WHEN FIRST ORDER T HAS LIMIT MODELS

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Abstract. We to a large extent sort out when does a (first order complete theory) $T$ have a superlimit model in a cardinal $\lambda$. Also we deal with related notions of being limit.
§0 Introduction, pg.3

[We give background and the basic definitions. We then present existence results for stable $T$ which have models which are saturated or closed to being saturated.]

§1 On countable superstable not $\aleph_0$-stable, pg.8

[Consistently $2^{\aleph_1} \geq \aleph_2$ and some such (complete first order) $T$ has a superlimit (non-saturated) model of cardinality $\aleph_1$. This shows that we cannot prove a non-existence result fully complementary to Lemma 0.9.]

§2 A strictly stable consistent example, pg.10

[Consistently $\aleph_1 < 2^{\aleph_0}$ and some countable stable not superstable $T$, has a (non-saturated) model of cardinality $\aleph_1$ which satisfies some relatives of being superlimit.]

§3 On the non-existence of limit models, pg.14

[The proofs here are in ZFC. If $T$ is unstable it has no superlimit models of cardinality $\lambda$ when $\lambda \geq \aleph_1 + |T|$. For unsuperstable $T$ we have similar results but with “few” exceptional cardinals $\lambda$ on which we do not know: $\lambda < \lambda^{\aleph_0}$ which are $< \beth_\omega$. Lastly, if $T$ is superstable and $\lambda \geq |T| + 2^{|T|}$ then $T$ has a superlimit model of cardinality $\lambda$ iff $|D(T)| \leq \lambda$ iff $T$ has a saturated model. Lastly, we get weaker results on weaker relatives of superlimit.]
§(0A) Background and Content

Recall that ([Sh:Ch.III]), if $T$ is (first order complete and) superstable then for $\lambda \geq 2^{|T|}$, $T$ has a saturated model $M$ of cardinality $\lambda$ and moreover

$(\ast)$ if $(M_\alpha : \alpha < \delta)$ is $\langle - \rangle$-increasing, $\delta$ a limit ordinal $< \lambda^+$ and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $\bigcup \{M_\alpha : \alpha < \delta\}$ is isomorphic to $M$.

When investigating categoricity of an a.e.c. (abstract elementary classes) $t = (K_t, \leq_t)$, the following property turns out to be central: $M$ is $\leq_t$-universal model of cardinality $\lambda$ with the property $(\ast)$ above (called superlimit) - possibly with addition parameter $\kappa = \text{cf}(\kappa) \leq \lambda$ (or stationary $S \subseteq \lambda^+$); we also consider some relatives, mainly limit, weakly limit and strongly limit. Those notions were suggested for a.e.c. in [Sh:SSN 3.1] or see the revised version [Sh:SSN 3.3] and see [Sh] or here in 0.7. But though coming from investigating non-elementary classes, they are meaningful for elementary classes and here we try to investigate them for elementary classes.

Recall that for a first order complete $T$, we know $\{\lambda : T$ has a saturated model of $T$ of cardinality $\lambda\}$, that is, it is $\{\lambda : \lambda <\lambda \geq |D(T)|$ or $T$ is stable in $\lambda\}$, on the definitions of $D(T)$ and other notions see §(0B) below. What if we replace saturated by superlimit (or some relative)? Let $EC_\lambda(T)$ be the class of models $M$ of $T$ of cardinality $\lambda$.

If there is a saturated $M \in EC_\lambda(T)$ we have considerable knowledge on the existence of limit model for cardinal $\lambda$, this was as mentioned in [Sh:SSN 3.6] by [Sh], see 0.9(1),(2). E.g. for superstable $T$ in $\lambda \geq 2^{|T|}$ there is a superlimit model (the saturated one). It seems a natural question on [Sh:SSN 3.6] whether it exhausts the possibilities of $(\lambda, \ast)$-superlimit and $(\lambda, \kappa)$-superlimit models for elementary classes.

Clearly the cases of the existence of such models of a (first order complete) theory $T$ where there are no saturated (or special) models are rare, because even the weakest version of Definition [Sh:SSN 3.1] = [Sh:SSN 3.3] or here Definition 0.7 for $\lambda$ implies that $T$ has a universal model of cardinality $\lambda$, which is rare (see Kojman Shelah [KjSh:409] which includes earlier history and recently Djamonzia [Mir]).

So the main question seems to be whether there are such cases at all. We naturally look at some of the previous cases of consistency of the existence of a universal model (for $\lambda < \lambda^{<\lambda}$), i.e., those for $\lambda = \aleph_1$.

E.g. a sufficient condition for some versions is the existence of $T_0 \supseteq T' \supseteq T$, where $T_0$ has the property $(\ast)$ and the consistency results for such $T_0'$ so naturally we first deal with the consistency results from [Sh:100]. In §1 we deal with the case of the countable superstable $T_0$ from [Sh:100] which is not $\aleph_0$-stable. By [Sh:100] consistently $\aleph_1 < 2^{\aleph_0}$ and for some $T_0 \supseteq T_0'$ of cardinality $\aleph_1$, $PC(T_0', T_0)$ is categorical in $\aleph_1$. We use this to get the consistency of “$T_0$ has a superlimit model of cardinality $\aleph_1$ and $\aleph_1 < 2^{\aleph_0}$”.

In §2 we prove that for some stable not superstable countable $T_1$ we have a parallel but weaker result. We relook at the old consistency results of “some $PC(T_1', T_1)$, $|T_1'| = \aleph_1 > |T_1|$, is categorical in $\aleph_1$” from [Sh:100]. From this we deduce that in this universe, $T_1$ has a strongly $(\aleph_1, \aleph_0)$-limit model.
It is a reasonable thought that we can similarly have a consistency result on the theory of linear order, but this is still not clear.

In §3 we show that if $T$ has a superlimit model in $\lambda \geq |T| + \aleph_1$ then $T$ is stable and $T$ is superstable except possibly under some severe restrictions on the cardinal $\lambda$ (i.e., $\lambda < \beth_\omega$ and $\lambda < \lambda^{\aleph_0}$). We then prove some restrictions on the existence of some (weaker) relatives.

Summing up our results on the strongest notion, superlimit, by §1.3 + §3.1 we have:

**Conclusion 0.1.** Assume $\lambda \geq |T| + \beth_\omega$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ is superstable and $\lambda \geq |D(T)|$.

In subsequent work we shall show that for some unstable $T$ (e.g. the theory of linear orders), if $\lambda = \lambda < \lambda > \kappa = \text{cf}(\kappa)$, then $T$ has a medium $(\lambda, \kappa)$-limit model, whereas if $T$ has the independence property even weak $(\lambda, \kappa)$-limit models do not exist; see [Sh:877] and more in [Sh:900, Sh:906, Sh:950, Sh:F1054].

We thank Alex Usvyatsov for urging us to resolve the question of the superlimit case and John Baldwin for comments and complaints.

§(0B) Basic Definitions

**Notation 0.2.** 1) Let $T$ denote a complete first order theory which has infinite models but $T_1, T'$, etc. are not necessarily complete.

2) Let $M, N$ denote models, $|M|$ the universe of $M$ and $|M|$ its cardinality and $M \prec N$ means $M$ is an elementary submodel of $N$.

3) Let $\tau_T = \tau(T), \tau_M = \tau(M)$ be the vocabulary of $T, M$ respectively.

4) Let $M \models \forall \vec{a}[\phi(\vec{a})]^\text{stat}$ means that the model $M$ satisfies $\phi[\vec{a}]$ iff the statement stat is true (or is 1 rather than 0).

**Definition 0.3.** 1) For $\vec{a} \in \omega^{|M|}$ and $B \subseteq M$ let $tp(\vec{a}, B, M) = \{\phi(\vec{x}, \vec{b}) : \phi = \phi(\vec{x}, \vec{y}) \in L(\tau_M), \vec{b} \in B \cap \omega B$ and $M \models \phi[\vec{a}, \vec{b}]\}$.  

2) Let $D(T) = \{tp(\vec{a}, 0, M) : M$ a model of $T$ and $\vec{a}$ a finite sequence from $M\}$.

