Magnetic Bianchi type II string cosmological model in loop quantum cosmology

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The loop quantum cosmology of the Bianchi type II string cosmological model in the presence of a homogeneous magnetic field is studied. We present the effective equations which provide modifications to the classical equations of motion due to quantum effects. The numerical simulations confirm that the big bang singularity is resolved by quantum gravity effects.

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I. INTRODUCTION

Loop quantum cosmology (LQC) arises from the application of the more general theory of loop quantum gravity (LQG) to cosmology. One of the most important predictions of LQC is that in the homogeneous and isotropic Friedmann-Robertson-Walker models the classical big bang singularity is avoided being replaced by a bouncing Universe.

More recently, it has been shown that the big bang singularity is also resolved for anisotropic Bianchi type I (BI) [1], II (BII) [2] and IX [3]. Usually the matter source that was considered is a massless scalar field that play the role of internal time. The investigations have been extended to more complicated models including a perfect fluid, magnetic fields [4], cosmological strings [5]. It is remarkable the fact that in all these studies it was observed that the bounce prediction is robust.

The purpose of this paper is to investigate the dynamics of a BII string cosmological model in the presence of a magnetic field in the framework of LQC. We show that a bounce occurs in a collapsing magnetized BII string cosmological model, thus extending the known cases of singularity resolution.

The plan of the paper is as follows: In Sect. 2 we outline the classical equations of a BII string cosmological model in the presence of a magnetic field. In Sect. 3 we discuss the quantum theory introducing the effective equations for the model. Sect. 4 is devoted to numerical calculations and it is shown that the classical singularity is resolved in the BII effective quantum dynamics. Finally, in Sect. 5 we summarize our results.

II. CLASSICAL DYNAMICS

A. Hamiltonian formulation

The spacetime metric of BII model is given by

$$ds^2 = -N(t)^2 dt^2 + a_1(t)^2 (dx - Kzdy)^2 + a_2(t)^2 dy^2 + a_3(t)^2 dz^2,$$

(2.1)

where $a_1, a_2$ and $a_3$ are the directional scale factors. The parameter $K$ makes the difference between BI ($K = 0$) and BII ($K = 1$) spacetimes.

Having in view a comparison between the classical theory and the effective theory from LQC it is useful to rewrite the theory in terms of triads and connections.

Taking into account that the spatial manifold is non-compact, one needs to introduce a fiducial cell $\mathcal{V}$ with the coordinate lengths $l_i$ and the fiducial volume $V_0 = l_1 l_2 l_3$ [2]. The fiducial metric is

$$\ddot{q}_{ab} := \delta_{ij} \dot{\omega}_a^i \dot{\omega}_b^j,$$

(2.2)

with the co-triads

$$\dot{\omega}_a^1 = (dx)_a - Kz(dy)_a, \quad \dot{\omega}_a^2 = (dy)_a, \quad \dot{\omega}_a^3 = (dz)_a,$$

(2.3)

and triads

$$\dot{e}_a^1 = \left( \frac{\partial}{\partial x} \right)^a, \quad \dot{e}_a^2 = Kz \left( \frac{\partial}{\partial x} \right)^a + \left( \frac{\partial}{\partial y} \right)^a, \quad \dot{e}_a^3 = \left( \frac{\partial}{\partial z} \right)^a.$$

(2.4)

In terms of the fiducial triads $\dot{e}_a^i$ and co-triads $\dot{\omega}_a^i$ a convenient parametrization of the phase space variables $E_i^a, A_i^a$ is

$$A_i^a = c^i (l_i)^{-1} \dot{\omega}_a^i, \quad E_i^a = p_i l_i V_o^{-1} \sqrt{q},$$

(2.5)

without sum over index $i$. 

The connection and triad components $c^i$ and $p_i$ satisfy the Poisson bracket
\[ \{c^i, p_j\} = 8\pi G\gamma\delta^i_j, \quad (2.6) \]
where $\gamma \approx 0.2375$ is Barbero-Immirzi parameter.

