ON VECTOR BUNDLES OVER REDUCIBLE CURVES WITH A NODE

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ABSTRACT. Let $C$ be a curve with two smooth components and a single node. Let $U_C(r, w, \chi)$ be the moduli space of $w$-semistable classes of depth one sheaves on $C$ having rank $r$ on both components and Euler characteristic $\chi$. In this paper, under suitable assumptions, we produce a projective bundle over the product of the moduli spaces of semistable vector bundles of rank $r$ on each components and we show that it is birational to an irreducible component of $U_C(r, w, \chi)$. Then we prove the rationality of the closed subset containing vector bundles with given fixed determinant.

INTRODUCTION

Moduli spaces of vector bundles on curves have always been a central topic in Algebraic Geometry. It is due to Mumford, by geometric invariant theory, see [Mum65], the construction of moduli space of isomorphism classes of stable vector bundle of rank $r$ and degree $d$ on a smooth projective curve of genus $g \geq 2$. It is a non singular quasi-projective variety, whose compactification was obtained by Seshadri in [Ses67], by introducing $S$-equivalence relation between semistable vector bundles, and it is denoted by $U_C(r, d)$. This is actually a normal irreducible projective variety of dimension $r^2(g - 1) + 1$. When $r$ and $d$ are coprime, the notion of semistability is the same of stability, so $U_C(r, d)$ parametrizes isomorphism classes of stable vector bundles. Moreover, in this case there exists a Poincaré bundle on $U_C(r, d)$, see [Ram73]. If $L \in \text{Pic}^d(C)$ is a line bundle, the moduli space $SU_C(r, L)$, parametrizing semistable vector bundles of rank $r$ and fixed determinant $L$, is also of great interest. Indeed, up to a finite étale covering, $U_C(r, d)$ is a product of $SU_C(r, L)$ and $\text{Pic}^0(C)$. Hence, a lot of the geometry of $U_C(r, d)$ is encoded in $SU_C(r, L)$. Moreover, $SU_C(r, L)$ is interesting on its own and it is a rational variety when $r$ and $d$ are coprime, see [KS99]. The geometry of these moduli spaces has been studied by many authors, in particular its relation with generalized theta functions, see [Bea13] for a survey, and [BF18] and [Bri18] and [Bri17] for recent works by the authors.

Unfortunately, as soon as the curve is singular, the above results do not apply anymore. For example, for a singular irreducible curve, in order to have a compact moduli space one

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possible approach consists in considering also torsion free sheaves, see [New78] and [Ses82]. This method was generalized for a reducible (but reduced) curve by Seshadri. Its idea was to include in the moduli space also depth one sheaves and to introduce the notion of polarization \( w \) and of \( w \)-semistability. More precisely, we denote by \( \mathcal{U}_C(w, r, \chi) \) the moduli space parametrizing \( w \)-semistable sheaves of depth one of rank \( r \) on each components and Euler characteristic \( \chi \).

In this paper we will assume that \( C \) is a nodal reducible curve with two smooth irreducible components \( C_1 \) and \( C_2 \), of genus \( g_i \geq 1 \) and a single node \( p \). We can get the curve by gluing \( C_1 \) and \( C_2 \) at the points \( q_1 \) and \( q_2 \). In this hypothesis, the moduli space \( \mathcal{U}_C(w, r, \chi) \) is a connected reducible projective variety, see [TiB95] and [TiB11]; each irreducible component has dimension \( r^2(p_a(C) - 1) + 1 \) and it corresponds to a possible pair of multidegree, see Section 2 for details.

In the above hypothesis, for any \( r \geq 2 \), fix a pair of integers \( (d_1, d_2) \) which are both coprime with \( r \). The existence of Poincaré vector bundles, on the moduli spaces \( \mathcal{U}_{C_i}(r, d_i) \), allows us to produce a projective bundle \( \pi : \mathbb{P}(\mathcal{F}) \to \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) \), whose fiber at \( ([E_1], [E_2]) \) is \( \mathbb{P}(\text{Hom}(E_{1,q_1}, E_{2,q_2})) \), see Lemma 3.1. Let \( u \in \mathbb{P}(\mathcal{F}) \), \( u = ([E_1], [E_2]), [\sigma] \), we can associate to it a depth one sheaf \( E_u \) which is obtained, roughly speaking, by gluing \( E_1 \) and \( E_2 \) along the fibers at \( q_1 \) and \( q_2 \) with \( \sigma \). This is a vector bundle if and only if \( \sigma \) is an isomorphism. Our first concern is to study when \( E_u \) turns out to be \( w \)-semistable for some polarization \( w \): we are able to give some necessary and sufficient conditions to ensure \( w \)-semistability (see section 3). Then we turn our attention to the map \( \phi \) sending \( u \to E_u \). This is a rational map from \( \mathbb{P}(\mathcal{F}) \) into a suitable moduli space \( \mathcal{U}_C(w, r, \chi) \). Our first result (Theorem 4.1) can be summarized in the following statement:

**Theorem A** Let \( C \) be a reducible nodal curve as above. Let \( r \geq 2 \) and \( d_1 \) and \( d_2 \) be integer coprime with \( r \). Set \( \chi_i = d_i + r(1 - g_i) \) and \( \chi = \chi_1 + \chi_2 - r \). For any pair \( (\chi_1, \chi_2) \) in a suitable non empty subset of \( \mathbb{Z}^2 \) there exists a polarization \( w \) such that \( \mathbb{P}(\mathcal{F}) \) is birational to the irreducible component of the moduli space \( \mathcal{U}_C(w, r, \chi) \) corresponding to bidegree \( (d_1, d_2) \).

The birational map of the statement is the map \( \phi \). We prove that it is an injective morphism on the open subset \( \mathcal{U} \subset \mathbb{P}(\mathcal{F}) \), given by points \( u \) where \( \sigma \) is an isomorphism. The image \( \phi(\mathcal{U}) \) is a dense subset of the moduli space and its points are classes of vector bundles whose restrictions to components are stable (see Theorem 4.1).

Moreover, when \( g_i > r + 1 \), we can give more informations about the domain of \( \phi \) as follows, see Theorem 4.3.

**Theorem B** Assume that the hypothesis of the Theorem A hold. If \( g_i > r + 1 \), for any pair \( (\chi_1, \chi_2) \) in a suitable non empty subset of \( \mathbb{Z}^2 \) there exist a non empty open subset \( V_1 \times V_2 \) of \( \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) \) and a polarization \( w \) such that \( \phi|_{\mathcal{U} \cap \mathcal{U}'} \) is a morphism, where we set \( \mathcal{U}' = \pi^{-1}(V_1 \times V_2) \).
Then, in analogy with the smooth case, for any \( L \in \text{Pic}(C) \) we define the variety \( SU_C(w, r, L) \) which, roughly, is the closure in \( U_C(w, r, \chi)_{d_1, d_2} \) of the locus parametrising classes of vector bundles with fixed determinant \( L \) where \( d_i = \deg(L|_{C_i}) \). When \( r \) and \( d_i \) are coprime, as in the smooth case, we obtain the following result, see Theorem 5.2:

**Theorem C** Under the hypothesis of Theorem A, \( SU_C(w, r, L) \) is a rational variety.

The paper is organized as follows. In Section 1 we fix notations about reducible nodal curves. In Section 2 we introduce the notion of depth one sheaves, of polarization and \( w \)-semistability and we recall general properties on their moduli spaces. In Section 3 we introduce the projective bundle \( P(F) \), we define the sheaf \( E_u \) associated to \( u \in P(F) \) and we study when it is \( w \)-semistable. In Section 4 we prove Theorems A and B. Finally, in Section 5 we deal with moduli spaces with fixed determinant and we prove Theorem C.

