COMPLEXES OF BIPARTITE GRAPHS, NEIGHBORHOOD COMPLEXES, AND BOX COMPLEXES

TAKAHIRO MATSUSHITA

Abstract. Neighborhood complexes and box complexes of graphs were constructed in the context of the graph coloring problem. In this paper, we investigate the relationships among graphs, their Hom complexes Hom\((K_2, G)\), and their neighborhood complexes. We prove that for graphs \(G\) and \(H\) having no isolated vertices, \(K_2 \times G \cong K_2 \times H\) if and only if \(\text{Hom}(K_2, G) \cong \text{Hom}(K_2, H)\) as posets. And \(G \cong H\) if and only if \(\text{Hom}(K_2, G) \cong \text{Hom}(K_2, H)\) as \(\mathbb{Z}_2\)-posets. In the proof of this fact, we construct a poset \(B_0(X)\) for a bipartite graph \(X\) satisfying \(\text{Hom}(K_2, G) \cong B_0(K_2 \times G)\) for a graph \(G\) as posets. As an application, we prove that there are connected graphs \(G\) and \(H\) such that \(\chi(G) \not\cong \chi(H)\), but \(\text{Hom}(K_2, G)\) and \(\text{Hom}(K_2, H)\) are isomorphic as posets, and their neighborhood complexes are isomorphic.

1. Introduction

Neighborhood complexes and box complexes of graphs were constructed in the context of the graph coloring problem. It is known that the homotopy types of neighborhood complexes and \(\mathbb{Z}_2\)-homotopy types of box complexes are closely related to the chromatic numbers of graphs. By using neighborhood complexes, Lovász determined the chromatic numbers of Kneser graphs. Box complexes are the \(\mathbb{Z}_2\)-complexes which were constructed in [1] for hypergraphs, and investigated in [3], [13], and [15].

In this paper, we investigate the relationships among graphs, the Hom complex \(\text{Hom}(K_2, G)\) from \(K_2\), and their neighborhood complexes. One of the main results obtained in this paper is the following theorem. This theorem states that the poset structure of \(\text{Hom}(K_2, G)\) is closely related with the bipartite graph \(K_2 \times G\), where “\(\times\)” implies the categorical product of graphs.

Theorem 1.1. Let \(G\) and \(H\) be graphs having no isolated vertices. Then the followings hold.

1. \(K_2 \times G \cong K_2 \times H\) if and only if \(B(G)\) is isomorphic to \(B(H)\) as posets.
2. \(G \cong H\) if and only if \(\text{Hom}(K_2, G)\) is isomorphic to \(\text{Hom}(K_2, H)\) as \(\mathbb{Z}_2\)-posets.

The case of the neighborhood complex is a little complicated.

Theorem 1.2. Let \(G\) and \(H\) be graphs having no isolated vertices. If \(K_2 \times G \cong K_2 \times H\), then \(N(G) \cong N(H)\) as simplicial complexes. On the other hand, if \(G\) and \(H\) are locally finite stiff graphs, then \(N(G) \cong N(H)\) implies \(K_2 \times G \cong K_2 \times H\).

To prove the above theorem, we construct a poset \(B_0(G)\) for a bipartite graph \(G\) which satisfies \(B_0(K_2 \times G) \cong \text{Hom}(K_2, G)\) for a graph \(G\) as posets. The important
property of the complex $B_0(G)$ is that its poset structure determines the graph up to isolated vertices, see Theorem 4.3.

By the application of the above theorems, we can construct examples of graphs which have different chromatic numbers, but whose neighborhood complexes are isomorphic.

**Theorem 1.3.** (Example 4.13) Let $m$ and $n$ be positive integers greater than 2. Then there are finite connected graphs $G$ and $H$ such that $\chi(G) = m$, $\chi(H) = n$, $N(G) \cong N(H)$ as simplicial complexes, and $\text{Hom}(K_2, G) \cong \text{Hom}(K_2, H)$ as posets.

Recall that Lovász proved in [9] that for a connected graph $G$, $N(G)$ is disconnected and non-empty if and only if $\chi(N(G)) = 2$. Namely, the $\chi(G) \neq 2$ is the topological property of the neighborhood complex. Lovász suggested in [9] a question asking whether there is a topological invariant of $G$ which is equivalent to the $m$-colorability. Theorem 1.3 suggested the negative answer to this question. On the other hand, Theorem 1.1.(2) implies that the $\mathbb{Z}_2$-poset $B(G)$ has all informations about the graph $G$. So we can expect that the colorability can be characterized from the viewpoint of $\mathbb{Z}_2$-posets.

We should remark that there is no $\mathbb{Z}_2$-homotopy invariant of $B(G)$ which is equivalent to the $m$-colorability of $G$. The author constructed in [11] a graph homomorphism $f : G \to H$ such that the $\mathbb{Z}_2$-map $B(G) \to B(H)$ induced by $f$ is a $\mathbb{Z}_2$-homotopy equivalence, but $\chi(G) \neq \chi(H)$, and, moreover, this difference can be arbitrarily large.

This paper is organized as follows. In Section 2, we review definitions and facts we need in this paper. In Section 3, we review the relationships about odd involutions of bipartite graphs and the double cover $K_2 \times G$ for a graph $G$. In Section 4, we construct the box complex $B_0(G)$ and prove main theorems. In Section 5, we observe that some deformations of $G$ do not change the homotopy type of $B_0(G)$. In Section 6, we construct the other complexes related to Hom complexes. In Section 7, we consider the case of box complexes investigated in [5] and [13].

2. Preliminaries

A graph we say in this paper is a pair $(V, E)$ where $V$ is a set and $E$ is a symmetric subset of $V \times V$, that is, $(x, y) \in E$ implies $(y, x) \in E$ for any $(x, y) \in V \times V$. For a graph $G = (V, E)$, we write $V(G)$ for $V$, called the vertex set of $G$, and $E(G)$ for $E$. A graph homomorphism from a graph $G$ to a graph $H$ is a map $f : V(G) \to V(H)$ such that $(f \times f)(E(G)) \subset E(H)$. We do not assume that $V(G)$ is finite. We write $\mathbb{N}$ for the set of all non-negative integers. For $n \in \mathbb{N}$, we write $K_n$ for the complete graph with $n$-vertices. The chromatic number of a graph $G$ is defined by the number

$$\chi(G) = \inf \{ n \in \mathbb{N} \mid \text{There is a graph homomorphism } G \to K_n. \}.$$ 

A simplicial complex is a pair $(V, \Delta)$ where $V$ is a set and $\Delta$ is a family of finite subsets of $V$ satisfying the followings:

1. For any $v \in V$, $\{v\} \in \Delta$.
2. Let $\sigma \in \Delta$ and $\tau \in 2^V$. If $\tau \subset \sigma$, then $\tau \in \Delta$.
For a simplicial complex \((V, \Delta)\), \(V\) is called the vertex set of \((V, \Delta)\). We often abbreviate the vertex set \(V\), and say “\(\Delta\) is a simplicial complex”. In this terminology, we write \(V(\Delta)\) for the vertex set of the simplicial complex \(\Delta\). For simplicial complexes \(\Delta_1\) and \(\Delta_2\), a map \(f : V(\Delta_1) \to V(\Delta_2)\) such that \(f(\sigma) \in \Delta_2\) for any \(\sigma \in \Delta_1\) is called a simplicial map.

A partially ordered set is called a poset. A subposet \(Q\) of a poset \(P\) is called an induced subposet if for any \(x, y \in Q\), \(x \leq y\) in \(Q\) if and only if \(x \leq y\) in \(P\). A chain of a poset \(P\) is a subset of \(P\) whose order induced by \(P\) is totally ordered. For a poset \(P\), we write \(\Delta(P)\), called the order complex of \(P\), for the simplicial complex whose simplices are all finite chains of \(\sigma \in \Delta_1\) is called a simplicial map.

There is a functor from the category of simplicial complexes to topological space, called geometric realization, see \([8]\) for the precise definition. We write \(|\Delta|\) for the geometric realization of a simplicial complex \(\Delta\). For a poset \(P\), we write \(|P|\) for the geometric realization of \(P\). We use topological terminologies for simplicial complexes and posets via taking geometric realization. For example, a simplicial map \(f : \Delta_1 \to \Delta_2\) is called a homotopy equivalence if the map \(|\Delta_1| \to |\Delta_2|\) induced by \(f\) is a homotopy equivalence.

**Theorem 2.1.** (Quillen’s lemma \([14]\)) Let \(f : P \to Q\) be an order preserving map. If \(f^{-1}(Q_{\leq x})\) is contractible for any \(x \in Q\), then \(f\) is a homotopy equivalence.

For the proof of the above, see \([4, 12, 13]\). The case of finite posets is proved in \([3]\) and \([8]\). But we need the case of infinite posets.

Let \(G\) be a graph. For \(v \in V(G)\), we write \(N(v)\) or \(N^G(v)\) for the subset \(\{w \in V(G) \mid (v, w) \in E(G)\}\). A vertex \(v \in V(G)\) is said to be isolated if \(N(v) = \emptyset\). A graph \(G\) is said to be locally finite if \(N(v)\) is finite for any \(v \in V(G)\). The neighborhood complex \(N(G)\) of \(G\) is a simplicial complex

\[N(G) = \{\sigma \subset V(G) \mid \sigma \text{ is finite and there is } v \in V(G) \text{ with } \sigma \subset N(v)\}.\]

It is known that \(B(G)\) and \(N(G)\) homotopy equivalent, which is naturally with respect to \(G\). Lovász proved the following theorem, and using this, he determined the chromatic number of graphs.

