First-passage time distribution for a random walker on a random forcing energy landscape

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Received 17 June 2010
Accepted 17 August 2010
Published 13 September 2010

Online at stacks.iop.org/JSTAT/2010/P09005
doi:10.1088/1742-5468/2010/09/P09005

Abstract. We present an analytical approximation scheme for the disordered average first-passage time distribution on a finite interval of a random walker on a random forcing energy landscape. The approximation scheme captures the behavior of the distribution over all timescales in the problem. The results are carefully checked against numerical simulations.

Keywords: disordered systems (theory), stochastic processes (theory)

ArXiv ePrint: 1006.3088
1. Introduction

The dynamics of a random walker on a one-dimensional random forcing (RF) energy landscape has been a subject of much interest over the last three decades. Part of the interest is due to dynamics in quenched disordered systems. Examples are the dynamics of a random-field Ising model [1]–[4] and the motion of dislocations in disordered crystals [5]. More recently, much interest has been due to many applications in biophysical settings. It seems that in such systems random forcing energy landscapes are the rule and not the exception. In this context, the dynamics of random walkers on random forcing energy landscapes have found applications in the mechanical unzipping of DNA [6]–[8], translocation of biomolecules through nanopores [9] and the dynamics of molecular motors [10]–[13]. In many of these experiments the first-passage time (FPT) distribution is directly measurable and in some, such as in the translocation of biomolecules through nanopores, it is the most direct measurement.

On a lattice the dynamics of the random walker is defined through the hopping rates between neighboring sites. We denote the hopping rate from site $i$ to site $i+1$ by $p_i$ and from site $i$ to site $i-1$ by $q_i$. To generate an RF energy landscape the sets of rates $\{p_i\}$ and $\{q_i\}$ are drawn randomly from distributions $p(p_i)$ and $q(q_i)$, respectively (both distributions are independent of $i$). It is convenient to introduce an energy difference variable $\{E_i\}$ such that $p_{i-1}/q_i = e^{-E_i}$ (measuring energies in units of $k_BT$). Thus each realization of the environment in the RF model can also be drawn as a set of i.i.d. random
variables \( \{q_i\} \) and \( \{E_i\} \). Note that the energy landscape in this model is itself described by a biased random walk in energy space (see figure 1 for an illustration).

It is well known [14]–[17] that the long time and infinite lattice asymptotics of a random walker on such an energy landscape is rather rich. In particular, the dynamics are controlled by a parameter, \( \mu = |\hat{\mu}| \), defined through the non-zero solution of the equation [15,16,18]

\[
\left\langle \left( \frac{p_i}{q_{i+1}} \right)^{\hat{\mu}} \right\rangle = 1,
\]

where the angular brackets represent an average of realizations of disorder or, equivalently, over the distributions of \( \{p_i\} \) and \( \{q_i\} \). For \( \mu > 2 \) the behavior is similar to that of a biased random walker on a flat energy landscape. Namely, for large times the mean position \( \langle \bar{x} \rangle \) and its variance \( \langle (x - \bar{x})^2 \rangle \) both grow linearly in time (here the overbar denotes an average over histories of the system starting from the same initial position). In this regime both quantities are self-averaging. When \( 0 < \mu < 2 \) the behavior is anomalous. For \( 1 < \mu < 2 \) the mean displacement is self-averaging and \( \bar{x} \sim \langle \bar{x} \rangle \sim t \). The diffusion, however, behaves as \( (x - \bar{x})^2 \sim t^{2/\mu} \) while its average over realizations of disorder behaves as \( \langle (x - \bar{x})^2 \rangle \sim t^{3-\mu} \). For \( 0 < \mu < 1 \) both the drift and the diffusion are anomalous. Asymptotically the velocity, defined through \( \lim_{t \to \infty} \langle \bar{x} \rangle / t \), vanishes. Specifically, the drift behaves as \( \bar{x} \sim \langle \bar{x} \rangle \sim t^\mu \) (note that, although \( \bar{x} \) and \( \langle \bar{x} \rangle \) have the same scaling, \( \bar{x} \) is not a self-averaging quantity [19]). In addition \( (x - \bar{x})^2 \sim t^{2/\mu} \) and its average over the realizations of disorder behaves as \( \langle (x - \bar{x})^2 \rangle \sim t^{3-\mu} \). Finally, when \( \mu = 0 \) (commonly referred to as Sinai diffusion) \( \bar{x} \sim \ln^2 t \), \( \langle \bar{x} \rangle = 0 \), \( \frac{\bar{x}}{t} \sim \langle \bar{x}^2 \rangle \sim \ln^4 t \) [14] and \( (x - \bar{x})^2 \sim \langle (x - \bar{x})^2 \rangle \sim t^0 \) [20,21].

The FPT properties on an RF energy landscape also exhibit rich behavior. For a given disorder realization the FPT distribution has an exponential tail [21]. However, the disorder-averaged distribution of the FPT may have a heavy tail that separates the
typical and the average FPTs [22, 23] and dramatically affects the behavior of the FPT moments [24]. The latter is the focus of this paper.

