THE SOBOLEV–POINCARÉ INEQUALITY AND
THE $L_{q,p}$-COHOMOLOGY OF TWISTED CYLINDERS

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Abstract. We establish a vanishing result for the $L_{q,p}$-cohomology ($q \geq p$) of a twisted cylinder, which is a generalization of a warped cylinder. The result is new even for warped cylinders. We base on the methods for proving the $(p,q)$ Sobolev–Poincaré inequality developed by L. Shartser.

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1. Introduction

The $L_{q,p}$-cohomology $H^k_{q,p}(M)$ of a Riemannian manifold $(M,g)$ is, by definition, the quotient of the space of closed $p$-integrable differential $k$-forms by the exterior differentials of $q$-integrable $k$-forms. If $p = q$ then $L_{q,p}$-cohomology is usually referred to simply as $L_p$-cohomology and the index $p$ is used instead of $p,p$ in all the notations.

A twisted product $X \times_h Y$ of two Riemannian manifolds $(X,g_X)$ and $(Y,g_Y)$ is the direct product manifold $X \times Y$ endowed with a Riemannian metric of the form

$$g := g_X + h^2(x,y)g_Y,$$

where $h : X \times Y \to \mathbb{R}$ is a smooth positive function (see [3]). If $X$ is a half-interval $[a,b)$ then the twisted product $X \times_h Y$ is called a twisted cylinder.

We refer to an $m$-dimensional Riemannian manifold $(M,g_M)$ as an asymptotic twisted product (respectively, as an asymptotic twisted cylinder) if, outside an $m$-dimensional compact submanifold, it is bi-Lipschitz equivalent to a twisted product (respectively, to a twisted cylinder).

In this paper, we prove some vanishing results for the $L_{q,p}$-cohomology of twisted cylinders $[a,b) \times_h N$ for a positive smooth function $h : [a,b) \times N \to \mathbb{R}$ in the case where the base $N$ is a closed manifold and $p \geq q > 1, \frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(q+1)(\dim N+1)}$.

If in [1] the function $h$ depends only on $x$ then we obtain the familiar notion of a warped product (see [1]). Twisted products were the object of recent investigations [4, 6, 8, 9, 10, 16, 20]. The $L_{q,p}$-cohomology of warped cylinders $[a,b) \times N$, i.e., of product manifolds $[a,b) \times N$ endowed with a warped product metric

$$g = dt^2 + h^2(t)g_N,$$

where $g_N$ is the Riemannian metric of $N$ and $h : [a,b) \to \mathbb{R}$ is a positive smooth function, was studied by Gol’dshtein, Kuz’minov, and Shvedov [11], Kuz’minov and Shvedov [18, 19] (for $p = q$), and Kopylov [17] for $p, q \in [1, \infty), \frac{1}{p} - \frac{1}{q} < \frac{1}{\dim N+1}$.

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The main result of the paper (Theorem 7.1) states that the $L_{q,p}$-cohomology $H^k_{q,p}(C^{h},N)$ of the twisted cylinder $C^{h}_a,N$ with $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{(\dim N+1)}$ is zero provided that the De Rham cohomology $H^k_{DR}(N)$ of the base $N$ is trivial and some integral conditions on the twisting function involving $p$, $q$ and an auxiliary parameter $\delta$ are fulfilled.

The paper is organized as follows: In Sec. 2 we recall some basic definitions concerning the $L_{q,p}$-cohomology of Riemannian manifolds. Sec. 3 describes the representations of differential forms on a warped product proposed by Gol’dshtein, Kuz’minov, and Shvedov in [12]. In Sec. 4 we develop a version of the weighted Sobolev–Poincaré inequality for convex sets in $\mathbb{R}^n$ by introducing a homotopy operator and consider some of its consequences; the exposition is based on the ideas of Shartser suggested in [21] and [22]. In Sec. 5 we consider a new homotopy operator $A_\alpha$ on differential forms defined on a convex domain in $\mathbb{R}^n$ and show that it guarantees the fulfillment of an inequality of Sobolev–Poincaré-type for $q \geq p \geq 1$ and $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$. In Sec. 6 using the ideas of Shartser’s article [22], we “glue” local homotopy operators on a twisted cylinder to obtain a global homotopy operator. In Sec. 7 we use this global homotopy operator for proving our above-mentioned main result on the triviality of the $L_{q,p}$-cohomology of a twisted cylinder (Theorem 7.1), and in Sec. 8 we extend this theorem to asymptotic twisted cylinders (Theorem 8.2). Sec. 9 contains some examples.

2. Basic Definitions

We recall the main definitions and notations.

Below we tacitly assume all manifolds to be oriented.

Let $M$ be a smooth oriented Riemannian manifold. Denote by $\mathcal{D}^k(M):=C^\infty_c(M,\Lambda^k)$ the space of all smooth differential $k$-forms with compact support contained in $M \setminus \partial M$ denote by $L^1_{loc}(M,\Lambda^k)$ the space of locally integrable differential forms.

Denote by $L^p(M,\Lambda^k)$ the Banach space of locally integrable differential $k$-forms endowed with the norm $\|\theta\|_{L^p(M,\Lambda^k)}:=\left(\int_M |\theta|^p dx\right)^{\frac{1}{p}}$ < $\infty$ (as usual, we identify forms coinciding outside a set of measure zero). Of course, we can add a positive (smooth) weight $\sigma : M \to \mathbb{R}$ and thus integrate $|\theta|^p \sigma^p$ to obtain the weighted $L^p$-space $L^p(M,\Lambda^k,\sigma)$.

**Definition 2.1.** We call a differential $(k+1)$-form $\theta \in L^1_{loc}(M,\Lambda^{k+1})$ the weak exterior derivative (or differential) of a differential $k$-form $\phi \in L^1_{loc}(M,\Lambda^k)$ and write $d\phi = \theta$ if

$$\int_M \theta \wedge \omega = (-1)^{k+1} \int_M \phi \wedge d\omega$$

for any $\omega \in \mathcal{D}^{n-k}(M)$.

**Remark 2.2.** Note that the orientability of $M$ is not substantial in this definition since one may take integrals over orientable domains on $M$ instead of integrals over $M$.

We then introduce an analog of Sobolev spaces for differential $k$-forms, i.e., the space of $q$-integrable forms with $p$-integrable weak exterior derivative:

$$\Omega^k_{q,p}(M) = \{ \omega \in L^q(M,\Lambda^k) \mid d\omega \in L^p(M,\Lambda^{k+1}) \}.$$
This is a Banach space for the graph norm
\[ \|\omega\|_{q,p} = \left( \|\omega\|^2_{L^q(M,\Lambda^k)} + \|d\omega\|^2_{L^p(M,\Lambda^{k+1})} \right)^{1/2}. \]
The space \( \Omega^k_{q,p}(M) \) is a reflexive Banach space for any \( 1 < q, p < \infty \). This can be proved using standard arguments of functional analysis.

We now define our basic ingredients (for three parameters \( r, q, p \)).

**Definition 2.3.** Put
(a) \( Z^k_{p,r}(M) = \text{Ker}[d : \Omega^k_{p,r}(M) \to L^r(M,\Lambda^{k+1})] \).
(b) \( B^k_{q,p}(M) = \text{Im}[d : \Omega^k_{q,p}(M) \to L^p(M,\Lambda^k)] \).

The subspace \( Z^k_{p,r}(M) \) does not depend on \( r \) and is a closed subspace in \( L^p(M,\Lambda^k) \) (see Lemma [14, Lemma 2.4(i)]). This allows us to use the notation \( Z^k_p(M) \) for all \( Z^k_{p,r}(M) \). Note that \( Z^k_p(M) \subset L^p(M,\Lambda^k) \) is always a closed subspace but that is in general not true for \( B^k_{q,p}(M) \). Denote by \( \overline{B}^k_{q,p}(M) \) its closure in the \( L^p \)-topology.

Observe also that since \( d \circ d = 0 \), one has \( \overline{B}^k_{q,p}(M) \subset Z^k_p(M) \). Thus,
\[ B^k_{q,p}(M) \subset \overline{B}^k_{q,p}(M) \subset Z^k_p(M) = \overline{Z}^k_p(M) \subset L^p(M,\Lambda^k). \]

**Definition 2.4.** Suppose that \( 1 \leq q, p \leq \infty \). The \( L_{q,p} \)-cohomology of \( (M,g) \) is defined as the quotient
\[ H^k_{q,p}(M) := Z^k_p(M)/B^k_{q,p}(M), \]
and the reduced \( L_{q,p} \)-cohomology of \( (M,g) \) is, by definition, the space
\[ \overline{H}^k_{q,p}(M) := Z^k_p(M)/\overline{B}^k_{q,p}(M). \]

Since \( B^k_{q,p} \) is not always closed, the \( L_{q,p} \)-cohomology is in general a (non-Hausdorff) semi-normed space, while the reduced \( L_{q,p} \)-cohomology is a Banach space.

