Loop Quantum Gravity and Asymptotically Flat Spaces

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Abstract

After motivating why the study of asymptotically flat spaces is important in loop quantum gravity, we review the extension of the standard framework of this theory to the asymptotically flat sector detailed in [1]. In particular, we provide a general procedure for constructing new Hilbert spaces for loop quantum gravity on non-compact spatial manifolds. States in these Hilbert spaces can be interpreted as describing fluctuations around fiducial fixed backgrounds. When the backgrounds are chosen to approximate classical asymptotically flat 3-geometries this gives a natural framework in which to discuss physical applications of loop quantum gravity, especially its semi-classical limit. We present three general proposals for the construction of suitable backgrounds, including one approach that can lead to quantum gravity on anti-DeSitter space as described by the Chern-Simons state.

1 Motivating Asymptotic Flatness

Remarkable progress has been made in the field of non-perturbative (loop) quantum gravity in the last decade or so and it is now a rigorously defined kinematical theory. We are at the stage where physical applications of loop quantum gravity can be studied and used to provide checks for the consistency of the quantisation programme. Equally, old fundamental problems of canonical quantum gravity such as the problem of time or the interpretation of quantum cosmology need to be reevaluated seriously. The purpose of this report is to suggest that all these issues can be tackled most profitably in the asymptotically flat sector of quantum gravity and

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to discuss the extension of loop quantum gravity to this sector that was developed in [4].

1.1 Canonical quantum gravity

One of the defining features of general relativity is its active diffeomorphism invariance. This implies that we can take all fields on the space-time manifolds and reassign their values to different points without changing predictions. Hence, points in the space-time have no significance of their own, or in other words there is no fixed reference frame. This leads to the relational nature of observables as is exemplified in Einstein’s hole argument: values of fields are only physically meaningful (observable) if they are given with reference to points defined by the values of other fields.

Physics in this context is difficult in practice. But usually, to study concrete gravitational systems and make predictions we do not need such generality. One can break the diffeomorphism invariance and introduce reference frames or introduce extra symmetries. A very convenient way to do this is to study the asymptotically flat sector of general relativity, as is discussed below.

Another set of reasons to study asymptotically flat spaces, originates in the structure of quantum theory. General relativity deals naturally with the physics of the entire universe, which is the definitive closed system. Quantum mechanics on the other hand is plagued with interpretational difficulties when attempting to describe genuine closed systems. More specifically, we need an external observer to make sense of predictions made for a given physical system.

For these reasons it is unlikely that we will succeed in a complete and comprehensive theory of quantum gravity, without major changes to the ingredient theories viz. general relativity and quantum mechanics. This motivates us to study sectors of the theory, where we can make progress in addressing physical problems, while avoiding many of the foundational concerns.

1.2 Asymptotically flat spaces

Many physical applications of general relativity involve the study of asymptotically flat spaces. Here one restricts attention to the subset of the phase space in which the variables approach the flat configuration at spatial infinity. Physically, this means that we are breaking diffeomorphism invariance at infinity and introducing a reference frame there. We can interpret this as the idealised description of an isolated gravitational system viewed by an observer in the ambient environment. This brings several technical advantages:
1. Since space is no longer compact we need to add boundary terms to the Hamiltonian, which are integrals over a two sphere at infinity. It follows that the Hamiltonian no longer vanishes but has the value of the ADM-energy: the total energy of the isolated gravitational region.

2. In addition, we recover momentum and angular momentum observables at infinity. Together these functions generate the Poincaré group at infinity, which is the symmetry group of the isolated region.

3. In particular, we can also introduce a notion of asymptotic time translations generated by the energy observable. This now makes sense since we can interpret this time variable with respect to the fixed reference frame at infinity.

4. The introduction of an external environment also allows a natural interpretation of a corresponding quantum theory in terms of observers at infinity. The setup essentially amounts to the study of a gravitational system in a box, which resembles the standard applications of quantum mechanics.

For these reasons the study of asymptotically flat spaces provides a very promising testing ground for any theory of quantum gravity. Indeed, it is likely that we should obtain a quantum theory for this special case even if it is not possible to quantise full general relativity. In addition, many of the physical applications of a quantum gravity theory fall into this sector.

