Some Comparisons of the Methods Applied to Solving the First-Order Integro-Differential Equations

G Mehdiyeva¹, V Ibrahimov¹,², M Imanova¹,²

¹Department of the Computational mathematics, Baku State University, Z. Khalilov 23 Azerbaijan, AZ1148
²Institute of Control Systems, Baku, Azerbaijan
E-mail: imn_bsu@mail.ru

Abstract. There are some classes of the methods for solving the initial-value problem of the integro-differential equations of Volterra type. Obviously, each class of these methods has its advantages and disadvantages. Therefore, here we consider a comparison of some known methods by which the region of the application of the certain methods is determined. The results obtained here make it possible to choose an appropriate method for solving the initial-value problem of the Volterra integro-differential equation.

1. Introduction

The solutions of integro-differential equations with variable boundaries are fundamentally investigated by V. Volterra. Many well-known scientists have studied integro-differential equations, constructed various methods for their solving, and also applied theories of the integro-differential equations in the study of many processes to take place in different fields of the natural science (see e.g. [1] - [11]). Many of them are devoted to the construction and application of approximate methods to the solving of the initial-value problem for integro-differential equations of Volterra type. In many of these studies, have constructed the numerical methods for solving the above-mentioned problem in which the integral is replaced by an integral sum or some other methods in which the value of computational works are increasing in the transition from the current point to the next. For example, consider the following quadrature method:

\[ \int_{x_0}^{x_n} K(x, s, y(s))ds = \sum_{i=0}^{m} K(x_m, x_i, y_i) + R_n. \]  (1)

As follows from equality (1), with increasing values of \( m \), the number of terms which are used in the sum in equality (1) also increases.

Here, the proposed methods are freed from the above mentioned shortcomings, and they are effective in applying them to solving some special cases of the first-order Volterra integro-differential equations.

We consider to the following problems, which are usually is called as the initial-value problem of the Volterra integro-differential equations of the first order:

\[ y' = F(x, y(x), \nu(x)), y(x_0) = y_0; x \in [x_0, X] \]  (2)

here the function \( \nu(x) \) is defined in the following form:
\[ v(x) = \lambda \int_{x_0}^{x} K(x, s, y(s))ds, x_0 \leq s \leq X. \] (3)

If the parameter \( \lambda = 0 \), then from the problem (2) we obtain the ordinary initial-value problem for the ODE of the first-order. Therefore, we assume that \( \lambda \neq 0 \) and put \( \lambda = 1 \). In this case the problem (2) can be written in the form:

\[ y' = f(x, y) + v(x), y(x_0) = y_0, x_0 \leq x \leq X. \] (4)

To find the numerical solution of the problem (4), we assume that the equation (1) has a unique continuous solution defined on the segment \([x_0, X]\). To find the approximately values of the solution of the problem (1) at some mesh points, let us divide the segment \([x_0, X]\) into \( N \) equal parts by the mesh points \( x_i = x_0 + ih(i = 0,1,\ldots,N) \). Here \( 0 < h \) - is a step size. Let us also denote by the \( y_i \) approximately values, but by the \( y(x_i) \) through, the exact values of the solution of the problem (4) at the mesh points \( x_i \) \( (i = 0,1,\ldots,N) \), respectively.

As is well known, the numerical method for solving of the problem (4) can be constructed in the following form (in one variant see e.g. [4] - [13])

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{k} \beta_i v_{n+i}, \] (5)

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i^{(j)} K(x_{n+j},x_{n+i},y_{n+i}). \] (6)

Here \( f_m = f(x_m, y_m), v_m = v(x_m) (m = 0,1,2,\ldots) \).

Note that the method (5) is an ordinary multistep method with constant coefficients, and the method (6) is a multistep method, proposed to solving of the Volterra integral equation. Thus, we obtain a system of difference methods for solving of the problem (4). It is obvious that the accuracy of the method determined by the proposed system of difference methods does not exceed the accuracy for each of these methods. Therefore, some scientists are constructed the methods consisting of the single formula.

