### 1. Introduction

Topological phases of matter have attracted broad interest over the past decades.\(^{[11-6]}\) A topological insulator displays the gapped bulk states and the gapless edge modes in the bulk gap, which can be characterized by a non-trivially topological number. The Su–Schrieffer–Heeger (SSH) model is the simplest two-band topological system initially introduced to research the polyacetylene that exhibits rich physical phenomena,\(^{[7]}\) such as fractional charge soliton excitation\(^{[8,9]}\) and nontrivial edge modes.\(^{[10]}\) Its chiral symmetry leads to nontrivial topology confirmed by a zero-non-zero winding number and the emergence of the zero-energy edge modes under open boundary conditions (OBCs).\(^{[11-13]}\)

On the other hand, Anderson localization in a disordered medium\(^{[14-17]}\) was first proposed in 1958, which has been realized in various experimental platforms such as cold atoms\(^{[18,19]}\) and microwaves cavity.\(^{[20,21]}\) Compared with traditional Anderson models, where even an infinitesimal random potential leads to localization in 1D and 2D cases, the quasi-periodic systems display distinctive localization properties. The 1D Aubry–André–Harper (AAH) model as one of the paradigmatic examples undergoes a metal–insulator transition characterized by the self-duality property when the strength of the quasi-periodic potential exceeds a finite critical value.\(^{[22-26]}\) Remarkably, the presence of the mobility edge, one of the essential concepts in disordered systems seen only in the traditional 3D Anderson model,\(^{[27]}\) has been demonstrated in various generalized AAH models even in 1D system. Some generalized AAH models displaying the mobility edges in compact analytic forms are proposed when the so-called self-dual symmetry is broken,\(^{[28-33]}\) or the system is in the form of slowly varying modulations.\(^{[34-37]}\) Recent advances in the mobility edges have been extended to the anomalous mobility edges separating the localized states from the critical ones in 1D disordered systems,\(^{[38]}\) which is different from the traditional mobility edges.

Recently, great effort has been devoted to understand the interplay of topology and disorder, which brings new perspectives, such as the topological phase transition can be introduced by the quasi-periodic disorder in the 1D SSH chain under the intercell hopping strength exceeds the intracell hopping strength,\(^{[37]}\) and topological phase with critical localization consisting of only critically localized states.\(^{[13,39,40]}\) One of the hallmark characteristics of a topological insulator is the robustness of the nontrivial edge states against weak disorder in the underlying lattice.\(^{[2,41]}\) When the amplitude of disorder is large enough, the topological features eventually disappear, accompanied by collapse of the nontrivial topological number.\(^{[42]}\) Conversely, a modest disorder added to a trivial system can lead to the emergence of protected edge modes and quantized topological charges. Such disorder-driven topological phase is named the topological Anderson insulator (TAI).\(^{[43-52]}\) The experimental observation of the disorder-induced TAI has been reported in 2D photonics,\(^{[53,54]}\) 1D engineering synthetic 1D chiral symmetric wires with a precisely controllable random disorder.\(^{[55]}\) Recent advances of the TAI induced by a random disorder have been extended to non-Hermitian systems\(^{[56,57]}\) and spin-orbit coupled superconductors.\(^{[58,59]}\) On the other hand, the TAI phase is also shown to appear in the SSH model with a quasi-periodic modulated intercell hopping term with all the eigenstates being localized.\(^{[60]}\)

Some interesting questions arise here: whether a slowly varying modulation can lead to a TAI phase, and what the localization properties in such cases are. To address these, we consider a toy model, which is the SSH model subjected to a slowly varying
intracellular hopping and open boundary condition, focusing on the topology and localization features in the system. The topological phase diagrams will be obtained by numerically calculating the topology and localization features in the system. The topological phase transition characterized by the emergence of zero-energy edge modes and nontrivial winding number. For $\Delta \neq 0$ and $u = 1$, $\Delta_j$ corresponds to an AA-type modulation illustrated in Figure 1c. The emergence of the TAI in the SSH model with an AA-type modulation has been reported.\cite{60} In the following, we testify the existence of the TAI in the SSH model with a slowly varying intracell modulation and discuss the corresponding localization properties. We fix $\beta = \frac{\sqrt{5}-1}{2}$ and $u = 0.7$ for our discussion.

