FRACTIONAL MAXIMAL OPERATORS WITH WEIGHTED HAUSDORFF CONTENT

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Abstract. Let $n \geq 2$ be the spatial dimension. The purpose of this note is to obtain some weighted estimates for the fractional maximal operator $\mathcal{M}_\alpha$ of order $\alpha$, $0 \leq \alpha < n$, on the weighted Choquet-Lorentz space $L^{p,q}(H^d_w)$, where the weight $w$ is arbitrary and the underlying measure is the weighted $d$-dimensional Hausdorff content $H^d_w$, $0 < d \leq n$. Concerning a dependence of two parameters $\alpha$ and $d$, we establish a general form of the Fefferman-Stein type inequalities for $\mathcal{M}_\alpha$. Our results contain the works of Adams, [1] and of Orobitg and Verdera [5] as the special cases. Our results also imply the Tang result [8], if we assume the weight $w$ is in the Muckenhoupt $A_1$-class.

1. Introduction

We will denote by $\mathcal{D}$ the family of all dyadic cubes $Q = 2^{-k}(j + [0,1]^n)$, $k \in \mathbb{Z}$, $j \in \mathbb{Z}^n$. For a locally integrable function $f$ on $\mathbb{R}^n$, we define the dyadic fractional maximal operator $\mathcal{M}_\alpha$, $0 \leq \alpha < n$, by

$$\mathcal{M}_\alpha f(x) := \sup_{Q \in \mathcal{D}} 1_Q(x) \int_Q |f| \, d\mathcal{l}(Q)^\alpha,$$

where $1_E$ denotes the characteristic function of $E$, the barred integral $\int_Q f \, dx$ stands for the usual integral average of $f$ over $Q$, and $l(Q)$ denotes the side length of the cube $Q$. When $\alpha = 0$, we simply write $\mathcal{M}_0 = \mathcal{M}$ which is the Hardy-Littlewood maximal operator. These maximal operators are fundamental tools to study Harmonic analysis, potential theory, and the theory of partial differential equations (cf. [2, 4]).

If $E \subset \mathbb{R}^n$ and $0 < d < n$, then the $d$-dimensional Hausdorff content $H^d$ of $E$ is defined by

$$H^d(E) := \inf \sum_{j=1}^\infty l(Q_j)^d,$$

where the infimum is taken over all coverings of $E$ by countable families of dyadic cubes $Q_j$. In [3], for the Hardy-Littlewood maximal operator $\mathcal{M}$, Orobitg and Verdera proved the strong type inequality

$$\int_{\mathbb{R}^n} (\mathcal{M} f)^p \, dH^d \leq C \int_{\mathbb{R}^n} |f|^p \, dH^d$$

for $d/n < p < \infty$, and the weak type inequality

$$\sup_{t > 0} t^{1/p} H^d \{ x \in \mathbb{R}^n : \mathcal{M} f(x) > t \}^{1/p} \leq C \int_{\mathbb{R}^n} |f|^p \, dH^d, \quad t > 0,$$

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for \( p = d/n \). Here, the integrals are taken in the Choquet sense, that is, the Choquet integral of \( f \geq 0 \) with respect to a set function \( C \) is defined by

\[
\int_{\mathbb{R}^n} f \, dC := \int_0^\infty C(\{x \in \mathbb{R}^n : f(x) > t\}) \, dt.
\]

By weights we will always mean nonnegative and locally integrable functions on \( \mathbb{R}^n \). Let \( w \) be a weight on \( \mathbb{R}^n \). If \( E \subset \mathbb{R}^n \) and \( 0 < d \leq n \), then the \( d \)-dimensional weighted Hausdorff content \( H_d^w \) of \( E \) is defined by

\[
H_d^w(E) := \inf \sum_{j=1}^\infty w_{Q_j} d_{Q_j}^d,
\]

where the infimum is taken over all coverings of \( E \) by countable families of dyadic cubes \( Q_j \). Notice that if we let \( w \equiv 1 \), then \( H_d^w = H_d \). The weighted Hausdorff content, which is also called the weighted Hausdorff capacity, plays a role for functions as a tool for measuring exceptional sets in the weighted Sobolev space \( W^{m,1}(\Omega, w) \). For details of the weighted Hausdorff content, see [2, 9].

