Novel Deformation Function Creating or Destroying any Number of Even Kink Solutions

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Abstract: We present a one-parameter family of deformation functions $f(\phi)$ which have novel properties. Firstly, the deformation function is its own inverse. We show that a class of potentials remains invariant under this deformation. Further, when applied to a certain class of kink bearing potentials, one obtains potentials which are not bounded from below. Besides, we show that there is a wide class of potentials having power law kink tails which are connected by this deformation but the corresponding kink tails of the two potentials have different asymptotic behavior. Finally, we show that when this deformation function is applied to an appropriate one-parameter family of potentials having two kink solutions, it creates new potentials with an arbitrary even number ($2n$) of kink solutions. Conversely, by starting from this potential with $2n$ kink solutions the deformation function annihilates $2n - 2$ kinks and we get back the potential which we started with having only two kink solutions.

1 Introduction

During the last few years there has been a resurgence in studying kink solutions with power-law tails in several higher (than 6th) order field theory models including $\phi^8$, $\phi^{10}$, and $\phi^{12}$ [1 2] and even $\phi^{14}$, $\phi^{16}$, and $\phi^{18}$ [3]. The study of higher order field theories, their attendant kink excitations as well as the associated kink interactions and scattering are important in a variety of physical contexts ranging from...
successive phase transitions \[1, 2, 4, 5\] to isostructural phase transitions \[6\] to models involving long-range interaction between massless mesons \[7\], as well as from protein crystallization \[8\] to successive phase transitions presumably driving the late time expansion of the Universe \[9\]. Thus, it is of interest to study kink solutions in additional higher order field theories and study the various properties of these solutions including the nature of the kink-kink and the kink-antikink forces and scattering in these theories.

In this context we mention that almost two decades ago, in an interesting paper, Bazeia, Losano and Malbonisson \[10\] introduced the idea of deformation function which takes one from a kink bearing potential to another kink bearing (or in some cases pulse bearing) potential. In a subsequent series of papers, these and other authors \[11, 12, 13, 14\] studied a variety of deformation functions and also discussed possible applications of some of these deformation functions.

We mention in particular the deformation function \(f(\phi) = (1 - \phi^2)^{1/2}\) introduced in \[12\] where the authors briefly discussed some of its properties. The purpose of this paper is to discuss a one-parameter family of deformation functions

\[f(\phi) = (1 - \phi^{2n})^{1/2n}, \quad n = 1, 2, 3, \ldots,\]  

where \(n\) is any positive integer. In particular we point out its several remarkable properties.

The plan of the paper is the following. In Sec. II we set up the formalism for the kink solutions in a neutral scalar field theory in 1 + 1 dimensions and then spell out the basic steps of the deformation prescription. We then discuss some of the interesting properties of the deformation function \[1\]. In particular, if under the deformation \[1]\), a scalar potential \(V_0(\phi)\) goes to another potential \(V_1(\phi)\), then the same deformation function acting on \(V_1(\phi)\) gives back the original potential \(V_0(\phi)\) i.e. \(V_0\) and \(V_1\) are deformation-dual potentials. In other words, the deformation \[1\] is inverse of itself. We then exhibit a wide class of potentials \(V_0(\phi)\) which are invariant under this deformation; we call such potentials as self-deformed potentials. Further, we exhibit a wide class of kink bearing potentials \(V_0(\phi)\) which under the deformation \[1\] go to potentials \(V_1(\phi)\) which are not bounded from below. In Sec. III we then discuss a wide class of deformation-dual potentials both of which admit 2 kink solutions and show that in case at least one of the kink tails of \(V_0\) and \(V_1\) is a power law tail, then the power law tail behavior of \(V_0\) and the corresponding tail of \(V_1\) are different. This is unlike most of the examples discussed so far in the literature. We illustrate this
feature by discussing one example in detail. In Sec. IV we exhibit perhaps the most remarkable property of the deformation \[ 1 \]. In particular, we show that by appropriately choosing the potential \( V_0(\phi) \) with two kink solutions and by applying the deformation \( f(\phi) \), one can obtain the potential \( V_1 \) with as many even number \( 2m \) of kink solutions as desired. We discuss one example in detail to illustrate this aspect. Needless to say that, if instead we apply the deformation \( f(\phi) \) on \( V_1 \) having \( 2m \) kink solutions, we obtain the potential \( V_0 \) with two kink solutions. In other words, by an appropriate choice of the potential, one can create or destroy as many even number of kink solutions as desired. Finally, in Sec. V we summarize the main results obtained in this paper and point out a few open questions.

