Weakly binary expansions of dense meet-trees

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Abstract

We compute the domination monoid in the theory DMT of dense meet-trees. In order to show that this monoid is well-defined, we prove weak binarity of DMT and, more generally, of certain expansions of it by binary relations on sets of open cones, a special case being the theory DTR from [EK19]. We then describe the domination monoids of such expansions in terms of those of the expanding relations.

If asked what a tree is, a mathematician has a number of options to choose from. The graph theorist’s answer will probably contain the words “acyclic” and “connected”, while the set theorist may have in mind certain sets of sequences of natural numbers. In this paper we are instead concerned with lower semilinear orders: posets where the set of predecessors of each element is linearly ordered.

More specifically, a meet-tree is a lower semilinear order $<$ in which each pair of elements $a, b$ has a greatest common lower bound, their meet $a \sqcap b$. When viewed as $\{<, \sqcap\}$-structures, finite meet-trees form an amalgamation class, hence have a Fraïssé limit, the universal homogeneous countable meet-tree. Its complete first-order theory DMT is that of dense meet-trees: dense lower semilinear orders with everywhere infinite ramification.

Such structures have received a certain amount of model-theoretic attention in the recent (and not so recent) past. They appear in the classification of countable 2-homogeneous trees from [Dro85], and have since been important in the theory of permutation groups, see for instance [Cam87, DHM89, AN98]. More recently, they were shown to be dp-minimal in [Sim11], and the automorphism group of the unique countable one was studied in [KRS19], while the interest in similar structures goes back at the very least to [Per73, Woo78], where they were used as a base to produce examples in the context of Ehrenfeucht theories.

Here we study DMT, and some of the expansions defined in [EK19], from the viewpoint of domination, in the sense of [Men20b].

One motivation for such a study comes from valuation theory. The nonzero points of a valued field $K$ can notoriously be identified with the branches of a meet-tree, that is, its maximal linearly ordered subsets. This identification is

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used, for instance, to endow $K$ with a C-relation; see \cite{Hol01,MS96}. Viewing the residue field $k$ of $K$ as a set of open valuation balls yields a correspondence between $k$ and, for an arbitrary but fixed point $g$ of the underlying tree, the set of open cones above $g$: the equivalence classes of the relation $E(x,y) = x \cap y > g$ defined on the set of points above $g$. If $K$ has pseudofinite residue field, then $k$ interprets a structure elementarily equivalent to the Random Graph (see \cite{Dur80,Bey10}); it is therefore interesting to study the theory of a dense meet-tree with a Random Graph structure on each set of open cones above a point. This theory was used in \cite{EK19}, where it was dubbed DTR, to show that restrictions to nonforking bases need not preserve NIP.

Another motivation is rooted in the study of invariant types: types over a saturated model $\mathfrak{U}$ of a first order theory which are fixed, under the natural action of $\text{Aut}(\mathfrak{U})$ on the space $S(\mathfrak{U})$ of types, by the stabiliser of some small set. The space of invariant types is a semigroup when equipped with the tensor product, and can be endowed with the preorder of domination, where a type $p(x)$ dominates a type $q(y)$ iff $q(y)$ is implied by the union of $p(x)$ with a small type $r(x,y)$ consistent with $p(x) \cup q(y)$. The induced equivalence relation, called domination-equivalence, may or may not be a congruence with respect to the tensor product, and some conditions ensuring this to be the case were isolated in \cite{Men20b}. One of the main results in the present work is a proof that one of them, weak binarity (Definition 2.1), is satisfied by DMT, and by certain expansions of the latter by binary structures on sets of open cones, a special case of which is DTR. This guarantees the semigroup operation to descend to the quotient, so we may proceed to calculate the domination monoid $\widetilde{\text{Inv}}(\mathfrak{U})$.

The paper is structured as follows. After briefly reviewing standard definitions and facts about dense meet-trees and invariant types in Section 1, we recall in Section 2 the definition of weak binarity and prove that, despite not being binary, DMT and all of its binary cone-expansions (Definition 2.5) are weakly binary. This is in particular the case for DTR.

**Theorem A** (Theorem 2.8). The theory of dense meet-trees is weakly binary, and so is each of its binary cone-expansions.

Hence the monoid $\widetilde{\text{Inv}}(\mathfrak{U})$ is well-defined in such theories, and we proceed to compute it. The case of pure dense meet-trees is handled in Section 3.

**Theorem B** (Theorem 3.11). If $\mathfrak{U}$ is a monster model of the theory of dense meet-trees, then there is a set $X$ such that

$$\widetilde{\text{Inv}}(\mathfrak{U}) \cong \mathcal{P}_\text{fin}(X) \oplus \bigoplus_{g \in \mathfrak{U}} \mathbb{N}$$

where $\mathcal{P}_\text{fin}(X)$ is the upper semilattice of finite subsets of $X$.

In the same section, we take the opportunity to record an instance of a theory where domination differs from $\text{F}_\kappa$-isolation in the sense of Shelah, Example 3.3. Theorem 3.11 is generalised in Section 4 to purely binary cone-expansions (Definition 2.5), such as DTR. If $\mathfrak{U}$ is a monster model of such an expansion, $X$ is given by applying Theorem 3.11 to the underlying tree and, for $g \in \mathfrak{U}$, we denote by $O_g$ the structure on the set of open cones above $g$, we obtain the following.

**Theorem C** (Theorem 4.10). If $T$ is a purely binary cone-expansion of DMT,

$$\text{Inv}(\mathfrak{U}) \cong \mathcal{P}_\text{fin}(X) \oplus \bigoplus_{g \in \mathfrak{U}} \widetilde{\text{Inv}}(O_g)$$


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1 Preliminaries

In what follows, lowercase Latin letters may denote tuples of variables or elements of a model. The length of a tuple is denoted by $|\cdot|$, and its coordinates will be denoted by subscripts, starting with 0; we may write, for example, $a = (a_0, \ldots, a_{|a|-1}) \in M^{|a|}$ or, with abuse of notation, simply $a \in M$. Concatenation is denoted by juxtaposition, and elements of a sequence of tuples by superscripts. For instance, if we write $a = a^n a^1$ then $a_{|a|}$ equals $a^1_0$, the first element of $a^1$.

Tuples may be treated as sets, in which case juxtaposition denotes union, as in $A \cup B$.

Example 3.3. Part of this research has been funded by a Leeds Anniversary Research Scholarship.

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Proofs regarding trees have a tendency to split in cases and subcases. As they become much easier to follow if the objects in them are drawn as soon as they appear in the proof, the reader is encouraged to reach for writing devices, preferably capable of producing different colours.

Invariant types

Fix a complete first-order theory $T$ with infinite models, a sufficiently large cardinal $\kappa$, and a $\kappa$-saturated and $\kappa$-strongly homogeneous $\mathfrak{U} \models T$. Small means “of cardinality strictly less than $\kappa$”; if $A$ is a small subset of $\mathfrak{U}$, we denote this by $A \subset^+ \mathfrak{U}$, or $A \prec^+ \mathfrak{U}$ if additionally $A \prec \mathfrak{U}$. Global type means “type over $\mathfrak{U}$”.

Definition 1.1.

1. Let $A \subseteq B$. A type $p(x) \in S(B)$ is $A$-invariant iff for all $\varphi(x; y) \in L$ and $a \equiv_A b$ in $B^{|y|}$ we have $p(x) \vdash \varphi(x; a) \leftrightarrow \varphi(x; b)$. A global type $p(x) \in S(\mathfrak{U})$ is invariant iff it is $A$-invariant for some $A \subset^+ \mathfrak{U}$. Such an $A$ is called a base for $p$.

2. If $p(x) \in S(\mathfrak{U})$ is $A$-invariant and $\varphi(x; y) \in L(A)$, write

$$(d_p \varphi(x; y))(y) := \{tp_y(b/A) \mid \varphi(x; b) \in p, b \in \mathfrak{U}\}$$

The map $\varphi \mapsto d_p \varphi$ is called the defining scheme of $p$ over $A$.

3. We denote by $S^{\text{inv}}(\mathfrak{U}, A)$ the space of global $A$-invariant types in variables $x$, with $A$ small, and by $S^{\text{inv}}_x(\mathfrak{U})$ the union of all $S^{\text{inv}}_x(\mathfrak{U}, A)$ as $A$ ranges among small subsets of $\mathfrak{U}$. Denote by $S(B)$ the union of all spaces of types over $B$ in a finite tuple of variables; similarly for, say, $S^{\text{inv}}(\mathfrak{U})$.

