DETECTING EINSTEIN GEODESICS: EINSTEIN METRICS IN PROJECTIVE AND CONFORMAL GEOMETRY

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Dedicated to Mike Eastwood on the occasion of his 60th birthday

Abstract. Here we treat the problem: given a torsion-free connection do its geodesics, as unparametrised curves, coincide with the geodesics of an Einstein metric? We find projective invariants such that the vanishing of these is necessary for the existence of such a metric, and in generic settings the vanishing of these is also sufficient. We also obtain results for the problem of metrisability (without the Einstein condition): We show that the odd Chern type invariants of an affine connection are projective invariants that obstruct the existence of a projectively related Levi-Civita connection. In addition we discuss a concrete link between projective and conformal geometry and the application of this to the projective-Einstein problem.

1. Introduction

Suppose that $\nabla$ is an affine connection on a manifold of dimension at least 2, and consider its geodesics as unparametrised curves. The collection of all such curves is called a projective structure. Two connections differing only by torsion share the same geodesics, so for our considerations there will be no loss of generality in assuming that $\nabla$ is torsion free. The problem of whether these paths agree with the (unparametrised) geodesics of a pseudo-Riemannian metric is a classical problem that is also attracting significant recent interest [5, 15, 24, 26, 27, 29]. In general dimensions this is a difficult problem. However a striking simplification occurs if we ask a variant of the original question:

Given a torsion-free connection $\nabla$, do its geodesics, as unparametrised curves, coincide with the geodesics of an Einstein metric?

The surprising point is that it is easier to treat this new problem directly without first investigating metrisability, and this problem is the main subject of our article. We develop a general theory that leads to ways to construct invariants with the property that they depend only on a projective structure and such that their vanishing is necessary for the projective structure to be compatible with an

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Einstein Levi-Civita connection (in the sense of the stated problem). In fact, in the generic case (where the notion of generic will later be defined precisely; see Remark 5.9) the vanishing of certain of these invariants is also *sufficient* for the projective structure to be compatible with an Einstein metric; see Section 5.

Although the Einstein metricity problem is our main aim, we also obtain new results for the metricity problem, that is where the Einstein condition is omitted. Both problems have a serious physical motivation in dimension 4. The observations of a part of spacetime may recover the path data of many bodies. If we assume that these bodies follow the paths of some affine connection we may use the invariants to test the hypothesis that the connection involved is the Levi-Civita connection of some Einstein metric. See [19, 20, 25] for further discussion and also related results. On the side of mathematics the ideas surrounding these problems have the potential to contribute deeply to understanding the links between pseudo-Riemannian geometry and projective geometry, a theme with strong classical tradition see e.g. [21, 23, 21] and references therein. The study of Einstein metrics is especially important and as we shall show they have a very special role in projective geometry. In this article we also describe in Section 8 a useful concrete link between conformal geometry and projective geometry; in particular this link led to several of the ideas used in the article. Finally the ideas developed here can provide a template for similar problems in the setting of, for example, $h$-projective geometry.

Our approach to the problems partly uses the projective tractor calculus following its treatment in [3, 8, 18], see Section 2.2. The problem of projective metricity is controlled by a projectively invariant overdetermined linear partial differential equation [26, 29]. The corresponding prolonged differential system is studied by Eastwood and Matveev in [15]. They show that the existence of a Levi-Civita connection in the projective class of a projective manifold is equivalent to the existence of a suitably nondegenerate section of a symmetric product of the basic tractor bundle that has the property that it is parallel for a certain projectively invariant connection that they also construct. The connection involved is not the usual (i.e. normal) tractor connection. However in [10] the authors show that the section involved is parallel for the normal tractor connection if and only if the solution corresponds to an Einstein metric (and there are strong connections with the works [7, 9, 8, 23] as explained there). This last fact is closely related to results of Armstrong [1], and it is this link between Einstein metricity and parallel tractors that leads to a considerable simplification, as the normal tractor connection is well understood. In Theorem 3.1 we give a new direct proof of the required fact, which extends a result of Armstrong.

Here are the main results in order of appearance. In Theorem 4.1 we show that the Chern type curvature forms of a manifold equipped with scale connection are projectively invariant. (A scale connection is simply a torsion-free affine connection with vanishing first Chern form, and there is always such a connection in a projective class; see Proposition 2.2 in Section 2.1) This parallels the
result of Chern-Simons that the same forms for a Levi-Civita connection are conformally invariant \[14\]. We show in Proposition 4.3 that they coincide with the curvature forms of the projectively invariant tractor connection; this is one way to understand the projective invariance. In Corollary 4.2 we conclude that the \(k\)-odd curvature forms obstruct the existence of a Levi-Civita connection in the projective class. To the best of our knowledge this is a new result for the question of projective metrisability.

In Section 5 we show that if \(\nabla\) is the Levi-Civita connection of an Einstein metric then its projective Cotton tensor (as defined in Section 2) vanishes. A first observation is that this solves the problem completely in dimension 2: see Remark 5.2. For other dimensions this suggests using the condition of projective-to-Cotton-flat; see Proposition 5.1. The key idea in Section 5 is that if an affine connection is projectively related to a Cotton-flat affine connection then one can obtain a natural formula for the change of connection required to achieve this (this combines expression (19) with the constructions of \(D\) in Section 5.3); at least this is the case if the projective Weyl curvature satisfies very mild conditions of genericity. That formula then leads to the construction, in Theorem 5.8, of sets of sharp obstructions to the (non-zero scalar curvature) Einstein metricity problem — a set of sharp obstructions means a collection of projective invariants the vanishing of which is necessary and sufficient for the existence of an Einstein Levi-Civita connection in the projective class. Along the way in our treatment we observe in Remark 5.4 that we also obtain projective invariants that sharply obstruct the existence of a Cotton flat connection in the projective class. These invariants are made natural by again using Theorem 5.8 to substitute for \(D\).

In Section 6 we describe a very general principle that may be used to proliferate further projective invariants which obstruct the existence of Einstein Levi-Civita connections in the projective class. Finally in Section 7 we show that when a projective class includes the Levi-Civita connection of an Einstein metric, the corresponding projective and conformal tractor connections are very simply related. This observation motivated many of the earlier constructions and is used to give alternative proofs of some of the results. Included in this section is an observation that on a projective 3-manifold the projective Weyl tensor itself provides a sharp obstruction to the Einstein metricity problem; see Corollary 8.3.

2. PROJECTIVE DIFFERENTIAL GEOMETRY

Projective structures were defined in the introduction. An equivalent definition is as follows. A projective structure \((M^n, p)\), \(n \geq 2\), is a smooth manifold equipped with an equivalence class \(p\) of torsion-free affine connections. The class is characterised by the fact that two connections \(\nabla\) and \(\hat{\nabla}\) in \(p\) have the same path structure, that is the same geodesics up to parametrisation. Explicitly the transformation relating these connections on \(TM\) and \(T^*M\) are given by

\[
\hat{\nabla}_a Y^b = \nabla_a Y^b + \Upsilon_a Y^b + \Upsilon_c Y^c \delta_a^b, \quad \text{and} \quad \hat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a,
\]
where $\Upsilon$ is some smooth section of $T^*M$. In the setting of a projective structure $(M, p)$ any connection $\nabla \in p$ is called a Weyl connection or Weyl structure on $M$.

2.0.1. Curvature for Weyl structures. Given a connection $\nabla \in p$ the curvature (on $TM$) is defined as usual by

$$ (\nabla_a \nabla_b - \nabla_b \nabla_a)v^c = R_{ab}^{\ c}d v^d. $$

Considering the tensor decomposition of this we see that it can be written uniquely as

$$ R_{ab}^{\ cd} = W_{ab}^{\ cd} + 2\delta^c_{[a}P_{b]}d + \beta_{ab}\delta^c_d, $$

where the projective Weyl tensor $W_{ab}^{\ cd}$ shares the algebraic symmetries of $R$, but in addition it is completely trace free, and $\beta_{ab}$ is skew. The tensor $P_{ab}$ is called the projective Schouten tensor and from the algebraic Bianchi identity $R_{[ab}^{\ cd]} = 0$, one finds $\beta_{ab} = -2P_{[ab]}$. The Ricci tensor is defined by

$$ \text{Ric}_{ab} := R_{ca}^{\ }c_b $$

and so $(n - 1)P_{ab} = \text{Ric}_{ab} + \beta_{ab}$.

2.1. Projective densities and their connections. On any smooth $n$-manifold $M$ the highest exterior power of the tangent bundle $(\Lambda^n TM)$ is a line bundle. The square of any real line bundle has a canonical positive orientation. It follows that its square $\Lambda^n TM$ has a canonical positive orientation and we shall fix that as its orientation. We may forget the tensorial structure of $\Lambda^n TM$ and view this purely as a line bundle. For our subsequent discussion it is convenient to take the positive $(2n+2)^{th}$ root of $\Lambda^n TM$ and we denote this $K$ or $E(1)$. Then for $w \in \mathbb{R}$ we denote $K^w$ by $E(w)$. Sections of $E(w)$ will be described as projective densities of weight $w$. Given any bundle $\mathcal{B}$ we shall write $\mathcal{B}(w)$ as a shorthand notation for $\mathcal{B} \otimes E(w)$.

Now we consider a projective manifold $(M, p)$. Each connection $\nabla \in p$ determines a connection, also denoted $\nabla$, on $\Lambda^n TM$ and hence on its roots $E(w)$, $w \in \mathbb{R}$. For $\nabla \in p$ let us (temporarily) denote the connection induced on $E(1)$ by $D^{\nabla}$, and write $F^{\nabla}$ for its curvature. (In fact $F^{\nabla} = \beta^{\nabla}$, see [18], however we shall not need that fact here.) It is easily verified that, under the transformation (1), $D^{\nabla}$ transforms according to

$$ D^{\nabla}_a = D^\xi_a + \Upsilon_a, $$

where $\Upsilon_a$ is some smooth section of $T^*M$. In the setting of a projective structure $(M, p)$ any connection $\nabla \in p$ is called a Weyl connection or Weyl structure on $M$. In dimension 2 this vanishes but the Cotton tensor is projectively invariant.
where we view \( \Upsilon_a \) as a multiplication operator. Since the connections on \( \mathcal{E}(1) \) form an affine space modelled on \( \Gamma(T^*M) \) it follows that by moving around in \( p \) we can hit any connection on \( \mathcal{E}(1) \), and conversely a choice of connection on \( \mathcal{E}(1) \) determines a connection in \( p \) (or this may be seen by an explicit formula, e.g. [10]). Let us summarise.

**Proposition 2.1.** On a projective manifold \( (M, p) \) a choice of Weyl structure is the same as a choice of connection on \( \mathcal{E}(1) \).

Thus in the setting of projective geometry we drop the notation \( D^\nabla \) and simply write \( \nabla \) for the connection on \( \mathcal{E}(1) \) equivalent to a Weyl structure \( \nabla \).

