THE REGULARITY WITH RESPECT TO DOMAINS OF THE ADDITIVE EIGENVALUES OF SUPERQUADRATIC HAMILTON–JACOBI EQUATION

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ABSTRACT. We study the additive eigenvalues on changing domains, along with the associated vanishing discount problems. We consider the convergence of the vanishing discount problem on changing domains for a general scaling type $\Omega_\lambda = (1 + r(\lambda))\Omega$ with a continuous function $r$ and a positive constant $\lambda$. We characterize all solutions to the ergodic problem on $\Omega$ in terms of $r$. In addition, we demonstrate that the additive eigenvalue $\lambda \mapsto c_{\Omega_\lambda}$ on a rescaled domain $\Omega_\lambda = (1 + \lambda)\Omega$ possesses one-sided derivatives everywhere. Additionally, the limiting solution can be parameterized by a real function, and we establish a connection between the regularity of this real function and the regularity of $\lambda \mapsto c_{\Omega_\lambda}$. We provide examples where higher regularity is achieved.

1. INTRODUCTION

The vanishing discount problem concerns the behavior of the family of solutions as the discount factor goes to 0. Let $v_\lambda$ be the solution to the state-constraint problem with the discount factor $\lambda > 0$,

$$
\begin{cases}
\lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \epsilon \Delta v_\lambda(x) \leq 0 & \text{in } \Omega, \\
\lambda v_\lambda(x) + |Dv_\lambda(x)|^p - f(x) - \epsilon \Delta v_\lambda(x) \geq 0 & \text{on } \overline{\Omega},
\end{cases}
$$

where $\Omega$ is an open, bounded, and connected domain in $\mathbb{R}^n$ with $C^2$ boundary, $\epsilon > 0$ is fixed, $p > 2$ and $f \in C^1(\overline{\Omega})$. The Hamiltonian is given by $H(x, \xi) = |\xi|^p - f(x)$ for $(x, \xi) \in \overline{\Omega} \times \mathbb{R}^n$. As shown in [25], we have $\lambda v_\lambda \to -c(0)$ and $v_\lambda - v_\lambda(x_0) \to v$ as $\lambda \to 0^+$ for a fixed $x_0 \in \Omega$ where $v$ solves the ergodic problem:

$$
\begin{cases}
H(x, Du(x)) - \epsilon \Delta u(x) \leq c_{\Omega} & \text{in } \Omega, \\
H(x, Du(x)) - \epsilon \Delta u(x) \geq c_{\Omega} & \text{on } \overline{\Omega}.
\end{cases}
$$

The additive eigenvalue denoted by $c(0)$ is defined as

$$
c_{\Omega} = \min \{ c \in \mathbb{R} : H(x, Du(x)) - \epsilon \Delta u(x) \leq c \text{ in } \Omega \text{ has a solution} \} \quad (1.1)
$$

and it is also the unique constant where $(0, \Omega)$ can be solved. Let $c_{\Omega_\lambda}$ be also the unique constant such that the cell problem on $\Omega_\lambda$ below can be solved:

$$
\begin{cases}
H(x, Du(x)) - \epsilon \Delta u(x) \leq c_{\Omega_\lambda} & \text{in } \Omega_\lambda, \\
H(x, Du(x)) - \epsilon \Delta u(x) \geq c_{\Omega_\lambda} & \text{on } \overline{\Omega_\lambda}.
\end{cases}
$$

The authors of [7] present some first properties of the additive eigenvalues with respect to the underlying domains when $1 < p \leq 2$. To be precise, let $c_{\Omega}$ be the additive eigenvalue of $H$ on
The domain \( \Omega \). They show that \( c_\Omega \leq c_{\Omega'} \) if \( \Omega \subset \Omega' \) and, in general, one has continuity with respect to the domain (under some appropriate perturbations). The deeper properties of this map (especially in the second-order case and with \( p > 2 \)) seem to remain uninvestigated.

In the linear case, the smoothness of \( \lambda \mapsto c_{\Omega_{\lambda}} \), the principal eigenvalue for elliptic equations, is vastly studied (see Section 6.3) for a general perturbation \( \Omega_{\lambda} = \{ x + \lambda V(x) : x \in \Omega \} \). When \( p = 2 \) and \( V = \text{Id} \), the problem we consider can be transformed into this linear problem, and we obtain the smoothness of this map via the so-called Hopf–Cole transform. We note that solutions to \((\lambda, \Omega)\) have different behaviors when \( p > 2 \) and \( 1 < p \leq 2 \), which solutions are also called large solution that explode on the boundary. The case \( 1 < p \leq 2 \) will be addressed in future work.

In the case of the first order Hamilton–Jacobi equations, using Mather measures from weak KAM theory, it is known that if \( \Omega_{\lambda} = (1 + \lambda) \Omega \) then \( \lambda \mapsto c_{\Omega_{\lambda}} \) is one-sided differentiable everywhere (see [33, Theorem 1.4]).

1.1. Assumptions and main results. Throughout the paper, we make the following assumptions.

\((A_1)\): The domain \( \Omega \) is a bounded star-shaped (with respect to the origin) open subset of \( \mathbb{R}^n \), and there exists some \( \kappa > 0 \) such that \( B(0, \kappa) \subset \subset \Omega \) and

\[
\text{dist}(x, \partial \Omega) \geq \kappa r \quad \text{for all } x \in (1 + r) \partial \Omega, \quad \text{for all } r > 0. \tag{1.2}
\]

The Hamiltonian is \( H(x, \xi) = |\xi|^p - f(x) \) where \( p > 2 \), \( f \in C^1(\mathbb{R}^n) \) and \( U \subset \mathbb{R}^n \) is an open, bounded set with smooth boundary such that \( \mathbb{R}^n \setminus U \).

Condition (1.2) is introduced in [9] to ensure the well-posedness of \((\lambda, \Omega)\). For Theorem 1.1, addressing vanishing discounts, we use the following changing domain setup.

\((A_2)\): We consider \( \Omega_{\lambda} := (1 + r(\lambda)) \Omega \) for \( \lambda > 0 \) such that \( \Omega_{\lambda} \subset U \), and \( r : [0, \infty) \to \mathbb{R} \) is continuous with \( r(0) = 0 \) and

\[
\lim_{\lambda \to 0^+} \frac{r(\lambda)}{\lambda} = \gamma \in (\infty, +\infty). \tag{1.3}
\]

We denote by \( L(x, v) \) the Legendre transform of \( H(x, \xi) \):

\[
L(x, v) = C_p |v|^q + f(x), \quad \text{where} \quad C_p = p^{-1/q}(p - 1), \quad p^{-1} + q^{-1} = 1. \tag{1.4}
\]

We write \( \nabla L(x, v) = (D_x L(x, v), D_v L(x, v)) \) for \( (x, v) \in \overline{\Omega} \times \mathbb{R}^n \). For a measure \( \mu \) on \( \overline{\Omega} \times \mathbb{R}^n \), we define

\[
\langle \mu, \varphi \rangle_{\Omega} := \int_{\overline{\Omega} \times \mathbb{R}^n} \varphi(x, v) \, d\mu(x, v), \quad \text{for } \varphi \in C(\overline{\Omega} \times \mathbb{R}^n) \cap L^1(\mu). \tag{1.5}
\]

Let \( \mathcal{M}(\Omega) \) be the set of viscosity Mather measures (Definition 3). We first show the convergence for the following vanishing discount problem,

\[
\begin{cases}
\lambda u_{\lambda}(x) + H(x, Du_{\lambda}(x)) - \varepsilon \Delta u_{\lambda}(x) \leq 0 & \text{in } \Omega_{\lambda}, \\
\lambda u_{\lambda}(x) + H(x, Du_{\lambda}(x)) - \varepsilon \Delta u_{\lambda}(x) \geq 0 & \text{on } \partial \Omega_{\lambda},
\end{cases}
\]

Similar to [33], if \( r(\lambda) \) and \( \gamma \) are given in \((A_2)\) then \( u_{\lambda} + \lambda^{-1} c_{\Omega} \) is bounded and convergent.

**Theorem 1.1.** Assume \((A_1)\) and \((A_2)\). Let \( u_{\lambda} \in C(\overline{\Omega}_{\lambda}) \) be the solution to \((\lambda, \Omega_{\lambda})\).

(i) We have \( u_{\lambda} + \lambda^{-1} c_{\Omega} \to u^\gamma \) as \( \lambda \to 0 \) uniformly on \( \overline{\Omega} \) and \( u^\gamma \) is a solution to \((0, \Omega)\).

(ii) Furthermore \( u^\gamma = \max_{w \in E^\gamma} w \) where \( E^\gamma \) denotes the family of subsolutions \( w \) of \((0, \Omega)\) such that

\[
\gamma \langle \mu, (\cdot, v) \cdot \nabla L(x, v) \rangle \Omega + \langle \mu, w \rangle \Omega \leq 0 \quad \text{for all } \mu \in \mathcal{M}(\Omega). \tag{1.6}
\]

We note that \( u^\gamma \) does not depend on \( r(\lambda) \), but only on the limit \( \gamma \in \mathbb{R} \) of \( \lim_{\lambda \to 0^+} r(\lambda)/\lambda \).

As a consequence, if \( r(\lambda)/\lambda \to 0 \) as \( \lambda \to 0^+ \) then \( u^\lambda + \lambda^{-1} c_{\Omega} \to u^0 \), the same limiting solution to the vanishing discount problems on the fixed domain \( \Omega \) as in [21, Theorem 3.7].
Remark 1.

(i) The term viscosity Mather measure is introduced in [20, 21], in contrast to stochastic Mather measure introduced in [15]. We should denote the effect of viscosity by $M_\epsilon(\Omega)$ in (2.3), but for simplicity, we did not include $\epsilon$ since the results in this paper concern the limit as $\lambda \to 0^+$. Nevertheless, it is evident that weak limits of measures in $M_\epsilon(\Omega)$ correspond to measures in $M_0(\Omega)$ due to the stability of viscosity solutions.

(ii) In the proof of Theorem 1.2 (where $\epsilon > 0$) and its corresponding first-order analog [33] (where $\epsilon = 0$), the scaling differs due to the appearance of $-\epsilon \lambda$. We plan to address this discrepancy in an upcoming work, where we explore this issue together with the approach to this problem using the framework of [15] instead.

The next result concerns the differentiability of $c_\Omega$ with respect to the scaling parameter $r(\lambda)$. We observe that the choice of $r(\lambda)$ and $\gamma \in \mathbb{R}$ does not affect the study of the derivatives of $r(\lambda) \mapsto c_{1+r(\lambda)\Omega}$. To simplify notation, we assume the following.

(A3): We assume $r(\lambda) = \lambda$ for $\lambda \in (-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$. As $\Omega_\lambda = (1+\lambda)\Omega$, we will write $c(\lambda) = c_\Omega$ and $c(0) = c_\Omega$ for simplicity.

It is important to note that in the context of (A3), $\lambda$ does not represent a discount factor.

**Theorem 1.2.** Assume (A1), and (A3). The map $\lambda \mapsto c(\lambda)$ is one-sided differentiable:

$$c_+(0) = \lim_{\lambda \to 0^+} \left( \frac{c(\lambda) - c(0)}{\lambda} \right) = \max_{\mu \in M(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega}, \quad (1.7)$$

$$c_-(0) = \lim_{\lambda \to 0^-} \left( \frac{c(\lambda) - c(0)}{\lambda} \right) = \min_{\mu \in M(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega}. \quad (1.8)$$

A similar result is established for first-order Hamilton–Jacobi equations (see [33, Theorem 1.4]).

**Theorem 1.2** gives us that the one-sided derivatives $c_+(\lambda)$ exists everywhere for $\lambda \in (-\epsilon_0, \epsilon_0)$. We emphasize that the set of $\lambda$ where $c'(\lambda)$ does not exist is at most countable. We refer to [8, Theorem 4.2, Chapter 4] or [19, Theorem 17.9] for this result.

**Corollary 1.3.** Assume (A1) and (A3).

(i) The map $\lambda \mapsto c(\lambda)$ is Lipschitz, increasing, the one-sided derivatives $c'_+(\lambda)$ and $c'_-(\lambda)$ exist.

(ii) $\lambda \mapsto c'_-(\lambda)$ is left-continuous and $\lambda \mapsto c'_+(\lambda)$ is right-continuous.

(iii) The set of points where $\lambda \mapsto c(\lambda)$ is not differentiable is almost countable.

**Remark 2.** By Theorem 1.1, for each $\gamma \in \mathbb{R}$ there exists a solution $u^\gamma \in C(\overline{\Omega})$ to $(0, \Omega)$ and this solution. In other words, we have a well-defined map $\Lambda : \mathbb{R} \to C(\overline{\Omega})$. In contrast to the first-order case, solutions to $(0, \Omega)$ are unique up to adding a constant ([25]). If $\gamma \in \mathbb{R}$ then there exists a unique constant, denoted by $\mathcal{C}(\gamma) \in \mathbb{R}$ such that $u^\gamma(x) - u^0(x) = \mathcal{C}(\gamma)$ for all $x \in \overline{\Omega}$. We therefore can define $\mathcal{C}(\gamma)$ as a function from $\mathbb{R}$ to $\mathbb{R}$ by $\mathcal{C}(\gamma) := u^\gamma(\cdot) - u^0(\cdot) \in \mathbb{R}$.

**Theorem 1.4.** Assume (A1), and that $c'(0)$ exists, in the sense that $\theta \mapsto c_{(1+\theta)\Omega}$ has derivative at $\theta = 0$. Then $\mathcal{C} : \mathbb{R} \to \mathbb{R}$ is differentiable at 0 with $\mathcal{C}'(0) = -c'(0)$, and $\mathcal{C}(\gamma) = -\gamma c'(0)$ for $\gamma \in \mathbb{R}$. In other words, $u^\gamma(\cdot) = u^0(\cdot) - \gamma c'(0)$ for all $\gamma \in \mathbb{R}$.

Theorems 1.1, 1.2 and Corollary 1.3 extend similar findings from [33] for the first-order case. Details on the encountered challenges and their resolution are discussed in Section 1.2.

We gain further insight into the regularity of the map $\lambda \mapsto c(\lambda)$ with more details about $f(\cdot)$. Recall that a function $u : I \to \mathbb{R}$ is semiconvex on an interval $I \subset \mathbb{R}$ if $x \mapsto u(x) + (2\tau)^{-1}|x|^2$ is convex for some $\tau > 0$.

**Theorem 1.5.** Assume (A1), (A3) and $f = \text{const}$. Then $\lambda \mapsto c(\lambda)$ belongs to $C^\infty$ in its domain.
Theorem 1.6. Assume \((A_1), (A_3)\) and \(f\) is semiconcave. Then \(\lambda \mapsto c(\lambda)\) is semiconvex in its domain. As a consequence, it is twice differentiable almost everywhere.

Assume \((A_1), (A_3)\) and \(p = 2\), the Hopf-Cole transform reveals that \(\lambda \mapsto c(\lambda)\) is smooth, and \(c'(0)\) has an integral representation (Section 6.3), known as Hadamard’s variational formula ([13, p. 369]). This holds for general perturbations \(\Omega_\lambda = (\Id + \lambda V)(x) : x \in \Omega\), where \(V\) is a smooth vector field \(\mathbb{R}^n \to \mathbb{R}^n\). Our scaling setting represents the special case where \(V = \Id\).

Theorem 1.4 and findings in Section 6, such as Theorem 1.5 and Corollary 6.1, are new and exclusive to second-order scenarios.

