THE TWISTED COHOMOLOGICAL EQUATION OVER THE
GEODESIC FLOW

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Abstract. We study the twisted cohomological equation over the geodesic flow
on $SL(2, \mathbb{R})/\Gamma$. We characterize the obstructions to solving the twisted coho-
mological equation, construct smooth solution and obtain the tame Sobolev
estimates for the solution, i.e, there is finite loss of regularity (with respect to
Sobolev norms) between the twisted coboundary and the solution. We also
give a tame splittings for non-homogeneous cohomological equations. The re-
sult can be viewed as a first step toward the application of KAM method in
obtaining differential rigidity for partially hyperbolic actions in products of
rank-one groups in future works.

1. Introduction.

1.1. Motivation and results. The cohomological equations of the horocycle flow
and the geodesic flow of the homogeneous spaces of $SL(2, \mathbb{R})$ have been well under-
stood, see [3] and [7]. In this paper, we extend the study to the twisted cohomolog-
ical equation of the geodesic flow.

In fact, the study of the twisted cohomological equation provides a tool for obtaining
local differentiable rigidity of algebraic actions by KAM type iteration scheme.
The KAM scheme was used by Damjanovic and Katok to prove local rigidity for genu-
inely higher-rank partially hyperbolic actions on torus in [1]. Later, an improved
version of the scheme was applied on homogeneous space of $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ to
obtain weak local rigidity for certain parabolic algebraic actions [2]. To carry out
the scheme, people need to solve the linearized equation:

$$\text{Ad}(\alpha) \Lambda - \Lambda \circ \alpha = 0$$

over the algebraic action $\alpha$, where $\Lambda$ is valued on the tangent space of the homoge-
neous space. The equation decomposes into the twisted cohomological equations of the form

$$\mu \Lambda_i - \Lambda_i \circ \alpha = 0$$

on the $\mu$-eigenspace of $\text{Ad}(\alpha)$. Hence a complete and detailed description of twisted
cohomological obstructions for the action $\alpha$ is necessary for the scheme.

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In this paper, we give a complete solution to the twisted cohomological equation over the geodesic flow. We construct the solution to the twisted coboundary equation, classify the obstructions and obtain tame estimates of the solution. The results in the present paper will be used to prove local differentiable rigidity of the left translations of the two-dimensional subgroup

\[ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \times \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \times \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \]

where \( s, t \in \mathbb{R} \) on \( SL(2, \mathbb{R})^3 / \Gamma \), see [11].

1.2. History and method. Results concerning the cohomology of horocycle flow are due to Flaminio and Forni in [3]. They used Fourier analysis in each irreducible unitary representations of \( PSL(2, \mathbb{R}) \) to obtain Sobolev estimates of the cohomological equation. These estimates satisfy a uniform upper bound condition, across the class of irreducible representations. Global estimates were then formed by glueing estimates together from each irreducible component. This scheme was further used in [10] to study the cohomological equation of the classical (discrete) horocycle map, and it was also used in [7] to study the cohomological equation of the classical geodesic flow.

In this paper, we follow the same general scheme as in [3] to study the twisted cohomological equation over the geodesic flow. In earlier papers, the obstructions to solving the equation can be constructed explicitly, which provides distributional solutions by Green’s function. For the twisted equations, the obstructions are much more complex, which results in explicit construction is mostly likely impossible. This does seem to require some new techniques for handling it; the same is true in an attempt at obtaining Sobolev estimates of the solution.

2. Statement of results.

2.1. Irreducible representations of \( SL(2, \mathbb{R}) \). We choose as generators for \( \mathfrak{sl}(2, \mathbb{R}) \) the elements

\[ X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \] \tag{2.1}

The Casimir operator is then given by

\[ \Box := -X^2 - 2(UV + VU), \]

which generates the center of the enveloping algebra of \( \mathfrak{sl}(2, \mathbb{R}) \). The Casimir operator \( \Box \) acts as a constant \( u \in \mathbb{R} \) on each irreducible unitary representation space and its value classifies them into three classes except the trivial representation. For Casimir parameter \( \mu \) of \( SL(2, \mathbb{R}) \), let \( \nu = \sqrt{1 - \mu} \) be a representation parameter. We denote by \( (\pi_\nu, \mathcal{H}_\nu) \) or \( (\pi_\mu, \mathcal{H}_\mu) \) the following models for the

1. principal series \( (\nu \in i\mathbb{R}) \);
2. complementary series \( (\nu \in (-1, 1) \setminus \{0\}) \);
3. the mock discrete series or the principal series \( (\nu = 0) \);
4. discrete series representation spaces \( (\nu \in \mathbb{Z} \setminus \{0\}) \).

For the principal series, we also use the notation \( (\pi^+, \mathcal{H}^+) \) for the spherical model and \( (\pi^-, \mathcal{H}^-) \) for the non-spherical model. For the discrete series we also use \( (\pi^+, \mathcal{H}^+) \) to denote the upper half-plane model and \( (\pi^-, \mathcal{H}^-) \) to denote the lower half-plane model.
Any unitary representation \((\pi, \mathcal{H})\) of \(SL(2, \mathbb{R})\) is decomposed into a direct integral (see [3] and [6])

\[
\mathcal{H} = \int_{\mathbb{R}} \ell(\mu)\mathcal{H}_{\mu}dS(\mu)
\]

with respect to a positive Stieltjes measure \(dS(\mu)\) over the spectrum \(\sigma(\square)\). The Casimir operator acts as the constant \(\mu \in \sigma(\square)\) on every Hilbert space \(\mathcal{H}_{\mu}\). Here \(\ell(\mu)\) is the (at most countable) multiplicity of the irreducible representation of \(SL(2, \mathbb{R})\) appearing in \(\pi\). We say that \(\pi\) has a spectral gap (of \(u_0\)) if \(u_0 > 0\) and \(S((0, u_0)) = 0\) and \(\pi\) contains no non-trivial \(SL(2, \mathbb{R})\)-fixed vectors.

In this paper, we only consider unitary representations of \(SL(2, \mathbb{R})\) with a spectral gap. That is, for complementary series, we assume there is \(0 < u_0 < 1\) such that \(\nu \in (-u_0, u_0) \setminus \{0\}\). For the proofs involving the discrete series, we only consider the holomorphic case \((\nu \geq 1)\) because there is a complex antilinear isomorphism between two series of the same Casimir parameter, but we list corresponding results for the anti-holomorphic case \((\nu \leq -1)\).

### 2.2. Statement of the results.

For any unitary representation \((\pi, \mathcal{H})\) of \(SL(2, \mathbb{R})\) let

\[
\mathcal{H}_{X-m}^{-k} = \{D \in \mathcal{H}^{-k} : (X - m)D = 0\}.
\]

The next result characterizes the obstructions to solving the twisted cohomological equation and obtains Sobolev estimates for the solution.

**Theorem 2.1.** Suppose \((\pi, \mathcal{H})\) is a unitary representation of \(SL(2, \mathbb{R})\) with a spectral gap. For the twisted cohomological equation \((X + m)f = g, \ m \in \mathbb{C} \setminus i\mathbb{R}\) we have

1. If \(g \in \mathcal{H}^s\) with \(s \geq \frac{|m|}{2} + 3\), and \(D(g) = 0\) for any \((X - m)\)-invariant distribution \(D\), then the equation has a solution \(f \in \mathcal{H}^{s - \frac{|m|}{2} - 3}\) with estimates
   \[
   \|f\|_t \leq C_{t, m}\|g\|_{t+2}, \quad t \leq s - \frac{|m|}{2} - 3.
   \]

2. If \(g \in \mathcal{H}^s\) with \(s \geq \frac{|m|}{2} + 3\), and the equation \((X + m)f = g\) has a solution \(f \in \mathcal{H}^{s - \frac{|m|}{2} + 3}\), then \(f \in \mathcal{H}^{s - \frac{|m|}{2} - 3}\) with estimates
   \[
   \|f\|_t \leq C_{t, m}\|g\|_{t+2}, \quad t \leq s - \frac{|m|}{2} - 3.
   \]

The next two theorems make a detailed study for the twisted equation in each non-trivial irreducible component of \(SL(2, \mathbb{R})\). Also, tame splittings are provided for non-homogeneous equations.

