UNIVERSAL GROUPS OF INTERMEDIATE GROWTH AND THEIR INVARIANT RANDOM SUBGROUPS

MUSTAFA GÖKHAN BENLI, ROSTISLAV GRIGORCHUK AND TATIANA NAGNIBEDA

To A.M. Vershik on the occasion of his 80-th birthday, with admiration and respect

ABSTRACT. We exhibit examples of groups of intermediate growth with $2^{\aleph_0}$ ergodic continuous invariant random subgroups. The examples are the universal groups associated with a family of groups of intermediate growth.

1. Introduction

The goal of this paper is to show the existence of groups of intermediate growth with $2^{\aleph_0}$ distinct ergodic continuous invariant random subgroups.

Invariant random subgroup (abbreviated IRS) is a convenient term that stands for a probability measure on the space of subgroups in a locally compact group, invariant under the action of the group by conjugation. In the case of a countable group $G$ (only such groups will be considered here), the space $S(G)$ of subgroups of $G$ is supplied with the topology induced from the Tychonoff topology on $\{0,1\}^G$ where a subgroup $H \leq G$ is identified with its characteristic function $\chi_H(g) = 1$ if $g \in H$ and 0 otherwise.

The delta mass corresponding to a normal subgroup is a trivial example of an IRS, as well as the average over a finite orbit of delta masses associated with groups in a finite conjugacy class. Hence, we are rather interested in continuous invariant probability measures on $S(G)$. Clearly, such a measure does not necessarily exist, for example if the group only has countably many subgroups.

Given a countable group $G$, a basic question is whether a continuous IRS exists. Ultimately one wants to describe the structure of the simplex of invariant probability measures of the topological dynamical system $(\text{Inn}(G), S(G))$ where $\text{Inn}(G)$ is the group of inner automorphisms of $G$ acting on $S(G)$. Of particular interest are ergodic measures, i.e., the extremal points in the simplex.

A more general problem is the identification of the simplex of invariant probability measures of the system $(\Phi, S(G))$ where $\Phi$ is a subgroup of the group $\text{Aut}(G)$ of automorphisms of $G$ (see [AGV12, Bow12, Ver12]). A closely related problem is the study of invariant measures on the space of rooted Schreier graphs of $G$, with $G$ acting by change of the root. This point of view is presented in [Gri11, Vor12].

Date: June 26, 2015.

The first and second authors were supported by NSF Grant DMS-1207699. The second and the third author were supported by the Swiss National Science Foundation.
A very fruitful idea in the subject belongs to Anatoly Vershik who introduced the notion of a totally non-free action of a locally compact group $G$ on a space $X$ with invariant measure $\mu$, i.e., an action with the property that different points $x \in X$ have different stabilizers $St_G(x)$ $\mu$-almost surely. Then the map $St : X \to S(G)$ defined by $x \mapsto St_G(x)$ is injective $\mu$-almost surely and the image of $\mu$ under this map is the law of an IRS on $G$ which is continuous and ergodic whenever $\mu$ is. In [Ver12], Vershik showed that a totally non-free action of a group $G$ provides us not only with an IRS but also with a factor representation of $G$. He also realized the plan outlined above and described all the ergodic $Aut(G)$-invariant measures on $S(G)$ in the case when $G$ is the infinite symmetric group, see [Ver10, Ver12].

Bowen showed in [Bow12] that non-abelian free groups of finite rank possess a whole “zoo” of ergodic continuous IRS, and that a big part of the simplex of IRS on a free group $F_r$, $r \geq 2$, is a Poulsen simplex. (A simplex is called a Poulsen if its extremal points are dense. It is unique up to affine isomorphism by [LOS78].)

As shown in [BGK12], already the so-called lamplighter group $L = \mathbb{Z}_2 \wr \mathbb{Z}$ (the "simplest" finitely generated group that has $2^{\aleph_0}$ subgroups) has a Poulsen simplex of IRS. Given a surjection $\phi : G \twoheadrightarrow H$, there is a natural homeomorphism $\bar{\phi} : S(H) \to S(G, Ker(\phi))$ where $S(G, Ker(\phi))$ denotes the subspace of $S(G)$ consisting of subgroups of $G$ containing $Ker(\phi)$. This allows to lift any IRS on $H$ to $G$ thus providing a large spectrum of IRS on $G$ from that on $H$. This applies in particular to the free group $F_2$ that covers $L$.

A finitely generated virtually nilpotent group has only countably many subgroups and therefore does not possess any continuous IRS. By Gromov’s theorem, the class of finitely generated virtually nilpotent groups coincide with the class of groups of polynomial growth. Recall that, given a finitely generated group $G$ with a system of generators $S$, one can consider its growth function $\gamma(n) = \gamma(G,S)(n)$ which counts the number of elements of length at most $n$. The growth type of this function when $n \to \infty$ does not depend on the generating set $S$ and can be polynomial, exponential or intermediate. The question of existence of groups of intermediate growth was raised by Milnor [Mil68] and was answered by the second author in [Gri84b]. The main construction associates with every sequence $\omega \in \Omega = \{0,1,2\}^\mathbb{N}$ a group $G_\omega$ generated by four involutions $a_\omega, b_\omega, c_\omega, d_\omega$ and if $\omega$ is not an eventually constant sequence, then $G_\omega$ has intermediate growth. Moreover, it was also observed in [Gri84b] that the groups $G_\omega$ fall into the class of just-infinite branch groups. A group is just infinite if it is infinite but every proper quotient is finite. A group is branch if it has a faithful level transitive action on a spherically homogeneous rooted tree with the property that rigid stabilizers of the levels of the tree are of finite index, see Section 4.2 for precise definitions. Just-infinite branch groups constitute one of three classes in which the class of just-infinite groups naturally splits [Gri00].

Since the groups $G_\omega$ are just-infinite, they only have countably many quotients. This raised the question of existence of groups of intermediate growth having $2^{\aleph_0}$
quotients, answered in [Gri84a]. The main idea was to take a suitable subset \( \Lambda \subset \Omega \) of cardinality \( 2^{\aleph_0} \) and consider the group \( U_\Lambda \) defined as the quotient of the free group \( F_4 \) by a normal subgroup \( N \) which is the intersection of normal subgroups \( N_\omega, \omega \in \Lambda \) where \( G_\omega = F_4/N_\omega \). In this paper we explore this idea further by using IRS on \( G_\omega \) and lift them to \( U_\Lambda \) deducing the main result.

Branch groups give us the most transparent examples of totally non-free actions and thus of IRS. Indeed, as shown in [BG02], the natural extension of a branch action on a spherically homogeneous tree \( T \) to its boundary \( \partial T \) is totally non free with respect to the uniform probability measure on \( \partial T \). It is even completely non free, i.e., different points have different stabilizers. The uniform probability measure on \( \partial T \) is ergodic and invariant. The groups \( G_\omega \) act on the binary rooted tree in a branch way. By lifting the uniform measure to \( S(G_\omega) \) and then to \( S(U_\Lambda) \), one obtains a host of IRS on \( U_\Lambda \). We then proceed to showing that the IRS obtained in this way are distinct. These considerations allow us to prove our main theorem:

**Main Theorem.** There exists a finitely generated group of intermediate growth with \( 2^{\aleph_0} \) distinct continuous ergodic invariant random subgroups.

We also investigate some additional properties of groups of the form \( U_\Lambda, \Lambda \subset \Omega \), including finite presentability, branching property and self-similarity.

## 2. Space of Marked Groups and Universal groups

**Definition 1.** A \( k \)-marked group is a pair \((G, S)\), where \( G \) is a group and \( S = (s_1, \ldots, s_k) \) is an ordered set of (not necessarily distinct) elements such that the set \( \{s_1, \ldots, s_k\} \) generates the group \( G \). The canonical map between two \( k \)-marked groups \((G, S)\) and \((H, T)\) is the map sending \( s_i \mapsto t_i \ i = 1, 2, \ldots, k \). If this map defines an epimorphism, it will be called the marked epimorphism and \((H, T)\) will be called a marked image of \((G, S)\). Two \( k \)-marked groups \((G, S)\) and \((H, T)\) are equivalent if the canonical map defines an isomorphism between \( G \) and \( H \).

The space of (equivalence classes of) \( k \)-marked groups will be denoted by \( \mathcal{M}_k \). This space has a natural topology, which for instance can be defined by the following metric: Two \( k \)-marked groups \((G, S)\) and \((H, K)\) are of distance \( 2^{-m} \), where \( m \) is the largest natural number such that the balls of radius \( m \) of the Cayley graphs of \((G, S)\) and \((H, K)\) are isomorphic (as directed labeled graphs). In [Gri84b] it was observed that this makes \( \mathcal{M}_k \) into a compact totally disconnected space.

