A NOTE ON SUBMANIFOLDS OF $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ WITH RESPECT TO CERTAIN CONNECTIONS

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Abstract. The present paper deals with some results of almost semi-invariant submanifolds of generalized Sasakian-space-forms in [3] with respect to semisymmetric metric connection, semisymmetric non-metric connection, Schouten-van Kampen connection and Tanaka-Webster connection.

1. Introduction

The notion of generalized Sasakian-space-form was introduced by Alegre et al. [2]. An almost contact metric manifold $\tilde{M}(\phi, \xi, \eta, g)$ whose curvature tensor $\tilde{R}$ satisfies

$$\tilde{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)X + \eta(Y)\eta(X)Z - g(Y, Z)\eta(X)\xi\}$$

for all vector fields $X, Y, Z$ on $\tilde{M}$ and $f_1, f_2, f_3$ are certain smooth functions on $\tilde{M}$ is said to be generalized Sasakian-space-form [2]. Such a manifold of dimension $(2n+1)$, $n > 1$ (the condition $n > 1$ is assumed throughout the paper), is denoted by $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ [2]. Many authors studied this space form with different aspects. For this, we may refer, ([9], [10], [11], [12], [13], [16], [17] and [22]). It reduces to Sasakian-space-form if $f_1 = c + \frac{3}{4}$, $f_2 = f_3 = c - \frac{1}{4}$ [2]. We denote Sasakian-space-form of dimension $(2n + 1)$ by $M^{2n+1}(c)$.

After introduced the semisymmetric linear connection by Friedman and Schouten [6], Hayden [8] gave the idea of metric connection with torsion on a Riemannian manifold. Later, Yano [29] and many others (see, [20], [21], [23] and references therein) studied semisymmetric metric connection in different context. The idea of semisymmetric non-metric connection was introduced by Agashe and Chafle [1].

The Schouten-van Kampen connection introduced for the study of non-holomorphic manifolds ([19], [27]). In 2006, Bejancu [5] studied Schouten-van Kampen connection on foliated manifolds. Recently Olszak [17] studied Schouten-van Kampen connection on almost(para) contact metric structure.

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The Tanaka-Webster connection ([24], [28]) is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold. Tanno [25] defined the Tanaka-Webster connection for contact metric manifolds. In [3], Alegre and Carriazo studied almost semi-invariant submanifolds of generalized Sasakian-space-form with respect to Levi-Civita connection. In this paper, we have studied the results of [3] with respect to certain connections, namely semi-symmetric metric connection, semi-symmetric non-metric connection, Schouten-van Kampen connection, Tanaka-Webster connection.

2. Preliminaries

In an almost contact metric manifold \( \bar{M} (\phi, \xi, \eta, g) \), we have [4]

\[
\phi^2 (X) = -X + \eta(X)\xi, \quad \phi \xi = 0,
\]

\[
\eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0.
\]

\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\]

\[
g(\phi X, Y) = -g(X, \phi Y).
\]

In \( \bar{M}^{2n+1} (f_1, f_2, f_3) \), we have [2]

\[
(\bar{\nabla} X \phi)(Y) = (f_1 - f_3)[g(X, Y)\xi - \eta(Y)X],
\]

\[
\bar{\nabla} X \xi = -(f_1 - f_3)\phi X,
\]

where \( \bar{\nabla} \) is the Levi-Civita connection of \( \bar{M}^{2n+1} (f_1, f_2, f_3) \).

The semi-symmetric metric connection \( \bar{\nabla}' \) and the Riemannian connection \( \bar{\nabla} \) on \( \bar{M}^{2n+1} (f_1, f_2, f_3) \) are related by [29]

\[
\bar{\nabla}'_X Y = \bar{\nabla}_X Y + \eta(Y)X - g(X, Y)\xi.
\]

