SEMI-MODULES AND IRREDUCIBLE COMPONENTS OF
AFFINE DELIGNE-LUSZTIG VARIETIES

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Abstract. Let $G$ be the Weil restriction of a general linear group. By extending the method of semi-modules developed by de Jong, Oort, Viehmann and Hamacher, we obtain a stratification of the affine Deligne-Lusztig varieties for $G$ (in the affine Grassmannian) attached to a minuscule coweight and a basic element. As an application, we verify a conjecture by Chen and Zhu on irreducible components of affine Deligne-Lusztig varieties for $G$.

Introduction

Affine Deligne-Lusztig varieties are closely related to the Rapoport-Zink moduli spaces of $p$-divisible groups (cf. [15], [16]), and play an important role in the study of Shimura varieties. There has been an extensive study on affine Deligne-Lusztig varieties. However, many basic aspects of their geometric structure are not fully understood yet. We refer to [9] and [6] for the current status of these topics.

In this note, we study the affine Deligne-Lusztig varieties $X^G_\mu(\gamma)$ in affine Grassmannians, where $G$ is the Weil restriction of a general linear group. By extending the method of semi-modules (or extended EL-charts) developed by de Jong-Oort [2], Viehmann [17] and Hamacher [5], we show that if $\gamma$ is basic and $\mu$ is minuscule, there is a stratification (in the loose sense) of $X^G_\mu(\gamma)$ parameterized by semi-modules. As an application, we verify a conjecture by Miaofen Chen and Xinwen Zhu concerned with the irreducible connected components of $X^G_\mu(\gamma)$.

To describe the results more precisely, we introduce some notation. Let $\mathbb{F}_q$ be a finite field with $q$ elements, and let $\mathbb{k}$ be an algebraic closure of $\mathbb{F}_q$. Denote by $F = \mathbb{F}_q((t))$ and $L = \mathbb{k}((t))$ the fields of Laurent series, whose integer rings are denoted by $\mathcal{O}_F = \mathbb{F}_q[[t]]$ and $\mathcal{O} = \mathbb{k}[[t]]$ respectively. Let $\sigma$ denote the Frobenius automorphism of $L/F$.

Let $G$ be a connected reductive group over $\mathbb{F}_q$. Fix $S \subseteq T \subseteq B \subseteq G$, where $S$ is a maximal split torus, $T$ a maximal torus and $B$ a Borel subgroup of $G$. Denote by $X_*(T)$ for the cocharacter group of $T$, and by $X_*(T)_G^{\text{dom}}$ the set of dominant cocharacters determined by $B$. Let $\leq$ denote the dominance partial order on $X_*(T)$ defined by $B$. Let $\leq$ denote the partial order on $X_*(T)_Q$ such that $v \leq v'$ if and only if $v' - v$ is a non-negative linear combination of positive coroots.

We have the Cartan decomposition $G(L) = \sqcup_{\lambda \in X_*(T)_G^{\text{dom}}} Kt^\lambda K$, where $K = G(\mathcal{O})$. For $\gamma \in G(L)$ and $\mu \in X_*(T)$, the attached affine Deligne-Lusztig variety

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is defined by

\[ X^G_\mu(\gamma) = \{ gK \in G(L)/K; g^{-1}\gamma\sigma(g) \in Kt^\mu K \} . \]

It carries a natural action by the group \( J^G_\gamma \), \( \{ g \in G(L); g^{-1}\gamma\sigma(g) = \gamma \} \). By definition, \( X^G_\mu(\gamma) \) only depends on the \( \sigma \)-conjugacy class \( [\gamma]_G \) of \( \gamma \). Thanks to Kottwitz [10], \([\gamma]_G \) is uniquely determined by two invariants: the Newton point \( \nu_G(\gamma) \in X_*(S)_G \) and the Kottwitz point \( \kappa_G(\gamma) \in \pi_1(G)_\sigma \); see [6, §2.1]. By [12] and [3], \( X^G_\mu(\gamma) \neq \emptyset \) if and only if \( \kappa_G(t^\mu) = \kappa_G(\gamma) \) and \( \nu_G(\gamma) \leq \mu^\circ \), where \( \mu^\circ \) denotes the \( \sigma \)-average of \( \mu \). Moreover, thanks to [4], [17], [5], [22] and [7], \( X^G_\mu(\gamma) \) is locally of finite type, equi-dimensional and

\[ \dim X^G_\mu(\gamma) = \langle \rho_G, \mu - \nu_G(\gamma) \rangle - \frac{1}{2}\text{def}_G(\gamma), \]

where \( \rho_G \) is the half-sum of positive roots of \( G \) and \( \text{def}_G(\gamma) \) denotes the defect of \( \gamma \); see [11, §1.9.1].

The stratification by semi-modules was first considered by de Jong and Oort [2] for \( X^G_\mu(\gamma) \) with \( G \) split, \( \mu \) minuscule and \( \gamma \) superbasic. It was latter extended by Viehmann [17] and Hamacher [5] to the case where \( \gamma \) is superbasic. Recently, Chen and Viehmann [1] defined a stratification for all affine Deligne-Lusztig varieties, which recovers the stratification by semi-modules in the superbasic case.

**Theorem 0.1.** (=Corollary 2.6) Suppose \( G = \text{Res}_{F'/F} \text{GL}_n \). If \( \mu \) is minuscule and \( \gamma \) is basic (i.e. \( \nu_G(\gamma) \) is central for \( G \)), then there is a decomposition

\[ X^G_\mu(\gamma) = \sqcup A \mathcal{L}(A), \]

where \( A \) ranges over all semi-modules of Hodge type \( \mu \) (see §2.1) and \( \mathcal{L}(A) \subseteq X^G_\mu(\gamma) \) is a locally closed subset, which is finite and smooth over an affine space.

Now we turn to the study of irreducible components of \( X^G_\mu(\gamma) \). In [6, §2.1], Hamacher and Viehmann defined the “best integral approximation” \( \lambda_G(\gamma) \) of the Newton point \( \nu_G(\gamma) \), which is the unique maximal element in the set

\[ \{ \lambda \in X_*(T)_\sigma; \kappa_G(t^\lambda) = \kappa_G(\gamma), \lambda^\circ \leq \nu_G(\gamma) \} . \]

Moreover, if \( M \supseteq T \) is a standard Levi subgroup such that \( \gamma \in M(L) \) and \( \nu_M(\gamma) = \nu_G(\gamma) \), then \( \lambda_M(\gamma) = \lambda_G(\gamma) \).

Let \( S \subseteq \hat{T} \subseteq \hat{B} \subseteq \hat{G} \) be the dual of \( S \subseteq T \subseteq B \subseteq G \) in the sense of Deligne and Lusztig. We have canonical identifications \( X_*(T) = X^*(\hat{T}) \), \( X_*(T)_G \) -dom = \( X^*(\hat{T})_G \) -dom and so on. For \( \mu \in X_*(T)_G \) -dom let \( V^G_\mu \) denote the irreducible \( \hat{G} \) -module with highest weight \( \mu \). Moreover, for \( \lambda \in X^*(\hat{S}) \), the \( \lambda \) -weight space of \( V^G_\mu \) is denoted by \( V^G_\mu(\lambda) \).

**Conjecture 0.1** (Chen, Zhu). Let notations be as above. Then there exists a natural bijection between \( \mathbb{F}_\gamma \backslash \text{Irr} X^G_\mu(\gamma) \) and a basis of \( V^G_\mu(\lambda_G(\gamma)) \) related to the Mirkovic-Vilonen cycles (see [13]). In particular,

\[ |\mathbb{F}_\gamma \backslash \text{Irr} X^G_\mu(\gamma)| = \dim V^G_\mu(\lambda_G(\gamma)) . \]

Here \( \text{Irr} X^G_\mu(\gamma) \) denotes the set of irreducible components of \( X^G_\mu(\gamma) \).
If $\mu$ is minuscule and $\gamma$ is superbasic, the conjecture is proved by Hamacher and Viehmann [6]. If $\gamma$ is unramified, that is, $\text{def}_G(\gamma) = 0$, it is proved by Xiao and Zhu [21]. In both cases, the authors obtained a complete description of $\text{Irr}_G^X(\mu)$.  

Remark. We mention that a complete description of $\text{Irr}_G^X(\gamma)$ was also known for the case where $G$ equals $\text{GL}_n$ or $\text{GSp}_{2n}$ and $\mu$ is minuscule (cf. [18], [19]), and for the case where $(G, \mu)$ is fully Hodge-Newton decomposable; see [20] for a typical example in this case.

**Theorem 0.2.** (=Corollary 3.5) Conjecture 0.1 holds for $G = \text{Res}_{F_q}^{\mathbb{F}_q} \text{GL}_n$.

To prove the theorem, we first use geometric Satake to reduce the problem to the case where $\mu$ is minuscule and $\gamma$ is basic. Then we can use Theorem 1.1 to show that the $\mathbb{F}_q$-orbits of $\text{Irr}_G^X(\mu)$ are parameterized by equivalence classes of rigid semi-modules $A$ (see §2.4) such that $\dim L(A) = \dim X_{\mu}^G(\gamma)$. Finally, we show the number of such rigid semi-modules coincides with the dimension of $V_{X,\mu}^G(\lambda_G(\gamma))$. This is accomplished by a reduction to the superbasic case, which has been solved by Hamacher and Viehmann [6, Theorem 1.5].

