Separability of Maxwell equation in Rotating black hole spacetime and its Geometric aspects

Norihiro Tanahashi [Kyushu U]

with
Tsuyoshi Houri [NIT, Maizuru College]
Yukinori Yasui [Setsunan U]
Recently, a progress was made about Maxwell field perturbation on Kerr BH spacetime and its separability. We try to find the geometric origin of this bland-new technique.

◆ Perturbations of Kerr black hole
◆ Recent breakthrough on separability
◆ Construction of commuting operators
◆ Summary
Perturbations of Kerr black hole

• Scalar field, Maxwell field, Metric perturbations on Kerr BH
• Important, but difficult

  Complicated PDE, many physical d.o.f. coupled with each other

• Teukolsky equation based on Newman-Penrose formalism

  EoM $\rightarrow$ decoupled PDEs that admit separation of variables
  $\rightarrow$ set of ODEs

[Teukolsky '72]
Teukolsky eq. for Maxwell perturbations on 4D Kerr BH

\[ ds^2 = \frac{1}{\Sigma} \left\{ -\Delta [dt - a \sin^2 \theta d\phi]^2 + \sin^2 \theta [(r^2 + a^2)d\phi - adt]^2 \right\} + \sum \left[ \frac{dr^2}{\Delta} + d\theta^2 \right] \]

\[ = -2\ell_{(\mu}n_{\nu)} + 2m_{(\mu}m_{\nu)} \]

\[ \left\{ \Delta = r^2 + a^2 - 2Mr, \Sigma = r^2 + a^2 \cos^2 \theta \right\} \]

\[ F_{\mu\nu} = 2 \left[ \phi_1 (n_{[\mu}l_{\nu]} + m_{[\mu}m_{\nu]} + \phi_2 l_{[\mu}m_{\nu]} + \phi_0 m_{[\mu}n_{\nu]} \right] + c.c. \]

\[ \psi_+ = \phi_0, \quad \psi_- = \bar{\rho}^2 \phi_2, \quad \psi_s = e^{i\omega t + im\phi} R_s(r) S_s(\theta) \]

Maxwell equation \( \nabla^\mu F_{\mu\nu} = 0 \rightarrow \) Teukolsky equation

\[ \frac{1}{\Delta^s} \frac{d}{dr} \left[ \Delta^{s+1} dR_s \right] + \left[ \frac{K(K - 2isr) + 2isMK}{\Delta} - 4is\omega r - \Lambda - (a\omega + m)^2 + m^2 \right] R_s = 0 \]

\[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{dS_s}{d\theta} \right] + \left[ (a\omega \cos \theta + s)^2 - \frac{(m + s \cos \theta)^2}{\sin^2 \theta} + s(1-s) + \Lambda \right] S_s = 0 \]

\( s = \pm 1, \quad \rho = r + ia \cos \theta, \quad K = (r^2 + a^2)\omega - am \)

✓ \( \phi_0 \) and \( \phi_2 \) are solved by Teukolsky eq. (while PDE for \( \phi_1 \) cannot be separated)

✓ Works only in 4D: separation of variables NOT achieved in higher dim.
Recent breakthrough on Separability

◆ Lunin’s new ansatz \^[Lunin ’17]

\[
\begin{align*}
\ell^\mu A_\mu &= G_+(r)\ell^\mu \partial_\mu \Psi \\
n^\mu A_\mu &= G_-(r)n^\mu \partial_\mu \Psi \\
m^\mu A_\mu &= F_+(\theta)m^\mu \partial_\mu \Psi \\
\bar{m}^\mu A_\mu &= F_-(\theta)\bar{m}^\mu \partial_\mu \Psi
\end{align*}
\]

✓ $G_\pm(r), F_\pm(\theta)$ chosen to achieve separation of variable
✓ Separable equations for all the variables $[\Psi = e^{i\omega t + im\phi} R(r) S(\theta)]$
✓ Works even in higher dimensions

◆ Covariant version of Lunin’s ansatz \^[Krtouš, Frolov, Kubizňák ’18]

\[
A^\mu = B^{\mu\nu} \nabla_\nu Z \quad \text{with} \quad B^{\mu\nu} = (g_{\mu\nu} - \beta \ h_{\mu\nu})^{-1}
\]

$h_{\mu\nu}$: Principal tensor = non-degenerate closed conformal Killing-Yano tensor

= “square root” of Killing tensor $K_{\mu\nu} = (\star h)_{\mu}^\rho (\star h)_{\rho\nu}$

\[
\begin{align*}
\text{Killing tensor} \quad K_{\mu\nu} (\nabla_{(\mu} K_{\nu\rho)} = 0) &\approx \text{“Hidden symmetry” of spacetime:} \quad K_{\mu\nu} p^\mu p^\nu = \text{(constant of motion)} \\
\text{Killing vector} \quad \xi^\mu (\nabla_{(\mu} \xi_{\nu)} = 0) &\approx \text{Symmetry of spacetime:} \quad \xi^\mu p_\mu = \text{(constant of motion)}
\end{align*}
\]