Alternatively, this space can be defined in the following way: Let \( F_k \) be a free group of rank \( k \) with a basis \( \{x_1, \ldots, x_k\} \). Let \( \mathcal{N}_k \) denote the set of all normal subgroups of \( F_k \), together with the topology inherited from the power set \( \mathcal{P}(F_k) \cong \{0, 1\}^F_k \) supplied with the Tychonoff topology. This topology has basis consisting of sets of the form \( \mathcal{O}_{A,B} = \{N < F_k \mid A \subset N, B \cap N = \emptyset\} \) where \( A \) and \( B \) are finite subsets of \( F_k \). Given \((G, S) \in \mathcal{M}_k\), let \( N_{(G,S)} \in \mathcal{N}_k \) be the kernel of the natural map \( \pi_{(G,S)} : F_k \rightarrow G \) sending \( x_i \mapsto s_i \). This gives a homeomorphism
between \( M_k \) and \( N_k \) (depending on the basis of \( F_k \)) (See [Cha00]). We will interchangeably use these two spaces.

**Definition 2.** Let \( \mathcal{C} = \{(G_i, S_i) \mid i \in I\} \) be a subset of \( M_k \). Let \( N_{\mathcal{C}} = \bigcap_{i \in I} N_{(G_i, S_i)} \). The universal group of the family \( \mathcal{C} \) is the \( k \)-marked group \( (U_{\mathcal{C}}, S_{\mathcal{C}}) \) where \( U_{\mathcal{C}} = F_k / N_{\mathcal{C}} \) and \( S_{\mathcal{C}} \) is the image of the basis \( \{x_1, \ldots, x_k\} \).

\( U_{\mathcal{C}} \) has the following universal property: If \( (H, T) \) is a marked group such that for all \( i \in I \) the canonical map from \( (H, T) \) to \( (G_i, S_i) \) defines a group homomorphism, then the canonical map from \( (H, T) \) to \( (U_{\mathcal{C}}, S_{\mathcal{C}}) \) defines a group homomorphism.

An alternative way to define the universal group is the following:

**Definition 3.** Given \( \mathcal{C} = \{(G_i, S_i) \mid i \in I\} \subseteq M_k \), write \( S_i = (s_{i_j}^{j_1}, \ldots, s_{i_j}^{j_k}) \). Let \( U^{\text{diag}}_{\mathcal{C}} \) be the subgroup of the (unrestricted) direct product \( \prod_{i \in I} G_i \) generated by the elements \( s_i = (s_{i_j}^{j_1})_{i \in I} j = 1, \ldots, k \). The \( k \)-marked group \( (U^{\text{diag}}_{\mathcal{C}}, S^{\text{diag}}_{\mathcal{C}}) \) is called the diagonal group of the family \( \mathcal{C} \).

It is straightforward to check that \( (U^{\text{diag}}_{\mathcal{C}}, S^{\text{diag}}_{\mathcal{C}}) \) equivalent (as a marked group) to the universal group \( (U_{\mathcal{C}}, S_{\mathcal{C}}) \) of Definition 2.

**Proposition 1.** Let \( \mathcal{C} \subseteq M_k \). Then the marked groups \( (U_{\mathcal{C}}, S_{\mathcal{C}}) \) and \( (U_{\overline{\mathcal{C}}}, S_{\overline{\mathcal{C}}}) \) are equivalent, where \( \overline{\mathcal{C}} \) denotes the closure of \( \mathcal{C} \) in \( M_k \).

**Proof.** We need to show that

\[
\bigcap_{(G, S) \in \mathcal{C}} N_{(G, S)} = \bigcap_{(G, S) \in \overline{\mathcal{C}}} N_{(G, S)}.
\]

Clearly, the right hand side is contained in the left. Suppose that some \( g \in F_k \) belongs to the left hand side but not to the right. Then there exists \( (G, S) \in \mathcal{C} \) such that \( g \notin N_{(G, S)} \). Let \( \{(G_n, S_n)\}_{n \geq 0} \) be a sequence in \( \mathcal{C} \) converging to \( (G, S) \). Since \( g \) belongs to the left hand side, \( g \) belongs to each \( N_{(G_n, S_n)} \) and by definition of the topology in \( N_k \), to \( N_{(G, S)} \) which gives a contradiction.

For an element \( w \in F_k, w \neq 1 \), denote \( O_w = \{N \lhd F_k \mid w \in N\} \).

**Lemma 1.** Let \( H \leq F_k \) be a subgroup and \( w_1, \ldots, w_m \in H, w_i \neq 1 \) for each \( i \). Then there exists \( w \in H, w \neq 1 \) such that \( \bigcup_{i=1}^m O_{w_i} \subset O_w \).

**Proof.** By induction on \( m \). The case \( m = 1 \) is clear, one can take \( w = w_1 \). So, assume \( m > 1 \).

**Case 1:** \([w_1, w_2] = 1 \) in \( F_k \). In this case there exists \( w \in F_k \) and \( s, t \in \mathbb{Z} \) such that \( w_1^s = w_2^t = w \) (all non-trivial abelian subgroups in a free group are cyclic). Therefore, \( O_{w_1} \cup O_{w_2} \subset O_w \) and hence we can apply the induction hypothesis by replacing \( O_{w_1} \) and \( O_{w_2} \) by \( O_w \).

**Case 2:** \([w_1, w_2] \neq 1 \) in \( F_k \). In this case we can replace \( O_{w_1} \) and \( O_{w_2} \) by \( O_{[w_1, w_2]} \) and apply the induction hypothesis.

\( \square \)
Proposition 2. Let $C \subset M_k$ be a closed subset and assume that no group in $C$ contains a nonabelian free subgroup. Then the universal group $U_C$ also has no nonabelian free subgroups.

Proof. Let $C = \{(G_i, S_i) \mid i \in I\}$. Let $a, b \in U_C$ be two distinct elements, given as words in the generators $S_i$. Let $w_a, w_b \in F_k$ such that $\pi_{(G, S)}(w_a) = a$ and $\pi_{(G, S)}(w_b) = b$. For each $i \in I$, since $G_i$ has no (non-abelian) free subgroups, there is nontrivial $w_i \in \langle w_a, w_b \rangle \leq F_k$ such that $\pi_{(G_i, S_i)}(w_i) = 1$, i.e., $w_i \in N_{G_i, S_i}$. Hence $\{O_w\}_{i \in I}$ is an open cover of $C$. Since $C$ is compact, there is a finite subcover $O_{w_1}, \ldots, O_{w_n}$. By Lemma [1] there exists non-trivial $w \in \langle w_a, w_b \rangle$ such that $C \subset O_w$. This shows that $w = 1$ in $U_C$. □

3. Grigorchuk 2-groups

We recall here the construction of [Gri84b]. Note that in the original construction in [Gri84b] the groups are defined as measure preserving transformations of the unit interval. Here we will define them as groups of automorphisms of the binary rooted tree.

Let $\Omega = \{0, 1, 2\}^\mathbb{N}$ be the space of infinite sequences $\omega = \omega_1 \omega_2 \ldots \omega_n \ldots$ where $\omega_i \in \{0, 1, 2\}$, considered with its natural product topology. Let $\tau$ be the shift transformation, i.e., if $\omega = \omega_1 \omega_2 \ldots \in \Omega$ then $\tau \omega = \omega_2 \omega_3 \ldots$. Let $T$ be the binary rooted tree whose vertices are identified with the set of all finite binary words $\{0, 1\}^*$ and edges defined in standard way: $E = \{(w, wx) \mid w \in \{0, 1\}^*, x \in \{0, 1\}\}$. For each $\omega \in \Omega$, consider the automorphisms $\{a, b_\omega, c_\omega, d_\omega\}$ of $T$ defined recursively as follows:

For $v \in \{0, 1\}^*$

$$a(0v) = 1v \text{ and } a(1v) = 0v$$

$$b_\omega(0v) = 0\beta(\omega_1)(v) \quad c_\omega(0v) = 0\zeta(\omega_1)(v) \quad d_\omega(0v) = 0\delta(\omega_1)(v)$$

$$b_\omega(1v) = 1b_{\tau \omega}(v) \quad c_\omega(1v) = 1c_{\tau \omega}(v) \quad d_\omega(1v) = 1d_{\tau \omega}(v),$$

where

$$\beta(0) = a \quad \zeta(0) = a \quad \delta(0) = e$$

$$\beta(1) = a \quad \zeta(1) = e \quad \delta(1) = a$$

$$\beta(2) = e \quad \zeta(2) = a \quad \delta(2) = a$$

and $e$ denotes the identity automorphism of $T$.

For each $\omega \in \Omega$, let $G_\omega$ be the subgroup of $Aut(T)$ generated by the set $S_\omega = \{a, b_\omega, c_\omega, d_\omega\}$ so that $G = \{(G_\omega, S_\omega) \mid \omega \in \Omega\}$ is a subset of $M_4$. In [Gri84b] it was observed that if two sequences $\omega, \eta \in \Omega$ which are not eventually constant, have long common beginning, then the 4-marked groups $(G_\omega, S_\omega)$ and $(G_\eta, S_\eta)$ are close to each other in $M_4$. It was also observed that the groups $(G_\omega, S_\omega)$ for eventually constant sequences $\omega$ are isolated in $\{(G_\omega, S_\omega) \mid \omega \in \Omega\}$. Hence, removing these isolated points from this set and taking its closure in $M_4$, one obtains a compact

---

1 The first and the third authors insist on using this standard terminology.
subset $\mathcal{G} = \{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega\} \subset \mathcal{M}_4$ which is homeomorphic to $\Omega$ (and hence to a Cantor set) via $\omega \mapsto (\tilde{G}_\omega, \tilde{S}_\omega)$. Note that $(\tilde{G}_\omega, \tilde{S}_\omega) = (G_\omega, S_\omega)$ if and only if $\omega$ is not eventually constant and $(\tilde{G}_\omega, \tilde{S}_\omega) = \lim_{n \to \infty} (G_{\omega(n)}, S_{\omega(n)})$ when $\omega$ is eventually constant and where $\{\omega^{(n)}\}_{n \geq 0}$ is a sequence of not eventually constant elements in $\Omega$ converging to $\omega$ (the limit does not depend on the choice of the sequence $\{\omega^{(n)}\}_{n \geq 0}$). In other words, the families $\mathcal{G}$ and $\mathcal{\tilde{G}}$ differ only on countably many points.

