Chaos and quantum scars in coupled top model

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We consider a coupled top model describing two interacting large spins, which is studied semiclassically as well as quantum mechanically. This model exhibits variety of interesting phenomena such as quantum phase transition (QPT), dynamical transition and excited state quantum phase transitions above a critical coupling strength. Both classical dynamics and entanglement entropy reveals ergodic behavior at the center of energy band for an intermediate range of coupling strength above QPT, where the level spacing distribution changes from Poissonian to Wigner-Dyson statistics. Interestingly, in this model we identify quantum scars as reminiscence of unstable collective dynamics even in presence of interaction. Statistical properties of such scarred states deviate from ergodic limit corresponding to random matrix theory and violate Berry’s conjecture. In contrast to ergodic evolution, oscillatory behavior in dynamics of unequal time commutator and survival probability is observed as dynamical signature of quantum scar, which can be relevant for its detection.

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Introduction: After the recent experiment on a chain of Rydberg atoms [1], quantum scar (QS) in many body systems has drawn significant interest due to its connection with ergodicity and non equilibrium dynamics. Observed athermal behavior and periodic revival phenomena in dynamics of a specific initial state has been attributed to the many body quantum scarring phenomena [2–8]. Many body quantum scar (MBQS) gives rise to the deviation from ergodicity [2–20] leading to the violation of eigenstate thermalization hypothesis (ETH) [21] in quantum systems [7–11]. Recent theoretical studies on interacting spin systems reveal that emergent symmetries and symmetry protected many body states are the key ingredient for MBQS [11–19]. Originally QS in non interacting quantum chaotic system has been identified as reminiscence of unstable classical orbits [22, 23]. However such connection between MBQS and underlying dynamics remains unclear for interacting many body systems. Similar problem also arises in understanding ergodicity in quantum many particle systems from the viewpoint of chaos and phase space mixing. However some studies on quantum collective models reveal a connection between ergodicity and underlying chaotic dynamics [24–26], as well elucidate the role of underlying unstable collective dynamics in the formation of MBQS [27].

In this work we consider coupled top (CT) model which allows us to study the collective dynamics of two large spins semiclassically, revealing interesting phenomena like quantum phase transition (QPT), dynamical transition (DT) [28–31] and chaotic behavior at an intermediate regime of coupling strength. More importantly, in the chaotic regime we identify MBQS as a signature of unstable orbits of collective spin dynamics within certain symmetry class. Statistical and dynamical properties of such scarred states exhibit clear deviation from ergodicity.

Model and semiclassical analysis: The coupled top model [32–36] describes the dynamics of ferromagnetically coupled two large spins similar to transverse field Ising model [37, 38] which is governed by the Hamiltonian,

\[ H = -\hat{S}_{1x} - \hat{S}_{2x} - \mu \frac{\hat{S}}{S} \hat{S}_{1z} \hat{S}_{2z}, \]  

where \( \hat{S}_{ia} \) represents components \( (a = x, y, z) \) of two large spins of equal magnitude \( S \), denoted by index \( i = 1, 2 \) and \( \mu \) is the ferromagnetic coupling strength.

Large magnitude of spin \( S \gg 1 \) allows us to analyze the model semiclassically, where the spin vectors are represented by \( \hat{S}_i = (S \sin \theta_i \cos \phi_i, S \sin \theta_i \sin \phi_i, S \cos \theta_i) \). In terms of these dynamical variables, corresponding classical Hamiltonian can be written as,

\[ H_{cl} = -\sqrt{1 - z_i^2} \cos \phi_1 - \sqrt{1 - z_2^2} \cos \phi_2 - \mu z_1 z_2 \]  

where \( z_i = \cos \theta_i \) is conjugate momentum corresponding to the variable \( \phi_i \). Here \( H_{cl} \) and classical energy are scaled by \( S \). Classical spin dynamics is described by following equations of motion (EOM),

\[ \dot{z}_i = -\sqrt{1 - z_i^2} \sin \phi_i; \quad \dot{\phi}_i = \frac{z_i \cos \phi_i}{\sqrt{1 - z_i^2}} - \mu z_i \]  

