Pseudo-Kähler and pseudo-Sasaki Einstein solvmanifolds

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Abstract

The aim of this paper is to construct left-invariant Einstein pseudo-Riemannian Sasaki metrics on solvable Lie groups.

We consider the class of $\mathfrak{z}$-standard Sasaki solvable Lie algebras of dimension $2n+3$, which are in one-to-one correspondence with pseudo-Kähler nilpotent Lie algebras of dimension $2n$ endowed with a compatible derivation, in a suitable sense. We characterize the pseudo-Kähler structures and derivations giving rise to Sasaki-Einstein metrics.

We classify $\mathfrak{z}$-standard Sasaki solvable Lie algebras of dimension $\leq 7$ and those whose pseudo-Kähler reduction is an abelian Lie algebra.

The Einstein metrics we obtain are standard, but not of pseudo-Iwasawa type.

Introduction

An effective method to construct Einstein metrics is by considering invariant metrics on a solvmanifold obtained by extending a suitable metric on a nilpotent Lie group of codimension one. Indeed, in the Riemannian case, Einstein solvmanifolds are described by a standard solvable Lie algebra $\tilde{\mathfrak{g}}$ of Iwasawa type ([19, 22]). In particular, this means that $\tilde{\mathfrak{g}}$ admits an orthogonal decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$, with $\mathfrak{g}$ nilpotent, $\mathfrak{a}$ abelian and $\text{ad} \, X$ self-adjoint whenever $X$ is in $\mathfrak{a}$. Furthermore, the restriction of the metric to $\mathfrak{g}$ satisfies the so-called nilsoliton equation ([21]).

Things are more complicated in the indefinite case (see e.g. [11]), but it is still possible to construct Einstein solvmanifolds by extending a nilsoliton; indeed, there is a correspondence between nilsolitons and a class of Einstein solvmanifolds for which $\tilde{\mathfrak{g}}$ admits a decomposition as above, called a pseudo-Iwasawa decomposition (see [12]).

In the non-invariant setting, Einstein metrics are often studied in the presence of additional structures, such as a Killing spinor or a restriction on the holonomy (see e.g. [2, 3]). It is then natural to ask whether Einstein metrics compatible with such special structures can be obtained in the invariant setting too.

This paper is focused on Sasaki metrics. More precisely, we consider a class of left-invariant pseudo-Riemannian Sasaki-Einstein metrics on solvable Lie groups. Sasaki-Einstein metrics admit a Killing spinor (see [18]), and may be viewed as the odd-dimensional counterpart of Kähler-Einstein geometry. We showed in [13] that Sasaki Lie algebras can never be of pseudo-Iwasawa type, regardless of whether they are Einstein; therefore, Sasaki-Einstein solvmanifolds cannot be obtained by extending a nilsoliton.

To work around this problem, we consider the more general class of $\mathfrak{z}$-standard Sasaki solvable Lie algebras introduced in [13]. This condition means that the Lie algebra $\tilde{\mathfrak{g}}$ is endowed with

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a Sasaki structure \((\phi, \xi, \eta, g)\) such that \(\tilde{\mathfrak{g}}\) takes the form of an orthogonal semidirect product \(\mathfrak{g} \rtimes \text{Span}\{v_0\}\), with \(\phi(v_0)\) central in \(\mathfrak{g}\). This implies that a reduction can be performed in the sense of contact geometry, whilst remaining in the category of Lie algebras; a further one-dimensional quotient by the center then yields a nilpotent pseudo-Kähler Lie algebra \(\mathfrak{g}\) in three dimensions less than the original Sasaki-Einstein Lie algebra. The pseudo-Kähler Lie algebra also inherits a derivation \(\tilde{D}\) satisfying a particular condition involving the metric. This process can be inverted: given a pseudo-Kähler nilpotent Lie algebra with an appropriate derivation \(\tilde{D}\), a 3-standard Sasaki solvable Lie algebra can be obtained in three dimensions higher.

In this paper, we introduce a generalization of the nilsoliton condition that enables one to construct Einstein solvmanifolds which are not of pseudo-Iwasawa type (Proposition 2.1). We then characterize 3-standard Sasaki-Einstein solvable Lie algebras in terms of their Kähler reduction \(\mathfrak{g}\), showing that the symmetric part of the derivation \(\tilde{D}\) is the identity and preserves the pseudo-Kähler structure. This implies that \(\tilde{D}\) lies in the Lie algebra \(\text{Der} \; \mathfrak{g} \cap \mathfrak{cu}(p, q)\), where \(\mathfrak{cu}(p, q) = \mathfrak{u}(p, q) \oplus \text{Span}\{\text{Id}\}\). We show that any two choices of \(\tilde{D}\) in this Lie algebra with symmetric part equal to the identity determine isometric Sasaki-Einstein extensions; the isometry is at the level of solvmanifolds as pseudo-Riemannian manifolds, and it does not preserve the Lie algebra structure (Theorem 4.1).

It turns out that \(\text{Der} \; \mathfrak{g} \cap \mathfrak{cu}(p, q)\) contains elements \(\tilde{D}\) with symmetric part equal to the identity if and only if it contains an element with nonzero trace. In that case, we show that the Lie algebra contains a canonical choice of \(\tilde{D}\).

This canonical element is obtained by adapting a construction of Nikolayevsky. Indeed, we fix an algebraic subalgebra \(\mathfrak{h}\) of \(\mathfrak{gl}(n, \mathbb{R})\), and define the \(\mathfrak{h}\)-Nikolayevsky derivation on a Lie algebra \(\mathfrak{g}\) with a \(H\)-structure as the unique semisimple derivation \(N\) in \(\mathfrak{h} \cap \text{Der} \; \mathfrak{g}\) such that

\[
\text{tr}(\psi N) = \text{tr} \; \psi, \quad \psi \in \mathfrak{h} \cap \text{Der} \; \mathfrak{g}.
\]

For \(\mathfrak{h} = \mathfrak{gl}(n, \mathbb{R})\), one obtains the Nikolayevsky derivation introduced in [23], and for \(\mathfrak{h} = \mathfrak{co}(p, q)\) the metric Nikolayevsky derivation of [9]. Existence and uniqueness of the \(\mathfrak{h}\)-Nikolayevsky derivation is proved similarly as in these particular cases (Proposition 2.6).

The relevant situation for this paper is the \(\mathfrak{cu}(p, q)\)-Nikolayevsky derivation of a pseudo-Kähler Lie algebra, which turns out to have rational eigenvalues, like the ordinary Nikolayevsky derivation. Thus, we see that the element of \(\text{Der} \; \mathfrak{g} \cap \mathfrak{cu}(p, q)\) that determines the Sasaki-Einstein extension can be assumed to be diagonalizable over \(\mathbb{Q}\) (Proposition 2.7).

We use the characterization of 3-standard Sasaki-Einstein solvmanifolds to classify all 3-standard Sasaki-Einstein solvmanifolds of dimension 7 (Theorem 4.4). In addition, we are able to write down all 3-standard Sasaki-Einstein solvmanifolds that reduce to a pseudo-Kähler \(\tilde{\mathfrak{g}}\) which is abelian as a Lie algebra (Corollary 4.2). This includes all Lorentzian 3-standard Sasaki-Einstein solvmanifolds, for which \(\tilde{\mathfrak{g}}\) is forced to be a nilpotent Kähler Lie algebra, hence abelian.

In particular, our results give rise to explicit pseudo-Kähler-Einstein and Sasaki-Einstein solvmanifolds in all dimensions \(\geq 4\). These metrics are not Ricci-flat, which is a general fact for Sasaki-Einstein metrics and their Kähler-Einstein quotients.

We point out that our Sasaki-Einstein metrics are examples of Einstein standard solvmanifolds that are not isometric to any Einstein solvmanifold of pseudo-Iwasa wa type. This is in sharp contrast to the Riemannian case, where [19] shows that all standard Einstein solvmanifolds are of Iwasawa type up to isometry.

We also show with an example that not all pseudo-Kähler Lie algebras can be extended to a 3-standard Sasaki-Einstein Lie algebra (Example 4.5).

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1 Preliminaries

In this section we introduce some general language relevant to the study of Sasaki-Einstein metrics, specialized to the invariant setting, and recall some results that will be needed in the sequel.

Given a Lie algebra $\mathfrak{g}$ of dimension $n$, we can think of a basis of $\mathfrak{g}$ as a frame $\mathbb{R}^n \cong \mathfrak{g}$. There is a natural right action of $\text{GL}(n, \mathbb{R})$ on the set of frames. Given a subgroup $H \subset \text{GL}(n, \mathbb{R})$, we will say that a $H$-structure on $\mathfrak{g}$ is a $H$-orbit in the space of frames. Given any frame $u$, the identification $u: \mathbb{R}^n \cong \mathfrak{g}$ induces a left action of $H$ on $\mathfrak{g}$. This induces an inclusion map $H \to \text{GL}(\mathfrak{g})$ that depends on the frame $u$, but the image of the inclusion only depends on the $H$-structure. Accordingly, whenever we have a $H$-structure on $\mathfrak{g}$, we will write $H \subset \text{GL}(\mathfrak{g})$, $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{g})$.

It is clear that a $H$-structure on a Lie algebra $\mathfrak{g}$ induces a left-invariant $H$-structure, in the usual sense, on any Lie group with Lie algebra $\mathfrak{g}$.

