Black holes and balanced metrics

Michael R. Douglas\textsuperscript{1,2,3} and Semyon Klevtsov\textsuperscript{1,2,4}

\textsuperscript{1} Simons Center for Geometry and Physics, Stony Brook University, Stony Brook, NY 11794–3840, USA

\textsuperscript{2} NHETC and Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08855–0849, USA

\textsuperscript{3} I.H.E.S., Le Bois-Marie, Bures-sur-Yvette, 91440, France

\textsuperscript{4} ITEP, Moscow, 117259, Russia

mrd@physics.rutgers.edu, klevtsov@physics.rutgers.edu

ABSTRACT: We consider a probe in a BPS black hole in type II strings compactified on Calabi-Yau manifolds, and conjecture that its moduli space metric is the balanced metric.
1. Introduction

A famous problem in quantum gravity is to derive the Bekenstein-Hawking entropy of a black hole by counting its microstates. In string theory, this was first done by Strominger and Vafa [1]. They counted the microstates of a BPS bound state of Dirichlet branes with the same charge as the black hole, and then argued that the number of states was invariant under varying the string coupling, turning the bound state into a black hole.

This line of argument has been the basis for a great deal of work, generalizing the result to other systems and away from the semiclassical limit. One important element in such results is the claim that entropies and numbers of microstates are independent of the moduli of the background. An argument to this effect is provided by the attractor mechanism [2]. This was originally stated for BPS black holes in type II strings compactified on a Calabi-Yau manifold $X$, but the idea is probably more general (see [3] for a recent discussion). The attractor mechanism is based on the observation that the equations of motion for the moduli in a black hole background can be written in the form of gradient flow equations for the area of a surface of fixed radius as a function of the moduli. This flow approaches an attracting fixed point at the event horizon, with a definite value of the moduli and area. Thus, these values are insensitive to small variations of the initial conditions. By the Bekenstein-Hawking relation, this implies that the entropy is invariant under such variations.

It is plausible that other properties of the black hole microsystem share this type of universal behavior. For example, we might conjecture that not only the Kähler moduli of the Calabi-Yau metric near a black hole take universal values, but that the entire metric is universal, determined only by the charge and structure of the black hole and independent of the asymptotic moduli.
What would this mean? In classical supergravity, of course the metric is determined by the Einstein equation, reducing to the Ricci flatness condition for the source-free case. Thus the stronger conjecture is quite reasonable and indeed follows directly from the validity of supergravity. On the other hand, for a finite charge black hole preserving eight or fewer supercharges, one knows that these equations will get string theoretic ($\alpha'$ or $g_s$) corrections. Thus, while the stronger conjecture is still reasonable, it is not \textit{a priori} clear either what the attractor CY metric should be, or what equations determine it.

Now, one reason the general question of finding exact metrics or even precisely defining corrected supergravity equations is hard, is that the metric and equations can be changed by field redefinitions, with no obvious preferred definition. For example, the metric $g_{ij}$ could be redefined as $g_{ij} \rightarrow g_{ij} + \alpha R_{ij} + \beta (R^2)_{ij} + \ldots$. Unless we postulate an observable which singles out one definition, say measurements done by a point-like observer who moves on geodesics, there is no way to say which definition is right. This problem shows up in computing $\alpha'$ corrections in the sigma model as the familiar question of renormalization scheme dependence; in general there is no preferred scheme. We must first answer this question, to give meaning to the “CY attractor metric.”

A nice way to answer this question is to introduce a probe brane, say a D0-brane, and study its world-volume theory. The kinetic term for its transverse coordinates is observable, and defines an unambiguous metric on the target space, including any $\alpha'$ corrections. While one can still make field redefinitions in the action, now these are just coordinate transformations. To make this argument straightforward, one requires that the mass (or tension) of the probe be larger than any other quantities under discussion, so that the action can be treated classically, and the metric read off from simple measurements.\footnote{This was the point of view taken in \cite{4, 5}. Actually, one can in principle reconstruct a manifold with metric from quantum measurements (the spectrum and some position space observables), so one can work without this assumption.} For example, this is true for D0-branes in weakly coupled string theory, as their mass goes as $1/g_s$. One can then (in principle) define any term in the $g_s$ expansion this way.

Both on general grounds \cite{6} and in examples \cite{6}, the moduli space metric seen by a D-brane probe gets $\alpha'$ corrections, and for a finite size Calabi-Yau background it is not Ricci flat. The existing results are consistent with the first such correction arising from the standard $\alpha'^3 R^4$ correction to supergravity \cite{7, 8}, but pushing this to higher orders seems difficult.