If we say that a type $p$ is invariant, and its domain is not specified and not clear from context, it is usually a safe bet to assume that $p \in S(\mathfrak{U})$. Similarly if we say that a tuple has invariant type without specifying over which set.
Dense meet-trees

A poset \((M, <)\) is a lower semilinear order iff every pair of elements from each set of the form \(\{x \in M \mid x < a\}\) is comparable. Let \(L_{\text{mt}} = \{<, \sqcap\}\), where \(<\) is a binary relation symbol and \(\sqcap\) is a binary function symbol. A meet-tree is an \(L_{\text{mt}}\)-structure \(M\) such that \((M, <)\) is a lower semilinear order where every pair of elements \(a, b\) has a greatest common lower bound, their meet \(a \sqcap b\). If \(M\) is a meet-tree and \(g \in M\), classes of the equivalence relation defined on \(\{x \in M \mid x > g\}\) by \(E(x, y) := x \sqcap y > g\) are called open cones above \(g\).

Figure 1: the point \(a\) is in the same open cone above \(g\) as the point \(b\), while \(c\) is in a different open cone above \(g\).

Finite meet-trees are well-known to form a Fraïssé class, hence have a Fraïssé limit, whose theory is complete and eliminates quantifiers.\(^1\) A dense meet-tree is a model of the theory \(\text{DMT}\) of the Fraïssé limit of finite meet-trees. It is also well-known that this theory can be axiomatised as follows.

**Fact 1.2.** The theory \(\text{DMT}\) of dense meet-trees is axiomatised by saying that

1. \((M, <, \sqcap)\) is a meet-tree;
2. for every \(a \in M\), the structure \((\{x \in M \mid x < a\}, <)\) is a dense linear order with no endpoints; and
3. for every \(a \in M\), there are infinitely many open cones above \(a\).

The following remark will be used throughout, sometimes tacitly.

**Remark 1.3.** The operation \(\sqcap\) is associative, idempotent, and commutative. Using this and quantifier elimination, and observing for example that for every \(a, b\) the set defined by \(x \sqcap a = b\) is either empty or infinite, it is easy to see that the definable closure \(\text{dcl}(A)\) of a set \(A\) coincides with its closure under meets. In particular, if \(A\) is finite, then so is \(\text{dcl}(A)\): by the properties of \(\sqcap\) we just pointed out, its size cannot exceed that of the powerset of \(A\).\(^2\)

When working in expansions of \(\text{DMT}\), we will denote the closure of a set \(A\) under meets by \(\text{dcl}^{\text{Lmt}}(A)\). This is justified by the previous remark.

**Definition 1.4.** Define the cut \(C_p\) of a type \(p(x) \in S_1(M)\) to be \(\{c \in M \mid p \vdash x \geq c\}\) and the cut in \(M\) of an element \(b \in N > M\) in some elementary extension of \(M\) to be \(C^M_b := C_{\text{tp}(b/M)}\). We say that \(C_p\) is bounded iff it is bounded from above in \(M\).

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\(^1\)For basic Fraïssé theory, see for instance [Hod93, Chapter 7].

\(^2\)While sufficient for our purposes, this upper bound is very far from optimal: one can show that \(|\text{dcl}(A)| \leq 2^{|A|}\). See [EK19, Remark 4.6].
This usage of the word “cut” is a bit more general than the one traditionally used for linear orders: our cuts have no upper part, only a lower one.

It can be shown by standard techniques that \( \text{DMT} \) is \( \text{NIP} \), in fact dp-minimal. This makes it amenable to an analysis of invariant types using indiscernible sequences, and it turns out that invariant \( 1 \)-types are necessarily of one of the six kinds below, as shown by using eventual types (see [Sim15, Subsection 2.2.3]). We refer the reader to [Sim11] and [Sim15, Subsection 2.3.1]. Alternatively, it is possible to prove this directly via quantifier elimination by considering, for a fixed \( p(x) \in S_1^{\text{inv}}(\mathcal{U}) \), what are the possible values of each \( d_p \varphi \), as \( \varphi(x;y) \) ranges among \( L \)-formulas.

**Definition 1.5.** Let \( \mathcal{U} \models \text{DMT} \) and \( p(x) \in S_1(\mathcal{U}) \). We say that \( p \) is of kind

1. (0) iff \( p \) is realised in \( \mathcal{U} \);
2. (Ia) iff there is a small linearly ordered set \( A \subset^+ \mathcal{U} \) such that \( p(x) \vdash \{ x < a \mid a \in A \} \cup \{ x > b \mid b \in \mathcal{U}, b < A \} \);
3. (Ib) iff there is a small linearly ordered set \( A \subset^+ \mathcal{U} \) with no maximum such that \( p(x) \vdash \{ x > a \mid a \in A \} \cup \{ x < b \mid b \in \mathcal{U}, b > A \} \), or there are \( a \) and \( c \) in \( \mathcal{U} \) such that \( p(x) \vdash \{ a < x < c \} \cup \{ x < b \mid b \in \mathcal{U}, a < b < c \} \);
4. (II) iff there is \( g \) such that \( p(x) \vdash \{ x > g \} \cup \{ x \sqcap b = g \mid b \in \mathcal{U}, b > g \} \);
5. (IIIa) iff \( p(x) \vdash \{ x \not\leq b \mid b \in \mathcal{U} \} \) and there is \( c \in \mathcal{U} \) such that \( \text{tp}(x \sqcap c/\mathcal{U}) \) is of kind (Ia);
6. (IIIb) iff \( p(x) \vdash \{ x \not\leq b \mid b \in \mathcal{U} \} \) and there is \( c \in \mathcal{U} \) such that \( \text{tp}(x \sqcap c/\mathcal{U}) \) is of kind (Ib).

So types of kind (0), (Ia), or (Ib) correspond to cuts in a linearly ordered subset of the tree, where in kind (Ib), if the cut of \( p \) has a maximum \( a \), we are specifying an existing open cone above \( a \). Kinds (II), (IIIa), and (IIIb) are the corresponding “branching” versions. Kind (II) is the type of an element in a new open cone above an existing point. See Figure 2.

We conclude this section by recording some easy observations for later use.

![Figure 2: some nonrealised B-invariant types, where points of B are denoted by triangles. In this picture, the set of triangles below x has no maximum, solid lines lie in U, and dotted lines lie in a bigger U_1^{+>} U. The type of x is of kind (Ib), that of y of kind (II), and that of z of kind (IIIb).](image-url)
Lemma 1.6.

1. Let \( b_0, b_1 \in N \succ M \). If \( C_{b_0}^M \subseteq C_{b_1}^M \), then \( C_{b_0/b_1}^M = C_{b_0}^M \). If none of \( C_{b_0}^M \) and \( C_{b_1}^M \) is included in the other, then \( b_0 \cap b_1 \in M \).

2. For all \( b_0, \ldots, b_n \in N \succ M \), points of \( \text{dcl}(Mb_0, \ldots, b_n) \) are either in \( M \) or have the same cut in \( M \) as one of the \( b_i \).

3. If \( p \in S_1(\mathcal{U}) \) then \( C_p \) is bounded.

Proof. The first part is clear from the definitions of cut and meet, and the second one follows by induction. The last part follows from the characterisation of invariant 1-types.

\[ \Box \]

2 Weak binarity

The main result of this section, Theorem 2.8, states that certain expansions of DMT are weakly binary. It applies for instance to the theory DTR from [EK19], obtained by equipping every set of open cones above a point with a structure elementarily equivalent to the Random Graph.

Recall that a theory \( T \) is binary iff every formula is equivalent modulo \( T \) to a Boolean combination of formulas with at most two free variables. Equivalently, for every set of parameters \( B \) and tuples \( a, b \),

\[ \text{tp}(a/B) \cup \text{tp}(b/B) \cup \text{tp}(ab/\emptyset) \vdash \text{tp}(ab/B) \]

Natural examples of such theories are those which eliminate quantifiers in a binary relational language. On the other hand, binary function symbols are usually an obstruction to binarity, as they can be used to write atomic formulas with an arbitrary number of free variables.