1. The study of the weak field limit of quantum gravity is a crucial test for quantum gravity. In the absence of concrete experimental evidence consistency with results of perturbative quantum gravity is essential. This involves the investigation of perturbations of flat space and the calculation of scattering amplitudes of gravitons.

2. Black Holes and more general isolated gravitational systems provide another fascinating application for quantum gravity. Again we need to recover results from quantum theory of curved spaces, which are theoretically very robust.

3. A genuine experimental test for quantum general relativity might be possible with the study of γ-ray bursts. The idea is to detect any breaking of Poincaré invariance by exploiting the accumulation of tiny effects over very long distances.

1.3 Loop quantum gravity and the GNS construction

Loop quantum gravity is a quantum theory of the 3-geometry of a spatial manifold Σ. In contrast to the geometrodynamical approach to canonical quantum gravity, the phase space coordinates in loop quantum gravity are given by connection and
triad fields defined on $\Sigma$. A well-defined kinematical quantum framework for this
theory has been developed, and several proposals for incorporating dynamics or the
Hamiltonian constraint into the theory are currently under study.

In this framework it turns out that fundamental excitations of geometry have 1-
dimensional support. More precisely, excitations are concentrated on graphs em-
bedded in $\Sigma$ providing a polymer like picture of space-time at the Planck scale.

The standard approach to loop quantum gravity is only applicable to the case where
the 3-manifold $\Sigma$ is compact. Roughly, this can be seen as follows. To describe
a geometry on $\Sigma$ with a quantum state we need excitations of geometry in every
macroscopic region of $\Sigma$. Since the excitations are concentrated on embedded graphs
this entails that underlying the definition of our state should be a graph that spreads
through all of $\Sigma$, i.e. every macroscopic region of $\Sigma$ should contain vertices and edges
of the graph.

For non-compact $\Sigma$ this implies that we need graphs with an infinite number of
vertices and edges and it turns out that states based on such graphs are not included
in the standard Hilbert space.

The solution we review in this report (c.f. [1]) is motivated by an analogy with
thermal field theory (TFT). Here a similar problem arises when considering fields
at a finite temperature. These are described by a condensate of an infinite number
of photons, which cannot be described within the standard Fock space. The issue
that needs to be addressed in both cases is how to describe an infinite number of
excitations.

To do this we take an algebraic approach to quantum theory. This means that
we regard algebras of observables as the primary objects, with states just arising
as elements on the spaces on which we choose to represent the algebra. Evolution
becomes an automorphism of the observable algebra and all physical questions then
involve the calculation of expectation values of the relevant operators. This view
point gives us the freedom to change between representations according to which
sector of the quantum theory we are interested in.

We will show how the Gel’fand Naimark Segal (GNS) construction provides us with
representations that can be interpreted naturally as describing the excitations of
fiducial fixed background states. In particular, these background states can be
chosen to describe asymptotically flat geometries on non-compact 3-manifolds.

This will be investigated in the last part of this report where we discuss three possible
frameworks for the construction of suitable backgrounds. Specific choices of these
backgrounds can be interpreted as semi-classical states, also referred to as vacua of
loop quantum gravity, and their excitations should describe field theory on curved
spaces.
2 An observable algebra for loop quantum gravity

In this section we review the standard kinematical framework of loop quantum gravity. In particular, we specify the algebra \( \mathcal{B}_{\text{aux}} \) of elementary classical observables that we wish to represent. We will take a brief look at how the standard representation of this algebra arises when space is compact before looking at new representations constructed via the GNS construction in the following sections.

The classical phase space for general relativity in the connection variables \( [2] \) is the co-tangent bundle of the configuration space, given by \( \mathfrak{su}(2) \) valued connection one-forms with co-ordinates \( A^i_a \) on a spatial 3-manifold \( \Sigma \). The conjugate variable is a desitised triad \( \tilde{E}^a_i \), which takes values in the dual of the Lie algebra \( \mathfrak{su}(2) \). These triads can be considered as duals to two forms \( \epsilon_{abi} \equiv \epsilon_{abc} \tilde{E}^c_i \). The dynamics of general relativity on spatially compact manifolds is then completely described by the Gauss constraints which generate \( \text{SU}(2) \)-gauge transformations, the diffeomorphism constraints which generate spatial diffeomorphisms on \( \Sigma \), and the Hamiltonian constraint, which is the generator of coordinate time evolution.

