CONSTRUCTIVE STRONG REGULARITY AND THE EXTENSION PROPERTY OF A COMPACTIFICATION

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ABSTRACT. In contexts in which the principle of dependent choice may not be available, as toposes or Constructive Set Theory, standard locale theoretic results related to complete regularity may fail to hold. To resolve this difficulty, B. Banaschewski and A. Pultr introduced strongly regular locales. Unfortunately, Banaschewski and Pultr’s notion relies on non-constructive set existence principles that hinder its use in Constructive Set Theory. In this article, a fully constructive formulation of strong regularity for locales is introduced by replacing non-constructive set existence with coinductive set definitions, and exploiting the Relation Reflection Scheme. As an application, every strongly regular locale \( L \) is proved to have a compact regular compactification. The construction of this compactification is then used to derive the main result of this article: a characterization of locale compactifications (and thus, classically, of the compactifications of a space) in terms of their ability of extending continuous functions with compact regular codomains. Finally, an open problem related to the existence of the compact regular reflection of a locale is presented.

INTRODUCTION

Among the different motivations for pursuing a point-free development of general topology \[13\], a pivotal one is that point-free spaces (locales) often allow for choice-free proofs of theorems that in their usual topological formulation require the application of some form of choice principle. This feature makes locale theory particularly suited for contexts in which no choice principle is available, as (the internal language of) toposes \[14\] and constructive set theories.

This ability of locales however fails in connection with the point-free version of a well-known standard topological result: the proof that a compact regular locale is completely regular relies indeed on the use of the principle of Dependent Choice (DC), as for the corresponding fact for spaces \[13\]. As already recalled, there are contexts in which DC is not available; specifically, there are toposes, that do not validate DC, in which a compact regular locale fails to be completely regular.

For obtaining choice-free analogues of this and other results involving complete regularity, Banaschewski and Pultr \[7\] introduced the notion of strongly regular locale. With DC, a locale is strongly regular if and only if it is completely regular. Without DC, every completely regular locale is strongly regular, and compact regular locales are strongly regular.

To formulate strong regularity, Banaschewski and Pultr use the existence of a largest interpolative relation contained in a given binary relation \( R \), defined as the

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union of all interpolative relations contained in $R$. Unfortunately, this definition is impredicative or circular, as the defined relation is itself an interpolative relation contained in $R$. For this construction, strong set existence principles, as the Powerset axiom or the unbounded Separation scheme, are implicitly used. While these principles are available in toposes and classical set theory, they are regarded as highly problematic in those settings that, beside using intuitionsistic logical principles (as the internal logic of toposes), also adopt a constructive notion of set.

The main set-theoretical system of this latter type is Constructive Set Theory, CST [3]. In this article, using coinductive methods, and exploiting the Relation Reflection Scheme in CST, I first formulate a constructive notion of strong regularity, that will allow for the choice-free proof that a compact regular locale is strongly regular, and that makes no use of non-constructive set existence principles. This appears in Section 3. In Section 4 as an application of this concept, I show that, as for toposes and ZF, in CST every strongly regular locale $L$ has a compact regular compactification, so that a locale has a compact regular compactification iff it is strongly regular (this result is largely a reformulation of topos-valid results I presented in [8], the missing ingredient there being indeed the formulation of strong regularity developed in this article).

Using the construction of this compactification, the core locale-theoretic (and topological) result of this article is then derived: as is well-known, Stone-Čech compactification is characterized by its ability to allow for the (unique) extension of every continuous function with compact and regular co-domain. It is shown here that, in a similar vein, every compactification $k$ can be characterized as the minimal compactification satisfying an extension property for an associated class $\mathcal{C}_k$ of continuous maps with compact regular co-domains, a fact that does not seem to have been noticed before, or adequately emphasized. It is also proved that for two compactifications $k_1, k_2$ of a locale $L$, $k_1 \leq k_2$ if and only if $\mathcal{C}_{k_1} \subseteq \mathcal{C}_{k_2}$, so that the larger the compactification, the larger the class of continuous maps that can be extended. In particular, if $k = \beta$ is Stone-Čech compactification, $\mathcal{C}_\beta$ coincides with the class of all continuous functions on $L$ with compact and regular co-domain.

In Section 5 an open problem is presented. In classical set theory and in toposes one can define the compact regular reflection of a locale [9] [7]. A natural question is then whether this result is also derivable in CST. That this is not the case for the compact regular reflection in its full generality follows from the proof in [10] that the compact completely regular reflection of a locale $L$ (Stone-Čech compactification of $L$) is independent from CST, also when extended with (various principles, including) the principle of dependent choice DC. However, one can construct in CST the compact completely regular reflection of a locale $L$ for a wide class of locales, and the locales for which this reflection is definable can be characterized [8]. So the question naturally arises whether analogous results may be obtained for the compact regular reflection. This appears to be an interesting open problem in CST.

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1 Constructive Set Theory is here used as a collective name for Myhill - Aczel systems, in particular for CZF, i.e., Constructive Zermelo Fraenkel set theory, and its standard subsystems and extensions; the specific system to be adopted is described informally in Section 1, and formally in the Appendix.

2 The corresponding results for compactifications of topological spaces, that follow from these in the presence of classical logic and the Prime Ideal Theorem, do not seem to have been noticed before as well.
This article has been written having in mind a reader familiar with locale theory, and possibly toposes, but not necessarily with CST. Constructive Set Theory is a subtheory of classical set theory, so the adopted formalism and many constructions are the familiar ones of classical mathematics. CST uses however principles (provable in classical set theory) with which the reader may not be familiar. For this reason, I included in Section 1 an informal presentation of the principles of CST to be exploited, focusing on those that differ the most from the standard classical ones (the formal system for CST is recalled in Appendix). In particular, the needed principles for inductive and coinductive definitions of sets are recalled: as examples of applications of these principles the inductive definition of the set of finite subsets of a given set $X$, and the coinductive definition of the largest interpolative relation $R_0$ contained in a relation $R$, are presented.

An ambition of this article is to show how, avoiding non-constructive set existence principles as Powerset and unrestricted Separation, inductive and coinductive definitions emerge as natural tools, that can be exploited for obtaining more explicit and informative constructions. Remarkably, principles of CST as the axioms needed for inductive and coinductive definitions of sets, or the Relation Reflection Scheme, that were introduced a priori, with no connection to a specific mathematical development, turn out to provide exactly what is needed for a constructive and choice-free theory of compactifications.

1. Informal Constructive Set Theory

The intended audience of this article includes readers with some familiarity with locale theory and intuitionistic logic (topos logic), but not necessarily with CST. In this section I give an informal presentation of the principles of constructive set theory to be exploited that may be unfamiliar to such audience. The last section contains a formal description of the axiom system(s) for CST. The reader is referred to [4] for a thorough introduction to the subject.

In the ordinary mathematical practice, one is rarely faced with the problem of showing that a certain collection of mathematical objects is a set. This is due to the fact that classical set theory, as well as the internal set theories of toposes, have strong set existence principles that trivially imply that the collections under consideration are indeed sets. For instance, given a set $X$, one considers the discrete topology on $X$ which is given by the class $\text{Pow}(X)$ of subsets of $X$. There is no need of a proof in those contexts that such a class is a set, since the Powerset Axiom exactly ensures that this is the case. The Powerset Axiom, as the unrestricted Separation axiom (see below), allow for circular non-constructive definitions, that are rejected in contexts which adopts (intuitionistic logic and) a constructive concept of set. In such constructive contexts, proving that a certain construction yields a set, rather than a mere class is, in general, much more demanding.

Constructive Set Theory, the setting of this article, is a subtheory of classical set theory, in particular the usual set-theoretical language and notation is adopted in this context. Several of the CST axioms and schemes, as the Union or Pairing or Infinity axioms, are familiar ZF axioms (cf. Appendix), so we here concentrate on those principles of CST that we shall exploit in the following, and that might be unfamiliar to the reader. Note that all such principles are provable in classical set theory.
As in classical set theory, given any formula \( \phi \), one has the class \( \{ x : \phi(x) \} \) of elements satisfying \( \phi \). For instance, given any set or class \( X \), one has the class \( \text{Pow}(X) = \{ x : x \subseteq X \} \) of subsets of \( X \). As already mentioned, in CST we do not have the Powerset axiom, so even when \( X \) is a set, if it is not empty, one will not be able to deduce that the class \( \text{Pow}(X) \) is a set.

Another principle of classical set theory that is not available in its full form in CST is the Separation axiom:

for every formula \( \phi \) and any set \( A \), there is a set \( X \) such that

\[
X = \{ x \in A : \phi(x) \}.
\]

In CST, this principle is replaced by Restricted Separation:

for every restricted formula \( \phi \) and any set \( A \), there is a set \( X \) such that

\[
X = \{ x \in A : \phi(x) \},
\]

where a formula \( \phi \) is \textit{restricted} if the quantifiers that occur in it (if any) are of the form \( \forall x \in B, \exists x \in C \), with \( B, C \) sets.

Note that the lack of full Separation, implies that one may have in CST a subclass of a set that need not be a set, precisely subclasses of a set \( A \) of the form \( C = \{ x \in A : \phi(x) \} \) where \( \phi \) is not restricted. Thus, the notation \( U \subseteq X \) merely says that \( U \) is a possibly proper subclass of \( X \), while \( U \in \text{Pow}(X) \) expresses the fact that \( U \) is a subset of \( X \), as \( \text{Pow}(X) \) is the class of subsets of \( X \).

Another principle to be exploited that might not be familiar to the reader is the Strong Collection axiom. Strong Collection is a strengthening of the more familiar ZF axiom of Replacement, stating that if \( f : A \to C \) is a function from a set \( A \) to a class \( C \), the image of \( A \) under \( f \) is a set.

Strong Collection states, more generally, that if \( A \) is a set and \( \phi(x, y) \) is a formula such that \( (\forall x \in A)(\exists y \phi(x, y)) \), then there is a set \( B \) such that

\[
(\forall x \in A)(\exists y \in B)\phi(x, y) \quad \& \quad (\forall y \in B)(\exists x \in A)\phi(x, y).
\]

Note that from Replacement it follows that if \( A \) is a set, \( f : A \to C \) a function, then \( f = \{ (x, f(x)) : x \in A \} \) is a set.

A key role in CST, and specifically in this article, is played by inductive and co-inductive definitions. An \textit{inductive definition} is any class \( \Phi \) of pairs. A class \( A \) is called \( \Phi \text{-closed} \) if:

\[
(a, X) \in \Phi, \text{ and } X \subseteq A \text{ implies } a \in A.
\]

CST enjoys the class \textit{induction property}:

for each class of pairs \( \Phi \), there is a smallest \( \Phi \text{-closed} \) class \( I(\Phi) \); i.e., there is a class \( I(\Phi) \) such that:

1. \( I(\Phi) \) is \( \Phi \text{-closed} \), and
2. if the class \( A \) is \( \Phi \text{-closed} \) then \( I(\Phi) \subseteq A \).