2 Formalism and Important Properties of the Deformation Function

Consider a relativistic neutral scalar field theory in \( 1 + 1 \) dimensions with the Lagrangian density being 
\[ L = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - \frac{1}{2} \left( \frac{d\phi}{dx} \right)^2 - V(\phi), \] (2)
which leads to the equation of motion
\[ \frac{d^2\phi}{dt^2} - \frac{d^2\phi}{dx^2} = \frac{dV}{d\phi}. \] (3)
We assume that the potential \( V(\phi) \) is smooth and nonnegative and attains its global minimum value of \( V = 0 \) for two or more values of \( \phi \) which are the global minima of the theory. Thus one has static kink and antikink solutions interpolating between the two adjoining global minima as \( x \) goes from \(-\infty\) to \(+\infty\).

While the field equation for a static kink is a second order ODE, it can be reduced to a first order ODE using the so called Bogomolnyi technique \[ 16 \]. The first order ODE is given by
\[ \frac{d\phi}{dx} = \pm \sqrt{2V(\phi)}. \] (4)

Let us now point out a few basic properties of the deformation formalism \[ 10, 14 \]. In particular, if we start with a potential \( V_0(\phi) \) and apply a deformation function \( f(\phi) \), then the new potential \( V_1(\phi) \) is related to the original potential \( V_0(\phi) \) by
\[ V_1(\phi) = \frac{V_0(\phi)}{(\frac{df(\phi)}{d\phi})^2}. \] (5)
Further, if the explicit kink solutions corresponding to the two potentials \( V_0(\phi) \) and \( V_1(\phi) \) are \( \phi^{(0)}_K(x) \) and \( \phi^{(1)}_K(x) \) respectively, then they are related by

\[
\phi^{(1)}_K(x) = f^{-1}[\phi^{(0)}_K(x)]. \tag{6}
\]

On the other hand, if instead of the explicit kink solution, only the implicit kink solutions are known and if corresponding to the potentials \( V_0(\phi) \) and \( V_1(\phi) \) they are given by \( x^{(0)}_K(\phi) \) and \( x^{(1)}_K(\phi) \) respectively, then

\[
x^{(1)}_K(\phi) = x^{(0)}_K(f[\phi]). \tag{7}
\]

Let us now discuss the various properties of the deformation function given by Eq. (1).

1. Notice that the inverse of this deformation is equal to itself, i.e.

\[
f^{-1}(\phi) = f(\phi). \tag{8}
\]

Stated differently, if we start with a potential \( V_0(\phi) \) and after applying the deformation function \( f(\phi) \) as given by Eq. (1) if we get the potential \( V_1(\phi) \), then we know that \( V_0 \) and \( V_1 \) are related to each other as given by Eq. (5). Similarly, if we apply the same deformation function to \( V_1(\phi) \), then we get back the potential \( V_0(\phi) \), i.e.

\[
V_0(\phi) = \frac{V_1(\phi)}{(df(\phi)/d\phi)^2}, \tag{9}
\]

where \( f(\phi) \) is given by Eq. (1). In that case, we call \( V_0 \) and \( V_1 \) as deformation-dual or deformation pair. As a result, the relation between the explicit kink solutions of the potentials \( V_0(\phi) \) and \( V_1(\phi) \) given by Eq. (6) can also be expressed as

\[
\phi^{(1)}_K(x) = f[\phi^{(0)}_K(x)]. \tag{10}
\]

2. For every integer \( n \), a wide class of potentials are invariant under the deformation (1). We call such potentials as self-dual potentials. Using Eq. (1) it is easy to check that for an arbitrary integer \( n \), the potential

\[
V_0(\phi) = \frac{1}{2} \phi^{4nm+2}[1 - \phi^{2n}]^{2m+2} = V_1(\phi), \quad m = 0, 1, 2, \ldots. \tag{11}
\]
We term these wide classes of potentials as self-deformed potentials. As an illustration, in the case of $n = 1$, the self-deformed one parameter family of potentials is

$$V_0(\phi) = \frac{1}{2} \phi^{4m+2}[1 - \phi^2]^{2m+2} = V_1(\phi), \quad m = 0, 1, 2, ... . \quad (12)$$

