The rainbow connection number of a watermill graph

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Abstract. Let $G$ be a simple, finite and undirected connected graph. An edge-colored graph $G$ is called rainbow connected, if any two vertices in the graph are connected by a path which the edges have distinct colors. Such a path is called rainbow path. An edge-coloring on a graph $G$ is a map $E(G) \to \{1, \cdots, k\}$, $k \in \mathbb{N}$. The smallest number $k$ of colors needed in order to make $G$ rainbow connected is called the rainbow connection number of $G$, denoted by $rc(G)$. The concepts of rainbow connection were introduced by Chartrand et. al. in 2008. Since then many classes of graphs are studied to find its rainbow connection number. The corona product $C_n \odot K_1$ is called a sun graph with $2n$ vertices. Let $P_m$ be a path with $m$ vertices. The watermill graph is a Cartesian product of graphs $C_n \odot K_1$ and $P_m$, which is denoted by $WM(m,n)$. In this paper, we determine the rainbow connection number of a watermill graph $WM(m,n)$.

1. Introduction

The concepts of rainbow connection were introduced by Chartrand et al. in 2008 [2]. An interesting application of the concepts is for communication of information between agencies of government. Ericksen [1] made the following observation: terrorist attacks on the World Trade Tower in New York and the Pentagon in Washington D.C, as crashing into a rural field in Pennsylvania caused law enforcement and intelligence agencies could not communicate with each other from the radio system to databases through regular channels. The technologies utilized were separate entities and prohibited shared access, meaning there was no way for officers and agents to cross check information between various organizations. This is behind the development of high technology in fields such as border protection, cyber security and disaster response support. This lead to the Department of Homeland Security (DHS) in 2003 in response to the weaknesses discovered in the transfer of classified information.

While the information needs to be protected since it relates to national security, there must also be procedures that permit access between appropriate parties. This two-fold issue can be addressed by assigning information transfer paths between agencies which may have other agencies as intermediaries while requiring a large enough number of passwords and firewalls that is prohibitive to intruders, yet small enough to manage (that is, enough so that one or more paths between every pair of agencies have no password repeated). An immediate question arises: What is the minimum number of passwords or firewalls needed that allows one or more secure paths between every two agencies so that the passwords along each path are distinct?

This situation can be modeled by graph-theoretic model. Let $G$ be a nontrivial connected graph on which is defined a coloring $c : E(G) \to \{1, \cdots, k\}$, $k \in \mathbb{N}$ of the edges of $G$, where
adjacent edges may be colored the same. In this case, the coloring $c$ is called rainbow coloring of $G$. An edge-colored graph $G$ is called rainbow connected, if any two vertices in the graph are connected by a path which the edges have distinct colors. Such a path is called rainbow path. The smallest number $k$ of colors needed in order to make $G$ rainbow connected is called the rainbow connection number of $G$, denoted by $rc(G)$. Furthermore, if $G$ is an undirected connected graph with size $|E(G)|$ and diameter (the largest distance between two vertices of $G$) $diam(G)$, then according to Chartrand et al. [2] we have

$$diam(G) \leq rc(G) \leq |E(G)|.$$  

Chartrand et al. [2] computed the precise rainbow connection number of several graph classes including cycles. The general strategy to approach the rainbow connection number seems to be in finding a coloring that is close to the trivial lower bound, since it is hard to raise the lower bound.

There are more results about rainbow connection numbers. Some of them are Septianto and Sugeng [3], Chartrand et al [2], and Syafrizal et al [7]. Septianto and Sugeng [3] determined the rainbow connection number and strong rainbow connection number of the $m$-splitting of the complete graph $K_n$ for several sets of values of $(m, n)$. In general, the following relationship holds for any $m$ and $n \geq 2$.

$$src(Spl_m(K_n)) \leq src(Spl_{m+1}(K_n)).$$

Chartrand et al [2] determined the rainbow connection number and strong rainbow connection number of the Cycle graph $C_n$ for each integer $n \geq 4$.

**Theorem 1.** [2] For each integer $n \geq 4$, $rc(C_n) = src(C_n) = \lfloor \frac{n}{2} \rfloor$.

In 2013, Syafrizal et al. [7] determined the rainbow connection number of fan and sun. A sun graph $S_n$ is defined as the graph obtained from a cycle $C_n := v_1, v_2, \cdots, v_n$ by adding $n$ vertices $v_{n+i}$ and $n$ pendant edges $e_{n+i} = v_i v_{n+i}$ for $i = 1, 2, \cdots, n$. For illustration, a graph $S_6$ is given in Figure 1.

