A NOTE ON THE NON-EXISTENCE OF SMALL COHEN-MACAULAY ALGEBRAS

BHARGAV BHATT

ABSTRACT. By finding a $p$-adic obstruction, we construct many examples of complete noetherian normal $\mathbf{F}_p$-algebras $R$ such that no module-finite extension $R \hookrightarrow S$ is Cohen-Macaulay. These examples should be contrasted with a result of Hochster-Huneke: the directed union of all such extensions is always Cohen-Macaulay.

1. Introduction

A noetherian ring is called Cohen-Macaulay (CM) if there are no non-trivial relations between elements of a system of parameters. These rings form an exceptionally well behaved class: local cohomology vanishes when it can, and duality works out beautifully. This note studies rings realisable as subrings of CM rings:

Definition 1.1. A noetherian ring $R$ admits a big CM algebra if there is an injective map $R \hookrightarrow S$ of rings with $S$ CM. If the ring $S$ can be chosen to be finitely generated as an $R$-module, then we say that $R$ admits a small CM algebra.

Under mild hypotheses, a fundamental result of Hochster and Huneke [HH92] shows that any ring $R$ that contains a field admits a big CM algebra; in fact, if $R$ is an $\mathbf{F}_p$-algebra, the absolute integral closure of $R$ does the job. Thus, one asks: does a ring $R$ always admit a small CM algebra? The answer is “no” in characteristic $0$ based on a local cohomological obstruction. In characteristic $p$, however, the results of loc. cit. immediately nullify any coherent cohomological obstructions: any non-trivial relation can be trivialised after a finite extension. In fact, by [HL07], there is a single finite extension $R \hookrightarrow S$ trivialising all unwanted relations. Nevertheless, our goal in this note is to discuss a negative answer to the above question in characteristic $p$ using a $p$-adic (rigid) cohomological obstruction.

Theorem 1.2. Let $(A, L)$ be polarised projective variety over a perfect field $k$; set $\hat{R}$ to be completion at the origin of $\bigoplus_{n \geq 0} H^0(A, L^n)$. Assume $H_{\text{rig}}^1(A)_{<1} \neq 0$ for some $0 < i < \dim(A)$. Then $\hat{R}$ does not admit a small CM algebra.

The hypothesis on $A$ is satisfied, for example, if $A$ is an abelian surface. This theorem generalises a calculation of Sannai-Singh [SS12] Example 5.3 (who showed the non-existence of a graded small CM algebra for a specific $A$). This theorem should also be contrasted with a result of Hartshorne: if $A$ is CM, then $R$ admits a small CM module, i.e., a module whose depth is $\dim(A)$ (see [Hoc75]). We end by noting that the proof of Theorem 1.2 shows a stronger statement: any local $\mathbf{F}_p$-algebra that admits a small CM algebra is “Witt Cohen-Macaulay,” see Remark 2.12.

A summary of the proof. Let us informally explain the proof in an example: $A$ is an ordinary abelian surface. The key idea is to work $p$-adically instead of modulo $p$ until the end to track the divisibility properties of local cohomology under finite extensions. More precisely, the $p$-rank of $A$ contributes free $\mathbf{Z}_p$-summands to $H^2_m(\text{Spec}(\hat{R})_{\text{et}}, \mathbf{Z}_p)$. Trace formalism then shows: for any finite extension $\hat{R} \hookrightarrow \hat{S}$, the group $H^2_m(\text{Spec}(\hat{S})_{\text{et}}, \mathbf{Z}_p)$ also contains free $\mathbf{Z}_p$-summands. Reducing these modulo $p$ and using the Artin-Schreier sequence shows $H^2_m(\hat{S}) \neq 0$, so $\hat{S}$ is not CM. In the body of the note, we work with Witt-vector cohomology instead of $p$-adic étale cohomology to apply the preceding argument with fewer ordinary constants, see Remark 2.2.

