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Sylvie Monniaux

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NAVIER-STOKES EQUATIONS IN ARBITRARY DOMAINS: 
THE FUJITA-KATO SCHEME

SYLVIE MONNIAUX

Abstract. Navier-Stokes equations are investigated in a functional setting in 3D open sets \( \Omega \), bounded or not, without assuming any regularity of the boundary \( \partial \Omega \). The main idea is to find a correct definition of the Stokes operator in a suitable Hilbert space of divergence-free vectors and apply the Fujita-Kato method, a fixed point procedure, to get a local strong solution.

1. Introduction

Since the pioneering work by Leray [3] in 1934, there have been several studies on solutions of Navier-Stokes equations

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \nabla \pi + (u \cdot \nabla) u &= 0 \quad \text{in} \quad ]0, T[ \times \Omega, \\
\text{div } u &= 0 \quad \text{in} \quad ]0, T[ \times \Omega, \\
u &= 0 \quad \text{on} \quad ]0, T[ \times \partial \Omega, \\
u(0) &= \nu_0 \quad \text{in} \quad \Omega.
\end{align*}
\]

Fujita and Kato [2] in 1964 gave a method to construct so called mild solutions in smooth domains \( \Omega \), producing local (in time) smooth solutions of \((NS)\) in a Hilbert space setting. These solutions are global in time if the initial value \( \nu_0 \) is small enough in a certain sense. The case of non smooth domains has been studied by Deuring and von Wahl [1] in 1995 where they considered domains \( \Omega \subset \mathbb{R}^3 \) with Lipschitz boundary \( \partial \Omega \). They found local smooth solutions using results contained in Shen’s PhD thesis [4]. Their method does not cover the critical space case as in [2]. One of the difficulty there was to understand the Stokes operator, and in particular its domain of definition.

In Section 2, we give a “universal” definition of the Stokes operator, for any domain \( \Omega \subset \mathbb{R}^3 \) (Definition 2.3). In Section 3, we construct a mild solution of \((NS)\) with a method similar to Fujita-Kato’s [2] (Theorem 3.2) for initial values \( \nu_0 \) in the critical space \( D(A^+) \). We show in Section 4 that this mild solution is a strong solution, i.e. \((NS)\) is satisfied almost everywhere.

2. The Stokes operator

Let \( \Omega \) be an open set in \( \mathbb{R}^3 \). The space

\[
L^2(\Omega)^3 = \{ u = (u_1, u_2, u_3); u_i \in L^2(\Omega), \ i = 1, 2, 3 \}
\]

endowed with the scalar product

\[
\langle u, v \rangle = \int_{\Omega} u \cdot v = \sum_{i=1}^{3} \int_{\Omega} u_i \overline{v_i}
\]

is a Hilbert space. Define \( G = \{ \nabla p; p \in L^2_{loc}(\Omega) \text{ and } \nabla p \in L^2(\Omega)^3 \} \).

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the set $\mathcal{G}$ is a closed subspace of $L^2(\Omega)^3$. Let
\[ \mathcal{H} = \mathcal{G}^\perp = \{ u \in L^2(\Omega)^3; \langle u, \nabla p \rangle = 0, \forall p \in H^1(\Omega) \} . \]
The space $\mathcal{H}$, endowed with the scalar product $\langle \cdot, \cdot \rangle$ is a Hilbert space. We have the following Hodge decomposition
\[ L^2(\Omega)^3 = \mathcal{H} \oplus \mathcal{G} . \]
We denote by $\mathbb{P}$ the projection from $L^2(\Omega)^3$ onto $\mathcal{H}$; $\mathbb{P}$ is the usual Helmholtz projection. We denote by $J$ the canonical injection $\mathcal{H} \hookrightarrow L^2(\Omega)^3$: $J = \mathbb{P}$ (see the adjoint of $J$) and $\mathbb{P} J$ is the identity on $\mathcal{H}$. Let now $\mathcal{D}(\Omega)^3 = \mathcal{C}_c^\infty(\Omega)^3$ and
\[ \mathcal{D} = \{ u \in \mathcal{D}(\Omega)^3; \text{div} \, u = 0 \} . \]

It is clear that $\mathcal{D}$ is a closed subspace of $\mathcal{D}(\Omega)^3$. We denote by $J_0 : \mathcal{D} \hookrightarrow \mathcal{D}(\Omega)^3$ the canonical injection: $J_0 \subset J$. Let $\mathbb{P}_1$ be the adjoint of $J_0 : \mathbb{P}_1 = J_0' : \mathcal{D}(\Omega)^3 \to \mathcal{D}'$. We have $\mathbb{P}_1 \subset \mathbb{P}$. The following theorem characterizes the elements in $\ker D$.

