Representation of harmonic functions with respect to subordinate Brownian motion

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Abstract

In this article we prove a representation formula for non-negative generalized harmonic functions with respect to a subordinate Brownian motion in a general open set $D \subset \mathbb{R}^d$. We also study oscillation properties of quotients of Poisson integrals and prove that oscillation can be uniformly tamed.

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1 Introduction

The goal of this article is to prove a representation formula for non-negative harmonic functions with respect to a class of subordinate Brownian motions in a general open set $D \subset \mathbb{R}^d$, $d \geq 2$, where the Laplace exponent of the corresponding subordinator is a complete Bernstein function satisfying certain weak scaling conditions. In this setting, the novelty is that we look at pairs $(f, \lambda)$ such that $f$ is a function on $D$ and $\lambda$ is a measure on $D^c$ that we call, following [8], functions with outer charge. We prove the following result: if $f$ is a non-negative harmonic function in $D$ with a non-negative outer charge $\lambda$, then there is a unique finite measure $\mu$ on $\partial D$ such that

$$f = P_D \lambda + M_D \mu, \quad \text{in } D. \quad (1)$$

Here $P_D \lambda$ denotes the Poisson integral of the measure $\lambda$ and $M_D \mu$ the Martin integral of the measure $\mu$, see Theorem 5.14. Such representation was proved for the case of the isotropic $\alpha$-stable process in [8] more than 10 years ago. A similar representation for functions (in the classical sense) was proved recently for more general Markov processes in bounded open sets in [20], and in nice and general open sets in [17]. Analogous result for non-negative classical harmonic functions on the ball $B(x, r)$, i.e. harmonic functions with respect to the Brownian motion,
is better known as Riesz-Herglotz theorem, cf. [1]. In the article the case $d = 1$ is
excluded since it would require somewhat different potential theoretic methods.

On the way to obtaining the representation, motivated by results in [8], we
study the relative oscillation of the quotient of Poisson integrals. The novelty
of this results is that we prove that the oscillation can be uniformly tamed.
To be more precise, for a positive function $f$ on a set $D$ we define the relative
oscillation of the function $f$ by

$$\text{RO}_D f := \frac{\sup_D f}{\inf_D f}.$$ 

We prove that for every $\eta > 0$ there is $\delta > 0$ such that for every $D \subset B(0, R)$
and measures $\lambda_1$ and $\lambda_2$ on $B(0, R)^c$ we have

$$\text{RO}_{D \cap B(0, \delta)} \frac{P_D \lambda_1}{P_D \lambda_2} \leq 1 + \eta,$$

see Lemma 5.5. Uniformity lies in the fact that $\delta$ is independent of the set $D$ and
the measures $\lambda_1$ and $\lambda_2$. Similar claims on the relative oscillation of harmonic
functions were recently proved for more general processes in [17, Proposition
2.5 & Proposition 2.11] and [16, Theorem 2.4 & Theorem 2.8] but the claims
lack the aforementioned uniformity.

In the article we also study the boundary trace operator $W_D$, see Definition
4.5. The operator $W_D$ was introduced in [7] building on results in [8]. In [7]
it plays a significant role in the semilinear Dirichlet problem for the fractional
Laplacian. We generalize the operator for the case of the subordinate Brownian
motion and use it as a tool to obtain the finite measure for the Martin integral
in the representation.

Motivated by the article [8] where harmonic functions with outer charge were
introduced for the case of the isotropic $\alpha$-stable process, we use the same concept
to define $L$-harmonic functions with outer charge, see Definition 3.7. The letter
$L$ stands for the integrodifferential operator $L$ which generates the subordinate
Brownian motion, see (5). In Theorem 3.16 we prove that $L$ annihilates all
$L$-harmonic functions in the weak sense. Also, the novelty of the study of $L$-
harmonic functions is that we prove that all such functions are continuous, see
Proposition 3.9, whereas in [8] the continuity condition was used as a part of the
definition. Moreover, motivated by results in [12], in Theorem 3.12 we prove
even stronger result which says that every $L$-harmonic function is infinitely
differentiable.

The article is organized as follows. Below this paragraph we introduce the
notation. In Section 2 we define the process of interest, introduce the Green
and the Poisson kernels, and state some well-known results on the process that
will be needed in the article. In Section 3 we prove basic results on the Poisson
kernel, define $L$-harmonic functions and study their basic properties. In Section
4 we recall already known facts on the theory of the Martin kernel and connect
them to $L$-harmonic functions. Section 5 begins with results on the boundary
trace operator $W_D$. After we prove results on the relative oscillations of the

\[ 2 \]
Throughout this article we suppose that $\mu$ is a non-negative measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t) \mu(dt) < \infty$. The measure $\mu$ is called the Lévy measure and $b$ the drift of the subordinator.

Throughout this article we suppose that $\phi$ is a complete Bernstein function. This assumption means that $\mu(dt)$ has a density $\mu(t)$ which is a completely monotone Poisson integrals, we finish the article by proving the representation formula for non-negative $L$-harmonic functions.

**Notation.** For an open set $D \subset \mathbb{R}^d$, $C(D)$ denotes the set of all continuous functions on $D$, $C^2(D)$ twice continuously differentiable functions on $D$, $C^\infty(D)$ infinitely differentiable functions on $D$, and $C^{\infty}_{\text{loc}}(D)$ infinitely differentiable functions with compact support on $D$. Furthermore, $L^1(D)$ is the set of all integrable functions on $D$, and $L^1_{\text{loc}}(D)$ the set of all locally integrable functions on $D$, with respect to the Lebesgue measure restricted on $D$. If $D = \mathbb{R}^d$ we write $L^1$ and $L^1_{\text{loc}}$ instead of $L^1(\mathbb{R}^d)$ and $L^1_{\text{loc}}(\mathbb{R}^d)$, respectively. The boundary of the set $D$ is denoted by $\partial D$. Notation $U \subset D$ means that $U$ is a nonempty bounded open set such that $U \subset \overline{U} \subset D$ where $\overline{U}$ denotes the closure of $U$. By $|x|$ we denote the Euclidean norm of $x \in \mathbb{R}^d$ and $B(x, r)$ denotes the ball around $x \in \mathbb{R}^d$ with radius $r > 0$. We abbreviate $B_r := B(0, r)$. For $A, B \subset \mathbb{R}^d$ let $\delta_A(x) = \inf\{|x - y| : y \in A^c\}$ and $\text{dist}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$. Unimportant constants in the article will be denoted by small letters $c, c_1, c_2, \ldots$, and their labeling starts anew in each new statement. By a big letter $C$ we denote some more important constants, where e.g. $C(a, b)$ means that the constant $C$ depends only on parameters $a$ and $b$. However, the dependence on the dimension $d$ will not be mentioned explicitly. All constants are positive finite numbers. Furthermore, in what follows when we say $\nu$ is a measure, we mean that $\nu$ is a non-negative measure on $\mathbb{R}^d$. By $|\nu|$ we denote the total variation of a signed measure $\nu$. When we say $\nu$ is a signed measure on $D \subset \mathbb{R}^d$, we mean that $\nu$ is a signed measure on $\mathbb{R}^d$ and $|\nu|(D^c) = 0$. The Dirac measure of a point $x \in \mathbb{R}^d$ is denoted by $\delta_x$. Finally, $\mathcal{B}(\mathbb{R}^d)$ denotes Borel measurable sets in $\mathbb{R}^d$, and we suppose that all functions in the article are Borel functions and all signed measures are Borel signed measures.

# 2 Preliminaries

## 2.1 Process and the jumping kernel

Let $S = (S_t)_{t \geq 0}$ be a subordinator with the Laplace exponent $\phi$, i.e. $S$ is an increasing Lévy process with $S_0 = 0$ and

$$
\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad \lambda, t \geq 0.
$$

It is well known that $\phi$ is a Bernstein function of the form

$$
\phi(\lambda) = b \lambda + \int_0^\infty (1 - e^{-t\lambda}) \mu(dt), \quad \lambda > 0,
$$

where $b \geq 0$ and $\mu$ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge t) \mu(dt) < \infty$. The measure $\mu$ is called the Lévy measure and $b$ the drift of the subordinator. Throughout this article we suppose that $\phi$ is a complete Bernstein function. This assumption means that $\mu(dt)$ has a density $\mu(t)$ which is a completely monotone
function. For details about Bernstein functions see [23]. Also, we suppose that \( \phi \) satisfies the following upper and lower scaling conditions at infinity:

**Remark 2.2**

Condition (H1) There exist constants \( \delta_1, \delta_2 \in (0, 1) \) and \( a_1, a_2 > 0 \) such that

\[
\phi(\lambda r) \geq a_1 \lambda^{\delta_1} \phi(r), \quad \lambda \geq 1, \ r \geq 1, \quad \text{(LSC)}
\]

\[
\phi(\lambda r) \leq a_2 \lambda^{\delta_2} \phi(r), \quad \lambda \geq 1, \ r \geq 1. \quad \text{(USC)}
\]

This assumption yields that \( b = 0 \).

Suppose that \( W = (W_t)_{t \geq 0} \) is a Brownian motion in \( \mathbb{R}^d, \ d \geq 2 \), independent of \( S \) with the characteristic exponent \( \xi \mapsto |\xi|^2 \), \( \xi \in \mathbb{R}^d \). The process \( X = ((X_t)_{t \geq 0}, (P_x)_{x \in \mathbb{R}^d}) \) defined as \( X_t = W_{S_t} \) is called a subordinate Brownian motion in \( \mathbb{R}^d \). Here \( P_x \) denotes the probability under which the process \( X \) starts from \( x \in \mathbb{R}^d \), and by \( E_x \) we denote the corresponding expectation. Under conditions above \( X \) is a pure-jump rotationally symmetric Lévy process with the characteristic exponent \( \xi \mapsto \Psi(\xi) = \phi(|\xi|^2) \). The exponent has the following form

\[
\Psi(\xi) = \phi(|\xi|^2) = \int_{\mathbb{R}^d} (1 - \cos(\xi \cdot x)) J(dx), \quad \xi \in \mathbb{R}^d,
\]

where the measure \( J \) satisfies \( \int_{\mathbb{R}^d} (1 \wedge |x|^2) J(dx) < \infty \) and it is called the Lévy measure of the process \( X \). Also, \( J \) has a density given by \( J(x) = j(|x|), \ x \in \mathbb{R}^d \), where

\[
j(r) := \int_0^{\infty} (4\pi t)^{-d/2} e^{-r^2/(4t)} \mu(t) dt, \quad r > 0.
\]

The density \( j \) is positive, continuous, decreasing and satisfies \( \lim_{r \to \infty} j(r) = 0 \).

It is well known that since \( \phi \) is a complete Bernstein function, there is a constant \( C = C(\phi) > 0 \) such that

\[
j(r) \leq Cj(r + 1), \quad r \geq 1, \quad (3)
\]

see e.g. [13, Eq. (2.12)]. Also, from [16, Lemma 4.3] we have that for every \( r_0 \in (0, 1) \)

\[
\limsup_{\delta \to 0} \frac{j(r)}{j(r + \delta)} = 1. \quad (4)
\]

Using (4) we can easily prove the following technical lemma.

**Lemma 2.1** Let \( R > 0, \ \varepsilon > 0, \) and \( 0 < q \leq 1 \). There exists \( p = p(q, \varepsilon, R) < q \) such that for all \( z \in B_{pR} \) and \( y \in B_{qR}^c \)

\[
\frac{1}{1 + \varepsilon} j(|y|) \leq j(|y - z|) \leq (1 + \varepsilon) j(|y|).
\]

**Remark 2.2** Condition \( r \geq 1 \) in (LSC) and (USC) is important in the sense that the scaling is true away from zero. Using the continuity of \( \phi \) it is easy to show that if \( R_0 > 0 \), then (LSC) and (USC) are also valid for \( r \geq R_0 \) but with different constants \( a_1 \) and \( a_2 \) (\( \delta_1 \) and \( \delta_2 \) remain the same). Similarly, since \( j \) is continuous, inequality (3) holds for \( r \geq R_0 \) with a different constant \( C \).
2.2 Additional assumptions

In some results dealing with unbounded sets we will occasionally make additional assumptions on the density \( j \) and the exponent \( \phi \). The first assumption strengthens (H1).

(H2) (Global scaling condition) There exist constants \( \delta_1, \delta_2 \in (0,1) \) and \( a_1, a_2 > 0 \) such that

\[
\phi(\lambda r) \geq a_1 \lambda^{\delta_1} \phi(r), \quad \lambda \geq 1, \ r > 0, \quad \text{(GLSC)}
\]

\[
\phi(\lambda r) \leq a_2 \lambda^{\delta_2} \phi(r), \quad \lambda \geq 1, \ r > 0. \quad \text{(GUSC)}
\]

The second assumption comes as an addition to Lemma 2.1.

(E) For every \( R \geq 1, \ \varepsilon > 0, \) and \( q \in (1, \infty) \), there exists \( p = p(q, \varepsilon, R) > q \) such that for all \( z \in B^c_{pR} \) and \( y \in B_{qR} \)

\[
\frac{1}{1+\varepsilon} j(|z|) \leq j(|y - z|) \leq (1+\varepsilon) j(|z|).
\]

To the best of our knowledge it is not clear if the assumption (E) is true for every density \( j \) generated by a complete Bernstein function. However, it is known that if for some \( \alpha \in (0,2) \) we have \( \lim_{\lambda \to 0} \frac{\phi(\lambda^2)}{\lambda^{\alpha} l(\lambda)} = 1 \), where \( l \) is a slowly varying function at 0, then the condition (E) is satisfied, see [16, Section 4.2].

