THE CENTER OF HECKE ALGEBRAS OF TYPES

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Abstract. We describe the center of the Hecke algebra of a type attached to a Bernstein block under some hypothesis. When $G$ is a connected reductive group over non-archimedean local field $F$ that splits over a tamely ramified extension of $F$ and the residue characteristic of $F$ does not divide the order of the absolute Weyl group of $G$, the works of Kim-Yu and Fintzen associate a type to each Bernstein block and our hypothesis is satisfied for such types. We use our results to give a description of the Bernstein center of the Hecke algebra $\mathcal{H}(G(F), K)$ when $K$ belongs to a nice family of compact open subgroups of $G(F)$ (which includes all the Moy-Prasad filtrations of an Iwahori subgroup) via the theory of types.

Introduction

Let $F$ be a non-archimedean local field. For a connected, reductive group $G$ over $F$, we write $G$ for its $F$-points.

Let $\mathfrak{R}(G)$ denote the category of smooth, complex representations of $G$. Let $\mathfrak{B}(G)$ for the set of all inertial equivalence classes in $G$ (this definition is recalled in Section 1.1). The Bernstein decomposition yields

$$\mathfrak{R}(G) = \prod_{s \in \mathfrak{B}(G)} \mathfrak{R}^s(G).$$

We are interested in describing the center of $\mathfrak{R}^s(G)$, $s \in \mathfrak{B}(G)$. Let $J$ be a compact open subgroup of $G$ and let $\rho$ be an irreducible representation of $J$ such that $(J, \rho)$ is an $s$-type (see Definition 1.1). Then the category $\mathfrak{R}^s(G)$ is equivalent to $\mathcal{H}(G, \rho) - \mathrm{mod}$. This leads us to the question of understanding the center of Hecke algebras of types.

First, suppose $\pi$ is an irreducible supercuspidal representation of $G$ of the form $c - \mathrm{ind}_J^G \tilde{\rho}$, where $\tilde{J}$ is an open, compact mod center subgroup of $G$ and $\tilde{\rho}$ is an irreducible representation of $\tilde{J}$. Let $0 G$ be the open normal subgroup of $G$ as in (1.1) and let $J = 0 G \cap \tilde{J}$ and let $\rho$ be an irreducible summand of $\tilde{\rho}|_J$. Then $(J, \rho)$ is an $s = [G, \pi]_G$-type. Assume that the intertwiners of $\rho$, denoted $\mathcal{I}_G(\rho)$, is contained in $\tilde{J}$. These requirements are satisfied for supercuspidal representations arising out of Yu’s construction (see [15]), which exhaust all supercuspidal representations of $G$ by [14] under the hypothesis that $G$ splits over a tamely ramified extension of $F$ and the residue characteristic of $F$ does not divide the order of the Weyl group of $G$. Let $\pi_0$ be an irreducible summand of $\pi|_G$. We show in Theorem 3.4 that multiplicity with which $\pi_0$ occurs in $\pi|_G$ is equal to the multiplicity with which $\rho$ occurs in $\tilde{\rho}|_J$. In fact, this theorem holds for slightly more general open normal subgroups $1 G$ of $G$. Next, we go on to describe the center of $\mathcal{H}(G, \rho)$ of the type

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\normalsize
(J, ρ): we show that the center \( Z(\mathcal{H}(G, ρ)) \simeq \mathbb{C}[[J/J]] \) where \( J = \bigcap_{v \in X_J(\rho)} \ker(v) \), with \( X_J(\rho) = \{ v \in \text{Hom}(\overline{J}/J, \mathbb{C}^\times) \mid \hat{\rho} \otimes v \simeq \hat{\rho} \} \) (See Lemma 2.2, Lemma 3.6 and Theorem 3.9). We deduce that the Hecke algebra \( \mathcal{H}(G, ρ) \) is commutative if and only if \( \pi|_{\nu} \) is multiplicity free (see Proposition 3.10). We then discuss some applications of this result to Yu’s supercuspidal representations in Section 3.7 - 3.9.

Next, we describe the center of non-supercuspidal blocks. Let \( s = [M, σ]_G \) and \( \mathfrak{s}_M = [M, σ]_M \). We assume that \( σ \) is an irreducible supercuspidal representation of \( M \) of the form \( c - \text{ind}^M_J \hat{ρ}_M \), where \( J_M \) is an open, compact mod center subgroup of \( M \) and \( \hat{ρ}_M \) is an irreducible representation of \( J_M \). Let \( (J_M, ρ_M) \) be the \( \mathfrak{s}_M \)-type as before. Again we assume that \( \mathcal{I}(\rho_M) \subset \mathcal{J}_M \). Let \( (J, ρ) \) be a \( G \)-cover of \( (J_M, ρ_M) \). Then \( (J, ρ) \) is an \( \mathfrak{s} \)-type. We show in Theorem 4.6 that that the center \( Z(\mathcal{H}(G, ρ)) \simeq \mathbb{C}[[J/M]/J]^{\text{W}(ρ_M)} \) where \( \text{W}(ρ_M) \) is described in Proposition 4.3.

Now, assume that \( G \) splits over a tamely ramified extension of \( F \) and the residue characteristic of \( F \) does not divide the order of the absolute Weyl group of \( G \). Then by [12, 10], every Bernstein block has a Kim-Yu type attached to it and our results in the preceding paragraphs hold for such types. We use this to give a description of the Bernstein center of \( \mathcal{H}(G, K) \) for certain nice compact open subgroups \( K \) of \( G \). Let us describe what these compact open subgroups are.

Let \( \mathcal{B}(G, F) \) denote the Bruhat–Tits building of \( G \) over \( F \). Let \( S \) be a maximal \( F \)-split torus in \( G \) and let \( \mathcal{A}(S, F) \) be the apartment of \( S \) over \( F \). For a compact open subgroup \( K \) of \( G \) and let \( \mathfrak{R}_K(G) \) be the full sub-category of \( \mathfrak{R}(G) \) consisting of representations \((π, V)\) that are generated by their \( K \)-fixed vectors. In [2], Section 3.7 - 3.9, the authors put criteria \( \triangledown_S \) (see Definition 5.1) on the compact open subgroup \( K \) and prove that the category \( \mathfrak{R}_K(G) \) is closed under taking sub-quotients when \( K \) satisfies these criteria. They further show that if \( x \) is a special point in \( \mathcal{A}(S, F) \), then \( G_{x,r} \) satisfies \( \triangledown_S \) for each \( r > 0 \). It was long expected that the category \( \mathfrak{R}_K(G) \) is closed under taking sub-quotients whenever \( K = G_{x,r} \) for all points \( x \in \mathcal{B}(G, F) \) and all \( r > 0 \). In [3], Bestvina–Savin put slightly different criteria \( \heartsuit_S \) (see Definition 5.4) on the compact open subgroup \( K \) for which the category \( \mathfrak{R}_K(G) \) is closed under taking sub-quotients. Further, they prove that \( G_{x,r} \) satisfies \( \heartsuit_S \) for each \( x \in \mathcal{A}(S, F) \) and each \( r > 0 \). If \( K \) satisfies \( \heartsuit \) or \( \heartsuit_S \), then there is a finite subset \( \mathcal{S}_K \subset \mathcal{B}(G) \) such that

\[
\mathfrak{R}_K(G) = \prod_{s \in \mathcal{S}_K} \mathfrak{R}^s(G).
\]

When \( K \) satisfies \( \heartsuit_S \), it is easy to see that for a Levi subgroup \( M \) of \( G \) that contains \( S \), and a representation \( σ \) of \( M \), if \((\text{Ind}_G^M σ)^K \neq 0 \), then \( σ^{K'} \neq 0 \) for a \( G \)-conjugate \( K' \) of \( K \) (that has an Iwahori factorization), where \( P = MN \) is a parabolic subgroup of \( G \) with Levi \( M \) and \( K_M = K \cap P/K \cap N \). On the other hand, when \( K \) satisfies \( \triangledown_S \) it follows that if \((\text{Ind}_G^M σ)^K \neq 0 \), then \( σ^{K_M} \neq 0 \). This property yields finer information about the set \( \mathcal{S}_K \) (see Lemma 5.3). For this reason, it is helpful to know which \( G_{x,r}, x \in \mathcal{A}(S, F), r > 0 \), also satisfy \( \triangledown_S \). We prove two results in this direction. First, we show that if \( a \) is an alcove in \( \mathcal{A}(S, F) \) and \( x \in a \), then \( G_{x,r}, r > 0 \), always satisfies \( \triangledown_S \) (See Proposition 5.5). Next, we give an example in \( G = \text{GL}_3 \), of a Moy-Prasad filtration subgroup \( G_{x,1} \), where \( x \) is a non-special point in the boundary of an alcove of \( \mathcal{A}(S, F) \), that does not satisfy \( \triangledown_S \) (see Example 5.11).
In Section 1, we use the results in the preceding sections and give a description of the Bernstein center of $\mathcal{H}(G, K)$ where $K$ is a compact open subgroup of $G$ that satisfies $\bullet_S$ or $\varnothing_S$, using the theory of types.

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1. The Bernstein center

1.1. The Bernstein decomposition. Let $F$ be a non-archimedean local field and let $G$ be a connected, reductive group over $F$. Let $G = G(F)$. Let $X_F(G)$ be the group of $F$-rational characters $\chi : G \to F^\times$ of $G$. For $\chi \in X_F(G)$ and $s \in \mathbb{C}$, we define a smooth one-dimensional representation $g \mapsto |\chi(g)|_F^s$ of $G$. Let $X_m(G)$ be the group unramified quasicharacters of $G$, generated by maps $G \to \mathbb{C}^\ast$ of this form. We write

$$0^0G = \bigcap_{\chi \in X_m(G)} \text{Ker}(\chi).$$  \hspace{1cm} (1.1)

The quotient $G/0^0G$ is free abelian of finite rank and

$$X_m(G) = \text{Hom}(G/0^0G, \mathbb{C}^\ast).$$

We consider pairs $(M, \sigma)$ where $M$ is an $F$-Levi subgroup of $G$ and $\sigma$ is a supercuspidal representation of $M$. Two pairs $(M_1, \sigma_1)$ and $(M_2, \sigma_2)$ are inertially equivalent if there exist $g \in G$ and $\chi \in X_G(M_2)$ such that

$$M_2 = M_1^g \text{ and } \sigma_2^g = \sigma_1 \otimes \chi.$$  

Here $M_1^g = g^{-1}M_1 g$ and $\sigma_1^g : x \mapsto \sigma_1(gxg^{-1})$, for $x \in M_1^\mathbb{C}$. We write $[M, \sigma]_G$ for the inertial equivalence class of the pair $(M, \sigma)$ and $\mathcal{B}(G)$ for the set of all inertial equivalence classes in $G$.

Now, let $(\pi, V)$ be an irreducible, smooth representation of $G$. There exists an $F$-parabolic subgroup $P$ of $G$ with Levi component $M$, and an irreducible, supercuspidal representation $\sigma$ of $M$ such that $\pi$ occurs as an irreducible subquotient of the parabolically induced representation $\text{Ind}_P^G \sigma$. The representation $\pi$ hence determines an inertial equivalence class $[M, \sigma]_G$ which we denote as $\mathfrak{I}(\pi)$ and call it the inertial support of $\pi$.

Let $\mathfrak{R}(G)$ denote the category of smooth, complex representations of $G$ and let $\text{Irr}(G)$ denoted the irreducible objects in $\mathfrak{R}(G)$. For $\mathfrak{s} \in \mathfrak{B}(G)$ we define a full subcategory $\mathfrak{K}(\mathfrak{s})$ of $\mathfrak{R}(G)$ as follows. Let $(\pi, V) \in \mathfrak{R}(G)$. Then $(\pi, V) \in \mathfrak{K}(\mathfrak{s})$ if and only if every irreducible subquotient of $\pi$ has inertial support $\mathfrak{s}$. The Bernstein decomposition yields

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{K}(\mathfrak{s}).$$

Let $\pi$ be an irreducible smooth representation of $G$. Then there exists a pair $(M, \sigma)$ consisting of a Levi subgroup $M$ of $G$ and a supercuspidal representation $\sigma$ of $M$ such that $\pi \in \text{Ind}_P^G \sigma$ for a suitable parabolic subgroup of $G$ with Levi
component $M$. The conjugacy class of $(M, \sigma)$ is unique and is called the cuspidal support of $\pi$ denoted $sc(\pi)$.

