Gossip Algorithms in Quantum Networks

Michael Siomau

Physics Department, Jazan University, P.O. Box 114, 45142 Jazan, Kingdom of Saudi Arabia and
Network Dynamics, Max Planck Institute for Dynamics and Self-Organization (MPIDS), 37077 Göttingen, Germany

(Dated: August 16, 2016)

Gossip algorithms is a common term to describe protocols for unreliable information dissemination in natural networks, which are not optimally designed for efficient communication between network entities. We consider application of gossip algorithms to quantum networks and show that any quantum network can be updated to optimal configuration with local operations and classical communication. This allows to seed-up – in the best case exponentially – the quantum information dissemination. Irrespective of the initial configuration of the quantum network, the update requites at most polynomial number of local operations and classical communication.

PACS numbers: 03.67.Ac, 03.67.Hk, 89.70.Hj

I. INTRODUCTION

Real-world networks are complex: natural social and brain networks as well as artificial technological and computer networks exhibit non-trivial structural features, which make complete simulation of the network dynamics practically impossible [1]. Complex non-stationary structure of modern artificial networks becomes a serious obstacle in the design of optimal protocols for information dissemination in such networks. Inspired by a natural way of rumor spreading in social networks, gossip algorithms [2] give a simple strategy for distributed and robust information dissemination in a network of unknown structure. These algorithms have found prominent applications in sensor, peer-to-peer and social networks.

Quantum networks [3] will be the next generation of complex structures for communication and advanced information processing [4]. Due to quantum superposition and nonlocality [5], quantum networks exhibit a number of structural and dynamical features that classical networks lack, among those are teleportation [6], quantum walks [7] and entanglement percolation [8, 9] to name just a few. Recently we showed that with local operations and classical communication (LOCC) [2] one may change connectivity of a given quantum network and simulate complex entanglement graphs on a simple underlying quantum network [10]. The structural modifications may radically improve the network capacity for information dissemination and performance of corresponding protocols, such as gossip algorithms.

In this paper we consider the problem of optimal information dissemination in quantum networks and analyze performance of gossip algorithms on the networks. As intuition suggests, the network where any pair of vertices is connected with an edge offers the most favorable conditions for information dissemination. Such a network is represented with a complete graph. We show that any quantum network represented with a connected graph, i.e. where any two vertices can be connected with a path of edges, may be updated to the complete graph using just polynomial number of LOCC. The update allows to dissipate information by means of quantum teleportation [5], thus radically improving the performance of the gossip algorithms on quantum networks.

This work is structured as follows. In the next section, we briefly describe classical gossip algorithms for single- and multi-piece information dissemination and introduce the quantities of interest, such as conductance, k-conductance and ε-dissimilation time. For a more detailed and mathematically rigorous treatment we suggest an excellent review by Shah [2]. In Section III we show how to improve the performance of gossip algorithms on quantum networks by LOCC. For sparse quantum networks the improvement in the information dissemination time due to the update is exponential, but still requires only polynomial number of LOCC. We conclude in Section IV.

II. CLASSICAL GOSSIP ALGORITHMS

From the structural viewpoint a network is a graph $G = (V, E)$ defined by sets of its vertices $V$ and edges $E$. The set $V = \{1, ..., n\}$ consists of a finite countable number of $n$ vertices. The edges represent connections between the vertices $E \subseteq V \times V$. The graph is called undirected if for any $(i, j) \in E$, $(j, i) \in E$ is also true. Here we impose no constraints on the direction of information dissimilation, hence consider only undirected graphs.

Information dissimilation on a graph may be studied with a discrete random walk technique, which requires definition of a $n \times n$ non-negative valued probability transition matrix $P = [P_{ij}]$, where $P_{ij}$ is the probability of information dissemination from vertex $i$ to $j$. Through the transition matrix, we may define an auxiliary function named conductance $\Phi(P)$, which characterizes the information dissemination capacity of a graph of particular configuration of vertices and edges. For symmetric $P$ – which is the case for undirected graphs – the con-

*Electronic address: siomau@nld.ds.mpg.de
ductance is defined as
\[
\Phi(P) = \min_{S \subset V, |S| \leq n/2} \frac{\sum_{i \in S} \sum_{j \in S} P_{ij}}{|S|},
\]
(1)
where \(S\) is the set of nodes that possess the information, while \(S^c\) is the set of those that doesn’t. The conductance is completely defined by the transition matrix of a graph, thus tells us how easy the information can be conducted through the graph. Also, the conductance is independent on a particular information dissemination protocol to be implemented on the graph.