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1. Reduction to the minuscule case

In this section, we reduce Theorem 0.2 to the minuscule case. Let $G$ be a split reductive group over $\mathbb{F}_q$. Let $d \in \mathbb{Z}_{\geq 1}$ and let $H = H_d$ be a reductive group over $k$ such that

$$H = \prod_{\tau \in \mathbb{Z}_d} H_\tau,$$

where $\mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z}$ for some $d \in \mathbb{Z}_{\geq 1}$ and $H_\tau = G \otimes_{\mathbb{F}_q} k$ for $\tau \in \mathbb{Z}_d$. Let $\sigma^H$ be an automorphism of $H(L)$ defined by $(g_1, g_2, \ldots, g_d) \mapsto (g_2, \ldots, g_r, \sigma(g_1))$, where $\sigma$ denotes the Frobenius automorphism $\sigma$ on $G(L)$.

Let $T$ be a split maximal torus of $G$. Then $T_H := \prod_\tau T$ is a maximal torus of $H$. Via diagonal embedding, we can identify $T$ with the maximal $\sigma^H$-split torus $S_H$ in $T_H$. The embedding $\lambda \mapsto \lambda^H = (0, \ldots, 0, \lambda)$ identifies $X_*(T_\sigma) = X_*(T)$ with $X_*(T_H)_{\sigma^H} = X^*(S_H)$. Similarly, the embedding $\gamma \mapsto \gamma^H = (1, \ldots, 1, \gamma)$ gives a bijection between $\sigma$-conjugacy classes of $G(L)$ and $\sigma^H$-conjugacy classes of $H(L)$. By abuse of notation, we will identify $\gamma$ (resp. $\lambda$) with $\gamma^H$ (resp. $\lambda^H$).

**Lemma 1.1.** Let $\gamma \in G(L)$. Then $\lambda_H(\gamma) = \lambda_G(\gamma) \in X_*(T) = X^*(S_H)$.

Let $\mu_\bullet = (\mu_1, \ldots, \mu_d) \in X_*(T_H)_{\text{dom}} = \prod_\tau X_*(T)_{G_{\text{dom}}}$. We consider the twisted product

$$Z_{\mu_\bullet} = Kt^{\mu_1}K \times_K \cdots \times_K Kt^{\mu_d}K/K$$

together with the convolution map

$$m_{\mu_\bullet} : Z_{\mu_\bullet} \to Kt^{w_{\bullet}K}/K = \cup_{\lambda \leq |\mu_\bullet|} Kt^{\lambda K}/K$$

given by $(g_1, \ldots, g_{d-1}, g_dK) \mapsto g_1 \cdots g_d K$, where $|\mu_\bullet| = \mu_1 + \cdots + \mu_d$. 

Consider the following decomposition of $\hat{G}$-modules
\[ V_{\mu_\bullet}^H = V_{\mu_1}^\hat{G} \otimes \cdots \otimes V_{\mu_d}^\hat{G} = \bigoplus_{\chi \in X_*(T)_{G-dom}} (V_{\chi}^\hat{G})^{\otimes a_{\chi}^\mu_\bullet}, \]
where $a_{\chi}^\mu_\bullet$ is the multiplicity of $V_{\chi}^\hat{G}$ in $V_{\mu_\bullet}^H$ (as $\hat{G}$-modules).

**Theorem 1.2** ([13], [14], [8]). Suppose $\mu_\bullet$ is a sum of dominant minuscule cocharacters. For each $y \in K t K / K$ with $\lambda \leq |\mu_\bullet|$ dominant, the fiber $m_{\mu_\bullet}^{-1}(y)$ is equidimensional of dimension $\langle \rho_G, |\mu_\bullet| - \lambda \rangle$ if $m_{\mu_\bullet}^{-1}(y) \neq \emptyset$. Moreover, the number of irreducible components of $m_{\mu_\bullet}^{-1}(y)$ equals $a_{\chi}^\mu_\bullet$.

For $\gamma \in G(L)$ and $\mu_\bullet \in X_*(T_H)_{H-dom}$, we define
\[ X_{\mu_\bullet}^H(\gamma) = \{ hK_H \in H(L)/K_H; h^{-1} \gamma \sigma^H(h) \in K_H t^{\mu_\bullet} K_H \}, \]
where $K_H = \prod \gamma K$ and $K = G(\emptyset)$. Thanks to Zhu [22, §3.13], there is a Cartesian square
\[
\begin{array}{ccc}
X_{\mu_\bullet}^H(\gamma) & \longrightarrow & G(L) \times_K Z_{\mu_\bullet} \\
\alpha \downarrow & & \downarrow \text{id} \times_K \text{dom}_{\mu_\bullet} \\
\cup_{\lambda \leq |\mu_\bullet|} X_{\mu_\bullet}^G(\gamma) & \longrightarrow & G(L) \times_K K t^{\mu_\bullet} K / K,
\end{array}
\]
where the bottom horizontal map is given by $gK \mapsto (g, g^{-1} \gamma \sigma(g) K)$, and the top horizontal map is given by $(g_1 K, \ldots, g_d K) \mapsto (g_1, g_1^{-1} g_2, \ldots, g_{d-1}^{-1} g_d, g_d^{-1} \gamma \sigma(g_1) K)$. Moreover, via the identification
\[ \mathbb{J}_\gamma^G \equiv \mathbb{J}_\gamma^H := \{ h \in H(L); h^{-1} \gamma \sigma^H(h) = \gamma \}, \quad g \mapsto (g, \ldots, g), \]
the above Cartesian square is $\mathbb{J}_\gamma^G$-equivariant (by left multiplication).

**Lemma 1.3.** Let notations be as above. Suppose $\mu_\bullet$ is sum of minuscule dominant coweights. Then
\[ \text{Irr} X_{\mu_\bullet}^H(\gamma) = \cup_{\lambda \leq |\mu_\bullet|} a_{\lambda}^\mu_\bullet \neq 0 \cup_{c \in \text{Irr} X_{\mu_\bullet}^G(\gamma)} \text{Irr}(\alpha^{-1}(c)). \]
In particular,
\[ \mathbb{J}_\gamma^H \setminus \text{Irr} X_{\mu_\bullet}^H(\gamma) = \cup_{\lambda \leq |\mu_\bullet|} a_{\lambda}^\mu_\bullet \neq 0 \cup_{c \in \mathbb{J}_\gamma^G \setminus \text{Irr} X_{\mu_\bullet}^G(\gamma)} \text{Irr}(\alpha^{-1}(c)) \]
and hence
\[ |\mathbb{J}_\gamma^H \setminus \text{Irr} X_{\mu_\bullet}^H(\gamma)| = \sum_{\lambda \leq |\mu_\bullet|} a_{\lambda}^\mu_\bullet |\mathbb{J}_\gamma^G \setminus \text{Irr} X_{\mu_\bullet}^G(\gamma)|. \]
Here $\lambda$ always denotes a dominant cocharacter.

**Proof.** Let $c \in \text{Irr} X_{\mu_\bullet}^G(\gamma)$. If $\alpha^{-1}(X_{\mu_\bullet}^G(\gamma)) \neq 0$, that is, $m_{\mu_\bullet}^{-1}(t^\lambda K) \neq \emptyset$, or equivalently, $a_{\lambda}^\mu_\bullet \neq 0$, then Theorem 1.2 tells that $\alpha^{-1}(c)$ is equidimensional and
\[ \dim \alpha^{-1}(c) = \dim c + \langle \rho_G, |\mu_\bullet| - \lambda \rangle = \dim X_{\lambda}^G(\gamma) + \langle \rho_G, |\mu_\bullet| - \lambda \rangle = \dim X_{\mu_\bullet}^H(\gamma). \]
The proof is finished. \qed

**Proposition 1.4.** Suppose $G = \text{GL}_n$. If $\gamma$ is basic and $\mu_\bullet \in X_*(T_H)_{H-dom}$ is minuscule. Then $|\mathbb{J}_\gamma^H \setminus \text{Irr} X_{\mu_\bullet}^H(\gamma)| = \dim V_{\mu_\bullet}^H(\lambda_H(\gamma))$.

The proof is given in §3; see Corollary 3.5.
Proposition 1.5. Suppose $G = \text{GL}_n$ and $\mu_\bullet \in X_*(T_H)_{H-\text{dom}}$ is minuscule. Then

$$|J^H_\gamma \setminus \text{Irr} X^H_{\mu_*}(\gamma)| = \dim V^H_{\mu_*} (\lambda_H(\gamma)).$$

Proof. Let $T_H \subseteq M_\gamma \subseteq P_\gamma \subseteq H$ be the standard Levi and parabolic subgroups associated to $\nu_H(\gamma)$. Let $X_*(T_H)_{M_\gamma-\text{dom}}$ denote the set of $M_\gamma$-dominant cocharacters $\chi_\bullet$ such that $\chi_\bullet \leq \mu_\bullet$. Let $I_{\mu_* \gamma} = I_{\mu_* \gamma, M_\gamma}$ be the set of cocharacters $\chi_\bullet \in X_*(T_H)_{M_\gamma-\text{dom}}$ such that $\kappa_{M_\gamma}(\chi_\bullet) = \kappa_{M_\gamma}(\gamma)$. By definition, $J^H_\gamma \subseteq M_\gamma(L)$. Then [6, Corollary 5.9] tells that there is a natural bijection

$$J^H_\gamma \setminus \text{Irr} X^H_{\mu_*}(\gamma) \sim \bigcup_{\chi_\bullet \in I_{\mu_* \gamma}} J^H_\gamma \setminus \text{Irr} X^M_{\chi_\bullet}(\gamma),$$

where $I_{\mu_* \gamma}$ is defined analogously as in [6, §5]. Therefore,

$$|J^H_\gamma \setminus \text{Irr} X^H_{\mu_*}(\gamma)| = \sum_{\chi_\bullet \in I_{\mu_* \gamma}} |J^H_\gamma \setminus \text{Irr} X^M_{\chi_\bullet}(\gamma)|$$

$$= \sum_{\chi_\bullet \in I_{\mu_* \gamma}} \dim V^M_{\chi_\bullet} (\lambda_{M_\gamma}(\gamma))$$

$$= \sum_{\chi_\bullet \in X_*(T_H)_{M_\gamma-\text{dom}}} \dim V^M_{\chi_\bullet} (\lambda_{M_\gamma}(\gamma))$$

$$= \dim V^H_{\mu_*} (\lambda_{M_\gamma}(\gamma))$$

$$= \dim V^H_{\mu_*} (\lambda_H(\gamma)),$$

where the second equality follows from Proposition 1.4 as $\gamma$ is basic in $M_\gamma(L)$; the third one follows from that $V^M_{\chi_\bullet} (\lambda_{M_\gamma}(\gamma)) = 0$ (for $\chi_\bullet \in X_*(T_H)_{M_\gamma-\text{dom}}$) unless $\chi_\bullet \in I_{\mu_* \gamma}$; the fourth one follows from that $V^H_{\mu_*} = \bigoplus_{\chi_\bullet \in X_*(T_H)_{M_\gamma-\text{dom}}} V^M_{\chi_\bullet}$ since $\mu_\bullet$ is minuscule.

