Analysis of the probability distribution of photocount number of the onemode stochastic radiation
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1. Introduction

For the description of the registration process of low-intensive electromagnetic radiation by counters of photons, it is necessary to take into account quantum effects. Thus, even in the case when the electromagnetic field does not contain any noise component, i.e. its amplitude represents the pure quantum state, the number $\tilde{n}$ of registered photons is random and there exists the problem of determination of the probability distribution of this random variable. This problem becomes very complicated if the registered electromagnetic field is the mixture of deterministic field and a noise. In this case, its state is statistically mixed and, therefore, it is necessary to use the appropriate density matrix. From theoretical point of view, the complexity of the description of such a state consists of the adequate choice of mathematical model of the electromagnetic noise and of the construction on its basis the appropriate density matrix. Simplification of this problem arises at sufficiently large typical frequencies of the electromagnetic field. In this case, one may use the quasi-classical approximation as it is shown in [1]. Therefore, probabilities of registered photon numbers are calculated on the basis of the classical (not quantum) probability distribution

$$P_n \equiv \Pr\{\tilde{n} = n\} = \frac{1}{n!} E \tilde{J}^n \exp[-\tilde{J}],$$

(1)

which represents the so-called composite Poisson distribution. It is referred to the Mandel distribution in quantum optics [2]. Here, $\tilde{J}$ is the random variable representing the energy of electromagnetic field absorbed during the registration time $T$.

Let us consider the model of the quantum photocounter [1] of the onemode electromagnetic radiation being completely noise. In this case, the appropriate model of the electromagnetic noise is the complex Ornstein-Uhlenbeck process as it is proposed in the photodetection theory. Then the random variable $\tilde{J}$ is represented by the formula [1]

$$\tilde{J} \equiv J[\tilde{\zeta}] = \int_0^T |\tilde{\zeta}(s)|^2 ds$$
where \( \tilde{\zeta}(s) = \tilde{\xi}(s) + i\tilde{\eta}(s) \), \( s \in \mathbb{R} \) are trajectories of the complex process connected with real Ornstein-Uhlenbeck’s processes \( \tilde{\xi} = \{\xi(t); t \in \mathbb{R}\} \), \( \tilde{\eta} = \{\eta(t); t \in \mathbb{R}\} \) being stochastically equivalent and independent. From the physical point of view, they correspond accordingly to electric and magnetic constituents of the noise electromagnetic field.

Ornstein-Uhlenbeck’s processes are markovian and gaussian and they are completely characterized by these properties and their stationary condition. This class of processes is parametrized by two numbers \( \nu > 0 \), \( \sigma > 0 \). Each Ornstein-Uhlenbeck process is completely determined by the following formula of the conditional probability density \( w(x_0, t_0|x, t) \) of the transition from the point \( x_0 \in \mathbb{R} \) at \( t_0 \in \mathbb{R} \) to the point \( x \in \mathbb{R} \) at \( t \in \mathbb{R} \). It depends on parameters \( \nu, \sigma \) and has the following form [3]

\[
w(x_0, t_0|x, t) = \left( \frac{\nu}{\pi \sigma (1 - e^{-2\nu|t-t_0|})} \right)^{1/2} \exp \left( -\frac{\nu [x - x_0 e^{-\nu|t-t_0|}]^2}{\sigma (1 - e^{-2\nu|t-t_0|})} \right).
\]

Thus, the one-point distribution density \( w(x), x \in \mathbb{R} \) of the process is determined by the formula

\[
w(x) = \lim_{t_0 \to -\infty} w(x_0, t_0|x, t) = \left( \frac{\nu}{\pi \sigma} \right)^{1/2} \exp \left( -\frac{\nu x^2}{\sigma} \right). \tag{3}
\]

The characteristic function \( Q(-i\lambda), \lambda \in \mathbb{R} \) of the random variable \( J[\tilde{\xi}] \) of the process \( \tilde{\xi} \) is given by the known Ziegert formula [4],

\[
Q_{\tilde{\xi}}(\lambda) = \mathbb{E} \exp(-\lambda J[\tilde{\xi}]) = \left( \frac{4r\nu \exp(\nu T)}{(r + \nu)^2 \exp(r T) - (r - \nu)^2 \exp(-r T)} \right)^{1/2}, \tag{4}
\]

where \( r = \sqrt{\nu^2 + 2\lambda \sigma} \).

Since processes \( \{\tilde{\xi}(t)\}, \{\tilde{\eta}(t)\} \) are independent and equivalent, the generating function of the random variable \( J[\tilde{\xi}] \) is found on the basis of equalities

\[
Q(\lambda) = Q_{\tilde{\xi}}(\lambda)Q_{\tilde{\eta}}(\lambda) = Q_{\tilde{\xi}}^2(\lambda).
\]

From here, we see that the Mandel distribution \( P_n \) determined by the formula (1) and by the probability distribution of the random variable \( \tilde{J} \) induced by the probability distribution of the process \( \tilde{\xi} = \{\z(t); t \in \mathbb{R}\} \) are very complex. It is easy to obtain the asymptotic formula of the probability
distribution $P_n$ at $T \to 0$. It has the form $P_n/P_n^{(0)} \to 1$ where $P_n^{(0)}$ is the following Poisson distribution [1]

$$P_n^{(0)} = \frac{1}{n!} \left( \frac{\sigma T}{\nu} \right)^n \exp \left( -\frac{\sigma T}{\nu} \right).$$

The purpose of the present work is the construction of the convenient calculation algorithm giving probabilities $P_n$ with the guaranteed accuracy at the sufficiently small value of $T$ on the basis of formulas (1)-(4). Our approximations are based on the simple idea of the series expansion of the Mandel distribution, i.e. we represent it in the form

$$P_n = \frac{1}{n!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \mathbb{E} \tilde{J}^{n+l}. \quad (5)$$