A related to the conductance auxiliary function is \(k\)-conductance, which minimizes \(1\) for \(k \leq n/2\), i.e.
\[
\Phi_k(P) = \min_{S \subset V, |S| \leq k} \frac{\sum_{i \in S} \sum_{j \in S} P_{ij}}{|S|}.
\]
(2)
Using the \(k\)-conductance, we may also define the mean conductance \(\Phi(P)\) as
\[
\Phi(P) = \sum_{k=1}^{n-1} \frac{k}{\Phi_k(P)}.
\]
(3)

In the following we will focus on two particular graphs: the complete graph, where each pair of nodes is connected with an edge, and the ring graph, where nodes are placed on a circle with edges between nearest neighbors only. These two graphs are chosen for comparison because of their radical difference in the capacity for information dissemination. With the probability matrix \(P_{ij} = 1/n\) for all \(i\) and \(j\), the complete graph has the best possible capacity to disseminate information, i.e. \(\Phi(P) = O(1)\) and \(\Phi_k(P) = O(n^2 \log n)\), where \(O(.)\) is the standard notation for asymptotic upper bound. The ring graph with the probability matrix \(P_{ii} = 1/2\) and \(P_{ij} = 1/4\) for \(i \neq j\), in contrast, has the strongest constrain for information dissemination leading to \(\Phi(P) = O(1/n)\) and \(\Phi_k(P) = O(n^3)\).

Analyzing gossip algorithms we will be interested in the value called \(\varepsilon\)-dissemination time \(T(\varepsilon)\). This value gives us time by which all nodes have the information with probability at least \(1 - \varepsilon\). The definition of the \(\varepsilon\)-dissemination time depends on the algorithm, thus will be given in the next sections for single- and multi-piece dissemination strategies separately. Our goal is to estimate the \(\varepsilon\)-dissemination time through the conductance, allowing general treatment of the algorithm efficiency for any graph structure.

### A. Single-Piece Dissemination

Let an arbitrary vertex \(v \in V\) has a piece of information that it wishes to spread to all the other vertices as quickly as possible. Let \(S(t) \subset V\) denotes the set of vertices that have the information at time \(t\), which is also assumed to be discrete. At each time step, each vertex \(i\) contacts at most one of its neighbors \(j\) with probability \(P_{ij}\). If either \(i\) or \(j\) has the information at \(t - 1\), then both vertices have it at time \(t\).

For the single-piece dissemination algorithm, the \(\varepsilon\)-dissemination time is defined as
\[
T_1(\varepsilon) = \sup_{v \in V} \{t : \Pr(S(t) \neq V | S(0) = v) \leq \varepsilon\}.
\]
(4)
The right hand side of this definition accounts for the maximal time at which the set \(S(t)\) is inequivalent to \(V\) with probability no greater then \(\varepsilon\), assuming that initially the set \(S(t = 0)\) consisted of a single vertex \(v\).

The \(\varepsilon\)-dissemination time for the single-piece dissemination algorithm may be expressed through the conductance \(\Phi\) as
\[
T_1(\varepsilon) = O\left(\frac{\log n + \log \varepsilon^{-1}}{\Phi(P)}\right).
\]
(5)
This expression tells us explicitly how the \(\varepsilon\)-dissemination time depends on the structure of underlying network, i.e. on its conductance. For the complete graph the \(\varepsilon\)-dissemination time is given by \(T_1^c(\varepsilon) = O(\log n)\), which is the upper bound for single-piece dissemination algorithm performance in any network. For the ring graph the \(\varepsilon\)-dissemination time is exponentially larger comparing to the previous case, i.e. \(T_1^r(\varepsilon) = O(n \log n)\). It is important to note that information dissemination on a ring can be performed as fast as \(O(n)\) by setting a simple intuitive rule, for example, ‘always send information to the left neighbor’. But, gossip algorithms has no account for network structure, which is the key for their universality. Moreover, the gossip algorithms on a ring are just logarithmically slower then the intuitive strategy, which is practical.

### B. Multi-Piece Dissemination

In contrast to single-piece dissemination algorithm, where just a single vertex has the information initially, in multi-piece dissemination each vertex wants to spread its own information to all the other vertices as quickly as possible. Let \(M = \{m_1, ..., m_n\}\) denotes the set of messages at time \(t = 0\). As before each vertex contacts at most one of its neighbors at each time step. During the contact, the vertices exchange all information they don’t have. The \(\varepsilon\)-dissemination time is defined as
\[
T_M(\varepsilon) = \inf\{t : \Pr\left(\bigcup_{i=1}^{n} S_i(t) \neq M | S_i(0) = m_i\right) \leq \varepsilon\},
\]
(6)
i.e. the maximal time at which the information at each vertex is inequivalent to the initial set \(M\) with probability no greater then \(\varepsilon\). The \(\varepsilon\)-dissemination time is expressed through the mean conductance \(\Phi\) as
\[
T_M(\varepsilon) = O\left(\frac{\Phi(P) \log \varepsilon^{-1}}{n}\right).
\]
(7)
The local characteristic of any graph is the average degree on LOCC to update a connected graph. An important to the complete graph using just polynomial number of dissemination.

