An interpretation of variational principles for gauge theories: a cyclic coordinate alternative to ADM split

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Abstract

I show that there is an ambiguity in how one treats auxiliary variables in gauge theories including general relativity cast as \( 3 + 1 \) geometrodynamics. Auxiliary variables may be treated pre-variationally as multiplier coordinates or as the velocities corresponding to cyclic coordinates. The latter treatment works through the physical meaninglessness of auxiliary variables’ values applying also to the end points (or end spatial hypersurfaces) of the variation, so that these are free rather than fixed. (This is also known as variation with natural boundary conditions.) Further principles of dynamics workings such as Routhian reduction and the Dirac procedure are shown to have parallel counterparts for this new formalism. One advantage of the new scheme is that the corresponding actions are more manifestly relational. While the electric potential is usually regarded as a multiplier coordinate and Arnowitt, Deser and Misner have regarded the lapse and shift likewise, this paper’s scheme considers new flux, instant and grid variables whose corresponding velocities are, respectively, the above-mentioned previously used variables. This paper’s way of thinking about gauge theory furthermore admits interesting generalizations as regards variational principles for the conformal form of general relativity’s initial value problem.

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1. Introduction

Let \((M, g_\lambda(x^\Omega))\) be the spacetime topology and metric with respect to the coordinates \(x^\Omega\) of a spacetime of general relativity (GR). Arnowitt, Deser and Misner (ADM) \([1]\) \((\text{also has a good exposition})\) split spacetime with respect to a family of spatial hypersurfaces, \(\Sigma_\lambda\). Under this split, \(x^\Omega\) splits into \((\lambda, x^\omega)\), where \(\lambda\) is a coordinate label time for each spatial...
hypersurface in the split and \(x^\omega(\lambda)\) is a coordinatization of that spatial hypersurface. Then ADM take \(g_{\Gamma\Delta}\) to split as follows:

\[
g_{\Gamma\Delta} = \left( \begin{array}{cc} \beta_\mu \beta^\mu - \alpha^2 & \beta_\lambda \\
\beta_\nu & \gamma_{\lambda\beta} \end{array} \right) .
\]

(1)

Here, \(h_{\gamma\delta}(\lambda, x^\omega)\) is the induced metric, i.e. the metric induced by the spacetime metric \(g_{\Gamma\Delta}\) on the spacelike \(\lambda\)-hypersurface. \(\alpha(\lambda, x^\omega)\) is the lapse, which relates the passage of coordinate time between infinitesimally adjacent surfaces \(\Sigma_\lambda\) and \(\Sigma_{\lambda+d\lambda}\) to the passage of proper time \(\tau\):

\[
d\tau = \alpha(\lambda, x^\omega) d\lambda .
\]

(2)

\(\beta^\mu(\lambda, x^\omega)\) is the shift, which relates the coordinates on \(\Sigma_{\lambda+d\lambda}\) to those on \(\Sigma_\lambda\):

\[
x^\omega(\lambda + d\lambda) - x^\omega(\lambda) = -\beta^\omega d\lambda ,
\]

(3)
i.e., the lapse is the change in spacing in how the spatial hypersurfaces are stacked, while the shift is the displacement made in identifying the spatial coordinates of one slice with those on an adjacent slice. Thus these two can be envisaged together as the determiners of what extrinsic geometry each spatial hypersurface \(\Sigma_\lambda\) is to have within spacetime (see, e.g., p 346 of [3]). The ADM split is then applied to the spacetime form of the GR action to produce the ADM action for split spacetime1,

\[
I_{ADM}[h_{\mu\nu}, K_{\mu\nu}, \alpha] = \int d\lambda \int d^3 x \sqrt{\alpha} \left\{ K_{\mu\nu} K_{\mu\nu} - K^2 + R \right\} .
\]

(4)

which, by the definition of the extrinsic curvature \(K_{\mu\nu} \equiv -[h_{\mu\nu} - \xi_\beta h_{\mu\nu}] / 2\alpha\), can be written in the explicitly Lagrangian form

\[
I_{ADM}[h_{\mu\nu}, \dot{h}_{\mu\nu}, \beta^\mu, \alpha] = \int d\lambda \int d^3 x \sqrt{\alpha} \left\{ \frac{G^{\mu\nu\rho\sigma} \left[ \dot{h}_{\mu\nu} - \xi_\beta h_{\mu\nu} \right] \left[ \dot{h}_{\rho\sigma} - \xi_\beta h_{\rho\sigma} \right]}{4\alpha^2} + R \right\} .
\]

(5)

The ADM split and variants have important applications in quantum gravity (see, e.g., [4, 6, 7]) and in the study of the compact binary problem in numerical GR (see, e.g., [8, 9]).

It is ‘standard lore’ in varying the ADM action to regard the lapse and the shift as coordinates; as the corresponding velocities do not feature in the action, these are Lagrange multiplier coordinates. Denoting Lagrange multiplier coordinates, in general, by \(m_B\), variation with respect to these produces simplified Euler–Lagrange equations of the form

\[
\frac{\partial L}{\partial m_B} = 0 .
\]

(6)

This is how the Hamiltonian and momentum constraints of GR arise in the ADM formulation (see the appendix).

1 I use ( ) for function dependence, [ ] for functional dependence, ( ; ) for a mixture of function dependence before the semi-colon and functional dependence after it. I use lower-case Greek letters for spatial indices, except that \(\alpha\) and \(\beta\) are reserved to denote lapse and shift. \(h = h(h_{\mu\nu}), \alpha, \nu, \) and \(R(x^\omega, h_{\mu\nu})\) are, respectively, the determinant, covariant derivative and Ricci scalar associated with the 3-metric \(h_{\mu\nu}(x^\omega)\). In the GR context, the dot denotes \(\partial / \partial \lambda\), \(\xi_\beta\) is the Lie derivative with respect to \(\beta^\mu\), \(G_{\mu\nu\rho\sigma}(h_{\gamma\delta}) = \frac{1}{\sqrt{g}} \left[ h_{\mu\rho} h_{\nu\sigma} - \frac{1}{2} h_{\rho\sigma} h_{\mu\nu} \right]\) is the DeWitt supermetric [4] with inverse \(G^{\mu\nu\rho\sigma}(h_{\gamma\delta}) = \sqrt{\alpha} \left[ h^{\mu\nu} h^{\rho\sigma} - h^{\mu\rho} h^{\nu\sigma} \right]\) which plays the role of kinetic metric (see, e.g., [5]) in GR. Overbar denotes densitization (multiplication by \(\sqrt{\alpha}\)) and underbar denotes dedensitization (division by \(\sqrt{\alpha}\)).
Now, it is a fairly standard observation (see, e.g., [10]) that the above expressions for the lapse and shift can, at the post-variational level, be straightforwardly rearranged to be interpreted as velocities,

\[ \alpha(\lambda, x^\omega) = \frac{\partial \tau}{\partial \lambda}, \quad \beta^\mu(\lambda, x^\omega) = -\frac{\partial x^\mu}{\partial \lambda}. \]  

(7)

What is new in this paper is to regard the objects of which the lapse and the shift are the velocities as new variables prior to variation. Reading off (44), these are the proper time \( \tau \) attributed to the spatial slice in question and the grid of coordinates \( x^\omega \) on that slice, but in their capacity as new variables to be varied, I term these the instant \( I(\lambda, x^\omega) \) and the grid \( F^\mu(\lambda, x^\omega) \). (I use an 'F' to denote this as it is a subcase of a more widely applicable concept of configurational frame variable). Through showing this to work out acceptably as an alternative interpretation, this paper shows that there is, in fact, an ambiguity in the long-standing, well-known and much-used statement that the lapse and shift are Lagrange multipliers.

