What is an Algorithm?: a Modern View

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Although algorithm is one of the central subjects, there have been little common understandings of what an algorithm is. For example, Gurevich[1] view algorithms as abstract state machines, while others view algorithms as recursors. We promote a third view: it is a combination to these two disparate views.

This approach – based on computability logic[3] – describes an algorithm as $A(I,O)$ where $I$ is a set of input services and $O$ an output service. It leads to the following modern definition:

An algorithm $A$ is a (tree of) sequence of legal moves for providing $O$ using $I$.

In the above, $A$ is written in an imperative language/abstract state machine and $I,O$ are written in recursors/logical specifications.

1 Introduction

There has been a declarative approach to algorithms. In this approach, algorithms are expressed using other algorithms. This approach includes recursive algorithms, logical algorithms and computability logical(CoL) algorithms[3,4]. Among these, CoL algorithms is the most expressive of all.

Unfortunately, executing declarative algorithms are often slow due to high nondeterminism in finding proofs. That is, it is often inefficient to automatically generate proof steps. For this reason, it is often useful for the programmer to specify proof scripts. In other words, combining imperative algorithms and declarative ones is often needed.

This paper proposes to use CoLi as a new algorithm language. The distinguishing feature of CoLi over CoL is that now the programmer is allowed to provide imperative features.

2 Turing machines or Japaridze machines?

The class of Turing machines (TMs) has been a standard model of computation. It describes an algorithm in the standard form of

$$A(i,o)$$

where $i,o$ is an input/output value, and $A$ is an internal imperative algorithm which maps $i$ to $o$. Thus TMs focuses on input-output mappings.

Japaridze[5,6,7] proposed a new computing model which we call Japaridze machines. It is a TM which focuses on its exchanging services(input services and output services). That is, it describes an algorithm with respect to their exchanging services. To be specific, it describes an algorithm in the form of

$$A(I,O)$$
where \( I \) is the set of input services and \( O \) is the output service and \( A \) is an internal imperative algorithm to accomplish \( O \) using \( I \). For example, consider a task "computes 3!". While the conventional TM would produce 6, Japaridze machines produce a service/knowledge which is \( \text{fact}(3, 6) \). We call this approach CoLi algorithms. Despite of its several advantages, it is quite unfortunate that CoLi algorithms have been largely ignored by academia and industry.

We compare these two models. First, TM preserve only the input/output behavior of a function. There is more to an algorithm than the function it computes. Japaridze machines provide services rather than function outputs. The notion of services is a big concept which includes knowledge, interactive services, other complex services.

Second, the single, low abstraction level of the Turing machine inhibits its ability to describe algorithms concisely. The author of Japaridze machines can choose an arbitrary set of input services and therefore has flexibility in choosing the level of abstraction.

Third, given \( I \) and \( O \), \( A \) can automatically be generated by Japaridze machines. We call the description \( (I, O) \) CoL algorithms.

Finally, it is easier to extend Japaridze machines to distributed computation. A distributed Japaridze machines (also know as computability-logic web[8]) is a set of Japaridze machines providing services to one another. It is a promising model for distributed computing with several attractive features such as local name space and service migrations.

Turing machines lead to the development of assembly languages and C. We now consider how Japaridze machines could be useful in a new language development. In the \( A(I, O) \) above, note that \( A \) is written in imperative languages whereas \( I, O \) are written in logic languages in CoL.

This lead us to a next-generation imperative language where imperative languages are used as implementation languages and logic languages are used as specification. This new language is thus closely related to the deep specification approach to software. An example may be Python with logical specification of input/output services. That is,

- Turing machines \( \Rightarrow \) Assembly languages, C, Python, . . .
- Japaridze machines \( \Rightarrow \) Assembly+deep specification, C+deep specification, . . .

We now consider when we need CoLi algorithms. CoL is a complex language with a huge yet of useful operations. The design of Japaridze machines aims at automatically generating an internal algorithm/strategy from given input and out services. This approach has been successful for various fragments of CoL. Yet, implementing the full CoL is a totally different story: it seems a daunting, almost impossible task due to its huge complexity. Accordingly, we have no other choice but to rely on CoLi algorithms to utilize the full CoL.

## 3 Preliminaries

In this section a brief overview of CoL is given.

There are two players: the machine \( \top \) and the environment \( \bot \).

There are two sorts of atoms: elementary atoms \( p, q, \ldots \) to represent elementary games, and general atoms \( P, Q, \ldots \) to represent any, not-necessarily-elementary, games.

**Constant elementary games** \( \top \) is always a true proposition, and \( \bot \) is always a false proposition.
Negation $\neg$ is a role-switch operation: For example, $\neg(0 = 1)$ is true, while $(0 = 1)$ is false.

Choice operations The choice group of operations: $\sqcap$, $\sqcup$, $\sqcap$ and $\sqcup$ are defined below.

