MEROMORPHIC CONTINUATION OF THE MEAN SIGNATURE OF FRACTIONAL BROWNIAN MOTION

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Abstract. It is proved that the mean signature of multi-dimensional fractional brownian motion admits a meromorphic continuation in the hurst parameter to the entire complex plane. Each constituent mean iterated integral is a sum of hypergeometric integrals indexed by the pair partitions which refine the partition arising from the sequential list of integrands which defines it. Furthermore, each such hypergeometric integral is holomorphic in the complement of a finite union of rational progressions determined by the combinatorial structure of the pair partition which defines it. It is not proved that these singularities actually exist, it is only proved that the singularities are of finite order and they can only occur in the specified discrete set of rational numbers.

INTRODUCTION

Let \( B = (B^1, \ldots, B^d) \) be a \( d \)-dimensional fractional brownian motion with hurst parameter \( H \in (0, 1) \) and let \( \{V_1, \ldots, V_d\} \) be a list of \( d \) smooth vector fields on \( \mathbb{R}^n \), bounded and having bounded derivatives of all orders. For fixed \( H > 1/2 \) and \( x \in \mathbb{R}^n \), let \( X^x \) denote the \( \mathbb{R}^n \)-valued stochastic process defined by

\[
X^x_t = x + \sum_{i=1}^d \int_0^t V_i(X^x_s)dB^i_s.
\]

Here the vector field \( V_i \) is viewed as a mapping \( V_i : \mathbb{R}^n \to \mathbb{R}^n \). Since \( H > 1/2 \) is assumed, (0.1) can be defined using Young’s theory of integration \[ FV10, Yon36 \].

Associated with the fractional brownian motion \( B \), and the vector fields \( \{V_1, \ldots, V_d\} \), one has the expectation operator \( P^x_t : C^\infty(\mathbb{R}^n; \mathbb{C}) \to \mathbb{C} \) given by \( P^x_t f = E[f(X^x_t)] \). In [BC97], Baudoin and Coutin proved that there is a family \( \{\Gamma_k\} \) of differential operators on \( \mathbb{R}^n \) such that for each \( N \),

\[
P^x_t f = \sum_{k=0}^N t^{2kH}\Gamma_k f(x) + o(t^{(2N+1)H})
\]

as \( t \downarrow 0 \). Moreover, they proved that

\[
\Gamma_k = \sum_{i_1, \ldots, i_{2k} \in \{1, \ldots, d\}} E \left[ \int_{\Delta^2[0,1]} dB_{i_1}^{i_1} \cdots dB_{i_{2k}}^{i_{2k}} \right] V_{i_1} \cdots V_{i_{2k}}
\]

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and

\[
E \left[ \int_{\Delta^{2k}|[0,1]} dB_{t_1}^{i_1} \cdots dB_{t_{2k}}^{j_{2k}} \right] = \frac{H^k(2H-1)^k}{k!^{2k}} 
\]

(0.3) \[ \times \sum_{\sigma \in \mathcal{S}_{2k}} \int_{\Delta^{2k}|[0,1]} \prod_{l=1}^{k} \delta_{i_{\sigma(2l-1)}, i_{\sigma(2l)}} |s_{\sigma(2l)} - s_{\sigma(2l-1)}|^{2H-2} \, ds_1 \land \ldots \land ds_{2k}.
\]

where \( \Delta^{2k}|[0,1] = \{ t \in \mathbb{R}^{2k} : 0 < t_1 < \ldots < t_{2k} < 1 \} \) is the standard increasing simplex. To clarify this expression, observe that the integrals which define the summands are encoded by pair partitions of the set \( \{1, \ldots, 2k\} \). In other words, if \( \mathcal{P} = \{ \{j_1^1, j_2^1\}, \ldots, \{j_1^k, j_2^k\} \} \) is a pair partition of \( \{1, \ldots, 2k\} \) then one can consider the hypergeometric integral

\[
L(\mathcal{P}; H) = \int_{\Delta^{2k}|[0,1]} \prod_{l=1}^{k} |s_{j_1^l} - s_{j_2^l}|^{2H-2} \, ds_1 \land \ldots \land ds_{2k},
\]

and the summands in (0.3) are such integrals. However, not all pair partitions occur - the factors \( \delta_{i_{\sigma(2l-1)}, i_{\sigma(2l)}} \) force only those pair partitions which refine the partition implied by the defining word \((i_1, \ldots, i_{2k})\) to occur in the sum. By this we mean that the integrand in a given summand of (0.3) is nonzero if and only if \( i_{\sigma(2l-1)} = i_{\sigma(2l)} \) for every index \( l \leq k \), but the word \((i_1, \ldots, i_{2k})\) already partitions the set \( \{1, \ldots, 2k\} \) according to the level sets of the map \( p \mapsto i_p \), so the integrand of a given summand in (0.3) is nonzero if and only if the permutation \( \sigma \) maps adjacent indices of the form \( \{2l-1, 2l\} \) into the blocks of the partition implied by the word \((i_1, \ldots, i_{2k})\).

For example, suppose \( d = 6, k = 5 \), \((i_1, \ldots, i_{10}) = (6, 3, 1, 3, 6, 1, 5, 6, 5)\) and consider the four diagrams in Figure 1. The coloring scheme indicates the partition defined by the word \((6, 3, 1, 3, 6, 1, 5, 6, 5)\), i.e., distinct numbers have a common color if and only if they belong to the same level set of the map \((1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \mapsto (6, 3, 1, 3, 6, 1, 5, 6, 5)\), or equivalently if they belong to the same block of the implied partition. The bracketings indicate pair partitions, or equivalently elements of the quotient \( \mathcal{S}_{2k}/M_{2k} \), where \( M_{2k} \subset \mathcal{S}_{2k} \) is the abelian subgroup of order \( 2^k \) generated by the adjacency transpositions \((1, 2), (3, 4), \ldots, (2k - 1, 2k)\).

![Figure 1](image-url)

\[ \mathcal{P}_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}, \{9, 10\}\} \]

\[ \mathcal{P}_2 = \{\{1, 6\}, \{2, 4\}, \{3, 7\}, \{5, 9\}, \{8, 10\}\} \]

\[ \mathcal{P}_3 = \{\{1, 9\}, \{2, 4\}, \{3, 7\}, \{5, 6\}, \{8, 10\}\} \]

\[ \mathcal{P}_4 = \{\{1, 9\}, \{2, 7\}, \{3, 4\}, \{5, 6\}, \{8, 10\}\} \]
Evidently \( \mathcal{P}_2 \) and \( \mathcal{P}_3 \) both refine the level set partition of the map

\[
(1, 2, 3, 4, 5, 6, 7, 8, 9, 10) \mapsto (6, 3, 1, 3, 6, 1, 5, 6, 5),
\]

so \( L(\mathcal{P}_2, H) \) and \( L(\mathcal{P}_3, H) \) as defined in (1.4) each occur with multiplicity \( 2^k = |M_{2k}| \) in the sum (1.3). However, \( \mathcal{P}_1 = 1M_{2k} \subset \mathcal{S}_{2k} \) and \( \mathcal{P}_4 \) do not refine the aforementioned level set partition, as can be easily seen from the coloring scheme since the constituent pairs \( \{1, 2\}, \{3, 4\}, \{7, 8\}, \{9, 10\} \) in \( \mathcal{P}_1 \) and \( \{3, 4\}, \{2, 7\} \) in \( \mathcal{P}_2 \) connect distinct colors. Therefore, \( L(\mathcal{P}_2, H) \) and \( L(\mathcal{P}_3, H) \) do not occur in the sum (1.3).

