Detection of a moving rigid solid in a perfect fluid

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Abstract
In this paper, we consider a moving rigid solid immersed in a potential fluid. The fluid–solid system fills the whole two-dimensional space and the fluid is assumed to be at rest at infinity. Our aim is to study the inverse problem, initially introduced in Conca \textit{et al} (2008 Inverse Problems \textbf{24} 045001), that consists in recovering the position and the velocity of the solid assuming that the potential function is known at a given time. We show that this problem is in general ill-posed by providing counterexamples for which the same potential corresponds to different positions and velocities of a same solid. However, it is also possible to find solids having a specific shape, such as ellipses for instance, for which the problem of detection admits a unique solution. Using complex analysis, we prove that the well-posedness of the inverse problem is equivalent to the solvability of an infinite set of nonlinear equations. This result allows us to show that when the solid enjoys some symmetry properties, it can be partially detected. Besides, for any solid, the velocity can always be recovered when both the potential function and the position are supposed to be known. Finally, we prove that by performing continuous measurements of the fluid potential over a time interval, we can always track the position of the solid.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

1.1. History
Sonars are the most common devices used to spot immersed bodies, such as submarines or banks of fish. These systems use acoustic waves: active sonars emit acoustic waves (making themselves detectable), while passive sonars only listen (and therefore are only able to detect targets that are noisy enough). To overcome these limitations, it would be interesting to design
systems imitating the lateral line systems of fish, a sense organ they use to detect movement and vibration in the surrounding water.

Most of the published results on inverse problems in fluid mechanics concern the detection of fixed immersed obstacles. For example in [1] the authors prove that a fixed smooth convex obstacle surrounded by a fluid governed by the Navier–Stokes equations can be identified via a localized boundary measurement of the velocity of the fluid and the Cauchy forces. In [4], the authors identify a single rigid obstacle immersed in a Navier–Stokes fluid by measuring both the gradient of the pressure and the velocity of the fluid on one part of the boundary. The distance from a given point to an obstacle is estimated in [5] from boundary measurements for a fluid governed by the stationary Stokes equations.

To our knowledge, the only work addressing the detection of moving bodies is [3]. In this paper, the authors consider a single moving disk in an ideal fluid and prove that the position and velocity of the body can be deduced from one single measurement of the potential along some part of the exterior boundary of the fluid. They obtain linear stability results as well, by using shape differentiation techniques.

1.2. Problem settings

1.2.1. Domains, frames, coordinates. At a given time $t$, we assume that a rigid solid occupies the domain $S \subset \mathbb{R}^3$, while the domain $\mathcal{F} := \mathbb{R}^3 \setminus S$ is filled by a perfect fluid. Let us assume that $S$ is a simply connected compact set. The unit normal vector to $\partial \mathcal{F}$ directed toward the exterior of $\mathcal{F}$ is denoted by $n$. As being a rigid solid, $S$ is the image by a rotation and a translation of a given reference domain $S_0$ which will be called the shape of the solid in the following. Therefore, at any time, there exist an angle $\alpha \in \mathbb{R}/2\pi$, a rotation matrix $R(\alpha) \in SO(2)$ of angle $\alpha$, a point $s := (s_1, s_2)^T \in \mathbb{R}^2$ (the center of the rotation) and a vector $r := (r_1, r_2)^T$ of $\mathbb{R}^3$ such that $S = R(\alpha)(S_0 - s) + r$. We will be concerned with recovering the position of the solid, so it is worth remarking that the triplet $(\alpha, s, r)$ is not unique. Two triplets $(\alpha_j, s_j, r_j) \in \mathbb{R}/2\pi \times \mathbb{R}^2 \times \mathbb{R}^2$ $(j = 1, 2)$ give the same position for any $S_0$ if and only if $R(\alpha_1) = R(\alpha_2) = R$ and $R(s_1 - s_2) = r_1 - r_2$. These equalities define an equivalence relation in $\mathbb{R}/2\pi \times \mathbb{R}^2 \times \mathbb{R}^2$. However, we would also like to take into account the possible symmetries of the solid. So, given $S_0$, we say that two triplets $(\alpha_j, s_j, r_j)$ are equivalent when $R(\alpha_1)(S_0 - s_1) + r_1 = R(\alpha_2)(S_0 - s_2) + r_2$. We denote by $\mathcal{P}$ the set of all of the equivalence class $\mathcal{P}$. We will make no difference in the notation between $p$ and any element $(\alpha, s, r)$ belonging to this class. In particular, we will write in short that for any $x \in \mathbb{R}^3$, $px = R(\alpha)(x - s) + r$. In the following, $p$ will be merely referred to as position of the solid.

Later on, we will use tools of complex analysis, so rather than $\mathbb{R}^2$, we will sometimes identify the plane with the complex field $\mathbb{C}$. For any complex number $z := z_1 + i z_2$ ($i^2 = -1$, $z_1, z_2 \in \mathbb{R}$), we will denote by $\bar{z} := z_1 - i z_2$ the conjugate of $z$ and $D$ will stand for the unit disk of $\mathbb{C}$.

1.2.2. Sequences of complex numbers. For any sequence of complex numbers $c := (c_k)_{k \in \mathbb{Z}}$, we can define $\tilde{c} := (\tilde{c}_k)_{k \in \mathbb{Z}}$ and $\overline{c} := (\overline{c}_k)_{k \in \mathbb{Z}}$. For any two sequences $a := (a_k)_{k \in \mathbb{Z}}$ and $b := (b_k)_{k \in \mathbb{Z}}$, we recall the definition of the convolution product: $a * b := \{ \sum_{j \in \mathbb{Z}} a_{k-j} b_j \}_{k \in \mathbb{Z}}$. The convolution product can be iterated $n$ times ($n$ an integer) to obtain $a^n := a * a * \cdots * a$.

1.2.3. Rigid velocity. The solid is moving. We denote by $v(x) := (v_1(x), v_2(x))^T \in \mathbb{R}^2$ the rigid Eulerian velocity field defined for all $x \in S$. This notation turns out to be $v(z) := v_1(z) + i v_2(z)$ in complex notation. It is well known in solid mechanics that $v$ can be decomposed into the sum of an instantaneous rotational velocity field and a translational
velocity field. For any \( x := (x_1, x_2)^T \in \mathbb{R}^2 \), we introduce the notation \( x^+ := (-x_2, x_1)^T \) and we have \( v(x) = \omega(x - s)^T + w \), where \( s \in \mathbb{R}^2 \) is the center of the instantaneous rotation, \( \omega \in \mathbb{R} \) is the angular velocity and \( w := (w_1, w_2)^T \in \mathbb{R}^2 \) is the translational velocity. Since we wish to recover these data, it is worth observing that the triplet \( (\omega, s, w) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \) is not unique. Both triplets \( (\omega_1, s_j, w_j) \) \( (j = 1, 2) \) give the same rigid velocity field \( v \) if and only if \( \omega_1 = \omega_2 = \omega \) and \( \omega(s_1^+ - s_2^+) = w_1 - w_2 \) (in particular, we can always choose for \( s \) any point of \( \mathbb{R}^2 \)). This is an equivalence relation and the velocity \( v \) can be seen as an equivalence class.

We denote by \( \mathcal{V} \) the set of all of the equivalence classes and we will not differentiate, in what follows, between the vector field, the class of equivalence and any element of this class. All of them will be denoted by \( v \).

**Definition 1.1** (Configurations). For any given shape \( \mathcal{S}_0 \), we define a configuration as any position–velocity pair \( (p, v) \in \mathcal{P} \times \mathcal{V} \).