It is easy to show that each moment $\mathbb{E} \tilde{J}^n$, $n \in \mathbb{N}$ is proportional to $T^n$ at $T \to 0$. Then one may expect that just such an expansion is appropriate for the solution of the above-mentioned problem at small values of $T$. On this way, we should solve two questions. The first, it is necessary to give the general foreseeable formula of moments $\mathbb{E} \tilde{J}^n$ depending on the number $n$ as on the parameter. It appears that it is rather problematic to obtain such a formula. Instead of this, it is possible, however, to point out the specific algorithm of the consecutive calculation of these values. One may think that this algorithm is the solution of the first problem. The second question consists of the estimation of the series remainder

$$\sum_{l=N+1}^{\infty} \frac{1}{l!} \mathbb{E} \tilde{J}^{n+l} \quad (6)$$

for any $N \in \mathbb{N}$. The presence of such an estimation permits to give the guaranteed accuracy of the approximation $P_n^{(N)}$ based on the account only of first $N$ terms in the series (5). Below, we solve these two problems.

2. Sequence algebra of power series coefficients

Let us introduce into consideration the linear manifold $\mathcal{L}_f = \{f(z)\}$ of functions depending on the variable $z \in \mathbb{C}$. Each of them is analytic in the appropriate circle having nonzero radius and the centre in the point 0.
Thus, each function of \( L_f \) is presented as the series
\[
f(z) = \sum_{k=0}^{\infty} a_k z^k
\] (7)
and each such a series is completely determined by the sequence of coefficients \( \{a_k \in \mathbb{C}; k \in \mathbb{N}_+\} \) which is regarded as the infinite ordered collection. In this connection, we shall consider also the linear manifold \( L = \{A\} \) of coefficient sequences \( A = \{a_k; k \in \mathbb{N}_+\} \) corresponding to serieses (7). We shall name elements of such sequences as *components* also. Linear operations on manifold \( L \) are introduced by the natural way. Namely, let \( A = \{a_k; k \in \mathbb{N}_+\} \) and \( B = \{b_k; k \in \mathbb{N}_+\} \) be two sequences. Then sequences \( A + B \) and \( \lambda A \) for any \( \lambda \in \mathbb{C} \) are determined by formulas
\[
A + B = \{a_k + b_k; k \in \mathbb{N}_+\} \quad \lambda A = \{\lambda a_k; k \in \mathbb{N}_+\}.
\]

There exists the natural one-to-one correspondence between introduced manifolds \( L \leftrightarrow L_f \) determined by the formula (7). We designate the mapping \( L \mapsto L_f \) generated by this correspondence using the letter \( F \). Further, let us designate the power series of \( f(z) \) being the image of the element \( A \in L \) under the mapping \( F : L \mapsto L_f \) by \( F[z|A] \). The mapping \( F : L \mapsto L_f \) is obviously linear, i.e. following relations take place
\[
F[z|A + B] = F[z|A] + F[z|B], \quad F[z|\lambda A] = \lambda F[z|A].
\]

For each sequence \( A \in L \), it is possible to consider each of its component \( a_k, k \in \mathbb{N}_+ \) as the appropriate projection of the infinite ordered collection \( A \). We shall write down this fact by means \( a_k = (A)_k, k \in \mathbb{N}_+ \).

Let us introduce the binary commutative operation on \( L \) which we shall name the *convolution* of sequences. We designate it by the symbol \( \circ \). This operation is determined as follows. With any pair of sequences \( A, B \in L \), we associate the sequence \( A \circ B \) having components
\[
(A \circ B)_k = \sum_{j=0}^{k} a_j b_{k-j} \quad k = 0, 1, 2, \ldots
\]
It is easy to see that the associativity property is fulfilled for the introduced convolution operation applied for any three elements \( A, B, C \in L \),
\[
(A \circ B) \circ C = A \circ (B \circ C).
\]
It concerns also the distributivity property relatively the addition,
\[(A + B) \circ C = A \circ C + B \circ C.\]

Besides, the following relation
\[(\lambda A) \circ B = \lambda (A \circ B)\]

takes place for any pair \(A, B \in \mathfrak{L}\) and for any \(\lambda \in \mathbb{C}\). Thus, these equalities together with the commutative property of the convolution operation permits to conclude that the linear manifold \(\mathfrak{L}\) equipped with the "multiplication" \(\circ\) turns into the commutative algebra which we shall designate by the same symbol \(\mathfrak{L}\).

In the algebra \(\mathfrak{L}\), there exists the unity \(E\) which is represented by the collection \(E = \langle 1, 0, 0, \ldots \rangle\) since for any \(A \in \mathfrak{L}\), the following relation takes place
\[A \circ E = E \circ A = A.\]

Let us notice that the subalgebra \(\mathfrak{L}_0 = \{ A : (A)_0 = 0 \}\) of the algebra \(\mathfrak{L}\) is its ideal, i.e. it takes place \(A \circ B \in \mathfrak{L}_0\) for any \(A \in \mathfrak{L}_0\) and for any \(B \in \mathfrak{L}\).

There exists the inverse element \(A^{-1}\) for any element \(A \notin \mathfrak{L}_0\) which has the property formulated by the following way
\[A^{-1} \circ A = A \circ A^{-1} = E.\]

One may find consecutively each its component by the equality system
\[
(A^{-1})_0 = a_0^{-1} \\
(A^{-1})_n = -a_0^{-1} \sum_{k=1}^{n-1} (A^{-1})_{n-k} a_k \quad n = 1, 2, \ldots.
\]

Let us introduce the following reduced designation of powers of any element \(A \in \mathfrak{L}\). We shall write
\[A^0_o = E, \quad A^1_o = A, \quad A \circ A = A^2_o, \quad \ldots, A^n_o \circ A = A^{n+1}_o, \quad n = 0, 1, 2, \ldots.
\]

Each component of the \(n\)th power of the element \(A = \langle a_m; m \in \mathbb{N}_+ \rangle\) is obtained by the formula
\[
(A^n_o)_m = \sum_{k_1 + \ldots + k_n = m} a_{k_1} \ldots a_{k_n} \quad m = 1, 2, \ldots
\]
Thus, it is obvious that if \( A \in \mathcal{L}_0 \) then the formula

\[
(A_n^n)_m = \sum_{n > j_1, \ldots, j_n \geq 1, \ j_1 + \cdots + j_n = m} a_{j_1} \cdots a_{j_n}
\]  

(9)
takes place. Therefore, the relation

\[
(A_n^n)_m = 0 \tag{10}
\]

is fulfilled for all \( m > n \).