Choosing the lapse function $N = \sqrt{|p_1 p_2 p_3|}$, in the Hamiltonian formulation of the model we have the Hamiltonian constraint $[2, 6, 7]$
\[ \mathcal{H}_{cl} = -\frac{1}{8\pi G\gamma^2} \left[ p_1 p_2 c_1 c_2 + p_2 p_3 c_2 c_3 + p_1 p_3 c_1 c_3 + K \epsilon p_2 p_3 c_1 ight. \
- \left. (1 + \gamma^2) \left( \frac{K p_2 p_3}{2p_1} \right)^2 \right] + \mathcal{H}_M V = 0, \quad (2.7) \]
where
\[ V = \sqrt{|p_1 p_2 p_3|}, \quad (2.8) \]
denotes the physical volume of the cell $\mathcal{V}$. $\epsilon = \pm 1$ depending on whether the frame $\hat{e}^a_i$ is right or left handed. Without any loss of generality, we will choose the orientation to be positive in the following. The Hamiltonian for the matter contribution $\mathcal{H}_M$ is proportional to the matter energy density
\[ \mathcal{H}_M = \rho_M V. \quad (2.9) \]

The triads $p_i$ are related to the directional scale factors as
\[ p_1 = l_2 l_3 a_2 a_3, \quad p_2 = l_1 l_3 a_1 a_3, \quad p_3 = l_2 l_1 a_1 a_1, \quad (2.10) \]
assuming $a_i > 0$, i.e. $p_i > 0$ with the positive orientation of the triads.

The relation between the phase space variables $c_i$ and the metric variables are [6]
\[ c_1 = \gamma l_1 a_1 H_1 + \frac{K a_1^2}{2 a_2 a_3 l_2 l_3}, \quad (2.11a) \]
\[ c_2 = \gamma l_2 a_2 H_2 - \frac{K a_2^2}{2 a_3 l_3}, \quad (2.11b) \]
\[ c_3 = \gamma l_3 a_3 H_3 - \frac{K a_3^2}{2 a_2 l_2}, \quad (2.11c) \]
in terms of the Hubble parameters
\[ H_i = \frac{\dot{a}_i}{a_i}, \quad i = 1, 2, 3, \quad (2.12) \]
where the 'dot' represents the derivative with respect to the harmonic time.

### B. Einstein’s equation

Einstein’s equations are derived from Hamilton’s equations:
\[ \dot{p}_i = \{p_i, \mathcal{H}_{cl}\} = -\kappa \gamma \frac{\partial \mathcal{H}_{cl}}{\partial c_i}, \quad \dot{c}_i = \{c_i, \mathcal{H}_{cl}\} = \kappa \gamma \frac{\partial \mathcal{H}_{cl}}{\partial p_i}, \quad (2.13) \]
where $\kappa = 8\pi G$.

Using the explicit form of the Hamiltonian $\mathcal{H}_{cl}$ we have the following equations $[2, 6]$: 
\[ \dot{p}_1 = \gamma (p_1 p_2 + p_1 p_3 + K p_3), \quad (2.14a) \]
\[ \dot{p}_2 = \gamma (p_2 p_1 + p_2 p_3), \quad (2.14b) \]
\[ \dot{p}_3 = \gamma (p_3 p_1 + p_3 p_2), \quad (2.14c) \]
\[ \dot{c}_1 = -\gamma \left( p_2 c_1 c_2 + p_3 c_1 c_3 + \frac{1}{2p_1} (1 + \gamma^2) \left( \frac{K p_3}{p_1} \right)^2 \right) + \kappa \gamma p_2 p_3 \left( \rho_M + p_1 \frac{\partial \rho_M}{\partial p_1} \right), \quad (2.15a) \]
\[ \dot{c}_2 = -\gamma \left( p_1 c_2 c_1 + p_3 c_2 c_3 + K p_3 c_1 - \frac{1}{2p_2} (1 + \gamma^2) \left( \frac{K p_2}{p_1} \right)^2 \right) + \kappa \gamma p_1 p_3 \left( \rho_M + p_2 \frac{\partial \rho_M}{\partial p_2} \right), \quad (2.15b) \]
\[ \dot{c}_3 = -\gamma \left( p_1 c_3 c_1 + p_2 c_3 c_2 + K p_2 c_1 - \frac{1}{2p_3} (1 + \gamma^2) \left( \frac{K p_3}{p_1} \right)^2 \right) + \kappa \gamma p_1 p_2 \left( \rho_M + p_3 \frac{\partial \rho_M}{\partial p_3} \right). \quad (2.15c) \]