In this paper we provide an approximation for the FPT distribution of a random walker on an RF energy landscape in a finite interval. Namely, we are interested in the disorder-averaged FPT probability density from site \( i_0 \geq 0 \) to the origin (site 0), \( F(t) = \langle F_{i_0 \rightarrow 0}(t)\{p_i, q_i\} \rangle \) with reflecting boundary conditions at lattice site \( d + 1 \) with \( d \geq i_0 \), so that \( p_d = 0 \) or \( E_{d+1} = \infty \) (see figure 1)\(^3\). The results are summarized in section 6. Note that this rather rich behavior makes it impossible to write the averaged propagator of the process as a scale-invariant function, except in the Sinai case \( \mu = 0 \). Therefore, the techniques developed in [25, 26] to calculate the distribution of FPTs for scale-invariant processes are not directly applicable. Our results are compared with numerical calculations and shown to agree very well. To obtain the approximation we present a ‘random tilt’ (RT) model, similar in spirit to that used in [27]. The parameters which define the RT model are given as functions of the parameters of a random walker on an RF energy landscape (for simplicity we assume a walker on a lattice) \textit{with no fitting parameters}. We stress that, in general, the dynamics on a disorder realization of the RT model do not reflect the dynamics on a given disorder realization of the RF model. Nevertheless, the RT model naturally exhibits all of the regimes exhibited by the RF model after a proper disorder average. Specifically, within the model we obtain an exact result for the Laplace transform

\[
\tilde{F}(s) = \int_0^\infty F(t)e^{-st} \, dt \tag{2}
\]
of the FPT between two given points with specified boundary conditions. The Laplace transform can be easily inverted numerically.

This paper is organized as follows: in section 2 we present a brief review of relevant known results for a random walker on an RF energy landscape. In section 3 we present and solve the RT model. In sections 5.2 and 5.1 we determine (as a function of the RF model parameters) the parameters of the RT model which are used to approximate the FPT probability density of the RF model. In section 5.3 we compare the approximation to numerical results.

2. Review of some known results for a random walker on an RF energy landscape

While the exact full FPT distribution, \( F(t) \), for which we provide an approximation, is not known, several related results are known. Specifically, the mean FPT, defined through \( \langle \tilde{t} \rangle = \int_0^\infty t F(t) \, dt \), can be obtained using the expression for a given realization of disorder (namely, a given set of \( \{E_i\} \) and \( \{q_i\} \)) [28]

\[
\tilde{t} = \sum_{i=1}^{i_0} \left( \frac{1}{q_d} \prod_{j=i_0}^{d-1} p_j + \sum_{k=1}^{d-1} \frac{1}{q_k} \prod_{j=k}^{k-1} p_j \right), \tag{3}
\]

\(^3\) To obtain \( i_0 < 0 \) with boundary condition \( q_{d<i_0} = 0 \) results one simply takes

\[
t_0 \rightarrow -i_0 \quad q_i \rightarrow p_i \quad E_i \rightarrow -E_i \quad d \rightarrow -d.
\]

doi:10.1088/1742-5468/2010/09/P09005
First-passage time distribution for a random walker on a random forcing energy landscape

where, as stated above, \( p_{i-1}/q_i = e^{-E_i} \). Higher moments of the FPT, \( \bar{t}^m \), may be obtained iteratively [29].

The disorder average of the mean FPT is

\[
\langle \bar{t} \rangle = \left( \frac{1}{q_i} \right) \frac{\langle e^{-E_i} \rangle^{d+1} - \langle e^{-E_i} \rangle^{d-i_0+1}}{(\langle e^{-E_i} \rangle - 1)^2} = \left( \frac{1}{q_i} \right) \frac{i_0}{\langle e^{-E_i} \rangle - 1}.
\]

Note that this quantity may be very different from the typical FPT, defined through \( \exp\langle \ln \bar{t} \rangle \) (see, for example, [23] for a discussion of the Sinai case \( \mu = 0 \)). From equation (3) one may see that in the large \( d \) limit the leading order of \( \langle \ln \bar{t} \rangle \) is \( \sum_{j=1}^{d} \ln(p_j/q_j) = -Ed \), where \( E = \langle E_i \rangle \) is the average tilt of the RF energy landscape. Therefore, the typical FPT scales as

\[
\exp\langle \ln \bar{t} \rangle \sim e^{-Ed}, \quad (5)
\]

increasing exponentially with \( d \) for \( E < 0 \) and remaining constant for \( E > 0 \).

In the limit \( d \to \infty \) the FPT probability density for \( E < 0 \) is not normalizable since the probability to never pass \( i = 0 \) is positive. For the \( E > 0 \) (and \( d \to \infty \)) case it is known [19, 30] that the mean FPT distribution density scales as \( \bar{t}^{-(1+\mu)} \) for large \( \bar{t} \). This implies that the value of \( \mu \) may be evaluated by finding the largest converging moment of the mean FPT for \( d \to \infty \). Equivalently, \( \mu \) is given by the smallest moment that diverges:

\[
\mu = \inf\{m: \langle \bar{t}^m \rangle = \infty \}. \quad (6)
\]

To obtain an approximation for the full FPT distribution, in the next section we introduce and solve a RT model. The parameters of the RT model are set by the known properties of the RF model.