An almost contact structure on a $(2n+1)$-dimensional Lie algebra $\mathfrak{g}$ is a triple $(\phi, \xi, \eta)$, where $\phi$ is a linear map from $\mathfrak{g}$ to itself, $\xi$ is an element of $\mathfrak{g}$, $\eta$ is in $\mathfrak{g}^\perp$ and

$$\eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi^2 = -\text{Id} + \eta \otimes \xi.$$ 

Given a nondegenerate scalar product $g$ on $\mathfrak{g}$, the quadruple $(\phi, \xi, \eta, g)$ is called an almost contact metric structure if $(\phi, \xi, \eta)$ is an almost contact structure and

$$g(\xi, \xi) = 1, \quad \eta = \xi^\perp, \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \mathfrak{g}$. One then defines a two-form $\Phi$ by $\Phi(X, Y) = g(X, \phi Y)$.

Given an almost contact metric structure of signature $(2p + 1, 2q)$ on $\mathfrak{g}$, one can find a frame $e_1, \ldots, e_{2n+1}$ with dual basis $\{e^i\}$ such that

$$g = e^1 \otimes e^1 + \ldots + e^{2p} \otimes e^{2p} - e^{2p+1} \otimes e^{2p+1} - \ldots - e^{2p+2q} \otimes e^{2p+2q} + e^{2p+2q+1} \otimes e^{2p+2q+1},$$

$$\eta = e^{2p+2q+1}, \quad \Phi = e^{12} + \ldots + e^{2p-1,2p} - e^{2p+1,2p+2} - \ldots - e^{2p+2q,2p+2q}.$$ 

The common stabilizer of these tensors in $\text{GL}(2p+2q+1, \mathbb{R})$ is $U(2p, 2q)$. Thus, an almost contact metric structure on a Lie algebra can be viewed as a $U(2p, 2q)$-structure.

An almost contact metric structure is called Sasaki if $N_\phi + d\eta \otimes \xi = 0$ and $d\eta = 2\Phi$, where $N_\phi$ denotes the Nijenhuis tensor.

These definitions mimick analogous definitions for structures on manifolds (see e.g. [5, 6]). It is clear that a Sasaki structure on $\mathfrak{g}$ defines a left-invariant almost Sasaki structure on any Lie group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ by left translation. In particular, $g$ defines a pseudo-Riemannian metric on $\tilde{G}$. We shall also refer to $g$ as a metric on $\tilde{\mathfrak{g}}$, and define its Levi-Civita connection, curvature and so on in terms of the corresponding objects on $\tilde{G}$.

Sasaki structures are an odd-dimensional analogue of (pseudo)-Kähler structures. In our invariant setting, a pseudo-Kähler structure on a Lie algebra $\mathfrak{g}$ is a triple $(g, J, \omega)$, where $g$ is a pseudo-Riemannian metric, $J: \mathfrak{g} \to \mathfrak{g}$ is an almost complex structure satisfying the compatibility condition $g(JX, JY) = g(X, Y)$, and $\omega(X, Y) = g(X, JY)$, where one further imposes that $N_J = 0$ and $d\omega = 0$.

Having fixed the metric, for any endomorphism $f: \mathfrak{g} \to \mathfrak{g}$ we write $f = f^g + f^a$, where $f^g$ is symmetric and $f^a$ is skew-symmetric relative to the metric, i.e.

$$f^g = \frac{1}{2}(f + f^*), \quad f^a = \frac{1}{2}(f - f^*).$$
With this notation, the Levi-Civita connection is given by
\[ \nabla_w v = -\text{ad}(v)^* w - \frac{1}{2}(\text{ad} w)^* v. \] (1)

If \( \mathfrak{g} \) is unimodular with Killing form equal to zero, the Ricci tensor satisfies
\[ 2 \text{ric}(v, w) = g(\text{d}v^\flat, dw^\flat) - g(\text{ad} v, \text{ad} w), \] (2)
see e.g. [10, Proposition 2.1].

We will need a result originally proved in [1] for Riemannian metrics and later adapted to standard indefinite metrics in [12], though the standard condition turned out not to be necessary (see [13]). The precise statement we are going to need is the following:

**Proposition 1.1** ([1, Proposition 2.1]). Let \( H \) be a subgroup of \( \text{SO}(r, s) \) with Lie algebra \( \mathfrak{h} \) and \( \mathfrak{g} \) a Lie algebra of the form \( \mathfrak{g} = \mathfrak{g} \times \mathfrak{a} \) endowed with a \( H \)-structure. Let \( \chi : \mathfrak{a} \to \text{Der}(\mathfrak{g}) \) be a Lie algebra homomorphism such that, extending \( \chi(X) \) to \( \mathfrak{g} \) by declaring it to be zero on \( \mathfrak{a} \),
\[ \chi(X) - \text{ad} X \in \mathfrak{h}, \quad [\chi(X), \text{ad} Y] = 0, \quad X, Y \in \mathfrak{a}. \] (3)

Let \( \mathfrak{g}^* \) be the Lie algebra \( \mathfrak{g} \rtimes \mathfrak{a} \). If \( \hat{G} \) and \( \hat{G}^* \) denote the connected, simply connected Lie groups with Lie algebras \( \hat{\mathfrak{g}} \) and \( \hat{\mathfrak{g}}^* \), with the corresponding left-invariant \( H \)-structures, there is an isometry from \( \hat{G} \) to \( \hat{G}^* \), whose differential at \( e \) is the identity of \( \mathfrak{g} \oplus \mathfrak{a} \) as a vector space, mapping the \( H \)-structure on \( \hat{G} \) into the \( H \)-structure on \( \hat{G}^* \).

**Proof.** For \( \mathfrak{h} = \text{so}(p, q) \), \( \chi(X) - \text{ad} X \) is skew-symmetric and the proof is identical to [13, Proposition 2.2]. In general, one uses that \( \chi(X) - \text{ad} X \) is in \( \mathfrak{h} \) to conclude that the action of \( G^* \) on \( \hat{G} \) preserves the \( H \)-structure.

We will say that two Lie algebras endowed with a \( H \)-structure are equivalent if there is an isometry between the corresponding simply-connected Lie groups mapping one \( H \)-structure into the other.

Recall from [12] that a **standard decomposition** of the Lie algebra with a fixed metric is an orthogonal splitting
\[ \hat{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}, \]
where \( \mathfrak{g} \) is a nilpotent ideal and \( \mathfrak{a} \) is an abelian subalgebra. This definition generalizes the definition given in [19] for positive-definite metrics.

In [13], we introduced a special class of standard Sasaki Lie algebras: if \( \hat{\mathfrak{g}} \) has both a Sasaki structure \( (\phi, \xi, \eta, g) \) and a standard decomposition of the form \( \hat{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\} \), it is called \( \mathfrak{s} \)-standard if \( \phi(e_0) \) is central in \( \mathfrak{g} \).

Combining Corollary 4.4 and Proposition 5.1 in [13], one sees that \( \mathfrak{s} \)-standard Sasaki Lie algebras can be characterized as follows:

**Proposition 1.2** ([13, Proposition 5.1 and Corollary 4.4]). Let \( (\hat{\mathfrak{g}}, J, \omega) \) be a pseudo-Kähler nilpotent Lie algebra. Let \( \hat{D} \) be a derivation of \( \hat{\mathfrak{g}} \), \( \tau = \pm 1 \), and \( \mathfrak{g} = \mathfrak{g} \oplus \text{Span} \{b, \xi\} \) a central extension of \( \mathfrak{g} \) with a metric of the form:
\[ g(x, y) = g(x, y), \quad g(x, b) = 0 = g(x, \xi), \quad g(\xi, \xi) = 1, \quad g(b, b) = \tau, \quad g(b, \xi) = 0, \]
where \( x, y \in \hat{\mathfrak{g}} \). Assume furthermore
\[ d\xi^\flat = 2\omega, \] where the right-hand-side is implicitly pulled back to \( \hat{\mathfrak{g}} \);
• \(db^\flat = \hat{D}\omega\), where the right-hand-side is implicitly pulled back to \(\tilde{g}\);

• \([J, \hat{D}] = 0\);

• \([\hat{D}^s, \hat{D}^a] = h\hat{D}^s - 2(\hat{D}^s)^2\) for some constant \(h\).

Let \(\tilde{g} = g \times \text{Span} \{e_0\}\), where

\[
[e_0, x] = \hat{D}x, \quad [e_0, b] = hb - 2\tau\xi, \quad [e_0, \xi] = 0;
\]

then \(\tilde{g}\) has a \(3\)-standard Sasaki structure \((\phi, \eta, \xi, \tilde{g})\) given by

\[\tilde{g} = g + \tau e^0 \otimes e^0, \quad \phi(x) = J(x) + \tau g(b, x)e_0, \quad \phi(e_0) = -b, \quad x \in g.\]

Conversely, every \(3\)-standard Sasaki Lie algebra arises in this way.

Notice that given a \(3\)-standard Sasaki Lie algebra \(\tilde{g} = g \times \text{Span}\{e_0\}\), the Lie algebra \(\tilde{g}\) and its pseudo-Kähler structure are determined; we say that \((\tilde{g}, J, \omega)\) is the Kähler reduction of \(\tilde{g}\).

It is well known that, given a Sasaki manifold \(M\), the space of leaves of the Reeb foliation has a pseudo-Kähler structure (see [24]), and the Ricci of the latter is determined by the Ricci tensor of \(M\) (see [5, Theorem 7.3.12]). In our invariant setting, this fact takes the following form:

**Proposition 1.3.** Let \(g\) be a Lie algebra with a Sasaki structure \((\phi, \xi, \eta, g)\). Suppose \(g\) has nonzero center. Then \(z(g) = \text{Span}\{\xi\}\) and the quotient \(\tilde{g} = g / \text{Span}\{\xi\}\) has an induced pseudo-Kähler structure \((\tilde{g}, J, \omega)\) with \(\mathring{\text{ric}} = \text{ric} + 2\tilde{g}\), where \(\text{ric}\) is restricted to \(\xi^\perp\) implicitly.