where \( i \neq i \). To understand the overall behavior of dynamics, we first analyze the fixed points (FP) and their stability for varying coupling strength \( \mu \) as charted in Fig.1(a). Due to the interaction between the spins, at a critical coupling strength \( \mu_c = 1 \), the CT model undergoes a QPT to ferromagnetically ordered ground state (GS) as well as DT corresponding to the highest excited state (ES) with anti-ferromagnetic ordering. For \( \mu < \mu_c \), two symmetry unbroken stable FPs are represented by: (I) \( \{ z_1 = 0, \phi_1 = 0, z_2 = 0, \phi_2 = 0 \} \) with energy \( E = -2 \) (GS) and (II) \( \{ z_1 = 0, \phi_1 = \pi, z_2 = 0, \phi_2 = \pi \} \) with energy \( E = 2 \) (ES). For symmetry unbroken phase FP-I (FP-II) both spins are aligned to positive (negative) x-axis without any magnetization along z-axis. Both FP-I and II undergoes a pitchfork bifurcation at \( \mu_c \),
and become unstable for \( \mu > \mu_c \). As a result of bifurcation, two pairs of symmetry broken stable steady states appear above \( \mu_c \), namely: (III) \( \{ z_1 = z_2 = \pm \sqrt{1 - 1/\mu^2}, \phi_1 = \phi_2 = 0 \} \) with energy \( E = - (\mu + 1/\mu) \) which corresponds to the ferromagnetic GS and (IV) \( \{ z_1 = -z_2 = \pm \sqrt{1 - 1/\mu^2}, \phi_1 = \phi_2 = \pi \} \) representing a dynamically stable anti-ferromagnetic state with energy \( E = (\mu + 1/\mu) \) corresponding to the ES. Apart from these states, there exists another pair of unstable FPs which we denote as \( \pi \)-phase: (V) \( \{ z_1 = z_2 = 0, \phi_1 = \pi, \phi_2 = 0 \} \) and \( \{ z_1 = z_2 = 0, \phi_1 = 0, \phi_2 = \pi \} \) with energy \( E = 0 \).

**Quantum chaos and ergodicity:** To analyze the system quantum mechanically, we diagonalize the Hamiltonian Eq.1 for fixed magnitude of spin \( S \) so that the dimensionality of Hilbert space becomes \( N = (2S + 1)^2 \). The eigenvalues \( \mathcal{E}_n \) and corresponding eigenfunctions \( |\psi_n\rangle \) of the Hamiltonian are obtained using the basis states \( |m_{1z}, m_{2z}\rangle \) where \( m_{1z} \) is eigenvalue of \( S_{1z} \). The eigenfunctions and eigenvalues can be classified according to the symmetry of the Hamiltonian (Eq.1) which remains invariant under parity \( \mathcal{P} = e^{-i\pi \hat{x}} \). From the quantum mechanical analysis of the model we notice for \( \mu > \mu_c \), both QPT and DT leads to excited state quantum phase transitions (ESQPT) [41–45] corresponding to the unstable symmetry unbroken FPs I and II with energy densities \( E = -2 \) and 2 respectively where the derivative of semiclassical density of state becomes singular [39, 41–45].

Next, we study the Entanglement entropy (EE) of the eigenstates which contains useful information related to ergodicity. The reduced density matrix of \( S \)-th spin \( \hat{\rho}_S = \text{Tr}_S |\psi_n\rangle \langle \psi_n| \) is obtained by tracing out other spin sector \( (\hat{S} \neq S) \) which yields the EE \( S_{\text{en}} = -\text{Tr}(\hat{\rho}_S \log \hat{\rho}_S) \) of the corresponding state \( |\psi_n\rangle \). We quantify the degree of ergodicity of a state by comparing it with the EE of maximally random state partitioned into subsystems \( \mathcal{A}(\mathcal{B}) \) with dimension \( D_A(D_B) \). Maximal EE corresponding to subsystem \( \mathcal{A} \) with \( D_A \leq D_B \) is given by [46]

\[
S_{\text{max}} \approx \log(D_A) - D_A/2D_B
\]

where \( D_A = D_B = 2S + 1 \) for the present model. As seen from Fig.2(b), EE of the eigenstates with ascending energy density \( E \) shows a peak at the center of the band with \( E \approx 0 \); moreover, for such states the EE attains the maximum possible value given in Eq.4, within a range of intermediate coupling around \( \mu \approx 2 \). Also from classical dynamics we observe chaotic behavior of phase space trajectory with \( E \approx 0 \) which fills up the Bloch sphere as depicted in Fig.2(a).