3. The random tilt model

Recently Oshanin and Redner [27] used an optimal fluctuation method [31] to study the splitting probability, \( P \), of a random walker on a random forcing energy landscape. The splitting probability is defined as the probability to reach site \( i = d \) before hitting site \( i = 0 \), starting from site \( i = i_0 \). Within this approach one replaces each specific realization of the random forcing energy landscape by its average slope, \( U \), which is specified by the energy difference between the origin and the last, \( i = d \), site divided by the total number of sites (see figure 2). The central limit theorem implies that the probability density of the average energy tilt is Gaussian:

\[
Pr(U) = \frac{e^{-d((U-E)^2/2\sigma^2)}}{\sqrt{2\pi(\sigma^2/d)}}, \quad (7)
\]

where \( E \), as stated before, is the average energy difference between two subsequent sites:

\[
E = \langle E_i \rangle = \left\langle \ln \frac{q_i}{p_{i-1}} \right\rangle \quad (8)
\]

and \( \sigma^2 \) is the variance of the energy difference between two subsequent sites:

\[
\sigma^2 = \langle E_i^2 \rangle - \langle E_i \rangle^2 = \left\langle \ln^2 \frac{q_i}{p_{i-1}} \right\rangle - \left\langle \ln \frac{q_i}{p_{i-1}} \right\rangle^2. \quad (9)
\]
Using the fact that on a constant energy tilt, $U$, the splitting probability is given by $(1 - e^{U_{\text{tilt}}})/(1 - e^{U_{\text{finite}}})$ [32] the average splitting probability can be well approximated by [27]

$$P \approx \int_{-\infty}^{\infty} \frac{e^{-d((U-E)^2/2\sigma^2)}}{\sqrt{2\pi d}} \frac{1 - e^{U_{\text{tilt}}}}{1 - e^{U_{\text{finite}}}} dU. \quad (10)$$

Inspired by this approach we introduce a random tilt (RT) model. Within the model the energy landscape of each realization is flat while its tilt is a random variable. Namely, defining hopping rates to the right and left by $u$ and $v$, respectively, with $u/v = e^{-\varepsilon}$, we take $\varepsilon$ to be a random variable with a Gaussian probability density

$$\Pr(\varepsilon) = \frac{1}{\sqrt{2\pi \sigma^2 / d}} e^{-d((\varepsilon - \langle\varepsilon\rangle)^2/2\sigma^2)} \quad (11)$$

with a mean $\langle\varepsilon\rangle$ and a variance $\sigma^2$. Here $d$, as before, is the size of the system. As we show below by fixing $\langle\varepsilon\rangle$, $\sigma^2$ and an overall time we can approximate very well the first-passage behavior of a random walker on an RF energy landscape in the limit $d \to \infty$ and large $t$. We refer to this as the universal regime. In this paper, however, we are interested in an approximation over any timescale when $d$ is finite. In this case the small $t$ and large $t$ behaviors are not universal. As we show, a very good approximation can be achieved by taking the value of $v$ as a random variable with its own distribution $\Pr(v)$. In sections 5.1 and 5.2 we show that, in order to approximate the FPT probability density of the RF model, the simplest choice one can make is

$$\Pr(v) = \pi_1 \delta(v - v_1) + \pi_2 \delta(v - v_2) + (1 - \pi_1 - \pi_2) \delta(v - v_3). \quad (12)$$

The values of $\langle\varepsilon\rangle$, $\sigma^2$, $v_1$, $v_2$, $v_3$, $\pi_1$ and $\pi_2$ are set by known properties of the RF model as a function of $E$, $\sigma$, $\mu$, $d$, $\langle\bar{t}\rangle$, $\langle q_1^{-1}\rangle^{-1}$ and $\langle q_i \rangle$. The final expression does not involve any free parameters. As stated above, the results are summarized at the end of the paper.
First-passage time distribution for a random walker on a random forcing energy landscape

Note that to approximate only part of the FPT distribution a simpler choice of \( \Pr(v) \) can be made. For example, if one is not interested in the short time behavior one can choose \( \Pr(v) = \delta(v - v_0) \), where \( v_0 \) is specified in what follows (see equation (33) below). If one is not interested in the long time behavior but wants to capture the short time behavior one can choose \( \Pr(v) = \delta(v - \langle q_i \rangle) \) (see equation (37) below).

4. The properties of the RT model

To approximate the FPT distribution on an RF energy landscape we evaluate the splitting probability of the RT model, \( P_{RT} \), the analog of the exponent \( \mu \) of the RT model, denoted by \( \mu_{RT} \), and the average FPT, denoted by \( \langle \bar{t} \rangle_{RT} \). In addition we also calculate the Laplace transform of the FPT probability density of the RT model, \( \tilde{F}_{RT}(s) \).

The splitting probability of the RT model can be easily obtained, similar to equation (10), and is given by

\[
P_{RT} = \int_{-\infty}^{\infty} \frac{e^{-(\varepsilon-\langle \varepsilon \rangle)^2/2\sigma^2 \varepsilon}}{\sqrt{2\pi\langle \varepsilon^2 \rangle/d}} \frac{1 - e^{\varepsilon \cos \theta}}{1 - e^{\varepsilon i}} d\varepsilon. \tag{13}
\]

Note that equation (13) is exact for the RT model.

