A new quantum analog of the Brauer algebra

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Abstract

We introduce a new algebra $B_l(z, q)$ depending on two nonzero complex parameters such that $B_l(q^n, q)$ at $q = 1$ coincides with the Brauer algebra $B_l(n)$. We establish an analog of the Brauer–Schur–Weyl duality where the action of the new algebra commutes with the representation of the twisted deformation $U'_q(\mathfrak{o}_n)$ of the enveloping algebra $U(\mathfrak{o}_n)$ in the tensor power of the vector representation.
1 Introduction

In the classical Schur–Weyl duality the actions of the general linear group $GL(n)$ and the symmetric group $\mathfrak{S}_l$ in the tensor power $(\mathbb{C}^n)^\otimes l$ are centralizers of each other. A quantum analog of this duality is provided by the actions of the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ and the Iwahori–Hecke algebra $\mathcal{H}_l(q)$ on the same space; see Jimbo [8], Leduc and Ram [10].

If the group $GL(n)$ is replaced with the orthogonal group $O(n)$ or symplectic group $Sp(n)$ (with even $n$) the corresponding centralizer is generated by the action of a larger algebra $B_l(n)$ called the Brauer algebra originally introduced by Brauer in [2]. The group algebra $\mathbb{C}[\mathfrak{S}_l]$ is a natural subalgebra of $B_l(n)$. When $U_q(\mathfrak{gl}_n)$ is replaced by the quantized enveloping algebra $U_q(\mathfrak{o}_n)$ or $U_q(\mathfrak{sp}_n)$ then the corresponding centralizer is generated by the action of the algebra $\mathcal{BMW}_l(z, q)$ introduced by Birman and Wenzl [1] and Murakami [12]; here the parameter $z$ should be appropriately specialized. The algebraic structure and representations of the Brauer algebra and the Birman–Wenzl–Murakami algebra were also studied by Wenzl [18], Ram and Wenzl [16], Leduc and Ram [10], Halverson–Ram [6], Nazarov [13].

Unlike the classical case, the algebras $U_q(\mathfrak{o}_n)$ and $U_q(\mathfrak{sp}_n)$ are not isomorphic to subalgebras of $U_q(\mathfrak{gl}_n)$. Accordingly, the Iwahori–Hecke algebra $\mathcal{H}_l(q)$ is not a natural subalgebra of $\mathcal{BMW}_l(z, q)$. On the other hand, there exist subalgebras $U_q'(\mathfrak{o}_n)$ and $U_q'(\mathfrak{sp}_n)$ of $U_q(\mathfrak{gl}_n)$ which specialize respectively to $U(\mathfrak{o}_n)$ and $U(\mathfrak{sp}_n)$ as $q \to 1$. They were first introduced by Gavrilyk and Klimyk [4] (orthogonal case) and by Noumi [14] (both cases). Following Noumi we call them twisted quantized enveloping algebras. Although $U_q'(\mathfrak{o}_n)$ and $U_q'(\mathfrak{sp}_n)$ are not Hopf algebras, they are coideal subalgebras of $U_q(\mathfrak{gl}_n)$. These algebras were studied in [14] in connection with the theory of quantum symmetric spaces. The algebra $U_q'(\mathfrak{o}_n)$ also appears as the symmetry algebra for the $q$-oscillator representation of the quantized enveloping algebra $U_q(\mathfrak{sp}_{2m})$; see Noumi, Umeda and Wakayama [15]. Central elements and representations of $U_q'(\mathfrak{o}_n)$ were studied by Gavrilyk and Iorgov [5], Havlíček, Klimyk and Pošta [7], Klimyk [8]. A relationship between the algebras $U_q'(\mathfrak{o}_n)$ and $U_q'(\mathfrak{sp}_n)$ and their affine analogs was studied in [11].

In this paper we only consider the algebra $U_q'(\mathfrak{o}_n)$ although the results can be easily extended to the symplectic case as well. We define a new algebra $B_l(z, q)$ and show that its action on the tensor power of the vector representation of $U_q(\mathfrak{gl}_n)$ (with $z = q^n$) commutes with that of the subalgebra $U_q'(\mathfrak{o}_n) \subseteq U_q(\mathfrak{gl}_n)$. We thus have an embedding (at least for generic $q$) of the Iwahori–Hecke algebra $\mathcal{H}_l(q)$ into $B_l(z, q)$. Moreover, the algebra $B_l(q^n, q)$ coincides with the Brauer algebra $B_l(n)$ for $q = 1$.