Refinement of partitions will be abbreviated with the “\( \leq \)” symbol, so the statement \( \{j^1, j^2\} \leq (i_1, \ldots, i_{2k}) \) is true if and only if \( i_l^1 = i_l^2 \) and \( \mathcal{P} \leq (i_1, \ldots, i_{2k}) \) if each constituent pair of \( \mathcal{P} \) is \( \leq (i_1, \ldots, i_{2k}) \). With this notation we can rewrite (1.3) as

\[
E \left[ \int_{\Delta^k[0,1]} dB_{i_1}^{j^1_1} \cdots dB_{i_{2k}}^{j^2_{2k}} \right] = \frac{H^k(2H-1)^k}{k!} \sum_{\mathcal{P} \leq (i_1, \ldots, i_{2k})} L(\mathcal{P}; H).
\]

A natural question to ask is that of the possibility of the meromorphic continuation of the operators \( \Gamma_{k} \) to a complex neighborhood of the interval \( H \in (1/2, 1) \), even for those \( H \) at which the stochastic integral (0.1) is ill-posed. With this in mind, the authors of [BC07] conjectured that the coefficients of the constituent operators \( \Gamma_k^H \) for \( k \geq 2 \) have meromorphic continuations with poles in the set \( \{1/2j : 2 \leq j \leq k\} \). In this note we prove that the \( \Gamma_k^H \) have meromorphic continuations to the entire complex plane in the variable \( H \), and we show that poles can only occur in certain specific sets of rational numbers which depend on the combinatorial structure of the words which define the constituent iterated integrals. For \( \Re H > 0 \), our results indicate that there may be more poles than those which are conjectured to exist in [BC07]. However, none of these poles is actually shown to exist - the laurent series coefficients are still prohibitively complicated so as of yet we cannot prove that they’re nonzero.

For any pair \( \{j^1, j^2\} \subset \{1, \ldots, 2k\} \), define

\[
I(\{j^1, j^2\}) = \{j^1 \land j^2 + 1, j^1 \lor j^2\} = \{j^1 \land j^2 + 1, \ldots, j^1 \lor j^2\} \subset \{2, \ldots, 2k\}.
\]

For example, if \( k = 5 \) then \( I(\{10, 6\}) = \{7, 10\} = \{7, 8, 9, 10\}, I(\{7, 3\}) = \{4, 7\} = \{4, 5, 6, 7\} \) and \( I(\{1, 2\}) = \{2\} = \{2, 5\} \). The significance of \( I(\{j^1, j^2\}) \) is that if without loss of generality \( j^1 < j^2 \) then

\[
s_{j^2} - s_{j^1} = (s_{j^2} - s_{j^2-1}) + (s_{j^2-1} - s_{j^2-2}) + \ldots + (s_{j^1+1} - s_{j^1})
\]

and \( I(\{j^1, j^2\}) \) represents the subset of variables \( \{x_1, \ldots, x_{2k}\} \) in the image of this sum under the change of variable \( x_i = s_i - s_{i-1} \) \((s_0 = 0)\).

For any pair partition \( \mathcal{P} = \{\{j^1_1, j^2_1\}, \ldots, \{j^1_k, j^2_k\}\} \), the map \( I \) can be applied to each of the pairs in \( \mathcal{P} \), thus obtaining a map

\[
I: \{\text{pair partitions of } \{1, \ldots, 2k\}\} \rightarrow \{\text{sets of } k \text{ subintervals of } \{2, \ldots, 2k\}\}.
\]

For example, if \( k = 3 \) and \( \mathcal{P}_1 = \{\{4, 6\}, \{5, 2\}, \{1, 3\}\} \) then \( I(\mathcal{P}_1) = \{5, 6\}, \{3, 5\}, \{2, 3\} \). On the other hand if \( \mathcal{P}_2 = \{\{1, 6\}, \{2, 5\}, \{3, 4\}\} \) then \( I(\mathcal{P}_2) = \{2, 6\}, \{3, 5\}, \{4\} \), and

\[
1 \text{Here and below we use the notation } [n] = [n, n) = \{n\} \text{ for an interval of natural numbers with only one element.}
\]
if \( P_3 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\} \) then \( I(P_3) = \{\{2, 4\}, \{3, 5\}, \{4, 6\}\} \). Now for any pair partition \( P \) of \( \{1, \ldots, 2k\} \) and any subset \( S \subset \{1, \ldots, 2k\} \) define

\[
[S|P] = |\mathcal{P}(S) \cap I(P)| = \text{number of subsets of } S \text{ which are elements of } I(P).
\]

**Theorem 0.1.** For any pair partition \( P = \{\{j_1^1, j_2^1\}, \ldots, \{j_k^1, j_k^2\}\} \) of the set \([1, 2k]\), the hypergeometric integral \( L(P; H) \) has a meromorphic continuation in the variable \( H \) to the entire plane \( \mathbb{C} \) which is holomorphic in the complement of the following union of rational progressions:

\[
\left\{ 1 - \frac{|S| + l}{2|S|P} : S \subset [1, 2k], [S|P] > 0, l = 0, 1, 2, \ldots \right\} \subset \mathbb{Q}.
\]

**Corollary 0.2.** For any word \((i_1, \ldots, i_{2k}) \in \{1, \ldots, d\}^{2k}\), the sum

\[
\frac{k!}{H^k(2H - 1)^k} E \left[ \int_{\Delta_{2k}[0,1]} dB_{i_1}^{2k} \cdots dB_{i_{2k}}^{2k} \right] = \sum_{P \leq (i_1, \ldots, i_{2k})} L(P; H)
\]

has a meromorphic continuation in the variable \( H \) to the entire plane \( \mathbb{C} \) which is holomorphic in the complement of the following union of rational progressions:

\[
\left\{ 1 - \frac{|S| + l}{2|S|P} : S \subset [1, 2k], P \leq (i_1, \ldots, i_{2k}), [S|P] > 0, l = 0, 1, 2, \ldots \right\} \subset \mathbb{Q}.
\]

The corollary follows immediately from the theorem by way of the equality \((0.5)\).

We stress yet again that since we’ve not proved that the meromorphic continuations are unbounded in every neighborhood of a point in the progression \( 1 - (|S| + l)/2|S|P| \), these numbers are merely the only candidates for the poles.