From the above equations of motion it can be observed that the classical solutions possess the following constants of motion:
\[ c_1 p_1 + c_2 p_2 := C_{12}, \quad (2.16a) \]
\[ c_1 p_1 + c_3 p_3 := C_{13}, \quad (2.16b) \]
\[ c_3 p_3 - c_2 p_2 = C_{32} = C_{13} - C_{12}, \quad (2.16c) \]

with \( C_{12}, C_{13} \) constants. These equations allow us to find exact analytically solutions for \( p_1 \) and \( p_2 \):
\[ \dot{p}_2 = \gamma^{-1} p_2 (c_1 p_1 + c_3 p_3) = \gamma^{-1} p_2 C_{13} \quad \Rightarrow \quad p_2 = p_2^0 \exp \left( \frac{C_{13} t}{\gamma} \right), \quad (2.17a) \]
\[ \dot{p}_3 = \gamma^{-1} p_3 (c_1 p_1 + c_2 p_2) = \gamma^{-1} p_3 C_{12} \quad \Rightarrow \quad p_3 = p_3^0 \exp \left( \frac{C_{12} t}{\gamma} \right), \quad (2.17b) \]
with \( p_2^0, p_3^0 \) the initial values at \( t = 0 \) of these variables.

**C. Cosmic strings in the presence of a magnetic field**

In our model the matter density \( \rho_M \) comprises the contribution of cosmological string density \( \rho_{\text{string}} \) and the energy density of the magnetic field
\[ \rho_M = \rho_{\text{string}} + \rho_{\text{mag}}. \quad (2.18) \]

The energy momentum tensor for a system of cosmic strings and magnetic field in a comoving coordinate system is given by
\[ T^{\nu}_{\mu} = \rho_{\text{string}} u_{\mu} u^{\nu} - \lambda x^{\mu} x^{\nu} + E^{\nu}_{\mu}. \quad (2.19) \]
where $\rho_{\text{string}}$ is the rest energy density of strings with massive particles attached to them [8]. It can be expressed as

$$\rho_{\text{string}} = \rho_p + \lambda,$$

(2.20)

where $\rho_p$ is the rest energy of the particles attached to the strings and $\lambda$ is the tension density of the system of strings. The four velocities $u_i$ and the direction of the string $x_i$ obey the relations

$$u_\mu u^\mu = -x_\mu x^\mu = -1, \quad u_\mu x^\mu = 0.$$

(2.21)

The electromagnetic field $E_{\mu \nu}$ is taken in the form given by Lichnerowich [9]. We assume that the magnetic field is homogeneous aligned along the $z$-direction, $B_\mu \sim B_3 \delta_\mu^3$ and consequently the energy density of the magnetic field is [4, 10]

$$\rho_{\text{mag}} = J^2 a_1 a_2^2 a_3^2,$$

(2.22)

where $J$ is a constant and $\mu$ is the magnetic permeability of the medium. Typically $\mu$ differs from unity only by a few parts in $10^{-5}$ and in our numerical simulations we shall take $\mu = 1$.

