TOPOLOGICAL PHASES AND THEIR DUALITY IN
ELECTROMAGNETIC AND GRAVITATIONAL FIELDS

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ABSTRACT

The duality found by Aharonov and Casher for topological phases in the electromagnetic field is generalized to an arbitrary linear interaction. This provides a heuristic principle for obtaining a new solution of the field equations from a known solution. This is applied to the general relativistic Sagnac phase shift due to the gravitational field in the interference of mass or energy around a line source that has angular momentum and the dual phase shift in the interference of a spin around a line mass. These topological phases are treated both in the linearized limit of general relativity and the exact solutions for which the gravitational sources are cosmic strings containing torsion and curvature, which do not have a Newtonian limit.

0. Introduction

As is well known, some of Yakir Aharonov’s most famous contributions concern topological phases due to the electromagnetic field. It is therefore fitting on this occasion of his sixtieth birthday for me to present him with some observations concerning these phases, which generalize naturally to the gravitational field. In particular, I shall examine the duality between the Aharonov-Bohm (AB) phase [1] and the phase shift in the interference of a magnetic moment in an electric field [2] which was found by Aharonov and Casher (AC) [3]. I shall show, by means of the linearized limit and an exact solution of the gravitational field equations, that both these phases have gravitational analogs and they satisfy this duality.

In section 1, I shall briefly summarize the phase shifts in the interference of a charge and a magnetic dipole (at low energies) due to the electromagnetic field. These phase shifts reveal, respectively, \( U(1) \) and \( SU(2) \) gauge field aspects of the electromagnetic field. But these two aspects are not independent: The \( SU(2) \) connection which gives the dipole phase shift depends on the electric and magnetic fields and as such are derived from the electromagnetic connection that gives the \( U(1) \) AB phase shift. It is nevertheless amusing to see a charged particle with a magnetic moment, such as an electron, interacting with an electromagnetic field as if it is a \( U(1) \times SU(2) \) gauge field. Two topological phase shifts due to electric and magnetic fields corresponding to two \( U(1) \) subgroups of \( SU(2) \) will be reviewed. The
duality between one of these and the AB phase, found by AC, will be generalized to an arbitrary interaction in section 2. I shall formulate a duality principle which states that any two dual phases are equal under certain conditions.

The gravitational phase shifts, obtained in section 3, are special cases of the phase shifts obtained previously [4, 5] due to the coupling of the mass and spin to the gravitational field. The key to the analogy with the electromagnetic phase shifts is that the mass or energy plays the role of the electric charge and spin the role of the magnetic dipole in the electromagnetic field. The gravitational phase shifts are the same as due to the phase shifts of a Poincare gauge field. The translational and Lorentz aspects of the Poincare group are respectively analogous to the U(1) and SU(2) aspects of the electromagnetic field, mentioned above.

If gravity contains torsion, as will be assumed here, the connection, which gives the phase shift of the spin, is independent of the metric or the vierbein, which gives the phase shift due to the mass or energy-momentum. Therefore, these two aspects are then complementary, unlike in the electromagnetic case in which the SU(2) connection depends on the U(1) connection as mentioned earlier. The electromagnetic field and its sources of course must satisfy the Maxwell’s equations. It is well known that the solenoid which produces the AB phase shift is a solution. Similarly, the gravitational field and its sources must satisfy Einstein’s field equations or a suitable generalization of it to include torsion. Fortunately, an exact solution corresponding to a spinning cosmic string with angular momentum and mass, which is the analog of the solenoid with a coaxial line charge in the electromagnetic case, can be obtained everywhere including the interior of the string.

There is a topological general relativistic Sagnac phase [4] which depends on the energy of a particle outside the string and the flux of torsion inside the string produced by its spin. This is analogous to the AB phase. There is another topological phase which depends on the spin of the particle and the flux of curvature inside the string, produced by its mass. This is the dual of the former phase. I shall show that this pair of dual phases satisfy the duality principle formulated in section 2.

1. Topological and Geometrical Phases due to the Electomagnetic Field

For simplicity, consider the non relativistic Hamiltonian of a charged particle in an electromagnetic field

\[ H = \frac{1}{2m} (\mathbf{p} - e\mathbf{A})^2 + eA_0, \]

(1.1)

where \( e \) and \( m \) are the charge and mass of the particle and \( A_\mu = (A_0, A_i) = (A_0, -A^i) \) is the electromagnetic potential representing the \( U(1) \) connection due to this field*. As is well known, (1.1) predicts the Aharonov-Bohm (AB) effect [1]. This is the electromagnetic phase difference between two interfering coherent beams which are entirely in a multiply connected region in which the field strength \( F_{\mu\nu} \) is zero. The phase factor that determines the electromagnetic shift in the interfering fringes is

\[ \Phi_C = \exp\left(-\frac{ie}{\hbar} \oint_C A_\mu dx^\mu\right), \]

(1.2)

where the closed curve \( C \) passes through the two interfering wave functions, and encloses a region in which the field strength is non zero. Consequently, this phase

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* Units in which the velocity of light \( c = 1 \) will be used throughout.
shift is constant as the beams are varied in the outside region in which the field strength is zero, which makes this effect topological. Conversely, the experimental observation of this phase shift may be used to infer that the electromagnetic field is a $U(1)$ gauge field [6,7].

Consider now the interaction of a neutral magnetic dipole, such as a neutron, with the electromagnetic field which at low energies is described by the Hamiltonian [2,3,8,9]

$$H = \frac{1}{2m}(p - \gamma A^k S_k)^2 + \gamma A_0^k S_k,$$

where $A_\mu^k = (A_0^k, A_i^k) = (-B^k, \epsilon_{ijk} E^j)$ in terms of the electric field $E$ and the magnetic field $B$, $S_k, k = 1, 2, 3$ are the spin components which generate the $SU(2)$ spin group. For a spin $\frac{1}{2}$ particle, the magnetic moment $\mu = \frac{\gamma h}{2}$. This interaction is like that of an isospin with an $SU(2)$ Yang-Mills field [8,9,10].

The phase shift due to both electric and magnetic fields, in the interference of a neutral dipole such as a neutron, was obtained by means of an explicit plane wave solution [2]. This result, of course, applies also to the more general situation when the interfering wave functions are locally approximate plane waves so that the WKB approximation is valid. Hence, the phase shift is determined by the non-abelian phase factor

$$\Phi_C = P \exp\left( -\frac{i\gamma}{h} \oint_C A_\mu^k S_k dx^\mu \right),$$

where $P$ denotes path ordering, and $C$ is a closed curve consisting of unperturbed trajectory [8]. Hence, $\Phi_C$ is an element of $SU(2)$, and this phase shift is like the phase shift due to an $SU(2)$ gauge field [5,6,11].