The functional \( F[z|A] \) is multiplicative relatively the introduced multiplication operation, i.e. the identity

\[
F[z|A \circ B] = F[z|A] F[z|B] \tag{11}
\]

takes place for any pair of collections \( A \) and \( B \). It is ascertained by the following transformations

\[
F[z|A \circ B] = \sum_{m=0}^{\infty} z^m (A \circ B)_m = \sum_{m=0}^{\infty} \left( \sum_{k=0}^{m} z^k a_k \right) \left( \sum_{m-k=0}^{\infty} z^{m-k} b_{m-k} \right) = \left( \sum_{k=0}^{\infty} z^k a_k \right) \left( \sum_{m=0}^{\infty} z^m b_m \right) = F[z|A] F[z|B].
\]

They are correct in the common convergence area of both series \( F[z|A] \) and \( F[A|B] \).

On the algebra \( \mathcal{L} \), it is possible to consider ”analytical” functions \( f_\circ(A) \). Each of them is defined for every \( A \in \mathcal{L} \) by means of the corresponding ”power” series

\[
f_\circ(A) = \sum_{n=0}^{\infty} c_n A_\circ^n, \quad c_n \in \mathbb{C}, \quad n \in \mathbb{N}_+.
\]  

(12)

Naturally, it has the sense under the condition of the componentwise convergence. In view of Eq.(10), such serieses are finite for each individual component of the function value \( f_\circ(A) \) if the element \( A \) is chosen in \( \mathcal{L}_0 \), i.e.

\[
(f_\circ(A))_n = \sum_{k=0}^{n} c_n (A_\circ^k)_n
\]
Therefore, the series (12) converges by the definition. In particular, we introduce the following function

$$(E - A)^{-1} = \sum_{n=0}^{\infty} A^n, \quad (13)$$

by means of the formula (12), which is defined at $A \in \mathcal{L}_0$. At last, for any function $f_\circ$ on elements of the algebra $\mathcal{L}$, the following formula is correct

$$F[z|f_\circ(A)] = f(F[z|A]). \quad (14)$$

It is due to linear and multiplicative properties of the functional $F[z|\cdot]$. Here, the function $f_\circ(\cdot)$ on $\mathcal{L}$ in the lefthand side is determined by means of Eq.(12) and the corresponding analytic function on $\mathbb{Z}$ determined by Eq.(7) in the righthand side is designated by $f(\cdot)$.

**3. Moments of the random variable $\tilde{J}$**

At first, let us consider the problem about the power expansion on $\lambda$ of the generating function $Q(\lambda) = \mathbb{E}e^{-\lambda \tilde{J}}$ of the random variable

$$\tilde{J} = \int_0^T |\tilde{\zeta}(t)|^2 \, dt.$$

This function is evaluated by the formula

$$Q(\lambda) = \frac{4r\nu \exp(\nu T)}{(r + \nu)^2 \exp(rT) - (r - \nu)^2 \exp(-rT)} \quad (15)$$

on the basis of Eq.(4). Here, $r = \sqrt{\nu^2 + 2\lambda\sigma} = \nu q$,

$$q = \left(1 + \frac{2\lambda\sigma}{\nu^2}\right)^{1/2}. \quad (16)$$

We represent the function $Q(\lambda)$ in the form

$$Q(\lambda) = e^T G^{-1}(\lambda)$$

where

$$G(\lambda) = Q^{-1}(\lambda)e^T = \frac{1}{4q} \left[ (1 + q)^2 e^{qT} - (q - 1)^2 e^{-qT} \right].$$
Here, we have introduced the "dimensionless" parameter $\nu T$ instead of $T$ which we designated hereafter by the same letter $T$ if it will not cause a misunderstanding.

At first, let us decompose the function $G(\lambda)$ in the series on $q$ powers. We have

$$G(\lambda) = (4q)^{-1} \left[ (1 + q)^2 \sum_{n=0}^{\infty} \frac{(qT)^n}{n!} - (q - 1)^2 \sum_{n=0}^{\infty} \frac{(-1)^n(qT)^n}{n!} \right] =$$

$$= (4q)^{-1} \left[ (1 + q^2) \sum_{n=0}^{\infty} \frac{(qT)^n}{n!} (1 - (-1)^n) + 2q \sum_{n=0}^{\infty} \frac{(-1)^n(qT)^n}{n!} (1 + (-1)^n) \right] =$$

$$= (2q)^{-1} \left[ (1 + q^2) \sum_{n=0}^{\infty} \frac{(qT)^{2n+1}}{(2n+1)!} + 2q \sum_{n=0}^{\infty} \frac{(qT)^{2n}}{(2n)!} \right] =$$

$$= \sum_{n=0}^{\infty} \frac{(qT)^{2n}}{(2n)!} \left[ 1 + \frac{T}{(2n+1)} \left( 1 + \frac{\lambda\sigma}{\nu^2} \right) \right].$$

Thus,

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} q^{2n} \left[ 1 + \frac{T}{(2n+1)} \left( 1 + \frac{\lambda\sigma}{\nu^2} \right) \right]. \quad (17)$$

Now, let us substitute the expression (16) of the variable $q = (1 + z)^{1/2}$, $z = 2\lambda\sigma/\nu^2$ into the formula (17),

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} (1 + z)^n \left[ 1 + \frac{T}{(2n+1)} \left( 1 + \frac{z}{2} \right) \right].$$