1. Nodal reducible curves

In this paper we will consider nodal reducible complex projective curves with two smooth irreducible components and one single node. Let \( C \) be such a curve, we consider a normalization map \( \nu : C_1 \sqcup C_2 \to C \), where \( C_i \) is a smooth irreducible curve of genus \( g_i \geq 1 \). Hence \( \nu^{-1}(x) \) is a single point except when \( x \) is the node \( p \) of \( C \), in this case \( \nu^{-1}(p) = \{q_1, q_2\} \) with \( q_j \in C_j \). Since the restriction \( \nu|_{C_i} \) is an isomorphism we will identify \( C_1 \) and \( C_2 \) with the irreducible components of \( C \).

Notice that \( C \) can be embedded in a smooth surface \( X \), on which \( C \) is an effective divisor \( C = C_1 + C_2 \) with \( C_1C_2 = 1 \). Let \( J_C = O_X(-C) \) and \( J_{C_i} = O_X(-C_i) \) be the ideal sheaves of \( C \) and \( C_i \) respectively in \( X \), then we have the inclusion \( J_C \subseteq J_{C_i} \). We have the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & O_X(-C) & \rightarrow & O_X(-C_2) & \rightarrow & O_{C_1}(-C_2) & \rightarrow & 0 \\
& & \uparrow & & \uparrow & \uparrow & & \uparrow & \\
& & O_X & \rightarrow & O_X & \rightarrow & O_{C_2} & \rightarrow & 0 \\
& & \uparrow \cong & & \uparrow & & \uparrow & \\
0 & \rightarrow & J_{C_2}/J_C & \rightarrow & O_C & \rightarrow & O_{C_2} & \rightarrow & 0 \\
\end{array}
\]

from which one deduce the isomorphism \( J_{C_2}/J_C \cong O_{C_1}(-C_2) \). This gives the exact sequence

\[(1.1) \quad 0 \rightarrow O_{C_1}(-C_2) \rightarrow O_C \rightarrow O_{C_2} \rightarrow 0.\]

which is called the **decomposition sequence of** \( C \). From it we can compute the Euler characteristic of \( O_C \):

\[\chi(O_C) = \chi(O_{C_1}(-C_2)) + \chi(O_{C_2}).\]

Let \( p_a(C) = 1 - \chi(O_C) \) be the **arithmetical genus** of \( C \), from the above relation we get that \( p_a(C) = g_1 + g_2 \).
Notation 1.1. We will denote by \( j_i : C_i \hookrightarrow C \) the natural inclusion of \( C_i \) in \( C \). We will denote by \( \mathcal{O}_{y_i} \) the stalk of \( (j_i)_* \mathcal{O}_{C_i} \) in \( p \) and by \( \mathcal{O}_p \) the stalk of \( \mathcal{O}_C \) in \( p \).

2. Moduli space of depth one sheaves

Let \( C \) be a smooth irreducible projective curve of genus \( g \geq 1 \). The moduli space of semistable vector bundles of rank \( r \) and degree \( d \) on \( C \) will be denoted by \( \mathcal{U}_C(r,d) \). Its points are \( S \)-equivalence classes of semistable vector bundles on the curve. We will denote by \([E]\) the class of a vector bundle \( E \). In \([Ses82]\) it is proved that \( \mathcal{U}_C(r,d) \) is an irreducible and projective variety. Moreover, see \([Ses82], [Lor93]\), we have:

\[
dim \mathcal{U}_C(r,d) = \begin{cases} r^2(g - 1) + 1 & g \geq 2 \\ \gcd(r,d) & g = 1 \end{cases}
\]

In particular, when \( r \) and \( d \) are coprime, \( \mathcal{U}_C(r,d) \) is a smooth variety, whose points parametrize isomorphism classes of stable vector bundles. For \( g = 1 \), we also have: \( \mathcal{U}_C(r,d) \simeq \mathbb{C} \), see \([Lor93]\).

Let \( C \) be a nodal curve with a single node \( p \) and two smooth irreducible components \( C_1 \) and \( C_2 \). To construct compactifications of moduli spaces of vector bundles on \( C \) we introduce depth one sheaves, following the approach of Seshadri, see \([Ses82]\).

Definition 2.1. A coherent sheaf \( E \) on \( C \) is of depth one if every torsion section vanishes identically on some components of \( C \).

A coherent sheaf \( E \) on \( C \) is of depth one if and only if the stalk at the node \( p \) is isomorphic to \( \mathcal{O}_p^0 \oplus \mathcal{O}_p^1 \oplus \mathcal{O}_p^2 \), see \([Ses82]\). In particular, any vector bundle \( E \) on \( C \) is a sheaf of depth one. If \( E \) is a sheaf of depth one on \( C \), then its restriction \( E|_{C_i} \) is a torsion free sheaf on \( C_i \setminus p \) (possibly identically zero). Moreover, any subsheaf of \( E \) is of depth one too.

Let \( E \) be a sheaf of depth one on \( C \). We define the relative rank of \( E \) on the component \( C_i \) as the rank of the restriction \( E_i = E|_{C_i} \) of \( E \) to \( C_i \)

\[
r_i = \text{Rk}(E_i)
\]

and the multirank of \( E \) as the pair \((r_1, r_2)\). We define the relative degree of \( E \) with respect to the component \( C_i \) as the degree of the restriction \( E_i \)

\[
d_i = \text{deg}(E_i) = \chi(E_i) - r_i\chi(O_{C_i}),
\]

where \( \chi(E_i) \) is the Euler characteristic of \( E_i \). The multidegree of \( E \) is the pair \((d_1, d_2)\).

Definition 2.2. A polarization \( w \) of \( C \) is given by a pair of rational weights \((w_1, w_2)\) such that \( 0 < w_i < 1 \) and \( w_1 + w_2 = 1 \). For any sheaf \( E \) of depth one on \( C \), of multirank \((r_1, r_2)\) and \( \chi(E) = \chi \), we define the polarized slope as

\[
\mu_w(E) = \frac{\chi}{w_1 r_1 + w_2 r_2}.
\]
Definition 2.3. Let $E$ be a sheaf of depth one on $C$. $E$ is said $w$-semistable if for any proper subsheaf $F \subset E$ we have $\mu_w(F) \leq \mu_w(E)$; $E$ is said $w$-stable if $\mu_w(F) < \mu_w(E)$ for all subsheaf $F$ of $E$.

For each $w$-semistable sheaf $E$ of depth one on $C$ there exists a finite filtration of sheaves of depth one on $C$:

$$0 = E^0 \subset E^1 \subset E^2 \subset \cdots \subset E^k = E,$$

such that each quotient $E^i/E^{i-1}$ is a $w$-stable sheaf of depth one on $C$ with polarized slope $\mu_w(E^i/E^{i-1}) = \mu_w(E)$. This is called a Jordan-Holder filtration of $E$. The sheaf

$$Gr_w(E) = \oplus_{i=1}^k E^i/E^{i-1}$$

is said the graduate sheaf associated to $E$ and it depends only on the isomorphism class of $E$. Let $E$ and $F$ be $w$-semistable sheaves of depth one on $C$. We say that $E$ and $F$ are $S(w)$-equivalent if and only if $Gr_w(E) \simeq Gr_w(F)$. If $E$ and $F$ are $w$-stable sheaves then $S(w)$-equivalence is just isomorphism, as in the smooth case.