As an example, we construct inductively the class of \textit{finite subsets} of a class \( C \). Following Kuratowski, we define \( \text{Pow}_{\text{fin}}(C) \) to be the class \( I(\Phi_{\text{fin}}) \), with

\[
\Phi_{\text{fin}} = \{ (\emptyset, \emptyset) \} \cup \{ (U \cup \{ y \}, \{ U \}) : U \in \text{Pow}(C), y \in C \}.
\]

A class \( X \) is then \( \Phi_{\text{fin}} \text{-closed} \) if: the empty set belongs to \( X \); if a subset \( U \) of \( C \) belongs to \( X \), the subset \( U \cup \{ y \} \) also belongs to \( X \), for every \( y \in C \).
A class $K$ is a bound for the inductive definition $\Phi$ if, for every $(x, X) \in \Phi$, there is a set $A \in K$ and an onto mapping $f : A \to X$. An inductive definition $\Phi$ is defined to be bounded if:

1. $\{x \mid (x, X) \in \Phi\}$ is a set for every $X$;
2. $\Phi$ is bounded by a set.

The system for CST we adopt satisfies the Bounded Induction Scheme (BIS):

For each bounded class of pairs $\Phi$, the class $I(\Phi)$ is a set. In particular, if $\Phi$ is a set, $I(\Phi)$ is a set.

As an example, using BIS we may prove that when $S$ is a set, $\text{Pow}_{\text{fin}}(S)$ is a set. Indeed, $\Phi$ is a set since $R \subseteq S$.

Let $\text{Pow}_{\text{fin}}(X)$, one may also prove that every $a \in \text{Pow}_{\text{fin}}(X)$ is the range of a (not necessarily one-one) function $f : n \to a$, for some $n \in \mathbb{N}$, $n$ the set with $n$ elements: one considers the class $\{u \in \text{Pow}_{\text{fin}}(X) : (\exists n \in \mathbb{N})(\exists f : n \to u\}$, and concludes showing that this class is $\Phi_{\text{fin}}$-closed.

Given an inductive definition $\Phi$ on a set $S$, i.e of the form $\Phi \subseteq S \times \text{Pow}(S)$, a class $C \subseteq S$ is said to be $\Phi$-inclusive if $C \subseteq \Gamma_{\Phi}(C)$, with

$$\Gamma_{\Phi}(C) = \{x \mid (\exists X) (x, X) \in \Phi \& X \subseteq C\}.$$ 

CST has the class co-induction property:

the class $C(\Phi) = \bigcup\{Y \in \text{Pow}(S) \mid Y \subseteq \Gamma_{\Phi}(Y)\}$ is the largest $\Phi$-inclusive subclass of $S$, the class co-inductively defined by $\Phi$.

In our system for CST the following principle is available.

Set Coinduction Scheme (SCS): when $S, \Phi$ are sets, $C(\Phi)$ is a set.

As an example, we show that one can construct coinductively the largest interpolative relation contained in a given binary set-relation on a set. This construction will be exploited in the definition of strong regularity to be introduced.

**Lemma 1.1.** Let $A$ and $R$ be sets, $R$ a binary relation on $A$, $R \subseteq A \times A$. Then the largest interpolative relation $R_0$ contained in $R$ is a set.

**Proof.** Let $R_0 = \bigcup\{Y \in \text{Pow}(R) : (\forall x, y)[(x, y) \in Y \to (\exists z \in A)(x, z) \in Y \& (z, y) \in Y]\}$ the union of all interpolative set-relations contained in $R$. A priori, this definition only defines a class. We now prove that $R_0$ can be constructed as the set co-inductively defined by an inductive definition. Let $\Phi$ be the inductive definition

$$\{(x, z), \{(x, y), (y, z)\} : ((x, z), (x, y), (y, z) \subseteq R\}.$$ 

Note that $\Phi$ is a set since $R$ is, so that the largest $\Phi$-inclusive class $C(\Phi) = \bigcup\{Z \in \text{Pow}(R) : Z \Phi$-inclusive$\}$ is a set by the SCS. We conclude showing that $R_0 = C(\Phi)$: let $Y$ be an interpolative relation contained in $R$. To show that $Y \subseteq C(\Phi)$ one just need to show that $Y$ is $\Phi$-inclusive: Let $(x, y) \in Y$. Then there is $z \in A$ such that $(x, z) \in Y \& (z, y) \in Y$. Since $Y \subseteq R$, we conclude that $Y$ is $\Phi$-inclusive. To prove the converse, i.e., that $C(\Phi) \subseteq R_0$, since $C(\Phi)$ is a set, it is enough to prove that $C(\Phi)$ is interpolative. Let $(x, y) \in C(\Phi)$. Then there is $Z \in \text{Pow}(R)$, $Z$ $\Phi$-inclusive, with $(x, y) \in Z$. There is thus $X$ such that $((x, y), X) \in \Phi \& X \subseteq Z$. 

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Therefore, an element \( z \in A \) exists such that \((x, z) \in R\), \((z, y) \in R\), and \((x, z) \in Z\), \((z, y) \in Z\). Since \( Z \subseteq C(\Phi) \) we conclude that \( C(\Phi) \) is interpolative. \( \square \)

Contrary to what one might expect, this result will not suffice for a constructive definition of the concept of strong regularity. Further work will be needed, in particular involving the application of a principle of CST called Relation Reflection Scheme, again provable from the ZF axioms ([2, Theorem 2.5]).

Relation Reflection Scheme, RRS:

For classes \( S, R \) with \( R \subseteq S \times S \), if \( a \in S \) and \( \forall x \in S \exists y \in S \ R(x, y) \) then there is a set \( S_0 \subseteq S \) such that \( a \in S_0 \) and \( \forall x \in S_0 \exists y \in S_0 \ R(x, y) \).

This scheme has proved useful in obtaining choice-free derivations of results that had first been proved using principles of Dependent Choice (see [2]).

2. Locales Constructively

We assume familiarity with the theory of frames (locales) [13]. In this section we illustrate how the notions we shall need are formulated in the present constructive context. Let \( X \) be a set. As already noted, the complete lattice \( \Omega(X) \) of open subsets of \( X \) endowed with the discrete topology is the power class of \( X \), \( \text{Pow}(X) \).

This example shows that, constructively, we cannot expect a frame to be carried by a set of elements (see also [12]). The definition of set-generated frame/locale to be recalled is the version of the frame/locale concept adopted in constructive set theory.

A pair of classes \((L, \leq)\), with \( \leq \) a partial order on \( L \), is a class locale (or class frame) if every finite subset of \( L \) has a least upper bound (meet), every subset has a greatest lower bound (join), and if the following infinite distributive law is satisfied:

\[
x \land \bigvee y U = \bigvee_{y \in U} (x \land y)
\]

for all \( x \in L \), \( U \in \text{Pow}(L) \). A class-locale \( L \) is said to be set-generated by a set \( B_L \subseteq L \) if, for every \( x \in L \),

i. the class \( D_x = \{ b \in B_L : b \leq x \} \) is a set,

ii. \( x = \bigvee D_x \).

\( B_L \) is called a basis of \( L \). In a fully impredicative context as classical set theory or the internal language of a topos, set-generated class-locales and ordinary locales come to the same thing.

In the following, a set-generated class-locale \((L, B_L)\) will simply be referred to as a locale \( L \), sometimes omitting the explicit mention of the basis \( B_L \). When no confusion may arise, we also write \( B \) for \( B_L \). Note that if \((L, B_L)\) is set-generated, by Restricted Separation (Section 1) the restriction of \( \leq \) to \( B_L \times B_L \) is a set: \( \leq \cap B_L \times B_L = \{(a, b) \in B_L \times B_L : a \in D_b\} \). Also, if \( B_L \) is a basis of \( L \), and \( B_L' \subseteq L \) is a set such that \( B_L \subseteq B_L' \), then \( B_L' \) also is a basis for \( L \), as is easy to verify again using Restricted Separation.

A continuous map of locales \( f : L \to M \) is a function \( f^- : B_M \to L \) (note the reverse direction) satisfying:

1. \( \bigvee_{a \in B_M} f^-(a) = 1 = \bigvee B_L \),
2. \( f^- (a) \land f^- (b) = \bigvee \{ f^-(c) : c \in B_M, c \leq a, c \leq b \} \), for all \( a, b \in B_M \),
3. \( f^- (a) \leq \bigvee_{b \in U} f^- (b) \), for all \( a \in B_M, U \in \text{Pow}(B_M) \) with \( a \leq \bigvee U \).
The (in general proper) class of these maps is denoted by $\text{Hom}_{\text{Loc}}(L, M)$. Observe that $\text{Hom}_{\text{Loc}}(L, M)$ is in a one-to-one correspondence with the collection of frame homomorphisms from $M$ to $L$, i.e., class functions from $M$ to $L$ preserving the frame structure: given $f : L \to M$, one extends $f^- : B_M \to L$ to $M$ by letting for $a \in M$, $f^-[a] = \bigvee \{f^-(b) : b \in B_M, b \leq a\}$. Trivially, for $b \in B_M$, $f^-[b] = f^-(b)$.

Note in particular that $f^-$ preserves meets and joins that exist in $B_M$. Composition $f \circ g : L \to N$ of two continuous maps $f : M \to N$ and $g : L \to M$ is defined as $(f \circ g)^- : N \to L$, $(f \circ g)^-(a) = \bigvee \{g^-(x) : x \in B_M & x \leq f^-(a)\} = g^-[f^-(a)]$.

A locale mapping $f : L \to M$ is dense if, for $a \in M$, $f^-[a] = 0$ implies $a = 0$; $f$ is an embedding (or a sublocale) if $f^-$ is onto, equivalently, if for every $a \in L$ there is $U \in \text{Pow}(B_M)$ such that $\bigvee_{b \in U} f^-[b] = a$.

A locale $L$ is compact iff every covering of $1 = \bigvee B_L$ by basic elements (i.e., every $U \in \text{Pow}(B_L)$ such that $1 = \bigvee U$) has a finite subcover. $L$ is regular if, for all $a \in B_L$, $a = \bigvee \{b \in B_L : b \prec a\}$, where, for $x, y \in L$, $y \prec x$ iff $1 = x \vee y^*$, with $y^* = \bigvee \{c \in B_L : c \land y = 0\}$ the pseudocomplement of $y$.

Note that $\{b \in B_L : b \prec a\} = \{x \in B_L : (\forall b \in B_L) b \in D(x, v_a)\}$ is a set, while we could not define, as in the classical or topos-theoretic context, a locale to be regular if $x \leq \bigvee \{y \in L : y \prec x\}$, for all $x \in L$. Indeed, the class $\{y \in L : y \prec x\}$ need not be constructively a set, so that its join need not exist. It is easy however to see that the given definition is in the classical/topos-theoretic context equivalent to the standard one. Similar considerations apply to the definition of compactness, and of the separation properties recalled below.

We shall need the following well-known facts.

**Lemma 2.1.** i. If $f : L \to M$ is a dense embedding, $f^-[: M \to L$ preserves pseudocomplements. ii. If $f : L \to M$ is a dense continuous map, $L$ is compact and $M$ is regular, then $f^[: M \to L is one-one. iii. If $h : L \to M$, $f, g : M \to N$ are continuous maps with $h$ dense, $N$ regular, and $f \circ h = g \circ h$, then $f = g$.

**Proof.** i. For every $a$ in $M$, one has $f^-[a*] = \bigvee \{f^-(x) : x \in B_M & x \land a = 0\} \leq f^-[a]^*$; for the converse, let $b \in M$ be such that $f^-[b] = f^-[a]^*$. By density, $f^-[b \land a] = f^-[b] \land f^-[a] = 0$ implies $b \land a = 0$, so that $b \leq a^*$, and $f^-[b] = f^-[a]^* \leq f^-[a^*]$.

ii. We first prove that $f^-[: M \to L is co-dense, i.e., that, for $a \in M$, $f^-[a] = 1$ implies $a = 1$. Indeed, $a = \bigvee \{x \in B_M : x \prec a\}$. Then, $f^-[a] = 1 = \bigvee \{f^-(x) : x \in B_M & x \prec a\}$. By compactness of $L$ there is a finite subset $U$ of $B_M$ such that $f^-[\bigvee U] = 1$ and $\bigvee U \prec a$, i.e., $1 = \bigvee U \prec a$. Since $f^-[\bigvee U] \leq f^-[\bigvee U] = 0$, by density $\bigvee U \prec a = 0$, so that $a = \bigvee U \prec a = a$.