The special case when the two waves interfere around a line charge was considered by AC [3]. In this case, $A_0^k = 0$ and the electric field $E^j$ and therefore $A_i^k = \epsilon_{ijk} E^j$ fall off inversely as the distance from the line charge. It follows immediately from (1.4) that, if the spin is polarized parallel to the line charge, then this phase shift is topological in the sense that it does not change when the curve $C$ surrounding the line charge is deformed. However, the Yang-Mills field strength $F_{\mu \nu}^k$ of $A_\mu^k$ is non-vanishing outside the line charge, which makes this effect fundamentally different from the AB effect in which the electromagnetic field strength $F_{\mu \nu} = 0$ along the beams. But if the line charge is in the 3-direction then $F_{3 \mu}^3 = 0$. That is the field strength corresponding to the $U(1)$ subgroup of the spin $SU(2)$ generated by $S_3$ is zero. So, for this subgroup, this phase shift is like the AB effect.

Another topological phase shift experienced by the dipole is the following: The wave packet of a neutron or an atom is split into two coherent wave packets, one of which enters a cylindrical solenoid. The homogeneous magnetic field of the solenoid is then turned on and is then turned off before the wave packet leaves the solenoid. Then there is a phase shift even though there is no force acting on the neutron. This phase shift, which is easily obtained from (1.4), is due to $A_0^k = -B^k$. Hence, this phase shift is due to the scalar potential of the gauge field of the $U(1)$ subgroup of the spin $SU(2)$ group generated by the component of $S$ in the direction of $B$. At the suggestion of Zeilinger [12] and the author [8] this experiment was performed for neutrons [13].

The general case of the phase shifts for a particle that has charge and magnetic moment interacting with an electromagnetic field was studied before [8,9], I shall restrict here, for simplicity, the special case of the particle being a Dirac electron with “g-factor” being two. Its Hamiltonian, at low energies in the inertial
frame of the laboratory, is

\[ H = \frac{1}{2m}(p - eA - \frac{1}{2}\gamma'A^kS_k)^2 + eA_o + \gamma'A_o^kS_k, \]  

(1.5)

This Hamiltonian is like as if the electron is interacting with a \( SU(2) \times U(1) \) gauge field represented by the gauge potentials \( A^k \) and \( A \). Note, however, the factor \( \frac{1}{2} \) in front of \( A^k \) compared to (1.3). This is due to the Thomas precession undergone by the electron when it accelerates in the electric field [2,9].

2. The Duality of AC and its Generalization

The major new contribution of AC, which is not contained in any earlier work, is the recognition that the phase shift due to the line charge is “dual” to the AB effect due to a solenoid. I shall now give a precise statement of this duality which would be general enough to apply to other interactions as well.

Suppose that an infinite uniform solenoid is situated along the z-axis of a Cartesian coordinate system. A charge of strength \( e \) is taken slowly around the solenoid along a circle in the \( xy \)-plane with its center at the solenoid, which is assumed to have negligible cross-section. The solenoid may be regarded as a magnetized medium with a constant magnetic moment per unit length equal to \( M \), say, which is parallel to the \( z \)-axis. The AB phase shift acquired by the charge is

\[ \Delta \phi = \frac{e}{\hbar} \int \Sigma B \cdot d\Sigma = \frac{eM}{\hbar}, \]  

(2.1)

where \( \Sigma \) is a cross-section of the solenoid, \( B \) is the magnetic field inside the solenoid, and \( M = |M| \).

Now, divide the solenoid into slices each of height \( \delta \ell \) bounded by cross-sections that are parallel to the \( z \)-axis. The magnetic moment of each slice is

\[ \mu = M\delta \ell. \]  

(2.2)

The linearity of Maxwell’s equations imply that the phase shift is linear in the sense that (2.1) is the sum of the phase shifts due to the influence of each slice of the solenoid on the charge. Consider a slice whose center is at \( z = Z \). Then taking the charge around the circle mentioned above is equivalent to keeping the charge fixed at \( z = -Z \) and taking the slice of magnetic moment \( \mu = |\mu| \) around the same circle in the \( xy \)-plane. The phase shifts acquired in both processes are the same. This has been shown using space-time translation and Galilei invariance of the Lagrangian, for the special case of charge-dipole interaction [3] and using Lorent invariance for the general case of an arbitrary interaction [9]. Now do this for each pairwise interaction between the charge \( e \) and each slice with magnetic moment \( \mu \). Then, as we account for all slices from \( z = +\infty \) to \( z = -\infty \), in the new situation, which will be called the dual of the original situation, there are charges from \( z = -\infty \) to \( z = +\infty \) along the \( z \)-axis, and the magnetic moment \( \mu \) circles around this line charge. Each charge \( e \) is contained in an interval of height \( \delta \ell \), and may be assumed to be spread uniformly in that interval. Therefore,

\[ e = \lambda \delta \ell, \]  

(2.3)

where \( \lambda \) is the charge per unit length. It follows that the magnetic moment which circles around this line charge, with its direction parallel to the \( z \)-axis, acquires a phase shift equal to \( \Delta \phi \) given by (2.1). From (2.2) and (2.3),

\[ \frac{e}{\lambda} = \frac{d}{M}. \]  

(2.4)
for these two dual situations. Using (2.2), (2.1) may be rewritten as

$$\Delta \phi = \frac{\lambda \mu}{\hbar}. \quad (2.5)$$

This phase shift may also be independently derived using (1.4) and the electric field of a line charge obtained by solving Maxwell's equations.

The above argument may be generalized to the case when the charge goes around an arbitrary closed curve \( r(t) \) which may or may not enclose the solenoid. Then relative to one of the above mentioned slices at say \( Z = (0, 0, Z) \) this curve is \( r(t) - Z \). Therefore, in the dual situation the slice with magnetic moment \( \mu \) moves around the closed curve \(-r(t) + Z\) relative to the charge. So, if the charge is placed at \(-Z\), the slice goes around the closed curve \(-r(t)\). By doing this for each pairwise interaction between the charge and the fixed magnetic moment of each of the slices into which the solenoid is divided, I obtain the dual situation in which a magnetic dipole of strength \( \mu \), and direction parallel to the \( z \)-axis, moves around the closed curve \(-r(t)\) with a line charge, whose charge per unit length is \( \lambda \), along the \( z \)-axis. Also, since this interaction is invariant under parity, the same phase shift is obtained for the situation obtained by parity transforming about the origin. This corresponds to the magnetic moment moving around the original curve \( r(t) \) traveled by the charge when in the presence of the solenoid.