1.2.4. Fluid dynamics. The dynamics of the fluid is described by means of its Eulerian velocity field \( u(x) := (u_1(x), u_2(x))^T \) defined for all \( x \in \mathcal{F} \). Since the fluid is assumed to be perfect (i.e. incompressible and inviscid) and the flow irrotational, there exists a potential function \( \varphi \), harmonic in \( \mathcal{F} \), such that \( u(x) = \nabla \varphi(x) \) \( (x \in \mathcal{F}) \). The fluid is assumed to be at rest at infinity so we impose the asymptotic behavior \( |\nabla \varphi(x)| \rightarrow 0 \) as \( |x| \rightarrow +\infty \). The classical slip boundary condition for inviscid fluid reads \( u \cdot n = v \cdot n \) on \( \partial \mathcal{S} \) and yields a Neumann boundary condition for \( \varphi \), namely \( \partial_n \varphi = v \cdot n \) on \( \partial \mathcal{S} \). Although the domain \( \mathcal{F} \) is not simply connected, we can still consider \( \varphi \), the harmonic conjugate function to \( \psi \), because \( \int_{\partial \mathcal{S}} \partial_n \varphi \, ds = 0 \). The functions \( \varphi \) and \( \psi \) satisfy the relation \( \nabla \varphi = (\nabla \psi)^T \) in \( \mathcal{F} \). In fluid mechanics, \( \psi \) is called the stream function and the complex function \( \xi = \varphi + i\psi \) is the holomorphic complex potential. As usual, we define \( u := u_1 + iu_2 = \xi' \) as the complex fluid velocity. Observe that the complex potential, as being the solution of a boundary value problem, depends on the domain \( \mathcal{F} \) and the velocity \( v \) only. With the notation introduced earlier, we deduce that \( \xi \) depends only on the shape \( \mathcal{S}_0 \) and the configuration \( (p, v) \in \mathcal{P} \times \mathcal{V} \).

The complex potential is defined up to an additive constant which can be chosen such that \( |\xi(z)| \rightarrow 0 \) as \( |z| \rightarrow +\infty \). For any \( v \in \mathbb{C} \), the complex potential can be expanded in the form of a Laurent series:

\[
\xi(z) := \sum_{j \geq 1} \frac{\lambda_j(v)}{(z - v)^j}, \quad |z - v| > R(v),
\]

where \( \lambda_j(v) \) \( (j \geq 1) \) are complex numbers and \( R(v) := \limsup_{j \rightarrow +\infty} |\lambda_j(v)|^{1/j} \). The series is uniformly convergent on the set \( \{z \in \mathbb{C} : |z - v| > R(v)\} \).

1.2.5. Measurements. We measure the complex velocity \( u \) of the fluid in some open subset of \( \mathcal{F} \). The analytic continuation theorem states that we can deduce the value of \( \xi' \) everywhere in the connected open set \( \mathcal{F} \) and then also the value of \( \xi \), up to an additive constant. In particular, we will assume that for all \( v \in \mathbb{C} \), we can always evaluate all of the terms of the complex sequence \( (\lambda(v))_{j \geq 1} \) arising in expression (1.1).

1.3. Main results

**Definition 1.2** (Detectability). A solid of shape \( \mathcal{S}_0 \) is said to be detectable if, for any configuration \( (p, v) \in \mathcal{P} \times \mathcal{V} \), the knowledge of the holomorphic potential function \( \xi \) suffices for recovering the pair \( (p, v) \).
Observe that this definition makes the property of being detectable independent of the configuration: detectability is a purely geometric property of the solid. Our first result is that not all the solids are detectable.

**Theorem 1.3.** For any integer \( n \geq 2 \), there exists a holomorphic function \( \xi \), a shape \( S_0 \), and \( n \) configurations \((p_j, v_j) \in \mathcal{P} \times \mathcal{V}, j = 1, \ldots, n\), satisfying \( p_j \neq p_k \) if \( j \neq k \) such that \( \xi \) is the potential of the fluid corresponding to the solid of shape \( S_0 \) with any of the configurations \((p_j, v_j), j = 1, \ldots, n\).

In other words, for any integer \( n \), there exists at least one solid that can occupy \( n \) different positions with \( n \) different velocities and for which the fluid potential is the same. This theorem shows that the result obtained in [3] for a disk cannot be generalized to any solid. However, the disk is not the only detectable body.

**Proposition 1.1.** Any ellipse is a detectable solid.

Going back to the general case, it is easy to see that the holomorphic potential never admits an analytic continuation over the whole complex plane. Furthermore, for any analytic continuation of the potential inside the solid, we will prove that the location of the singularities provides clues allowing one in many cases to determine the position of the solid. This discussion is carried out in subsection 6.1.

According to theorem 1.3, the problem of detection is ill-posed in the general case. However, we claim that when the solid enjoys some symmetry properties, it can be partially detected (i.e. some but not all of the parameters among \( r, \alpha, w, \omega \) can be deduced from the potential). The following proposition illustrates this idea.

**Proposition 1.2.** If the shape of the solid is invariant under a rotation of angle \( \pi/2 \), then \( r, w \) and \( |\omega| \) can be deduced from the potential function.

We refer to propositions 6.5 and 6.6 for a more precise statement of this result.

In a general case, we can also try to determine less parameters with more information. For instance, we can prove

**Proposition 1.3.** For any solid with configuration \((p, v) \in \mathcal{P} \times \mathcal{V}\), the knowledge of both the potential function and the position \( p \) suffices for recovering \( v \).

Finally, we can also measure the potential function, not only at a given instant, but over a time interval. In this case, we obtain

**Theorem 1.4** (Tracking). For any solid \( S_0 \) if we know its position at the time \( t = 0 \) and we perform continuous measurements of the complex potential over the time interval \([0, T]\) for some \( T > 0 \), then we can deduce the configuration of the solid at any time \( t \in [0, T] \).

### 1.4. Outline of the paper

In the next section, we provide examples of non-detectable solids and prove theorem 1.3. In section 3, we derive the expression of the complex potential. In section 4, we determine all the *stealth* solids, i.e. all the solids that can move in the fluid without disturbing it. The detection of a moving ellipse is discussed in section 5. Section 6 is split into three parts: the first one is dedicated to the study of the singularities of the potential function and the second to its asymptotic expansion and how these results can be used for the detection problem we are dealing with. The third part deals with an example of detection. In section 7 we give the proof of theorem 1.7 and at last in section 8, we indicate some remaining open problems.
2. Examples of non-detectable solids

This section is devoted to the proof of theorem 1.3.

2.1. Expression of the stream function

Let a shape $S_0$ and a configuration $(p, v)$ be given with $v = \omega(x - s)^2 + w$ (for some real number $\omega$ and some vector $s$) and remember that $S = p(S_0)$. Then, let us introduce $\gamma : [0, \ell] \rightarrow \gamma(s) = (\gamma_1(s), \gamma_2(s))^T \in \mathbb{R}^2$ a parameterization of $\partial S$ satisfying $|\gamma'(s)| = 1$ for all $s \in [0, \ell]$ ($\ell > 0$). We assume that $\partial S$ is described positively (counterclockwise parameterization), we denote $\tau = \gamma'$ (the unit tangent vector to $\partial S$) and we obtain $n = \tau \perp$. We deduce that $\partial_n \phi = -\partial_\tau \psi$ and hence that $\partial_\tau \psi(\gamma) = -w_1\gamma_2' + w_2\gamma_1' + \omega \gamma' \cdot (\gamma - s)$. We can integrate along $\partial S$ to obtain $\psi(\gamma) = -w_1\gamma_2 + w_2\gamma_1 + (\omega/2)|\gamma - s|^2 + C$ on $\partial S$, where $C$ is the real constant. This Dirichlet boundary condition for the stream function also reads: $\psi(x) = -w_1x_2 + w_2x_1 + (\omega/2)|x - s|^2 + C$ on $\partial S$. In this form, the boundary of the solid turns out to be a level set of the function $g(x) := (\omega/2)|x - s|^2 - w_1x_2 + w_2x_1 - \psi(x)$, an observation we will now take advantage of.