**Proposition 2.2.** Since \( \mathcal{E}(1) \) is a trivial bundle, any chosen trivialisation determines a flat connection on \( \mathcal{E}(1) \) in the obvious way. Such a connection will be called a scale connection. That is there is a special class \( s \) of connections in \( p \): \( \nabla \in s \) if and only if it preserves a section of \( \mathcal{E}(1) \). Again using that \( \mathcal{E}(1) \) is a trivial bundle it follows that \( \nabla \) is a scale connection if and only if \( F^{\nabla} = 0 \).

It follows from (5) that

\[
F^{\hat{\nabla}} = F^{\nabla} + d\Upsilon
\]

and hence it is clear that if \( \nabla \) and \( \hat{\nabla} \) are both scale connections then \( d\Upsilon = 0 \). In fact since from the definition of scale connections \( \Upsilon \) is then exact, as a change of scale connection is equivalent to a change of trivialisation of the globally trivial bundle \( \mathcal{E}(1) \). (Scale connections are called exact Weyl structures in [11].)

### 2.2. Projective tractor calculus.

By the definition of a projective structure \( (M, p) \), there is no preferred connection on the tangent bundle to \( M \). This potentially impedes calculation and understanding. However there is a canonical connection, known as the tractor connection, on a rank \( n + 1 \) bundle that is closely related to \( TM \). This is due independently to Cartan and Thomas [12, 30], here we follow [3, 7] and the conventions there. In an abstract index notation let us write \( \mathcal{E}_A \) for the first jet prolongation of \( \mathcal{E}(1) \). Canonically we have the jet exact sequence

\[
0 \rightarrow \mathcal{E}_a(1) \xrightarrow{Z_A^a} \mathcal{E}_A \xrightarrow{X_A} \mathcal{E}(1) \rightarrow 0,
\]

where we have written \( X^A \in \Gamma\mathcal{E}^A(1) \) for the jet projection, and \( Z_A^a \) for the map inserting \( \mathcal{E}_a(1) \); these are both canonical [28]. We write \( \mathcal{E}_A = \mathcal{E}_a(1) \hookrightarrow \mathcal{E}(1) \) to summarise the composition structure in (7). As mentioned, any connection \( \nabla \in p \) determines a connection on \( \mathcal{E}(1) \). On the other hand a connection on a vector bundle is the same as a splitting of its 1-jet prolongation. Thus, in particular, a choice of connection on \( \mathcal{E}(1) \) is a splitting of \( \mathcal{E}(7) \). If the tractor \( U_A \) splits, with respect to the connection \( \nabla \), as \( (\mu_i \mid \sigma) \), then it splits with respect to a different \( \nabla \) related to \( \nabla \) by (1) as

\[
(\mu_i + \Upsilon_i \sigma \mid \sigma).
\]
With respect to the direct sum decomposition \( \mathcal{E}_A \cong \mathcal{E}_a(1) \oplus \mathcal{E}(1) \) from a choice of splitting \( \nabla \), we define a connection \( \nabla^T_a \) on \( T^* \) by,

\[
(9) \quad \nabla^T_a (\mu_b \mid \sigma) := (\nabla_a \mu_b + P_{ab} \sigma \mid \nabla_a \sigma - \mu_a).
\]

It turns out that (9) is independent of the choice \( \nabla \in p \), and so \( \nabla^T \) is determined canonically by the projective structure \( p \). This is the cotractor connection as defined in [3] and is equivalent to the normal Cartan connection for the Cartan structure of type \((G, P)\), see [6]. Thus we shall also term \( \mathcal{E}_A \) the cotractor bundle, and we note the dual bundle, called tractor bundle and denoted \( \mathcal{E}^A \) (or in index free notation \( T \)), has canonically the dual tractor connection: in terms of a splitting dual to that above this is given by

\[
(10) \quad \nabla_a^T \left( \frac{\nu^b}{\rho} \right) = \left( \frac{\nabla_a \nu^b + \rho \delta_a^b}{\nabla_a \rho - P_{ab} \nu^b} \right).
\]

In the following we shall normally use simply \( \nabla \) to denote \( \nabla^T \), \( \nabla^T_* \), or any of the connections these induce on tensors products and powers of \( T \) and \( T^* \). We may use the same notation for a connection in \( p \) and the coupling of this with the tractor connection, but the meaning will clear by context.

In the next Sections we will need the curvature \( \Omega \) of the projective tractor connection \( \nabla \). This is defined by \((\nabla_a \nabla_b - \nabla_b \nabla_a)U^C = \Omega_{ab}^C D U^D \). This is easily computed in terms of the curvature of any Weyl connection in the projective class. First we need some notation.

Let us denote by

\[
Y_A : \mathcal{E}(1) \rightarrow \mathcal{E}_A \quad \text{and} \quad Y^A_a : \mathcal{E}_A \rightarrow \mathcal{E}_a(1)
\]

the bundle maps splitting (7) as determined by (and equivalent to) some connection \( \nabla \) on \( \mathcal{E}(1) \). So we have \( X^A Y_A = 1 \), \( Z_A^a Y_A^a = \delta_a^b \) (the section of \( \text{End}(TM) \) that is the identity at every point) and all other tractor index contractions of a pair from \( X^A, Y_A, Z_A^a, Y^A_a \) results in zero. Thus, for example, if \( \Gamma(\mathcal{E}_A) \ni U_A \cong (\mu_b \mid \sigma) \in \Gamma(\mathcal{E}_a(1) \oplus \mathcal{E}(1)) \) then this means

\[
U_A = Z_A^a \mu_a + Y_A \sigma.
\]

In this notation the curvature of the tractor connection is easily calculated to be (cf. [3])

\[
(11) \quad \Omega_{ab}^C D = W_{ab}^c d Z_D^d Y^C_c - C_{dab} Z_D^d X^C.
\]

That is, schematically, for a tractor \( U^A = \left( \frac{\nu^a}{\rho} \right) \),

\[
\Omega_{ab}^C D U^D = \begin{pmatrix} W_{ab}^c d & 0 \\ -C_{dab} & 0 \end{pmatrix} \begin{pmatrix} \nu^a \\ \rho \end{pmatrix}.
\]

It follows that

\[
(12) \quad \Omega_{ab}^C D X^D = 0.
\]
3. CHARACTERISING EINSTEIN IN PROJECTIVE GEOMETRY

We call a symmetric form $h^{AB}$ on $\mathcal{T}^*$ a sub-metric, if the restriction of $h^{AB}$ to the invariant sub-bundle $\mathcal{E}_a(1)$ (see (7)) is genuinely a metric, i.e. non-degenerate. Sub-metrics that are parallel for the tractor connection have an important interpretation.

**Theorem 3.1.** The sub-metrics on $\mathcal{T}^*$, compatible with the tractor connection $\nabla$, are in one-to-one correspondence with the Einstein metrics on $M$ whose Levi-Civita connection is in $p$.

In the case when the Einstein metric is not Ricci-flat, the sub-metric on $\mathcal{T}^*$ is a metric and hence its inverse is a compatible metric on the projective tractor bundle, $\mathcal{T}$. This compatible metric, together with the analogous correspondence theorem, has been described by Armstrong [1]. On the other hand [10, Theorem 3.3] establishes an equivalent of Theorem 3.1 by characterising what is called a normal solution of a certain overdetermined partial differential equation, each suitably non-degenerate solution of which corresponds to a metric giving a Levi-Civita connection in the projective class. The remainder of this Section is used to give a simple direct proof of the Theorem using the projective tractor calculus.

Recall that a connection $\nabla \in p$ is the same as a splitting of the sequence (7).

**Lemma 3.2.** Let $\nabla$ be some splitting of (7). Suppose that $h^{BC}$ is a section of $\text{Sym}^2(\mathcal{T})$ which is diagonal with respect to the splitting $\nabla$, 

$$
\left(\begin{array}{cc}
g^{ab} & 0 \\
0 & \tau \end{array}\right).
$$

Then in the splitting $\nabla$ the tractor connection on $h^{BC}$ is given by

$$
\nabla_a h^{BC} = \left(\begin{array}{c}
\nabla_a g^{bc} \\
\tau \delta^c_a - g^{bc} \lambda_{ab} \tau \\
\nabla_a \tau \end{array}\right).
$$

**Proof.** This is immediate from (10). \qed

First suppose that $g_{ab}$ is Einstein, with $\lambda_{ab} = \lambda g_{ab}$. We define a metric $h^{AB}$ on the cotractor bundle as follows. We work in the splitting of $g$’s Levi-Civita connection $\nabla$. For $U_A \overset{\nabla}{=} (\mu_a \mid \sigma)$, $\overset{\nabla}{U_A} \equiv (\mu_a \mid \sigma)$, $h^{AB}$ is defined by

$$
h^{AB} U_A \overset{\nabla}{U_B} = g^{ab} \mu_a \mu_b + \lambda \sigma \sigma.
$$

From the fact that $\nabla$ is a Levi-Civita connection for an Einstein metric $g$ we have $\nabla c = 0$, $\nabla \lambda = 0$, and $\lambda g_{ab} = P_{ab}$. So by Lemma 3.2 $h^{AB}$ is compatible with the tractor connection.

Conversely, suppose that $p$ is a projective equivalence class such that there exists a sub-metric $h^{AB}$ on $\mathcal{T}^*$, compatible with the tractor connection $\nabla$. The restriction of $h^{AB}$ to $\mathcal{T}^*$’s invariant sub-bundle $\mathcal{E}_a(1)$ is a metric $g^{ab}$. This induces a metric on

$$
(\Lambda^n[T^*M(1)])^2 = (\Lambda^n T^* M \otimes \mathcal{E}(1)^n)^2 = \mathcal{E}(-2n - 2) \otimes \mathcal{E}(2n) = \mathcal{E}(-2),
$$

...
hence also a metric on $E_a = E_a(1) \otimes E(-1)$. Thus we obtain a metric on all weighted tensor bundles and this is compatible with trivialising the density bundles using positively oriented unit length sections. We will use the notation $g$ to mean any of these metrics, with the meaning clear from context.

**Lemma 3.3.** There is a unique connection $\nabla$ in $p$ which splits the Euler sequence

$$0 \rightarrow E_a(1) \rightarrow T^* \rightarrow E(1) \rightarrow 0$$

in agreement with the orthogonal splitting via $h$.