Perhaps this problem becomes simpler in a black hole background. Rather than the D0, the probe brane we will use is a D2 or M2-brane wrapped on the black hole horizon. As discussed in the works \cite{6, 10, 11, 12}, such a brane, and D0-branes as well, in a near horizon BPS black hole background can preserve $SU(1, 1|2)$ superconformal invariance. This is a symmetry of the $AdS_2 \times S^2$ near horizon geometry and thus this is as expected if multi-D0 quantum mechanics can be used as a dual gauge theory of the black hole. In these works, this quantum mechanics was argued to factorize into a space-time part, and an internal (Calabi-Yau) part; this second part describes motion of the probe in the Calabi-Yau and can be used to define a probe metric.
Given this system and its relation to the black hole, we will give a physical argument, based on the idea that a black hole must have “maximal entropy” no matter how this is defined, that suggests that the probe metric in such a black hole background is in fact the “balanced metric” introduced and studied in the mathematics literature \cite{13, 14, 15}. As we explain, this is a condition which determines a unique Kähler metric, in a way which \textit{a priori} is unrelated to the Ricci flatness condition. Nevertheless, one can show that in a certain large charge scaling limit $k \to \infty$, the balanced metric approaches the Ricci flat metric, with computable corrections in inverse powers in $k$.

Since our physical argument for the balanced metric does not assume the equations of motion, it illustrates a way to derive equations of motion from a maximum entropy principle. As pointed out to us by Vijay Balasubramanian, this idea was suggested some time ago by Jacobson \cite{16, 17} and might have more general application.

2. BPS black holes and probes

Let us consider a BPS black hole solution in IIa theory compactified on a Calabi-Yau manifold $X$. Such a solution is characterized by discrete and continuous parameters. The discrete parameters are its electric and magnetic charges, which we take to be those of a system of D0, D2 and D4-branes. The continuous parameters are the values of the hypermultiplet moduli, namely the dilaton, complex structure moduli and their $N = 2$ supersymmetry partners. The vector multiplet (Kähler) moduli are determined by the attractor mechanism, as we review shortly.

As is well known, by varying the dilaton to strong coupling, this theory is continuously connected to M theory compactified on $X \times S^1$. In this theory, the black hole can be thought of as a black string wrapped on $S^1$, and carrying $S^1$ momentum \cite{18}. It will eventually turn out that our conjecture appears more natural in M theory, so let us start from that limit. To get a black string, we can wrap M5-branes on a four-cycle $[P] \in H_4(X, \mathbb{Z})$. By Poincaré duality $[P]$ can also be thought of as a class $p^A \omega_A$ in $H^2(X, \mathbb{Z})$, where we introduce a basis $\omega_A$ of $H_2(X, \mathbb{Z})$. In general, there are also electric charges $q_A$, corresponding to M2-branes wrapping dual two-cycles. We will set these to zero in the subsequent discussion.

According to the attractor mechanism, the Kähler class $J_5$ of the CY at the horizon of the black string is determined in terms of the charges $p^A$. Unlike in $d = 4$, in the black string solution, the volume $V$ of the CY is a free parameter; thus we have (using 11d conventions of \cite{19})

\[
\frac{J_5}{V^{1/3}} = \frac{p^A \omega_A}{D^{1/3}},
\]

where $D \equiv D_{ABC} p^A p^B p^C$ and $D_{ABC}$ are the triple intersection numbers on the Calabi-Yau. We recall that $V = D_{ABC} J^A J^B J^C / 6$ where $J = J^A \omega_A$. It will also be useful to define

\[
J_{\text{CY}} \equiv \frac{p^A \omega_A}{D^{1/3}}
\]

which is independent of the overall scale of the charges. The corresponding supergravity solutions simplify in the near-horizon limit: the M theory solution approaches $\text{AdS}_3 \times S^2$.
geometry
\[ ds^2 = L^2 \left( \frac{-dt^2 + dx^2_i + d\sigma^2}{\sigma^2} + d\Omega^2_{S^2} \right) . \]  

(2.2)

with the following 4-form flux sourced by M5-branes
\[ F_4 \sim \frac{1}{L} \omega_{S^2} \wedge p^A \omega_A . \]

