Case’s Eigenvalues by Markel’s Matrix

Manabu Machida*
Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

Abstract
A method of computing Case’s eigenvalues is proposed. The eigenvalues are obtained as eigenvalues of a tridiagonal matrix.

1 Introduction
In this paper we consider eigenvalues in Case’s method [1] of solving the radiative transport equation. For isotropic scattering, discrete eigenvalues $\nu$ of Case’s method are given by solutions to $1 - c\nu\tanh^{-1}(1/\nu) = 0$, where $c$ is a constant [9]. In the general case of anisotropic scattering, the largest eigenvalue can be found by Holte’s expansion [2, 3, 4, 5]. The behavior of roots has been intensively studied [6, 7, 5]. However, we have many discrete eigenvalues when scattering is highly anisotropic such as the Henyey-Greenstein model [8]. Then, finding discrete eigenvalues as roots is not easy. Here we compute eigenvalues making use of a tridiagonal matrix, which we refer to as Markel’s matrix $B^m$ [10, 11]. All eigenvalues are obtained by one diagonalization.

2 Case’s method
Let $u(x, \hat{s})$ be the specific intensity at position $x \in \mathbb{R}$ in direction $\hat{s} \in \mathbb{S}^2$. Let $\mu$ be the cosine of the polar angle with respect to the positive $z$ axis, and $\varphi$ be the azimuthal angle. The parameter $c > 0$. The (time-independent) radiative transport equation is given by

$$\mu \frac{\partial}{\partial x} u(x, \hat{s}) + u(x, \hat{s}) = c \int_{\mathbb{S}^2} f(\hat{s} \cdot \hat{s}') u(x, \hat{s}') \, d\hat{s}' ,$$  \hspace{1cm} (1)

We give the phase function $f(\hat{s} \cdot \hat{s}')$ as a polynomial of order $N$:

$$f(\hat{s} \cdot \hat{s}') = \sum_{l=0}^{N} \sum_{m=-l}^{l} f_l Y_{lm}(\hat{s}) Y_{lm}^*(\hat{s}') .$$  \hspace{1cm} (2)

*Email: mmachida@umich.edu
In the case of isotropic scattering, \( N = 0 \) and \( f_0 = 1 \). In the Henyey-Greenstein model \[8\], \( N = \infty \) and \( f_l = g^l \), where \( g \in [-1, 1] \) is a constant.

We seek particular solutions of (1) of the form
\[ u(x, \hat{s}) = \Phi^m_v(\hat{s}) e^{-x/\nu}, \] (3)
where \[12\]
\[ \Phi^m_v(\hat{s}) = \phi^m(\nu, \mu) \left(1 - \mu^2\right)^{|m|/2} e^{im\varphi} \] (4)

We obtain
\[ \left(1 - \frac{\mu}{\nu}\right) \Phi^m_v(\hat{s}) = c \int_{S^2} f(\hat{s} \cdot \hat{s}') \Phi^m_v(\hat{s}') \, d\hat{s}'. \] (5)

We define
\[ \sigma_l = 1 - cf_l \Theta(N - l), \] (6)
where the step function \( \Theta(\cdot) \) is defined as \( \Theta(x) = 1 \) for \( x \geq 0 \) and \( = 0 \) for \( x < 0 \).

We introduce polynomials \( h^m_l(\nu) \) (\( 0 \leq l \) and \(|m| \leq N\)) as \[12, 13\]
\[ h^m_l(\nu) = \int_{-1}^{1} \phi^m(\nu, \mu) \left(1 - \mu^2\right)^{|m|/2} (-1)^m P^m_l(\mu) \, d\mu. \] (7)

We then have the recurrence relation,
\[ \nu(2l + 1)\sigma_l h^m_l(\nu) - (l - m + 1)h^m_{l+1}(\nu) - (l + m)h^m_{l-1}(\nu) = 0, \] (8)

with
\[ h^{|m|}_{|m|}(\nu) = (2|m|-1)!! = \frac{(2|m|)!}{2^{2|m|} |m|!}, \quad h^{|m|}_{|m|+1}(\nu) = (2|m|+1)\nu\sigma_{|m|}h^{|m|}_{|m|}(\nu). \] (9)

We also have
\[ h^{-|m|}_l(\nu) = (-1)^{|m|} \frac{(l - |m|)!}{(l + |m|)!} h^{|m|}_l(\nu). \] (10)

Let us define
\[ \gamma^m(\nu, \mu) = (-1)^m \sum_{l'=|m|}^{N} f_{l'}(2l' + 1) \frac{(l' - m)!}{(l' + m)!} P^m_{l'}(\mu) \left(1 - \mu^2\right)^{-|m|/2} h^m_{l'}(\nu). \] (11)

Singular eigenfunctions \( \phi^m(\nu, \mu) \) are obtained as
\[ \phi^m(\nu, \mu) = \frac{c \nu}{2} \mathcal{P} \gamma^m(\nu, \mu) + \lambda^m(\nu)(1 - \nu^2)^{-|m|/2} \delta(\nu - \mu), \] (12)

where
\[ \lambda^m(\nu) = 1 - \frac{c \nu}{2} \mathcal{P} \int_{-1}^{1} \gamma^m(\nu, \mu) \left(1 - \mu^2\right)^{|m|} \, d\mu. \] (13)
The continuous spectrum lies on \([-1, 1]\), and discrete eigenvalues \(\nu_j^m\) \((j = 0, \ldots, M - 1)\) are computed as roots of \(\Lambda^m(z), z \in \mathbb{C}\):

\[
\Lambda^m(z) = 0,
\]

where

\[
\Lambda^m(z) = 1 - \frac{cz}{2} \int_{-1}^{1} \frac{\gamma^m(z, \mu)}{z - \mu} (1 - \mu^2)^{|m|} d\mu.
\]

