Action functionals of single scalar fields and arbitrary–weight gravitational constraints that generate a genuine Lie algebra.

I. Kouletsis

Theoretical Physics Group,
Blackett Laboratory
Imperial College of Science, Technology & Medicine
London SW7 2BZ, U.K.

Abstract

We discuss the issue initiated by Kuchař et al, of replacing the usual Hamiltonian constraint by alternative combinations of the gravitational constraints (scalar densities of arbitrary weight), whose Poisson brackets strongly vanish and cast the standard constraint–system for vacuum gravity into a form that generates a true Lie algebra. It is shown that any such combination—that satisfies certain reality conditions—may be derived from an action principle involving a single scalar field and a single Lagrange multiplier with a non–derivative coupling to gravity.

1email: tpmisc17@ic.ac.uk.
1 Introduction.

The canonical formulation of general relativity is based on the requirement that the spacetime manifold $\mathcal{M}$ is diffeomorphic with $\Sigma \times \mathbb{R}$. The real line $\mathbb{R}$ plays the role of a global time while the three–space $\Sigma$—which is assumed to be compact for simplicity—represents physical space. The foliation map from $\Sigma \times \mathbb{R}$ to $\mathcal{M}$ induces an one–parameter family of embeddings from $\Sigma$ to $\mathcal{M}$ that are spacelike with respect to the four–metric on $\mathcal{M}$.

As a result, when the four metric (written as $\gamma_{\alpha\beta}(X)$ in local coordinates) is pulled–back to $\Sigma \times \mathbb{R}$, a symmetric tensor field is induced on $\Sigma$ (the spatial metric with local components $g_{ij}(x)$) that has signature $(1,1,1)$ and is positive definite. The other four independent components of the pull–back are related to the lapse function and the shift vector which determine how the hypersurface $\Sigma$ is displaced in spacetime under an infinitesimal change of global time.

The canonical version of general relativity, for example see [1], is derived by decomposing the usual Einstein Lagrangian density with respect to the spatial metric, the shift vector and the lapse function, and then performing a Legendre transformation to replace the time derivative of the three–metric with its conjugate momentum $p^{ij}(x)$. In doing so, the lapse function and the shift vector become non–dynamical Lagrange multipliers, enforcing on the canonical variables $(g_{ij}(x), p^{ij}(x))$ the Hamiltonian and momentum constraints

\begin{equation}
\mathcal{H}_\perp(x) = G_{ijkl}(x)p^{ij}(x)p^{kl}(x) - \frac{1}{2}D_t p^i_i(x),
\end{equation}

where $g(x)$ denotes the determinant of the spatial metric $g_{ij}(x)$.

The constraints

\begin{equation}
\mathcal{H}_\perp(x) = 0 = \mathcal{H}_i(x)
\end{equation}

satisfy the Dirac algebra [1],

\begin{align}
\{\mathcal{H}_\perp(x), \mathcal{H}_\perp(x')\} &= g^{ij}(x)\mathcal{H}_i(x)\delta_j(x, x') - (x \leftrightarrow x') \quad (1.3) \\
\{\mathcal{H}_\perp(x), \mathcal{H}_i(x')\} &= \mathcal{H}_\perp(x)\delta_i(x, x') + \mathcal{H}_\perp(x')\delta_i(x, x') \quad (1.4) \\
\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} &= \mathcal{H}_j(x)\delta_i(x, x') - (ix \leftrightarrow jx'), \quad (1.5)
\end{align}

and are first–class since the right hand side of equations (1.3)–(1.5) vanishes on the constraint surface (1.2). The appearance of the $g^{ij}(x)$ factor in (1.3), however, implies that the algebra they generate is not a genuine Lie algebra—a property that creates various problems in a possible quantum version of the theory.

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2 Spacetime points are denoted by $X$ and spatial points by $x$. Greek letters $\alpha, \beta, ...$ are used as spacetime indices and run from 0 to 3 while Latin characters $i, j, ...$ are used as spatial indices and run from 1 to 3. The notation for functions, functionals, ...e.t.c., is not consistent but rather obeys the requirement of maximum notational simplicity under each particular situation.
While studying the coupling of gravity to dust, Brown and Kuchař [5] came across a weight–two scalar combination of the gravitational constraints,

$$G(x) := \mathcal{H}_1^2(x) - g^{ij}\mathcal{H}_i(x)\mathcal{H}_j(x), \quad (1.6)$$

that has strongly vanishing Poisson brackets with itself. The combination was found equal to the square of the momentum conjugate to the dust time and, as such, it had to be positive; equivalently, the gravitational sector of the phase space of the theory was limited to certain appropriate regions.

If $G(x)$ replaces the usual Hamiltonian constraint to form an—at least locally—equivalent new set of constraints for vacuum general relativity,

$$G(x) = 0 = \mathcal{H}_i(x), \quad (1.7)$$

a genuine Lie algebra is created:

$$\{G(x), G(x')\} = 0 \quad (1.8)$$
$$\{G(x), \mathcal{H}_i(x')\} = G_{,i}(x)\delta(x, x') + 2G(x)\delta_{,i}(x, x') \quad (1.9)$$
$$\{\mathcal{H}_i(x), \mathcal{H}_j(x')\} = \mathcal{H}_j(x)\delta_{,i}(x, x') - (ix \leftrightarrow jx'). \quad (1.10)$$

It corresponds to the semidirect product of the Abelian algebra generated by $G(x)$, equation (1.8), and the algebra of spatial diffeomorphisms $\text{LDiff}\Sigma$ produced by $\mathcal{H}_i(x)$, equation (1.10). The Poisson bracket (1.9) reflects the transformation of $G(x)$ as a weight–two scalar density under $\text{Diff}\Sigma$.

A similar result was obtained by Kuchař and Romano [6]. They coupled gravity to a single massless scalar field and extracted the following weight–two scalar combination of the gravitational constraints:

$$\Lambda_{\pm}(x) := g^{12}(x)\left(-\mathcal{H}_1(x) \pm \sqrt{G(x)}\right). \quad (1.11)$$

The combination was found to be equal to the square of the momentum conjugate to the scalar field and has exactly the same Poisson brackets as $G(x)$, given by equations (1.8)–(1.10). If a proper choice of sign for the square root is made, $\Lambda_{\pm}(x)$ may be used to replace the Hamiltonian constraint in a new locally equivalent system of constraints for pure gravity. Again, the theory allows only certain regions of the gravitational phase space so that the quantity inside the square root and the combination $\Lambda_{\pm}(x)$ itself be positive.

The question that arises [6] is whether some other couplings of gravity to fields can also lead to combinations of the gravitational constraints with strongly vanishing Poisson brackets, and whether the multiplicity of such possible alternative combinations conveys any general message about the structure of canonical general relativity.

A significant advance was made recently by Markopoulou [7], who constructed a unique nonlinear partial differential equation satisfied by any scalar combination of arbitrary $\Lambda_{\pm}(x)$.

\[^3\text{See the end of sections 2 and 3.}\]
weight that obeys the abelian algebra (1.8). Because of the extensive overlap between [7] and the present paper, a brief review of Markopoulou’s results is presented.

The basic observation was that any arbitrary–weight scalar density (which in [7] was collectively denoted by $W(x)$) can be written as a general function of the simplest possible scalar combinations of the gravitational constraints and the determinant of the spatial metric[4].

$$W_\omega[\tilde{H}(x), \tilde{F}(x), g(x)] = g^{\frac{\omega}{2}}(x)W_\omega[\tilde{H}(x), \tilde{F}(x)],$$

(1.12)
on the assumption that $W_\omega[\tilde{H}(x), \tilde{F}(x)]$ is an ultralocal [8] function of them. The two basic weight–zero combinations of the constraints are defined by $\tilde{H}(x) := g^{-\frac{1}{2}}(x)\mathcal{H}_\perp(x)$ and $\tilde{F}(x) := g^{-1}(x)g^{ij}(x)\mathcal{H}_i(x)\mathcal{H}_j(x)$, while the parameter $\omega$ denotes the weight of the relevant scalar densities, and is now also used as a subscript.

The requirement of strongly vanishing Poisson brackets was then imposed on $W_\omega(x)$, leading to an unexpectedly compact and exactly solvable differential equation for the weightless part of (1.12) (the subscript $\omega$ has been dropped),

$$\frac{\omega}{2}W(x)W_F(x) = \tilde{F}(x)W_\omega^2(x) - \frac{1}{4}W_\omega^4(x),$$

(1.13)

which must hold at every spatial point $x$. The uniqueness of equation (1.13) is assured by the kind of assumption made on the form of $W_\omega(x)$, which is the least restrictive one. Its general solution is found to be

$$W_\omega[\tilde{H}, \tilde{F}, B(\alpha[\tilde{H}, \tilde{F}])] = \pm \left[ (\tilde{H} - \frac{1}{2}B'(\alpha[\tilde{H}, \tilde{F}])) + \sqrt{(\tilde{H} - \frac{1}{2}B'(\alpha[\tilde{H}, \tilde{F}]) - \tilde{F})} \right]^{\frac{\omega}{2}}$$

$$\times \exp \left( B(\alpha[\tilde{H}, \tilde{F}]) + \frac{\omega}{2} \frac{\frac{1}{2}B'(\alpha[\tilde{H}, \tilde{F}])}{\sqrt{(\tilde{H} - \frac{1}{2}B'(\alpha[\tilde{H}, \tilde{F}]) - \tilde{F})}} \right),$$

(1.14)

where the $x$’s have been omitted, and $\alpha[\tilde{H}, \tilde{F}]$ is determined by algebraically solving the equation

$$\alpha = -\frac{\omega}{4\sqrt{(\tilde{H} - \frac{1}{2}B'(\alpha))^2 - \tilde{F}}}$$

(1.15)

for a given choice of $B(\alpha)$. Complex solutions for $W_\omega(x)$ can in general exist.

