Bulk-edge correspondence in topological transport and pumping

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Abstract. The bulk-edge correspondence (BEC) refers to a one-to-one relation between the bulk and edge properties ubiquitous in topologically nontrivial systems. Depending on the setup, BEC manifests in different forms and govern the spectral and transport properties of topological insulators and semimetals. Although the topological pump is theoretically old, BEC in the pump has been established just recently [1] motivated by the state-of-the-art experiments using cold atoms [2, 3]. The center of mass (CM) of a system with boundaries shows a sequence of quantized jumps in the adiabatic limit associated with the edge states. Despite that the bulk is adiabatic, the edge is inevitably non-adiabatic in the experimental setup or in any numerical simulations. Still the pumped charge is quantized and carried by the bulk. Its quantization is guaranteed by a compensation between the bulk and edges. We show that in the presence of disorder the pumped charge continues to be quantized despite the appearance of non-quantized jumps.

1. Introduction

Recently, the role of topology is often highlighted in condensed-matter physics. This new trend in condensed-matter dates back to a few papers published three decades ago. One of them is the so-called TKNN paper [4], in which topological interpretation was given to quantum Hall effect (QHE), which had been discovered experimentally a few years earlier. The concept of topological pump is a variant of the idea of TKNN, applied to a temporal evolution of the system, instead of to the Brillouin zone. In Ref. [4], quantization of the Hall plateaus was given an interpretation as manifestation of an underlying topological order, which encodes a nontrivial phase property of the bulk wave function. Almost at the same time, Thouless, one of the authors of Ref. [4] had the idea of applying the same scenario to the system of adiabatic pump [5]. Since then, there have been both theoretical [6] and experimental [7, 8] efforts to realize and quantify quantum mechanical pumping in electronic systems. Yet, another experimental breakthrough was brought about recently in the field of cold atoms [2, 3]. In these papers two experimental groups reported experimental realization of the idea of topological pumping a la Thouless in the system of cold atoms. There have been some attempts to realize quantized pumping in other physical systems [9, 10, 11]. One of the issues yet to be investigated in the experimental studies is on the role of disorder in pumping, which we focus on in this paper.
2. Contribution of the bulk vs. the edge: the bulk-edge correspondence

A natural way to quantify pumping is to keep track of a temporal evolution of the many body wave function. In case of topological pump this temporal evolution is adiabatic so that following the snapshot $\bar{x}(t)$ of the center of mass (CM) of the ground state

$$\bar{x}(t) = \sum_i x_i n_i$$

is enough to describe the pumping where $n_i$ is a many body particle number at $x_i$ [1]. Here taking a rescaling as

$$x_i = \frac{i - i_0}{L}, \quad i_0 = \frac{L}{2}, \quad (i = 1, \ldots, L)$$

is essential ($L$ is the system size). To demonstrate this we consider a 1 + 1-dimensional (i.e., 1 spatial + 1 temporal dimensions) model; a one-dimensional (1D) model with an (effective) time-dependent potential [see Eq. (7)]. For the practical numerical simulation we employ the pump version of the so-called Harper, or Aubry-Andre model. [1] Then, we impose the boundary condition such that the system is periodic in time, while it is open i.e., with boundaries $1$ in the space direction. This means that the snapshots of the CM is well-defined and periodic in time:

$$\bar{x}(t_0 + T) - \bar{x}(t_0) = 0,$$

where $t_0$: initial time, $T$: pumping cycle. This might seem to imply that it is impossible to quantify pumping in this way. Let us recall, however, in the typical situation we consider in topological pump, the (one-body) spectrum of the system is characterized by the existence of edge modes traversing the bulk energy gap [see FIG. 1 (a)]. Therefore, the Fermi energy set typically in the gap intersects with such edge modes in the course of the time evolution. Then, if we consider an evolution of CM of the (many-body) ground state $\bar{x}(t)$, it has two distinct parts which can be readily separable; patches of continuous curves and discrete jumps [see FIG. 1 (b)]. The jumps are necessarily associated with edge modes in the clean limit, and their magnitudes $\Delta \bar{x}_{\text{jump}}$ are always quantized to be half integral: $\Delta \bar{x}_{\text{jump}} = \pm 1/2$. [1]

In topological insulators and related systems a one-to-one relation can be established between the appearance of edge/surface modes and the topological non-triviality in the bulk. The bulk-edge correspondence (BEC) refers to this one-to-one relation. [12] Here, in our choice of the boundary condition (open in one and periodic in the other), which was also the case in the so-called Laughlin’s argument, [13] the effects of the bulk and edges are superposed and interconnected. The evolution of the CM in continuous patches is due to the bulk, while the jumps are due to the edge modes. To quantify pumping and reveal the compensating roles of the bulk and the edge, we attempt to separate the effect of the bulk and that of the edges.