Moreover, the alternative better complies with relational first principles, which is of interest along the lines of establishing GR to have multiple foundations, along the lines of Wheeler’s arguments for ‘many routes to GR’ [2, 6, 11]. The relational perspective, implements ideas of Leibniz [12] and Mach [13] to modern physics along the lines of Barbour’s work [14–16]. One starts with a configuration space \( Q \) of (models of) whole-universe systems.

Then configurational relationalism is the declaration that certain motions acting on \( Q \) are to be physically meaningless. One way of implementing this is to use arbitrary-\( G \)-frame-corrected quantities rather than bare \( Q \) configurations, where \( G \) is the group of physically meaningless motions. Despite this augmenting \( Q \) to \( Q \times G \), variation with respect to each adjoined independent auxiliary \( G \)-variable produces a constraint which wipes out one \( G \) variable and one redundancy among the \( Q \) variables, so that one ends up on the quotient space \( Q/G \) (the desired reduced configuration space). This is widely a necessity in theoretical physics through working on the various reduced spaces directly often being technically unmanageable.

Temporal relationalism is the notion that there is no meaningful primary notion of time for the universe as a whole. One implementation of temporal relationalism is through using manifestly reparametrization invariant actions, though Barbour would prefer this to be attained without any extraneous time variables either. If one is working towards deriving GR in this picture, one would, e.g., take \( Q \) to be the space of positive-definite 3-metrics on a fixed spatial topology, \( \text{Riem}(\Sigma) \), and \( G \) to be the 3-diffeomorphisms \( \text{Diff}(\Sigma) \) that correspond to the coordinatization of space being physically meaningless. It is then clear upon inspecting the split spacetime GR action that the instant and grid interpretations of the auxiliaries therein give the action a manifest reparametrization invariance while the lapse and shift multiplier coordinate interpretations obstruct this by inhomogenizing the Lagrangian’s dependence on \( \partial / \partial \lambda \). To have no extraneous time variables either, one would require not an ADM-type action with its manifest lapse or velocity of the instant, but a Baierlein–Sharp–Wheeler (BSW) [20] or BFO-A\(^5\) type action in which this has been eliminated (or, alternatively, such an action could be adopted as one’s starting point). This paper completes this work by mapping out

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2 There is also brief mention in a specific example in [3].

3 For, different routes offer different techniques and different insights, suggest different alternatives or generalizations against which the established theory can be tested, and different routes to a given classical theory may be quantum-mechanically inequivalent.

4 This is along the lines of [17], which is closely connected with well-known approaches to gauge theory (see, e.g., [18]). For Barbour’s own and somewhat conceptually different way of thinking about configurational relationalism (‘best matching’), see, e.g., [14, 19].

5 This action (44) first appeared in [17], building on [16] and insights in [21]. See also [22] for discussion.
how the instant can be eliminated from the instant–grid version of the split GR action so as to produce the BFO-A action.

There is an obstruction that explains why this kind of alternative interpretation of auxiliary variables has not been spotted until recently. At first sight, it looks preposterous to treat the lapse and shift as velocities prior to variation. This is because cyclic coordinates $c_B$ are well known to lead to simplified Euler–Lagrange equations of the form

$$\frac{\delta L}{\delta c_B} = \text{constant},$$

which are generally not equivalent to (6) produced by the preceding formalism. However, there is a subtlety. In a gauge formulation of a physical theory, the configuration variables contain both dynamical degrees of freedom and nondynamical (auxiliary) variables. Then by the gauge principle (or configurational relationalism), the reference frame variables are to be taken to be unphysical, i.e. arbitrarily specifiable at each instant, with no measurable quantities depending on which choice of these is made. This includes among the things that are to be arbitrary the values of the auxiliary arbitrary $G$-frame variables at the end points (or end spatial hypersurfaces in the case of field theory) of the variation. Thus it is free end point value variation (also known as variation with natural boundary conditions) [23, 24] that turns out to be the correct type of variation as regards reference frame variables in gauge theories. In section 2, I show that this in no way affects the outcome of varying with respect to multiplier variables, but does alter the outcome of varying with respect to cyclic variables precisely so as to remove the above inequivalence.

In section 3, I then show that there is also no difference between multiplier elimination in the auxiliary multiplier formulation and cyclic velocity elimination (by the more subtle procedure of Routhian reduction, see, e.g., [5]) in the above auxiliary cyclic coordinate formulation. I also explain how one subsequently gets an alternative to Dirac’s appending procedure [25]. These sections are supported by relational particle mechanics toy models [14, 15, 26–29]—zero total angular momentum universes formulated by frame correcting with respect to the rotations with a rotational auxiliary analogue of the grid–shift object. Armed with all this, in section 4, I give the manifestly reparametrization-invariant alternative to the ADM formulation of Einstein–Maxwell theory. This requires, in addition to the lapse and shift being re-interpreted pre-variationally as instant and grid variables, for the conventional electric potential $\Phi(\lambda, x^\omega)$ to be re-interpreted as the velocity of a new (corrected magnetic) flux variable $\Psi(\lambda, x^\omega)$ (which is another example of frame variable). Then performing Routhian reduction on this action so as to eliminate the instant leads to the recovery of the BFO-A action that complies with the first principles of the relational approach, which working is in direct parallel to how multiplier elimination of the lapse from the ADM action gives the BSW action. I conclude in section 5 with a discussion of classical and quantum applications and extensions. These include (1) using auxiliaries more general than multipliers or cyclic coordinates, which is applied to forming a variational principle for the conformal form of the GR initial value problem in [30]. (2) I comment on extending the present work, which is for spatial sections that are compact without boundary, to asymptotically flat spatial sections and spatial sections with boundaries [31]. The appendix provides Einstein–Maxwell theory’s ADM and BSW formulations (and the BSW elimination that links them) for useful comparison.

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6 In sections 1–3 the dot denotes $d/d\lambda$.

7 That the range of spacetimes covered by the present paper is substantial, is illustrated by the following examples. Some Robertson–Walker spacetimes are spatially $S^3$, and many known anisotropic cosmology solutions are spatially $S^3$ or $T^3$ [31] (e.g., the Kantowski–Sachs solution and many of the Bianchi models). As regards inhomogenous cosmology solutions, there are Gowdy cosmologies (and many less well-known solutions) that are spatially $S^3$ or $T^3$ which are documented in [31].
2. An alternative picture for gauge formulations

One common way of formulating gauge theories is in the Lagrangian picture; the Lagrangians that one builds are to possess a particular group of symmetries, \( G \). Another common way of formulating gauge theories is in the Hamiltonian picture, in which one appends certain constraints (e.g., in the sense of Dirac [25]) that are related to the action of the gauge group. The Dirac procedure has the additional advantage that even if one does not know what symmetries the theory is to possess, the procedure nevertheless systematically derives the theory’s constraints (see, e.g., [32]). One can move between the two pictures by Legendre transformation. One approach within the Lagrangian picture is to attempt to impose a group \( G \) of symmetries on a Lagrangian, and then use the Dirac procedure to systematically determine whether this imposition is consistent and complete.