2.2. Proof of theorem 1.3

Pick some integer $n \geq 2$ and consider the harmonic function whose expression in polar coordinates is $\psi(r, \theta) := \cos(n\theta)r^{-n}$. Since, in Cartesian coordinates, $|(\partial \psi/\partial x_j)(x)| \leq |\nabla \psi(x)| = n|x|^{n-1}$ ($j = 1, 2$), we deduce that for any $\omega > 0$ and $s \in \mathbb{R}^2$, there exists $\delta > 0$ such that $|\partial\psi/\partial x_1|(x) - \omega(x_1 - s_1)$ and $|\partial\psi/\partial x_2|(x) - \omega(x_2 - s_2)$ cannot be simultaneously null providing $|x - s| > \delta$. Applying the local inversion theorem, we deduce that for any $\lambda \in \mathbb{R}$, the solutions of

$$\frac{\omega}{2}|x - s|^2 - \psi(x) - \lambda = 0,$$

satisfying $|x - s| > \delta$ (if any) are locally smooth curves.

From the estimate $|\psi(x)| \leq |x|^{-n}$ ($x \in \mathbb{R}^2$), we deduce that for all $s := (s_1, s_2)^T$ and all $\omega > \varepsilon > 0$, there exists $\delta' > 0$ such that

$$\frac{\omega - \varepsilon}{2}|x - s|^2 - \psi(x) - \lambda \leq \frac{\omega}{2}|x - s|^2 - \psi(x) - \lambda \leq \frac{\omega + \varepsilon}{2}|x - s|^2 - \lambda$$

for all $\lambda \in \mathbb{R}$ providing $|x - s| > \delta'$. If we choose for instance $\lambda = \max(\delta', \delta)^2(\omega + \varepsilon)$, there is a zero level set of the function $g(x) := \omega|x - s|^2/2 - \psi(x) - \lambda$ between the circles $|x - s| = \sqrt{2\lambda}/\sqrt{\omega + \varepsilon}$ and $|x - s| = \sqrt{2\lambda}/\sqrt{\omega + \varepsilon}$ (because $\sqrt{2\lambda}/\sqrt{\omega + \varepsilon} \geq \sqrt{2}\delta > \delta$). It remains to choose $s$ properly, in order to take advantage of the symmetry of the function $\psi$. Let $\rho$ be any positive number and denote by $S_1$ the zero level set of $g$ obtained as described above by specifying $s_1 := (\rho, 0)$. This level set defines the smooth boundary of a solid for which $\psi$ is the stream function (and $\xi(z) := i/z^n$ the holomorphic potential) associated with the velocity $v_1 := \omega(x - s_1)\perp$. Next by choosing $s_k := (\rho \cos(2(k - 1)\pi/n), \rho \sin(2(k - 1)\pi/n))^T$ for $k = 2, \ldots, n$, we obtain $n - 1$ copies of $S_1$ at $n - 1$ different positions with respective velocities $v_k := \omega(x - s_k)\perp$. Some examples of such solids with the associated rigid velocity fields are shown in figures 1, 2 and 3.

3. The complex potential

Before proceeding further, we need to describe the shape $S_0$. Actually, for convenience, rather than $S_0$ we shall describe $\mathcal{F}_0 := C \setminus S_0$. Thus, assume that $\mathcal{F}_0$ is the image by a conformal
Figure 1. For both configurations, the stream function is the same. It reads $\psi(r, \theta) = \cos(2\theta)/r^2$ in polar coordinates. The holomorphic potential is $\xi(z) = i/z^2$.

Figure 2. For both configurations, the stream function and the holomorphic potential are the same as in figure 1.

mapping $f$ of $\Omega := C \setminus \bar{D}$, the exterior of the unit disk. For any simply connected shape $S_0$ and corresponding domain $F_0$, the Riemann mapping theorem states that $f$ can be written in the form

$$f(z) = c_1 z + c_0 + \sum_{k \leq -1} c_k z^k, \quad (z \in \Omega),$$

(3.1)

where $c_k \in C$ for $k = 1$ and all $k \leq -1$ and $c_1 \neq 0$. We can assume, without loss of generality, that $c_0 = 0$. To simplify forthcoming computations, we will also assume that $c_k$ is actually defined for all $k \in \mathbb{Z}$ and that $c_k = 0$ for $k = 0$ and $k \geq 2$. We denote by $c := (c_k)_{k \in \mathbb{Z}}$ the complex sequence of elements $c_k$ and the area theorem (see [7, theorem 14.13]) states that the area of $S_0$ is equal to $\pi \sum_{k \leq 1} k |c_k|^2$. Since $S_0$ is of finite extent, it means that this sum has to
Figure 3. The stream function is \( \psi(r, \theta) = \cos(6\theta)/r^6 \), the holomorphic potential is \( \xi = \frac{i}{z^6} \) and \( \omega = 0.7, \rho = 0.9, \lambda = -2.5 \) and \( s_k = (\rho \cos(2k\pi/6), \rho \sin(2k\pi/6)) \) for \( k = 1, \ldots, 6 \).

Figure 4. Level sets of the holomorphic potential \( \xi_0 \), for \( a = 2, b = 1, w_0 = e^{\pi/3} \) and \( \omega = -2 \). The boundary of the ellipse (dashed line) can hardly be directly detected but the branch points \(-\sqrt{3}\) and \(\sqrt{3}\) are clearly identifiable. (a) Level sets of the potential function \( \phi_0 = \Re(\xi_0) \); (b) level sets of the stream function \( \psi_0 = \Im(\xi_0) \).
be finite. Actually, we will also assume that $c \in \ell^1 (C)$, which entails in particular that $f$ is continuous in the closed set $\bar{\Omega}$.

Such a description allows us to consider a broad set of solids. In particular, the boundary of the solid can be very rough. Degenerated cases can be considered as well (for instance $S_0$ can be a segment modeling a one-dimensional beam).

For any position $p := (a, 0, r)$, we recall that $S := R(\alpha)S_0 + r$ is the actual domain occupied by the solid. Let us introduce then the functions $\varphi_0(x) := \varphi(R(\alpha)x + r)$ and $\psi_0(x) := \psi(R(\alpha)x + r)$ which are harmonic (and defined) over the fixed domain $F_0$. For any velocity $v := (\omega, r, w)$ (we choose here $s = r$), the Dirichlet boundary condition for $\psi$ turns out to be in complex notation $2i\psi_0 := w_0 \bar{z} - w_0 z + i|z|^2$ where $w_0(z) := w(R(\alpha)z + r)$. We introduce $\xi$ as the holomorphic complex potential of the fluid defined for any $z \in \Omega$ by $\xi(z) = \varphi_0(f(z)) + i\psi_0(f(z))$. Since $\xi = 1/z$ on $\partial \Omega$, we get the identity $2i\psi_0(f(z)) = -\bar{w}f(z) + w\bar{f}(1/z) + i\alpha f(z)f(1/z)$. For any $z \in \partial \Omega$, we also have $f(1/z) = \sum_{k \in \mathbb{Z}} \xi_k z^k$ and $f(z)\bar{f}(1/z) = \sum_{k \in \mathbb{Z}} (\bar{\xi}_k + c_k)z^k$. So we get

$$2i\psi_0(f(z)) = \sum_{k \in \mathbb{Z}} [-\bar{w}_0 \xi_k + w_0 \bar{\xi}_k + i\alpha (\bar{c}_k + c_k)]z^k, \quad (z \in \partial \Omega).$$