We recall the description of the Bernstein center of $\mathcal{R}(G)$. Let $t = [M, \sigma]_M \in \mathfrak{B}(M)$ and let $s = [M, \sigma]_G \in \mathfrak{B}(G)$. The action of $N_G(M)$ on $M$ by conjugation induces an action of $W(M)$ on $\mathfrak{B}(M)$. Let $W^t$ denote the stabilizer of $t$. Thus $W^t = N^t/M$ where

$$N^t = \{ n \in N_G(M) \mid n\sigma \simeq \sigma n, \text{ for some } \nu \in X_{nr}(M) \}.$$ 

Let $\mathfrak{Z}^t$ denote the center of $\mathcal{R}^t(M)$ and $\mathfrak{Z}^s$ denote the center of $\mathcal{R}^s(G)$. Then (see [16, Theorem 1.9.1.1])

$$\mathfrak{Z}^s = (\mathfrak{Z}^t)^{W^t}.$$ 

1.2. Type associated to a Bernstein block.

**Definition 1.1.** Let $J$ be a compact open subgroup of $G$ and $\rho$ be an irreducible representation of $J$. The pair $(J, \rho)$ is a $\mathfrak{z}$-type if: for all irreducible representations $\pi$ of $G$, $sc(\pi) \subset \mathfrak{z}$ if and only if $\text{Hom}_J(\rho, \pi) \neq 0$.

Let $H(G, \rho) = \text{End}_G(\text{ind}_J^G \rho)$. If $(J, \rho)$ is an $\mathfrak{z}$-type, then the functor

$$\mathcal{M}_\rho : \mathcal{R}^s(G) \to H(G, \rho) - \text{ mod } \pi \to \text{Hom}_J(\rho, \pi)$$

is an equivalence of categories (see [14, Section 4.2]).

Let $\pi$ be an irreducible, supercuspidal representation of $G$ of the form $\pi = \text{ind}_J^G \tilde{\rho}$, that is $\pi$ is compactly induced from an irreducible representation $\tilde{\rho}$ of an open, compact mod center subgroup $\tilde{J}$ of $G$. Write $J = \tilde{J} \cap G^0$. Write $J = \tilde{J} \cap G^0$ for the unique maximal compact subgroup of $\tilde{J}$. Let $G^0$ be any irreducible summand of $\tilde{\rho}|_{\tilde{J}}$. By [14] Proposition 5.4], the pair $(J, \rho)$ is a $[G, \pi]_G$-type.

**Definition 1.2.** The pair $(G^0, \tilde{\rho})$ is called a cuspidal type if

(A1) The representation $\tilde{\rho}|_{\tilde{J}}$ is irreducible.

(A2) Any $g \in G$ which intertwines the representation $\rho$ lies in $\tilde{J}$.

For cuspidal types, it is shown in [14] Proposition 5.6] that the Hecke algebra $H(G, \rho)$ is commutative. As noted in [14], (A1) and (A2) is satisfied when $G = G^0$. The existence of a type that satisfies (A1) and (A2) is known when $G = \text{GL}_n$ or its inner forms and when $G$ is a classical group provided the residue characteristic of $F$ is odd.

2. SOME CLIFFORD THEORY

In this section, we collect some facts from Clifford theory that will be used later in the work. Let $\pi$ be an irreducible, supercuspidal representation of $G$ of the form $\pi = \text{ind}_J^G \tilde{\rho}$ for a representation $\tilde{\rho}$ of an open, compact mod center subgroup $\tilde{J}$ of $G$.

2.1. $^t$-world. Let $^tG \leq G$ of finite index in $G^0$ containing $G^{\text{der}}$ and some open subgroup. So obviously, $^tG$ is open. In the latter part of the article, we will restrict to the case $^tG = G^0$. But for instance, one can take $^tG = \ker(\kappa_G)$ the kernel of the Kottwitz map.
2.2. Since $Z \triangleleft G$ is a normal subgroup of finite index in $G$, we have the following as in usual Clifford theory.

**Lemma 2.1.** Let $\tilde{H}$ be any open subgroup of $G$ containing $Z$ and set $H = \tilde{H} \cap ^{1}G$. Let $\tilde{\sigma}$ be an irreducible representation of $\tilde{H}$. Let $X(\tilde{H}/H) = \text{Hom}(\tilde{H}/H, \mathbb{C}^{*})$ and let $X_{\tilde{H}}(\tilde{\sigma}) = \{ \nu \in X(\tilde{H}/H) \mid \tilde{\sigma} \otimes \nu \cong \tilde{\sigma} \}.$

1. Denote by $O_{\tilde{H}}(\tilde{\sigma})$ the set of all $\sigma \in \text{Irr}(H)$, which are isomorphic to a subrepresentation of $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma})$. Write $\text{Int}_{\tilde{H}}(\tilde{\sigma}) = \{ g \in \tilde{H} \mid \sigma \cong \sigma^{g} \}$ for the inertia group of $\sigma$ in $\tilde{H}$. We have $O_{\tilde{H}}(\tilde{\sigma}) \simeq \tilde{H}/\text{Int}_{\tilde{H}}(\tilde{\sigma})$.

2. Let $\sigma \in \text{Irr}(H)$ contained in $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma})$. There exists a positive integer $m_{H}(\tilde{\sigma})$ such that

$$\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma}) \simeq m_{H}(\tilde{\sigma}) \cdot \bigoplus_{g \in \tilde{H}/\text{Int}_{\tilde{H}}(\tilde{\sigma})} \sigma^{g}$$

and $\{ \sigma^{g} : g \in \tilde{H}/\text{Int}_{\tilde{H}}(\tilde{\sigma}) \}$ are all nonisomorphic conjugates of $\sigma$. In particular, if $\dim_{\mathbb{C}}(\sigma) < \infty$ then

$$\dim_{\mathbb{C}}(\tilde{\sigma}) = |\tilde{H}/\text{Int}_{\tilde{H}}(\tilde{\sigma})| \cdot m_{H}(\tilde{\sigma}) \cdot \dim_{\mathbb{C}}(\sigma).$$

3. Let $\tilde{\sigma}$ be the sum of all subrepresentations of $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma})$ that are isomorphic to $\sigma$. Then $\tilde{\sigma}$ is an irreducible representation for $\text{Int}_{\tilde{H}}(\tilde{\sigma})$ and

$$\text{Res}_{\tilde{H}}^{\text{Int}_{\tilde{H}}(\tilde{\sigma})}(\tilde{\sigma}) \simeq m_{H}(\tilde{\sigma}) \cdot \sigma$$

and $\tilde{\sigma} \simeq \text{Ind}_{\text{Int}_{\tilde{H}}(\tilde{\sigma})}^{\tilde{H}}(\tilde{\sigma})$.

4. For any $\tilde{\sigma}' \in \text{Irr}(\tilde{H})$, the following properties are equivalent:

i) $\tilde{\sigma}' \cong \chi \otimes \tilde{\sigma}$ for some character $\chi \in X(\tilde{H}/H)$,

ii) $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma}') \cong \text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma})$,

iii) $O_{\tilde{H}}(\tilde{\sigma}) = O_{\tilde{H}}(\tilde{\sigma}')$, 

iv) $O_{\tilde{H}}(\tilde{\sigma})$ and $O_{\tilde{H}}(\tilde{\sigma}')$ has direct irreducible factor that are isomorphic.

2.2.1. Let $^{1}H := \cap_{\chi \in X_{\tilde{H}}(\tilde{\sigma})} \ker(\chi)$. This is an open normal finite index subgroup of $\tilde{H}$. We choose an irreducible $H$-subspace $W$ of $\tilde{\sigma}$ whose $\tilde{H}$-stabilizer $^{s}H$ is maximal. We let $^{s}\sigma$ denote the natural representation of $^{s}H$ on $W$. Replacing by a $\tilde{H}$-conjugate, we can assume that $\text{Res}_{\tilde{H}}^{\tilde{H}}(^{s}\sigma) = \sigma$.

**Lemma 2.2.**

1. $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma}) = \Theta_{\tilde{H}^{1}H^{1}}^{\tilde{H}}(^{s}\sigma^{h})$.

2. We have $^{1}H \leq {^{s}H} \leq \text{Int}_{\tilde{H}}(\tilde{\sigma})$ and $[\text{Int}_{\tilde{H}}(\tilde{\sigma}) : {^{s}H}] = [{^{s}H} : {^{1}H}] = m_{H}(\tilde{\sigma})$.

3. $\tilde{\sigma} \cong \text{ind}_{^{1}H}^{\tilde{H}}(^{s}\sigma)$.

4. The representation $^{1}\sigma := \text{Res}_{\tilde{H}}^{\tilde{H}}(^{s}\sigma)$ is the unique irreducible representation which occurs in $\text{Res}_{\tilde{H}}^{\tilde{H}}(\tilde{\sigma})$ and satisfies $\text{Res}_{\tilde{H}}^{\tilde{H}}(^{1}\sigma) = \sigma$.

5. $\text{ind}_{^{1}H}^{\tilde{H}}(^{1}\sigma) \cong m_{H}(\tilde{\sigma}) \cdot \tilde{\sigma}$.

6. $X_{\tilde{H}}(\tilde{\sigma}) \cong X(\tilde{H}/^{1}H)$.

**Proof.** This is [10, Lemma 8.3] when $\tilde{H} = G$ and $H = ^{1}G = ^{0}G$, whose detailed proof can be found in [16, Lemma 1.6.3.1]. The exact same proof works out in this more general set up. $\Box$
3. Some results on Hecke algebras of types

3.1. Isomorphism of Hecke algebras. Let \( \tilde{J} \) be an open subgroup of \( G \) containing and compact mod \( Z \) and set \( J = \tilde{J} \cap \gamma G \).

Let \( \tilde{\rho} \in \text{Irr}(\tilde{J}) \) and \( \rho \in \mathcal{O}_J(\tilde{\rho}) \) contained in \( \tilde{\rho} \). Write

\[
\mathcal{I}_G(\rho) := \{ g \in G \mid \text{Hom}_{J_\rho} (\rho, \rho^g \neq 0) \}.
\]

By Schur’s lemma we have an equality \( \mathcal{I}_G(\rho) \cap N_G(J) = \text{Int}_{N_G(J)}(\rho) \).

**Lemma 3.1.** We have an isomorphism of algebras \( \mathcal{H}(G, \rho) \cong \mathcal{H}(\mathcal{I}_G(\rho), \rho) \).

**Proof.** Consider the morphism given by induction

\[
\text{End}_{\mathcal{I}_G(\rho)}(\text{ind}^G_{\mathcal{I}_G(\rho)}(\rho)) \xrightarrow{\text{ind}^G_{\mathcal{I}_G(\rho)}} \text{End}_G(\text{ind}^G_J(\rho)).
\]

By [17] §15.8, we have a canonical \( \mathcal{I}_G(\rho) \)-injective \( \text{ind}^G_{\mathcal{I}_G(\rho)}(\rho) \rightarrow \text{ind}^G_J(\rho) \) and the image is a direct summand. So by definition of \( \text{ind}^G_{\mathcal{I}_G(\rho)} \) (see [17] §15.1), the above morphism must be injective. Now, by Mackey decomposition [17, 5.5], we have an isomorphism of vector spaces

\[
\text{End}_G(\text{ind}^G_J(\rho)) \cong \bigoplus_{g \in X_t \cap G/J} \text{Hom}_{J_\rho}(\rho, \rho^g).
\]

By definition of \( \mathcal{I}_G(\rho) \), the only nonzero summands of this direct sum are precisely elements \( J \setminus \mathcal{I}_G(\rho) / J \). Therefore, we get an isomorphism of vector spaces

\[
\text{End}_{\mathcal{I}_G(\rho)}(\text{ind}^G_{\mathcal{I}_G(\rho)}(\rho)) \cong \text{End}_G(\text{ind}^G_J(\rho)).
\]

This ends the proof of the lemma.