**Theorem 2.2.** Let \(G\) be a graph and \(n \geq -1\). If \(N(G)\) is \(n\)-connected\(^1\), then \(\chi(G) \geq n + 3\).

Next we recall the Hom complex following \([2, 8, 6]\). Our definition of Hom complex is due to \([6]\). Let \(G\) and \(H\) be graphs. A multi-homomorphism from \(G\) to \(H\) is a map \(\eta : V(G) \to 2^{V(H)} \setminus \{\emptyset\}\) such that \(\eta(v) \times \eta(w) \subset E(H)\) for \((v, w) \in E(G)\). The Hom complex \(\text{Hom}(G, H)\) from \(G\) to \(H\) is the poset of all multi-homomorphisms from \(G\) to \(H\) with the order \(\eta \leq \eta' \iff \eta(v) \subset \eta'(v)\) for any \(v \in V(G)\). It is obvious that the box complex we say in this paper is isomorphic to \(\text{Hom}(K_2, G)\).

In the case of infinite graphs, it should be noticed that this definition of Hom complex \(\text{Hom}(G, H)\) is not homeomorphism given in some references, for example, \([2, 7, 8]\). Kozlov defined the Hom complex as a cell complex of a direct product of canonical simplexes with some properties, and its face poset is isomorphic to the induced subposet

\[\text{Hom}_f(G, H) = \{\eta \in \text{Hom}(G, H) \mid \sum_{v \in V(G)} (2\eta(v) - 1) < \infty\}.\]

\(^1\)(-1)-connected means non-empty.
It can be proved that the inclusion $\text{Hom}_f(G,H) \to \text{Hom}(G,H)$ is a homotopy equivalence, see [12] for the proof.

We define the poset $B(G)$ as

$$B(G) = \{ (\sigma, \tau) \mid \sigma, \tau \neq \emptyset, \sigma \times \tau \subseteq E(G) \}.$$  

with the ordering $(\sigma, \tau) \leq (\sigma', \tau')$ if and only if $\sigma \subseteq \sigma'$ and $\tau \subseteq \tau'$. Obviously $B(G) \cong \text{Hom}(K_2,G)$. $B(G)$ has the natural involution $(\sigma, \tau) \mapsto (\tau, \sigma)$, and becomes a free $\mathbb{Z}_2$-complex. It is known that $B(G)$ is naturally homotopy equivalent to the neighborhood complex $N(G)$ of $G$.

Notice that this $B(G)$ is not equal to the box complex investigated in [5] and [13] although they are the same notation.

3. Odd involutions of bipartite graphs

The double cover $K_2 \times G$ over a graph $G$ often appears and have the fundamental role in this paper. The graph $K_2 \times G$ is closely related to the odd involution of a bipartite graph. So, in this section, we investigate odd involutions of bipartite graphs, and the relation of a graph $G$ and $K_2 \times G$.

A graph $G$ is said to be bipartite if $\chi(G) \leq 2$. Let $G$ be a bipartite graph. Then there are two subsets $A, B$ of $G$ satisfying the following.

- $A$ and $B$ are independent, that is, $(A \times A) \cap E(G) = (B \times B) \cap E(G) = \emptyset$.
- $A \cap B = \emptyset$ and $A \cup B = V(G)$.

For a bipartite graph $G$, an ordered pair $(A, B)$ of $V(G)$ satisfying the above properties is called a 2-partition of $G$. If $G$ is connected (in this paper, we assume that connected graphs are non-empty), then the unordered pair $\{A, B\}$ of a 2-partition $(A, B)$ of $G$ is uniquely determined.

For a non-negative integer $n$, the graph $L_n$ is defined by $V(L_n) = \{0, 1, \ldots, n\}$ and $E(L_n) = \{ (x, y) \mid |x - y| = 1 \}$. A path $\gamma$ of a graph $G$ with length $n$ is a graph homomorphism from $L_n$ to $G$, and $\gamma(0)$ is the initial point of $\gamma$ and $\gamma(n)$ is the terminal point of $\gamma$. Let $G$ be a bipartite graph with decomposition $\{A, B\}$, and let $x \in A$. Then for any $y \in V(G)$ belonging to the same connected component of $x$, then the length of any path connects $x$ to $y$ is even if and only if $y \in A$, and is odd if and only if $y \in B$.

An involution of a graph $G$ is a graph homomorphism $\tau : G \to G$ such that $\tau^2 = \text{id}_G$. Then the involution determines the $\mathbb{Z}_2$-action on the graph. For a graph $G$ with its involution $\tau$, we write $G/\tau$ for the quotient $G/\mathbb{Z}_2$, that is, $V(G/\tau) = \{ \{v, \tau(v)\} \mid v \in V(G) \}$, and $E(G/\tau) = \{ (\alpha, \beta) \mid (\alpha \times \beta) \cap E(G) \neq \emptyset \}$.

**Definition 3.1.** Let $G$ be a bipartite graph. An involution $\tau$ of $G$ is said to be odd if for any $v \in V(G)$, $\tau(v)$ is connected with a path with odd length, or is not contained in the connected component $v$ belongs to.

The central example of odd involution is that for a graph $G$, the involution of $K_2 \times G$ defined by

$$(1, v) \leftrightarrow (2, v) \ (v \in V(G)).$$

The following lemma shows that all odd involutions can be written as this form.

**Lemma 3.2.** Let $G$ be a bipartite graph with an odd involution $\tau$. Then we have $K_2 \times (G/\tau) \cong G$ as graphs.
Then we have

\[ I_0 = \{ i \in I \mid \tau(G_i) = G_i \}, \]

\[ I_1 = \{ i \in I \mid \tau(G_i) = G_j, (j \neq i), \text{ and } \tau(A' \cap V(G_i)) \subset B' \}, \]

\[ I_2 = \{ i \in I \mid \tau(G_i) = G_j, (j \neq i), \text{ and } \tau(A' \cap V(G_i)) \subset B' \}. \]

Then we have \( I = I_0 \cup I_1 \cup I_2 \). \( \tau \) defines an involution on \( I \) defined by \( G_{\tau(i)} = \tau(G_i) \), and \( I_0, I_1, \) and \( I_2 \) are closed under this involution. Write \( I_2/\tau \) for the orbit set of this \( \mathbb{Z}_2 \)-action on \( I_2 \). For each \( \alpha \in I_2/\tau \), let \( i_\alpha \in \alpha \). Let \( J \) denote the subset \( I \) defined by \( J = I \setminus \{ i_\alpha \mid \alpha \in I_2/\mathbb{Z}_2 \} \). Put the 2-partition \((A, B)\) by

\[ A = \coprod_{j \in J} (A' \cap V(G_j)) \cup \coprod_{\alpha \in I_2/\tau} (B' \cap V(G_{i_\alpha})), \]

\[ B = \coprod_{j \in J} (B' \cap V(G_j)) \cup \coprod_{\alpha \in I_2/\tau} (A' \cap V(G_{i_\alpha})). \]

Then we have that \( \tau(A) \subset B \) as desired.

Let \( \varepsilon : G \to K_2 \) be a graph homomorphism defined by \( \varepsilon^{-1}(1) = A \) and \( \varepsilon^{-1}(2) = B \). Let \( \tau : G \to G/\tau \) denote the quotient. Define the graph homomorphism \( f : G \to K_2 \times (G/\tau) \) by \( f(v) = (\varepsilon(v), \pi(v)) \). On the other hand, define the graph homomorphism \( g : K_2 \times (G/\tau) \to G \) by \( g((i, \{ v, \tau(v) \})) = \varepsilon^{-1}(i) \cap \{ v, \tau(v) \} \).

First we prove that the map \( g \) is a graph homomorphism. Let \((1, \alpha) \sim (2, \beta)\) in \( K_2 \times (G/\tau) \). Let \( v, w \in V(G) \) such that \( \{ v \} = A \cap \alpha \) and \( \{ w \} = A \cap \beta \). Then \( g(1, \alpha) = v \), and \( g(2, \beta) = \tau(w) \). Remark that \( v \not\sim w \) and \( \tau(v) \not\sim \tau(w) \) in \( G \) since \( v, w \) (or \( \tau(v), \tau(w) \)) is in the same part of the 2-partition. Remark that \( (v, \tau(w)) \in E(G) \) if and only if \( (\tau(v), w) \) since \( \tau \) is a graph homomorphism and \( \tau^2 = \text{id} \). Since \( \alpha \) is adjacent to \( \beta \) in \( G/\tau \), we have \( (v, \tau(w)) \in G \). This proves that \( g \) is a graph homomorphism.

Let us show that \( g \) is the inverse of \( f \). Let \( v \in V(G) \). Then \( \{ g \circ f(v) \} = \{ g(\varepsilon(v), \pi(v)) \} = \varepsilon^{-1}(\varepsilon(v)) \cap \pi(v) = \{ v \} \), and hence we have \( g \circ f(v) = v \). Next let \( \alpha V(G/\tau) \). Let \( v \in V(G) \) with \( \{ v \} = \alpha \cap A \). Then we have

\[ f \circ g(1, \alpha) = f(v) = (1, \alpha), \]

\[ f \circ g(2, v) = f(\tau(v)) = (2, \alpha). \]

Hence \( f \circ g = \text{id} \). This completes the proof. \( \square \)

Remark 3.3. Let \( G \) denote the category of graphs, and \( G' \) the category of bipartite graphs with odd involution. Then we have functors \( F : G \to G' \), \( F'G \to K_2 \times G \), and \( G' \to G, (G, \tau) \to G/\tau \). But these are not categorical equivalence. \( F \) is faithful and essentially surjective, but is not full. Indeed, for a graph \( G \), there is no graph homomorphism \( f : G \to G \) such that the involution of \( K_2 \times G \). On the other hand, \( F' \) is full and essentially surjective, but is not faithful.