Then we use the binomial formula. As a result, we obtain

$$G(\lambda) = \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} \left[ \sum_{m=0}^{n} \binom{n}{m} z^m \right] \left[ 1 + \frac{T}{(2n+1)} \left( 1 + \frac{z}{2} \right) \right] =$$

$$= \sum_{m=0}^{\infty} \frac{z^m}{m!} (u_m + v_m) + \frac{1}{2} \sum_{m=0}^{\infty} \frac{z^{m+1}}{m!} v_m = \sum_{m=0}^{\infty} \frac{z^m}{m!} \left[ u_m + \frac{m}{2} v_{m-1} + v_m \right],$$

where

$$u_m = \sum_{n=m}^{\infty} \frac{T^{2n}}{(2n)! (n-m)!} \quad m \in \mathbb{N}_+,$$  \hfill (18)
\[ v_m = \sum_{n=m}^{\infty} \frac{T^{2n+1}}{(2n+1)!} \frac{n!}{(n-m)!} \quad m \in \mathbb{N}_+. \]  

Consider the sequence \( W = \langle w_m; m \in \mathbb{N} \rangle \) with components \( w_0 = 0 \) and

\[ w_m = \frac{1}{m!} e^{-T} (u_m + m v_{m-1}/2 + v_m), \quad m \in \mathbb{N}. \]  

One may note that \( e^{-T}(u_0 + v_0) = 1 \). Therefore, the expansion of the function \( G(\lambda) \) in the series on \( \lambda \) powers is presented in the following form

\[ G(\lambda) = e^T \left[ 1 + \sum_{m=1}^{\infty} z^m w_m \right] = e^T \left( 1 + F[z|W] \right), \quad z = \frac{2\lambda \sigma}{\nu^2}. \]  

Since \( (W)_0 = 0 \), the sequence \( W \) is the element of the algebra \( \mathfrak{L}_0 \). Therefore, having written down the decomposition (21) in the form

\[ G(\lambda) = e^T \left[ 1 - F[z|W] \right] \]  

and using firstly Eq.(22) and secondly Eq.(13), we obtain the following expansion of the function \( G^{-1}(\lambda) \) on \( \lambda \) powers,

\[ G^{-1}(\lambda) = e^{-T} (1 - F[z|W])^{-1} = e^{-T} F[z|X], \]  

where \( X = (E + W)^{-1} = \langle x_n; n \in \mathbb{N}_+ \rangle, \quad x_0 = 1, \)

\[ x_n = \sum_{l=1}^{n} (-1)^l \left( W^{(l)}_0 \right)_n, \quad n \in \mathbb{N}. \]  

Thus, we find

\[ Q(\lambda) = e^{T} G^{-1}(\lambda) = F[z|X] \]  

substituting the expansion (24) in the expression of the function \( Q(\lambda) \). Formula (24), in the combination with definition of the variable \( z \), permits to us to be convinced that the following statement is valid.

Theorem 1. Moments \( M_n, n \in \mathbb{N} \) of the random variable \( \bar{J} \) are presented by the formula

\[ M_n = (-1)^n n! \left( \frac{2\sigma}{\nu^2} \right)^n x_n, \quad n \in \mathbb{N} \]  

where coefficients \( x_n \) are determined by formulas (18) - (20), (23).
The formula (25) follows directly from the definition of moments $M_n$ on the basis of their generated function $Q(\lambda)$. Let us use the expansion (24) of $Q(\lambda)$. Then, for any $n \in \mathbb{N}$, we have

$$M_n = (-1)^n \left( \frac{\partial^n Q(\lambda)}{\partial \lambda^n} \right)_{\lambda=0} = (-1)^n \left[ \frac{\partial^n F[z|X]}{\partial z^n} \right]_{z=0} \left( \frac{dz}{d\lambda} \right)^n = (-1)^n n! \left( \frac{2\sigma}{\nu^2} \right)^n x_n. \quad \blacksquare$$

We note that each coefficient $x_n$ has the sign $(-1)^n$, $n = 1, 2, \ldots$ due to the positivity of moments $M_n$.

4. Estimations of moments

For the solution of the accuracy estimation problem of Mandel’s distribution approximations, it is necessary to find some a priori estimations of moments $M_n$ of the random variable $\tilde{J}$. This section is devoted to the obtaining of such estimations. We divide the obtaining in some simple steps.

Lemma 1. At $\alpha \in [0, 1/2]$, the following inequality takes place

$$\ln(1 - \alpha) \geq -2\alpha. \quad (26)$$

□ At $\alpha = 0$, Eq.(26) turns into the exact equality. At $\alpha \in [0, 1/2]$, the inequality

$$-\frac{1}{1 - \alpha} \geq -2$$

of derivatives of Eq.(26) both parts is valid. We obtain Eq.(26) integrating last inequality using the equality condition at $\alpha = 0$. ■

Lemma 2. The inequality

$$\frac{1}{(2n)!} \leq \frac{en}{2^{2n}(n!)^2} \quad (27)$$

takes place.

□ Following identical transformations are valid

$$(2n)! = 2^n n!(2n - 1)!! = 2^{2n}(n!)^2 \prod_{l=1}^{n} \left( 1 - \frac{1}{2l} \right) =$$
\begin{align}
\frac{2^n}{(n!)^2} \exp \left( \sum_{l=1}^{\infty} \ln \left( 1 - (2l)^{-1} \right) \right). \quad (28)
\end{align}

Let us use the inequality (26) for the estimation of the logarithm from below,
\begin{align}
\ln \left( 1 - (2l)^{-1} \right) \geq -l^{-1}, \quad l = 1, 2, \ldots, n. \quad (29)
\end{align}

We give upper estimation of the expression \( \sum_{l=1}^{n} \frac{1}{l} \) considering it as the integral sum of the function \( \varphi(\alpha) = \alpha^{-1} \),
\begin{align}
\sum_{l=1}^{n} \frac{1}{l} < 1 + \int_{1}^{n} \frac{d\alpha}{\alpha} = 1 + \ln n. \quad (30)
\end{align}

The inequality follows from the low estimation of the integral in the right-hand side by the rectangular method taking into account that the function \( \varphi(\alpha) \) is decreasing at \( \alpha > 0 \).