Now, by compactifying the black string on \( S^1 \) with the radius \( R_{10} \), we obtain a 4d black hole with additional charge \( q_0 \), corresponding to momentum along the string. At this level, the discussion is simply mapped into IIa string theory, and the charges \((q_0, q_A, p^A)\) correspond to D0, D2 and D4 brane charges. The radius \( L \) of \( S^2 \) and \( AdS_3 \) is related to the radius \( R_{10} \) of \( S^1 \) as \( L \sim R_{10} \sqrt{\frac{D}{q_0}} \) and the volume of Calabi-Yau scales as \( V \sim \alpha'^3 q_0 \sqrt{q_0/D} \). In \( d = 4 \), the overall scale of \( J \) and thus the volume \( V \) is also determined by the attractor mechanism, and Eq. (2.1) becomes
\[ J_4 = \alpha' \sqrt{q_0/D} p^A \omega_A , \]

(2.3)

where as usual \( \alpha' = l_p^3 / R_{10} \) and \( Q \) is the graviphoton charge (10d conventions correspond to those of [11]). The IIa supergravity solution in four dimensions approaches the \( AdS_2 \times S^2 \) near horizon geometry.

The metric on the CY \( X \) is determined by the Einstein equations,
\[ R_{ij} - \frac{1}{2} R g_{ij} \sim (F^2)_{ij} + \delta + \mathcal{O}(l_p) \]  

(2.4)

where \((F^2)_{ij}\) schematically denotes the contribution to stress-energy density due to the RR fields of the charged black hole, and \( \delta \) denotes the contribution from brane sources [20]. The last term on the rhs represents string or M-theoretic corrections to this equation [8, 21].

In general, this metric depends on details of the state of the black hole. For example, if there are localized brane sources, the metric will depend on their location. However, we can simplify it by considering an appropriate state of the black hole. For general \( X \), despite the non-zero rhs in Eq. (2.4), there exists a black hole state in which the sources are averaged in an analogous way, removing the localized terms, and leading to a Ricci flat metric on \( X \) in the supergravity limit.

At first it may seem that the field strength \( F \) would lead to a complicated position-dependent source; say concentrated on the cycles wrapped by the M5-branes. However, as observed in [12], the combination of the attractor mechanism and the equations of motion force the field strength to be proportional to the Kähler form on \( X \),
\[ F = dA = p^A \omega_A = QJ , \]

(2.5)

where \( Q \) is the graviphoton charge \( Q \sim \sqrt{D/q_0} \). This is true before adding \( \alpha' \) or \( g_s \) corrections, and all the simple candidate corrections one can write down (such as powers of \( F \) and its derivatives, for example those which appear in the MMMS equation [22]),
preserve Eq. (2.5). While there are still $F^2$ sources in Eq. (2.4), one can now check that these are canceled by terms coming from the space-time dependence of radius $R_{10}$ of 11th dimension (in M theory) or the dilaton (in IIA). We omit the explicit check, instead pointing out that since the sources are constructed only from the metric tensor on $X$, they can at most add a cosmological constant term in Eq. (2.4), leading to a Kähler-Einstein metric. However, standard arguments imply that a solution exists only if this term is proportional to $c_1(X)$, which is zero for a CY.

We have thus defined a preferred state of the black hole, for which the metric on $X$ would be Ricci flat in the supergravity limit. To study corrections, we need to define this preferred state in a way which does not assume that we know the correction equations of motion or their solution in advance. Since the supergravity argument required us to average over all of the internal structure of the black hole, it is natural to define it as a mixed state of maximal entropy, as we will do below.

3. The probe theory

Having defined the state of the black hole under consideration, we now proceed to define the metric with $l_p$ (11d Planck length) or $\alpha'$ corrections. As we discussed earlier, this can be made precise by introducing a probe, whose moduli space (space of zero energy configurations) includes $X$. This typically requires that the probe preserves some supersymmetry.

Now, the BPS black hole preserves an $N = 1$ supersymmetry, determined by the phase of its central charge $Z$, which is determined by the charges and attractor moduli. In asymptotically Minkowski space-time, introducing another BPS brane will typically break all of the supersymmetry. However, it was shown in [9] that in the near-horizon limit, a probe zero brane can nevertheless preserve space-time supersymmetry, if it follows its "charged geodesic" (i.e. trajectory determined by the background metric and RR field). Even a collection of such branes with misaligned charges can preserve supersymmetry; consistent with this, the combined gravitational and RR potential energy of such a collection is additive.

