Non-stationary resonance dynamics of weakly coupled pendula

L. I. Manevitch\textsuperscript{1} and F. Romeo\textsuperscript{2}

\textsuperscript{1} Institute of Chemical Physics, Russian Academy of Science - Kosygin str. 4, Moscow, Russia
\textsuperscript{2} Department of Structural and Geotechnical Engineering, Sapienza University of Rome
Via Gramsci 53, Rome, Italy

received 11 August 2015; accepted in final form 2 November 2015
published online 25 November 2015

PACS 05.45.-a – Nonlinear dynamics and chaos
PACS 63.20.Pw – Localized modes

Abstract – In this letter we fill the gap in understanding the non-stationary Hamiltonian dynamics of the weakly coupled pendula model having significant applications in numerous fields of physics. While common knowledge of this model is predominantly based on the stationary theory and quasi-linear approach to non-stationary dynamics, we consider a strongly nonlinear system without any polynomial approximation of the anharmonic potential. In the adopted asymptotics only closeness to any inter-pendulum resonance frequency is assumed. Being able to explore the whole diapason of initial conditions, two key nonlinear features are revealed by means of the Limiting Phase Trajectories concept: the conditions of intense energy exchange between the pendula and transition to energy localization. The roots and the domain of chaotic behavior are clarified as they are associated with the latter, purely non-stationary, topological transition.

Copyright © EPLA, 2015

Introduction. – The model of coupled pendula and some of its modifications play a significant role in mechanics [1], solid-state physics [2] including superconducting Josephson junctions [3–5], photonics, including Bose-Einstein condensates [6], biophysics, including DNA functioning [7]. The majority of the studies in all these fields relate to stationary Hamiltonian dynamics or its extension to damped and forced models [8] (and references therein) in which several peculiar dynamical phenomena can arise, such as synchronised [9] (and references therein) and chimera [10] (and references therein) states. They are based on the system fundamental regimes, namely the Nonlinear Normal Modes (NNMs) in finite systems [11] and solitons (breathers) in infinite models [1,2]. As for non-stationary processes, NNMs can also be used for their description provided that the intermodal resonance is absent [12]. However, the non-stationary resonance dynamics of finite systems turns out to be much more complicated. Therefore, only isolated and predominantly numerical results were obtained in this field [13]. The recently developed concept of Limiting Phase Trajectories (LPTs) allowed for a systematic analytical approach to describe non-stationary resonance regimes in quasi-linear approximations [14–17]. This concept introduces a fundamental non-stationary process of new type which corresponds to the maximum possible energy exchange between the oscillators (here pendula) or clusters of oscillators. In essence, the LPTs play in the non-stationary resonance dynamics of finite systems a role similar to that of the NNMs in the stationary theory and in the study of non-stationary yet non-resonant regimes. In terms of LPT, the transition from intense energy exchange between some clusters of oscillators (coherence domains), in particular, weakly coupled pendula, to energy localization in the initially excited cluster (oscillator) can also be predicted [13,18].

However, all existing analytical results in the non-stationary resonance dynamics of finite-dimensional systems were so far related to perturbations of some asymptotic limits which are either linear systems [14], as mentioned above, or separated oscillators or pendula [13]. Moreover, the results obtained in the latter case were mostly qualitative. As far as numerical approaches are concerned, they have also been extensively adopted; however, their benefit was limited since they could only be interpreted by means of the mentioned analytical approximations.

In this study we apply the LPT concept to weakly coupled pendula. The sine-type nonlinearity of the considered model differs significantly from the polynomial nonlinearity of the previously studied coupled oscillators models. As a matter of fact, willing to consider arbitrary oscillations amplitude of the pendula, polynomial approximations of power series expansions of the nonlinear term are
not applicable. This difficulty is overcome by resorting to a modified asymptotic procedure in which only the closeness to any possible inter-pendulum resonance frequency is required in the framework of multiple scale expansions [12,19]. It is shown that such an extension is crucial for revealing the main topological transitions in the large amplitude dynamics and for predicting the necessary conditions for chaotic regimes onset. The analytical findings are confirmed by numerical simulations based on direct integration of the starting equations and construction of Poincaré sections.