Now, applying inequalities (29) and (30) for the low estimation of the righthand side of Eq.(28), we find
\begin{align}
(2n)! > \frac{2^{2n}(n!)^2}{en}. \quad \blacksquare
\end{align}

\textbf{Lemma 3.} \textit{The following upper estimation of coefficients \( u_m, m = 1, 2, \ldots \) is valid,}
\begin{align}
u_m \leq e \frac{(T/2)^{2m}}{(m-1)!} I_0(T) \quad (31)
\end{align}

where
\begin{align}
I_0(T) = \sum_{n=0}^{\infty} \frac{(T/2)^{2n}}{(n!)^2}
\end{align}
is the zero order Bessel function on the imaginary variable.

\( \square \) Using the inequality (27), from the definition formula (18), we have for any \( m \in \mathbb{N} \) that
\begin{align}
u_m = \sum_{n=m}^{\infty} \frac{T^{2n}}{(2n)!} \frac{n!}{(n-m)!} \leq e \sum_{n=m}^{\infty} \frac{(T/2)^{2n}}{(n-1)!(n-m)!} \leq
\end{align}
\[ \leq e^{(T/2)^{2m}} \frac{(T/2)^{2m}}{(m-1)!} \sum_{n=m}^{\infty} \frac{(T/2)^{2(n-m)}}{[(n-m)!]^2} = e^{(T/2)^{2m}} I_0(T), \]

We apply here the elementary inequality \((n-1)! \geq (m-1)!(n-m)!\) which takes place at \(n \geq m\). □

**Corollary.** From Eq.(31) it follows the upper estimation

\[
v_m \leq \frac{T}{2m+1} u_m = \frac{e}{m!} (T/2)^{2m+1} I_0(T) \quad (32)
\]

of coefficients \(v_m, m = 1, 2, ...\) since \((2l+1)! \geq (2m+1)(2l)!\) for all \(l \geq m\).

Summarizing, it is possible to assert on the basis of Eqs.(20), (31),(32) that the following lemma takes place.

**Lemma 4.** For coefficients \(w_m\), the following estimation is valid

\[
w_m < \frac{e}{m!(m-1)!} (T/2)^{2m} I_0(T) \varphi(m),
\]

\[
\varphi(m) = \left[1 + \frac{T}{2m} + \frac{m}{T}\right].
\]

**Lemma 5.** For modules of coefficients \(x_m\), at \(m > 1\), following estimations take place

\[
|x_m| < \frac{e \psi(T) I_0(T)}{e \psi(T) I_0(T) - T} \left(eT \psi(T) I_0(T)/4\right)^m
\]

where the function \(\psi(T)\) is determined by the equality

\[
\psi(T) \equiv TI_1(2) + \frac{T^2}{2} (I_0(2) - 1) + I_0(2)
\]

and

\[
I_1(T) = \frac{dT}{dT} = \sum_{m=1}^{\infty} \frac{(T/2)^{2m-1}}{m!(m-1)!}
\]

is the Bessel function of first order.

□ At \(m > 1\), the formula (23) rewrites as follows

\[
x_m = -w_m + \sum_{k=1}^{m-1} (-1)^{k-1} \sum_{\substack{l_1, ..., l_k > 0 \ \ \ \ l_1 + ... + l_k < m}} w_{m-l_1-...-l_k} \prod_{j=1}^{k} w_{l_j}.
\]

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Then, using the estimation (33), we have

\[ |x_m| \leq |w_m| + \sum_{k=1}^{m-1} \sum_{l_1+\ldots+l_k< m} |w_{m-l_1-\ldots-l_k}| \prod_{j=1}^{k} |w_{l_j}| \leq \]

\[ \leq (T/2)^{2m} \sum_{k=1}^{m} \sum_{l_1+\ldots+l_k= m} \prod_{j=1}^{k} \frac{eI_0(T)}{l_j!(l_j-1)!} \varphi(l_j) \leq \]

\[ \leq (T/2)^{2m} \sum_{k=1}^{m} \prod_{j=1}^{k} \sum_{l=1}^{m} \frac{eI_0(T)}{l!(l-1)!} \varphi(l) \leq \]

\[ \leq (T/2)^{2m} \sum_{k=1}^{m} (eI_0(T))^k \left( \sum_{l=1}^{\infty} \frac{\phi(l)}{l!(l-1)!} \right)^k. \]

According to the definition of Bessel’s functions on the imaginary variable, the sum in brackets in the last expression is equal to

\[ \sum_{l=1}^{\infty} \frac{\phi(l)}{l!(l-1)!} = I_1(2) + T (I_0(2) - 1)/2 + I_0(2)/T \equiv \frac{\psi(T)}{T}. \]

Then, taking in mind of the above obtained estimation and the inequality \( e\psi(T) > 1 \), we find

\[ |x_m| < (T/2)^{2m} \sum_{k=1}^{m} (eI_0(T))^k \left( \psi(T)/T \right)^k = \frac{e\psi(T)I_0(T)}{e\psi(T)I_0(T) - T} \left( eT\psi(T)I_0(T)/4 \right)^n. \]

Corollary. Basing on Theorem 1, from obtained estimations (34), we find following estimations of moments \( M_n \),

\[ M_n = n! \left( \frac{2\sigma}{\nu^2} \right)^n |x_n| < \frac{e\psi(T)I_0(T)}{e\psi(T)I_0(T) - T} n! \left( \frac{e\sigma T\psi(T)I_0(T)}{2\nu^2} \right)^n. \]

Remark. At \( T \to 0 \), following asymptotic formulas take place

\[ u_m = \left[ T^{2m}/(2m)! \right] (1 + o(1)), v_m = \left[ T^{2m+1}/(2m+1)! \right] (1 + o(1)). \]

Then

\[ w_m = \left[ T^{2m-1}/2(m-1)! (2m-1)! \right] (1 + o(1)) \]
and, according to the formula (36), the basic contribution in $x_m$ gives the term with $k = 1$ at $T \to 0$, i.e. $x_m = [(-1)^m(T/2)^m] (1 + o(1))$. Therefore, since $\psi(0) = \text{const}$, the obtained estimation (34) is asymptotically exact at $T \to 0$.