3) If $A \subseteq M$ then $S^n(A, M) = \{tp(\vec{a}, A, N) : M \prec N$ and $\vec{a} \in \vec{a} \cap \omega M\}$, if $m = 1$ we may omit it.

4) A model $M$ is $\lambda$-saturated when: if $A \subseteq M, |A| < \lambda$ and $p \in S(A, M)$ then $p$ is realized by some $a \in M$, i.e. $p \subseteq tp(a, A, M)$; if $\lambda = |M|$ we may omit it.

5) A model $M$ is special when letting $\lambda = |M|$, there is an increasing sequence $\langle \lambda_i : i < \text{cf}(\lambda) \rangle$ of cardinals with limit $\lambda$ and a $\prec$-increasing sequence $\langle M_i : i < \text{cf}(\lambda) \rangle$ of models with union $M$ such that $M_{i+1}$ is $\lambda_i$-saturated of cardinality $\lambda_{i+1}$ for $i < \text{cf}(\lambda)$.

**Definition 0.4.** 1) For any $T$ let $EC(T) = \{M : M$ is a $\tau_T$-model of $T\}$.  

2) $EC_\lambda(T) = \{M \in EC(T) : M$ is of cardinality $\lambda\}$.

3) For $T \subseteq T'$ let

\[ PC(T', T) = \{M \mid \tau_T : M$ is model of $T'\} \]

\[ PC_\lambda(T', T) = \{M \in PC(T', T) : M$ is of cardinality $\lambda\}. \]

4) We say $M$ is $\lambda$-universal for $T_1$ when it is a model of $T_1$ and every $N \in EC_\lambda(T)$ can be elementarily embedded into $M$; if $T_1 = \text{Th}(M)$ we may omit it.

5) We say $M \in EC(T)$ is universal when it is $\lambda$-universal for $\lambda = |M|$.
We are here mainly interested in

**Definition 0.5.** Given \( T \) and \( M \in \text{EC}_\lambda(T) \) we say that \( M \) is a superlimit or \( \lambda \)-superlimit model when: \( M \) is universal and if \( \delta < \lambda^+ \) is a limit ordinal, \( \langle M_\alpha : \alpha \leq \delta \rangle \) is \( \prec \)-increasing continuous, and \( M_\alpha \) is isomorphic to \( M \) for every \( \alpha < \delta \) then \( M_\delta \) is isomorphic to \( M \).

**Remark 0.6.** Concerning the following definition we shall use strongly limit in \( [2.14]^1 \), medium limit in \( [2.14]^2 \).

**Definition 0.7.** Let \( \lambda \) be a cardinal \( \geq |T| \). For parts 3) - 7) but not 8), for simplifying the presentation we assume the axiom of global choice and \( F \) is a class function; alternatively restrict yourself to models with universe an ordinal \( \in [\lambda, \lambda^+]. \)

1) For non-empty \( \Theta \subseteq \{ \mu : \mu \leq \lambda < \lambda \) and \( \mu \) is regular \} and \( M \in \text{EC}_\lambda(T) \) we say that \( M \) is a \( (\lambda, \Theta) \)-superlimit when: \( M \) is universal and 
   if \( \langle M_i : i \leq \mu \rangle \) is \( \prec \)-increasing, \( M_i \cong M \) for \( i < \mu \) and \( \mu \in \Theta \) 
   then \( \bigcup \{ M_i : i < \mu \} \cong M \).

2) If \( \Theta \) is a singleton, say \( \Theta = \{ \theta \} \), we may say that \( M \) is \( (\lambda, \theta) \)-superlimit.

3) Let \( S \subseteq \lambda^+ \) be stationary. A model \( M \in \text{EC}_\lambda(T) \) is called \( S \)-strongly limit or \( (\lambda, S) \)-strongly limit when for some function: \( F : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T) \) we have:
   
   \begin{enumerate}
   
   \begin{itemize}
   
   \item[(a)] for \( N \in \text{EC}_\lambda(T) \) we have \( N \prec F(N) \)
   
   \item[(b)] if \( \delta \in S \) is a limit ordinal and \( \langle M_i : i < \delta \rangle \) is a \( \prec \)-increasing continuous sequence in \( \text{EC}_\lambda(T) \) and \( i < \delta \Rightarrow F(M_{i+1}) \prec M_{i+2} \); then \( M \cong \bigcup \{ M_i : i < \delta \} \).
   
   \end{itemize}
   
   \end{enumerate}

4) Let \( S \subseteq \lambda^+ \) be stationary. \( M \in \text{EC}_\lambda(T) \) is called \( S \)-limit or \( (\lambda, S) \)-limit if for some function \( F : \text{EC}_\lambda(T) \rightarrow \text{EC}_\lambda(T) \) we have:
   
   \begin{enumerate}
   
   \begin{itemize}
   
   \item[(a)] for every \( N \in \text{EC}_\lambda(T) \) we have \( N \prec F(N) \)
   
   \item[(b)] if \( \langle M_i : i < \lambda^+ \rangle \) is a \( \prec \)-increasing continuous sequence of members of \( \text{EC}_\lambda(T) \) such that \( F(M_{i+1}) \prec M_{i+2} \) for \( i < \lambda^+ \) then for some closed unbounded \( \mathcal{C} \) subset \( C \) of \( \lambda^+ \), 
      
      \[ \delta \in S \cap C \Rightarrow M_\delta \cong M \].
   
   \end{itemize}
   
   \end{enumerate}

5) We define\(^4^4\) “\( S \)-weakly limit”, “\( S \)-medium limit” like “\( S \)-limit”, “\( S \)-strongly limit” respectively by demanding that the domain of \( F \) is the family of \( \prec \)-increasing continuous sequence of members of \( \text{EC}_\lambda(T) \) of length \( < \lambda^+ \) and replacing “\( F(M_{i+1}) \prec M_{i+2}^\prime \)” by “\( M_{i+1} \prec F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}^\prime \)”.

6) If \( S = \lambda^+ \) then we may omit \( S \) (in parts (3), (4), (5)).

7) For non-empty \( \Theta \subseteq \{ \mu : \mu \leq \lambda < \lambda \) and \( \mu \) is regular \}, \( M \in (\lambda, \Theta) \)-strongly limit\(^4\) if \( M \) is \( \{ \delta < \lambda^+ : c(\delta) \in \Theta \} \)-strongly limit. Similarly for the other notions. If we do not write \( \lambda \) we mean \( \lambda = |M| \).

\(^1\)no loss if we add \( M_{i+1} \cong M \), so this simplifies the demand on \( F \), i.e., only \( F(M') \) for \( M' \cong M \) is required

\(^2\)alternatively, we can use as a parameter a filter on \( \lambda^+ \) extending the co-bounded filter

\(^3\)Note that \( M \) is \( (\lambda, S) \)-strongly limit iff \( M \) is \( \{ \lambda, c(\delta) : \delta \in S \} \)-strongly limit.

\(^4\)in \( [\text{Sh:88}] \) we consider: we replace “\( \text{limit} \)” by “\( \text{limit} \)” if “\( F(M_{i+1}) \prec M_{i+2}' \), “\( M_{i+1} \prec F(\langle M_j : j \leq i+1 \rangle) \prec M_{i+2}' \)” are replaced by “\( F(M_i) \prec M_{i+1}' \), “\( M_i \prec F(\langle M_j : j \leq i \rangle) \prec M_{i+1}' \)” respectively. But \( (\text{EC}(T), \prec) \) has amalgamation.
8) We say that $M \in K\lambda$ is invariably strong limit when in part (3), $F$ is just a subset of $\{(M,N)\} \equiv M \prec N$ are from $EC\lambda(T)$ and in clause (b) of part (3) we replace “$F(M_{i+1}) \prec M_{i+2}$” by “$(\exists N)(M_{i+1} \prec N \prec M_{i+2} \land ((M,N)\equiv F)$”. But abusing notation we still write $N = F(M)$ instead $((M,N)\equiv F)$. Similarly with the other notions, so we use the isomorphism type of $M^{\sim}(N)$ for “weakly limit” and “medium limit”.

9) In the definitions above we may say “$F$ witness $M$ is ...”

**Observation 0.8.** 1) Assume $F_1, F_2$ are as above and $F_1(N) \prec F_2(N)$ (or $F_1(\bar{N}) \prec F_2(\bar{N})$) whenever defined. If $F_1$ is a witness then so is $F_2$.