This is for instance the case for DMT, whose language contains the binary function symbol \( \sqcap \): it is easy to see that DMT is not binary, nor is any of its expansions by constants. Even though DMT is known to be ternary (see [Sim11 Corollary 4.6]), this is not sufficient for our purposes: the theory from [Men20b Proposition 2.3] where \( \tilde{\text{Inv}}(U) \) is not well-defined is ternary as well.

Definition 2.1. A theory is weakly binary iff, whenever \( a, b \) are tuples such that \( \text{tp}(a/\mathcal{U}) \) and \( \text{tp}(b/\mathcal{U}) \) are invariant, there is \( A \subset + \mathcal{U} \) such that

\[ \text{tp}(a/\mathcal{U}) \cup \text{tp}(b/\mathcal{U}) \cup \text{tp}(a, b/A) \vdash \text{tp}(a, b/\mathcal{U}) \]

Weak binarity was introduced in [Men20b] as a sufficient condition for well-definedness of the domination monoid. The class of weakly binary theories clearly contains any theory which happens to have an expansion by constants which is binary. Examples with no binary expansion by constants include the theory of a generic equivalence relation where every equivalence class carries a circular order and, as we will shortly see, DMT.

It follows immediately from the definitions that, in every theory, if \( p \in S(\mathcal{U}) \) is invariant then each of its 1-subtypes, that is, each of the restrictions of \( p \) to one of its variables, is invariant as well. It is easily seen, say by using [Men20b Lemma 1.27] and induction, that if \( T \) is weakly binary then the converse holds as well. We record this here for later reference.
Remark 2.2. Let $T$ be weakly binary and $p \in S(\mathfrak{U})$. Then $p$ is invariant if and only if every $1$-subtype of $p$ is invariant.

Before returning to trees, note that this converse is in general false. For example, in the theory of Divisible Ordered Abelian Groups, let $p(x_0, x_1)$ be a 2-type prescribing $x_0, x_1$ to be larger than $\mathfrak{U}$, and such that both the cofinality of $\{d \in \mathfrak{U} \mid p \vdash x_0 - x_1 > d\}$ and the coinitiality of its complement are not small. Then $p$ is not invariant, even if both of its 1-subtypes are.

Notation 2.3. We write $x \parallel y$ to mean that $x \not\leq y$ and $y \not\leq x$.

Lemma 2.4. In the theory $\text{DMT}$, let $b$ be a finite tuple. There is a finite tuple $d$ such that $\mathfrak{U}bd$ is closed under meets. Moreover, $d$ can be chosen such that additionally, if we let $c := \mathfrak{U} \cap d$, then $d \in \text{dcl}(bc)$, and for every $e \in bd \setminus \mathfrak{U}$ such that $C_{c \setminus e}$ is bounded, the following happens.

1. There is $a_e \in c$ such that $a_e > C_{c \setminus e}$.
2. If $C_{c \setminus e}$ has a maximum $g$ and $e$ is in an existing open cone above $g$, then this is the open cone of $a_e$.

Proof. Define a tuple $a$ as follows. If $C_{c \setminus e}$ is not bounded, choose $a_i$ to be an arbitrary point of $\mathfrak{U}$ (or, if the reader prefers, leave $a_i$ undefined). If $C_{c \setminus e}$ has a maximum $g$ and $i$ is in an open cone above $g$ which intersects $\mathfrak{U}$, let $a_i \in \mathfrak{U}$ be such that $a_i \cap b_i > g$ (see first half of Figure 3); otherwise (second half of the same figure), choose any $a_i > C_{c \setminus e}$. Note that the closure $\text{dcl}(ba)$ of $ba$ under meets is finite by Remark 2.3 and let $d$ be a tuple enumerating $\text{dcl}(ba) \setminus b$. Recall that we defined $c := \mathfrak{U} \cap d$, and note that, by construction, $d \in \text{dcl}(bc)$.

We now prove the “moreover” part, and then show how closure under meets of $\mathfrak{U}bd$ follows. Let $e \in bd \setminus \mathfrak{U}$ have bounded cut. By Lemma 1.6, construction, and the fact that $e \notin \mathfrak{U}$, there is $i < |b|$ such that $e$ can be written as the meet of $b_i$ with other points (possibly none), either with the same cut as $b_i$, or in $\mathfrak{U}$ and above $C_{c \setminus e}$. In particular $e \leq b_i$ and $C_{c \setminus e} = C_{c \setminus b_i}$.

1. Let $i$ be as above. Since $C_{c \setminus a_i} = C_{c \setminus b_i}$, we have $a_e := a_i > C_{c \setminus e}$.
2. Let $i$ and $a_e$ be as above. By choice of $a_i = a_e$, we have $a_i \cap b_i > g$. By construction and the fact that $e \notin \mathfrak{U}$, we have $g < e \leq b_i$, so $e \cap b_i = e > g$ and $e$ and $b_i$ are in the same open cone above $g$, which is that of $a_i$. This completes the proof of 2] hence of the “moreover” part.

![Figure 3: how to choose $a_i$ in the proof of Lemma 2.4](image-url) In the first three pictures, $C_{b_i}$ has a maximum, $g$, denoted by a triangle. In the last picture it does not have one. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1 ^{^+} \succ \mathfrak{U}$. 
We are left to prove that $\mathcal{U}bd$ is closed under meets. As both $\mathcal{U}$ and $bd$ are, and $\cap$ is commutative, all we need to show is that if $e \in bd \setminus \mathcal{U}$ and $f \in \mathcal{U}$ then $f \cap e \in \mathcal{U}bd$. If $e$ and $f$ and comparable there is nothing to prove, so assume they are not, i.e. that $e \parallel f$. It is immediate to notice that if $C_e^\mathcal{U}$ is unbounded, since $f \in \mathcal{U}$ and $e \parallel f$, none of $C_e^\mathcal{U}$ and $C_f^\mathcal{U}$ is included in the other. Hence, by the first point of Lemma 1.6 we have $e \cap f \in \mathcal{U}$. Assume now that $C_e^\mathcal{U}$ is bounded.

**Claim.** To conclude, it is enough to show that $f \cap e \leq f \cap a_e$. 

**Proof of Claim.** By assumption, commutativity, and idempotency of $\cap$ we have $f \cap e = (f \cap e) \cap (f \cap a_e) = (f \cap a_e) \cap (a_e \cap e)$. Since $f \cap a_e$ and $a_e \cap e$ are both predecessors of $a_e$ they are comparable, so their meet is one of them. But $a_e \cap e \in bd$ and $f \cap a_e \in \mathcal{U}$, so $f \cap e \in \mathcal{U}bd$. \hfill $\Box$

We prove that $f \cap e \leq f \cap a_e$ by cases. Note that, since $f \cap a_e$ and $f \cap e$ are both predecessors of $f$, they are comparable.

1. If $f > C_e^\mathcal{U}$ then $C_f^\mathcal{U} \cap e = C_f^\mathcal{U}$. Suppose additionally that $f \cap a_e > C_e^\mathcal{U} = C_f^\mathcal{U}$. Since $f \cap a_e \in \mathcal{U}$, having $f \cap a_e \leq f \cap e$ would contradict $f \cap a_e > C_f^\mathcal{U}$, and therefore $f \cap e < f \cap a_e$.

2. If $f > C_e^\mathcal{U}$ and we are not in the previous case, then $C_e$ has a maximum $g$ and $f \cap a_e = g$, i.e. $f$ and $a_e$ are in different open cones above $g$. Now, $e$ can be either in the same open cone as $a_e$, or in a new one, but in both cases $f \cap e = g = f \cap a_e$.

3. If $f \not> C_e^\mathcal{U}$ then there is $h \in C_e^\mathcal{U}$ such that $f \not> h$, and then $f \cap h = f \cap (h \cap e) = f \cap e$. As $a_e > C_e^\mathcal{U}$ in particular $a_e > h$, hence by definition of meet we must have $f \cap a_e = f \cap h = f \cap e$. \hfill $\Box$

**Definition 2.5.** A *binary cone-expansion* of DMT is a theory $T$ in a language $L = L_{\text{un}} \cup \{R_j, P_{j'} \mid j \in J, j' \in J'\}$ satisfying the following properties.

1. Every $P_{j'}$ is a unary relation symbol; every $R_j$ is a binary relation symbol.

2. $T$ is a completion of DMT and eliminates quantifiers in $L$.

3. Every $R_j$ is on open cones, in the sense that

   (a) $R_j(x, y) \rightarrow x \parallel y$, and

   (b) if $x \parallel y$ and $x', y'$ are such that $x \cap x' > x \cap y$ and $y \cap y' > x \cap y$ then
   
   $R_j(x, y) \leftrightarrow R_j(x', y')$.