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2 Brauer algebra and its quantum analog

Let \( l \) be a positive integer and \( \eta \) a complex number. An \( l \)-diagram \( d \) is a collection of \( 2l \) dots arranged into two rows with \( l \) dots in each row connected by \( l \) edges such that any dot belongs to only one edge. The product of two diagrams \( d_1 \) and \( d_2 \) is determined by placing \( d_1 \) above \( d_2 \) and identifying the vertices of the bottom row of \( d_1 \) with the corresponding vertices in the top row of \( d_2 \). Let \( s \) be the number of loops in the picture. The product \( d_1d_2 \) is given by \( \eta^s \) times the resulting diagram without loops. The Brauer algebra \( \mathcal{B}_l(\eta) \) is defined as the \( \mathbb{C} \)-linear span of the \( l \)-diagrams with the multiplication defined above. The dimension of the algebra is \( 1 \cdot 3 \cdots (2l-1) \).

The following presentation of the Brauer algebra is well-known; see, e.g., \([I]\).

**Proposition 2.1** The Brauer algebra \( \mathcal{B}_l(\eta) \) is isomorphic to the algebra with \( 2l-2 \) generators \( \sigma_1, \ldots, \sigma_{l-1}, e_1, \ldots, e_{l-1} \) and the defining relations

\[
\begin{align*}
\sigma_i^2 &= 1, & e_i^2 &= \eta e_i, & \sigma_i e_i &= e_i \sigma_i = e_i, & i = 1, \ldots, l-1, \\
\sigma_i \sigma_j &= \sigma_j \sigma_i, & e_i e_j &= e_j e_i, & \sigma_i e_j &= e_j \sigma_i, & |i-j| > 1, \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & e_i e_{i+1} e_i &= e_i, & e_{i+1} e_i e_{i+1} &= e_{i+1}, \\
\sigma_i e_{i+1} e_i &= e_{i+1} e_i, & e_{i+1} e_i e_{i+1} &= e_{i+1} \sigma_i, & i = 1, \ldots, l-2.
\end{align*}
\]

The generators \( \sigma_i \) and \( e_i \) correspond to the following diagrams respectively:

\[
\begin{align*}
\begin{array}{cccc}
1 & 2 & \cdots & i \\
\end{array}
\begin{array}{cccc}
i+1 & l-1 & l \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\end{align*}
\]

and

\[
\begin{align*}
\begin{array}{cccc}
1 & 2 & \cdots & i \\
\end{array}
\begin{array}{cccc}
i+1 & l-1 & l \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\begin{array}{cccc}
& & & \\
\end{array}
\end{align*}
\]

The subalgebra of \( \mathcal{B}_l(\eta) \) generated by \( \sigma_1, \ldots, \sigma_{l-1} \) is isomorphic to the group algebra \( \mathbb{C}[S_l] \) so that \( \sigma_i \) can be identified with the transposition \((i, i+1)\). It is clear from the presentation that the algebra \( \mathcal{B}_l(\eta) \) is generated by \( \sigma_1, \ldots, \sigma_{l-1} \) and one of the elements \( e_i \). The following proposition is easy to verify. We put \( k = l-1 \) to make the formulas more readable.

**Proposition 2.2** The Brauer algebra \( \mathcal{B}_l(\eta) \) is isomorphic to the algebra with generators \( \sigma_1, \ldots, \sigma_{l-1}, e_{l-1} \) and the defining relations

\[
\begin{align*}
\sigma_i^2 &= 1, & \sigma_i \sigma_j &= \sigma_j \sigma_i, & \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, \\
e_k^2 &= \eta e_k, & \sigma_k e_k &= e_k \sigma_k = e_k, \\
e_k \sigma_{k-1} e_k &= e_k, & \sigma_i e_k &= e_k \sigma_i, & i = 1, \ldots, k-2, \\
e_k \tau e_k \tau &= \tau e_k \tau e_k,
\end{align*}
\]

where \( \tau = \sigma_{k-1} \sigma_{k-2} \sigma_k \sigma_{k-1} \).
Note that is the permutation \((k-2,k)(k-1,k+1)\) and so the last relation of Proposition 2.2 is equivalent to the relation \(e_{k-2}e_k = e_k e_{k-2}\) in the presentation of Proposition 2.1.

Suppose now that \(q \) and \(z\) are nonzero complex numbers.

**Definition 2.3** The algebra \(\mathcal{B}_l(z,q)\) is defined to be the algebra over \(\mathbb{C}\) with generators \(\sigma_1, \ldots, \sigma_{l-1}, e_{l-1}\) and the defining relations
\[
\sigma_i^2 = (q - q^{-1})\sigma_i + 1, \quad \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},
\]
\[
e_k^2 = \frac{z - z^{-1}}{q - q^{-1}} e_k, \quad \sigma_k e_k = e_k \sigma_k = q e_k,
\]
\[
e_k \sigma_{k-1} e_k = z e_k, \quad \sigma_i e_k = e_k \sigma_i, \quad i = 1, \ldots, k - 2,
\]
\[
e_k (zq \tau^{-1} + z^{-1} q^{-1} \tau) e_k (q \tau^{-1} + q^{-1} \tau) = (q \tau^{-1} + q^{-1} \tau) e_k (zq \tau^{-1} + z^{-1} q^{-1} \tau) e_k,
\]
where \(k = l - 1\) and \(\tau = \sigma_{k-1} \sigma_{k-2} \sigma_k \sigma_{k-1}\).