Teukolsky’s ansatz

\[
\begin{align*}
\ell^\mu A_\mu &= \frac{2ia}{r} l^\mu \partial_\mu [e^{i\omega t + im\phi} g_+(r)f_+(\theta)] \\
n^\mu A_\mu &= \frac{2ia}{r} n^\mu \partial_\mu [e^{i\omega t - im\phi} g_-(r)f_- (\theta)] \\
m^\mu A_\mu &= -\frac{2ia}{ia \cos \theta} m^\mu \partial_\mu [e^{i\omega t + im\phi} f_+(\theta)g_+(r)] \\
\bar{m}^\mu A_\mu &= -\frac{2ia}{ia \cos \theta} \bar{m}^\mu \partial_\mu [e^{i\omega t + im\phi} f_- (\theta)g_-(r)]
\end{align*}
\]
**Covariant ansatz** [Krtouš, Frolov, Kubizňák ’18]

- Most-general $2N$ dim. spacetime admitting $h_{\mu\nu} = \text{Kerr-NUT-(A)dS}$

\[
d s^2 = \sum_{\mu=1}^{N} \left[ \frac{U_{\mu}}{X_{\mu}} \, d x_{\mu}^2 + \frac{X_{\mu}}{U_{\mu}} \left( \sum_{j=0}^{N-1} A^{(j)}_{\mu} \, d \psi_{j} \right)^2 \right]
\]

- $x^{\mu} = \{r, y(\theta), \ldots\}$ nontrivial directions
- $\psi^{i} = \{\tau, \phi, \ldots\}$ Killing directions

\[
\begin{align*}
\text{Maxwell equation} & \quad C_0 Z \equiv (\Box + 2 \beta \xi_k B^{kn} \nabla_n) \, Z = 0 \\
\text{Lorenz gauge} & \quad C Z \equiv \nabla_m (B^{mn} \nabla_n Z) = 0
\end{align*}
\]

\[
A^\mu = B^{\mu\nu} \nabla_\nu Z
\]

- Both equations given by commuting operators $[C_k, C_l] = [C_k, L_l] = [L_k, L_l] = 0$
- $Z$ is given by simultaneous eigenfunctions of $C_k, L_k$

\[
\begin{align*}
C_k Z &= C_k Z \\
L_k Z &= L_k Z
\end{align*}
\]

$Z = Z(\beta; C_0, C_1, \ldots, C_{N-1}, L_0, \ldots, L_{N-1})$

- Eigenvalues $C_k, L_k \approx \text{Separation constants}$
- **?: What is the geometric origin & covariant form of the commuting operators?**
Construction of commuting operators

1. Express perturbation equations in terms of gauged Laplacian:
   \((\nabla^\mu - iqA^\mu)(\nabla_\mu - iqA_\mu) + \cdots = 0\)

2. Express it as \(\hat{\Box}\psi = 0\) by applying the Eisenhart-Duval lift:
   \(g_{\mu\nu} \rightarrow \hat{g}_{AB}\)
   [Eisenhart 1928, Duval+ 1985]

3. It turns out that the geodesic equation for the lifted metric \(\hat{g}_{AB}\) admits
   separation of variables completely. [Benenti ’91]
   Then, there exists the Killing tensors \(\hat{K}_{AB}\) s.t.
   \(\left\{\hat{g}_{AB} p^A p^B, \hat{K}_{AB} p^A p^B\right\} = 0\).

4. By quantization \(p_\mu \rightarrow -i\nabla_\mu\) it follows
   \[\hat{g}^{AB} \hat{\nabla}_A \hat{\nabla}_B, \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B\] \(= \frac{4}{3} \nabla_A \left(\hat{K}^{[A} \hat{R}^{B]C}\right) \hat{\nabla}_B\).