To concretize this BEC we reconnect the discrete patches of the CM curve by eliminating the discontinuities, and form a continuous CM curve over the cycle. Note that the resulting continuous curve is no longer periodic in time but it acquires a net gain (or loss) $\Delta \bar{x}_{\text{net}}$ per cycle. One can interpret this $\Delta \bar{x}_{\text{net}}$ as the net pumped charge, transported through the bulk.

This is a polarization of the bulk. Since the net effect of bulk and edge contributions cancel after a cycle [see Eq. (3)],

$$\Delta \bar{x}_{\text{net}} = - \sum_{\{j_n\}} \Delta \bar{x}_{\text{jump}}(t_{j_n}),$$

where the summation is over the jumps, i.e., discontinuities of $\bar{x}(t)$ due to the “appearance” or “disappearance” of an edge mode in the ground state subspace. Here, we are in a “grand-canonical point of view,” [1] in which all the states below $\epsilon_F$ is occupied (in the ground state).
\{j_n\} = \{j_1, j_2, \cdots \} represents a set of time slices where the jumps occur (see Sec. 3 for its precise definition). The number \(N_e\) of the occupied states below \(\epsilon_F\), i.e., the number of electrons changes by 1 at such jumps. Since after a complete cycle of time \(T\) this number must get back to the original value, the number of times an edge state “appears” must be equal to the number of times an edge state “disappears” so that the total number \(N_{jump}\) of jumps is even. This immediately results in that the pumped charge \(\Delta x_{\text{net}}\) per cycle is quantized to be an integer[1].

This integral quantization of \(\Delta x_{\text{net}}\) has a profound mathematical meaning. In parallel with the case of QHE one can directly relate \(\Delta x_{\text{net}}\) to a topological (Chern) number [5]. In QHE, quantization of the Hall conductance \(\sigma_{xy}\) was attributed to the existence of an underlying topological number. [4, 14] Here, quantization of \(\Delta x_{\text{net}}\) has the same mathematical origin. In case of FIG. 1 more than two: \(n_F \geq 2\) bands are fully occupied below the Fermi energy \(\epsilon_F\), which is set to be between the \(n_F\)-th and \(n_F + 1\)-th bands. Then, \(\Delta x_{\text{net}}\) becomes the sum of all the Chern numbers associated with a filled band:

\[
\Delta x_{\text{net}} = C(n_F), \quad C(n_F) = \sum_{n=1}^{n_F} C_n, \tag{5}
\]

where \(C_n\) is a Chern number associated with \(n\)-th band. Or if one rather defines \(I(n_F) = -\sum_{j_n} \Delta x_{\text{jump}}(j_n)\), then \(I(n_F)\) represents the number of (the pair of) edge modes with suitable sign that appear at \(\epsilon_F\). The edge quantity \(I(n_F)\) is connected to the bulk quantity \(C(n_F)\) through Eq.(4). This is the BEC relation[1, 12, 15] in the topological pump:

\[
I(n_F) = C(n_F), \quad I(n_F) - I(n_F - 1) = C_{n_F}. \tag{6}
\]

3. Quantify pumping by the snapshots

To implement the features of topological pump, it is convenient to work on a simple theoretical model. Even experimentally, such an approach is proven to be useful, e.g., in the system of cold atoms, [2, 3] already referred to. In Refs. [2, 3] the so-called Rice-Mele model was presumed, and an effective situation in which a description by the Rice-Meile model has been realized experimentally in an optical lattice. On the other hand, Ref. [1] considers the case of Harper model, in which situations represented by a high \((\geq 2)\) Chern number can be readily realized. In the practical implementation we also consider this model, but for the time being, we can still work on a general 1D model with a time-dependent potential:

\[
H(k_y) = \sum_{i=1}^{\ell} \left( |x_{i+1}\rangle t_x |x_i\rangle + |x_i\rangle t^*_x |x_{i+1}\rangle + |x_i\rangle (V(x_i, k_y) + W(x_i)) |x_i\rangle \right), \tag{7}
\]

where the replacement: \(k_y \rightarrow 2\pi t / T\) is presumed. [1, 5] In case of the Harper model the potential term becomes \(V(x_i, k_y) = 2t_y \cos(k_y - 2\pi \phi)\), where in this original 2D representation, \(\phi\) represents the strength of a magnetic flux piercing a plaquette, while \(\phi\) enters the 2D hopping Hamiltonian through Peierls substitution. \(W(x)\) is a site random potential distributed uniformly in the range \(W(x) \in [-W_0/2, W_0/2]\). Note that \(W(x)\) is randomly distributed in space, while once this distribution is chosen, it stays static. \(W_0\) specify the strength of disorder. Eq. (7) is a Fourier transform of that 2D-hopping Hamiltonian. In Eq. (7), \(|x\rangle\) represents a Bloch state \(|x, k_y\rangle = \sum_y e^{i k_y y} |x, y\rangle\), while we make the replacement: \(k_y \rightarrow 2\pi t / T\), in order to make Eq. (7) a 1D pump Hamiltonian.

