Fermion Currents on Asymmetric Orbifolds

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Abstract

We study whether orbifold models are equivalently rewritten into torus models in the case of fermionic string theories. It is pointed out that symmetric orbifold models cannot be rewritten into torus models in the case of fermionic string theories because of the absence of twist-untwist intertwining currents on the orbifold models. We present a list of current algebras on asymmetric $\mathbb{Z}_N$-orbifold models of type II superstring theories with inner automorphisms of Lie algebra lattices of the $A_n$ series. It turns out that whether an asymmetric orbifold model is rewritten into a torus model depends on the specific choice of a momentum lattice and an inner automorphism of the lattice.
Various methods have been used to construct four-dimensional string models. However, the relationship between different methods has not been so fully understood. One of the examples showing such relationship between different constructions is the torus-orbifold equivalence \[1\]–\[7\] in bosonic string theories. If the $\mathbb{Z}_N$-transformation of an orbifold model is an inner automorphism of the momentum lattice, then the $\mathbb{Z}_N$-transformation is equivalent to a shift on the lattice. The orbifold model associated with a shift is equivalent to a torus model in the case of bosonic string theories\[1\]. In this paper, we shall investigate whether orbifold models are equivalently rewritten into torus models in the case of fermionic (i.e. heterotic or type II) string theories, which has not been studied in detail before.

We shall define the fermion currents on an orbifold model as the currents of the Kač-Moody algebra generated by the NSR fermions on the orbifold model. Such fermion currents will generate a Kač-Moody algebra which contain the $SO(2)$ Kač-Moody algebra generated by the transverse space-time NSR fermions in the light-cone gauge. On the other hand, fermions on a torus model are all NSR fermions and will generate an $SO(8)$ Kač-Moody algebra in the light-cone gauge. Thus, the necessary condition for the orbifold model to be rewritten into a torus model is that the fermion currents on the orbifold model should generate an $SO(8)$ Kač-Moody algebra and it turns out that this is also the sufficient condition for the orbifold model with an inner automorphism of the momentum lattice to be rewritten into a torus model.

It should be noted that, in order for the fermion currents on an orbifold model to generate an $SO(8)$ Kač-Moody algebra, it is necessary to exist the twist-untwist intertwining currents \[4\]–\[6\] which convert untwisted string states to twisted ones in the orbifold model. The reason for this is that, in each twisted or untwisted sector, the unbroken Kač-Moody algebra generated by the fermions on the orbifold model is always smaller than the $SO(8)$ algebra generated by the fermions on a torus model. Therefore, symmetric orbifold models cannot be rewritten into torus models in the case of fermionic string theories since there is no twist-untwist intertwining current on symmetric orbifold models. In the following, we will construct asymmetric $\mathbb{Z}_N$-orbifold models of type II

\[1\] Examples which suggest strongly the torus-orbifold equivalence with outer automorphisms of the momentum lattices are discussed in ref. \[7\] in the case of bosonic string theories.
superstring theories with inner automorphisms of Lie algebra lattices of the $A_n$ series and investigate whether such asymmetric orbifold models could equivalently be rewritten into torus models.

In the construction of an orbifold model, we start with a 6-dimensional torus compactification of type II superstring theories which is specified by a $(6 + 6)$-dimensional even self-dual lattice $\Gamma_{6,6}$. The left- and right-moving momentum $(p^i_L, p^i_R)$ $(i = 1, \ldots, 6)$ lies on the lattice $\Gamma_{6,6}$. Let $g$ be a group element which generates a cyclic group $Z_N$. The $g$ is defined to act on the left- and right-moving string coordinate $(X^i_L, X^i_R)$ $(i = 1, \ldots, 6)$ by

$$g : (X^i_L, X^i_R) \rightarrow (U^{ij}_L X^j_L, U^{ij}_R X^j_R),$$

where $U_L$ and $U_R$ are rotation matrices which satisfy $U^N_L = U^N_R = 1$. The $g$ acts on the left-movers and the right-movers differently. The $Z_N$-transformation must be an automorphism of the lattice $\Gamma_{6,6}$, i.e.,

$$(U^{ij}_L p^j_L, U^{ij}_R p^j_R) \in \Gamma_{6,6} \quad \text{for all } (p^i_L, p^i_R) \in \Gamma_{6,6}.$$  

The action of the operator $g$ on the left- and right-moving fermions on the orbifold is given by $U_L$ and $U_R$ rotations, respectively.