In order to understand the structure of these rational progressions we observe that \([S|P]\) is additive with respect to any pairwise monotone and nonadjacent disjoint decomposition. By this we mean that if \( S = \bigcup_{r=1}^s S_r \) and for \( 1 \leq r < \rho \leq s \) the greatest element of \( S_r \) is at least two less than the least element of \( S_\rho \) then \([S|P] = \sum_{r=1}^s [S_r|P]\). This is easy to see, because the existence of a nontrivial gap between \( S_r \) and \( S_\rho \) means that any subinterval of \([1, 2k]\) contained in \( S_r \cup S_\rho \) must be contained in either \( S_r \) or in \( S_\rho \). In particular this is true of the maximal interval decomposition of \( S \), i.e. \( S = \bigcup_{r=1}^k I_r \) where each \( I_r \) is an interval and if \( 1 \leq r < \rho \leq s \) then the right endpoint of \( I_r \) is at least two less than the left endpoint of \( I_\rho \).

Thus, apparently \([S|P] = \sum_{r=1}^k |I_r|P\) so it is enough to understand the expression \([I|P]\) where \( I \) is an interval. For this, we define two functions on the subintervals of \([1, 2k]\) taking values in the subsets of \([1, 2k]\), the \( P \) augmentation of \( I\):

\[
\text{Aug}_P(I) = \begin{cases} 
I \cup \{\text{the element immediately left adjacent to } I\} & \text{if the element immediately left adjacent to } I \text{ is paired by } P \text{ with an element of } I \\
I & \text{in all other cases,}
\end{cases}
\]

and the \( P \) deficiency of \( I\):

\[
\text{Def}_P(I) = \{\text{elements of } I \text{ which are not paired by } P \text{ with elements of } \text{Aug}_P(I)\}.
\]

With these definitions in place, it is straightforward to see that \( \text{Aug}_P(I) \setminus \text{Def}_P(I) \) is precisely the union of \( P \)-pairs \( \{j^1, j^2\} \) such that \( I(\{j^1, j^2\}) \subset I \), and therefore

\[
2[I|P] = |\text{Aug}_P(I)| - |\text{Def}_P(I)|.
\]
Consequently,

\[ 2[S|\mathcal{P}] = \sum_{r=1}^{s} |\text{Aug}_{\mathcal{P}}(I_r)| - |\text{Def}_{\mathcal{P}}(I_r)| \]

where \( S = \cup_{r=1}^{s} I_r \) still denotes the maximal interval decomposition of \( S \). The following corollary follows immediately from Theorem 0.1 by decomposing all subsets of \([1, 2k]\) into maximal connected intervals.

**Corollary 0.3.** For any pair partition \( \mathcal{P} = \{ \{j_1^1, j_1^2\}, \ldots, \{j_k^1, j_k^2\} \} \) of the set \([1, 2k]\), the hypergeometric integral \( L(\mathcal{P}; H) \) has a meromorphic continuation in the variable \( H \) to the entire plane \( \mathbb{C} \) which is holomorphic in the complement of the union of rational progressions of the form

\[
\left\{ 1 - \frac{\sum_{r=1}^{s} |I_r| + l}{\sum_{r=1}^{s} |\text{Aug}_{\mathcal{P}}(I_r)| - |\text{Def}_{\mathcal{P}}(I_r)|} : l = 0, 1, 2, \ldots \right\} \subset \mathbb{Q}
\]

with \( \{I_1, \ldots, I_s\} \) equal to any collection of pairwise nonadjacent subintervals of \([1, 2k]\) such that the denominator is nonzero.

The size \(|I_r|\) of any of the intervals in Corollary 0.3 can be extracted from the expression in the denominator \(|\text{Aug}_{\mathcal{P}}(I_r)| - |\text{Def}_{\mathcal{P}}(I_r)|\) by observing that \(|\text{Aug}_{\mathcal{P}}(I_r)|\) is either equal to \(|I_r|\) or \(|I_r| + 1\) depending on whether or not \(\text{Aug}_{\mathcal{P}}(I_r) = I_r\) or not. In the latter case we say that \( I_r \) is \( \mathcal{P} \)-augmented, and

\[
\sum_{r=1}^{s} |\text{Aug}_{\mathcal{P}}(I_r)| = \sum_{r=1}^{s} |I_r| + (\text{number of } \mathcal{P} - \text{augmented intervals in } \{I_1, \ldots, I_s\}).
\]

Likewise, an element of \(\text{Def}_{\mathcal{P}}(I_r)\) is said to be \( \mathcal{P} \)-deficient and

\[
\sum_{r=1}^{s} |\text{Def}_{\mathcal{P}}(I_r)| = (\text{number of } \mathcal{P} - \text{deficient points in } \cup_{r=1}^{s} I_r)).
\]

Therefore, by decomposing an arbitrary subset \( S \subset [1, 2k] \) uniquely as a union \( S = \cup_{r=1}^{s} I_r \) of pairwise nonadjacent intervals and defining the \( \mathcal{P} \)-deficient points of \( S \) as the union of those in the \( I_r \) individually, the expression in (0.6) can be rewritten as

\[
(0.7) \quad 1 - \frac{|S| + l}{|S| + (\text{number of } \mathcal{P} - \text{augmented components in } S) - (\text{number of } \mathcal{P} - \text{deficient points in } S)}.
\]

Note that for any \( S \) and any \( \mathcal{P} \), the number of \( \mathcal{P} \)-augmented components in \( S \) is at most \(|S|\) and for this reason the fraction written above is not less than 1/2. With this in mind, the following corollary follows immediately from Theorem 0.1 and Corollary 0.3.

**Corollary 0.4.** For any pair partition \( \mathcal{P} = \{ \{j_1^1, j_1^2\}, \ldots, \{j_k^1, j_k^2\} \} \) of the set \([1, 2k]\), the meromorphic continuation of the hypergeometric integral \( L(\mathcal{P}; H) \) is holomorphic at all nonreal points and all real points greater than the maximal value of (0.7) over all subsets \( S \subset [1, 2k] \) such that the denominator is nonzero. In particular it is holomorphic at all points greater than 1/2 and the only possible poles in the interval \((0, 1/2)\) are at those values of (0.3) arising from the subsets \( S \subset [1, 2k] \) such that the number of \( \mathcal{P} \)-augmented components in \( S \) exceeds the number of \( \mathcal{P} \)-deficiencies in \( S \).
The pair partition integrals $L$ of similar types of integrals have preoccupied many prominent researchers [Ati70, Bar86, BG69, GS64, Var95]. The pair partition integrals $L(\{j_1^1, j_1^2\}, \ldots, \{j_k^1, j_k^2\}; H)$ bear a strong resemblance to the standard beta integral $\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, along with its higher dimensional analog

$$\int_{\Delta^n[0,1]} t_1^{\lambda_1-1}(t_2 - t_1)^{\lambda_2-1} \cdots (1 - t_n)^{\lambda_{n+1}-1}dt_1 \cdots dt_n = \frac{\Gamma(\lambda_1) \cdots \Gamma(\lambda_{n+1})}{\Gamma(\lambda_1 + \cdots + \lambda_{n+1})}.$$
Computation of these integrals is a straightforward matter, i.e., reduction of the multi-dimensional case to the one-dimensional case can be achieved by computing it as an iterated integral. Alternatively, one can make a change of variable to express it as an integral over the standard simplex \( \Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_i > 0, \sum x_i = 1\} \), multiply by the convergence factor \( e^{-\sum x_i} \), and use a homogeneity argument to obtain the desired equality.