Now the statement that the AB phase shift is topological may be expressed as follows: If the curve \( r(t) \) goes around the solenoid \( n \) times, \( n = 0, 1, 2, 3, \ldots \), then the phase shift acquired by the charge going around this curve is \( n \Delta \phi \), independent of the shape of this curve. (Topology has to do with integers. So, a topological phase shift should, strictly speaking, be expressed in terms of integers.) Then the curve \(-r(t)\), which is the parity transform of the original curve, goes around the line charge \( n \) times in the dual situation also. Therefore, the phase shift acquired in the dual situation is \( n \Delta \phi \), independent of the shape of this curve. Hence, the latter phase shift is also topological. This may also be seen from the fact that the expressions (2.1) and (2.5) for these phase shifts are independent of the shape of the curve traveled by the particle. But notice that the argument above which establishes the equality of phase shifts in the two situations that are dual to each other does not assume that the phase shift is topological. It is valid for the phase shift due to any interaction, which may or may not be topological. Also, the above duality can be generalized to the case of the charge moving around a closed curve \( C \) and acquiring a phase in the field of an arbitrary distribution of dipoles, each having the same magnetic moment in both direction and magnitude. A little thought, by considering each pairwise interaction of the charge and each dipole, shows that in the dual situation, in which the dipole moves along the parity transformed curve with the charges in the parity transformed positions of the dipoles of the original situation, the same phase is acquired by the dipole. Again using the invariance of this interaction under parity, it is concluded that the same phase shift is obtained when the dipole travels the original closed curve \( C \) with the charges in the positions of the dipoles in the original situation [9]. This argument may be generalized to an arbitrary linear interaction, but the interaction needs to be invariant under parity for the last step to be valid. The equality between the two phases in the two dual situations will be called the duality principle.

This duality principle enables us to obtain from the known phase shift due to a line source a new phase shift. Alternatively, if both phase shifts are known then this principle may be used heuristically to obtain a new solution of the field equations from the old solution that gave the old phase shift. For example, suppose
we know the magnetic field of a solenoid and the AB phase shift \( [1] \) of a charge due to it, and the phase shift of a magnetic moment due to a general electric field \([2]\). Then according to the duality principle, the phase shift in the situation dual to the AB effect in which a charge is interfering in the electric field \( E \) due to a line charge is the same and is given by (2.5). Then using the phase shift of the magnetic moment due to \( E \) implied by (1.4) and the axial symmetry one obtains \( E = \frac{\lambda}{2\pi\rho}\hat{\rho} \), where \( \rho \) is the distance from the line charge, \( \hat{\rho} \) is a unit vector in the radial direction and \( \lambda \) is the charge per unit length. Thus \( E \) of a line charge is obtained without solving Maxwell’s equations. In the next section, I shall apply this general argument to phase shifts produced by gravity.

3. Topological Phase Shifts in the Gravitational Field

It is well known that the mass and the spin angular momentum in a gravitational field are, respectively, analogous to the charge and magnetic moment in an electromagnetic field. Therefore the analog of the AB effect for gravity is the phase shift \( \Delta \phi_G \) acquired by a mass going around a string that has angular momentum (analog of the solenoid). The dual situation then is a spin going around a string or a rod having mass only, and acquiring a phase \( \Delta \phi'_G \). Then according to the general argument in section 1, if the field equations are linear,

\[
\Delta \phi'_G = \Delta \phi_G. \tag{3.1}
\]

The actual values of \( \Delta \phi_G \) and \( \Delta \phi'_G \) depends on the gravitational theory used to compute them. I shall study these phase shifts in the following theories: A. Newtonian gravity, B. linearized limit of Einstein’s theory of general relativity, and C. The Einstein-Cartan-Sciama-Kibble (ECSK) theory of the gravitational field with torsion [14]. In all three cases (3.1) will be shown to be satisfied. The differences between these phase shifts provide a way of distinguishing, in principle, between these theories, although in practice the predicted effects are too small for realistic experimental tests at the present time.

A. Newtonian Gravity

In this case, only the mass, not the angular momentum, acts as the source of gravity and is acted upon by gravity. Therefore, both \( \Delta \phi_G \) and \( \Delta \phi'_G \) are zero. Hence, (3.1) is trivially satisfied.

B. Linearized General Relativity

Consider now the low energy weak field limit of general relativity. Write the metric as \( g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \), where \( \gamma_{\mu\nu} \ll 1 \). In this subsection, all terms which are second order in \( \gamma_{\mu\nu} \) will be neglected. On writing \( \tau_{\mu\nu} = \gamma_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta}\gamma_{\alpha\beta} \), the well known linearized Einstein field equations are

\[
\partial^\alpha \partial_\alpha \tau_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{3.2}
\]

in the gauge defined by \( \partial^\nu \tau_{\mu\nu} = 0 \). I neglect stresses so that we have

\[
T_{ij} = 0, \tau_{ij} = 0, i, j = 1, 2, 3. \tag{3.3}
\]

Consider now a particle with mass \( m \) and intrinsic spin \( \mathbf{S} \) at low energies. In the stationary situation, a coordinate system may be chosen so that \( \gamma_{\mu\nu} \) are time
independent. Then the acceleration due to gravity is \( g = -\frac{1}{2} \nabla \gamma_{00} \) and the ‘Coriolis’ vector potential is \( \gamma_0 = -(\gamma_{01}, \gamma_{02}, \gamma_{03}) \), as can be seen easily from the geodesic equation. The ‘gravimagnetic field’ \( H = \nabla \times \gamma_0 = 2\Omega \), where \( \Omega \) is the angular velocity of the coordinate basis relative to the local inertial frame. To couple the spin to the gravitational field introduce the vierbein \( e^\mu_a \) and its inverse \( e_a^\mu \):

\[
e^\mu_a = \delta^\mu_a - \frac{1}{2} \gamma^\mu_a, \quad e_a^\mu = \delta_a^\mu + \frac{1}{2} \gamma_a^\mu, \tag{3.4}
\]

which satisfy (3.20) and (3.22) below. The latin indices, which take values 0, 1, 2, 3 may now be lowered and raised using the Minkowski metric \( \eta_{ab} \) and its inverse \( \eta^{ab} \).