To obtain a category which is categorical equivalent to \( G' \), we need to assume that more structure on the bipartite graphs with odd involutions. Let \( G'' \) be the category of triples \((G, \varepsilon, \tau)\) where \( \varepsilon : G \to K_2 \) is a graph homomorphism and \( \tau \) is an involution of \( G \) such that \( \varepsilon(\tau(v)) = \neq \varepsilon(v) \) for \( v \in V(G) \). The morphism from \((G, \varepsilon, \tau)\) to \((G', \varepsilon', \tau')\) is a graph homomorphism \( f : G \to G' \) such that \( \varepsilon' \circ f = f \circ \varepsilon \)
and \( \tau' \circ f = f \circ \tau \). Then this category \( \mathcal{G}' \) is categorical equivalent to \( \mathcal{G} \) with \( \mathcal{G} \to \mathcal{G}' \), \( G \mapsto K_2 \times G \), and \( \mathcal{G}' \to \mathcal{G}, (G, \varepsilon, \tau) \mapsto G/\tau \).

4. Complexes of bipartite graphs

In this section, we construct a poset \( B_0(G) \) which is similar to a poset \( B(G) \), and prove main theorems announced in Section 1.

**Definition 4.1.** Let \( G \) be a bipartite graph. Then the poset \( B_0(G) \) is defined by

\[
B_0(G) = \{ \{X, Y\} \mid X \neq \emptyset, Y \neq \emptyset, X \times Y \subset E(G) \}
\]

with the order relation \( X \leq Y \) if for any \( X \in X \), there is \( Y \in Y \) such that \( X \subset Y \), where \( X, Y \in B_0(G) \).

Let \((A, B)\) be a 2-partition of \( G \). Then the induced subposet \[
\{ (\sigma, \tau) \mid \sigma \subset A \text{ and } \tau \subset B \}
\]
of \( B(G) \) is isomorphic to \( B_0(G) \). But we should regard that \( G \to B_0(G) \) is a functor from the category of bipartite graphs.

If \( \tau \) is an odd involution of a bipartite graph \( G \), then \( \tau \) induced an involution on \( B_0(G) \).

**Lemma 4.2.** Let \( G \) be a graph. Then \( B_0(K_2 \times G) \cong B(G) \) as \( \mathbb{Z}_2 \)-posets, where the involution of \( B_0(K_2 \times G) \) is the natural involution on \( K_2 \times G \).

**Proof.** It can be easily shown that for \( \{\sigma, \tau\} \in B_0(K_2 \times G) \), one of \( \{\sigma, \tau\} \) is in \( \{1\} \times V(G) \) and the other is in \( \{2\} \times V(G) \). In the proof of Theorem 6.1, we prove this more general situation.

Let \( p_2 : K_2 \times G \to G \) denote the 2nd projection. The map \( \Phi : B_0(K_2 \times G) \to B(G) \) is defined as that \( \Phi(\mathcal{X}) = (p_2(\sigma), p_2(\tau)) \) where \( \mathcal{X} = \{\sigma, \tau\} \) and \( \sigma \subset \{1\} \times V(G) \) and \( \tau \subset \{2\} \times V(G) \). On the other hand, a map \( \Psi : B(G) \to B_0(K_2 \times G) \) is defined by \( (\sigma, \tau) \mapsto \{\{1\} \times \sigma, \{2\} \times \tau\} \). Then it can be easily shown that \( \Phi \) and \( \Psi \) are the inverse of each other, and \( \Psi \) is a \( \mathbb{Z}_2 \)-equivariant map.

It is obvious that if \( G \cong H \), then \( B_0(G) \cong B_0(H) \) as posets. But the important fact is that the inverse also holds.

**Theorem 4.3.** Let \( G \) and \( H \) be bipartite graphs having no isolated vertices. If \( B_0(G) \cong B_0(H) \) as posets, then \( G \cong H \).

**Proof.** Let \( f \) be an isomorphism \( B_0(G) \to B_0(H) \) of posets.

**Claim 1.** Let \( \{\sigma, \tau\} \in B_0(G) \). Then \( \{\sigma, \tau\} \) is minimal if and only if there are \( v, v' \in V(G) \) such that \( \sigma = \{v\} \) and \( \tau = \{v'\} \). Especially, \( f(\{\{v\}, \{v'\}\}) \) is the form \( \{\{w\}, \{w'\}\} \) \((w, w' \in V(H))\).

**Claim 2.** Let \( v \in V(G) \) and suppose \( \sharp N(v) \geq 2 \). Then \( f(\{\{v\}, N(v)\}) \) is the form \( \{\{w\}, N(w)\} \) and \( \sharp N(v) = \sharp N(w) \) \((w \in V(H))\).

First we prove the above claims. Claim 1 is obvious. So we only give the proof of Claim 2.

Let \( v \in V(G) \). First suppose \( \sharp N(v) \geq 2 \). Then \( f(\{\{v\}, N(v)\}) \) is a form of \( \{w, N(w)\} \in B_0(H) \) with \( \sharp N(v) = \sharp N(w) \), and put \( \hat{f}(v) = w \). Let \( v \in V(G) \) such that \( \sharp N(v) \geq 2 \). Then \( B_0(G)_{\leq \hat{f}(v), N(v)} \) satisfies the following property \((*)\) of poset. We say that an element \( x \) of a poset \( X \) has the finite level if \( P_{\leq x} \) is finite.
(\ast) For any finite level element $x \in P$, $P_{\leq x}$ is isomorphic to the face poset of a $d$-simplex, where $d+1$ is the number of minimal elements of $P_{\leq x}$.

Since $f$ is an isomorphism of posets, $f(B_0(G) \leq \{v\},N(v)) = B_0(H) \leq f(\{v\},N(v))$ has the property (\ast). Put $\{\{v\},N(v)\}$ does not have the property (\ast). So we assume that $\sigma = \{w\}$. Then $\tau \subset N(w)$. Suppose $\tau \neq N(w)$. Then $\{\{w\},\tau\} < \{\{w\},N(w)\}$ and $B_0(H) \leq \{\{w\},N(w)\}$ has the property (\ast). Hence there is $v \in B_0(G)$ such that $\{v\},N(v)\} < \alpha$ and $B_0(G) \leq \alpha$ has the property (\ast). This is contradiction. Hence $\hat{\sigma} \hat{\tau}$. We can not say that $\hat{\sigma}$ is uniquely determined. This is contradiction. Hence $\hat{\sigma} \hat{\tau}$ completed the proof of $f(\hat{\sigma} \hat{\tau})$. This implies that $f(\{\{v\},\{v'\}\}) = \{\{w\},\{w'\}\}$. So we put $\hat{f}(v)$ one of $w$ and $w'$ and $\hat{f}(v)$ the other.

Next suppose $\hat{\tau} = \{v'\}$ and $\hat{\sigma} \hat{\tau} = 1$. Put $N(v) = \{v'\}$. Suppose $\hat{\tau} \hat{\sigma} \hat{\tau} = 1$. The minimal element of $B_0(G)$, so is $f(\{\{v\},\{v'\}\})$. Hence we can write $f(\{\{v\},\{v'\}\}) = \{\{w\},\{w'\}\} \in B_0(H)$. Since $\{\{w\},\{w'\}\} \leq f(\{\{v\},\{v'\}\}) = \{\hat{f}(v),\hat{f}(v')\}$, one of $w_1,w_2$ is equal to $\hat{f}(v)$. Put $\hat{f}(v)$ the one of $w,w'$ which is not $\hat{f}(v)$. This completes the construction of $f : V(G) \rightarrow V(H)$ since $G$ has no isolated vertices.

Let us show that $f$ is a graph homomorphism. Let $(v,v') \in E(G)$. We want to show that $(\hat{f}(v),\hat{f}(v')) \in E(H)$. Since the case $N(v) = \{v'\}$ is obvious from the definition of $f$, we only give the proof of $\hat{\tau} N(v) \geq 2$ and $\hat{\sigma} N(v') \geq 2$. Then we have

$$f(\{\{v\},N(v)\}) = \{\hat{f}(v),N(\hat{f}(v))\},$$

$$f(\{\{v'\},N(v')\}) = \{\hat{f}(v'),N(\hat{f}(v'))\},$$

and put $f(\{\{v\},\{v'\}\}) = \{\{w\},\{w'\}\}$. Since $\{\{w\},\{w'\}\} \in B_0(H)$, we have $(w,w') \in E(H)$. Since $\{\{w\},\{w'\}\} \leq \{\{v\},N(\hat{f}(v))\}$ and $\{\{w\},\{w'\}\} \leq \{\{f(v'),N(\hat{f}(v'))\}$, we have $\{\hat{f}(v),\hat{f}(v')\} \subseteq \{w,w'\}$. If $f(v) = f(v')$, then an element which is smaller than $\{\{f(v),N(\hat{f}(v))\}$ and $\{\{f(v'),N(\hat{f}(v'))\}$ is not unique although an element which is smaller than $\{\{v\},N(v)\}$ and $\{\{v'\},N(v')\}$ is uniquely determined. This is contradiction. Hence $\hat{f}(v) \neq \hat{f}(v')$ and we have completed the proof of $(\hat{f}(v),\hat{f}(v')) \in E(H)$.