Note that the isotropic \( \alpha \)-stable process, \( \alpha \in (0,2) \), satisfies all mentioned assumptions, since in this case we have \( \phi(\lambda) = \lambda^{\alpha/2} \) and \( j(r) = c(d, \alpha) \frac{1}{r^{d+\alpha}} \).

2.3 Operator \( L \)

For \( x \in \mathbb{R}^d \) and \( u : \mathbb{R}^d \to \mathbb{R} \) we let

\[
Lu(x) := \text{P.V.} \int_{\mathbb{R}^d} (u(y) - u(x)) j(|y - x|) dy \quad \text{(5)}
\]

\[
:= \lim_{\varepsilon \to 0^+} \int_{|y-x| > \varepsilon} (u(y) - u(x)) j(|y - x|) dy,
\]

whenever the limit above exists. In the case of the isotropic \( \alpha \)-stable process the operator \( L \) is the fractional Laplacian \( \Delta^{\alpha/2} \).

If \( \varphi \in C^2_c(\mathbb{R}^d) \), i.e. \( \varphi \) is a twice continuously differentiable function with compact support, then \( L \varphi(x) \) exists for every \( x \in \mathbb{R}^d \). In fact, if \( \varphi \in C^2_c(\mathbb{R}^d) \), then using Taylor’s theorem it is easy to see that there is a constant \( C = C(K, \phi) > 0 \), where \( \text{supp} \ \varphi \subset K \subset \mathbb{R}^d \), such that

\[
|L \varphi(x)| \leq C ||\varphi||_{C^2(\mathbb{R}^d)} (1 \wedge j(|x|)), \quad x \in \mathbb{R}^d.
\]

(6)

Here \( || \cdot ||_{C^2(\mathbb{R}^d)} \) denotes the standard norm for twice differentiable functions.
For functions $u \in \mathcal{L}^1 := L^1(\mathbb{R}^d, (1 \wedge j(|x|))dx)$ we define the distribution $\tilde{L}$ as
\[
\langle \tilde{L}u, \varphi \rangle := \langle u, L\varphi \rangle := \int_{\mathbb{R}^d} u(x)L\varphi(x)dx, \quad \varphi \in C_c^\infty(\mathbb{R}^d).
\]
The condition $u \in \mathcal{L}^1$ is needed to ensure that the integral above is well defined, see (6). Also, note that since $j$ is positive, we have $\mathcal{L}^1 \subset L^1_{loc}$. Following [3, Section 3], it is easy to show that if $u \in C^2(D) \cap \mathcal{L}^1$, then $Lu(x)$ exists for every $x \in D$ and $\tilde{L}u = Lu$ as distributions on $D$, i.e.
\[
\langle \tilde{L}u, \varphi \rangle = \langle Lu, \varphi \rangle, \quad \varphi \in C_c^\infty(D).
\]
Furthermore, we extend the definition of $\tilde{L}$ to measures in the following way
\[
\langle \tilde{L}\lambda, \varphi \rangle := \langle \lambda, L\varphi \rangle := \int_{\mathbb{R}^d} L\varphi(x)\lambda(dx), \quad \lambda \text{ such that } \int_{\mathbb{R}^d} (1 \wedge j(|x|))|\lambda|(dx) < \infty.
\]

2.4 Green and Poisson kernel

Since we assume (H1) throughout the article, we have $\int_{\mathbb{R}^d} e^{-t\Phi(\xi)|\xi|^n}d\xi < \infty$, for $t > 0$ and $n \in \mathbb{N}$, see [18, Eq. (3.5)], so $X$ has transition densities $p(t, x, y) = p(t, y - x)$ given by
\[
p(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \cos(x \cdot \xi)e^{-t\Phi(\xi)}d\xi, \quad t > 0, \quad x \in \mathbb{R}^d.
\]

We assume that the process $X$ is transient, i.e. $\mathbb{P}_x(\lim_{t \to \infty} |X_t| = \infty) = 1$, $x \in \mathbb{R}^d$. When $d \geq 3$ this is always true, and for $d = 2$ by the Chung-Fuchs condition this means that
\[
\int_0^1 \frac{1}{\phi(\lambda)}d\lambda < \infty.
\]

We define the potential kernel of $X$, i.e. the Green function of $X$, by
\[
G(x) := \int_0^\infty p(t, x)dt, \quad x \in \mathbb{R}^d.
\]
The kernel $G$ is the density of the mean occupation time for $X$, i.e. for $f \geq 0$ we have
\[
\int_{\mathbb{R}^d} G(x - y)f(y)dy = \mathbb{E}_x \left[ \int_0^\infty f(X_t)dt \right], \quad x \in \mathbb{R}^d.
\]
From [15, Lemma 3.2(b)] it follows that for every $M > 0$ there is a constant $C = C(\phi, M) > 0$ such that
\[
C^{-1} \frac{1}{|x|^d}\phi(|x|^{-2}) \leq G(x) \leq C \frac{1}{|x|^d}\phi(|x|^{-2}), \quad |x| \leq M.
\]
In particular, $G$ is finite for $x \neq 0$. Further, $G$ is rotationally symmetric and radially decreasing so we will slightly abuse notation by denoting $G(x, y) = G(x - y) = g(|x - y|)$.

For an open $D \subset \mathbb{R}^d$ set $\tau_D = \inf\{t > 0 : X_t \notin D\}$. We define the killed process $X_D^\tau$ by

$$X_D^\tau_t := \begin{cases} X_{\tau_D}, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

where $\partial$ is an adjoint point to $\mathbb{R}^d$ called the cemetery. The process $X_D^\tau$ has a transition density which is for $t > 0$ and $x, y \in \mathbb{R}^d$ given by

$$p_D^\tau(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - \tau_D, X_{\tau_D}, y)1_{\{\tau_D < t\}}].$$

(9)

It follows that $0 \leq p_D^\tau \leq p$ and by repeating the proof of [9, Theorem 2.4] we get that $p_D^\tau$ is symmetric. Since the process $X$ has right continuous paths, it follows that $p_D^\tau(t, x, y) = 0$ if $x \in \overline{D}$ or $y \in \overline{D}$. The Green function of $X_D^\tau$ is defined by $G_D(x, y) := \int_0^\infty p_D^\tau(t, x, y)dt$, $x, y \in \mathbb{R}^d$, which is the density of the mean occupation time for $X_D^\tau$, i.e. for $f \geq 0$ we have

$$\int_D G_D(x, y)f(y)dy = \mathbb{E}_x\left[\int_0^{\tau_D} f(X_t)dt\right], \quad x \in \mathbb{R}^d.$$  \hspace{1cm} (10)

Note that $G = G_\mathbb{R}^d$.

For $x \in \mathbb{R}^d$ the distribution of $X_{\tau_D}$ is denoted by $\omega_\tau^D$, i.e.

$$\mathbb{P}_x(X_{\tau_D} \in A) = \omega_\tau^D(A), \quad A \in \mathcal{B}(\mathbb{R}^d).$$

The measure $\omega_\tau^D$ is concentrated on $D^c$ and since we are in the transient case, we have the following formula for $x, y \in \mathbb{R}^d$

$$G_D(x, y) = G(x, y) - \mathbb{E}_x[G(X_{\tau_D}, y)] = G(x, y) - \int_{D^c} G(w, y)\omega_\tau^D(dw).$$ \hspace{1cm} (11)

It follows from (9) that $G_D$ is symmetric and non-negative. On $(D \times D) \setminus \{(x, x) : x \in D\}$ the kernel $G_D$ is jointly continuous which can be easily seen via well-known representation of the densities $p(t, x)$ in [22, Eq. (2.8)]. Using the strong Markov property and (11) one can easily show that for all open $U \subset D$ and $x, y \in \mathbb{R}^d$ it holds

$$G_D(x, y) = G_U(x, y) + \int_{U^c} G_D(w, y)\omega_\tau^D(dw).$$ \hspace{1cm} (12)

Equation (9) also yields that $G_D(x, y) = 0$ if $x \in \overline{D}$ or $y \in \overline{D}$. Furthermore, if $y \in \partial D$, then $G_D(x, y) = 0$ for all $x \in D$ if and only if $y$ is a regular point for $D$. A point $x \in \partial D$ is regular for $D$ if $\mathbb{P}_x(\tau_D = 0) = 1$, i.e. if $\omega_\tau^D = \delta_x$. A point at $\partial D$ which is not regular is called irregular and it is well-known that the set of irregular points is polar. This property will be used many times throughout the article.
Equation (12) yields that for every \( x, y \in \mathbb{R}^d \) and open \( U \subset D \) we have \( G_U(x, y) \leq G_D(x, y) \). In fact, if we have open sets \( D_1 \subset D_2 \subset \cdots \subset D \) and \( \cup_n D_n = D \), then \( G_{D_n}(x, y) \uparrow G_D(x, y) \), for every \( x, y \in \mathbb{R}^d \) except if \( x \) or \( y \) are irregular for \( D \). This follows from (11), the continuity of \( G \) off the diagonal and the quasi-left-continuity of \( X \).

For an open \( D \subset \mathbb{R}^d \), we define \( P_D \), the Poisson kernel of \( D \) with respect to \( X \), by

\[
P_D(x, y) := \int_{\mathbb{R}^d} G_D(x, w) j(|w - y|) dw, \quad (x, y) \in \mathbb{R}^d \times D^c. \tag{13}
\]

If \( x \in D \) the measure \( \omega^x_D \) is absolutely continuous with respect to the Lebesgue measure in the interior of \( D^c \). Its Radon-Nikodym derivative is \( P_D(x, \cdot) \), see [16, Eq. (1.1)]. Further, if the boundary of \( D \) possesses enough regularity, e.g. if \( D \) is a Lipschitz set, then

\[
\omega^x_D(dy) = P_D(x, y)dy, \quad \text{on the whole} \ D^c, \tag{14}
\]

see [21, Proposition 4.1].

By integrating (12) with respect to \( j(|y - z|)dy \) on \( \mathbb{R}^d \), with \( z \in D^c \), and using Fubini’s theorem we get

\[
P_D(x, z) = P_U(x, z) + \int_{D \cup U} P_D(w, z) \omega_U^w(dw), \quad (x, z) \in U \times D^c, \tag{15}
\]

where we used that the sets of irregular points at \( \partial D \) and \( \partial U \) are polar.

**Definition 2.3** Let \( D \subset \mathbb{R}^d \) be an open set and \( f: D \to [-\infty, \infty] \). The **Green potential of** \( f \) **is defined by**

\[
G_D f(x) := \int_D G_D(x, y)f(y)dy, \tag{16}
\]

for all \( x \in \mathbb{R}^d \) such that the integral above converges absolutely.

**Lemma 2.4** Let \( f \geq 0 \). If the integral \( \int_D G_D(x_0, y)f(y)dy \) converges at one point \( x_0 \in D \), then \( G_D f < \infty \) a.e., \( G_D f \in \mathcal{L}^1 \) and \( f \in L_{loc}^1(D) \).

**Proof.** Let \( 0 < s < \delta_D(x_0) \), and denote just for this proof \( B = B(x_0, s) \). Using the strong Markov property we have

\[
\infty > G_D f(x_0) \geq \mathbb{E}_{x_0} \left[ \int_0^{\tau_B} f(X_t)dt \right] = \mathbb{E}_{x_0} \left[ \mathbb{E}_{X_{\tau_B}} \left[ \int_0^{\tau_D} f(X_t)dt \right] \right] = \mathbb{E}_{x_0}[G_D f(X_{\tau_B})] = \int_{B^c} G_D f(y)P_B(x_0, y)dy. \tag{17}
\]

From [12, Lemma 2.2] we have that \( P_B(x_0, y) \geq c_1 j(|x_0 - y|), \ y \in \overline{D^c} \). Furthermore, let \( r_0 \in (1, \infty) \) such that \( j(|y|) \leq 1 \), for \( |y| \geq r_0 \). Inequality (3) implies
that there is a constant $c_2 > 0$ such that $j(|y|) \leq c_2 j(|x_0 - y|)$, for all $|y| \geq r_0$. Let $m := \inf\{j(|x_0 - y|) : y \in B^c, |y| \leq r_0\} > 0$. Thus, for $y \in B^c$ we have
\[1 \land j(|y|) \leq \max\{c_2, 1/m\} j(|x_0 - y|).
\]
Therefore, there is $c_3 > 0$ such that $P_B(x_0, y) \geq c_3(1 \land j(|y|)) > 0$, $y \in B^c$. This yields
\[\int_{B^c} G_D f(y)(1 \land j(|y|))\,dy < \infty,
\]
hence $G_D f < \infty$ a.e. on $B^c$. Starting the calculations again from the point $\hat{x} \in D \setminus B$ such that $G_D f(\hat{x}) < \infty$, we also get $\int_B G_D f(y)(1 \land j(|y|))\,dy < \infty$. Hence, $G_D f < \infty$ a.e. and $G_D f \in L^1$.

To prove that $f \in L^1_{loc}(D)$ take $U \subset D$ and $x \in D \setminus U$ such that $G_D f(x) < \infty$. Since the function $y \rightarrow G_D(x, y)$ is bounded from below and above on $U$ by the Harnack inequality, see [10, Theorem 7], we have the claim.

The following proposition is an extension of [4, Lemma 5.3] to more general non-local operators.

**Proposition 2.5** Let $D$ be an open set. If $f : D \rightarrow [-\infty, \infty]$ satisfies $G_D[f](x) < \infty$ for some $x \in D$, then $L_i(G_D f) = -f$ in $D$.

**Proof.** In [11, Lemma 3.5] the claim was proved for bounded $D$ and for $f \in L^1(D)$. Recall that Lemma 2.4 yields that $G_Df$ is well defined almost everywhere and $f \in L^1_{loc}(D)$. Without loss of generality we can assume that $f \geq 0$.