If additionally $J' = \emptyset$ we say that $T$ is a *purely binary cone-expansion* of DMT.

**Example 2.6.** A purely binary cone-expansion of DMT is DTR, axiomatised by taking $J = \{R\}$, $J' = \emptyset$, and saying that, for all $g$, the structure induced by $R$ on the (imaginary sort of) open cones above $g$ is elementarily equivalent to the Random Graph. See [EK19] for DTR, and for a more general analysis of theories of trees with relations on sets of open cones.
Example 2.7. Another theory examined in [EK19], called $\text{DTE}_2$, is defined similarly to $\text{DTR}$, but instead of the Random Graph it uses the Fraïssé limit of all finite structures with two equivalence relations. More generally, one can define $\text{DTE}_n$ in an analogous fashion. The results of this paper apply to these theories as well even if, strictly speaking, they do not satisfy Definition 2.5 since the latter requires the $R_j$ to be irreflexive. This can easily be circumvented by observing that, if $E$ is an equivalence relation and $\Delta$ is the diagonal, then $E$ and $E \setminus \Delta$ are interdefinable.

Theorem 2.8. Every binary cone-expansion of $\text{DMT}$ is weakly binary.

Proof. Let $b^0, b^1$ be tuples each having invariant global type. By quantifier elimination it is enough to find a finite tuple $c \in \mathcal{U}$ such that $\text{tp}_\mathcal{U}(b^0/\mathcal{U}) \cup \text{tp}_\mathcal{U}(b^1/\mathcal{U}) \cup \text{tp}_\mathcal{U}(b^0b^1/c)$ decides all the atomic relations in $L$ between points of $b^0, b^1, \mathcal{U}$, and their meets. Apply Lemma 2.4 to $b := b^0b^1$, let $d$ be the resulting tuple and set $c := d \cap \mathcal{U}$. We want to show that

$$\pi := \text{tp}(b^0/\mathcal{U}) \cup \text{tp}(b^1/\mathcal{U}) \cup \text{tp}(b/c) \vdash \text{tp}(b/\mathcal{U})$$

If $e$ and $f$ are both in $bd$ then $e, f \in \text{dcl}^{\text{mut}}(bc)$, hence $\text{tp}(b/c)$ entails $\text{tp}(e/f \emptyset)$, and in particular decides all formulas of the forms $R_j(e, f)$ and $P_j(e)$.

Claim. We have $\pi \vdash \text{tp}^{\text{mut}}(b/(\mathcal{U} \mid L_{\text{mut}}))$.

Proof of Claim. Since $\mathcal{U}bd$ is closed under meets we only need to show that the position of all the $e \in d \setminus \mathcal{U}b$ with respect to $\mathcal{U}$ is determined. By Lemma 1.6 and the fact that $e \in \text{dcl}^{\text{mut}}(bc) \setminus \mathcal{U}b$ there is $i < |b|$ such that $e < b_i$ and $C^\mathcal{U}_e = C^\mathcal{U}_{b_i}$; note that this information is deduced by $\pi$, because $e$ is a meet of points in $bc$. If $C^\mathcal{U}_e$ is unbounded, we are done. Otherwise, if $a_e \in e$ is as in Lemma 2.4 all we need to decide is whether $e$ is below or incomparable with $\{h \in \mathcal{U} \mid h > a_e \cap e\}$. This is decided by whether $a_e > e$ or not, and this information is in $\text{tp}(b/c)$.

We then need to take care of formulas of the form $R_j(e, f)$ for $e \in d \setminus \mathcal{U}b$ and $f \in \mathcal{U}$: the argument for formulas of the form $R_j(f, e)$ is identical mutatis mutandis. If $e \leq f$ or $f \leq e$, by hypothesis we must have $\neg R_j(e, f)$, so we may assume that $e \parallel f$. We distinguish three cases; the fact that, by the Claim, $\pi$ implies the position of $e$ with respect to $\mathcal{U}$ will be used tacitly.

1. Assume first $e \cap f > C^\mathcal{U}_e$. Some subcases of this case are depicted in Figure 4. By assumption $C^\mathcal{U}_e$ is bounded and, if $a_e \in e$ is as in Lemma 2.4 we have $a_e \cap f > C^\mathcal{U}_e = C^\mathcal{U}_{e\cap f}$. Since $a_e \cap f$ and $e \cap f$ must be comparable, and $a_e \cap f \in \mathcal{U}$, this implies $a_e \cap f > e \cap f$, so $a_e$ and $f$ are in the same open cone above $e \cap f$. By our hypotheses on $T$ then $R_j(e, f) \iff R_j(e, a_e)$, but $a_e \in c$ and $e \in \text{dcl}^{\text{mut}}(bc)$, so since $\pi \vdash \text{tp}(b/c)$ we are done.

2. Assume now that $e \cap f \not\supset C^\mathcal{U}_e$ and there is $h \in \mathcal{U}$ such that $e \cap h > e \cap f$. Then $e$ is in the same open cone above $e \cap f$ as $h$, hence $R_j(e, f) \iff R_j(h, f)$. Since $f, h \in \mathcal{U}$ we are done.

3. If $e \cap f \not\supset C^\mathcal{U}_e$ but there is no $h$ as in the previous point, then $C^\mathcal{U}_e$ must have a maximum $g$, which needs to equal $e \cap f$, and since $e \parallel f$ we need to have $f > g$. If $e$ is in an existing open cone above $g$, since the $R_j$ are on open
The domination monoid: pure trees

We now compute the domination monoid in $\text{DMT}$. We first recall briefly its definition and some of its basic properties for the reader's convenience, and otherwise refer to [Men20b]. See also [Men20a] for a more extensive treatment.

It is well-known that, if $A \subseteq B$ and $p \in S^{\text{inv}}(\mathfrak{U}, A)$, then there is a unique $b \models q$ for each $\varphi(x,y) \in L(A)$ and $b \in B$,

$$\varphi(x;b) \in p \mid B \iff \text{tp}(b/A) \in (d_{\varphi} \varphi(x;y))(y)$$

This canonical extension to bigger parameter sets allows to define the tensor product of $p \in S^{\text{inv}}(\mathfrak{U}, A)$ with any $q \in S_y(\mathfrak{U})$ as follows. Fix $b \models q$; for each $\varphi(x,y) \in L(\mathfrak{U})$, define

$$\varphi(x,y) \in p(x) \otimes q(y) \iff \varphi(x,b) \in p \mid \mathfrak{U}b$$

Some authors denote by $q(y) \otimes p(x)$ what we denote by $p(x) \otimes q(y)$.

It is an easy exercise to show that the product $\otimes$ does not depend on $b \models q$, nor on the choice of a base of invariance for $p$, that it is associative, and that if $p, q$ are both $A$-invariant, then so is $p \otimes q$.

In what follows, when considering types $p(x), q(y)$, say, we assume without loss of generality that the tuples of variables $x$ and $y$ are disjoint.

Figure 4: two subcases of case 1 in the proof of Theorem 2.8, where $e \cap f > C^u_e$. In the first picture, $C^u_e$ does not have a maximum. In the second picture it has one, denoted by a triangle. Solid lines lie in $\mathfrak{U}$, and dotted lines lie in a bigger $\mathfrak{U}_1$. Other subcases are similar, and correspond to different arrangements of $a_e$ and $f$, e.g. $a_e > f$.

cones, we are done, so assume it is in a new one. Since $e \in \text{dcl}(bc)$, by Lemma 1.6 this can only happen if there is $i < |b|$ such that $e \leq b_i$, hence $e$ shares the same open cone above $g$ as $b_i$. Again, since the $R_j$ are on open cones, we are done.

Remark 2.9. Weak binarity was introduced as a sufficient condition for $\tilde{\text{Inv}}(\mathfrak{U})$ to be well-defined. Consequently, in Definition 2.1, we only require (†) to hold for tuples $a, b$ such that $\text{tp}(a/U)$ and $\text{tp}(b/U)$ are invariant. Nevertheless, in the proofs above we never used invariance, hence binary cone-expansions of $\text{DMT}$ satisfy a condition slightly stronger than weak binarity, obtained by requiring (†) from Definition 2.1 to hold for all tuples $a, b$.