Corollary 1.6. Suppose $G = \text{GL}_n$. Then $|J^G_\gamma \setminus X^G_{\mu}(\gamma)| = \dim V^G_{\mu} (\lambda_G(\gamma))$. 

Proof. If $\mu$ is minuscule, it is proved in [6]. Assume it is true for all dominant cocharacters $\mu'$ such that $\mu' < \mu$. We show it is also true for $\mu$. Since $G = \text{GL}_n$, there exist $d' \in \mathbb{Z}_{>1}$ and a $d'$-tuple $\mu_\bullet$ of dominant minuscule cocharacters (of $G$) such that $\mu = |\mu_\bullet|$. Set $H = H_{d'}$. Thus

$$\dim V^H_{\mu_*} (\lambda_G(\gamma)) = |J^H_\gamma \setminus \text{Irr} X^H_{\mu_*}(\gamma)|$$

$$= \sum_{\chi \leq \mu} a_{\mu_\bullet}^\chi \cdot |J^G_\gamma \setminus \text{Irr} X^G_{\chi}(\gamma)|$$

$$= |J^G_\gamma \setminus \text{Irr} X^G_{\mu}(\gamma)| + \sum_{\chi \leq \mu} a_{\mu_\bullet}^\chi \cdot |J^G_\gamma \setminus \text{Irr} X^G_{\chi}(\gamma)|$$

$$= |J^G_\gamma \setminus \text{Irr} X^G_{\mu}(\gamma)| + \sum_{\chi \leq \mu} a_{\mu_\bullet}^\chi \cdot \dim V^G_{\chi} (\lambda_G(\gamma)),$$

where the first equality follows from Proposition 1.5; the second one follows from Lemma 1.3; the last one follows from induction hypothesis. On the other hand, we have $\dim V^H_{\mu_*} (\lambda_H(\gamma)) = \sum_{\chi \leq \mu} a_{\mu_\bullet}^\chi \cdot \dim V^G_{\chi} (\lambda_G(\gamma))$ by Lemma 1.1. Therefore, $|J^G_\gamma \setminus X^G_{\mu}(\gamma)| = \dim V^G_{\mu}(\lambda_G(\gamma))$ as desired.

$\square$
Combining Lemma 1.3 with Corollary 1.6, we deduce that

**Corollary 1.7.** Suppose $G = GL_n$. Then $|\mathbb{P}^H \backslash X_\mu^H(\gamma)| = V_{\mu^\gamma}(\lambda_H(\gamma))$.

2. **Decomposition by semi-modules**

Let notations be as in §1, except that we use $\chi$, instead of $\chi_\bullet$, to denote a cocharacter in $X_*(T_H)$. Moreover, without loss of generality, we can assume $H = Res_{\mathbb{G}_a^d/F_q} GL_n$ for simplicity. We will use the method of semi-modules to give a decomposition for $X_\mu^H(\gamma)$, where $\mu$ is minuscule and $\gamma$ is basic.

For each $\tau \in \mathbb{Z}_d$ let $N_\tau = \bigoplus_{i \leq n} L e_{\tau,i}$. Then $H_\tau(L) \cong GL_n(N_\tau)$ and

$$H(L) = \prod_{\tau \in \mathbb{Z}_d} H_\tau(L) \cong \prod_{\tau \in \mathbb{Z}_d} GL_n(N_\tau).$$

Via this identification, we can specify the notations in §1 as follows.

- $\sigma = \sigma^H : H(L) \to H(L)$ is induced by $e_{\tau,i} \mapsto e_{\tau-1,i}$.

- $\gamma \in H(L)$ is given by $e_{\tau,i} \mapsto e_{\tau,i+m}$ for some $m \in \mathbb{Z}_{\geq 0}$ if $\tau = 0 \in \mathbb{Z}_d$ and $e_{\tau,i} \mapsto e_{\tau,i}$ otherwise. Here we adopt the convention that $e_{\tau,i+n} = te_{\tau,i}$.

- $S_H \subseteq T_H \subseteq B_H$ denote the split diagonal torus, the diagonal torus and the upper triangular Borel subgroup, respectively.

- $X_*(T_H) = (\mathbb{Z}^n)^{\mathbb{Z}_d}$ and $\chi = (\chi_\tau)_{\tau \in \mathbb{Z}_d} \in X_*(T_H)$ is dominant if and only if each $\chi_\tau$ is dominant, that is, $\chi_\tau(1) \geq \cdots \geq \chi_\tau(n)$.

- $\mu = (\mu_\tau)_{\tau \in \mathbb{Z}_d}$ is the fixed dominant minuscule cocharacter. Let $m_\tau = \sum_{i=1}^n \mu_\tau(i)$ for $\tau \in \mathbb{Z}_d$. As $X_\mu^H(\gamma) \neq \emptyset$, we have $m = \sum_{\tau} m_\tau$.

- $K_H = \prod_{\tau} GL_n(\Delta_\tau)$, where $\Delta_\tau = \bigoplus_{i=1}^n o e_{\tau,i}$ is the standard $o$-lattice in $N_\tau$.

The map $g_\tau \mapsto g_\tau \Delta_\tau$ induces a bijection

$$H(L)/K_H \cong \{\Lambda = (\Lambda_\tau)_{\tau \in \mathbb{Z}_d} ; \Lambda_\tau \text{ is a lattice in } N_\tau\}.$$

Let $\Lambda, \Lambda' \in H(L)/K_H$. Denote by $\text{inv}(\Lambda, \Lambda')$ the unique dominant cocharacter $\chi$ such that $g(\Lambda, \Lambda') = (\Delta, t^x \Delta)$ for some element $g \in G(L)$. Now the affine Deligne-Lusztig variety $X_\mu^H(\gamma)$ is given by

$$X_\mu(\gamma) = X_\mu^H(\gamma) = \{\Lambda \in H(L)/K_H ; \text{inv}(\Lambda, \gamma \sigma(\Lambda)) = \mu\}.$$

2.1. Let $O = \mathbb{Z}_d \times \mathbb{Z}$. Let $(\tau', i'), (\tau, i) \in O$. We write $(\tau, i) \leq (\tau', i')$ if $\tau = \tau'$ and $i \leq i'$. For $k \in \mathbb{Z}$ we set $k + (\tau, i) = (\tau, i) + k = (\tau, i+k) \in O$. For $a = (\tau, i) \in O$ we set $e_a = e_{\tau,i} \in N_\tau$. For $v \in N_\tau$ we define $h(v) = \max_{\tau} \{a ; v \in \sum_{j=0}^\infty ke_{a+j}\}$.

Let $h \in \mathbb{Z}_{\geq 1}$ be the greatest common divisor of $m$ and $n$. Set $n' = n/h$ and $m' = m/h$. Let $\tau \in \mathbb{Z}_d$ and $k \in \mathbb{Z}$. We set $O^k = \{(\tau, j) \in O ; j = k \mod h\}$ and $O_\tau = \{(\tau, j) \in O ; j \in \mathbb{Z}\}$. For any subset $E \subseteq O$, we set $E^k = E \cap O^k$, $E_\tau = E \cap O_\tau$ and $E_{\tau,k} = E \cap O_\tau \cap O^k$. Define $f : O \to O$ by $(\tau, i) \mapsto (\tau - 1, i + m)$ if $\tau = 1$ and $(\tau, i) \mapsto (\tau - 1, i)$ otherwise. Notice that $\gamma \sigma(e_a) = e_{f(\tau)}$ for $a \in O$.

We say a subset $A \subseteq O$ is a **semi-module** (for $H$) if $A$ is bounded below, $n + A, f(A) \subseteq A$ and $O = n\mathbb{Z} + A$. Set $\bar{A} = A \setminus (n + A)$. For $a \in A$, let
By induction on the partial order \( (3) \) hold. Moreover, the coefficients there exists a unique collection of vectors \((\varphi_A(b))_{b \in A, \tau} \) for \( \tau \in \mathbb{Z}_d \).