Suppose that $D$ is bounded and $f \in L^1_{loc}(D)$. There is an increasing sequence of precompact sets $(K_n)_n$ in $D$ such that $\bigcup K_n = D$. Define $f_n := f1_{K_n} \in L^1$. Obviously, $G_D f_n \uparrow G_D f$ a.e. and also in $L^1$ due to Lemma 2.4 and the dominated convergence theorem. Hence, due to (6) for all $\varphi \in C^\infty_c(D)$ we get
\[\langle \tilde{L}G_D f, \varphi \rangle = \langle G_D f, L\varphi \rangle = \lim_{n \to \infty} \langle G_D f_n, L\varphi \rangle = \lim_{n \to \infty} -\langle f_n, \varphi \rangle = -\langle f, \varphi \rangle.
\]

Now take $D$ unbounded and $f \in L^1_{loc}(D)$. There is an increasing sequence of open precompact sets $(D_n)_n$ in $D$ such that $\bigcup D_n = D$. Obviously, $G_{D_n} f \uparrow G_D f$ a.e. and in $L^1$. Take any $\varphi \in C^\infty_c(D)$. There is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\varphi \in C^\infty_c(D_n)$. Hence,
\[\langle \tilde{L}G_D f, \varphi \rangle = \langle G_D f, L\varphi \rangle = \lim_{n \to \infty} \langle G_{D_n} f, L\varphi \rangle = \lim_{n \to \infty} -\langle f_{D_n}, \varphi \rangle = -\langle f, \varphi \rangle.
\]

□
3 Poisson kernel and $L$-harmonic functions

Proposition 3.1 Let $D$ be an open set. Then $P_D : D \times \overline{D} \to (0, \infty)$ is jointly continuous.

Proof. We imitate the proof of the similar claim for the isotropic $\alpha$-stable process, see [24, Theorem 5.7]. Let $(x_n)_n \subset D$ and $(z_n)_n \subset \overline{D}$ such that $x_n \to x \in D$, and $z_n \to z \in \overline{D}$. Let $0 < \varepsilon, \delta < 1$ such that $\delta_D(x) > 2\delta$ and $\delta_D(z) > 2\varepsilon$. Then for all large enough $n \in \mathbb{N}$ we have $\delta_D(x_n) > \delta$ and $\delta_D(z_n) > \varepsilon$. We have by (13)

$$|P_D(x_n,z_n) - P_D(x,z)| = \left| \int_D G_D(x_n,y)j(|y-z_n|)dy - \int_D G_D(x,y)j(|y-z|)dy \right|$$

$$\leq \left| \int_{D \cap B(x,2\delta)^c} G_D(x_n,y)j(|y-z_n|)dy - \int_{D \cap B(x,2\delta)^c} G_D(x,y)j(|y-z|)dy \right|$$

$$+ \int_{B(x,\delta)} G_D(x_n,y)j(|y-z_n|)dy + \int_{B(x,\delta)} G_D(x,y)j(|y-z|)dy.$$}

Recall that $j$ is continuous and that $G_D$ is continuous off the diagonal. Thus, for the first term we have by the dominated convergence theorem

$$\lim_{n \to \infty} \int_{D \cap B(x,2\delta)^c} G_D(x_n,y)j(|y-z_n|)dy = \int_{D \cap B(x,2\delta)^c} G_D(x,y)j(|y-z|)dy.$$}

Indeed, we can apply the dominated convergence theorem since $G_{\mathbb{R}^d}$ is radially decreasing so there is $c_1 > 0$ such that $G_D(w,y) \leq G_{\mathbb{R}^d}(w,y) \leq c_1$ for all $w \in B(x,\delta)$ and $y \in B(x,2\delta)^c$. Also, using (3) there is $c_2 > 0$ such that $j(|y-q|) \leq c_2j(|y-z|)$ for $q \in B(z,\varepsilon)$ and $y \in D \cap B(x,2\delta)^c$.

For the other two integrals we use the estimate (8), i.e. we use

$$G_D(x,y) \leq G_{\mathbb{R}^d}(x,y) \leq c_3 \frac{1}{|x-y|^d \phi(|x-y|^{-2})}, \quad |x-y| < 3,$$

where $c_3 = c_3(\phi) > 0$. Now for all $w \in B(x,\delta)$, and $q \in B(z,\varepsilon)$ we have

$$\int_{B(x,\delta)} G_D(w,y)j(|y-q|)dy \leq j(\varepsilon) \int_{B(x,\delta)} G_D(w,y)dy$$

$$\leq j(\varepsilon) \left( \int_{B(x,\delta) \cap B(w,\delta)} G_D(w,y)dy + \int_{B(x,2\delta) \cap B(w,\delta)^c} G_D(w,y)dy \right)$$

$$\leq j(\varepsilon)c_3 \left( \int_0^\delta \frac{dr}{r \phi(r^{-2})} + \int_\delta^{3\delta} \frac{dr}{r \phi(r^{-2})} \right) \leq j(\varepsilon)c_3 \left( \int_0^\delta \frac{dr}{r \phi(r^{-2})} \right) \delta \to 0,$$
where we use (LSC) for the convergence of the integral part.

**Definition 3.2** Let $D \subset \mathbb{R}^d$ be an open set and let $\lambda$ be a $\sigma$-finite signed measure on $D^c$ such that for all $x \in D$

$$\int_{D^c} P_D(x, y) |\lambda|(dy) < \infty. \quad (18)$$

The Poisson integral of $\lambda$ is defined by

$$P_D \lambda(x) := \int_{D^c} P_D(x, y) \lambda(dy), \quad x \in D.$$ 

We extend the definition of the Poisson integral for non-negative $\sigma$-finite measures by the same formula, i.e. for $\sigma$-finite measure $\lambda$ we define

$$P_D \lambda(x) := \int_{D^c} P_D(x, y) \lambda(dy) \in [0, \infty], \quad x \in D.$$ 

Although this seems as an extension of the definition, it will follow from Theorem 3.5 that either $P_D |\lambda| \equiv \infty$ or $P_D |\lambda| < \infty$ in $D$, see Remark 3.6.

It will be of considerable interest to extend $P_D \lambda$ to the whole $\mathbb{R}^d$ in the following sense. We define the (signed) measure $P^*_D \lambda$ by

$$P^*_D \lambda(dy) = P_D \lambda(y) 1_D(y)dy + 1_{D^c}(y) \lambda(dy), \quad (19)$$

i.e. $P^*_D \lambda$ is on $D$ the (signed) measure with the density function $P_D \lambda$ and on $D^c$ it is the (signed) measure $\lambda$. This extension was introduced in [8, Eq. (25)] for the case of the isotropic $\alpha$-stable process.

**Remark 3.3** Suppose that $P_D |\lambda|(x) < \infty$ for all $x \in D$. Then $\lambda$ is finite on compact subsets of $\overline{D}$. Indeed, let $K$ be a compact subset of $\overline{D}$ and let $s \in (0, 1)$ such that $B(x, s) \subset D$. For $y \in \overline{D}$ by [13, Proposition 4.7] we have $P_D(x, y) \geq P_{B(x,s)}(x, y) \geq c_1 j(|x - y|)$, where $c_1 > 0$. Thus, since $j$ is continuous and strictly positive, we have

$$\infty > \int_{D^c} P_D(x, y) |\lambda|(dy) \geq c_2 |\lambda|(K),$$

where $c_2 > 0$. Furthermore, in Remark 4.2 we will see that $\lambda$ can have some mass on $\partial D$ but only on the specific part of the boundary at so-called inaccessible points.

**Lemma 3.4** (a) Let $R \in (0, 1)$. There is a constant $C = C(\phi) > 0$ such that if $\lambda$ is a $\sigma$-finite measure supported on $B^c_R$, and $D \subset B_R$, then for all $x \in D \cap B_{R/2}$ it holds

$$C^{-1} \mathbb{E}_x \tau_D \int_{B_{R/2}} j(|y|) P_D \lambda(dy) \leq P_D \lambda(x) \leq C \mathbb{E}_x \tau_D \int_{B_{R/2}} j(|y|) P^*_D \lambda(dy). \quad (20)$$
(b) Suppose (H2) and let $R \geq 1$. There is a constant $C = C(\phi) > 0$ such that if $\lambda$ is a $\sigma$-finite measure supported on $\overline{B}_R$, and $D \subset \overline{B}_R$, then for all $x \in D \cap \overline{B}_{2R}$ it holds

$$C^{-1}P_D(x, 0) \int_{\overline{B}_R} P_D^* \lambda(dy) \leq P_D \lambda(x) \leq C P_D(x, 0) \int_{\overline{B}_R} P_D^* \lambda(dy). \quad (21)$$

**Proof.** For part (a) we will use [13, Lemma 5.4]. Notice that the inequality from the statement of [13, Lemma 5.4] is valid for any $(x, y) \in (U \cap B(z_0, r/2)) \times B(z_0, r)^c$. This can be seen by inspecting the proof of the lemma since [13, Eq. (5.1)] can be extended to (15). Hence, to finish the proof we just need to integrate the mentioned inequality with respect to the measure $\lambda(dy)$, where $z_0 = 0$, $U = D$ and $r = R$.

For part (b) we will use [14, Lemma 3.4]. Similarly as above, the inequality from the statement of [14, Lemma 3.4] is valid for any $(x, z) \in (U \cap B(0, 2r)) \times B(0, r)$. This can be checked by inspecting the proof. Again, the only difference is in the fact that [14, Eq. (3.10)] can be extended to (15). To finish the proof we need to integrate the mentioned inequality with respect to the measure $\lambda(dz)$ where $a = 2$, $U = D$ and $r = R$.

Lemma 3.4 yields the following version of a uniform boundary Harnack principle.

**Theorem 3.5 (a)** There is a constant $C = C(\phi) > 1$ such that for every $R \in (0, 1)$, for all open $D \subset \mathbb{R}^d$, $x_1, x_2 \in D \cap B_{R/2}$, $y_1, y_2 \in D^c \cap B_R$, and for all $\sigma$-finite measures $\rho, \lambda$ on $B_R^c$ we have

$$P_D(x_1, y_1)P_D(x_2, y_2) \leq C P_D(x_1, y_2)P_D(x_2, y_1) \quad (22)$$

and

$$P_D(x_1)P_D(x_2) \leq C P_D(x_2)P_D(x_1). \quad (23)$$

(b) Suppose (H2). There is a constant $C = C(\phi) > 1$ such that for every $R \geq 1$, for all open $D \subset \mathbb{R}^d$, $x_1, x_2 \in D \cap \overline{B}_{2R}$, $y_1, y_2 \in D^c \cap \overline{B}_R$, and for all $\sigma$-finite measures $\rho, \lambda$ on $\overline{B}_R^c$ we have

$$P_D(x_1, y_1)P_D(x_2, y_2) \leq C P_D(x_1, y_2)P_D(x_2, y_1) \quad (24)$$

and

$$P_D(x_1)P_D(x_2) \leq C P_D(x_2)P_D(x_1). \quad (25)$$

The first part of this theorem is an extension of [13, Theorem 1.1(ii)] with $D^c$ being replaced by $D^c$, i.e. the difference is that points $y_1$ and $y_2$ in (22) can be at $\partial D$. This subtle difference comes as a consequence of Lemma 3.3 and will play a very important role in proving the results on the relative oscillation of Poisson integrals, e.g. Lemma 5.5.
Proof of Theorem 3.5. We give the proof of the first claim. The second claim follows similarly.

Let $D_R = D \cap B_R$. It is easy to see from (15) that for $x_i \in D \cap B_{R/2}$, $i \in \{1, 2\}$, and $y_j \in D^c \cap B_{R/2}^c$, $j \in \{1, 2\}$, we have that $P_D(x_i, y_j) = P_{D_R} \lambda_j(x_i)$ for some measure $\lambda_j$ supported on $B_R$. Now (22) follows from Lemma 3.4. By integrating (22) with respect to the measures $\rho(dy_1)$ and $\lambda(dy_2)$ we get (23).

Remark 3.6 Note that for the $\sigma$-finite measures $\rho$ and $\lambda$ appearing in Theorem 3.5 we do not assume (18). However, by fixing $\rho = \delta_{y_2}$, where $y_2 \in \overline{D}$, it follows from (23) that if for a $\sigma$-finite signed measure $\lambda$ on $D^c$ we have $P_D|\lambda|(x) < \infty$ for some $x \in D$, then we have $P_D|\lambda|(x) < \infty$ for all $x \in D$. This means that either $P_D|\lambda| \equiv \infty$ or $P_D|\lambda| < \infty$ in $D$.

Before we define $L$-harmonic functions we recall that a function $u : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic with respect to the process $X$ in an open set $D \subset \mathbb{R}^d$ if for every open $U \subset D$ and all $x \in U$ it holds that $\mathbb{E}_x[|u(X_{\tau_U})|] < \infty$ and

$$u(x) = \mathbb{E}_x[u(X_{\tau_U})]. \quad (26)$$

We say that $u$ is regular harmonic in $D$ if (26) holds with $U = D$. If $u$ is harmonic in $D$ and $u = 0$ in $\overline{D}$, then $u$ is said to be singular harmonic. From (12) we can see that for $y \in D$ the function $x \mapsto G_D(x, y)$ is harmonic in $D \setminus \{y\}$ and regular harmonic in $D \setminus B(y, \varepsilon)$ for every $\varepsilon > 0$.

Definition 3.7 Let $D \subset \mathbb{R}^d$ be an open set. We say that $f : D \to \mathbb{R}$ is $L$-harmonic in $D$ with outer charge $\lambda$ if $\lambda$ is a $\sigma$-finite (signed) measure on $D^c$ and if for every $U \subset D$ and $x \in U$ we have

$$f(x) = \int_{D^c} P_U(x, y)\lambda(dy) + \int_{D \setminus U} f(y)\omega^{-}_U(dy), \quad (27)$$

where the integrals converge absolutely.