**Proof.** Any element of the center satisfies \(v \downnabla d\eta = 0\), so it is a multiple of \(\xi\). Thus, the kernel coincides with \(\text{Span}\{\xi\}\).

As a vector space, we identify \(\tilde{g}\) with \(\xi^\perp\), so that the metric \(\tilde{g}\) is the restriction of \(g\). The Lie algebra structure of \(\tilde{g}\) is given by a projection, i.e.

\[\text{ad}(v)(w) = \widetilde{\text{ad}}(v)(w) - d\eta(v, w)\xi.\]

Therefore

\[\text{ad}(v)^*(w) = \widetilde{\text{ad}}(v)^*(w), \quad \text{ad}(v)^*(\xi) = -(v \downnabla d\eta)^\sharp.\]

Then, from equation (1), the Levi-Civita connections \(\nabla, \tilde{\nabla}\) are related by

\[\nabla_w v = -\text{ad}(v)^*(w) - \frac{1}{2} \text{ad}(w)^*(v)\]

\[= -\widetilde{\text{ad}}(v)^*(w) + \frac{1}{2} d\eta(v, w)\xi - \frac{1}{2} \widetilde{\text{ad}}(w)^*(v) = \tilde{\nabla}_w v + \frac{1}{2} d\eta(v, w)\xi;\]

\[\nabla_w \xi = -\text{ad}(w)^*(\xi) - \frac{1}{2} \text{ad}(\xi)^*(w) = \frac{1}{2} (w \downnabla d\eta)^\sharp.\]

If \(\alpha \in \text{Ann}(\xi)\), we have

\[(\nabla_w \alpha)(v) = -\alpha(\nabla_w v) = -\alpha(\tilde{\nabla}_w v) = (\tilde{\nabla}_w \alpha)(v),\]

\[(\nabla_w \alpha)(\xi) = -\alpha(\nabla_w \xi) = -\frac{1}{2} g(w \downnabla d\eta, \alpha).\]
The Sasaki condition implies
\[ \nabla_v d\eta = 2\eta \wedge v^\flat, \]
so \( \check{\nabla}_v d\eta = 0 \). This implies that \( d\eta \) defines a pseudo-Kähler structure on \( \check{\mathfrak{g}} \).

The exterior derivative \( \check{d} \) on \( \check{\mathfrak{g}} \) can be identified with the restriction of \( d \). By (2), we obtain
\[
2 \text{ric}(v, w) = g(dv^\flat, dw^\flat) - g(ad v, ad w) = g(dv^\flat, \check{d}w^\flat) - g(ad v, \check{d}w) - g(v, d\eta, w, d\eta)
= 2\check{\text{ric}}(v, w) - g(2\phi(v), 2\phi(w)) = 2\check{\text{ric}}(v, w) - 4g(v, w). \]

Proposition 1.4. There is no nilpotent Einstein-Sasaki Lie algebra.

Proof. Let \( \mathfrak{g} \) be a nilpotent Lie algebra with an Einstein-Sasaki structure. We know that its center is spanned by \( \xi \). By Proposition 1.3, the quotient \( \mathfrak{g}/\text{Span} \{\xi\} \) is pseudo-Kähler and Einstein with positive scalar curvature. Since it is also nilpotent, it must be Ricci-flat by [17, Lemma 6.3], which is absurd.

Another link between Sasaki-Einstein and Kähler-Einstein geometry is given by the following:

Proposition 1.5 ([5, Corollary 11.1.8]). Let \( (\phi, \xi, \eta, g) \) be an almost contact pseudo-Riemannian metric structure on a manifold \( M \) of dimension \( 2n + 1 \). The following are equivalent:

1. \( (\phi, \xi, \eta, g) \) is Sasaki-Einstein;
2. the cone \( (\mathbb{R}^+ \times M, J, \omega) \) is pseudo-Kähler and Ricci-flat.

2 Einstein standard Lie algebras

In this section we study the Einstein condition on standard Lie algebras \( \mathfrak{g} \ltimes \text{Span} \{e_0\} \), without assuming the pseudo-Iwasawa condition (see [13, Proposition 2.6]). We write down the conditions that the induced metric \( g \) and the derivation \( D = ad e_0 \) must satisfy, generalizing the nilsoliton equation. In particular, the conditions are satisfied if \( g \) is Ricci-flat and the symmetric part of \( D \) is an appropriate multiple of the identity.

We then recall and generalize the construction of the Nikolayevsky and metric Nikolayevsky derivation ([23, 9]). We show that a nilpotent Lie algebra admits a standard Einstein extension with the symmetric part of \( D \) equal to a multiple of the identity if and only if it is Ricci-flat and the metric Nikolayevsky derivation is nonzero. In this case, the extension is unique up to isometry.

Recall that given endomorphisms \( f, g \) of \( \mathfrak{g} \), we have
\[ g(f, g) = \text{tr}(fg^*) = \text{tr}(f(g^* - g^a)). \]

Proposition 2.1. Let \( \mathfrak{g} \) be a nilpotent Lie algebra with a pseudo-Riemannian metric \( g \), \( D \) a derivation and \( \tau = \pm 1 \). Then the metric \( \tilde{g} = g + \tau e^0 \otimes e^0 \) on \( \tilde{\mathfrak{g}} = \mathfrak{g} \ltimes D \text{Span} \{e_0\} \) is Einstein if and only if
\[ \text{Ric} = \tau(-\text{tr}((D^*)^2) \text{Id} - \frac{1}{2}[D, D^*] + (\text{tr} D) D^*), \quad \text{tr}(ad v \circ D^*) = 0, \quad v \in \mathfrak{g}; \]
in this case, \( \check{\text{ric}} = -\tau \text{tr}((D^*)^2) \tilde{g} \).
Proof. By [12, Proposition 1.10], we have

\[
\tilde{\text{ric}}(v, w) = \text{ric}(v, w) + \tau \tilde{g}(\frac{1}{2}[D, D^*](v), w) - \tau(\text{tr} D)\tilde{g}(D^*(v), w)
\]

\[
\tilde{\text{ric}}(v, e_0) = \frac{1}{2} \tilde{g}(\text{ad} v, D)
\]

\[
\tilde{\text{ric}}(e_0, e_0) = -\frac{1}{2} \tilde{g}(D, D) - \frac{1}{2} \text{tr} D^2 = -\frac{1}{2} \text{tr} D(D^* - D^a) - \frac{1}{2} \text{tr} D(D^a + D^s)
\]

Thus, the Einstein condition \(\tilde{\text{Ric}} = \lambda \text{Id}\) holds if and only if

\[
\lambda \text{Id} = \text{Ric} + \frac{1}{2} \tau[D, D^*] - \tau(\text{tr} D)D^a, \quad \text{tr}(\text{ad} v \circ D^*) = 0, \quad \lambda = -\tau \text{tr}(D^a)^2.
\]

Remark 2.2. If \(h = -g\), then \(h, g\) have the same Ricci tensor and opposite Ricci operators; the operators \(D \mapsto D^*\) and \(D \mapsto D^a\) are identical. Therefore, if \(g\) satisfies

\[
\text{Ric}^g = \tau(-\text{tr}((D^a)^2)\text{Id} - \frac{1}{2}[D, D^*] + (\text{tr} D)D^s), \quad \text{tr}(\text{ad} v \circ D^*) = 0, \quad v \in g,
\]

then

\[
\text{Ric}^h = (-\tau)(-\text{tr}((D^a)^2)\text{Id} - \frac{1}{2}[D, D^*] + (\text{tr} D)D^s), \quad \text{tr}(\text{ad} v \circ D^*) = 0, \quad v \in g.
\]

This amounts to the fact that \(g + \tau e^0 \otimes e^0\) is Einstein if and only if so is \(h - \tau e^0 \otimes e^0\).

Remark 2.3. We can write

\[
[D, D^*] = [D^a + D^s, -D^a + D^s] = 2[D^a, D^s].
\]

Although we will not need it in the rest of the paper, we show for completeness that the condition that \(\text{tr}(\text{ad} v \circ D^*)\) vanish can be eschewed under a suitable assumption on the eigenvalues of \(D\).

Corollary 2.4. Let \(\mathfrak{g}\) be a nilpotent Lie algebra with a pseudo-Riemannian metric \(g\), \(D\) a derivation such that \(-\text{tr} D\) is not an eigenvalue of \(D\) and \(\tau = \pm 1\). Then the metric \(\tilde{g} = g + \tau e^0 \otimes e^0\) on \(\tilde{g} = \mathfrak{g} \times_D \text{Span}\{e_0\}\) is Einstein if and only if

\[
\text{Ric} = \tau(-\text{tr}((D^a)^2)\text{Id} - \frac{1}{2}[D, D^*] + (\text{tr} D)D^s);
\]

in this case, \(\tilde{\text{ric}} = -\tau \text{tr}((D^a)^2)\tilde{g}\).