The Riemannian curvature tensor \( \bar{R} \) of \( \bar{M}^{2n+1} (f_1, f_2, f_3) \) with respect to \( \bar{\nabla} \) is

\[
\bar{R}(X, Y)Z = (f_1 - 1)\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y
- g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\}
+ (f_3 - 1)\{\eta(X)\eta(Z)Y
- \eta(Y)\eta(Z)X + g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\}
+ (f_1 - f_3)\{g(X, \phi Z)Y - g(X, Z)\phi Y\}
+ g(Y, Z)\phi X - g(X, Z)\phi Y\}.
\]

The semi-symmetric non-metric connection \( \bar{\nabla}' \) and the Riemannian connection \( \bar{\nabla} \) on \( \bar{M}^{2n+1} (f_1, f_2, f_3) \) are related by [1]

\[
\bar{\nabla}'_X Y = \bar{\nabla}_X Y + \eta(Y)X.
\]
The Riemannian curvature tensor $\bar{R}$ of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}$ is

$$\bar{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\} + f_3\{\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X - g(Y, Z)\eta(X)\xi\} + \{(f_1 - f_3)g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\}.$$ \hfill (2.10)

The Schouten-van Kampen connection $\hat{\nabla}$ and the Riemannian connection $\bar{\nabla}$ of $M^{2n+1}(f_1, f_2, f_3)$ are related by [18]

$$\hat{\nabla}_X Y = \bar{\nabla}_X Y + (f_1 - f_3)\eta(Y)\phi X - (f_1 - f_3)g(\phi X, Y)\xi.$$ \hfill (2.11)

The Riemannian curvature tensor $\hat{R}$ of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\nabla}$ is

$$\hat{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + \{(f_1 - f_3)g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\}.$$ \hfill (2.12)

The Tanaka-Webster connection $\tilde{\nabla}$ and the Riemannian connection $\bar{\nabla}$ of $M^{2n+1}(f_1, f_2, f_3)$ are related by [7]

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + \eta(X)\phi Y + (f_1 - f_3)\eta(Y)\phi X - (f_1 - f_3)g(\phi X, Y)\xi.$$ \hfill (2.13)

The Riemannian curvature tensor $\tilde{R}$ of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\nabla}$ is

$$\tilde{R}(X, Y)Z = f_1\{g(Y, Z)X - g(X, Z)Y\} + f_2\{g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X\} + \{(f_1 - f_3)g(X, \phi Z)\phi Y - g(Y, \phi Z)\phi X\} + 2(f_1 - f_3)g(X, \phi Y)\phi Z.$$ \hfill (2.14)

Let $M$ be a $(m + 1)$-dimensional submanifold of $M^{2n+1}(f_1, f_2, f_3)$. If $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively then the Gauss and Weingarten formulae are given by [30]

$$\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X V = -A_V X + \nabla^\perp_X V$$ \hfill (2.15)

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $h$ and $A_V$ are second fundamental form and shape operator (corresponding to the normal vector field $V$), respectively and they are related by $g(h(X, Y), V) = g(A_V X, Y)$ [30].

Moreover, if $h(X, Y) = 0$ for all $X, Y \in \Gamma(TM)$ then $M$ is said to be totally geodesic and if $H = 0$ then $M$ is minimal in $M$, where $H$ is the mean curvature tensor.
From (2.15), we have the Gauss equations as

\[ \hat{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) + g(h(X, Z), \hat{h}(Y, W)), \]

where \( R \) is the curvature tensor of \( M \). Let \( \hat{\nabla}, \nabla', \hat{\nabla} \) and \( \hat{\nabla} \) are the induced connection of \( M \) from the connection \( \hat{\nabla}, \nabla', \hat{\nabla} \) of \( M^{2n+1}(f_1, f_2, f_3) \) respectively.