Note that we have the following: $N((\tilde{G}_\omega, \tilde{S}_\omega)) = N(G_\omega, S_\omega)$ if $\omega$ is not eventually constant and $N((\tilde{G}_\omega, \tilde{S}_\omega)) \subset N(G_\omega, S_\omega)$ for eventually constant $\omega \in \Omega$.

Let $\Omega^\infty$ be the set of sequences in $\Omega$ in which all three letters $\{0, 1, 2\}$ occur infinitely often and $\Omega_0$ be the set of eventually constant sequences. Regarding the groups in $\mathcal{G}$ and $\mathcal{\tilde{G}}$ the following are known:

Theorem 1 ([Gri84b]).

1. All groups $G_\omega$, $\omega \in \Omega$ are infinite residually finite groups.
2. $G_\omega$ is virtually $\mathbb{Z}^m$ if $\omega$ becomes constant starting with $n$-th coordinate.
3. If $\omega \notin \Omega_0$ then $G_\omega$ has intermediate growth between polynomial and exponential.
4. If $\omega \in \Omega_0$ then $\tilde{G}_\omega$ is virtually metabelian, infinitely presented and has exponential growth.
5. If $\omega \in \Omega^\infty$ then $G_\omega$ is a torsion 2-group.
6. If $\omega \in \Omega^\infty$ then $G_\omega$ is just-infinite, i.e., all its nontrivial quotients are finite.
7. For $\omega_1, \omega_2 \in \Omega^\infty$ we have $G_{\omega_1} \cong G_{\omega_2}$ if and only if $\omega_1$ can be obtained from $\omega_2$ by applying a permutation from $\text{Sym}(\{0, 1, 2\})$ letter by letter.

Proof. For proofs of (1),(2),(3) and (5) see [Gri84b] Theorem 2.1. (4) is proven in [Gri84b] Theorem 6.1,6.2 and (6) in [Gri84b] Theorem 8.1. (7) is proven in [Nek05] Theorem 2.10.13. \hfill \Box

4. SOME PROPERTIES OF THE FULL UNIVERSAL GROUP $U$

Regarding the universal groups corresponding to the families $\mathcal{G}$ and $\mathcal{\tilde{G}}$ we have the following:

Proposition 3. $U_{\mathcal{G}} = U_{\mathcal{\tilde{G}}}.$

Proof. Referring to the notation of Definition 2 we need to show the following equality:

$$N_{\mathcal{G}} := \bigcap_{\omega \in \Omega} N(G_\omega, S_\omega) = N_{\mathcal{\tilde{G}}} := \bigcap_{\omega \in \Omega} N(\tilde{G}_\omega, \tilde{S}_\omega).$$

Since $N(\tilde{G}_\omega, \tilde{S}_\omega) \subset N(G_\omega, S_\omega)$ for each $\omega \in \Omega$, the right-hand side of the above equation is contained in the left-hand side. Since $\{(\tilde{G}_\omega, \tilde{S}_\omega) \mid \omega \in \Omega \setminus \Omega_0\}$ is dense in $\mathcal{\tilde{G}}$, by
Proposition: we have
\[ N_{\tilde{g}} = \bigcap_{\omega \in \Omega \setminus \Omega_0} N((\tilde{g}_\omega, \tilde{s}_\omega)). \]

Therefore,
\[ N_{\tilde{g}} \subset \bigcap_{\omega \in \Omega \setminus \Omega_0} N((\tilde{g}_\omega, S_\omega)) = \bigcap_{\omega \in \Omega \setminus \Omega_0} N((\tilde{g}_\omega, \tilde{s}_\omega)) = N_{\tilde{g}}. \]

We will use the notation \( U = U_G \) for the full universal group and denote by \( S = \{a, b, c, d\} \) its canonical generators. Note that the basic relations \( a^2 = b^2 = c^2 = d^2 = bcd = 1 \) hold in \( U \).

**Theorem 2.** \( U \) contains no nonabelian free subgroups, has uniformly exponential growth and is not finitely presented.

*Proof.* Since all groups in \( \tilde{G} \) are amenable (and hence cannot contain nonabelian free subgroups), the first assertion follows from Proposition 2. By Theorem 1 part (4), the group \( \tilde{G}_\eta \) for \( \eta = 000 \ldots \) is an elementary amenable group of exponential growth, and hence of uniformly exponential growth by [Osi04]. Therefore \( U \) has uniformly exponential growth. By [BGdH13, Theorem 1.10], any finitely presented group mapping onto the groups \( G_\omega, \omega \in \Omega \) must be large, i.e., has a finite index subgroup mapping onto a nonabelian free group. In particular, such group contains a nonabelian free subgroup. Therefore \( U \) cannot be finitely presented. \( \square \)

4.1. \( U \) as an automaton group.

In this section we will realize \( U \) as an automaton group and explore further properties. Firstly, we will recall some basics.

Let \( T_d \) denote the \( d \)-ary rooted tree with vertex set \( \{0, 1, 2, \ldots, d - 1\}^* \). For an automorphism \( g \in Aut(T_d) \) and \( x \in \{0, 1, \ldots, d - 1\} \), the section of \( g \) at \( x \) (denoted by \( g_x \)) is the automorphism of \( T_d \) defined uniquely by
\[ g(xv) = g(x)g_x(v) \quad \text{for all} \quad v \in \{0, 1, \ldots, d - 1\}. \]

This gives an isomorphism
\[ Aut(T_d) \rightarrow S_d \times (Aut(T_d) \times \cdots \times Aut(T_d)) \]
\[ g \mapsto \sigma_g(g_0, \ldots, g_{d-1}) \]
where \( \sigma_g \) describes how \( g \) permutes the first level subtrees and \( g_i \) describes its action within each subtree. (Here \( S_d \) is the symmetric group on \( d \) letters).

**Definition 4.** A subgroup \( G \leq Aut(T_d) \) is called self-similar if for all \( g \in G \) and \( x \in \{0, 1, \ldots, d - 1\} \), \( g_x \in G \).

For an overview of self-similar groups and related topics we refer to [GS07].
A standard way to construct self-similar groups is to start with a list of symbols $S = \{s^1, \ldots, s^m\}$ and permutations $\sigma_1, \ldots, \sigma_m \in S_d$ and consider the system

$s^1 = \sigma_1(s^1_0, \ldots, s^1_{d-1})$

$\vdots$

$s^m = \sigma_m(s^m_0, \ldots, s^m_{d-1})$

where $s^i_j \in S$. If $\sigma_i = id$, we will omit writing it. Such a system defines a unique set of $m$ automorphisms of $T_d$. Clearly the group $G = \langle S \rangle$ will be self-similar. Since in this case the generating set $S$ is closed under taking sections, the action of the group can be modeled by a Mealy type automaton where each generator will correspond to a state of the automaton (see the figure below for an example). Such groups, i.e., groups generated by the states of a Mealy type automaton are called automata groups. We refer to [GNS00] for a detailed account on automata groups.

Consider the tree $T_6$ determined by the alphabet $A = \{0, 1\} \times \{0, 1, 2\} = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ whose elements are enumerated as $0, 1, \ldots, 5$. Let $V \leq Aut(T_6)$ be generated by the following elements:

\begin{equation}
\begin{array}{ll}
A & = (14)(25)(36) \quad (E, E, E, E, E) \\
B & = (A, A, E, B, B, B) \\
C & = (A, E, A, C, C, C) \\
D & = (E, A, A, D, D, D)
\end{array}
\end{equation}

where, $(14)(25)(36)$ is an element of the symmetric group $S_6$ and $E$ corresponds to the identity automorphism. Observe that $A^2 = B^2 = C^2 = D^2 = BCD = 1$. The corresponding automaton is as follows:
We will show that the group $V$ is isomorphic to $U$ (as a marked group).