1.2. Contributions. We mention our contributions here, mainly in two parts:

(i) The technical development in Theorems 1.1, 1.2 and Corollary 1.3.
(ii) The new connection in Theorem 1.4 and observations in Theorems 1.5 and 1.6.

For (i), the challenge arises from the new scaling with the second-order term, along with the absence of finite speed of propagation, in contrast to the first-order case. We address this by employing the tool developed in [20, 21] and carefully tracking all conditions where the representation formula can be applied. Another potential way to study this problem is using the stochastic analogue of Aubry-Mather theory from [15]. The advantage of the duality framework in [20, 21] is its compatibility with state-constraint boundary conditions, whereas tools from [15] require some technical adaptations for different boundary conditions. This aspect will be addressed in future work.

• For the first-order case, using the finite speed of propagation, one can replace the space of the test function as \(\Phi^+(\Omega) = C(\bar{\Omega} \times \mathbb{R}^n)\) by \(\Phi^+(\Omega) = C(\bar{\Omega} \times \bar{B}_h)\) for some \(h > 0\).
• For the second-order case, the duality framework in [20, 21] works if one restrict the space of test functions to \(\Phi^+(\Omega) = \{tL(x, v) + \chi(x) : t > 0, \chi \in C(\bar{\Omega})\}\).

It turns out that the new scaling of the equation in the second-order case works perfectly with this new space of test functions, which results in the following differences:

• We must assume that the Hamiltonian is separable, as we employ \(H(x, \xi) = |\xi|^p - f(x)\) in this paper. The exploration of the general Hamiltonian will be pursued in future work.
• In contrast to the first-order case, the scaling structure is captured by \(\nabla_{(x,v)} L\) instead of just the \(x\)-gradient \(D_x L\) as seen in the first-order case ([33]). This can be roughly explained as a variation in the scaling structure of the equation and the utilization of the space of test functions.

For (ii), this is a new result that is only available in the second-order case. The result also contains a connection to the problem of characterization of minimizing measures, i.e., how all minimizing measures can be approximated. At the moment, we cannot show the differentiability of this map yet unless some special cases, as in Theorems 1.5 or the case where \(p = 2\) (Corollary 6.1).

Remark 3. According to the [20, 21], the dependence of minimizing measures \(\mathcal{M}(\Omega)\) on the choice of the space of test functions is unknown. This creates a difference in the limiting procedure as \(\varepsilon\) approaches 0 (in the first-order case) in representations (1.7), (1.8), (1.6) when a different space of test functions is used (as in this paper and [33]). This aspect will be considered in future work.

Remark 4. The problem of characterizing Mather measures has been studied, as in [14]. One approach to proving their existence is through approximation from a related vanishing discount problem. A key question arises: can all viscosity Mather measures be described by \(\bigcup_{\gamma \in \mathbb{R}} \cup_{\gamma}(\Omega)\) (Definition 7)?
1.3. Literature. A general form of $(\lambda, \Omega)$ is studied in [20, 21], where the authors develop viscosity Mather Measures using duality. A different notion of viscosity Mather measures for second-order equation (in the periodic setting) defined using a stochastic analogue of the Aubry-Mather theory has been studied earlier in [15]. In contrast to the first-order case, solutions to the ergodic problem $(0, \Omega)$ are unique up to adding a constant (see [25]). This makes the vanishing discount problem on a fixed domain simpler to show, and it was shown in [25] using a pure PDE approach. The first-order case is harder, and it is a non-trivial problem to characterize the limiting solution ([11, 20, 21, 32]).

If $p > 2$, there exists a unique bounded viscosity solution $u \in C(\overline{\Omega})$ that is Hölder continuous, and it is a maximal solution in $W^{2,p}_{\text{loc}}(\Omega)$ (for all $1 < r < \infty$) of
\[
\delta u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) = 0 \quad \text{in } \Omega.
\] (1.9)

When $1 < p \leq 2$, equation $(\lambda, \Omega)$ can be written as (see [25]) (1.9) coupled with $= +\infty$ on $\partial \Omega$, and solutions can be understood in the classical sense. When $p > 2$, a solution is uniformly bounded up to $\partial \Omega$, and the equation is understood in the viscosity sense ([1, 5, 25]). We refer the readers to [27] (elliptic problem with subquadratic growth), [2, 6, 7, 31] (time-dependent problem), [1, 25] (elliptic problem) and the references therein for more properties of solutions and related problems. The rate of convergence as $\varepsilon \to 0$ is studied in [16].

The vanishing discount problem on fixed domain concerns the convergence of solutions is considered in [22] for the case, the domain is $\mathbb{R}^n$, [11, 20, 26, 32, 35] for the case the domain is torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$, and [21] for a general domain with various boundary conditions. We refer the readers also to [34] for a vanishing discount problem with more general nonlinear dependence on the monotone term. The case of changing domain for $\varepsilon = 0$ is studied in [33].

1.4. Organization of the paper. The paper is organized in the following way. In Section 2, we provide the background results on state-constraint Hamilton–Jacobi equations, estimates on solutions, the duality representation, definitions, and properties of minimizing measures that will be needed throughout the paper. The proof of Theorem 1.1 is provided in Section 3. Section 4 is devoted to proving the main result of Theorem 1.2 and its consequences on getting the estimate on the nested domain. The connection between $c'(0)$ and $\mathcal{C}'(0)$ as in Theorem 1.4 is discussed in Section 5. The last section records some discussions (Theorems 1.5, 1.6) and open questions, as well as future research related to the theme of the paper. In particular, we investigate a very special case where $p = 2$ (it falls into the subquadratic case $1 < p \leq 2$) using a tool from elliptic theory, which provides a different perspective on the derivative $c'(0)$ of the additive eigenvalue.

2. Preliminaries

We say $u$ solves $(\delta, 0)$ if it solves the state-constraint problem on $\mathcal{O}$ with discount factor $\delta \geq 0$ as in Definition 4. We make the assumption regarding the geometry of $\Omega$ as specified in $(A_1)$.

2.1. State-constraint solutions and the representation formula. Proofs for certain results in this section, as well as background information on viscosity solutions, can be found in Appendix A.

Definition 1. We say that $u$ is a viscosity state-constraint solution of $(\lambda, \Omega)$ if $u$ is a subsolution of the equation in $\Omega$ and is a supersolution of the equation on $\overline{\Omega}$.

The state-constraint problem was first studied in [29] for the first-order equation using the interior sphere condition. Assumption $(A_1)$ is introduced in [9] as a simpler way to obtain uniqueness. The well-posedness for $(\lambda, \Omega)$ and the properties of its solution can be summarized as follows.

Theorem 2.1. Assume $(A_1)$ and $\lambda \geq 0$. If $u$ is a viscosity solution to $(\lambda, \Omega)$ then $u \in C^{0,\alpha}(\overline{\Omega})$ where $\alpha = (p - 2)/(p - 1)$. Furthermore:
(i) If $\lambda > 0$ then there exists a unique continuous viscosity solution $u$, and it is the maximal subsolution of $\lambda u + H(x, Du) - \varepsilon \Delta u = 0$ in $\Omega$ in the class of upper semicontinuous subsolutions.

(ii) If $\lambda = 0$, then up to adding a constant, there exists a unique continuous viscosity solution.

**Theorem 2.2** (Comparison principle). Assume $(A_1)$ and $\lambda > 0$. Let $u, v \in C(\overline{\Omega})$ be a viscosity subsolution in $\Omega$ and supersolution on $\overline{\Omega}$ of $\lambda u + H(x, Du) - \varepsilon \Delta u = 0$, respectively. Assume that $u, v$ are Hölder continuous, i.e., belong to $C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1]$, then $u(x) \leq v(x)$ for all $x \in \overline{\Omega}$.

An important consequence is the following bound (consequence from Proposition A.6).

**Proposition 2.3.** Assume $(A_1)$ and $\lambda > 0$. The solution $u_\lambda$ of $(\lambda, \Omega)$ satisfies

(i) $-\max_{x \in \Omega} H(x, 0) \leq \lambda u_\lambda(x)$ for $x \in \Omega$.

(ii) if $B(0, \kappa) \subset \Omega$ then $\lambda u_\lambda(x) \leq C(\kappa)$ for $x \in \overline{B(0, \kappa)}$, where $C(\kappa)$ depends only on $\kappa$.

We employ the representations in [20, 21] (see also [32, Theorem 6.3]). We note that a different representation not using duality is given in [15] for the periodic setting.

We denote by $\mathcal{P}$ the set of all probability measures, and likewise, if $E$ is a metric space, we denote by $\mathcal{P}_E$ the set of all probability measures on $E$. The set of all Radon measures (resp., nonnegative Radon measures) in $E$ is denoted by $\mathcal{R}(E)$ (resp., $\mathcal{R}^+(E)$).

**Definition 2.** Let us define

$$\mathcal{F}_\delta, \Omega = \left\{ (\phi, u) \in \mathcal{F}(\overline{\Omega} \times \mathbb{R}^n) : \delta u + H(x, Du) - \varepsilon \Delta u \leq 0 \text{ in } \Omega \right\},$$

$$G_{z, \delta, \Omega} = \left\{ \phi - \delta u(z) : (\phi, u) \in \mathcal{F}_\delta, \Omega \right\},$$

$$G'_{z, \delta, \Omega} = \left\{ \mu \in \mathcal{R}(\overline{\Omega} \times \mathbb{R}^n) : \langle \mu, g \rangle_\Omega \geq 0 \text{ for all } g \in G_{z, \delta, \Omega} \right\}.$$

When $\delta = 0$, we note that $G_{z, 0, \Omega}$ and $G'_{z, 0, \Omega}$ are independent of $z \in \overline{\Omega}$. Therefore we denote

$$G_{0, \Omega} := G_{z, 0, \Omega} \quad \text{and} \quad G'_{0, \Omega} := G'_{z, 0, \Omega} \quad \text{for all } z \in \overline{\Omega}.$$

Precisely, we have

$$G_{0, \Omega} = \left\{ \phi \in \mathcal{F}^+(\overline{\Omega} \times \mathbb{R}^n) : \text{there exists } u \in C(\overline{\Omega}) : H(x, Du) - \varepsilon \Delta u \leq 0 \text{ in } \Omega \right\},$$

$$G'_{0, \Omega} = \left\{ \mu \in \mathcal{R}(\overline{\Omega} \times \mathbb{R}^n) : \langle \mu, g \rangle_\Omega \geq 0 \text{ for all } g \in G_{0, \Omega} \right\}.$$

We observe that $\Phi^+(\overline{\Omega} \times \mathbb{R}^n)$ is a convex cone in $C(\overline{\Omega} \times \mathbb{R}^n)$ and $(x, \xi) \mapsto H_\phi(x, \xi)$ is well-defined and continuous for $\phi \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n)$.

**Theorem 2.4** ([21, Theorem 3.3]). Assume $(A_1)$. Let $(z, \lambda) \in \overline{\Omega} \times (0, \infty)$ and $u_\lambda \in C(\overline{\Omega})$ be a solution of $(\lambda, \Omega)$. Then for $\lambda > 0$ there holds

$$\lambda u_\lambda(z) = \min_{\mathcal{P} \cap G'_{0, \Omega}} \langle \mu, L \rangle_\Omega \quad \text{and} \quad -c_\Omega = \min_{\mathcal{P} \cap G_{0, \Omega}} \langle \mu, L \rangle_\Omega.$$ 

**Definition 3** (Viscosity Mather measures, [20, 21]). We define

$$\mathcal{M}(\Omega) := \left\{ \mu \in \mathcal{P} \cap G'_{0, \Omega} : -c_\Omega = \langle \mu, L \rangle_\Omega \right\}.$$
We will use Definition 2 mostly in the following way.

**Corollary 2.5.** Let $\phi \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n)$ and $\delta \geq 0, z \in \overline{\Omega}$ and $u \in C(\overline{\Omega})$ such that

$$\delta u + H_\phi(x, Du) - \varepsilon \Delta u \leq 0 \text{ in } \Omega.$$ 

(i) If $\mu_\delta \in \mathcal{G}_{z, \delta, \Omega}$ then $\langle \mu_\delta, \phi - \delta u(z) \rangle = 0.$

(ii) If $\mu \in \mathcal{G}_{0, \delta, \Omega}$ then $\langle \mu, \phi - \delta u \rangle \geq 0.$

**Proof.** The conclusion follows from $H_{\phi - \delta u}(x, Du) - \varepsilon \Delta u = \delta u + H_\phi(x, Du) - \varepsilon \Delta u \leq 0.$

2.2. **Settings for problems on changing domains.**

**Definition 4.** For an open subset $\Omega \subset \mathbb{R}^n$ and $\delta > 0$, we say $u \in C(\Omega)$ solves $(\delta, \Omega)$ if $u$ solves

$$\begin{cases}
\delta u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) \leq 0 & \text{in } \Omega, \\
\delta u(x) + |Du(x)|^p - f(x) - \varepsilon \Delta u(x) \geq 0 & \text{on } \overline{\Omega}
\end{cases}
$$

$(\delta, \Omega)$

If $\delta = 0$, we say $u \in C(\Omega)$ solves $(0, \Omega)$ if $u$ solves

$$\begin{cases}
|Du(x)|^p - f(x) - \varepsilon \Delta u(x) \leq c(\Omega) & \text{in } \Omega, \\
|Du(x)|^p - f(x) - \varepsilon \Delta u(x) \geq c(\Omega) & \text{on } \overline{\Omega}
\end{cases}
$$

$(0, \Omega)$

where $c(\Omega)$ is the additive eigenvalue on $\Omega$, as defined in (1.1).

**Lemma 2.6.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume $(A_1)$ for $\Omega$.

(i) For $\delta > 0$, the unique solution $u_\delta$ to $(\delta, \Omega)$ satisfies

$$|\delta u_\delta(x) + c(\Omega)| \leq C_2 \delta$$

for all $\delta > 0, x \in \overline{\Omega}$,

where $C_2$ depends only on diam$(\Omega), \|f\|_{L^\infty(\Omega)}$ and $p$.

(ii) There holds $|c(\Omega)| \leq C$ where $C$ depends only on $\min_{\overline{\Omega}} f$ on $\Omega$ and $\kappa$ as defined in $(A_1)$.

**Proof.** Thanks to the H"older estimate of $u_\delta$ in Theorem 2.1, we have

$$u_\delta(x) - u_\delta(0) \rightarrow w(x), \quad \delta u_\delta(x) \rightarrow -c(0)$$

uniformly on $\Omega$ and $w$ solves the ergodic problem $(0, \Omega)$ with the bound $|w(x)| \leq C|x|^\alpha \leq C$, for all $x \in \Omega$ where $C = C(\text{diam}(\Omega), \|f\|_{L^\infty(\Omega)}, p)$. Now $w(\cdot) + \delta^{-1} c(\Omega)$ and $w(\cdot) - \delta^{-1} c(\Omega)$ are supersolution and subsolution to $(\delta, \Omega)$, respectively. By comparison principle we deduce that $|\delta u_\delta(\cdot) + c(\Omega)| \leq \delta \|w\|_{L^\infty(\Omega)}.$

For (ii), let $u_\delta \in C(\Omega)$ solves $(\delta, \Omega)$, i.e., the state-constraint problem with discount factor $\delta$ on $\Omega$. By Proposition 2.3, $|\delta u_\delta(0)| \leq C$ where $C$ depends only on $\min\{f(x) : x \in \overline{\Omega}\}$ and $\kappa$. Let $\delta \rightarrow 0^+$ we deduce that $|c(\Omega)| \leq C.$

We obtain an improved bound for $\delta u_\delta(\cdot)$ that for solution $u_\delta(\cdot)$ of $(\delta, \Omega)$.