To further investigate the degree of chaos at quantum level, we sort the eigenvalues \( \mathcal{E}_n \) in ascending order, and study the distribution \( P(\delta) \) of the level spacing \( \delta_n = \mathcal{E}_{n+1} - \mathcal{E}_n \) corresponding to same sector of parity and exchange symmetries following the usual prescription [47]. According to BGS conjecture, the level spacing distribution of classically chaotic system follows Wigner-Dyson (WD) statistics [48] whereas Poissonian statistics \( P(\delta) = \exp(-\delta) \) is observed in regular (integrable) regime [49]. For weak coupling \( (\mu < \mu_c) \), the level spacing follows Poissonian distribution as evident from Fig.3(c) whereas in the ergodic regime within a
range of intermediate coupling strength, it approaches to WD statistics $P(\delta) = \frac{\pi}{2} \delta \exp(-\pi \delta^2)$ corresponding to Gaussian orthogonal ensemble (GOE) of random matrix (see Fig.3(d)). Also, the average value of ratio of level spacing $(r) = (\min(\delta_n, \delta_{n+1})/\max(\delta_n, \delta_{n+1}))$ [50–52] confirms such crossover of statistics. Additionally, a detailed analysis of eigenvectors provides more information related to ergodicity.

Quantum many body scars: Existence of dynamically unstable ‘\(\pi\)-phase’ (FP-V) within the maximally ergodic regime around the energy density $E = 0$, motivates us to search for MBQS near $\mu \approx 2$ where EE approaches the upper limit set by Eq.4, forming a band like structure. Next, we investigate the statistical properties of the wavefunction of scarred states $|\psi_n\rangle = \sum_i \psi_n^i |i\rangle$ and compute the probability distribution $P(\eta)$ of the scaled elements $\eta \equiv |\psi_n^i|^2/\sigma$ where $\sigma$ denotes the standard deviation of $|\psi_n^i|^2$. The elements of the eigenvectors of GOE matrices follow Porter-Thomas (PT) distribution $P(\eta) = (1/\sqrt{2\pi \eta}) \exp(-\eta/2)$ [47], which is in accordance with Berry’s conjecture for higher energy eigenstates of a quantum systems whose classical counterpart is chaotic [54]. As seen from Fig.3(b) corresponding distribution $P(\eta)$ for scarred eigenstates deviates from the PT distribution in contrast to the ergodic states which also signifies athermal behaviour [55]. Also comparing Fig.3(c) and (d), we observe that degree of scarring reduces with increasing coupling strength $\mu$ and $P(\eta)$ approaches to PT distribution (see Fig.3(b)). This is a consequence of enhanced dynamical (Lyapunov) instability of ‘\(\pi\)-phase’ (FP-V) with $\mu$ [39].

Second type of scarred states are identified from the Shannon entropy (SE) $S_{Sh} = -\sum_i |\psi_n^i|^2 \log |\psi_n^i|^2$ of the eigenstates $|\psi_n\rangle$ at the band centred around $E = 0$. The SE of most states form a band like structure close to the GOE limit $\log(0.48N)$ [56, 57]; however a few states strongly deviate from it, as shown in Fig.4(a). The

![FIG. 2: (a) Chaotic behaviour of a trajectory on Bloch sphere corresponding to $E \approx 0$ and $\mu = 1.85$. (b) Variation of EE of eigenstates with energy density $E$ for different $\mu$. Horizontal pink dashed line represents the maximum EE corresponding to Eq.4 and vertical black dashed-dotted lines indicate ES-QPT. Level spacing distribution for (c) regular and (d) ergodic regime. Parameters chosen: $S = 30$ for (b), (c), (d) and all other figures unless otherwise mentioned.](image-url)
Husimi distribution of most deviated states reveal high degree of scarring as depicted in Fig.4(c),(d). Second type of scars resemble the shape of periodic orbits and are quite different from scar of ‘π-phase’. As observed earlier, these scarred states also violate Berry’s conjecture and the distribution of elements of the wavefunction deviates from GOE limit (see Fig.4(f)). We notice that weakly visible scars can also be present in Husimi distribution of ergodic states in the upper band (see Fig.4(b)) however the statistical properties of the wavefunction follow the GOE limit due to much lower degree of scarring compared to the deviated states.

