Exact Liouvillian Spectrum of a One-Dimensional Dissipative Hubbard Model

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A one-dimensional dissipative Hubbard model with two-body loss is shown to be exactly solvable. We obtain an exact eigenspectrum of a Liouvillian superoperator using a non-Hermitian extension of the Bethe-ansatz method. We find steady states, the Liouvillian gap, and an exceptional point that is accompanied by the divergence of the correlation length. A dissipative version of spin-charge separation induced by the quantum Zeno effect is also demonstrated. Our result shows a new class of exactly solvable Liouvlillians of open quantum many-body systems.

In quantum physics, no realistic system can avoid the coupling to an environment. The problem of decoherence and dissipation due to an environment is crucial even for small quantum systems. Furthermore, recent remarkable progress in quantum simulations with a large number of atoms, molecules, and ions has raised a fundamental and practical problem of understanding open quantum many-body systems, where interparticle correlations are essential for intended purposes [1–4]. Within the Markovian approximation, the nonunitary dynamics of an open quantum system is generated by a Liouvillian superoperator acting on the density matrix of the system [5–7]. While interesting solvable examples have been found [8–18], the diagonalization of a Liouvillian in quantum many-body settings is in general more difficult than that of a Hamiltonian. Extending the class of exactly solvable models to the realm of dissipative systems and a discovery of a prototypical solvable model that can be realized experimentally should serve as an important step towards deepening our understanding of strongly correlated open quantum systems.

The Hubbard model provides a quintessential Hamiltonian in quantum many-body physics, where the interplay between quantum-mechanical hopping and interactions plays a key role. In particular, equilibrium properties of the one-dimensional case are well understood with the help of the exact solution [19–21]. The Hubbard model has been experimentally realized with ultracold fermionic atoms in optical lattices [22], and the high controllability in such systems has recently invigorated the investigation of the effect of dissipation due to particle losses [23]. In this Letter, we show that the one-dimensional Hubbard model subject to two-body particle losses is exactly solvable. On the basis of the exact solution, we obtain an eigenspectrum of the Liouvillian, and elucidate how dissipation fundamentally alters the physics of the Hubbard model. Our main findings are threefold. First, we obtain the steady states and slowly decaying eigenmodes which govern the relaxation dynamics after a long time. Second, we show that the excitations above the Hubbard gap are significantly affected by dissipation, and find that the model shows novel critical behavior near an exceptional point [24], which stems from non-diagonalizability of the Liouvillian. Third, we demonstrate that spin-charge separation, which is a salient property of one-dimensional systems [25], is extended to dissipative systems, by exploiting the fact that the strong correlation is induced by dissipation even in the absence of an interaction. Our result shows that a number of exactly solvable Liouvillians can be constructed from quantum integrable systems subject to particle losses.

Setup.– We consider an open quantum many-body system described by a quantum master equation in the Gorini-Kossakowski-Sudarshan-Lindblad form [5–7]

$$\frac{dp}{d\tau} = -i[H, p] + \sum_{j=1}^{L} (L_j \rho L_j^\dagger - \frac{1}{2} \{L_j^\dagger L_j, \rho\}) \equiv \mathcal{L} \rho, \quad (1)$$

where $\rho(\tau)$ is the density matrix of a system at time $\tau$. The system Hamiltonian $H$ is given by the Hubbard model on an $L$-site chain

$$H = -t \sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} (c_j^\dagger \sigma c_{j+1,\sigma} + \text{H.c.}) + U \sum_{j=1}^{L} n_{j,\uparrow} n_{j,\downarrow}, \quad (2)$$

where $c^\dagger_{j,\sigma}$ is the annihilation operator of a spin-$\sigma$ fermion at site $j$, and $n_{j,\sigma} \equiv c^\dagger_{j,\sigma} c_{j,\sigma}$. The Lindblad operator $L_j = \sqrt{2} \gamma c^\dagger_{j,\downarrow} c_{j,\uparrow}$ describes a two-body loss at site $j$ with rate $\gamma > 0$, which is caused by on-site inelastic collisions between fermions [23, 26, 28]. The formal solution of the quantum master equation can be written down in terms of the eigensystem of the Liouvillian superoperator $\mathcal{L}$ defined in Eq. (1). In this Letter, we aim at diagonalizing the Liouvillian and obtain exact results for the effect of dissipation on correlated many-body systems.