It then follows that the Ricci rotation coefficients \( \omega_{\mu a b} \equiv e_\nu^a \nabla_\mu e_\nu^b \) are given by

\[
\omega_{\mu a b} = \frac{1}{2} (\gamma_{\mu b a} - \gamma_{\mu a b}), \tag{3.5}
\]

where ,a denotes partial differentiation with respect to \( x^a \).

The phase shift in interference due to the gravitational field may be obtained in the present approximation by taking the low energy weak field limit of the phase shift obtained in reference 5. In particular, the phase shift due to spin alone is obtained by parallel transporting the spin wave function by acting on it by the operator

\[
\Phi_S = P \exp \left[-\frac{i}{\hbar} \int_C \frac{1}{2} \omega_{\mu}^a S^b d\mu^a \right] = P \exp \left[-\frac{i}{\hbar} \int_C \frac{1}{2} \gamma_{\mu a b} S^{ab} d\mu^a \right], \tag{3.6}
\]

where the integral is along the unperturbed classical trajectory with \( P \) denoting path ordering, and \( S^b \), which generate Lorentz transformations in spin space, are related to the spin vector \( S^a \) and the 4-velocity \( v^b \) by

\[
S^{ab} = \epsilon^{abcd} v_c S_d, \tag{3.7}
\]

with all components being with respect to the vierbein. The subsidiary condition \( S^a v_a = 0 \) is assumed here.

Rewrite (3.6) as

\[
\Phi_S = P \exp \left[-\frac{i}{\hbar} \int_C \frac{1}{2} \gamma_{0 a b} S^{ab} dt - \frac{i}{\hbar} \int_C \frac{1}{2} \gamma_{i a b} S^{ab} dx^i \right]. \tag{3.6'}
\]

The first integral in the exponent of (3.6') is

\[
\int_C \frac{1}{2} \gamma_{0 i} S^{0i} dt + \int_C \frac{1}{2} \gamma_{0i,j} S^{ij} dt = \int_C (g \times S \cdot v - \Omega \cdot S) dt. \tag{3.8}
\]

The second integral of (3.6') is approximately

\[
\int_C \frac{1}{2} \gamma_{ij,k} S^{ijk} dx^i = \int_C g \times S \cdot dr. \tag{3.9}
\]

Combining these results, (3.6) reads

\[
\Phi_S = P \exp \left[-\frac{i}{\hbar} \int_C g \times S \cdot dr + \frac{i}{\hbar} \int_C \Omega \cdot S dt \right]. \tag{3.10}
\]
The precession represented by the last term of the exponent in (3.10) corresponds to the interaction energy \(-\boldsymbol{\Omega} \cdot \mathbf{S} = -4\mathbf{S} \cdot \mathbf{H}\) in the Hamiltonian. This may be understood from the fact that when we transform to a frame rotating with an angular velocity \(\boldsymbol{\Omega}\) relative to the local inertial frame, the spin that is constant in the inertial frame obviously rotates with angular velocity \(-\boldsymbol{\Omega}\) relative to the new frame [15]. The ratio \(\gamma\) of the magnetic moment to spin introduced in section 1 for the electromagnetic interaction is, for a particle with charge \(e\) and mass \(m\), \(\gamma = \frac{ge^2}{2m}\), where \(g\) is the gyromagnetic ratio. For the gravitational field, the principle of equivalence implies that the ‘charge’ density equals mass density. Therefore, \(g = 1\) and \(e = m\). Hence, \(\gamma = \frac{1}{2}\) and the ‘gravimagnetic moment’ \(\mu_G = \frac{1}{2}\mathbf{S}\), consistent with the above interaction energy.

Equations (3.2) subject to (3.3) are

\[
\partial^\alpha \partial_\alpha \gamma_0 = 8 \pi G T_0.\]

These are like Maxwell’s equations in the Lorentz gauge, and may be solved in the same way. Consider the specific case of an infinite uniform hollow cylinder of radius \(\rho_0\) and mass per unit length \(\mu\) rotating about its axis with angular momentum per unit length \(J\) parallel to the axis of the cylinder along the \(x^3\)-axis. This is analogous to a rotating charged cylinder in electromagnetism. So, on defining \(\mathbf{r} = (x^1, x^2, x^3)\), \(\rho = (x^1, x^2, 0)\), and \(\rho = |\rho|\), the solution exterior to the cylinder \((\rho > \rho_0)\) is obtained to be

\[
\gamma_{00} = \gamma_{11} = \gamma_{22} = \gamma_{33} = 4G\mu \log \rho, \gamma_0 = -\frac{4G}{\rho^2} \mathbf{J} \times \mathbf{r} = -\frac{4G}{\rho} \mathbf{J} \times \hat{\rho}, \quad (3.11)
\]

where \(\hat{\rho}\) is a unit vector in the direction of \(\rho\). The solution in the interior to the cylinder is

\[
\gamma_{00} = \gamma_{11} = \gamma_{22} = \gamma_{33} = 4G\mu \log \rho_0, \gamma_0 = -\frac{4G}{\rho_0^2} \mathbf{J} \times \mathbf{r} = -\frac{4G}{\rho_0^2} \mathbf{J} \times \rho. \quad (3.12)
\]

Suppose at first that \(J = 0\). Then, from (3.11), \(\Omega = 0\) and \(g = -\frac{2G\mu}{\rho} \hat{\rho}\). Consider the interference around the cylinder of a particle whose spin is polarized in the \(x^3\)-direction with the axis of the cylinder lying along the \(x^3\)-axis. Then the phase shift due to the coupling of spin to curvature [5,16] is obtained from (3.10) to be

\[
\Delta \phi_G = -\frac{2}{\hbar} \oint_C \mathbf{g} \times \mathbf{S} \cdot d\mathbf{r} = -8\pi \frac{G}{\hbar} \mu \mathbf{S}. \quad (3.13)
\]

This phase shift is independent of \(C\) and is therefore topological.

Consider now the dual situation that is constructed as follows. Divide the cylinder into small segments of length \(\delta \ell\). The mass of each segment is \(m = \mu \delta \ell\). In performing the duality operation, each segment is replaced by a segment whose spin is the same as \(S\) and the particle is replaced with another particle with mass \(m\). Then the cylinder has angular momentum per unit length \(J = \frac{S}{\delta \ell}\). Therefore,

\[
\frac{m}{\mu} = \frac{S}{J}. \quad (3.14)
\]

So, in the dual situation, the mass \(m\) is interfering around the cylinder with angular momentum per unit length \(J\), which is a gravitational analog of the AB experiment. From (3.11), the phase shift due to \(J\) is the Sagnac phase shift [4]

\[
\Delta \phi'_G = \frac{m}{\hbar} \oint_C \gamma_0 \cdot d\mathbf{r} = -8\pi \frac{G}{\hbar} mJ. \quad (3.15)
\]
It follows from (3.13), (3.14) and (3.15) that (3.1) is satisfied.