We have used the same symbols for the generators of the both algebras \(\mathcal{B}_l(z,q)\) and \(\mathcal{B}_l(\eta)\) since \(\mathcal{B}_l(z,q)\) becomes the Brauer algebra for the special case \(q = 1\) where \(z\) is chosen in such a way that the ratio \((z - z^{-1})/(q - q^{-1})\) takes value \(\eta\) at \(q = 1\). In particular, if \(\eta = n\) is a positive integer then \(\mathcal{B}_l(q^n, q)\) coincides with \(\mathcal{B}_l(n)\) for \(q = 1\).

The relations in the first line of Definition 2.3 are precisely the defining relation of the Iwahori–Hecke algebra \(\mathcal{H}_l(q)\). So we have a natural homomorphism
\[
\mathcal{H}_l(q) \rightarrow \mathcal{B}_l(z,q).
\]

(2.1)

Its injectivity for generic \(q\) (not a root of unity) can be deduced from the Schur–Weyl duality between \(U_q(\mathfrak{g}_n)\) and \(\mathcal{H}_l(q)\); see Section 4. We can therefore regard \(\mathcal{H}_l(q)\) as a subalgebra of \(\mathcal{B}_l(z,q)\).

The algebra \(\mathcal{B}_l(z,q)\) has a presentation analogous to the one given in Proposition 2.1. However, it does not seem to exist an obvious choice of a distinguished family of generators analogous to the \(e_i\). As an example of such a family we can take the elements \(e_i\) of the algebra \(\mathcal{B}_l(z,q)\) defined inductively by the formulas
\[
e_i = \sigma_{i+1} \sigma_i \sigma_{i+1}^{-1} \sigma_i^{-1}, \quad i = 1, \ldots, l - 2.
\]

(2.2)

Then the following relations are easily deduced from Definition 2.3:
\[
e_i^2 = \frac{z - z^{-1}}{q - q^{-1}} e_i, \quad \sigma_i e_i = e_i \sigma_i = q e_i, \quad i = 1, \ldots, l - 1,
\]
\[
\sigma_i e_j = e_j \sigma_i, \quad \sigma_i e_j = e_j \sigma_i, \quad |i - j| > 1, \quad e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} = e_{i+1},
\]
\[
e_i \sigma_{i+1} e_i = e_i \sigma_{i+1} e_i = z e_i, \quad e_i \sigma_{i+1}^{-1} e_i = e_i \sigma_{i+1}^{-1} e_i = z^{-1} e_i,
\]
\[
\sigma_i e_{i+1} e_i = z q \sigma_{i+1}^{-1} e_i, \quad e_{i+1} e_i \sigma_{i+1} = z q e_{i+1}^{-1} \sigma_i.
\]
The analogs of the Brauer algebra relations \( e_i e_j = e_j e_i \) where \(|i - j| > 1\) have a complicated form and we shall not write them down.

### 3 Quantized enveloping algebras

We shall use an \( R \)-matrix presentation of the algebra \( U_q(\mathfrak{gl}_n) \); see Jimbo \[8\] and Reshetikhin, Takhtajan and Faddeev \[17\]. As before, \( q \) is a nonzero complex number. Consider the \( R \)-matrix

\[
R = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i < j} E_{ij} \otimes E_{ji} \quad (3.1)
\]

which is an element of \( \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n \), where the \( E_{ij} \) denote the standard matrix units and the indices run over the set \( \{1, \ldots, n\} \). The \( R \)-matrix satisfies the Yang–Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (3.2)
\]

where both sides take values in \( \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n \) and the subindices indicate the copies of \( \text{End} \mathbb{C}^n \), e.g., \( R_{12} = R \otimes 1 \) etc.