For the sake of clarity we will discuss a mechanical interpretation of the problem in terms of two weakly coupled pendula undergoing planar motion. The corresponding dimensionless equations of motion can be written as

\[ \frac{d^2 q_j}{dt^2} + \sin q_j + \varepsilon \beta (q_j - q_{3-j}) = 0, \quad j = 1, 2, \quad (1) \]

where \( q_j \) is the angular coordinate of the \( j \)-th pendulum with unit mass and length \( l \); \( \tau_0 = \omega_0 t \), \( \omega_0 = \sqrt{g/l} \) is its linear natural frequency, \( g \) is the gravitational acceleration, \( \varepsilon \beta \) is the coupling parameter with \( \varepsilon \ll 1 \). As known, these equations describe also a particular case of the two Josephson junctions (two-junction interferometer) [3] as well as of the Frenkel-Kontorova models, having numerous applications in solid-state physics and photonics [2,6].

Dealing with pendula oscillations under internal 1 : 1 resonance conditions, we rewrite eqs. (1) in the form

\[ \frac{d^2 q_j}{dt^2} + \omega^2 q_j + \varepsilon \beta (q_j - q_{3-j}) + \varepsilon \mu (\sin q_j - \omega^2 q_j) = 0, \quad j = 1, 2 \quad (2) \]

where \( 0 < \omega \leq 1 \) (the lower limit will be specified in the following) is the resonance oscillation frequency. Under internal 1 : 1 resonance conditions the combination of two terms in the second brackets has to be small (we suppose of order \( \varepsilon \)), and \( \mu = \varepsilon^{-1} \) is a book-keeping parameter. Thus, we assume the closeness to resonance but we do not impose any restrictions to the oscillations amplitude and the ensuing resonance frequency.

**Main asymptotic approach.** — Passing to complex variables, given by \( \varphi_j = (\frac{dq_j}{dt} + i \omega q_j) e^{i \omega \tau_0} \); \( \varphi_j^* = (\frac{dq_j}{dt} - i \omega q_j) e^{-i \omega \tau_0} \), we apply the two-scale expansion procedure [12]. Accordingly, \( \varphi_j = \varphi_j(\tau_0, \tau_1) + \varepsilon \varphi_j(1, \tau_0, \tau_1) + \ldots \), in which \( \tau_1 = \varepsilon \tau_0 \) is a slow-time scale, then taking into account that \( \frac{d^2 q_j}{dt^2} = \frac{d^2 q_j}{d\tau_0^2} + \varepsilon \frac{d^2 q_j}{d\tau_1^2} \) and substituting in eq. (2), we come to a set of partial differential equations corresponding to different-order approximations with respect to parameter \( \varepsilon \). In the zeroth-order approximation it turns out that \( \frac{d^2 q_j}{d\tau_0^2} = 0 \) and, consequently, \( \varphi_j = \varphi_j(\tau_0) \). Therefore, in the first-order approximation, we come to ordinary differential equations for functions \( \varphi_j(\tau_0) \) with respect to fast \( \tau_0 \) leading to the main asymptotic equations in slow-time scale with respect to \( \varphi_{j,0} \).

\[ \frac{d \varphi_{j,0}}{d\tau_1} + \frac{i \beta}{\omega} (\varphi_{j,0} - \varphi_{3-j,0}) + \mu \left[ \frac{i}{2\omega} \varphi_{j,0} + \sum_{n=1}^{\infty} (-1)^n \frac{\alpha_n}{\omega^{2n+1}} \left( \frac{\varphi_{j,0}}{\omega} \right)^n \right] = \frac{i \omega}{2} \varphi_{j,0} \quad (3) \]

in which \( \alpha_n = 1/(2\alpha_n) \) and \( \alpha_n \) is obtained by the following recurrence relations: \( \alpha_n = b_n \alpha_{n-1} \), with \( \alpha_0 = 1 \), and \( b_n = b_{n-1} + n \), with \( b_0 = 0 \).

Contrary to the original system (1) which possesses only one integral, eqs. (3) admit two integrals of motion:

\[ H = \left| \frac{i \beta}{\omega} + \mu \left( \frac{i \omega}{2} - \frac{i \omega}{2\omega} \right) \right|^2 \sum_{j=1}^{2} |\varphi_{j,0}|^2 \]

\[ N = \sum_{j=1}^{2} |\varphi_{j,0}|^2 \quad (4) \]