**Proof.** Pick some $\nabla \in p$. By the dual of (8) a projective change of $\nabla \mapsto \nabla'$ by a 1-form $\Upsilon$ induces the change of splitting

$$(v')^a \rightarrow (v^b - g^{ab}_\Upsilon_g v^a - g^{ab}_\Upsilon_g \Upsilon_b)$$

of the bundle $\text{Sym}^2(T)$. (Here $g^{ab}$, $v^a$, $\tau$ denote sections of $E^{ab}(-2)$, $E^a(-2)$, $E(-2)$.) Since $g^{ab}$ is nondegenerate, there is thus a unique change of connection $\Upsilon_b$ such that $(v')^a = v^a - g^{ab}_\Upsilon_g \Upsilon_b$ vanishes. \hfill $\Box$

With respect to the splitting $\nabla$ determined by Lemma 3.3 the metric $h$ is given on cotractors

$$U_A = (\mu_a \mid \sigma), \quad U_A = (\mu_a \mid \sigma),$$

by

$$h^{AB} U_A U_B = g^{ab} \mu_a \mu_b + \tau \sigma \sigma.$$

Since $\nabla$ and $h^{BC}$ are compatible, by Lemma 3.2 we therefore have that $\nabla_c g^{ac} = 0$, $\nabla_c \tau = 0$, and $\tau g_{ab} = P_{ab}$. Using these identities in turn we conclude first (from $\nabla_c g^{ac} = 0$) that $\nabla$ is $g$'s Levi-Civita connection, next (using $\nabla \tau = 0$) that $\tau$ has constant length with respect to $g$, and hence finally (from $\tau g_{ab} = P_{ab}$) that $g$ is Einstein.

4. **Obstructions to projectively metric connections**

Here we construct projective invariants the vanishing of which is necessary for a projective class to include a Levi-Civita connection.

4.1. **The obstructions.** Here we work on an arbitrary manifold with affine connection $(M, \nabla)$ except with symmetric Schouten tensor, in other words $\nabla$ is an arbitrary scale connection. Recall the decomposition (2) of the full curvature tensor as

$$R_{ab}^\ c\ d = W_{ab}^\ c\ d + \delta_a^c P_{bd} - \delta_b^c P_{ad},$$

where $W$ denotes the projective Weyl curvature. From this we obtain the following theorem.
Theorem 4.1. Consider an manifold $M$ equipped with a scale connection $\nabla$, and for each $k \in \mathbb{Z}_{>0}$ define a curvature form $p_k$ on $(M, \nabla)$, by
\[
(p_k)_{a_1a_2 \cdots a_{2k}} := R_{a_1a_2}^{\phantom{a_1a_2}c_1} R_{a_3a_4}^{\phantom{a_3a_4}c_2} \cdots R_{a_{2k-1}a_{2k}}^{\phantom{a_{2k-1}a_{2k}}c_k} c_1,
\]
where (here and below) the $a_1, \ldots, a_{2k}$ are skewed over. Then
\[
(p_k)_{a_1a_2 \cdots a_{2k}} = W_{a_1a_2}^{\phantom{a_1a_2}c_1} c_2 W_{a_3a_4}^{\phantom{a_3a_4}c_2} c_3 \cdots W_{a_{2k-1}a_{2k}}^{\phantom{a_{2k-1}a_{2k}}c_k} c_1,
\]
and so $p_k$ depends only on the projective class $[\nabla]$ of $\nabla$.

Proof. The other terms that arise when calculating $p_k$ all contain multiplicands of at least one of the following two forms:
\[
W_{a_2a_1}^{\phantom{a_2a_1}c_1} c_i + \delta_{a_2a_1}^{\phantom{a_2a_1}c_1} \delta_{a_2a_i}^{\phantom{a_2a_i}c_i} P_{a_2a_i} \quad = \quad W_{a_2a_1}^{\phantom{a_2a_1}c_1} c_i + \delta_{a_2a_1}^{\phantom{a_2a_1}c_1} \delta_{a_2a_i}^{\phantom{a_2a_i}c_i} P_{a_2a_i}
\]
When the full skew over $a_i$’s is taken, both these must vanish: the former because of the symmetry $0 = W_{[i}^{\phantom{i}k]l}$ (the skew over all three bottom indices), the latter because $0 = P_{[ij]}$. \qed

Thus $p_k$ is a well-defined natural and canonical form on any projective manifold $(M, p)$. For $k$ odd $p_k$ has an interpretation: it obstructs the existence of a Levi-Civita connection in $p$.

Corollary 4.2. Let $(M, p)$ be a projective manifold. For odd positive integers $k$, if $p_k \neq 0$ then there is no Levi-Civita connection in $p$.

Proof. Suppose that there is a Levi-Civita connection $\nabla \in p$. Since $p_k$ is projectively invariant we may calculate $p_k$ using this Levi-Civita connection $\nabla$, which we note is a scale connection. We have
\[
p_k = R_{a_1a_2}^{\phantom{a_1a_2}c_1} c_2 R_{a_3a_4}^{\phantom{a_3a_4}c_2} c_3 \cdots R_{a_{2k-1}a_{2k}}^{\phantom{a_{2k-1}a_{2k}}c_k} c_1
\]
where $R_{ab}^{\phantom{ab}cd}$ is the (Riemann) curvature of $\nabla$. But for $k$ odd this means $p_k = 0$, as $R_{abcd} = -R_{abdc}$. \qed

For completeness, we establish that the above corollary is not vacuous:

Proposition 4.3. For each odd $k \geq 3$, there exist projective equivalence classes $p$, such that the curvature form $p_k$ does not uniformly vanish.

Proof. Let $\mathcal{V}$ be the sub-bundle of $\Lambda^2 TM \otimes \text{End}(TM)$ consisting of tensors $A_{ab}^{\phantom{ab}cd}$ which are tracefree and satisfy the Bianchi identity:
\[
A_{ab}^{\phantom{ab}cd} = A_{ac}^{\phantom{ac}bd} = 0, \quad A_{ab}^{\phantom{ab}cd} + A_{bd}^{\phantom{bd}ac} + A_{da}^{\phantom{da}bc} = 0.
\]
Note that if a connection $\nabla$ is torsion-free with symmetric Schouten tensor, then its projective Weyl curvature $W$ is a section of $\mathcal{V}$.

Let $x \in M$. First, we show that every element of the vector space $\mathcal{V}_x$ is the projective Weyl curvature at $x$, $W|_x$, say, of some torsion-free connection $\nabla$ with
symmetric Schouten tensor. Indeed, let $\nabla$ be the connection whose Christoffel symbols in fixed co-ordinates $(x^i)$ centred at $x$ are

$$\Gamma^i_{jk} = \frac{1}{3} S^i_{ajk} x^a,$$

where $S^i_{jk} = 2 A_{i(j} k)$. Since $\Gamma^i_{jk}$ is symmetric in $j, k$, the connection $\nabla$ is torsion-free. The curvature of $\nabla$ is

$$R^i_{jk} l = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^a_{ji} \Gamma^k_{al} - \Gamma^a_{jl} \Gamma^k_{ia},$$

$$= \frac{1}{3} \left[ S^i_{jk} l - S^i_{jl} k + (S^a_{bj} i S^k_{ai} a - S^a_{bi} i S^k_{aj} a) x^b x^c \right],$$

$$= A^i_{jk} l + \frac{1}{3} \left( S^a_{bj} i S^k_{ai} a - S^a_{bi} i S^k_{aj} a \right) x^b x^c.$$

Since $A$, thus $S$, is totally tracefree, the Ricci curvature of $\nabla$ simplifies to

$$R_{jl} = \frac{1}{3} S^a_{bj} i S^k_{ai} a x^b x^c,$$

where the last step is by the $a, j$ symmetry of $S$. This is clearly symmetric in $j$ and $l$, as required. Finally, at the point $0 = x$, we have

$$R^i_{jk} l = A^i_{jk} l,$$

$$R_{jl} = 0,$$

so as required the projective Weyl curvature is $A^i_{jk} l$.

Suppose now that $n \geq 2k + 2$. There then exist a form $\omega \in \Lambda^2 T_x^* M$ and an endomorphism $B \in \text{End}(T_x M)$, such that

$$\omega^{\cdot k} \neq 0, \quad \text{tr}(B^k) \neq 0, \quad \text{tr}(B) = 0, \quad \omega(\cdot, B \cdot) = 0.$$

The tensor $A \in \Lambda^2 T_x^* M \otimes \text{End}(T_x M)$,

$$A_{ab}^c d := 2 \omega_{ab} B_d^c - \omega_{da} B_b^c - \omega_{bd} B_a^c,$$

belongs to the subspace $\mathcal{V}_x$. Thus there exists a connection $\nabla$ whose projective Weyl curvature at $x$ is $A$.

Split $A$ as $A^{(0)} + A^{(1)} + A^{(2)}$, where

$$(A^{(0)})_{ab}^c d = 2 \omega_{ab} B_d^c, \quad (A^{(1)})_{ab}^c d = -\omega_{da} B_b^c, \quad (A^{(2)})_{ab}^c d = -\omega_{bd} B_a^c,$$

and observe that the assumption $\omega(\cdot, B \cdot) = 0$ makes all traces starting with $A^{(1)}$ or $A^{(2)}$ in $p_k$’s construction vanish:

$$\omega_{ra_1} B_{a_2}^c \omega_{b_{i+1}} B_{b_{i+2}}^r = 0,$$

$$\omega_{r a_2} B_{a_1}^c \omega_{b_{i+1}} B_{b_{i+2}}^r = 0.$$
We thus have
\[
(p_k)|_x = \sum_{\sigma \in \Sigma_{2k}} (-1)^\sigma \prod_{i=1}^{k} (A^{(0)})_{a_{\sigma(2i-1)}a_{\sigma(2i)}} c_i c_{i+1}
\]
\[= \frac{2^k}{(2k)!} \omega^{\wedge k} \text{tr}(A^k),\]
which is nonzero.

Remark 4.4. For odd positive integers \(k\) the \(p_k\) are necessarily exact. Although finding a primitive will involve making a choice, it should be that this has some interesting geometric applications. For example, for a given such primitive, on a cycle that is homologically trivial its integral is independent of choice (obviously projectively invariant) and is a global obstruction to the projective metricity of the ambient space.

4.2. A tractor motivation. On a projective manifold \((M, p)\) let us define the tractor connection curvature forms
\[
q_k := \Omega_{a_1 a_2}^c c_1 c_2 \Omega_{a_3 a_4}^c c_3 c_4 \cdots \Omega_{a_{2k} a_{2k}}^c c_{2k} c_{2k+1},
\]
where the \(a_1, \ldots, a_{2k}\) are skewed over. Since the tractor curvature \(\Omega\) is determined by the projective structure, by construction the \(q_k\) are projectively invariant (i.e. determined by \((M, p)\)). Of course these are characteristic type forms for a connection, and so are closed forms by the Bianchi identity (here on the tractor curvature). Since as a vector bundle the projective tractor bundle is a trivial extension of the tangent bundle (twisted by a trivial line bundle) it follows at once that, for each \(k\), the \(q_k\) must represent the same de Rham cohomology class as \(p_k\) (and this is of course trivial if \(k\) is odd). In fact a much stronger result is true, as follows.

Proposition 4.5. Let \((M, p)\) be a projective manifold. Then \(q_k = p_k\).

Proof. Recall from (11), the Cotton term in the tractor curvature \(\Omega_{ab}^C D\) has coefficient \(X^C\), but from (12) we have \(\Omega_{ab}^C D X^D = 0\). Thus using the identity \(Z_A a Y^A_b = \delta^a_b\) the result follows from Theorem 4.1.