Then Gauss equation with respect to \( \hat{\nabla}, \nabla', \hat{\nabla} \) and \( \hat{\nabla} \) are

\[ \hat{R}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) - g(h'(X, W), h'(Y, Z)) + g(h'(X, Z), \hat{h}'(Y, W)), \]

\[ \hat{R}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) - g(h(X, W), \hat{h}(Y, Z)) + g(h(X, Z), \hat{h}(Y, W)), \]

\[ \hat{R}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) - g(\hat{h}(X, W), \hat{h}(Y, Z)) + g(\hat{h}(X, Z), \hat{h}(Y, W)), \]

where \( \hat{h}, h', \hat{h}, \hat{h} \) are the second fundamental forms with respect to \( \hat{\nabla}, \nabla', \hat{\nabla} \) and \( \hat{\nabla} \) respectively. Also \( \hat{H}, H', \hat{H}, \hat{H} \) be the mean curvature of \( M \) with respect to \( \hat{\nabla}, \nabla', \hat{\nabla} \) and \( \hat{\nabla} \) respectively.

For any \( X \in \Gamma(TM) \), we may write

\[ \phi X = TX + FX, \]

where \( TX \) is the tangential component and \( FX \) is the normal component of \( \phi X \).

**Definition 2.1.** ([3], [26]) A submanifold \( M \) of an almost contact metric manifold \( M, \xi \) tangent to \( M \), is said to be an almost semi-invariant submanifold if their exist \( l \) functions \( \lambda_1, \ldots, \lambda_l \), defined on \( M \) with values in \((0,1)\), such that

(i) \(-\lambda_1^2(p), \ldots, -\lambda_l^2(p)\) are distinct eigenvalues of \( T^2|_D \) at \( p \in M \), with

\[ T_pM = D^1_p \oplus D^0_p \oplus D^{\lambda_1}_p \oplus \cdots \oplus D^{\lambda_l}_p \oplus \text{span}\{\xi_p\}, \]

where \( D^\lambda_p, \lambda \in \{1,0,\lambda_1(p), \ldots, \lambda_l(p)\} \), denotes the eigenspace associated to the eigenvalue \(-\lambda^2\).

(ii) the dimension of \( D^1_p, D^0_p, D^{\lambda_1}_p, \ldots, D^{\lambda_l}_p \) are independent of \( p \in M \).

Let the orthogonal projection from \( TM \) on \( D^\lambda \) be \( U^\lambda \). Then we have

\[ g(TX, TY) = \sum_\lambda \lambda^2 g(U^\lambda X, U^\lambda Y). \]

Let us consider \( \{E_1, \ldots, E_m, E_{m+1} = \xi\} \) and \( \{F_1, \ldots, F_{2n-m}\} \) local orthonormal basis of \( TM \) and \( T^\perp M \) respectively, and denote \( A_{F_k} = A_k \).
3. Ricci tensor on $M$ of $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ with respects to $\tilde{\nabla}$

Lemma 3.1. The Ricci tensor $\tilde{S}$ of submanifold $M$ of $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\nabla}$ is

\begin{equation}
\tilde{S}(X,Y) = m f_1 g(X,Y) + 3 f_2 g(TX,TY) - (f_3 - 1)\{g(X,Y) + (m - 1)\eta(X)\eta(Y)\} + (f_1 - f_3)(m - 1)g(TX,Y)
+ \sum_{k=1}^{2n-m} \{(m + 1)\text{trace} \tilde{A}_k g(\tilde{A}_kX,Y) - g(\tilde{A}_kX,\tilde{A}_kY)\}
\end{equation}

for any vector fields $X, Y$ on $M$.

Proof. Using (2.8) and (2.17) we have the above Lemma. \hfill \Box

Lemma 3.2. The Ricci tensor $\tilde{S}$ of almost semi-invariant submanifolds $M$ of $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\nabla}$ is

\begin{equation}
\tilde{S}(X,Y) = \sum_{\lambda} \left( m f_1 + 3 f_2 \lambda^2 - f_3 + 1 \right) g(U^\lambda X, U^\lambda Y)
+ m(f_1 - f_3 + 1)\eta(X)\eta(Y) + (f_1 - f_3)(m - 1)g(TX,Y)
+ \sum_{k=1}^{2n-m} \{(m + 1)\text{trace} \tilde{A}_k g(\tilde{A}_kX,Y) - g(\tilde{A}_kX,\tilde{A}_kY)\}
\end{equation}

for any vector fields $X, Y$ on $M$.