Diagonalization of the Liouvillian.– The one-dimensional Hubbard model (2) is known to be solvable with the Bethe ansatz [19–21]. Here, we generalize the solvability of the Hubbard Hamiltonian to that of the Liouvillian using the existence of a
conserved quantity in the Hamiltonian \[29\]. We first decompose the Liouvillian into two parts as \( \mathcal{L} = \mathcal{K} + \mathcal{J} \), where \( \mathcal{K} \rho \equiv -i(H_{\text{eff}} \rho - \rho H_{\text{eff}}^\dagger) \) and \( \mathcal{J} \rho \equiv \sum_{j=1}^{L} J_{ij} \rho L_{j}^\dagger \). The effective non-Hermitian Hubbard Hamiltonian \( H_{\text{eff}} \) is given by \( H_{\text{eff}} \equiv H - \frac{i}{2} \sum_{j=1}^{L} J_{ij}^\dagger J_{ij} \), and its explicit form is obtained by replacing \( U \) in \( H \) with \( U - i \gamma \), making the interaction strength complex-valued \[33, 34\]. Notably, the one-dimensional Hubbard model with a complex-valued interaction strength is still integrable \[12, 13\]. Even if the interaction strength becomes a complex number, the SO(4) symmetry of the Hubbard Hamiltonian \[37, 39\] remains intact. In particular, an eigenstate of the non-Hermitian Hubbard model can be labeled by the number of particles. Let \( |N, a\rangle_R \) be a right eigenstate of \( H_{\text{eff}} \) with \( N \) particles: \( H_{\text{eff}} |N, a\rangle_R = E_{N,a} |N, a\rangle_R \) (a labels eigenstates having the same particle number). Then, one can diagonalize the superoperator \( \mathcal{K} \) as \( \mathcal{K}_Q^{(N)} = \lambda_{ab}^{(N)} \rho_{ab}^{(N)} \), where \( \lambda_{ab}^{(N)} = -i(E_{N,a} - E_{N,b}) \) and \( \rho_{ab}^{(N)} = |N, a\rangle_R \langle N, b| \[40\]. The superoperator \( \mathcal{J} \) lowers the particle number, but never increases it. Thus, in the representation with the basis \( \{|\rho_{ab}^{(N)}\rangle_{N,a,b}\} \), the Liouvillian \( \mathcal{L} \) is a triangular matrix, which can easily be diagonalized. This fact was pointed out in Ref. \[29\] for a class of Liouvillians under appropriate conditions. Indeed, because eigenvalues of a triangular matrix are given by its diagonal elements, the eigenvalues of the Liouvillian are given by \( \lambda_{ab}^{(N)} \). The corresponding right eigenvector is given by a linear combination of the basis as \( C_{ab}^{(N)} \rho_{ab}^{(N)} = \sum_{n=0}^{N} C_{n,a}^{(n)} C_{n,b}^{(n)} \), where the coefficients \( C_{n,a}^{(n)} \) are obtained from the matrix elements \( L |n-2, r\rangle_L |n, r\rangle_R \) of the Lindblad operator \( L \) with \( |n, r\rangle_L \) being the left eigenstate dual to \( |n, r\rangle_R \[29, 41\]. Thus, we conclude that given that the non-Hermitian Hubbard Hamiltonian \( H_{\text{eff}} \) is integrable, the Liouvillian \( \mathcal{L} \) is also solvable. Note that this does not mean that the Liouvillian itself has the integrable structure such as the Yang-Baxter relation. Therefore, the mechanism of the solvability here is different from previous work on Yang-Baxter integrable Liouvillians \[12, 16, 18\].

**Steady states.**—A steady state of the system is characterized by an eigenvector of \( \mathcal{L} \) with zero eigenvalue. If a state \( |\Psi\rangle \) is a right eigenstate of \( H_{\text{eff}} \) with a real eigenvalue, one can show \( L_{j} |\Psi\rangle = 0 \) and hence \( |\Psi\rangle \langle \Psi| \) is a steady state. For example, the fermion vacuum \( |0\rangle \) is trivially a steady state. Also, in the Hilbert subspace with no spin-down particles, all eigenstates of \( H_{\text{eff}} \) coincide with those in the non-interacting \( (U = \gamma = 0) \) case and thus describe steady states. By acting the spin-lowering operator on the spin-polarized eigenstates, one can construct many steady states owing to the spin SU(2) symmetry of \( H_{\text{eff}} \). Clearly, these steady states are ferromagnetic and far from conventional equilibrium states of the one-dimensional Hubbard model. Physically, the steadiness of the ferromagnetic states can be understood from the Fermi statistics, because the spin wavefunction that is fully symmetric with respect to a particle exchange requires antisymmetry in the real-space wavefunction and forbids doubly occupied sites that cause a decay, as observed in Refs. \[35, 42\]. In general, a steady state realized after a time evolution becomes a mixed state of the above steady states, depending on the initial condition.

**Bethe ansatz.**—We use the Bethe ansatz to obtain the eigenspectrum of the non-Hermitian Hubbard model \( H_{\text{eff}} \). The Bethe equations are \[19, 21\]

\[
k_j L = \Phi + 2\pi I_j - \sum_{\beta=1}^{2M} \Theta \left( \frac{\sin k_j - \lambda_\beta}{u} \right),
\]

\[
- \sum_{j=1}^{N} \Theta \left( \frac{\sin k_j - \lambda_\alpha}{u} \right) = 2\pi J_\alpha + \sum_{\beta=1}^{M} \Theta \left( \frac{\lambda_\alpha - \lambda_\beta}{2u} \right),
\]

where \( N \) is the number of particles, \( M \) is the number of down spins, \( k_j \) \((j = 1, \cdots, N)\) is a quasi-momentum, \( \lambda_\alpha \) \((\alpha = 1, \cdots, M)\) is a spin rapidity, \( u = (U - i\gamma)/4\Pi\) is a dimensionless complex interaction coefficient, and \( \Theta(z) = \text{arctan} \, z \). The quantum number \( I_j \) takes an integer (half-integer) value for even (odd) \( M \), and \( J_\alpha \) takes an integer (half-integer) value for odd (even) \( N - M \). Here we employ a twisted boundary condition \( c_{L+1, \sigma} = e^{-i\Phi_{1, \sigma}} c_{1, \sigma} \) for later convenience, but basically set \( \Phi = 0 \) (i.e. the periodic boundary condition) unless otherwise specified.