Let us consider the $g^\ell$-twisted sector in which strings close up to the $g^\ell$-action. We denote the eigenvalues of $U^\ell_L$ and $U^\ell_R$ by $\{e^{i2\pi r^a_L}; a = 1, 2, 3\}$ and $\{e^{i2\pi r^a_R}; a = 1, 2, 3\}$, respectively. Let $N_\ell$ be the minimum positive integer such that $(g^\ell)^{N_\ell} = 1$. The necessary conditions for one-loop modular invariance are for $N_\ell$ even

$$N_\ell \sum_{a=1}^3 r^a_L = 0 \mod 2,$$

$$N_\ell \sum_{a=1}^3 r^a_R = 0 \mod 2,$$

$$p^i_L (U^{N_\ell}_L)^{ij} p^j_L - p^i_R (U^{N_\ell}_R)^{ij} p^j_R = 0 \mod 2$$

for all $(p^i_L, p^i_R) \in \Gamma_{6,6}$; for $N_\ell$ odd, there is no condition for one-loop modular invariance \[3\]. These are called the left-right level matching conditions and it has been proved that these are also sufficient conditions for one-loop modular invariance \[11, 12\].
Suppose that the $Z_N$-transformation is an inner automorphism of the momentum lattice. Then, it can be shown that the $Z_N$-transformation is equivalent to a shift as follows \cite{6}: Let us consider the $g^\ell$-sector (the untwisted sector for $\ell = 0$ and the twisted sector for $\ell = 1, \ldots, N - 1$). Since the rank of the unbroken current algebra in each sector is equal to the dimension of the orbifold, we can always construct the $Z_N$-invariant operators $P_L^i(z)$ and $P_R^i(\bar{z})$ ($i = 1, \ldots, 6$) such that

\[ g(P_L^i(z), P_R^i(\bar{z}))g^{-1} = (P_L^i(z), P_R^i(\bar{z})), \]  

and

\[ P_L^i(w)P_L^j(z) = \frac{\delta^{ij}}{(w - z)^2} + \text{(regular terms)}, \]  

\[ P_R^i(\bar{w})P_R^j(\bar{z}) = \frac{\delta^{ij}}{(\bar{w} - \bar{z})^2} + \text{(regular terms)}, \]

where $g$ is the operator which generates the $Z_N$-transformation in the $g^\ell$-sector. It follows from (7) and (8) that $P_L^i(z)$ and $P_R^i(\bar{z})$ can be expanded as

\[ P_L^i(z) \equiv i\partial_z X^i_L(z) \equiv \sum_{n \in \mathbb{Z}} \alpha^i_L n z^{n-1}, \]  

\[ P_R^i(\bar{z}) \equiv i\partial_{\bar{z}} X^i_R(\bar{z}) \equiv \sum_{n \in \mathbb{Z}} \alpha^i_R n \bar{z}^{n-1}, \]

with

\[ [\alpha^i_L m, \alpha^j_L n] = m\delta^{ij} \delta_{m+n,0}, \]

\[ [\alpha^i_R m, \alpha^j_R n] = m\delta^{ij} \delta_{m+n,0}, \]

where $\alpha^i_{L0} = p^i_L$ and $\alpha^i_{R0} = p^i_R$. Since $P_L^i(z)$ and $P_R^i(\bar{z})$ are invariant under the $Z_N$-transformation, the string coordinate in the new basis transforms as

\[ g(X_L^i(z), X_R^i(\bar{z}))g^{-1} = (X_L^i(z) + 2\pi v^i_L, X_R^i(\bar{z}) - 2\pi v^i_R), \]

for some constant vector $(v^i_L, v^i_R)$. This implies that the string coordinate $(X_L^i(z), X_R^i(\bar{z}))$ in the $g^\ell$-sector obeys the following boundary condition:

\[ (X_L^i(e^{2\pi i}z), X_R^i(e^{-2\pi i}\bar{z})) = (X_L^i(z) + 2\pi \ell v^i_L, X_R^i(\bar{z}) - 2\pi \ell v^i_R) + \text{(torus shift)}, \]  

for some constant $\ell$. This implies that the string coordinate $(X_L^i(z), X_R^i(\bar{z}))$ in the $g^\ell$-sector obeys the following boundary condition:

\[ (X_L^i(e^{2\pi i}z), X_R^i(e^{-2\pi i}\bar{z})) = (X_L^i(z) + 2\pi \ell v^i_L, X_R^i(\bar{z}) - 2\pi \ell v^i_R) + \text{(torus shift)}, \]
and hence that the eigenvalues of the momentum \((p_{\ell}^L, p_{\ell}^R)\) in the new basis are of the form
\[
(p_{\ell}^L, p_{\ell}^R) \in \Gamma^{6,6} + \ell(v_{\ell}^L, v_{\ell}^R).
\] (15)

