MOMENT ESTIMATES AND WELL-POSEDNESS OF THE BINARY-TERNARY BOLTZMANN EQUATION

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Abstract. In this paper, we show generation and propagation of polynomial and exponential moments, as well as global well-posedness of the homogeneous binary-ternary Boltzmann equation. We also show that the co-existence of binary and ternary collisions yields better generation properties and time decay, than when only binary or ternary collisions are considered. To address these questions, we develop for the first time angular averaging estimates for ternary interactions. This is the first paper which discusses this type of questions for the binary-ternary Boltzmann equation and opens the door for studying moments properties of gases with higher collisional density.

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1. Introduction

The Boltzmann equation (cf. [21, 22, 13, 30]), given by
\[ \partial_t f + v \cdot \nabla_x f = Q_2(f, f), \]
for the position \( x \in \mathbb{R}^d \), the time \( t \in \mathbb{R}^+ \) and velocity \( v \in \mathbb{R}^d \) describes the evolution of the density \( f(t, x, v) \) of gas particles, where \( Q_2(f, f) \) is a quadratic integral operator that expresses the change of \( f \) due to instantaneous binary collisions of particles. The exact form of \( Q_2 \) depends on the type of interaction between particles. However, when the gas is dense enough, higher order interactions
become relevant too. For example, in the case of a colloid\footnote{which is a homogeneous non-crystalline substance consisting of ultramicroscopic particles of one substance dispersed through a second substance.} it was noted by Russ - Von-Günberg\cite{38} that multi interactions among particles significantly contribute to the grand potential of a colloidal gas and are modeled by a sum of higher order interaction terms. Motivated in part by this observation, in a sequence of works\cite{9,8,7} Ampatzoglou and Pavlović proposed a toy model for a non-ideal gas of the form:

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f = \sum_{k=2}^{m} Q_k(f,f,\cdots,f), & (t,x,v) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \\
f(0,x,v) = f_0(x,v), & (x,v) \in \mathbb{R}^d \times \mathbb{R}^d,
\end{cases}
\] (1.2)

where \(Q_k(f,f,\cdots,f)\) denotes the \(k\)-th order collisional operator, and \(m \in \mathbb{N}\) is the parameter that reflects the order of the highest collision allowed. Also equations similar to (1.2) were studied for Maxwell molecules in the works of Bobylev, Cercignani and Gamba in \cite{16,15}.

In this paper we continue our analysis of (1.2) with \(m = 3\). We refer to this equation as the binary-ternary Boltzmann equation. While in our previous work\cite{10} we established global well-posedness near vacuum for the binary-ternary equation, the paper at hand focuses on understanding behavior of moments for the probability density associated with the spatially homogeneous version of the binary ternary equation which we write as:

\[
\begin{cases}
\partial_t f = Q[f] & (t,v) \in (0, +\infty) \times \mathbb{R}^d, \\
f(t = 0) = f_0,
\end{cases}
\] (1.3)

where the binary-ternary collision operator is given by

\[
Q[f] := Q_2(f,f) + Q_3(f,f,f),
\] (1.4)

with operators \(Q_2\) and \(Q_3\) given in Section 2.1. Importantly, we show that the homogeneous binary-ternary equation (1.3) is “better behaved” compared to the homogeneous version of the Boltzmann equation (1.1), namely

\[
\partial_t f = Q_2(f,f),
\] (1.5)

or the homogeneous version of the purely ternary equation

\[
\partial_t f = Q_3(f,f,f).
\] (1.6)

In order to describe what we mean by the phrase “better behaved”, we recall definitions of polynomial and exponential moments of a measurable function and present a brief summary of results on generation and propagation in time of moments associated with solutions to (1.5).

Given \(k \geq 0\), we define the \(k\)-th order polynomial moment of a measurable non-negative function \(f(t,v)\) as

\[
m_k[f](t) := \int_{\mathbb{R}^d} f(t,v) \langle v \rangle^k \, dv,
\] (1.7)
where $\langle v \rangle = \sqrt{1 + |v|^2}$.

Analogously, given $s > 0$ and $z > 0$, we define exponential moment of order $s$ and rate $z$ of a measurable non-negative function $f(t, v)$ as

$$E_s[f](t, z) := \int_{\mathbb{R}^d} f(t, v) e^{z \langle v \rangle^s} dv.$$  \hspace{1cm} (1.8)

When it is clear from the context which function we are referring to, we will just write $m_k$ instead of $m_k[f]$, and $E_s$ instead of $E_s[f]$.

For each of these two types of moments, one can consider proving either generation or propagation of moments in time. In particular, by introducing

$$\phi(v) = \begin{cases} \langle v \rangle^k & \text{in the case of polynomial moments}, \\ e^{z \langle v \rangle^s} & \text{in the case of exponential moments} \end{cases},$$

by generation in time of $\phi$-moments one means

$$\int_{\mathbb{R}^d} f_0(v) \langle v \rangle^2 dv < \infty \quad \Rightarrow \quad \int_{\mathbb{R}^d} f(t, v) \phi(v) dv < \infty \text{ for all } t > 0. \hspace{1cm} (1.9)$$

On the other hand, by propagation in time of $\phi$-moments one means

$$\int_{\mathbb{R}^d} f_0(v) \phi(v) dv < \infty \quad \Rightarrow \quad \sup_{t \geq 0} \int_{\mathbb{R}^d} f(t, v) \phi(v) dv < \infty. \hspace{1cm} (1.10)$$

Moment estimates for the spatially homogeneous Boltzmann equation (1.5) have been studied for decades. In the setting that is similar to ours - the case of an integrable angular kernel $b_2$, sometimes referred to as the cutoff case, with variable hard potentials, that is when the potential rate $\gamma_2$ of the angular kernel is strictly positive (see (2.7) for the definition of $b_2$ and $\gamma_2$) - it is known (cf. [33, 1]) that generation of polynomial moments, (1.9) with $\phi(v) = \langle v \rangle^k$, as well as propagation of polynomial moments, (1.10) with $\phi(v) = \langle v \rangle^k$, hold for any $k > 2$, and once moments become finite they remain uniformly bounded in time. Similarly, (cf. [1]) exponential moment of order $\gamma_2$ is generated instantaneously, i.e. (1.9) holds with $\phi(v) = e^{z \langle v \rangle^{2\gamma_2}}$ for some $z > 0$. Also, exponential moments of orders $s \in (0, 2]$ propagate in time i.e., (1.10) with $\phi(v) = e^{z \langle v \rangle^{2\gamma_2}}$ holds for $s \in (0, 2]$. For more results regarding moment estimates for the spatially homogeneous Boltzmann equation and its variations, see [2, 6, 11, 12, 14, 17, 20, 23, 24, 25, 26, 27, 29, 32, 34, 35, 36, 37, 39, 40, 41].

One of the main results of this paper demonstrates that the spatially homogeneous binary-ternary Boltzmann equation (1.3) behaves better that the classical homogeneous Boltzmann equation (1.5) in the following sense (see Theorem 2.7):

- Adding the ternary operator $Q_3$ to the classical Boltzmann equation (1.5) can improve the order of the exponential moment that is generated. In other words, the binary-ternary Boltzmann equation generates higher order exponential than the binary Boltzmann equation (1.5) or ternary Boltzmann equation (1.6) alone.

- Generation of exponential moments happens even if one of $\gamma_2, \gamma_3$ is zero (corresponding to the Maxwell molecules case) as long as the other one is strictly positive. This is in contrast with the binary Boltzmann equation for the Maxwell molecules, in which case generation of exponential moments is not known to happen.

\footnote{Note that moments are increasing with respect to their order i.e. $m_{k_2}[f] \leq m_{k_1}[f]$, when $k_1 \leq k_2$.}
Additionally, we show that exponential moments of solutions to (1.3) of order \( s \in (0, 2) \) as well as polynomial moments of order \( k > 2 \) propagate in time (see Theorem 2.7 and Theorem 2.8), as was the case with the Boltzmann equation (1.5). Finally, polynomial moments of any order are generated in time as long as initial mass and energy are finite (see Theorem 2.6).

The proof of propagation and generation of moments estimates is done in two phases:

1. Phase 1: proving that polynomial moments are finite. More precisely, in Theorem 5.1 we show that any solution corresponding to initial data with finite mass and energy has finite and differentiable moments of any order \( k > 2 \). This is proven by an inductive argument that relies on the new decomposition of the collision operator and the novel angular averaging estimate (see Lemmata 3.2 and 3.5).

2. Phase 2: proving quantitative moment estimates (see Subsection 5.2). Here one uses Povzner-type angular averaging estimates on the binary and ternary gain operators (3.3), (3.32), (3.33), and Lemma 3.1 to obtain an ordinary differential inequality for polynomial moments, which results in quantitative estimates after comparison to a Bernoulli-type ODE.

Although the moments analysis described above provides explicit bounds on the generation and propagation of polynomial and exponential moments respectively, the results proved are still a-priori, in the sense that they assume existence of a solution to the equation (1.3). However, since it is the first time that the binary-ternary homogeneous equation (1.3) is studied in the literature, we need to address its well-posedness as well. That is exactly what we do. Namely we prove that, as long as the initial data have \( 2 + \varepsilon \) finite moments, there exists a unique, global in time, solution to the equation (1.3); see Theorem 2.8 for more details. To prove this existence theorem, we will rely on techniques of the general theory for ODEs in Banach spaces, namely Theorem A.8, which was implemented in the context of the Boltzmann equation for the first time by Bressan in [19]. Subsequently, versions of this technique have been used in the context of the Boltzmann equation [19], systems of Boltzmann equations for gas mixtures [27], as well as the quantum Boltzmann equation [5].

The instruments that are essential for obtaining moments as well as well-posedness results in this paper are the angular averaging estimates. For both binary and ternary collision operators, we prove two types of such estimates:

1. The first type provides an upper bound on the angular averaging part of the gain operator (3.3), (3.32) in terms of the total energy of the interaction (see Lemma 3.1 and Lemma 3.5). These estimates are used to prove the second type of angular averaging described below, and to establish the propagation and generation of moments. In the binary case, this type of estimate was obtained in [14, 17] and is typically referred to as Povzner-type estimate. To the best of our knowledge, this is the first paper that establishes such an estimate for the ternary collision operator.

2. The second type introduces a new decomposition of the angular averaging part of the collision operator, which we refer to as the modified gain and modified loss terms (see (3.13), (3.14), (3.41), (3.42)) and consequently provides upper bounds on the modified gain terms and lower bounds on the corresponding modified loss terms (see Lemma 3.2 and Lemma 3.5). These estimates are then applied in an inductive argument that establishes finiteness of moments (for details see Subsection 5.1).
While these estimates are inspired by the work of Mischler and Wennberg [33], we emphasize that they are novel even in the binary case. More precisely, results in [33] rely on the representation of post-collisional energies of particles as a sum of a convex combination of pre-collisional energies and a remainder. We, on the other hand, base our estimates on representing post-collisional energies as a fraction of the total energy of the interaction. This representation is especially suitable for higher order interactions, such as ternary, where pre-post collisional laws are more complex.

**Organization of the paper.** In Section 2 we provide the notation pertaining to the binary-ternary Boltzmann equation and some of its basic properties, and we state our main results. In Section 3 we derive angular averaging estimates, while in Section 4 we establish estimates on the collision operator for a function that is not necessarily solution to the binary-ternary Boltzmann equation. In Section 5 we focus on propagation and generation of polynomial moments, while in Section 6 we prove propagation and generation of exponential moments. In Section 7 we prove well-posedness of the equation. Finally, in the Appendix we gather properties of the collision operator and its kernel, as well as provide some general results such as: estimates for polynomials and convex functions, auxiliary moment estimates, a general well-posedness theorem for ODEs in Banach spaces, and a lower convolution type bound for generic functions of uniformly nonnegative mass and bounded energy.

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2. Notation and main results

2.1. Vocabulary. We begin this section by introducing notation that will be used throughout the paper. After that, we will state our main results.

**The binary-ternary Boltzmann equation.** We study the well-posedness and generation and propagation of polynomial and exponential moments for the homogeneous binary-ternary Boltzmann equation

\[
\begin{aligned}
\partial_t f &= Q[f] := Q_2(f, f) + Q_3(f, f, f), \quad (t, v) \in (0, +\infty) \times \mathbb{R}^d, \\
f(t = 0) &= f_0.
\end{aligned}
\] (2.1)

In (2.1), \(Q_2(f, f)\) is the binary collisional operator given by

\[
Q_2(f, f) = \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left( f' f'_1 - f f_1 \right) \, d\omega \, dv_1,
\] (2.2)

where \(\omega \in S^{d-1}\) is the relative position and \(u := v_1 - v\) is the relative velocity of the colliding particles, \(f' := f(t, v'), f'_1 := f(t, v'_1), f := f(t, v), f_1 := f(t, v_1)\), and the post-collisional velocities \(v', v'_1\) are related to the pre-collisional velocities \(v, v_1\) via the binary collisional law:

\[
\begin{aligned}
v' &= v + (\omega \cdot u) \omega, \\
v'_1 &= v_1 - (\omega \cdot u) \omega.
\end{aligned}
\] (2.3)
Clearly the binary momentum-energy conservation system is satisfied i.e.
\[
v' + v'_1 = v + v_1, \quad |v'|^2 + |v'_1|^2 = |v|^2 + |v_1|^2. \tag{2.4}
\]
Denoting \( u' := v'_1 - v' \) the post-collisional relative velocity, one can also easily verify that the binary micro-reversibility condition holds
\[
u' \cdot \omega = -u \cdot \omega, \tag{2.5}
\]
as well as the conservation of binary relative velocities magnitude
\[
|u'| = |u|. \tag{2.6}
\]
Moreover, given \( \omega \in S^{d-1} \), the transformation \((v, v_1) \rightarrow (v', v'_1)\) is a linear measure-preserving involution of \( \mathbb{R}^{2d} \).

The binary cross-section \( B_2(u, \omega) \), which expresses the statistical repartition of binary collisions, is assumed of the form
\[
B_2(u, \omega) = |u|^\gamma_2 b_2(\hat{u} \cdot \omega), \quad u \neq 0, \quad \gamma_2 \in [0, 2], \tag{2.7}
\]
where the binary angular cross-section \( b_2 : [-1, 1] \rightarrow [0, +\infty) \) is an even function. Then relations \(2.6, 2.7\) yield
\[
B_2(u', \omega) = B_2(u, \omega), \quad \forall u \neq 0, \quad \omega \in S^{d-1}. \tag{2.8}
\]
We assume \( b_2 \) satisfies the cut-off assumption \( b_2(z)(1 - z^2)^{\frac{d-3}{2}} \in L^1([-1, 1], dz) \). We then define
\[
\|b_2\| := \int_{S^{d-1}} b_2(\hat{u} \cdot \omega) \, d\omega < \infty. \tag{2.9}
\]
Indeed, by a spherical change of coordinates, \( \|b_2\| \) is independent of the direction \( \hat{u} \) and finite since
\[
\|b_2\| = \int_{S^{d-1}} b_2(\hat{u} \cdot \omega) \, d\omega = \omega_{d-2} \int_{-1}^{1} b_2(z)(1 - z^2)^{\frac{d-3}{2}} \, dz < \infty,
\]
where \( \omega_{d-2} \) denotes the surface measure of \( S^{d-2} \).

The ternary collisional operator \( Q_3(f, f, f) \), introduced for the first time in \([7, 9]\), is given by
\[
Q_3(f, f, f) = \int_{S^{d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) \left( f^* f_1^* f_2^* - f f_1 f_2 \right) \, d\omega \, dv_{1,2} \tag{2.10}
\]
\[
+ 2 \int_{S^{d-1} \times \mathbb{R}^{2d}} B_3(u_1, \omega) \left( f^* f_1^* f_2^* - f f_1 f_2 \right) \, d\omega \, dv_{1,2},
\]
where \( \omega = (\omega_1, \omega_2) \in S^{2d-1} \) is the impact directions vector and \( u := (v_1 - v, u_1 = (v_1 - v_1, v_2 - v_2) \) are the relative velocities vectors of the colliding particles when the tracked particle \((x, v)\) is central or adjacent respectively for the ternary interaction happening. When the tracked particle is central, the collisional formulas are
\[
\begin{align*}
v^* &= v + \frac{u \cdot \omega}{1 + \omega_1 \cdot \omega_2} (\omega_1 + \omega_2), \\
v_1^* &= v_1 - \frac{u \cdot \omega}{1 + \omega_1 \cdot \omega_2} \omega_1, \\
v_2^* &= v_2 - \frac{u \cdot \omega}{1 + \omega_1 \cdot \omega_2} \omega_2. \tag{2.11}
\end{align*}
\]
When the tracked particle is adjacent, the collisional formulas are
\[
\begin{align*}
v^* &= v + \frac{\mathbf{u}_1 \cdot \mathbf{\omega}}{1 + \omega_1 \cdot \omega_2} (\omega_1 + \omega_2), \\
v_1^* &= v_1 - \frac{\mathbf{u}_1 \cdot \mathbf{\omega}}{1 + \omega_1 \cdot \omega_2} \omega_1, \\
v_2^* &= v_2 - \frac{\mathbf{u}_1 \cdot \mathbf{\omega}}{1 + \omega_1 \cdot \omega_2} \omega_2.
\end{align*}
\] (2.12)

One can easily verify that the ternary momentum-energy conservation system is satisfied i.e.
\[
|v^*|^2 + |v_1^*|^2 + |v_2^*|^2 = |v_1|^2 + |v_2|^2,
\] (2.13)
as well as the ternary micro-reversibility conditions:
\[
\mathbf{u}^* \cdot \mathbf{\omega} = -\mathbf{u} \cdot \mathbf{\omega}, \quad \mathbf{u}_1^* \cdot \mathbf{\omega} = -\mathbf{u}_1 \cdot \mathbf{\omega},
\] (2.14)
where we denote \(\mathbf{u}^* := (v_1^* - v_2^*)\) and \(\mathbf{u}_1^* := (v_1^* - v_1^*)\). Moreover, given \(\mathbf{\omega} = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1}\), the transformations \((v, v_1, v_2) \rightarrow (v^*, v_1^*, v_2^*)\) and \((v, v_1, v_2) \rightarrow (v_1^* + v_1^*, v_2^*)\) are linear measure-preserving involutions of \(\mathbb{R}^{3d}\).

Let us define the symmetric quantity
\[
|\tilde{\mathbf{u}}| := (|v - v_1|^2 + |v - v_2|^2 + |v_1 - v_2|^2)^{1/2}.
\] (2.15)
One immediately observes the inequality
\[
\frac{1}{\sqrt{3}} |\tilde{\mathbf{u}}| \leq |\mathbf{u}|, \quad |\mathbf{u}_1| \leq |\tilde{\mathbf{u}}|.
\] (2.16)
By conservation of momentum and energy, one can also easily verify that
\[
|\tilde{\mathbf{u}}^*| = |\tilde{\mathbf{u}}_1^*| = |\tilde{\mathbf{u}}|,
\] (2.17)
where we denote \(|\tilde{\mathbf{u}}^*| = (|v^* - v_1|^2 + |v^* - v_2|^2 + |v_1^* - v_2^*|^2)^{1/2}\) and \(|\tilde{\mathbf{u}}_1^*| = (|v_1^* - v_1^*|^2 + |v_1^* - v_2^*|^2)^{1/2}\).

Defining as well the quantities
\[
\tilde{\mathbf{u}} := |\tilde{\mathbf{u}}|^{-1} \mathbf{u}, \quad \tilde{\mathbf{u}}_1 := |\tilde{\mathbf{u}}|^{-1} \mathbf{u}_1,
\] (2.18)
we notice that \(\tilde{\mathbf{u}}, \tilde{\mathbf{u}}_1 \in \mathbb{E}^{2d-1}_1\), where \(\mathbb{E}^{2d-1}_1\) is the \((2d - 1)\)-dimensional ellipsoid
\[
\mathbb{E}^{2d-1}_1 = \{ (\nu_1, \nu_2) : |\nu_1|^2 + |\nu_2|^2 + |\nu_1 - \nu_2|^2 = 1 \}.
\] (2.19)

The ternary cross-section, which expresses the statistical repartition of ternary collisions, is assumed to be of the form
\[
B_3(\mathbf{u}, \mathbf{\omega}) = |\tilde{\mathbf{u}}|^{\gamma_3 - \theta_3} |\mathbf{u}|^{\theta_3} b_3(\tilde{\mathbf{u}} \cdot \mathbf{\omega}, \omega_1 \cdot \omega_2), \quad \mathbf{u} \neq 0, \quad \gamma_3, \theta_3 \in [0, 2], \quad \theta_3 \geq 0,
\] (2.20)
where \(b_3 : [-1, 1] \times [-1/2, 1/2] \rightarrow [0, \infty)\) is of the form
\[
b_3(x, y) = |x|^\theta_3 \phi(y)
\]
where \( \phi \in L^\infty([-1/2, 1/2]) \). Note that, since \( \omega = (\omega_1, \omega_2) \in \mathbb{S}^{2d-1} \), Cauchy-Schwarz inequality implies \( \omega_1 \cdot \omega_2 \in [-1/2, 1/2] \). Moreover, notice that due to the form of \( b_3 \), we may as well write

\[
B_3(u, \omega) = |\bar{u}|^{\gamma_3} b_3(\bar{u} \cdot \omega, \omega_1 \cdot \omega_2), \quad B_3(u_1, \omega) = |\bar{u}|^{\gamma_3} b_3(\bar{u}_1 \cdot \omega, \omega_1 \cdot \omega_2). \tag{2.21}
\]

thus by \((2.13), (2.17)\), we obtain

\[
B_3(u^*, \omega) = B_3(u, \omega), \quad B_3(u_1^*, \omega) = B_3(u_1, \omega). \tag{2.22}
\]

We also define

\[
||b_3|| := \int_{\mathbb{S}^{2d-1}} b_3(\hat{u} \cdot \omega, \omega_1 \cdot \omega_2) \, d\omega < \infty. \tag{2.23}
\]

Indeed, by spherical symmetry, the quantity \( ||b_3|| \) is independent of the direction \( \hat{u} \) and finite since

\[
||b_3|| = \int_{\mathbb{S}^{2d-1}} |\hat{u} \cdot \omega|^\theta \phi(\omega_1 \cdot \omega_2) \, d\omega \\
\leq \|\phi\|_{L^\infty} \int_{\mathbb{S}^{2d-1}} |\hat{u} \cdot \omega|^\theta \, d\omega \\
= \omega_{2d-2} \|\phi\|_{L^\infty} \int_{-1}^{1} |z|^\theta (1 - z^2)^{d-\frac{3}{2}} \, dz < \infty,
\]

where \( \omega_{2d-2} \) denotes the surface measure of \( \mathbb{S}^{2d-2} \).

Throughout this paper we assume that the binary collisional kernel \( B_2 \) satisfies \((2.7)\) with \((2.9)\), ternary collisional kernel \( B_3 \) satisfies \((2.20)\) with \((2.23)\), and that

\[
\gamma := \max \{\gamma_2, \gamma_3\} > 0. \tag{2.24}
\]

Equivalently, at least one of \( \gamma_2, \gamma_3 \) is strictly positive, and this will play a crucial role in the paper.

**Spaces relevant for this paper.** The relevant spaces for the study of properties of moments that will be used throughout the paper are polynomially weighted \( L^1 \)-spaces, in particular, given \( q \geq 0 \), we define the Banach spaces

\[
L^1_q := \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ such that } f \text{ is measurable and } \int_{\mathbb{R}^d} |f(v)| v^q \, dv < \infty \right\},
\]

with norm

\[
\|f\|_{L^1_q} = \int_{\mathbb{R}^d} |f(v)| v^q \, dv.
\]

Notice that \( L^1_{q_2} \subset L^1_{q_1} \), whenever \( q_1 < q_2 \) and for a non-negative function \( f \), \( m_q[f] < \infty \iff f \in L^1_q \).

Now, consider a time interval \( I \subset \mathbb{R} \) and \((X, \| \cdot \|)\) a Banach space. We will denote

\[
C(I, L^1_q) := \left\{ f : I \to L^1_q \text{ such that } f \text{ is continuous} \right\}.
\]

In case \( I \) is compact, the above linear space becomes a Banach space with norm

\[
\|f\|_{C(I, L^1_q)} := \sup_{t \in I} \|f(t)\|_{L^1_q} < \infty.
\]

If \( S \subset X \) with \( S \neq \emptyset \), we will write

\[
C(I, S) := \{ f \in C(I, X) \text{ such that } f(t) \in S, \quad \forall t \in I \}.
\]

Finally, we will denote

\[
L^1(I, X) := \left\{ f : I \to X \text{ measurable, such that } \int_I \|f(\tau)\|_X \, d\tau < \infty \right\},
\]
which is a Banach space with norm

$$\|f\|_{L^1(I,X)} := \int_I \|f(\tau)\|_X \, d\tau.$$  

Clearly, if \(I\) is compact, then \(C(I,X) \subset L^1(I,X)\). We also denote

$$L^1_{loc}(I,X) := \{ f : I \to X \text{ with } f \in L^1(K,X) \text{ for any } K \subset I \}.$$  

2.2. Weak form of the collisional operators. We will now write the collisional operators \(Q_2(f,f), Q_3(f,f,f)\) and \(Q[f] = Q_2(f,f) + Q_3(f,f,f)\) in weak formulation. Assuming sufficient integrability for \(f\) and a test function \(\phi\) for all integrals involved to make sense (see Remark 2.1 below), the binary collisional operator can be written in weak form as (see Appendix A.7)

$$\int_{\mathbb{R}^d} Q_2(f,f)\phi \, dv = \frac{1}{2} \int_{S^d-1 \times \mathbb{R}^{2d}} B_2(u,\omega) f f_1(\phi' + \phi_1 - \phi_1) \, d\omega \, dv_1 \, dv$$

$$= \frac{1}{2} \int_{S^d-1 \times \mathbb{R}^{2d}} |u|^{-2} b_2(\hat{u} \cdot \omega) f f_1(\phi' + \phi_1 - \phi_1) \, d\omega \, dv_1 \, dv, \quad (2.25)$$  

while the ternary collision operator can be written in weak form as (see Appendix A.16)

$$\int_{\mathbb{R}^d} Q_3(f,f,f)\phi \, dv = \frac{1}{6} \int_{S^d-1 \times \mathbb{R}^{3d}} B_3(u,\omega) f f_1 f_2(\phi^* + \phi_1^* + \phi_2^* - \phi_1 - \phi_2) \, d\omega \, dv_{1,2} \, dv$$

$$= \frac{1}{6} \int_{S^d-1 \times \mathbb{R}^{3d}} |\hat{u}|^{-3-\theta} |u|^\theta b_3(\hat{u} \cdot \omega,\omega_1 \cdot \omega_2) f f_1 f_2(\phi^* + \phi_1^* + \phi_2^* - \phi_1 - \phi_2) \, d\omega \, dv_{1,2} \, dv. \quad (2.26)$$  

Combining (2.25), (2.26), we conclude that \(Q[f]\) can be written in weak form as:

$$\int_{\mathbb{R}^d} Q[f]\phi \, dv = \frac{1}{2} \int_{S^d-1 \times \mathbb{R}^{2d}} |u|^{-2} b_2(\hat{u} \cdot \sigma) f f_1(\phi' + \phi_1 - \phi_1) \, d\omega \, dv_1 \, dv$$

$$+ \frac{1}{6} \int_{S^d-1 \times \mathbb{R}^{3d}} |\hat{u}|^{-3-\theta} |u|^\theta b_3(\hat{u} \cdot \omega, (\omega_1,\omega_2)) f f_1 f_2(\phi^* + \phi_1^* + \phi_2^* - \phi_1 - \phi_2) \, d\omega \, dv_{1,2} \, dv. \quad (2.27)$$  

Remark 2.1. When studying moments, the relevant test functions \(\phi\) for (2.27) are of the form \(|\phi(v)| \leq \psi(v)\psi(q)\), where \(q \geq 0, \psi \in L^\infty\). In that case, a sufficient condition for (2.27) to hold in the cut-off regime (2.19), (2.23), is \(f \in L^1_{q+\gamma}\), where \(\gamma = \max\{\gamma_2, \gamma_3\}\). This is a consequence of Lemmata A.1, A.2, A.3 in the Appendix. In particular \(Q : L^1_{q+\gamma} \to L^1_q\). However, the fact \(f \in L^1_{q+\gamma}\) is typically not a-priori known, therefore one of the main difficulties one has to face when studying moments is to actually prove that the required integrability is guaranteed under time evolution, see Theorem 5.1 where we resolve this issue for solutions to the equation (2.1).  

Collisional averaging. By Remark 2.1 the weak formulation (2.27) combined with the microscopic conservation laws of binary and ternary collisions (2.1), (2.13), yield the mass, momentum...
and energy averaging properties of the collisional operator:

\[
\int_{\mathbb{R}^d} Q[f] \, dv = 0, \quad \text{when } f \in L^1_\gamma, \tag{2.28}
\]

\[
\int_{\mathbb{R}^d} Q[f]v \, dv = 0, \quad \text{when } f \in L^1_{1+\gamma}, \tag{2.29}
\]

\[
\int_{\mathbb{R}^d} Q[f]\lvert v \rvert^2 \, dv = 0, \quad \text{when } f \in L^1_{2+\gamma}. \tag{2.30}
\]

where we denote \(\gamma = \max\{\gamma_2, \gamma_3\}\).