First we demonstrate that an analog of \( \mu \), denoted by \( \mu_{RT} \), exists for the RT model when \( d \to \infty \). This shows that indeed the model can reproduce a qualitative first-passage behavior similar to a random walker on an RF energy landscape. To do this we calculate the scaling behavior of the moments of the average FPT, \( \bar{t}_{\varepsilon} \). For a given realization of \( \varepsilon \) (the equivalent of a random realization of the RF energy landscape) this scales as

\[
\bar{t}_{\varepsilon} = \frac{1}{v} \left( \frac{u/v}{d} - 1 \right) \sim \begin{cases} 
1 & \varepsilon < 0 \\
\frac{1}{v} \frac{1 - e^{-\varepsilon d}}{1 - e^{-\varepsilon}} & \varepsilon > 0
\end{cases} \tag{14}
\]

Recalling that the probability density of \( \varepsilon \) is Gaussian

\[
\Pr(\varepsilon) \sim e^{-d(\varepsilon-\langle \varepsilon \rangle)^2/2\sigma^2 \varepsilon}, \tag{15}
\]

and in analogy with the RF model (see equation (6)), \( \mu_{RT} \) is given by

\[
\mu_{RT} = \inf \{ m : \langle \bar{t}_{\varepsilon}^m \rangle = \infty \}, \tag{16}
\]

where now the angular brackets denote an average over \( \varepsilon \) values. In the \( d \to \infty \) limit, when \( \langle \bar{t}_{\varepsilon}^m \rangle = \infty \), the average of the \( m \) th moment is controlled by the contributions where \( \varepsilon < 0 \) so that

\[
\langle \bar{t}_{\varepsilon}^m \rangle \sim \nu^{-m(\varepsilon)+m^2(\sigma^2/2)d}. \tag{17}
\]

We therefore obtain

\[
\mu_{RT} = \left| \frac{2\langle \varepsilon \rangle}{\sigma^2 \varepsilon} \right|. \tag{18}
\]

Thus, as stated above, an analog of \( \mu \) exists within the RT model after averaging over realizations of disorder. It, of course, does not exist for each realization of the model. Note that this expression is identical to that obtained for a full random forcing model where the random force is drawn from a Gaussian distribution with a mean \( \langle \varepsilon \rangle \) and variance...
\[ \sigma_\varepsilon^2 [19] \]. Furthermore, equation (14) implies that the scaling behavior of the typical FPT is given by
\[ \exp(\ln \bar{t})_{RT} \sim e^{-\langle \varepsilon \rangle d}, \] (19)
increasing exponentially with \( d \) for \( \langle \varepsilon \rangle < 0 \) and remaining constant when \( \langle \varepsilon \rangle > 0 \).

Next we consider the mean FPT of the RT model (averaged over realizations of the tilt of the landscape) to the origin from site \( i_0 \). For a given energy tilt, \( \varepsilon \), and a given left hopping rate, \( v \), the thermally averaged FPT may be obtained using equation (3) with the substitutions \( \langle 1/q_i \rangle = 1/v \) and \( \langle e^{-E_i} \rangle = e^{-\varepsilon} \). Averaging over \( \varepsilon \) and \( v \) one obtains
\[ \langle \bar{t} \rangle_{RT} = \left\langle \frac{1}{v} \right\rangle I \] (20)
where
\[ I = \int_{-\infty}^{\infty} \left[ \frac{e^{-\varepsilon(d+1)} - e^{-\varepsilon(d-i_0+1)}}{(e^{-\varepsilon} - 1)^2} - \frac{i_0}{e^{-\varepsilon} - 1} \right] \frac{e^{-d(\varepsilon-\langle \varepsilon \rangle)^2/2\sigma_\varepsilon^2}}{\sqrt{2\pi(\sigma_\varepsilon^2/d)}} \, d\varepsilon \] (21)
and
\[ \left\langle \frac{1}{v} \right\rangle = \int_0^\infty \frac{1}{v} \Pr(v) \, dv. \] (22)

The Laplace transform of the FPT probability density can also be obtained exactly and is given by
\[ \tilde{F}_{RT}(s) = \int \int \Pr(\varepsilon) \Pr(v) \Phi \left( \frac{s}{v}, \varepsilon \right) \, d\varepsilon \, dv, \] (23)
where \( \Phi(s/v, \varepsilon) \) is the FPT probability density on a flat energy landscape with a tilt \( \varepsilon \) and a left hopping rate \( v \). Using standard FPT results [32] one has
\[ \Phi \left( \frac{s}{v}, \varepsilon \right) = \frac{\lambda_2^{i_0} \left( \frac{s}{v}, \varepsilon \right)}{2^{i_0}} \left[ 1 - \left( \frac{\lambda_2 \left( \frac{s}{v}, \varepsilon \right)}{\lambda_1 \left( \frac{s}{v}, \varepsilon \right)} \right)^{d-i_0-1} \frac{(\varepsilon+1)\lambda_2 \left( \frac{s}{v}, \varepsilon \right)-2}{(\varepsilon+1)\lambda_1 \left( \frac{s}{v}, \varepsilon \right)-2} \right] \] (24)
with
\[ \lambda_{1,2} \left( \frac{s}{v}, \varepsilon \right) = 1 + \left( 1 + \frac{s}{v} \right) e^\varepsilon \pm \sqrt{1 + 2e^\varepsilon \left( \frac{s}{v} - 1 \right) + e^{2\varepsilon} \left( \frac{s}{v} + 1 \right)^2}. \] (25)

Next we use these results to approximate the FPT distribution in the RF model.