I shall now describe the gravitational analog to the topological phase shift of the neutron due to the magnetic field described towards the end of section 1. Suppose, in a neutron or atomic interference experiment each of the two interfering beams passes along the axis of each of two identical very massive cylinders. One of the cylinders rotates as the wave packet of each neutron enters the cylinder and stops rotating before the neutron leaves the cylinder. Then from (3.12),

$$\Omega = \frac{1}{2} \nabla \times \gamma_0 = -\frac{4G}{\rho_0^2} J. \tag{3.16}$$

Suppose, for simplicity, that the spin of the neutron or atom is polarized along the axis of the cylinder. The first integral in (3.10) would be the same for both beams. Therefore, the phase shift between them is given by the second integral of (3.10) due to the rotating cylinder to be

$$\Delta \phi_\Omega = \frac{1}{\hbar} \oint_C \Omega \cdot S \, dt = -\frac{4GJS\tau}{\hbar \rho_0^2}, \tag{3.17}$$
on using (3.16), where $\tau$ is the time spent by the neutron inside the hollow cylinder, assuming that the time intervals during which the rotation of the cylinder is turned on and off is negligible compared to $\tau$. As in the electromagnetic case, the phase shift (3.17) is not accompanied by a force, apart from transient effects when the field is turned on and off which occurs also for the AB effect due to the scalar potential [1]. Hence, (3.17) is a topological phase shift.

It is interesting that $\Delta \Phi_S$ obtained here by parallel transport with respect to the gravitational connection is analogous to how the electromagnetic phase shift experienced by the dipole was obtained by parallel transport with respect to a corresponding connection [8,9]. The three gravitational phase shifts obtained above using (3.10) and (3.15) are the low energy weak field limit of the phase shifts obtained previously using Dirac’s equation [5]. However, these phase factors correspond to the tentative Hamiltonian

$$H = \frac{1}{2m} (p - m\gamma_0 - 2S \times g)^2 + m\gamma_0^0 \frac{1}{2} S \cdot H, \tag{3.18}$$
in the sense that they may be derived from this Hamiltonian. This is a generalization of the Hamiltonian found by DeWitt [17] to include spin. One way of confirming (3.18) is to directly take the low energy weak field limit of Dirac’s equation, which will be studied in a future paper.

C. General Relativity with Torsion

The phase shift in general relativity may be obtained from the action on the wave function of the gravitational phase factor [18]

$$\Phi_C = P \exp[-i \oint_C (e_\mu^a P_a + \frac{1}{2} \omega_\mu^a b M^b_a) dx^\mu], \tag{3.19}$$

where $C$ goes through the interfering beams. Here, $P_a$ and $M^b_a, a, b = 0, 1, 2, 3$ are the translation and Lorentz transformation generators which generate the Poincare group that acts on the Hilbert space. Then $e_\mu^a$ and $\omega_\mu^a b$ have the interpretation of the gauge potentials of a Poincare gauge field. In an interferometry experiment the
two beams need to be brought together by means of mirrors which gives rise to the Thomas precession [19], which will be treated elsewhere [20].

It was shown by means of the WKB approximation of Dirac’s equation that (3.19) determines the gravitational phase [5,18]. This may also be realized for a particle with arbitrary spin as follows: The Lorentz part of (3.19) ensures that the wave packet is parallel transported infinitesimally, while it acquires a phase, which is a good approximation for the locally approximate plane wave being considered here. To find the phase acquired due to energy-momentum, note first that $e^a_\mu$ depends on the observer. A Lorentz transformation of the observer results in $e^a_\mu$ transforming as a contravariant vector in the index $a$ while $P^a$ transforms as a covariant vector. Suppose that a particle is in a state $|\psi\rangle$ that is approximately an eigenstate of $P^a$ with eigenvalues $p^a$. The fact that the gravitational phase is observable along an open curve implies that the “wave vector” $p^a = e^a_\mu p^\mu$ is observable [21,22]. Requiring that the correspondence between $p^a$ and $p^\mu$ is $(1-1)$ implies that $e^a_\mu$ is a non singular matrix. Therefore, it has an inverse $e_b^a$:

$$e^a_\mu e^\mu_b = \delta^a_b. \quad (3.20)$$

Hence, $p^a = e^a_\mu p^\mu$. The Casimir operator $\eta^{ab}P^aP^b$ of the Poincare group has a definite value, say $m^2$, for the given particle. Therefore,

$$\eta^{ab}p^ap^b = g^{\mu\nu}p^\mu p^\nu = m^2, \quad (3.21)$$

where $g^{\mu\nu} \equiv \eta^{ab}e^a_\mu e^b_\nu$ is a non singular matrix. Its inverse $g_{\mu\nu}$ defines a pseudo-Riemannian metric of Lorentzian signature on space-time. On using (3.20),

$$g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu. \quad (3.22)$$

Thus the definiteness of the mass (which may be zero) ensures the definiteness of the phase that depends on $e^a_\mu$ even for open curves. In this way the space-time metric is deduced from the gravitational phase corresponding to the translational part of (3.19) which is observable along an open curve. Conversely, the metric determines the latter phase to be observable along an open curve.

The field strength or curvature of this Poincare gauge field is obtained by evaluating (3.19) for an infinitesimal closed curve $C$:

$$\Phi_C = 1 - \frac{i}{2}(Q_{\mu\nu}^a P^a + \frac{1}{2}R_{\mu\nu}^a b M^b_a)ds^{\mu\nu}, \quad (3.23)$$

using the Poincare Lie algebra, where

$$Q^a = de^a + \omega^a_b \wedge e^b \quad (3.24)$$

is the torsion and

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b \quad (3.25)$$

is the linear curvature.

