Rank and Nielsen equivalence in hyperbolic extensions

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Abstract

In this note, we generalize a theorem of Juan Souto on rank and Nielsen equivalence in the fundamental group of a hyperbolic fibered 3–manifold to a large class of hyperbolic group extensions. This includes all hyperbolic extensions of surfaces groups as well as hyperbolic extensions of free groups by convex cocompact subgroups of Out(F_n).

1 Introduction

Perhaps the most basic invariant of a finitely generated group is its rank, that is, the minimal cardinality of a generating set. Despite its simple definition, rank is notoriously difficult to calculate even for well-behaved groups. For example, work of Baumslag, Miller, and Short [BMS] shows that the rank problem is unsolvable for hyperbolic groups. In this note we calculate the rank for a large class of hyperbolic group extensions and furthermore show that, up to Nielsen equivalence, all minimal-cardinality generating sets are of a standard form.

Let 1 → H → G → Γ → 1 be an exact sequence of infinite hyperbolic groups. We say that the extension has the Scott–Swarup property if each finitely generated, infinite index subgroup of H is quasiconvex as a subgroup of G. Every subgroup ∆ ≤ Γ induces a new short exact sequence 1 → H → G_∆ → ∆ → 1, where G_∆ is the full preimage of ∆ under the surjection G → Γ. Our main theorem is the following; for the statement ℓΓ(·) denotes conjugacy length with respect to any finite generating set for Γ.

Theorem 1.1. Let 1 → H → G → Γ → 1 be a sequence of infinite hyperbolic groups that has the Scott–Swarup property and torsion-free kernel H. For every r ≥ 0 there is an N ≥ 0 such that if ∆ ≤ Γ is a finitely generated subgroup with rank(∆) ≤ r and ℓΓ(δ) ≥ N for each δ ∈ ∆ \ {1}, then

\[ \text{rank}(G_∆) = \text{rank}(H) + \text{rank}(∆). \]

Moreover, every minimal generating set for G_∆ is Nielsen equivalent to a generating set which contains a minimal generating set for H and projects to a minimal generating set for ∆.

Examples of subgroups ∆ ≤ Γ satisfying these conditions can easily be constructed. Indeed, for any set δ_1, . . . , δ_r of pairwise independent infinite order elements of Γ, Theorem 1.1 applies to ∆ = ⟨δ^{m_1}_1, . . . , δ^{m_r}_r⟩ for all sufficiently large m. Alternately, one can build finite-index subgroups K ≤ Γ such that Theorem 1.1 applies to every rank r subgroup of K.

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Theorem 1.1 generalizes a theorem of Juan Souto [Sou], who established this result when \( G \cong \mathbb{Z} \) and \( H \) is the fundamental group of a closed orientable surface \( S_g \) of genus \( g \geq 2 \). Here the extension is induced by a hyperbolic \( S_g \)-bundle over \( S^1 \) with pseudo-Anosov monodromy \( f : S_g \to S_g \), so that \( G \) is the fundamental group of the mapping torus \( M_f \) of \( f \). In this language, Souto proves that the rank of \( \pi_1(M_f) \cong G_{\curvearrowleft f} \) is equal to \( 2g + 1 \) for \( N \) sufficiently large. Moreover, any two minimal generating sets in this situation are Nielsen equivalent. See also the work of Biringer–Souto [BS] for more on this special case. In this paper, we use techniques previously established by Kapovich and Weidmann [KW2, KW1] to generalize Souto’s result to Theorem 1.1.

Theorem 1.1 applies to all hyperbolic extensions of surface groups [FM, Ham, KL] as well as all hyperbolic extensions of free groups by convex cocompact subgroups of Out\( (F_n) \) [DT1, HH, DT2]. We thus obtain the following corollary:

**Corollary 1.2.** The conclusions of Theorem 1.1 hold for all extensions of the following forms:

i. Extensions \( 1 \to \pi_1(S_g) \to G \to \Gamma \to 1 \) with \( G \) and \( \Gamma \) both infinite and hyperbolic.

ii. Extensions \( 1 \to F_g \to G \to \Gamma \to 1 \) such that \( G \) is hyperbolic and the induced outer action \( \Gamma \to \text{Out}(F_g) \) has convex cocompact image.