**Corollary 2.7.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and assume $(A_1)$ for $\Omega$. There exists a constant $C$ that depends only on $\min_{\overline{\Omega}} f$ and $\kappa$ such that:

(i) If $|r(\lambda)|$ is small so that $B(0, \kappa) \subset \Omega_{\Omega_1}$ then $|c_{\Omega_1}| \leq C$.

(ii) If $u_\delta(\cdot)$ solves $(\delta, \Omega)$ then $|\delta u_\delta(\cdot)| \leq C + C_2 \delta$, where $C_2$ is defined in Lemma 2.6.

The error estimate for discount solutions on slightly different domains implies that the map $\lambda \mapsto c(\lambda)$ is H"older continuous with degree $\alpha = (p - 2)/(p - 1)$. However, it could be of interest on its own, as seen in the setting of [23]. We provide the proof in Appendix B for completeness.
Theorem 2.8 (Nested domains estimate). Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain, and assume \((A_1)\) for \( \Omega \). Let \( \theta > 0 \), \( 0 < \delta < 1 \) and \( u \in C((1 + \theta)\Omega) \) and \( v \in C(\Omega) \) be solutions to \((\delta, (1 + \theta)\Omega)\) and \((\delta, \Omega)\), respectively. Then, it holds that

\[
0 \leq \delta v(x) - \delta u(x) \leq C \theta^\alpha, \quad \text{for all } x \in \overline{\Omega},
\]

where \( \alpha = (p - 2)/(p - 1) \) and \( C \) depends on \( \text{diam}(\Omega) \), the Lipschitz constant of \( f \) on \( U \) and \( p \).

Corollary 2.9. Assume \((A_1)\). If \( \Omega' = (1 + \theta)\Omega \) then

\[
|c_{\Omega'} - c_\Omega| \leq C (\lambda + |\theta|^\alpha) \quad \text{where } \alpha = (p - 2)/(p - 1).
\]

Consequently, under \((A_3)\) then \( \lambda \mapsto c(\lambda) \) is Hölder continuous of degree \( \alpha \).

Proof. Let \( u_\lambda \in C(\Omega') \) be a solution to \((\lambda, \Omega')\) and \( v_\lambda \in C(\Omega) \) be a solution to \((\lambda, \Omega)\), respectively. From Theorem 2.8 and Lemma 2.6 we obtain

\[
|c_{\Omega'} - c_\Omega| \leq |\lambda u_\lambda(0) + c_{\Omega'}| + |\lambda v_\lambda(0) + c_\Omega| + \lambda |u_\lambda(0) - v_\lambda(0)| \leq 2C\lambda + C|\theta|^\alpha.
\]

Therefore \( \lambda \mapsto c(\lambda) \) is Hölder continuous of degree \( \alpha \) if we assume \((A_3)\).

We can improve that \( \lambda \mapsto c(\lambda) \) is Lipschitz in Theorem 1.2.

2.3. Limit of minimizing measures.

Definition 5. Let \( \sigma \) be a measure on \((1 + r)\overline{\Omega} \times \mathbb{R}^n\), we define \( \tilde{\sigma} \) as a measure on \( \overline{\Omega} \times \mathbb{R}^n \) by

\[
\int_{\overline{\Omega} \times \mathbb{R}^n} \phi(x, v) \, d\tilde{\sigma}(x, v) = \int_{(1 + r)\overline{\Omega} \times \mathbb{R}^n} \phi \left( \frac{x}{1 + r}, v \right) \, d\sigma(x, v), \quad \phi \in C(\overline{\Omega} \times \mathbb{R}^n). \tag{2.4}
\]

Using \( \phi \equiv 1 \) we see that this scaling is mass-preserving. Recall that \( \Omega_\lambda = (1 + r(\lambda))\Omega \).

Lemma 2.10. Assume \((A_1)\) and \((A_2)\).

(i) Let \( \{\mu_\lambda\} \) be a sequence of measures in \( M(\lambda, \Omega_\lambda) \). Let \( \{\overline{\mu}_\lambda\} \) be its scaling defined on \( \overline{\Omega} \times \mathbb{R}^n \) and \( \mu_\lambda \rightharpoonup \mu \) along a subsequence to some measure \( \mu \), then \( \mu \in M(\Omega) \).

(ii) Let \( \{\nu_\lambda\} \) be a sequence of measures in \( M(0, \Omega_\lambda) \). Let \( \{\overline{\nu}_\lambda\} \) be its scaling defined on \( \overline{\Omega} \times \mathbb{R}^n \) and \( \nu_\lambda \rightharpoonup \mu \) along a subsequence to some measure \( \mu \), then \( \mu \in M(\Omega) \).

The proof of this Lemma is ommitted as is quite standard (adaptation of [33, Lemma 3.6]).

3. THE VANISHING DISCOUNT PROBLEM

Let \( u_\lambda \in C(\overline{\Omega}_\lambda) \) be the solution to \((\lambda, \Omega_\lambda)\). In this section, we show \( u_\lambda + \lambda^{-1}c_\Omega \to u^\gamma \) as \( \lambda \to 0^+ \).

We scale \( u_\lambda \) into \( \tilde{u}_\lambda \in C(\overline{\Omega}) \) for convenience by

\[
\tilde{u}_\lambda(x) = \frac{u_\lambda \left( (1 + r(\lambda))x \right)}{(1 + r(\lambda))^2}, \quad x \in \overline{\Omega}. \tag{3.1}
\]

We then have:

\[
\lambda (1 + r(\lambda))^2 \tilde{u}_\lambda(x) + H \left( (1 + r(\lambda))x, (1 + r(\lambda)) D\tilde{u}_\lambda(x) \right) - \varepsilon \Delta \tilde{u}_\lambda(x) \leq 0 \quad \text{in } \Omega. \tag{3.2}
\]

The main idea is scaling equation from a domain to a different domain while maintaining the following (equivalent) forms

\[
\delta u + H_\phi(x, Du) - \varepsilon \Delta u \leq 0 \quad \text{in } \emptyset \quad \text{or} \quad H_{\phi - \delta u}(x, Du) - \varepsilon \Delta u \leq 0 \quad \text{in } \emptyset, \tag{3.3}
\]

where \( \emptyset = \Omega \) or \( \Omega_\lambda \), and then applying Corollary 2.5 in an appropriate way. More importantly, for \( \alpha, \beta \in \mathbb{R} \setminus \{0\} \) we have

\[
H(\alpha x, \beta Du) = H_\phi(x, Du) \quad \text{where} \quad \phi(x, v) = L \left( \alpha x, \beta^{-1}v \right). \tag{3.4}
\]
Indeed, we have
\[
H_\phi(x, Du(x)) = \sup_v \left( v \cdot Du(x) - L(\alpha x, \beta^{-1}v) \right)
= \sup_{\beta \hat{\phi}} \left( \hat{\phi} \cdot \beta Du(x) - L(\alpha x, \hat{\phi}) \right)
= \sup_\phi \left( \phi \cdot \beta Du(x) - L(\alpha x, \phi) \right) = H(\alpha x, \beta Du(x))
\]
due to Legendre transform.

**Lemma 3.1.** Assume \((A_1)\) and \((A_2)\). If \(u_\lambda \in C(\overline{\Omega}_\lambda)\) solves \((\lambda, \Omega_\lambda)\) then,
\[
|\lambda u_\lambda(x) + c_{\Omega_\lambda}| \leq C\lambda \quad \text{and} \quad |\lambda u_\lambda(x) + c_\Omega| \leq C(\lambda + |r(\lambda)|),
\]
for all \(x \in \overline{\Omega}_\lambda\), where \(C\) is independent of \(\lambda\).

**Proof.** The first inequality is a consequence of Lemma 2.6. The second inequality is the consequence of the first inequality and \(|c_{\Omega_\lambda} - c_\Omega| \leq C|r(\lambda)|\) as in Corollary 1.3. \(\square\)

We use the normalization \(\hat{u}_\lambda(\cdot) + \lambda^{-1}c_\Omega\) to make it consistent with the first-order case [33].

**Proof of Theorem 1.1.** We split the proof into 3 steps. In steps 1 and 2, we establish two inequalities that, when used together, yield the limit of \(\hat{u}_\lambda + \lambda^{-1}c_\Omega \to u^T\). In step 3, we give a characterization of \(u^T\). We start by observing that the sequence
\[
\hat{u}_\lambda(\cdot) + \frac{c_\Omega}{\lambda} \tag{3.5}
\]
is bounded and equicontinuous, thus it has convergence subsequence. We show that such a limit is unique. Assume that there exist \(\lambda_j \to 0\) and \(\delta_j \to 0\) such that
\[
\lim_{\lambda_j \to 0} \left( \hat{u}_{\lambda_j}(x) + \frac{c_\Omega}{\lambda_j} \right) = u(x) \quad \text{and} \quad \lim_{\delta_j \to 0} \left( \hat{u}_{\delta_j}(x) + \frac{c_\Omega}{\delta_j} \right) = w(x) \tag{3.6}
\]
locally uniformly as \(j \to \infty\). For given \(z \in \overline{\Omega}\), we show that \(u(z) = w(z)\). We will show the following inequalities,
\[
\gamma \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \langle \mu, w \rangle_\Omega - 2\gamma c_\Omega \leq 0, \quad \forall \mu \in M(\Omega), \tag{3.7}
\gamma \langle \mu_0, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \langle \mu_0, w \rangle_\Omega - 2\gamma c_\Omega + u(z) - w(z) \geq 0, \quad \text{for some } \mu_0 \in M(\Omega). \tag{3.8}
\]
Then, consequently \(u(z) \geq w(z)\). We can reverse the roles of \(u(z)\) and \(w(z)\) to obtain \(u(z) \leq w(z)\) and thus \(u(z) \equiv w(z)\).

**Step 1.** We show (3.7).

We start with equation of \(\hat{u}_\lambda\) as in (3.2). Using the strategy in (3.3), (3.4) we obtain
\[
\begin{cases}
H_\phi(x, D\hat{u}_\lambda(x)) - \epsilon \Delta \hat{u}_\lambda(x) \leq 0 & \text{in } \Omega \\
\phi(x, v) = L \left( (1 + r(\lambda)) x, \frac{v}{1 + r(\lambda)} \right) - \lambda (1 + r(\lambda))^2 \hat{u}_\lambda(x) \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n).
\end{cases}
\]

Using Corollary 2.5 (ii) and definition of \(M(\Omega)\) we have
\[
\langle \mu, \phi \rangle_\Omega \geq 0 \quad \mu \in M(\Omega),
-\langle \mu, L \rangle_\Omega = c_\Omega \quad \mu \in M(\Omega).
\]
Therefore
\[
\left\langle \mu, L \left[ (1 + r(\lambda)) x, \frac{v}{1 + r(\lambda)} \right] - \lambda (1 + r(\lambda))^2 \bar{u}_\lambda(x) - L(x, v) \right\rangle_{\Omega} \geq c_\Omega \quad \forall \mu \in \mathcal{M}(\Omega).
\]
We rearrange the left-hand side to match the form of (3.5):
\[
\left\langle \mu, L \left[ (1 + r(\lambda)) x, \frac{v}{1 + r(\lambda)} \right] - L(x, v) \right\rangle_{\Omega} \geq \left\langle \mu, \lambda (1 + r(\lambda))^2 \bar{u}_\lambda(x) + c_\Omega \right\rangle_{\Omega}
\]
\[
= \lambda(1 + r(\lambda))^2 \left( \mu, \bar{\mu}_\lambda + \frac{c_\Omega}{\lambda} \right)_{\Omega} + (1 - (1 + r(\lambda))^2) c_\Omega, \quad \forall \mu \in \mathcal{M}(\Omega).
\]
Divide both sides by \(\lambda\) and let \(\lambda = \delta \rightarrow 0^+\), by (3.6) we obtain
\[-\gamma \left\langle \mu, (-x, v) \cdot \nabla L(x, v) \right\rangle_{\Omega} \geq \left\langle \mu, w \right\rangle_{\Omega} - 2\gamma c_\Omega, \quad \forall \mu \in \mathcal{M}(\Omega).
\]
Therefore (3.7) follows.

**Step 2.** We show (3.8).

We have
\[H(x, Dw(x)) - \epsilon \Delta w(x) \leq c_\Omega, \quad \text{in } \Omega.
\]
We scale this equation into an equation on \(\Omega_\lambda\) as follows:
\[\begin{cases}
\tilde{w}_\lambda(x) = (1 + r(\lambda))^2 w \left( \frac{x}{1 + r(\lambda)} \right) & \text{on } \overline{\Omega}_\lambda, \\
H \left( \frac{x}{1 + r(\lambda)}, \frac{1}{1 + r(\lambda)} \right) Dw_\lambda(x) - \epsilon \Delta w_\lambda(x) \leq c_\Omega & \text{in } \Omega_\lambda.
\end{cases}
\]
Using the strategy in (3.3), (3.4) we obtain
\[\begin{cases}
\lambda \tilde{w}_\lambda(x) + H_\phi(x, Dw_\lambda(x)) - \epsilon \Delta w_\lambda(x) \leq 0 & \text{in } \Omega_\lambda \\
\psi(x, v) = L \left( \frac{x}{1 + r(\lambda)}, (1 + r(\lambda)) v \right) + \lambda \tilde{w}_\lambda(x) + c_\Omega \in \Phi^+ (\overline{\Omega}_\lambda \times \mathbb{R}^n)
\end{cases}
\]
Using Corollary 2.5 (i) we have
\[\left\langle \sigma, \phi - \lambda \tilde{w}_\lambda(z) \right\rangle_{\Omega_\lambda} \geq 0, \quad \sigma \in \mathcal{P} \cap \mathcal{G}_{z, \lambda, \Omega_\lambda}^\prime.
\]
For all \(\lambda > 0\), by Theorem 2.4 there exist \(\mu_\lambda \in \mathcal{P} \cap \mathcal{G}_{z, \lambda, \Omega_\lambda}^\prime\) such that
\[\left\langle \mu_\lambda, L \right\rangle_{\Omega_\lambda} = \lambda u_\lambda(z).
\]
In (3.9), we set \(\sigma = \mu_\lambda\), and by combining it with (3.10), we obtain:
\[\left\langle \mu_\lambda, L \left( \frac{x}{1 + r(\lambda)}, (1 + r(\lambda)) v \right) - L(x, v) + \lambda u_\lambda(z) - \lambda \tilde{w}_\lambda(z) + c(0) + \lambda \tilde{w}_\lambda \right\rangle_{\Omega_\lambda} \geq 0.
\]
Let \(\bar{\mu}_\lambda\) be the measure obtained from \(\mu_\lambda\) defined as in Definition 5. We write (3.11) as
\[\left\langle \bar{\mu}_\lambda, L \left( x, (1 + r(\lambda)) v \right) - L((1 + r(\lambda)) x, v) \right\rangle_{\Omega} + \lambda u_\lambda(z) - \lambda \tilde{w}_\lambda(z) + c_\Omega
\]
\[+ \lambda(1 + r(\lambda))^2 \left\langle \bar{\mu}_\lambda, w \right\rangle_{\Omega} \geq 0.
\]
Divide both sides by \(\lambda > 0\) we have
\[\left\langle \bar{\mu}_\lambda, \frac{L \left( x, (1 + r(\lambda)) v \right) - L((1 + r(\lambda)) x, v)}{\lambda} \right\rangle_{\Omega} + u_\lambda(z) - \tilde{w}_\lambda(z) + c_\Omega \lambda
\]
\[+ (1 + r(\lambda))^2 \left\langle \bar{\mu}_\lambda, w \right\rangle_{\Omega} \geq 0.
\]
By Lemma 2.10 there exists (up to subsequence) \( \mu_0 \in M(\Omega) \) such that \( \tilde{\mu}_\lambda \rightharpoonup \mu_0 \) in measure. Divide both sides by \( \lambda \) and let \( \lambda = \lambda_j \to 0 \). We note that
\[
u_\lambda(z) + \frac{c_\Omega}{\lambda} = (1 + r(\lambda))^2 \left[ \varphi_\lambda(x) \frac{z}{1 + r(\lambda)} + \frac{c(0)}{\lambda} \right] + \left( \frac{1 - (1 + r(\lambda))^2}{\lambda} \right) c_\Omega \to u(z) - 2\gamma c_\Omega
\]
as \( \lambda_j \to 0 \), thanks to (3.6). We obtain from (3.12) that
\[
\gamma \langle \mu_0, (-x, v) \cdot \nabla L(x, v) \rangle + u(z) - w(z) - 2\gamma c_\Omega + \langle \mu_0, w \rangle \geq 0.
\]
and thus (3.8) follows.