Tang [8] proved the following.

**Theorem (8).** Suppose that a weight \( w \) belongs to \( A_1(\mathbb{R}^n) \), if \( \mathcal{M}w(x) \leq Cw(x) \) holds for almost every \( x \in \mathbb{R}^n \).

Let \( \tilde{D} \) be a family of measurable sets having certain dyadic structure. For \( E \subset \mathbb{R}^n \), let \( H_{\mu}^d(E) \) be the modified Hausdorff content defined by

\[
H_{\mu}^d(E) := \inf \sum_{j=1}^\infty \mu(Q_j)^{d/n},
\]

where \( \mu \) is a locally finite Borel measure on \( \mathbb{R}^n \) and the infimum is taken over all countable coverings of \( E \) by \( \{Q_j\} \subset \tilde{D} \). Let \( M_{\tilde{D}}^\mu \) be the maximal operator adopted to the family \( \tilde{D} \) by

\[
M_{\tilde{D}}^\mu f(x) := \sup_{Q \in \tilde{D}} 1_{Q}(x) \int_Q |f| \, d\mu.
\]

In [3], the authors established the strong and the weak estimates of \( M_{\tilde{D}}^\mu \) on the Choquet space \( L^p(H_d^w) \).

To state our main results, we introduce the Choquet-Lorentz space associated with the weighted Hausdorff content.

For \( 0 < p \leq q \leq \infty \), we define the quasinorm

\[
\|f\|_{L^{p,q}(H_d^w)} := \left( \int_0^\infty \left( \int_0^{t^p H_d^w(\{ f > t \})} \frac{dt}{t} \right)^{q/p} dt \right)^{1/q}
\]

and we denote by \( L^{p,q}(H_d^w) \) the set of all functions for which the above quasinorms are finite. When \( d = n \) and \( w \equiv 1 \), this is classical Lorentz space. If \( p = q \), then \( L^{p,q}(H_d^w) = L^p(H_d^w) \). Further, \( L^{p,\infty}(H_d^w) \) is the set of all functions \( f \) such that

\[
\sup_{t>0} t^{1/p} H_d^w(\{x \in \mathbb{R}^n : f(x) > t\}) < \infty.
\]
Remark 1.1. It is well known that the classical Lorentz quasinorm $\|f\|_{L^p,q(\mathbb{R}^n)}$ does not satisfy the triangle inequality. However, if $p > 1$, then there exists a certain norm which is equivalent to this quasinorm; see [1] for details. In the case of Choquet-Lorentz spaces, one can prove this fact analogously, which enables us that the quasinorm satisfies the countable subadditivity up to appropriate positive multiplicative constant.

In [1], Adams investigated the boundedness of fractional maximal operators on unweighted Choquet-Lorentz spaces.

**Theorem (1).** Let $0 < d \leq n$, $0 \leq \alpha < n$, and $p \leq q$.

(i) Let $d/n < p < d/\alpha$ and set $\delta = \frac{2}{p}(d - \alpha p)$, then there is a constant $C$ independent of $f$ such that

$$\|\mathcal{M}_\alpha f\|_{L^{p,q}(H^d)} \leq C\|f\|_{L^p(H^d)}.$$  

(ii) For $p = d/n$, there is a constant $C$ independent of $f$ such that

$$\|\mathcal{M}_\alpha f\|_{L^{p,\infty}(H^d)} \leq C\|f\|_{L^p(H^d)}$$

with $\delta = q(n - \alpha)$.

In this paper, as an extension of this result to weighted settings, we establish the following theorems.

**Theorem 1.2.** Let $w$ be any weight on $\mathbb{R}^n$. Let $0 < d \leq n$, $0 \leq \alpha < n$, and $\delta = \frac{2}{p}(d - (\alpha - \gamma)p)$. We assume that $d/n < p < q < n/\gamma$ and that $p < d/\alpha$. Then there exists a constant $C$ independent of $f$ and $w$ such that

$$\|\mathcal{M}_\alpha f\|_{L^{p,q}(H^d_w)} \leq C\|f\|_{L^p(H^d_w)}.$$  

Notice that the above inequality is

$$\int_0^\infty (t^\delta H^d_w(\{\mathcal{M}_\alpha f > t\}))^{p/q} \frac{dt}{t} \leq C \int_{\mathbb{R}^n} |f|^p \, dH^d_{(\mathcal{M}_\gamma w)^{p/q}}.$$  