The number \(M\) of discrete eigenvalues depends on \(|m|\) and we have \([14, 12]\)

\[
M \leq N - |m| + 1.
\]

3 Expansion by spherical harmonics

Finding roots by (14) is not easy when scattering is highly anisotropic (\(N\) is large). Here we compute eigenvalues making use of Markel’s matrix \(B^m\) \([10, 11]\), which was first introduced in the method of rotated reference frames. Let us expand singular eigenfunctions as \([15, 16]\)

\[
\Phi^m_{\nu}(\hat{s}) = \sum_{l=|m|}^{\infty} c^m_{\nu l}(\nu) Y_{lm}(\hat{s}).
\]

The coefficients \(c^m_{\nu l}(\nu)\) are calculated using (12). We multiply \(Y^*_{lm}(\hat{s})\) on both sides of (5) and integrate them with respect to \(\hat{s}\).

\[
\int_{S^2} \left(1 - \frac{\mu}{\nu}\right) \Phi^m_{\nu}(\hat{s}) Y^*_{lm}(\hat{s}) d\hat{s} = \frac{c}{\sqrt{\sigma_l \sigma_l'}} \int_{S^2} \int_{S^2} f(\hat{s} \cdot \hat{s}') \Phi^m_{\nu}(\hat{s}') Y^*_{lm}(\hat{s}) d\hat{s} d\hat{s}'.
\]

We obtain

\[
c^m_{\nu l}(\nu) = \frac{1}{\nu} \sum_{\nu' = |m|}^{\infty} \left(\int_{S^2} \mu Y^*_{l'm}(\hat{s}) Y^*_{lm}(\hat{s}) d\hat{s}\right) c^m_{\nu' l}(\nu) = c f_l \Theta(N - l - 1) c^m_l(\nu).
\]

Hence we arrive at an eigenproblem:

\[
B^m \psi^m(\nu) = \nu \psi^m(\nu),
\]

where

\[
B^m_{ll'} = \frac{1}{\sqrt{\sigma_l \sigma_{l'}}} \int_{S^2} \mu Y^*_{l'm}(\hat{s}) Y^*_{lm}(\hat{s}) d\hat{s}
\]

\[
= \sqrt{\frac{l^2 - m^2}{(4l^2 - 1)\sigma_l \sigma_{l-1}}} \delta_{l, l-1} + \sqrt{\frac{(l + 1)^2 - m^2}{(4(l + 1)^2 - 1)\sigma_{l+1} \sigma_l}} \delta_{l, l+1},
\]

\[
\langle l | \psi^m(\nu) \rangle = \frac{1}{\sqrt{Z^m(\nu)}} \sqrt{\sigma_l} c^m_l(\nu),
\]

3
where $Z^m(\nu)$ is the normalization constant to guarantee $\langle \psi^m(\nu) | \psi^m(\nu) \rangle = 1$. Therefore, we can compute $\nu$ as eigenvalues of $B^m$. If $0 < c < 1$, $\nu$ are real values because $B^m$ is a real symmetric matrix. If $\nu$ is an eigenvalue, then $-\nu$ is also an eigenvalue [10]. Note that $\nu$ depends on $m$, and $|\nu^m_j| > 1$. In particular, discrete eigenvalues $\nu^m_j$ are obtained using $B^m$.

To numerically calculate discrete eigenvalues, we can diagonalize an $N_B \times N_B$ matrix $B^m$ with $N_B$ sufficiently large that discrete eigenvalues are obtained with desired accuracy. Figures 1 and 2 show eigenvalues for different $N_B$ in the cases of isotropic scattering and the Henyey-Greenstein model with $g = 0.9$ and $m = 0$, respectively. In Fig. 3 eigenvalues for $g = 0.9$ are plotted for different $m$. As is shown in [18], the value of the largest eigenvalue sharply drops as $m$ increases.

4 Conclusions

In this paper, we have proposed a novel method of computing Case’s discrete eigenvalues using a tridiagonal matrix $B^m$. Discrete eigenvalues are computed by one diagonalization. In Figs. 1 and 2 we see that discrete eigenvalues are
Figure 2: The Henyey-Greenstein model with $N = 9$, $g = 0.9$. We set $c = 0.9$. Eigenvalues $\nu$ for different $N_B$ are shown for $m = 0$. 
Figure 3: The Henyey-Greenstein model with $N = 9, g = 0.9$. We set $c = 0.9$. Eigenvalues $\nu$ for different $m$ are shown for $N_B = 501$. 
quite stable as $N_B$ increases; relatively small matrices give accurate discrete eigenvalues. For isotropic scattering $N = 0$, there are two discrete eigenvalues $\pm \nu_0$. If $c = 0.9$, then we find $
u_0 = 1.9032$ from the transcendental equation $1 - cv_0 \tanh^{-1}(1/\nu_0) = 0$. On the other hand, the largest eigenvalue of $B^0$ is obtained as 1.8257, 1.9027, 1.9032, 1.9032 for $N_B = 1, 3, 5, 501$, respectively.

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