Finally, let us prove that $f$ is an isomorphism. Let $g : B(H) \rightarrow B(G)$ denote the inverse of $f$, and $\hat{g}$ denote the graph homomorphism $K_2 \times H \rightarrow K_2 \times G$ constructed by the above way. We claim that $\hat{g} \circ f$ is an isomorphism. Let $v \in V(G)$. If $\hat{\tau} N(v) \geq 2$, then we have

$$\{\hat{g}\hat{f}(v),\hat{g}(N(\hat{f}(v)))\} = g(\{\hat{f}(v),N(\hat{f}(v))\}) = gf(\{\{v\},N(v)\}) = \{\{v\},N(v)\},$$

and hence $gf(v) = v$. Next suppose $N(v) = \{v'\}$ but $\hat{\tau} N(v') \geq 2$. Then we have

$$\{\{v\},\{v'\}\} = gf(\{\{v\},\{v'\}\}) = g(\{\hat{f}(v),\hat{f}(v')\}) = \{\hat{g}\hat{f}(v),\hat{g}(\hat{f}(v'))\}.$$
Corollary 4.4. Let $G$ and $H$ be graphs having no isolated vertices. If $B(G) \cong B(H)$ as posets, then $K_2 \times G \cong K_2 \times H$ as graphs.

Next we consider the $\mathbb{Z}_2$-poset structure of $B(G)$.

Theorem 4.5. Let $G$ and $H$ be graphs having no isolated vertices. If $B(G) \cong B(H)$ as $\mathbb{Z}_2$-posets, then $G \cong H$ as graphs.

Proof. It can be similarly proved, but is more complicated, as the previous theorem. Here, we give the proof using the map $\hat{f}$ constructed in the proof of Theorem 4.3.

Let $G$ and $H$ be bipartite graphs having no isolated vertices, with odd involutions $\tau_G$ and $\tau_H$. $\tau_G$ and $\tau_H$ induce poset involutions $B_0(G)$ and $B_0(H)$ respectively, and have $B(G/\tau_G) \cong B_0(G)$ and $B(H/\tau_H) \cong B_0(H)$ as $\mathbb{Z}_2$-posets from the viewpoint of Lemma 3.2. Suppose $B_0(G) \cong B_0(H)$ as $\mathbb{Z}_2$-posets, and let $f : B_0(G) \to B_0(H)$ a $\mathbb{Z}_2$-poset isomorphism. Let $\hat{f} : K_2 \times G \to K_2 \times H$ be an isomorphism of graphs which is constructed in Theorem 4.3. What we must prove is that after a little modification, $\hat{f}$ can be a $\mathbb{Z}_2$-graph isomorphism.

Let $v \in V(G)$ such that $\sharp N(v) \geq 2$. Then we have
\[
\{\hat{f}(\tau_G(v)), N(\hat{f}(\tau_G(v)))\} = f(\tau_G(\{v\}, N(v))) = \tau_H f(\{v\}, N(v)) = \{\tau_H \hat{f}(v), N(\tau_H \hat{f}(v))\}.
\]

Hence $\hat{f}(\tau_G(v)) = \tau_H \hat{f}(v)$.

Next suppose $N(v) = \{v'\}$ and $\sharp N(v') \geq 2$. Then we have
\[
\{\tau_G \hat{f}(v), \tau_G \hat{f}(v')\} = \{\hat{f}(\tau_H(v)), \hat{f}(\tau_H(v'))\}.
\]

Since $\hat{f}(\tau_H(v')) = \tau_G \hat{f}(v')$, we have $\hat{f}(\tau_H(v)) = \tau_G \hat{f}(v)$.

Finally, consider the case $N(v) = \{v\}$ and $N(v') = \{v\}$. Suppose that $\{\{v\}, \{v'\}\} \in B_0(G)$ is a fixed point of $\tau_G$. Then $f(\{v\}, \{v'\}) = \{\hat{f}(v), \hat{f}(v')\}$ is a fixed point of $\tau_H$. Then $\tau_G(v) = v', \tau_G(v') = v$, $\tau_H f(v) = f(v')$, and $\tau_H f(v') = f(v)$ since $\tau_G$ and $\tau_H$ are odd. Hence we have $\hat{f}(\tau_G(v)) = \tau_G \hat{f}(v)$.

Next suppose that $\{\{v\}, \{v'\}\} \in B_0(G)$ is not a fixed point of $\tau_G$. If $\hat{f}(\tau_G(v)) \neq \tau_H \hat{f}(v)$, we modify $f$ to the map $g$ such that $g(\tau_G(v)) = \tau_H \hat{f}(v)$ and $g(\tau_G(v')) = \tau_H \hat{f}(v')$, and $g(x) = \hat{f}(x)$ for the other $x \in V(G)$. Then
\[
g(\tau_G v) = g(v) = \hat{f}(v) = \tau_H \hat{f}(v) = \tau_H g(\tau_G v),
\]
and
\[
g(\tau_G v') = \tau_H g(\tau_G v').
\]
Hence $g(\tau_G x) = \tau_H g(x)$ holds for $x = v, v', \tau_G v, \tau_G v'$. After these modifications, we can modify $\hat{f}$ to a $\mathbb{Z}_2$-graph isomorphism (in the case of infinite graphs, we need the transfinite induction). This completes the proof of the theorem. \hfill \QED

Then Theorem 1.1 have been proved by Lemma 4.2, Corollary 4.4, and Theorem 4.5.

Next we consider the case of neighborhood complexes.

For a bipartite graph $G$ with its 2-partition $(A, B)$, we can construct abstract simplicial complexes $N_A(G)$ and $N_B(G)$ by
\[
N_A(G) = \{\sigma \subset A : \sharp \sigma < \infty, \text{and there is } v \in B \text{ such that } \sigma \subset N(v)\},
\]
\[
N_B(G) = \{\sigma \subset B : \sharp \sigma < \infty, \text{and there is } v \in A \text{ such that } \sigma \subset N(v)\}.
\]
Notice that the correspondence \( G \to B_0(G) \) is a functor from the category of bipartite graphs, on the other hand, the correspondence \( G \to N_A(G) \) and \( G \to N_B(G) \) are a functor from the category of bipartite graphs with 2-partitions whose morphisms are graph homomorphisms preserving 2-partitions. But the homotopy types of \( N_A(G) \) and \( N_B(G) \) are uniquely determined as follows.

**Lemma 4.6.** Let \( G \) be a bipartite graph with its 2-partition \((A,B)\). Then \( B_0(G), N_A(G), \) and \( N_B(G) \) are all homotopy equivalent.

*Proof.* We can regard \( B_0(G) \) as the poset \( \{(\sigma,\tau) \in B(G) \mid \sigma \subset A, \tau \subset B\} \). Let \( B^t_0(G) \) be the induced subposet of \( B_0(G) \) defined by

\[
B^t_0(G) = \{(\sigma,\tau) \in B_0(G) \mid \sigma \text{ and } \tau \text{ are finite.}\}
\]

It can be shown that for \( \{(\sigma,\tau) \in B_0(G), B_0(G) \leq(\sigma,\tau) \cap B_0(G) \) is contractible (see Lemma 2.6 of [12]). Hence applying the Quillen’s lemma A, we have that \( B^t_0(G) \to B_0(G) \) is a homotopy equivalence.

It is sufficient to show that \( B^t_0(G) \) is homotopy equivalent to \( N_A(G) \) and \( N_B(G) \). We only give the proof of the case \( N_A(G) \), the other is similarly proved.

Define the map \( p : B^t_0(G) \to FN_A(G), (\sigma,\tau) \to \sigma \). Then

\[
p^{-1}(FN_A(G) \leq \sigma) = \{(\sigma,\tau) \in B^t_0(G) \mid \sigma \times \tau \subset E(G)\}.
\]

Then by Lemma 2.6 of [12], we have this is contractible. By the Quillen’s lemma A, we have \( N_A(G) \) is homotopy equivalent to \( B^t_0(G) \).

In general, \( N_A(G) \) and \( N_B(G) \) are not homeomorphic. But we have the following, whose proof is trivial.

**Lemma 4.7.** Let \( G \) be a bipartite graph with a 2-partition \((A,B)\). If there is an involution \( \tau \) of \( G \) such that \( \tau(A) \subset B \), then \( N_A(G) \) is isomorphic to \( N_B(G) \).

**Lemma 4.8.** Let \( G \) be a graph and \((A,B)\) a 2-partition of \( K_2 \times G \) such that \( A = \{(1,v) \mid v \in V(G)\} \) and \( B = \{(2,v) \mid v \in V(G)\} \). Then \( N_A(K_2 \times G) \cong N_B(K_2 \times G) \cong N(G) \) as simplicial complexes.

*Proof.* We only give the proof of the case \( N_A \). Let \( p_2 : K_2 \times G \to G \) denote the 2nd projection. Then \( p_2 \) induces a simplicial map \( N_A(G) \to N(G) \). The inverse \( N(G) \to N_A(G) \) is given by \( \sigma \to \sigma \times \{1\} \). The details are left to the reader.

**Proposition 4.9.** Let \( G \) and \( H \) be graphs. If \( K_2 \times G \cong K_2 \times H \), then \( N(G) \cong N(H) \) as simplicial complexes.