Quantisation proceeds in two steps. First we seek a representation of the algebra of classical variables \( \mathcal{B}_{\text{aux}} \) on some auxiliary Hilbert space \( \mathcal{H}_{\text{aux}} \). The second step is to obtain operator versions of the classical constraints and to then impose these on the Hilbert space to obtain a reduced space of physical states along with a representation of the subalgebra of observables that commute with the constraints. In this review we will restrict ourselves to the kinematical sector and focus on the first step of this procedure, more details can be found in \([1]\).

To obtain the classical algebra of elementary functions which can be implemented in the quantum theory, we need to integrate the canonically conjugate variables, \( A^i_a \) and \( \epsilon_{abi} \), against suitable smearing fields. In usual quantum field theory, these test fields are three-dimensional. However, in canonical quantum general relativity, due to the absence of a background metric, it is more convenient to smear \( n \)-forms against \( n \)-dimensional surfaces \([15, 3, 3]\).

Configuration observables (depending only on the connection) can be constructed through holonomies of connections. Given an embedded graph \( \Gamma \) which is a collection of paths \( \{ \gamma_1, \ldots, \gamma_n \} \in \Sigma \), and a smooth function \( f \) from \( \text{SU}(2)^n \) to \( \mathbb{C} \), we can construct cylindrical functions of the connection:

\[
\psi_{f,\Gamma}(A) = f(H(A, \gamma_1), \ldots, H(A, \gamma_n)).
\]

\( H(A, \gamma_i) \in \text{SU}(2) \) is the holonomy assigned to the edge \( \gamma_i \) of \( \Gamma \) by the connection \( A \). We denote by \( \mathcal{C} \), the algebra generated by all the functions of this form. To obtain momentum variables, we smear the two-forms \( \epsilon_{abi} \) against distributional test fields \( t^i \) which take values in the dual of \( \mathfrak{su}(2) \) and have two dimensional support. This

\(^1\)Here \( a \) denotes a spatial index and \( i \) a Lie-algebra index.
gives us

\[ E_{t,S} \equiv \int_S e_{ab} t^i dS_{ab}, \]

where \( S \) is a two-dimensional surface embedded in \( \Sigma \). More precisely (c.f. \([3]\))
we require that \( S = \bar{S} - \partial \bar{S} \), where \( \bar{S} \) is any *compact*, analytic, two dimensional submanifold of \( \Sigma \).

The elements of \( \mathcal{C} \) and the functions \( E_{t,S} \) are the variables that we wish to promote to quantum operators. They form a large enough subset of all classical observables in the sense that they suffice to distinguish phase space points. The algebra of elementary observables, \( \mathcal{B}_{aux} \), is the algebra generated by the cylindrical functions and vector fields on \( \mathcal{C} \) associated to the momentum variables, details can be found in \([3]\).

2.1 The standard representation

We now describe briefly how \( \mathcal{H}_{aux} \) with its representation of \( \mathcal{B}_{aux} \) is constructed. The Hilbert space \( \mathcal{H}_{aux} \) is chosen to be the completion of the space of cylindrical functions \( \mathcal{C} \), first in the sup norm:

\[ \| \psi_{f,\Gamma} \|_\infty = \sup |\psi(A)_{f,\Gamma}|, \]

and then in the norm based on the (standard) inner product \([3]\):

\[ \langle \psi_1, \psi_2 \rangle_s = \int d\mu(A)\psi_1^*(A)\psi_2(A) \]

\[ \equiv \int_{SU(2)^n} f_1^*(g_1, \ldots, g_n)f_2(g_1, \ldots, g_n)dg_1 \cdots dg_n, \tag{1} \]

where \( g_i \in SU(2) \) and \( dg \) is the Haar measure on \( SU(2) \). Here, \( \psi_1 \) and \( \psi_2 \) are defined with respect to the same graph. This does not present a loss of generality since given two states supported on different graphs \( \Gamma_1 \) and \( \Gamma_2 \) we can view them as being both defined on \( \Gamma_1 \cap \Gamma_2 \), with the states having trivial dependence on the edges that do not belong to their respective original graphs. \( \mathcal{H}_{aux} \) can also be regarded as space of square integrable functions defined with respect to a genuine measure on some completion \( \bar{\mathcal{A}} \) of \( \mathcal{A} \) as is done in \([4]\).