**Step 3.** We show the characterization (1.6).

From (3.7) and (3.8) we conclude that \( u \equiv w \). Let us denote the unique limit of \( \tilde{\nu}_\lambda(x) + \lambda^{-1} c(0) \) by \( \tilde{\nu}^\gamma \). We can get the relation between \( \tilde{\nu}^\gamma \) and \( u^\gamma \) by writing
\[
\tilde{\nu}_\lambda(x) + \frac{c_\Omega}{\lambda} = \frac{1}{(1 + r(\lambda))^2} \left( \varphi_\lambda(x) + \frac{c_\Omega}{\lambda} \right) + \frac{\varphi_\lambda((1 + r(\lambda))x) - \varphi_\lambda(x)}{(1 + r(\lambda))^2} + \frac{c_\Omega}{\lambda} \left( 1 - \frac{1}{(1 + r(\lambda))^2} \right).
\]
Let \( \lambda \to 0 \) we obtain
\[
\tilde{\nu}^\gamma = u^\gamma + 2\gamma c_\Omega.
\]
In (3.8) let \( u = \tilde{\nu}^\gamma \) we obtain
\[
\gamma \langle \mu_0, (-x, v) \cdot \nabla L(x, v) \rangle + \langle \mu_0, w \rangle + u^\gamma(z) - w(z) \geq 0.
\]
If \( w \in E^\gamma \) then by definition it implies \( u^\gamma(z) \geq w(z) \). On the other hand, (3.7) reads \( u^\gamma \in E^\gamma \). Therefore \( u^\gamma = \max E^\gamma \). □

We omit the proof of the following result as it is similar to the first-order case in [33, Corollary 1.3] once Theorem 1.1 is established.

**Corollary 3.2.** The mapping \( \gamma \mapsto u^\gamma(\cdot) \) is concave and decreasing. Precisely, if \( \alpha \leq \beta \) then \( u^\beta \leq u^\alpha \) and \( (1 - s)u^\alpha + su^\beta \leq u^{(1-s)\alpha + s\beta} \) for \( s \in (0, 1) \).

4. **The Differentiability of the Additive Eigenvalue**

In this section, we prove Theorem 1.2 and Corollary 1.3.

4.1. **The one-sided derivatives of the eigenvalue.**

**Proof of Theorem 1.2.** We will divide the proof of the limit in (1.7) into three parts. Let us consider \( \lambda_0 \) first and prove (1.7) as (1.8) is similar. In steps 1 and 2, we establish inequalities for lim inf and lim sup of the limit (1.7). Then, we combine these inequalities together in step 3 to obtain the conclusion.
Step 1. We show that

\[
\liminf_{\lambda \to 0^+} \left( \frac{c(\lambda) - c(0)}{\lambda} \right) \geq \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} \quad \text{for all } \mu \in \mathcal{M}(\Omega). \tag{4.1}
\]

Let \( w \) be a subsolution to

\[
H(x, Dw(x)) - \epsilon \Delta w(x) \leq c(\lambda) \quad \text{in } \Omega_\lambda.
\tag{4.2}
\]

We scale (4.2) into an equation on \( \Omega \) as follows:

\[
\begin{cases}
\tilde{w}(x) = \frac{w(1 + \lambda)}{(1 + \lambda)^2} & \text{on } \overline{\Omega}.
\end{cases}
\tag{4.3}
\]

Then, using the strategy in (3.3), (3.4) we obtain

\[
\begin{cases}
H_{\phi}(x, D\tilde{w}(x)) - \epsilon \Delta \tilde{w}(x) \leq 0 & \text{in } \Omega
\end{cases}
\]

\[
\phi(x, v) = L \left( (1 + \lambda) x, \frac{v}{1 + \lambda} \right) + c(\lambda) \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n).
\]

Using Corollary 2.5 (ii) and the definition of \( \mathcal{M}(\Omega) \) we obtain

\[
\langle \mu, \phi \rangle_{\Omega} \geq 0, \quad \mu \in \mathcal{M}(\Omega),
\]

\[
-\langle \mu, L \rangle_{\Omega} = c(0), \quad \mu \in \mathcal{M}(\Omega).
\]

Therefore

\[
\langle \mu, L \left( (1 + \lambda) x, \frac{v}{1 + \lambda} \right) - L(x, v) \rangle_{\Omega} + c(\lambda) - c(0) \geq 0, \quad \forall \mu \in \mathcal{M}(\Omega). \tag{4.4}
\]

Thus if \( \lambda > 0 \) we deduce that

\[
-\langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} + \liminf_{\lambda \to 0^+} \left( \frac{c(\lambda) - c(0)}{\lambda} \right) \geq 0 \quad \forall \mu \in \mathcal{M}(\Omega). \tag{4.5}
\]

Therefore, we obtain our conclusion of the first step, equation (4.1).

Step 2. We will show that there exists a measure \( \sigma_0 \in \mathcal{M}(\Omega) \) such that

\[
\limsup_{\lambda \to 0^+} \left( \frac{c(\lambda) - \sigma_0}{\lambda} \right) \leq \langle \sigma_0, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega}. \tag{4.6}
\]

To see that, we choose \( \lambda_j \to 0^+ \) as a subsequence along which the lim sup is attained:

\[
\limsup_{\lambda \to 0^+} \left( \frac{c(\lambda) - c(0)}{\lambda} \right) = \lim_{j \to \infty} \left( \frac{c(\lambda_j) - c(0)}{\lambda_j} \right). \tag{4.7}
\]

Let \( u \in C(\overline{\Omega}) \) be such that

\[
H(x, Du(x)) - \epsilon \Delta u(x) \leq c(0) \quad \text{in } \Omega. \tag{4.8}
\]

We scale (4.8) into an equation on \( \Omega_\lambda \) as follows:

\[
\begin{cases}
\tilde{u}(x) = (1 + \lambda)^2 u \left( \frac{x}{1 + \lambda} \right) & \text{on } \overline{\Omega}_\lambda,
\end{cases}
\tag{4.9}
\]

\[
H \left( \frac{x}{1 + \lambda}, \frac{1}{1 + \lambda} D\tilde{u}(x) \right) - \epsilon \Delta \tilde{u}(x) \leq c(0) \quad \text{in } \Omega_\lambda.
\]
Then, using the strategy in (3.3), (3.4) we obtain
\[
\begin{align*}
H_p(x, D\sigma(x)) - \epsilon \Delta \sigma(x) \leq 0 & \quad \text{in } \Omega, \\
\psi(x, v) = L \left( \frac{x}{1+\lambda}, (1+\lambda)v \right) + c(0) \in \Phi^+(\bar{\Omega}_\lambda \times \mathbb{R}^n)
\end{align*}
\]
Using Corollary 2.5 (ii) and definition of \( M(\Omega_\lambda) \) we have
\[
\langle \sigma, \phi \rangle_{\Omega_\lambda} \geq 0 \quad \sigma \in M(\Omega_\lambda) \\
- \langle \sigma, L \rangle_{\Omega_\lambda} = c(\lambda) \quad \sigma \in M(\Omega_\lambda).
\]
Therefore
\[
\langle \sigma, L \left( \frac{x}{1+\lambda}, (1+\lambda)v \right) - L(x, v) \rangle_{\Omega_\lambda} - c(\lambda) + c(0) \geq 0, \quad \forall \sigma \in M(\Omega_\lambda).
\] (4.10)
Let \( \sigma_\lambda \) be a measure in \( M(\Omega_\lambda) \) and \( \tilde{\sigma}_\lambda \) be its scaling as in Definition 5. We write (4.10) as
\[
\langle \tilde{\sigma}_\lambda, L(D, (1+\lambda)v) - L((1+\lambda)x, v) \rangle_{\Omega} \geq c(\lambda) - c(0).
\] (4.11)
If \( \lambda > 0 \), we have
\[
\langle \tilde{\sigma}_\lambda, L(D, (1+\lambda)v) - L((1+\lambda)x, v) \rangle_{\Omega} \geq \frac{c(\lambda) - c(0)}{\lambda}.
\] (4.12)
Along the sequence \( \lambda_j \) that the limsup in (4.7) is attained, we can assume up to subsequence that \( \tilde{\sigma}_{\lambda_j} \to \sigma_0 \) and \( \sigma_0 \in M(\Omega) \). Let \( \lambda_j \to 0^+ \) in (4.12) we deduce that
\[
\langle \sigma_0, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} \geq \limsup_{\lambda \to 0^+} \frac{c(\lambda) - c(0)}{\lambda}.
\] (4.13)
Thus we obtain the conclusion (4.6) for our second steps.

**Step 3.** We combine (4.1) (take \( \mu = \sigma_0 \)) and (4.6) to obtain the conclusion (1.7), as:
\[
\lim_{\lambda \to 0^+} \frac{c(\lambda) - c(0)}{\lambda} = \langle \sigma_0, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} = \max_{\mu \in M(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega}.
\]
Similarly, if \( \lambda \leq 0 \) as \( \lambda \to 0^+ \) then we conclude (1.8),
\[
\lim_{\lambda \to 0^+} \frac{c(\lambda) - c(0)}{\lambda} = \min_{\mu \in M(\Omega)} \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega}.
\]
For an oscillating \( \lambda \) such that neither \( r^{-}(\lambda) = \min\{0, \lambda\} \) nor \( r^{+}(\lambda) = \max\{0, \lambda\} \) is identical to zero as \( \lambda \to 0^+ \), by applying the previous results we obtain the final conclusion. \( \square \)

4.2. Additional properties of the eigenvalue map.

**Lemma 4.1.** If \( \Omega, \Omega' \) are two domains that satisfy \((A_1)\). If \( \Omega \subset \Omega' \) then \( c_\Omega \leq c_{\Omega'} \).

**Proof.** If \( c \in \mathbb{R} \) and \( u \) is a subsolution for \( H(x, Du) - \epsilon \Delta u \leq c \) in \( \Omega' \) then it is also a subsolution to the same equation in \( \Omega \). Therefore, by (1.1), we obtain the conclusion. \( \square \)

**Lemma 4.2.** Assume \((A_1), (A_2)\). We have
\[
\langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} \geq 0 \quad \forall \mu \in M(\Omega).
\] (4.14)

**Proof.** The proof of (4.14) is a minor adaptation of [33, Lemma 3.5]. \( \square \)

From the proof of Theorem 1.2, we see that equations (4.4) and (4.12) imply that \( \lambda \mapsto c(\lambda) \) is indeed Lipschitz, which is stronger than Corollary 2.9. We verify that in the following Lemma.
Lemma 4.3. Assume $(A_1), (A_2)$. There exists $C = C(p, \text{diam}(U), \min_{U} f, \kappa)$ where $\kappa$ is defined in $(A_1)$ such that
\[
\langle \mu, |v|^q \rangle_{\Omega_\lambda} \leq C \quad \text{for all } \mu \in \mathcal{M}(\Omega_\lambda).
\]
Consequently, there exists a constant $C$ such that
\[
|c(\lambda) - c(0)| \leq C|\lambda|.
\]
(4.15)

Proof. For $\mu \in \mathcal{M}(\Omega_\lambda)$ we have
\[
-c(\lambda) = \langle \mu, \lambda \rangle_{\Omega_\lambda} = \langle \mu, C_q |v|^q + f(x) \rangle_{\Omega_\lambda} = C_q \langle \mu, |v|^q \rangle_{\Omega_\lambda} + \langle \mu, f \rangle_{\Omega_\lambda}.
\]
Therefore
\[
\langle \mu, |v|^q \rangle_{\Omega_\lambda} = -\frac{c(\lambda) + \langle \mu, f \rangle_{\Omega_\lambda}}{C_q} \leq \frac{1}{C_q} \left( \max_{U} |f| + C \right)
\]
where $C$ is the constant defined in Lemma 2.6 and $U$ is defined in $(A_1)$. To prove (4.15), using $L(x, v) = C_p |v|^q + f(x)$ we observe that for $s \in (-1, 1)$ then $(1 + s)^q - 1 \leq q^2$ for $s \in (-1, 1)$ and $q \in (1, 2]$, therefore
\[
|L(x, (1 + s)v) - L((1 + s)x, v)| \\
\leq C_p ((1 + s)^q - 1) |v|^q + |f(x) - f((1 + s)x)| \leq (C_p q^2 |v|^q + C) |s|
\]
where $C$ depends on the Lipschitz constant of $f$ on $U$ and $\text{diam}(U)$. Let us consider $\lambda > 0$ first, then from (4.12) and the fact that $c_{\Omega} \leq c(\lambda)$ (Lemma 4.1) we have
\[
0 \leq c(\lambda) - c(0) \leq \langle \nu_\lambda, L(x, (1 + \lambda)v) - L((1 + \lambda)x, v) \rangle_{\Omega} \\
\leq (C_p q^2 \langle \nu_\lambda, |v|^q \rangle_{\Omega} + C) |\lambda| \\
= \left( C_p q^2 \langle \nu_\lambda, |v|^q \rangle_{\Omega} + C \right) |\lambda| \leq C(C_p q^2 + 1)|\lambda|.
\]
The case where $\lambda \leq 0$ is similar. We conclude that $\lambda \mapsto c(\lambda)$ is Lipschitz.

We finish this section by proving Corollary 1.3.

Proof of Corollary 1.3. Under $(A_3)$, the fact that $\lambda \mapsto c(\lambda)$ is Lipschitz, increasing and $c'_\pm(\lambda)$ exists are consequences of Lemmas 4.1, 4.2 and 4.3. Parts (ii) is similar to the first-order case in [33, Theorem 4.4]. Thus, we omit the proof of this fact. Part (iii) is a result from [8, Theorem 4.2, Chapter 4] or [19, Theorem 17.9], where if $c'_\pm(\lambda)$ exists everywhere then the set where they are different is at most countable.

Together with Theorem 2.8, we use the fact that $\lambda \mapsto c(\lambda)$ is Lipschitz to improve the error estimate in the nested domain setting but with some trade-off by a discount factor.

Corollary 4.4 (Error estimate on nested domain). Assume $(A_1)$ and $(A_2)$, $\theta > 0$ and $0 < \delta < 1$. Let $u \in C((1 + \theta)\overline{\Omega})$ and $v \in C(\overline{\Omega})$ be solutions to $(\delta, (1 + \theta)\Omega)$ and $(\delta, \Omega)$, respectively. Then, it holds that
\[
0 \leq \delta v(x) - \delta u(x) \leq C \min \{ \delta + \theta, \theta^a \}, \quad \text{for all } x \in \overline{\Omega}.
\]
where $a = (p - 2)/(p - 1)$ and $C$ depends on $\text{diam}(U)$, the Lipschitz constant of $f$ on $U$ and $p$. 