**Liouvillian gap.**—The late-time dynamics of the system near a steady state is governed by slowly decaying eigenmodes that have a small negative real part of eigenvalues of \( \mathcal{L} \[43\]. By construction of the steady states, the slowly decaying eigenmodes correspond to Bethe eigenstates in the \( M = 1 \) case and their descendants derived from the spin SU(2) symmetry. They consist of ferromagnetic spin-wave-type excitations, and their dispersion relation is obtained by a standard calculation with the Bethe ansatz \[41\]. Assuming a specific configuration of the quantum number \( I_j = -(N+1)/2 + j \), we obtain an analytic form of the excitation energy

\[
\Delta E \simeq -\frac{t}{\pi u} \left( Q_0 - \frac{1}{2} \sin 2Q_0 \right) \left( 1 - \cos \frac{\pi \Delta P}{Q_0} \right)
\]

for the momentum \( \Delta P \approx 0 \), where \( Q_0 = \pi N/L \) is the Fermi momentum. Since the momentum is discretized in units of \( 2\pi/L \), the gapless quadratic dispersion around \( \Delta P = 0 \) leads to the smallest imaginary part of the excitation energy \(|\text{Im}(\Delta E)| \) proportional to \( 1/L^2 \). Thus, the Liouvillian gap, which is defined by the smallest nonzero real part of eigenvalues of the Liouvillian, vanishes in the \( L \to \infty \) limit, implying a power-law time dependence in the decay dynamics \[43\].

**Hubbard gap, correlation length, and exceptional point.**—Next, we consider the half-filling case \( L = N = \)}
2M and focus on the solution that can be adiabatically connected to the ground state if one takes the \( \gamma \to 0 \) limit. Such a solution may not contribute to the late-time behavior due to a short lifetime, but it can be used to study the early-time decay dynamics of a Mott insulator. We here assume that \( U > 0 \) and \( N (M) \) is even (odd), and set \( J_j = -((N + 1)/2 + j) \) and \( J_m = -((M + 1)/2 + \alpha) \). In the \( L \to \infty \) limit, the Bethe equations (3) and (4) are rewritten in the form of integral equations for distribution functions \( \rho(k) \) and \( \sigma(\lambda) \) as

\[
\rho(k) = \frac{1}{2\pi} + \cos k \int_S d\lambda a_1(\sin k - \lambda)\sigma(\lambda), \tag{6}
\]

\[
\sigma(\lambda) = \int_{\mathcal{C}} dk a_1(\sin k - \lambda)\rho(k) - \int_S d\lambda' a_2(\lambda - \lambda')\sigma(\lambda'), \tag{7}
\]

where \( a_n(z) = (1/\pi)[nu/(z^2 + n^2 u^2)] \), and \( \mathcal{C} \) and \( S \) denote the trajectories of quasimomenta and spin rapidities, respectively [21]. Figures 4 (a) and (b) show typical distributions of \( \{k_j\}_{j=1,\ldots,N} \) and \( \{\lambda_n\}_{n=1,\ldots,M} \) which are obtained from the solution of the Bethe equations (3) and (4). They indicate that if the trajectories \( \mathcal{C} \) and \( S \) do not enclose a pole in the integrands of Eqs. (6) and (7), the trajectories can continuously be deformed onto those of the \( \gamma = 0 \) case, i.e. \( \mathcal{C} = [-\pi, \pi] \) and \( S = (-\infty,\infty) \). Thus we obtain the eigenvalue \( E_0 \) in the \( L \to \infty \) limit from analytic continuation of the solution in the \( \gamma = 0 \) case [19] as

\[
E_0/L = -2t \int_{-\infty}^{\infty} d\omega J_0(\omega)J_1(\omega)/(1 + e^{2u|\omega|}), \tag{8}
\]

where \( J_n(x) \) is the \( n \)th Bessel function. Similarly, the Hubbard gap \( \Delta_c \) [19 [24] is given as

\[
\Delta_c = 4tu - 4t \left[ 1 - \int_{-\infty}^{\infty} d\omega J_1(\omega)/(1 + e^{2u|\omega|}) \right]. \tag{9}
\]

Here \( E_0 \) and \( \Delta_c \) take complex values in general, the lifetime of an eigenmode can be extracted from the imaginary part of the eigenvalue. The absolute value of \( \text{Im}[E_0] \leq 0 \) first increases with increasing \( \gamma \), takes the maximum at some point, and then decreases [31]. The decreasing behavior at large \( \gamma \) is attributed to the continuous quantum Zeno effect [26, 27, 45-48], which prevents the creation of doubly occupied sites in eigenstates due to a large cost of the imaginary part of energy. By contrast, the absolute value of \( \text{Im}[\Delta_c] \leq 0 \) monotonically increases with increasing \( \gamma \) [31], since the excitation corresponding to the Hubbard gap creates doubly occupied sites.

