Covering $B_1$*) All agents originally placed in $B_1$ are informed.

**Phase III** (*Covering $G_n$*) All remaining uninformed agents are informed.

![Figure 1: Broadcasting scenario.](image)

The following lemma bounds from above the completion time of Phase I.

**Lemma 4.** Let $\ell_1 = \sqrt{(4q \log^2 n)/(c_2 p(n))}$, for constants $c_2 > 0$ (defined in Lemma 3) and $q > 0$. Let $T_1 = 3\ell_1^2 \log n$. With high probability, at time $T_1$ there is a set $A_1$ of informed agents such that:

1. $|A_1| \geq q \log^2 n$;

2. for each pair of agents $a_1, a_2 \in A_1$, their initial positions at $t = 0$ are within distance $8\ell_1 \log^{3/2} n$. 

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**Proof.** Partition the $T_1$ time steps into $3 \log n$ disjoint intervals of length $\ell_1^2$, and consider the path of the source agent during one such interval. Let $v_0$ be the location of the agent at the beginning of the interval.

By Lemma 3, with probability $2e^{-\log^3 n} \leq 1/n^2$ all the nodes visited by the walk are within distance $4\ell_1 \log^{3/2} n$ from $v_0$, and thus within distance $8\ell_1 \log^{3/2} n$ of one other. Moreover, the same lemma implies that with probability $\geq 1/2$ at least $c_2\ell_1^2/\log \ell_1$ distinct nodes are visited.

Conditioning on visiting these many nodes in the interval, the expected number of agents informed by the walk is $c_2\ell_1^2/\log \ell_1 p(n) \geq 2q \log^2 n$.

Applying the Chernoff bound, with probability at least $1 - 1/n$, the number of agents informed by the path is at least $q \log^2 n$.

Thus, with probability at least $1/2 - 2/n$ one segment of the path satisfies the claim, hence the probability that the claim holds for one of the $3 \log n$ segments is at least $1 - 1/n$ for $n$ large enough.

Consider now an arbitrary square area $B_1$ with side $\ell_2 = 8\ell_1 \log^{3/2} n$, containing all initial locations of the agents of the set $A_1$ of Lemma 4. Next lemma bounds the time taken to complete Phase II.

**Lemma 5.** Let $T_2 = 4\ell_2^2$. With high probability, every agent initially located in $B_1$ at $t = 0$ is informed by time $\tau_2 = T_1 + T_2$.

**Proof.** We show that every node in $B_1$ is visited by at least one agent in $A_1$ before time $T_2$.

There are $\ell_2^2$ nodes in $B_1$, and the diameter of $B_1$ is at most $2\ell_2$. Applying Lemma 2 and the fact that the $|A_1|$ walks are independent, the probability that any node in $B_1$ is not visited by some agent in $A_1$ until time $T_2$ is at most

$$\ell_2^2 \left(1 - \frac{c_1}{\log(2\ell_2)}\right)^{|A_1|} \leq \exp\left(-\frac{c_1 q \log^2 n}{\log(2\ell_2)} + 2 \log \ell_2\right) \leq \frac{1}{n},$$

by selecting a suitably large value for the positive constant $q$.

To analyze the completion time of Phase III, we consider a tessellation of $G_n$ into cells of side $\ell_3 = \sqrt{q' \log^3 n/p(n)}$, for some suitable constant $q' > 0$, such that one of the cells is entirely contained in $B_1$. We show that once all agents placed in one cell are informed, then with high probability all agents placed in the adjacent cells are informed within $20\ell_3^2$ steps.

**Lemma 6.** Consider a cell $C$, and let $\tau$ be the first time when all the agents originating in $C$ are informed. With probability greater than $1 - 1/n^2$, all
agents originating in the (at most) 4 cells adjacent to C are informed at time $\tau + 20\ell_3^2$.