Below |\( X \)| stands for the volume of a Riemannian manifold \( (X,g) \).

It follows from the results of [13] that, under suitable assumptions on \( p, q \), the \( L_{q,p} \)-cohomology of a Riemannian manifold \( M \) can be expressed in terms of smooth forms.

Let \( C^\infty(M,\Lambda^k) \) be the space of smooth \( k \)-forms on \( M \).

Introduce the notations:
\[ C^\infty L^p(M,\Lambda^k) := C^\infty(M,\Lambda^k) \cap L^p(M,\Lambda^k); \]
\[ C^\infty L^p(M,\Lambda^k,\sigma) := C^\infty(M,\Lambda^k) \cap L^p(M,\Lambda^k,\sigma); \]
\[ C^\infty \Omega^k_{q,p}(M) := C^\infty(M,\Lambda^k) \cap \Omega^k_{q,p}(M); \]
\[ C^\infty H^k_{q,p}(M) := \frac{C^\infty(M,\Lambda^k) \cap Z^k_p(M)}{C^\infty(M,\Lambda^k) \cap B^k_{q,p}(M)}; \]
\[ C^\infty \overline{H}^k_{q,p}(M) := \frac{C^\infty(M,\Lambda^k) \cap \overline{Z}^k_p(M)}{C^\infty(M,\Lambda^k) \cap \overline{B}^k_{q,p}(M)}. \]

**Theorem 2.5.** [13, Theorem 12.5 and 12.8, Corollary 12.9]. Let \( (M,g) \) be a \( n \)-dimensional Riemannian manifold and suppose the fulfillment of one of the following conditions:
- \( p, q \in (1, \infty) \) and \( \frac{1}{p} - \frac{1}{q} \leq \frac{1}{n} \);
Then the cohomology $H^*_{q,p}(M)$ can be represented by smooth forms, and thus $H^*_{q,p}(M) = C^\infty H^*_{q,p}(M)$.

More exactly, any closed form in $Z^k_p(M)$ is cohomologous to a smooth form in $L^p(M)$. Furthermore, if two smooth closed forms $\alpha, \beta \in C^\infty(M, \Lambda^k) \cap Z^k_p(M)$ are cohomologous modulo $dC^\infty \Omega^{k-1}_q(M)$ then they are cohomologous modulo $dC^\infty \Omega^{k-1}_q(M)$.

Similarly, any reduced cohomology class can be represented by a smooth form.

3. Differential Forms on a Twisted Cylinder

From now on, $C^h_{a,b}N$ is the twisted cylinder $[a, b) \times hN$, that is, the product of a half-interval $[a, b)$ and a closed smooth $n$-dimensional Riemannian manifold $(N, g_N)$ equipped with the Riemannian metric $dt^2 + h^2(t, x) g_N$, where $h : [a, b) \times N \to \mathbb{R}$ is a smooth positive function.

Every differential form on $[a, b) \times N$ admits a unique representation of the form $\omega = \omega_A + dt \wedge \omega_B$, where the forms $\omega_0$ and $\omega_1$ do not contain $dt$ (cf. [12]). It means that $\omega_0$ and $\omega_1$ can be viewed as one-parameter families $\omega_A(t)$ and $\omega_B(t)$, $t \in I$, of differential forms on $N$.

The modulus of a form $\omega$ of degree $k$ on $C^h_{a,b}N$ is expressed via the moduli of $\omega_A(t)$ and $\omega_B(t)$ on $N$ as follows:

$$|\omega(t, x)|_{C^h_{a,b}N} = \left[ h^{-2k}(t, x)|\omega_A(t, x)|^2_N + h^{-2(k+1)}(t, x)|\omega_B(t, x)|^2_N \right]^{1/2}$$

Consequently,

$$\|\omega\|_{L^p(C^h_{a,b}N, \Lambda^k)} = \left[ \int_a^b \int_N \left( h^{2(k-\frac{k}{p})}(t, x)|\omega_A(t, x)|^2_N + h^{2(k+1)-\frac{k}{p}}(t, x)|\omega_B(t, x)|^2_N \right)^{\frac{p}{2}} dx dt \right]^{\frac{1}{p}}$$

Put

$$f_{k,p}(t) = \min_{x \in N} \left\{ h^{\frac{k}{p}-k}(t, x) \right\}$$

and

$$F_{k,p}(t) = \max_{x \in N} \left\{ h^{\frac{k}{p}-k}(t, x) \right\}.$$

4. The Weighted Sobolev–Poincare Inequality for Convex Sets in $\mathbb{R}^n$

Denote by $\Lambda^\infty_{loc}(M)$ the space all locally integrable differential forms with locally integrable weak differential.

Suppose that $D \subset \mathbb{R}^n$ is a convex set and $\psi_y : D \times [0, 1] \to D$, $\psi_y(x, t) := tx + (1 - t)y$, is the homotopy induced by the convex structure. For a $k$-form $\omega \in \Omega^k_{loc}(D)$ the pullback $\psi^*_y \omega$ can be written in the form

$$\psi^*_y \omega(x, t) = \left( \psi^*_y \omega \right)_0 (x, t) + dt \wedge \left( \psi^*_y \omega \right)_1 (x, t),$$

where $\left( \psi^*_y \omega \right)_0$ and $\left( \psi^*_y \omega \right)_1$ do not contain $dt$.

For each $y \in D$ define a homotopy operator

$$K_y : \Omega^k_{loc}(D) \to \Omega^{k-1}_{loc}(D)$$

as follows:

$$K_y \omega(x) := \int_0^1 \left( \psi^*_y \omega \right)_1 (t) dt$$
It is easy to see that $K_y$ takes smooth forms to smooth forms. It is proved in [15] that $K_y d\omega + dK_y \omega = \omega$. The following proposition is a generalization of results from [2] and Shartser's thesis [21] (see also [22]) to the weighted case and to unbounded convex domains.

**Proposition 4.1.** Suppose that $D$ is a convex set in $\mathbb{R}^n$, $q \geq p \geq 1$, and $\beta : D \to \mathbb{R}$ is a positive smooth function.

If the inequality

$$C(k, p, q, n, \beta) := \int_0^1 \sup_{x \in D} \|\beta(x)1_{xt+(1-t)D}(z)\|_{L^q(D, dx)} t^k(1-t)^{-n/p} dt < \infty$$

holds then the inequality

$$\left\| \beta(x) \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^p(D, dy)} \right\|_{L^q(D, dx)} \leq C(k, p, q, n, \beta) \|d\omega\|_{L^p(D, \Lambda^{k+1})}.$$ 

is valid for every $\omega \in \Omega^{k}_{\text{loc}}(D)$ such that $d\omega \in L^p(D, \Lambda^{k+1})$. Here $1_{xt+(1-t)D}$ is the characteristic function of the set $xt+(1-t)D$.

**Proof.** By the definition of $K_y$, we have

$$\left\| \beta(x) \left\| \frac{K_y d\omega(x)}{|x-y|} \right\|_{L^p(D, dy)} \right\|_{L^q(D, dx)} = \beta(x) \left\| \int_0^1 \left( \frac{\psi_y^* d\omega}{|x-y|} \right) (x, t) dt \right\|_{L^p(D, dy)} \right\|_{L^q(D, dx)}$$

$$\leq \int_0^1 \beta(x) \left\| \left( \frac{\psi_y^* d\omega}{|x-y|} \right) (x, t) \right\|_{L^p(D, dy)} \right\|_{L^q(D, dx)} dt$$

$$\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[ \int_D \left( \frac{\psi_y^* d\omega}{|x-y|^p} \right) (x, t) \right|^p dy \right] dx \right\}^{1/q} dt.$$

As usual, we identify the tangent space to $\mathbb{R}^n$ at any of its points with $\mathbb{R}^n$. By easy calculations,

$$\left( \psi_y^* d\omega \right) (x, t) \leq |x-y| t^k |d\omega (\psi_y(x, t))|.$$ 

Therefore,

$$\int_0^1 \left\{ \int_D \beta^q(x) \left[ \int_D \left( \frac{\psi_y^* d\omega}{|x-y|^p} \right) (x, t) \right|^p dy \right] dx \right\}^{1/q} dt$$

$$\leq \int_0^1 \left\{ \int_D \beta^q(x) \left[ \int_D t^{kp} |d\omega (\psi_y(x, t))|^p dy \right] dx \right\}^{1/q} dt$$

$$= \int_0^1 \left\{ \int_D \beta^q(x) \left[ \int_D t^{kp} |d\omega (tx + (1-t)y)|^p dy \right] dx \right\}^{1/q} dt := I.$$
The change of variables $z = tx + (1 - t)y$ in the inner integral yields

$$I = \int_0^1 \left\{ \int_{D} \beta^q(x) \left[ \int_{tx+(1-t)D} |d\omega(z)|^p dz \right]^{q/p} dx \right\}^{1/q} t^k(1-t)^{-n/p} dt$$