Given $\omega \in \Omega$ and $u \in \{0,1\}^*$ let $\omega^u \in \{0,1,2\}^*$ be the beginning of $\omega$ of length $|u|$. Note that
\[
\omega^{uv} = \omega^u (\tau^{u|\omega})^v
\]
for all $u, v \in \{0,1\}^*$

For any $\omega \in \Omega$ let $T_\omega = \{(u, v) \in T_6 \mid u \in \{0,1\}^*, v = \omega^u\}$. Clearly $T_\omega$ is a binary subtree of $T_6$. Denote $\{0,1\}^*$ by $T_2$ and let $\phi_\omega : T_\omega \to T_2$ be defined as
\[
\phi_\omega(u, v) = u
\]
which clearly is a bijection. For $(u, v) \in T_\omega$ and $(u', v') \in T_{\tau |\omega}$ we have
\[
(2) \quad \phi_\omega(uu', vv') = \phi_\omega(u, v) \phi_{\tau|\omega}(u', v')
\]

Given $g \in V$ and $\omega \in \Omega$ define a group homomorphism $\psi_\omega : V \to \text{Aut}(T_2)$ by
\[
\psi_\omega(g)(u) = \phi_\omega(g(u, \omega^u)) \quad \text{for all} \quad u \in T_2
\]

It is straightforward to verify that $\psi_\omega(g) \in \text{Aut}(T_2)$ and that $\psi_\omega$ defines a group homomorphism.

**Lemma 2.** For all $u \in T_2$ with $|u| = n$ have
\[
\psi_\omega(g)_u = \psi_{\tau^n \omega}(g_{(u, \omega^u)})
\]

**Proof.** Let $u, z \in \{0,1\}^*, |u| = n$ and denote $\omega^u = v, (\tau^n \omega)^z = v'$
\[
g(uz, \omega^{uz}) = g(uz, vv') = g((u, v)(z, v')) = g(u, v)g(u,v)(z, v')
\]
Hence, by Equation (2)
\[
\psi_\omega(g)(uz) = \phi_\omega(g(uz, \omega^{uz})) = \phi_\omega(g(u, v)g(u,v)(z, v')) = \phi_\omega(g(u, v)) \phi_{\tau^n \omega}(g_{(u,v)}(z, v'))
\]
\[
= \psi_\omega(g)(u) \psi_{\tau^n \omega}(g_{(u,v)})(z)
\]
The result follows. \qed

**Lemma 3.** For any $\omega \in \Omega$, $\psi_\omega$ defines a marked surjective homomorphism $\psi_\omega : V \to G_\omega$.

**Proof.** It is enough to show that $\psi_\omega$ maps generators of $V$ to the generators of $G_\omega$.
Firstly, by definition of $A$ we have
\[
\psi_\omega(A)(u) = \phi_\omega(A(u, \omega^u)) = \phi_\omega((a(u), \omega^u)) = a(u) \quad \text{for all} \quad u.
\]
We will show by induction on $|u|$ that $B, C, D$ are mapped to $b_\omega, c_\omega, d_\omega$ respectively. If $|u| = 1$ it is straightforward to check this. Using Lemma 2 and induction assumption we have for $u \in \{0,1\}^*$
\[
\psi_\omega(B)(0u) = 0\psi_\omega(B)(0u) = 0\psi_\omega(B_{(0,\omega^0)})(u) = \begin{cases} 0a(u) & \text{if } \omega^0 = 0, 1 \\
0u & \text{if } \omega^0 = 2 \end{cases} = b_\omega(0u)
\]
Similarly one can check that $\psi_\omega(B)(1u) = b_\omega(1u)$ for all $u \in \{0, 1\}^*$ and hence $\psi_\omega(B) = b_\omega$. Repeating the argument shows that $\psi_\omega(C) = c_\omega, \psi_\omega(D) = d_\omega$.

\[\square\]

**Theorem 3.** The group $V$ is isomorphic to the universal group $U$ (as a marked group).

**Proof.** By Lemma 3, for each $\omega \in \Omega$ there exists a marked surjection $\psi_\omega : V \to G_\omega$, and hence there exists a marked surjection $\psi : V \to U$. If $g \in V$ is a nontrivial, let $v \in T_\delta$ such that $gv \neq v$. Let $\omega \in \Omega$ be such that $v \in T_\omega$. This shows that $\psi_\omega(g) \neq 1$ and hence $\psi(g) \neq 1$. This shows that $\psi$ is a marked isomorphism. \[\square\]

From now on we will identify $U$ with $V$.

Sidki in [Sid00] classified automata groups according to their activity growth and conjectured that automata groups of polynomial growth are amenable. Note that the automaton defining $U$ has exponential activity growth in Sidki’s classification. The question of amenability of the group $U$ remains open. The note [Muc05] claiming amenability of this group unfortunately contains a mistake.

### 4.2. Branch Structure of $U$

Let $G$ be a group acting on a rooted $d$-ary tree $T_d$. For a vertex $v$ of $T_d$, let $T_v$ denote the subtree hanging down at vertex $v$ and for an element $g \in G$ let $\text{supp}(g)$ be the support of $g$ i.e., the set of vertices not fixed by $g$. The stabilizer of a vertex $v$ is the subgroup $\text{St}_G(v) = \{g \in G \mid g(v) = v\}$. The rigid stabilizer of a vertex $v$ is the subgroup $\text{Rist}_G(v) = \{g \in G \mid \text{supp}(g) \subset T_v\}$. The rigid stabilizer of level $n$ is the subgroup $\text{Rist}_G(n) = \langle \text{Rist}_G(v) \mid |v| = n \rangle$. Since rigid stabilizer of distinct vertices of the same level commute, we have $\text{Rist}_G(n) = \prod_{|v|=n} \text{Rist}_G(v)$.

**Definition 5.** Let $G$ be group of automorphisms of a rooted tree $T$. $G$ is said to be a near branch group (resp. weakly near branch group) if for all $n \geq 1$, the subgroup $\text{Rist}_G(n)$ has finite index in $G$ (resp. is nontrivial). If in addition $G$ acts level transitively (i.e., transitively on each level of the tree) then $G$ is called a branch group (weakly branch group) respectively.

The class of (weakly) branch groups is interesting from various points of view and plays an important role in the classification of just-infinite groups, i.e., infinite groups whose proper homomorphic images are all finite (see [Gri00] for a detailed account on branch groups and just-infinite groups).

Let us mention the following fact which will be used in the forthcoming sections. We will also give an alternative proof of this fact later.

**Theorem 4.** [Gri84b] For $\omega \in \Omega_\infty$, the group $G_\omega$ is a branch group.

Note that at the terminology “branch group” was not used in [Gri84b].

If $G$ is a self-similar group, a standard way to show near branch property (resp. weakly near branch property) is to find a finite index subgroup $K$ (resp. nontrivial subgroup) of $G$ such that the image $\phi(K)$ contains the subgroup $K \times \cdots \times K$ where
$\phi : Aut(T_d) \to S_d \ltimes (Aut(T_d) \times \cdots \times Aut(T_d))$ is as defined in the previous section. This inclusion is denoted by $K \ni K \times \cdots \times K$. In this case the group is said to be a regular ((weakly) near) branch group over the subgroup $K$.

**Definition 6.** Let $G$ be a self-similar group of automorphisms of a $d$-ary rooted tree $T$. $G$ is said to be self-replicating if for all $g \in G$ and all $x \in \{0, 1, 2, \ldots, d-1\}$, there exists an element $h \in St_G(1)$ such that $h_x = g$.

Regarding the action of $U$ on $T_0$ we have the following:

**Theorem 5.** $U$ is a self-replicating weakly near branch group, regular branching over the third commutator subgroup $U''$.

**Proof.** Note that $St_U(1)$ is generated by the elements $\{b, c, d, aba, aca, ada\}$. Since we have
\[
\begin{align*}
  b &= (a, a, 1, b, b, b) \\
  c &= (a, 1, a, c, c, c) \\
  d &= (1, a, a, d, d, d) \\
  aba &= (b, b, b, a, a, 1) \\
  aca &= (c, c, c, a, 1, a) \\
  ada &= (d, d, d, 1, a, a)
\end{align*}
\]

it follows that $U$ is self-replicating. We claim that the derived subgroup $U'$ is generated by $(ab)^2, (ac)^2, (ad)^2$. From the basic relations we have that $a, b, c, d$ are of order 2 and $b, c, d$ commute with each other. Hence $U'$ is generated as a normal subgroup by
\[
[a, b] = (ab)^2, [a, c] = (ac)^2, [a, d] = (ad)^2
\]

Therefore it is enough to show that the subgroup generated by $(ab)^2, (ac)^2, (ad)^2$ is normal in $U$. Clearly conjugation by $a$ inverts the elements $(ab)^2, (ac)^2, (ad)^2$. For other conjugations we have (using the relation $bcd = 1$):
\[
x(ax)^2 = (xa)^2 = ((ax)^2)^{-1}
\]

and
\[
y(ax)^2y = (ya)^2(az)^2 = ((ay)^2)^{-1}(az)^2
\]

where $x, y, z \in \{b, c, d\}$ are distinct. Therefore $U'$ is generated by $(ab)^2, (ac)^2, (ad)^2$.