Comparison of (3.19) with (1.2) and (1.4), and (3.23) suggest that there may be topological phase shifts due to interference of coherent beams that enclose a region that contains curvature and torsion, but which are zero along the beams. Such an example is provided by the cosmic string whose metric exterior to the string is given cylindrical coordinates as [23, 24, 25].

$$ds^2 = (dt + \beta d\phi)^2 - d\rho^2 - \alpha^2 \rho^2 d\phi^2 - dz^2, \quad (3.26)$$
where $\alpha$ and $\beta$ constants. Then the metric $g_{\mu\nu}$ satisfies (3.22) for the following orthonormal co-frame field \{${e}^a$\} adapted to the above coordinate system:

$$e^0 = dt + \beta d\phi, e^1 = d\rho, e^2 = \alpha \rho d\phi, e^3 = dz.$$  \tag{3.27}

The connection coefficients in this basis are $\omega^a_{\ \beta} \equiv e^a_{\nu} \nabla_{\mu} e^\nu_{\beta} = 0$, for all $a, b, \mu$ except for

$$\omega^1_{\ 2} = -\omega^2_{\ 1} = -\alpha d\phi.$$  \tag{3.28}

It follows, on using (3.24) and (3.25), that $Q^a = 0, R^a_{\ b} = 0$ outside the string. The scattering cross section of particles with definite energy in the above geometry has been obtained before [26].

In the appendix it is shown that this solution may be extended to an interior solution that has uniform energy and spin densities and which generate curvature and torsion according to the ECSK equations [14]. The constants $\alpha$ and $\beta$ are then determined by matching the interior and exterior solutions to be

$$\alpha = 1 - 4G\mu, \beta = 4GJ.$$  \tag{3.29}

where $\mu$ is the mass per unit length and $J$ is the angular momentum per unit length due to the intrinsic spin density inside the string.

If $C$ is a closed curve around the cosmic string then from (3.19),

$$\Phi_C = \exp \left( -i \oint_C e_{\mu}^0 P_0 dx^\mu \right) P \exp \left[ -i \oint_C \left( \sum_{k=1}^3 e_{\mu}^k P_k + \omega_{\mu}^1 M^2 \right) dx^\mu \right],$$  \tag{3.30}

using the fact that $e_{\mu}^0 P_0$ commutes with the other terms that occur in (3.30). The first exponential is a time translation second is a spatial Euclidean transformation. Hence, if (3.23) is valid, then the time translation would correspond to torsion being non zero inside the string. Suppose that the surface of the string is given by $\rho = \rho_0$, where $\rho_0$ is a small constant. Then, substituting (3.27), (3.28) into (3.30), and using (3.23), the flux of torsion through a cross-section $\Sigma$ of the string is

$$\int_\Sigma Q^0 = 2\pi \beta, \int_\Sigma R^1_{\ 2} = 2\pi (1 - \alpha)$$  \tag{3.31}

This is independent of the particular geometry inside the string so long as $\Sigma$ is “infinitesimal” so that (3.23) is valid. In particular, (3.31) is easily verified for the solution in the appendix, independently of the value of $\rho_-$, using (A.12), (A.15) and (A.16).

For simplicity, consider a circular interferometer with constant radius $r > \rho_0$ in a plane normal to the string, with its center on the axis of the string. It may be a superconducting interferometer, e. g. a superconducting ring interrupted by a Josephson junction. Or it may be an electron interferometer, or a wave guide, such as an optical fiber, at one point of which is the beam splitter that splits a beam into two which travel in opposite senses and interfere at a mirror that is at another point in the interferometer. The interferometer does not rotate relative to the distant stars, which may be ensured by requiring that telescopes rigidly attached to this interferometer are focused on the distant stars.

The phase shift may be obtained using (3.30) with $C$ along integral curves of $\rho^\mu$ which lie on a 2 dimensional submanifold $\sigma$ with constant $z$ and $\rho = r$. Hence,
C may be chosen to be along a circle around \( \sigma \) with constant \( t \). Suppose \( E \) is the energy of the wave function which is assumed to be constant in time at the beam splitter. Therefore, in this WKB approximation, the magnitude of the momentum \( p = (E^2 - m^2)^{1/2} \) is also a constant along the beam. By taking into account the Fermi-Walker transport of vectors associated with \( |\psi\rangle \), \( M^2 \) in (3.30) may be replaced by the spin operator \( S^2 \) in the present coordinate basis [19]. The spin is assumed to be polarized in the \( z \)-direction, i.e. \( |\psi\rangle \) is an eigenvector of \( S^2 \) with eigenvalue \( S/\hbar \).

Now, the three operators in (3.30) commute with one another and their actions on \( |\psi\rangle \) give rise to the following topological phase shifts: (i) The general relativistic Sagnac phase shift [4] is obtained from the first factor to be

\[
\Delta \phi_E = -\int_C \frac{E}{\hbar} \omega_1^0 \, d\phi = -\int_C \frac{E}{\hbar} Q_{\rho \phi}^0 \, d\rho \wedge d\phi = -8\pi \frac{G}{\hbar} E J, \tag{3.32}
\]

where \( \Sigma \) is a 2-surface spanned by \( C \). (ii) The phase shift due to the coupling of spin to curvature [5] which is obtained by the second factor in (3.30)*

\[
\Delta \phi_S = -\frac{S}{\hbar} \left( \int_C \omega_2^1 \, d\phi - 2\pi \right) = -\int\Sigma \frac{R_{\rho \phi}^1}{\hbar} \, d\rho \wedge d\phi = -8\pi \frac{G}{\hbar} S \mu. \tag{3.33}
\]

The phase shifts (3.32) and (3.33) are expressed as flux integrals of torsion and curvature because they could have also been obtained from (3.19) which depends only on the affine connection. (The torsion and curvature fluxes contained in (3.32) and (3.33) are the same as (3.31) which is independent of the particular geometry interior to the string considered here.) It follows that these phase shifts are independent of the shape of the interferometer enclosing the string and therefore may be called topological.

I shall now show that the above topological effects satisfy the principle of duality formulated in section 2. Consider first the Sagnac effect on a particle with energy \( E \) due to the spinning string with angular momentum per unit length \( J \). This is like the AB effect due to a solenoid. Divide the string into small segments of length \( \delta \ell \). The spin of each segment is \( S = J \delta \ell \). In performing the duality operation, each segment is replaced by a segment whose mass is the same as \( E \) and the particle is replaced with another particle with spin \( S \). Then the solenoid has been replaced by a rod with mass per unit length \( \mu = E \frac{\mu}{J} \). Therefore,

\[
\frac{E}{\mu} = \frac{S}{J}. \tag{3.34}
\]

Conversely, if (3.34) is valid then the two situations may be obtained from each other by performing the duality operation. Hence, by the duality principle, the phase shifts for the two situations should be equal. Indeed, the phase shifts (3.32) and (3.33) which were derived without paying any attention to the duality principle are equal if and only if (3.34) is valid.