Proof. One direction follows from Proposition 2.1. For the other direction, assume that \(f = (\text{tr} D)\text{Id} + D\) is invertible and (4) holds. Since \(\text{ad} v\) is a derivation,

\[
0 = \text{tr}(\text{ad} v \circ \text{Ric}) = -\text{tr}((D^a)^2)\text{tr} \text{ad} v - \frac{1}{2} \text{tr}([D, D^*] \circ \text{ad} v) + (\text{tr} D) \text{tr}(\text{ad} v \circ D^s)
\]

\[
= -\frac{1}{2} \text{tr}(\text{ad} v, D^s) + \frac{1}{2} (\text{tr} D) \text{tr}(\text{ad} v \circ (D + D^s))
\]

\[
= \frac{1}{2} \text{tr}(\text{ad} Dv \circ D^s) + \frac{1}{2} (\text{tr} D) \text{tr}(\text{ad} v \circ D^s) = \frac{1}{2} \text{tr}((f(v)) \circ D^s),
\]

where we have used \(\text{tr}(\text{ad} v \circ D) = 0\) (see e.g. [4, Chapter 1, Section 5.5]). Since \(f\) is invertible, this implies that \(\text{tr}(\text{ad} w \circ D^*) = 0\) for all \(w\), so \(\tilde{g}\) is Einstein by Proposition 2.1. \(\Box\)
Example 2.5. Fix the Lie algebra \( g = (0, 0, e^{12}, 0) \), which is the direct sum of the Heisenberg Lie algebra and \( \mathbb{R} \); the notation, inspired by [27], means that \( g^* \) has a fixed basis of 1-forms \( e^1, e^2, e^3, e^4 \) with \( de^3 = e^1 \wedge e^2 \) and the other forms closed. Consider the two-parameter family of metrics \( g = ae^1 \circ e^2 + be^3 \circ e^4 \) and the derivation

\[
D = \begin{pmatrix}
-\frac{a}{4} & \lambda & 0 & 0 \\
-\frac{a}{8} & -\frac{b}{4} & 0 & 0 \\
0 & 0 & -\frac{b}{2} & -\frac{b}{3a^2 \mu \tau} \\
0 & 0 & 0 & -\frac{b}{4}
\end{pmatrix},
\]

where \( \lambda \) and \( \mu \) are nonzero parameters.

Then equation (4) is satisfied with \( \tau = -1 \). In this case

\[
D^s = \begin{pmatrix}
-\frac{a}{4} & \lambda & 0 & 0 \\
-\frac{a}{8} & -\frac{b}{4} & 0 & 0 \\
0 & 0 & \mu & -\frac{b}{3a^2 \mu \tau} \\
0 & 0 & 0 & -\frac{b}{4}
\end{pmatrix},
\]

hence \( \text{tr}(D^s) = 0 \) and \( \text{tr}((D^s)^2) = -\frac{9}{8} \mu^2 \), thus equation (4) becomes

\[
\text{Ric} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{b}{2a^2} \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

In order to obtain a standard Einstein metric, it is sufficient, thanks to Corollary 2.4, to show that \( \text{tr} \, D = 0 \) is not an eigenvalue. Since \( \mu \) is assumed not to be zero, \( D \) is not singular and 0 cannot be an eigenvalue.

As a particular case, consider solutions of (4) such that \( D^s = a \text{Id} \). The case \( a = 0 \) corresponds to a standard extension by a skew-symmetric derivation of a Ricci-flat metric, which by Proposition 1.1 yields a Ricci-flat metric isometric to a product with a line.

In the case \( a \neq 0 \), we have that \( D \) is a derivation in the Lie algebra

\[
\mathfrak{co}(r, s) = \mathfrak{so}(r, s) \oplus \text{Span} \{ \text{Id} \},
\]

where \( (r, s) \) is the signature of \( g \), and the inclusion \( \mathfrak{co}(r, s) \subset \mathfrak{gl}(g) \) is determined by fixing an orthonormal frame. Additionally, \( D \) has nonzero trace. This implies that the metric Nikolayevsky derivation \( N \) is nonzero. We proceed to recall the construction of \( N \), giving a slight generalization for use in later sections. For the proof, we refer to [23] and [9, Theorem 4.9].

Proposition 2.6. Let \( \mathfrak{h} \) be an algebraic subalgebra of \( \mathfrak{gl}(n, \mathbb{R}) \). There exists a semisimple derivation \( N \) in \( \mathfrak{h} \cap \text{Der} \, g \) such that

\[
\text{tr}(N \psi) = \text{tr} \, \psi, \quad \psi \in \mathfrak{h} \cap \text{Der} \, g.
\]

The derivation \( N \) is unique up to automorphisms of \( \mathfrak{h} \).

For \( \mathfrak{h} = \mathfrak{gl}(n, \mathbb{R}) \) the derivation \( N \) of Proposition 2.6 corresponds to the pre-Einstein or Nikolayevsky derivation introduced in [23]; accordingly, we will refer to the derivation \( N \) of Proposition 2.6 as the \( \mathfrak{h} \)-Nikolayevsky derivation. For \( \mathfrak{h} = \mathfrak{co}(r, s) \), the \( \mathfrak{h} \)-Nikolayevsky derivation is the metric Nikolayevsky derivation introduced in [9].
Notice that the $\mathfrak{h}$-Nikolayevsky derivation is zero if and only if all derivations in $\mathfrak{h}$ are traceless (i.e. $\mathfrak{h}$ is contained in $\mathfrak{sl}(n, \mathbb{R})$). In particular, we see that that there is derivation with $D^0 = \text{Id}$ if and only if the metric Nikolayevsky is nonzero.

In later sections, we will consider Lie algebras with an almost pseudo-Hermitian structure and use the $\mathfrak{cu}(p, q)$-Nikolayevsky derivation, where

$$\mathfrak{cu}(p, q) = \mathfrak{u}(p, q) \oplus \text{Span} \{\text{Id}\}.$$ 

Like the Nikolayevsky and the metric Nikolayevsky, the $\mathfrak{cu}(p, q)$-Nikolayevsky derivation turns out to have rational eigenvalues:

**Proposition 2.7.** Let $\mathfrak{g}$ be a Lie algebra with an almost pseudo-Hermitian structure. Then the $\mathfrak{cu}(p, q)$-Nikolayevsky derivation of $\mathfrak{g}$ has rational eigenvalues.

**Proof.** The proof follows [23] and [9, Theorem 4.9]. We can characterize elements of $\mathfrak{cu}(p, q)$ as elements of $\mathfrak{co}(2p, 2q)$ that commute with the complex structure $J$.

If $N$ is the $\mathfrak{cu}(p, q)$-Nikolayevsky, let $\mathfrak{g}^\mathbb{C} = \bigoplus \mathfrak{b}_t$ be the decomposition into eigenspaces and let $\pi_t: \mathfrak{g}^\mathbb{C} \to \mathfrak{b}_t$ denote the projections. Define

$$n = \left\{ \sum \nu_t \pi_t \mid \sum \nu_t \pi_t \in (\text{Der} \mathfrak{g} \cap \mathfrak{co}(2p, 2q))^\mathbb{C} \right\}.$$ 

Since $N$ commutes with $J$, each $\mathfrak{b}_t$ is $J$-invariant. Therefore, $J$ commutes with projections, and we can write 

$$n = \left\{ \sum \nu_t \pi_t \mid \sum \nu_t \pi_t \in (\text{Der} \mathfrak{g} \cap \mathfrak{h})^\mathbb{C} \right\}.$$ 

One can now proceed as in [9, Theorem 4.9] and show that $N$ is the unique element of $n$ such that $\text{tr} \bar{N} \psi = \psi$ for all $\psi \in n$, and its coefficients $\nu_t$ are rational numbers. \[\square\]

**Lemma 2.8.** Let $H$ be an algebraic subgroup of $\text{SO}(r, s)$ with Lie algebra $\mathfrak{h}$ and let $\mathfrak{g}$ be a nilpotent Lie algebra with a $H$-structure. If $D, D'$ are two elements of $(\mathfrak{h} \oplus \text{Span} \{\text{Id}\}) \cap \text{Der} \mathfrak{g}$ with the same trace, then the $H$-structures on $\mathfrak{g} \times_D \text{Span} \{e_0\}$ and $\mathfrak{g} \times_{D'} \text{Span} \{e_0\}$ are equivalent.

**Proof.** The Lie algebra $\mathfrak{t} = (\mathfrak{h} \oplus \text{Span} \{\text{Id}\}) \cap \text{Der} \mathfrak{g}$ is algebraic. Observe that that two commuting derivations of $\mathfrak{t}$ with the same trace determine equivalent extensions by Proposition 1.1, as their difference is in $\mathfrak{h} \cap \mathfrak{so}(p, q)$. We will use this fact repeatedly.

Denote by $\mathfrak{r}$ the radical of $\mathfrak{t}$. By [8], the fact that $\mathfrak{t}$ is algebraic implies that $\mathfrak{r}$ is also algebraic, and we can write $\mathfrak{r} = n \times \mathfrak{a}$, where $\mathfrak{a}$ is an abelian Lie algebra consisting of semisimple elements and $n$ is the nilradical. Since $\mathfrak{a}$ is abelian, any two derivations in $\mathfrak{a}$ with the same trace determine isometric extensions. Thus, we only need to show that for any $D \in \mathfrak{t}$ there is an element of $\mathfrak{a}$ determining an equivalent extension.

Since $\mathfrak{t}$ is algebraic, we can write $D = D_{ss} + D_n$, where $D_{ss}$ is semisimple, $D_n$ is nilpotent, and $[D_{ss}, D_n] = 0$. Since $D_n$ has trace zero, $D$ and $D_{ss}$ determine isometric extensions. Since $D_{ss}$ is semisimple, so are

$$\text{ad} D_{ss}: \mathfrak{t} \to \mathfrak{t}, \quad \text{ad} D_{ss}: \mathfrak{t}_0 \to \mathfrak{t}_0,$$

where we have set $\mathfrak{t}_0 = \mathfrak{r} \cap \mathfrak{so}(p, q)$. We can choose a decomposition 

$$\mathfrak{t} = \mathfrak{r} \oplus W,$$

where $W$ is contained in $\mathfrak{h}$ and $\text{ad} D_{ss}$-invariant. Indeed, it suffices to choose for $W$ an $\text{ad} D_{ss}$-invariant complement of $\mathfrak{t}_0 \cap \mathfrak{r}$ in $\mathfrak{t}_0$. 