5. Approximations of the Mandel distribution and estimations of their accuracy

Let us consider the Mandel distribution (1). We present it as the expansion on moments

$$P_n = \frac{1}{n!} E \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} J^{l+n} = \frac{1}{n!} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} M^{l+n},$$

or, on the basis of the representation of moments (25), it gives

$$P_n = \frac{(-1)^n}{n!} \sum_{l=0}^{\infty} \frac{(n+l)!}{l!} \left( \frac{2\sigma}{\nu^2} \right)^{n+l} x^{l+n}. \tag{37}$$

We determine the sequence of approximations $P_n^{(N)}$, $N = 1, 2, ...$ of the probability distribution which present it with the accuracy up to the $N$th power of the parameter $(\sigma T/\nu^2)$ (see Remark of Lemma 5) when $n \leq N$,

$$P_n^{(N)} = \frac{(-1)^n}{n!} \sum_{l=0}^{N-n} \frac{(n+l)!}{l!} \left( \frac{2\sigma}{\nu^2} \right)^{n+l} x^{l+n}. \tag{38}$$

Further, we estimate the deviation of the $(N-1)$th approximation from the exact distribution (1). On the basis Eq.(37), we have

$$|P_n - P_n^{(N-1)}| \leq \frac{1}{n!} \sum_{l=N-n}^{\infty} \frac{(n+l)!}{l!} \left( \frac{2\sigma}{\nu^2} \right)^{n+l} |x^{l+n}| =$$

$$= \frac{1}{n!} \sum_{l=N}^{\infty} \frac{l!}{(l-n)!} \left( \frac{2\sigma}{\nu^2} \right)^l |x^l|.$$

Then we use the estimation (34),

$$|P_n - P_n^{(N-1)}| \leq \frac{e^{\psi I_0(T)}}{e^{\psi I_0(T)} - T} \sum_{l=N}^{\infty} \frac{l!}{(l-n)!} \left( \frac{2\sigma}{\nu^2} \right)^l \left( e^{T \psi I_0(T)} \right)^{\frac{l}{2^l}}, \tag{39}$$
where the variable of the function $\psi$ hereinafter is not pointed out explicitly. Let us give separately the estimation of the series

$$R_N(\zeta, n) = \sum_{l=N}^{\infty} \frac{l!}{(l-n)!} \zeta^l$$

for positive values of $\zeta$ and for $n \geq N$. The following transformations are valid

$$R_N(\zeta, n) = \zeta^n \sum_{l=N}^{\infty} \frac{l!}{(l-n)!} \zeta^{l-n} = \zeta^n \sum_{l=N}^{\infty} \zeta^l = \zeta^n \frac{\zeta^n}{1-\zeta} = \zeta^n \sum_{l=0}^{n} \binom{n}{l} \frac{N!}{(N-l)!} \zeta^{N-l} \frac{(n-l)!}{(1-\zeta)^{1+n-l}} = \frac{\zeta^n}{1-\zeta} \sum_{l=0}^{n} n! \binom{N}{l} \left( \frac{\zeta}{1-\zeta} \right)^{n-l} \leq n! \frac{(2\zeta)^N}{1-\zeta} \left( \frac{\zeta}{1-\zeta} \right)^{n-l} \leq n! \frac{(2\zeta)^N}{1-2\zeta}.$$

Applying the obtained estimation to the inequality (39), we come to the following statement.

**Theorem 2.** The $N$th approximation $P_n^{(N)}$ of the Mandel distribution presents this distribution with the guaranteed accuracy determined by the inequality

$$|P_n - P_n^{(N)}| \leq \left( \frac{e\psi I_0(T)}{e\psi I_0(T) - T} \right) \left( \frac{(2\zeta)^{N+1}}{1-2\zeta} \right)$$

if

$$\zeta = \frac{e\sigma T \psi}{2\nu^2} I_0(T) < \frac{1}{2}.$$

In order to obtain the effective algorithm of consecutive calculation of $P_n^{(N)}$, it is necessary to solve the last problem. It consists of the creation of the method which permits to find consecutively all components of the sequence $X$.

**Lemma 6.** The following formula takes place

$$\left( \frac{\partial^m}{\partial \alpha^m} \exp(\pm \alpha^{1/2}T) \right)_{\alpha=1} = m! e^{\pm T} R_m^\pm(T)$$

(41)
where polynomials $R_m^\pm(T)$ of the $m$ degree on the variable $T$ are determined by the recurrent relation

$$R_{m+1}^\pm(T) = \pm \frac{T}{2(m+1)} \sum_{l=0}^{m} (-1)^l \frac{(2l)!}{2^{2l} l!^2} R_{m-l}^\pm(T), \quad m \in \mathbb{N}_+ \quad (42)$$

and by the condition at $m = 0$

$$R_0^\pm(T) = 1. \quad (43)$$

\[ \square \] The proof is realized by the induction on $m$. At $m = 0$, we obtain (42). We shall build the induction step. Let Eq. (41) takes place at the given value $m \in \mathbb{N}_+$. Then we shall introduce the function $R_{m+1}(T)$ according to

$$(m + 1)! e^{\pm T} R_{m+1}(T) = \left( \frac{\partial^{m+1}}{\partial \alpha^{m+1}} \exp \left( \pm \alpha^{1/2} T \right) \right)_{\alpha = 1} =$$

$$= \pm \left( \frac{\partial^m}{\partial \alpha^m} \left[ \frac{T}{2 \alpha^{1/2}} \exp(\pm \alpha^{1/2} T) \right] \right)_{\alpha = 1} \cdot$$