(3.2)

According to [6, chapter IX, section 9.63], we keep only the negative powers in (3.2) to obtain the expression of $\xi$. Defining the coefficients $\xi_k(w_0, \omega) := [-\bar{w}_0 \xi_k + w_0 \bar{\xi}_k + i\alpha (\bar{c}_k + c_k)]$ for all $k \leq -1$, we obtain

$$\xi(z) = \sum_{k \leq -1} \xi_k(w_0, \omega)z^k, \quad (z \in \Omega).$$

(3.3)

Eventually, the expression of the measured complex potential, defined in $F$, is

$$\xi(z) = \xi(f^{-1}(z - r)e^{-i\omega}), \quad (z \in F).$$

(3.4)

According to our rule of notation, we also introduce

$$\xi_0(z) = \varphi_0(z) + \psi_0(z) = \xi(f^{-1}(z)), \quad (z \in F_0).$$

(3.5)

4. Stealth rigid solids

In this section we wish to determine all possible shapes and configurations of solids for which the complex potential $\xi$ is identically null. Such a displacement will be termed **stealth**.

**Theorem 4.1.** The only solids $S$ that can undergo stealth motions in a fluid are

- disks rotating about their centers;
- arc of circles and segments with velocity field everywhere tangent to $S$.

The arc of circles and segments are one-dimensional solids and can be considered as degenerated cases.

**Proof.** Let us assume that $\xi = 0$. Then we also have $\zeta = 0$, which means that $-2\Re (\bar{u}f(z)) + |f(z)|^2 = 0$ for all $z \in \partial \Omega$. If $\omega \neq 0$, some easy computations tell us that for all $z \in \partial \Omega$, $f(z)$ belongs to the circle of center $i\omega_0/2\alpha$ and radius $|\omega_0/(2|\alpha|)$. Since $f$ is a homeomorphism from $\partial \Omega$ onto $f(\partial \Omega)$, $f(\partial \Omega)$ is a connected compact subset of this circle.

- If $f(\partial \Omega)$ is the complete circle, it means that $c_1 = 1$ and $c_k = 0$ for all $k \neq 1$. In this case, since $\zeta_1 = 0$ and $(\bar{c}_k + c_k) = 0$, we deduce that $\omega_0 = 0$ and hence that the circle is just rotating about its center.
Up to a translation and a rotation, all the conformal mappings that map the circle onto an arc of circle have the form \( f(z) = z + (1 - h^2)/(z + ih) \) where \( h \) is any real number such that \( 0 < h < 1 \). We can put \( f \) into the general form (3.1) by setting: \( c_1 = 1 \), \( c_{-1} = 1 - h^2 \) and \( c_k = (1 - h^2)(-ih)^{-k-1} \) for all \( k \leq -2 \). Some simple computations lead to \((\bar{c} * c)_{-1} = -ihc_{-1} \) and \((\bar{c} * c)_k = (ih^{-1} - ih)c_k \) for all \( k \leq -2 \). Substituting these expressions into (3.3) and writing that \( \zeta_k(w_0, \omega) = 0 \) for all \( k \leq -1 \) we obtain the same equation for all \( k \) which yields the relation \( w_0 = \omega(h - 1/h) \). We can then easily prove that this motion corresponds to the case where the velocity field is tangent to the solid.

Let us now assume that \( \omega \) is zero (and \( w_0 \neq 0 \)). In this case, we deduce with (3.3) that \( c_k = 0 \) for all \( k \leq -2 \). For \( k = -1 \), we get \( c_{-1} = \bar{c}_1w_0/u_0 \). We set \( w_0 = R e^{i\theta} \), \( c_1 = \bar{R} e^{i\theta} \) and rewrite \( f \) in the form

\[
f(z) = \bar{R}[e^{i\theta}z + e^{i(-\beta+2\theta)/z}] = 2\bar{R} e^{i\theta}[e^{i(\beta-\theta)/z} + e^{-i(\beta-\theta)/z}].
\]

We seek the image of the unit circle by \( f \). We specify \( z = e^{it} \) with \( t \in \mathbb{R}/2\pi \) and we get \( f(e^{it}) = 2\bar{R} e^{i\theta} \cos(\beta - \theta + t) \). So the image of the unit circle is the segment \([-\bar{R}, \bar{R}] \) turned by an angle \( \theta \). The velocity \( w_0 \) is collinear to the segment.

5. Detection of a moving ellipse

When \( \mathcal{S}_0 \) is an ellipse, the function \( f \) has the form \( f(z) = (a + b)z/2 + (a - b)/2z \), where \( a, b \in \mathbb{R}_+, a > b > 0 \). We can now give the proof of proposition 1.4.

Proof. First, we can explicitly compute the inverse function

\[
f^{-1}(z) = \frac{z}{(a + b)} \left( 1 + \frac{1}{1 - \frac{(a^2 - b^2)}{z^2}} \right), \quad (z \in \mathcal{F}_0).
\]

In this expression, \(-\sqrt{a^2 - b^2}\) and \(\sqrt{a^2 - b^2}\) are the branch points and the function is holomorphic everywhere but on the segment \([-\sqrt{a^2 - b^2}, \sqrt{a^2 - b^2}] \) which is a branch cut. Next, we obtain

\[
\xi(z) = \left[ -\bar{w}_0 \frac{a - b}{2} + \frac{a + b}{2} \right] \frac{1}{z} + \frac{1}{4} \frac{a^2 - b^2}{z^2}, \quad (z \in \Omega),
\]

and then

\[
\xi(z) = \frac{-[a^2 - b^2]\bar{w}_0 + (a + b)^2 w_0}{2(z - r)} \left[ 1 + \sqrt{1 - \frac{(a^2 - b^2)^2}{4(z - r)^2}} \right] + \frac{i(a^2 - b^2)(a + b)^2 e^{2i\alpha}}{4(z - r)^2 \left[ 1 + \sqrt{1 - \frac{(a^2 - b^2)^2}{4(z - r)^2}} \right]}. \quad (z \in \mathcal{F}).
\]

The real and imaginary parts of the complex potential \( \xi \) are drawn on figure 4. Observe that, due to the symmetry of the ellipse, we can change \( \alpha \) into \( \alpha + \pi \) and accordingly \( w_0 \) into \(-\bar{w}_0 \) without changing the expression of \( \xi \). The potential \( \xi \) is holomorphic everywhere but on the branch cut \([r - \sqrt{a^2 - b^2} e^{i\alpha}, r + \sqrt{a^2 - b^2} e^{i\alpha}] \). So if we thoroughly know \( \xi \), we can determine the location of the branch points \( r - \sqrt{a^2 - b^2} e^{i\alpha} \) and \( r + \sqrt{a^2 - b^2} e^{i\alpha} \) and hence also the position of the center \( r \) and the orientation \( \alpha \) (up to \( \pi \) only). Next we compute the limit \( \mu := \lim_{r \rightarrow \infty} e^{-i\alpha}\xi(z)/(a + b) = \frac{[-(a - b)\bar{w}_0 + (a + b)w_0]}{4} \) and we deduce the expression of \( w_0 \), namely \( w_0 = \frac{(\mu + \bar{\mu})/b + (\mu - \bar{\mu})/a} \). The only remaining unknown quantity \( \omega \), which is easily obtained following the same idea. \qed
6. Detection: general case

6.1. Singularities of the holomorphic potential

In the preceding example, the branch points of the potential \( \xi \) played a crucial role in determining the position of the solid in the fluid. Note that the existence of these points did not depend on the configuration but only on the shape of the solid (they came from definition \((5.1)\) of the inverse function \( f^{-1} \) and were subsequently just translated and rotated according to the position). We shall prove that this result can be generalized to any solid: there is no singularity in the potential function, that does not come from the conformal mapping \( f^{-1} \) (but unfortunately, the potential function may have less singular points than the function \( f^{-1} \)). Let us make this statement precise.