3. Topological Phase Diagram

A TAI is characterized by the emergence of the protected edge states and the quantized topological charge induced by the addition of sufficient disorder or incommensurate modulation to a trivial band structure. To detect the TAI phase, we first utilize the open-bulk winding number to characterize the topological properties for our slowly varying SSH model. For a given modulation configuration, we can diagonalize the open-chain Hamiltonian as $H(|\psi^\alpha\rangle) = E_\alpha|\psi^\alpha\rangle$ and $H(|\tilde{\psi}^\alpha\rangle) = -E_\alpha|\tilde{\psi}^\alpha\rangle$ to obtain a pair of chiral-symmetric partners $|\psi^\alpha\rangle$ and $|\tilde{\psi}^\alpha\rangle$ with the relation $|\tilde{\psi}^\alpha\rangle = S_i|\psi^\alpha\rangle$, where the entries of $S$ are $S_{\alpha\gamma} = \delta_{ij}\sigma_{\alpha\gamma}$ with $i,j$ referring to the unit cell and $\alpha,\gamma$ to the sublattice. We introduce an open-boundary $Q$ matrix given by

$$Q = \sum_\alpha (|\psi^\alpha\rangle\langle\psi^\alpha| - |\tilde{\psi}^\alpha\rangle\langle\tilde{\psi}^\alpha|)$$

where $\sum_\alpha$ is the sum over the eigenstates in the bulk spectrum without the edge modes. The open-bulk winding number in real space is defined as\cite{11}

$$W_c = \frac{1}{2L'} Tr'[S_{ij} Q_{ij}]$$

with the slowly varying parameter $0 < u < 1$. The result indicates the difference of $\Delta_j$ tends to 0 when $j$ is large enough as shown in Figure 1b, which corresponds to a slowly varying modulation.\cite{34,35} For $\Delta = 0$ or $u = 0$, the system reduces to a standard SSH model\cite{7} with a uniform intracell hopping amplitude. When the intercell hopping strength exceeds the intracell hopping strength, system undergoes a topological phase transition characterized by the emergence of zero-energy edge modes and nontrivial winding number. For $\Delta \neq 0$ and $u = 1$, $\Delta_j$ corresponds to an AA-type modulation illustrated in Figure 1c. The emergence of the TAI in the SSH model with an AA-type modulation has been reported.\cite{60} In the following, we testify the existence of the TAI in the SSH model with a slowly varying intracell modulation and discuss the corresponding localization properties. We fix $\beta = \frac{\sqrt{5}-1}{2}$ and $u = 0.7$ for our discussion.

3. Topological Phase Diagram

A TAI is characterized by the emergence of the protected edge states and the quantized topological charge induced by the addition of sufficient disorder or incommensurate modulation to a trivial band structure. To detect the TAI phase, we first utilize the open-bulk winding number to characterize the topological properties for our slowly varying SSH model. For a given modulation configuration, we can diagonalize the open-chain Hamiltonian as $H(|\psi^\alpha\rangle) = E_\alpha|\psi^\alpha\rangle$ and $H(|\tilde{\psi}^\alpha\rangle) = -E_\alpha|\tilde{\psi}^\alpha\rangle$ to obtain a pair of chiral-symmetric partners $|\psi^\alpha\rangle$ and $|\tilde{\psi}^\alpha\rangle$ with the relation $|\tilde{\psi}^\alpha\rangle = S_i|\psi^\alpha\rangle$, where the entries of $S$ are $S_{\alpha\gamma} = \delta_{ij}\sigma_{\alpha\gamma}$ with $i,j$ referring to the unit cell and $\alpha,\gamma$ to the sublattice. We introduce an open-boundary $Q$ matrix given by

$$Q = \sum_\alpha (|\psi^\alpha\rangle\langle\psi^\alpha| - |\tilde{\psi}^\alpha\rangle\langle\tilde{\psi}^\alpha|)$$

where $\sum_\alpha$ is the sum over the eigenstates in the bulk spectrum without the edge modes. The open-bulk winding number in real space is defined as\cite{11}

$$W_c = \frac{1}{2L'} Tr'[S_{ij} Q_{ij}]$$

with the slowly varying parameter $0 < u < 1$. The result indicates the difference of $\Delta_j$ tends to 0 when $j$ is large enough as shown in Figure 1b, which corresponds to a slowly varying modulation.\cite{34,35} For $\Delta = 0$ or $u = 0$, the system reduces to a standard SSH model\cite{7} with a uniform intracell hopping amplitude. When the intercell hopping strength exceeds the intracell hopping strength, system undergoes a topological phase transition characterized by the emergence of zero-energy edge modes and nontrivial winding number. For $\Delta \neq 0$ and $u = 1$, $\Delta_j$ corresponds to an AA-type modulation illustrated in Figure 1c. The emergence of the TAI in the SSH model with an AA-type modulation has been reported.\cite{60} In the following, we testify the existence of the TAI in the SSH model with a slowly varying intracell modulation and discuss the corresponding localization properties. We fix $\beta = \frac{\sqrt{5}-1}{2}$ and $u = 0.7$ for our discussion.
system deviated from periodicity, and $W_i$ is quantized to an integer, while a modest size is enough in the practical calculation. We also define the disorder-averaged winding number $\overline{W}$ as $1/N_c \sum_{j=1}^{N_c} W_i$ with the configuration number $N_c$. Here, we take $N_c = 100$ disorder realizations for different $\phi_0 = [0, 2\pi]$ throughout the work.

