Near Optimal Algorithm for the Directed Single Source Replacement Paths Problem

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Abstract

In the Single Source Replacement Paths (SSRP) problem we are given a graph $G = (V, E)$, and a shortest paths tree $\tilde{K}$ rooted at a node $s$, and the goal is to output for every node $t \in V$ and for every edge $e$ in $\tilde{K}$ the length of the shortest path from $s$ to $t$ avoiding $e$.

We present an $\tilde{O}(m\sqrt{n} + n^2)$ time randomized combinatorial algorithm for unweighted directed graphs. Previously such a bound was known in the directed case only for the seemingly easier problem of replacement path where both the source and the target nodes are fixed.

Our new upper bound for this problem matches the existing conditional combinatorial lower bounds. Hence, (assuming these conditional lower bounds) our result is essentially optimal and completes the picture of the SSRP problem in the combinatorial setting.

Our algorithm extends to the case of small, rational edge weights. We strengthen the existing conditional lower bounds in this case by showing that any $O(mn^{1/2-\epsilon})$ time (combinatorial or algebraic) algorithm for some fixed $\epsilon > 0$ yields a truly subcubic algorithm for the weighted All Pairs Shortest Paths problem (previously such a bound was known only for the combinatorial setting).

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As usual, $n$ is the number of vertices, $m$ is the number of edges and the $\tilde{O}$ notation suppresses poly-logarithmic factors.
1 Introduction

In the replacement paths (RP) problem, we are given a graph $G$ and a shortest path $P$ between two vertices $s$ and $t$ and the goal is to return for every edge $e$ in $P$ the length $d(s, t, G - e)$, where $G - e$ is the graph obtained by removing the edge $e$ from $G$, and $d(s, t, G - e)$ is the distance between $s$ and $t$ in the resulted graph. In some cases the goal is to provide the shortest path itself and not only its length. The interest in replacement path problems stems from the fact that failures and changes in real world networks are inevitable, and in many cases we would like to have a solution or a data structure that can adapt to these failures. The replacement paths problem is a notable example where we would like to have backup paths between two distinguished vertices in the event of edge failures. The replacement paths problem is also very well motivated as it is used as an important ingredient in other applications such as the Vickrey pricing of edges owned by selfish agents from auction theory [13, 7]. Another application of the replacement path problem is finding the $k$ shortest simple paths between a pair of vertices. The $k$ shortest simple paths problem can be solved by invoking the replacement paths algorithm $k$ times and adding a very small weight to the path found in each invocation. The $k$ shortest simple paths problem has many applications by itself [5]. The replacement paths problem has been extensively studied and the literature covers many aspects of this problem with many near optimal solutions in many of the cases (see e.g. [12, 14, 13, 11, 10, 21, 17, 3]).

In this paper we consider a natural and important generalization of the replacement paths problem, referred to as the single source replacement paths (SSRP) problem, which is defined as follows. Given a graph $G$ and a shortest paths tree $K$ rooted at a node $s$, the SSRP problem is to compute the values of $d(s, t, G - e)$ for every vertex $t \in V(G)$ and for every edge $e \in E(K)$. Note that as the number of edges in $K$ is $n - 1$, there are $O(n^2)$ different distances we need to evaluate. It follows that the size of the SSRP output is $O(n^2)$. Despite of its natural flavor, the picture of the SSRP problem is not yet complete in many of the cases. To the best of our knowledge the first paper that considered the SSRP problem is by Hershberger et al. [8] who referred to the problem as edge-replacement shortest paths trees and showed that in the path-comparison model of computation of Karger et al. [9], there is a lower bound of $\Omega(mn)$ comparisons in order to solve the SSRP problem for arbitrarily weighted directed graphs. For the directed weighted case it was shown by Vassilevska Williams and Williams [19] that any truly sub-cubic algorithm for the simpler problem of RP in directed, arbitrarily weighted graph admits a truly sub-cubic algorithm for the arbitrarily weighted All Pairs Shortest Paths (APSP) problem. The conditional lower bound from [19] holds only for the directed case, and quite interestingly for the undirected arbitrarily weighted case, the classical RP problem admits a near linear time algorithm [12, 14, 13]. However, the SSRP problem in undirected graphs appears to be much harder than the RP problem. In [8] it was shown by Chechik and Cohen that any truly sub-cubic solution for the SSRP problem in undirected arbitrarily weighted graphs, admits a truly sub-cubic algorithm for the arbitrarily weighted APSP problem. Therefore, it seems there is no hope to solve the SSRP problem in weighted graphs, both in the directed and undirected case. Meaning that if we seek for truly sub-cubic algorithms for the SSRP problem we must either consider unweighted graphs or restrict the edge weights in some other way.

One way to restrict the weights is to consider only bounded integer edge weights. This restriction was considered by Grandoni and Vassilevska Williams [6], who were also the ones to name this problem the single source replacement paths problem. Grandoni and Vassilevska Williams [6] gave the first non trivial upper bound for the SSRP problem. They showed that one can bypass the cubic lower bounds by using fast matrix multiplications and by restricting the weights to be integers in a bounded range. More precisely, they showed that for graphs with positive integer edge weights in the range $[1, M]$, SSRP can be computed in $O(Mn^\omega)$ time (here $\omega < 2.373$ is the matrix multiplication exponent [18, 11]). This matches the current best known bound for the simpler problem of RP for directed graph with weights $[1, M]$, by Vassilevska Williams [17]. Quite interestingly, for integer edge weights in the range $[-M, M]$, the authors of [6] gave a higher upper bound of $O(Mn^{\omega} + n^2 + \frac{M}{n})$ time, which creates an interesting gap between the SSRP problem and the RP problem for negative integer weights. Grandoni and Vassilevska Williams [6] conjectured that the gap between these two problems is essential and in fact they conjectured that the SSRP problem with negative weights is as hard as the directed APSP problem. The algorithm described in [6] uses fast matrix multiplication tricks in order to break the trivial cubic upper bound, such algorithms are known as “algebraic algorithms”. Algorithms that do not use any matrix multiplication tricks are known as “combinatorial algorithms”. The interest in combinatorial algorithms mainly stems from the assumption that in practice combinatorial algorithms are much more efficient since the constants and sub-polynomial factors hidden in the matrix multiplication bounds are considered to be very high.
The SSRP problem was also recently considered in the combinatorial setting by Chechik and Cohen in [3] for undirected unweighted graphs. Specifically, Chechik and Cohen in [3] gave an $\tilde{O}(m\sqrt{n} + n^2)$ time randomized algorithm for SSRP in undirected unweighted graphs. Moreover, using conditional lower bounds Chechik and Cohen also showed that under some reasonable assumptions any combinatorial algorithm for the SSRP problem in unweighted undirected graphs requires $\tilde{\Omega}(m\sqrt{n})$ time.

Since there is little hope to solve the weighted case, the only missing piece in the picture of combinatorial SSRP is the case of directed unweighted graphs.

For the directed unweighted case it was shown earlier by Vassilevska Williams and Williams [14], using a conditional combinatorial lower bound that under some reasonable assumptions any combinatorial algorithm for the directed unweighted RP (and hence SSRP) problem requires $\tilde{\Omega}(m\sqrt{n})$ time. For the seemingly easier problem of replacement paths Roditty and Zwick [16] showed a near optimal solution of $\tilde{O}(m\sqrt{n})$ time for directed unweighted graphs.

Note that in the undirected unweighted case there is an essential gap between the RP and the SSRP problems. A natural question is whether such a gap also exists in the directed unweighted case. In this paper we show that this is not the case by providing a combinatorial near optimal $\tilde{O}(m\sqrt{n} + n^2)$ time algorithm for the case of directed unweighted graphs, which up to the $n^2$ factor (that is unavoidable as the output itself is of size $O(n^2)$) matches the running time of the algorithm in [14] (and also matches the running time of the undirected case in [3]). We therefore (up to poly-logarithmic factors) complete the picture of combinatorial SSRP.

Our main result is as follows.

**Theorem 1.1.** There exists an $\tilde{O}(m\sqrt{n} + n^2)$ time combinatorial algorithm for the SSRP problem on unweighted directed graphs. Our randomized algorithm is Monte Carlo with a one-sided error, as we always output distances which are at least the exact distances, and with high probability (of at least $1 - n^{-C}$ for any constant $C > 0$) we output the exact distance.

Note that for unweighted directed graphs where $m = \tilde{O}(n^{1.5})$ our algorithm runs in $\tilde{O}(n^2)$ time, which is the time it takes just to output the result. Namely, in this range of density our algorithm surpasses the current best algebraic SSRP algorithm [6] (which has a running time complexity of $\tilde{O}(n^2)$) as long as $\omega > 2$.

We will note that while we focus on the case of edge failures, in the directed case there is a well known reduction showing that edge failures can be used to simulate vertex failures. The reduction is as follows, replace every vertex $v$ with two vertices $v_{in}$ and $v_{out}$, and connect them by a direct edge ($v_{in}, v_{out}$). Then, for every incoming edge $(u, v)$ add the edge $(u, v_{in})$, and for every outgoing edge $(v, u)$ add the edge $(v_{out}, u)$. The failure of the vertex $v$ is now simulated by the failure of the edge ($v_{in}, v_{out}$).

Our main novelty is in the introduction of a tool which we refer to as *weight functions*. This tool proved to be very useful in order to apply a divide and conquer approach and could perhaps be utilized in other related problems.

### 1.1 Rational Weights

While we describe an algorithm for the problem of SSRP in unweighted graphs, our algorithm (much like the directed RP algorithm [16]) can be easily generalized to solve the case of weighted graphs for *rational* edge weights in the range $[1, C]$, for every constant $C \geq 1$, in the same time complexity. This is because the only place our algorithm (and the algorithm from [16]) uses the fact that the graph is unweighted is in the claim that a path of length $l$ contains $\Theta(l)$ vertices, which is used in order to utilize sampling techniques. As this is also true for rational weights in the range $[1, C]$, our algorithm generalizes for this case trivially.

Algebraic algorithms inherently can not perform on graphs with rational weights. This is since algebraic algorithms use a reduction from a problem known as min-plus product to the problem of matrix product, and this reduction works only for *integer* weights. Since in some use-cases (like the $k$-simple paths problem) it is very useful to have rational weights, this shows another potential interest in combinatorial algorithms.

We note that in order to store rational numbers, we must make some common assumptions regarding the model of computation. More specifically, we assume that computing the summation of $n$ edge weights can be performed in $\tilde{O}(n)$ time and that all numbers we are dealing with can be stored in one (or $O(1)$) space unit. A

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2 Also known as funny matrix multiplication or distance product, see [3, 22]
realistic option is working in a word-RAM model, and considering only rational edge weights which are of the form \( \frac{m}{2^k} \), where the two integers \( m \) and \( 2^k \) fit in the size of \( O(1) \) computer words. This way, the summation of \( O(n) \) numbers also fits in \( O(1) \) computer words. This way of representing rational numbers is reminiscent of the floating-point representation, that is commonly used in practical applications.

In Section 7 we show that any algorithm (combinatorial or not) for the SSRP problem for graphs with rational edge weights from the range \([1, 2]\), that runs in \( O(mn^{1/2-\varepsilon}) \) time for any fixed \( \varepsilon > 0 \) implies a truly sub-cubic algorithm for APSP over graphs with arbitrary integer weights. The claim is formally stated in Theorem 7.1. Previously such a conditional lower bound was only known for combinatorial algorithms using a reduction from Boolean Matrix Multiplication (see 8).

## 2 Preliminaries

We will use the following notation: \( \mathbb{N} = \{0, 1, 2, ...\}, \mathbb{N}^+ = \{1, 2, 3, ...\}, \forall a \in \mathbb{N}^+: [a] = \{1, 2, ..., a\} \). Let \( G \) be a weighted directed graph then \( E(G) \) denotes the set of edges in \( G \) and \( V(G) \) the set of nodes. For a vertex \( v \) we say that \( v \in G \) if \( v \in V(G) \) and for an edge \( e \) we say that \( e \in G \) if \( e \in E(G) \). Let \( u, v \in V(G) \) be two vertices, we denote by \( d(u, v, G) \) the distance from \( u \) to \( v \) in the graph \( G \), and denote by \( R(u, v, G) \) some shortest path from \( u \) to \( v \) in \( G \). Let \( P \subseteq G \) be a path from \( u \) to \( v \), we define \( |P| = |E(P)| = |V(P)| - 1 \). We also denote the length of \( P \) by \( d(P) \). Note that \( d(R(u, v, G)) = d(u, v, G) \). For a set of edges \( A \subseteq E(G) \) we denote the graph \( (V(G), E(G) \setminus A) \) by \( G - A \). For an edge \( e \) we denote \( G - \{e\} \) by \( G - e \), and for a path \( P \) we shortly denote \( G - E(P) \) by \( G - P \).

We denote by \( G^R \) the graph obtained by reversing the directions of all edges - that is the graph obtained by replacing each edge \((v, u) \in E(G) \) with the edge \((u, v) \) with the same weight. Given a sub-graph \( H \subseteq G \) we denote by \( G[H] \) the sub-graph of \( G \) induced by the nodes in \( V(H) \).

Let \( P \subseteq G \) be a shortest path from a node \( s \) to a node \( t \). Let \( u, v \in V(P) \) be two nodes in \( P \), we say that \( u \) is before \( v \) in \( P \) if \( d(u, t, G) \geq d(v, t, G) \) and that \( u \) is after \( v \) in \( P \) if \( d(u, t, G) \leq d(v, t, G) \). For an edge \( e = (u, v) \in E(P) \) and a node \( a \in V(P) \) we say that \( a \) is before \( e \) in the path \( P \) if \( a \) is before \( u \) in \( P \) and say that \( a \) is after \( e \) in \( P \) if \( a \) is after \( v \) in the path \( P \).

The following sampling Lemma is a folklore.

**Lemma 2.1 (Sampling Lemma).** Consider \( n \) balls of which \( R \) are red and \( n - R \) are blue. Let \( C > 0, N > 0 \) be two numbers such that \( R > C \cdot \ln(N) \). Let \( B \) be a random set of balls such that each ball is chosen to be in \( B \) independently at random with probability \( \frac{C \cdot \ln(N)}{R} \). Then \( w.h.p \) (with probability at least \( 1 - \frac{1}{N^2} \)) there is a red ball in \( B \) and the size of \( B \) is \( O(n/R) \).

The following separation Lemma was used extensively in many divide and conquer algorithms on graphs including the algebraic SSRP algorithm from 8.

**Lemma 2.2 (Separator Lemma).** Given a tree \( K \) with \( n \) nodes rooted at a node \( s \), one can find in \( O(n) \) time a node \( t \) that separates the tree \( K \) into 2 edge disjoint sub-trees \( S, T \) such that \( E(S) \cup E(T) = E(K) \), \( V(T) \cap V(S) = \{t\} \) and \( \frac{n}{2} \leq |V(T)|, |V(S)| \leq \frac{2n}{3} \).

WLOG we always assume that \( s \in V(S) \), which implies that \( t \) must be the root of \( T \). Note that it might be the case that \( t = s \).

### 2.1 The Generalized SSRP Problem

We next describe a generalization of the directed-SSRP problem that our algorithm works with. We start by describing the notation of weight functions, a new concept we developed that allows us to compress a lot of information into one recursive call of the algorithm. In the next section we will give more intuition about the weight functions and this specific generalization.

**Definition 2.1 (Weight Function).** Let \( G \) be an unweighted directed graph. Let \( s \in V(G) \) be some special source node. A function \( w : V(G) \rightarrow \mathbb{N} \cup \{\infty\} \) is a weight function (with respect to the source node \( s \)) if \( w(v) \geq d(s, v, G) \) for every vertex \( v \in V(G) \). We refer to this requirement as the weight requirement.
For a source node $s \in V(G)$ and a weight function $w$ (with respect to the source node $s$) we define the weighted directed graph $G_w$ by taking the unweighted graph $G$, and assigning each edge the weight 1. We then add for every node $v \in V(G)$ the edge $(s, v)$ and assign to it the weight $w(v)$. Note that $G$ is a sub-graph of $G_w$. Also, note that by the weight requirement, for every two nodes $u, v \in V(G) : d(u, v, G) = d(u, v, G_w)$.

The generalized SSRP problem is now defined as follows. The input consists of the following:

- An unweighted directed graph $H$ and a source vertex $s \in V(H)$
- A BFS tree $K$ in $H$ rooted at the source $s$ ($E(K) \subseteq E(H)$)
- A set of weight functions $W$ (with respect to source node $s$)
- A set of queries $Q \subseteq E(K) \times V(H) \times W$

The goal is to output for every $(e, x, w) \in Q$ the distance $d(s, x, H_w - e)$. Note that this problem is indeed a generalization of the classic SSRP problem. In order to solve the SSRP problem on the initial graph $G$ and the BFS tree $K$, we simply define a single weight function $w : V(G) \rightarrow \mathbb{N} \cup \{\infty\}$ that is defined to be $w \equiv \infty$. We then invoke our algorithm with the graph $G$, the BFS tree $K$, the set of weight functions $W = \{w\}$, and the query set $Q = E(K) \times V(G) \times W$. Note that $G$ and $G_w$ are the same graph in the sense that for every edge $e \in E(G)$ and destination $x \in V(G)$ we have that $d(s, x, G - e) = d(s, x, G_w - e)$. Hence, invoking our algorithm for the generalized SSRP would suffice. As we will only work with the generalized SSRP problem, we hereafter refer to it as the SSRP problem for simplicity.

3 Overview

Our algorithm uses a divide and conquer approach. Each recursive call works on a different sub-tree ($K$) of the original BFS tree ($\hat{K}$), where both the destination and the edge failure are within this sub-tree (for the case when the edge failure and the destination are not in the same sub-tree our algorithm solves this in a non recursive manner to be described later in case 1 of the algorithm overview). The vertices of the sub-tree $K$ induces a sub-graph $H$ of the original graph $G$. Denote by $n = |V(G)|, m = |E(G)|, n_H = |V(H)|, m_H = |E(H)|$.

The first step of our algorithm is to separate the input BFS tree $K$ into two edge disjoint sub-trees $S$ and $T$ using a balanced tree separator (see Lemma 22). We denote the root of the BFS tree $K$ by $s$. We assume WLOG that the root of $S$ is $s$ and the root of $T$ is some node $t$. It might be the case that $s = t$. We define $P$ as the path from $s$ to $t$ in the BFS tree $K$. Note that $P \subseteq S$. An illustration of this separation can be found in Figure 2 in the appendix.

Let $K'$ be one of the two sub-trees of $K$ (that is $S$ or $T$). If a replacement path is fully contained in the graph induced by $K'$ then simply using the recursive call is enough in order to compute its length. The more challenging case is when the replacement path contains vertices that are not in $K'$.

In [6] the authors used a somewhat similar divide and conquer approach. Consider a recursive call on a sub-tree $K'$ and consider the case when the edge failure and destination node are both in $K'$. In their algorithm, the authors of [6] used sampling techniques and a truncated version of the algebraic APSP algorithm (as presented in [23]) in order to create a compressed version of the subgraph induced over $K'$ (by adding shortcuts between vertices in $K'$), which (w.h.p) preserves all information needed in order to compute the true distance.

However, in the combinatorial setting, one cannot use this sort of compression process for several reasons. Firstly after the first call to the compression step (as described in the algorithm in [6]), the resulted graph could be very dense, maybe even complete. Since the conditional lower bound of $\Omega(m/\sqrt{n} + n^2)$ for a combinatorial SSRP ($G, [19]$) depends on the number of edges, we do not want to receive such dense graphs. Secondly, as we are in the combinatorial setting, we cannot use fast matrix multiplication in the compression step, which is a critical part of the algorithm described in [6]. Lastly, after the compression step the resulted graph is weighted which leaves us with a substantially more difficult problem. In fact in the combinatorial setting there is no sub-cubic time algorithm that solves the even seemingly easier problem of weighted RP (see [19] for conditional lower bounds).
So in the combinatorial setting we must devise a new, more restricted, compression technique. We will essentially show that if we add weighted edges only from the source s to all other vertices, and restrict the weights to be such that the weight of the edge \((s, v)\) is at least \(d(s, v, H)\), then solving replacement path on such a graph still requires only \(\tilde{O}(m \sqrt{n} + n^2)\) time. We therefore would like to add only edges between \(s\) and all other nodes. However, this quickly proves to be difficult, and it seems that if we add only weighted edges from \(s\) to the compressed graph we either “under-shoot” and do not represent all replacement paths, or we “over-shoot” and represent replacement paths that does not really exist in the graph \(H - e\) (for some edge failure \(e\)) - such paths will be called untruthful paths.