5. Then, if the anomaly-free condition \(\nabla_A \left(\hat{K}^{[A} \hat{R}^{B]C}\right) = 0\) is satisfied,
   it follows \(\left[\hat{g}^{AB} \hat{\nabla}_A \hat{\nabla}_B, \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B\right] = 0\). [Carter 1977]

6. The commuting operators s.t. \(\left[\hat{\Box}, \hat{C}_k\right] = 0\) is then given by \(\hat{C}_k = \hat{\nabla}_A \hat{K}^{AB} \hat{\nabla}_B\).
Construction of commuting operators

- Lunin’s equation \((\Box + 2\beta \xi_k B^{kn} \nabla_n) Z = 0\)
- Teukolsky equation \((\Box + f_1^{\mu} \nabla_\mu + f_2) \psi = 0\)

Both given by gauged wave equation \((\nabla^\mu - i q A^\mu)(\nabla_\mu - i q A_\mu) + \cdots = 0\)

Can be expressed as wave operator in higher dimensions \(\hat{\Box} \psi = 0\) by lifting the metric \(g_{\mu\nu}\) to a higher-dimensional one \(\hat{g}_{\mu\nu}\) [Eisenhart 1928, Duval+ 1985]

\[
ds^2 = \hat{g}_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + 2q A_\mu dx^\mu du + 2du dv - 2V du^2, \quad \hat{\psi}(x^A) = \psi(x^\mu)e^{i uv} \]

\[\hat{\Box} \psi = e^{i uv} (\Box_A - 2V) \psi \quad \Box_A = -2i q A^\mu \nabla_\mu - q^2 A^\mu A_\mu - i q \nabla_\mu A^\mu\]

- It turns out that the geodesic eq. for the uplifted metric \(\hat{g}_{AB}\) admits separation of variable completely. [Benenti '91]

- Separation of variable for geodesics
  \(\rightarrow \exists \) Constants of motion
  \(\rightarrow \exists \) Killing tensor satisfying \(\{\hat{H}, \hat{K}_{AB} p^A p^B\} = 0\)
  \(\rightarrow \) Commuting operators \([\Box, \hat{\nabla}_A (\hat{K}^{AB} \hat{\nabla}_B)] = 0\) by quantization \(p_\mu \rightarrow -i \hat{\nabla}_\mu\)
  if anomaly-free condition \(\nabla_A (\hat{K}_{C, [A} \hat{R}^{B]C}) = 0\) is satisfied.
Construction of commuting operators

ex.) Teukolsky eq. for 4D Kerr BH

\[
\begin{align*}
\left[ \frac{1}{\Delta^s} \frac{\partial}{\partial r} \left( \Delta^{s-1} \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{K(K - 2isr) + 2isMK}{\Delta} \right. \\
- (a\omega + m)^2 + m^2 + (a\omega \cos \theta + s)^2 - \left( \frac{m + s \cos \theta}{\sin^2 \theta} \right)^2 + s(1 - s) \left. \right] \psi = 0
\end{align*}
\]

\[
(\Box - 2iqA^\mu \nabla_\mu - q^2 A^\mu A_\mu - iq \nabla_\mu A^\mu - 2V) \psi = 0
\]

\[
ds^2 = \hat{g}_{AB} dx^A dx^B = g_{\mu\nu} dx^\mu dx^\nu + 2qA_\mu dx^\mu du + 2du dv - 2V du^2
\]

**Metric**

\[
\begin{align*}
\hat{g}^{\mu\nu} &= g^{\mu\nu} \\
\hat{g}^{\mu u} &= 0 \\
\hat{g}^{uv} &= g^{rr} \frac{is(M - r)}{\Delta} \\
\hat{g}^{\theta\nu} &= g^{rr} \frac{is(M - r)}{\Delta^2} + g^{\theta\theta} \frac{s \cos \theta}{\sin^2 \theta} \\
\hat{g}^{tu} &= g^{rr} is \left( \frac{r}{\Delta} + \frac{M(a^2 - r^2)}{\Delta^2} \right) + g^{\theta\theta} (-sa \cos \theta) \\
\hat{g}^{u\nu} &= 1 \\
\hat{g}^{vv} &= g^{\theta\theta} s^2 \cot^2 \theta
\end{align*}
\]

**Killing tensor**

\[
\begin{align*}
\hat{K}^{\mu\nu} &= K_{\text{Kerr}}^{\mu\nu} \\
\hat{K}^{iu} &= 0 \\
\hat{K}^{rv} &= K_{rr}^{rv} \frac{is(M - r)}{\Delta} \\
K^{\theta\nu} &= 0 \\
K^{\phi\nu} &= K_{rr}^{\phi\nu} \frac{is(M - r)}{\Delta^2} + K^{\theta\theta} \frac{s \cos \theta}{\sin^2 \theta} \\
K^{tv} &= K_{rr}^{tv} \frac{is \left( \frac{r}{\Delta} + \frac{M(a^2 - r^2)}{\Delta^2} \right)}{\Delta^2} + K^{\theta\theta} (-sa \cos \theta) \\
K^{uv} &= 0 \\
K^{vv} &= K^{\theta\theta} s^2 \cot^2 \theta
\end{align*}
\]

\[
\left\{ \hat{g}^{AB} p_{APB}, \hat{K}_{AB} p^A p^B \right\} = 0
\]