Proof. By the Chernoff bound, at least $\ell_3^2 p(n)/2 = q' \log^3 n/2$ agents originate from a cell C, with probability at least $1 - 1/2n^2$. The maximum distance between a node in C and a node in an adjacent cell is bounded by $3\ell_3$. Thus, the probability that any node in the adjacent cells is not visited by one of the agents placed in C within $20\ell_3^2$ steps after time $\tau$ is bounded by

$$4\ell_3^2 \left(1 - \frac{c_1}{\log(3\ell_3)}\right)^{q' \log^3 n/2} \leq \exp\left(-\frac{c_1 q' \log^2 n}{2 \log(3\ell_3)} + 2 \log 2\ell_3\right) \leq \frac{1}{n^3},$$

by selecting a suitably large value for the constant $q'$.

We are now ready to prove the main result of the subsection:

**Theorem 1.** With high probability,

$$T_B = \tilde{O}\left(\frac{n}{\sqrt{m}}\right).$$

Proof. The case $m = O\left(\log^3 n\right)$ immediately follows from the observation made at the beginning of the subsection. Consider now the case $m = \Omega\left(\log^3 n\right)$.

With probability $1 - 1/n$, Phase I terminates in $\ell_1^2 \log n$ steps (Lemma 4) and with probability $1 - 1/n$, Phase II terminates in $4\ell_2^2$ steps (Lemma 5).

In Phase III, with probability $1 - 1/n^2$, all agents in a given cell are informed no later than $20\ell_3^2$ steps after all agents in one of its adjacent cells are informed. Since the “cell distance” (number of cells) between any two cells is at most $2\sqrt{n}/\ell_3$, Phase III terminates in $(2\sqrt{n}/\ell_3)20\ell_3^2 = 40\ell_3\sqrt{n}$ steps with probability at least $1 - 1/n$.

Putting it all together, we obtain that, with probability at least $1 - O\left(1/n\right)$,

$$T_B \leq 3\ell_1^2 \log n + 4\ell_2^2 + 40\ell_3\sqrt{n} = O\left(\frac{n \log^{3/2} n}{\sqrt{m}}\right).$$

However, we need to prove a lower bound on the broadcasting time.

### 3.2 A lower bound on the broadcasting time

We develop a lower bound of $\tilde{\Omega}\left(n/\sqrt{m}\right)$ to the broadcasting time. The argument relies on the intuition that many agents are initially located at a large distance from one another: this limits the rate at which the rumor may spread. We need the following definition (see also Figure 2):
Definition 2 (Island). Let \( A \) be the set of agents. For any parameter \( \gamma > 0 \), let \( G_t(\gamma) \) be the graph with vertex set \( A \) and such that there is an edge between two vertices iff the corresponding agents are within distance \( \gamma \) at time \( t \). The island of parameter \( \gamma \) of an agent \( a \) at time \( t \), denoted by \( I_t(a; \gamma) \), is the connected component of \( G_t(\gamma) \) containing \( a \).

\[
\begin{array}{c}
\cdot \quad I_3 \\
I_1 \\
I_2 \\
\cdots \\
I_8 \\
\end{array}
\]

Figure 2: A partition of agents into islands with \( \gamma = 3 \).

In this section, we only consider the islands at time \( t = 0 \). Islands at \( t > 0 \) are used for the lower bound on the gossiping time in the next section. We first prove that with high probability there are no big islands.

Lemma 7. Let \( \gamma = \sqrt{n/(4e^3m)} \) and write \( A = \bigcup_{j=1}^{s} I_j \) as the disjoint union of \( s \) islands of parameter \( \gamma \). Then, with high probability, \( |I_j| \leq \log n \), for all \( 1 \leq j \leq s \).

Proof. Let \( B_k \) denote the event that there exists an island with at least \( k \) agents. It is easy to see that \( \Pr(B_k) \) is upper bounded by the probability that \( G_t(\gamma) \) contains a tree of \( k \) vertices of \( A \) as a subgraph. Since \( k^{k-2} \) is the number of unrooted labeled trees on \( k \) nodes, and \( 4\gamma^2/n \) is (an upper bound to) the probability that a given agent lies within distance \( \gamma \) from another given agent, we have that

\[
\Pr(B_k) \leq \left( \frac{|A|}{k} \right)^{k-2} \left( \frac{4\gamma^2}{n} \right)^{k-1} \leq \left( \frac{|A|}{k} \right)^{k-2} \left( \frac{4\gamma^2}{n} \right)^{k-2} \left( \frac{4\gamma^2}{n} \right)^{-1}.
\]