2) All versions of limit models implies being a universal model in $EC\lambda(T)$.

3) The Obvious implications diagram: For non-empty $\Theta \subseteq \{\theta : \theta$ is regular $\leq \lambda\}$ and stationary $S_1 \subseteq \{\delta < \lambda^+ : \text{cf}(\delta) \in \Theta\}$:

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superlimit = ($\lambda, \{\mu : \mu \leq \lambda$ regular$\}$)-superlimit
↓
($\lambda, \Theta$)-superlimit
↓
$S_1$-strongly limit
↓  ↓
$S_1$-medium limit, $S_1$-limit
↓  ↓
$S_1$-weakly limit.
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**Lemma 0.9.** Let $T$ be a first order complete theory.

1) If $\lambda$ is regular, $M$ a saturated model of $T$ of cardinality $\lambda$, then $M$ is ($\lambda, \lambda$)-superlimit.

2) If $T$ is stable, and $M$ is a saturated model of $T$ of cardinality $\lambda \geq \aleph_1 + |T|$ and $\Theta = \{\mu : \kappa(T) \leq \mu \leq \lambda$ and $\mu$ is regular$\}$, then $M$ is ($\lambda, \Theta$)-superlimit (on $\kappa(T)$-see [Shc] III,$\S 3$).

3) If $T$ is stable in $\lambda$ and $\kappa = \text{cf}(\kappa) \leq \lambda$ then $T$ has an invariably strongly ($\lambda, \kappa$)-limit model.

**Remark 0.10.** Concerning [0.9] 2), note that by [Shc] if $\lambda$ is singular or just $\lambda < \lambda^{< \lambda}$ and $T$ has a saturated model of cardinality $\lambda$ then $T$ is stable (even stable in $\lambda$) and $\text{cf}(\lambda) \geq \kappa(T)$.

**Proof.** 1) Let $M_i$ be a $\lambda$-saturated model of $T$ of cardinality $\lambda$ for $i < \lambda$ and $\langle M_i : i < \lambda\rangle$ is $\prec$-increasing and $M_\lambda = \bigcup_{i<\lambda} M_i$. Now for every $A \subseteq M_\lambda$ of cardinality $\lambda < \lambda$ there is $i < \lambda$ such that $A \subseteq M_i$ hence every $p \in S(A, M_\lambda)$ is realized in $M_i$ hence in $M_\lambda$; so clearly $M_\lambda$ is $\lambda$-saturated. Remembering the uniqueness of a $\lambda$-saturated model of $T$ of cardinality $\lambda$ we finish.

2) Use [Shc] III,3.11]: if $M_i$ is a $\lambda$-saturated model of $T, \langle M_i : i < \delta\rangle$ increasing $\text{cf}(\delta) \geq \kappa(T)$ then $\bigcup_{i<\delta} M_i$ is $\lambda$-saturated.

3) Let $K_{\lambda,\kappa} = \{M : M = (M_i : i \leq \kappa) \text{ is } \prec$-increasing continuous, $M_i \in EC\lambda(T)$ and $(M_{i+2}, c)_{c \in M_{i+1}}$ is saturated for every $i < \kappa\}$. Clearly $M, \bar{N} \in K_{\lambda,\kappa} \Rightarrow M_\kappa \cong \bar{N}$.
$N_{\kappa}$. Also for every $M \in \text{EC}_{\lambda}(T)$ there is $N$ such that $M \prec N$ and $(N,c)_{c \in \mathcal{M}}$ is saturated, as also $\text{Th}((M,c)_{c \in \mathcal{M}})$ is stable in $\lambda$; so there is an invariant $\mathbf{F} : \text{EC}_{\lambda}(T) \to \text{EC}_{\lambda}(T)$ such that $M \prec \mathbf{F}(M)$ and $(\mathbf{F}(M),c)_{c \in \mathcal{M}}$ is saturated; such $\mathbf{F}$ witness the desired conclusion. \hfill \Box

**Definition 0.11.**

0) For regular $\kappa < \lambda$ let $S_\theta^\lambda = \{ \delta < \lambda : \text{cf}(\delta) = \lambda \}$.

1) For a regular uncountable cardinal $\lambda$ let $\check{I}[\lambda] = \{ S \subseteq \lambda : \text{some pair } (E,\bar{u}) \text{ witnesses } S \in \check{I}[\lambda], \text{ see below} \}$.

2) We say that $(E,\bar{u})$ is a witness for $S \in \check{I}[\lambda]$ iff:

(a) $E$ is a club of the regular cardinal $\lambda$

(b) $\bar{u} = \langle u_\alpha : \alpha < \lambda \rangle$, $u_\alpha \subseteq \alpha$ and $\beta \in u_\alpha \Rightarrow u_\beta = \beta \cap u_\alpha$

(c) for every $\delta \in E \cap S$, $u_\delta$ is an unbounded subset of $\delta$ of order-type $\text{cf}(\delta)$ (and $\delta$ is a limit ordinal).

By [Sh:420, §1]

**Claim 0.12.** If $\kappa^+ < \lambda$ and $\kappa,\lambda$ are regular then some stationary $S \subseteq \{ \delta < \lambda : \text{cf}(\delta) = \kappa \}$ belongs to $\check{I}[\lambda]$.

By [Sh:108]

**Claim 0.13.** If $\lambda = \mu^+, \theta = \text{cf}(\theta) \leq \text{cf}(\mu)$ and $\alpha < \mu \Rightarrow |\alpha|^<\theta \leq \mu$ then $S_\theta^\lambda \in \check{I}[\lambda]$. 

1. On superstable not $\aleph_0$-stable $T$

We first note that superstable $T$ tend to have superlimit models.

Claim 1.1. Assume $T$ is superstable and $\lambda \geq |T| + 2^{\aleph_0}$. Then $T$ has a superlimit model of cardinality $\lambda$ iff $T$ has a saturated model of cardinality $\lambda$ iff $T$ has a universal model of cardinality $\lambda$ iff $\lambda \geq |D(T)|$.

Proof. By [Sh:c, III, 5] we know that $T$ is stable in $\lambda$ iff $\lambda \geq |D(T)|$. Now if $|T| \leq \lambda < |D(T)|$ trivially there is no universal model of $T$ of cardinality $\lambda$ hence no saturated model and no superlimit model, etc., recalling [0.8](2). If $\lambda \geq |D(T)|$, then $T$ is stable in $\lambda$ hence has a saturated model of cardinality $\lambda$ by [Sh:c, III] (hence universal) and the class of $\lambda$-saturated models of $T$ is closed under increasing elementary chains by [Sh:c, III] so we are done. □

The following are the prototypical theories which we shall consider.

Definition 1.2. The following are the prototypical theories which we shall consider.

1) $T_0 = \text{Th}(\langle 2, E^0_n \rangle_{n < \omega})$ when $\eta E^0_n \nu \iff \eta \upharpoonright n = \nu \upharpoonright n$.
2) $T_1 = \text{Th}(\langle \omega_1, E^1_n \rangle_{n < \omega})$ where $\eta E^1_n \nu \iff \eta \upharpoonright n = \nu \upharpoonright n$.
3) $T_2 = \text{Th}(\mathbb{R}, <)$.

Recall

Observation 1.3. 0) $T_\ell$ is a countable complete first order theory for $\ell = 0, 1, 2$.
1) $T_0$ is superstable not $\aleph_0$-stable.
2) $T_1$ is strictly stable, that is, stable not superstable.
3) $T_2$ is unstable.
4) $T_\ell$ has elimination of quantifiers for $\ell = 0, 1, 2$.

Claim 1.4. It is consistent with ZFC that $\aleph_1 < 2^{\aleph_0}$ and some $M \in \text{EC}_{\aleph_1}(T_0)$ is a superlimit model.

Proof. By [Sh:100], for notational simplicity we start with $\mathbf{V} = \mathbf{L}$.