**Proof.** Since the kernels of the above extensions are torsion-free, it suffices to verify the Scott–Swarup property. For the surface group extensions in (i), this was established by Scott and Swarup in the case that \( \Gamma \cong \mathbb{Z} \) [SS] and by Dowdall–Kent–Leininger in the general case [DKL] (see also [MR]). For the free group extensions in (ii), Mitra [Mit] established the Scott–Swarup property when \( \Gamma \cong \mathbb{Z} \) and the general case was proven by the authors in [DT2] and by Mj–Rafi in [MR].

We note that Souto’s theorem is exactly case (i) above with \( \Gamma \) a cyclic group; the other cases of Corollary 1.2 are all new. In particular, the result is new even for free-by-cyclic groups \( G = F \rtimes_\phi \mathbb{Z} \) with fully irreducible and atoroidal monodromy \( \phi \in \text{Out}(F_g) \), where the conclusion is that \( F_g \rtimes_{\phi^N} \mathbb{Z} \) has rank \( g + 1 \) for all sufficiently large \( N \).

The following shows that the Scott–Swarup hypothesis cannot be dropped from Theorem 1.1

**Example 1.3** (Removing the Scott–Swarup property). In [Bri, Section 1.1.1], Brinkmann builds a hyperbolic automorphism \( \phi \) of the free group \( F = F_m \ast \langle a_0, \ldots, a_{n-1} \rangle \), where \( m \geq 3 \), of the form

\[
\phi(F_m) = F_m, \\
\phi(a_i) = \begin{cases} 
  a_{i+1} & \text{if } 0 \leq i < n - 1 \\
  w_0 v & \text{if } i = n - 1,
\end{cases}
\]

where \( w, v \in F_m \). Notice that the induced extension \( G_{\phi} = F \rtimes_\phi \mathbb{Z} \) does not have the Scott–Swarup property: \( F_m \) is not quasiconvex in \( F_m \rtimes_\phi \mathbb{Z} \) (which is hyperbolic) and hence not quasiconvex in \( G_{\phi} \). Focusing on the case where \( n = 2 \), one sees that for each \( k \) odd, \( \phi^k \) has the property that \( \phi^k(a_0) = w_k a_1 v_k \) and \( \phi^k(a_1) = w_k' a_0 v_k' \) for some \( w_k, v_k, w_k', v_k' \in F_m \). Hence, when \( k \) is odd, \( G_{\phi^k} \) is generated by \( F_m, a_0, \) and a generator of \( \mathbb{Z} \), making its rank at most \( m + 2 < \text{rank}(F) + 1 \).

**Acknowledgments:** This work drew inspiration from Souto’s paper [Sou] and owe’s an intellectual debt to the powerful machinery provided by Kapovich and Weidmann [KW1, KW2].

## 2 Setup

Fix a group \( G \) with a finite, symmetric generating set \( S \) and let \( X = \text{Cay}(G, S) \) be its Cayley graph. Equip \( X \) with the path metric \( d \) in which each edge has length 1, making \( (X, d) \) into a proper,
geodesic metric space. For subsets $A, B \subset X$, define $d(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\}$ and declare the $\varepsilon$-neighborhood of $A$ to be $N_\varepsilon(A) = \{x \in X \mid d(\{x\}, A) < \varepsilon\}$. The Hausdorff distance between sets is defined as

$$d_{\text{Haus}}(A, B) = \inf\{\varepsilon > 0 \mid A \subset N_\varepsilon(B) \text{ and } B \subset N_\varepsilon(A)\}.$$ 

We identify $G$ with the vertices of $X$ and define the wordlength of $g \in G$ by $|g|_\varepsilon = d(e, g)$, where $e$ is the identity element of $G$. A tuple in $G$ is a (possibly empty) ordered list $L = (g_1, \ldots, g_n)$ of $g$. The length of a tuple $L = (g_1, \ldots, g_n)$ is the number $\ell(L) = n$ of entries of the list, and its magnitude is defined to be $\|L\| = \max_i |g_i|_\varepsilon$. We define the conjugacy magnitude of a tuple $L$ to be $\mathcal{C}(L) = \min_{h \in G} \|hLh^{-1}\|$. The following three operations are called elementary Nielsen moves on a tuple $L = (g_1, \ldots, g_n)$:

- For some $i \in \{1, \ldots, n\}$, replace $g_i$ by $g_i^{-1}$ in $L$.
- For some $i, j \in \{1, \ldots, n\}$ with $i \neq j$, interchange $g_i$ and $g_j$ in $L$.
- For some $i, j \in \{1, \ldots, n\}$ with $i \neq j$, replace $g_i$ by $g_i g_j$ in $L$.