**Conservation laws.** The averaging properties of the collisional operator \(2.28)-(2.30)\) applied in time, formally yield the conservation of mass, momentum, and energy for a solution \(f\) to the binary-ternary Boltzmann equation \((2.1)\) with initial data \(f_0 \in L^1_2\):}

\[
\int_{\mathbb{R}^d} f(t, v) \left( \frac{1}{v} \right) \, dv = \int_{\mathbb{R}^d} f_0(v) \left( \frac{1}{v} \right) \, dv. \tag{2.31}
\]

**2.3. Definition of solutions to the binary-ternary Boltzmann equation and conservation laws.** In this section, we give a precise definition of a solution to the binary-ternary Boltzmann equation \((2.1)\).

**Definition 2.2.** Let \(T > 0\), \(\gamma = \max\{\gamma_2, \gamma_3\}\) and \(f_0 \in L^1_2\) with \(f_0 \geq 0\). We say that a nonnegative function \(f \in C([0,T], L^1_2) \cap C^1((0,T), L^1_{1+\gamma}) \cap L^1_{\text{loc}}(0,T), L^1_{2+\gamma})\) is a solution of the binary-ternary Boltzmann equation \((2.1)\) with initial data \(f_0\) in \([0,T]\) if \((2.31)\) hold, and if

\[
\begin{cases}
\frac{df}{dt} = Q[f], & t \in (0,T], \\
f(t = 0) = f_0.
\end{cases}
\tag{2.32}
\]

**Remark 2.3.** Although Definition 2.2 allows the initial data to be in \(L^1_2\), we prove well-posedness of the equation \((2.1)\) when \(f_0 \in L^1_{2+\varepsilon}\), where \(\varepsilon > 0\) can be arbitrarily small, see Theorem 2.8 for more details.

**Remark 2.4.** The reason conservation laws are included in the definition of a solution is because we construct solutions for initial data with less than \(2 + \gamma\) moments. In that case we cannot guarantee that \(f \in L^1((0,T), L^1_{2+\gamma})\), which together with \(f_0 \in L^1_{2+\gamma}\) would automatically imply conservation laws by collisional averaging \((2.28) - (2.30)\). Instead, the conservation laws will hold for our solutions by construction.

**Remark 2.5.** If initial data has mass zero i.e. \(\int_{\mathbb{R}^d} f_0(v) \, dv = 0\), then by the conservation of mass, the unique solution is trivially zero. Thus, in the rest of the paper we will assume that mass of the initial data is non-zero, i.e. \(\int_{\mathbb{R}^d} f_0(v) \, dv > 0\).

**2.4. Main results.** We prove several results regarding the spatially homogeneous binary-ternary Boltzmann equation \((2.1)\) with integrable angular kernels (see \((2.9)\), \((2.23)\)) and hard potentials in the sense that \(\gamma_2, \gamma_3 \geq 0\) and \(\gamma = \max\{\gamma_2, \gamma_3\} > 0\) i.e. either \(\gamma_2 > 0\) or \(\gamma_3 > 0\). Our first main result in this paper is propagation and generation of polynomial moments of solutions to the equation \((2.1)\).
Theorem 2.6. Let \( T > 0 \), and let \( f \geq 0 \) be a solution to the binary-ternary Boltzmann equation (2.1) corresponding to the initial data \( f_0 \in L^1_2, f_0 \geq 0 \).

(1) (Generation) Assume \( i \in \{2, 3\} \) with \( \gamma_i > 0 \). For \( q > 2 \), we have the estimate
\[
m_q[f](t) \leq K_{q,i} \max \left\{ 1, t^{\frac{2q}{q-2}} \right\}, \quad \forall t \in (0, T],
\]
for a constant \( K_{q,i} > 0 \), depending on \( q, \gamma_i, m_0, m_2 \), and independent of \( T \).

If \( \gamma_2, \gamma_3 > 0 \), we have the bound
\[
m_q[f](t) \leq K_q \max \left\{ 1, \min \left\{ t^{\frac{2q}{q-2}}, t^{\frac{2q}{q-3}} \right\} \right\}, \quad \forall t \in (0, T],
\]
where \( K_q = \max\{K_{q,2}, K_{q,3}\} > 0 \) depends on \( q, \gamma_2, \gamma_3, m_0, m_2 \), and is independent of \( T \).

(2) (Propagation) Let \( q > 2 \) and \( m_q(0) < \infty \). Then
\[
\sup_{t \in [0,T]} m_q[f](t) \leq M_q,
\]
for some constant \( M_q > 0 \) depending on \( q, \gamma_2, \gamma_3, m_0, m_2, m_q(0) \), and independent of \( T \).

Our second main result, and the one that exhibits the better behavior of the binary-ternary equation compared to the classical Boltzmann equation, is propagation and generation of exponential moments of weak solutions of the binary-ternary equation.

Theorem 2.7. Let \( T > 0 \) and let \( f \geq 0 \) a solution to the binary-ternary Boltzmann equation (2.1) corresponding to the initial data \( f_0 \in L^1_2, f_0 \geq 0 \).

(a) (Generation of exponential moments) Then, there exist \( a_1, C > 0 \), depending on \( b_2, b_3, \gamma_2, \gamma_3 \), initial mass and initial energy, such that
\[
\int_{\mathbb{R}^d} f(t,v) e^{a_1 \min\{1,t\}\gamma}\, dv \leq C, \quad \text{for } t \in [0,T]
\]
(b) (Propagation of exponential moments) Let \( s \in (0,2] \) and suppose that initial data \( f_0 \) satisfies
\[
\int_{\mathbb{R}^d} f_0(v) e^{a_0 (v)^s} < C_0,
\]
for some positive constants \( a_0 \) and \( C_0 \). Then there exist positive constant \( a \) depends on \( C_0, a_0, \|b_2\|, \|b_3\|, \gamma_2, \gamma_3 \) and initial mass and initial energy such that
\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^d} f(t,v) e^{a (v)^s} < 6C_0.
\]

Finally, we establish existence of a unique global solution, when the initial data are in \( L^1_{2+\varepsilon} \), for some \( \varepsilon > 0 \).

Theorem 2.8. Let \( T > 0, \varepsilon > 0 \), and consider initial data \( f_0 \in L^1_{2+\varepsilon} \) with \( f_0 \geq 0 \). Then the binary-ternary Boltzmann equation (2.1) has a unique solution \( f \geq 0 \), in the sense of Definition 2.2. Moreover, \( f \in C^1((0,T), L^1_k) \) for any \( k > 2 \).

All of our main results crucially rely on new angular averaging estimates for the binary-ternary collision operator. Many of the estimates that we derive are novel even for the binary equation. For more details, see Section 3.

\[ \text{such an } i \text{ always exists since } \gamma = \max\{\gamma_2, \gamma_3\} > 0. \]
3. Angular averaging estimates

In this section we introduce several angular averaging estimates, which will be crucial for obtaining moment estimates, and consequently allow us to prove generation and propagation of polynomial and exponential moments, as well as global well-posedness. The first subsection provides estimates for the binary part of the collision operator, while the second subsection addresses the ternary part. In each of the subsections there are two angular averaging estimates - one on the gain part of the collision operator, and the other for the newly introduced modified gain and modified loss.

3.1. Binary angular averaging estimates. We now work on establishing two types of angular averaging estimates for the collision operator, and we begin by considering the binary collision operator. The first estimate has been established in earlier works [14, 17], but is stated here for completeness, and will be used to obtain the second type of the averaging estimates for the binary kernel, see Lemma 3.2.

Lemma 3.1 ([14, 17]). Suppose $b_2$ satisfies (2.9). Let $k \geq 2$. Then there exists a strictly decreasing mapping $\{\alpha_{k/2}\}_{k \geq 2}$ with $\alpha_1 = \|b_2\|$ and $\alpha_{k/2} \rightarrow 0$ as $k \rightarrow \infty$, such that for all $v, v_1 \in \mathbb{R}^d$ with $u = v_1 - v \neq 0$, we have

$$\int_{\mathbb{S}^{d-1}} b_2(\hat{u} \cdot \omega) \left( \langle v \rangle^k + \langle v_1 \rangle^k \right) \, d\omega \leq \alpha_{k/2}^2 E_2^{k/2}, \quad (3.1)$$

where $E_2 = \langle v \rangle^2 + \langle v_1 \rangle^2$ is the binary kinetic energy. The mapping $\{\alpha_{k/2}\}_{k \geq 2}$ is called the binary coercive map.

In order to state the second type of angular averaging estimate for the binary collisional operator, Lemma 3.2, we first introduce the following notation. For a given non-negative function $\psi$, and $v, v_1 \in \mathbb{R}^d$ with $u = v_1 - v \neq 0$, we write

$$K_{2,\psi}(v, v_1) := G_{2,\psi}(v, v_1) - L_{2,\psi}(v, v_1), \quad (3.2)$$

where

$$G_{2,\psi}(v, v_1) = \int_{\mathbb{S}^{d-1}} b_2(\hat{u} \cdot \omega) \left( \psi(\langle v \rangle^2) + \psi(\langle v_1 \rangle^2) \right) \, d\omega, \quad (3.3)$$

$$L_{2,\psi}(v, v_1) = \int_{\mathbb{S}^{d-1}} b_2(\hat{u} \cdot \omega) \left( \psi(\langle v \rangle^2) + \psi(\langle v_1 \rangle^2) \right) \, d\omega = \|b_2\| \left( \psi(\langle v \rangle^2) + \psi(\langle v_1 \rangle^2) \right). \quad (3.4)$$

We refer to $G_{2,\psi}$ as the binary gain operator and to $L_{2,\psi}$ as the binary loss operator. In the lemma below, we will construct a new decomposition of the collision operator $K_{2,\psi}$. We emphasize that, to the best of our knowledge, this result is novel for the binary collision operator.

Lemma 3.2. Suppose $b_2$ satisfies (2.9). Let $k > 2$ and $\psi(x) = x^{k/2}$. Then, for all $v, v_1 \in \mathbb{R}^d$ with $u = v_1 - v \neq 0$, we can write $K_{2,\psi} = G_{2,\psi} - L_{2,\psi}$, where $\tilde{G}_{2,\psi}, \tilde{L}_{2,\psi}$ satisfy the following estimates:

$$0 \leq \tilde{G}_{2,\psi} \leq \alpha_{k/2} C_k \langle v \rangle \left( \langle v \rangle^{k-2} + \langle v_1 \rangle^{k-2} \right), \quad (3.5)$$

$$\tilde{L}_{2,\psi} \geq (\|b_2\| - \alpha_{k/2}) \left( \langle v \rangle^k + \langle v_1 \rangle^k \right), \quad (3.6)$$

and where $(\alpha_{k/2})_{k \in \mathbb{N}}$ is the corresponding binary coercive term from Lemma 3.1, and $C_k > 1$ is an appropriate constant.
Additionally, if \( \psi_n \not\rightarrow \psi \) is the approximating sequence given in Lemma \ref{lemma:approximation}, then for every \( n \), we can write \( K_{2,\psi_n} = G_{2,\psi_n} - \bar{L}_{2,\psi_n} \), where \( G_{2,\psi_n}, \bar{L}_{2,\psi_n} \) satisfy the following:

\[
0 \leq G_{2,\psi_n} \leq \|b_2\|C_k \langle v \rangle \left( \langle v \rangle^{k-2} + \langle v_1 \rangle^{k-2} \right),
\]

\[
\bar{L}_{2,\psi_n} \geq 0, \quad \bar{L}_{2,\psi_n} \to \bar{L}_{2,\psi},
\]

where \( C_k > 1 \) is the same constant as in \ref{lemma:approximation}.

**Proof.** Let us define the sets

\[
A_0 := \{ v, v_1 \in \mathbb{R}^d : \langle v \rangle \leq 2 \langle v_1 \rangle \},
\]

\[
A_1 := \{ v, v_1 \in \mathbb{R}^d : \langle v_1 \rangle \leq 2 \langle v \rangle \},
\]

\[
A := A_0 \cap A_1 = \{ v, v_1 \in \mathbb{R}^d : 2^{-1} \langle v \rangle \leq \langle v_1 \rangle \leq 2 \langle v \rangle \}.
\]

Then \( A^c = A_0 \cup A_1 \), and \( A_0 \cap A_1 = \emptyset \). Therefore

\[
1_A + 1_{A_0} + 1_{A_1} = 1.
\]

Let \( k > 2 \) and \( \psi(x) = x^{k/2} \). Recalling \( G_{2,\psi}, \bar{L}_{2,\psi} \) from \ref{equation:binary_gain} and \ref{equation:binary_loss}, we define

\[
\bar{G}_{2,\psi} := \left( 1 + \frac{\psi(\langle v \rangle^2)}{\psi(E_2)} \right) 1_{A_0} + \left( 1 - \frac{\psi(\langle v_1 \rangle^2)}{\psi(E_2)} \right) 1_{A_1} \right) G_{2,\psi}
\]

\[
\bar{L}_{2,\psi} := L_{2,\psi} - \left( \frac{\psi(\langle v \rangle^2)}{\psi(E_2)} \right) 1_{A_0} + \left( \frac{\psi(\langle v_1 \rangle^2)}{\psi(E_2)} \right) 1_{A_1} \right) G_{2,\psi},
\]

and we refer to \( \bar{G}_{2,\psi} \) as the modified binary gain operator and to \( \bar{L}_{2,\psi} \) as the modified binary loss operator. Note that by \ref{equation:binary_gain}, \( \bar{G}_{2,\psi} - \bar{L}_{2,\psi} = G_{2,\psi} - L_{2,\psi} = K_{2,\psi} \). In order to estimate \( \bar{G}_{2,\psi} \), we apply \ref{lemma:trilinear} to obtain

\[
\bar{G}_{2,\psi} \leq \alpha_{k/2} \left( E_2^{k/2} 1_A + \left( E_2^{k/2} - \langle v \rangle^k \right) 1_{A_0} + \left( E_2^{k/2} - \langle v_1 \rangle^k \right) 1_{A_1} \right) =: \alpha_{k/2} \bar{G}_{2,\psi}.
\]

Since \( \langle v \rangle^2 + \langle v_1 \rangle^2 \leq (\langle v \rangle + \langle v_1 \rangle)^2 \), we have

\[
\left( E_2^{k/2} - \langle v \rangle^k \right) 1_{A_0} \leq \left( (\langle v \rangle + \langle v_1 \rangle)^k - \langle v \rangle^k \right) 1_{A_0}.
\]

Then an application of Lemma \ref{lemma:trilinear} implies

\[
\left( E_2^{k/2} - \langle v \rangle^k \right) 1_{A_0} \leq \left( (\langle v \rangle^k + C_{2,k} \langle v \rangle^k \langle v_1 \rangle^k) \right) 1_{A_0}
\]

\[
\leq 2^{-1} \langle v \rangle \langle v_1 \rangle^{k-1} + C_{2,k} \langle v \langle v_1 \rangle^{k-1} \rangle + \langle v \rangle \langle v_1 \rangle^{k-1}
\]

\[
= \langle v \rangle \langle v_1 \rangle \left( 2^{-1} + C_{2,k} \langle v_1 \rangle^{k-2} + C_{2,k} \langle v \rangle^{k-2} \right)
\]

where \( C_{2,k} = k \max\{1, 2^{k-3}\} \). A similar estimate holds for \( \left( E_2^{k/2} - \langle v_1 \rangle^k \right) 1_{A_1} \).
Another application of $\langle v \rangle^2 + \langle v^2 \rangle^2 \leq (\langle v \rangle + \langle v^1 \rangle)^2$ and Lemma A.3 yields

\[
E_k^{2/2} \mathbb{I}_A \leq (\langle v \rangle + \langle v^1 \rangle)^k \mathbb{I}_A \leq \left( \langle v \rangle^k + \langle v^1 \rangle^k + C_{2,k} \left( \langle v \rangle^{k-1} \langle v^1 \rangle + \langle v \rangle^k \langle v^1 \rangle^{k-1} \right) \right) \mathbb{I}_A
\]

\[
\leq (2 + C_{2,k}) \left( \langle v \rangle^{k-1} \langle v^1 \rangle + \langle v \rangle^k \langle v^1 \rangle^{k-1} \right)
\]

\[
= (2 + C_{2,k}) \langle v \rangle (v_1) \left( \langle v \rangle^{k-2} + \langle v \rangle^{k-2} \right),
\]

and so

\[
\overline{G}_{2,\psi} \leq (2 + 2^{-1} + 3C_{2,k}) \langle v \rangle \langle v^1 \rangle \left( \langle v \rangle^{k-2} + \langle v \rangle^{k-2} \right),
\]

which proves (3.15) with $C_k = 2 + 2^{-1} + 3C_{2,k} > 1$.

The lower bound on $L_{2,\psi}$ (3.9), follows immediately from (3.1) since:

\[
\overline{L}_{2,\psi} \geq L_{2,\psi} - \alpha_{k/2} \left( \langle v \rangle^k \mathbb{I}_{A_0^k} + \langle v^1 \rangle^k \mathbb{I}_{A_1^k} \right) \geq (\|b_2\| - \alpha_{k/2}) \left( \langle v \rangle^k + \langle v^1 \rangle^k \right).
\]

Consider now the approximating sequence $\psi_n \nearrow \psi$ from Lemma A.3 and let $\overline{G}_{2,\psi_n}, \overline{L}_{2,\psi_n}$ be defined as in (3.13)-(3.14). Then, clearly, $\overline{G}_{2,\psi_n} - \overline{L}_{2,\psi_n} = K_{2,\psi_n}$. To obtain non-negativity of $\overline{L}_{2,\psi_n}$, recall the fact that $\langle v \rangle^2 = \nu E_2$, $\langle v^1 \rangle^2 = \nu_1 E_2$, for some $\nu, \nu_1 \in [0,1]$ with $\nu + \nu_1 = 1$. Since $\psi_n$ is convex and satisfies $\psi_n(0) = 0$, Lemma A.4 implies that for all $x \geq 0$ and $\mu \in [0,1]$ we have $\psi_n(\mu x) \leq \mu \psi_n(x)$. Therefore,

\[
G_{2,\psi_n} = \int_{\mathbb{R}^d \setminus 1} b_2(\hat{u} \cdot \omega)(\psi_n(\nu E_2) + \psi_n(\nu_1 E_2)) \, d\omega
\]

\[
\leq \int_{\mathbb{R}^d \setminus 1} b_2(\hat{u} \cdot \omega)(\nu + \nu_1) \psi_n(E_2) \, d\omega = \|b_2\| \psi_n(E_2),
\]

and hence

\[
\overline{L}_{2,\psi_n} \geq L_{2,\psi_n} - \|b_2\| \left( \langle \psi_n(\langle v \rangle^2) \rangle + \langle \psi_n(\langle v^1 \rangle^2) \rangle \right) = 0.
\]

The convergence $\overline{L}_{2,\psi_n} \to \overline{L}_{2,\psi}$ is immediate, since $\psi_n \nearrow \psi$. It remains to prove the bound (3.14). Notice that by (3.16) we have

\[
\overline{G}_{2,\psi_n} \leq \|b_2\| \left( \psi_n(E_2) \mathbb{I}_A + \left( \psi_n(E_2) - \psi_n(\langle v \rangle^2) \right) \mathbb{I}_{A_0^k} + \left( \psi_n(E_2) - \psi_n(\langle v^1 \rangle^2) \right) \mathbb{I}_{A_1^k} \right)
\]

\[
= \|b_2\| \overline{G}_{2,\psi_n}.
\]

In order to establish the bound (3.17), it suffices to prove $\overline{G}_{2,\psi_n} \nearrow \overline{G}_{2,\psi}$. For this purpose, we first show that the sequences $a_n = \psi_n(E_2) - \psi_n(\langle v \rangle^2)$ and $b_n = \psi_n(E_2) - \psi_n(\langle v^1 \rangle^2)$ are increasing in $n$. Namely, seeing $n$ as a continuous variable for the moment, we compute

\[
\partial_n a_n = \partial_n \left( \psi_n(E_2) \right) - \partial_n \left( \psi_n(\langle v \rangle^2) \right),
\]

\[
\partial_n \psi_n(x) = \begin{cases} 0, & x \leq n \\
\frac{k}{n^{k/2-1}} \left( \frac{k}{2} - 1 \right) \left( \frac{n}{k} - 1 \right), & x > n. \end{cases}
\]

If $E_2 \leq n$ then $\langle v \rangle^2 \leq n$ as well, so $\partial_n a_n = 0$. If $E_2 > n$ and $\langle v \rangle^2 \leq n$, then

\[
\partial_n a_n = \partial_n \left( \psi_n(E_2) \right) = \frac{k}{2} n^{k/2-1} \left( \frac{k}{2} - 1 \right) \left( \frac{E_2}{n} - 1 \right) > 0.
\]
If $E_2 > n$ and $(v)^2 > n$, then
\[
\partial_n a_n = \frac{k}{2} n^{k/2-1} \left( \frac{k}{2} - 1 \right) \left( \frac{E_2 - (v)^2}{n} \right) \geq 0.
\]

In any case $\partial_n a_n \geq 0$, and so $(a_n)_n$ is increasing. Similarly, $(b_n)_n$ is increasing as well. Therefore, $\tilde{G}_{2,\psi} \to \tilde{G}_{2,\psi}$ and $\mathbf{8.7}$ follows thanks to $\mathbf{8.14}$.

\[\square\]

### 3.2. Ternary angular averaging estimates

We now state and prove angular averaging estimates for the ternary collision operator, and we start with the estimate on the ternary gain operator.

**Lemma 3.3.** Suppose $b_3$ satisfies $\mathbf{2.8}$ and $b_3$. Let $k \geq 2$ and $v, v_1, v_2 \in \mathbb{R}^d$. Then there is a strictly decreasing mapping $\{\lambda_{k/2}\}_{k \geq 2}$ with $\lambda_1 = \|b_3\|$ and $\lambda_{k/2} \to 0$ as $k \to \infty$, such that for all $v, v_1, v_2 \in \mathbb{R}^d$ with $u = (v_1 - v) \neq 0$, we have
\[
\int_{S^{2d-1}} b_3(\tilde{u} \cdot \omega, (\omega_1, \omega_2)) \left( \langle v^* \rangle^k + \langle v_1^* \rangle^k + \langle v_2^* \rangle^k \right) \, d\omega \leq \lambda_{k/2} E_3^{k/2},
\]
where $E_3 := \langle v \rangle^2 + \langle v_1 \rangle^2 + \langle v_2 \rangle^2$ denotes the ternary kinetic energy. The mapping $\{\lambda_{k/2}\}_{k \geq 2}$ is called the ternary coercive map.

**Proof.** Let $v, v_1, v_2 \in \mathbb{R}^d$, with $u \neq 0$. Let us define the scattering direction as:
\[
\sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} := \frac{1}{|u|} \begin{pmatrix} v^* - v_1^* \\ v^* - v_2^* \end{pmatrix}.
\]

Notice that, due to $\mathbf{2.17}$, $\sigma$ belongs to the ellipsoid $E_1^{2d-1}$, given by $\mathbf{2.19}$. Moreover, formulas $\mathbf{2.11}$ imply that $\sigma$ depends smoothly on $\omega \in S^{2d-1}$. Let us also denote the center of mass of the velocities $v, v_1, v_2$ by
\[
V_3 = \frac{v + v_1 + v_2}{3}.
\]

Then we obtain the following energy identity
\[
1 + |V_3|^2 = \frac{|u|^2}{9} = \frac{E_3}{3}.
\]

By the conservation of momentum, the post-collisional velocities can be written in terms of the scattering direction $\sigma = (\sigma_1, \sigma_2) \in E_1^{2d-1}$ and the center of mass $V_3$ as follows:
\[
v^* = V_3 - \frac{|\sigma|}{3} (\sigma_1 + \sigma_2), \quad v_1^* = V_3 + \frac{|u|}{3} (2\sigma_1 - \sigma_2), \quad v_2^* = V_3 + \frac{|u|}{3} (-\sigma_1 + 2\sigma_2),
\]
and therefore we have
\[
\langle v^* \rangle^2 = E_3 \frac{1 - \xi_1 V_3 \cdot (\sigma_1 + \sigma_2) + \xi_1 (|\sigma_1 + \sigma_2|^2 - 1)}{3},
\]
\[
\langle v_1^* \rangle^2 = E_3 \frac{1 + \xi_1 V_3 \cdot (2\sigma_1 - \sigma_2) + \xi_1 (|2\sigma_1 - \sigma_2|^2 - 1)}{3},
\]
\[
\langle v_2^* \rangle^2 = E_3 \frac{1 + \xi_1 V_3 \cdot (-\sigma_1 + 2\sigma_2) + \xi_1 (|\sigma_1 + 2\sigma_2|^2 - 1)}{3}.
\]

\[\square\]
where
\[
\xi_1' := 2|\tilde{u}|V_3/E_3 \quad \text{and} \quad \xi_1 := |\tilde{u}|^2/(3E_3). \tag{3.23}
\]
Note that $E_3 \neq 0$ since $\tilde{u} \neq 0$. Also note that $0 < \xi_1, \xi_1' < 1$. Indeed, $\xi_1 < 1$ follows immediately from (3.22), while Young’s inequality and (3.20) imply
\[
\xi_1' = \frac{6}{E_3} \frac{|\tilde{u}|}{3} |V_3| \leq \frac{3}{E_3} \left( \frac{|\tilde{u}|^2}{9} + |V_3|^2 \right) < 1.
\]
Parameters $\xi_1$ and $\xi_1'$ are related, due to (3.20), by the following identity:
\[
\frac{(\xi_1')^2}{4} + \xi_1^2 = \alpha \xi_1,
\]
where $\alpha = 1 - \frac{3}{E_3} \in (0, 1)$. Since $\xi_1'$ is nonnegative, we obtain the formula
\[
\xi_1' = 2\sqrt{\alpha \xi_1 - \xi_1^2}.
\]
For a general $\xi \in [0, 1]$, and for $\hat{V}_3 \in \mathbb{R}^{d-1}$ and $(\sigma_1, \sigma_2) \in \mathbb{R}_{1}^{2d-1}$, we define
\[
\mu(\xi) = \frac{1 - 2\sqrt{\alpha \xi - \xi^2}}{3} \hat{V}_3 \cdot (\sigma_1 + \sigma_2) + \xi (|\sigma_1 + \sigma_2|^2 - 1),
\]
\[
\mu_1(\xi) = \frac{1 + 2\sqrt{\alpha \xi - \xi^2}}{3} \hat{V}_3 \cdot (2\sigma_1 - \sigma_2) + \xi (|2\sigma_1 - \sigma_2|^2 - 1),
\]
\[
\mu_2(\xi) = \frac{1 + 2\sqrt{\alpha \xi - \xi^2}}{3} \hat{V}_3 \cdot (-\sigma_1 + 2\sigma_2) + \xi (|\sigma_1 + 2\sigma_2|^2 - 1).
\tag{3.24}
\]
Since $|\sigma_1 + \sigma_2|^2 + |2\sigma_1 - \sigma_2|^2 + |\sigma_1 + 2\sigma_2|^2 = 3$, which is true due to $(\sigma_1, \sigma_2) \in \mathbb{R}_{1}^{2d-1}$, we have
\[
\mu + \mu_1 + \mu_2 = 1, \quad \forall \xi \in [0, 1].
\]
Moreover, by Cauchy-Schwarz inequality, and the fact that $\alpha < 1$, we have
\[
\mu \geq \frac{1 - (\xi|\sigma_1 + \sigma_2|^2 + (\alpha - \xi)) + \xi (|\sigma_1 + \sigma_2|^2 - 1)}{3} = \frac{1 - \alpha}{3} > 0, \quad \forall \xi \in [0, 1].
\]
Similarly, one can show that $\mu_1, \mu_2 > 0$ for all $\xi \in [0, 1]$. Therefore, we conclude
\[
\begin{cases}
0 < \mu, \mu_1, \mu_2 < 1 \\
\mu + \mu_1 + \mu_2 = 1, \quad \forall \xi \in [0, 1].
\end{cases}
\tag{3.25}
\]
Now, let
\[
I_k(\xi_1, \tilde{u}, \hat{V}_3) := \int_{S_{\xi_1}^{d-1}} b_3(\tilde{u}, \omega, (\omega_1, \omega_2))(|v^*|^k + |v_1^*|^k + |v_2^*|^k) \, d\omega. \tag{3.26}
\]
and note from (3.22) that the post-collisional velocities can be represented as:
\[
\langle v^* \rangle^2 = \mu(\xi_1)E_3, \quad \langle v_1^* \rangle^2 = \mu_1(\xi_1)E_3, \quad \langle v_2^* \rangle^2 = \mu_2(\xi_1)E_3, \quad \text{with} \quad \xi_1 = |\tilde{u}|^2/(3E_3). \tag{3.27}
\]
Therefore,
\[
I_k(\xi_1, \tilde{u}, \hat{V}_3) = E_3^{k/2} \int_{S_{\xi_1} \cap \omega \in \omega_3} b_3(\tilde{u}, \omega, (\omega_1, \omega_2))(\mu(\xi_1)^{k/2} + \mu_1(\xi_1)^{k/2} + \mu_2(\xi_1)^{k/2}) \, d\omega
\]
\[
= E_3^{k/2} J_k(\xi_1, \tilde{u}, \hat{V}_3).
\]
Let
\[ \lambda_{k/2} := \sup_{(\xi, \bar{u}, \bar{V}_3) \in [0,1] \times \mathbb{E}^{d-1} \times \mathbb{S}^{d-1}} J_k(\xi, \bar{u}, \bar{V}_3). \]  
(3.28)

Then we have
\[ I_k(\xi_1, \bar{u}, \bar{V}_3) \leq \lambda_{k/2} E_3^{k/2}. \]

It remains to check the properties of the sequence \( \{\lambda_{k/2}\}_k \) for \( k \geq 2 \). Since \( \mu(\xi) + \mu_1(\xi) + \mu_2(\xi) = 1 \), (2.23) implies that \( \lambda_1 = \|b_3\| \). Moreover, since \( \mu, \mu_1, \mu_2 < 1 \), the following strict inequality holds for any \( 2 \leq k_1 < k_2 \)
\[ J_{k_2}(\xi, \bar{u}, \bar{V}_3) < J_{k_1}(\xi, \bar{u}, \bar{V}_3). \]
(3.29)

Since the map \( (\xi, \bar{u}, \bar{V}_3) \mapsto J_k(\xi, \bar{u}, \bar{V}_3) \) is continuous for each \( k \geq 2 \) on its compact domain \([0,1] \times \mathbb{E}_1^{2d-1} \times \mathbb{S}^{d-1} \), the supremum in the definition of \( \lambda_{k/2} \) (3.28) is attained. Therefore, from (3.29), since \( J_{k_1}(\xi, \bar{u}, \bar{V}_3) \leq \lambda_{k_1/2}, \) we have
\[ J_{k_2}(\xi, \bar{u}, \bar{V}_3) < \lambda_{k_1/2}, \quad \forall (\xi, \bar{u}, \bar{V}_3) \in [0,1] \times \mathbb{E}_1^{2d-1} \times \mathbb{S}^{d-1}. \]
(3.30)

Again, since the supremum in (3.28) is attained, there exists \( (\xi_2, \bar{u}_2, (\bar{V}_3)_2) \in [0,1] \times \mathbb{E}_1^{2d-1} \times \mathbb{S}^{d-1}, \) such that
\[ \lambda_{k_2/2} = J_{k_2}(\xi_2, \bar{u}_2, (\bar{V}_3)_2) < \lambda_{k_1/2}, \]
where the last inequality holds by (3.30). Therefore, \( \{\lambda_{k/2}\}_k \) is strictly decreasing in \( k \) for \( k \geq 2 \).

