Transversal and cotransversal matroids via the Lindström lemma.

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Abstract

It is known that the duals of transversal matroids are precisely the strict gammoids. The purpose of this short note is to show how the Lindström-Gessel-Viennot lemma gives a simple proof of this result.

1

Matroids and duality. A matroid $M = (E, B)$ is a finite set $E$, together with a non-empty collection $B$ of subsets of $E$, called the bases of $M$, which satisfy the following axiom: If $B_1, B_2$ are bases and $e$ is in $B_1 - B_2$, there exists $f$ in $B_2 - B_1$ such that $B_1 - e \cup f$ is a basis.

If $M = (E, B)$ is a matroid, then $B^* = \{ E - B \mid B \in B \}$ is also the collection of bases of a matroid $M^* = (E, B^*)$, called the dual of $M$.

Representable matroids. Matroids can be thought of as a combinatorial abstraction of linear independence. If $V$ is a set of vectors in $\mathbb{R}^n$ and $B$ is the collection of maximal linearly independent sets of $V$, then $M = (V, B)$ is a matroid. Such a matroid is called representable over $\mathbb{R}$, and $V$ is called a representation of $M$.

Transversal matroids. Let $A_1, \ldots, A_r$ be subsets of $[n] = \{1, \ldots, n\}$. A transversal (also known as system of distinct representatives) of $(A_1, \ldots, A_r)$ is a subset $\{e_1, \ldots, e_r\}$ of $[n]$ such that $e_i$ is in $A_i$ for each $i$. The transversals of $(A_1, \ldots, A_r)$ are the bases of a matroid on $[n]$. Such a matroid is called a transversal matroid, and $(A_1, \ldots, A_r)$ is called a presentation of the matroid. This presentation can be encoded in the bipartite graph $H$ with “left” vertex set $L = [n]$, “right” vertex set $R = \{\hat{1}, \ldots, \hat{r}\}$, and an edge joining $j$ and $\hat{i}$ whenever $j$ is in $A_i$.

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transversals are the $r$-sets in $L$ which can be matched to $R$. We will denote this transversal matroid by $M[H]$.

**Strict gammoids.** Let $G$ be a directed graph with vertex set $[n]$, and let $A = \{v_1, \ldots, v_r\}$ be a subset of $[n]$. We say that an $r$-subset $B$ of $[n]$ *can be linked to* $A$ if there exist $r$ vertex-disjoint directed paths whose initial vertex is in $B$ and whose final vertex is in $A$. We will call these $r$ paths a *routing* from $B$ to $A$. The collection of $r$-subsets which can be linked to $A$ are the bases of a matroid denoted $L(G, A)$. Such a matroid is called a *strict gammoid*.

We can assume that the vertices in $A$ are sinks of $G$; *i.e.*, that there are no edges coming out of them. This is because the removal of those edges does not affect the matroid $L(G, A)$.

2

**Representations of transversal matroids.** Consider a collection of algebraically independent $\alpha_{ij}$s for $1 \leq i \leq r, 1 \leq j \leq n$. Let $M$ be a transversal matroid on the set $[n]$ with presentation $(A_1, \ldots, A_r)$. Let $X$ be the $r \times n$ matrix whose $(i, j)$ entry is $-\alpha_{ij}$ if $j \in A_i$ and 0 otherwise. The columns of $X$ are a representation of $M$.

To see this, consider the columns $j_1, \ldots, j_r$. They are independent when their determinant is non-zero. As soon as one of the $r!$ summands in the determinant is non-zero, the determinant itself will be non-zero, by the algebraic independence of the $\alpha_{ij}$s. But the summand $\pm X_{\sigma_1 j_1} \cdots X_{\sigma_r j_r}$ (where $\sigma$ is a permutation of $[r]$) is non-zero if and only if $j_1 \in A_{\sigma_1}, \ldots, j_r \in A_{\sigma_r}$. So the determinant is non-zero if and only if $\{j_1, \ldots, j_r\}$ is a transversal. The desired result follows.

We will find it convenient to choose a transversal $j_1 \in A_1, \ldots, j_r \in A_r$ ahead of time, and normalize the rows to have $-\alpha_{iji} = 1$ for $1 \leq i \leq r$.

**Example 1.** Let $n = 6$ and $A_1 = \{1, 2, 3\}, A_2 = \{2, 4, 5\}, A_3 = \{3, 5, 6\}$. The corresponding bipartite graph $H$ is shown below.

![Bipartite Graph](image.png)
If we choose the transversal $1 \in A_1$, $2 \in A_2$, $3 \in A_3$, we obtain a representation for the transversal matroid $M[H]$, given by the columns of the following matrix:

$$X = \begin{pmatrix}
1 & -a & -b & 0 & 0 & 0 \\
0 & 1 & 0 & -c & -d & 0 \\
0 & 0 & 1 & 0 & -e & -f
\end{pmatrix}$$

**Representations of strict gammoids.** Let $M = L(G, A)$ be a strict gammoid. Say $G$ has vertex set $\{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_{n-r}\}$. Assign algebraically independent weights smaller than 1 to the edges of $G_n$. For $1 \leq i \leq n-r$ and $1 \leq j \leq n$, let $p_{ij}$ be the sum of the weights of all finite paths from vertex $i$ to vertex $j$. Let $Y$ be the $(n-r) \times n$ matrix whose $(i,j)$ entry is $p_{ji}$. The columns of $Y$ are a representation of $M$.