Choosing a probe brane which preserves supersymmetry, one expects its configuration space to be some moduli space associated with the compactification space $X$. In the simplest example of a D0-brane, the moduli space is $X$ itself. Another example for which the moduli space is $X$ is a D2-brane wrapped around the $S^2$ horizon. Other choices, for example a D$p$-brane wrapped on a $p$-cycle of $X$, would lead to different moduli spaces, related to the geometry of $X$.

While there will be a probe world-volume potential, this is determined by superconformal invariance [10] to be a function of the radius $\sigma$, but independent of the other coordinates. In particular, it is independent of position on $X$ or other "internal" coordinates. Thus the world-volume theory includes a supersymmetric quantum mechanics on the moduli space $X$. Ground states of this quantum mechanics will correspond in the usual way to differential forms on $X$.

The argument we are about to make is clearest for the case of a probe D2 wrapping the $S^2$ horizon, so let us consider that. According to [10], the supersymmetry condition for
such a brane forces it to the center of $AdS_2$ (in global coordinates), so there are no other moduli on which the probe metric on $X$ can depend.

At leading order, the probe will see both the metric on $X$, and a magnetic field on $X$. The latter follows (in IIA language) from the D4 charge of the black hole: a probe D2 wrapping the horizon will see a background magnetic field on the CY $X$ $[1, 12]$,

$$F_{CY} = \int_{S^2} F(4) = p^A \omega_A.$$  

Mathematically, such a magnetic field defines a line bundle $\mathcal{L}$ over $X$; whose first Chern class is the D4 charge $p^A$. From Eq. (2.3), the Kähler class $J$ is proportional to the first Chern class of $\mathcal{L}$,

$$J = \frac{1}{Q} c_1(\mathcal{L}).$$

4. Maximal entropy argument for the probe metric

Our argument will be based on two assumptions. First, the most symmetric state of a BPS black hole, and thus the state corresponding to the simplest metric on $X$, is a state of maximal entropy. Second, that there is a sense in which the black hole can be regarded as made up of constituents with “the same dynamics” as the probe. We will use this to argue that the probe should also be in a state of maximal entropy to get a simple result.

The first assumption is very natural and straightforward to explain. To define “maximal entropy,” we look at the Hilbert space of BPS states of the black hole, call this $\mathcal{H}_{BH}$. By standard arguments going back to $[1]$, these are BPS states of the quantum system describing the black hole, here a bound state of D0 and D4 branes. Let us denote an orthonormal basis of $\mathcal{H}_{BH}$ as $|h_\alpha\rangle$. Now, the states $|h_\alpha\rangle$ are pure states in the usual sense of quantum mechanics. The maximal entropy state of such a system is a mixed state, described by the density matrix

$$P_{BH} = \frac{1}{\dim \mathcal{H}_{BH}} \sum_\alpha |h_\alpha\rangle \langle h_\alpha|,$$  

(4.1)

in which each pure state appears with equal probability. Thus, we have a clear definition of “maximal entropy” of the black hole.

The original description of the black hole Hilbert space $\mathcal{H}_{BH}$ $[23]$ was in terms of a postulated bound state of D0-branes at each triple intersection of D4-brane on the Calabi-Yau. Denoting the number of triple intersections as $k$, one finds that the supergravity entropy formula can be matched if there is one D0 bound state for each value $n$ of eleven-dimensional momentum, with 4 bosonic and 4 fermionic degrees of freedom, leading to a partition function

$$Z_{BH} = \prod_{i=1}^{k} \prod_{n \geq 1} \left( \frac{1 + q^n}{1 - q^n} \right)^4$$  

(4.2)

A later argument to the same effect $[18]$ proceeds by lifting the black hole to M theory on $X \times S^1$, in which it becomes a wrapped M5-brane. First compactifying on $X$, a
wrapped five-brane on a 4-cycle (or divisor) \( D \) becomes a black string. The string is then compactified on \( S^1 \) to obtain the black hole.