We have devised a technique to fix the over-shooting. That is, we give the recursive call weights that may represent untruthful replacement paths in \(H - e\), but we force the recursive call to restrict the replacement paths it searches for, so we will be able to fix them before the algorithm outputs them, while maintaining optimality. The way we do so is by a novel concept we call weight functions. The idea is that the unweighted graph \(H\) will come equipped with a set of functions \(W\), such that every \(e \in W\) is a function from \(V(H)\) to \(\mathbb{N} \cup \{\infty\}\) and for every vertex \(v\) it holds that \(w(v) \geq d(s, v, H)\). For every weight function \(w \in W\) the weighted graph \(H_w\) is defined by adding for every node \(v \in V\) the edge \((s, v)\) with weight \(w(v)\). The goal of the algorithm is then outputting \(d(s, v, H_w - e)\) for every triplet \(x \in V, w \in W, e \in E(K)\). By restricting the algorithm to only use a single, specific weight function we achieve enough “control” to fix the untruthful paths. In order to maintain the desired running time it will be critical to keep the number of weight functions (|\(W|\)) at most \(\tilde{O}(\sqrt{n})\) (where \(n\) is the number of nodes of the original graph \(G\)).

### 3.1 Algorithm Overview

In the remaining of this section we sketch the ideas of our algorithm in high level. For the sake of simplicity, the algorithm in this section runs in \(\tilde{O}(n^{2.5})\) time rather than \(\tilde{O}(m \sqrt{n} + n^2)\) time. At the end of this section we will briefly describe how one can use some simple techniques to reduce the running time to the near optimal of \(\tilde{O}(m \sqrt{n} + n^2)\). While sketching the algorithm, we also ignore the query set \(Q\), as it is only necessary when reducing the running time of the algorithm to \(\tilde{O}(m \sqrt{n} + n^2)\). So the goal of the algorithm in this section is to estimate \(d(s, x, H_w - e)\) for every \(e \in E(K), x \in V(H)\) and \(w \in W\). The complete algorithm and proof of correctness can be found in Sections 4 and 5 correspondingly.

In our algorithm we distinguish between a few cases according to where the edge failure and the destination are with respect to \(S\) and \(T\). Note that for each edge failure \(e\) and destination node \(x\) we clearly know in which case we are. In each such case we distinguish between different sub-cases according to different properties of the replacement path. Clearly we do not know the replacement path a-priori, meaning that we do not know in which sub-case we are. So when proving the correctness of our algorithm in Section 5 we show that the estimation created for every sub-case is always at least the real value of \(d(s, x, H_w - e)\), that is, we do not underestimate. Then we show that for the true sub-case (the sub-case describing the true replacement path) our estimation matches the true value of \(d(s, x, H_w - e)\) w.h.p. By returning the minimum estimation from all of the sub-cases we are guaranteed to return the true distance w.h.p. To distinguish between the different sub-cases we first define two useful characterization of replacement paths in \(H_w\).

**Definition 3.1 (Weighted paths).** Let \(e \in E(K), x \in V(H), w \in W\), and let \(R\) be a path from \(s\) to \(x\) in the graph \(H_w - e\). The path \(R\) will be called weighted if it uses some edge from \(E(H_w) - E(H)\). \(R\) will be called unweighted if it is fully contained in \(H - e\).

The following crucial observation allows us to handle many cases involving weighted replacement paths

**Observation 1.** Let \(e \in P\) be an edge failure, \(x \in H\) be a destination node and \(w \in W\) be a weight function. If the replacement path \(R(s, x, H_w - e)\) is weighted then it leaves \(P\) at \(s\) and does not intersect with \(P\) until after the edge failure.

To see why this observation is true, first note that all the edges in \(E(H_w) - E(H)\) begin at \(s\) by definition, so \(R(s, x, H_w - e)\) indeed leaves \(P\) at \(s\). Also, \(R(s, x, H_w - e)\) does not intersect with \(P\) until after the edge failure as otherwise \(R(s, x, H_w - e)\) could have used the path \(P\) to get from \(s\) to the intersection point, which is a shortest path by the weight requirements. In other words, the use of the weighted edge is unnecessary. An illustration of such path can be seen in Figure 4.
For edge failures from $P$ we also define the following useful characterization

**Definition 3.2** (Jumping and Departing Paths). Let $e \in E(P), x \in V(H), w \in W$, and let $R$ be a path from $s$ to $x$ in the graph $H_w - e$. The path $R$ will be called *jumping* if it uses some node $u$ such that $u \in P$ and $u$ is after the edge failure $e$ in the path $P$. A path that is not jumping will be called *departing*.

First case - the failure is in $P$ and the destination is in $T$: This case can be solved in a non-recursive manner, using observation 1 and somewhat similar observations to those that were used in $T$. We distinguish between 3 different forms the path $R(s,x,H_w - e)$ can take:

Case 1.1: $R(s,x,H_w - e)$ is departing and weighted. Using observation 1 we can conclude that $R(s,x,H_w - e)$ is edge-disjoint from $P$ as it does not intersect with $P$ before the edge failure nor after (since it is departing). This implies that the length of $R(s,x,H_w - e)$ is $d(s,x,H_w - P)$. This value can easily be computed by running Dijkstra’s algorithm from $s$ in the graph $H_w - P$ for every weight function $w$.

Case 1.2: $R(s,x,H_w - e)$ is departing and unweighted. An illustration of this case can be seen in Figure 1.

In this case one can use a technique similar to the one used in $T$ in order to compute length of the replacement path w.h.p. That is, if $e$ is among the last $\sqrt{m_H}$ edges of $P$, then we can use a brute force solution to compute $d(s,x,H - e)$. If $e$ is of distance at least $\sqrt{m_H}$ from $t$, then the length of the detour of $R(s,x,H_w - e)$ is at least $\sqrt{m_H}$ as this path departs before $e$ and gets to $T$. So by sampling a set of nodes $B$ of size $\tilde{O}(\sqrt{m_H})$, we hit every such detour w.h.p. Assuming we hit the detour using the pivot node $b \in B$, we can compute $d(s,b,H - e)$ rather easily, and have that $d(s,x,H - e) = d(s,b,H - e) + d(b,x,H - P)$.

In the full algorithm we denote the estimation obtained by the pivots sampling by $\text{Depart}(s,x,e)$. We show how to compute this estimation in step 5 of the algorithm and prove its correctness in Claims 5.1, 5.2.

Case 1.3: $R(s,x,H_w - e)$ is jumping. As observed by the authors of $T$, taking care of jumping replacement paths in the case when $x \in T$ essentially reduces to solving the RP problem, where the source node is $s$ and the destination node is $t$. This is since a jumping replacement path passes WLOG through the separator node $t$.

So we focus on computing the length of $R(s,t,H_w - e)$ for every $e \in P$ and $w \in W$. Using observation 1 if $R(s,t,H_w - e)$ is weighted then its length is $\min_{u \text{ after } e} \{d(s,u,H_w - P) + d(u,t,H)\}$ as it does not intersect with $P$ until after the edge failure and from the intersection node the replacement path can go to $t$ using the shortest path $P$ (as this subpath does not contain the edge failure $e$). Computing this value naively for every $w \in W$ and $e \in P$ takes $\tilde{O}(|W|n_H^2)$ time.

If $R(s,t,H_w - e)$ is unweighted, the algorithm of 16 can be used to compute its length.

Second case - the failure is in $T$ and the destination is in $T$: We solve this case recursively. The recursive call will be invoked over the subgraph $H[T]$. Because the root of the tree $T$ is $t$ and not $s$, we must change the source of our SSRP. This implies that replacement paths that use the path $P$ to get from $s$ to $t$
will be \(d(s, t, H)\) units shorter in the recursive call than they truly are. So when we compress different forms of replacement paths using weight functions, for normalization reasons we must also subtract \(d(s, t, H)\) from the weight function. For simplicity, we ignore this issue in the overview, but keep in mind that we always need to subtract \(d(s, t, H)\) from every weight function before the algorithm passes them to the recursion call, and add this value back when it receives the recursion’s estimation.

We distinguish between two possible forms of the replacement path: weighted and unweighted. Rather interestingly we will see that this separation provides enough information about the structure of the replacement path in order to compress it, and find its length recursively.

**Case 2.1:** The path \(R(s, x, H_w - e)\) is weighted. We claim that in this case, the only node from \(S\) that \(R(s, x, H_w - e)\) uses is \(s\). To see this note that for every node \(u\) from \(S\) that is not \(s\), the path from \(s\) to \(u\) in the BFS tree \(K\) does not contain the edge failure \(e\), since \(e \in T\). By the weight requirement this path is a shortest path in \(H_w\). Hence, if a weighted replacement path uses a node \(u\) from \(S\), it can use the path from \(s\) to \(u\) in \(K\). In other words, the use of a weighted edge was unnecessary. So in this case the replacement path is almost completely contained within \(H[T]\). Therefore, in order to take care of this case, we simply need to pass the weight function \(w\) to the recursive call. We formally prove the correctness of this case in Claim 5.20.

**Case 2.2:** The path \(R(s, x, H_w - e)\) is unweighted. Let \(u\) be the last node of \(R(s, x, H_w - e)\) that is from \(S\). If \(u\) is \(t\), then \(u\) will be able to obtain the length of the replacement path. We formally prove the correctness of this case in Claim 5.19.

We note that \(c_T\) is truthful, in the sense that for every edge failure \(e \in T\), \(c_T(v)\) is the length of some path from \(s\) to \(v\) in \(H - e\). This is since the path from \(s\) to \(u\) in \(K\) is of length \(d(s, u, H)\) and does not contain \(e\) (as previously claimed), and the edge \((u, v)\) is of length 1 and is not in \(T\) because \(u\) is not in \(T\). We formally prove that \(c_T\) is truthful as part of Claim 5.18.

**Third case - the failure is in \(E(S) - E(P)\) and the destination is in \(S\):** We handle this case similarly to the way we handled the second case. However we still sketch the algorithm for this case as it will introduce the notation of a “help from bellow” replacement path, which will be useful in the fourth case. We solve this case recursively. The recursive call will be invoked over the subgraph \(H[S]\). In order to take care of this case we distinguish between two forms of the replacement path \(R(s, x, H_w - e)\).

**Case 3.1:** The path \(R(s, x, H_w - e)\) uses only nodes from \(S\). In this case simply passing the weight function \(w\) to the recursive call would suffice in order to compute the length of \(R(s, x, H_w - e)\).

**Case 3.2:** \(R(s, x, H_w - e)\) uses a node from \(V(T) - \{t\}\). Let \(u\) be the last node in \(R(s, x, H_w - e)\) that belongs to \(V(T) - \{t\}\). Note that the path from \(s\) to \(u\) in the BFS tree \(K\) uses only edges from \(P\) and \(T\), meaning that it does not use the edge failure \(e \in E(S) - E(P)\). Hence, WLOG we may assume that the sub-path from \(s\) to \(u\) in \(R(s, x, H_w - e)\) is the path from \(s\) to \(u\) in the BFS tree \(K\) as this is a shortest path by the weight requirements. Note that this implies in particular that \(R(s, x, H_w - e)\) is unweighted. An illustration of this case can be found in Figure 2. We name this kind of paths “help from bellow” replacement paths.
Let \( v \) be the node right after \( u \) in \( R(s, x, H_w - e) \). Since \( u \) was the last node in \( R(s, x, H_w - e) \) that belongs to \( V(T) - \{ t \} \) the sub-path of \( R(s, x, H_w - e) \) from \( v \) to \( x \) is fully contained in \( H[S] - e \).

So we only need to compress the sub-path of \( R(s, x, H_w - e) \) from \( s \) to \( v \). In order to do so we define a new weight function \( c_S : V(S) \rightarrow \mathbb{N} \cup \{ \infty \} \) where for every vertex \( v \), \( c_S(v) \) is defined to be \( \min_{w \in V(T) - \{ t \}, (u, v) \in E(H)} \{ d(s, u, H) + 1 \} \). The sub-path of \( R(s, x, H_w - e) \) from \( s \) to \( v \) is represented in the graph \( H[S]_{c_S} \) by the weighted edge \( (s, v) \).

So by passing the weight function \( c_S \) to the recursive call over \( H[S] \), and computing \( d(s, x, H[S]_{c_S} - e) \), we will be able to compute the length of \( R(s, x, H_w - e) \). We formally prove the correctness of this case in Claim 5.22.

We also claim that this function is truthful for edge failures from \( E(S) - E(P) \) in the sense that for every \( e \in E(S) - E(P) \), \( c_S(v) \) is the weight of some path from \( s \) to \( v \) in \( H - e \). This is since the path from \( s \) to \( u \) in \( K \) is of length \( d(s, u, H) \) and does not contain \( e \) (as previously claimed), and the edge \( (u, v) \) is of length 1 and is not in \( S \) because \( u \) is not in \( S \). We formally prove this fact as part of Claim 5.23.

Note that the weight function \( c_S \) is untruthful for edge failures from \( P \), as the path from \( s \) to \( u \) in \( K \) contains the entire path \( P \). But if we consider the recursion’s estimation for \( d(s, x, H[S]_{c_S} - e) \) only for an edge failure \( e \in E(S) - E(P) \), we are promised that this estimation represents the length of a true path in \( H - e \). If we were to add weighted edges instead of weight functions, we would lose the ability to consider \( d(s, x, H[S]_{c_S} - e) \) as an estimation for \( d(s, x, H_w - e) \) only for specific edge failures.

**Fourth case - the failure is in \( P \) and the destination is in \( S \):** This case is the most complicated case in our algorithm. Since we cannot allow three recursive calls (in order to obtain the desired running time) and because we see no efficient way to solve this case in a non-recursive manner, we will need to use the same recursive call over \( H[S] \) as in the previous case (the third case). We will do so by adding more weight functions.

We begin by making two simple observations that take care of some easy cases, so we could focus on the more involved ones.

- If the replacement path \( R(s, x, H_w - e) \) uses no nodes from \( T \) then one can simply use a recursive call over the graph \( H[S] \) to compute its length.

- If \( R(s, x, H_w - e) \) is departing, then since we may assume it contains nodes from \( T \), we can use observations similar to those made in cases (1.1) and (1.2) in order to compute its length.

So we now focus on the more interesting case when \( R(s, x, H_w - e) \) uses nodes from \( T \) and is jumping. Note that since \( R(s, x, H_w - e) \) is jumping it must leave the path \( P \) at some node \( v_i \) before the edge failure and return to \( P \) at some node \( v_j \) after the edge failure.

We will in fact still need to separate this case into 3 further sub-cases, depending on the order \( R(s, x, H_w - e) \) uses nodes from \( T \). These 3 cases present the true power of weight functions, and their ability to compress graphs in a way that is sometimes untruthful but fixable.
Case 4.1: \( R(s, x, H_w - e) \) uses a node from \( T \) after it uses \( v_j \). An illustration for this case can be found in Figure 3.

![Figure 3: Case 4.1: \( R(s, x, H_w - e) \) uses a node from \( T \) after it uses \( v_j \)](image)

We claim that in this case the length of \( R(s, x, H_w - e) \) is \( d(s, x, H[S]_{c_S - e}) - d(s, t, H) + d(s, t, H_w - e) \). While formally proving the correctness of this claim is rather technical we attempt to give some intuition for this claim. Note that since \( R(s, x, H_w - e) \) uses a node from \( T \) after it uses \( v_j \), it passes WLOG through \( t \) (as \( v_j \) is after the edge failure). So we can split \( R(s, x, H_w - e) \) into two sub-paths: the replacement path from \( s \) to \( t \) - which is of length \( d(s, t, H_w - e) \), and the path from \( t \) to \( x \) - which we denote by \( R[t, x] \).

Let us consider the path \( P \circ R[t, x] \). We claim that even though the path \( R[t, x] \) contains nodes from \( T \), the recursive call over \( H[S]_c \) can evaluate the length of the path \( P \circ R[t, x] \). This is because, roughly speaking, the path \( P \circ R[t, x] \) is a sort of "help from below" replacement path - as described in the the third case in which \( e \in E(S) - E(P) \). So like in the "help from below" case, the path \( P \circ R[t, x] \) would be represented in \( H[S]_{c_S - e} \) as a weighted replacement path. When we receive the length of this weighted replacement path we remove \( P \) and replace it with the replacement path from \( s \) to \( t \). That is, we subtract \( d(s, t, H) \) and add \( d(s, t, H_w - e) \). We formally prove the correctness of this estimation in Claim 5.12. As stated in the beginning of the overview, we do not know a-priori if the replacement path indeed falls in this sub-case, so we have to make sure that we never underestimate \( d(s, x, H_w - e) \). We formally prove this in Claim 5.6. In this case we see that weight functions allow us to assign weights that are untruthful for some edge failures, but give us enough control in order to fix the untruthful replacement paths.

Note that \( d(s, x, H[S]_{c_S - e}) \) is used regardless of which weight function \( w \) the true replacement path uses. The fact that we use one recursive call over all weight functions, allows us to compute this term only once, which we could not do if the algorithm would have used a different recursive call for each weight function.

Case 4.2: \( R(s, x, H_w - e) \) is weighted and it uses no nodes from \( T \) after \( v_j \). An illustration for this case can be found in Figure 4.
Let \((s, v)\) be the weighted edge in the replacement path \(R(s, x, H_w - e)\). Note that \((s, v)\) is not in \(P\) (as \(P\) contains only unweighted edges from \(H\)). Hence, by definition the replacement path leaves the path \(P\) at \(s\), that is, \(v_i = s\).

This implies that the sub-path from \(s\) to \(v_j\) is edge disjoint to \(P\) and so its length is \(d(s, v_j, H_w - P)\). So for every weight function \(w \in W\), we would have wished to define a new weight function \(w|\mathcal{S}\) such that \(w|\mathcal{S}(v_j) = d(s, v_j, H_w - P)\) for every \(v_j \in P\). We will then ask the recursive call to estimate \(d(s, x, H[\mathcal{S}|\mathcal{P}] - e)\).

This will indeed suffice in order to compute the length of the replacement path recursively, as \(R(s, x, H_w - e)\) uses no nodes from \(T\) after \(v_j\).

However, by doing so we increase the number of weight function passed to the recursive call by a factor of 2. This sort of exponential growth will prevent us from achieving the desired running time. So instead we define a new weight function \(w|\mathcal{S}\) such that \(w|\mathcal{S}(x) = d(s, x, H_w - P)\) if \(x \in P\) and \(w|\mathcal{S}(x) = w(x)\) if \(x \notin P\). Note that for every \(x\) it holds that \(w|\mathcal{S}(x) \leq w(x)\), since the distance \(d(s, x, H_w - P)\) is at most the weight of the edge \((s, x) \in H_w\) which is \(w(x)\). This implies that the \(w|\mathcal{S}\) function preserves information from both \(w\) and \(w|\mathcal{P}\). So instead of passing \(w\) to the recursive call, we pass \(w|\mathcal{S}\). Later in Claim 5.13 we prove that the new \(w|\mathcal{S}\) function is truthful in the sense that for every \(e \in \mathcal{S}, x \in \mathcal{S}\) it holds that \(d(s, x, H[\mathcal{S}|\mathcal{S} - e]\) is at least \(d(s, x, H_w - e)\), meaning we do not create underestimations by using \(w|\mathcal{S}\) instead of \(w\). In the full version of the algorithm, we prove the correctness of this case in Claim 5.13.

So as one can see, weight functions allow us to specifically choose special nodes and decrease their weights in order to compress more information, without sacrificing the truthfulness of the weight function.

**Case 4.3:** \(R(s, x, H_w - e)\) is unweighted, it uses no nodes from \(T\) after \(v_j\).

This is the most involved and interesting case our algorithm handles. Note that since we assume \(R(s, x, H_w - e)\) uses a node from \(T\), and since \(R(s, x, H_w - e)\) uses no nodes from \(T\) after \(v_j\), then the sub-path of \(R(s, x, H_w - e)\) from \(v_i\) to \(v_j\) must contain a node from \(T\). An illustration for this case can be found in Figure 5.

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**Figure 4:** Case 4.2: \(R(s, x, H_w - e)\) is weighted, it uses no nodes from \(T\) after \(v_j\).  

**Figure 5:** Case 4.3: \(R(s, x, H_w - e)\) is unweighted, it uses a node from \(T\) in the sub-path from \(v_i\) to \(v_j\), and uses no nodes from \(T\) after \(v_j\).
Similarly to Case 1.2, we may assume that the edge failure is not among the last \(\sqrt{n_H}\) edges of \(P\) as otherwise we can use a brute force solution to compute the length of the replacement path. Since the sub-path from \(v_i\) to \(v_j\) uses a node from \(T\), its length is at least \(d(v_i, t, H)\) that is at least \(\sqrt{n_H}\). So w.l.o.g we have sampled some pivot node \(b \in B\) on this sub-path. Note that the sub-path from \(s\) to \(b\) as departing as the replacement path returns to \(P\) only at \(v_j\). So we can easily compute \(d(s, b, H - e)\) as stated before in Case 1.2.