3 The domination monoid: pure trees
Definition 3.1. Let $p \in S_a(\Omega)$ and $q \in S_q(\Omega)$. We say that $p$ dominates $q$, and write $p \geq_D q$, iff there are some small $A$ and some $r \in S_{pq}(A)$ such that

- $r \in S_{pq}(A) := \{r \in S_{xy}(A) \mid r \geq (p \upharpoonright A) \cup (q \upharpoonright A)\}$, and
- $p(x) \cup r(x, y) \vdash q(y)$.

In this case, we say that $r$ is a witness to, or witnesses $p \geq_D q$. We say that $p$ and $q$ are domination-equivalent, and write $p \sim_D q$, if $p \geq_D q$ and $q \geq_D p$.

Example 3.2. Suppose that $q(y)$ is the pushforward of $p(x)$ under the $A$-definable function $f$, namely $q(y) := \{\varphi(y) \mid p(x) \vdash \varphi(f(x))\}$. In this case, and in the more general one where $|y| > 1$ and $f$ is a tuple of definable functions, we have $p \geq_D q$, witnessed by any completion of $(p(x) \upharpoonright A) \cup (q(y) \upharpoonright A) \cup \{y = f(x)\}$.

In Definition 3.1 we are not requiring $p \cup r$ to be a complete global type in variables $xy$; in other words, domination is “small-type semi-isolation”, as opposed to “small-type isolation”, i.e. $F^*_s$-isolation in the notation of [She90, Chapter IV]. While it is easy to see that $F^*_s$-isolation is the same as domination in every weakly binary theory, the two relations are in general distinct. This can be seen in the theory below; the reader who dislikes random digraphs may feel free to replace them with generic equivalence relations.

Example 3.3. Work in a 2-sorted language, with sorts $O$ (“objects”) and $D$ (“digraphs”). Let $L := \{E^{(O)}, P^{(O)}, R^{(O \times D)}\}$, a relational language with arities indicated as superscripts. Consider the following universal axioms.

1. $E$ is an equivalence relation.
2. $R(x, y, w) \rightarrow E(x, y)$.
3. $R(x, y, w) \rightarrow \neg R(y, x, w)$.

The finite structures satisfying these axioms form a Fraïssé class; let $T$ be the theory of its limit. In a model of $T$, the sort $O$ carries an equivalence relation with infinitely many classes. On each class $a/E$ the predicate $P$ is infinite and coinfinite, and each point of $D$ induces a random digraph on each $a/E$. Different random digraphs, on the same $a/E$ or on different ones, interact generically with $P$ and with each other, but no digraph has an edge across different classes.

Let $x$ be a variable of sort $O$, define $\pi(x) := \{\neg E(x, d) \mid d \in \Omega\}$, and let $p(x) := \pi(x) \cup \{P(x)\}$ and $q(y) := \pi(y) \cup \{\neg P(y)\}$. By quantifier elimination and the lack of edges across different classes, $p$ and $q$ are complete global types, in fact $\emptyset$-invariant ones. Any $r \in S_{pq}(\emptyset)$ containing $p(x, y) := E(x, y) \wedge P(x) \wedge \neg P(y)$ witnesses simultaneously that $p \geq_D q$ and that $q \geq_D p$, since $p \cup \{\rho\} \vdash q$ and $q \cup \{\rho\} \vdash p$. Note that the predicate $P$ forbids $r$ from containing $x = y$. By genericity, there is no small $A$ such that for some $r \in S_{pq}(A)$ the partial type $p \cup r$ decides, for all $d \in \Omega$, whether $R(x, y, d)$ holds, and similarly for $q \cup r$. Hence, for all $a \Vdash p$ and $b \Vdash q$, neither $\text{tp}(a/\Omega\emptyset)$ nor $\text{tp}(b/\Omega\emptyset)$ is $F^*_s$-isolated.

It can be shown that $\sim_D$ is a preorder, hence $\sim_D$ is an equivalence relation. Let $\text{Inv}(\Omega)$ be the quotient of $S^{\text{inv}}(\Omega)$ by $\sim_D$. The partial order induced by $\geq_D$ on $\text{Inv}(\Omega)$ will, with abuse of notation, still be denoted by $\geq_D$, and we call $(\text{Inv}(\Omega), \geq_D)$ the domination poset. This poset has a minimum, the (unique) class of realised types, i.e. global types realised in $\Omega$, denoted by $\{0\}$.

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In this case, we say that $r$ is a witness to, or witnesses $p \geq_D q$. We say that $p$ and $q$ are domination-equivalent, and write $p \sim_D q$, if $p \geq_D q$ and $q \geq_D p$.

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The finite structures satisfying these axioms form a Fraïssé class; let $T$ be the theory of its limit. In a model of $T$, the sort $O$ carries an equivalence relation with infinitely many classes. On each class $a/E$ the predicate $P$ is infinite and coinfinite, and each point of $D$ induces a random digraph on each $a/E$. Different random digraphs, on the same $a/E$ or on different ones, interact generically with $P$ and with each other, but no digraph has an edge across different classes.

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If $T$ is such that $(\text{Inv}(\Omega), \otimes, \geq_D)$ is a preorder semigroup, we say that $\otimes$ respects $\geq_D$. In particular, then $\sim_D$ is a congruence with respect to $\otimes$, and induces a well-defined operation on $\text{Inv}(\Omega)$, still denoted by $\otimes$, easily seen to have neutral element $[0]$. Call the structure $(\text{Inv}(\Omega), \otimes, [0], \geq_D)$ the domination monoid. We usually denote it simply by $\text{Inv}(\Omega)$, and say that $\text{Inv}(\Omega)$ is well-defined to mean that $\otimes$ respects $\geq_D$; this should cause no confusion since $\text{Inv}(\Omega)$ is always well-defined as a poset.

As shown in [Men20b], $\text{Inv}(\Omega)$ need not be well-defined in general, but it is in certain classes of theories, such as stable ones. More relevantly to the present endeavour, we recall the following.

**Fact 3.4 ([Men20b] Corollary 1.30).** In every weakly binary theory, the partially ordered monoid $(\text{Inv}(\Omega), \otimes, [0], \geq_D)$ is well-defined.

Recall that two types $p(x), q(y) \in S(B)$ are weakly orthogonal, denoted by $p \perp^{\text{w}} q$, if $p(x) \cup q(y)$ is a complete type in $S_{xy}(B)$. In particular, if $p, q \in S^{\text{inv}}(\Omega)$, then $p(x) \oplus q(y) = q(y) \otimes p(x)$, since both extend $p(x) \cup q(y)$. We will also need the following two facts.

**Fact 3.5 ([Sim14] Corollary 4.7).** Let $T$ be NIP and \( \{p_i \mid i \in I\} \) be a family of types $p_i(x^i) \in S^{\text{inv}}(\Omega)$ such that if $i \neq j$ then $p_i \perp^{\text{w}} p_j$. Then $\bigcup_{i \in I} p_i(x^i)$ is complete.

**Fact 3.6 ([Men20b] Proposition 3.13 and Corollary 3.14]).** Let $p_0, p_1 \in S^{\text{inv}}(\Omega)$, $q \in S(\Omega)$, and assume that $p_0 \geq_D p_1$. If $p_0 \perp^{\text{w}} q$, then $p_1 \perp^{\text{w}} q$. If $p_0 \geq_D q$ and $p_0 \perp^{\text{w}} q$, then $q$ is realised.

In particular we may endow $\text{Inv}(\Omega)$ with an additional relation, induced by $\perp^{\text{w}}$ and denoted by the same symbol.

In what follows, if $r \in S_{pq}(A)$ witnesses $p \geq_D q$, by passing to a suitable extension of $r$ there is no harm in enlarging $A$, provided it stays small, which we may do tacitly; if $p, q$ are invariant, we will furthermore assume $A$ to be large enough so that $p, q \in S^{\text{inv}}(\Omega, A)$. Sometimes, we say that $r$ witnesses domination even if it is not complete, but merely consistent with $(p(x) \cup q(y)) \models A$. In that case, we mean that any of its completions to a type in $S_{pq}(A)$ does. Similarly, we sometimes just write e.g. “put in $r$ the formula $\varphi(x,y)$”.

**Proposition 3.7.** The following statements hold in DMT.

1. Suppose all coordinates of $p \in S(\Omega)$ have the same cut $C_0$, all coordinates of $q \in S(\Omega)$ have the same cut $C_1$, and $C_0 \neq C_1$. Then $p \perp^{\text{w}} q$.

