KÄHLER MANIFOLDS AND MIXED CURVATURE

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Abstract. In this work we consider compact Kähler manifolds with non-positive mixed curvature which is a “convex combination” of Ricci curvature and holomorphic sectional curvature. We show that in this case, the canonical line bundle is nef. Moreover, if the curvature is negative at some point, then the manifold is projective with canonical line bundle being big and nef. If in addition the curvature is negative, then the canonical line bundle is ample. As an application, we answer a question of Ni concerning manifolds with negative -Ricci curvature and generalize a result of Wu-Yau and Diverio-Trapani to the conformally Kähler case. We also show that the compact Kähler manifold is projective and simply connected if the mixed curvature is positive.

1. Introduction

In this work, we will study the structure of compact Kähler manifolds \((M^n, h)\) with certain mixed curvature conditions. More precisely, for any constants \(\alpha, \beta\), we consider the following mixed curvature defined on \(T^{1,0}(M)\):

\[(1.1) \quad C_{\alpha,\beta}(X) = a\text{Ric}(X, X) + \beta |X|^2_R(X, X, X, X)\]

for \(X \in T^{1,0}(M)\) where \(|X|^2_R \neq 0\). Here \(R\) and \(\text{Ric}\) denote the curvature tensor and Ricci tensor of \(h\) respectively. In most cases, the dependency of \(C_{\alpha,\beta}\) on \(h\) is clear. If we want to emphasis the dependency of \(C_{\alpha,\beta}\) on \(h\), we will denote \(C_{\alpha,\beta}\) by \(C_{\alpha,\beta}(h)\), or \(C_{\alpha,\beta,h}\). The study is motivated by the following:

(i) \(C_{1,0,h}(X)\) is the standard Ricci curvature.
(ii) \(C_{0,1,h}(X)\) is the holomorphic sectional curvature up to scaling.
(iii) \(C_{1,1,h}(X)\) is the notion \(\text{Ric}^+(X, X)\) introduced by Ni [23]. See Corollary 7.1 for the definition.
(iv) \(C_{1,-1,h}(X)\) is the orthogonal Ricci curvature \(\text{Ric}^+(X, X)\) introduced by Ni-Zheng [25], and is an analogue of the orthogonal bisectional curvature.
(v) $C_{k-1,n-k,h}(X)$ is closely related to the $k$-Ricci curvature introduced by Ni [24], Lemma 2.1.

Motivated by the previous studies on different notions of curvature as above, we would like to understand the structure of $(M, h)$ in which $C_{\alpha,\beta,h} \leq 0$, or more generally, for all non-zero $X \in T^1_0M$,

$$C_{\alpha,\beta}(X) + \sqrt{-1}\partial\bar{\partial}\phi(X, X) \leq \lambda |X|^2_h$$

for some smooth function $\phi$ and continuous function $\lambda$ which is nonpositive, or quasi-negative, or negative. We can also consider its counterpart: there is a smooth function $\phi$ so that

$$C_{\alpha,\beta}(X) + \sqrt{-1}\partial\bar{\partial}\phi(X, X) \geq \lambda |X|^2_h$$

for some continuous function $\lambda$ which is positive.

In addition to the fact that $C_{\alpha,\beta}$ is a generalization of some known notions of curvature, it can also be defined in general for Hermitian metric if we replace the Ricci curvature by the first Ricci curvature of the Chern connection. Moreover, the above conditions are invariant under conformal changes (in the Hermitian category) in the sense that if $h$ satisfies (1.2) for some $\phi$ and $\lambda$ with $\lambda \leq 0$, then a conformal metric $e^{2f}h$ also satisfies the same condition with the same $\alpha, \beta$ but with another $\tilde{\phi}$ and $\tilde{\lambda}$ which is still nonpositive, etc. We refer readers to Lemma 6.1 for details.

Under the nonpositive condition (1.2), we have the following:

**Theorem 1.1.** Let $(M^n, h)$ be a compact Kähler manifold satisfying (1.2) for some constants $\alpha \geq 0, \beta > 0, \phi \in C^\infty(M)$ and a continuous function $\lambda$.

(a) Suppose $\lambda \leq 0$, then the canonical bundle $K_M$ of $M$ is numerically effective (nef).

(b) Suppose $\lambda < 0$, then $M$ is projective with ample $K_M$ and $M$ supports a Kähler-Einstein metric with negative scalar curvature.

(c) Suppose $\lambda$ is quasi-negative, then $M$ is projective with $\int_M c_1(K_M)^n > 0$. In particular, together with (a), $K_M$ is big. If in addition $M$ does not contain any rational curve, then $K_M$ is ample.

Recall that $K_M$ is numerically effective if for any $\varepsilon > 0$, there is a smooth function $f$ so that $-\text{Ric}(h) + \sqrt{-1}\partial\bar{\partial}f + \varepsilon \omega_h > 0$ where $\omega_h$ is the Kähler form of $h$. $K_M$ is big if its Kodaira-Iitaka dimension of $K_M$ is equal to the dimension of $M$.

In case of $C_{1,0}$, the existence of Kähler-Einstein metric in Theorem 1.1 (b) is the celebrated result of Aubin [1] and Yau [44]. In the case $C_{0,1}$, (1.2) with $\phi = 0$ is equivalent to the condition that the holomorphic sectional curvature $H$ being bounded from above by $\lambda$. In this case, Theorem 1.1 was proved by Wu-Yau [37, 38], Tosatti-Yang [35] and Diverio-Trapani [5]. Note that $H(X) \leq 0$ implies that $M$ does not contain any rational curve by the work of Royden [28]. There have been important contributions on compact Kähler manifolds with $H \leq 0$ by many other people, we refer readers to [9, 10, 11, 17, 27, 42].
and the reference therein for further details. Note that in our case, even for $C_{0,1}$, we may allow $\phi$ to be nonzero in the condition (1.2).

As an application, Theorem 1.1 gives an affirmative answer to a question of Ni [24] asking whether a compact Kähler manifold with $\text{Ric}_k < 0$ is projective. On the other hand, in the Hermitian category, it was conjectured by Yang-Zheng [42, Conjecture 1.1] that a compact Hermitian manifold with quasi-negative holomorphic sectional curvature is projective with ample $K_M$.

Since condition (1.2) is invariant under conformal deformation, one can use Theorem 1.1 to verify the conjecture is true for the special case that the Hermitian metric is conformally Kähler.

We will use the twisted Kähler-Ricci flow\(^1\) to study the nefness and ampleness of $K_M$. Namely, we will study the following

\begin{equation}
\begin{cases}
\partial_t \omega(t) = -\text{Ric}(\omega(t)) - \eta; \\
\omega(0) = \omega_h,
\end{cases}
\end{equation}

where $\omega(t)$ is a smooth family of Kähler forms, $\omega_h$ is the Kähler form of $h$, and $\eta$ is a smooth closed real $(1,1)$ form on $M$. When $\eta = 0$, this coincides with the usual Kähler-Ricci flow. We will prove Theorem 1.1(a) by showing the long-time existence. We will prove Theorem 1.1(b) by studying its long-time behaviour. The method may have independent interest. Finally, we will modify the continuity method of Wu-Yau [38] to obtain Theorem 1.1(c).

Next, let us discuss condition (1.3). There are many results on the structure of compact Kähler manifolds related to positive curvature recently. In [39], Yang proved a conjecture of Yau [45] that a compact Kähler manifold with $\text{Ric}_1 > 0$, i.e. positive holomorphic sectional curvature, must be projective and rationally connected, see also [8]. The result is generalized further by Ni [23] and Ni-Zheng [25] recently. In particular, in [23, 25], it was shown that compact Kähler manifolds with $\text{Ric}^+ > 0$, $\text{Ric}^\perp > 0$ or $\text{Ric}_k > 0$ are simply connected and projective. For more recent development, we refer interested readers to [8, 20, 21, 22, 25, 26, 40, 41] and references therein. In this regard, following the argument in [23, Theorem 2.7], we show that Kähler manifolds satisfying (1.3) with $\phi = 0$ are also simply connected and projective.

**Theorem 1.2.** Suppose $(M,h)$ is a compact Kähler manifold with

\begin{equation}
C_{\alpha,\beta}(X) > 0
\end{equation}

for some $\alpha, \beta$ with $\alpha > 0, \alpha + \beta \geq 0$ and all non-zero $X \in T^{1,0}M$, then $h^{p,0} = 0$ for all $1 \leq p \leq n$. In particular, $M$ is simply connected and projective.

