Regularity results for solutions of mixed local and nonlocal elliptic equations

Xifeng Su¹ · Enrico Valdinoci² · Yuanhong Wei³ · Jiwen Zhang⁴

Received: 11 July 2022 / Accepted: 29 July 2022 / Published online: 6 September 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
We consider the mixed local-nonlocal semi-linear elliptic equations driven by the superposition of Brownian and Lévy processes
\[
\begin{align*}
-\Delta u + (-\Delta)^su &= g(x, u) \text{ in } \Omega, \\
u &= 0 \text{ in } \mathbb{R}^n \setminus \Omega.
\end{align*}
\]
Under mild assumptions on the nonlinear term \(g\), we show the \(L^\infty\) boundedness of any weak solution (either not changing sign or sign-changing) by the Moser iteration method. Moreover, when \(s \in \left(0, \frac{1}{2}\right]\), we obtain that the solution is unique and actually belongs to \(C^{1,\alpha}(\Omega)\) for any \(\alpha \in (0, 1)\).

Keywords Operators of mixed order · Regularity · \(L^\infty\) boundedness · Hölder estimate for the gradient

Mathematics Subject Classification 35B65 · 35R11 · 35J67

1 Introduction

The present paper is concerned with the regularity results of elliptic equations driven by a special subclass of mixed differential and pseudo-differential elliptic operators
\[
\mathcal{L} = -\Delta + (-\Delta)^s, \quad \text{for some } s \in (0, 1).
\]
Here, \((-\Delta)^s\) is defined as
\[
(-\Delta)^s u(x) = c_{n,s} \text{ PV} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy,
\]
Both X. Su and Y. Wei are supported by the National Natural Science Foundation of China (Grant no. 11971060, 11871242). Y. Wei is supported by Natural Science Foundation of Jilin Province (Grant no. 20200201248JC), and Scientific Research Project of Education Department of Jilin Province (Grant no. JJKH20220964KJ).

✉ Jiwen Zhang
jwzhang628@mail.bnu.edu.cn

Extended author information available on the last page of the article
where $c_{n,s}$ is a suitable normalization constant, whose explicit value only plays a minor role in this paper, and PV means that the integral is taken in the Cauchy Principal Value sense.

The operator $L$ naturally arises as the superposition of a classical random walk and a Lévy flight. For instance, as observed in [21], these operators describe a biological species whose individuals diffuse either by a random walk or by a jump process, according to prescribed probabilities. The analysis of different types of mixed operators motivated by biological questions has also carried out in [19, 26, 27].

Moreover, mixed operators have recently received a great attention from different points of view, including regularity theory [2, 3, 8, 10, 12, 14, 15, 17, 25], existence and non-existence results [1, 6, 28], eigenvalue problems [13, 16, 22], shape optimization and calculus of variations [7, 11], symmetry and rigidity results [9], etc.

One interesting challenging aspect of this topic is that it combines the classical setting and the features typical of nonlocal operators in a framework that is not scale-invariant. Hence, at different scales, different features of the classical/nonlocal world tend to prevail, or they end up coexisting into new interesting phenomena.

The goal of this paper is to show the $L^\infty$- and $C^{1,\alpha}$-regularity for the weak solutions of

$$\begin{cases}
Lu = g(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

where a suitable notion of weak solutions associated to (1.1) on the space $X^1_{\alpha}$ will be detailed in Definition 2.1 of Sect. 2.

In order to get right to the point, we will suppose in the present paper that the existence of weak solutions of (1.1) is known. As a matter of fact, weak solutions are often obtained via variational methods and nonlinear analysis tools, e.g. one may refer [29] for the existence of weak solutions (both not changing sign and sign-changing) under standard nonlinear analysis assumptions.

We assume once and for all that $s \in (0, 1)$ is given, $n > 2s$, and $\Omega \subset \mathbb{R}^n$ is an open bounded set satisfying some smooth boundary conditions, which will be specified as needed.

We will first derive an $L^\infty$-regularity result for any weak solution of the elliptic problem (1.1). More precisely, denoting $2^*_s := \frac{2n}{n - 2s}$, we have:

**Theorem 1.1** Let $u \in X^1_0$ be a weak solution of (1.1) and $s \in (0, 1)$. Assume that there exist $c > 0$ and $q \in [2, 2^*_s]$ such that

$$|g(x, t)| \leq c(1 + |t|^{q-1}) \quad \text{for a.e. } x \in \Omega, t \in \mathbb{R}. \quad (1.2)$$

Then, $u \in L^\infty(\Omega)$. More precisely, there exists a constant $C_0 = C_0(c, n, s, \Omega) > 0$ independent of $u$, such that

$$\|u\|_\infty \leq C_0 \left(1 + \int_{\Omega} |u|^{2^*_s \beta_1} \, dx\right)^{\frac{1}{2^*_s (\beta_1 - 1)}},$$

where $\beta_1 := \frac{2^*_s + 1}{2}$ and $\| \cdot \|_\infty := \| \cdot \|_{L^\infty(\Omega)}$.

We remark that the weak solutions here could be both not changing sign and sign-changing. In the case of the fractional Laplace operator $(-\Delta)^s$, the $L^\infty$ boundedness is also proved

\footnote{As a technical remark, the additional presence of the Laplace operator will produce an extra term (that is the first term in (3.3)). We will check that this term is nonnegative by using a suitable integration by parts for the second-order generalized derivative, which will allow us to successfully complete the estimate produced by the full mixed order operator.}
respectively in [4, 20] for a positive weak solution and [30, 31] for a weak solution which could change sign.

As is standard in the literature (see e.g. [5, 24]), the \( L^\infty \)-regularity of weak solutions allows one to obtain a global \( C^{1,\alpha} \)-regularity theory, which relies on the \( W^{2,p} \)-regularity theory.

We will pursue this direction and our \( C^{1,\alpha} \)-regularity theorem of the mixed elliptic equation (1.1) with \( s \in (0, \frac{1}{2}] \) goes as follows.

**Theorem 1.2** Let us assume, in addition to the hypothesis of Theorem 1.1, that \( \partial \Omega_1 \) is of class \( C^{1,1} \) and \( s \in (0, \frac{1}{2}] \), with \( n > 2s \). Then,

\[
\|u\|_{C^{1,\alpha}(\Omega_1)} \quad \text{for any } \alpha \in (0, 1).
\]

**Remark 1.3** As observed e.g. in [7, Theorem 2.7] and in the references therein, Hölder estimates for the gradient of the solution remain valid for all \( s \in (0, 1) \), in the sense that one can prove in such a generality that \( u \in C^{1,\beta}(\Omega_1) \) for some \( \beta \in (0, 1) \) (for instance, such a result would follow by combining Theorem 1.2 and Lemma 4.4). The interest of Theorem 1.2 is in finding a precise Hölder exponent for this type of regularity theory in the range \( s \in (0, \frac{1}{2}] \).

Note that Theorem 1.2 is just an immediate corollary of the following \( W^{2,p} \)-regularity theorem.

**Theorem 1.4** Let \( \Omega \) be \( C^{1,1} \) domain in \( \mathbb{R}^n \) and \( s \in (0, \frac{1}{2}] \), \( n > 2s \). Then if \( f \in L^p(\Omega) \) with \( 1 < p < +\infty \), the problem

\[
-\Delta u + (-\Delta)^s u = f, \quad \text{in } \Omega
\]

has a unique solution \( u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \). Furthermore,

\[
\|u\|_{W^{2,p}(\Omega)} \leq C_1 \left( \|u\|_p + \|f\|_p \right),
\]

where the constant \( C_1 \) depends on \( \Omega, n, s, p \) and \( \|\cdot\|_p := \|\cdot\|_{L^p(\Omega)} \) for short.

## 2 Some preliminary facts

In this section, we provide several definitions and basic facts on the weak solutions of the Dirichlet problem associated with the mixed operator \( \mathcal{L} \) in (1.1), that is

\[
\begin{aligned}
\mathcal{L} u &= g(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega.
\end{aligned}
\]

See also [29] for the existence and multiplicity results of weak solutions.

Let \( s \in (0, 1) \) be given and \( \Omega \subset \mathbb{R}^n \) be an open bounded set with \( C^1 \) boundary where \( n > 2s \). A “natural” space to consider is the following (see e.g. [10]):

\[
X^1 := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ is Lebesgue measurable: } u|_\Omega \in H^1(\Omega); \frac{|u(x) - u(y)|}{|x - y|^{n+2s}} \in L^2(\mathbb{R}^n) \right\}.
\]

The norm of \( u \in X^1 \) is defined as follows:

\[
\|u\|_{X^1} = \left( \|u\|_{H^1(\Omega)}^2 + \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2}.
\]
Our working space for the weak solutions would be:

\[ X^1_0 \equiv \{ u \in X^1 : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}. \]

**Remark 2.1** Since \( \Omega \) has \( C^1 \) boundary, any function \( u \in X^1_0 \) satisfies

\[ u|_{\Omega} \in H^1_0(\Omega). \]

Due to Remark 2.1, the norm in \( X^1_0 \) is also equivalent to

\[ \|u\|_{X^1_0} := \left( \|\nabla u\|_2^2 + \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dxdy \right)^{1/2}. \]

Obviously, \( X^1_0 \) is a Hilbert space equipped with the inner product

\[ \langle u, v \rangle_{X^1_0} = \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dxdy, \quad \forall u, v \in X^1_0. \]

**Definition 2.1** (Weak solution) We say that \( u \in X^1_0 \) is a weak solution of the mixed elliptic equation (1.1), if \( u \) satisfies

\[ \int_{\Omega} \langle \nabla u, \nabla \varphi \rangle + \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dxdy = \int_{\Omega} g(x, u(x))\varphi(x) \, dx, \quad \forall \varphi \in X^1_0. \]

To begin with, we investigate some key facts of \( X^1_0 \) in the following embedding lemma.

**Lemma 2.2** The embedding \( X^1_0 \hookrightarrow L^{2^*_s}(\Omega) \) is continuous where \( 2^*_s = \frac{2n}{n-2s} \).