**Proposition 3.2.** If \( m_J(\tilde{\rho}) = 1 \) and \( \mathcal{I}_G(\rho) \subset \mathcal{I}_G(\rho) \) then the Hecke algebra \( \mathcal{H}(G, \rho) \) is commutative.

**Remark 3.3.** Note that if \( \mathcal{I}_G(\rho) = \mathcal{I}_G(\rho) \) then all irreducible \( J \)-subrepresentations of \( \text{Res}_J(\tilde{\rho}) \) are isomorphic. If in addition \( m_J(\tilde{\rho}) = 1 \) then \( \tilde{\rho} \) is an extension of \( \rho \).

**Proof.** Note that by [17] (I) 5.2 d) we have (using Lemma 2.7 (3))

\[
\text{ind}^G_{\mathcal{I}_G(\rho)}(\rho) \cong \text{ind}^G_{\mathcal{I}_G(\rho)} \circ \text{Res}_J^\mathcal{I}_G(\rho)(\tilde{\rho}) \cong \tilde{\rho} \otimes_{\mathcal{C}} \text{ind}^G_{\mathcal{I}_G(\rho)}(1) \cong \tilde{\rho} \otimes_{\mathcal{C}} \mathcal{C}[\mathcal{I}_G(\rho) / J],
\]

with a diagonal action of \( \mathcal{I}_G(\rho) \), which is the canonical one on the second factor. Now, since \( \tilde{\rho} \) is \( \mathcal{I}_G(\rho) \)-irreducible, we have

\[
\text{End}_{\mathcal{I}_G(\rho)}(\text{ind}^G_{\mathcal{I}_G(\rho)}(\rho)) \cong \text{End}_{\mathcal{I}_G(\rho)}(\mathcal{C}[\mathcal{I}_G(\rho) / J]) \cong \mathcal{C}[\mathcal{I}_G(\rho) / J].
\]

3.2. Multiplicites for types. Let \( \pi \) be an irreducible, supercuspidal representation of \( G \) of the form \( \pi = c - \text{ind}^G_J \rho \) for an open compact mod center \( \tilde{J} \) and an irreducible representation \( \tilde{\rho} \) of \( \tilde{J} \). Let \( J = \tilde{J} \cap \gamma G \) and let \( \rho \) be an irreducible summand of \( \text{Res}_{\tilde{J}}^J(\tilde{\rho}) \). Assume \( \mathcal{I}_G(\rho) \subset \mathcal{I}_G(\rho) \). In particular, we have \( \mathcal{I}_G(\rho) = \mathcal{I}_J(\rho) = \text{Int}_J(\rho) \). In this subsection, we prove that \( m_J(\pi) = m_J(\tilde{\rho}) \). We will deduce several consequences of this result in the subsequent subsections.
3.2.1. If \( g \in G \) satisfies \( \text{Hom}_{J \cap J\sigma}(\hat{\rho}, \hat{\rho}^g) \neq 0 \), then \( \text{Hom}_{J \cap J\sigma}(\text{Res}_{J}^{J}(\hat{\rho}), \text{Res}_{J}^{J}(\hat{\rho})^g) \neq 0 \). Using Mackey and Clifford theories we deduce that there exists \( h, h' \in J \) such that \( h'gh^{-1} \in \mathcal{I}_G(\hat{\rho}) \), which shows

\[
I_G(\hat{\rho}) = \mathcal{J}I_G(\rho)\mathcal{J}.
\]

In particular, the assumption \( I_G(\rho) \subseteq \mathcal{J} \) implies that \( I_G(\hat{\rho}) = \mathcal{J} \).

3.2.2. We then recall the following statements are equivalent by [17, §8.3]: Let \( \sigma \in \text{Irr}(K) \) for an open compact mod center \( K \).

(i) \( \text{ind}_K^G(\sigma) \) is irreducible,

(ii) \( \text{End}_G(\text{ind}_K^G(\sigma)) = \mathbb{C} \),

(iii) \( I_G(\sigma) = K \),

(iv) \( \sigma \) is not contained in \( \text{ind}_K^G \text{Res}_{K \cap K_\sigma}^{K}(\hat{\rho}^g) \) for any \( g \notin K \).

This implies that \( \pi := \text{ind}_J^G(\hat{\rho}) \in \text{Irr}(G) \), \( ^b\pi := \text{ind}_{k}^G(\hat{\rho}) \in \text{Irr}(^bGJ) \) and \( ^b\pi := \text{ind}_J^G(\rho) \in \text{Irr}(^bG) \).

Indeed, for any \( g \in G \) we have

\[
\text{Hom}_{C_G}(^b\pi, ^b\pi^g) \simeq \text{Hom}_{C_G}(\text{ind}_J^G(\rho), \text{ind}_J^G(\rho^g)) \\
\simeq \bigoplus_{h \in J \cap J\sigma} \text{Hom}_{J \cap J\sigma}(\rho, \rho^{g})
\]

So since \( ^b\pi \) is irreducible the left \( \text{Hom} \) space is non zero if and only if \( g \in ^bG\mathcal{I}_G(\rho) \).

Accordingly,

\[
\text{Int}_G(^b\pi^g) = \mathcal{I}_G(^b\pi^g) = ^bG\mathcal{I}_G(\rho)^g, \quad \forall g \in G.
\]

3.2.3.

**Theorem 3.4.** We have

\[
m_{\mathcal{G}}(\pi) = m_{\mathcal{G}}(^b\pi) = m_J(\hat{\rho}).
\]

**Proof.** Using Mackey theory we can easily see that \( ^b\pi \) contains \( ^b\pi \) and that \( \pi \) contains \( ^b\pi \), so using Lemma \ref{lemma:1} we have

\[
\text{Res}_{G}^{GJ}(^b\pi) = m_{\mathcal{G}}(^b\pi) \bigoplus_{h \in J \cap J\sigma} ^b\pi^h \quad \text{and} \quad \text{Res}_{GJ}^{G}(\pi) = m_{\mathcal{G}}(^b\pi) \bigoplus_{h \in G \setminus GJ} ^b\pi^h.
\]

Similarly

\[
\text{Res}_{G}^{G}(\pi) = m_{\mathcal{G}}(^b\pi) \bigoplus_{h \in G \setminus G\mathcal{I}_G(\rho)} ^b\pi^h.
\]

We are going to compute the dimension of \( \Phi := \text{Hom}_G(\text{ind}_J^G(\rho), \text{ind}_J^G(\hat{\rho})) \) in various ways, mainly by playing with Mackey Theory and Frobenius reciprocities.

- First, we have \( \Phi \neq 0 \). Indeed,

\[
\text{Hom}_J(\rho, \text{Res}_{J}^{J}(\text{ind}_J^G(\hat{\rho}))) = \text{Hom}_J(\rho, \text{Res}_{J}^{J}(\text{Res}_{J}^{G}(\text{ind}_J^G(\hat{\rho})))) \\
= \text{Hom}_J(\rho, \text{Res}_{J}^{J}(\hat{\rho} \oplus \hat{\rho})) \quad [17, \S 5.8] \\
= \text{Hom}_J(\rho, \text{Res}_{J}^{J}(\hat{\rho})) \oplus \text{Hom}_J(\rho, \text{Res}_{J}^{J}(\hat{\rho}^c)).
\]

In particular, \( \text{dim}_C(\Phi) \geq m_J(\hat{\rho}) \).
Finally, \( \Phi \simeq \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\pi)). \) So (since \( \Phi \neq 0 \)) by the irreducibility of \( \mathcal{V}, \) and Clifford theory

\[
\dim_C(\Phi) = \bigoplus_{G/\text{Int}_G(\mathcal{V})} m_G(\pi) \dim_C(\text{Hom}_G(\mathcal{V}, \mathcal{V})) = m_G(\pi). \quad (3.1)
\]

- Recall that \( G/\mathcal{V}J \) is finite, so \( \text{ind}_G^G(\mathcal{V}) = \text{Ind}_G^G(\mathcal{V}) \) is admissible. Using Frobenius reciprocity and Mackey theory

\[
\Phi \simeq \text{Hom}_G(\text{Res}_G^G(\text{ind}_G^G(\rho)), \mathcal{V})
\]

\[
\simeq \bigoplus_{g \in J \mathcal{V}GJ} \text{Hom}_G(\text{ind}_{G \cap J}^G(J\pi), \text{Res}_J^G(J\pi))
\]

\[
\simeq \bigoplus_{g \in J \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\pi))
\]

Observe that \( \text{ind}_{G}^G(\mathcal{V}) = \text{ind}_{G}^G(\mathcal{V}) = \text{ind}_{G}^G(\mathcal{V}). \) So

\[
\Phi \simeq \bigoplus_{g \in G \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \mathcal{V}) = \text{End}_G(\mathcal{V})
\]

For any \( j \in \mathcal{V}/J \mathcal{V}GJ. \) The last Hom space is nonzero only if \( g^{-1}j \in J \mathcal{V}GJ = G \mathcal{V}GJ/\mathcal{V}GJ = \mathcal{V}GJ(\pi), \) hence \( g \in \mathcal{V}GJ \subset \text{Int}_G(\mathcal{V}) \) \( \mathcal{V}GJ(\pi). \) So,

\[
\Phi \simeq m_G(\mathcal{V}) \bigoplus_{j \in \mathcal{V}/J \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \mathcal{V}) = m_G(\mathcal{V}) \text{End}_G(\mathcal{V})
\]

Therefore,

\[
\dim_C(\Phi) = m_G(\mathcal{V}). \quad (3.2)
\]

- Finally,

\[
\Phi \simeq \bigoplus_{g \in G \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\mathcal{V}))
\]

\[
\simeq \bigoplus_{g \in G \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\mathcal{V}))
\]

\[
\simeq \bigoplus_{g \in G \mathcal{V}GJ} \bigoplus_{g \in \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\mathcal{V}))
\]

\[
\simeq \bigoplus_{g \in G \mathcal{V}GJ} \text{Hom}_G(\mathcal{V}, \text{Res}_G^G(\mathcal{V}))
\]

Accordingly \( \Phi \simeq m_J(\mathcal{V}) \text{End}_G(\mathcal{V}). \) In conclusion

\[
\dim_C(\Phi) = m_J(\mathcal{V}). \quad (3.3)
\]
Corollary 3.5. We have
\[ \text{Res}_G(\pi) = m_J(\tilde{\rho}) \bigoplus_{h \in G/\text{Int}_G(\rho)} h. \]

If in addition \( m_V(\pi) = 1 \), i.e. \( m_J(\tilde{\rho}) = 1 \) then
\[ \text{Res}_G(\pi) = \bigoplus_{\chi \in X^*(G/\text{Int}_G(\rho))} \chi \otimes h. \]

Lemma 3.6. Let \( J \) be as in Lemma 2.2 where \( \tilde{\mathcal{I}} = \tilde{J} \) and \( \tilde{\sigma} = \tilde{\rho} \). Then
\[ \tilde{J} = \text{Int}_G(\rho) \cap \bigcap_{\nu \in X_G(\pi)} \ker(\nu). \]

In particular \( \bigcap_{\nu \in X_G(\pi)} \ker(\nu) = ^V G \tilde{J} \) and \( \pi^\dagger = \text{ind}_{J}^{\nabla G}(\tilde{\rho}). \)

Proof. The inclusion \( \psi : \text{Int}_G(\rho)/J \hookrightarrow \tilde{\nabla} G \) shows that
\[ \bigcap_{\nu \in X_G(\pi)} \ker(\nu) \subset \bigcap_{\nu \psi(\tilde{\rho})} \ker(\nu) = \tilde{J} \tilde{G}. \]

Hence \( \text{Int}_G(\rho) \cap \bigcap_{\nu \in X_G(\pi)} \ker(\nu) \subset \tilde{J}. \) But we already know that
\[ [\text{Int}_G(\rho) : \text{Int}_G(\rho) \cap \bigcap_{\nu \in X_G(\pi)} \ker(\nu)] = [\text{Int}_G(1) : \bigcap_{\nu \in X_G(\pi)} \ker(\nu)] \]
\[ = m_G(\pi) = m_J(\tilde{\rho}) \]
\[ = [\text{Int}_G(\rho) : \tilde{J}]. \]

Therefore, \( \text{Int}_G(\rho) \cap \bigcap_{\nu \in X_G(\pi)} \ker(\nu) \) and \( \tilde{J} \) must be equal. \( \square \)

3.2.4.