Expressions (1.14) and (1.15)—which provide a family of self–commuting constraint combinations parametrised by an arbitrary function of one variable—were based on purely algebraic considerations [6] and, consequently, their physical relevance is not clear. However, at least for the special case of weight two, a lot of insight into their origin could be gained if they were shown to be related to some phenomenological physical system similar to the ones discussed in [3] and [8]. An attractive possibility—supported by the fact that both combinations $G(x)$ and $\Lambda_{\perp}(x)$ appear in (1.14), (1.15) as genuine subcases of the same

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[4] For partial differentials the notation $A_v(b) := \partial A(b)/\partial b$ is used. Very often in what follows, partial differentials with respect to $\mathcal{H}(x) := g^{-\frac{1}{2}}(x)\mathcal{H}_\perp(x)$ and $\mathcal{F}(x) := g^{-1}(x)g^{ij}(x)\mathcal{H}_i(x)\mathcal{H}_j(x)$ appear; they are denoted by $A_H(x) := \partial A[\mathcal{H}(x), \mathcal{F}(x)]/\partial \mathcal{H}(x)$ and $A_F(x) := \partial A[\mathcal{H}(x), \mathcal{F}(x)]/\partial \mathcal{F}(x)$ respectively.
weight–two solution—would be for the action functionals for dust \[5\] and for a massless scalar field \[6\] to both arise as different versions of a wider, generalised action functional parametrised by an arbitrary function of one variable.

An obvious objection to such a proposal comes from the fact that the dust action involved four scalar fields compared to the single scalar field that was considered in \[6\]. A closer analysis however suggests that, practically speaking, it was the “one–term–factorisation” property of the dust action that was responsible for the successful extraction of the $G(x)$, or, to put it in another way, it was the fact that the dust action strongly resembled the simpler action for a massless scalar field. Indeed, it is straightforward to show that $G(x)$ can also be derived from a variety of single–scalar–field actions which—although perhaps not as convincingly interpretable as the one for dust—are still equally effective.

Motivated by this observation, one may use some action functional of a single scalar field—having the two important properties of being parametrised by an arbitrary function of one variable and reducing to the actions of \[5\] and \[6\] for certain choices of this function—and hope that, when coupled to gravity, it will provide the general solution of equation (1.13) for weight two (at least the subset of the general solution that can have real positive values in some regions of the gravitational phase space, as in the cases of \[5\] and \[6\]). In addition, due to a remarkable feature of equation (1.13)—namely, if $W_\omega(x)$ is a solution of weight $\omega$ then $W_{\omega'}(x) := W_\omega \frac{\omega'}{\omega}(x)$ is a corresponding solution of weight $\omega'$—one expects that if the weight–two procedure proves to be successful, may be naturally extendable to arbitrary weight. The above argument actually works and is the key to the present analysis, which is organised as follows:

In the introduction to section 2, we make some preliminary remarks about differential equation (1.13) and the existence of complex solutions (1.14), (1.15), as well as some general comments about the present paper. As claimed above, equation (1.13) is for every weight directly related to a generalised action functional of a single scalar field, parametrised by an arbitrary function of one variable. This is presented in section 3. As it stands, however, the equation does not make this connection clear and therefore a suitable ansatz is used (in section 2) to convert the weight–zero part of the scalar densities $W_\omega(x)$ into a form that is better suited for our purposes.

The so–called $\omega$–ansatz (since there is one for each weight) expresses $W_\omega(x)$ in terms of two weight–zero complex scalar densities $\lambda(x)$ and $\mu(x)$ which, as $W_\omega(x)$, are ultralocal functions of the simplest possible scalar constraint–combinations $\tilde{H}(x)$ and $\tilde{F}(x)$. The ansatz essentially transforms the nonlinear arbitrary–weight differential equation (1.13)\(^5\) provided that both $\omega$ and $\omega'$ are different from zero, so that the algorithm is well–defined and invertible. In particular, if $\omega$ equals zero, the left hand side of (1.13) vanishes and the equation becomes a homogeneous one, i.e., a different equation—this degenerate case is therefore excluded from the present discussion.

\(^6\)Although the differential equation (1.13) admits complex–valued solutions that cannot be reconciled to the idea of the physical system of section 3, it is chosen—for the sake of uniformity—not to impose at this stage any reality–conditions on $W_\omega(x)$, leaving the necessary adjustments for the physical relevance of these solutions to be done in section 4. It should be mentioned here—for more details see the introduction of section 2—that the terms “complex” and “complex–valued” do not imply each other.
into four different pairs of coupled quasilinear partial differential equations for $\lambda(x)$ and $\mu(x)$. This transformed version of the “$\omega$–equation” has the property of being weight–independent with all the information about the relevant weight contained in the $\omega$–ansatz.

More precisely, any (weight–independent) solution of a particular pair of equations for $\mu(x)$ and $\lambda(x)$ is mapped—through the (weight–dependent) $\omega$–ansatz—to a solution of the differential equation (1.13) of the corresponding weight $\omega$–ansatz. A crucial feature is that the ansatz map, denoted by $a_\omega$, is onto and it can be made one–to–one if certain (weight–dependent) equivalence classes of solutions in the domain set of each $a_\omega$ are defined. This implies that any one of the four “linearised” pairs of equations for $\lambda(x)$ and $\mu(x)$ is exactly equivalent to the original nonlinear $\omega$–equation for $W_\omega(x)$, through the appropriate $a_\omega$ map. The most symmetric pair of the four is then singled out and considered as the “representative” one. It is the pair of equations that is going to be related to the action principle.

In section 3, the relevant scalar–field action is introduced and involves a single scalar field and two initially arbitrary real–valued functions $\lambda(M(X))$ and $\mu(M(X))$ of a single (real or complex) Lagrange multiplier $M(X)$. It is the simplest possible action–candidate that includes as genuine subcases both actions of \cite{5} and \cite{6}—the former in a single–field version—and possesses the required parametrisation by an arbitrary function of one variable, after the elimination of the non–dynamical multiplier. The scalar–field action is coupled to the Einstein–Hilbert one, and the Hamiltonian form of the total action is obtained by the usual A.D.M. decomposition. It follows the general rule that is common to all theories with a non–derivative coupling to gravity.

When the $\frac{\omega^2}{2}$–power of the square of the weightless field momentum $(\pi/g\frac{\omega^2}{2})(x)$—in the coupled Hamiltonian system—is solved with respect to the gravitational variables alone, it takes the form of the $\omega$–ansatz of section 2, with $\lambda(x)$ and $\mu(x)$ replaced by the two functions of the multiplier $\lambda(M(x))$ and $\mu(M(x))$ (being now in canonical form). Furthermore, when the multiplier $M(x)$ and hence these two functions are also solved with respect to the gravitational variables, $\lambda(M(x))$ and $\mu(M(x))$ become functionally dependent and are surprisingly shown to satisfy the “representative” pair of equations! In other words, $(\pi^2/g\frac{\omega^2}{2})(x)$ is bound to produce solutions $W_\omega(x)$ of the corresponding $\omega$–equation, for every initial choice of functions $\lambda(M(x))$ and $\mu(M(x))$.

In section 4, the inverse procedure is analysed and it is shown that any solution $W_\omega(x)$ that satisfies certain conditions of reality—suggested by the qualitative predictions of the preceding sections—can be derived from the scalar–field action principle of section 3; under these conditions therefore, the latter provides the general solution of the $\omega$–equation. Considering the diverse origins of the calculations in sections 2 and 3, this is a striking

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7Excluding some trivial cases where the partial derivatives of $W_\omega(x)$ identically vanish; see introduction of section 2.

8The inclusion of complex multipliers is because they are allowed by expressions (1.14), (1.15) and because they do not spoil the physical “realizability” of the scalar–field system, as it is properly explained in the introduction to section 2

9Apart from a few choices that lead either to a constraint between $\tilde{H}$ and $\tilde{F}$ (see end of section 3) or to solutions for the multiplier $M(x)$ that are not consistent with the requirement that $\lambda(M(X))$ and $\mu(M(X))$ be real–valued. Both these problems, however, are finally avoided in section 4.
result and suggests that the rôle of scalar fields in classical general relativity deserves to be investigated further.

The connection between the $\lambda$'s, $\mu$'s and the notation used in the explicit solutions (1.14) and (1.15) is established in one of the appendices.

2 An equivalent form of the weight–$\omega$ differential equation.

2.1 General remarks.

A minimum prerequisite concerning the physical relevance in vacuum gravity of the gravitational combinations that satisfy equation (1.13), is for them to create a constraint surface that is locally identical to the usual one. Then, by replacing the Hamiltonian constraint, they can form an at least locally equivalent system of constraints, that has the additional property of generating a true Lie algebra. However, by concentrating on the actual form of equation (1.13), one observes that in the particular case when the partial derivatives of some solution $W_\omega$ with respect to either $\hat{H}$ or $\tilde{F}$ happen to be identically zero, this solution is either a function of $\tilde{F}$ alone or a constant. In both cases the solution $W_\omega$ does not depend on the Hamiltonian constraint and all information concerning the latter is lost. These special solutions of equation (1.13)—when the quantities $W_{\omega F}$, $W_{\omega H}$ and of course $W_\omega$ identically vanish—can therefore be safely ignored as inadequate for any physical application.

On the other hand, the existence of complex solutions of equation (1.13) is a feature that certainly deserves some attention. It is true that such solutions cannot be easily reconciled to the idea of a physical system—especially in the present discussion when, later, they are related to the $\frac{\omega}{2}$ power of the square of some conjugate field–momentum. However, a complex combination of the gravitational constraints is not necessarily complex–valued, as the simple example $i\sqrt{\hat{H}^2 - \tilde{F}}$ (that solves the $\omega=1$ equation) demonstrates in an appropriate region of the phase space where $\tilde{F} > \hat{H}^2$. Indeed, the above complex combination is merely an equivalent way of writing $\sqrt{\tilde{F} - \hat{H}^2}$ which—in the same region of the phase space—is both real and real–valued.

For the sake of uniformity, therefore, and in order to avoid using adjectives like “complex” and “complex–valued” throughout this section, it seems preferable to accept any generality offered by the two expressions (1.14) and (1.15) that solve the differential equation (1.13), and allow—at this stage—negative and even complex–valued combinations of the gravitational constraints. The necessary amendments for the physical relevance of the combinations is reserved for section 4.