So $T_0$ is defined in [1,2](1) and it is the $T$ from Theorem [Sh:100, 1.1] and let $S$ be the set of $\eta \in (\langle 2 \rangle)^\mathbb{L}$. We define $T'$ (called $T_1$ there) as the following theory:

@1 (i) $T_0$, or just for each $n$ the sentence saying $E_n$ is an equivalence relation with $2^n$ equivalence classes, each $E_n$ equivalence class divided to two by $E_{n+1}$, $E_{n+1}$ refine $E_n$, $E_0$ is trivial

(ii) the sentences saying that

(a) for every $x$, the function $z \mapsto F(x, z)$ is one-to-one and

(b) $x E_n(F(x, z))$ for each $n < \omega$

(iii) $E_n(c_\eta, c_\nu)^{\forall[\eta \upharpoonright n = \nu \upharpoonright n]}$ for $\eta, \nu \in S$.

In [Sh:100] it is proved that in some forcing5 extension $\mathbf{L}^p$ of $\mathbf{L}$, $\mathbb{P}$ an $\aleph_2$-c.c. proper forcing of cardinality $\aleph_2$, in $\mathbf{V} = \mathbf{L}^p$, the class $\text{PC}(T', T_0) = \{ M \upharpoonright \tau_{T_0} : M \text{ is a } \tau\text{-model of } T' \}$ is categorical in $\aleph_1$.

However, letting $M^*$ be any model from $\text{PC}(T', T_0)$ of cardinality $\aleph_1$, it is easy to see that (in $\mathbf{V} = \mathbf{L}^p$):

@2 the following conditions on $M$ are equivalent

(a) $M$ is isomorphic to $M^*$

5We can replace $\mathbf{L}$ by any $\mathbf{V}_0$ which satisfies $2^{\aleph_0} = \aleph_1, 2^{\aleph_1} = \aleph_2$. 
(b) \( M \in \text{PC}(T', T_0) \)
(c) (a) \( M \) is a model of \( T_0 \) of cardinality \( \aleph_1 \)
(\( \beta \)) \( M^* \) can be elementarily embedded into \( M \)
(\( \gamma \)) for every \( a \in M \) the set \( \cap\{a/E^n_M : n < \omega\} \) has cardinality \( \aleph_1 \).

But

\( \circ \circ \circ \) every model \( M_1 \) of \( T \) of cardinality \( \leq \aleph_1 \) has a proper elementary extension to a model satisfying (c), i.e., (a), (\( \beta \)), (\( \gamma \)) of \( \circ \circ \circ \) above

\( \circ \circ \circ \) if \( \langle M_\alpha : \alpha < \delta \rangle \) is an increasing chain of models satisfying (c) of \( \circ \circ \circ \) and \\
\( \delta < \omega_2 \) then also \( \cup\{M_\alpha : \alpha < \delta\} \) does.

Together we are done.

Naturally we ask

**Question 1.5.** What occurs to \( T_0 \) for \( \lambda > \aleph_1 \) but \( \lambda < 2^{\aleph_0} \)?

**Question 1.6.** Does the theory \( T_2 \) of linear order consistently have an \( (\aleph_1, \aleph_0) \)-superlimit? (or only strongly limit?) but see §3.

**Question 1.7.** What is the answer for \( T \) when \( T \) is countable superstable not \( \aleph_0 \)-stable and \( D(T) \) countable for \( \aleph_1 < 2^{\aleph_0} \) for \( \aleph_2 < 2^{\aleph_0} \)\

So by the above for some such \( T \), in some universe, for \( \aleph_1 \) the answer is yes, there is a superlimit.
2. A strictly stable consistent example

We now look at models of $T_1$ (redefined below) in cardinality $\aleph_1$; recall

**Definition 2.1.** $T_1 = \text{Th}(\omega(\omega_1), E_n)_{n<\omega}$ where $E_n = \{(\eta, \nu) : \eta, \nu \in \omega(\omega_1) \text{ and } \eta \upharpoonright n = \nu \upharpoonright n\}$.

**Remark 2.2.**
(a) Note that $T_1$ has elimination of quantifiers.
(b) If $\lambda = \Sigma(\lambda_n : n < \omega)$ and $\lambda_n = \lambda_n^{\aleph_0}$, then $T_1$ has a $(\lambda, \aleph_0)$-superlimit model in $\lambda$ (see [Sh:100]).

**Definition/Claim 2.3.** 1) Any model of $T_1$ of cardinality $\lambda$ is isomorphic to $M_{\lambda, h} := \{(\eta, \varepsilon) : \eta \in A, \varepsilon < h(\eta)\}, E_n\_{n<\omega}$ for some $A \subseteq \omega\lambda$ and $h : \omega\lambda \to (\text{Car} \cap \lambda^+) \setminus \{0\}$ where $(\eta_1, \varepsilon_1)E_n(\eta_2, \varepsilon_2) \iff \eta_1 \upharpoonright n = \eta_2 \upharpoonright n$, pedantically we should write $E_n^{M_{\lambda, h}} = E_n\|_{M_{\lambda, h}}$.
2) We write $M_A$ for $M_{\lambda, h}$ when $A$ is as above and $h : A \to \{|A|\}$, so constantly $|A|$ when $A$ is infinite.
3) For $A \subseteq \omega\lambda$ and $h$ as above the model $M_{\lambda, h}$ is a model of $T_1$ iff $A$ is non-empty and $(\forall \eta \in A)(\forall \nu < \omega) (\exists \eta, \nu \in A)(|\nu| = n \wedge \nu(n) \neq \eta(n))$.
4) Above $M_{\lambda, h}$ has cardinality $\lambda$ iff $\exists\{h(\eta) : \eta \in A\} = \lambda$.

**Definition 2.4.** 1) We say that $A$ is a $(T_1, \lambda)$-witness when
(a) $A \subseteq \omega\lambda$ has cardinality $\lambda$.
(b) if $B_1, B_2 \subseteq \omega\lambda$ are $(T_1, A)$-big (see below) of cardinality $\lambda$ then $(B_1 \cup \omega \lambda, \triangleleft)$ is isomorphic to $(B_2 \cup \omega \lambda, \triangleleft)$.

2) A set $B \subseteq \omega\lambda$ is called $(T_1, A)$-big when it is $(\lambda, \lambda) - (T_1, A)$-big; see below.
3) $B$ is $(\mu, \lambda) - (T_1, A)$-big means: $B \subseteq \omega\lambda, |B| = |A| = \mu$ and for every $\eta \in \omega\lambda$ there is an isomorphism $f$ from $(\omega \lambda, \triangleleft)$ onto $(\{\eta\nu : \nu \in \omega \lambda\}, \triangleleft)$ mapping $A$ into $\{\nu : \eta \nu \in B\}$.
4) $A \subseteq \omega(\omega_1)$ is $\aleph_1$-suitable when:
(a) $|A| = \aleph_1$.
(b) for a club of $\delta < \omega_1, A \cap \omega \delta$ is everywhere not meagre in the space $\omega \delta$, i.e., for every $\eta \in \omega \delta$ the set $\{\nu \in A \cap \omega \delta : \eta \triangleleft \nu\}$ is a non-meagre subset of $\omega \delta$ (that is what really is used in [Sh:100]).

Claim 2.5. It is consistent with ZFC that $2^{\aleph_0} > \aleph_1+$ there is a $(T_1, \aleph_1)$-witness; moreover every $\aleph_1$-suitable set is a $(T_1, \aleph_1)$-witness.

**Proof.** By [Sh:100] [2].

**Remark 2.6.** The witness does not give rise to an $(\aleph_1, \aleph_0)$-limit model, as for the union of any “fast enough” $\triangleleft$-increasing $\omega$-chain of members of $E_{\aleph_1}(T_1)$, the relevant sets are meagre.