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Since $|A| = m$, setting $k = 1 + \log n$ and substituting the definition of $\gamma$, we obtain

$$\Pr (B_k) \leq \frac{e|A|}{k^2} e^{-2(k-1)} \leq \frac{em}{k^2} \frac{1}{n^2} \leq \frac{1}{n}. \quad \square$$

Since the agents are distributed uniformly and independently at random, with high probability there exists an agent placed at distance $\Omega (\sqrt{n})$ from the source of the rumor:

**Lemma 8.** With probability $1 - 1/n^2$, at time 0 at least one agent is placed at $L_1$ distance $\geq \sqrt{n}/2$ from the source agent.

**Proof.** Given node $v$ of a grid, there are up to $4i$ nodes at distance $i$ from $v$, so there are up to $n/2$ nodes at distance $< \sqrt{n}/2$ from the source agent. The expected number of agents outside this area is $np(n)/2 = \Omega (\log^3 n)$, thus the probability that no agent is placed at distance at least $\sqrt{n}/2$ from the source agent is bounded by $1/n^2$. \quad \square

We are now ready to prove the lower bound on the broadcasting time:

**Theorem 2.** With high probability,

$$T_B = \Omega \left( \frac{n}{\sqrt{m \log^2 n}} \right).$$

**Proof.** Let $v_0$ denote the agent placed at distance at least $\sqrt{n}/2$, whose existence is guaranteed with probability $1 - 1/n^2$ by Lemma 8.

By setting the parameter $\gamma$ as in Lemma 7, with high probability no island has more than $\log n$ agents, hence the maximum distance between two agents in the same island is at most $\gamma (\log n - 1)$. Thus, with high probability, the rumor must traverse at least $H = \sqrt{n}/(2\gamma (\log n - 1))$ islands to reach node $v_0$.

Hence, for $H - 1$ non-overlapping time periods, the rumor must cover the inter-island distance which is at least $\gamma$. By applying Lemma 3, the probability that one inter-island distance be covered in at most $\tau = (1/(24e^3))n/(m \log n)$ steps is at most $1/n^2$, therefore, with probability at least $1 - 1/n$ the time to spread the rumor is at least $(H - 1)\tau = \Omega (n/(\sqrt{m \log^2 n}))$. \quad \square

### 4 The Gossiping scenario

The Gossiping scenario differs from the Broadcasting scenario in that each agent has initially a distinct rumor to spread, and all agents perform independent random walks starting at time 0.
4.1 The meeting probability of two random walks

The main new ingredient in analyzing the Gossiping scenario, which is also a result of independent interest, is a lower bound on the probability that two random walks on the grid meet within a given time interval. The following lemma is the analogous of Lemma 2 for the case of two walks. The proof, however, requires a different argument.

**Lemma 9.** Consider two independent, simple random walks $\bar{a}$ and $\bar{b}$, starting at time 0 at node $a_0$ and $b_0$, respectively, with $a_0 \neq b_0$. Let $a_t$ and $b_t$ be the locations of the walks at time $t$ and let $T \geq ||a_0 - b_0||^2$. Then, there exists a constant $c_3 > 0$ such that

$$P_{\bar{a},\bar{b}}(T) \triangleq \Pr (\exists t \leq T \text{ such that } a_t = b_t) \geq \frac{c_3}{\max \{1, \log(||a_0 - b_0||)\}}.$$ 

**Proof.** The case $||a_0 - b_0|| = 1$ is immediate. Consider now the case $||a_0 - b_0|| > 1$. Let $P_t(w, x)$ denote the probability that a walk that started at node $w$ at time 0 is at node $x$ at time $t$, and let $R(w, u, s)$ be the expected number of times that two walks which started at nodes $w$ and $u$ at time 0 meet during the time interval $[0, s]$, then

$$R(w, u, s) = \sum_{t=0}^{s} \sum_{x} P_t(w, x) P_t(u, x).$$

Let $\tau(a, b)$ be the first meeting time of the walks $\bar{a}$ and $\bar{b}$. Then

$$R(a_0, b_0, T) = \sum_{t=0}^{T} \Pr (\tau(a, b) = t) R(x_t, x_t, T - t) \leq P_{\bar{a},\bar{b}}(T) \max_{x} R(x, x, T).$$