Now let $a, b \in M$ be such that $f^-[a] = f^-[b]$. If $x \in B_M$ is such that $x \prec a$, i.e., $x^* \lor a = 1$, then $1_L = f^-[x^* \lor a] = f^-[x^*] \lor f^-[a] = f^-[x*] \lor f^-[b] = f^-[x^* \lor b]$. Since $f^-$ is codense, this gives $x^* \lor b = 1$, i.e., $x \prec b$. Thus, $x \prec a \iff x \prec b$ for all $x \in B_M$, and since by regularity for every $a \in M$, $a = \bigvee \{x \in B_M : x \prec a\}$, we conclude $a = b$.

iii. We have to show that if $h^-[f^-(c)] = h^-[g^-(c)]$ for all $c \in B_N$, then $f^- = h^-$. By regularity of $N$, for every $c \in B_N$, $c = \bigvee \{b : b \prec c\}$. If $b \prec c$ one has $b^* \lor c = 1$, so that $g^-[b^* \lor c] = g^-[b^*] \lor g^-(c) = 1$. Then, $f^-[b] = f^-[b] \lor (g^-[b^*] \lor g^-(c)) = (f^-[b] \lor g^-[b^*]) \lor (f^-[b] \lor g^-(c))$. On the other hand, since $f \circ h = g \circ h$,
\[ h^{-}(f^{-}(b) \land g^{-}([b^*])) = h^{-}(f^{-}(b)) \land h^{-}(g^{-}([b^*])) = h^{-}(f^{-}(b)) \land h^{-}(f^{-}(b^*)) = h^{-}(f^{-}(b \land b^*)) = 0, \]

and since \( h \) is dense, we get \( f^{-}(b) \land g^{-}([b^*]) = 0 \). Together with the above, this gives \( f^{-}(b) = f^{-}(b) \land g^{-}(c) \), whence \( f^{-}(b) \leq [g^{-}(c)] \) for all \( b \prec c \), so that \( f^{-}(c) \leq g^{-}(c) \).

Symmetrically \( g^{-}(c) \leq f^{-}(c) \).

\[ \square \]

Compared with regularity, a constructive concept of complete regularity requires some more work. The standard definition of complete regularity for locales demands that for every \( x \) in \( L \), \( x \leq \bigvee \{ y \in L : y \prec x \} \), where \( y \prec x \), \( y \) really inside (or completely below) \( x \), if there is a scale from \( y \) to \( x \). A map \( s : \mathbb{I} \to L \) from the rational unit interval \( \mathbb{I} \) to \( L \) is called a scale from \( y \) to \( x \) if \( s(0) = y, s(1) = x \) and, for \( p < q \), \( s(p) \prec s(q) \).

In this case we have the additional difficulty that, even restricting to basic elements as we did in redefining regularity, we still not have that \( \{ b \in B : b \prec \prec a \} \) is a set, because, by contrast with \( \prec \), the really inside relation \( \prec \prec \) is not a set when restricted to \( B \times B \) (as the class of scales is not a set). This difficulty is circumvented as follows.

A locale \( L \) will be defined completely regular if a family \( r_i : B \to \text{Pow}(B) \) of subsets of \( B \) exists such that \( i. \) for all \( a \in B \), \( a = \bigvee r_i(a) \), and \( ii. \) for all \( b \in r_i(a) \) a scale exists from \( b \) to \( a \). The idea is that, given a basic element \( a \), one does not try to construct the set of all basic elements \( b \) for which a scale exists from \( b \) to \( a \); to be able to establish that \( L \) completely regular, it will suffice to have a set of such elements that is sufficiently rich to cover \( a \).

As said for regularity, it is not difficult to verify that this concept is classically (or topos-theoretically) equivalent to the standard one, as well as that this notion allows for the derivation of standard desirable properties of complete regularity: it is hereditary, invariant under isomorphisms, and allows to derive Tychonoff embedding theorem [11].

The given constructive definition of complete regularity circumvents the use of the really inside relation. However, this relation enjoys the important interpolation property that is in particular useful in the construction of completely regular compactifications. It is possible to reconstruct a satisfactory set-analogue of the really inside relation, satisfying the interpolation property, adding enough scales to the basis as follows.

For \( a, b \in B \), let \( b \prec \prec_B a \) iff a scale of basic elements exists from \( b \) to \( a \). Note that this relation is interpolative on \( B \). We say that \( L \) has a completely regular basis \( B \) if, for all \( a \in B \), \( a = \bigvee \{ b \in B : b \prec \prec_B a \} \).

Of course, a locale with a completely regular basis is completely regular. For the converse the following holds true.

**Lemma 2.2.** If \((L, B)\) is a completely regular locale, then \( L \) has a completely regular basis.

**Proof.** By definition of complete regularity, for every pair \( a, b \) such that \( b \in r_i(a) \) a scale of elements of \( L \) exists from \( b \) to \( a \). We may then apply Strong Collection (Section 1) to collect a set \( V' \) of scales so that for all \( a, b \in B \) such that \( b \in r_i(a) \) there is a scale \( s : \mathbb{I} \to L \) in \( V' \) from \( b \) to \( a \), and conversely for all \( s : \mathbb{I} \to L \) in \( V' \) there are \( a, b \in B \) such that \( s : \mathbb{I} \to L \) is a scale from \( b \) to \( a \). Let \( V \) be the set of elements of \( L \) that belong to a scale in \( V' \), \( V = \bigcup_{s \in V'} \text{Range}(s) \). Since \( V \) is a set, we can construct a new basis \( B' = B \cup V \) for \( L \). To prove that \( B' \) is a completely regular
base, i.e., that for all \( u \in B' \), \( u = \bigvee \{ v \in B' : u \prec u' \} \), note that we have, for all \( u \in B' \), \( u = \bigvee \{ x \in B : (\exists b \in B) b \in D \land x \in ri(b) \} \). Since for every \( c \in ri(b) \) there is in \( V' \) a scale from \( c \) to \( b \), we have \( u = \bigvee \{ c \in B : (\exists b \in B) b \leq u \land c \prec u' b \} \), and a fortiori \( u = \bigvee \{ v \in B' : (\exists b \in B) b \leq u \land v \prec u' b \} \). But if \( v \prec u' b \) for \( b \leq u \), then also \( v \prec u' u \): given the scale \( s : \emptyset \to L \) from \( v \) to \( b \), replace \( s(1) = b \) with \( s(1) = u \) to get a scale from \( v \) to \( u \).

An important property of completely regular bases is that any extension of a completely regular basis is again a completely regular basis.

### 3. Strong Regularity

As is well-known \(^3\), in a compact regular locale the well-inside relation interpolates. Then, if the principle of countable dependent choice (DC) is available, whenever \( x \prec y \), a scale from \( x \) to \( y \) can be constructed\(^4\). Thus, assuming DC, a compact regular locale is completely regular. In \(^7\) Banaschewski and Pultr note that this result may fail to hold in toposes that do not validate DC. For resolving this and related difficulties, they introduce strongly regular locales.

Banaschewski and Pultr begin by noting that any binary relation \( R \) contains a largest interpolative relation. Indeed, as the union of any set of interpolative relations is an interpolative relation, defining \( R_0 \) to be the union of all interpolative relations contained in \( R \) will yield the largest interpolative relation contained in \( R \).

Taking \( R = \prec \) one thus obtains the largest interpolative relation \( \prec_0 \) contained in the well-inside relation. A locale \( L \) is then defined strongly regular if \( x \leq \bigvee \{ y \in L : y \prec_0 x \} \), for all \( x \in L \).

Unfortunately, this definition, albeit adequate to the choice-free context of toposes, is impredicative or circular: \( R_0 \) is indeed itself an interpolative relation contained in \( R \), so that it belongs to the collection of interpolative relations of which it is the union. In the present constructive setting, defining \( R_0 = \bigcup \{ R' \in \text{Pow}(R) : R' \text{ interpolative} \} = \{ (x, y) \in R : \exists R' \in \text{Pow}(R)[R' \text{ interpolative } \land (x, y) \in R'] \} \) a priori only yields a proper class, since \( \text{Pow}(R) \) cannot be proved to be a set in CST when \( R \) is not empty \(^{10}\), and the unbounded separation scheme is not available. Furthermore, in this context, as already noted, \( R = \prec \subseteq L \times L \) is a proper class, rather than a set.

Exploiting the Set Coinduction Scheme and the Relation Reflection Scheme (Section\(^1\)), in this section I present an alternative method for defining strong regularity, that is both non-circular and adequate to a choice-free intuitionistic setting.

On a locale \( L \), let the class

\[
\prec_0 = \bigcup \{ R \in \text{Pow}(\prec) : (\forall x, y)[R(x, y) \to (\exists z \in L)R(x, z) \land R(z, y)] \}
\]

be the union of all interpolative set-relations contained in the well-inside relation \( \prec \) on \( L \). Note again that \( \prec_0 \) is a class, so that it need not belong to \( \text{Pow}(\prec) \). Similarly to what has been said before for regularity and complete regularity, one cannot define \( L \) to be strongly regular if \( x \leq \bigvee \{ y \in L : y \prec_0 x \} \) for all \( x \in L \), since the class \( \{ y \in L : y \prec_0 x \} \) is not a set in general. Again, we circumvent the

\(^3\)This holds true for systems as ZF or the internal logic of toposes. In CST, due to the absence of Powerset, a stronger version of the axiom of Dependent Choice, the axiom of Relativized Dependent Choice, RDC (see e.g. \(^1\) for this principle) is needed. If the locale is compact and regular, DC suffices for the construction of a scale, since Lemma\(^3\) below ensures that the element \( z \) such that \( x \prec z \prec y \) belongs to a set (a base), rather than a class.
problem by defining $L$ to be **strongly regular** if a family $si : B \rightarrow \mathsf{Pow}(B)$ of subsets of $B$ exists such that

i. for all $a \in B$, $a = \bigvee si(a)$, and

ii. if $b \in si(a)$, then $b \prec_0 a$.

Trivially, in the classical and topos-theoretic context, $L$ is strongly regular according to the Banaschewski and Pultr notion if and only if it is strongly regular for this definition. Also, one has that a completely regular locale in the sense of the previous section is strongly regular (just let $si(a) = ri(a)$ for all $a \in B$). Conversely, assuming the axiom of (relativized) dependent choice, one proves that a strongly regular locale is completely regular, with $ri(a) = si(a)$ for all $a \in B$ (cf. note 3).

As for complete regularity, this definition is however not completely satisfactory. It avoids the circularity in the definition of the largest interpolative set-relation contained in $\prec$, but the important interpolation property of this relation is lost. To recover this property we shall argue as follows. First we construct a larger basis $B'$ containing 'enough interpolants'. Over this new basis, we then construct coinductively the largest set-relation contained in the well-inside relation. This construction is the analogue for strong regularity of the construction of completely regular bases for completely regular locales.