**Figure 2.** (a) Disorder-averaged winding number $\overline{W}$ as a function of the modulation strength $\Delta$ and the amplitude of the intracell hopping $v$ with $L = 300$ and $N_c = 100$ disorder realizations. The colorbar shows the value of disorder-averaged winding number $\overline{W}$. Two red dot lines represent the analytic critical lines Equation (11) for the divergence of localization length $\lambda$. (b) $\lambda^{-1}$ as a function of $\Delta$ and $v$ for $L = 600$ and $N_c = 100$ disorder realizations. The colorbar shows the value of $\lambda^{-1}$. (c, d) Two disorder-averaged energies $\overline{E}_L$ and $\overline{E}_{L+1}$ in the center of the spectrum, and the disorder-averaged winding number $\overline{W}$ as a function of the modulation strength $\Delta$ under OBCs with $v = 0.5$ (c) and $v = 1.1$ (d), respectively.

change of the nontrivial winding number in the region $\Delta \in (1, 2)$, meanwhile the disorder-averaged zero modes are also detected. When $v > 2$, the TAI phase disappears, no matter how strong the modulation amplitude is, as can be seen in Figure 2a. Our numerical results show that the slowly varying intracell modulation will evoke the TAI phase in a finite region for $v > 1$.

According to refs. [49, 55, 56, 60], the topological transition is accompanied by the divergence of the localization length of the zero modes. Hence, one can obtain the phase diagram by studying the localization length of the zero modes. For the zero modes, the Schrödinger equation of the SSH model, Equation (1), $H\psi = 0$ reads:

$$w\psi_{j+1,B} + v_{j+1}\psi_{j+1,B} = 0$$
$$v_j\psi_{j,A} + w\psi_{j+1,A} = 0$$

(7)

where $\psi_{j,A}(\psi_{j,B})$ is the probability amplitude of the zero mode on the sublattice site $A(B)$ in the $j$th lattice cell. By solving the coupled equations, one has $\psi_{j+1,A} = (-1)^j \prod_{j=1}^{n} (v_j/w_j)\psi_{j,A}$, leading to the localization length $\lambda$ of the zero modes given by:

$$\lambda^{-1} = \lim_{L \to \infty} \frac{1}{L} \ln \left| \frac{\psi_{L+1,A}}{\psi_{L,A}} \right|$$

(8)

According to Weyl’s equidistribution theorem,[61,62] we can use the ensemble average to evaluate the last expression

$$\lambda^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dq \ln |v + \Delta \cos q|$$

(9)

The integration can be performed straightforwardly as

$$\lambda^{-1} = \begin{cases} \ln \frac{\Delta}{2} & v > \Delta \\ \ln \frac{v + \Delta}{2} & v < \Delta \end{cases}$$

(10)

The divergence of this localization length $\lambda$, that is, $\lambda^{-1} \to 0$, gives the two critical lines

$$v = \Delta^2/4 + 1 \quad v > \Delta$$
$$\Delta = 2 \quad v < \Delta$$

(11)

The localized critical points match the topological phase transition points shown in Figure 2a. Numerically the value of $\lambda^{-1}$ for the $i$th eigenstate is computed by

$$\lambda^{-1} = \lim_{L \to \infty} \frac{1}{L} \ln \| T \|$$

(12)

where $\| \cdot \|$ denotes the norm of the total transfer matrix $T = \prod_{j=1}^{L} T_j T_{j-1}$ with

$$T_j = \begin{bmatrix} \frac{E_j - \Delta_{j-1}}{v_j} & -1 \\ 1 & 0 \end{bmatrix}$$

(13)
and
\[
T_i = \begin{bmatrix} \left(\frac{\varepsilon^2}{\varepsilon-i} \right) & 1 \\ i & 0 \end{bmatrix}
\]  

(14)

The numerical result of \(\lambda^{-1}\) as a function of \(\nu\) and \(\Delta\) by means of the transfer matrix method\(^{[45,63]}\) is shown in Figure 2b, and the two diverging critical lines match with the analytic solutions Equation (11) pretty well. Moreover, for \(u = 1\), the modulation is reduced to the AA-type, the topological phase diagram is the same (see Appendix A).