One way of building a formulation in which the group of symmetries \( G \) (whether actual, expected or imposed) is present and manifest is to enlarge the configuration space \( Q \) to \( Q \times G \) by adjoining auxiliary variables \( g_B \). These show up in the Lagrangian formalism as arbitrary \( G \)-frame corrections to the \( Q \) variables [17, 22]. To see how this gives gauge theories, consider for the moment that the auxiliary variables take their usual guise as multipliers in the generalized sense (coordinates whose velocities do not occur in the action). For example, one conventionally views in this way the electric potential \( \Phi_1 \), and the shift \( \beta^\mu \) and lapse \( \alpha \) of GR split with respect to spatial hypersurfaces. Further examples are the translation and rotation auxiliaries, \( a \) and \( b \) in mechanics models that are spatially relational with respect to the Euclidean group [14, 27, 28] (as opposed to being based on the supposition of absolute space).

Variation with respect to the auxiliaries \( g_B \) then produces

\[
\frac{\delta L}{\delta g_B} = 0,
\]

which are the constraints that one is expecting if one is indeed to obtain thus a gauge theory with gauge group \( G \). These constraints then use up both the auxiliary variables’ degrees of freedom and an equal amount of degrees of freedom from the \( Q \), thus indeed leaving one with a theory in which the dynamical variables pertain to the quotient space \( Q/G \) (i.e. variables among the original \( Q \) which are entirely unaffected by which choice of \( G \)-frame is made).

As a simple example, consider the translation and rotation invariant mechanics [14, 27, 28] that follows from the Jacobi-type action

\[
\mathcal{I}(q_I, \dot{q}_I, a, b) = \int d\lambda \, L(q_I, \dot{q}_I, a, b) = 2 \int d\lambda \sqrt{T\{E - V\}}
\]

with kinetic term

\[
T(q_I, \dot{q}_I, a, b) = \frac{1}{2} \sum_{I=1}^{N} m_I |\dot{q}_I - a - b \times q_I|^2,
\]

potential term

\[
V = V(|q_I - q_J| \text{ alone})
\]

and total energy \( E \). Then variation with respect to \( a \) gives the multiplier equation

\[
0 = \frac{\partial L}{\partial a} = \sum_{I=1}^{N} p_I, \text{ i.e. zero total momentum},
\]

and variation with respect to \( b \) gives

\[
0 = \frac{\partial L}{\partial b} = \sum_{I=1}^{N} p_I, \text{ i.e. zero total momentum}.
\]

Constraints that do not correspond to \( G \) could arise as integrabilities for other constraints in the theory, signalling that \( G \) need be extended if one is to get a consistent theory. See the end of section 4 for an example of this. So many constraints can arise in this way that the resulting ‘theory’ ends up trivial (all degrees of freedom used up) or inconsistent (more independent constraints than degrees of freedom) [16].
the multiplier equation $0 = \partial L / \partial b = \sum_{i=1}^{N} q_i \times p_i$, i.e. zero total angular momentum. See
the appendix for another example (the standard ADM split of Einstein–Maxwell theory).

The main point of the present paper, however, is that taking $a, b, \Phi, \beta^\alpha$ and $\alpha$ to be
multipliers is not the only viable interpretation. They may also be interpreted as velocities
associated with an auxiliary cyclic coordinate. This is possible because the variables in
question are auxiliary. Thus by the gauge principle, their values are entirely arbitrary. Thus,
in particular, they take arbitrary rather than fixed values at the end points of the variation. Thus
the appropriate type of variation is free end point (FEP) variation (also known as variation
with natural boundary conditions) \[5, 23, 24\]^9.

FEP variation involves more freedom than standard variation, i.e. it involves a larger space
of varied curves,

\[ q_A(\lambda, \mu, \nu) = \tilde{q}_A(\lambda) + \mu Y(\lambda) + \nu Z(\lambda) \quad \text{such that} \quad Y(\lambda_i) = 0 \quad \text{and} \quad Z(\lambda_f) = 0 \quad (13) \]

(where $\mu$, $\nu$ are further parameters). Then consider configurations \(q_A, g_B\) for $g_B$ entirely
arbitrary through being unphysical. With this in mind, I start from scratch, with a more
general Lagrangian in which both the auxiliaries $g_B$ and their velocities $\dot{g}_B$ in general occur:

\[ L(\lambda, q_A, g_B, \dot{q}_A, \dot{g}_B). \]

Then variation with respect to $g_B$ gives

\[ 0 = \delta l = \int_{\lambda_i}^{\lambda_f} \delta \dot{\lambda} L(\dot{\lambda}, q_A, g_B, \dot{q}_A, \dot{g}_B) = \int_{\lambda_i}^{\lambda_f} \delta \dot{\lambda} \frac{\partial L}{\partial \dot{g}_B} \left\{ \frac{\partial L}{\partial g_B} \right\}_{\lambda_i}^{\lambda_f}, \]

(14)

Because $g_B$ is auxiliary, this variation is free, so $\delta g_B|_{\lambda_i}$ and $\delta g_B|_{\lambda_f}$ are not controllable. Thus,
one obtains three conditions per variation:

\[ \frac{\partial L}{\partial \dot{g}_B} = \frac{d}{d\dot{\lambda}} \left\{ \frac{\partial L}{\partial \dot{g}_B} \right\}_{\lambda_i}^{\lambda_f} = 0 = \left. \frac{\partial L}{\partial \dot{g}_B} \right|_{\lambda_i}, \]

(15)

or, in terms of momenta $p^B \equiv \partial L / \partial \dot{g}_B$,

\[ \frac{\partial L}{\partial \dot{g}_B} = p^B, \quad p^B|_{\lambda_i} = 0 = p^B|_{\lambda_f}. \]

(16)

9 In general, use of natural boundary conditions amounts to a lack of ‘artificial’ boundary conditions \[5, 23, 33\]
imposed in modelling, e.g., rods or hot plates on a) physical grounds—to represent objects of (semi)finite extent as
observed or set up in the laboratory, or b) numerical grounds—so that numerical simulations do not break down or
exhibit artefacts due to limitations in the modelling (such as totally absorbing boundary conditions to stop such as
unphysical finiteness of a numerical grid resulting in noticeable amounts of unphysical reflection or scattering). This
lack of ‘artificial’ boundary conditions reflects a lack of constraints (e.g., forces in the case of rods) being applied
in a prescribed manner at the endpoints of the variation. In the case of GR, in some situations one might likewise
choose to impose boundary conditions to study in isolation (semi)finite pieces of a spatial section or spacetime
\[33–35\]. Some of these situations, such as (1) studying a star occupying a finite region \[8\] (2) studying a black
hole exterior with some kind of excision boundary \[36, 37\] condition preventing the coordinate pathologies of the
horizon, and actual pathologies within the black hole itself, from causing the breakdown of numerical simulation,
or (3) using such as a totally absorbing outer edge \[37, 38\] to one’s grid in studying a region of space, can involve
additional departures from natural boundary conditions. Natural boundary conditions remain, however, important,
both in being the simplest prescription and in providing the requisite number of remaining boundary conditions in
the cases in which nature or the physicist does not supply enough boundary conditions of other kinds \[5\]. On both
of these grounds they make for a good first port of call, while I leave the inclusion of other boundary conditions by
consideration of boundary terms in the various actions for a future occasion. Another reason for boundary conditions
on the instant and grid (or lapse and shift) not to have physical content is that these variables are auxiliary or ‘gauge’.
Certain choices of such, nevertheless, can be convenient both analytically and in aiding the longevity and accuracy of
numerical integrations of the Einstein evolution equations (e.g., the constant mean curvature slicing preserving lapse
\[8, 39\] or such as the minimal distortion choice of shift \[8, 39\]).
If the auxiliaries $g^B$ are multipliers $m_B$, (16) reduces to
\[
p_B = 0, \quad \frac{\partial L}{\partial m_B} = 0
\]
and redundant equations. That means the $\lambda_i$ and $\lambda_f$ terms automatically vanish in this case by applying the multiplier equation to the first factor of each. This is the case regardless of whether the multiplier is not auxiliary and thus standardly varied, or auxiliary and thus FEP varied, as this difference translates to whether or not the cofactors of the above zero factors are themselves zero or not. Thus the FEP subtlety in no way affects the outcome in the multiplier coordinate case.