To further elucidate the physics of the dissipative Mott insulator, we calculate the correlation length \( \xi \) of the above eigenstate from an asymptotic behavior of the charge stiffness as \( \left. \frac{d^2 E_0(\phi)}{d\phi^2} \right|_{\phi=0} \sim \exp[-L/\xi] \) \( (L \to \infty) \) [49]. We find that the correlation length is obtained from

\[
\frac{1}{\xi} = \text{Re} \left[ \frac{1}{u} \int_{-\infty}^{\infty} dy \frac{\ln(y + \sqrt{y^2 - 1})}{\cosh(\pi y/2u)} \right]. \tag{10}
\]

Figures 2 (a)-(c) show the correlation length for different values of the repulsive interaction. For large \( \gamma \), the correlation length decreases in all cases, indicating that particles are more localized due to dissipation. This behavior is consistent with the quantum Zeno effect [26, 27, 45-48]. On the other hand, when \( U \) is small, the correlation length grows at an intermediate dissipation strength [see Fig. 2 (b)], implying that dissipation facilitates delocalization of particles. More surprisingly, the correlation length even diverges for small \( U \), and takes negative values in between the divergence points [see Fig. 2 (c)], which signal the breakdown of analytic continuation since the negative correlation length is unphysical. In fact, when the correlation length diverges, the trajectory \( \mathcal{C} \) crosses poles in the integrand of Eq. (7), thereby preventing the trajectory from deforming onto the real axis. This fact can be seen numerically (see red crosses on (off) the trajectory \( \mathcal{C} (S) \) in Fig. 1 (c) [(d)]), and can also be shown analytically from the Bethe equations (see the Supplemental Material [41]). Similar transitions in Bethe-ansatz solutions have been found in other non-Hermitian integrable models [33, 50, 51].

The poles in the integrand in the first term on the right-hand side of Eq. (7) are given by \( \sin k = \lambda \pm iu \). The same condition appears in the construction of the \( k \)-\( \lambda \) string excitations in the Hubbard model [24, 52], in which a pair of quasimomenta \( k^{(1)}, k^{(2)} \) form a string configuration around a center \( \lambda \) as \( \sin k^{(1)} = \lambda + iu \) and \( \sin k^{(2)} = \lambda - iu \). Physically, such string excitations

\[
\text{FIG. 1. Numerical solutions of Bethe equations (3) and (4) for } L = N = 2M = 250. (a) (c) Blue dots show quasimomenta \( \{k_j\} \), and red crosses show the locations of poles at \( k = \pm \pi - \arcsin(\pm iu) \). (b) (d) Green dots show spin rapidities \( \{\lambda_n\} \), and red crosses show the locations of poles at \( \lambda = \pm 2iu \). The interaction strength is set to (a) (b) \( u = 1 - 0.5i \) and (c) (d) \( u = 0.6 - 0.469i \). Points on the real axis show the solutions for the case of \( \gamma = 0 \) with the same \( U \) for comparison.
describe the creation of a doublon-holon pair from the ground state $|21\rangle$. The existence of the poles on trajectory $C$ indicates that the solution in the $L \to \infty$ limit becomes degenerate with a $k$-$\lambda$ string solution. In fact, the excitation energy of a $k$-$\lambda$ string is given by $|21\rangle$.

$$\varepsilon(k) = 2tu + 2t \cos k + 2t \int_0^\infty d\omega J_1(\omega) \cos(\omega \sin k) e^{-u\omega} \omega \cosh u\omega,$$

which vanishes at the poles $k = \pm \pi - \arcsin(\pm u)$. Here not only the eigenvalues but also the eigenstates are the same. This means that the critical point where the correlation length diverges is an exceptional point $|24\rangle$, at which the non-Hermitian Hamiltonian $H_{\text{eff}}$ cannot be diagonalized. Importantly, we can show that non-diagonalizability of $H_{\text{eff}}$ leads to non-diagonalizability of the Liouvillian $L$ $|41\rangle$. Thus, the exceptional point is the same for both the non-Hermitian Hamiltonian and the Liouvillian; however, this does not hold for general Liouvillians $|53\rangle$.

The solid curve in Fig. 3 shows the position of the exceptional point as a function of $U$ and $\gamma$. Outside the shaded region, the analytic continuation of the Bethe-ansatz solution from the $\gamma = 0$ case remains valid. For a large repulsive interaction $U > 0$, a Mott insulator is formed as in the Hermitian Hubbard model and it has a finite lifetime due to nonzero $\gamma$. On the other hand, for small $U > 0$ and large $\gamma$, particles are localized due to dissipation. Because the Hubbard gap becomes negative $\text{Re}[\Delta_c] < 0$ in this region, the localization should be attributed to the quantum Zeno effect rather than the repulsive interaction, and therefore this localized state may be called a Zeno insulator. Interestingly, the phase diagram looks qualitatively similar to that obtained from a mean-field theory for a three-dimensional non-Hermitian attractive Hubbard model $|54\rangle$ after changing the sign of $U$ via the Shiba transformation $|51\rangle$.