Since $D$ is convex, the set $tx + (1 - t)D$ is contained in $D$ for all $x \in D$ and $t \in [0, 1]$. Using Minkowski’s integral inequality, we infer

$$\left\{ \int_{D} \beta^q(x) \left[ \int_{tx+(1-t)D} |d\omega(z)|^p dz \right]^{q/p} dx \right\}^{1/q} \leq \left\{ \int_{D} \| \beta^p(x)1_{tx+(1-t)D}(z) \|_{L^q(D,Dx)} dz \right\}^{1/p}$$

$$= \left\{ \int_{D} \left( \int_{D} \beta^q(x)1_{tx+(1-t)D}(z) |d\omega(z)|^q dx \right)^{p/q} dz \right\}^{1/p}$$

$$= \left\{ \int_{D} \left( \int_{D} \beta^q(x)1_{tx+(1-t)D}(z) d\omega(z) \|_{L^p(D,Dx)} dz \right) \right\}^{1/p}$$

$$\leq \left( \sup_{z \in D} \int_{D} \beta^q(x)1_{tx+(1-t)D}(z) dx \right)^{1/q} \left( \int_{D} |d\omega(z)|^p dz \right)^{1/p}$$

$$= \sup_{z \in D} \| \beta(x)1_{tx+(1-t)D}(z) \|_{L^q(D,Dx)} \| d\omega \|_{L^p(D,\Lambda_{k+1})}.$$ 

The proposition follows. 

Theorem 4.2. Suppose that $D$ is a convex set of finite measure in $\mathbb{R}^n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, and the weight $\beta(x) \equiv 1$. Then

$$C(k,p,q,n,\beta) = \int_0^1 \sup_{z \in D} \| \beta(x)1_{tx+(1-t)D}(z) \|_{L^q(D,Dx)} t^k(1-t)^{-n/p} dt$$

in particular cases.

**Corollary 4.2.** Suppose that $D$ is a convex set of finite measure in $\mathbb{R}^n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{n}$, and the weight $\beta(x) \equiv 1$. Then

$$C(k,p,q,n,1) \leq |D|^{1/q} \int_0^1 t^{k-n/q}(1-t)^{-n/p} \min(t^{n/q},(1-t)^{n/q}) dt.$$ 

**Remark 4.3.** It is easy to see that the integral of the corollary exists because of the conditions imposed on $p$ and $q$. 


Proof. Using the change of variables $u = tx$, we obtain
\[
\int_0^1 \sup_{z \in D} \left\| 1_{tx+(1-t)D}(z) \right\|_{L^p(D, dx)} t^k(1-t)^{-n/p} dt = \int_0^1 \sup_{z \in D} \left\| 1_{u+(1-t)D}(z) \right\|_{L^p(tD, du)} t^{k-n/q}(1-t)^{-n/p} dt.
\]
Note that $|tD \cap \{u+(1-t)D\}| \leq |D| \min(t^n, (1-t)^n)$. It follows that
\[
\left\| 1_{u+(1-t)D}(z) \right\|_{L^p(tD, du)} \leq |D|^{1/q} \min(t^{n/q}, (1-t)^{n/q});
\]
\[
C(k, p, q, n, 1) \leq |D|^{1/q} \int_0^1 t^{k-n/q}(1-t)^{-n/p} \min(t^{n/q}, (1-t)^{n/q}) dt.
\]

Corollary 4.4. Suppose that $U$ is a convex set of finite measure $|U|$ in $\mathbb{R}^n$, $D = [a, b] \times U$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{1}{q(n+1)}$, and $\beta : [a, b) \rightarrow \mathbb{R}$ is an integrable positive function. If $\|\beta\|_{L^p([a, b))} < \infty$ then
\[
C(k, p, q, n, \beta) \leq |U|^{1/q} \|\beta\|_{L^p([a, b))}.
\]

Proof. If $x \in D$ then $x = (\tau, w)$, where $\tau \in [a, b)$ and $w \in U$. Using the special type of the weight $\beta(x) := \beta(\tau)$ and representing $z \in D$ as $z = (\eta, \zeta)$ with $\eta \in [a, b)$ and $\zeta \in U$, we obtain
\[
\int_0^1 \sup_{z \in D} \left\| \beta(x)1_{tx+(1-t)D}(z) \right\|_{L^p(D, dx)} t^k(1-t)^{-\frac{n+1}{p}} dt \leq \int_0^1 \sup_{a \leq \eta < b} \left( \int_a^b \beta'(\tau) 1_{t\tau+(1-t)[a, b]}(\eta) d\tau \right)^{\frac{1}{p}} \sup_{\zeta \in U} \left( \int_U 1_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k(1-t)^{-\frac{n+1}{p}} dt,
\]
where $x = (\tau, w)$.

Using the change of variables $u = tw$ and the estimate
\[
|tU \cap \{u+(1-t)U\}| \leq |U| \min(t^n, (1-t)^n),
\]
we finally get
\[
\int_0^1 \sup_{a \leq \eta < b} \left( \int_a^b \beta'(\tau) 1_{t\tau+(1-t)[a, b]}(\eta) d\tau \right)^{\frac{1}{p}} \sup_{\zeta \in U} \left( \int_U 1_{tw+(1-t)U}(\zeta) dw \right)^{\frac{1}{q}} t^k(1-t)^{-\frac{n+1}{p}} dt \leq |U|^{1/q} \|\beta\|_{L^p([a, b))} \int_0^1 t^{k-n/q}(1-t)^{-(n+1)/p} \min(t^{n/q}, (1-t)^{n/q}) dt
\]

The conditions on $p$ and $q$ imply the finiteness of the last integral. \qed

Corollary 4.4 is a key ingredient in the proof of our main result, Theorem 7.1. Unfortunately, for being able to “separate” the variable $t$, we have to impose the stronger constraint $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1} - \frac{1}{q(n+1)}$ than the condition $\frac{1}{p} - \frac{1}{q} < \frac{1}{n+1}$ given by Proposition 4.4.
5. A New Homotopy Operator for $q \geq p$.

The Case of a Convex Domain in $\mathbb{R}^n$

In the previous section, we considered the homotopy operator on $\Omega^*_{\text{loc}}$ of the form

$$A_\alpha = \int_D \alpha(y)K_y \omega(x)dy$$

for a convex set $D$ in $\mathbb{R}^n$. We will need to modify $A$ for obtaining some estimates.

Consider the same operator $K_y$ as in the previous section:

$$\psi_y(x,t) = tx + (1-t)y, \quad K_y \omega(x) = \int_0^1 (\psi_y)_t \omega dt.$$  \hspace{1cm}

Recall that $dK_y \omega + K_y d\omega = \omega$. Choose a smooth positive function $\alpha : D \to \mathbb{R}$ such that $\int_D \alpha(x)dx = 1$ and put

$$A_\alpha \omega(x) := \int_D \alpha(y)K_y \omega(x)dy, \quad \omega \in \Omega^*_{\text{loc}}.$$  \hspace{1cm}

By a straightforward calculation,

$$dA_\alpha \omega = \partial \left( \int_D \alpha(y)K_y \omega(x)dy \right) = \int_D \alpha(y)d_x K_y \omega(x)dy;$$

$$A_\alpha d\omega = \int_D \alpha(y)K_y d\omega(x)dy;$$

$$dA_\alpha \omega + A_\alpha d\omega = \int_D \alpha(y)[d_x K_y \omega(x) + K_y d\omega(x)]dy = \int_D \alpha(y)\omega(x)dy = \omega.$$

In particular, if $d\omega = 0$ then

$$dA_\alpha \omega = \omega.$$  \hspace{1cm}

The definition of $A_\alpha$ easily implies the following

**Proposition 5.1.** The homotopy operator $A_\alpha$ takes smooth forms to smooth forms.

**Definition 5.2.** Call a smooth positive function $\alpha : D \to \mathbb{R}$ an admissible weight for a convex domain $D \subset \mathbb{R}^n$ and $p \geq 1$ if

$$\int_D \alpha(x)dx = 1; \quad \|\alpha\|_{L^{p'}(D)} < \infty; \quad \|\alpha(y)y\|_{L^{p'}(D)} < \infty.$$  \hspace{1cm}

For $p \geq 1$, we as usual put

$$p' = \begin{cases} \frac{n}{p-1} & \text{if } p > 1, \\ \infty & \text{if } p = 1 \end{cases}$$