**Definition 2.7.** Let $A$ be a $(T_1, \lambda)$-witness. We define $K_{T_1, A}^1$ as the family of $M = (|M|, <^M, P^M_\alpha)_{\alpha \leq \omega}$ such that:

(a) $(|M|, <^M)$ is a tree with $(\omega + 1)$ levels
(b) $P^M_{\alpha}$ is the $\alpha$-th level; let $P^M_{<\omega} = \cup\{P^M_n : n < \omega\}$
(γ) \( M \) is isomorphic to \( M^1_B \) for some \( B \subseteq \omega \lambda \) of cardinality \( \lambda \) where \( M^1_B \) is defined by \( |M^1_B| = (\omega \lambda) \cup B, P^M_B = n \lambda, P^M_B = B \) and \( <^M_B = \omega |M^1_B| \), i.e., being an initial segment

(δ) moreover \( B \) is such that some \( f \) satisfies:

⊕ (a) \( f : \omega \lambda \to \omega \) and \( f(\langle \rangle) = 0 \) for simplicity

(b) \( \eta \leq \nu \in \omega \lambda \Rightarrow f(\eta) \leq f(\nu) \)

(c) if \( \eta \in B \) then \( f(\eta \restriction n) : n < \omega \) is eventually constant

(d) if \( \eta \in \omega \lambda \) then \( \{ \nu \in \omega \lambda : \eta \restriction \nu \in B \text{ and } m < \omega \Rightarrow f(\eta \restriction (\nu \restriction m)) = f(\eta) \} \) is \((T_1, A)\)-big

(e) for \( \eta \in \omega \lambda \) and \( n \in \{ f(\eta), \omega \} \) for \( \lambda \) ordinals \( \alpha < \lambda \), we have \( f(\eta \restriction \langle \alpha \rangle) = n \).

**Claim 2.8.** [The Global Axiom of Choice] If \( A \) is a \((T_1, \mathcal{N})\)-witness then

(a) \( K^1_{T_1, A} \neq \emptyset \)

(b) any two members of \( K^1_{T_1, A} \) are isomorphic

(c) there is a function \( F \) from \( K^1_{T_1, A} \) to itself (up to isomorphism, i.e., \((M, F(M))\) is defined only up to isomorphism) satisfying \( M \subseteq F(M) \) such that \( K^1_{T_1, A} \) is closed under increasing unions of sequence \( \langle M_n : n < \omega \rangle \) such that \( F(M_n) \subseteq M_{n+1} \).

**Proof.**

Clause (a): Trivial.

Clause (b): By the definition of “\( A \) is a \((T_1, \mathcal{N})\)-witness” and of \( K^1_{T_1, A} \).

Clause (c):

We choose \( F \) such that

⊕ if \( M \in K^1_{A, T_1} \), then \( M \subseteq F(M) \in K^1_{A, T_1} \) and for every \( k < \omega \) and \( a \in P^M_k \), the set \( \{ b \in P^{F(M)}_k : a <_{F(M)} b \text{ and } b \not\in M \} \) has cardinality \( \aleph_1 \).

Assume \( M = \bigcup \{ M_n : n < \omega \} \) where \( \{ M_n : n < \omega \} \) is \( \subseteq \)-increasing, \( M_n \in K^1_{A, T_1} \), \( F(M_n) \subseteq M_{n+1} \). Clearly \( M \) is as required in the beginning of Definition 2.7, that is, satisfies clauses (α), (β), (γ) there. To prove clause (δ), we define \( f : P^M_\omega \to \omega \) by \( f(a) = \min \{ n : a \in M_n \} \). Pendentically, \( F \) is defined only up to isomorphism.

So we are done.

**Claim 2.9.** [The Global Axiom of Choice]

If \( A \) is a \((T_1, \lambda)\)-witness then

(a) \( K^1_{T_1, A} \neq \emptyset \)

(b) any two members of \( K^1_{T_1, A} \) are isomorphic

(c) if \( M_n \in K^1_{T_1, A} \) and \( n < \omega \Rightarrow M_n \subseteq M_{n+1} \) then \( M := \bigcup \{ M_n : n < \omega \} \in K^1_{T_1, A} \).

**Remark 2.10.** If we omit clause (b), we can weaken the demand on the set \( A \).
Proof. Assume \( M = \cup \{ M_n : n < \omega \} \), \( M_n \subseteq M_{n+1} \), \( M_n \in K_{T_1}^1 \), and \( f_n \) witnesses \( M_n \in K_{T_1}^1 \). Clearly \( M \) satisfies clauses \((\alpha), (\beta), (\gamma)\) from Definition \( 2.7 \), we just have to find a witness \( f \) as in clause \((\delta)\) there.

For each \( a \in M \) let \( n(a) = \text{Min} \{ n : a \in M_n \} \), clearly if \( M \models \text{“} a < b < c \text{”} \) then \( n(a) \leq n(b) \) and \( n(a) = n(c) \) \( \Rightarrow \) \( n(a) = n(b) \). Let \( g_n : M \rightarrow M \) be defined by: \( g_n(a) = b \) if \( b \leq M a, b \in M_n \) and \( b \) is \( \leq M \)-maximal under those restrictions; clearly it is well defined. Now we define \( f'_n : M_n \rightarrow \omega \) by induction on \( n < \omega \) such that \( m < n \Rightarrow f'_m \subseteq f'_n \), as follows.

If \( n = 0 \) let \( f'_0 = f_n \).

If \( n = m + 1 \) and \( a \in M_n \) we let \( f'_n(a) \) be \( f'_{m}(a) \) if \( a \in M_m \) and be \( (f_n(a) - f_n(g_m(a)))+f'_m(g_m(a)) \) if \( a \in M_n \setminus M_m \). Clearly \( f := \cup \{ f'_n : n < \omega \} \) is a function from \( M \) to \( \omega \), \( a \leq M b \Rightarrow f(a) \leq f(b) \), and for any \( a \in M \) the set \( \{ b \in M : a \leq M b \text{ and } f(b) = f(a) \} \) is equal to \( \{ b \in M_n(a) : f_n(a)(b) \text{ and } a \leq M b \} \).

So we are done. \( \square \)

Definition 2.11. Let \( A \) be a \( (T_1, \lambda) \)-witness. We define \( K_{T_1}^2 \) as in Definition \( 2.7 \), but \( f \) is constantly zero.

Claim 2.12. [The Global Axiom of Choice] If \( A \) is a \( (T_1, \aleph_1) \)-witness then

\begin{itemize}
  \item[(a)] \( K_{T_1}^2 \neq \emptyset \)
  \item[(b)] any two members of \( K_{T_1}^2 \) are isomorphic
  \item[(c)] there is a function \( F \) from \( \cup \{ \alpha + 2 : (K_{T_1}^2) : \alpha < \omega_1 \} \) to \( K_{T_1}^2 \) which satisfies:
    \begin{itemize}
      \item[(\alpha)] if \( \bar{M} = \langle M_i : i \leq \alpha + 1 \rangle \) is an \( \prec \)-increasing sequence of models of \( T \) then \( M_{\alpha+1} = F(M) \in K_{T_1}^2 \)
      \item[(\beta)] the union of any increasing \( \omega_1 \)-sequence \( \bar{M} = \langle M_\alpha : \alpha < \omega_1 \rangle \)
        of members of \( K_{T_1}^2 \) belongs to \( K_{T_1}^2 \) when \( \omega_1 = \sup \{ \alpha : F(M_\alpha) \models T(\alpha + 2) \} \subseteq M_{\alpha+2} \) and is a well defined embedding of \( M_\alpha \) into \( M_{\alpha+2} \).
    \end{itemize}
\end{itemize}

Remark 2.13. Instead of the global axiom of choice, we can restrict the models to have universe a subset of \( \lambda^+ \) (or just a set of ordinals).

Proof. Clause \((a)\): Easy.

Clause \((b)\): By the definition.

Clause \((c)\): Let \( \{ \mathcal{Z}_\varepsilon : \varepsilon < \omega_1 \} \) be an increasing sequence of subsets of \( \omega_1 \) with union \( \omega_1 \) such that \( \varepsilon < \omega_1 \Rightarrow |\mathcal{Z}_\varepsilon| \leq \sum_{\zeta < \varepsilon} |\mathcal{Z}_\zeta| = \aleph_1 \). Let \( M^* \in K_{T_1}^2 \) be such that \( \omega^>(\omega_1) \subseteq |M^*| \subseteq \omega^>(\omega_1) \) and \( M^* \models \omega^>(\mathcal{Z}_\varepsilon) \) belongs to \( K_{T_1}^2 \) for every \( \varepsilon < \omega_1 \).