Next we claim that $U$ is near weakly branch over the third derived subgroup $U''$, that is: $U'' \ni U'' \times U'' \times U'' \times U'' \times U'' \times U''$. Let
\[
t = [(ab)^2, (ac)^2], \quad v = [(ab)^2, (ad)^2], \quad w = [(ac)^2, (ad)^2]
\]

$U''$ is generated as a normal subgroup by $t, v$ and $w$. Hence $U''$ is generated by the set
\[
\{t^g, v^g, w^g \mid g \in U\}
\]

It follows that $U''$ is generated as a normal subgroup by the set
\[
S = \{[t^g, v^g], [t^g, w^g], [v^g, w^g] \mid g \in U\}
\]

We have the following equalities (this can best be checked with the GAP package...
Doing same thing in second and third coordinates and using other elements of $U,\gamma, g,\gamma'\gamma, g,\gamma''\gamma, g,\gamma'''\gamma, g$ we see that

$$h_1 = \left[ (ab)^2, [b, (ca)^2] \right] = (t, *, 1, 1, 1, 1)$$
$$h_2 = \left[ (ab)^2, [c, (da)^2] \right] = (v, 1, 1, 1, *, 1)$$
$$h_3 = \left[ c, (ca)^2, [b, (da)^2] \right] = (w, 1, 1, 1, 1, *)$$
$$h_4 = \left[ b, (ba)^2, [d, (ca)^2] \right] = (1, t, 1, *, 1, 1)$$
$$h_5 = \left[ d, (ad)^2, [b, (ba)^2] \right] = (1, v, 1, 1, *, 1)$$
$$h_6 = \left[ d, (ca)^2, [b, (da)^2] \right] = (1, w, 1, 1, 1, *)$$
$$h_7 = \left[ c, (ba)^2, [d, (ca)^2] \right] = (1, 1, t, *, 1, 1)$$
$$h_8 = \left[ d, (ba)^2, [c, (da)^2] \right] = (1, 1, v, 1, *, 1)$$
$$h_9 = \left[ c, (ca)^2, [d, (da)^2] \right] = (1, 1, w, 1, 1, *)$$

where * are elements of $U$ not of importance. Clearly $h_i \in U''$ for $i = 1, 2, 3, 4, 5, 6$.

Given $g_1, g_2, g_3, g_4, g_5, g_6 \in U$, due the fact that $U$ is self-replicating, there are elements $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6 \in U$ such that

$$\gamma_1 = (g_1, *, *, *, *, *)$$
$$\gamma_2 = (g_2, *, *, *, *, *)$$
$$\gamma_3 = (g_3, *, *, *, *, *)$$
$$\gamma_4 = (g_4, *, *, *, *, *)$$
$$\gamma_5 = (g_5, *, *, *, *, *)$$
$$\gamma_6 = (g_6, *, *, *, *, *)$$

So,

$$h_1^{\gamma_1}, h_2^{\gamma_2} = (t^{g_1}, v^{g_2}, 1, 1, 1, 1, 1)$$
$$h_1^{\gamma_4}, h_3^{\gamma_4} = (t^{g_3}, w^{g_4}, 1, 1, 1, 1, 1)$$
$$h_2^{\gamma_5}, h_3^{\gamma_6} = (v^{g_5}, w^{g_6}, 1, 1, 1, 1, 1)$$

and clearly left hand sides are elements of $U'''$. Using the fact that $U$ is self-replicating we see that

$$U''' \not\cong U'' \times 1 \times 1 \times 1 \times 1 \times 1.$$ 

Doing same thing in second and third coordinates and using other $h_i$ we see that

$$U''' \not\cong 1 \times U'' \times 1 \times 1 \times 1 \times 1$$

and

$$U''' \not\cong 1 \times 1 \times U'' \times 1 \times 1 \times 1$$

and finally conjugating with $a$ we also have

$$U''' \not\cong 1 \times 1 \times 1 \times U''' \times 1 \times 1$$
$$U''' \not\cong 1 \times 1 \times 1 \times 1 \times U''' \times 1$$
$$U''' \not\cong 1 \times 1 \times 1 \times 1 \times 1 \times U'''$$

which shows that

$$U''' \not\cong U'' \times U'' \times U''' \times U''' \times U''' \times U'''$$.

Clearly $U'''$ is non-trivial since $U$ has non-solvable quotients. □
Note that $U/U'''$ maps onto the group $\tilde{G}_{000\ldots}$ and hence is infinite. Also, $U$ cannot have a branch type action (on any rooted tree) since all non-trivial quotients of branch groups are virtually abelian, a fact proven in [Gri00].

4.3. **Branch structure of general universal groups.**

In this subsection we will investigate the branch structure of arbitrary universal groups.

For $\omega \in \Omega$ we have an injection

$$\phi_\omega : G_\omega \rightarrow S_2 \ltimes (G_{\tau\omega} \times G_{\tau\omega})$$

$$a \mapsto (01)(1,1)$$

$$b_\omega \mapsto (\beta(\omega_0), b_{\tau\omega})$$

$$c_\omega \mapsto (\zeta(\omega_0), c_{\tau\omega})$$

$$d_\omega \mapsto (\delta(\omega_0), d_{\tau\omega})$$

For subgroups $H \leq G_\omega$ and $H \leq G_{\tau\omega}$ let us write $K \ltimes K \leq H$ if $K \ltimes K \leq \phi_\omega(H)$. Note that this means $H$ contains a subgroup isomorphic to $K \times K$.

**Proposition 4.** For $\omega \in \Omega$ we have $G'''_{\tau\omega} \times G'''_{\tau\omega} \leq G'''_\omega$

**Proof.** Let us assume that $\omega_0 = 0$. Define

$$\pi : U \times U \times U \times U \times U \times U \rightarrow G_\omega \times G_\omega$$

by $\pi(u_1, u_2, u_3, u_4, u_5, u_6) = (\psi_\omega(u_1), \psi_\omega(u_4))$ where $\psi_\omega$ is as defined in section 4.1.

Let $\phi : U \rightarrow S_6 \ltimes U \times U \times U \times U \times U \times U$ be the canonical map.

Then the following diagram commutes.

$$\begin{array}{cccc}
St_U(1) & \xrightarrow{\phi} & U \times U \times U \times U \times U
\downarrow \psi_\omega & \downarrow \pi \\
\phi_{G_\omega} \downarrow & & & \\
St_{G_\omega}(1) & \xrightarrow{\phi_\omega} & G_{\tau\omega} \times G_{\tau\omega}
\end{array}$$

By Theorem 5 we have $U''' \times U''' \times U''' \times U''' \times U''' \times U''' \leq U'''$. Since $\psi_\omega(U''') = G'''_{\tau\omega} \times G'''_{\tau\omega} \leq G'''_\omega$ we see that $G'''_{\tau\omega} \times G'''_{\tau\omega} \leq G'''_\omega$.

The case when $\omega_0 = 1$ or $\omega_0 = 2$ can be proven similarly by modifying $\pi$. \hfill \Box

**Corollary 1.** For $\omega \in \Omega_{\infty}$, $G_\omega$ is a branch group.

**Proof.** It follows by Proposition 4 and an induction argument that for any $n \geq 1$ we have

$$\prod_{1}^{2^n} G'''_{\tau\omega} \leq G'''_\omega$$

It follows that for any $n \geq 1$, $\prod_{1}^{2^n} G'''_{\tau\omega} \leq Rist_{G_\omega}(n)$. Note that for any $\omega \in \Omega \setminus \Omega_0$, $G'''_\omega$ is nontrivial (since $G_\omega$ is not solvable) and also have finite index (since $G_\omega$ are just-infinite.) It follows that $Rist_{G_\omega}(n)$ has finite index for all $n \geq 1$. \hfill \Box
For a non-empty subset $\Lambda \subset \Omega$, let us denote the universal group corresponding to the family $\{(G_\omega, S_\omega) \mid \omega \in \Lambda\}$ by $U_\Lambda$. Given $\Lambda \subset \Omega$ let $T_\Lambda = \bigcup_{\omega \in \Lambda} T_\omega$ and note that $T_\Lambda$ is a (not necessarily regular) subtree of $T_\Omega$. Also note that $T_\Lambda$ is $U$ invariant (since each $T_\omega$ is so) and the restriction of $U$ onto $T_\Lambda$ gives the universal group $U_\Lambda$.

**Proposition 5.** If $\Lambda \subset \Omega \setminus \Omega_0$ then with the action onto $T_\Lambda$, $U_\Lambda$ is a weakly near branch group.

**Proof.** Let $v \in T_\Lambda$ and let $v \in T_\omega$ for some $\omega \in \Lambda$. Let $g$ be a non-trivial element of $\text{Rist}^{G_\omega}(v)$. Then by the proof of Proposition 4, there exists $h \in \text{Rist}^{U}(v)$ such that $\psi_\omega(h) = g$. The restriction of $h$ onto $T_\Lambda$ gives a non-trivial element in $\text{Rist}^{U_\Lambda}(v)$. □

## 5. Universal groups of intermediate growth

The aim of this section is to show that there exists a subset $\Lambda \subset \Omega$ of cardinality $2^{\aleph_0}$ such that $U_\Lambda$ has intermediate growth. This fact was first established in [Gri84a], we fix some inaccuracy in the proof of this fact.

First, let us briefly recall basic notions related to the growth of groups. We refer to [dlH00, Man12, Gri13] for a detailed account on growth and related topics.