Lemma 3.7. Let \( \tilde{\rho} \) be an irreducible representation of an open, compact mod center subgroup \( \tilde{J} \) of \( G \). Write \( J = \tilde{J} \cap \nabla G \). Let \( \rho \) be any irreducible summand of \( \text{Res}_G^J(\tilde{\rho}) \). Assume that \( \text{Int}_G(\rho) \subset \tilde{J} \). Then \( \pi = \text{ind}_{J}^{\nabla G}(\tilde{\rho}) \) is irreducible supercuspidal.

Proof. It suffices to show that \( \pi \) is irreducible, or in other words that, \( \text{Int}_G(\rho) \subset \tilde{J} \). This holds by the assumption \( \text{Int}_G(\rho) \subset \tilde{J} \) (see Section 3.2.1). \( \square \)

Lemma 3.8. Let \( \pi = \text{ind}_{J}^{\nabla G}(\tilde{\rho}) \), where \( \tilde{\rho} \) is an irreducible representation of an open, compact mod center subgroup \( \tilde{J} \) of \( G \). Write \( J = \tilde{J} \cap \nabla G \). Let \( \rho \) be any irreducible summand of \( \text{Res}_G^J(\tilde{\rho}) \). Assume that \( \text{Int}_G(\rho) \subset \tilde{J} \). Any irreducible quotient of \( \text{ind}_{J}^{\nabla G}(\rho) \) is of the form \( \pi \otimes \chi \) for some \( \chi \in X^*(\nabla G/\nabla G) \).

Proof. An irreducible representation \( \sigma \) is a quotient of \( \text{ind}_{J}^{\nabla G}(\rho) \) if and only if \( \text{Hom}_G(\text{ind}_{J}^{\nabla G}(\rho), \sigma) \neq 0 \). By Frobenius reciprocity this Hom space equals
\[ \text{Hom}_G(\text{ind}_{J}^{\nabla G}(\rho), \text{Res}_G^\nabla(\sigma)) \neq 0 \]
So \( \text{Res}_G^G(\sigma) \) contains \( \text{ind}_{J}^{\nabla G}(\rho) \), the latter being an irreducible component of \( \text{Res}_G^G(\pi) \). We deduce from (5) Theorem 2.1 that
\[ \sigma \cong \pi \otimes \chi \text{ for some } \chi \in X^*(\nabla G/\nabla G). \]
\( \square \)
3.3. Center of Hecke algebras. The following result describes the center of the Hecke algebra of a supercuspidal type.

**Theorem 3.9.**

\[ Z(\mathcal{H}(G, \rho)) \simeq Z(\mathcal{H}(\mathcal{I}_G(\rho), \rho)) \simeq \mathcal{H}(\mathring{\mathbf{J}}, \rho) \simeq \mathbb{C}[\mathring{\mathbf{J}}/\mathbf{J}]. \]

**Proof.** The first equality follows readily from Lemma 3.1. Now to prove the second isomorphism we use [16, Proposition 1.6.3.2] which says that

\[ Z(\mathcal{H}(G, \mathring{\mathbf{J}})) \simeq \mathcal{H}(T, \mathring{\mathbf{J}}), \]

where \( T = \cap_{\nu \in \mathbf{X}_G(\pi)} \ker(\nu) \). Lemma 3.1 applied to \( T \) shows that

\[ \mathcal{H}(T, \text{ind}\mathring{\mathbf{J}}(\rho)) = \mathcal{H}(\mathcal{I}_T(\rho), \rho) = \mathcal{H}(T \cap \mathcal{I}_G(\rho), \rho) \]

Now applying Lemma 3.6 we get

\[ Z(\mathcal{H}(G, \mathring{\mathbf{J}})) \simeq \mathcal{H}(\mathring{\mathbf{J}}, \rho). \]

We could have also reproduced the same argument of [16, Proposition 1.6.3.2] with \( \mathring{\mathbf{J}} \) now playing the role of \( T \) and \( \mathcal{I}_G(\rho) \) that of \( G \) and prove directly the second isomorphism above. \( \square \)

3.4. Criterion for the Hecke algebra of a supercuspidal type to be commutative.

**Proposition 3.10.** Let \( \pi \) be an irreducible supercuspidal representation of \( G \) of the form \( \pi = \text{ind}_J^G \tilde{\rho} \) for an irreducible representation \( \tilde{\rho} \) of an open, compact mod center subgroup \( \tilde{J} \) of \( G \). Write \( J = \tilde{J} \cap \mathring{\mathbf{J}}. \) Let \( \rho \) be any irreducible summand of \( \text{Res}_J^G(\tilde{\rho}) \). Assume that any \( \mathcal{I}_G(\rho) \subset \tilde{J} \). The following statements are equivalent

1. The representation \( \text{Res}_G(\pi) \) is multiplicity free.
2. The representation \( \text{Res}_J^\mathring{\mathbf{J}}(\tilde{\rho}) \) is also multiplicity free.
3. The Hecke algebra \( \mathcal{H}(G, \rho) \) is commutative.

**Proof.** Given that \( \mathring{\mathbf{J}} \) is open we know that \( \mathcal{H}(G, \mathring{\mathbf{J}}) \simeq \mathcal{H}(G, \rho) \simeq \mathcal{H}(\mathcal{I}_G(\rho), \rho) \) thanks to the transitivity of the induction. So by Theorem 3.9 we have the equivalence \( (1) \iff (3) \) and Proposition 3.4 gives \( (2) \iff (1) \iff (2) \). \( \square \)

3.5. Yu’s construction of supercuspidal representations and a corollary.

In this section, we study the commutativity of Hecke algebras of types attached to supercuspidal representations arising from Yu’s construction. To do this, let us recall Yu’s construction of supercuspidal representations and introduce the necessary notation along the way.

Let \( F \) be a non-archimedean local field and \( F^t \) its maximal tamely ramified extension. Let \( G \) be a connected, reductive group over \( F \). The Yu datum consists of a 5-tuple \( (\tilde{G}, y, \tilde{r}, \rho_{-1}, \tilde{\phi}) \) where

**D1:** \( \tilde{G} = (G^0, \ldots, G^d) \) is a tower of algebraic subgroups of \( G \),

\[ G^0 \varsubsetneq \cdots \varsubsetneq G^d = G \]

such that \( Z(G^0)/Z(G) \) is anisotropic over \( F \) and \( \tilde{G} \) is a tamely ramified twisted Levi sequence in \( G \) in the section of [18, Section 1]. In particular, \( G^t \otimes F^t \) is split and is a Levi factor of a parabolic subgroup of \( G \otimes F^t \).
and Yu also defines subgroups $J_i$ of depth $d$. Starting with such a datum, Yu’s construction gives a supercuspidal representation $\rho_{-1}$ of $K^0 = G^0_{[y]}$ such that $\rho_{-1} \mid G^0_{y,\beta}$ is 1-isotypic and the compactly induced representation $\pi_{-1} = e - \text{ind}_{K^0} G^0_{y,\beta}$ is irreducible, supercuspidal. Here $[y]$ is the projection of $y$ on the reduced building and $G^0_{[y]}$ is the subgroup of $G$ fixing $[y]$.

Yu also defines subgroups $J_i$ for $1 \leq i \leq d$ such that each $\phi_i$ is a quasi-character of $G_i$ for each $i$. We assume that $\phi_i$ is trivial on $G^i_{r_i}$ but not on $G^i_{r_i}$ that is, that $\text{depth}(\phi_i) = r_i$ for $0 \leq i \leq d - 1$. Here that we have used the convention $G^i_{r_i} = G^i_{y,r_i}$ and similarly for $G^i_{r_i}$. If $r_{d-1} < r_d$ we assume that $\phi_d$ is trivial on $G^d_{r_{d+1}}$ but not on $G^d_{r_d}$ otherwise assume that $\phi_d = 1$. We also assume that $\phi_i$ is $G^{i+1}$ generic (see [18] Section 9).

Starting with such a datum, Yu’s construction gives a supercuspidal representation of depth $r_d$. Let us summarize this construction. Let $K^0_+ = G^0_{0,}$ and for $1 \leq i \leq d$, let $s_i = r_i/2$ and let

$$K^i = K^0_i G^1_{s_0} \cdots G^i_{s_i}$$

and

$$K^i_+ = K^0_i G^1_{s_0} \cdots G^i_{s_i+1}$$

Yu’s construction of the supercuspidal representation of $H$ from this data is done inductively and includes the following steps.

(a) In [18] Section 11, for $0 \leq i \leq d - 1$, Yu constructs an irreducible representation $\hat{\phi}_{i-1}$ of $K^{i-1}_+ \times J^i$ using the character $\phi_{i-1}$ of $G^{i-1}$, that satisfies condition SC2, in [18] Section 4, Page 592. Let us recall this construction. Let $\hat{\phi}_{i-1}$ be the character of $K^0_i G^{0}_0 G^1_{y,s_{i-1}+1}$ as in [18] Section 4, Page 591]. Using $\hat{\phi}_{i-1}$, he defines a non-degenerate $\mathbb{F}_p$-valued pairing on the $\mathbb{F}_p$-vector space $J^i_+ / J^i_+$ making $J^i_+ / J^i_+$ a symplectic space over $\mathbb{F}_p$. Let $(J^i_+ / J^i_+)^#$ be the Heisenberg group of $J^i_+ / J^i_+$. Yu constructs a canonical isomorphism

$$j : J^i_+ / \ker(\hat{\phi}_{i-1}) \to (J^i_+ / J^i_+)^#$$

in [18] Proposition 11.4. Note that $K^i$ act on $J^i_+ / J^i_+$ by conjugation and this gives a homomorphism from $K^i \to Sp(J^i_+ / J^i_+)$. Let $\hat{\phi}_{i-1}$ be the pull back of the Weil representation of $Sp(J^i_+ / J^i_+) \times (J^i_+ / J^i_+)^#$ via the map $K^{i-1} \times J^i \to Sp(J^i_+ / J^i_+) \times (J^i_+ / J^i_+)^#$. He shows in [18] Theorem 11.5 that $\hat{\phi}_{i-1} | J^i_+$ is $\hat{\phi}_{i-1} | J^i_+$ is 1-isotypic and that the restriction of $\hat{\phi}_{i-1}$ to $K^{i+1}_+$ is 1-isotypic.
(b) Next, Yu constructs a representation $\rho'_i$ of $K^i$ such that $\rho'_i|_{G^i_{\phi}}$ is 1-isotypic. He then sets $\rho_i = \rho'_i \otimes \phi_i|_{K^i}$. First, put $\rho'_0 = \rho_{-1}$ and $\rho_0 = \rho_0 \otimes (\phi_0|_{K^0})$. Now suppose that $\rho'_{i-1}$ and $\rho_{i-1}$ have already been constructed. Inflating $\phi_{i-1}|_{K^{i-1}}$ to a representation $\inf(\phi_{i-1})$ of $K^{i-1} \rtimes J^i$. He shows that the representation $\inf(\phi_{i-1}) \circ \phi_{i-1}$ factors through the natural map $K^{i-1} \times J^i \to K^{i-1}J^i = K^i$. Let $\phi'_{i-1}$ be the representation of $K^i$ whose inflation to $K^{i-1} \rtimes J^i$ is $\inf(\phi_{i-1}) \circ \phi_{i-1}$. Inflating $\rho'_{i-1}$ to a representation $\inf(\rho'_{i-1})$ of $K^i = K^{i-1}J^i$ via the map $K^i \to K^{i-1}J^i/J^i = K^{i-1}/K^{i-1} \cap J^i$ (This can be done $\rho'_{i-1}$ restricted to $K^{i-1} \cap J^i$ is 1-isotypic). Set $\rho'_i = \inf(\rho'_{i-1}) \otimes \phi'_{i-1}$ and $\rho_i = \rho'_i \otimes (\phi_i|_{K^i})$.