The paper is organized as follows: In Section 2 using Royden’s idea [28] we show that $\text{Ric}_k \leq \lambda$ (respectively $\geq \lambda$) implies (1.2) (respectively (1.3)) for some $\alpha, \beta$, which in turns will give a useful upper bound for the trace of the bisectional curvature. In Section 3 we will prove the nefness and ampleness of the canonical line bundle using the twisted Kähler-Ricci flow. In Section 3

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\(^1\)See also [48] for an elliptic proof which is based on the earlier version of this preprint.
we will consider the quasi-negative case using compactness argument in [38] together with the twisted Monge-Ampère equation. In Section 5, we apply Theorem 1.1 to prove the structural theory of manifolds with non-positive Ric\(k\) and Ric\(+\). In Section 6, we will show that (1.2) is conformally invariant and a special case of [42, Conjecture 1.1]. In Section 7, we will prove Theorem 1.2.

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2. ON SYMMETRIC BIHERMITIAN FORMS

In this section, we will first prove that the condition Ric\(k\leq\lambda\) will imply (1.2) for some suitable \(\alpha, \beta\). This is essentially a result in linear algebra on symmetric bihermitian form on a Hermitian vector space. We will generalize the result of Royden [28] where an upper bound on the trace of bisectional curvature in terms of an upper bound of holomorphic sectional curvature was obtained.

Let \((V^n, h)\) be a Hermitian vector space with Hermitian metric \(h\). Let \(S(X, \overline{Y}, Z, \overline{W})\) be a bihermitian form on \(V^n\), namely \(S(X, \overline{Y}, \cdot, \cdot)\) and \(S(\cdot, \cdot, W, \overline{Z})\) are Hermitian forms for fixed \(X, Y\) or \(W, Z\). Moreover, we assume that the bihermitian form \(S\) satisfies the following symmetry:

\[
S(X, \overline{Y}, Z, \overline{W}) = S(Z, \overline{Y}, X, \overline{W}), \quad S(X, \overline{Y}, Z, \overline{W}) = S(Y, X, W, Z).
\]

For \(S\), we define its holomorphic sectional curvature, Ricci tensor and scalar curvature by

\[
H^S(X) = \frac{S(X, \overline{X}, X, \overline{X})}{|X|^4_h}, \quad \text{Ric}^S(X, \overline{Y}) = \text{tr}_h(S(X, \overline{Y}, \cdot, \cdot)), \quad S = \text{tr}_h \text{Ric}^S.
\]

For any \(k\)-dimensional subspace \(U\) of \(V\), define the \(\text{Ric}^S_{k,U}\) to be the Ricci tensor for \(S|_U\). Note that \(\text{Ric}^S_{k,U}\) depends also on the subspace \(U\). We say that the \(k\)-Ricci curvature of \(S\) is less than or equal to \(\lambda\), denoted by \(\text{Ric}^S_{k} \leq \lambda\), for some \(\lambda \in \mathbb{R}\) if for any \(X \in V\) and any \(k\)-dimensional subspace \(U\) containing \(X\), we have \(\text{Ric}^S_{k,U}(X, \overline{X}) \leq \lambda |X|^2_h\). The notion \(\text{Ric}^S_{k} \geq \lambda\) is defined analogously. It is easy to see that \(\text{Ric}^S_{k}\) on a unit vector is the holomorphic sectional curvature if \(k = 1\) and is the Ricci tensor for \(V\) if \(k = n\). We first give some relation between \(\text{Ric}^k\) and \(C_{\alpha, \beta}\).

Lemma 2.1. Suppose \((V^n, h)\) is a Hermitian vector space and \(S\) is a bihermitian form on \(V\) satisfying (2.1). For \(1 < k < n\), let \(\{e_1, \ldots, e_n\}\) be a unitary
basis and for any \( I \subset \{2, \ldots, n\} \) with \(|I| = k - 1\), let \( U_I \) be the subspace spanned by \( \{e_1\} \cup \{e_i | i \in I\} \). Then

\[
(k-1) \sum_{I: I \subset \{2, \ldots, n\}, |I| = k-1} \text{Ric}^S_{k,U_I}(e_1, \overline{e}_1) = C_{k-2}^{n-2} \left((n-k)H^S(e_1) + (k-1)\text{Ric}^S(e_1, \overline{e}_1)\right)
\]

**Proof.** For notational convenience, we use \( S_{ijkl} \) to denote \( S(e_i, e_j, e_k, e_l) \). Let \( I \) be a subset of \( \{2, \ldots, n\} \) such that \(|I| = k - 1\). Then there are \( C_{k-1}^{n-1} \) different such \( I \)'s. For each \( i \neq 1 \), there are \( C_{k-2}^{n-2} \) different such \( I \)'s which also contain \( i \). Hence

\[
\sum_{I: I \subset \{2, \ldots, n\}, |I| = k-1} \text{Ric}^S_{k,U_I}(e_1, \overline{e}_1) = \sum_{I: I \subset \{2, \ldots, n\}, |I| = k-1} \left(S_{1111} + \sum_{j \in I} S_{11jj}\right)
\]

\[
=C_{k-1}^{n-1} S_{1111} + C_{k-2}^{n-2} \sum_{i=2}^n S_{i1i1}
\]

\[
=(C_{k-1}^{n-1} - C_{k-2}^{n-2}) S_{1111} + C_{k-2}^{n-2} \sum_{i=1}^n S_{i1i1}
\]

\[
=C_{k-2}^{n-2} \left(\frac{n-k}{k-1} H^S(e_1) + \text{Ric}^S(e_1, \overline{e}_1)\right).
\]

From this, the result follows. \(\square\)

**Lemma 2.2.** Suppose \((V^n, h)\) is a Hermitian vector space and \( S \) is a bihermitian form on \( V \) satisfying (2.1). Suppose \( \text{Ric}^S \leq (k+1)\sigma \) for some integer \( 1 \leq k \leq n \) and \( \sigma \in \mathbb{R} \). Then for any \( X \in V \), \( X \neq 0 \), we have

\[
(k-1)\text{Ric}^S(X, \overline{X}) + (n-k)|X|^2_h H^S(X) \leq (n-1)(k+1)\sigma |X|^2_h.
\]

Similarly, if \( \text{Ric}^S \geq (k+1)\sigma \), then

\[
(k-1)\text{Ric}^S(X, \overline{X}) + (n-k)|X|^2_h H^S(X) \geq (n-1)(k+1)\sigma |X|^2_h.
\]

**Proof.** We may assume \(|X|^2_h = 1\). If \( k = 1 \) or \( n \), the result is obviously true. In case \( 1 < k < n \), we extend \( e_1 \) to be a unitary frame \( \{e_1, \ldots, e_n\} \). Suppose \( \text{Ric}^S \leq (k+1)\sigma \). With the notation as in Lemma 2.1, we have

\[
(k-1)(k+1)C_{k-1}^{n-1} \sigma \sum_{I: I \subset \{2, \ldots, n\}, |I| = k-1} \text{Ric}^S_{k,U_I}(e_1, \overline{e}_1)
\]

\[
=C_{k-2}^{n-2} \left((n-k)H^S(e_1) + (k-1)\text{Ric}^S(e_1, \overline{e}_1)\right).
\]

From this, it is easy to see that the first part of the lemma is true. The case that \( \text{Ric}^S \geq (k+1)\sigma \) can be proved similarly. \(\square\)

Now we adapt the trick of Royden [28] to deduce an upper bound on the trace of bisectional curvature with respect to another Hermitian metric \( g \).
Motivated by Lemma 2.2, we consider bihermitian form $S$ such that there is a real $(1,1)$ form $\rho$ satisfying the following:

$$h(X, \bar{X}) \cdot \rho(X, \bar{X}) + \beta S(X, \bar{X}, X, \bar{X}) \leq \lambda |X|^4$$

for all $X \in V^{1,0}$ for some constants $\beta, \lambda$ where $\beta$ are positive.