There exists a moduli space $U_C^w((r_1, r_2), \chi)$ parametrizing isomorphism classes of $w$-stable sheaves of depth one on $C$ of multirank $(r_1, r_2)$ and given Euler characteristic $\chi$, see [Ses82]. It has a natural compactification $\overline{U}_C^w((r_1, r_2), \chi)$, whose points correspond to $S(w)$-equivalence classes of $w$-semistable sheaves of depth one on $C$ of multirank $(r_1, r_2)$ and given Euler characteristic $\chi$. In particular, when $r_1 = r_2 = r$, we denote by $U_C^w(w, r, \chi)$ the corresponding moduli space. In this case we have the following result (see [TiB95] and [TiB11]):

Theorem 2.1. Let $C$ be a nodal curve with a single node $p$ and two smooth irreducible components $C_i$ of genus $g_i \geq 1$, $i = 1, 2$. For a generic polarization $w$ we have the following properties:

1. any $w$-stable vector bundle $E \in U_C^w(w, r, \chi)$ satisfies the following condition:

$$w_i \chi(E) \leq \chi(E_i) \leq w_i \chi(E) + r,$$

where $E_i$ is the restriction of $E$ to $C_i$;

2. if a vector bundle $E$ on $C$ satisfies the above condition for $i = 1, 2$ and the restrictions $E_1$ and $E_2$ are semistable vector bundles, then $E$ is $w$-semistable. Moreover, if at least one of the restrictions is stable, then $E$ is $w$-stable;

3. the moduli space $U_C^w(w, r, \chi)$ is connected, each irreducible component has dimension $r^2(p_a(C) - 1) + 1$ and it corresponds to the choice of a multidegree $(d_1, d_2)$ satisfying conditions 2.4.

Definition 2.4. We denote by $U_C^w(w, r, \chi)_{d_1, d_2}$ the irreducible component of $U_C^w(w, r, \chi)$ corresponding to the multidegree $(d_1, d_2)$. 
3. Construction of depth one sheaves.

In this section we deal with construction of depth one sheaves on a nodal curve $C$ with two irreducible components and a single node. We begin with the following lemma:

**Lemma 3.1.** Let $C_1$ and $C_2$ be smooth complex projective curves of genus $g_i \geq 1$, $i = 1, 2$, and $q_i \in C_i$. Fix $r \geq 2$ and $d_1, d_2 \in \mathbb{Z}$ such that $r$ is coprime with both $d_1$ and $d_2$. Then, there exists a projective bundle

$$\pi : \mathbb{P}(\mathcal{F}) \rightarrow \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$$

such that the fiber over $([E_1], [E_2])$ is $\mathbb{P}(\text{Hom}(E_{1,q_1}, E_{2,q_2}))$, where $E_{i,q_i}$ is the fiber of $E_i$ at the point $q_i$.

**Proof.** We recall that, as $r$ and $d_i$ are coprime, there exists a Poincaré bundle $\mathcal{P}_i$ for the moduli space of semistable vector bundles on $C_i$ of rank $r$ and degree $d_i$, i.e. a vector bundle $\mathcal{P}_i$ on $\mathcal{U}_{C_i}(r, d_i) \times C_i$ such that $\mathcal{P}_i|_{[E_i] \times C_i} \cong E_i$, under the identification $[E_i] \times C_i \cong C_i$. This follows from a result of [Ram73] if $g_i \geq 2$ and from the isomorphism $\mathcal{U}_{C_i}(r, d_i) \cong C_i$ when $g_i = 1$.

For $i = 1, 2$, consider the natural inclusion

$$\iota_i : \mathcal{U}_{C_i}(r, d_i) \times q_i \hookrightarrow \mathcal{U}_{C_i}(r, d_i) \times C_i,$$

and the pull back $\iota_i^*(\mathcal{P}_i)$ of the Poincaré bundle. Since $\mathcal{U}_{C_i}(r, d_i) \times q_i$ is isomorphic to $\mathcal{U}_{C_i}(r, d_i)$, $\iota_i^*(\mathcal{P}_i)$ can be seen as a vector bundle on $\mathcal{U}_{C_i}(r, d_i)$ of rank $r$ whose fiber at $[E_i]$ is actually $E_{i,q_i}$.

Note that the product $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$ is a smooth irreducible variety. Let $p_1$ and $p_2$ denote the projections of the product onto factors. We define on $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$ the following sheaf:

$$\mathcal{F} : = \mathcal{H}om(p_1^*(\iota_1^*(\mathcal{P}_1)), p_2^*(\iota_2^*(\mathcal{P}_2))).$$

By construction, $\mathcal{F}$ is a vector bundle of rank $r^2$ whose fiber at the point $([E_1], [E_2])$ is $\text{Hom}(E_{1,q_1}, E_{2,q_2})$. By taking the associated projective bundle we conclude the proof. $\square$

Let $C_1$ and $C_2$ be smooth irreducible curves, we consider a nodal curve $C$ with two smooth components and a single node $p$ which is obtained by identifying the points $q_1 \in C_1$ and $q_2 \in C_2$. Let $E_i$ be a stable vector bundle of rank $r$ and degree $d_i$ on $C_i$ and consider a non zero homomorphism $\sigma : E_{1,q_1} \rightarrow E_{2,q_2}$ between the fibres. Assume that the rank of $\sigma$ is $k$, with $1 \leq k \leq r$. We can associate to these data a depth one sheaf on the nodal curve $C$, roughly speaking, by gluing the vector bundles $E_1$ and $E_2$ along the fibers (at $q_1$ and $q_2$ respectively) with the homomorphism $\sigma$, as follows.

Let $j_i : C_i \rightarrow C$ be the inclusion, the sheaf $j_{i*}E_i$ is a depth one sheaf on $C$ whose stalk at $p$ is the stalk of $E_i$ at $q_i$. Hence, there is a natural surjective map onto the fiber of $E_i$ at $q_i$:

$$\rho_i : j_{i*}E_i \rightarrow E_{i,q_i}.$$  

The sheaves $j_{1*}(E_1) \oplus j_{2*}(E_2)$ and $j_{p*}j_{p}^*j_{2*}(E_2)$ are depth one on $C$ too.
In particular, the second sheaf is a skyscraper sheaf over \( p \) whose stalk is \( E_{2,q_2} \). So we can consider the following commutative diagram which defines the map \( \tilde{\sigma} \) and its kernel.

\[
\begin{array}{ccc}
K_1 \oplus K_2 & \xrightarrow{j_{1*}(E_1) \oplus j_{2*}(E_2)} & K_1 \oplus K_2 \\
0 & \xrightarrow{\ker \tilde{\sigma}} & 0 \\
\mathbb{C}_p^k & \xrightarrow{id} & \mathbb{C}_p^k \\
\end{array}
\]

\[
\begin{array}{ccc}
E_{1,q_1} \oplus E_{2,q_2} & \xrightarrow{\delta} & E_{2,q_2} \\
\mathbb{C}_p^k & \xrightarrow{\sigma \oplus id} & \text{Im}(\sigma) \oplus E_{2,q_2} \\
0 & \xrightarrow{\rho_1 \oplus \rho_2} & 0 \\
\end{array}
\]

where \( \delta \) sends \((u, v) \mapsto u - v \) and \( \mathbb{C}_p^k \) is the diagonal \( \Delta \subset \text{Im}(\sigma) \oplus \text{Im}(\sigma) \).