We choose a pair \( (F, f) \) of functions with domain \( \{ \bar{M} : \bar{M} \text{ an increasing sequence of members of } K_{T_1}^2 \text{ of length } < \omega_1 \} \) such that:

\begin{itemize}
  \item[(\alpha)] \( F(M) \) is an extension of \( \cup \{ M_i : i < \ell g(M) \} \) from \( K_{T_1}^2 \),
  \item[(\beta)] \( f(M) \) is an embedding from \( M^*_{g(M)} \) into \( F(M) \),
  \item[(\gamma)] if \( \bar{M}^\ell = \langle M_\alpha : \alpha < \alpha_2 \rangle \) for \( \ell = 1, 2 \) and \( \alpha_1 < \alpha_2, \bar{M}^1 = \bar{M}^2 \models \alpha_1\) and \( F(M^1) \subseteq M^1_{\alpha_1} \) then \( F(M^1) \subseteq F(M^2) \),
  \item[(\delta)] if \( a \in F(M) \) and \( n < \omega \) then for some \( b \in M^*_{g(M)} \) we have \( F(M) \models aE_n(f(M)(b)) \).
\end{itemize}
Now check. \[\square\]

**Conclusion 2.14.** Assume there is a \((T_1, R_1)\)-witness (see Definition 2.4) for the first-order complete theory \(T_1\) from 2.7.

1) \(T_1\) has an \((\aleph_1, \aleph_0)\)-strongly limit model.
2) \(T_1\) has an \((\aleph_1, \aleph_1)\)-medium limit model.
3) \(T_1\) has a \((\aleph_1, \aleph_0)\)-superlimit model.

**Proof.**

1) By 2.8 the reduction of problems on \((EC(T_1), \prec)\) to \(K_{T_1, A}\) (which is easy) is exactly as in [Sh:100].

2) By 2.12.

3) Like part (1) using claim 2.9. \[\square\]

**Claim 2.15.** If \(\lambda = \Sigma \{\lambda_n : n < \omega\}\) and \(\lambda_n = \lambda_n^{R_n}\), then \(T_1\) has a \((\lambda, \aleph_0)\)-superlimit model in \(\lambda\).

**Proof.** Let \(M_n\) be the model \(M_{A_n, h_n}\) where \(A_n = \prec(\lambda_n)\) and \(h_n : A_n \to \lambda_n^+\) is constantly \(\lambda_n\).

Clearly

\((*)_1\) \(M_n\) is a saturated model of \(T_1\) of cardinality \(\lambda_n\)

\((*)_2\) \(M_n \prec M_{n+1}\)

\((*)_3\) \(M_\omega = \cup\{M_n : n < \omega\}\) is a special model of \(T_1\) of cardinality \(\lambda\).

The main point:

\((*)_4\) \(M_\omega\) is \((\lambda, R_0)\)-superlimit model of \(T_1\).

[Why? Toward this assume]

(a) \(N_n\) is isomorphic to \(M_n\) say \(f_n : M_\omega \to N_n\) is such isomorphic

(b) \(N_n \prec N_{n+1}\) for \(n < \omega\).

Let \(N_\omega = \cup\{N_n : n < \omega\}\) and we should prove \(N_\omega \cong M_\omega\), so just \(N_\omega\) is a special model of \(T_1\) of cardinality \(\lambda\) suffice.

Let \(N_n' = N_\omega |(\cup\{f_n(M_k) : k \leq n\})\). Easily \(N_n \prec N_{n+1} \prec N_\omega\) and \(\cup\{N_n' : n < \omega\} = N_\omega\) and \(\|N_n'\| = \lambda_n\). So it suffices to prove that \(N_n'\) is saturated and by direct inspection shows this. \[\square\]
3. ON NON-EXISTENCE OF LIMIT MODELS

Naturally we assume that non-existence of superlimit models for unstable $T$ is easier to prove. For other versions we need to look more. We first show that for $\lambda \geq |T| + \aleph_1$, if $T$ is unstable then it does not have a superlimit model of cardinality $\lambda$ and if $T$ is unsuperstable, we show this for “most” cardinals $\lambda$. On “$\Phi$ proper for $K_\alpha$ or $K_\alpha^\omega$”, see [Sh:829] VII or [Sh:E59] or hopefully some day in [Sh:829]. We assume some knowledge on stability.

Claim 3.1. 1) If $T$ is unstable, $\lambda \geq |T| + \aleph_1$, then $T$ has no superlimit model of cardinality $\lambda$.
2) If $T$ is stable not superstable and $\lambda \geq |T| + 2^\omega$ or $\lambda = \lambda^{\aleph_0} \geq |T|$ then $T$ has no superlimit model of cardinality $\lambda$.

Remark 3.2. 1) We assume some knowledge on EM models for linear orders $I$ and members of $K_\alpha^\omega$ as index models, see, e.g. [Shvč VII].
2) We use the following definition in the proof, as well as a result from [Sh:460] or [Sh829].

Definition 3.3. For cardinals $\lambda > \kappa$ let $\lambda^{[\kappa]}$ be the minimal $\mu$ such that for some, equivalently for every set $A$ of cardinality $\lambda$ there is $\mathcal{R}_A \subseteq [A]^{\leq \kappa} = \{ B \subseteq A : |B| \leq \kappa \}$ of cardinality $\lambda$ such that any $B \in [\lambda]^{\leq \kappa}$ is the union of $< \kappa$ members of $\mathcal{R}_A$.

Proof. 1) Towards a contradiction assume $M^*$ is a superlimit model of $T$ of cardinality $\lambda$. As $T$ is unstable we can find $F_\lambda$ such that any $\tau(I, \Phi)$ with $|I| \leq n$ is a superlimit model of $T$ for $\lambda^*$. Then $\Phi$ in the proof (see [Shvč VII]) and $F_\lambda(\ell < m)$ such that $F_\lambda \in \tau_\lambda \setminus \tau_\lambda$ is a unary function symbol for $\ell < m$, $\tau_\lambda \subseteq \tau(\Phi)$ and for every linear order $I$, $EM(I, \Phi)$ has Skolem functions and its $\tau_\lambda$-reduct $EM_{\tau_\lambda}(I, \Phi)$ is a model of $T$ of cardinality $|T| + |I|$ and $\tau(\Phi)$ is of cardinality $|T| + \aleph_0$ and $\langle a_s : s \in I \rangle$ is the Skeleton of $EM(I, \Phi)$, that is, it is an indiscernible sequence in $EM(I, \Phi)$ and $EM(I, \Phi)$ is the Skolem hull of $\{ a_s : s \in I \}$, and letting $\bar{a}_s = \langle F_\lambda(a_s) : \ell < m \rangle$ in $EM(I, \Phi)$ we have $EM_{\tau_\lambda}(I, \Phi) = EM_{\tau_\lambda}(I, \Phi) \models \forall [s < t] \forall \bar{a}_{\bar{x}} \bar{a}_t)[[\bar{x}, \bar{y}] = m].$

Next we can find $\Phi_n$ (for $n < \omega$) such that:

1) $\Phi_n$ is proper for linear order and $\Phi_0 = \Phi$
2) $EM_{\tau(\Phi)}(I, \Phi_n) \equiv EM_{\tau(\Phi)}(I, \Phi_{n+1})$ for every linear order $I$ and $n < \omega$; moreover
3) $\tau(\Phi_n) \subseteq \tau(\Phi_{n+1})$ and $EM(I, \Phi_n) \equiv EM_{\tau(\Phi_n)}(I, \Phi_{n+1})$ for every $n < \omega$ and linear order $I$
4) if $|I| \leq n$ then $EM_{\tau(\Phi)}(I, \Phi_n) = EM_{\tau(\Phi)}(I, \Phi_{n+1})$ and $EM_{\tau_\lambda}(I, \Phi_n) \equiv M^*$
5) $|\tau(\Phi_n)| = \lambda$.