**Theorem 2.1 (de Rham).** Let $T \in \mathcal{D}(\Omega)^3$ such that $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$. Then there exists $S \in (\mathcal{C}_c^\infty(\Omega))'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in (\mathcal{C}_c^\infty(\Omega))'$, then $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$.

We denote by $H^1_0(\Omega)$ the closure of $\mathcal{D}(\Omega)^3$ with respect to the scalar product $\langle u, v \rangle = \langle u, v \rangle_1 = \langle u, v \rangle + \sum_{i=1}^3 \langle \partial_i u, \partial_i v \rangle$. By Sobolev embeddings, we have $H^1_0(\Omega)^3 \hookrightarrow L^6(\Omega)^3$. Define
\[ V = \mathcal{H} \cap H^1_0(\Omega)^3 . \]
The space $V$ is a closed subspace of $H^1_0(\Omega)^3$; endowed with the scalar product $\langle \cdot, \cdot \rangle_1$, $V$ is a Hilbert space. The canonical injection $\tilde{J} : V \hookrightarrow H^1_0(\Omega)^3$ is the restriction of $J$ to $V$. Let $H^{-1}(\Omega)^3 = (H^1_0(\Omega)^3)'$; $\mathbb{P}_1$ maps $H^{-1}(\Omega)^3$ to $V'$; the restriction of $\mathbb{P}_1$ to $H^{-1}(\Omega)^3$ is $\mathbb{P}$, the adjoint of $\tilde{J}$. On $V \times V$ we define now the form $a$ by
\[ a(u, v) = \sum_{i=1}^3 \langle \partial_i \tilde{J} u, \partial_i \tilde{J} v \rangle : a \text{ is a bilinear, symmetric, } \delta + a \text{ is a coercive form on } V \times V \text{ for all } \delta > 0, \text{ then defines a bounded self-adjoint operator } A_0 : V \to V' \text{ by } (A_0 u)(v) = a(u, v) \text{ with } \delta + A_0 \text{ invertible for all } \delta > 0 .

**Proposition 2.2.** For all $u, v \in V$, $A_0 u = \tilde{P}(-\Delta_D^B) \tilde{J} u$, where $\Delta_D^B$ denotes the Dirichlet-Laplacian on $H^1_0(\Omega)^3$.

**Proof.** For all $u, v \in V$, we have
\[ (A_0 u)(v) \overset{(1)}{=} a(u, v) \overset{(2)}{=} \sum_{i=1}^3 \langle \partial_i \tilde{J} u, \partial_i \tilde{J} v \rangle \overset{(3)}{=} \langle (-\Delta_D^B) \tilde{J} u, \tilde{J} v \rangle_{H^{-1}, H^1_0} \overset{(4)}{=} \langle \tilde{P}(-\Delta_D^B) \tilde{J} u, v \rangle_{V', V} . \]
The first two equalities come from the definition of $A_0$ and $a$. The third equality comes from the definition of the Dirichlet-Laplacian on $H^1_0(\Omega)^3$ and the fact that for $v \in V$, $\tilde{J} v = v$. The last equality is due to $\tilde{J} \varphi = \tilde{P} \varphi$ in $V'$ for all $\varphi \in H^{-1}(\Omega)^3$. This shows that $A_0 u$ and $\tilde{P}(-\Delta_D^B) \tilde{J} u$ are two continuous linear forms on $V$ which coincide on $V$, they are then equal.

**Definition 2.3.** The operator $A$ defined on its domain $D(A) = \{ u \in V; A_0 u \in \mathcal{H} \}$ by $A u = A_0 u$ is called the Stokes operator.
Remark that particular, \( \langle \delta u, v \rangle_{H^1} \) endowed with the norm and therefore is bounded on \((0, T)\). Let \( \langle \delta + A_0 \rangle^{-\frac{1}{2}}P_1 \) maps \( L^2(\Omega)^3 \) into \( \mathcal{H} \).

3. Mild solution to the Navier-Stokes system

Let \( T > 0 \).

Define the following Banach space

\[
\mathcal{E}_T = \left\{ u \in C^0([0, T]; D(A^\frac{1}{2} \cap C^1([0, T]; D(A^\frac{1}{2}))) \right\}
\]

such that

\[
\sup_{0 < t < T} \| s^\frac{1}{2} A^\frac{1}{2} u(s) \|_H + \sup_{0 < t < T} \| s A^\frac{1}{2} u'(s) \|_H < \infty
\]

endowed with the norm

\[
\| u \|_{\mathcal{E}_T} = \sup_{0 < t < T} \| s^\frac{1}{2} A^\frac{1}{2} u(s) \|_H + \sup_{0 < t < T} \| s^\frac{1}{2} A^\frac{1}{2} u'(s) \|_H.
\]