This illustrates also again how the duality principle may be used to obtain the phase shift for the dual situation: From (3.32), we may obtain (3.33), or vice

\*

The phase shifts (3.32) and (3.33) may be evaluated using the line integral outside the string using (A.16) or the surface integral inside the string using (A.11). In (3.33), \( 2\pi \) has been subtracted from the line integral to remove the purely coordinate effect due to the rotation of \( e^2 \) by \( 2\pi \) as one goes around \( C \), consistent with the Gauss-Bonnet theorem.
versa, on using (3.34). Even though the general relativistic equations are in general non linear, the equations that are solved in the appendix to obtain the exact solution are all linear, so that there must be duality in the present case according to the general arguments of section 2. If this duality is assumed then a new gravitational solution may be obtained from an old solution both in the present case and in the low energy weak field case considered earlier, similar to how this was done in the electromagnetic case at the end of section 2.

4. Concluding Remarks

As already mentioned, the AB effect shows that the field strength is insufficient to describe the electromagnetic field, whereas the phase factor (1.2), which is called a holonomy transformation because it parallel transports around a closed curve, adequately describes the field. More generally, for an arbitrary gauge field, the holonomy transformations are of the form (1.4), with \( A^k_\mu \) now being the corresponding vector potential. The fact that these are sufficient to determine the field uniquely is shown by the following theorem [27]: Given the holonomy transformations (2) for piece-wise differentiable curves which begin and end at a given point in space-time, the gauge potential \( A^k_\mu \) may be reconstructed, and it is then unique up to gauge transformations.

But since the set of such curves form an infinite dimensional manifold \( L \), the corresponding operators (2) have a great deal of redundancy. Indeed, the gauge field in space-time may be reconstructed from a minimal set of these operators defined on a four dimensional submanifold of \( L \) [7]. This is mathematically equivalent to working with the gauge potential defined in a particular gauge on the four dimensional space-time. Therefore, once the redundancy in the loop space \( L \) has been removed, there is no advantage to using the holonomy transformation (2) as opposed to the gauge potential in a particular gauge.

It follows that in quantizing the electromagnetic or more general gauge fields, one must quantize the gauge potential instead of the field strength. Similarly, the topological effects due to the gravitational field described above suggest that in quantizing the gravitational field, it is the ‘gauge potentials’ \( e^a_\mu \) and \( M^a_\mu \) which should be quantized, and the metric (3.22) is obtained from them as a secondary variable [26,22]. However, there is a breaking of gauge symmetry which makes \( e^a_\mu \) a tensor field instead of a connection [22]. This is like how in a superconductor the \( U(1) \) gauge symmetry is spontaneously broken, which makes \( A_\mu \) a covariant vector field instead of a connection.

So, it may well be that in the early universe there was the full Poincaré gauge symmetry with \( e^a_\mu \) and \( M^b_\mu \) having vacuum expectation value zero in an appropriate gauge. As a result of spontaneous symmetry breaking of the translational part of the Poincaré group, \( e^a_\mu \) may have acquired a vacuum expectation value equal to \( \delta^a_\mu \) corresponding to the Minkowski geometry. But I emphasize that these are speculative remarks, and need justification by a detailed theory.

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Appendix: Spinning Torsion String

The simplest gravitational field equations in the presence of torsion are the Einstein-Cartan-Sciama-Kibble (ECSK) equations [14], which may be written in the form [29]

\[
\frac{1}{2} \epsilon_{ijkl} \theta^l \wedge R^{lk} = -8\pi G t_i, \\
\epsilon_{ijkl} \theta^l \wedge Q^k = 8\pi G s_{ij},
\]

(A.1) (A.2)

where \( t_i \) and \( s_{ij} \) are 3-form fields representing the energy-momentum and spin densities. I shall now obtain an exact solution of these equations for the interior of the cosmic string which matches the exterior solution (3.26). This will then give physical and geometrical meaning to the parameters \( \alpha \) and \( \beta \) in (3.26). This solution will be different from earlier torsion string solutions [31] that have static interior simplicies. I make the following ansatz in the interior:

\[
\theta^0 = u(\rho)dt + v(\rho)d\phi, \theta^1 = d\rho, \theta^2 = f(\rho)d\phi, \theta^3 = dz, \omega^1 = k(\rho)d\phi = -\omega^1_{2},
\]

(A.3)

all other components of \( \omega^a \) being zero, and \( ds^2 = \eta_{\alpha\beta} \theta^\alpha \wedge \theta^\beta = g_{\mu\nu} dx^\mu dx^\nu \). Suppose also that the energy density \( \epsilon \) and spin density \( \sigma \), polarized in the \( z \)-direction, are constant and classical fluid at rest. I.e.

\[
t_0 = \epsilon \theta^1 \wedge \theta^2 \wedge \theta^3 = \epsilon f(\rho) d\rho \wedge d\phi \wedge dz,
\]

\[
s_{12} = -s_{21} = \sigma \theta^1 \wedge \theta^2 \wedge \theta^3 = \sigma f(\rho) d\rho \wedge d\phi \wedge dz,
\]

(A.4)

the other components of \( s_{ij} \) being zero. In terms of the components of the energy-momentum and spin tensors in the present basis, this means that \( t^0_0 = \epsilon = \text{constant} \) and \( s^0_{12} = \sigma = \text{constant} \).

It is assumed that there is no surface energy-momentum or spin for the string. Then the metric must satisfy the junction conditions [30], which in the present case are

\[
g_{\mu\nu}\Big|_+ = g_{\mu\nu}\Big|_-, \quad \partial^\rho g_{\mu\nu}\Big|_+ = \partial^\rho g_{\mu\nu}\Big|_- + 2K_{(\mu\nu)}\hat{\beta},
\]

(A.5)

where \( K_{\alpha\beta\gamma} = \frac{1}{\rho}(Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta}) \) is the contorsion or the defect tensor, \( |_+ \) and \( |_- \) refer to the limiting values as the boundary of the string is approached from outside and inside the string, respectively, and the hat denotes the corresponding coordinate component.