This Hilbert space carries a multiplicative representation of the configuration variables \( \mathcal{C} \) and we are left with the task of representing the momentum variables. This done by constructing essentially self adjoint operators \( \hat{E}_{t,S} \) on \( \mathcal{C} \) which can be extended to \( \mathcal{H}_{aux} \). These operators are derivations on \( \mathcal{C} \) i.e. linear maps satisfying the Leibnitz rule, which act on functions \( \psi_{f,\Gamma} \in \mathcal{C} \) only at points where \( \Gamma \) intersects the oriented surface \( S \). The precise definition of these operators is not needed for our purposes, but it can be found, e.g., in \([5]\). This choice of operators gives the correct representation of the classical algebra \( \mathcal{B}_{aux} \), which provides us with a kinematical framework for canonical quantum gravity. In the following this representation will be referred to as the standard representation \( \pi_s \).
2.2 Problems with describing non-compact geometries

Are main interest lies in studying the asymptotically flat sector of loop quantum gravity and hence we need to have states describing geometries on non-compact spatial manifolds. In the following we describe briefly why this is generically not possible in the standard framework. As mentioned in the introduction we need cylindrical functions based on graphs with an infinite number of edges and vertices to describe excitations everywhere on a non-compact space. Hence, one first needs to specify what one means by a cylindrical function based on such an infinite graph. A natural approach is to consider sequences of cylindrical functions based on larger and larger but finite graphs. One then finds that in general these sequences are not Cauchy in the available norms on the Hilbert space $H_{\text{aux}}$ and hence they do not lie in the standard state space.

3 Representations induced by background states

As the standard framework of loop quantum gravity is not applicable to the asymptotically flat sector we look at how we can construct new representations of the observable algebra of loop quantum gravity. The main aim will be to obtain quantum theories that describe excitations of asymptotically flat backgrounds. In this section we will look at how this can be done in general using the GNS construction before applying these techniques to loop quantum gravity.

When faced with an infinite dimensional algebra of observables that we wish to quantise, we can no longer rely on the Stone - Von Neumann theorem to guarantee us the existence of a unique irreducible representation. It is in fact well-known that in quantum field theory there are an infinite number of representations to choose from. The obvious questions that arise are how to construct these representations, how to select an appropriate representation for the physical situation at hand and how to interpret physically the states that arise in this representation. As we will motivate in this section, there is a satisfactory answer to these questions if we are given a suitable background state as is defined more explicitly below.

To motivate the definition that follows in the next section, we imagine that we are given a representation $\pi$ of an algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$. In addition we require the existence of a preferred “vacuum” state $\Psi$ that is cyclical, which means that the set of states $\{\pi(a)\Psi | a \in \mathfrak{A}\}$ is dense in $\mathcal{H}$. The first immediate observation is that essentially all states can be identified with an algebra element, i.e. $a \leftrightarrow \phi$ if $\phi = \pi(a)\Psi$. We say that $\phi$ is an excitation of the vacuum corresponding to $a$. This should be seen in close analogy to the construction of the Fock space by repeated action of creation operators on the vacuum.

The next observation follows from the introduction of the positive linear form $\omega$ which assigns to every algebra element its vacuum expectation value (VEV): $\omega(a) = \langle \Psi | \pi(a)\Psi \rangle$. When the state is cyclical the vacuum expectation value is well-defined and we can construct the state representation $\pi_\omega$ of $\mathfrak{A}$ as follows: $\pi_\omega(a) = \langle \Psi | \pi(a)\Psi \rangle$.
\[ \langle \Psi, \pi(a) \Psi \rangle \]. Once \( \omega \) is specified we can express the inner product between essentially any two states \( \phi_1, \phi_2 \) as \( \langle \phi_1, \phi_2 \rangle = \omega(a_1^* a_2) \), where \( \pi(a_{1/2}) \Psi = \phi_{1/2} \). So in summary, given \( \omega \) we can answer any physical question about the system of interest.