Finally, by the dominated convergence theorem, for every \( (\xi, \bar{u}, \bar{V}_3) \in [0,1] \times \mathbb{E}_1^{2d-1} \times \mathbb{S}^{d-1}, \) we have
\[ \lim_{k \to \infty} J_k(\xi_1, \bar{u}, \bar{V}_3) = 0. \]
Therefore, Dimi’s theorem implies that \( \lambda_{k/2} \to 0 \), as \( k \to \infty. \)

\[ \square \]

Remark 3.4. We note that in the above proof we take the advantage of working with the scattering direction representation \( \sigma \) (see (3.21)) which allows us to use the energy identity (3.20). However, we still integrate with respect to the impact direction \( \omega \) (see (2.10)).

For a given non-negative function \( \psi \), and \( v, v_1, v_2 \in \mathbb{R}^d \) with \( u = (v_2 - v) \neq 0 \), we write
\[ K_{3, \psi}(v, v_1, v_2) := G_{3, \psi}(v, v_1, v_2) - L_{3, \psi}(v, v_1, v_2), \]
(3.31)

where
\[ G_{3, \psi}(v, v_1, v_2) = \int_{\mathbb{S}_d} b_3(\bar{u} \cdot \omega, \omega_1 \cdot \omega_2) \left( \psi((v)^2) + \psi((v_1)^2) + \psi((v_2)^2) \right) d\omega, \]
(3.32)
\[ L_{3, \psi}(v, v_1, v_2) = \int_{\mathbb{S}_d} b_3(\bar{u} \cdot \omega, \omega_1 \cdot \omega_2) \left( \psi((v)^2) + \psi((v_1)^2) + \psi((v_2)^2) \right) d\omega \]
(3.33)
\[ = \|b_3\| \left( \psi((v)^2) + \psi((v_1)^2) + \psi((v_2)^2) \right). \]

We refer to \( G_{3, \psi} \) as the ternary gain operator and to \( L_{3, \psi} \) as the ternary loss operator. In the lemma below, we will construct a modified decomposition of the collision operator.
Lemma 3.5. Suppose $b_3$ satisfies (2.23). Let $k > 2$ and $\psi(x) = x^{k/2}$. Then, for all $v, v_1, v_2 \in \mathbb{R}^d$ with $u = (v_1 - v) \neq 0$, we can write $K_{3,\psi} = \widetilde{G}_{3,\psi} - \widetilde{L}_{3,\psi}$, where $\widetilde{G}_{3,\psi}, \widetilde{L}_{3,\psi}$ satisfy the following:

$$0 \leq \widetilde{G}_{3,\psi} \leq \lambda_{k/2} C_k \left( \langle v \rangle v_1 + \langle v \rangle v_2 + \langle v_1 \rangle v_2 \right) \left( \langle v \rangle k^{-2} + \langle v_1 \rangle k^{-2} + \langle v_2 \rangle k^{-2} \right),$$

(3.34)

$$\widetilde{L}_{3,\psi} \geq \left( \|b_3\| - \lambda_{k/2} \right) \left( \langle v \rangle k + \langle v_1 \rangle k + \langle v_2 \rangle k \right),$$

(3.35)

where $\lambda_{k/2}$ is the corresponding ternary coercive term from Lemma A.3 and $C_k > 1$ is an appropriate constant.

Additionally, if $\psi_n \supsetneq \psi$ is the sequence given in Lemma A.3, then for every $n$, we can write $K_{3,\psi_n} = \widetilde{G}_{3,\psi_n} - \widetilde{L}_{3,\psi_n}$, where $\widetilde{G}_{3,\psi_n}, \widetilde{L}_{3,\psi_n}$ satisfy the following:

$$0 \leq \widetilde{G}_{3,\psi_n} \leq C_k \left( \langle v \rangle v_1 + \langle v \rangle v_2 + \langle v_1 \rangle v_2 \right) \left( \langle v \rangle k^{-2} + \langle v_1 \rangle k^{-2} + \langle v_2 \rangle k^{-2} \right),$$

(3.36)

$$\widetilde{L}_{3,\psi_n} \geq 0, \quad L_{3,\psi_n} = L_{3,\psi} + \widetilde{L}_{3,\psi}.$$  

(3.37)

Proof. We define the sets

$$A_0 := \left\{ v, v_1, v_2 \in \mathbb{R}^d : \langle v \rangle \leq 2 \langle v_1 \rangle \lor \langle v \rangle \leq 2 \langle v_2 \rangle \right\},$$

$$A_1 := \left\{ v, v_1, v_2 \in \mathbb{R}^d : \langle v \rangle \leq 2 \langle v_1 \rangle \lor \langle v \rangle \leq 2 \langle v_2 \rangle \right\},$$

(3.38)

$$A_2 := \left\{ v, v_1, v_2 \in \mathbb{R}^d : \langle v \rangle \leq 2 \langle v_1 \rangle \lor \langle v \rangle \leq 2 \langle v_2 \rangle \right\},$$

and

$$A := \bigcap_{i=0,1,2} A_i.$$  

(3.39)

Then $A^c = \bigcup_{i=0,1,2} A_i^c$, and the above union is disjoint so

$$\mathbb{1}_A + \sum_{i=0,1,2} \mathbb{1}_{A_i^c} = 1.$$  

(3.40)

Let $k > 2$ and $\psi(x) = x^{k/2}$. Recalling $G_{3,\psi}, L_{3,\psi}$ from (3.32)-(3.33), we define

$$\widetilde{G}_{3,\psi} := \mathbb{1}_A + \sum_{i=0,1,2} \left( \frac{1 - \psi(v_i^2)}{\psi(E_3)} \mathbb{1}_{A_i^c} \right) G_{3,\psi},$$

(3.41)

$$\widetilde{L}_{3,\psi} := L_{3,\psi} - \sum_{i=0,1,2} \frac{\psi(v_i^2)}{\psi(E_3)} \mathbb{1}_{A_i^c} G_{3,\psi},$$

(3.42)

and refer to $\widetilde{G}_{3,\psi}$ as the modified ternary gain operator and to $\widetilde{L}_{3,\psi}$ as the modified ternary loss operator. By (3.40), we have $\widetilde{G}_{3,\psi} - \widetilde{L}_{3,\psi} = G_{3,\psi} - L_{3,\psi} = K_{3,\psi}$. In order to estimate $\widetilde{G}_{3,\psi}$, we apply (3.18) to obtain

$$\widetilde{G}_{3,\psi} \leq \lambda_{k/2} \left( E_3^{k/2} \mathbb{1}_A + \sum_{i=0,1,2} \left( E_3^{k/2} - \langle v_i \rangle^k \right) \mathbb{1}_{A_i^c} \right) =: \lambda_{k/2} \widetilde{G}_{3,\psi}.$$
Since \( E_3 \leq (\langle v \rangle + \langle v_1 \rangle + \langle v_2 \rangle)^2 \), we have

\[
\left( E_3^{k/2} - \langle v \rangle^k \right) I_A^6 \leq \left( (\langle v \rangle + \langle v_1 \rangle + \langle v_2 \rangle)^k - \langle v \rangle^k \right) I_A^6.
\]

Lemma 3.3 then yields

\[
\left( E_3^{k/2} - \langle v \rangle^k \right) I_A^6 \leq \left( \langle v \rangle^k + \langle v_1 \rangle^k + \langle v_2 \rangle^k + C_{3,k} \sum_{i \neq j \in \{0, 1, 2\}} \langle v_i \rangle^{k-1} \langle v_j \rangle \right) I_A^6
\]

\[
\leq 2^{-1} \left( \langle v \rangle \langle v_1 \rangle^{k-1} + \langle v \rangle \langle v_2 \rangle^{k-1} \right) + C_{3,k} \sum_{i \neq j \in \{0, 1, 2\}} \langle v_i \rangle^{k-1} \langle v_j \rangle
\]

\[
\leq (2^{-1} + C_{3,k}) \left( \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v \rangle \langle v_1 \rangle \langle v_2 \rangle \right) \left( \langle v \rangle^{k-2} + \langle v_1 \rangle^{k-2} + \langle v_2 \rangle^{k-2} \right)
\]

where \( C_{3,k} = k \max\{1, 2k^{-3}\} + \frac{k(k-1)}{2} \max\{1, 2^{k-4}\} \). A similar calculation yields the same upper bound on \( \left( E_3^{k/2} - \langle v \rangle^k \right) I_A^6 \) and \( \left( E_3^{k/2} - \langle v \rangle^k \right) I_A^5 \). On the other hand, another application of \( E_3 \leq (\langle v \rangle + \langle v_1 \rangle + \langle v_2 \rangle)^2 \), Lemma 3.3 yields and the definition of the set \( A \) in (3.39) yields

\[
E_3^{k/2} I_A \leq (\langle v \rangle + \langle v_1 \rangle + \langle v_2 \rangle)^k I_A
\]

\[
\leq \left( \langle v \rangle^k + \langle v_1 \rangle^k + \langle v_2 \rangle^k + C_{3,k} \sum_{i \neq j \in \{0, 1, 2\}} \langle v_i \rangle^{k-1} \langle v_j \rangle \right) I_A
\]

\[
\leq 2 \left( \langle v \rangle^{k-1} \langle v_1 \rangle + \langle v \rangle^{k-1} \langle v_2 \rangle + \langle v \rangle \langle v_2 \rangle^{k-1} \right)
\]

\[
+ C_{3,k} \left( \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v \rangle \langle v_1 \rangle \langle v_2 \rangle \right) \left( \langle v \rangle^{k-2} + \langle v_1 \rangle^{k-2} + \langle v_2 \rangle^{k-2} \right)
\]

\[
\leq (2 + C_{3,k}) \left( \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v \rangle \langle v_1 \rangle \langle v_2 \rangle \right) \sum_{i=0,1,2} \langle v_i \rangle^{k-2},
\]

and so

\[
\tilde{G}_{3, \psi} \leq (2 + 3/2 + 4C_{2,k}) \left( \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v \rangle \langle v_1 \rangle \langle v_2 \rangle \right) \sum_{i=0,1,2} \langle v_i \rangle^{k-2}
\]

(3.43)

which proves (3.44) with \( C_k = 2 + 3/2 + 4C_{2,k} > 1 \).

The lower bound on \( L_{2, \psi} \), (3.34), is an immediate application of (3.18) since

\[
L_{3, \psi} \geq L_{3, \psi} - \lambda_k/2 \left( \sum_{i=0,1,2} \langle v_i \rangle^k I_A^i \right) \geq (\|b_3\| - \lambda_k/2) \sum_{i=0,1,2} \langle v_i \rangle^k.
\]

Consider now the approximating sequence \( \psi_n \nearrow \psi \) from Lemma 3.5 and let \( \tilde{G}_{3, \psi_n}, \tilde{L}_{3, \psi_n} \) be defined as in (3.41)-(3.42). Then, clearly, \( \tilde{G}_{3, \psi_n} - \tilde{L}_{3, \psi_n} = K_{3, \psi_n} \). To obtain non-negativity of \( \tilde{L}_{2, \psi_n} \), observe that by the convexity of \( \psi_n \) and the fact that \( \psi_n(0) = 0 \), we have \( \psi_n(\mu x) \leq \mu \psi(x) \) for all \( x \geq 0 \) and \( \mu \in [0, 1] \). Now, recall the fact that \( \langle v^* \rangle^2 = \mu E_3 \), \( \langle v_1^* \rangle^2 = \mu_1 E_3 \) and \( \langle v_2^* \rangle^2 = \mu_2 E_3 \) for
some $\mu, \mu_1, \mu_2 \in [0, 1]$ with $\mu + \mu_1 + \mu_2 = 1$. Using the above observation, we obtain

$$G_{3, \psi_n} = \int_{S^{d-1}} b_3(\mathbf{u} \cdot \omega, \omega_1 \cdot \omega_2)(\psi_n(\mu E_3) + \psi_n(\mu_1 E_3) + \psi_n(\mu_2 E_3)) \, d\omega$$

$$\leq \int_{S^{d-1}} b_3(\mathbf{u} \cdot \omega, \omega_1 \cdot \omega_2)(\mu + \mu_1 + \mu_2)\psi_n(E_3) \, d\omega = \|b_3\| \psi_n(E_3),$$

(3.44)

and hence

$$\bar{L}_{3, \psi_n} \geq L_{3, \psi_n} - \|b_3\| \left(\psi_n(\langle \psi \rangle^2) + \psi_n(\langle v_1 \rangle^2) + \psi_n(\langle v_2 \rangle^2)\right) = 0.$$  

The convergence $\bar{L}_{2, \psi_n} \to \bar{L}_{2, \psi}$ is immediate, since $\psi_n \nearrow \psi$. In order to prove the bound (3.36), first notice that (3.44) implies

$$\bar{G}_{3, \psi_n} \leq \|b_3\| \left(\psi_n(E_3) I_A + \sum_{i=0,1,2} (\psi_n(E_3) - \psi_n(\langle v_i \rangle^2)) I_{A_i^c}\right) =: \|b_3\| \bar{G}_{3, \psi_n}.$$

(3.45)

We aim to show that $\bar{G}_{3, \psi_n} \nearrow \bar{G}_{3, \psi}$. For this purpose, we first show that the sequences $a_{n,i} = \psi_n(E_3) - \psi_n(\langle v_i \rangle^2)$ are increasing in $n$. Namely, seeing $n$ as a continuous variable for the moment, we compute

$$\partial_n a_{n,i} = \partial_n (\psi_n(E_3)) - \partial_n (\psi_n(\langle v_i \rangle^2)),
$$

$$\partial_n \psi_n(x) = \begin{cases} k/2 n^{k/2-1} \left(\frac{k}{2} - 1\right) \left(\frac{x}{n}\right) - 1, & x \leq n \\ 0, & x > n. \end{cases}$$

If $E_3 \leq n$ then $\langle v_i \rangle^2 \leq n$ as well, so $\partial_n a_{n,i} = 0$. If $E_3 > n$ and $\langle v_i \rangle^2 \leq n$, then

$$\partial_n a_{n,i} = \partial_n (\psi_n(E_3)) = \frac{k}{2} n^{k/2-1} \left(\frac{k}{2} - 1\right) \left(\frac{E_3}{n} - 1\right) > 0.$$  

If $E_3 > n$ and $\langle v_i \rangle^2 > n$, then

$$\partial_n a_{n,i} = \frac{k}{2} n^{k/2-1} \left(\frac{k}{2} - 1\right) \left(\frac{E_3 - \langle v_i \rangle^2}{n}\right) \geq 0.$$  

In any case, $\partial_n a_{n,i} \geq 0$, and so $(a_{n,i})_n$ is increasing. Therefore $\bar{G}_{2, \psi_n} \nearrow \bar{G}_{2, \psi}$ and (3.36) follows thanks to (3.43).

\[\square\]

4. Collision operator estimates

In this section we present estimates on the weak form of the collision operator applied on a function $f \in L^1_q$. Without assuming that $f$ is a solution to the binary-ternary Boltzmann equation. This estimate will have twofold purpose - it will be used for proving quantitative generation and propagation estimates on polynomial moments as stated in Theorem 2.6 and the existence of solutions to the binary-ternary Boltzmann equation (see Section 7).
Proposition 4.1. Suppose \( b_2, b_3 \) satisfy (2.14) and (2.20)–(2.23). Let \( q > 2 \) and suppose \( f(v) \geq 0, f \in L^1_{q^+} \), where \( \gamma = \max\{\gamma_2, \gamma_3\} \). Then the following estimates hold:

\[
\int_{\mathbb{R}^d} Q[f](v)^q \, dv \leq C_q(m_0[f], m_2[f])m_q[f] - C_q(m_0[f])(m_{q+\gamma}[f] + m_{q+\gamma_3}[f]),
\] (4.1)

and

\[
\int_{\mathbb{R}^d} Q[f](v)^q \, dv \leq C_q(m_0[f], m_2[f])m_q[f] - \bar{C}_q(m_0[f], m_2[f]) \left( m_q[f]^{1+\frac{2}{q}} + m_q[f]^{1+\frac{2}{q'}} \right),
\] (4.2)

where \( C_q(m_0[f], m_2[f]) > 0 \) is continuous with respect to \( m_0[f], m_2[f] \) and depends on \( q, \gamma_2, \gamma_3, ||b_2|| \) and \( ||b_3|| \), while the two coercive factors, \( C_q(m_0[f]) \) and \( \bar{C}_q(m_0[f], m_2[f]) \) have the following formulas:

\[
C_q(m_0[f]) = \min \left\{ (||b_2|| - \frac{\alpha_q}{2})^{-1} - \frac{2}{q} m_0[f], \frac{1}{4}(||b_3|| - \lambda_q)^3 \frac{6}{q} \left( \frac{2}{3} \right)^{\frac{2}{3}} m_0[f]^2 \right\},
\]

\[
\bar{C}_q(m_0[f]) = C_q(m_2[f])^{\frac{1}{2}} + m_2[f]^{\frac{1}{2}} + \frac{2}{q'},
\]

with \( \alpha_{q/2} \) and \( \lambda_{q/2} \) as in (3.1) and (3.18), respectively.

Proof. Let \( q > 2 \) and write it as \( q = 2r \), with \( r > 1 \). The cross-section representation (2.7) and the binary angular averaging Lemma 3.1 yield

\[
K_{2,2r}(v, v_1) := \int_{S^{d-1}} B_2(u, \omega) \left( \langle v' \rangle^{2r} + \langle v'_1 \rangle^{2r} - \langle v \rangle^{2r} - \langle v_1 \rangle^{2r} \right) \, d\omega
\]

\[
\leq |u|^\gamma \left( \alpha_r \left( E_2^r - \langle v \rangle^{2r} - \langle v_1 \rangle^{2r} \right) - (||b_2|| - \alpha_r) \left( \langle v \rangle^{2r} + \langle v_1 \rangle^{2r} \right) \right).
\]

Applying Lemma A.3 to the first term yields

\[
K_{2,2r}(v, v_1) \leq |u|^\gamma \left( \alpha_r C_{2,r} \left( \langle v \rangle^{2r} - \langle v_2 \rangle^{2r} \right) - (||b_2|| - \alpha_r) \langle v \rangle^{2r} \langle v_1 \rangle^{2r} \right),
\] (4.3)

where \( C_{2,r} = r \max\{1, 2r^{-3}\} \). Similarly, by (2.20), Lemma 3.3 and the ternary angular averaging Lemma 3.3 and Lemma A.3 imply

\[
K_{3,2r}(v, v_1, v_2) := \int_{S^{2d-1}} B_3(u, \omega, \omega') \left( \langle v' \rangle^{2r} + \langle v'_1 \rangle^{2r} + \langle v'_2 \rangle^{2r} - \langle v \rangle^{2r} - \langle v_1 \rangle^{2r} - \langle v_2 \rangle^{2r} \right) \, d\omega
\]

\[
\leq |u|^\gamma_3 \left| \frac{\theta_3}{\theta_1} \left( \lambda_r C_{3,r} \left( \langle v \rangle^{2r-2} \langle v_1 \rangle^2 + \langle v \rangle^2 \langle v_1 \rangle^{2r-2} + \langle v \rangle^{2r} \langle v_2 \rangle^2 
\right.
\right.
\]

\[
+ \langle v \rangle^2 \langle v_2 \rangle^{2r-2} + \langle v_1 \rangle^{2r-2} \langle v_2 \rangle^2 + \langle v_1 \rangle^2 \langle v_2 \rangle^{2r-2}
\]

\[
- (||b_3|| - \lambda_r) \langle v \rangle^{2r} + \langle v_1 \rangle^{2r} + \langle v_2 \rangle^{2r} \right),
\] (4.4)
where $C_{3,r} = r \max \{1, 2^{r-3}\} + \frac{r(r-1)}{2} \max \{1, 2^{r-4}\} \leq 2^{2r}$. Plugging estimates [4.3] of [4.4] into the weak form (2.27) yields

$$
\int_{\mathbb{R}^d} Q[f]\langle v \rangle^{2r} dv = \frac{1}{2} \int_{\mathbb{R}^d} f f_1 K_{2,2r}(v, v_1) dv \; dv_1 + \frac{1}{6} \int_{\mathbb{R}^d} f f_1 f_2 K_{3,2r}(v, v_1, v_2) dv \; dv_1 \; dv_2
$$

$$
\leq \alpha r C_{2,r} \int_{\mathbb{R}^d} |u|^{\gamma_2} f f_1 \langle v \rangle^{2r-2} \langle v_1 \rangle^2 dv \; dv_1 - (\|b_2\| - \alpha r) \int_{\mathbb{R}^d} |u|^{\gamma_2} f f_1 \langle v \rangle^{2r} dv \; dv_1
$$

$$
+ \lambda_r C_{3,r} \int_{\mathbb{R}^d} |u|^{\gamma_3 - \theta_3} |u|^{\theta_3} f f_1 f_2 \langle v \rangle^{2r-2} \langle v_1 \rangle^2 dv \; dv_1 \; dv_2
$$

$$
- \frac{1}{2} (\|b_3\| - \lambda_r) \int_{\mathbb{R}^d} |u|^{\gamma_3 - \theta_3} |u|^{\theta_3} f f_1 f_2 \langle v \rangle^{2r} dv \; dv_1 \; dv_2, \quad (4.5)
$$

At this point we apply upper and lower bounds on the binary and ternary potentials [A.29] - [A.32], and switch back to the $q$ notation to obtain

$$
\int_{\mathbb{R}^d} Q[f]\langle v \rangle^q dv \leq \alpha_r C_{2,q} C_{\gamma_2} (m_{q-\gamma_2} m_2 + m_{q-2} m_{\gamma_2+2}) - (\|b_2\| - \alpha_r) \left(2^{-\frac{\gamma_2}{2}} m_{q+\gamma_2} m_0 - m_0 \right)
$$

$$
+ \lambda_r C_{3,q} C_{\gamma_3} (m_{q-2+\gamma_3} m_2 + m_{q-2} m_{\gamma_3+2} m_0 + m_{q-2} m_{\gamma_3} m_2)
$$

$$
- \frac{1}{2} (\|b_3\| - \lambda_r) 3^{-\frac{\theta_3}{2}} \left(\frac{2}{3}\right)^{q-\gamma_3} m_{q+\gamma_3} m_0 - m_0 - 2m_0 m_{\gamma_3} m_0,
$$

where $C_{\gamma_2} = \max \{1, 2^{\gamma_2-1}\}$ and $C_{\gamma_3} = 2^{\gamma_3} \max \{1, 3^{\gamma_3-1}\}$. In order to estimate the term $m_{q-2} m_{\gamma_2+2}$ (and $m_{q-2} m_{\gamma_3+2}$), motivated by [3] we combine interpolation estimates together with the $\varepsilon$-Young’s inequality. Namely, by the interpolation Lemma [A.6] we have

$$
m_{q-2} \leq m_0 \frac{\gamma_2+2}{q+\gamma_2} \frac{q-2}{q+\gamma_2} \frac{q+\gamma_2}{q-\gamma_2} \quad \text{and} \quad m_{\gamma_2+2} \leq m_2 \frac{q-2}{q+\gamma_2-2} \frac{\gamma_2-2}{\gamma_2} \frac{\gamma_2-2}{q+\gamma_2-2} \frac{\gamma_2}{q-\gamma_2}.
$$

Therefore,

$$
m_{q-2} m_{\gamma_2+2} \leq A_{q,\gamma_2} m_{q+\gamma_2}^{\theta_{q,\gamma_2}},
$$

where

$$
A_{q,\gamma_2} = m_0 \frac{\gamma_2+2}{q+\gamma_2} \frac{q-2}{q+\gamma_2} \frac{q+\gamma_2}{q+\gamma_2-2},
$$

$$
\theta_{q,\gamma_2} = \frac{q-2}{q+\gamma_2} + \frac{\gamma_2}{q+\gamma_2-2} < \frac{q-2}{q+\gamma_2-2} + \frac{\gamma_2}{q+\gamma_2-2} = 1.
$$

Next, we use $\varepsilon$-Young’s inequality to obtain

$$
m_{q-2} m_{\gamma_2+2} \leq \varepsilon_{q,\gamma_2} B_{q,\gamma_2} + \varepsilon_{q,\gamma_2} \theta_{q,\gamma_2} m_{q+\gamma_2},
$$

where

$$
B_{q,\gamma_2} = (1 - \theta_{q,\gamma_2}) A_{q,\gamma_2}^{-\frac{1}{q+\gamma_2}} \leq A_{q,\gamma_2}^{-\frac{1}{q+\gamma_2}} \leq \left(\frac{\gamma_2+2}{q+\gamma_2} \frac{q-2}{q+\gamma_2-2} \right)^{\frac{(q+\gamma_2)(q+\gamma_2-2)}{2(q-2)}} m_2 \frac{(q+\gamma_2)(q+\gamma_2-2) + (q-2)(q+\gamma_2)}{2(q-2)},
$$

and where $\varepsilon_{q,\gamma_2} > 0$ will be chosen in a moment. Analogous estimates holds for $m_{q-2} m_{\gamma_3+2}$, where $\gamma_2$ is replaced with $\gamma_3$ appropriately.
Now, using that $\gamma_2, \gamma_3 \leq 2$ and the fact that moments are monotone increasing with respect to their order, we obtain

$$\int_{\mathbb{R}^d} Q[f](v)^q \, dv \leq \alpha_2 C_{2,2} C_{\gamma_2} \left( m_q m_2 + \varepsilon_{q,\gamma_2} \varepsilon_{q,\gamma_3} B_{q,\gamma_2} + \varepsilon_{q,\gamma_2} \theta_{q,\gamma_2} m_{q+\gamma_2} \right)$$

$$- \left( \|b_2\| - \alpha_2 \right) \left( 2^{-\frac{q}{4}} m_{q+\gamma_2} m_0 - m_q m_2 \right)$$

$$+ \lambda_2 C_{3,2} C_{\gamma_3} \left( 2m_q m_2^2 + \varepsilon_{q,\gamma_3} \varepsilon_{q,\gamma_3} B_{q,\gamma_3} + \varepsilon_{q,\gamma_3} \theta_{q,\gamma_3} m_{q+\gamma_3} \right)$$

$$- \frac{1}{2}(\|b_3\| - \alpha_2^3) \varepsilon_{q,\gamma_3} \left( 2^{-\frac{q}{3}} m_{q+\gamma_3} m_0^2 - 2m_q m_2^2 \right).$$

In order to ensure that $m_{q+\gamma_2}$ and $m_{q+\gamma_3}$ terms are negative, we choose $\varepsilon_{q,\gamma_2}$ and $\varepsilon_{q,\gamma_3}$ to be

$$\varepsilon_{q,\gamma_2} = \frac{(1 - \alpha_2^2)2^{\frac{q}{2}} m_0}{2\alpha_2 C_{2,2} C_{\gamma_2} \theta_{q,\gamma_2}}$$

and

$$\varepsilon_{q,\gamma_3} = \frac{1}{2}(1 - \lambda_2^3)3^{-\frac{q}{3}} \left( \frac{2}{3} \right)^{\frac{q}{3}} m_0^2.$$

With such a choice of $\varepsilon_{q,\gamma_2}$ and $\varepsilon_{q,\gamma_3}$, we have

$$\int_{\mathbb{R}^d} Q[f](v)^q \, dv \leq E_q + D_q m_q - (\|b_2\| - \alpha_2) \varepsilon_{q,\gamma_3} \left( 2^{-1-\frac{q}{2}} m_0 m_{q+\gamma_2} - \frac{1}{4}(\|b_3\| - \lambda_2^3)3^{-\frac{q}{3}} \left( \frac{2}{3} \right)^{\frac{q}{3}} m_0^2 m_{q+\gamma_3} \right),$$

where the coefficients $E_q$ and $D_q$ are given by

$$E_q = \alpha_2 C_{2,2} C_{\gamma_2} \varepsilon_{q,\gamma_2} + \lambda_2 C_{3,2} C_{\gamma_3} \varepsilon_{q,\gamma_3} B_{q,\gamma_2},$$

$$D_q = \alpha_2 C_{2,2} C_{\gamma_2} m_2 + (\|b_2\| - \alpha_2^3) m_2 + 2\lambda_2 C_{3,2} C_{\gamma_3} m_2 + (\|b_3\| - \lambda_2^3)3^{-\frac{q}{3}} m_2.$$
and differentiability of polynomial moments (as stated in Theorem 5.1). This, in turn, will enable us to obtain an ordinary differential inequality for moments (see Proposition 4.1) which will yield quantitative estimates of polynomial moments, including their propagation and generation in time, thanks to a comparison principle with a Bernoulli-type ODE.