5. Approximating the FPT distribution of the RF model

To use the results to approximate the FPT probability density of the RF model here we set the parameters \( \langle \varepsilon \rangle \) and \( \sigma_\varepsilon^2 \) and the probability distribution \( \Pr(v) \) as a function of the RF model parameters such that \( \tilde{F}_{RT}(s) \) yields a good approximation to \( \tilde{F}(s) \). Namely, we work with the Laplace transform. The matching is done so the distributions agree in different ranges of \( s \) and can be done by matching between known quantities of both the RT and the RF models. We first determine the parameters \( \langle \varepsilon \rangle \) and \( \sigma_\varepsilon \).

doi:10.1088/1742-5468/2010/09/P09005

8
5.1. Determining $\langle \varepsilon \rangle$ and $\sigma_\varepsilon$

Here we find expressions for $\langle \varepsilon \rangle$ and $\sigma_\varepsilon$ by matching the scaling properties of the FPT probability densities. The value of $\langle \varepsilon \rangle$ may be found by matching the scaling behavior of the typical FPT of the two models in the large $d$ limit. We showed above (see equation (5)) that in the RF model the typical FPT grows as $e^{-Ed}$ for $E < 0$ and remains constant while $E > 0$ (see equation (14)). The same is true (see equation (19)) for the RT model when $E$ is replaced by $\langle \varepsilon \rangle$. Thus to match the scaling behavior of the typical FPT in both models we set

$$\langle \varepsilon \rangle = E. \quad (26)$$

The scaling properties in the intermediate $s$ regime for a finite value of $d$ can be expected to behave identically to the small $s$ behavior of the $d \to \infty$ limit. Since the scaling behavior is different for negative and positive average tilts we separate our discussion into two cases.

5.1.1. The case $E < 0$. In this case the splitting probability, $P$, remains finite in the $d \to \infty$ limit. Thus, the small $s$ behavior of the FPT probability density in the $d \to \infty$ limit is $1 - P + O(s)$. To match the intermediate $s$ behavior of FPT probability density of RF and RT models we match the splitting probabilities (i.e. the leading, $O(s^0)$, order of the FPT probability densities) of these two models.

Comparing (10) with (13) and using equation (26) yields

$$\sigma_\varepsilon = \sigma. \quad (27)$$

Note that both the average tilt and the variance in the RF and RT models are equal in this regime.

5.1.2. The case $E > 0$. As studied in section 1, in this case the mean FPT distribution for $d \to \infty$ in the small $s$ region scales as $1 - O(s^\mu)$ for $\mu < 1$ (or, equivalently in time, as $\tilde{t}^{-(1+\mu)}$). Therefore, we first match $\mu_{RT}$ of the RT model with $\mu$ of the RF model. Namely, we set $\mu_{RT} = \mu$. Using equation (18) this yields

$$\frac{2 \langle \varepsilon \rangle}{\sigma_\varepsilon^2} = \mu. \quad (28)$$

Solving equations (26) and (28) we get

$$\sigma_\varepsilon = \sqrt{\frac{2E}{\mu}}. \quad (29)$$

Note when the distribution of $E_i$ is Gaussian equations (27) and (29) become identical. Namely, both the average tilt and the variance of the RF and RT models are equivalent. When the distribution is not Gaussian the average tilt is the same in both models but the variances can be different. In this regime it is important to match the precise value of $\mu$ rather than the splitting probability as in the $E < 0$ case.
5.2. Determination of \( \Pr(v) \)

In this section we choose the distribution \( \Pr(v) \) by matching the FPT probability density of the RT model to that of the original RF system in the small and large \( s \) limits. We show that for each limit one may choose \( v \) to be a constant. However, this constant depends on the limit. Thus, to match the full \( s \) behavior \( v \) cannot be a constant and, as we show below, the simplest choice for \( \Pr(v) \) is

\[
\Pr(v) = \pi_1 \delta(v - v_1) + \pi_2 \delta(v - v_2) + (1 - \pi_1 - \pi_2) \delta(v - v_3) \tag{30}
\]

with \( v_{1,2,3} \) set according to relevant timescales in the problem and \( \pi_{1,2} \) determined by the matching procedure, described below.

5.2.1. The small \( s \) approximation. The \( s \rightarrow 0 \) behavior of the FPT probability densities of the RF and the RT models is

\[
\tilde{F}(s \rightarrow 0) = 1 - \langle \bar{t} \rangle s \tag{31}
\]

and

\[
\tilde{F}_{RT}(s \rightarrow 0) = 1 - \langle \bar{t}_{RT} \rangle s, \tag{32}
\]

respectively. Therefore, to match \( \tilde{F}(s \rightarrow 0) \) and \( \tilde{F}_{RT}(s \rightarrow 0) \) one has to match the mean FPTs of both models. Demanding \( \langle \bar{t} \rangle_{RT} = \langle \bar{t} \rangle \) and using equation (20) one has

\[
\langle \frac{1}{v} \rangle = \frac{\langle \bar{t} \rangle}{I} \tag{33}
\]

where \( I \) is defined in equation (21) and \( \langle \bar{t} \rangle \) is given by equation (4).