To compress the sub-path from \(b\) to \(v_j\) we define a weight function \(w_0\) for every pivot node. We would have wished to define \(w_0(v_j) = d(b, v_j, H - P)\), recursively compute \(d(s, x, H[S]_{w_0} - e)\) and add \(d(s, b, H - e)\) when receiving the answer from the recursion. This will indeed suffice in order to compress the length of the sub-path from \(v_i\) to \(v_j\) as it is edge disjoint to \(P\). However this is not a valid weight function as it does not necessarily fulfill the weight requirements. So instead we define \(w_0(v_j) = d(s, b, H) + d(b, v_j, H - P)\) which is a valid weight function, and we fix the output of the recursion by replacing \(d(s, b, H)\) with \(d(s, b, H - e)\), i.e. subtracting the former and adding the latter.

As in case 4.1, we need to prove that we never underestimate \(d(s, x, H - e)\). This is formally done in Claim 5.14.

As we can see, while the weight function \(w_0\) is untruthful, in the sense that \(w_0(v_j)\) is not necessarily the distance of a path from \(s\) to \(v_j\) in \(H - e\), we are able to fix this untruthfulness as we know what pivot \(b \in B\) is used in each weight function \(w_0\). In a sense if we could have used a different recursive call for every \(b \in B\) we could have used edges from \(s\) rather than weight functions but this would be very inefficient. The weight functions allow us to compress all these recursive calls into one.

In the full version of the algorithm we denote the estimation made using the \(w_0\) functions by \(\text{Pivot}(s, x, e)\). We show how to compute this estimation in step 9 of the algorithm and prove the correctness of this case in Claim 5.14.

### 3.2 Running Time Analysis

First we note that the number of weight functions in each recursive call increases by \(\tilde{O}(\sqrt{n_H})\) in each level of the recursion, as we add the \(c_S, c_T\) and \(\{u_B\}_{B \in B}\) weight functions, and \(|B| = \tilde{O}(\sqrt{n_H})\). Since the number of the weight functions in the first call to the algorithm is 1 (the function \(w \equiv \infty\)), and since the depth of the recursion is logarithmic we have that \(|W| = \tilde{O}(\sqrt{n})\) at all times. So if we simply analyze the non-recursive parts of the algorithm, we can conclude the algorithm spends \(\tilde{O}(n_H^2\sqrt{n})\) times on the recursive call over the sub-graph \(H\).

One can rather easily see that since the BFS trees in each level of the recursion are edge disjoint sub-trees of the original BFS tree \(\tilde{K}\), the total number of vertices in each level of the recursion is at most \(2n\). We prove this formally in Section 5.14. So the total time the algorithm spends on each level of the recursion is \(\tilde{O}(n^{2.5})\). Since the depth of the recursion is logarithmic the running time of the algorithm is \(\tilde{O}(n^{2.5})\).

### 3.3 Going From \(\tilde{O}(n^{2.5})\) to \(\tilde{O}(m\sqrt{n} + n^2)\)

In this section we sketch the ideas of improving the running time from \(\tilde{O}(n^{2.5})\) to \(\tilde{O}(m\sqrt{n} + n^2)\).

We first note that as the number of weight functions in our algorithm is \(\tilde{O}(\sqrt{n})\) in each recursive call then even outputting \(d(s, x, H_w - e)\) for every triplet \(w \in W, x \in V(H), e \in E(K)\) is impossible (in the desired running time) as there are \(\tilde{O}(n_H^{2.5})\) such triplets. In order to overcome this issue, the algorithm does not output distances to all such triplets but rather each recursive call is given as input a set of queries \(Q\) that is a small subset of all possible triplets (that is \(Q \subseteq E(K) \times V(H) \times W\) and the goal is to output the distances only for the given set of queries. Initially, \(Q\) is set to be \(E(\tilde{K}) \times V(G) \times \{w\}\), where \(w \equiv \infty\), and so its size is \(|Q| = \tilde{O}(n^2)\). Each recursive call over a graph \(H\) will make sure to ask only \(\tilde{O}(n_H^{2.5})\) new queries (queries which it didn’t received). Since the total number of vertices in each level is at most \(2n\), the number of new queries added at each level of the recursion is \(\tilde{O}(n^2)\). Since the depth of the recursion is logarithmic, the total number of queries asked is \(\tilde{O}(n^2)\).

Secondly, recall that in Case 1.3, where the edge failure is in \(E(P)\) and the destination is \(t\), the algorithm computes the value of \(\min_{u\text{ is after }e\text{ in }P} \{d(s,u,H_w - P) + d(u,t,H)\}\) naïvely for every \(e \in E(P), w \in W\). This computation costs \(\tilde{O}(|W|n_H^{2.5})\) time. However for a specific function \(w \in W\), this value can be computed for all \(e \in E(P)\) in \(\tilde{O}(n_H^{2.5})\) time using a simple dynamic programming argument which will be shown in Section 5.2 in the complete algorithm. Hence, we can reduce the running time of this part to \(\tilde{O}(|W|n_H^{2.5})\).
Finally, and most importantly, when handling departing unweighted paths (Case 4.3) the algorithm samples a set $B$ of pivots of size $\tilde{O}(\sqrt{mH})$. Then for every edge failure $e \in P$ and destination node $x \in V(H)$ we iterate over $B$ and find the pivot that provides the smallest distance estimation. This implies that the algorithm spends $\tilde{O}(|B||P||V(H)|)$ time to find these pivots, which is again $\tilde{O}(\sqrt{mH})$. The problem is that our estimation for the distance between an edge failure and the separator node $t$ is too loose. On the one hand when sampling $B$ we say that this distance is at least $\sqrt{mH}$, but on the other hand when bounding $|P|$ we say that it is at most $O(nH)$.

In order to solve this problem we use a standard scaling trick. More specifically, we consider a logarithmic number of sub-paths $\{P_k\}$, where $P_k$ is the sub-path of $P$ induced by the vertices $\{v \in V(P) : 2^{k+1}\sqrt{mH} \geq d(v, t, H)\}$. $P_k$ is defined to be the sub-path of $P$ induced by the last $2\sqrt{mH}$ vertices of $P$. Note that the set of paths $\{P_k\}$ is an edge disjoint partition of $P$, and that $|P_k| = O(2^k\sqrt{mH})$. An illustration for this partition can be seen in Figure 7 in the appendix. For every index $k \neq 0$ we then sample a random set $B_k$ of size $\tilde{O}(\sqrt{mH})$ using the sampling lemma (2). Now, if we consider an edge failure $e \in P_k$ for $k \neq 0$, we know that the distance from $e$ to $t$ is at least $2^k\sqrt{mH}$. So when we wish to estimate the length of the departing replacement path in Case 1.2 or send the query $(e, x, w_b)$ to the recursive call over $H[V]$ in Case 4.3, we only need consider pivot nodes $b$ that are from $B_k$.

4 An $\tilde{O}(m\sqrt{n} + n^2)$ Algorithm for SSRP in Unweighted Directed Graphs

In this section we describe in details our $\tilde{O}(m\sqrt{n} + n^2)$ time algorithm for SSRP in unweighted directed graphs. Our algorithm is recursive and uses a balance tree separator (formally stated in Lemma 2) in order to divide its input into two smaller inputs. In our algorithm we utilize the following LCA data structure presented by Bender and Farach-Colton in [2].

Lemma 4.1 (LCA Data Structure [2]). Given a rooted tree $T$ containing $n$ vertices, one can construct an LCA data structure in linear $O(n)$ time and answer LCA queries in constant time.

We now describe the algorithm whose input is a sub-graph $H$ of the original graph $G$, a BFS tree $K$ over $H$ rooted at a node $s$, a set of weight functions $W$ and a set of queries $Q$. The goal of the algorithm is to create an estimation $\tilde{d}(s, x, H_w - e)$ for every query $(e, x, w) \in Q$, such that the estimation matches the real distance $d(s, x, H_w - e)$. We denote by $n_H = |V(H)|$, $m_H = |E(H)|$ and $n = |V(G)|$, $m = |E(G)|$. The pseudo code code for the algorithm can be found in Algorithm 1.

Step 1: Base case:
If $n_H \leq 6$ the algorithm constructs the graph $H_w - e$ for every weight function $w \in W$ and edge failure $e \in E(K)$. The algorithm then runs Dijkstra’s algorithm from $s$ in the graph $H_w - e$ to compute $d(s, x, H_w - e)$ for every $e \in E(K), x \in V(H), w \in W$. The algorithm then returns $d(s, x, H_w - e)$ for every $(e, x, w) \in Q$.

Step 2: Tree Separation $(S, T)$
In this step, the algorithm finds (using Lemma 2) a balanced tree separator node $t \in V(H)$ that separates the BFS tree $K$ into two edge disjoint sub-trees $S, T$, such that $E(S) \cup E(T) = E(K)$ and $|V(S)|, |V(T)| \leq \frac{2m_H}{n_H}$. An illustration of the separation can be seen in Figure 8. Let $P$ denote the path from $s$ to $t$ in the BFS tree $K$. The algorithm next computes the distances $d(t, x, H), d(x, t, H)$ for every node $x \in V(H)$ by running the BFS algorithm from the node $t$ in the graphs $H, H^R$.

Step 3: Computing $d(s, x, H_w - P)$
The algorithm computes $d(s, x, H_w - P)$ for every node $x \in V(H)$ and weight function $w \in W$, by constructing the graph $H_w - P$ for every weight function $w \in W$, and running Dijkstra’s algorithm from $s$ in the resulted graph.

Step 4: Sampling Pivots ($B_k$) and Defining Path Intervals ($P_k$)
For $k \in \{\log(n_H)\}$ the algorithm constructs a set $B_k$ by sampling every vertex in $V(H)$ independently at random with probability $\frac{C\ln(n)}{2^{k+1}\sqrt{mH}}$ (where $C \geq 3$ is a constant that will be fixed later on). If the algorithm sampled too
many vertices and it does not hold that $|B_k| = \tilde{O}(\sqrt{n/\log(n)})$, the algorithm re-samples $B_k$. The algorithm sets $\mathcal{B} = \bigcup_{k \in \llbracket\log(n_H)\rrbracket} B_k$ For every pivot $\forall b \in \mathcal{B}$ the algorithm then runs the BFS algorithm from $b$ in the graph $H - P$ and in the graph $(H - P)^B$.

Also, $\forall k \in \llbracket\log(n_H)\rrbracket$ the algorithm sets the path $P_k$ to be the sub-path of $P$ induced by the vertices $\{v \in V(P) : 2^{k+1}\lfloor\sqrt{n_H}/2\rfloor \geq d(v,t,H) \geq 2^k\lfloor\sqrt{n_H}/2\rfloor\}$. The algorithm then sets $P_0$ to be the sub-path of $P$ induced by the vertices, $\{v \in V(P) : 2\lfloor\sqrt{n_H}/2\rfloor \geq d(v,t,H)\}$.

An illustration of this separation can be seen in Figure 7, note that all the edges and nodes in $P_k$ are after all the edges and nodes in $P_{k+1}$ and that the set of subpaths $\{P_k\}_{k \in \llbracket\log(n_H)\rrbracket \cup \{0\}}$ is an edge disjoint partition of $P$.

Step 5: Computing Departing Paths (Depart($s, x, e$))

$\forall k \in \llbracket\log(n_H)\rrbracket, \forall b \in \mathcal{B}_k, \forall e \in E(P_k)$ the algorithm na"ively computes $\text{Depart}(s, b, e) = \min_{u \in V(P)} \{d(s, u, H) + d(u, b, H - P)\}$. Afterwards, $\forall k \in \llbracket\log(n_H)\rrbracket, \forall e \in E(P_k), \forall x \in V(H) - B_k$, the algorithm na"ively computes $\text{Depart}(s, x, e) = \min_{i \in \mathcal{B}_k} \{\text{Depart}(s, b, e) + d(b, x, H - P)\}$. Also, $\forall e \in P_0, \forall x \in V(H)$ the algorithm sets $\text{Depart}(s, x, e) = d(s, x, H - e)$, where $d(s, x, H - e)$ is computed by running the BFS algorithm from $s$ in the graph $H - e$ for every $e \in P_0$.

Step 6: Computing $\hat{d}(s, t, H_w - e)$ When $e \in E(P)$

The algorithm runs the replacement path algorithm from $[10]$, for the unweighted directed graph $H$ and the path $P$ (from $s$ to $t$), to find (w.h.p) a replacement path for every edge failure $e \in E(P)$. Let $\hat{d}_{RZ}(s, t, H - e)$ be the length returned by the algorithm of $[10]$ for the edge failure $e \in E(P)$. For every $w \in W$ and for every $e \in E(P)$ the algorithm computes $A_w[e] \overset{def}{=} \min_{u \text{ after } e \text{ in } P} \{d(s, u, H_w - P) + d(u, t, H)\}$.

For efficiency reasons, the algorithm uses the following dynamic programming to compute the values $A_w[e]$ for $e \in E$. Let $u_0, u_1, ..., u_0$ be the path from $s$ to $t$, that is $u_0 = s$ and $u_0 = t$ and let $e_i = (u_{i+1}, u_i)$. For every $w \in W$ the algorithm sets $A_w[e_0] \leftarrow d(s, t, H_w - P)$ and then for every $1 \leq i \leq |P| - 1$ (in increasing order) the algorithm sets $A_w[e_i] \leftarrow \min\{A_w[e_{i-1}], d(s, u_{i-1}, H_w - P) + d(u_{i-1}, t, H)\}$.

$\forall w \in W, \forall e \in E(P)$, the algorithm sets

$$\hat{d}(s, t, H_w - e) = \min\{\hat{d}_{RZ}(s, t, H - e), A_w[e]\}$$

Step 7: Computing $\hat{d}(s, x, H_w - e)$ When $e \in E(P), x \in V(T) - \{t\}$

$\forall (e, x, w) \in Q : e \in E(P), x \in V(T) - \{t\}, w \in W$ the algorithm sets

$$\hat{d}(s, x, H_w - e) = \min\{d(s, x, H_w - P), \ \hat{d}(s, t, H_w - e) + d(t, x, H), \text{Depart}(s, x, e)\}$$

Step 8: Computing $\hat{d}(s, x, H_w - e)$ When $e \in E(T), x \in V(T) - \{t\}$

Defining the Recursive Input:

$\forall w \in W$ the algorithm defines a new weight function $w|_T : V(T) \rightarrow \mathbb{N} \cup \{\infty\}$ as follows $\forall v \in V(T) : w|_T(v) = w(v) - |P|$. The algorithm also defines a new weight function $c_T : V(T) \rightarrow \mathbb{N} \cup \{\infty\}$ as follows $\forall v \in V(T) : c_T(v) = \min_{u \in V(S), u \neq t, (u,v) \in E(H)} \{d(s, u, H) + 1\} - |P|$.

The algorithm sets the new set of weight functions to be $W_T = \{w|_T : w \in W\} \cup \{c_T\}$. The algorithm sets the new query set $Q_T$ to be as follows: $\forall (e, x, w) \in Q : e \in E(T), x \in V(T), w \in W$ the algorithm adds the query $(e, x, w|_T)$ to $Q_T$. Also $\forall e \in E(T), x \in V(T)$ the algorithm adds to $Q_T$ the query $(e, x, c_T)$.

The algorithm invokes recursively on the following input: the induced graph $H[T]$, the BFS tree $T$, the set of weight functions $W_T$ and the query set $Q_T$. Let $\hat{d}(s, x, H[T] - e)$ be the output values the recursive call returns.
Computing the Results
\(\forall (e, x, w) \in Q : e \in E(T), x \in V(T) - \{t\}, w \in W\) the algorithm sets
\[
\hat{d}(s, x, H_w - e) = \min \{ \hat{d}(t, x, H[T]_{\partial T} - e) + |P|, \hat{d}(t, x, H[T]_{\partial T} - e) + |P| \}
\]

Step 9: Computing \(\hat{d}(s, x, H_w - e)\) When \(e \in E(S), x \in V(S) - \{t\}\)

Defining the Recursive Input
\(\forall w \in W\) the algorithm defines \(w|_S : V(S) \rightarrow \mathbb{N} \cup \{\infty\}\) as follows: \(\forall v \in V(P) : w|_S(v) = d(s, v, H_w - P)\) and \(\forall v \in V(S) - V(P) : w|_S(v) = w(v)\). The algorithm then defines the new weight function \(c_S : V(S) \rightarrow \mathbb{N} \cup \{\infty\}\) as follows: \(\forall v \in V(S) : c_S(v) = \min_{u \in V(T): u \neq v} \{d(s, u, H) + 1\}\). \(\forall k \in [\log(n_H)]\), \(\forall b \in B_k\) the algorithm defines a weight function \(w_b : V(S) \rightarrow \mathbb{N} \cup \{\infty\}\) as follows: \(\forall v \in V(S) : w_b(v) = d(s, b, H) + \max(b, v, H - P)\). The algorithm sets the new set of weight functions \(W_S\) to be \(\{w|_S : w \in W\} \cup \{c_S\} \cup \{w_b : b \in B_k\}\) for every query \((e, x, w)\). Note that if \(e, x, w\) is not on the shortest path \((e, x, w)e\) is not on the shortest path \((e, x, w)e\) in the BFS tree \(S\), the set of weight functions \(W_S\) and the query set \(Q_S\). Let \(\hat{d}(s, x, H[S]_w - e)\) be the output values it received from the recursive call.

Computing Pivot\((s, x, e)\)
\(\forall k \in [\log(n_H)]\), \(\forall e \in E(P_k)\), \(\forall x \in V(S)\) the algorithm computes \(\text{Pivot}(s, x, e) = \min_{b \in B_k} \{d(s, x, H[S]_{w_b} - e) - d(s, b, H) + \text{Depart}(s, b, e)\}\).

Then \(\forall e \in E(P_k)\), \(\forall x \in V(S)\) the algorithm specially sets \(\text{Pivot}(s, x, e) = \text{Depart}(s, x, e) = d(s, x, H - e)\).

Computing the Results
\(\forall (e, x, w) \in Q : e \in E(S) - E(P), x \in V(S) - \{t\}, w \in W\) the algorithm sets
\[
\hat{d}(s, x, H_w - e) = \min \{ \hat{d}(s, x, H[S]_{w|_S} - e) - d(s, b, H) + \text{Depart}(s, b, e), \hat{d}(s, x, H[S]_{w|_S} - e) - d(s, b, H) + \text{Depart}(s, b, e) \}
\]

\(\forall (e, x, w) \in Q : e \in E(P), x \in V(S) - \{t\}, w \in W\) the algorithm sets
\[
\hat{d}(s, x, H_w - e) = \min \{ \hat{d}(s, x, H[S]_{w|_S} - e) - d(s, t, H), \hat{d}(s, x, H[S]_{w|_S} - e) - d(s, t, H) \}
\]

Step 10: Outputting the results
For every query \((e, x, w) \in Q\), the algorithm checks if \(e\) is on the shortest path from \(s\) to \(x\) in \(K\) (this can be done easily by computing an LCA data structure (see e.g. [1,1]) on the BFS tree \(K\)). If \(e\) is not on the shortest path from \(s\) to \(x\) in \(K\) then the algorithm simply returns \(d(s, x, H)\). Otherwise the algorithm returns \(\hat{d}(s, x, H_w - e)\).

Note that if \(e \in E(T), x \in V(S)\), then \(e\) is not on the path from \(s\) to \(x\) in \(K\) since this path is contained in the BFS tree \(S\). Similarly if \(e \in E(S) - E(P)\) and \(x \in V(T)\) then \(e\) is not on the path from \(s\) to \(x\) in \(K\) since this path uses only edges from \(E(P) \cup E(T)\). For intuition see Figure[6]. So we ensured that algorithm has computed all the \(\hat{d}\) values it needs to.