3. For every integer $n$, there is a two-parameter family of kink bearing potentials $V_0$ for which when we apply the kink deformation function (1) to them, the resulting deformed-dual potential $V_1$ is not bounded from below. In particular, for an arbitrary integer $n$, consider the two-parameter family of potentials

$$V_0(\phi) = \frac{1}{2} \phi^{4nm+2(2n-1)(n-1)(1 - \phi^{2n})^{2p+2}}, \quad m, p = 0, 1, 2, ..., \quad (13)$$

and if we apply the deformation function (1) to this $V_0$, then we find that the corresponding potential $V_1$ is given by

$$V_1(\phi) = \frac{1}{2} \phi^{4np+2(1 - \phi^{2n})^{2m+2n-1}}. \quad (14)$$

Notice that $V_1(\phi)$ is not bounded from below. As an illustration, consider the special case when $n = m = 1, p = 0$, in which case the deformed-dual potentials are

$$V_0(\phi) = \frac{1}{2} \phi^4(1 - \phi^2)^2, \quad V_1(\phi) = \frac{1}{2} \phi^2(1 - \phi^2)^3. \quad (15)$$

It is worth pointing out that the potential $V_0$ in Eq. (15) has recently received a lot of attention [7 17 18 19 20 21] in the context of kink solutions with a power law tail around $\phi = 0$ while an exponential tail around $\phi = \pm 1$. As is clear from Eq. (15), the corresponding dual potential $V_1(\phi)$ is not bounded from below.

### 3 Comparison of Kink Tails of the Deformed-Dual Potentials

We now consider for any arbitrary integer $n$, a two-parameter family of deformed-dual potentials given by

$$V_0(\phi) = \frac{1}{2} \phi^{4np+2(1 - \phi^{2n})^{2m+2}}, \quad p, m = 0, 1, 2, ... , \quad (16)$$

and

$$V_1(\phi) = \frac{1}{2} \phi^{4np+2(1 - \phi^{2n})^{2p+2}}. \quad (17)$$
Note that here \( p \neq m \) since in that case the two potentials \( V_0, V_1 \) are one and the same, i.e. it is the self-deformed potential. We note that for any set of integers \( p, m, n \), both the deformed-dual potentials (16) and (17) have two kinks and two antikink solutions. We now show that (for \( p \neq m \)) at least one of the kink (and hence the antikink) tails of \( V_0 \) and the corresponding kink tails of \( V_1 \) have different asymptotic behavior. In this context it is worth reminding that for most of the examples discussed in the literature, the kink tails of \( V_0 \) and the corresponding kink tails of \( V_1 \) are the same up to an overall factor.

In this context let us note the well known result that as \( x \) goes from \(-\infty\) to \(+\infty\), if a kink solution for a potential \( V(\phi) \) goes from 0 to \( b \) and if \( V(\phi) \) behaves as \( \phi^q \) around \( \phi = 0 \), then the kink tail around \( \phi = 0 \) has the asymptotic behavior

\[
\lim_{x \to -\infty} \phi_K(x) = x^{-(q-2)/2},
\]

where \( q \) is a real number. Thus if \( q = 2 \) then the kink tail is exponential while for \( q > 2 \) the kink has a power law tail. Let us now use this result and look at the behavior of the kink solution around \( \phi = 0 \) and \( \phi = 1 \) for the dual potentials given above. Using Eq. (18), it is straightforward to show that for the potential \( V_0(\phi) \) as given by Eq. (16), for the kink solution between 0 and 1, the behavior of the kink tail is given by

\[
\begin{align*}
\lim_{x \to -\infty} \phi_{K}^{(0)}(x) &= \frac{A_0}{x^p}, \\
\lim_{x \to \infty} \phi_{K}^{(0)}(x) &= 1 - \frac{A_1}{x^m}.
\end{align*}
\]

On the other hand, for the potential \( V_1(\phi) \) as given by Eq. (17), for the kink solution between 0 and 1, the behavior of the kink tail is given by

\[
\begin{align*}
\lim_{x \to -\infty} \phi_{K}^{(0)}(x) &= \frac{B_0}{x^m}, \\
\lim_{x \to \infty} \phi_{K}^{(0)}(x) &= 1 - \frac{B_1}{x^p}.
\end{align*}
\]