Two tuples are Nielsen equivalent if one may be transformed into the other via a finite chain of elementary Nielsen moves. Nielsen proved that any two minimal generating sets of a finitely generate free group are Nielsen equivalent [Nie]. Hence, two tuples $L_1$ and $L_2$ of length $n$ are Nielsen equivalent if and only if there is an automorphism $\psi : F_n \rightarrow F_n$ such that $\phi_1 = \phi_2 \circ \psi$, where $\phi_i : F_n \rightarrow G$ is the homomorphism taking the $j$th element of a (fixed) basis for $F_n$ to the $j$th element of $L_i$. Note that Nielsen equivalent tuples generate the same subgroup of $G$.

Following Kapovich–Weidmann [KW2, Definition 6.2], we consider the following variation:

**Definition 2.1.** A partitioned tuple in $G$ is a list $M = (Y_1, \ldots, Y_s; T)$ of tuples $Y_1, \ldots, Y_s, T$ of $G$ with $s \geq 0$ such that (1) either $s > 0$ or $\ell(T) > 0$, and (2) $\{Y_i\} \neq \{e\}$ for each $i > 0$. Thus $;T$ (where $\ell(T) > 0$) and $\langle Y_i \rangle$ (where $\langle Y_i \rangle \neq \{e\}$) are examples of partitioned tuples. The length of $M$ is defined to be $\ell(M) = \ell(Y_1) + \cdots + \ell(Y_s) + \ell(T)$. The underlying tuple of $M$ is the The $\ell(M)$–tuple $\ll(M) = (Y_1, \ldots, Y_s, T)$ obtained by concatenating $Y_1, \ldots, Y_s, T$. The elementary moves on a partitioned tuple $M = (Y_1, \ldots, Y_s; (t_1, \ldots, t_n))$ consist of:

- For some $i \in \{1, \ldots, s\}$ and $g \in \langle \cup_{j \neq i} Y_j \cup \{t_1, \ldots, t_n\} \rangle$, replace $Y_i$ by $gY_ig^{-1}$.
- For some $k \in \{1, \ldots, n\}$ and elements $u, u' \in \langle \cup_{j \neq k} Y_j \cup \{t_1, \ldots, t_{k-1}, t_{k+1}, \ldots, t_n\} \rangle$, replace $t_k$ by $ut_ku'$.

Two partitioned tuples $M$ and $M'$ are equivalent if $M$ can be transformed into $M'$ via a finite chain of elementary moves. In this case, it is easy to see that the underlying tuple $\ll(M)$ and $\ll(M')$ are Nielsen equivalent.

We henceforth assume that $G$ is a hyperbolic group, which is equivalent to requiring that $X$ be $\delta$–hyperbolic for some fixed $\delta \geq 0$. This means that every geodesic triangle $\triangle(a, b, c)$ in $X$ is $\delta$–thin in the sense that each side is contained in the $\delta$–neighborhood of the union of the other two. A geodesic in $X$ is a map $\gamma : J \rightarrow X$ of an interval $J \subset \mathbb{R}$ such that $|s - t| = d(\gamma(s), \gamma(t))$ for all $s, t \in J$. Two geodesic rays $\gamma_1, \gamma_2 : \mathbb{R}^+ \rightarrow X$ are asymptotic if $d_{\text{Haus}}(\gamma_1, \gamma_2) < \infty$. The Gromov boundary of $X$ is defined to be the set $\partial X$ of equivalence classes of geodesic rays in $X$. Note that every isometry of $X$ induces a self-bijection of $\partial X$. The equivalence class or endpoint of a ray $\gamma : \mathbb{R}^+ \rightarrow X$ is denoted $\gamma(\infty) \in \partial X$, and $\gamma$ is said to join $\gamma(0)$ to $\gamma(\infty)$. A biinfinite geodesic $\gamma : \mathbb{R} \rightarrow X$ determines two rays and is said to join their respective endpoints $\gamma(-\infty)$ and $\gamma(\infty)$. The
fact that $X$ is a proper and $\delta$–hyberbolic ensures that any two points of $X \cup \partial X$ can be joined by a geodesic segment, ray, or line; see [KB, KW1]. The convex hull of a set $Y \subset X \cup \partial X$ is the union $\text{Conv}(Y)$ of all geodesics joining points of $Y$ (including degenerate geodesics of the form $\{0\} \to Y$). The set $Y$ is $\epsilon$–quasiconvex if $\text{Conv}(Y) \subset \mathcal{N}_\epsilon(Y)$. A subgroup $U \leq G$ is $\epsilon$–quasiconvex if it is so when viewed as a subset of $X$. We refer the reader to [Gro, GdlH, BH] for further background on hyperbolic groups.