**Definition 1** (Analytic continuation). An holomorphic function \( \tilde{\xi} \) (respectively \( \tilde{\xi}_0 \)) defined in a connected open set \( \tilde{\mathcal{F}} \) (respectively \( \tilde{\mathcal{F}}_0 \)) containing \( \mathcal{F} \) (respectively \( \mathcal{F}_0 \)) is called an analytic continuation of \( \xi \) (respectively \( \xi_0 \)) when \( \tilde{\xi} = \xi \) in \( \mathcal{F} \) (respectively \( \tilde{\xi}_0 = \xi_0 \) in \( \mathcal{F}_0 \)).

There may exist several analytic continuations of \( \xi \) that do not coincide everywhere. Assume that \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) are two such functions defined respectively on \( \tilde{\mathcal{F}}_1 \) and \( \tilde{\mathcal{F}}_2 \). So the analytic continuation theorem ensures only that \( \tilde{\xi}_1 = \tilde{\xi}_2 \) on the connected component of \( \tilde{\mathcal{F}}_1 \cap \tilde{\mathcal{F}}_2 \) containing \( \mathcal{F} \). In section \( 5 \) for instance, we cannot choose where the branch points are, but there are many different possible choices for the branch cut, each corresponding to a different analytic continuation of \( \xi \).

Assume that for some potential function \( \xi \), there exists an analytic continuation \( \tilde{\xi} \) such that \( \tilde{\mathcal{F}} = \mathbb{C} \). Since, by construction, \( \xi(z) \) tends to 0 as \( |z| \) goes to infinity, \( \tilde{\xi} \) is an entire and bounded function. According to Liouville’s theorem, this function is constant, equal to 0. This case was treated in section \( 4 \) and is possible only for solids listed in theorem \( 4.1 \). For all of the other solids and for any analytic continuation \( \tilde{\xi} \), there exists at least one point, located inside the solid, which does not belong to \( \tilde{\mathcal{F}} \). This very simple observation allows one to locate the solid in a very first approximation.

Let us now prove that the singularities of \( \tilde{\xi} \) come from the singularities of \( f^{-1} \).

**Proposition 6.1.** If there exists an analytic continuation \( \tilde{g} \) of \( f^{-1} \) defined over an open connected set \( \tilde{\mathcal{F}}_0 \) containing \( \mathcal{F}_0 \), then for any configuration \((p, v) \in \mathcal{P} \times \mathcal{V}\), there exists an analytic continuation \( \tilde{\xi} \) of \( \xi \) defined over \( \tilde{\mathcal{F}} := p(\tilde{\mathcal{F}}_0 \setminus \tilde{g}^{-1}((0))) \).

In this proposition, the notation \( \tilde{g}^{-1}((0)) \) stands for the preimage of \( [0] \) under \( \tilde{g} \) and does not mean that \( \tilde{g} \) is invertible. Since \( \tilde{g} \) is holomorphic, the set \( \tilde{g}^{-1}((0)) \) consists only in isolated points and \( \tilde{\mathcal{F}}_0 \setminus \tilde{g}^{-1}((0)) \) is still connected and still contains \( \mathcal{F}_0 \).

**Proof.** For all \( z \in \partial D \), we can rewrite \( \xi \) in the form

\[
\xi(z) = -u_0(f(z) - c_1z) + w_0\tilde{c}_1z^{-1} + \text{io}[\tilde{c}_1z^{-1}(f(z) - c_1z) + \sum_{k \geq 1} \tilde{c}_{-k}z^{k} \left( f(z) - \sum_{-k \leq j \leq 1} c_jz^{j} \right)].
\]

Expanding the right-hand side and recombining terms, we get the identity

\[
\xi(z) = -u_0f(z) + c_1u_0z + u\tilde{c}_1z^{-1} + \text{io}[f(z)\tilde{f}(z^{-1}) - H_1(z)],
\]

where \( H_1(z) := \sum_{j \geq 0} (\tilde{c} \ast c)_{j}z^{j} \) and \( \tilde{f}(z) := \tilde{c}_1z + \sum_{k \leq -1} \tilde{c}_kz^{k} \). Classical results for the convolution product ensure that this series is uniformly convergent for \( |z| \leq 1 \) since
\[ \| \hat{c} \|_{L^1(C)} \leq \| \hat{c} \|_{L^1(C)} \| \hat{c} \|_{L^1(C)}. \]

Next we obtain
\[
\zeta(f^{-1}(z)) = -u_0 z + c_1 u_0 f^{-1}(z) + u c_1 / f^{-1}(z) + i \omega(z) f^{-1}(z) - H_1(f^{-1}(z)), \quad (z \in \partial S_0).
\]

Let \( \tilde{g} \) be any analytic continuation of \( f^{-1} \) (not necessary invertible), defined in an open set \( \tilde{F}_0 \). It can be split into three parts: \( \tilde{F}_0^+ := \{ z \in \tilde{F}_0 : |\tilde{g}(z)| > 1 \} \), \( \tilde{F}_0^- := \{ z \in \tilde{F}_0 : 0 < |\tilde{g}(z)| < 1 \} \) and \( \tilde{F}_0^0 := \{ z \in C : \tilde{g}(z) = 1 \} \). We can next define
\[
\xi_0(z) := -u_0 z + c_1 u_0 \tilde{g}(z) + u_0 c_1 / \tilde{g}(z) + i \omega(z) f^{-1}(\tilde{g}(z)) - H_1(\tilde{g}(z)), \quad z \in \tilde{F}_0^0 \cup \tilde{F}_0^1,
\]
\[
\xi_0(z) := -u_0 z + c_1 u_0 \tilde{g}(z) + u_0 c_1 / \tilde{g}(z) + i \omega(\tilde{H}_1(\tilde{g}(z))), \quad z \in \tilde{F}_0^-.
\]

where \( H_2(z) := \sum_{j \leq 1} (\hat{c} * e) j z^j \) is uniformly convergent for \( |z| \geq 1 \). We deduce that the function \( \tilde{g}_0 \) is holomorphic in \( \tilde{F}_0^- \) and in \( \tilde{F}_0^+ \) and continuous in \( \tilde{F}_0^0 \setminus \tilde{g}^{-1}(0) \). Let \( z_0 \in \tilde{F}_0^1 \) and denote \( z_1 := \tilde{g}(z_0) \). Since \( \tilde{g} \) is holomorphic at the point \( z_0 \), there exists \( R > 0, n \geq 1 \) and a function \( h \) holomorphic in the disk \( D(0, R) \) such that \( h(0) \neq 0 \) and
\[
\tilde{g}(z) = z_1 + (z - z_0)^n h(z - z_0), \quad z \in D(0, R).
\]