**Proof** Thanks to [18], we know for all \( v \in X^1_0, v \in H^s(\mathbb{R}^n) \), and

\[ \|v\|^2_{L^{2^*_s}(\mathbb{R}^n)} \leq S \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dxdy, \]

where \( S \) is a positive constant depending on \( n \) and \( s \). It follows that

\[ \|v\|^2_{2^*_s} = \|v\|^2_{L^{2^*_s}(\mathbb{R}^n)} \leq S \int_{\mathbb{R}^{2n}} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}} \, dxdy \leq S\|v\|^2_{X^1_0}. \]

Thus, the embedding \( X^1_0 \hookrightarrow L^{2^*_s}(\Omega) \) is continuous. \( \square \)

Let

\[ C^\infty := \{ u \in C(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \, u|_{\Omega} \in C^\infty(\Omega) \} \]

and denote by \( \overline{C^\infty} \) the closure of \( C^\infty \) with respect to \( X^1_0 \)-norm. Then we have

**Lemma 2.3** \( \overline{C^\infty} = X^1_0 \).

**Proof** (i). We claim that

\[ \mathcal{C}^\infty \subset X^1_0. \quad (2.1) \]

Note that this and the fact that \( X^1_0 \) is complete would imply that \( \overline{C^\infty} \subset X^1_0 \).
To check (2.1) we proceed as follows. For any \( u \in C_0^\infty \), there exists a compact subset \( K \subset \Omega \) such that \( u \equiv 0 \) in \( \mathbb{R}^n \setminus K \). We split the square of the norm \( \| u \|_{X_0^1}^2 \) into three parts

\[
\| u \|_{X_0^1}^2 = \| u \|_{H_0^1(\Omega)}^2 + \iint_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy = \| u \|_{H_0^1(\Omega)}^2 + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy + 2 \iint_{\Omega} \iint_{\mathbb{R}^n} \frac{|u(x)|^2}{|x - y|^{n+2\alpha}} \, dy \, dx = I + II + III.
\]

(2.2)

From [18, Proposition 2.2], we obtain\(^2\) that

\[
II \leq c_0 \| u \|_{H_0^1(\Omega)}^2, \quad \text{where } c_0 = c_0(n, s) \text{ is a constant}. \tag{2.3}
\]

It remains to show that \( III \) is bounded. For any \( y \in \mathbb{R}^n \setminus \Omega \), we have that

\[
\frac{|u(x)|^2}{|x - y|^{n+2\alpha}} \leq \chi_K(x)|u(x)|^2 \sup_{x \in K} \frac{1}{|x - y|^{n+2\alpha}},
\]

and so

\[
\iint_{\mathbb{R}^n \setminus \Omega} \frac{|u(x)|^2}{|x - y|^{n+2\alpha}} \, dy \, dx \leq \iint_{\mathbb{R}^n \setminus \Omega} \frac{1}{\text{dist}(y, \partial K)^{n+2\alpha}} \, dy \, \| u \|_{H_0^1(\Omega)}^2. \tag{2.4}
\]

We stress that the integral in (2.4) is finite since \( \text{dist}(\partial \Omega, \partial K) \geq \alpha > 0 \). Combining (2.2) and (2.4), we conclude that \( u \in X_0^1 \), thus proving (2.1) as desired.

(ii). On the other hand, since \( C_0^\infty(\Omega) \) is dense in \( H_0^1(\Omega) \), for all \( u \in X_0^1 \subset H_0^1(\Omega) \), there exists \( \{ u_m \} \subset C_0^\infty(\Omega) \) such that

\[
\| u_m - u \|_{H_0^1(\Omega)} \to 0 \quad \text{and} \quad u_m \to u \text{ a.e. in } \Omega, \quad m \to +\infty.
\]

We define \( u_m \equiv 0 \) in \( \mathbb{R}^n \setminus \Omega \) for every \( m \in \mathbb{N} \). Then \( \{ u_m \} \subset C_0^\infty \).

We claim that

\[
\{ u_m \} \text{ is a Cauchy sequence in } X_0^1. \tag{2.5}
\]

Indeed, notice that half of \( III \) in (2.2) could be divided into the following two integrals and be estimated respectively by

\[
\iint_{\Omega} \iint_{\mathbb{R}^n \setminus \Omega \cap |x-y| \geq 1} \frac{|(u_j - u_k)(x)|^2}{|x - y|^{n+2\alpha}} \, dy \, dx \leq \int_{\Omega} \left( \int_{|z| \geq 1} \frac{1}{|z|^{n+2\alpha}} \, dz \right) |(u_j - u_k)(x)|^2 \, dx < c_1 \| (u_j - u_k) \|_{H_0^1(\Omega)}^2. \tag{2.6}
\]

\(^2\) As a notation remark, the convention used here is that

\[
\int_A \int_B f(x, y) \, dy \, dx := \int_A \left( \int_B f(x, y) \, dy \right) \, dx.
\]
and
\[
\int_{\Omega} \int_{(\mathbb{R}^n \setminus \Omega) \cap \{|x-y|<1\}} \frac{|(u_j - u_k)(x)|^2}{|x-y|^{n+2s}} \, dy \, dx 
\leq \int_{\Omega} \int_{(\mathbb{R}^n \setminus \Omega) \cap \{|z|<1\}} \frac{\int_0^1 |\nabla (u_j - u_k)(x + tz)|^2 \, dt}{|z|^{n+2s-2}} \, dz \, dx
\leq \int_{\mathbb{R}^n} \int_{(\mathbb{R}^n \setminus \Omega) \cap \{|z|<1\}} \frac{\int_0^1 |\nabla (u_j - u_k)(x + tz)|^2 \, dt}{|z|^{n+2s-2}} \, dz \, dx
\leq \|u_j - u_k\|_{H^1(\mathbb{R}^n)}^2 \int_{|z|<1} \frac{1}{|z|^{n+2s-2}} \, dz \leq c_2 \|u_j - u_k\|_{H^1_0(\Omega)}^2,
\]
where $c_1, c_2 > 0$ are constants independent of $\{|u_m|\}$. Due to (2.2), (2.3) and (2.4), we thereby obtain (2.5) as desired. This yields that $X_0^1 \subset C^{\infty}_0$.

**Remark 2.4** Thanks to (2.3), (2.6) and (2.7), we see that, for every $u \in X_0^1$,
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^2 \, dx \, dy \leq C \|u\|_{H^1_0(\Omega)},
\]
which implies that the norm $\| \cdot \|_{X_0^1}$ is equivalent to $\| \cdot \|_{H^1_0(\Omega)}$ in the space $X_0^1$.

### 3 $L^\infty$-regularity for any weak solution

In this section, we show that any weak solution $u \in X_0^1$ of (1.1) is actually of $L^\infty(\Omega)$ class. To prove this result, we will use the Moser iteration method.

**Proof of Theorem 1.1** We will divide the proof into the following three steps.

**Step 1.** We construct an auxiliary function $\varphi$ and provide several fundamental properties of $\varphi$, which are useful for the iterative procedure.

Given $\beta > 1$ and $T > 0$, we define
\[
\varphi(t) = \begin{cases} 
-\beta T^{\beta-1}(t+T) + T^\beta, & \text{if } t \leq -T, \\
|t|^{\beta}, & \text{if } -T < t < T, \\
\beta T^{\beta-1}(t-T) + T^\beta, & \text{if } t \geq T.
\end{cases}
\]

From the definition of $\varphi(t)$, we may compute $\varphi'$ and $\varphi''$ (in the sense of distributions), that is
\[
\varphi'(t) = \begin{cases} 
-\beta T^{\beta-1}, & \text{if } t < -T, \\
-\beta(-t)^{\beta-1}, & \text{if } -T \leq t < 0, \\
\beta t^{\beta-1}, & \text{if } 0 \leq t < T, \\
\beta T^{\beta-1}, & \text{if } t > T.
\end{cases}
\]

and
\[
\varphi''(t) = \begin{cases} 
\beta(\beta-1)t^{\beta-2}, & 0 < t < T, \\
\beta(\beta-1)(-t)^{\beta-2}, & -T < t < 0, \\
0, & \text{otherwise}.
\end{cases}
\]

We observe that $\varphi(u) \in X_0^1$. As a matter of fact, from the definition of $\varphi$, we have

- $\varphi(u) = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$;
- $\varphi$ is convex and Lipschitz with Lipschitz constant $L = \beta T^{\beta-1}$;

\(\square\) Springer
one can calculate that
\[
\| \varphi(u) \|_{X_0^1}^2 = \| \nabla(\varphi(u)) \|_2^2 + \iint_{\mathbb{R}^{2n}} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x - y|^{n+2s}} \, dx dy
\]
\[
\leq L^2 \left( \| \nabla u \|_2^2 + \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx dy \right) < \infty.
\]
Moreover, for given \( x, y \in \mathbb{R}^n \), we deduce from the convexity of \( \varphi \) that
\[
\varphi(u(x)) - \varphi(u(y)) \leq \varphi'(u(x))(u(x) - u(y)),
\]
which implies that
\[
(-\Delta)^s \varphi(u) \leq \varphi'(u)(-\Delta)^s u
\] (3.1)
in the sense of distribution.

**Step 2.** Now we give an estimate of \( \| \varphi(u) \|_{2^s} \).

For this, we recall that
\[
\|u\|_{L^{2^*_s}(\mathbb{R}^n)}^2 \leq S \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx dy,
\]
where \( S \) is a constant depending only on \( n \) and \( s \).