**Theorem 5.3.** Suppose that $q \geq p \geq 1$, $D \subset \mathbb{R}^n$ is a convex set, $\beta : D \to \mathbb{R}$ is a positive smooth function, and $\alpha : D \to \mathbb{R}$ is an admissible weight. If

$$C_1(k,p,q,n,\beta) := \int_0^1 \sup_{z \in D} \|\beta(z)\|_{L^q(D,z)}t^k(1-t)^{-n/p}dt < \infty;$$

$$C_2(k,p,q,n,\beta) := \int_0^1 \sup_{z \in D} \|x\|_{L^q(D,z)}t^k(1-t)^{-n/p}dt < \infty$$

then for any $\omega \in C^\infty L^p(D,\Lambda^k)$ we have

$$\|A_\alpha \omega\|_{L^q(D,\Lambda^{k-1},\beta)} \leq C(k,p,q,\alpha,\beta,n)\|\omega\|_{L^p(D,\Lambda^k)}$$
where

\[ C(k, p, q, \alpha, \beta, n) = \|\alpha(y)|y|\|_{L^p(D)} C_1(k, p, q, n, \beta) + \|\alpha\|_{L^p(D)} C_2(k, p, q, n, \beta). \]

**Proof.** Put \( \xi := A_\omega \). If \( p > 1 \) then, by Hölder’s inequality, we infer

\[
\|A_\omega\|_{L^q(\mathbb{D}, \mathbb{A}^{k-1}, \beta)} = \left\| \beta(x) \int_D \alpha(y) K_y \omega(x) dy \right\|_{L^q(\mathbb{D}, \mathbb{A}^{k-1}, dx)}
\leq \|A_\omega\|_{L^q(\mathbb{D}, \mathbb{A}^{k-1}, \beta)} \|K_y \omega(x)\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, dy)} \|\alpha(y)|y|\|_{L^p(D)} \|x-y\|_{L^p(D)}.
\]

The above estimate also obviously holds for \( p = 1 \). By the triangle inequality,

\[
\|\alpha(y)|x-y|\|_{L^p(D, dy)} \leq |x|\|\alpha(y)\|_{L^p(D, dy)} + \|\alpha(y)|y|\|_{L^p(D, dy)}.
\]

Therefore,

\[
\|A_\omega\|_{L^q(\mathbb{D}, \mathbb{A}^{k-1})} \leq \|\alpha(y)|y|\|_{L^p(D, dy)} \|\beta(x)\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, dx)} \|K_y \omega(x)\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, dy)} \|\alpha(y)|y|\|_{L^p(D)} \|x-y\|_{L^p(D)}.
\]

By Proposition 4.1,

\[
\|\beta(x)\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, dx)} \leq C_1(k, p, q, n, \beta) \|\omega\|_{L^p(\mathbb{D}, \mathbb{A}^{k})};
\]

\[
\|\beta(x)|x|\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, dx)} \leq C_2(k, p, q, n, \beta) \|\omega\|_{L^p(\mathbb{D}, \mathbb{A}^{k})}.
\]

The theorem is proved. \( \square \)

**Corollary 5.4.** Suppose that \( q \geq p \geq 1 \), \( D \subset \mathbb{R}^n \) is a convex set, \( \alpha: [a, b) \rightarrow \mathbb{R} \) is an admissible weight, \( \beta, \gamma: D \rightarrow \mathbb{R} \) are positive smooth functions. If the conditions

\[
C_1(k, \overline{\tau}, q, n, \beta) := \int_0^1 \sup_{z \in D} \|\beta(x)1_{tx+(1-t)D}(z)\|_{L^q(D, dz)} t^k (1-t)^{-n/\overline{\tau}} dt < \infty;
\]

\[
C_2(k, \overline{\tau}, q, n, \beta) := \int_0^1 \sup_{z \in D} \||x|\beta(x)1_{tx+(1-t)D}(z)\|_{L^q(D, dz)} t^k (1-t)^{-n/\overline{\tau}} dt < \infty;
\]

\[
Q(k, \overline{\tau}, p, \gamma) := \|\gamma^{-1}\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, \overline{\tau})} < \infty
\]

are fulfilled for some \( \overline{\tau} \), \( 1 \leq \overline{\tau} \leq p \) (for \( \overline{\tau} = p \), we put \( \overline{\tau} = \overline{\tau} = \infty \)), then the inequality

\[
\|A_\omega\|_{L^q(\mathbb{D}, \mathbb{A}^{k-1}, \beta)} \leq C(k, p, q, \alpha, \beta, \gamma, n) \|\omega\|_{L^p(\mathbb{D}, \mathbb{A}^{k-1}, \gamma)},
\]

where

\[
C(k, p, q, \alpha, \beta, \gamma, n) = Q(k, \overline{\tau}, p, \gamma) C(k, \overline{\tau}, q, n, \alpha, \beta),
\]

holds for any \( \omega \in C^\infty L^p(D, \mathbb{A}^{k}) \).
Proof. By Theorem 5.3,
\[ \| A_0 \omega \|_{L^q(D, \Lambda^{k-1}, \beta)} \leq C(k, p, q, n, \alpha, \beta) \| \omega \|_{L^p(D, \Lambda^k)}. \]
If \( p < p \) then, using Hölder’s inequality, we have
\[ \| \omega \|_{L^p(D, \Lambda^k)} \leq \| \gamma \omega \|_{L^p(D, \Lambda^k)} \| \gamma^{-1} \|_{L^p/(p-p)(D)}. \] (4)
Inequality 4 also holds for \( p = p \).

The corollary follows. \( \square \)

Corollary 5.5. Suppose that \( q \geq p \geq 1, \frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}, U \) is a bounded convex set in \( \mathbb{R}^n, D = [a, b] \times U, \alpha : [a, b) \rightarrow \mathbb{R} \) is an admissible weight, and \( \beta, \gamma : [a, b) \rightarrow \mathbb{R} \) are positive smooth functions. If the conditions \( \| \beta \|_{L^q([a, b))} < \infty, \| \gamma \beta(\tau) \|_{L^q([a, b))} < \infty, \) and \( \| \gamma^{-1} \|_{L^q(p-p, \gamma(\tau) \langle a, b \rangle)} < \infty \) are fulfilled for some \( \bar{p}, 1 \leq \bar{p} \leq p \) (for \( \bar{p} = p \), we put \( \bar{p} = \infty \)), then the inequality
\[ \| A_0 \omega \|_{L^q(D, \Lambda^{k-1}, \beta)} \leq \text{const} \| \omega \|_{L^p(D, \Lambda^k, \gamma)} \]
with some constant depending \( k, p, q, n, \alpha, \beta, \gamma \) holds for any \( \omega \in C^\infty L^p(D, \Lambda^k, \gamma) ). \]

Proof. Suppose that a number \( \bar{p} \leq p \) satisfies the conditions of the corollary.

If \( x \in D \) then \( x = (\tau, w) \), where \( \tau \in [a, b) \) and \( w \in U \). By Corollary 4.4, since \( \frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)} \) and \( \| \beta \|_{L^q([a, b))} < \infty \), we have
\[ \int_0^1 \sup_{z \in D} \| \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \leq \| U \|^{1/q} \| \beta \|_{L^q([a, b))} \int_0^1 t^{-n/q} (1-t)^{-(n+1)/p} \text{min}(t^{n/q}, (1-t)^{n/q}) dt. \]

On the other hand, since \( \| \gamma \beta(\tau) \|_{L^q([a, b))} < \infty \), we have by Corollary 4.4
\[ \int_0^1 \sup_{z \in D} \| \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \]
\[ = \int_0^1 \sup_{z \in D} \| \sqrt{\tau^2 + w^2} \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \]
\[ \leq \sqrt{2} \int_0^1 \sup_{z \in D} \| |\tau| + |w| \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \]
\[ \leq \sqrt{2} \int_0^1 \sup_{z \in D} \| |w| \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \]
\[ + \sqrt{2} \int_0^1 \sup_{z \in D} \| w \beta(\tau) 1_{(1-t)D}(z) \|_{L^q(D, dx)} t^k (1-t)^{-(n+1)/\bar{p}} dt \]
\[ \leq \sqrt{2} \| U \|^{1/q} \| \beta(\tau) \|_{L^q([a, b))} \int_0^1 t^{-n/q} (1-t)^{-(n+1)/\bar{p}} \text{min}(t^{n/q}, (1-t)^{n/q}) dt \]
\[ + \sqrt{2} \sup_{w \in U} \| w \beta(\tau) \|_{L^q([a, b))} \int_0^1 t^{-n/q} (1-t)^{-(n+1)/\bar{p}} \text{min}(t^{n/q}, (1-t)^{n/q}) dt < \infty. \]

The relations \( \| \beta \|_{L^q([a, b))} < \infty \) and \( \| \beta(\tau) \|_{L^q(D)} < \infty \) enable us to apply Corollary 4.4 and obtain the desired assertion. \( \square \)
6. Globalization: the Sobolev–Poincare Inequality on a Cylinder

Here we globalize the Sobolev–Poincare inequality to cylinders. The main assertion of the section is

**Theorem 6.1.** Suppose that $M$ is the cylinder $[a, b) \times N$, where $N$ is a closed manifold of dimension $n$, $q \geq p \geq 1$, $\frac{1}{p} - \frac{1}{q} < \frac{q-1}{q(n+1)}$, and $\beta, \gamma : [a, b) \to \mathbb{R}$ be positive smooth functions. Let $\omega$ be an exact $k$-form in $\mathcal{C}^\infty L^p(M, \Lambda^k, \gamma)$. If the conditions $\|\beta\|_{L^q([a, b))} < \infty$, $\|t\beta(t)\|_{L^q([a, b))} < \infty$, and $\|\gamma^{-1}\|_{L^p([a, b))} < \infty$ are fulfilled for some $\beta$, $1 \leq p \leq \infty$ (for $\beta = p$, we put $\frac{\beta(p)}{p^2} = \infty$), then there exists a $(k - 1)$-form $\xi \in \mathcal{C}^\infty L^q(M, \Lambda^{k-1}, \beta)$ such that

$$d\xi = \omega$$

and $\|\xi\|_{L^q(M, \Lambda^{k-1}, \beta)} \leq \text{const} \|\omega\|_{L^q(M, \Lambda^k, \gamma)}$.  