In this analysis, the string has world-sheet fields parameterizing the moduli space of degree \( N \) hypersurfaces \( P_N \), which is precisely the projectivization of the space of sections of \( \mathcal{L} \). The resulting black string Hilbert space is that of a symmetrized orbifold \( \text{Sym}^M(\mathcal{M}(P_N)) \) of the moduli space, constructed as

\[
\prod_{i=1}^k \alpha^{A_i}_{\sum n_i}|0\rangle
\]  

with \( \sum n_i = M \). Along with these moduli are additional fields (the dimensional reduction of the fivebrane two-form, and fermions), combining into \((0, 4)\) supersymmetry multiplets with \(4 + 4\) components. Finally, using the standard result for the density of states of a conformal field theory with central charge \( c = 6k \), the entropy is

\[
S = 2\pi \sqrt{\frac{c \cdot q_0}{6}}.
\]

Another, more mathematical way to derive the multiplicity \( 4 + 4 \) for each modulus, is to observe that the BPS states of \((0, 4)\) supersymmetric quantum mechanics include all \((p, q)\) forms taking values in the target space (here the moduli space of divisors),

\[
\mathcal{H} = \oplus_{0 \leq p, q \leq 3} H^p(X, \Omega^q \otimes \mathcal{L}).
\]  

Since the divisor is ample, these vanish for \( p > 0 \), while the \( q = 0, 1, 2, 3 \) terms have multiplicities \( k, 3k, 3k, k \) (for large \( D \)). The even and odd \( q \)-forms then give rise to bosonic and fermionic moduli (respectively), whose quantization reproduces Eq. (4.2) or Eq. (4.3).

More recently, a related but not obviously identical description of the black hole Hilbert space has been developed, motivated by the idea that the black hole should be described by a superconformal matrix quantum mechanics of \( n \) D0-branes in the D4 background. \[11, 12\] In this picture, the basic object is a bound state of \( n \) D0-branes which can be thought of as a “fuzzy D2-brane,” which arises from the matrix D0 theory by a Myers-type effect \[24\]. The general form of Eq. (4.2) then arises by summing over all partitions of the total D0 charge \( q_0 \).

Note that this second description is in terms of a supersymmetric quantum mechanics with target space the Calabi-Yau manifold \( X \), very much like our probe theory. Indeed, the background RR field is postulated to appear as a non-trivial \( U(1) \) magnetic field, of topological type exactly that of the bundle \( \mathcal{L} \).

A strategy to get the D0 matrix quantum mechanics on this background, pursued in \[11\], is to consider a D2-brane wrapped on the black hole horizon, an \( S^2 \). As is familiar (for example) in M(atrix) theory \[25, 26\], D0 matrix quantum mechanics contains bound configurations of \( N \) D0’s which represent a wrapped or stretched D2. If we can reverse this identification, we can derive the matrix quantum mechanics from the D2 theory.

Of course, the D2 theory in this background is precisely the probe theory we discussed in section \[3\]. Its full low-energy hamiltonian was found in \[11\]. It factorizes into an \( AdS_2 \)
part and a $CY$ part, with the latter being

$$H_{CY} = g^{a\bar{a}}(P_a - A_a)(P_{\bar{a}} - A_{\bar{a}}).$$

(4.5)

Here the metric $g$ is built from the Kähler form $J$ and the gauge field has the field strength proportional to $J$ as in Eq. (2.5). The general idea is then that, by promoting this quantum mechanics to matrix quantum mechanics, one would obtain a description of the black hole.

This brings us to our second assumption, that there is a sense in which the black hole can be regarded as made up of constituents with “the same dynamics” as the probe. If we grant the second description of the black hole, in terms of D0 matrix quantum mechanics, then clearly we can identify constituents with the same dynamics as our probe. As we mentioned, reproducing the black hole partition function Eq. (4.2) requires summing over configurations each labelled by a partition $\{n_i\}$ of the total D0 charge. Such a configuration is obtained by considering the matrix variables as a direct sum of blocks, each a matrix of dimension $n_i$. The dynamics of such a block is described by the $U(n_i)$ reduction of the matrix quantum mechanics, with interactions with the rest of the black hole produced by integrating out off-diagonal degrees of freedom. The supersymmetry of the combined system will cancel the relative potential between the blocks, and presumably makes the other induced interactions small.

Let us consider a sector with $n_1 = 1$, in other words containing single unbound D0. The dynamics of this D0 is approximately described by the $U(1)$ version of matrix quantum mechanics, in other words the theory discussed in [11]. The BPS Hilbert space of this theory is $\mathcal{H}$ defined in Eq. (4.4), and we see that this sits naturally in $\mathcal{H}_{BH}$.