Neither of these strategies works universally for the pair partition integrals \( L((\{j_1, j_1^2\}, \ldots, \{j_k, j_k^2\}); H) \) and the obstacle is easy to identify, the linear factors \( s_{i1} - s_{i2} \) of the integrand are not necessarily differences of coordinates having adjacent indices so they do not vanish on the codimension one boundary strata of the simplex - instead, they vanish on lower dimensional boundary strata. To overcome this obstacle, we will generalize the problem and compute instead the possible locations of the singularities in \( \mathbb{C}^{2^n-1} \) of the meromorphic continuation of the integral

\[
\mathcal{H}^n((\lambda_S)_{S \neq \varnothing}) = \int_{\Delta^n} \prod_{S \neq \varnothing} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n.
\]

Here the vectors \( 1_S \) are the characteristic functions of the subsets of \( \{1, \ldots, n\} \), so \( x \cdot 1_S = \sum_{i \in S} x_i \), and \( \Delta_n \) is the \( d_1 \wedge \ldots \wedge dx_n \)-oriented simplex \( \Delta_n = \{x \in \mathbb{R}^n : x_i > 0, \sum x_i < 1\} \). Analysis of the meromorphic continuation of (1.8) will be achieved by blowing up the simplex \( \Delta_n \) in the following manner. In section 1 we define for \( S \subset \{1, n\} \) the affine form \( f_S(y) = -q(S) + y \cdot 1_S \) for \( y \in \mathbb{R}^n \) and a strategically chosen function \( q : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0} \). It is then proved that the map \( F : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( F_i = \prod_{S \ni i} f_S \) is a polynomial diffeomorphism from the region \( \Omega = \bigcap_{S \neq \varnothing} f_S > 0 \subset \mathbb{R}^n \) onto the orthant \((0, \infty)^n \subset \mathbb{R}^n \) (Lemma 1.5), which maps \( \Omega' = \Omega \cap \{\sum_i F_i < 1\} \) onto the simplex \( \Delta_n \). For \( \text{Re} \lambda_S > 0 \), \( \mathcal{H}^n((\lambda_S)_{S \neq \varnothing}) \) is therefore equal to the integral of the n-form pullback \( F^\ast((\prod_{S \neq \varnothing} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n)) \) over the region \( \Omega' \). Evidently after rearranging terms,

\[
F^\ast\left(\prod_{S \neq \varnothing} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n\right) = \prod_{S \neq \varnothing} f_S^{\sum_{T \subset S} \lambda_T} P_S^{\lambda_S} \det(\partial F/\partial y) dy_1 \wedge \ldots \wedge dy_n
\]

\[
= \prod_{S \neq \varnothing} f_S^{-|S|-1+\sum_{T \subset S} \lambda_T} P_S^{\lambda_S} \det(\partial F/\partial y) \prod_{S \neq \varnothing} f_S^{-|S|-1} dy_1 \wedge \ldots \wedge dy_n.
\]

where \( P_S = \sum_{i \in S} \prod_{S \ni T \ni i} f_T \). However, in section 1 it is proved that the rational quotient \( R = \det(\partial F/\partial y)/\prod_{S \neq \varnothing} f_S^{-|S|-1} \), written in (1.5), is a polynomial which is strictly positive on the closure of \( \Omega' \) (Lemma 1.4 and related commentary). Furthermore, in section 2 it is proved that for each nonempty \( S, P_S \) is strictly positive on the closure of \( \Omega' \). Therefore,

\[
\mathcal{H}^n((\lambda_S)_{S \neq \varnothing}) = \int_{\Omega'} \prod_{S \neq \varnothing} f_S^{-|S|-1+\sum_{T \subset S} \lambda_T} P_S^{\lambda_S} Rd_1 \wedge \ldots \wedge dy_n.
\]

The integral will converge absolutely in an open subset of \( \mathbb{C}^{2^n-1} \) (the product of half planes with positive real part, for instance). The meromorphic continuation arises from this expression by dissecting \( \Omega' \) in such a way that the closure
of every boundary component of the dissection intersects exactly one connected boundary stratum of \( \Omega' \) of minimal dimension (for that component). The factor 
\[ f_S^{|S|−1+|\sum_{T \subseteq S} \lambda_T|} \] 
will contribute a progression of singularities along the hyperplanes defined by 
\[ |S| + \sum_{T \subseteq S} \lambda_T \in \mathbb{N}_{\leq 0}. \] 
This procedure is described in section 2.

The pair partition integrals \( L(\mathcal{P}; H) \) can easily be converted to special forms of the more general simplex integrals \( \tau^{2k}((\lambda_S)_{S \neq \emptyset}) \) by making the change of variable 
\[ x_i = s_i, \] 
and \( x_i = s_i - s_{i-1} \) for \( i \in \{2, \ldots, 2k\} \) which maps the increasing simplex \( \Delta^{2k}[0,1] \) bijectively onto the solid simplex \( \Delta_{2k} = \{ x \in \mathbb{R}^{2k} : x_i > 0, \sum x_i < 1 \} \). Therefore, with \( \mathcal{P} = \{ \{j_1^1, j_1^2\}, \ldots, \{j_k^1, j_k^2\} \} \),

\[
L(\mathcal{P}; H) = \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{2k} |s_{j_l^1} - s_{j_l^2}|^{2H-2} ds_1 \ldots ds_{2k}
\]

\[
= \int_{\Delta^{2k}[0,1]} \prod_{l=1}^{2k} \left( \sum_{i=j_l^1 \wedge j_l^2+1}^{\lambda} x_i \right)^{2H-2} dx_1 \ldots dx_{2k}
\]

\[
(0.10)
\]

which is precisely the integral \( \tau^{2k} \) evaluated at the parameter \( (\lambda_S)_{S \neq \emptyset} \) defined by

\[
\lambda_S = \begin{cases} 
2H - 2 & \text{if } S \in \mathcal{I}(\mathcal{P}) \\
0 & \text{otherwise}
\end{cases}
\]