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Accordingly, write \( D_{ss} = D_t + D_W \). Then
\[
[D_{ss}, D_W] = [D_t, D_W];
\]
the left-hand side belongs to the \( \text{ad} \, D_{ss} \)-invariant space \( W \), and the right-hand side to the ideal \( \mathfrak{r} \), so both must vanish.

Therefore, \( D_{ss} \) and \( D_t \) are commuting derivations with the same trace, and they determine equivalent extensions.

Using the Jordan decomposition in the algebraic Lie algebra \( \mathfrak{r} \), we see that \( D_t \) determines, up to equivalence, the same standard extension as its semisimple part. On the other hand, the latter is conjugate in \( \mathfrak{r} \) to an element of \( \mathfrak{a} \) by [20, Section 19.3]. The conjugation is realized by an element of the Lie group with Lie algebra \( \mathfrak{t} \) which can be assumed to have determinant one, and therefore by an element of \( H \).

**Theorem 2.9.** Let \( \mathfrak{g} \) be a nilpotent Lie algebra with a pseudo-Riemannian metric \( g \) such that the metric Nikolayevsky derivation \( N \) is nonzero. Then \( g \) is Ricci-flat and \( \mathfrak{g} \) has an Einstein standard extension \( \mathfrak{g} \rtimes \mathbb{R} \) or \( \mathfrak{g} \rtimes \text{Span} \{e_0\} \) according to whether \( a \) is zero or not.

**Proof.** Let \( D \) be a multiple of \( N \) such that \( D^* = \text{Id} \). Every metric of the form \( e^* g \) can be written as \( g(\exp(tD), \exp(tD^*)) \), i.e. it is related to \( g \) by an isomorphism. The Ricci tensor transforms accordingly; however, the Ricci tensor of \( e^* g \) coincides with that of \( g \), and this forces it to be zero. Then \( [D, D^*] = [D, 2\text{Id} - D] = 0 \) and (4) holds. In addition,
\[
\text{tr}(\text{ad} \, v \circ D^*) = \text{tr}(2 \text{ad} \, v - \text{ad} \, v \circ D) = 0,
\]
where \( \text{ad} \, v \) and \( \text{ad} \, v \circ D \) are traceless because \( \mathfrak{g} \) is nilpotent and \( D \) is a multiple of the Nikolayevsky derivation. Thus, Proposition 2.1 implies that \( \mathfrak{g} \rtimes \text{Span} \{e_0\} \) is Einstein.

We claim that replacing \( D \) with a nonzero multiple, say \( D' = kD \), has the effect of giving the same standard extension up to isometry and rescaling. Indeed, observe that \( \{\exp tD\} \) acts on the metric \( g \) by rescaling while leaving \( D \) unchanged. This means that the \( \tilde{g} = g + e^0 \otimes e^0 \) and \( \tilde{g}' = k^2 \tilde{g} + e^0 \otimes e^0 \) are isometric metrics on \( \tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\} \). Setting \( e_0' = ke_0 \), we can write \( \tilde{g}' = k^2 (g + (e^0)') \otimes (e^0)' \), and \( \tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\} \).

Now suppose that \( \mathfrak{g} \) has a standard Einstein extension with \( D^* = a \text{Id} \). In this case, if \( \mathfrak{g} \) has dimension \( n \) and \( D^* = a \text{Id} \), then \( [D, D^*] = 2[D^a, D^a] = 0 \) and (4) becomes
\[
\text{Ric} = \tau (-a^2 n \text{Id} + na^2 \text{Id}) = 0.
\]
If \( a = 0 \), \( D \) is skew-symmetric; by Proposition 1.1, we can assume \( D = 0 \) up to isometry, obtaining a direct product \( \mathfrak{g} \times \mathbb{R} \).

If \( a \neq 0 \), \( D \) has nonzero trace and the metric Nikolayevsky \( N \) is nonzero, so it too has nonzero trace. We already observed that rescaling \( N \) yields an isometric extension up to isometry. Therefore, we can assume that \( D \) and \( N \) have the same trace and conclude by Lemma 2.8. □

### 3 z-standard Sasaki-Einstein Lie algebras

In this section we study the Ricci curvature of z-standard Sasaki-Lie algebras, characterizing the Einstein metrics in terms of their Kähler reduction.
Let $\mathfrak{g}$ be a central extension of a nilpotent Lie algebra $\tilde{\mathfrak{g}}$, i.e.\[ 0 \to \mathbb{R}^k \to \mathfrak{g} \to \tilde{\mathfrak{g}} \to 0.\]

As vector spaces, $\mathfrak{g} = \tilde{\mathfrak{g}} \oplus \mathbb{R}^k$. Let $\{e_s\}$ be a basis of $\mathbb{R}^k$; the elements $\{e^s\}$ of the dual basis can be viewed as elements of $\tilde{\mathfrak{g}}^*$, and the Lie algebra structure of $\mathfrak{g}$ is entirely determined by $\tilde{\mathfrak{g}}$ and the exterior derivatives $\{de^s\}$. Explicitly,\[ [v, w]_\mathfrak{g} = [v, w]_\tilde{\mathfrak{g}} + \sum_s de^s(v, w)e_s, \quad v, w \in \mathfrak{g}.\]

**Lemma 3.1.** Let $\tilde{\mathfrak{g}}$ be a nilpotent Lie algebra with a metric $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{R}^k$, fix a metric of the form\[ g = \tilde{g} + \sum_s \epsilon_s e^s \otimes e^s, \quad \epsilon_s = \pm 1.\]

Then, for $v, w \in \tilde{\mathfrak{g}}$, the Ricci tensors of $g$ and $\tilde{g}$ are related by\[ \text{ric}(v, w) = \tilde{\text{ric}}(v, w) - \frac{1}{2} \sum_s \epsilon_s g(\nu_s, \nu_s), \quad \text{ric}(v, e_s) = \frac{1}{2} \epsilon_s g(\nu_s, \nu_s), \quad \text{ric}(e_s, e_t) = \frac{1}{2} \epsilon_s \epsilon_t g(\nu_s, \nu_t).\]

**Proof.** By construction, $\text{ad} v = \tilde{\text{ad}} v - \sum_s \nu_s \otimes e_s$. For one-forms $\alpha$ on $\tilde{\mathfrak{g}}$, zero-extended to $\mathfrak{g}$, we have $d\alpha = d\alpha$. We use the fact that the musical isomorphisms relative to $g$ and $\tilde{g}$ are compatible, so using [10, Proposition 2.1] we obtain\[ \text{ric}(v, w) = \frac{1}{2} g(\nu_s, \nu_s) - \frac{1}{2} g(\text{ad} v, \text{ad} w) = \frac{1}{2} g(\nu_s, \nu_s) - \frac{1}{2} g(\nu_s, \nu_s) - \frac{1}{2} g(\sum_s \nu_s \otimes e_s, \sum_t \nu_t \otimes e_t) = \tilde{\text{ric}}(v, w) - \frac{1}{2} \sum_s \epsilon_s g(\nu_s, \nu_s). \]

**Lemma 3.2.** The Ricci tensor of the metric on $\mathfrak{g}$ constructed in Proposition 1.2 is \[ \text{Ric}(v) = -2(\tau (\bar{\mathcal{D}}^s)^2 + \text{Id})v, \quad v \in \text{Span} \{b, \xi\}^\perp, \]
\[ \text{Ric}(b) = \tau \text{tr}((\mathcal{D}^s)^2)b - (\text{tr} \bar{\mathcal{D}})\xi, \quad \text{Ric}(\xi) = (2n - 2)\xi - \tau(\text{tr} \bar{\mathcal{D}})b.\]

where $\text{dim} \mathfrak{g} = 2n$.

**Proof.** Since $\tilde{\mathfrak{g}}$ is pseudo-Kähler and nilpotent, $\tilde{\text{ric}}$ is zero by [17, Lemma 6.3]. By Lemma 3.1, we have\[ \text{ric}(v, w) = -\frac{1}{2} g(\nu_s, \nu_s) - \frac{1}{2} g(\nu_s, \nu_s) = -\frac{1}{2} g(\bar{\mathcal{D}}^s(v), \bar{\mathcal{D}}^s(w)) - 2g(Jv, Jw) = -2g(JD^s(v), D^s(w)) - 2g(v, w).\]
Then
\[ \text{ric}(v, b) = \frac{1}{2} \tau g(dv^b, db^\flat) = \frac{1}{2} \tau g(db^\flat, \bar{D}\omega), \quad \text{ric}(v, \xi) = \frac{1}{2} \tau g(dv^b, d\eta) = \tau g(db^\flat, \omega), \]
\[ \text{ric}(b, b) = \frac{1}{2} g(db^\flat, db^\flat) = \frac{1}{2} g(\bar{D}\omega, \bar{D}\omega), \quad \text{ric}(b, \xi) = \frac{1}{2} g(db^\flat, d\eta) = g(\bar{D}\omega, \omega), \]
\[ \text{ric}(\xi, \xi) = \frac{1}{2} \tau g(d\eta, d\eta) = 2g(\omega, \omega). \]