Hence, by (3.1), we have that
\[
\| \varphi(u) \|_{2^s}^2 \leq S \iint_{\mathbb{R}^{2n}} \frac{|\varphi(u(x)) - \varphi(u(y))|^2}{|x - y|^{n+2s}} \, dx dy
\]
\[
= S \iint_{\mathbb{R}^{2n}} \left[ \frac{\varphi(u(x))}{|x - y|^{n+2s}} - \frac{\varphi(u(y))}{|y - x|^{n+2s}} \right] \frac{\varphi(u(x)) - \varphi(u(y))}{|x - y|^{n+2s}} \, dx dy
\]
\[
= S \iint_{\mathbb{R}^{2n}} \left[ \frac{\varphi(u(x))}{|x - y|^{n+2s}} - \frac{\varphi(u(y))}{|y - x|^{n+2s}} \right] \frac{\varphi(u(y)) - \varphi(u(x))}{|y - x|^{n+2s}} \, dx dy
\]
\[
= 2S \iint_{\mathbb{R}^{2n}} \varphi(u(x)) \frac{\varphi(u(x)) - \varphi(u(y))}{|x - y|^{n+2s}} \, dx dy
\]
\[
= 2S \int_{\mathbb{R}^n} \varphi(u)(-\Delta)^s \varphi(u) \, dx
\]
\[
\leq 2S \int_{\mathbb{R}^n} \varphi(u) \varphi'(u) (-\Delta)^s u \, dx.
\] (3.2)

We multiply the first equation of (1.1) by \( \varphi(u) \varphi'(u) \) and integrate over \( \mathbb{R}^n \)
\[
- \int_{\mathbb{R}^n} \varphi(u) \varphi'(u) \Delta u \, dx + \int_{\mathbb{R}^n} \varphi(u) \varphi'(u) (-\Delta)^s u \, dx = \int_{\mathbb{R}^n} \varphi(u) \varphi'(u) g(x, u) \, dx.
\] (3.3)
The first term on the left side of (3.3) above could be rewritten as
\[
- \int_{\Omega} \varphi(u) \varphi'(u) \Delta u \, dx = \int_{\Omega} \nabla u \cdot \nabla (\varphi(u) \varphi'(u)) \, dx
\]
\[
= \int_{\Omega} |\nabla u|^2 |\varphi'(u)|^2 \, dx + \int_{\Omega} |\nabla u|^2 \varphi(u) \varphi''(u) \, dx \geq 0.
\] (3.4)
The equations (3.4) and (3.3) lead to
\[
\int_{\mathbb{R}^n} \varphi(u) \varphi'(u) (-\Delta)^s u \, dx \leq \int_{\Omega} \varphi(u) \varphi'(u) g(x, u) \, dx.
\]
Hence, recalling (1.2) and (3.2), we obtain
\begin{equation}
\|\varphi(u)\|^2_{L^2} \leq 2S \int_{\Omega} \varphi(u) \varphi'(u) g(x, u) \, dx \\
= 2S \int_{\Omega} \varphi(u) \varphi'(u) |g(x, u)| \, dx \\
\leq C \int_{\Omega} \varphi(u) \varphi'(u) |1 + |u|^{2s-1}| \, dx \\
= C \left( \int_{\Omega} \varphi(u) |\varphi'(u)| \, dx + \int_{\Omega} \varphi(u) |\varphi'(u)| |u|^{2s-1} \, dx \right),
\end{equation}
where the constant \(C\) depends on \(n, s, \Omega\) and \(c\).

Using the estimates \(\varphi(u) \leq |u|^\beta, \varphi'(u) \leq \beta |u|^{\beta-1}\) and \(\varphi \varphi'(u) \leq \beta \varphi(u)\), we have
\begin{equation}
\left( \int_{\Omega} (\varphi(u))^{2s} \, dx \right)^{2/2s} \leq C \beta \left( \int_{\Omega} |u|^{2\beta-1} \, dx + \int_{\Omega} (\varphi(u))^{2} |u|^{2s-2} \, dx \right). \tag{3.5}
\end{equation}
Notice that \(C\) is a positive constant that does not depend on \(\beta\) and that the last integral on the right-hand-side of the inequality (3.5) is well defined for every \(T > 0\), since
\begin{align*}
\int_{\Omega} (\varphi(u))^2 |u|^{2s-2} \, dx &= \int_{\{|u| \leq T\}} (\varphi(u))^2 |u|^{2s-2} \, dx + \int_{\{|u| > T\}} (\varphi(u))^2 |u|^{2s-2} \, dx \\
&\leq T^{2\beta-2} \int_{\Omega} |u|^{2s} \, dx + (\beta + 1)T^{\beta-1} \int_{\Omega} |u|^{2s} \, dx < +\infty.
\end{align*}

**Step 3.** We apply Moser iteration method and the limiting arguments of \(L^p\)-norm to prove the result.

First, we claim if \(\beta_1\) is such that \(2\beta_1 - 1 = 2s\), then \(u \in L^{2s, \beta_1}(\Omega)\). To see this, for any given \(R > 0\), we apply the H"older inequality to the last integral in (3.5) and get
\begin{align*}
\int_{\Omega} (\varphi(u))^2 |u|^{2s-2} \, dx &= \int_{\{|u| \leq R\}} (\varphi(u))^2 |u|^{2s-2} \, dx + \int_{\{|u| > R\}} (\varphi(u))^2 |u|^{2s-2} \, dx \\
&\leq \int_{\{|u| \leq R\}} \frac{(\varphi(u))^2}{|u|^\beta} R^{2s-1} \, dx + \left( \int_{\Omega} (\varphi(u))^{2s} \, dx \right)^{2/2s} \left( \int_{\{|u| > R\}} |u|^{2s} \, dx \right)^{2s-2/2s} \tag{3.6}
\end{align*}
By the Monotone Convergence Theorem, one can choose \(R > 0\) large enough such that
\begin{equation}
\left( \int_{\{|u| > R\}} |u|^{2s} \, dx \right)^\frac{2s-2}{2s} \leq \frac{1}{2C\beta_1},
\end{equation}
where \(C\) is the constant in (3.5). Therefore one can reabsorb the last term in (3.6) into the left hand side of (3.5) to get
\begin{equation}
\left( \int_{\Omega} (\varphi(u))^{2s} \, dx \right)^{2/2s} \leq 2C\beta_1 \left( \int_{\Omega} |u|^{2s} \, dx + R^{2s-1} \int_{\Omega} \frac{(\varphi(u))^2}{|u|^\beta} \, dx \right),
\end{equation}
Now, using \(\varphi(u) \leq |u|^\beta_1\), we get that the terms in the right hand side of the above inequality is bounded and independent of \(T\). Sending \(T \to +\infty\), we obtain that
\begin{equation}
\left( \int_{\Omega} |u|^{2s, \beta_1} \, dx \right)^{2/2s} \leq 2C\beta_1 \left( \int_{\Omega} |u|^{2s} \, dx + R^{2s-1} \int_{\Omega} |u|^{2s} \, dx \right) < +\infty, \tag{3.7}
\end{equation}
which proves the claim.
Next, we will find an increasing unbounded sequence $\beta_m$ such that

$$u \in L^{2s, \beta_m} (\Omega), \quad \forall m > 1.$$  

To this end, let us suppose that $\beta > \beta_1$. Thus, using that $\varphi (u) \leq |u|^\beta$ in the right hand side of (3.5) and letting $T \to \infty$ we get

$$\left( \int_\Omega |u|^{2s, \beta_m} \, dx \right)^{2/2s} \leq C \beta \left( \int_\Omega |u|^{2\beta - 1} \, dx + \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right). \tag{3.8}$$

We also remark that

$$\int_\Omega |u|^{2\beta - 1} \, dx \leq \left( \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right)^{2\beta - 1 \over 2\beta + 2s - 2} |\Omega|^{2s - 1 \over 2\beta + 2s - 2} \tag{3.9}$$

Hence, by combining (3.8) with (3.9), we conclude that

$$\left( \int_\Omega |u|^{2s, \beta_m} \, dx \right)^{2/2s} \leq C \beta \left( |\Omega| + 2 \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right) \leq 2C \beta (|\Omega| + 1) \left( 1 + \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right).$$

Moreover, by the formula $(a + b)^2 \leq 2(a^2 + b^2)$, we see that

$$\left( 1 + \int_\Omega |u|^{2s, \beta_m} \, dx \right)^2 \leq 2 + 2 \left[ 2C \beta (|\Omega| + 1) \left( 1 + \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right) \right]^{2s}.$$  

Therefore,

$$\left( 1 + \int_\Omega |u|^{2s, \beta_m} \, dx \right)^{\frac{1}{2s(\beta_m - 1)}} \leq (C \beta)^{\frac{1}{2s(\beta_m - 1)}} \left( 1 + \int_\Omega |u|^{2\beta + 2s - 2} \, dx \right)^{\frac{1}{2s(\beta_m - 1)}}, \tag{3.10}$$

where $C$ is renamed independently of $\beta$.

For $m \geq 1$, we define $\beta_{m+1}$ such that

$$2\beta_{m+1} + 2s - 2 = 2s \beta_m.$$  

Thus,

$$\beta_{m+1} - 1 = \left( \frac{2s}{2} \right)^m (\beta_1 - 1)$$

and (3.10) becomes

$$\left( 1 + \int_\Omega |u|^{2s, \beta_{m+1}} \, dx \right)^{\frac{1}{2s(\beta_{m+1} - 1)}} \leq (C \beta_{m+1})^{\frac{1}{2s(\beta_{m+1} - 1)}} \left( 1 + \int_\Omega |u|^{2s, \beta_m} \, dx \right)^{\frac{1}{2s(\beta_m - 1)}}.$$  

We now define $C_{m+1} := C \beta_{m+1}$ and

$$A_m := \left( 1 + \int_\Omega |u|^{2s, \beta_m} \, dx \right)^{\frac{1}{2s(\beta_m - 1)}}.$$
In particular, note that \( A_1 = \left( 1 + \int_{\Omega} |u|^{2(\beta_1)} \, dx \right)^{\frac{1}{2\beta_1(\beta_1-1)}} \) is bounded by (3.7).

Now we claim that there exists a constant \( C_0 > 0 \) independent of \( m \), such that
\[
A_{m+1} \leq \prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} A_1 \leq C_0 A_1. \tag{3.11}
\]

We stress that once (3.11) is established, then by the Hölder inequality, we conclude that \( u \in L^p(\Omega) \), for every \( p \in [1, +\infty) \). Furthermore, a limiting argument implies that
\[
\|u\|_\infty \leq C_0 A_1 < +\infty,
\]
which would complete the proof of Theorem 1.1.