5.1. Phase 1: Finiteness and differentiability of moments. Our first goal is to prove the following result on finiteness and differentiability of polynomial moments for solutions to the binary-ternary Boltzmann equation.

**Theorem 5.1.** Let $T > 0$, and let $f \geq 0$ be a solution to the binary-ternary Boltzmann equation (2.1) corresponding to the initial data $f_0 \in L^1_k$, $f_0 \geq 0$. Then $f \in C^1((0, T], L^1_k)$ for any $k > 2$. Additionally,

$$m_k[f] = \int_{\mathbb{R}^d} Q[f](v)^k \, dv, \quad \forall t \in (0, T].$$

(5.1)

The proof of the theorem is based on the following lemma, which establishes time integrability of moments via an inductive argument. The lemma is inspired by a related result in [33] for the homogeneous Boltzmann equation (1.5). However, in the proof of the lemma we use novel angular averaging estimates of Section 3.

**Lemma 5.2.** Let $T > 0$ and let $f \geq 0$ be a solution to the binary-ternary Boltzmann equation (2.1) corresponding to the initial data $f_0 \in L^1_k$, $f_0 \geq 0$. Then for any $0 \leq s < t < T$, any $k \geq 2 + \frac{\gamma}{2}$, and any $\varepsilon \in (0, t - s)$,

$$\int_s^t m_k(\tau) \, d\tau < \infty \quad \Rightarrow \quad m_{k-\frac{\gamma}{2}}(t) + \int_{s+\varepsilon}^t m_{k+\frac{\gamma}{2}}(\tau) \, d\tau < \infty. \quad (5.2)$$

Moreover, for any $k > 2$ and any $0 < s < t < T$, we have $m_k(t) < \infty$ and

$$\int_s^t m_k(\tau) \, d\tau < \infty. \quad (5.3)$$

Before proving the lemma, we show how we use it to prove Theorem 5.1.

**Proof of Theorem 5.1.** Let $k > 2$ and $t_0 \in (0, T)$ arbitrarily small. Now, Lemmata A.1-A.2, the conservation laws, and estimate (5.3) yield that $\int_{t_0}^T \|Q[f(\tau)]\|_{L^1_k} \, d\tau < \infty$, so $Q[f] \in L^1([t_0, T], L^1_k)$. Now, integrating (2.32) in time, testing with $\langle v \rangle^k$, and using Fubini’s theorem, we obtain

$$m_k(t) = m_k(t_0) + \int_{t_0}^t \int_{\mathbb{R}^d} Q[f](v)^k \, dv \, d\tau, \quad \forall t \in [t_0, T]. \quad (5.4)$$

In particular, $f \in C([t_0, T], L^1_k)$. Since $k > 2$ was arbitrary, we have $f \in C([t_0, T], L^1_k)$, for all $k > 2$, which in turn implies (by Lemmata A.1-A.2 respectively) that $Q[f] \in C^1([t_0, T], L^1_k)$, for all $k > 2$. Then, differentiating (5.4), we obtain (5.1) in $[t_0, T]$ and that $f \in C^1([t_0, T], L^1_k)$. Since $t_0$ was chosen arbitrarily small, (5.1) holds $(0, T)$ and $f \in C^1((0, T], L^1_k)$.

□

We next prove Lemma 5.2.
Proof of Lemma 5.2. Let $0 \leq s < t < T$, $\varepsilon \in (0, t-s)$, $k \geq 2 + \frac{1}{2}$ and assume that $\int_s^t m_k(\tau)d\tau < \infty$. Then, by the monotonicity of moments

$$
\int_s^t m_k(\tau)d\tau < \infty,
$$

and so there exists $s_0$ such that $0 < s < s_0 < s + \varepsilon < t$ and $m_k(\tau)(s_0) < \infty$. Let $\psi(x) = x^{1+\varepsilon}$. Since $\frac{1}{t} - \frac{1}{s} \geq 1$, this is a convex function, so by Lemma A.5 there exists a sequence of functions $\phi_n \uparrow \psi$, defined by

$$
\phi_n(x) = \begin{cases} 
\psi(x), & x \leq n \\
\phi_n(x), & x > n,
\end{cases}
$$

where $\phi_n$ is a polynomial of degree one given by $\phi_n(x) = \psi_n(x) + \psi_n(x) - n\psi_n(x)$. Since for each $n \in \mathbb{N}$, $\psi_n((v)^2) \leq C_n(v)^2$, and $f$ is a solution to the binary-ternary Boltzmann equation, by Lemmata A.1 and A.2 $\psi_n((v)^2)$ can be used as a test function in the weak formulation to obtain

$$
\int_{\mathbb{R}^d} (f(t,v) - f(s_0,v)) \psi_n((v)^2)dv = \frac{1}{2} \int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} K_{2,\psi_n}(v,v_1) dvdv d\tau \\
+ \frac{1}{6} \int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 ff_2 |u|^{\gamma_3 - \theta_3} |u|^{\theta_3} K_{3,\psi_n}(v,v_1) dvedv d\tau,
$$

where $K_{2,\psi_n}$ and $K_{3,\psi_n}$ are defined as in (3.2) and (3.31). Using Lemma 3.2 and Lemma 3.3, we write $K_{2,\psi_n} = \tilde{G}_{2,\psi_n} - \tilde{L}_{2,\psi_n}$ and $K_{3,\psi_n} = \tilde{G}_{3,\psi_n} - \tilde{L}_{3,\psi_n}$, where these quantities are defined as in (3.13), (3.14), (3.31) and (3.42). Next, we show that integrals $\int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \tilde{L}_{2,\psi_n} dvedv d\tau$ and $\int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 ff_2 |u|^{\gamma_3 - \theta_3} |u|^{\theta_3} \tilde{L}_{3,\psi_n} dvedv d\tau$ are finite, and thus can be added to both sides of the above equation. Namely,

$$
\int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \tilde{L}_{2,\psi_n} dvedv d\tau = \parallel b_2 \parallel \int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \left(\psi_n((v)^2) + \psi_n((v_1)^2)\right) dvedv.
$$

Due to the symmetry of this expression with respect to $v \leftrightarrow v_1$, it suffices to show that the integral $\int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \psi_n((v)^2) dvedv d\tau$ is finite. In order to establish this, we use the definition of the approximation function $\psi_n$ given in (A.20) to obtain

$$
\int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \psi_n((v)^2) dvedv
\leq n^k \int_{\mathbb{R}^d} \int_{\{(v)^2 \leq n\}} ff_1 |v - v_1|^{\gamma_2} \psi((v)^2) dvedv + \int_{\mathbb{R}^d} \int_{\{(v)^2 > n\}} ff_1 |v - v_1|^{\gamma_2} p_n((v)^2) dvedv
\leq n^k \int_{\mathbb{R}^d} \int_{\{(v)^2 \leq n\}} ff_1 |v - v_1|^{\gamma_2} dvedv + \int_{\mathbb{R}^d} \int_{\{(v)^2 > n\}} ff_1 |v - v_1|^{\gamma_2} (A(v)^2 + B) dvedv,
$$

where $A, B$ are coefficients (that depend on $n$) of the first order polynomial $p_n$ defined in (A.20). Therefore,

$$
\int_{s_0}^t \int_{\mathbb{R}^{2d}} ff_1 |v - v_1|^{\gamma_2} \psi_n((v)^2) dvedv \leq Cm_2 \int_{s_0}^t m_2 + \gamma_1(\tau)d\tau,
$$

and therefore the proof of Lemma 5.2 is complete.
where $C > 0$ is a constant that depends on $n$ and $\gamma_2$. This is a finite quantity since $f$ is solution. Similarly, one can show that $\int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2 - \theta_3} |u|^{\theta_3} \tilde{L}_{3, \psi_n} \, dv \, d\tau$ is finite. Additionally, by an analogous domain-splitting one can show that $\int_{\mathbb{R}^d} f(s_0, v) \psi_n(\langle v \rangle^2) \, dv \leq C m_2(s_0) \leq C m_{k-\frac{1}{2}}(s_0) < \infty$ by the choice of time $s_0$, where $C > 0$ depends on $n$ and $\gamma_2$. Therefore, adding these finite integrals to both sides of the equation (5.6) yields

\[
\int_{\mathbb{R}^d} f(t, v) \psi_n(\langle v \rangle^2) \, dv + \frac{1}{2} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \tilde{L}_{2, \psi_n} \, dv \, d\tau + \frac{1}{6} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2 - \theta_3} |u|^{\theta_3} \tilde{L}_{3, \psi_n} \, dv \, d\tau \\
= \int f(s_0, v) \psi_n(\langle v \rangle^2) \, dv + \frac{1}{2} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \tilde{G}_{2, \psi_n} \, dv \, d\tau + \frac{1}{6} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2 - \theta_3} |u|^{\theta_3} \tilde{G}_{3, \psi_n} \, dv \, d\tau.
\]

In the rest of the proof we abuse notation and denote by $C_k$ various positive constants that depend on $k, \gamma, \gamma_2, \gamma_3, \beta, b_3, m_0$ and $m_2$. Since $\frac{5}{2} - \frac{2}{n} \geq 1$, estimates (5.7) and (5.36) can be used to obtain

\[
\int_{\mathbb{R}^d} f(t, v) \psi_n(\langle v \rangle^2) \, dv + \frac{1}{2} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \tilde{L}_{2, \psi_n} + \frac{1}{6} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2 - \theta_3} |u|^{\theta_3} \tilde{L}_{3, \psi_n} \, dv \, d\tau \\
\leq \int_{\mathbb{R}^d} f(s_0, v) \psi_n(\langle v \rangle^2) \, dv + C_k \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \theta_2(\langle v \rangle \langle v_1 \rangle) (\langle v_1 \rangle^{k-\frac{2}{n}-2} + \langle v \rangle^{k-\frac{2}{n}-2}) \, dv \, d\tau \\
+ C_k \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2} \langle \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v_1 \rangle \langle v_2 \rangle \rangle \sum_{i=0,1,2} \langle v_i \rangle^{k-\frac{2}{n}-2} \, dv \, d\tau.
\]

Since $\tilde{L}_{2, \psi_n}, \tilde{L}_{3, \psi_n} \geq 0$, and $\tilde{L}_{2, \psi_n} \rightarrow \tilde{L}_{2, \psi}, \tilde{L}_{3, \psi_n} \rightarrow \tilde{L}_{3, \psi}$, Fatou’s lemma can be applied on the integrals containing $\tilde{L}_{2, \psi_n}$ and $\tilde{L}_{3, \psi_n}$. Therefore, using the monotone convergence theorem on the other two terms containing $\psi_n$, letting $n \rightarrow \infty$ yields

\[
\int_{\mathbb{R}^d} f(t, v) \psi_n(\langle v \rangle^2) \, dv + \frac{1}{2} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \tilde{L}_{2, \psi} + \frac{1}{6} \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2 - \theta_3} |u|^{\theta_3} \tilde{L}_{3, \psi} \, dv \, d\tau \\
\leq \int_{\mathbb{R}^d} f(s_0, v) \psi(\langle v \rangle^2) \, dv + C_k \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \theta_2(\langle v \rangle \langle v_1 \rangle) (\langle v_1 \rangle^{k-\frac{2}{n}-2} + \langle v \rangle^{k-\frac{2}{n}-2}) \, dv \, d\tau \\
+ C_k \int_{s_0}^t \int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2} \langle \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v_1 \rangle \langle v_2 \rangle \rangle \sum_{i=0,1,2} \langle v_i \rangle^{k-\frac{2}{n}-2} \, dv \, d\tau. 
\]

Using the upper bounds (A.29) and (A.31) on the potentials $|v - v_1|^{\gamma_2}$ and $|v - v_1|^{\gamma_3}$, one obtains

\[
\int_{\mathbb{R}^d} f f_1 |v - v_1|^{\gamma_2} \theta_2(\langle v \rangle \langle v_1 \rangle) (\langle v_1 \rangle^{k-\frac{2}{n}-2} + \langle v \rangle^{k-\frac{2}{n}-2}) \, dv \leq C_k m_{k-1+\frac{2}{n}}, 
\]

\[
\int_{\mathbb{R}^d} f f_1 f_2 |\overline{u}|^{\gamma_2} \langle \langle v \rangle \langle v_1 \rangle + \langle v \rangle \langle v_2 \rangle + \langle v_1 \rangle \langle v_2 \rangle \rangle \sum_{i=0,1,2} \langle v_i \rangle^{k-\frac{2}{n}-2} \, dv \leq C_k m_{k-1+\frac{2}{n}}.
\]
On the other hand, lower bounds (3.6) and (A.30) yield the following lower bound:

\[
\int_{\mathbb{R}^{2d}} f f_1 |v - v_1|^{\gamma_2} \tilde{L}_{2,\psi} \, dv dv_1 \\
\geq (\|b_2\| - \alpha \frac{1}{k}) \int_{\mathbb{R}^{2d}} f f_1 \left( 2^{\gamma_2} \sum_{i=0,1} (v_i)^{k-\frac{\gamma_2}{2}} - (v)^{\gamma_2} (v_1)^{k-\frac{\gamma_2}{2}} - (v)^{k-\frac{\gamma_2}{2}} (v_1)^{\gamma_2} \right) \, dv dv_1 \\
\geq C_k \left( m_0 m_{k-\frac{\gamma_2}{2}+\gamma_2} - m_{\gamma_2} m_{k-\frac{\gamma_2}{2}} \right), \tag{5.10}
\]

while bounds (3.35), (A.32) and (2.16) imply

\[
\int_{\mathbb{R}^{2d}} f f_1 f_2 |\tilde{u}|^{\gamma_3-\theta_3} |u|^{\theta_3} \tilde{L}_{3,\psi} \, dv dv_1 dv_2 \\
\geq (\|b_3\| - \lambda \frac{1}{k}) 3^{\gamma_3} \int_{\mathbb{R}^{2d}} f f_1 f_2 \left( \frac{2}{3} (v_1)^{\gamma_3} - \sum_{j \in \{0,1,2\} \setminus \{i\}} (v_j)^{\gamma_3} \right) \, dv dv_1 dv_2 \\
\geq C_k \left( m_0 m_{k-\frac{\gamma_3}{2}+\gamma_3} - m_{\gamma_3} m_{k-\frac{\gamma_3}{2}} \right). \tag{5.11}
\]

Combining estimates (5.8) - (5.11) with (5.7) yields

\[
m_{k-\frac{\gamma_2}{2}}(t) + C_k \int_{s_0}^t \left( m_{k-\frac{\gamma_2}{2}+\gamma_2} + m_{k-\frac{\gamma_2}{2}+\gamma_3} \right) \, d\tau \leq m_{k-\frac{\gamma_2}{2}}(s_0) + C_k \int_{s_0}^t m_k \, d\tau.
\]

Therefore, since one of \(\gamma_2\) or \(\gamma_3\) coincides with \(\gamma\), we have

\[
m_{k-\frac{\gamma_2}{2}}(t) + C_k \int_{s_0}^t m_{k-\frac{\gamma_2}{2}} \, d\tau \leq m_{k-\frac{\gamma_2}{2}}(s_0) + C_k \int_{s_0}^t m_k \, d\tau < \infty,
\]

which proves (5.2).

Finally, to prove (5.3), let \(k > 2\) and fix any \(0 < s < t\). Let \(n \in \mathbb{N}\) be the smallest positive integer so that \(k \leq 2 + \frac{n}{2}\), and choose \(\varepsilon_0 > 0\) so that \(\frac{\gamma_2}{2} + n \varepsilon_0 \leq s\). By the definition of solutions

\[
\int_{\frac{s}{2}}^t m_{2+\gamma}(\tau) \, d\tau < \infty.
\]

Then by (5.2), we have

\[
m_{2+\gamma}(t) + \int_{\frac{s}{2}+\varepsilon_0}^t m_{2+\gamma}(\tau) \, d\tau < \infty.
\]

In fact, applying (5.2) inductively yields

\[
m_{2+\gamma}(t) + \int_{\frac{s}{2}+n\varepsilon_0}^t m_{2+\gamma}(\tau) \, d\tau < \infty.
\]

Since \(k \leq 2 + \frac{(n+2)\gamma}{2}\) and \(\frac{\gamma}{2} + n \varepsilon_0 \leq s\), we conclude that \(m_k(t) < \infty\) and \(\int_t s m_k(\tau) \, d\tau < \infty\). \(\square\)
5.2. Phase 2: Quantitative estimates of moments. Now that Phase 1 is completed and finiteness of moments is established, we proceed to prove quantitative moment estimates on generation and propagation in time of polynomial moments of solutions to the binary-ternary Boltzmann equation as stated in Theorem 2.6. The proof relies on the already established finiteness and differentiability of moments Theorem 5.1, as well as the estimate on the weak form of the collision operator Proposition 4.1.

Proof of Theorem 2.6 Without loss of generality, we assume that \( m_0[f_0] > 0 \), otherwise by the conservation of mass the only solution is zero, so the claim trivially holds.

Proof of (i): Fix \( q > 2 \). By Theorem 5.1, \( f \in C^1((0, T], L^1_q) \). Testing (2.1) against \( (v)^q \) and integrating, differentiability of \( m_q \) and estimate (5.12) yield

\[
m'_q[f](t) \leq C_q m_q[f](t) - \tilde{C}_q \left( m_q[f](t)^{1 + \frac{2q}{q-2}} + m_q[f](t)^{1 + \frac{q}{q-2}} \right).
\]  

(5.12)

In particular, (5.12) implies

\[
m'_q[f](t) \leq C_q m_q[f](t) - \tilde{C}_q m_q[f](t)^{1 + \frac{q}{q-2}},
\]

(5.13)

for each \( i \in \{2, 3\} \). Now by Lemma 3.8 in [29], \( m_q[f] \) satisfying (5.13), with the additional constraint that \( \gamma_i > 0 \), is a sub-solution to the Bernoulli-type initial value problem:

\[
\begin{align*}
y' &= C_q y - \tilde{C}_q y^{1 + \frac{q}{q-2}} , \quad t > 0 \\
\lim_{t \to 0^+} y(t) &= +\infty,
\end{align*}
\]

(5.14)

and we have

\[
m_q[f](t) \leq y(t) = \left( \frac{C_q}{\tilde{C}_q} \right)^{\frac{2q}{q-1}} \left( 1 - e^{-t C_i \frac{q}{q-2}} \right)^{\frac{2-q}{q}} , \quad t > 0.
\]

(5.15)

For \( t > 1 \), estimate (5.15) implies

\[
m_q[f](t) \leq \left( \frac{C_q}{\tilde{C}_q} \right)^{\frac{2q}{q-1}} \left( 1 - e^{-t C_i \frac{q}{q-2}} \right)^{\frac{2-q}{q}}.
\]

If \( t \leq 1 \), then for any \( A > 0 \), we have \( 1 - e^{-t A} \geq A t e^{-A} \). In particular, for \( A = \frac{C_i \gamma_i}{q-2} \), we have

\[
1 - e^{-t C_i \frac{q}{q-2}} \geq \frac{C_q \gamma_i t}{q-2} e^{-C_q \gamma_i / q-2}
\]

for all \( t \in (0, 1] \). Therefore, (5.15) implies that for all \( t \in (0, 1] \) we have

\[
m_q[f](t) \leq \left( \frac{C_q}{\tilde{C}_q} \right)^{\frac{2q}{q-1}} e^{C_q \gamma_i} \left( \frac{C_q \gamma_i}{q-2} \right)^{\frac{2-q}{q}} \left( 1 - e^{-t C_i \frac{q}{q-2}} \right)^{\frac{2-q}{q}}.
\]

Defining

\[
K_{q,i} = \left( \frac{C_q}{\tilde{C}_q} \right)^{\frac{2q}{q-1}} \max \left\{ 1 - e^{-t C_i \frac{q}{q-2}} \right\},
\]

(5.16)

we obtain

\[
m_q[f](t) \leq K_{q,i} \max \{1, t^\frac{2-q}{q-1}\}.
\]
Therefore, estimate (2.33) has been shown. Now if both $\gamma_2, \gamma_3 > 0$, we have
\[
m_q[f](t) \leq K_q, \max\{1, t^{\frac{2}{\gamma_2}}, t^{\frac{3}{\gamma_3}}\}, \quad \forall t > 0, \quad i = 2, 3,
\]
which implies bound (2.34) for $K_q = \max\{K_{q,2}, K_{q,3}\}$.

Proof of (ii): Now assume $m_q(0) < \infty$. To control the behavior of $m_q$ for $t \in [0, \min\{1, T\}]$, we will use the fact that $m_q$ is initially finite. Indeed, estimate (5.12) yields
\[
m_q'[f](t) \leq C_q m_q[f](t) - \tilde{C}_q m_q[f](t)^{1 + \frac{2}{\gamma_q}},
\]
where $\gamma = \max\{\gamma_2, \gamma_3\} > 0$. Consider now the Bernoulli IVP
\[
\begin{cases}
y' = C_q y - \tilde{C}_q y^{1 + \frac{2}{\gamma_q}}, & t \in [0, \min\{1, T\}] \\
y(0) = m_q(0),
\end{cases}
\]
which has the solution
\[
y(t) = \left( m_q(0)^{-\frac{\tilde{C}_q}{C_q}} e^{-tC_q} + \frac{\tilde{C}_q}{C_q} \left(1 - e^{-tC_q}\right)\right)^{\frac{\gamma_q}{2}} \leq m_q(0) e^{tC_q}.
\]
By the comparison principle, we have $m_q[f](t) \leq y(t)$, thus
\[
\sup_{t \in [0, \min\{1, T\}]} m_q(t) \leq m_q(0) e^{tC_q}. \quad (5.17)
\]
If $T \leq 1$, this completes the proof of the lemma. Now if, $T > 1$ by the generation estimate (2.34), we have that
\[
\sup_{1 < t \leq T} m_q(t) \leq K_q, \quad (5.18)
\]
which implies estimate (2.35) for $M_q = \max\{m_q(0) e^{C_q}, K_q\}$. □

6. Generation and propagation of exponential moments

In this section we prove generation and propagation of exponential moments of solutions to the spatially homogeneous binary-ternary Boltzmann equation (2.1) as formulated in Theorem 2.7. Our proof is inspired by [1], where an analogous result was established for the homogeneous binary Boltzmann equation (1.5). As [1], we rely on angular averaging estimates, but we use those derived specifically for the binary-ternary operator in Section 3. In fact, angular averaging estimates are first used to obtain an upper bound on the binary-ternary collision operator $Q[f]$, (1.4) (see Lemma 6.1) for a general function which is not necessarily a solution to the binary-ternary Boltzmann equation, and in addition to being used in this section, it will play an important role in the proof of well-posedness in Section 4. We note that compared to Section 5 where there were two phases of the proof, since finiteness of moments is already established (see Theorem 6.1), in this section there is only one phase of proving the quantitative propagation and generation of exponential moments.

Recall the definition of exponential moments (1.8), and as first exploited by Bobylev [13], note that they can be expressed as an infinite sum of weighted polynomial moments thanks the the Taylor expansion of the exponential weight $e^{z(v_s)z}$ as follows
\[
E_s(t, z) = \sum_{p=0}^{\infty} m_{sp}(t) \frac{z^p}{p!}.
\]
Let us denote the partial sum of this expansion by $E^n_s$, and the partial sum shifted by a parameter $\tilde{\gamma} > 0$ by $I^n_{s,\tilde{\gamma}}$, where

$$ E^n_s(t, z) := \sum_{p=0}^{n} m_{sp}(t) \frac{z^p}{p!}, \quad (6.1) $$

$$ I^n_{s,\tilde{\gamma}}(t, z) := \sum_{p=0}^{n} m_{sp+\tilde{\gamma}}(t) \frac{z^p}{p!}. \quad (6.2) $$

Throughout the paper, the shift $\tilde{\gamma}$ will be $\gamma_2$ or $\gamma_3$.

We also use the following notation for binomial coefficients

$$ \binom{p}{k, k_1} = \frac{p!}{k!k_1!}, \quad (6.3) $$

where $p = k + k_1$, in order to observe the similarity with the calculations for the ternary term. Similarly, we use the notation for trinomial coefficients, for $p = k + k_1 + k_2$,

$$ \binom{p}{k, k_1, k_2} = \frac{p!}{k!k_1!k_2!}. \quad (6.4) $$

Due to the expansion (6.1) and the use of averaging Lemmata 3.1, 3.3, in this section we work with polynomial weights of order $sp > 2$, where $p$ is an integer and $s \in (0, 2]$. The following lemma provides an estimate of the collision operator integrated against a polynomial weight of order $sp$.  