Using the induction assumption and the formula

$$(\alpha^{-1/2})^{(l)} = ( -1 )^l \frac{(2l)!}{2^{2l} l!} \alpha^{-(2l+1)/2},$$

we accomplish the $m$-fold differentiation,

$$(m + 1)! e^{\pm T} R_{m+1}(T) =$$

$$= \pm \frac{T}{2} \sum_{l=0}^{m} \binom{m}{l} \left( \frac{\partial^{m-l}}{\partial \alpha^{m-l}} \exp \left( \pm \alpha^{-1/2} T \right) \right)_{\alpha = 1} \left( \frac{d^l}{d \alpha^l} \alpha^{-1/2} \right)_{\alpha = 1} =$$

$$= \pm m! T e^{\pm T} \frac{\alpha^{-1/2}}{2} \sum_{l=0}^{m} \frac{(-1)^l}{l!} R_{m-l}^\pm(T) \frac{(2l)!}{2^{2l} l!}.$$ 

Whence formulas (41), (42) follow and although the fact that the function $R_{m+1}(T)$ is the polynomial of the $m + 1$ degree is proved. \[ \square \]

**Theorem 3.** Components of sequences $U = \langle u_m; m \in \mathbb{N} \rangle$ and $V = \langle v_m; m \in \mathbb{N} \rangle$ are represented in the form

$$u_m = \frac{1}{2} m! \left( e^T R_m^+(T) + e^{-T} R_m^-(T) \right), \quad (44)$$
\[ v_m = \frac{1}{T} (m + 1)! \left( e^T R_{m+1}^+(T) + e^{-T} R_{m+1}^-(T) \right) \]  \hspace{2cm} (45)

where polynoms \( R_{m}^\pm(T) \) are calculated by the formula (42).

□ Using Eq.(41), from the definition of coefficients \( u_m \), we have

\[
u_m = \sum_{n=m}^{\infty} \frac{T^{2n}}{(2n)!} \frac{n!}{(n-m)!} = \left( \frac{\partial^m}{\partial \alpha^m} \sum_{n=m}^{\infty} \frac{\alpha^n T^{2n}}{(2n)!} \right)_{\alpha=1} = \left( \frac{\partial^m}{\partial \alpha^m} \text{ch}(\alpha^{1/2}T) \right)_{\alpha=1} = \frac{1}{2} \left( \frac{\partial^m}{\partial \alpha^m} \left( \exp(\alpha^{1/2}T) + \exp(-\alpha^{1/2}T) \right) \right)_{\alpha=1} = \frac{1}{2} m! \left( e^T R_{m}^+(T) + e^{-T} R_{m}^-(T) \right)
\]

On the basis of the formula (41), we have although

\[
\left( \frac{\partial^m}{\partial \alpha^m} \left[ \alpha^{-1/2} \exp \left( \pm \alpha^{1/2}T \right) \right] \right)_{\alpha=1} = \sum_{l=0}^{m} \binom{m}{l} \left( \frac{\partial^{m-l}}{\partial \alpha^{m-l}} \exp \left( \pm \alpha^{1/2}T \right) \right) \left( \frac{\partial^l}{\partial \alpha^l} \alpha^{-1/2} \right)_{\alpha=1} = m! e^{\pm T} \sum_{l=0}^{m} (-1)^l R_{m-l}^\pm(T) \frac{(2l)!}{2^{2l}(l!)^2} = \pm \frac{2}{T} (m + 1)! e^{\pm T} R_{m+1}^\pm(T).
\]

Further, on the basis of the obtained formula, from the definition of coefficients \( v_m \), we have

\[
v_m = \sum_{n=m}^{\infty} \frac{T^{2n+1}}{(2n + 1)!} \frac{n!}{(n-m)!} = \left( \frac{\partial^m}{\partial \alpha^m} \sum_{n=m}^{\infty} \frac{\alpha^n T^{2n+1}}{(2n + 1)!} \right)_{\alpha=1} = \left( \frac{\partial^m}{\partial \alpha^m} \text{sh}(\alpha^{1/2}T) \right)_{\alpha=1} = \frac{1}{2} \left( \frac{\partial^m}{\partial \alpha^m} \left[ \frac{\exp(\alpha^{1/2}T)}{\alpha^{1/2}} - \frac{\exp(-\alpha^{1/2}T)}{\alpha^{1/2}} \right] \right)_{\alpha=1} = \frac{1}{T} (m + 1)! \left( e^T R_{m+1}^+(T) + e^{-T} R_{m+1}^-(T) \right).
\]

Corollary. Taking into account Eqs.(44),(45) and Eq.(20), components of the sequence \( W \) are represented as follows

\[
w_m = \frac{1}{2} R_{m}^+(T) \left( 1 + \frac{m}{T} \right) + \frac{m + 1}{T} R_{m+1}^+(T) +
\]

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6. Approximate formulas

Thus, Theorem 3 permits to calculate successively all approximations $P_n^{(N)}$ of probabilities $P_n$ using formulas (46). In this section, we shall give some first approximations of the probability distribution $P_n$ by means of this algorithm.