It follows immediately by construction that \( \ker \tilde{\sigma} \) is a sheaf of depth one on \( C \), whose restriction to \( C_i \setminus p \) is \( E_i \). One can easily see that the isomorphism class of \( \ker \tilde{\sigma} \) does not depend on the isomorphism class of \( E_i \). Moreover, the same happens if one uses \( \sigma' = \lambda \sigma \) with \( \lambda \in \mathbb{C}^* \), instead of \( \sigma \).

From now on, we will assume to be under the hypothesis of Lemma 3.1. Let \( \mathbb{P}(\mathcal{F}) \) be the projective bundle on \( \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) \). We can conclude that the construction of \( \ker \tilde{\sigma} \) depends on the data contained in \( u \) and not on the particular choices of \( E_1, E_2 \) and \( \sigma \).

**Definition 3.1.** We will denote by \( E_u \) the kernel of \( \tilde{\sigma} \) defined by \( u = ([E_1], [E_2], [\sigma]) \in \mathbb{P}(\mathcal{F}) \).

The above construction gives the following:

**Proposition 3.2.** Let \( E_u \) be the sheaf defined by \( u = ([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F}) \). Then \( E_u \) is a depth one sheaf on \( C \) with \( \chi(E_u) = \chi(E_1) + \chi(E_2) - r \) and multirank \((r, r)\). It is a vector bundle if and only if \( \sigma \) is an isomorphism. In this case, \( E_{u|C_i} = E_i \).

**Proof.** Let \( \text{Rk}(\sigma) = k \), from the diagram 3.2, it follows that the stalk of \( E_u \) at the node \( p \) is isomorphic to \( \mathcal{O}_p^k \oplus \mathcal{O}_{q_1}^{r-k} \oplus \mathcal{O}_{q_2}^{r-k} \), hence \( E_u \) is a vector bundle if and only if \( k = r \), i.e., exactly when \( \sigma \) is an isomorphism.

In order to obtain a \( w \)-semistable sheaf, for some polarization \( w \), we have the following necessary condition:

**Lemma 3.3.** Let \( E = E_u \) be the sheaf defined by \( u = ([E_1], [E_2], [\sigma]) \in \mathbb{P}(\mathcal{F}) \) and let \( k \) be the rank of \( \sigma \). Then, if \( E \) is \( w \)-semistable for some \( w \), the following conditions are satisfied:

\[
\begin{align*}
\chi(E)w_1 & \leq \chi(E_1) \leq \chi(E)w_1 + k, \\
\chi(E)w_2 + r - k \leq \chi(E_2)\chi(E)w_2 + r.
\end{align*}
\]
Proof. Assume that $E$ is $w$-semistable for a polarization $w$. Let $K_1$ be the kernel of the map
\[ \sigma \circ \rho_1 : j_1^* E_1 \to \text{Im}\sigma, \]
and $K_2$ be the kernel of the map $\rho_2 : j_2^* E_2 \to E_{2,q_2}$, as in diagram 3.2. Since $K_i$ is a subsheaf of $E$, then by $w$-semistability of $E$ we must have $\mu_w(K_i) \leq \mu_w(E)$. We have:
\[ \mu_w(K_1) = \frac{\chi(K_1)}{w_1 r} \leq \frac{\chi(E_1)}{w_1 r} \leq \frac{\chi(E)}{r}, \]
which implies
\[ \chi(E_1) \leq \chi(E)w_1 + k. \]
By replacing $\chi(E_1) = \chi(E) - \chi(E_2) + r$ in the above inequality, we obtain:
\[ \chi(E_2) \geq \chi(E)w_2 + r - k. \]
Finally, we have $\mu_w(K_2) = \frac{\chi(K_2)}{w_2 r} = \frac{\chi(E_2) - r}{w_2 r} \leq \frac{\chi(E)}{r}$, which implies
\[ \chi(E_2) \leq \chi(E)w_2 + r. \]
Again, by replacing $\chi(E_2) = \chi(E) - \chi(E_1) + r$ we obtain $\chi(E_1) \geq \chi(E)w_1$. \hfill \Box

Given $u = ((E_1), (E_2), [\sigma])$ and $E_u$ defined by $u$, we wonder if there exists a polarization $w$ such that the above conditions 3.3 hold. The answer depends only on numerical assumptions on $(\chi(E_1), \chi(E_2))$ and $\text{Rk}\sigma$ as it is shown in the following lemma.

**Lemma 3.4.** Let $r \geq 2$ and $1 \leq k \leq r$ be integers. There exists a non empty subset $\mathcal{W}_{r,k} \subset \mathbb{Z}^2$ such that for any pair $(\chi_1, \chi_2) \in \mathcal{W}_{r,k}$ we can find a polarization $w$ satisfying the following conditions:

\[
\begin{align*}
\chi w_1 & \leq \chi_1 \leq \chi w_1 + k, \\
\chi w_2 + r - k & \leq \chi_2 \leq \chi w_2 + r,
\end{align*}
\]

where $\chi = \chi_1 + \chi_2 - r$.

**Proof.** First of all note that if $\chi = 0$, i.e. $\chi_1 + \chi_2 = r$ and we assume that $0 \leq \chi_1 \leq r$, then any polarization $w$ satisfies conditions 3.4.

We distinguish two cases according to the sign of $\chi$. Assume that $\chi > 0$. Then there exists a polarization $w$ satisfying conditions 3.4, if and only if the following system has solutions:

\[
\begin{align*}
\frac{\chi_1 - k}{\chi} & \leq w_1 \leq \frac{\chi_1}{\chi} \\
\frac{\chi_2 - r}{\chi} & \leq w_2 \leq \frac{\chi_2 + k - r}{\chi} \\
w_1 + w_2 & = 1 \\
0 & < w_i < 1, w_i \in \mathbb{Q}
\end{align*}
\]
This occurs if and only if $\chi_1 > 0$ and $\chi_2 > r - k$. Likewise, if $\chi < 0$, then we have the following system:

\[
\begin{align*}
\frac{\chi_1}{\chi} & \leq w_1 \leq \frac{\chi_1 - k}{\chi} \\
\frac{\chi_2 - r + k}{\chi} & \leq w_2 \leq \frac{\chi_2 - r}{\chi} \\
w_1 + w_2 &= 1 \\
0 & < w_i < 1, w_i \in \mathbb{Q}
\end{align*}
\]

which has solutions if and only if $\chi_1 < k$ and $\chi_2 < r$. \hfill \Box

**Remark 3.4.1.** Let $W_r = \bigcap_{k=1}^{r} W_{r,k}$. Note that it is a non empty subset and it is actually $W_{r,1}$. Moreover, if $(\chi_1, \chi_2) \in W_r$, then by the proof of lemma 3.4 it follows that we can find a polarization $w$ which satisfies the conditions 3.4 for all $k = 1, \ldots, r$.

Assume that $Rk\sigma = r$, i.e. $E$ is a vector bundle, then the necessary conditions of Lemma 3.3 are the same of Theorem 2.1. Hence, by the above Theorem, they are also sufficient to give $w$-semistability of $E$. So we obtain the following:

**Corollary 3.5.** Let $E = E_u$ be the sheaf defined by $u = ([E_1], [E_2], [\sigma]) \in \mathbb{P}(F)$. Assume that $Rk\sigma = r$ and $(\chi(E_1), \chi(E_2)) \in W_{r,r}$, then there exists a polarization $w$ such that $E$ is $w$-semistable. In particular, since $E_i$ are stable, then $E$ is $w$-stable too.