Below we use this fact to match the behavior of the FPT distribution of the RF model over the whole \( s \) range. Note, however, that if one is not interested in the short time behavior the procedure is simplified. One can then choose \( \Pr(v) = \delta(v - v_1) \), where \( v_1 = I/\langle \bar{t} \rangle \). Then, using equation (23), the Laplace transform of the FPT probability density is given by

\[
\tilde{F}_{RT}(s) = \int_{-\infty}^{\infty} \frac{e^{-d((\epsilon - \langle \tilde{\epsilon} \rangle)^2/2\sigma^2)}}{\sqrt{2\pi(\sigma^2/d)}} \Phi \left( \frac{s}{v_1}, \frac{\epsilon}{\sqrt{2\sigma^2/d}} \right) \, d\epsilon. \tag{34}
\]

5.2.2. The large \( s \) approximation. The large \( s \) limit corresponds to the small \( t \) limit. This regime of the FPT distribution is controlled by walks which hop only to the left before reaching the origin. Therefore, in the \( s \rightarrow \infty \) limit the behavior of the FPT probability densities of the RF and the RT models is

\[
\tilde{F}(s \rightarrow \infty) = \langle q \rangle^{io} \tag{35}
\]

and

\[
\tilde{F}_{RT}(s \rightarrow \infty) = \langle v^{io} \rangle, \tag{36}
\]

respectively. Thus, to match the large \( s \) behavior of \( \tilde{F}_{RT}(s) \) and \( \tilde{F}(s) \) we demand

\[
\langle v^{io} \rangle = \langle q \rangle^{io}. \tag{37}
\]
Figure 3. In this graph a comparison between the analytical approximation (47) and the numerical results for a Bernoulli disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: \( r = 0.5, \varepsilon_1 = -0.4, \varepsilon_2 = 0.4 \) and \( d = 100 \), such that \( \mu = 0 \). Circles represent the choice \( i_0 = 1 \) while squares represent the choice \( i_0 = 50 \) cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

Next we match the behaviors in the RT and RF models over the whole range of \( s \). However, we comment that if one is not interested in the approximate FPT distribution in the long time limit one can choose \( \Pr(v) = \delta(v - v_2) \), where \( v_2 = \langle q \rangle \). Then, using equation (23), the Laplace transform of the FPT probability density is given by

\[
\tilde{F}_{\text{RT}}(s) = \int_{-\infty}^{\infty} \frac{e^{-d(\varepsilon-\langle \varepsilon \rangle)^2/2\sigma_\varepsilon^2}}{\sqrt{2\pi\sigma_\varepsilon^2/d}} \Phi \left( \frac{s}{v_2}, \varepsilon \right) \mathrm{d}\varepsilon. \tag{38}
\]

5.2.3. An approximation for the whole \( s \) range. To match both the small and large \( s \) limit one should supply \( \Pr(v) \) such that equations (33) and (37) are satisfied. However, these equations do not determine \( \Pr(v) \) uniquely and there is a lot of freedom. As stated above, the simplest choice that satisfies these equations is\(^4\)

\[
\Pr(v) = \pi_1 \delta(v - v_1) + \pi_2 \delta(v - v_2) + (1 - \pi_1 - \pi_2) \delta(v - v_3). \tag{39}
\]

\(^4\) Other, simpler choices lead to nonlinear equations. For example, using \( \Pr(v) = \frac{1}{2} \delta(v - v_1) + \frac{1}{2} \delta(v - v_2) \), equations (33) and (37) lead to \( 1/v_1 + 1/v_2 = 2\langle T \rangle/I \) and \( v_1^{\alpha_0} + v_2^{\alpha_0} = 2\langle q \rangle^{\alpha_0} \).
First-passage time distribution for a random walker on a random forcing energy landscape

Figure 4. In this graph a comparison between the analytical approximation (47) and the numerical results for a Bernoulli disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $r = 0.49, \varepsilon_1 = -0.1, \varepsilon_2 = 0.1$ and $d = 550$, such that $\mu = 0.4$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 100$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

However, since one has to satisfy only two equations ((33) and (37)) there is still a lot of freedom with five free parameters. Therefore, first we set $v_{1,2,3}$ such that they represent relevant timescales in the problem. Equation (33) suggests

$$
\frac{1}{v_1} = \langle \tilde{t} \rangle_I,
$$

while equation (37) suggests

$$
v_2 = \langle q \rangle.
$$

The third timescale is chosen to represent the average time of a hop to the left:

$$
v_3 = \left\langle \frac{1}{q} \right\rangle^{-1}.
$$

Given these three quantities, $v_{1,2,3}$, equations (33) and (37) give

$$
\pi_1 v_1^{i_0} + \pi_2 v_2^{i_0} + (1 - \pi_1 - \pi_2) v_3^{i_0} = \langle q \rangle^{i_0}
$$
Using these quantities, the probability density for $\pi$ and $\pi$ and the numerical results for a Bernoulli disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $r = 0.58$, $\varepsilon_1 = -0.3$, $\varepsilon_2 = 0.3$ and $d = 150$, such that $\mu = -1.08$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 50$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