Taking into account the conservation of the energy-momentum tensor, i.e., $T^\nu_{\mu \nu} = 0$, after a little manipulation one obtains [5, 10, 11]:

$$\dot{\rho}_{\text{string}} + \dot{V} \rho_{\text{string}} - \frac{\dot{a}_1}{a_1} \lambda = 0.$$

(2.23)

Here we take into account that the conservation law for magnetic field fulfills identically. Usually it is assumed that $\rho_{\text{string}}$ and $\lambda$ are proportional [8]:

$$\rho_{\text{string}} = \alpha \lambda,$$

(2.24)

where the constant $\alpha$ is 1 for the so called geometric string, greater than 1 for Takabayasi string and $-1$ for Reddy string.

The solution of (2.23) with the proportionality relation (2.24) is

$$\rho_{\text{string}} = Ra_1^{1-\alpha} a_2^{-1} a_3^{-1},$$

(2.25)

with $R$ a constant of integration.

**III. EFFECTIVE DYNAMICS WITHIN LQC**

In LQC the connection variables $c_i$ do not have direct quantum analogues and are replaced by holonomies. The quantum effects are incorporated in the effective Hamiltonian, $\mathcal{H}_{\text{eff}}$, constructed from the classical one, $\mathcal{H}_{\text{cl}}$, by replacing the connection components $c_i$ with sine functions:

$$c_i \rightarrow \frac{\sin(\bar{\mu}_i c_i)}{\bar{\mu}_i}.$$

(3.1)

where $\bar{\mu}_i$ are real valued functions of the triad coefficients $p_i$.

In what follows we shall use the so called $\bar{\mu}'$-scheme in which the parameters $\bar{\mu}_i'$ are chosen as follows [12, 13]:

$$\bar{\mu}_1' = \sqrt{\frac{p_1 \Delta}{p_2 p_3}}, \quad \bar{\mu}_2' = \sqrt{\frac{p_2 \Delta}{p_1 p_3}}, \quad \bar{\mu}_3' = \sqrt{\frac{p_3 \Delta}{p_1 p_2}},$$

(3.2)
where $\Delta = 4\sqrt{3}\pi \gamma l_{Pl}^2$ is the area gap in the LQC with the Planck length $l_{Pl} := \sqrt{\frac{G\hbar}{c^3}}$.

The equations of motion (2.13) that incorporate loop quantum modifications (3.1) are deduced accordingly [2, 6, 7]. For the equations for $c_i$ in r.h.s. we shall take into account the contribution of the matter density (2.18).

The equations for the effective theory are given by Poisson brackets with the Hamiltonian constraint ($\mathcal{H}_{eff} = 0$):

$$\dot{p}_1 = \frac{\gamma}{\bar{\mu}_1^2} (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3 + \eta) \cos \bar{\mu}_1 c_1, \quad (3.3a)$$

$$\dot{p}_2 = \frac{\gamma^2}{\bar{\mu}_2^2} (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_3 c_3) \cos \bar{\mu}_2 c_2, \quad (3.3b)$$

$$\dot{p}_3 = \frac{\gamma^2}{\bar{\mu}_3^2} (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_2 c_2) \cos \bar{\mu}_3 c_3, \quad (3.3c)$$

$$\dot{c}_1 = -\frac{p_2 p_3}{2\gamma \Delta} \left[ 2 (\sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3) + \mu_1 \cos \bar{\mu}_1 c_1 (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3) - \bar{\mu}_2 \cos \bar{\mu}_2 c_2 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_3 c_3) - \mu_3 \cos \bar{\mu}_3 c_3 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_2 c_2) + \eta^2 (1 + \gamma^2) + \eta (\mu_1 \cos \bar{\mu}_1 c_1 - \sin \bar{\mu}_1 c_1) \right] + R \frac{\alpha - 1}{2\alpha} p_1 \frac{1 - \alpha}{2\alpha} p_2 \frac{1 + \alpha}{2\alpha} p_3 + \frac{\gamma^2}{2\mu} p_1 p_2 p_3^{-1}, \quad (3.4)$$