Let $G$ be a finitely generated group and $S$ a finite generating set. The length of an element (with respect to $S$) is given by $\ell_S(g) = \min \{n \mid g = s_1s_2\ldots s_n, \ i \in S^\pm\}$. The growth function of $G$ (with respect to $S$) is $\gamma_{G,S}(n) = \#B(G,S,n)$ where $B(G,S,n) = \{g \in G \mid \ell_S(g) \leq n\}$ is the ball of radius $n$. For two increasing functions $f_1, f_2$ defined on the set of natural numbers, let us write $f_1 \preceq f_2$ if there exists $C > 0$ such that $f_1(n) \leq f_2(Cn)$ for all $n$. Let us also write $f_1 \sim f_2$ if $f_1 \preceq f_2$ and $f_2 \preceq f_1$, which defines an equivalence relation. It can be observed that the growth functions of a group with respect to different finite generating sets are $\sim$ equivalent and hence the asymptotic behavior of the growth functions of a group is an invariant of the group.

There are three types of growth for groups: If $\gamma_G \preceq n^d$ for some $d \geq 0$ then $G$ is said to be of polynomial growth, if $\gamma_G \sim e^n$ then it is said to have exponential growth. If neither of this happens then the group is said to have *intermediate growth*.

If we are talking about the growth of a marked group $(G, S)$, we will simply write $\gamma_G$ for the growth function of $G$ with respect to $S$.

**Lemma 4.** Let $F = \{(G_i, S_i) \mid i \in I\} \subset \mathcal{M}_k$ be a non-empty subset. Denote by $\gamma_F$ the growth function of the diagonal group $(U^{\text{diag}}_F, S^{\text{diag}}_F)$ of Definition 3. Then

1. For all $i \in I$, $\gamma_F(n) \geq \gamma_i(n)$ for all $n$,
2. If $I$ is finite then, $\gamma_F(n) \leq \prod_{i \in I} \gamma_i(n)$ for all $n$. 

Proof. In general, if \((H, K)\) is a marked image of \((G, S)\), then \(\gamma_G(n) \geq \gamma_H(n)\) for every \(n\). Since all \((G_i, S_i)\) are marked images of the diagonal group, we obtain the first assertion. For the second assertion, observe that \(B(U^{\text{diag}}_F, S^{\text{diag}}_F, n) \subset \prod_{i \in I} B(G_i, S_i, n)\). □

For a positive integer \(M\) let \(\Omega_M \subset \Omega_{\infty}\) be the set of all sequences for which every subword of length \(M\) contains all symbols 0, 1, 2.

Theorem 6. [Gri84b, Theorem 3.3] There exist constants \(C\) and \(\alpha < 1\) depending only on \(M\), such that if \(\omega \in \Omega_M\) then
\[
\gamma_{G_\omega}(n) \leq C n^\alpha \text{ for all } n.
\]

Given natural numbers \(r_1, \ldots, r_k\) let
\[
\Lambda_{r_1, \ldots, r_k} = \{(012)^{r_1} \eta_1 (012)^{r_2} \eta_2 \ldots (012)^{r_k} \eta_k (012)^\infty \mid \eta_i \in \{0, 1, 2\}\} \subset \Omega.
\]
where \((012)^\infty\) stands for the periodic sequence 012012012\ldots.

For a sequence of natural numbers \(r = \{r_k\}\), let
\[
\Lambda_r = \{(012)^{r_1} \eta_1 (012)^{r_2} \eta_2 \ldots (012)^{r_k} \eta_k \ldots \mid \eta_i \in \{0, 1, 2\}\} \subset \Omega.
\]
Note that both \(\Lambda_{r_1, \ldots, r_k}\) and \(\Lambda_r\) are subsets of \(\Omega_4\). Let us denote the universal groups \(U_{\Lambda_{r_1, \ldots, r_k}}\) and \(U_{\Lambda_r}\) by \(U_{r_1, \ldots, r_k}\) and \(U_r\) respectively. Let \(\gamma_{r_1, \ldots, r_k}\) and \(\gamma_r\) denote the growth functions (with respect to the canonical generating sets) of \(U_{r_1, \ldots, r_k}\) and \(U_r\) respectively.

Lemma 5. Given natural numbers \(r_1, \ldots, r_k\), there exists a natural number \(m\) such that
\[
\gamma_{r_1, \ldots, r_k, x}(m) \leq \left(1 + \frac{1}{k}\right)^m \text{ for any } x \in \mathbb{N}.
\]

Proof. Since \(\Lambda_{r_1, \ldots, r_k, x} \subset \Omega_4\), by Theorem 6 there exists \(C\) and \(\alpha < 1\) (not depending on \(x\)) such that for all \(\omega \in \Lambda_{r_1, \ldots, r_k, x}\) we have
\[
\gamma_\omega(n) \leq C n^\alpha \text{ for all } n.
\]
Therefore, by Lemma 4 (using the fact that \(|\Lambda_{r_1, \ldots, r_k, x}| = 3^{k+1}\) we have
\[
\gamma_{r_1, \ldots, r_k, x}(n) \leq (C n^\alpha)^{3^{k+1}} = D n^\alpha \text{ for all } n
\]
where \(D = C^{3^{k+1}}\) does not depend on \(x\). Therefore there exists a natural number \(m\) such that
\[
\gamma_{r_1, \ldots, r_k, x}(m) \leq \left(1 + \frac{1}{k}\right)^m \text{ for any } x \in \mathbb{N}.
\]

Lemma 6. [Gri84a, Lemma 3] Let \(r = \{r_k\}\) be a sequence of natural numbers. If for some \(k\)
\[
k + r_1 + r_2 + \ldots + r_k \geq \log_2 2n
\]
then \(\gamma_{r_1, \ldots, r_k}(n) = \gamma_r(n)\).
Theorem 7. [Gri84a, Theorem 1] There exists a sequence \( r = \{r_k\} \) such that \( U_r \) has intermediate growth.

Proof. Let \( r_1 = 1 \). By Lemma 5 there exists a natural number \( n_1 \) such that

\[
\gamma_{r_1,x}(n_1) \leq \left(1 + \frac{1}{1}\right)^{n_1} \text{ for any } x.
\]

Choose \( r_2 \) such that \( 2 + r_1 + r_2 \geq \log_2 2n_1 \). Again by Lemma 5 there exists \( n_2 > n_1 \) such that

\[
\gamma_{r_1,r_2,x}(n_2) \leq \left(1 + \frac{1}{2}\right)^{n_2} \text{ for any } x.
\]

Assume \( r_1, \ldots, r_k \) has been already chosen. By Lemma 5 there exists \( n_k > n_{k-1} \) such that

\[
\gamma_{r_1,\ldots,r_k,x}(n_k) \leq \left(1 + \frac{1}{k}\right)^{n_k} \text{ for any } x.
\]

Choose \( r_{k+1} \) such that

\[
k + 1 + r_1 + \ldots + r_{k+1} \geq \log_2 2n_k.
\]

Continuing in this manner we construct sequences \( r = \{r_k\} \) and \( \{n_k\} \) for which Equations 3 and 4 are satisfied. Lemma 6 and Equation 4 shows that for all \( k \) we have

\[
\gamma_{r_1,\ldots,r_{k+1}}(n_k) = \gamma_r(n_k).
\]

Using this and Equation 3 we have,

\[
\lim_{n \to \infty} \gamma_r(n) \frac{1}{n} = \lim_{k \to \infty} \gamma_r(n_k) \frac{1}{n_k} = \lim_{k \to \infty} \gamma_{r_1,\ldots,r_{k+1}}(n_k) \frac{1}{n_k} \leq \lim_{k \to \infty} \left(1 + \frac{1}{k}\right) = 1
\]

\[\square\]

Corollary 2. There exists a finitely generated group of intermediate growth with \( 2^{\aleph_0} \) non-isomorphic homomorphic images.

As mentioned in the beginning of this section, this fact was established in [Gri84a] with a small inaccuracy. Our proof mainly follows the lines of [Gri84a] only difference being that one needs Lemma 5.

6. Invariant random subgroups of universal groups

The aim of this section is to show that there are universal groups with many invariant random subgroups.
6.1. Preliminaries About Invariant Random Subgroups.

Let $G$ be a countable group and let $S(G)$ be the space of subgroups of $G$ endowed with the topology having sets of the form $\mathcal{O}_{A,B} = \{N \leq G \mid A \subset N, B \cap N = \emptyset\}$ where $A, B$ are finite subsets of $G$, as basis. $S(G)$ can be identified with a closed subspace of $\{0,1\}^G$ supplied with by the topology induced from the Tychonoff topology. The group $G$ acts on $S(G)$ by conjugation and hence forming a topological dynamical system $(G, S(G))$. We are interested in dynamical system of the form $(G, S(G), \mu)$ where $\mu$ is a conjugation invariant probability measure on $S(G)$.

**Definition 7.** A conjugation invariant Borel probability measure on $S(G)$ is called an invariant random subgroup (IRS in short).