**Theorem 1.3.** Let $w$ be any weight on $\mathbb{R}^n$. Let $0 < d \leq n$, $0 \leq \alpha < n$, and $0 \leq \gamma \leq \alpha$. For $d/n \leq q < n/\alpha$ and $p = d/n$, there is a constant $C$ independent of $f$ and $w$ such that

$$\sup_{t > 0} t H^d_w(\mathcal{M}_\alpha f > t)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f|^p \, dH^d_{(\mathcal{M}_\gamma w)^{p/q}} \right)^{1/p}$$

with $\delta = q(n - \alpha + \gamma)$.

Remark. In Theorem 1.2 when $\gamma = 0$, the condition $q < n/\gamma$ should be interpreted as $q < \infty$. In both theorems 1.2 and 1.3, if we take $\gamma = 0$ and $w \equiv 1$, we obtain the Adams theorem. If we take $p = q$ and $\alpha = 0$, then $\delta$ coincides $d$ and we obtain weighted estimates of the Orobitg and Verdera theorem. In Theorem 1.2 if we take $p = q$ and $\gamma = 0$ and we assume $w \in A_1(\mathbb{R}^n)$, then we obtain the Tang theorem. Finally, let $p = q$, $\gamma = \alpha$, and $d = n$, then $\delta$ becomes $n$ and we obtain the Fefferman-Stein inequality for the fractional maximal operator due to Sawyer [7] (see also [3]).

Remark 1.4. One may expect that the following strong and weak type inequalities also hold:

$$\int_0^\infty (t^\delta H^d_w(\{\mathcal{M}_\alpha f > t\}))^{p/q} \frac{dt}{t} \leq C \int_{\mathbb{R}^n} |f|^p \, dH^d_{(\mathcal{M}_\gamma w)^{p/q}}$$

and

$$\sup_{t > 0} t H^d_w(\mathcal{M}_\alpha f > t)^{1/q} \leq C \left( \int_{\mathbb{R}^n} |f|^p \, dH^d_{(\mathcal{M}_\gamma w)^{p/q}} \right)^{1/p}.$$  

However, since it is difficult to compare $dH^d_w$ and $wdH^d$, we have no idea to show the both inequalities until now.
The letter $C$ will be used for the positive finite constants that may change from one occurrence to another. Constants with subscripts, such as $C_1$, $C_2$, do not change in different occurrences.

2. Proof of Theorems

We begin by proving strong type estimate Theorem 1.2. To do this, we need the following lemma due to Lemma 1 in [5]. Hereafter, we always assume $p < q$.

**Lemma 2.1.** Let $0 < d < n$, $0 < \alpha < n$, $0 \leq \gamma \leq \alpha$, and $\delta = \frac{\gamma}{p}(d - (\alpha - \gamma)p)$. We assume $d/n < p < q < n/\gamma$ and $p < d/\alpha$. Then we have that

\[
\|M_{\alpha}\|_{L^p(H^d_\#)}^p \leq C_{n,p,d}\int_Q (M_{\gamma_d}w)^{p/q} \, dx(Q)^d
\]

for all dyadic cubes $Q \in \mathcal{D}$ and for any weight $w$.

**Proof.** For fixed dyadic cube $Q$, we let $\pi^0(Q) = Q$ and $\pi^j(Q)$ is the smallest dyadic cube containing $\pi^{j-1}(Q)$, $j = 1, 2, \ldots$. Now, we see that

\[M_{\alpha}(1_Q)(x) = \alpha_0 1_Q(x) + \sum_{j=1}^{\infty} \alpha_j 1_{\pi^j(Q) \setminus \pi^{j-1}(Q)}(x),\]

where

\[\alpha_j = \frac{|Q|}{|\pi^j(Q)|} l(\pi^j(Q))^\alpha = l(Q)^\alpha 2^{(\alpha-n)j}, \quad j = 0, 1, \ldots.\]