Unfortunately, when $E_u$ fails to be a vector bundle, the necessary conditions of Lemma 3.3 are not enough to ensure $w$-semistability, see [TiB11] for an example. Nevertheless, we are able to produce an open subset of $U_{C_1}(r, d_1) \times U_{C_1}(r, d_1)$ such that for every $u$ over this open, the sheaf $E_u$ is $w$-semistable.

We recall the following definition, see [NMRS78].
Definition 3.2. Let $G$ be a vector bundle on a smooth curve. For any integer $k$ we set:

$$\mu_k(G) = \frac{\deg(G) + k}{rk(G)}.$$ 

A vector bundle $G$ is said $(m,k)$-semistable (resp. stable) if for any subsheaf $F$ we have:

$$\mu_m(F) \leq \mu_{m-k}(G) \quad (\text{resp. } <).$$

Proposition 3.6. Let $E = E_u$ be the sheaf defined by $u = ([E_1], [E_2], [\sigma]) \in \mathbb{P}(F)$. Assume that $\text{Rk} \sigma = k \leq r - 1$. If $(\chi(E_1), \chi(E_2)) \in W_{r,k}$, $E_1$ is $(0,k)$-semistable and $E_2$ is $(0,r)$-semistable, then there exists a polarization $w$ such that $E$ is $w$-semistable. Moreover, if $E_1$ is $(0,k)$-stable or $E_2$ is $(0,r)$-stable then $E$ is $w$-stable too.

Proof. Since $(\chi(E_1), \chi(E_2)) \in W_{r,k}$, by Lemma 3.4 there exists a polarization $w$ such that the necessary conditions 3.3 hold. We claim that if $E_1$ is $(0,k)$-semistable and $E_2$ is $(0,r)$-semistable, then $E$ is $w$-semistable.

Let $F \subset E$ be a subsheaf, it is a sheaf of depth one too. Assume that $F$ has multirank $(s_1, s_2)$ and that at the node $p$ the stalk of $F$ is $\mathcal{O}_p^s \oplus \mathcal{O}_{q_1}^a \oplus \mathcal{O}_{q_2}^b$, with $s \geq 0$, $s_1 = s + a \leq r$ and $s_2 = s + b \leq r$. Since $\text{Rk} \sigma = k$, by construction the free part of the stalk of $E$ at $p$ is $\mathcal{O}_p^k$.

This implies that $0 \leq s \leq k$.

By construction, there exists two vector bundles $F_1 \subseteq E_1$ and $F_2 \subseteq E_2$ such that $F$ is the kernel of the restriction of $\sigma$ to the subsheaf $j_{1*}(F_1) \oplus j_{2*}(F_2)$:

$$\tilde{\sigma}(j_{1*}(F_1) \oplus j_{2*}(F_2)) : j_{1*}(F_1) \oplus j_{2*}(F_2) \to j_{p,*}j_p^*j_{2*}(E_2).$$

By proceeding as in the diagram 3.2, we deduce that $F$ fits into an exact sequence as follows:

$$0 \to G_1 \oplus G_2 \to F \to \mathbb{C}_p^s \to 0,$$

where $G_1$ is the kernel of $(\sigma \circ \rho_1)|_{F_1}$ and $G_2$ is the kernel of $\rho_2|_{F_2}$. Hence $G_i \subseteq K_i$. Note that if $s = 0$, then actually $F \simeq G_1 \oplus G_2$.

For any $s$, we compute the $w$-slope of $F$:

$$\mu_w(F) = \frac{\chi(F)}{w_1s_1 + w_2s_2} = \frac{\chi(G_1) + \chi(G_2) + s}{w_1s_1 + w_2s_2} =$$

$$= \frac{\deg(G_1) + s_1(1 - g_1) + \deg(G_2) + s_2(1 - g_2) + s}{w_1s_1 + w_2s_2}.$$ 

Since $E_1$ is $(0,k)$-semistable, then we have:

$$\frac{\deg(G_1)}{s_1} \leq \frac{d_1 - k}{r},$$

since $E_2$ is $(0,r)$-semistable, then $E_2(-q_2)$ is $(0,r)$-semistable too, so we have:

$$\frac{\deg(G_2)}{s_2} \leq \frac{d_2 - 2r}{r}. $$

By replacing we obtain:

\[
\mu_w(F) \leq \frac{1}{w_1 s_1 + w_2 s_2} \left[ s_1 w_1 \left( \frac{(d_1 - k) + r(1 - g_1)}{w_1 r} \right) + s_2 w_2 \left( \frac{(d_2 - r) + r(1 - g_2)}{w_2 r} \right) + s - s_2 \right] = \\
= \frac{s_1 w_1}{w_1 s_1 + w_2 s_2} \mu_w(K_1) + \frac{s_2 w_2}{w_1 s_1 + w_2 s_2} \mu_w(K_2) + \frac{s - s_2}{w_1 s_1 + w_2 s_2}.
\]

By Lemma 3.3, we have that \( \mu_w(K_i) \leq \mu_w(E) \), so we obtain:

\[
\mu_w(F) \leq \mu_w(E) + \frac{s - s_2}{w_1 s_1 + w_2 s_2}.
\]

Since \( s - s_2 \leq 0 \), we have that \( \mu_w(F) \leq \mu_w(E) \).

Finally, if \( E_1 \) is \((0,k)\)-stable or \( E_2 \) is \((0,r)\)-stable, then the above inequality is strict. This concludes the proof. \( \square \)

Note that, by definition, if \( E_i \) is \((0,r)\)-stable, then it is also \((0,k)\)-stable for all \( k \leq r \).

**Lemma 3.7.** Let \( \mathcal{U}_C(r, d_i) \) be the moduli space of semistable vector bundles of rank \( r \) and degree \( d_i \) on a smooth curve \( C_i \) of genus \( g_i \). If \( g_i > r + 1 \), then the locus of vector bundles of \( \mathcal{U}_C(r, d_i) \) which are \((0,r)\)-stable is a non empty open subset of \( \mathcal{U}_C(r, d_i) \).

**Proof.** Let’s consider the following locus:

\[
Y = \{ [E] \in \mathcal{U}_C(r, d_i) \mid E \text{ is not } (0,r) \text{- stable} \}.
\]

Let \( E \in Y \), then there exists a subbundle \( F \) such that \( \mu_0(F) \geq \mu_{-r}(E) \). We can assume that both \( F \) and the quotient bundle \( Q \simeq E/F \) are stable. We can consider the component \( Y_{a,s} \) of \( Y \) given by all semistable vector bundles which can be written as

\[
0 \to F \to E \to Q \to 0,
\]

with \( F \) and \( Q \) stable vector bundles and \( \deg F = a \) and \( \text{Rk } F = s \leq r - 1 \). Since for \( Q \) and \( F \) general in their moduli spaces one has \( \text{Hom}(Q, F) = 0 \), we have

\[
\dim Y_{a,s} \leq \dim \mathcal{U}_C(s, a) + \dim \mathcal{U}_C(r - s, d_i - a) + \dim H^1(C_i, \mathcal{H}om(Q, F)) - 1 = \\
(g_i - 1)(r^2 - rs + s^2) + 1 + (d_i s - ar).
\]

Hence:

\[
\dim \mathcal{U}_C(r, d_i) - \dim Y_{a,s} \geq (g_i - 1)(rs - s^2) - (d_i s - ar).
\]

Since \( E \in Y \), \( \mu_0(F) \geq \mu_{-r}(E) \), i.e.

\[
\frac{a}{s} \geq \frac{d_i - r}{r},
\]

which implies

\[
d_i s - ar \leq rs.
\]

Finally, if \( g_i > 1 + r \), for all \( s \leq r - 1 \), we have:

\[
\dim \mathcal{U}_C(r, d_i) - \dim Y_{a,s} \geq s[(g_i - 1)(r - s) - r] > 0
\]

which concludes the proof. \( \square \)
4. Main results

In this section we prove our main results. We assume that the hypothesis of Lemma 3.1 are satisfied.