For \(\Delta = 0\), the system reduces to the original SSH model. When the intracell hopping strength \(v\) exceeds the intercell hopping strength \(w\), the system undergoes a topological phase transition from a topologically nontrivial phase into a trivial phase accompanied by a jump of the winding number from 1 to 0. However, when one introduces a slowly varying modulation for \(u \neq 0\) and \(\Delta \neq 0\), the topologically nontrivial phase, that is the TAI phase, emerges in the trivial regime of the original SSH model, which is shown in Figure 2a for non-zero \(\Delta\).

4. Localization Transition and Mobility Edges

A common opinion is that when the system is in the TAI phase, all states are localized, showing Anderson’s localization phenomenon. In this section, we study another aspect of the localized nature of the TAI phase, that is, the access of mobility edges to the TAI phase in the SSH model under a slowly varying incommensurate modulation. We follow the method in ref. \[64\] to derive the expression of mobility edges for our slowly varying model. Due to the chiral symmetry, the system’s spectrum is symmetric about \(E = 0\). To simplify, we only focus on the upper band \((E > 0)\), and the lower one displays identical behaviors.

The localization in the slowly varying model exhibits different features for two cases of the site-independent intracell hopping strength \(v\). As following results imply, the delocalization–localization transition is not connected to a topological phase transition in both regimes.

a) For \(\nu > 1\), the mobility edges are given by (see Appendix B for details):
\[
\varepsilon_c = \nu \pm (\Delta - 1)
\]  

(15)

Here and in Equation (17) \((\sim)\) denotes the upper(lower) mobility edge marked by the black(red) solid line shown in Figure 3a, respectively. We find that \(\Delta_c = 1\) is the localization transition point. When \(\Delta > \Delta_c\), all eigenstates become localized, and there exist mobility edges in the regime \(0 < \Delta < \Delta_c\).

According to the topological phase diagram Figure 2a, we find that the TAI phase is compatible with the regime hosting mobility edges. To characterize the mobility edges, we define the fractal dimension of the wave function as following
\[
\Gamma = -\lim_{\varepsilon \to 0} \frac{\ln \text{IPR}}{\ln (2L)}
\]  

(16)

where the inverse participation ratio \(\text{IPR} = \sum_{j=1}^{N_c} \sum_{\alpha} |\psi_j^{\alpha}|^4\). Further we can define the disorder-averaged fractal dimension \(\bar{\Gamma} = 1/N_c \sum_{\alpha} \Gamma^{\alpha}\). It is known that \(\bar{\Gamma} \to 1\) for extended states and \(\bar{\Gamma} \to 0\) for localized ones\(^{[30]}\). In Figure 3a, we show the disorder-averaged fractal dimension \(\bar{\Gamma}\) of the eigenstates as a function of the modulation strength \(\Delta\) for \(\nu = 1.1\). The solid lines represent the mobility edges defined by Equation (15), and the shaded region denotes the region with TAI. One can see that there exists an overlap between the TAI regime and the extended states regime with mobility edges. To clarify the mobility edges more clearly, Figure 3b plots the IPR as a function of the eigenenergy \(E\) for different \(\Delta\) with \(\nu = 1.1\). The magnitude of the IPR jumps from the order of magnitude of \(10^{-2}\) (a typical value for the localized states\(^{[16]}\)) to \(10^{-3}\) (a typical value scales as \(1/L\) for the extended states\(^{[30]}\), when the eigenenergy \(E\) crosses some critical values for \(\Delta = 0.3\) and 0.6. This clearly signals the transition between localized and extended states, and these critical values correspond to the positions of the mobility edges in the spectrum marked by the black dotted lines in Figure 3b defined by Equation (15). When \(\Delta = 1.2 > \Delta_c\), the IPR magnitude for all eigenstates is up to \(10^{-2}\), which indicates there is no mobility edges in the spectrum. Furthermore, the mobility edges can also be confirmed intuitively by the spatial distributions of the wave functions in Figure 3c,d. The wave functions are localized or extended when their eigenvalues satisfy \(E > \nu - (\Delta - 1)\) and \(\nu + (\Delta - 1) < E < \nu - (\Delta - 1)\), respectively.
5. Conclusion

In summary, we study the topological phase transition and localization transition of a SSH model with a slowly incommensurate modulation. We numerically and analytically obtain the topological phase diagram, and our results imply that a slowly varying incommensurate modulation can induce the TAI. Different from the random disorder induced the TAI, the mobility edges can enter into the TAI regime in our case.