(5)

Let $\hat{U} = \{\hat{U}_x\}, x \in N$, be a coordinate open cover of the base $N$. At each point $x \in N$, consider a geodesic ball $U_x$ that is geodesically convex (small balls are geodesically convex, see [7, Proposition 4.2]) and such that its closure (a compact set) is in $\hat{U}_x$. Then $U' = \{U_x\}_x$ is a good cover of $N$. Extract a finite subcover $U = \{U_i\}, i = 1, \ldots, l$, from $U'$. Since $U$ consists of geodesic balls, it is a good cover, i.e., all finite intersections $U_I = U_{i_0} \cap \cdots \cap U_{i_{s-1}}, I = (i_0, \ldots, i_{s-1})$, are bi-Lipschitz diffeomorphic to convex open sets with compact closure in $\mathbb{R}^n$. With such a cover $U$, we associate the corresponding cover $V = \{V_i = [a, b) \times U_i\}, i = 1, \ldots, l$, of $M$ and put $V_I = V_{i_0} \cap \cdots \cap V_{i_{s-1}}$ for $I = (i_0, \ldots, i_{s-1})$. Then each intersection $V_I$ is a convex set with compact closure in $\mathbb{R}^n$. By analogy with [22], we put

$$K^{k,0} := \mathcal{C}^\infty (M, \Lambda^k); \quad K^{k,s} := \bigoplus_{i_0 < \cdots < i_{s-1}} \mathcal{C}^\infty (V_I, \Lambda^k).$$

Given $\zeta \in K^{r,s}$, denote by $\zeta_I, I = (i_0, \ldots, i_{s-1})$, $i_0 < \cdots < i_s$, the components of $\zeta$. Define a coboundary operator $\delta : K^{k,s} \to K^{k,s+1}$ as follows:

$$(\delta \zeta)_{\nu} = \sum_{\tau=0}^{s} (-1)^{\tau} \zeta_{\rho_{\tau},\cdots,\rho_{s}} \bigg|_{V_{\nu}}, \quad J = (j_0, \ldots, j_s).$$

Let $L^q(K^{k,s})$ be the space of elements $\zeta \in K^{k,s}$ with the finite norm

$$\|\zeta\|_{L^q(K^{k,s}, \beta)} = \sum_{i_0 < \cdots < i_{s-1}} \|\zeta_I\|_{L^q(V_I, \Lambda^k, \beta)}.$$

As usual, if $\zeta \in K^{k,s}$ has components $\zeta_I, I = (i_0, \ldots, i_{s-1}), i_0 < \cdots < i_s$, and $\nu$ is a permutation of the set $\{0, \ldots, s - 1\}$ then $\alpha_{\nu(I)} = \alpha_I \text{sign} \nu$.

The following proposition is a modification for our case of [22] Proposition 3.6], which is in turn an adaptation of [3] Propositions 8.3 and 8.5].

**Proposition 6.2.** $(K^{k,\bullet}, \delta)$ is an exact complex. Moreover, if $\lambda \in L^q(K^{k,s+1}, \beta)$ satisfies $\delta \lambda = 0$ then there exists $\zeta \in L^q(K^{k,s}, \beta)$ such that $\lambda = \delta \zeta$ and

- $\|\zeta\|_{L^q(K^{k,s}, \beta)} \leq \text{const} \|\lambda\|_{L^q(K^{k,s+1}, \beta)}$
- $\|d\zeta\|_{L^q(K^{k,s+1}, \beta)} \leq \text{const} \left( \|\lambda\|_{L^q(K^{k,s+1}, \beta)} + \|d\lambda\|_{L^q(K^{k,s+1}, \beta)} \right)$.

**Proof.** The fact that $(K^{k,\bullet}, \delta)$ is an exact complex was established in [3] Propositions 8.3 and 8.5] but we will give the standard argument for completeness. If
THE SOBOLEV–POINCARÉ INEQUALITY AND THE $L_{q,p}$-COHOMOLOGY

\[ \kappa \in L_q(K^{k,s}, \beta) \text{ then} \]

\[ (\delta(\delta \kappa))_{i_0 \ldots i_{s+1}} = \sum_r (-1)^i (\delta \kappa)_{i_0 \ldots \hat{i}_r \ldots i_{s+1}} \]

\[ = \sum_{i < r} (-1)^i \kappa_{i_0 \ldots \hat{i}_r \ldots i_{s+1}} + \sum_{i < r} (-1)^{i-1} \kappa_{i_0 \ldots \hat{i}_r \ldots i_{s+1}} = 0. \]

Suppose that $\lambda \in L_q(K^{k,s+1}, \beta)$ is such that $\delta \lambda = 0$. Let $\tilde{\rho}_j$ be a partition of unity subordinate to the cover \{U_i\} of N. Then the functions $\rho_j : M \to \mathbb{R}$, $\rho_j(t, x) = \tilde{\rho}_j(x)$ for all $(t, x) \in M = [a, b] \times N$, constitute a partition of unity subordinate to the cover \{V_i\} of M. Put

\[ \kappa_{i_0 \ldots i_{s-1}} := \sum_j \rho_j \lambda_{i_0 \ldots i_{s-1}}. \]

(6)

Show that $\delta \kappa = \lambda$.

We have

\[ (\delta \kappa)_{i_0 \ldots i_s} = \sum_r (-1)^r \kappa_{i_0 \ldots \hat{i}_r \ldots i_s} = \sum_{r, j} (-1)^r \rho_j \lambda_{i_0 \ldots i_s}. \]

Since $\lambda$ is a cocycle,

\[ (\delta \lambda)_{i_0 \ldots i_s} = \lambda_{i_0 \ldots i_s} + \sum_r (-1)^{r+1} \lambda_{i_0 \ldots i_s} = 0. \]

Hence,

\[ (\delta \kappa)_{i_0 \ldots i_s} = \sum_j \rho_j \sum_r (-1)^r \lambda_{i_0 \ldots \hat{i}_r \ldots i_s} = \sum_j \rho_j \lambda_{i_0 \ldots i_s} = \lambda_{i_0 \ldots i_s}. \]

Thus, $(K^{k, \bullet}, \delta)$ is indeed an exact complex.

The element $\kappa$ defined by (6) admits the estimates of the norms mentioned in the proposition.

Indeed, we infer

\[ \|\kappa\|_{L_q(K^{k,s}, \beta)} = \sum_{i_0 < \ldots < i_{s-1}} \left\| \sum_j \rho_j \lambda_{i_0 \ldots i_{s-1}} \right\|_{L^q(U_i)} \]

\[ \leq \sum_{i_0 < \ldots < i_{s-1}} \sum_j \|\rho_j \lambda_{i_0 \ldots i_{s-1}}\|_{L^q(U_i)} \]

\[ \leq \sum_{i_0 < \ldots < i_{s-1}} \sum_j \|\lambda_{i_0 \ldots i_{s-1}}\|_{L^q(U_j, t)} \leq \|\lambda\|_{L_q(K^{k,s+1}, \beta)}, \]

which gives the first estimate of the proposition.

Let us prove the second estimate. We have

\[ d\kappa_{i_0 \ldots i_{s-1}} = \sum_j (d\rho_j \wedge \lambda_{i_0 \ldots i_{s-1}} + \rho_j d\lambda_{i_0 \ldots i_{s-1}}). \]
Therefore,

$$
\| \mathcal{X} \|_{L^q(K^{k+1, \beta})} = \sum_{i_0 < \cdots < i_{s-1}} \left( \sum_j \| d\rho_j \wedge \lambda_{j i_0 \ldots i_{s-1}} + \rho_j d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right)
$$

$$
\leq \sum_{i_0 < \cdots < i_{s-1}} \left( \sum_j \| d\rho_j \wedge \lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} + \| \rho_j d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right)
$$

$$
\leq \text{const} \sum_{i_0 < \cdots < i_{s-1}} \left( \| \lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} + \| d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right)
$$

$$
= \text{const} \left( \| \lambda \|_{L^q(K^{k+1, \beta})} + \| d\lambda \|_{L^q(K^{k+1, \beta})} \right).
$$

Now, applying the general scheme of [22], we first construct some elements $\xi^s \in L^q(K^{k-s-1, s+1, \beta})$ and then elements $x^s \in L^q(K^{k-s-1, \beta})$ such that $x = x^0 \in C^\infty L^q(M, K^{k, \beta})$ is an element satisfying the claim of Theorem 6.1.