(c) The main theorem of Yu’s paper [18] says that the compactly induced representation $\pi_i = \text{ind}_{K^i}^{G^i}\rho_i$ of $G^i$ is irreducible and supercuspidal of depth $r_i$, $0 \leq i \leq d$.

Let $0K^0 = G^0_y$ and $0K^i = (0K^0)G^1_{s_0} \cdots G^i_{s_{i-1}}$. Let $0\rho_i$ be an irreducible summand of $\rho_i|_{0K^i}$. As noted in [18 Corollary 15.3], we have that $(0K^i, 0\rho_i)$ is a $[G^i, \pi_i]_{G^i}$-type for $0 \leq i \leq d$.

**Lemma 3.11.** Let $\pi_i = c - \text{ind}_{K^0}^{H^0}\rho_{-1}$ be a depth zero irreducible, supercuspidal representation of $G^0$ that is part of the Yu datum discussed above. Then, for each $0 \leq i \leq d$,

1. Let $0\rho_i$ be any irreducible summand of $\rho_i|_{0K^i}$. Then every $g \in G^i$ that intertwines $0\rho_i$ lies in $K^i$, and
2. $m_{0K^0}(\rho_{-1}) = m_{0K^i}(\rho_i)$.

**Proof.** Note that by [13 Theorem 4.15], every $h \in H^0$ that intertwines $0\rho_{-1}$ lies in $K^0$. Now, (1) is a consequence of [18 Corollary 15.5], whose corrected proof can be found in [14 Proposition 4.4]. In more detail, it is shown in loc. cit that if $g \in G^i$ intertwines $0\rho_i$ then $g \in (0K^i)G^0(0K^i)$ and that $g \in G^0$ intertwines $0\rho_i$ if and only if $g$ intertwines $0\rho_0$. We know that if $g \in G^0$ intertwines $0\rho_0$, then $g \in K^0$. Hence $g \in G^0$ intertwines $0\rho_i$, then $g \in (0K^i)(0K^i)$. By definition $(0K^i)(0K^i) = K^i$ and clearly $(0K^i)(0K^i) = K^i$, hence (1) follows.

Let us prove (2). Recall that for $0 \leq i \leq d$, $\rho_i = \rho'_i \otimes (\phi_i|_{K^i})$, so $\rho_i|_{0K^i} = \rho'_i|_{0K^i} \otimes (\phi_i|_{K^i})$. Hence $m_{0K^i}(\rho_i) = m_{0K^i}(\rho'_i)$. To prove (2), it suffices to show that $m_{0K^0}(\rho_{-1}) = m_{0K^i}(\rho_{-1})$. Note that $0K^0 = 0K^{i-1}J^i$, and hence under the map $K^i \to K^{i-1}J^i/J^i = K^{i-1}/K^{i-1} \cap J^i$, we have $0K^i \to 0K^0 \cap K^{i-1}J^i/J^i = 0K^{i-1}J^{i-1} \cap J^i$. Hence $\inf(\rho_{i-1})|_{0K^i} = \inf(\rho_{i-1}|_{0K^{i-1}})$ as representations of $0K^i$. Hence $m_{0K^0}(\rho_{-1}) = m_{0K^0}(\rho_{i-1}) = m_{0K^i}(\rho_{i-1})$. Now $\rho'_i|_{0K^i} = \inf(\rho_{i-1})|_{0K^i} \otimes (\phi'_i|_{0K^{i-1}})$. So, to show that $m_{0K^{i-1}}(\rho_{i-1}) = m_{0K^i}(\rho'_i)$, it suffices to observe that that $(\phi'_i|_{0K^{i-1}})$ is irreducible. This is in fact clear from Yu’s construction of $\phi_{i-1}$; in more detail, Note that $0K^i$ also acts on $J^i/J^i$ by conjugation and this gives a homomorphism from $0K^i \to Sp(J^i/J^i)$. Let $0\phi_{i-1}$ be the pull back of the Weil representation of $Sp(J^i/J^i) \times (J^i/J^i)^\#$ by the map $0K^{i-1} \times J^i \to Sp(J^i/J^i) \times (J^i/J^i)^\#$. Then clearly, $(\phi_{i-1}|_{0K^i}) = 0\phi_{i-1}$ is irreducible. This proves that $(0\phi_{i-1})|_{0K^i} = \inf(\phi_{i-1})|_{0K^i} \otimes (\phi'_i|_{0K^{i-1}}) = \inf(\phi_{i-1})|_{0K^i} \otimes (0\phi_{i-1})$ is irreducible. Hence (2) is proved.

\[ \square \]

**Corollary 3.12.** Let $G$ be a connected, reductive group over $F$ and let $\pi_\gamma = c - \text{ind}_{K^0}^{H^0}\rho_{-1}$ be the depth zero supercuspidal representation of $G^0$ that is part of the Yu datum. Let $\pi = \pi_d$ be a tame supercuspidal representation of $G$ of depth
type, i.e. Definition 4.1.

4.1.1. is an equivalence of categories (see [4, Section 4.2]).

π inertial support contains ρ that contain ρ in inertial support contains π.

following properties are equivalent.

Lemma 4.2. Let J, ρ be the subset of irreducible representations of G whose inertial support contains π. By [3] Proposition 5.4 the pair (J, ρ) is a σ = [G, π]_G-type, i.e. [ρ] = [σ]_G. Accordingly, the functor

\[ M_\rho : R^\sigma(G) \to H(G, \rho) - \mod, \pi \mapsto \text{Hom}_J(\rho, \pi) \]

is an equivalence of categories (see [4, Section 4.2]).

4.1.1.

Definition 4.1. We say that two types (J, ρ) and (J', ρ') are G-equivalent and write it as (J, ρ) \equiv_G (J', ρ') if ind^G_J(ρ) \equiv ind^G_{J'}(ρ')

Lemma 4.2. Let (J, ρ) and (J', ρ') be an s-type and s'-type respectively. The following properties are equivalent.

(1) [ρ] ∩ [ρ'] ≠ ∅.
(2) [ρ] = [ρ']
(3) Hom_G(ind^G_J(ρ), ind^G_{J'}(ρ')) ≠ 0.
(4) (J, ρ) \equiv_G (J', ρ').

Proof. (1) \iff (2) is clear since any two orbits in Irr(G) under the action of X^*(G^0/\rho)G are disjoint or equal.

(2) \implies (3). Let \{x_i | i \in I\} be a system of representatives of J' \setminus G\uparrow J. Using Mackey formula and Frobenius reciprocity we have

\[ \text{Hom}_G(\text{ind}^G_J(\rho'), \text{ind}^G_J(\rho)) \simeq \text{Hom}_G(\text{ind}^G_J(\rho'), \text{ind}^G_J(\text{Res}^G_J(1/\rho))) \]
\[ \quad \simeq \bigoplus_{i \in I} \text{Hom}_{\rho \cap J'^+(\rho'}, (\rho \otimes_{\mathbb{C}[J'/J]} \mathbb{C}[1/\rho]) \text{mod} \mathbb{C}[1/\rho]) \]
\[ \quad \simeq \bigoplus_{i \in I} \text{Hom}_{\rho \cap J'^+(\rho'}, (\rho \otimes_{\mathbb{C}[J'/J]} \mathbb{C}[1/\rho]) \text{mod} \mathbb{C}[1/\rho]) \]
\[ \quad \simeq \bigoplus_{i \in I} \text{Hom}_{\rho \cap J'^+(\rho'}, (\rho \otimes_{\mathbb{C}[J'/J]} \mathbb{C}[1/\rho]) \text{mod} \mathbb{C}[1/\rho]) \]
\[ \quad \simeq \text{Hom}_G(\text{ind}^G_J(\rho'), \text{ind}^G_J(\rho)) \otimes_{\mathbb{C}[J'/J]} \mathbb{C}[1/\rho]. \]
The commutation of the tensor product in the fifth equation comes from the fact that $J' \cap J^x = J' \cap G^i \cap J^x = J' \cap J^x \subset J^x$ by maximality of $J^x$ in $\dagger$, hence $J' \cap J^x$ acts trivially on $\mathbb{C}[\dagger; J^x]$. The isomorphism from the fifth to sixth equation is obtained as follows. Let $\psi \in \text{Hom}_G(\text{ind}_{J'}^G(\rho'), \text{ind}_{J'}^G(\rho))$ map to $(\psi_i)_{i \epsilon I} \in \left(\oplus_{i \epsilon I} \text{Hom}_{J_i \cap J^x} (\rho', \dagger \rho_i)\right)$. The isomorphism is then given by sending $\psi \otimes \chi \rightarrow (\psi_i \otimes \chi^{\rho_i})_{i \epsilon I}$.

Now by Lemma 22 we deduce

$$\text{Hom}_G(\text{ind}_{J'}^G(\rho), \text{ind}_{J'}^G(\rho)) = \text{Hom}_G(\text{ind}_{J'}^G(\rho'), \text{ind}_{J'}^G(\rho)) \otimes \mathbb{C}[\dagger; J/J]$$

Let $\psi \in \text{Hom}_G(\text{ind}_{J'}^G(\rho'), \pi)$ non zero, such morphism exists since $\pi_{\rho'} \neq 0$ by assumption (2); therefore, $\psi \otimes 1_{\mathbb{C}[\dagger; J/J]}$ yields a non zero morphism $\text{ind}_{J'}^G(\rho') \rightarrow \text{ind}_{J'}^G(\rho)$.

(3) $\Rightarrow$ (2): The representation $\text{ind}_{J'}^G(\rho')$ is of finite type; so its image by a nonzero $G$-morphism in $\text{ind}_{J'}^G(\rho)$ admits an irreducible quotient which must be of the form $\pi \otimes \chi$ by Lemma 3.8 Hence $[\rho] \cap [\rho'] \neq 0$, which proves the implication.

(2) $\Rightarrow$ (4): By assumption, $[G, \pi]_G = [G, \pi']_G$, so we must have $\pi = \pi' \otimes \chi$ for some $\chi \in X^*(G/\chi G)$. Hence, there exists $x, y \in G$ such that

$$\text{Hom}_G((\text{ind}_{J'}^G(\rho')^x), (\text{ind}_{J'}^G(\rho)^y)) \neq 0.$$ 

Accordingly, $\text{ind}_{J'}^G(\rho'^x) \simeq \text{ind}_{J'}^G(\rho^y)$ and so $\text{ind}_{J}^G(\rho'^x) \simeq \text{ind}_{J}^G(\rho^y)$ which yields (4).

(4) $\Rightarrow$ (1) This is clear. □

4.2. Normalizer of a type. Let $M$ be a Levi subgroup of $G$ and let $\sigma = \text{ind}^G_M(\tilde{\rho}_M)$ be a cuspidal representation where $\tilde{\rho}_M$ is an irreducible representation of an open, compact mod center subgroup $\tilde{J}_M$ of $M$. Let $s = [M, \sigma]_G$ and $t = [M, \sigma]_M$. Let $J_M = M^o \cap J_M$ and let $\rho_M$ be any irreducible summand of $\tilde{\rho}_M$.

We assume as usual

$$\mathcal{I}_M(\rho_M) \subset \tilde{J}_M.$$

Let $(J, \rho)$ be a $G$-cover of $(J_M, \rho_M)$, which implies that $(J, \rho)$ is a $s$-type [4, Theorem 8.3].

