Existence and Multiplicity of Bound State Solutions to a Kirchhoff Type Equation with a General Nonlinearity

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Abstract
In this paper, we consider the following Kirchhoff type equation

\[- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3,\]

where \(a, b > 0\) and \(f \in C(\mathbb{R}, \mathbb{R})\), and the potential \(V \in C^1(\mathbb{R}^3, \mathbb{R})\) is positive, bounded and satisfies suitable decay assumptions. By using a perturbation approach together with a new version of global compactness lemma of Kirchhoff type, we prove the existence and multiplicity of bound state solutions for the above problem with a general nonlinearity. We especially point out that neither the Ambrosetti-Rabinowitz condition with \(\mu > 4\) nor any monotonicity assumption on \(f\) is required. Moreover,

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the potential $V$ may not be radially symmetry or coercive. As a prototype, the nonlinear term involves the power-type nonlinearity $f(u) = |u|^{p-2}u$ for $p \in (2, 6)$.

**Keywords** Kirchhoff type equation · Perturbation method · Multiplicity · Variational method

**Mathematics Subject Classification** 35J50 · 35J65 · 35J60

1 Introduction

In the present paper, we investigate the existence and multiplicity of bound state solutions to the following Kirchhoff equation

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3), \quad (K)$$

where $V \in C^1(\mathbb{R}^3, \mathbb{R})$ and $a, b > 0$ are positive constants. Problem (K) arises in an interesting physical context. Precisely, if we set $V(x) = 0$ and a domain $\Omega \subset \mathbb{R}^3$ and replace $f(u)$ by $f(x, u)$, problem (K) becomes the following Dirichlet problem:

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial \Omega, \end{cases} \quad (1.1)$$

which is the general form of the stationary counterpart of the hyperbolic Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \left[\frac{p_0}{h} + \frac{E}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 \, dx \right] \frac{\partial^2 u}{\partial x^2} + f(t, x, u). \quad (1.2)$$

This equation is related to the classical D’Alembert’s wave equations for free vibration of elastic strings. Such type of problems were proposed by Kirchhoff in [21], in which the model takes into account the changes in length of the string produced by transverse vibrations. In (1.2), $L$ denotes the length of the string, $E$ the Young modulus of the material, $h$ is the area of the cross section, $\rho$ stands for mass density and $p_0$ is the initial tension, $f(t, x, u)$ stands for the external force. The function $u$ in (1.1) denotes the displacement, $b$ is the initial tension while $a$ is related to the intrinsic properties. Besides, we also point out that Kirchhoff problems appear in other fields like biological systems, such as population density, where $u$ describes a process which depends on its own average. For the further physical background, we refer the readers to [5, 8, 11].

1.1 Overview and Motivation

From the mathematical point of view, due to the presence of the term $\int_{\mathbb{R}^3} |\nabla u|^2$, Kirchhoff equations are no longer a pointwise identity and therefore, are viewed as being nonlocal. This leads to a difficulty in applying the weak convergence method and brings mathematical challenges to the analysis. Meanwhile, such feature makes
the study of such a problem particularly interesting. In the past decades, Kirchhoff problems have been receiving extensive attention. In particular, initiated by Lions [25], the solvability of Kirchhoff type Eq. (1.1) has been investigated in many studies, see [2,3,24,34,35,37,39,41,44,50] and the references therein. There also have been many interesting works about the existence and multiplicity of bound state solutions to Kirchhoff type equation (K) via variational methods, see for instance [1,6,7,13,14,16–18,22,23,26,30,36,45,47–49] and the references therein. We note that minimax methods are used to study the existence and multiplicity in a typical way. In this process, one has to overcome the difficulties arising from the effect of nonlocal property and in showing the boundedness and compactness of Palais–Smale [(PS) for short] sequences. For this aim, one usually assumes that the function $f$ satisfies either the $4$-superlinear growth condition:

$$
\lim_{|t| \to +\infty} F(t)/t^4 = +\infty,
$$

(4-superlinear)

where $F(u) = \int_0^u f(s)ds$, or the well-known Ambrosetti–Rabinowitz type condition

$$
0 < F(t) \leq \frac{1}{\mu} f(t)t, \quad \mu > 4, \ t \neq 0
$$

(AR)

or the monotonicity condition

$$
\frac{f(t)}{t^3} \text{ is strictly increasing in } (0, +\infty).
$$

(Ne)

The above conditions are crucial in proving the existence and boundedness of (PS) sequences. Furthermore, nontrivial solutions can be obtained by providing some further conditions on $f$ and $V$ to guarantee the compactness of (PS) sequences, such as the radial symmetric setting or the coercive condition on $V$. It worth of pointing out that, without the above conditions, Li and Ye [22] proved the existence of positive ground state solutions to problem (K) with $f(u) = |u|^{p-2}u, p \in (3, 6)$ by using the method of Nehari-Pohozaev manifold together with the concentration compactness arguments. Recently, the results of [22] were extended in [26] to a more general case, see also [15,43].

Compared with the existence results on nontrivial solutions, there are few works in the literature on the multiplicity of solutions to Kirchhoff type problem in $\mathbb{R}^3$, see [10,19,36,47]. As mentioned above, (AR)-condition or 4-superlinear growth condition or some compactness conditions play important roles in the literature. More specifically, Sun et al. [42] obtained infinitely many sign-changing solutions to problem (K) without 4-superlinear growth condition but the coercive condition of $V$, by using methods of invariant sets to descending flow and the Ljusternik–Schnirelman type minimax method. Under some weaker compactness assumptions on $V$ without radial symmetry setting or the coercive hypotheses, Zhang et al. [51] established the existence of infinitely many solutions to problem (K) with $f$ satisfying 4-superlinear growth condition. Very recently, Liu et al. [33] employed a perturbation approach and the method of invariant sets of descending flow to prove the existence of infinitely
many sign-changing solutions to problem (K) with a general nonlinearity in the radial symmetry setting.

1.2 Our Problem

These results above left one question:

Does problem (K) admit infinitely many nontrivial solutions without the radial symmetry condition or coercive condition in the case

\[ f(u) = |u|^{p-2}u, \quad p \in (2, 4)? \]

Obviously, this type of nonlinearity \( f \) does not satisfy (AR) or (4-superlinear) or (Ne). To the best of our knowledge, so far there has been no result known in this aspect. The main interest of the present paper is to give an affirmative answer to this question.

1.3 Our Results

Throughout this paper, we assume nonlinearity \( f \) satisfies the following hypotheses

\[ (f) \quad f \in C(\mathbb{R}, \mathbb{R}) \text{ and } \lim_{t \to 0} \frac{f(t)}{t} = 0; \]

\[ (f_2) \quad \limsup_{|t| \to \infty} \frac{|f(t)|}{|t|^{p-1}} < \infty \text{ for some } p \in (2, 6); \]

\[ (f_3) \quad \text{there exists } \mu > 2 \text{ such that } tf(t) \geq \mu F(t) > 0 \text{ for } t \neq 0, \text{ where } F(t) = \int_0^t f(s)ds. \]

These are quite natural assumptions when dealing with general subcritical nonlinearities. In particular, by \((f_1)-(f_2)\) it follows that for any \( \varepsilon > 0, \) there exists \( C_\varepsilon > 0 \) such that

\[ |f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^{p-1} \quad \text{and} \quad |F(t)| \leq \varepsilon t^2 + C_\varepsilon |t|^p, \quad t \in \mathbb{R}. \quad (1.3) \]

**Remark 1.1** It follows from \((f)-(f_3)\) that \( 2 < \mu \leq p < 6. \) As a reference model, \( f(t) = |t|^{p-2}t \) satisfies \((f)-(f_3)\) for \( p \in (2, 6). \)

Moreover, the potential \( V \in C^1(\mathbb{R}^3, \mathbb{R}) \) enjoys the following conditions:

\[ (V_1) \quad \text{there exist } V_0, V_1 > 0 \text{ such that } V_0 \leq V(x) \leq V_1 \text{ for all } x \in \mathbb{R}^3; \]

\[ (V_2) \quad \text{for all } \gamma > 0, \lim_{|x| \to \infty} \frac{\partial V}{\partial r}(x)e^{\gamma |x|} = +\infty, \text{ where } \frac{\partial V}{\partial r}(x) = \left( \frac{x}{|x|}, \nabla V(x) \right); \]

\[ (V_3) \quad \text{there exists } \tilde{c} > 1 \text{ such that } |\nabla V(x)| \leq \tilde{c} \frac{\partial V}{\partial r}(x) \text{ for } |x| \geq \tilde{c}; \]

\[ (V_4) \quad (\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3) \text{ and } \frac{\mu-2}{\mu} V(x) \geq (\nabla V(x), x) \geq 0 \text{ for a.e. } x \in \mathbb{R}^3. \]