Notation. For any $\mathbf{F}_p$-scheme $X$, we write $\{W_n \mathcal{O}_X\}$ of the projective system of (Zariski) sheaves defined by the truncated Witt vector functors (see [HL79] Ser58), and $W^0_X := \lim_{\leftarrow} W_n \mathcal{O}_X$ for its inverse limit (which is also the derived limit: the transition maps $W_n \mathcal{O}_X \to W_{n-1} \mathcal{O}_X$ are surjective for all $n$, and $W_n \mathcal{O}_X$ has no higher cohomology on affines). For a closed subset $Z \subset X$ and any abelian sheaf $A$, we write $R\Gamma_Z(X, A)$ for the homotopy-kernel of $R\Gamma(X, A) \to R\Gamma(X - Z, A)$. In particular, since cohomology commutes with derived limits, we have $R\Gamma_Z(X, W^0 \mathcal{O}_X) \simeq R \lim_{\leftarrow} R\Gamma_Z(X, W_n \mathcal{O}_X)$. Recall that $R\Gamma(X, -)$ commutes with filtered colimits of sheaves if $X$ is quasi-compact and quasi-separated. In particular, if both $X$ and $X - Z$ are quasi-compact and quasi-separated, then $R\Gamma_Z(X, W^0 \mathcal{O}_X, \mathbb{Q}) \simeq \left(R \lim_{\leftarrow} R\Gamma_Z(X, W_n \mathcal{O}_X)\right) \otimes \mathbb{Q}$.

\footnote{A normal local noetherian $\mathbf{Q}$-algebra $R$ that admits a small CM algebra $S$ is itself CM: $H^i_m(R)$ is a summand of $H^i_m(S)$ via the trace splitting.}
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2. Proof

2.1. Remarks on Witt vectors. We use \( k \) to denote a fixed perfect base field of characteristic \( p \). We start by recalling a fundamental result identifying the slope \( < 1 \) part of rigid cohomology with Witt vector cohomology; the smooth case is due to Bloch-Deligne-Illusie (see [Bli77] Theorem 0.2] and [Ill79] [II.3]), while the singular case is newer:

**Theorem 2.1** ([Bli77] Theorem 1.1). Let \( A \) be a proper \( k \)-variety. Then \( H^i(A, W\mathcal{O}_A)_{\mathbb{Q}} \simeq H^{i+1}_{d\mathbb{S}}(A)_{< 1} \).

**Remark 2.2.** Theorem 2.1 is the main reason we work with Witt-vector cohomology instead of \( A \).

**Notation 2.7.** Let \( \mathcal{O} \) be a fundamental result identifying the slope \( \geq 1 \) part of rigid cohomology in this note: the latter only describes the slope \( 0 \) part of rigid cohomology, while the former describes the (potentially much larger) slope \( < 1 \) part. In particular, if \( A \) is an abelian scheme, then \( H^1(A, W\mathcal{O}_A, \mathbb{Q}) \) is always non-zero for weight reasons, while \( H^1(A, \mathcal{Z}_p) \) vanishes if \( A \) is supersingular and \( k = \overline{k} \).

The next lemma allows deduction of modulo \( p \) consequences from rational assumptions.

**Lemma 2.3.** Let \( Y \) be a \( k \)-scheme with a closed subscheme \( Z \). If \( H^j_Z(Y, W\mathcal{O}_Y)_{\mathbb{Q}} \neq 0 \) for some \( i \), then \( H^j_Z(Y, \mathcal{O}_Y) \neq 0 \) for \( j = i \) or \( j = i - 1 \).

**Proof.** The assumption on \( H^j_Z(Y, W\mathcal{O}_Y)_{\mathbb{Q}} \) and the formula

\[
R\Gamma_Z(Y, W\mathcal{O}_Y)_{\mathbb{Q}} \simeq \left( R\lim_n R\Gamma_Z(Y, W_n\mathcal{O}_Y) \right) \otimes_{\mathbb{Q}} \mathbb{Q}
\]

show that \( H^j_Z(Y, W_n\mathcal{O}_Y) \neq 0 \) for some \( n > 0 \) and some \( j \in \{i - 1, i\} \) (as \( R^i \lim_n = 0 \) for \( i > 1 \)). The rest follows by standard exact sequences expressing \( W_n\mathcal{O}_Y \) as an iterated extension of copies of \( \mathcal{O}_Y \).

We need trace maps in Witt vector cohomology, so we recall a direct construction (essentially due to [SV96]).

**Lemma 2.4.** Let \( f : Y \to X \) be a finite surjective morphism of noetherian normal schemes. For any abelian sheaf \( A \) on \( X_{\text{et}} \) that is representable by an algebraic space, there is a functorial trace map \( \text{Tr} : f_*f^*A \to A \) such that the composite \( A \xrightarrow{f_*} f_*f^*A \xrightarrow{\text{Tr}} A \) is multiplication by the generic degree of \( f \).