Given a strongly regular locale $(L, B)$, with $si : B \rightarrow \mathsf{Pow}(B)$ as above, let $S$ be the class

$$\{(x_0, \ldots, x_n) : n \in \mathbb{N}_{>0} \& x_0, \ldots, x_n \in L \& (\forall i < n)x_i \prec_0 x_{i+1}\}.$$  

Let $R \subseteq S \times S$ be defined by

$$R(\{x_0, \ldots, x_n\}, \{y_0, \ldots, y_m\}) \iff x_0 = y_0 \& x_n = y_m \& \{x_0, \ldots, x_n\} \subseteq \{y_0, \ldots, y_m\} \& (\forall k < n)(\exists k' < m) x_k \prec_0 y_{k'} \prec_0 x_{k+1}.$$  

We then have $(\forall l_1 \in S)(\exists l_2 \in S)R(l_1, l_2)$, since the class $\prec_0$ is interpolative. Given a pair $(c, b) \in S$, i.e., such that $c \prec_0 b$, using the Relation Reflection Scheme (Section 1) one has a set $S_0$ such that $S_0 \subseteq S$, and

$$(c, b) \in S_0 \& (\forall l_1 \in S_0)(\exists l_2 \in S_0)R(l_1, l_2).$$

Thus, for all pairs $(c, b)$ such that $c \in si(b)$ there is such a set $S_0$. Let $SI = \{(c, b) : c \in si(b)\}$. By Strong Collection (Section 1), a set $V'$ exists such that

1. $(\forall(c, b) \in SI)(\exists S_0 \in V')(c, b) \in S_0 \subseteq S \& (\forall l_1 \in S)(\exists l_2 \in S)R(l_1, l_2)$

and

$$(\forall S_0 \in V')(\exists(c, b) \in SI)(c, b) \in S_0 \subseteq S \& (\forall l_1 \in S)(\exists l_2 \in S)R(l_1, l_2).$$

Let $V = \{z \in L : (\exists S_0 \in V')(\exists z \in S_0)z \in l\}$. Define $B' = B \cup V$. Note that $B'$ is a set, and a basis of $L$.

Let then $\prec'$ be the restriction of the well-inside relation to $B' \times B'$, $\prec' = \{(y, x) \in B' \times B' : 1 \leq x \vee y^*\}$. The relation $\prec'$ is a set: one has $1 \leq x \vee y^* \iff (\forall b \in B)b \in D_{x \vee y^*}$. Since $D_{x \vee y^*} = \{c \in B : c \leq x \vee y^*\}$ is a set, as $L$ is set-generated by $B$, by Restricted Separation (Section 1) we can conclude that $\prec'$ is a set. By Lemma 1.1, we can then construct the largest interpolative set-relation $\prec_0'$ contained in $\prec'$.  

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**Lemma 3.1.** Let $u, v, x, y \in B'$. If $x \leq u \prec_b v \leq y$, then $x \prec_b y$.

**Proof.** Consider the relation $R = \{(x, y) \in B' \times B' : (\exists(u, v) \in B' \times B')x \leq u \prec_b v \leq y\}$.

By Restricted Separation $R$ is a set. Moreover, $R$ is interpolative and $R \subseteq \prec'$. Since $\prec\_b$ is the largest relation with these properties, $R = \prec\_b$.

We shall say that a locale $L$ has a strongly regular basis $B$ if the largest interpolative relation $\prec\_b$ contained in the restriction of the well-inside relation to $B$ is compatible, i.e., if, for all $a \in B$, $a = \bigvee\{b \in B : b \prec\_b a\}$. A locale with a strongly regular basis is strongly regular. Conversely, the following obtains.

**Proposition 3.2.** If $(L, B)$ is a strongly regular locale, then $L$ has a strongly regular basis $B'$ extending $B$.

**Proof.** We prove that the basis $B' = B \cup V$ we have constructed is a strongly regular basis for $L$. We have to show that for all $u \in B' = B \cup V$, $u = \bigvee\{v \in B' : v \prec\_b u\}$. Let $u \in B'$. Then $u = b \in B$, or $u \in V$.

In the first case, as $L$ is strongly regular, $b \leq \bigvee sib(b)$. We show that $sib(b) \subseteq \{v \in B' : v \prec\_b u\}$. If $c \in sib(b)$ by $\dagger$ there is $S_0 \in V'$ such that

$$\text{(*)} \quad (c, b) \in S_0 \& (\forall l_1 \in S_0)(\exists l_2 \in S_0)R(l_1, l_2).$$

On $B'$ consider the binary relation

$$R'(u, v) \iff \exists \{x_0, \ldots, x_n\} \in S_0(\exists i)i > j) x_i = u \& x_j = v.$$ 

Note that $R'(u, v)$ implies $u \prec' v$, and that $R'$ interpolates: assume $R'(u, v)$, then, by $(\ast)$, there is $y$ such that $R'(u, y)$ and $R'(y, v)$. As a consequence, $R' \subseteq \prec\_b$, and since in particular $R'(c, b)$, we conclude $sib(b) \subseteq \{v \in B' : v \prec\_b b\}$, whence $b = \bigvee\{v \in B' : v \prec\_b b\}$.

If, on the other hand, $u \in V$, then $u = \bigvee\{b \in B : b \leq u\}$. For every $b \in B$ such that $b \leq u$, one has $b \leq \bigvee\{v \in B' : v \prec\_b b\}$. Since then $u \leq \bigvee\{v \in B' : (\exists b \in B)b \in D_0 \& v \prec\_b b\}$, we conclude observing that by the previous lemma $v \prec\_b b \leq u$ implies $v \prec\_b u$.

We leave to the reader to verify that every basis extending a strongly regular base is strongly regular.

This section ends with a proof that a compact regular locale is strongly regular. This will be the consequence of the following lemma, to be often invoked in Section 3. Given a locale $(L, B)$, we can always extend its basis $B$ to a basis $B^\ast$ that is (a set and) a sub-pseudocomplemented distributive lattice of $L$, as follows. $B^\ast$ is defined inductively by the following clauses:

- if $b \in B$ then $b \in B^\ast$;
- if $u \in B^\ast$ then $u^\ast \in B^\ast$;
- if $V$ is a finite subset of $B^\ast$, then $\land V, \lor V \in B^\ast$,

where $\land, \lor, \ast$ are the operations in $L$. In other terms, $B^\ast$ is generated by the inductive definition

$$\Phi_{B^\ast} = \{(b, \emptyset) : b \in B\} \cup \{(u^\ast, \{u\}) : u \in L\} \cup$$

$$\cup \{(\land V, V) : V \in \text{Pow}_{\text{fin}}(L)\} \cup \{(\lor V, V) : V \in \text{Pow}_{\text{fin}}(L)\}$$

Since this inductive definition is bounded, $B^\ast = I(\Phi_{B^\ast})$ is a set (Section 1).
The following lemma is proved in [9]. The proof is reformulated here for the reader’s convenience.

**Lemma 3.3.** Let \( L \) be a compact regular locale, \( P \) a set, a sub-lattice and a basis of \( L \). Assume, for \( b \in L \), \( U \in \text{Pow}(P) \), that \( b \prec \bigvee U \). Then \( b = 0 \), or finite subsets \( \{p_1, ..., p_n\} \), \( \{p_1', ..., p_n'\} \), and \( \{p''_1, ..., p''_n\} \) of \( P \) exist with \( p_i \prec p_i' \prec p''_i \), \( p_i' \in U \), and such that

\[
b \leq \bigvee p_i \prec \bigvee p_i' \prec \bigvee U.
\]

**Proof.** Since \( L \) is regular, and since \( P \) is a basis, by \( b \prec \bigvee U \), one obtains \( 1 = b^* \vee \bigvee \{p \in P : (\exists p'' \in U)(\exists p' \in P)p' \prec p'', \ p \prec p'\} \). By compactness, thus, \( 1 = \bigvee s_0 \), with \( s_0 \in \text{Pow}_{\text{fin}}(P) \) a finite subset of \( \{b^*\} \cup \{p \in P : (\exists p'' \in U)(\exists p' \in P)p' \prec p'', \ p \prec p'\} \). By induction on \( \text{Pow}_{\text{fin}}(P) \) (Section 1), \( s_0 = u_0 \cup v_0 \), with \( u_0 \subseteq \{b^*\} \), \( v_0 \subseteq \{p \in P : (\exists p'' \in U)(\exists p' \in P)p' \prec p'', \ p \prec p'\} \), and both finite. Then \( b \leq \bigvee v_0 \). Again by induction, \( v_0 \) is either empty or inhabited. In the first case \( b = 0 \); in the second, let \( v_0 = \{p_1, ..., p_n\} \) be an enumeration of \( v_0 \) (Section 1). Thus, \( b \leq \bigvee \{p_1, ..., p_n\} \). For all \( i \leq n \), there are \( y \in P \) and \( z \in U \) such that \( p_i \prec y \prec z \). Using induction on the length \( n \) of the enumeration of \( v_0 \) (see also [4, Lemma 8.2.8], provable choice for finite sets), one constructs the required subsets \( \{p_1', ..., p_n'\} \), \( \{p''_1, ..., p''_n\} \). \( \Box \)

In particular, if for \( a, b \in P \) one has \( b \prec a \), then there are \( p, q \in P \) with \( b \leq p \prec q \prec a \). Therefore, a compact regular locale \( L \) is strongly regular, with \( P \) a strongly regular basis for \( L \).

4. **Compact regular compactifications**

In contexts as toposes or constructive set theory, in which the principle of Dependent Choice is not assumed, compact regular compactifications do not coincide with compact completely regular compactifications. In this section we deal with compactifications of the first type. Part of the material in this section, formulated for the topos-theoretic context, is presented in [9]. It is here reproduced in a version adequate to CST.

A compact regular compactification of a locale \( L \), in the following simply a compactification, is a dense embedding \( k : L \to M \), with compact regular co-domain. Exploiting the coinductive formulation of strong regularity derived in the previous section, we shall show how, given a compactifiable locale \( L \), and any set-indexed family of locale mappings \( \mathcal{F} \equiv \{f_i\}_{i \in I} \), \( f_i : L \to L_i \), with compact and regular co-domain, a compactification \( \gamma L \) of \( L \) can be constructed with the property that each function in \( \mathcal{F} \) has a unique extension to \( \gamma L \).

The construction of the compactification to be presented will then allow us to show that any compactification can be characterized in terms of its ability of extending continuous functions with compact regular codomain, an aspect of compactifications that does not seem to have been emphasized before. The relation of these results with the compact regular reflection of \( L \) (the ‘weak’ Stone-Čech compactification) is discussed in the next section.

As a compact regular locale is strongly regular, and since, as is easy to verify, strong regularity is hereditary (if \( (L, B_s) \) is a strongly regular locale with strongly regular basis \( B_s \), and \( f : L' \to L \) is an embedding, \( B'_s = \{f^{-1}(b) : b \in B_s\} \) is a strongly regular basis for \( L' \)), a compactifiable locale must be strongly regular. We now show that the converse also holds constructively.
In [5] Banaschewski showed that relations of strong inclusion on a frame \( L \) may be used for defining the compactifications of that frame. The whole frame structure is not needed for carrying out Banaschewski’s construction. In particular, strong inclusions can be considered more generally over pcd-lattices [9]. This fact is particularly relevant in a constructive context as the present one, since, while in such a context a (non-trivial) frame is carried by a proper class, a pcd-lattice may well be carried by a set. Recall in particular from the previous section that every set \( B \) generating a frame can be extended to a pcd-lattice \( P = B^* \) that is a set; recall also that, as noted earlier, the partial order restricted to any set of generators is a set.