This is easy. Let $\Phi_\omega$ be the limit of $\langle \Phi_n : n < \omega \rangle$, i.e. $\tau(\Phi_\omega) = \cup \{ \tau(\Phi_n) : n < \omega \}$ and if $k < \omega$ then $EM_{\tau(\Phi_k)}(I, \Phi_n) = EM_{\tau(\Phi_k)}(I, \Phi_{n+1}) : n \in [k, \omega])$. So as $M^*$ is a superlimit model, for any linear order $I$ of cardinality $\lambda$, $EM_{\tau_\lambda}(I, \Phi_\omega)$ is the direct limit of $EM_{\tau_\lambda}(I, \Phi_n) : n \leq I < \omega$ finite, each isomorphic to $M^*$, so as we have assumed that $M^*$ is a superlimit model it follows that $EM_{\tau_\lambda}(I, \Phi_\omega)$ is isomorphic.
to $M^*$. But by $[\text{Sh:300}, \text{III}]$ or $[\text{Sh:E59}]$ which may eventually be $[\text{Sh:E}]$ III] there are $2^\lambda$ many pairwise non-isomorphic models of this form varying $I$ on the linear orders of cardinality $\lambda$, contradiction.

2) First assume $\lambda = \lambda^\aleph_0$. Let $\tau \subseteq T_\tau$ be countable such that $T' = T \cap L(\tau)$ is not superstable. Clearly if $M^*$ is $(\lambda, \aleph_0)$-limit model then $M^* \upharpoonright \tau'$ is not $\aleph_1$-saturated.

[Why? As in $[\text{Sh:a}]$ Ch.VI, §6, but we shall give full details. There are $N_\aleph = \tau, p = \{ \varphi_n(\lambda, \bar{a}_n) : n < \omega \}$ a type in $N_\aleph, \bar{a}_n \triangleleft \bar{a}_{n+1}, \bar{a}_{<\omega}$ empty and $\varphi_{n+1}(x, \bar{a}_{n+1})$ forks over $\bar{a}_n$. Let $F(M)$ be such that if $n < \omega$ and $\bar{b}_n \subseteq M$ realizes $\text{tp}(\bar{a}_n, \emptyset, N_\aleph)$ then for some $\bar{b}_{n+1}$ from $F$, $M$ realizing $\text{tp}(\bar{a}_{n+1}, \emptyset, N_\aleph, b)$, the type $\text{tp}(\bar{b}_{n+1}, M, F(M))$ does not fork over $\bar{b}_n$.] But if $\kappa = \text{cf}(\kappa) \in [\aleph_1, \lambda]$ and $M^*$ is a $(\lambda, \kappa)$-limit then $M^* \upharpoonright \tau'$ is $\aleph_1$-saturated, contradiction.

The case $\lambda \geq |T| + \aleph_0$ is more complicated (the assumption $\lambda \geq \aleph_0$ is to enable us to use $[\text{Sh:460}]$ or see $[\text{Sh:829}]$ for a simpler proof; we can use weaker but less transparent assumptions; maybe $\lambda \geq 2^{\aleph_0}$ suffices).

As $T$ is stable not superstable by $[\text{Sh:a}]$ for some $\Delta$: 

$\oplus_1$ for any $\mu$ there are $M$ and $\langle \eta_{\alpha, \varphi} : \eta, \alpha \in \eta, \alpha \in \omega \rangle$ such that

(a) $M$ is a model of $T$

(b) $I_{\eta} = \{ \eta_{\alpha, \varphi} : \alpha < \mu \} \subseteq M$ is an indiscernible set (and $\alpha < \beta < \mu \Rightarrow \alpha_{\eta, \omega} \neq \alpha_{\eta, \beta}$)

(c) $\Delta = (\Delta_n : n < \omega)$ and $\Delta_n \subseteq L_{r(T)}$ infinite

(d) for $\eta, \nu \in \omega$ we have $\text{Av}_{\Delta_n}(M, I_\eta) = \text{Av}_{\Delta_n}(M, I_\nu)$ iff $\eta \upharpoonright n = \nu \upharpoonright n$.

Hence by $[\text{Sh:a}]$ VIII], or see $[\text{Sh:E59}]$ assuming $M^*$ is a universal model of $T$ of cardinality $\lambda$:

$\oplus_2.1$ there is $\Phi$ such that

(a) $\Phi$ is proper for $K_\tau^\omega$, $\tau \subseteq r(\Phi), |r(\Phi)| = \lambda \geq \aleph_1 + \aleph_0$

(b) for $I \subseteq 2^{\omega \geq \lambda}$, $\text{EM}_r(\Phi)(I, \Phi)$ is a model of $T$ and $I \subseteq J \Rightarrow \text{EM}(I, \Phi) \subseteq \text{EM}(J, \Phi)$

(c) for some two-place function symbol $F$ if for $I \subseteq K_\omega^\omega$ and $\eta \in P_\omega^I, I$ a subtrival $\omega \geq \lambda$ for transparency we let $I_{\eta, I} = \{ F(\eta, a) : a \in I \}$ then $\langle I_{\eta, I} : \eta \in P_\omega^I \rangle$ are as in $\oplus_1(b), (d)$.

Also $\oplus_2.2$ if $\Phi_1$ satisfies (a),(b),(c) of $\oplus_2.1$ and $M$ is a universal model of $T$ then there is $\Phi_2$ satisfying (a),(b),(c) of $\oplus_2.1$ and $\Phi_1 \leq \Phi_2$ see $\oplus_2.3(a)$ and for every finitely generated $J \subseteq K^\omega$, see $\oplus_2.3(b)$ below, there is $M' \cong M$ such that $\text{EM}_r(T)(J, \Phi_1) \prec M' \prec \text{EM}_r(T)(J, \Phi_2)$

$\oplus_2.3$ (a) we say $\Phi_1 \leq \Phi_2$ when $r(\Phi_1) \subseteq r(\Phi_2)$ and $J \subseteq K^\omega$ implies $\text{EM}(J, \Phi_1) \prec \text{EM}(J, \Phi_2)$

(b) we say $J \subseteq I$ is finitely generated if it has the form $\{ \eta_k : \ell < n \} \cup \{ \rho_k : \text{for some } n, \ell \text{ we have } \rho_k \in P_\eta^I \text{ and } \rho_k < \ell \}$ for some $\eta_0, \ldots, \eta_n-1 \in I$

$\oplus_2.4$ if $M_\ast \in \text{EC}_\lambda(T)$ is superlimit (or just weakly $S$-limit, $S \subseteq \lambda^+$ stationary) then there is $\Phi$ as in $\oplus_2.1$ above such that $\text{EM}_r(T)(J, \Phi) \cong M_\ast$ for every finitely generated $J \subseteq K^\omega$

$\oplus_2.5$ we fix $\Phi$ as in $\oplus_2.4$ for $M_\ast \in \text{EC}_\lambda(T)$ superlimit.
holds, the conclusion holds for every $\lambda$ then $E M_{r}(\Phi)(I, \Phi)$ is isomorphic to $M^*$. 

Now by [Sh:460], we can find regular uncountable $\kappa < \sum_\omega$ such that $\lambda = \lambda[\kappa]$, see Definition 3.3.

Let $S = \{ \delta < \kappa : cf(\delta) = \aleph_0 \}$ and $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ be such that $\eta_\delta$ is an increasing sequence of length $\omega$ with limit $\delta$.

For a model $M$ of $T$ let $O B_\eta(M) = \{ \vec{a} : \vec{a} = (a_{\eta_\delta, \alpha} : \delta \in W \text{ and } \alpha < \kappa), W \subseteq S \text{ and } (a_{\eta_\delta, \alpha} : \alpha < \kappa) \}. \text{ For } \vec{a} \in O B_\eta(M) \text{ let } W[\vec{a}] \text{ be } W \text{ as above and let }

$$\Xi(\vec{a}, M) = \{ \eta \in ^{\omega} \kappa : \text{ there is an indiscernible set } \ I = \{ a_\alpha : \alpha < \kappa \} \text{ in } M \text{ such that for every } n \text{ for some } \delta \in W[\vec{a}], \eta \mid n = \eta_\delta \mid n \text{ and } Av_{\Delta_n}(M, \ I) = Av_{\Delta_n}(M, \{ a_{\eta_\delta, \alpha} : \alpha < \kappa \}). \$$

Clearly

$\circledast 4$ (a) if $M \prec N$ then $O B_\eta(M) \subseteq O B_\eta(N)$

(b) if $M \prec N$ and $\vec{a} \in O B_\eta(M)$ then $\Xi(\vec{a}, M) \subseteq \Xi(\vec{a}, N)$.