Thus, setting $T_0 = ||a_0 - b_0||^2$, we have

$$P_{\bar{a},\bar{b}}(T) \geq P_{\bar{a},\bar{b}}(T_0) \geq \frac{R(a_0, b_0, T_0)}{\max_{x} R(x, x, T_0)}.$$ 

Let $D(a_0, b_0)$ be the set of nodes with distance up to $||a_0 - b_0||$ from both $a_0$ and $b_0$, i.e.,

$$D(a_0, b_0) = \{x \mid ||x - a_0|| \leq ||a_0 - b_0|| \text{ and } ||x - b_0|| \leq ||a_0 - b_0||\}.$$
It is easy to verify that \(|D(a_0, b_0)| \geq \frac{1}{4}||a_0 - b_0||^2\). Applying Theorem 1.2.1 in [14] we have:

\[
R(a_0, b_0, T_0) \geq \sum_{t=0}^{T_0} \sum_{x \in D(a_0, b_0)} P_t(a_0, x)P_t(b_0, x) \\
\geq \sum_{t=\frac{T_0}{2}+1}^{T_0} \sum_{x \in D(a_0, b_0)} 4 \left( \frac{1}{\pi t} \right)^2 e^{-\frac{||x-a_0||^2 + ||x-b_0||^2}{t}}.
\]

By bounding \(||x - a_0||^2\) and \(||x - b_0||^2\) from above with \(T_0\) in the formula, easy calculations show that \(R(a_0, b_0, T_0) = \Omega(1)\). Similarly, using the fact that there are no more than \(4i\) nodes at distance exactly \(i\) from \(x\), we have:

\[
\max_x R(x, x, T) \leq 1 + \sum_{t=1}^{T} \sum_{i=1}^{4i} 4i \left( \frac{1}{\pi t} \right)^2 2e^{-\frac{i^2}{t}} \\
\leq 1 + \left( \frac{4}{\pi} \right)^2 \sum_{t=1}^{T} \frac{1}{t^2} \left( \sum_{i=1}^{\sqrt{t}} i \right) + \left( \sum_{i=1+\sqrt{t}}^{t} ie^{-i^2/t} \right) \\
\leq 1 + \left( \frac{4}{\pi} \right)^2 \sum_{t=1}^{T} \frac{1}{t^2} \left( \frac{t}{2} + \left( \sum_{i=1+\sqrt{t}}^{t} i^2e^{-i^2/t} \right) \right) \\
\leq 1 + \left( \frac{4}{\pi} \right)^2 \sum_{t=1}^{T} \frac{1}{t^2} \left( \frac{t}{2} + \frac{e}{(e-1)^2t} \right) = O(\log T).
\]

We conclude that there is a constant \(c_3 > 0\) such that

\[
P_{\bar{a}, \bar{b}}(T) \geq \frac{R(a_0, b_0, T_0)}{\max_x R(x, x, T_0)} \geq \frac{c_3}{\log(||a_0 - b_0||)}.
\]

### 4.2 An upper bound on the gossiping time

We observe that since the \(L_1\) diameter of \(G_n\) is \(2\sqrt{n} - 2\), we can use Lemma 9 to show that with probability \(1 - 1/n^2\), at time \(8n \log^2 n\) an agent has met all other agents walking in \(G_n\). Thus, the theorem trivially holds for \(m\) polylogarithmic in \(n\).