**Remark 1.2** We note that \((V_2)\) and \((V_3)\) were firstly given in Cerami et al. [9] to study the existence of infinitely many bound state solutions for nonlinear scalar field equations. These assumptions are key in recovering the compactness of solution sequences when one uses a local Pohozaev identity together with decay estimates to study the behavior of solutions, see also Liu and Wang [28]. Of course, \((V_4)\) is also a mild condition to
ensure the boundedness of solution sequences, see Li and Ye [22]. It is not difficult to find some concrete function \( V \) satisfying assumptions (V1)-(V4), such as
\[
V(x) = V_1 - \frac{1}{1 + |x|}, \quad V_1 > 4,
\]

or
\[
V(x) = V_0 + e^{-\frac{1}{1 + |x|}}, \quad V_0 > \frac{\mu + 4}{2(\mu - 2)}.
\]

Our main result is as follows.

**Theorem 1.3** If (V1)-(V4) and (f1)-(f3) hold, then problem (K) admits at least one least energy solution in \( H^1(\mathbb{R}^3) \).

**Theorem 1.4** If (V1)-(V4) and (f1)-(f3) hold, then problem (K) has infinitely many bound state solutions in \( H^1(\mathbb{R}^3) \) provided that \( f \) is odd.

Now we summarize the main difficulties in finding bound state solutions to problem (K) under the effect of nonlocal term \( \int_{\mathbb{R}^3} |\nabla u|^2 \). On the one hand, when \( p \in (2, 4) \), neither (AR) nor (Ne) holds, which makes tough to get the boundedness of (PS) sequences. On the other hand, it is also hard to prove the convergence of (PS) sequences without radial symmetry setting or the coercive hypotheses for \( V \). It is mainly motivated by [9,22,27–29,31,32] that we make use of a new perturbation approach together with the symmetric mountain-pass theorem to study problem (K). More precisely, in order to get boundedness and compactness of (PS) sequences, we modify problem (K) by adding a coercive term and a nonlinear term of order larger than 4 (see (K_\lambda)), and then the corresponding Pohozaev type identity enables us to get a bounded solution sequence independent of the parameter \( \lambda \). As a result, by passing to the limit, a convergence argument allows us to get nontrivial solutions of the original problem (K). In this process, we also need to establish a version of global decomposition of solution sequences (such solutions maybe are not positive), which seems new for Kirchhoff type equations. This decomposition is crucial in using the local Pohozaev identity and some decay estimates of solutions to prove the compactness of the solution sequences. Moreover, we believe that this perturbation approach should be of independent interest in other problems.

**Remark 1.5** Theorem 1.3 is not surprising. Indeed, the authors in [26] proved the existence of positive ground states to problem (K) with a general nonlinearity. Moreover, with some more general assumptions of \( f \), the existence of ground state solutions was also obtained in [14,15,43]. However, the methods used in this paper are different from ones in [14,15,22,26,43]. The main ingredient of this paper is proving the existence of infinite many solutions to problem (K). But it seems difficult to obtain infinitely many solutions by using those arguments in [14,15,22,26,43].

Hereafter, the letter \( C \) will be repeatedly used to denote various positive constants whose exact values are irrelevant. We omit the symbol \( dx \) in the integrals when no confusion can arise. This paper is organized as follows. Firstly, some notations are
given in Sects. 2, and 3 is devoted to the existence of positive ground state solutions. Then in Sect. 4, we investigate the existence of infinitely many bound state solutions.

2 Preliminary Results

To proceed, we first consider the Hilbert space \( H^1(\mathbb{R}^3) \) (\( H \) for short) with the inner product

\[
\langle u, v \rangle = \int_{\mathbb{R}^3} a \nabla u \nabla v + V(x)uv
\]

and the norm

\[
\|u\| := \sqrt{\langle u, u \rangle} = \left( \int_{\mathbb{R}^3} a |\nabla u|^2 + V(x)u^2 \right)^{\frac{1}{2}}.
\]

The associated energy functional \( I : H \to \mathbb{R} \) is given by

\[
I(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u).
\]

It is a well-defined \( C^1 \) functional in \( H \) and its derivative is given by

\[
I'(u)v = \int_{\mathbb{R}^3} (a \nabla u \nabla v + V(x)uv) + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla v - \int_{\mathbb{R}^3} f(u)v, \; u, v \in H.
\]

We introduce the following coercive function which will be of use

\[
W(x) := 1 + |x|^{\alpha}, \quad 0 < \alpha < \frac{\mu - 2}{\mu}, \; x \in \mathbb{R}^3. \tag{2.1}
\]

Obviously,

\[
W(x) \geq 1 > 0, \quad \lim_{|x| \to \infty} W(x) \to \infty, \tag{2.2}
\]

and

\[
\frac{\mu - 2}{\mu} W(x) \geq (\nabla W(x), x) \geq 0 \quad x \in \mathbb{R}^3. \tag{2.3}
\]

For any \( \lambda \in (0, 1] \), let \( E_\lambda := \{ u \in H : \int_{\mathbb{R}^3} \lambda W(x)u^2 \, dx < \infty \} \) equipped with the norm

\[
\|u\|_{E_\lambda} = \left( \int_{\mathbb{R}^3} (a |\nabla u|^2 + V(x)u^2 + \lambda W(x)u^2) \right)^{\frac{1}{2}}.
\]

Note that \( E_1 = E_\lambda \subseteq H \) for any \( \lambda \in (0, 1] \).
3 Existence

3.1 The Perturbed Problem

It is known that the boundedness of the Palais-Smale sequence is not easy to get for the case $p \in (2, 4)$. To overcome this difficulty, we introduce a perturbation technique to problem (K). We now give more details to describe such a technique. Fix $\lambda \in (0, 1]$ and $r \in (\max\{p, 4\}, 6)$, we consider the modified problem

$$
\begin{cases}
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + (x)u + \lambda W(x)u = f_\lambda(u), & \text{in } \mathbb{R}^3, \\
u \in E_\lambda,
\end{cases}
$$

where

$$f_\lambda(u) = f(u) + \lambda |u|^{r-2} u.$$ 

An associated functional can be constructed as

$$I_\lambda(u) = I(u) + \frac{\lambda}{2} \int_{\mathbb{R}^3} W(x)u^2 - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r, \quad u \in E_\lambda,$$

and for $u, v \in E_\lambda$,

$$I'_\lambda(u)v = I'(u)v + \lambda \int_{\mathbb{R}^3} W(x)uv - \lambda \int_{\mathbb{R}^3} |u|^{r-2} uv. \quad (3.1)$$

It is known that $I_\lambda$ belongs to $C^1(E_\lambda, \mathbb{R})$ and $C^1(E, \mathbb{R})$, and a critical point of $I_\lambda$ is a weak solution of problem $((K_\lambda))$. The original problem can be seen as the limit equation of $((K_\lambda))$ as $\lambda \to 0^+$. 

We will make use of the following Pohozaev type identity, whose proof is standard and can be found in [12].

**Lemma 3.1** Let $u$ be a critical point of $I_\lambda$ in $E_\lambda$ for $\lambda \in (0, 1]$, then

$$a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} (V(x) + \lambda W(x))u^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)u^2$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R}^3} (\nabla W(x), x)u^2 + \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - 3 \int_{\mathbb{R}^3} \left( F(u) + \frac{\lambda}{r} |u|^r \right) = 0.$$

We now verify that the functional $I_\lambda$ has the mountain pass geometry uniformly in $\lambda$.