We will try to compare methods similar to the above proposed, which are used to solve the problem (4), in special cases.

2. An investigation of the problem (4) in the special case

Here we will investigate the following problem, which is a particular case of the problem (4):

\[ y' = f(x) + \int_{x_0}^{x} K(s, y(s))ds, y(x_0) = y_0, x \in [x_0, X] \] (7)

If we apply methods (5) and (6) to solving of the problem (7), then we get:

\[ \sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} (\beta_i - \beta_i \alpha_i) y_{n+i} + h^2 \sum_{i=0}^{k} \beta_i K_{n+i}, (\alpha_i = 1), \] (8)

Here \( K_m = K(x_m, y_m), (m = 0,1,2,\ldots) \).

Remark that by the Dahlqvist theorem one can obtain that the order of accuracy of the method (8) satisfies the condition \( p \leq k + 2 \) (see for example [14]). To constructed more accuracies methods, the specialists proposed different ways. One of them is the use of the higher derivatives of the function \( y(x) \), which is a solution of the problem (7) (see [15] - [19]). Euler himself proposed to use the calculation of the subsequent terms in the expansion of a sufficiently smooth function in a Taylor
series to increase the order of accuracy of his famous method (see [20], [21]). One such method is the following:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}. \quad (9)$$

If method (9) is stable, then its order of accuracy is determined by the formula: $$p \leq 2k + 2.$$

It is said that the method (9) is stable if the roots of the following polynomial

$$\rho(\lambda) = \alpha_k \lambda^k + \alpha_{k-1} \lambda^{k-1} + \ldots + \alpha_0 \lambda + \alpha_0$$

lie inside the unit circle, on the boundary of which there are no multiple roots. And the integer value $$P$$ is the degree of the method (9) if the following is holds:

$$\sum_{i=0}^{k} (\alpha_i y(x + ih) - h \beta_i y'(x + ih) - h^2 \gamma_i y''(x + ih)) = O(h^{p+1}), h \to 0.$$

It is easy to understand that to increase the values of the degree of the methods defined by the formula of type (5), it is sufficient to add to its right-hand side new sums using the values of the higher derivatives of the function $$y(x)$$. It is known that there exist stable multistep methods of Obreshkov’s type with the degree $$p = r(k = 1) + 1$$ for even $$k$$ and odd $$r$$, and $$p = r(k + 1)$$ in other cases. Here, is denotes by the the order of the method, and by $$r$$ the order of the ordinary differential equations (see for example [22]). Consequently, the method (9) is a method of Obreshkov type with the second derivative and therefore $$P_{\text{max}} = 2k + 2$$.

Let us consider to the application of the method (9) to solving of the problem (7). Then we have:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i (f'_{n+i} + K_{n+i}). \quad (10)$$

If we take into account that the derivative of the function $$y(x)$$ does not exist on the right-hand side of the integro-differential equation, then (10) can be replaced by the following:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h^2 \sum_{i=0}^{k} \gamma_i f'_{n+i} + K_{n+i}. \quad (11)$$

Note that, the use of the method (11) is simpler than the using of the method (10). However, the maximum value of the degree of the method (11) is less than the value of the degree of the method (10). It does not follows from this that if a method is used to calculate the values of a quantity $$y'_m (m \geq k)$$, then a stable method of type (10) will have the degree $$p = 2k + 2$$. For example, if to calculation of the values $$y'_m (m \geq k)$$, proposed to use the following method:

$$\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i (f'_{n+i} + K_{n+i}), \quad (12)$$

then the degree of the stable method obtained from the formula (12) will be satisfies the condition: $$p \leq 2[k/2] + 2$$. Thus, if we construct a method based on the formulas (10), (12) and applied them to solving of the problem (7), then its degree satisfies the condition$$p \leq k + 3$$. Consequently, the use of the method (11) is preferable than to using the method constructed on the basis of the formulas (10) and (12).