**Lemma 6.1.** Consider the binary-ternary collision operator (1.4) with (2.7), (2.8), (2.9), and (2.20) - (2.23). Let $s \in (0, 2]$, $p \in \mathbb{N}$ with $sp > 2$, and suppose a non-negative function $f \in L^1_{sp+\gamma}$ conserves mass and energy, i.e. for some $0 < m_0 < m_2 < \infty$ and any $t \geq 0$, $m_0(t) = m_0$, $m_2(t) = m_2$. Then the following estimate holds:

$$ \int_{\mathbb{R}^d} Q[f](v)^{sp} \, dv \leq -K_{1,sp} m_{sp+\gamma_2} - K_{2,sp} m_{sp+\gamma_3} + K_{3,sp} m_{sp} + 2C_{\gamma_2} \alpha_{sp/2} S^p_{2,s,\gamma_2} + 3C_{\gamma_3} \lambda_{sp/2} S^p_{3,s,\gamma_3}, \quad (6.5) $$

where

$$ K_{1,sp} = 2^{1-\frac{2s}{p}} m_0 (\|b_2\| - \alpha_{sp/2}), $$

$$ K_{2,sp} = 3 \left( \frac{2}{3} \right)^{\gamma_3/2} m_2 (\|b_3\| - \lambda_{sp/2}), \quad (6.6) $$

$$ K_{3,sp} = 2m_2 (\|b_2\| - \alpha_{sp/2}) + 6m_0m_2 (\|b_3\| - \lambda_{sp/2}) $$

are positive constants that are increasing in $sp$, and where

$$ S^p_{2,s,\gamma_2} = \sum_{0 < k, k_1 < p \atop k + k_1 = p} \binom{p}{k, k_1} m_{sk+\gamma_2} m_{sk_1}, \quad (6.7) $$

$$ S^p_{3,s,\gamma_3} = \sum_{0 < k, k_1, k_2 < p \atop k + k_1 + k_2 = p} \binom{p}{k, k_1, k_2} m_{sk+\gamma_3} m_{sk_1} m_{sk_2}. \quad (6.8) $$
Proof. By the properties of the binary cross-section (2.6) - (2.9) and the binary angular averaging Lemma 3.1, we have

\[
G_{2,sp}(v, v_1) := \int_{\mathbb{S}^{d-1}} B_2(u, \omega) \left( \langle u' \rangle^{sp} + \langle v_1' \rangle^{sp} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} \right) d\omega
\]

\[
\leq |u|^{\gamma_2} \left( \alpha_{sp/2} \left( E_2^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} \right) - \|b_2\| - \alpha_{sp/2} \right) \cdot (\langle v \rangle^{sp} + \langle v_1 \rangle^{sp}) .
\]

(6.9)

Similarly, properties of the ternary cross-section (2.20), (2.21) - (2.23) and the ternary angular averaging Lemma 3.3 imply that

\[
G_{3,sp}(v, v_1, v_2) := \int_{\mathbb{S}^{d-1}} B_3(u, \omega) \left( \langle v' \rangle^{sp} + \langle v_1' \rangle^{sp} + \langle v_2' \rangle^{sp} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} - \langle v_2 \rangle^{sp} \right) d\omega
\]

\[
\leq |\bar{u}|^{\gamma_3} \left( \lambda_{sp/2} \left( E_3^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} - \langle v_2 \rangle^{sp} \right) - \|b_3\| - \lambda_{sp/2} \right) \cdot (\langle v \rangle^{sp} + \langle v_1 \rangle^{sp} + \langle v_2 \rangle^{sp})
\]

(6.10)

By the weak form (2.27), estimates (6.9) - (6.10), and symmetry of $|\bar{u}|$ with respect to renaming velocities, for a general non-negative function $f \in L^1_{sp+\gamma}$ conserves mass and energy, we obtain

\[
\int_{\mathbb{R}^d} Q[f]|\langle v \rangle|^{sp} dv = \int_{\mathbb{R}^d} f f_1 G_{2,sp}(v, v_1) dv dv_1 + \int_{\mathbb{R}^d} f f_1 f_2 G_{3,sp}(v, v_1, v_2) dv dv_1 dv_{1,2}
\]

\[
\leq \alpha_{sp/2} \int_{\mathbb{R}^d} |u|^{\gamma_2} \left( \int_{\mathbb{R}^d} f f_1 \left( E_2^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} \right) dv dv_1 - 2\|b_2\| - \alpha_{sp/2} \right) \cdot (\langle v \rangle^{sp} + \langle v_1 \rangle^{sp})
\]

\[
+ \lambda_{sp/2} \int_{\mathbb{R}^d} |\bar{u}|^{\gamma_3} \left( \int_{\mathbb{R}^d} f f_1 f_2 \left( E_3^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} - \langle v_2 \rangle^{sp} \right) dv dv_1 dv_1,2
\]

\[
- 3(\|b_3\| - \lambda_{sp/2}) \int_{\mathbb{R}^d} |\bar{u}|^{\gamma_3} \left( \int_{\mathbb{R}^d} f f_1 f_2 \left( v \right)^{sp} dv dv_1 dv_{1,2}
\]

\[
: = \alpha_{sp/2} A_1 + 2(\|b_2\| - \alpha_{sp/2}) A_2 + \lambda_{sp/2} B_1 - 3(\|b_3\| - \lambda_{sp/2}) B_2.
\]

(6.11)

**Upper bound on $A_1$:** Since $s \leq 2$, $(\langle v \rangle^2 + \langle v_1 \rangle^2)^{s/2} \leq (\langle v \rangle^s + \langle v_1 \rangle^s)$, so by the binomial theorem,

\[
E_2^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} \leq (\langle v \rangle^s + \langle v_1 \rangle^s)^{(s-1)/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} = \sum_{0<k,k_1<p \atop k+k_1=p} \binom{p}{k,k_1} (\langle v \rangle^k (v_1)^{k_1}.
\]

Therefore, (A.29) yields, with $C_{\gamma_2} = \max\{1, 2^{\gamma_2-1}\}$,

\[
A_1 \leq C_{\gamma_2} \sum_{0<k,k_1<p \atop k+k_1=p} \binom{p}{k,k_1} \int_{\mathbb{R}^d} f f_1 \left( \langle v \rangle^{sk+\gamma_2} (v_1)^{sk_1} + \langle v \rangle^{sk} (v_1)^{sk_1+\gamma_2} \right) dv dv_1
\]

\[
= 2C_{\gamma_2} \sum_{0<k,k_1<p \atop k+k_1=p} \binom{p}{k,k_1} m_{sk+\gamma_2} m_{sk_1}.
\]

(6.12)

**Lower bound on $A_2$:** Using (A.30), we obtain

\[
A_2 = \int_{\mathbb{R}^d} |u|^{\gamma_2} f f_1 \left( v \right)^{sp} dv dv_1 \geq \frac{1}{2^{\gamma_2/2}} m_{sp+\gamma_2} m_0 - m_{sp} m_{\gamma_2}.
\]

(6.13)
Therefore, using the multinomial expansion, we have

\[ E^{sp/2} - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} - \langle v_2 \rangle^{sp} \leq \left( \langle v \rangle^s + \langle v_1 \rangle^s + \langle v_2 \rangle^s \right)^p - \langle v \rangle^{sp} - \langle v_1 \rangle^{sp} - \langle v_2 \rangle^{sp} \]

\[ \sum_{0 \leq k_1, k_2 < p} \binom{p}{k_1, k_2} \langle v \rangle^{sk_1} \langle v_1 \rangle^{sk_1} \langle v_2 \rangle^{sk_2}. \]

Therefore, (A.31) yields, with \( C_{\gamma_3} = 2^{\gamma_3} \max\{1, 3^{\gamma_3} - 1\}, \)

\[ B_1 \leq C_{\gamma_3} \sum_{0 \leq k_1, k_2 < p} \binom{p}{k_1, k_2} (m_{sk_1 + \gamma_3} m_{sk_2} + m_{sk_1} + m_{sk_2} + m_{sk_1 + \gamma_3} m_{sk_2} + m_{sk_1 + \gamma_3} m_{sk_2} + m_{sk_1 + \gamma_3} m_{sk_2} + m_{sk_1 + \gamma_3} m_{sk_2}) \]

\[ = 3C_{\gamma_3} \sum_{0 \leq k_1, k_2 < p} \binom{p}{k_1, k_2} m_{sk_1 + \gamma_3} m_{sk_2}. \]

(6.14)

**Lower bound on term \( B_2 \):** Estimate (A.32) implies

\[ B_2 = \int_{\mathbb{R}^d} |\mathbf{u}|^{\gamma_3} f \cdot f_2 (\langle v \rangle^{sp}) dv \geq 2^{\gamma_3/2} \frac{(2)}{3} m_{sp+\gamma_3} m_0^2 - 2m_{sp} m_{\gamma_3} m_0. \]

(6.15)

Plugging (6.12)–(6.14) into (6.11), we conclude that for \( sp > 2 \), we have

\[ \int_{\mathbb{R}^d} Q[f] (\langle v \rangle^{sp}) dv \leq 2C_{\gamma_2} \alpha_{sp/2} S_{2, s, \gamma_2} - 2\|b_2\| - \alpha_{sp/2} \left( \frac{1}{2^{\gamma_2/2}} m_{sp+\gamma_2} m_0 - m_{sp} m_{\gamma_2} \right) \]

\[ + 3C_{\gamma_3} \lambda_{sp/2} S_{3, s, \gamma_3} - 3\|b_3\| - \lambda_{sp/2} \left( \frac{2}{3} \right) \frac{\gamma_3/2}{m_{sp+\gamma_3} m_0^2 - 2m_{sp} m_{\gamma_3} m_0}. \]

Monotonicity of moments and the fact that \( \gamma_2, \gamma_3 \leq 2 \) imply (6.5). \( \square \)

**Lemma 6.2.** For \( E^n_s(t, z), I^n_s(t, z) \) and \( I^n_s(t, z) \) defined as in (6.1)–(6.2), we have:

\[ \sum_{p=p_0}^n \frac{z^p}{p!} S_{2, s, \gamma_2}^p (t) \leq I^n_s(t, z) E^n_s(t, z), \]

(6.16)

\[ \sum_{p=p_0}^n \frac{z^p}{p!} S_{3, s, \gamma_3}^p (t) \leq I^n_s(t, z) (E^n_s(t, z))^2. \]

(6.17)

**Proof.** Inequality (6.16) is proven by exchanging the order of summations and observing that the inner sum can be bounded by \( E^n_s \). Namely, recalling (6.7), we have

\[ \sum_{p=p_0}^n \frac{z^p}{p!} \sum_{0 \leq k_1, k_2 < p} \binom{p}{k_1, k_2} m_{sk_1} m_{sk_2} = \sum_{k=1}^{n-1} \sum_{p=\max\{p_0, k+1\}}^n \frac{z^p}{p!} \binom{p}{k, p-k} m_{sk_1+\gamma_2} m_{sk_2} \]

\[ = \sum_{k=1}^{n-1} \frac{z^k}{k!} m_{sk_1+\gamma_2} \sum_{p=\max\{p_0, k+1\}}^n \frac{z^{p-k}}{(p-k)!} m_{sp-k} \leq I^n_s(t, z) E^n_s(t, z). \]
To prove inequality (6.17), we begin by rewriting the summation in \( k, k_1, k_2 \) as two sums - one in \( k \) and the other one in \( k_1 \) and then we exchange the order of summation in \( p \) and \( k \):

\[
\sum_{p=p_0}^n \frac{z^p}{p!} \sum_{0 \leq k, k_1, k_2 < p \atop k + k_1 + k_2 = p} \binom{p}{k, k_1, k_2} m_{sk+\gamma_3} m_{sk_1} m_{sk_2}
\]

\[
= \sum_{p=p_0}^n \sum_{k=0}^{p-1} \sum_{k_1=0}^{p-1} \frac{z^p}{p!} \binom{p}{k, k_1, p-k-k_1} m_{sk+\gamma_3} m_{sk_1} m_{s(p-k-k_1)}
\]

\[
= \sum_{p=p_0}^n \sum_{k=0}^{p-1} \sum_{k_1=0}^{\min\{p-1, p-k\}} \frac{z^p}{p!} \binom{p}{k, k_1, p-k-k_1} m_{sk+\gamma_3} m_{sk_1} m_{s(p-k-k_1)}
\]

In order to exchange the second two sums, we first separate the term \( k = 0 \) from the rest of the sum, and then exchange summations in \( k_1 \) and \( p \):

\[
\sum_{k=0}^{n-1} \sum_{p=\max\{p_0, k+1\}}^n \sum_{k_1=0}^{\min\{p-1, p-k\}} \sum_{0 < k + k_1}^n = \sum_{k=1}^{\max\{p_0+1, k_1+1\}} \sum_{k_1=0}^{\max\{p_0, k+1\}} \sum_{k=1}^{\max\{p_0, k+1, k_1+k\}} \sum_{k_1=0}^{p-k} \sum_{k=0}^{n-k} \sum_{k_1=0}^{p-k} \sum_{k_1=0}^{\max\{p_0, k+1, k_1+k\}}
\]

Therefore,

\[
\sum_{p=p_0}^n \frac{z^p}{p!} \sum_{0 \leq k, k_1, k_2 < p \atop k + k_1 + k_2 = p} \binom{p}{k, k_1, k_2} m_{sk+\gamma_3} m_{sk_1} m_{sk_2}
\]

\[
\leq \sum_{k=0}^{n-k} \sum_{k_1=0}^{\max\{p_0, k+1, k_1+k\}} \frac{z^p}{p!} \binom{p}{k, k_1, p-k-k_1} m_{sk+\gamma_3} m_{sk_1} m_{s(p-k-k_1)}
\]

\[
= \sum_{k=0}^n \frac{z^k}{k!} m_{sk+\gamma_3} \sum_{k_1=0}^{n-k} \frac{z^{k_1}}{k_1!} m_{sk_1} \sum_{p=\max\{p_0, k+1, k_1+k\}}^n \frac{z^{p-k-k_1}}{(p-k-k_1)!} m_{s(p-k-k_1)}
\]

\[
\leq I_{n, \gamma_3}^{n} E_{n}^{n} E_{n}^{n}.
\]
We are now ready to prove one of our main results, Theorem 2.7 on propagation and generation of exponential moments.

**Proof of Theorem 2.7.** Without loss of generality, we assume that \( a_0 > 0 \), otherwise by the conservation of mass the only solution is zero, so the claim trivially holds. We first prove propagation of exponential moments since it will be used in the proof of the generation of moments. In addition to the notation introduced in (6.1) and (6.2), let us also introduce the following truncated partial sum notation:

\[
P^n_{s,p_0}(t,z) := \sum_{p=p_0}^{n} m^p_{s,p} z^p / p!.
\]

(b) (**Proof of exponential moments’ propagation**) Let \( \gamma_2, \gamma_3 \) be as in (2.7) and (2.20), and suppose \( s \in (0, 2] \). For a positive constant \( a < a_0 \) that will be fixed later, and fixed \( n \in \mathbb{N} \), we define \( T_n > 0 \) as

\[
T_n := \sup \{ t \in (0, T) : E^n_s(t, a) < 6 C_0 \},
\]

where \( C_0 \) is the constant in (2.36). The goal is to show that \( T_n = T \) and then let \( n \to +\infty \). Let us fix \( n \in \mathbb{N} \). Since \( E^n_s(0, a) \leq \int_{\mathbb{R}^d} f(0, t) e^{\alpha(t) r} \, dv \leq \int_{\mathbb{R}^d} f(0, t) e^{\alpha_0(t) r} \, dv < C_0 \), we have that \( T_n > 0 \) by the continuity of \( E^n_s \).

For an integer \( p_0 > 2/s \), to be chosen later, the moment differential inequality (6.3) yields

\[
\sum_{p=p_0}^{n} m^{p}_{s,p} \frac{a^p}{p!} \leq \sum_{p=p_0}^{n} \frac{a^p}{p!} \left( -K_{1,s} m_{sp+\gamma_2} - K_{2,s} m_{sp+\gamma_3} + K_{3,s} m_{sp} + 2 C_{\gamma_2} \alpha_{sp/2} S^p_{2,s,\gamma_2} + 3 C_{\gamma_3} \lambda_{sp/2} S^p_{3,s,\gamma_3} \right)
\]

\[
\leq -\tilde{K}_1 I^n_{s,\gamma_2}(t, a) + \tilde{K}_1 \sum_{p=0}^{p_0-1} m_{sp+\gamma_2} \frac{a^p}{p!} - \tilde{K}_2 I^n_{s,\gamma_3}(t, a) + \tilde{K}_2 \sum_{p=0}^{p_0-1} m_{sp+\gamma_3} \frac{a^p}{p!}
\]

\[
+ \tilde{K}_3 E^n_s(t, a) + 2 C_{\gamma_2} \sum_{p=p_0}^{n} \alpha_{sp/2} S^p_{2,s,\gamma_2} \frac{a^p}{p!} + 3 C_{\gamma_3} \sum_{p=p_0}^{n} \lambda_{sp/2} S^p_{3,s,\gamma_3} \frac{a^p}{p!},
\]

(6.18)

where, we used the fact that \(-K_{1,s}\) and \(-K_{2,s}\) in (6.9) are decreasing in \( sp \). The positive constants \( \tilde{K}_1, \tilde{K}_2, \tilde{K}_3 \) above are defined by

\[
\tilde{K}_1 = 2^{1-\frac{2}{s}} m_0 (\|b_2\| - \alpha_{sp_0/2}),
\]

\[
\tilde{K}_2 = 3 \left( \frac{2}{3} \right) \frac{\gamma_3}{2} m_0^2 (\|b_3\| - \lambda_{sp_0/2}),
\]

(6.19)

\[
\tilde{K}_3 = 2m_2 \|b_2\| + 6m_0 m_2 \|b_3\|.
\]

We note that for \( a < 1 \) (this will be one of conditions on \( a \)), the propagation of polynomial moments (Theorem 2.6) implies that

\[
\tilde{K}_1 \sum_{p=0}^{p_0-1} m_{sp+\gamma_2} \frac{a^p}{p!} + \tilde{K}_2 \sum_{p=0}^{p_0-1} m_{sp+\gamma_3} \frac{a^p}{p!} \leq \tilde{K}_1 \sum_{p=0}^{p_0-1} m_{sp+\gamma_2} + \tilde{K}_2 \sum_{p=0}^{p_0-1} m_{sp+\gamma_3} \leq \tilde{K}_4,
\]

(6.20)

where \( \tilde{K}_4 \) depends on \( s, p_0, \gamma_2, \gamma_3, m_0, m_2, \|b_2\|, \|b_3\| \).
Since $\gamma_{sp/2}$ and $\lambda_{sp/2}$ are decreasing in $p$, we can apply Lemma 6.2 to obtain

$$2C_{\gamma_2} \sum_{p=p_0}^{n} \alpha_{sp/2} S_{2,s,\gamma_2} \frac{a^p}{p!} + 3C_{\gamma_3} \sum_{p=p_0}^{n} \lambda_{sp/2} S_{3,s,\gamma_3} \frac{a^p}{p!}$$

$$\leq 2C_{\gamma_2} \alpha_{sp_0/2} I^n_{s,\gamma_2}(t,a) E^n_s(t,a) + 3C_{\gamma_3} \lambda_{sp_0/2} I^n_{s,\gamma_3}(t,a) (E^n_s(t,a))^2.$$  (6.21)

Combining [6.18] - [6.21], we obtain

$$\sum_{p=p_0}^{n} m'_{sp} \frac{a^p}{p!} \leq -\bar{K}_1 I^n_{s,\gamma_2}(t,a) - \bar{K}_2 I^n_{s,\gamma_3}(t,a) + \bar{K}_3 E^n_s(t,a) + \bar{K}_4$$

$$+ 2C_{\gamma_3} \lambda_{sp_0/2} I^n_{s,\gamma_3}(t,a) E^n_s(t,a) + 3C_{\gamma_3} \lambda_{sp_0/2} I^n_{s,\gamma_3}(t,a) (E^n_s(t,a))^2.$$  (6.22)

By further regrouping the terms, we have

$$\sum_{p=p_0}^{n} m'_{sp} \frac{a^p}{p!} \leq I^n_{s,\gamma_2}(t,a) \left(2C_{\gamma_2} \alpha_{sp_0/2} E^n_s(t,a) - \bar{K}_1 \right) + \bar{K}_3 E^n_s(t,a) + \bar{K}_4$$

$$+ I^n_{s,\gamma_3}(t,a) \left(3C_{\gamma_3} \lambda_{sp_0/2} (E^n_s(t,a))^2 - \bar{K}_2 \right).$$  (6.22)

Now we choose $p_0$ large enough so that

$$12 C_{\gamma_3} \alpha_{sp_0/2} C_0 \leq \bar{K}_1 / 2 \quad \text{and} \quad 108 C_{\gamma_3} \lambda_{sp_0/2} (C_0)^2 \leq \bar{K}_2 / 2.$$  (6.23)

For such a choice of $p_0$ we then have

$$\sum_{p=p_0}^{n} m'_{sp} \frac{a^p}{p!} \leq \bar{K}_1 \frac{1}{2} I^n_{s,\gamma_2}(t,a) - \frac{\bar{K}_2}{2} I^n_{s,\gamma_3}(t,a) + 6\bar{K}_3 C_0 + \bar{K}_4$$

$$\leq \frac{\bar{K}_1}{2} I^n_{s,\gamma_2}(t,a) - \frac{\bar{K}_2}{2} I^n_{s,\gamma_3}(t,a) + \bar{K}_5,$$  (6.24)

where $\bar{K}_5 = 6\bar{K}_3 C_0 + \bar{K}_4$ and so it depends on $s, p_0, \gamma_2, \gamma_3, m_0, m_2, \|b_2\|, \|b_3\|.$

Next we need a lower bound on $I^n_{s,\gamma_2}(t,a)$ and $I^n_{s,\gamma_3}(t,a)$ in terms of $E^n_s(t,a)$.

$$I^n_{s,\gamma_2}(t,a) \geq \frac{1}{a^{\gamma_2/2}} \sum_{p=0}^{n} \int_{\{v\geq \frac{1}{a}\}} f(t,v) \langle v \rangle^{sp} \frac{a^p}{p!} dv$$

$$= \frac{1}{a^{\gamma_2/2}} \left( \sum_{p=0}^{n} \int_{\mathbb{R}^d} f(t,v) \langle v \rangle^{sp} \frac{a^p}{p!} dv - \sum_{p=0}^{n} \int_{\{v\geq \frac{1}{a}\}} f(t,v) \langle v \rangle^{sp} \frac{a^p}{p!} dv \right)$$

$$\geq \frac{1}{a^{\gamma_2/2}} \left( E^n_s(t,a) - \sum_{p=0}^{n} \int_{\mathbb{R}^d} f(t,v) \frac{a^p(1-\frac{1}{a^2})}{p!} dv \right)$$

$$\geq \frac{1}{a^{\gamma_2/2}} \left( E^n_s(t,a) - m_0 \sum_{p=0}^{\infty} \frac{a^p(1-\frac{1}{a^2})}{p!} \right)$$

$$= \frac{1}{a^{\gamma_2/2}} \left( E^n_s(t,a) - m_0 e^{a^{-1}} \right).$$
Similarly, we have
\[ I^n_{s,3}(t, a) \geq \frac{1}{a^{\gamma_3/2}} \left( E^n_s(t, a) - m_0 e^{a^{-\gamma_2}} \right). \]

Therefore, plugging the lower bounds for \( I^n_{s,3} \) and \( I^n_{s,3}(t, a) \) into (6.24) yields
\[ \sum_{p=p_0}^n m_{sp} a^p p! \leq - \left( \frac{\bar{K}_1}{2a^{\gamma_2/2}} + \frac{\bar{K}_2}{2a^{\gamma_3/2}} \right) E^n_s(t, a) + \frac{m_0 \bar{K}_1 e^{a^{1-\gamma_2}}}{2a^{\gamma_2/2}} + \frac{m_0 \bar{K}_2 e^{a^{1-\gamma_3}}}{2a^{\gamma_3/2}} + \bar{K}_5. \quad (6.25) \]

Since clearly, \( E^n_s(t, a) \geq P^n_{s,p_0}(t, a) \), we have
\[ \frac{d}{dt} P^n_{s,p_0}(t, a) \leq - \left( \frac{\bar{K}_1}{2a^{\gamma_2/2}} + \frac{\bar{K}_2}{2a^{\gamma_3/2}} \right) P^n_{s,p_0}(t, a) + \frac{m_0 \bar{K}_1 e^{a^{1-\gamma_2}}}{2a^{\gamma_2/2}} + \frac{m_0 \bar{K}_2 e^{a^{1-\gamma_3}}}{2a^{\gamma_3/2}} + \bar{K}_5. \quad (6.26) \]

Therefore,
\[ P^n_{s,p_0}(t, a) \leq P^n_{s,p_0}(0, a) + m_0 e^{a^{1-\gamma_2}} + \frac{\bar{K}_5}{\bar{K}_1} \frac{m_0 \bar{K}_1 e^{a^{1-\gamma_2}}}{2a^{\gamma_2/2}} + \frac{m_0 \bar{K}_2 e^{a^{1-\gamma_3}}}{2a^{\gamma_3/2}} \leq C_0 + m_0 e^{a^{1-\gamma_2}} + \frac{\bar{K}_5}{\bar{K}_1} \frac{m_0 \bar{K}_1 e^{a^{1-\gamma_2}}}{2a^{\gamma_2/2}} + \frac{m_0 \bar{K}_2 e^{a^{1-\gamma_3}}}{2a^{\gamma_3/2}} \]
\[ \leq C_0 + m_0 e^{a^{1-\gamma_2}} + \frac{\bar{K}_5}{\bar{K}_1} \frac{m_0 \bar{K}_1 e^{a^{1-\gamma_2}}}{2a^{\gamma_2/2}} + \frac{m_0 \bar{K}_2 e^{a^{1-\gamma_3}}}{2a^{\gamma_3/2}} = C_0 + m_0 e^{a^{1-\gamma_2}} + a^{\gamma/2} \bar{K}_6, \quad (6.27) \]

where \( \bar{K}_6 = \frac{2 \bar{K}_5}{\bar{K}_1} \) depends on \( s, p_0, \gamma_2, \gamma_3, m_0, m_2, \|b_2\|, \|b_3\| \).

In order to have an estimate on \( E^n_s \), we also need an estimate on \( \sum_{p=p_0}^{p_0-1} m_{sp} a^p p! \). Recalling the propagation of polynomial moments result (Theorem 4.1), we have \( m_{sp} \leq C_{s,p_0,m_0,m_2} \) and so
\[ \sum_{p=0}^{p_0-1} m_{sp} a^p p! \leq m_0 + a \sum_{p=1}^{p_0-1} m_{sp} a^{p-1} p! \leq C_0 + a C_{s,p_0,m_0,m_2}. \quad (6.28) \]

Combining (6.24) with (6.28), we obtain
\[ E^n_s(t, a) \leq 2C_0 + a C_{s,p_0,m_0,m_2} + m_0 e^{a^{1-\gamma_2}} + a^{\gamma/2} \bar{K}_6. \]

We can choose \( a < \min\{a_0, 1\} \) small enough so that
\[ a C_{s,p_0,m_0,m_2} + m_0 e^{a^{1-\gamma_2}} + a^{\gamma/2} \bar{K}_6 < 4C_0, \quad (6.29) \]
which, in turn, implies that for such \( a \) we have
\[ E^n_s(t, a) < 6C_0. \]

In conclusion, if \( p_0 \) is chosen according to (6.24), and if \( a \) is small enough that it satisfies (6.24), we have that the strict inequality \( E^n_s(t, a) < 6C_0 \) holds on the closed interval \([0, T_n]\). The continuity of \( E^n_s(t, a) \) then implies that \( E^n_s(t, a) < 6C_0 \) holds on a larger time interval which would contradict the maximality of \( T_n \) unless \( T_n = T \). Thus, we conclude \( T_n = T \) for all \( n \in \mathbb{N} \). Therefore, in fact, we have
\[ E^n_s(t, a) < 6C_0, \quad \text{for all } t \in [0, T], \quad \text{for all } n \in \mathbb{N}. \]
Letting $n \to +\infty$, we conclude
\[
\int_{\mathbb{R}^d} f(t,v) e^{\alpha(v)t} \, dv \leq 6C_0, \quad \text{for all } t \in [0,T].
\]

(a) (Proof of exponential moments’ generation) Let us write $z(t) = at$, where $a < 1$ is a positive constant that will be fixed later. For this constant $a$ and for a fixed $n \in \mathbb{N}$, we define $T_n > 0$ as
\[
T_n := \min \left\{ 1, \sup \{ t \in [0,T] : E^n_\gamma(t,at) < 4m_0 \text{ for all } \tau \in [0,t] \} \right\}. \tag{6.30}
\]
Indeed, $T_n > 0$ for every $n$ because $E^n_\gamma(0,0) = m_0 < 4m_0$. Our goal is to prove that, in fact, $T_n = \min\{1,T\}$ for all $n$, at which point one can restart the argument at $t = \min\{1,T\}$ and use the propagation result to conclude finiteness of the exponential moment for all times. More details will be provided below. All equations below that depend on time are valid for $t \in (0,T_n]$ unless noted otherwise.

For $p_0 > 2/\gamma$, we have
\[
\frac{d}{dt} P^n_{\gamma,p_0}(t,z(t)) = \sum_{p=p_0}^n m_{\gamma p}(t) \frac{z^p(t)}{p!} + a \sum_{p=p_0-1}^{n-1} m_{\gamma p+\gamma}(t) \frac{z^p(t)}{p!}.
\]
The last term can be bounded by $aI^n_{\gamma,\gamma}(t,z(t))$. Therefore, applying (6.5) to the first term yields
\[
\frac{d}{dt} P^n_{\gamma,p_0}(t,z(t)) \leq aI^n_{\gamma,\gamma}(t,z(t)) - \bar{K}_1 \sum_{p=p_0}^n m_{\gamma p+\gamma}(t) \frac{z^p(t)}{p!} - \bar{K}_2 \sum_{p=p_0}^n m_{\gamma p+\gamma}(t) \frac{z^p(t)}{p!}
+ \bar{K}_3 \sum_{p=p_0}^n m_{\gamma p}(t) \frac{z^p(t)}{p!} + 2C_{\gamma_2} \sum_{p=p_0}^n \alpha_{\gamma p/2} S_{2,\gamma_2}^p \frac{z^p(t)}{p!} + 3C_{\gamma_3} \sum_{p=p_0}^n \lambda_{\gamma p/2} S_{3,\gamma_3}^p \frac{z^p(t)}{p!}
=: aI^n_{\gamma,\gamma}(t,z(t)) - \bar{K}_1 S_1 - \bar{K}_2 S_2 + \bar{K}_3 S_3 + S_4 + S_5, \tag{6.31}
\]
where, by (6.31), the coefficients are given by
\[
\bar{K}_1 = 2^{1-\gamma_3/2} m_0 (\|b_2\| - \alpha_{\gamma p_0/2}),
\bar{K}_2 = 3 \left( \frac{2}{3} \right)^{\gamma_3/2} m_0^2 (\|b_3\| - \lambda_{\gamma p_0/2}),
\bar{K}_3 = 2 m_{\gamma_2} \|b_2\| + 6 m_0 m_{\gamma_3} \|b_3\|.
\]

Bound on $S_3$: The sum $S_3$ can be simply bounded as follows
\[
S_3 := \sum_{p=p_0}^n m_{\gamma p}(t) \frac{z^p(t)}{p!} \leq E^n_\gamma(t,z(t)) = E^n_\gamma(t,at) < 4m_0, \tag{6.32}
\]
where the last inequality holds by the definition of $T_n$ given in (6.30).
Bound on $-S_1$: By the generation of polynomial moments estimate (2.33), we have

\[-S_1 = -I^n_{\gamma,\gamma_2}(t,z(t)) + \sum_{p=0}^{p_0-1} m_{\gamma p+\gamma_2}(t) \frac{z^p(t)}{p!} \]
\[\leq -I^n_{\gamma,\gamma_2}(t,z(t)) + C_{p_0} \sum_{p=0}^{p_0-1} t^{2-(\gamma_2+\gamma)} \frac{z^p(t)}{p!} \]
\[= -I^n_{\gamma,\gamma_2}(t,z(t)) + C_{p_0} t^{2-\gamma} \sum_{p=0}^{p_0-1} \frac{\alpha^p}{p!} \]
\[\leq -I^n_{\gamma,\gamma_2}(t,z(t)) + C_{p_0} t^{2-\gamma} e, \quad (6.33)\]

since $t \leq 1$, $\gamma_2 \leq \gamma$ and $a < 1$.