**Lemma 2.3.** Suppose $(V^n, h)$ is a Hermitian vector space and $S$ is a bihermitian form on $V$ satisfying (2.1) and (2.2). If $g$ is another Hermitian metric on $V$, then we have:

$$2g^{ij}g^{kl}S_{ijkl} \leq \frac{1}{\beta} (\lambda (tr_g h)^2 - tr_g h \cdot tr_g \rho) + \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i})$$

$$\leq \frac{\lambda}{\beta} ((tr_g h)^2 + |h|^g_2) - \frac{1}{\beta} tr_g h \cdot tr_g \rho - \frac{1}{\beta} \langle \omega_h, \rho \rangle_g$$

where $\{E_i\}_{i=1}^{n}$ is a unitary frame with respect to $g$ so that $h$ is diagonal. Here $S_{ijkl} = S(E_i, E_j, E_k, E_l)$ and $\omega_h$ is the Kähler form for $h$.

**Proof.** We follow closely the argument of Royden [28]. Let $\{E_i\}_{i=1}^{n}$ be a frame such that $g(E_i, \overline{E_j}) = \delta_{ij}$ and $h(E_i, \overline{E_j}) = \tau_i \delta_{ij}$. Let $\eta_A = \sum_{i=1}^{n} \epsilon_i^A E_i$ where $A = (\epsilon_i^A) \in \mathbb{Z}_4^n$ with $\mathbb{Z}_4$ being the finite group consisting of 4-th roots of unity. Note that for any $A \in \mathbb{Z}_4^n$, we have

$$h(\eta_A, \overline{\eta_A}) = \sum_{i,j=1}^{n} \epsilon_i^A \overline{\epsilon_j} h(E_i, \overline{E_j}) = \sum_{i=1}^{n} \tau_i = tr_g h.$$

Also since $\sum_{A \in \mathbb{Z}_4^n} \epsilon_i^A \overline{\epsilon_j} = 0$ for all $i \neq j$, by symmetry we obtain

$$\sum_{A \in \mathbb{Z}_4^n} \rho(\eta_A, \overline{\eta_A}) = \sum_{A \in \mathbb{Z}_4^n} \sum_{i,j=1}^{n} \epsilon_i^A \overline{\epsilon_j} \rho(E_i, \overline{E_j}) = 4^n \sum_{i=1}^{n} \rho(E_i, \overline{E_i}) = 4^n tr_g \rho.$$

Similarly,

$$\sum_{A \in \mathbb{Z}_4^n} S(\eta_A, \overline{\eta_A}, \eta_A, \overline{\eta_A}) = \sum_{A \in \mathbb{Z}_4^n} \sum_{i,j,k,l} \epsilon_i^A \overline{\epsilon_j} \epsilon_k \overline{\epsilon_l} S(E_i, \overline{E_j}, E_k, \overline{E_l})$$

$$= 4^n \sum_{i \neq j} (S(E_i, \overline{E_j}, E_j, \overline{E_i}) + S(E_i, \overline{E_i}, E_j, \overline{E_j})) + 4^n \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i})$$

$$= 4^n g^{ij} g^{kl} (S_{ijkl} + S_{ijlk}) - 4^n \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i})$$

$$= 4^n \left( 2g^{ij} g^{kl} S_{ijkl} - \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i}) \right)$$

Thus we have:

$$2g^{ij} g^{kl} S_{ijkl} \leq \frac{1}{\beta} (\lambda (tr_g h)^2 - tr_g h \cdot tr_g \rho) + \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i})$$

$$\leq \frac{\lambda}{\beta} ((tr_g h)^2 + |h|^g_2) - \frac{1}{\beta} tr_g h \cdot tr_g \rho - \frac{1}{\beta} \langle \omega_h, \rho \rangle_g.$$
and
\[
\sum_{A \in \mathbb{Z}_4^4} |\eta_A|^4_h = \sum_{i,j,\gamma,\delta=1}^{n} \varepsilon_i^A \varepsilon_j^A \varepsilon_{\gamma}^A \varepsilon_{\delta}^A h(E_i, \overline{E_j})h(E_{\gamma}, \overline{E_{\delta}})
\]
\[
= 4^n \sum_{i \neq j} (h(E_i, \overline{E_j})h(E_j, \overline{E_i}) + h(E_i, \overline{E_i})h(E_j, \overline{E_j})) + 4^n \sum_{i=1}^{n} h(E_i, \overline{E_i})h(E_i, \overline{E_i})
\]
\[
= 4^n \left( \sum_{i \neq j} \tau_i \tau_j + \sum_{i=1}^{n} \tau_i^2 \right) = 4^n (\text{tr}_g h)^2.
\]
Applying (2.2) for each \(\eta_A\) and summing over \(A \in \mathbb{Z}_4^n\), we conclude that
\[
\lambda(\text{tr}_g h)^2 \geq \text{tr}_g h \cdot \text{tr}_g \rho + \beta \left( 2 g_{ij} g^{kl} S_{ijkl} - \sum_{i=1}^{n} S(E_i, \overline{E_i}, E_i, \overline{E_i}) \right).
\]
This implies the first inequality in the Lemma. The second follows from (2.2).

3. Canonical line bundle under non-positive curvature

In this section, we are going to prove Theorem 1.1 (a), (b). Consider a compact Kähler manifold \((M, h)\) with curvature \(R_h\) satisfying (1.2). Hence it satisfies (2.2) for \(\rho = \alpha \text{Ric}_h + \sqrt{-1} \partial \bar{\partial} \phi\). That is:
\[
(3.1) \quad h(X, \overline{X}) \cdot (\alpha \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi)(X, \overline{X}) + \beta R^h(X, \overline{X}, X, \overline{X}) \leq \lambda |X|^4
\]
for some \(\alpha \geq 0, \beta > 0\) and for some function \(\phi \in C^\infty(M)\) and non-positive function \(\lambda\). We first show that \(K_M\) is nef using (1.4). This flow is equivalent to the following Monge-Ampère type flow:
\[
(3.2) \quad \begin{cases}
\partial_t \varphi = \log \left( \frac{\omega_h - t \text{Ric}(\omega_h) - t \eta + \sqrt{-1} \partial \bar{\partial} \varphi}{\omega_h^n} \right); \\
\varphi(0) = 0,
\end{cases}
\]
in the sense that if \(\varphi\) satisfies (3.2) on \(M \times [0, T]\) so that
\[
(3.3) \quad \omega_h - t \text{Ric}(\omega_h) - t \eta + \sqrt{-1} \partial \bar{\partial} \varphi > 0,
\]
then \(\omega(t) = \omega_h - t \text{Ric}(\omega_h) - t \eta + \sqrt{-1} \partial \bar{\partial} \varphi\) will satisfy (1.4). Moreover if \(\omega(t)\) satisfies (1.4), then
\[
\varphi(t) = \int_0^t \log \frac{\omega(s)^n}{\omega_h^n} \, ds
\]
satisfies (3.2). In the following, \(\varphi\) will always denote the solution of (3.2) which corresponds to (1.4). Moreover, \(g(t)\) will be the Kähler metric with respect to the Kähler form \(\omega(t)\).

Since \(M\) is closed, the twisted Kähler-Ricci flow \(\omega(t)\) admits a short time solution, for example see [7]. In this section, we will estimate the existence time of the flow \(g(t)\) under the assumption (3.1). We need the following fact,
which states that if the solution $g(t)$ to (1.4) is uniformly equivalent to a fixed metric $h$ on $M \times [0, T_0)$, then we have higher order regularity of $g(t)$ up to $T_0$ and hence the solution can be extended beyond $T_0$, see [3, 6, 15, 29, 34]. To summarize, we have the following existence time criteria for the parabolic complex Monge-Ampère equation.

**Lemma 3.1.** Let $g(t)$ be a smooth solution to (1.4) on $M \times [0, T_0)$. Suppose there is a positive constant $C > 0$ such that

$$C^{-1}h \leq g(t) \leq Ch$$

on $M \times [0, T_0)$. Then there is $\varepsilon > 0$ such that $g(t)$ can be extended to $M \times [0, T_0 + \varepsilon)$ which satisfies (3.2).

**Proof.** We sketch the proof as follows. The condition that $C^{-1}h \leq g(t) \leq Ch$ ensure that we have uniform $C^0$ estimate up on $[0, T_0)$. If $\eta = 0$, (1.4) is the ordinary Kähler-Ricci flow. By [29, Corollary 1.1], all the derivatives of $g(t)$ with respect to $h$ are uniformly bounded on $[\frac{T_0}{2}, T_0)$. One can combine the method in [29] with the techniques for more general fully nonlinear elliptic equations as in [3, 6, 15, 34] to show that the same estimates are true for general $\eta$. From this it is easy to see that the solution $g(t)$ can be extended beyond $T_0$.

By direct calculation, we have the following well-known formulas, for example see [33] and [31].