5. THE ERGODIC PROBLEM AND THE DIFFERENTIABILITY OF THE ADDITIVE EIGENVALUE

In this section, we prove Theorem 1.4. From Theorem 1.2, when talking about $c'_{\pm}(0)$ we think about the derivatives of the map $\theta \mapsto c_{(1+\theta)\Omega}$ for $\theta \in (-\epsilon_0, \epsilon_0)$, with the formulas provided by (1.7) and (1.8).

We first recall the definition of $\mathcal{C}(\gamma) = u^\gamma(\cdot) - u^0(\cdot)$ as a function from $\mathbb{R} \to \mathbb{R}$, from Remark 2. We collect some definitions on the minimizing measures following (2.1) in Theorem 2.4.

**Definition 6** (Discount measures). For $\lambda > 0$ and $z \in \overline{\Omega}$, we define

$$M(\lambda, z, \Omega) = \{ \mu \in \mathcal{P} \cap \mathcal{G}^{\lambda}_{\gamma, \lambda, \Omega} : \lambda u_\lambda(z) = \langle \mu, L \rangle_\Omega \},$$

$$M(\lambda, \Omega) = \bigcup_{z \in \overline{\Omega}} M(\lambda, z, \Omega).$$

We say measures in $M(\lambda, \Omega)$ are discount measures.

It is clear that weak limits of measures in $M(\lambda, \Omega)$ as $\delta \to 0^+$ are members of $M(\Omega)$. Regarding Lemma 5.2, we define the following sets of measures.

**Definition 7.** For $\gamma \in \mathbb{R}$, $z \in \overline{\Omega}$, we define

$$r_\gamma(\lambda) = \gamma \lambda, \quad \Omega_\lambda = (1 + \gamma \lambda) \Omega, \quad \lambda > 0$$

and

$$\mathcal{U}_\gamma(\Omega, z) = \{ \mu \in M(\Omega) : \exists \mu_\lambda \in M(\lambda, z, \Omega) \text{ such that } \mu_\lambda \rightharpoonup \mu \text{ along a subsequence} \},$$

$$\mathcal{U}_\gamma(\Omega) = \{ \mu \in M(\Omega) : \exists \mu_\lambda \in M(\lambda, \Omega_\lambda) \text{ such that } \mu_\lambda \rightharpoonup \mu \text{ along a subsequence} \},$$

$$\mathcal{V}_\gamma(\Omega) = \{ v \in M(\Omega) : \exists v_\lambda \in M(\Omega_\lambda) \text{ such that } v_\lambda \rightharpoonup v \text{ along a subsequence} \},$$

where $\mu_\lambda, v_\lambda$ are the scaling measure defined on $\overline{\Omega} \times \mathbb{R}^n$ as in Definition 5.

We note that this choice of $r$ does not affect $u^\gamma$ as indicated by Theorem 1.1. It is clear that $\bigcup_{z \in \overline{\Omega}} \mathcal{U}_\gamma(\Omega, z) \subset \mathcal{U}_\gamma(\Omega)$, and $\mathcal{U}_\gamma(\Omega), \mathcal{V}_\gamma(\Omega)$ are independent of the vertex $z \in \overline{\Omega}$ in the way we defined $M(\lambda, \Omega)$. Using [21, Lemma 2.1 and Lemma 2.2], we have $\emptyset \neq \mathcal{V}_\gamma(\Omega), \mathcal{U}_\gamma(\Omega) \subset M(\Omega)$. If $\gamma = 0$ then we work on a fixed domain $\Omega$, and

$$\mathcal{U}_0(\Omega) = \{ \mu \in M(\Omega) : \mu \text{ is a weak limit of a sequence of measures in } M(\lambda, \Omega) \}.$$

Here $M(\lambda, \Omega)$ is defined Definition 6. In other words, measures in $\mathcal{U}_0(\Omega)$ are weak limits of discount measures on $\Omega$, while measures in $\mathcal{U}_\gamma(\Omega)$ are weak limits of discount measures on $\Omega_\lambda = (1 + \gamma \lambda) \Omega$.

**Corollary 5.1.** Assume $(A_1)$. We have

$$\gamma \langle u_\gamma, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} + \langle \mu, u^\gamma \rangle_{\Omega} = 0 \quad \text{for any } \mu \in \mathcal{U}_\gamma(\Omega).$$

**Proof.** Take $\mu \in \mathcal{U}_\gamma(\Omega)$, we can find a sequence of points $z_\lambda \in \Omega_\lambda$ and $\mu_\lambda \in M(\lambda, z_\lambda, \Omega_\lambda)$ such that $\lambda u_\lambda(z_\lambda) = \langle \mu_\lambda, L \rangle_{\Omega_\lambda}$ and $\mu_\lambda \rightharpoonup \mu$ weakly in measures where $\mu_\lambda$ is the measure obtained from $\mu_\lambda$ defined as in Definition 5. Going through Step 2 as in the proof of Theorem 1.2 with $w = \tilde{u}^\gamma$ as in (3.12), we obtain

$$\gamma \langle u_\gamma, (-x, v) \cdot \nabla L(x, v) \rangle_{\Omega} + \langle \mu, u^\gamma \rangle_{\Omega} \geq 0$$

and thus, the conclusion follows, as the other side of the inequality is given in (3.7). Note that we use (3.13) to connect $\tilde{u}^\gamma$ to $u^\gamma$. \qed

**Lemma 5.2.** Assume $(A_1)$.  

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(i) We have $\langle \sigma, u^0 \rangle_\Omega = 0$ for all $\sigma \in U_0(\Omega)$. Furthermore 
\[ \mathcal{E}(\gamma) = u^\gamma(z) - u^0(z) = \langle \sigma, u^\gamma \rangle_\Omega \quad \text{for all } z \in \overline{\Omega}, \sigma \in U_0(\Omega). \]

As a consequence, we have 
\[ \gamma \langle \sigma, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \mathcal{E}(\gamma) \leq 0 \quad \text{for all } \sigma \in U_0(\Omega). \]

(ii) We have 
\[ \gamma \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \mathcal{E}(\gamma) \geq 0 \quad \text{for all } \mu \in U_\gamma(\Omega). \tag{5.7} \]

(iii) The map $\mathcal{E} : \gamma \mapsto \mathcal{E}(\gamma)$ is decreasing, concave with $\mathcal{E}(0) = 0$.

(iv) We have 
\[ \langle \sigma, u^\gamma \rangle_\Omega + \gamma c^\prime_\lambda(0) \leq 0 \quad \text{for all } \sigma \in V_\gamma(\Omega), \gamma > 0, \]
\[ \langle \sigma, u^\gamma \rangle_\Omega + \gamma c^\prime_\lambda(0) \leq 0 \quad \text{for all } \sigma \in V_\gamma(\Omega), \gamma < 0. \]

Proof. We recall that $U_0, U_\gamma \subset M(\Omega)$.

(i) By Theorem 1.1 we have $\langle \sigma, u^0 \rangle \leq 0$ for all $\sigma \in M(\Omega)$. Take $\sigma \in U_0(\Omega)$, there exist $z_\lambda \in \overline{\Omega}$ and $\nu_\lambda \in M(\nu, z_\lambda, \Omega)$ such that (up to a subsequence) $\nu_\lambda \rightharpoonup \sigma$ weakly in measures. Let $w$ be a solution to $H(x, Dw(x)) - \varepsilon \Delta w(x) \leq c_\Omega \in \Omega$. Using the strategy in (3.3), (3.4) we write
\[
\begin{cases}
\lambda w + H_{\nu}(x, Dw) - \varepsilon \Delta w \leq 0 & \text{in } \Omega, \\
\phi(x, v) = L(x, v) + \lambda w(x) + c_\Omega \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n). 
\end{cases}
\]

Using Corollary 2.5 (i) we have 
\[ \langle \sigma, \phi - \lambda w(z) \rangle_\Omega \geq 0. \]

Using the definition of $\sigma_\lambda$ we have $\langle \sigma_\lambda, L \rangle_\Omega = \lambda \nu_\lambda(z_\lambda)$ where $\nu_\lambda$ solves $(\lambda, \Omega)$. Therefore 
\[ \lambda \nu_\lambda(z_\lambda) + c_\Omega + \lambda \langle \sigma_\lambda, w \rangle_\Omega - \lambda w(z_\lambda) \geq 0. \]

Divide both sides by $\lambda$ we obtain 
\[ \left( \frac{\nu_\lambda(z_\lambda) + c_\Omega}{\lambda} \right) + \langle \sigma_\lambda, w \rangle_\Omega - w(z_\lambda) \geq 0. \]

We can assume $z_\lambda \to z_0$ for some $z_0 \in \overline{\Omega}$. Take $\lambda \to 0^+$, thanks to Theorem 1.1 we have 
\[ u^0(z_0) + \langle \sigma, w \rangle_\Omega \geq w(z_0) \tag{5.8} \]

○ Let $w = u^0$ in (5.8) then $\langle \sigma, u^0 \rangle \geq 0$, thus 
\[ \langle \sigma, u^0 \rangle = 0 \quad \text{for } \sigma \in U_0(\Omega), \tag{5.9} \]

since $\langle \mu, u^0 \rangle \leq 0$ for all $\mu \in M(\Omega)$.

○ On the other hand, using (5.9) we have 
\[ \langle \sigma, u^\gamma \rangle_\Omega = \langle \sigma, u^\gamma \rangle_\Omega - \langle \sigma, u^0 \rangle_\Omega = \langle \sigma, u^\gamma - u^0 \rangle_\Omega = \langle \sigma, \mathcal{E}(\gamma) \rangle_\Omega = \mathcal{E}(\gamma). \tag{5.10} \]

In equation (1.6) of Theorem 1.1 (ii), let $w = u^\gamma$ and use (5.10) we obtain 
\[ \gamma \langle \sigma, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \mathcal{E}(\gamma) = \gamma \langle \sigma, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \langle \sigma, u^\gamma \rangle_\Omega \leq 0. \]

(ii) Since $\mathcal{E}(\gamma) = u^\gamma(\cdot) - u^0(\cdot)$ is a constant, we have 
\[ \mathcal{E}(\gamma) = \langle \mu, \mathcal{E}(\gamma) \rangle_\Omega = \langle \mu, u^\gamma - u^0 \rangle_\Omega = \looparrowright \langle \mu, u^\gamma \rangle_\Omega = \mathcal{E}(\gamma) + \langle \mu, u^0 \rangle_\Omega. \]

Using this in Corollary 5.1 we have 
\[ 0 = \gamma \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \langle \mu, u^\gamma \rangle_\Omega \]
\[ = \gamma \langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega + \mathcal{E}(\gamma) + \langle \mu, u^0 \rangle_\Omega \quad \text{for all } \mu \in U_\gamma(\Omega). \]
Since \( \langle \mu, u^0 \rangle \leq 0 \) for all \( \mu \in \mathcal{M}(\Omega) \) (Theorem 1.1) and \( \mathcal{U}_\gamma \subset \mathcal{M}(\Omega) \), we obtain the desired conclusion.

(iii) From part (i) we have \( \langle \mu, u^\gamma \rangle = \mathcal{C}(\gamma) \) for all \( \mu \in \mathcal{U}_0(\Omega) \). By Corollary 3.2 \( \gamma \mapsto u^\gamma(\cdot) \) is decreasing and concave. Take \( \sigma \in \mathcal{U}_0(\Omega) \), then \( \langle \sigma, u^0 \rangle = 0 \) from part (i). If \( a \leq b \) as real numbers then \( u^a - u^b \geq 0 \), thus \( \langle \sigma, u^a - u^b \rangle \geq 0 \), hence

\[
\langle \sigma, u^a - u^0 \rangle - \langle \sigma, u^b - u^0 \rangle \geq 0 \quad \implies \quad \mathcal{C}(a) \geq \mathcal{C}(b).
\]

Therefore \( \gamma \mapsto \mathcal{C}(\gamma) \) is decreasing. On the other hand, for \( s \in (0,1) \) and \( a, b \in \mathbb{R} \) then \( su^a + (1-s)u^b \leq u^{sa+(1-s)b} \), hence the concavity of \( \gamma \mapsto \mathcal{C}(\gamma) \) follows from part (i).

(iv) Let \( u_\lambda \in C(\overline{\Omega}_\lambda) \) be the solution to \( (\lambda, \Omega_\lambda) \) then \( \lambda u_\lambda + H(x, Du_\lambda) - \varepsilon \Delta u_\lambda \leq 0 \) in \( \Omega_\lambda \). Using the strategy in (3.3), (3.4) we write

\[
\begin{cases}
H_\phi(x, Du_\lambda(x)) - \varepsilon \Delta u_\lambda(x) \leq 0 & \text{in } \Omega_\lambda, \\
\phi(x,v) = L(x,v) - \lambda u_\lambda(x) \in \Phi^+(\overline{\Omega} \times \mathbb{R}^n).
\end{cases}
\]

Using Corollary 2.5 (ii) we have \( \langle \sigma_\lambda, \phi \rangle_{\Omega_\lambda} \geq 0 \) and \( \langle \sigma_\lambda, L \rangle_{\Omega_\lambda} = -c_{\Omega_\lambda} \) if \( \sigma_\lambda \in \mathcal{M}(\Omega_\lambda) \), therefore

\[
\lambda \langle \sigma_\lambda, -c_{\Omega_\lambda} - \lambda u_\lambda \rangle_{\Omega_\lambda} = \langle \sigma_\lambda, L(x,v) - \lambda u_\lambda(x) \rangle_{\Omega_\lambda} \geq 0 \quad \text{for all } \sigma_\lambda \in \mathcal{M}(\Omega_\lambda).
\]

In other words, we have

\[
\langle \sigma_\lambda, u_\lambda(x) + \frac{c_{\Omega_\lambda}}{\lambda} \rangle_{\Omega_\lambda} \leq 0 \quad \text{for all } \sigma_\lambda \in \mathcal{M}(\Omega_\lambda).
\]

Let \( \sigma_\lambda \) be the scaling measure obtained from \( \sigma_\lambda \) as in Definition 5. We have

\[
\langle \sigma_\lambda, u_\lambda((1+r(\lambda))x) + \frac{c_{\Omega_\lambda}}{\lambda} \rangle_{\Omega} + \frac{c_{\Omega_\lambda} - c_{\Omega_\lambda}}{\lambda} \leq 0 \quad \text{for all } \sigma_\lambda \in \mathcal{M}(\Omega_\lambda). \tag{5.11}
\]

If \( \sigma \in \mathcal{V}_\gamma(\Omega) \), we can find a sequence \( \sigma_\lambda \in \mathcal{M}(\Omega_\lambda) \) such that \( \sigma_\lambda \to \sigma \) in measures (thanks to Lemma 2.10). Let \( \sigma = \sigma_\lambda \) in (5.11), then as \( \lambda \to 0^+ \) we obtain the conclusion.

\[
\square
\]

**Proof of Theorem 1.4.** Using the concavity of \( \gamma \mapsto \mathcal{C}(\gamma) \), the one-sided derivatives \( \mathcal{C}_\pm'(\gamma) \) exists everywhere. From Lemma 5.2 we have

\[
\langle \mu_\gamma, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} \geq -\frac{\mathcal{C}(\gamma)}{\gamma} \quad \forall \mu \in \mathcal{U}_\gamma(\Omega), \gamma > 0 \tag{5.12}
\]

\[
\langle \mu_\gamma, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} \leq -\frac{\mathcal{C}(\gamma)}{\gamma} \quad \forall \mu \in \mathcal{U}_\gamma(\Omega), \gamma < 0. \tag{5.13}
\]

Take a subsequence \( \mu_\gamma \to \mu_+ \) for some \( \mu_+ \in \mathcal{M}(\Omega) \) as \( \gamma \to 0^+ \) we obtain

\[
\langle \mu_+, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} \geq -\mathcal{C}_+(0). \]

Similarly, take a subsequence \( \mu_\gamma \to \mu_- \) for some \( \mu_- \in \mathcal{M}(\Omega) \) as \( \gamma \to 0^- \) we obtain

\[
\langle \mu_-, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} \leq -\mathcal{C}_-(0). \]

As a concave function, we have \( \mathcal{C}_+(0) \leq \mathcal{C}_-(0) \), therefore

\[
\langle \mu_-, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} \leq -\mathcal{C}_-(0) \leq -\mathcal{C}_+(0) \leq \langle \mu_+, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega}.
\]

If \( c'(0) \) exists, then since

\[
\langle \mu, (-x,v) \cdot \nabla L(x,v) \rangle_{\Omega} = c'(0) \quad \text{for all } \mu \in \mathcal{M}(0), \tag{5.14}
\]
we deduce that $-c'(0) = c'(0)$. Furthermore, from (5.12), (5.13) and (5.14) we obtain

$$\frac{-c(\gamma)}{\gamma} = c'(0) \quad \text{for all } \gamma.$$ 

Therefore $c(\gamma) = -\gamma c'(0)$. □

### 6. Remarks on the Differentiability of the Additive Eigenvalue Map

Let $r(\lambda) = \lambda$ for $\lambda \in (-\varepsilon_0, \varepsilon_0)$. While Corollary 1.3 guarantees that $\lambda \mapsto c(\lambda)$ is differentiable except for a countable set, the following question remains open.