And now let us to compares the above proposed methods with the stable methods, constructed on the basis of the methods (5) and (6). If the methods defined by the formulas (5) and (6) are applicable to solving of the problem (4), then as a result we obtain method (8). But in this case, the degree of the methods constructed according to the scheme described above will coincide with the degree of the methods constructed by the formula (5). Consequently, the degree of the stable methods obtained from
the formula (8) satisfies the condition \( p \leq 2\lceil k/2 \rceil + 2 \). Thus, we obtain that the degree of all methods defined by one of the formulas (8), (11) and (12), satisfies the condition \( p \leq 2\lceil k/2 \rceil + 2 \). Therefore, to construct methods with the higher accuracy, we can use formula (10). However, the accuracy of the method (10) depends from the accuracy of the methods proposed for calculating the values of the quantity \( y'_m (m = 1, 2, \ldots) \). If we use to calculation of the values of \( y'_m \) the stable methods determined by formula (12), then their degree satisfies the condition \( p \leq k + 2 \). But the degree of a stable method, defined by the formula (10), satisfies the condition \( p \leq 2k + 2 \). Hence, we need to construct more exact methods by using the formula (10) to calculating the values of the quantity \( y'_m \) and having the degree \( p \leq 2k + 1 \). Those methods are in the class of hybrid methods. For the purpose of constructing such methods, consider to the following formula:

\[
\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y''_{n+i} + h \sum_{i=0}^{k} \gamma_i y''_{n+i+v_i}, \quad (|v_i| < 1, i = 0, 1, \ldots, k). \tag{13}
\]

In the class of methods (13), there exist stable methods with degree \( p \geq 2k + 2 \) (see, for example, [10]). We apply formula (13) to solving of the problem (7) and obtain the following:

\[
\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i (f'_{n+i} + K_{n+i}) + h \sum_{i=0}^{k} \gamma_i (f''_{n+i+v_i} + K_{n+i+v_i}). \tag{14}
\]

Note that the calculation of the values of the function \( f'(x) \) does not cause any difficulties. But when calculating the values of the function \( K(x, y) \), some difficulties arise related to the calculation of the values \( y'_{n+i+v_i} (0 \leq i \leq k) \). Thus, we get that the exactness of the methods obtained from the formula (14), depends on the exactness of the approximate values of the solution of problem (7), calculated at hybrid points. Therefore, constructed here some specific methods for using formula (14). For this aim have proposed specific methods for calculating the values of \( y'_{n+i+v_i} \), the degree of which agrees with the degree of the proposed methods. Below are some specific methods.

Let us put \( k = 1 \) in the formula (14). Then from here one can receive the following methods:

\[
y'_{n+1} = y'_n + h(f'_{n+1} + K_{n+1} + f''_n + K_n)/12 +
+ 5h(f'_{n+1} + K_{n+1} + f''_n + K_n)/12, (\alpha = \sqrt{5}/10), \tag{15}
\]

\[
y'_{n+1} = y'_n + h(f'_{n+1} + K_{n+1})/9 + h((16 + \sqrt{6})(f'_{n+1} + K_{n+1}))
+ (16 - \sqrt{6})(f'_{n+1} + K_{n+1})/36, \gamma_0 = (6 - \sqrt{6})/10, \gamma_1 = (6 + \sqrt{6})/10. \tag{16}
\]

Note that these methods are stable. Method (15) has the degree \( p = 6 \), and the method (16) has the degree \( p = 5 \). The following method is also stable and has the degree \( p = 4 \):

\[
y'_{n+1} = y'_n + h(f'_{n+1} + K_{n+1} + f''_{n+1} + K_{n+1})/2, (l_0 = -l_1; l_0 = 3 - \sqrt{3}/6; l_1 = 3 + \sqrt{3}/6). \tag{17}
\]