\[
[\Box, \hat{\nabla}_A (\hat{K}^{AB} \hat{\nabla}_B)] = 0 \quad \text{if} \quad \nabla_A \left( K_{[C}^{[A} \hat{R}^{B]C]} \right) = 0
\]

\[
\nabla_A \left( \hat{K}_{C}^{[A} \hat{R}^{B]C]} \right) = 0 \quad \text{is indeed satisfied, hence} \quad \nabla_A \hat{K}^{AB} \nabla_B \quad \text{becomes a commuting op.}
\]

\[
\text{This procedure works also for 4D Kerr-NUT-AdS spacetime}
\]
Construction of commuting operators

ex.) $D$-dim. Kerr-NUT-AdS spacetime ($D = 2n + \varepsilon$)

$$g^{-1} = \sum_{\mu=1}^{n} (X_\mu \otimes X_\mu + X_\mu \otimes X_\mu) + \epsilon X_0 \otimes X_0 = \sum_{\mu=1}^{n} g^{\mu\mu} \frac{\partial}{\partial x_\mu} \right)^2 + \sum_{k,l=0}^{n-1+\epsilon} g^{k\ell} \frac{\partial}{\partial \psi_k} \frac{\partial}{\partial \psi_\ell}$$

$$g^{\mu\mu} = Q_\mu, \quad g^{k\ell} = \sum_{\mu=1}^{n-1+\epsilon} \zeta^{k\ell}_{(\mu)}(x_\mu) Q_\mu, \quad \zeta^{k\ell}_{(\mu)} = \frac{(-1)^{k+\ell} x_{2n-k-\ell}}{X_\mu^2} \delta_{nk} \delta_{n\ell}$$

Perturbation eq.  \( (\Box + 2\beta \xi^k B^{kn} \nabla_n) Z = 0 \)

\( \rightarrow \) gauge field $q A^a = 2i \beta \xi_b B^{ba}$

Lifted metric

$$\hat{g}^{\mu\mu} = g^{\mu\mu}, \quad \hat{g}^{k\ell} = g^{k\ell}, \quad \hat{g}^{\mu\nu} = -q g^{\mu\mu} A_\mu, \quad \hat{g}^{k\nu} = -q g^{k\ell} A_\ell, \quad \hat{g}^{\nu\nu} = -i q \text{div} A, \quad \hat{g}^{\mu\nu} = 1$$

$$\Leftrightarrow \hat{g}^{\mu\mu} = Q_\mu, \quad \hat{g}^{AB} = \sum_{\mu=1}^{n} \zeta^{AB}_{(\mu)}(x_\mu) \sigma_j(\hat{x}_\mu) Q_\mu \quad (A = B \neq \mu)$$

By Benenti’s construction, the Killing tensor $\hat{K}^{(j)AB}$ of the lifted metric $\hat{g}^{AB}$ is given by

$$\hat{K}^{\mu\mu}_{(j)} = \sigma_j(\hat{x}_\mu) Q_\mu, \quad \hat{K}^{AB}_{(j)} = \sum_{\mu=1}^{n} \zeta^{AB}_{(\mu)}(x_\mu) \sigma_j(\hat{x}_\mu) Q_\mu \quad (A = B \neq \mu)$$

The anomaly-free condition $\nabla_A \left( \hat{K}^{[A}_{(j) B^{C]} C \right) = 0$ turns out to be satisfied by $\hat{K}^{(j)AB}$, hence the operator $\hat{\nabla}_A(\hat{K}^{AB} \hat{\nabla}_B)$ commutes with the Laplacian $\hat{\Box} : [\hat{\Box}, \hat{\nabla}_A(\hat{K}^{AB} \hat{\nabla}_B)] = 0$

\( \checkmark \) The operator $\hat{\nabla}_A(\hat{K}^{AB} \hat{\nabla}_B)$ coincides with the commuting operators $\mathcal{C}_k$ up to (Killing vector)$^\mu \nabla_\mu$
Summary

✓ New ansatz for Maxwell perturbations on Kerr BH

✓ EoMs given by commuting operators ➞ Separability for all variables

◆ Tried to give geometric interpretation to the commuting operators
  
  • Master eq. = scalar field eq. with gauged wave operator
    
    = scalar eq. with (non-gauged) wave op. in higher dimensions
  
  • Uplifted higher-dimensional metric possesses Killing tensors
  
  • This Killing tensor generates commuting operators  \([\Box, \hat{\nabla}_A (\hat{K}^{AB} \hat{\nabla}_B)] = 0\)
  
  • Procedure above works for Teukolsky eq. and also Lunin’s eq.

■ Future tasks

  • Uplifted spacetimes corresponding to Teukolsky eq. and Lunin’s eq. are apparently different. What is the essential difference?
  
  • Can we apply this procedure to gravitational perturbations in higher D?