The problem faced in field theory is that we are not given a vacuum as a state in some Hilbert space. In general, all we have initially is the algebra \( \mathfrak{A} \) of observables. The question thus is whether we can reverse the above and construct a representation of \( \mathfrak{A} \) just given a positive linear form \( \omega \) on \( \mathfrak{A} \), which should be interpreted as giving the VEV’s of all observables. States arising in this representation should be thought of as excitations of the vacuum corresponding to \( \omega \). The answer is in the affirmative and the procedure is given by the GNS construction detailed below.

### 3.1 The GNS construction

The GNS construction (see, e.g., [9, 12, 10] for more detailed expositions) allows us to construct a representation of any \( * \)-algebra \( \mathfrak{A} \) for any given positive linear form (also called a state) \( \omega \) on this algebra. This is done in three steps:

1. Using \( \omega \), define a scalar product on \( \mathfrak{A} \), regarded as a linear space over \( \mathbb{C} \), by
   \[
   \langle a_1, a_2 \rangle = \omega(a_1^* a_2),
   \]
   for \( a_1, a_2 \in \mathfrak{A} \). The positivity of \( \omega \) implies \( \langle a, a \rangle \geq 0 \). Hence, we are interpreting the elements of the algebra directly as excitations of the vacuum, as motivated above.

2. To obtain a positive definite scalar product, we construct the quotient \( \mathfrak{A}/\mathcal{J} \) of \( \mathfrak{A} \) by the null space \( \mathcal{J} = \{ a \in \mathfrak{A} | \omega(a^* a) = 0 \} \). We denote the equivalence classes in \( \mathfrak{A}/\mathcal{J} \) by \([a]\) and we have:
   \[
   \langle [a], [a] \rangle \equiv \|a\|^2 > 0
   \]
   The completion of \( \mathfrak{A}/\mathcal{J} \) in the above norm is the carrier Hilbert space \( \mathcal{H}_\omega \) for our representation.

3. Finally it can be shown that a representation \( \pi_\omega \) of \( \mathfrak{A} \) on \( \mathfrak{A}/\mathcal{J} \) (which, if \( \mathfrak{A} \) is a Banach \( * \)-algebra, can be extended continuously to \( \mathcal{H}_\omega \)) is given by:
   \[
   \pi_\omega(a) \Psi = [ab],
   \]
   for \( \Psi = b \in \mathcal{H}_\omega \) and \( a \in \mathfrak{A} \).

Hence we have a procedure at our disposal that provides representations of observable algebras describing fluctuations around fiducial backgrounds. Note crucially that this “vacuum” is not defined via a state in some Hilbert space. Indeed, the background need not be realised at all as an element of \( \mathcal{H}_\omega \). Although, if \( \mathfrak{A} \) has a unit \( \mathbb{I} \) then the background corresponds to the cyclic state \([\mathbb{I}]\).
4 New representations for loop quantum gravity

We will now use this technique to construct new representations of loop quantum gravity. The main technical input will be to construct positive linear forms on the algebra of observables that can be interpreted as giving vacuum expectation values of some background.

4.1 Cylindrical function backgrounds

We consider first the most straightforward extension of the standard loop quantum gravity framework. As we have seen the problem there is that sequences of cylindrical functions based on increasingly large graphs are not physical states. The crucial point to this subsection is that these sequences can nevertheless be used to provide us with a definition of a positive linear form on the algebra of observables.

In the following we will look at this construction in more detail. Since this will depend on the precise choice of sequence that one is interested in, we will for simplicity restrict our attention to a specific class of states. Let us assume that we are given an infinite graph $\Gamma_Q$, i.e. a graph that extends throughout a non-compact spatial manifold $\Sigma$ but that has (at most) a finite number of edges and vertices in any compact subregion of $\Sigma$. Associated to each edge $e_i$ of $\Gamma_Q$ there should be a normalised cylindrical function $q_i$ that is a function of the holonomy along $e_i$. This allows to define the states:

$$Q_n = \prod_{i=1}^{n} q_i,$$

for any $n$. As described of above we are interested in the limit $Q = Q_\infty$ of this sequence as $n$ tends to infinity which in general is not an element on $H_{aux}$. Nevertheless, we can define the action of an element of $B_{aux}$ on $Q$. The crucial point is that the elementary quantum observables — the elements of $C$ and the derivations on $C$ — have support on a compact spatial region, which is a direct consequence of the smearing needed to make sense of the classical expressions. Hence given an arbitrary element $a \in B_{aux}$ we proceed as follows:

1. Denote the closure of the support of $a$ by $R \subset \Sigma$.
2. Construct the graph $\Gamma_Q|_R$ of $\Gamma_Q$ restricted to $R$:

$$\Gamma_Q|_R \equiv \bigcup_{e_i \cap R \neq \emptyset} e_i.$$

In other words, consider the union of all edges which have a non-zero intersection with the support of $a$. This graph is finite, since $R$ is compact and
we obtain the state $Q|_R \in \mathcal{H}_{\text{aux}}$ which is given by restricting $Q$ to the graph $\Gamma_Q|_R$:

$$Q|_R \equiv \mathbb{I} \cdot \prod_{e_i \in \Gamma_Q|_R} q_i,$$

where $\mathbb{I}(A) = 1$ for all $A$ is the identity function. This state has unit norm in $\mathcal{H}_{\text{aux}}$ since all the $q_i$’s are normalised.

3. Since $Q|_R \in \mathcal{H}_{\text{aux}}$, the action of $a$ on $Q|_R$ denoted by $\pi(a)|_R$ is well-defined. It is understood that the region $R$ will depend on $a$.

This allows us to define the positive linear form $\omega_Q(a)$:

$$\omega_Q(a) = \langle Q|_R, \pi_s(a) Q|_R \rangle_s. \quad (2)$$

This is well-defined since $Q|_R$ is an element of $\mathcal{H}_{\text{aux}}$ for all $a$. It follows from the fact that equation (1) defines a true inner product $\langle \cdot, \cdot \rangle_s$ on $\mathcal{H}_{\text{aux}}$ that $\omega$ is indeed a positive (not necessarily strictly positive) linear form on $\mathfrak{B}_{\text{aux}}$.

Given $\omega_Q$, we can proceed with steps 2 and 3 of the GNS construction outlined in section 3.1 to obtain a representation of the algebra $\mathfrak{B}_{\text{aux}}$. It turns out, as described in detail in [1], that the representation is equivalent to a very intuitive representation $\pi_Q$ on $\mathcal{H}_{\text{aux}}$:

$$\pi_Q(a)\psi = Q|_R^{-1} \pi_s(a)(Q|_R\psi), \quad (3)$$

where $a \in \mathfrak{B}_{\text{aux}}$, $\psi \in \mathcal{H}_{\text{aux}}$ and $Q|_R^{-1}$ denotes the inverse function\footnote{At this point we note an additional requirement for the background state: $Q(A) \neq 0$ for all $A$. This invertability property is motivated physically since our background state is meant to represent an infinite ‘condensate of gravitons’. We should be able to annihilate as well as create these gravitons, which motivates invertability.} of $Q|_R$ i.e., $Q|_R^{-1} Q|_R = \mathbb{I}$.

Intuitively, the above representation has a clear interpretation. We can regard the algebra of cylindrical functions $\mathfrak{C}$ as creating and annihilating excitations on the background state. More general operators then act on this excited “vacuum”. Hence, we have constructed a Hilbert space and representation of observables on it that describes fluctuations restricted to essentially compact regions around some fixed infinite background state. Note that this representation is truly inequivalent to the standard one. Roughly, since $Q|_R$ depends on the algebra element $a$, equation (3) does not define a unitary map.