Thus,

\[
L(\mathcal{P}; H) = \int_{\Omega'} \prod_{S \neq \emptyset} f_S^{|S|−1+2(H−1)|S|} \prod_{S \in \mathcal{I}(\mathcal{P})} P_S^{2H−2} Rdy_1 \ldots dy_n,
\]

and (0.5) can be rewritten as

\[
(0.11)
\]

\[
E \left[ \int_{\Delta^{2k}[0,1]} dB_{t_1} \ldots dB_{t_{2k}} \right] = \frac{H^k(2H-1)^k}{k!}
\]

\[
(0.12)
\]

\[
\times \sum_{\mathcal{P} \subseteq \{t_1, \ldots, t_{2k}\}} \int_{\Omega'} \prod_{S \neq \emptyset} f_S^{|S|−1+2(H−1)|S|} \prod_{S \in \mathcal{I}(\mathcal{P})} P_S^{2H−2} Rdy_1 \ldots dy_n.
\]

For any \( z \in \mathbb{C} \), \( 1 - (|S| + z)/2|S| \) is the value of \( H \) such that the exponent \( |S| - 1 + 2(H−1)|S| \) written above is equal to \(-1 - z\), so by requiring \(-1 - z \in \{-1, -2, \ldots\} \) one obtains the rational progressions specified in Theorem 0.1. In this manner, Theorem 0.1 and therefore also Corollaries 0.2, 0.3 and 0.4 follow from the analysis of the singularities of \( \Omega' \) given in section 2.

1. Blowup of the orthant \( \mathbb{R}_+^n \)

In this section we will give a constructive resolution of singularities for the orthant \( \mathbb{R}_+^n = (0, \infty)^n \). To begin, for each subset \( S \subseteq \{1, \ldots, n\} \), define the vector \( 1_S \) to be the characteristic function of the set \( S \) (i.e. if \( n = 5 \) and \( S = \{2, 3, 5\} \) then \( 1_S = (0, 1, 1, 0, 1) \)) and the affine form

\[
(1.1)
\]

\[
f_S(y_1, \ldots, y_n) = -q(|S|) + y \cdot 1_S = -q(|S|) + \sum_{i \in S} y_i
\]
where \( q : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0} \) is an arbitrary function that satisfies
\[
(1.2) \quad q(0) = 1 \quad \text{and} \quad q(a) + q(b) < q(\max\{a, b\} + 1) \quad \text{and} \quad 3q(a) \leq q(a + 1)
\]
\( q(r) = 3^r \) seems to be a natural choice. A finite collection \( \{S_1, \ldots, S_r\} \subset \mathbb{P}\{\{1, \ldots, n\}\} \) of distinct subsets will be called \textit{monotone} if for any two of them one is fully contained in the other, or equivalently if there exists a permutation \( \sigma \in \mathfrak{S}_r \) such that \( S_{\sigma(1)} \subset \cdots \subset S_{\sigma(r)} \).

**Lemma 1.1.** If \( S_0, S_1 \subset \{1, \ldots, n\} \) and \( y \in \mathbb{R}^n \) is any point such that \( f_{S_0}(y) = f_{S_1}(y) = 0 \) then \( f_{S_0 \cup S_1}(y) + y \cdot 1_{S_0 \cap S_1} \) is positive if \( \{S_0, S_1\} \) is a monotone pair and is otherwise negative. In particular, if \( f_{S_0}(y) = f_{S_1}(y) = 0 \), \( y \cdot 1_{S_0 \cap S_1} \geq 0 \) and \( \{S_0, S_1\} \) is not a monotone pair, then \( f_{S_0 \cup S_1}(y) \) is negative.

**Proof.** The hypothesis \( f_{S_0}(y) = f_{S_1}(y) = 0 \) implies \( y \cdot 1_{S_0} = q(|S_0|) \) and \( y \cdot 1_{S_1} = q(|S_1|) \), thus
\[
f_{S_0 \cup S_1}(y) + y \cdot 1_{S_0 \cap S_1} = -q(|S_0 \cup S_1|) + y \cdot 1_{S_0 \cup S_1} + y \cdot 1_{S_0 \cap S_1} = -q(|S_0 \cup S_1|) + y \cdot 1_{S_0} + y \cdot 1_{S_1} = -q(|S_0| + |S_1|) + q(|S_0|) + q(|S_1|).
\]
Set \( M = \max\{|S_0|, |S_1|\} \) and \( m = \min\{|S_0|, |S_1|\} \) so that
\[
f_{S_0 \cup S_1}(y) + y \cdot 1_{S_0 \cap S_1} = -q(M + m - |S_0 \cap S_1|) + q(M) + q(m).
\]
If \( S_0, S_1 \) is a monotone pair then \( M = |S_0 \cap S_1| \) so
\[
f_{S_0 \cup S_1}(y) + y \cdot 1_{S_0 \cap S_1} = -q(M) + q(M) + q(m) = q(m) > 0.
\]
If \( S_0, S_1 \) is not a monotone pair then \( m - |S_0 \cap S_1| \geq 1 \) so
\[
f_{S_0 \cup S_1}(y) + y \cdot 1_{S_0 \cap S_1} \leq -q(M + 1) + q(M) + q(m) < 0
\]
according to the assumed hypotheses (1.2) on \( q \).

Define the region \( \Omega = \cap_{S \neq \emptyset} \{f_S > 0\} \subset \mathbb{R}^n \) and the affine hyperplane \( L_S = \{f_S = 0\} \subset \mathbb{R}^n \) for each nonempty \( S \subset \{1, \ldots, n\} \).

**Lemma 1.2.** A set \( \{f_{S_1}, \ldots, f_{S_r}\} \) of affine forms of type (1.1) can vanish simultaneously at a boundary point of the region \( \Omega \) only if \( \{S_1, \ldots, S_r\} \) is monotone and does not contain \( \emptyset \), in which case the set
\[
\Omega(\{S_1, \ldots, S_r\}) = \partial \Omega \cap \bigcap_{S \in \{S_1, \ldots, S_r\}} \{f_S \leq 0\} \cap \bigcup_{S \neq \emptyset, S \notin \{S_1, \ldots, S_r\}} L_S^c
\]
is a nonempty convex and relatively open subset of the codimension \( r \) affine hyperplane \( \cap_{S \in \{S_1, \ldots, S_r\}} L_S \subset \mathbb{R}^n \).