Now, we will implement our second assumption, by deriving a natural maximal entropy state for the probe. By starting with the maximal entropy state Eq. (4.1) of the black hole in $\mathcal{H}_{BH}$ and tracing over all of the other degrees of freedom, we obtain a density matrix $P$ over the Hilbert space $\mathcal{H}$. The result will be the standard quantum state of maximal entropy for this quantum mechanics, which assigns equal probability to each state in $\mathcal{H}$, given by the expression

$$P = \frac{1}{\dim \mathcal{H}} \sum_\alpha |h_\alpha\rangle\langle h_\alpha|,$$

(4.6)

Indeed, one might regard this choice of quantum state as the natural one whatever the probe is, without calling upon any relation to the black hole. However we spell out this step as it explains how we could, given a precise D0 quantum mechanics for the black hole, compute the probe state and observables.

Now, given $P$, we can ask, what is the probability to find the D2 probe at a given point $z \in X$. This will be

$$P(z, \bar{z}) = \langle z | P | z \rangle.$$

(4.7)

In general, the D2 will have “spin” degrees of freedom as well, corresponding to the degrees $(p, q)$ of cohomology; let us fix these in the $p = q = 0$ sector.$^2$ By inserting explicit wave

$^2$We comment on this point in section [1].
functions $\psi_\alpha(z, \bar{z})$, the density matrix can be written in position space as a kernel,
\[
    P(z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{1}{\dim H^0} \sum_\alpha \psi_\alpha^*(z_1, \bar{z}_1) \psi_\alpha(z_2, \bar{z}_2),
\]
with Eq. (4.7) its values on the diagonal $z_1 = \bar{z}_1 = z_2 = \bar{z}_2$.

Note that, although the lowest Landau level wavefunctions satisfy the metric-independent linear differential equation $Dh = 0$, their normalizations depend on the metric. Thus the kernel Eq. (4.8) depends on the specific choice of metric, not just the Kähler class.

Now, since the probe has maximal entropy, one would expect that this probability does not favor any particular point in moduli space, in other words
\[
    P(z, \bar{z}) = \text{constant}.
\]
But this is not at all obvious from what we have said so far. We might regard it as a second, independent interpretation of the claim that the black hole has maximal entropy.

While from the point of view of an asymptotic observer, the first definition Eq. (4.6) of maximal entropy seems more natural, if we can only make measurements with the probe, the second definition seems more natural. Going further, to the extent that (following the arguments above) the probe can also be thought of as a constituent of the black hole, we might be able to reformulate black hole thermodynamics in terms of the second definition. In particular, the postulate that the black hole has maximal entropy, should imply that its constituents are equidistributed in moduli space. Otherwise, there would be a simple way for the system to increase its entropy.

To summarize, while not self-evident, it is an attractive hypothesis that the entropy should be maximal in both senses. Actually, the two definitions of maximal entropy are not directly in conflict. Indeed, we could compute Eq. (4.7) from the definition Eq. (4.8), and check whether they agree. But since the actual wave functions and thus Eq. (4.8) depend on the details of the probe world-volume theory, in particular the metric, we need to know the probe metric to make this check.

Turning around this logic, we can regard the conjunction of Eq. (4.6) and Eq. (4.9) as a non-trivial condition on the probe metric. In fact, this is a known condition: it implies that the probe metric is the balanced metric.

5. Balanced metrics

We begin by recalling the relation between a “physical” wave function $\psi(z, \bar{z})$, for which the inner product is
\[
    \langle \psi | \psi' \rangle = \int_M d\text{vol} \, \psi^*(z, \bar{z}) \psi'(z, \bar{z}),
\]
and a holomorphic section $s(z)$ of a line bundle. We assume that $\psi$ couples minimally to a $U(1)$ vector potential $A_a$, so that the Hamiltonian is written in terms of covariant derivatives
\[
    D_a = \frac{\partial}{\partial \bar{z}^a} + iA_a
\]
and
\[
    \bar{D}_a = \frac{\partial}{\partial \bar{z}^a} - iA_\bar{a}.
\]
Now, if $0 = F^{0,2} = \bar{\partial}\bar{A}$, there will exist a complex gauge transformation $g(z, \bar{z})$ such that

$$g^{-1} \bar{D}_u g = \bar{\partial}_u .$$

Defining

$$\psi = g \cdot s,$$ (5.4)

we convert the condition $\bar{D}\psi = 0$ to the holomorphy condition $\bar{\partial}s = 0$, at the cost of turning the inner product Eq. (5.1) into

$$\langle s|s' \rangle = \int_M d\text{vol} \ e^{-K} s^* (\bar{z}) s'(z)$$ (5.5)

with

$$e^{-K(z, \bar{z})} = |g(z, \bar{z})|^2 .$$ (5.6)

Mathematically, the factor $e^{-K}$ is referred to as a hermitian fiber metric on $\mathcal{L}$; it allows one to multiply two sections $s$ and $s'$ pointwise to get a function.