Here \( A_0, A_1, B_0, B_1 \) are constants. Since under the deformation (1), while \( \phi = 0 \) and \( \phi = 1 \) of \( V_0(\phi) \) get mapped to \( \phi = 1 \) and \( \phi = 0 \) respectively of \( V_1(\phi) \), one would expect that the behavior of the kink tail around \( \phi = 0(1) \) for the potential \( V_0(\phi) \) should be the same as that of the kink tail around \( \phi = 1(0) \) for the deformation-dual potential \( V_1(\phi) \). However it is clear from Eqs. (19) and (20) that this is not so and the asymptotic behavior of the kink tails in the two dual potentials (16) and (17) is very different unless \( p = m \) but in that case the two potentials are self-deformed (i.e. \( V_1(\phi) = V_0(\phi) \)). It is worth pointing out that in case, say \( p = 0 \) but \( m \neq 0 \), then the kink tail of \( V_0(\phi) \) around \( \phi = 0 \) and the kink tail of \( V_1(\phi) \)
around $\phi = 1$ are both exponential while the kink tail of $V_0(\phi)$ around $\phi = 1$ and the kink tail of $V_1(\phi)$ around $\phi = 0$ are both power law tails having different asymptotic behavior. On the other hand when both $p, m > 0$ but $p \neq m$, then both the kink tails of $V_0(\phi)$ are different from the corresponding kink tails of $V_1(\phi)$.

For simplicity we now explicitly discuss the case of $n = m = 1, p = 0$ and demonstrate that indeed the appropriate kink tails of the two deformed-dual potentials are different. In case $n = m = 1, p = 0$, $V_0(\phi)$ and $V_1(\phi)$ are given by

$$V_0(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^4,$$

$$V_1(\phi) = \frac{1}{2} \phi^6 (1 - \phi^{2n})^2.$$  

These two potentials are depicted in Fig. 1 with three degenerate minima each. The kink solutions in both the cases can be implicitly obtained.

![Figure 1: Potential $V_0(\phi)$ in Eq. (21) and its dual potential $V_1(\phi)$ in Eq. (22) for $n = 1$.](image)

For example, in order to obtain the kink solution of $V_0(\phi)$ as given by Eq. (21) from 0 to 1, we need to solve the self-dual equation

$$\frac{d\phi}{dx} = \phi(1 - \phi^2)^2.$$  

(23)
This equation is easily integrated using partial fractions and we obtain

\[ x^{(0)} = \frac{1}{2(1 - \phi^2)} + \frac{1}{2} \ln \left[ \frac{\phi^2}{(1 - \phi^2)} \right]. \]  

(24)

It then immediately follows that for the potential \( V_0(\phi) \) as given by Eq. (21), the asymptotic behavior of the kink solution from 0 to 1 is given by

\[ \lim_{x \to -\infty} \phi^{(0)}_K(x) = e^{(x-1/2)}, \quad \lim_{x \to \infty} \phi^{(0)}_K(x) = 1 - \frac{1}{4x}. \]  

(25)

The kink solution given by Eq. (24) is shown in Fig. 2 with the exponential and power law tails clearly visible at the two ends.

![Figure 2: Kink solution obtained by inverting implicit Eq. (24).](image)

In order to obtain the kink solution of \( V_1(\phi) \) as given by Eq. (22) from 0 to 1, we need to solve the self-dual equation

\[ \frac{d\phi}{dx} = \phi^3(1 - \phi^2). \]  

(26)
This equation is easily integrated using partial fractions and we obtain

\[ x^{(1)} = -\frac{1}{2\phi^2} + \frac{1}{2} \ln \left( \frac{\phi^2}{(1 - \phi^2)} \right). \]  

(27)

It then immediately follows that for the potential \( V_1(\phi) \) as given by Eq. (22), the asymptotic behavior of the kink solution from 0 to 1 is given by

\[ \lim_{x \to -\infty} \phi_K^{(1)}(x) = \frac{1}{\sqrt{-2x}}, \quad \lim_{x \to \infty} \phi_K^{(1)}(x) = 1 - \frac{1}{2} e^{-2(x+1/2)}, \]  

(28)

so that while the kink tail of \( V_0 \) goes like \( x^{-1} \) around \( \phi = 1 \), the kink tail of \( V_1 \) around \( \phi = 0 \) goes like \( (-x)^{1/2} \). The kink solution given by Eq. (27) is shown in Fig. 3 with the power law and exponential tails clearly visible at the two ends. Needless to say that alternatively we can also calculate the kink tails of the potential (22) by using Eqs. (7) and (24) and we would get the same kink tails as given by Eq. (28) up to an overall factor.