We can simplify these formulae by observing that
\[ \bar{D}\omega(x, y) = -\omega(\bar{D}x, y) - \omega(x, \bar{D}y) = -\bar{g}(\bar{D}x, Jy) - \bar{g}(x, \bar{D}y) \]
\[ = -\bar{g}(x, (\bar{J}\bar{D} + \bar{D}^*J)y) = -\bar{g}(x, (\bar{D} + \bar{D}^*)Jy), \]
so we can view \( \bar{D}\omega \) as a \((1,1)\) tensor \((\bar{D}\omega)^{\sharp} = -(\bar{D} + \bar{D}^*)J\). Similarly, we have \( \omega^\sharp = J \). Then
\[ g(\omega, \omega) = \frac{1}{2} g(J, J) = n - 1, \]
\[ g(\omega, \bar{D}\omega) = \frac{1}{2} g(J, -(\bar{D} + \bar{D}^*)J) = \frac{1}{2} \text{tr}((\bar{D} + \bar{D}^*)J^2) = -\frac{1}{2} \text{tr}(\bar{D} + \bar{D}^*) = -\text{tr} \bar{D} = -\text{tr} \bar{D}, \]
\[ g(\bar{D}\omega, \bar{D}\omega) = \frac{1}{2} g((\bar{D} + \bar{D}^*)J, (\bar{D} + \bar{D}^*)J) = \frac{1}{2} \text{tr}((\bar{D} + \bar{D}^*)J)^2 = 2\text{tr}(\bar{D}^*)^2. \]

Finally, observe that \( \omega \) and \( \bar{D}\omega \) are \( d^* \)-closed, so (since \( \bar{g} \) is unimodular),
\[ g(dv^b, \omega) = g(v^b, d^*\omega) = 0, \quad g(dv^b, \bar{D}\omega) = g(v^b, d^*\bar{D}\omega) = 0. \]

Summing up,
\[ \text{ric}(v, w) = -2\tau g(\bar{D}^*(v), \bar{D}^*(w)) - 2g(v, w), \quad \text{ric}(v, b) = 0, \]
\[ \text{ric}(v, \xi) = 0, \quad \text{ric}(b, b) = \text{tr}((\bar{D}^*)^2), \]
\[ \text{ric}(b, \xi) = -\text{tr} \bar{D}, \quad \text{ric}(\xi, \xi) = (2n - 2). \quad \Box \]

**Lemma 3.3.** With the hypothesis of Proposition 1.2, the metric \( \bar{g} = g + \tau e^0 \otimes e^0 \) on \( \bar{g} = g \times_D \text{Span } \{e_0\} \) is Einstein if and only if
\[ \tau = -1, \quad \bar{D}^* = \pm \text{Id}, \quad h = \pm 2. \]

**Proof.** By Proposition 2.1, \( \bar{g} \) is Einstein if and only if
\[ \text{Ric} = \tau( -\text{tr}((\bar{D}^*)^2) \text{Id} + [\bar{D}^*, D^a] + (\text{tr} D) D^a), \quad \text{tr} (\text{ad } v \circ D^*) = 0, \quad v \in \mathfrak{g}. \]

We have
\[
D = \begin{pmatrix} \bar{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix}, \quad D^* = \begin{pmatrix} \bar{D}^* & 0 & 0 \\ 0 & h & -2 \\ 0 & 0 & 0 \end{pmatrix}, \]
\[
D^a = \begin{pmatrix} \bar{D}^a & 0 & 0 \\ 0 & h & -1 \\ 0 & -\tau & 0 \end{pmatrix}, \quad D^a = \begin{pmatrix} \bar{D}^a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\tau & 0 \end{pmatrix}.
\]
So

\[ [D^*, D^*] = \begin{pmatrix} h D^* - 2(D^*)^2 & 0 & 0 \\ 0 & 2\tau & h \\ 0 & -h\tau & -2\tau \end{pmatrix}. \]

Multiplying by \( \tau \) each side of (4) and using Lemma 3.2, we get

\[
\begin{pmatrix} -2(D^*)^2 - 2\tau I & 0 & 0 \\ 0 & \text{tr}((D^*)^2) & -\text{tr}(D D^*) \tau(2n-2) \\ 0 & -\tau \text{tr} D & \tau(2n-2) \end{pmatrix} = -(\text{tr}((D^*)^2) + h^2 + 2\tau) \begin{pmatrix} I & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
+ \begin{pmatrix} h D^* - 2(D^*)^2 & 0 & 0 \\ 0 & 2\tau & h \\ 0 & h\tau & -2\tau \end{pmatrix} + (\text{tr} D^* + h) \begin{pmatrix} D^* & 0 & 0 \\ 0 & h & -1 \\ 0 & -\tau & 0 \end{pmatrix},
\]

i.e.

\[
(\text{tr}((D^*)^2) + h^2) I = (\text{tr} D^* + 2h) D^*, \\
2\text{tr}((D^*)^2) = (\text{tr} D^*) h, \\
\tau(2n+2) = -(\text{tr}((D^*)^2) + h^2).
\]

If this system of equations holds, \( D^* \) is a multiple of the identity; setting \( \text{tr} D^* = \lambda \), so that \( \text{tr}((D^*)^2) = \frac{\lambda^2}{2n-2} \), we get

\[
\lambda = \pm(2n-2).
\]

So the system holds if and only if \( D^* = \pm I \) and \( h = \pm 2 \). This condition also implies \( \text{tr}(\text{ad} v \circ D^*) = 0 \) because \( \mathfrak{g} \) is unimodular and \( \text{tr}(\text{ad} v \circ D) = 0 \) by [4, Chapter 1, Section 5.5], proving the equivalence in the statement.

Remark 3.4. As observed in [13, Remark 5.2], changing the sign of \( h, D, e_0 \) and \( b \) yields an isometric metric. Therefore, we will only consider the case \( h = 2 \) and \( D^* = I \).

The construction of Proposition 1.2 can be specialized to the Sasaki-Einstein case as follows:

**Proposition 3.5.** Let \((\hat{\mathfrak{g}}, J, \omega)\) be a pseudo-Kähler nilpotent Lie algebra and let \( \hat{D} \) be an element of coder\((\hat{\mathfrak{g}}, \hat{\mathfrak{g}}) \) with \( \hat{D}^* = I \) and commuting with \( J \). If \( \hat{\mathfrak{g}} = \mathfrak{g} \oplus \text{Span}\{b, \xi\} \) is the central extension of \( \mathfrak{g} \) characterized by \( d \xi^* = 2\omega = db^* \), where \( \{b^*, \xi^*\} \) is the basis dual to \( \text{Span}\{b, \xi\} \), with the metric \( \hat{g} = g - b^* \otimes b^* + \xi^* \otimes \xi^* \), then the semidirect product \( \hat{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span}\{e_0\} \), where

\[
[e_0, x] = \hat{D} x, \quad [e_0, b] = 2b + 2\xi, \quad [e_0, \xi] = 0
\]

has a Sasaki-Einstein structure \((\phi, \eta, \xi, \hat{\mathfrak{g}})\) given by

\[
\hat{g} = g - e^0 \otimes e^0, \quad \phi(w) = J(w) - g(h, w)e_0, \quad \phi(e_0) = -b, \quad w \in \mathfrak{g}.
\]

**Proof.** We have \( \hat{D} \omega = D^* \omega = -2\omega \); applying Proposition 1.2 with \( h = 2 \) and \( \tau = -1 \) we obtain a Sasaki extension as in the statement, which is Einstein by Lemma 3.3.

**Example 3.6.** Let \( \hat{\mathfrak{g}} = \mathbb{R}^{2n-2} \), with

\[
J e_1 = e_2, \ldots, J e_{2n-3} = e_{2n-2}, \quad \omega = \epsilon_1 e_1^{12} + \cdots + \epsilon_{n-1} e_2^{2n-3,2n-2}, \quad \epsilon_i = \pm 1,
\]

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and set $D = \text{Id}$. We get

$$dc^* = db^* = 2\omega, \quad \text{ad} e_0 = 2b^* \otimes (b + \xi) + \sum e_i \otimes e_i.$$  

The extension $\tilde{g}$ has a basis $\{e_0, e_1, \ldots, e_{2n}\}$ such that

$$de^0 = 0,$$
$$de^i = e_i^0, \quad i = 1, \ldots, 2n - 2,$$
$$de^{2n} = de^{2n-1} = 2\varepsilon_1 e_2^{12} + \cdots + 2\varepsilon_{n-1} e_{2n-3}^{2n-2} + 2e^{2n-1,0},$$

and the Einstein-Sasaki metric is

$$g = \sum_{i=1}^{n-1} \varepsilon_i (e^{2i-1} \otimes e^{2i-1} + e^{2i} \otimes e^{2i}) - e^{2n-1} \otimes e^{2n-1} + e^{2n} \otimes e^{2n} - e^0 \otimes e^0.$$  

The quotient by $\xi = e_{2n}$ yields the Kähler Lie algebra

$$de^0 = 0,$$
$$de^i = e_i^0, \quad i = 1, \ldots, 2n - 2,$$
$$de^{2n-1} = 2\varepsilon_1 e_2^{12} + \cdots + 2\varepsilon_{n-1} e_{2n-3}^{2n-2} + 2e^{2n-1,0},$$

with the pseudo-Kähler-Einstein metric

$$g = \sum_{i=1}^{n-1} \varepsilon_i (e^{2i-1} \otimes e^{2i-1} + e^{2i} \otimes e^{2i}) - e^{2n-1} \otimes e^{2n-1} - e^0 \otimes e^0.$$  

When the $\varepsilon_i$ are equal to $-1$, this is the negative definite symmetric metric on the Iwasawa subgroup of $SU(1, n+2)$. Suggestively, this Lie group and metric appear as the fibre of quaternion-Kähler manifolds obtained via the $c$-map (see [16]).