It follows that

\[
\|M_{\alpha}(1_Q)\|_{L^p(H^d_\#)}^p = \int_0^\infty (t^d H^\delta_w(\{M_{\alpha}(1_Q) > t\}))^{p/q} \frac{dt}{t}
\]

\[= \sum_{j=1}^{\infty} \int_{a_j}^{a_{j-1}} (t^d H^\delta_w(\{M_{\alpha}(1_Q) > t\}))^{p/q} \frac{dt}{t}
\]

\[\leq \sum_{j=1}^{\infty} H^\delta_w(\pi^j(Q))^{p/q} \int_{a_j}^{a_{j-1}} t^{p-1} \, dt
\]

\[\leq p^{-1} \sum_{j=1}^{\infty} \left( \int_{\pi^j(Q)} w \, dx(\pi^j(Q))^\delta \right)^{p/q} (a_{j-1})^p.
\]

We notice that

\[(a_{j-1})^p = (l(Q)^\alpha 2^{(\alpha-n)(j-1)})^p = 2^{(n-\alpha)p} (l(Q)^\alpha 2^{(\alpha-n)j})^p.
\]

Thus,

\[
\left( \int_{\pi^j(Q)} w \, dx(\pi^j(Q))^\delta \right)^{p/q} (a_{j-1})^p
\]

\[= 2^{(n-\alpha)p} \left( \int_{\pi^j(Q)} w \, dx(\pi^j(Q))^\gamma q \right)^{p/q} l(\pi^j(Q))^{d-\alpha p} l(Q)^{\gamma q} 2^{(\alpha-n)j}
\]

\[= 2^{(n-\alpha)p} \left( \int_{\pi^j(Q)} w \, dx(\pi^j(Q))^\gamma q \right)^{p/q} l(Q)^{d-\alpha p} 2^{(d-n)j}
\]

\[\leq C_{n,p} \sup_{P \in \mathcal{D}, P \supseteq Q} \left( \int_P w \, dx(P)^\gamma q \right)^{p/q} l(Q)^{d-\alpha p} 2^{(d-n)j},
\]
where we have used the definition of $\delta$ and this implies

$$
\|M_\alpha[1_Q]\|^p_{L^{p,q}(H^\delta)} \leq C_{n,p} \sup_{P \in D : P \supset Q} \left( \int_P w \, dx \, l(P)^q \right)^{p/q} \cdot l(Q)^d \cdot \sum_{j=1}^{\infty} 2^{(d-\alpha)pj}
$$

$$
\leq C_{n,p,d} \int_Q (\mathcal{M}_\gamma w)^{p/q} \, dx \, d(\gamma)
$$

where we have used $d/n < p$. We notice that the condition $p < d/\alpha$ needs to be $\delta > 0$. \qed

**Proof of Theorem 1.2.** First we notice that for $0 < p \leq q < \infty$, we have

$$
\|M_\alpha f\|^p_{L^{p,q}(H^\delta)} = \frac{1}{p} \| (M_\alpha f)^p \|_{L^{q/p,1}(H^\delta)}.
$$

Indeed,

$$
\|M_\alpha f\|^p_{L^{p,q}(H^\delta)} = \int_0^\infty (t^{q/p} H_w^\delta(\{M_\alpha f > t\}))^{p/q} \, dt
$$

$$
= \frac{1}{p} \int_0^\infty (t^{q/p} H_w^\delta(\{(M_\alpha f)^p > t\}))^{p/q} \, dt
$$

$$
= \frac{1}{p} \int_0^\infty (t^{q/p} H_w^\delta(\{(M_\alpha f)^p > t\}))^{1/(q/p)} \, dt
$$

$$
= \frac{1}{p} \| (M_\alpha f)^p \|_{L^{q/p,1}(H^\delta)}.
$$

We may assume that $f \geq 0$. For each integer $k$, let $\{Q^k_j\}_j$ be a family of nonoverlapping dyadic cubes $Q^k_j$ such that

$$
\{x \in \mathbb{R}^n : 2^k < f(x) \leq 2^{k+1}\} \subset \bigcup_j Q^k_j
$$

and

$$
\sum_j \int_{Q^k_j} (\mathcal{M}_\gamma w)^{p/q} \, dx \, d(\gamma) \leq 2 H_w^d(\{x \in \mathbb{R}^n : 2^k < f(x) \leq 2^{k+1}\})
$$

Set $g = \sum_k 2^{p(k+1)} 1_{A_k}$, where $A_k = \bigcup_j Q^k_j$. Thus, $f^p \leq g$.