The quantized enveloping algebra \( U_q(\mathfrak{gl}_n) \) is generated by elements \( t_{ij} \) and \( \bar{t}_{ij} \) with \( 1 \leq i, j \leq n \) subject to the relations

\[
t_{ij} = \bar{t}_{ji} = 0, \quad 1 \leq i < j \leq n,
\]

\[
t_{ii} \bar{t}_{ii} = t_{ii} t_{ii} = 1, \quad 1 \leq i \leq n,
\]

\[
R T_1 T_2 = T_2 T_1 R, \quad R \overline{T_1 T_2} = \overline{T_2 T_1} R, \quad R \overline{T_1 T_2} = T_2 \overline{T_1} R. \quad (3.3)
\]

Here \( T \) and \( \overline{T} \) are the matrices

\[
T = \sum_{i,j} E_{ij} \otimes t_{ij}, \quad \overline{T} = \sum_{i,j} E_{ij} \otimes \bar{t}_{ij}, \quad (3.4)
\]

which are regarded as elements of the algebra \( \text{End} \mathbb{C}^n \otimes U_q(\mathfrak{gl}_n) \). Both sides of each of the \( R \)-matrix relations in \((3.3)\) are elements of \( \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n \otimes U_q(\mathfrak{gl}_n) \) and the subindices of \( T \) and \( \overline{T} \) indicate the copies of \( \text{End} \mathbb{C}^n \) where \( T \) or \( \overline{T} \) acts.

We shall also use another \( R \)-matrix \( \tilde{R} \) given by

\[
\tilde{R} = q^{-1} \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1} - q) \sum_{i > j} E_{ij} \otimes E_{ji}. \quad (3.5)
\]

We have the relation

\[
\tilde{R} = P R^{-1} P, \quad (3.6)
\]
where
\[ P = \sum_{i,j} E_{ij} \otimes E_{ji} \] (3.7)
is the permutation operator.

The coproduct \( \Delta \) on \( U_q(\mathfrak{g}l_n) \) is defined by the relations
\[ \Delta(t_{ij}) = \sum_{k=1}^{n} t_{ik} \otimes t_{kj}, \quad \Delta(\bar{t}_{ij}) = \sum_{k=1}^{n} \bar{t}_{ik} \otimes \bar{t}_{kj}. \] (3.8)

The universal enveloping algebra \( U(\mathfrak{g}l_n) \) can be regarded as a limit specialization of \( U_q(\mathfrak{g}l_n) \) as \( q \rightarrow 1 \) so that
\[ \frac{t_{ij} - \delta_{ij}}{q - q^{-1}} \rightarrow E_{ij} \quad \text{for} \quad i \geq j \] (3.9)
and
\[ \frac{\bar{t}_{ij} - \delta_{ij}}{q - q^{-1}} \rightarrow -E_{ij} \quad \text{for} \quad i \leq j. \] (3.10)

Following Noumi [14] introduce the twisted quantized enveloping algebra \( U'_q(\mathfrak{o}_n) \) as the subalgebra of \( U_q(\mathfrak{g}l_n) \) generated by the matrix elements \( s_{ij} \) of the matrix \( S = T \bar{T}' \). More explicitly,
\[ s_{ij} = \sum_{a=1}^{n} t_{ia} \bar{t}_{ja}. \] (3.11)

It can be easily derived from (3.3) that the matrix \( S \) satisfies the relations
\[ s_{ij} = 0, \quad 1 \leq i < j \leq n, \] (3.12)
\[ s_{ii} = 1, \quad 1 \leq i \leq n, \] (3.13)
\[ R S_1 'R S_2 = S_2 'R S_1 R, \] (3.14)
where \( 'R \) denotes the element obtained from \( R \) by the transposition in the first tensor factor:
\[ 'R = q \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i<j} E_{ji} \otimes E_{ji}. \] (3.15)

It can be shown that (3.12)–(3.14) are defining relations for the algebra \( U'_q(\mathfrak{o}_n) \); see [15], [11]. A different presentation of \( U'_q(\mathfrak{o}_n) \) is given in the original paper by Gavrilik and Klimyk [4]. An isomorphism between the presentations is provided by Noumi [14].

As \( q \rightarrow 1 \) the algebra specializes to \( U(\mathfrak{o}_n) \) so that
\[ \frac{s_{ij}}{q - q^{-1}} \rightarrow E_{ij} - E_{ji} \quad \text{for} \quad i > j. \] (3.16)
4 Quantum Brauer duality

We start by recalling the well-known quantum analog of the Schur–Weyl duality between the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ and the Iwahori–Hecke algebra $\mathcal{H}_l(q)$; see [8], [10], [3, Chapter 10]. Consider the vector representation $U_q(\mathfrak{gl}_n) \to \text{End } \mathbb{C}^n$ of the algebra $U_q(\mathfrak{gl}_n)$ defined by

$$
t_{ii} \mapsto \sum_{a=1}^n q^{\delta_{ia}} E_{aa}, \quad \bar{t}_{ii} \mapsto \sum_{a=1}^n q^{-\delta_{ia}} E_{aa},$$