Algorithm 1 The SSRP algorithm

1: function GENERALIZED SSRP\((H, K, W, Q)\)
2: if \(|V(H)| \leq 6\) then \(\triangleright\) Naively computing the SSRP for every edge failure and weight function
3: for \(e \in E(K)\) do
4: for \(w \in W\) do

14
\[
\hat{d}(s, o, H_w - e) \leftarrow \text{Dijkstra}(s, H_w - e)
\]

return \( \hat{d} \)

\[ S, T \leftarrow \text{Balanced Separation}(K) \]

\[ \triangleright \text{See Lemma 2.2} \]

\[ \triangleright \text{Recall That } s \text{ is the root of } S \text{ and } K, t \text{ is the root of } T \text{ and } P \text{ is the path from } s \text{ to } t \text{ in } K \]

\[ d(t, o, H) \leftarrow \text{Dijkstra}(t, H) \]

\[ d(o, t, H) \leftarrow \text{Dijkstra}(t, H^R) \]

for \( w \in W \) do

\[ d(s, o, H_w - P) \leftarrow \text{Dijkstra}(s, H_w - P) \]

for \( k := 1 \) to \( \lfloor \log(n_H) \rfloor \) do

\[ B_k \leftarrow \text{Sample}\left(V(H), \frac{C \ln n}{2^k \sqrt{\ln ||V(H)||}}\right) \]

for \( b \in B_k \) do

\[ d(b, o, H - P) \leftarrow \text{Dijkstra}(b, H - P) \]

\[ d(o, b, H - P) \leftarrow \text{Dijkstra}(b, (H - P)^R) \]

\[ \text{Computation of Depart}(s, x, e) \]

\[ \text{for } k := 1 \text{ to } \lfloor \log(n_H) \rfloor \text{ do } \]

\[ \text{for } e \in P_k \text{ do } \]

\[ \text{for } b \in B_k \text{ do } \]

\[ \text{Depart}(s, b, e) \leftarrow \min_u \{d(s, u, H) + d(u, b, H - P)\} \]

\[ \text{for } x \in V(H) - \{B_k\} \text{ do } \]

\[ \text{Depart}(s, x, e) \leftarrow \min_{b \in B_k} \{\text{Depart}(s, b, e) + d(b, x, H - P)\} \]

\[ \text{for } e \in P_0 \text{ do } \]

\[ \text{Depart}(s, o, e) \leftarrow \text{Dijkstra}(s, H - e) \]

The case when the failure is in \( P \) and the destination is \( t \)

\[ \langle u_0, u_1, \ldots, u_{|P|} \rangle \leftarrow P^R \]

\[ \text{for } i := 0 \text{ to } |P| - 1 \text{ do } \]

\[ e_i \leftarrow (u_{i+1}, u_i) \]

for \( w \in W \) do

\[ A_w[e_i] \leftarrow d(s, t, H_w - P) \]

for \( i := 1 \) to \( |P| - 1 \) do

\[ A_w[e_i] \leftarrow \min\{A_w[e_{i-1}], d(s, u_i, H_w - P) + d(u_i, t, H)\} \]

\[ \hat{d}_{RZ} \leftarrow \text{RP}(H, P) \]

\[ \text{for } e \in P, w \in W \text{ do } \]

\[ \hat{d}(s, t, H_w - e) \leftarrow \min\{\hat{d}_{RZ}(s, t, H - e), A_w[e]\} \]

The case when the failure is in \( P \) and the destination is in \( V(T) - \{t\} \)

for \( (e, x, w) \in Q \) do

if \( e \in P \) and \( x \in V(T) - \{t\} \) then

\[ \hat{d}(s, x, H_w - e) \leftarrow \min\{\hat{d}(s, t, H_w - e) + d(t, x, H), d(s, x, H_w - P), \text{Depart}(s, x, e)\} \]

\[ \text{RP is the algorithm from } \text{[16]} \]

\[ \triangleright \text{Note that } u_0 = t \text{ and } u_{|P|} = s \]
The case when the failure is in $T$ and the destination is in $V(T) - \{t\}$

$W_T \leftarrow \emptyset$  \hspace{1cm} \text{\textarrow{ DEFINING THE RESTRICTED WEIGHT FUNCTIONS}}

\begin{algorithmic}
  \State \textbf{for} $w \in W$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{Let } w|_T : V(T) \rightarrow \mathbb{N} \cup \{\infty\} \text{ be a new function}  \\
  \hspace{2cm} \textbf{for} $v \in V(T)$ \textbf{do} \hspace{1cm}  \\
  \hspace{3cm} $w|_T(v) \leftarrow w(v) - d(s, t, H)$  \\
  \hspace{2cm} $W_T \leftarrow W_T \cup \{w|_T\}  \\
  \hspace{1cm} \text{\textarrow{ DEFINING THE "HELP FROM ABOVE" WEIGHT FUNCTION}}

\end{algorithmic}

\begin{algorithmic}
  \State $Q_T \leftarrow E(T) \times V(T) \times \{c_T\}$  \\
  \State $\text{for} \ (e, x, w) \in Q$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{if } e \in V(T) \text{ and } x \in V(T) - \{t\} \text{ then}  \\
  \hspace{3cm} $Q_T \leftarrow Q_T \cup \{(e, x, w|_T)\}$  \\
  \hspace{1cm} \text{\textarrow{ DEFINING THE NEW SET OF QUERIES}}

\end{algorithmic}

\begin{algorithmic}
  \State $\tilde{d}(v, o, o) \leftarrow \text{GENERALIZED SSRP}(H[T], T, W_T, Q_T)$  \\
  \State $\text{for} \ (e, x, w) \in Q$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{if } e \in T \text{ and } x \in V(T) - \{t\} \text{ then}  \\
  \hspace{3cm} $\tilde{d}(s, x, H_w - e) \leftarrow \min \{\tilde{d}(t, x, H[T]_{w|_T} - e) + d(s, t, H), \tilde{d}(t, x, H[T]_{c_T} - e) + d(s, t, H)\}$  \\

\end{algorithmic}

The case when the failure is in $S$ and the destination is in $V(S) - \{t\}$

$W_S \leftarrow \emptyset$  \hspace{1cm} \text{\textarrow{ DEFINING THE RESTRICTED WEIGHT FUNCTIONS}}

\begin{algorithmic}
  \State $W_S \leftarrow \emptyset$ \hspace{1cm}  \\
  \State $\text{for} \ w \in W$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{Let } w|_S : V(S) \rightarrow \mathbb{N} \cup \{\infty\} \text{ be a new function}  \\
  \hspace{2cm} \textbf{for} \ v \in V(S) \textbf{do} \hspace{1cm}  \\
  \hspace{3cm} \text{if } v \in P \text{ then}  \\
  \hspace{4cm} $w|_S(v) \leftarrow d(s, v, H_w - P)$  \\
  \hspace{3cm} \text{else}  \\
  \hspace{4cm} $w|_S(v) \leftarrow w(v)$  \\
  \hspace{2cm} $W_S \leftarrow W_S \cup \{w|_S\}$  \\
  \hspace{1cm} \text{\textarrow{ DEFINING THE "HELP FROM BELOW" WEIGHT FUNCTION}}

\end{algorithmic}

\begin{algorithmic}
  \State $\text{for} \ v \in V(S)$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{Let } c_S : V(S) \rightarrow \mathbb{N} \cup \{\infty\} \text{ be a new function}  \\
  \hspace{2cm} $c_S(v) \leftarrow \min \{d(s, u, H) + 1\}$  \\
  \hspace{2cm} $W_S \leftarrow W_S \cup \{c_S\}$  \\
  \hspace{1cm} \text{\textarrow{ DEFINING THE NEW SET OF QUERIES}}

\end{algorithmic}

\begin{algorithmic}
  \State $Q_S \leftarrow E(S) \times V(S) \times \{c_S\}$  \\
  \State $\text{for} \ (e, x, w) \in Q$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \text{if } e \in V(S) \text{ and } x \in V(S) - \{t\} \text{ then}  \\
  \hspace{3cm} $Q_S \leftarrow Q_S \cup \{(e, x, w|_S)\}$  \\
  \hspace{1cm} \text{\textarrow{ DEFINING THE PIVOTS WEIGHT FUNCTIONS}}

\end{algorithmic}

\begin{algorithmic}
  \State $k := 1 \hspace{1cm} \text{to} \hspace{1cm} \lfloor \log(n_H) \rfloor$ \textbf{do} \hspace{1cm}  \\
  \hspace{2cm} \textbf{for} \ b \in B_k \textbf{do} \hspace{1cm}  \\
  \hspace{3cm} \text{Let } w_b : V(S) \rightarrow \mathbb{N} \cup \{\infty\} \text{ be a new function}  \\
  \hspace{3cm} \textbf{for} \ v \in V(S) \textbf{do} \hspace{1cm}  \\
  \hspace{4cm} $w_b(v) \leftarrow d(s, b, H) + d(b, v, H - P)$  \\
  \hspace{3cm} $W_S \leftarrow W_S \cup \{w_b\}$  \\

\end{algorithmic}
\[ Q_S \leftarrow Q_S \cup E(P_k) \times V(S) \times \{w_b\} \]  
\[ \hat{d}(o, o, o) \leftarrow \text{GENERALIZED SSRP}(H[S], S, W_S, Q_S) \]

\[ \text{for } k := 1 \text{ to } \lfloor \log(n_H) \rfloor \text{ do} \]
\[ \text{for } e \in E(P_k) \text{ do} \]
\[ \text{for } x \in V(S) \text{ do} \]
\[ \text{Pivot}(s, x, e) = \min_{b \in B_k} \{ \hat{d}(s, x, H[S]_{w_k} - e) - d(s, b, H) + \text{Depart}(s, b, e) \} \]

\[ \text{for } e \in P_0 \text{ do} \]
\[ \text{for } x \in V(S) \text{ do} \]
\[ \text{Pivot}(s, x, e) = \text{Depart}(s, x, e) \]

\[ \text{for } (e, x, w) \in Q \text{ do} \]
\[ \text{if } e \in E(S) - E(P) \text{ and } x \in V(S) - \{t\} \text{ then} \]
\[ \hat{d}(s, x, H[w] - e) \leftarrow \min\{\hat{d}(s, x, H[S]_{w} - e), \hat{d}(s, x, H[S]_{cS} - e)\} \]

\[ \text{else if } e \in P \text{ and } x \in V(S) - \{t\} \text{ then} \]
\[ \hat{d}(s, x, H[w] - e) \leftarrow \min\{\hat{d}(s, x, H[S]_{w} - e), d(s, x, H[w] - P), \text{Depart}(s, x, e), \hat{d}(s, x, e), \hat{d}(s, x, H[S]_{cS} - e) + \hat{d}(s, t, H[w] - e) - d(s, t, H)\} \]

\[ \text{Handling the trivial case and outputting the results} \]

\[ \text{Construct an LCA data structure for the tree } K \]
\[ \text{for } (e, x, w) \in Q \text{ do} \]
\[ \text{if } e \text{ is not on the path from } s \text{ to } x \text{ in } K \text{ then} \]
\[ \hat{d}(s, x, H[w] - e) \leftarrow d(s, x, H) \]
\[ \text{return } \hat{d} \]

5 Proof of correctness

We now prove the correctness of our algorithm. To do so we define the following properties for a call to our algorithm.

**Definition 5.1** (Complete and Sound calls). We say that a call to our algorithm is complete if for every query \((e, x, w) \in Q\) the output of the algorithm to the query is at least \(d(s, x, H[w] - e)\). Similarly we say that the call is sound if the answer for every query \((e, x, w) \in Q\) is at most \(d(s, x, H[w] - e)\).

We will prove the correctness of our algorithm by showing that it is complete and that w.h.p it is also sound.

The proof of both the soundness and completeness will be done by induction on the height of the recursive calls. The base case of the induction is a call to the algorithm that makes no recursive calls, this happens if and only if \(n_H\) is at most 6. In this case the completeness and soundness of our algorithm holds trivially as the algorithm computes all of the distances naively. In the induction step, we assume that both recursive calls the algorithm makes (in steps \(8\) and \(9\) of the algorithm or lines \(75\) and \(110\) in the pseudo code) are complete and sound, and show that the current call is also complete and sound, as stated in claims \(5.25, 5.26\). The proof of these claims will be rather involved and is comprised of an extensive case analysis that depends on the relation between \(e, x\) (the edge failure and destination node) and \(S, T\) (the tree separation). The different cases are presented in Sections \(5.2, 5.3, 5.4, 5.5,\) and \(5.6\).

For the proof of soundness, we define the following property of paths.
Definition 5.2 (K-simple path). Let $e \in E(K), x \in V(H), w \in W$ and let $R = (s = v_1, v_2, ..., v_t = x)$ be a path in the graph $H_w - e$. $R$ will be called K-simple if for every $i \in [t]$, the following holds:

If the path from $s$ to $v_i$ in the BFS tree $K$ does not contain $e$, then the path $(v_1, v_2, ..., v_i)$ is the path from $s$ to $v_i$ in the BFS tree $K$.

Note that by the weight requirement, every path from $s$ to $x$ in the graph $H_w - e$ can be easily transformed into a K-simple path from $s$ to $x$ in the graph $H_w - e$ without increasing its length. And so, we can assume (WLOG) that for every $e \in E(K), x \in V(H), w \in W$, the path $R(s, x, H_w - e)$ is K-simple.

We remind the reader of the definition of jumping and departing replacement paths, which is defined only for edge failures from $P$, and was first defined in the overview. For an edge failure $e \in P$ and a destination node $x \in H$ the path $R(s, x, H_w - e)$ will be called jumping if it uses some node $u$ such that $u \in P$ and $u$ is after the edge failure $e$ in the path $P$. A path which is not jumping will be called departing.

5.1 The Depart($s, x, e$) value

The Depart($s, o, o$) values are used by the algorithm in several cases, so we now state some auxiliary claims regarding these values that will be used later on. The proofs of these claims are deferred to the appendix.

Claim 5.1 (Proof of Completeness). Let $e \in P$ be an edge failure and let $x \in V$ be a destination, then Depart($s, x, e$) $\geq d(s, x, H - e)$.

Claim 5.2 (Proof of Soundness). Let $e \in E(P), x \in V(H), w \in W$, assume $R(s, x, H_w - e)$ is departing and unweighted, and assume that $R(s, x, H_w - e)$ contains some node $v_t \in V(T) - \{t\}$. Then w.h.p the length of $R(s, x, H_w - e)$ is at least Depart($s, x, e$).

5.2 The case when $e \in P$ and $x = t$

The proof of correctness of this case can be found in the appendix. We state here the specific claims proved in the appendix as they will be used in later cases.

Claim 5.3 (Proof of Completeness). Let $e \in P$ be an edge failure and $w \in W$ a weight function, then $d(s, t, H_w - e) \geq d(s, t, H_w - e)$.

Claim 5.4 (Proof of Soundness). Let $e \in E(P), w \in W$, w.h.p $d(s, t, H_w - e) \geq d(s, t, H_w - e)$.

5.3 The case when $e \in P$ and $x \in V(S) - \{t\}$

We now consider the most involved case, in which the edge failure is from $P$ and the destination node is from $V(S) - \{t\}$. Recall that in this case the algorithm sets $d(s, x, H_w - e)$ to be the minimum between: $d(s, x, H[S]_{w|S} - e), d(s, x, H_w - P)$, Depart($s, x, e$), Pivot($s, x, e$) and $d(e, s, H[S]_{w|S} - x) + d(s, t, H_w - e) - d(s, t, H)$, in step 3 of the algorithm (lines 125-126 of the pseudocode).

Proof of Completeness

We begin by showing that the weight functions $w|S$ are truthful for edge failures from both $P$ and $E(S) - E(P)$. More formally we prove the following claim:

Claim 5.5. Let $w \in W, e \in E(S), x \in V(S) - \{t\}$, then $d(s, x, H[S]_{w|S} - e) \geq d(s, x, H_w - e)$.

Proof. Let us denote the shortest path from $s$ to $x$ in the graph $H[S]_{w|S} - e$ by $R$. Note that if $R \subseteq H[S] - e$ then we have that $R$ is a path from $s$ to $x$ in the graph $H - e \subseteq H_w - e$. This implies that $d(s, x, H_w - e) \leq d(R) = d(s, x, H[S]_{w|S} - e)$ as required. So we may assume that $R$ uses some weighted edge $(s, v) \in$
If $v \notin P$ then $w_S(s, v) = w(v)$, and so we can simply replace the edge $(s, v)$ with the edge $(s, v) \in E(H_w) - E(H)$. If $v \in P$ then $w_S(s, v) = d(s, v, H_w - P)$. If $e \in P$ we can simply replace the edge $(s, v)$ with shortest path from $s$ to $v$ in the graph $H_w - P$. Note that we can do this since $e \in P$ and so $H_w - P$ does not contain $e$.

If $e \notin P$ we can replace the edge $(s, v)$ with the subpath of $P$ from $s$ to $v$. Since $P$ is a shortest path in $H$ the length of this path is $d(s, v, H) = d(s, v, H_w) \leq d(s, v, H_w - P) = w_S(s, v)$, where the first equality holds by the weight requirements.

Let $\overline{R}$ denote the path resulted after one of these replacements. The path $\overline{R}$ is a path from $s$ to $x$ in the graph $H_w - e$, this implies that $d(s, x, H_w - e) \leq d(\overline{R})$. Since we have shown that in all of the cases $d(\overline{R}) \leq d(R)$, and since $d(R) = d(s, x, H[S]_{w|[S]} - e)$, the claim holds.

Claim 5.6. Let $e \in E(P), x \in V(S) - \{t\}, w \in W$. Assuming that the recursive call over $H[S]$ is complete, then $d(s, x, H[S]_{c_S} - e) + d(s, t, H_w - e) - d(s, t, H) \geq d(s, x, H_w - e)$.\hfill\qed

Proof. A crucial observation is that $d(s, t, H_w - e) \geq d(s, t, H)$. To see this note that $d(s, t, H_w - e) \geq d(s, t, H_w) = d(s, t, H)$, where the first inequality holds by Claim 5.5 and the last equality holds by the weight requirements.

Recall that $d(s, x, H[S]_{c_S} - e)$ is the result obtained by the recursive call of the algorithm over the sub-graph $H[S]$ and the query $(e, x, c_S)$. Let $R$ be the shortest path from $s$ to $x$ in the graph $H[S]_{c_S} - e$. By the assumption that the recursive call over $H[S]$ is complete, we have that $d(R) \leq d(s, x, H[S]_{c_S} - e)$. If $R$ uses no edges from $E(H[S]_{c_S}) - E(H[S])$, then $R \subseteq H[S] - e \subseteq H - e \subseteq H_w - e$. So $R$ is a path from $s$ to $x$ in the graph $H_w - e$, meaning that $d(s, x, H_w - e) - d(s, t, H) \geq d(R)$. Since we have shown that $d(s, t, H_w - e) \geq d(s, t, H)$ we get that $d(s, x, H_w - e) \leq d(R) + d(s, t, H_w - e) - d(s, t, H) \leq d(s, x, H[S]_{c_S} - e) + d(s, t, H_w - e) - d(s, t, H)$ which implies the claim.

So we may assume that $R$ uses some edge $(s, v) \in E(H[S]_{c_S} - e) - E(H[S])$. As in the proof of Claim 5.5 we wish to slightly fix $R$ by replacing $(s, v)$ with some path from $s$ to $v$ in the graph $H_w - e$. Recall that the weight of the edge $(s, v)$ is $c_S(v)$. By definition of $c_S$ we have that $c_S(v) = d(s, u, H) + 1$ for some $u \in V(T)$ such that $u \neq t$ and $(u, v) \in E$. Since $u \in T$ we can denote by $R(t, u, H)$ the path from $t$ to $u$ in the BFS tree $T$. Since $T$ is a BFS tree the length of $R(t, u, H)$ is $d(t, u, H)$. Note that since $e \in P \subseteq S$ we have that $e \notin R(t, u, H)$. Also, since $u \in V(T), u \neq t$ we have that $u \notin S$ and so $(u, v) \notin S$, and so $(u, v) \neq e$. Let $R(s, t, H_w - e)$ denote the shortest path from $s$ to $t$ in the graph $H_w - e$. Note that by Claim 5.3 we have that $d(R(s, t, H_w - e)) \leq d(s, t, H_w - e)$. Overall we have that $\overline{R} = R(s, t, H_w - e) \circ R(t, u, H) \circ (u, v)$ is a path from $s$ to $v$ in the graph $H_w - e$ of length at most $d(s, t, H_w - e) + d(t, u, H) + 1$.

Note that since $u \in V(T)$, we have that $s$ is an ancestor of $t$ which is an ancestor of $u$ in the BFS tree $K$, which implies that $d(s, u, H) = d(s, t, H) + d(u, t, H)$. This implies that $d(t, u, H) + 1 = d(s, u, H) - d(s, t, H) + 1 = c_S(v) + 1$. So we have that $d(R) \leq c_S(v) + d(s, t, H_w - e) - d(s, t, H)$. So if we replace the edge $(s, v)$ with the path $\overline{R}$ we get a path from $s$ to $x$ in the graph $H_w - e$ of length at most $d(R) + d(s, t, H_w - e) - d(s, t, H)$ (since we increased the length of $R$ by $d(s, t, H_w - e) - d(s, t, H))$. This implies that $d(s, x, H_w - e) \leq d(R) + d(s, t, H_w - e) - d(s, t, H)$.