Proof. Using (2.22) and (3.1) we have the above Lemma. \hfill \Box

Corollary 3.1. For an almost semi-invariant submanifolds $M$ of Sasakian-space-form $\tilde{M}^{2n+1}(c)$ with respect to $\tilde{\nabla}$ is

\begin{equation}
\tilde{S}(X,Y) = \frac{(m - 1 + 3\lambda^2)c + 3(m - \lambda^2) + 5}{4} g(U^\lambda X, U^\lambda Y)
+ (m - 1)g(TX,Y)
+ \sum_{k=1}^{2n-m} \{(m + 1)\text{trace} \tilde{A}_k g(\tilde{A}_kX,Y) - g(\tilde{A}_kX,\tilde{A}_kY)\}
\end{equation}

for any vector fields $X, Y$ on $M$.

Proof. Putting $f_1 = \frac{c + 3}{4}$, $f_2 = f_3 = \frac{c - 1}{2}$ in (3.2) we obtain the result. \hfill \Box

Lemma 3.3. The scalar curvature $\tilde{\varphi}$ of an almost semi-invariant submanifold $M$ of $\tilde{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\tilde{\nabla}$ is

\begin{equation}
\tilde{\varphi} = f_1 + \frac{1}{m(m + 1)} \left( 3 f_2 \sum_{\lambda} n(\lambda)\lambda^2 - 2m f_3 + 2m \right)
+ (m + 1)^2 ||\tilde{H}||^2 - ||\tilde{h}||^2.
\end{equation}

Proof. Let us consider an orthonormal frame $\{E_1, \cdots, E_{n(\lambda)}\}$ in $D^\lambda$. Then we have

\begin{equation}
\tilde{\varphi} = \frac{1}{m(m + 1)} \sum_{i,j=1}^{m+1} \tilde{R}(E_i, E_j, E_{j}, E_i).
\end{equation}

Using (2.8), (2.17) in (3.5) we get (3.4). \hfill \Box
Theorem 3.1. If $M$ is an almost semi-invariant minimal submanifold of \( M^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla} \), then the following relation holds:

(i) \( \tilde{S}(X, X) \leq \sum_{\lambda}(mf_1 + 3f_2\lambda^2 - f_3 + 1)g(U^\lambda X, U^\lambda X) + m(f_1 - f_3 + 1)\eta(X)\eta(X) + (f_1 - f_3)(m - 1)g(TX, X) \),

(ii) \( \tilde{\tau} \leq f_1 + \frac{1}{m(m+1)}\{3f_2\sum_{\lambda}n(\lambda)\lambda^2 - 2m(f_3 - 1) \} \).

Proof. Since $M$ is minimal submanifold with respect to \( \tilde{\nabla} \), then we have

\[
(3.6) \quad \sum_{k=1}^{2n-m} (\text{trace} \, \tilde{A}_k)g(\tilde{A}_k X, X) = \sum_{i=1}^{m+1} g(\tilde{h}(X, X), \tilde{h}(E_i, E_i)) = (m + 1)g(\tilde{h}(X, X), \tilde{H}) = 0.
\]

Using (3.10) in (3.2) we have

\[
(3.7) \quad \tilde{S}(X, X) - \sum_{\lambda}(mf_1 + 3f_2\lambda^2 - f_3 + 1)g(U^\lambda X, U^\lambda X) - m(f_1 - f_3 + 1)\eta(X)\eta(X) - (f_1 - f_3)(m - 1)g(TX, X)
\]

\[
= - \sum_{k=1}^{2n-m} g(\tilde{A}_k X, \tilde{A}_k X) \leq 0,
\]

which proves (i).

The second part (ii) comes directly from Lemma 3.3. \( \square \)

Remark 3.1. The equality of (i) and (ii) in Theorem 3.1 holds if $M$ is almost semi-invariant totally geodesic submanifolds of \( M^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla} \).