**Lemma 3.2.** Let $\dot{\varphi} = \partial_t \varphi$. We have

$$\begin{cases} 
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \dot{\varphi} = -\operatorname{tr}_g (\operatorname{Ric}(h) + \eta); \\
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) (t\dot{\varphi} - \varphi - nt) = -\operatorname{tr}_g h.
\end{cases}$$

**Proof.** The formulas are standard, we include the proof for reader’s convenience. By differentiating (3.2) with respect to time, we have

$$\partial_t \dot{\varphi} = \operatorname{tr}_g \left( \frac{\partial}{\partial t} \left( \omega_h - t\operatorname{Ric}(h) - t\eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi \right) \right)$$

$$= \operatorname{tr}_g \left( -\operatorname{Ric}(h) - \eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi \right)$$

$$= -\operatorname{tr}_g (\operatorname{Ric}(h) + \eta) + \Delta_{g(t)} \dot{\varphi}.$$

This proved the first equation. For the second equation, taking trace with respect to $g(t)$ of the equality

$$\omega(t) = \omega_h - t\operatorname{Ric}(\omega_h) - t\eta + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi,$$
we have \(-t \text{tr}_g(\text{Ric}(h) + \eta) = n - \Delta_g \varphi - \text{tr}_g(h)\). Hence
\[
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right)(t \varphi) = -t \text{tr}_g(\text{Ric}(h) + \eta) + \varphi
\]
\[
= n - \text{tr}_g h + \left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \varphi.
\]
This proved the second equation. \(\square\)

Along the Kähler-Ricci flow, it is well-known that the scalar curvature is bounded from below. We have the following analogy for the twisted Kähler-Ricci flow.

**Lemma 3.3.** Let \(g(t), t \in [0, T)\) be a solution to the twisted Kähler-Ricci flow on \(M\) with initial metric \(g(0) = h\). Then the scalar curvature \(S(g(t))\) satisfies
\[
S(g(t)) + \text{tr}_g \eta \geq -\frac{n}{t + \sigma}
\]
on \(M \times [0, T)\) where \(\sigma > 0\) so that \(\inf_M (S(h) + \text{tr}_h \eta) \geq -n\sigma^{-1}\). In particular,
\[
\sup_M \left(\log \frac{\det g(t)}{\det h}\right) = \sup_M \varphi(\cdot, t) \leq n \log \left(\frac{t + \sigma}{\sigma}\right).
\]

**Proof.** Direct calculation shows
\[
\partial_t (S + \text{tr}_g \eta) = g^{ik} \eta_{ij} \partial_i g_{kl} + g^{ij} \partial_i \text{Ric}_{ij}
\]
\[
= |\text{Ric} + \eta|^2 + \Delta_{g(t)}(S + \text{tr}_g \eta)
\]
\[
\geq \frac{1}{n} (S + \text{tr}_g \eta)^2 + \Delta_{g(t)}(S + \text{tr}_g \eta).
\]
The lower bound of \(S + \text{tr}_g \eta\) follows from the maximum principle. The upper bound of \(\varphi\) follows from the fact that \(\partial_t \varphi = -S - \text{tr}_g \eta\) and \(\varphi(0) = 0\). \(\square\)

The lemma gives an upper bound of \((\det g)/(\det h)\). Next, we want to estimate the upper bound of \(\text{tr}_g h\). Combining these two bounds, one can obtain \(C^0\) estimates along the twisted Kähler-Ricci flow. This turns will give an estimate on the existence time. We need the following parabolic Schwarz Lemma which is a straightforward application of Yau’s Schwarz Lemma [43].

**Lemma 3.4.** Let \(g(t)\) be a solution to the twisted Kähler-Ricci flow \([14]\), then
\[
\left(\frac{\partial}{\partial t} - \Delta_{g(t)}\right) \log \text{tr}_g h \leq \frac{1}{\text{tr}_g h} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\text{tr}_g h} g^{ij} g^{kl} h g_{kl} \eta_{ij}.
\]

**Proof.** The proof is similar to the parabolic Schwarz Lemma in Kähler-Ricci flow [30], see also [31, Theorem 2.6]. By Yau’s Schwarz Lemma [43], for two Kähler metrics \(g\) and \(h\) we have
\[
\Delta_g \log \text{tr}_g h \geq \frac{1}{\text{tr}_g h} \left( R_{ijkl}(g) h_{ij} - g^{ij} g^{kl} R_{ijkl}(h) \right).
\]
Applying (3.7) with \( g = g(t) \), we conclude that

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \text{tr}_g h 
\leq \frac{1}{\text{tr}_g h} \left( g^{ij} g^{kl} R_{ijkl}(h) - R^{ij}(g) h_{ij} \right) + \frac{1}{\text{tr}_g h} h_{ij} \partial_k g^{ij} 
= \frac{1}{\text{tr}_g h} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\text{tr}_g h} g^{il} g^{kj} h_{ij} \eta_{kl},
\]

where we have used (1.4) in the last step.

\( \square \)

We are now ready to prove Theorem 1.1 (a).

**Proof of Theorem 1.1 (a).** For the Kähler metric \( h \), we let

\[
S = \inf \{ s \in \mathbb{R} : \exists f \in C^\infty(M), \text{Ric}(h) < s\omega_h + \sqrt{-1} \partial \bar{\partial} f \}.
\]

We claim that \( S \leq 0 \). If the claim is true, then for any \( \varepsilon > 0 \) we can find smooth function \( f \) so that \(-\text{Ric}(h) - \sqrt{-1} \partial \bar{\partial} f \geq -\varepsilon \omega_h \). Since the first Chern class of the canonical line bundle \( K_M \) is represented by \(-\text{Ric}(h) \), we see that the canonical bundle is nef.

Suppose on the contrary that \( S > 0 \). Since \( \alpha \geq 0 \), for \( \mu > S \) to be chosen later, we can find \( v \in C^\infty(M) \) such that

\[
(3.9) \quad \alpha \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi = \rho \leq \mu \omega_h + \sqrt{-1} \partial \bar{\partial} v.
\]

Let \( g(t) \) be the twisted Kähler-Ricci flow with \( \eta = \sqrt{-1} \partial \bar{\partial} u \) and \( g(0) = h \) where \( u = (2\beta)^{-1} v \). We want to show that the maximal existence time \( T_{\max} > S^{-1} \). If this is true, then (3.3) implies that

\[
\omega(t) = \omega_h - t\text{Ric}(h) - t\sqrt{-1} \partial \bar{\partial} u + \sqrt{-1} \partial \bar{\partial} \varphi
\]

is a Kähler metric for some \( t > S^{-1} \) where \( \varphi \) is the solution to (3.2). But this contradicts the definition of \( S \). See also [32, 36] for the existence time characterization of the Kähler-Ricci flow.

On \( M \times [0, T_{\max}) \), we are going to estimate \( \Lambda := \text{tr}_g h \). Let \( E_i \) be a unitary frame with respect to \( g(t) \) which diagonalizes \( h \) at a point. By Lemma 3.4 and
Lemma 2.3 with \( g = g(t) \) and \( \rho = \alpha \text{Ric}(h) + \sqrt{-1} \partial \bar{\partial} \phi \), we have

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \Lambda \\
\leq \frac{1}{\Lambda} g^{ij} g^{kl} R_{ijkl}(h) + \frac{1}{\Lambda} g^{ij} g^{kl} h_{ij} u_{kl}
\leq \frac{\lambda}{2 \beta} \Lambda + \frac{\lambda}{2 \beta \Lambda} |h|^2 + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle - \frac{1}{2 \beta} \text{tr}_g \rho - \frac{1}{2 \beta \Lambda} \langle \rho, \omega_h \rangle
\leq \frac{\lambda}{2 \beta} \Lambda + \frac{\lambda}{2 \beta \Lambda} |h|^2 - \frac{1}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle + \frac{1}{2 \beta \Lambda} (\Lambda \text{tr}_g \rho - \langle \rho, \omega_h \rangle)
= \frac{\lambda}{2 \beta} \Lambda + \frac{\lambda}{2 \beta \Lambda} |h|^2 - \frac{1}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle + \frac{1}{2 \beta \Lambda} \sum_{i=1}^{n} \rho(E_i, E_i) (\Lambda - h(E_i, E_i))
\leq \frac{\lambda}{2 \beta} \Lambda + \frac{\lambda}{2 \beta \Lambda} |h|^2 - \frac{1}{\beta} \text{tr}_g \rho + \frac{1}{\Lambda} \langle \omega_h, \sqrt{-1} \partial \bar{\partial} u \rangle
+ \frac{1}{2 \beta \Lambda} \sum_{i=1}^{n} \left( \alpha u h(E_i, E_i) + (\sqrt{-1} \partial \bar{\partial} v)(E_i, E_i) \right) (\Lambda - h(E_i, E_i))
= \left( \frac{\lambda + \alpha u}{2 \beta} \right) \Lambda + \left( \frac{\lambda - \alpha u}{2 \beta} \right) \frac{|h|^2}{\Lambda} + \frac{1}{2 \beta} \Delta_{g(t)} v - \frac{1}{\beta} \text{tr}_g \rho.
\]