Claim 5.7. Let $e \in E(P), x \in V(S) - \{t\}$. Assuming that the recursive call over $H[S]$ is complete, then $\text{Pivot}(s, b, e) \geq d(s, x, H - e)$.\hfill\qed

Proof. Let $k \in [\log(n_H)] \cup \{0\}$ be the unique integer such that $e \in P_k$. If $k = 0$ we have that $\text{Pivot}(s, x, e) = \text{Depart}(s, x, e) = d(s, x, H - e)$ which implies the claim. Otherwise $k \neq 0$, and so $\text{Pivot}(s, x, e) = d(s, x, H[S]_{w_e} - e) - d(s, b, H) + \text{Depart}(s, b, e)$ for some $b \in B_k$.\hfill\qed
Similarly to the proof of Claim 5.3, a crucial observation is that \( \text{Depart}(s, b, e) \geq d(s, b, H) \). To see this note that \( \text{Depart}(s, b, e) \geq d(s, b, H - e) \geq d(s, b, H) \), where the first inequality is by Claim 5.1.

Recall that \( \hat{d}(s, x, H[S]_{w_b} - e) \) is the result obtained by the recursive call of the algorithm over the subgraph \( H[S] \) and the query \((e, x, w_b)\). Let \( R \) be the shortest path from \( s \) to \( x \) in the graph \( H[S]_{w_b} - e \). Since the recursive call over \( H[S] \) is complete we have that \( d(R) \leq \hat{d}(s, x, H[S]_{w_b} - e) \). If \( R \) uses no edges from \( E(H[S]_{w_b}) \) then \( R \subseteq H[S] - e \subseteq H - e \). So \( R \) is a path from \( s \) to \( x \) in the graph \( H - e \), meaning that \( d(R) \geq d(s, x, H - e) \). Since we have shown that \( \text{Depart}(s, b, e) \geq d(s, b, H) \) we get that \( d(s, x, H - e) \leq d(R) \leq d(R) + \text{Depart}(s, b, e) - d(s, b, H) \leq d(s, x, H[S]_{w_b} - e) + \text{Depart}(s, b, e) - d(s, b, H) = \text{Pivot}(s, x, e) \) as required.

So we may assume \( R \) uses some edge \((s, v) \in E(H[S]_{w_b}) - E(H[S])\). As in previous cases we wish to slightly fix \( R \) by replacing \((s, v)\) with some path from \( s \) to \( v \) in the graph \( H - e \). Recall that the weight of the edge \((s, v)\) is \( w_b(v) = d(s, b, H) + d(b, v, H - P) \). Let \( R(s, b, H - e) \) denote the shortest path from \( s \) to \( b \) in the graph \( H - e \), note that by Claim 5.1 \( d(R(s, b, H - e)) \leq \text{Depart}(s, b, e) \). Let \( R(b, v, H - P) \) denote the shortest path from \( b \) to \( v \) in the graph \( H - P \), note that since \( e \in P \) we have that \( e \notin R(b, v, H - P) \). So the path \( \tilde{R} = R(s, b, H - e) \circ R(b, v, H - P) \) is a path from \( s \) to \( v \) in the graph \( H - e \) of length at most \( \text{Depart}(s, b, e) + d(b, v, H - P) \) \( + \text{Depart}(s, b, e) - d(s, b, H) = w_b(v) + \text{Depart}(s, b, e) - d(s, b, H) \). So if we replace the edge \((s, v)\) with the path \( \tilde{R} \) we get a path from \( s \) to \( x \) in the graph \( H - e \) of length at most \( d(R) + \text{Depart}(s, b, e) - d(s, b, H) \). This implies that \( d(s, x, H - e) \leq d(R) + \text{Depart}(s, b, e) - d(s, b, H) \leq d(s, x, H[S]_{w_b} - e) + \text{Depart}(s, b, e) - d(s, b, H) = \text{Pivot}(s, x, e) \) as required. 

**Claim 5.8.** Let \((e, x, w) \in Q\) be a query such that \( e \in P \) and \( x \in V(S) - \{t\} \). Assuming that the recursive call over \( H[S] \) is complete, then \( d(s, x, H[w]_w - e) \geq d(s, x, H[w]_w - e) \).

**Proof.** We show that each one of the elements in the minimum that defines \( \hat{d}(s, x, H[w]_w - e) \) is at least \( d(s, x, H[w]_w - e) \). This will suffice to show that \( d(s, x, H[w]_w - e) \) is at least \( d(s, x, H[w]_w - e) \).

Note that since \( H - e \subseteq H[w]_w - e \) we have that \( d(s, x, H - e) \geq d(s, x, H[w]_w - e) \). So for the cases of \( \text{Depart}(s, x, e) \), \( d(e, s, H[S]_{v < x} - x) + \hat{d}(s, t, H[w]_w - e) - d(s, t, H) \) and Pivot\( (s, x, e) \) we have by Claims 5.1, 5.6, and 5.7 (correspondingly) that each one of these values is at least \( d(s, x, H[w]_w - e) \).

For the case of \( d(s, x, H[w]_w - P) \), since \( e \in P \) we have that \( H[w]_w - P \subseteq H[w]_w - e \) and so \( d(s, x, H[w]_w - P) \geq d(s, x, H[w]_w - e) \).

For the case of \( \hat{d}(s, x, H[S]_{w} - e) \) since the recursive call over \( H[S] \) is complete we have that \( \hat{d}(s, x, H[S]_{w} - e) \geq d(s, x, H[S]_{w} - e) \) and by Claim 5.5 we have that \( \hat{d}(s, x, H[S]_{w} - e) \geq d(s, x, H[w]_w - e) \), these implies the claim.

**Proof of Soundness - Departing Paths**

We again start with a couple of general claim which will be useful in future cases.

**Claim 5.9.** Let \( e \in E(S), x \in V(S) - \{t\}, w \in W \), assume \( R(s, x, H[w]_w - e) \) uses no nodes from \( V(T) - \{t\} \), then its length is at least \( d(s, x, H[S]_{w|w} - e) \).

**Proof.** If \( R(s, x, H[w]_w - e) \) is unweighted then since it does not use any edges from \( V(T) - \{t\} \) it is a path from \( s \) to \( x \) in the graph \( H[S] - e \). Since \( H[S] - e \subseteq H[S]_{w|w} - e \) it is also a path in \( H[S]_{w|w} - e \) and so its length must be at least \( d(s, x, H[S]_{w|w} - e) \) as required.

The more interesting case is when \( R(s, x, H[w]_w - e) \) is weighted, meaning the first edge is the weighted edge \((s, v) \in E(H[w]_w) - E(H)\). Since \( R(s, x, H[w]_w - e) \) uses no nodes from \( V(T) - \{t\} \) the rest of the edges are from \( H[S] \). So if we replace the edge \((s, v) \in E(H[w]_w) - E(H)\) with the edge \((s, v) \in E(H[S]_{w|w}) - E(H[S])\), we get a new path from \( s \) to \( x \) in the graph \( H[S]_{w|w} - e \), and so its length is at least \( d(s, x, H[S]_{w|w} - e) \). We claim that by replacing the edge we do not increase the length of \( R(s, x, H[w]_w - e) \), this will imply that the length of \( R(s, x, H[w]_w - e) \) is at least \( d(s, x, H[S]_{w|w} - e) \) as well, which implies the claim.

To see why the length does not increase note that the length difference between \( R(s, x, H[w]_w - e) \) and the new path is exactly \( w|w|(v) - w(v) \), so we only need to show that \( \forall v \in V(S) : w|w|(v) \leq w(v) \). To see this note that is \( v \notin P \) we have that \( w|w|(v) = w(v) \) by definition and as required. If \( v \in P \) the we have \( w|w|(v) = d(s, v, H[w]_w - P) \).
Since \((s,v) \in E(H_w) - E(H)\) is a path of length \(w(v)\) from \(s\) to \(v\) in the graph \(H_w - P\), we have that \(d(s,v,H_w - P) \leq w(v)\), as required.

**Claim 5.10.** Let \(e \in E(P), x \in V(H), w \in W\), assume \(R(s,x,H_w - e)\) is departing and weighted. Then the length of \(R(s,x,H_w - e)\) is \(d(s,x,H_w - P)\).

*Proof.* We claim that \(V(R(s,x,H_w - e)) \cap V(P) = \{s\}\). To see this assume for the sake of contradiction that there is some node \(v \in R(s,x,H_w - e)\) such that \(v \in P - \{s\}\). If \(v\) is after the edge failure \(e\) in \(P\) then \(R(s,x,H_w - e)\) is jumping and hence not departing. So \(v\) must be before the edge failure \(e\) in \(P\), this implies that the path from \(s\) to \(v\) in the BFS tree \(K\) does not contain \(e\). Since we assume \(R(s,x,H_w - e)\) is \(K\)-simple, this implies that the subpath of \(R(s,x,H_w - e)\) from \(s\) to \(v\) is the path from \(s\) to \(v\) in \(K\). However since \(R(s,x,H_w - e)\) is weighted, the very first edge of \(R(s,x,H_w - e)\) is not from \(H\), and hence not from \(K\), contradiction.

So we conclude that \(V(R(s,x,H_w - e)) \cap V(P) = \{s\}\), and so \(E(R(s,x,H_w - e)) \cap E(P) = \emptyset\). Since \(e \in P\) this implies that \(R(s,x,H_w - e)\) is a shortest path from \(s\) to \(x\) in the graph \(H_w - P\), and so its length is \(d(s,x,H_w - P)\).

Now, using previously made claims, we show that if a replacement path is departing, we have successfully computed its weight (w.h.p). We show this by proving the following claim:

**Claim 5.11.** Let \((e,x,w) \in Q\) be a query such that \(e \in P\) and \(x \in V(S) - \{t\}\). Assume that the recursive call over the subgraph \(H[S]\) is sound and assume that \(R(s,x,H_w - e)\) is departing. Then w.h.p \(d(s,x,H_w - e) \geq d(s,x,H_w - e)\).

*Proof.* Note that if \(R(s,x,H_w - e)\) uses no nodes from \(V(T) - \{t\}\) then we fall in the case of Claim 5.9 and so its length is at least \(d(s,x,H[S]_{\bar{w} \setminus \bar{e}})\). Since the recursive call over \(H[S]\) is sound we have that \(d(s,x,H[S]_{\bar{w} \setminus \bar{e}}) \geq d(s,x,H[S]_{\bar{w} \setminus \bar{e}})\) which is at least \(d(s,x,H_w - e)\). This implies the claim.

Otherwise if \(R(s,x,H_w - e)\) does use a node from \(V(T) - \{t\}\) and is unweighted, then we fall in the case of Claim 5.2 and so the length of \(R(s,x,H_w - e)\) is w.h.p at least \(\text{Depart}(s,x,e)\) which is at least \(d(s,x,H_w - e)\). This implies the claim.

Finally if \(R(s,x,H_w - e)\) is weighted then we fall in the case of Claim 5.10 and so the length of \(R(s,x,H_w - e)\) is at least \(d(s,x,H_w - P)\) which is at least \(d(s,x,H_w - e)\). This implies the claim.

**Proof of Soundness - Jumping Paths**

So we may assume that the path \(R(s,x,H_w - e)\) is jumping. We define the following notation:

Since \(R(s,x,H_w - e)\) is jumping it must first leave the path \(P\) at some node \(v_j \in P\) before the edge failure \(e\) and then return to \(P\) at some node \(v_j \in P\). Let us denote the subpath of \(R(s,x,H_w - e)\) from \(v_j\) to \(v_j\) by \(R[v_j,v_j]\). Note that \(V(R[v_j,v_j]) \cap P = \{v_j\}\) (since \(R(s,x,H_w - e)\) returns to \(P\) at \(v_j\)) and that the path \(R[v_j,v_j]\) is edge disjoint to \(P\). We will use this notation for the rest of this section. So it is important to visually grasp it. Illustrations for this notation can be seen in Figures 3, 5, 11.

Note that all the nodes from \(R(s,x,H_w - e)\) which after \(v_j\) and are from \(P\) (in particular \(v_j\)) must be after the edge failure \(e\) as otherwise \(R(s,x,H_w - e)\) was not \(K\)-simple.

**Claim 5.12.** Under the above notation, assume that \(R(s,x,H_w - e)\) does use a node \(v_r \in V(T) - \{t\}\) outside of the subpath \(R[v_j,v_j]\). Then the length of \(R(s,x,H_w - e)\) is at least \(d(s,x,H[S]_{\bar{w} \setminus \bar{e}}) - d(s,t,H) + d(s,t,H_w - e)\).

We would like to clarify some important characterizations of the current case. Note that \(R(s,x,H_w - e)\) uses \(v_r\) and after it uses \(v_j\) (it cannot use it before \(v_j\) since the subpath from \(s\) to \(v_j\) is a subpath of \(P\)). Note also that since \(v_j\) is after the edge failure \(e\) in \(P\), and since \(v_j \in T\), then when \(R(s,x,H_w - e)\) goes from \(v_j\) to \(v_r\) it passes through (WLOG) the separator \(t\). An illustration of this case can be seen in Figure 3.

*Proof.* We can split the path \(R[s,t]\) into two subpaths: \(R[s,t]\) - the subpath from \(s\) to \(t\), and \(R[t,x]\) - the subpath from \(t\) to \(x\). Note that \(R[s,t]\) is a replacement path from \(s\) to \(t\) in the graph \(H_w - e\), and so its length is \(d(s,t,H_w - e)\).

We claim that the length of \(R[t,x]\) is at least \(d(s,x,H[S]_{\bar{w} \setminus \bar{e}}) - d(s,t,H)\). Clearly this will imply the claim. To see why this is true, recall that \(R[t,x]\) contains a node from \(V(T) - \{t\}\) (namely \(v_r\)). Let \(u\) be the last node
in $R[t, x]$ which is from $V(T) - \{t\}$ and let $v$ be the node right after it. This implies that $(u, v) \in E$, and so by definition $c_S$ we have that $c_S(v) \leq d(s, u, H) + 1$. Let us denote the subpath of $R[t, x]$ from $t$ to $v$ by $R[t, v]$. The length of $R[t, v]$ is at least $d(s, u, H) + 1 = d(s, t, H) + d(t, u, H) + 1 - d(s, t, H) \geq d(s, u, H) + 1 - d(s, t, H) \geq c_S(v) - d(s, t, H)$. So we conclude that $d(R[t, v]) \geq c_S(v) - d(s, t, H)$.

Let us denote by $R[v, x]$ the subpath of $R[t, x]$ from $v$ to $x$. Note that since $u$ was the last node from $V(T) - \{t\}$, then $R[v, x]$ is fully contained in $H[S]$. Consider the edge $(s, v) \in E(H[S]) \subseteq E(H)$, its weight is $c_S(v)$. The path $(s, v) \in R[v, x]$ is a path from $s$ to $x$ in the graph $H[S] \subseteq E(H)$ and so its length is at least $d(s, x, H[S]) - e$. This implies that $d(R[v, x]) + c_S(v) \geq d(s, x, H[S]) - e$. Since we have shown that $d(R[t, v]) \geq c_S(v) - d(s, t, H)$ by simple arithmetics we get that $d(R[t, v]) + d(R[v, x]) \geq d(s, x, H[S]) - e - d(s, t, H)$, which concludes the proof.

**Claim 5.13.** Under the above notation, assume that $R(s, x, H_w - e)$ does not use nodes from $V(T) - \{t\}$ outside of the subpath $R[v_i, v_j]$. Also assume that $R(s, x, H_w - e)$ is weighted. Then the length of $R(s, x, H_w - e)$ is at least $d(s, x, H[S]_{\mid s, g} - e)$.

Note that in this case, $R(s, x, H_w - e)$ leaves the path $P$ at $s$ (since all weighted edges begin at $s$) and so $v_i = s$ by definition. An illustration of this case can be seen in Figure 4.

**Proof.** Let us rename $R[v_i, v_j]$ to $R(s, v_j)$ for clarity. Recall that by a previously made observation $R[s, v_j]$ is edge disjoint to the path $P$. Since $e \in P$, this implies that the length of $R[s, v_j]$ is exactly $d(s, v_j, H_w - P)$ which is exactly $w_{|s, v_j|}$ since $v_j \in P$. So we have that $d(R[s, v_j]) = w_{|s, v_j|}$.

Let us denote by $R[v_j, x]$ the subpath of $R(s, x, H_w - e)$ from $v_j$ to $x$. We claim that the length of $R[v_j, x]$ is at least $d(s, x, H[S]_{\mid |v_j, x|} - e) - w_{|s, v_j|}$. In fact, showing this will suffice in order to prove the claim, since the length of $R(s, x, H_w - e)$ is $d(R[s, v_j]) + d(R[v_j, x])$ and we have shown that $d(R[s, v_j]) = w_{|s, v_j|}$.

So we aim to show that the length of $R[v_j, x]$ is at least $d(s, x, H[S]_{\mid |v_j, x|} - e) - w_{|s, v_j|}$, to see this consider the edge $(s, v_j) \in E(H[S]_{\mid |v_j, x|}) - E(H[S])$, its weight is $w_{|s, v_j|}$. Since $R(s, x, H_w - e)$ does not use any nodes from $V(T) - \{t\}$ outside of the subpath $R[s, v_j]$, the path $R[v_j, x]$ is fully contained in $H[S]$. And so the path $(s, v_j) \circ R[v_j, x]$ is a path from $s$ to $x$ in the graph $H[S]_{\mid s, g} - e$, and so its length is at least $d(s, x, H[S]_{\mid s, g} - e)$. This implies that the length of $R[v_j, x]$ is at least $d(s, x, H[S]_{\mid s, g} - e) - w_{|s, v_j|}$.

**Claim 5.14.** Under the above notation, assume that $R(s, x, H_w - e)$ does not use nodes from $V(T) - \{t\}$ outside of the subpath $R[v_i, v_j]$, but does use a node $v_t \in V(T) - \{t\}$ inside the subpath $R[v_i, v_j]$. Also, assume that $R(s, x, H_w - e)$ is unweighted and that the recursive call over the subgraph $H[S]$ is sound. Then w.h.p, the length of $R(s, x, H_w - e)$ is at least $\text{Pivot}(s, x, e)$.

An illustration of this case can be seen in Figure 5.

**Proof.** The proof for this claim is rather similar to the proofs for Claims 5.12 and 5.2. Note that since $R(s, x, H_w - e)$ is unweighted its length is exactly $d(s, x, H_w - e)$. And so if $e \in P_k$ we have by the definition of $\text{Pivot}(s, x, e)$ that $\text{Pivot}(s, x, e) = \text{Depart}(s, x, e) = d(s, x, H_w - e)$ and so the claim holds. The more interesting case would be when $e \in P_k$ for some $k \in ([\log(n_H)])$. By similar observations to those made in the proof of Claim 5.2 we can show that the length of the path from $v_i \in P$ to $v_r \in T$ is at least $2^k \cdot \sqrt{|R|}$, and so w.h.p we have sampled some $b \in B_k$ such that $b \in R[v_i, v_j]$. So we can split the path $R(s, x, H_w - e)$ into two subpaths: $R[s, b]$ - the subpath of $R(s, x, H_w - e)$ from $s$ to $b$, and $R[b, x]$ - the subpath of $R(s, x, H_w - e)$ from $b$ to $x$.

Note that since the subpath $R[v_i, v_j]$ is edge-disjoint to $P$, we have that the path from $v_i$ to $b$ uses no edges from $P$. Since $e \in P$, this implies that the length of $R[s, b]$ is exactly $d(s, v_i, H) + d(v_i, b, H - P)$. Since $v_i$ is before $e$, by the definition of $\text{Depart}(s, b, e)$ we get that the length of $R[s, b]$ is at least $\text{Depart}(s, b, e)$.

We now show that the length of $R[b, x]$ is at least $d(s, x, H[S]_{\mid w, e}) - d(s, b, H)$. In-fact we claim that showing this will suffice in order to prove the claim. To see this note that by the assumption that the recursive call over the subgraph $H[S]$ is sound we have that $\hat{d}(s, x, H[S]_{\mid w, e}) \leq d(s, x, H[S]_{\mid w, e})$. Hence, the length of $R(s, x, H_w - e)$ is $d(R[s, b]) + d(R[b, x])$ which is at least $d(s, x, H[S]_{\mid w, e}) - d(s, b, H) + \text{Depart}(s, b, e)$ which is at least $\text{Pivot}(s, x, e)$ by its definition.
So we aim to prove that the length of $R[b, x]$ is at least $d(s, x, H[S]_{w_b} - e) - d(s, b, H)$. Recall that $v_j \in R[b, x]$, and so let us denote by $R[b, v_j]$ the subpath of $R[b, x]$ from $b$ to $v_j$, and by $R[v_j, x]$ the subpath of $R[b, x]$ from $v_j$ to $x$. Note that since the path $R[v_j, v_j]$ is edge-disjoint to $P$, and since $e \in P$, the length of $R[b, v_j]$ is exactly $d(b, v_j, H - P) + d(b, v_j, H - P) - d(s, b, H) = w_b(v_j) - d(s, b, H)$.