Since the lattice \(\Gamma^{6,6}\) is self-dual, the shift vector \((v_{\ell}^L, v_{\ell}^R)\) must satisfy
\[
N(v_{\ell}^L, v_{\ell}^R) \in \Gamma^{6,6},
\] (16)
and
\[
\ell(v_{\ell}^L, v_{\ell}^R) \not\in \Gamma^{6,6} (\ell = 1, \ldots, N - 1).
\] (17)

We may bosonize the fermions on the orbifold. The GSO projected left- and right-moving NSR fermions are represented by the bosons \(\phi_L^t(z)\) and \(\phi_R^t(\bar{z})\) \((t = 1, \ldots, 4)\), respectively. The momentum \((p_{\ell}^L, p_{\ell}^R)\) of the bosons each lies on the weight lattice of \(SO(8)\). The momentum in the vector conjugacy class corresponds to the state in the NS sector. The momentum in the spinor or conjugate spinor conjugacy class corresponds to the state in the R sector. We denote the eigenvalues of \(U_L^t\) and \(U_R^t\) as \(\{e^{i2\pi \zeta_L^a}; e^{-i2\pi \zeta_L^a}; a = 1, 2, 3\}\) and \(\{e^{i2\pi \zeta_R^a}; e^{-i2\pi \zeta_R^a}; a = 1, 2, 3\}\), respectively. The \(Z_N\)-transformation acts on \(\phi_L^t(z)\) and \(\phi_R^t(\bar{z})\) as a shift:
\[
g(\phi_L^t(z), \phi_R^t(\bar{z}))g^{-1} = (\phi_L^t(z) + 2\pi v_L^t, \phi_R^t(\bar{z}) - 2\pi v_R^t),
\] (18)
where the shift vector \((v_L^t, v_R^t)\) are given by
\[
v_L^t = (0, \zeta_L^t),
\] (19)
\[
v_R^t = (0, \zeta_R^t).
\] (20)

Thereby, the eigenvalues of the momentum \((p_{\ell}^L, p_{\ell}^R)\) in the \(g^\ell\)-sector are shifted from the ones in the untwisted sector by the constant vector \(\ell(v_L^t, v_R^t)\).

In this bosonized form, the operator \(g\) will be given by
\[
g = \eta(\ell) \exp[i2\pi (p_{\ell}^L v_{\ell}^L - p_{\ell}^R v_{\ell}^R + p_{\ell}^L v_{\ell}^L - p_{\ell}^R v_{\ell}^R)],
\] (21)
where \(\eta(\ell)\) is a constant phase with \((\eta(\ell))^N = 1\). The phase \(\eta(\ell)\) is determined in exactly the same way as in ref. \(\[\]\) by one-loop modular invariance:
\[
\eta(\ell) = \exp[-i\pi \ell((v_L^t)^2 - (v_R^t)^2 + (v_L^t)^2 - (v_R^t)^2)].
\] (22)
Every physical state must obey the condition $g = 1$ because it must be invariant under the $Z_N$-transformation. Thus, the allowed momentum eigenvalues $(p_L^i, p_R^i; p_L^j, p_R^j)$ of the physical states in the $g^j$-sector must satisfy the following condition:

$$p_L^i v_L^i - p_R^i v_R^i + p_L^j v_L^j - p_R^j v_R^j - \frac{1}{2} \ell((v_L^i)^2 - (v_R^i)^2 + (v_L^j)^2 - (v_R^j)^2) = 0 \mod 1. \quad (23)$$

We may now discuss the asymmetric $Z_N$-orbifold models with inner automorphisms of Lie algebra lattices of the $A_n$ series. We take the lattice $\Gamma^{6,6}$ of an asymmetric $Z_N$-orbifold model to be of the form:

$$\Gamma^{6,6} = \{(p_L^i, p_R^i)|p_L^i, p_R^i \in \Lambda^*, p_L^i - p_R^i \in \Lambda\}, \quad (24)$$

where $\Lambda$ is a 6-dimensional lattice and $\Lambda^*$ is the dual lattice of $\Lambda$. It turns out that $\Gamma^{6,6}$ is Lorentzian even self-dual if $\Lambda$ is even integral. In the following, we will take $\Lambda$ in eq. (24) to be the products of root lattices of $A_n$ algebras with the squared length of the root vectors normalized to two [13]. The left- and right-rotation matrices of the $Z_N$-transformation in eq. (1) are taken from the Weyl group elements of the root lattices of the $A_n$ algebras. Then it is easy to see that such a $Z_N$-transformation is an automorphism of the lattice $\Gamma^{6,6}$ in eq. (24).