In the following, by pcd-lattice we mean a partial order \((P, \leq)\) with \( P, \leq \) sets, satisfying the standard axioms of a pseudo-complemented distributive lattice. A strong inclusion on a pcd-lattice \( P \) is a binary set-relation \( \prec \) on \( P \) satisfying:

1. \( 0 \prec 0, 1 \prec 1; \)
2. if \( x \leq a \prec b \leq y \) then \( x \prec y; \)
3. if \( x \prec a, x \prec b \) then \( x \prec a \land b; \)
4. if \( x \prec a, y \prec a \) then \( x \lor y \prec a; \)
5. if \( a \prec b \) then \( b^* \prec a^*; \)
6. \( \prec \subseteq \leq; \)
7. if \( x \prec y \) then \( x \prec z \prec y, \) for some \( z \in P. \)

Note that from 6. it follows \( \leq \subseteq \leq. \) On every pcd-lattice \( P \) the well inside relation \( \prec \) (is a set and) satisfies conditions 1 to 6, while \( \prec'_0 \) is a strong inclusion: that \( \prec'_0 \) is a set is due to Proposition 1.1 that it is a strong inclusion is proven as in [7]. 1. consider \( R = \{(0,0),(1,1)\}. \) \( R \) is interpolative and contained in \( \prec. \) Since \( \prec'_0 \) is the largest such relation, \( \{(0,0),(1,1)\} \subseteq \prec'_0. \) 2. Let \( R = \{(x,y) \in P \times P : (\exists a \in P)(\exists b \in P)x \leq a \prec_0 b \leq y\}. \) \( R \) is a set, interpolative and contained in \( \prec, \) so \( R \subseteq \prec'_0. \) Properties from 3 to 5 are proven similarly. 6. and 7. hold by definition.

If \( P \) is a sub-pcd-lattice of a frame \( L, \) and \( L \) is set-generated by \( P, \) a strong inclusion \( \prec \) on \( P \) is said to be compatible with \( L \) if \( a = \bigvee_L \{x \in P : x \prec a\}, \) for all \( a \in P. \) The following lemma provides a method for constructing strong inclusions on \( P. \)

**Lemma 4.1.** Let \( P \) be a pcd-lattice, and \( R \) be an interpolative binary set-relation on \( P \) with \( R \subseteq \leq. \) The relation \( \prec_R \) defined inductively on \( P \) as the least relation containing \( R \) and closed under conditions 1-5 is a strong inclusion on \( P. \)

**Proof.** Let \( \prec_R = I(\Phi_R), \) with

\[
\Phi_R = \{(x,y) : (x,y) \in R\} \cup \\
\cup \{(0,0),(1,1)\} \cup \\
\cup \{(x,y),(a,b) : x,y,a,b \in P, x \leq a, b \leq y\} \cup \\
\cup \{(x,a \land b),(x,a),(x,b) : x,a,b \in P\} \cup \\
\cup \{(x \lor y,a),(x,a),(y,a) : x,y,a \in P\} \cup \\
\cup \{(b^*,a^*),(a,b) : a,b \in P\}.
\]

This inductive definition is a set, and so is bounded. Thus, \( \prec_R = I(\Phi_R) \) is a set. Obviously, \( \prec_R \) satisfies conditions from 1 to 5. Indeed, assume for instance \( x \leq a \prec_R b \leq y \) as in condition 2. Then, since \( \prec_R \) is \( \Phi_R \)-closed, also \( x \prec_R y. \)
The proof of 6 and 7 is by induction. So, to prove 6, one shows that \( \prec \) is \( \Phi_R \)-closed. Since \( \triangleleft_R \) is the least \( \Phi_R \)-closed class, we conclude \( \triangleleft_R \subseteq \prec \): the base cases \( (x, y) \in R \) and \( 1 \) are trivial (as \( R \subseteq \prec \) by hypothesis); easy calculations show that \( \prec \) is closed for the other conditions in \( \Phi_R \).

Proof of 7: we prove that the set \( K = \{ (x, y) \in \triangleleft_R : (\exists z \in P)(x, z) \in \triangleleft_R \land (z, y) \in \triangleleft_R \} \) is \( \Phi_R \)-closed, so that it must contain \( \triangleleft_R \). The base cases are immediate. Now assume \( x \leq a, b \leq y \) for \( x, y, a, b \in P \), and let \( (a, b) \in K \). Then, by the proof that condition 2 on strong inclusions holds, \( (x, y) \in \triangleleft_R \) and \( x \triangleleft_R z \triangleleft_R y \), for some \( z \in P \), so \( (x, y) \in K \), as wished. The closure for the other conditions in \( \Phi_R \) is proved similarly, as a further example we show that if \( (x, a), (x, b) \in K \) then also \( (x, a \land b) \in K \): from \( (x, a), (x, b) \in K \) we have \( x \triangleleft_R c_1 \triangleleft_R a, x \triangleleft_R c_2 \triangleleft_R b \) for some \( c_1, c_2 \in P \). Then, by 3 and 2, \( x \triangleleft_R c_1 \land c_2 \triangleleft_R a \land b \).

Note that, for \( R = \emptyset \), one gets the least strong inclusion on \( P \): \( x \triangleleft_R y \) iff \( x = 0 \) or \( y = 1 \). Recall that an ideal in a join-semilattice is a set \( I \) such that

1. if \( a \in I, b \leq a \), then \( b \in I \).
2. if \( u \subseteq I \) is finite, then \( \forall u \in I \).

A **round ideal** of a pcd-lattice \( P \) with a strong inclusion \( \triangleleft \) is an ideal \( I \) such that for all \( b \in I \) there is \( a \in I \) with \( b \triangleleft a \).

Ordered by inclusion, the class \( \mathcal{R}(P) \) (or \( \mathcal{R}(P, \triangleleft) \)) when confusion may arise) of such ideals is a class-frame, in fact a subframe of the frame of ideals of \( P \). Indeed, the meet of a finite set \( F \) of round ideals \( \land F = \cap F \) is round, as is the join \( \lor U = \{ x \in P : (\exists u \in \text{Pow}_{fia}(P))(\forall y \in u)(\exists I \in U) y \in I \land x \leq \lor u \} \), for \( U \subseteq \mathcal{R}(P) \) a set of round ideals.

Observe also that for all \( a \in P \), \( \forall a = \{ b \in P : b \triangleleft a \} \) is a round ideal, and that the set \( B_d = \{ \forall a : a \in P \} \) is a basis of \( \mathcal{R}(P) \). With this basis, \( (\mathcal{R}(P), B_d) \) is a set-generated frame.

In [5] Banaschewski proved that given a compatible strong inclusion on a frame \( L \), the locale map \( h : L \to \mathcal{R}(L) \), with \( h^{-1}(I) = \lor I \), is a compactification of \( L \). The round ideal construction cannot be performed over a non-trivial frame constructively (since a frame is carried by a proper class, see also Section 5). Fortunately, the whole frame structure is not needed for obtaining a compact regular locale.

**Lemma 4.2.** Given a pcd-lattice \( P \) and a strong inclusion \( \triangleleft \) on \( P \), \( (\mathcal{R}(P), B_d) \) is a compact regular locale.

**Proof.** Compactness is trivial. Regularity: with minor adjustments, the proof in [5] Lemma 2] is valid in CST if one replaces \( L \) with \( P \). We reproduce it here for the reader’s convenience. Note first that, if \( x \triangleleft a \), then \( \forall x \triangleleft a \) in \( \mathcal{R}(P) \): indeed if \( x \triangleleft a \) there are \( b, c \) such that \( x \triangleleft c \triangleleft b \triangleleft a \). By 5., also \( a^* \triangleleft b^* \triangleleft c^* \triangleleft x^* \). Moreover, as \( b \lor c^* = 1 \) (the greatest element in \( P \)), \( P = \forall a \lor (x^*) \leq \forall a \lor (\forall x) = P \) (the latter since trivially \( \forall (x^*) \lor (\forall x) = 0 \)). By 7. and 4. we have \( \forall a = \forall \{ \forall x : x \triangleleft a \} \), so that, in conclusion, \( \forall a = \forall \{ \forall x \in B_d : x \triangleleft a \} \), proving the regularity of \( \mathcal{R}(P) \).
is onto. We want to show that $j^{-1} : \mathcal{R}(B^*) \to L$ is one-one; note that $j^{-1}[I] = \bigvee I$, so we have to prove that for all $I, J \in \mathcal{R}(B^*)$, if $\bigvee I = \bigvee J$ then $I = J$. This follows from the observation that if $L$ is compact, then a strongly regular ideal $I$ on $B^*$ is such that $x \in I \iff 1_L = x^* \lor \bigvee I$ (cf [2] Lemma 5).

Thus, by the above lemma, compact regular frames are precisely the frames of round ideals over some pcd-lattice endowed with a strong inclusion.

Let $\preceq$ be a strong inclusion on a sub-pcd-lattice $P$ of a locale $L$. Given a locale map $f : L \to L'$ we say that $\preceq$ is finer than $f^{-} \times f^{-} [\preceq]$ if $y \preceq x$ on $L'$ implies $f^{-}[y] \preceq f^{-}[x]$, for some $p, p' \in P$. Denote by $C$ the class of mappings $f : L \to L'$ with compact and regular codomain such that $\preceq$ is finer than $f^{-} \times f^{-} [\preceq]$.

**Theorem 4.3.** Let $L$ be a locale, $P$ any sub-pcd-lattice of $L$, and $\preceq$ a strong inclusion on $P$. Then, the locale mapping $\mu : L \to \mathcal{R}(P)$, defined, for $I \in B_d$, by $\mu^{-}(I) = \bigvee I$, satisfies the following property: for every $f : L \to L'$ in $C$ a unique locale mapping $g : \mathcal{R}(P) \to L'$ exists such that $g \circ \mu = f$:

$$
\begin{array}{c}
L \\
\downarrow f
\end{array}
\begin{array}{c}
\mu \\
\downarrow g
\end{array}
\begin{array}{c}
\mathcal{R}(P) \\
L'
\end{array}
$$

If $\preceq$ is compatible, $\mu : L \to \mathcal{R}(P)$ is a compactification of $L$.

**Proof.** Standard arguments show that $\mu^{-}$ satisfies the conditions on continuous maps. To prove the extension property, if $f : L \to L'$ is in $C$, let $B'$ be a sub-pcd-lattice and basis of $L'$ ($B'$ always exists, cf. Section 3). For $a \in B'$, set

$$g^{-}(a) = \{ c \in P : (\exists b \in B') b \prec c, c \leq f^{-}(b) \}.$$

First we prove that $g^{-}(a) \in \mathcal{R}(P)$. By Restricted Separation, $g^{-}(a)$ is a set, and trivially an ideal. Let $c \in g^{-}(a)$. Then $c \leq f^{-}(b)$, for $b \prec a$. By the interpolation property of $\prec$ on $B'$ (Lemma [3,3]), there is $d \in B'$ such that $b \prec d \prec a$. By hypothesis, there are $p, p' \in P$ such that $f^{-}(b) \preceq p \preceq p' \leq f^{-}(d)$. Then $c \preceq p' \leq f^{-}(d)$; since $d \prec a$, one concludes $p' \in g^{-}(a)$. This gives $g^{-}(a) \in \mathcal{R}(P)$.