We now aim to lower bound the length of $R[v_j, x]$. Since $R(s, x, H_w - e)$ uses no nodes from $V(T) - \{t\}$ outside of the subpath $R[v_j, v_j]$, the path $R[v_j, x]$ is fully contained in $H[S]$. Consider the edge $(s, v_j) \in H[S]_{w_b}$, its weight is $w_b(v_j)$. The path $(s, v_j, v_j, x)$ is then a path from $s$ to $x$ in the graph $H[S]_{w_b} - e$, and so its length is at least $d(s, x, H[S]_{w_b} - e)$. This implies that $d(R[v_j, x]) + w_b(v_j) \geq d(s, x, H[S]_{w_b} - e)$. Since we have shown that $d(R[b, v_j]) = w_b(v_j) - d(s, b, H)$ by simple arithmetics we get that $d(R[b, v_j]) + d(R[v_j, x]) \geq d(s, x, H[S]_{w_b} - e) - d(s, b, H)$. Since the length of $R[b, x]$ is $d(R[b, v_j]) + d(R[v_j, x])$ this concludes the proof.

\begin{claim}
Let $(e, x, w) \in Q$ be a query such that $e \in P, x \in V(S) - \{t\}$. Assume that the recursive call over $H[S]$ is sound and assume $R(s, x, H_w - e)$ is jumping. Then w.h.p $d(s, x, H_w - e) \geq \tilde{d}(s, x, H_w - e)$.
\end{claim}

\begin{proof}
Note that if $R(s, x, H_w - e)$ uses no nodes from $V(T) - \{t\}$ then we fall in the case of Claim 5.13 meaning that the length of $R(s, x, H_w - e)$ is at least $d(s, x, H[S]_{w_b} - e)$. Since the recursive call over $H[S]$ is sound, we have that $d(s, x, H[S]_{w_b} - e) \geq \tilde{d}(s, x, H[S]_{w_b} - e)$, and so the length of $R(s, x, H_w - e)$ is at least $\tilde{d}(s, x, H_w - e)$. If $R(s, x, H_w - e)$ does use a node from $V(T) - \{t\}$ we must fall in the one of the cases of Claims 5.11, 5.12 or 5.13. In total we have proven in each case that the length of $R(s, x, H_w - e)$ is w.h.p at least $\tilde{d}(s, x, H_w - e)$. 

\end{proof}

\section{The case when $e \in P$ and $x \in V(T) - \{t\}$}

Recall that in the case when $e \in E(P)$ and $x \in V(T)$, the algorithm sets $\tilde{d}(s, x, H_w - e)$ to be the minimum between $d(s, x, H_w - P), \text{Depart}(s, x, e)$ and $d(s, t, H_w - e) + d(t, t, H)$ in step 7 of the algorithm (line 54 of the pseudocode).

\begin{proof}[Proof of Completeness]
Let $(e, x, w) \in Q$ be a query such that $e \in P$ and $x \in V(T) - \{t\}$, then $\tilde{d}(s, x, H_w - e) \geq d(s, x, H_w - e)$. 

\end{proof}

\begin{proof}[Proof of Soundness]
Let $(e, x, w) \in Q$ be a query such that $e \in P$ and $x \in V(T) - \{t\}$. Then w.h.p $d(s, x, H_w - e) \geq \tilde{d}(s, x, H_w - e)$.

\end{proof}
since $u$ is after the edge failure $e$ in $P$, the path from $u$ to $x$ in $K$ does not contain $e$. Since $K$ is a BFS tree in $H_w$ (by the weight requirements) we can assume (WLOG) that when $R(s, x, H_w - e)$ goes from $u$ to $x$ it takes the path from $u$ to $x$ in $K$. Since this path passes through the separator $t$ we can split $R(s, x, H_w - e)$ into two subpaths: $R(s, t)$ a subpath of $R(s, x, H_w - e)$ from $s$ to $t$, and $R(t, x)$ the subpath of $R(s, x, H_w - e)$ from $t$ to $x$. The first path $R(s, t)$ is a path from $s$ to $t$ in the graph $H_w - e$, and so its length is at least $d(s, t, H_w - e)$, and the length of the second path is at least $d(t, x, H_w)$ which is $d(t, x, H)$ (by the weight requirements). So the length of the path $R(s, x, H_w - e)$ is at least $d(s, t, H_w - e) + d(t, x, H)$ as required.

\section{The case when $e \in T$ and $x \in V(T) - \{t\}$}

Recall that in the case when $e \in E(T)$ and $x \in V(T) - \{t\}$, the algorithm sets $\hat{d}(s, x, H_w - e)$ to be the minimum between $\hat{d}(s, x, H[T]_{w\upharpoonright_T} - e)$ and $\hat{d}(s, x, H[T]_{c_T} - e)$ in step $S$ of the algorithm (line 75 of the pseudocode).

Recall that $\hat{d}(s, x, H[T]_{w\upharpoonright_T} - e)$ and $\hat{d}(s, x, H[T]_{c_T} - e)$ are the result obtained by the recursive calls over the subgraph $H[T]$ for the queries $(e, x, w\upharpoonright_T)$ and $(e, x, c_T)$ correspondingly.

**Proof of Completeness**

Claim 5.18. Let $(e, x, w) \in Q$ be a query such that $e \in T, x \in V(T) - \{t\}$. Assuming that the recursive call over the subgraph $H[T]$ is complete, then $\hat{d}(s, x, H_w - e) \geq d(s, x, H_w - e)$.

Proof. As in previous cases, we again show that both $\hat{d}(t, x, H[T]_{w\upharpoonright_T} - e) + d(s, t, H)$ and $\hat{d}(t, x, H[T]_{c_T} - e) + d(s, t, H)$ are at least $d(s, x, H_w - e)$.

We now handle the term $\hat{d}(t, x, H[T]_{w\upharpoonright_T} - e) + d(s, t, H)$. Recall that $\hat{d}(t, x, H[T]_{w\upharpoonright_T} - e)$ is the result obtained by the recursive call over the graph $H[T]$ for the query $(e, x, w\upharpoonright_T)$. Let $R$ be the shortest path from $t$ to $x$ in the graph $H[T]_{w\upharpoonright_T} - e$. Since the recursive call over $H[T]$ is complete, we have that $d(R) \leq \hat{d}(t, x, H[T]_{w\upharpoonright_T} - e)$.

Note that if $R \subseteq H[T] - e$, then $R \subseteq H - e$. Since $e \notin P$, the path $P \circ R$ is a path from $s$ to $x$ in the graph $H - e \subseteq H_w - e$. This implies that $d(s, x, H_w - e) \leq d(R) + d(P) \leq \hat{d}(t, x, H[T]_{w\upharpoonright_T} - e) + d(s, t, H)$, as required.

If $R \not\subseteq H[T] - e$ then $R$ uses some weighted edge $(t, v) \in E(H[T]_{w\upharpoonright_T}) - E(H[T])$. Recall that other than this edge, the path $R$ is fully contained in $H[T] - e$. Let $\overline{R}$ be the path obtained by replacing the edge $(t, v)$ in $R$ with the edge $(s, v) \in E(H_w)$. The length of $\overline{R}$ is then larger than the length of $R$ by $w(v) - w\upharpoonright_T(v)$. So the definition $w\upharpoonright_T(v) = w(v) - d(s, t, H)$ we have that $d(\overline{R}) = d(R) + d(s, t, H) \leq \hat{d}(t, x, H[T]_{w\upharpoonright_T} - e) + d(s, t, H)$. And obviously $\overline{R}$ is a path from $s$ to $x$ in the graph $H_w - e$, meaning that $d(s, x, H_w - e) \leq d(\overline{R}) \leq \hat{d}(t, x, H[T]_{w\upharpoonright_T} - e) + d(s, t, H)$ as required.

We now handle the term $\hat{d}(t, x, H[T]_{c_T} - e) + d(s, t, H)$. Again we denote by $R$ the shortest path from $t$ to $x$ in the graph $H[T]_{w\upharpoonright_T} - e$, and by the assumption of completeness we have that $d(R) \leq \hat{d}(t, x, H[T]_{w\upharpoonright_T} - e)$.

Again if $R \subseteq H[T] - e$ we can concatenate $P$ before $R$ and show that $d(s, x, H_w - e) \leq d(R) + d(s, t, H) \leq \hat{d}(t, x, H[T]_{c_T} - e) + d(s, t, H)$. So we may assume that $R$ uses some weighted edge $(t, v) \in E(H[T]_{c_T}) - E(H[T])$. We again want to change $R$ by replacing the edge $(t, v)$ with some path from $s$ to $v$ in the graph $H_w - e$.

By definition of $c_T$ there is some vertex $u \in V(S), u \neq t, (u, v) \in E$ such that $c_T(v) = d(s, u, H) + 1 - d(s, t, H)$. Note that the path from $s$ to $u$ in the BFS tree $K$ is contained in $S$ and so does not contain $e$. Let us denote this path by $R(s, u, H)$. Also since $u \notin T$ we have that $(u, v) \notin T$ and so $e \notin (u, v)$. We can conclude that $e \notin R(s, u, H) \circ (u, v)$. Let $\overline{R}$ be the path obtained by replacing the edge $(s, v)$ in $R$ with the path $R(s, u, H) \circ (u, v)$. We have that $\overline{R}$ is a path from $s$ to $x$ in the graph $H - e \subseteq H_w - e$. Meaning that $d(s, x, H_w - e) \leq d(\overline{R})$. The length of $\overline{R}$ is larger than the length of $R$ by $d(R(s, u, H)) + 1 - c_T(v) = d(s, t, H)$.

So we have that $d(\overline{R}) = d(R) + d(s, t, H) \leq d(t, x, H[T]_{c_T} - e) + d(s, t, H)$ which implies the claim.

**Proof of soundness**

Claim 5.19. Let $e \in E(T), x \in V(T) - \{t\}, w \in W$, if $R(s, x, H_w - e)$ is unweighted then its length is at least $d(t, x, H[T]_{c_T} - e) + d(s, t, H)$.

Proof. Note that $R(s, x, H_w - e)$ uses nodes from $V(S)$ as it uses $s$. Let $u$ be the last node in $R(s, x, H_w - e)$ which is from $V(S)$. We fall in to two cases:
If \( u = t \) then since \( R(s, x, H_w - e) \) is \( K \)-simple, the path from \( s \) to \( t \) in \( R(s, x, H_w - e) \) is the shortest path \( P \). Meaning we can split the path \( R(s, x, H_w - e) \) into two parts: \( P - \) the path from \( s \) to \( t \), and \( R[t, x] \) the sub-path of \( R(s, x, H_w - e) \) from \( t \) to \( x \). Note that since \( t = u \) was the last node in \( R(s, x, H_w - e) \) which is from \( V(S) \), and since \( R(s, x, H_w - e) \) is unweighted we have that \( R[t, x] \) is fully contained in \( H[T] \). This implies that \( R[t, x] \) is a path from \( t \) to \( x \) in the graph \( H[T] - e \) and so \( d(t, x, H[T] - e) \leq d(R[t, x]) \). Since \( H[T] - e \subseteq H[T]_{T,e} - e \) we have that \( d(t, x, H[T]_{T,e} - e) \leq d(t, x, H[T] - e) \). Since the length of \( R(s, x, H_w - e) \) is \( d(P) + d(R[t, x]) \) we can conclude that the length of \( R(s, x, H_w - e) \) is at least \( d(t, x, H[T]_{T,e} - e) + d(s, t, H) \) as required.

Otherwise we have that \( u \neq t \), and so \( u \in V(S) - \{t\} \). An illustration of this case can be seen in Figure 5. Let \( v \) denote the node right after \( u \) in \( R(s, x, H_w - e) \). We can split the path \( R(s, x, H_w - e) \) into two parts: \( R[v, x] \) the subpath of \( R(s, x, H_w - e) \) from \( s \) to \( v \), and \( R[v, x] \) the subpath of \( R(s, x, H_w - e) \) from \( v \) to \( x \). Note that the length of \( R[v, x] \) is at least \( d(s, u, H) + 1 \). Since \( u, v \in E \) and \( u \in V(S) - \{t\} \) by the definition of \( c_T \) we have that \( c_T(v) \leq d(s, u, H) + 1 - d(s, t, H) \). So we can conclude that \( d(R[v, x]) - d(s, t, H) \geq c_T(v) \). Consider the edge \( (t, v) \in E(H[T]_{T,e}) - E(H[T]) \), its weight is \( c_T(v) \). Since \( u \) was the last node in \( R(s, x, H_w - e) \) that is from \( V(S) \), the path \( (t, v) \circ R[v, x] \) is a path from \( t \) to \( x \) in the graph \( H[T]_{T,e} - e \). So the length of \( (t, v) \circ R[v, x] \) is at least \( d(t, x, H[T]_{T,e} - e) \). So we have that \( c_T(v) + d(R[v, x]) \geq d(t, x, H[T]_{T,e} - e) \) and that \( d(R[v, x]) - d(s, t, H) \geq c_T(v) \). By simple arithmetic this implies that \( d(R[v, x]) + d(R[v, x]) \geq d(t, x, H[T]_{T,e} - e) + d(s, t, H) \) which implies the claim.

Claim 5.20. Let \( e \in E(T), x \in V(T) - \{t\} \), \( w \in W \), if \( R(s, x, H_w - e) \) is weighted then its length is at least \( d(t, x, H[T]_{T,e} - e) + d(s, t, H) \).

Proof. We claim that in this case the only node from \( V(S) \) that \( R(s, x, H_w - e) \) uses is \( s \). To see this assume towards contradiction that \( R(s, x, H_w - e) \) uses a node \( u \in V(S) \) such that \( u \neq s \). Since \( u \in S \) the path from \( s \) to \( u \) in the BFS tree is contained in \( S \) and so does not contain the edge failure \( e \in T \). Since \( R(s, x, H_w - e) \) is \( K \)-simple this implies that the subpath of \( R(s, x, H_w - e) \) from \( s \) to \( u \) is the path from \( s \) to \( u \) in \( K \). However since \( R(s, x, H_w - e) \) is weighted the first edge of \( R(s, x, H_w - e) \) is from \( E(H_w) - E(H) \) and hence not from \( K \), contradiction.

So we conclude that \( V(R(s, x, H_w - e)) \cap V(S) = \{s\} \). Let \( (s, v) \) be the first edge of \( R(s, x, H_w - e) \). Note that \( (s, v) \in E(H_w) - E(H) \) since \( R(s, x, H_w - e) \) is weighted, and that the weight of \( (s, v) \) is \( w(v) \). Consider the path \( \overline{R} \) obtained by replacing the edge \( (s, v) \in E(H_w) - E(H) \) with the edge \( (t, v) \in E(H[T]_{T,e}) - E(H[T]) \) (which is of weight \( w(T(v)) \)). Since \( V(R(s, x, H_w - e)) \cap V(S) = \{s\} \) the path \( \overline{R} \) is a path from \( t \) to \( x \) in the graph \( H[T]_{T,e} - e \). This implies that \( d(\overline{R}) \geq d(t, x, H[T]_{T,e} - e) \). Since the length of \( \overline{R} \) is smaller then the length of \( R(s, x, H_w - e) \) by exactly \( w(v) - w(T(v)) \). By the definition of \( w[T] \) we have that \( w(v) - w(T(v)) = d(s, t, H) \). So we have that \( d(\overline{R}) = d(s, x, H_w - e) - d(s, t, H) \) and that \( d(\overline{R}) \geq d(t, x, H[T]_{T,e} - e) - d(s, t, H) \), these implies the claim.

Claim 5.21. Let \( e, x, w \in Q \) be a query such that \( e \in T \) and \( x \in V(T) - \{t\} \). Assuming that the recursive call over the subgraph \( H[T] \) is sound, then \( d(s, x, H_w - e) \geq d(s, x, H_w - e) \).

Proof. Recall that in the case when \( e \in T \) and \( x \in V(T) - \{t\} \) the algorithm sets \( \hat{d}(s, x, H_w - e) \) to be the minimum between \( \hat{d}(t, x, H[T]_{T,e} - e) + d(s, t, H) \) and \( \hat{d}(t, x, H[T]_{T,e} - e) + d(s, t, H) \).

Let us look at the replacement path \( R(s, x, H_w - e) \). If \( R(s, x, H_w - e) \) is unweighted then by Claim 5.19 its length is at least \( d(t, x, H[T]_{T,e} - e) + d(s, t, H) \). Since the recursive call over \( H[T] \) is sound we have that \( d(t, x, H[T]_{T,e} - e) \geq \hat{d}(t, x, H[T]_{T,e} - e) \) and so \( d(s, x, H_w - e) \geq \hat{d}(t, x, H[T]_{T,e} - e) + d(s, t, H) \). This implies the claim.

Similarly we can use Claim 5.20 to show that if \( R(s, x, H_w - e) \) is weighted it holds that \( d(s, x, H_w - e) \geq \hat{d}(t, x, H[T]_{T,e} - e) + d(s, t, H) \geq \hat{d}(s, x, H_w - e) \) which implies the claim.

5.6 The case when \( e \in E(S) - E(P) \) and \( x \in V(S) - \{t\} \)

Recall that in the case when \( e \in E(S) - E(P) \) and \( x \in V(S) \), the algorithm sets \( \hat{d}(s, x, H_s - e) \) to be the minimum between \( \hat{d}(s, x, H[S]_{S,s} - e) \) and \( \hat{d}(s, x, H[S]_{S,s} - e) \) in step 3 of the algorithm (line 122 of the pseudocode).

25
Recall that $\tilde{d}(s, x, H[S]_{w|s} - e)$ and $\hat{d}(s, x, H[S]_{c_{S}} - e)$ are the result obtained by the recursive calls over the subgraph $H[S]$ for the queries $(e, x, w|s)$ and $(e, x, c_{S})$ correspondingly.

**Proof of Completeness**

**Claim 5.22.** Let $(e, x, w) \in Q$ be a query such that $e \in E(S) - E(P)$ and $x \in V(S) - \{t\}$. Assuming that the recursive call over $H[S]$ is complete, then $\hat{d}(s, x, H[w] - e) \geq d(s, x, H[w] - e)$.

**Proof.** Recall that in the case when $e \in E(S) - E(P)$ and $x \in V(S) - \{t\}$, the algorithm sets $\tilde{d}(s, t, H[w] - e)$ to be the minimum between $\tilde{d}(s, x, H[S]_{w|s} - e)$ and $\hat{d}(s, x, H[S]_{c_{S}} - e)$. As in previous cases, we show that both terms are at least $d(s, x, H[w] - e)$.

We now handle the term $\tilde{d}(s, x, H[S]_{w|s} - e)$. Note that since the recursive call over $H[S]$ is complete we have that $\tilde{d}(s, x, H[S]_{w|s} - e) \geq d(s, x, H[S]_{c_{S}} - e)$, and by Claim 5.23, we have that $\hat{d}(s, x, H[S]_{w|s} - e) \geq d(s, x, H[w] - e)$, these implies the claim.

We now handle the term $\hat{d}(s, x, H[S]_{w|s} - e)$. We denote by $d(s, x, H[S]_{w|s} - e)$ the shortest path from $s$ to $x$ in the graph $H[S]_{w|s} - e$.

Recall that the weight of the edge $(s, v)$ is $c_{S}(v)$ which is equal to $d(s, u, H) + 1$ for some $u \in V(T) - \{t\}$ such that $(u, v) \in E(H)$. Let $R(s, u, H)$ be the path from $s$ to $u$ in the BFS tree $K$. Since $K$ is a BFS tree the length of $R(s, u, H)$ is $d(s, u, H)$. Note that since $u \in V(T)$ the path $R(s, u, H)$ contains only edges from $E(P) \cup E(T)$ and so does not contain the edge $e \in E(S) - E(P)$. Also note that since $u \in V(T)$ we have that $u \notin S$ and so the edge $(u, v)$ is not from $E(S)$ and hence is not $e$. So we have that the path $R(s, u, H) \circ (u, v)$ is a path from $s$ to $x$ in the graph $H - e$ of length $\hat{d}(s, x, H[S]_{w|s} - e)$ as required.

So we may assume $R$ uses some edge $(s, v) \in E(H[S]_{c_{S}} - E(H[S]))$. As in previous claims, we wish to replace the edge $(s, v)$ with some path from $s$ to $v$ in the graph $H[w] - e$ without changing the length of $R$.

Recall that the weight of the edge $(s, v)$ is $c_{S}(v)$ which is equal to $d(s, u, H) + 1$. Then we fall in the case of Claim 5.9, and so the edge $(u, v)$ is not from $E(S)$ and hence is not $e$. So we have that the path $R(s, u, H) \circ (u, v)$ is a path from $s$ to $x$ in the graph $H - e$ of length $\hat{d}(s, x, H[S]_{w|s} - e)$ as required.