Let \(I_\nu = 2\mathbb{Z}\) or \(2\mathbb{Z} + 1\) if \(\mu\) parametrizes the principal series, or let \(I_\nu = 2\mathbb{Z}\) if \(\mu\) parametrizes the complementary series or let \(I_\nu = [n, \infty] \subset \mathbb{Z}^+\) or \(I_\nu = [-\infty, n] \subset \mathbb{Z}^-\) if \(\mu\) parametrizes the holomorphic discrete series.

For any \(f = \sum_{k \in I_\nu} f_k u_k \in \mathcal{H}_\nu\) (see Section 3.2) and \(n \in I_\nu \setminus \{0\}\), set

\[
f|_n = \begin{cases} 
  \sum_{k \in I_\nu, k \geq n} f_k u_k, & n > 0 \\
  \sum_{k \in I_\nu, k \leq n} f_k u_k, & n < 0.
\end{cases}
\]
Suppose Theorem 2.2. where \( \delta \) is \( t \) is \( s \) is \( n \) is \( \nu \) in \( \mathbb{R} \); \( S_\nu^+ = \{0, 2\} \) (resp. \( S_\nu^- = \emptyset \)) if \( \nu \in (-\nu_0, 0) \); and \( S_\nu^+ = \{0, 2\} \) (resp. \( S_\nu^- = \emptyset \)) if \( \nu \in \mathbb{Z}^+ \cup 0 \) (resp. \( \nu \in \mathbb{Z}^- \cup 0 \)).

**Theorem 2.2.** Suppose \( m \in \mathbb{C} \setminus \mathbb{R} \). In any non-trivial irreducible representation \((\pi_\nu, \mathcal{H}_\nu^0), \delta = \pm, \nu \in \mathbb{R} \cup \mathbb{Z} \cup (-\nu_0, 0) \) of \( SL(2, \mathbb{R}) \), there exists \( D_{\nu,n}^{\delta,m} \in (\mathcal{H}_\nu^0)_{X-\frac{m}{2}} \) where \( \delta = \pm, n \in S_\nu^0 \) such that for any \( g \in (\mathcal{H}_\nu^0)^*, \) \( s \geq \frac{|m|}{2} + 4 \), \( \delta = \pm \) we have

1. the equation

\[
(X + m)f = g + \sum_{n \in S_\nu^0} D_{\nu,n}^{\delta,m}(g)u_n
\]

has a solution \( f \in (\mathcal{H}_\nu^0)^{s-\frac{|m|}{2}-3} \) with estimates

\[
\|f\|_t \leq C_m \|g\|_{t+|m|/2+3}, \quad t \leq s - \frac{|m|}{2} - 3.
\]

Furthermore, if we write \( f = \sum_{n \in \mathcal{T}_\nu} f_n u_n \in \mathcal{H}_\nu \) we have

\[
\|f\|_t \leq \begin{cases} C_m \|\Theta^{\frac{|m|}{2}+3} (g)\|_t, & \text{if } n < 0, \\ C_m \|\Theta^{\frac{|m|}{2}+3} (g)\|_t, & \text{if } n > 0, \end{cases}
\]

where \( \Theta = U - V \), for any \( t \leq s - \frac{|m|}{2} - 3 \);

2. if the equation \( (X + m)f = g \) has a solution \( f \in (\mathcal{H}_\nu^0)^{\frac{|m|}{2}+4} \) then for \( n \in S_\nu^0 \), \( D_{\nu,n}^{\delta,m}(g) = 0 \);

3. for any \( g \in (\mathcal{H}_\nu^0)^*, s \geq \frac{|m|}{2} + 6 \) and \( n \in S_\nu^\pm \) we have

\[
\|D_{\nu,n}^{\delta,m}(g)u_n\|_t \leq C_m \|\Theta^{\frac{|m|}{2}+6} g\|_t,
\]

for any \( 0 \leq t \leq s - \frac{|m|}{2} - 6 \).

The case of \( m = 0 \) is proved in [7], see Theorem 4.1 and Theorem 4.2.

**Theorem 2.3.** Suppose \( g \in (\mathcal{H}_\nu)^*, s \geq \frac{|m|}{2} + 8, m \in \mathbb{C} \setminus \mathbb{R} \). For any \( n \in \mathcal{I}_\nu \setminus 0 \) with \( |n| \geq \frac{|Re(\nu)|}{2} + 2 \) there exists

\[
\tilde{g} = \begin{cases} a_1 u_n + a_2 u_{n-2}, & \text{if } n > 0, \\ b_1 u_n + b_2 u_{n+2}, & \text{if } n < 0, \end{cases}
\]

where \( a_1, a_2, b_1, b_2 \in \mathbb{C} \) such that the equation

\[
(X + m)f = g|_n - \tilde{g}
\]

has a solution \( f \in (\mathcal{H}_\nu)^{s-\frac{|m|}{2}-3} \) such that \( f = f|_n \) with estimates

\[
\|f\|_t \leq \|g|_n\|_{t+\frac{|m|}{2}+3}, \quad t \leq s - \frac{|m|}{2} - 3.
\]

Moreover, we have

\[
\|\tilde{g}\|_t \leq C_m \|g|_n\|_{t+\frac{|m|}{2}+4}
\]

if \( t \leq s - \frac{|m|}{2} - 4 \).
Remark 2.4. In [12] by using general representation theory, we can show that the solutions in (1) and (2) of Theorem 2.1 are in $H^s$, with estimates

$$||f||_t \leq C_{t,m} ||g||_{\min(t+2,s)}, \quad t \leq s;$$

and the order of the invariant distribution is $\frac{|m|}{2}$ if $m \in \mathbb{R}\setminus 0$. The results of the current paper are not optimal, but are good enough for the application of the KAM method to obtain local rigidity.

3. Preliminaries on representation theory of $SL(2, \mathbb{R})$.

3.1. Sobolev spaces. In this part we follow the nations, symbols and computations from Flaminio-Forni [3]. The Laplacian gives unitary representation spaces a natural Sobolev structure. Let $\pi$ be a unitary representation of $SL(2, \mathbb{R})$ on a Hilbert space $\mathcal{H}$. The Sobolev space of order $s > 0$ is the Hilbert space $H^s \subset \mathcal{H}$ that is the maximal domain determined by the inner product

$$\langle v_1, v_2 \rangle_s = \langle (I + \Delta^s)v_1, v_2 \rangle$$

for any $v_1, v_2 \in \mathcal{H}$.

The subspace $\mathcal{H}^\infty$ coincides with the intersection of the spaces $H^s$ for all $s \geq 0$. $\mathcal{H}^{-s}$, defined as the Hilbert space duals of the spaces $H^s$, are subspaces of the space $\mathcal{E}(\mathcal{H})$ of distributions, defined as the dual space of $\mathcal{H}^\infty$.