Taking into account the second integral in eq. (5), we set \( \varphi_{1,0} = \sqrt{N} \cos \theta(\tau_1) e^{i \phi(\tau_1)} \), \( \varphi_{2,0} = \sqrt{N} \sin \theta(\tau_1) e^{i \beta \theta(\tau_1)} \); then, the first integral, eq. (4), after series summation, takes the following form:

\[ H = \frac{\beta}{\omega} \cos \Delta \sin 2\theta + \frac{\mu}{2N\omega} \left\{ N - 8\omega^2 + 4\omega^2 \left| J_1 (k_2 \cos \theta) + J_0 (k_2 \sin \theta) \right|^2 \right\} \quad (6) \]

where \( \Delta = \delta_2 - \delta_1 \) and \( k_2 = \sqrt{N}/\omega \). The corresponding equations of motion can be written as follows:

\[ \dot{\theta} = -\frac{\beta}{\omega} \sin \Delta \quad (7) \]

\[ \Delta \sin 2\theta = \frac{2\beta}{\omega} \cos 2\theta \cos \Delta \]

\[ = -\frac{2\mu}{\sqrt{N}} \left| J_1 (k_2 \cos \theta) \sin \theta - J_1 (k_2 \sin \theta) \cos \theta \right| \quad (8) \]

where \( J_0() \), \( J_1() \) are Bessel functions of the first kind and the derivative with respect to \( \tau_1 \) is considered. This system describes the slow dynamics of weakly coupled pendula for arbitrary initial conditions.

Having obtained the integral (6) and equations of motions (7), (8) we have the possibility to perform the analytical study of the considered problem. We begin by analyzing the phase portrait described by equation \( H = \text{const} \). It is convenient to present the evolution of the \( \Delta-\theta \) phase plane by changing the parameter \( \varepsilon \), characterising inter-pendulum coupling, for a given value of the resonance frequency that determines also the maximum value
Non-stationary resonance dynamics of weakly coupled pendula

Local and global dynamic transitions. – There exist two dynamical transitions, relating stationary and highly non-stationary dynamics, that are manifested by changing the parameter $\varepsilon$; the phase portraits shown in fig. 1 highlight the topological changes associated with these transitions. The stationary points in fig. 1(a) correspond to NNMs of the considered system; the closed phase trajectories surrounding them imply weak energy exchange between the pendula. The Limiting Phase Trajectory encircling all trajectories corresponds to a full energy exchange between the pendula. While the threshold of the first transition corresponds to instability and bifurcation of the in-phase NNM, the second transition can be determined by means of the linearized version of eqs. (7), (8):

$$\varepsilon = \frac{1}{4\beta} \left[ -J_0 \left( \frac{k_2}{\sqrt{2}} \right) + 2\omega \sqrt{\frac{2}{N}} J_1 \left( \frac{k_2}{\sqrt{2}} \right) + J_2 \left( \frac{k_2}{\sqrt{2}} \right) \right]$$

Fig. 1: (Color online) Evolution of the $\Delta-\theta$ phase portrait for $\omega = 0.65$ and maximum angles $q_{j,\text{max}} \simeq 3\pi/4$ for decreasing parameter $\varepsilon$; LPTs (red), separatrix (blue). (a) Before the first transition, $\varepsilon = 0.2$; (b) after the first transition, $\varepsilon = 0.1$; (c) second transition, $\varepsilon = 0.0695$; (d) after the second transition, $\varepsilon = 0.05$.

$q_{j,\text{max}}$ of the pendulum oscillation angle $q_j$. In fig. 1 the phase portrait corresponding to $\omega = 0.65$ ($q_{j,\text{max}} \simeq 3\pi/4$) and $\beta = 1.0$ is shown.

The analytical conditions for both transitions are reported below and are confirmed by the numerical solution of eqs. (1). For the first transition prediction we resort to the solution of eqs. (7), (8) in the vicinity of the stationary point $\Delta = 0$, $\theta = \pi/4$ (in-phase NNM) as $\Delta = \Delta_1$, $\theta = \pi/4 + \theta_1$. Assuming that $\Delta_1$ and $\theta_1$ are small perturbations, the solution can be determined by means of the linearized version of eqs. (7), (8):

$$\dot{\Delta}_1 = \frac{4\beta}{\omega} - \sum_{k,j} \left[ J_0 \left( \frac{k_2}{\sqrt{2}} \right) - 2\mu \sqrt{\frac{2}{N}} J_1 \left( \frac{k_2}{\sqrt{2}} \right) \right] \theta_1.$$

