Estimation of phase diffusion rates in a condensate interferometer using the Gross–Pitaevskii equation

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Abstract
Atom interferometers using Bose–Einstein condensates are fundamentally limited by a phase diffusion process that arises from atomic interactions. Numerical solution of the Gross–Pitaevskii equation is here used to accurately calculate the diffusion rate for a Bragg interferometer. It is seen to agree with a Thomas–Fermi approximation at large atom numbers and a perturbative approximation at low atom numbers. Most experiments to date operate in the crossover region between these two regimes. Even in this case, it is found that that the diffusion time is governed in a simple way by the ratio of the healing length to the size of the condensate.

Keywords: atom interferometry, Bose–Einstein condensate, phase diffusion, coherence time

1. Introduction

Atom interferometry is a useful technique for a variety of precision measurements [1, 2]. The extension of atom interferometry to Bose–Einstein condensates offers several potential advantages, particularly the ability to create an interferometer using trapped atoms [3–10]. This can permit long interaction times while avoiding the complication of having the atoms fall in gravity. In addition, the trapping potential can shape the atomic trajectories into rings or other geometries so as to optimize the interferometer performance for a particular purpose [10, 11].

Different approaches to condensate interferometry can be considered, based on the method used to split the initial condensate into two spatially distinct wave packets. One successful method is to expose the condensate to an off-resonant standing wave laser beam with wave number k [6, 7, 12]. Bragg scattering can then generate momentum kicks of ±2ℏk, producing two packets moving in opposite directions. The light pulse is short and non-adiabatic with respect to the atomic motion. This provides a simple and robust splitting scheme that is similar to techniques often used with thermal atoms [1, 2].

A known drawback of condensate interferometry is that atoms in a trapped condensate interact relatively strongly due to their high density. These interactions can lead to degradation from phase diffusion [13–19]. Briefly, when a condensate with N atoms is optically split, the resulting packets will have a mean atom number of N/2 and Poissonian fluctuations ΔN ≈ N^{1/2}. Since the energy of an interacting packet depends on its atom number, the number fluctuations make the energy uncertain. This leads in turn to uncertainty in the quantum phase evolution and thus noise in the atom interferometer measurement. The interaction phase increases with time, so phase diffusion imposes a limit on the usable measurement time. As the ability to achieve long interaction times is one of the key advantages of a condensate interferometer, it is important to have accurate and quantitative estimates of the effect. We focus here on optical splitting methods, but we note that similar considerations apply in double-well interferometer schemes [20].

A simple estimate for the diffusion time can be obtained as follows [14]: if the initial condensate has N atoms and chemical potential μ, then a packet with N_p ≈ N/2 atoms will have an interaction energy U ≈ μN_p/N, and an interaction phase of approximately UT/ℏ after interaction time T. If N_p
varies by $N^{3/2}$, then the interaction phase will vary by $\mu TN^{1/2}/\hbar$, leading to a decoherence time $\tau \approx \hbar N^{1/2}/\mu$.

A more accurate calculation can be obtained using results by Ilo-Okeke and Zozulya (IZ) [21]. They establish that in the limit of atom number $N \gg 1$, the interferometer visibility $V$ is

$$V(T) = \exp\left(-2\xi^2N\right),$$

where $\xi(T)$ is an interaction phase per atom

$$\xi = \frac{1}{\hbar} \int_0^T g(t)dt$$

and

$$g(t) = \frac{2\pi\hbar^2a}{m} \int |\psi_p(\mathbf{r}, t)|^4 d^3r.$$

Here $T$ is the interferometer measurement time, $a$ is the s-wave scattering length, $m$ is the atomic mass, and $\psi_p$ is the wave function of a packet in the interferometer, normalized to unity.

To estimate the diffusion rate, IZ used a simple Thomas–Fermi approximation for $\psi_p$ in (3). This neglects possible effects of packet dynamics, and is valid only in the large $N$ limit. In this paper, we present higher accuracy calculations by numerically solving the Gross–Pitaevskii equation to obtain a realistic $\psi(\mathbf{r}, t)$. This provides confirmation of some assumptions in the IZ derivation and demonstrates the Thomas–Fermi result in the appropriate limit. We also illustrate how the diffusion rate approaches a simple form in the weak interaction limit. Most experiments to date have in fact operated in the crossover regime, where the diffusion rate is lower than either approximation would suggest. Our main results in figure 5 provide an accurate determination under any conditions for a trap that is not extremely asymmetric.