The space $S(G)$ is a compact, metrizable, totally disconnected space which (applying the Cantor-Bendixson procedure \cite[I.6]{Kec95}) consists of a perfect kernel $\kappa(G)$ and its complement $S(G) \setminus \kappa(G)$ which is countable. The perfect kernel $\kappa(G)$ is either empty or is homeomorphic to a Cantor set, and it is empty if and only if $S(G)$ is countable, that is $G$ has only countably many subgroups. This is the case, for instance, for finitely generated virtually nilpotent groups, virtually polycyclic groups, some metabelian groups like Baumslag-Solitar groups $B(1,n)$, or Tarski monsters \cite{Ol'80}.\footnote{One can find a nice survey for IRS in the case of groups with finitely many subgroups in \cite{LiPZ15}.}

As $\kappa(G)$ is an invariant subset of $S(G)$ with respect to the action of $\text{Aut}(G)$ and as the complement $S(G) \setminus \kappa(G)$ is countable, it is clear that a continuous IRS has law $\mu$ supported on $\kappa(G)$.

Given a subgroup $L \leq G$, let $S(G,L) \subset S(G)$ be the set of subgroups containing $L$, which clearly is closed. Note that, if $L$ is a normal subgroup of $G$, then $S(G,L)$ is invariant under the action of $G$.

Let $\varphi : G \rightarrow H$ be a homomorphism. It induces two maps

\[ \bar{\varphi} : S(G) \rightarrow S(H) \]
\[ N \mapsto \varphi(N) \]

and

\[ \tilde{\varphi} : S(H) \rightarrow S(G, \text{Ker}(\varphi)) \]
\[ K \mapsto \varphi^{-1}(K) \]

**Lemma 7.**

1. $\tilde{\varphi}$ is Borel.
2. $\tilde{\varphi}$ is continuous.
3. $\tilde{\varphi}(K^{g}) = \varphi(K)^{g}$ for all $g \in G$ and $K \leq H$.
4. $\tilde{\varphi}^{-1}(C^{g}) = \varphi^{-1}(C)^{g}$ for all $g \in G$ and $C \subset S(G, \text{Ker}(\varphi))$.
5. If $\varphi$ is surjective, then $\tilde{\varphi}$ is a homeomorphism.

**Proof.**
Corollary 3. If $\mu$ is an IRS of $H$ then the measure $\nu = \overline{\varphi}_*(\mu)$ is an IRS of $G$ supported on the set $\{\varphi^{-1}(K) \mid K \in \text{supp}(\mu)\}$. If moreover $\mu$ is continuous, ergodic with respect to the action of $H$ and $\varphi$ is surjective, then $\nu$ is continuous and ergodic with respect to the action of $G$.

Proof. The first part is immediate consequence of Lemma\[7\] parts (1) and (3). Note that the measure $\overline{\varphi}_*(\mu)$ is defined on the closed subset $S(G, \text{Ker}(\varphi))$ of $S(G)$, and hence can be considered as a measure on $S(G)$ with support in $S(G, \text{Ker}(\varphi))$. Suppose that $\mu$ is continuous, ergodic and $\varphi$ is surjective. Since $\overline{\varphi}$ is a homeomorphism the measure $\nu$ is continuous. Let $C \subset S(G, \text{Ker}(\varphi))$ be $G$-invariant. Given $h \in H$, pick $g \in G$ such that $\varphi(g) = h$. By Lemma\[7\] part (3), $\overline{\varphi}^{-1}(C)^h = \overline{\varphi}^{-1}(C)^{\varphi(g)} = \overline{\varphi}^{-1}(C^g) = \overline{\varphi}^{-1}(C)$. Therefore $\overline{\varphi}^{-1}(C)$ is $H$ invariant, from which it follows that $\nu(C) = \mu(\overline{\varphi}^{-1}(C)) \in \{0,1\}$.
Proposition 6. Let $X$ be a metrizable Hausdorff topological space and let $\mu$ be a Borel measure on $X$. Suppose also that a group $G$ acts on the Borel space $(X, \mu)$ by measure preserving transformations. Then the map $St : X \rightarrow S(G)$ given by $x \mapsto St_G(x)$ is Borel. Moreover, the measure $\nu = St_*(\mu)$ is an IRS supported on \{St_G(x) | x \in X\}.

Proof. Observe that the Borel $\sigma$-algebra on $S(G)$ is generated by sets of the form $O_g = \{N \leq G \mid g \in N\}$. Also observe that $St^{-1}(O_g) = Fix(\varphi_g)$ where $\varphi_g : X \rightarrow X$ given by $\varphi_g(x) = g.x$. Therefore $St^{-1}(O_g)$ is a Borel set (see e.g. [Par05, Chapter 1] on sections of Borel maps). This shows that the measure $\nu = St_*(\mu)$ is a Borel measure on $S(G)$ with support $\{St_G(x) \mid x \in X\}$. The relation $St_G(g.x) = St_G(x)g^{-1}$ and the $G$ invariance of $\mu$ show that $\nu$ is conjugation invariant.

\[ \square \]

It is known (see [AGV12]) that every IRS of a finitely generated group arises from a measure preserving action on a Borel probability space $(X, \mu)$.

If $T_d$ is the rooted $d$-ary tree, its boundary $\partial T_d$ is the set of all infinite rays emanating from the root vertex. $\partial T_d$ is in bijection with infinite sequences over the alphabet $\{0, 1, \ldots, d-1\}$ and hence homeomorphic to a Cantor Set. If $G$ is a group of automorphisms of a rooted tree $T_d$, its action on $T_d$ extends to an action onto the boundary $\partial T_d$ and this action is by homeomorphisms. Let $\mu$ be the uniform Bernoulli measure on $\partial T_d$, (i.e., the product of uniform measures on the set $\{0, 1, \ldots, d-1\}$). Observe that $\mu$ is continuous and invariant under the action of $\text{Aut}(T_d)$ and hence invariant under the action of any subgroup $G \leq \text{Aut}(T_d)$. Regarding the the dynamics of such actions the following is known:

Proposition 7. [Gri11] Let $G$ be a countable group of automorphisms of a regular rooted tree $T_d$. Then, the following are equivalent:

1. the group $G$ acts transitively on the levels of $T_d$,
2. the action of $G$ on $\partial T_d$ is minimal (i.e., orbits are dense),
3. the action of $G$ on $\partial T_d$ is ergodic with respect to the uniform Bernoulli measure on $\partial T_d$.
4. the action is uniquely ergodic.

An action of weakly branch type on $T$ gives a totally non-free action on the boundary $\partial T$.

Proposition 8. [BG02, Gri11] Let $G \leq \text{Aut}(T)$ be weakly branch. Then the map $St : \partial T \rightarrow S(G)$ given by $\xi \mapsto St_G(\xi)$ is injective.

Proof. Let $\xi, \eta \in \partial T$ be distinct elements and let $u, v$ be distinct prefixes of $\xi$ and $\eta$ respectively, of same length, say, $n$. Infinite sequences starting with $u$ (respectively with $v$) form a neighbourhood of $\xi$ (respectively of $\eta$) in $\partial T$. We will show that the stabilizer of the neighbourhood of $\eta$ (which is a subgroup in $St_G(\eta)$) is not contained in $St_G(\xi)$.
Let $u$ and $v$ be distinct prefixes of length $n$ of $\xi$ and $\eta$ respectively. Let $g \in Rist_G(u)$ be nontrivial. Since $v$ is not contained in the subtree $T_u$, $g$ fixes every infinite sequence starting with $v$. Since $g$ is nontrivial it moves some vertex in $uu_1 \in T_u$, say $g(uu_1) = uu_2$ for some $u_1 \neq u_2$ of lengths $m$. Let $uu'$ be the prefix of $\xi$ of length $n + m$.

If $u' = u$ or $u' = u_2$, then $g(uu') \neq uu'$ and hence $g \notin St_G(\xi)$. If both $u' \neq u_1$ and $u' \neq u_2$, by level transitivity let $h \in G$ such that $h(uu_1) = uu'$. Then

$$(hgh^{-1})(uu') = (hg)(uu_1) = h(uu_2) \neq uu'$$

because $u_1 \neq u_2$. Therefore $hgh^{-1} \notin St_G(\xi)$. Since $h(uu_1) = uu_2$, we have $h \in St_G(u)$ and hence $hgh^{-1} \in Rist_G(u)$. It follows that $hgh^{-1} \in St_G(\eta)$. \qed

As explained in Introduction, this readily provides us with a continuous ergodic IRS on $G$. See for example [DDMN10] for a detailed study of this and related measures on the space of Schreier graphs of the Basilica group.

Regarding the action of the Grigorchuk groups $G_\omega, \omega \in \Omega$ on the boundary $\partial T_2$ of the binary tree we obtain the following.

**Proposition 9.** For $\omega \in \Omega$ the action of $G_\omega$ on $T_2$ is level transitive and hence the action of $G_\omega$ on $(\partial T_2, \mu)$ is ergodic. Therefore, the induced IRS on $G_\omega$ is continuous and ergodic.

**Proof.** By Proposition 9 the action of $G_\omega$ on $(\partial T_2, \mu)$ induces an IRS on $G_\omega$. This IRS will be continuous by Proposition 8 and ergodic by Proposition 7. \qed

### 6.2. IRS on universal groups.

Given $\omega_1, \omega_2 \in \Omega$, let us write $\omega_1 \sim \omega_2$ if there exists $\sigma \in Sym\{0, 1, 2\}$ such that $\omega_2$ is obtained from $\omega_1$ by application of $\sigma$ to each letter of $\omega_1$. Recall that by Theorem 1 part (7) we have that for $\omega_1, \omega_2 \in \Omega_\infty$, $G_{\omega_1} \cong G_{\omega_2}$ if and only if $\omega_1 \sim \omega_2$.