Note added: on submission of this manuscript, we notice a very recent preprint arXiv:2201:00988 entitled “Topological Anderson insulators with different bulk states in quasiperiodic chains”. The authors studied a similar problem, and main results of the TAI regime with the intermediate phase were obtained. In this paper, we derive the exact expression of mobility edges in this TAI phase.

Appendix A: AA-Type Disorder

When $\nu = 1$, $\Delta_i$ is reduced to the AA-type modulation. Figure A1 shows the topological phase diagram characterized by $\overline{W}$ versus the disorder strength $\Delta$ and intracell hopping $\nu$, which is the same with the result in Figure 2a. As shown in Figure A1, the TAI phase also can be introduced by AA-type modulation and topological critical points still can be described by Equation (11), which labeled by two red dot lines, shown in Figure A1.

To figure out the localization property in the TAI phase, we display the fractal dimension $\Gamma'$ of different eigenstates as a function of the corresponding $E$ and $\Delta$ for $\nu = 1.1$ in Figure A2a. One can see that some eigenstates are localized and the rest are extended, which indicates the mobility edges exist in system. By increasing $\Delta$, we can find an intermediate region consisting of both extended and localized, and even critical eigenstates exists in spectrum. For a clear comparison, the $\Gamma'$ associated to eigenstates as a function of $\Delta$ is shown in Figure A2b. With the increase of $\Delta$, more extended states become localized, shown in Figure A2a,b. And an intermediate region with a mixture of extended, localized, and critical eigenstates emerge in $1.1 < \Delta < 2$. When the AA-type modulation is introduced in the system, the localization feature becomes very complex. So we cannot get the analytic expression of mobility edges.

**Figure 4.** a) The disorder-averaged fractal dimension $\overline{\Gamma}$ of different eigenstates as a function of the corresponding $E$ and $\Delta$ with $N_t = 100$ disorder realizations. The colorbar indicates the magnitude of $\overline{\Gamma}$. The black and red solid lines represent the mobility edges given in Equation (17). b) The distribution of the IPR as a function of the eigenenergy $E$ with different $\Delta$ with $\phi = 0$. The black dashed lines correspond to the positions of mobility edges given in Equation (17). Here, only $E > 0$ states are considered, $\beta = (\sqrt{5} - 1)/2$, $\nu = 0.3$, and $L = 600$. 

b) For $0 < \nu < 1$, the mobility edges are instead

$$E_c = 1 \pm (\nu - \Delta)$$

Equation (17)

**Figure 4** shows the disorder-averaged fractal dimension $\overline{\Gamma}$ of the eigenstates as a function of disorder-averaged eigenenergy $\overline{E}$ and modulation strengthen $\Delta$ for $\nu = 0.3$. In the regime $0 < \Delta < \nu$, as expected from the analytical results, $\overline{\Gamma}$ approximately exhibits sharp jumps from 1 to 0 for energies upper or lower lines defined by Equation (17). When $\Delta > \nu$, the mobility edges vanish, and all the states are localized. Figure 4b plots the IPR as a function of the eigenenergy for different $\Delta$ with $\nu = 0.3$, where the jumping points denote the positions of the mobility edges.

**Figure A1.** Disorder-averaged winding number $\overline{W}$ as a function of the modulation strength $\Delta$ and the amplitude of the intracell hopping $\nu$ with $L = 300$ and 100 disorder realization. Two red dot lines represent the analytic critical lines Equation (11). Other parameters: $\beta = (\sqrt{5} - 1)/2$ and $\nu = 1$.

**Figure A2.** a) Fractal dimension $\Gamma'$ of different eigenstates as a function of the corresponding eigenvalues and $\Delta$. b) The $\Gamma'$ associated to the eigenstate number indices as a function $\Delta$. Other parameters: $\beta = (\sqrt{5} - 1)/2$, $\nu = 1$, $\phi = 0$, $\nu = 1.1$, and $L = 600$.
Appendix B: Derivation of the Analytical Expression for Mobility Edges

In this Appendix, we present a detailed derivation of the expression of mobility edges Equations (15) and (17) in details following the method proposed in ref. [64]. To do this, we approximate the slowly varying modulation model with an ensemble of different periodic models $M_\phi$. The extended state regime in the spectrum of the slowly varying modulation model corresponds to the range of energy bands of these periodic models $M_\phi$. Thus, the mobility edges can be determined as the smallest overlaps of these ranges.

We first assume the slowly varying incommensurate modulation $\Delta_\nu$ may take a series of constants $\Delta_\nu \Delta_\nu \in [-\Delta, \Delta]$ in the thermodynamic limit, and the system becomes an ensemble of periodic models $M_\phi$. Under periodic boundary conditions (PBCs) with $c_{i+1} \bar{a} = c_{i}\bar{a}$, the energy spectra of $M_\phi$ can be obtained by the tight-binding approximation as follows

$$E = \pm \sqrt{(\nu + \Delta)\nu + 1 + 2(\nu + \Delta)\cos k} \quad (B1)$$

where $k \in [0, 2\pi]$ is the wave number. Here we only focus on the upper band ($E > 0$) for positive $\nu$ and $\Delta$.