From the latter equations (9), (10) it can be seen that the instability of in-phase NNM occurs when the coefficient of $\theta_1$ is equal to zero. The latter condition leads to the following expression for the first transition threshold:

$$\varepsilon = \frac{1}{4\beta} \left[ -J_0 \left( \frac{k_2}{\sqrt{2}} \right) + 2\omega \sqrt{\frac{2}{N}} J_1 \left( \frac{k_2}{\sqrt{2}} \right) + J_2 \left( \frac{k_2}{\sqrt{2}} \right) \right].$$

(11)

after which the phase portrait becomes qualitatively similar to the one shown in fig. 1(b). To reveal the condition for the occurrence of the second transition we derive the equation describing LPTs by considering that they possess the point $\Delta = 0$, $\theta = \pi/4$ as $\Delta = \Delta_2$, $\theta = \pi/4 + \theta_2$. The Limiting Phase Trajectory encircling all trajectories corresponds to a full energy exchange between the pendula. While the threshold of the first transition is given by eq. (11) (curve I in fig. 2(a)), the condition of the second transition can be found by taking into account that its occurrence implies that LPT possesses the unstable stationary point $\Delta = 0$, $\theta = \pi/4$. Therefore, we get the sought threshold for the second transition (eq. II in fig. 2(a)), namely

$$\varepsilon = \frac{2\omega^2}{\beta N} \left[ 1 + J_0 \left( k_2 \right) - 2J_0 \left( \frac{k_2}{\sqrt{2}} \right) \right].$$

(12)

30005-p3
It is worth emphasizing that the described scenario is observed not only for small angles (quasi-linear case) but also for values of the angles close to \( \pi \). However, the threshold values of the parameters corresponding to both transitions change strongly with the resonance frequency, as shown in fig. 2(a). The perfect agreement between analytical prediction of curves I, II and their numerical counterparts \( \Gamma^*, \Pi^* \) is observed for \( 0 < q \lesssim 3\pi/4 \) (\( 0 < \omega \lesssim 0.65 \)). The top horizontal axis labels refer to the amplitude \( q \) corresponding to the values of the resonance frequency \( \omega \) reported on the bottom horizontal axis. It can be seen that the lowest value considered for the latter resonance frequency is \( \omega = 0.2 \) (\( q_{\text{max}} = 3.14 \)); this lower limit stems from the resonance assumption which implies the existence of two time scales. Moreover, both boundaries shown in fig. 1(a) refer to the parameter \( \varepsilon \) ranging from 0 to 0.2, in agreement with our initial assumption concerning the smallness of the sum in the second bracket in eqs. (2). The quantitative difference for larger angles (which reaches at most 10\% and 20\% for curve I and II, respectively) can be reduced by considering the next-order approximation in the multiple-scale expansion procedure. The structure of the phase plane depicted in fig. 2(b) allows to predict the qualitative difference in the temporal behavior of the angle variable \( \theta \). The trajectories situated far from the separatix correspond to almost straight lines. However, due to the restriction \( 0 \leq \theta \leq \pi/2 \), they become saw-tooth type functions. The analytical solution of the problem in terms of non-smooth functions can be obtained after the change of temporal variable through the procedure proposed in [20,21] and used for the study of non-stationary resonance processes in [15,16]. It is worth emphasizing that fig. 2(a) clarifies the relation among the obtained results and conventional approximations used for the description of coupled pendula dynamics: the quasi-linear approaches can be applied only in the right part of the parametric plane (\( 0 < q \lesssim \pi/4 \)), whereas the independent pendula approximation holds only for the bottom part of the parametric plane (\( 0 < \varepsilon \lesssim 0.03 \)). On the contrary, the proposed approach based on resonance asymptotics turns out to be valid for the description of regular motion in all the parametric plane (\( \omega - \varepsilon \)).