Consider now the case \(m = \Omega(\log^3 n)\). As in the derivation of the upper bound to the Broadcasting time, we resort to the binomial model introduced in Section 2. Let \(\tilde{m}\) be the number of agents in a given instance of the process and recall that \(\tilde{m} = \Theta(m)\) w.h.p.
We prove the theorem by bounding from above the spreading time of any fixed rumor. Specifically, let $T^i_B$ be the time for spreading of the $i$-th rumor. We show that there is a constant $c_4 > 0$ such that, for $1 \leq i \leq \tilde{m}$,

$$
\Pr \left( T^i_B \geq c_4 n \frac{\log^{3/2} n}{\sqrt{m}} \right) \leq \frac{1}{n^2}.
$$

The argument proceeds as follows. We tessellate $G_n$ into cells of suitable side $\ell_1$ (defined in Lemma 10) and say that a cell $Q$ is reached at time $t_Q$ if $t_Q$ is the first time when a node of the cell hosts an agent informed of the rumor. We call this first visitor the explorer of $Q$, and let $A_Q$ denote the set of agents inside $Q$ at time $t_Q$.

We first show that, after a suitably chosen number of steps after $t_Q$, cell $Q$ is conquered, in the sense that a large number of agents of $A_Q$ have been informed. Also, we show that by the time cell $Q$ is conquered, each of its neighboring cells has been reached, and that a large number of informed agents are within a short distance from $Q$. These facts imply that the conquering process proceeds smoothly. Finally, we prove that all agents are informed of the rumor shortly after all cells are conquered. This argument is made rigorous in the following sequence of lemmas.

**Lemma 10.** Let $\ell_1 = \sqrt{4q n \log^3 n / (c_3 m)}$, where $q > 0$ and $c_3$ is defined in Lemma 9. Let also $T_1 = 4\ell_1^2$. Then, for any cell $Q$ of the tessellation, by time $\tau_1 = t_Q + T_1$, at least $q \log^2 n$ agents of $A_Q$ are informed with probability $1 - 1/n^6$, for sufficiently large $n$.

**Proof.** Since at any given time the agents are at random and independent locations, the expected number of agents in $Q$ at time $t_Q$ is $4q \log^3 n / c_3$. Thus, by the Chernoff bound at least $4q \log^3 n / (2c_3)$ agents are inside $Q$ at time $t_Q$ with probability at least $1 - 1/n^6$.

For each agent $a \in A_Q$, define an indicator variable $X_a$ such that $X_a = 1$ iff agent $a$ has met the explorer of $Q$ by time $\tau_1$. The $X_a$’s are independent and $X = \sum_{a \in A_Q} X_a$ is a lower bound to the number of informed agents in $A_Q$ by time $\tau_1$. Applying Lemma 9, we have $\Pr (X_a = 1) \geq \frac{c_3}{\log^2 \ell_1}$, hence $E(X) \geq 2q \log^2 n$. Using the Chernoff bound, we finally obtain $\Pr (X < q \log^2 n) \leq \exp (-q \log^2 n / 4) \leq 1/n^6$, for a sufficiently large $n$. \hfill \Box

**Lemma 11.** Consider a cell $Q$, and define $R$ to be the square of side $\ell_2 = (1 + 2\sqrt{2})\ell_1$ centered at $Q$ (see Figure 3). Let $\tau_1$ and $T_1$ be defined as in Lemma 10, and let $T_2 = 4\ell_2^2$. Let $A_R \subseteq A_Q$ denote the set of agents
residing inside Q at time $t_Q$ and contained in R at time $\tau_1$. Finally, let $\tau_2 = \tau_1 + T_2 = t_Q + T_1 + T_2$. Then, with probability at least $1 - 1/n^5$,

1. at least $\frac{1}{2}(1 - \frac{2}{e})q\log^2 n$ informed agents are within distance $\ell_2$ from Q at time $\tau_1$;

2. $\frac{1}{4}(1 - \frac{2}{e})\frac{4q\log^3 n}{c_3} \leq |A_R| \leq \frac{9}{4}(1 - \frac{2}{e})\frac{4q\log^3 n}{c_3}$;

3. every agent $a \in A_R$ is informed by time $\tau_2$; and

4. each of the neighboring cells of Q has been reached by time $\tau_2$.