Proof. If $M$ is totally geodesic submanifold with respect to \( \tilde{\nabla} \), then $M$ is minimal submanifold with respect to \( \tilde{\nabla} \). Then by virtue of Lemma 3.2 we have the equality case (i) and by virtue of Lemma 3.3 we have equality case of (ii). \( \square \)

4. Submanifolds of \( M^{2n+1}(f_1, f_2, f_3) \) with \( \tilde{\nabla}' \)

Lemma 4.1. The Ricci tensor \( S' \) of submanifold $M$ of \( M^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla}' \) is

\[
(4.1) \quad S'(X, Y) = mf_1g(X, Y) + 3f_2g(TX, TY) - (f_3 - 1)\{g(X, Y) + (m - 1)\eta(X)\eta(Y) - m\eta(X)\eta(Y) + (f_1 - f_3)g(TX, Y) + \sum_{k=1}^{2n-m} \{(m + 1)(\text{trace} A'_k)g(A'_k X, Y) - g(A'_k X, A'_k Y)\}.
\]

for any vector fields $X, Y$ on $M$.

Proof. Using (2.10) and (2.18) we have the above Lemma. \( \square \)
Lemma 4.2. The Ricci tensor $\dot{S}'$ of almost semi-invariant submanifolds $M$ of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\nabla'$ is

\[
\dot{S}'(X, Y) = \sum_\lambda (mf_1 + 3f_2\lambda^2 - f_3 + 1)g(U^\lambda X, U^\lambda Y) \\
+ m(f_1 - f_3)\eta(X)\eta(Y) + (f_1 - f_3)g(TX, Y) \\
+ \sum_{k=1}^{2n-m} \{(m + 1)(\text{trace}A_k)g(A'_kX, Y) - g(A'_kX, A'_kY)\}
\]

for any vector fields $X, Y$ on $M$.

Proof. Using (4.2) and (4.1) we have the above Lemma.

Corollary 4.1. The Ricci tensor $S'$ of almost semi-invariant submanifolds $M$ of $M^{2n+1}(c)$ with respect to $\nabla'$ is

\[
S'(X, Y) = \frac{(m - 1 + 3\lambda^2)c + 3(m - \lambda^2) + 5}{4}g(U^\lambda X, U^\lambda Y) + g(TX, Y) \\
+ \sum_{k=1}^{2n-m} \{(m + 1)(\text{trace}A_k)g(A'_kX, Y) - g(A'_kX, A'_kY)\}
\]

for any vector fields $X, Y$ on $M$.

Proof. Putting $f_1 = \frac{c+3}{4}, \ f_2 = f_3 = \frac{c-1}{4}$ in (4.2) we obtain the result.

Lemma 4.3. The scalar curvature $\tau'$ of an almost semi-invariant submanifold $M$ of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\nabla'$ is

\[
\tau' = f_1 + \frac{1}{m(m+1)}\{3f_2\sum_\lambda n(\lambda)\lambda^2 - 2mf_3 + m\} \\
+ (m + 1)^2||H'||^2 - ||h'||^2.
\]

Proof. It is known that

\[
\tau' = \frac{1}{m(m+1)}\sum_{i,j=1}^{m+1} R'(E_i, E_j, E_j, E_i).
\]

Using (2.10), (2.18) in (4.5) we get (4.4).

Theorem 4.1. If $M$ is an almost semi-invariant minimal submanifolds of $M^{2n+1}(f_1, f_2, f_3)$ with respect to $\nabla'$, then the following condition holds:

(i) $S'(X, X) \leq \sum_\lambda (mf_1 + 3f_2\lambda^2 - f_3 + 1)g(U^\lambda X, U^\lambda X) + m(f_1 - f_3)\eta(X)\eta(X) + (f_1 - f_3)g(TX, X)$,

(ii) $\tau' \leq f_1 + \frac{1}{m(m+1)}\{3f_2\sum_\lambda n(\lambda)\lambda^2 - 2mf_3 + m\}$.