**Proof.** First, since \( f_{\emptyset}(y) = -1 \) for all \( y \), if \( y \in \partial \Omega \) and \( f_{S_i}(y) = \ldots = f_{S_r}(y) = 0 \) then none of the \( S_i \) can be empty. Next, choose distinct indices \( i, j \in \{1, \ldots, r\} \). Since all boundary points of \( \Omega \) must have strictly positive coordinates, evidently \( \{S_i, S_j\} \) must be a monotone pair, for if not then \( f_{S_i \cup S_j}(y) < 0 \) by Lemma 1.1 but this impossible on \( \partial \Omega \). Thus, we can conclude that either \( S_i \subset S_j \) or \( S_j \subset S_i \). Since this is true for any pair \( i, j \in \{1, \ldots, r\} \) of distinct indices, it is clear that the set \( \{S_1, \ldots, S_r\} \) must be monotone. Now the set \( \Omega(\{S_1, \ldots, S_r\}) \) can alternatively be written as
\[
\Omega(\{S_1, \ldots, S_r\}) = \cap_{S \in \{S_1, \ldots, \}} \{y \cdot 1_S = q(|S|)\} \cap \{y \cdot 1_S > q(|S|)\}
\]
and each of the factors in the intersection is convex, so the intersection must be as well. Also, it is clear that the intersection is relatively open in \( \cap_{S \in \{S_1, \ldots, S_r\}} L_S \). It remains to prove that \( \Omega(\{S_1, \ldots, S_r\}) \) is nonempty. For this we can extend the monotone set \( \{S_1, \ldots, S_r\} \) into a monotone set \( \{S_1, \ldots, S_r, S_{r+1}, \ldots, S_n\} \) of maximal length \( n \). For \( k \in \{1, \ldots, n\} \) let \( i_k \in \{1, \ldots, n\} \) be the unique index such that \( |S_{i_k}| = k \), let \( j_k \) be the unique index such that \( \{j_k\} = S_{i_k} \setminus S_{i_{k-1}}(S_{i_0} = \emptyset) \), and let \( \alpha \in \mathbb{R}^n \) be any vector such that
\[
\begin{cases}
\alpha_k = q(k) \\
q(k) < \alpha_k < \frac{q(k)+q(k+1)}{2} \quad &i_k \in \{1, \ldots, r\}, \\
q(k) < \alpha_k \quad &i_k \in \{r+1, \ldots, n\}.
\end{cases}
\]
We claim that the vector \( y \) defined by \( y_{j_k} = \alpha_1 \) and \( y_{j_k} = |S_{i_k}| - q(k) \) for \( k \in \{2, \ldots, n\} \) is an element of \( \Omega(\{S_1, \ldots, S_r\}) \). To prove this, observe that \( y \cdot 1_{S_{i_k}} = \alpha_k \) so we only need to check that \( y \cdot 1_S > q(|S|) \) for \( S \notin \{S_1, \ldots, S_n\} \). This is straightforward: let \( k_S \) be the minimal number such that \( S \subset S_{i_{k_S}} \), since \( S \neq S_{i_{k_S}} \) evidently \( |S| \leq k_S - 1 \) but also \( S \cap S_{i_{k_S}} = S_{i_{k_S}} \) by the minimality of \( k_S \) so
\[
y \cdot 1_S \geq y \cdot 1_{S_{i_{k_S}}} \\
= \alpha_{k_S} - \alpha_{k_S-1} \\
> q(k_S) - \frac{q(k_S - 1) + q(k_S)}{2} \\
= \frac{q(k_S) - q(k_S - 1)}{2} \\
\geq q(k_S - 1) \\
\geq q(|S|)
\]
according to the assumed hypotheses (1.2) on \( q \) (this is where \( 3q(a) \leq q(a+1) \) is used).

The following corollary follows immediately.

**Corollary 1.3.** The boundary of \( \Omega \) admits a decreasing filtration in closed sets given by
\[
\partial_i \Omega = \bigcup_{r \leq \rho \leq n} \bigcup_{S_1 \supset \ldots \supset S_\rho} \Omega(\{S_1, \ldots, S_\rho\})
\]
where the union is taken over all length not less than \( r \) monotone lists of subsets which do not contain \( \emptyset \). The top-dimensional stratum \( \partial_i \Omega \setminus \partial_{i+1} \Omega \) in each filtration degree is relatively open in \( \partial_i \Omega \) and its connected components are in bijection with the length-\( r \) monotone lists of subsets which do not contain \( \emptyset \).

Next, define the map \( F : \mathbb{R}^n \to \mathbb{R}^n \) by \( F = (F_1, \ldots, F_n) \) with
\[
F_i(y_1, \ldots, y_n) = \prod_{S \ni i} f_S(y_1, \ldots, y_n)
\]

**Lemma 1.4.** The Jacobian determinant of \( F \) is given by
\[
\det(\partial F/\partial y) = \prod_{S \neq \emptyset} f_S^{1-|S|} \left( \sum_{\emptyset \neq P \subset \{1, \ldots, n\}} \det(M^1_{S_1, \ldots, S_n})^{2} \prod_{\emptyset \neq P \subset \{S_1, \ldots, S_n\}} f_P \right).
\]
Before the proving the lemma, a few remarks are in order. First, $M^{s_1,\ldots,s_n}$
denotes the matrix having the given ordered list of vectors as columns. The matrix $M^{s_1,\ldots,s_n}$ itself depends on the order of the list $S_1,\ldots,S_n$ but the determinant of its square does not, so the fact that the summands are indexed by subcollections and not ordered subcollections of $\Psi(\{1,\ldots,n\}) \setminus \emptyset$ is not an issue. Most importantly, the factor

$$R = \sum_{\emptyset \neq \{S_1,\ldots,S_n\} \subset \Psi(\{1,\ldots,n\})} \det(M^{s_1,\ldots,s_n})^2 \prod_{\emptyset \neq P \subset \{S_1,\ldots,S_n\}} f_P$$

in $\det(\partial F/\partial y)$ must be strictly positive at all boundary points of $\Omega$, for it is a sum of functions which are all nonnegative on $\Omega$ so if it vanishes at a boundary point then all summands must vanish at that point, but by Lemma [1.2] there exists a monotone list $\{S_1,\ldots,S_r\}$ of nonempty subsets of $\{1,\ldots,n\}$ such that $f_S$ vanishes at the point in question if and only if $S \in \{S_1,\ldots,S_r\}$ so we can enlarge the given list to a monotone list $\{S_1,\ldots,S_n\}$ and thus conclude that for this particular list the summand

$$\det(M^{s_1,\ldots,s_n})^2 \prod_{\emptyset \neq P \subset \{S_1,\ldots,S_n\}} f_P,$$

and therefore the entire sum, does not vanish at the specified boundary point.

This being true of every point in $\partial \Omega$, we conclude that according to Lemma [1.3] $\det(\partial F/\partial y)$ is equal to the product $\prod_{S \neq \emptyset} f_S^{\lvert S \rvert - 1}$ times the polynomial $R$ defined in (1.5) which is strictly positive on the closure $\overline{\Omega}$.