Figure 3: Kink solution obtained by inverting implicit Eq. (27).
4 Kink Creating and Kink Annihilating Deformation

Finally, we discuss perhaps the most interesting property of the deformation function (1). In particular, by a judicious choice of the potential $V_0(\phi)$ having two kink solutions, this deformation can create a new potential $V_1(\phi)$ having as many even number of kink solutions as desired. We give below three wide classes of $V_0$ and $V_1$ satisfying this property.

Let us start from the potential

$$V_0(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^2 (b_1^{2n} + \phi^{2n})^2 (b_2^{2n} + \phi^{2n})^2 (b_3^{2n} + \phi^{2n})^2 ... (b_m^{2n} + \phi^{2n})^2,$$  

which has two kink and two antikink solutions. On applying the deformation function (1) on this $V_0(\phi)$ we obtain the deformed potential

$$V_1(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^2 (b_1^{2n} + 1 - \phi^{2n})^2 (b_2^{2n} + 1 - \phi^{2n})^2 ... (b_m^{2n} + 1 - \phi^{2n})^2,$$

which clearly has $2m + 2$ kink solutions and $2m + 2$ antikink solutions. Thus in this case, for $V_0(\phi)$ as given by Eq. (29), the deformation function (1) acts like a kink (and antikink) creating operator, creating extra $2m$ kink solutions. From here it immediately follows that, as far as the potential $V_1(\phi)$ as given by Eq. (30) is concerned, the deformation function (1) acts like a kink (and antikink) annihilating operator destroying $2m$ kinks out of its $2m + 2$ kinks.

One can generalize the potential $V_0$ given by Eq. (29) and choose more general potentials of the form

$$V_0(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^2 (b_1^{2n} + \phi^{2n})^{2p_1} (b_2^{2n} + \phi^{2n})^{2p_2} ... (b_m^{2n} + \phi^{2n})^{2p_m},$$

where $p_1, p_2, ..., p_m$ are arbitrary positive integers. This $V_0$ obviously has has two kink and two antikink solutions. On applying the deformation function (1) on this $V_0(\phi)$ we obtain the deformed potential

$$V_1(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^2 (b_1^{2n} + 1 - \phi^{2n})^{2p_1} (b_2^{2n} + 1 - \phi^{2n})^{2p_2} ... (b_m^{2n} + 1 - \phi^{2n})^{2p_m},$$

which clearly has $2m + 2$ kinks and $2m + 2$ antikink solutions.

We can further generalize the potential $V_0(\phi)$ of Eq. (31) and choose more general potentials of the form

$$V_0(\phi) = \frac{1}{2} \phi^{4np+2} (1 - \phi^{2n})^{2k+2} (b_1^{2n} + \phi^{2n})^{2p_1} (b_2^{2n} + \phi^{2n})^{2p_2} ... (b_m^{2n} + \phi^{2n})^{2p_m},$$

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where $p, k = 0, 1, 2, \ldots$ while $p_1, p_2, \ldots, p_m$ are arbitrary positive integers. Clearly this $V_0$ has two kinks and two antikink solutions. On applying the deformation function $\mathcal{D}$ on this $V_0(\phi)$, we obtain the deformed potential

$$V_1(\phi) = \frac{1}{2} \phi^{4n_k+2}(1 - \phi^{2n})^{2p+2}(b_{1}^{2n} + 1 - \phi^{2n})^{2p_1}(b_{2}^{2n} + 1 - \phi^{2n})^{2p_2} \ldots (b_{m}^{2n} + 1 - \phi^{2n})^{2p_m},$$

which clearly has $2m + 2$ kink solutions.

Remarkably, there are several potentials $V_0$ and $V_1$ in which kink solutions are explicitly or implicitly known and they fall in the above category. As an illustration we discuss two such examples, in one of which while $V_0$ has two kink solutions which are explicitly known, the corresponding $V_1$ has four kink solutions which are all explicitly known as well. In another case, while $V_0$ has two kink solutions which are implicitly known, the corresponding $V_1$ has six kink solutions all of which are implicitly known as well.