**Lemma 3.2** Suppose that $(V_1)$-$(V_4)$ hold. Then

1. there exist $\rho, \delta > 0$ such that, for any $\lambda \in (0, 1]$, $I_\lambda(u) \geq \delta$ for every $u \in S_\rho = \{u \in E_\lambda : \|u\|_{E_\lambda} = \rho\}$;
2. there is $e_0 \in E \setminus \{0\}$ with $\|e_0\|_{E_\lambda} > \rho$ such that, for any $\lambda \in (0, 1]$, $I_\lambda(v) < 0.$
**Proof** (1) For any \( u \in E_\lambda \), by the definition of \( I_\lambda \), (1.3) and Sobolev’s inequality, one has

\[
I_\lambda(u) \geq \frac{1}{4} \| u \|_{E_\lambda}^2 - C \int_{\mathbb{R}^3} |u|^p - \frac{1}{r} \int_{\mathbb{R}^3} |u|^r \\
\geq \frac{1}{4} \| u \|_{E_\lambda}^2 - C \| u \|_{E_\lambda}^p - \frac{C}{r} \| u \|_{E_\lambda}^r.
\]

Taking \( \rho > 0 \) small enough, it is easy to check that there exists \( \delta > 0 \) such that \( I_\lambda(u) \geq \delta \) for every \( u \in S_\rho \).

(2) For \( e \in E \setminus \{0\} \), let \( e_t = t^{1/2} e(\frac{x}{t}) \). Observe that

\[
\int_{\mathbb{R}^3} F(e_t) = t^3 \int_{\mathbb{R}^3} F(t^{1/2} e) =: t^3 \Phi(t).
\]

By \((f_3)\), a straightforward computation yields

\[
\frac{\Phi'(t)}{\Phi(t)} \geq \frac{\mu}{2t}, \quad \forall t > 0
\]

and then, by integrating on \([1, t]\), with \( t > 1 \), we have \( \Phi(t) \geq \Phi(1)t^{\frac{\mu}{2}} \), implying that

\[
\int_{\mathbb{R}^3} F(e_t) \geq t^{\frac{\mu+6}{2}} \int_{\mathbb{R}^3} F(e). \tag{3.2}
\]

Then by the definition of \( I_\lambda \) and \((V_1)\) and (2.1), one has

\[
I_\lambda(e_t) \leq t^2 \| \nabla e \|^2_2 + \frac{t^4}{4} \| \nabla e \|^4_2 + \lambda \int_{\mathbb{R}^3} \frac{V(t)e^2}{2} + \int_{\mathbb{R}^3} W(t(x)e^2 - t^{\frac{\mu+6}{2}} \int_{\mathbb{R}^3} F(e) \\
\leq t^2 \| \nabla e \|^2_2 + \frac{t^4}{4} \| \nabla e \|^4_2 + \frac{t^4 V_1}{2} \int_{\mathbb{R}^3} e^2 + \frac{t^4}{2} \int_{\mathbb{R}^3} W(x)e^2 - t^{\frac{\mu+6}{2}} \int_{\mathbb{R}^3} F(e) \lessgtr 0,
\]

which holds for \( t > 1 \) large enough, owing to \( \alpha < \frac{\mu-2}{\mu} \). The proof is complete.

\[\square\]

By recalling the well-known mountain-pass theorem (see \([4,46]\)), there exists a \((PS)_{c_\lambda}\) sequence \( \{u_n\} \subset E_\lambda \), that is,

\[
I_\lambda(u_n) \to c_\lambda \quad \text{and} \quad I_\lambda'(u_n) \to 0.
\]

We stress that \( \{u_n\} \) depends on \( \lambda \) but we omit this dependence in the sequel for convenience. Here \( c_\lambda \) is the mountain pass level characterized by

\[
c_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t))
\]
with

$$\Gamma_\lambda := \left\{ \gamma \in C^1([0, 1], E_\lambda) : \gamma(0) = 0 \text{ and } \gamma(1) = e_0 \right\},$$

where $e_0$ has been given in Lemma 3.2.

**Remark 3.3** Observe from Lemma 3.2 that there exist two constants $m_1, m_2 > 0$ independent of $\lambda$ small such that $m_1 < c_\lambda < m_2$.

In what follows, we prove the functional $I_\lambda$ satisfies the (PS)-condition.

**Lemma 3.4** For fixed $\lambda \in (0, 1)$, assume that there exists a sequence $\{u_n\} \subset E_\lambda$ such that $I_\lambda(u_n) \to c_\lambda \in \mathbb{R}$ and $I'_\lambda(u_n) \to 0$ as $n \to \infty$, then there exists a convergence subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $u_n \to u$ in $E_\lambda$ for some $u \in E_\lambda$.

**Proof** For $\gamma \in (4, r)$, by (1.3) we have

$$\gamma I_\lambda(u_n) - \langle I'_\lambda(u_n), u_n \rangle = \frac{\gamma - 2}{2} \|u_n\|_{E_\lambda}^2 + \frac{b(\gamma - 4)}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \int_{\mathbb{R}^3} \left( f(u_n)u_n - \gamma F(u_n) \right) + \lambda \frac{r - \gamma}{r} \int_{\mathbb{R}^3} |u|^r.$$

Then it follows from (1.3) that

$$\|u_n\|_{E_\lambda}^2 + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \int_{\mathbb{R}^3} |u|^r \leq C \left( 1 + \|u_n\|_{E_\lambda} + \|u_n\|_p^p \right) \quad (3.4)$$

for large $n$. We claim that $\{u_n\}$ is uniformly bounded in $E_\lambda$. Assume by contradiction that $\|u_n\|_{E_\lambda} \to \infty$, then by (3.4) we have

$$\|u_n\|_{E_\lambda}^2 + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \|u_n\|_r^r \leq C \|u_n\|_p^p, \quad (3.5)$$

which implies that

$$\|u_n\|_{E_\lambda}^2 + \|u_n\|_r^r \leq C \|u_n\|_p^p.$$

Let $t \in (0, 1)$ be such that $\frac{1}{p} = \frac{t}{2} + \frac{1-t}{r}$. From the interpolation inequality, we deduce that

$$\|u_n\|_2^2 + \|u_n\|_r^r \leq C \|u_n\|_p^p \leq C \|u_n\|_2^{pt} \|u_n\|_r^{(1-t)} \quad (3.6)$$

It follows from (3.6) that there exist $C_1, C_2 > 0$ such that

$$C_1 \|u_n\|_2^2 \leq \|u_n\|_r \leq C_2 \|u_n\|_2^2. \quad (3.7)$$

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In view of (3.6) and (3.7), we have \( \|u_n\|_p^p \leq C_3 \|u_n\|_2^2 \) for some \( C_3 > 0 \). Therefore, by (3.5), we have for some \( C_4 > 0 \) such that

\[
\|u_n\|_{E_\lambda}^2 + b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \|u_n\|_{r}^r \leq C_4 \|u_n\|_2^2.
\]

Let \( v_n = \frac{u_n}{\|u_n\|_{E_\lambda}} \), then

\[
\|v_n\|_2^2 \geq \frac{1}{C_4}
\]

and

\[
b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 \right)^2 \leq C_4 \|u_n\|_{E_\lambda}^2,
\]

which implies that \( \int_{\mathbb{R}^3} |\nabla v_n|^2 \to 0 \) as \( n \to \infty \). By \( \|v_n\|_{E_\lambda} = 1 \), we assume \( v_n \rightharpoonup v \) in \( E_\lambda \). By Fatou’s lemma we have

\[
\int_{\mathbb{R}^3} |\nabla v|^2 \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} |\nabla v_n|^2 = 0,
\]

which implies \( v = 0 \). Then by (3.8) we have \( \|v\|_2^2 \geq \frac{1}{C_4} \), a contradiction. Thus, we finish the proof of the claim. Without loss of generality, we assume that there exists \( u \in E_\lambda \) such that

\[
u_n \rightharpoonup u \text{ weakly in } E_\lambda,
\]

\[
u_n \to u \text{ strongly in } L^q(\mathbb{R}^3) \text{ for } q \in [2, 6).
\]

Note that

\[
\sigma(1) = \left( I_{\lambda}'(u_n) - I_{\lambda}'(u) \right) (u_n - u) \\
= \|u_n - u\|_{E_\lambda}^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 \\
+ b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) - \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \\
- \lambda \int_{\mathbb{R}^3} (|u_n|^{-2} u_n - |u|^{-2} u)(u_n - u).
\]

According to the boundedness of \( \{u_n\} \) in \( E_\lambda \), one has

\[
b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) \to 0.
\]
Similarly, we also have
\[
\int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \to 0,
\]
\[
\lambda \int_{\mathbb{R}^3} (|u_n|^r - |u|^r)(u_n - u) \to 0, \quad \text{as } n \to \infty.
\]

Based on the above facts, from (3.9) we deduce that \( u_n \to u \) in \( E_\lambda \). \qed

It follows from Lemma 3.4 that for each \( \lambda \in (0, 1] \), there exists \( u_\lambda \in E_\lambda \) such that
\[
I_\lambda(u_\lambda) = c_\lambda \quad \text{and} \quad I_\lambda'(u_\lambda) = 0.
\]

That is to say, \( u_\lambda \) is a nontrivial solution of \((K_\lambda)\). We now expect that \( \{u_\lambda\} \) converges to a nontrivial solution of \((K)\) as \( \lambda \to 0 \) by controlling \( \{u_\lambda\} \) in a proper way.