The extended state regime for $M_\phi$ is just the range of the energy band $|\nu + \Delta - 1| < E < |\nu + \Delta + 1|$ (B2)

or $|\nu + \Delta + 1| < E < |\nu + \Delta - 1|$ (B3)

There exists two limit situations for $\Delta_\nu$. When $\Delta_\nu = \Delta$, we have

$$|E_1| < E < |E_2|$$

with $E_1 = \nu + \Delta - 1$ and $E_2 = \nu + \Delta + 1$. The other limit is $\Delta_\nu = -\Delta$, for that we have

$$|E_3| < E < |E_4|$$

with $E_2 = \nu - \Delta + 1$ and $E_3 = \nu - \Delta - 1$. To determine the smallest overlaps of energy bands of $M_\phi$, one shall first sort the values of $|E_1|$, $|E_2|$, and $|E_3|$. We first determine the sign of $E_1$, $E_2$, and $E_3$ in the $\nu$-$\Delta$ plane. As shown in Figure B1, the $\nu$-$\Delta$ plane is divided into four regions by three lines corresponding to $E_1 = 0$, $E_2 = 0$, and $E_3 = 0$, respectively, each labeled with sign combinations of $E_1$, $E_2$, and $E_3$. In region I, for instance, we have $|E_1| < |E_2|$ and $|E_2| < |E_3|$. When $\Delta > \nu$, $|E_1| < |E_2| < |E_3|$, and the smallest energy overlaps for the extended states is $E \in (|E_2|, |E_3|)$. Thus, the mobility edges reads

$$E_c = 1 \pm (\nu - \Delta) \quad (B7)$$

When $\Delta > \nu$, $|E_1| < |E_2| < |E_3| < |E_4|$. The smallest energy overlaps of the extended states is given by $E \in (|E_2|, |E_3|)$, and the expression of mobility edges is

$$E_c = 1 \pm (\nu - \Delta) \quad (B8)$$

For the regions II, III, and IV, the smallest energy overlaps can be obtained by the same procedure. Table B1 lists the smallest energy overlaps for different $\nu$ and $\Delta$ in four regions.

According to Table B1, there are three forms for mobility edges, that is, $E_{c1} = |E_1|$, $E_{c2} = |E_2|$, and $E_{c3} = |E_3|$. In Figure B2, we show the region of the smallest energy overlaps of the extended states in the $\nu$-$\Delta$ plane for $M_\phi$. We find that the smallest energy overlaps of the extended states change from $(E_{c2}, E_{c3})$ to $(E_{c3}, E_{c1})$, when $\Delta$ goes beyond $\nu$ in the region I. At $E_{c2} = E_{c3}$, the extended state regime shrinks to a point with $\Delta = \nu$. In the regions II and III, the localization transition occurs at $\Delta = 1$ and $\Delta = \nu$. The shaded area in Figure B2 indicates the regime of the system where the mobility edges actually exist. Outside this region, all states are kept localized. As a conclusion, when $\nu > 1$, $E_c = \nu \pm (\Delta - 1)$, and the localization transition occurs at $\Delta = 1$. And for $0 < \nu < 1$, $E_c = 1 \pm (\nu - \Delta)$, and the localization transition occurs at $\Delta = \nu$.

Table B1. The smallest energy overlaps of extended states in I, II, III, and IV regions.

| $\nu$ | $\Delta$ | $|\nu, \Delta|$ | The smallest energy overlaps |
|------|----------|----------------|----------------------------|
| I    | $\nu < 1$ | $\Delta < 1$  | $(|E_1|, |E_2|)$          |
|      | $\nu > 1$ | $\Delta > 1$  | $(|E_2|, |E_3|)$          |
| II   | $\nu < 1$ | $\Delta < 1$  | $(|E_2|, |E_3|)$          |
|      | $\nu > 1$ | $\Delta > 1$  | $(|E_2|, |E_3|)$          |
| III  | $\nu < 1$ | $\Delta > 1$  | $(|E_3|, |E_4|)$          |
|      | $\nu > 1$ | $\Delta > 1$  | $(|E_3|, |E_4|)$          |
| IV   | $\nu < 1$ | $\Delta > 1$  | $(|E_4|, |E_3|)$          |
|      | $\nu > 1$ | $\Delta > 1$  | $(|E_4|, |E_3|)$          |
Acknowledgements

This work was supported by the National Natural Science Foundation of China (12074340), the NSFC (Grants No. 11604188 and No. 12147215), Fundamental Research Program of Shanxi Province (Grant No. 20210302123442), Beijing National Laboratory for Condensed Matter Physics, and STIP of Higher Education Institutions in Shanxi under Grant No. 2019L0097. This work is also supported by NSF for Shaxi Province Grant No. 1331KSC.

Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

Research data are not shared.

Keywords

mobility edges, quasi-periodic disorder, topological Anderson insulators

Figure B2. The distribution of the smallest energy overlaps for different $v$ and $\Delta$ in different regions. The shaded area indicates the regime of the system with mobility edges.

Received: May 3, 2022
Revised: June 12, 2022
Published online: June 29, 2022

[1] D. J. Thouless, M. Kohmoto, M. P. Nightingale, M. den Nijs, Rev. Mod. Phys. 1982, 49, 405.
[2] M. Z. Hasan, C. L. Kane, Rev. Mod. Phys. 2010, 82, 3045.
[3] X.-L. Qi, S.-C. Zhang, Rev. Mod. Phys. 2011, 83, 1057.
[4] A. Bansil, H. Lin, T. Das, Rev. Mod. Phys. 2016, 88, 021004.
[5] C. K. Chiu, J. C. Y. Teo, A. P. Schnyder, S. Ryu, Rev. Mod. Phys. 2016, 88, 035005.
[6] N. P. Armitage, E. J. Mele, A. Vishwanath, Rev. Mod. Phys. 2018, 90, 015001.
[7] W. P. Su, J. R. Schrieffer, A. J. Heeger, Phys. Rev. Lett. 1979, 42, 1698.
[8] R. Jackiw, C. Rebbi, Phys. Rev. D 1976, 13, 3398.
[9] A. J. Heeger, S. Kivelson, J. R. Schrieffer, W. P. Su, Rev. Mod. Phys. 1988, 60, 781.
[10] S. Ganeshan, K. Sun, S. Das Sarma, Phys. Rev. Lett. 2013, 110, 180403.
[11] F. Song, S. Yao, Z. Wang, Phys. Rev. Lett. 2019, 123, 246801.
[12] Z. Xu, R. Zhang, S. Chen, L. Fu, Y. Zhang, Phys. Rev. A 2020, 101, 013635.
[13] T. Xiao, D. Xie, Z. Dong, T. Chen, W. Yi, B. Yan, Sci. Bull. 2021, 66, 2175.
[14] P. W. Anderson, Phys. Rev. 1958, 109, 1492.
[15] P. A. Lee, T. V. Ramakrishnan, Rev. Mod. Phys. 1985, 57, 287.
[16] D. A. Abanin, Z. Papic, Ann. Phys. (Berlin, Ger.) 2017, 529, 1700169.
[17] X. Chen, T. Zhou, D. A. Huse, E. Fradkin, Ann. Phys. (Berlin, Ger.) 2017, 529, 1600332.
[18] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer, A. Aspect, Nature 2008, 453, 891.
[19] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, M. Inguscio, Nature 2008, 453, 895.
[20] A. A. Chabanov, M. Stoytchev, A. Z. Genack, Nature 2000, 404, 850.
[21] P. Pradhan, S. Sridhar, Phys. Rev. Lett. 2000, 85, 2360.
[22] S. Aubry, G. André, Ann. Isr. Phys. Soc. 1980, 3, 133.
[23] P. G. Harper, Proc. Phys. Soc. London Sect. A 1955, 68, 874.
[24] S. Longhi, Phys. Rev. Lett. 2019, 122, 237601.
[25] S. Longhi, Phys. Rev. B 2019, 100, 125157.
[26] Y. Xing, L. Qi, X. D. Zhao, Z. Lü, S. T. Liu, S. Zhang, H. F. Wang, Ann. Phys. (Berlin, Ger.) 2021, 533, 1021007.
[27] N. Mott, J. Phys. C 1987, 20, 3075.
[28] J. Biddle, S. Das Sarma, Phys. Rev. Lett. 2010, 104, 076601.
[29] Y. Liu, X.-P. Jiang, J. Cao, S. Chen, Phys. Rev. B 2020, 101, 174205.
[30] Y. Wang, X. Xia, L. Zhang, H. Yao, S. Chen, J. You, Q. Zhou, X.-J. Liu, Phys. Rev. Lett. 2020, 125, 196604.
[31] T. Liu, H. Guo, Y. Pu, S. Longhi, Phys. Rev. B 2020, 102, 024205.