If the auxiliaries $g^B$ are cyclic coordinates $c^B$, the above reduces to
\[
p^B|_{\lambda_i} = 0 = p^B|_{\lambda_f}
\]
and
\[
\dot{p}^B = 0,
\]
which implies that
\[
p^B = C,
\]
but $C$ is identified as zero at either of the two end points (18), and, being constant, is therefore zero everywhere. Thus (19) and the definition of momentum give
\[
\frac{\partial L}{\partial \dot{c}^B} \equiv p^B = 0.
\]
So, after all, one does get equations that are equivalent to the multiplier equations (9) at the classical level, albeit there are conceptual and foundational reasons to favour the latter, as laid out in the introduction. For previous discussion of this case in the literature, see [17, 21, 29, 40].

For the translation and rotation invariant mechanics example, this works out as follows. The appropriate action is
\[
I[q_I, \dot{q}_I, \dot{u}, \dot{b}] = \int d\lambda \ L(q_I, \dot{q}_I, \dot{u}, \dot{b}) = 2 \int d\lambda \sqrt{T(U + E)},
\]
with
\[
T(q_I, \dot{q}_I, \dot{u}, \dot{b}) = \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{q}_I - \dot{u} - \dot{b} \times q_I|^2
\]
and $U = -V$. Then variation with respect to $q$ gives $C = \partial L/\partial \dot{u} = \sum_{j=1}^{N} p_j$ and the FEP conditions $\partial L/\partial \dot{u}|_{\lambda_i} = 0 = \partial L/\partial \dot{u}|_{\lambda_f}$, so $C = \partial L/\partial \dot{u} = 0$ at the end point, but $C$ is constant, so $C = 0$ everywhere, so one obtains $\sum_{i=1}^{N} p_j = 0$ again. And variation with respect to $b$ gives $D = \partial L/\partial \dot{b} = \sum_{j=1}^{N} q_I \times p_j$ and FEP conditions $\partial L/\partial \dot{b}|_{\lambda_i} = 0 = \partial L/\partial \dot{b}|_{\lambda_f}$, so $D = \partial L/\partial \dot{b} = 0$ at the end point, but $D$ is constant, so $D = 0$ everywhere, so one obtains $\sum_{i=1}^{N} q_I \times p_j = 0$ again. For further examples of this kind of approach to gauge theories, see e.g., section 4 (3 + 1 split GR), section 5 (3 + 1 split of Einstein–Maxwell theory), [28, 29] (mechanics that is scale invariant as well as translation and rotation invariant), and [21, 40, 41] alongside examples in [30].

The case of (15) in which the auxiliary is neither cyclic nor a multiplier I leave for [30], as the conformal geometrodynamics formulations in which it applies [21, 40] are rather more complicated than the present paper’s examples.
As regards how one determines in the first place which variables are the auxiliaries that one is to FEP vary with respect to, it turns out that for the type of examples in this suffices to FEP vary with respect to the \( g_B \), as follows. While some sets of variables contain partly physically relevant and partly physically irrelevant degrees of freedom, the gauge principle only gives license to vary in the FEP way with respect to sets of variables that are purely physically irrelevant. So FEP variation should be applied only to isolated physically irrelevant variables. Moreover, one does not have to isolate all of these, for the following reason. In gauge theory, physically irrelevant variables come in pairs that are associated with a single constraint, which, when taken onto account, ensures that neither member of the pair remains in the theory’s equations. Variation with respect to either member of the pair produces the same constraint, thus to get all of the constraints, one needs to isolate a full half set of auxiliary variables. Moreover, one does not have to isolate all of these, for the following reason. In the arbitrary \( G \)-frame method as used in this paper, the frame variables \( g_B \) that one adjoins are clearly both such a half set and already separate. Thus for the type of theories considered in this paper, it suffices to FEP vary with respect to the \( g_B \).

3. Establishing further equivalences between auxiliary cyclic coordinates and auxiliary multiplier coordinates

We also need to establish that the \( a \) \( p \) \( r \) \( i \) \( o \) \( r \) \( i \) \( t \) \( e \) distinct procedures of cyclic velocity elimination (known as Routhian reduction [5]) and multiplier elimination are equivalent. Consider \( L(q_A, \dot{q}_A, g_B) \) where \( g_B \) are auxiliary coordinates. If these are taken to be multiplier coordinates \( m_B \), then variation yields \( 0 = \partial L / \partial m_B \). If this is soluble for \( m_B \), one can replace it by \( m_B = m_B(q_A, \dot{q}_A) \), and then substitute that into \( L \). If the \( g_B \) are taken to be velocities corresponding to cyclic coordinates \( c_B \), then FEP variation yields

\[
0 = \frac{\partial L}{\partial c_B} = p_B = \dot{c}_B. \tag{24}
\]

This is soluble for \( \dot{c}_B \) iff the above is soluble for \( m_B \). However one now requires passage to the Routhian in eliminating \( \dot{c}_B \) from the action: \( R(q_A, \dot{q}_A) = L(q_A, \dot{q}_A, c_A) - \dot{c}_B p_B \). But the last term is zero, by (24), so Routhian reduction in the cyclic velocity interpretation is equivalent to multiplier elimination in the multiplier coordinate interpretation.

Next, I consider how the Dirac procedure implies encodement by auxiliaries at the level of the Lagrangian. From the bare (\( G \)-uncorrected Lagrangian) \( L^{\text{bare}}(q_A, \dot{q}_A) \), Legendre transformation gives \( H(q_A, p^A) = \dot{q}_A p^A - L^{\text{bare}}(q_A, \dot{q}_A) \) for which \( c_B = 0 \) are discovered by the first part of the Dirac procedure. Then, in the multiplier interpretation,

\[
H(q_A, p^A, m_B) = \dot{q}_A p^A - L^{\text{bare}}(q_A, \dot{q}_A) + m_B c^B. \tag{25}
\]

so that this procedure standardly follows from the standard Dirac procedure of appending constraints multiplied by Lagrange multipliers. In the cyclic velocity interpretation however, one has a new alternative to Dirac appending that involves appending constraints multiplied by velocities of cyclic coordinates. Then one is not forming a Hamiltonian, but rather an ‘almost Hamiltonian’, in that it depends not only on positions and momenta but also on the velocities of the auxiliary cyclic coordinates,

\[
A(q_A, p^A, \dot{c}^B) = \dot{q}_A p^A + \dot{c}_B c^B - L^{\text{bare}}(q_A, \dot{q}_A). \tag{26}
\]

There is no problem uplifting section 2 or 3 to field theory. (Now one has free end \( \text{spatial hypersurfaces (FESH)} \) and \( C(x^{\mu}) \) in place of \( C \), constant, but all the above arguments straightforwardly carry through).
4. Manifestly reparametrization invariant alternatives to the ADM and BSW-type formulations of Einstein–Maxwell theory

As delineated in the introduction and appendix, in the ADM split, lapse and shift variables are used and have the status of Lagrange multipliers. In this paper, instead, I use the new split

$$g_{\Gamma\Delta} = \begin{pmatrix} F_{\mu} F^\mu - I^2 & F_\gamma \\ F_\delta & h_{\gamma\delta} \end{pmatrix}.$$  (27)

where $F^{\mu}$ is the frame (grid) and $I$ is the instant, the meanings of which are explained in the introduction. The considerations in section 2 shall ensure that the subsequent variation works out.