Hence, it remains to check (3.11). To this end, we write \( \tilde{q} = \frac{2}{2^s} < 1 \), and observe that
\[
\beta_{m+1} = \left( \frac{1}{\tilde{q}} \right)^m (\beta_1 - 1) + 1 = \left( \frac{1}{\tilde{q}} \right)^m - \frac{1}{2} \left( \frac{1}{\tilde{q}} \right)^m + 1 \leq 2 \left( \frac{1}{\tilde{q}} \right)^m,
\]
thus, \( C_k = C \beta_k \leq 2C \left( \frac{1}{\tilde{q}} \right)^k \), and
\[
\prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} \leq \prod_{k=2}^{m+1} \left( 2C \left( \frac{1}{\tilde{q}} \right)^k \right)^{\frac{1}{2\beta_k-1}} = \left[ \left( 2C \left( \frac{1}{\tilde{q}} \right)^{(m+1)} \right)^{\tilde{q}^m} \left( 2C \left( \frac{1}{\tilde{q}} \right)^m \right)^{\tilde{q}^{m-1}} \cdots \left( 2C \left( \frac{1}{\tilde{q}} \right)^2 \right)^{\tilde{q}} \right]^{\frac{1}{2\beta_1-1}}.
\]

We consider
\[
\left( 2C \left( \frac{1}{\tilde{q}} \right)^{(m+1)} \right)^{\tilde{q}^m} \left( 2C \left( \frac{1}{\tilde{q}} \right)^m \right)^{\tilde{q}^{m-1}} \cdots \left( 2C \left( \frac{1}{\tilde{q}} \right)^2 \right)^{\tilde{q}} = (2C)^{\sum_{k=1}^{m} \tilde{q}^k} \left( \frac{1}{\tilde{q}} \right)^{\tilde{q}^m(m+1)+\tilde{q}^{m-1}m+\cdots+2}\tilde{q},
\]
where
\[
\sum_{k=1}^{m} \tilde{q}^k = \tilde{q} \frac{(1 - \tilde{q}^m)}{1 - \tilde{q}} \leq \frac{\tilde{q}}{1 - \tilde{q}},
\]
and
\[
0 < \tilde{q}^m(m+1)+\tilde{q}^{m-1}m+\cdots+2\tilde{q} = \frac{\tilde{q}(1 - \tilde{q}^m)}{(1 - \tilde{q})^2} + \frac{\tilde{q}}{1 - \tilde{q}} - \frac{\tilde{q}^m(m+1)}{(1 - \tilde{q})^2} < \frac{\tilde{q}}{(1 - \tilde{q})^2} + \frac{\tilde{q}}{1 - \tilde{q}},
\]
which implies that there exists \( C_0 > 0 \) independent of \( m \) such that
\[
\prod_{k=2}^{m+1} C_k^{\frac{1}{2(\beta_k-1)}} \leq C_0,
\]
where \( C_0 \) depends on \( c, n, s, \Omega \). The proof is completed. \( \Box \)
4 $C^{1,\alpha}$-regularity

In order to obtain the $C^{1,\alpha}$-regularity of weak solutions of (1.1), it suffices to prove a general $W^{2,p}$-regularity result for the mixed operator $L$ in Theorem 1.4.

4.1 $W^{2,p}$-regularity

We begin by recalling the following classical results about the $W^{2,p}$-regularity of the operator $-\Delta$. Let $p$ be given. Then, there exists $\lambda_0 \geq 0$ such that the problem

$$-\Delta u + \lambda u = f$$

has a unique solution $u_\lambda \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for any $\lambda \geq \lambda_0$, $f \in L^p(\Omega)$ and

$$\|u_\lambda\|_{W^{2,p}(\Omega)} \leq C\|f\|_p \quad (4.1)$$

$$\|\lambda - \lambda_0\|_p \leq C\|f\|_p \quad (4.2)$$

where the positive constant $C$ is independent of $u_\lambda$ and $\lambda$.

According to the above results, the proof of Theorem 1.4 will be divided into the following subsections: Sect. 4.1.1 is devoted to the basic $L^p$ estimates of the operator $(-\Delta)^s$; in Sect. 4.1.2, we apply the fixed point theorem to obtain that the problem

$$-\Delta u + \lambda u = f - (-\Delta)^s u, \quad \text{in } \Omega$$

has a unique solution $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for $\lambda > 0$ large enough; in Sect. 4.1.3, thanks to the Maximum Principle in [10, Theorem 1.2] and the bootstrap method, one can obtain the result as desired.

4.1.1 $L^p$ estimate of $(-\Delta)^s$

We introduce the following Extension Theorem which is useful for the $L^p$ estimate of $(-\Delta)^s$.

**Theorem 4.1** (See e.g. [23, Chapter 7]) Let $\Omega$ be a $C^{k-1,1}$ domain in $\mathbb{R}^n$, $k \geq 1$. Then

(i) $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$, $1 \leq p < \infty$.

(ii) For any open set $\Omega' \supset \Omega$, there exists a bounded linear extension operator $E$ from $W^{k,p}(\Omega)$ into $W^{k,p}_0(\Omega')$ such that $Eu = u$ in $\Omega$ and

$$\|Eu\|_{W^{k,p}(\Omega')} \leq C\|u\|_{W^{k,p}(\Omega)} \quad \text{for all } u \in W^{k,p}(\Omega)$$

where $C = C(k, \Omega, \Omega')$.

Now we state some useful interpolation results.

**Lemma 4.2** Let $\Omega$ be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $s \in (0, \frac{1}{2})$. Then the operator $(-\Delta)^s \in \mathcal{L}(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), L^p(\Omega))$ with $p > 1$, and we have

$$\|(-\Delta)^s u\|_p \leq C \left[ \epsilon\|u\|_{W^{2,p}(\Omega)} + \tau(\epsilon)\|u\|_p \right], \quad \text{for every } \epsilon > 0, \quad (4.3)$$

where $C$ is a constant independent of $u$.  

$\text{Springer}$
Proof Let \( u \in W^{1,p}_0(\Omega) \). We split the term \((-\Delta)^s u(x)\) into three parts and estimate their \(L^p\)-norms one by one.

\[
(-\Delta)^s u(x) = \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \ dy = \int_{\{y \in \Omega \cap |x-y| \leq 1\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \ dy \\
+ \int_{\{y \notin \Omega \cap |x-y| \leq 1\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \ dy + \int_{\{|x-y| > 1\}} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \ dz \\
= I + II + III.
\]

Now, using the change of variable \( z = y - x \), and applying the Extension Theorem 4.1 and the Hölder inequality, we see that

\[
\|I\|_p \leq \left( \int_{\Omega} \left[ \int_{\{|z| \leq 1\}} \left| \frac{u(x) - u(x+z)}{|z|^{n+2s}} \ dz \right|^p \right]^{1/p} \right) \\
= \left( \int_{\Omega} \left[ \int_{\{|z| \leq 1\}} \left| \frac{\nabla u(x + \theta z) d\theta}{|z|^{n+2s-1}} \right|^p \right]^{1/p} \right) \\
\leq \left( \int_{\mathbb{R}^n} \left[ \int_{\{|z| \leq 1\}} \left( \frac{\nabla u(x + \theta z) d\theta}{|z|^{n+2s-1}} \right)^p \ dz \right] \left( \int_{\{|z| \leq 1\}} \frac{1}{|z|^{n+2s-1}} d\theta d\tau \right)^{p/q} \right) \\
\leq \| Eu \|_{W^{1,p} (\mathbb{R}^n)} \left( \int_{\{|z| \leq 1\}} \frac{1}{|z|^{n+2s}} d\tau \right) \leq C(n, s, p) \| u \|_{W^{1,p} (\Omega)}.
\]

Also, we estimate

\[
\|III\|_p \leq \left( \int_{\mathbb{R}^n} \left[ \int_{\{|z| > 1\}} \left| \frac{|u(x)| + |u(x+z)|}{|z|^{n+2s}} \ dz \right|^p \right]^{1/p} \right) \\
\leq C(p) \left( \int_{\mathbb{R}^n} |Eu(x)|^p + |Eu(x+z)|^p \ dx \right)^{1/p} \left( \int_{\{|z| > 1\}} \frac{1}{|z|^{n+2s}} d\tau d\theta \right) \\
\leq C(p, s) \| Eu \|_{L^p (\mathbb{R}^n)} \leq C(p, s, \Omega) \| u \|_{W^{1,p} (\Omega)}.
\]

Set \( d(x) := \text{dist}(x, \partial \Omega) \) for any \( x \in \Omega \). Then note that for \( u \in W^{1,p}_0(\Omega) \), we have

\[
\|II\|_p \leq \left( \int_{\Omega} \left[ \int_{\{|z| \leq 1\}} \left| \frac{u(x)}{|z|^{n+2s-1} d(x)} \ dz \right|^p \right]^{1/p} \right) \\
\leq C(s, n, p) \left( \int_{\Omega} \left| \frac{u(x)}{d(x)} \right|^p \ dx \right)^{1/p} \\
\leq C(n, s, p) \left( \int_{\Omega} \int_0^1 |\nabla u(x + \theta (x - x_0))| \ d\theta \ dx \right)^{1/p} \leq C(n, s, p, \Omega) \| u \|_{W^{1,p} (\Omega)},
\]

where \( x_0 \in \partial \Omega \) such that \( d(x) = \text{dist}(x, x_0) \). Here, the constant \( C \) changes from line to line, but remains independent of \( u \).

As a consequence, the Sobolev interpolation inequality implies that for every \( \epsilon > 0 \), we have

\[
\|(-\Delta)^s u\|_p \leq C \left[ \epsilon \| u \|_{W^{2,p} (\Omega)} + \tau(\epsilon) \| u \|_p \right].
\]

Next, we extend the preceding result to the case \( s = \frac{1}{2} \).
Lemma 4.3 Let $\Omega$ be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $s = \frac{1}{2}$. Then the operator $(-\Delta)^{\frac{1}{2}} \in \mathcal{L}(W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega), L^p(\Omega))$ with $p > 1$, and

$$
\|(-\Delta)^{\frac{1}{2}} u\|_p \leq C \left[ o(\epsilon) \|u\|_{W^{2,p}(\Omega)} + \tau(\epsilon) \|u\|_{W^{1,p}(\Omega)} \right], \quad \text{for every } \epsilon > 0 \quad (4.4)
$$

where $C$ is a constant independent of $u$, $o(\epsilon) \to 0$ as $\epsilon \to 0$, and $\tau(\epsilon)$ is unbounded as $\epsilon \to 0$.

**Proof** Case 1. $p > n$. Then $u \in C^1(\overline{\Omega})$.

Let $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$. We split the term $-(-\Delta)^{\frac{1}{2}} u$ into five parts and estimate their $L^p$-norms one by one.

$$
-(-\Delta)^{\frac{1}{2}} u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+z) - 2u(x) + u(x-z)}{|z|^{n+1}} \, dz \\
= \frac{1}{2} \int_{\Omega_I} \frac{u(x+z) + u(x-z) - 2u(x)}{|z|^{n+1}} \, dz \\
+ \frac{1}{2} \int_{\Omega_{II}} \left[ \frac{u(x+z) - u(x)}{|z|^{n+1}} + \frac{-u(x)}{|z|^{n+1}} \right] \, dz \\
+ \frac{1}{2} \int_{\Omega_{III}} \left[ \frac{u(x-z) - u(x)}{|z|^{n+1}} + \frac{-u(x)}{|z|^{n+1}} \right] \, dz \\
+ \int_{\Omega_{IV}} \frac{-u(x)}{|z|^{n+1}} \, dz + \int_{|z| > 1} \left[ \frac{u(x+z) + u(x-z) - 2u(x)}{|z|^{n+1}} \right] \, dz \\
= I + II + III + IV + V,
$$

where

- $\Omega_I := \{x + z \in \Omega \cap \{x \in \Omega \cap \{x \leq 1\}\}$,
- $\Omega_{II} := \{x + z \in \Omega \cap \{x \in \Omega \cap \{x \leq 1\}\}$,
- $\Omega_{III} := \{x + z \in \Omega \cap \{x \in \Omega \cap \{x \leq 1\}\}$,
- $\Omega_{IV} := \{x + z \in \Omega \cap \{x \in \Omega \cap \{x \leq 1\}\}$.