![Figure 1. Sun Graph $S_6$](image)

**Theorem 2.** [7] The rainbow connection number and strong rainbow connection number of a graph $S_n$ for $n \geq 2$ are $rc(S_n) = src(S_n) = \lfloor \frac{n}{2} \rfloor + n$. 

2
The cartesian product $G \times H$ of graphs $G$ and $H$ is a graph such that the vertex set of $G \times H$ is the cartesian product $V(G) \times V(H)$ and two vertices $(u, v)$ and $(u', v')$ are adjacent in $G \times H$ if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$ [4]. There are some results on the cartesian product graphs, as example is the $P_n \times K_{2,2}$ for $n \geq 2$.

**Theorem 3.** [6] The rainbow connection number of a graph $P_n \times K_{2,2}$ for $n \geq 2$ are $rc(P_n \times K_{2,2}) = n + 1$.

In this paper, we determine the rainbow connection number of a watermill graph $WM(m, n)$. For simplifying, define $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$.

2. Main Results

**Definition 1.** [5] Let $S_n$ be a sun graph on $n$ vertices and $P_m$ be a path graph on $m$ vertices. Watermill graph is a graph formed by cartesian product of graphs $S_n$ and $P_m$, $S_n \times P_m$. Watermill graph is denoted by $WM(m, n)$.

Let $m$ and $n$ be two positive integers with $m \geq 2$ and $n \geq 4$. We define the vertex set and the edge set of $WM(m, n)$ as follows.

$$WM(m, n) = (V(WM(m, n)), E(WM(m, n)))$$

with,

$$V(WM(m, n)) = \{u^i_j \mid i \in [1, 2n], j \in [1, m]\}$$

and

$$E(WM(m, n)) = \{u^iju^i_{i+1} \mid i \in [1, n - 1], j \in [1, m]\} \cup \{u^i_1u^i_n \mid j \in [1, m]\} \cup \{u^i_ju^{i+1}_j \mid i \in [1, n], j \in [1, m]\} \cup \{u^i_1u^{i+1}_j \mid i \in [1, 2n], j \in [1, m - 1]\}.$$

For illustration, a watermill graph ($WM(m, 5)$) is given in Figure 2.

**Theorem 4.** Let $m$ and $n$ be two positive integers with $m \geq 2$ and $n \geq 4$. Then rainbow connection number of $WM(m, n)$ is

$$rc(WM(m, n)) = \begin{cases} \left[\frac{n}{2}\right] + n, & \text{for } m \leq n - 1, \\ \left[\frac{n}{2}\right] + m + 1, & \text{for } m \geq n. \end{cases}$$

We shall consider two cases.

**Case 1.** $m \leq n - 1$

If $m = 1$ then $WM(1, n) = S_n$. From Theorem 1, we know that for $n \geq 2$ are $rc(S_n) = \left[\frac{n}{2}\right] + n$. First construct $c : E(WM(m, n)) \rightarrow [1, \left[\frac{n}{2}\right] + n]$. Since the path that is connected from the leaf vertex in level 1, $u^i_1$ for $i \in [1, n]$, to any leaf vertex in level $m$, $u^m_i$ for $i \in [1, n]$, is the shortest path, then the rainbow connection number $rc(WM(m, n)) \geq \left[\frac{n}{2}\right] + n$. The color for every edge of copies of sun can be the same with the edge color of the sun in level 1. The color for the vertical edges such in Figure 2 can use the same color for the pendant edges of the sun. Next, it remains to show that $rc(WM(m, n)) \leq \left[\frac{n}{2}\right] + n$. We define a coloring by using $\left[\frac{n}{2}\right] + n$ colors as follows.
We define a coloring by using $rc$ the vertical edges such as shown in Figure 2, with the same color as pendant edges. Thus a copy of the sun can be use the same color as the one in level 1. However, we cannot color first construct $c(e)$ as follows.