The construction we have presented is very general and can be applied to a large class of background states. The advantages of this approach are that the final formalism is very simple. One can use the same (separable) Hilbert space as in the standard representation and in particular the reduction by the constraints can be carried out as in the standard approach. To study quantum gravity on semi-classical, asymptotically flat geometries one need states that approximate phase space points. There is now a variety of such cylindrical functions available that can be used in the above approach, c.f. [1, 17, 8].
4.2 Mixed backgrounds

Here we look at a possible improvement of the above approach. A difficulty in studying and interpreting the classical limit of loop quantum gravity is the fact that states are supported on graphs or more physically that we have a quantum picture of polymer like excitations of geometry. The familiar continuum picture has to be recovered from the study of coarse grained observables. In particular, we would like to approximate classical values of observable functions at a particular phase space point to increasing accuracy with the expectation values of corresponding observable operators in some semi-classical state as \( \hbar \to 0 \). These conditions are not enough to specify a unique state. In particular, the graph on which the semi-classical state is to be based is left largely undetermined. This is due to the fact that in studying the classical limit we are using operators that are too coarse grained to determine the micro structure of the quantum state completely.

This suggests naturally that we should really be considering a statistical mixture of states, which is in fact analogous to what is done in thermal field theory, where the vacuum state is only specified by the macroscopic temperature.

Hence, the “gravitational vacuum” is composed of many subsystems, each described by their own micro-state \( |\phi_i\rangle \). Given the set of macroscopic variables that characterise the vacuum let us denote the probability that a particular subsystem will be in the state \( |\phi_i\rangle \) by \( P(\phi_i) \). The gravitational vacuum is then given by the density matrix:

\[
\rho = \sum_i P(\phi_i) |\phi_i\rangle \langle \phi_i|,
\]

where \( |\phi_i\rangle \langle \phi_i| \) denotes the projector onto the micro-state \( |\phi_i\rangle \). This can be used to define a positive linear functional on the algebra of observables:

\[
\omega(a) \equiv Tr[\rho a],
\]

which gives rise to the desired quantum theory. Again we are especially interested in the case that the density matrix describes asymptotically flat geometries. In this case the micro-states \( \phi_i \) will be based on infinite graphs and we need to make use of the techniques of the preceding section to make sense of the above linear form.

In the case that the macroscopic observable characterising the gravitational background is the volume of regions of the spatial manifold \( \Sigma \) there is a natural construction to implement the above based on random lattices, c.f. [18, 7]. This gives us a mixture of states based on a large class of graphs.

4.3 Chern Simons backgrounds

The preceding constructions depend on the ability to approximate classical phase space points with cylindrical function states. While, progress has been made in
this direction a major unaddressed issue is still the dynamics. No semi-classical
cylindrical function state has been proposed so far that solves all the constraints,
which should be a necessary criterion for any true physical state.

The power of the GNS construction proposed here is that one is not tied to using
cylindrical functions to define suitable approximations of 3-geometries. As we have
seen all that is needed is the definition of a positive linear form on the observable
algebra, which can be interpreted as giving the vacuum expectation values in some
preferred state. This section will be devoted to the study of one such alternative
based on the Chern Simons state. While more heuristic at present we believe that
the following approach has many promising features.

The Chern Simons state was discovered early on \[ \text{[11]} \] as an exact solution to all the
constraints of quantum general relativity. The term “state” here is used here in a
heuristic sense, as it is not an element of a known Hilbert space. Rather, it is defined
as a function on the classical configuration space, the space of connections:

\[
\Psi_{CS}(A) \equiv \exp \left( -\frac{6}{\Lambda} \int_{\Sigma} Tr[A \wedge dA + \frac{2}{3} A \wedge A \wedge A] \right),
\]

where $\Lambda$ is the cosmological constant. This state has generated a lot of interest as
it has a well-defined classical limit corresponding to anti-DeSitter spa-

te. Study of this state has been mainly within the loop representation of loop quantum

gravity, which is in a precise sense dual to the one described in this report. Here
states are functions of loops or, more generally, graphs embedded in $\Sigma$ instead of
connections. Given a cylindrical function $\Psi_{f,\gamma}$ based (for simplicity, generalisations
to graphs are straightforward) on a loop $\gamma$ we can transform to the following state
in the loop representation:

\[
\tilde{\Psi}(\gamma) = \int Tr[H(\gamma, A)]\Psi_{f,\gamma}(A)d\mu(A),
\]

using the cylindrical function measure defined in eq. \((4)\). One can attempt to do
the same with the Chern-Simons state and define:

\[
\tilde{\Psi}_{CS}(\gamma) = \int Tr[H(\gamma, A)]\Psi_{CS}(A)dA
\]  

Since no appropriate measure on the space of connections is known the above is
only heuristic. But crucially, one can nevertheless define the integral by using Wit-

ten’s celebrated result that the above transformation defines knot invariants of any
embedded loop in $\Sigma$ (see e.g. \[ \text{[14]} \] for an overview). This implies that if we want
to interpret the Chern-Simons state as a cylindrical function then it would have
support on all possible loops and graphs in $\Sigma$.