$$\dot{c}_2 = -\frac{p_1 p_3}{2\gamma \Delta} \left[ 2 (\sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3) - \mu_1 \cos \bar{\mu}_1 c_1 (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3) + \bar{\mu}_2 \cos \bar{\mu}_2 c_2 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_3 c_3) - \mu_3 \cos \bar{\mu}_3 c_3 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_2 c_2) + \eta^2 (1 + \gamma^2) - \eta (\mu_1 \cos \bar{\mu}_1 c_1 - 3 \sin \bar{\mu}_1 c_1) \right] + R \frac{\alpha + 1}{2\alpha} p_1 \frac{1 - \alpha}{2\alpha} p_2 \frac{1 + \alpha}{2\alpha} p_3 + \frac{\gamma^2}{2\mu} p_1 p_3^{-1}, \quad (3.5)$$

$$\dot{c}_3 = -\frac{p_1 p_2}{2\gamma \Delta} \left[ 2 (\sin \bar{\mu}_1 c_1 \sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_1 c_1 \sin \bar{\mu}_3 c_3 + \sin \bar{\mu}_2 c_2 \sin \bar{\mu}_3 c_3) - \mu_1 \cos \bar{\mu}_1 c_1 (\sin \bar{\mu}_2 c_2 + \sin \bar{\mu}_3 c_3) - \bar{\mu}_2 \cos \bar{\mu}_2 c_2 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_3 c_3) + \mu_3 \cos \bar{\mu}_3 c_3 (\sin \bar{\mu}_1 c_1 + \sin \bar{\mu}_2 c_2) + \eta^2 (1 + \gamma^2) - \eta (\mu_1 \cos \bar{\mu}_1 c_1 - 3 \sin \bar{\mu}_1 c_1) \right] + R \frac{\alpha + 1}{2\alpha} p_1 \frac{1 - \alpha}{2\alpha} p_2 \frac{1 + \alpha}{2\alpha} p_3 + \frac{\gamma^2}{2\mu} p_1 p_2 p_3^{-2}. \quad (3.6)$$

where $\eta = K \sqrt{\Delta p_{Pl}^{-3} p_{Pl}^{-2}}$. The complexity of these equations does not allow for analytic solutions and imposes numerical simulations. However, taking into account that $\sin \theta$ is bounded by 1 for all $\theta$ it is possible to infer an upper bound for the density of matter [2]:

$$\rho_M \leq \frac{3 + (1 + \gamma^2)^{-1}}{8\pi G \gamma^2 \Delta} \approx 0.54 \rho_{Pl}. \quad (3.7)$$

Let us note that in the case of BI cosmology, the matter density is bounded by 0.41 $\rho_{Pl}$.
In our numerical analysis we choose the units: $h = c = G = 1$, $\gamma = 0.2375$, $l_1 = l_2 = l_3 = 1$, $\epsilon = 1$, $\mu = 1$, $K = 1$. The graphics are plotted as functions of the harmonic time (lapse function $N = \sqrt{|p_1 p_2 p_3|}$).

In order to compare the classical solutions with those of the effective equations we investigate the evolution of the directional Hubble parameters (2.12), the matter density (2.18), the volume (2.8) and the shear

$$\Sigma^2 = \frac{1}{6}[(H_1 - H_2)^2 + (H_2 - H_3)^2 + (H_1 - H_3)^2]. \quad (4.1)$$

In the figures red solid line shows the volume scale $V = a_1 a_2 a_3$, blue dashed line presents the density of matter $\rho_M$, and black dotted lines is for the shear $\Sigma^2$.

The initial conditions for effective and classical solutions should be chosen in accordance with the Hamiltonian constraints. In order to simplify the analysis we choose as initial conditions $c_2(t_0) = c_3(t_0) = c_0$ and $p_1(t_0) = p_2(t_0) = p_3(t_0) = p_0$ at the initial moment $t = t_0$. In Fig. 1 we plot the admissible values of $c_0, p_0$ which provide an acceptable value for the initial data $c_1(t_0)$.