This identity allows us to describe the set \( \tilde{F}_0^1 \) around the point \( z_0 \). For instance if \( n = 3 \), and for \( R \) small enough, we get something like that in figure 5 where \( z_1 = \tilde{g}(z_0) \), \( \cup_{j=1}^3 A_j^+ = D(z_0, R) \cap \tilde{F}_0^+ \) and the curves radiating from \( z_0 \) correspond to the set \( D(z_0, R) \cap \tilde{F}_0^1 \). Except at the point \( z_0 \), the boundaries shared by the regions \( A_j^+ \) and \( A_j^- \) are smooth. The function \( \tilde{g} \) maps \( A_j^+ \) onto \( U^+ \) (an open set located outside the unit disk) and \( A_j^- \) onto \( U^- \) (an open set located inside the unit disk). Furthermore, the function \( \tilde{g} \mid_{A_j^+ \cup A_j^-} : A_j^+ \cup A_j^- \to U^- \cup U^+ \) is a conformal mapping. Consider next a conformal mapping \( \phi \) which maps a neighborhood of \( z_1 \) onto a neighborhood of \( 0 \) as shown in figure 5 and such that the image of the unit circle is the imaginary axis. We can then apply [7, theorem 16.8]: the function \( \xi_0 \circ \tilde{g}^{-1} \circ \phi^{-1} \) is holomorphic on both sides of the imaginary axis and continuous across this boundary, so it is holomorphic over the whole domain. We deduce that \( \xi_0 \) is holomorphic across the boundary between \( A_j^+ \) and \( A_j^- \). We can repeat this process with the domains \( A_j^+ \cup A_j^- \), \( A_k^+ \cup A_k^- \) and so on. Finally, we obtain that \( \xi_0 \) is holomorphic on the whole disk \( D(z_0, R) \) except maybe at the point \( z_0 \). But once more, the continuity of the function \( \xi_0 \) does not allow this possibility.

As already mentioned, the potential function can make some singularities of \( f^{-1} \) to vanish. This is illustrated by the example in figure 6.
Figure 6. The solid consists in a disk of center $C$ with a segment $[A, B]$. When this solid is moving to the left, parallel to the segment, one can easily check that the potential coincides with the potential of the disk. Although any analytic continuation of $f^{-1}$ has singularities at $A$ and $B$, the complex potential does not see these points. In this case, the singularities do not allow one to determine the orientation of the solid.

6.2. Asymptotic expansion of the holomorphic potential

In this section, we shall compute the asymptotic expansion of $\xi$ in terms of the geometrical data of $S_0$ and the configuration $(p, v) \in P \times V$. As explained in the preceding section, we know that if $S_0$ is not one of the solids listed in theorem 4.1, the potential function admits no analytic continuation on the whole complex plane. It allows one to deduce approximately where the solid is. For all $\nu \in C$, we can next consider $\Gamma_1$, a contour large enough to encircle the solid and the point $\nu$. The contour $\tilde{\Gamma}_1 := f(e^{-i\alpha}(\Gamma - r))$ encircles the unit disk. According to expression (1.1) of the potential as a Laurent series, we obtain that, for all $n \geq 1$,

$$\lambda_n(\nu) = \frac{1}{2i\pi} \oint_{\tilde{\Gamma}_1} \xi(z)(z - \nu)^{n-1} \, dz = \frac{e^{i\alpha}}{2i\pi} \oint_{\tilde{\Gamma}_1} \xi(z)(e^{i\alpha} f(z) + r - \nu)^{n-1} f'(z) \, dz$$

$$= 1 \cdot 1 \frac{1}{2i\pi} \oint_{\tilde{\Gamma}_1} \xi(z) \frac{d}{dz} (e^{i\alpha} f(z) + r - \nu)^{n} \, dz$$

$$= -1 \cdot 1 \frac{1}{2i\pi} \oint_{\tilde{\Gamma}_1} \xi'(z)(e^{i\alpha} f(z) + r - \nu)^{n} \, dz$$

$$= -1 \cdot 1 \frac{1}{2i\pi} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) e^{i\alpha n} (r - \nu)^{n-k} \oint_{\tilde{\Gamma}_1} \xi'(z) f(z)^{n-k} \, dz.$$ 

If we define the complex sequence $d := (d_k)_{k \in \mathbb{Z}}$ by $d_k := (k + 1)\xi_{k+1}$ for all $k \leq -1$ and $d_k = 0$ for $k \geq 0$, we obtain

$$\lambda_n(\nu) = -\frac{1}{n} \sum_{k=1}^{n} \left(\begin{array}{c} n \\ k \end{array}\right) e^{i\alpha} (r - \nu)^{n-k} (d \ast e^k)_{-k}, \quad (\nu \in C).$$

We can rewrite the last term as

$$(d \ast e^k)_{-k} = \sum_{i_1 + \ldots + i_{k+1} = -1} c_{i_1} \ldots c_{i_{k+1}} d_{i_{k+1}}$$

$$= -A_k \bar{u}_0 + B_k u_0 + i\alpha C_k,$$

where

$$A_k := \sum_{i_1 + \ldots + i_{k+1} = 0} i_1 c_{i_1} \ldots c_{i_{k+1}}, \quad B_k := \sum_{i_1 + \ldots + i_{k+1} = 0} i_1 \bar{c}_{i_1} c_{i_2} \ldots c_{i_{k+1}},$$

$$C_k := \sum_{i_1 + \ldots + i_{k+1} = 0} (i_1 + i_2) \bar{c}_{i_1} c_{i_2} \ldots c_{i_{k+1}}.$$
In the following, to simplify the notation, we consider the quantities $A_k := -A_k/k$, $B_k := -B_k/k$ and $\mathcal{C}_k := -C_k/k$. Indeed, we get, for all $n \geq 1$,

$$\lambda_n(\nu) = \sum_{k=1}^{n} \left(\frac{n-1}{k-1}\right) C_{n-k}^{k-1} \left(\nu - v\right)^{n-k} \left[-A_k \bar{w}_0 + B_k w_0 + i\alpha \mathcal{C}_k\right]. \quad (6.2)$$

The problem of detection can now be reformulated as a purely algebraic problem: the complex sequence $(\lambda_j(\nu))_{j \geq 1}$ being given for all $\nu \in \mathbb{C}$, as well as the complex numbers $A_k, B_k$, and $\mathcal{C}_k (k \geq 1)$, can we solve the infinite nonlinear system of equations (6.2) and find the values of $\nu, \alpha, w_0$ and $\omega$? According to the results of sections 2.4, we already know that there exist cases (namely, coefficients $(c_k)_{k \in \mathbb{Z}}$) for which the answer is negative.

In order to rewrite this infinite set of equations in a convenient short form, we introduce some linear operators: let us denote by $\mathcal{G}_N : \mathbb{R}^3 \to \mathbb{C}^N$ by $(\mathcal{G}_N U)_k := (-A_k + B_k) U_1 + i(A_k + B_k) U_2 + i\mathcal{C}_k U_3$ for all $U = (U_1, U_2, U_3)^T \in \mathbb{R}^3$ and all $1 \leq k \leq N$. We define $D_N : \mathbb{C}^N \to \mathbb{C}^N$ and $S_N : \mathbb{C}^N \to \mathbb{C}^N$ as well, by respectively $(D_N Z)_k := kZ_k$ (for all $k \leq N$) and $(S_N Z)_1 := 0$ and $(S_N Z)_k := Z_{k-1}$ (for all $k \leq N$) for all $Z := (Z_1, \ldots, Z_N) \in \mathbb{C}^N$. The first $N$ equations (6.2) can now be rewritten as

$$\Lambda_N(\nu) = \frac{\delta \log(r-\nu)D_N}{\delta \nu} e^{\delta \log(r-\nu)D_N} e^{i\alpha D_N} \mathcal{G}_N U, \quad (6.3a)$$

$$= \Theta_N (r - \nu, \alpha) \mathcal{G}_N U, \quad (6.3b)$$

where $U := (\Re(w_0), \Im(w_0), \alpha)^T$ and $\Lambda_N(\nu) := (\lambda_1(\nu), \ldots, \lambda_N(\nu))^T$. The operator $e^{S_N D_N}$ is lower triangular and the identity $(e^{S_N D_N})_{k,n} = \binom{n}{k}$ for all $1 \leq k \leq n \leq N$, not so obvious, can be found in [2]. Considering expressions (6.3), it is worth noting that