Now assume $a \in B'$, $U \in \text{Pow}(B')$ and $a \leq \bigvee U$. Then $g^{-}(a) \leq \bigvee_{q \in U} g^{-}(q)$; let $c \in g^{-}(a)$, i.e., assume there is $b \in B'$ such that $c \leq f^{-}(b)$, $b \prec a$. By Lemma [3,3] either $b = 0$, whence $c = 0$, or there are finite subsets $\{q_1, \ldots, q_n\}$, $\{q'_1, \ldots, q'_m\}$ of $B'$, with $q_i \prec q'_i \prec q_i \in U$, and such that $b \leq \bigvee q_i \prec \bigvee q'_i \in U$. By hypothesis, $p_i, p'_i$ exist such that $f^{-}(q'_i) \preceq p_i \prec p'_i \leq f^{-}(q'_i)$. Thus, as $q'_i \prec q_i$, $p'_i \in g^{-}(q_i)$. Since $c \leq f^{-}(b) \leq f^{-}(\bigvee q'_i) \leq \bigvee p'_i$, one has $c \in \bigvee_{q \in U} g^{-}(q)$. Therefore $g^{-}(a) \leq \bigvee_{q \in U} g^{-}(q)$.

By the properties of $\prec$, and the fact that $f^{-}$ defines a continuous map, it immediately follows that $g^{-}$ also satisfies conditions 1, 2 on continuous maps (since $B'$ is a pcd-lattice, to verify condition 2 it is enough to show that $g^{-}(a) \land g^{-}(b) = g^{-}(a \land b)$, as $g^{-}(a \land b) = \bigvee \{g^{-}(c) : c \in B', c \leq a, c \leq b\}$, for all $a, b \in B'$). So $g : \mathcal{R}(P) \to L'$ is a continuous map.

To show that $f^{-}(a) = \mu^{-}[g^{-}(a)]$, for all $a \in B'$, note first that, for $I \in \mathcal{R}(P)$, $\mu^{-}(I) = \bigvee I$; to prove then $\bigvee g^{-}(a) = f^{-}(a)$, let $c \in g^{-}(a)$, i.e., assume there is $b \in B'$ such that $c \leq f^{-}(b)$, $b \prec a$. This implies $c \leq f^{-}(a)$, so that $\bigvee g^{-}(a) \leq f^{-}(a)$. Conversely, since $a = \bigvee \{b \in B' : b \prec a\}$, $f^{-}(a) = f^{-}(\bigvee \{b \in B' : b \prec a\}) = \bigvee_{b \prec a, b \in B'} f^{-}(b)$. Let $b \prec a$; by Lemma [3,3] there is $d \in B'$ such that $b \prec d \prec a$. 


By hypothesis, there are $p, p' \in P$ such that $f^-(b) \leq p \prec p' \leq f^-(d)$. Since $d \prec a$, one gets $p' \in g^-(a)$. Thus $f^-(b) \leq \vee g^-(a)$ for all $b \prec a$, so that $f^-(a) \leq \vee g^-(a)$. Finally, uniqueness of $g$ follows from Lemma 2.1 iii, since $\mu$ is trivially dense.

If $\prec$ is compatible, recalling Lemma 4.2 to prove that $\mu$ is a compactification we only have to show that $\mu^-$ is onto: by compatibility of $\prec$, $L$ is set-generated by $P$, and given $a \in P$, the set $\{b \in P : b \prec a\}$ is a round ideal of $P$ whose join is $a$. \qed

Now let $F \equiv \{f_i\}_{i \in I}$ be a (possibly empty) family of locale mappings, $f_i : L \to L_i$, with $L_i$ compact and regular, and let $S$ be a subset of $L$. Set $S_F = S \cup \{f_i^-(b) : i \in I, b \in B_i\}$, where $B_i$ is a set of generators and sub-pcd-lattice of $L_i$. With $S_F^*$ we shall abbreviate $(S_F)^*$.

**Lemma 4.4.** The relation $\prec_F \subseteq S_F^* \times S_F^*$ defined inductively as the least relation containing the set $\{(f_i^-(b), f_i^-(a)) : i \in I, a, b \in B_i, b \prec a\}$, and closed under conditions from 1 to 5 on strong inclusions, is a strong inclusion on $S_F^*$.

**Proof.** The set of pairs $(f_i^-(b), f_i^-(a))$ is a relation satisfying the hypotheses of Lemma 4.1 (as $f_i^-$ preserves $\prec$, and since, by Lemma 3.3 $\prec$ interpolates on $B_i$). \qed

Starting from $S = \emptyset$, one obtains the least strong inclusion $\prec_F$ containing the indicated set, on the least (sub)structure on which it may be considered, i.e., $\emptyset_F^*$.

**Proposition 4.5.** Let $L$ be a locale, $S$ any subset of $L$, $F$ a family of locale mappings as above. Let $\prec$ be a strong inclusion on $S_F^*$ containing $\prec_F$. Then the locale mapping $\mu : L \to \mathcal{R}(S_F^*, \prec), \mu^-(I) = \vee I$, is such that each $f_i$ factors uniquely through $\mu$: for all $i$ a unique $g_i : \mathcal{R}(S_F^*, \prec) \to L_i$ exists such that $g_i \circ \mu = f_i$.

**Proof.** By Lemma 3.3 if $b \prec a$ on $L_i$, one has $b \leq b' \prec a'$, for $a', b' \in B_i$. Then, by construction of $S_F^*$ and $\prec_F$, each $f_i$ is such that $\prec$ is finer than $f_i^* \times f_i^-[\prec]$, and hence, by Theorem 4.3 each $f_i$ factors uniquely through $\mu$. \qed

The compactification to be described next was defined in [9] using topos-valid, but impredicative, comprehension principles. There I claimed that co-inductive definitions would probably have allowed for a predicative construction\footnote{At the time of writing [9], [10] Theorem 13.2.3, on the existence of the largest set coinductively defined by a given set-sized inductive definition, had not been (proved or) published yet.}. The following shows that this is indeed the case.

In general, the strong inclusion $\prec_F$ one obtains by Lemma 4.4 need not be compatible, so that $\mu : L \to \mathcal{R}(B_F^*, \prec_F)$ need not be a compactification. For obtaining a compatible strong inclusion, we shall have to enlarge $\prec_F$.

**Theorem 4.6.** Let $L$ be a strongly regular locale, and let $F \equiv \{f_i\}_{i \in I}$ be a (possibly empty) family of locale mappings $f_i : L \to L_i$, with $L_i$ compact and regular for all $i \in I$. Let $B$ be a strongly regular basis for $L$, and $B_i$ any basis for $L_i$. Define $B_F = B \cup \bigcup_{i \in I} \{f_i^-(b_i) : b_i \in B_i^*\}$. On $B_F^*$ let $\prec_\lambda$ be the largest interpolative relation contained in $\prec \cap B_F^* \times B_F^*$. Then the following hold:

i. $\prec_\lambda$ is a set and a strong inclusion on $B_F^*$ compatible with $L$;

ii. $\mu : L \to \gamma L = \mathcal{R}(B_F^*, \prec_\lambda)$ is a compactification of $L$ such that for all $f_i$ there is a unique $g_i : \gamma L \to L_i$ such that $g_i \circ \mu = f_i$:
Lemma 4.7. Let $\mathcal{F} = \{f\}$, $f : L \to M$ with $M$ compact and regular, $S$ any subset of $L$, and let $B_M$ be a base and sub-pcd-lattice of $M$. If $f^{-1} : M \to L$ is $\ast$-preserving, then for all $x, y \in S^*_f$, $x \prec_f y \iff x \leq \gamma f^{-1}(b), f^{-1}(a) \leq y$ for some $a, b \in B_M$.

Proof. On $P = S^*_f$, $\prec_f$ is defined inductively as in Lemma 4.1 as the least relation containing the set $R = \{(f^{-1}(b), f^{-1}(a))) : a, b \in B_M, b \prec a\}$, and closed under conditions from 1 to 5 on strong inclusions. I.e., $\prec_f = I(\Phi_R)$, with $\Phi_R$ as in Lemma 4.1. We prove the claim by induction. Let $T \subseteq P \times P$, $T = \{(x, y) \in P \times P : (\exists a, b \in B_M)(\exists b' \in B_M) \forall a, b, c, d \in B_M d \ast c, f^{-1}(a) \leq f^{-1}(d), f^{-1}(c) \leq f^{-1}(a)\}$. One simply takes $d = b, c = a$. The remaining cases are as simple, we only prove two of them: assume $x \leq f^{-1}(b'), f^{-1}(a') \leq a$ with $b' \prec a'$, and $y \leq f^{-1}(b''), f^{-1}(a'') \leq a$ with $b'' \prec a''$. Then, $x \land y \leq f^{-1}(b') \lor f^{-1}(b'') = f^{-1}(b' \lor b''), f^{-1}(a' \lor a') = f^{-1}(a') \lor f^{-1}(a'') \leq a$, with $b' \lor b'' \prec a' \lor a''$, so that $(x \land y, a) \in T$. If instead $a \leq f^{-1}(b'), f^{-1}(a') \leq b$, with $b' \prec a'$, then $b' \ast f^{-1}(a') = f^{-1}(a'\ast)$ and $f^{-1}(b') \ast f^{-1}(b'\ast) \leq a'$, with $a' \ast \prec b'$, since $f^{-1}$ is $\ast$-preserving.

Theorem 4.8. Let $k : L \to kL$ be a compactification of a locale $L$. Then a sub-pcd-lattice $P_k$ of $L$ and a compatible strong inclusion $\prec_k$ on $P_k$ exist such that $\mathcal{R}(P_k, \prec_k)$ is isomorphic with $kL$.

Proof. Let $P_k = \emptyset^*_k$, and $\prec_k$ be the strong inclusion given by Lemma 4.4, where $\mathcal{F} = \{k\}$, i.e., the least relation on $\emptyset^*_k$ containing the set $\{(k^{-1}(b), k^{-1}(a)) : a, b \in B_{kL}, b \prec a\}$, and closed under conditions from 1 to 5 on strong inclusions ($B_{kL}$ sub-pcd-lattice and basis of $kL$). Compatibility of $\prec_k$: for all $a \in P_k$, $a \leq \bigvee_{b \in U} k^{-1}(b)$.
for some \( U \in \text{Pow}(B_{kL}) \), as \( k^- \) is onto. For \( b \in U \), one has \( b = \bigvee_{c \in E B_{kL}} \{ c : c \prec b \} \). Thus, \( a = \bigvee_{b \in U} \bigvee_{c \in E B_{kL}} \{ k^-(c) : c \prec b \} \). Since \( k^-(c) \prec_k k^-(b) \leq a \), one concludes that \( \prec_k \) is compatible.

It remains to prove that \( kL \) is isomorphic with \( \mathcal{R}(P_k, \prec_k) \). By Theorem 4.3, the mapping \( \mu : L \to \mathcal{R}(P_k) \), defined by \( \mu^{-1}(I) = \bigvee I \), is a compactification of \( L \) that satisfies: for every \( f : L \to L' \) in \( C_k \) a unique mapping \( g : \mathcal{R}(P_k) \to L' \) exists such that \( g \circ \mu = f \), with \( C_k \) the class of mappings \( f : L \to L' \) with compact and regular codomain such that \( \prec_k \) is finer than \( f^- \times f^- [-\prec] \). Now, compactification \( k \) is in \( C_k \).