2. Let $C$ be a cut with maximum $g$. Suppose that all 1-subtypes of $p$ are of kind (Ib) with cut $C$ and all 1-subtypes of $q$ are of kind (II) with cut $C$, or that all 1-subtypes of $p, q$ are of kind (Ib) with cut $C$, but no open cone above $g$ contains both a coordinate of $p$ and one of $q$. Then $p \perp^{\text{w}} q$.

3. Every 1-type of kind (IIIa) is domination-equivalent to the unique 1-type of kind (Ia) with the same cut. Every 1-type of kind (IIIb) is domination-equivalent to the unique 1-type of kind (Ib) with the same cut and, if this cut has a maximum $g$, the same open cone above $g$.

In particular, if $p, q \in S_{1}^{\text{inv}}(\Omega)$, then either $p \perp^{\text{w}} q$ or $p \sim_D q$. 


Proof. ① By quantifier elimination and the first two points of Lemma 1.6.
② This does not follow from the previous point because such types have the same cut, but it is still easy from quantifier elimination and the fact that the open cones in which types of kind (II) concentrate are new, while those of types of kind (Ib) are realised.
③ We give a proof for kind (IIIa) which may be easily modified to yield one for kind (IIIb). Suppose that \( c \in \mathcal{U} \) and \( A \subset \mathcal{U} \) are such that
\[
\forall x (x \not\leq a \or b \in \mathcal{U}, b < A)
\]
Let \( q \) be the pushforward of \( p \) under the definable function \( x \mapsto x \cap c \). By this very description \( p(x) \geq_D q(y) \) (cf. Example 3.2) and, by definition of kind (IIIa), \( q \) is of kind (Ia), and clearly has the same cut as \( p \). To prove \( q(y) \geq_D p(x) \), use some \( r \in S_{pq}(A) \) extending \( \{ a > (x \cap c) > y \mid a \in A \} \); since \( r \) contains \( p(x) \upharpoonright A \), which proves \( x \not\leq a \) for all \( a \in A \), we are done. See Figure 5. ②

Proposition 3.9. In the theory of dense meet-trees the following hold.

1. Types of kind (Ia) and (Ib) are idempotent modulo equidominance.
2. If \( p \) is of kind (II) and \( m < n \in \omega \) then \( p^{(m)} \not\leq_D p^{(n)} \).

Remark 3.8. Let \( p(x) \) and \( q(y) \) be the types respectively of kind (IIIb) and (Ib) with cut 0. Then \( p \not\equiv_D q \).

Proof. Suppose that \( r(x, y) \) witnesses equidominance. If \( r(x, y) \vdash x \cap y < y \), then \( p(x) \cup r(x, y) \not\equiv q(y) \), since by quantifier elimination and compactness it cannot prove all formulas \( y < d \), for \( d \in \mathcal{U} \). If \( r(x, y) \vdash x \cap y = y \), then \( q(y) \cup r(x, y) \not\equiv p(x) \), since it cannot prove all formulas \( x \not\leq d \).

Figure 5: proof of Proposition 3.7, how to show that \( q(y) \geq_D p(x) \). In this picture \( A \) only contains the point \( c \), denoted by a triangle. Solid lines lie in \( \mathcal{U} \), and dotted lines lie in a bigger \( \mathcal{U}^1 \).
Proof. (1) Let $A$ be such that $p$ is $A$-invariant. It follows easily from quantifier elimination that in order to show $p(x_1) \otimes p(x_0) \equiv_D p(y)$ it is enough to put in $r \in S_{pq}(A)$ the formula $x_0 = y$.

(2) For notational simplicity we show the case $m = 1$, $n = 2$, the general case being analogous. Suppose that $p$ is the type of an element in a new open cone above $g$, i.e. $p(y) \vdash \{y > g\} \cup \{y \cap b = g \mid b \in \mathcal{U}, b > g\}$. We want to show that there is no small $r(y, x_0, x_1)$ such that $p(y) \cup r \vdash p(x_1) \otimes p(x_0)$. Since $p^{(2)} \mid \{g\}$ proves $x_0 \cap x_1 = g$, i.e. that the cones of $x_0$ and $x_1$ are distinct, there is $i < 2$ such that $r \vdash y \cap x_i = g$. Since $r$ is small there is $d > g$ in $\mathcal{U}$ such that $p(y) \cup r \not\vdash x_i \cap d = g$; in other words it is not possible, with a small type, to say that $x_i$ is in a new open cone, unless it is the same cone as $y$, but $y$ cannot be in the open cones of $x_0$ and $x_1$ simultaneously. \hfill\qed

Since by Theorem \text{2.8} dense meet-trees are weakly binary, $\widehat{\text{Inv}}(\mathcal{U})$ is well-defined by Fact \text{3.4}. By the results above, (domination-equivalence classes of) 1-types of kind (II) generate a copy of $\mathbb{N}$, while all other (classes of) 1-types are idempotent. We have also seen that if $p, q$ are nonrealised 1-types, then either $p \not\equiv q^0$ or $p \sim_D q$. In particular, all pairs of 1-types commute modulo domination-equivalence. To complete our study we need one last ingredient.

**Proposition 3.10.** In DMT, every invariant type is domination-equivalent to a product of invariant 1-types.

**Proof.** By Fact \text{3.8} and Proposition \text{3.7} we reduce to showing the conclusion for types $p(x)$ consisting of elements all with the same cut $C_p$.

Assume first that $C_p$ does not have a maximum, let $c \models p(x)$ and let $d \in \mathcal{U}$ be such that $d > C_p$, which exists by the last point of Lemma \text{1.6}. Let $H = \{h_0(c), \ldots, h_n(c)\}$ be the (finite, by Remark \text{1.3}) set of points in $\text{dcl}(cd)$ such that $d > h_i(c)$, where each $h_i(x)$ is a $\{d\}$-definable function. By semilinearity $H$ is linearly ordered; suppose, up to reindexing, that $h_0(c) = \min H$ and $h_n(c) = \max H$. We have two subcases. If $C_p$ has small cofinality, let $q(y)$ be of kind (Ib) with $C_p = C_q$. Let $A$ be such that $p, q \in S^\text{inv}(\mathcal{U}, A)$, let $r(x, y) \in S_{pq}(A)$ contain the formula $h_0(x) < y$, and note that $q(y) \cup r(x, y)$ implies the type over $\mathcal{U}$ of each point of $\text{dcl}(xd)$, i.e. of the closure of $x$ under meets. It follows from quantifier elimination that $q \cup r \vdash p$. To prove $p \cup r \vdash q$, use instead some $r$ containing the formula $y < h_0(x)$. In the other subcase, $\{c \in \mathcal{U} \mid C_p < c < d\}$ has small cofinality. The argument is analogous, except we use an $r$ containing $h_0(x) > y$ to show $q \cup r \vdash p$ and one containing $h_0(x) < y$ to show $p \cup r \vdash q$.

Suppose now that $C_p$ has maximum $g$. Assume without loss of generality that $c_0, \ldots, c_{k-1}$ are the points of $c$ such that there is $d_i \in \mathcal{U}$ with $d_i \cap c_i > g$. In other words, these are the points in existing open cones above $g$, and $c_k, \ldots, c_{|c|-1}$ are in new open cones. Again by quantifier elimination, we have $\text{tp}(c_0, \ldots, c_{k-1}/\mathcal{U}) \not\equiv \text{tp}(c_k, \ldots, c_{|c|-1}/\mathcal{U})$, so we can deal with the two subtypes separately. Similarly, by using weak orthogonality we may split $c_{<k}$ further, and we may assume that for $i < \ell$, say, all $c_i$ are in the same open cone, say that of the point $d \in \mathcal{U}$. It is now enough to proceed as in the previous case, by taking $q(y)$ to be the type of kind (Ib) with the same cut and open cone above $g$. As for $c_{k}, \ldots, c_{|c|-1}$, let $H$ be the set of minimal elements of $\text{dcl}(c_k, \ldots, c_{|c|-1}) \setminus \mathcal{U}$. Let $q(y)$ be the type of kind (II) above $g$. To conclude, let $r$ identify elements of $H$ with coordinates of a realisation of $q(\mathcal{U})$. \hfill\qed
The previous results yield the following characterisation of $\widetilde{\text{Inv}}(\mathcal{U})$ in DMT.