The GR kinetic term is now

$$T' = G^{\mu\nu\rho\sigma} \{ \dot{h}_{\mu\nu} - \mathcal{E} \dot{F} h_{\mu\nu} \} \{ \dot{h}_{\rho\sigma} - \mathcal{E} \dot{F} h_{\rho\sigma} \},$$  (28)

and the GR potential term is $R$ as before. The prime symbol here and in (35) and (34) is to distinguish these objects built with $F_{\mu}$ (and, where appropriate, $\Psi$) corrections from their unprimed counterparts built with $\beta_{\mu}$ (and, where appropriate, $\Phi$) corrections.

For greater generality, I include the electromagnetic field (massless spin 1 field) as an example of matter field. In Maxwell electrodynamics, the usually employed electric potential variable $\Phi(\lambda, x^\alpha)$ is defined by

$$\nabla \Phi = -E - A.$$  (29)

Here $A(\lambda, x^\alpha)$ is the magnetic potential such that $B = \nabla \times A$, where $B(\lambda, x^\alpha)$ is the magnetic field, and $E(\lambda, x^\alpha)$ is the electric field. The variable I use in this paper is, rather, $\Psi(\lambda, x^\alpha)$ such that $\dot{\Psi} = \Phi$. In the same way that $\Phi$ is a ‘work per unit charge’ $\int \int_{S(\lambda)} F_\mu dx^\mu / q$ (where $F(\lambda, x^\alpha)$ denotes force), $\Psi$ is an ‘action per unit charge’ $\int_\Gamma(\lambda) F_\mu dx^\mu / q$. Another interpretation is as follows. In the flat spacetime context, integrating (29) and applying the Faraday–Lenz law to the first factor and Stokes’s theorem, mixed partial equality and the definition of $A$ to the second factor,

$$\Phi = -\int_{\Gamma(\lambda)} [E_\mu + A_\mu] dx^\mu = \frac{d}{d\lambda} \int \int_{S(\lambda)} B \cdot dS - \int \int_{S(\lambda)} B \cdot dS = \left( \text{a corrected rate of change of magnetic flux} \right)$$  (30)

(for surface $S$ with boundary curve $\Gamma$). By Leibniz’s rule and theorem this is also

$$\Phi = \int \int_{S(\lambda)} B \cdot dS = \left( \text{change in magnetic flux due to change in shape of the surface over time} \right).$$  (31)

Thus

$$\Psi = \int \int_{S(\lambda)} B \cdot dS - \int d\lambda \int \int_{S(\lambda)} B \cdot dS = \text{(corrected magnetic flux) },$$  (32)

the first term being magnetic flux and the second term being the correction; this is also

$$\Psi = \int d\lambda \int \int_{S(\lambda)} B \cdot dS = \left( \text{time integral of magnetic flux due to change in shape of the surface over time} \right).$$  (33)

10 Including scalar fields is straightforward, and inclusion of spin-1/2 fermions follows along standard lines. This paper’s treatment of electromagnetism generalizes to Yang–Mills theory and to the various associated scalar and spin-1/2 gauge theories.
Electromagnetism in this formulation has kinetic and potential terms,
\[ T(\Lambda_\mu; \Psi) = \frac{1}{2} |A_\mu - \partial_\mu \Psi|^2, \quad \forall [A_\mu] = \frac{1}{2} F_{\mu\nu}, \] (34)
for \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) the spatial part of the electromagnetic field strength tensor. This formulation does indeed work in the flat spacetime context: \( C(x^\mu) = \delta L/\delta \Psi = \partial_\mu \Pi^\mu \) (where \( \Pi^\mu(\lambda, x^\nu) \) is the momentum conjugate to \( A_\mu \)) but \( C \) on the end spatial hypersurface is zero and \( C \) is time independent, so \( C \) is zero everywhere, and thus we recover the Gauss constraint, \( \partial_\mu \Pi^\mu = 0 \).

I then obtain the suitable manifestly reparametrization invariant action for Einstein–Maxwell theory by building the above up to hold on curved spaces and then coupling it to \( \mathcal{L} \) so that it holds on dynamically changing curved spaces. The complete Einstein–Maxwell theory by building the above up to hold on curved spaces and then coupling it to \( \mathcal{L} \) where the third equality implicitly defines this formulation’s Maxwell kinetic term,
\[ \frac{\partial}{\partial \mu} \{ \Pi^\mu | (\text{end spatial hypersurface}) \} = 0. \]

Particular to this starting point, the conjugate momenta are
\[ \pi^{\mu\nu}(\lambda, x^\alpha) = \frac{\delta L^A}{\delta \dot{h}_{\mu\nu}} = \frac{\sqrt{h}}{2} \frac{G^{\mu\nu\rho\sigma}}{2} \left( \dot{h}_{\rho\sigma} - \dot{\xi}_{\rho} h_{\rho\sigma} \right) \quad \text{and} \]
\[ \Pi^\mu(\lambda, x^\alpha) = \frac{\partial L^A}{\partial A_\mu} = \frac{\sqrt{h}}{I} \left( \dot{A}_\mu - \dot{\xi}_F A_\mu - \partial_\mu \Psi \right). \] (36)

Variation with respect to \( I \) gives as a secondary constraint
\[ C(x^\alpha) = \frac{\delta L^A}{\delta I} = G_{\mu\nu\rho\sigma} \pi^{\mu\nu\rho\sigma} + \frac{1}{2} \frac{\sqrt{h}}{2} \Pi^\mu + \frac{\sqrt{h}}{4} F_{\mu\nu} F^{\mu\nu} - \sqrt{h} R, \] (37)
and the FESH conditions \( \partial L^A/\partial I \big|_{\lambda} = 0 = \partial L^A/\partial I \big|_{\lambda}, \) so \( C(x^\alpha) = \partial L^A / \partial I = 0 \) at the end spatial hypersurface, but \( C \) is hypersurface independent, so \( C = 0 \) everywhere. Thus one arrives at
\[ G_{\mu\nu\rho\sigma} \pi^{\mu\nu\rho\sigma} + \frac{1}{2} \frac{\sqrt{h}}{2} \Pi^\mu + \frac{\sqrt{h}}{4} F_{\mu\nu} F^{\mu\nu} - \sqrt{h} R = 0, \] (38)
i.e. the Einstein–Maxwell Hamiltonian constraint, \( \mathcal{H} \). Variation with respect to \( \Psi \) gives
\[ G(x^\alpha) = \frac{\partial L^A}{\partial \Psi} = D_\mu \Pi^\mu = \partial_\mu \Pi^\mu, \] (39)
and the FESH conditions \( \partial L^A / \partial \Psi \big|_{\lambda} = 0 = \partial L^A / \partial \Psi \big|_{\lambda}, \) so \( G(x^\alpha) = \partial L^A / \partial \Psi = 0 \) at the end spatial hypersurface, but \( G \) is hypersurface independent, so \( G = 0 \) everywhere. Thus one arrives at
\[ D_\mu \Pi^\mu = 0, \] (40)
which is the electromagnetic Gauss constraint, \( G \), of Einstein–Maxwell theory. Variation with respect to \( F^\mu \) gives (modulo a Gauss constraint term) also as a secondary constraint.
\[ E_\mu(x^\nu) = \frac{\partial \bar{L}_B}{\partial \dot{F}^\mu} = -2D_\nu \pi_\mu^\nu - \Pi^\nu [D_\nu A_\mu - D_\mu A_\nu] \]  