In this sense, the update means exploring and improving network structure for the purpose of future gossip dissemination. This assumption seems reasonable, because the update of the quanton step – applies to the quantum information. This assumption for the gossip algorithm – namely that each vertices may freely communicate classically, while the classical communication [6]. Therefore, we assume that the teleportation of quantum information also requires transmission by means of teleportation. However, an entangled state may be subsequently used for information entanglement swapping operations and doubled number of longest non-local entanglement edge using.

The procedure of the update begins with creating perfect entanglement between any pair of physically connected vertices of the quantum network. To be precise, let us assume that a pair of qubits in a Bell state [3] is to be distributed between any pair of physically connected vertices. A perfect Bell state can be created between two neighboring vertices by exchanging photons through the edges and, if necessary, purification [11]. The perfect entanglement can be distributed on arbitrary distance with entanglement swapping [8]. This distribution creates a single non-local edge that connects physically disconnected vertices. Let \( |a⟩ = \sum_{i,j} a_{ij} |ij⟩ \) be a two-qubit state in the computational basis \{\{0\}, \{1\}\}. The entanglement of this state can be described with concurrence \( K \) defined as \( C(a) = 2|\det A| \), where \( A = [a_{ij}] \). After \( K \) entanglement swapping operations the concurrence reads as

\[
C_K = \sup_M \sum_i 2|\det (A_1 M_1 A_2 M_2 \cdots A_K M_K)|
\]

where \( M_i \) for \( i = 1..K \) are \( 2 \times 2 \) matrices that denote the choice of measurements. The entanglement of the qubit pair after \( K \) entanglement swapping remains perfect, i.e. \( C_K = 1 \), iff the initial entangled states \( |a⟩ \), where maximally entangled \( C(a_i) = 1 \).

Summing up all the considerations above, the ring graph can be updated to the complete graph by sharing multiple copies of perfect entanglement between neighboring vertices and creating non-local entanglement edges with (multiple) entanglement swapping. The update of the ring graph gives the upper bound on LOCC for any connected graph. In the following we will estimate the bound for the single- and multi-piece gossip algorithms.

### A. Quantum Network Update

Let us show that any connected graph can be updated to the complete graph using just polynomial number of LOCC. To do so we need to estimate the upper bound on LOCC to update a connected graph. An important local characteristic of any graph is the average degree \( k = 2E/V \), i.e. the average number of edges \( E \) connected to vertex \( V \). A global characteristic that measures the efficiency of the information transport in a network is the average path length

\[
L_G = \frac{1}{n(n-1)} \sum_{i\neq j} d(v_i, v_j),
\]

where \( d(v_i, v_j) \) is the shortest distance between vertices \( v_i, v_j \in V \). The graph with the smallest average degree and the largest average path length is the most constrained for information dissemination, thus the update of the graph requires maximal number of LOCC. The ring graph and the 1D chain with \( k = 2 \) and \( L_G = O(1) \) are the desired graphs [13] to estimate the upper bound on LOCC. In the following we will focus on the ring graph noticing that all considerations remain valid also for the 1D chain.

### B. Single-Piece Dissemination in Quantum Networks

Let vertex \( v \in V \) has a piece of quantum information encoded into a qubit state \( |ψ⟩ \) to disseminate among the others. Because there is just one piece of information to disseminate, each entangled edge is to be used just once to send the information. Let us suppose that we have a ring graph with \( n \) vertices. For sake of clarity let us assume that \( n \) is even, noticing that the results remain valid for odd \( n \). To update the ring graph to the complete graph we need to create for a chosen vertex the longest non-local entanglement edge using \( n/2 - 1 \) entanglement swapping operations and doubled number of edges using \( n/2 - i \) for \( i = 2..(n/2 - 1) \). This procedure is to be repeated for all vertices excluding duplications.