**Proof.** We refer the reader to [Bha, Proposition 6.2] for a proof.

**Remark 2.5.** The trace constructed in Lemma 2.4 is non-standard and slightly ad hoc. For example, if \( f : X \to X \) is the Frobenius map, then \( f^* : \text{Shv}(X_{\text{et}}) \to \text{Shv}(X_{\text{et}}) \) is an equivalence, so the “correct” trace map should be an equivalence, while the one from Lemma 2.4 is multiplication by \( \deg(f) \) (composed with the inverse of \( \text{id} \xrightarrow{\cong} f_*f^* \)).

In particular, the trace map constructed in Lemma 2.4 does not furnish a right adjoint to \( f_* \approx f_* \).

Using trace maps, we show that (rational) Witt vector cohomology cannot be killed by finite covers.

**Corollary 2.6.** Let \( f : Y \to X \) be a finite surjective morphism of noetherian normal \( k \)-schemes, and let \( Z \subset X \) be a closed subset. Then \( H^j_Z(X, W\mathcal{O}_X)_{\mathbb{Q}} \to H^{j-1}_{d\mathbb{S}}(Y, W\mathcal{O}_Y)_{\mathbb{Q}} \) is a direct summand.

**Proof.** Let \( d \) be the generic degree of \( f \). As \( W_n(-) \) is representable for each \( n > 0 \), Lemma 2.4 gives maps \( f_*W_n\mathcal{O}_Y \to W_n\mathcal{O}_X \) whose composition with the pullbacks \( W_n\mathcal{O}_X \to f_*W_n\mathcal{O}_Y \) is multiplication by \( d \). These maps are compatible as \( n \) varies, so taking limits (and commuting them with \( f_* \)) gives a map \( f_*W\mathcal{O}_Y \to W\mathcal{O}_X \) whose composition with \( W\mathcal{O}_X \to f_*W\mathcal{O}_Y \) is multiplication by \( d \). The claim follows by applying \( H^j_Z(X, W\mathcal{O}_X)_{\mathbb{Q}} \).

2.2. The main theorem. To prove Theorem 1.2 we first establish some notation.

**Notation 2.7.** Fix a polarised projective variety \( (A, L) \) of a perfect characteristic \( p \) field \( k \); set \( R = \oplus_{n>0} H^0(A, L^n) \) to be the section ring with \( m \subset R \) the homogeneous maximal ideal. Set \( X = \text{Spec}(R) \), \( Z = \{m\} \subset X \) and \( U = X - Z \) with \( \pi : U \to A \) realising \( U \) as the total space of \( L^{-1} \) over \( A \). Set \( \tilde{X} = \text{Spec}(\tilde{R}) \), \( \tilde{U} = \tilde{X} \times_X U \), and abusively let \( Z \subset \tilde{X} \) denote the closed point.

Next, we record the expected relation between the cohomology of \( A \) and \( U \):
Lemma 2.8. The natural map $W_n O_A \to \pi_* W_n O_U$ is a direct summand for all $n$. In particular, $H^i(A, W_n O_A) \to H^i(U, W_n O_U)$ is a direct summand for all $i$ and $n$.

Proof. As $\pi$ is a $G_m$-torsor, the natural map $O_A \to \pi_* O_U$ realises the source as the weight 0 eigenspace of the target. The rest follows by taking products and observing that $W_n(R) \simeq R^n$ as sets functorially in $R$. 

We can now prove the main theorem:

Proof of Theorem 1.2. As $W_n O_X$ is an extension of $O_X$ by $W_{m-1} O_X$, induction and the affineness of $X$ show $H^i(X, W_n O_X) = 0$ for $i > 0$. Standard sequences then identify $H^i(U, W_n O_U) \simeq H^i_{Z}(X, W_n O_X)$ for all $i, n > 0$. Since $H^0(U, W_n O_U) = W_n(H^0(U, O_U))$, the system $\{H^0(U, W_n O_U)\}$ has no $\lim^1$ (as $W_n(R) \to W_{n-1}(R)$ is surjective for any ring $R$), so

$$H^i(U, W \otimes O_U)_{\Q} \simeq H^i_{Z}(X, W \otimes O_X)_{\Q} \simeq H^i_{Z}(\widehat{X}, W \otimes O_{\widehat{X}})_{\Q}$$