*Proof.* Put \( A = \{1\} \times V(G), B = \{2\} \times V(G), A' = \{1\} \times V(H), \) and \( B' = \{2\} \times V(H) \). By Lemma 4.8 we have that

\[
N(G) \cup N(G) \cong N_A(K_2 \times G) \cup N_B(K_2 \times G) \cong N(K_2 \times G),
\]

\[
N(H) \cup N(H) \cong N_A(K_2 \times H) \cup N_B(K_2 \times H) \cong N(K_2 \times H).
\]

Since \( K_2 \times G \cong K_2 \times H \), we have \( N(G) \cup N(G) \cong N(H) \cup N(H) \). This implies that \( N(G) \cong N(H) \).

\[\footnotetext{2}{\text{In the word of Section 6, } N_A \text{ and } N_B \text{ are functors from } \mathcal{G}/K_2.}\]
Next we consider the inverse of the above theorem. Let $G$ be a graph. Before giving the precise statement, we review some definitions and facts about the $\times$-homotopy theory established in [6]. For the proof of the following facts, we refer [6] or [12]. For a negative integer $n$, we write $I_n$ for the graph defined by $V(I_n) = \{0,1,\cdots,n\}$ and $E(I_n) = \{(x,y) \mid |x-y| \leq 1\}$.

Let $f$ and $g$ be graph homomorphisms from a graph $G$ to a graph $H$. Then $f$ is said to be $\times$-homotopic to $g$, and written by $f \simeq\times g$ if there is $n \in \mathbb{N}$ and a graph homomorphism $F : G \times I_n \rightarrow H$ such that $F(x,0) = f(x)$ and $F(x,n) = g(x)$ for $x \in V(G)$. A graph homomorphism $f : G \rightarrow H$ is said to be $\times$-homotopy equivalence if there is a graph homomorphism $g : H \rightarrow G$ such that $gf \simeq\times id_G$ and $fg \simeq\times id_H$. If there is a $\times$-homotopy equivalence from $G$ to $H$, then we write $G \simeq\times H$.

If $f : G \rightarrow H$ is a $\times$-homotopy equivalence, then $f$ induces a homotopy equivalence from $N(G)$ to $N(H)$, and $B(G)$ to $B(H)$.

A vertex $v$ of $G$ is said to be dismantlable if there is $w \in V(G) \setminus \{v\}$ such that $N(v) \subset N(w)$. A graph $G$ is said to be stiff if $G$ has no dismantlable vertices. If $f : G \rightarrow H$ is a $\times$-homotopy equivalence between stiff graphs, then $f$ is an isomorphism.

Remark that for a finite graph $G$, a vertex $v$ of $G$ has no dismantlable vertices if and only if the number of facets of $N(G)$ is equal to the number of vertices of $G$.

**Proposition 4.10.** Let $G$ and $H$ be non-empty locally finite graphs. If $N(G) \cong N(H)$ as simplicial complexes, then $K_2 \times G \cong\times K_2 \times H$.

**Proof.** Let $f : N(G) \rightarrow N(H)$ be an isomorphism of simplicial complexes. We construct a graph homomorphism $\hat{f} : K_2 \times G \rightarrow K_2 \times H$ as follows.

Let $v \in V(G)$. Put $\hat{f}(1, v) = (1,f(v))$. Because $G$ is locally finite, $N(v)$ is finite and hence is a simplex of $N(G)$. Hence $f(N(v))$ is a simplex of $N(H)$. Hence there is $f'(v) \in V(H)$ such that $f(N(v)) \subset N(f'(v))$. Put $\hat{f}(2,v) = (2,f'(v))$. This is the definition of $\hat{f} : V(K_2 \times G) \rightarrow V(K_2 \times H)$.

Let us show that $\hat{f}$ is a graph homomorphism. Let $(v, w) \in E(G)$. Then $v \in N(w)$, and hence we have $f(v) \in f(N(w)) \subset N(f'(w))$, in other word, we have $(f(v), f'(w)) \in E(H)$. This implies $(\hat{f}(1,v), \hat{f}(2,w)) \in E(K_2 \times H)$ and $\hat{f}$ is a graph homomorphism.

Put $g = f^{-1} : N(H) \rightarrow N(G)$ and construct $\hat{g}'$ and $\hat{g}$ similarly. What we must show is that $\hat{g}$ is a $\times$-homotopy inverse of $\hat{f}$. We only give the proof of $\hat{g} \circ \hat{f} \simeq\times id_{K_2 \times G}$, and the other $\hat{f} \circ \hat{g} \simeq\times id_{K_2 \times H}$ is similarly proved.

By Lemma 4.11, it is sufficient to show that for any $(i, v) \in V(G)$, $N(i, v) \subset N(\hat{g} \circ \hat{f}(i,v))$. If $i = 1$, then we have $\hat{g} \circ \hat{f}(1,v) = (1,v)$, we must show in the case of $i = 2$. Remark that for $v \in V(G)$, we have

$$N(v) = g \circ f(N(v)) \subset g(N(f'(v))) \subset N(g' \circ f'(v)).$$

Since $N(2,v) = \{1\} \times N(v)$, we have $N(2,v) \subset N(2, g' \circ f'(v)) = N(\hat{g} \circ \hat{f}(2,v))$. \hfill $\square$

**Lemma 4.11.** Let $f$ and $g$ be graph homomorphisms $G \rightarrow H$. If $N(f(v)) \subset N(g(v))$ for any $v \in V(G)$, then we have $f \simeq\times g$.

**Proof.** If $(v,w) \in E(G)$, then $f(v) \in N(f(w)) \subset N(g(w))$, namely, $(f(v), g(w)) \in E(H)$. This implies $(f \times g)(E(G)) \subset E(H)$. It is easy to show that the map
Let \( \tau \) be a graph homomorphism from \( G \times I_1 \) to \( H \).

**Conjecture 4.13.** Let \( \tau \) be a stiff graph having no isolated vertices. Then \( K \) is defined by the following.

**Corollary 4.12.** Let \( G \) and \( H \) be locally finite stiff graphs having no isolated vertices. If \( N(G) \cong N(H) \) as simplicial complexes, then \( K \times G \cong K \times H \).

**Proof.** This is because of the fact that if a graph \( G \) is stiff and has no isolated vertices, then so is \( K \times G \).

The proof of Theorem 1.2 is completed by Proposition 4.10 and Corollary 4.12.

In the above corollary, it is obvious that the assumption \( G \) and \( H \) have no isolated vertices can not be omitted, but I have no idea that the locally finite and stiff conditions can be omitted. So I suggest the following conjecture.

**Conjecture 4.13.** Let \( G \) and \( H \) be graphs having no isolated vertices. If \( N(G) \cong N(H) \), then \( K \times G \cong K \times H \).

An application of Theorem 1.1 and Theorem 1.2, we prove Theorem 1.3.

**Example 4.14.** Given positive integers \( n \) and \( m \) greater than 2. The goal of this example is that we construct graphs \( G \) and \( H \) such that \( \chi(G) = n \) and \( \chi(H) = m \).

First we construct a bipartite graph \( K \) as follows. Put \( X_1 = X_2 = K_2 \times K_n \) and \( Y_1 = Y_2 = K_2 \times K_m \). Then \( K \) is the graph obtained by the disjoint union \( X_1 \uplus Y_1 \uplus X_2 \uplus Y_2 \) of graphs by connecting the following vertices.

- \((2, 1) \in V(X_1)\) is connected with \((1, 1) \in V(Y_1)\) by an edge.
- \((2, 1) \in V(Y_1)\) is connected with \((1, 1) \in V(X_2)\) by an edge.
- \((2, 1) \in V(X_2)\) is connected with \((1, 1) \in V(Y_2)\) by an edge.
- \((2, 1) \in V(Y_2)\) is connected with \((1, 1) \in V(X_1)\) by an edge.

The following figure show the case of \( n = 4 \) and \( m = 3 \).

![Graph K for n = 4 and m = 3](image)

Define the odd involutions \( \tau \) and \( \tau' \) of \( K \) as follows. First \( \tau \) is defined by the following correspondence.

- \( V(X_1) \ni (1, v) \leftrightarrow (2, v) \in V(X_1) \),
- \( V(X_2) \ni (1, v) \leftrightarrow (2, v) \in V(X_2) \),
- \( V(Y_1) \ni (\varepsilon, v) \leftrightarrow (\varepsilon, v) \in V(Y_2) \), \( (\varepsilon = 1, 2) \).

And \( \tau' \) is defined by the following.

- \( V(Y_1) \ni (1, v) \leftrightarrow (2, v) \in V(Y_1) \),
Put $G = K/\tau$ and $H = K/\tau'$. Then $G$ is a graph which has two $K_n$ and one $K_2 \times K_m$ as subgraphs, and each $K_n$ and $K_2 \times K_m$ are connected by one edge. Similarly, $H$ is a graph which has two $K_m$ and one $K_2 \times K_n$ as subgraphs, and each $K_m$ is connected with $K_2 \times K_n$ by one edge. See the following figure which shows $G$ and $H$ in the case $n = 4$ and $m = 3$.

The graph $G$ for $n = 4$ and $m = 3$.

The graph $H$ for $n = 4$ and $m = 3$.

It can be shown that $\chi(G) = n$ and $\chi(H) = m$. By Lemma 3.2, we have $K_2 \times G \cong K \cong K_2 \times H$. Hence $N(G) \cong N(H)$ and $B(G) \cong B(H)$.

5. Deformations of $B_0(G)$

In this section, we prove further properties of $B_0(G)$. These properties investigated in this section allows us to determine the homotopy type of $B_0(G)$ from the geometric observations.

Let $G$ be a graph. A vertex $v$ of $G$ is said to be dismantlable if there is $w \in V(G) \setminus \{v\}$ such that $N(v) \subset N(w)$. In this case, $G \setminus v$ is called a folding of $G$. It is easy to see that the inclusion $G \setminus v \hookrightarrow G$ is a $\times$-homotopy equivalence, by using Lemma 4.11.