Let $\mathbb{P}(\mathcal{F})$ be the projective bundle on $\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$. For each $1 \leq k \leq r - 1$, let $B_k$ be the subset of $\mathbb{P}(\mathcal{F})$ such that

$$B_k \cap \pi^{-1}([E_1], [E_2]) = \{[\sigma] \in \mathbb{P}(\text{Hom}(E_{1,q_1}, E_{2,q_2})) \mid \text{Rk}(\sigma) \leq k\}.$$ 

It is a proper closed subvariety of $\mathbb{P}(\mathcal{F})$.

**Definition 4.1.** We will denote by $\mathcal{W}$ the open subset given by the complementary of $B_{r - 1}$ in $\mathbb{P}(\mathcal{F})$.

**Remark 4.0.1.** Note that $\dim \mathcal{W} = \dim \mathbb{P}(\mathcal{F}) = r^2(g_1 + g_2 - 1) + 1$. If we denote by $\pi_{\mathcal{W}}$ the restriction of $\pi$ to $\mathcal{W}$ we have that, by construction,

$$\pi_{\mathcal{W}} : \mathcal{W} \to \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2)$$

is a fiber bundle whose fibers are isomorphic to $\text{PGL}(r)$. More precisely

$$\pi_{\mathcal{W}}^{-1}([E_1], [E_2]) = \mathbb{P}((\text{GL}(E_{1,q_1}, E_{2,q_2}))).$$

For $\chi = d_1 + d_2 + r(1 - g_1 - g_2)$, let $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ be the irreducible component of the moduli space of depth one sheaves on $C$ of rank $r$ and characteristic $\chi$ corresponding to the multidegree $(d_1, d_2)$, see Section 2. Let $\mathcal{V}_C(w, r, \chi)_{d_1,d_2} \subset \mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ be the subset parametrizing classes of vector bundles.

**Theorem 4.1.** Let $C$ be a nodal curve with a single node $p$ and two smooth irreducible components $C_1$ of genus $g_1 \geq 1$. Fix $r \geq 2$, for any $d_i \in \mathbb{Z}$ we set $\chi_i = d_i + r(1 - g_i)$ and $\chi = d_1 + d_2 + r(1 - g_1 - g_2)$. Assume that $r$ is coprime with both $d_1$ and $d_2$ and $(\chi_1, \chi_2) \in \mathcal{W}_{r,r}$. Then there exists a polarization $w$ such that the map

$$\phi : \mathbb{P}(\mathcal{F}) \dashrightarrow \mathcal{U}_C(w, r, \chi)_{d_1,d_2}$$

sending $u \to [E_u]$ is birational. In particular, the restriction $\phi|_{\mathcal{W}}$ is an injective morphism and the image $\Phi(\mathcal{W})$ is contained in $\mathcal{V}_C(w, r, \chi)_{d_1,d_2}$.

**Proof.** Let $u = (([E_1], [E_2]), [\sigma]) \in \mathbb{P}(\mathcal{F})$ and consider the sheaf $E = E_u$ defined by $u$, as in Section 3. Since $(\chi_1, \chi_2) \in \mathcal{W}_{r,r}$, then, as a consequence of Lemma 3.4 and Corollary 3.5 there exists a polarization $w$ such that $E_u$ is $w$-semistable for every $u \in \mathcal{W}$. This gives a point in the moduli space $\mathcal{U}_C(w, r, \chi)_{d_1,d_2}$ and it shows that $\phi$ is well defined at least on $\mathcal{W}$.

Now we will prove that $\phi|_{\mathcal{W}}$ is injective. Let $u = (([E_1], [E_2]), [\sigma])$ and $u' = (([E'_1], [E'_2]), [\sigma'])$ in $\mathcal{W}$ with $\phi(u) = [E]$ and $\phi(u') = [E']$. Assume that $\phi(u) = \phi(u')$. Since $E$ and $E'$ are both $w$-stable, they have to be isomorphic. Let $\tau : E \to E'$ be an isomorphism. This induces an isomorphism $\tau_i : E_i \to E'_i$. So we can assume, that $E'_i = E_i$ and, thus $\sigma, \sigma' : E_{1,q_1} \to E_{2,q_2}$ and $\tau_i : E_i \to E_i$ are isomorphism. As $E_p$ (respectively $E'_p$) is obtained by glueing $E_{1,q_1}$
with $E_{2,q_2}$ along the isomorphism $\sigma$ (respectively along $\sigma'$), $\tau_i$ have to satisfy a compatibility condition, summarized in the following commutative diagram:

$$
\begin{array}{ccc}
E_{1,q_1} & \xrightarrow{\sigma} & E_{2,q_2} \\
(\tau_1)_{q_1} \downarrow & & \downarrow (\tau_2)_{q_2} \\
E_{1,q_1} & \xrightarrow{\sigma'} & E_{2,q_2}
\end{array}
$$

Since $E_i$ is stable we have $\text{Hom}(E_i, E_i) \simeq \mathbb{C} \cdot id_{E_i}$. Hence $(\tau_i)_{q_i}$ is the multiplication by some $\lambda_i \in \mathbb{C}^*$. In particular, $\sigma'$ is a non zero multiple of $\sigma$ and thus $[\sigma] = [\sigma']$.

Now we prove that $\phi|_U$ is a morphism. It is enough to prove that $\phi$ is regular at $u_0$, for any $u_0 \in \mathcal{U}$. At this hand, we claim that there exists a non empty open subset $W \subseteq \mathcal{U}$ with $u_0 \in W$ and a vector bundle $E$ on $W \times C$ such that

$$
[\mathcal{E}|_{u \times C}] = \phi(u), \quad \forall u \in W.
$$

Step 1: There exist two sheaves $Q$ and $R$ on $\mathcal{U} \times C$ such that, for each $u = ([E_1], [E_2], [\sigma])$, with $u \in \mathcal{U}$, we have

$$
Q|_{u \times C} \simeq j_1^*(E_1) \oplus j_2^*(E_2), \quad R|_{u \times C} \simeq j_p^*(j_p^*(j_2^*(E_2))),
$$

where $j_p: p \hookrightarrow C$ and $j_i: C_i \hookrightarrow C$ are the natural inclusions.