**Question 1.** Can we show that $\lambda \mapsto c(\lambda)$ is indeed differentiable everywhere?

In this section, we demonstrate the smoothness of $\lambda \mapsto c(\lambda)$ for constant $f$, discuss its semiconvexity for semi-concave data, and briefly outline the smoothness of $\lambda \mapsto c(\lambda)$ for $p = 2$, with an explicit formula for $c'(0)$ using the Hopf-Cole transformation. Investigation of the full regime $1 < p \leq 2$ is planned for future work.

#### 6.1. Smoothness with constant data.

**Proof of Theorem 1.5.** Without loss of generality we can assume $f \equiv 0$, then $L(x, v) = C_p|v|^q$ where $C_p = p^{-1/q}(p - 1)$ and $p^{-1} + q^{-1} = 1$. By definition of $\mathcal{M}(\Omega)$ we have

$$-c(0) = \langle \mu, L \rangle_\Omega = C_p \langle \mu, |v|^q \rangle_\Omega \quad \text{for any } \mu \in \mathcal{M}(\Omega).$$

Since $(-x, v) \cdot \nabla L(x, v) = qC_p|v|^q$, we have

$$\langle \mu, (-x, v) \cdot \nabla L(x, v) \rangle_\Omega = qC_p \langle \mu, |v|^q \rangle_\Omega = -qc(0) \quad \text{for all } \mu \in \mathcal{M}(\Omega).$$

In view of Theorem 1.2 we conclude that $c'(0)$ exists and $c'(0) = -p(p - 1)c(0)$. As the argument can be done for any $\Omega_\lambda$ we obtain $\lambda \mapsto c(\lambda)$ is differentiable everywhere and

$$c'(\lambda) = -p(p - 1)c(\lambda) = -qc(\lambda). \quad (6.1)$$

From Corollary 1.3, we see that $\lambda \mapsto c(\lambda)$ is continuously differentiable. Equation (6.1) implies that $\lambda \mapsto c(\lambda)$ is $C^\infty$, with $c^{(k)}(\lambda) = (-q)^k c(\lambda)$ for $k \in \mathbb{N}$. □

#### 6.2. Semiconvexity with semiconcave data.

Another equivalent definition of a semiconvex function is as follows: there is some $\tau > 0$ such that

$$u(sx + (1-s)y) \leq s u(x) + (1-s)u(y) + s(1-s)\frac{|x-y|^2}{2\tau}, \quad x, y \in I, s \in [0,1].$$

**Proof of Theorem 1.6.** As usual we only need to consider $r(\lambda) = \lambda$. Let $\alpha < \eta < \beta$ and the corresponding $(w_\alpha, c_\alpha), (w_\eta, c_\eta), (w_\beta, c_\beta)$ be solutions and eigenvalues to the ergodic problem where $\eta = s\alpha + (1-s)\beta$ for some $s \in [0,1]$. Let us define for $\lambda \in \mathbb{R}$ the function

$$\tilde{w}_\lambda(x) = \left( \frac{1+\eta}{1+\lambda} \right)^2 w_\lambda \left( \frac{1+\lambda}{1+\eta} x \right), \quad x \in \overline{\Gamma}_\lambda.$$ 

Using that with $\lambda = \alpha, \beta$ we obtain

$$\begin{cases} 
H \left( \left( \frac{1+\eta}{1+\lambda} \right) x, \left( \frac{1+\eta}{1+\lambda} \right) D\tilde{w}_\alpha(x) \right) - \varepsilon \Delta \tilde{w}_\alpha(x) \leq c(\alpha) & \text{in } (1+\eta)\Omega, \\
H \left( \left( \frac{1+\beta}{1+\lambda} \right) x, \left( \frac{1+\beta}{1+\lambda} \right) D\tilde{w}_\beta(x) \right) - \varepsilon \Delta \tilde{w}_\beta(x) \leq c(\beta) & \text{in } (1+\eta)\Omega.
\end{cases}$$

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Let $\tilde{\alpha} = \frac{1+\alpha}{1+\eta}$ and $\tilde{\beta} = \frac{1+\beta}{1+\eta}$ then $s\tilde{\alpha} + (1-s)\tilde{\beta} = 1$. We can write the equations as

$$
\begin{cases}
H \left( \tilde{\alpha} x, \frac{1}{\tilde{\alpha}} D\tilde{\omega}_\alpha(x) \right) - \varepsilon \Delta \tilde{\omega}_\alpha(x) \leq c(\alpha) & \text{in } (1+\eta)\Omega, \\
H \left( \tilde{\beta} x, \frac{1}{\tilde{\beta}} D\tilde{\omega}_\beta(x) \right) - \varepsilon \Delta \tilde{\omega}_\beta(x) \leq c(\beta) & \text{in } (1+\eta)\Omega.
\end{cases}
$$

Let us define $\tilde{\omega} = s\tilde{\omega}_\alpha + (1-s)\tilde{\omega}_\beta \in C(\overline{\Omega}_\eta)$. Using $H(x, \xi) = |\xi|^p - f(x)$, we compute heuristically, assuming $\tilde{\omega}_\alpha$ and $\tilde{\omega}_\beta$ are differentiable

$$
sc(\alpha) + (1-s)c(\beta) \geq sH \left( \tilde{\alpha} x, \frac{1}{\tilde{\alpha}} D\tilde{\omega}_\alpha(x) \right) + (1-s)H \left( \tilde{\beta} x, \frac{1}{\tilde{\beta}} D\tilde{\omega}_\beta(x) \right) - \varepsilon \Delta \tilde{\omega}(x)
$$

where we use the semi-concavity of $f(\cdot)$ and $s\tilde{\alpha} + (1-s)\tilde{\beta} = 1$ in the last inequality. We note that the rigorous argument using viscosity solution can be done, for example, following the strategy of [20, Lemma 2.7]. We use Hölder inequality to get, for $p > 2$ and $a, b, x, y > 0$ and $s \in (0,1)$ that

$$
\left( s \frac{a^p}{x^p} + (1-s) \frac{b^p}{y^p} \right) (sx^2 + (1-s)y^2)(sx + (1-s)y)^{p-2} \geq (sa + (1-s)b)^p.
$$

Applying this inequality, we obtain

$$
\left( s \left| \frac{D\tilde{\omega}_\alpha(x)}{\tilde{\alpha}} \right|^p + (1-s) \left| \frac{D\tilde{\omega}_\beta(x)}{\tilde{\beta}} \right|^p \right) \geq \frac{|D\tilde{\omega}(x)|^p}{s \tilde{\alpha}^2 + (1-s)\tilde{\beta}^2} = \left| \frac{1}{\theta_s} D\tilde{\omega}(x) \right|^p
$$

where

$$
\theta_s = (s \tilde{\alpha}^2 + (1-s)\tilde{\beta}^2)^{1/p} = \left( 1 + s(1-s) \frac{(\beta - \alpha)^2}{(1+\eta)^2} \right)^{1/p} \in (1, +\infty). \quad (6.2)
$$

We deduce that

$$
\left| \frac{1}{\theta_s} D\tilde{\omega}(x) \right|^p - f(x) - \varepsilon \Delta \tilde{\omega}(x) \leq sc(\alpha) + (1-s)c(\beta) + s(1-s) \left| \frac{\alpha - \beta}{1+\eta} \right|^2 |x|^2 \quad \text{in } \Omega_\eta
$$

or in other words, we have

$$
H \left( x, \frac{1}{\theta_s} D\tilde{\omega}(x) \right) - \varepsilon \Delta \tilde{\omega}(x) \leq sc(\alpha) + (1-s)c(\beta) + s(1-s) \left| \frac{\alpha - \beta}{1+\eta} \right|^2 |x|^2 \quad \text{in } \Omega_\eta.
$$

Therefore $H_{\phi}(x, D\tilde{\omega}(x)) - \varepsilon \Delta \tilde{\omega}(x) \leq 0$ in $\Omega_\eta$ where

$$
\phi(x, v) = L(x, \theta_s v) + sc(\alpha) + (1-s)c(\beta) + s(1-s) \left| \frac{\alpha - \beta}{1+\eta} \right|^2 |x|^2 \in \Phi^+(\overline{\Omega}_\eta \times \mathbb{R}^n)
$$

thanks to the separable form $H(x, \xi) = |\xi|^p - f(x)$. We, therefore, deduce that

$$
\langle \mu, L(x, \theta_s v) \rangle_{\Omega_\eta} + sc(\alpha) + (1-s)c(\beta) + s(1-s) \left| \frac{\alpha - \beta}{1+\eta} \right|^2 \langle \mu, \frac{|x|^2}{1+\eta} \rangle_{\Omega_\eta} \geq 0
$$

for all $\mu \in \mathcal{M}(\Omega_\eta)$, where $\mathcal{M}(\Omega_\eta)$ is the set of minimizing Mather measures on $\overline{\Omega}_\eta \times \mathbb{R}^n$ as defined in Definition 3. We observe that

$$
\langle \mu, \frac{|x|^2}{1+\eta} \rangle_{\Omega_\eta} = \int_{\overline{\Omega}_\eta \times \mathbb{R}^n} \frac{x}{1+\eta} \frac{|x|^2}{1+\eta} d\mu(x, v) \leq \tilde{C}_1(\overline{\Omega})
$$
where \( \hat{C}_1(\Omega) \) depends only on the size of \( \Omega \). Using \( \langle \mu, L(x,v) \rangle_{\Omega_q} = -c(\eta) \) we have
\[
\langle \mu, L(x,\theta_s v) - L(x,v) \rangle_{\Omega_q} + sc(\alpha) + (1-s)c(\beta) + C_1(\Omega) s(1-s) |\alpha - \beta|^2 \geq c(\eta)
\]
for all \( \mu \in M(\Omega_q) \). Using (6.2), we estimate
\[
L(x,\theta_s v) - L(x,v) = |\theta_s v|^q - |v|^q = |v|^q (|\theta_s v|^q - 1) = |v|^q \left[ \left( 1 + s(1-s) \frac{|\beta - \alpha|^2}{(1+\eta)^2} \right)^{\frac{q}{p}} - 1 \right]
\]
Therefore
\[
\langle \mu, L(x,\theta_s v) - L(x,v) \rangle_{\Omega_q} = \langle \mu, |v|^q \rangle_{\Omega_q} \left[ \left( 1 + s(1-s) \frac{|\beta - \alpha|^2}{(1+\eta)^2} \right)^{\frac{q}{p}} - 1 \right].
\]
Using Bernoulli’s inequality \((1 + x)^r \leq 1 + rx\) if \( r \in (0,1) \) and \( x > -1 \), we deduce that
\[
\left[ \left( 1 + s(1-s) \frac{|\beta - \alpha|^2}{(1+\eta)^2} \right)^{\frac{q}{p}} - 1 \right] \leq \left( \frac{q}{p} \right) s(1-s) \frac{|\beta - \alpha|^2}{(1+\eta)^2}
\]
since \( p/q < 1 \) as \( q \leq 2 < p \).

Using the fact that \( \langle \mu, L \rangle_{\Omega_q} = -c(\eta) \) for \( \mu \in M(\Omega_q) \) and \( L(x,v) = C_p|v|^q + f(x) \) we deduce that \( \langle \mu, C_q|v|^q \rangle_{\Omega_q} + \langle \mu, f \rangle_{\Omega_q} = -c(\eta) \) and thus
\[
\langle \mu, |v|^q \rangle_{\Omega_q} = -\frac{c(\eta)}{C_q} + \frac{\langle \mu, f \rangle_{\Omega_q}}{C_q} \leq \frac{1}{C_q} \left( \max_{\Omega} |f| + C_M(\Omega) \right)
\]
where \( C_M(\Omega) \) is the constant defined in Corollary A.5 and \( U \) is the bigger domain where all variations \( \Omega_q \subset U \). Therefore we obtain
\[
c(\eta) \leq sc(\alpha) + (1-s)c(\beta) + \left[ \hat{C}_2(\Omega, f, q) \left( \frac{q}{p} \right) + \hat{C}_1(\Omega) \right] s(1-s) |\beta - \alpha|^2
\]
for all \( \alpha < \beta \) and \( \eta \in (\alpha, \beta) \) with \( \eta = s\alpha + (1-s)\beta \) for \( s \in [0,1] \). \( \square \)

6.3. The quadratic case. The differentiability of \( \lambda \mapsto c(\lambda) \) is simpler when \( p = 2 \), owing to the connection between the quadratic Hamilton-Jacobi equation and linear elliptic equation via the Hopf-Cole transformation. For the existence of an eigenvalue and related well-posedness in the \( p = 2 \) case, see [25] or the Appendix of [16]. Here, we present a formal argument for the existence and representation of \( c'(0) \). A comprehensive examination of \( 1 < p \leq 2 \) is planned for future investigation.

We recall that \( r(\lambda) = \lambda \) and \( \Omega_\lambda = (1+\lambda)\Omega \). If \( p = 2 \), the equation that defines the eigenvalue on \( \Omega_\lambda \) is
\[
\left\{
\begin{aligned}
|Dv(x)|^2 - f(x) - \epsilon \Delta v(x) &= c(\lambda) &\text{in } \Omega_\lambda, \\
v(x) &= +\infty &\text{on } \partial \Omega_\lambda.
\end{aligned}
\right.
\] (6.3)
Solutions to (6.3) are unique up to adding a constant and bounded from below (see [25]). Let us choose the solution \( \hat{v}_{\lambda} \) of (6.3) such that
\[
\int_{\Omega_\lambda} e^{-\hat{\theta}_{\lambda}(x)/\epsilon} \, dx = 1.
\] (6.4)
Using the Hopf-Cole transform, we define \( w : \Omega_\lambda \to \mathbb{R} \) as \( w(x) = e^{-\hat{\theta}_{\lambda}(x)/\epsilon} \) for \( x \in \Omega_\lambda \), leading to a linear problem
\[
\left\{
\begin{aligned}
-\epsilon^2 \Delta w(x) + f(x)w(x) &= c(\lambda)w(x) &\text{in } \Omega_\lambda, \\
w(x) &= 0 &\text{on } \partial \Omega_\lambda.
\end{aligned}
\right.
\] (6.5)
Therefore, \( c(\lambda) \) is an eigenvalue of the linear operator \( \mathcal{L}[w] = (-\varepsilon^2 \Delta + f)w \) on \( \Omega_\lambda \) with Dirichlet boundary condition. Furthermore, by using (1.1), it is clear that \( c(\lambda) \) is the principal eigenvalue of the linear problem (6.5). Thus, it is a simple eigenvalue, and it admits a variational presentation

\[
c(\lambda) = \min \left\{ \int_{\Omega_\lambda} (\varepsilon^2 |Du(x)|^2 + f(x)|u(x)|^2) \, dx : u \in H^1_0(\Omega_\lambda), \|u\|_{L^2(\Omega_\lambda)} = 1 \right\}. \tag{6.6}
\]

Let \( w_\lambda \) be the unique solution to (6.5) with \( \|w_\lambda\|_{L^2(\Omega_\lambda)} = 1 \).