This result puts the curvature forms \(p_k\) in a nice setting. As defined originally they are the curvature forms of affine connection, and so not obviously projectively invariant. In Theorem 4.1 they are seen to arise from contractions of the projective Weyl curvature so the projective invariance is then clear, but in that formulation they are not manifestly from a connection. On the other hand by the Proposition 4.5 here we see they do arise as characteristic type polynomials from the curvature of a projectively invariant connection. The situation here is parallel to a number of conformal results: When \(\nabla\) is a Levi-Civita connection then it is an important result of Chern-Simons that the \(p_k\) are conformally invariant [14] (see also [2]). In that case they can be viewed to have a conformal tractor origin [3].

This tractor interpretation of the \(p_k\)’s was our initial motivation for their study. That the odd \(p_k\)’s are obstructions to projective-metricity, is less surprising when
it is observed that they are at least obstructions to projective-Einstein-metricity. This argument goes as follows:

If $\nabla$ preserves a metric $h$ on $\mathcal{T}$ then the curvature 2-form $\Omega$ necessarily takes values in the $\mathfrak{so}(h)$ subbundle of $\text{End}(\mathcal{T})$. That means exactly that,

$$(15) \quad \Omega_{abCD} = -\Omega_{abDC},$$

where we used $h$ to identify $\mathcal{T}$ with $\mathcal{T}^*$ and, in particular, lower a tractor index of $\Omega_{ab}^C \cdot D$.

It follows that when $k$ is odd the $q_k$ are projectively invariant obstructions to the existence of a non-Ricci-flat Einstein metric compatible with the projective structure. In fact it is not hard to strengthen the result by dropping the “non-Ricci-flat” restriction by using also the results of Section 8. But we leave this as an exercise since, in any case, the result in Corollary 4.2 supersedes this.

5. Obstructions to projectively Einstein connections

Here we show and exploit the fact that if a Levi-Civita connection is Einstein then its Cotton tensor must vanish. This idea is partly inspired by the approach to analogous questions in conformal geometry as developed in [22, 17].

5.1. A projective C-space condition.

**Proposition 5.1.** Let $\nabla$ be a torsion-free affine connection. If $\nabla$ is projectively equivalent to some Einstein metric’s Levi-Civita connection, via (as in Equation (7)) the 1-form $\Upsilon_i$, then

$$(16) \quad C_{kij} + W_{ij}^l \Upsilon_l = 0.$$ 

**Remark 5.2.** In dimension 2 the situation is rather special. The projective Weyl tensor $W_{ij}^l \Upsilon_l$ is necessarily zero and this shows that the Cotton tensor is a projective invariant that obstructs the existence of an Einstein Levi-Civita connection in the projective class. In fact, as is well known, the Cotton tensor is a sharp obstruction to the existence of a flat connection in the projective class. Hence it is a sharp obstruction to the existence of an Einstein metric’s Levi-Civita connection in the projective class.

In all dimensions a connection $\nabla$ satisfies (16) if and only if $\nabla$ is projectively related to a connection with vanishing Cotton tensor, and this is a precondition to $\nabla$ being the Levi-Civita connection of an Einstein metric.

In this section we give two proofs of this proposition: one conceptual using the tractor language, and one by direct computation.

**Tractor proof.** Let $p$ be a projective equivalence class of connections which contains some Einstein metric’s Levi-Civita connection. First we observe the existence of a distinguished rank-1 sub-bundle of the cotractor bundle.
In the case when the Einstein metric contained in \( p \) is not Ricci-flat, this proceeds as follows. By Theorem 3.1, we have an induced metric \( h^{AB} \) on the cotractor bundle, compatible with the tractor connection \( \nabla \). Inverting, this induces a metric \( h_{AB} \) on the tractor bundle. We then have a distinguished nonvanishing section of the weighted cotractor bundle \( \mathcal{E}_A(1) \), given by
\[
U_A := h_{AB} X^B.
\]

In the case when the Einstein metric is Ricci-flat, by Theorem 3.1, we have an induced sub-metric \( h^{AB} \) on the cotractor bundle, compatible with the tractor connection \( \nabla \). The nullspace of the sub-metric \( h^{AB} \) is everywhere of rank 1. This defines a rank-1 sub-bundle of the cotractor bundle which is preserved by the tractor connection.

We note that, for either case, it is immediate from (13) that in the splitting determined by an Einstein Levi-Civita connection in \( p \) the distinguished line sub-bundle consists of the tractors
\[
V_A = (0 \mid \sigma),
\]
where \( \sigma \) is a section of \( \mathcal{E}(1) \). Hence, by (13), in an arbitrary splitting \( \nabla \in p \), the distinguished line sub-bundle consists of the tractors
\[
V_A = (-\sigma \Upsilon_i \mid \sigma),
\]
where \( \sigma \) is a section of \( \mathcal{E}(1) \), and \( \Upsilon_i \) is the 1-form which defines the projective change to an Einstein metric’s Levi-Civita connection.

Next we observe that the rank-1 subbundle lies in the nullity of the cotractor curvature.

In the case when the Einstein metric is not Ricci-flat, note that using (12) we have
\[
0 = \Omega_{ab} C^D D^E X^D = \Omega_{ab} C^D h^{DE} U_E.
\]
Since the tractor connection preserves \( h^{DE} \), we have
\[
\Omega_{ab} C^D h^{DE} = -\Omega_{ab} E^D h^{DC}.
\]
Substituting back,
\[
0 = \Omega_{ab} C^D h^{DE} U_E = -\Omega_{ab} E^D h^{DC} U_E.
\]
Since \( h \) is nondegenerate we conclude that \( \Omega_{ab} E^D U_E = 0 \).

In the Ricci-flat case this follows from (14), as in the Einstein scale \( \mathcal{P}_{ab} = 0 \) and hence the Cotton tensor vanishes everywhere. Thus in this scale \( \Omega_{ab} C^D V_C = 0 \), for all sections of the distinguished subbundle. But that equation is projectively invariant.

It follows from this that if \( \nabla \) is some representative of \( p \), and
\[
V_A = (-\sigma \Upsilon_i \mid \sigma),
\]
is a section of the rank-1 sub-bundle, then we have that
\[
0 = \Omega_{ab} C^D V_C = (W_{ab} i^k \Upsilon_i + C_{kab}) \sigma.
\]
Computational proof. We show that for an arbitrary change of connection
\[ \nabla'_i \omega_j = \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i, \]
the Cotton tensor changes by
\[ 2 \nabla'_i P'_{jk} = 2 \nabla_i P_{jk} + W_{ij}^l \Upsilon_l. \]
If the new connection \( \nabla' \) is the Levi-Civita connection of some Einstein metric, the tensor \( P' \) is a constant multiple of that metric, hence parallel, and we then obtain
\[ 0 = 2 \nabla_i P_{jk} + W_{ij}^l \Upsilon_l \]
as required.

Indeed, recall from (4) that under such a projective change of connection, the projective Weyl tensor \( W_{ij}^k \) is invariant, and the projective Schouten tensor changes by
\[ P'_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j, \]
So by (1) its covariant derivative changes by
\[ \nabla'_i P'_{jk} = \nabla_i P_{jk} - 2 P_{jk} \Upsilon_i - P'_{ji} \Upsilon_k - P'_{ik} \Upsilon_j. \]
Skewing over \( i \) and \( j \) gives
\[ \nabla'_i P'_{jk} = \nabla'_i P'_{jk} - \Upsilon_i P'_{jk} + \frac{1}{2} \beta'_{ij} \Upsilon_k. \]
We expand the three terms separately (using the transformation formula (4) for the last one):
\[ \nabla'_i P_{jk} = \nabla'_i P_{jk} - \nabla_i \nabla_j \Upsilon_k + \Upsilon_k \nabla_i \Upsilon_j + \Upsilon_j \nabla_i \Upsilon_k \]
\[ = \nabla_i P_{jk} + \frac{1}{2} R_{ij}^l \Upsilon_l + \Upsilon_k \nabla_i \Upsilon_j + \Upsilon_j \nabla_i \Upsilon_k \]
\[ = \nabla_i P_{jk} + \frac{1}{2} W_{ij}^l \Upsilon_l + \Upsilon_k \nabla_i \Upsilon_j + \Upsilon_j \nabla_i \Upsilon_k \]
\[ - \Upsilon_i P_{jk} = - \Upsilon_i P_{jk} + \Upsilon_i \nabla_j \Upsilon_k - \Upsilon_j \nabla_i \Upsilon_k \]
\[ = - \Upsilon_i P_{jk} + \Upsilon_i \nabla_j \Upsilon_k \]
\[ = \frac{1}{2} \beta_{ij} \Upsilon_k + \nabla_{[i} \Upsilon_{j]} \Upsilon_k. \]
Summing, we obtain:
\[ 2 \nabla'_i P_{jk} = 2 \nabla_i P_{jk} + W_{ij}^l \Upsilon_l \]
as required.
5.2. Obstructions in the generic setting. By analogy with [17], we call a torsion-free connection \( \text{weakly generic} \), if the bundle map

\[
W : T^*M \to \Lambda^2 T^*M \otimes T^*M
\]
defined by its projective Weyl tensor \( W_{ij}^l \) has trivial kernel at every point. We observe that this condition is projectively invariant.

If \( \nabla \) is weakly generic, we may locally choose a left inverse field \( D_{ij}^{mk} \) for \( W_{ij}^l \), so that

\[
D_{ij}^{mk}W_{ij}^l = \delta_{mk}.
\]

A weakly generic connection has at most one solution \( \Upsilon_i \) to the C-space equation (and thus on weakly generic structures there is at most one Einstein Levi-Civita connection in \( p \)). In terms of \( D_{ij}^{mk} \) this solution, if it exists, is the covector field

\[
\Upsilon_i = -D^{ab}_{i} \epsilon C_{cab}.
\]

In the following we shall construct tensors using \( D \) but which are otherwise canonical to the projective structure \((M, p)\). We shall refer to these as projective invariants even though by dint of the choice \( D \) they are not natural. In some cases there exist natural \( D \), whence in these cases we obtain natural projective invariants. Funding such \( D \) is taken up in Section 5.3 below.

Now for each \( D \) satisfying (19) we may construct tensors that partly capture the projective-Einstein condition as follows. First observe that we may use (19) to replace \( \Upsilon_s \) in the expression

\[
P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j
\]

for \( P'_{ij} \) to define a 2-tensor

\[
G_{ij} := P_{ij} + \nabla_i(D^{ab}\epsilon C_{cab}) + D^{ab}_{i} \epsilon C_{cab}D^{ef}_{j} \epsilon C_{gef}.
\]

**Lemma 5.3.** For a fixed choice of \( D \), the tensor \( G_{ij} \) is projectively invariant.

**Proof.** Consider an arbitrary projective change of connection

\[
\nabla'_{i} \omega_j = \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i.
\]

The expression (20) for the tensor \( G'_{ij} \) has three terms; we calculate it term by term.