Proof. Observe that by Lemma 3, the probability that an agent $a$ contained in Q at time $t_Q$ is outside R at time $\tau_1 = t_Q + T_1$ is at most $2/e$, whence $\Pr(a \in A_R|a \in A_Q) \geq (1 - \frac{2}{e}) \triangleq r$. Since (by Lemma 10) at least $q\log^2 n$ agents of $A_Q$ are informed at time $\tau_1$, and each of them belongs to $A_R$ independently with probability $r$, we can apply the Chernoff-Hoeffding bound to show that with probability $1 - 1/n^6$ the number of informed agents in $R$ at time $\tau_1$ is at least $I_R(\tau_1) = \frac{r}{2}q\log^2 n$. This proves Point (1), and Point (2) follows by a similar argument.

For what concerns Point (3), consider an agent $a \in A_R$, not informed at time $\tau_1$. Since $a$ is within distance $2\ell_2$ from the informed nodes belonging to $A_R$, by Lemma 9, the probability that $a$ is not informed at time $\tau_2$ is at most

$$\left(1 - \frac{c_3}{\log 2\ell_2}\right)^{I_R(\tau_1)} \leq \exp\left(-\frac{c_3I_R(\tau_1)}{\log 2\ell_2}\right) < \frac{1}{n^6}$$

by selecting a suitably large constant $q$. The point is proven by applying the union bound on $|A_R| = O(\log^3 n)$ agents.

Finally, for Point (4), consider one of the neighboring cells of Q, say $Q'$. Since $Q' \subseteq R$, each point of $Q'$ is at distance at most $2\ell_2$ from the actual location of the $I_R(\tau_1)$ informed agents which are inside $R$ at time $\tau_1$. Therefore, applying Lemma 2, we can conclude that the probability that $Q'$ is not reached by time $\tau_2$ is at most

$$\left(1 - \frac{c_1}{\log 2\ell_2}\right)^{|Q'|I_R(\tau_1)} \leq \exp\left(-\frac{c_1}{\log 2\ell_2} \frac{rq\log^2 n}{2} \frac{4q\log^3 n}{c_3m/n}\right) < \frac{1}{n^6},$$

for a sufficiently large $n$. We conclude the proof by applying the union bound over the neighboring cells. 

A cell Q is conquered when all agents in the set $A_R$, defined in the previous lemma, are informed.
Lemma 12. With the same notation of Lemma 11, with probability $1 - 1/n^3$, for any time instant $\tau_2 \leq t \leq 8n \log^2 n$, there are at least $\Omega(\log^3 n)$ informed agents at distance at most $\ell_2$ from any node of $Q$.

Proof. Consider an agent $a \in A_R$ and let $I_Q(t)$ denote the set of informed agents inside $Q$ at time $t \geq \tau_2$.

By Lemma 11, the number $N_R(\tau_1)$ of agents inside $R$ at time $\tau_1$ is at least $|A_R| = \Omega(\log^3 n)$. Since each of the agents of $A_R$ was inside cell $Q$ at time $\ell_Q$, the probability $r'$ that $a$ is still inside $Q$ at time $\tau_2$ is $\Theta(\ell_1^2/\ell_2^2) = \Theta(1)$. Since the agents perform independent random walks, we can apply a concentration bound and get that

$$|I_Q(\tau_2)| \geq \frac{1}{2} r' N_R(\tau_1) \geq \frac{1}{8} \left(1 - \frac{2}{e}\right) r' \frac{4q \log^3 n}{c_3}$$

with probability greater than $1 - 1/n^5$ for a sufficiently large $n$. Next, observe that the probability that one of the agents of $I_Q(\tau_2)$ is still inside $R$ (i.e., is within distance $\ell_2$ from $Q$) at time $\tau_3 = \tau_2 + T_1 + T_2$ is at least $r = (1 - \frac{2}{e})$ by Lemma 3. Therefore, applying again the concentration bound, we have that, with probability greater than $1 - 1/n^5$ (for a sufficiently large $n$), there
are at least
\[ N_R(t) \geq \frac{r't^2}{4c_2}q \log^3 n \]  
\[ \text{(2)} \]

informed nodes within distance \( \ell_2 \) from \( Q \) in each of the time instants \( \tau_2 \leq t \leq \tau_3 \).