Bound on $-S_2$: Similarly we obtain

\[-S_2 \leq -I^n_{\gamma,\gamma_3}(t,z(t)) + C_{p_0} t^{2-\gamma_3} e. \quad (6.34)\]

Bound on $S_4$: By (6.16), we have

\[S_4 \leq 2C_{\gamma_2} \alpha_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t,z(t)) E^n_{\gamma}(t,z(t)) = 2C_{\gamma_2} \alpha_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t) E^n_{\gamma}(t). \quad (6.35)\]

Bound on $S_5$: Similarly, by (6.17), we have

\[S_5 \leq 3C_{\gamma_2} \lambda_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t,z(t)) (E^n_{\gamma}(t,z(t)))^2 = 3C_{\gamma_2} \lambda_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t) (E^n_{\gamma}(t))^2. \quad (6.36)\]

Combining estimates (6.31) - (6.36), and using $E^n_{\gamma}(t) < 4m_0$ for $t \in [0,T_4]$, we obtain

\[\frac{d}{dt} I^n_{\gamma,p_0}(t,at) \leq a I^n_{\gamma}(t,at) - \bar{K}_1 I^n_{\gamma,\gamma_2}(t,at) - \bar{K}_2 I^n_{\gamma,\gamma_3}(t,at) + (\bar{K}_1 + \bar{K}_2) C_{p_0} e^{2-\gamma} + 4m_0 \bar{K}_3 \]
\[+ 2C_{\gamma_2} \alpha_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t,at) E^n_{\gamma}(t,at) + 3C_{\gamma_2} \lambda_{\gamma p_0/2} I^n_{\gamma,\gamma_2}(t,at) (E^n_{\gamma}(t,at))^2 \]
\[\leq a I^n_{\gamma}(t,at) + (\bar{K}_1 + \bar{K}_2) C_{p_0} e^{2-\gamma} + 4m_0 \bar{K}_3 \]
\[- (\bar{K}_1 - 8m_0 C_{\gamma_2} \alpha_{\gamma p_0/2}) I^n_{\gamma,\gamma_2}(t,at) - (\bar{K}_2 - 48m_0^2 C_{\gamma_2} \lambda_{\gamma p_0/2}) I^n_{\gamma,\gamma_3}(t,at). \]

Since the sequences $p \mapsto \alpha_{\gamma p/2}$, $p \mapsto \lambda_{\gamma p/2}$ tend to zero, so for $p_0$ large enough, we have

\[\frac{d}{dt} I^n_{\gamma,p_0}(t,at) \leq a I^n_{\gamma}(t,at) + (\bar{K}_1 + \bar{K}_2) C_{p_0} e^{2-\gamma} + 4m_0 \bar{K}_3 \]
\[- \bar{K}_1 \frac{1}{2} I^n_{\gamma,\gamma_2}(t,at) - \bar{K}_2 \frac{1}{2} I^n_{\gamma,\gamma_3}(t,at) \leq - \frac{1}{2} \min\{\bar{K}_1, \bar{K}_2\} I^n_{\gamma,\gamma}(t,at), \]

and thus

\[\frac{d}{dt} I^n_{\gamma,p_0}(t,at) \leq \left( a - \frac{1}{2} \min\{\bar{K}_1, \bar{K}_2\} \right) I^n_{\gamma,\gamma}(t,at) + (\bar{K}_1 + \bar{K}_2) C_{p_0} e^{2-\gamma} + 4m_0 \bar{K}_3. \]
For a small enough, we have
\[
\frac{d}{dt} P^n_{\gamma,p_0}(t, at) \leq -\frac{1}{4} \min\{\bar{K}_1, \bar{K}_2\} P^n_{\gamma}(t, at) + (\bar{K}_1 + \bar{K}_2) C_{p_0} e t^{\frac{2 - \gamma}{\gamma}} + 4m_0 \bar{K}_3.
\]

Now notice that
\[
I^n_{\gamma,p}(t, z) = \frac{n}{p!} \sum_{p=0}^{n} m_{\gamma(p+1)} \frac{z^p}{p!} = \frac{1}{z} \sum_{p=1}^{n+1} p m_{\gamma p}(t) \frac{z^p}{p!} \geq \frac{1}{z} \sum_{p=1}^{n} m_{\gamma p}(t) \frac{z^p}{p!} = \frac{1}{z} (E^n_{\gamma}(t, z) - m_0).
\]

Using this inequality with \(z = at\), we have
\[
\frac{d}{dt} P^n_{\gamma,p_0}(t, at) \leq -\frac{\min\{\bar{K}_1, \bar{K}_2\}}{4at} \left( E^n_{\gamma}(t, at) - m_0 - \frac{4a}{\min\{\bar{K}_1, \bar{K}_2\}} \left[ (\bar{K}_1 + \bar{K}_2) C_{p_0} e t^{\frac{2 - \gamma}{\gamma}} + 4m_0 \bar{K}_3 \right] \right)
\]
\[
\leq -\frac{\min\{\bar{K}_1, \bar{K}_2\}}{4at} \left( E^n_{\gamma}(t, at) - m_0 - \frac{4a}{\min\{\bar{K}_1, \bar{K}_2\}} \left[ (\bar{K}_1 + \bar{K}_2) C_{p_0} e + 4m_0 \bar{K}_3 \right] \right),
\]
where to obtain the last inequality we used the fact that \(t \leq 1\). Using that \(E^n_{\gamma}(t, at) \geq P^n_{\gamma,p_0}(t, at)\), for a small enough, we have
\[
\frac{d}{dt} P^n_{\gamma,p_0}(t, at) \leq -\frac{\min\{\bar{K}_1, \bar{K}_2\}}{4at} \left( P^n_{\gamma,p_0}(t, at) - 2m_0 \right).
\]

Therefore, whenever \(P^n_{\gamma,p_0}(t, at) > 2m_0\), \(P^n_{\gamma,p_0}(t, at)\) decreases in time. Since \(P^n_{\gamma,p_0}(0, 0) = 0 < 2m_0\), we conclude that
\[
P^n_{\gamma,p_0}(t, at) \leq 2m_0 \tag{6.37}
\]
holds uniformly on the closed interval \([0, T_n]\).

On the other hand, the first \(p_0\) terms can be bounded as follows:
\[
\sum_{p=0}^{p_0-1} m_{\gamma p}(t) \frac{(at)^p}{p!} = m_0 + \sum_{p=1}^{p_0-1} m_{\gamma p}(t) \frac{(at)^p}{p!} \leq m_0 + ae C^*_p, \tag{6.38}
\]
for \(t \in [0, T_n]\). Namely, from Theorem 2.6 and the fact that \(t \leq 1\), we have
\[
m_{\gamma p}(t) \leq C^*_p t^{-p} = C^*_p t^{-p}.
\]

If we define \(C^*_p := \max_{p \in \{0, 1, \ldots, p_0-1\}} C_{\gamma p}\), we have
\[
m_{\gamma p}(t) \leq C^*_p t^{-p} \quad \text{for all } p \in \{0, 1, \ldots, p_0-1\}.
\]

Therefore, since \(0 < t \leq T_n \leq 1\),
\[
\sum_{p=1}^{p_0-1} m_{\gamma p}(t) \frac{(at)^p}{p!} \leq C^*_p \sum_{p=1}^{p_0-1} \frac{t^{-p} a^p}{p!} \leq a C^*_p \sum_{p=1}^{p_0-1} \frac{a^{p-1}}{(p-1)!} \leq a C^*_p e,
\]
since \(a < 1\).

Finally, combining (6.37) with (6.38) we obtain:
\[
E^n_{\gamma}(t, at) \leq 3m_0 + ae C^*_p,
\]
therefore for a small enough we have
\[
E^n_{\gamma}(t, at) < 4m_0, \quad \text{for all } t \in [0, T_n].
\]
Now, the continuity of the partial sum implies that the strict inequality $E_n^\gamma(t, at) < 4m_0$ holds beyond $T_n$, which contradicts the maximality of $T_n$ unless $T_n = \min\{1, T\}$. Therefore, $T_n = \min\{1, T\}$ for all $n$. This implies that

$$E_n^\gamma(t, at) < 4m_0, \quad \text{for all } t \in [0, \min\{1, T\}] \text{ for all } n \in \mathbb{N}. $$

Let $n \to \infty$ to conclude:

$$\int_{\mathbb{R}^d} f(t, v) e^{at(v)\gamma} \, dv \leq 4m_0, \quad \text{for all } t \in [0, \min\{1, T\}]. $$

This implies that at time $t = \min\{1, T\}$, the exponential moment of order $\gamma$ and rate $a$ is finite. The propagation of exponential moments result then implies that there exists $0 < a_1 < a$ such that the exponential moment of the same order $\gamma$ and a rate $a_1$ remains bounded uniformly for all $t \geq \min\{1, T\}$. In conclusion

$$\int_{\mathbb{R}^d} f(t, v) e^{a_1\min(1, t)(v)\gamma} \, dv < C, \quad \text{for all } t \in (0, T]. $$

\[\square\]

7. Global well-posedness of the binary-ternary Boltzmann equation

In this last section, we prove Theorem 2.8, which establishes existence and uniqueness of global in time solutions for the binary-ternary Boltzmann equation (2.1) for initial data in $L^{1+2\gamma}$, where $\varepsilon > 0$. Without loss of generality, we assume that $m_0[f_0] > 0$, otherwise by the conservation of mass the unique solution is trivially zero.

As mentioned in the introduction, motivated by analogous results \[19, 3, 27, 5\] in the context of the Boltzmann equation, system of Boltzmann equations for gas mixtures, and the quantum Boltzmann equation, we will rely on the general theory for ODEs in Banach spaces, namely Theorem A.8 stated in the appendix. The idea is to first construct a unique solution assuming the initial data have $2 + 2\gamma$ moments, where $\gamma = \max\{\gamma_2, \gamma_3\} > 0$, see Proposition 7.1 below. Then, in order to prove Theorem 2.8, we will relax the assumption on the initial data using generation and propagation of polynomial moments (Theorem 2.6).

More specifically, given $f_0 \in L^{1+2\gamma}_2$, we first prove that all the conditions of Theorem A.8 are satisfied for the operator $Q := Q$, the Banach space $X := L^1_2$ and the invariant subset $S := \Omega[f_0] \subset L^1_2$, where $\Omega[f_0]$ is given by:

$$\Omega[f_0] = \left\{ f \in L^1_2 : f \geq 0, \ m_0[f] = m_0[f_0], \ m_2[f] = m_2[f_0], \ m_{2+2\gamma}[f] \leq \max\{A_{2+2\gamma}, m_{2+2\gamma}[f_0]\} \right\}, \quad (7.1)$$

where $A_{2+2\gamma}$ is an appropriate constant defined in (7.2).
Proposition 7.1. Let \( T > 0 \), \( \gamma = \max\{\gamma_2,\gamma_3\} \), and \( f_0 \in L^2_{2+2} \), with \( f_0 \geq 0 \). Then the binary-ternary Boltzmann equation \((2.1)\) has a solution \( f \geq 0 \), in the sense of Definition \((2.2)\), with \( f \in C([0,T],\Omega[f_0]) \cap C^1((0,T),L^1_4) \). In particular, the solution \( f \) conserves the mass, momentum and energy of the initial data \( f_0 \). Moreover \( f \in C^1((0,T),L^1_k) \) for all \( k > 2 \).

Proof. Let us first note that \( \Omega[f_0] \) is clearly convex and bounded and \( f_0 \in \Omega[f_0] \). Moreover, by Fatou’s Lemma, \( \Omega[f_0] \) is also a closed subset of \( L^1_2 \). Additionally, by the definition of \( \Omega[f_0] \), we have \( C([0,T],\Omega[f_0]) \subseteq L^1((0,T),L^1_{2+\gamma}) \). Therefore, to prove existence of a solution, it remains to prove that the assumptions (1)-(3) of Theorem \((A.8)\) are satisfied. For that, we will strongly rely on the generalized description of the collisional operators \( Q_2 \) and \( Q_3 \), and their weak formulation, both of which are presented in the Appendix. In the following, we denote \( m_0 := m_0[f_0], m_2 := m_2[f_0] \).

Proof of condition (1): We show that \( Q \) satisfies \((A.33)\) in the set \( \tilde{\Omega}[f_0] \supseteq \Omega[f_0] \) where

\[
\tilde{\Omega}[f_0] = \{ f \in L^2_2 : f \geq 0, m_{2+2\gamma}[f] \leq \max\{A_{2+2\gamma},m_{2+2\gamma}[f_0]\} \},
\]

and therefore in \( \Omega[f_0] \) as well. Note that proving \((A.33)\) in \( \tilde{\Omega}[f_0] \) also proves that \( Q : \tilde{\Omega}[f_0] \to L^1_2 \).

We first prove an estimate on \( \|Q[f] - Q[g]\|_{L^1_2} \) for any two functions \( f, g \in L^2_{2+\gamma} \) (see \((7.4)\)), that in addition to being used to prove Hölder continuity condition \((A.33)\), will also be needed to relax the initial data condition at the end of this section. So, let \( f, g \in L^1_{2+\gamma} \). Due to bilinearity-trilinearity and symmetry of the operators (see \((A.4)\), \((A.13)\)), we have:

\[
Q[f] - Q[g] = Q_2(f - g, f + g) + Q_3(f - g, f, f) + Q_3(f - g, f, g) + Q_3(f - g, g, g).
\]

Let \( s_0 \) be the sign of \( Q_2(f - g, f + g) \), \( s_1 \) the sign of \( Q_3(f - g, f, f) \), \( s_2 \) the sign of \( Q_3(f - g, f, g) \) and \( s_3 \) the sign of \( Q_3(f - g, g, g) \). Then, by the triangle inequality, we have

\[
\|Q[f] - Q[g]\|_{L^1_2} \leq \int_{\mathbb{R}^d} Q_2(f - g, f + g)s_0\langle v \rangle^2 \, dv + \int_{\mathbb{R}^d} Q_3(f - g, f, f)s_1\langle v \rangle^2 \, dv
+
\int_{\mathbb{R}^d} Q_3(f - g, f, g)s_2\langle v \rangle^2 \, dv + \int_{\mathbb{R}^d} Q_3(f - g, g, g)s_3\langle v \rangle^2 \, dv.
\]

Since \( f, g \in L^1_{2+\gamma} \), we use the weak formulations \((A.7)\), \((A.16)\) with \( s_i\langle v \rangle^2, i = 0, 1, 2, 3 \) as test functions. We also use the triangle inequality, bound all signs by one, and we use conservation of energy by the collisions to obtain

\[
\|Q[f] - Q[g]\|_{L^1_2} \leq \int_{\mathbb{R}^{d+1} \times \mathbb{R}^d} B_2(u,\omega)(f - g)((f_1 + g_1)(\langle v \rangle^2 + \langle v_1 \rangle^2) \, d\omega dv dv_1
+
\frac{1}{18} \sum_{\pi \in S_{0,1,2}} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0} - g_{\pi_0}(f_{\pi_1}f_{\pi_2} + f_{\pi_1}g_{\pi_2} + g_{\pi_1}g_{\pi_2})
\times (\langle v \rangle^2 + \langle v_1 \rangle^2 + \langle v_2 \rangle^2) \, d\omega dv dv_1,2.
\]

\footnote{writing \( f \in C([0,T],\Omega[f_0]) \), we mean that \( f \in C([0,T],L^1_2) \), and that \( f(t) \in \Omega[f_0] \) for all \( t \in [0,T] \).}
Now, using the form of the cross-sections $\Omega_i$, the cut-offs $\Omega_f$, and the potential bounds $\Omega_a$, as well as the symmetry with respect to the integration variables, we obtain

$$
\|Q[f] - Q[g]\|_{L^1_\gamma} \leq C_{\gamma_2} \|b_2\| \int_{\mathbb{R}^d} |f - g|(f_1 + g_1) \left( \langle v \rangle^{2+\gamma_2} + \langle v \rangle^{\gamma_3} \langle v_1 \rangle^2 + \langle v \rangle^2 \langle v_1 \rangle^{\gamma_2} + \langle v_1 \rangle^{2+\gamma_2} \right) dv \, dv_1
$$

$$+ \frac{C_{\gamma_2}}{3} \|b_3\| \int_{\mathbb{R}^d} |f - g| (f_1 f_2 + f_1 g_2 + g_1 g_2) \left( \langle v \rangle^{2+\gamma_2} + \langle v \rangle^{2} \langle v_1 \rangle^{\gamma_3} + \langle v \rangle^2 \langle v_2 \rangle^{\gamma_3} + \langle v \rangle^{\gamma_3} \langle v_1 \rangle^2 + \langle v \rangle^{\gamma_3} \langle v_2 \rangle^2 + \langle v_1 \rangle^{\gamma_3} \langle v_2 \rangle^2 + \langle v_2 \rangle^{2+\gamma_3} \right) dv \, dv_1 dv_2,
$$

$$\leq 3C_{\gamma_2} \|b_2\| \|f - g\|_{L^2_{2+\gamma_2}} \left( \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right) + C_{\gamma_2} \|b_2\| \|f - g\|_{L^1_\gamma} \left( \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right) + \frac{7C_{\gamma_2}}{3} \|b_3\| \|f - g\|_{L^2_{2+\gamma_3}} \left( \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right)^2 + \frac{2C_{\gamma_2}}{3} \|b_3\| \|f - g\|_{L^2_{2+\gamma_3}} \left( \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right)^2 + C_2 \|f - g\|_{L^1_\gamma} \left( 1 + \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right) \left( \|f\|_{L^1_\gamma} + \|g\|_{L^1_\gamma} \right),
$$

where $C_1 = \max\{3C_{\gamma_2} \|b_2\|, \frac{7C_{\gamma_2}}{3} \|b_3\|\}$ and $C_2 = \max\{C_{\gamma_2} \|b_2\|, \frac{2C_{\gamma_2}}{3} \|b_3\|\}$. Let us note, again, that estimate (7.3) is valid for any $f, g \in L^1_{2+\gamma}$.

Now to complete the proof of the Hölder condition (A.33), assume $f, g \in \tilde{\Omega}[f_0]$. Then we have $\|f\|_{L^2_{2+\gamma}} \|g\|_{L^2_{2+\gamma}} \leq \max\{A_{2+\gamma}, m_{2+\gamma}[f_0]\}$. Therefore, estimate (7.3) and the interpolation inequality (A.27) yield

$$
\|Q[f] - Q[g]\|_{L^1_\gamma} \leq C_H \|f - g\|_{L^1_\gamma}^{1/2},
$$

for some appropriate constant $C_H > 0$, depending on $\max\{A_{2+\gamma}, m_{2+\gamma}[f_0]\}$. Condition (1) is proved.

Proof of condition (2): We will now show that $Q$ satisfies (A.34) in $\tilde{\Omega}[f_0]$, and therefore in $\Omega[f_0]$ as well. First notice that for $u, v \in L^1_{\gamma}$ we have

$$[u, w] := \lim_{h \to 0} \frac{\|w + hu\|_{L^1_\gamma} - \|w\|_{L^1_\gamma}}{h} = \int_{\mathbb{R}^d} u \, sgn\langle v \rangle^2 dv.
$$

(7.5)

Indeed, for $h \neq 0$, triangle inequality implies

$$
\left| \frac{|w + hu| - |w|}{h} \right| \leq |u| \in L^1_{\gamma}.
$$

Moreover, we have

$$
\lim_{h \to 0} \frac{|w + hu| - |w|}{h} = \lim_{h \to 0} \frac{2wu + hu^2}{|w + hu| + |w|} = u \, sgn\langle w \rangle.
$$

Therefore, by the dominated convergence theorem, we take

$$
[u, w] = \lim_{h \to 0} \frac{\|w + hu\|_{L^1_\gamma} - \|w\|_{L^1_\gamma}}{h} = \lim_{h \to 0} \int_{\mathbb{R}^d} \frac{|w + hu| - |w|}{h} \langle v \rangle^2 dv = \int_{\mathbb{R}^d} u \, sgn\langle w \rangle^2 dv.
$$

(7.6)
We first prove an estimate on \([Q[f] - Q[g], f - g]\) for any two functions \(f, g \in L^1_{2+\gamma}\) (see (7.6)), that in addition to being used to prove one-sided Lipschitz condition (A.34), will also be needed to relax the initial data condition at the end of this section. So, let \(f, g \in L^1_{2+\gamma}\) and let us write \(s := \text{sgn}(f - g)\). Since \(f, g \in L^1_{2+\gamma}\), we have \([Q[f] - Q[g]] \in L^1_{2}\), due to (7.3) (or Lemmata 1.1 and 2 in the Appendix). Thus (7.5) and (7.3) yield

\[
[Q[f] - Q[g], f - g] = \int_{\mathbb{R}^d} Q_2(f - g, f + g) s \langle v \rangle^2 dv + \int_{\mathbb{R}^d} Q_3(f - g, f, f) s \langle v \rangle^2 dv
\]

Since \(s \in L^1_{2+\gamma}\), using the weak formulations (A.7), (A.16) with \(s \langle v \rangle^2\) as a test function, we obtain

\[
[Q[f] - Q[g], f - g] = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^{2d}} B_2(u, \omega) |f - g| (f_1 + g_1) \left( s'(v')^2 + s'(v')^2 - s_1(v_1)^2 \right) d\omega dv dv_1
\]

\[
+ \frac{1}{36} \sum_{\pi \in S_{0,1,2}} \int_{S^{d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) |f_\pi_0 - g_\pi_0| (f_\pi_1 f_\pi_2 + f_\pi_1 g_\pi_2 + g_\pi_1 g_\pi_2) \times \left( s^*_{\pi_0} (v^*_{\pi_0})^2 + s^*_{\pi_1} (v^*_{\pi_1})^2 + s^*_{\pi_2} (v^*_{\pi_2})^2 - s_{\pi_0} (v_{\pi_0})^2 - s_{\pi_1} (v_{\pi_1})^2 - s_{\pi_2} (v_{\pi_2})^2 \right) d\omega dv dv_{1,2}.
\]

Using the fact \((f - g)s = |f - g|\), \((f_\pi_0 - g_\pi_0) s_{\pi_0} = |f_\pi_0 - g_\pi_0|\), bounding the rest of the signs by one, and using the conservation of energy by the collisions, we obtain

\[
[Q[f] - Q[g], f - g] \leq \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^{2d}} B_2(u, \omega) |f - g| |f_1 + g_1| \left( (v')^2 + (v')^2 - (v)^2 + (v_1)^2 \right) d\omega dv dv_1
\]

\[
+ \frac{1}{36} \sum_{\pi \in S_{0,1,2}} \int_{S^{d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) |f_\pi_0 - g_\pi_0| |f_\pi_1 f_\pi_2 + f_\pi_1 g_\pi_2 + g_\pi_1 g_\pi_2| \times \left( (v_{\pi_0})^2 + (v_{\pi_1})^2 + (v_{\pi_2})^2 - (v_{\pi_0})^2 + (v_{\pi_1})^2 + (v_{\pi_2})^2 \right) d\omega dv dv_{1,2}
\]

\[
= \int_{S^{d-1} \times \mathbb{R}^{2d}} B_2(u, \omega) |f - g| |f_1 + g_1| (v_1)^2 d\omega dv dv_1
\]

\[
+ \frac{1}{18} \sum_{\pi \in S_{0,1,2}} \int_{S^{d-1} \times \mathbb{R}^{2d}} B_3(u, \omega) |f_\pi_0 - g_\pi_0| |f_\pi_1 f_\pi_2 + f_\pi_1 g_\pi_2 + g_\pi_1 g_\pi_2| \left( (v_{\pi_1})^2 + (v_{\pi_2})^2 \right) d\omega dv dv_{1,2}
\]
Now, using the form of the cross-sections (2.7), (2.20), the cut-offs (2.9), (2.23), and the potential for some constant $\nu$, we get

$$
\nu_C \text{tonicity of moments implies }
$$

By (A.29), monotonicity of moments, and the fact that

$$
\beta_2 \frac{b_2}{1 + \|f\|_{L_{2+\gamma}^1} + \|g\|_{L_{2+\gamma}^1}} \left(\|f\|_{L_{2+\gamma}^1} + \|g\|_{L_{2+\gamma}^1}\right),
$$

(7.6)

where $\beta_2 = 2 \max\{C_{\gamma_2} b_2, C_{\gamma_3} b_3\}$. Let us note that estimate (7.6) is valid for any $f, g \in L_{2+\gamma}^1$.

Now, assume $f, g \in \Omega[f_0]$, hence $\|f\|_{L_{2+\gamma}^1}, \|g\|_{L_{2+\gamma}^1} \leq \max\{A_{2+\gamma}, A_{2+\gamma}[f_0]\}$. Then, monotonicity of moments implies

$$
[Q[f] - Q[g], f - g] \leq C_L \|f - g\|_{L_{2+\gamma}^1},
$$

for some appropriate constant $C_L > 0$, depending on $\max\{A_{2+\gamma}, A_{2+\gamma}[f_0]\}$. Condition (2) is proved.

**Proof of condition (3):** First we bound the collision frequency for the binary-ternary Boltzmann equation. For $f \in \Omega[f_0]$, one can represent $Q[f]$ in gain and loss form as follows:

$$
Q[f] = Q^+_3(f, f) + Q^+_3(f, f, f) - \nu_2(f, f) = Q^+ [f] - f\nu[f],
$$

(7.7)

where $Q^+[f] \geq 0$ and

$$
\nu_2(f) = \int_{\mathbb{R}^{d-1} \times \mathbb{R}^d} |u|^\gamma \beta_2 (\hat{u} \cdot \omega) f_1 \ d\omega \ dv_1 = \|b_2\| \int_{\mathbb{R}^d} |u|^\gamma f_1 \ dv_1,
$$

$$
\nu_3(f, f, f) = \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} |\hat{u}|^{\gamma - \theta_3} |\hat{u}|^{\theta_3} b_3 (\hat{u} \cdot \omega_1 \cdot \omega_2) f_1 f_2 f_2 d\omega d\omega_1 dv_1.2 = \|b_3\| \int_{\mathbb{R}^d} |\hat{u}|^{\gamma - \theta_3} |\hat{u}|^{\theta_3} f_1 f_2 f_2 dv_1.2.
$$

By (A.29), monotonicity of moments, and the fact that $f \in \Omega[f_0]$, we have

$$
\nu_2(f) \leq C_{\gamma_2} \|b_2\| \int_{\mathbb{R}^d} (\langle v \rangle^\gamma + \langle v_1 \rangle^\gamma) f_1 dv_1 = C_{\gamma_2} \|b_2\| (m_{\gamma_2} + m_0 |v|^\gamma) \leq C_2 (1 + |v|^\gamma),
$$

(7.8)

for some constant $C_2 > 0$, depending on $\gamma_2, m_0, m_2, \|b_2\|$. Similarly, by (A.31) we have

$$
\nu_3(f, f, f) \leq C_{\gamma_3} \|b_3\| \int_{\mathbb{R}^{2d}} (\langle v \rangle^\gamma + \langle v_1 \rangle^\gamma + \langle v_2 \rangle^\gamma) f_1 f_2 f_2 dv_1.2 = C_{\gamma_3} \|b_3\| (2m_0 m_{\gamma_3} + m_0^2 |v|^\gamma) \leq C_3 (1 + |v|^\gamma),
$$

for some constant $C_3 > 0$ depending on $\gamma_3, m_0, m_2, \|b_3\|$. Combining estimates for $\nu_2$ and $\nu_3$, we get

$$
\nu[f] = \nu_2(f) + 3\nu_3(f, f) \leq C (1 + \langle v \rangle^\gamma + \langle v \rangle^\gamma),
$$

(7.9)

for some constant $C > 0$, depending on $\gamma_2, \gamma_3, m_0, m_2, \|b_2\|, \|b_3\|$.
Now, in order to prove the sub-tangent condition for \( f \in \Omega[f_0] \), it suffices to prove that for any \( \varepsilon > 0 \) there exists \( h^* = h^*(\varepsilon) > 0 \) so that for any \( h \in (0, h^*) \)

\[
B_{L^1_h}(f + hQ[f], h\varepsilon) \cap \Omega[f_0] \neq \emptyset.
\]

Let \( 0 < h < 1 \) and \( R > 1 \). We define

\[
f_R := 1_{|v| \leq R} f, \quad W_{R,h} = f + hQ[f_R].
\]

First notice that, since \( f \in \Omega[f_0] \subset L^1_{2+2\gamma} \) and \( f_R \) is compactly supported, we have \( W_{R,h} \in L^1_{2+2\gamma} \). Our goal is to choose \( R \) large enough and \( h \) small enough so that \( W_{R,h} \in B_{L^1_h}(f + hQ[f], h\varepsilon) \cap \Omega[f_0] \).