Using tools from shape analysis, for instance, see [12, 17, 18, 28] and the references therein, we can obtain some information about the regularity of \( \lambda \mapsto c(\lambda) \) and \( \lambda \mapsto w_\lambda(\cdot) \). This problem is often examined in the literature within a slightly broader context. We assume that each point \( x \) in the domain \( \overline{\Omega} \) is changed under a smooth, autonomous (time-independent) velocity field \( \mathbf{V}(x) \in \mathbb{C}^k(\overline{\Omega}; \mathbb{R}^n) \). Consider the following transformation, which is close to a perturbation of the identity (see [28]), defined by \( T_\lambda(x) = x + \lambda \mathbf{V}(x) \) for \( x \in \overline{\Omega} \) where \( \lambda \in I_0 \subset \mathbb{R} \) is a neighborhood of 0. We denote \( \Omega_\lambda = T_\lambda(\Omega) \) (our setting is a special case with \( \mathbf{V}(x) = x \) for \( x \in \mathbb{R}^n \)).

For simplicity, we henceforth assume that \( f = 0 \). Using [17, Theorem 2.5.1] we have \( c'(0) \) exists and also \( \lambda \mapsto w_\lambda(\cdot) \) is differentiable, as well as \( w'(x) = \frac{d}{d\lambda} (w_\lambda(x)) \big|_{\lambda=0} \) exists and is a function in \( H^1(\Omega) \). We will derive a formula for \( c'(0) \) using solution \( w_0 \) to (6.5).

Denote by \( \mathbf{n}(x) \) the unit outward normal vector at \( x \in \partial \Omega \). We use differentiation with respect to \( \lambda \) in [28, Section 2.31]. We differentiate \( \|w_\lambda\|_{L^2(\Omega_\lambda)} = 1 \) with respect to \( \lambda \) to obtain that

\[
0 = \int_\Omega 2w_0(x)w'(x) \, dx + \int_{\partial \Omega} |w_0(x)|^2 \mathbf{V}(x) \cdot \mathbf{n}(x) \, dS(x) = \int_\Omega 2w_0(x)w'(x) \, dx, \tag{6.7}
\]

since \( w_0 = 0 \) on \( \partial \Omega \). Next, we differentiate \( c(\lambda) \) in (6.6) with \( f = 0 \) to obtain that

\[
c'(0) = 2\varepsilon^2 \int_\Omega Dw_0(x) \cdot Dw'(x) \, dx + \varepsilon^2 \int_{\partial \Omega} |Dw_0(x)|^2 \mathbf{V}(x) \cdot \mathbf{n}(x) \, dS(x)
= 2\varepsilon^2 \int_{\partial \Omega} w'(x) \frac{\partial w_0}{\partial \mathbf{n}}(x) \, dS(x) + \varepsilon^2 \int_{\partial \Omega} |Dw_0(x)|^2 \mathbf{V}(x) \cdot \mathbf{n}(x) \, dS(x), \tag{6.8}
\]

where we use integration by parts for \( -\varepsilon^2 \Delta w_0 = c(0)w_0 \) and (6.7). On the other hand, by differentiating \( -\varepsilon^2 \Delta w_\lambda = c(\lambda)w_\lambda \), the equation for \( w' \) reads

\[-\varepsilon \Delta w' = c'(0)w_0 + c(0)w' \quad \text{in} \; \Omega. \]

Multiply both sides by \( w_0 \), using integration by parts with (6.7) and \( \|w_0\|_{L^2(\Omega)} = 1 \) we obtain

\[
c'(0) = -\varepsilon^2 \int_\Omega w_0(x)\Delta w'(x) \, dx = \varepsilon^2 \int_\Omega w' \frac{\partial w_0}{\partial \mathbf{n}} \, dS(x). \tag{6.9}
\]

From (6.8) and (6.9) we deduce that

\[
\int_{\partial \Omega} w' \frac{\partial w_0}{\partial \mathbf{n}} \, dS(x) = - \int_{\partial \Omega} |Dw_0|^2 \mathbf{V} \cdot \mathbf{n} \, dS(x)
\]

and thus, we conclude that

\[
c'(0) = -\varepsilon^2 \int_{\partial \Omega} |Dw_0|^2 \mathbf{V} \cdot \mathbf{n} \, dS(x) = -\varepsilon^2 \int_{\partial \Omega} \left| \frac{\partial w_0}{\partial \mathbf{n}}(x) \right|^2 (x \cdot \mathbf{n}) \, dS(x).
\]

We also have higher derivatives of \( \lambda \mapsto c(\lambda) \), as in [17].

**Corollary 6.1.** If \( p = 2, f \equiv 0 \) and \( r(\lambda) = \lambda \), then the map \( c \mapsto c(\lambda) \) is twice differentiable everywhere.
A.1. **Assumptions.** We present this section as a self-contained section. For generality, we state the results under a more general set of assumptions.

1. $\mathcal{H}_1$: $\xi \mapsto H(x, \xi)$ is convex for every fixed $x \in \overline{\Omega}$.
2. $\mathcal{H}_2$: There exists $p > 2$ such that, for every $R > 0$, there are $0 < a_R \leq b_R$ such that for $x, y \in B_R$ and all $\xi, \xi_1, \xi_2 \in \mathbb{R}^n$ there holds
   \[
   a_R|\xi|^p - b_R \leq H(x, \xi) \leq \Lambda_1(|\xi|^p + 1) \tag{A.1}
   \]
   \[
   |H(x, \xi) - H(y, \xi)| \leq (\Lambda_1|\xi|^p + b_R)|x - y| \tag{A.2}
   \]
   \[
   |H(x, \xi_1) - H(x, \xi_2)| \leq \Lambda_1(|\xi_1| + |\xi_2| + 1)^{p-1}|\xi_1 - \xi_2|. \tag{A.3}
   \]
3. $\mathcal{H}_3$: The matrix $a(x) : \mathbb{R}^n \to \mathbb{M}^n = \mathbb{R}^{n \times n}$ has a Lipschitz square root $\sigma : \mathbb{R}^n \to \mathbb{R}^{k \times n}$ for $k \in \mathbb{N}$ such that $a(x) = \sigma^T(x)\sigma(x)$ where
   \[
   |\sigma(x)| \leq \Lambda_2, \quad |\sigma(x) - \sigma(y)| \leq \Lambda_2|x - y|, \quad x, y \in \mathbb{R}^n \tag{A.4}
   \]
   for some constant $\Lambda_2 > 0$.

A.2. **Existence and Hölder estimate.**

**Definition 8.** We consider the following equation with $\delta \geq 0$:

\[
\delta u(x) + H(x, Du(x)) - \varepsilon \text{Tr}(a(x)D^2u(x)) = 0 \quad \text{in } \Omega. \tag{HJ}
\]

We say that

(i) $v \in \text{BUC}(\Omega; \mathbb{R})$ is a viscosity subsolution of (HJ) in $\Omega$ if for every $x \in \Omega$ and $\varphi \in C^2(\Omega)$ such that $v - \varphi$ has a local maximum over $\Omega$ at $x$ then

\[
\delta v(x) + H(x, D\varphi(x)) - \varepsilon \text{Tr}(a(x)D^2\varphi(x)) \leq 0.
\]

(ii) $v \in \text{BUC}(\overline{\Omega}; \mathbb{R})$ is a viscosity supersolution of (HJ) on $\overline{\Omega}$ if for every $x \in \overline{\Omega}$ and $\varphi \in C^2(\overline{\Omega})$ such that $v - \varphi$ has a local minimum over $\overline{\Omega}$ at $x$ then

\[
\delta v(x) + H(x, D\varphi(x)) - \varepsilon \text{Tr}(a(x)D^2\varphi(x)) \geq 0.
\]

If $v$ is a viscosity subsolution to (HJ) in $\Omega$, and is a viscosity supersolution to (HJ) on $\overline{\Omega}$, i.e.,

\[
\begin{align*}
\delta v(x) + H(x, Dv(x)) - \varepsilon \text{Tr}(a(x)D^2v(x)) &\leq 0 \quad \text{in } \Omega, \\
\delta v(x) + H(x, Dv(x)) - \varepsilon \text{Tr}(a(x)D^2v(x)) &\geq 0 \quad \text{on } \overline{\Omega},
\end{align*}
\]

then we say that $v$ is a state-constraint viscosity solution of (HJ).

We refer the readers to [3, 4, 32] for the equivalent definition of viscosity solution using super-differential $D^2_{\varepsilon_0}$ and sub-differential $D^{-}$. Existence, wellposedness, and gradient bound on the solution of (A.5) and a description of state-constraint boundary condition can be found in [9, 23, 30]. The existence of solutions to (A.5) can be established using Perron’s method. We omit the proof of the following Theorem and refer the readers to [1, Theorem 4.2].

**Theorem A.1.** Assume $(\mathcal{H}_1), (\mathcal{H}_2), (\mathcal{H}_3)$. Define

\[
u^\delta(x) = \sup \left\{ w(x) : w \in \text{USC}(\overline{\Omega}) \text{ is a subsolution of (A.5) in } \Omega \right\}. \tag{A.6}
\]

Then $\nu^\delta \in C_{\text{loc}}^{0,1}(\Omega) \cap \text{LSC}(\overline{\Omega})$ and $\nu^0$ is a solution of (A.5).

For the superquadratic Hamiltonian, any subsolution is at least Hölder continuous.
**Theorem A.2.** Under assumptions $(H_1), (H_2), (H_3)$, any upper semicontinuous subsolution \( u \) of \((HJ)\) is uniformly Hölder continuous up to the boundary, i.e., \( u \in C^{0,a}(\overline{\Omega}) \) with
\[
|u(x) - u(y)| \leq C_0 |x - y|^\alpha, \quad \alpha = \frac{p - 2}{p - 1} \in (0, 1)
\]
where \( C_0 \) is a constant depending on the diameter of \( \Omega \), \( p \) and constants from \((H_2), (H_3)\).

We refer the reader to [1] or [5] for the proof of this Theorem. We also refer the readers to [25] for a proof in the case \( a(x) \equiv 1 \) and \( H(x, \xi) = |\xi|^p - f(x) \).

**A.3. Comparison principle for Hölder continuous solutions.** We further assume that
\[(H_4): \ H(x, \xi) = \mathcal{H}(\xi) - f(x) \text{ where } f \in C^1(\overline{\Omega}) \text{ and } \mathcal{H} \text{ is homogeneous of degree } p > 2.\]

The uniqueness of the state-constraint solution to \((HJ)\) follows from a comparison principle. It was first studied in [29] for the first-order equation under an interior sphere assumption (see also [5] for a general nonlinear case with different assumptions). For our purpose of studying the scaling domains, we provide a simple proof using \((A_1)\) here for the readers’ convenience. We note that in \((A_1)\), the assumption \( \Omega \) is star-shaped can be removed, that is, any bounded, open subset of \( \mathbb{R}^n \) containing the origin that satisfies \((1.2)\) for some \( \kappa > 0 \) is star-shaped (see [33]).

**Theorem A.3** (Comparison principle). Assume \((1.2), (H_1), (H_2), (H_3), (H_4)\) and \( \lambda > 0 \). Let \( u, v \in C_0(\overline{\Omega}) \) be a viscosity subsolution in \( \Omega \) and supersolution on \( \overline{\Omega} \) of \((HJ)\), respectively. Assume that \( u, v \) are Hölder continuous, i.e., belong to \( C^{0,a}(\overline{\Omega}) \) for some \( a \in (0, 1) \), then \( u(x) \leq v(x) \) for all \( x \in \overline{\Omega} \).

**Proof.** For \( \delta > 0 \) we define \( C_1 = (2/\kappa)^{2-a}C_0 \) where \( \kappa \) is defined as in \((A_1)\) and \( C_0 \) is the Hölder constant of \( u \). Let us define
\[
\Phi(\hat{x}, y) = u \left( \frac{x}{1 + \theta} \right) - v(y) - \frac{C_1|x - y|^2}{\theta^{2-a}}, \quad (\hat{x}, y) \in (1 + \theta)\Omega \times \overline{\Omega}.
\]
Assume \( \Phi \) achieves its maximum over \((1 + \theta)\Omega \times \overline{\Omega}\) at \((\hat{x}_\theta, y_\theta)\) \( \in (1 + \theta)\Omega \times \overline{\Omega}\). We claim that \( \hat{x}_\theta \in (1 + \theta)\Omega \). From \( \Phi(\hat{x}_\theta, y_\theta) \geq \Phi(y_\theta, y_\theta) \) we obtain
\[
\frac{C_1|\hat{x}_\theta - y_\theta|^2}{\theta^{2-a}} \leq u \left( \frac{\hat{x}_\theta}{1 + \theta} \right) - u \left( \frac{y_\theta}{1 + \theta} \right) \leq \frac{C_0|\hat{x}_\theta - y_\theta|^a}{(1 + \theta)^a} \leq C_0|\hat{x}_\theta - y_\theta|^a.
\]
As a consequence
\[
\text{dist}(\hat{x}_\theta, \overline{\Omega}) \leq |\hat{x}_\theta - y_\theta| \leq \left( \frac{C_0}{C_1} \right)^{\frac{1}{a}} \theta = \left( \frac{\kappa}{2} \right) \theta. \tag{A.7}
\]
From \((A_1)\) we conclude \( \hat{x}_\theta \in (1 + \theta)\Omega \). This yields that
\[
(\hat{x}, y) \mapsto u \left( \frac{x}{1 + \theta} \right) - v(y) - \frac{C_1|x - y|^2}{\theta^{2-a}}, \quad \text{has a maximum over } (1 + \theta)\Omega \times \overline{\Omega}.
\]
at \((\hat{x}_\theta, y_\theta) \in (1 + \theta)\Omega \times \overline{\Omega}\). Let \( \hat{x} = (1 + \theta)x \) for \( x \in \Omega \) then \( \hat{x} \mapsto x \) is a bijection from \((1 + \theta)\Omega\) to \( \overline{\Omega} \). Equivalently, we deduce that
\[
(x, y) \mapsto u(x) - v(y) - \frac{C_1|(1 + \theta)x - y|^2}{\theta^{2-a}}, \quad \text{has a maximum over } \overline{\Omega} \times \overline{\Omega}
\]
at \((x_\theta, y_\theta) \in \Omega \times \overline{\Omega}\). Let
\[
\phi(x, y) = \frac{C_1|(1 + \theta)x - y|^2}{\theta^{2-a}}, \quad (x, y) \in \overline{\Omega} \times \overline{\Omega}.
\]
From Lemma A.4 below, we can find \( X_\theta, Y_\theta \in S^n \) such that
\[
\left( u(x_\theta), D_x \phi(x_\theta, y_\theta), X_\theta \right) \in \mathcal{T}^+_{\Omega} u(x_\theta), \quad \left( v(y_\theta), -D_y \phi(x_\theta, y_\theta), Y_\theta \right) \in \mathcal{T}^-_{\Omega} v(y_\theta)
\]
\[ \xi^T (X_\theta - Y_\theta) \xi \leq 2C_1 \theta^a |\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \]