**Lemma 3.5** Suppose that \( \lambda_n \to 0^+ \) as \( n \to \infty \), \( \{u_n\} \subset E_{\lambda_n} \) are nontrivial solutions of \((K_{\lambda_n})\) with \( \sup_{n \in \mathbb{N}} |I_{\lambda_n}(u_n)| \leq C \). Then there exists \( M > 0 \) such that \( \|u_n\|_{E_{\lambda_n}} \leq M \) for some \( M > 0 \) independent of \( n \).

**Proof** By sequence \( \{\lambda_n\} \subset (0, 1] \) satisfying \( \lambda_n \to 0^+ \) and \( \{u_{\lambda_n}\} \) (still denoted by \( \{u_n\} \)) of \( I_{\lambda_n} \), with \( I_{\lambda_n}(u_n) = c_{\lambda_n} \), we claim that \( \{u_n\} \) is bounded in \( E_{\lambda_n} \). Observe that
\[
C \geq I_{\lambda_n}(u_n) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2
\]
\[
+ \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - \int_{\mathbb{R}^3} F(u_n) - \frac{\lambda_n}{r} \int_{\mathbb{R}^3} |u_n|^r
\]
\[
= 0 = a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2
\]
\[
+ b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - \int_{\mathbb{R}^3} F(u_n) - \lambda_n \int_{\mathbb{R}^3} |u_n|^r.
\]

Moreover, from Lemma 3.1, the following identity holds
\[
\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{3}{2} \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x) + \lambda_n \nabla W(x), x)u_n^2
\]
\[
+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - 3 \int_{\mathbb{R}^3} (F(u_n) + \frac{\lambda_n}{r} |u_n|^r) = 0.
\]
Multiplying (3.10), (3.11) and (3.12) by four, $-\frac{1}{\mu}$ and $-1$ respectively and adding them up, we get

$$4C \geq a \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\mu - 2}{2\mu} \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2$$

$$- \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x) + \lambda_n \nabla W(x), x)u_n^2$$

$$+ \frac{\mu - 2}{2\mu} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda_n \frac{r - \mu}{\mu r} \int_{\mathbb{R}^3} |u_n|^r + \int_{\mathbb{R}^3} \left( \frac{1}{\mu} f(u_n)u_n - F(u_n) \right).$$

It then follows from (V4) and (2.3) that

$$4C \geq a \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\mu - 2}{2\mu} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda_n \frac{r - \mu}{\mu r} \int_{\mathbb{R}^3} |u_n|^r,$$

which implies that there exists $C_5 > 0$ independent of $\lambda_n$ such that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 < C_5. \quad (3.13)$$

Moreover, combining (1.3), (3.10) and hypotheses (V1), we obtain that for small $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$C > a \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2 - \int_{\mathbb{R}^3} F(u_n) - \frac{\lambda_n}{r} \int_{\mathbb{R}^3} |u_n|^r$$

$$> \frac{1 - \varepsilon}{2} \int_{\mathbb{R}^3} V(x)u_n^2 - C_\varepsilon \int_{\mathbb{R}^3} u_n^6 + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} W(x)u_n^2$$

$$> \frac{1 - \varepsilon}{2} \int_{\mathbb{R}^3} V(x)u_n^2 - C_\varepsilon S^{-3} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^3 + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} W(x)u_n^2. \quad (3.14)$$

Combining (3.13) and (3.14), there exists $C_6 > 0$ independent of $\lambda_n$ such that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} (V(x) + \lambda_n W(x))u_n^2 \leq C_6. \quad (3.15)$$

The conclusions follow immediately. $\square$

The following lemma is devoted to the behavior of solution sequences to problem $(K_\lambda)$.

**Lemma 3.6** Let $\{u_n\} \subset E_\lambda$ be a solution sequence of problem $(K_\lambda)$ with $\lambda = \lambda_n \geq 0$ and $\lambda_n \to 0$, and $\|u_n\|_{E_{\lambda_n}} \leq M$ for $M > 0$ independent of $n$. Then there exist a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, a number $k \in \mathbb{N} \cup \{0\}$, and finite sequences

$$(a_1, \ldots, a_k) \subset \mathbb{R}, \quad \{u_0, w_1, \ldots, w_k\} \subset H, \quad a_j \geq 0, \quad w_j \neq 0,$$

and $A \geq 0$ and $k$ sequences of points $\{y_n^j\} \subset \mathbb{R}^3$, $1 \leq j \leq k$, such that
(i) $u_n \rightarrow u_0$, $u_n (\cdot + x_n^{(i)}) \rightarrow w_j$ in $H$ as $n \rightarrow \infty$,
(ii) $|y_n^{(j)}| \rightarrow +\infty$, $|y_n^{(j)} - y_n^{(i)}| \rightarrow +\infty$ if $i \neq j$, $n \rightarrow +\infty$,
(iii) $\|u_n - u_0 - \sum_{i=1}^{k} w_i (\cdot - y_n^{(i)})\| \rightarrow 0$,
(iv) $A = \|\nabla u_0\|_2^2 + \sum_{k=1}^{k} \|\nabla w_i\|_2^2$,
(v) for any $\varphi \in C_0^\infty (\mathbb{R}^3)$ with $\varphi \geq 0$
\begin{equation}
(a + bA) \int_{\mathbb{R}^3} \nabla |w_j| \nabla \varphi + (V_0 + a_j) \int_{\mathbb{R}^3} |w_j| \varphi \leq \int_{\mathbb{R}^3} f(w_j)|\varphi|.
\end{equation}

**Proof**  Note that $\{u_n\}$ is a bounded sequence in $H$. There exists $u_0 \in H$ and $A > 0$ such that $u_n \rightarrow u_0$ weakly in $H$ and $\|\nabla u_n\|_2^2 \rightarrow A$ as $n \rightarrow \infty$ after extracting a subsequence. For any $\psi \in C_0^\infty (\mathbb{R}^3)$, we have $J'_{\lambda_n} (u_n) \psi \equiv 0$, where
\[ J_{\lambda_n} (u) := \frac{1}{2} \|u\|^2 + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} W(x) u^2 + \frac{A b}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \int_{\mathbb{R}^3} F(u) - \frac{\lambda_n}{r} \int_{\mathbb{R}^3} |u|^r. \]
Moreover, one has for any $\psi \in C_0^\infty (\mathbb{R}^3)$
\[ |\lambda_n \int_{\mathbb{R}^3} W(x) u_n \psi| \leq \left( \lambda_n \int_{\mathbb{R}^3} W(x) u_n^2 |\psi| \right)^{\frac{1}{2}} \left( \lambda_n \int_{\mathbb{R}^3} W(x) |\psi| \right)^{\frac{1}{2}} \leq C \lambda_n^{\frac{1}{2}} \rightarrow 0, \]
which, together with the fact that $J'_{\lambda_n} (u_n) = 0$, implies that
\[ \lim_{n \rightarrow \infty} J' (u_n) \psi = \lim_{n \rightarrow \infty} \left( J'_{\lambda_n} (u_n) \psi - \lambda_n \int_{\mathbb{R}^3} W(x) u_n \psi + \lambda_n \int_{\mathbb{R}^3} |u_n|^{r-2} u_n \psi \right) = 0, \]
with the functional $J = J_{\lambda}$ with $\lambda = 0$. It then follows that $J' (u_0) = 0$, that is,
\begin{equation}
\int_{\mathbb{R}^3} (a \nabla u_0 \nabla \psi + V(x) u_0 \psi) + b A \int_{\mathbb{R}^3} \nabla u_0 \nabla \psi = \int_{\mathbb{R}^3} f(u_0) \psi.
\end{equation}
As a consequence of Kato’s Inequality [20, Lemma A], we know the following differential inequality holds
\begin{equation}
\int_{\mathbb{R}^3} (a \nabla |u_0| \nabla \varphi + V_0 |u_0| \varphi) + b A \int_{\mathbb{R}^3} \nabla |u_0| \nabla \varphi \leq \int_{\mathbb{R}^3} |f(u_0)| \varphi,
\end{equation}
for any $\varphi \in C_0^\infty (\mathbb{R}^3)$ with $\varphi \geq 0$. We now apply the concentration compactness principle to the sequence of $\{v_{1,n}\}$ with $v_{1,n} = u_n - u_0$. Clearly, $v_{1,n} \rightarrow 0$ weakly in $H$. If vanishing occurs,
\[ \sup_{y \in \mathbb{R}^3} \int_{B_1 (y)} |u_n - u_0|^2 \rightarrow 0, \quad as \ n \rightarrow \infty. \]
Then \( v_{1,n} \to 0 \) in \( L^s(\mathbb{R}^3) \) for \( s \in (2, 6) \). By the fact that \( J'(u_0) = J'_{\lambda_n}(u_n) = 0 \), we arrive at