Now consider the time interval \( [\tau_2, 8n \log^2 n] \) and partition it into consecutive, non-overlapping subintervals of length \( T_1 + T_2 \). Let \( t_k = \tau_2 + (k - 1)(T_1 + T_2) \) denote the beginning of the \( k \)-th subinterval. For any subinterval \( k \), we show that (a) \( |I_Q(t_k)| \geq 1 \) and (b) at least \( \Omega(\log^3 n) \) informed agents are inside \( R \) during the \( k \)-th subinterval. Note that, by (1) and (2), the two properties hold for the first subinterval \( [\tau_2, \tau_3) \) with probability at least \( 1 - 1/n^5 \).

The key idea for extending the properties to \( k > 1 \) is that we can pick an arbitrary informed agent \( a \) inside \( Q \) at time \( t_k \) and make it play the role of the explorer of \( Q \). More precisely, by mimicking the analyses of Lemmas 10 and 11, provided that we replace \( t_Q \) with \( t_k \), we can show that the presence of the explorer ensures the existence of \( \Omega(\log^3 n) \) informed agents at the beginning of the next interval. This observation, combined with (1) and (2), proves that if properties (a)-(b) hold for the \( k \)-th subinterval, they will hold also for the \((k + 1)\)-th subinterval.

Applying the union bound over \( O(n \log^2 n) \) subintervals concludes the proof.

We are now ready to prove the main theorem of this subsection:

**Theorem 3.** With high probability,

\[ T_G = \tilde{O}\left(\frac{n}{\sqrt{m}}\right). \]

**Proof.** As observed at the beginning of the subsection, we can limit ourselves to the case \( m = \Omega(\log^3 n) \). Consider first the spreading of the \( i \)-th rumor, and tessellate the domain with cells of side \( \ell_1 \), specified in Lemma 10.

Consider a cell \( Q \) reached for the first time at time \( t_Q \). By Lemma 11, we know that each of the neighboring cells of \( Q \) is reached within time \( \tau_2 = t_Q + T_1 + T_2 \) with probability \( 1 - 1/n^5 \). Therefore, all cells are reached within time \( (2\sqrt{n}/\ell_1)(T_1 + T_2) \) with probability \( 1 - 1/n^4 \).

For any agent \( a \), let \( t_a \) be the first time when the agent is inside a cell that is either reached or conquered at time \( t_a \), and observe that \( t_a \leq (2\sqrt{n}/\ell_1 + 1)(T_1 + T_2) \) with probability \( 1 - 1/n^4 \). Using the result of Lemma 12 about the number of informed agents near the cell where agent \( a \) resides at
time $t_a$, an argument similar to the one used in the proof of Point (3) of Lemma 11 shows that $a$ will be informed by time $t_a + T_2$ with probability $1 - 1/n^3$.

Putting it all together, by setting $c_4 = 64(5 + 2\sqrt{2})\sqrt{q/c_3}$, we have proved that the broadcasting time $T_B^i$ of the $i$-th rumor satisfies

$$\Pr\left(T_B^i \geq \frac{c_4 n \log^{3/2} n}{\sqrt{m}} \right) \leq \frac{1}{n^2}.$$  

The theorem follows by observing that the broadcasting time of a rumor does not depend on the origins of the other rumors, so we can take the union bound over all $m = O(n)$ different rumors and conclude that $T_G = O\left(n \log^{3/2} n / \sqrt{m}\right)$ with probability $1 - 1/n$.  

4.3 A lower bound on the gossiping time

In the Gossiping scenario uninformed agents move, hence we need to resort to a lower bound argument which takes such movements into account. We define the informed area $\mathcal{I}(t)$ at time $t$ as the set of places visited by an informed agent up to time $t$. The frontier of $\mathcal{I}(t)$ is the border separating the informed area from the remaining places. To simplify the exposition, we imagine that the rumor travels from left to right and consider the grid node $x(t)$ of the rightmost informed agent at time $t$. By definition, the informed area lies to the left of $x(t)$ and we need to show that there is a sufficiently large value $T$ such that, at time $T$, there is at least one uniformed agent right of $x(T)$.