**Example I:**

Few years ago, we along with Christov [2] had explicitly obtained two kink and two antikink solutions for the one-parameter family of potentials

$$V_0(\phi) = \frac{1}{2} \phi^2(1 - \phi^{4n})^{2}, \quad n = 1, 2, 3, \ldots.$$  

(35)

On the other hand, very recently, two of us [22] have obtained explicit four kink and four antikink solutions of the model

$$V_1(\phi) = \frac{1}{2} \phi^2(1 - \phi^{2n})^{2}(2 - \phi^{2n})^{2}, \quad n = 1, 2, 3, \ldots.$$  

(36)

Note that these $V_0$ and $V_1$ are in fact special cases of the potentials given in Eqs. (29) and (30), respectively, in case $b_1 = 1$ while $b_2 = \ldots = b_m = 0$ and hence $V_0(\phi)$ and $V_1(\phi)$ as given by Eqs. (35) and (36) are deformation-dual. The potentials given by Eqs. (35) and (36) for $n = 1$ are depicted in Fig. 4 with three and five degenerate minima, respectively.

As has been shown in [1], for the potential (35) with arbitrary $n$, the kink from 0 to 1, the corresponding mirror kink from $-1$ to 0 and the corresponding two antikink solutions are given by

$$\phi_{0K,0aK}(x) = \pm \left[1 \pm \tanh(2nx)\right]^{1/4n}.$$  

(37)
Figure 4: Potential $V_0(\phi)$ in Eq. (35) and its dual potential $V_1(\phi)$ in Eq. (36) for $n = 1$.

For $n = 1$ this kink solution is depicted in Fig. 5. Having obtained all the kink and the antikink solutions of the potential (35) from 0 to 1, the corresponding kink, mirror kink and the two anti-kink solutions of the dual potential (36) are then immediately obtained from the deformation formalism. In particular using Eq. (10) we obtain

$$\phi_{1K,1aK}(x) = \pm \left[ 1 - \phi_{0K,0aK}(x) \right]^{1/4n} = \pm \left[ 1 - \sqrt{\frac{1 \pm \tanh(2nx)}{2}} \right]^{1/4n}.$$  \hfill (38)

For $n = 1$ his kink solution is depicted in Fig. 6. It is worth pointing out that these are precisely the explicit kink solutions that we have recently obtained [22] for the potential (36). Thus, one has found a novel relationship between the two seemingly very different potentials, one admitting two while the other admitting four kink solutions.

Unfortunately, at present we do not know how to obtain the kink solution from 1 to $(2)^{1/2n}$, the corresponding mirror kink and the corresponding two antikink solutions of potential (36) from the kink solution of the deformed-dual potential (35) since it does not have a kink solution from 1 to $(2)^{1/2n}$.

Example II:
We have recently obtained [22] implicit six kink and six antikink solutions of the model characterized by the potential

\[
V_1(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{2n})^2 (2 - \phi^{2n})^2 (3 - \phi^{2n})^2, \quad n = 1, 2, 3, ..., \tag{39}
\]

without realizing that this potential is deformation-dual to the potential

\[
V_0(\phi) = \frac{1}{2} \phi^2 (1 - \phi^{4n})^2 (2 + \phi^{2n})^2, \quad n = 1, 2, 3, ..., \tag{40}
\]

i.e. the above \(V_1(\phi)\) can be obtained from the \(V_0(\phi)\) as given by Eq. (40) by applying the deformation function \([\Pi]\) on it. Potentials given by Eqs. (39) and (40) are shown in Fig. 7 for \(n = 1\) with seven and three degenerate minima, respectively.

In order to obtain the kink solution of \(V_0(\phi)\) as given by Eq. (40) from 0 to 1, we need to solve the self-dual equation

\[
\frac{d\phi}{dx} = \phi(1 - \phi^{4n})(2 + \phi^{2n}). \tag{41}
\]
This equation is easily integrated using partial fractions and we obtain
\[ x^{(0)} = \frac{1}{12} \ln \left[ \frac{\phi^{6n}(2 + \phi^{2n})}{(1 - \phi^{2n})(1 + \phi^{2n})^3} \right]. \tag{42} \]

It then immediately follows that for the potential \( V_0(\phi) \) as given by Eq. (40), the asymptotic behavior of the kink solution from 0 to 1 is given by
\[ \lim_{x \to -\infty} \phi_K^{(0)}(x) = (2)^{-1/6}e^{2x/n}, \quad \lim_{x \to \infty} \phi_K^{(0)}(x) = 1 - \frac{3}{16n}e^{-12x}. \tag{43} \]

The kink solution given by Eq. (42) for \( n = 1 \) is depicted in Fig. 8 with both exponential tails clearly visible at the two ends.