Figure 5. In this graph a comparison between the analytical approximation (47) and the numerical results for a Bernoulli disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $r = 0.58$, $\varepsilon_1 = -0.3$, $\varepsilon_2 = 0.3$ and $d = 150$, such that $\mu = -1.08$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 50$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

and

$$\frac{\pi_1}{v_1} + \frac{\pi_2}{v_2} + \frac{1 - \pi_1 - \pi_2}{v_3} = \frac{\langle \bar{t} \rangle}{T}.$$  \hspace{1cm} (44)

The solution for $\pi_{1,2}$ is

$$\pi_1 = \frac{v_1 \langle \bar{t} \rangle v_2 v_3 (v_1^{0} - v_3^{0}) + (I \langle q \rangle / \langle \bar{t} \rangle)(v_2 - v_3) + (I / \langle \bar{t} \rangle)(v_3^{-1} - v_2^{-1})}{v_2 v_3 (v_2^{0} - v_3^{0}) + v_1^{0} (v_2 - v_3) + v_1 (v_3^{-1} - v_2^{-1})}$$  \hspace{1cm} (45)

and

$$\pi_2 = -\frac{v_2 \langle \bar{t} \rangle v_1 v_3 (v_1^{0} - v_3^{0}) + (I \langle q \rangle / \langle \bar{t} \rangle)(v_1 - v_3) + I / \langle \bar{t} \rangle(v_3^{1} - v_1^{1})}{v_2 v_3 (v_2^{0} - v_3^{0}) + v_1^{1} (v_2 - v_3) + v_1 (v_3^{1} - v_2^{1})}.$$  \hspace{1cm} (46)

Using these quantities, the probability density (39) for $v$ and equation (23), the Laplace transform of the FPT probability density is given by

$$\tilde{F}_{\text{RT}}(s) = \int_{-\infty}^{\infty} \frac{e^{-d(\varepsilon - \langle \varepsilon \rangle)^2 / 2\sigma_\varepsilon^2}}{\sqrt{2\pi(\sigma_\varepsilon^2/d)}} \left[ \pi_1 \Phi\left( \frac{s}{v_1}, \varepsilon \right) + \pi_2 \Phi\left( \frac{s}{v_2}, \varepsilon \right) + (1 - \pi_2 - \pi_3) \Phi\left( \frac{s}{v_3}, \varepsilon \right) \right] d\varepsilon.$$  \hspace{1cm} (47)

doi:10.1088/1742-5468/2010/09/P09005
In this graph a comparison between the analytical approximation (47) and the numerical results for a Gaussian disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $E = 0$, $\sigma = 0.3$ and $d = 150$, such that $\mu = 0$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 50$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

In section 5.3 we compare the obtained result with the numerical simulation. This expression with the above chosen values of parameters is the main result of our paper. Note that while quite a few parameters have to be set there are no free fitting parameters. As we show, the agreement is very good.

5.3. FPT probability density and comparison with numerical result

In this section we compare between equation (47) and numerical simulations. Namely, we compare the Laplace transform of the FPT probability, density, $\tilde{F}_{RT}(s)$, and the survival probability

$$ F_S(t) = 1 - \int_0^t F(t') \, dt' $$

of the RF and RT models. We checked the results for RF models with both Bernoulli and Gaussian disorders.

For the RF system with Bernoulli disorder the energy difference on each site, $E_i$, is drawn from the distribution

$$ \Pr(E_i) = \begin{cases} 
    r & E_i = \varepsilon_1 \\
    1 - r & E_i = \varepsilon_2.
\end{cases} $$

\[\text{doi:10.1088/1742-5468/2010/09/P09005}\]
First-passage time distribution for a random walker on a random forcing energy landscape

Figure 7. In this graph a comparison between the analytical approximation (47) and the numerical results for a Gaussian disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $E = 0.01$, $\sigma = 0.3$ and $d = 250$, such that $\mu = 0.2222$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 50$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

In figures 3–5 one may see the comparison between equation (47) and the numerical calculation in different parameter ranges. As can be seen the agreement is very satisfying.

For an RF model with a Gaussian disorder the energy difference between sites, $E_i$, is drawn from a normal distribution with a mean $E$ and a variance $\sigma^2$:

$$\Pr(E_i) = \frac{e^{-d((E_i-E)^2/2\sigma^2)}}{\sqrt{2\pi(\sigma^2/d)}}. \quad (50)$$

In figures 6–8 we show a comparison between equation (47) and numerical calculations in different parameter ranges. Again, the results of the approximation are very good.

6. Summary

In this paper we presented a random tilt model and solved it analytically. We showed that the model can be used to obtain an approximate expression for the disorder-averaged FPT distribution of a random walker on a random forcing energy landscape. To do this several parameters of the RT model have to be set as a function of the RF model parameters. As we showed, this can be done with no free fitting parameters. For convenience we summarize the results below.

doi:10.1088/1742-5468/2010/09/P09005
First-passage time distribution for a random walker on a random forcing energy landscape

![Graphs showing first-passage times](image)

**Figure 8.** In this graph a comparison between the analytical approximation (47) and the numerical results for a Gaussian disorder is shown in Laplace (top) and time (bottom) spaces. The parameters for these plots are: $E = -0.01$, $\sigma = 0.2$ and $d = 350$, such that $\mu = -0.5$. Circles represent the choice $i_0 = 1$ while squares represent the choice $i_0 = 50$ cases in both, left and right, figures. The red lines are analytical approximations based on equation (47).