Consider the diagram

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\sim} & \mathcal{U} \times p \\
\Pi_{\mathcal{U}} \downarrow & & \downarrow \pi_{\mathcal{U}} \\
\mathcal{U} \times C & \xrightarrow{J_i} & \mathcal{U} \times C
\end{array}
$$

where the morphisms which appear have been defined as

$$
(4.2) \quad J_i = \text{id}_{\mathcal{U}_C(r,d_i)} \times j_i, \quad P_i = p_i \times \text{id}_C, \quad \Pi_{\mathcal{U}} = \pi_{\mathcal{U}} \times \text{id}_C, \quad J_p = \text{id}_{\mathcal{U}} \times j_p.
$$

If, as before, we denote with $\mathcal{P}_i$ the Poincaré bundle on $\mathcal{U}_C(r,d_i) \times C_i$ we can set

$$
\mathcal{Q}_i = \Pi_{\mathcal{U}}^*(P_i^*(J_i^*(\mathcal{P}_i))) \quad \mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2,
$$

and

$$
\mathcal{R} = J_p^*(J_p^*(\mathcal{Q}_2)).
$$
Note that $\text{Supp}(R) = \mathcal{U} \times p$. Moreover, one can verify that if we identify $\mathcal{U} \times p$ with $\mathcal{U}$ we have:

\begin{equation}
\tag{4.3}
J^\ast_p(Q_i) \simeq \pi^\ast_{\mathcal{U}}(p^\ast_1(i^\ast_1(P_1))),
\end{equation}

where $i_1: \mathcal{U}_i(r, d_i) \times q_i \hookrightarrow \mathcal{U}_i(r, d_i) \times C_i$.

Step 2: There is an open subset $W \subset \mathcal{U}$ containing $u_0$ and a surjective map of sheaves

$$Q_1 \oplus Q_2\big|_{W \times C} \xrightarrow{\Sigma_W} \mathcal{R}\big|_{W \times C}$$

whose kernel is the desired vector bundle $E$ on $W \times C$.

Let $\pi: \mathbb{P}(F) \rightarrow \mathcal{U}_i(r, d_i) \times \mathcal{C}_j(r, d_j)$ be the projective bundle defined in Lemma 3.1. First of all consider on $\mathbb{P}(F)$ the tautological line bundle $\mathcal{O}_{\mathbb{P}(F)}(-1)$ which is, by definition, the subsheaf of $\pi^\ast(F)$ whose fiber at $u \in \mathbb{P}(F)$ is

$$\text{Span}(\sigma) \subset \text{Hom}(E_{1, q_1}, E_{2, q_2}),$$

where $u = ([E_1], [E_2], [\sigma])$. We can choose $W$ to be an open subset of $\mathcal{U}$ containing the point $u_0$ and admitting a section $s \in \mathcal{O}_{\mathbb{P}(F)}(-1)(W)$ with $s(u) \neq 0$, for any $u \in W$.

In particular $s$ induces a map of sheaves

\begin{equation}
\tag{4.4}
s: \pi^\ast_{\mathcal{U}}p^\ast_1(i^\ast_1(P_1)))|W \rightarrow \pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W.
\end{equation}

such that $s_\alpha: E_{1, q_1} \rightarrow E_{2, q_2}$ is an isomorphism and $[s_\alpha] = [\sigma]$ in $\mathbb{P}(\text{Hom}(E_{1, q_1}, E_{2, q_2}))$.

We can also define a morphism of sheaves

\begin{equation}
\tag{4.5}
s - id_2 : \pi^\ast_{\mathcal{U}}p^\ast_1(i^\ast_1(P_1)))|W \oplus \pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W \rightarrow \pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W
\end{equation}

where $id_2$ is the identity of $\pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W$.

This allows us to define the map $\Sigma_W$ we are looking for. Indeed, since $\text{Supp}(R|_{W \times C}) = W \times p$, it is enough to give the map on $W \times p$, which can be identified with $W$. Using the isomorphism \ref{4.3}, we have a diagram which defines $\Sigma_W$:

$$
\begin{array}{c}
Q_1 \oplus Q_2\big|_{W \times C} \\
\downarrow|W \times p
\end{array}
\xrightarrow{\Sigma_W}
\begin{array}{c}
R|_{W \times C} \\
\downarrow|W \times p
\end{array}
\xrightarrow{J^\ast_p}\begin{array}{c}
J^\ast_p(Q_1 \oplus Q_2\big|_{W \times C}) \\
\downarrow \pi^\ast_{\mathcal{U}}p^\ast_1(i^\ast_1(P_1)))|W \oplus \pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W \\
\end{array}
\xrightarrow{s - id_2}
\pi^\ast_{\mathcal{U}}p^\ast_2(i^\ast_2(P_2)))|W
$$

By taking the kernel $E$ of this map we concludes the second step of the proof of the claim.

In particular, $\phi|_{\mathcal{U}}$ is a morphism.

By construction, $\phi(\mathcal{U})$ is contained in $\mathcal{V}_C(w, r, \chi)_{d_1, d_2}$ and it coincide with the open subset of $w$-semistable vector bundles whose restrictions are semistable. Moreover, $\mathcal{V}_C(w, r, \chi)_{d_1, d_2}$
is a dense open subset of \( \mathcal{U}_C(w, r, \chi)_{d_1, d_2} \) (see [Ses82]). Notice that, by 4.0.1, we have:
\[
\dim(\phi(\mathcal{W})) = \dim(\mathcal{W}) = r^2(g_1 + g_2 - 1) + 1
\]
which is the dimension of \( \mathcal{U}_C(w, r, \chi)_{d_1, d_2} \), see Theorem 2.1. This implies that \( \phi \) is a dominant map. Hence, by a generic smoothness argument, we can conclude that \( \phi|_\mathcal{W} \) is a birational morphism.

**Corollary 4.2.** Let \( C \) be a nodal curve with a single node \( p \) and two smooth irreducible components \( C_i \) of genus \( g_i \geq 1 \). Assume that the moduli space \( \mathcal{U}_{C_i}(w, r, \chi) \) has an irreducible component corresponding to bidegree \( (d_1, d_2) \) with \( d_1 \) and \( d_2 \) coprime with \( r \). Then this component is birational to a projective bundle over the smooth variety \( \mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) \).

Note that \( \phi \) provides a desingularization of the component \( \mathcal{U}_C(w, r, \chi)_{d_1, d_2} \).

If the genus of the curve \( C_i \) is big enough, we can be more precise about the domain of the rational map \( \phi \).

Assume that \( g_i > r + 1 \), then by Lemma 3.7 the locus of vector bundles of \( \mathcal{U}_{C_i}(r, d_i) \) which are \((0, r)\)-stable is a non empty open subset of \( \mathcal{U}_{C_i}(r, d_i) \), let’s denote it by \( V_i \).

**Definition 4.2.** We will denote by \( \mathcal{V} \) the open subset \( \pi^{-1}(V_1 \times V_2) \) in \( \mathbb{P}(\mathcal{F}) \).

By construction, \( \mathcal{V} \) is a projective bundle over \( V_1 \times V_2 \).

**Theorem 4.3.** Assume that the hypothesis of Theorem 4.1 hold. Moreover, assume that \( g_i > r + 1 \) and \((\chi_1, \chi_2) \in \mathcal{W}_r \). Then there exists a polarization \( w \) such that the map \( \phi \) sending \( u \) to \([E_u] \) is a birational map such that \( \phi|_{\mathcal{W} \cup \mathcal{V}} \) is a morphism.

**Proof.** Since \((\chi_1, \chi_2) \in \mathcal{W}_r \) then, by Remark 3.4.1, there exists a polarization \( w \) such that conditions 3.4 hold for any \( k = 1, \cdots, r \). In particular, as \( \mathcal{W}_r \subset \mathcal{W}_{r,r} \), Theorem 4.1 holds: \( \phi \) is a birational map which is defined on the open subset \( \mathcal{W} \).