2. Theoretical methods

In the mean field approximation, the packet wave functions are governed by the Gross–Pitaevskii equation [22]

$$i\hbar \frac{d\psi}{dt} = -\frac{\hbar^2}{2m} \nabla^2 + V_{\text{trap}}(\mathbf{r}) + \frac{4\pi\hbar^2aN}{m} |\psi|^2 \psi.$$  (4)

Here $V_{\text{trap}}(\mathbf{r})$ represents the three dimensional trapping potential in which the packets move. We take this to be harmonic

$$V_{\text{trap}}(\mathbf{r}) = \frac{m}{2} \left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right).$$  (5)

The mean trap frequency is then $\omega = (\omega_x, \omega_y, \omega_z)^{1/2}$. We consider $^{87}\text{Rb}$ atoms with scattering length $a = 5.77$ nm.

In the interferometers considered here, the initial condensate is the equilibrium solution to (4), $\psi_0(\mathbf{r})$. The optical pulse splits this into two packets $\psi_+$. and $\psi_-$ moving at velocities $\pm 2\hbar k/m$ along $z$. As the packets move apart, they initially interact with each other. Since the interaction energy is proportional to the product of the two packet densities, the overall interaction phase will be the same for both packets, even if the atom numbers are different. The interactions between the packets do lead to a phase gradient across the packets that can be estimated as $d\varphi/dz = m\mu/(\hbar^2k)$, where $\mu$ is the initial chemical potential [23]. We neglect this effect here, as it will typically be small for a useful interferometer.

Instead we focus on the intra-packet interactions and solve (4) for a single packet $\psi_p(\mathbf{r}, t)$ with an atom number $N_p$ near $N/2$, and an initial state $\psi_0(\mathbf{r}, 0) = \psi_0(\mathbf{r})$. By using a Galilean transformation to the frame of the moving packet, we can treat $\psi_p$ as being at rest. However, since the equilibrium wave function depends on the atom number, the time evolution of $\psi_p$ remains nontrivial.

To perform the calculation, we first determine $\psi_0$ using an imaginary-time propagation technique [24], starting from a Thomas–Fermi or non-interacting wave function initial guess. The atom number is then suddenly reduced to $N_p$, and the equation is propagated forward in real time using the Strang-splitting technique outlined in [25]. We use a three-dimensional grid of 128$^3$ points, restricted to one octant by symmetry. Depending on the parameters, a simulation takes about an hour to run on a desktop computer.

Figure 1 shows a typical result for the rms packet width as a function of time. Since the packet is initially out of equilibrium, we observe large amplitude oscillations in all three dimensions. The dominant observed frequencies are the quadrupole modes [22].

We used the solution for $\psi_p$ to evaluate the IZ visibility, equation (1). In fact, the IZ calculation assumes that the packet wave functions $\psi_p$ do not depend on the precise value of $N_p$, and that the number sensitivity enters only through the phase evolution. This assumption is essential for obtaining the simple result (1). On the one hand, it is reasonable
is the initial chemical potential relative to the zero-zero.  

\[ \mu = \mu_0 - \frac{\hbar^2}{2m} \left( \frac{1}{a_0} \right)^2 \frac{N}{2} \omega_T^2 \]

\[ \omega_T = \frac{\hbar}{\sqrt{2m}} \left( \frac{1}{a_0} \right)^2 \]

\[ \xi \approx \frac{1}{2} \frac{\hbar}{\mu_0} \frac{N}{2} \omega_T \]

The Gross–Pitaevskii equation does depend on the atom number, where the Thomas–Fermi approximation gives

\[ \tau = N \frac{\hbar^2}{\sqrt{2m}} \frac{1}{\omega_T^2} \]

\[ \tau = N \frac{\hbar^2}{\sqrt{2m}} \frac{1}{\omega_T^2} \]

where \( \mu_0 \) is the initial chemical potential relative to the zero-point energy of the trap.