### III. GOSSIP ALGORITHMS IN QUANTUM NETWORKS

Eqs. [5] and [7] unambiguously define performance of gossip algorithms through conductance [1] and it’s mean [3] for any classical network. In the classical case, there is no option to change the conductance of a network without addition of physical connections between vertices. In quantum networks, in contrast, entanglement swapping allows physically disconnected vertices to become connected with an entangled state, i.e. an entangled edge, without direct interaction between the vertices [3]. The entangled state may be subsequently used for information transmission by means of teleportation. However, the teleportation of quantum information also requires classical communication [3]. Therefore, we assume that vertices may freely communicate classically, while the condition for the gossip algorithm – namely that each vertex contacts at most one of its neighbors at each time step – applies to the quantum information. This assumption seems reasonable, because the update of the quantum network with the entangled edges may be done in advance to gossip algorithm run as we explain below. In this sense, the update means exploring and improving network structure for the purpose of future gossip dissemination.
The total number of the non-local edges to establish is \((n - 1)(n/2 - 1)\). Thus the total number of the entanglement swapping operations scales as \(O(n^3)\) [12]. This is the upper bound on the LOCC for single-piece dissemination algorithm in any quantum network. As we showed in Section III.A, the \(\varepsilon\)-dissemination time is exponentially larger in the complete graph \(T^n_\varepsilon(\varepsilon) = O(\log n)\) comparing to the ring graph \(T^n_\varepsilon(\varepsilon) = O(n\log n)\). Thus, the update gives the exponential benefit in information dissemination time requiring just \(O(n^3)\) LOCC.

Let us consider a ring graph of just eight vertices as shown in Fig. 1. Starting from an arbitrary vertex, we may disseminate information through the others at best in seven steps. If the ring is updated to the complete graph by adding entangled edges, the same information can be spread in just three steps. This difference in the information dissemination capacity growth radically with the network size. For a ring with \(n = 2^k\) nodes, the fastest classical dissemination is possible with \(2^k - 1\) steps, while in the updated network, the information can be disseminated as fast as in \(k\) steps. At the same time, the update requires \(n^3\) LOCC, i.e. scales polynomially with the network size.

An interesting aspect of the information dissemination in the updated network is that this dissemination is secure. In the ring, each vertex may corrupt information it receives: even though the information is encoded in quantum states, it is possible to copy the information partially [13]. In the updated network, in contrast, each pair of vertices is connected making the gossip secure at each step from the other vertices. Overall faster gossip dissemination reduces the number of potential information modifications due to previous hosts.

C. Multi-Piece Dissemination in Quantum Networks

In contrast to the previous case, in multi-piece dissemination algorithm each vertex has its own qubit state to disseminate, i.e. \(M_{|\psi\rangle} = \{|\psi\rangle_1, \ldots, |\psi\rangle_n\}\), thus each edge is to be used \(n\) times. But, the non-local entanglement edges are destroyed after the state teleportation. The simplest way to overcome this complication is to create the complete graph with \(n\) replicas of the non-local edges. This requires just \(O(n^4)\) local operations to update the ring graph, which is still appropriate cost in our opinion and gives the upper bound on LOCC. The update of the ring graph allows to improve exponentially the \(\varepsilon\)-dissemination time from \(T^n_M(\varepsilon) = O(n^2 \log n)\) to \(T^n_M(\varepsilon) = O(n \log^2 n)\).

IV. CONCLUSION

We suggested a new way to speed-up information distribution in quantum networks by structural update, which requires at most \(O(n^3)\) and \(O(n^4)\) LOCC for single- and multi-piece dissemination gossip algorithms respectively. Our approach is based solely on quantum non-locality, i.e. the ability to connect physically disconnected vertices with entangled states and quantum teleportation. But, because (classical) gossip algorithms are based on random walk, we believe that our approach is compatible with quantum walks [14]. Taking into account that gossip algorithms have applications not only in information dissemination but in linear and separable function computation [2], the combination of our structural approach with the quantum walks may lead to new model of quantum computing [3] and quantum machine learning [14] in complex quantum networks.

Acknowledgments

This work was supported by KACST.

References

[1] S.N. Dorogovtsev and J.F.F. Mendes, Evolution of Networks: From biological networks to the Internet and WWW (Oxford University Press, 2003).
[2] D. Shah, Found. & Trends in Networking 3, 1 (2008).
[3] H.J. Kimble, Nature 453, 1023 (2008).
[4] J.P. Dowling and G.J. Milburn, Phil. Trans. R. Soc. Lond. A 361, 1655 (2003)
[5] M.A. Nielsen and I.L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[6] C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres and W.K. Wooters, Phys. Rev. Lett. 70 1895 (1993).
[7] S.E. Venegas-Andraca, Quant. Inf. Proc. 11, 1015 (2012).
[8] A. Acin, J.I. Cirac and M. Lewenstein, Nature Phys. 3, 256 (2007).
[9] M. Siomau, arXiv:1602.06152 forthcoming in J. Phys. B (2016).
[10] M. Siomau, AIP Conf. Proc. 1742, 030017 (2016).
[11] G. Vidal, Phys. Rev. Lett. 83, 1046 (1999).
[12] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[13] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D.-U. Hwang, Phys. Rep. 424, 175 (2006).
[14] M. Siomau, Quant. Inf. Proc. 13, 1211 (2014).
[15] H. Fan, Y.-N. Wang, L. Jing, J.-D. Yue, H.-D. Shi, Y.-L. Zhang and L.-Z. Mu, Phys. Rep. 544, 241 (2014).