**Poincaré sections.** – The dependence on the resonance frequency (or initial angle) of the thresholds corresponding to both dynamical transitions singles out in parametric space the domains of regular motion of the pendula. As will be shown, all regular motions revealed in the main asymptotic approximation as well as the analytical predictions of both dynamic transitions are confirmed by direct numerical integration of the starting equations of motion (1). However, it must be underlined that, contrary to the asymptotic approximation, the initial system is not integrable. Therefore, it is of interest to clarify the onset of chaotic behavior and the role played by LPTs in the general behavior of the pendula. Towards this goal, Poincaré sections constructed on the basis of the starting equations of motion (1) are reported in this section. In fig. 3 and fig. 4 Poincaré sections are shown for maximum angles \( q_{j,\text{max}} \simeq 3\pi/4 \) and \( q_{j,\text{max}} \simeq 9\pi/10 \), respectively. The four sections correspond to different dynamic regimes (see fig. 1) for decreasing values of \( \varepsilon \), according to the points highlighted in fig. 2(a). Figure 3(a) and fig. 4(a) refer to the dynamics before the first transition; the LPTs (red curve) encircling the in-phase NNM are also depicted. Figure 3(b) and fig. 4(b) refer to the case between the two transitions, where the new stationary states born as a result of instability of in-phase NNM can be seen; moreover, the associated homoclinic separatix encircles the corresponding stationary points. Figure 4(c) and fig. 5(c) reflect the conditions at the second transition, where LPT becomes a separatix; as indicated in fig. 2(a), the manifestation of chaotic behavior can be observed in the vicinity of the second dynamic transition for large enough angles.
Non-stationary resonance dynamics of weakly coupled pendula

Fig. 3: (Color online) Poincaré sections for $\omega = 0.65$ and angles $q_{j,\text{max}} \simeq 3\pi/4$ for decreasing parameter $\epsilon$: LPTs (red), separatrix (blue). (a) Before the first transition, $\epsilon = 0.2$; (b) after the first transition, $\epsilon = 0.1$; (c) second transition, $\epsilon = 0.0695$; (d) after the second transition, $\epsilon = 0.05$.

Fig. 4: (Color online) Poincaré sections for $\omega = 0.482$ and angles $q_{j,\text{max}} \simeq 9\pi/10$ for decreasing parameter $\epsilon$: LPTs (red), separatrix (blue). (a) Before the first transition, $\epsilon = 0.225$; (b) after the first transition, $\epsilon = 0.145$; (c) second transition, $\epsilon = 0.104$; (d) after the second transition, $\epsilon = 0.03$.

$(q_{j,\text{max}} \simeq 9\pi/10$, fig. 4(c)). In fig. 3(d) and fig. 4(d) the localized LPTs as well as the heteroclinic separatrix (blue curve) are well seen; it is worth noticing that, for small enough $\epsilon$, the motion remains regular in all the phase space (see also fig. 2(a)).

Examples of the pendula oscillations temporal evolution corresponding to the parameters considered in fig. 3 and fig. 4 are reported in fig. 5 and fig. 6. The reported responses are obtained by direct numerical integration of the initial dimensionless equations of motion (1). More specifically, fig. 5 refers to the case $\omega = 0.65$ and the maximum angles $q_{j,\text{max}} = 3/4\pi$; differently, in fig. 6, $\omega = 0.482$ and $q_{j,\text{max}} = 9/10\pi$. The numerical integrations confirm the analytical predictions reported in fig. 1 and in fig. 2(a). In particular, in fig. 5(a) and fig. 6(a) the pendula regular response is characterized by nonlinear beatings. The latter involves slow- and fast-time scales and implies the energy exchange between the pendula. A qualitative similar response can be observed in fig. 5(b) and fig. 6(b) which represent the pendula behavior after the first (local) topological transition. The response shown in fig. 5(c) corresponds to the second (global) topological transition occurring at $\epsilon = 0.0695$ for $q_{j,\text{max}} = 3/4\pi$. As for $q_{j,\text{max}} = 9/10\pi$, the LPT chaotization arising in the neighborhood of this threshold can be seen in fig. 6(c) where random intervals of energy exchange can be observed. This behavior is in agreement with the Poincaré section shown in fig. 4(c). As the value of $\epsilon$ decreases below the second threshold, the pendula behavior is characterized by predominant energy localization in the initially excited pendulum as shown in fig. 5(d) and fig. 6(d). This is also in full agreement with the analytical predictions and the numerical evidence provided by the Poincaré sections shown in fig. 3(d) and fig. 4(d).
While only regular motion is observed for angles less than 135°, for larger angles signs of chaotization appear in the vicinity of the second dynamic transition triggered by heteroclinic chaos, as anticipated in the discussion of fig. 2(a). As the maximum oscillation angle grows, the chaotic region in the parameter space increases on both sides of the second transition threshold. Then, for angles greater than 170°, this region approaches the first dynamic transition threshold where homoclinic chaos occurs. The absence of chaotization for maximum oscillation angles smaller than 135° means that the system is close to integrable separated pendula. For larger angles the system is far enough from being integrable and chaos appears. Such behavior is caused by the interaction of the dynamic separatix, coinciding with LPT in the conditions of the second dynamic transition, and the conventional pendulum separatix.