We achieve that in the following steps:

- Given \( R > 0 \), there exists \( h_1 = h_1(R) \) such that for any \( h \in (0, h_1) \) we have \( W_{R,h} \geq 0 \).
  - Indeed, if \( \langle v \rangle > R \), we have \( f_R = 0 \), thus \( W_{R,h} = f \geq 0 \) since \( f \in \Omega[f_0] \). If \( \langle v \rangle \leq R \), we have \( f_R = f \), thus decomposition \( (2.17) \), positivity of \( Q^+ \), and bound \( (2.49) \) imply
    
    \[
    W_{R,h} = f + hQ^+[f] - hf\nu[f] \geq f(1 - h\nu[f]) \geq f(1 - hC(1 + R^{2\gamma} + R^{3\gamma})),
    \]
    
    where \( C \) is the constant appearing in \( (2.49) \). Therefore, defining \( h_1 = h_1(R) = \frac{1}{C(1 + R^{2\gamma} + R^{3\gamma})} \), we have \( W_{R,h} \geq 0 \) for all \( h \in (0, h_1) \).

- Since \( f_R \) is compactly supported, by \( (2.28), (2.30) \), we have
  
  \[
  \int_{\mathbb{R}^d} Q[f_R] \, dv = \int_{\mathbb{R}^d} Q[f_R](v) \, dv = 0.
  \]
  
  Therefore, since \( f \in \Omega[f_0] \), we have \( m_0[W_{R,h}] = m_0[f_0] \) and \( m_2[W_{R,h}] = m_2[f_0] \).

- Finally, since \( f_R \) is compactly supported, bound \( (4.2) \) and the fact that \( m_0[f_R] \to m_0 \), \( m_2[f_R] \to m_2 \) as \( R \to \infty \), yield the estimate
  
  \[
  \int_{\mathbb{R}^d} Q[f_R](v)(v) \, dv \\
  \leq C_{2+2\gamma}(m_0[f_R], m_2[f_R])m_{2+2\gamma}[f_R] - \tilde{C}_{2+2\gamma}(m_0[f_R], m_2[f_R])m_{2+2\gamma}[f_R]^{3/2} \\
  \leq 2C_{2+2\gamma}(m_0, m_2)m_{2+2\gamma}[f_R] - \frac{1}{2}\tilde{C}_{2+2\gamma}(m_0, m_2)m_{2+2\gamma}[f_R]^{3/2}, \tag{7.10}
  \]
  
  for \( R > R^* \), where \( R^* \) is large enough. Consider the mapping \( \mathcal{L} : [0, \infty) \to \mathbb{R} \), defined by
  
  \[
  \mathcal{L}(x) = 2C_{2+2\gamma}(m_0, m_2)x - \frac{1}{2}\tilde{C}_{2+2\gamma}(m_0, m_2)x^{3/2}. \tag{7.11}
  \]

Besides zero, the map \( \mathcal{L} \) has a unique positive root

\[
x^* = \frac{4C_{2+2\gamma}(m_0, m_2)}{\tilde{C}_{2+2\gamma}(m_0, m_2)}^2,
\]

where it changes from positive to negative, and a global maximum

\[
\mathcal{L}^* = \frac{8}{27} \left( \frac{4C_{2+2\gamma}(m_0, m_2)}{\tilde{C}_{2+2\gamma}(m_0, m_2)} \right)^2 C_{2+2\gamma}(m_0, m_2),
\]

achieved in \([0, x^*] \). We define \( A_{2+2\gamma} \) as follows:

\[
A_{2+2\gamma} := x^* + \mathcal{L}^* = \left( \frac{4C_{2+2\gamma}(m_0, m_2)}{\tilde{C}_{2+2\gamma}(m_0, m_2)} \right)^2 \left( 1 + \frac{8}{27} C_{2+2\gamma}(m_0, m_2) \right). \tag{7.12}
\]
We now show that, for appropriate \( h, R \), we have
\[
m_{2+2_\gamma}(W_{h,R}) \leq \max\{A_{2+2_\gamma}, m_{2+2_\gamma}[f_0]\}.
\]
Indeed, for \( R \) large enough, (7.10)–(7.11) imply
\[
\int_{\mathbb{R}^d} Q[f_R] \langle \nu \rangle^{2+2_\gamma} \, dv \leq \mathcal{L}(m_{2+2_\gamma}[f_R]) \leq \mathcal{L}^*,
\]
so if \( m_{2+2_\gamma}[f] \leq x^* \), we have
\[
m_{2+2_\gamma}(W_{h,R}) \leq x^* + h \int_{\mathbb{R}^d} Q[f_R] \langle \nu \rangle^{2+2_\gamma} \, dv \leq x^* + h\mathcal{L}^* < x^* + \mathcal{L}^* = A_{2+2_\gamma},
\]
if, on the other hand, \( m_{2+2_\gamma}[f] > x^* \), we choose \( R^{**} \) large enough so that \( m_{2+2_\gamma}[f_R] > x^* \) for all \( R > R^{**} \). Hence, \( \mathcal{L}(m_{2+2_\gamma}[f_R]) \leq 0 \), which yields
\[
m_{2+2_\gamma}(W_{h,R}) \leq m_{2+2_\gamma}[f] \leq \max\{A_{2+2_\gamma}, m_{2+2_\gamma}[f_0]\}.
\]
We conclude that for \( R > \max\{R^*, R^{**}\} \) and \( h < h_1(R) \), we have \( W_{h,R} \in \Omega[f_0] \).

Moreover, by Hölder continuity in \( \Omega[f_0] \), we have
\[
\frac{\|f + hQ[f] - W_{h,R}\|_{L^1}}{h} = \|Q[f] - Q[f_R]\|_{L^1} \leq C_R \|f - f_R\|_{L^1}^{1/2} \leq \varepsilon,
\]
for \( R \geq R(\varepsilon) \) sufficiently large. For such \( R \), we have \( W_{h,R} \in B(f + hQ[f], h\varepsilon) \).

Finally, let \( R = \max\{R^*, R^{**}, R(\varepsilon)\} \) and \( h_1 = \frac{1}{C(1 + R^3 + R^{3/2})} \), where \( C \) is determined by (7.31). Then \( W_{h,R} \in B(f + hQ[f], h\varepsilon) \cap \Omega \) for all \( h \in (0, h_1) \), and condition (3) follows.

By Theorem A.8 we conclude that there exists a unique strong solution \( f \geq 0 \) (since \( f(t) \in \Omega[f_0] \) for all \( t \in [0,T] \)) to the binary-ternary Boltzmann equation (2.1) with \( f \in C([0,T], \Omega[f_0] \cap C^1(\mathbb{R}^d, L^1_{\varepsilon})) \). Note that the conservation of mass and energy hold by the definition of \( \Omega[f_0] \), while the conservation of momentum holds due to collision averaging (2.29) which can be applied since \( f \in C([0,T], L^1_{\varepsilon}) \). Moreover, by Theorem 5.1 \( f \in C^1((0,T), L^1_{\varepsilon}) \) for any \( k > 2 \), which completes the proof of Proposition 7.1.

Now, we will prove Theorem 2.8 by relaxing the assumption on the initial data to \( f_0 \in L^1_{\varepsilon} \), where \( \varepsilon > 0 \). Inspired by the relaxation of initial data argument for the classical Boltzmann equation in [3], we will rely on the generation and propagation of polynomial moments (Theorem 2.6).

**Proof of Theorem 2.8** Assume that \( f_0 \in L^1_{\varepsilon + \delta} \), where \( \varepsilon > 0 \). Without loss of generality, we may assume that \( \varepsilon < \gamma \). Let \( (f^j_0)_j \) be a sequence such that \( f^j_0 \in \Omega[f_0] \subset L^1_{\varepsilon + \delta} \) with \( f^j_0 \to f_0 \) in \( L^1_{\varepsilon + \delta} \). Such a sequence exists, take for instance \( f^j_0 = 1_{(v) < j} f_0 \). Let \( f^j \in C([0,T], \Omega[f^j_0]) \cap C^1((0,T), L^1_{\varepsilon + \delta}) \) be the solution of equation (2.1) with initial data \( f^j_0 \) obtained by Proposition 7.1. We aim to construct the solution by taking the limit of \( f^j \) as \( j \to \infty \). To do that, we will first show that for fixed \( \delta < \varepsilon \), the sequence \( (f^j)_j \) converges to some function \( f \) in \( C([0,T], L^1_{\varepsilon + \delta}) \).

Note that, by a standard regularization argument, for any \( j, l \in \mathbb{N} \) we have
\[
\frac{d}{dt}\|f^j(t) - f^l(t)\|_{L^1_{\varepsilon + \delta}} = \|Q[f^j(t)] - Q[f^l(t)], f^j(t) - f^l(t)\|_{L^1_{\varepsilon + \delta}},
\]
where the bracket notation is defined in (7.13). Then from (7.6), we have
\[
\frac{d}{dt}\|f^j(t) - f^l(t)\|_{L^1_{\varepsilon + \delta}} \leq A(t)\|f^j(t) - f^l(t)\|_{L^1_{\varepsilon + \delta}},
\]
for all \( t \in [0,T] \) and \( j, l \in \mathbb{N} \), where
\[
A(t) = \frac{1}{\varepsilon + \delta} \left( \frac{1}{(\varepsilon + \delta)^2} + \frac{1}{(\varepsilon + \delta)^3} \right).
\]
where
\[
\mathcal{A}(t) = \tilde{C} \left(1 + \|f_0\|_{L^2} + \|f_0^j\|_{L^2}^j\right) \left(\|f^j(t)\|_{L^{2+\gamma}_{2+\delta}} + \|f^l(t)\|_{L^{2+\gamma}_{2+\delta}}\right),
\] (7.15)
and \(\tilde{C} > 0\) is a constant that depends on \(b_2, b_3, \gamma_2, \gamma_3\). In order to estimate \(\mathcal{A}(t)\), we use interpolation to obtain
\[
\|f^k(t)\|_{L^{2+\gamma}_{2+\delta}} \leq \|f^k(t)\|_{L^{2+\gamma}_{1+\delta}}^{\frac{\delta}{\gamma}} \|f^k(t)\|_{L^{2+\gamma}_{2+\delta}}^{\frac{(\gamma-\delta)}{\gamma}}, \quad k = j, l.
\] (7.16)
Since \(f_0^j \to f_0\) in \(L^{1+\varepsilon}_{2+\varepsilon}\) as \(j \to \infty\), the propagation estimate (2.35) yields
\[
\|f^j(t)\|_{L^{1+\varepsilon}_{2+\varepsilon}} \leq \|f^j(t)\|_{L^{1+\varepsilon}_{2+\varepsilon}} \leq M, \quad \forall t \in [0, T),
\] (7.17)
where \(M\) is a constant, uniform in \(k\), that depends on \(f_0, \gamma, \varepsilon, b_2,\) and \(b_3\). Invoking the generation estimate (2.33) as well, for all \(0 < t \leq T\), we have
\[
\|f^k(t)\|_{L^{2+\gamma}_{2+\delta}} \leq M^\theta \|f^k(t)\|_{L^{2+\gamma}_{1+\delta}}^{\frac{\delta}{\gamma}} \leq M^\theta K \left(1 + t^{-\frac{2+\delta}{\gamma}}\right)^{\frac{\delta}{\gamma}} \leq M^\theta K \left(1 + t^{-\theta}\right),
\] (7.18)
where \(\theta = \frac{2-\delta^2}{\gamma^2} \in (0, 1)\) since \(0 < \delta < \varepsilon < \gamma\), and \(K\) is a constant, uniform in \(k\), that depends on \(\varepsilon, \gamma_2, \gamma_3, b_2,\) and \(b_3\). Therefore, for all \(0 < t \leq T\), we have
\[
\mathcal{A}(t) \leq C \left(1 + t^{-\theta}\right),
\] (7.19)
for some uniform constant \(C\). Now, (7.14), (7.19) and Gronwall’s inequality yield
\[
\|f^j(t) - f^l(t)\|_{L^2} \leq \|f_0^j - f_0^l\|_{L^2} \exp \left(\frac{C}{1-\theta} (T + T^{1-\theta})\right), \quad \forall t \in [0, T],
\] (7.20)
since \(\theta \in (0, 1)\). Hence, interpolating again and using the right hand side inequality of (7.17), for all \(t \in [0, T]\), we have
\[
\|f^j(t) - f^l(t)\|_{L^{1+\varepsilon}_{2+\varepsilon}} \leq \|f^j(t) - f^l(t)\|_{L^2}^{\frac{\varepsilon^{1-\delta}}{\varepsilon}} \left(\|f^j(t)\|_{L^{2+\gamma}_{2+\delta}} + \|f^l(t)\|_{L^{2+\gamma}_{2+\delta}}\right)^{\frac{\delta}{\gamma}}
\leq (2C_0)^{\delta/\varepsilon}\|f_0^j - f_0^l\|_{L^2}^{\frac{\varepsilon^{1-\delta}}{\varepsilon}} \exp \left(\frac{C(\varepsilon - \delta)}{\varepsilon(1-\theta)} (T + T^{1-\theta})\right).
\] (7.21)
Since \(f_0^j \to f_0\) in \(L^{1+\varepsilon}_{2+\varepsilon}\), bound (7.21) implies that \((f^j)_j\) is a Cauchy sequence in \(C([0, T], L^{1+\varepsilon}_{2+\varepsilon})\), thus it converges to some \(f \in C([0, T], L^{1+\varepsilon}_{2+\varepsilon})\). Clearly, \(f \geq 0\) and \(f(t = 0) = f_0\), and conservation laws hold.

Next, we show that for arbitrarily small \(t_0 \in (0, T)\), we have that \(Q[f^j], Q[f] \in C([t_0, T], L^2)\) and \(Q[f^j] \to Q[f]\) in \(C([t_0, T], L^2)\). Indeed fix such a \(t_0\). By Proposition (7.4) for any \(j \in \mathbb{N}\), we have that \(f^j \in C^1((0, T], L^{2+\gamma}_{2+\delta})\), thus Lemmata A.1, A.2 imply that \(Q[f^j] \in C([t_0, T], L^1)\). Now, for \(j, l \in \mathbb{N}\),
and $t \in [t_0, T]$, estimates (7.14) and (7.18) and the triangle inequality yield

$$
\|Q[f^j(t)] - Q[f^i(t)]\|_{L^1_T} \\
\leq C \|f^j(t) - f^i(t)\|_{L_{2+\gamma}^1} + C \left( \|f^j(t)\|_{L_{2+\gamma}^1} + \|f^i(t)\|_{L_{2+\gamma}^1} \right) \|f^j(t) - f^i(t)\|_{L^0_T} \\
\leq C \|f^j(t) - f^i(t)\|_{L_{2+\delta}^2} \|f^j(t) - f^i(t)\|_{L_{2+\gamma+\delta}^1} + C \left( \|f^j(t)\|_{L_{2+\gamma}^1} + \|f^i(t)\|_{L_{2+\gamma}^1} \right) \|f^j(t) - f^i(t)\|_{L^0_T} \\
\leq C(1 + t^{-\theta}) \left( \|f^j(t) - f^i(t)\|_{L_{1+\gamma}^1} + \|f^j(t) - f^i(t)\|_{L^0_T} \right) \\
\leq C(1 + t_0^{-\theta}) \left( \|f^j - f^i\|_{C([t_0,T], L_{2+\delta}^1)} + \|f^j - f^i\|_{C([t_0,T], L^1_T)} \right). 
$$

(7.22)

Since the sequence $(f^j)_j$ converges in $C([0,T], L_{2+\delta}^1)$, estimate (7.22) implies that the sequence $(Q[f^j])_j$ is Cauchy in $C([t_0,T], L_{2+\delta}^1)$ so it converges to some element $\alpha \in C([t_0,T], L_{2}^1)$. At the same time, Lemmata A.1 and A.2 imply that $Q[f^j] \to Q[f]$ in $C([t_0,T], L_{2+\gamma+\delta}^1)$. Therefore, we conclude that $\alpha = Q[f]$, so $Q[f] \in C([t_0,T], L_{2}^1)$ and $Q[f^j] \to Q[f]$ in $C([t_0,T], L_{2}^1)$.

Now we show that $f$ is a solution to (2.1) with initial data $f_0$. First, we have already shown that $f \in C([0,T], L_{2}^1)$ and that $f(t = 0) = f_0$. Moreover, $f$ satisfies conservation laws (2.31) since $(f^j)_j$ does as well. Also $f \in L_{1loc}([0,T], L_{2+\gamma}^1)$. Namely, estimates (7.14) and (7.22) are also valid if the collision operator $Q$ is replaced with the loss operator $L$. By the same reasoning applied to $Q$ one can conclude that $L[f] \in C([t_0,T], L_{2}^1)$, and therefore $L[f] \in L^1([t_0,T], L_{2}^1)$. Thus, applying Lemma A.3 we get

$$
\begin{align*}
\infty > & \int_{t_0}^{T} \|L[f(t)]\|_{L^2_T} dt \\
= & \int_{t_0}^{T} \left( \|b_2\| \int_{\mathbb{R}^d} f(t, v) f(t, v_1) |v - v_1|^2 \langle v \rangle^2 dv dv_1 + \|b_3\| \int_{\mathbb{R}^d} f(t, v) f(t, v_1) f(t, v_2) |\bar{u}|^2 \langle v \rangle^2 dv_1 dv_2 \right) dt \\
\geq & \int_{t_0}^{T} \left( \|b_2\| C_T \int_{\mathbb{R}^d} f(t, v) \langle v \rangle^{2+\gamma_2} dv + \|b_3\| \int_{\mathbb{R}^d} f(t, v) f(t, v_1) f(t, v_2) |v - v_2|^2 \langle v \rangle^2 dv_1 dv_2 \right) dt \\
\geq & \int_{t_0}^{T} \left( \|b_2\| C_T \int_{\mathbb{R}^d} f(t, v) \langle v \rangle^{2+\gamma_2} dv + C_T \|b_3\| \int_{\mathbb{R}^d} f(t, v) f(t, v_1) \langle v \rangle^{2+\gamma_3} dv_1 \right) dt \\
= & \int_{t_0}^{T} \left( \|b_2\| C_T \int_{\mathbb{R}^d} f(t, v) \langle v \rangle^{2+\gamma_2} dv + C_T \|b_3\| \|f\|_{L^0_T} \int_{\mathbb{R}^d} f(t, v) \langle v \rangle^{2+\gamma_3} dv \right) dt \\
\geq & C(T) \int_{t_0}^{T} \int_{\mathbb{R}^d} f(t, v) \langle v \rangle^{2+\gamma} dv dt,
\end{align*}
$$

where $C(T) = \|b_2\| C_T$ if $\gamma_2 = \gamma$ and $C = \|b_3\| C_T \|f\|_{L^0_T}$ if $\gamma_3 = \gamma$. If $\gamma_2 = \gamma_3 = \gamma$, one can set $C(T)$ to be either of the two values. Since $t_0 \in (0,T)$ was arbitrary, we conclude $f \in L_{1loc}([0,T], L_{2+\gamma}^1)$.

Finally, since $f^j$ solves (2.1), for arbitrarily small $t_0 \in (0,T)$ we have

$$
f^j(t) = f^j(t_0) + \int_{t_0}^{t} Q[f^j(\tau)] d\tau, \quad \forall t \in [t_0,T].
$$
Letting $j \to \infty$, and using the fact that for any $t_0 \in (0, T)$, $Q[f] \to Q[f]$ in $C([t_0, T], L^1_2)$, we obtain
\[
f(t) = f(t_0) + \int_{t_0}^{t} Q[f(\tau)] \, d\tau, \quad \forall \, t \in [t_0, T],
\]
thus differentiating, we obtain $\partial_t f = Q[f]$ for all $t \in [t_0, T]$ and $f \in C^1([t_0, T], L^1_2)$. Since $t_0$ was chosen arbitrarily small, we conclude that $\partial_t f = Q[f]$ for all $t \in (0, T]$ and $f \in C^1((0, T], L^1_2)$.

Uniqueness of the solution follows by a similar Gronwall’s inequality type of argument. Finally, by Theorem 2.8 $f \in C^1((0, T), L^1_2)$ for any $k > 2$, which completes the proof of Theorem 2.8. 

\[\Box\]

**Appendix A.**

**A.1. Multilinear collisional operators and weak formulation.** Here we recall the notation introduced in Section 2.1.

**The generalized binary collisional operator.** The generalized binary collisional operator is given by
\[
Q_2(f, g) = Q^+_2(f, g) - Q^-_2(f, g), \tag{A.1}
\]
where the gain and loss operators $Q^+_2, Q^-_2$ are given by
\[
Q^+_2(f, g) = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left( f' g' + f g' \right) \, d\omega dv_1, \tag{A.2}
\]
\[
Q^-_2(f, g) = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) \left( f g + f' g \right) \, d\omega dv_1, \tag{A.3}
\]
$u = v_1 - v$ and $B_2$ is given by (2.7). Notice that for $f = g$ in (A.1), one recovers (2.2). Clearly $Q_2$ is symmetric and bilinear and the following identity holds:
\[
Q_2(f, f) - Q_2(g, g) = Q_2(f - g, f + g). \tag{A.4}
\]

Assuming sufficient integrability conditions for $f$ and a test function $\phi$ for all integrals involved to make sense, we have the weak formulations (see 2.2)
\[
\int_{\mathbb{R}^d} Q^-_2(f, g) \phi dv = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) fg_1(\phi + \phi_1) \, d\omega dv_1 dv, \tag{A.5}
\]
\[
\int_{\mathbb{R}^d} Q^+_2(f, g) \phi dv = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) fg_1(\phi' + \phi'_1) \, d\omega dv_1 dv, \tag{A.6}
\]
which yield
\[
\int_{\mathbb{R}^d} Q_2(f, g) \phi dv = \frac{1}{2} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) fg_1(\phi' + \phi'_1 - \phi - \phi_1) \, d\omega dv_1 dv, \tag{A.7}
\]
as well as
\[
\int_{\mathbb{R}^d} Q_2(f, g) \phi dv = \frac{1}{4} \int_{S^{d-1} \times \mathbb{R}^d} B_2(u, \omega) (f' g'_1 - f g_1)(\phi + \phi_1 - \phi' - \phi'_1) \, d\omega dv_1 dv. \tag{A.8}
\]
Although we will not use (A.8), it is worth mentioning, since it implies entropy dissipation for the binary collisional operator, see 2.2.
A sufficient integrability condition for \( \mathbf{A.3} - \mathbf{A.8} \) to hold is \( f, g \in L^1_{q + \gamma_2} \) and \( |\phi(v)| \leq \psi(v)(v)^q \), where \( q \geq 0 \) and \( \psi \in L^\infty \). In particular, when \( f, g, h \in L^1_{q + \gamma_2} \), we can have \( \phi \in C_c(\mathbb{R}^d) \). This is justified by the following Lemma:

**Lemma A.1.** Let \( q \geq 0 \). Then for any test function \( \phi \) satisfying \( |\phi(v)| \leq \psi(v)(v)^q \), where \( \psi \in L^\infty \), the weak formulations \( \mathbf{A.5} - \mathbf{A.8} \) hold. In particular, \( Q_2^-, Q_2^+ : L^1_{q + \gamma_2} \times L^1_{q + \gamma_2} \rightarrow L^1_{q} \), and there hold the bounds:

\[
\|Q_2^-(f, g)\|_{L^1_{q + \gamma_2}}, \|Q_2^+(f, g)\|_{L^1_{q + \gamma_2}}, \|Q_2(f, g)\|_{L^1_{q}} \leq C\|b_2\|_2\|f\|_{L^1_{q + \gamma_2}} + \|g\|_{L^1_{q + \gamma_2}}\|f\|_{L^1_{q}} + \|g\|_{L^1_{q}}\).
\]

(A.9)

for some constant \( C > 0 \) depending on \( \gamma_2 \). Additionally, if \( I \subset \mathbb{R} \) is a time interval, and \( f, g \in C(\mathbb{T}, L^1_{q + \gamma_2}) \cap L^1(\mathbb{T}, L^1_{q + \gamma_2}) \) then \( Q_2^-(f, g), Q_2^+(f, g), Q_2(f, g) \in L^1(\mathbb{T}, L^1_{q + \gamma_2}) \). If \( f, g \in C(\mathbb{T}, L^1_{q + \gamma_2}) \), then \( Q_2^-(f, g), Q_2^+(f, g), Q_2(f, g) \in C(\mathbb{T}, L^1_{q + \gamma_2}). \)

The proof essentially reduces to proving estimate \( \mathbf{A.9} \). To achieve that, one needs to use \( \mathbf{A.20} \), for the loss term, while for the gain term one needs to use \( \mathbf{A.29} \) and Lemma \( \mathbf{A.41} \) instead.

**The generalized ternary collisional operator.** The properties of the ternary collisional operator have been studied for the first time in [7], for hard ternary interactions. Here, we discuss these properties in more generality. Denoting by \( \mathbf{S}_0,1,2 \) the set of permutations of the set \( \{0, 1, 2\} \), the generalized ternary collisional operator is given by

\[
Q_3(f, g, h) = Q_3^+(f, g, h) - Q_3^-(f, g, h)
\]

where the gain and loss operators \( Q_3^+, Q_3^- \) are given by

\[
Q_3^+(f, g, h) = \frac{1}{36} \sum_{\pi \in \mathbf{S}_{0,1,2}} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} B_3(u, \omega) f_{\pi_0}^* g_{\pi_1}^* h_{\pi_2}^* \omega dv_1 \omega dv_{1,2}
\]

\[+ \frac{1}{18} \sum_{\pi \in \mathbf{S}_{0,1,2}} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} B_3(u_1, \omega) f_{\pi_0} g_{\pi_1}^* h_{\pi_2}^\ast \omega dv_1 \omega dv_{1,2}
\]

(A.11)

\[
Q_3^-(f, g, h) = \frac{1}{36} \sum_{\pi \in \mathbf{S}_{0,1,2}} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} B_3(u, \omega) f_{\pi_0} g_{\pi_1} h_{\pi_2} \omega dv_1 \omega dv_{1,2}
\]

\[+ \frac{1}{18} \sum_{\pi \in \mathbf{S}_{0,1,2}} \int_{\mathbb{R}^{2d-1} \times \mathbb{R}^d} B_3(u_1, \omega) f_{\pi_0} g_{\pi_1}^* h_{\pi_2} \omega dv_1 \omega dv_{1,2}
\]

(A.12)

where \( u = (u_1, u_2 - v), u_1 = (u_2 - v_1) \) and \( B_3 \) is given by (2.20). The operators \( Q_3^{+} \), \( Q_3^{-} \) correspond to the situation where the tracked particle is the central particle of a ternary interaction, while the operators \( Q_3^{(a)}, Q_3^{(c)} \) correspond to the situation where the tracked particle is one of the adjacent particles of the interaction. Notice that for \( f = g = h \) in (A.10), one recovers expression (2.10). Clearly \( Q_3 \) is symmetric and trilinear and there holds the identity

\[
Q_3(f, f, f) - Q_3(g, g, g) = Q_3(f - g, f, f) + Q_3(f - g, f, g) + Q_3(f - g, g, g).
\]

(A.13)
Assuming sufficient integrability conditions for $f$ and a test function $\phi$ for all integrals involved to make sense, we have the weak formulations:

$$\int_{\mathbb{R}^d} Q_3^-(f,g,h)\phi\,dv = \frac{1}{36} \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}(\phi + \phi_1 + \phi_2)\,d\omega\,dv_{1,2}\,dv, \quad (A.14)$$

$$\int_{\mathbb{R}^d} Q_3^+(f,g,h)\phi\,dv = \frac{1}{36} \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}(\phi^* + \phi_1^* + \phi_2^*)\,d\omega\,dv_{1,2}\,dv, \quad (A.15)$$

which imply

$$\int_{\mathbb{R}^d} Q_3(f,g,h)\phi\,dv = \frac{1}{36} \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}(\phi^* + \phi_1^* + \phi_2^* - \phi - \phi_1 - \phi_2)\,d\omega\,dv_{1,2}\,dv \quad (A.16)$$

as well as

$$\int_{\mathbb{R}^d} Q_3^-(f,g,h)\phi\,dv = \frac{1}{7} \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)(f_{\pi_0g_{\pi_1}h_{\pi_2}}^* - f_{\pi_0g_{\pi_1}h_{\pi_2}})\,d\omega\,dv_{1,2}\,dv \quad (A.17)$$

Although we will not use (A.17), it is worth mentioning it, since it implies entropy dissipation for the ternary collisional operator, see [S] for more details.