10As, for example, the solutions $-(\hat{H}^2 - \tilde{F})^2$ and $i(\hat{H}^2 - \tilde{F})$ that solve the $\omega = 4$ and $\omega = 2$ equations, respectively.
2.2 The $\omega$-ansatz and the “representative” equation.

As already mentioned, equation (1.13) will be related in section 3 to a quite general action principle involving a single scalar field and two arbitrary functions of a Lagrange multiplier. For this reason, an appropriate $\omega$-ansatz is introduced—one for each weight—that transforms the corresponding $\omega$-equation (1.13) into a form that is easily connected to this action principle. The ansatz is:

$$W_{\omega}[\tilde{\mathcal{H}}, \tilde{F}] = \lambda_{\omega}^2 \tilde{\mathcal{H}}^{\frac{1}{2}} \left( \tilde{\mathcal{H}} - \mu_{[\tilde{\mathcal{H}}, \tilde{F}]} + \sqrt{(\tilde{\mathcal{H}} - \mu_{[\tilde{\mathcal{H}}, \tilde{F}])^2} - \tilde{F} \right)^{-\frac{1}{2}}, \quad (2.1)$$

where $W_{\omega}[\tilde{\mathcal{H}}, \tilde{F}]$, $\mu_{[\tilde{\mathcal{H}}, \tilde{F}]}$ and $\lambda_{[\tilde{\mathcal{H}}, \tilde{F}]}$ are ultralocal functions of $\tilde{\mathcal{H}}$ and $\tilde{F}$, and are allowed—in general—to be complex-valued.

Both signs for the square root in equation (2.1) are permitted; this is not denoted by a $\pm$ sign just for notational simplicity. Note that any solution, say $v_{\omega}[\tilde{\mathcal{H}}, \tilde{F}]$, of the differential equation (1.13)—obtained by any means—can always be brought into the form of the ansatz by choosing

$$\lambda_{\omega}^2 [\tilde{\mathcal{H}}, \tilde{F}] = v_{\omega} [\tilde{\mathcal{H}}, \tilde{F}] \left( \tilde{\mathcal{H}} - \mu_{[\tilde{\mathcal{H}}, \tilde{F}]} + \sqrt{(\tilde{\mathcal{H}} - \mu_{[\tilde{\mathcal{H}}, \tilde{F}])^2} - \tilde{F} \right)^{-\frac{1}{2}}, \quad (2.2)$$

while keeping $\mu_{[\tilde{\mathcal{H}}, \tilde{F}]}$ arbitrary. There is no loss of generality, therefore, in writing any solution $v_{\omega}$ of (1.13) in the form of equation (2.1).

It must be said here that, strictly speaking, our arguments in all that follows must be restricted to regions of the $\tilde{\mathcal{H}}, \tilde{F}$ plane where every function of the constraints as well as its partial derivatives are well-defined. This implies the necessity—at least for most of the cases—to perform all calculations away from the usual constraint surface of vacuum general relativity, and return to it only after the completion of the calculations. It also clearly emphasizes the difficulty in assigning any proper meaning to the differential equation (1.13) without invoking some medium to which gravity is presumably coupled.

Having mentioned that, the square root in (2.1) is defined as

$$R_{[\tilde{\mathcal{H}}, \tilde{F}]} := \sqrt{(\tilde{\mathcal{H}} - \mu_{[\tilde{\mathcal{H}}, \tilde{F}]})^2 - \tilde{F}}, \quad (2.3)$$

and the two partial derivatives of $W_{\omega}[\tilde{\mathcal{H}}, \tilde{F}]$ in terms of $\lambda_{[\tilde{\mathcal{H}}, \tilde{F}]}$, $\mu_{[\tilde{\mathcal{H}}, \tilde{F}]}$ and $R_{[\tilde{\mathcal{H}}, \tilde{F}]}$ are obtained (the square bracket notation will be omitted from now on):

$$W_{\omega H} = \frac{\omega}{2} \lambda_{\omega}^2 (\tilde{\mathcal{H}} - R + R)^{\frac{1}{2}} \left( \frac{1}{\lambda} \lambda_{H} - \frac{1}{R} \mu_{H} + \frac{1}{R} \right),$$

$$W_{\omega F} = \frac{\omega}{2} \lambda_{\omega}^2 (\tilde{\mathcal{H}} - R + R)^{\frac{1}{2}} \left( \frac{1}{\lambda} \lambda_{F} - \frac{1}{R} \mu_{F} - \frac{1}{2R} \left( \frac{1}{R} \right) \right). \quad (2.4)$$
When the ansatz (2.1) and the derivatives (2.4) are substituted into the differential equation (1.13), the latter becomes:

\[
\left( \frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F \right) \left[ -\tilde{F} \left( \frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F \right) + \frac{\tilde{H} - \mu}{R} \right] + \frac{1}{4} \left( \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H \right) \left[ \left( \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H \right) + \frac{2}{R} \right] = 0,
\]

(2.5)

where, quite noticeably, the arbitrary weight \( \omega \) is no longer present.

It can be seen, by inspection, that there exist four obvious solutions of (2.5), corresponding to four different pairs of coupled quasilinear equations for the functions \( \mu \) and \( \lambda \). These quasilinear equations are written explicitly below:

\[
\begin{align*}
\frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F &= 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H = 0, \\
\frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F &= 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H = -\frac{2}{R}, \\
\frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F &= \frac{\tilde{H} - \mu}{R} \quad \text{and} \quad \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H = 0, \\
\frac{1}{\lambda} \lambda F - \frac{1}{R} \mu F &= \frac{\tilde{H} - \mu}{R} \quad \text{and} \quad \frac{1}{\lambda} \lambda H - \frac{1}{R} \mu H = -\frac{2}{R}.
\end{align*}
\]

(2.6) \hspace{1cm} (2.7) \hspace{1cm} (2.8) \hspace{1cm} (2.9)

Given a \( \mu \), any of the above pairs of equations can be solved for the corresponding \( \lambda \)—provided that the system of the two partial equations for \( \lambda \) is not contradictory—and then the \( \omega \)–ansatz (2.1) can be used to obtain solutions \( W_\omega \) of equation (1.13) of the corresponding weight. An equivalent procedure can be followed if, instead of \( \mu \), a \( \lambda \) is initially chosen. In other words, each of the above four pairs of equations (2.6)–(2.9) provides—for each weight—a family of solutions of the corresponding differential equation, that is parametrized by an arbitrary function \( \mu \) or \( \lambda \) and is subject to the condition that the relevant pair of equations from the set (2.6)–(2.9) is not self–contradictory.

Furthermore, the above pairs are all \textit{equivalent} in the sense that, for each weight, they all lead to exactly the same family of solutions of the differential equation (1.13). More precisely, for every solution \( W_\omega \) of (1.13) that can be reached through an \( \omega \)–ansatz (2.1) by some \( \mu \), \( \lambda \), \( R \) satisfying, say, equation (2.6), there always exist three corresponding functions \( \mu \), \( \lambda \) and \( R \) that satisfy the other three equations (2.7) to (2.9) respectively, and lead to the same solution \( W_\omega \) of the differential equation (1.13). For a proof the reader is referred to Appendix A.

Having shown the equivalence of the four “linearized components” of the full non-linear equation (2.5), we concentrate on the most symmetric of them, equation (2.6), and think of it as the “representative” of the whole set of equivalent equations (2.6)–(2.9). The reason we single out (2.6) is its direct relation to the action principle of the next section; it is therefore important for the proper foundation of any further developments to know the exact range of equation (2.6), or, in other words, to compare its solutions with the general solution of each \( \omega \)–equation (1.13). The rather surprising result of such a comparison is that—for all weights—equations (1.13) and (2.6) are exactly equivalent; the remaining of this section is devoted to the relevant proof.
2.3 Equivalence between the “representative” and each weight–ω equation.

The ansatz relation (2.1) can be considered as a parametrised map, \( a_\omega \), from the set of functions \((\mu, \lambda, R)\) satisfying equation (2.6) to the set of all solutions \( W_\omega \) of the corresponding weight–ω differential equation, with \( W_\omega, W_\omega H \) and \( W_\omega F \) being different from zero. The reason we have included \( R \) in the set of functions \((\mu, \lambda, R)\) is that although \( R \) is a function of \( \mu \)—defined by equation (2.3)—is not fully specified by \( \mu \) due to the sign ambiguity.

One would like to know whether the map \( a_\omega \) is one-to-one and, most importantly, whether it is onto. To check the latter, one supposes that \( v \) is any solution of the ω–equation (1.13), where—for simplicity—the subscript ω of \( v \) is omitted. A set of functions \((\mu, \lambda, R)\) obeying the “representative” equation (2.6) is therefore required, with the property of producing through the \( a_\omega \) map the given solution \( v \). This is similar to the procedure followed in Appendix A in order to show that equations (2.6)–(2.9) are equivalent; the difference is that the requirement that at least one of the cases (2.6)–(2.9) leads to \( v \) is lifted—\( v \) is now only confined to obey equation (1.13) for some weight ω.