The normalizer $N_G(M)$ of a Levi $M$ acts naturally by conjugation on $\text{Irr}(M)$. Denote by either $w(\sigma)$ or $\sigma^w$ the conjugate of any $\sigma \in \text{Irr}(M)$ by an element $w \in N_G(M)$ and the pair $(J_M^w, \rho_M^w)$ the conjugate of $(J_M, \rho_M)$; it is clear that since $(J_M, \rho_M)$ is a $t$-type then $(J_M^w, \rho_M^w)$ is a $w(t)$-type. Moreover, $\mathcal{I}_M(\rho_M^w) = w(\mathcal{I}_M(\rho_M))$ is a group such that $\mathcal{H}(M, \rho_M) \simeq \mathcal{H}(M, w(\rho_M)).$

Proposition 4.3. In the setup above, the following are equivalent for $n \in N_G(M)$.

(1) $n[J_M] \cap [J_M] \neq \emptyset$.

(2) $\text{Hom}_M(n(\text{ind}^M_{J_M}(\rho_M)), \text{ind}^M_{J_M}(\rho_M)) \neq 0$.

(3) $(J_M^w, \rho_M^w) \simeq_M (J_M, \rho_M)$.

(4) There exists $m \in M$ such that $mn \in \mathcal{I}_G(\rho)$.

The group $N_G(\rho_M) := \{n \in N_G(M) : (J_M^w, \rho_M^w) \simeq_M (J_M, \rho_M)\}$ is called the normalizer of the type $(J_M, \rho_M)$. In particular, the stabilizer of $t$ in the Weyl group is $W = W(\rho_M) := N_G(\rho_M)/M$. 


Proof. (1) $\iff$ (2) $\iff$ (3). As in [9, Proposition 1.9.1], we first observe that $n(\text{ind}_{J_M}^M(\rho_M)) \cong (\text{ind}_{J_M}^M(\rho_M'))$, so these equivalences follow readily from Lemma 4.2

(2) $\iff$ (4) As we saw in the proof of Lemma 4.2 we have

$$\text{Hom}_M(\text{ind}_{J_M}^M(\rho_M), \sigma^n) \neq 0$$

Using Frobenius reciprocity and then Mackey formula we show the existence of a $m' \in M$ such that

$$\text{Hom}_{J_M \cap J'_M}(\text{Res}_{J_M \cap J'_M}^J(\rho_M), \text{Res}_{J_M \cap J'_M}^J(\rho_M')) \neq 0.$$ 

Now since $J_M \cap J'_M = J_M \cap J'_M$ we deduce using Clifford theory that

$$\text{Hom}_{J_M \cap J'_M}(\text{Res}_{J_M \cap J'_M}^J(\rho_M), \text{Res}_{J_M \cap J'_M}^J(\rho_M')) \neq 0$$

for some $m \in M$.

Let $P$ be any parabolic subgroup with Levi factor $M$ and a radical unipotent $U$, $U$ its opposite. By definition of a cover, we have an Iwahori decomposition for $J$ with respect to any parabolic subgroup with Levi component $M$. Now it suffices to observe that

$$J \cap J' = (J \cap J' \cap U) \cdot (J_M \cap J'') \cdot (J_M \cap J'' \cap U)$$

and that $\rho$ and $\rho'$ are both trivial on both unipotent factors $J \cap J' \cap U$ and $J_M \cap J'' \cap U$. Therefore,

$$\text{Hom}_{J_M \cap J''}(\text{Res}_{J_M \cap J''}(\rho), \text{Res}_{J_M \cap J''}(\rho')) = \text{Hom}_{J_M \cap J''}(\text{Res}_{J_M \cap J''}^J(\rho_M), \text{Res}_{J_M \cap J''}^J(\rho'_M)) \neq 0.$$ 

Which shows (4).

(4) $\iff$ (1) It is easy to invert this argument to see that if $n$ verifies (4), then it verifies $\text{Hom}_M(\text{ind}_{J_M}^M(\rho_M), \sigma^n) \neq 0$ which is sufficient to prove (1). 

Corollary 4.4. For any two $G$-covers $(J, \rho)$ and $(J', \rho')$ of two types $(J_M, \rho_M)$ and $(J'_M, \rho'_M)$ the following is equivalent

(1) $[\rho] \cap [\rho'] \neq \emptyset$,

(2) $(J, \rho) \cong_G (J', \rho').$

Proof. Given Proposition 4.3 this is the same as [9, Proposition 4.5.1]. 

4.3. Weyl action on the center.

Definition 4.5. Let $n \in N_G(M)$, Proposition 4.3 shows that $n$ normalizes $t$ if and only if $(\text{ind}_{J_M}^M(\rho_M))^n \cong \text{ind}_{J_M}^M(\rho_M)$. Write $V_i$ the underline space of $\sigma_{un} = \text{ind}_{J_M}^M(\rho_M)$. For any $w \in W_i$ choose any representative $n_w \in N_G(\rho_M)$ and an element $\bar{w} \in \text{Aut}_C(V_i)$ which realizes the isomorphism $\sigma_{un} \leadsto \sigma_{un}^{n_w}$ meaning

$$\sigma_{un}^{n_w}(m)(v) = \sigma_{un}(m^w)(v) = \bar{w}(\sigma_{un}(m))(v), \quad \forall m \in M, \forall v \in V_i.$$ 

Note that if $n_w'$ is another representatives for the class $w \in W_i$ and $\bar{w}'$ the corresponding automorphism in $\text{Aut}_C(V_i)$ then $\bar{w}^{-1}\bar{w}' \in \text{End}_M(\sigma_{un})^\times = \mathcal{H}(M, \rho_M)^\times$ since $n_w^{-1}n'_w \in M$.

Consider the following algebra isomorphism

$$\text{End}_M(\sigma_{un}) \longrightarrow \text{End}_M(\sigma_{un}), \quad \phi \longmapsto \bar{w}\phi\bar{w}^{-1}.$$
Although this depends on the choice of $n_w$ and $\bar{w}$, its restriction to the center does not depend on either of the two. Which defines an action of $W = W(\rho_M)$ on $Z(\mathcal{H}(M, \rho_M))$:

$$w : Z(\mathcal{H}(M, \rho_M)) \rightarrow Z(\mathcal{H}(M, \rho_M)), \ \phi \mapsto \bar{w}\phi\bar{w}^{-1}.$$  

We can make precise the action of the normalizer of a type on the center of the Hecke algebra:

**Theorem 4.6.**

$$Z(\mathcal{H}(G, \rho)) \simeq \mathbb{C}[\uparrow J_M/J_M]^{W(\rho_M)}.$$  

**Proof.** For any $j \in 0M^\uparrow J_M$, let $\psi_j$ be the unique element in $\mathcal{H}(M, \rho_M)$ with support $J_Mj$. One can verify that

$$\bar{w}\psi_j\bar{w}^{-1} = \psi_{w(j)}.$$  

This ends the proof since by Theorem 3.9

$$Z(\mathcal{H}(M, \rho_M)) = Z(\mathcal{H}(0M^\uparrow J_M, \text{ind}_{J_M}^0(\rho_M))) \simeq \mathbb{C}[0M^\uparrow J_M/\uparrow J_M] \simeq \mathbb{C}[\uparrow J_M/J_M]. \quad \square$$

**Example 4.7.** Assume that $M$ is a minimal Levi and that $(0M, 1_{0M})$ is its maximal open compact subgroup together with its trivial character. This is a cuspidal type for $t = [M, 1]$ and $Wt = W_0 = N_G(M)/M$. Clearly $M = \mathcal{I}_M(1) = \mathcal{I}_M$. Let $I$ be an (maximal) Iwahori open compact subgroup containing $0M$. The pair $(I, 1_I)$ is a $G$-cover for $(0M, 1)$. So Theorem 4.6 shows in this particular case the classical Satake isomorphism

$$\mathcal{Z}(\mathcal{H}(G, I)) \simeq \mathbb{C}[M^{0M}]^{W_0}.$$  

5. SOME NICE FAMILIES OF COMPACT OPEN SUBGROUPS

Let $K$ be a compact open subgroup of $G$ and let $\mathcal{R}_K(G)$ be the full sub-category of $\mathcal{R}(G)$ consisting of representations $(\pi, V)$ that are generated by their $K$-fixed vectors.

Let $S$ be a maximal split torus in $G$. In [2 Section 3.7] the authors introduce criteria on $K$, which we call $\sigma_S$ and recall now.

**Definition 5.1.** Let $K$ be a compact open subgroup of $G$. We say $K$ satisfies $\sigma_S$ if

1. Let $P$ be a parabolic subgroup of $G$ that contains $S$. Write $P = MN$ with $M$.

   Let $K'$ be a $G$-conjugate of $K$ and let $K'_P = K' \cap P/K' \cap N$. For any parabolic subgroup $Q$ of $G$ with the same Levi subgroup $M$ and any other $G$-conjugate $K_1$ of $K$, $(K_1)_Q$ is a conjugate of $K'_P$ in $M$.

2. Let $(\sigma, V)$ be a representation of $G$. Let $V(N) = \text{Span}(\sigma(n)v - v|v \in V, n \in N)$ and let $V_N = V/V(N)$. Then the canonical map $V^K \rightarrow V_N^{M\cap K}$ is surjective.

Let $\mathcal{K}_S(S, G)$ be the collection of all compact open subgroups of $G$ that satisfies $\sigma_S$. Let us recall the following proposition.

**Proposition 5.2** (Corollary 3.9 of [2]). Let $S$ be a maximal $F$-split torus in $G$ and let $K \in \mathcal{K}_S(S, G)$. The category $\mathcal{R}_K(G)$ is closed under sub-quotients. There exists a finite subset $\mathfrak{S}_K \subset \mathcal{B}(G)$ such that

$$\mathcal{R}_K(G) = \prod_{s \in \mathfrak{S}_K} \mathcal{R}_s(G).$$
The functor
\[ \mathcal{R}_K(G) \rightarrow \mathcal{H}(G, K) - \text{mod}, \]
\[ (\sigma, V) \rightarrow V^K, \]
is an equivalence of categories with left adjoint
\[ \mathcal{H}(G, K) - \text{mod} \rightarrow \mathcal{R}_K(G), \]
\[ V^K \rightarrow (\mathcal{H}(G) * e_K) \otimes_{\mathcal{H}(G, K)} V^K. \]

Lemma 5.3. Let \( K \in K^\circ(S, G) \). We have \( s = [M, \sigma]_G \in \mathfrak{S}_K \) if and only if \( \sigma^{K \cap M} \neq 0 \).

Proof. The proof given in [5, Proposition 4] goes through verbatim. \( \square \)

5.1. Some compact open subgroups that live in \( K^\circ(S, G) \). In [3, Section 5], the following condition is considered in place of \( \forall_S \) above.

Definition 5.4. Let \( S \) be a maximal \( F \)-split torus in \( G \). Let \( K \) be a compact open subgroup of \( G \) and let \( K^G \) be the set of \( G \)-conjugates of \( K \). We say \( K \) satisfies \( \bullet_S \) if, for any parabolic subgroup \( P \) of \( G \) that contains \( S \), any \( P \)-conjugacy class of \( K^G \) contains a \( K' \) that admits an Iwahori decomposition with respect to \( P \):
\[ K' = (K' \cap N^-)(K' \cap M)(K' \cap N). \]

Let \( K^\bullet(S, G) \) be the collection of compact open subgroups of \( G \) that satisfy \( \bullet_S \). It is shown in [3, Proposition 5.1] that Proposition 5.2 holds for all \( K \in K^\bullet(S, G) \).

By [3, Proposition 5.2], we have \( G_{x,r} \in K^\bullet(S, G) \) for all \( x \in \mathcal{A}(S, F) \) and \( r > 0 \).

We are interested in compact open subgroups for which Lemma 5.3 holds, that is in compact open subgroups that lie in \( K^\circ(S, G) \). To this end, we prove the following proposition.