Here we have used the fact that:

\[
\frac{1}{2 \beta} \sum_{i=1}^{n} (\sqrt{-1} \partial \bar{\partial} v)(E_i, E_i) h(E_i, E_i) = \langle \sqrt{-1} \partial \bar{\partial} u, \omega_h \rangle.
\]

Since \( n |h|^2 \geq \Lambda^2 \), then we have

\[
\left( \frac{\lambda + \alpha u}{2 \beta} \right) \Lambda + \left( \frac{\lambda - \alpha u}{2 \beta} \right) \frac{|h|^2}{\Lambda} \leq \frac{(n+1)\lambda + \alpha u (n-1)}{2n \beta} \Lambda.
\]

Let \( w = t \dot{\phi} - \varphi - nt \) and \( B = \frac{(n+1)\lambda + \alpha u (n-1)}{2n \beta} \). Combining the above with Lemma 3.2 we deduce that

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \log \Lambda \leq B \Lambda + \frac{1}{2 \beta} \Delta_{g(t)} v - \frac{1}{\beta} \text{tr}_g \rho
\]

\[
\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) w + \frac{1}{2 \beta} \Delta_{g(t)} v + \frac{1}{\beta} \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) (\alpha \dot{\phi} + \phi - \alpha u)
= \left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) \left[ -B w - \frac{1}{2 \beta} v + \frac{1}{\beta} (\alpha \dot{\phi} + \phi - \alpha u) \right].
\]

Since at \( t = 0 \),

\[
\log \Lambda + B w + \frac{1}{2 \beta} v + \frac{1}{\beta} (\alpha \dot{\phi} - \phi + \alpha u) \leq C
\]
for some constant $C$ depending only on $n, \alpha, \beta, \sup_M |u|, \sup_M |\phi|$ and the upper bound of $\mu$. It follows from the maximum principle that

$$\log \Lambda \leq C_1 + \left( \frac{\alpha}{\beta} - Bt \right) \phi + B\varphi + Bnt, \tag{3.12}$$

where $C_1$ depends only on $n, \alpha, \beta, \sup_M |v|, \sup_M |\varphi|$ and the upper bound of $\mu$. Since $\lambda \leq 0$, $(\frac{\alpha}{\beta} - Bt) \geq 0$ for $t < \min\{T_{\max}, \frac{2n}{(n-1)\mu}\}$. Combining this with (3.12) and Lemma 3.3, we conclude that there exists a constant $C_2 > 1$ such that for all $t < \min\{T_{\max}, \frac{2n}{(n-1)\mu}\}$,

$$C_2^{-1}h \leq g(t) \leq C_2h.$$

By Lemma 3.1 we conclude that $T_{\max} \geq \frac{2n}{(n-1)\mu\beta}$. Hence $T_{\max} > S^{-1}$ if we choose $\mu$ sufficiently close to $S$. This completes the proof. $\square$

Next we want to prove that $K_M$ is ample if $\lambda < 0$.

**Proof of Theorem 1.1 (b).** By part (a) of the theorem, the canonical line bundle $K_M$ is nef. Therefore, for all $\varepsilon > 0$, we can find $v \in C^\infty(M)$ such that

$$\alpha \text{Ric}(h) + \sqrt{-1} \partial \overline{\partial} \phi = \rho \leq \varepsilon \omega_h + \sqrt{-1} \partial \overline{\partial} v. \tag{3.13}$$

Choose $\varepsilon, \sigma > 0$ small enough so that

$$B = \frac{(n+1)\lambda + (n-1)\varepsilon\alpha}{2\beta n} \leq -\sigma.$$

Let $g(t)$ be the twisted Kähler-Ricci flow with $\eta = \sqrt{-1} \partial \overline{\partial} u$ where $u = (2\beta)^{-1} v$. By Theorem 1.1 (a) and the existence time estimates in [32], the flow exists for all time. Let $\Lambda = \text{tr}_g h$ and $F = \log \Lambda + \alpha (1 + \beta^{-1}) u - \beta^{-1}(\alpha \dot{\phi} + \phi)$. By Lemma 3.2 and the computation in (3.11), we have

$$\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) F \leq -\sigma \Lambda \tag{3.14}$$

for some $\sigma > 0$. Now let $G = F + \left( 1 + \frac{\alpha n}{\beta} \right) \log t$. Then

$$\left( \frac{\partial}{\partial t} - \Delta_{g(t)} \right) G \leq -\sigma \Lambda + \left( 1 + \frac{\alpha n}{\beta} \right) t^{-1}. \tag{3.15}$$

For any $T_0 > 0$, suppose that $G(x_0, t_0) = \sup_{M \times [0, T_0]} G$. Since $G \to -\infty$ as $t \to 0$, then $t_0 > 0$. At $(x_0, t_0)$, (3.15) shows $t_0 \Lambda(x_0, t_0) \leq C_1$ for some $C_1 > 0$.\/
depending only on \( \alpha, \beta, \sigma \) and \( n \). Hence by AM-GM inequality,

\[
\sup_{M \times [0, T_0]} G(x, t) \leq G(x_0, t_0)
\]

\[
= \left( \log \Lambda - \frac{1}{\beta} (\alpha \dot{\phi} + \phi) + \alpha \left( 1 + \frac{1}{\beta} \right) u + \left( 1 + \frac{\alpha n}{\beta} \right) \log t \right) \big|_{(x_0, t_0)}
\]

\[
= \left( \log \Lambda + \frac{\alpha n}{\beta} \log \left( \frac{\det h}{\det g} \right)^{1/n} + \left( 1 + \frac{\alpha n}{\beta} \right) \log t \right) \big|_{(x_0, t_0)} + C_2
\]

\[
\leq \left( 1 + \frac{\alpha n}{\beta} \right) (\log \Lambda + \log t) \big|_{(x_0, t_0)} + C_2
\]

\[
\leq \left( 1 + \frac{\alpha n}{\beta} \right) \log C_1 + C_2 =: C_3
\]

for some constant \( C_3 > 0 \) independent of \( T_0 \). By letting \( T_0 \to +\infty \), we have an uniform upper bound of \( \sup_{M \times [0, +\infty)} G \) and hence for \( t \) large enough,

\[
\Lambda \leq C_4 t^{-1}
\]

for some constant \( C_4 \) independent of \( t \) where we have used Lemma 3.3. Therefore, \( g(t) \geq C_4 t h \) for some \( C_4 > 0 \) for \( t \) large enough. With the uniform lower bound, the normalized Kähler-Ricci flow will converge to the unique negative Kähler-Einstein metric, see [12] for example. One can also argue as follows. By rewriting \( g(t) \) using potential, the lower bound implies that for sufficiently large \( t \),

\[
-\text{Ric}(h) - \sqrt{-1} \partial \bar{\partial} f \geq \varepsilon \omega_h
\]

for some \( \varepsilon > 0 \) and \( f(t) \in C^\infty(M) \). By the result of Aubin [1] and Yau [44], \( M \) supports a Kähler-Einstein metric with negative Ricci curvature. In particular, the canonical line bundle \( K_M \) is ample and \( M \) is projective. \( \square \)

4. QUASI-NEGATIVE CASE

In this section, we will prove Theorem 1.1 (c). We consider compact Kähler manifolds which satisfies (3.1) for some quasi-negative function \( \lambda \). We will follow closely the arguments in [38]. Our main contribution is the following:

**Lemma 4.1.** Let \((M^n, h)\) be a compact Kähler manifold satisfying (3.1) for some quasi-negative function \( \lambda \). Then \( \int_M c_1(K_M)^n > 0 \).