A sequence $\{x_n\}$ in $X$ is said to converge to $\zeta \in \partial X$ if for some (equivalently every) geodesic $\gamma: \mathbb{R}_+ \to X$ in the class $\zeta$ and sequence $\{t_m\}$ in $\mathbb{R}_+$ with $t_m \to \infty$, one has

$$\lim_{n,m} (d(x_n,x_0) + d(\gamma(t_m),x_0) - d(x_n,\gamma(t_m))) = \infty.$$ 

The limit set of a subgroup $U \leq G$ is the set $\Lambda(U)$ accumulation points $\zeta \in \partial X$ of an orbit $U \cdot x_0 \subset X$; the fact that any two orbits of $U$ have finite Hausdorff distance implies that this is independent of the point $x_0$. Following Kapovich–Weidmann [KW1, Definition 4.2] we define the hull of a subgroup $U$ to be

$$\mathcal{H}(U) = \text{Conv} \left( \text{Conv} \left( \Lambda(U) \cup \{ x \in X \mid d(x,u \cdot x) \leq 100\delta \text{ for some } x \in X \text{ and } u \in U \setminus \{ e \} \} \right) \right).$$

We leave the following fact as an exercise for the reader. Alternatively, it follows from a slight modification of [KW1, Lemma 4.10 and Lemma 10.3].

**Lemma 2.2.** There is a constant $A = A(\epsilon)$ for each $\epsilon \geq 0$ such that $d_{\text{Haus}}(U, \mathcal{H}(U)) \leq A$ for every torsion-free $\epsilon$–quasiconvex subgroup $U \leq G$ of $G$.

By noting that there are only finitely many subgroups of $G$ that may be generated by elements from the finite set $\mathcal{N}_r(\{ e \})$, we have the following lemma:

**Lemma 2.3.** There is a constant $c = c(r)$ for each $r > 0$ such that every quasiconvex subgroup $U \leq G$ generated by elements from the $r$–ball $\mathcal{N}_r(\{ e \})$ is $c$–quasiconvex.

The following technical result of Kapovich and Weidmann is a key ingredient in our argument:

**Theorem 2.4** (Kapovich–Weidmann [KW2, Theorem 6.7], c.f. [KW1, Theorem 2.4]). For every $m \geq 1$ there exists a constant $K = K(m) \geq 0$ with the following property. Suppose that $M = \langle Y_1, \ldots, Y_s; T \rangle$ is a partitioned tuple in $G$ with $\ell(M) = m$ and let $H = \langle \mathcal{U}(M) \rangle$ be the subgroup generated by the underlying tuple of $M$. Then either

$$H = \langle Y_1 \rangle \ast \cdots \ast \langle Y_s \rangle \ast \langle T \rangle,$$

with $\langle T \rangle$ free on the basis $T$, or else $M$ is equivalent to a partitioned tuple $M' = \langle Y_1', \ldots, Y_s'; T' \rangle$ for which one of the following occurs:

1. There are $i, j \in \{1, \ldots, s\}$ with $i \neq j$ and $d_{\text{Haus}}(\mathcal{H}(\langle Y_i' \rangle), \mathcal{H}(\langle Y_j' \rangle)) \leq K$.
2. There is some $i \in \{1, \ldots, s\}$ and $t \in T'$ such that $d_{\text{Haus}}(\mathcal{H}(\langle Y_i' \rangle), t \cdot \mathcal{H}(\langle Y_i' \rangle)) \leq K$.
3. There exists an element $t \in T'$ with a conjugate in $G$ of wordlength at most $K$.

We conclude this section with the following lemma, which ties into the conclusions of Theorem 2.4 and is an adaptation of [KW2, Propositions 7.3–7.4] to our context. Since the hypotheses of [KW2] are not satisfied here, we include a short proof.

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Lemma 2.5. For every $K, r > 0$ there is a constant $B = B(K, r)$ with the following property: Let $Y_1, Y_2, Y_3$ be tuples in $G$ generating torsion-free quasiconvex subgroups $U_i = \langle Y_i \rangle$ and satisfying $C(Y_i) \leq r$ for each $i = 1, 2, 3$.