**Construction of the elements** $\xi^s \in L^q(K^{k-s-1, s+1, \beta})$.

Put $\xi_{-1} = \omega$ and define (by induction) $\xi^s$ by setting its component $(\xi^s)_I$ to be a solution to the equation

$$
\delta \xi^s_I = (\delta \xi^{s-1})_I
$$

in $V_I$, $I = (i_0, \ldots, i_s)$ such that

$$
\| \xi^s_I \|_{L^q(V_I, K^{k-s-1, \beta})} \leq \text{const} \| (\delta \xi^{s-1})_I \|_{L^q(V_I, K^{k-s, \beta})}
$$

for $0 \leq s \leq k - 1$.

Note that such a solution always exists due to the local Sobolev–Poincaré inequality (Corollary 6.3) since $V_I$ is bi-Lipschitz diffeomorphic to a cylinder over a convex subset in $\mathbb{R}^n$ of finite volume.

We have the following estimate of the weighted $q$-norm of $\xi^s$:

**Proposition 6.3.** If $I = (i_0, \ldots, i_s)$ then

$$
\| \xi^s_I \|_{L^q(V_I, K^{k-s-1, \beta})} \leq \text{const} \| \omega \|_{L^q(M, K^{k, \gamma})}.
$$

**Proof.** Use induction on $s$. For $s = 0$, the assertion follows from the local Sobolev–Poincaré inequality. Let now $s > 0$. We infer

$$
\| \xi_I^s \|_{L^q(V_I, K^{k-s-1, \beta})} \leq \text{const} \| (\delta \xi^{s-1})_I \|_{L^q(V_I, K^{k-s, \beta})}
$$

$$
\leq \text{const} \sum_{r=0}^{s} \| \xi^{s-1}_{i_0 \ldots i_r \ldots i_{s-1}} \|_{L^q(V_I, K^{k-s, \beta})}
$$

$$
\leq \text{const} \sum_{r=0}^{s} \| \omega \|_{L^p(M, \gamma)} \leq \text{const} \| \omega \|_{L^p(M, \gamma)}
$$

Note that $\xi^{k-1}$ is a collection of 0-forms satisfying the condition $d\delta \xi^{k-1} = 0$. Thus, the functions $(\delta \xi^{k-1})_I$ are constants on each set $V_I$, $I = (i_0, \ldots, i_k)$. The global constant functions $\delta \xi^{k-1}_I$ on $M$ belong to $L^q(M, \beta)$ due to the hypotheses on $\beta$.

The following assertion is Theorem 3.10 in [22]:

$$
\begin{align*}
\| \mathcal{X} \|_{L^q(V_I, K^{k+1, \beta})} &= \sum_{i_0 < \cdots < i_{s-1}} \left( \sum_j \| d\rho_j \wedge \lambda_{j i_0 \ldots i_{s-1}} + \rho_j d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right) \\
&\leq \sum_{i_0 < \cdots < i_{s-1}} \left( \sum_j \| d\rho_j \wedge \lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} + \| \rho_j d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right) \\
&\leq \text{const} \sum_{i_0 < \cdots < i_{s-1}} \left( \| \lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} + \| d\lambda_{j i_0 \ldots i_{s-1}} \|_{L^q(U_j)} \right) \\
&= \text{const} \left( \| \lambda \|_{L^q(K^{k+1, \beta})} + \| d\lambda \|_{L^q(K^{k+1, \beta})} \right),
\end{align*}
$$

\[\square\]
Lemma 6.4. There exists $c \in K^{0,k}$ with constant components $c_I$, $I = (i_0, \ldots, i_{k-1})$, such that

$$(\delta c)_I = \sum_{r=0}^{k} (-1)^r c_{i_0, \ldots, i_r, \ldots, i_{k-1}} (\delta \xi^{k-1})_I, \quad I = (i_0, \ldots, i_k).$$

In addition, there exist numbers $b_{I,L} \in \mathbb{R}$, $I = (i_0, \ldots, i_{k-1})$, $L = (i_0, \ldots, i_k)$, such that

$$c_I = \sum_L b_{I,L} (\delta \xi^{k-1})_L,$$

where $b_{I,L}$ depend on the chosen cover $\mathcal{U}$ of $N$.

We have

Proposition 6.5. The constants $c_I$ of Lemma 6.4 satisfy the estimate

$$\|c_I\|_{L^q(V_1, \beta)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^{k,\gamma})}.$$

Proof. By Lemma 6.3 each $c_I$ is representable as $c_I = \sum_L b_{I,L} (\delta \xi^{k-1})_L$. Hence,

$$\|c_I\|_{L^q(V_1, \beta)} \leq \sum_L |b_{I,L}| \|\delta \xi^{k-1}_L\|_{L^q(V_1, \beta)}.$$

Since $(\delta \xi^{k-1})_L$ is a globally defined constant function on $M$ as in the proof of Proposition 6.2 we have

$$\|\delta \xi^{k-1}_L\|_{L^q(V_1, \beta)} = \frac{\|\beta\|_{L^q(V_1, \beta)} (\text{Vol}(U_I))^{1/q}}{\beta \|L^q(V_1, \beta)} \|\delta \xi^{k-1}_L\|_{L^q(V_1, \beta)} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^{k,\gamma})}.$$

This gives the estimate of the proposition.

Construction of the elements $x^s \in L^q(K^{k-s-1,\beta})$.

Let us now glue all the forms $\xi^s$, $s = 0, \ldots, k-1$, into a global form $\xi$ satisfying (9). Construct by induction elements $x^s \in L^q(K^{k-s-1,\beta})$, $s = k-1, \ldots, 1, 0$, such that $\xi = x^0$ is a desired form on $M$.

Put $\xi^{k-1}_I = \xi_I - c_I$, where $c_I$ is as in Lemma 6.3 $I = (i_0, \ldots, i_{k-1})$. We have $d\xi^{k-1}_I = d\xi^k_I$ and $\delta \xi^{k-1}_I = 0$. By Proposition 6.2 there exists $x^{k-1} \in L^q(K^{0,k-1,\beta})$ such that $d\xi^{k-1} = \xi^{k-1}$ and

$$\|dx^{k-1}\|_{L^q(K^{0,k-1,\beta})} \leq \text{const} \|\xi^{k-1}\|_{L^q(K^{0,k-1,\beta})},$$

$$\|x^{k-1}\|_{L^q(K^{0,k-1,\beta})} \leq \text{const} \left( \|\xi^{k-1}\|_{L^q(K^{0,k-1,\beta})} + \|d\xi^{k-1}\|_{L^q(K^{1,k-1,\beta})} \right).$$

Propositions 6.3 and 6.5 yield

$$\|x^{k-1}\|_{L^q(K^{0,k-1,\beta})} \leq \text{const} \|\omega\|_{L^p(M, \Lambda^{k,\gamma})}$$

and

$$\|dx^{k-1}\|_{L^q(K^{0,k-1,\beta})} \leq \text{const} \left( \|\omega\|_{L^p(M, \Lambda^{k,\gamma})} + \|\delta \xi^{k-2}\|_{L^q(K^{1,k-1,\beta})} \right) \leq \text{const} \left( \|\omega\|_{L^p(M, \Lambda^{k,\gamma})} + \|\xi^{k-2}\|_{L^q(K^{k-1,\beta})} \right) \leq \text{const} \|\omega\|_{L^p(M, \Lambda^{k,\gamma})}. \quad (10)$$

Suppose that $x^{k-(r-1)}$ is already constructed. By Proposition 6.2 there exists $x^{k-r}$ such that

$$\delta x^{k-r} = \xi^{k-r} - dx^{k-r+1},$$

and

$$\|dx^{k-r}\|_{L^q(K^{0,k-r-1,\beta})} \leq \text{const} \left( \|\omega\|_{L^p(M, \Lambda^{k-r,\gamma})} + \|\delta \xi^{k-r-1}\|_{L^q(K^{1,k-r-1,\beta})} \right) \leq \text{const} \left( \|\omega\|_{L^p(M, \Lambda^{k-r,\gamma})} + \|\xi^{k-r-1}\|_{L^q(K^{k-r,\beta})} \right) \leq \text{const} \|\omega\|_{L^p(M, \Lambda^{k-r,\gamma})}.$$
Since \( \delta \) \( \omega \) assume that \( x^* \) admit the estimates:

1. \( \| x^{k-r} \|_{L^s(K^{r-1,k-r},\beta)} \leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)} \)
2. \( \| dx^{k-r} \|_{L^s(K^{r-1,k-r},\beta)} \leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)} \).