In addition to the decomposition (2.2), all the operators in the enveloping algebra are decomposable with respect to the direct integral decomposition (2.2). Hence there exists for all $s \in \mathbb{R}$ an induced direct decomposition of the Sobolev spaces:

$$H^s = \int_{\mathbb{R}} \ell(\mu)H^s_\mu dS(\mu)$$

with respect to the measure $dS(\mu)$ (we refer to [13, Chapter 2.3] or [5] for more detailed account for the direct integral theory).

The existence of the direct integral decompositions (2.2), (3.1) allows us to reduce our analysis of the cohomological equation to irreducible unitary representations. This point of view is essential for our purposes.

3.2. Sobolev norms. There exists an orthogonal basis $\{u_k\}$ in $H_\nu$, basis of eigenvectors of the operator $\Theta = U - V$ and hence of the Laplacian operator $\Delta = \Box - 2\Theta^2$, satisfying:

$$\Theta u_k = ik u_k, \quad \Delta u_k = (\mu + 2k^2)u_k;$$

and the norms of the $u_k$ are given recursively by

$$\|u_k\|^2 = \begin{cases} \|u_{k-2}\|^2, & \nu \in i\mathbb{R} \\ \frac{|k| - 1 - \nu}{|k| - 1 + \nu} \|u_{k-2}\|^2, & \nu \in \mathbb{R}. \end{cases}$$

Here we make a slight change of the normalizations of the basis in [3]. For example for the spherical series, $u_k$ here is indexed by $2\mathbb{Z}$ while in [3] is indexed by $\mathbb{Z}$.

By defining $\Pi_{\nu,k} = \Pi_j^{\ell=\nu}\frac{|k| - 1 - \nu}{|k| - 1 + \nu}$, for any integer $k \geq i_\nu = 1 + \text{Re}(\nu)$ (Empty products are set equal to 1; hence, if $k = i_\nu$, then $\Pi_{\nu,k} = 1$ in all cases) we get that

$$\|u_k\|^2 = |\Pi_{\nu,k}|.$$
From Section 3.1 the Sobolev norms of the vectors of the orthogonal basis \( \{u_k\} \) are given by the identities

\[
\|u_k\|^2_s = \langle (I + \Delta^*) u_k, u_k \rangle = (1 + \mu + 2k^2) \|u_k\|_s^2.
\]

Then the Sobolev norm of a vector \( f = \sum_k f_k u_k \in \mathcal{H}^s \) is:

\[
\|f\|_s = \left( \sum_k (1 + \mu + 2k^2) |\Pi_{\nu,k}| |f_k|^2 \right)^{1/2}.
\] (3.4)

**Lemma 3.1.** (Lemma 2.1 of [3]) If \( \nu \in i\mathbb{R} \), the for all \( k \geq 0 \),

\[
|\Pi_{\nu,k}| = 1.
\]

There exists \( C > 0 \) such that, if \( \nu \in (-1, 1) \setminus \{0\} \) for all \( k > 0 \), we have

\[
C^{-1} \left( \frac{1 - \nu}{1 + \nu} \right) (1 + k)^{-\nu} \leq |\Pi_{\nu,k}| \leq C \left( \frac{1 - \nu}{1 + \nu} \right) (1 + k)^{-\nu};
\]

if \( \nu \in \mathbb{Z}^+ \) for all \( k \geq \ell \geq i_\nu = \nu + 1 = n \), we have

\[
C^{-1} \left( \frac{k - n + 2}{\ell - n + 2} \right)^{-\nu} \leq |\Pi_{\nu,k}| \leq C \left( \frac{k - n + 2}{\ell - n + 2} \right)^{-\nu}.
\]

By the above lemma, \( \|u_k\|^2_s \approx (1 + |k|)^{2s-\text{Re}(\nu)} \). So it follows that,

\[
\|f\|_s \approx \left( \sum_k (1 + |k|)^{2s-\text{Re}(\nu)} |f_k|^2 \right)^{1/2}.
\] (3.5)

4. The basic solutions of the twisted equation. In this section we study the twisted cohomological equation

\[
(X + m)f = g
\] (4.1)

\( m \in \mathbb{C} \setminus i\mathbb{R} \) of the classical geodesic flow defined by the \( \mathfrak{sl}(2, \mathbb{R}) \)-matrix

\[
X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

in each \( \mathcal{H}_\nu \). The action of \( X \) on the basis element \( \{u_k\} \) is given by:

**Lemma 4.1.** (Lemma 3.4 of [3]) We have

\[
(X + m)u_k = \frac{k + 1 + \nu}{2} u_{k+2} + mu_k - \frac{k - 1 - \nu}{2} u_{k-2}, \quad \forall k \in I_\nu.
\]

For \( n \in \mathbb{N} \) \( \nu = n - 1 \) and \( k = n \), the above equation must be read as \( (X + m)u_n = mu_n + nu_{n+2} \).

Let \( f = \sum_k f_k u_k \) and \( g = \sum_k g_k u_k \) be the Fourier expansions of the distributions \( f, g \) with respect to the adapted basis of \( \mathcal{H}_\nu \). So the twisted equation (4.1) becomes

\[
g_k = -\frac{k + 1 - \nu}{2} f_{k+2} + mf_k + \frac{k - 1 + \nu}{2} f_{k-2}
\] (4.2)

for all \( k \in I_\nu \); for \( \nu = n - 1 \) (discrete series) and \( k = n \) equation (4.2) should be read as

\[
g_n = mf_n - f_{n+2}.
\]

**Definition 4.2.** We say that a vector \( f = \sum_k f_k u_k \in \mathcal{H}_\nu \) is \( \Theta \)-finite if where exists \( \ell \in \mathbb{N} \) such that \( f_k = 0 \) if \( |k| \geq \ell \). It is clear that if \( f \) is \( \Theta \)-finite then \( f \in \mathcal{H}_\nu^{\ell, infinite} \).

From (4.2) we have a simple **observation**: if \( (X + m)f = g \), where \( g = \sum_{k \in S_\nu} g_k u_k \) and \( f \) is \( \Theta \)-finite, then \( f = g = 0 \).
4.1. **Uniqueness of the solution.** For the twisted equation (4.1), if \( m = 0 \), in each non-trivial irreducible component of \( SL(2, \mathbb{R}) \), the uniqueness of the solution is guaranteed by the ergodicity of the geodesic flow [4]. For the case of \( m \neq 0 \), the next results show that it is unique in any representations.

4.2. **Explicit construction of basic solutions.** For any \( n \in I_\nu \setminus S_\nu \), we want to find the vector

\[
\mathbf{U}_n = \begin{cases} 
  u_n + d_0 u_0 + d_2 u_2; & \text{if } u_n \in \mathcal{H}_\nu^+, \nu \in i\mathbb{R} \cup (-1, 1) \\
  u_n + d_{-1} u_{-1} + d_1 u_1; & \text{if } u_n \in \mathcal{H}_\nu^-, \nu \in i\mathbb{R} \cup (-1, 1) \\
  u_n + d_{\nu+1} u_{\nu+1}; & \text{if } u_n \in \mathcal{H}_\nu^+, \nu \in \mathbb{Z}^+ \cup 0,
\end{cases}
\]

such that there is a \( \Theta \)-finite \( f_{(n)} \) such that

\[
(X + m) f_{(n)} = \mathbf{U}_n.
\]

We write

\[
f_{(n)} = \begin{cases} 
  \sum_{2 \leq 2k \leq n} b_{n,n+2k} u_{n+2k}; & \text{if } n < 0 \\
  \sum_{2 \leq 2k < n} b_{n,n-2k} u_{n-2k}; & \text{if } n > 0.
\end{cases}
\]

By (4.2), if \( n \geq 1 \) it is easy to see that

\[
b_{n,n-2} = \frac{2}{n-1 + \nu}, \quad b_{n,n-4} = \frac{4m}{(n-1+\nu)(n-3+\nu)},
\]

and for any \( k \geq 1 \) with \( n - 2k \geq 0 \), we can obtain the sequence \( b_{n,n-2k} \) using a recursive rule along with the two initial elements \( b_{n,n-2} \) and \( b_{n,n-4} \):

\[
b_{n,n-2k} = \frac{2m}{n-2k+1+\nu} b_{n,n-2k+2} + \frac{n-2k+3-\nu}{n-2k+1+\nu} b_{n,n-2k+4}.
\]

If \( n \leq -1 \) we get that

\[
b_{n,n+2k} = b_{n,-n-2k}, \quad \text{if } n < 0, k \geq 1, \text{ and } n + 2k \leq 0.
\]

Set

\[
\mathbf{U}_n = (X + m) f_{(n)}.
\]

It is clear that \( \mathbf{U}_n \) has the same form as in (4.3).