**(i). Estimate of the term $I$ above.** Given $\epsilon \in (0, 1)$, set $I = I_{|z| \leq \epsilon} + I_{\epsilon < |z| \leq 1}$, and then we have

$$
\|I_{|z| \leq \epsilon}\|_p = \frac{1}{2} \left( \int_{\Omega} dx \left[ \int_{|z| \leq \epsilon} \frac{f_0^1 d\theta' D_{ij} u(x + \theta' z) z_i z_j}{|z|^{n+1}} \, dz \right]^p \right)^{1/p} \\
\leq \left( \int_{\mathbb{R}^n} dx \left[ \int_{|z| \leq \epsilon} \frac{f_0^1 d\theta' D_{ij} u(x + \theta' z) z_i z_j}{|z|^{n+1}} \, dz \right]^p \right)^{1/p} \left( \int_{|z| \leq \epsilon} \frac{1}{|z|^{n-1}} \, dz \right)^{p/(p-1)} \\
\leq \|u\|_{W^{2,p}((\mathbb{R}^n)} \left( \int_{|z| \leq \epsilon} \frac{1}{|z|^{n-1}} \, dz \right)^{1/p} \leq C(n, p, o(\epsilon)) \|u\|_{W^{2,p}(\Omega)},
$$

$$
\|I_{\epsilon < |z| \leq 1}\|_p = \left( \int_{\Omega} dx \left[ \int_{\Omega \cap \{\epsilon < |z| \leq 1\}} \frac{u(x+z) - u(x)}{|z|^{n+1}} \, dz \right]^p \right)^{1/p} \\
\leq \left( \int_{\mathbb{R}^n} dx \left[ \int_{\epsilon < |z| \leq 1} \frac{|\nabla u(x + \theta z)|^p}{|z|^n} \, d\theta \, dz \right]^p \right)^{1/p} \left( \int_{\epsilon < |z| \leq 1} \frac{1}{|z|^n} \, dz \right)^{p/(q-p)} \\
\leq \|u\|_{W^{1,p}((\mathbb{R}^n)} \left( \int_{\epsilon < |z| \leq 1} \frac{1}{|z|^n} \, dz \right)^{1/p} \leq C(n, p) \tau(\epsilon) \|u\|_{W^{1,p}(\Omega)},
$$

\(\text{Springer}\)
where \( o(\epsilon) \to 0 \) as \( \epsilon \to 0 \), and \( \tau(\epsilon) \) is arbitrary and unbounded, as \( \epsilon \to 0 \). Gathering the above estimates, we obtain that

\[
\| I \|_p \leq C(n, p) \left[ o(\epsilon) \| u \|_{W^{2,p}(\Omega)} + \tau(\epsilon) \| u \|_{W^{1,p}(\Omega)} \right].
\]

(ii). Estimates of the terms \( II \sim IV \). Set \( d(x) := \text{dist}(x, \partial \Omega) \) for any \( x \in \Omega \). Noting that \( u \in C^1(\overline{\Omega}) \), \( u|_{\partial \Omega} = 0 \), and for all \( p > n \), there exists \( \alpha > 0 \) such that \( p < \frac{1}{\alpha} \), then, given \( \epsilon \in (0, 1) \), we see that

\[
\int_{\Omega \setminus \{|z| \leq \epsilon\}} \frac{u(x+z) - u(x)}{|z|^{n+1}} \, dz \leq \int_{\Omega \setminus \{|z| \leq \epsilon\}} \| \nabla u(x) \|_\infty \, dz,
\]

and

\[
\int_{\Omega \setminus \{|z| \leq \epsilon\}} \frac{-u(x)}{|z|^{n+1}} \, dz \leq \int_{\Omega \setminus \{|z| \leq \epsilon\}} \| \nabla u(x) \|_\infty \, dz.
\]

Furthermore, using the well-known Sobolev inequality, we obtain that

\[
\| II_{|z| \leq \epsilon} \|_p \leq \left( \int_{\Omega} dx \left[ \int_{\Omega \setminus \{|z| \leq \epsilon\}} \| \nabla u(x) \|_\infty \, dz \right]^p \right)^{1/p} \leq C(n, p) o(\epsilon) \| u \|_{W^{2,p}(\Omega)},
\]

and by the estimate of \( \| I_{\epsilon<|z| \leq 1} \|_p \), we have

\[
\| II_{\epsilon<|z| \leq 1} \|_p \leq C(n, p) \tau(\epsilon) \| u \|_{W^{1,p}(\Omega)}.
\]

Thus,

\[
\| II \|_p \leq C(n, p) \left[ o(\epsilon) \| u \|_{W^{2,p}(\Omega)} + \tau(\epsilon) \| u \|_{W^{1,p}(\Omega)} \right].
\]

By a similar argument to the estimate of \( \| II \|_p \), we see that

\[
\| III \|_p \leq C(n, p) \left[ o(\epsilon) \| u \|_{W^{2,p}(\Omega)} + \tau(\epsilon) \| u \|_{W^{1,p}(\Omega)} \right],
\]

and

\[
\| IV \|_p \leq C(n, p) \left[ o(\epsilon) \| u \|_{W^{2,p}(\Omega)} + \tau(\epsilon) \| u \|_{W^{1,p}(\Omega)} \right].
\]

(iii). Estimate of the term \( V \). Thanks to \( u = 0 \), a.e. in \( \mathbb{R}^n \setminus \Omega \), we get

\[
\| V \|_p \leq \frac{1}{2} \left( \int_{\mathbb{R}^n} dx \left[ \int_{|z| \leq 1} \frac{2|u(x) + |u(x+z) + |u(x-z)| \, dz}{|z|^{n+1}} \right]^p \right)^{1/p} \leq C(p, n) \| u \|_{L^p(\mathbb{R}^n)} \leq C(p, n) \| u \|_{W^{1,p}(\Omega)}.
\]

As a consequence of all these estimates, we obtain (4.4) as desired.

Case 2. \( p = n \). Then \( u \in C^{0,1}(\overline{\Omega}) \). The Sobolev inequality implies that

\[
\| \nabla u(x) \|_\infty \leq C(p, n) \| u \|_{W^{2,n}(\Omega)}.
\]

Thus, (4.4) follows from the Case 1.

\[\square\]
Case 3. $p < n$. According to the proof of Case 1, we only need to verify the estimate of $II_{|z| \leq \epsilon}$. Next, using the Hölder and Sobolev inequalities, we find

$$
\left( \int_{\Omega} dx \left[ \int_{\Omega_{I\cap|z| \leq \epsilon}} \frac{u(x + z) - u(x)}{|z|^{n+1}} dz \right]^p \right)^{1/p} 
\leq \left( \int_{\Omega} dx \left[ \int_{\Omega_{I\cap|z| \leq \epsilon}} \frac{f_0^1 |\nabla u(x + \theta z)| d\theta}{|z|^{n-\alpha} d(x)^\alpha} dz \right]^p \right)^{1/p} 
\leq \left( \int_{\mathbb{R}^n} dx \left[ \int_{\Omega_{I\cap|z| \leq \epsilon}} \frac{f_0^1 |\nabla Eu(x + \theta z)| d\theta}{|z|^{n-\alpha} d(x)^\alpha} dz \right]^p \right)^{1/p} 
\leq \left\| \frac{|\nabla Eu|}{|d(x)^\alpha \chi + (d(x) + 1) \chi_{\mathbb{R}^n \setminus \Omega}|} \right\|_{L^p(\mathbb{R}^n)} \left( \int_{|z| \leq \epsilon} \frac{1}{|z|^{n-\alpha}} dz \right)^{1/q}, 
$$

for $p < n < q < \frac{1}{\alpha}$.

$$
\leq C(n, p, \Omega)e^\alpha \left( \int_{\mathbb{R}^n} |\nabla Eu|^{\frac{pq}{q-p}} dx \right)^{\frac{q-p}{pq}} \leq C(n, p, \Omega)e^\alpha \|Eu\|_{W^{2,p}(\mathbb{R}^n)} 
\leq C(n, p, \Omega)\omega(\epsilon)\|u\|_{W^{2,p}(\Omega)}.
$$

Also,

$$
\left( \int_{\Omega} dx \left[ \int_{\Omega_{I\cap|z| \leq \epsilon}} \frac{-u(x)}{|z|^{n+1}} dz \right]^p \right)^{1/p} 
\leq \left( \int_{\Omega} dx \left[ \int_{\Omega_{I\cap|z| \leq \epsilon}} \frac{|u(x)|}{|z|^{n-\alpha} d(x)^{1+\alpha} dz} \right]^p \right)^{1/p} 
\leq e^\alpha \left( \int_{\Omega} \left| \frac{u(x)}{d(x)^\alpha} \right|^p \frac{1}{d(x)^\alpha} dx \right)^{1/p} 
\leq C(n, p, \Omega)e^\alpha \left( \int_{\mathbb{R}^n} \frac{|u(x)|}{d(x)^\alpha} \left| \frac{1}{d(x)^\alpha} \right| dx \right)^{1/q}, 
$$

for $p < n < q < \frac{1}{\alpha}$.

$$
\leq C(n, p, \Omega)e^\alpha \left( \int_{\mathbb{R}^n} |\nabla Eu|^{\frac{pq}{q-p}} dx \right)^{\frac{q-p}{pq}} \leq C(n, p, \Omega)\omega(\epsilon)\|u\|_{W^{2,p}(\Omega)}.
$$

Combining the above two estimates, we get

$$
\|II_{|z| \leq \epsilon}\|^p_p \leq C(n, p, \Omega)\omega(\epsilon)\|u\|_{W^{2,p}(\Omega)}.
$$

Therefore, we obtain (4.4) for all cases.