Next, we search for the origin of second type of QS from the underlying classical dynamics. From Husimi distribution we observe that the structure of scar remains same for both the spin space resembling the shape of closed orbits with reflection symmetry. Also the classical Hamiltonian Eq.2 remains invariant under the transformation $\phi_1 \leftrightarrow \pm \phi_2$ and $z_1 \leftrightarrow \pm z_2$, which leads us to search for integrable motion within two symmetry classes (i): $\{z_1 = -z_2, \phi_1 = -\phi_2\}$ or (ii): $\{z_1 = z_2, \phi_1 = \phi_2\}$. Redefining the collective coordinates as $s \pm = (z_1 \pm z_2)/2$ and $\phi_\pm = (\phi_1 \pm \phi_2)/2$, the condition $\{z_+ = 0, \phi_+ = 0\}$ corresponding to symmetry class (ii), remains as steady state and the dynamics of remaining tw coordinates is effectively reduced to that of Lipkin Meshkov Glick (LMG) [58] model with single spin.

For a given energy $E$, the classical orbits of LMG model can be obtained analytically [30, 39] as shown in Fig.4(e) for different values of $E$ corresponding to two symmetry classes. For $E \approx 0$, the trajectories resemble the shape of QS, moreover the orbits of two symmetry classes touch at $z = 0, \phi = \pm \pi/2$, where the Husimi distribution shows large phase space density. The time period of such orbits with $E \approx 0$ is given by $T = 4K(k)/(1 + \mu^2)^{1/4}$ [39] where $K(k)$ is complete elliptic integral of the first kind [59] with $k^2 = 1/2(1 - 1/\sqrt{1 + \mu^2})$.

We also study the stability of these orbits using the method of monodromy matrix [60, 61] and find instability of the orbits for $\mu \geq 1.23$ where Lyapunov exponent increases with $\mu$ [39]. In this region, we also observe, for initial values slightly violating the symmetry condition (i) or (ii), the trajectories deviate from the closed orbit and diffuse in phase space indicating instability. This confirms that the second type of QS is a signature of unstable orbits preserving the symmetry classes. We notice that the degree of scarring reduces with increasing dynamical instability; as a result the deviated scarred states disappears as $\mu$ increases.

At this point we emphasize that the CT model exhibits integrable behavior in two extreme limits of coupling $\mu$, which is manifested by Poissonian level spacing distribution for $\mu \ll 1$, whereas the interaction term in the Hamiltonian (Eqs.1,2) dominates for $\mu \gg 1$ leading towards integrability [39] and resulting in a deviation from ergodicity.

**Dynamics of scarred states:** To investigate the dynamical signature of two types of QS as mentioned above, we consider two different methods. We study the time evolution of $|\pi_+\rangle$ state (Eq.5) representing first type of QS and compare it with the dynamics of arbitrary coherent state with similar energy $E \approx 0$. The survival probability $F(t) = \langle |\psi(t)\rangle \langle \psi(0)\rangle^2$ computed for $|\pi_+\rangle$ state shows an oscillatory behaviour and significant deviation from the GOE limit $3/N$ at long time [57, 62] in contrast to other states (see Fig.5(b)), thus indicating non-Markovian dynamics. Even after long time the Husimi distribution of initial $|\pi_\pm\rangle$ state retains the QS of ‘π-phase’ [27].

Next, we analyze the dynamics of the second type of scarred states by studying the “out-of-time-ordered correlator” (OTOC) which has recently become a useful tool to diagnose many body quantum chaos [63–71]. Here we investigate the dynamics of unequal time commutator of the spins $C(t) = \text{Tr}[\hat{S}_{i_1}(t)/S, \hat{S}_{i_2}(0)/S]^2$, which is related to the OTOC [72] and first introduced in the context of superconductivity [73]. To distinguish the second
type of scarred states we construct two initial density matrices $\rho(0)$ namely, $\hat{\rho}_{mc} \equiv \sum |\psi_j\rangle \langle \psi_j|/N_\Delta$ representing micro canonical ensemble of $N_\Delta$ ergodic states within the small window of energy density $\Delta E \sim 0.1$ around $E = 0$ and $\hat{\rho}_{dev} = |\psi_n\rangle \langle \psi_n|$ corresponding to the deviated state $|\psi_n\rangle$ with scar (see Fig.4(c)). As seen from Fig.5(a), for ergodic states $C(t)$ grows at a faster rate compared to $\hat{\rho}_{dev}$ and eventually saturates [70, 71], whereas it exhibits oscillations for $\hat{\rho}_{dev}$ reflecting non-ergodic behavior due to QS. Moreover, the period of such oscillation matches with the time period $T$ of classical orbit reflecting its underlying connection with QS.