$$c(e) = \begin{cases} 
  i, ife = u^j_i u^j_{i+1}; & i \in [1, \lfloor \frac{n}{2} \rfloor]; j \in [1, m], \\
  i - \lfloor \frac{n}{2} \rfloor, ife = u^j_i u^j_{i+1}; & i \in [\lceil \frac{n}{2} \rceil + 1, n-1]; j \in [1, m], \\
  \lfloor \frac{n}{2} \rfloor, ife = w^j_i w^j_{i+1}; & j \in [1, m], \\
  \frac{n}{2} + j, ife = w^j_i w^j_{i+1} + 1 = w^j_{i+1} w^j_{i+1}; & i \in [1, \frac{n}{2}]; j \in [1, m-1]; i \text{ odd}; n \text{ even}, \\
  \frac{n}{2} + 2j, ife = w^j_i w^j_{i+1} + 1 = w^j_{i+1} w^j_{i+1}; & i \in [1, \frac{n}{2}]; j \in [1, m-1]; i \text{ odd}; n \text{ even}, \\
  \frac{n}{2} + 2j - 1, ife = w^j_i w^j_{i+1} + 1 = w^j_{i+1} w^j_{i+1}; & i \in [1, \frac{n}{2}]; j \in [1, m-1]; i \text{ even}; n \text{ odd}, \\
  i + j, ife = w^j_i w^j_{i+1}; & i \in [1, n-1]; j \in [1, m-1]; i + j \leq n; n \text{ odd}, \\
  i - n, ife = w^j_i w^j_{i+1}; & i \in [1, n-1]; j \in [1, m-1]; i + j > n; n \text{ odd}, \\
  kife = u^j_{i+n} w^j_{i+n}; & i \in [1, \lfloor \frac{n}{2} \rfloor]; j \in [1, m-1]; k \in [\lceil \frac{n}{2} \rceil + j, \lfloor \frac{n}{2} \rfloor + m - 1]; \\
  k \neq n; i, n \text{ odd}, \\
  k - 1, ife = u^j_{i+n} w^j_{i+n}; & i \in [1, \lfloor \frac{n}{2} \rfloor]; j \in [1, m-1]; k \in [\lceil \frac{n}{2} \rceil + j, \lfloor \frac{n}{2} \rfloor + m - 1]; \\
  k \neq n; i \text{ even}, n \text{ odd}, \\
  i + ji, fe = u^j_{i+n} w^j_{i+n}; & i \in [\lceil \frac{n}{2} \rceil, n]; j \in [1, m-1]; i = even, n \text{ odd}. 
\end{cases}$$

It follows that $rc(WM(m, n)) \leq \lceil \frac{n}{2} \rceil + n$. So, we can conclude that $rc(WM(m, n)) = \lceil \frac{n}{2} \rceil + n$.

**Case 2.** $m \geq n$.

First construct $c : E(WM(m, n)) \rightarrow [1, \lfloor \frac{n}{2} \rfloor + m + 1]$. Similar with Case 1, every edge of the copy of the sun can be use the same color as the one in level 1. However, we cannot color the vertical edges such as shown in Figure 2, with the same color as pendant edges. Thus $rc(WM(m, n)) \geq \lfloor \frac{n}{2} \rfloor + m + 1$. Next, it remains to show that $rc(WM(m, n)) \leq \lceil \frac{n}{2} \rceil + m + 1$. We define a coloring by using $\lfloor \frac{n}{2} \rfloor + m + 1$ colors as follows.

![Watermill graph](image_url)
given in Figure 3, respectively.

Figure 3. (a) A rainbow 6-coloring on $WM(3, 4)$ and (b) A rainbow 7-coloring on $WM(4, 4)$. 

It follows that $rc(WM(m, n)) \leq \left\lfloor \frac{3n}{2} \right\rfloor + m + 1$. So, we can conclude that $rc(WM(m, n)) = \left\lceil \frac{3n}{2} \right\rceil + m + 1$. 

For illustration, rainbow 6-colorings on $WM(3, 4)$ and a rainbow 7-coloring on $WM(4, 4)$ are given in Figure 3, respectively.
3. Conclusion and future works
We have studied the rainbow connection number of Watermill graph $WM(m,n)$ that was modeled by graph-theoretic model. Let $WM(m,n)$ be a nontrivial connected graph on which is defined a coloring $c : E(G) \rightarrow \{1, \cdots, k\}, k \in \mathbb{N}$ of the edges of $G$. The result show that the minimum number of passwords or firewalls needed is $\left\lfloor \frac{n}{2} \right\rfloor + n$ for $m \leq n - 1$ or $\left\lfloor \frac{n}{2} \right\rfloor + m + 1$ for $m \geq n$ that allows one or more secure paths between every two agencies so that the passwords along each path are distinct.

For the future works, we can study the rainbow connection number of the graphs $G \times P_m$, for any graph $G$, and also the rainbow connection of other product of graphs.

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