At first, we find the explicit form of polynomials $R_m^\pm(T), m = 0, 1, 2, 3, 4,$

\[ R_0^\pm(T) = 1, \quad R_1^\pm(T) = \pm T/2, \]
\[ R_2^\pm(T) = \pm (T/2^3) (R_1^\pm(T) - 1/2) = (T/2^3)(T \mp 1), \]
\[ R_3^\pm(T) = \pm (T/2 \cdot 3) (R_2^\pm(T) - R_1^\pm(T)/2 + 3/8) = \pm (T/2^4\cdot 3) (T^2 \mp 3T + 3), \]
\[ R_4^\pm(T) = \pm (T/2^3) (R_3^\pm(T) - R_2^\pm(T)/2 + 3R_1^\pm(T)/8 - 5/16) = \]
\[ = (T/2^7 \cdot 3) (T^3 \mp 6T^2 + 15T \mp 15). \]

On their basis, we calculate components $w_m, m = 1, 2, 3.$

\[ w_1 = \frac{T}{2}, \quad w_2 = \frac{1}{2^4} (2T^2 - 2T + 1 - e^{-2T}), \quad (47) \]
\[ w_3 = \frac{1}{3 \cdot 2^5} (2T^3 - 6T^2 + 9T - 6 + 3(T + 2)e^{-2T}). \quad (48) \]

At last, we find components $x_m, m = 1, 2, 3,$ by the formula (36),

\[ x_0 = 1, \quad x_1 = -w_1, \quad x_2 = -w_2 + w_1^2, \quad x_3 = -w_3 + 2w_1w_2 - w_1^3 \]

or after the substitution (46), (47), we have

\[ x_1 = -\frac{T}{2}, \quad x_2 = \frac{1}{2^4} (2T^2 + 2T - 1 + e^{-2T}), \]
\[ x_3 = -\frac{1}{3 \cdot 2^5} \left[ 2T^3 + 6T^2 + 3T - 6 + 3(3T + 2)e^{-2T} \right]. \]

Explicit expressions of components $x_m, m = 0, 1, 2, 3$ permit to us to write the approximated expressions of $n$-photon registration probabilities for $n = 0, 1, 2, 3$ up to the third order ($N = 0, 1, 2, 3$). All probabilities $P_n^{(N)}$
of some higher values \( n > 3 \) are equal to zero in this approximation. We result consecutively those expressions for each order of the approximation.

For \( N = 0 \), we have that only the probability \( P^{(0)}_0 = 1 \) differs from zero.

At \( N = 1 \), we obtain

\[
P^{(1)}_0 = \sum_{l=0,1} \left( \frac{2\sigma}{\nu^2} \right)^l x_l = 1 - \frac{\sigma T}{\nu^2}, \quad P^{(1)}_1 = \frac{\sigma T}{\nu^2}.
\]

At \( N = 2 \), we have correspondingly

\[
P^{(2)}_0 = \sum_{l=0,1,2} \left( \frac{2\sigma}{\nu^2} \right)^l x_l = 1 - \frac{\sigma T}{\nu^2} + \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}),
\]

\[
P^{(2)}_1 = \frac{\sigma T}{\nu^2} - 2 \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}),
\]

\[
P^{(2)}_2 = \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}).
\]

At last, for \( N = 3 \), we get following approximate formulas

\[
P^{(3)}_0 = \sum_{l=0,1,2,3} \left( \frac{2\sigma}{\nu^2} \right)^l x_l =
\]

\[
= 1 - \frac{\sigma T}{\nu^2} + \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}) -
\]

\[
- \frac{2}{3} \left( \frac{\sigma}{2\nu^2} \right)^3 \left[ 2T^3 + 6T^2 + 3T - 6 + 3(3T + 2)e^{-2T} \right],
\]

\[
P^{(3)}_1 = - \sum_{l=0,1,2} (l + 1) \left( \frac{2\sigma}{\nu^2} \right)^{l+1} x_{l+1} =
\]

\[
= \frac{\sigma T}{\nu^2} - 2 \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}) +
\]

\[
\quad + 2 \left( \frac{\sigma}{2\nu^2} \right)^3 \left[ 2T^3 + 6T^2 + 3T - 6 + 3(3T + 2)e^{-2T} \right],
\]

\[
P^{(3)}_2 = \frac{1}{2} \sum_{l=0,1} (2 + l)(1 + l) \left( \frac{2\sigma}{\nu^2} \right)^{l+2} x_{l+2} =
\]

\[
= \left( \frac{\sigma}{2\nu^2} \right)^2 (2T^2 + 2T - 1 + e^{-2T}) -
\]
\[-2 \left( \frac{\sigma}{2
u^2} \right)^3 \left[ 2T^3 + 6T^2 + 3T - 6 + 3(3T + 2)e^{-2T} \right], \]

\[P_3^{(3)} = \frac{2}{3} \left( \frac{\sigma}{2
u^2} \right)^3 \left[ 2T^3 + 6T^2 + 3T - 6 + 3(3T + 2)e^{-2T} \right].\]

7. Conclusion

It is necessary to point out some lacks of the problem setting and its solution in the given work.

Our research is based on the expansion (5) which is directly connected with the expansion of the generating function (15) on \(\lambda\) powers. The convergence radius of last expansion is equal to the convergence radius \(r(T, \nu, \sigma)\) of the power series on \(\lambda\) of the function \(Q(\lambda)\) which is meromorphic only. It has poles in the complex plane \(\lambda \in \mathbb{C}\). It means that the distance to nearest of these poles is equal to \(r(T, \nu, \sigma)\). It is clear that \(r(T, \nu, \sigma) \rightarrow 0\) when parameters \(T\) or \(\sigma\) increasing. At the same time, such restriction of the convergence domain of the expansion (5) is not caused by a physical reason, i.e. the divergence of the series (5) is not connected with the presence of anything qualitative change of the probability distribution \(P_n\) behavior when parameters \(T, \nu, \sigma\) varying. Therefore, it is desirable to get rid of this lack of the expansion (5).

Other lack is connected with the above mentioned one. With irreversibility, we make a mistake when calculating the convergence radius by the estimation of the residual series (6) basing on a priori estimations of moments \(E\tilde{J}^m, m \in \mathbb{N}\). Actually, only its low bound is obtained by this way and it is sufficiently coarse as one may see from our work.

It is possible to consider mentioned lacks as ones generated by our method when the problem solving. The following lack concern with the unsuccessful problem setting.

Let us pay attention that obtained formulas permit to calculate the registration probability of everyone concrete photon number \(n\). However, they do not permit to calculate (to estimate) the registration probability in the case when this number is indefinite, i.e. in the case when it is not known exactly and, therefore, the number \(n\) is the parameter of the problem. It is connected with the fact that their analytical representation becomes very tedious when the approximation order increasing.
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