In order to show (A.14)-(A.17), for $i \in \{0, 1, 2\}$, we write

$$A_i = \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}\phi_i\,d\omega\,dv_{1,2}\,dv$$

$$A_i^* = \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}\phi_i^*\,d\omega\,dv_{1,2}\,dv$$

It suffices to show that

$$\int_{\mathbb{R}^d} Q_3^-(f,g,h)\phi\,dv = \frac{1}{36} (A_0 + A_1 + A_2), \quad \int_{\mathbb{R}^d} Q_3^+(f,g,h)\phi\,dv = \frac{1}{36} (A_0^* + A_1^* + A_2^*). \quad (A.18)$$

By (2.22), we clearly have

$$\int_{\mathbb{R}^d} Q_3^{-(c)}(f,g,h)\phi\,dv = \frac{A_0}{36}, \quad \int_{\mathbb{R}^d} Q_3^{+(c)}(f,g,h)\phi\,dv = \frac{A_0^*}{36}$$

Notice that interchanging $v$ with $v_1$, we obtain

$$A_1 = \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u_1,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}\phi\,d\omega\,dv_{1,2}\,dv$$

$$= \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u_1,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}\phi\,d\omega\,dv_{1,2}\,dv.$$ 

Also, performing the involutory change of variables $(v,v_1,v_2) \to (v^{1*},v_1^{1*},v_2^{1*})$ and using the micro-reversibility condition (2.22), we have

$$A_1^* = \sum_{\pi \in \mathbb{S}_{0,1,2}} \int_{S_1^{d-1} \times \mathbb{R}^d} B_3(u_1,\omega)f_{\pi_0g_{\pi_1}h_{\pi_2}}^{1*}\phi_1\,d\omega\,dv_{1,2}\,dv.$$
Hence
\[
\int_{\mathbb{R}^d} Q_3^{-}(f,g,h) \phi \, dv = \frac{A_1}{36}, \quad \int_{\mathbb{R}^d} Q_3^{+}(f,g,h) \phi \, dv = \frac{A_1^*}{36}.
\]

Finally, by symmetry of the adjacent particles, we have \( A_1^* = A_1^* \) and \( A_2 = A_1^* \), and \( A.18 \) follows.

A sufficient integrability condition for \( (A.5)-(A.8) \) to hold is \( f,g,h \in L^1_{q+\gamma} \) and \( |\phi(v)| \leq \psi(v)\beta^q \), where \( q \geq 0 \) and \( \psi \in L^\infty \). In particular, when \( f,g,h \in L^1_{2} \), we can have \( \phi \in C_c(\mathbb{R}^d) \). This is justified by the following Lemma:

**Lemma A.2.** Let \( q \geq 0 \). Then for any test function \( \phi \) satisfying \( |\phi(v)| \leq \psi(v)\beta^q \), where \( \psi \in L^\infty \), the weak formulations \( A.14, A.17 \) hold. In particular, \( Q_3^{-}, Q_3^{+}, Q_3 : L^1_{q+\gamma} \times L^1_{q+\gamma} \times L^1_{q+\gamma} \rightarrow L^1_{q} \), and there hold the bounds:

\[
\|Q_3^{-}(f,g,h)\|_{L^1_{q}} + \|Q_3^{+}(f,g,h)\|_{L^1_{q}} + \|Q_3(f,g,h)\|_{L^1_{q}} \leq C\|b_3\| \left( \|f\|_{L^1_{q+\gamma}} + \|g\|_{L^1_{q+\gamma}} + \|h\|_{L^1_{q+\gamma}} \right),
\]

for some constant \( C > 0 \) depending on \( \gamma \). Additionally, if \( I \subseteq \mathbb{R} \) is a time interval, and \( f,g,h \in C(\overline{T}, L^1_{q+\gamma}) \) then \( Q_3^{-}(f,g,h) + Q_3^{+}(f,g,h) + Q_3(f,g,h) \in L^1(T,L^1_q) \). If \( f,g,h \in C(\overline{T}, L^1_{q+\gamma}) \), then \( Q_3^{-}(f,g,h) + Q_3^{+}(f,g,h) + Q_3(f,g,h) \in C(\overline{T}, L^1_q) \).

The proof essentially reduces to proving estimate \( A.19 \). To achieve that, one needs to use \( 2.20 \) and \( A.31, 2.9 \) for the loss term, while for the gain term one needs to use \( A.31 \) and Lemma 3.3 instead.

### A.2. Binomial and trinomial estimates.

**Lemma A.3.** The following polynomial estimates hold:

(a) If \( p > 1 \), then for all \( x, y > 0 \), we have
\[
(x + y)^p - x^p - y^p \leq C_{2,p}(x^{p-1}y + xy^{p-1}),
\]
where
\[
C_{2,p} = p \max\{1, 2^{p-3}\}.
\]  
(b) If \( p > 2 \), then for all \( x, y, z > 0 \), we have
\[
(x + y + z)^p - x^p - y^p - z^p \leq C_{3,p}(x^{p-1}y + xy^{p-1} + xz^{p-1} + yz^{p-1} + y^{p-1}z + z^{p-1}y)
\]
where
\[
C_{3,p} = p \max\{1, 2^{p-3}\} + \frac{p(p-1)}{2} \max\{1, 2^{p-4}\}.
\]
(c) If \( p > 0 \), then for all \( x, y, z \geq 0 \), we have
\[
(x + y)^p \leq \max\{1, 2^{p-1}\}(x^p + y^p)
\]
\[
(x + y + z)^p \leq \max\{1, 3^{p-1}\}(x^p + y^p + z^p)
\]

Proof.
Lemma A.4. Consider \( f : [0, +\infty) \to \mathbb{R} \) with \( f(0) = 0 \). The following hold:

1. If \( \mu \in [0, 1] \) and \( f \) is convex, then \( f(\mu x) \leq \mu f(x) \) for all \( x \in [0, +\infty) \).
2. If \( \mu \geq 1 \) and \( f \) is concave, then \( f(\mu x) \geq \mu f(x) \) for all \( x \in [0, +\infty) \).

(a) Let \( p > 1 \) and \( x, y > 0 \). Since \( p - 2 > -1 \), we have

\[
(x + y)^p - x^p - y^p = \int_0^x \int_0^y p(p-1)(r+s)^{p-2} dr ds.
\]

Then using that \( (r+s)^{p-2} \leq A_p(r^{p-2} + s^{p-2}) \), where \( A_p = \max\{1, 2^{p-3}\} \), we have

\[
(x + y)^p - x^p - y^p \leq A_p \int_0^x \int_0^y p(p-1)(r^{p-2} + s^{p-2}) dr ds.
\]

After integration, we have

\[
(x + y)^p - x^p - y^p \leq A_p (p x y^{p-1} + p y x^{p-1}) = C_{2,p}(x^{p-1} y + x y^{p-1}).
\]

(b) Let \( p > 2 \) and \( x, y, z > 0 \). Since \( p - 3 > -1 \), we have

\[
(x+y+z)^p - x^p - y^p - z^p
\]

\[
= \int_0^x \int_0^y \int_0^z p(p-1)(p-2)(r+s+t)^{p-3} dr ds dt + ((x+y)^p - x^p - y^p)
\]

\[
+ ((x+z)^p - x^p - z^p) + ((y+z)^p - y^p - z^p).
\]

Using the estimate from part (a) and that \( (r+s+t)^{p-3} \leq B_p(r^{p-3} + s^{p-3} + t^{p-3}) \), where \( B_p = \max\{1, 2^{p-4}\} \), we have

\[
(x+y+z)^p - x^p - y^p - z^p
\]

\[
\leq B_p \int_0^x \int_0^y \int_0^z p(p-1)(p-2)(r^{p-3} + s^{p-3} + t^{p-3}) dr ds dt
\]

\[
+ C_{2,p}(x^{p-1} y + x y^{p-1} + x^{p-1} x + x z^{p-1} + y^{p-1} z + y z^{p-1}).
\]

After integration, we get

\[
(x+y+z)^p - x^p - y^p - z^p = p(p-1) B_p (x y z^{p-2} + x y^{p-2} z + x^{p-2} y z)
\]

\[
+ C_{2,p}(x^{p-1} y + x y^{p-1} + x^{p-1} x + x z^{p-1} + y^{p-1} z + y z^{p-1}). \tag{A.24}
\]

Since \( p > 2 \), for any \( a, b > 0 \) we have that \( (a - b)(a^{p-2} - b^{p-2}) \geq 0 \), which implies \( a^{p-2} b + a b^{p-2} \leq a^{p-1} + b^{p-1} \). Therefore,

\[
2(x y z^{p-2} + x y^{p-2} z + x^{p-2} y z)
\]

\[
= (x y z^{p-2} + x y^{p-2} z) + (x y^{p-2} z + x^{p-2} y z)
\]

\[
\leq (x y^{p-1} + x z^{p-1}) + (x^{p-1} y + y z^{p-1}) + (x^{p-1} z + y^{p-1} z)
\]

\[
= x^{p-1} y + x y^{p-1} + x^{p-1} x + x z^{p-1} + y^{p-1} z + y z^{p-1}.
\]

Combining the last inequality with (A.24), completes the proof of part (b) of the Lemma.

\[\square\]
Proof. If \( x = 0 \) or \( \mu = 0, 1 \), the inequality is trivially satisfied. So, assume \( x > 0 \) and \( 0 < \mu < 1 \). By the definition of convexity for all \( x_1 < x_2 \leq x_3 < x_4 \in [0, \infty) \), we have
\[
\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(x_4) - f(x_3)}{x_4 - x_3}
\]
Since \( x > 0 \) and \( 0 < \mu < 1 \), take \( x_1 = 0, x_2 = x_3 = \mu x, x_4 = x \). Then
\[
\frac{f(\mu x) - f(0)}{\mu x} \leq \frac{f(x) - f(\mu x)}{(1 - \mu)x},
\]
so, since \( f(0) = 0 \) and \( x > 0, 0 < \mu < 1 \)
\[
(1 - \mu)f(\mu x) \leq \mu f(x) - f(\mu x) \Rightarrow f(\mu x) \leq \mu f(x)
\]
\[\Box\]

Lemma A.5. [Approximation of convex functions] Let \( \psi : [0, \infty) \to \mathbb{R} \) be a differentiable, increasing and convex function. Then there exists a sequence of functions \((\psi_n)_n\) such that:

(1) \( \psi_n \) is differentiable, increasing and convex for all \( n \).
(2) \( \psi_{n+1} - \psi_n \) is convex.
(3) \( \psi_n \not\nearrow \psi \) as \( n \to \infty \).
(4) \( \psi_n - p_n \) has compact support for some appropriate sequence of first degree polynomials \((p_n)_n\).
(5) If, in addition, \( \psi(x) = x\phi(x) \), where \( \phi \) is concave, increasing and differentiable, then for all \( n \), we can write \( \psi_n(x) = x\phi_n(x) \), where \( \phi_n \) is concave, increasing, differentiable and \( \phi_n(0) = 0 \).

Proof. We define the first degree polynomial
\[
p_n(x) = \psi'(n)x + \psi(n) - n\psi'(n), \tag{A.25}
\]
and
\[
\psi_n(x) = \begin{cases} 
\psi(x), & x \leq n \\
p_n(x), & x > n
\end{cases} \tag{A.26}
\]
For any \( n \), \( \psi_n \) is differentiable, since \( \psi, p_n \) are so and
\[
\lim_{x \to n^-} \frac{\psi_n(x) - \psi_n(n)}{x - n} = \lim_{x \to n^-} \frac{\psi(x) - \psi(n)}{x - n} = \psi'(n),
\]
\[
\lim_{x \to n^+} \frac{\psi_n(x) - \psi_n(n)}{x - n} = \lim_{x \to n^+} \frac{\psi'(n)x + \psi(n) - n\psi'(n) - \psi(n)}{x - n} = \lim_{x \to n^+} \frac{(x - n)\psi'(n)}{x - n} = \psi'(n).
\]
The derivative of \( \psi_n \) is given by:
\[
\psi'_n(x) = \begin{cases} 
\psi'(x), & x < n \\
\psi'(n), & x \geq n
\end{cases}
\]
In particular, since \( \psi \) is increasing, we have \( \psi' \geq 0 \), thus \( \psi'_n \geq 0 \), therefore \( \psi_n \) is increasing for all \( n \). Moreover, since \( \psi \) is convex, we have that \( \psi' \) is increasing, thus \( \psi'_n \) is increasing, therefore \( \psi_n \) is convex for all \( n \).
(2) For fixed \( n \), we compute

\[
(\psi_{n+1}' - \psi_n')(x) = \begin{cases}
\psi'(x), & x < n, \\
\psi'(x) - \psi'(n), & n \leq x < n + 1, \\
\psi'(n + 1), & x \geq n + 1.
\end{cases}
\]

Since \( \psi' \) is increasing, we have that \( \psi_{n+1}' - \psi_n' \geq 0 \), so \( \psi_{n+1} - \psi_n \) is convex.

(3) Fix \( x \in [0, \infty) \). It is immediate that \( \psi_n(x) \to \psi(x) \) as \( n \to \infty \). To show that \( (\psi_n)_n \) is increasing, fix \( n \). We have the following cases:

- \( x \leq n \): We have \( \psi_{n+1}(x) - \psi_n(x) = \psi(x) - \psi(x) = 0 \).
- \( n < x \leq n + 1 \): We have

\[
\psi_{n+1}(x) - \psi_n(x) = \psi(x) - p_n(x)
= (\psi'(x) - \psi'(n))(x - n)
= (\psi'(\xi) - \psi'(n))(x - n),
\]

for some \( \xi \in (n, x) \). Since \( \psi' \) is increasing, we obtain that \( \psi_{n+1}(x) \geq \psi_n(x) \).

- \( x \geq n + 1 \): Since \( \psi \) and \( \psi' \) are increasing, we take

\[
\psi_{n+1}(x) - \psi_n(x) = p_{n+1}(x) - p_n(x)
= \psi'(n + 1)(x - n - 1) - \psi'(n)(x - n) + \psi(n + 1) - \psi(n)
\geq (\psi'(n + 1) - \psi'(n))(x - n - 1) + \psi(n + 1) - \psi(n)
\geq 0
\]

In any case, we have \( \psi_n(x) \leq \psi_{n+1}(x) \), so \( (\psi_n)_n \) is increasing.

(4) It is clear that \( \psi - p_n \) is supported in \([0, n]\).

(5) Define

\[
\phi_n(x) = \begin{cases}
0, & x = 0 \\
\phi(x), & 0 < x \leq n, \\
\psi'(n) + \frac{\psi(n) - n\psi'(n)}{x}, & x > n.
\end{cases}
\]

Clearly, \( \psi_n(x) = x\phi_n(x) \). Moreover, since \( \psi = x\phi \), we have \( \psi' = \phi + x\phi' \), and so \( x\psi' = \psi + x^2\phi' \). Thus, \( \psi(n) - n\psi'(n) = -n^2\phi'(n) \leq 0 \) since \( \phi \) is increasing. Therefore,

\[
\phi_n'(x) = \begin{cases}
\phi'(x), & x < n, \\
\frac{n^2\phi(n)}{x^2}, & x > n.
\end{cases}
\]

Also,

\[
\lim_{x \to n^-} \frac{\phi_n(x) - \phi_n(n)}{x - n} = \lim_{x \to n^-} \frac{\phi(x) - \phi(n)}{x - n} = \phi'(n),
\]

\[
\lim_{x \to n^+} \frac{\phi_n(x) - \phi_n(n)}{x - n} = \lim_{x \to n^+} \frac{1}{x - n}(\psi(n) - n\psi'(n)) \left( \frac{1}{x} - \frac{1}{n} \right) = \lim_{x \to n^+} \frac{\psi(n) - n\psi'(n)}{-x n} = \phi'(n).
\]

Therefore, \( \phi_n' \geq 0 \), and so \( \phi_n \) is an increasing function for all \( n \). Moreover, since \( \phi \) is concave, we have that \( \phi' \) is decreasing, thus \( \phi_n' \) is decreasing, therefore \( \phi_n \) is concave for all \( n \).

□
A.3. Auxiliary moment estimates. Here we provide some auxiliary moments estimates. First we present the basic interpolation estimate which is used extensively throughout the manuscript. For the proof see e.g. [3].

Lemma A.6. Let $0 \leq s_1 \leq s_2$ and $s = \tau s_1 + (1 - \tau)s_2$, $\tau \in [0,1]$. Then, for $f \geq 0$, we have

$$m_s[f] \leq m_{s_1}[f]^\tau m_{s_2}[f]^{1-\tau}. \quad (A.27)$$

In particular, when $s_2 > 0$, the following estimate holds

$$m_s[f]^{\frac{1}{s}} \leq m_0[f]^{\frac{1}{s}} - \frac{m_{s_2}[f]}{m_0[f]}. \quad (A.28)$$

We also use the following product of moments estimate

Lemma A.7 ([10], Lemma A.1). Let $\ell > 0$ and $0 \leq i,j,k,l \leq \ell$ such that $i+j = k+l = \ell$. Assume that $\min\{k,l\} \leq \min\{i,j\}$. Then, given $f \geq 0$, we have

$$m_i[f]m_j[f] \leq m_k[f]m_l[f].$$

A.4. Estimates on binary and ternary potentials. Here we establish upper and lower bounds on $|u|^{\gamma_2}$ and $|\tilde{u}|^{\gamma_3-\theta_3}|u|^{\theta_3}$ which we will rely on throughout the paper:

- **Upper bound on $|u|^{\gamma_2}$:** Since $\gamma_2 \geq 0$, we have

  $$|u|^{\gamma_2} \leq (|v| + |v_1|)^{\gamma_2} \leq C_{\gamma_2} (|v|^{\gamma_2} + |v_1|^{\gamma_2}), \quad C_{\gamma_2} = \max\{1, 2^{\gamma_2-1}\}. \quad (A.29)$$

- **Lower bound on $|u|^{\gamma_2}$:** Since $|v| = |u - v| \leq |u| + |v_1|$, for any permutation $(\pi_0, \pi_1)$ of $\{0,1\}$, we have $\frac{1}{2}(v_{\pi_0})^2 \leq \frac{1}{2} + |u|^2 + |v_1|^2 \leq |u|^2 + |v_2|^2$. Raising both sides of the inequality to the power $\gamma_2/2$ and using the fact that $\gamma_2 \in [0,2]$, we obtain the following lower bound

  $$|u|^{\gamma_2} \geq 2^{-\frac{\pi_1}{2}} (v_{\pi_0})^{\gamma_2} - \langle v_{\pi_1} \rangle^{\gamma_2}. \quad (A.30)$$

- **Upper bound on $|\tilde{u}|^{\gamma_3-\theta_3}|u|^{\theta_3}$:** Since $|u| \leq |\tilde{u}| \leq |v - v_1| + |v - v_2| + |v_1 - v_2| \leq 2(|v| + |v_1| + |v_2|)$, and $\gamma_3, \theta_3 \geq 0$, we have the following upper bound

  $$|\tilde{u}|^{\gamma_3} \leq C_{\gamma_3} (|v|^{\gamma_3} + |v_1|^{\gamma_3} + |v_2|^{\gamma_3}), \quad C_{\gamma_3} = 2^{\gamma_3} \max\{1, 3^{\gamma_3-1}\}. \quad (A.31)$$

- **Lower bound on $|\tilde{u}|^{\gamma_3-\theta_3}|u|^{\theta_3}$:** Note that for any permutation $(\pi_0, \pi_1, \pi_2)$ of $\{0,1,2\}$, we have

  $$\sqrt{2}|v_{\pi_0}| = \left| \begin{pmatrix} v_{\pi_0} \\ v_{\pi_0} \\ v_{\pi_0} \\ v_{\pi_0} \end{pmatrix} \right| \leq \left| \begin{pmatrix} v_{\pi_0} - v_{\pi_1} \\ v_{\pi_0} - v_{\pi_2} \end{pmatrix} \right| \leq |\tilde{u}| + |v_{\pi_1}| + |v_{\pi_2}|,$

  and thus $\frac{1}{2}(v_{\pi_0})^2 \leq \frac{1}{2} + |\tilde{u}|^2 + |v_{\pi_1}|^2 + |v_{\pi_2}|^2 \leq |\tilde{u}|^2 + |\langle v_{\pi_1} \rangle^2 + |\langle v_{\pi_2} \rangle^2|$. Raising this inequality to the power $\gamma_3/2$ and using the fact that $\gamma_3 \in [0,2]$ and $|u| \geq \frac{1}{\sqrt{3}}|\tilde{u}|$, we obtain

  $$|\tilde{u}|^{\gamma_3-\theta_3}|u|^{\theta_3} \geq 3^{-\frac{\theta_3}{2}} |\tilde{u}|^{\gamma_3} \geq 3^{-\frac{\theta_3}{2}} \left( \frac{2}{3} \right)^{\gamma_3} (v_{\pi_0})^{\gamma_3} - \langle v_{\pi_1} \rangle^{\gamma_3} - \langle v_{\pi_2} \rangle^{\gamma_3}. \quad (A.32)$$
A.5. ODE theory in Banach spaces. We now present a general well-posedness theorem for ODEs in Banach spaces:

**Theorem A.8.** Let \( X = (X, \| \cdot \|) \) be a Banach space and \( S \) a bounded, convex and closed subset of \( X \). Consider a mapping \( Q : S \to X \) that satisfies the following properties:

1. **Hölder continuity:** There exists \( \alpha \in (0, 1) \) and \( C_H > 0 \) such that for all \( u, v \in S \) we have
   \[
   \|Q[u] - Q[v]\| \leq C_H \|u - v\|^\alpha.
   \] (A.33)

2. **One-sided Lipschitz condition:** There exists \( C_L > 0 \) such that for all \( u, v \in S \) we have
   \[
   [Q[u] - Q[v], u - v] \leq C_L \|u - v\|,
   \] (A.34)
   where
   \[
   [f, g] := \lim_{h \to 0^-} \frac{\|g + hf\| - \|g\|}{h}.
   \]

3. **Sub-tangent condition:** For any \( u \in S \), we have
   \[
   \lim_{h \to 0^+} \frac{\text{dist}(u + hQ[u], S)}{h} = 0,
   \] (A.35)
   where for \( x \in X \) we denote \( \text{dist}(x, S) = \inf\{\|x - y\| : y \in S\} \).

Let \( T > 0 \). Then for any \( u_0 \in S \), the abstract Cauchy problem:
\[
\begin{aligned}
\partial_t u &= Q[u], & t \in (0, T) \\
|u|_{t=0} &= u_0,
\end{aligned}
\] (A.36)
has a unique solution \( u \in C([0, T], S) \cap C^1((0, T), X) \).

The above statement of the theorem and the corresponding proof can be found in [3], which in turn are inspired by [19]. We note that an extension of this theorem and its proof can be found in [5].

A.6. A functional inequality. Here, we prove a lower bound on the convolution of a function \( f \) and the potential function \( |\cdot|^\gamma \). Estimates of this kind have been known and used in kinetic theory for a long time, see e.g. [3, 4, 13, 14, 26, 31].

The estimate provided in the lemma below is a general functional inequality. Unlike previous results, this estimate does not require finite entropy or zero momentum, and no assumptions are made on moments of order higher than two. However, the constant in the lower bound depends on the end-time \( T \).

**Lemma A.9.** Let \( T > 0 \) and \( \gamma \in (0, 2) \). Suppose a nonnegative function \( f \in C([0, T], L^1_2) \) satisfies
\[
0 < c_0 \leq \int_{\mathbb{R}^d} f(t, v) dv \leq \int_{\mathbb{R}^d} f(t, v)(v)^2 dv \leq C_2, \quad \text{for all } t \in [0, T],
\]
where \( c_0, C_2 \) are constants. Then there exists a constant \( C_T = C_T(T, c_0, C_2, \gamma) > 0 \) so that
\[
\int_{\mathbb{R}^d} f(t, v_1)|v - v_1|^\gamma dv_1 \geq C_T(v)^\gamma, \quad \text{for all } t \in [0, T], \ v \in \mathbb{R}^d.
\] (A.37)

---

5this limit always exists since it is the side derivative of the convex map \( x \to \|x\| \).
Proof. Estimate (A.30) implies that for all $t \in [0, T]$ and all $v \in \mathbb{R}^d$ we have
\[
\int_{\mathbb{R}^d} f(t, v_1) |v - v_1|^\gamma dv_1 \geq \int_{\mathbb{R}^d} f(t, v_1) \left( 2^{-\gamma} \langle v \rangle^\gamma - \langle v_1 \rangle^\gamma \right) dv_1 \geq 2^{-\gamma} c_0 \langle v \rangle^\gamma - C_2. \tag{A.38}
\]
Therefore, if $v$ is sufficiently large so that $2^{-\gamma} c_0 \langle v \rangle^\gamma \geq C_2$, that is
\[
|v| \geq \left( \frac{2^{1+\gamma} C_2}{c_0} \right)^{2/\gamma} - 1 =: C_*, \tag{A.39}
\]
then
\[
\int_{\mathbb{R}^d} f(t, v_1) |v - v_1|^\gamma dv_1 \geq 2^{-1-\gamma} c_0 \langle v \rangle^\gamma, \quad \text{for } |v| \geq C_*. \tag{A.40}
\]
It remains to prove the bound (A.37) for $|v| \leq C_*$. To achieve that, we first prove the following claim: for fixed $T > 0$, $\gamma \in (0, 2]$, $C_*$ and $c_0$,
\[
\exists R > 0 \text{ so that } \forall t \in [0, T] \forall v \in B(0, C_*) \text{ we have } \int_{\{v_1: |v - v_1| \leq R\}} f(t, v_1)dv_1 \leq \frac{c_0}{2}. \tag{A.41}
\]
To prove the claim (A.41), suppose it is not true. Then we would have
\[
\forall n \in \mathbb{N}, \exists n \in [0, T] \exists \bar{v}_n \in B(0, C_*) \text{ so that } \int_{B(\bar{v}_n, \frac{\bar{v}}{\bar{v}_1})} f(t, v_1)dv_1 > \frac{c_0}{2}. \tag{A.42}
\]
Up to subsequences, there exits $\bar{t} \in [0, T]$ and $\bar{v} \in B(0, C_*)$ so that $t_n \to \bar{t}$ and $\bar{v}_n \to \bar{v}$ as $n \to \infty$. Using triangle inequality, we get
\[
\frac{c_0}{2} \leq \int_{\mathbb{R}^d} \langle v \rangle^2 \left| f(t_n, v_1) - f(\bar{t}, \bar{v}_1) \right| dv_1 + \int_{\mathbb{R}^d} \chi_{B(\bar{v}_n, \frac{\bar{v}}{\bar{v}_1})} f(\bar{t}, \bar{v}_1)dv_1.
\]
Since $t_n \to \bar{t}$ and $f \in C([0, T], L^1)$, $\exists N \in \mathbb{N}$ so that $\forall n \geq N$, $\int_{\mathbb{R}^d} \langle v \rangle^2 \left| f(t_n, v_1) - f(\bar{t}, \bar{v}_1) \right| dv_1 < \frac{c_0}{4}$. Therefore,
\[
\forall n \geq N \text{ we have that } \int_{\mathbb{R}^d} \chi_{B(\bar{v}_n, \frac{\bar{v}}{\bar{v}_1})} f(\bar{t}, \bar{v}_1)dv_1 > \frac{c_0}{4}.
\]
Letting $n \to \infty$ and using the dominated convergence theorem, we get $0 > \frac{c_0}{4}$, which contradicts the fact that $c_0 > 0$. Thus, the proof of the claim (A.41) is completed.

Now suppose $|v| \leq C_*$, then by the claim (A.41) there exists a constant $R = R(T, \gamma, c_0, C_2)$ so that
\[
\int_{\{v_1: |v - v_1| \leq R\}} f(t, v_1)dv_1 \leq \frac{c_0}{2}, \quad \forall t \in [0, T], \forall v \in B(0, C_*)..
\]

Then
\[
\int_{\mathbb{R}^d} f(t, v_1) |v - v_1|^\gamma dv_1 \geq R \int_{\{v_1: |v - v_1| > R\}} f(t, v_1)dv_1 \geq R \frac{c_0}{2}, \quad \forall t \in [0, T], \forall v \in B(0, C_*).. \tag{A.43}
\]
Combining (A.38) and (A.43), we get that for any $\varepsilon > 0$, $t \in [0, T]$ and $v \in B(0, C_*)$, we have
\[
\int_{\mathbb{R}^d} f(t, v_1) |v - v_1|^\gamma dv_1 \geq \varepsilon \left( 2^{-\gamma} c_0 \langle v \rangle^\gamma - C_2 \right) + (1 - \varepsilon) R \frac{c_0}{2}.
\]
By choosing $\varepsilon$ small enough so that $(1 - \varepsilon)R^\gamma \frac{c_0}{C_2} - \varepsilon C_2 \geq 0$, i.e. $\varepsilon := \frac{R^\gamma \frac{c_0}{C_2} - \varepsilon C_2}{R^\gamma \frac{c_0}{C_2} + R^\gamma \frac{c_0}{C_2}}$, we get that

$$\int_{\mathbb{R}^d} f(t, v_1)|v - v_1|^\gamma \, dv_1 \geq 2^{-\gamma} c_0 \frac{R^\gamma \frac{c_0}{C_2} + R^\gamma \frac{c_0}{C_2}}{R^\gamma \frac{c_0}{C_2}} \langle v \rangle^\gamma, \quad \forall t \in [0, T], \forall v \in B(0, C_*). \quad (A.44)$$

The lemma follows from estimates (A.40) and (A.44), with $C_T = \max\{2^{-1-\gamma} c_0, c_0 2^{-\gamma} \frac{R^\gamma \frac{c_0}{C_2} + R^\gamma \frac{c_0}{C_2}}{R^\gamma \frac{c_0}{C_2}}\}$. □

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