Proof. Since $M$ is minimal submanifold with respect to $\nabla'$, then we have

\[
\sum_{k=1}^{2n-m} (\text{trace}A'_k)g(A'_kX, X) = \sum_{i=1}^{m+1} g(h'(X, X), h'(E_i, E_i)) \\
= (m + 1)g(h'(X, X), H') = 0.
\]
Using (4.2) and (4.6) we have

\begin{equation}
S'(X, X) - \sum \lambda \left( m f_1 + 3 f_2 \lambda^2 - f_3 + 1 \right) g(U^\lambda X, U^\lambda X) - m(f_1 - f_3) \eta(X) \eta(X) - (f_1 - f_3) g(TX, X) \nonumber \\
= - \sum_{k=1}^{2n-m} g(A'_k X, A'_k X) \leq 0.
\end{equation}

This proves (i).

The second part (ii) is comes directly from Lemma 4.3. \qed

Remark 4.1. The equality of (i) and (ii) in Theorem 4.1 holds if \( M \) is almost semi-invariant totally geodesic submanifolds of \( \tilde{M}^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla}' \).

Proof. If \( M \) is totally geodesic submanifold with respect to \( \tilde{\nabla}' \), then \( M \) is minimal submanifold with respect to \( \tilde{\nabla}' \). Then by virtue of Lemma 4.2 we have the equality case of (i) and by virtue of Lemma 4.3 we have the equality case (ii). \qed

5. Submanifolds of \( \tilde{M}^{2n+1}(f_1, f_2, f_3) \) with \( \tilde{\nabla} \)

Lemma 5.1. The Ricci tensor \( \tilde{S} \) of submanifold \( M \) of \( \tilde{M}^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla} \) is

\begin{equation}
\tilde{S}(X, Y) = \sum \lambda \left[ m f_1 + 3 f_2 \lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1) \right] g(U^\lambda X, U^\lambda Y) - m \{ f_3 + (f_1 - f_3)^2 \} \eta(X) \eta(Y) \nonumber \\
+ \sum_{k=1}^{2n-m} \{ (m + 1)(\text{trace} \tilde{A}_k) g(\tilde{A}_k X, Y) - g(\tilde{A}_k X, \tilde{A}_k Y) \}
\end{equation}

for any vector fields \( X, Y \) on \( M \).

Proof. Using (2.12) and (2.19) we have the above Lemma. \qed

Lemma 5.2. The Ricci tensor \( \tilde{S} \) of almost semi-invariant submanifolds \( M \) of \( \tilde{M}^{2n+1}(f_1, f_2, f_3) \) with respect to \( \tilde{\nabla} \) is

\begin{equation}
\tilde{S} = \sum \lambda \left[ m f_1 + 3 f_2 \lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1) \right] g(U^\lambda X, U^\lambda Y) \\
+ \sum_{k=1}^{2n-m} \{ (m + 1)(\text{trace} \tilde{A}_k) g(\tilde{A}_k X, Y) - g(\tilde{A}_k X, \tilde{A}_k Y) \}
\end{equation}

for any vector fields \( X, Y \) on \( M \).

Proof. Using (2.22) and (5.1) we have the above Lemma. \qed
Corollary 5.1. The Ricci tensor $\hat{S}$ of almost semi-invariant submanifolds $M$ of $\hat{M}^{2n+1}(\epsilon)$ with respect to $\hat{\nabla}$ is

\begin{equation}
\hat{S}(X,Y) = \frac{(m-1+3\lambda^2)c+3(m-1)+\lambda^2}{4}g(U^\lambda X,U^\lambda Y) + 2m\eta(X)\eta(Y) + \sum_{k=1}^{2n-m} \{(m+1)(\text{trace}_k\hat{A}_k)g(\hat{A}_kX,Y) - g(\hat{A}_kX,\hat{A}_kY)\}.
\end{equation}

Proof. Putting $f_1 = \frac{c+2}{4}$, $f_2 = f_3 = \frac{c-1}{4}$ in (5.2) we obtain the result. \qed

Lemma 5.3. The scalar curvature $\hat{\tau}$ of an almost semi-invariant submanifold $M$ of $\hat{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\hat{\nabla}$ is

\begin{equation}
\hat{\tau} = f_1 + \frac{1}{m(m+1)}\sum_{\lambda} n(\lambda)\lambda^2 \left\{3f_2 + (f_1 - f_3)^2\right\} + 2m\left\{(f_1 - f_3)^2\right\} + (m+1)^2\left\|\hat{H}\right\|^2 - \left\|\hat{h}\right\|^2.
\end{equation}