For the first term,

\[
P'_{ij} = P_{ij} - \nabla_i \Upsilon_j + \Upsilon_i \Upsilon_j.
\]

Before continuing, note that by equation (17), in the second proof of Proposition 5.1 we know that \( C_{cab} \) changes by

\[
C'_{cab} = C_{cab} + W_{ab}^l \epsilon \Upsilon_l.
\]

So \( D^{ab}_{i} \epsilon C_{cab} \) changes by

\[
D^{ab}_{i} \epsilon C_{cab}' = D^{ab}_{i} \epsilon C_{cab} + D^{ab}_{i} \epsilon W_{ab}^l \epsilon \Upsilon_l \]

\[
= D^{ab}_{i} \epsilon C_{cab} + \Upsilon_i.
\]
Now, for the second term,
\[ \nabla_i(D_{ab}^{\prime} C_{cab}^\prime) = \nabla_i(D_{ab}^{\prime} C_{cab}^\prime) - \Upsilon_i D_{ab}^{\prime} C_{cab}^\prime - D_{ab}^{\prime} C_{cab}^\prime \Upsilon_j. \]
\[ = \nabla_i(D_{ab}^{\prime} C_{cab}^\prime) + \nabla_i \Upsilon_j - \Upsilon_i D_{ab}^{\prime} C_{cab}^\prime - D_{ab}^{\prime} C_{cab}^\prime \Upsilon_j - 2 \Upsilon_i \Upsilon_j. \]

For the third term,
\[ D_{ab}^{\prime} C_{cab}^\prime D_{ef}^{\prime} g_{gef} = D_{ab}^{\prime} C_{cab}^\prime D_{ef}^{\prime} g_{gef} + D_{ab}^{\prime} C_{cab}^\prime \Upsilon_j + \Upsilon_i D_{ef}^{\prime} g_{gef} + \Upsilon_i \Upsilon_j. \]

Summing, we find that
\[ G_{ij}^{\prime} = P_{ij} + \nabla_i(D_{ab}^{\prime} C_{cab}^\prime) + D_{ab}^{\prime} C_{cab}^\prime D_{ef}^{\prime} g_{gef} = G_{ij}, \]
as required.

Now if \( p \) contains a Levi-Civita connection \( \nabla' \) for some Einstein metric \( g \) then (19) finds the 1-form which by (1) gives the projective transformation to that connection. Thus by construction in that case \( G_{ij} = \lambda g_{ij} \) for some constant \( \lambda \in \mathbb{R} \). On the other hand starting with a weakly generic \((M, p)\) the properties of the invariant \( G_{ij} \) may forbid the existence of Einstein Levi-Civita connection in \( p \): if the skew part \( G_{[ij]} \) is non-zero then \((M, p)\) is not projectively Einstein; if \( G_{ij} \) is non-zero but degenerate then \((M, p)\) is not projectively Einstein; \( G_{ij} \) is zero if and only if \((M, p)\) is projectively Ricci-flat (but not necessarily projectively metric). In relation to the second of these points let us define the density \( \gamma \in \mathcal{E}(-2(n + 1)) \) by
\[ G_{a_1 b_1} G_{a_2 b_2} \cdots G_{a_n b_n} \]
(and the usual identification of \((\Lambda^n T^* M)^2 \) with \( \mathcal{E}(-2(n + 1)) \)) where the sequentially labelled indices are skewed over.

To construct the next obstruction we again consider \( \nabla' \in p \). We want to test whether this is an Einstein Levi-Civita connection. If so then by the argument above then \( G_{ij} \) is a constant times the Einstein metric whence \( \nabla_i G_{jk} \) would be zero. We calculate at some \( \nabla \in p \). The first step is that we expand the formula which expresses \( \nabla_i G_{jk} \) in terms of \( \nabla \), and the 1-form \( \Upsilon_i \) that relates \( \nabla \) and \( \nabla' \) as in (11). Using (11) we have
\[ \nabla_i G_{jk} = \nabla_i G_{jk} - 2 G_{jk} \Upsilon_i - G_{ji} \Upsilon_k - G_{ik} \Upsilon_j \]
and in here we use (19) to replace the \( \Upsilon \)s. This yields a projective invariant
\[ E_{ijk} := \nabla_i G_{jk} + (2 G_{jk} D_{ib}^{ab} c + G_{ji} D_{kb}^{ab} c + G_{ik} D_{jb}^{ab} c) C_{cab}, \]
which obstructs the existence of an Einstein connection projectively equivalent to \( \nabla \). This polynomially involves \( D, W \) and \( C \) and their \( \nabla \)-covariant derivatives to second order.

Remark 5.4. Since \( E_{ijk} \) is projectively invariant so is \( 2 E_{[ij]k} \). This has a nice interpretation: it is exactly the obstruction to the existence of a Cotton flat affine connection in the projective class. Indeed there is a simple formula for this
\[ 2 E_{[ij]k} = C_{kij} - W_{ij} \ell_k D_{ell}^{ab} c C_{cab}, \]
on obtained by inserting (19) into (17).
Lemma 5.5. For a fixed choice of $D$, the tensor $E_{ijk}$ is projectively invariant.

Proof. Consider an arbitrary projective change of connection
\[ \nabla_i' \omega_j = \nabla_i \omega_j - \Upsilon_i \omega_j - \Upsilon_j \omega_i. \]
The expression (21) for the tensor $E'_{ijk}$ has four terms; we calculate it term by term. This is simplified by the projective invariance of $G_{ij}$ (Lemma 5.3).

For the first term,
\[ \nabla_i' G_{jk} = \nabla_i G_{jk} - 2G_{jk} \Upsilon_i - G_{j} \Upsilon_k - G_{i} \Upsilon_j. \]

Before continuing, note as in the proof of Lemma 5.3 that by equation (17), the quantity $D_{ab}C_{cab}$ transforms by
\[ D_{ab}C_{cab} = D_{ab}C_{cab} + \Upsilon_i. \]

Now, for the second, third and fourth terms,
\[ 2G_{jk}D_{ab}C_{cab} = 2G_{jk}D_{ab}C_{cab} + 2G_{jk} \Upsilon_i, \]
\[ G_{ji}D_{ab}C_{cab} = G_{ji}D_{ab}C_{cab} + G_{ji} \Upsilon_k, \]
\[ G_{ki}D_{ab}C_{cab} = G_{ki}D_{ab}C_{cab} + G_{ki} \Upsilon_j. \]

Summing, we find that
\[ E'_{ijk} = \nabla_i G_{jk} + (2G_{jk}D_{ab}C_{cab} + G_{ji}D_{ab}C_{cab} + G_{ki}D_{ab}C_{cab})C_{cab} = E_{ijk}, \]
as required. □

Proposition 5.6. Let $\nabla$ be a weakly generic torsion-free connection, and let $D$ be a left inverse for $W$ in the sense of (18). Then $\nabla$ is projectively equivalent to a Ricci-flat affine connection if and only if $G_{ij}$ is zero. Moreover the following are equivalent:

1. $\nabla$ is projectively equivalent to the Levi-Civita connection of an Einstein metric with nonvanishing Einstein constant.
2. $E_{ijk}$ and $G_{ij}$ are each zero while $\gamma$ is nowhere zero (i.e. $G_{ij}$ is everywhere nondegenerate).

Proof. The first claim was discussed earlier. The discussion before the Proposition shows that (1) implies (2). For the converse, given a connection for which $E_{ijk}$ vanishes and $G_{ij}$ is symmetric and nondegenerate, we define a tensor $\Upsilon_i$ by (19). But if $G_{ij}$ is symmetric and nondegenerate then it is a metric of some signature. We define $\nabla'$ to be the connection obtained from $\nabla$ and the projective change (1) using $\Upsilon_i$.

The vanishing of $E_{ijk}$ implies that, in the projective change of connection defined by $\Upsilon_i$,
\[ \nabla_i' G_{jk} = E_{ijk} = 0. \]
So $\nabla'$ is the Levi-Civita connection of $G_{ij}$. On the other hand by construction $G_{ij}$ is equal to $\mathcal{P}_{ij}'$, the Schouten tensor for $\nabla'$. Thus
\[ \text{Ric}(G_{ij}) = (n-1)\mathcal{P}_{ij}' = (n-1)G_{ij}, \]
so $G_{ij}$ and all of its non-zero multiples are Einstein, and their common Levi-Civita connection $\nabla'$ is projectively equivalent to $\nabla$.

Again, for completeness, we check that these obstructions are nontrivial.

**Proposition 5.7.** There exist torsion-free weakly generic connections.

**Proof.** We prove this in dimension 4. Let $\nabla$ be the Levi-Civita connection of a metric which is Ricci-flat but not locally conformally flat. The conformal Weyl curvature $	ilde{W}$ satisfies,

$$4\tilde{W}^k_i j \tilde{W}^i j k m = |\tilde{W}|^2 \delta^m_l,$$

so the map

$$\tilde{W} : T^* M \to \Lambda^2 T^* M \otimes T^* M$$

is invertible. Since $\nabla$ is Ricci-flat, the two Weyl curvatures agree: $W = \tilde{W}$; see Proposition 8.2 below.

5.3. **Natural left inverses.** Proposition 5.6 reduces the problem of finding a sharp obstruction to the projective Einstein problem to that of finding natural left inverses for $W_{abcd}$, the tensors $D$ in the Proposition. More precisely, this is true in the case of nonzero Einstein constant. In this section we construct such natural tensors.

First, suppose the dimension of the manifold is even, say $n = 2m$. Write $(Q^k)_{a_1 a_2 \cdots a_{2k} \ i j}$ for the totally alternating part of

$$W_{a_1 a_2 c_1 j} W_{a_3 a_4 c_2 c_1} \cdots W_{a_{2k-1} a_{2k} \ i c_{k-1}}$$

(that is, the result of skewing over all $a_i$-indices), so that its trace $(Q^k)_{a_1 a_2 \cdots a_{2k} \ i}$ is the $k$-th curvature form. Each tensor $(Q^m)_{a_1 a_2 \cdots a_{2k} \ i}$ is a section of $\Lambda^{2k} T^* M \otimes \text{End}(TM)$. For the two highest $k$ among such tensors we have the following: Via the natural (“Hodge-star”) isomorphism

$$\Lambda^k (T^* M) \to \Lambda^{2m-k} (TM) \otimes \Lambda^{2m} (T^* M),$$

we may in fact treat $Q^{m-1}$ as a section $(Q^{m-1})_{b_1 b_2 \ i \ r}$ of the bundle

$$\Lambda^2 (TM) \otimes \text{End}(TM) \otimes \Lambda^{2m} (T^* M)$$

and $Q^m$ as a section $(Q^m)_r \ i$ of the bundle

$$\text{End}(TM) \otimes \Lambda^{2m} (T^* M).$$

We note that

$$(Q^{m-1})_{b_1 b_2 \ i} W_{b_1 b_2 s} = (Q^m)_r \ i.$$ 

Since $\text{End}(TM) \otimes \Lambda^{2m} (T^* M)$ is just a twisting of the endomorphism bundle by the line bundle $\Lambda^{2m} (T^* M)$, we may meaningfully speak

- of $Q^m$’s determinant $||Q^m||$, which is a section of $(\Lambda^{2m} (T^* M))^{2m}$;
- of $Q^m$’s pointwise adjugate $(\tilde{Q^m})_r \ i$, which is a section of

$$\text{End}(TM) \otimes (\Lambda^{2m} (T^* M))^{2m-1};$$
of $Q^m$’s invertibility, which occurs precisely where $||Q^m|| \neq 0$; the inverse is then the section $||Q^m||^{-1}(Q^m)^i_j$ of $\text{End}(TM) \otimes (\Lambda^{2m}(TM))$.