Let \( \alpha \) be defined by \( \alpha(t) = e^{-tA}u_0 \) where \( u_0 \in D(A^\frac{1}{2}) \). Then \( \alpha \in \mathcal{E}_T \). Indeed, it is clear that \( \alpha \in C^0([0, T]; D(A^\frac{1}{2})) \). We also have that \( t^\frac{1}{2}A^\frac{1}{2}\alpha(t) = t^\frac{1}{2}A^\frac{1}{2}e^{-tA}A^\frac{1}{2}u_0 \) is bounded on \((0, T)\) since \( e^{-tA} \rangle \leq 0 \) is an analytic semigroup. Moreover, one has \( \alpha'(t) = -Ae^{-tA}u_0 \) which yields to \( tA^\frac{1}{2}\alpha'(t) = -tAe^{-tA}A^\frac{1}{2}u_0 \) continuous on \([0, T], \) bounded in \( \mathcal{H} \). For \( u, v \in \mathcal{E}_T \), we define now

\[
\Phi(u, v)(t) = \int_0^t e^{-(t-s)A}\left(-\frac{1}{2}P_1\right)((u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s))ds,
\]

\( 0 < t < T \).

Proposition 3.1. The transform \( \Phi \) is bilinear, symmetric, continuous from \( \mathcal{E}_T \times \mathcal{E}_T \) to \( \mathcal{E}_T \) and the norm of \( \Phi \) is independent of \( T \).

Proof. The fact that \( \Phi \) is bilinear and symmetric is clear. Moreover, \( \Phi(u, v) = e^{-tA}f \) where \( f \) is defined by

\[
f(s) = (\frac{1}{2}P_1)((u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s)), \quad s \in [0, T].
\]

For \( u, v \in \mathcal{E}_T \), it is clear that \( (u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s) \in L^2(\Omega)^3 \) and therefore \( (\delta + A_0)^{\frac{1}{2}}f(s) \in \mathcal{H} \) with

\[
\sup_{0 < s < T} s^\frac{1}{2} \| (\delta + A_0)^{\frac{1}{2}}f(s) \|_H \leq c\| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T}.
\]

We have then

\[
\Phi(u, v) = e^{-A}f = (\delta + A)^{\frac{1}{2}}e^{-A}((\delta + A_0)^{\frac{1}{2}}f)
\]

and therefore

\[
\| A^\frac{1}{2}\Phi(u, v)(t) \|_H \leq \int_0^t \| A^\frac{1}{2}(\delta + A)^{\frac{1}{2}}e^{-(t-s)A}f(s) \|_H ds
\]

\[
\leq c \left( \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} ds \right) \| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T}
\]

\[
\leq c \left( \int_0^\infty \frac{1}{\sqrt{1-\sigma}} \frac{1}{\sqrt{\sigma}} d\sigma \right) \| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T}
\]

\[
\leq c \| u \|_{\mathcal{E}_T} \| v \|_{\mathcal{E}_T} .
\]
Continuity with respect to $t \in [0,T]$ of $t \mapsto \mathcal{A}^T \Phi(u,v)(t)$ is clear once we have proved the boundedness. We also have
\[
\|A^T \Phi(u,v)(t)\|_H \leq \int_0^t \|A^T (\delta + A)^{\frac{t}{2}} e^{-((t-s)A)}\|_{\mathcal{L}(H)} \| (\delta + A_0)^{-\frac{t}{2}} f(s)\|_H ds \\
\leq c \left( \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} \sqrt{s}} \, ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq ct^{-\frac{1}{2}} \left( \int_0^1 \frac{1}{(1-\sigma)^{\frac{1}{2}} \sqrt{\sigma}} \, d\sigma \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq ct^{-\frac{1}{2}} \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]
Continuity with respect to $t \in [0,T]$ is clear once we have proved the boundedness. To prove the last part of the norm of $\Phi(u,v)$ in $\mathcal{E}_T$, we have for $s \in [0,T]$ \[
f'(s) = (-\frac{1}{2} \mathcal{P}_1)((u'(s) \cdot \nabla)v(s) + (u(s) \cdot \nabla)v'(s) + (v'(s) \cdot \nabla)u(s) + (v(s) \cdot \nabla)u'(s))
\]
and therefore
\[
\sup_{0 < s < T} \|s^\frac{1}{2} (\delta + A_0)^{-\frac{t}{2}} f'(s)\|_H \leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]
We have
\[
\Phi(u,v)(t) = \int_0^t e^{-sA} f(t-s) ds + \int_0^t e^{-((t-s)A)} f(s) ds \quad t \in [0,T],
\]
and therefore
\[
\Phi(u,v)'(t) = e^{-\frac{1}{2} A} f(t) + \int_0^\frac{1}{2} (\delta + A)^{\frac{1}{2}} e^{-sA} (\delta + A_0)^{-\frac{1}{2}} f'(t-s) ds \\
+ \int_0^{\frac{1}{2}} -A(\delta + A)^{\frac{1}{2}} e^{-(t-s)A} (\delta + A_0)^{-\frac{1}{2}} f(s) ds,
\]
which yields
\[
\|A^T \Phi(u,v)'(t)\|_H \leq c \|\delta + A_0\|_H \|f(t)\|_H + c \left( \int_0^{\frac{1}{2}} \frac{1}{s^{\frac{1}{2}} (t-s)^{\frac{1}{2}}} \, ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
+ c \left( \int_0^{\frac{1}{2}} \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \, ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T} \\
\leq c \left( \int_0^{\frac{1}{2}} \frac{ds}{(t-s)^{\frac{1}{2}} s^{\frac{1}{2}}} \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
\]
This last inequality ensures that $\Phi(u,v) \in \mathcal{E}_T$ whenever $u,v \in \mathcal{E}_T$. \hfill \Box