Proposition 3.5 has a Kähler analogue:

**Corollary 3.7.** Let $(\tilde{g}, J, \omega)$ be a pseudo-Kähler nilpotent Lie algebra with nonzero metric Nikolayevsky derivation, and let $D$ be an element of $\text{coker}(\tilde{g}, \tilde{J})$ with $D^s = \text{Id}$. If $\tilde{g} = \tilde{g} \oplus \text{Span} \{b\}$ is the central extension of $g$ characterized by $DB^* = 2\omega$, where $\{b^*\}$ is the basis dual to $\text{Span} \{b\}$, with the metric $g = \tilde{g} - b^* \otimes b^*$, then the semidirect product $\tilde{g} = \tilde{g} \times \text{Span} \{e_0\}$, where

$$[e_0, x] = \tilde{D} x, \quad [e_0, b] = 2b$$  

has a pseudo-Kähler-Einstein structure $(\tilde{g}, \tilde{J}, \tilde{\omega})$ given by

$$\tilde{g} = g - e^0 \otimes e^0, \quad \tilde{J}(w) = J(w) - g(b, w)e_0, \quad \tilde{J}(e_0) = -b, \quad w \in \tilde{g},$$

with $\tilde{\text{ric}} = (2n + 2)\tilde{g}$, with $2n$ the dimension of $\tilde{g}$.

**Proof.** Take the Lie algebra constructed in Proposition 3.5 and take the quotient by $\xi$. Then by Proposition 1.3 it is Kähler-Einstein with $\text{ric} = (2n + 2)\tilde{g}$. $\square$

**Remark 3.8.** If the Lie algebra $\tilde{g}$ is not abelian, then Corollary 3.7 produces pseudo-Kähler-Einstein rank-one extensions which are not pseudo-Iwasawa, unlike the method presented in [26], where one constructs pseudo-Kähler-Einstein rank-one extensions of pseudo-Iwasawa-type.

Indeed, the derivation $D = \text{ad} e_0$ of Corollary 3.7 is self-adjoint with respect to the metric if and only if $D^s = \frac{1}{2}(D + D^*)$ is a derivation, but since $D^s = \text{Id}$, this happens only if the identity is a derivation, i.e. if $\tilde{g}$ is an abelian Lie algebra.

**Remark 3.9.** The pseudo-Kähler-Einstein quotient constructed in Example 3.6 is precisely the family of [26, Example 7.6], and since $\tilde{g}$ is abelian, this is consistent with the previous Remark 3.8.
4 Classification results

In this section we characterize 3-standard Sasaki-Einstein Lie algebras in terms of their reduction using the \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation introduced in Section 2. We also classify 3-standard Sasaki-Einstein Lie algebras of dimension \(\leq 7\).

**Theorem 4.1.** If \(\mathfrak{g} = \mathfrak{g} \rtimes \text{Span} \{e_0\}\) is a 3-standard Sasaki-Einstein Lie algebra, the \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation of its Kähler reduction is nonzero.

Conversely, if \(\mathfrak{g}\) is a pseudo-Kähler Lie algebra with nonzero \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation, it extends to a 3-standard Sasaki-Einstein Lie algebra \(\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\}\), uniquely determined up to equivalence.

**Proof.** If \(\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \text{Span} \{e_0\}\) is a 3-standard Sasaki-Einstein Lie algebra, Proposition 1.2 asserts that \(\tilde{\mathfrak{g}}\) can be realized as an extension of its Kähler reduction \(\mathfrak{g}\). By Proposition 3.5, \(\tilde{D}\) is a derivation commuting with \(J\) such that \(\tilde{D}^* = \text{Id}\). This implies that \(\tilde{D}\) is an element of

\[
\mathfrak{so}(2p,2q) \cap \mathfrak{gl}(p+q,\mathbb{C}) = \mathfrak{cu}(p,q)
\]

with nonzero trace; if such a \(\tilde{D}\) exists, the \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation is nonzero.

Now assume \(\tilde{\mathfrak{g}}\) is pseudo-Kähler and \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation is nonzero. By rescaling, we obtain a derivation \(\tilde{D}\) whose symmetric part is the identity; this yields a Sasaki-Einstein extension by Proposition 3.5.

To prove uniqueness, fix two derivations \(\tilde{D}, \tilde{D}'\) commuting with \(J\), \(\tilde{D}'^* = \text{Id} = (\tilde{D}')^*\). The Lie algebras \(\tilde{\mathfrak{g}} \rtimes \mathfrak{sp}(e_0)\) and \(\tilde{\mathfrak{g}} \rtimes \mathfrak{sp}(e_0)\) have a natural \(U(p,q)\)-structure. By Lemma 2.8, they are equivalent.

We can view \(\tilde{\mathfrak{g}}\) as a central extension \((\tilde{\mathfrak{g}} \rtimes \mathfrak{sp}(e_0)) \oplus \text{Span} \{b, \xi\}\), where \(db^*\) and \(d\xi^*\) are determined by the \(U(p,q)\)-invariant form \(\omega\). Therefore, \(\tilde{\mathfrak{g}}\) and its counterpart obtained using \(\tilde{D}'\) are equivalent.

In the case that \(\tilde{\mathfrak{g}}\) is abelian, we obtain:

**Corollary 4.2.** Every 3-standard Sasaki-Einstein Lie algebra such that the Kähler reduction is an abelian Lie algebra is equivalent to one of those constructed in Example 3.6.

**Proof.** If \(\tilde{\mathfrak{g}}\) is an abelian Lie algebra, we can assume \(\tilde{\mathfrak{g}} = \mathbb{R}^{2n−2}\), with

\[
Je_1 = e_2, \ldots, Je_{2n−3} = e_{2n−2}, \quad \omega = e_1e_2 + \cdots + e_{n−1}e_{2n−3}, e_i = \pm 1;
\]

the \(\mathfrak{cu}(p,q)\)-Nikolayevsky derivation is Id, so by Theorem 4.1 the extension is equivalent to one of those constructed in Example 3.6.

In dimension 3, 3-standard Sasaki-Einstein Lie algebras take the form \(\mathbb{R}^2 \rtimes \text{Span} \{e_3\}\), with \(\text{ad} e_3\) acting on \(\mathbb{R}^2\) as the identity. In dimension 5, 3-standard Sasaki-Einstein Lie algebras determine a reduction of dimension 2, which is abelian. Therefore, these metrics have the form given in Example 3.6, and we obtain:

**Proposition 4.3.** Let \(\tilde{\mathfrak{g}}\) be a 3-standard Sasaki-Einstein Lie algebra of dimension \(\leq 5\). Then \(\tilde{\mathfrak{g}}\) is equivalent to one of

\[
\begin{align*}
(2e_{13}, 2e_{13}, 0), & \quad \tilde{g} = -e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3, \\
(e_{15}, e_{25}, 2e_{12} + 2e_{35}, 2e_{12} + 2e_{35}, 0), & \quad \tilde{g} = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4 - e^5 \otimes e^5, \\
(e_{15}, e_{25}, -2e_{12} + 2e_{35}, -2e_{12} + 2e_{35}, 0), & \quad \tilde{g} = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4 - e^5 \otimes e^5.
\end{align*}
\]
Note that the 5-dimensional solvable Lie algebras appearing in Proposition 4.3 are isomorphic; up to a sign, the metric of signature (1, 4) is isometric to [14, Example 5.6].

In dimension 7, we can classify 3-standard Sasaki–Einstein Lie algebras by using the classification of four-dimensional Lie algebras with a pseudo-Kähler metric in [25]:

**Theorem 4.4.** Let \( \mathfrak{g} \) be a 3-standard Sasaki–Einstein Lie algebra of dimension 7. Then \( \mathfrak{g} \) is equivalent to one of the following:

1. \( \mathfrak{g} \) is the solvable Lie algebra

\[
(e^{17}, e^{27}, e^{37}, e^{47}, 2\epsilon_1 e^{12} + 2\epsilon_2 e^{34} + 2\epsilon_5 e^{57}, 2\epsilon_1 e^{12} + 2\epsilon_2 e^{34} + 2\epsilon_5 e^{57}, 0)
\]

with metric

\[
\tilde{g} = \epsilon_1(e^1 \otimes e^1 + e^2 \otimes e^2) + \epsilon_2(e^3 \otimes e^3 + e^4 \otimes e^4) + \gamma, \quad \epsilon_1, \epsilon_2 \in \{+1, -1\};
\]

2. \( \mathfrak{g} \) is the solvable Lie algebra

\[
\left(\frac{2}{3} e^{17}, \frac{2}{3} e^{27}, \frac{a}{3} e^{17} + \frac{4}{3} e^{37} + e^{12}, \frac{a}{3} e^{17} + \frac{4}{3} e^{47}, 2(e^{13} + e^{24} + ae^{12} + e^{57}), 2(e^{13} + e^{24} + ae^{12} + e^{57}, 0)\right)
\]

with metric

\[
\tilde{g} = -a(e^1 \otimes e^1 + e^2 \otimes e^2) + e^1 \otimes e^4 - e^2 \otimes e^3 + \gamma, \quad a \in \mathbb{R};
\]

3. \( \mathfrak{g} \) is the solvable Lie algebra

\[
\left(\frac{2}{3} e^{17}, \frac{2}{3} e^{27}, \frac{b}{3} e^{17} + \frac{4}{3} e^{37} + e^{12}, \frac{b}{3} e^{27} + \frac{4}{3} e^{47}, 2a(e^{13} + e^{24}) + 2(e^{14} - e^{23} + be^{12} + e^{57}), 2a(e^{13} + e^{24}) + 2(e^{14} - e^{23} + be^{12} + e^{57}, 0)\right)
\]

with metric

\[
\tilde{g} = -b(e^1 \otimes e^1 + e^2 \otimes e^2) + a(e^1 \otimes e^4 - e^2 \otimes e^3) - e^1 \otimes e^3 - e^2 \otimes e^4 + \gamma, \quad a, b \in \mathbb{R};
\]

where we have set \( \gamma = -e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7 \).