**Proof of Lemma [1.4]** Evidently

$$\frac{\partial f_i}{\partial y_j} = \sum_{S \ni i} \frac{\lvert \{j\} \cap S \rvert}{S'} \prod_{S' \ni j, S' \neq S} f_{S'} = \sum_{S \ni i} \prod_{S' \ni j, S' \neq S} f_{S'} = \frac{1}{f_S} \prod_{S \ni i} f_{S'}.$$

Factoring out the $j$-independent product gives $\frac{\partial F}{\partial y_j} = \left(\prod_{S' \ni i} f_{S'}\right) \left(\sum_{S \ni i} f_S^{-1}\right)$ from which we conclude that the Jacobian matrix $\partial F/\partial y$ factors as $\partial F/\partial y = DA$ where $D$ is the diagonal matrix with $i$-th diagonal entry $\prod_{S' \ni i} f_{S'}$ and $A$ is the matrix with $\sum_{S \ni i} f_S^{-1}$ in row $i$ and column $j$. Since $D$ is diagonal, its determinant is easily computed by observing that the factor $f_S$ appears in precisely $\lvert S \rvert$ entries on the diagonal, and therefore $\det(\partial F/\partial y) = \prod_{S \neq \emptyset} f_S^{\lvert S \rvert - 1} \det A$. The computation thus reduces to that of $\det A$. For this, we observe that the matrix $A$ is the sum $A = \sum_S f_S^{-1} \mathbf{1}_{S}^{\otimes 2}$ where $\mathbf{1}_{S}^{\otimes 2}$ denotes the matrix with 1 in entry $i,j$ if and only if $i$ and $j$ are elements of $S$ (i.e. the matrix $\mathbf{1}_{S}^{\otimes 2}$, viewed as a function on the set $\{1,\ldots,n\}$, takes value 1 if $i,j \in S$, 0 otherwise, and 0 otherwise, thus

$$Ae_1 \wedge \ldots \wedge Ae_n = \sum_{S_1,\ldots,S_n \subset \{1,\ldots,n\}} \left(\prod_{i=1}^n \frac{1}{f_{S_i}}\right) (\mathbf{1}_{S_1}^{\otimes 2} e_1 \wedge \ldots \wedge \mathbf{1}_{S_n}^{\otimes 2} e_n).$$

Here the sum is taken over all lists of $n$ nonempty subsets of $\{1,\ldots,n\}$, with or without repetitions. Now $\mathbf{1}_{S_i}^{\otimes 2} e_i$ is $\mathbf{1}_{S_i}$, if $i \in S_i$ and is zero otherwise, thus
\(1_{S_i}^\otimes e_i = (e_i \cdot 1_{S_i})1_{S_i}\) and therefore

\[
Ae_1 \land \ldots \land Ae_n = \sum_{S_1,\ldots,S_n \subset \{1,\ldots,n\}} \left( \prod_{i=1}^n \frac{1}{f_{S_i}} \right) ((1_{S_1} \cdot e_1)1_{S_1} \land \ldots \land (1_{S_n} \cdot e_n)1_{S_n})
\]

\[
= \sum_{S_1,\ldots,S_n \subset \{1,\ldots,n\}} \left( \prod_{i=1}^n \frac{e_i \cdot 1_{S_i}}{f_{S_i}} \right) \det(M^{1_{S_1},\ldots,1_{S_n}})(e_1 \land \ldots \land e_n)
\]

where \(e_i = 1_{\{i\}}\) is the \(i\)-th standard basis vector. If \(S_1,\ldots,S_{2^n-1}\) is an enumeration of the nonempty subsets of \(\{1,\ldots,n\}\) then evidently

\[
\det A = \sum_{S_1,\ldots,S_n \subset \{1,\ldots,n\}} \left( \prod_{i=1}^n \frac{e_i \cdot 1_{S_i}}{f_{S_i}} \right) \det(M^{1_{S_1},\ldots,1_{S_n}})
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_n \leq 2^n-1} \sum_{\sigma \in \mathfrak{S}_n} \det(M^{1_{S_1},\ldots,1_{S_n}}) \prod_{j=1}^n \left( e_{j} \cdot 1_{S_{i_j}} \right)
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_n \leq 2^n-1} \frac{\det(M^{1_{S_1},\ldots,1_{S_n}})}{\prod_{j=1}^n f_{S_{i_j}}} \sum_{\sigma \in \mathfrak{S}_n} \sgn(\sigma) \prod_{j=1}^n \left( e_{j} \cdot 1_{S_{i_j}} \right)
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_n \leq 2^n-1} \frac{1}{\prod_{j=1}^n f_{S_{i_j}}} \det(M^{1_{S_1},\ldots,1_{S_n}})^2
\]

\[
= \sum_{\emptyset \neq p \subset \{1,\ldots,n\}} \frac{\prod_{p \neq s \in \{1,\ldots,n\}} f_{S'}}{\prod_{S \neq \emptyset} f_{S}} \det(M^{1_{S_1},\ldots,1_{S_n}})^2.
\]

Therefore,

\[
\det(\partial F/\partial y)
\]

\[
= \prod_{S \neq \emptyset} f_{S}^{|S|} \left( \sum_{\{S_1,\ldots,S_n\} \subset \mathfrak{P} \{1,\ldots,n\}\setminus\emptyset} \frac{\prod_{p \neq s \in \{1,\ldots,n\}} f_{S'}}{\prod_{S \neq \emptyset} f_{S}} \det(M^{1_{S_1},\ldots,1_{S_n}})^2 \right)
\]

and the lemma follows after factoring \(\prod_{S \neq \emptyset} f_{S}^{-1}\) out of the sum. \(\square\)

**Lemma 1.5.** The map \(F\) is a polynomial diffeomorphism from the region \(\Omega\) onto the orthant \(\mathbb{R}^n_+\).

**Proof.** Injectivity is easily proved, since the partial derivatives \(\partial F_i/\partial y_j\) are positive on \(\Omega\). To prove surjectivity, choose any fixed element of \(\Omega\) such as \((t,\ldots,t)\) with \(t > q(n)/n\). If \((x_1,\ldots,x_n) \in \mathbb{R}^n_+\) is arbitrarily chosen and \(v = (x_1,\ldots,x_n) - F(t,\ldots,t)\) then \((\partial F/\partial y)^{-1}v\) is a vector field on \(\Omega\). Flowing through this vector field in one unit of time will carry \((t,\ldots,t)\) into a point \((y_1,\ldots,y_n)\) such that \(F(y_1,\ldots,y_n) = (x_1,\ldots,x_n)\). Moreover, the resulting integral curve cannot leave \(\Omega\), for if it did then its image would have to leave the orthant \(\mathbb{R}^n_+\) but the image of this curve is the line segment connecting \(F(t,\ldots,t)\) to \((x_1,\ldots,x_n)\) and the orthant is convex so this line segment cannot intersect the boundary of the orthant. This proves that the \(F\)-preimage \((y_1,\ldots,y_n)\) of \((x_1,\ldots,x_n)\) is indeed an element of \(\Omega\). Since \((x_1,\ldots,x_n)\) was chosen arbitrarily, \(F\) must map \(\Omega\) surjectively onto \(\mathbb{R}^n_+\). \(\square\)
2. Pullback of $\mathcal{J}^n$ through $F$