Let \( \Lambda = (\Lambda_{\tau})_{\tau \in \mathbb{Z}_d} \in H(L)/K_H \). We set
\[
A(\Lambda) = \{ h(v); 0 \neq v \in \Lambda_{\tau} \text{ for some } \tau \in \mathbb{Z}_d \} \subseteq O.
\]

For \( a \in A(\Lambda) \) define \( \varphi_A(a) \) to be the maximal integer \( l \) such that \( t^{-l} \gamma \sigma(v) \in \Lambda \) for some \( v \in \Lambda \) with \( h(v) = a \). We set \( \bar{A}(\Lambda) = \bar{A}(\Lambda), \bar{A}_c(\Lambda) = \bar{A}(\Lambda)_{c} \) and so on.

**Lemma 2.1.** [5, Corollary 5.10] We have \( \Lambda \in X(\gamma) \) if and only if \( A(\Lambda) \) is a semi-module of Hodge type \( \mu \) and \( \varphi_A = \varphi_{A(\Lambda)} \).

2.2. Let \( A \) be a semi-module of Hodge type \( \mu \). Let \( \iota \in \mathbb{Z}_d \). Set \( Y_\iota = \{ \max \leq \bar{A}_c; 1 \leq k \leq h \} \). Let
\[
V(A) = \{(b, j) \in \bar{A} \times \mathbb{Z}_{\geq 1}; b + j \in \bar{A}, \varphi_A(b) > \varphi_A(b + j)\};
\]
\[
W(A, \iota) = \{(b, j) \in Y_\iota \times \mathbb{Z}_{\geq 1}; b + j \notin A\};
\]
\[
D(A, \iota) = V(A) \cup W(A, \iota).
\]

Notice that \( V(A) \cap W(A, \iota) = \emptyset \). Let \( \leq_{\iota} \) be a partial order on \( \bar{A} \) such that \( r^{-1}(b) \leq_{\iota} b \) for \( b \in \bar{A} \setminus Y_\iota \). This induces a partial order on \( \bar{A} \times \mathbb{Z}_{\geq 0} \), which is still denoted by \( \leq_{\iota} \), such that \( (b, j) \leq_{\iota} (b', j') \) if either \( j < j' \) or \( j = j' \) and \( b \leq_{\iota} b' \).

For \( x = (x_{b, j})_{(b, j) \in D(A, \iota)} \in \mathbb{K}^{D(A, \iota)} \) we consider the following equations:
1. \( v(a) \leq \sum_{j=0}^{n} \alpha_{a, j} x_{a+j} \) with coefficients \( \alpha_{a, 0} = 1 \);
2. \( v(b) = c_b + \sum_{(b, j) \in W(A, \iota)} x_{b, j} v(b + j) + \sum_{(b, j) \in V(A)} x_{b, j} v(b + j) \) for \( b \in Y_\iota \);
3. \( v(a) = t v(a - n) \) if \( a \in n + A \);

Thanks to Lemma 2.2 below, such vectors \( v(a) \) for \( a \in A \) always exist and are unique. We set \( \Lambda(x) = (\Lambda_{\tau}(x))_{\tau \in \mathbb{Z}_d} \) where \( \Lambda_{\tau}(x) \) is the \( \Omega \)-lattice spanned by \( v(b) \) for \( b \in A_{\tau} \). We say \( (v(a))_{a \in A} \) is the normalized basis for \( \Lambda(x) \) or \( x \in \mathbb{K}^{D(A, \iota)} \). Moreover, we denote by \( L(A, \iota) \) the set of points \( x \in \mathbb{K}^{D(A, \iota)} \) such that
\[
(4) t^{-\varphi_A(b)} \gamma \sigma(v(b)) \in \Lambda(x) \text{ for } b \in \bar{A}.
\]
Thanks to (2), the condition (4) is equivalent to
\[
(4') t^{-\varphi_A(r^{-1}(b))} \gamma \sigma(v(r^{-1}(b))) \in \Lambda(x) \text{ for } b \in Y_\iota.
\]

**Lemma 2.2 (cf. Claim 1 of [17, Theorem 4.3]).** For each point \( x \in \mathbb{K}^{D(A, \iota)} \), there exists a unique collection of vectors \( (v(a))_{a \in A} \) for which the equations (0)-(3) hold. Moreover, the coefficients \( \alpha_{b, j} \), viewed as functions on \( \mathbb{K}^{D(A, \iota)} \), belong to the polynomial ring \( P_{b, j} := \mathbb{K}[X_{c, i}] \) \((c, i) \in D(A, \iota), (c, i) \leq_{\iota} (b, j) \).

**Proof.** By induction on the partial order \( \leq_{\iota} \) on \( \bar{A} \times \mathbb{Z}_{\geq 0} \), we show that there exist unique coefficients \( \alpha_{b, j} \in P_{b, j} \) satisfying (0)-(2), modulo the lattice \( \sum_{i=1}^{\infty} k e_{b+i} \).

If \( j = 0 \), then \( \alpha_{b, j} \equiv 1 \in k = P_{b, j} \). Suppose \( j \geq 1 \) and the statement holds for all pairs \((b', j')\) such that \((b', j') \leq_{\iota} (b, j)\). We show it also holds for \((b, j)\). By induction hypothesis, the equations (1) and (2) holds modulo \( \sum_{i=1}^{\infty} k e_{b+i} \) if and
only if
\[
(\ast) \quad \alpha_{b,j} = \begin{cases} 
X_{b,j} + \sum_{1 \leq i < j-1} (b,i) \in V(A) X_{b,i} \alpha_{b+i,j-i} & \text{if } b \in Y_\iota, (b,j) \in W(A,\iota); \\
\sum_{1 \leq i < j, (b,i) \in V(A)} X_{b,i} \alpha_{b+i,j-i} & \text{if } b \in Y_\iota, (b,j) \notin W(A,\iota); \\
\alpha_{b,j}^q - \sum_{1 \leq i < j, (b,i) \in V(A)} X_{b,i} \alpha_{b+i,j-i} & \text{otherwise.}
\end{cases}
\]

As \((b,i) \preceq (b,j), (b+i,j-i) \prec_i (b,j)\) for \(1 \leq i \leq j\) and \((v_{\bar{A}}^{-1}(b),j) \prec_i (b,j)\) for \(b \in \bar{A} \setminus Y_\iota\), the coefficients \(\alpha_{b+i,j-i}, \alpha_{b,j}^q \) are uniquely determined by induction hypothesis. So \(\alpha_{b,j} \) is also uniquely determined.

**Lemma 2.3** (cf. Claim 2 & 3 of [17, Theorem 4.3]). For \(x \in \mathbb{A}^{D(A,\iota)}\) we have \(A(\Lambda(x)) = A\). Moreover, \(\Lambda(x) \in X_\mu(\gamma)\) if \(x \in \mathcal{L}(A,\iota)\).

**Proof.** The equality \(A(\Lambda(x)) = A\) follows from the observation that \(\bar{A} \subseteq A(\Lambda(x))\) and that \(b \notin b' + n\mathbb{Z}\) if \(b \neq b' \in \bar{A}\).

Assume \(x \in \mathcal{L}(A,\iota)\). By (2) and (4) we have \(\varphi_{\Lambda(x)} = \varphi_A\). So \(\Lambda(x) \in X_\mu(\gamma)\) as \(A\) is a semi-module of Hodge type \(\mu\) (see Lemma 2.1). \(\square\)

**Lemma 2.4.** The natural projection \(\mathcal{L}(A,\iota) \to \mathbb{A}^{V(A)}\) is finite and smooth.

**Proof.** Set \(\Lambda = \Lambda(x)\). Let \(b \in Y_\iota\) and \(b' = r_{\bar{A}}^{-1}(b)\). Write
\[
t^{-\varphi_A(b')} \gamma_\iota(v(b')) = v(b) + \sum_{j=1}^\infty \beta_{b,j} v(b+j),
\]
where \(\beta_{b,j} \in k[\mathbb{A}^{D(A,\iota)}]\) and \(v(b+j) = e_{b+j}\) if \(b+j \notin A\), that is, \((b,j) \in W(A,\iota)\).

By definition, \(\mathcal{L}(A,\iota) \subseteq \mathbb{A}^{D(A,\iota)}\) is the zero locus of the coefficients \(\beta_{b,j}\) such that \((b,j) \in W(A,\iota)\).

Let \(s = n'd = |\bar{A}|\) for any \(k \in \mathbb{Z}\). By Lemma 2.2 \((\ast)\) and \(\S 2.2\) (3) we have

(i) \(\beta_{b,j} \in \alpha_{b,j}^q - \alpha_{b,j} + P_{j-1}\), where \(P_{j-1} = k[X_{c,i}; (c,i) \in D(A,\iota), 1 \leq i \leq j-1]\).

(ii) \(\alpha_{b,j} \in \alpha_{b,j}^q + P_{j-1}[X_{c,i}; (c,i) \in V(A), b \prec_i c]\).

(iii) \(\alpha_{b,j} \in X_{b,j} + P_{j-1}\) if \((b,j) \in W(A,\iota)\).

Therefore, the coefficients \(\beta_{b,j}\) for \((b,j) \in W(A,\iota)\) are of the form
\[
\beta_{b,j} = X_{b,j}^q - X_{b,j} + \delta_{b,j}
\]
for some \(\delta_{b,j} \in P_{j-1}[X_{c,i}; (c,i) \in V(A), b \prec_i c]\). Using the partial order \(\preceq\) on \(W(A,\iota)\), we see that the the Jacobian matrix \((\frac{\partial \beta_{b,j}}{\partial v(b',j)})\)\((b,j), (b',j') \in W(A,\iota))\) is invertible. So the projection \(\mathcal{L}(A,\iota) \to \mathbb{A}^{V(A)}\) is finite and smooth. \(\square\)

**Lemma 2.5** (cf. Claim 4 of [17, Theorem 4.3]). Let \(\Lambda \in X_\mu(\gamma)\) and \(\iota \in \mathbb{Z}_d\). Then there exists a unique point \(x \in \mathcal{L}(A(\Lambda),\iota)\) such that \(\Lambda(x) = \Lambda\).