In general, \( \xi \) can have a more complicated time dependence since \( g(t) \) typically oscillates as the packet evolves. However, we find \( \xi(T) \) to be quite linear under all the conditions we considered, as illustrated in figure 3. This justifies the general notion of a well-defined diffusion time, where \( \tau \) is linear with \( t \).

\[ \tau = N^{-1/2} \frac{\hbar \sqrt{N}}{\omega_T^2} \frac{d\xi}{dT} \]

We obtain \( \tau \) from a linear fit of \( \xi(T) \).

3. Results

We investigated how the diffusion time depends on various parameters. Figure 4 illustrates the dependence on atom number \( N \), trap frequency \( \omega \), and trap asymmetry \( \lambda = \omega_y/\omega_x \) for a cylindrical trap with \( \omega_y \) and \( \omega_x \). The general trends agree...
with expectations: the diffusion time scales rapidly with $\omega$ since a tighter trap leads to higher atom density. For large atom numbers, $\tau$ increases slowly with $N$ owing to the interplay between the increasing density in the Thomas–Fermi wave function and the decreasing relative impact of the $N$ number fluctuations. The diffusion time also increases at low atom numbers as the interactions are reduced. As seen, the results agree well with the Thomas–Fermi approximation at larger $N$, where the approximation is expected to hold. At low $N$, the results converge to the weak-interaction result. The crossover between these regimes is governed by the healing length $\zeta = (8\pi n)^{-1/2}$, for the atomic density $n$. The Thomas–Fermi approximation is good when $\zeta$ is small compared to the wavepacket size, and interactions are weak when $\zeta$ is large. To illustrate this, figure 5 plots the ratio $\tau/\tau_{TF}$ as a function of $\zeta/L$, where $L = (L_x L_y L_z)^{1/3}$ is the Thomas–Fermi condensate size

$$L = (15Na^4)^{1/5}. \quad (12)$$

In $\zeta$ we used the peak Thomas–Fermi density, which gives $\zeta/L = (\zeta/15Na)^{2/3}$.

In both the strong and weak interaction limit, the diffusion time is independent of the trap symmetry. A weak dependence is observed in the crossover region, illustrated further by figure 6. However, for traps that do not have extreme asymmetry, the curve in figure 5 can be used to accurately determine the diffusion time for a system with any interaction strength. We expect this result to be useful for future experimental design.

4. Impact and conclusions

Table 1 shows the parameters used and coherence times obtained in several experimental implementations, including recent results from our own group. As seen from the $\zeta/L$ values, all of these experiments are in the crossover regime where the predicted coherence time is longer than either the Thomas–Fermi or weak interaction formulas would suggest.

A long phase diffusion time is of course beneficial, but as can be seen from the table, experiments to date have mostly been limited by technical noise sources. The work by Segal et al [28, 29] is a notable exception. The measurement time in that experiment is close to the predicted diffusion time, and in
fact exceeds the Thomas–Fermi result of 0.19 s. In that experiment, the interferometer did not exhibit a stable output, but fluctuated from one run to the next. This is expected when the measurement time approaches the diffusion time. However, the experiment also suffered from vibrational noise large enough to explain the observed behavior, so it was not possible to conclusively attribute the noise to phase diffusion [29].

Our results do indicate how much performance improvement is possible. For instance, a system similar to our current one could have a 10 s coherence time. If configured in a loop geometry with a 2 cm radius, a single shot-noise limited measurement would have a Sagnac rotation sensitivity of about $3 \times 10^{-9}$ rad s$^{-1}$.

In conclusion, we have applied IZ’s analysis of phase diffusion to a numerical analysis of the effect in a full range of trapping and interaction strength parameters. The numerical solutions support the approximations used in IZ’s work, and provide what should be nearly exact predictions for the diffusion time in these systems. We find that the Thomas–Fermi and perturbative approximation are accurate in their appropriate limits, but that most experiments to date fall in an intermediate regime where the numerical results shown in figure 5 are most appropriate. We hope that these results will be of use in designing and interpreting future experiments based on condensate interferometry.

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