Clearly, the three conditions that must be satisfied are:

1. The original differential equation

\[
\frac{\omega}{2} v v_F = \tilde{F} v_H^2 - \frac{1}{4} v_H^2, \quad v \neq 0 \quad v_F \neq 0 \quad v_H \neq 0.
\] (2.10)

2. The “representative” equation

\[
\frac{1}{\lambda} \lambda_F - \frac{1}{R} \mu_F = 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda_H - \frac{1}{R} \mu_H = 0; \quad R = \sqrt{(\tilde{H} - \mu)^2 - \tilde{F}}.
\] (2.11)

3. The ansatz relation

\[
v = \lambda \left( \tilde{H} - \mu + \sqrt{(\tilde{H} - \mu)^2 - \tilde{F}} \right)^{\frac{1}{2}}.
\] (2.12)

The third condition can be written as

\[
v = \lambda \left( \tilde{H} - \mu + \sqrt{(\tilde{H} - \mu)^2 - \tilde{F}} \right)^{\frac{1}{2}}.
\] (2.13)

where this relation is valid up to a \( \frac{1}{\omega} \) power of unity. Equation (2.13) can now be solved for \( \mu \), resulting in

\[
\mu = \tilde{H} - \frac{1}{2} \left( \frac{v}{\lambda} + \frac{\lambda}{v} \tilde{F} \right).
\] (2.14)

Differentiating \( \mu \) with respect to both \( \tilde{H} \) and \( \tilde{F} \) gives

\[
\mu_H = 1 - \frac{1}{\omega} \left( \frac{v}{\lambda} - \frac{\lambda}{v} \tilde{F} \right) v_H + \frac{1}{2} \left( \frac{v}{\lambda^2} - \frac{1}{v} \tilde{F} \right) \lambda_H \quad \text{and}
\]

\[
\mu_F = -\frac{1}{2} \frac{\lambda}{v} - \frac{1}{\omega} \left( \frac{v}{\lambda} - \frac{\lambda}{v} \tilde{F} \right) v_F + \frac{1}{2} \left( \frac{v}{\lambda^2} - \frac{1}{v} \tilde{F} \right) \lambda_F.
\] (2.15)
Conditions 2 and 3—being now in the same form—can easily be compared; more precisely, substituting equation (2.14) into the expression for \( R \)—used in condition 2—one finds that
\[
R = \frac{1}{2} \left( \frac{v^2}{\lambda} - \frac{\lambda}{v^2} \tilde{F} \right),
\]
and hence condition 2 becomes
\[
\mu_H = \frac{1}{2} \left( \frac{v^2}{\lambda^2} - \frac{1}{v^2} \tilde{F} \right) \lambda_H \quad \text{and} \quad \mu_F = \frac{1}{2} \left( \frac{v^2}{\lambda^2} - \frac{1}{v^2} \tilde{F} \right) \lambda_F. \quad (2.17)
\]

When equations (2.17)—derived from the second condition—are compared to equations (2.15)—derived from the third condition—they lead to the following pair of equations for \( \lambda \):
\[
\omega - \left( \frac{v^2 - \omega}{\lambda} - \frac{\lambda}{v^2} \tilde{F} \right) v_H = 0 \quad \text{and} \quad \frac{\lambda}{v^2} + \frac{2}{\omega} \left( \frac{v^2}{\lambda} - \frac{\lambda}{v^2} \tilde{F} \right) v_F = 0. \quad (2.18)
\]
The above set of equations admits a common solution for \( \lambda \), we call it \( \bar{\lambda} \), given by
\[
\bar{\lambda} = -\frac{2v^2 v_F}{v_H}. \quad (2.19)
\]
Equations (2.18) and (2.19) are all well defined since \( v, v_H \), and \( v_F \), are not allowed to vanish identically, but for the two equations in (2.18) to be consistent, they must lead either to an identity or at least to a valid equation when \( \bar{\lambda} \) is substituted into them—indeed, by doing so, they both reduce to
\[
\frac{\omega}{2} v_F = \tilde{F} v_F^2 - \frac{1}{4} v_H^2, \quad (2.20)
\]
which is of course true by virtue of condition 1. This proves that the \( a_\omega \) map (2.1) (from the set of functions \( (\mu, \lambda, R) \) to the set of solutions of the corresponding \( \omega \)–equation (1.13)) is onto.

The expression for \( \bar{\lambda} \) is now substituted back into equation (2.14) to give the relevant expression for \( \mu \),
\[
\bar{\mu} = \tilde{\mu} + \frac{v_F}{v_H} \tilde{F} + \frac{1}{4} \frac{v_H}{v_F}, \quad (2.21)
\]
and the one for \( R \),
\[
\bar{R} = \frac{v_F}{v_H} \tilde{F} - \frac{1}{4} \frac{v_H}{v_F}. \quad (2.22)
\]
Note that the expression for \( \bar{R} \) is sign–unambiguous.

To check if the map \( a_\omega \) is one–to–one, two sets of functions \( (\mu_1, \lambda_1, R_1) \) and \( (\mu_2, \lambda_2, R_2) \) are considered. They are required to satisfy the “representative” pair of equations (2.1) and provide—through the \( a_\omega \) map—the same solution \( v \) of the original \( \omega \)–equation. The problem is readily solved in Appendix A and leads to the following three conditions,
\[ \lambda_1^2 (\tilde{H} - \mu_1 + R_1)^{\tilde{\omega}} = \lambda_2^2 (\tilde{H} - \mu_2 + R_2)^{\tilde{\omega}}, \]
\[ \frac{\lambda_1^2}{R_1} (\tilde{H} - \mu_1 + R_1)^{\tilde{\omega}} = \frac{\lambda_2^2}{R_2} (\tilde{H} - \mu_2 + R_2)^{\tilde{\omega}}, \]
\[ \frac{\lambda_1^2}{R_1} (\tilde{H} - \mu_1 + R_1)^{\tilde{\omega}^2} = \frac{\lambda_2^2}{R_2} (\tilde{H} - \mu_2 + R_2)^{\tilde{\omega}^2}, \] (2.23)

which admit the almost trivial solution

\[ \mu_1 = \mu_2 \quad \lambda_1^{\tilde{\omega}} = \lambda_2^{\tilde{\omega}} \quad R_1 = R_2. \] (2.24)

The word “almost” is used because of the ambiguity in the expression for the \( \lambda \)'s. However, if the equivalence class of \( \lambda \) is defined as the set of functions that differ from \( \lambda \) by an \( \tilde{\omega} \) power of unity (it can be easily shown that this defines an equivalence relation) then each anzatz–map \((2.1)\) becomes one–to–one. Hence \( \bar{\mu}, \bar{R} \) and the equivalence class of \( (\bar{\lambda}) \)—given respectively by equations \((2.21), (2.22)\) and \((2.19)\)—are unique.

We have thus shown that the general solution of the representative equation \((2.0)\) is indeed equivalent to the general solution of the original arbitrary–weight differential equation \((1.13)\) (with \( W_{\omega}, W_{\omega H}, W_{\omega F} \) not identically zero), and this completes the proof.

### 2.4 On the local equivalence between the constraints.

As a final remark, we return to one to the introductory comments of this section, concerning the local equivalence of the old and new constraints—\( (\mathcal{H}_\perp, \mathcal{H}_i) \) and \( (\mathcal{W}_\omega, \mathcal{H}_i) \) respectively—in case that the combinations \( W_{\omega}[\mathcal{H}, \tilde{F}] \) are used in vacuum gravity. Specifically, for the constraint surface to be locally mapped onto itself under the replacement of \( \mathcal{H}_\perp \) by \( \mathcal{W}_\omega[\mathcal{H}, \tilde{F}] \), the statement \( W_{\omega}[\mathcal{H}, \tilde{F}] = 0 \) must be equivalent to the condition \( \tilde{\mathcal{H}} = 0 \) when \( \tilde{F} = 0 \). Looking back in equation \((2.1)\) however, it is fairly transparent that at least for one choice of sign for the square root, the constraint \( W_{\omega}[\mathcal{H}, \tilde{F}] = 0 \) is satisfied identically at \( \tilde{F} = 0 \), and hence does not enforce the necessary for equivalence Hamiltonian constraint.

One would thus roughly expect—for each weight—half of the candidate combinations \( W_{\omega}[\mathcal{H}, \tilde{F}] \) to be inadequate for use in vacuum gravity. Fortunately, the sign–unambiguous expression for \( \bar{R} \), equation \((2.22)\), makes certain that this is not the case since—for \( \tilde{F} = 0 \) and by virtue of equation \((2.21)\)—the quantities \( \bar{R} \) and \( \mathcal{H} - \bar{\mu} \) always come with the same sign and do not identically vanish.

In particular, for any given solution \( v[\mathcal{H}, \tilde{F}] \) of the weight–\( \omega \) differential equation, the corresponding expression \( \bar{\mathcal{H}} - \bar{\mu} + \bar{R} \) reduces to \(-v_H/2v_F\) at \( \tilde{F} = 0 \) and, therefore, there is no other reason for \( v[\mathcal{H}, \tilde{F}] \) to identically vanish at \( \tilde{F} = 0 \), provided that \( v_H[\mathcal{H},0], v_F[\mathcal{H},0] \) and \( (\bar{\lambda})[\mathcal{H},0] \) are not identically zero themselves. It is of course still possible that \( v[\mathcal{H},0] = 0 \) will not imply the Hamiltonian constraint \( \bar{\mathcal{H}} = 0 \), but this cannot be known unless the explicit form of a solution \( v[\tilde{H}, \tilde{F}] \) is given. An example of the above considerations concerns the Kuchař–Romano combination and is presented towards the end of the next section.
3 An action functional for the weight–ω differential equation.

The “representative” pair of equations (2.6) (being equivalent to the original differential equation (1.13) under the restriction that \( W_\omega, W_{\omega H} \) and \( W_{\omega F} \) do not vanish identically) is of particular interest because—as it will be shown in this section—can be naturally derived from an action principle involving a single scalar field and a single Lagrange multiplier with a non–derivative coupling to gravity.

For the physical relevance of the “gravity plus scalar–field” system, however, certain reality conditions have to be taken into account. They are imposed on almost every quantity used here and—because of the ultimately close relation between the previous and the present section—they demand for an appropriate modification of the results of the first; this is done in section 4. Since no other reference to some underlying physical interpretation is present, the following construction should be viewed mainly as a mathematical one.

3.1 The scalar–field action.

The relevant action functional \( S^\phi \) is introduced as:

\[
S^\phi[\phi, M, \gamma^{\alpha\beta}] = \int d^4X |\gamma|^\frac{1}{2} \left( \frac{1}{2} \lambda(M) \gamma^{\alpha\beta} \phi,\alpha \phi,\beta + \mu(M) \right),
\]

(3.1)

where the dependence of \( \phi, M \) and \( \gamma^{\alpha\beta} \) on the spacetime points \( X \) is not explicitly denoted. The signature of the spacetime metric is taken to be (–, +, +, +) and \( \lambda(M) \) and \( \mu(M) \) are some given but otherwise arbitrary continuous functions of the Lagrange multiplier \( M \). Extension to an arbitrary number of scalar fields and multipliers is possible, but seems rather redundant.