Assume that \( u \in \mathcal{V} \) and \( u \notin \mathcal{U} \). Then \( u = ((|E_1|, |E_2|), [\sigma]) \), with \((|E_1|, |E_2|) \in V_1 \times V_2 \) and \( \text{Rk} \sigma \leq r - 1 \). Since \([E_1] \in V_i \), then by lemma 3.6, \( E_u \) is \( w \)-semistable, hence \( \phi \) is defined all over the open subset \( \mathcal{V} \) too.

To prove that \( \phi|_{\mathcal{V}} \) is a morphism, we can proceed as in the proof of Theorem 4.1, just by replacing \( \mathcal{W} \) with \( \mathcal{V} \) and \( \mathcal{U}_{C_i}(r, d_i) \) with \( V_i \). \( \square \)

## 5. Fixed-determinant moduli space

Let \( C \) be a smooth curve of genus \( g \geq 1 \) and \( L \in \text{Pic}^d(C) \). We recall that the moduli space of semistable vector bundles of rank \( r \) and determinant \( L \) on \( C \) is denoted by \( SU_C(r, L) \) and it is an irreducible and projective variety. It is the fiber of the determinant map:

\[
\text{det}: \mathcal{U}_C(r, d) \to \text{Pic}^d(C).
\]
In this section we will investigate a similar subvariety of the moduli space $\mathcal{U}_C(w, r, \chi)_{d_1, d_2}$ for a nodal reducible curve with two irreducible component $C_i$. Fix a pair $(L_1, L_2)$ with $L_i \in \text{Pic}^{d_i}(C_i)$. Note that there exists a unique line bundle $L$ on the nodal curve $C$ whose restriction to the component $C_i$ is $L_i$. Recall that $\mathcal{V}_C(w, r, \chi)_{d_1, d_2} \subset \mathcal{U}_C(w, r, \chi)_{d_1, d_2}$ is the open subset parametrizing $w$-semistable classes which are represented by vector bundles.

**Definition 5.1.** Let $L$ be the line bundle on $C$ induced by the pair $(L_1, L_2)$. We define $S_{\mathcal{U}_C(w, r, L)}$ as the closure of $\{[E] \in \mathcal{V}_C(w, r, \chi)_{d_1, d_2} \mid \det E = L\}$ in $\mathcal{U}_C(w, r, \chi)_{d_1, d_2}$.

If we assume that $r$ and $d_i$ are coprime, then $S_{\mathcal{U}_C(r, L_i)}$ is a smooth irreducible projective variety of dimension $(r^2 - 1)(g_i - 1)$. As in Lemma 3.1, we can define a vector bundle $\mathcal{F}_L$ on $S_{\mathcal{U}_C(r, L_1)} \times S_{\mathcal{U}_C(r, L_2)}$ just by restricting $\mathcal{F}$. Then we can consider the associated projective bundle $\mathbb{P}(\mathcal{F}_L)$ and

$$\mathcal{U}_L = \mathcal{U} \cap \mathbb{P}(\mathcal{F}_L),$$

a $\text{PGL}(r)$-bundle on $S_{\mathcal{U}_C(r, L_1)} \times S_{\mathcal{U}_C(r, L_2)}$. We denote by $\phi_L$ the restriction of the morphism $\phi$ defined in Theorem 4.1 to $\mathcal{U}_L$. As a consequences of Theorem 4.1, we have the following:

**Corollary 5.1.** In the hypothesis of Theorem 4.1, the map $\phi_L: \mathbb{P}(\mathcal{F}_L) \rightarrow S_{\mathcal{U}_C(w, r, L)}$ is a birational map, whose restriction $\phi_L|_{\mathcal{U}_L}$ is an injective morphism.

**Proof.** Note that $\phi_L|_{\mathcal{U}_L}$ is a morphism and its image is the following subset:

$$\text{Im} \phi_L = \{E \in \mathcal{V}_C(w, r, \chi)_{d_1, d_2} \mid [E|_{C_i}] \in S_{\mathcal{U}_C(r, L_i)}\}.$$

In particular, $\text{Im} \phi_L \subseteq S_{\mathcal{U}_C(w, r, L)}$.

Consider the following map:

$$\psi: \mathcal{V}_C(w, r, \chi)_{d_1, d_2} \rightarrow \text{Pic}^{d_1}(C_1) \times \text{Pic}^{d_2}(C_2),$$

sending $E \rightarrow (\det(E|_{C_1}), \det(E|_{C_2}))$, which fit into the following commutative diagramm:

(5.1) $\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\phi} & \mathcal{V}_C(w, r, \chi)_{d_1, d_2} \\
\pi \downarrow & & \downarrow \psi \\
\mathcal{U}_{C_1}(r, d_1) \times \mathcal{U}_{C_2}(r, d_2) & \xrightarrow{\text{det}_1 \times \text{det}_2} & \text{Pic}^{d_1}(C_1) \times \text{Pic}^{d_2}(C_2)
\end{array}$

It follows immediately that $\psi$ is a surjective morphism and $\text{Im} \phi_L \subseteq \psi^{-1}(L_1, L_2)$.

We claim that $\psi$ has irreducible fibers of dimension $(r^2 - 1)(g_1 + g_2 - 1)$. 

First of all we prove that any two fibers of $\psi$ are isomorphic. Let $(L_1, L_2)$ and $(L'_1, L'_2)$ in $\text{Pic}^d(C_1) \times \text{Pic}^d(C_2)$, then there exists $\xi_i \in \text{Pic}^d(C_i)$ such that $L_i \otimes \xi_i \simeq L'_i$. Let $\xi$ be the unique line bundle on $C$ such that $\xi_{|C_1} \simeq \xi_i$. The natural map 

$$
\psi^{-1}(L_1, L_2) \rightarrow \psi^{-1}(L'_1, L'_2)
$$

sending $E$ to $E \otimes \xi$ preserves $w$-semistability and it gives an isomorphism of the fibers. In particular, from fiber dimension Theorem (see [Har77], p.95), this implies that any fiber has pure dimension $(r^2 - 1)(g_1 + g_2 - 1)$.

Finally we prove that any fiber is irreducible. Let $Y = V_{C}(w, r, \chi)_{d_1, d_2} \setminus \phi(\mathcal{M})$, it is a proper subvariety of $V_{C}(w, r, \chi)_{d_1, d_2}$. Assume that the fiber of $\psi$ over $(L_1, L_2)$ is reducible. Let $F_1$ be the irreducible component containing $\phi(\mathcal{M})$, then there exists an irreducible component $F_2 \subset Y$. So the restriction of $\psi$ to $Y$ is a surjective morphism whose fibers have dimension $(r^2 - 1)(g_1 + g_2 - 1)$. This implies that $\dim Y = \dim V_{C}(w, r, \chi)_{d_1, d_2}$, which is impossible.

This allows us to conclude that $SU_C(w, r, L)$ is irreducible too and $\phi_L$ is a birational morphism. □

**Theorem 5.2.** In the hypothesis of Theorem 4.1, $SU_C(w, r, L)$ is a rational variety.

**Proof.** By hypothesis $d_i$ and $r$ are coprime, then the moduli space $SU_{C_i}(r, L_i)$ is rational for any line bundle $L_i \in \text{Pic}^{d_i}(C_i)$, see [KS99], [New75] and [New80]. Since $\mathcal{M}$ is a $\mathbb{P}^{r^2-1}$-bundle over the product $SU_{C_1}(r, L_1) \times SU_{C_2}(r, L_2)$, then it is a rational variety too. The assertion follows from corollary 5.1. □

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