- If $d(\mathcal{H}(U_1), \mathcal{H}(U_2)) \leq K$, then $(Y_1, Y_2)$ is Nielsen equivalent to a tuple $Y$ satisfying $C(Y) \leq B$.
- If $d(\mathcal{H}(U_3), g \cdot \mathcal{H}(U_3)) \leq K$ for $g \in G$, then $(Y_3, (g))$ is Nielsen equivalent to a tuple $Z$ with $C(Z) \leq B$.

Proof. For brevity, we prove the claims simultaneously. By assumption, we may choose points $x_1 \in \mathcal{H}(U_1)$, $x_2 \in \mathcal{H}(U_2)$ and $z_3, z_4 \in \mathcal{H}(U_3)$ with $d(x_1, x_2) \leq K$ and $d(z_3, g z_4) \leq K$. For $i = 1, 2, 3$, we also choose $h_i \in G$ such that $\|h_i Y_i^{-1}\| \leq r$. The subgroups $U'_i = h_i U_i h_i^{-1}$ are then $c(r)$-quasiconvex by Lemma 2.3 and hence satisfy $d_{\text{Baum}}(U'_i, \mathcal{H}(U'_i)) \leq A(c(r))$ by Lemma 2.2. Noting that $\mathcal{H}(U'_i) = h_i \mathcal{H}(U_i)$, we may choose $u_i \in U_i$ for $i = 1, 2$ such that $d(h_i u_i h_i^{-1}, h_i x_i) \leq A(c(r))$. Similarly choose $w_j \in U_3$ so that $d(h_3 w_j h_3^{-1}, h_3 z_j) \leq A(c(r))$ for $j = 3, 4$. Set $B = 4A(c(r)) + 2K + r$.

To conclude the second claim, observe that

$$
\begin{align*}
\|h_3(w_3^{-1}gw_4)h_3^{-1}\| &= d(w_3 h_3^{-1}, gw_4 h_3^{-1}) \\
&\leq d(w_3 h_3^{-1}, z_3) + d(z_3, g z_4) + d(g z_4, gw_4 h_3^{-1}) \\
&\leq 2A(c(r)) + K.
\end{align*}
$$

Since $\|h_3 Y_3 h_3^{-1}\| \leq r$ as well, the concatenated tuple $Z = (Y_3, (w_3^{-1}gw_4))$ clearly satisfies $C(Y') \leq B$. Further, since $w_3, w_4 \in \langle Y_3 \rangle$, it is immediate that $Z$ is Nielsen equivalent to $(Y_3, (g))$.

For the first claim, set $f = h_1 u_1^{-1} u_2 h_2^{-1}$ and use the triangle inequality to observe

$$
\begin{align*}
\|f\| &= d(u_1 h_1^{-1}, u_2 h_2^{-1}) \\
&\leq d(u_1 h_1^{-1}, x_1) + d(x_1, x_2) + d(x_2, u_2 h_2^{-1}) \\
&\leq 2A(c(r)) + K.
\end{align*}
$$

Since $\|h_1 Y_1 h_1^{-1}\| \leq r$, another use of the triangle inequality gives

$$
\|h_1 (u_1^{-1} u_2 Y_2 h_2 u_1^{-1}) h_1^{-1}\| = \|f(h_2 Y_2 h_2^{-1}) f^{-1}\| \leq 4A(c(r)) + 2K + r = B.
$$

The concatenated tuple $Y = (Y_1, u_1^{-1} u_2 Y_2 u_1^{-1})$ thus evidently satisfies $C(Y) \leq B$. To complete the proof, it only remains to show that $(Y_1, Y_2)$ is Nielsen equivalent to $Y$. But this is clear: since $u_2 \in \langle Y_2 \rangle$, the tuple $(Y_1, Y_2)$ is equivalent to $(Y_1, u_2 Y_2 u_2^{-1})$ which, since $u_1^{-1} \in \langle Y_1 \rangle$, is in turn equivalent to $Y$.