In [2], [7] and [6], foldings do not change the homotopy types of Hom complexes and neighborhood complexes. This holds for $B_0$ too.

**Lemma 5.1.** Let $G$ be a bipartite graph and $v$ a dismantlable vertex of $G$. Then $B_0(G \setminus v)$ is a deformation retract of $B_0(G)$.

**Proof.** This is proved in more general situation in Lemma 6.4. □

The 4-cycle $C_4$ is the graph defined by $V(C_4) = \mathbb{Z}/4$, $E(C_4) = \{(x, x + 1), (x + 1, x) \mid x \in V(C_4)\}$. A square of a graph $G$ is a graph homomorphism from $C_4$ to $G$. A square $f : C_4 \rightarrow G$ is said to be non-degenerate if $f(0) \neq f(2)$ and $f(1) \neq f(3)$.

**Proposition 5.2.** Let $G$ be a graph and $X, Y$ induced subgraphs of $G$ satisfying the following properties.

1. $G = X \cup Y$.
2. $X \cap Y$ has no isolated vertices.
3. Any non-degenerate square $C_4 \rightarrow G$ of $G$ factors through $X$ or $Y$. 

V($Y_2$) $\ni (1, v) \leftrightarrow (2, v) \in V(Y_2)$,  
V($X_1$) $\ni (\varepsilon, v) \leftrightarrow (\varepsilon, v) \in V(X_2)$,  ($\varepsilon = 1, 2$)
Then the inclusion $B_0(X) \cup B_0(Y) \hookrightarrow B_0(G)$ is a homotopy equivalence.

**Proof.** For $v \in V(X) \cap V(Y)$, we write $P_v$ for the induced subposet of $B_0(G)$ such that

$$P_v = \{ \{ \sigma, \{ v \} \} \mid \sigma \subset N(\{ v \}) \}.$$  

Since any non-degenerate square of $G$ factors through $X$ or $Y$, we have

$$B_0(G) = B_0(X) \cup B_0(Y) \cup \bigcup_{v \in V(X) \cap V(Y)} P_v.$$  

Remark that $P_v \cap P_w \subset B_0(X) \cup B_0(Y)$ for $v \neq w$. Since $P_v$ is contractible, what we must prove is that $P_v \cap (B_0(X) \cup B_0(Y))$ is contractible. Since $P_v \cap B_0(X)$, $P_v \cap B_0(Y)$, and $P_v \cap B_0(X) \cap B_0(Y)$ has the maximums $\{ N(\{ v \} \cap V(X), \{ v \} \}$, $\{ N(\{ v \} \cap V(Y), \{ v \} \}$, and $\{ N(\{ v \} \cap V(X) \cap V(Y), \{ v \} \}$ respectively (remark that $N(\{ v \} \cap V(X) \cap V(Y) \neq \emptyset$ since $X \cap Y$ has no isolated vertices), we have that $P_v \cap (B_0(X) \cup B_0(Y))$ is contractible.

**Corollary 5.3.** Under the above assumption, we have

$$|B_0(G)| \cong |B_0(X)| \cup |B_0(X) \cap B_0(Y)| \cup B_0(Y)|.$$  

Recall that we write $L_3$ for the graph defined by $V(L_3) = \{0,1,2,3\}$, $E(L_3) = \{(x,y) \mid |x-y| = 1\}$, see Section 3.

**Proposition 5.4.** Let $G$ be a bipartite graph and $x,w \in V(G)$. Let $H$ denote the graph obtained from $G$ by adding one edge connecting $x$ with $w$. Suppose that there is only one graph homomorphism $f : L_3 \rightarrow G$ such that $f(0) = x$ and $f(3) = w$. Then the inclusion $B_0(G)$ is a deformation retract of $B_0(H)$.

**Proof.** Put $f(1) = y$ and $f(2) = z$. We can assume that $(x,w) \notin E(G)$. Let $(A,B)$ be a 2-partition of $G$ such that $x \in A$ and $w \in B$. Then $(A,B)$ is also a 2-partition of $H$. What we must prove is that $N_A(G) \rightarrow N_A(H)$ is a homotopy equivalence. Let $\sigma \in N_A(H)$. Then $\sigma \notin N_A(G)$ if and only if $x \in \sigma$ and $\sigma$ includes an element of $N^G(w)$. Indeed, suppose $\sigma \notin N_A(G)$. There is $v \in B$ such that $\sigma \in N^H(v)$. If $v \neq x,w$, then $N^H(v) = N^G(v)$, and since $v \in B$, we have $v = w$, hence $\sigma \subset N^H(w)$. Since $N^H(w) = N^G(x) \cup \{ x \}$, we have that $\sigma$ must contain $x$. If $\sigma = \{ x,z \}$ or $\{ x \}$, then $\sigma \subset N^G(z)$ and is contradiction. Hence $\sigma$ must contain an element of $N^G(w) \setminus \{ z \}$. On the other hand, we assume that $\sigma$ contains $x$ and an element of $N^G(w) \setminus \{ z \}$, and suppose $\sigma \subset N(G)$. Then there is $y'$ such that $\sigma \subset N^G(y')$. Let $z' \in \sigma \cap (N^G(w) \setminus \{ z \})$. Then the map $f' : V(L_3) \rightarrow V(G)$ such that

$$f'(0) = x, f'(1) = y', f'(2) = z', f'(3) = w$$

is a graph homomorphism from $L_3$ to $G$ which is not equal to $f$. This contradicts to the assumption.

Hence we have the acyclic matching

$$\sigma \leftrightarrow \sigma \cup \{ z \}, (\sigma \in N(H) \setminus N(G), z \notin \sigma)$$

on $FN(H) \setminus FN(G)$ which has no critical points. Therefore by the fundamental theorem of discrete Morse theory (see [8]), we have $N(G)$ is a deformation retract of $N(H)$.

The properties of $B_0(G)$ investigated in this section allows us to determine the homotopy type of $B_0(G)$ by geometric observations.
Example 5.5. Let \( n \) be a positive integer. Let \( G_n \) denote the graph be defined by

\[
V(G_n) = \{(a, b, c) \in \mathbb{Z}^3 \mid 0 \leq a, b, c \leq n, \text{ one of } a, b, c \text{ is equal to } 0 \text{ or } n\},
\]

\[
E(G_n) = \{(a, b, c), (a', b', c') \mid |a' - a| + |b' - b| + |c' - c| = 1\}.
\]

The subgraph \( X_n \) and \( Y_n \) be the induced subgraphs of \( G_n \) with the vertex sets,

\[
V(X_n) = \{(a, b, c) \in V(G) \mid \text{One of } a, b, c \text{ is equal to } 0\},
\]

\[
V(Y_n) = \{(a, b, c) \in V(G) \mid \text{One of } a, b, c \text{ is equal to } n\}.
\]

The following figures are the case of \( n = 3 \).

By the folding lemma, we have that \( B_0(X_n) \cong B_0(K_2) = * \) and \( B_0(Y_n) \cong B_0(K_2) = * \). Since any non-degenerate square \( C_4 \rightarrow G_n \) factors through \( X_n \) or \( Y_n \), we have that \( B_0(G_n) \cong B_0(X_n) \cup_{B_0(X_n \cap Y_n)} B_0(Y_n) \). Since \( B_0(X_n \cap Y_n) = B_0(C_{10}) \cong S^1 \), we have that \( B_0(G_n) \cong S^2 \).

Let \( \tau \) be an involution \( \tau(a, b, c) = (n - a, n - b, n - c) \). Then \( \tau \) is odd if and only if \( n \) is odd. Suppose \( n \) is odd. Then \( B(G_n/\tau) = B_0(G_n) \cong S^2 \), and hence we have \( \chi(G_n) \geq 4 \) by Theorem 2.2. It can be easily shown that \( G_n/\tau \) is 4-colorable, and hence we have \( \chi(G_n/\tau) = 4 \) for odd \( n \).

6. The case of the other Hom complexes

The box complex \( B(G) \) we say in this paper is isomorphic to the Hom complex \( \text{Hom}(K_2, G) \). In this section, we ask whether we can construct the “similar” construction with \( B_0(G) \) for the other Hom complexes. Before mentioning the precise statement about this, we need some notations. The elementary terminologies of category theory is due to [10]. For a category \( \mathcal{C} \), we write \( A \in \mathcal{C} \) if \( A \) is an object of \( \mathcal{C} \).
• Let $G$ denote the category of graphs. Namely, an object of $G$ is a graph, and a morphism of $G$ is a graph homomorphism.
• Let $T$ be a graph. Let $G_T$ denote the full subcategory of $G$ whose object is a graph $G$ such that there is a graph homomorphism $G \to T$.
• Let $T$ be a graph. Let $G_T$ denote the category of graphs over $T$, that is, an object of $G_T$ is a pair $(G, \varphi)$ such that $G$ is a graph and $\varphi$ is a graph homomorphism from $G$ to $T$, and a morphism from $(G, \varphi_G)$ to $(H, \varphi_H)$ is a graph homomorphism $f : G \to H$ such that $\varphi_H \circ f = \varphi_G$.

We often abbreviate the homomorphism $G \to T$ and written by “$G$ is a graph over $T$”. In this terminology, we write $\varphi_G$ for the given homomorphism of $G$.
• Let $P$ denote the category of posets.