The definition above was first used in [8] for the isotropic $\alpha$-stable process with an additional assumption of continuity of the function $f$. We prove in Proposition 3.9 that this additional assumption can be dropped, and in Theorem 3.12 we prove that $f \in C^\infty(D)$. Furthermore, note that a function $u : \mathbb{R}^d \to \mathbb{R}$ which is harmonic in $D$ is $L$-harmonic in $D$ with outer charge $\lambda(dy) = u(y)dy$. Indeed, take $U \subset D$ and $x \in U$. Equation (26) implies

$$u(x) = \mathbb{E}_x[u(X_{\tau_U})] = \int_{U^c} u(y)\omega_U^{-}(dy)$$

$$= \int_{D^c} P_U(x, y)u(y)dy + \int_{D \setminus U} u(y)\omega_U^{-}(dy), \quad (28)$$

where we used that $P_U(x, \cdot)$ is the density of $\omega_U^{-}$ in the interior of $U^c$. Hence, every harmonic function is $L$-harmonic. Furthermore, if $u$ is $L$-harmonic in $D$ with outer charge $\lambda$ such that $\lambda$ is absolutely continuous with respect to the
Lebesgue measure on $D^c$, then $u$ is harmonic in $D$. In particular, if $u$ has zero outer charge, i.e. $\lambda \equiv 0$, then $u$ is a singular harmonic function.

If $f$ is $L$-harmonic in $D$ with outer charge $\lambda$ we sometimes abbreviate notation by saying $(f, \lambda)$ is $L$-harmonic in $D$. Property (27) is often referred to as the mean-value property because of the connection with (28). Similarly as in (19), integrating with respect to $(f, \lambda)$ means that we integrate with respect to the measure $f(y)\mathbf{1}_D(y)dy + \mathbf{1}_{D^c}(y)\lambda(dy)$. We continue with a few properties of $L$-harmonic functions.

**Lemma 3.8** Let $D$ be an open set. If $(f, \lambda)$ is $L$-harmonic in $D$, then

$$
\int_{\mathbb{R}^d} (1 \wedge j(|y|))(|f|, |\lambda|)(dy) < \infty. 
$$

(29)

In particular, $f \in L^1_{\text{loc}}(D)$ and if $D$ is bounded, we have $f \in L^1(D)$.

**Proof.** Let $B(x, s) \subset \subset D$ with $s < 1$. From (27) with $U = B(x, s)$ and with the same calculations as in Lemma 2.4 we get that

$$
\begin{align*}
\infty &> \int_{D^c} (1 \wedge j(|y|))\lambda(dy) + \int_{D^c} f(y)(1 \wedge j(|y|))(dy).
\end{align*}
$$

Since $j$ is continuous and $j > 0$ we see that $f \in L^1_{\text{loc}}(D)$ and we get (29). Obviously, if $D$ is bounded, we have $f \in L^1(D)$. \hfill \Box

**Proposition 3.9** Let $D$ be an open set. If $(f, \lambda)$ is $L$-harmonic in $D$, then $f \in C(D)$.

**Proof.** Let $x \in D$ and $(x_n)_n \subset D$ such that $x_n \to x$. Let $0 < \varepsilon < 1$ be such that $\delta_D(x) > \varepsilon$. Without loss of generality suppose that for all $n \in \mathbb{N}$ we have $x_n \in B(x, \varepsilon/2)$. Using (27) with $U = B(x, \varepsilon)$ and applying (14) we have

$$
\begin{align*}
f(x) &= \int_{D^c} P_{B(x, \varepsilon)}(x, y)\lambda(dy) + \int_{D \setminus B(x, \varepsilon)} f(y)P_{B(x, \varepsilon)}(x, y)dy, \\
f(x_n) &= \int_{D^c} P_{B(x, \varepsilon)}(x_n, y)\lambda(dy) + \int_{D \setminus B(x, \varepsilon)} f(y)P_{B(x, \varepsilon)}(x_n, y)dy.
\end{align*}
$$

Note that Proposition 3.1 yields $P_{B(x, \varepsilon)}(x_n, y) \to P_{B(x, \varepsilon)}(x, y)$. Also, inequality (22) for $D = B(x, \varepsilon)$ implies that there is a constant $c > 0$ such that $P_{B(x, \varepsilon)}(x_n, y) \leq cP_{B(x, \varepsilon)}(x, y)$, for all $n \in \mathbb{N}$ and all $y \in B(x, s)^c$. Now by the dominated convergence theorem we have $f(x_n) \to f(x)$. \hfill \Box

In Theorem 3.12 we strengthen the previous proposition by proving that $f \in C^\infty(D)$. This is achieved using the same technique as in [12, Proposition 3.2 & Theorem 1.7]. First we invoke [12, Proposition 3.2] and its consequences.

**Lemma 3.10** Let $0 \leq q < r < \infty$. There is a radial kernel function $\mathcal{P}_{q, r} : \mathbb{R}^d \to \mathbb{R}$, a constant $C = C(\phi, q, r) > 0$, and a probability measure $\mu_{q, r}$ on $[q, r]$ with the following properties:
Proof. The claim can be proved in the same way as in [λ charge c where

\[ \frac{\partial}{\partial t} u = \Delta u - \lambda u \]

and let

\[ v(t, x) = u(t, x) + \frac{\lambda}{2} x \cdot \nabla u(t, x) \]

be a non-negative smooth radial function which takes values in [0, 1], which is equal to 1 in \( B_{3r/2} \), and which is equal to 0 in \( B_{2r}^c \). Define \( \pi_r(z) = \mathcal{P}_{0,r}(z)/r \), and \( \Pi_r(z) = \mathcal{P}_{0,r}(z)/(1 - \kappa(z)) \).

Note that Proposition 3.9 yields that \( f \) is bounded on \( B_{2(k+1)r} \) so set \( m := \sup_{B_{2(k+1)r}} |f| < \infty \). From Lemma 3.10(a) for \( x \in B_{2kr} \) we have

\[
\langle [f], [\lambda] \rangle \prod_{0,r}(x) = \int_{B(x, 2r)} |f(z)| P_{0,r}(x - z) dz + \int_{B^c(x, 2r)} P_{0,r}(x - z) |f| dz \\
\leq c_1 m C_r + \int_{B^c(x, 2r)} P_{B(x,r)}(x, z) |f| dz,
\]

(31)

where \( c_1 = c_1(r) > 0 \) is the volume of a ball with radius \( r \). From [13, Proposition 4.7] we have \( P_{B(x,r)}(x, z) \leq c_2 f(|x - z| - r) \), where \( c_2 = c_2(\phi, r) > 0 \). Thus,
using (3), we get that there is \( c_3 = c_3(\phi, k, r) > 0 \) such that for all \( x \in B_{2kr} \) and \( z \in B^c(x, 2r) \) it holds

\[
P_{B(x,r)}(x,z) \leq c_3(1 \wedge j(|z|)).
\]

Applying this inequality in (31) and recalling Lemma 3.8 we get that there is \( M = M(\phi, k, r, f, \lambda) < \infty \) such that

\[
(|f|, |\lambda|) * \mathcal{P}_{0,r} \leq M, \quad \text{in } B_{2kr}.
\]

(32)

Obviously, since \( \mathcal{P}_{0,r} = \pi_r + \Pi_r \), we have \(|f| * \pi_r \leq M \) and \((|f|, |\lambda|) * \Pi_r \leq M \) in \( B_{2kr} \). Also, since \( f = (f, \lambda) * \mathcal{P}_{0,r} \) in \( B_{2kr} \), we have

\[
f = f * \pi_r + (f, \lambda) * \Pi_r, \quad \text{in } B_{2kr}.
\]

(33)

Finally, inequality (32) implies that the convolution property (33) can be used iteratively to get that for \( x \in B_r \) it holds

\[
f = (\delta_0 + \pi_r + \pi_r^{*2} + \ldots, \pi_r^{*(k-1)}) * \Pi_r * (f, \lambda) + \pi_{r^k} * f.
\]

Since all derivatives of the jumping kernel \( j \) exist and are absolutely integrable in \( B_c^\varepsilon \), for every \( \varepsilon > 0 \), see [5, Proposition 7.2], we may proceed with the proof in the same way as in [12, Theorem 1.7]. \( \square \)

**Corollary 3.13** Let \( D \) be an open set. If \( \lambda \) is a \( \sigma \)-finite signed measure on \( D^c \) satisfying (18), then for every \( x \in U \subset D \)

\[
P_D \lambda(x) = \int_{D^c} P_U(x,y) \lambda(dy) + \int_{D \setminus U} P_D \lambda(y) \omega_U(y) dy.
\]

(34)

In particular, \( P_D \lambda \) is \( L \)-harmonic in \( D \) with outer charge \( \lambda \) and \( P_D \lambda \in C^\infty(D) \cap L^1_{loc}(D) \). Also, if \( D \) is bounded, then \( P_D \lambda \in L^1(D) \).

**Proof.** Take \( U \subset D \) and \( x \in U \). By integrating (15) with respect to \( \lambda(dz) \) we get (34). In particular, \( P_D \lambda \) is \( L \)-harmonic in \( D \) with outer charge \( \lambda \). Hence by Theorem 3.12 and Lemma 3.8 we have \( P_D \lambda \in C^\infty(D) \cap L^1_{loc}(D) \) and if \( D \) is bounded, then \( P_D \lambda \in L^1(D) \). \( \square \)

**Remark 3.14** Note that (34) holds for every \( U \subset D \) which is a lot stronger than needed in (27). This property will be heavily used in proving results on the relative oscillation of Poisson integrals.

We finish this section by proving two theorems about the connection between harmonic functions and the operator \( L \). First we prove an auxiliary result.

**Lemma 3.15** Let \( D \) be an open set and \( \lambda \) be a \( \sigma \)-finite signed measure on \( D^c \) such that (18) is satisfied. Then \( \hat{L}(P_D^* \lambda) = 0 \) in \( D \).
Proof. First recall that for $\varphi \in C_c^\infty(D)$ we have

$$L\varphi(x) = \text{P.V.} \int_{\mathbb{R}^d} (\varphi(y) - \varphi(x)) j(|x - y|) dy,$$

and

$$\langle \tilde{L}(P_D^* \lambda), \varphi \rangle = \langle P_D^* \lambda, L\varphi \rangle = \int_D P_D \lambda(x) L\varphi(x) dx + \int_{D^c} L\varphi(x) \lambda(dx) =: I_1 + I_2.$$

Note that $P_D \lambda(x) = G_D f(x), \; x \in D$, where $f(z) = \int_{D^c} j(|z - y|) \lambda(dy)$. For the integral $I_1$ by Proposition 2.5 we have

$$\int_D P_D \lambda(x) L\varphi(x) dx = - \int_D \left( \int_{D^c} j(|x - y|) \lambda(dy) \right) \varphi(x) dx.$$

For the integral $I_2$ recall that $\text{supp } \varphi \subset D$ and $\varphi = 0$ on $D^c$. Hence

$$\int_{D^c} L\varphi(x) \lambda(dx) = \int_{D^c} \left( \int_D \varphi(y) j(|x - y|) dy \right) \lambda(dx)$$

$$= \int_D \varphi(y) \left( \int_{D^c} j(|x - y|) \lambda(dx) \right) dy,$$

where we can change the order of integration by Fubini’s theorem since $f \in L_{1,\text{loc}}(D)$. Thus, $\langle \tilde{L}(P_D^* \lambda), \varphi \rangle = 0$ for all $\varphi \in C_c^\infty(D)$. 

Theorem 3.16 Let $D$ be an open set and $u$ $L$-harmonic in $D$ with outer charge $\lambda$. Then $\tilde{L}(u, \lambda) = 0$ in $D$.

Proof. Let $\varphi \in C_c^\infty(D)$. There is $U \subset D$ with Lipschitz boundary such that $\text{supp } \varphi \subset U$, i.e. $\varphi \in C_c^\infty(U)$. From (27) for $u$ we have $u = P_U^* \lambda$ in $U$, where $\tilde{\lambda}(dy) = u(y) 1_{D^c}(y) \lambda(dy) + 1_{D^c}(y) \lambda(dy)$. This means that $u$ is the Poisson integral on $U$ so by Lemma 3.15 we have

$$\int_D L\varphi(x) u(x) dx + \int_{D^c} L\varphi(x) \lambda(dx) = \int_U L\varphi(x) P_U^* \tilde{\lambda}(dx) + \int_{D^c} L\varphi(x) \tilde{\lambda}(dx) = 0.$$

Since $\varphi$ was arbitrary, we have the claim.

Remark 3.17 The proof of the previous theorem is valid in a much greater generality. Indeed, the only non-trivial part of the proof was the property $\tilde{L}(G_D f) = -f$ in $D$ proved in Proposition 2.5. One can check that Proposition 2.5 is true with the same proof for the isotropic unimodal Lévy process with the condition (3) on the jumping kernel since the auxiliary results [11, Lemma 3.5] and Lemma 2.4 also hold in this setting.

We can extract a weakened converse claim of Theorem 3.16 using [11, Lemma 3.3]:

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Theorem 3.18 Let $D$ be an open set and $u \in L^1$. If $\tilde{L}u = 0$ in $D$, then $u$ has a modification that is $L$-harmonic in $D$.

Proof. If $\tilde{L}u = 0$ in $D$, then it is proved in [11, Lemma 3.3] that for every Lipschitz $U \subset D$ we have $u(\cdot) = P_U u(\cdot) = \int_{D^c} u(y) P_U (\cdot, y) dy$ a.e. in $\mathcal{U}$.