**Proof.** Set \(A = A(\Lambda)\). For each \(k \in \mathbb{Z}_{\geq 0}\) we define a collection \((v_k(a))_{a \in A}\) of vectors in \(A\) and parameters \(x_{b,j} \in k\) with \((b,j) \in D(A,\iota)\) and \(1 \leq j \leq k\) such that

(i) \(h(v_k(b)) = b\) and \(t^{-\varphi_A(b)} \gamma_\iota(v_k(b)) \in \Lambda\) for \(b \in A\);

(ii) the equations (0)-(3) hold for \(v_k(b)\) modulo \(\sum_{\ell=k+1}^\infty k e_{b+\ell}\);

(iii) \(v_k(b) - v_{k-1}(b) \in \sum_{\ell=k}^\infty k e_{b+\ell}\) for \(b \in \bar{A}\);

(iv) \(v_k(a) = t^{\varepsilon_k(a-n)}\) for \(a \in n + A\).

If \(k = 0\), as \(\varphi_A = \varphi_\Lambda\) (see Lemma 2.1) we can take \(v_0(b) \in \Lambda\) for \(b \in \bar{A}\) such that (i)-(iv) hold. Assume \((v_{k-1}(a))_{a \in A}\) and \(x_{b,j}\) for \(1 \leq j \leq k-1\) are
already constructed. Then by induction on the partial order \( \preceq \), on \( \bar{A} \), one can construct \((v_k(a))_{c \in A}\) and \(x_{b,k}\) such that (i)-(iv) hold. By (iii) and (iv) the limit 
\[ v(a) = \lim_{k \to \infty} v_k(a) \in \Lambda \] 
exists for \( a \in A \). Since \( A(\Lambda(x)) = A \) and \( \Lambda(x) \subseteq \Lambda \), we have \( \Lambda(x) = \Lambda \). Moreover, by (i) and (ii), \((v(a))_{a \in A}\) satisfies (0)-(4) for \( x \). So \( x \in \mathcal{L}(A, \iota) \) as desired.

Let \( x' \in \mathcal{L}(A, \iota) \) such that \( \Lambda(x') = \Lambda \). Let \((v'(a))_{a \in A}\) be the normalized basis for \( x' \). Suppose \( x' \neq x \). Let 
\[ (b, j) \in \min\{(c, i) \in D(A, \iota); x_{c,i} \neq a \}. \]
By Lemma 2.2, we have \( \alpha'_{c,i} = \alpha_{c,i} \) if \( (c, i) \prec (b, j) \). Thus Lemma 2.2 (*) tells that \( h(v(b) - v'(b)) = b + j \). On the other hand, as \( v(b) - v'(b) \in \Lambda \) and \( t^{-v_{A(\Lambda)}(b)\gamma}\sigma(v(b) - v'(b)) \in \Lambda \), we deduce that \( b + j \in A \) and \( \varphi_A(b + j) \geq \varphi_A(b) \). This contradicts the fact that \( (b, j) \in D(A, \iota) \).

Let \( A_\mu \) be the set of semi-modules of Hodge type \( \mu \), and let \( A_\mu^{\text{top}} \) be the set of semi-modules \( A \in A \) such that \( \dim \mathcal{L}(A) = |V(A)| = \dim X_\mu(\gamma) \). Here \( \mathcal{L}(A) = \mathcal{L}(A, \iota) \) for some/any \( \iota \in \mathbb{Z}_h \).

**Corollary 2.6.** We have the following decompositions 

\[
X_\mu(\gamma) = \sqcup_{A \in A_\mu} \mathcal{L}(A) \\
\text{Irr}X_\mu(\gamma) = \sqcup_{A \in A_\mu^{\text{top}}} \text{Irr}\mathcal{L}(A),
\]

where each \( \mathcal{L}(A) \) is a locally closed subvariety of \( X_\mu(\gamma) \).

**Proof.** The first decomposition follows from Lemma 2.1, Lemma 2.3 and Lemma 2.5. The second decomposition follows from the first one and the fact that each locally closed subvariety, which is a priori bounded and of finite type, intersects only finitely many strata in the decomposition. The last claim follows from Lemma 2.2. \( \square \)

### 2.3.

For \( 1 \leq k \leq h \) define \( \omega_k \in \mathbb{J}_\gamma = \mathbb{J}^H_\gamma \) such that 
\[
\omega_k(e_a) = \begin{cases} 
eq a+k, & \text{if } a \in O^k; \\
eq a, & \text{otherwise.} \end{cases}
\]
We denote by \( \Omega_\gamma \subseteq \mathbb{J}_\gamma \) the subgroup generated by \( \omega_k \) for \( 1 \leq k \leq h \). Then \( \Omega_\gamma \) is a free abelian group of rank \( h \).

For \( X, X' \subseteq O \), we write \( X \preceq X' \) if \( X\tau \subseteq X'\tau \) for each \( \tau \in \mathbb{Z}_d \). We say a semi-module \( A \) is ordered if \( \bar{A}^1 \preceq \bar{A}^2 \preceq \cdots \preceq \bar{A}^h \).

Let \( c \in \text{Irr}X_\mu(\gamma) \). Denote by \( A(c) \in A_\mu^{\text{top}} \) the unique semi-module such that \( \mathcal{L}(A(c)) \) contains an open dense subset of \( c \). For \( 1 \leq i, j \leq h \) we write \( \bar{A}^i(c) \preceq \bar{A}^j(c) \) if there exists \( \omega \in \Omega_\gamma \), such that \( \bar{A}(\omega\Delta^i) = A(\omega\Delta^i) \setminus (n + A(\omega\Delta^i)) \preceq \bar{A}^i(c) \) and \( \omega\Delta^i \subseteq \Lambda \) for \( \lambda \in c \). Here \( \Delta^i = (\Delta^i_\tau)_{\tau \in \mathbb{Z}_d} \) with \( \Delta^i_\tau = \bigoplus_{k=0}^\infty e_{\tau,i+kh} \). We have \( \bar{A}^i(c) \preceq \bar{A}^j(c) \) if \( \bar{A}^i(c) \preceq \bar{A}^j(c) \).

**Lemma 2.7.** Let \( \omega = \prod_k \omega_k^p_k \in \Omega_\gamma \) with \( p_k \in \mathbb{Z}_{\geq 0} \). Let \( 1 \leq i \neq j \leq h \) such that \( 1 \leq p_l = \max\{p_k; 1 \leq k \leq h\} \). Then for \( c \in \text{Irr}X_\mu(\gamma) \) we have 

1. \( \bar{A}^i(\omega^l c) \preceq \bar{A}^j(\omega^l c) \) if \( p_l = 0 \) and \( l \gg 0 \);
2. \( \bar{A}^i(\omega c) \preceq \bar{A}^j(\omega c) \) if \( \bar{A}^i(c) \preceq \bar{A}^j(c) \).
Proof. As \( p_j = \max \{ p_k; 1 \leq k \leq h \} \), we have \( A^j(\omega^j\Lambda) \subseteq lhp_j + A^j(\Lambda) \) for \( l \in \mathbb{Z}_{\geq 0} \).

(1) By [5, Lemma 6.1], there exists \( r \gg 0 \) such that \( \omega^r_i \Delta^i \subseteq \Lambda \) for \( \Lambda \in \mathcal{L}(\mathcal{A}(c)) \). Since \( p_i = 0 \), \( \omega^r_i \Delta^i = \omega^r_i \Delta^i \). Thus, if \( l \gg 0 \), we have \( \omega^r_i \Delta^i \subseteq \omega^r_i \Delta^i \) and \( \overline{A}^r_i(\omega^r_i \Delta^i) \subseteq lhp_j + \overline{A}^r_i(\Lambda) \). So \( \overline{A}^r_i(\omega^r_i \Delta^i) \subseteq \overline{A}^r_i(\omega^r_i \Delta^i) \) since \( \overline{A}^r_i(\omega^r_i \Delta^i) \subseteq lhp_j + \overline{A}^r_i(\Lambda) \). This means \( \overline{A}^r_i(c) \ll \overline{A}^r_i(c) \) as desired.

(2) Let \( \omega' \in \Omega \), such that \( \overline{A}(\omega' \Delta^i) \subseteq \overline{A}(c) \) and \( \omega' \Delta^i \subseteq \Lambda \) for \( \Lambda \in \mathcal{A} \). Notice that \( \overline{A}(\omega' \Delta^i) = h p_i + \overline{A}(\omega' \Delta^i) \). So \( \omega' \Delta^i \subseteq \omega \Lambda \) and

\[
\overline{A}(\omega' \Delta^i) = h p_i + \overline{A}(\omega' \Delta^i) = p_j h + \overline{A}(\Lambda),
\]

which means \( \overline{A}(\omega' \Delta^i) \subseteq \overline{A}(\omega \Lambda) \) and hence \( A^r_i(\omega c) \ll A^r_i(\omega c) \) as desired. \( \square \)

**Corollary 2.8.** Let \( c \in \text{Irr}_\mu(\gamma) \). Then there exist \( j \in \Omega \) and an ordered semi-module \( A \in \mathcal{A}_\mu \) such that \( j c \in \text{Irr}\mathcal{L}(A) \).