**Proof.** Use induction on \( r \). For \( r = 1 \), (1) and (2) are just estimates \( \text{[9]} \) and \( \text{[10]} \). Assume that \( r > 1 \). For proving estimate (2), observe that, by the induction hypothesis and (12),

\[
\| dx^{k-r} \|_{L^s(K^{r-1,k-r},\beta)} \leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)} + \| dx^{k-r+1} \|_{L^s(K^{r-1,k-r+1},\beta)} \\
\leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)}.
\]

Now, Proposition \( \text{[6.3]} \) and estimates (11) and (2) yield

\[
\| x^{k-r} \|_{L^s(K^{r-1,k-r},\beta)} \leq \text{const} \| x^{k-r} - dx^{k-r+1} \|_{L^s(K^{r-1,k-r+1},\beta)} \\
\leq \text{const} \| x^{k-r} - dx^{k-r+1} \|_{L^s(K^{r-1,k-r+1},\beta)} \\
\leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)}.
\]

Finally, put \( x = x^0 \). Then \( dx = \omega \). Indeed, we have

\[
\delta(\omega - dx^0) = \delta \omega - d\delta x^0 = \delta \omega - d(\xi^0 - dx^1) = \delta \omega - d\xi^0 = 0.
\]

Since \( \delta(\omega - dx^0) \mid_i = (\omega - dx^0) \mid_i \), we infer that \( \omega = dx^0 \) on \( M \). By Proposition \( \text{[6.6]} \)

\[
\| \xi \|_{L^s(M,\Lambda^{k-1},\beta)} = \| x^0 \|_{L^s(K^{k-1,0},\beta)} \leq \text{const} \| \omega \|_{L^p(M,\Lambda^k,\beta)}.
\]

Theorem \( \text{[6.1]} \) is completely proved.

7. \( L_{q,p} \)-Cohomology of a Twisted Cylinder

**Theorem 7.1.** Suppose that \( N \) is a closed manifold of dimension \( n \), \( H^k_{\text{DR}}(N) = 0 \), \( q \geq p \geq 1 \), and \( \frac{1}{p} + \frac{1}{q} < \frac{n}{n+1} \). If

\[
\| \max(F_{k-2,q},F_{k-1,q}) \|_{L^s([a,b])} < \infty, \quad \| t \max(F_{k-2,q},F_{k-1,q})(t) \|_{L^s([a,b])} < \infty
\]

and

\[
\| \min(f_{k-1,p},f_{k,p}) \|_{L^p([a,b])} < \infty
\]

for some \( \overline{p} \), \( 1 \leq \overline{p} \leq p \) (for \( \overline{p} = p \), we put \( \frac{1}{p-\overline{p}} = \infty \)), then \( H^k_{q,p}(C_{a,b}N) = 0 \).
Proof. Let $\overline{M}$ be the cylinder $[a, b] \times N$ with the usual product metric. By the Künneth formula for the de Rham cohomology, we have

$$H^k_{DR}(\overline{M}) = H^k_{DR}(N) = 0.$$  

Using expression (3) for the norm and the definition of $f_{l,p}$, we infer

$$\|\omega\|_{L^p(\overline{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))}$$

$$= \left[ \int_a^b \left\{ \min(f_{k-1,p}(t), f_{k,p}(t)) \right\} \int_N (|\omega_A(t, x)|^2_N + |\omega_B(t, x)|^2_N)^{\frac{p}{2}} \, dx \, dt \right]^{\frac{1}{p}}$$

$$\leq \left[ \int_a^b \int_N \left( h^{2(\frac{k}{2} - 1)}(t, x)|\omega_A(t, x)|^2_N + h^{2(\frac{k}{2} - 1)}(t, x)|\omega_B(t, x)|^2_N \right)^{\frac{p}{2}} \, dx \, dt \right]^{\frac{1}{p}}$$

$$= \|\omega\|_{L^p(C^h_{a,b}N, \Lambda^k)}.$$  

Thus, $\omega \in C^\infty L^p(\overline{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p}))$. Since the de Rham cohomology $H^k_{DR}(\overline{M})$ is trivial, $\omega$ is exact, and we can apply Theorem 6.11 by which there exists $\xi \in C^\infty L^q(\overline{M}, \Lambda^k; \max(F_{k-2,q}(t), F_{k-1,q}(t))$ with

$$\|\xi\|_{L^q(\overline{M}, \Lambda^k, \max(F_{k-2,q}, F_{k-1,q}))} \leq \|\omega\|_{L^p(\overline{M}, \Lambda^k, \min(f_{k-1,p}, f_{k,p})).}$$  

For this form $\xi$, we have

$$\|\xi\|_{L^q(C^h_{a,b}N, \Lambda^{k-1})}$$

$$= \left[ \int_a^b \int_N \left( h^{2(\frac{k}{2} - 1)}(t, x)|\xi_A(t, x)|^2_N + h^{2(\frac{k}{2} - 1)}(t, x)|\xi_B(t, x)|^2_N \right)^{\frac{q}{2}} \, dx \, dt \right]^{\frac{1}{q}}$$

$$\leq \left[ \int_a^b \int_N \left( \max(F_{k-2,q}(t), F_{k-1,q}(t)) \right)^q \int_N (|\xi_A(t, x)|^2_N + |\xi_B(t, x)|^2_N)^{\frac{q}{2}} \, dx \, dt \right]^{\frac{1}{q}}$$

$$= \|\xi\|_{L^q(\overline{M}, \Lambda^{k-1}, \max(F_{k-2,q}, F_{k-1,q}))}.$$  

Combining (8), (9), and (10), we obtain

$$\|\xi\|_{L^q(C^h_{a,b}N, \Lambda^{k-1})} \leq \|\omega\|_{L^p(C^h_{a,b}N, \Lambda^k)}.$$  

Thus, $C^\infty H^k_{q,p}(C^h_{a,b}N) = 0$, and hence, by Theorem 2.14 also $H^k_{q,p}(C^h_{a,b}N) = 0$. \qed

8. $L_{q,p}$-COHOMOLOGY OF AN ASYMPTOTIC TWISTED CYLINDER

Recall the following definition, given in [10]:

**Definition 8.1.** We refer to a pair $(M, X)$ consisting of an $m$-dimensional manifold $M$ and an $m$-dimensional compact submanifold $X$ with boundary as an asymptotic twisted cylinder $AC^h_{a,b} \partial X$ if $M \setminus X$ is bi-Lipschitz diffeomorphically equivalent to the twisted cylinder $C^h_{a,b} \partial X$.

For asymptotic twisted cylinders, Theorem 7.1 gives:

**Theorem 8.2.** Let $(M, X) = AC^h_{a,b} \partial X$ be an asymptotic twisted cylinder with $\dim M = \dim X = m = n + 1$. Assume that $q \geq p \geq 1, \frac{1}{p} - \frac{1}{q} < \frac{2 - 1}{qm}$, and $H^k_{DR}(X) = 0$. If

$$\|\max(F_{k-2,q}, F_{k-1,q})\|_{L^q([a,b])} < \infty, \quad \|t \max(F_{k-2,q}, F_{k-1,q})(t)\|_{L^q([a,b])} < \infty$$

then

$$\|\Pi_{AC^h_{a,b} \partial X}^k\|_{L^q([a,b])} < \infty.$$
and
\[
\|\{\min(f_{k-1,p}, f_{k,p})\}^{-1}\|_{L^\infty([a,b])} < \infty,
\]
for some \(p\), \(1 \leq p \leq \overline{p}\) (for \(p = \overline{p}\), we put \(\frac{\overline{p}}{p} = \infty\)), then \(H_{\ast,p}^k(M) = 0\).

**Proof.** Since bi-Lipschitz diffeomorphisms preserve \(L_p\) and \(L_q\), and extension by zero gives a topological isomorphism between the spaces \(W_{p_1,p_2}(C_h^k a,b, \partial X)\) and \(W_{p_1,p_2}(M)\) for all \(p_1, p_2\), we have a topological isomorphism
\[
H_{\ast,p_1}^k(M) \cong H_{\ast,p_2}^k(C_h^k a,b, \partial X)
\]
for all \(p_1, p_2\). The theorem now follows from Theorem 7.1. \(\Box\)

9. Examples

Let us analyze the conditions of the last theorems for comparatively simple cases. Suppose that \(N\) is the \(n\)-dimensional sphere \(S^n\). Then \(H_{\ast,p}^k(N) = 0\) for any \(k \neq n\). By the hypothesis of the theorems, \(q \geq p \geq 1\) and \(\frac{1}{p} - \frac{1}{q} + \frac{1}{\overline{p}} < \frac{1}{q(n+1)}\). Put
\[
s(t) := \max_{x \in S^n} h(t,x) \quad \text{and} \quad g(t) := \min_{x \in S^n} h(t,x).
\]
Then, by definition,
\[
I_{1,q,k} := \max (F_{k-2,q}, F_{k-1,q}) = \max(s_{\overline{p}}^{q-k+2}, s_{\overline{p}}^{q-k+1}),
\]
\[
I_{2,q,k}(t) := t \max (F_{k-2,q}, F_{k-1,q})(t) = t \max(s_{\overline{p}}^{q-k+2}(t), s_{\overline{p}}^{q-k+1}(t))
\]
and
\[
I_{3,p,k} := \{\min(f_{k-1,p}, f_{k,p})\}^{-1} = \{\max(g_{\overline{p}}^{q-k+1}, g_{\overline{p}}^{q-k})\}^{-1}.
\]
By the hypotheses of the theorems, we must check the integrability of these three functions in the corresponding degrees under the above-mentioned restrictions on \(p\) and \(q\).