For connections $\nabla$ satisfying the further “genericity” condition that $||Q^m||$ does not vanish, we thus obtain a natural left inverse $D$ for the Weyl curvature $W$: the tensor

$$D_{(Q)}^{b_1b_2} k := ||Q^m||^{-1}(Q^m)^r_k (Q^{m-1})^{b_1b_2} r.$$  

Indeed,

$$D_{(Q)}^{b_1b_2} k W_{b_1b_2}^j = ||Q^m||^{-1}(Q^m)^r_k (Q^{m-1})^{b_1b_2} r W_{b_1b_2}^j = ||Q^m||^{-1}(Q^m)^r_k (Q^m)^j_r = \delta^j_r.$$  

This is just one example of a large family of natural left inverses for $W_{ab}$. A general procedure which works in either dimension parity is as follows. First construct a family of tensors $(Q^m)^{b_1...-b_r}_{s}^s$: the inputs are

1. A nonnegative integer $R$, with $R < n$, and an even natural number $2N$, such that $2N + R$ is a multiple of $n$, the dimension of the manifold;
2. A partition $N_0 + \cdots + N_r = N$ of $N$;
3. A function

$$F : \prod_{j=0}^r \{1, \ldots, 2N_j\} \to \{1, \ldots, (2N + R)/n\}$$

with the property that for each $i \in \{2, \ldots, (2N + R)/n\}$ (but not for $i = 1$), $|F^{-1}(i)| = n$. Thus $|F^{-1}(1)| = n - R$.

We introduce the notation

$$(P^k)_{a_1...-a_{2k}}^s := W_{a_1a_2} c_1 W_{a_3a_4} c_2 \cdots W_{a_{2k-1}a_{2k}} s c_{k-1}.$$

(Thus the full skew of such a tensor’s trace, $(P^k)_{a_1...-a_{2k}}^s$, is the $k$-th curvature form $(p_k)_{a_1...-a_{2k}}$.) Consider the tensor

$$(P^N)_{a_1...-a_{2N}}^s \prod_{j=1}^r (P^N)_{a_1...-a_{2N_j}}^{s_j c_j}.$$  

For each $i \in \{2, \ldots, (2N + R)/n\}$, skew the $n$ indices $\{a^i_\alpha : F(j, \alpha) = i\}$ of this tensor, and also (for $i = 1$) skew the $n - R$ indices $\{a^i_\alpha : F(j, \alpha) = 1\}$. Finally, again via the natural (“Hodge-star”) isomorphism, we may identify the result with a section of

$$\Lambda^R(TM) \otimes \text{End}(TM) \otimes (\Lambda^n(T^*M))^{(2N+R)/n}.$$  

This section is the tensor $(Q^{m,F})^{b_1...-b_r}_{s}^s$.

To construct obstructions, now take a valid set of inputs $(N, 0, N, F)$ as above, i.e. a valid set of inputs in which $R = 0$. This means:

1. $2N$ is an even number which is a multiple of $n$, the dimension of the manifold.
(2) $N_0 + \cdots + N_r = N$ is a partition of $N$.

(3) $F : \prod_{j=0}^{r} \{1, \ldots, 2N_j\} \to \{1, \ldots, 2N/n\}$

is a function with, for each $i \in \{1, \ldots, 2N/n\}$, $|F^{-1}(i)| = n$.

(The inputs $N = m$, $N = (m)$, $F \equiv 1$ will yield the special case considered at the start of this subsection.) Write

$$N' = (N_0 - 1, N_1, \ldots, N_r), \quad F' = F|_{\{1, \ldots, 2N_0-2\} \cup (\prod_{j=1}^{r} \{1, \ldots, 2N_j\})}.$$  

With $N' = N - 1$ and $R' = 2$, the set of inputs $(N', 2, N', F')$ is then also valid.

We note that

$$(Q_{N', F'})^{b_1 b_2 k} W_{b_1 b_2}^i = (Q_{N, F})^{s}.$$  

Since $\text{End}(TM) \otimes (\Lambda^n(T^*M))^{2N/n}$ is just a twisting of the endomorphism bundle by the line bundle $(\Lambda^n(T^*M))^{2N/n}$, we may meaningfully speak

- of $Q_{N, F}$’s determinant $||Q_{N, F}||$, which is a section of $(\Lambda^n(T^*M))^{2N}$;

- of $Q_{N, F}$’s pointwise adjugate $(Q_{N, F})^{s}_r$, which is a section of $\text{End}(TM) \otimes (\Lambda^n(T^*M))^{2N(n-1)/n}$.

For connections $\nabla$ satisfying the further “genericity” condition that $||Q_{N, F}||$ does not vanish, we thus obtain a natural left inverse $D$ for the Weyl curvature $W$: the tensor

$$D^{b_1 b_2 k}_{(Q)} := ||Q_{N, F}||^{-1}(Q_{N, F})^{r}_{i}(Q_{N', F'})^{b_1 b_2} W_{b_1 b_2}^{i}.$$  

Indeed,

$$D^{b_1 b_2 k}_{(Q)} W_{b_1 b_2}^{j} = ||Q_{N, F}||^{-1}(Q_{N, F})^{r}_{i}(Q_{N', F'})^{b_1 b_2} W_{b_1 b_2}^{j} = ||Q_{N, F}||^{-1}(Q_{N, F})^{r}_{i}(Q_{N, F})^{j} = \delta^i_j.$$  

In summary, adapting Proposition 5.6 we have the following.

**Theorem 5.8.** For a $||Q_{N, F}||$-nowhere-zero torsion-free connection $\nabla$, the natural projective invariant $G_{ij}^{(Q)}$ completely obstructs the projective class containing a Ricci-flat connection. Moreover the following are equivalent:

1. $\nabla$ is projectively equivalent to the Levi-Civita connection of an Einstein metric with nonvanishing Einstein constant.

2. The natural projective invariants $E_{ijk}^{(Q)}$ and $G_{ij}^{(Q)}$ are each zero while $\gamma^{(Q)}$ is nowhere zero (i.e. $G_{ij}^{(Q)}$ is everywhere nondegenerate).
Remark 5.9. It would be interesting to determine which valid input sets \((N, 0, \mathcal{N}, F)\) yield nontrivial obstructions, i.e. have the property that \(\|Q^N_F\|\) is generically non-vanishing. We expect that in both dimension parities this happens frequently. The non-vanishing of any such \(\|Q^N_F\|\) defines a notion of *generic* Weyl curvature.

Remark 5.10. By multiplying through by a suitable power of \(\|Q^N_F\|\), these invariants can be made polynomial rather than rational in the jets of the underlying affine connection.

6. Obstructions proliferating

We now note that using the results established above there are many ways available for producing projective obstructions to Einstein Levi-Civita connections. In fact there is a general principle that is very effective. We describe this here. The principle exploits linear identities, whereas the theory above provides a systematic approach to producing such identities. The idea behind this principle is well-known. For instance, in [17, Corollary 3.5] it is used to obtain obstructions to a conformal class being conformally Einstein, and in [5, Section 8] (mentioned again in [27, Theorem 2.13]) it is used to obtain an obstruction to a projective class being metric. This idea and some strategies from [17] are also used effectively by Case in [13].

We formalise the principle in the following lemma.

**Lemma 6.1.** Let \((M, p)\) be a projective manifold, let \(E\) and \(F\) be natural vector bundles on \((M, p)\), and let the field \(A\) be a natural section of \(E \otimes F\). Let \(k\) be the rank of \(E\).

Then the corresponding section \(A^k\) of \(\Lambda^k(E) \otimes \Lambda^k(F)\), sharply obstructs the existence of nonvanishing local sections \(\xi\) of \(E^*\) such that

\[
\langle A, \xi \rangle = 0.
\]

**Proof.** The section \(A\) of \(E \otimes F\) induces a natural bundle map \(A : E^* \to F\). Its \(k\)-th exterior power

\[
A^k : \Lambda^k(E^*) \to \Lambda^k(F),
\]

the “top-dimensional minors” bundle map, which may be identified with a section of

\[
\Lambda^k(E) \otimes \Lambda^k(F),
\]

vanishes precisely if \(A : E^* \to F\) has nontrivial kernel. \(\square\)

**Remark 6.2.** The Lemma is useful when \(k \leq \text{Rank}(F)\). Otherwise if \(\text{Rank}(E) > \text{Rank}(F)\) then the bundle \(\Lambda^k(E) \otimes \Lambda^k(F)\) has rank zero.

We have already obtained several linear relationships to which this lemma may be applied. We recall them here. Let \(p\) be a projective equivalence class which contains the Levi-Civita \(\nabla\) connection of an Einstein metric \(g_{ab}\). Then we have the following:
(1) It will be proved later (Proposition 5.2) that the projective Weyl curvature $W$ of the class $p$ must agree with the conformal Weyl curvature $\tilde{W}$ of $\nabla$. This latter has the symmetries of $g_{ab}$’s Riemann curvature tensor; therefore

$$W_{ab}^c (d g^c) = 0.$$ 

(2) Recall from the “Tractor Proof” of Proposition 5.1 the existence of a distinguished nonvanishing section $V_A$ of $\mathcal{T}^*$ such that

$$0 = \Omega_{ab}^C D V_C.$$ 

(3) Recall from Section 3 the existence of a section $h$ of $\text{Sym}^2 (\mathcal{T})$, given in the scale connection $\nabla$ and the trivialisation of density bundles $d Vol_g$ by

$$h^{AB} = \left( \begin{array}{cc} g^{ab} & 0 \\ 0 & \frac{1}{n} \mathbb{P} \end{array} \right),$$

which is parallel with respect to the tractor connection:

$$\nabla h^{AB} = 0.$$ 

Differentiating again and skewing yields a linear relationship involving tractor curvature:

$$\Omega_{ab}^C D h^{E}D.$$ 

(4) Since the tractor $h^{AB}$ is parallel, taking jets of the linear relationship just obtained yields arbitrarily many further linear relationships.