Proof. We argue by induction on \( k \) that there exist \( \omega \in \Omega \) such that \( \overline{A}^i(\omega c) \ll \overline{A}^i(\omega c) \ll \cdots \ll \overline{A}^i(\omega c) \) for \( 1 \leq i \leq k \). If \( k = h + 1 \), there is nothing to prove. Suppose the statement is true for \( k = j + 1 \) for some \( 1 \leq j \leq h \). We show it is also true for \( k = j \). By induction hypothesis, there exists \( \omega \in \Omega \) such that \( \overline{A}^i(\omega c) \ll \overline{A}^{i+1}(\omega c) \ll \cdots \ll \overline{A}^h(\omega c) \) for \( 1 \leq i \leq j \). Let \( \omega'' = \omega_j \cdots \omega_h \in \Omega \). Then Lemma 2.7 tells that for \( l \gg 0 \) one has \( \overline{A}^l((\omega'')^l(\omega c)) \ll \overline{A}^l((\omega'')^l(\omega c)) \ll \cdots \ll \overline{A}^h((\omega'')^l(\omega c)) \) for \( 1 \leq i \leq j - 1 \). So the induction is finished. \( \square \)

### 2.4. Let \( A \) be an ordered semi-module of Hodge type \( \mu \). We fix \( i \in \mathbb{Z}_d \).

**Lemma 2.9.** Let \( x \in \mathcal{L}(A,i) \) and let \( (v(a))_{a \in A} \) be the corresponding normalised basis. For \( 1 \leq k \leq h \) and \( a \in A^k \) we have

\[
v(a) \in \sum_{i=0}^{h-k} \sum_{j=0}^{\infty} k e_{a+i+j}.\]

Proof. The statement follows from the following two facts:

(1) \( A \) is ordered and hence \( \overline{A}^1 \leq \overline{A}^2 \leq \cdots \leq \overline{A}^h \);

(2) for \( 1 \leq i \leq h \) and \( b \in \overline{A}^i \) we have \( (b,j) \notin D(A,i) \) if \( \overline{A}^i \geq \overline{A}^{i+j}. \) \( \square \)

For \( 1 \leq k \leq h \) we set \( y_k^i = \max \overline{A}^k \) and \( s = n'd = |\overline{A}^k| \). For \( \xi = (\xi)_1 \leq i \leq k \in \mathbb{F}_{q^s} \), we define \( n_{i,k} \in \mathbb{F}_q \) by \( e_a \mapsto e_{a+i} \) if \( a \notin \mathcal{O}^k \) and \( e_a \mapsto e_{a+i} + \sum_{i=1}^{k} \xi^j \) if \( a \in f^l(y_k^i) + nZ \) for some \( l \in \mathbb{Z} \).

**Lemma 2.10.** If \( \overline{A}^{k+1} \geq h + \overline{A}^k \), then \( (y_k^i, i) \in W(A,i) \) for \( 1 \leq i \leq h - k \) and \( x_{y_k^i, i} \in \mathbb{F}_q^s \) for each \( x \in \mathcal{L}(A,i) \). As a consequence,

\[
\mathcal{L}(A,i) = \sqcup_{\xi \in \mathbb{F}_{q^s}} \mathcal{L}(A,i,k,\xi),
\]

where \( \mathcal{L}(A,i,k,\xi) = \{ x \in \mathcal{L}(A,i); x_{y_k^i, i} = \xi_i \} \) for \( 1 \leq i \leq h - k \). Moreover,

\[
n_{i,k,\xi} \mathcal{L}(A,i,k,\xi) = \mathcal{L}(A,i,k,\xi + \xi').
\]

Proof. As \( A \) is ordered and \( \overline{A}^{k+1} \geq h + \overline{A}^k \), we have \( b + i \notin A \) for \( b \in \overline{A}^k \) and \( 1 \leq i \leq h - k \). So \( (y_k^i, i) \in W(A,i) \) if \( b = y_k^i \) and \( (b,i) \notin D(A,i) \) if \( b \in \overline{A}^k \setminus \{ y_k^i \} \). Then Lemma 2.2 (*) tells that \( \alpha_{b,i} = X_{y_k^i, i}^q \) if \( b \in f^l(y_k^i) + nZ \) for some \( 0 \leq l \leq s - 1 \).

In view of Lemma 2.9 and the requirement §2.2 (4'), we deduce that \( x_{y_k^i, i} \in \mathbb{F}_q^s \) as desired. \( \square \)
Lemma 2.11. Let \(1 \leq k \leq h - 1\) such that \(\overline{A}^{k+1} \geq h + \overline{A}^{k}\). For \(\Lambda \subseteq L(A, \iota, k, 0)\) we have \(\overline{A}^{i}(\omega_{k}\Lambda) = h + \overline{A}^{i}(\Lambda)\) if \(\overline{A}^{1} = \overline{A}^{k}\) and \(\overline{A}^{i}(\omega_{k}\Lambda) = \overline{A}^{i}(\Lambda)\) otherwise. Here \(\omega_{k} \in \Omega_{\gamma}\) is defined in \(\S 2.3\).

Proof. Set \(A = A(\Lambda)\). Let \((v(a))_{a \in A}\) be the normalized basis for \(\Lambda \subseteq L(A, \iota, k, 0)\).

By Lemma 2.9 and Lemma 2.10 we deduce that

\[
\begin{align*}
\text{Corollary 2.12.} & \quad \text{Let } c \in \text{Irr}X_{\mu}(\gamma). \quad \text{Then there exist } j \in \mathbb{J}_{\gamma} \text{ and a rigid semi-module } A \text{ such that } jc \in \text{Irr}L(A).
\end{align*}
\]

For \(1 \leq k \leq h - 1\) we define \(s_{k} \in \mathbb{J}_{\gamma}\) by

\[
s_{k}(e_{a}) = \begin{cases} 
eq e_{a-1}, & \text{if } a \in O^{k+1}; \\ e_{a+1}, & \text{if } a \in O^{k}; \\ e_{a}, & \text{otherwise.} \end{cases}
\]

We denote by \(S_{\gamma} \subseteq \mathbb{J}_{\gamma}\) the subgroup generated by \(s_{k}\) for \(1 \leq k \leq h - 1\). Then \(S_{\gamma}\) is isomorphic to the symmetry group of \(h\) letters.

Lemma 2.13. If \(A\) is rigid, then \(S_{\gamma}\) preserves \(\text{Irr}L(A)\).

Proof. It suffices to show \(s_{k}\) preserves \(\text{Irr}L(A)\) for each \(1 \leq k \leq h - 1\).

Case(1): \(\overline{A}^{k+1} = 1 + \overline{A}^{k}\).

We claim that \(s_{k}L(A) = L(A)\). Indeed, let \(\tau \in \mathbb{Z}_{d}\) and \(x \in L(A, \tau)\). Let \((v(a))_{a \in A}\) be the normalized basis for \(x\). Since \(\overline{A}^{k} \leq \overline{A}^{k+1} = 1 + \overline{A}^{k}\), we have \(h = n\) and \(\varphi_{A}(b+1) = \varphi_{A}(b)\) for \(b \in \overline{A}^{k}\). Thus \((\overline{A}^{k}, 1) \cap D(A, \tau) = \emptyset\) and \(v(a) \in e_{a} + \sum_{j=2}^{\infty} ke_{a+j}\) for \(a \in \overline{A}^{k}\). In particular,

\[
h(s_{k}(v(a))) = \begin{cases} a + 1, & \text{if } a \in \overline{A}^{k}; \\ a - 1, & \text{if } a \in \overline{A}^{k+1}; \\ a, & \text{otherwise.} \end{cases}
\]

Therefore, \(A = A(\Lambda(x)) \subseteq A(s_{k}\Lambda(x))\). On the other hand, let \(v \in \Lambda(x)\) and \(a = hv \in A\). If \(a \in A \setminus (A^{k} \cup A^{k+1})\), then \(h(s_{k}(v)) = a \in A\). Otherwise,
h(s_k(v)) = a - 1 \in A^k if a \in A^{k+1} and h(s_k(v)) \in \{a, a+1\} \subseteq A^k \cup A^{k+1} if a \in A^k. Therefore, A(s_k \Lambda(x)) = A and the claim is proved.

Case(2): \(\bar{A}^{k+1} \neq 1 + \bar{A}^k\).

First we claim that

(i) there exist \(i_0 \in \mathbb{Z}_d\) and \(y \in \bar{A}_{i_0}^k\) such that \(y + 1 \in \bar{A}_{i_0}^{k+1}\) and \((y, 1) \in V(A)\). In particular, \(y = \max \bar{A}_{i_0}^k\) as \(\bar{A}_{i_0}^{k+1} \supseteq \bar{A}_{i_0}^k\).

Indeed, since \(A\) is rigid, there exists \(b \in \bar{A}^k \) with \(b + 1 \in \bar{A}^{k+1}\). Assume (i) fails. Then \((b, 1) \notin V(A)\), that is, \(\varphi_A(b) \leq \varphi_A(b + 1)\). If \(\varphi_A(b) < \varphi_A(b + 1)\), then \(r_A(b) > r_A(b + 1)\), contradicting that \(\bar{A}^{k+1} \supseteq \bar{A}^k\). So we have \(\varphi_A(b) = \varphi_A(b + 1)\) and hence \(r_A(b) + 1 = r_A(b + 1)\). Repeating this argument, we deduce that \(\bar{A}_r^\ell = 1 + \bar{A}^k\), contradicting our assumption. So (i) is proved.