**Conclusion:** We analyzed the coupled top model both semiclassically and quantum mechanically showing a series of interesting phenomena, such as, quantum phase transition, dynamical transition and excited state quantum phase transition at two edges of the energy band above a critical coupling. From phase space dynamics and EE, ergodic behavior at the central part of energy band is observed within an intermediate range of coupling. In this region we identify states with two types of QSs arising from unstable steady state and unstable orbits corresponding to symmetry preserved integrable motion. Such scarred states violate Berry’s conjecture and their statistical properties deviates from the ergodic limit corresponding to GOE. Dynamical signature of QS can also be observed from oscillatory behaviour of OTOC and survival probability contrasting the ergodic evolution.

In conclusion, we identified QS as reminiscence of underlying unstable collective dynamics even in presence of interaction. Such scarred states can be diagnosed from the statistical properties as well from dynamical behavior exhibiting a clear deviation from ergodicity.

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I. QUANTUM PHASE TRANSITION AND DYNAMICAL TRANSITION

The coupled top (CT) model undergoes a Quantum phase transition (QPT) as well as a Dynamical transition (DT) at the critical coupling $\mu_c = 1$ which is evident from the bifurcation of steady states obtained from semiclassical analysis as given in the main text. Quantum mechanically both QPT and DT can be identified from the change in the relevant physical quantities associated with the ground state (GS) and highest excited state (ES). Both DT and QPT are related under the transformation $\hat{S}_x \rightarrow -\hat{S}_x$ and $\mu \rightarrow -\mu$ of the Hamiltonian changing the ES to GS of anti-ferromagnetic CT model which is reflected from the energy density, averages of $\hat{S}_{1z}/S$ and $\hat{S}_{1z}\hat{S}_{2z}/S^2$ as shown in Fig.1. Here we point out that the magnetization along the z-axis of individual spins in the broken symmetry sector can not be obtained directly from the parity conserving eigenstates, however both QPT and DT are distinctly visible from the sharp change in the quantity $\langle \hat{S}_{1z}\hat{S}_{2z}/S^2 \rangle$ with increasing coupling strength $\mu$ as well as ferromagnetic and anti-ferromagnetic spin ordering of respective symmetry broken phase (see Fig.1(c),(f)). When spin operators are scaled by its magnitude $S$, the physical quantities converges to the semiclassically obtained analytical result for increasing value of $S$ revealing sharp change at the critical point as shown in Fig.1.

![Fig. 1](image-url)

**FIG. 1:** Signature of QPT is captured as a function of coupling strength $\mu$ by (a) energy $E_{GS}$ corresponding to ground state, (b) $\langle \hat{S}_{1z}/S \rangle$, c) $\langle \hat{S}_{1z}\hat{S}_{2z}/S^2 \rangle$ whereas that of DT is evident from (d) energy $E_{ES}$ corresponding to highest excited state, (e) $\langle \hat{S}_{1z}/S \rangle$ and (f) $\langle \hat{S}_{1z}\hat{S}_{2z}/S^2 \rangle$ for different values of spin $S$. The black solid line in all the figures denote the semiclassically obtained analytical result of the respective quantities.

- **Oscillation frequency:** We perform linear stability analysis of the fixed points (FP) obtained from the equations of motion (EOM) (Eq.3 in the main text) of the collective coordinates $\mathbf{X} = \{z_1, \phi_1, z_2, \phi_2\}$. From the time evolution of small fluctuation $\delta \mathbf{X}(0)$ around the steady state $\bar{\mathbf{X}}$, we obtain the frequency of small amplitude oscillation $\omega$ and the stability of FP is ensured from the condition $\text{Im}(\omega) = 0$. The oscillation frequency corresponding to a steady state $\bar{\mathbf{X}}$ can be written as,

$$\omega^2 = \frac{1}{2} \left( A_1 + A_2 - \sqrt{(A_1 - A_2)^2 + 4\mu^2 \cos \phi_1 \cos \phi_2} \right)$$  

(1)
\[ \omega = \begin{cases} \sqrt{1 - \mu} & \text{for } \mu < 1 \\ \sqrt{\mu^2 - 1} & \text{for } \mu \geq 1 \end{cases} \] (2)

which becomes gapless at \( \mu = \mu_c \) and vanishes as \( \sim (\mu - \mu_c)^{1/2} \) indicating mean field like behaviour of the transitions.