\[
(a + bA) \int_{\mathbb{R}^3} |\nabla u_0|^2 + \int_{\mathbb{R}^3} V(x)u_0^2 \\
\leq \liminf_{n \to \infty} \left( (a + bA) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x)u_n^2 \right) \\
\leq \limsup_{n \to \infty} \left( (a + bA) \int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x)u_n^2 + \lambda_n \int_{\mathbb{R}^3} W(x)u_n^2 \right) \\
= \limsup_{n \to \infty} \left( \int_{\mathbb{R}^3} f(u_n)u_n + \lambda_n \int_{\mathbb{R}^3} |u_n|^r \right) \\
\leq \int_{\mathbb{R}^3} f(u_0)u_0 = (a + bA) \int_{\mathbb{R}^3} |\nabla u_0|^2 + \int_{\mathbb{R}^3} V(x)u_0^2,
\]

which implies that \( u_n \to u_0 \) strongly in \( H \). So the conclusions of Lemma 3.6 hold for \( k = 0 \). If non-vanishing occurs, then there exist \( m > 0 \) and a sequence \( \{y_n^1\} \subset \mathbb{R}^3 \) such that

\[
\liminf_{n \to \infty} \int_{B_1(y_n^1)} |v_{1,n}(x)|^2 \geq m > 0. \tag{3.19}
\]

Let us consider the sequence \( \{v_{1,n}(\cdot + y_n^1)\} \). The boundedness of \( \{v_{1,n}\} \) in \( H \) implies that there exists \( w_1 \) such that \( v_{1,n}(\cdot + y_n^1) \to w_1 \) in \( H \). Furthermore, by (3.19) one has

\[
\int_{B_1(0)} |w_1(x)|^2 > \frac{m}{2},
\]

and thus, \( w_1 \neq 0 \). Recalling the fact that \( v_{1,n} \to 0 \) in \( H \), we know that \( \{y_n^1\} \) must be unbounded and, up to a subsequence, we suppose that \( |y_n^1| \to +\infty \).

Now we show the following inequality holds:

\[
(a + bA) \int_{\mathbb{R}^3} \nabla |w_1| \nabla \varphi + \int_{\mathbb{R}^3} (a_1 + V_0)|w_1| \varphi \leq \int_{\mathbb{R}^3} |f(w_1)| \varphi \tag{3.20}
\]

for \( \varphi \in C_0^\infty(\mathbb{R}^3) \) with \( \varphi \geq 0 \). Recalling (3.15), we have \( \lambda_n \int_{\mathbb{R}^3} W(x)u_n^2 \leq C \). So, (3.19) implies that

\[
C \geq \lambda_n \int_{\mathbb{R}^3} W(x)|v_{1,n}(x)|^2 \\
\geq \lambda_n W(y_n^1) \int_{B_1(y_n^1)} |v_{1,n}(x)|^2 - \lambda_n \int_{B_1(y_n^1)} |W(x) - W(y_n^1)||v_{1,n}(x)|^2 \\
\geq \lambda_n W(y_n^1)m - \lambda_n C.
\]
which implies that, up to subsequence, \( \lambda_n W(y_n^1) \to a_1 \in [0, +\infty) \). Based on the above facts, we have for \( \varphi \in C_0^\infty(\mathbb{R}^3) \) with \( \varphi \geq 0 

\begin{align*}
\lambda_n \int_{\mathbb{R}^3} W(x + y_n^1) v_{1,n}(x + y_n^1) \varphi &= \lambda_n W(y_n^1) \int_{\mathbb{R}^3} v_{1,n}(x + y_n^1) \varphi + \lambda_n \int_{\mathbb{R}^3} (W(x + y_n^1) - W(y_n^1)) v_{1,n}(x + y_n^1) \varphi \\
&= a_1 \int_{\mathbb{R}^3} v_{1,n}(x + y_n^1) \varphi + o(1) \\
&= a_1 \int_{\mathbb{R}^3} w_1 \varphi + o(1). \quad (3.21)
\end{align*}

Recalling the fact that \( v_{1,n} \to 0 \) in \( H \) as \( n \to \infty \), we have \( J'_{\lambda_n}(v_{1,n})\varphi(\cdot - y_n^1) \to 0 \) for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \) with \( \varphi \geq 0 \), and

\begin{align*}
J'_{\lambda_n}(v_{1,n})\varphi(\cdot - y_n^1) &= (a + bA) \int_{\mathbb{R}^3} \nabla v_{1,n}(x + y_n^1) \nabla \varphi + \int_{\mathbb{R}^3} V(x + y_n^1) v_{1,n}(x + y_n^1) \varphi \\
+ \int_{\mathbb{R}^3} \lambda_n W(x + y_n^1) v_{1,n}(x + y_n^1) \varphi - \int_{\mathbb{R}^3} f(v_{1,n}(x + y_n^1)) \varphi = o(1), \quad (3.22)
\end{align*}

which implies by (3.21) that

\begin{align*}
(a + bA) \int_{\mathbb{R}^3} \nabla w_1 \nabla \varphi + \int_{\mathbb{R}^3} V(x + y_n^1) v_{1,n}(x + y_n^1) \varphi \\
+ a_1 \int_{\mathbb{R}^3} w_1 \varphi - \int_{\mathbb{R}^3} f(w_1) \varphi = o(1). \quad (3.23)
\end{align*}

Set \( w_\varepsilon = \sqrt{|w_1|^2 + \varepsilon^2} - \varepsilon \), \( \varepsilon > 0 \). It is clear that \( w_\varepsilon \to |w_1| \) in \( H \) as \( \varepsilon \to 0 \). As Kato’s Inequality [21, Lemma A], we obtain (3.20). Let us set

\[ v_{2,n}(x) = v_{1,n}(x) - w_1(x - y_n^1), \quad (3.24) \]

then \( v_{2,n}(\cdot + y_n^1) \to 0 \) weakly in \( H \). It follows from the Brezis-Lieb lemma that

\[ \|v_{2,n}\|_s^s = \|u_n\|_s^s - \|u_0\|_s^s - \|w_1\|_s^s + o(1), \quad \text{for } s \in [2, 6]. \quad (3.25) \]

Applying the concentration compactness principle to \( \{v_{2,n}\} \), we have two possibilities: either vanishing or non-vanishing. If vanishing occurs we have

\[ \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |v_{2,n}(x)|^2 \to 0, \]
then $v_{2,n} \to 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$, and Lemma 3.6 holds with $k = 1$. Otherwise, \{v_{2,n}\} is non-vanishing, there exist $m' > 0$ and a sequence $\{y^0_n\} \subset \mathbb{R}^3$ such that

$$\liminf_{n \to \infty} \int_{B_1(y^0_n)} |v_{2,n}(x)|^2 \geq m' > 0. \quad (3.26)$$

We repeat the arguments. By iterating this procedure we obtain sequences of points $\{y^j_n\} \subset \mathbb{R}^3$ such that $|y^j_n| \to +\infty$, $|y^i_n - y^j_n| \to +\infty$ if $i \neq j$ as $n \to +\infty$ and $v_{j,n} = v_{j-1,n} - w_{j-1}(x - y^j_{n-1})$ (like (3.24)) with $j \geq 2$ such that $v^j_n \rightharpoonup 0$ in $H$.