First, we can prove that, with probability $1 - 1/n^2$, at each time instant $0 \leq t \leq 8n \log^2 n$, every island of parameter $\gamma = \sqrt{n/(4e^6m)}$ has no more than $\log n$ agents. The proof is similar to the argument used in Lemma 7 for the broadcasting scenario:

**Lemma 13.** Let $\gamma = \sqrt{n/(4e^6m)}$. Then, the probability that there exists an island of parameter $\gamma$ in any time instant $0 \leq t \leq 8n \log^2 n$ with more than $\log n$ agents is at most $1/n^2$.

**Proof.** Since at any given time the agents are uniformly distributed in $G_n$, the probability that a given agent is within distance $\gamma$ of another given agent at time $t_0$ is bounded by $4\gamma^2/n$. Fix a time instant $t_0$ and let $B_k(t_0)$ denote the event that there exists an island with at least $k > \log n$ elements at time
Then,

\[
\Pr (B_k(t_0)) \leq \left( \frac{m}{k} \right)^{k-2} \left( \frac{4\gamma^2}{n} \right)^{k-1} \leq \left( \frac{em}{k} \right)^{k-2} \left( \frac{4\gamma^2}{n} \right)^{k-1}.
\]

Using definition of \( \gamma \) and the bound \( k \geq 1 + \log n \) and \( m \leq n \), we have

\[
\Pr (B_k(t_0)) \leq \frac{em}{k^2} e^{-5(k-1)} \leq \frac{1}{k^2 n^3} \leq \frac{1}{n^4},
\]

for a sufficiently large \( n \). Applying the union bound on \( O(n \log n) \) time instants concludes the proof.

Next we show that, with high probability, the frontier of the informed area cannot advance too fast.

**Lemma 14.** Let \( \gamma = \sqrt{n/(4e^6m)} \) and let \( t_0 \) and \( t_1 = t_0 + \gamma^2/(36 \log n) \) be two time steps. Then, with probability \( 1 - 1/n^2 \),

\[
||x(t_1) - x(t_0)|| \leq (\gamma \log n)/2.
\]

**Proof.** Consider the island \( I = I_{t_0}(a_0; \gamma) \) of the informed agent \( a_0 \) located at node \( x(t_0) \) at time \( t_0 \).

By Lemma 3, with probability \( 1 - 2/n^3 \) an agent cannot cover a distance of more than \( \gamma/2 \) in \( \gamma^2/(36 \log n) \) time steps. Thus, with probability \( 1 - 1/n^2 \), up to time \( t_1 \) the rumor cannot propagate (directly or through intermediate agents) outside the set of agents \( I \), and the farthest it can get from \( x(t_0) \) by members of \( I \) is bounded by \( (\gamma \log n)/2 \) since the island has no more than \( \log n \) agents by Lemma 13.

Then, \( ||x(t_1) - x(t_0)|| \leq (\gamma \log n)/2 \) with probability \( 1 - 1/n^2 \).

Finally, we can prove the main theorem of the subsection:

**Theorem 4.** With high probability,

\[
T_G = \Omega \left( \frac{n}{\sqrt{m} \log^2 n} \right).
\]

**Proof.** By Lemma 8, with high probability there exists an agent \( a \) whose distance at time 0 from the source was at least \( \sqrt{n}/2 \).

Let \( T = n/(144e^3 \sqrt{m} \log^2 n) \) and \( \gamma = \sqrt{n/(4e^6m)} \). By Lemma 14, with probability \( 1 - 1/n \) the frontier cannot move right in \( T \) steps more than \( (\gamma \log n/2)T/(\gamma^2/(36 \log n)) = \sqrt{n}/4 \).
By Lemma 3, with probability $1 - 2/n^2$, agent $a$ cannot move left more than $2\sqrt{T \log n} < \sqrt{n}/4$ and thus that agent is not informed at time $T$. Hence, the gossiping time is at least $T_G > T = \Omega(n/(\sqrt{m} \log^2 n))$ with probability $1 - 1/n$.

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