By (i) we have \((y, 1) \in V(A)\) and hence all the coefficients \(a_{b, 1}\) for \(b \in \bar{A}^k\) are non-zero polynomials in \(k[X_{a, 1}; (a, 1) \in V(A), a \in A^k]\). Let

\[ U = \{x \in A^{V(A)}; a_{b, 1}(x) \neq 0 \text{ for } b \in \bar{A}^k\}, \]

which is an open dense subset of \(A^{V(A)}\). Let \(U' \subseteq \mathcal{L}(A, i_0)\) be the preimage of \(U\) under the natural projection \(\mathcal{L}(A, i_0) \to A^{V(A)}\). By Lemma 2.4, \(U'\) is open dense in \(\mathcal{L}(A, i_0) = \mathcal{L}(A)\). Let \(x \in U'\) and let \((v(a))_{a \in A}\) the corresponding normalized basis. By definition,

\[ v(a) \in e_a + k^x e_{a+1} + \sum_{k=2}^{\infty} k e_{a+k} \text{ for } a \in A^k. \]

In particular, \(h(s_k(v(a))) = a \text{ for } a \in A \setminus A^{k+1}\). Moreover, for each \(a \in A^{k+1}\), there exists \(c \in \bar{A}^k\) such that \(a \equiv c + 1 \mod n\). Moreover, since \(\bar{A}^{k+1} \supseteq \bar{A}^k\), \(a - (c + 1) = i_0 n\) for some \(i_0 \in \mathbb{Z}_{\geq 0}\). Then

\[ v(a) - t^{i_0} \alpha_{c, 1}^{-1} v(c) \in \alpha_{c, 1}^{-1} e_{a-1} + \sum_{k=1}^{\infty} k e_{a+k}. \]

So \(h(s_k(v(a) - t^{i_0} \alpha_{c, 1}^{-1} v(c))) = a\). Therefore, \(A \subseteq A(s_k \Lambda(x))\). On the other hand, let \(v \in \Lambda(x)\) such that \(v = v(a) + \sum_{j=1}^{\infty} \beta_j v(a+j)\) for some \(\beta_j \in k\). We can assume \(\beta_j = 0\) if \(a + j \notin A\). If \(a \in A \setminus (A^{k+1} \cup A^k)\), then \(h(s_k(v)) = a \in A\). If \(a \in A^{k+1}\), then \(h(s_k(v)) = a - 1 \in A^k\). If \(a \in A^k\), then \(h(s_k(v))\) equals either \(a \in A^k\) or \(a + 1\). In the latter case, we have \(\beta_1 \neq 0\) (as \(a_{a, 1} \neq 0\)) and hence \(a + 1 \in A^{k+1}\). Therefore, \(A(s_k \Lambda(x)) = A\) and hence \(s_k(U') \subseteq \mathcal{L}(A)\). The proof is finished. \(\square\)

3. Orbits of irreducible components

Let notations be as in §2. Let \(M = \prod_{1 \leq k \leq h} M^k\) with each \(M^k \cong \text{Res}_{\mathbb{Q}_a/\mathbb{Q}} GL_{n^k}\). Moreover precisely, \(M^k(L) = \prod_{\tau \in \mathbb{Z}_d} GL(N^k_{\tau}), \) where \(N^k_{\tau} = \bigoplus_{j=1}^{n'} Le_{r,k+jh}. \) Then \(M \supseteq T_H\) is a semi-standard Levi subgroup of \(H\). Notice that \(\gamma\) is superbasic in \(M(L)\).

3.1. We say \(C = (C^k)_{1 \leq k \leq h}\) is a semi-module for \(M\) if \(C^k \subseteq O^k\) is bounded below, \(n + C^k, f(C^k) \subseteq C^k\) and \(n\mathbb{Z} + C^k = O^k\) for \(1 \leq k \leq h\). Set \(\bar{C} = C \setminus (n + C)\) and define \(\varphi_C : C \to \mathbb{Z}\) and \(r_C : \bar{C} \sim \bar{C}\) analogously as in §2.1. Let \(\lambda = \ldots\)
Since \( \lambda \) is ordered, Lemma 2.9 tells that \( (v(a))_{a \in A} \) is the normalized basis for \( X_\mu(T_M) = X_\mu(T_H) \) be an \( M \)-dominant cocharacter (with respect to the Borel subgroup \( B_M = B_H \cap M \)). We say a semi-module \( C \) is of Hodge type \( \lambda \) if \( (\lambda^i)(k + ih)^{0 \leq i < n'} - 1 \) is a permutation of \( (\varphi_C(b))_{b \in \mathcal{C}_{\tau \lambda}} \) for \( 1 \leq k \leq h \) and \( \tau \in \mathbb{Z}_d \).

Let \( I_{\mu, \gamma} = I_{\mu, \gamma, M} \) be the set of \( M \)-dominant cocharacters \( \lambda \) such that \( \lambda \) is conjugate to \( \mu \) under \( W_H \) and \( X_\lambda^M(\gamma) \neq \emptyset \). As \( \mu \) is minuscule, we have \( \lambda \in I_{\mu, \gamma} \) if and only if \( \lambda \leq \mu \) is \( M \)-dominant and \( m' = \sum_{\tau \in \mathbb{Z}_d} \sum_{i=1}^{n'} \lambda^i_{\tau}(k + ih) \) for \( 1 \leq k \leq h \).

On the other hand, let \( \Lambda \in I_{\mu, \gamma} \). Suppose \( A \) is ordered. Let \( \iota \in \mathbb{Z}_d \) and \( x \in \mathcal{L}(A, \iota) \). Let \( (v(a))_{a \in A} \) denote the normalized basis for \( x \). For \( 1 \leq k \leq h \) and \( a \in A^k \) we set \( v^k(a) = \sum_{i=0}^\infty \alpha_{a, ih}e_{a + ih} \). Since \( A \) is ordered, Lemma 2.9 tells that \( (v^k(a))_{a \in A^k} \) is the normalized basis for \( x^k \in \mathcal{L}(A^k, \iota) \), where \( x^k_{b, \iota} = x_{b, \iota} \) for \( (b, j) \in D(A^k, \iota) = V(A^k) := \{(b, j) \in V(A); b, b + j \in A^k\} \). The map \( x \mapsto (x^k)_{1 \leq k \leq h} \) gives a morphism

\[
\beta_{A, \iota} : \mathcal{L}(A) = \mathcal{L}(A, \iota) \to \mathcal{L}(\gamma A) = \mathcal{L}(A^1, \ldots, \mathcal{L}(A^h)) \subseteq X^M_\gamma(\gamma).
\]

On the other hand, let \( N \subseteq H \) be the unipotent subgroup such that

\[
N(L)e_{\tau, i} = e_{\tau, i} + \sum_{l=i+1}^{h} \sum_{k=0}^{n'} L e_{\tau, j + kh}
\]

for \( 1 \leq i \leq h \). Then \( P := MN = NM \subseteq H \) is a semi-standard parabolic subgroup. The Iwasawa decomposition gives a natural projection

\[
\beta_{\gamma} : H(L)/K = P(L)K/K \to M(L)/M(0).
\]

Observing that \( v(a) \in P(L)e_a \) for \( a \in A \), we have

**Lemma 3.1.** If \( A \) is ordered, then \( \beta_{A, \iota} = \beta_{\gamma} |_{\mathcal{L}(A, \iota)} = \beta_{\gamma} |_{\mathcal{L}(A)} \). In particular, the morphism \( \beta_{A, \iota} \) is independent of the choice of \( \iota \in \mathbb{Z}_d \).

**3.2.** Let \( \mathcal{R}^\text{top} \) be the set of rigid semi-modules \( A \) of Hodge type \( \mu \) such that \( \dim \mathcal{L}(A) = \dim X_\mu(\gamma) \). For \( A', A'' \in \mathcal{R}^\text{top} \) we write \( A' \sim A'' \) if \( A' = A'' + kh \) for some \( k \in \mathbb{Z} \). We set \( \mathcal{R}^\text{top} = \mathcal{R}^\text{top} / \sim \).

Analogously, for each \( M \)-dominant cocharacter \( \lambda \leq \mu \) we denote by \( \mathcal{C}^\text{top} \) the set of semi-modules \( C \) for \( M \) of Hodge type \( \lambda \) such that \( \dim \mathcal{L}(C) = \dim X^M_\lambda(\gamma) \). Then \( \mathcal{C}^\text{top} \) admits an action by \( \Omega_\gamma \) (see §2.3) such that

\[
(\omega^1 \cdots \omega^h)(p^1, \ldots, C^h) = (p_1 + C^1, \ldots, p_h + C^h).
\]

Actually, we have \( \omega C(\Lambda) = C(\omega \Lambda) \) for \( \Lambda \in X^M_\lambda(\gamma) \) and \( \omega \in \Omega_\gamma \). Here \( C(\Lambda) \) denotes the semi-module for \( M \) associated to \( \Lambda \). Let \( \mathcal{C}^\text{top} \) be the set of \( \Omega_\gamma \)-orbits in \( \mathcal{C}^\text{top} \).

**Lemma 3.2.** Let \( A \) be a rigid semi-module of Hodge type \( \mu \). Then \( A \in \mathcal{R}^\text{top} \) if and only if \( \gamma A = (A^1, \ldots, A^h) \in \mathcal{C}^\text{top} \).