These \( f_{(n)}, n \in \mathbb{Z} \) are called basic solutions and will be used to construct explicit solution of the twisted equation (4.1) in next Section 5.2. We now make a slight digression to obtain upperbounds of

\[
|b_{n,n-2k}| \frac{\|u_{n-2k}\|}{\|u_n\|},
\]

which will be used to estimate the Sobolev orders of the basic solutions, see Proposition 5.1.

**Proposition 4.3.** For \( \nu \in i\mathbb{R} \cup \mathbb{Z}^+ \cup (-u_0, u_0) \) and \( n - 2k \in I_\nu \) we have

\[
|b_{n,n-2k}| \frac{\|u_{n-2k}\|}{\|u_n\|} \leq C_m \left| \frac{n}{|\nu| + 1} \right|^{\frac{|m|+2}{2}}.
\]

From (4.6) it suffices to consider the case of \( n \geq 0 \) and \( n - 2k \geq 0 \). The remaining part of this section will be dedicated to the proof of this proposition.
4.3. Upperbounds for $\nu \in i\mathbb{R}$. We set
\[
c_{n-2k} = \frac{2}{|n-2k+1+\nu|} \prod_{\ell=1}^{k} (1 + \frac{|m|}{|n-2\ell+1+\nu|}). \tag{4.8}
\]

Lemma 4.4. Suppose $\nu \in i\mathbb{R}$ and $n-2k \geq \max\{6, |m|^2\}$, then
\[
|b_{n,n-2k}| \leq c_{n-2k}.
\]

Proof. We prove by induction. From (4.5) we see that that
\[
|b_{n,n-2k}| \leq c_{n-2k}, \quad k = 1, 2.
\]
Suppose
\[
|b_{n,n-2k}| \leq c_{n-2k}, \quad 2 \leq k \leq r, \tag{4.9}
\]
where $n-2r \geq \max\{6, m^2\}$.

If $n-2(r+1) \geq \max\{6, m^2\}$ then
\[
|b_{n,n-2r-2}| \leq \frac{n-2r+1-\nu}{n-2r-1+\nu} b_{n,n-2r-2} + \left| \frac{2}{n-2r-1+\nu} c_{n-2r-2} \right| \tag{4.10}
\]
\[
\leq \frac{n-2r+1-\nu}{n-2r-1+\nu} c_{n-2r-2} + \left| \frac{2}{n-2r-1+\nu} c_{n-2r} \right| c_{n-2r-2} \tag{4.10}
\]
\[
= \frac{n-2r+1-\nu}{n-2r-1+\nu} \cdot \frac{2}{|n-2r+3+\nu|} \prod_{\ell=1}^{r-1} (1 + \frac{|m|}{|n-2\ell+1+\nu|})
\]
\[
+ \frac{2}{|n-2r-1+\nu|} \prod_{\ell=1}^{r-1} (1 + \frac{|m|}{|n-2r+1+\nu|})
\]
\[
\cdot \left( \frac{n-2r+1-\nu}{n-2r+3+\nu} + \frac{2|m|}{|n-2r+1+\nu|} \right).
\]

In (1) we use the assumption (4.9).

Hence to show that $|b_{n,n-2r-2}| \leq c_{n-2r-2}$, it suffices to prove that
\[
\frac{n-2r+1-\nu}{n-2r+3+\nu} + \frac{2|m|}{|n-2r+1+\nu|} (1 + \frac{|m|}{|n-2r+1+\nu|}) \tag{4.10}
\]
\[
\leq \prod_{\ell=1}^{r+1} (1 + \frac{|m|}{|n-2\ell+1+\nu|}).
\]

In fact, for any $x \geq \max\{5, |m|^2-1\}$ and $b \in \mathbb{R}$ we have
\[
\frac{\sqrt{x^2+b^2}}{(x+2)^2+b^2} \leq \sqrt{1 - \frac{4x+4}{(x+2)^2+b^2}} \leq 1 - \frac{2x+2}{(x+2)^2+b^2}
\]
\[
\leq 1 - \frac{|m|^2}{x^2+b^2} = 1 - \frac{|m|}{\sqrt{x^2+b^2}} (1 + \frac{|m|}{\sqrt{x^2+b^2}})
\]
\[
\leq (1 + \frac{|m|}{\sqrt{x^2+b^2}}) (1 + \frac{|m|}{\sqrt{(x-2)^2+b^2}} - \frac{2|m|}{\sqrt{x^2+b^2}}).
\]

In (1) we use the inequality $\sqrt{1-y} \leq 1 - \frac{1}{2}y$ if $0 \leq y \leq 1$; in (2) we use the assumption that $x \geq \max\{5, |m|^2-1\}$.

Then (4.10) follow immediately from the above inequality by letting $b = |\nu|$ and $x = n-2r+1$. Hence we finish the proof. □
Corollary 4.5. Suppose \( \nu \in i\mathbb{R} \). If \( n \geq 0 \) and \( n - 2k \geq 0 \) then

\[
|b_{n,n-2k}\|\frac{\|u_{n-2k}\|}{\|u_n\|} \leq C_m \frac{|m|}{\nu + 1}.
\]

Proof. We note that if \( n \geq 0 \) and \( n - 2k \geq 2 \) then

\[
\log \left( \prod_{\ell=1}^{k} \left( 1 + \frac{|m|}{n-2\ell+1+\nu} \right) \right) = \sum_{\ell=1}^{k} \log \left( 1 + \frac{|m|}{n-2\ell+1+\nu} \right).
\]

By Lemma 3.1 we have

\[
|\Pi_{\ell=1}^{k+1} \left( 1 + \frac{|m|}{n-2\ell+1+\nu} \right) \| \leq \int_{1}^{k+1} \frac{|m|}{n-2\ell+1} d\ell = \frac{|m|}{2} \log \left( \frac{n-1}{n-2k-1} \right).
\]

In (1) we use the inequality \( \log(1+x) \leq x \), for any \( x \geq 0 \) and integral inequalities.

The above inequality shows that if \( n - 2k \geq 2 \) then

\[
|c_{n-2k}| \leq \frac{2\sqrt{2}}{n-2k+1+\nu} \left( \frac{n-1}{n-2k-1} \right)^{|m|}.
\]

By Lemma 3.1 we have \( \|u_{n-2k}\| = \|u_n\| \). Then the above estimates and Lemma 4.4 imply the conclusion. \( \square \)

4.4. Upperbounds for \( \nu \in (-1,1) \setminus \{0 \} \). Let

\[
c_{n-2k} = \frac{4}{n-2k+1+\nu} \Pi_{\ell=1}^{k} \left( 1 + \frac{|m|+2-\nu}{n-2\ell-1+\nu} \right).
\]

Lemma 4.6. Suppose \( \nu \in (-1,1) \setminus \{0 \} \) and \( n - 2k - \nu > \max\{7, 2|m|(|m|+2)\} \). Then \( |b_{n,n-2k}| \leq c_{n-2k} \).