In each case we have a nonvanishing field (e.g. the metric in (1), $V$ in (2)) which satisfies a homogeneous linear equation whose coefficients are natural projectively invariant fields.

The point is that from each of these geometrically obtained linear identities we obtain obstructions to the projective-Einstein problem. For instance, from the first relationship we obtain a projectively invariant $W^{n(n+1)}_{\frac{n^2}{2}}$, a section of the bundle

$$\Lambda^{\frac{n(n+1)}{2}} (\mathcal{E}^{(ab)} \otimes \Lambda^{\frac{n(n+1)}{2}} (\mathcal{E}_{[ab]} \otimes \mathcal{E}_{(de)}),$$

which obstructs the existence of an Einstein connection in the projective equivalence class.

From the other relationships, the projective invariants we obtain are initially sections of mixed tensor-tractor bundles, but these may be expanded into collections of (individually non-invariant) tensor obstructions if desired.

7. CONFORMAL DIFFERENTIAL GEOMETRY

A conformal structure (of signature $(p, q)$) $(M^n, c)$, $n \geq 3$, is a smooth manifold equipped with an equivalence class $c$ of signature $(p, q)$-metrics, where two metrics $g$ and $g'$ in $c$ are equivalent if there is some positive smooth function $\Omega$ such that $g' = \Omega^2 g$. That is, an equivalence class is a maximal set of metrics which are mutually pointwise homothetic on each tangent space.
Let $g$ and $g' = \Omega^2 g$ be conformally related metrics, and define a 1-form $\Upsilon_a := \Omega^{-1} \nabla_a \Omega$ from their conformal factor. The two metrics determine Levi-Civita connections $\nabla$, $\nabla'$, which are related by, for $u_b \in \mathcal{E}_b$,

$$\nabla'_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a + g_{ab} \Upsilon_c u_c. \quad (22)$$

7.0.1. **Curvature tensors arising in conformal geometry.** Given a metric $g \in \mathcal{C}$, letting $\nabla$ be its Levi-Civita connection, the (Riemannian) curvature is defined as usual by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) v^c = R_{abcd} v^d. \quad (23)$$

This can be decomposed into the totally trace-free conformal Weyl tensor $\tilde{W}_{abcd}$ and a remaining part described by the symmetric conformal Schouten tensor $\tilde{P}_{ab}$, according to

$$R_{abcd} = \tilde{W}_{abcd} + 2 g_{ca} \tilde{P}_{bd} + 2 g_{db} \tilde{P}_{ac},$$

where $[\cdots]$ indicates the antisymmetrisation over the enclosed indices. We write $\tilde{J}$ for the trace $g^{ab} \tilde{P}_{ab}$. The Schouten tensor is a trace modification of the Ricci tensor $\text{Ric}_{bd} = R_{abad}$:

$$\text{Ric}_{ab} = (n - 2) \tilde{P}_{ab} + \tilde{J}_{ab}. \quad (24)$$

Under a conformal change of metric, one computes that the Weyl curvature $\tilde{W}_{abcd}$ is unchanged. Thus it is an invariant of the conformal structure $(M, c)$. (In dimension 3 this vanishes.)

7.1. **Conformal densities and connections thereon.** For our subsequent discussion it is convenient to take the positive $(2n)^{th}$ root of $(\Lambda^n TM)^2$ and we denote this $\mathcal{E}[1]$. Then for $w \in \mathbb{R}$ we denote by $\mathcal{E}[w]$ its $w^{th}$-power. Sections of $\mathcal{E}[w]$ will be described as *conformal densities* of weight $w$. Given any bundle $\mathcal{B}$ we shall write $\mathcal{B}[w]$ as a shorthand notation for $\mathcal{B} \otimes \mathcal{E}[w]$.

Now we consider a conformal manifold $(M, c)$. Each metric $g \in \mathcal{C}$ determines a metric (also denoted $g$) on $(\Lambda^n TM)^2$ and hence on its roots $\mathcal{E}[w]$, $w \in \mathbb{R}$. Moreover the Levi-Civita connection $\nabla$ of $g$ determines compatible, flat connections (also denoted $\nabla$) on $(\Lambda^n TM)^2$ and on its roots $\mathcal{E}[w]$, $w \in \mathbb{R}$.

The conformal class $c$ determines canonical sections $g_{ab}$ of $\mathcal{E}_{(ab)}[2]$ and $g^{ab}$ of $\mathcal{E}(ab)[-2]$.

7.2. **Conformal tractor calculus.** In analogy with the case of projective manifolds, on a conformal manifolds there is a canonical tractor bundle equipped with metric and connection $\mathbb{R}$, with historical precedents as in the projective case. This is a rank-$(n + 2)$ bundle that is closely related to $TM$.

To construct this bundle (in fact we construct the dual of the bundle usually considered), we consider the jet exact sequence at 2-jets of the density bundle $\mathcal{E}[1]$: $0 \to \mathcal{E}_{(ab)}[1] \to J^2(\mathcal{E}[1]) \to J^1(\mathcal{E}[1]) \to 0,$

where $(\cdots)$ indicates symmetrisation over the enclosed indices. Note we have a bundle homomorphism $\mathcal{E}_{(ab)}[1] \to \mathcal{E}[-1]$ given by complete contraction with the
conformal class $g^{ab}$. This is split via $\rho \mapsto \frac{1}{n} \rho g^{ab}$ and so the conformal structure decomposes $E_{(ab)[1]}$ into the direct sum $E_{(ab)[1]} \oplus E[-1]$. Clearly then the c-tracefree bundle $E_{(ab)[1]}$ is a smooth subbundle of $J^2(E[1])$, and we define $E_\alpha$ to be the quotient bundle. That is, the conformal cotractor bundle $\tilde{T}^*$ or $E_\alpha$ is defined by the exact sequence

$$0 \to E_{(ab)[1]} \to J^2(E[1]) \to E_\alpha \to 0.$$  \hfill (25)

The jet exact sequence at 2-jets, and the corresponding sequence at 1-jets, viz

$$0 \to E_a[1] \to J^1(E[1]) \to E[1] \to 0,$$

determine a composition series for $E_\alpha$ which we can summarise via a self-explanatory semi-direct sum notation $E_\alpha = E[-1] \triangleright E_a[1] \triangleright E[1]$. A choice of metric $g$ from the conformal class determines canonical splittings of this exact sequence, and hence a canonical identification of $E_\alpha$ with the direct sum $E[-1] \oplus E_a[1] \oplus E[1]$.

The conformal cotractor bundle has an invariant metric $\tilde{h}^{ab}$ of signature $(p+1, q+1)$, the (conformal) tractor metric, and an invariant connection $\tilde{\nabla}_a$ preserving $\tilde{h}^{ab}$, the conformal tractor connection. If for a metric $g$ from the conformal class $V_\alpha, V_\beta \in E_\alpha$ are given by

$$V_\alpha \overset{g}{=} (\tau \mid \mu_a \mid \sigma), \quad V_\beta \overset{g}{=} (\tau \mid \mu_b \mid \sigma),$$

then the tractor metric is given by

$$\tilde{h}^{ab}V_\alpha V_\beta = g^{ab}\mu_a\mu_b + \sigma\tau + \tau\sigma,$$  \hfill (26)

and the tractor connection is given by

$$\tilde{\nabla}_a V_\beta \overset{g}{=} \begin{pmatrix} \nabla_a \tau - \tilde{P}_{ab}g^{be}\mu_c \\ \nabla_a \mu_b + g_{ab}\tau + \tilde{P}_{ab}\sigma \\ \nabla_a \sigma - \mu_a \end{pmatrix}^T.$$  \hfill (27)

8. The Projective-Conformal Connection

Given a metric $g$, with Levi-Civita connection $\nabla$, one may compare the conformal geometry of the class $[g]$ and the projective geometry of the class $[\nabla]$. The main result of this section is that, when $g$ is Einstein, the connection between these two geometries is particularly simple. These ideas motivated our original (re-)construction of Armstrong’s projective tractor sub-metrics, Theorem 3.1.

For use in this section, we note the relationships between corresponding conformal and projective curvature tensors of a metric $g_{ij}$ which is Einstein. Let $\lambda$ be the constant such that $\text{Ric}_{ij} = \lambda g_{ij}$.

**Lemma 8.1.** The conformal Schouten tensor of $g$ and projective Schouten tensor of $\nabla$ simplify to, respectively, $\tilde{P}_{ij} = \frac{1}{2(n-1)}\lambda g_{ij}$ and $P_{ij} = \frac{1}{n-1}\lambda g_{ij}$. In particular, $\tilde{P}_{ij} = \frac{1}{2}P_{ij}$. 


Proof. This follows from the definitions (24) and (3).

For the following see e.g. [27, Corollary 2.6].

**Proposition 8.2.** The conformal Weyl curvature of \( g \) is the same as the projective Weyl curvature of its Levi-Civita connection \( \nabla \).

**Proof.** By Lemma 8.1, and the definitions (23) and (2),

\[
\tilde{W}_{ijkl} = R_{ijkl} - 2g_{k[i} \tilde{P}_{j]l} - 2g_{l[j} \tilde{P}_{i]k} = R_{ijkl} - \frac{2}{n-1} \lambda g_{k[i} g_{j]l};
\]

\[
W_{ij}^k = R_{ij}^k - 2\delta_i^k \tilde{P}_j^l = R_{ij}^k - \frac{2}{n-1} \lambda \delta_i^k g_{j]l}.
\]

These agree up to the raising of an index of \( \tilde{W} \).

**Corollary 8.3.** On projective 3-manifolds \((M, p)\) the projective Weyl curvature sharply obstructs the existence of a Einstein Levi-Civita connection in the projective class.

**Proof.** In dimension 3 the conformal Weyl tensor is identically zero.

**Remark 8.4.** One can also see this from the fact that in dimension 3, a metric is Einstein if and only if it has constant sectional curvature. Hence any connection projectively related to the Levi-Civita connection of an Einstein metric is also projectively related to a flat connection.

8.1. **Conformal and projective Einstein models.** By Proposition 8.2, an Einstein metric is conformally flat precisely if its Levi-Civita connection is projectively flat. Such a metric has constant sectional curvature.

The model conformally flat manifold of dimension \( n \) and its tractor bundle are constructed as follows [16]. Let \( \tilde{V} \) be a vector space of dimension \( (n + 2) \), equipped with a nondegenerate symmetric bilinear form \( \langle \cdot, \cdot \rangle \) of indefinite signature \( (p + 1, q + 1) \). Let

\[
C = \{ w \in \tilde{V} : \langle w, w \rangle = 0 \}
\]

be the null cone of \( \tilde{V} \). The ray projectivisation \( \mathbb{P}C \) is diffeomorphic to \( S^p \times S^q \) and the bilinear form \( \langle \cdot, \cdot \rangle \) descends to a well-defined, flat conformal equivalence class \( c \) of metrics on \( \mathbb{P}C \). For any open, ray-closed subset \( U \subseteq C \), choices of metric \( g \in c|_{\mathbb{P}U} \) are in bijection with sections of the \((\mathbb{R}^+)\)-principal bundle \( U \to \mathbb{P}U \).