Define the function $\tilde{u} : \mathbb{R}^d \rightarrow \mathbb{R}$ as $\tilde{u} = u$ on $D^c$ and for $x \in D$ choose some Lipschitz $U \subset D$ such that $x \in U$ and define $\tilde{u}(x) = P_U u(x)$. Let us show that $\tilde{u}$ is well defined. Suppose that we have Lipschitz sets $U_1 \subset D$ and $U_2 \subset D$ such that $x \in U_1 \cap U_2$ and $P_{U_1} u(x) > P_{U_2} u(x)$. Since by Corollary 3.13 $P_{U_j} u$ is continuous in $U_j$, $j \in \{1, 2\}$, there is $\varepsilon > 0$ such that for every $y \in B(x, \varepsilon) \subset U_1 \cap U_2$ we have $P_{U_1} u(y) > P_{U_2} u(y) + \varepsilon$. But $u = P_{U_1} u = P_{U_2} u$ a.e. in $U_1 \cap U_2$ so we have a contradiction. Hence, $\tilde{u}$ is well defined.

Recall that since $D$ is an open set, it is a countable union of balls. Also, every ball is a Lipschitz set so it is obvious from the construction of $\tilde{u}$ and the beginning of the proof that $u = \tilde{u}$ a.e. in $\mathbb{R}^d$.

Now we prove that $\tilde{u}$ is harmonic in $D$. Note that since $u = \tilde{u}$ a.e., we have for all Lipschitz sets $V \subset D$ and all $x \in V$

$$\tilde{u}(x) = \mathbb{E}_x [u(X_{\tau_V})] = \int_{V^c} u(y) P_V (x, y) dy = \int_{V^c} \tilde{u}(y) P_V (x, y) dy = \mathbb{E}_x [\tilde{u}(X_{\tau_V})].$$

Let $x \in U \subset D$ and take a Lipschitz set $V$ such that $U \subset V \subset D$. We have by the strong Markov property and the previous equality

$$\tilde{u}(x) = \mathbb{E}_x [\tilde{u}(X_{\tau_V})] = \mathbb{E}_x [\mathbb{E}_{X_{\tau_V}} [\tilde{u}(X_{\tau_V})]] = \mathbb{E}_x [\tilde{u}(X_{\tau_V})].$$

\[\square\]

4 Accessible points and Martin kernel

In this section we give a summary of results concerning the Martin boundary. All of the results are already known but some are not plainly stated. Our goal is to state and prove results that are important for our article for the reader’s convenience.

In the case where only (H1) holds many results concerning the Martin kernel can be proved only for bounded sets so the additional assumptions (H2) and (E) will be occasionally assumed to get results for unbounded sets.

For $D \subset \mathbb{R}^d$ let us denote

$$D^* := \begin{cases} \partial D, & \text{if } D \text{ is bounded,} \\ \partial D \cup \{\infty\}, & \text{if } D \text{ is unbounded,} \end{cases} \quad \partial^* D := \begin{cases} \partial D, & \text{if } D \text{ is bounded,} \\ \partial D \cup \{\infty\}, & \text{if } D \text{ is unbounded,} \end{cases}$$

where $\infty$ is an additional point in Alexandroff compactification and it is called the point at infinity.
Definition 4.1 Let $D$ be an open set. A point $y \in \partial D$ is called accessible from $D$ if

$$P_D(x_0, y) = \int_D G_D(x_0, z) j(|z - y|) dz = \infty, \quad \text{for some } x_0 \in D.$$ 

The point at infinity is accessible from $D$ if

$$\mathbb{E}_{x_0} \tau_D = \int_D G_D(x_0, y) dy = \infty, \quad \text{for some } x_0 \in D.$$ 

If $y \in \partial^* D$ is not accessible it is called inaccessible. The set of all accessible points is denoted by $\partial M_D$.

Remark 4.2 In [17, Proposition 4.1 & Remark 4.2] the following claims were proved.

(a) Let $y \in \partial D$. If $P_D(x_0, y) < \infty$ for some $x_0 \in D$, then $P_D(x, y) < \infty$ for all $x \in D$.

(b) Assume (H2). If $\mathbb{E}_{x_0} \tau_D < \infty$ for some $x_0 \in D$, then $\mathbb{E}_x \tau_D < \infty$ for all $x \in D$.

Note that we could get the claim (a) directly from Theorem 3.5 (a). Also, from the definition of accessible points it is clear that if $\lambda$ is a signed measure on $D^c$ such that $P_D|\lambda < \infty$, then $\lambda$ is concentrated on $\mathbb{R}^d \setminus (D \cup \partial M D)$, i.e. $\lambda$ can have no mass on the set of accessible points.

For an open $D \subset \mathbb{R}^d$ we fix an arbitrary point $x_0 \in D$ and define the Martin kernel on $D$ by

$$M_D(x, y) := \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x, y \in D, y \neq x_0,$$

$$M_D(x, z_0) := \lim_{D \ni v \to z_0} \frac{G_D(x, v)}{G_D(x_0, v)}, \quad x \in D, z_0 \in \partial^* D. \quad (35)$$

In [16] and [17] many important and useful results about the Martin kernel of more general processes than the subordinate Brownian motion were proved. E.g. it was proved that $M_D(x, z_0)$ exists, is finite and strictly positive for every $z_0 \in \partial^* D$ (with the additional assumptions (H2) and (E) if $z_0$ is the point at infinity). We summarize some of those results in the following theorem.

Theorem 4.3 Let $D$ be an open set, and $z_0 \in \partial^* D$.

(a) Let $z_0 \in \partial M D$ (for $z_0 = \infty$ assume (H2)). The function $x \mapsto M_D(x, z_0)$ is $L$-harmonic in $D$ with zero outer charge and for every open $U \subset D$ it holds

$$M_D(x, z_0) = \int_{D \setminus U} M_D(y, z_0) \omega_U^z(dy), \quad x \in U.$$
Let $z_0 \notin \partial M D$ (for $z_0 = \infty$ assume (H2) and (E)). The function $x \mapsto M_D(x, z_0)$ is not $L$-harmonic in $D$ with zero outer charge and for every open $U \subset \subset D$ it holds

$$M_D(x, z_0) > \int_{D \setminus U} M_D(y, z_0)\omega^+_U(dy), \quad x \in U.$$ 

Proof. First notice that by adding the assumptions (H2) and (E) where needed all assumptions of claims from [16] and [17] are satisfied, see [16, Section 4.1] and [17, Section 4.1]. Furthermore, recall Lemma 2.1 for the assumption $E1$ of [16].

Suppose that $z_0 \notin \partial M D$. From [16, Theorem 3.1] we have that

$$M_D(x, z_0) = \begin{cases} \frac{P_D(x, z_0)}{P_D(x_0, z_0)}, & \text{if } z_0 \in \partial D, \\ \frac{E_{x_0} \tau_D}{E_{x_0} \tau_D}, & \text{if } z_0 = \infty. \end{cases}$$

Hence for finite $z_0 \notin \partial M D$, $x \mapsto M_D(x, z_0)$ is $L$-harmonic with outer charge $\delta_{z_0}/P_D(x_0, z_0)$ but it is not $L$-harmonic with zero outer charge, see Corollary 3.13. Also, for every $x \in U \subset \subset D$ we have by the mean-value property of $L$-harmonic functions

$$M_D(x, z_0) = \int_{D \setminus U} M_D(y, z_0)\omega^+_U(dy) + \frac{P_U(x, z_0)}{P_D(x_0, z_0)} \int_{\tau_U}^{\tau_D} \frac{dt}{t}.$$ 

If $z_0 = \infty$, then $M_D(x, \infty)$ is not $L$-harmonic with zero outer charge because for $x \in U \subset \subset D$ we have

$$\int_{D \setminus U} M_D(y, \infty)\omega^+_U(dy) = \frac{1}{E_{x_0} \tau_D} \mathbb{E}_{x_0} \left[ \mathbb{E}_{X_{\tau_U}} \tau_D \right] = \frac{1}{E_{x_0} \tau_D} \mathbb{E}_{x_0} \left[ \int_{\tau_U}^{\tau_D} 1 dt \right]$$

$$< \frac{E_{x_0} \tau_D}{E_{x_0} \tau_D} = M_D(x, \infty),$$

where the strict inequality comes from the fact that for $x \in U$ there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset U$ and $\mathbb{E}_{x} \tau_U \geq \mathbb{E}_{x} \tau_{B(x, \varepsilon)} > 0$ by [13, Lemma 4.3].

Suppose now that $z_0 \in \partial M D$. Then we have that $x \mapsto M_D(x, z_0)$ is $L$-harmonic with zero outer charge. For the finite point $z_0$ this follows from [16, Theorem 1.2(b)] (see the proof), or [17, Theorem 1.1], and for the point at infinity we apply [16, Theorem 1.4(b)], or [17, Theorem 1.3]. In either case by the mean-value property of $L$-harmonic functions we get for every $U \subset \subset D$ and all $x \in U$

$$M_D(x, z_0) = \int_{D \setminus U} M_D(y, z_0)\omega^+_U(dy).$$

\qed
Remark 4.4 It will be very useful to note that in [17] two specific mean-value formulae were proved. If \( z_0 \in \partial M \setminus \{ \infty \} \), then for every \( r < \frac{1}{4}|z_0 - x_0| \) and \( U_r := D \setminus B(z_0, r) \)

\[
M_D(x, z_0) = \int_{U_r} M_D(y, z_0) \omega_U^x(dy), \quad x \in U_r, \tag{36}
\]

see [17, (3.14)].

Also, if \( z_0 = \infty \in \partial M \) and we additionally assume (H2), then for every \( R > 4|x_0| \) and \( U_R := D \cap B(0, R) \)

\[
M_D(x, \infty) = \int_{U_R} M_D(y, \infty) \omega_U^x(dy), \quad x \in U_R, \tag{37}
\]

see [17, (3.4)].

In fact, from (36) it follows by using the strong Markov property that (36) is true for every \( U \subset D \) open such that \( z_0 \notin \overline{U} \). By similar reasoning (37) holds for every \( U \subset D \) open and bounded such that \( U_R \subset U \) for some \( R > 4|x_0| \).

Definition 4.5 Let \( D \subset \mathbb{R}^d \) be an open set and \( \mu \) a finite signed measure on \( \partial^* D \) concentrated on \( \partial M D \). The Martin integral of \( \mu \) is defined by

\[
M_D \mu(x) := \int_{\partial M D} M_D(x, y) \mu(dy), \quad x \in \mathbb{R}^d.
\]

Remark 4.6 Let \( \mu \) be a finite measure concentrated on \( \partial M D \). From \( M_D(x_0, z) = 1, \ z \in \partial^* D \), we see that \( M_D \mu(x_0) = \mu(\partial M D) \). It will follow from Corollary 5.13 that \( M_D \mu \) is finite at some point (or all points) if and only if \( \mu \) is finite. Also, due to harmonicity of \( x \mapsto M_D(x, z_0) \) for \( z_0 \in \partial M D \), it is easy to check that \( M_D \mu \) is \( L \)-harmonic in \( D \) with outer charge zero. That is the reason why we look, regarding the Martin integral, at finite measures concentrated on \( \partial M D \) in what follows.

5 Boundary trace operator \( W_D \) and representation of \( L \)-harmonic functions

Let \( D \) be an open set, \( u : D \to [\infty, \infty] \), and let \( U \subset D \) be a set with Lipschitz boundary such that \( x_0 \in U \), where \( x_0 \) is the fixed point from the definition of the Martin kernel. We define the signed measure \( \eta_U u \) by

\[
\eta_U u(A) = \int_A G_U(x_0, z) \left( \int_{\partial^* U} j(|z - y|) u(y) dy \right) dz, \quad A \in \mathcal{B}(\mathbb{R}^d).
\]

Definition 5.1 If \( (\eta_U u|(D))_U \) is bounded as \( U \uparrow D \) and \( (\eta_U u)_U \) weakly converges to a signed measure \( \mu \) as \( U \uparrow D \), then we denote \( W_D u = \mu \), i.e. \( W_D u := \lim_{U \uparrow D} \eta_U u \).
The boundary trace operator $W_D$ was used in [7] as the boundary condition in the Dirichlet problem for the fractional Laplacian and it was used as a tool to get the representation of non-negative $\alpha$-harmonic functions in [8]. As one can see, the definition of $W_D$ is rather delicate. It can be easily seen that for a bounded function $f$ we have $W_Df = 0$ since $\partial f / \partial N(\partial D) \downarrow 0$. However, $W_D$ can be applied to many more functions, e.g. we will show that $W_D[M_D\mu] = \mu$ and $W_D[G_Df] = W_D[P_{D}\lambda] = 0$, see also [2, Proposition 4.8]. In what follows, we prove that some important properties of $W_D$ are also true in the case of subordinate Brownian motions and at the end of the article we will use the operator to get the representation of non-negative $L$-harmonic functions.

**Lemma 5.2** $W_Du$ is concentrated on $\partial^* D$.

*Proof.* Let $A \subset \subset D$. Then there is a Lipschitz set $U_A \subset \subset D$ such that $x_0 \in U_A$ and $A \subset \subset U_A$. Now we will show that $G_U(x_0, y) = G_D(x_0, y)$, for all $y \in A$ and for all Lipschitz $U$ such that $U_A \subset \subset U \subset \subset D$.