Proof. We prove by induction. From (4.5) it is easy to check that

\[
|b_{n,n-2k}| \leq c_{n-2k}, \quad k = 1, 2.
\]

if \( n - 2k - \nu > \max\{7, 2|m|(|m|+2)\} \).

Suppose

\[
|b_{n,n-2r}| \leq c_{n-2r}, \quad 2 \leq r \leq k - 1,
\]

(4.11)

where \( n - 2(k-1) > \max\{7, 2|m|(|m|+2)\} \).

If \( n - 2k > \max\{7, 2|m|(|m|+2)\} \) we have

\[
|b_{n,n-2k}| \leq \frac{2|m|}{n-2k+1+\nu} |b_{n,n-2k+2}| + \frac{n-2k+3-\nu}{n-2k+1+\nu} |b_{n,n-2k+4}|
\]

\[
\leq \frac{2|m|}{n-2k+1+\nu} |c_{n-2k+2}| + \frac{n-2k+3-\nu}{n-2k+1+\nu} |c_{n-2k+4}|
\]

\[
= \frac{2|m| \cdot 4}{(n-2k+1+\nu)(n-2k+3+\nu)} \Pi_{\ell=1}^{k-1} \left( 1 + \frac{|m|+2-\nu}{n-2\ell-1+\nu} \right)
\]

\[
+ \frac{4(n-2k+3-\nu)}{(n-2k+1+\nu)(n-2k+5+\nu)} \Pi_{\ell=2}^{k-2} \left( 1 + \frac{|m|+2-\nu}{n-2\ell-1+\nu} \right)
\]

\[
= \frac{4}{n-2k+1+\nu} \Pi_{\ell=3}^{k-2} \left( 1 + \frac{|m|+2-\nu}{n-2\ell-1+\nu} \right)
\]

\[
\left( \frac{2|m|}{n-2k+3+\nu} \left( 1 + \frac{m+2-\nu}{n-2k+1+\nu} \right) + \frac{n-2k+3-\nu}{n-2k+5+\nu} \right).
\]

In (1) we use the assumption (4.11).
Then to prove that $|b_{n, n-2k}| \leq c_{n-2k}$ it suffices to show that
\[
\frac{2|m|}{n - 2k + 3 + \nu} (1 + \frac{|m| + 2 - \nu}{n - 2k + 1 + \nu}) + \frac{n - 2k + 3 - \nu}{n - 2k + 5 + \nu}
\leq \Pi_{k=1}^k (1 + \frac{|m| + 2 - \nu}{n - 2\ell - 1 + \nu}).
\] (4.12)

If $2 - |m| \geq \nu$ we have
\[
\frac{n - 2k + 3 - \nu}{n - 2k + 5 + \nu} = 1 - \frac{2\nu + 2}{n - 2k - 1 + \nu} \leq 1 - \frac{2\nu - 4}{n - 2k - 1 + \nu}
\leq (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu}) (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu} - \frac{2|m|}{n - 2k - 1 + \nu})
\leq (1 + \frac{|m| + 2 - \nu}{n - 2k + 1 + \nu}) (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu} - \frac{2|m|}{n - 2k - 1 + \nu})
\]
Here in (1) we use the relation: $(1 + a)(1 + b) \geq 1 + a + b$ if $ab > 0$. Hence we get (4.12).

If $\nu > 2 - |m|$, we have
\[
1 - \frac{2\nu + 2}{n - 2k - 1 + \nu} = 1 - \frac{2\nu - 4}{n - 2k - 1 + \nu} + \frac{2\nu + 2}{n - 2k + 5 + \nu}
= 1 - \frac{2\nu - 4}{n - 2k - 1 + \nu} + \frac{6(n - 2k - \nu + 3)}{(n - 2k - 1 + \nu)(n - 2k + 5 + \nu)}
\leq 1 - \frac{2\nu - 4}{n - 2k - 1 + \nu} + \frac{(|m| + 2 - \nu)(|m| - 2 + \nu)}{(n - 2k - 1 + \nu)^2}
= (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu}) (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu} - \frac{2|m|}{n - 2k - 1 + \nu})
\leq (1 + \frac{|m| + 2 - \nu}{n - 2k + 1 + \nu}) (1 + \frac{|m| + 2 - \nu}{n - 2k - 1 + \nu} - \frac{2|m|}{n - 2k - 1 + \nu})
\]
Here in (1) we use that
\[
n - 2k + 5 + \nu \leq 2(n - 2k - 1 + \nu),
\]
if $n - 2k - \nu > 7$; and
\[
n - 2k - \nu > (|m| + 2 - \nu)(|m| - 2 + \nu),
\]
if $n - 2k - \nu > 2|m|(|m| + 2)$. Hence we finish the proof.

**Corollary 4.7.** Suppose $\nu \in (-1, 1) \setminus 0$. If $n \geq 0$ and $n - 2k \geq 0$ then
\[
|b_{n, n-2k}| \|u_{n-2k}\| \|u_n\| \leq C_{m} n^{\frac{|m|+2}{|\nu|+1}}.
\]

**Proof.** Similar to the proof of Corollary 4.5, by using $\log(1 + x) \leq x$ and integral inequalities we have
\[
|c_{n-2k}| \leq C_{m} \left( \frac{n - 1 + \nu}{n - 2k - 1 + \nu} \right)^{\frac{|m|+2-\nu}{2}}.
\]
By Lemma 3.1 we have
\[
\|u_{n-2k}\| \|u_n\| \leq C \left( \frac{n - 2k + 1}{n + 1} \right)^{-\frac{\nu}{2}}.
\]
The above estimates and Lemma 4.6 imply the conclusion.

4.5. Upperbounds for \( \nu \in \mathbb{Z}^+ \). Let

\[
c_{n-2k} = \frac{2}{[(n - 2k + 1)^2 - \nu^2]^2} \prod_{\ell=1}^{k} (1 + \frac{|m|}{n - 2\ell - 1 - \nu}).
\]

Lemma 4.8. Suppose \( \nu \in \mathbb{Z}^+ \) and \( n - 2k - \nu > \max(6, 2|m|^2 + 2) \). Then

\[
|b_{n,n-2k}| \frac{\|u_{n-2k}\|}{\|u_n\|} \leq c_{n-2k}.
\]

Proof. We prove by induction. By (3.3) and (4.5) it is easy to check that

\[
|b_{n,n-2k}| \frac{\|u_{n-2k}\|}{\|u_n\|} \leq c_{n-2k}, \quad k = 1, 2.
\]

if \( n - 2k - \nu > \max\{7, 2|m|(|m| + 2)\} \). Suppose

\[
|b_{n,n-2r}| \frac{\|u_{n-2r}\|}{\|u_n\|} \leq c_{n-2r}, \quad 1 \leq r \leq k - 1,
\]

where \( n - 2k - \nu > \max\{7, 2|m|(|m| + 2)\} \). Then

\[
|b_{n,n-2k}| \frac{\|u_{n-2k}\|}{\|u_n\|} \leq \frac{2|m|}{n - 2k + 1 + \nu} |b_{n,n-2r}| \frac{\|u_{n-2r}\|}{\|u_n\|} \leq \frac{2|m|}{n - 2k + 1 + \nu} \left( \frac{n - 2k + 3 + \nu}{n - 2k + 1 + \nu} \right)^\frac{1}{2} |b_{n,n-2k+2}| \frac{\|u_{n-2k+2}\|}{\|u_n\|}.
\]
For any $b \in \mathbb{R}$ and $x \geq \max\{2|m|^2 + 2, |b| + 6\}$ and we have

$$\frac{\sqrt{x^2 - b^2}}{(x + 2)^2 - b^2} \leq \sqrt{1 - \frac{4x + 4}{(x + 2)^2 - b^2}} \leq 1 - \frac{2x + 2}{(x + 2)^2 - b^2} \leq 1 - \frac{m^2}{(x + 2)^2 - b^2} \leq 1 - \frac{|m|^2}{x^2 - b^2}$$