In order to obtain the kink solution of \( V_1(\phi) \) as given by Eq. (39) from 0 to 1, we need to solve the self-dual equation
\[ \frac{d\phi}{dx} = \phi(1 - \phi^{2n})(2 - \phi^{2n})(3 - \phi^{2n}). \tag{44} \]

This equation is easily integrated using partial fractions and as we have shown in our recent paper [22] we
obtain

\[ x^{(1)} = \frac{1}{12} \ln \left[ \frac{\phi^{2n}(2 - \phi^{2n})^3}{(1 - \phi^{2n})^3(3 - \phi^{2n})} \right]. \tag{45} \]

It then immediately follows that for the potential \( V_1(\phi) \) as given by Eq. (39), the asymptotic behavior of the kink solution from 0 to 1 is given by

\[ \lim_{x \to -\infty} \phi^{(0)}_{K}(x) = (3/8)^{1/2n} e^{6x/n}, \quad \lim_{x \to \infty} \phi^{(0)}_{K}(x) = 1 - \frac{1}{24/3n} e^{-4x}. \tag{46} \]

The kink solution given by Eq. (45) for \( n = 1 \) is depicted in Fig. 9 with both exponential tails clearly visible at the two ends.

Unfortunately, at present we do not know how to obtain the kink solutions from 1 to \( (2)^{1/2n} \) and from \( (2)^{1/2n} \) to \( (3)^{1/2n} \), the corresponding mirror kinks and the corresponding four antikink solutions of the potential (39) from the kink solutions of the deformed-dual potential (40), since it only has a kink solution from 0 to 1 and the corresponding mirror kink solution.
5 Conclusions and Open Questions

In this paper we have discussed several novel properties of the deformation function as given by Eq. (1). We have shown that this deformation function is its own inverse. As a result, as shown by us, apart from a wide class of self-deformed potentials (which are invariant under this deformation), all other potentials break up into deformed pairs. As also shown by us, for a wide class of kink bearing potentials, the corresponding deformed-dual potential is not bounded from below. On the other hand, there is a wide class of deformed pair of potentials both of which admit two kink and two antikink solutions and for which at least one of the kink tails has a power law tail. In such cases, as shown by us, the appropriate kink tails of the two deformed pairs with power law tail have different asymptotic behavior. As a result the kink-kink and the kink-antikink forces in the two deformed pair of potentials are very different. Finally, we have pointed out what we consider to be the most remarkable feature of the deformation function (1), i.e. it can act as an arbitrary even number $2m$ of kink creating or kink annihilating function. That is,
starting from a wide class of potentials with two kinks, this deformation can take one to the corresponding deformed-dual potential having arbitrary even \(2m + 2\) number of kink solutions. Conversely, by starting from the deformed-dual potential with \(2m + 2\) kink solutions and applying the deformed function \(f\), one goes to its partner potential having only two kink solutions.

This paper raises several interesting questions, some of which are as follows.

1. In this paper we have considered a deformation function which is its own inverse, i.e. naively it is a deformation with period two. Generalizing, are there deformation functions of period \(2^n\), where \(n\) is any positive integer? If yes, what are its key properties? Can it also act as a kink creating and/or kink annihilating deformation? How does this deformation function behave for large \(n\)?

2. Are there deformation functions of prime number (like 3, 5, 7, 11, ...) period as well as an arbitrary multiple of them? If yes, what are the properties of such deformation functions?

3. Are there deformations which can act as a kink creating function with creating, say \(n, 2n, 3n \ldots\) kink
solutions or conversely can it act as a kink annihilating function with reducing \( n, 2n, 3n, \ldots \) number of kink solutions?

4. So far the various deformation functions suggested in the literature are such that the qualitative nature of the kink tail in a given potential and the corresponding appropriate kink tail of the potential obtained after the deformation are similar, i.e. both could be exponential or both could be power law tail or other tails, such as super-exponential \[23\], etc. Is there a deformation for which the qualitative nature of the appropriate kink tails are qualitatively different?

It would be quite insightful if one could answer some of the questions raised here.

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