Substitute (A.3), (A.4) into the Cartan equations (A.2). The \((i,j) = (0,2), (0,3), (2,3)\) eqs. are automatically satisfied. The \((i,j) = (0,1), (1,3), (1,2)\) eqs. yield

\[
f'(\rho) - k(\rho) = 0, u'(\rho) = 0, v'(\rho) = 8\pi G\sigma f(\rho),
\]

(A.6)

where the prime denotes differentiation with respect to \( \rho \). Therefore, the continuity of the metric (eq. (A.5)) implies that, since \( u = 1 \) at the boundary, \( u(\rho) = 1 \) everywhere. Now substitute (A.3), (A.4) into the Einstein equations (A.1). The \( i = 0 \) eq. yields

\[
k'(\rho) = -8\pi G\epsilon f(\rho).
\]

(A.7)
The \( i = 1, 2, 3 \) equations yield, respectively

\[
t_1 = 0, t_2 = 0, t_3 = \frac{k'}{8\pi G} dt \wedge d\rho \wedge d\phi = -\epsilon \theta^0 \wedge \theta^1 \wedge \theta^2, \quad (A.8)
\]

using (A.7). Hence, \( t^3 = \epsilon = t^0_0 \). From (A.6) and (A.7),

\[
f''(\rho) + \frac{1}{\rho^2} f(\rho) = 0, \quad (A.9)
\]

where \( \rho_* = (8\pi G \epsilon)^{-1/2} \). In order for there not to be a metrical “cone” singularity at \( \rho = 0 \), it is necessary that \( \theta^2 \sim \rho d\phi \) near \( \rho = 0 \). Hence, the solution of (A.9) is \( f(\rho) = \rho_* \sin \left( \frac{\rho}{\rho_*} \right) \). This gives the metric in the interior of the string to be

\[
ds^2 = \left[ dt + 8\pi G \sigma \rho_*^2 \left( 1 - \cos \left( \frac{\rho}{\rho_*} \right) \right) \right]^2 - d\rho^2 - \rho_*^2 \sin^2 \left( \frac{\rho}{\rho_*} \right) d\phi^2 - dz^2. \quad (A.10)
\]

The only non vanishing components of curvature and torsion are

\[
Q^0 = 8\pi G \sigma \rho_* \sin \left( \frac{\rho}{\rho_*} \right) \rho \wedge d\phi, R_{12} = 1 \rho_* \sin \left( \frac{\rho}{\rho_*} \right) \rho \wedge d\phi = -R_{21}. \quad (A.11)
\]

I apply now the junction conditions (A.5), which will show that \( \rho \) is discontinuous across the boundary. Denote the values of \( \rho \) for the boundary in the internal and external coordinate systems by \( \rho_- \) and \( \rho_+ \) respectively. From (3.22) and (A.10), \( g_{i\phi} \) and \( g_{\phi\phi} \) are respectively continuous iff

\[
\beta = 8\pi G \sigma \rho_*^2 \left( 1 - \cos \left( \frac{\rho}{\rho_*} \right) \right), \quad (A.12)
\]

\[
\alpha \rho_+ = \rho_+ \sin \left( \frac{\rho_-}{\rho_*} \right). \quad (A.13)
\]

The remaining metric coefficients are clearly continuous. The only non zero contorsion terms which enter into (A.5) are obtained from (A.11) to be

\[
K_{(\phi\phi)\hat{\rho}} = -4\pi G \sigma \rho_* \sin \left( \frac{\rho}{\rho_*} \right), \quad K_{\phi\phi\rho} = -(8\pi G \sigma)^2 \rho_*^3 \left( 1 - \cos \left( \frac{\rho}{\rho_*} \right) \right) \sin \left( \frac{\rho}{\rho_*} \right). \quad (A.14)
\]

Using (A.13) and (A.14), it can now be verified that the remaining junction conditions (A.5) are satisfied provided \( \alpha = \cos \left( \frac{\rho_-}{\rho_*} \right) \). The mass per unit length is

\[
\mu \equiv \int_{\Sigma} \epsilon \theta^1 \wedge \theta^2 = \frac{1}{4G} \left( 1 - \cos \left( \frac{\rho_-}{\rho_*} \right) \right) = \frac{1}{8\pi G} \int_{\Sigma} R_{12}, \quad (A.15)
\]

where \( \Sigma \) is a cross-section of the string (constant \( t, z \)). Therefore, \( \alpha = 1 - 4G \mu \). The angular momentum per unit length due to the spin density is

\[
J \equiv \int_{\Sigma} \sigma \theta^1 \wedge \theta^2 = 2\pi \sigma \rho_*^2 \left( 1 - \cos \left( \frac{\rho_-}{\rho_*} \right) \right) = \frac{1}{8\pi G} \int_{\Sigma} Q^0. \quad (A.16)
\]
Hence, from (A.12), $\beta = 4GJ$. The Sagnac phase shift obtained earlier is therefore $\Delta\phi = ET$, where $T$ is the flux of $Q^0$ through $\Sigma$. In the special case when torsion is absent, which in the ECSK theory means that spin density is zero, $\beta = 0$, and the above solution reduces to the exact static solution of Einstein’s theory found by Gott [32] and others [33], whose linearized limit was previously found by Vilenkin [34]. After this work was completed I learned that Tod [35] has studied torsion singularities using affine holonomy and the ECSK equations analogous to the present approach.

Note added in proofs: The phase shift due to spin in the interference around a rod and a cosmic string has also been studied by B. Reznik (PhD thesis, Tel Aviv Univ., 1994, and preprint to be published in Phys. Rev. D), by using the contribution to the Lagrangian due to the gravitational interaction energy $U = \frac{1}{2} \int T^\mu\nu \gamma_{\mu\nu} d^3x$. This amounts to treating gravity as a spin 2 field, compared to the present geometric approach which begins with the full general relativistic theory. However, the above mentioned paper assumes that $T^{oi}$ in the rest frame of the particle is the curl of spin density, which is then boosted to the laboratory frame. This assumption corresponds to setting the ‘gravimagnetic moment’ $\mu_G$ equal to the spin. This differs from the result in section 3 of the present paper that $\mu_G$ is half the spin in accordance with the principle of equivalence. The latter result implies that $T^{oi}$ in the rest frame is half the curl of spin density. Then, integrating by parts, it is easy to show that the Lagrangian for a particle with mass $m$, velocity $v$ and spin $S$ in the laboratory frame is

$$L = \frac{1}{2}mv^2 - U = \frac{1}{2}mv^2 - m\gamma_{00} + \gamma v + 2v \times S + \frac{1}{2}S \cdot H.$$

This confirms the Hamiltonian (3.18) of the present paper. Also, the present paper studies an additional spin interaction represented by the last term of (3.18), or the last term of the above Lagrangian, which gives rise to a new topological phase shift (3.17).

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