We observe that Lemmata 4.2 and 4.3 cover the range $s \in \left( 0, \frac{1}{2} \right]$ which is of interest in the statements of Theorems 1.2 and 1.4. For completeness, though it will not be really utilized in this paper (apart from the comment in Remark 1.3), we also point out a similar result when $s \in \left( \frac{1}{2}, 1 \right)$.

**Lemma 4.4** Let $\Omega$ be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $s \in \left( \frac{1}{2}, 1 \right)$. Then for $n < p < \frac{n}{2s-1}$, we have

$$
\|(-\Delta)^s u\| \leq C \left[ o(\epsilon)\|u\|_{W^{2,p}(\Omega)} + \tau(\epsilon)\|u\|_{W^{1,p}(\Omega)} \right], \quad \text{for every } \epsilon > 0,
$$

where $C$ is a constant independent of $u$. 

\[\text{Springer}\]
Combining the above two estimates, we get
\[
\left( \int_{\Omega} \left( \int_{\Omega_I \cap \{ |z| \leq \epsilon \}} \frac{u(x+z) - u(x)}{|z|^{n+2s}} \, dz \right)^p \right)^{1/p} = 0,
\]
and
\[
\left( \int_{\Omega} \left( \int_{\Omega_I \cap \{ |z| \leq \epsilon \}} \frac{-u(x)}{|z|^{n+2s}} \, dz \right)^p \right)^{1/p} = 0.
\]
Thus, we get that
\[
\left( \int_{\Omega} \left( \int_{\Omega_I \cap \{ |z| \leq \epsilon \}} \frac{u(x+z) - u(x)}{|z|^{n+2s}} \, dz \right)^p \right)^{1/p} \leq C(n, p, s, \gamma) \| \nabla u(x) \|_\infty^\gamma + 1 - 2s \left( \int_{\Omega} \frac{1}{d(x)^{\gamma p}} \, dx \right)^{1/p}
\]
\[
\leq C(n, p, s, \gamma, \Omega) \| \nabla u(x) \|_\infty^\gamma + 1 - 2s \epsilon^{-\gamma + \frac{n}{\gamma}}
\]
\[
\leq C(n, p, s, \gamma, \Omega, \epsilon) \| u \|_{W^{2,p}(\Omega)}^\epsilon + 2s + \frac{n}{\gamma}
\]
Combining the above two estimates, we get
\[
\| I_{|z| \leq \epsilon} \|_p \leq C(n, p, \gamma, \Omega) \epsilon \| u \|_{W^{2,p}(\Omega)}.
\]
Therefore, as argued in the proof of Lemma 4.3 for $I, II_{|z| < 1}$, and $III \sim V$, we obtain the desired result.

4.1.2 Towards the existence of the solution to $-\Delta u + (-\Delta)^s u + \lambda u = f$

Now we develop some preliminary material needed to establish the existence result in Theorem 1.4. Here, we look at a linear perturbation of the equation in Theorem 1.4.

\textbf{Lemma 4.5} Under the assumptions of Theorem 1.4 and $s \in (0, \frac{1}{2})$, let $\lambda_1 > 0$ be given, large enough and independent of $f$. Then, the problem
\[
-\Delta u + (-\Delta)^s u + \lambda u = f, \quad \text{in } \Omega
\]
has a unique solution $u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for any $\lambda \geq \lambda_1$.\phantomsection\footnote{Springer}
Proof For a given \( w \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \), the problem
\[
-\Delta u + \lambda u = f - (\Delta)^s w, \quad \text{in } \Omega
\]
has a unique solution \( u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) since \( f - (\Delta)^s w \in L^p(\Omega) \). Thus, we can define a mapping \( T_\lambda : w \to u \) of \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) into itself. We are going to prove that it is possible to choose a positive number \( \lambda_1 \) large enough in such a way that \( T_\lambda \) is a contraction mapping for any \( \lambda \geq \lambda_1 \).

Indeed, let \( w_1, w_2 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) and \( T_\lambda w_1 = u_1, T_\lambda w_2 = u_2 \). Then,
\[
-\Delta(u_1 - u_2) + \lambda(u_1 - u_2) = -(\Delta)^s (w_1 - w_2), \quad \text{in } \Omega.
\]

Due to (4.1), Lemma 4.2 and the Gagliardo-Nirenberg interpolation inequality, we have that
\[
\|u_1 - u_2\|_{W^{2,p}(\Omega)} \leq C \|-(\Delta)^s(w_1 - w_2)\|_p
\]
\[
\leq C \left[ \epsilon \|w_1 - w_2\|_{W^{2,p}(\Omega)} + \tau(\epsilon)\|w_1 - w_2\|_p \right].
\]

Using (4.2) instead of (4.1), we can see that
\[
(\lambda - \lambda_0)\|u_1 - u_2\|_p \leq C \left[ \epsilon \|w_1 - w_2\|_{W^{2,p}(\Omega)} + \tau(\epsilon)\|w_1 - w_2\|_p \right].
\]

Since \( C \) is independent of \( \lambda \), then we can provide \( W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) with the equivalent norm
\[
\|\cdot\|_{W^{2,p}} := \|\cdot\|_{W^{2,p}(\Omega)} + (\lambda - \lambda_0)\|\cdot\|_p.
\]

First, let \( k \in (0, 1) \) be given. We choose \( \epsilon > 0 \) small enough such that
\[
C \epsilon \leq \frac{k}{2}
\]

Next, take \( \lambda_1 > 0 \) large enough such that
\[
C \tau(\epsilon) \leq (\lambda_1 - \lambda_0) \frac{k}{2}.
\]

From this, for \( \lambda \geq \lambda_1 \) it follows that
\[
\|T_\lambda w_1 - T_\lambda w_2\|_{W^{2,p}} \leq k \|w_1 - w_2\|_{W^{2,p}},
\]
i.e., \( T_\lambda \) is a contraction, hence the result holds as desired. \( \square \)

Next, we consider the case \( s = \frac{1}{2} \).

Lemma 4.6 Under the assumptions of Theorem 1.4, let \( \lambda_1' > 0 \) be given, large enough and independent of \( f \). Then the problem
\[
-\Delta u + (\Delta)^{\frac{1}{2}} u + \lambda u = f, \quad \text{in } \Omega \quad (4.6)
\]
has a unique solution \( u \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \) for any \( \lambda \geq \lambda_1' \).

Proof The proof is close in spirit to that of Lemma 4.5, relying here on Lemma 4.3 instead of Lemma 4.2. We provide full details for the reader’s convenience.

For a \( w \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \), the problem
\[
-\Delta u + \lambda u = f - (\Delta)^{\frac{1}{2}} w, \quad \text{in } \Omega
\]
Due to (4.1), Lemma 4.3 and the Gagliardo-Nirenberg interpolation inequality, we have that

\[-\Delta (u_1 - u_2) + \lambda (u_1 - u_2) = -(-\Delta)^{\frac{1}{2}}(w_1 - w_2), \quad \text{in } \Omega.\]

To begin with, we give some auxiliary results related to the Maximum Principle. Let

\[\|u_1 - u_2\|_{W^2,p(\Omega)} \leq C\|(-\Delta)^{\frac{s}{2}}(w_1 - w_2)\|_p\]

and

\[\leq C \left[ (o(\epsilon) + \tau(\epsilon)\delta)\|w_1 - w_2\|_{W^2,p(\Omega)} + \frac{\tau(\epsilon)}{\delta}\|w_1 - w_2\|_p \right].\]

Using (4.2) instead of (4.1), one can get that

\[(\lambda - \lambda_0)\|u_1 - u_2\|_p \leq C \left[ (o(\epsilon) + \tau(\epsilon)\delta)\|w_1 - w_2\|_{W^2,p(\Omega)} + \frac{\tau(\epsilon)}{\delta}\|w_1 - w_2\|_p \right].\]

We observe that \(C\) is independent of \(\lambda\), hence we can provide \(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\) with the equivalent norm

\[\|\cdot\|_{W^2,p} = \|\cdot\|_{W^2,p(\Omega)} + (\lambda - \lambda_0)\|\cdot\|_p.\]

First, let \(k \in (0, 1)\). We can choose \(\epsilon, \delta > 0\) small enough such that

\[C [o(\epsilon) + \tau(\epsilon)\delta] \leq \frac{k}{2},\]

Next, take \(\lambda_1' > 0\) large enough such that

\[C \frac{\tau(\epsilon)}{\delta} < (\lambda_1 - \lambda_0)\frac{k}{2}.\]

From this, for \(\lambda \geq \lambda_1'\) it follows that

\[\|T_\lambda w_1 - T_\lambda w_2\|_{W^2,p} \leq k\|w_1 - w_2\|_{W^2,p},\]

i.e., \(T_\lambda\) is a contraction, hence the result holds as desired.

\[\Box\]

### 4.1.3 Proof of Theorem 1.4

Our objective is now to complete the proof of the existence result stated in Theorem 1.4. For this, to begin with, we give some auxiliary results related to the Maximum Principle.

**Lemma 4.7** Let \(\Omega \subset \mathbb{R}^n\) be \(C^{1,1}\) domain and \(s \in (0, 1)\). Supposing that \(h \in L^\infty(\Omega)\) and \(\lambda \geq \max\{\lambda_1, \lambda_1'\}\), where the positive numbers \(\lambda_1\) and \(\lambda_1'\) are from Lemmata 4.5 and 4.6 respectively. Then, the problem

\[-\Delta u + (-\Delta)^s u + \lambda u = h, \quad \text{in } \Omega\]

has a unique solution \(u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\) and

\[\|u\|_\infty \leq c(\lambda) \|h\|_\infty,\]

where \(c(\lambda) < \frac{1}{\lambda}\).
Before that, we point out that by combining Lemma 4.5 with the Sobolev embedding theorem, we infer that \( u \in C^0(\overline{\Omega}) \subset L^\infty(\Omega) \). In order to complete the proof of Lemma 4.7, it remains to prove the next two Lemmata 4.8 and 4.9.