$$(4.1)$$

$$t_{ij} \mapsto (q - q^{-1}) E_{ij}, \quad i > j,$$

$$\bar{t}_{ij} \mapsto (q^{-1} - q) E_{ij}, \quad i < j.$$  

It will be convenient to interpret the representation in a matrix form. We shall regard it as the homomorphism

$$\text{End } \mathbb{C}^n \otimes U_q(\mathfrak{gl}_n) \to \text{End } \mathbb{C}^n \otimes \text{End } \mathbb{C}^n$$  

so that the images of the matrices $T$ and $\bar{T}$ are given by

$$T \mapsto \ 'R_{01}, \quad \bar{T} \mapsto \ '\bar{R}_{01},$$  

where we label the copies of $\text{End } \mathbb{C}^n$ in the tensor product $\text{End } \mathbb{C}^n \otimes \text{End } \mathbb{C}^n$ by the indices 0 and 1, respectively, and the left prime of the $R$-matrices denotes the transposition in the first tensor factor; see (3.13).

Using the coproduct (3.8) we consider the representation of $U_q(\mathfrak{gl}_n)$ in the tensor product space $(\mathbb{C}^n)^{\otimes l}$. In the matrix interpretation it takes the form

$$\text{End } \mathbb{C}^n \otimes U_q(\mathfrak{gl}_n) \to \text{End } \mathbb{C}^n \otimes \text{End } \mathbb{C}^n \otimes \cdots \otimes \text{End } \mathbb{C}^n,$$  

with

$$T \mapsto R_{01} \cdots R_{0l}, \quad \bar{T} \mapsto \bar{R}_{01} \cdots \bar{R}_{0l},$$  

where the multiple tensor product in (4.4) contains $l+1$ factors labelled by 0, 1, . . . , $l$.

We let $\tilde{R}$ denote the element of the algebra $\text{End } \mathbb{C}^n \otimes \text{End } \mathbb{C}^n$ defined by

$$\tilde{R} = PR = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + (q - q^{-1}) \sum_{i < j} E_{jj} \otimes E_{ii}. $$  

(4.6)

The mapping

$$\sigma_i \mapsto \tilde{R}_{i,i+1}, \quad i = 1, \ldots, l - 1$$  

defines a representation of the Iwahori–Hecke algebra $\mathcal{H}_l(q)$ in the space $(\mathbb{C}^n)^{\otimes l}$. If the parameter $q \in \mathbb{C}$ is generic (nonzero and not a root of unity) then the actions of
$U_q(\mathfrak{gl}_n)$ and $\mathcal{H}_l(q)$ on $(\mathbb{C}^n)^{\otimes l}$ are centralizers of each other. Moreover, if $l < n$ then $\mathcal{H}_l(q)$ is isomorphic to the centralizer of $U_q(\mathfrak{gl}_n)$.

Consider now the restriction of the $U_q(\mathfrak{gl}_n)$-module $(\mathbb{C}^n)^{\otimes l}$ to the subalgebra $U_q'(\mathfrak{o}_n)$. The action of the elements $s_{ij}$ is given by the formula

$$S \mapsto 'R_{01} \cdots 'R_{0l} \tilde{R}_{0l} \cdots \tilde{R}_{01}.$$ (4.8)

Clearly, the operators $\tilde{R}_{i,i+1}$ belong to the centralizer of the action of $U_q'(\mathfrak{o}_n)$. Introduce an operator $Q \in \text{End} \mathbb{C}^n \otimes \text{End} \mathbb{C}^n$ by

$$Q = \sum_{i,j=1}^n q^{n-2i+1} E_{ij} \otimes E_{ij}. \quad (4.9)$$

**Proposition 4.1** The operator $Q_{l-1,l}$ commutes with the action of $U_q'(\mathfrak{o}_n)$ on $(\mathbb{C}^n)^{\otimes l}$.

**Proof.** We need to verify that $Q_{l-1,l}$ commutes with the image of the matrix $S$ given in (4.8). It is sufficient to consider the case $l = 2$ since $Q_{l-1,l}$ commutes with $'R_{0i}$ and $\tilde{R}_{0i}$ for $i \leq l - 2$. The result will follow from the relations

$$Q_{12}'R_{01}'R_{02} \tilde{R}_{02} \tilde{R}_{01} = Q_{12} = 'R_{01}'R_{02} \tilde{R}_{02} \tilde{R}_{01}Q_{12}. \quad (4.10)$$

Let us prove the first equality. We verify directly that the operators $'R$ and $\tilde{R}$ commute with each other and so the left hand side of (4.10) equals $Q_{12}'R_{01}\tilde{R}_{02}'R_{02} \tilde{R}_{01}$. We also have $R_{20} = \tilde{R}_{02}^{-1}$ by (3.6) and

$$Q_{12}'R_{01} = Q_{12}R_{20}. \quad (4.11)$$

Therefore,

$$Q_{12}'R_{01} \tilde{R}_{02}'R_{02} \tilde{R}_{01} = Q_{12}'R_{02} \tilde{R}_{01} = Q_{12}. \quad (4.12)$$