Now by the choice of $\kappa$ it should be clear that

$\circledast 5$ if $M \models T$ is of cardinality $\lambda$ then we can find an elementary extension $N$ of $M$ of cardinality $\lambda$ such that for every $\vec{a} \in O B_\eta(M)$ with $W[\vec{a}]$ a stationary subset of $\kappa$, for some stationary $W' \subseteq W[\vec{a}]$ the set $\Xi[\vec{a}, N]$ includes $\{ \eta \in ^{\omega} \kappa : (\forall n)(\exists \delta \in W')(\eta \mid n = \eta_\delta \mid n) \}$, (moreover we can even find $\varepsilon^* < \kappa$ and $W_\varepsilon \subseteq W$ for $\varepsilon < \varepsilon^*$ satisfying $W[\vec{a}] = \cup \{ W_\varepsilon : \varepsilon < \varepsilon^* \}$).

$\circledast 6$ we can find $M \in E C_\lambda(T)$ isomorphic to $M^*$ such that for every $\vec{a} \in O B_\eta(M)$ with $W[\vec{a}]$ a stationary subset of $\kappa$, we can find a stationary subset $W'$ of $W[\vec{a}]$ such that the set $\Xi[\vec{a}, M]$ includes $\{ \eta \in ^{\omega} \mu : (\forall n)(\exists \delta \in W')(\eta \mid n = \eta_\delta \mid n) \}.$

[Why? We choose $(M_i, N_i)$ for $i < \kappa^+$ such that

(a) $M_i \in E C_\lambda(T)$ is $\prec$-increasing continuous

(a) $M_i$ is isomorphic to $M^*$

(a) $M_i \prec N_i < M_{i+1}$

(a) $(M_i, N_i)$ are like $(M, N)$ in $\circledast 5.$

Now $M = \cup \{ M_i : i < \kappa^+ \}$ is as required.

Now the model $M$ is isomorphic to $M^*$ as $M^*$ is superlimit.]

Now the model from $\circledast 6$ is not isomorphic to $M' = E M_{r}(T)^{(\omega^\omega, \lambda)} \cup \{ \eta_\delta : \delta \in S \}, \Phi$ where $\Phi$ is from $\circledast 2,1$. But $M' \cong M^*$ by $\circledast 3$.

Together we are done.

The following claim says in particular that if some not unreasonable pcf conjectures holds, the conclusion holds for every $\lambda \geq 2^{\aleph_0}$. 

\[\square\]
Claim 3.4. Assume $T$ is stable not superstable, $\lambda \geq |T|$ and $\lambda \geq \kappa = \text{cf}(\kappa) > \aleph_0$. 
1) $T$ has no $(\lambda, \kappa)$-superlimit model provided that $\kappa = \text{cf}(\kappa) > \aleph_0, \lambda \geq \kappa^{\aleph_0}$ and $\lambda = U_D(\lambda) := \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\kappa}\}$ and for every $f : \kappa \rightarrow \lambda$ for some $u \in \mathcal{P}$ we have $\{\alpha < \kappa : f(\alpha) \in u\} \in D^+$, where $D$ is a normal filter on $\kappa$ to which $\{\delta < \kappa : \text{cf}(\delta) = \aleph_0\}$ belongs.
2) Similarly if $\lambda \geq 2^{\aleph_0}$ and letting $J_0 = \{u \subseteq \kappa : |u| \leq \aleph_0\}, J_1 = \{u \subseteq \kappa : u \cap S^{ \aleph_0}_\kappa \text{ non-stationary}\}$ we have $\lambda = U_{J_1, J_0}(\lambda) := \min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\aleph_0}\}$, if $u \in J_1, f : (\kappa \setminus u) \rightarrow \lambda$ then for some countable infinite $w \subseteq \kappa(u)$ and $v \in \mathcal{P}$, $\text{Rang}(f[w]) \subseteq v$.

Proof. Like 3.1(2).

Claim 3.5. 1) Assume $T$ is unstable and $\lambda \geq |T| + \beth_\omega$. Then for at most one regular $\kappa \leq \lambda$ does $T$ have a weakly $(\lambda, \kappa)$-limit model and even a weakly $(\lambda, S)$-limit model for some stationary $S \subseteq S^\kappa_\lambda$.
2) Assume $T$ is unsuperstable and $\lambda \geq |T| + \beth_\omega(\kappa_2)$ and $\kappa_1 = \aleph_0 < \kappa_2 = \text{cf}(\kappa_2)$. Then $T$ has no model which is a weak $(\lambda, S)$-limit where $S \subseteq \lambda$ and $S \cap S^\kappa_\lambda$ is stationary for $\ell = 1, 2$.

Proof. 1) Assume $\kappa_1 \neq \kappa_2$ form a counterexample. Let $\kappa < \beth_\omega$ be regular large enough such that $\lambda = \lambda^{\kappa_1}$, see Definition 3.3 and $\kappa \notin \{\kappa_1, \kappa_2\}$. Let $m, \varphi(\bar{x}, \bar{y})$ be as in the proof of 3.1

\text{(a)} if $M \in \text{EC}_\lambda(T)$ then there is $N$ such that
\text{(b)} $M \prec N$
\text{(c)} if $\bar{a} = (\bar{a}_i : i < \kappa) \in \kappa^m(M)$ for $\alpha < \kappa$ then for some $\mathcal{U} \in [\kappa]^\kappa$ for every uniform ultrafilter $D$ on $\kappa$ to which $\mathcal{U}$ belongs there is $\bar{a}_D \in [\kappa]^\kappa$ such that $\text{tp}(\bar{a}_D, N, N) = \text{Av}(\bar{a}_D/M) = \{\psi(\bar{x}, \bar{c}) : \psi(\bar{x}, \bar{c}) \in \mathcal{L}(\kappa_T), \bar{c} \in \text{tg}(\bar{z})M \text{ and } \{\alpha < \kappa : N \models \psi[\bar{a}_i, \bar{c}]\} \in D\}$.

Similarly

\text{(1) for every function $F$ with domain $\{M : M \text{ an } \varphi\text{-increasing sequence of models of } T \text{ of length } < \lambda^+ \text{ each with universe in } \lambda^+\}$ such that $M_i \prec F(M)$ for $i < \text{tg}(\bar{M})$ and $F(\bar{M})$ has universe in $\lambda^+$ there is a sequence $\langle M_\varepsilon : \varepsilon < \lambda^+ \rangle$ obeying $F$ such that: for every $\varepsilon < \lambda^+$ and $\bar{a} \in \kappa^m(M_\varepsilon)$ for $\alpha < \kappa$, there is $\mathcal{U} \in [\kappa]^\kappa$ such that for every ultrafilter $D$ on $\kappa$ to which $\mathcal{U}$ belongs, for every $\zeta \in (\varepsilon, \lambda^+)$ there is $\bar{a}_{D, \zeta} \in \kappa^m(M_{\zeta+1})$ realizing $\text{Av}(\bar{a}/D, M_{\zeta})$ in $M_{\zeta+1}$.

Hence

\text{(2) for } (M_\alpha : \alpha < \lambda^+) \text{ as in } \text{(1)} \text{ for every limit } \delta < \lambda^+ \text{ of cofinality } \neq \kappa \text{ for every } \bar{a} = (\bar{a}_i : i < \kappa) \in \kappa^m(M_\delta), \text{ there is } \mathcal{U} \in [\kappa]^\kappa \text{ such that for every ultrafilter } D \text{ on } \kappa \text{ to which } \mathcal{U} \text{ belongs, there is a sequence } (\bar{b}_\varepsilon : \varepsilon < \text{cf}(\delta)) \in \text{cf}(\delta)^m(M_\delta) \text{ such that for every } \psi(\bar{x}, \bar{z}) \in \mathcal{L}(\kappa_T) \text{ and } \bar{c} \in \text{tg}(\bar{z})(M_\delta) \text{ for every } \varepsilon < \text{cf}(\delta) \text{ large enough, } M_\delta \models \psi[\bar{b}_\varepsilon, \bar{c}] \iff \psi(\bar{x}, \bar{c}) \in \text{Av}(\bar{a}/D, M_\delta)$. The rest should be clear.

2) Combine the above and the proof of 3.1(2).
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