**Proof.** It is equivalent to prove \( \int_M (-\text{Ric}(h))^n > 0 \). Since \( K_M \) is nef by Theorem 1.1 (a), for all \( 1 \geq \varepsilon > 0 \) there exists \( u_\varepsilon \in C^\infty(M) \) such that

\[
(4.1) \quad -\text{Ric}(h) + \varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} u_\varepsilon > 0.
\]

By Yau’s Theorem [44] Theorem 4, p.383, we can find \( v_\varepsilon \in C^\infty(M) \) such that

\[
(4.2) \quad \begin{cases}
( -\text{Ric}(h) + \varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + v_\varepsilon) )^n = \exp \left( v_\varepsilon + u_\varepsilon + \frac{1}{2\pi} (\phi + \alpha u_\varepsilon) \right) \omega_h^n; \\
-\text{Ric}(h) + \varepsilon \omega_h + \sqrt{-1} \partial \bar{\partial} (u_\varepsilon + v_\varepsilon) > 0.
\end{cases}
\]
For notational convenience, we denote
\[ f_\varepsilon := v_\varepsilon + u_\varepsilon + \frac{1}{2\beta}(\phi + \alpha u_\varepsilon); \quad \omega_\varepsilon := -\text{Ric}(h) + \varepsilon \omega_h + \sqrt{-1}\partial\bar{\partial}(u_\varepsilon + v_\varepsilon) \]
so that
\[ \omega_\varepsilon^n = \exp(f_\varepsilon)\omega_h^n; \quad -\text{Ric}(\omega_\varepsilon) + \text{Ric}(\omega_h) = \sqrt{-1}\partial\bar{\partial}f_\varepsilon. \]
By Stokes’ theorem,
\[ \lim_{\varepsilon \to 0} \int_M \omega_\varepsilon^n = \lim_{\varepsilon \to 0} \int_M (-\text{Ric}(\omega_\varepsilon) + \varepsilon \omega_h)^n = \int_M (-\text{Ric}(h))^n. \]
Therefore, it suffices to estimate the lower bound of \( \int_M \omega_\varepsilon^n \) for some \( \varepsilon_i \to 0. \)

In order to use Wu-Yau’s method \[38\], we will derive a differential inequality involving of \( \Lambda = \text{tr}_g h \) where \( g \) is the Kähler metric associated to the Kähler form \( \omega_\varepsilon \). By Yau’s Schwarz Lemma \[43\],
\[ -\Delta_g \log \Lambda \leq \frac{1}{\Lambda} g^{ij} g^{kl} R(h)_{ijkl} - \frac{1}{\Lambda} \langle \text{Ric}(g), h \rangle. \]
Here and below the inner product is taken with respect to \( g \). We choose a unitary frame \( E_i \) with respect to \( g \) so that \( \rho \) is diagonal at a point. By (4.1) and similar computation of (3.10), for \( \rho = \alpha \text{Ric}(h) + \sqrt{-1}\partial\bar{\partial}\phi \), we see that
\[ \frac{1}{\Lambda} g^{ij} g^{kl} R(h)_{ijkl} \leq \frac{\lambda + \alpha \varepsilon}{2\beta} \Lambda + \frac{\lambda - \alpha \varepsilon}{2\beta} |h|^2 - \frac{\alpha}{\beta} \text{tr}_g \text{Ric}(h) \]
\[ + \frac{1}{2\beta} \Delta_g (\alpha u_\varepsilon - \phi) - \frac{1}{2\beta \Lambda} \langle \sqrt{-1}\partial\bar{\partial}(\alpha u_\varepsilon + \phi), \omega_h \rangle. \]
On the other hand,
\[ -\frac{1}{\Lambda} \langle \text{Ric}(g), h \rangle = -\frac{1}{\Lambda} \langle \text{Ric}(h) - \sqrt{-1}\partial\bar{\partial}f_\varepsilon, \omega_h \rangle \]
\[ = \frac{1}{2\beta \Lambda} \langle \sqrt{-1}\partial\bar{\partial}(\alpha u_\varepsilon + \phi), \omega_h \rangle + \frac{1}{\Lambda} \langle \omega_\varepsilon - \varepsilon \omega_h, \omega_h \rangle \]
because \( \sqrt{-1}\partial\bar{\partial}f_\varepsilon = \frac{2\beta}{\Lambda} \sqrt{-1}\partial\bar{\partial}(\alpha u_\varepsilon + \phi) + \omega_\varepsilon - \varepsilon \omega_h + \text{Ric}(h) \). By (4.5)–(4.7), we conclude that
\[ -\Delta_g \log \Lambda \leq \left( \frac{\lambda + \varepsilon \alpha}{2\beta} \right) \Lambda + \left( \frac{\lambda - \varepsilon \alpha}{2\beta} \right) |h|^2 + \frac{\alpha}{2\beta} \Delta_g (\alpha u_\varepsilon - \phi) \]
\[ - \frac{\alpha}{\beta} \text{tr}_g \text{Ric}(h) + \frac{1}{\Lambda} \langle \omega_\varepsilon - \varepsilon \omega_h, \omega_h \rangle. \]
Since \( \lambda \leq 0 \), by rewriting \( -\text{Ric}(h) = \omega_\varepsilon - \varepsilon \omega_h - \sqrt{-1}\partial\bar{\partial}(u_\varepsilon + v_\varepsilon) \), we can see that the function \( F = -\log \Lambda - \frac{1}{2\beta} (\alpha u_\varepsilon - \phi) + \frac{n}{\beta} (u_\varepsilon + v_\varepsilon) \) satisfies
\[ \Delta_g F \leq \left( 1 + \frac{n\alpha}{\beta} \right) + \left( \frac{\lambda}{2\beta} \right) \Lambda \leq \left( 1 + \frac{n\alpha}{\beta} \right) + \left( \frac{n\lambda}{2\beta} \right) \exp \left( -\frac{\max_M f_\varepsilon}{n} \right). \]
Here we have used AM-GM inequality at the last inequality. From this, one can proceed as in the proof of [38, Theorem 2]. We sketch the arguments for the convenience of the readers. First one can estimate $\sup_M f_\varepsilon$ as follows. Since $M$ is compact, for all $1 \geq \varepsilon > 0$, there is $x_0 \in M$ such that $f_\varepsilon(x_0) = \max_M f_\varepsilon$. At $x_0$, we have $\sqrt{-1}\partial\bar{\partial}(u_\varepsilon + v_\varepsilon) \leq -\frac{1}{2\beta} \sqrt{-1}\partial\bar{\partial}(\phi + \alpha u_\varepsilon)$

\begin{equation}
\leq -\frac{1}{2\beta} \sqrt{-1}\partial\bar{\partial}(\phi + \frac{\alpha}{2\beta} (-\text{Ric}(h) + \varepsilon \omega_h)) \leq C_0(\alpha, \beta, \phi, n, h) \omega_h
\end{equation}

where we have used (4.1) in the second inequality. By substituting it back to the Monge-Ampère equation (4.2), we conclude that

\begin{equation}
\sup_M f_\varepsilon \leq C_1(\alpha, \beta, \phi, n, h),
\end{equation}

which is independent of $\varepsilon$. On the other hand, from (4.1) and (4.2), we see that $C_1 \omega_h + \sqrt{-1}\partial\bar{\partial} f_\varepsilon > 0$ for some sufficiently large $C_1$ independent of $\varepsilon$. By [38, Lemma 7], we can find $\varepsilon_i \to 0$ such that the sequence $\{\exp(1 + \sup_M f_\varepsilon - f_\varepsilon_i)\}_{i=1}^\infty$ converges to $\exp(\varepsilon w)$ for some function $w$ almost everywhere. Since $(1 + \sup_M f_\varepsilon - f_\varepsilon_i) \geq 1$, integrating (4.9) and using Lebesgue dominated convergence theorem, we obtain

$$
\exp\left(-\frac{\max_M f_{\varepsilon_i}}{n}\right) \leq \left(1 + \frac{n\alpha}{\beta}\right) \frac{\int_M \omega^n_{\varepsilon_i}}{\int_M -\frac{n\lambda}{2\beta} \omega^n_{\varepsilon_i}} \to \left(1 + \frac{n\alpha}{\beta}\right) \frac{\int_M \exp(-\varepsilon w) \omega^n_h}{\int_M -\frac{n\lambda}{2\beta} \exp(-\varepsilon w) \omega^n_h}
$$

and so $\sup_M f_{\varepsilon_i} \geq -C_3$ for some $C_3 > 0$ independent of $i$. Together with the upper bound (4.11), passing to a subsequence $f_{\varepsilon_i} \to -\varepsilon w + c$ for some constant $c$. This implies that

$$
\int_M \omega^n_{\varepsilon_i} \to \int_M \exp(-\varepsilon w + c) \omega^n_h > 0.
$$

This completes the proof. □

Proof of Theorem 5.1 (c). By part (a) of Theorem 5.1, $K_M$ is nef. Combine it with Lemma 4.1 it follows that $K_M$ is big using [19, Corollary 2.3.38, p.114]. Hence $M$ is Moishezon by [4, Theorem 0.5]. Since $M$ is Kähler, $M$ is projective by Moishezon’s Theorem. If in addition that $M$ does not contain any rational curve, then $K_M$ is ample by the proof of [37, Lemma 5], see also [5]. □

5. Applications on Kähler manifolds with non-positive curvature

Now we are in a position to apply Theorem 5.1 to prove the following structural result which answer a question of Ni [23].
Theorem 5.1. Suppose $(M^n, h)$ is a compact Kähler manifold with $\text{Ric}_k(h) \leq -(k+1)\sigma$ for some non-negative continuous function $\sigma$ and integer $k$ with $1 \leq k \leq n$.