**Proof of Soundness**

**Claim 5.23.** Let $e \in E(S) - E(P), x \in V(S) - \{t\}, w \in W$, assume $R(s, x, H[w] - e)$ uses a node from $V(T) - \{t\}$, then its length is at least $d(s, x, H[S]_{c_{S}} - e)$.

**Proof.** Let us denote by $u$ the last node in $R(s, x, H[w] - e)$ which is also in $V(T) - \{t\}$ and let $v$ be the node right after it. We can then split the path $R(s, x, H[w] - e)$ into two subpaths: $R[s, v]$ - the subpath of $(s, x, H[w] - e)$ from $s$ to $v$, and $R[v, x]$ - the subpath of $(s, x, H[w] - e)$ from $v$ to $x$. We claim that the length of $R[s, v]$ is at least $c_{S}(v)$. To see this note that $R[s, v]$ goes from $s$ to $u$, and then from $u$ to $v$ through the edge $(u, v) \in E(H)$. Meaning the length of $R[s, v]$ is at least $d(s, u, H) + 1$ (recall that $d(s, u, H) = d(s, u, H[w])$ by the weight requirement). By the definition of $c_{S}$ we have that $c_{S}(v) \leq d(s, u, H) + 1$. So we can conclude that $d(R[s, v]) \geq c_{S}(v)$.

We now attempt to lower bound the length of $R[v, x]$. Note that since $u$ was the last node in $R(s, x, H[w] - e)$ which is from $V(T) - \{t\}$, the subpath $R[v, x]$ is fully contained in $H[S]$. Consider the edge $(s, v) \in E(H[S]_{c_{S}}) - E(H[S])$, its weight is $c_{S}(v)$. The path $(s, v) \circ R[v, x]$ is then a path from $s$ to $x$ in the graph $H[S]_{c_{S}} - e$ and so its length is at least $d(s, x, H[S]_{c_{S}} - e)$. So we have that $c_{S}(v) + d(R[v, x]) \geq d(s, x, H[S]_{c_{S}} - e)$ and that $d(R[v, x]) \geq c_{S}(v)$. By simple arithmetics this implies that the length of $R(s, x, H[w] - e)$ is at least $d(s, x, H[S]_{c_{S}} - e)$.

**Claim 5.24.** Let $(e, x, w) \in Q$ be a query such that $e \in E(S) - E(P), x \in V(S) - \{t\}$. Assuming that the recursive call over the subgraph $H[S]$ is sound, then $d(s, x, H[w] - e) \geq \hat{d}(s, x, H[w] - e)$.

**Proof.** Note that if $R(s, x, H[w] - e)$ uses no nodes from $V(T) - \{t\}$ then we fall in the case of Claim 5.33 and so the length of $R(s, x, H[w] - e)$ is at least $d(s, x, H[S]_{w|s} - e)$. Since the recursive call over $H[S]$ is sound (by
assumption) we have that \( d(s, x, H[S]_{w-e}) \geq \tilde{d}(s, x, H[S]_{w-e}) \), which is at least \( \tilde{d}(s, x, H_w - e) \), this implies the claim. Otherwise \( R(s, x, H_w - e) \) does use a node from \( V(T) - \{t\} \), meaning we fall to the case of Claim 5.23 and so by the same arguments we get that the length of \( R(s, x, H_w - e) \) is at least \( \tilde{d}(s, x, H[S]_{w-e}) \), which again implies the claim.

Combined together claims 5.3, 5.10, 5.14, 5.22 and 5.8 imply the following claim:

**Claim 5.25.** Assuming that \( |V(H)| > 6 \) and assuming that the recursive calls over \( H[T] \) and \( H[S] \) are complete, the call over \( H \) is also complete.

Similarly, combined together claims 5.4, 5.17, 5.21, 5.24, 5.11 and 5.15 imply the following claim:

**Claim 5.26.** Assuming that \( |V(H)| > 6 \) and assuming that the recursive calls over \( H[T] \) and \( H[S] \) are sound, the call over \( H \) is w.h.p also sound.

Since the algorithm is both sound and complete for graphs of size at most 6 we can use these to claims to prove by induction that the algorithm is complete, and w.h.p is also sound.

### 6 Running Time analysis

We now prove that the running time of the above algorithm is indeed \( \tilde{O}(m \sqrt{n} + n^2) \). In order to upper bound the algorithm’s running time, we prove the following claim.

**Claim 6.1.** Let \( T(H, W, Q) \) denote the running time of the algorithm over the sub-graph \( H \) of the original graph \( G \), weight functions set \( W \) and queries set \( Q \). Denote \( n_H = |V(H)|, m_H = |E(H)| \).

Then \( T(H, W, Q) = T(H[S], W_S, Q_S) + T(H[T], W_T, Q_T) + \tilde{O}(n_H |W| + m_H \sqrt{n_H} + n_H^2 + |Q|) \).

And it holds that \( |Q_S| + |Q_T| = |Q| + \tilde{O}(n_H^2) \) and \( |W_S| = |W| + \tilde{O}(\sqrt{n_H}) \).

**Proof.** The proof of this claim will be done by simply going through the different steps of the algorithm, upper bounding their running time and upper bounding the size of the recursive invocations constructed.

**Step 2** Tree separation \((S, T)\):

Since by Lemma 2.2 one can find a balanced tree separator in linear time, the algorithm takes \( \tilde{O}(n_H + m_H) \) to find the separator and preform the two BFS invocations described in this section.

**Step 3** Computing \( d(s, x, H_w - P) \):

In this section for every \( w \in V \) the algorithm constructs the graph \( H_w - E(P) \), this obviously can be done in \( \tilde{O}(n_H + m_H) \) time. Then the algorithm runs Dijkstra’s algorithm on the generated graph. For each \( w \in V \) this take \( \tilde{O}(n_H + m_H) = \tilde{O}(m_H) \) time so in total this section of the algorithm will take \( \tilde{O}(|W| \cdot m_H) \) time.

**Step 4** Sampling Pivots \((B_k)\) and Defining Path Intervals \((P_k)\):

We show that the running time of this step of the algorithm is \( \tilde{O}(m_H \sqrt{n_H}) \) time. Note that by Lemma 2.1 \( |B_k| = \tilde{O}(\sqrt{n_H}) \) w.h.p and so w.h.p the algorithm samples each \( B_k \) only once - resulting in \( \tilde{O}(n) \) time for the sampling step. Also note that \( |B| = \tilde{O}(\sqrt{n_H}) \) and that \( \forall k \in [\log(n_H)] \cdot |V(P_k)| = O(2^k \sqrt{n_H}) \).

Invoking the BFS algorithm for every \( b \in B \) then takes \( \tilde{O}(m|B|) \) time which is \( \tilde{O}(m \sqrt{n}) \) time.

**Step 5** Computing Departing Paths \((\text{Depart}(s, x, e))\):

We show that the running time of this step of the algorithm is \( \tilde{O}(n_H^2 + m_H \sqrt{n_H}) \) time. To see that let us fix some \( k \in [\log(n_H)] \) and \( e \in P_k \). It takes \( \tilde{O}(n_H) \) time to compute \( \text{Depart}(s, b, e) \) for every pivot node \( b \in B_k \) and \( \tilde{O}(|B_k|) \) time to compute \( \text{Depart}(s, x, e) \) for every non-pivot node \( x \in V(H) - B_k \). This implies that computing \( \text{Depart}(s, v, e) \) for every node \( v \in V(H) \) takes \( \tilde{O}(n_H \cdot |B_k|) \) time. So for some \( k \in [\log(n_H)] \) it takes \( \tilde{O}(n_H \cdot |B_k|) \) to take care of edge failures from \( P_k \). Note that \( |P_k| = O(2^k \sqrt{n_H}) \) and that \( |B_k| = O(\sqrt{n_H}) \).

This implies that \( |P_k| \cdot |B_k| = \tilde{O}(n_H) \). So it takes \( \tilde{O}(n_H^2) \) to care of some \( k \in [\log(n_H)] \), meaning that it takes \( \tilde{O}(n_H^2) \) time to take care of edge failures from \( E(P) - E(P_0) \).

Taking care of an edge failure from \( P_0 \) involves invoking Dijkstra’s algorithm a single time meaning it takes \( \tilde{O}(m_H |P_0|) \) time to take care of edge failures from \( P_0 \), which is \( \tilde{O}(m_H \sqrt{n_H}) \) time since \( |P_0| = O(\sqrt{n_H}) \).

27
Step 6. Computing $\hat{d}(s,t,H_w-e)$ When $e \in E(P)$
We show that the running time of this step of the algorithm is $\tilde{O}(n_H|W| + m\sqrt{n_H})$ time. To see this note that defining the nodes $u_t$ and edges $e_i$ takes $\tilde{O}(n_H)$ time. For every $w \in W$ the algorithm then computes $A_w$ using dynamic programming, by iterating over the set $\{e_i\}_{i=0}^{P-1}$ once. This implies that the algorithm takes $\tilde{O}(|P|) = \tilde{O}(n_H)$ time to compute each $A_w$, meaning it takes $\tilde{O}(n_H|W|)$ to compute all $A_w$. Invoking the algorithm from [16] takes $\tilde{O}(m\sqrt{n_H})$ time, and computing the final results takes $\tilde{O}(n_H|W|)$ time.

Step 7. Computing $\hat{d}(s,x,H_w-e)$ When $e \in E(P), x \in V(T) - \{t\}$
It can be easily verified that the running time of this step of the algorithm is $\tilde{O}(|Q|)$ time.

Step 8. Computing $\hat{d}(s,x,H_w-e)$ When $e \in E(T), x \in V(T) - \{t\}$
We now show that the non recursive part of this step runs in $\tilde{O}(n_H|W| + n_H^2 + |Q|)$ time. To see this note that computing the restricted weight function $w|_T$ for every $w \in W$ takes $\tilde{O}(n_H)$ time, resulting in a total $\tilde{O}(n_H|W|)$ time to define all of them. Computing $c_T$ naïvely takes $\tilde{O}(n_H^2)$ time and constructing $Q_T$ takes $|Q| + \tilde{O}(n_H^2)$ time. Finally computing the final distance estimations takes $\tilde{O}(|Q|)$ time.

We also upper bound the size of the recursive input this step constructs (as stated by the claim). It can be easily seen that the new set of weight functions $W_T$ is of size $|W| + 1$ (we only added the weight function $c_T$). Also the new queries added to $Q_T$ (queries that do not originate from a query in $Q$) are the queries $E(T) \times V(T) \times \{c_T\}$. Meaning we only add $\tilde{O}(n_H^2)$ new queries.

Step 9. Computing $\hat{d}(s,x,H_w-e)$ When $e \in E(S), x \in V(S) - \{t\}$
We now show that the non recursive part of this step runs in $\tilde{O}(n_H|W| + n_H^2 + |Q|)$ time. We will also upper bound the size of the recursive input this step constructs.

Computing the new weight functions set: Computing the restricted weight function $w|_S$ for every $w \in W$ takes $\tilde{O}(n_H)$ time, resulting in a total $\tilde{O}(n_H|W|)$ time to define all of them. Computing $c_S$ naïvely takes $\tilde{O}(n_H^2)$ time. Defining and computing the new weight functions $w_b$ takes $\tilde{O}(n_H)$ time for each $b \in B$. Since $|B| = \tilde{O}(\sqrt{n_H})$ defining and computing all the $w_b$ weight functions take $\tilde{O}(n_H^2)$ time. So it takes $\tilde{O}(n_H^2 + n_H|W|)$ time to define the new weight functions set $W_S$ and it is of size $|W| + 1 + |B| = |W| + \tilde{O}(\sqrt{n_H})$.

Computing the new queries set: The algorithm first add the query $(e,x,w|_S)$ for every $(e,x,w) \in Q$ such that $e \in S, x \in V(S) - \{t\}$, this takes $\tilde{O}(|Q|)$ time. Then it adds the queries $E(S) \times V(S) \times \{c_S\}$, which takes $\tilde{O}(n_H^2)$ time. Then, for every $k \in [[\log(n_H)]]$ the algorithm adds to $Q_S$ the queries $(e,x,w_b)$ for every $(e,x,b) \in E(P_k) \times V(S) \times B_k$. This implies that the algorithm adds $n_H \cdot |P_k| \cdot |B_k|$ new queries to $Q_S$. Since $|P_k| = \tilde{O}(2^k \sqrt{n_H})$ and $|B_k| = \tilde{O}(\sqrt{n_H^2})$ this implies that for every $k \in [[\log(n_H)]]$ the algorithm adds $\tilde{O}(n_H^2)$ new queries to $Q_S$. So it takes $\tilde{O}(n_H^2 + |Q|)$ time to compute the new set of queries $Q_S$ and it contains $\tilde{O}(n_H^2)$ new queries (queries that do not originate from a query in $Q$).

Computing $\text{Pivot}(s,x,e)$: Let us fix some $k \in [[\log(n_H)]]$ and $e \in P_k$. Computing $\text{Pivot}(s,x,e)$ for each $x \in V(S)$ takes $\tilde{O}(|B_k|)$ time. Meaning that computing $\text{Pivot}(s,x,e)$ for every $e \in E(P) - E(P_0)$ and $x \in V(S)$ takes $\sum_{k\in [[\log(n_H)]]} \hat{O}(|P_k| \cdot n_H \cdot |B_k|) = \sum_{k\in [[\log(n_H)]]} \hat{O}(\sqrt{n_H}^k \cdot n_H \cdot \sqrt{n_H}) = \sum_{k\in [[\log(n_H)]]} \hat{O}(n_H^2) = \tilde{O}(n_H^2)$. Computing $\text{Pivot}(s,x,e)$ for every $e \in E(P_0)$ and $x \in V(S)$ takes $\hat{O}(|P_0| \cdot n_H) = \hat{O}(n_H^2)$ time.

Finally computing the final distance estimations takes $\hat{O}(|Q|)$ time.

Step 10. Outputting the Results:
In this section the algorithm constructs a LCA data structure over the BFS tree $K$, which takes $O(n_H)$ time by Lemma 6. Then for every $(e,x,w) \in Q$ the algorithm checks in $O(1)$ time if $e$ is on the path from $s$ to $x$ in the BFS tree $K$, if so the algorithm sets the estimated value $\hat{d}(s,x,H_w-e)$ as the result for the query $(e,x,w)$, otherwise it sets $\hat{d}(s,x,H)$ as the result for the query. Overall the running time for this section is $\hat{O}(|Q| + n_H)$.
6.1 Running Time Analysis of the Recursion

We now use Claim 6.1 to perform an analysis of our recursive algorithm. The input to the algorithm is a graph $G$, a BFS tree $K$, a weight functions set of size 1 $W_0 = \{w_\infty\}$ (where $w_\infty$ assigns $\infty$ to each vertex), and the complete set of queries $Q_0 = E(\tilde{K}) \times V(G) \times W_0$.

Let $L$ denote the number of levels of the recursion, note that $L = O(\log n)$ since the size of the graphs decays exponentially. Let $\mathcal{H}_i$ denote the set of graphs in the $i$’th level of the recursion. Note that $\sum_{H \in \mathcal{H}_i} |E(H)| \leq m$ since the sets $\{E(H)\}_{H \in \mathcal{H}_i}$ are disjoint subsets of $E(G)$. We also claim that $\sum_{H \in \mathcal{H}_i} |V(H)| \leq 2n$.

To see this note that every graph $H \in \mathcal{H}_i$ arrives to the recursive call with a BFS tree $K$. Let $\mathcal{K}_i = \{K$ the BFS tree of $H : H \in \mathcal{H}_i\}$. Note that by construction $\mathcal{K}_i$ is an edge disjoint family of sub-trees of $\tilde{K}$ - the original BFS tree of $G$. Note also that every tree in $\mathcal{K}_i$ has at least 2 vertices as otherwise the tree which called it recursively $K'$, had at most 6 vertices (since $\frac{|V(K')|^2}{2} - 1 \leq |V(K)|$ by Lemma 2.2). If $K'$ had at most 6 vertices it would not have preform any recursive call, since it is in the base case of the recursion.

Let $v \in V(G)$ be some node. We claim that the number of $K \in \mathcal{K}_i$ such that $v \in K$ is at most $\deg_{\tilde{K}}(v)$ - where $\deg_{\tilde{K}}(v)$ is the degree of $v$ in the BFS tree $\tilde{K}$. To see this note that if $v \in K$ then since $K$ is a tree of size at least 2, $v$ has some neighbor $u \in K$. Since $K_i$ is an edge disjoint family of sub-trees, no other $K'$ will have both $v$ and $u$ as nodes outside $K$ and $K'$ will share the edge $(u, v)$ or $(v, u)$. So by the pigeonhole principle there are at least $\deg_{\tilde{K}}(v)$ trees $K \in \mathcal{K}_i$ such that $v \in K$. And so $\sum_{H \in \mathcal{H}_i} |V(H)| = \sum_{K \in \mathcal{K}_i} |V(K)| = \sum_{v \in V(G)} \deg_{\tilde{K}}(v) = 2 \cdot |E(\tilde{K})| = 2 \cdot (n - 1) < 2n$.

Let $Q_i$ denote the set of all query sets in the $i$’th level. Let $Q_i = |\bigcup_{Q \in Q_i} Q|$ denote the total number of queries in the $i$’th level. Note that by Claim 6.1 every recursive call of over a sub-graph $G \in \mathcal{H}_i$ only increase the total number of queries in it is recursive calls by at most $\tilde{O}(|V(H)|^2)$. So we have that $Q_{i+1} = Q_i + \tilde{O}\left(\sum_{H \in \mathcal{H}_i} |V(H)|^2\right)$. Note that $\sum_{H \in \mathcal{H}_i} |V(H)|^2 \leq \left(\sum_{H \in \mathcal{H}_i} |V(H)|\right)^2 \leq (2n)^2$. And so $Q_{i+1} = Q_i + \tilde{O}(n^2)$. Since $Q_0 = (n - 1) \cdot n \cdot 1 \leq n^2$ and $L = O(\log(n))$, we have that $\forall i \in [L] : Q_i = \tilde{O}(n^2)$. Note that this implies that the total number of queries asked by the algorithm is $\sum_{i=0}^{L} Q_i = \sum_{i=0}^{L} \tilde{O}(n^2) = \tilde{O}(n^2)$.

Let $W_i$ denote the set of all weight functions sets in the $i$’th level. Let $W_i = \max_{W \in W_i} |W|$. Note that by Claim 6.1 every recursive call of over a sub-graph $H \in \mathcal{H}_i$ only increase the number of weight functions for each of its recursive calls by at most $\tilde{O}(\sqrt{|V(H)|}) = \tilde{O}(\sqrt{n})$. And so $W_{i+1} = W_i + \tilde{O}(\sqrt{n})$, since $Q_0 = 1$ we have that $\forall i \in [L] : W_i = \tilde{O}(\sqrt{n})$.

Let us fix some $i \in [L]$. Note that by Claim 6.1 the total time the algorithm spends for some recursive call over a graph $G \in \mathcal{H}_i$ with a query set $Q \in Q_i$ can be bounded by $\tilde{O}(|E(H)| \cdot \sqrt{|V(H)|} + |E(H)| \cdot W_i + |V(H)|^2 + |Q|) = \tilde{O}(|E(H)| \cdot \sqrt{n} + |V(H)|^2 + |Q|)$. And so the total time the algorithm spends for the $i$’th level of the recursion is $\tilde{O}\left(\sum_{H \in \mathcal{H}_i} |E(H)| \cdot \sqrt{n}\right) + \tilde{O}\left(\sum_{H \in \mathcal{H}_i} |V(H)|^2\right) + \tilde{O}\left(\sum_{Q \in Q_i} |Q|\right) = \tilde{O}(\sqrt{n} \cdot \sum_{H \in \mathcal{H}_i} |E(H)|) + \tilde{O}\left(\sum_{H \in \mathcal{H}_i} |V(H)|^2\right) + \tilde{O}(Q_i) = \tilde{O}(\sqrt{n} \cdot m) + \tilde{O}(n^2) + \tilde{O}(n^2) = \tilde{O}(m \sqrt{n} + n^2)$. Since the total number of layers is $O(\log(n))$ we have that the total running time of our algorithm is $\tilde{O}(m \sqrt{n} + n^2)$.

7 Conditional Lower Bound for SSRP With Rational Weights

In this section we present our conditional lower bound for the SSRP problem for graphs with rational weights in $[1, 2]$. Let $A$ and $B$ be two $n \times n$ matrices with entries from $\mathbb{R} \cup \{\infty\}$. The min-plus product of these two matrices $A \bullet B$ is defined to be an $n \times n$ matrix $C$ such that $C_{i,j} = \min_{k=1}^{n} \{A_{i,k} + B_{k,j}\}$. Finding efficient algorithms for computing the min-plus product of two matrices is a well-studied and active area of work (see 1, 22, 20). For a subset of numbers $S \subseteq \mathbb{R}$ we denote by $MP(n, m, S)$ the problem of computing the min-plus product of two $n$ by $m$ matrices with entries from $S \cup \{\infty\}$, such that there is a total of at most $m$ entries to the two matrices which are not $\infty$.