$\sqcap x.A(x)$ is the game where, in the initial position, only $\bot$ has a legal move which consists in choosing a value for $x$. After $\bot$ makes a move $c \in \{0, 1, \ldots\}$, the game continues as $A(c)$. $A \sqcap B$ is similar, only here the choice is made between “left” and “right”. $\sqcap$ and $\sqcup$ are symmetric to $\sqcap$ and $\sqcap$, with the difference that now it is $\top$ who makes an initial move.

Parallel operations Playing $A_1 \land \ldots \land A_n$ means playing the $n$ games concurrently. In order to win, $\top$ needs to win in each of $n$ games. Playing $A_1 \lor \ldots \lor A_n$ also means playing the $n$ games concurrently. In order to win, $\top$ needs to win one of the games. To indicate that a given move is made in the $i$th component, the player should prefix it with the string “$i$.”. The operations $\lambda A$ means an infinite parallel game $A \land \ldots \land A \land \ldots$ To indicate that a given move is made in the $i(i > 1)$th component, we assume the player should first replicate $A$ and then prefix it with the string “$i$.”.

Reduction $A \rightarrow B$ is defined by $\neg A \lor B$. Intuitively, $A \rightarrow B$ is the problem of reducing $B$ (consequent) to $A$ (antecedent).

4 Introducing Directories

Logical formulas are inadequate for locating subformulas. Our approach to achieving this effect is through the use of directories. For example, consider the following directory definition.

$$/m = p(a)$$

where $/m$ is a directory name and $p(a)$ is a formula. In this case, we call $p(a)$ its “content”. Alternatively, we can view $/m$ as an agent and $p(a)$ as its knowledgebase.

Our directory system is very flexible and is designed to represent both formulas and cirquents. For example, $/n = !/m! /m$ represents that the directory $/n$ contains $p(a) \land p(a)$. Here $!/m$ is intended to read as “a copy of the content of $/m$. In contrast, $/o = /m \land /m$ represents that $/o$ contains a cirquent $p(a) \land p(a)$ where two $p(a)$s in $/o$ and $p(a)$ in $/m$ are shared.

As another example, consider the following recursive directory definition.

$$/m(0) = q$$
$$/m(s(X)) = p \land /m(X)$$

Given this definition, $p \land (p \land (p \land q)))$ can be represented simply as $/m(s(s(s(0)))).$ We assume in the above that $s$ is the number-successor function.

Thus, we propose the notion of directorized formulas. They are formulas enhanced with directories. These formulas are better-suited to structuring large formulas such as pigeonhole principle formulas. It is interesting to note that directories also play the role of global variables in imperative languages and much more.
5 CoLi Algorithms

A CoLi algorithm is of the form

\[ A(I, O) \]

where \( I = \{I_1, \ldots, I_n\} \) is a set of input services and \( O \) is an output service, both written in CoL. \( A \) is a winning strategy, i.e., a tree of runs written in pseudocode. A run is a sequence of \( S_1; S_2; \ldots \) where \( S \) is of several kinds. We list some of them below.

1. \texttt{l.read}(N) replaces a subformula \( \Box \! x F \) at location \( l \) by \( F(X/x) \) where \( X \) is a new variable. For example, suppose \( \Box \! x p(x) \) is at \( l \). \texttt{l.read}(N) replaces it by \( p(X) \) and stores \( X \) in the global variable \( N \).

2. \texttt{l.write} replaces \( \exists \! x F \) at \( l \) by \( F(W/x) \) where \( W \) is a new global variable. For example, suppose \( \exists \! x p(x) \) is at \( l \). \texttt{l.write} replaces \( \exists \! x p(x) \) by \( p(W) \). The value of \( W \) is unknown and will later be determined via the unification process. This technique is well-known in the logic programming community.

3. \texttt{choose}(l_1 : r_1, \ldots, l_n : r_n) limits the proof search space to the given \( n \) choices, where each \( l_i \) denotes a location of some subformula \( H \) and \( r_i \) is a list of rule candidates to apply to \( H \). We often omit \( r_i \) when it is obvious.

4. \texttt{schoose}(l_1 : r_1, \ldots, l_n : r_n) is identical to \texttt{choose}(l_1 : r_1, \ldots, l_n : r_n) with the difference that \( l_1 : r_1 \) has the highest priority and \( l_n : r_n \) has the lowest.

5. Conditional statements include the if-then-else. Iterational statements include the for-loop.

6. \texttt{prove} tries to extract a winning strategy \( S \) from the current configuration. \texttt{execute} executes \( S \).

An internal algorithm \( A \) typically is very complex and challenging. It requires two stages:

- The stage 1 extracts a winning strategy \( S \) from the given \((I, O)\). \( S \) is typically non-deterministic and has the form of a tree of runs due to interactive services. This stage is very difficult due to the complexity of the proof procedure.
- The stage 2 executes a branch of \( S \) which is obtained by interacting with the user. This stage is easy.