(41)

and the FESH conditions \( \partial \bar{L}_A / \partial F^\mu |_{x^\nu} = 0 = \partial \bar{L}_A / \partial F^\nu |_{x^\nu} \), so \( E_\mu(x^\nu) = \partial \bar{L}_A / \partial F^\mu = 0 \) at the end spatial hypersurface, but \( E_\mu \) is hypersurface independent, so \( E_\mu = 0 \) everywhere. Thus one arrives at the momentum constraint of Einstein–Maxwell theory, \( \mathcal{H}_\mu: \)

\[-2D_\nu \pi_\mu^\nu = \Pi^\nu [D_\nu A_\mu - D_\mu A_\nu]. \]  

(42)

This formulation’s equations of motion \( \pi_\mu^\nu = \partial \bar{L}_A / \partial h_\mu^\nu \), \( \Pi^\mu = \partial \bar{L}_A / \partial A_\mu \) propagate the above constraints without giving rise to any further ones.

Taking the Lagrangian form of (37), one can now solve for \( \dot{I} \):

\[ \dot{I} = \pm \sqrt{\frac{T/4 + T_A}{R + U_A}}. \]  

(43)

Then take the + sign by convention of the direction of forward march of time, from which \( \dot{I} \) is straightforwardly algebraically eliminable from the action (35) by Routhian reduction. Thus one recovers the ‘BFO–A’ action

\[ I_{\text{BFO–A}}[h_\mu^\nu, h_\nu^\sigma, A_\mu, \dot{A}_\mu, F^\mu, \Psi] = \int \mathcal{D} \lambda \int d^3 x \bar{L}_{\text{BFO–A}}(\lambda, x^\mu, h_\mu^\nu, \dot{A}_\mu; h_\mu^\nu, A_\mu, F^\mu, \Psi) \]

\[ = \int \mathcal{D} \lambda \int d^3 x \sqrt{\left[T + 4T_A\right]} \left[\begin{array}{c}
{G}^{\mu_\nu\sigma\rho} [h_\mu^\nu - \xi_F h_\mu^\rho] [h_\rho^\sigma - \xi_F h_\rho^\nu] \\
+ 2h_\mu^\nu [A_\mu - \xi_F A_\mu - \partial_\nu \Psi] [\dot{A}_\nu - \xi_F A_\nu - \partial_\nu \Psi] \end{array}\right] \left\{R - \frac{1}{4} F_\mu^\nu F^\mu_\nu \right\} \right]. \]  

(44)

This is my formulation’s counterpart and equivalent of BSW’s multiplier elimination procedure [20], thus revealing [42]’s ‘explicit use is made of multiplier status’ to be an alternative rather than an obligation.

Taking this action as starting point, the momenta are

\[ \pi^{\mu\nu} = \frac{\partial \bar{L}_{\text{BFO–A}}}{\partial h_\mu^\nu} = \sqrt{\frac{R + U_A}{T + 4T_A}} G^{\mu_\nu\sigma\rho} [h_\mu^\nu - \xi_F h_\mu^\rho], \]

\[ \Pi^\mu = \frac{\partial \bar{L}_{\text{BFO–A}}}{\partial A_\mu} = \sqrt{\hbar} \sqrt{\frac{R + U_A}{T + 4T_A} \left[2h_\mu^\nu [A_\mu - \xi_F A_\nu - \partial_\nu \Psi] \right]} \]  

(45)

Now one does not have not a priori an instant \( I \) of which the rate appears in the equations, but rather an emergent quantity \( \sqrt{R + U_A / T + 4T_A} \). Then there is a primary constraint

\[ G_{\mu_\nu\rho\sigma} \pi^{\mu_\nu} \pi^{\rho_\sigma} + \frac{1}{2 \sqrt{\hbar}} h_\mu^\nu \Pi^\mu = G_{\mu_\nu\rho\sigma} G^{\nu_\sigma\phi\eta} \sqrt{\frac{R + U_A}{T + 4T_A} \left[h_\phi^\eta - \xi_F h_\phi^\eta \right]} \]

\[ \times \sqrt{\frac{R + U_A}{T + 4T_A} \left[h_\eta^\phi - \xi_F h_\eta^\phi \right]} + \frac{1}{2 \sqrt{\hbar}} h_\mu^\nu h_{\nu\phi} \sqrt{\hbar} \sqrt{\frac{R + U_A}{T + 4T_A} \left[2A_\sigma - \xi_F A_\sigma - \partial_\sigma \Psi \right]} \]

\[ \times 2 [\dot{A}_\rho - \xi_F A_\rho - \partial_\rho \Psi] h^{\nu_\rho} \sqrt{\hbar} \sqrt{\frac{R + U_A}{T + 4T_A} \left[2A_\sigma - \xi_F A_\sigma - \partial_\sigma \Psi \right]} \]

\[ = \sqrt{\hbar} \sqrt{\frac{R + U_A}{T + 4T_A} \left[T + 4T_A \right]} = \sqrt{\hbar} \left( R + U_A \right). \]  

(46)
(by the definition of momentum, and using that $G_{\mu\nu\rho\sigma}$ is the inverse of $G^{\mu\nu\rho\sigma}$), the Hamiltonian constraint. Variation with respect to $\Psi$ gives

$$G(x^{\omega}) = \frac{\partial \mathcal{L}_{BFO-A}}{\partial \dot{\Psi}} = D_{\mu} \Pi^\mu = \partial^\mu \Pi^\mu,$$

and the FESH conditions $\partial \mathcal{L}_{BFO-A}/\partial \dot{\Psi}\big|_{\lambda_i} = 0 = \partial \mathcal{L}_{BFO-A}/\partial \dot{\Psi}\big|_{\lambda_f}$, so $G(x^{\omega}) = \partial \mathcal{L}_{BFO-A}/\partial \dot{\Psi}$ is hypersurface independent, so $G = 0$ everywhere. Variation with respect to $F^\mu$ gives (modulo a Gauss constraint term) the secondary constraint

$$E^\mu(x^{\omega}) = \frac{\partial \mathcal{L}_{BFO-A}}{\partial \dot{F}^\mu} = -2D_\nu \pi^{\mu\nu} = \Pi^\nu \{D_\nu A^\mu - D^\mu A_\nu\},$$

and the FESH conditions $\partial \mathcal{L}_{BFO-A}/\partial \dot{F}^\mu\big|_{\lambda_i} = 0 = \partial \mathcal{L}_{BFO-A}/\partial \dot{F}^\mu\big|_{\lambda_f}$, so $E^\mu(x^{\omega}) = \partial \mathcal{L}_{BFO-A}/\partial \dot{F}^\mu = 0$ at the end spatial hypersurface, but $F^\mu$ is hypersurface independent, so $F^\mu = 0$ everywhere. Thus one arrives at

$$-2D_\nu \pi^{\mu\nu} = \Pi^\nu \{D_\nu A^\mu - D^\mu A_\nu\},$$

the momentum constraint of GR, $H^\mu$. The BFO–A equations of motion $\dot{\pi}^{\mu\nu} = \partial \mathcal{L}_{BFO-A}/\partial h_{\mu\nu}$ propagate these constraints without giving rise to any further ones.