(a) The canonical bundle $K_M$ of $M$ is numerically effective (nef).
(b) Suppose $\sigma > 0$, then $K_M$ is ample. In particular, $M$ supports a Kähler-Einstein metric with negative Ricci curvature and $M$ is projective.
(c) Suppose $\sigma$ is quasi-positive, then $\int_M c_1(K_M)^n > 0$. In particular, $K_M$ is big and $M$ is projective. If in addition $M$ does not contain any rational curve, then $K_M$ is ample.

Proof. If $k = 1$ or $n$, the result is well-known. It suffices to consider $1 < k < n$. By Lemma 2.2, the curvature of $g$ satisfies (3.1) with $\alpha = k - 1$, $\beta = n - k$ and $\lambda = -(n - 1)(k + 1)\sigma$. The result follows from Theorem 1.1. \qed

The condition (2.2) is also related to curvature $\text{Ric}^+$ introduced in [23] which is defined to be $\text{Ric}^+(X, \overline{X}) = \text{Ric}(X, \overline{X}) + \frac{R(X, \overline{X}, X, \overline{X})}{|X|^2}$ and is equivalent to the left hand side of (1.2) with $\alpha = \beta = 1$ and $\phi = 0$. It was proved in [23, Proposition 6.2] that a compact Kähler manifold with $\text{Ric}^+ < 0$ has no nontrivial holomorphic vector field. By Theorem 1.1, we have the following stronger results.

Corollary 5.1. Suppose $(M^n, g)$ is a compact Kähler manifold with $\text{Ric}^+ \leq -(n + 2)\sigma$ for some nonnegative function $\sigma$, then the canonical line bundle $K_M$ is nef. If $\sigma > 0$ on $M$, then $K_M$ is ample. If $\sigma$ is quasi-positive, then $M$ is projective with $K_M$ being big and nef.

6. Conformal invariant of curvature condition

For a Hermitian manifold $(M, g, J)$, the Chern curvature $R$ is the curvature induced by the Chern connection $\nabla^C$ which is defined to be the connection such that $\nabla^C g = \nabla^C J = 0$ with no $(1, 1)$ components on the torsion. The first Chern Ricci curvature $\text{Ric}^1$ is defined by $\text{Ric}^1 = -\sqrt{-1} \partial \bar{\partial} \log \det g$.

Since $\text{Ric}^1$ coincides with the Ricci curvature in the Kähler case, we can naturally extend the notion of $C_{\alpha, \beta}$ to general Hermitian metrics:

$C_{\alpha, \beta}(X) = \alpha \text{Ric}^1(X, \overline{X}) + \beta |X|^{-2} R(X, \overline{X}, X, \overline{X})$.

Using this, we show that condition (1.2) is conformally invariant.

Lemma 6.1. If a Hermitian metric $g$ satisfies $C_{\alpha, \beta}(g)(X) + \sqrt{-1} \partial \bar{\partial} \phi(X, \bar{X}) \leq \lambda |X|^2_g$
for some $\alpha, \beta \in \mathbb{R}$, $\phi \in C^\infty(M)$ and $\lambda \in C^0(M)$ and for all non-zero $X \in T^{1,0}M$, then $\tilde{g} = e^{-2F}g$ satisfies
\[
C_{\alpha, \beta}(\tilde{g})(X) + \sqrt{-1}\partial \bar{\partial} \tilde{\phi}(X, X) \leq \tilde{\lambda}|X|^2_{\tilde{g}}
\]
for $\tilde{\phi} = \phi + 2(n\alpha + \beta)F$, $\tilde{\lambda} = e^{-2F}\lambda$ and for all non-zero $X \in T^{1,0}(M)$.

**Proof.** Denote the curvature and the first Chern Ricci curvature of the Chern connection simply by $R$ and $\text{Ric}$. By the conformal formula for Hermitian metric, for example see [16, Appendix B.1], the Hermitian metric $\tilde{g} = e^{2F}g$ satisfies
\[
\begin{align*}
\{ & \tilde{R}_{k\bar{l}i\bar{j}} = e^{2F}(R_{k\bar{l}i\bar{j}} - 2g_{i\bar{j}}F_{k\bar{l}}); \\
& \tilde{R}_{ij} = R_{ij} - 2nF_{ij}. \\
\end{align*}
\]
Hence for $X \in T^{1,0}M$,
\[
C_{\alpha, \beta}(\tilde{g})(X) = \alpha\text{Ric}(X, \bar{X}) + \beta|X|^2_{\tilde{g}}\tilde{R}(X, \bar{X}, X, \bar{X})
\]
\[
= \alpha(\text{Ric}_{X\bar{X}} - 2nF_{X\bar{X}}) + |X|^2_{\tilde{g}}\beta(R_{X\bar{X}X\bar{X}} - 2|X|^2_{\tilde{g}}F_{X\bar{X}})
\]
\[
= C_{\alpha, \beta}(g)(X) - 2(n\alpha + \beta)F_{X\bar{X}}.
\]
The assertion follows. $\square$

By Lemma 6.1 it is clear that Theorem 1.1 also holds if the metric $h$ is Hermitian metric which is conformally Kähler. This in particular proves a special case of [42, Conjecture 1.1].

**Theorem 6.1.** Suppose $(M, g)$ is a compact Hermitian manifold which is conformally Kähler such that the holomorphic sectional curvature $H_g \leq 0$, then $K_M$ is nef. Moreover if $H_g$ is negative at some point, then $M$ is projective with ample $K_M$.

**Proof.** Since $g$ is conformally Kähler, there is a Kähler metric $h$ and a smooth function $F$ such that $h = e^{-2F}g$. The assumption $H_g \leq 0$ and Lemma 6.1 imply that $C_{0,1}(h)(X) + \sqrt{-1}\partial \bar{\partial} \phi(X, \bar{X}) \leq 0$ for some $\phi \in C^\infty(M)$ and all non-zero $X \in T^{1,0}M$. Namely, the Kähler metric $h$ satisfies the assumption in Theorem 1.1. Now, the nefness of $K_M$ follows from (a) of Theorem 1.1. Moreover, since $H_g \leq 0$ and the standard metric of $\mathbb{CP}^1$ is a positive constant, by a result of Royden [28, p.558], there is no nonconstant holomorphic map from $\mathbb{CP}^1$ into $M$, and so $M$ does not contain rational curve.

If $H_g$ is negative somewhere, then $C_{0,1}(h) + \sqrt{-1}\partial \bar{\partial} \phi$ is negative at some point by Lemma 6.1. The projectivity of $M$ and ampleness of $K_M$ follows from (c) of Theorem 1.1 and absence of rational curve. $\square$

7. **Kähler manifolds with positive mixed curvature**

In [23, 25], it was shown that compact Kähler manifolds with $\text{Ric}^+ > 0$, $\text{Ric}^\perp > 0$ or $\text{Ric}_k > 0$ are simply connected and projective. Following the argument in [23, Theorem 2.7], we show that Kähler manifolds satisfying the positive counterpart of (1.2) are also simply connected and projective.
Theorem 7.1. Suppose \((M, g)\) is a compact Kähler manifold with
\[
\alpha |X|^2_g \cdot \text{Ric}(X, \bar{X}) + \beta R(X, \bar{X}, X, \bar{X}) > 0
\]
for some \(\alpha, \beta\) with \(\alpha > 0, \alpha + \beta \geq 0\) and all non-zero \(X \in T^{1,0}M\), then \(h^{p,0} = 0\) for all \(1 \leq p \leq n\). In particular, \(M\) is simply connected and projective.