Note that if the function \( K(s, y(s)) \) depends on \( x \), i.e. \( K(s, y(s)) = \varphi(x, s, y(s)) \), then the above methods are written in the differently form. For example, method (17) in one variant can be rewritten in the form:

\[
y'_{n+1} = y'_n h(2f'_{n+1} + \varphi(x_{n+1}, x_{n+1}, y_{n+1}) + \varphi(x_{n+1}, x_{n+1}, y_{n+1}) + 2f'_{n+1} +
+ \varphi(x_{n+1}, x_{n+1}, y_{n+1}) + \varphi(x_{n+1}, x_{n+1}, y_{n+1}))/4. \tag{18}
\]
Note that the hybrid methods, generally speaking, are not implicit. For example, methods (16) and (17) are not implicit. The method (15) is implicit, since the term in the form \( K(x_{n+1}, y_{n+1}) \) participates in it. The hybrid method can be constructed so that participates in it the term in the form \( K(x_{n+v}, y_{n+v})(v \geq 1) \). If \( v = 1 \), then the point \( x_{n+v} \) is not a hybrid point. Therefore, for the construction of implicit hybrid methods, the condition \( 1 < v < 2 \) must be satisfied. But in this case we get the forward-jumping methods. Therefore, the hybrid methods, which have the following form:

\[
\sum_{i=0}^{k} \alpha_i y'_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i+v}, \quad (|v| < 1; i = 0, 1, \ldots, k)
\]

(19)

are not implicit. However, they can’t be called explicit. Usually the method is called explicit if it can be used to solve proposed problems without using other methods. As follows from here, for using hybrid methods, it is necessary to find the values of the quantities \( y'_{n+i+v}, (|v| < 1; i \geq 0) \), which are the values of the solution of the considered problem at the hybrid points. Note that it is not easy to calculate these values. Thus, we get that a lot of hybrid methods constitute a separate class of the methods. Usually they are called a class of hybrid methods. Note that the methods, which have composed from the multi-step and hybrid methods are also, constitute the new classes of numerical methods, which include in it the methods constructed at the junction of multi-step methods with constant coefficients (explicit, implicit and forward-jumping methods) and hybrid methods. Note that these classes of methods include methods of both type (14) and (10) (methods of type (10) are constructed with the addition of the quantity of type: \( y'_{n+i+v} \), and \( y_{n+i+v} (|v| < 1; i = 0, 1, \ldots, k) \).

Remark, that here have constructed the stable methods with the degree \( p = 4k + 2 \) and give the way for application of it to solving problem (7).

3. Conclusions

Here, have considered to investigation multistep hybrid methods with the higher order of exactness. For this aim has been used one of general form of multistep hybrid methods with constant coefficients. For the demonstration of the received results have construct the concrete methods, which have applied to solving of model problem. For the compares results obtained here, with the known to us, has used the special case of Volterra integro-differential equations, which has used as the model problem in the investigation of many scientific problems, what arises in the studies of the natural sciences (see for example[3], [9], p.269 [25]).

Acknowledgments

The authors express their thanks to the academicians Telman Aliyev, Abel Maharramov and Ali Abbasov for their suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan – Grant № EIF-KETPL-2-20151(25)-56/07/1.

References

[1] Volterra V. Theory of functional and of integral and integro-differential equations, Moskow, Nauka, 1982.
[2] Linz P. Linear Multistep methods for Volterra Integro-Differential equations, Journal of the Association for Computing Machinery, Vol.16, No.2, April 1969, pp.295-301.
[3] Feldstein, A., & Sopka, J.R. Numerical methods for nonlinear Volterra integro differential equations. SIAM J. Numer. Anal. V. 11., 1974, 826-846.
[4] Brunner H. Implicit Runge-Kutta Methods of Optimal order for Volterra integro-differential equation. Mathematics Of Computation Volume 43, Number 165 January 1984, p. 95-109.
[5] Lubich Ch., “Runge-Kutta theory for Volterra and Abel integral equations of the second kind” Mathematics of computation volume 41, number 163, July 1983, 87-102.