**Dissipation-induced spin-charge separation.**—Finally, we address an intriguing connection between strong correlations and dissipation. The Bethe equations $|3\rangle$ and $|4\rangle$ are greatly simplified when one takes the large $|u|$ limit, in which one can expand the equations as (here we

![Graph](image)

FIG. 2. Correlation length $\xi$ [Eq. (10)] for (a) $U/4t = 1$, (b) $U/4t = 0.7$, and (c) $U/4t = 0.6$.

![Graph](image)

FIG. 3. “Phase diagram” of the one-dimensional non-Hermitian Hubbard model for $U > 0$ at half filling. The solid curve indicates the location of the exceptional point at which the non-Hermitian Hubbard model cannot be diagonalized. In the shaded region, the analytic continuation from the case of $\gamma = 0$ breaks down. The dashed curve shows where the real part of the Hubbard gap $\text{Re}[\Delta_c]$ vanishes.

set $\Phi = 0$)

$$k_j L = 2\pi I_j + O(1/u),$$

$$N\Theta\left(\frac{\lambda_\alpha}{u}\right) + O(1/u^2) = 2\pi J_\alpha + \sum_{\beta=1}^M \Theta\left(\frac{\lambda_\alpha - \lambda_\beta}{2u}\right).$$

These equations indicate that quasimomenta and spin rapidities are completely decoupled in the $|u| \to \infty$ limit $|44\rangle$ $|55\rangle$. The quasimomenta in this limit are identical to those of free fermions, and Eq. (13) gives the same Bethe equation as that of the Heisenberg chain after rescaling $\lambda_\alpha \equiv \lambda_\alpha/u$. This leads to a remarkable fact that the Bethe wavefunction is factorized into the charge part and the spin part $|55\rangle$. This argument is parallel to that for the spin-charge separation in the Hermitian Hubbard model, which is a salient property of one-dimensional systems $|25\rangle$. However, the unique feature here is that the spin-charge separation can occur due to large $\gamma$, even in the absence of the repulsive interaction $U$. Thus, in a Zeno insulator, the strong dissipation itself induces a strongly correlated state, and holes created by a loss behave as almost free fermions, whereas the spin excitations are described by a non-Hermitian Heisenberg chain with the exchange coupling $4\pi^2/(U - i\gamma)$ $|55\rangle$. As spin-charge separation in a Hermitian Hubbard chain has recently been observed in experiments with ultracold atoms $|56\rangle$ $|57\rangle$, the dissipation-induced spin-charge separation should be observed with current experimental techniques.

**Conclusion.**—We have shown that the one-dimensional dissipative Hubbard model is exactly solvable. The exact solution has enabled us to elucidate how strongly correlated states of the Hubbard model are fundamentally altered by dissipation, as experimentally realized with ul-
endeavor to uncover as yet unexplored exactly solvable models in open quantum many-body systems.

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[40] Here, $|N,a\rangle_R |N',b\rangle$ with $N \neq N'$ is also an eigenvector of $K$, but we neglect it because the off-diagonal part of the density matrix for different particle numbers cannot be generated in a time evolution if the initial state has a definite particle number.

[41] See Supplemental Material for the explicit form of the eigensystem of the Liouvillian, derivation of the dispersion relation of spin-wave excitations, dependences of the eigenvalues $E_0$ and $\Delta_+$ on dissipation, and the divergence of the correlation length at the exceptional point.

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Supplemental Material for “Exact Liouvillian Spectrum of a One-Dimensional Dissipative Hubbard Model”

Eigensystem of the Liouvillian

As mentioned in the main text, the eigenvalues of the Liouvillian $L$ are given by $\lambda_{ab}^{(N)} = -i(E_{N,a} - E_{N,b}^*)$, where $E_{N,a}$ and $E_{N,b}$ are eigenvalues of the non-Hermitian Hubbard model $H_{\text{eff}}$. The corresponding eigenvector $\sigma_{ab}^{(N)}$ can be expanded in terms of the basis set $\{\varrho_{cd} = |n,c\rangle_R \langle n,d|\}_{n,c,d}$ as

$$\sigma_{ab}^{(N)} = C_{ab}^{(N)} \varrho_{ab}^{(N)} + \sum_{n=0}^{N-2} \sum_{a',b'} C_{a'b'}^{(n)} \varrho_{a'b'}^{(n)}.$$  \hspace{1cm} (S1)

Here, we follow Ref. [29] to determine the coefficients $C_{a'b'}^{(n)}$. We first expand a right eigenstate $|n,a\rangle_R$ acted on by the Lindblad operator $L_j$ as

$$L_j |n,a\rangle_R = \sum_r v_{j,r}^{(n,a)} |n-2,r\rangle_R,$$

where $v_{j,r}^{(n,a)} = L \langle n-2,r | L_j |n,a\rangle_R$ under the biorthonormal condition $L \langle n, a|n', b\rangle_R = \delta_{n,n'} \delta_{a,b}$. Then, we have

$$\mathcal{J} \varrho_{ab}^{(n)} = \sum_j L_j \varrho_{ab}^{(n)} L_j^\dagger = \sum_{j} \sum_{r,r'} v_{j,r}^{(n,a)} (v_{j,r'}^{(n,b)})^* |n-2,r\rangle_R \langle n-2,r'|.$$ \hspace{1cm} (S2)