Assume first that $1 \leq p$. Then

$$
(M_\alpha f)^p \leq M_{\alpha p}(f^p) \leq M_{\alpha p}(g) \leq \sum_k 2^{p(k+1)} \sum_j M_{\alpha p}(1_{Q^k_j}).
$$

Recalling Remark 1.1 by $q/p > 1$, we have that

$$
\frac{1}{p} \| (M_\alpha f)^p \|_{L^{q/p,1}(H^\delta)} \leq C \frac{1}{p} \sum_k 2^{p(k+1)} \sum_j \| M_{\alpha p}(1_{Q^k_j}) \|_{L^{q/p,1}(H^\delta)}.
$$
By Lemma 2.1

\[
\frac{1}{p} \sum_k 2^{p(k+1)} \sum_j \|M_{\alpha p}(1_{Q_j})\|_{L^{p/n}(H^d_\delta^k)} \\
\leq \frac{1}{p} \sum_k 2^{p(k+1)} \sum_j C_{n,p,d} \int_{Q_j^k} (M_{\gamma q^k} w)^{p/q} \, dx \cdot l(Q_j^k)^d \\
\leq C \sum_k 2^{p(k+1)} H^d_{\gamma q^k} \left( \{ x : 2^k < f(x) \leq 2^{k+1} \} \right) \\
\leq C \int_{\mathbb{R}^n} f^p \, dH^d_{\gamma q^k} \\
\leq C \int_{\mathbb{R}^n} f^p \, dH^d_{\gamma q^k},
\]

which proves this case.

Assume now that \(d/n < p < 1\). Since \(f \leq \sum_k 2^{k+1} 1_{A_k}\),
\[
M_{\alpha} f \leq \sum_k 2^{k+1} \sum_j M_{\alpha}[1_{Q_j^k}].
\]

We have that, since \(p < 1\),
\[
(M_{\alpha} f)^p \leq \sum_k 2^{p(k+1)} \sum_j (M_{\alpha}[1_{Q_j^k}])^p \\
\text{and, hence,}
\]

\[
\|M_{\alpha} f\|_{L^{p/n}(H^d_\delta^k)}^p = \frac{1}{p} \| (M_{\alpha} f)^p \|_{L^{p/n}(H^d_\delta^k)} \\
\leq C \frac{1}{p} \sum_k 2^{p(k+1)} \sum_j \|M_{\alpha}[1_{Q_j^k}]^p\|_{L^{p/n}(H^d_\delta^k)} \\
= C \sum_k 2^{p(k+1)} \sum_j \|M_{\alpha}[1_{Q_j^k}]\|^p_{L^{p/n}(H^d_\delta^k)} \\
\leq C_{n,p,d} \sum_k 2^{p(k+1)} \sum_j \int_{Q_j^k} (M_{\gamma q^k} w)^{p/q} \, dx \cdot l(Q_j^k)^d \\
\leq C \int_{\mathbb{R}^n} f^p \, dH^d_{\gamma q^k}. 
\]

This completes the proof. \[\square\]

Next, we prove the weak type estimate Theorem 1.3. The strategy of the proof is also due to [5].

**Proof of Theorem 1.3** Given \(t > 0\), let \(\{Q_j\}\) be the family of maximal dyadic cubes \(Q_j\) such that
\[
\int_{Q_j} f \, dx l(Q_j)^\alpha > t
\]
(we assume again, without loss of generality, that \(f \geq 0\)). Then
\[
\{ x : M_{\alpha} f(x) > t \} = \bigcup_j Q_j.
\]
By Lemma 3 of [5],

\[(2.1) \quad t^q l(Q_j)^q \leq \left( l(Q_j) \right)^q \int_{Q_j} f \, dx \leq C l(Q_j)^{\gamma q} \left( \int_{Q_j} f^p \, dH^d \right)^{\frac{q}{p}}.
\]

Again, applying the proof of Lemma 2 of [5] to the family \(\{Q_j\}\), we see the following.