[32] Z. Xu, X. Xia, S. Chen, Sci. China: Phys., Mech. Astron. 2022, 65, 227211.
[33] Z. Xu, X. Xia, S. Chen, Phys. Rev. B 2021, 104, 224204.
[34] S. Das Sarma, S. He, X. C. Xie, Phys. Rev. Lett. 1988, 61, 2144.
[35] S. Das Sarma, S. He, X. C. Xie, Phys. Rev. B 1990, 41, 5544.
[36] T. Liu, H. Guo, Phys. Rev. B 2018, 98, 104201.
[37] T. Liu, H. Guo, Phys. Lett. A 2018, 382, 3287.
[38] T. Liu, X. Xia, S. Longhi, L. Sanchez-Palencia, SciPost Phys. 2022, 12, 027.
[39] T. Liu, X. Xia, Phys. Rev. B 2021, 104, 134202.
[40] J. Wang, X.-J. Liu, G. Xianlong, H. Hu, Phys. Rev. B 2016, 93, 104504.
[41] E. Prodan, T. L. Hughes, B. A. Bernevig, Phys. Rev. Lett. 2010, 105, 115501.
[42] X. Cai, L.-J. Lang, S. Chen, Y. Wang, Phys. Rev. Lett. 2013, 110, 176403.
[43] J. Li, R.-L. Chu, J. K. Jain, S.-Q. Shen, Phys. Rev. Lett. 2009, 102, 136806.
[44] C. W. Groth, M. Wimmer, A. R. Akhmerov, J. Tworzydlo, C. W. J. Beenakker, Phys. Rev. Lett. 2009, 103, 196805.
[45] H.-M. Guo, G. Rosenberg, C. Refael, M. Franz, Phys. Rev. Lett. 2010, 105, 216601.
[46] Y.-Y. Zhang, R.-L. Chu, F.-C. Zhang, S.-Q. Shen, Phys. Rev. B 2012, 85, 035107.
[47] J. Song, H. Liu, H. Jiang, Q.-F. Sun, X. C. Xie, Phys. Rev. B 2012, 85, 195125.
[48] A. Girschik, F. Libisch, S. Rotter, Phys. Rev. B 2013, 88, 014201.
[49] I. Mondragon-Shem, T. L. Hughes, J. Song, E. Prodan, Phys. Rev. Lett. 2014, 113, 046802.
[50] Z. Q. Zhang, B. L. Wu, J. Song, H. Jiang, Phys. Rev. B 2019, 100, 184202.
[51] S. Velury, B. Bradlyn, T. L. Hughes, Phys. Rev. B 2021, 103, 024205.
[52] G.-Q. Zhang, L.-Z. Tang, L.-F. Zhang, D.-W. Zhang, S.-L. Zhu, Phys. Rev. B 2021, 104, L161118.
[53] S. Stützer, Y. Plotnik, Y. Lumer, P. Titum, N. H. Lindner, M. Segev, M. C. Rechtsman, A. Szameit, Nature 2018, 560, 461.
[54] G. G. Liu, Y. Yang, X. Ren, H. Xue, X. Lin, Y. H. Hu, H. X. Sun, B. Peng, P. Zhou, Y. Chong, B. Zhang, Phys. Rev. Lett. 2020, 125, 133603.
[55] E. J. Meier, F. A. An, A. Dauphin, M. Maffei, P. Massignan, T. L. Hughes, B. Gadway, Science 2018, 362, 929.
[56] D.-W. Zhang, L.-Z. Tang, L.-J. Lang, H. Yan, S.-L. Zhu, Sci. China: Phys., Mech. Astron. 2020, 63, 267062.
[57] L.-Z. Tang, L.-F. Zhang, G.-Q. Zhang, D.-W. Zhang, Phys. Rev. A 2020, 101, 063612.
[58] J. Borchmann, A. Farrell, T. Pereg-Barnea, Phys. Rev. B 2016, 93, 125133.
[59] C.-B. Hua, R. Chen, D.-H. Xu, B. Zhou, Phys. Rev. B 2019, 100, 205302.
[60] S. Longhi, Opt. Lett. 2020, 45, 4036.
[61] H. Weyl, Math. Ann. 1916, 77, 313.
[62] G. H. Choe, Proc. R. Ir. Acad., Sect. A 1993, 93A, 193.
[63] A. MacKinnon, B. Kramer, Z. Phys. B: Condens. Matter 1983, 53, 1.
[64] Q. Tang, Y. He, J. Phys.: Condens. Matter 2021, 33, 185505.
[65] L.-Z. Tang, S.-N. Liu, G.-Q. Zhang, D.-W. Zhang, arXiv:2201.00988, 2022.