In (1) we use the inequality $\sqrt{1 - \frac{y}{2}} \leq 1 - \frac{1}{2}y$ if $0 \leq y \leq 1$; in (2) we use the fact that $\frac{2x + 2}{(x + 2)^2 - b^2} < \frac{1}{3}$ and $(x + 2)^2 - 3(2(x + 2) - m^2) - b^2 > 0$; if $x \geq |b| + 6$; in (3) we use the fact that if $x > 2m^2$ then $(x + 2)^2 - 3(2(x + 2) - m^2) < x^2$;

in (4) we use that $\frac{1}{\sqrt{x^2 - b^2}} \leq \frac{1}{x^2}$ if $x > b > 0$.

Then (4.14) follows immediately from the above inequality by letting $b = \nu$ and $x = n - 2k + 3$. Hence we finish the proof.

**Corollary 4.9.** Suppose $\nu \in \mathbb{Z}^+$. If $n \geq 1 + \nu$ and $n - 2k \geq 1 + \nu$ then

$$|b_{n,n-2k}||u_{n-2k}| \leq C_m \frac{n^{|m|+2}}{[\nu]+1}.$$

**Proof.** Similar to the proof of Corollary 4.5, by using $\log(1 + x) \leq x$ and integral inequalities we have

$$|c_{n-2k}| \leq C_m \left(\frac{n - \nu}{n - 2k - \nu}\right)^{|m|},$$

if $1 \leq k \leq \frac{1}{2}(n - 1 - \nu)$. This and Lemma 4.8 imply the conclusion.

It is clear that Proposition 4.3 is a direct consequence of Corollary 4.5, 4.7 and 4.9.

5. Invariant distributions and Sobolev norms of the solution.

**Proposition 5.1.** For any $n \in \mathbb{Z}\setminus0$ and $t \geq 0$ we have

$$\|f_{\{n\}}\|_1 \leq C_m (1 + \mu + 2n^2)^{\frac{1}{2}} \frac{|n|^{\frac{|m|+3}{|\nu|}}}{[\nu]+1}\|u_n\|.$$  

**Proof.** By (3.4) and (4.6) we have

$$\|f_{\{n\}}\|^2 \leq \sum_{2 \leq 2k \leq |n| \quad |n|-2k \in I_t} (1 + \mu + 2(|n| - 2k)^2)^t |b_{|n|,|n|-2k}|^2 |u_{|n|-2k}|^2$$
We have proof.

Lemma 5.2. Suppose from (4.7) we have

\[ 5.1. \]

\((2)\) is a direct consequence of Proposition 4.3; in \((3)\) we use \((3.3)\); in \((1)\) and \((4)\) we

\[ \begin{align*}
\end{align*} \]

Here \((1)\) is a direct consequence of Proposition 4.3. Then we finish the proof. \(\square\)

5.1. Invariant distributions in \(H_\nu\). For \(n \in S^s_\nu\) the linear functional

\[ \mathcal{D}^{\delta,m}_{\nu,n} : g = \sum_{k \in I_\nu} g_k u_k \rightarrow -g_n + \frac{\nu - n - 1}{2} \sum_{j \in I_\nu \setminus S^s_\nu} b_{j,n+2} g_j + \frac{\nu + n - 1}{2} \sum_{j \in I_\nu \setminus S^s_\nu} b_{j,n-2} g_j \]

is defined for any \(g \in H_{\nu}^{\frac{|m|+s}{2}}\). We note that if \(\nu \in \mathbb{Z}^+\), then \(b_{\nu+1,\nu-1} = 0\). In fact, from (4.7) we have

\[ g + \sum_{n \in S^s_\nu} \mathcal{D}^{\delta,m}_{\nu,n}(g) u_n = (X + m) \left( \sum_{n \in I_\nu \setminus S^s_\nu} gn f(n) \right). \]

\[ (5.1) \]

Lemma 5.2. Suppose \(g \in H_{\nu}^s\), \(s \geq \frac{|m|+6}{2}\). Then

\[ \|\mathcal{D}^{\delta,m}_{\nu,n}(g) u_n\|_t \leq C_m \|g\|_t, \quad t \leq s - \frac{|m|+6}{2}. \]

Proof. We have

\[ \begin{align*}
\|\mathcal{D}^{\delta,m}_{\nu,n}(g) u_n\|_t & \leq C(1 + u + 2n^2)^\frac{1}{2} (|m| + |\nu|) \sum_{\ell = -2, 0, 2} \sum_{j \in I_\nu \setminus S^s_\nu} |b_{j,n+\ell} g_j| \|u_n\| \\
& \leq C(1 + u)^\frac{1}{2} (|m| + |\nu|) \sum_{\ell = -2, 0, 2} \sum_{j \in I_\nu \setminus S^s_\nu} \left(|b_{j,n+\ell}| \frac{\|u_n\|}{\|u_j\|}\right) \left(\frac{\|u_n\|}{\|u_{n+\ell}\|}\right) \left(\frac{\|u_n\|}{\|u_j\|}\right) \left(\frac{\|g_j\|}{\|u_j\|}\right) \\
& \leq C_m (1 + u)^\frac{1}{2} \sum_{\ell = -2, 0, 2} \sum_{j \in I_\nu \setminus S^s_\nu} \left(|j| |g_j| \|u_j\|\right) \left(|j| \frac{|m|+6}{2} \|g_j\| \|u_j\|\right) \\
& \leq C_m (1 + u)^\frac{1}{2} \sum_{j \in I_\nu \setminus S^s_\nu} \left(|j| \frac{|m|+6}{2} \|g_j\| \|u_j\|\right) \left(|j| \frac{|m|+6}{2} \|g_j\| \|u_j\|\right) \\
& \leq C_m (1 + u)^\frac{1}{2} \left(\sum_{j \in I_\nu \setminus S^s_\nu} \left(|j| \frac{|m|+6}{2} \|g_j\| \|u_j\|\right) \right)^\frac{1}{2} \\
& \leq C_m \|g\|_t
\end{align*} \]

(2) is a direct consequence of Proposition 4.3; in (3) we use (3.3); in (1) and (4) we use (3.4). Hence we finish the proof. \(\square\)

In fact, these \(\mathcal{D}^{\delta,m}_{\nu,n}(g)\) are \((X - m)\)-invariant distributions.

Proposition 5.3. If there is \(f \in H_{\nu}^{\frac{|m|+s}{2}}\) such that \((X + m) f = g\), then for \(n \in S^s_\nu\), \(\mathcal{D}^{\delta,m}_{\nu,n}(g) = 0\).

We postpone the proof to Section 5.3.
5.2. Sobolev estimates of the solution. We are now ready to give the explicit solution of the twisted equation.