2 This one dimensional randomness was discussed as a fictitious one in the original 2D problem[15]. It is now realized as it is in the 1D topological pump.
Let us consider snapshots of such a Hamiltonian as given in Eq. (7) with $k_y = 2\pi t/T$ at $t = j\Delta t$ ($j = 1, 2, 3, \cdots$). At each time slice $t_j = j\Delta t$ we diagonalize the Hamiltonian Eq. (7) and find eigenstates, which include both the bulk and the edge states. Then, we consider the ground state of the system in which all the states below $\epsilon_F$ is occupied; both at the edge and in the bulk. We typically consider the case in which the Fermi energy $\epsilon_F$ is in the gap, since the pumped charge is topologically quantized in this case. Since the present case is non interacting, the center of mass $\bar{x}(t_j)$ of the many body ground state is given as

$$\bar{x}(t_j) = \sum_{i=1}^{L} x_i n_i(t_j), \quad n_i(t_j) = \sum'_{\alpha} |\psi_\alpha(x_i, t_j)|^2,$$

at different time slices $t_j$, where the summation $\sum'$ is taken over all the occupied states $\alpha$ in the ground state. $\psi_\alpha(x_i, t_j)$ represents the eigenwavefunction corresponding to the eigenenergy $\epsilon_\alpha(t_j)$.

FIG. 1 shows the evolution of the spectrum $\epsilon_\alpha(t)$ [panel (a)]; here, the erratic behavior of the spectrum is due to the disorder potential, and that of the center of mass $\bar{x}(t)$ in the ground state [panel (b)]. In practice, we plot simply the spectrum $\epsilon_\alpha(t_j)$ and $\bar{x}(t_j)$ at different time slices to visualize their evolution. In panel (a) one can observe that four branches of edges modes appear and traverse the energy gap. In panel (b) the CM curve $\bar{x}(t)$ shows predominantly a continuous evolution except at a few [actually, six in the specific case of panel (b)] discontinuities (jumps), which occur typically when the equi-energy line at $\epsilon_F$ intersects with either of these edge branches. In panel (b) this is the case at the first four jumps, while the remaining jumps are due to impurities. Suppose that at $t = t_{j_1}$ a “right” edge mode localized in the vicinity of the right edge at $x = +1/2$. becomes available in the subspace of the ground state: $\epsilon \leq \epsilon_F$. 

![Figure 1](image_url)
Then, the number \( N_e(t) = \sum_i n_i(t) \) of occupied states increases by one at this time slice: \( N_e(t_{j1}) - N_e(t_{j1-1}) = 1 \). Correspondingly, \( \bar{x}(t) \) shows a quantized jump of +1/2; i.e., \( \Delta \bar{x}_{j1} = +1/2 \). In panel (b) this seems to happen at the second and at the fourth jump. Generally, such a change of the occupied states \( N_e(t) \) occurs at a set of a finite number of time slices: \( t = t_{j1}, t_{j2}, \cdots \), and there, \( \bar{x}(t) \) possibly shows discontinuities. \( \{j_n\} \) in Eq. (4) specify the set of time slices \( \{j_1, j_2, \cdots \} \) at which the jumps occur. In the clean limit and if \( \epsilon_F \) is in the gap, all of such intersections are associated with an edge mode, and at each \( t_{j_n} \) \( (n = 1, 2, \cdots) \) \( \bar{x}(t) \) shows a quantized jump of magnitude 1/2, since the localization length tends to zero after rescaling Eq.(2) at the extremity of the system. The sign of the jump depends on the location and the slope of the edge mode: [1]

\[
\Delta \bar{x} = \frac{1}{2} \text{sgn}(x_{\text{edge}})[-\text{sgn} \text{(slope)}],
\]

where \( x_{\text{edge}} = \pm 1/2 \) represents the location of the edge state, while when its slope is positive (negative) the state becomes empty (occupied) at the time slice in question. In other words, the factor \([-\text{sgn} \text{(slope)}]\) is a measure of the appearance/disappearance of the state in question in/from the ground state.