The proof of the second equality in (4.10) is similar and follows from the relations

$$'R_{02} \tilde{R}_{01}Q_{12} = Q_{12} \quad \text{and} \quad 'R_{01} \tilde{R}_{02}Q_{12} = Q_{12}. \quad (4.13)$$

The following theorem establishes a quantum analog of the Brauer duality between the algebras $U_q'(\mathfrak{o}_n)$ and $\mathcal{B}_l(z, q)$ at $z = q^n$.

**Theorem 4.2** The mappings

$$\sigma_i \mapsto \tilde{R}_{i,i+1}, \quad i = 1, \ldots, l - 1 \quad (4.14)$$

and

$$e_{l-1} \mapsto Q_{l-1,l} \quad (4.15)$$

define a representation of the algebra $\mathcal{B}_l(q^n, q)$ on the space $(\mathbb{C}^n)^{\otimes l}$. Moreover, the actions of the algebras $U_q'(\mathfrak{o}_n)$ and $\mathcal{B}_l(q^n, q)$ on this space commute with each other.
Proof. The second statement follows from Proposition 1.1. We now verify the relations of Definition 2.3 with \( z = q^n \) for the operators \( \hat{R}_{i,i+1} \) and \( Q_{l-1,l} \). This is done by an easy calculation for all relations except for the last one whose proof is more involved. Clearly, in order to verify the latter we may assume that \( l = 4 \). We start by proving the following relations

\[
Q_{34} \hat{R}_{23} \hat{R}_{34} \hat{R}_{12} \hat{R}_{23} Q_{34} = Q_{12} Q_{34} + q^{n+1} (q - q^{-1}) Q_{34} (\hat{R}_{12} + I q^{-1}) \tag{4.16}
\]

and

\[
Q_{34} \hat{R}^{-1}_{23} \hat{R}^{-1}_{34} \hat{R}^{-1}_{12} \hat{R}^{-1}_{23} Q_{34} = Q_{12} Q_{34} + q^{-n-1} (q^{-1} - q) Q_{34} (\hat{R}_{12} + I q^{-1}) \tag{4.17}
\]

where \( I \) is the identity operator. Replacing \( \hat{R} \) by \( PR \) we can rewrite the left hand side of (4.16) as

\[
P_{13} P_{24} Q_{12} R_{14} R_{24} R_{13} R_{23} Q_{34}. \tag{4.18}
\]

Introducing the diagonal matrix

\[
D = \sum_{i=1}^{n} q^{n-2i+1} E_{ii} \in \text{End} \, \mathbb{C}^n, \tag{4.19}
\]

we can present the operator \( Q_{12} \in \text{End} \, \mathbb{C}^n \otimes \text{End} \, \mathbb{C}^n \) as

\[
Q_{12} = D_1 \overline{Q}_{12} = D_2 \overline{Q}_{12}, \tag{4.20}
\]

where

\[
\overline{Q}_{12} = \sum_{i,j=1}^{n} E_{ij} \otimes E_{ij} = 'P_{12} = P_{12}', \tag{4.21}
\]

where for any operator

\[
A = \sum_{i,j,r,s} a_{ijrs} E_{ij} \otimes E_{rs} \in \text{End} \, \mathbb{C}^n \otimes \text{End} \, \mathbb{C}^n \tag{4.22}
\]

we denote by \( 'A \) and \( A' \) the corresponding transposed operators with respect to the first or second tensor factor:

\[
' A = \sum_{i,j,r,s} a_{ijrs} E_{ji} \otimes E_{rs} \tag{4.23}
\]

and

\[
A' = \sum_{i,j,r,s} a_{ijrs} E_{ij} \otimes E_{sr}. \tag{4.24}
\]

We have

\[
Q_{12} R_{14} = Q_{12} ' R_{24}, \quad R_{23} Q_{34} = D_4 R_{24}' \overline{Q}_{34}. \tag{4.25}
\]

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Now the operator (4.18) takes the form
\[ P_{13} P_{24} Q_{12} R_{13} ' R_{24} R_{24} D_4 R_{24}' Q_{34}. \] (4.26)

Multiplying directly the matrices, we get
\[ ' R_{24} R_{24} D_4 R_{24}' = R_{24} D_4 + q^{n+1} (q - q^{-1}) Q_{24}. \] (4.27)

Furthermore,
\[ Q_{12} R_{13} = Q_{12} ' R_{23}, \quad Q_{24} Q_{34} = P_{23} Q_{34}, \quad R_{24} D_4 Q_{34} = D_3 R_{23}' Q_{34}. \] (4.28)