Physically, this is useful as a minimal energy (or supersymmetric) gauge field will satisfy $F^{0,2} = 0$, while ground state wave functions will satisfy $\bar{D}\psi = 0$ (the lowest Landau level, here BPS states).3

One reason to make the definition Eq. (5.6) is that the condition Eq. (2.5) becomes

$$J \propto F^{(1,1)} = \bar{\partial}\bar{\partial}K,$$

in other words $K$ is proportional to the Kähler potential on $X$.

Now, let us recall the definition of the balanced metric from [13, 14, 15] (see also [28, 29] for discussions with more physical introduction). An ample line bundle $\mathcal{L}$ over $X$ defines an embedding of $X$ into $\mathbb{P}(H^0(X, \mathcal{L}))$, the projective space parameterized by a basis of sections of $\mathcal{L}$. Let us choose a basis $s^\alpha$ of these sections, and define it by fiat to be orthonormal. Having done this, we have a Fubini-Study metric $J_{FS}$ on this projective space, with Kähler potential given by the standard expression in terms of the homogeneous coordinates $s^\alpha$,

$$K = \log \sum_\alpha |s^\alpha|^2 .$$ (5.7)

We can then pull this back to $X$ to get a Kähler metric $J_X$ on $X$ (formally with the same Kähler potential, but evaluated on the subset $X$).

As in Eq. (5.7), this metric can then be used to define an inner product on sections,

$$\langle s'|s \rangle = \int_X d\text{vol}(J_X) \frac{(s')^* s}{\sum_\alpha |s^\alpha|^2} ,$$ (5.8)

where $\text{vol}(J_X) = J^n_X/n!$ is just the usual volume form $\sqrt{g}$ for this Kähler metric.

Now, there are two senses of an “orthonormal basis of sections,” the one we introduced in the beginning by fiat, and a second one defined by Eq. (5.8). In general, these will not

3As is well known, these states can also be used to define “fuzzy” or noncommutative versions of a space. This was tried in the present context in [14]; there may be connections to [27], or to other appearances of noncommutative geometry in string and M theory.
agree. But, if they do, we refer to this as a “balanced” embedding, and the resulting metric $J_X$ as the balanced metric.

Now, going back to Eq. (4.8), this is a squared sum over a basis of orthogonal wave functions in the usual physical inner product Eq. (5.1). However, tracing through the definitions, Eq. (5.8) is the same as Eq. (5.1), just rewritten using Eq. (5.4) and the definition of the pull-back metric. Now, the balanced condition implies that this orthonormal basis, is the same as the orthonormal basis $s^\alpha$ we chose at the start.

Thus, using Eq. (5.7), we have

$$P(z, \bar{z}) = \sum_\alpha s^\alpha(z) \bar{s^\alpha(z')} \left| \sum_\alpha |s^\alpha(z)|^2 \right|_{z'=z} = 1$$

so indeed the kernel, and thus the probability distribution for the probe brane, is constant precisely for the balanced metric.

Conversely, it is a theorem that (given suitable assumptions, which hold here), the balanced metric exists and is unique [14]; thus this is the only metric satisfying both Eq. (4.9) and Eq. (2.5). Thus, granting Eq. (2.5), our physical consistency condition between the two definitions of “maximal entropy,” precisely picks out the balanced metric associated to the line bundle $\mathcal{L}$ whose first Chern class is the D4 charge.

Now, in the limit of a large charge black hole, in which the local curvatures and field strengths near the black hole become small, one would expect the probe metric to be approximately Ricci flat. To take the large charge limit, we scale up the D4 charge by a factor $k$, and take $A \to kA$. Mathematically, this corresponds to replacing the line bundle $\mathcal{L}$ by the line bundle $\mathcal{L}^k$ (defined as a tensor product).

The claim is now that, in the large $k$ limit, the balanced metric, defined by the maximal entropy property Eq. (5.9), should satisfy the supergravity equations of motion. By the discussion following Eq. (2.4), these equation imply that the metric $X$ will be Ricci flat. But a priori, the condition Eq. (5.9) has no evident connection with Ricci flatness or any other equation of motion. Thus this claim is in fact a nontrivial test of the conjecture. And it passes this test, as we explain in the next section.