Proof. It is known that

\begin{equation}
\hat{\tau} = \frac{1}{m(m+1)}\sum_{i,j=1}^{m+1} \hat{R}(E_i,E_j,E_j,E_i).
\end{equation}

Using (2.8), (2.19) in (5.5) we get (5.4). \qed

Theorem 5.1. If $M$ is an almost semi-invariant minimal submanifolds of $\hat{M}^{2n+1}(f_1,f_2,f_3)$ with respect to $\hat{\nabla}$, then the following condition holds:

(i) $\hat{S} \leq \sum_{\lambda} \left\{mf_1 + 3f_2\lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1)\right\}g(U^\lambda X,U^\lambda Y) + m\left\{(f_3 + (f_1 - f_3)^2)\right\} \eta(X)\eta(Y),$

(ii) $\hat{\tau} \leq f_1 + \frac{1}{m(m+1)}\sum_{\lambda} n(\lambda)\lambda^2 - 2m\left\{(f_1 - f_3)^2\right\}.$

Proof. Since $M$ is minimal submanifold with respect to with respect to $\hat{\nabla}$, then we have

\begin{equation}
\sum_{k=1}^{2n-m} (\text{trace}_k\hat{A}_k)g(\hat{A}_kX,\hat{A}_kX) = \sum_{i=1}^{m+1} g(\hat{h}(X,X),\hat{h}(E_i,E_i)) = (m+1)g(\hat{h}(X,X),\hat{H}) = 0.
\end{equation}

Using (5.2) and (5.6) we have

\begin{equation}
\hat{S} - \sum_{\lambda} \left\{mf_1 + 3f_2\lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1)\right\}g(U^\lambda X,U^\lambda Y) - m\left\{(f_1 - (f_1 - f_3)^2)\right\} \eta(X)\eta(Y)
\end{equation}

\begin{equation}
\leq - \sum_{k=1}^{2n-m} g(\hat{A}_kX,\hat{A}_kX) \leq 0.
\end{equation}

This complete the proves (i).

The second part (ii) is comes directly from Lemma 5.3. \qed
Lemma 6.1. The Ricci tensor $\bar{\mathcal{S}}$ of submanifold $M$ of $\bar{\mathcal{M}}^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}$ is

$$\bar{\mathcal{S}}(X, Y) = m f_1 g(X, Y) + \{3 f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2\} g(TX, TY)$$

- $\{f_3 + (f_1 - f_3)^2\} \{g(X, Y) + (m - 1)\eta(X)\eta(Y)\}$

+ $\sum_{k=1}^{2n-m} \{((m + 1)(trace \, A_k)g(A_k X, Y) - g(A_k X, A_k Y)\}

for any vector fields $X, Y$ on $M$.

Proof. Using (2.14) and (2.20) we have the above Lemma.

Lemma 6.2. The Ricci tensor $\hat{\mathcal{S}}$ of almost semi-invariant submanifolds $M$ of $\bar{\mathcal{M}}^{2n+1}(f_1, f_2, f_3)$ with respect to $\hat{\nabla}$ is

$$\hat{\mathcal{S}}(X, Y) = \sum_{\lambda} \{m f_1 + \{3 f_2 + 2(f_1 - f_3)\}\lambda^2 - f_3$$

+ $\{f_3 + (f_1 - f_3)^2\} \{\lambda^2 - 1\}\nonumber\}$

+ $\{f_1 - \{f_3 + (f_1 - f_3)^2\}\} \{\eta(X)\eta(Y)\}$

+ $\sum_{k=1}^{2n-m} \{((m + 1)(trace \, \hat{A}_k)g(\hat{A}_k X, Y) - g(\hat{A}_k X, \hat{A}_k Y)\}

for any vector fields $X, Y$ on $M$.

Proof. Using (2.22) and (6.1) we have the above Lemma.