**Lemma 4.8** There exists \( w \in C^\infty(\overline{\Omega}) \cap W^{1,\infty}(\mathbb{R}^n) \) such that for every \( s \in (0, 1) \)
\[
-\Delta w + (-\Delta)^s w \geq 1, \quad \text{in } \Omega, \quad w > 0 \text{ in } \overline{\Omega}, \quad w \geq 0 \text{ in } \mathbb{R}^n.
\]

**Proof** Since \( \Omega \) is bounded in \( \mathbb{R}^n \), there exist \( x_0 \in \mathbb{R}^n \) and a positive constant \( R \) such that \( \overline{\Omega} \subset B_3(x_0) \setminus B_R(x_0) \) where \( B_R(x_0) := \{ x \in \mathbb{R}^n : |x - x_0| < R \} \). Without loss of generality, one could assume \( x_0 = 0 \) by the translation-invariance of the mixed operator \( L \) and denote \( B_R = B_R(0) \).

Let
\[
w(x) := \begin{cases} 1 - e^{\beta(|x|^2 - R^2)}, & |x| \leq R, \\
0, & |x| > R,
\end{cases}
\]
where \( \beta > 0 \) will be determined below.

One can calculate that
\[
-\Delta w(x) = e^{\beta(|x|^2 - R^2)}(2n\beta + 4\beta^2|x|^2).
\]
Also, \( w \) is concave on \( \overline{B}_R \), and for every \( x \in \Omega \), if \( |z| \leq 1 \), then \( x \pm z \in B_R \).

Therefore,
\[
(-\Delta)^s w(x) = -\frac{1}{2} \int_{|z| \leq 1} \frac{w(x + z) + w(x - z) - 2w(x)}{|z|^{n+2s}} \, dz
\geq -\frac{1}{2} \int_{|z| > 1} \frac{w(x + z) + w(x - z) - 2w(x)}{|z|^{n+2s}} \, dz
= -\int_{|z| > 1} \frac{w(x + z) - w(x)}{|z|^{n+2s}} \, dz
\geq -C(s)e^{\beta(|x|^2 - R^2)}.
\]

As a consequence of this, we get
\[
-\Delta w + (-\Delta)^s w \geq e^{\beta(|x|^2 - R^2)} \left( 2n\beta + 4\beta^2|x|^2 - C(s) \right)
\geq e^{-\frac{15\beta R^2}{16}} \left( 2n\beta + \frac{\beta^2R^2}{4} - C(s) \right) \geq \alpha_0,
\]
for some \( \alpha_0 > 0 \) if \( \beta \) is large enough, which can immediately imply Lemma 4.8. \( \square \)

**Lemma 4.9** The problem
\[
-\Delta u_\lambda + (-\Delta)^s u_\lambda + \lambda u_\lambda = 1, \quad \text{in } \Omega
\]
has a unique solution \( w_\lambda \in W^{2,p}(\Omega) \cap L^\infty(\Omega) \) and \( \|w_\lambda\|_\infty < \frac{1}{\lambda} \).
Proof. The existence and uniqueness of \( w_\lambda \) are guaranteed by Lemmata 4.5 and 4.6.

Then, the Sobolev embedding theorem implies that \( w_\lambda \in C^0(\Omega) \subset L^\infty(\Omega) \). Also, by Lemma 4.8, it follows that there exists \( w \in C^2(\Omega) \cap W^{1,\infty}(\mathbb{R}^n) \) such that

\[
-\Delta u + (-\Delta)^s w \geq 1, \quad \text{in } \Omega, \quad w > 0 \text{ in } \Omega, \quad w \geq 0 \text{ in } \mathbb{R}^n.
\]

We set \( \varphi(w) = \frac{1}{\lambda}(1 - e^{-\lambda w}) \), and note that \( \varphi \) is concave for all \( w \geq 0 \). Thus, for \( x, y \in \mathbb{R}^n \),

\[
\varphi(w(x)) - \varphi(w(y)) \geq \varphi'(w(x))(w(x) - w(y)).
\]

Computing \((-\Delta)^s \varphi(w(x))\) for \( x \in \Omega \)

\[
(-\Delta)^s \varphi(w(x)) \geq \int_{\mathbb{R}^n} \frac{\varphi'(w(x))(w(x) - w(y))}{|x - y|^{n+2s}} \, dy = e^{-\lambda w} (-\Delta)^s w(x).
\]

As a consequence,

\[
-\Delta \varphi(w(x)) + (-\Delta)^s \varphi(w(x)) + \lambda \varphi(w(x)) \geq e^{-\lambda w} \left( -\Delta u + \lambda \sum_{i=1}^n (D_i u)^2 + (-\Delta)^s w \right) + 1 - e^{-\lambda w} \geq 1.
\] (4.8)

Combining with (4.8), we get

\[
-\Delta (\varphi(w) - w_\lambda) + (-\Delta)^s (\varphi(w) - w_\lambda) + \lambda (\varphi(w) - w_\lambda) \geq 0, \quad \text{in } \Omega.
\]

The fact \( \varphi(w), w_\lambda \in C^0(\Omega) \cap W^{2,p}(\Omega) \) with \( p > n \) implies that \( \varphi(w), w_\lambda \in H^1(\mathbb{R}^n) \).

Applying the Maximum Principle (see [10, Theorem 1.2]), we obtain

\[
\varphi(w(x)) \geq w_\lambda(x), \quad \text{a.e. in } \Omega,
\]

since \( \varphi(w(x)) - w_\lambda(x) \geq 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \).

Hence, due to the fact that \( \max_{\Omega} \varphi(w(x)) < \frac{1}{\lambda} \), we conclude that \( \|w_\lambda\|_\infty < \frac{1}{\lambda} \) in \( \Omega \), as desired. \qed

Proof of Lemma 4.7. Recalling (4.7), we get

\[
-\Delta \left( \frac{u}{\|h\|_\infty} \right) + (-\Delta)^s \left( \frac{u}{\|h\|_\infty} \right) + \lambda \left( \frac{u}{\|h\|_\infty} \right) \leq 1, \quad \text{in } \Omega,
\]

and, combining with the Lemma 4.9, we see that

\[
-\Delta \left( w_\lambda - \frac{u}{\|h\|_\infty} \right) + (-\Delta)^s \left( w_\lambda - \frac{u}{\|h\|_\infty} \right) + \lambda \left( w_\lambda - \frac{u}{\|h\|_\infty} \right) \geq 0, \quad \text{in } \Omega.
\]

The Maximum Principle implies that

\[
\|u\|_\infty \leq w_\lambda \|h\|_\infty.
\]

The desired result thus follows by Lemma 4.9. \qed

We are finally ready to prove Theorem 1.4.
Proof of Theorem 1.4  Fixed $\lambda \geq \max\{\lambda_1, \lambda_1'\}$ such that we can solve the equation (4.5) or (4.6) with $f \in L^p(\Omega)$. We consider the sequence of functions defined in the following way.

$$-\Delta u_0 + (-\Delta)^s u_0 + \lambda u_0 = f, \quad u_0 \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega),$$

and as $u_k$ is defined in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, $u_{k+1}$ is the solution of

$$-\Delta u_{k+1} + (-\Delta)^s u_{k+1} + \lambda u_{k+1} = f + \lambda u_k, \quad u_{k+1} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega). \quad (4.9)$$

We can consider that $u_{k+1} - u_k$ is the solution of

$$-\Delta(u_{k+1} - u_k) + (-\Delta)^s(u_{k+1} - u_k) + \lambda(u_{k+1} - u_k) = \lambda(u_k - u_{k-1}). \quad (4.10)$$

Now, let

$$j = \left\lfloor \frac{n}{2p} \right\rfloor, \quad \frac{1}{p_1} = \frac{1}{p} - \frac{2i}{n}.$$

Assuming that $u_{-1} = 0$, $u_{k+1} - u_k$ is the solution of equation (4.5) or (4.6) with $f = \lambda(u_k - u_{k-1})$. By the preceding we can deduce other regularity properties of $u_{k+1} - u_k$.

Indeed, since $u_0 \in W^{2,p}(\Omega)$, we also have $u_0 \in L^{p_1}(\Omega)$ with $rac{1}{p_1} = \frac{1}{p} - \frac{2}{n}$. Hence,

$$u_1 - u_0 \in W^{2,p}(\Omega) \cap W^{2,p_1}(\Omega), \quad \frac{1}{p_1} = \frac{1}{p} - \frac{2}{n}.$$

Then $u_1 - u_0 \in L^{p_2}(\Omega)$. Hence

$$u_2 - u_1 \in W^{2,p}(\Omega) \cap W^{2,p_2}(\Omega), \quad \frac{1}{p_2} = \frac{1}{p} - \frac{4}{n}.$$

By induction we prove

$$u_k - u_{k-1} \in W^{2,p}(\Omega) \cap W^{2,p_k}(\Omega), \quad \frac{1}{p_k} = \frac{1}{p} - \frac{2k}{n},$$

as long as $k < \frac{n}{2p}$.

Case 1. If $j = \left\lfloor \frac{n}{2p} \right\rfloor = 0$, then $0 < \frac{n}{2p} < 1$. Hence $u_0 \in C^0(\Omega) \subset L^\infty(\Omega)$.

Case 2. If $\frac{n}{2p} < j + 1$, then $u_j - u_{j-1} \in W^{2,p}(\Omega) \cap W^{2,p_j}(\Omega)$ with $p > \frac{n}{2j}$. Hence

$$u_j - u_{j-1} \in C^0(\Omega) \subset L^\infty(\Omega).$$

Case 3. If $j = \frac{n}{2p} \geq 1$, then $u_{j-1} - u_{j-2} \in W^{2,p}(\Omega) \cap W^{2,q}(\Omega) \subset W^{1,\frac{q}{2}}(\Omega)$. Hence

$$u_{j-1} - u_{j-2} \in L^\infty(\Omega).$$

Therefore, $u_j - u_{j-1} \in W^{2,p}(\Omega) \cap W^{2,n}(\Omega)$. Due to the interpolation inequality, we know $u_j - u_{j-1} \in W^{2,\theta p + (1-\theta)n}(\Omega)$ with $\theta \in [0, 1]$. Let

$$q := \theta p + (1-\theta)n = n(1 - \theta(1 - \frac{1}{2j})).$$

In particular, we observe that

$$2 - \frac{n}{q} = 2 - \frac{1}{1 - \theta(1 - \frac{1}{2j})} = \frac{1 - 2\theta\left(1 - \frac{1}{2j}\right)}{1 - \theta\left(1 - \frac{1}{2j}\right)} = \alpha,$$

and we can choose $\theta > 0$ small enough such that $\alpha \in (0, 1)$. From this, it follows that $u_j - u_{j-1} \in C^{0,\alpha}(\Omega) \subset L^\infty(\Omega)$.  