As described in the introduction, we are concerned with the singularities of the meromorphic continuation to $\mathbb{C}^{n-1}$ of the integral

$$\mathcal{J}^n((\lambda_S)_{S \neq \emptyset}) = \int_{\Delta_n} \prod_{S \neq \emptyset} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n$$

which converges and defines a holomorphic function in the region $\cap_{S \neq \emptyset} \{ \Re \lambda_S > 0 \}$. This integral can now be pulled back to $\Omega' = F^{-1}(\Delta_n) = \Omega \cap \{ \sum_i F_i < 1 \}$ through the map $F$ defined in the previous section, allowing a detailed study of the singularities:

$$\mathcal{J}^n((\lambda_S)_{S \neq \emptyset}) = \int_{\Omega'} \prod_{S \neq \emptyset} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n$$

$$= \int_{\Omega'} F^* \left( \prod_{S \neq \emptyset} (x \cdot 1_S)^{\lambda_S} dx_1 \wedge \ldots \wedge dx_n \right)$$

$$= \int_{\Omega'} \prod_{S \neq \emptyset} (F \cdot 1_S)^{\lambda_S} (\det \partial F/\partial y) dy_1 \wedge \ldots \wedge dy_n. \quad (2.1)$$

The integrand in (2.1) can be further simplified as follows,

$$\prod_{S \neq \emptyset} (F \cdot 1_S)^{\lambda_S} = \prod_{S \neq \emptyset} \left( \prod_{T \supset S} f_T \right)^{\lambda_S} \left( \sum_{i \in S} \prod_{T \supset \emptyset} f_T \right)^{\lambda_S} = \prod_{S \neq \emptyset} f_S^{\sum_{T \supset S} \lambda_T} P_S^{\lambda_S}$$

where $P_S = \sum_{i \in S} \prod_{T \supset \emptyset} f_T$ as in the introduction.

Regarding $P_S$, we observe that it must be strictly positive on $\partial \Omega$. The only points in question are the boundary points, and if the sum vanishes at a boundary point then all the summands must vanish. This cannot happen, for as a result of the monotonicity property of the boundary (Corollary 1.3), at any given boundary point $x \in \partial \Omega'$ there is a unique monotone list $T_1 \subset \cdots \subset T_r$ such that $f_U(x) = 0$ if and only if $U$ is one of the $T_j$. For any $S$, either $S \subset T_1$ in which case every summand in $P_S$ is positive, or from among these $T_j$ there is a maximal choice $T_p$ such that $T_p \subset S$, i.e. $T_1 \subset \cdots \subset T_p \subset S \subset T_{p+1} \subset \cdots \subset T_r$. In the latter case, as $T_p$ is a proper subset of $S$ we are free to choose an element $i_x \in S$ such that $i_x \notin T_p \supset \cdots \supset T_1$. If $i_x$ is in $T$ then either $f_T(x) > 0$ or $T \supset S$. Consequently, the summand $\prod_{S \supset \emptyset} f_T$ is positive at $x$. This being true for any $x \in \partial \Omega'$, evidently $P_S$ is positive on $\partial \Omega$ for every $S$.

From this we obtain the expression (1.9) from the introduction:

$$\mathcal{J}^n((\lambda_S)_{S \neq \emptyset}) = \int_{\Omega'} \prod_{S \neq \emptyset} f_S^{|S|-1+\sum_{T \supset S} \lambda_T} P_S^{\lambda_S} R dy_1 \wedge \ldots \wedge dy_n.$$

To complete the proof of Theorem 0.1, the boundary of $\Omega'$ can be covered by a partition of unity in such a way that in an open neighborhood of the support of every element of the partition there is a monotone list $S_1 \subset \cdots \subset S_r$ such that $df_{S_1} \wedge \ldots \wedge df_{S_r}$ is a nonzero $r$-form and such that every boundary hyperplane in the support of $\varphi$ is defined by one of the $f_{S_i}$. The list can then be completed to a full coordinate system by adding in other $f_{S_i}$ and possibly also $F_1 + \ldots + F_n$ if the
support of $\varphi$ intersects the boundary hypersurface defined by $F_1 + \ldots + F_n = 1$. This reduces the computation to that of a finite sum of integrals of the form

$$\int_{\Omega} \prod_{S \neq \emptyset} f_S^{|S|-1+\sum_{T \subset S} \lambda_T} P_S^{\lambda_S} R \varphi dy_1 \wedge \ldots \wedge dy_n$$

where $\varphi$ is an element of the chosen partition of unity and the integrand can be rewritten using the coordinates $f_{S_1}, \ldots, f_{S_n}$ in most cases (or $f_{S_1}, \ldots, f_{S_{n-1}}, F_1 + \ldots + F_n$ in other cases) as an integral of the form

$$\int_{0 < f_{S_i} < 1, i \leq n} f_{S_1}^{|S_1|-1+\sum_{T \subset S_1} \lambda_T} \ldots f_{S_n}^{|S_n|-1+\sum_{T \subset S_n} \lambda_T} \Phi((\lambda_S)_{S \neq \emptyset}) \varphi df_{S_1} \wedge \ldots \wedge df_{S_n}$$

where $\Phi(f_{S_1}, \ldots, f_{S_n}, (\lambda_S)_{S \neq \emptyset})$ is an entire function of $(\lambda_S)_{S \neq \emptyset} \in \mathbb{C}^{2^n-1}$ taking values in the analytic and nonvanishing functions on the cube $0 < f_{S_i} < 1, i \leq n$.

Now we can define a new set of variables $(\psi_S)_{S \neq \emptyset} \in \mathbb{C}^{2^n-1}$ and consider the integral

$$\int_{0 < f_{S_i} < 1, i \leq n} f_{S_1}^{|S_1|-1+\sum_{T \subset S_1} \lambda_T} \ldots f_{S_n}^{|S_n|-1+\sum_{T \subset S_n} \lambda_T} \Phi((\psi_S)_{S \neq \emptyset}) \varphi df_{S_1} \wedge \ldots \wedge df_{S_n}$$

which for $\text{Re} \lambda_S > 0$ is apparently a holomorphic function of the $2^{n+1} - 2$ complex variables $(\lambda_S)_{S \neq \emptyset}, (\psi_S)_{S \neq \emptyset}$. For $(\psi_S)_{S \neq \emptyset} \in \mathbb{C}^{2^n-1}$ fixed, this integral evidently extends to a meromorphic function on $\mathbb{C}^{2^n-1}$ in the variables $(\lambda_S)_{S \neq \emptyset}$ with poles in the union of hyperplanes defined by $|S_i| - 1 + \sum_{T \subset S_i} \lambda_T \in \{-1, -2, \ldots\}$ for $1 \leq i \leq n$ as can be shown by expanding the factor $\Phi((\psi_S)_{S \neq \emptyset})\varphi$ into its taylor series in the coordinates $f_{S_1}, \ldots, f_{S_n}$ and integrating termwise.

Thus, the given integral extends to a meromorphic function on $\mathbb{C}^{2^{n+1}-2}$ with poles in the given list of hyperplanes and the original integral is the restriction of this meromorphic function to the diagonal subspace $\{\lambda_S - \psi_S = 0\}_{S \neq \emptyset}$. This completes the proof of Theorem [C].

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