**Theorem 3.11.** In dense meet-trees, $\widetilde{\text{Inv}}(\mathcal{U}) \cong \mathcal{P}_{\text{fin}}(X) \oplus \bigoplus_{g \in \mathcal{U}} \mathbb{N}$. Generators of copies of $\mathbb{N}$ correspond to types of elements in a new open cone above a point $g \in \mathcal{U}$, i.e. to types of kind (II), while each point of $X$ corresponds to, either:

1. a linearly ordered subset of $\mathcal{U}$ with small cofinality; this corresponds to types of kind (Ia)/(IIa);

2. a cut with no maximum, but with small cofinality; this corresponds to some types of kind (Ib)/(IIb);

3. an existing open cone above an existing point; this corresponds to the rest of the types of kind (Ib)/(IIb).

**4 The domination monoid: expansions**

In this section we generalise Theorem 3.11 to purely binary cone-expansions of DMT, such as DTR, by replacing the direct summands isomorphic to $\mathbb{N}$ with the domination monoids of the structures induced on sets of open cones. In DMT these are pure sets, which are easily seen to have domination monoid isomorphic to $\mathbb{N}$ and generated by the $\sim_\text{DTR}$-class of the unique nonrealised 1-type.

Before restricting our attention to purely binary cone-expansions, we observe a phenomenon which can arise in the presence of unary predicates. Suppose for instance that $L = L_{\text{nat}} \cup \{P\}$, where $P$ is a unary predicate symbol interpreted as a branch of $\mathcal{U}$, i.e. a maximal linearly ordered subset. In this case, there is an $\emptyset$-invariant type $p$ with cut $c_p = P(\mathcal{U})$, and by the last point of Lemma 1.6 $p \upharpoonright L_{\text{nat}}$ is not invariant. Another binary cone-expansion of DMT where there is an invariant type $p$ such that $p \upharpoonright L_{\text{nat}}$ is not invariant can be obtained by taking as $P(\mathcal{U})$ a bounded linearly ordered subset with no supremum. However, using unary predicates is the only way to obtain such behaviour in a binary cone-expansion of DMT, as we are about to show. We refer the reader interested to preservation of invariance under reducts to [RS17].

Denote by $G_g$ the closed cone above $g$, namely $\{b \in \mathcal{U} \mid b \geq g\}$.

**Definition 4.1.** Let $T$ be an expansion of DMT. We call a formula $\varphi(x)$ with $|x| = 1$ tame iff it has the following property: there is a finite set $D \subseteq \mathcal{U}$ such that, for every $a \in \varphi(\mathcal{U})$, either there is $d \in D$ such that $a \leq d$, or $G_a \subseteq \varphi(\mathcal{U})$.

**Proposition 4.2.** If $T$ is a purely binary cone-expansion of DMT, then every formula in one free variable is tame.

*Proof.* It is clear that every atomic and negated atomic $\varphi(x) \in L_{\text{nat}}(\mathcal{U})$ is tame. Fix a point $c$, and consider $\varphi(x) := R_j(x, c)$; if $a \in \varphi(\mathcal{U})$, by assumption we also have $\varphi(b)$ for every $b > a$, hence $G_a \subseteq \varphi(\mathcal{U})$. Consider now $\varphi(x) := \neg R_j(x, c)$, and let $D = \{c\}$. Suppose that $a \not\leq c$. If $a \parallel c$ and $\varphi(a)$ holds, we can argue as above, so assume that $a > c$. For any $b \geq a$ we have in particular $b > c$, hence $\varphi(b)$ holds by assumption and $G_a \subseteq \varphi(\mathcal{U})$; therefore $\neg R_j(x, c)$ is tame. The formula $R_j(x, x \cap c)$ and its negation are tame, because $R_j(x, x \cap c)$ is always false. As for the formula $\varphi(x) := R_j(x \cap c_0, x \cap c_1)$, take $D = \{c_0, c_1\}$. If $a \not\leq c_0 \wedge a \not\leq c_1$, then for every $b > a$ and $i < 2$ we have $a \cap c_i = b \cap c_i$, hence $b > a \rightarrow (\varphi(a) \leftrightarrow \varphi(b))$, proving tameness of both $\varphi(x)$ and $\neg \varphi(x)$. Since the
The domination monoid: expansions

Assumption 4.4. Let \( \Phi(x) \) be an invariant \( T \)-formula with \( \ell \)-types over \( \mathcal{U} \). As noted above, we work in a purely binary cone-expansion \( T \) of \( \text{DMT} \), in a language \( L = L_{\text{inv}} \cup \{ R_j \mid j \in J \} \).

We saw in Theorem 3.11 that, in \( \text{DMT} \), domination-equivalence classes of invariant \( 1 \)-types correspond to either new open cones above existing points, or to certain cuts in linearly ordered subsets of \( \mathcal{U} \). In what follows, restrictions of invariant \( 1 \)-types to \( L_{\text{inv}} \), which are still invariant by Corollary 4.3, will play a special role; we therefore introduce some terminology for these cones and cuts.

Definition 4.5. Let \( p,q \in S_{1}^{\text{inv}}(\mathcal{U}) \) be nonrealised, and suppose that \( (p \upharpoonright L_{\text{inv}}) \sim_{D} (q \upharpoonright L_{\text{inv}}) \) in \( \text{DMT} \). If these restrictions are of kind (II), in a new open cone above \( g \in \mathcal{U} \), we say that \( p,q \) have the same sprout, and that each of them sprouts from \( g \). If the restrictions are of another kind, we say that \( p,q \) have the same graft.

So, in Theorem 3.11, \( X \) corresponds to the set of grafts, and there is a copy of \( \mathbb{N} \) for each sprout. The reason behind the choice of terminology should be clear from Figure 2.

Lemma 4.6. Let \( p(x),q(y) \in S_{1}^{\text{inv}}(\mathcal{U}) \). Denote by \( q \upharpoonright i \) the restriction of \( q \) to the variable \( x_i \), and similarly for \( p \). If, for all \( i < |y| \) and \( i' < |x| \), the types \( q \upharpoonright i \) and \( p \upharpoonright i' \) have the same graft, then \( p \sim_{D} q \).

Proof. As the roles of \( p,q \) are symmetric, it is enough to prove \( p \geq_{D} q \). By assumption and Theorem 3.11 we have \( (p \upharpoonright L_{\text{inv}}) \geq_{D} (q \upharpoonright L_{\text{inv}}) \), witnessed by some \( r' \) over a small set \( A \), and all coordinates of \( p,q \) have the same cut \( C \). Recall that \( C \) must be bounded by Corollary 4.3 and the last point of Lemma 4.6. Up to enlarging \( A \), we may assume that (cf. Lemma 2.4)

1. there is \( a \in \mathcal{U} \) such that \( a > C \) and, if \( C \) has a maximum \( g \), such that \( a \) is in the same open cone above \( g \) of each coordinate of \( p \) and \( q \); and

2. \( \mathcal{U}g \) is closed under meets and \( \text{dcl}^{L_{\text{inv}}}((\mathcal{U}g) \setminus \mathcal{U}) \subseteq \text{dcl}^{L_{\text{inv}}}(Ay) \).
Furthermore, up to adjoining to \( y \) finitely many points of \( \text{dcl}^{\text{mt}}(Ay) \), we may assume \( \text{dcl}^{\text{mt}}(Uy) \setminus U = y \). By the second point of Lemma 1.6, this assumption does not break the hypothesis that all points of \( y \) have the same graft. Fix any \( r \in S_pq(A) \) extending \( r' \), and recall that \( p \cup r \vdash (q \mid L_{\text{mt}}) \cup (q \mid A) \). By quantifier elimination and our assumptions on \( y \), we are only left to deal with the formulas \( R_j(y_i, f) \) and \( R_j(f, y_i) \), where \( f \in U \) and \( i < |y| \). We have three possibilities for \( y_i \cap f \). If \( y_i \cap f = y_i \), then \( y_i \leq f \). If instead \( y_i \cap f \in U \), then there must be a point of \( U \) in the same open cone as \( y_i \) above \( y_i \cap f \), because otherwise \( (q \mid i) \upharpoonright L_{\text{mt}} \) would be of kind (II). In the only other possible case, which can only arise if \( (q \mid i) \upharpoonright L_{\text{mt}} \) is of kind (IIIa) or (IIIb), it is easy to see that \( f \) must be in the same open cone above \( y_i \cap f \) as \( a \). In each case, since the \( R_j \) are on open cones, it is enough to show that at least one between \( R_j(y_i, f) \) and \( R_j(f, y_i) \) hold, and we are done. \( \square \)

**Remark 4.7.** The set of grafts of types in \( S_1^{\text{inv}}(U) \) can be identified with that of grafts of types in \( S_1^{\text{inv}}(U \upharpoonright L_{\text{mt}}) \).