In the case of panel (b) in FIG. 1 the contribution of the first four jumps (associated with an edge state) to the summation on the right hand side of Eq. (4) is close to +2, i.e., \( \sum'_{\{j_n\}} \Delta x_{\text{jump}}(t_{j_n}) = +2 \), where we used the notation \( \sum'_{\{j_n\}} \) to make explicit that only the contribution from the edge states is considered (by just counting the discontinuities). Recall that these half-integral quantized jumps always appear \textit{in pairs}. Accordingly, the pumped charge in the bulk becomes \( \Delta \bar{x}_{\text{net}} = -2 \), which is indeed identical to the bulk topological (Chern) number. In the situation of panel (b) the flux \( \phi \) and \( \epsilon_F \) are chosen such that \( \phi = 1/7, \ L = 351 \) and \( \epsilon_F = 1.4 \) with \( t_x = t_y = 0 \). As a result, 5 of 7 bands are fully occupied; \( \epsilon_F \) is in the gap between the 5th and 6th bands. The Chern number \( C_n \) characterizing the occupied bands are +1, +1, +1, −6, +1, +1, +1 from the bottom to the valence band so that they sum up to −2.

**4. Role of disorder: non-quantized jumps vs. quantized pumped charge**

In FIG. 1(b) one can observe that in addition to the quantized jumps we have focused on so far, there are additional jumps which are not quantized and appear “trivially” in pairs. The fifth and sixth jumps in FIG. 1(b) fall on this category. These additional jumps are due to impurity states. The direction and magnitude of such jumps are such that

\[
\Delta \bar{x} = x_{\text{imp}}[-\text{sgn} \text{(slope)}],
\]

where \( x_{\text{imp}} \) represents the location (CM) of the impurity state, while the factor \([-\text{sgn} \text{(slope)}]\) indicates whether that appear or disappear in/from the ground state at the particular time slice. Another implication of this factor is that a given impurity state gives a pair of contributions to the summation on the right hand side of Eq. (4) with the same magnitude but with opposite signs, so that their contributions simply cancel each other. This is contrasting to the case of half-integral quantized jumps due to edge modes which also appear “in pairs” but in a different sense. In the case of quantized jumps the factor \text{sgn}(x_{\text{edge}})\) allows them to evade this cancellation and can still give a non-vanishing (though integral quantized) contribution to \( \Delta \bar{x}_{\text{net}} \). Thanks to this cancellation, the calculated value of \( \Delta \bar{x}_{\text{net}} \) in the case of FIG. 1(b) is \( \Delta \bar{x}_{\text{net}} = -1.97588 \), which is close to the ideal value −2 in the clean limit [in the case of two panels in FIG. 1 the strength of disorder \( W_0 \) is set as \( W_0 = 1 \)]. This example shows that despite the appearance of non-quantized jumps the pumped charge can still be quantized in the presence of disorder.

\[1\] Due to the scaling Eq.(2), the position of the localized state is unambiguously specified as far as the localization length is finite.
5. Concluding remarks: comments on the experimental situations

Let us summarize what we have argued so far. For the actual time evolution to be strictly identical to the collection of snapshots, the system must be adiabatic. This is the case when the pumping cycle $T$ is long enough, satisfying the inequality $T \gg h/\Delta \epsilon$, where $\Delta \epsilon$ is the characteristic energy scale. In the “bulk regime” in which $\bar{x}(t)$ shows a continuous evolution, $\Delta \epsilon = \epsilon_g$ (scale of the bulk energy gap), i.e., the adiabaticity is controlled by $\epsilon_g$. In the vicinity of the jumps, on contrary, or in the “edge regime” $\Delta \epsilon_{edge} \to 0$, since the edge is gapless, so that the typical time $t_{edge} = h/\Delta \epsilon_{edge}$ tends to be infinity. This means that for the adiabatic condition to be strictly satisfied at the edge the pumping cycle $T$ must be infinite. Fortunately, this condition will not be satisfied experimentally; the pumping cycle $T = T_{exp}$ is well between the above two time scales: $t_{bulk} \ll T_{exp} \ll t_{edge}$, i.e., the bulk is adiabatic, while the edges in experiments are non adiabatic, that is, described by the “sudden” approximation. Therefore, the jumps are not seen in the the experiments, even not in the numerical simulations of time evolution, while the “reconnection” is justified; i.e., one can safely skip the jumps. This is because in the regime of $T_{exp} \ll t_{edge}$, the sudden approximation is fully justified at the edge. The system behaves before and after the jump as if the jump does not exist.

Finally, let us recall that the CM is only well-defined for an open system although the pumped charge is described by the polarization of the bulk, which is compensated by the discontinuities due to the edge states in the extreme adiabatic limit. Even though we may not see the jumps in the experiment, this part must underlie for the whole phenomenon to occur.

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