II. EXCITED STATE QUANTUM PHASE TRANSITION (ESQPT)

The CT model exhibits excited state quantum phase transition (ESQPT) above the critical coupling \( \mu_c = 1 \) corresponding to the unstable symmetry unbroken FPs I and II with energy density \( E = \pm 2 \) respectively. The ESQPT is associated with the singularity of semiclassical density of states (DOS) at critical energy densities [2–6]. Using the semiclassical Hamiltonian \( \mathcal{H}_{cl} \) (Eq.(2) in main text) the DOS can be written as,

\[ \rho(E) = \frac{C}{(2\pi)^2} \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \delta(E - \mathcal{H}_{cl}) \, d\phi_1 \, d\phi_2 \, dz_1 \, dz_2 \] (3)

where we use the normalization condition \( \int \rho(E) \, dE = 1 \), which yields \( C = 1/4 \). We also calculate DOS from quantum mechanical spectrum which is in good agreement with semiclassical result as compared in Fig.2(a). For \( \mu > \mu_c \), the derivative of the semiclassical DOS shown in the inset of Fig.2(a) revealing the singularities at critical energy densities \( E = \pm 2 \). Physically above \( \mu_c \), the symmetry unbroken FPs I and II at critical energy densities \( E = \pm 2 \) separates symmetry unbroken states within the range \( -2 < E < 2 \) from symmetry broken states \( E_{GS} < E < -2 \) and \( 2 < E < E_{ES} \) corresponding to QPT and DT respectively. As shown in Fig.2(b), the pair gap defined as \( \Delta_n = E_{2n} - E_{2n-1} \) \( (n = 1, 2, ...) \) vanishes exponentially with \( S \) for \( E < -2 \) and \( E > 2 \) revealing symmetry broken nature of quasi degenerate consecutive even odd parity states. The average values of the observables like \( \langle \hat{S}_{1z}/S \rangle \) and \( \langle \hat{S}_{1x}/S \rangle \) clearly distinguish symmetry unbroken states from symmetry broken sector with a significant change at \( E = \pm 2 \) indicating ESQPT (see Fig.2(c),(d)).

FIG. 2: Manifestation of ESQPT using (a) \( \rho(E) \) and its derivative \( d\rho/dE \) given in the inset, (b) pair gap \( \Delta \), (c) \( \langle \hat{S}_{1z}/S \rangle \) and (d) \( \langle \hat{S}_{1x}/S \rangle \) as a function of energy density \( E \) at \( \mu = 3 \). The red dashed lines marked at \( E = \pm 2 \) indicate the critical energy densities corresponding to ESQPT.
III. CLASSICAL PERIODIC ORBITS

Due to the symmetry of the Hamiltonian $H_{cl}$ ($z_1 \leftrightarrow \pm z_2, \phi_1 \leftrightarrow \pm \phi_2$), we explore the integrable motion within two symmetry classes: I $\{z_1 = -z_2, \phi_1 = -\phi_2\}$ and II $\{z_1 = z_2, \phi_1 = \phi_2\}$. These constraints remain as steady states $\{\phi_+ = 0; z_+ = 0\}$ and $\{\phi_- = 0; z_- = 0\}$ corresponding to symmetry class I and II respectively, where we define new coordinates as $z_{\pm} = (z_1 \pm z_2)/2$ and $\phi_{\pm} = (\phi_1 \pm \phi_2)/2$. Using the conditions mentioned above, the remaining degrees will satisfy the EOM of an effective anti-ferromagnetic (for I) or ferromagnetic (for II) LMG model,

$\dot{z}_- = -\sqrt{1 - z_-^2} \sin \phi_-; \quad \dot{\phi}_- = \frac{z_-}{\sqrt{1 - z_-^2}} \cos \phi_- + \mu z_- \quad \text{(For class I)}$  

$\dot{z}_+ = -\sqrt{1 - z_+^2} \sin \phi_+; \quad \dot{\phi}_+ = \frac{z_+}{\sqrt{1 - z_+^2}} \cos \phi_+ - \mu z_+ \quad \text{(For class II)}$  