- in (6.3), the coefficients $\lambda_j(\nu)$ in the asymptotic expansion of $\xi$ are obtained by applying to the vector $U$ (the velocity) first the operator $\mathcal{G}_N$ encapsulating the information relating to the geometry of the solid and next the operator $\Theta_N (r - \nu, \alpha)$ depending only on the position;
- the linear operator $\mathcal{G}_N$ depends on the complex sequence $c$ only, i.e. on the shape of the solid. Moreover, the complex quantities $A_k - c_{-k} c_{k}^{*} (k \geq 1)$, $B_{k+1} - c_{-k+1} c_{k}^{*} (k \geq 2)$ and $\mathcal{C}_k - c_{-k} c_{-k}^{*} (k \geq 1)$ do not depend on $c_{-n}$ for all $n \geq k$. In other words, for all $N \geq 1$, $\mathcal{G}_N$ depends on $c_1, c_{-1}, c_{-2}, \ldots, c_{-N-1}$ only. We deduce

**Proposition 6.2.** Let $S^1_0$ and $S^2_0$ be two shapes described by means of the complex sequences $(c^1_k)_{k \geq 1}$ and $(c^2_k)_{k \geq 1}$, such that $c^1_k = c^2_k$ for all $1 \leq k \leq N$. Then if both solids have the same configuration, their complex potentials will have the same asymptotic expansion up to the order $N - 1$.

- The solutions $(r_1, r_2, \alpha, \Re(w_0), \Im(w_0), \alpha)^T$ of all of equations (6.3) (for all $N \geq 1$) form a sub-analytic set of $\mathbb{R}^d$ with $0 \leq d \leq 6$. However, because the dependence in $(\Re(w_0), \Im(w_0), \alpha)^T$ is linear, if it had dimension $d \geq 4$, it would entail the existence of a position $(r_1, r_2, \alpha)^T$ and a non-zero velocity $U_0 \in \mathbb{R}^3$ such that $\Theta_N (r - \nu, \alpha) \mathcal{G}_N U_0 = 0$ for all $N \geq 1$. As mentioned before, this case is only possible if the solid is in the list of theorem 4.1.

We do not know if there exist solids such that $0 < d \leq 3$. Observe that in section 2, we have only given examples for which the solids can occupy a finite number of different positions, so $d = 0$ in these cases.
For all $N \geq 1$, we can invert the system (6.3) to obtain
\[ G_N U = e^{-i\alpha D_N} e^{\log(r-v)D_N} e^{-S_N D_N} e^{-\log(r-v)D_N} \Lambda_N(v), \]  
(6.4)
\[ = \Theta_N(r-v, \alpha) \Lambda_N(v), \]  
(6.5)
or equivalently, with the notation of equation (6.2),
\[ -A_n \bar{u}_0 + B_n u_0 + \alpha i C_n = e^{-i n\alpha} \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) (v-r)^{-k} \lambda_k(v), \]  
(6.6)
for all $n \in \mathbb{N}$. In this form, we can easily prove

**Proposition 6.3.** If the solid does not occur in theorem 4.1 and its position is given, then we can deduce its velocity.

**Proof.** Denote by $G^j$ ($j = 1, 2, 3$) the complex sequences respectively defined by $G^1_j := (-A_k + B_k)_{k \geq 1}$, $G^2_j := (i(A_k + B_k))_{k \geq 1}$, and $G^3_j := (iC_k)_{k \geq 1}$. If these sequences were not $\mathbb{R}$-linearly independent in $\mathbb{C}^N$, it would exist $(U_1, U_2, U_3)^T \neq 0$ in $\mathbb{R}^3$ such that $\sum_j U_j G^j = 0$ and then, for any position $(r, \alpha)$ and any $N \geq 1$, we would have $\Theta_N(r-v, \alpha) G_N U = 0$ which contradicts the assumption that the solid is not listed in theorem 4.1. Conversely, if the sequences $G^j$ are $\mathbb{R}$-linearly independent, then there exists $N_0 \geq 3$ such that for all $N \geq N_0$, $G_N$ is of rank 3 and the proof is completed. \hfill \Box

**Proposition 6.4.** Assume that the shape $S_0$ of the solid, described by the conformal mapping (3.1), is such that $e^{\pi/2} S_0 = S_0$ (the shape of the solid is invariant by rotation of angle $\pi/2$ and center 0). Then, from the holomorphic potential $\xi$, we can always deduce the values of $r$, $w_0 e^{\bar{\alpha}}$ (the linear velocity expressed in a reference fixed frame) and $|\omega|$ (the absolute value of the rotational velocity).

**Proof.** The assumption on the shape $S_0$ means that $f(\Omega) = \tilde{f}(\Omega)$ where $\tilde{f}$ is the conformal mapping defined by
\[ \tilde{f}(z) := \tilde{c}_1 z + \sum_{k \leq -1} \tilde{c}_k z^k, \quad (z \in \Omega), \]  
with $\tilde{c}_k = e^{\pi/2} c_k$ for any $k = 1$ and $k \leq -1$. Replacing the function $f$ by $\tilde{f}$ in the computations, we obtain that
\[ \lambda_n(v) = \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) e^{i\alpha} (r-v)^{-k} [-A_k e^{i(k+1)\pi/2} \bar{u}_0 + B_k u_0 e^{i(k-1)\pi/2} + \alpha i C_k e^{i(k)\pi/2}], \]  
(6.7)
and the coefficients $\lambda_k(v)$, defined equivalently by (6.2) and (6.7), must be equal for all $r, v \in \mathbb{C}$, $\alpha \in \mathbb{R}/2\pi$ and all $u_0 \in \mathbb{C}$ and $\omega \in \mathbb{R}$. In particular, for $r = v = 0$ and $\alpha = 0$, we obtain that $A_n (1 - e^{i(n+1)\pi/2}) u_0 - B_n (1 - e^{i(n-1)\pi/2}) \bar{u}_0 = 0$ and $C_n (1 - e^{i\alpha \pi/2}) \omega = 0$ for all $u_0 \in \mathbb{C}$, $\omega \in \mathbb{R}$ and $n \geq 1$. We deduce that $(A_n \neq 0) \Leftrightarrow (n \equiv -1 \ [4])$, $(B_n \neq 0) \Leftrightarrow (n \equiv 1 \ [4])$ and $(C_n \neq 0) \Leftrightarrow (n \equiv 0 \ [4])$.

Let us now go back to the problem of detection and take advantage of this extra information about $A_n$, $B_n$ and $C_n$. Equation (6.6) with $n = 2$ gives $\lambda_1(v-r) + \lambda_2(v) = 0$ for all $v \in \mathbb{C}$. Two cases have to be considered.

- Either $\lambda_1 = 0$, which also means that $\lambda_2(v) = 0$ for all $v \in \mathbb{C}$. It entails, according to equation (6.6) with $n = 1$, that $B_1 u_0 = 0$. But $B_1 = |c_1|^2 \neq 0$ and hence $u_0 = 0$. We get $\omega \neq 0$ otherwise we would have $\xi = 0$. Now let $m$ be the smallest index such
that $C_m \neq 0$. We know that such an index exists, otherwise it would mean that $C_k = 0$ for all $k \geq 1$ and hence that any rotational motion of the solid generates a complex potential equal to 0. This is impossible for a solid not listed in theorem 4.1. By induction on $n$ with equation (6.6), we must now have $\lambda_n(\nu) = 0$ for all $n < m$.