3 Proof of the main result

Suppose now that our fixed group $G$ fits into a short exact sequence

$$
1 \rightarrow H \rightarrow G \xrightarrow{\phi} \Gamma \rightarrow 1
$$

of infinite hyperbolic groups that enjoys the Scott–Swarup property with torsion-free kernel $H$. Recall that the conjugation action of $G$ on $H$ induces a homomorphism $\Phi: \Gamma \rightarrow \text{Out}(H)$ and that, since $G$ is hyperbolic, $\Phi$ has finite kernel. For any subgroup $\Delta \leq \Gamma$, we set $G_{\Delta} = \phi^{-1}(\Delta) \leq G$, and note that this subgroup of $G$ fits into the sequence $1 \rightarrow H \rightarrow G_{\Delta} \rightarrow \Delta \rightarrow 1$.

The follow lemma summarizes some of the basic properties we will require.
Lemma 3.1. For the sequence (1), we have the following:

i. For every infinite order \( g \in \Gamma \), \( \Phi(g) \in \text{Out}(H) \) does not fix the conjugacy class of any infinite index, finitely generated subgroup of \( H \).

ii. The kernel \( H \) is either free of rank at least 3 or else isomorphic to the fundamental group of a closed surface of genus at least 2.

iii. Every proper subgroup \( U \leq H \) is either quasiconvex or else has \( \text{rank}(U) > \text{rank}(H) \).

Proof. To prove item (i), suppose towards a contradiction that \( g \in \Gamma \) of infinite order fixes the conjugacy class of an infinite index, finitely generated subgroup \( A \) of \( H \). Then, after applying an inner automorphism of \( H \), we see that the semidirect product \( A \rtimes \varphi \mathbb{Z} \) is contained in \( G \), where \( \varphi \) is an automorphism in the class \( \Phi(g) \). However, it is well-known that the subgroup \( A \) is distorted (i.e. not quasi-isometrically embedded) in \( A \rtimes \varphi \mathbb{Z} \) and hence distorted in \( G \). This, however, contradicts the Scott–Swarup property and proves item (i).

Next, the theory of JSJ decompositions for hyperbolic groups [RS] (see also [Lev]) shows that a sequence of hyperbolic groups as in (1) with torsion-free kernel \( H \) must have \( H \) isomorphic to the free product \( \ast_{i=1}^k \Sigma_i \ast F_n \), where \( F_n \) is free of rank \( n \) and each \( \Sigma_i \) is the fundamental group of a closed surface. We must show that this factorization is trivial, i.e. either \( k = 0 \) or \( n = 0 \). This follows from the fact that such a nontrivial free product decomposition is canonical (e.g. [SW, Theorem 3.5]) and so is preserved under any automorphism of \( H \) (up to permuting the factors). Hence, for each infinite order \( g \in \Gamma \), some power of \( \Phi(g) \) fixes the conjugacy class of a surface group factor of \( H \), contradicting item (i) above unless \( k = 0 \) or \( n = 0 \). This proves (ii).

For (iii), let \( J = [U : H] > 1 \). If \( J = \infty \), then \( U \) is quasiconvex in \( G \) by the Scott–Swarup property. Otherwise basic covering space theory implies \( \text{rank}(U) = m(1 - J) + J \text{rank}(H) \) for \( m \in \{1, 2\} \) depending, respectively, on whether \( H \) is free or the fundamental group of a closed surface.

The following lemma is essential proven in [KK, Corollary 11] in the case where \( H \) is free and \( \Gamma \) is cyclic. We sketch the argument for the reader.

Lemma 3.2. If \( 1 \to H \to G \to \Gamma \to 1 \) is sequence of infinite hyperbolic groups such that \( H \) is torsion-free and \( G \) has the Scott–Swarup property, then \( G \) does not split over a cyclic (or trivial) group. Moreover, the same holds for \( G_\Delta \leq G \) whenever the subgroup \( \Delta \leq \Gamma \) is infinite.

Proof. We prove the moreover statement since it is clearly stronger. Let \( \Delta \leq \Gamma \) be an infinite subgroup. Suppose towards a contradiction that \( G_\Delta \) has a minimal, nontrivial action on a simplicial tree \( T \) with cyclic (or trivial) edge stabilizers. Since \( H \) is normal in \( G_\Delta \), the action \( H \acts T \) is also minimal. Hence the main theorem of [BF], implies that \( T/H \) is a finite graph. Notice that \( \Delta \) acts on the corresponding graph of groups decomposition of \( H \) (via \( \Phi : \Gamma \to \text{Out}(H) \)). First, this decomposition must have trivial edge groups: an infinite cyclic edge stabilizer would be fixed under some infinite order \( g \in \Delta \leq \Gamma \), contradicting that \( G \) is hyperbolic. Hence, the nontrivial graph of groups \( T/H \) has trivial edge stabilizers, but this implies that \( \Delta \) virtually fixes this splitting of \( H \). From this we obtain an infinite order element \( g \in \Delta \leq \Gamma \) which fixes a vertex group \( A \) of the splitting. Since \( A \) is finitely generated and has infinite index in \( H \), we have a contradiction to Lemma 3.1.i. This completes the proof.