Indeed, if \( y \prec x \) in \( kL \), by Lemma 3.3 there are \( q, q' \in B_{kL} \) with \( y \leq q < q' \prec x \). Therefore, \( k^-(y) \leq k^-(q) \prec k^-(q') \prec k^-(x) \), with \( k^-(q), k^-(q') \in P_k \). We then have a unique \( g : \mathcal{R}(P_k) \to kL \) such that \( g \circ \mu = k \), defined by, for \( a \in B_{kL} \),

\[
g^-(a) = \{ c \in P_k : (\exists b \in B_{kL}) b \prec a, \ c \leq k^-(b) \}.
\]

We prove that \( g \) provides the required isomorphism. Since \( k \) is dense, \( g \) is dense too (as \( k = g \circ \mu \)). By Lemma 2.1, then \( g^- [-] \) in one-one. To conclude thus we have to prove that \( g^- \) is onto: if \( I \in \mathcal{R}(P_k, \prec_k) \) there is \( U \in \text{Pow}(B_{kL}) \) with \( \bigvee_{b \in U} g^- (b) = I \). It is enough to show that for \( a \in P_k \) there is \( U \in \text{Pow}(B_{kL}) \) with \( \bigvee_{b \in U} g^- (b) = \downarrow a \). Let \( U = \{ a' \in B_{kL} : k^- (a') \leq a \} \). If \( d \in \downarrow a \), then \( d \prec_k a \), so that by the previous Lemma, \( d \leq k^- (b') \), \( k^- (a') \leq a \) for some \( a', b' \in B_{kL} \), \( b' \prec a' \) (\( k^- \) is \( \ast \)-preserving by Lemma 2.1 i). Thus, \( \downarrow a \subseteq \bigcup_{a' \in U} g^- (a') = \bigcup_{c \in P_k} (\exists b' \in B_{kL}) b' \prec a', \ c \leq k^- (b')) \), that gives \( \downarrow a \leq \bigvee_{a' \in U} g^- (a') \). On the other hand, for all \( a' \in U \), \( g^- (a') \leq \downarrow a \) again by the previous Lemma. This proves that \( g^- \) is onto, as wished. \( \square \)

Note that the class \( C_k \) of continuous maps \( f : L \to L' \) with compact and regular codomain such that \( \prec_k \) is finer than \( f^- \times f^- [-\prec] \) (recall that \( \prec_k \) is finer than \( f^- \times f^- [-\prec] \) if \( y \prec x \) on \( L' \) implies \( f^- (y) \leq p \prec_k p' \leq f^- (x) \), for some \( p, p' \in P_k \), is completely determined by the compactification \( k : L \to kL \), i.e., it does not depend on the choice of the base \( B_{kL} \) of \( kL \) for the construction of \( (P_k, \prec_k) \). Indeed, assume \( B'_{kL} \) is a different sub-pcd-lattice and basis of \( kL \), and let \( P'_k = \emptyset_{kL} \) and \( \prec'_k \) be the sub-pcd-lattice of \( L \) and strong inclusion constructed from \( B'_{kL} \). We prove that if \( \prec'_k \) is finer than \( f^- \times f^- [-\prec] \), then also \( \prec_k \) is finer than \( f^- \times f^- [-\prec] \): let \( y \prec x \) on \( L' \), so that \( f^- (y) \leq p \prec_k p' \leq f^- (x) \) for \( p, p' \in P'_k \). By Lemma 4.7 if \( p \prec_k p' \) then \( a', b' \in B'_{kL} \) exist with \( b' \prec a' \), \( p \leq k^- (b') \), \( k^- (a') \leq p' \), as \( k^- \) is \( \ast \)-preserving. Since \( kL \) is compact, by Lemma 3.3 \( a, b \) exist in \( B_{kL} \) such that \( b' \prec b \prec a \prec a' \). Therefore, \( p \leq k^- (b) = q, q' = k^- (a) \leq p' \), whence \( q \prec_k q' \), and \( f^- (y) \leq q \prec_k q' \leq f^- (x) \).

Banaschewski [5] proves (in our notation) that every compactification \( k : L \to kL \) is associated with a compatible strong inclusion \( \prec_k \) on \( L \) determining an isomorphic compactification \( \mathcal{R}(L, \prec_k) \), by defining, for \( x, y \in L, x \prec_k y \iff \bigvee \{ u \in kL : k^- (u) = x \} \prec \bigvee \{ v \in kL : k^- (v) = y \} \). As already expounded, Banaschewski’s construction is not viable in the present constructive setting. Theorem 4.8 provides then a constructive version of Banaschewski’s result.

The construction of \( \mathcal{R}(P_k, \prec_k) \) in the previous theorem, aside from its constructive character, allows us to characterize a compactification in terms of its ability to extend continuous mappings with compact and regular codomain. Recall indeed that the compactifications of a locale \( L \) are ordered as follows: if \( k : L \to kL \) and \( k' : L \to k'L \) are compactifications of \( L \), \( k \leq k' \) if and only if \( k = h \circ k' \), for a continuous \( h : k'L \to kL \):
The following is the announced characterization.

**Corollary 4.9.** Let \( k : L \to kL \) be a compactification of a strongly regular locale \( L \), and \( C_k \) the associated class of continuous maps \( f : L \to L' \) with compact and regular codomain such that \( C_k \) is finer than \( f^- \times f^-[\prec] \). Then, for every \( f \in C_k \) a unique continuous \( g : kL \to L' \) exists such that \( g \circ k = f \), and \( k : L \to kL \) is the minimal compactification with this property.

**Proof.** The first part of this corollary follows directly from Theorems 4.8 and 4.9. For the second, assume \( k' : L \to k'L \) has the same property with respect to the class \( C_k \). Then, since \( k : L \to kL \) is in \( C_k \) (cf. proof of Theorem 4.8), there is a (unique) \( g : k'L \to kL \) with \( g \circ k' = k \), so that \( k \leq k' \).

Observe that, given a strong inclusion \( \prec \) on a sub-pcd-lattice \( P \) of a locale \( L \), the composite \( \overline{\sim} \equiv \circ \prec \circ \leq \circ \) on \( L \) yields a class relation satisfying the axioms of a strong inclusion, the least one on \( L \) extending \( \prec \). One then has that \( \circ \prec \) on \( P \) is finer than \( f^- \times f^-[\prec] \) iff \( f^- \times f^-[\prec] \subseteq \overline{\sim} \). So the extension property in the previous corollary may be expressed as follows: every continuous \( f : L \to L' \) with compact and regular codomain such that \( f^- \times f^-[\prec] \subseteq \overline{\sim} \) has a unique continuous extension \( g : kL \to L' \) to \( kL \). In \[9\] we showed that Alexandroff compactification \( \alpha L \) of a locally compact regular locale \( (L, B) \) can be defined by constructing inductively the least strong inclusion \( \prec \prec \) on \( B^* \) that contains the restriction of the familiar way-below relation to \( B^* \times B^* \). Once the concepts of local compactness and way-below relation are treated as indicated in \[10\], the given proof can be formulated in CST. So by the above Corollary, Alexandroff compactification \( \alpha L \) can be characterized as the minimal compactification that allows for the extension of all those continuous mappings from \( L \) to a compact and regular codomain whose inverse images ‘send the well-inside relation into the least strong inclusion containing the way-below relation’.

Finally, we observe that for compactifications \( k_1 : L \to k_1 L, k_2 : L \to k_2 L \) one has

\[
k_1 \leq k_2 \iff C_{k_1} \subseteq C_{k_2}
\]

Indeed, assume \( k_1 \leq k_2 \), i.e., assume there is \( h : k_2 L \to k_1 L \) such that \( k_1 = h \circ k_2 \). Let \( f : L \to L' \) in \( C_{k_1} \), so that, if \( y \prec x \) in \( L' \), there are \( p, p' \in P_{k_1} \) with \( f^-[y] \leq p <_1 p' \leq f^-[x] \). To conclude that \( f \in C_{k_2} \), we want \( q, q' \in P_{k_2} \) such that \( f^-[y] \leq q <_2 q' \leq f^-[x] \). By \( p <_1 p' \), recalling Lemma 4.7, we have \( a, b \in B_{k_1 L}, b \prec a \) with \( p \leq k^-_1(b) = k^-_2[h^-[b]], k^-_2[h^-[a]] = k^-_1(a) \leq p' \) (\( B_{k_1 L} \) sub-pcd-lattices and bases for \( k_1 L \)). Since \( k_2 L \) is compact and regular, and since \( h^-[b] \prec h^-[a] \), by Lemma 4.8, we have \( a', b' \in B_{k_2 L} \) with \( h^-[b] \prec b' \prec a' \prec h^-[a] \), so that \( f^-[y] \leq p \leq k^-_1(b) = k^-_2[h^-[b]] \leq k^-_2(b') \leq k^-_2(a') \leq k^-_1(a) \leq p' \leq f^-[x] \). Thus we conclude with \( q = k^-_2(b'), q' = k^-_2(a') \in P_{k_2} \). Conversely, if \( C_{k_1} \subseteq C_{k_2} \), we conclude by Corollary 4.9 that \( k_1 \leq k_2 \) observing that \( k_1 \in C_{k_1} \).

Therefore, if the sublattice \( P_k \) of \( L \) and strong inclusion \( \prec \) are ‘sufficiently large’ to be such that \( \prec \) is finer than every continuous function with compact regular codomain, so that \( C_k \) coincides with the class of all such functions, \( k : L \to kL \) is the largest compactification of \( L \), i.e., the compact regular reflection of \( L \) (‘weak’
Stone–Čech compactification). In this sense, the ability of a compactification \( k \) to extend the continuous functions in \( C_k \) can be regarded as an approximation of the characterizing property of Stone–Čech compactification (when it exists, cf. next section).

5. AN INTERESTING OPEN QUESTION: THE CONSTRUCTIVE COMPACT REGULAR REFLECTION OF A LOCALE

One might be tempted to think that, choosing in Theorem 4.6 as \( F \) the family of all mappings \( f : L \to L' \), with \( L' \) compact regular, one would get the ‘weak’ Stone–Čech compactification of \( L \) (i.e., its compact regular reflection, ‘weak’ as opposed to the compact completely regular reflection). Unfortunately, this is not the case, since those mappings are too many to be set-indexed (even in classical set theory).

In [7], Banaschewski and Pultr define the compact regular reflection of a strongly regular locale \( L \) considering the strongly regular ideals of \( L \) with respect to the largest strong inclusion contained in the well-inside relation on \( L \). Previously, Banaschewski and Mulvey [6] used the completely regular ideals on \( L \) (i.e., the strongly regular ideals with respect to the really inside relation as strong inclusion) to define the compact completely regular reflection of \( L \) (‘strong’ Stone–Čech compactification).

None of these reflections can be defined constructively in the present sense in full generality. This is not just because the specific constructions cannot be carried out in CST (a non-trivial frame is a proper class [10], and the ideals of this class do not yield a frame). It is proved in [10] that the compact completely regular reflection is independent from CST also when extended with (various principles including) the principle of dependent choice. With DC the compact regular and compact completely regular reflection coincide, so that if we could define the compact regular reflection of every locale \( L \) in CST, we could also define the compact completely regular reflection of every locale \( L \) in CST+DC (see the Appendix for the formal version of these statements).

In [8] it is however proved that one may define constructively the compact completely regular reflection \( \beta L \) of \( L \), for every locale \( L \) such that the class of continuous mappings \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) from \( L \) to the localic real unit interval is a set (while this is always the case in classical set theory or in a topos, proving that \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) is a set is in general non-trivial, and not the case for every locale in CST). This condition in fact characterizes the locales \( L \) of which the compact completely regular reflection can be constructed.