There is a functor $T \times (-) : G \to G_T$, $G \mapsto T \times G$ where we regard $T \times G$ as a graph over $T$ with the 1st projection $T \times G \to T$. And there is a forgetful functor $\Phi_T : G_T \to G_T$, $(G, \varphi) \mapsto G$.

Then $B_0$ is a functor from $G_{K_n}$ to $P$. Hence the functor $B_n(-)(\cong \text{Hom}(K_2, -))$ is naturally isomorphic to the composition of the following functors
\[
G \xrightarrow{K_n \times (-)} G_{K_n} \xrightarrow{\Phi_{K_n}} G_{K_n} \xrightarrow{B_n(-)} P.
\]

In this section, we consider that for a graph $T$, there is a functor $G_T \to P$ such that the functor $G \mapsto \text{Hom}(T, G)$ is decomposed as the compositions
\[
G \xrightarrow{T \times (-)} G_T \xrightarrow{\Phi_T} G_T \xrightarrow{?} P.
\]

The main results obtained in this section are followings.

**Theorem 6.1.** Let $n$ be a non-negative integer. Then there is a functor $B_n^0(-) : G_T \to P$ such that the composition of
\[
G \xrightarrow{K_n \times (-)} G_{K_n} \xrightarrow{\Phi_{K_n}} G_{K_n} \xrightarrow{B_n^0(-)} P
\]
is naturally isomorphic to the functor $G \mapsto \text{Hom}(K_n, G)$.

**Theorem 6.2.** Let $T$ be an arbitrary graph. Then there is a functor $\text{Hom}_T(T, -) : G_T \to P$ such that the composition of
\[
G \xrightarrow{T \times (-)} G_T \xrightarrow{\text{Hom}_T(T, -)} P
\]
is naturally isomorphic to the functor $G \mapsto \text{Hom}(T, G)$.

From the above two theorems, we have the following.

**Corollary 6.3.** The followings hold.

1. Let $G$ and $H$ be graphs and $n$ a non-negative integer. If $K_n \times G \cong K_n \times H$ as graphs, then we have $\text{Hom}(K_n, G) \cong \text{Hom}(K_n, H)$ as posets.
2. Let $G$ and $H$ be graphs. If $T \times G \cong T \times H$ as graphs over $T$, then we have $\text{Hom}(T, G) \cong \text{Hom}(T, H)$ as posets.

Let $\mathbf{1}$ denote one looped vertex graph. Then the functor $G \to P$, $G \mapsto \text{Hom}(\mathbf{1}, G)$ factors through $G \to G_{\mathbf{1}} \to \mathbf{1} \to P$ since in this case, $G = G_{\mathbf{1}} = \mathbf{1}$. I wonder that $\mathbf{1}$ and $K_n$ for $n \geq 0$ are only graphs such that the $\text{Hom}(T, -)$-functor is factored as $G \to G_T \to G_T \to P$.

First we prove Theorem 6.2. Let $G$ and $H$ be graphs over $T$. A multi-homomorphism from $G$ to $H$ over $T$ is a map $\eta : V(G) \to 2^{V(H)}$ satisfying the following properties:
(1) \( \eta(v) \times \eta(w) \subseteq E(G) \) for \((v, w) \in E(G)\).

(2) \( \varphi_{\mathcal{H}}(\eta(v)) = \{ \varphi_G(v) \} \) for \( v \in V(G) \).

We write \( \text{Hom}_{/T}(G, H) \) for the poset of all multi-homomorphisms from \( G \) to \( H \) over \( T \) with the order relation \( \eta \leq \eta' \iff \eta(v) \subseteq \eta'(v) \) for any \( v \in V(G) \).

Then we can easily show that \( \text{Hom}_{/T}(G, H) \) is covariant functorial with respect to \( H \), and contravariant functorial with respect to \( G \), as is the case of \( \text{Hom} \) complexes.

**Proof of Theorem 6.2.** Let \( G \) be a graph. Let \( p_i \) denote the \( i \)-th projection of \( T \times G \) \((i = 1, 2)\). The map

\[
\Phi : \text{Hom}(T, G) \to \text{Hom}_{/T}(T, T \times G)
\]

is defined by \( \Phi(\eta)(v) = \{ v \} \times \eta(v) \), and the map

\[
\Psi : \text{Hom}_{/T}(T, T \times G) \to \text{Hom}(T, G)
\]

is defined by \( \Psi(\eta)(v) = p_2(\eta(v)) \). What we want to show is that \( \Phi \) and \( \Psi \) are inverse of each other. It is obvious that \( \Psi \circ \Phi = \text{id} \). On the other hand, let \( \eta \in \text{Hom}_{/T}(T, T \times G) \). Then \( \eta(v) \subseteq \{ v \} \times V(G) \) since \( p_1 \circ \eta(v) = \{ v \} \). Hence there is \( \eta' : V(T) \to 2^V(G) \setminus \{ \emptyset \} \) such that \( \eta(v) = \{ v \} \times \eta'(v) \). Then \( \Psi(\eta) = \eta' \) and \( \Phi \circ \Psi(\eta)(v) = \{ v \} \times \eta'(v) = \eta(v) \). Hence we have \( \Phi \circ \Psi = \text{id} \), and \( \text{Hom}(T, G) \cong \text{Hom}_{/T}(T, T \times G) \).

The naturality of \( \Phi \) is easily proved and the details are left to the reader. \( \square \)

Next let us prove Theorem 6.1. The construction of \( B^n_0(G) \) is similar to \( B_0 \). Namely, for \( G \in \mathcal{G}_{K_n} \), we put

\[
B^n_0(G) = \{ \{ X_1, \ldots, X_n \} \mid \emptyset \neq X_i \subseteq V(G), X_i \times X_j \subseteq E(G) (i \neq j) \}\]

with the ordering defined as follows. For \( \mathcal{X}, \mathcal{Y} \in B^n_0(G) \), \( \mathcal{X} \leq \mathcal{Y} \) if and only if for any \( X \in \mathcal{X} \), there is \( Y \in \mathcal{Y} \) such that \( X \subseteq Y \). It is easy to show that \( B^n_0(G) \) is actually a poset.

Let us prove Theorem 6.1.

**Proof of Theorem 6.1.** Let \( G \) be a graph. For \( i \in \{1, \ldots, n\} \), put \( A_i = \{ i \} \times V(G) \subseteq V(K_n \times G) \). Let \( \mathcal{X} \in B^n_0(K_n \times G) \). Let us prove that for \( X_0 \in \mathcal{X} \), there is \( i \in \{1, \ldots, n\} \) such that \( X_0 \subseteq A_i \). Suppose such \( i \) does not exist. Then there are \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \) such that \( X_0 \cap A_i \neq \emptyset \) and \( X_0 \cap A_j \neq \emptyset \). Define the map

\[
\varphi : \mathcal{X} \to 2^{\{1, \ldots, n\}}
\]

by \( \varphi(X) = \{ i \mid X \cap A_i \neq \emptyset \} \). Then \( \sharp \varphi(X_0) \geq 2 \). Hence if \( \varphi(X) \subseteq \mathcal{X} \) are disjoint, then we have \( \sharp \{ \bigcup_{i=1}^n \varphi(i) \} > n \), and this is contradiction. Hence there are \( X', X'' \in \mathcal{X} \) such that \( X' \neq X'' \) but \( \varphi(X') \cap \varphi(X'') \neq \emptyset \). Then let \( i \in \varphi(X') \cap \varphi(X'') \), \( v' \in X' \cap A_i \), and \( v'' \in X'' \cap A_i \). Then we have \( (v', v'') \in X' \times X'' \subseteq E(K_n \times G) \), but this is contradiction since \( A_i \) is independent subset of \( K_n \times G \).

Hence for \( X \in \mathcal{X} \), there is \( i \in \{1, \ldots, n\} \) such that \( X \subseteq A_i \). Define a map

\[
\psi : \{1, \ldots, n\} \to \mathcal{X}
\]

by \( \psi(i) \) is \( X \in \mathcal{X} \) such that \( X \subseteq A_i \). It is easy to show that \( \psi \) is well-defined bijection. Remark that

\[
\text{Hom}_{/K_n}(K_n, K_n \times G) \cong \{ (X_1, \ldots, X_n) \mid X_i \subseteq A_i, X_i \times X_j \subseteq E(K_n \times G) (i \neq j) \}
\]

as posets. Define the map \( \Phi : B^n_0(G) \to \text{Hom}_{/K_n}(K_n, K_n \times X) \) by

\[
\Phi(\mathcal{X}) = (\psi_\mathcal{X}(1), \ldots, \psi_\mathcal{X}(n)).
\]
Then it can be easily shown that $\Psi : \text{ is the inverse of } \Phi$, and hence we have that

$$B_0^n(G) \cong \text{Hom}_{/K_n}(K_n, K_n \times G) \cong \text{Hom}(K_n, G)$$

by Theorem 6.2.

However, there are properties which hold for $n = 2$ but do not hold for $n \geq 3$.

1. An $n$-partition of a graph $G$ is an ordered $n$-tuple $(A_1, \cdots, A_n)$ of independent subsets of $G$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_1 \cup \cdots \cup A_n = V(G)$.
   It is easy to show that $G$ has an $n$-partition if and only if $G$ is $n$-colorable.
   Fix an $n$-partition $(A_1, \cdots, A_n)$ of $n$-colorable graph $G$. Unlike the case $n = 2$, $B_0^n(G)$ is not isomorphic to the following induced subposet
   $$\{(X_1, \cdots, X_n) \mid X_i \subset A_i \ (i \in \{1, \cdots, n\})\}$$
   of $\text{Hom}(K_n, G)$ in general. Indeed, for $\mathcal{X} \in B_0^n(G)$ and $X \in \mathcal{X}$, it does not hold.