The solution of above equations [7] is given in terms of elliptic function as

$$z_{\pm}(t) = C \, \text{cn} \left( \frac{C \mu}{2k} (t + t_0), k \right), \quad \cos(\phi_{\pm}(t)) = \frac{-E + \xi \mu z_{\pm}(t)^2}{2 \sqrt{1 - z_{\pm}(t)^2}}$$

where $\text{cn}$ is the Jacobi elliptic function with elliptic modulus $k$ and the constants are defined in the following way

$$C^2 = \frac{2}{\mu^2} \left[ -\frac{\xi \mu E}{2} - 1 + \Omega \right], \quad k^2 = \frac{1}{2} \left[ 1 - \frac{\xi \mu E/2 + 1}{\Omega} \right], \quad t_0 = \frac{F \left( \cos^{-1} \left( \frac{z_{\pm}(0)}{C} \right), k \right)}{\Omega^{1/2}}, \quad \Omega = \sqrt{\mu^2 + 1 + \xi E \mu}$$

The parameter $\xi \mu$ where $\xi = -1(\pm1)$ for class I (II) acts as the coupling constant for the effective anti-ferromagnetic (ferromagnetic) LMG as seen from Eq.4 (Eq.5). Within the energy range $-2 < E < 2$, the elliptic modulus is always $k < 1$. Interestingly the constants $C, k, \Omega$ are same for both the periodic orbits formed close to the unstable FP-I (II) belonging to class I (II) corresponding to energy $E < 0$ ($E > 0$). We find that the shape of the orbits belonging to two different symmetry classes having equal and opposite energy are same but with a $\pi$ shift along $\phi$. The time period $T$ for oscillations of $z_{\pm}(t)$ is given by

$$T = \frac{8kK(k)}{C \mu}$$

where $K(k) = F(\pi/2, k)$ is the complete elliptic integral of first kind and $F(\phi, k) = \int_0^\phi dx (1 - k^2 \sin^2 x)^{-1/2}$. To analyze the stability of these orbits, we calculate the Lyapunov exponent (LE). The LE is a measure which characterizes the growth of a small perturbation given to the solution of the system. If the small perturbation $\delta X(t = 0)$ to the collective coordinates $X = \{z_1, \phi_1, z_2, \phi_2\}$ grows exponentially in time, then one can write $||\delta X(t)|| = e^{\lambda t} ||\delta X(0)||$ and characterize the LE ($\lambda$) as

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \ln \left( \frac{||\delta X(t)||}{||\delta X(0)||} \right)$$

When the limit in Eq.8 exists and is positive, the trajectory is extremely sensitive to the initial condition and thus becomes chaotic in nature. To obtain LE, we calculate Monodromy matrix (MM) [8, 9] $M$ such that $\delta X(t) = M(t) \delta X(0)$, and the differential equation governing the evolution of MM is

$$\frac{dM(t)}{dt} = J(X(t)) M(t)$$

Where $J(X(t))$ is the Jacobian matrix. In order to obtain the LE for autonomous system, it is necessary to solve the dynamical equations and the elements of MM $M(t)$ simultaneously where the initial MM is identity matrix. By evolving the MM upto one time period $T$ given by the Eq.7, we obtain the largest eigenvalue of the MM. Then the largest lyapunov exponent (LLE) is given by

$$\lambda_l = \ln(m_l)/T$$

where $m_l$ is the largest eigenvalue of MM at $t = T$. 

FIG. 3: Instability of periodic orbits: (a) LLE ($\lambda_l$) as a function of $\mu$ for the periodic orbits with different energies, (b) time period of periodic orbits corresponding to energy $E = 0$ as a function of $\mu$. Evolution of a periodic orbit with energy $E \approx 0$ corresponding to class II under small symmetry breaking perturbation: (c) in the stable region with $\mu = 1.2$ and (d) unstable region with $\mu = 1.4$.