The conjugacy classes of the Weyl groups of all simple Lie algebras have been classified in ref. [14]. In the following, we will give the results concerning the conjugacy classes of the Weyl groups of Lie algebras of the $A_n$ series. We may use the orthonormal basis $\{e_i; i = 1, \ldots, n + 1\}$ to describe the roots of the $A_n$ algebra. Then the roots of the $A_n$ algebra is given by $\{\pm (e_i - e_j)\}$. In this basis, any element of the Weyl group of the $A_n$ algebra is given by a permutation of the $n + 1$ basis vectors. This permutation can be expressed as a product of the disjoint cyclic permutations of the basis vectors. Let $[m_1, m_2, \ldots]$ be the structure of such cyclic permutations. Then there is a one-to-one correspondence between the structures of the cyclic permutations and the conjugacy classes of the Weyl group elements. Hereafter we shall call $[m_1, m_2, \ldots]$ the cycle-type of the automorphism of the Lie algebra lattice of the $A_n$ series.

Although the left- and right-rotations of the $Z_N$-transformations in eq. (1) can be defined for arbitrary elements of the Weyl groups, the conditions (3), (4) and (5) for the modular invariance put severe restrictions on the allowed left- and right-rotations
of the \(Z_N\)-transformations. All the models we have to consider are shown in table 1. The root lattice \(\Lambda\) associated with the momentum lattice \(\Gamma\) in eq. (24) is given in the second column of table 1. The left- and right-moving cycle-types \(C_L\) and \(C_R\) of the automorphism of the momentum lattice are given in the third and fourth columns of table 1, respectively.

Now we will investigate fermion currents on the asymmetric \(Z_N\)-orbifold models listed in table 1 in order to study whether such orbifold models are equivalently rewritten into torus models in the case of fermionic string theories. However, it is not easy in general to examine current algebras on fermionic string theories due to the (generalized) GSO projections. Then we may use the bosonic string map [15] and investigate current algebras on the corresponding bosonic string models. In the case of four-dimensional string models, the bosonic string map works as follows: The light-cone \(SO(2)\) Kač-Moody algebra generated by the NSR fermions in the fermionic string theory is replaced by an \(SO(10) \times E_8\) Kač-Moody algebra in the bosonic string theory. This is done in such a way that the \(SO(2)\) Kač-Moody characters of the fermionic string theory are mapped to the \(SO(10) \times E_8\) Kač-Moody characters of the bosonic string theory preserving the modular transformation properties of the Kač-Moody characters. Under this map, the gravitino state (i.e. the space-time supercharges) in the fermionic string theory becomes a set of operators of conformal weight one transforming as a spinor of the \(SO(10)\) in the bosonic string theory. Then it can be shown that, in the bosonic string theory, these operators of weight one should extend the \(SO(10)\) Kač-Moody algebra to one of the exceptional algebras of \(E_6\), \(E_7\) and \(E_8\) [16]. On the corresponding bosonic string models, we can easily see that the orbifold models must be asymmetric in order to possess twist-untwist intertwining currents. The reason for this is that the left- and right-conformal weights, \(h\) and \(\bar{h}\), of the ground states of any twisted sector are both positive and equal in the case of the symmetric orbifold models.

Using the bosonic string map, we investigate the fermion currents on the asymmetric orbifold models listed in table 1. It should be noted that we must investigate the full current algebras on the orbifold models which possess twist-untwist intertwining currents in order to determine fermion currents on the orbifold models. The results are summarized in table 2. The fermion currents \(F_L\) and \(F_R\) in the left- and right-moving
degrees of freedom are given in the first and second columns of table 2, respectively. The numbers of space-time supercharges $N_L$ and $N_R$ from the left- and right-moving degrees of freedom are given in the third and fourth columns of table 2, respectively. The left- and right-moving remaining current algebras $G_L$ and $G_R$ of the asymmetric orbifold models are given in the fifth and sixth columns of table 2, respectively. The model numbers of the corresponding asymmetric orbifold models are given in the last column of table 2. In order for an asymmetric orbifold model to be rewritten into a torus model, it is necessary that both the left- and right-moving fermion currents $F_L$ and $F_R$ should generate the $SO(8)$ Kač-Moody algebras. In fact, this is also the sufficient condition for the asymmetric orbifold model with the inner automorphism of the momentum lattice to be rewritten into a torus model since the remaining current algebras $G_L$ and $G_R$ can be constructed in terms of the string coordinate on the 6-dimensional torus whose momentum lies on the lattice determined by the physical momentum of the asymmetric orbifold models.