There exist a subfamily \(\{Q_{jm}\} \subset \{Q_j\}\) and a set of non-overlapping dyadic cubes \(\{\tilde{Q}_k\}\) such that

(i) for each dyadic cube \(Q_j\),

\[\sum_{Q_{jm} \subset Q} l(Q_{jm})^d \leq 2 l(Q)^d;\]

(ii) for each \(j\),

\[\bigcup_j Q_j \subseteq \left( \bigcup_{m} Q_{jm} \right) \cup \left( \bigcup_{k} \tilde{Q}_k \right);\]

(iii) for each \(k\),

\[l(\tilde{Q}_k)^d \leq \sum_{Q_{jm} \subset \tilde{Q}_k} l(Q_{jm})^d.\]

By (ii), we have that

\[t^q H_w^\delta \left( \bigcup_j Q_j \right) \leq t^q \sum_m \int_{Q_{jm}} w \, dH(\tilde{Q}_{jm})^\delta + t^q \sum_k \int_{\tilde{Q}_k} w \, dH(\tilde{Q}_k)^\delta.\]

For all \(Q_{jm}\), by (2.1) we see that

\[t^q \int_{Q_{jm}} w \, dH(\tilde{Q}_{jm})^\delta \leq C \int_{Q_{jm}} w \, dH(\tilde{Q}_{jm})^{\gamma q} \left( \int_{Q_{jm}} f^p \, dH^d \right)^{\frac{q}{p}} \leq C \left( \int_{Q_{jm}} f^p(\mathcal{M} w)^{p/q} \, dH^d \right)^{\frac{q}{p}}.\]

For \(\tilde{Q}_k\), we separate the argument.

**Case** \(\delta \leq d\). By (iii), for each \(k\),

\[l(\tilde{Q}_k)^\delta = \left( l(\tilde{Q}_k) \right)^{\delta/d} \leq \left( \sum_{Q_{jm} \subset \tilde{Q}_k} l(Q_{jm})^d \right)^{\delta/d} \leq \sum_{Q_{jm} \subset \tilde{Q}_k} l(Q_{jm})^\delta.
\]

Thus, by (2.1) we have that

\[t^q \int_{\tilde{Q}_k} w \, dH(\tilde{Q}_k)^\delta \leq C \sum_{Q_{jm} \subset \tilde{Q}_k} \left( \int_{Q_{jm}} f^p(\mathcal{M} w)^{p/q} \, dH^d \right)^{\frac{q}{p}}.\]
Case $\delta \geq d$. By (iii), for each $k$,
\[
\begin{align*}
     t^q \ell (\tilde{Q}_k)^\delta & = t^q (\ell (\tilde{Q}_k)^d)^{\delta / d} \\
     & \leq \left( \sum_{Q_{jm} \subset \tilde{Q}_k} (t^q \ell (Q_{jm})^{\delta d / \delta}) \right)^{\delta / d}.
\end{align*}
\]
Thus, by (2.1) we have that
\[
\begin{align*}
     t^q \int_{\tilde{Q}_k} w \, d x (\tilde{Q}_k)^\delta & \leq C \left[ \sum_{Q_{jm} \subset \tilde{Q}_k} \left( t^q \int_{Q_{jm}} w \, d x (Q_{jm})^{\delta d / \delta} \right)^{q / q} d H^d \right]^{\delta / d} \\
     & \leq C \left( \sum_{Q_{jm} \subset \tilde{Q}_k} \int_{Q_{jm}} f^p (\mathcal{M}_{\gamma q} w)^{p / q} \, d H^d \right)^{q / p},
\end{align*}
\]
where we have used $qd / p\delta = n / (n - \alpha + \gamma) > 1$. Combining altogether, we obtain
\[
\begin{align*}
     t^q H^d \left( \bigcup_j Q_j \right) & \leq C \left( \sum_m \int_{Q_{jm}} f^p (\mathcal{M}_{\gamma q} w)^{p / q} \, d H^d \right)^{q / p} \\
     & \leq C \left( \int_{\mathbb{R}^n} f^p (\mathcal{M}_{\gamma q} w)^{p / q} \, d H^d \right)^{q / p},
\end{align*}
\]
where the last inequality is due to the packing condition (i). This completes the proof. \hfill \Box

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