Substituting Eq. [S1] into the eigenvalue equation $\mathcal{L} \sigma_{ab}^{(N)} = \lambda_{ab}^{(N)} \sigma_{ab}^{(N)}$, we obtain

$$\mathcal{L} \sigma_{ab}^{(N)} = \lambda_{ab}^{(N)} \sigma_{ab}^{(N)} = C_{ab}^{(N)} \varrho_{ab}^{(N)} + \sum_{n=0}^{N-2} \sum_{r,r'} C_{a'b'}^{(n)} (v_{j,r}^{(n,a)} (v_{j,r'}^{(n,b)})^* \varrho_{r,r'}^{(n-2)},$$ \hspace{1cm} (S3)

Comparing the right-hand side of Eqs. [S4] and [S5], we find that the first terms are equal. Comparing the coefficient of $\varrho_{a'b'}^{(N-2)}$, we have

$$\sum_j C_{ab}^{(N)} v_{j,a}^{(N,a)} (v_{j,b'}^{(N,b')}^*) + \lambda_{a'b'}^{(N-2)} C_{a'b'}^{(N-2)} = \lambda_{ab}^{(N)} C_{a'b'}^{(N-2)}.$$ \hspace{1cm} (S6)

Thus, the coefficient $C_{a'b'}^{(N-2)}$ is given by

$$C_{a'b'}^{(N-2)} = \frac{1}{\lambda_{ab}^{(N)} - \lambda_{a'b'}^{(N-2)}} \left( \sum_j v_{j,a}^{(N,a)} (v_{j,b'}^{(N,b')}^*) \right) C_{ab}^{(N)}.$$ \hspace{1cm} (S7)

Similarly, by comparing the coefficient of $\varrho_{a'b'}^{(N-4)}$, we have

$$\lambda_{a'b'}^{(N-4)} C_{a'b'}^{(N-4)} + \sum_j \sum_{a'',b''} C_{a''b''}^{(N-2)} v_{j,a''}^{(N-2,a'')} (v_{j,b''}^{(N-2,b'')}^*) = \lambda_{ab}^{(N)} C_{a'b'}^{(N-4)}.$$ \hspace{1cm} (S8)
which leads to

\[ C_{a'b'}^{(N-4)} = \frac{1}{\lambda_{ab}^{(N)} - \lambda_{ab'}^{(N-4)}} \left( \sum_{j} \sum_{a''b''} C_{a''b''}^{(N-2,a')} v_{j,a'}^{(N-2,a''')} (v_{j,b''}^{(N-2,b')})^* \right). \] (S9)

Here, we have assumed \( \lambda_{ab}^{(N)} - \lambda_{a'b'}^{(N)} \neq 0 \) \((n = 0, 2, \cdots, N-2)\), which serve as necessary conditions for diagonalizability of the Liouvillian. Repeating the above procedures, one can recursively obtain all the coefficients \( C_{a'b'}^{(N)} \) \((n = 0, \cdots, N-2)\) from \( C_{ab}^{(N)} \). The overall coefficient \( C_{ab}^{(N)} \) is determined from the normalization condition for \( \sigma_{ab}^{(N)} \).

It follows from the above construction of the eigenvectors \( \sigma_{ab}^{(N)} \) that if the non-Hermitian Hamiltonian \( H_{ab} \) is at an exceptional point (i.e. it cannot be diagonalized), so is the Liouvillian \( \mathcal{L} \). To see this, let us assume that the non-Hermitian Hamiltonian is parameterized as \( H_{ab}(g) \) and that it is at an exceptional point for \( g = g_{\text{EP}} \). The eigenvalue equation is given by \( H_{ab}(g)|N,a,g \rangle_R = E_{N,a}(g)|N,a,g \rangle_R \). At the exceptional point, at least two eigenstates and the corresponding eigenvalues are degenerate:

\[ \lim_{g \to g_{\text{EP}}} (E_{N,a_1}(g) - E_{N,a_2}(g)) = 0, \] (S10)
\[ \lim_{g \to g_{\text{EP}}} (|N,a_1,g \rangle_R - |N,a_2,g \rangle_R) = 0. \] (S11)