This is a direct consequence of the Lindström lemma or Gessel-Viennot method, which tells us that the determinant of the matrix with columns $j_1, \ldots, j_{n-r}$ is equal to the signed sum of the routings from $\{j_1, \ldots, j_{n-r}\}$ to $\{a_1, \ldots, a_{n-r}\}$. This signed sum is non-zero if and only if it is non-empty.

**Example 2.** Consider the graph $G$ shown below, where all edges point down, and the set of sinks $A = \{4, 5, 6\}$.

![Graph](image)

The representation we obtain for the strict gammoid $L(G, A)$ is given by the columns of the following matrix:

$$Y = \begin{pmatrix}
ac & c & 0 & 1 & 0 & 0 \\
ad + be & d & e & 0 & 1 & 0 \\
b & 0 & f & 0 & 0 & 1
\end{pmatrix}$$

Notice that the rowspaces of $X$ and $Y$ are orthogonally complementary in $\mathbb{R}^6$. That is, essentially, the punchline of this story.

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1 The weight of a path is defined to be the product of the weights of its edges. The sum converges since the weights are less than 1.

2 The sign is determined by the permutation that matches the starting and ending points of the paths in the routing.
Representations of dual matroids. If a rank $r$ matroid $M$ is represented by the columns of an $r \times n$ matrix $A$, we can think of $M$ as being represented by the $r$-dimensional subspace $V = \text{rowspace}(A)$ in $\mathbb{R}^n$. The reason is that, if we consider any other $r \times n$ matrix $A'$ with $V = \text{rowspace}(A')$, the columns of $A'$ also represent $M$.

This point of view is very amenable to matroid duality. If $M$ is represented by the $r$-dimensional subspace $V$ of $\mathbb{R}^n$, then the dual matroid $M^*$ is represented by the $(n-r)$-dimensional orthogonal complement $V^*$ of $\mathbb{R}^n$.

Digraphs with sinks and bipartite graphs with complete matchings. From a directed graph $G$ on the set $[n]$ and a set of $n-r$ sinks $A \subseteq [n]$ of $G$, we can construct a bipartite graph $H$ as follows. The left vertex set is $[n]$, and the right vertex set is a copy $\hat{n} - \hat{A}$ of $n - A$. We join $\hat{u}$ and $u$ for each $u \in [n] - A$, and we join $\hat{u}$ and $v$ whenever $u \rightarrow v$ is an edge of $G$. This graph $H$ has the obvious complete matching between $\hat{u}$ and $u$. Conversely, if we are given the bipartite graph $H$ with a complete matching, it is clear how to recover $G$ and $A$.

Observe that if we start with the directed graph $G$ and sinks $A$ of Example 1, we obtain the bipartite graph $H$ of Example 2.

Duality of transversal matroids and strict gammoids. Now we show that, in the above correspondence between a graph $G$ with sinks $A$ and a bipartite graph $H$ with a complete matching, the strict gammoid $L(G, A)$ is dual to the transversal matroid $M[H]$. We have constructed a subspace of $\mathbb{R}^n$ representing each one of them, and now we will see that they are orthogonally complementary, as observed in Examples 1 and 2.

Our representation of $M[H]$ is given by the columns of the $r \times n$ matrix $X$ whose $(i, i)$ entry is 1, and whose $(i, j)$ entry is $-\alpha_{ij}$ if $i \rightarrow j$ is an edge of $G$ and 0 otherwise. Think of the $\alpha_{ij}$s as weights on the edges of $G$. A vector $y \in \mathbb{C}^n$ is in the $(n-r)$-dimensional null space of $X$ when, for each vertex $i$ of $G$,

$$y_i = \sum_{j \in N(i)} \alpha_{ij} y_j. \tag{1}$$

Here $N(i)$ denotes the set of vertices $j$ such that $i \rightarrow j$ is an edge of $G$.

As before, let $p_{ia}$ be the sum of the weights of the finite paths from $i$ to $a$ in $G$. Our representation $Y$ of $L(G, A)$ has rows $(y_1, \ldots, y_n) = (p_{1a}, \ldots, p_{na})$ (for $a \in A$). Clearly, each row of $Y$ is a solution to (1), so rowspace($Y$) $\subseteq$ nullspace($X$). But these two subspaces are $(n-r)$-dimensional, so they must be equal, as we wished to show. This completes our proof of the theorem that the strict gammoids are precisely the cotransversal matroids.
For more information on matroid theory, Oxley’s book [8] is a wonderful place to start. The representation of transversal matroids shown here is due to Mirsky and Perfect [7]. The representation of strict gammoids that we use was constructed by Mason [6] and further explained by Lindström [5]. The theorem that strict gammoids are precisely the cotransversal matroids is due to Ingleton and Piff [3]. Our proof of this result appears to be new.

This note is a small side project of [1]. While studying the geometry of flag arrangements and its implications on the Schubert calculus, we were led to study a specific family of strict gammoids which starts with Example 2. I would like to thank Sara Billey for several helpful discussions, and Laci Lovasz and Jim Oxley for help with the references.

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3It is in this context that he discovered what is now known as the Lindström lemma or Gessel-Viennot method [2]. This method was also used earlier by Karlin and MacGregor [4].