\[\varepsilon\] Springer
Thus in all cases, Lemma 4.7 and (4.10) can imply that, for all \( k \geq j \), \( u_k - u_{k-1} \in L^\infty(\Omega) \). Moreover,
\[
\|u_{k+1} - u_k\|_\infty \leq K \|u_k - u_{k-1}\|_\infty,
\]
where \( 0 < K < 1 \). It is then immediate to see that \( \{u_k\} \) is a Cauchy sequence in \( L^\infty(\Omega) \) and is bounded. Recalling (4.9), utilizing (4.1) and (4.2), we see that
\[
\|u_{k+1}\|_{W^2,p(\Omega)} + (\lambda - \lambda_0)\|u_{k+1}\|_p \leq 2C\left(\|u_{k+1}\|_{W^2,p(\Omega)} + \|f\|_p + \|\lambda u_k\|_p\right)
\leq k\left(\|u_{k+1}\|_{W^2,p(\Omega)} + (\lambda - \lambda_0)\|u_{k+1}\|_p\right) + C\|f\|_p + C\|\lambda u_k\|_\infty,
\]
where the constant \( C \) is independent of \( \lambda \) and \( u_k \). It follows that \( \{u_k\} \) is bounded in \( W^2,p(\Omega) \cap W_0^{1,p}(\Omega) \). Thus there exists a subsequence, which we relabel as \( \{u_k\} \), converging weakly to a function \( u \in W^2,p(\Omega) \cap W_0^{1,p}(\Omega) \). Since
\[
\int_\Omega gD^\alpha u_k \to \int_\Omega gD^\alpha u
\]
for all \( |\alpha| \leq 2 \) and \( g \in L^{p/(p-1)}(\Omega) \), we must have
\[
\int_\Omega g(-\Delta u_k) \to \int_\Omega g(-\Delta u).
\]
Similarly,
\[
\int_\Omega g(-\Delta)^s u_k \to \int_\Omega g(-\Delta)^s u
\]
since \( (-\Delta)^s \in L^p(W^2,p(\Omega) \cap W_0^{1,p}(\Omega), L^p(\Omega)) \). Combining the above results and (4.9), we obtain
\[
-\Delta u + (-\Delta)^s u = f, \quad u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).
\]
In particular, we can set (i) \( \epsilon \in (0, \frac{1}{2\pi}) \) in (4.3) for \( s \in \left(0, \frac{1}{2}\right) \) and (ii) \( \epsilon > 0 \) such \( o(\epsilon) < \frac{1}{2\pi} \) in (4.4) for \( s = \frac{1}{2} \), one can obtain
\[
\|u\|_{W^{2,p}(\Omega)} \leq C_1\left(\|u\|_p + \|f\|_p\right),
\]
where the constant \( C_1 \) is independent of \( u \). The result holds as desired. \( \square \)

### 4.2 Proof of \( C^{1,\alpha} \)-regularity

Now we complete the proof of Theorem 1.2.

**Proof of Theorem 1.2** Let \( u \in X_0^1 \) be the weak solution of (1.1). Theorem 1.1 implies that \( g(x, u) \in L^\infty(\Omega) \).

Moreover, due to Theorem 1.4, we deduce that there exists a unique solution \( v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \) for every \( p > n \). By combining this with the Sobolev embedding inequality, we obtain \( v \in C^{1,\alpha}(\Omega) \subset X_0^1 \) for any \( \alpha \in (0, 1) \).

Finally, the Lax-Milgram Theorem yields that \( u = v \). Hence, the proof of Theorem 1.2 is completed. \( \square \)
Acknowledgements The authors would like to thank the anonymous referee for carefully reading the manuscript and the valuable comments and suggestions on it.

References

1. Abatangelo, N., Cozzi, M.: An elliptic boundary value problem with fractional nonlinearity. SIAM J. Math. Anal. 53(3), 3577–3601 (2021)
2. Barles, G., Imbert, C.: Second-order elliptic integro-differential equations: viscosity solutions’ theory revisited. Ann. Inst. H. Poincaré C Anal. Non Linéaire 25(3), 567–585 (2008)
3. Barles, G., Chasseigne, E., Ciomaga, A., Imbert, C.: Lipschitz regularity of solutions for mixed integro-differential equations. J. Differ. Equ. 252(11), 6012–6060 (2012)
4. Barrios, B., Colorado, E., Servadei, R., Soria, F.: A critical fractional equation with concave-convex power nonlinearities. Ann. Inst. H. Poincaré C Anal. Non Linéaire 32(4), 875–900 (2015)
5. Bensoussan, A., Lions, J.-L.: Impulse Control and Quasivariational Inequalities. μ. Gauthier-Villars, Montrouge; Heyden & Son, Inc., Philadelphia (1984). Translated from the French by J. M. Cole
6. Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E.: A Brezis-Nirenberg type result for mixed local and nonlocal operators. (2022) (Preprint)
7. Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E.: A Faber-Krahn inequality for mixed local and nonlocal operators. J. Anal. Math. (2021). https://arxiv.org/abs/2104.00830
8. Biagi, S., Mugnai, D., Vecchi, E.: Global boundedness and maximum principle for a Brezis-Oswald approach to mixed local and nonlocal operators. (2022) (Preprint)
9. Biagi, S., Vecchi, E., Dipierro, S., Valdinoci, E.: Semilinear elliptic equations involving mixed local and nonlocal operators. Proc. R. Soc. Edinb. Sect. A 151(5), 1611–1641 (2021)
10. Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E.: Mixed local and nonlocal elliptic operators: regularity and maximum principles. Comm. Partial Differ. Equ. 47(3), 585–629 (2022)
11. Biagi, S., Dipierro, S., Valdinoci, E., Vecchi, E.: A Hong-Krahn-Szegö inequality for mixed local and nonlocal operators. Math. Eng. 5(1), 25 (2023)
12. Biswas, I.H., Jakobsen, E.R., Karlsen, K.H.: Viscosity solutions for a system of integro-PDEs and connections to optimal switching and control of jump-diffusion processes. Appl. Math. Optim. 62(1), 47–80 (2010)
13. Buccheri, S., da Silva, J.V., de Miranda, L.H.: A system of local/nonlocal p-Laplacians: the eigenvalue problem and its asymptotic limit as p → ∞. Asymptot. Anal. 128(2), 149–181 (2022)
14. Cabré, X., Dipierro, S., Valdinoci, E.: The Bernstein technique for integro-differential equations. Arch. Ration. Mech. Anal. 243(3), 1597–1652 (2022)
15. Chen, Z.-Q., Kim, P., Song, R., Vondraček, Z.: Boundary Harnack principle for Δ + Δα/2. Trans. Am. Math. Soc. 364(8), 4169–4205 (2012)
16. Del Pezzo, L.M., Ferreira, R., Rossi, J.D.: Eigenvalues for a combination between local and nonlocal p-Laplacians. Fract. Calc. Appl. Anal. 22(5), 1414–1436 (2019)
17. del Teso, F., Endal, J., Jakobsen, E.R.: On distributional solutions of local and nonlocal problems of porous medium type. C. R. Math. Acad. Sci. Paris 355(11), 1154–1160 (2017)
18. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136(5), 521–573 (2012)
19. Dipierro, S., Lippi, E.P., Valdinoci, E.: (Non)local logistic equations with Neumann conditions. Ann. Inst. H. Poincaré C Anal. Non Linéaire (2021). https://arxiv.org/abs/2101.02315
20. Dipierro, S., Medina, M., Valdinoci, E.: Fractional Elliptic Problems with Critical Growth in the Whole of Rd. Volume 15 of Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa (2017)
21. Dipierro, S., Valdinoci, E.: Description of an ecological niche for a mixed local/nonlocal dispersal: an evolution equation and a new Neumann condition arising from the superposition of Brownian and Lévy processes. Phys. A 575, 20 (2021)
22. Dipierro, S., Lippi, E.P., Valdinoci, E.: Linear theory for a mixed operator with Neumann conditions. Asymptot. Anal. 128(4), 571–594 (2022)
23. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Classics in Mathematics. Springer, Berlin (2001). (Reprint of the 1998 edition)
24. Gimbert, F., Lions, P.L.: Existence and regularity results for solutions of second-order, elliptic integro-differential operators. Ricerche Mat. 33(2), 315–358 (1984)
25. Jakobsen, E.R., Karlsen, K.H.: Continuous dependence estimates for viscosity solutions of integro-PDEs. J. Differ. Equ. 212(2), 278–318 (2005)
26. Montefusco, E., Pellacci, B., Verzini, G.: Fractional diffusion with Neumann boundary conditions: the logistic equation. Discrete Contin. Dyn. Syst. Ser. B 18(8), 2175–2202 (2013)

27. Pellacci, B., Verzini, G.: Best dispersal strategies in spatially heterogeneous environments: optimization of the principal eigenvalue for indefinite fractional Neumann problems. J. Math. Biol. 76(6), 1357–1386 (2018)

28. Salort, A., Vecchi, E.: On the mixed local-nonlocal Hénon equation (Preprint)

29. Su, X., Valdinoci, E., Wei, Y., Zhang, J.: Multiple solutions for mixed local and nonlocal elliptic equations arising from the Lévy type processes (2022) (Preprint)

30. Wei, Y., Xifeng, S.: Multiplicity of solutions for non-local elliptic equations driven by the fractional Laplacian. Calc. Var. Partial Differ. Equ. 52(1–2), 95–124 (2015)

31. Wei, Y., Xifeng, S.: On a class of non-local elliptic equations with asymptotically linear term. Discrete Contin. Dyn. Syst. 38(12), 6287–6304 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.

Authors and Affiliations

Xifeng Su1 · Enrico Valdinoci2 · Yuanhong Wei3 · Jiwen Zhang4

Xifeng Su
xfsu@bnu.edu.cn; billy3492@gmail.com

Enrico Valdinoci
enrico.valdinoci@uwa.edu.au

Yuanhong Wei
weiyuanhong@jlu.edu.cn

1 Laboratory of Mathematics and Complex Systems (Ministry of Education), School of Mathematical Sciences, Beijing Normal University, No. 19, XinJieKouWai St., HaiDian District, Beijing 100875, People’s Republic of China

2 Department of Mathematics and Statistics, University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia

3 School of Mathematics, Jilin University, No. 2699, Qianjin St., Changchun 130012, People’s Republic of China

4 School of Mathematical Sciences, Beijing Normal University, No. 19, XinJieKouWai St., HaiDian District, Beijing 100875, People’s Republic of China