For a subset $\Lambda \subset \Omega$ let $|\Lambda|_\sim$ denote the cardinality of the set of $\sim$ equivalence classes in $\Lambda$.

**Proposition 10.** For $\Lambda \subset \Omega_\infty$, $U_\Lambda$ has at least $|\Lambda|_\sim$ distinct continuous, ergodic invariant random subgroups.

**Proof.** Fix $\Lambda \subset \Omega_\infty$. Let $\varphi_\omega : U_\Lambda \longrightarrow G_\omega$ be the canonical surjection and let $N_\omega = Ker(\varphi_\omega)$. Note that if $\omega \sim \eta$, then by Theorem 1 part (7) and the fact that $G_\eta$ is just infinite, we have $N_\eta \not\cong N_\omega$. For $\omega \in \Omega$ and $\xi \in \partial T_2$ let $W_{\omega, \xi} = St_{G_\omega}(\xi)$. By Proposition 9 the canonical action of $G_\omega$ onto $(\partial T_2, \mu)$ induces a continuous, ergodic IRS $\mu_\omega$ on $G_\omega$. Moreover, $\mu_\omega$ is supported on $\{W_{\omega, \xi} \mid \xi \in \partial T_2\}$.

Let $\nu_\omega$ denote the induced IRS on $U_\Lambda$ obtained as described in Corollary 3 (i.e., $\nu_\omega = (\varphi_\omega)_*(\mu_\omega)$). Again by Corollary 3 $\nu_\omega$ is continuous and ergodic. Let $L_{\omega, \xi} = \varphi_\omega^{-1}(W_{\omega, \xi})$ and note that $\nu_\omega$ is supported on $Y_\omega = \{L_{\omega, \xi} \mid \xi \in \partial T_2\}$. Observe that for all $\xi \in \partial T_2$, $L_{\omega, \xi}$ contains $N_\omega$. 

Suppose that for some \( \omega \not\sim \eta \in \Lambda \) and \( \xi, \rho \in \partial T_2 \) we have \( L_{\omega, \xi} = L_{\eta, \rho} \). Then \( N_{\omega, \eta} \leq L_{\omega, \xi} \) and hence \( L_{\omega, \xi} \) contains the subgroup \( N = N_{\omega, \eta} \). Since \( N_{\eta} \not\leq N_{\omega} \), \( N \) contains \( N_{\omega} \) as a proper subgroup. It follows that the group \( U_{\Lambda}/N \) is a proper quotient of the group \( U_{\Lambda}/N_{\omega} \cong G_{\omega} \). Since \( G_{\omega} \) is a just infinite group it follows that \( N \) and hence \( L_{\omega, \xi} \) has finite index in \( U_{\Lambda} \). This, in turn shows that \( St_{G_{\omega}}(\xi) \) has finite index in \( G_{\omega} \) which is a contradiction. Therefore if \( \omega \sim \eta \) we see that the measures \( \nu_{\omega} \) and \( \nu_{\eta} \) have disjoint supports and are in particular distinct. □

Combining this with results from Section 5 we obtain the main theorem:

**Main Theorem.** There is a subset \( \Lambda \subset \Omega \) such that the corresponding universal group \( U_{\Lambda} \) has intermediate growth and has \( 2^{\aleph_0} \) distinct continuous ergodic invariant random subgroups.

**Acknowledgements:**

The authors wish to thank Anatoly Vershik and Yaroslav Vorobets for useful discussions and the referee for the careful reading of the manuscript.

**References**

[AGV12] Miklos Abert, Yair Glasner, and Balint Virag. Kesten’s theorem for invariant random subgroups, 2012. (available at [http://arxiv.org/abs/1201.3399](http://arxiv.org/abs/1201.3399)).

[BG02] Laurent Bartholdi and Rostislav I. Grigorchuk. On parabolic subgroups and Hecke algebras of some fractal groups. *Serdica Math. J.*, 28(1):47–90, 2002.

[BGdlH13] Mustafa Gökhan Benli, Rostislav Grigorchuk, and Pierre de la Harpe. Amenable groups without finitely presented amenable covers. *Bull. Math. Sci.*, 3(1):73–131, 2013.

[BGK12] Lewis Bowen, Rostislav Grigorchuk, and Rotislav Kravchenko. Invariant random subgroups of the lamplighter group, 2012. (available at [http://arxiv.org/abs/1206.6780](http://arxiv.org/abs/1206.6780)).

[Bow12] Lewis Bowen. Invariant random subgroups of the free group, 2012. (available at [http://arxiv.org/abs/1204.5939](http://arxiv.org/abs/1204.5939)).

[Cha00] Christophe Champetier. L’espace des groupes de type fini. *Topology*, 39(4):657–680, 2000.

[DDMN10] Daniele D’Angeli, Alfredo Donno, Michel Matter, and Tatiana Nagnibeda. Schreier graphs of the Basilica group. *J. Mod. Dyn.*, 4(1):167–205, 2010.

[dlH00] Pierre de la Harpe. *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000.

[GNS00] Rostislav I. Grigorchuk, Volodymyr V. Nekrashevich, and V. I. Sushchanskiĭ. Automata, dynamical systems, and groups. *Tr. Mat. Inst. Steklova*, 231(Din. Sist., Avtom. i Beskon. Gruppy):134–214, 2000.

[Gri13] Rostislav I. Grigorchuk. Milnor’s problem on the growth of groups and its consequences, 2013. (available at [http://arxiv.org/pdf/1111.0512.pdf](http://arxiv.org/pdf/1111.0512.pdf)).

[Gri84a] Rostislav I. Grigorchuk. Construction of \( p \)-groups of intermediate growth that have a continuum of factor-groups. *Algebra i Logika*, 23(4):383–394, 478, 1984.

[Gri84b] Rostislav I. Grigorchuk. Degrees of growth of finitely generated groups and the theory of invariant means. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):939–985, 1984.

[Gri00] Rostislav I. Grigorchuk. Just infinite branch groups. In *New horizons in pro-\( p \) groups*, volume 184 of *Progr. Math.*, pages 121–179. Birkhäuser Boston, Boston, MA, 2000.
[Gri11] Rostislav I. Grigorchuk. Some problems of the dynamics of group actions on rooted trees. *Tr. Mat. Inst. Steklova*, 273(Sovremennye Problemy Matematiki):72–191, 2011.

[GŠ07] Rostislav Grigorchuk and Zoran Šunić. Self-similarity and branching in group theory. In *Groups St. Andrews 2005. Vol. 1*, volume 339 of *London Math. Soc. Lecture Note Ser.*, pages 36–95. Cambridge Univ. Press, Cambridge, 2007.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[LOŠ78] Joram Lindenstrauss, Gunnar Hans Olsen, and Yaki Sternfeld. The Poulsen simplex. *Ann. Inst. Fourier (Grenoble)*, 28(1):vi, 91–114, 1978.

[Man12] Avinoam Mann. *How groups grow*, volume 395 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2012.

[Mil68] John Milnor. Problem 5603, *amer. math. monthly* 75(3), 685–686, 1968.

[Muc05] Roman Muchnik. Amenability of universal 2-grigorchuk group, 2005. (available at http://arxiv.org/abs/math/0505572).

[Nek05] Volodymyr Nekrashevych. *Self-similar groups*, volume 117 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[Oļ’80] Alexander Ju. Ol’šanskiĭ. An infinite group with subgroups of prime orders. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(2):309–321, 479, 1980.

[Osi04] Denis. V. Osin. Algebraic entropy of elementary amenable groups. *Geom. Dedicata*, 107:133–151, 2004.

[Par05] Kalyanapuram R. Parthasarathy. *Probability measures on metric spaces*. AMS Chelsea Publishing, Providence, RI, 2005. Reprint of the 1967 original.

[Sid00] Said Sidki. Automorphisms of one-rooted trees: growth, circuit structure, and acyclicity. *J. Math. Sci. (New York)*, 100(1):1925–1943, 2000. Algebra, 12.

[Ver10] Anatoly M. Vershik. Nonfree actions of countable groups and their characters. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 378(Teoriya Predstavlenii, Dinamicheskie Sistemy, Kombinatornye Metody. XVIII):5–16, 228, 2010.

[Ver12] Anatoly M. Vershik. Totally nonfree actions and the infinite symmetric group. *Mosc. Math. J.*, 12(1):193–212, 216, 2012.

[Vor12] Yaroslav Vorobets. Notes on the Schreier graphs of the Grigorchuk group. In *Dynamical systems and group actions*, volume 567 of *Contemp. Math.*, pages 221–248. Amer. Math. Soc., Providence, RI, 2012.

Mustafa Gökhan Benli, Middle East Technical University, Ankara, Turkey

E-mail address: benli@metu.edu.tr

Rostislav Grigorchuk, Texas A&M University, College Station, TX, USA

E-mail address: grigorch@math.tamu.edu

Tatiana Nagnibeda, Section de mathématiques, Université de Genève, C.P. 64, CH–1211 Genève 4, Suisse.

E-mail address: tatiana.smirnova-nagnibeda@unige.ch