[6] Christopher T.H. Baker and Mir S. Derakhshan R-K Formulae applied to Volterra equations with delay. Journal of Comp. And Applied Math 29, 1990, 293-310.

[7] Makroglou A. Hybrid methods in the numerical solution of Volterra integro-differential equations. Journal of Numerical Analysis 2, 1982, 21-35

[8] Mehdiyeva G.Yu., Ibrahimov V.R., Imanova M.N., Research of a multistep method applied to numerical solution of Volterra integro-differential equation, World Academy of Science, engineering and Technology, Amsterdam, 2010 (Scopus), pp. 349-352

[9] Makroglou, A.A. Block - by-block method for the numerical solution of Volterra delay integro-differential equations. Computing 3, 30, №1, 1983, 49-62.

[10] Mehdiyeva G.Yu., Ibrahimov V.R., Imanova M.N., Solving Volterra Integro-Differential Equation by the Second Derivative Methods, Applied Mathematics and Information Sciences, Volume 9, No. 5, Sep. 2015, pp. 2521-2527.

[11] Mehdiyeva G.Yu., Ibrahimov V.R., Imanova M.N., Guliyeva A. One a way for constructing hybrid methods with the constant coefficients and their applied, Alloys and Experimental Mechanics, ICMAEM-2017 IOP Publishing, IOP Conf. Series: Materials Science and Engineering 225 (2017) 012042 doi:10.1088/1757-899X/225/1/012042

[12] Budnikova, O.S. Bulatov, M.V. (2012). The numerical solution of integroalgebraicheskih multistep methods. Journal of Comput. Math. and mat.fiziki, т.52, №5, 829-839 (Russian).

[13] Mehdiyeva G., Ibrahimov V., Imanova M. The application of the hybrid method to solving the Volterra integro-differential equation. World Congress on Engineering 2013, London, U.K., 3-5 July, 2013, 186-190.

[14] Dahlquist G. Convergence and stability in the numerical integration of ordinary differential equations, Math.Scand, 1956, No.4, 33-53.

[15] Dahlquist G. Stability and error bounds in the numerical integration of ordinary differential equation. Trans. Of the Royal Inst. Of Techn., Stockholm, Sweden, Nr. 130, 1959, 3-87.

[16] Ibrahimov V.R. On a relation between order and degree for stable forward jumping formula. Zh. Vychis. Mat., № 7, 1990, p.1045-1056.

[17] Iserles A. and Norset S.P. Two-step method and Bi-orthogonality. Math. of Comput., 180 (1987) 543-552.

[18] Kobza J. Second derivative methods of Adams type. Applikace Mathematicky, 1975, №20, 389-405.

[19] Huta A.A. An a priori bound of the discretization error in the integration by multistep difference method for the differential equations \( y(x) = f(x, y) \), Acta F.R.N. Univer. Comen. Math., 1979, №34, pp.51-56.

[20] Euler L. Integral calculus, vol. II. Moscow, Gostehus-Dates, 1957.

[21] Krylov A.N. Lectures on approximate calculations. Moscow, Goctech-izdat, 1950.

[22] Mehdiyeva G., Imanova M., Ibrahimov V. A way to construct an algorithm that uses hybrid methods. Applied Mathematical Sciences, HIKARI Ltd, Vol. 7, 2013, no. 98, 4875-4890.

[23] Mehdiyeva G.Yu., Ibrahimov V.R., Imanova M.N. On a technique of construction of hybrid methods with the higher order of accuracy. Transactions of Azerbaijan national academy of sciences, (2011)pp. 112-118.

[24] Ibrahimov V.R. On the maximum degree of the k-step Obrechkoff’s method. Bulletin of Iranian Mathematical Society, 20, №1 (2002) p.1-28.

[25] Modern Numerical Methods for Ordinary Differential Equations, edited by G.Hall and J.M.Watt, Clarendon Press, Oxford 1976, 312 p.