Thus, we have

\[ \lim_{g \to g_{\text{EP}}} (\lambda_{a_1b}(g) - \lambda_{a_2b}(g)) = \lim_{g \to g_{\text{EP}}} (\lambda_{ba_1}(g) - \lambda_{ba_2}(g)) = 0, \] (S12)
\[ \lim_{g \to g_{\text{EP}}} (\theta_{a_1b}(g) - \theta_{a_2b}(g)) = \lim_{g \to g_{\text{EP}}} (\theta_{ba_1}(g) - \theta_{ba_2}(g)) = 0, \] (S13)
\[ \lim_{g \to g_{\text{EP}}} (v_{j,a_1}^{(N,a_1)}(g) - v_{j,a_2}^{(N,a_2)}(g)) = 0, \] (S14)

where \( \lambda_{ab}^{(N)}(g) = -i(E_{N,a}(g) - E_{N,b}(g)), \theta_{ab}^{(N)}(g) = |N,a,g \rangle_R \langle N,b,g |, \) and \( v_{j,a}^{(N,a)}(g) = L \langle N - 2, r \mid L_j | N, a, g \rangle_R \). Then, the above construction of the eigensystem of the Liouvillian shows that the Liouvillian eigenvectors corresponding to eigenvalues \( \lambda_{a_1b}^{(N)}(g) \) and \( \lambda_{a_2b}^{(N)}(g) \) are degenerate for \( g = g_{\text{EP}} \), indicating that the Liouvillian is at an exceptional point. Note that, for \( g = g_{\text{EP}} \), the Liouvillian eigenvectors corresponding to eigenvalues \( \lambda_{ba_1}^{(N)}(g) \) and \( \lambda_{ba_2}^{(N)}(g) \) are also degenerate. Thus, at the exceptional point, the degeneracy of eigenstates of the non-Hermitian Hamiltonian leads to a large number of degenerate eigenvectors of the Liouvillian.

### Derivation of dispersion relation of spin-wave excitations

Here, we derive the dispersion relation [Eq. (5) in the main text] of the spin-wave-type excitations which provide the slowly decaying eigenmodes in relaxation towards steady states. To this end, we consider the Bethe equations (3) and (4) in the main text with \( M = 1 \). From Eq. (3), we define the counting function \( z_c(k) \) as

\[ z_c(k) = \frac{2 \pi I_j}{L} = k + \frac{1}{L} \Theta \left( \frac{\sin k - \lambda_1}{u} \right). \] (S15)

The distribution function \( \rho(k) \) is then given by

\[ \rho(k) = \frac{1}{2 \pi} \frac{dz_c(k)}{dk} = \frac{1}{2 \pi} + \frac{\cos k}{L} a_1(\sin k - \lambda_1) \]
\[ = \rho_0 + \frac{1}{L} \bar{\rho}(k), \] (S16)

where \( \rho_0 \equiv 1/2\pi \) and \( \bar{\rho}(k) \equiv \cos k \cdot a_1(\sin k - \lambda_1) \). \( \bar{\rho}(k) \) gives a \( 1/L \) correction to the distribution function due to the excitation. For simplicity, here we assume that the quantum numbers take consecutive values as \( I_j = -(N+1)/2 + j \). Then, in the large-\( L \) limit, the quasimomenta are densely distributed on an interval \([-Q, Q] \), and \( Q \) is determined...
FIG. S1. Dispersion relation of the spin-wave-type excitations for $N/L = 1/3$ and $u = 0.8 - 0.5i$. (a) Real part of the excitation energy as a function of the momentum of the excitation. (b) Imaginary part of the excitation energy as a function of the momentum of the excitation. The dots are excitation energies calculated from numerical solutions of the Bethe equations and for $L = 240, N = 80$. The black curves are dispersion relations obtained from Eqs. (S19) and (S20). The green curves show approximate results [Eq. (S21)] that become accurate for $\Delta P \approx 0$.

from the particle density as

$$\frac{N}{L} = \int_{-Q}^{Q} dk \rho(k)$$

$$= \int_{-Q_0}^{Q_0} dk \rho_0 + \frac{1}{L} \int_{-Q_0}^{Q_0} dk \tilde{\rho}(k) + 2(Q - Q_0)\rho_0 + \mathcal{O}(1/L^2),$$

(S17)

where $Q_0 = \pi N/L$. Thus, we have

$$Q - Q_0 = -\frac{1}{L} \int_{-Q_0}^{Q_0} \frac{u \cos k}{(\sin k - \lambda_1)^2 + u^2} + \mathcal{O}(1/L^2).$$

(S18)

The energy of the excitation is given by

$$\Delta E = -2tL \int_{-Q}^{Q} dk \rho(k) \cos k + 2tL \int_{-Q_0}^{Q_0} dk \rho_0 \cos k$$

$$= -2t \int_{-Q_0}^{Q_0} dk \tilde{\rho}(k) \cos k - 2t \cdot 2(Q - Q_0)L \cdot \rho_0 \cos Q_0$$

$$= -\frac{2t}{\pi} \int_{-Q_0}^{Q_0} dk \frac{u \cos k(\cos k - \cos Q_0)}{(\sin k - \lambda_1)^2 + u^2},$$

(S19)

and the momentum of the excitation is

$$\Delta P = \frac{2\pi}{L} \left( \sum_{j=0}^{N} I_j + J_1 \right) - \frac{2\pi}{L} \sum_{j=1}^{N} \left( -\frac{N}{2} + j \right)$$

$$= -Q_0 - \frac{1}{\pi} \int_{-Q_0}^{Q_0} dk \arctan \left( \frac{\sin k - \lambda_1}{u} \right).$$