**Proof.** The natural map from the former to the latter, well-defined by Corollary 4.3, is injective by definition of graft, and is surjective because, since \( T \) is a purely binary cone-expansion of DMT, if \( p \in S_1^{\text{inv}}(U \upharpoonright L_{\text{mt}}) \) is of kind (Ia) or (Ib), then \( p \) implies a unique type in \( S_1(U) \), easily seen to be invariant. \( \square \)

**Lemma 4.8.** Let \( p_0, \ldots, p_n \in S^{\text{inv}}(U) \) be such that \( \Phi := \bigcup_{i \leq n} (p_i(x^i) \mid L_{\text{mt}}) \) is a complete type in DMT. Then \( \bigcup_{i \leq n} p_i(x^i) \) is a complete type in \( T \).

**Proof.** Let \( b^i \vdash p_i \). In order for \( \Phi \) to be complete in DMT, given \( i < i' \leq n \), no coordinate of \( p_i \) can have the same graft as a coordinate of \( p_{i'} \); if this was the case for \( x_0^i \) and \( x_0^{i'} \), say, then there would be \( a \in U \) such that \( \Phi \) does not decide whether \( x_0^i \cap a = x_0^{i'} \cap a \) holds. Similarly, no coordinate of \( p_i \), say \( x_0^i \) again, can have the same sprout as a coordinate of \( p_{i'} \), say \( x_0^{i'} \), otherwise \( \Phi \) does not decide whether \( x_0^i = x_0^{i'} \) holds. It follows from this observation and Lemma 1.6 that \( \text{dcl}^{\text{mt}}(Uy^0, \ldots, b^n) = \bigcup_{i \leq n} \text{dcl}^{\text{mt}}(Uy^i) \). Therefore, we only need to show that, for each \( i < i' \leq n \), each \( y \in \text{dcl}^{\text{mt}}(Uy^i) \setminus U \), and each \( z \in \text{dcl}^{\text{mt}}(Uy^{i'}) \setminus U \), every formula of the form \( R_j(y, z) \) is decided by \( p_i(x^i) \cup p_{i'}(x^{i'}) \). Since the \( R_j \) are on open cones, it is enough to show that at least one between \( y \) and \( z \) must be in the same open cone above \( y \cap z \) as a point of \( U \). Again because \( \Phi \) is complete, \( y \) and \( z \) cannot have the same graft, nor the same sprout.

We have two cases. Suppose first that \( \Phi \vdash y \cap z = g \) for some \( g \in U \). This happens for example if \( \text{tp}(y/U) \) sprouts from \( g \) and the graft of \( \text{tp}(z/U) \) is in an existing open cone above \( g \), or if none of \( C^y \) and \( C^z \) is included in the other. Then, at least one between \( y \) and \( z \) must be in an open cone above \( g \) represented in \( U \), because otherwise both would be sprouting from \( g \), contradicting completeness of \( \Phi \).

If instead \( \Phi \vdash "y \cap z \notin U" \) then, up to swapping \( y \) and \( z \), we must have \( \Phi \vdash "C^y \subseteq C^z \cap U" \), because otherwise \( y \) and \( z \) have the same graft. It follows that for some \( a \in U \) we have \( \Phi \vdash y > a > y \cap z \), and in particular \( y \) is in the same open cone above \( y \cap z \) as \( a \). \( \square \)

Recall that a sort \( Y \) of a multi-sorted \( U \) is said to be *stably embedded* iff, whenever \( D \subseteq U^m \) is definable, then \( D \cap Y^m \) is definable with parameters from \( Y \), in the sense that it is definable with parameters when we view \( Y \) as a structure
on its own, the atomic relations being the traces on $Y$ of $\emptyset$-definable relations of $\mathfrak{U}$. It is easy to obtain a proof of the following fact; the reader may find one in \cite[Proposition 2.3.3]{Men20n}.

**Fact 4.9** ($T$ arbitrary). Let $Y$ be a stably embedded sort of $\mathfrak{U}$. There is an embedding of posets $\Inv(Y) \hookrightarrow \Inv(\mathfrak{U})$. This embedding is a $\mathcal{P}^\emptyset$-homomorphism, a $\mathcal{P}^\emptyset$-homomorphism, and, if $\otimes$ respects $\geq$, an embedding of monoids.

For $g \in \mathfrak{U}$, denote by $O_g$ the set of open cones above $g$ equipped with the $\{R_j \mid j \in J\}$-structure induced by $\mathfrak{U}$. This may be regarded as an imaginary sort of the expansion of $\mathfrak{U}$ obtained by naming the point $g$, and each type $p \in S_n(O_g)$ may be seen as the pushforward under the projection map of a suitable $q \in S_n(\mathfrak{U})$ with all non-realised coordinates sprouting from $g$. Since $T$ eliminates quantifiers, $q$ is axiomatised by its quantifier-free part. It follows easily that the same is true of $p$, and therefore $\Th(O_g)$ eliminates quantifiers in a binary language, hence is (weakly) binary. By Fact 3.4, $I_g := \Inv(O_g)$ is well-defined.

**Theorem 4.10.** Let $T$ be a purely binary cone-expansion of $\mathcal{DMT}$, and let $X$ be the set of grafts of types in $S^\text{fin}(\mathfrak{U})$. Then

$$\widetilde{\Inv}(\mathfrak{U}) \cong \mathcal{P}_\text{fin}(X) \oplus \bigoplus_{g \in \mathfrak{U}} I_g$$

**Proof.** Recall that by Theorem 2.8, $\widetilde{\Inv}(\mathfrak{U})$ is well-defined, and that by Corollary 4.3 taking restrictions to $L_{\text{nat}}$ preserves invariance. By Lemma 4.8, $\widetilde{\Inv}(\mathfrak{U})$ is generated by the $\sim_D$-classes of those types $p$ where coordinates of $p$ have all the same graft, or have all the same sprout. If all coordinates of $p$ have the same graft, by Lemma 4.6, $g$ is domination-equivalent to any 1-type with such a graft, and by using Lemma 4.8 a second time we see that $\mathcal{P}_\text{fin}(X)$ embeds in $\Inv(\mathfrak{U})$. Again by Lemma 4.8, $\Inv(\mathfrak{U}) = \mathcal{P}_\text{fin}(X) \oplus \bigoplus_{g \in \mathfrak{U}} I_g$, where $I_g$ is the monoid of types whose every coordinate sprouts from $g$.

We are only left to show that $I_g \cong I_g$. Fix $g \in \mathfrak{U}$. Since $\widetilde{\Inv}(\mathfrak{U})$ does not change after naming a small number of constants, we may add to $L$ a constant symbol to be interpreted as $g$. For the time being, we also adjoin to the language a sort for $O_g$ and its natural projection map $\pi_g$. Call the resulting structure $\mathfrak{U}_g$. Clearly $\mathfrak{U}$ is stably embedded in $\mathfrak{U}_g$, so by Fact 4.9 we have an embedding $\Inv(\mathfrak{U}) \hookrightarrow \Inv(\mathfrak{U}_g)$. Similarly, $O_g$ is stably embedded, hence $\Inv(O_g) = I_g$ embeds in $\Inv(O_g)$. Let $p$ be a type with all coordinates sprouting from $g$ (different coordinates might be in different open cones), and let $q$ be the pushforward of $p$ along $\pi_g$. Clearly $p \geq_D q$, and if we show that $q \geq_D p$ we may simply conclude by discarding the sort $O_g$ and forgetting about the new constant symbol. That $q \geq_D p$ is easily seen to be witnessed by any $r$ containing all the formulas $y_i = \pi_g x_i$ for $i < \langle x \rangle$: the only information lost when taking the projection concerns points in the same new open cone, but this information is in $r$. For instance, if $x_0 \cap x_1 > g$, we need to recover whether $R_j(x_0, x_1)$ holds, and whether any inequality holds between $x_0$ and $x_1$. More generally, the information we need to recover is implied by $p \mid \emptyset$, which is included in $r$ by Definition 3.1.\[\square\]
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