Proposition 5.5. Let \( \mathfrak{a} \) be an alcove in \( \mathcal{A}(S, F) \). Let \( x \in \mathfrak{a} \) and let \( r > 0 \). Then \( G_{x,r} \in K^\circ(S, G) \).

Proof. We only need to verify that \( G_{x,r} \) satisfies (1). Choose a special vertex \( x_0 \) in the closure of \( \mathfrak{a} \). Let \( N \) be the normalizer of \( S \) in \( G \). Then, using the Iwasawa decomposition, we know that \( gG_{x_0, r}g^{-1} \) is \( P \)-conjugate to \( nG_{x_0, r}n^{-1} \) for a suitable \( n \) in \( N \). But \( nG_{x_0, r}n^{-1} = G_{n(x), r} \). So, we only need to verify that \( G_{n(x), r} \cap M \) is \( M \)-conjugate to \( G_{x,r} \cap M \). The alcove \( \mathfrak{a} \) and the choice of \( x_0 \) determines a set of simple roots \( \Delta \in \Phi(G, S) \). We may and do assume that \( M = M_\theta \) for a suitable \( \theta \in \Delta \). Let \( \Phi_\theta \) be the set of roots of \( \Phi(G, S) \) that lie in the \( \mathbb{Q} \)-span of \( \theta \). We accordingly have \( \Phi_\theta^+ \) and \( \Phi_\theta^- \). Then \( W_\theta = \{ s_a \mid a \in \theta \} = W(M, S) \). Every element \( w \in W(G, S) \) can be written as \( w_1w_2 \) where \( w_1 \in W_\theta \) and \( w_2^{-1}(\theta) > 0 \).

To prove (2), it suffices to show that \( G_{w_1w_2x, r} \cap M \) is \( M \)-conjugate to \( G_{x, r} \cap M \). Since \( w_1 \in W_\theta \), we see that \( G_{w_1w_2x, r} \cap M \) is \( M \)-conjugate to \( G_{w_2x, r} \cap M \). To finish the proof, it suffices to show that \( G_{w_2x, r} \cap M = G_{x, r} \cap M \). Since \( G_{x, r} = \langle U_\alpha(F)_{x, r} \mid \alpha \in \Phi_\theta \rangle \), it suffices show that for each \( \alpha \in \Phi_\theta \), \( U_\alpha(F)_{w_2x, r} = U_\alpha(F)_{x, r} \). Let us recall the definition of \( U_\alpha(F)_{x,r} \).

Recall that we have chosen a special point \( x_0 \) in \( \mathcal{A}(S, F) \). Let \( (\phi_{a})_{a \in \Phi(G, S)} \) be the corresponding valuation of root datum of \( (T, (U_\alpha)_{a \in \Phi(G, S)}) \) (see [7, §6.2]). For
as \( \theta \) the argument to go through. This suggests how to look for points \( x \) that if \( a \in \Phi(G,S) \) and \( r \in \mathbb{R} \), let \( U_a(F)_{x,r} \) denote the filtration of the root subgroup \( U_a \) (see \[3\] §4.3 - §4.6, §5.1]). More precisely, \[
U_a(F)_{x,r} = \{ u \in U_a(F) \mid a(x - x_0) + \phi_a(u) \geq r \}.
\]

For a root \( b \in \Phi(G,S) \), let \( \Gamma_{b}, \Gamma'_{b} \) be the groups attached to the root \( b \) as in \[3\] Section 4.2.21 and Proposition 5.1.19. We know that all the affine roots with gradient \( b \) are of the form \( b + k \), with \( k \in \Gamma'_{b} \). Let \( e_\theta = \inf \{ k \mid k \in \Gamma'_{b} \} \).

Now, we want to prove that \( U_a(F)_{x,r} = U_a(F)_{w_2 \cdot x, r} \). Note that \( u \in U_a(F)_{x,r} \) if and only if \( \phi_a(u) \geq r - a(x - x_0) \) and \( u \in U_a(F)_{w_2 \cdot x, r} \) if and only if \( \phi_a(u) \geq r - a(w_2 \cdot x - x_0) \). Hence, to prove that \( U_a(F)_{x,r} = U_a(F)_{w_2 \cdot x, r} \), it suffices to prove that \( 0 \leq a(w_2 \cdot x - x) < e_a \). Note that \( a(w_2 \cdot x - x) = (w_2^{-1} \cdot a - a)(x - x_0) \).

Now, we claim that for each \( a \in \Phi^+_{\theta}, w_2^{-1}(a) - a \), which apriori lies in \( \Phi(G,S) \), in fact lies in \( \Phi^*(G,S) \). It suffices to prove this claim for \( a \in \theta \). In this case, \( w_2^{-1}(a) - a > 0 \) since \( w_2^{-1}(a) > 0 \).

Note that \( \Gamma'_{w_2^{-1} \cdot a} = \Gamma_a \) and this implies that \( e_a \in \Gamma'_{w_2^{-1} \cdot a} \). Since \( w_2^{-1} \cdot a - a \) is a positive root, this implies that \( 0 \leq (w_2^{-1} \cdot a - a)(x - x_0) \leq e_a \), so we only need to prove that \( (w_2^{-1} \cdot a - a)(x - x_0) = e_a \).

Again, since \( e_a \in \Gamma'_{w_2^{-1} \cdot a} \), we have \( e_a - (w_2^{-1} \cdot a - a) \) is an affine root. Since \( x \) lies in the interior of \( \mathcal{A} \), it cannot lie in the vanishing locus of any affine root, which implies that \((e_a - (w_2^{-1} \cdot a - a))(x - x_0) \neq 0 \), proving that \( a(w_2 \cdot x - x) < e_a \), which implies that \( U_a(F)_{x,r} = U_a(F)_{w_2 \cdot x, r} \). This finishes the proof of the proposition. \( \square \)

### 5.1.1. An example

It is shown in \[3\] Section 5], that for each \( x \in A(S,F) \) and each \( r > 0, G_{x,r} \in K^\bullet(S,G) \). In this subsection, we give an example of such a \( G_{x,r} \) that does not lie in \( K^\circ(S,G) \). In the proof of Proposition 5.3.5, we had used crucially that if \( x \) is not already a special point, then it lies in the interior of the alcove for the argument to go through. This suggests how to look for points \( x \in A(S,F) \) for which \( G_{x,r} \notin K^\circ(S,G) \). We will give a concrete example.

Let \( G = GL_3 \) with the diagonal matrices as \( T \) and upper-triangular matrices as \( B \). With this choice, let \( \Delta = \{ e_1 - e_2, e_2 - e_3 \} \). Let \( M = GL_1 \times GL_2 \). Then \( \theta = \{ e_2 - e_3 \} \). Let \( a = e_2 - e_3 \), let \( w = s_{e_1 - e_2} \) and let \( x = e_3^2/2 \). Then

\[
G_{x,1} = \begin{bmatrix}
1 + p_F & p_F & p_F \\
p_F & 1 + p_F & p_F \\
p_F & p_F & 1 + p_F
\end{bmatrix}.
\]

With

\[
n = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

we know that \( n \) is a representative of \( w \) in \( GL_3 \). Now,

\[
nG_{x,1}n^{-1} = \begin{bmatrix}
1 + p_F & p_F^2 & p_F \\
p_F & 1 + p_F & p_F \\
p_F & p_F^2 & 1 + p_F
\end{bmatrix}.
\]

Then

\[
G_{x,1} \cap M = \begin{bmatrix}
1 + p_F & 0 & 0 \\
0 & 1 + p_F & p_F \\
0 & p_F & 1 + p_F
\end{bmatrix}.
\]
We claim that $nG_{x,1}n^{-1} \cap M = \left[ \begin{array}{ccc} 1 + p_F & 0 & 0 \\ 0 & 1 + p_F & p_F \\ 0 & p_F^2 & 1 + p_F \end{array} \right]$. We claim that $nG_{x,1}n^{-1} \cap M$ and $G_{x,1} \cap M$ are not $M$-conjugate. To see this, it suffices to prove that the groups

$$K_1 = \left[ \begin{array}{cc} 1 + p_F & p_F \\ p_F & 1 + p_F \end{array} \right] \text{ and } I_1 = \left[ \begin{array}{cc} 1 + p_F & p_F \\ p_F^2 & 1 + p_F \end{array} \right]$$

are not $GL_2$-conjugate. This is intuitively clear since $K_1 \subset GL_2(\mathcal{O}_F)$ which corresponds to the parahoric subgroup of a hyperspecial vertex in the building and $I_1 \subset I = \left[ \begin{array}{cc} \mathcal{O}_F^* & \mathcal{O}_F^* \\ \mathcal{O}_F^* & \mathcal{O}_F^* \end{array} \right]$ which is an Iwahori subgroup of $GL_2(F)$ and corresponds to the parahoric subgroup of the interior of an alcove. We justify this as follows. Normalize the Haar measure on $GL_2(F)$ so that $\text{vol}(I) = 1$. Note that $K_1$ and $I_1$ are both normal subgroups of $I$ and $I_1 \not\subset K_1 \not\subset I$. Now, if $K_1$ and $I_1$ are $GL_2$-conjugate, then $\text{vol}(K_1) = \text{vol}(I_1)$, which then implies that $[I : K_1] = \text{vol}(K_1)^{-1} = \text{vol}(I_1)^{-1} = [I : I_1]$, which is not possible. This proves that with $x = c_1^\vee/2$, $G_{x,1} \not\subset K^\vee(S,G)$.

6. A SPECIAL CASE

Let $G$ be a connected, reductive group over $F$, $S$ a maximal split torus in $G$, $A(S,F)$ an apartment of $S$ over $F$ and let $I$ be the Iwahori subgroup corresponding to an alcove in $A(S,F)$. Let $M_0$ be a minimal Levi subgroup of $G$ that contains $S$ and $T$ a maximal torus in $M_0$ that contains $S$. We know that the center of the Iwahori Hecke algebra $H(G,I)$ is given by $\mathbb{C}[\Omega_{M_0}]^{W_0}$ where $W_0$ is the Weyl group of $G$. When $G$ is split, we have $M_0 = S = T$ and the center is given by $\mathbb{C}[X_*(T)]^{W_0}$ or in other words is $H(T,T_0)^{W_0}$. However, for Hecke algebras at deeper level, one cannot hope for such a simple description of the center. In fact, we will now show that $H(T,T_0)^{W_0}$ captures only a part of the center of only the principal series blocks. We illustrate this when $G$ is split.

Assume now that $G$ is a split connected, reductive group over $\mathbb{Z}$. In this case, $M = T$ is a maximal split torus in $G$. Let $\Phi = \Phi(G,T)$ be the set of roots of $T$ in $G$, $\Phi^\vee$ the set of coroots and let $W_0 = W(G,T)$ denote the absolute Weyl group. Let $W$ denote the Iwahori Weyl group of $G$, $I$ be an Iwahori subgroup of $G$ and let $I_n$ be the $n$-th Moy–Prasad filtration subgroup of $I$. Let $T_n = T \cap I_n$. The goal of this section is to prove the following proposition. All the crucial ingredients needed for this proposition are contained in the work of Roche [15].

**Proposition 6.1.** Let $p = \text{char}(k_F)$ be large enough satisfying the hypothesis for Theorem 4.15 of [15]. There exists an algebra embedding $t_{T,n} : H(T,T_n) \rightarrow H(G,I_0)$ with the following properties:

1. $t_{T,n}(H(T,T_0)) \subset H(G,I_0)$.
2. Identify the subalgebra $H(T,T_0) \subset H(T,T_n)$ with $\mathbb{C}[X_*(T)]$. For $\lambda \in X_*(T)$, write $\lambda = \lambda_1 - \lambda_2$ for $\lambda_i \in X_*(T)_+$ Then $t_{T,n}(\lambda) = c_{\lambda} t_{T,n}(\lambda_1)t_{T,n}(\lambda_2)^{-1}$, where $c_{\lambda} = |T_n x T_n|^{-1/2} |I_n x I_n|^{-1/2}$. 
(3) The sub-algebra \( t_{T,n}(\mathcal{H}(T,T_n)^{W_0}) \subset \mathfrak{J}(\mathcal{H}(G,I_n)) \). In fact

\[
t_{T,n}(\mathcal{H}(T,T_n))^{W_0} \subset \bigcap_{\chi \in (T_0/T_n)^{\vee}} t_{T,\chi}(\mathcal{H}(T,\chi))^{W_\chi} \cong \bigcap_{\chi \in (T_0/T_n)^{\vee}} \mathfrak{J}(\mathcal{R}_\chi(G))
\]

where \( \mathfrak{R}_\chi(G) \) is the principal series block determined by the inertial equivalence class of the character \( \chi \) of \( T_0/T_n \), and \( W_\chi = \{ w \in W_0 \mid w \cdot \chi = \chi \} \).