Based on the properties of the weak convergence, we have

(a) $\|u_n\|_s^s - \|u_0\|_s^s - \sum_{i=1}^{j-1} \|w_i\|_s^s = \|u_n - u_0 - \sum_{i=1}^{j-1} w_i(\cdot - y^i_n)\|_s^s + o(1) \geq 0$,

(b) for any $\varphi \in C^\infty_0(\mathbb{R}^3)$ with $\varphi \geq 0$ and $i = 1, ..., j - 1$,

$$(a + bA) \int_{\mathbb{R}^3} \nabla |w_i| \nabla \varphi + (V_0 + a_i) \int_{\mathbb{R}^3} |w_i| \varphi \leq \int_{\mathbb{R}^3} |f(w_i)| \varphi.$$ 

By the Sobolev embedding theorem and conclusion (b), we have for $i = 1, ..., j - 1$

$$\|w_i\|_p^2 \leq S_p \int_{\mathbb{R}^3} (|\nabla |w_i||^2 + |w_i|^2) \leq C \|w_i\|_p^p,$$

where $S_p$ is the Sobolev constant of embedding from $H^1(\mathbb{R}^3)$ to $L^p(\mathbb{R}^3)$. Hence, there exists $c_0 > 0$ independent of $w_i$ such that $|w_i|_p^2 \geq c_0$. Since $\{u_n\}$ is bounded sequence in $H$, conclusion (a) implies that the iteration stops at some finite index $k$. The proof is complete. \( \square \)

**Remark 3.7** The proof of Lemma 3.6 is in the spirit of Struwe [40] and Li and Ye [22]. It is worth of pointing out that this is the first result on decomposition of (PS) sequences (families of approximating solutions, maybe sign-changing solutions) with general energy level for Kirchhoff type equation. Similar decompositions of positive solution sequences can be found in [22,26,43] to recover the compactness.

In the following, let $u_n \in E_{\lambda_n}$ be a nontrivial solution of $(K_{\lambda_n})$ with $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $\lambda_n \to 0^+$ as $n \to \infty$. Now we investigate the exponential decay property of sequence $\{u_n\}$. For the sake of notation simplicity, in Lemma 3.6, we define $\gamma^0_n = 0$, $a_0 = 0$ and $u_0 = w_0$.

Thus the conclusion in Lemma 3.6 can be restated as $|y^i_n - y^j_n| \to \infty$, $0 \leq i < j \leq k$,

$$\|u_n - \sum_{i=0}^k w_i(\cdot - y^i_n)\| \to 0.$$
for any \( \varphi \in C_0^\infty(\mathbb{R}^3) \) with \( \varphi \geq 0 \)

\[
(a + bA) \int_{\mathbb{R}^3} \nabla |w_i| \nabla \varphi + (V_0 + a_i) \int_{\mathbb{R}^3} |w_i| \varphi \leq \int_{\mathbb{R}^3} |f(w_i)| \varphi, \; i = 0, 1, \ldots, k.
\]

(3.27)

**Lemma 3.8** There exists \( \delta, R_0 > 0 \) and \( n_0 \in \mathbb{N} \) such that for \( R > R_0 \) and \( n > n_0 \),

\[
\int_{\Omega_R^{(n)}} (|\nabla u_n|^2 + |u_n|^2) \leq C e^{-\delta R}, \quad \lambda_n \int_{\Omega_R^{(n)}} W(x) |u_n|^2 \leq C e^{-\delta R},
\]

(3.28)

where \( \Omega_R^{(n)} = \mathbb{R}^3 \setminus \bigcup_{i=1}^k B_R(y_n^i) \) and \( C > 0 \) is independent of \( n, R \).

**Proof** Using Moser’s iteration and the comparison principle to the differential inequality (3.27), we have, for some \( \delta > 0 \) and \( i = 1, \ldots, k \),

\[
\left( \int_{\mathbb{R}^3 \setminus B_R(0)} (|\nabla w_i|^2 + |w_i|^2) \right)^{1/2} \leq C e^{-\delta R}, \quad \|w_i\|_{L^\infty(\mathbb{R}^3 \setminus B_R(0))} \leq C e^{-\delta R}.
\]

So by property (iii) of Lemma 3.6, we have for \( s \in [2, 6] \)

\[
\left( \int_{\Omega_R^{(n)}} |u_n|^s \right)^{1/s} \leq \|u_n - \sum_{i=0}^k w_i (\cdot - y_n^i)\|_{L^s(\Omega_R^{(n)})} + \sum_{i=0}^k \left( \int_{\mathbb{R}^3 \setminus B_R(0)} w_i^s \right)^{1/s} \leq o(1) + C e^{-\delta R}.
\]

So we use Moser’s iteration to prove the \( L^\infty \)-estimate

\[
|u_n(x)| \leq o(1) + C e^{-\delta R}, \quad \text{for all } x \in \Omega_R^{(n)}.
\]

Define

\[
a_n := a + b \int_{\mathbb{R}^3} |\nabla u_n|^2.
\]

For any \( R > 0 \), define \( \varphi_R \in C_0^\infty(\mathbb{R}^3) \) as \( \varphi_R(x) = 0 \) for \( x \notin \Omega_R^{(n)} \), \( \varphi_R(x) = 1 \) for \( x \in \Omega_{R+1}^{(n)} \) and \( |\nabla \varphi_R| \leq 2 \). Thus, in view of \((V_1)\) and \((f_1)\), by choosing \( n_0 \in \mathbb{N}, R_0 \) such that for \( R > R_0 \) and \( n > n_0 \), we have

\[
\int_{\Omega_R^{(n)}} \left( a_n \nabla u_n (\varphi_R^2 \nabla u_n + 2 u_n \varphi_R \nabla \varphi_R) + \left( \lambda_n W(x) + \frac{V_0}{2} \right) a_n^2 \varphi_R^2 \right) \leq 0,
\]

(3.29)
which implies
\[
\int_{\Omega^{(n)}_R} \left( a \| \nabla u_n \|^2 + \frac{V_0}{2} u_n^2 \right) \varphi_R^2 \leq C \int_{\Omega^{(n)}_R} |u_n \nabla u_n \varphi R \nabla \varphi R| \\
\leq C \int_{\Omega^{(n)}_R \setminus \Omega^{(n)}_{R+1}} \left( a \| \nabla u_n \|^2 + \frac{V_0}{2} u_n^2 \right), \tag{3.30}
\]
where \( C > 0 \) does not depend on \( n, R \). From (3.30) we infer that
\[
\int_{\Omega^{(n)}_R} \left( a \| \nabla u_n \|^2 + \frac{V_0}{2} u_n^2 \right) \leq \frac{C}{1 + C} \int_{\Omega^{(n)}_R} \left( a \| \nabla u_n \|^2 + \frac{V_0}{2} u_n^2 \right).
\]
Thus, there exist \( C > 0 \) and \( \delta > 0 \) (independent of \( n, R \)) such that
\[
\int_{\Omega^{(n)}_R} (| \nabla u_n |^2 + | u_n |^2) \leq C e^{-\delta R}
\]
for \( R > R_0 \) and \( n > n_0 \). Combining (3.29) with (3.30) we also have
\[
\lambda_n \int_{\Omega^{(n)}_R} W(x) | u_n |^2 \leq C e^{-\delta R}.
\]
The proof is complete. \( \square \)

Motivated by [9], we derive a local Pohozaev-type identity which is of use in proving the convergence of solution sequences.

**Lemma 3.9** If \( u \in E_\lambda \) solves equation \((K_\lambda)\), then the following identity holds:
\[
\frac{1}{2} \int_{\mathbb{R}^3} t \cdot \nabla V(x) |u|^2 \psi + \frac{\lambda}{2} \int_{\mathbb{R}^3} t \cdot \nabla W(x) |u|^2 \psi = -\frac{1}{2} \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \\
\times \int_{\mathbb{R}^3} |\nabla u|^2 t \cdot \nabla \psi + \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} (t \cdot \nabla u)(\nabla u \cdot \nabla \psi) \\
- \frac{1}{2} \int_{\mathbb{R}^3} (V(x) + \lambda W(x)) |u|^2 t \cdot \nabla \psi + \int_{\mathbb{R}^3} (F(u) + \frac{\lambda}{r} |u|^r) t \cdot \nabla \psi
\]
for \( t \in \mathbb{R}^3 \) and \( \psi \in C_0^\infty (\mathbb{R}^3) \).