The operator (4.26) now becomes
\[ P_{13} P_{24} Q_{12} X_{23} Q_{34}. \] (4.29)

Performing another multiplication, we write this as
\[ P_{13} P_{24} Q_{12} X_{23} Q_{34}. \] (4.30)

with
\[ X_{23} = D_3 + q^{n+1} (q - q^{-1})(q^{-1} Q_{23} + ' R_{23} P_{23}). \] (4.31)

Furthermore,
\[ P_{13} P_{24} Q_{12} X_{23} Q_{34} = Q_{34} P_{13} P_{24} X_{23} Q_{34} = Q_{34} P_{13} Y_{23}' Q_{34}. \] (4.32)

where \( Y_{23} = P_{23} X_{23}' \). Finally, since
\[ P_{13} Y_{23}' = Y_{21}' P_{13} \quad \text{and} \quad Q_{34} P_{13} Q_{34} = Q_{34}, \] (4.33)

the left hand side of (4.16) takes the form \( Y_{21}' Q_{34} \) where
\[ Y_{21}' = Q_{12} + q^{n+1} (q - q^{-1})(\tilde{R}_{12} + I q^{-1}) \] (4.34)

thus completing the proof of (4.16).

Without giving all the details, we note for the proof of (4.17) that \( \tilde{R}_{12}^{-1} = \tilde{R}_{21} P_{12} \) by (3.6) and so the left hand side of the relation can be written as
\[ Q_{34} \tilde{R}_{32} \tilde{R}_{42} \tilde{R}_{31} \tilde{R}_{41} Q_{12} P_{13} P_{24}. \] (4.35)

Then we proceed in the same manner as for the proof of (4.16) modifying the argument appropriately.
Now, (4.16) and (4.17) give
\[ Q_{34}(q^{-n-1}\tilde{R}_{23}\tilde{R}_{34}\tilde{R}_{12}\tilde{R}_{23} + q^{n+1}\tilde{R}^{-1}_{23}\tilde{R}^{-1}_{34}\tilde{R}^{-1}_{12}\tilde{R}^{-1}_{23})Q_{34} = (q^{-n-1} + q^{n+1})Q_{12}Q_{34}. \]
The proof of the theorem will be completed by verifying the following relations
\[ Q_{12}Q_{34}(q^{-1}\tilde{R}_{23}\tilde{R}_{34}\tilde{R}_{12}\tilde{R}_{23} + q \tilde{R}^{-1}_{23}\tilde{R}^{-1}_{34}\tilde{R}^{-1}_{12}\tilde{R}^{-1}_{23}) = (q^{-3} + q^3)Q_{12}Q_{34} \tag{4.36} \]
and
\[ (q^{-1}\tilde{R}_{23}\tilde{R}_{34}\tilde{R}_{12}\tilde{R}_{23} + q \tilde{R}^{-1}_{23}\tilde{R}^{-1}_{34}\tilde{R}^{-1}_{12}\tilde{R}^{-1}_{23})Q_{12}Q_{34} = (q^{-3} + q^3)Q_{12}Q_{34}. \tag{4.37} \]
The arguments are similar for both relations, so we only give a proof of (4.36). Since \(Q_{12}Q_{34} = D_1D_3\overline{Q}_{12}\overline{Q}_{34}\) we may replace \(Q_{12}Q_{34}\) with \(\overline{Q}_{12}\overline{Q}_{34}\) in (4.36). We have
\[ \overline{Q}_{12}\overline{Q}_{34}\tilde{R}_{23}\tilde{R}_{34}\tilde{R}_{12}\tilde{R}_{23} = \overline{Q}_{12}\overline{Q}_{34}P_{13}P_{24}R_{14}R_{24}R_{13}R_{23}. \tag{4.38} \]
Furthermore,
\[ \overline{Q}_{12}\overline{Q}_{34}P_{13}P_{24} = \overline{Q}_{12}P_{13}P_{24}\overline{Q}_{12} = \overline{Q}_{12}\overline{Q}_{34}P_{24}\overline{Q}_{12} = \overline{Q}_{12}P_{24}\overline{Q}_{12}\overline{Q}_{34} = \overline{Q}_{12}\overline{Q}_{34}. \tag{4.39} \]
The right hand side of (4.38) now takes the form \(\overline{Q}_{12}\overline{Q}_{34}R_{14}R_{24}R_{13}R_{23}\). Using the relations
\[ \overline{Q}_{12}R_{14} = \overline{Q}_{12}'R_{24}, \quad \overline{Q}_{12}\overline{Q}_{34}R_{13} = \overline{Q}_{12}\overline{Q}_{34}'R_{14}' = \overline{Q}_{12}\overline{Q}_{34}'R_{24}', \tag{4.40} \]
we can write it as
\[ \overline{Q}_{12}\overline{Q}_{34}V_{23}R_{23} = \overline{Q}_{12}\overline{Q}_{34}'V_{23}'R_{23}, \tag{4.41} \]
where
\[ V_{23} = 'R_{23}'R_{23}R_{23}. \tag{4.42} \]
Now we transform the second summand on the left hand side of (4.36) in a similar manner. Since \(\tilde{R}^{-1} = P\tilde{R}\) we have
\[ \overline{Q}_{12}\overline{Q}_{34}\tilde{R}_{23}^{-1}\tilde{R}_{34}^{-1}\tilde{R}_{12}^{-1}\tilde{R}_{23}^{-1} = \overline{Q}_{12}\overline{Q}_{34}\tilde{R}_{14}\tilde{R}_{24}\tilde{R}_{13}\tilde{R}_{23}. \tag{4.43} \]
Exactly as above, we write this operator in the form
\[ \overline{Q}_{12}\overline{Q}_{34}W_{23}'\tilde{R}_{23}, \tag{4.44} \]
where
\[ W_{23} = 'R_{23}'R_{23}R_{23}. \tag{4.45} \]
A direct calculation shows that
\[ q^{-1}V'R + qW'\tilde{R} = (q^{-3} + q^3)I, \tag{4.46} \]
completing the proof.
References