for $i > 0$, where the last isomorphism comes from the excision identification $\{RG_{Z}(X, W_n O_X)\} \simeq \{RG_{Z}(\widehat{X}, W_n O_{\widehat{X}})\}$ of projective systems. Theorem 2.1 and Lemma 2.8 then show $H^i_{\rig}(A)_{< 1} \simeq H^i(A, W \otimes O_A)_{\Q}$ is a direct summand of $H^i_{Z}(\widehat{X}, W \otimes O_{\widehat{X}})_{\Q}$ for $i > 0$. Choose $0 < i < \dim(A)$ such that $H^i_{\rig}(A)_{< 1} \neq 0$. Let $f : \widehat{Y} \to \widehat{X}$ be a finite surjective morphism of noetherian normal schemes; we will show that $\widehat{Y}$ is not CM along $f^{-1} Z$. Corollary 2.6 shows $H^i_{f^{-1} Z}(\widehat{Y}, W \otimes O_{\widehat{Y}})_{\Q} \neq 0$. Lemma 2.3 then gives $H^i_{f^{-1} Z}(\widehat{Y}, O_{\widehat{Y}}) \neq 0$ for some $j \in \{i, i + 1\}$, which proves the claim as $i + 1 = \dim(\widehat{Y})$. 

Remark 2.9. As rigid cohomology is a Weil cohomology theory, the hypothesis on $A$ in Theorem 1.2 may be reformulated topologically, at least when $A$ is smooth and $k$ is finite, to say: for some $0 < i < \dim(A)$, there is at least one Frobenius eigenvalue on $H^i_{\rig}(A, \Q_\ell)$ which is not divisible by $p$ (for some auxiliary prime $\ell$ invertible on $k$). Indeed, the eigenvalues occurring in $H^i_{\rig}(A)$ coincide with those on $H^i(A, \Q_\ell)$ and are algebraic integers.

Example 2.10. Some elementary examples of projective varieties $A$ to which Theorem 1.2 applies include: any projective variety of dimension $\geq 2$ dominating a positive dimensional abelian variety, any projective variety of dimension $\geq 3$ dominating a $K3$ surface of finite height, etc.

We give an example showing that the presence of non-trivial middle cohomology of the structure sheaf does not force the non-existence of a small CM algebra for the section ring, even for smooth projective varieties; this example also shows the necessity of making a $p$-adic (rather than modulo $p$) assumption on $A$ in Theorem 1.2.

Example 2.11. Assume $k$ is algebraically closed, and let $X$ be a smooth complete intersection in some $\mathbb{P}^n_k$ with $\dim(X) \geq 2$ with the property that a suitable subgroup $G \simeq F_p \subset \GL_{n+1}(k)$ preserves $X$ and acts fixed point freely on $X$; such examples were constructed by Serre, see [905] [6]. Let $Y = X/G$ denote the quotient. Then $X \to Y$ is a finite étale $G$-torsor, and $Y$ is a smooth projective $k$-variety; explicitly, the line bundle $O(1) \in \Pic(X)$ is equivariant for the $G$-action, and hence descends to an ample line bundle $L$ on $Y$. Let $R$ and $S$ denote the section rings of $(Y, L)$ and $(X, O(1))$ respectively. Then $R \to S$ is a finite extension. We will show that $S$ is CM (even lci), but $R$ is not. The former follows immediately as $(X, O(1))$ is a complete intersection. On the other hand, the Lefschetz formalism (see [Gro61], Corollary XII.3.5)] shows that $X$ is simply connected. The Leray spectral sequence for $X \to Y$ then yields $F_p \simeq G \simeq \pi^0(Y) \simeq H^1(Y, \mathbb{F}_p^2)^Y$, and hence $H^1(Y, O_Y) = 0$ by the Artin-Schreier sequence. The latter group is a direct summand of $H^2_m(R)$, so $R$ is not CM (as $\dim(R) = \dim(X) + 1 \geq 3$).

Remark 2.12. Let $(R, m)$ be a noetherian local $k$-algebra. The proof of Theorem 1.2 shows that being “Witt Cohen-Macaulay” (i.e., having $H^i_m(\Spec(R), W \otimes R, \Q)$ for $i < \dim(R)$) is necessary for the existence of small CM algebras. We do not know if it is a sufficient condition. A (weakened) graded analogue asks: any projective variety $X$ with $H^i(X, W \otimes O_X, \Q) = 0$ for $0 < i < \dim(X)$ admits an alteration $\pi : Y \to X$ with $H^i(Y, O_Y) = 0$ for $0 < i < \dim(Y)$. The simplest non-trivial instance of this question is when $X = S \times \mathbb{P}^1$ with $S$ a supersingular $K3$ surface, where a positive answer is implied by Artin’s conjecture on unirationality of supersingular $K3$ surfaces.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, ANN ARBOR 48109, USA

E-mail address: bhattb@umich.edu

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