**Proof** It follows by the classical bootstrap method that \( u \in L^p (\mathbb{R}^3) \) for any \( p \geq 1 \). Then we deduce that \( u \in W^{2,p}_{\text{loc}} (\mathbb{R}^3) \) for any \( p \geq 1 \). Choosing \( \psi \in C_0^\infty (\mathbb{R}^3) \), \( t \in \mathbb{R}^3 \) and taking \( t \cdot \nabla u \psi \) as a test function in equation \((K_\lambda)\), we get the local Pohozaev-type identity. We can see [9] for the details of proof. \( \square \)

Without loss of generality, we assume that \( |y_{n_i}^1| = \min \{|y_{n_i}^i|, i = 1, \ldots, k\} \). Denote \( y_n = y_{n_1}^1 \) for simplicity of notations. Borrowing some idea in [9], we construct a
sequence of cones $C_n$, having vertex $\frac{1}{2}y_n$ and generated by a ball $B_{R_n}(y_n)$ as follows:

$$C_n = \left\{ z \in \mathbb{R}^3 | z = \frac{1}{2}y_n + l(x - \frac{1}{2}y_n), x \in B_{R_n}(y_n), l \in [0, \infty) \right\},$$

where $R_n$ satisfies

$$\frac{\gamma}{k} \cdot \frac{|y_n|}{2} = r_n \leq R_n \leq kr_n = \gamma \cdot \frac{|y_n|}{2}, \quad \gamma = \frac{1}{5(c + 1)},$$

where $c$ is the constant in $(V_4)$. It is known in [9] that the cone $C_n$ has the following property:

$$\partial C_n \cap \bigcup_{i=1}^{k} B_{\frac{\gamma_i}{2}}(y_n) = \emptyset. \quad (3.31)$$

**Lemma 3.10** Let $u_n \in E_{\lambda_n}$ be a nontrivial solution of $(K_{\lambda_n})$ with $I_{\lambda_n}(u_n) = c_{\lambda_n}$ and $\lambda_n \to 0^+$ as $n \to \infty$, then there exists $u_0 \in H$ such that, up to subsequence, $u_n \to u_0$ in $H$.

**Proof** Similarly to Lemma 3.5, there exists $u_0 \in H$ such that, up to subsequence, $u_n \to u_0$ in $H$. Take $u = u_n$, $t = t_n = \frac{y_n}{|y_n|}$ and $\psi = \eta \varphi_R \in C_0^\infty(\mathbb{R}^3)$, where $\eta, \varphi_R \in C_\infty(\mathbb{R}^3)$ such that $\eta(x) = 0$ for $x \notin C_n$, $\eta(x) = 1$ for $x \in C_n$ and $\text{dist}(x, \partial C_n) \geq 1$, $\varphi_R(x) = 1$ for $x \in B_R(0)$, and $\varphi_R(x) = 0$ for $x \in \mathbb{R}^3 \setminus B_2(0)$. It follows from Lemma 3.9 that by letting $R \to \infty$, we have

$$\frac{1}{2} \int_{\mathbb{R}^3} t_n \cdot \nabla V(x)|u_n|^2 \eta + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} t_n \cdot \nabla W(x)|u_n|^2 \eta$$

$$= -\frac{\alpha_n}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 t_n \cdot \nabla \eta + a_n \int_{\mathbb{R}^3} (t_n \cdot \nabla u_n)(\nabla u_n \cdot \nabla \eta) - \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u_n|^2 t_n \cdot \nabla \eta$$

$$+ \int_{\mathbb{R}^3} (F(u_n) + \frac{\lambda_n}{r}|u_n|^r) t_n \cdot \nabla \eta - \frac{\lambda_n}{2} \int_{\mathbb{R}^3} W(x)|u_n|^2 t_n \cdot \nabla \eta. \quad (3.32)$$

From (3.31) and the definition of $\eta$, we see that the support of $\nabla \eta$ is contained in the domain $\Omega = \Omega_R^{(n)}$ with $R = \frac{1}{2}r_n - 1$. In view of Lemma 3.8, we know that the right-hand side of (3.32) decays exponentially, say less than $C e^{-\delta|y_n|}$. Observe that by Lemma 4.2 of [9], we have $t_n \cdot \nabla V \geq \frac{1}{2} \frac{\partial V}{\partial r}$ for $x \in C_n$. Besides, by the definition of $W$, we see that $\int_{\mathbb{R}^3} t_n \cdot \nabla W(x)|u_n|^2 \eta$ is bounded uniformly for $n$. So the left-hand side of can be estimated as

$$\frac{1}{2} \int_{\mathbb{R}^3} t_n \cdot \nabla V(x)|u_n|^2 \eta + \frac{\lambda_n}{2} \int_{\mathbb{R}^3} t_n \cdot \nabla W(x)|u_n|^2 \eta$$

$$= \frac{1}{2} \int_{\mathbb{R}^3} t_n \cdot \nabla V(x)|u_n|^2 \eta + o(1) \geq \frac{1}{2} \inf_{x \in B_1(y_n)} \frac{\partial V(x)}{\partial r} \int_{\mathbb{R}^3} |u_n|^2 + o(1)$$

$$\geq \frac{m}{4} \inf_{x \in B_1(y_n)} \frac{\partial V(x)}{\partial r}, \quad (3.33)$$

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for large \( n \), where \( \int_{B_1(y_n)} u_n^2 \, dx \geq \frac{m}{2} > 0 \). Thus, together (3.32) and (3.33), we obtain

\[
\frac{m}{4} \inf_{x \in B_1(y_n)} \frac{\partial V(x)}{\partial r} \leq C e^{-\delta |y_n|},
\]

which contradicts \((V_3)\). Thus \( k = 0 \) and by Lemma 3.6 (iii), we have \( u_n \to u_0 \) in \( H \). The proof is complete. \( \square \)

Since \( u_n \) is a nontrivial solution of \((K_\lambda)\) with \( \lambda = \lambda_n \) and \( I_{\lambda_n}(u_n) = c_{\lambda_n} \), by Sobolev’s inequality and \((f_1)\), one can get that \( \|u_n\| \geq c \), where \( c > 0 \) is independent of \( n \). In view of Lemma 3.10, \( \|u_0\| \geq c \) which implies that \( u_0 \) is a nontrivial solution of problem \((K)\). Actually we have proved the following fact.

**Proposition 3.11** Assume \( \{u_{\lambda}\}_{\lambda \in (0,1]} \) satisfies \( I'_{\lambda}(u_{\lambda}) = 0 \) and \( c_{\lambda} = I_{\lambda}(u_{\lambda}) \in [m_1, m_2] \) (see Remark 3.3), then there exist \( u_0 \in H \setminus \{0\} \) and a sequence \( \{\lambda_n\} \) tending to zero, such that

\[
u_{\lambda_n} \to u_0 \text{ in } H, \quad c_{\lambda_n} \to c_0, \quad I(u_0) = c_0 \quad \text{and} \quad I'(u_0) = 0.
\]

### 3.2 Proof of Theorem 1.3

Define the set of solutions

\[
S := \{u \in H \setminus \{0\} : I'(u) = 0\}
\]

that, for what we have proved, is nonempty. For \( u \in S \), by Sobolev’s inequality, for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that

\[
\|u\|^2 + b \|\nabla u\|^2 \leq \varepsilon \int_{\mathbb{R}^3} u^2 + C_\varepsilon \int_{\mathbb{R}^3} |u|^6
\]

which implies that \( S \) is bounded away from zero. Besides, we can also see from the above inequality that \( \inf_{u \in S} \|\nabla u\|^2 \geq C > 0 \). By recalling (3.10)-(3.12), there exists some \( C > 0 \) satisfying \( I(u) \geq C \|\nabla u\|^2 \) for all \( u \in S \). So we infer that

\[
c_* := \inf_{u \in S} I(u) > 0.
\]

Choose finally a minimising sequence \( \{u_n\} \subset S \) so that \( I(u_n) \to c_* \). Similarly to Lemma 3.5 we know that \( \{u_n\} \) is bounded in \( H \). Like the modified functional \( I_{\lambda} \), we can also prove some facts for critical point sequence \( \{u_n\} \) of \( I \) corresponding to Lemmas 3.6-3.10. As a consequence, there exists \( u_* \in H \) so that \( u_n \to u_* \) in \( H \) and \( I'(u_*) = 0 \). Then \( u_* \) is a ground state solution of \((K)\). The proof is complete. \( \square \)
4 Multiplicity

In this section, we are attempt to use the perturbation approach together with the symmetric mountain pass theorem to prove the existence of infinitely many high energy solutions to problem (K).