First of all we choose the initial conditions corresponding to a classically collapsing universe approaching the classical big bang singularity. For this purpose we take all initial $c_i$ as being negative. In Fig. 2 we have plotted the classical evolution of volume scale $V$, matter density $\rho_M$ and shear $\Sigma^2$ for a positive proportionality constant $\alpha = +1$ with nonvanishing magnetic field ($\mathcal{F} \neq 0$) and string density ($\epsilon \neq 0$). The Fig. 3 shows quantum analog of Fig. 2. The Fig. 4 is the classical picture of evolution with a negative proportionality constant $\alpha = -1$. The quantum counterpart of Fig. 4 is given in Fig. 5. As it is seen from the Figs. 2 - 5 while the classical evolution ends in Big Crunch in quantum case the spacetime avoids singularity and after attaining some minimum value the Universe again begins to expand. It should be noted that the qualitative pictures of evolution are almost the same for positive or negative values of $\alpha$. Accordingly, in what follows we shall present only the plots for a single value of the parameter $\alpha$, namely $\alpha = +1$.

In Figs. 6 and 7 we present the effective quantum evolution of the BII universe with the initial conditions $c_0$ and $c_1(t_0)$ of different signs. We remark the appearance of a few oscillations in the vicinity of the classical singularity and from which the solutions evolve smooth in time.

In Fig. 8 we present the evolution of the classical evolution of volume scale $V$, matter density $\rho_M$ and shear $\Sigma^2$ choosing all initial conditions $c_1(t_0) > 0$. Their quantum counterpart is plotted in Fig. 9. Again we get that the classical BII universe evolves from a singularity, while in the LQC approach all observable are finite. In the vicinity of the classical singularity the effective solution presents a bounce and subsequently the evolution is similar with the classical one.

V. CONCLUSIONS

In this paper we analyzed the numerical solutions of the effective equations of the LQC dynamics for BII model. We have extended the effective LQC treatment of BII cosmologies by including cosmic strings and a homogeneous magnetic field.

We considered the analytical and numerical solutions of BII model at the classical and effective level. The main objective of the paper was the investigation how the classical big bang singularity is resolved and how the effective equations evolve. For this purpose we chose as a set of observable quantities like volume, string density, shear expansion and studied their evolution numerically. We showed that a big bounce occurs in a collapsing magnetized BII string universe, thus extending the known cosmological models of singularity avoidance. After the bounce the universe enters a classical regime.

The numerical simulations are quite sensitive to the initial conditions regarding the directional Hubble parameters. Choosing a negative $c_0$, the numerical simulations are appropriate to describe the evolution towards the big bang singularity. In the classical case the evolution stops at the singularity, while in the quantum case we have bounces and the singularity is eluded. On the other hand the choice of the initial conditions with a positive $c_0$ is adequate to have in view the evolution
after the big bang. In the classical case, starting from the vicinity of the singularity, the volume of the universe tends to infinity, while the string density is smaller and smaller. In the quantum case, after a few small bounces we have for large time the same behavior as in the classical case.

As a final comment regarding the numerical simulations we note that they are stable in respect of reasonable variations of the parameters describing the cosmic strings.

The study of anisotropic models with different kinds of matter sources in the framework of LQC deserves further investigations. These studies will contribute to answer the challenge question if the bouncing non-singular behavior of the effective solutions is generic.

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FIG. 6. Quantum case for $c_0 = 4$, $J = 0.5$, $R = 0.2$, $\alpha = +1$ and $c_1(t_0) = -0.4993069119$.

FIG. 7. Quantum case for $c_0 = -1$, $J = 0.5$, $R = 0.2$, $\alpha = +1$ and $c_1(t_0) = 3.535164226$.

FIG. 8. Classical case for $c_0 = 4$, $J = 0.5$, $R = 0.2$, $\alpha = +1$ and $c_1(t_0) = 0.8576784788$.

FIG. 9. Quantum case for $c_0 = 4$, $J = 0.5$, $R = 0.2$, $\alpha = +1$ and $c_1(t_0) = 3.589017650$.

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