Choose \( \xi = \sigma_i(x) \) in (A.12) where \( \sigma_i(x) \) is the \( i \)-th row vector of \( \sigma(x) \) for \( i = 1, 2, \ldots, k \) and taking the sum we deduce that
\[
\text{Tr}(a(x)(X_\theta - Y_\theta)) = \frac{1}{2} \text{Tr}(\sigma^T(x)(X_\theta - Y_\theta)\sigma(x)) = \sum_{i=1}^{k} \sigma_i^T(x)(X_\theta - Y_\theta)\sigma_i(x) \leq k\Lambda_2 C_1 \theta^a.
\]

Here, \( \Lambda_2 \) is given in (A.4). The subsolution test of \( u \) at \( x_\theta \) and the supersolution test of \( v \) at \( y_\theta \) give
\[
\delta u(x_\theta) + (1 + \theta)^p \mathcal{H} \left( \frac{2C_1((1 + \theta)x_\theta - y_\theta)}{\theta^{2-a}} \right) - f(x_\theta) - \epsilon \text{Tr}(a(x)X_\theta) \leq 0, \tag{A.8}
\]
\[
\delta v(y_\theta) + \mathcal{H} \left( \frac{2C_1((1 + \theta)x_\theta - y_\theta)}{\theta^{2-a}} \right) - f(y_\theta) - \epsilon \text{Tr}(a(x)Y_\theta) \geq 0. \tag{A.9}
\]

Subtract (A.9) from (A.8) and use the fact that \( 1 \leq (1 + \theta)^p \) we deduce that
\[
\delta u(x_\theta) - \delta v(y_\theta) \leq f(x_\theta) - f(y_\theta) + \epsilon \text{Tr}(a(x)(X_\theta - Y_\theta)) + \left( 1 - (1 + \theta)^p \right) \mathcal{H} \left( \frac{2C_1((1 + \theta)x_\theta - y_\theta)}{\theta^{2-a}} \right)
\]
\[
\leq \omega_f(|x_\theta - y_\theta|) + \epsilon \left( k\Lambda_2 C_1 \theta^a \right). \tag{A.11}
\]

where \( \omega_f(\cdot) \) is the modulus of continuity of \( f \). By compactness of \( \overline{T} \) we can assume \( (x_\theta, y_\theta) \to (z, z) \) as \( \theta \to 0 \). Assume \( x_0 \in \overline{T} \) such that \( \max_{x \in \overline{T}} (u - v) = u(x_0) - v(x_0) \). Using \( \Phi(x_\theta, y_\theta) \geq \Phi(x_0, x_0) \) where \( x_\theta = (1 + \theta)x_0 \) we have
\[
u(x_\theta) - v(y_\theta) \geq u(x_0) - v(y_\theta) - C_1 \theta^a |x_0|^2.
\]

Let \( \theta \to 0 \) we deduce that \( u(x_0) - v(y_\theta) \geq u(z) - v(z) \geq u(x_0) - v(y_\theta) \). Therefore we obtain
\[
(u - v)(z) = \max_{\overline{T}} (u - v), \quad \lim_{\theta \to 0} (u(x_\theta) - v(y_\theta)) = \max_{\overline{T}} (u - v).
\]

Using this in (A.11) we obtain \( \max_{\overline{T}} (u - v) \leq 0 \) and thus \( u \leq v \) on \( \overline{T} \).

**Lemma A.4.** Under the setting in the proof of Theorem A.3, there exists \( X_\theta, Y_\theta \in S^n \) such that
\[
(u(x_\theta), +D_2 \phi(x_\theta, y_\theta), X_\theta) \in \overline{T}_2^+, u(x_\theta), \quad (v(y_\theta), -D_2 \phi(x_\theta, y_\theta), Y_\theta) \in \overline{T}_2^- v(y_\theta),
\]
and
\[
\xi^T (X_\theta - Y_\theta) \xi \leq 2C_1 \theta^a |\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^n. \tag{A.12}
\]

**Proof.** By Ishii’s Lemma (see [10] or [24, Theorem 3.6]), for each \( \mu > 1 \) there are \( X_\mu, Y_\mu \in S^n \) such that
\[
(u(x_\theta), D_2 \phi(x_\theta, y_\theta), X_\mu) \in \overline{T}_2^+ u(x_\theta), \quad (v(y_\theta), -D_2 \phi(x_\theta, y_\theta), Y_\theta) \in \overline{T}_2^- v(y_\theta),
\]
and
\[
-(\mu + \|A\|) \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \leq \begin{pmatrix} X_\mu & 0 \\ 0 & -Y_\mu \end{pmatrix} \leq A \begin{pmatrix} I_{2n} & 1 \\ -1 & \mu \end{pmatrix}, \tag{A.13}
\]
where \( A = D^2 \phi(x, y) \in S^{2n} \). We compute from the definition of \( \phi \) that
\[
A = \frac{C_1}{\theta^{2-a}} \begin{pmatrix} (1 + \theta)^2 I_n & -(1 + \theta)I_n \\ -(1 + \theta)I_n & I_n \end{pmatrix}
\]
\[
= \frac{C_1}{\theta^{2-a}} \left[ \begin{pmatrix} I_n & -I_n \\ -I_n & I_n \end{pmatrix} + \theta \begin{pmatrix} 2I_n & -I_n \\ -I_n & 0 \end{pmatrix} + \theta^2 \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \right].
\]
The first two terms annihilate \((\zeta, \bar{\zeta})^T\) for \(\zeta \in \mathbb{R}^N\). Therefore
\[
(\zeta, \bar{\zeta}) \mathbf{A} (\zeta, \bar{\zeta})^T = \frac{C_1}{\theta^{2-\alpha}} \left[ \theta^2 |\zeta|^2 \right] = C_1 \theta^\alpha |\zeta|^2.
\]
Similarly, we have
\[
\mathbf{A}^2 = \left( \frac{C_1}{\theta^{2-\alpha}} \right)^2 \left( \begin{array}{cc}
\theta^4 + 4\theta^3 + 7\theta^2 + 6\theta + 1 & I_n \\
-\theta^3 + 3\theta^2 + 4\theta + 1 & 0
\end{array} \right) - \left( \begin{array}{cc}
\theta^3 + 3\theta^2 + 4\theta + 1 & 0 \\
(\theta^2 + 2\theta + 2)I_n & I_n
\end{array} \right)
\]
\[
= \frac{C_1^2}{\theta^{4-2\alpha}} \left( \begin{array}{cc}
0 & I_n \\
-I_n & 2I_n
\end{array} \right) + 2\theta \left( \begin{array}{cc}
3I_n & -2I_n \\
-2I_n & I_n
\end{array} \right) + \theta^2 \left( \begin{array}{cc}
7I_n & -3I_n \\
-3I_n & I_n
\end{array} \right)
\]
\[
+ \theta^3 \left( \begin{array}{cc}
4I_n & -1I_n \\
-1I_n & 0
\end{array} \right) + \theta^4 \left( \begin{array}{cc}
I_n & 0 \\
0 & 0
\end{array} \right)
\]
Using \((\zeta, \bar{\zeta})^T\) for \(\zeta \in \mathbb{R}^n\) the previous equation we deduce that
\[
(\zeta, \bar{\zeta}) \mathbf{A}^2 (\zeta, \bar{\zeta})^T = \frac{C_1^2}{\theta^{4-2\alpha}} \left( 1 + 0 + 2\theta^2 + 2\theta^3 + \theta^4 \right) |\zeta|^2.
\]
We conclude that for \(\theta\) small then
\[
\zeta^T (X_{\mu} - Y_{\mu}) \zeta \leq \left( C_1 \theta^\alpha + \frac{2C_1^2}{\mu \theta^{4-2\alpha}} \right) |\zeta|^2.
\]
Choose \(\mu = \mu(\theta)\) such that \(\mu = 2C_1 \theta^{\alpha-4} > 1\) we obtain the conclusion (A.12).

From Theorem A.1, the local H"older continuity of solutions, and the comparison principle, we obtain the following corollary.

**Corollary A.5.** Assume (1.2), (\(\mathcal{H}_1\)), (\(\mathcal{H}_2\)), (\(\mathcal{H}_3\)), (\(\mathcal{H}_4\)) and \(\lambda > 0\). There exists a unique solution \(u_\delta \in C^{0,\alpha} (\overline{\Omega})\) of (HJ).

**Proof.** Let \(u_\delta \in C(\overline{\Omega})\) be the viscosity solution defined as in (A.6). By Theorem A.2 we have \(u_\delta \in C^{0,\alpha}(\overline{\Omega})\) where \(\alpha = (p-2)/(p-1)\). If \(v \in C(\overline{\Omega})\) is another viscosity solution to (HJ) then by Theorem A.2 \(v \in C^{0,\alpha}(\overline{\Omega})\) as well, thus by the comparison principle in Theorem A.3 we obtain \(u_\delta \equiv v\).

**Proposition A.6.** Assume (1.2), (\(\mathcal{H}_1\)), (\(\mathcal{H}_2\)), (\(\mathcal{H}_3\)), (\(\mathcal{H}_4\)) and \(\lambda > 0\).

(i) We have the uniform bound from below \(\delta u_\delta (\cdot) \geq -\max_{x \in \Omega} H(x,0)\) for all \(\delta > 0\).

(ii) If \(B(0,\kappa) \subset \Omega\) then \(\delta u_\delta (x) \leq C(\kappa)\) for \(x \in B(0,\kappa)\), where \(C\) depends on \(\kappa\) and constants from (\(\mathcal{H}_1\)),(\(\mathcal{H}_2\)), (\(\mathcal{H}_3\)).

**Proof.** Since \(u \equiv -\delta^{-1} \max\{H(x,0) : x \in \overline{\Omega}\}\) is a classical subsolution in \(\Omega\), the comparison principle concludes the lower bound.

Let \(v \in C(\overline{B(0,\kappa)})\) be a solution to the state-constraint solution to the problem (HJ) with discount factor \(\delta = 1\) and on the domain \(B(0,\kappa)\), i.e.,
\[
\begin{cases}
v(x) + H(x, Dv(x)) - \varepsilon \text{Tr}(a(x)D^2v(x)) \leq 0 & \text{in } B(0,\kappa), \\
v(x) + H(x, Dv(x)) - \varepsilon \text{Tr}(a(x)D^2v(x)) \geq 0 & \text{in } \overline{B(0,\kappa)}.
\end{cases}
\]
We observe that
\[
w(x) = v(x) + (1 + \delta^{-1})\|v\|_{L^\infty(B(0,\kappa))} \quad x \in \overline{B(0,\kappa)}
\]
is a supersolution to (A.5), thus by comparison principle on \(B(0,\kappa)\) we have the conclusion.
Appendix B. Proofs of some technical results

Proof of Theorem 2.8. Since \( u \) is a subsolution to \((\delta, 0)\), by comparison principle we have \( u \leq v \) on \( \overline{\Omega} \). Let \( \bar{u}(x) = (1 + \theta)^{-2}u((1 + \theta)x) \) for \( x \in \Omega \). We define

\[
\Psi(\hat{x}, y) = v \left( \frac{\hat{x}}{1 + \theta} \right) - \bar{u}(y) - \frac{C_1|\hat{x} - y|^2}{\theta^{2-a}}, \quad (\hat{x}, y) \in (1 + \theta)\overline{\Omega} \times \overline{\Omega}
\]

where \( C_1 = (2/\kappa)^{2-a}C_0 \). Assume \( \Psi \) has a maximum at \((\hat{x}_\theta, y_\theta) \in (1 + \theta)\overline{\Omega} \times \overline{\Omega}\). Using \( \Psi(\hat{x}_\theta, y_\theta) \geq (y_\theta, y_\theta) \) we deduce that

\[
\frac{C_1|\hat{x}_\theta - y_\theta|^2}{\theta^{2-a}} \leq v \left( \frac{\hat{x}_\theta}{1 + \theta} \right) - v \left( \frac{\hat{y}_\theta}{1 + \theta} \right) \leq C_0|\hat{x}_\theta - y_\theta|^{2-a}
\]

where \( v \) is Hölder continuous (Theorem 2.1). Therefore dist\((\hat{x}_\theta, \overline{\Omega}) \leq |\hat{x}_\theta - y_\theta| \leq \frac{\theta}{\kappa} \). By (A1) we obtain \( \hat{x}_\theta \in (1 + \theta)\Omega \). Let \( \hat{x} = (1 + \theta)x \) for \( x \in \Omega \), we conclude that

\[
(x, y) \mapsto v(x) - \bar{u}(y) - \frac{C_1|(1 + \theta)x - y|^2}{\theta^{2-a}}
\]

has a maximum over \( \overline{\Omega} \times \overline{\Omega} \) at \( (x_\theta, y_\theta) \in \Omega \times \overline{\Omega} \). From Ishii’s lemma there exist matrices \( X_\theta, Y_\theta \in S^n \) such that \( \xi^T(X_\theta - Y_\theta)\xi \leq 2C_1\theta^a|\xi|^2 \) for all \( \xi \in \mathbb{R}^n \) and

\[
\begin{align*}
\delta v(x_\theta) + (1 + \theta)^p \left| \frac{2C_1((1 + \theta)x_\theta - y_\theta)}{\theta^{2-a}} \right|^p - f(x_\theta) - \varepsilon \text{ Tr}(X_\theta) & \leq 0, \\
\delta(1 + \theta)^2 \bar{u}(y_\theta) + (1 + \theta)^p \left| \frac{2C_1((1 + \theta)x_\theta - y_\theta)}{\theta^{2-a}} \right|^p - f((1 + \theta)y_\theta) - \varepsilon \text{ Tr}(Y_\theta) & \geq 0.
\end{align*}
\]

Therefore

\[
\delta v(x_\theta) - \delta u((1 + \theta)y_\theta) \leq f(x_\theta) - f((1 + \theta)y_\theta) + \varepsilon \text{ Tr}(X_\theta - Y_\theta) \leq C\theta + \varepsilon C\theta^a
\]

where \( C \) depends on diam\((\Omega)\), the Lipschitz constant of \( f \) on \( U \) and \( p \). Finally, for \( x \in \overline{\Omega} \) then \( \Psi(x_\theta, y_\theta) \geq \Psi((1 + \theta)x, x) \) implies that

\[
v(x) - \frac{1}{(1 + \theta)^2}u((1 + \theta)x) \leq v(x_\theta) - \frac{1}{(1 + \theta)^2}u((1 + \theta)x_\theta) + C_1(1 + \theta)^2|x|^2\theta^a.
\]

Therefore

\[
\delta v(x) - \delta u(x) - C|\theta|^a \leq (1 + \theta)^2\delta v(x) - \delta u((1 + \theta)x)
\]

\[
\leq \left( \delta v(x_\theta) - \delta u((1 + \theta)x_\theta) \right) + \left( (1 + \theta)^2 - 1 \right)\delta v(x_\theta) + 4C_1\delta^a|x|^2
\]

\[
\leq C\theta + C\theta^a + 2C\theta + 4C_1\delta^a|x|^2
\]

where we invoke Corollary 2.7 to obtain \( |\delta v(x_\theta)| \leq C \). □

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