Lemma 5.4. Suppose \( g \in (H^\delta_{\nu})^s \) with \( s > \frac{|m|+6}{2} \). If \( D^{h,m}_{\nu,n}(g) = 0 \) for any \( n \in S_{\nu}^{\delta} \), then the equation (4.1) has a solution \( f \in (H^\delta_{\nu})^{s-|m|/2-3} \) with estimates

\[
\|f\|_t \leq C_m \|g\|_{t+\frac{|m|}{2}+3}, \quad t \leq s - \frac{|m|}{2} - 3. \tag{5.2}
\]

Furthermore, we have

\[
\|f\|_n \leq \begin{cases} 
C_m \|g|_{n-2}\|_t & \text{if } n \leq 0 \\
C_m \|g|_{n+2}\|_t & \text{if } n > 0,
\end{cases} \tag{5.3}
\]

for any \( t \leq s - \frac{|m|}{2} - 3 \) (see (2.3)).

Proof. Let

\[
f = \sum_{n \in I_{\nu}\setminus S_{\nu}} g_n f_{\{n\}} \tag{5.4}
\]

(see (4.4)). From (5.1) we see that the assumption \( D^{h,m}_{\nu,n}(g) = 0 \) for any \( n \in S_{\nu}^{\delta} \) implies that \( f \) is a formal solution of the equation (4.1).

Furthermore, from (4.4) we have

\[
f|_{n} = \begin{cases} 
\sum_{n \in I_{\nu}\setminus S_{\nu}, \atop k \leq -1} g_{n+2k} f_{\{n+2k\}} & \text{if } n \leq 0 \\
\sum_{n \in I_{\nu}\setminus S_{\nu}, \atop k \geq 1} g_{n+2k} f_{\{n+2k\}} & \text{if } n > 0,
\end{cases}
\]

By Proposition 5.1 for any \( n \in \mathbb{N} \), any \( \delta > 0 \) and \( t < s - \frac{|m|+5}{2} \) we have

\[
\|f\|_n \leq \sum_{n \in I_{\nu}\setminus S_{\nu}, \atop k \geq 1} |g_{n+2k}| \|f_{\{n+2k\}}\|_t
\]

\[
\leq \sum_{n+2k \in I_{\nu}, \atop k \geq 1} C_m (1 + u + 2(n + 2k)^2)^{\frac{|m|+3}{2}} (n + 2k)^{\frac{|m|+5}{2}} \|g_{n+2k} u_{n+2k}\|
\]

\[
\leq C_m \left( \sum_{k \geq 1} (1 + u + 2(n + 2k)^2)^{\frac{|m|+5}{2}} \|g_{n+2k} u_{n+2k}\|^2 \right)^{\frac{1}{2}}
\]

\[
(1) \leq C_m \|g|_{n+2}\|_t. \tag{5.5}
\]

Here in (1) we use (3.4).

Hence we prove (5.3) for \( n > 0 \). The proof of the case of \( n < 0 \) is similar. Then we get (5.3). It is clear that (5.2) is a direct consequence of (5.3). Hence we finish the proof.

5.3. Proof of proposition 5.3. We need an additional step to get to the proof.

Lemma 5.5. We have

\[
D^{h,m}_{\nu,n}((X + m)u_k) = 0, \quad \forall k \in I_{\nu}.
\]
Proof. Set \(g = (X + m)u_k\). Let

\[
g' = g + \sum_{n \in S^2_\nu} D_{\nu,n}^\delta (g)u_n = (X + m)u_k + \sum_{n \in S^2_\nu} D_{\nu,n}^\delta (g)u_n. \tag{5.6}
\]

Then \(D_{\nu,n}^\delta (g') = 0\), for any \(n \in S^2_\nu\). Since \(g'\) is \(\Theta\)-finite (see Definition 4.2), by Lemma 5.4, we have \(f \in \mathcal{H}^\infty\) such that

\[
(X + m)f = g'
\]

moreover, \(f\) is also \(\Theta\)-finite.

Hence, it follows from (5.6) and (5.7) that

\[
\sum_{n \in S^2_\nu} D_{\nu,n}^\delta (g)u_n = (X + m)(f - u_k).
\]

Since \(f - u_k\) is also \(\Theta\)-finite, the observation after Definition 4.2 implies that \(f - u_k = 0\) and \(\sum_{n \in S^2_\nu} D_{\nu,n}^\delta (g)u_n = 0\). Hence we finish the proof.

Now we are ready to prove Proposition 5.3. Write \(f = \sum_{k \in I_\nu} f_k u_k\). For any \(\ell \in \mathbb{N}\) let

\[
g_\ell = (X + m)(\sum_{k \in I_\nu, |k| \leq \ell} f_k u_k).
\]

From (3.4) we see that

\[
\left\| \sum_{k \in I_\nu, |k| > \ell} f_k u_k \right\|_{|m|+\frac{3}{2}} \to 0, \quad \text{as } \ell \to \infty.
\]

Hence we have

\[
\left\| g_\ell - g \right\|_{|m|+\frac{3}{2}} \to 0, \quad \text{as } \ell \to \infty.
\]

Then it follows from Lemma 5.2 that

\[
D_{\nu,n}^\delta (g) = \lim_{\ell \to \infty} D_{\nu,n}^\delta (g_\ell) = 0.
\]

Hence we finish the proof.

5.4. **Extended distributions and solutions.** For any \(n \in \mathbb{Z} \setminus \{0\} \) with \(|n| \geq |\text{Re}(\nu)| + 2\) we consider the subspace \(\mathcal{F}_{n, \nu} = \{ g \in (\mathcal{H}_\nu)^s : g = g_{|n|}, s \geq \frac{|m|+8}{2} \}\). For any \(g \in \mathcal{F}_{n, \nu}\) let \(\tilde{f} = (\sum_{\ell \in I_\nu} g_{\ell} f(\ell))_{|n|}\). Then it follows from (5.5) of the proof of Lemma 5.4 immediately that

\[
||\tilde{f}||_t \leq \|g\|_{t+\frac{|m|}{2}+3}, \quad t \leq s - \frac{|m|}{2} - 3. \tag{5.8}
\]

It is clear that

\[
(X + m)(\sum_{\ell} g_{\ell} f(\ell))_{|n|} = g - \tilde{g},
\]

where

\[
\tilde{g} = \begin{cases} 
\mathcal{E}_{\nu,n+1}(g)u_n + \mathcal{E}_{\nu,n,2}(g)u_{n-2}; & \text{if } n > 0 \\
\mathcal{E}_{\nu,n+1}(g)u_n + \mathcal{E}_{\nu,n,2}(g)u_{n+2}; & \text{if } n < 0,
\end{cases}
\]

with

\[
\mathcal{E}_{\nu,n,1}(g)u_n = \begin{cases} 
\mathcal{E}_{\nu,n+1}(g)u_{n-1}; & \text{if } n > 0 \\
\mathcal{E}_{\nu,n+1}(g)u_{n+1}; & \text{if } n < 0.
\end{cases}
\]

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Suppose (\pi, \mathcal{H}) is a unitary representation of SL(2, \mathbb{R}). Then for any \( g \in \mathcal{H} \) and any \( m \in \mathbb{C} \setminus \mathbb{R} \), the twisted equation

\((X + m)f = g\)  \hspace{1cm} (6.1)

has a unique solution \( f \in \mathcal{H} \) with \( \|f\| \leq |\Re(m)|^{-1}\|g\| \). Further, if \( f \in \mathcal{H}^s \) and \( g \in \mathcal{H}^{s+2} \), then

\( \|f\| \leq C_{t,m}\|g\|_{t+2}, \quad t \leq s. \)  \hspace{1cm} (6.2)

**Proof.** In SL(2, \mathbb{R}) let \( X_t \) denote the classical geodesic flow \( \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\}_{t \in \mathbb{R}} \). Since \( X_t \) is isomorphic to \( \mathbb{R} \) we have a direct integral decomposition

\[ \pi |_{X_t} = \int_{\mathbb{R}} \chi(t)du(\chi) \]

where \( u \) is a regular Borel measure and

\[ v = \int_{\mathbb{R}} v \chi du(\chi), \quad \forall v \in \mathcal{H}. \]

Set

\[ f_\chi = (m + \chi'(0))^{-1}g_\chi, \quad \chi \in \hat{\mathbb{R}}. \]

We see that \( f = \int_{\mathbb{R}} (m + \chi'(0))^{-1}g_\chi du(\chi) \) is a formal solution of the equation \((X + m)f = g\).