The pieces are now in place to prove our main theorem:

Proof of Theorem 1.1. Let \( \bar{S} \subset \Gamma \) be the image of our fixed generating set \( S \subset G \). We assume that \( \ell_\Gamma(\cdot) \) is conjugacy length in \( \Gamma \) with respect to \( \bar{S} \). For the given \( r \), let \( K \) be the maximum of the
constants $K(1), \ldots, K(rank(H) + r)$ provided by Theorem 2.4. Set $D_0 = K$ and use Lemma 2.5 recursively to define $D_{n+1} = \max\{D_n, B(K, D_n)\}$ for each $n \in \mathbb{N}$. Set $N = 1 + D_{2\text{rank}(H)}$ and suppose that $\Delta \leq \Gamma$ is any subgroup with $\text{rank}(\Delta) \leq r$ and $\ell_{\Gamma}(\delta) \geq N$ for all $\delta \in \Delta \setminus \{1\}$. Let $G_{\Delta}$ be the preimage of $\Delta$ under the projection $p: G \to \Gamma$. We make the following observations:

**Claim 3.3.** If $Y$ is a tuple in $G$ with $Y \subset G_{\Delta}$ and $\mathcal{C}(Y) < N$, then $\langle Y \rangle \leq H$.

**Proof.** Choose $g \in G$ so that $\|gYg^{-1}\| < N$. Then for each $y \in Y$ we have

$$\|p(g)p(y)p(g)^{-1}\|_S = \|p(gyg^{-1})\|_S \leq \|gyg^{-1}\|_S < N$$

which shows that $\ell_{\Gamma}(g) < N$. Since we also have $p(y) \in \Delta$ by assumption, this gives $p(y) = 1$ and hence $y \in H$ by the hypothesis on $\Delta$. Thus $\langle Y \rangle \leq H$. \hfill $\square$

**Claim 3.4.** Fix $n \in \{0, \ldots, 2\text{rank}(H) - 1\}$ and suppose that $M = (Y_1, \ldots, Y_s; T)$ is a partitioned tuple with $\langle \mathcal{U}(M) \rangle = G_{\Delta}$ and $\ell(M) \leq \text{rank}(H) + r$ such that for each $i \in \{1, \ldots, s\}$ we have $\mathcal{C}(Y_i) \leq D_n$ with $\langle Y_i \rangle$ quasiconvex. Then there is a partitioned tuple $M = (\tilde{Y}_1, \ldots, \tilde{Y}_s; T)$ satisfying $\mathcal{C}(\tilde{Y}_i) \leq D_{n+1}$ for each $j \in \{1, \ldots, s\}$ such that $\langle \mathcal{U}(M) \rangle$ is Nielsen equivalent to $\langle \mathcal{U}(M) \rangle$ and either

a. $\ell(T) < \ell(T)$ with $\tilde{s} \leq s + 1$ or else

b. $\ell(T) = \ell(T)$ with $\tilde{s} < s$.

**Proof.** Since $\ell(M) \leq \text{rank}(H) + r$ and $\langle \mathcal{U}(M) \rangle = G_{\Delta}$ does not split as a nontrivial free product (Lemma 3.2), we may apply Theorem 2.4 to obtain partitioned tuple $M' = (Y'_1, \ldots, Y'_s; T')$ that is equivalent to $M$ and satisfies one of the three conclusions of that theorem. Since all elementary moves on a partitioned tuple $(W_1, \ldots, W_p; V)$ preserve the conjugacy class of each tuple $W_i$, we have $\mathcal{C}(Y'_i) \leq D_n$ with $\langle Y'_i \rangle$ quasiconvex for each $i$. As $D_n < N$, Claim 3.3 gives $\langle Y'_i \rangle \leq H$ and so ensures that $\langle Y'_i \rangle$ is torsion-free.