To ensure that the scalar field \( \phi \) will always be present in the action functional, \( \lambda(M) \) is required to be different from zero—no such restriction is imposed on \( \mu(M) \). The notation for the two functions of the multiplier is intentionally chosen to reflect the notation used in section 2; unlike the corresponding quantities there, however, \( \lambda(M) \) and \( \mu(M) \) must now be real–valued\(^{11}\) and they must also come with the correct sign—this is further discussed in subsection 3.2.

If the “dimension” of \( \gamma^{\alpha\beta}dX^\alpha dX^\beta \) is defined as length squared,

\[
[\gamma^{\alpha\beta}dX^\alpha dX^\beta] = L^2,
\]

(3.2)

and the action \( S^\phi \) is required to be dimensionless, the only consistent attribution of dimensions to the various terms appearing in (3.1)—keeping the conventional dimensions of inverse length for the scalar field—is the following:

\[
[\phi] = L^{-1}, \quad [M] = [\lambda(M)] = L^0 = 1, \quad [\mu(M)] = L^{-4}.
\]

(3.3)

\(^{11}\)Recall that this does not mean that \( \lambda(M) \) and \( \mu(M) \) must necessarily be real functions of \( M \).
This means that $\mu(M(X))$ can be considered as a function of the multiplier $M(X)$ scaled by a constant scalar function $C(X)$,

$$
\mu(M(X)) = C(X)\rho(M(X)), \tag{3.4}
$$

where $[C] = L^{-4}$ and $[\rho(M)] = L^0 = 1$. For simplicity, appropriate units can be chosen so that the value of $C(X)$ equals 1.

### 3.2 A.D.M. decomposition of the coupled system.

By coupling the field action $S^\phi[\phi, M, \gamma_{\alpha\beta}]$ to the gravitational Einstein-Hilbert action $S[\gamma_{\alpha\beta}]$

$$
S[\gamma_{\alpha\beta}] = \int d^4X |\gamma|^\frac{1}{2} R[\gamma_{\alpha\beta}], \tag{3.5}
$$

and by proceeding with the usual A.D.M. decomposition \[2\] of the total action,

$$
S^T := S + S^\phi, \tag{3.6}
$$

one obtains the constraints $H^T_\perp$ and $H^T_i$, whose form is common to any theory with a non-derivative coupling to gravity \[9\]:

$$
H^T_\perp := H_\perp + H^\phi_\perp = 0 \tag{3.7}
$$

$$
H^T_i := H_i + H^\phi_i = 0. \tag{3.8}
$$

The gravitational parts of the constraints, $H_\perp$ and $H_i$, are identical to the constraints of vacuum general relativity (written out in equations \[1.1\]), while the field contributions $H^\phi_\perp$ and $H^\phi_i$ are like the ones of a massless scalar field, only extended by the presence of the two functions $\lambda(M)$ and $\mu(M)$,

$$
H^\phi_i = \pi \phi, \tag{3.9}
$$

$$
H^\phi_\perp = g^\frac{i}{2} \left( -\frac{1}{2} \frac{\pi^2}{g_\lambda(M)} - \mu(M) - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} H_i^\phi H_j^\phi \right). \tag{3.10}
$$

where $\pi$ is the momentum conjugate to the field $\phi$.

From equation \[3.10\] one observes that $\lambda(M)$ must be restricted to be negative-valued, in order for the scalar field to have positive kinetic energy. On the other hand, and in search of a proper interpretation for $\mu(M)$, we make use of the fact that $\mu(M)$ appears as a cosmological constant in equation \[3.10\], and allow it to have any real value at all. It must be said, however, that this uncertainty about the correct sign of $\mu(M)$ is rather unimportant (if such thing as a “correct” sign really exists), since care is also taken to make the remaining of this discussion essentially independent of the actual restrictions imposed. The “negativity” of $\lambda(M)$ and the “reality” of $\mu(M)$ constitute the first reality condition presented in this section.
3.3 The two equations for $M$ and $\pi$.

At this stage, the total action $S^T$ can be varied with respect to the multiplier. This only appears in the field action $S^\phi$ and in particular in the $\mathcal{H}^\phi_\perp$–term, when the latter is written in A.D.M. form:

$$S^\phi = \int d^3x \, dt \left( \pi \dot{\phi} + N \mathcal{H}^\phi_\perp - N^i \mathcal{H}^\phi_i \right). \tag{3.11}$$

As a result, $M$ can be equivalently determined by requiring that

$$\frac{d\mathcal{H}^\phi_\perp}{dM} = 0, \tag{3.12}$$

which produces the following condition:

$$\frac{1}{2} \frac{\pi^2}{g \lambda^2(M)} \lambda'(M) - \mu'(M) - \frac{1}{2} \frac{\lambda'(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j = 0. \tag{3.13}$$

In a usual notation, $\lambda'(M)$ and $\mu'(M)$ denote the total derivatives of $\lambda(M)$ and $\mu(M)$ with respect to $M$. The constraints (3.7) and (3.8) can now be used to rewrite equations (3.10), (3.13) in terms of the gravitational contributions to these constraints,

$$\frac{1}{2} \frac{\pi^2}{g \lambda(M)} \lambda'(M) - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j = \mathcal{H} - \mu(M), \tag{3.14}$$

$$\frac{1}{2} \frac{\pi^2}{g \lambda^2(M)} \lambda'(M) - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j \lambda'(M) = \mu'(M), \tag{3.15}$$

where $\mathcal{H}$ is the zero–weight scalar density defined in [7].

The pair of equations (3.14) and (3.15)—relating the quantities $\pi$, $M$, $\mathcal{H}_\perp$, and $\mathcal{H}_i$—is the starting point of the main discussion of this section. The aim is to solve these equations for $\pi$ and $M$ in terms of $\mathcal{H}_\perp$ and $\mathcal{H}_i$, regarding the functions $\lambda(M)$ and $\mu(M)$ as known. Because the analysis depends on the actual form of the derivatives, one should distinguish some special cases—in which either $\lambda'(M)$ or $\mu'(M)$ is identically zero or both $\lambda'(M)$ or $\mu'(M)$ are identically zero—and treat them separately from the general case where the derivatives do not vanish.

3.4 Solving for $M$ and $\pi$.

3.4.1 The general case. Recovery of the $\omega$–ansatz and reconstruction of the “representative” equation.

The general case then occurs when:

$$\lambda'(M) \neq 0 \quad \mu'(M) \neq 0. \tag{3.16}$$

When this condition holds, one can multiply equation (3.13) by $\left(\lambda(M)/\lambda'(M)\right)$ and obtain its equivalent version

$$\frac{1}{2} \frac{\pi^2}{g \lambda(M)} - \frac{1}{2} \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j = \frac{\mu'(M) \lambda(M)}{\lambda'(M)}. \tag{3.17}$$
As already stated, the aim is to solve equations (3.14) and (3.17) for \( \pi \) and \( M \) in terms of the gravitational contributions to the constraints. This can be done by adding and subtracting (3.14) and (3.17), and then cross-multiplying the resulting equations to eliminate the field momenta. One is left with an algebraic equation determining the multiplier \( M \) in terms of \( H_\perp, H_i \),

\[
\frac{\mu'(M)\lambda(M)}{\lambda'(M)} = \sqrt{\left(\tilde{H} - \mu(M)\right)^2 - \tilde{F}},
\]

(3.18)

where both choices of sign for the square root are allowed, and \( \tilde{F} \) is the quantity defined in [7]. To get the corresponding expression for the field momenta \( \pi \) in terms of \( H_\perp \) and \( H_i \), equation (3.14) is solved to give an expression of \( \pi \) as a function of \( M, H_\perp, \) and \( H_i \), and then a solution of (3.18) is substituted in this expression, to replace \( M \). This is subject to the problem of existence of a proper (real or complex) solution for \( M \) so that the functions \( \lambda(M) \) and \( \mu(M) \) are negative-valued and real-valued respectively; this problem, however, is finally avoided in section 4.

In particular, if \( M[H_\perp, H_i] \) denotes such a proper solution, the corresponding expression for the square of the field momentum—written in weightless form—is given by

\[
\frac{1}{(g^2)^2} \pi^2[H_\perp, H_i] = \lambda(M[H_\perp, H_i]) \left(\tilde{H} - \mu(M[H_\perp, H_i]) + \sqrt{\left(\tilde{H} - \mu(M[H_\perp, H_i])\right)^2 - \tilde{F}}\right),
\]

(3.19)

which, together with (3.18), provide the required set of solutions of the original system of equations (3.14) and (3.15).