As a corollary, we recover the above mentioned results by Ni and Zheng [23, 25]:

Corollary 7.1. Let \((M^n, g)\) be a compact Kähler manifold with (i) \(\text{Ric}_k > 0\) for some \(1 \leq k \leq n\); or (ii) \(\text{Ric}^+ > 0\); or (iii) \(\text{Ric}^\perp > 0\), then \(h^{p,0} = 0\) for all \(1 \leq p \leq n\). In particular, \(M\) is simply connected and projective. Here \(\text{Ric}^\perp\) is defined as
\[
\text{Ric}^\perp(X, \bar{X}) =: \text{Ric}(X, \bar{X}) - H(X)\frac{|X|^2}{|X|^2}
\]
for all non-zero \(X \in T^{1,0}M\).

Proof. We only need to check the condition (7.1). If \(\text{Ric}_k > 0\) for some \(1 < k < n\), then the condition is satisfied with \(\alpha = (k - 1), \beta = (n - k)\) by Lemma 2.2. If \(k = 1\), the condition is satisfied for small enough \(\alpha > 0\) and \(\beta = 1\). If \(k = n\), then the condition is satisfied for \(\alpha = 1\) and small enough \(\beta > 0\). If \(\text{Ric}^+ > 0\), then the condition is satisfied with \(\alpha = \beta = 1\). If \(\text{Ric}^\perp > 0\), then the condition is satisfied with \(\alpha = 1, \beta = -1\). \(\square\)

To prove Theorem 7.1, we first show that the Hodge numbers vanish. This will follow from a slight modification of argument in [23, Section 6].

Proposition 7.1. Suppose \((M, g)\) is a compact Kähler manifold satisfying (7.1) for some \(\alpha > 0\) and \(\beta \in \mathbb{R}\) such that \(\alpha + \frac{2\beta}{p+1} \geq 0\) for an integer \(1 \leq p \leq n\), then \(h^{p,0} = 0\). In particular if \(\alpha + \beta \geq 0\), then \(h^{p,0} = 0\) for all \(1 \leq p \leq n\).

Proof. The first part of proof follows similarly as in that of [23, Theorem 2.2]. Assuming the existence of a nonzero holomorphic \((p,0)\)-form \(\phi\), we may conclude that at the point \(x_0\) where the maximum of the comass \(\|\phi\|_0\) is attained,
\[
0 \geq \sum_{i=1}^{p} R_{v\bar{v}i}
\]
for any \(v \in T^{1,0}M\), for some choice of unitary frame \(\left\{ \frac{\partial}{\partial z_l} \right\}_{l=1}^{p}\). Denote \(\Sigma = \text{span}\left\{ \frac{\partial}{\partial z_l} : l = 1, ..., p \right\}\). On the other hand, assumption (7.1) implies
\[
0 < \int_{Z \in \Sigma, |Z| = 1} \alpha \text{Ric}(Z, \bar{Z}) + \beta H(Z) d\theta(Z)
\]
\[
= \frac{\alpha}{p} \sum_{i=1}^{p} R_{ii} + \frac{2\beta}{p(p+1)} S_p(x_0, \Sigma) = \frac{\alpha}{p} \sum_{j=p+1}^{n} \sum_{i=1}^{p} R_{ijj} + \frac{1}{p} \left( \alpha + \frac{2\beta}{p+1} \right) \sum_{i,j=1}^{p} R_{ijj}.
\]
By (7.2), if \( \alpha + \frac{2\beta}{p+1} \geq 0 \) and \( \alpha \geq 0 \), then the right hand side is non-positive which is impossible. This completes the proof.

The next ingredient is the compactness of positively curved Kähler manifold. This was done by the second variational argument in the proof of Bonnet-Meyer theorem.

**Proposition 7.2.** Let \((M^n, g)\) be a complete Kähler manifold satisfying (7.1) for some \( \alpha, \beta, \lambda \) with \( \alpha, \lambda > 0 \), \( \alpha + \beta \geq 0 \). Then \((M, g)\) is a compact manifold with \( \text{diam}(M, g) \leq \pi \sqrt{\frac{(2n-1)+\beta}{\lambda}} \).

**Proof.** For any \( p, q \in M \), let \( \gamma : [0, \ell] \to M \) be a minimizing geodesic connecting \( p \) and \( q \). It suffices to show \( \ell \leq \frac{\pi}{\sqrt{\frac{(2n-1)+\beta}{\lambda}}} \).

Let \( \{e_i\}_{i=1}^{2n} \) be an orthonormal parallel vector fields along \( \gamma \) with \( e_{2n-1} = J' \gamma \) and \( e_{2n} = \gamma' \), where \( J \) is the complex structure of \((M, g)\). For \( 1 \leq i \leq 2n-1 \), define

\[
V_i(t) = \sin \left( \frac{\pi t}{\ell} \right) e_i(t), \quad \phi_i(t, s) = \exp_{\gamma(t)}(sV_i(t)), \quad L_i(s) = \text{length}(\phi_i(\cdot, s)).
\]

Since \( \phi_i(t, 0) = \gamma(t) \) and \( \gamma \) is minimizing, then \( L_i \) has a minimum point at 0. Using second variation formula of arc length, it is clear that

\[
0 \leq \left. \frac{d^2}{ds^2} \right|_{s=0} L_i(s) = \int_0^\ell \left( |\nabla V_i|_g^2 - R(V_i, \gamma', \gamma', V_i) \right) dt
\]

\[
\text{(7.3)} \quad = \int_0^\ell \left( \left( \frac{\pi}{\ell} t \right)^2 \cos^2 \left( \frac{\pi t}{\ell} \right) - \sin^2 \left( \frac{\pi t}{\ell} \right) R(e_i, \gamma', \gamma', e_i) \right) dt.
\]

Applying (7.1) to \( \gamma' = \frac{1}{\sqrt{2}}(\gamma' - \sqrt{-1}J' \gamma') \), we see that

\[
\alpha \text{Ric}(\gamma', \gamma') + \beta R(J' \gamma', \gamma', J' \gamma') \geq \lambda.
\]

Since \( \alpha > 0 \) and \( \alpha + \beta \geq 0 \), (7.3) shows

\[
0 \leq \left. \frac{d^2}{ds^2} \right|_{s=0} \left( \alpha \sum_{i=1}^{2n-2} L_i(s) + (\alpha + \beta)L_{2n-1}(s) \right)
\]

\[
= \int_0^\ell \left( \alpha (2n-2) + (\alpha + \beta) \right) \left( \frac{\pi}{\ell} t \right)^2 \cos^2 \left( \frac{\pi t}{\ell} \right) dt
\]

\[= - \int_0^\ell \sin^2 \left( \frac{\pi t}{\ell} \right) \left( \alpha \sum_{i=1}^{2n-2} R(e_i, \gamma', \gamma', e_i) + (\alpha + \beta) R(e_{2n-1}, \gamma', \gamma', e_{2n-1}) \right) dt\]

\[= \frac{\ell}{2} \left( \alpha (2n-1) + \beta \right) \left( \frac{\pi}{\ell} \right)^2 - \int_0^\ell \sin^2 \left( \frac{\pi t}{\ell} \right) \left( \alpha \text{Ric}(\gamma', \gamma') + \beta R(J' \gamma', \gamma', J' \gamma') \right) dt\]

\[\leq \frac{\ell}{2} \left( \alpha (2n-1) + \beta \right) \left( \frac{\pi}{\ell} \right)^2 - \lambda,
\]

where we used (7.4) in the last inequality. This completes the proof. \( \square \)
Proof of Theorem 7.1. By Proposition 7.1, \( h^{p,0} = 0 \) for \( 1 \leq p \leq n \). Hence \( M \) is projective by a result of Kodaira. By Proposition 7.2, the universal cover \( \tilde{M} \) is also compact with zero Hodge numbers \( \tilde{h}^{p,0} \) for \( 1 \leq p \leq n \) and is projective. Then one can conclude that \( M \) is simply connected by comparing the Euler characteristic numbers of \( M \) and \( \tilde{M} \) using [13, Lemma 1]. □

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