Thus in this scheme temporal relationalism as implemented by manifest reparametrization invariance without extraneous variables gives the Hamiltonian constraint of GR and spatial relationalism as regards 3-diffeomorphisms as implemented by arbitrary $G$-frame cyclic coordinate velocity corrections to the metric velocities gives the GR momentum constraint. In fact, even if a number of features of $\mathcal{L}_{BFO-A}$ are not known, if it is built according to temporal relationalism, the Dirac procedure is applied and consistency is demanded, $\mathcal{L}_{BFO-A}$ emerges as one of very few possibilities [16, 17, 22, 43]. The above very clear way of obtaining the GR constraints (and lack of prior knowledge of the form of the GR equations to do so) and the accommodability into this scheme of a sufficiently wide range of fundamental matter fields to describe nature at the classical level [17, 22, 43] make (44)11 itself interesting as a starting point for gravitational physics. In fact, starting with temporal relationalism (and a configuration space of spatial 3-metrics) alone suffices [44] as that produces $\mathcal{H}$ as above, which then happens to dictate $\mathcal{H}$ and $G$ as integrability conditions (see also [45]), whose subsequent appending or encoding leads to the recovery of (44)11. Though this may be considered to involve luck, as, in other cases [44], relying on integrability can cause constraints to get ‘missed out’.

5. Conclusion

In this paper, I have explained that variation with respect to auxiliary variables is free end point (FEP) in finite cases or free end spatial hypersurface (FESH) in field-theoretic cases. This makes a difference to the outcome of varying cyclic coordinates if they are auxiliary, whereby interpreting an auxiliary present in one’s action as a cyclic velocity is equivalent to interpreting it as a multiplier coordinate Thus one has an alternative to the ADM split: instead of lapse $\alpha$ and shift $\beta^\mu$ multiplier coordinates, one has instant $I$ and grid $F^\mu$ cyclic coordinates, which are related to the $\alpha$ and $\beta^\mu$ by $I = \alpha$ and $F^\mu = \beta^\mu$. Also one has an alternative to the usual electric–magnetic split formulation: instead of an electric potential multiplier coordinate $\phi$, one has a flux cyclic coordinate $\Psi$ that is related to it by $\Psi = \phi$.

Further principles of dynamics manipulations continue to give the same answers in the new scheme. For example, Routhian reduction now works out to be equivalent to what

11 Strictly speaking, these references consider the BSW formulation counterpart of these results; the BFO-A counterpart to these results is a new result of the present paper.
was previously multiplier elimination, e.g., the new scheme has a Routhian reduction instant-
velocity elimination counterpart of Baierlein, Sharp and Wheeler’s multiplier elimination of the
lapse. As another example, this scheme supplants Hamiltonians \( H \) by ‘almost Hamiltonians’ \( A \)
which depend on positions and momenta for the usual variables but on positions and velocities
for manifestly pure-frame variables. There is then no difference in constraints arising from
the multiplier picture’s \( H \) and the cyclic coordinate picture’s \( A \) in the position–momentum
language for the theories in this paper (for which the auxiliary variables, however represented,
do not enter the position–momentum form of the constraints). Nor is there any difference in
the thin sandwich conjecture (solving the Lagrangian form of \( H^\mu \) with matter struck out for
\( \beta^\mu \) [46] or now for \( F^\mu \), or the Lagrangian form of \( H^\mu \) and \( G \) for \( \beta^\mu \) and \( \Phi \) [47] or now for \( F^\mu \)
and \( \Psi \)).

One advantage of this paper’s formulation is that it is more manifestly temporally relational
than ADM’s. It is true that the instant-less BFO-A formalism is ‘even more’ manifestly
temporally relational in Barbour’s sense of not containing an extraneous time variable, but,
on the other hand, thinking along the lines of this paper’s formalism and then eliminating
the extraneous time variable permits the relational program to go beyond its current spatially
compact without boundary setting to cases with boundaries or that are open.

The cyclic velocity picture also gives a different account of how the frozen formalism
problem of GR (see, e.g., [7, 26, 48]) arises.

\[
H = \frac{\partial O}{\partial \dot{I}} = -\frac{\partial L}{\partial \dot{I}} = -\Pi^I
\]  

(50)

by, respectively, trivial direct computation, the chain rule applied to the Legendre
transformation relation and the definition of momentum. Thus

\[
\Pi^I + H = 0.
\]  

(51)

Similarly,

\[
\Pi^F_\mu + H^\mu = 0.
\]  

(52)

At this stage, these contain linear momenta, so, in particular, the first quadratic equation has
no trace of the frozen formalism. It would amount to

\[
\frac{i\hbar}{\partial t} |\Psi\rangle - \hat{H} |\Psi\rangle \equiv 0
\]  

(53)

at the quantum level, which is a (instant, or proper) time-dependent Schrödinger equation
rather than a stationary equation like the Wheeler–DeWitt equation. However, one has also
the FESH condition \( \Pi^I = 0 \), which precisely wipes out the linear momentum in the crucial,
quadratic constraint, leaving

\[
H = 0.
\]  

(54)

At the quantum level this amounts to the usual stationary Wheeler–DeWitt equation

\[
\hat{H} |\Psi\rangle = 0. \tag{55}
\]

However, in subcases or simplified models in which there is a privileged background, the
situation here is, very transparently, that FEP is then inappropriate, whereby some kind
of time-dependent Schrödinger equation survives. As many quantum schemes proceed via
the position–momentum form of the constraints that is unchanged by passing to this paper’s
formalism, these schemes are likewise unchanged. I leave investigation of whether this paper’s
reformulation leads to any other differences in quantization procedures for a future occasion.

As regards classical applications, I explained above how auxiliary variables can be more
general than either multiplier coordinates or the velocities corresponding to cyclic coordinates.
This affords further extension of how gauge theories are treated in this paper. In particular, I establish a formal equivalence between treating such general auxiliaries and considering both them and their velocities to be a doubled number of independent Lagrange multipliers. I leave presentation of this in detail to [30], since the applications of this take one further afield than standard formulations of standard theories of physics. Examples of applications are careful justifications for the derivations of alternative theories of gravitation [21] and of the variational principle for GR in its favoured initial value problem formulation [40], both of which rest on conformal splits of 3 + 1 split GR along the lines of York’s work [34, 49]). In particular, my present investigation clarifies that the latter case bears the hallmarks of a gauge fixing rather than being an entirely clean isolation of true dynamical degrees of freedom for spatially compact without boundary GR.