This is appealing but also leads directly to the fact that the Chern Simons state is
not a physical state of loop quantum gravity as it cannot be normalised with respect
to any known inner product. The key to the remainder of this section will again
be the fact that we can still use the Chern-Simons state to define a positive linear
function on the algebra of observables and thus define a quantum theory describing
fluctuations of the Chern-Simons state, which following arguments in [16] should
correspond to field theory on Anti-DeSitter space in the semi-classical limit.

So to proceed we propose to define $\omega$ via:

$$\omega(a) \equiv \int dA \Psi_{CS}^*(A) \hat{a} \Psi_{CS}(A)$$

and we make the following comments:

1. The above integral should be interpreted as giving the expectation value of the
algebra element $a$ in the state $\Psi_{CS}$. Note that this differs from eq. (4) by the
inclusion of the complex conjugate term. This should not be confused with the
calculation of vacuum expectation values evaluated in Chern-Simons theory,
which is 3-dimensional and where eq. (4) is interpreted as a path integral.

2. Because we need to include the complex conjugate term it is crucial that
the exponent in the Chern-Simons state is real. This implies that we use
the original complex, $SL(2, \mathbb{C})$, connection formulation of general relativity
and also that this approach is only likely to be interesting in the Lorentzian
framework.

3. The above integral is of course only heuristic as we have no suitable measure
on the space of connections. Again we want to make contact with the fact
that if the algebra element $a$ is a cylindrical function, i.e. corresponds to a
configuration variable then the above should define a diffeomorphism invariant
of the graph underlying the definition of the cylindrical function.

4. When attempting to define the above integral in this way we need to take
care that we are dealing with the non-compact gauge group $SL(2, \mathbb{C})$, which
has to our knowledge not been studied in the context of Chern-Simons knot
invariants. A possible way to avoid this difficulty by splitting the connection
into real and complex parts is suggested in [16]. Note also that a rigorous
treatment of the above integrals requires both framings of the manifold $\Sigma$ and
of the graphs supporting the cylindrical functions. Progress in this direction
has been made in [13].

5. To complete this approach we also need to define the integral when $a$ is a
momentum variable, i.e. a triad smeared over a two surface. To our knowledge
this has not been investigated in the literature so far. Intuitively, one expects
to get a diffeomorphism invariant of the two surfaces on which the momentum
variables are defined.
These techniques provide a possible approach to study the Chern-Simons state as a physical state of a well-defined theory of quantum gravity. The intimate relation between loop quantum gravity, knot theory and Chern-Simons theory gives strong support for the study of this state.

5 Conclusions

In this report we have focussed on three main themes. Firstly, we have motivated why the study of the asymptotically flat sector of quantum general relativity is important and should be pursued actively at the present stage in the non-perturbative quantum gravity programme. Restricting our attention to asymptotically flat space allows us to avoid many conceptual problems facing quantum gravity while at the same time enabling the study of a large number of physical applications. In short, the quantisation of general relativity is most likely to succeed and produce meaningful results in the asymptotically flat sector.

We then looked at how the standard framework of loop quantum gravity can be extended to the asymptotically flat sector or more generally, to the case where the spatial slice Σ is non-compact. Here we followed an analogy with thermal field theory and used the GNS construction to provide us with new representations of the observable algebra of loop quantum gravity. This gave rise to quantum theories that can be interpreted as describing excitations of fiducial fixed background states — vacua of loop quantum gravity.

In the last part of this report we discussed three possible approaches in constructing such background states. In particular this addresses the issue of how classical phase space data or classical 3-geometries can be approximated quantum mechanically. Together with the GNS construction these backgrounds give us a very natural approach to study physical applications of loop quantum gravity, especially the semi-classical limit. Hopefully, this will enable us to make essential progress in uncovering the quantum picture of space and time that loop quantum gravity provides us.

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