**Proof.** Let \( m_{A, \tau} = \sum_{b \in \mathcal{C}_{\tau A+1}} \varphi_A(b) \) for \( \tau \in \mathbb{Z}_d \) and \( 1 \leq k \leq h \). Since \( A \) is a semi-module of Hodge type \( \mu \), we have \( m_\tau = \sum_{k'} m_{A, \tau}^k \) and \( m' = \sum_{\tau} m_{A, \tau}^k \) for \( \tau \in \mathbb{Z}_d \).
and $1 \leq k \leq h$. For $1 \leq i, j \leq h$ set

$$V_{i,j}(A) = \{(b, k) \in V(A); b \in \bar{A}^i, b + k \in \bar{A}^j\}.$$  

Then $V(A) = \biguplus_{i,j} V_{i,j}(A)$ and $V_{i,j}(A) = \emptyset$ unless $i \leq j$ since $A$ is ordered. As $\mu$ is minuscule, we can assume $\varphi_A(b) \in \{0, 1\}$ for $b \in \bar{A}$. In particular, $m_{A,\tau}^k = \sharp \{b \in \bar{A}_\tau^{k+1}; \varphi_A(b) = 1\}$. Thus

$$\sum_{i=1}^{h} \sharp V_{i,i}(A) = \sum_{i=1}^{h} \dim \mathcal{L}(A^i) = \dim \mathcal{L}(\gamma A) \\ \leq \dim X_{\lambda_A}^M(\gamma) \\ = -\frac{1}{2} h(n' - 1) + \frac{1}{2} \sum_{1 \leq k \leq h} \sum_{\tau \in \mathbb{Z}_d} (n' - m_{A,\tau}^k)m_{A,\tau}^k \\ = -\frac{1}{2} h(n' - 1) + \frac{1}{2} n' \sum_{k} \sum_{\tau} m_{A,\tau}^k - \frac{1}{2} \sum_{k} \sum_{\tau} (m_{A,\tau}^k)^2 \\ = -\frac{1}{2} h(n' - 1) + \frac{1}{2} h n'm' - \frac{1}{2} \sum_{k} \sum_{\tau} (m_{A,\tau}^k)^2.$$  

Moreover, for $1 \leq i < j \leq h$ we have

$$\sharp V_{i,j}(A) = \{(b, b') \in \bar{A}^i \times \bar{A}^j, \varphi_A(b) = 0, \varphi_A(b') = 1\} \\ = \sum_{\tau \in \mathbb{Z}_d} m_{A,\tau}^j (n' - m_{A,\tau}^i).$$  

Hence

$$\sum_{1 \leq i < j \leq h} \sharp V_{i,j}(A) = n' \sum_{\tau \in \mathbb{Z}_d} \sum_{1 \leq k \leq h} (k - 1)m_{A,\tau}^k - \sum_{1 \leq i < j \leq h} \sum_{\tau} m_{A,\tau}^i m_{A,\tau}^j \\ = \frac{1}{2} h(h - 1)n'm' - \sum_{1 \leq i < j \leq h} \sum_{\tau} m_{A,\tau}^i m_{A,\tau}^j.$$  

Thus

$$\dim \mathcal{L}(A) = \sharp V(A) = \sum_{1 \leq i < j \leq h} \sharp V_{i,j}(A) \\ = \dim \mathcal{L}(\gamma A) - \dim X_{\lambda_A}^M(\gamma) - \frac{1}{2} h(n' - 1) \\ + \frac{1}{2} h^2 n'm' - \frac{1}{2} \sum_{\tau} (\sum_{k=1}^{h} m_{A,\tau}^k)^2 \\ = \dim \mathcal{L}(\gamma A) - \dim X_{\lambda_A}^M(\gamma) - \frac{1}{2} h(n' - 1) \\ + \frac{1}{2} h^2 n'm' - \frac{1}{2} \sum_{\tau} (m_{\tau})^2.$$
Therefore, the equality holds if and only if \( \dim X_\mu(\gamma) = \frac{1}{2}(n - h) + \frac{1}{2} \sum_\tau (n - m_\tau)m_\tau \)
\[ = \frac{1}{2}(n - h) + \frac{1}{2}n m - \frac{1}{2} \sum_\tau (m_\tau)^2 \]
\[ = \frac{1}{2}(n - h) + \frac{1}{2}n m - \frac{1}{2} \sum_\tau (m_\tau)^2 \]
\[ = \dim \mathcal{L}(A) + \dim X_{\lambda_A}^M(\gamma) - \dim \mathcal{L}(\gamma A) \]
\[ \geq \dim \mathcal{L}(A). \]

Therefore, the equality holds if and only if \( \dim \mathcal{L}(\gamma A) = \dim X_{\lambda_A}^M(\gamma) \), that is, \( \gamma A \in \mathcal{C}_{\lambda_A}^{\top}\).

**Lemma 3.3.** If \( A \in \mathcal{R}_{\mu}^{\top} \), then all the irreducible components of \( \mathcal{L}(A) \) are conjugate by \( N(L) \cap J_\gamma \).

**Proof.** Consider the surjective projection \( \beta_\gamma|_{\mathcal{L}(A)} : \mathcal{L}(A) \rightarrow \mathcal{L}(\gamma A) \). By Lemma 3.2, \( \mathcal{L}(\gamma A) \cong \prod_k A[\mathcal{V}(A^k)] \) is an irreducible component of \( X_{\lambda_A}^M(\gamma) \). Then [6, Proposition 5.6] tells that all the irreducible components of \( X_{\mu}(\gamma) \cap \beta_\gamma^{-1}(\mathcal{L}(\gamma A)) \) are conjugate by \( N(L) \cap J_\gamma \). So the statement follows from the inclusion \( \operatorname{Irr} \mathcal{L}(A) \subseteq \operatorname{Irr}(X_{\mu}(\gamma) \cap \beta_\gamma^{-1}(\mathcal{L}(\gamma A))) \). \( \square \)

**Lemma 3.4.** Let \( A, A' \in \mathcal{R}_{\mu}^{\top} \). If \( J_\gamma \operatorname{Irr} \mathcal{L}(A) = J_\gamma \operatorname{Irr} \mathcal{L}(A') \neq \emptyset \), then \( A \sim A' \).

**Proof.** Let \( I \subseteq H(L) \) be the (standard) Iwahori subgroup such that
\[ I(e_a) \subseteq k^\times e_a + \sum_{j=1}^\infty ke_{a+j}. \]

By Bruhat decomposition we have
\[ J_\gamma = (J_\gamma \cap I)\Omega_{\gamma} \mathcal{G}_\gamma (J_\gamma \cap I) = (J_\gamma \cap I)\mathcal{G}_\gamma \Omega_{\gamma} (J_\gamma \cap I), \]
where \( \Omega_{\gamma}, \mathcal{G}_\gamma \subseteq J_\gamma \) are defined in §2.3 and §2.11, respectively. By definition, \( J_\gamma \cap I \) preserves \( \mathcal{L}(A) \) and \( \mathcal{L}(A') \). Thanks to Lemma 2.13, \( \mathcal{G}_\gamma \) preserves \( \operatorname{Irr} \mathcal{L}(A) \) and \( \operatorname{Irr} \mathcal{L}(A') \). Thus there exist \( \Lambda \in \mathcal{L}(A) \) and \( \omega \in \Omega_{\gamma} \) such that \( \omega \Lambda \in \mathcal{L}(A') \). So
\[ \gamma(A') = C(\beta_\gamma(\omega \Lambda)) = C(\omega \beta_\gamma(\Lambda)) = \omega C(\beta_\gamma(\Lambda)) = \omega(\gamma A). \]

Thus \( A \sim A' \) since \( A \) and \( A' \) are rigid. \( \square \)

By the construction of \( P = MN \), there exits \( z = \sigma(z) \) in the Weyl group \( W_H \) of \( T_H \) in \( H \) such that \( z^*P = z^*M^zN \) is a standard parabolic subgroup. Moreover, we can assume further that \( z(B_H \cap M) = B_H \cap z^*M \). In particular, \( z(\lambda_M(\gamma)) = \lambda_{M}(z^*\gamma) = \lambda_H(z^*\gamma) \) and \( zI_{\mu,\gamma,M} = I_{\mu,z^*\gamma,z^*M} \).

**Corollary 3.5.** We have natural bijections
\[ J_\gamma \setminus \operatorname{Irr} X_{\mu}(\gamma) \sim \tilde{\mathcal{R}}_{\mu}^{\top}(G) \sim \sqcup_{\lambda \in I_{\mu,\gamma}} \mathcal{C}_{\lambda_{A}}^{\top}. \]

In particular,
\[ |J_\gamma \setminus \operatorname{Irr} X_{\mu}(\gamma)| = \dim V_{\mu,H}(\lambda_H(\gamma)). \]
Proof. The first bijection is due to Corollary 2.12 and Lemma 3.3 and Lemma 3.4. The second bijection is induced by the map $A \mapsto \gamma A$ (see Lemma 3.2). As $\gamma$ is superbasic in $M(L)$, [6, Thoerem 1.5] tells that $|\bar{\mathcal{C}}_{\rm top}^{\lambda}| = \dim V_{\bar{\lambda}}(\lambda_M(\gamma))$. Thus

$$
|\mathcal{J}_\gamma \backslash \operatorname{Irr} X_\mu(\gamma)| = \sum_{\lambda \in I_{\mu, \gamma}} |\bar{\mathcal{C}}_{\lambda}^{\lambda_{\mu}}| = \sum_{\lambda \in I_{\mu, \gamma}} \dim V_{\bar{\lambda}}(\lambda_M(\gamma)) = \dim V_{\mu}^H(\lambda_M(\gamma)),
$$

where the last equality follows from a similar argument in Proposition 1.5. Notice that $\lambda_H(\gamma) = z(\lambda_M(\gamma))$ for some element $z = \sigma(z) \in W_H$. Therefore, $\dim V_{\mu}^H(\lambda_M(\gamma)) = \dim V_{\mu}^H(\lambda_H(\gamma))$ and the last statement follows.

□

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