We have discussed whether orbifold models are equivalently rewritten into torus models in the case of fermionic string theories and presented a list of fermion currents on asymmetric $Z_N$-orbifold models of type II superstring theories with inner automorphisms of Lie algebra lattices of the $A_n$ series. We have found that some of the asymmetric orbifold models can be rewritten into torus models owing to the existence of the twist-untwist intertwining currents and that whether an asymmetric orbifold model is rewritten into a torus model depends on the specific choice of the momentum lattice and an inner automorphism of the lattice. These results in the fermionic string theories differ entirely from the ones in the bosonic string theories because all the symmetric and asymmetric bosonic orbifold models with inner automorphisms of momentum lattices can equivalently be rewritten into the torus models.

It will be straightforward to apply our analysis to heterotic string theories. On the other hand, it would be of interest to classify all asymmetric $Z_N$-orbifold models of type II superstring theories with inner automorphisms of the momentum lattices. Work in this direction is in progress and will be reported elsewhere.

The author would like to thank Dr. M. Sakamoto for reading the manuscript and
useful comments.
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Table 1: Asymmetric $Z_N$-orbifold models with inner automorphisms of Lie algebra lattices of the $A_n$ series. $\Lambda$ denotes the root lattice associated with the momentum lattice. $C_L$ and $C_R$ denote the left- and right-moving cycle-types of the automorphism of the momentum lattice, respectively, where semicolons separate the direct products of the lattices.

| No | $\Lambda$ | $C_L$ | $C_R$ |
|----|-----------|-------|-------|
| 1  | $A_2 \times A_1 \times A_1 \times A_1 \times A_1$ | $[1^3; 1^2; 1^2; 1^2] \times 1^2$ | $[3; 1^2; 1^2; 1^2; 1^2]$ |
| 2  | $A_2 \times A_2 \times A_3 \times A_1$ | $[1^3; 1^3; 1^2; 1^2]$ | $[3; 3; 1^2; 1^2]$ |
| 3  | $A_2 \times A_2 \times A_1 \times A_1$ | $[1^3; 1^3; 1^2; 1^2]$ | $[3; 3; 1^2; 1^2]$ |
| 4  | $A_2 \times A_2 \times A_1 \times A_1$ | $[1^3; 3; 1^2; 1^2]$ | $[3; 1^3; 1^2; 1^2]$ |
| 5  | $A_2 \times A_2 \times A_1 \times A_1$ | $[1^3; 3; 1^2; 1^2]$ | $[3; 3; 1^2; 1^2]$ |
| 6  | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[1^3; 1^3; 3]$ |
| 7  | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[1^3; 3; 3]$ |
| 8  | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[1^3; 3; 3]$ |
| 9  | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[1^3; 3; 3]$ |
| 10 | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[3; 3; 3]$ |
| 11 | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[3; 3; 3]$ |
| 12 | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[3; 3; 3]$ |
| 13 | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[3; 3; 3]$ |
| 14 | $A_2 \times A_2 \times A_2$ | $[1^3; 1^3; 1^3]$ | $[3; 3; 3]$ |
| 15 | $A_3 \times A_1 \times A_1 \times A_1$ | $[1^4; 1^2; 1^2; 1^2]$ | $[3; 1^2; 1^2; 1^2; 1^2]$ |
| 16 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[1^4; 3; 1^2]$ |
| 17 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[3; 1^3; 1^2]$ |
| 18 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[1^4; 3; 1^2]$ |
| 19 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[1^4; 3; 1^2]$ |
| 20 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[3; 1^3; 1^2]$ |
| 21 | $A_3 \times A_2 \times A_1$ | $[1^4; 1^3; 1^2]$ | $[3; 1^3; 1^2]$ |
| 22 | $A_3 \times A_3$ | $[1^4; 1^4]$ | $[1^4; 3; 1]$ |
| 23 | $A_3 \times A_3$ | $[1^4; 1^4]$ | $[3; 1; 3; 1]$ |
| 24 | $A_3 \times A_3$ | $[1^4; 3; 1]$ | $[3; 1; 1^4]$ |
| 25 | $A_3 \times A_3$ | $[1^4; 3; 1]$ | $[3; 1; 3; 1]$ |
| 26 | $A_4 \times A_1 \times A_1$ | $[1^5; 1^2; 1^2]$ | $[3; 1^2; 1^2; 1^2]$ |
| 27 | $A_4 \times A_1 \times A_1$ | $[1^5; 1^2; 1^2]$ | $[5; 1^2; 1^2]$ |
| 28 | $A_4 \times A_1 \times A_1$ | $[3; 1^2; 1^2; 1^2]$ | $[5; 1^2; 1^2]$ |
| No. | $A$ | $C_L$ | $C_R$ |
|-----|-----|-------|-------|
| 29  | $A_4 \times A_2$ | [1; 1] | [3; 3] |
| 30  | $A_4 \times A_2$ | [5; 1] | [3; 1] |
| 31  | $A_4 \times A_2$ | [1; 1] | [3; 1] |
| 32  | $A_4 \times A_2$ | [5; 1] | [3; 3] |
| 33  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 34  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 35  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 36  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 37  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 38  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 39  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 40  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 41  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 42  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 43  | $A_4 \times A_2$ | [5; 1] | [1; 3] |
| 44  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 45  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 46  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 47  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 48  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 49  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 50  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 51  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 52  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 53  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 54  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 55  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 56  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 57  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 58  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
| 59  | $A_5 \times A_1$ | [6; 2] | [3; 1] |
Table 2: Current algebras on asymmetric $Z_N$-orbifold models. $F_L$ and $F_R$ denote the left- and right-moving fermion currents on asymmetric orbifold models, respectively. $N_L$ and $N_R$ denote the numbers of space-time supercharges from left- and right-moving degrees of freedom, respectively. $G_L$ and $G_R$ denote the left- and right-moving remaining current algebras on the asymmetric orbifold models, respectively. In the last column, the model numbers of the corresponding asymmetric orbifold models are indicated.