**Proof.** By Proposition 1.2, every 3-standard Sasaki Lie algebra can be obtained by extending a four-dimensional pseudo-Kähler Lie algebra \( \mathfrak{g} \). By the classification of [25], we have the following possibilities:

1. \( \mathfrak{g} \) is abelian; we can assume that the metric is either positive-definite or neutral. Then we obtain the Lie algebras of Example 3.6, i.e.

\[
\mathfrak{g} = (e^{17}, e^{27}, e^{37}, e^{47}, 2\epsilon_1 e^{12} + 2\epsilon_2 e^{34} + 2\epsilon_5 e^{57}, 2\epsilon_1 e^{12} + 2\epsilon_2 e^{34} + 2\epsilon_5 e^{57}, 0)
\]

with metric

\[
\tilde{g} = \epsilon_1(e^1 \otimes e^1 + e^2 \otimes e^2) + \epsilon_2(e^3 \otimes e^3 + e^4 \otimes e^4) - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7,
\]

where \( \epsilon_1, \epsilon_2 = \pm 1 \).
2. \( \tilde{g} = (0, 0, e^{12}, 0), \) with \( Je_1 = e_2, Je_3 = e_4, \) and \( \omega = e^{13} + e^{24} + ae^{12} \) for \( a \in \mathbb{R}. \) Then
\[
\tilde{g} = -a(e^1 \otimes e^1 + e^2 \otimes e^2) + e^1 \otimes e^4 - e^2 \otimes e^3.
\]
The generic \( \tilde{D} \) satisfying the hypothesis of Proposition 3.5 is
\[
\tilde{D} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 \\
\lambda & \frac{1}{3} & \frac{1}{3} & 0 \\
-\frac{1}{3} & \lambda & 0 & \frac{2}{3}
\end{pmatrix}.
\]
By Theorem 4.1, we can assume \( \lambda = 0. \) Therefore, we obtain the extension
\[
\tilde{g} = \left( \frac{2}{3} e^{17}, \frac{2}{3} e^{27}, \frac{a}{3} e^{27} + \frac{4}{3} e^{37} + e^{12}, -\frac{a}{3} e^{17} + \frac{4}{3} e^{47}, 2e^{13} + 2e^{24} + 2ae^{12} + 2e^{57}, 2e^{13} + 2e^{24} + 2ae^{12} + 2e^{57}, 0 \right)
\]
with the metric
\[
\tilde{g} = \tilde{g} - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7.
\]
3. \( \tilde{g} = (0, 0, e^{12}, 0) \) with \( Je_1 = e_2, Je_3 = e_4, \) and \( \omega = a(e^{13} + e^{24}) + e^{14} - e^{24} + be^{12} \) for \( a, b \in \mathbb{R}. \) Then
\[
\tilde{g} = -b(e^1 \otimes e^1 + e^2 \otimes e^2) + a(e^1 \otimes e^4 - e^2 \otimes e^3) - e^1 \otimes e^3 - e^2 \otimes e^4.
\]
The generic \( \tilde{D} \) satisfying the hypothesis of Proposition 3.5 is
\[
\tilde{D} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 \\
a\lambda + \frac{b}{3} & -\lambda & \frac{1}{3} & 0 \\
\lambda & a\lambda + \frac{b}{3} & 0 & \frac{2}{3}
\end{pmatrix}.
\]
Again, we may assume \( \lambda = 0 \) and obtain
\[
\tilde{g} = \left( \frac{2}{3} e^{17}, \frac{2}{3} e^{27}, \frac{b}{3} e^{17} + \frac{4}{3} e^{37} + e^{12}, \frac{b}{3} e^{27} + \frac{4}{3} e^{47}, 2a(e^{13} + e^{24}) + 2e^{14} - 2e^{24} + 2be^{12} + 2e^{57}, 2a(e^{13} + e^{24}) + 2e^{14} - 2e^{24} + 2be^{12} + 2e^{57}, 0 \right)
\]
with the metric
\[
\tilde{g} = \tilde{g} - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7.
\]

**Example 4.5.** Consider the 6-dimensional Lie algebra \( \mathfrak{g} = (0, 0, 0, e^{12}, e^{13}, e^{14} - e^{23}), \) denoted by \( \mathfrak{h}_{11} \) in [15]; by [27, 7], it admits a one-parameter family of complex structures. By the work of [15], we know that it has a four-dimensional space of compatible pseudo-Kähler metrics.

Instead of fixing the complex structure, we use the explicit form of the two families of pseudo-Kähler structures given in [28].

The first one is \( \omega_1 = e^{16} - \lambda e^{25} - (\lambda - 1) e^{34}, \) which has as compatible canonical complex structure
\[
J_1(e_2) = (1 + b)ae_1, \quad J_1(e_4) = ae_3, \quad J_1(e_6) = \left( \frac{1 + b}{b} \right) e_5
\]
and metric $g_1 = \omega_1 J_1$; while the second one is $\omega_2 = e^{16} + e^{24} - \frac{1}{2}(e^{25} - e^{34})$ which has as canonical complex structure compatible

$$J_2(e_2) = -ae_1, \quad J_2(e_3) = \frac{3}{2a} e_4 + \frac{3}{a} e_5, \quad J_2(e_4) = \frac{2}{3} ae_3 - \frac{1}{a} e_6, \quad J_2(e_6) = -2ae_5$$

and metric $g_2 = \omega_2 J_2$.

The first case, imposing $[D, J_1] = 0$ gives

$$D = \begin{pmatrix}
\frac{2\mu_2}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -a^2 \mu_2 (b + 1) & \frac{2\mu_1}{3} & 0 & 0 & 0 \\
\frac{\mu_2}{\mu} & \frac{2\mu_2}{\mu} & \mu_2 + \frac{2\mu}{\mu} & 0 & 0 & 0 \\
\frac{\mu_2}{\mu} & -a^2 \mu_2 (b + 1)^2 & \mu_2 + \frac{2\mu}{\mu} & -a^2 \mu_2 (b + 1) & \mu_1 & 0 \\
\mu_5 & \mu_4 & \mu_3 & \mu_2 & 0 & \mu_1
\end{pmatrix}$$

and imposing $D^* = \text{Id}$ gives $\mu_1 = \frac{3}{5}$ and $\mu_i = 0$ for $i = 2, \ldots, 5$, that is

$$D = \begin{pmatrix}
\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{3}{5} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{3}{5}
\end{pmatrix}.$$ 

On the other hand, $[D, J_2] = 0$ gives

$$D = \begin{pmatrix}
\frac{2\mu_1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2}{3}(2a^2 \mu_2 + \mu_1) & \frac{2\mu_1}{3} & 0 & 0 & 0 \\
2\mu_2 & -\frac{2}{3}(2a^2 \mu_3 + \mu_1) & \frac{2\mu_1}{3} & 0 & 0 & 0 \\
3\mu_2 & -\frac{2}{3}(2a^2 \mu_5 + 3\mu_2) & 3\mu_2 & -\frac{2}{3}(2a^2 \mu_3 + \mu_1) & \mu_1 & 0 \\
2(\mu_4 + 2\mu_3 + \frac{2\mu_5}{\mu_5}) & -2(2a^2 \mu_5 + 3\mu_2) & 3\mu_2 & -\frac{2}{3}(2a^2 \mu_3 + \mu_1) & \mu_1 & 0 \\
\mu_5 & \mu_4 & \mu_3 & \mu_2 & 0 & \mu_1
\end{pmatrix}$$

but imposing $D^* = \text{Id}$ does not yield any solution for the $\mu_i$.

**Example 4.6.** The following example shows a 3-standard Sasaki-Einstein $\tilde{g}$ obtained by extending a 6-dimensional pseudo-Kähler Lie algebra with a derivation $\tilde{D}$ which is not a multiple of the $\text{cu}(p, q)$-Nikolayevsky derivation. Consider the Lie algebra $\tilde{g} = (0, 0, e^{12}, 0, 0, 0)$ with symplectic form $\omega = e^{13} + e^{24} + e^{36}$ and complex structure $J(e_1) = e_2$, $J(e_3) = e_4$ and $J(e_5) = e_6$. Then

$$\tilde{D} = \begin{pmatrix}
\frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\
\mu & 0 & \frac{1}{2} & 0 & \lambda & -\nu \\
0 & \mu & 0 & \frac{1}{2} & \nu & \lambda \\
\nu & -\lambda & 0 & 0 & 1 & -\rho \\
\lambda & \nu & 0 & 0 & \rho & 1
\end{pmatrix}$$

satisfies the hypothesis of Proposition 3.5, and therefore determines a 3-standard Sasaki-Einstein $\tilde{g}$ Lie algebra of dimension 9. The derivation $\tilde{D}$ is not diagonalizable over $\mathbb{R}$, but has eigenvalues $(\frac{2}{3}, \frac{2}{3}, 1 - \nu, 1 + \nu, \frac{1}{2}, \frac{1}{2})$; therefore, $\tilde{D}$ is only a multiple of the $\text{cu}(p, q)$-Nikolayevsky derivation when $\rho$ is zero. Note, however, that all the resulting extensions are isometric by Theorem 4.1.
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