In most cases, stage 1 can be automated by the machine. Unfortunately, there are cases when \( S \) is difficult to extract by the machine but the programmer knows \( S \). In such cases, CoLi would be useful. In writing CoLi algorithms, it would be painful for the programmer to write \( A \) from scratch. Instead, \( A \) is typically written by the programmer in a minimal way, i.e., in the style of proof scripts. The rest will then be automatically generated by the machine.

\footnote{A run is a sequence of moves.}
\footnote{Global variables are different from variables. The unification process only deals with global variables.}
6 Examples

As an example, we present the factorial algorithm to help understand this notion. The factorial algorithm can be defined using two input services \( /c, /d \) whose tasks are described below: where the recurrence action is preceded with \( \land \).

\[
\begin{align*}
/c &= f act(0, 1) \\
/d &= \land \exists x \exists y (f act(x, y) \rightarrow f act(x + 1, xy + y)) \\
/query &= \land y \sqcup z f act(y, z) \% \text{read } y \text{ and compute } z \text{ (which is } y! \text{)}
\end{align*}
\]

Suppose computation tries to solve the \( /query \) with respect to \( /c, /d \). Observe that \( /query \) is a logical consequence of \( /c \) and \( /d \) and proving this fact requires mathematical induction. Extracting a winning strategy for this problem is quite difficult and nontrivial. Unlike the machine, however, a winning strategy is obvious to human and is the following: read \( y \) in the \( /query \), make \( y \) copies of \( /d \), instantiate \( x, y \) in each copy of \( /d \) and then compute \( z \) which is a logical consequence of these knowledgebase\(^3\). The corresponding proof script \( F act \) is shown below:

Algorithm \( F act(\{/c, /d\}, /query) \) % An algorithm for computing factorial.
\( /query.\text{read}(n); \)
for \( i=1 \) to \( n; \)
\( /d.\text{i}.\text{write}; \% \text{process } \land x \text{ in the } i\text{th copy of } /d. \)
\( /d.\text{i}.\text{write}; \% \text{process } \land y \text{ in the } i\text{th copy of } /d. \)
endfor;
\( /query.\text{write}; \% \text{process } \sqcup z \text{ in the query} \)
execute; % invoke the unification procedure

In the above, note that \( \land y \) in the \( /query \) is the major obstacle in extracting a winning strategy \( S \). Once \( \land y \) has been processed, extracting the rest of \( S \) poses no problem. Therefore, \( F act \) can be simplified to the following:

Algorithm \( F act(\{/c, /d\}, /query) \) % A shortened algorithm for computing factorial.
\( /query.\text{read}(n); \)
prove; % extract a winning strategy from the current configuration.
execute; % invoke the unification procedure.

Proof scripts are also useful for dealing with semidecidable problems. For example, consider the following:

\( /q = \land \sqcup x p(x) \lor q(a) \).

The above formula is invalid. Unfortunately, a typical proof procedure does not terminate for this formula, as it repeatedly replicates \( \land \sqcup x p(x) \). We can avoid this unpleasant situation

\(^3\)In fact, making a copy of \( /d \) occurs on demand, i.e., when the corresponding \( i\)th component is not available yet.
by providing the following proof script which tells the machine not to replicate it.

/q.1:⊔-rule; % replicate is disallowed in ⊓xp(x).

In the beginning, the machine first chooses a term t for ⊓x and obtains p(t) ∨ q(a). Then the proof procedure terminates with the failure.

7 Conclusion

Our intention is to raise awareness of the CoLi algorithms as a new tool for expressing algorithms. We believe it is a tool of real value. Another interest is in designing a more flexible proof script language.

References

[1] Yuri Grevich. What is an algorithm?. Microsoft TR-MSR-TR-2011-116, 2011.
[2] G. Japaridze. The logic of tasks. Annals of Pure and Applied Logic, 117:263–295, 2002.
[3] G. Japaridze. Introduction to computability logic. Annals of Pure and Applied Logic, 123:1–99, 2003.
[4] G. Japaridze. Sequential operators in computability logic. Information and Computation, vol.206, No.12, pp.1443-1475, 2008.
[5] G. Japaridze. Computability logic: a formal theory of interaction. In Interactive Computation: The New Paradigm, Goldin D, Smolka S A, Wegner P (eds.), Springer, 2006, pp. 183-223.
[6] G. Japaridze. In the beginning was game semantics. In Games: Unifying Logic, Language and Philosophy, Majer O, Pietarinen A -V, Tulenheimo T (eds.), Springer, 2009, pp. 249-350.
[7] G. Japaridze. Towards applied theories based on computability logic. Journal of Symbolic Logic, 2010, 75(2): 565-601.
[8] K. Kwon. Computability-logic web: an alternative to deep learning. arXiv:2101.09222, 2020.