**Theorem 3.2.** For all $u_0 \in D(A^\frac{1}{2})$, there exists $T > 0$ such that there exists a unique $u \in \mathcal{E}_T$ solution of $u = \alpha + \Phi(u,u)$ on $[0,T]$. This function $u$ is called the mild solution to the Navier-Stokes system.

**Proof.** Let $T > 0$. Since $\Phi : \mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$ is bilinear continuous, it suffices to apply Picard fixed point theorem, as in [3]. The sequence in $\mathcal{E}_T$ $(v_n)_{n \in \mathbb{N}}$ defined by $v_0 = \alpha$ as first term and
\[
v_{n+1} = \alpha + \Phi(v_n,v_n), \quad n \in \mathbb{N}
\]
converges to the unique solution $u \in \mathcal{E}_T$ of $u = \alpha + \Phi(u,u)$ provided $\|A^T u_0\|_H$ is small enough ($\|\alpha\|_{\mathcal{E}_T} < \frac{1}{\|\mathcal{A}^T\|_{\mathcal{L}(\mathcal{E}_T \times \mathcal{E}_T , \mathcal{E}_T)}}$). In the case where $\|A^T u_0\|_H$ is not small...
(that is, if $\|\alpha\|_{\mathcal{E}_T} \geq \frac{1}{4\|\Phi\|_{L^2(\mathcal{E}_T \times \mathcal{E}_T)}}$) then for $\varepsilon > 0$, there exists $u_{0,\varepsilon} \in D(A)$ such that $\|A^\frac{1}{2}(u_0 - u_{0,\varepsilon})\|_{\mathcal{H}} \leq \varepsilon$. If we take as initial value $u_{0,\varepsilon} \in D(A)$, we have $\|\alpha\|_{\mathcal{E}_T} \leq c_T \|\Phi\|_{L^2(\mathcal{E}_T \times \mathcal{E}_T)}$. Therefore, we can find $T > 0$ such that $\|\alpha\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{L^2(\mathcal{E}_T \times \mathcal{E}_T)}}$. □

4. Strong solutions

Let $u$ be the mild solution to the Navier-Stokes system. We show in this section that $u$ in fact satisfies the equations of the Navier-Stokes system in an $L^p$-sense (for a suitable $p$). To begin with, we know that $u \in \mathcal{E}_T$ and satisfies

$$u = \alpha + \Phi(u, u) = \alpha + e^{-A \cdot \varphi(u)},$$

where $\varphi(u) = -P_1((u \cdot \nabla)u)$ and we have $\|t^\frac{1}{2}(u(t) \cdot \nabla)u(t)\|_{L^2} \leq c\|u\|_{\mathcal{E}_T}^2$. Therefore, we get

$$(4.1) \quad u(0) = \alpha(0) = u_0,$$

$$(4.2) \quad \text{div}u(t) = 0 \text{ in the } L^2 - \text{sense for } t \in ]0, T],$$

and

$$u' + Au = f \quad \text{in } \mathcal{C}([0, T]; \mathcal{V}),$$

which means that for all $t \in ]0, T]$, $P_1 (u'(t) - \Delta_D u(t) + (u(t) \cdot \nabla)u(t)) = 0$.

Then, by Theorem [2], there exists $(-\pi)(t) \in (\mathcal{C}_0^\infty(\Omega))^3$ such that $\nabla\pi(t) \in H^{-1}(\Omega)^3$ and

$$(4.3) \quad \nabla(-\pi)(t) = u'(t) - \Delta_D u(t) + (u(t) \cdot \nabla)u(t)$$

and we have for $0 < t < T$

$$-\Delta_D u(t) + \nabla\pi(t) = -u'(t) - (u(t) \cdot \nabla)u(t) \in L^3(\Omega)^3 + L^2(\Omega)^3.$$

The equation (4.3), together with (4.1) and (4.2), give the usual Navier-Stokes equations which are fulfilled in a strong sense (a.e.) where we consider the expression $-\Delta u + \nabla\pi$ undecoupled.

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