Let $\varepsilon > 0$ be such that $B(x_0, 2\varepsilon) \subset U_A$. For $y \in B(x_0, \varepsilon)$ and all Lipschitz $U$ such that $U_A \subset \subset U \subset \subset D$ we have

\begin{equation}
G_{B(x_0, 2\varepsilon)}(x_0, y) \leq G_U(x_0, y) \leq G_{\mathbb{R}^d}(x_0, y) \leq C G_{B(x_0, 2\varepsilon)}(x_0, y) \tag{38}
\end{equation}

where $C > 1$ is independent of $U$. Indeed, by (8) and [12, Theorem 1.3] we have for $y \in B(x_0, \varepsilon)$

\begin{equation}
G_{\mathbb{R}^d}(x_0, y) \leq c_1 \frac{1}{|x_0 - y|^d \phi(|x_0 - y|^2)}\tag{39}
\end{equation}

\begin{equation}
G_{B(x_0, 2\varepsilon)}(x_0, y) \geq \frac{1}{c_2 (K(|x_0 - y|) + L(|x_0 - y|))^2}
\end{equation}

where $K(r) = \int_{B(0,r)} \frac{|z|^2}{j(|z|)} |z| dz$ and $L(r) = \int_{B(0,r)} j(|z|) dz$. Define $h(r) = K(r) + L(r) = \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|^2}{j(|z|)}\right) j(|z|) dz$. By [6, Eq. (6) and Lemma 1] we have that $h(r) \asymp \phi(\frac{1}{r^2})$ so by using [13, Theorem 2.3] for all small enough $q > 0$ we have that

\begin{equation}
K(q) \leq K(q) + L(q) = h(q) \leq c_3 \phi(\frac{1}{q^2}) \leq c_4 j(q) q^d.
\end{equation}

Using this inequality with inequalities (39) we get (38).

For $y \in B(x_0, \varepsilon) \cap A$ notice that $0 < c_5 \leq G_{U_A}(x_0, y) \leq G_U(x_0, y) \leq G_D(x_0, y) \leq c_6 < \infty$ because Green functions are continuous and strictly positive on $B(x_0, r) \cap A$ since $A \subset \subset U_A \subset \subset D$. Thus, $G_U(x_0, y) \asymp G_D(x_0, y)$, for all $y \in A$ and for all Lipschitz $U$ such that $U_A \subset \subset U \subset \subset D$.

Hence, for all such $U$ we have

\begin{equation}
\eta_U \int_A G_D(x_0, y) \int_{D \setminus U} j(|z - y|) |u(z)| dz dy \underbrace{\to}_{\downarrow 0 \text{ as } U \uparrow D} 0
\end{equation}

by the dominated convergence theorem. \qed
Remark 5.3 (a) If we take a closer look at the proof of the previous lemma, we have actually proved that if \( (\eta_U|u|(D))_U \) is bounded as \( U \uparrow D \), then for every \( A \subset D \) we have

\[
\lim_{U \uparrow D} \eta_U|u|(A) = 0.
\]

(b) The measures \( (\eta_U|u|)_U \) depend on \( x_0 \in D \) but we can prove quite simply that for any other \( x \in D \), the measures

\[
\eta^x_U |u|(dy) := G_U(x,y) \left( \int_{D \setminus U} j(|z-y|)|u(z)|dz \right) dy
\]

are also bounded as \( U \uparrow D \) if \( (\eta_U|u|)_U \) are. Indeed, let \( M := \lim sup \eta_U|u|(D) \). Notice that by Fubini’s theorem

\[
\eta_U|u|(D) = \int_D G_U(x_0,z) \left( \int_{D \setminus U} j(|z-y|)|u(y)|dy \right) dz
\]

\[
= \int_{D \setminus U} P_U(x_0,y)|u(y)|dy.
\]

Find \( R \in (0,1) \) such that \( \delta_D(x_0) > 2R \) and let \( (U_n)_n \) be some increasing sequence of Lipschitz sets such that \( x_0 \in U_1 \), \( \delta_{U_1}(x_0) > R \), and such that for all \( n \in \mathbb{N} \) it holds \( U_n \subset D \) and \( \cup_n U_n = D \). Also, fix some \( \tilde{y} \in \overline{D} \).

Theorem 3.5 yields that there is \( C > 0 \) such that for all \( n \in \mathbb{N} \), all \( x \in B(x_0,R/2) \), and all \( y \in U_n^c \)

\[
P_{U_n}(x,y) \leq C \frac{P_{U_n}(x,\tilde{y})}{P_{U_n}(x_0,\tilde{y})} P_{U_n}(x_0,y).
\]

Notice that

\[
\frac{P_{U_n}(x,\tilde{y})}{P_{U_n}(x_0,\tilde{y})} \leq \frac{P_D(x,\tilde{y})}{P_{U_1}(x_0,\tilde{y})} \leq \frac{\max_{x \in B(x_0,R/2)} P_D(z,\tilde{y})}{P_{B(x_0,R/2)}(x_0,\tilde{y})} \leq c_1 < \infty,
\]

where \( c_1 > 0 \) depends on \( x_0 \), \( R \) and \( \tilde{y} \) but it is independent of \( n \in \mathbb{N} \) and \( x \in B(x_0,R/2) \). Finiteness of \( c_1 \) is due to the continuity of the Poisson kernel. Thus, there is \( c_2 > 0 \) such that for all \( n \in \mathbb{N} \), all \( x \in B(x_0,R/2) \) and all \( y \in U_n^c \) we have \( P_{U_n}(x,y) \leq c_2 P_{U_n}(x_0,y) \). Hence

\[
\eta^x_{U_n}|u|(D) = \int_{D \setminus U_n} P_{U_n}(x,y)|u(y)|dy 
\]

\[
\leq c_2 \int_{D \setminus U_n} P_{U_n}(x_0,y)|u(y)|dy \leq c_2 \cdot M,
\]

i.e. \( (\eta_U^x|u|(D))_U \) is bounded as \( U \uparrow D \), for all \( x \in D \).
Proposition 5.4  Let $D$ be an open set, $f : D \to [-\infty, \infty]$ such that $G_D[f](x) < \infty$ for some $x \in D$, and $\lambda$ a $\sigma$-finite signed measure on $D^c$ such that (18) holds. Then

$$W_D[G_Df] = W_D[P_D\lambda] = 0.$$  

Proof. The proof is the same as in the isotropic $\alpha$-stable case, see [7, Lemma 1.17].

We now focus on proving the mentioned property $W_D[M_D\mu] = \mu$. We use an adaptation of the technique used in [8] where the property was shown for the isotropic $\alpha$-stable process. In the next few results we have twofold statements - for sets near the origin, and for sets away of the origin. In the isotropic $\alpha$-stable case the Kelvin transform allowed the authors to deal only with sets near the origin but in our setting this is not the case.

Let us recall the definition of the relative oscillation of a positive function $f$ on a nonempty set $D$

$$\text{RO}_Df := \frac{\sup_{x \in D} f(x)}{\inf_{x \in D} f(x)}.$$  

If $D = \emptyset$ we put $\text{RO}_Df = 1$.

The first lemma is the one that generalizes [8, Lemma 8].

Lemma 5.5  (a) For every $R \in (0,1)$ and $\eta > 0$ there exists $\delta > 0$ such that for all open $D \subset B_R$ and all $\sigma$-finite measures $\lambda_1, \lambda_2$ on $B_R^c$ satisfying (18) we have

$$\text{RO}_{D \cap B_R} \frac{P_D\lambda_1}{P_D\lambda_2} \leq 1 + \eta. \tag{40}$$

(b) Assume (H2) and (E). For every $R \geq 1$ and $\eta > 0$ there exists $\delta > 0$ such that for all open $D \subset \overline{B_R}$ and all $\sigma$-finite measures $\lambda_1, \lambda_2$ on $\overline{B_R}$ satisfying (18) we have

$$\text{RO}_{D \cap \overline{B_R}} \frac{P_D\lambda_1}{P_D\lambda_2} \leq 1 + \eta. \tag{41}$$

Before we bring the proof let us emphasize the results of the previous lemma. In both parts of the lemma $\delta$ is chosen independently of the set $D$, and the measures $\lambda_1$ and $\lambda_2$. In similar results on the relative oscillation of harmonic functions, e.g. [17, Proposition 2.5, Proposition 2.11], $\delta$ is dependent on the set $D$, see also the proofs of [16, Theorem 2.4, Theorem 2.8]. This subtle but big difference will be used as a crucial and indispensable step in proving $W_D[M_D\mu] = \mu$, see (52).

Moreover, the previous lemma yields that the Martin kernel $M_D(x,z)$ is well defined and strictly positive for $x \in D$ and $z \in \partial^*D$. To this end, recall $M_D(x,z) = \lim_{y \to z} \frac{G_D(x,y)}{G_D(x_0,y)}$, where if $z = \infty$ we look at the limit as $|y| \to \infty$.  

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Since the process $X$ is translation invariant, we can assume that for the finite point $z$ it holds $z = 0$. Further, notice that from (12) we have for $\rho > 0$

$$G_D(\tilde{x}, y) = P_{D \cap B_\rho}[G_D(\tilde{x}, v)dv](y), \quad \tilde{x} \in D \setminus \overline{B}_\rho, \; y \in D \cap B_\rho,$$

and

$$G_D(\tilde{x}, y) = P_{D \cap \overline{B}_\rho}[G_D(\tilde{x}, v)dv](y), \quad \tilde{x} \in D \cap B_\rho, \; y \in B \setminus \overline{B}_\rho.$$ 

Now the claim follows from (40) and (41). However, for this result the uniformity of $\delta$ was not important.

**Proof of Lemma 5.5.** We prove only part (b). The proof of part (a) is almost identical to the proof of the [8, Lemma 8]. The only difference is that instead of the unit ball $B$ we look at the ball $B_R$ and instead of [8, Eq. (48)] we use Lemma 2.1. The proof of part (b) follows the same idea and we present the proof to emphasize the differences. To establish a connection between our proof and the proof of [8, Lemma 8] we will keep a similar notation.

For an open set $D$ and $R, p, q > 0$ denote by

$$D_p = D \cap \overline{B}_p,$$

$$D_p^R = (D \setminus D_p) \cup \overline{B}_R,$$

$$D_{p,q} = D_q \setminus D_p.$$ 

For a measure $\mu$ let

$$\Lambda_{0,p}(\mu) = \int_{\overline{B}_p} \mu(dy),$$

$$\Lambda_{0,p,q}(\mu) = \int_{D_{p,q}} \mu(dy).$$

Fix $R \geq 1, D \subset \overline{B}_R^c$, and $\sigma$-finite measures $\lambda_1$ and $\lambda_2$ on $\overline{B}_R$ satisfying (18). We will see at the end of the proof that $\delta$ will not depend on $D$, $\lambda_1$ or $\lambda_2$, so this is not a loss of generality. Let $c$ denote $C(\phi) > 1$ of Lemma 3.4(b) and notice that Theorem 3.5(b) holds with the constant $C = c^4$. Thus, (41) holds for $\delta = \frac{1}{2}$ with $1 + \eta$ replaced by $c^4$. We denote

$$f_i = P_{D \lambda_i}, \quad f_i^{pR,qR} = P_{D_p}[1_{D_{p,q}=R}P_D^{*}\lambda_i], \quad \overline{f}_i^{pR,qR} = P_{D_p}[1_{D_{p,R}} P^{*}_D \lambda_i],$$

$$f_i^* = P^{*}_{D \lambda_i}, \quad f_i^{pR,qR*} = P_{D_{p,\eta}}[1_{D_{p,\eta,q}=R} P^{*}_D \lambda_i], \quad \overline{f}_i^{pR,qR*} = P_{D_{p,\eta}}[1_{D_{p,R}} P^{*}_D \lambda_i].$$

Recall that $P_D \lambda$ satisfies the mean-value formula for every $U \subset D$ by Corollary 3.13. Hence, using (14) we have $f_i = f_i^{pR,qR} + \overline{f}_i^{pR,qR}$ and $f_i^* = f_i^{pR,qR*} + \overline{f}_i^{pR,qR*}$, for $i = 1, 2$. For $\delta \in (0, \frac{4}{3}]$ we denote $m_{R/\delta} = \inf_{D_{p,R/\delta}} (f_1/f_2)$ and $M_{R/\delta} = \sup_{D_{p,R/\delta}} (f_1/f_2)$. As we have already noted we have $M_{R/\delta} \leq c^4 m_{R/\delta}$.

Let $\varepsilon > 0$ such that $1 + \varepsilon < c$ and let $q \geq 2$. Assumption (E) yields that there is $p = p(q, \varepsilon, R) > 2q$ such that for $z \in D_{p,R/2}$ and $y \in \overline{B}_Q$ we have

$$\frac{1}{1 + \varepsilon} j(|z|) \leq j(|z - y|) \leq (1 + \varepsilon) j(|z|).$$

(42)
Thus, for \( x \in D_{\bar{p}R/2} \) we have
\[
\tilde{f}_i^{\bar{p}R/2,qR}(x) = \int_{D_{\bar{q}R/2}} G_{D_{\bar{p}R/2}}(x,y) dy f_i^{*}(dy) \\
\leq (1 + \varepsilon) \Lambda_{0,qR}(f_i^{*}) P_{D_{\bar{p}R/2}}(x,0),
\]
and similarly
\[
\tilde{f}_i^{\bar{p}R/2,qR}(x) \geq (1 + \varepsilon)^{-1} \Lambda_{0,qR}(f_i^{*}) P_{D_{\bar{p}R/2}}(x,0).
\]

Let us examine consequences of the following assumption:
\[
\Lambda_{0,pR,qR}(f_i^{*}) \leq \varepsilon \Lambda_{0,qR}(f_i^{*}), \quad i = 1, 2. \tag{43}
\]

If (43) is true, then using Lemma 3.4(b) we have for \( x \in D_{\bar{p}R} \)
\[
f_i^{\bar{p}R/2,qR}(x) \leq c P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,pR}(f_i^{\bar{p}R/2,qR}) \leq c P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,pR,qR}(f_i^{*}) \leq c \varepsilon P_{D_{\bar{p}R/2}}(x,0) \Lambda_{0,qR}(f_i^{*}).
\]

Recall that \( f_i = f_i^{\bar{p}R/2,qR} + \tilde{f}_i^{\bar{p}R/2,qR} \) so if (43) holds, we have for \( x \in D_{\bar{p}R} \)
\[
\frac{1 + \varepsilon}{1 + \varepsilon} \Lambda_{0,qR}(f_i^{*}) \leq f_i(x) \leq \frac{(1 + \varepsilon) \Lambda_{0,qR}(f_i^{*})}{(1 + \varepsilon)^{-1} \Lambda_{0,qR}(f_i^{*})} \tag{44}
\]
and finally
\[
RO_{D_{\bar{p}R}} f_1 \leq (1 + \varepsilon)^2 (1 + \varepsilon)^2. \tag{45}
\]

We are satisfied with (45) for now.