In order to represent rational numbers, we assume that we work in the word-RAM model with $w$-bit size words.
Let $N_w$ denote the the set of integers representable as a single $w$-bit computer word. Let $Q_w = \{ \frac{a}{2^k} : a, 2^k \in N_w \}$ be the set of (efficiently) representable rationals. Finally, let $[1,2)_w = Q_w \cap [1,2)$ be the set of representable rationals between 1 and 2.

We denote by $A_{MP}(1,2)(n,m)$ an algorithm for the $MP(n,m,[1,2)_w)$ problem. We prove the following claim.

**Claim 7.1.** Given an algorithm $A_{MP}(1,2)(n,m)$ whose running time is $T(n,m)$, there is a $T(n,m) + O(n^2)$ time algorithm for the $MP(n,m,N_w)$ problem.

**Proof.** The proof of this claim is rather trivial. Let $A, B$ be two matrices that are our input to the $MP(n,m,N_w)$ problem. Assume at least one of the matrices has an entry which is not efficiently representable rationals between 1 and 2. Let $M$ denote the maximum entry which is not $\infty$ among all entries of both $A$ and $B$. Let $M$ denote the smallest power of 2 which is greater than $M$.

We normalize $A$ by dividing it by $M$ and adding 1 to all entries. We denote the resulted matrix by $\overline{A}$. We normalize $B$ in the same way to get the normalized matrix $\overline{B}$. It can be easily verified that $\overline{A}, \overline{B}$ is a valid input to the $MP(n,m,[1,2)_w)$ problem.

We invoke the algorithm $A_{MP}(1,2)$ over the normalized matrices, and obtain a result $\overline{C}$. We then subtract a value of 2 from each entry of $\overline{C}$ and multiply the resulted matrix by $M$. Let $C$ be the resulted matrix. One can easily verify that $C$ is indeed the distance product of the two matrices $A$ and $B$. \qed

We denote by $A_{SSRP}(1,2)(n,m)$ an algorithm for the undirected SSRP problem, for a graph with at most $n$ nodes and $m$ edges, and edge weights from $[1,2)_w$. The following claim is obtained by a slight generalization of the construction presented in [3].

**Claim 7.2.** Given an algorithm $A_{SSRP}(1,2)(n,m)$ whose running time is $T(n,m)$ there is a $O(\sqrt{n}T(O(n), O(m)))$ time algorithm for the $MP(n,m,[1,2)_w)$ problem.

**Proof.** Let $X, Y$ be two $n \times n$ matrices with entries from $[1,2)_w \cup \{\infty\}$ such that there is a total of $m$ entries that are not $\infty$. Denote $Z = X \ast Y$. We will show how using a single invocation of $A_{SSRP}(1,2)$ over a graph with $O(n)$ vertices and $O(m)$ edges and edge weights from $[1,2)_w$, we can compute $Z_{i,j}$ for every $1 \leq i \leq \sqrt{n}$ and $1 \leq j \leq n$. We can then use $O(\sqrt{n})$ invocations of $A_{SSRP}(1,2)$ and "shift" the rows of the matrix $X$ in each invocation to compute the entire matrix $Z$. Hence showing this will imply the claim.

Let us denote $L = \sqrt{n} + 1$. To compute $Z_{i,j}$ for every $1 \leq i < L$ and $1 \leq j \leq n$ we construct the following undirected graph $G$. Let $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_n\}$, $C = \{c_1, c_2, ..., c_n\}$ be three independent sets of vertices. For all $1 \leq i, k \leq n$ if $X_{i,k} \neq \infty$ we add the edge $(a_i, b_k)$ with weight $X_{i,k}$. Similarly for all $1 \leq j, k \leq n$ if $Y_{k,j} \neq \infty$ we add the edge $(b_k, c_j)$ with weight $Y_{j,k}$. We then add a path of new vertices $P = \{x_1, x_2, ..., x_L\}$. For every $1 \leq i \leq L$ we add a path from $x_i$ to $a_i$ of length $8 \cdot (L - i) + 1$ using auxiliary vertices and edges of length 1. Note that for every $1 \leq i \leq L$ it holds that $d(x_i, a_i, G) = i + 8 \cdot (L - i) + 1 = 8L - 7i + 1$. An illustration of this construction can be seen in Figure 3 in the appendix.

We now invoke the algorithm $A_{SSRP}(1,2)$ over $G$ with source node $x_1$. For every $1 \leq i < L$ and $1 \leq j \leq n$ let us denote by $\alpha_{i,j}$ the distance from $x_1$ to $c_j$ with the edge failure $(x_i, x_{i+1})$.

We claim that if $\alpha_{i,j} < 8L - 7i + 5$ then $Z_{i,j} = \alpha_{i,j} - (8L - 7i + 1)$, and otherwise $Z_{i,j} = \infty$. To see why this is true note that the distance in $G$ from $x_1$ to $a_i$ such that $t < i$ is at least $8L - 7i + 8$. Also, note that any path from $x_1$ to $c_j$ in the graph $G - (x_i, x_{i+1})$, that passes through $x_i$ such that $t > i$ is of length at least $d(x_1, a_i, G) + 4 = 8L - 7i + 5$. So we have that $\alpha_{i,j} < 8L - 7i + 5$ if and only if the replacement path from $x_i$ to $c_j$ with edge failure $(x_i, x_{i+1})$ goes from $x_1$ to $a_i$ and then preforms a 3 vertex path: $a_i \rightarrow b_k \rightarrow c_j$ for some index $1 \leq k \leq n$.

If $\alpha_{i,j} < 8L - 7i + 5$ then $k$ must be a index which minimizes the length of this 3 vertex path. Since this length is $X_{i,k} + Y_{k,j}$ we must have that the length of this 3 vertex path is $Z_{i,j}$. Meaning that $Z_{i,j} = \alpha_{i,j} - (8L - 7i + 1)$. Otherwise $\alpha_{i,j} \geq 8L - 7i + 5$, so we have that for every index $k$ there is no such 3 vertex path, meaning that for every $1 \leq k \leq n$ it holds that $X_{i,k} + Y_{k,j} = \infty$, meaning that indeed $Z_{i,j} = \infty$. \qed

We denote by $APSP(n,m,N_w)$ the problem of computing the APSP of a graph with $n$ nodes an $m$ edges with edge weights from $N_w$. We denote by $A_{MP}(n,m)$ an algorithm for the $MP(n,m,N_w)$ problem. The following claim is a well known reduction, presented in [1].
Claim 7.3. Let $A_{MP}(n, m)$ be an algorithm whose running time is $T(n, m)$. Then there is an algorithm for the $APSP(n, m, \mathbb{N}_w)$ problem that runs in time $O(T(n, n^2) \cdot \log(n))$.

We now turn to prove the conditional lower bound, using the above 3 reductions.

Theorem 7.1. Let $A_{SSRP}[1,2](n, m)$ be an algorithm whose running time is $T(n, m)$. If $T(n, m) = O(m \cdot n^{1/2 - \epsilon})$ for some $0 < \epsilon \leq \frac{1}{2}$ then there is a $O(n^{3-\epsilon} \cdot \log(n))$ time algorithm for the $APSP(n, m, \mathbb{N}_w)$ problem.

Proof. Note that by a combination of Claims 7.1 and 7.2 we can use the SSRP algorithm $A_{SSRP}[1,2](n, m)$ to construct an algorithm $A_{MP}(n, m)$ that solves the MP$(n, m, \mathbb{N}_w)$ problem in $O(n^{3-\epsilon})$ time. And so by Claim 7.3 we can construct an algorithm for the $APSP(n, m, \mathbb{N}_w)$ problem in $O(n^{3-\epsilon} \cdot \log(n))$ time.

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8 Appendix

Figures

Figure 6: Tree separation

Figure 7: The $P_k$ partition

Figure 8: $R(s, x, H_w - e)$ is a "help from above" path
Figure 9: The reduction from min-plus product to rationally weighted SSRP. Using a single invocation of SSRP from the source $x_1$ over a graph with $O(n)$ vertices and $O(m)$ edges, we can compute $Z_{i,j}$ for every $1 \leq i < L$ and $1 \leq j \leq n$ (where $L = \sqrt{n} + 1$).
The \(\text{Depart}(s, x, e)\) value - Proof of Correctness

Proof of Completeness
We first prove the following claim

Claim 8.1 (Proof of Completeness for Pivots). Consider some \(k \in [\lfloor \log(n_H) \rfloor]\). Let \(b \in P_k\) be a pivot node and let \(e \in P_k\) be an edge failure, then \(\text{Depart}(b, e) \geq d(s, b, H - e)\)

Proof. The proof is rather trivial. Recall that by definition (Step 5) we have that \(\text{Depart}(b, e) = \min_{u \in V(P) \text{ before } e} \{d(s, u, H) + d(u, b, H - P)\}\). Let \(u \in V(P)\) be the vertex such that \(u\) is before \(e\) in \(P\) that minimizes the expression \(d(s, u, H) + d(u, b, H - P)\).

Note that since \(u\) is before \(e\) in the subpath of \(P\) from \(s\) to \(u\) (denoted \(P(s, u)\)) does not contain \(e\). Note also that since \(e \in P\), the shortest path from \(u\) to \(b\) in the graph \(H - P\) (denoted \(R(u, b, H - P)\)) does not contain \(e\). And so \(P(s, u) \circ R(u, b, H - P)\) is a path from \(s\) to \(b\) in the graph \(H - e\) of length \(d(s, u, H) + d(u, b, H - P)\) = \(\text{Depart}(s, b, e)\). This implies that \(\text{Depart}(b, e) \geq d(s, b, H - e)\).

We now turn to prove the completeness for non-pivot nodes as well.

Claim 5.1 (Proof of Completeness). Let \(e \in P\) be an edge failure and let \(x \in V\) be a destination, then \(\text{Depart}(s, x, e) \geq d(s, x, H - e)\).

Proof. Let \(k \in [\lfloor \log(n_H) \rfloor] \cup \{0\}\) be the unique integer such that \(e \in P_k\). Note that \(k\) exists and is unique since the family \(\{P_k\}_{k \in [\lfloor \log(n_H) \rfloor]} \cup \{0\}\) is an edge disjoint partition of \(P\). If \(k = 0\) then \(\text{Depart}(s, x, e) = d(s, x, H - e)\) by definition and so we are done. Otherwise \(k \neq 0\) and so \(\text{Depart}(s, x, e) = \text{Depart}(s, b, e) + d(x, b, H - P)\) for some \(b \in P_k\). By Claim 8.1 we have that \(\text{Depart}(s, b, e) \geq d(s, b, H - e)\) and since \(e \in P\) we have that \(d(x, b, H - P) \geq d(x, b, H - e)\), so we get that \(\text{Depart}(s, x, e) \geq d(s, b, H - e) + d(x, b, H - e)\). By the triangle inequality we have that \(d(s, b, H - e) + d(x, b, H - e) \geq d(s, x, H - e)\) and so \(\text{Depart}(s, x, e) \geq d(s, x, H - e)\).

Proof of Soundness

Claim 5.2 (Proof of Soundness). Let \(e \in E(P), x \in V(H), w \in W\), assume \(R(s, x, H_w - e)\) is departing and unweighted, and assume that \(R(s, x, H_w - e)\) contains some node \(v_r \in V(T) - \{t\}\). Then w.h.p the length of \(R(s, x, H_w - e)\) is at least \(\text{Depart}(s, x, e)\).

An illustration for this case can be seen in Figure 4.

Proof. Note that since \(R(s, x, H_w - e)\) is an unweighted replacement path, its length is \(d(s, x, H - e)\). If \(e \in P_0\) we have by the definition of \(\text{Depart}(s, x, e)\) that \(\text{Depart}(s, x, e) = d(s, x, H - e)\) and so the claim holds.

The more interesting case would be where \(e \in P_k\) for some \(k \in [\lfloor \log(n_H) \rfloor]\). Note that \(R(s, x, H_w - e)\) must leave the path \(P\) at some node \(u \in P\) before \(e\). Let us denote by \(R[u, x]\) the subpath of \(R(s, x, H_w - e)\) from \(u\) to \(x\), and let us denote by \(R[u, b]\) the subpath of \(R(s, x, H_w - e)\) from \(s\) to \(u\).

Note that since \(u\) is before \(e\) in \(P\) the path from \(s\) to \(u\) in the BFS tree \(K\) does not contain \(e\). Since we assume \(R(s, x, H_w - e)\) is \(K\)-simple this implies that \(R[u, b]\) is the subpath of \(P\) from \(s\) to \(u\). In particular its length is \(d(s, u, H)\) and it is fully contained in \(S\).

Since \(v_r \in V(T) - \{t\}\) we have that \(v_r \notin S\). And so it must be the case that \(v_r \notin R[u, x]\). Since \(v_r \in R[u, x]\) its length is at least \(d(u, v_r, H)\). Note that since \(u \in P\) and \(v_r \in T\) we have that \(u\) is an ancestor of \(t\) which is an ancestor of \(v_r\) in the BFS tree \(K\). And so \(d(R[u, x]) \geq d(u, v_r, H) = d(u, t, H) + d(t, v_r, H) \geq d(u, t, H)\).

Since \(u\) is before \(e\) in \(P\), and since \(e \in P_k\) the distance from \(u\) to \(t\) is at least \(2^k|\sqrt{n_H}|\). So the length of \(R[u, x]\) is at least \(2^k|\sqrt{n_H}|\), and since \(R[u, b]\) is unweighted it must contain at least \(2^k|\sqrt{n_H}|\) vertices. And so by the sampling Lemma 2.4.1 w.h.p we have sampled some node \(b \in B_k\) such that \(b \in R[u, x]\).

Note that since \(R(s, x, H_w - e)\) is departing it will not return to \(P\) after it leaves it at the node \(u\). In other words \(V(R[u, x]) \cap V(P) = \{u\}\), and so \(E(R[u, x]) \cap E(P) = \emptyset\). Since \(R(s, x, H_w - e)\) is unweighted this implies that \(R[u, x]\) is a shortest path from \(u\) to \(x\) in \(H - P\). Since we have shown that \(b \in R[u, x]\) we have that the length of \(R[u, x]\) is \(d(u, b, H - P) + d(b, x, H - P)\).

To conclude, the length of \(R(s, x, H_w - e)\) is exactly \(d(u, b, H - P) + d(b, x, H - P)\). By the definition of \(\text{Depart}(s, b, e)\) we have that \(\text{Depart}(s, b, e) \leq d(s, u, H) + d(u, b, H - P)\) and by the definition of \(\text{Depart}(s, x, e)\)
we have that $\text{Depart}(s, x, e) \leq \text{Depart}(s, b, e) + d(b, x, H - P)$. So we get that the length of $R(s, x, H_w - e)$ is w.h.p at least $\text{Depart}(s, x, e)$.

The case when $e \in P$ and $x = t$ - Proof of correctness

Recall that by the definition of $\hat{d}(s, t, H_w - e)$ in step 8 of the algorithm (line 15 in the pseudocode), $\hat{d}(s, t, H_w - e)$ is set to be the minimum between $A_w[e]$ and $\hat{d}_{RZ}(s, t, H - e)$. Where $\hat{d}_{RZ}(s, t, H - e)$ is the distance estimation obtained by the RP algorithm from [16] and $A_w[e] = \min_{u \text{ is after } e \in P} \{d(s, u, H_w - P) + d(u, t, H)\}$.

Proof of Completeness

**Claim 5.3** (Proof of Completeness). Let $e \in P$ be an edge failure and $w \in W$ a weight function, then $\hat{d}(s, t, H_w - e) \geq d(s, t, H_w - e)$.

**Proof.** We wish to prove that each one of the two values $\hat{d}(s, t, H_w - e)$ can receive ($A_w[e]$ and $\hat{d}_{RZ}(s, t, H - e)$) is at least $d(s, t, H_w - e)$. This will suffice to show that $\hat{d}(s, t, H_w - e) \geq d(s, t, H_w - e)$.

Recall that $\hat{d}_{RZ}(s, t, H - e)$ is the result obtained by the algorithm from [16], over the unweighted directed graph $H$ and the edge failure $e$. By the one-sided error property of the algorithm from [16] we have that $\hat{d}_{RZ}(s, t, H - e) \geq d(s, t, H - e)$, and since $H - e \subseteq H_w - e$ we get that $d(s, t, H - e) \geq d(s, t, H_w - e)$, and so $\hat{d}_{RZ}(s, t, H - e) \geq d(s, t, H_w - e)$ as required.

We now handle the term $A_w[e]$. Let $u \in V(P)$ be some vertex such that $u$ is after $e$ in $P$. Note that since $u$ is after $e$ in $P$, the subpath of $P$ from $u$ to $t$ (denoted by $P(u, t)$) does not contain $e$. Since $P$ is a shortest path in $H$ we have that the length of $P(u, t)$ is $d(u, t, H)$.

Since $e \in P$ we can by concatenate the shortest path from $s$ to $u$ in the graph $H_w - P$ with the path $P(u, t)$ and get a path from $s$ to $t$ in the graph $H_w - e$ of length $d(s, u, H_w - P) + d(u, t, H)$. This implies that $d(s, u, H_w - P) + d(u, t, H) \geq d(s, t, H_w - e)$ which implies in particular that $A_w[e] \geq d(s, t, H_w - e)$ as required.

Proof of Soundness

**Claim 8.2.** Let $e \in E(P), w \in W$, if $R(s, t, H_w - e)$ is weighted then its length is at least $A_w[e]$.

**Proof.** The proof for this case is rather simple. Let $u$ be the first node in $R(s, t, H_w - e)$ such that $u \in P$ and $u$ is after $e$. Note that $u$ exists since $t \in P$ and $t$ is after $e$.

Let $R[s, u]$ denote the subpath of $R(s, t, H_w - e)$ from $s$ to $u$. We claim that $V(R[s, u]) \cap P = \{s, u\}$. To see this assume for the sake of contradiction there is some node $v \in V(s, u)$ such that $v \in P - \{s, u\}$. If $v$ is after $e$, then $u$ is not the first node in $R(s, t, H_w - e)$ which is after $e$, contradiction. If $v$ is before $e$, then the path from $s$ to $v$ in $K$ does not contain $e$. Since we assume $R(s, t, H_w - e)$ is $K$-simple this means that the subpath of $R(s, t, H_w - e)$ from $s$ to $v$ is contained in $K$. However since $R(s, t, H_w - e)$ is weighted the first edge in it is from $E(H_w) - E(H)$ and so is not from $K$, contradiction. We conclude that $V(R[s, u]) \cap P = \{s, u\}$, meaning that $E(R[s, u]) \cap P \subseteq \{s, u\}$. Since $u$ is not the first node in $R(s, t, H_w - e)$, and so (since $A_w[e]$ is weighted) $(s, u) \in E(H_w) - E(H)$, meaning in particular that $(s, u) \notin P$. So we can conclude that $E(R[s, u]) \cap P = \emptyset$. Since $R[s, u]$ is a shortest path from $s$ to $u$ in $H_w - e$ and $e \in P$ we can conclude that the length of $R[s, u]$ is exactly $d(s, u, H_w - P)$.

Let $R[u, t]$ denote the subpath of $R(s, t, H_w - e)$ from $u$ to $t$. Its length is $d(u, t, H_w - e)$ which is at least $d(u, t, H)$ which is equal to (by the weight requirements) $d(u, t, H)$. So the we can conclude that the length of $R(s, t, H_w - e)$ is at least $d(s, u, H_w - P) + d(u, t, H)$. Since $A_w[e] = \min_{u \text{ is after } e \in P} \{d(s, u, H_w - P) + d(u, t, H)\}$ this implies the claim.

**Claim 5.4** (Proof of Soundness). Let $e \in E(P), w \in W$, w.h.p $d(s, t, H_w - e) \geq \hat{d}(s, t, H_w - e)$.

**Proof.** If $R(s, x, H_w - e)$ is weighted then by Claim 8.2 we have that its length is at least $A_w[e]$ meaning that its length is at least $\hat{d}(s, t, H_w - e)$. Otherwise $R(s, x, H_w - e)$ is unweighted, and so $R(s, x, H_w - e)$ is a replacement path from $s$ to $x$ in the graph $H - e$. So by the proof of correctness of the algorithm from [16] we have that the
length of $R(s, x, H_w - e)$ is equal to $\hat{d}_{RZ}(s, x, H - e)$ w.h.p. Meaning that w.h.p the length of $R(s, x, H_w - e)$ is at least $d(s, t, H_w - e)$. □