The approximation for the random walker’s Laplace-transformed FPT probability density from site $i_0 > 0$ to the origin on a random forcing energy landscape with i.i.d. random variables $\{p_1\}$ and $\{q_1\}$ is

$$
\tilde{F}(s) = \frac{e^{-d((s-\langle \epsilon \rangle)^2/2\sigma^2) \Phi} \left( \frac{s}{v_1}, \frac{\epsilon}{\sigma^2} \right) + \pi_2 \Phi \left( \frac{s}{v_2}, \frac{\epsilon}{\sigma^2} \right)}{\sqrt{2\pi}} \left[ \pi_1 \Phi \left( \frac{s}{v_1}, \frac{\epsilon}{\sigma^2} \right) \phi \left( \frac{s}{v_2}, \frac{\epsilon}{\sigma^2} \right) \right] d\epsilon \quad \text{(equation (47))},
$$

where

$$
\Phi \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right) = \frac{\lambda_2 \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right)}{\lambda_1 \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right)} \left[ \lambda_1 \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right) + 1 \right]^{-d-1} \left[ \lambda_1 \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right) + 1 \right]^{-2} \quad \text{(equation (24))},
$$

$$
\lambda_{1,2} \left( \frac{s}{v}, \frac{\epsilon}{\sigma^2} \right) = 1 + \left( 1 + \frac{s}{v} \right) e^\epsilon \sqrt{1 + 2e^\epsilon \left( \frac{s}{v} - 1 \right) + e^{2\epsilon} \left( \frac{s}{v} + 1 \right)^2} \quad \text{(equation (25))},
$$

doi:10.1088/1742-5468/2010/09/P09005 16
First-passage time distribution for a random walker on a random forcing energy landscape

\[
\langle \bar{t} \rangle = \frac{1}{\langle q_i \rangle} \left( \frac{\langle e^{-E_i} \rangle^{d+1} - \langle e^{-E_i} \rangle^{d-i_0+1}}{\langle e^{-E_i} \rangle - 1} \right) - \frac{1}{\langle q_i \rangle} \left( \frac{i_0}{\langle e^{-E_i} \rangle - 1} \right) \quad \text{(equation (4))},
\]

\[
I = \int_{-\infty}^{\infty} \left[ \frac{\langle e^{-\varepsilon(d+1)} - e^{-\varepsilon(d-i_0+1)} \rangle}{(e^{-\varepsilon} - 1)^2} - \frac{i_0}{e^{-\varepsilon} - 1} \right] \frac{e^{-d(\varepsilon - \langle \varepsilon \rangle)^2/2\sigma_d^2}}{\sqrt{2\pi(\sigma_d^2/d)}} \, d\varepsilon \quad \text{(equation (21))},
\]

\[
v_1 = \frac{I}{\langle \bar{t} \rangle} \quad \text{(equation (40))},
\]

\[
v_2 = \langle q_i \rangle \quad \text{(equation (41))},
\]

\[
v_3 = \frac{1}{\langle q_i \rangle}^{-1} \quad \text{(equation (42))},
\]

\[
\pi_1 = \frac{v_1 \langle \bar{t} \rangle v_2 v_3 (v_2^{i_0} - v_3^{i_0}) + (I \langle q_i \rangle^{i_0} / \langle \bar{t} \rangle)(v_2 - v_3) + I / \langle \bar{t} \rangle (v_3^{i_0+1} - v_2^{i_0+1})}{v_2 v_3 (v_2^{i_0} - v_3^{i_0}) + v_1^{i_0+1} (v_2 - v_3) + v_1 (v_3^{i_0+1} - v_2^{i_0+1})} \quad \text{(equation (45))},
\]

\[
\pi_2 = -\frac{v_2 \langle \bar{t} \rangle v_1 v_3 (v_1^{i_0} - v_3^{i_0}) + (I \langle q_i \rangle^{i_0} / \langle \bar{t} \rangle)(v_1 - v_3) + I / \langle \bar{t} \rangle (v_3^{i_0+1} - v_1^{i_0+1})}{v_2 v_3 (v_1^{i_0} - v_3^{i_0}) + v_1^{i_0+1} (v_2 - v_3) + v_1 (v_3^{i_0+1} - v_2^{i_0+1})} \quad \text{(equation (46))},
\]

\[
\langle \varepsilon \rangle = E \quad \text{(equation (26))}
\]

and

\[
\sigma_\varepsilon = \left\{ \begin{array}{ll}
\sigma & \quad E \leq 0 \\
\sqrt{\frac{2E}{\mu}} & \quad E > 0
\end{array} \right. \quad \text{(equations (27) and (29))}.
\]

If one is not interested in the large or in the small \( s \) behavior one may use the much simpler equations (34) or (38), respectively, instead of equation (47).

Comparing this approximation with the numerically calculated FPT distribution of the RF model we showed that the first may serve as a good approximation to the second. Finally, similar methods can be used to approximate other, say absorbing, boundary conditions at \( i = d + 1 \).

**Acknowledgments**

We thank S Redner for useful discussions. This work was supported by the High Council for Scientific and Technological Cooperation between France and Israel. MS and YK were also supported by the Israeli Science Foundation, and OB and RV by ANR grant ‘Dyoptri’.

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