The construction of \( \beta L \) in [8] can be summarized as follows: if \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) is a set, one can expand a base \( B \) of \( L \) to a base \( B_F \) containing counterimages of basic elements of the localic real unit interval. Since \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) is a set, \( B_F \) is a set, too. Then, on the pcd-lattice \( B_F \) one takes the completely regular ideals. The resulting frame \( R(B_F) \) is compact and completely regular, and allows for the extension of each mapping in \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) to \( R(B_F) \).

One then proves that (Čech’s theorem implies Stone’s theorem, i.e., that) a compactification of \( L \) that allows for the extension of each mapping in \( \text{Hom}_{\text{Loc}}(L, [0, 1]) \) in fact allows for the extension of each continuous mapping in \( \text{Hom}_{\text{Loc}}(L, L') \), for every \( L' \) compact and completely regular. This latter result in turn is proved exploiting Tychonoff embedding theorem for exhibiting the completely regular locale \( L' \) as a sublocale of a set-indexed product of the localic real unit interval.
A similar procedure does not seem to work for the compact regular reflection of a strongly regular locale. Indeed, when $\text{Hom}_{\text{Loc}}(L, [0,1])$ is a set for $L$ a strongly regular locale, by Theorem 4.6 we are able to construct a compact regular compactification that allows for the extension of all mappings in $\mathcal{F} = \text{Hom}_{\text{Loc}}(L, [0,1])$, but we cannot use Tychonoff embedding theorem for embedding a compact regular (not necessarily completely regular without DC) $L$ in a set-indexed product of $[0,1]$, for showing that the obtained compactification is in fact the compact regular reflection of $L$.

In [6] Banaschewski and Mulvey also provided a definition of the compact regular reflection of a locale, but remarked that a concrete description of the compact regular reflection of a strongly regular locale using strongly regular ideals would have been desirable. As said such a description, when not appealing to strongly non constructive set existence principles, cannot be derived in full generality, but the question whether a restricted version of it similar to that obtained for the compact completely regular reflection in [8] can be constructed is open.

**Open Problem:** Can the compact regular reflection of a locale $L$ be defined in CST in non-trivial cases? Can the locales of which the compact regular reflection exists in CST be characterized?

A related open problem is the following. In [9] we proved that, when $\text{Hom}_{\text{Loc}}(L, [0,1])$ is a set, the sub-pcd-lattice $B^*_F$ of $L$ used for constructing the compact completely regular reflection is sufficiently large for fully replacing $L$ in the construction of every compactification, in the sense that every compactification of a frame $L$ can be obtained as the frame of round ideals over $B^*_F$, for a certain inductively defined strong inclusion on $B^*_F$. A similar result for compact regular compactifications would be desirable.

6. **Appendix: Constructive Set Theory. Inductive and co-inductive definitions**

As pointed out before, we used CST as a collective name for Aczel-Myhill formal systems for constructive set theory. In this appendix the specific formal system in which we have been working in is described to make the article self-contained. The reader may consult [1, 3, 4] for a thorough introduction to the subject. A core formal system for CST is the choice-free *Constructive Zermelo-Fraenkel Set Theory* (CZF). This system is often extended by principles (described below) ensuring that certain inductively and co-inductively defined classes are sets. Note that CZF, extended or not by those principles, is a subtheory of classical set theory. As already pointed out, in contrast to ZF, CZF does not have the impredicative unrestricted Separation scheme and the Powerset axiom.

The language of CZF is the same as that of Zermelo-Fraenkel Set Theory, ZF, with $\in$ as the only non-logical symbol. Beside the rules and axioms of a standard calculus for intuitionistic predicate logic with equality (e.g., [19]), CZF has the following axioms and axiom schemes:

1. Extensionality: $\forall a \forall b (\forall y (y \in a \iff y \in b) \rightarrow a = b)$.
2. Pair: $\forall a \forall b \exists x \forall y (y \in x \iff y = a \lor y = b)$.
3. Union: $\forall a \exists x \forall y (y \in x \iff \exists z \in a)(y \in z)$.
4. Restricted Separation scheme: $\forall a \exists x \forall y (y \in x \iff y \in a \& \phi(y))$,
We shall denote by \( \text{CZF}^- \) the class of axioms and schemes; for this article it suffices to note that using it on \( e \) proves that Collection scheme. Subset Collection is perhaps the most unusual of the CZF Axiom. We did not make use of Subset Collection or Exponentiation in this article.

As shown in this article, a major role in constructive set theory is played by inductive definitions. An inductive definition is any class \( \Phi \) of pairs. A class \( A \) is \( \Phi \)-closed if:

\[
(a, X) \in \Phi, \text{ and } X \subseteq A \text{ implies } a \in A.
\]

The following theorem is called the class inductive definition theorem \( \text{[3]} \). It says that (any extension of) the system \( \text{CZF}^- \) has the class induction property CIP recalled in Section 1.

**Theorem 6.1 (CZF\(^-\)).** Given any class \( \Phi \) of ordered pairs, there exists a least \( \Phi \)-closed class \( I(\Phi) \), the class inductively defined by \( \Phi \).

Even when \( \Phi \) is a set, \( I(\Phi) \) need not be a set in CZF. For this reason, CZF is often extended with the Regular Extension Axiom, REA.

REA: every set is a subset of a regular set.

A set \( c \) is regular if it is transitive, inhabited, and for any \( u \in c \) and any set \( R \subseteq u \times c \), if \( (\forall x \in u)(\exists y)(x, y) \in R \), then there is a set \( v \in c \) such that

\[
(\forall x \in u)(\exists y \in v)((x, y) \in R) \& (\forall y \in v)(\exists x \in u)((x, y) \in R).
\]

c is said to be weakly regular if in the above definition of regularity the second conjunct in (1) is omitted. The weak regular extension axiom, wREA, is the statement that every set is the subset of a weakly regular set.

A class \( K \) is a bound for \( \Phi \) if, for every \( (x, X) \in \Phi \), there is a set \( k \in K \) and an onto mapping \( f : k \to X \). The inductive definition \( \Phi \) is defined to be bounded if:

1. \( \{x \mid (x, X) \in \Phi \} \) is a set for every set \( X \);
2. \( \Phi \) is bounded by a set.

The following theorem states that in the system CZF + wREA the Bounded Induction Scheme (BIS) is derivable.

for \( \phi \) a restricted formula. A formula \( \phi \) is restricted if the quantifiers that occur in it are of the form \( \forall x \in b, \exists x \in c \).

(5) Subset Collection scheme:

\[
\forall a \forall b \exists c \forall u((\forall x \in a)(\exists y \in b)\phi(x, y, u) \rightarrow (\exists d \in c)((\forall x \in a)(\exists y \in d)\phi(x, y, u) \& (\forall y \in e)(\exists x \in a)\phi(x, y, u))).
\]

(6) Strong Collection scheme:

\[
\forall a((\forall x \in a)\exists y \phi(x, y) \rightarrow \exists b((\forall x \in a)(\exists y \in b)\phi(x, y) \& (\forall y \in b)(\exists x \in a)\phi(x, y))).
\]

(7) Infinity: \( \exists x(\forall x \in a)(\exists y \in a)(\exists y \in a)x \in y \).

(8) Set Induction scheme: \( \forall a((\forall x \in a)\phi(x) \rightarrow \phi(a)) \rightarrow \forall a\phi(a) \).

We shall denote by \( \text{CZF}^- \) the system obtained from CZF by leaving out the Subset Collection scheme. Subset Collection is perhaps the most unusual of the CZF axioms and schemes; for this article it suffices to note that using it one proves that the class \( b^a \) of functions from a set \( a \) to a set \( b \) is a set, i.e., the Exponentiation Axiom. We did not make use of Subset Collection or Exponentiation in this article.

Note that the theory obtained from CZF by adding the Law of Excluded Middle has the same theorems as ZF.
Theorem 6.2 (CZF + wREA). If $\Phi$ is bounded, in particular if $\Phi$ is a set, then $I(\Phi)$ is a set.

Given an inductive definition $\Phi$ on a class $S$, $\Phi \subseteq S \times \text{Pow}(S)$, a class $C \subseteq S$ is said to be $\Phi$-inclusive if $C \subseteq \Gamma_\Phi(C)$, with $\Gamma_\Phi$ the operator on subclasses associated with $\Phi$: $\Gamma_\Phi(C) = \{ x \mid (\exists X) (x, X) \in \Phi \land X \subseteq C \}$. Aczel showed that the class $J = \bigcup \{ Y \in \text{Pow}(S) \mid Y \subseteq \Gamma_\Phi(Y) \}$ is the largest $\Phi$-inclusive subclass of $S$, the class co-inductively defined by $\Phi$, denoted by $C(\Phi)$. This result is proven in the system CZF$^-+$RRS, where the Relation Reflection Scheme RRS is the following axiom scheme.

Relation Reflection Scheme, RRS:

For classes $S, R$ with $R \subseteq S \times S$, if $a \in S$ and $\forall x \in S \exists y \in S R(x, y)$ then there is a set $S_0 \subseteq S$ such that $a \in S_0$ and $\forall x \in S_0 \exists y \in S_0 R(x, y)$.

This scheme can be regarded as a weakening of the Relativized Dependent Choices Axiom, RDC. By contrast with RDC, RRS is valid in all topological models (all cHa-valued models). Note also that RRS is a theorem of ZF (see [2] for a proof of these facts).

The following strengthening of RRS and REA is used to show that $J$ is a set when $S, \Phi$ are sets. A regular set $A$ is strongly regular if it is closed under the union operation, i.e., if $\forall x \in A \cup x \in A$. Let $A$ be a strongly regular set. $A$ is defined to be RRS-strongly regular if also, for all sets $A' \subseteq A$ and $R \subseteq A' \times A'$, if $a_0 \in A'$ and $\forall x \in A' \exists y \in A' xRy$ then there is $A_0 \in A$ such that $a_0 \in A_0 \subseteq A'$ and $\forall x \in A_0 \exists y \in A_0 xRy$.

RRS-∪REA: Every set is the subset of a RRS-strongly regular set.

The following theorem is proved in [1].

Theorem 6.3 (CZF + RRS-∪REA). If $S$ and $\Phi \subseteq S \times \text{Pow}(S)$ are sets, the largest $\Phi$-inclusive subclass of $S$, i.e., the class $C(\Phi) = J = \bigcup \{ Y \in \text{Pow}(S) \mid Y \subseteq \Gamma_\Phi(Y) \}$ co-inductively defined by $\Phi$, is a set.

This is the Set Coinduction Scheme recalled in Section 1. CZF + RRS-∪REA is then the formal system for CST adopted in this article.

We conclude this Appendix describing more formally the independence - mentioned in Section 4 - of the weak Stone-Čech compactification of locales in its full form, from the formal system we have adopted. In [10], the compact completely regular reflection of a Boolean locale is proved to be independent from the formal system CZF, and from every extension of CZF that is consistent with the Generalized Uniformity Principle (GUP), as e.g. CZF+REA+RDC. As pointed out to me by M. Rathjen, using realizability [15], one shows that one such system is also CZF+∪REA+RDC. This system extends the choice-free system CZF+RRS-∪REA, described above, and adopted in this article as formal system for CST. So, if the compact regular reflection of a Boolean locale were derivable in CZF+RRS-∪REA, it would also be derivable in CZF+∪REA+RDC. However, in the presence of DC (and a fortiori of RDC), a locale is compact regular iff it is compact completely regular, so that the compact regular and compact completely regular reflection of a locale coincide.

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