(S20)

By eliminating $\lambda_1$ from Eqs. (S19) and (S20), we obtain the dispersion relation. For $\Delta P \approx 0$, the spin rapidity satisfies $|\lambda_1| \gg |\sin k|$, and hence $\Delta P \approx -Q_0 + \frac{2\pi}{\lambda_1} \arctan \frac{\lambda_1}{u}$. Thus, for $\Delta P \approx 0$, we have

$$\Delta E \approx -\frac{2t}{\pi} \int_{-Q_0}^{Q_0} dk (\cos^2 k - \cos Q_0 \cos k) \cdot \frac{u}{\lambda_1^2 + u^2}$$

$$\approx -\frac{t}{\pi u} \left( Q_0 - \frac{1}{2} \sin 2Q_0 \right) \left( 1 - \cos \frac{\pi \Delta P}{Q_0} \right),$$

(S21)

which is Eq. (5) in the main text. In Fig. S1, we plot the dispersion relation of the excitations for $-Q_0 \leq \Delta P \leq Q_0$. 
Dependence of the Hubbard gap on dissipation

In Figs. S2 (a) (b) and S3 (a) (b), we show the dependences of the energy eigenvalue $E_0$ [Eq. (8) in the main text] and the Hubbard gap $\Delta_c$ [Eq. (9) in the main text] on dissipation for $U/4t = 0.8$ and $U/4t = 2$, respectively. The real part of the energy eigenvalue $\text{Re}[E_0]$ monotonically increases with increasing the dissipation strength. The absolute value of the imaginary part of the energy eigenvalue $|\text{Im}[E_0]|$ first increases with increasing $\gamma$, indicating that the dissipation causes a decay of the eigenmode. However, $|\text{Im}[E_0]|$ takes the maximum at an intermediate dissipation strength, and decreases for large $\gamma$. The decreasing behavior of $|\text{Im}[E_0]|$ signals the onset of the quantum Zeno effect [26, 27, 45–48]. While the qualitative behavior of the energy eigenvalue $E_0$ does not significantly depend on the magnitude of the repulsive interaction $U$, the Hubbard gap $\Delta_c$ shows a nontrivial dependence on $U$. For a weak repulsive interaction [see Fig. S2 (c)], the real part of the Hubbard gap $\text{Re}[\Delta_c]$ monotonically decreases with increasing the dissipation.
strength, and becomes negative when \( \gamma \) exceeds a certain value. On the other hand, for a strong repulsive interaction [see Fig. S3 (c)], the real part of the Hubbard gap remains positive for any \( \gamma \). The qualitative difference of \( \text{Re}[\Delta_c] \) for small and large \( U \) is attributed to the competition between the repulsive interaction and the quantum Zeno effect. For small \( U \), particles are not well localized in the Mott insulator formed at \( \gamma = 0 \), and therefore the Hubbard gap is significantly affected by the localization due to the quantum Zeno effect. On the other hand, for large \( U \), the Mott insulating state at \( \gamma = 0 \) is not largely changed by dissipation, since particles are already well localized in the Mott insulator. Therefore, the real part of the Hubbard gap \( \text{Re}[\Delta_c] \) remains almost unchanged by increasing the dissipation strength. By contrast, the absolute value of the imaginary part of the Hubbard gap \( \text{Im}[\Delta_c] \) monotonically increases with increasing the dissipation strength [see Fig. S2 (d) and Fig. S3 (d)], because the excitation corresponding to the Hubbard gap creates doubly occupied sites and leads to \( \text{Im}[\Delta_c] \propto -\gamma \) for large \( |u| \).

**Divergence of the correlation length at the exceptional point**

We follow Ref. [49] to calculate the correlation length \( \xi \) as

\[
\frac{1}{\xi} = \text{Im}[z_c(k_*)],
\]  

(S22)

where

\[
z_c(k) = k + 2 \int_0^\infty d\omega \frac{J_0(\omega) \sin(\omega \sin k)}{\omega (1 + e^{2\omega})}
\]  

(S23)

is the counting function derived from the Bethe equations. Here \( k_* = \pi - \arcsin(iu) \) denotes the stationary point of the counting function as

\[
\frac{dz_c}{dk}(k_*) = 1 + 2 \cos k_* \int_0^\infty d\omega \frac{J_0(\omega) \cos(\omega \sin k_*)}{1 + e^{2\omega}}
\]

\[
= 1 - \sqrt{1 + u^2} \int_0^\infty d\omega J_0(\omega) e^{-u\omega}
\]

\[
= 0.
\]  

(S24)

Note that \( k_* \) also gives a pole of the integrand in the Bethe equation [7] in the main text. Thus, if the pole \( k_* \) is located on the trajectory \( \mathcal{C} \) of quasimomenta, there exists a quasimomentum \( k_j = k_* \) for which

\[
z_c(k_*) = \frac{2\pi I_j}{L}
\]  

(S25)

is satisfied in the large-\( L \) limit. Since the quantum number \( I_j \) is real, we have the divergence of the correlation length \( \xi = \infty \) from Eq. (S22).