Next, we will show that \( f \in \mathcal{H} \). Since \( \chi'(0) \in i\mathbb{R} \) and \( |m + \chi'(0)| \geq |\Re(m)| \) for any \( \chi \in \hat{\mathbb{R}} \), we have

\[ \|f\|^2 = \int_{\mathbb{R}} |m + \chi'(0)|^{-2}\|g_\chi\|^2du(\chi) \leq |\Re(m)|^{-2} \int_{\mathbb{R}} \|g_\chi\|^2du(\chi) = |\Re(m)|^{-2}\|g\|^2. \]

This shows that \( f \in \mathcal{H} \).

On the other hand, if \((X + m)f = 0\) with \( f \in \mathcal{H} \), then we have

\[(m + \chi'(0))f_\chi = 0 \]
for almost every $\chi \in \hat{\mathbb{R}}$ with respect to $u$. This implies that $f_\chi = 0$ for almost every $\chi \in \hat{\mathbb{R}}$. Then we have $f = 0$. Hence we showed the uniqueness of the solution of the twisted equation. This completes the proof of the first part.

If $f \in \mathcal{H}^s$, from equation (6.2) for $0 \leq n \leq s$ we have

$$(X + m)X^nf = X^n g, \ (X + m + 2n)V^n f = V^n g, \ (X + m - 2n)U^n f = U^n g.$$ 

From the arguments in the first part of the proof, we have

$$\|Z_i f\| \leq C_{i,m} \|Z_i g\|,$$

where $0 \leq i \leq s$ and $i \neq \frac{m(n)}{2}$; and $Z$ stands for $X, U,$ and $V$.

Then (6.2) is a direct consequence of the above estimates and the elliptic regularity theorem (see [9, Chapter I, Corollary 6.5 and 6.6]).

7. Proof of Theorem 2.1, 2.2 and 2.3. Proof of Theorem 2.2. Let

$$g_1 = g + \sum_{n \in S_{\nu}^\delta} D_{\nu,n}^\delta (g) u_n.$$ 

We see that $D_{\nu,n}^\delta (g_1) = 0$ for any $n \in S_{\nu}^\delta$. Then by Lemma 5.4, the equation

$$(X + m)f = g_1$$ 

has a solution $f \in \mathcal{H}_\nu$, where $f = \sum_{n \in I_\nu \setminus S_\nu} g_n f \{n\}$. Hence (1) follows from Lemma 5.4. (2) is a direct consequence of Proposition 5.3. (3) is Lemma 5.2.

Proof of Theorem 2.3. It follows directly from Section 5.4.

Proof of Theorem 2.1. We consider the decomposition of $\pi$ as in (2.2) and the Sobolev spaces decomposition as in (3.1). We write $g = \int g_\mu dS(\mu)$.

(1): Arguments in Section 3.1 allow us to apply (1) of Theorem 2.2 to each $g_\mu$. Hence the equation

$$(X + m)f_\mu = g_\mu$$

has a solution $f_\mu$ with estimates

$$\|f_\mu\|_t \leq C_m \|g_\mu\|_{t + \frac{|m|}{2} + 3}$$

if $t \leq s - \frac{|m|}{2} - 3$.

Let $f = \int f_\mu dS(\mu)$. Then

$$\|f\|_t^2 = \int \ell(\mu) \|f_\mu\|_t^2 dS(\mu) \leq C_m \int \ell(\mu) \|g_\mu\|_{t + \frac{|m|}{2} + 3}^2 dS(\mu)$$

$$= \|g\|_{t + \frac{|m|}{2} + 3}^2$$

(7.1)

if $t \leq s - \frac{|m|}{2} - 3$. This shows that $f \in \mathcal{H}^{s - \frac{|m|}{2} - 3}$. Then we can use Lemma 6.1 to get a refinement of the estimates:

$$\|f\|_t \leq C_{t,m} \|g\|_{t + \frac{|m|}{2} + 3}, \quad t \leq s - \frac{|m|}{2} - 3.$$ 

(7.2)

Hence we finish the proof.

(2): By above arguments, we write $f = \int f_\mu dS(\mu)$. The assumption implies that $f_\mu \in \mathcal{H}_{\mu}^{s - \frac{|m|}{2} + 3}$ for almost all $\mu$. Then it follows from (2) and (1) of Theorem 2.2
that \( f_\mu \in \mathcal{H}^{s-\frac{|m|}{2}-3}_\mu \) with the estimate
\[
\|f_\mu\|_{s-\frac{|m|}{2}-3} \leq C_m\|g_\mu\|_s
\]
for almost all \( \mu \).

Following the same way as in (7.1) and (7.2), we have
\[
\|f\|_t \leq C_{t,m}\|g\|_{t+2}, \quad t \leq s - \frac{|m|}{2} - 3.
\]
Hence we finish the proof.

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REFERENCES

[1] D. Damjanovic and A. Katok, Local Rigidity of Partially Hyperbolic Actions. I. KAM method and \( \mathbb{Z}^k \) actions on the torus, Annals of Mathematics, 172 (2010), 1805–1858.
[2] D. Damjanovic and A. Katok, Local rigidity of homogeneous parabolic actions: I. A model case, J. Modern Dyn., 5 (2011), 203–235.
[3] L. Flaminio and G. Forni, Invariant distributions and time averages for horocycle flows, Duke Math J., 119 (2003), 465–526.
[4] R. Howe and C. C. Moore, Asymptotic properties of unitary representations, J. Func. Anal., 32 (1979), Kluwer Acad., 72–96.
[5] G. A. Margulis, Discrete Subgroups of Semisimple Lie Groups, Springer-Verlag, 1991.
[6] F. I. Mautner, Unitary representations of locally compact groups, II, Ann. of Math., (2) 52 (1950), 528–556.
[7] D. Mieczkowski, The Cohomological Equation and Representation Theory, Ph.D thesis, The Pennsylvania State University, 2006.
[8] F. A. Ramirez, Cocycles over higher-rank abelian actions on quotients of semisimple Lie groups, Journal of Modern Dynamics, 3 (2009), 335–357.
[9] D. W. Robinson, Elliptic Operators and Lie Groups, Oxford Mathematical Monographs, 1991.
[10] J. Tanis, The cohomological equation and invariant distributions for horocycle maps, Ergodic Theory and Dynamical systems, 34 (2014), 299–340.
[11] Z. J. Wang, Various smooth rigidity examples in \( SL(2, \mathbb{R}) \times \cdots SL(2, \mathbb{R})/\Gamma \), in preparation.
[12] Z. J. Wang, The twisted cohomological equation over the partially hyperbolic flow, submitted, arXiv:1809.04672
[13] R. J. Zimmer, Ergodic Theory and Semisimple Groups, Birkhäuser, Boston, 1984.

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