We next deduce $e^{i4\alpha}$ and hence the orientation of the solid (because of the symmetry property, $e^{i4\alpha}$ suffices to provide the orientation). Since $m \equiv 0 \ [4]$ (because $C_m \neq 0$) we next deduce the values of $e^{i\alpha a}$ and $\omega$. We know that there exists at least one index $p$ such that $A_p \neq 0$. We use equation (6.6) with $n = p$ to deduce the value of $A_p w_0 e^{ip\alpha}$ or equivalently $\hat{A}_p w_0 e^{-ip\alpha}$. Since necessarily $p \equiv -1 \ [4]$ then $p = 1 \ [4]$ and we can deduce the values of both $\omega$ and $e^{i4\alpha}$ with $w_0 e^{i\alpha a}$.

To conclude this section, we give examples of solids without any symmetry, which are detectable (one of which is shown in figure 7).

Figure 7. Example of a detectable solid as described in subsection 6.3.
6.3. Examples of detection

We consider a shape $S_0$ described by a complex sequence $c$ such that $c_1 \neq 0$, $c_{-4} \neq 0$ and $c_{-7} \neq 0$, all the other coefficients $c_k$ being null. Direct computations lead to

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|
| $A_k$ | 0 | 0 | 0 | $c_1^2 c_{-4}$ | 0 | 0 | $c_1^2 c_{-7}$ | 0 |
| $B_k$ | $|c_1|^2$ | 0 | 0 | 0 | 0 | $|c_1|^2 c_1^4 c_{-4}$ | 0 | 0 |
| $C_k$ | 0 | 0 | $c_1^2 c_{-4} c_{-7}$ | 0 | 2$c_1 |c_1^2| c_1^4 c_{-4}$ | 0 | 0 | (2$|c_1|^2 + 3|c_{-4}|^2$)$c_1^3 c_{-7}$ |

Substituting these values into the system (6.6) we obtain that

\[ B_1 u_0 = e^{-i\omega} \lambda_1, \]  
\[ 0 = e^{-i\alpha} [\lambda_1 (v - r) + \lambda_2 (v)], \]  
\[ C_3 i \omega = e^{-i\alpha} \left[ \sum_{k=1}^{3} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right]. \]  
\[ -A_4 \bar{u}_0 = e^{-i\alpha} \left[ \sum_{k=1}^{4} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right]. \]

- If $\omega \neq 0$ and $u_0 \neq 0$: from equation (6.8a) we deduce that $\lambda_1 \neq 0$ and from (6.8b) we deduce the value of $r$. Equation (6.8c) allows us to determine $|\omega|$ and $3\alpha$ up to $\pi$. Combining next equation (6.8a) and (6.8d), we get $5\alpha$. Using Bezout’s identity: $\alpha = u3\alpha + v5\alpha$ with $u = 2$ and $v = -1$, we get $\alpha$. We determine $u_0$ with equation (6.8a) and $\omega$ with equation (6.8c).
- If $\omega = 0$, $u_0 \neq 0$: we compute $r$ as in the preceding case. Then, we need further calculations:

\[ B_5 u_0 = e^{-i\alpha} \left[ \sum_{k=1}^{6} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right]. \]  
\[ -A_7 \bar{u}_0 = e^{-i\alpha} \left[ \sum_{k=1}^{7} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right]. \]

With (6.9a) and (6.8a) we get $5\alpha$ and with (6.9b) and (6.8a) we get $8\alpha$. Since 5 and 8 are coprime numbers, we deduce the value of $\alpha$. We conclude as in the preceding case.
- If $\omega \neq 0$ and $u_0 = 0$: with (6.8a) and (6.8b) we deduce that $\lambda_1 = \lambda_2 (v) = 0$ for all $v \in \mathbb{C}$. We rewrite (6.8c) and (6.8d) as

\[ C_3 i \omega = e^{-i\alpha} \lambda_3, \]  
\[ 0 = \lambda_4 (v) + 3(v - r)\lambda_3, \]

and we deduce first that $\lambda_3 \neq 0$ and then the value of $r$. We next add the equations

\[ C_5 i \omega = e^{-i\alpha} \left[ \sum_{k=1}^{5} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right], \]  
\[ C_8 i \omega = e^{-i\alpha} \left[ \sum_{k=1}^{8} \frac{(n-1)}{(k-1)} (v - r)^{n-k} \lambda_k (v) \right]. \]
Equations (6.8c) and (6.10a) give us 2α and (6.10a) and (6.10b) give us 3α. Since 2 and 5 are coprime numbers we get α and then ω with (6.8c).

7. Tracking

In this section, we perform the proof of theorem 1.7. So we assume that we know the complex potential for all t in a time interval [0, T] (T > 0). We have the expression

\[ \xi(t, z) := \sum_{j \geq 1} \frac{\lambda_j(t, v)}{(z - v)^j} \cdot \quad |z - v| > R(t, v), \quad (7.1) \]

where \( \lambda_j(t, v) \) are complex numbers and \( R(t, v) := \lim \sup_{j \to +\infty} |\lambda_j(t, v)|^{1/j} \). At any time t, the series is uniformly convergent on \( \{ z \in \mathbb{C} : |z - v| > R(t, v) \} \). For all \( N \geq 1 \), we denote \( \Lambda_N(t, v) := (\lambda_1(t, v), \ldots, \lambda_N(t, v))^T \) and we have, according to the results of the preceding section,

\[ G_N U(t) = \Theta_N(r(t) - v, \alpha(t))^{-1} \Lambda_N(t, v), \]

where we recall that \( U(t) := (\Re(w_0(t)), \Im(w_0(t)), \omega(t))^T \). In the proof of proposition 6.4 we have shown that if the solid is not one of those described in theorem 4.1, then there exists \( N \geq 1 \) such that \( G_N \) has rank 3. It means that there exists (at least) one inverse \( G_N^{-1} \) allowing one to express the velocity as

\[ U(t) := G_N^{-1} \Theta_N(r(t) - v, \alpha(t))^{-1} \Lambda_N(t, v). \]

We next get

\[ \begin{pmatrix} \omega_0(t) \\ \omega \end{pmatrix} = \begin{pmatrix} \omega_0(t) \\ \omega \end{pmatrix} = \begin{pmatrix} i \omega_0(t) \\ 0 \end{pmatrix} G_N^{-1} \Theta_N(r(t) - v, \alpha(t))^{-1} \Lambda_N(t, v). \]

This equation can be rewritten as

\[ \frac{d}{dt} \begin{pmatrix} r(t) \\ \alpha(t) \end{pmatrix} = \begin{pmatrix} i \omega_0(t) \\ 0 \end{pmatrix} G_N^{-1} \Theta_N(r(t) - v, \alpha(t))^{-1} \Lambda_N(t, v), \]

to which we can apply the Cauchy–Lipschitz theorem. The proof is then completed.

8. Conclusion

In this paper, we have proved that not all solids moving in a perfect fluid can be detected by measuring the potential of the fluid. This observation has led us to define the notion of detectable solids, which is a purely geometric property. When the geometry is described by means of a conformal mapping, we were able to exhibit examples of detectable (or partially detectable) solids. However, the complete characterization of such solids in terms of the complex sequence \( (c_k)_{k \in \mathbb{Z}} \) remains to be done.

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