2. Unlike the case $n \neq 2$, if $n \geq 3$, then $B_0^n(G) \cong B_0^n(H)$ for $n$-colorable graphs $G$ and $H$ having no isolated vertices does not imply $G \cong H$. Indeed, if $G$ has no subgraph which is isomorphic to $K_n$, then $B_0^n(G) = \emptyset$.

By the construction of $B_0^n$ and $\text{Hom}_{/T}(T, G)$, we have that $B_0^n(G) = B_0(G)$ and the functor $\text{Hom}_{/K_n}(K_n, -)$ is naturally isomorphic to the composition of

$$\mathcal{G}_T \xrightarrow{\Phi_{K_n}} \mathcal{G}_{/T} \xrightarrow{B_0^n(-)} \mathcal{P}.$$ 

The folding lemma for $\text{Hom}_{/T}(G, H)$ and $B_0^n(G)$.

**Lemma 6.4.** Let $G$ and $H$ be graphs over a graph $T$. Let $v \in V(H)$. If there is $w \in V(H) \setminus \{v\}$ such that $N(v) \subset N(w)$ and $\varphi_H(w) = \varphi_G(v)$. Then the maps

$$\text{Hom}_{/T}(G, H \setminus v) \to \text{Hom}_{/T}(G, H),$$

$$\text{Hom}_{/T}(H, G) \to \text{Hom}_{/T}(H \setminus v, G)$$

induced by $H \setminus v \hookrightarrow H$ are homotopy equivalences.

**Proof.** This can be shown by the proof given in [7] and [8] with trivial modifications. So we give only that $\text{Hom}(G, H \setminus v) \to \text{Hom}(G, H)$ is a homotopy equivalence.

Define the order preserving map $c : \text{Hom}_{/T}(G, H) \to \text{Hom}(G, H)$ by, for $\eta \in \text{Hom}_{/T}(G, H),$

$$c(\eta)(x) = \begin{cases} 
\eta(x) \cup \{w\} & (v \in \eta(x)) \\
\eta(x) & (v \notin \eta(x)).
\end{cases}$$

Then $c$ is an ascending closure operator and hence induces a homotopy equivalence. The order preserving map $c' : c(\text{Hom}_{/T}(G, H)) \to c(\text{Hom}_{/T}(G, H))$ is defined by

$$c'(\eta)(x) = \begin{cases} 
\eta(x) \setminus \{v\} & (v \in \eta'(x)) \\
\eta(x) & (v \notin \eta'(x)).
\end{cases}$$

Then $c'$ is a descending closure operator, we have that $c'c(\text{Hom}_{/T}(G, H))$ is a deformation retract of $\text{Hom}_{/T}(G, H)$. Since $c'c(\text{Hom}_{/T}(G, H)) \cong \text{Hom}_{/T}(G, H \setminus v)$, this completes the proof.

**Lemma 6.5.** Let $G$ be an $n$-colorable graph and $v \in V(G)$ an dismantlable vertex. Then the inclusion $G \setminus v \hookrightarrow G$ induces a homotopy equivalence $B_0^n(G \setminus v) \to B_0^n(G)$.

**Proof.** This is easily proved by the fact $B_0^n(G) \cong \text{Hom}_{/K_n}(K_n, K_n \times G)$ and the previous lemma.
The proofs of the above two lemmas are fulfilled by the proofs of Lemma 5.1 and the folding lemma of Hom complexes given in [7] or [8] with easy modifications. And the details are left to the reader.

It is easy to show that Lemma 5.2 and Proposition 5.3 do not hold for $B^0(G)$ and $\text{Hom}_{/T}(G, H)$ without non-trivial modifications.

7. The case of the other definition of box complexes

In this section, we consider the case of the other definition of the box complexes. In [5], and [13], a box complex for a bipartite graph is defined as a subcomplex of $N(G) \ast N(G)$. For $v \in N(G)$, we write $(1, v)$ for the vertex of $N(G) \ast N(G)$ the left $N(G)$-part, and $(2, v)$ for the vertex of the right. $B'(G)$ has a natural involution, such that $(1, v) \leftrightarrow (2, v)$, which is free in the case of non-looped graphs. It is known that $B'(G)$ is $\mathbb{Z}_2$-homotopy equivalent to $B(G)$, and hence homotopy equivalent to $N(G)$.

We can construct a simplicial complex $B'_0(G)$ for a bipartite graph $G$, such that $B'_0(K_2 \times H) \cong B'(H)$ as simplicial complexes.

**Definition 7.1.** Let $G$ be a bipartite graph. The simplicial complex $B'_0(G)$ is defined as follows. Let $(A, B)$ be a 2-partition of $G$. The vertex set of $B'_0(G)$ is the set of all non-isolated vertices of $G$, and a simplex $\sigma$ is a finite subset of $V(B'_0(G))$ satisfying the followings.

- $\sigma \cap A \subset N_A(G)$ and $\sigma \cap B \subset N_B(G)$.
- The subgraph induced by $\sigma$ is a complete bipartite graph, namely, $\{ \sigma \cap A \} \times \{ \sigma \cap B \} \subset E(G)$.

It is easy to show that the definition of $B'_0(G)$ does not depend on the choice of 2-partition of $G$, and $G \mapsto B'_0(G)$ is a functor from the category of bipartite graphs to the category of simplicial complexes.

**Lemma 7.2.** Let $G$ be a graph. Then $B'(G) \cong B'_0(K_2 \times G)$ as simplicial complexes.

**Proof.** Put $A = \{1\} \times V(G)$ and $B = \{2\} \times V(H)$. Let $p_2 : K_2 \times G \to G$ denote the 2nd projection. Then the map $\Phi : B'_0(K_2 \times G) \to B'(G)$ by $\Phi(1, v) = (1, v)$ and $\Phi(2, v) = (2, v)$. If $\sigma$ is a simplex of $B'_0(K_2 \times G)$, then $\Phi(\sigma) = (\sigma \cap A) \cup (\sigma \cap B)$, and hence $(\sigma \cap A) \times (\sigma \cap B) \subset G$. This implies $\Phi(\sigma)$ is a simplex of $B'(G)$ and $\Phi$ is a simplicial map.

On the other hand, $\Psi : B'(G) \to B'_0(K_2 \times G)$ is defined by $\Psi(\sigma \cup \tau) = \{(1) \times \sigma\} \cup \{(2) \times \tau\}$. We can show that $\Psi$ is a simplicial map and is the inverse of $\Phi$. □

The following is an easy consequence of the above lemma.

**Proposition 7.3.** Let $G$ and $H$ be graphs. If $K_2 \times G \cong K_2 \times H$, then $B(G) \cong B(H)$ as simplicial complexes.

But $B'_0(G) \cong B'_0(H)$ does not imply $G \cong H$ for bipartite graphs $G$ and $H$ having no isolated vertices. The following graphs $G$ and $H$ satisfy $B'_0(G) \cong B'_0(H)$ as simplicial complexes but $G \not\cong H$. 

The author wonders whether there are graphs $G$ and $H$ and $B_0(G) \cong B_0(H)$ as simplicial complexes but $G \not\cong H$.

**Acknowledgement.** The author would like to express his gratitude to Mikio Furuta for his insightful viewpoints, and helpful suggestions. This work was supported by the Program for Leading Graduate Schools, MEXT, Japan.

**References**

[1] N. Alon, P. Frankl, L. Lovász, *The chromatic number of Kneser hypergraphs*, Trans. Amer. Math. Soc. 298 (1986), no. 1, pp. 359-370

[2] E. Babson, D. N. Kozlov, *Complexes of graph homomorphisms*, Israel J. Math. 152, 285-312 (2006)

[3] J. A. Barmak, *On Quillen’s Theorem A for posets*, J. Combin. Theory Ser. A 118, 2445-2453 (2011)

[4] A. Björner, *Topological Methods*, Handbook of Combinatorics, vol. 1,2, (eds. R. Graham, M. Grötschel and L. Lovász), Elsevier, Amsterdam, 1819-1872 (1995)

[5] P. Csorba, *Homotopy types of box complexes*, Combinatorica, 27 (6):669-682 (2007)

[6] A. Dochtermann, *Hom complexes and homotopy theory in the category of graphs*, European J. Combin. 30 (2) 490-509 (2009)

[7] D. N. Kozlov, *A simple proof for folds on both sides in complexes of graph homomorphisms*, Proc. Amer. Math. Soc. 134, (5) 1265-1270 (2006)

[8] D. N. Kozlov. *Combinatorial algebraic topology*. Springer, Berlin (2008)

[9] L. Lovász, *Kneser conjecture, chromatic number, and homotopy*. J. Combin. Theory Ser. A, 25 (3):319-324 (1978)

[10] S. Mac Lane, *Categories for the working mathematician*, volume 5 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition (1998)

[11] T. Matsushita, *Deformations of the neighborhood complex*, arXiv 1312.3052

[12] T. Matsushita, *Cell complexes obtained from sets with relations*, arXiv 1402.0311.

[13] J. Matoušek, G. M. Ziegler, *Topological lower bounds for the chromatic number: A hierarchy.*

[14] D. Quillen, *Higher algebraic K-theory I*, Lecture Notes in Mathematics 341, Springer, 85-147 (1973)

[15] R. T. Živaljević, *WI-posets, graph complexes and Z2-equivalences*, J. Combin. Ser. A 111 (2005), no. 2, 204-223