Let us finally comment on the appropriate large $k$ scaling limit in terms of black holes charges. Looking at Eq. (2.1) one sees that in M-theory settings the above scaling corresponds just to rescaling of the magnetic charges $p^A \to kp^A$. In the IIA set up, Eq. (2.3) and Eq. (2.4) tell us that in addition to rescaling the magnetic charges $p \to kp$, one also has to scale $g_0 \to kg_0$, so that the curvature of line bundle $\mathcal{L}^k$ scales as $k$ times the metric. Such a scaling limit is described in [14] as the natural limit scaling up Kähler moduli.

6. Comparisons and conclusions

As it turns out, we can get precise results for the limit $k \to \infty$ of the density matrix Eq. (4.6), using an asymptotic expansion for the diagonal of the kernel Eq. (4.8) developed in [13, 14, 15, 16, 17]. We refer to [29] for a physical explanation and derivation of this expansion, and merely cite the result here:

$$P(z, \bar{z}) = \frac{k^3}{\dim H^0} \left( 1 + \frac{1}{k} R + \frac{1}{k^2} \left( \frac{1}{3} \Delta R + \frac{1}{24} (|R_{a\bar{a}b}|^2 - 4 |R_{a\bar{a}a}|^2 + 3 R^2) \right) + \ldots \right).$$

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(6.1)
where $R, R_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ are the scalar curvature, Ricci and Riemann tensors of the metric whose Kähler form is the curvature of $\mathcal{L}$, i.e. $J = F$.

The expansion (6.1) is a general statement, but combining it with Eq. (5.9) implies that, for sufficiently large $k$, the balanced metric will have constant scalar curvature, up to corrections of order $1/k$. In the case at hand with $c_1(X) = 0$, this implies Ricci flatness, and the probe metric satisfies this test at leading order. Thus the basic consistency of our conjecture with supergravity is clear.

Another variation of the conjecture is that, because the probe is a superparticle, we should also sum over spin states in the density matrix Eq. (4.8), and enforce Eq. (5.9) on the diagonal of this density matrix. In [29], we show how to define and compute the leading terms of this kernel using supersymmetric quantum mechanics. The resulting expansion is very similar. For example, the leading nontrivial term, proportional to the Ricci scalar, now comes with a coefficient $(1 - N)$, where $N$ is the number of supercharges.

In the case at hand, the probe is an $N = 2$ supersymmetric quantum mechanics. Thus incorporating the spin states leads to the same basic result, that in the large volume limit the conjecture agrees with supergravity.

However, we have not been able to identify the subleading terms in Eq. (6.1), nor those in its supersymmetric analogs, with any known physical corrections. In particular, one might expect the famous coefficient $\zeta(3)$ of the $R^4$ correction to show up in this expansion, from both the IIA string and M theory points of view. On the other hand, it is clear from the nature of the expansion Eq. (6.1) that such transcendental coefficients will not appear.

It seems possible that the limits involved kill these particular terms, but one still needs to explain the other corrections. One might speculate that these are related to the much studied higher genus superpotential terms in RR backgrounds [34, 35], but we did not find evidence for this either. Or, it could be that the problem is essentially nonperturbative from the supergravity point of view, and that these terms are seeing another regime.

One might ask if the expression Eq. (2.7) for the magnetic field could also get stringy or M-theoretic corrections, which while preserving its cohomology class, nevertheless modify Eq. (6.1). This is not possible because, as also shown in [29], the degeneracy of the lowest Landau level for the Hamiltonian Eq. (4.7) requires the gauge connection to satisfy the hermitian Yang-Mills equation, which for a $U(1)$ gauge field implies Eq. (2.5).

Another possibility is that additional couplings on the probe world-volume are important. We assumed that the probe can be described purely in terms of a particle in a background metric and magnetic field, and then derived the balanced metric from the maximal entropy condition. Of course, there could be higher derivative terms, perhaps induced by interactions with the other constituents of the black hole. These might modify the wave functions so as to achieve Eq. (5.9) with a different metric.

Of course, the maximal entropy assumptions might not hold for these black holes. We nevertheless feel that our argument is making an important point. The assumptions do seem very natural in the context of this problem. Indeed, if we had a precise definition of the D0 matrix quantum mechanics suggested in [11, 12], we could in principle use it to compute the probe metric, and find out where the contradiction arises. Indeed, understanding this point might be a useful hint to how this (still mysterious) quantum mechanics works.
Furthermore our argument is very simple, indeed far simpler than the supergravity or topological string considerations one might compare it with. We believe it will find applications regardless of the fate of this conjecture.

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