| $F_L$ | $F_R$ | $N_L$ | $N_R$ | $G_L$ | $G_R$ | models       |
|-------|-------|-------|-------|-------|-------|--------------|
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(7)$ | $SU(7)$ | 50–53, 55–59 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(6) \times SU(2)$ | $SU(6) \times SU(2)$ | 44–46, 48–49 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(5) \times SU(3)$ | $SU(5) \times SU(3)$ | 29–36, 34–36 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(5) \times SU(2)^2$ | $SU(5) \times SU(2)^2$ | 26–28 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(4)^2$ | $SU(4)^2$ | 22,23 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(4) \times SU(3) \times SU(2)$ | $SU(4) \times SU(3) \times SU(2)$ | 16–18 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(4) \times SU(2)^3$ | $SU(4) \times SU(2)^3$ | 15 |
| $SO(8)$ | $SO(8)$ | 4     | 4     | $SU(3)^3$ | $SU(3)^3$ | 6,7 |
| $SO(8)$ | $SO(8)$ | 4     | 2     | $SU(3)^3$ | $SU(3)^3$ | 2,3 |
| $SO(8)$ | $SU(6)$ | 4     | 0     | $SU(5) \times U(1)^3$ | $SU(5) \times U(1)^3$ | 37 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(5) \times U(1)^3$ | $SU(5) \times U(1)^3$ | 32,40 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(5) \times U(1)^3$ | $SU(5) \times U(1)^3$ | 31,40 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(4) \times SU(2) \times U(1)^3$ | $SU(4) \times SU(2) \times U(1)^3$ | 24 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(4) \times SU(2) \times U(1)^3$ | $SU(4) \times SU(2) \times U(1)^3$ | 19 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 8 |
| $SO(6)$ | $SU(6)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 4 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(5) \times U(1)^3$ | $SU(5) \times U(1)^3$ | 54 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(5) \times U(1)^3$ | $SU(5) \times U(1)^3$ | 54 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(4) \times SU(2) \times U(1)^3$ | $SU(4) \times SU(2) \times U(1)^3$ | 47 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(4) \times SU(2) \times U(1)^3$ | $SU(4) \times SU(2) \times U(1)^3$ | 20,25 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 9,11,38 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 12 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 13 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 43 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 43 |
| $SO(6)$ | $SU(4)$ | 4     | 0     | $SU(3)^2 \times U(1)^3$ | $SU(3)^2 \times U(1)^3$ | 14 |