One then observes that the actual form of equations (3.18) and (3.19) ensures that the Hamiltonian and momentum contributions to the constraints, \( H_\perp \) and \( H_i \), only appear in the form of the scalar combinations \( \tilde{H} \) and \( \tilde{F} \) respectively. This means that the solutions \( M[H_\perp, H_i] \) and \( \pi^2[H_\perp, H_i] \) can be written as \( M[\tilde{H}, \tilde{F}] \) and \( \pi^2[\tilde{H}, \tilde{F}] \), and also by regarding \( \lambda \) and \( \mu \) as functions of \( \tilde{H} \) and \( \tilde{F} \), according to

\[
\lambda[\tilde{H}, \tilde{F}] := \lambda(M[H_\perp, H_i]) \quad \text{and} \quad \mu[\tilde{H}, \tilde{F}] := \mu(M[H_\perp, H_i]),
\]

(3.20)

we can bring equation (3.19) into the equivalent form

\[
\frac{1}{(g^2)^2} \pi^2[\tilde{H}, \tilde{F}] = \lambda[\tilde{H}, \tilde{F}] \left(\tilde{H} - \mu[\tilde{H}, \tilde{F}] + \sqrt{\left(\tilde{H} - \mu[\tilde{H}, \tilde{F}]\right)^2 - \tilde{F}}\right).
\]

(3.21)

If the above expression for the field momentum is raised to the power of \( \frac{\omega^2}{2} \), it is immediately recognised as the ansatz equation (2.1) of section 2. It satisfies the differential equation (1.13), provided that \( \lambda[\tilde{H}, \tilde{F}], \mu[\tilde{H}, \tilde{F}] \) and the square root in equation (3.21) obey the—common to all weights—“representative” equation (2.6). Quite remarkably, the last statement is true as the following argument demonstrates:

Since (3.18) is an algebraic equation for \( M \), it must hold identically when written in terms of an actual (real or complex) solution \( M[\tilde{H}, \tilde{F}] \). As a result, it automatically
becomes a differential equation for \( \lambda(M[\tilde{H}, \tilde{F}]) \equiv \lambda[\tilde{H}, \tilde{F}] \) and \( \mu(M[\tilde{H}, \tilde{F}]) \equiv \mu[\tilde{H}, \tilde{F}] \), whatever the functions \( \lambda(M) \) and \( \mu(M) \) were initially chosen to be. Furthermore, equation (3.18) makes certain that its solution \( M[\tilde{H}, \tilde{F}] \) will always satisfy\footnote{Again, this means that \( M_H \) and \( M_F \) must not identically vanish, but proper care should be taken to restrict to regions of the \( \tilde{H}, \tilde{F} \) plane where \( M_H \) and \( M_F \) do not even take the value zero.}

\[
M_H \neq 0 \quad \text{and} \quad M_F \neq 0 \tag{3.22}
\]
and, therefore, by multiplying equation (3.18) with \( M_H \) and \( M_F \), we get a pair of two partial differential equations,

\[
\frac{1}{\lambda} \lambda_H - \frac{1}{\sqrt{(\mathcal{H} - \mu)^2 - \tilde{F}}} \mu_H = 0 \quad \text{and} \quad \frac{1}{\lambda} \lambda_F - \frac{1}{\sqrt{(\mathcal{H} - \mu)^2 - \tilde{F}}} \mu_F = 0, \tag{3.23}
\]
which is exactly the “representative” equation (2.6).

In other words, for any initial choice (3.16) of functions \( \lambda(M) \) and \( \mu(M) \) (subject to the condition that there exist a proper solution for \( M \) that makes \( \lambda(M) \) and \( \mu(M) \) negative–valued and real–valued) the action (3.1) necessarily produces arbitrary–weight combinations of the gravitational constraints that have the property of generating a true Lie algebra; they are explicitly given by the \( \frac{\pi^2}{2} \) power of the square of the field momentum, equation (3.21).

Note, however, that for equation (3.21) to make sense, there must exist some regions of the \( \tilde{H}, \tilde{F} \) plane—equivalently, regions of the gravitational phase space—such that the quantity inside the square root, as well as the whole right hand side of equation (3.21) are positive–valued. This is the second reality–condition of the section which—as the first one—is taken into proper account in section 4.

We can now return to our starting point—equations (3.14), (3.15)— and treat the special cases where either \( \lambda'(M) \) or \( \mu'(M) \) is identically zero, or both of them are identically zero. There are three possibilities:

### 3.4.2 Special case 1. The “Kuchař–Romano” family.

\[
\lambda'(M) = 0 \quad \text{and} \quad \mu'(M) = 0. \tag{3.24}
\]

Suppose that \( \mu(M) = C_1 \) and \( \lambda(M) = C_2 \), where \( C_1 \) and \( C_2 \) are, respectively, real and negative constants, according to the first reality condition. Now there is no multiplier present in the total action and, therefore, equation (3.15) is trivially satisfied, both sides being equal to zero. Correspondingly, the coupled system of equations (3.14), (3.15) for \( M \) and \( \pi^2 \) reduces to the single equation (3.14), giving \( \pi^2 \) directly as a function of \( \mathcal{H} \) and \( \tilde{F} \):

\[
\frac{\pi^2}{(g^{1/2})^2} = C_2 \left( \tilde{H} - C_1 \right) + \sqrt{\left( \tilde{H} - C_1 \right)^2 - \tilde{F}}. \tag{3.25}
\]
When this is raised to the $\frac{\omega}{2}$ power, it has the form of the $\omega$-ansatz equation (2.1), provided that the identification
\[
\lambda[\tilde{H}, \tilde{F}] \equiv C_2 \quad (3.26)
\]
\[
\mu[\tilde{H}, \tilde{F}] \equiv C_1 \quad (3.27)
\]
is made. The “representative” equation, (3.23) or (2.4), is satisfied trivially for these $\lambda[\tilde{H}, \tilde{F}]$ and $\mu[\tilde{H}, \tilde{F}]$, and therefore expression (3.25) provides further solutions of the differential equation (1.13); they are also required to be positive–valued, according to the second reality condition.

This case is interesting because it reduces to the Kuchař–Romano combination under the identification $\omega = 2$, $C_1 = 0$ and $C_2 = -1$. It also illustrates our previous comment—made at the end of section 2—concerning the equivalence of the old and new constraints, in case that the Hamiltonian constraint is replaced by any self–commuting constraint combination. More precisely, from the two different weight–two solutions defined by equation (3.25) (corresponding to the two choices of sign for the square root) one obeys $v_H [\tilde{H},0] = 0$ and according to subsection 2.4 is excluded, while the other one—although it satisfies $v_H [\tilde{H},0] \neq 0$, $v_F [\tilde{H},0] \neq 0$ and $(\lambda)[\tilde{H},0] \neq 0$—does not imply the Hamiltonian constraint when evaluated at $F = 0$, unless the (real) constant $C_1$ is chosen to be zero. This surviving combination is of course one of the two $\Lambda_{\pm}$.

**3.4.3 Special case 2. The “pseudomultiplier”**.

\[
\lambda'(M) = 0 \quad \text{and} \quad \mu'(M) \neq 0. \quad (3.28)
\]

Suppose that $\lambda(M) = C_2$, where $C_2$ is negative. Equation (3.15) becomes
\[
\mu'(M) = 0. \quad (3.29)
\]

Any possible solution of this equation can only produce a numerical value for $M$ and, therefore, $M$ is not a proper multiplier but merely “fixes itself a value”. However, one still proceeds by solving (3.14) for $\pi^2$ and finds
\[
\frac{\pi^2}{(g^{1/2})^2} = -C_2 \left( (\tilde{H} - C_1) + \sqrt{(\tilde{H} - C_1)^2 - F} \right), \quad (3.30)
\]
where now $C_1$ is the real numerical value of $\mu(M)$ after the elimination of the “pseudomultiplier” $M$. Therefore, provided that a proper solution for $M$ (leading to a real–valued $\mu(M)$) exists, case 2 is essentially equivalent to case 1.

---

13To avoid any possible confusion we point out that equations (3.28) and (3.29) are not contradictory; the first means that $\mu'(M)$ must not be identically zero while the second is an algebraic equation for determining $M$ if one regards $\mu(M)$ as known.
3.4.4 Special case 3. The “null vector” family.

\[ \lambda'(M) \neq 0 \quad \text{and} \quad \mu'(M) = 0. \quad (3.31) \]

Suppose that \( \mu(M) = C_1 \), \( C_1 \) being real. Now equation (3.15) becomes

\[ \left( \frac{\pi^2}{g\lambda(M)} - \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j \right) \lambda'(M) = 0, \quad (3.32) \]

and, as a result, \( M \) either has a real numerical value (thus producing exactly the same combinations as in special cases 1 and 2) or it satisfies

\[ \frac{\pi^2}{g\lambda(M)} - \frac{\lambda(M)}{\pi^2} g^{ij} \mathcal{H}_i \mathcal{H}_j = 0. \quad (3.33) \]

When (3.33) is combined with equation (3.14), leads to a constraint between the two variables \( \tilde{F} \) and \( \tilde{\mathcal{H}} \), namely

\[ \tilde{F} = (\tilde{\mathcal{H}} - C_1)^2. \quad (3.34) \]

An example is when \( \lambda(M) = M \) and \( \mu(M) = 0 \). It can be interpreted as a coordinate condition on \( \gamma_{\alpha\beta} \) such that \( \phi_\alpha \) becomes a null vector,

\[ \gamma^{\alpha\beta} \phi_\alpha \phi_\beta = 0, \quad (3.35) \]

provided that \( \phi \) is considered as an externally fixed field.

4 The inverse procedure.

It was demonstrated in the previous section that apart from some improper choices for \( \mu(M) \) and \( \lambda(M) \)—resulting either in constraints (3.34) between the two gravitational variables \( \tilde{\mathcal{H}} \) and \( \tilde{F} \), or in solutions of equation (3.18) that lead to complex–valued quantities—all other choices yield, in principle, a solution of the \( \omega \)-differential equation (1.13). It is given by the \( \frac{\pi^2}{g} \) power of the square of the scalar field momentum when the latter is evaluated in terms of the gravitational variables alone and—because of that—must necessarily admit real positive values in some regions of the \( \tilde{\mathcal{H}}, \tilde{F} \) plane.

The important question that we deal with in this section is whether the totality of solutions \( (\pi^2)\tilde{\mathcal{H}}[\tilde{\mathcal{H}}, \tilde{F}] \) provides the complete “real” subset of solutions of the weight–\( \omega \) equation; the latter is meant to be the set of all solutions of (1.13) that satisfy the reality conditions of section 3.

Using the “onto” property of each \( \omega \)-ansatz map (discussed in section 2), the above question is equivalent to asking if the set of functions \( (\lambda(M[\tilde{\mathcal{H}}, \tilde{F}]), \mu(M[\tilde{\mathcal{H}}, \tilde{F}]), R(M[\tilde{\mathcal{H}}, \tilde{F}])) \) (obtained by the action principle of the previous section) can provide the complete “real–sending” subset of solutions of the “representative” equation; i.e., the (weight–dependent)

\[ ^{14} \text{Excluding the physically irrelevant cases when } W_\omega[\tilde{\mathcal{H}}, \tilde{F}] = 0, W_{\omega\mathcal{H}}[\tilde{\mathcal{H}}, \tilde{F}] = 0 \text{ and } W_{\omega F}[\tilde{\mathcal{H}}, \tilde{F}] = 0. \]
set of solutions of the “representative” equation whose image under each \( \omega \)–ansatz map is the “real” subset of the corresponding \( \omega \)–equation. The re-phrasing is essential because it allows a comparison between linear equations to take place, which greatly simplifies the whole problem.