4.1 Proof of Theorem 1.4

We recall that $I_\lambda$ belongs to $C^1(E, \mathbb{R})$. Denote $B_R$ by the ball of radius $R > 0$ of $E$. Choose a sequence of finite dimensional subspaces $E_j$ of $E$ such that $\dim E_j = j$ and $E_j^\perp$ denotes the orthogonal complement of $E_j$. We define $\partial P$ by

$$
\partial P := \left\{ u \in E \setminus \{0\} \left| \frac{(\mu + 2)a}{2\mu} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{2 + 3\mu}{2\mu} \int_{\mathbb{R}^3} V(x)u^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)u^2 
+ \frac{(2 + 3\mu)\lambda}{2\mu} \int_{\mathbb{R}^3} W(x)u^2 + \frac{\lambda}{2} \int_{\mathbb{R}^3} (\nabla W(x), x)u^2 
+ \frac{(\mu + 2)b}{2\mu} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 
= \int_{\mathbb{R}^3} \left( \frac{1}{\mu} f(u)u + 3F(u) \right) + \frac{(r + 3\mu)\lambda}{\mu r} \int_{\mathbb{R}^3} |u|^r \right\}.
$$

Recalling assumption $(V_4)$ and (2.3), it follows from Sobolev’s inequality that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
\frac{(\mu + 2)a}{2\mu} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{2 + 3\mu}{2\mu} \int_{\mathbb{R}^3} V(x)u^2 
\leq \int_{\mathbb{R}^3} \left( \frac{1}{\mu} f(u)u + 3F(u) \right) + \frac{(r + 3\mu)\lambda}{\mu r} \int_{\mathbb{R}^3} |u|^r 
\leq \varepsilon \int_{\mathbb{R}^3} |u|^2 + C_\varepsilon \int_{\mathbb{R}^3} |u|^6, \quad \forall u \in \partial P \cap E_j^\perp,
$$

which implies that there exists $m_3 > 0$ independent of $\lambda$ such that $\|\nabla u\|_2^2 \geq m_3$. For any $u \in \partial P \cap E_j^\perp$, using the definition of $I_\lambda$ and (4.1), we arrive at

$$
I_\lambda(u) \geq a \frac{3\mu - 2}{8\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\mu - 2}{8\mu} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \frac{r - \mu}{4\mu r} \int_{\mathbb{R}^3} |u_n|^r 
\geq a \frac{3\mu - 2}{8\mu} m_3 + \frac{\mu - 2}{8\mu} bm_3^2 =: \delta.
$$

Moreover, we can choose $R_j > 0$ such that $I_\lambda(u) < 0$ for $u \in E_j \cap \partial B_{R_j}$. Actually, such an $R_j$ can be found by the fact that in the proof of (2) of Lemma 3.2 the element $e \in C_0^\infty(\mathbb{R}^3)$ is arbitrary. Note that $R_j$ does not depends on $\lambda$, that is to say,

$$
\forall \lambda \in (0, 1): I_\lambda(u) < 0 \quad \text{for any } u \in E_j \cap \partial B_{R_j}.
$$

(4.3)
Thus, the functional $I_\lambda$ satisfies all the assumptions of the symmetric mountain pass
theorem [38], and we define the minimax values

$$c_\lambda(j) = \inf_{B \in \Gamma_j} \sup_{u \in B} I_\lambda(u) \geq \delta,$$

where

$$\Gamma_j = \left\{ B = \phi(E_j \cap B_{R_j}) | \phi \in C(E_j \cap B_{R_j}, E), \ \phi \text{ is odd, } \phi = \text{Id on } E_j \cap \partial B_{R_j} \right\}.$$

It is easy to prove that the following intersection property holds (see [38, Proposition 9.23]): for $B \in \Gamma_j$, $B \cap \partial \mathcal{P} \cap E_{j-1}^\perp \neq \emptyset$.

More precisely, let us define $Y := \emptyset$, $m := j$, $k := j - 1$ and $X := E_{j-1}^\perp$, then (4.4) follows immediately from Proposition 9.23 of [38]. For any fixed $j$, by the definition of $c_\lambda(j)$, we have, in view of (2) of Lemma 3.2,

$$c_\lambda(j) \leq \sup_{u \in E_j \cap B_{R_j}} I_\lambda(u) \leq \sup_{u \in E_j \cap B_{R_j}} \left\{ C_1 \|u\|_E^2 + C_2 \|u\|_E^4 \right\} = C_{R_j},$$

where $C_{R_j}$ is indeed independent of $\lambda \in (0, 1]$ and $\| \cdot \|_E$ is a norm in $E_j$. Based on the above arguments, one has $c_\lambda(j) \in [\delta, C_{R_j}]$. Hence, there exists $u_\lambda(j) \in E_\lambda$ such that $J'_\lambda(u_\lambda(j)) = 0$ with $J_\lambda(u_\lambda(j)) = c_\lambda(j)$. Moreover, the symmetric mountain pass theorem yields $c_\lambda(j) \to +\infty$ as $j \to +\infty$. Using again Lemmas 3.6-3.10 and Proposition 3.11, there exists $u_0(j) \in H \setminus \{0\}$ such that for each fixed $j$, as $\lambda_n \to 0^+$,

$$u_{\lambda_n}(j) \to u_0(j) \text{ in } H, \ c_{\lambda_n}(j) \to c_0(j) \geq \delta, \ I(u_0(j)) = c_0(j)$$

and

$$I'(u_0(j)) = 0,$$

that is, $u_0(j)$ is a nontrivial solution of problem (K).

Once we show that $c_0(j) \to +\infty$ as $j \to +\infty$, problem (K) has infinitely many bounded state solutions and the proof of Theorem 1.4 is finished.

Obviously, for any $u \in E_\lambda$, we have

$$I_\lambda(u) = I(u) + \frac{\lambda}{2} \int_{\mathbb{R}^3} W(x)u^2 - \frac{\lambda}{r} \int_{\mathbb{R}^3} |u|^r \geq \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) - \frac{1}{r} \int_{\mathbb{R}^3} |u|^r := L(u).$$

Define the set $\partial \Theta \subset H$ by

$$\partial \Theta := \left\{ u \in H \setminus \{0\} : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2) = \int_{\mathbb{R}^3} |u|^r \right\},$$
which is the Nehari manifold associated to energy functional $L$, which, by classical arguments, is bounded away from zero and homeomorphic to the unit sphere. Define \( \gamma(\cdot) \) denotes the Krasnoselski genus of a symmetric set. Then, for $B \in \Gamma_j$, an easy modification of the proof of [38, Proposition 9.23] shows that $\gamma(B \cap \partial \Theta) \geq j$, for all $j \in \mathbb{N}$. Indeed, using the property of genus, one has

$$\gamma((E_j \cap B_{R_j}) \cap \partial \Theta) \geq \gamma((E_j \cap B_{R_j}) \cap \phi^{-1}(\partial \Theta)).$$

Observe that $\phi = \text{Id}$ on $E_j \cap \partial B_{R_j}$ and $(E_j \cap B_{R_j}) \cap \phi^{-1}(\partial \Theta) \neq \emptyset$. It is easy to check that $(E_j \cap B_{R_j}) \cap \phi^{-1}(\partial \Theta)$ bounds a symmetric neighborhood of the origin in $E_j$. Thus it follows from the property of genus that

$$\gamma((E_j \cap B_{R_j}) \cap \phi^{-1}(\partial \Theta)) = j.$$

So, we have immediately $\gamma(B \cap \partial \Theta) \geq j$. Hence,

$$c_\lambda(j) = \inf_{B \in \Gamma_j} \sup_{u \in B} I_\lambda(u) \geq \inf_{A \subset \partial \Theta, \gamma(A) \geq j} \sup_{u \in A} L(u) := b(j).$$

As a consequence of Theorem 1.1 in [9], one can get that $b(j) \to +\infty$ as $j \to +\infty$. Therefore,

$$c_0(j) = \lim_{\lambda \to 0^+} c_\lambda(j) \geq b(j) \to +\infty, \quad \text{as} \ j \to +\infty.$$

That is to say, problem (K) has infinitely many higher energy solutions. The proof is complete.

\[\square\]

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