The conformal tractor bundle may be identified with the space \( \tilde{V} \), and parallel tractors with elements of \( \tilde{V} \).

Einstein (pseudo-)metrics on regions of \( \mathbb{P}C \) are in bijection with choices of non-vanishing vector \( I \in \tilde{V} \). Indeed, such a vector determines a section of some part of \( C \to \mathbb{P}C \), via the intersection of the hyperplane \( \langle I, \cdot \rangle = 1 \) with the null cone.

Einstein metrics of positive, zero and negative curvature correspond respectively to timelike, null and spacelike vectors \( I \).
The model projectively flat manifold of dimension $n$ is the ray projectivisation of a dimension-$(n + 1)$ vector space $V$. The total space of the projective tractor bundle may be identified with the space $V$.

Thus, for a choice $I$ of nonvanishing vector, determining a choice of Einstein metric in the model conformal geometry, the projective tractor geometry of this Einstein metric is naturally embedded in the conformal tractor bundle $\tilde{V}$ as the hyperplane $I^\perp$.

8.2. A relationship of tractor bundles. Let $g_{ab}$ be an Einstein metric, with Levi-Civita connection $\nabla$. In this section we compare the conformal tractor geometry of $[g_{ab}]$ and the projective tractor geometry of $[\nabla]$. We obtain an elegant curved analogue of Subsection 8.1.

The metric induces canonical trivialisations of the density bundles $\mathcal{E}(w)$ and $\mathcal{E}[w]$, so that we may unambiguously omit all weights. It thus also induces canonical co-tractor bundle isomorphisms, via the choices $g \in [g]$ and $\nabla \in [\nabla]$,

$$\mathcal{E}_a \cong \mathcal{E} \oplus \mathcal{E}_a \oplus \mathcal{E}; \quad \mathcal{E}_A \cong \mathcal{E}_a \oplus \mathcal{E}.$$

Lemma 8.5 ([3],[17]). The tractor $I^\alpha := \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{n} \tilde{J} \end{pmatrix}$ is parallel.

Thus the kernel of $I^\alpha$ (a rank-$(n + 1)$ subbundle of the co-tractor bundle $\mathcal{E}_a$) is preserved by the tractor connection.

Proposition 8.6. Define a bundle inclusion $\iota : T^* \to \tilde{T}^*$ of the projective into the conformal co-tractor bundle by, for a projective co-tractor $U_A = (\mu_a \mid \sigma)$,

$$\iota_a^A U_A = \begin{pmatrix} \frac{1}{n} \tilde{J} \sigma & | & \mu_a & | & \sigma \end{pmatrix}.$$ 

(1) The image of $\iota$ is the kernel of the parallel tractor $I^\alpha$; thus, $\iota_a^A I^\alpha = 0$.

(2) The bundle inclusion $\iota$ is connection-preserving.

Proof. (1) Calculate the pairing of $I^\alpha$ with a tractor in the image of this inclusion: for sections $\sigma$ of $\mathcal{E}$ and $\mu_a$ of $\mathcal{E}_a$, the co-tractor $U_A = (\mu_a \mid \sigma)$ indeed satisfies

$$\iota_a^A U_A I^\alpha = \frac{1}{n} \tilde{J} \sigma \cdot 1 - \frac{1}{n} \tilde{J} \cdot \sigma = 0.$$ 

(2) Compare the restriction to this sub-bundle of the conformal tractor connection ([27]) with the projective tractor connection ([10]) on its preimage:
for sections $\sigma$ of $\mathcal{E}$ and $\mu_a$ of $\mathcal{E}_a$,

$$\nabla_b \left( \frac{1}{n} \tilde{J} \sigma \mid \mu_a \mid \sigma \right) = \left( \begin{array}{c} \nabla_b \left( \frac{1}{n} \tilde{J} \sigma \right) - \tilde{P}_{bc} g^{ca} \cdot \mu_a \\ \nabla_b \mu_a + g_{ba} \cdot \frac{1}{n} \tilde{J} \sigma + \tilde{P}_{ba} \cdot \sigma \\ \nabla_b \sigma - \mu_b \end{array} \right)^T,$$

$$= \left( \begin{array}{c} \frac{1}{n} \tilde{J} \nabla_b \sigma - \mu_b \\ \nabla_b \mu_a + 2 \tilde{P}_{ab} \sigma \\ \nabla_b \sigma - \mu_b \end{array} \right);$$

$$\nabla_b (\mu_a \mid \sigma) = (\nabla_b \mu_a + \tilde{P}_{ab} \sigma \mid \nabla_b \sigma - \mu_b);$$

for the former calculation using the Einstein properties that $\tilde{P}_{ab} = \frac{1}{n} \tilde{J} g_{ab}$ and that $\tilde{J}$ is constant.

Using that $\tilde{P}_{ab} = 2 \tilde{P}_{ab}$ (Lemma 8.1) to compare the two right-hand sides, we obtain, as required, that for $U_A = (\mu_a \mid \sigma)$,

$$\nabla_b (\iota_A U_A) = \iota_A \left( \nabla_b U_A \right).$$

□

**Proposition 8.7.** The projective tractor sub-metric $h^{AB}$ (Theorem 3.1) is the pullback under the inclusion $\iota$ of the conformal tractor metric $\tilde{h}^{\alpha\beta}$. That is, $h^{AB} = \iota^A_A \iota^B_B \tilde{h}^{\alpha\beta}$.

**Proof.** Define two projective co-tractors $U_A, U_B$, by, in the splitting $\nabla$,

$$U_A = (\mu_a \mid \sigma), \quad U_B = (\tilde{\mu}_a \mid \tilde{\sigma}).$$

The conformal inner product $\tilde{h}^{\alpha\beta}$ (26) of their images under $\iota$ is then

$$\tilde{h}^{\alpha\beta} (\iota_A U_A) (\iota_B U_B) = g^{ab} (\mu_a \tilde{\mu}_b) + \frac{2}{n} \tilde{J} \sigma \tilde{\sigma}.$$

By Lemma 8.1, this is exactly (13). □

**References**

[1] S. Armstrong, *Projective holonomy. I. Principles and properties*, Ann. Global Anal. Geom., 33 (2008), 47–69.
[2] A. Avez, *Characteristic classes and Weyl Tensor: Applications to general relativity*, Proc. Nat. Acad. Sci., 66 (1970) 265–268.
[3] T.N. Bailey, M.G. Eastwood, and A.R. Gover, *Thomas’s structure bundle for conformal, projective and related structures*, Rocky Mountain J. Math. 24 (1994), 1191–1217.
[4] T. Branson, and A.R. Gover, *Pontrjagin forms and invariant objects related to the Q-curvature*, Commun. Contemp. Math., 9 (2007), 335–358.
[5] R. Bryant, M.G. Eastwood, and M. Dunajski, *Metrizability of two-dimensional projective structures*, J. Differential Geom. 83 (2009), 465–499.
[6] A. Čap, and A.R. Gover, *Tractor calculi for parabolic geometries*, Trans. Amer. Math. Soc. 354 (2002), 1511–1548.
[7] A. Čap, A.R. Gover, and M. Hammerl, *Projective BGG equations, algebraic sets, and compactifications of Einstein geometries*, J. London Math. Soc. (2), (2012), doi: 10.1112/jlms/jds002.

[8] A. Čap, A.R. Gover, and M. Hammerl, *Holonomy reductions of Cartan geometries and curved orbit decompositions*, arXiv:1103.4497.

[9] A. Čap, A.R. Gover, and M. Hammerl, *Normal BGG solutions and polynomials*, Internat. J. Math., 23 (2012), DOI 10.1142/S0129167X12501170. arXiv:1201.0799.

[10] A. Čap, A.R. Gover, and H.R. Macbeth, *Einstein metrics in projective geometry*, arXiv:1207.0128.

[11] A. Čap, J. Slovák, *Parabolic Geometries I: Background and General Theory*, Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2009.

[12] Cartan, *Sur les variétés à connexion projective*, Bull. Soc. Math. France 52 (1924), 205–241.

[13] J.S. Case, *Sharp metric obstructions for quasi-Einstein metrics*, arXiv:1110.3010.

[14] S.-S. Chern, and J. Simons, *Characteristic forms and geometric invariants*, Ann. Math., 99 (1974) 48–69.

[15] M.G. Eastwood, and V. Matveev, *Metric connections in projective differential geometry* in “Symmetries and overdetermined systems of partial differential equations”, 339–350, IMA Vol. Math. Appl., 144, Springer, New York, 2008.

[16] A.R. Gover, *Almost Einstein and Poincare-Einstein manifolds in Riemannian signature*, J. Geometry and Physics, 60, (2010), 182–204.

[17] A. R. Gover and P. Nurowski, *Obstructions to conformally Einstein metrics in n dimensions*, J. Geom. Phys., 56, (2006), 450–484.

[18] A. R. Gover and P. Nurowski, *Calculus and invariants on almost complex manifolds, including projective and conformal geometry*, arXiv:1208.6448.

[19] G.S. Hall, D.P. Lonie, *The principle of equivalence and projective structure in spacetimes*, Classical Quantum Gravity, 24 (2007), 14, 3617–3636.

[20] G.S. Hall, D.P. Lonie, *Projective equivalence of Einstein spaces in general relativity*, Classical Quantum Gravity, 26 (2009), 125009, 10 pp.

[21] V. Kiosak, and V. Matveev, *Complete Einstein metrics are geodesically rigid*, Comm. Math. Phys., 289, (2009), 383–400.

[22] C. Kozameh, E.T. Newman, K.P. Tod, *Conformal Einstein spaces*, Gen. Relativity Gravitation, 17 (1985) 343–352.

[23] F. Leitner, *Conformal Killing forms with normalisation condition*, Rend. Circ. Mat. Palermo (2) Suppl. No. 75 (2005) 279–292.

[24] R. Lionville, *Sur les invariants de certaines équations différentielles et sur leurs applications*, Journ. de l'Ecole Polytechnique, Cah.59 (1889) 7–76.

[25] V.S. Matveev, *Geodesically equivalent metrics in general relativity*, J. Geom. Phys., 62 (2012), 675691.

[26] J. Mikes, *Geodesic mappings of affine-connected and Riemannian spaces*, Journ. Math. Sci. 78 (1996) 311–333.

[27] P. Nurowski, *Projective vs metric structures*, arXiv:1003.1469.

[28] R.S. Palais, Seminar on the Atiyah-Singer index theorem. With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57 Princeton University Press, Princeton, N.J. 1965 x+366 pp.

[29] N.S. Sinjukov, *geodesic mappings of Riemannian spaces* (Russian), “Nauka,” Moscow 1979.

[30] T.Y. Thomas, *Announcement of a projective theory of affinely connected manifolds*, Proc. Nat. Acad. Sci., 11 (1925), 588–589.
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