Let \( 2 \leq \bar{q} < \bar{p}/4 < \infty, g = f_1^{\bar{p}R/2,\bar{q}R} - m_{\bar{q}R} f_2^{\bar{p}R/2,\bar{q}R} \), and \( h = M_{\bar{q}R} f_2^{\bar{p}R/2,\bar{q}R} - f_1^{\bar{p}R/2,\bar{q}R} \). Note that on \( D_{\bar{p}R/2} \) the functions \( g \) and \( h \) are the Poisson integrals of non-negative measures. If \( D_{\bar{p}R} \neq \emptyset \), then by (25)
\[
\sup_{D_{\bar{p}R}} f_1^{\bar{p}R/2,\bar{q}R} - m_{\bar{q}R} = \sup_{D_{\bar{p}R}} \frac{g}{f_2^{\bar{p}R/2,\bar{q}R}} \leq c^4 \inf_{D_{\bar{p}R}} \frac{g}{f_2^{\bar{p}R/2,\bar{q}R}} \leq c^4 \left( \inf_{D_{\bar{p}R}} \frac{f_1^{\bar{p}R/2,\bar{q}R}}{f_2^{\bar{p}R/2,\bar{q}R}} - m_{\bar{q}R} \right),
\]
and similarly
\[
M_{\bar{q}R} - \inf_{D_{\bar{p}R}} f_1^{\bar{p}R/2,\bar{q}R} \leq c^4 \left( M_{\bar{q}R} - \sup_{D_{\bar{p}R}} f_1^{\bar{p}R/2,\bar{q}R} \right).
\]

By adding these two inequalities we obtain
\[
(c^4 + 1) \left( \sup_{D_{\bar{p}R}} f_1^{\bar{p}R/2,\bar{q}R} - \inf_{D_{\bar{p}R}} f_1^{\bar{p}R/2,\bar{q}R} \right) \leq (c^4 - 1)(M_{\bar{q}R} - m_{\bar{q}R}). \tag{46}
\]

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Let us examine consequences of the following assumption:
\[ \Lambda_0, \bar{q}_R(f_i^\star) \leq \varepsilon \Lambda_{0, \bar{p}_R/2, \bar{q}_R}(f_i^\star), \]  
(47)
for \( \bar{p} \) big enough such that \( j(|z - y|) \leq cj(|z|) \) for all \( z \in \bar{D}_{\bar{p}_R/2} \) and \( y \in \bar{B}_{\bar{q}_R} \) (see \((42)\)). We have for all \( x \in D_{\bar{p}_R/2} \) and \( y \in \bar{B}_{\bar{q}_R} \)
\[ P_{D_{\bar{p}_R/2}}(x, y) = \int_{D_{\bar{p}_R/2}} G_{D_{\bar{p}_R/2}}(x, z) j(|z - y|)dz \leq cP_{D_{\bar{p}_R/2}}(x, 0), \]

hence
\[ f_i^{\bar{p}_R/2, \bar{q}_R}(x) = \int_{D_{\bar{p}_R}} P_{D_{\bar{p}_R/2}}(x, y)f_i^\star(dy) \leq cP_{D_{\bar{p}_R/2}}(x, 0)\Lambda_0, \bar{q}_R(f_i^\star). \]

From the previous inequality using the assumption \((47)\) and Lemma \(3.4(b)\) we have for \( x \in D_{\bar{p}_R} \)
\[ f_i^{\bar{p}_R/2, \bar{q}_R}(x) \leq c \varepsilon P_{D_{\bar{p}_R/2}}(x, 0)\Lambda_0, \bar{p}_R/2, \bar{q}_R(f_i^\star) \leq c \varepsilon P_{D_{\bar{p}_R/2}}(x, 0)\Lambda_0, \bar{p}_R(f_i^{\bar{p}_R/2, \bar{q}_R \star}) \]
\[ \leq c^2 \varepsilon f_i^{\bar{p}_R/2, \bar{q}_R}(x). \]

Recall \( f_i = f_i^{\bar{p}_R/2, \bar{q}_R} + f_i^{\bar{p}_R/2, \bar{q}_R} \) on \( D_{\bar{p}_R/2} \) so the previous inequality and \((46)\) yield
\[ (c^4 + 1) \left( M_{\bar{p}_R}/(1 + c^2 \varepsilon) - m_{\bar{p}_R}(1 + c^2 \varepsilon) \right) \leq (c^4 - 1) (M_{\bar{q}_R} - m_{\bar{q}_R}). \]

Since \( m_{\bar{p}_R} \geq m_{\bar{q}_R} \), dividing by \( m_{\bar{q}_R} \) we finally get
\[ RO_{D_{\bar{p}_R}} f_1/f_2 \leq (1 + c^2 \varepsilon)^2 + (1 - 1 - 1) \left( RO_{D_{\bar{p}_R}} f_1/f_2 - 1 \right). \]  
(48)

We now come to the conclusion of our considerations. Let \( \eta > 0 \). If \( \varepsilon \) is small enough, then the right hand side of \((45)\) is smaller than \( 1 + \eta \) and the right hand side of \((48)\) does not exceed \( \varphi(RO_{D_{\bar{p}_R}}(f_1/f_2)) \), where
\[ \varphi(t) = 1 + \frac{\eta}{2} + \frac{c^4}{c^4 + 1} (t - 1), \quad t \geq 1. \]

Let \( \varphi^l = \varphi, \varphi^{l+1} = \varphi \circ \varphi^l, l \in \mathbb{N} \). Observe that \( \varphi \) is an increasing linear contraction with a fixed point \( t = 1 + \eta(c^4 + 1)/2 \). Thus the \( l \)-fold compositions \( \varphi^l(c^4) \) converge to \( 1 + \eta(c^4 + 1)/2 \) as \( l \to \infty \). In what follows let \( l \) be such that
\[ \varphi^l(c^4) < 1 + \eta(c^4 + 1). \]

Let \( k \) be the smallest integer such that \( k - 1 > c^2/\varepsilon^2 \). We denote \( n = lk \). Note that \( n \) depends only on \( \eta \) and \( \phi \). Let \( q_0 = 2, q_{j+1} = p(q_j, \varepsilon, R) \) for
$j = 0, 1, \ldots, n-1$, from (42), and $\delta = \frac{1}{b_0}$. Note that $\delta$ depends only on $\eta$, $R$ and $\phi$. If for any $j < n$, (43) holds with $q = q_j$ and $p = p(q) = q_{j+1}$, then

$$\text{RO}_{D_R/\varepsilon} \frac{f_1}{f_2} \leq \text{RO}_{D_{q_{j+1}R}} \frac{f_1}{f_2} \leq 1 + \eta,$$

by the definition of $\varepsilon$ and (45). Otherwise for $j = 0, \ldots, n-1$, we have $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*) > \varepsilon \Lambda_{0,q_jR}(f_i^*)$ for $i = 1$ or $i = 2$. Note that by Lemma 3.4(b)

$$c^{-1} \frac{f_1(x)}{\Lambda_{0,q_jR}(f_i^*)} \leq P_{D_{q_jR/2}}(x, 0) \leq c \frac{f_{a-1}(x)}{\Lambda_{0,q_jR}(f_i^*)}, \quad x \in D_{q_{j+1}R,q_jR}.$$

Hence $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*)/\Lambda_{0,q_jR}(f_i^*) \leq c^2 \Lambda_{0,q_{j+1}R,q_jR}(f_i^*)/\Lambda_{0,q_jR}(f_i^*)$ and so $\Lambda_{0,q_{j+1}R,q_jR}(f_i^*) \geq c^2 \varepsilon \Lambda_{0,q_jR}(f_i^*)$ for both $i = 1$ and $i = 2$ (and all $j = 0, \ldots, n-1$).

If $0 \leq j < l$ and $\bar{p} = q_{(j+1)k}$, $\bar{q} = q_{jk}$, then

$$\Lambda_{0,R/2,\bar{q}R}(f_i^*) \geq \Lambda_{0,q_{(j+1)k-1}R,q_{jk}R}(f_i^*) \geq (k-1) \frac{\varepsilon}{c^2} \Lambda_{0,\bar{q}R}(f_i^*) \geq \varepsilon^{-1} \Lambda_{0,\bar{q}R}(f_i^*),$$

so that (47) is satisfied. We conclude that (48) holds. Recall that $q_0 = 2$ and $\text{RO}_{D_{2R}}(f_1/f_2) \leq c^4$. By the definition of $l$ and the monotonicity of $\varphi$

$$\text{RO}_{D_{q_{jk}R}} \frac{f_1}{f_2} \leq \varphi \left( \text{RO}_{D_{(q_{(j-1)k})R}} \frac{f_1}{f_2} \right) \leq \cdots \leq \varphi^l \left( \text{RO}_{D_{q_kR}} \frac{f_1}{f_2} \right) \leq 1 + \eta(c^4 + 1),$$

i.e. $\text{RO}_{D_{R/\varepsilon}} \frac{f_1}{f_2} \leq 1 + \eta(e^4 + 1)$. Since $\eta > 0$ was arbitrary and $\delta$ is dependant only on $\eta$, $R$ and $\phi$, the proof is complete.

**Corollary 5.6** Let $D$ be an open set, $D_{\text{reg}}$ the set of all regular points for $D$, $z \in \partial D$, and $0 < r < 1 \leq R$.

(a) Let $f_1$ and $f_2$ be non-negative functions which are regular harmonic in $D \cap B(z,r)$ and $f_i = 0$ on $(D^c \cup D_{\text{reg}}) \cap B(z,r)$, $i = 1, 2$. Then

$$\lim_{D \ni x \rightarrow z} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

(b) Assume (H2) and (E). If $f_1$ and $f_2$ are non-negative functions which are regular harmonic in $D \cap \overline{D_R}$ and $f_i = 0$ on $(D^c \cup D_{\text{reg}}) \cap \overline{D_R}$, $i = 1, 2$, then

$$\lim_{D \ni x \rightarrow \infty} \frac{f_1(x)}{f_2(x)}$$

exists and is finite.

Moreover, the speed of convergence in the limits above does not depend on the set $D$. 

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The previous corollary is an immediate consequence of Lemma 5.5, cf. [16, Theorem 2.4, Theorem 2.8] and [17, Corollary 2.6, Corollary 2.12] where the speed of convergence depends on the set \( D \).

**Proof of Corollary 5.6.** For part (a) it is enough to notice that from the assumptions of the corollary we have for \( x \in D \cap B(z, r) \) and both \( i = 1, 2 \)

\[
f_i(x) = \int_{D \cup B(z, r)} f_i(y) \omega_{D \cap B(z, r)}^x(dy) = \int_{B(z, r)} P_{D \cap B(z, r)}(x, y)f_i(y)dy.
\]

The claim now follows from Lemma 5.5(a). Part (b) follows similarly. \( \square \)

The following results generalize [8, Lemma 12].

**Lemma 5.7** For every \( 0 < \rho < 1 \) and \( \eta > 0 \) there is \( r > 0 \) such that for all open \( D \) it holds

\[
\operatorname{RO}_{y \in \overline{D} \cap B_r} M_D(x, y) \leq 1 + \eta, \quad \text{if } x, x_0 \in D \setminus \overline{B}_\rho, \quad (49)
\]

and with the additional assumptions (H2) and (E) it holds

\[
\operatorname{RO}_{y \in \overline{D} \setminus \overline{B}_{1/r}} M_D(x, y) \leq 1 + \eta, \quad \text{if } x, x_0 \in D \cap B_{1/\rho}. \quad (50)
\]

**Proof.** Let \( 1 > \rho > r > 0 \). Note that

\[
\sup_{y \in \overline{D} \cap B_r} M_D(x, y) = \sup_{y \in D \cap B_r} \frac{G_D(x, y)}{G_D(x_0, y)}, \quad \inf_{y \in \overline{D} \cap B_r} M_D(x, y) = \inf_{y \in D \cap B_r} \frac{G_D(x, y)}{G_D(x_0, y)}.
\]

Since \( G_D(\tilde{x}, y) = P_{D \cap B_r}[G_D(\tilde{x}, v)dv](y) \) for \( \tilde{x} \in D \setminus \overline{B}_\rho \), the claim (a) follows from Lemma 5.5(a). For part (b) we apply Lemma 5.5(b) in a similar way. \( \square \)

**Remark 5.8** From the previous lemma it is clear that the function \( z \mapsto M_D(x, z) \) is continuous for every \( x \in D \).

Now we state two lemmas that appeared in [8] for the case of the isotropic \( \alpha \)-stable process. The lemmas will be useful for proving uniqueness of representation of non-negative \( L \)-harmonic functions with zero outer charge.

**Lemma 5.9** Let \( D \) be an open set. Suppose that \( 0 \leq g \leq f \) on \( D \), and that \( f, g \) are \( L \)-harmonic in \( D \) with zero outer charge. If \( U \subset D \) and \( f(x) = \int_U f(y) \omega_U^x(dy), x \in U \), then \( g(x) = \int_U g(y) \omega_U^x(dy) \), \( x \in U \).