Conclusions. – Summarizing, the analytical description of highly non-stationary Hamiltonian dynamics in a system of weakly coupled pendula without any restrictions on the amplitude of oscillations was presented. It is shown that such processes can be adequately described by LPTs corresponding to the maximum possible energy exchange between pendula. These regimes encircle the domains of regular motion which are determined for all initial angles in the oscillation dynamic regime. The threshold coupling governing the transition from intense inter-pendulum energy exchange to predominant energy localization in one of the two pendula was identified. The manifestation of chaotic behavior in the considered model is also shown to be strongly connected with this, purely non-stationary, dynamic transition.

The obtained results can be applied in a variety of fields where the model of weakly coupled pendula plays a basic role. For example, in the application to Josephson junctions these results correspond to the lowest-order approximation in the presence of both damping and external forces that could be taken into account in the next-order approximation [3]. Therefore, the revealed dynamical regimes can be experimentally verified and exploited in numerous applications of Josephson junctions.

By considering the simplest case of two weakly coupled pendula, we pave the way for new opening possibilities of significant extensions in both fundamental and applied directions. The results can indeed be naturally extended to non-homogeneous systems as well as to chains with an arbitrary finite number of pendula. Moreover, damped, forced and self-sustained oscillations can be also considered within the proposed analytical framework.

***

LM would like to acknowledge the RFBR grant 14-01-00284 and the Sapienza University of Rome visiting professors grant 2014.

REFERENCES

[1] Scott A., Nonlinear Science (Oxford University Press, New York) 2003.
[2] Braun O. M. and Kivshar Y. S., The Frenkel-Kontorova Model (Springer-Verlag, Berlin, Heidelberg) 2004.
[3] Likharev K. K., Dynamics of Josephson Junctions and Circuits (Gordon and Breach Science Publishers, Amsterdam) 1986.
[4] Hadley P., Beasley M. R. and Wiensfeld K., Phys. Rev. B, 38 (1988) 8712.
[5] Braun O. M., Surf. Sci., 250 (1990) 262.
[6] Cataliotti F. S., Burger S., Fort C., Maddaloni P., Minardi F., Trombettoni A., Smerzi A. and Inguscio M., Science, 293 (2001) 843.
[7] Yakushevich L. V., Savin A. V. and Manevitch L. I., Phys. Rev. E, 66 (2002) 016614.
[8] Golubitsky M. and Stewart I., in Geometry, Mechanics, and Dynamics, edited by Newton P., Holmes P. and Weinstein A. (Springer, New York) 2002, p. 243.
[9] Olusola O. I., Vincent U. E. and Njah A. N., J. Sound Vib., 329 (2010) 443.
[10] Kapitaniak T., Kuzma P., Wojewoda J., Czolczynski K. and Maistrenko Y., Sci. Rep., 4 (2014) 6379.
[11] Vakakis A. F., Manevitch L. I., Mikhailin Y. V., Pilipchuk V. N. and Zevin A. A., Normal Modes and Localization in Nonlinear Systems (Wiley, New York) 1996.
[12] Nayfeh A. H. and Mook D. T., Nonlinear Oscillations (Wiley, New York) 2008.
[13] Sepulchre R., in Localization and Energy Transfer in Nonlinear Systems, edited by Velazquez L., MacKay R. S. and Zorzano M. P. (World Scientific, Singapore) 2003, p. 102.
[14] Manevitch L. I. and Smirnov V. V., Phys. Rev. E, 82 (2010) 036602.
[15] Manevitch L. I., Arch. Appl. Mech., 77 (2007) 301.
[16] Manevitch L. I., Kovaleva M. A. and Pilipchuk V. N., EPL, 101 (2013) 50002.
[17] Smirnov V. V., Shepelev D. S. and Manevitch L. I., Phys. Rev. Lett., 113 (2014) 135502.
[18] Aubry S., Kopidakis G., Morgeante A. M. and Tsironis G. B., Physica B, 296 (2001) 222.
[19] Kuehn C., Multiple Time Scale Dynamics (Springer, New York) 2015.
[20] Pilipchuk V. N., J. Sound Vib., 192 (1996) 43.
[21] Pilipchuk V. N., Nonlinear Dynamics: Between Linear and Impact Limits (Springer-Verlag, Berlin) 2011.