We now analyze the conclusions of Theorem 2.4: If $M'$ satisfies conclusion (1), then after reordering we may assume $\text{d}_\text{Haus}(\mathcal{H}(\langle Y'_i \rangle), \mathcal{H}(\langle Y'_j \rangle)) \leq K$ and use Lemma 2.5 to find a tuple $Y$ Nielsen equivalent to $\langle Y'_i, Y'_j \rangle$ with $\mathcal{C}(Y) \leq D_{n+1}$. The partitioned tuple $\langle Y, Y'_i, Y'_j; T' \rangle$ then satisfies the claim. If $M$ satisfies (2), then after reordering we have $\text{d}_\text{Haus}(\mathcal{H}(\langle Y'_i \rangle), \mathcal{H}(\langle Y'_j \rangle)) \leq K$ for some $i \in T'$ and so may use Lemma 2.5 to find a tuple $Z$ equivalent to $\langle Y'_i, \langle i \rangle \rangle$ with $\mathcal{C}(Z) \leq D_{n+1}$. Here we take $Z = (Z, Y'_i, \ldots, Y'_j; T' \setminus \{i\})$ to complete the claim. If $M'$ satisfies (3), then $T'$ contains an element $i$ with $\mathcal{C}(\langle i \rangle) \leq K \leq D_{n+1}$ and the partitioned tuple $\langle Y'_i, Y'_j, \langle i \rangle; T' \setminus \{i\} \rangle$ satisfies the claim. \hfill $\square$

We now complete the proof of the theorem: Let $L$ be any minimal-length tuple with $\langle L \rangle = G_{\Delta}$. Since $G_{\Delta}$ has a standard generating set of size $\text{rank}(H) + \text{rank}(\Delta)$, we have $\ell(L) \leq \text{rank}(H) + \text{rank}(\Delta)$. Set $M_0 = \langle L \rangle$ and observe that $M_0$ satisfies Claim 3.4 with $n = 0$. We may therefore inductively apply Claim 3.4 (with $n = 0, 1, \ldots$) to obtain a sequence $M_0, M_1, \ldots$ of partitioned tuples each with $\mathcal{U}(M_i)$ Nielsen equivalent to $L$. After inducting as many times as possible, we obtain a partitioned tuple $M_k = (Y_1, \ldots, Y_s; T)$ that satisfies $\mathcal{C}(Y_i) \leq D_k$ for each $i$ (by construction) but violates the hypotheses of Claim 3.4, either because $k = 2\text{rank}(H)$ or because some $\langle Y_i \rangle$ fails to be quasiconvex. Since $\mathcal{C}(Y_i) \leq D_k < N$, Claim 3.3 ensures that $\langle Y_i \rangle \leq H$ for each $i$. Since $G_{\Delta} = \langle \mathcal{U}(M_k) \rangle$ surjects onto $\Delta$, it follows that $\ell(T) \geq \text{rank}(\Delta)$. Thus at most $\ell(L) - \text{rank}(\Delta)$ applications of Claim 3.4 could have reduced the length of $T$ (option a) and so at least $k - \ell(L) + \text{rank}(\Delta)$ applications must have combined $Y_i$'s (option b). It now follows that $k < 2\text{rank}(H)$, for otherwise $k$ applications of the claim would necessarily produce a tuple $Y$ with $\ell(Y) > \text{rank}(H)$, contradicting $\ell(Y) + \ell(T) \leq \text{rank}(H) + \text{rank}(\Delta)$.\hfill $\square$
Since $M_k$ violates Claim 3.4 but $k < 2\text{rank}(H)$, it must be that some $\langle Y_i \rangle$ fails to be quasiconvex. After reordering, let us assume $\langle Y_1 \rangle \leq H$ is not quasiconvex. Note that we also cannot have $\text{rank}(\langle Y_i \rangle) > \text{rank}(H)$, for otherwise $\ell(Y_i) + \ell(T) > \text{rank}(H) + \text{rank}(\Delta)$ contradicting our choice of $L$. The only possibility afforded by Lemma 3.1.iii is therefore $\langle Y_1 \rangle = H$ with $\ell(Y_1) = \text{rank}(H)$. Since $\ell(M_k) \leq \text{rank}(H) + \text{rank}(\Delta)$, it follows that $M_k$ is of the form $M_k = (Y_1; T)$ with $\ell(Y_1) = \text{rank}(H)$ and $\ell(T) = \text{rank}(\Delta)$. Therefore $M_k$ is a standard generating set for $G_\Delta$ that is Nielsen equivalent to $L$.\hfill \Box

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