4.1 The “real” subset of solutions.

The exact definition of the “real” subset of solutions involves the following procedure: For every solution \( W_\omega[\tilde{H}, \tilde{F}] \) of the \( \omega \)–equation, one obtains the corresponding quantities \( \tilde{\lambda}[\tilde{H}, \tilde{F}] \) and \( \tilde{\mu}[\tilde{H}, \tilde{F}] \), that are given respectively by equations (2.11) and (2.21). For our present purposes, two triplets of functions \( (W_\omega, \tilde{\lambda}, \tilde{\mu}) \) and \( (W_\omega, \tilde{\lambda}', \tilde{\mu}) \) are considered to be different if \( \tilde{\lambda} \) and \( \tilde{\lambda}' \) differ by an \( \omega^2 \) power of unity, unless of course this \( \omega^2 \) power is unity itself.

The label \( Q_{W_\omega} \) is then attached to every possible triplet \( (W_\omega, \tilde{\lambda}, \tilde{\mu}) \), and is defined as the largest set of values of \( \tilde{H} \) and \( \tilde{F} \) for which \( W_\omega \) and the corresponding quantities \( \tilde{\lambda} \) and \( \tilde{\mu} \) are positive, negative and real–valued, respectively. In other words, given any solution \( W_\omega \), its label \( Q_{W_\omega} \) provides the regions of the gravitational phase space where all the reality conditions of section 3 are satisfied. It should be mentioned, however, that although the labeling is suggested by the results of section 3, it does not depend on it in any way—if it did, the whole procedure would simply be inconsistent.

The “real” subset of solutions of each \( \omega \)–equation can then be defined as the set of solutions \( W_\omega \), for which the label \( Q_{W_\omega} \) of the corresponding triplet \( (W_\omega, \tilde{\lambda}, \tilde{\mu}) \) is not the empty set. The definition is clearly independent of the actual reality conditions imposed—in the sense that it can be trivially adapted to more refined physical conditions—and thus justifies a previous comment in subsection 3.2. In addition, because of the requirement on \( \tilde{\lambda} \) to have a unique sign, only one member (at most) of each former equivalence class can survive the definition of reality, which means that the “real–sending” subset is still mapped onto the “real” subset in an one–to–one fashion.

4.2 The inverse procedure: Action functionals for given “real” solutions.

Using the above terminology, the main question of the section is addressed once more, whether the functions \( (\lambda(M[\tilde{H}, \tilde{F}]), \mu(M[\tilde{H}, \tilde{F}]), R(M[\tilde{H}, \tilde{F}])) \)—obtained by the action prescription of section 3—can generate the complete “real–sending” subset of the “representative” equation.

In case they did generate the complete “real–sending” subset, all problems concerning the improper choices for \( \mu(M) \) and \( \lambda(M) \)—mentioned in the beginning of this section—could be easily considered as irrelevant, since now for every “real” solution \( W_\omega[\tilde{H}, \tilde{F}] \) of the weight–\( \omega \) differential equation there would always exist a proper choice of \( \mu(M) \) and \( \lambda(M) \) that would lead to this solution.

In particular, for the above statement to be true, there must exist some (real or complex)
functions $M[\tilde{\mathcal{H}}, \tilde{F}]$, $\mu(M)$ and $\lambda(M)$ that satisfy the conditions
\[
\begin{align*}
\mu(M[\tilde{\mathcal{H}}, \tilde{F}]) &= \bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}] \quad (4.1) \\
\lambda(M[\tilde{\mathcal{H}}, \tilde{F}]) &= \bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}] \quad \text{and} \quad (4.2) \\
\sqrt{(\tilde{\mathcal{H}} - \mu(M[\tilde{\mathcal{H}}, \tilde{F}]))^2 - \tilde{F}} &= \bar{R}[\tilde{\mathcal{H}}, \tilde{F}], \quad (4.3)
\end{align*}
\]
for every “real” solution $W_\omega[\tilde{\mathcal{H}}, \tilde{F}]$. The overbar symbol on the right-hand side of equations (4.1), (4.2), (4.3) is a reminder of the uniqueness of these expressions for each given $W_\omega[\tilde{\mathcal{H}}, \tilde{F}]$, as explained in subsection 4.1.

By inspecting the representative equation (2.6)—which holds for any such set of functions $(\lambda, \bar{\mu}, \bar{R})$—one observes that $\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]$ and $\bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}]$ can either be functions of both $\tilde{\mathcal{H}}$ and $\tilde{F}$ or constants; if this is not so, the system of the two partial differential equations in (2.6) is self-contradictory. Correspondingly, we have to distinguish between the two cases:

4.2.1 The special case. Constant functions $\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]$ and $\bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}]$.

If $\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]$ and $\bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}]$ are negative and real-valued constants, say $C_2$ and $C_1$ respectively, then special case 1 in subsection 3.4.2 suggests that—without needing to specify the Lagrange multiplier as a function $M[\tilde{\mathcal{H}}, \tilde{F}]$ of the gravitational variables—one can directly identify the required functions
\[
\lambda(M) = \bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}] = C_2 \quad \text{and} \quad \mu(M) = \bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}] = C_1. \quad (4.4)
\]
The above relations satisfy equation (4.3) for an appropriate choice of sign and, therefore, the problem of finding an action functional is solved.

4.2.2 The general case. Non–trivial functions $\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]$ and $\bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}]$ and the unique “$\kappa$”.

In the general case, when $\bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}]$ and $\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]$ are non–trivial functions of $\tilde{\mathcal{H}}$ and $\tilde{F}$, the situation is more complicated and the following four conditions must be satisfied:
\[
\begin{align*}
\bar{\mu}_H[\tilde{\mathcal{H}}, \tilde{F}] &= \mu'(M) M_H[\tilde{\mathcal{H}}, \tilde{F}] \\
\bar{\lambda}_H[\tilde{\mathcal{H}}, \tilde{F}] &= \lambda'(M) M_H[\tilde{\mathcal{H}}, \tilde{F}] \\
\bar{\mu}_F[\tilde{\mathcal{H}}, \tilde{F}] &= \mu'(M) M_F[\tilde{\mathcal{H}}, \tilde{F}] \\
\bar{\lambda}_F[\tilde{\mathcal{H}}, \tilde{F}] &= \lambda'(M) M_F[\tilde{\mathcal{H}}, \tilde{F}]. 
\end{align*} \quad (4.5)
\]
From (4.3) one gets a differential equation for the required $M[\tilde{\mathcal{H}}, \tilde{F}]$,
\[
M_H[\tilde{\mathcal{H}}, \tilde{F}] - \mathcal{A}[\tilde{\mathcal{H}}, \tilde{F}] M_F[\tilde{\mathcal{H}}, \tilde{F}] = 0, \quad (4.6)
\]
where
\[
\frac{\bar{\mu}_H}{\bar{\mu}_F} = \frac{\bar{\lambda}_H}{\bar{\lambda}_F}. \quad (4.7)
\]
Note that because of the properties of the “representative” equation, the denominator and the numerator in the above expression do not identically vanish and therefore—provided
that all “improper” regions of the \((\tilde{\mathcal{H}}, \tilde{\mathcal{F}})\) plane are excluded—equation (4.7) is in general well-defined.

An explicit expression for \(A[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\) (in terms of a given weight–\(\omega\) solution \(v[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\) and its derivatives) is found by substituting equations (2.21, 2.19) for \(\bar{\mu}\) and \(\bar{\lambda}\) into equation (4.7):

\[
A[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}] = \frac{v^2_H v_F + v v_H v_{HF} - vv_F v_{HH}}{v^2_F v_H + v v_H v_{FF} - vv_F v_{HF}},
\]

where \(v_{HH}, v_{HF}\) and \(v_{FF}\) denote the second partial derivatives of \(v\) with respect to \(\tilde{\mathcal{H}}\) and \(\tilde{\mathcal{F}}\). Note that (4.8) is \(\omega\)-independent.

By examining equation (4.6), it can be seen that any arbitrary (real or complex) function \(f\) of \(\bar{\mu}, \bar{\lambda}\) provides a solution to it:

\[
M[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}] = f(\bar{\mu}[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}], \bar{\lambda}[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]).
\]

(4.9)

An important observation, however, is that—due to the “representative” equation (2.6)—the Jacobian of \(\bar{\mu}[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\) and \(\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\) with respect to the variables \(\tilde{\mathcal{H}}, \tilde{\mathcal{F}}\) is identically zero; hence there are at least some local regions of the \((\mathcal{H}, \mathcal{F})\) plane where \(\bar{\mu}[\mathcal{H}, \mathcal{F}]\) is solvable as some unique real “\(\kappa\)” function of \(\bar{\lambda}[\mathcal{H}, \mathcal{F}]\):

\[
\bar{\mu}[\mathcal{H}, \mathcal{F}] = \kappa(\bar{\lambda}[\mathcal{H}, \mathcal{F}]).
\]

(4.10)

Since \(\bar{\mu}\) and \(\bar{\lambda}\) only depend on the specific solution \(W_\omega[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\), the same is true for the uniquely defined \(\kappa\). Equation (4.3) then reduces to

\[
M[\mathcal{H}, \mathcal{F}] = h(\bar{\lambda}[\mathcal{H}, \mathcal{F}]),
\]

(4.11)

where \(h\) is some arbitrary (real or complex) function which, in addition, can be chosen invertible.