**Proof.** The idea is to decompose \( \mathcal{H}(T,T_n) \) and \( \mathcal{H}(G,I_n) \) in accordance with the Bernstein decomposition and construct the required embedding block wise. First, note that there is a \( C \)-algebra isomorphism

\[
\mathcal{H}(T,T_n) \cong \prod_{\chi \in (T_0/T_n)^{\vee}} \mathcal{H}(T,\chi).
\]

In fact, we have an exact sequence \( 1 \to T_0/T_n \to T/T_n \to X_*(T) \to 1 \). Fixing a uniformizer \( \varpi_F \) of \( F \) yields the splitting \( \mu \mapsto \mu(\varpi_F) \mod T_n \), hence

\[
T/T_n \cong X_*(T) \times T_0/T_n.
\]

Accordingly, \( \mathcal{H}(T,T_n) \cong \mathbb{C}[X_*(T)] \otimes_{\mathbb{C}} \mathbb{C}[T_0/T_n] \). Now decomposing the \( C \)-algebra \( \mathbb{C}[T_0/T_n] \) using Artin–Wedderburn theorem proves (6.1).

Fix a smooth character \( \chi : T_0 \to \mathbb{C}^* \) that is trivial on \( T_n \). Let \( s_\chi \) be the principal series element of \( \mathfrak{B}(G) \) determined by \( \chi \). Roche has constructed a \( s_\chi \)-type \( (J_\chi, \rho_\chi) \) where \( J_\chi \) is a compact open subgroup of \( G \) which contains \( T_0 \) and \( \rho_\chi \) is a smooth character of \( J \) whose restriction to \( T_0 \) is \( \chi \).

It is clear from the construction above that \( I_n \subset J_\chi \) and \( \rho_\chi|_{I_n} = 1 \). In particular, \( \mathcal{H}(G,\rho_\chi) \subset \mathcal{H}(G,I_n) \).

The category \( \mathfrak{R}_\chi(G) \) corresponding to the inertial equivalence class \( s_\chi \) is equivalent to the category \( \mathcal{H}(G,\rho_\chi) \mod \). The character \( \chi \) also canonically determines a component \( \mathfrak{R}_\chi(T) \) of the Bernstein decomposition of \( T \). This is the full subcategory of \( \mathfrak{R}(T) \) consisting of all smooth representations of \( T \) whose restriction to \( T_0 \) is a multiple of \( \chi \). Further, the category \( \mathfrak{R}_\chi(T) \) is equivalent to \( \mathcal{H}(T,\chi) \mod \).

There is an algebra embedding \( t_{T,\chi} : \mathcal{H}(T,\chi) \to \mathcal{H}(G,\rho_\chi) \) such that the natural parabolic induction functors \( \mathfrak{R}_\chi(T) \to \mathfrak{R}_\chi(G) \) correspond via equivalence of categories mentioned above to algebraic induction functors between \( \mathcal{H}(T,\chi) \mod \) to \( \mathcal{H}(G,\rho_\chi) \mod \) induced by appropriate twists of the algebra embedding \( t_{T,\chi} \).

Using (6.1), we get the required embedding \( t_{T,n} : \mathcal{H}(T,T_n) \to \mathcal{H}(G,I_n) \).

Note that \( \mathcal{H}(T,T_0) = \mathcal{H}(T,\chi_0) \) where \( \chi_0 \) is the trivial character of \( T_0 \). Then \( J_{\chi_0} = I, \rho_{\chi_0} \) is the trivial character of \( J_{\chi_0} \) and \( \mathcal{H}(G,\rho_{\chi_0}) = \mathcal{H}(G,I) \) is the Iwahori Hecke algebra. Further, \( t_{T,n}|_{\mathcal{H}(T,T_0)} = t_{T,\chi_0} \) has the properties stated in (1) and (2) of the theorem by construction.

Let us prove (3). Recall that \( \mathfrak{R}_\chi(G) \) is equivalent to \( \mathcal{H}(G,\rho_\chi) \mod \) and \( \mathfrak{R}_\chi(T) \) is equivalent to \( \mathcal{H}(T,\chi) \mod \). Let \( \tilde{\chi} \) be any extension of \( \chi \) to \( T \) and let \( t = [T,\tilde{\chi}]_T \in \mathfrak{B}(T) \) be the inertial equivalence class of \( \mathfrak{R}_\chi(T) \). Then

\[ N^1 = \{ n \in N_G(T) \mid n \cdot \tilde{\chi} \cong \nu \cdot \tilde{\chi} \text{ for some } \nu \in X_*(T) \}. \]

Note that \( n \cdot \tilde{\chi} \cong \nu \cdot \tilde{\chi} \) for some \( \nu \in X_*(T) \) if and only if \( n \cdot \chi = \chi \). Hence

\[ N^1 = \{ n \in N_G(T) \mid n \cdot \chi = \chi \}. \]

So \( W^1 = N^1/T = W_\chi = \{ w \in W_0 \mid w \cdot \chi = \chi \} \). This implies that

\[ \mathfrak{J}(\mathcal{R}_\chi(G)) = (t_{T,\chi}(\mathcal{H}(T,\chi))^{W_\chi}). \]
It remains to see that
\[ t_{T,n}(\mathcal{H}(T,T_n)^{W_0}) \subset \prod_{\chi \in (T_0/T_n)^{\vee}} t_{T,\chi}(\mathcal{H}(T,\chi)^{W_0}). \]

The elements \( \{ f_{\lambda,\chi} \mid \lambda \in X_*(T), \chi \in (T_0/T_n)^{\vee} \} \) forms a basis for \( \mathcal{H}(T,T_n) \). The elements \( z_{\mathcal{O}} = \sum_{(\lambda,\chi) \in \mathcal{O}} f_{\lambda,\chi} \) for all \( W_0 \)-orbits \( \mathcal{O} \) in \( X_*(T) \times (T_0/T_n)^{\vee} \) then forms a basis for \( \mathcal{H}(T,T_n)^{W_0} \). So it suffices to prove that for each such \( W_0 \)-orbit \( \mathcal{O} \), \( z_{\mathcal{O}} \in \prod_{\chi \in (T_0/T_n)^{\vee}} \mathcal{H}(T,\chi)^{W_0} \).

For \( \chi \in (T_0/T_n)^{\vee} \), let \( \mathcal{O}_{\chi} = \{ (w(\lambda),\chi) \mid w \in W_\chi \} \) and let \( z_{\mathcal{O}_{\chi}} = \sum_{(\lambda',\chi') \in \mathcal{O}_{\chi}} f_{\lambda',\chi'} \). Note that \( z_{\mathcal{O}_{\chi}} \in \mathcal{H}(T,\chi)^{W_\chi} \). For each \( w \in W_0 \), it is easy to see that \( w \cdot \mathcal{O}_{\chi} = \mathcal{O}_{w\cdot \chi} \).

Write \( \mathcal{O} = \{ (w(\lambda_1),w(\chi_1)) \mid w \in W_0 \} \) for a fixed \( \lambda_1 \) and \( \chi_1 \). It is clear that
\[ \mathcal{O} = \bigcup_{\chi \in W_0-\text{orbit of } \chi_1} \mathcal{O}_{\chi}. \]

Then
\[ z_{\mathcal{O}} = \sum_{\chi \in W_0-\text{orbit of } \chi_1} z_{\mathcal{O}_{\chi}}. \]

Since \( z_{\mathcal{O}_{\chi}} \in \mathcal{H}(T,\chi')^{W_{\chi'}} \), it follows that \( z_{\mathcal{O}} \in \prod_{\chi \in (T_0/T_n)^{\vee}} \mathcal{H}(T,\chi)^{W_{\chi}} \). This finishes the proof of the proposition.

\[ \square \]

7. The Bernstein center of \( \mathcal{H}(G,K) \)

Let \( F \) be a non-archimedean local field and let \( G \) be a connected, reductive group over \( F \). We assume the following.

**Assumption 7.1.** \( G \) splits over a tamely ramified extension of \( F \), and the residue characteristic \( p \) of \( F \) does not divide the order of the Weyl group of \( G \).

Then

1. By [10, Theorem 8.1] Every irreducible, supercuspidal representation of \( G \) arises from Yu’s construction, that was recalled in Section 3.5.
2. Let \((\pi,V)\) be an irreducible, smooth representation of \( G \) and let \( \mathfrak{s} = [M,\sigma]_G \) be the inertial class of \( \pi \). Let \((J_M,\rho_M)\) be the supercuspidal type of the Bernstein block corresponding to \( \mathfrak{s}_M = [M,\sigma]_M \). Then there exists a \( G \)-cover \((K,\rho)\) of \((K_M,\rho_M)\), which in particular says that \((J,\rho)\) is a \( \mathfrak{s} \)-type. This construction is carried out in [12, Theorem 9.1] under some additional hypothesis, but in [10, Theorem 7.12], the author has proved that this holds merely under Assumption 7.1.

For the remainder of this section we assume that Assumption 7.1 holds.

**Corollary 7.2.** Let \( S \) be a maximal \( F \)-split torus in \( G \) and let \( K \in K^\bullet(S,G) \). We have
\[ \mathcal{Z}(\mathcal{H}(G,K)) = \prod_{s \in \mathcal{E}(K)} \mathcal{Z}(\mathcal{H}(G,\rho)) \cong \prod_{s \in \mathcal{E}(K)} \mathbb{C}[\Gamma J_M/J_M]^{W(\rho_M)} \]
where, for \( \mathfrak{s} = [M,\sigma]_G \), \((J_M,\rho_M)\) is the supercuspidal type of \( \mathfrak{s}_M \), and \((J,\rho)\) is a \( G \)-cover of \((J_M,\rho_M)\).

**Proof.** This is a consequence of Theorem 4.6 Note that we can apply Theorem 4.6 since the assumption \( \mathcal{I}_M(\rho_M) \subset \mathcal{J}_M \) is satisfied by Lemma 3.11. \( \square \)
Now, we describe the center of $\mathcal{H}(G,K)$ for $K \in \mathcal{K}^\circ(S,G)$. Let $M$ be a Levi subgroup of $G$ that contains $S$ and let

$$l_M^G : \mathfrak{B}(M) \to \mathfrak{B}(G), \quad [M,\sigma]_M \to [M,\sigma]_G.$$ 

Let $\mathfrak{S}(K \cap M)_{sc} = \{[M,\sigma]_M \in \mathfrak{S}(K \cap M) | \sigma \text{ a supercuspidal representation of } M\}.$

**Corollary 7.3.** Let $K \in \mathcal{K}^\circ(S,G)$. We have

$$\mathfrak{Z}(\mathcal{H}(G,K)) \cong \prod_{[M] \in \mathfrak{S}(K \cap M)_{sc}} \mathbb{C}[\hat{J}_M/J_M]^{W(\rho_M)}$$

where $[M]$ runs through the $G$-conjugacy classes of $F$-Levi subgroups of $G$ and $M$ is a representative in this conjugacy class that contains $S$.

**Proof.** Note that by Lemma 5.3, we have

$$\mathfrak{S}(K) = \bigsqcup_M l_M^G(\mathfrak{S}(K \cap M)_{sc}).$$

Hence the corollary follows from Corollary 7.2. □

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