The extension of the present paper to asymptotically flat GR is also of technical and conceptual interest. Now, one would like the lapse and shift at spatial infinity to take fixed values in this context, so that it is not a priori clear that FESH variation is applicable in this case as the values taken at spatial infinity by the \( \delta I \) and \( \delta F^\mu \) factors in the initial and final boundary terms are then zero so that the hypersurface-independent functions \( C \) and \( E^\mu \) are not forced to be zero at spatial infinity. However, at least for relatively simple function spaces, as \( C \) and \( E^\mu \) vanish throughout the interior of the boundaries by continuity they ought to vanish at spatial infinity on the boundaries too. [33] also provides an argument for equivalence of free variation with reference fields chosen to take desired boundary values, which would apply to asymptotically flat situations in at least some situations (see also [50] for similar considerations including an explicit example). The extension to further types of boundary condition [34, 35] would likewise be of interest. These extensions will allow one to consider from a Machian perspective (in the sense of Barbour) a wide range of situations of interest in modern mathematical and numerical relativity.

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Appendix. ADM and BSW-type formalisms of Einstein–Maxwell theory

Paralleling the working between (4) and (5), the 3 + 1 split action for Einstein–Maxwell theory can be cast in the Lagrangian variables form

\[
I_{ADM}[h_{\mu\nu}, \dot{h}_{\mu\nu}, A_\mu, \dot{A}_\mu, \beta^\mu, \alpha, \Phi] = \int d\lambda \int d^3x \sqrt{h} \left\{ \frac{T_{\mu\nu}}{4\alpha^2} + \frac{T_A}{\alpha^2} + R + U_A \right\}
\]

\[
= \int d\lambda \int d^3x \sqrt{h} \left\{ \frac{G_{\mu\nu\sigma\tau} [\dot{h}_{\mu\nu} - \xi_\beta h_{\mu\nu}] [\dot{h}_{\sigma\tau} - \xi_\beta h_{\sigma\tau}]}{4\alpha^2} + \frac{h_{\mu\nu} [\dot{A}_\mu - \xi_\beta A_\mu - \partial_\mu \Phi] [\dot{A}_\nu - \xi_\beta A_\nu - \partial_\nu \Phi]}{2\alpha^2} \right\} + R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (A.1)
\]
Here, $\alpha$ is the lapse, $\beta^\mu$ is the shift and $\Phi$ is the electric potential, which are all taken here to be Lagrange multipliers, and the third equality implicitly defines this formulation’s Maxwell kinetic term, $T_A$, and GR kinetic term, $T$. See footnote 1 and section 4 for the rest of the notation.

From this, the conjugate momenta are
\[
\pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{h}_{\mu\nu}} = \frac{1}{2\alpha} G^{\mu\nu\rho\sigma} \{ \dot{h}_{\rho\sigma} - \xi_\rho h_{\rho\sigma} \} \quad \text{and}
\]
\[
\Pi^{\mu} = \frac{\partial L}{\partial \dot{A}_\mu} = \sqrt{\alpha} h^{\mu\nu} \{ \dot{A}_\nu - \xi_\nu A_\nu - \partial_\nu \Phi \}.
\] (A.2)

Variation with respect to $\alpha$ gives the Einstein–Maxwell Hamiltonian constraint, $H_A$:
\[
G_{\mu\nu\rho\sigma} \pi^{\mu\nu} \pi^{\rho\sigma} + \frac{1}{2\alpha} \alpha^{\mu} \Pi^{\mu} + \frac{\sqrt{\alpha}}{4} F_{\mu\nu} F^{\mu\nu} - \sqrt{\alpha} R = 0.
\] (A.3)

Variation with respect to $\Phi$ gives the Gauss constraint, $G$:
\[
D_\mu /\Phi_1 = 0.
\] (A.4)

Variation with respect to $F^\mu$ gives (modulo a Gauss constraint term) the Einstein–Maxwell momentum constraint $H_A^{\mu}$:
\[
-2 D_\nu \pi^{\mu\nu} = \Pi^\nu [D_\nu A_\mu - D_\mu A_\nu].
\] (A.5)

One then finds that propagation of these by the ADM equations of motion for Einstein–Maxwell theory,
\[
\dot{\pi}^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{h}_{\mu\nu}} \quad \text{and} \quad \dot{\Pi}^\mu = \frac{\partial \mathcal{L}_{\text{ADM}}}{\partial \dot{A}_\mu},
\]
gives no further constraints.

Using the Lagrangian variables version of the $\alpha$-multiplier equation,
\[
R + U_A - \frac{T}{4\alpha^2} - \frac{T_A}{\alpha^2} = 0 \quad \Rightarrow \quad \alpha = \pm \sqrt{\frac{\frac{T}{4} + \frac{T_A}{T + U_A}}{R + U_A}}
\] (A.6)

(taking the + sign by convention of direction of forward march of time), $\alpha$ is straightforwardly algebraically eliminable from the ADM action giving the Baierlein–Sharp–Wheeler (BSW)-type action for Einstein–Maxwell theory:
\[
l_{\text{BSW}}[h_{\mu\nu}, \dot{h}_{\mu\nu}, A_\mu, \dot{A}_\mu, \beta^\mu, \Phi] = \int d\lambda \int d^3 x \mathcal{L}_{\text{BSW}}(\lambda, x^\mu, h_{\mu\nu}, \dot{h}_{\mu\nu}, A_\mu, \dot{A}_\mu, \beta^\mu, \Phi)
\]
\[
= \int d\lambda \int d^3 x \sqrt{\alpha} \left\{ G^{\mu\nu\rho\sigma} \{ \dot{h}_{\rho\sigma} - \xi_\rho h_{\rho\sigma} \} [\dot{h}_{\mu\nu} - \xi_\mu h_{\mu\nu}] \right\} + 2 h^{\mu\nu} [\dot{A}_\mu - \xi_\mu A_\mu - \partial_\mu \Phi] [\dot{A}_\nu - \xi_\nu A_\nu - \partial_\nu \Phi] \left\{ R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} \right.^{\frac{1}{2}}.
\] (A.7)

From this starting point, the conjugate momenta are then
\[
\pi^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{BSW}}}{\partial \dot{h}_{\mu\nu}} = \sqrt{\alpha} \frac{R + U_A}{T + 4T_A} G^{\mu\nu\rho\sigma} \{ \dot{h}_{\rho\sigma} - \xi_\rho h_{\rho\sigma} \} \quad \text{and}
\]
\[
\Pi^\mu = \frac{\partial \mathcal{L}_{\text{BSW}}}{\partial \dot{A}_\mu} = \sqrt{\alpha} \frac{R + U_A}{T + 4T_A} 2 h^{\mu\nu} [\dot{A}_\mu - \xi_\mu A_\mu - \xi_\nu A_\nu - \partial_\nu \Phi].
\] (A.8)

Then there arises as a primary constraint the Hamiltonian constraint of Einstein–Maxwell theory (A.3). Variation with respect to $\Phi$ gives the Gauss constraint (A.4). Variation with
respect to $F$ gives the Einstein–Maxwell momentum constraint (A.5) (again modulo a Gauss constraint term). One then finds that propagation of these by the BSW equations of motion

$$\pi^{\mu\nu} = \partial L_{BSW} / \partial h_{\mu\nu}, \quad \Pi^\mu = \partial L_{BSW} / \partial A_\mu,$$

again gives no further constraints.

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