The LLE ($\lambda_l$) of the periodic orbits corresponding to energy $E = 0$ become positive for $\mu > \mu_u = 1.23$ (see Fig.3(a)), which shows that the orbits in the ergodic regime around $\mu \approx 2$ (as discussed in main text) are indeed unstable. As seen from Fig.3(a), the stability region decreases for periodic orbits with increasing magnitude of energy $|E|$ (from $E = 0$) as LLE becomes positive for $\mu < \mu_u$; finally it terminates at $\mu_c = 1$ for $E = \pm 2$ corresponding to unstable symmetry unbroken FPs I and II. Moreover, the LLE for $E = \pm 2$ coincides with the imaginary part of the excitation frequency $\text{Im}(\omega) = \sqrt{\mu - 1}$ (Eq.2) corresponding to FP I and II. This clearly indicates that the periodic orbits corresponding to class I(II) with energy $E < 0 (E > 0)$ formed around the unstable FP-I(FP-II) as shown in Fig.4(e) in the main text.

To confirm the stability of orbits with $E \approx 0$ in dynamics, we study time evolution of an initial point on a periodic orbit under small perturbation deviating from the symmetry class of corresponding orbit. In the stable region $\mu < \mu_u$, the periodic orbit is formed even in presence of symmetry breaking perturbation as seen from Fig.3(c). On the contrary, for $\mu > \mu_u$ the trajectory of perturbed initial point diffuses in phase space without forming closed orbit (see Fig.3(d)) indicating instability. Although such perturbed trajectory does not form periodic orbit of particular symmetry class in the unstable region, it evolves around the orbits of both the symmetry classes diffusively, as seen from Fig.3(d). Interestingly, such phase space structure of unstable orbit resembles the shape of QS observed in Husimi distribution shown in Fig.4(c) of main text.

IV. INTEGRABILITY AT LARGE COUPLING STRENGTH

It is interesting to note that CT model becomes integrable in two extreme limit of coupling strength $\mu$. For $\mu \rightarrow 0$, the Hamiltonian describes two non interacting spins precessing around the x-axis. At the quantum level this is manifested by Poissonian level spacing distribution for $\mu < 1$. In the opposite limit $\mu \gg 1$, we scale the Hamiltonian (Eq.1 in the main text) by $\mu$ which can be rewritten as,

$$\hat{H} = -\epsilon(\hat{S}_{1z} + \hat{S}_{2z}) - \frac{1}{S}\hat{S}_{1z}\hat{S}_{2z}$$

where $\epsilon \equiv 1/\mu$ and $\epsilon \rightarrow 0$ ($\mu \rightarrow \infty$). Semiclassically, the above Hamiltonian can be described by the collective coordinates as,

$$\mathcal{H}_{cl} = -\epsilon \left(\sqrt{1 - z_1^2} \cos \phi_1 + \sqrt{1 - z_2^2} \cos \phi_2\right) - z_1z_2.$$
For $\epsilon = 0$, the classical Hamiltonian $H_{cl}$ is integrable and independent of angle variable. The EOM yields the solution $\phi_i = z_i t + C$ with constant $z_i$ which represent precession of spins around the $z$-axis as depicted in Fig.4.

FIG. 4: Bloch sphere trajectories shown in a particular spin sector at different energies (a) for $\epsilon = 0$ and (b) for $\epsilon = 0.01$. Diffusive behaviour in the trajectories is observed for $\epsilon = 0.01$ near $E = 0$ whereas a regular behaviour is observed away from the equator.

The above results obtained from semiclassical analysis at large $\mu$ are also supported by quantum analysis as well. To see this, we sort the energy levels obtained from exact diagonalization of the above Hamiltonian (Eq.11) in ascending order belonging to a particular parity as well as exchange symmetry sector (mentioned in main text). Next, we compute the level spacing (LS) $\delta_n = \epsilon_n - \epsilon_{n-1}$ which is shown with increasing value of the index $n$ in Fig.5. For the integrable limit with $\epsilon = 0$, the LS are distributed in a regular fashion compared to the that for $\epsilon \neq 0$. However, for $\epsilon \ll 1$, the LS corresponding to very low as well as high energy states show almost regular structure, whereas the LS at the middle of the spectrum ($E \approx 0$) exhibits significant deviation which is consistent with the classical picture (see Fig.4). Also such deviation increases as the Hamiltonian differs from the integrable limit with increasing value of $\epsilon$. A similar analysis has also been done in the context of Dicke model [10].

FIG. 5: (a) Plot of level spacing $\delta_n$ with index $n$ for different values of $\epsilon$ showing a regular behavior at $\epsilon = 0$ and almost regular behavior for very low as well as high energy eigenstates. Parameter chosen: $S = 30$. The symmetry sector is considered even for both the symmetry operators.

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