Corollary 6.1. The Ricci tensor $\hat{\mathcal{S}}$ of almost semi-invariant submanifolds $M$ of $\bar{\mathcal{M}}^{2n+1}(c)$ with respect to $\hat{\nabla}$ is

$$\hat{\mathcal{S}}(X, Y) = \frac{(m - 1 + 3\lambda^2)c + 3(m - 1 + 3\lambda^2)}{4} g(U^\lambda X, U^\lambda Y)$$

+ $2mn\eta(X)\eta(Y) + \sum_{k=1}^{2n-m} \{((m + 1)(trace \, \hat{A}_k)g(\hat{A}_k X, Y) - g(\hat{A}_k X, \hat{A}_k Y)\}

for any vector fields $X, Y$ on $M$.

Proof. Putting $f_1 = \frac{m+1}{4}$, $f_2 = f_3 = \frac{m-1}{4}$ in (6.2) we obtain the result.
Lemma 6.3. The scalar curvature $\bar{\tau}$ of an almost semi-invariant submanifold $M$ of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}$ is

$$
\bar{\tau} = f_1 + \frac{1}{m(m + 1)} \left\{ 3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2 \right\} \sum \lambda n(\lambda) \lambda^2 - 2m\{f_3 + (f_1 - f_3)^2\} + (m + 1)^2 \left\| \bar{H} \right\|^2 - \left\| \bar{h} \right\|^2.
$$

Proof. It is known that

$$\bar{\tau} = \frac{1}{m(m + 1)} \sum_{i,j=1}^{m+1} \bar{R}(E_i, E_j, E_i).$$

Using (2.8), (2.20) in (6.5) we get (6.4).

Theorem 6.1. Let $M$ be an almost semi-invariant minimal submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}$, then the following condition holds:

(i) $\bar{S} (X, Y) \leq \sum \left\{ mf_1 + \left\{ 3f_2 + 2(f_1 - f_3) \right\} \lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1) \right\} g(U^\lambda X, U^\lambda Y) + m\left\{ f_3 + (f_1 - f_3)^2 \right\} \eta(X)\eta(Y)$.

(ii) $\bar{S} \leq f_1 + \frac{1}{m(m + 1)} \left\{ 3f_2 + 2(f_1 - f_3) + (f_1 - f_3)^2 \right\} \sum \lambda n(\lambda) \lambda^2 - 2m\{f_3 + (f_1 - f_3)^2\}$.

Proof. Since $M$ is minimal submanifold with respect to with respect to $\bar{\nabla}$, then we have

$$\sum_{k=1}^{2n-m} \left( \text{trace} \ \bar{A}_k \right) g(\bar{A}_k X, X) = \sum_{i=1}^{m+1} \bar{g}(\bar{h} (X, X), \bar{h} (E_i, E_i)) = (m + 1)\bar{g}(\bar{h} (X, X), \bar{h} X) = 0.$$

Using (6.2) and (6.6) we have

$$\bar{S} = \sum \left\{ mf_1 + \left\{ 3f_2 + 2(f_1 - f_3) \right\} \lambda^2 - f_3 + (f_1 - f_3)^2(\lambda^2 - 1) \right\} g(U^\lambda X, U^\lambda Y) - m\left\{ f_3 + (f_1 - f_3)^2 \right\} \eta(X)\eta(Y) - \sum_{k=1}^{2n-m} \bar{g}(\bar{A}_k X, \bar{A}_k X) \leq 0,$$

which proves (i).

The proof of (ii) comes directly from Lemma 6.3.

Remark 6.1. The equality of (i) and (ii) in Theorem 6.1 holds if $M$ is almost semi-invariant totally geodesic submanifolds of $\bar{M}^{2n+1}(f_1, f_2, f_3)$ with respect to $\bar{\nabla}$.

Proof. If $M$ is totally geodesic submanifold with respect to $\bar{\nabla}$, then $M$ is minimal submanifold with respect to $\bar{\nabla}$. Then by virtue of Lemma 6.2 we have the equality case (i) and by virtue of Lemma 6.3 we have the equality case of (ii).
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