[1] J. Birman and H. Wenzl, *Braids, link polynomials and a new algebra*, Trans. AMS 313 (1989), 249–273.

[2] R. Brauer, *On algebras which are connected with the semisimple continuous groups*, Ann. Math. 38 (1937), 857–872.

[3] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, 1994.

[4] A. M. Gavrilik and A. U. Klimyk, *q-deformed orthogonal and pseudo-orthogonal algebras and their representations*, Lett. Math. Phys. 21 (1991), 215–220.

[5] A. M. Gavrilik and N. Z. Iorgov, *On Casimir elements of q-algebras U_q'(so_n) and their eigenvalues in representations*, in ‘Symmetry in nonlinear mathematical physics’, Proc. Inst. Mat. Ukr. Nat. Acad. Sci. 30, Kyiv, 1999, pp. 310–314.

[6] T. Halverson and A. Ram, *Characters of algebras containing a Jones basic construction: the Temperley–Lieb, Okada, Brauer, and Birman–Wenzl algebras*, Adv. Math. 116 (1995), 263–321.

[7] M. Havlíček, A. U. Klimyk and S. Pošta, *Central elements of the algebras U_q'(so_m) and U_q(iso_m)*, Czechoslovak J. Phys. 50 (2000), 79–84.

[8] M. Jimbo, *A q-analogue of U_q(gl(N + 1)), Hecke algebra and the Yang–Baxter equation*, Lett. Math. Phys. 11 (1986), 247–252.

[9] A. U. Klimyk, *Classification of irreducible representations of the q-deformed algebra U_q'(so_n)*, preprint [math.QA/0110038](http://www.math.ucf.edu/~mathphys/).

[10] R. Leduc and A. Ram, *A ribbon Hopf algebra approach to the irreducible representations of centralizer algebras: the Brauer, Birman–Wenzl, and type A Iwahori–Hecke algebras*, Adv. Math. 125 (1997), 1–94.

[11] A. I. Molev, E. Ragoucy and P. Sorba, *Coideal subalgebras of quantum affine algebras*, preprint.

[12] J. Murakami, *The representations of the q-analogue of Brauer’s centralizer algebras and the Kauffman polynomial of links*, Publ. Res. Inst. Math. Sci. 26 (1990), 935–945.

[13] M. Nazarov, *Young’s orthogonal form for Brauer’s centralizer algebra*, J. Algebra 182 (1996), 664–693.
[14] M. Noumi, *Macdonald’s symmetric polynomials as zonal spherical functions on quantum homogeneous spaces*, Adv. Math. 123 (1996), 16–77.

[15] M. Noumi, T. Umeda and M. Wakayama, *Dual pairs, spherical harmonics and a Capelli identity in quantum group theory*, Compos. Math. 104 (1996), 227–277.

[16] A. Ram and H. Wenzl, *Matrix units for centralizer algebras*, J. Algebra 145 (1992), 378–395.

[17] N. Yu. Reshetikhin, L. A. Takhtajan and L. D. Faddeev, *Quantization of Lie Groups and Lie algebras*, Leningrad Math. J. 1 (1990), 193–225.

[18] H. Wenzl, *On the structure of Brauer’s centralizer algebras*, Ann. Math. 128 (1988), 173–193.