Having obtained an expression (4.11) for the multiplier \(M[\tilde{\mathcal{H}}, \tilde{\mathcal{F}}]\), one only needs to find the remaining required functions \(\mu(M), \lambda(M)\), with the property that

\[
\mu\left(h(\bar{\lambda}[\mathcal{H}, \mathcal{F}])\right) = \bar{\mu}[\mathcal{H}, \mathcal{F}] = \kappa(\bar{\lambda}[\mathcal{H}, \mathcal{F}])
\]

\[
\lambda\left(h(\bar{\lambda}[\mathcal{H}, \mathcal{F}])\right) = \bar{\lambda}[\mathcal{H}, \mathcal{F}].
\]

(4.12)

From (4.12) these functions are easily found to be

\[
\mu(M) = \kappa \circ h^{-1}(M)
\]

\[
\lambda(M) = h^{-1}(M),
\]

(4.13)

which are well-defined since \(h\) can be always chosen invertible.

An equivalent way of writing (4.13)—which can be seen directly from equation (4.10)—is

\[
\mu(M) = \kappa(\lambda(M)),
\]

(4.14)

where \(\lambda\) is kept arbitrary. This expression—which now only involves the \(\kappa\) function—is the required solution to our problem and justifies the title of the section. It is reminded that \(\kappa\) is specified uniquely by the given solution \(W_\omega[\mathcal{H}, \mathcal{F}]\) of the weight–\(\omega\) differential equation (at least locally).
4.2.3 An application. The Brown and Kuchař combination:

As an example, the Brown and Kuchař combination $W_2[\tilde{\mathcal{H}}, \tilde{F}] = \tilde{\mathcal{H}}^2 - \tilde{F}$ (in its weightless form) is considered. By using equations (2.21) and (2.19), one finds the corresponding unique expressions $\bar{\mu}$ and $\bar{\lambda}$ as

$$\bar{\mu} = \frac{1}{2} \left( \tilde{\mathcal{H}} - \frac{\tilde{F}}{\tilde{\mathcal{H}}} \right) \quad \text{and} \quad \bar{\lambda} = \left( \tilde{\mathcal{H}} - \frac{\tilde{F}}{\tilde{\mathcal{H}}} \right). \quad (4.15)$$

The set of values of $\tilde{\mathcal{H}}$ and $\tilde{F}$ for which the two reality conditions are satisfied (i.e., the label $Q_{W_2}$) is given by the two inequalities $\tilde{\mathcal{H}}^2 > \tilde{F}$ and $\tilde{\mathcal{H}} < 0$ and, therefore, the combination $W_2$ indeed belongs to the “real” subset of solutions of the weight–two equation. Furthermore, for this range of values, the quantities $W_2$, $\tilde{\mathcal{H}}$, $\tilde{F}$, as well as their derivatives are all well defined, and thus the same also applies to the whole procedure in sections 2 and 3.

The uniquely specified real function $\kappa$ is then determined as

$$\kappa(\bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}]) = \frac{1}{2} \bar{\lambda}[\tilde{\mathcal{H}}, \tilde{F}] = \bar{\mu}[\tilde{\mathcal{H}}, \tilde{F}] \quad (4.16)$$

and, therefore, any choice of $\mu(M) = \kappa(\lambda(M)) = \lambda(M)/2$ is fine, with $\lambda(M)$ being an arbitrary (real or complex) function of $M$. This illustrates a statement in the introduction of the paper, that there is actually a variety of single–scalar–field actions which lead to the combination $G(x)$\textsuperscript{15}.

5 Future possibilities.

In this paper, we attempted to find a phenomenological medium that would produce constraint–combinations similar to the ones obtained by algebraic means in [7], hoping that such a procedure would throw more light on their origin. For this reason, we transformed the nonlinear arbitrary–weight differential equation of [7] into a set of coupled quasilinear equations—called the “representative” set—whose integrability condition is exactly the original nonlinear equation. We then tried the simplest possible action functional of a single scalar field—including both combinations $G(x)$ and $\Lambda_{\pm}(x)$ as special cases—and coupled it to gravity. It quite remarkably reconstructed the “representative” equation and produced the general solution of the differential equation of [7], modulo certain reality conditions. The result strongly suggests that the role of scalar fields in classical general relativity deserves to be investigated further.

Such an investigation should be partly related to the physical interpretation of the action discussed here, as well as to some geometrical understanding of the correlation between pure gravity—i.e., Markopoulou’s equation—and scalar fields—the “representative”\textsuperscript{15}

\textsuperscript{15}It seems interesting to mention here that $G(x)$ is a very special kind of combination, since the family it generates—by the algorithm $G_\omega(x) := G^\omega(x)$—provides the only solutions of equation (1.13) that are polynomial in $\mathcal{H}$ and $\tilde{F}$. 


equation. In addition, the possibility of connecting the above results with the more general problem of representing spacetime diffeomorphisms in canonical gravity is certainly not excluded. It would also be interesting to know whether these commuting combinations can arise from a canonical transformation of the geometric data, or even if they can be related to the Ashtekar program in quantum gravity, where complex combinations of the geometric data are also allowed. We hope to be able to discuss these questions in the future.

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A Appendix

We shall explicitly demonstrate equivalence between cases (2.6) and (2.7); the same argument applies when comparing any two cases from the set (2.6)–(2.9). We respectively denote the set of functions obeying equations (2.6) and (2.7), by \( (\mu_1, \lambda_1, R_1) \) and \( (\mu_2, \lambda_2, R_2) \). For the two cases to be equivalent (in the sense described in subsection 2.2) the following three conditions must be satisfied:

1. \( \mu_1, \lambda_1, R_1 \) should obey (2.6)—the functions \( \mu_1, \lambda_1 \) regarded as known—
2. \( \mu_2, \lambda_2, R_2 \) should obey (2.7) and
3. \( v[\tilde{H}, \tilde{F}] = \lambda_1 \tilde{\xi}(\tilde{H} - \mu_1 + R_1)\tilde{\xi} = \lambda_2 \tilde{\xi}(\tilde{H} - \mu_2 + R_2)\tilde{\xi} \).

To avoid having to compare the differential equations of conditions 1 and 2, we do the following “trick”. We successively insert equations (2.6) and (2.7) into equations (2.4) determining the partial derivatives of \( v[\tilde{H}, \tilde{F}] \), which by condition 3 must be the same for both cases. As a result, we turn the differential equations (2.6) and (2.7) into the pair of algebraic equations

\[
\begin{align*}
v_H &= \frac{\lambda_1 \tilde{\xi}(\tilde{H} - \mu_1 + R_1)\tilde{\xi}}{R_1} = -\frac{(\lambda_2) \tilde{\xi}(\tilde{H} - \mu_2 + R_2)\tilde{\xi}}{R_2} \quad (A.1) \\
v_F &= \frac{\lambda_1 \tilde{\xi}(\tilde{H} - \mu_1 + R_1)\tilde{\xi}^2}{R_1} = \frac{\lambda_2 \tilde{\xi}(\tilde{H} - \mu_2 + R_2)\tilde{\xi}^2}{R_2} \quad (A.2)
\end{align*}
\]

and—by comparing condition 3 with equations (A.1) and (A.2)—we get the consistent solution (since the system is over-determined for \( \mu_2, \lambda_2 \) in terms of \( \mu_1, \lambda_1 \)):

\[
\begin{align*}
\mu_2 &= 2\tilde{H} - \mu_1 \\
\lambda_2 \tilde{\xi} &= (-\lambda_1) \tilde{\xi} \\
R_2 &= -R_1, \quad (A.3)
\end{align*}
\]
which proves equivalence.

For completeness, we write down the corresponding results from comparing cases (2.8) with (2.6) and (2.8) with (2.9); we still consider the pair $(\mu_1, \lambda_1)$—that obeys (2.6)—as given:

\[
\mu_3 = 2\tilde{H} - \mu_1, \quad (\lambda_3) = \frac{(-\lambda_1)\tilde{H} + \mu_1 + R_1}{(\tilde{H} - \mu_1 - R_1)}, \quad R_3 = R_1,
\]

\[
\mu_4 = \mu_1, \quad (\lambda_4) = \frac{(-\lambda_1)\tilde{H} + \mu_1 + R_1}{(\tilde{H} - \mu_1 - R_1)}, \quad R_4 = -R_1,
\]

where $(\mu_3, \lambda_3)$ and $(\mu_4, \lambda_4)$ respectively satisfy equations (2.8) and (2.9).

**B Appendix**

The exact relation between the quantities $\lambda[\tilde{H}, \tilde{F}]$ and $\mu[\tilde{H}, \tilde{F}]$—used in the present paper—and the quantities $B(\alpha)$, $B'(\alpha)$ and $\alpha[\tilde{H}, \tilde{F}]$—appearing in [7]—is established.

By comparing equation (1.14) with the “$\omega$-ansatz” equation (2.1), one immediately identifies

\[
\mu[\tilde{H}, \tilde{F}] = \frac{1}{2} B'(\alpha[\tilde{H}, \tilde{F}])
\]

and

\[
\lambda[\tilde{H}, \tilde{F}] = \pm \exp\left( B(\alpha[\tilde{H}, \tilde{F}]) + \frac{\omega}{2} \frac{B'(\alpha[\tilde{H}, \tilde{F}])}{\sqrt{(\tilde{H} - \frac{1}{2} B'(\alpha[\tilde{H}, \tilde{F}]))^2 - \tilde{F}} \right)
\]

which, using (1.15), can be written as

\[
\mu[\tilde{H}, \tilde{F}] = \frac{1}{2} B'\left( -\frac{\omega}{4R[\tilde{H}, \tilde{F}] \right)
\]

and

\[
\lambda[\tilde{H}, \tilde{F}] = \pm \exp\left[ \frac{\omega}{2} \frac{\mu}{R[\tilde{H}, \tilde{F}] \right] + B\left( -\frac{\omega}{4R[\tilde{H}, \tilde{F}]} \right) \right].
\]

$B'(-1/2R)$ denotes the total derivative of $B(-1/2R)$ with respect to $(-1/2R)$. Equations (B.3) and (B.4) provide an implicit solution of the representative equation (2.6)—depending on one arbitrary function of one variable. From the arguments in the previous sections one expects this solution to be the general solution of (2.6)—independently of the value of $\omega$. In the present form—and using the weight-two ansatz map—the choice $B(-1/2R) = 0$ gives the Kuchař–Romano combination, while the choice $B(-1/2R) = -\ln(-1/2R)$ produces the Brown–Kuchař one.
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