**Proof.** The proof is the same as in [8, Lemma 9]. \( \square \)

**Lemma 5.10** Let \( D_1 \) and \( D_2 \) be open sets such that

\[
\operatorname{dist}(D_1 \setminus D_2, D_2 \setminus D_1) > 0.
\]
Set $D = D_1 \cup D_2$ and assume that $\omega^*_D(D^c) > 0$ for one (and therefore for all) $x \in D$. Let $f \geq 0$ be a function on $\mathbb{R}^d$ such that $f = 0$ on $D^c$, and for $i = 1, 2$ and all $x \in D_i$ we have

$$f(x) = \int f(y) \omega^*_D(dy).$$

Let $D_1$ be bounded and if $D_2$ is unbounded assume (H2). Then $f = 0$ on the whole of $D$.

Proof. The proof is the same as in [8, Lemma 10] where for inequalities (70) and (71) we use the Harnack inequality for the subordinate Brownian motion [10, Theorem 7].

Now we have a generalization of [8, Lemma 14].

Proposition 5.11 (Martin representation) Let $D$ be an open set. If $D$ is unbounded we additionally assume (H2) and (E). Suppose $f \geq 0$ is $L$-harmonic on $D$ with zero outer charge. Then there is a unique finite measure $\mu \geq 0$ on $\partial_M D$ such that

$$f(x) = \int_{\partial_M D} M_D(x, y) \mu(dy),$$

and we have $W_D f = \mu$. Conversely, if $\mu$ is a finite measure on $\partial_M D$ and $f(x) := \int_{\partial_M D} M_D(x, y) \mu(dy)$, then $f$ is $L$-harmonic with zero outer charge.

Before we prove the proposition we connect the result with the Martin boundary of $D$ with respect to $X^D$ in the sense of Kunita-Watanabe, see [19]. From [16, 17] it follows that in our setting the (abstract) Martin boundary of the set $D$ can be identified with $\partial^* D$. Also, the minimal Martin boundary can be identified with $\partial_M D$. However, in [17, Corollary 1.2 & Corollary 1.4] the Martin representation of harmonic functions with respect to $X^D$ was proved only for the case $\partial_M D = \partial^* D$, cf. [19, Theorem 4]. Hence, the proposition above extends [17, Corollary 1.2 & Corollary 1.4] on more general sets but on less general processes.

Proof of Proposition 5.11. The second claim is almost trivial. Since $\mu$ is a finite measure on $\partial_M D$, we have that $f := M_D \mu$ is $L$-harmonic in $D$ with zero outer charge because of the harmonicity of the Martin kernel.

The first claim is proved similarly as in [8, Lemma 14] but because of some differences at the end of the proof we give the full proof for the reader’s convenience. Let $(D_n)_n$ denote an increasing sequence of open sets with Lipschitz boundary such that for all $n \in \mathbb{N}$ we have $D_n \subset \subset D$ and $D = \bigcup_{n=1}^{\infty} D_n$. By the
mean-value property we have for \( x \in D \):
\[
f(x) = \int_{D \setminus D_n} P_{D_n}(x, y)f(y)dy
\]
\[
= \int_{D_n} M_{D_n}(x, v) \left( G_{D_n}(x_0, v) \int_{D \setminus D_n} j(|v - y|)f(y)dy \right) dv
\]
\[
= \int_{D_n} M_{D_n}(x, v)\eta_{D_n}f(dv),
\]
where \( \eta_{D_n}f \) is the measure from Definition 5.1. For brevity’s sake, we write \( \eta_n \) for \( \eta_{D_n}f \). Since \( \eta_n(D) = f(x_0) < \infty \), by considering a subsequence we may assume that the sequence \( (\eta_n)_n \) weakly converges on \( D^* \) to a finite non-negative measure \( \mu^* \). It follows from Lemma 5.2, more precisely Remark 5.3(a), that \( \mu^* \) is supported on \( \partial^* D \).

Let \( \varepsilon > 0 \) and \( x \in D \). By Lemma 5.7 for every \( y \in \partial^* D \) there exists a neighbourhood \( V_y \) of \( y \) such that
\[
\text{RO}_{V_y \cap U} M_U(x, \cdot) \leq 1 + \varepsilon,
\]
for all \( U \in \{ D, D_1, D_2, \ldots \} \). From \( \{ V_y : y \in \partial^* D \} \), we select a finite family \( \{ V_j : j = 1, \ldots, m \} \) such that \( V \coloneqq V_1 \cup \cdots \cup V_m \supset \partial^* D \). For \( j \in \{1, \ldots, m\} \) let \( z_j \in D \cap V_j \). Let \( k \) be so large that for \( n > k \) we have \( z_j \in D_n \) and
\[
(1 + \varepsilon)^{-1} \leq \frac{M_D(x, z_j)}{M_{D_n}(x, z_j)} \leq (1 + \varepsilon), \quad j = 1, \ldots, m.
\]
The last inequality can be achieved because \( G_{D_n} \uparrow G_D \) pointwise in \( D \) as \( n \to \infty \).

If \( v \in D_n \cap V_j \), then by (52) and the last inequality we get
\[
(1 + \varepsilon)^{-3} \leq \frac{M_D(x, v)}{M_{D_n}(x, v)} \cdot \frac{M_D(x, z_j)}{M_{D_n}(x, z_j)} \cdot \frac{M_{D_n}(x, z_j)}{M_{D_n}(x, v)} \leq (1 + \varepsilon)^3.
\]
Therefore
\[
(1 + \varepsilon)^{-3} \leq \frac{\int_{D \cap V} M_D(x, y)\eta_n(dy)}{\int_{D \cap V} M_{D_n}(x, y)\eta_n(dy)} \leq (1 + \varepsilon)^3, \quad n > k.
\]
Notice that \( (\eta_n)_n \) also weakly converges to \( \mu^* \) on \( D^* \cap V \) and that \( x, x_0 \notin D \cap V \). Recall that \( M_D(x, \cdot) \) is continuous and bounded on \( D^* \cap V \) (see Lemma 5.7).

Therefore
\[
\int_{D \cap V} M_D(x, y)\eta_n(dy) \to \int_{D^* \cap V} M_D(x, y)\mu^*(dy) = \int_{\partial^* D} M_D(x, y)\mu^*(dy).
\]
Also, note that \( f(x) = \int_{D \cap V} M_{D_n}(x, y) \eta_n(dy) + \int_{D \cap V^c} M_{D_n}(x, y) \eta_n(dy) \) and that there is \( k \) so large such that \( D \cap V^c \subset D_k \). Hence
\[
\int_{D \cap V^c} M_{D_n}(x, y) \eta_n(dy) \leq \int_{D_k} M_{D_n}(x, y) \eta_n(dy)
\]
\[
= \int_{D_k} G_{D_n}(x, v) \int_{D \setminus D_n} j(|v - y|) f(y) dy dv
\]
\[
\leq c_k \left( \int_{D_k} G_D(x, v) dv \right) \left( \int_{D \setminus D_n} f(y)(1 \wedge |y|) dy \right) \to 0,
\]
since \( f \in L^1 \) by Lemma 3.8 and since \( G_D(x, \cdot) \in L^1_{loc} \) which we get from (8).
By letting \( n \to \infty \) in (53) we obtain
\[
(1 + \varepsilon)^{-3} \leq \frac{\int_{\partial^* D} M_D(x, y) \mu^*(dy)}{f(x)} \leq (1 + \varepsilon)^3.
\]

i.e. \( f(x) = \int_{\partial^* D} M_D(x, y) \mu^*(dy). \)

We now prove that the measure \( \mu^* \) is concentrated on \( \partial M D \). Let \( x \in U \subset \subset D \). If \( y \in \partial^* D \), then by Theorem 4.3 \( M_D(x, y) \geq \int_{D \setminus U} M_D(z, y) \omega_D^x(dz) \) and equality holds if and only if \( y \in \partial M D \). By Fubini’s theorem
\[
0 = f(x) - \int_{D \setminus U} f(z) \omega_D^x(dz) = \int_{\partial^* D} \left( M_D(x, y) - \int_{D \setminus U} M_D(z, y) \omega_D^x(dz) \right) \mu^*(dy),
\]
hence \( \mu^*(\partial^* D \setminus \partial M D) = 0. \)

Now we prove uniqueness. Consider first the case \( f(\cdot) = M_D(\cdot, z_0) = M_D \delta_{z_0}(\cdot) \) and suppose that there is another measure \( \mu \) on \( \partial M D \) such that \( f = M_D \mu \). If \( z_0 \) is finite, then the uniqueness is proved in the same way as in [8]. Therefore, we deal with the case \( z_0 = \infty \). For \( s > 0 \) define \( D_s = D \cap B_s \) and take \( R > 0 \) such that (37) is true, i.e. \( M_D(x, \infty) = \mathbb{E}_x[M_D(X_{\tau_{DR}}, \infty)] \), \( x \in D_R \). Define the function \( g : \mathbb{R}^d \to [0, \infty) \) as \( g(x) = \int_{|y| < R} M_D(x, y) \mu(dy). \) For \( x \in D \setminus D_{2R} \), by Fubini’s theorem and the comment about (36) in Remark 4.4, we have that
\[
\int_{D_{2R}} g(z) \omega_{D \setminus D_{2R}}^x(dz) = \int_{|y| < R} \left( \int_{D_{2R}} M_D(z, y) \omega_{D \setminus D_{2R}}^x(dz) \right) \mu(dy)
\]
\[
= \int_{|y| < R} M_D(x, y) \mu(dy) = g(x).
\]

Also, for \( x \in D_{3R} \) we have \( g(x) = \int_{D \setminus D_{3R}} g(z) \omega_{D \setminus D_{3R}}^x(dz) \). Indeed, \( g \leq f \) and \( f(x) = \int_{D \setminus D_{3R}} f(z) \omega_{D \setminus D_{3R}}^x(dz) \) because of (37) so Lemma 5.9 yields the claim. Lemma 5.10 yields \( g = 0 \) on whole \( D \), in particular \( g(x_0) = \mu(\{|y| < R\}) = 0. \)
Since this is true for all big \( R > 0 \), we see that \( \mu \) is concentrated at the point at infinity. Thus, we have uniqueness for the function \( f(\cdot) = M_D(\cdot, \infty) \).

Consider now \( f = M_D\mu \) for a finite measure \( \mu \) on \( \partial M D \) and let \( (\eta_{D_n}, f)_n \) be the corresponding sequence of measures for \( f \) from the beginning of the proof. We want to show that \( \mu^* = \mu \). Since \( (\eta_{D_n}, f)_n \) converges weakly to \( \mu^* \), by uniqueness of the weak limit it is enough to show that for every relatively open set \( A \subset \overline{D} \) we have \( \liminf_n \eta_{D_n}(A) \geq \mu(A) \). To this end, using Fubini’s theorem, Fatou’s lemma, and what was already proven for the case of the Dirac measures we have

\[
\liminf_{n \to \infty} \eta_{D_n}(A) = \liminf_{n \to \infty} \int_A G_{D_n}(x_0, v) \left( \int_{D \setminus D_n} j(v, y) M_D(y) dy \right) dv \\
= \liminf_{n \to \infty} \int_{\partial M D} \left( \int_A G_{D_n}(x_0, v) \left( \int_{D \setminus D_n} j(v, y) M_D(y, z) dy \right) dv \right) \mu(dz) \\
\geq \int_{\partial M D} \liminf_{n \to \infty} \left( \int_A G_{D_n}(x_0, v) \left( \int_{D \setminus D_n} j(v, y) M_D(y, z) dy \right) dv \right) \mu(dz) \\
= \int_{\partial M D} \liminf_{n \to \infty} \eta_{D_n}(M_D(\cdot, z))(A) \mu(dz) \geq \int_{\partial M D} \delta_z(A) \mu(dz) = \mu(A).
\]

Thus, we have proved uniqueness.

Notice that due to uniqueness of the measure \( \mu \), any choice of the sequence \( (D_n)_n \) from the beginning of the proof gives \( \mu \) as the limit of \( \eta_{D_n} f \) so we have proved that \( W_D f \) is well defined and that \( W_D f = \mu \). \( \square \)

**Remark 5.12** Since for a finite measure \( \mu \) on \( \partial M D \) we have that \( M_D \mu \) is \( L \)-harmonic with zero outer charge, we have that \( M_D \mu \in C^\infty(D) \cap L^1 \) and if \( D \) is bounded we have \( M_D \mu \in L^1(D) \), see Lemma 3.8, and Theorem 3.12.

Combining Propositions 5.4 and 5.11, we get that (under the additional assumptions (H2) and (E) if \( D \) is unbounded)

\[
W_D[G_D f + P_D \lambda + M_D \mu] = \mu. \tag{54}
\]

**Corollary 5.13** Let \( D \) be an open set. If \( D \) is unbounded suppose (H2) and (E). Let \( \mu \) be a measure on \( \partial M D \). \( M_D \mu(x) = \infty \) for some \( x \in D \) if and only if \( M_D \mu = \infty \) in \( D \), and in that case \( \mu \) is an infinite measure.

**Proof.** Lemma 5.7 yields that \( \partial^* D \equiv z \mapsto M_D(x, z) \) is bounded from below and above for every \( x \in \partial D \). Now the claim easily follows. \( \square \)

**Theorem 5.14** (Representation of non-negative \( L \)-harmonic functions) Let \( D \) be an open set. If \( D \) is unbounded additionally assume (H2) and (E). If \( f \) is
a non-negative function, \( L \)-harmonic in \( D \) with a non-negative outer charge \( \lambda \), then there is a unique finite measure \( \mu_f \) on \( \partial M_D \) such that \( f = P_D \lambda + M_D \mu_f \) on \( D \).

Proof. The proof is the same as in [8, Lemma 13]. □

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