Suppose for simplicity that \(s(t)\) and \(g(t)\) are smooth increasing functions tending to \(\infty\) as \(t \to b - 0\). Denote the maximal integrability intervals for \(s^u\) and \(g^v\) by \((-\infty, \alpha)\) and \((-\infty, \beta)\), i.e. \(s^u\) is integrable on \([a,b]\) for every \(u < \alpha\) and is not integrable for every \(u > \alpha\) and similarly for \(g^v\). Let also \(\alpha_1\) be the supremum of \(\mu\) such that \(ts^\mu(t)\) is integrable on \([a,b]\).

For this case \(I_{1,q,k} = s_{\overline{p}}^{q-k+2}\), \(I_{2,q,k}(t) = t s_{\overline{p}}^{q-k+2}(t)\), and \(I_{3,p,k} = g_{\overline{p}}^{q-k}\).

The conditions of the theorems are fulfilled if
\[
\frac{n}{q} - k + 2 < \min(\alpha, \alpha_1), \quad \frac{n}{p} - k > -\beta.
\]
Note that these inequalities cannot hold simultaneously if \(b = \infty\). In this case, \(\alpha, \alpha_1,\) and \(\beta\) are all negative, whence \(\frac{n}{p} - k > \frac{n}{q} - k + 2\). We thus have \(\frac{1}{p} - \frac{1}{q} > \frac{2}{n}\), which contradicts the hypotheses.

Examine more closely the case of \(0 \leq a < b < \infty\). The function \(t\) is bounded, and hence \(\alpha = \alpha_1\). Therefore, the inequalities for \(I_{1,q,k}\) and \(I_{3,p,k}\) can be combined into one inequality
\[
\frac{k - 2 + \alpha}{n} < \frac{1}{q} < \frac{1}{p} < \frac{k - \beta}{n}.
\]
It means that the additional condition \(k - 2 + \alpha \leq k - \beta\), i.e., \(\alpha + \beta \leq 2\), must be fulfilled.

The last condition is \(\frac{1}{p} - \frac{1}{q} < \frac{q - 1}{q(n+1)}\), i.e., \(p \leq q < \frac{np}{n+1-p}\).
Summarizing, we conclude that for known integrability limits $\alpha$ and $\beta$, we need to check two simple conditions for $p$ and $q$:

$$\alpha + \beta \leq 2, \ p < q < \frac{np}{n + 1 - p}$$

and the inequality

$$\frac{k - 2 + \alpha}{n} < \frac{1}{q} < \frac{1}{p} < \frac{k - \beta}{n}.$$  

for the degree $k$.

Under these conditions, the cohomology of the warped product $C^{f}_{[a,b]}S^n$ vanishes.

For example, if $f(t) = g(t) = (b - t)^{-2}$ then $\alpha = \beta = 1/2$. For $p = 2$ we have $2 \leq q < 2\frac{2n}{n-1}$.

The last inequality yields

$$\frac{k - 3/2}{n} < \frac{1}{q} < \frac{1}{2} < \frac{k - 1/2}{n}.$$  

(16)

Let $q$ be an arbitrary number in $(2, \frac{4}{3})$. Then the second inequality $q < \frac{2n}{n-1}$ gives us the constraint $n < q/(q - 2)$. Since $q/(q - 2) < 4$, we can take $n = 4$. We have $1/2 < (2k - 1)/8$, i.e., $k > 2$. For $k = 3$, the leftmost inequality gives us the fulfilled condition $3/8 < 1/q$. Note that if $q = 2$ then always $q < \frac{2n}{n-1}$. If $n$ is even and $k = \frac{n}{2} + 1$ then all inequalities in (16) are fulfilled. Thus, we have

$$H^3_{q,2}(C^{f}_{[a,b]}S^4) = 0 \quad \text{if} \quad q \in \left[2, \frac{8}{3}\right)$$

and

$$H^{l+1}_{2,2}(C^{f}_{[a,b]}S^{2l}) = 0, \quad l \geq 2.$$

REFERENCES

[1] R. L. Bishop and B. O’Neill: Manifolds of negative curvature, Trans. Am. Math. Soc. 145, 1–49 (1969).

[2] L. P. Bos and P. D. Milman: Sobolev–Gagliardo–Nirenberg and Markov type inequalities on subanalytic domains, Geom. Funct. Anal. 5 (1995), no. 6, 853–923.

[3] R. Bott and L. W. Tu: Differential Forms in Algebraic Topology. Graduate Texts in Mathematics, 82. New York–Heidelberg–Berlin: Springer-Verlag (1982).

[4] A. Boulal, N. E. H. Djaa, M. Djaa and S. Ouakkas: Harmonic maps on generalized warped product manifolds, Bull. Math. Anal. Appl. 4, no. 1, 156–165 (2012).

[5] E. Y. Chen: Geometry of Submanifolds and Its Applications, Tokyo: Science University of Tokyo (1981).

[6] N. E. H. Djaa, A. Boulal, and A. Zagane: Generalized warped product manifolds and biharmonic maps, Acta Math. Univ. Comen., New Ser. 81, no. 2, 283–298 (2012).

[7] M. P. do Carmo: Riemannian Geometry, Boston, MA etc.: Birkhäuser (1992).

[8] M. Falcitelli: A class of almost contact metric manifolds and double-twisted products, Math. Sci. Appl. E-Notes 1, no. 1, 36–57 (2013).

[9] M. Fernández-López, E. García-Río, D. N. Kupeli, and B. Ünal: A curvature condition for a twisted product to be a warped product, Manuscr. Math. 106, no. 2, 213–217 (2001).

[10] V. M. Gol’dshtein and Ya. A. Kopylov: Reduced $L_{q,p}$-cohomology of some twisted products, Annales Math. Blaise Pascal 23 (2016), no. 2, 151–169.

[11] V. M. Gol’dshtein, V. I. Kuz’minov, and I. A. Shvedov: $L_p$-cohomology of warped cylinders, Siber. Mat. Zh. 31 (1990), no. 6, 55–63; English translation in: Siberian Math. J. 31 (1990), no. 6, 919–925.
[12] V. M. Gol’dshtein, V. I. Kuz’minov, and I. A. Shvedov: Reduced $L_p$-cohomology of warped cylinders, Siberian Mat. Zh. 31 (1990), no. 5, 31–42; English translation: Siberian Math. J. 31, no. 5, 716–727.

[13] V. Gol’dshein and M. Troyanov: Sobolev inequalities for differential forms and $L_{q,p}$-cohomology, J. Geom. Anal. 16 (2006), no. 4, 597–632.

[14] V. Gol’dshein and M. Troyanov: The Hölder–Poincaré duality for $L_{q,p}$-cohomology, Ann. Global Anal. Geom. 41 (2012), no. 1, 25–45.

[15] T. Iwaniec and A. Lutoborski: Integral estimates for null Lagrangians, Arch. Rational Mech. Anal. 125 (1993), no. 1, 25–79.

[16] B. H. Kim, D. J. Shounge, T. H. Kang, and H. K. Pak, Conformal transformations in a twisted product space, Bull. Korean Math. Soc. 42, no. 1, 5–15 (2005).

[17] Ya. A. Kopylov, $L_{p,q}$-cohomology and normal solvability, Arch. Math. 89 (2007), no. 1, 87–96.

[18] V. I. Kuz’minov and I. A. Shvedov: On normal solvability of the exterior differentiation on a warped cylinder, Siberian Mat. Zh. 34 (1993), 85–95. English translation in: Siberian Math. J. 34 (1993), no. 1, 73–82.

[19] V. I. Kuz’minov and I. A. Shvedov: On normal solvability of the operator of exterior derivation on warped products, Siberian Mat. Zh. 37 (1996), no. 2, 324–337. English translation in: Siberian Math. J. 37 (1996), no. 2, 276–287.

[20] R. Ponge and H. Reckziegel: Twisted products in pseudo-Riemannian geometry, Geom. Dedicata 48, no. 1, 15–25 (1993).

[21] L. Shartser: De Rham Theory and Semialgebraic Geometry, Thesis (Ph.D.)-University of Toronto (Canada). 2011

[22] L. Shartser: Explicit proof of Poincare inequality for differential forms on manifolds, C. R. Math. Acad. Sci. Soc. R. Can. 33 (2011), no. 1, 21–32.

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