AN INDECOMPOSABLE AND UNCONDITIONALLY SATURATED BANACH SPACE

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Abstract. We construct an indecomposable reflexive Banach space $X_{ius}$ such that every infinite dimensional closed subspace contains an unconditional basic sequence. We also show that every operator $T \in B(X_{ius})$ is of the form $\lambda I + S$ with $S$ a strictly singular operator.

1. Introduction

The aim of this paper is to present a Banach space which is not the sum of two infinite dimensional closed subspaces $Y$, $Z$ with $Y \cap Z = \{0\}$ and every closed subspace of it contains an unconditional basic sequence. We shall denote this space as $X_{ius}$. W.T. Gowers’ famous dichotomy, [G3], provides an alternative description of this space. Namely $X_{ius}$ is an Indecomposable Banach space not containing any Hereditarily Indecomposable (H.I.) subspace. The problem of the existence of such spaces was posed by H.P. Rosenthal and it is stated in [G2]. The interest for such spaces arises from the coexistence of conditional (indecomposable) and unconditional (unconditionally saturated) structure on them. This is a free translation of W.T.Gowers’ comments before stating the problem of the existence of such spaces in [G2] (Problem 5.11). We should mention that Indecomposable spaces which are not H.I. are already known. For example, [AF] provides reflexive H.I. spaces $X$ such that $X^*$ contains an unconditional basic sequence. The methods used in [AF] do not seem to be able to provide H.I. spaces $X$ with $X^*$ unconditionally saturated.

The space presented in this paper is built following ideas used for the construction of H.I. Banach spaces. The method we follow is an adaptation of [AD] constructions as they were extended in [AT]. Both are variations of the fundamental discovery of W.T. Gowers and B. Maurey, [GM]. In our case we use as an unconditional frame a mixed Tsirelson space $T[(A_{n_j}, \frac{1}{m_j})_j]$ which is a space sharing similar properties with Th. Schlumprecht’s space $S$, [S]. The norming set $K$ of the space $X_{ius}$ is a subset of the unit ball of the dual of $T[(A_{n_j}, \frac{1}{m_j})_j]$. The only difference that the space $X_{ius}$ has from a corresponding construction of a H.I. space concerns the definition of the special functionals. The key observation that changing the special functionals one could obtain interesting non H.I. spaces is due to W.T.Gowers and it was used for the solution of important and long standing problems in the theory of Banach space, [G].

For the space $X_{ius}$ we need the special functionals to be defined such that the following geometric property holds in the space. For every $Y = \langle e_n \rangle_{n \in M}$, $M \in [\mathbb{N}]$, and $(e_n)_{n \in \mathbb{N}}$ the natural basis of $X_{ius}$, the quotient map $Q : X_{ius} \rightarrow X_{ius}/Y$ is strictly singular. This is equivalent to say that $\text{dist}(S_Z, S_Y) = 0$ for all $Z$ infinite dimensional subspace of $X_{ius}$. This property clearly holds in the case of H.I. spaces. In our case we define the special

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functionals such that the aforementioned property holds and on the other hand we have attempted to keep the dependence inside of each special functional as small as possible. Thus going deeper in the structure of any subspace of $X_{ius}$ the action of the special functionals becomes negligible, which permits us to find unconditional basic sequences. Another property of the space $X_{ius}$ concerns the bounded linear operators. Namely every $T : X_{ius} \to X_{ius}$ is of the form $T = \lambda I + S$, where $S$ is strictly singular. Thus $X_{ius}$ is not isomorphic to any of its proper subspaces.

2. Definition of the space $X_{ius}$

We shall use the standard notation. Thus $c_{00}$ denotes the linear space of all eventually zero sequences and for $x \in c_{00}$ we denote by $\text{supp } x = \{ n : x(n) \neq 0 \}$ and by $\text{range}(x)$ the minimal interval of $\mathbb{N}$ containing $\text{supp } x$. Also for $x, y \in c_{00}$ by $x < y$ we mean that $\max \text{supp } x < \min \text{supp } y$. We shall also use the standard results from the theory of bases of Banach spaces as they are described in [LT].

We choose two strictly increasing sequences $(n_j)_j$, $(m_j)_j$ of positive integers, such that

(i) $m_1 = 2$ and $m_{j+1} = m_j^3$

(ii) $n_1 = 4$ and $n_{j+1} = (4n_j)^{n_j}$ where $2^{n_j} \geq m_{j+1}^3$.

Let $Q$ be the set of scalars sequences with finite nonempty support, rational coordinates and maximum at most 1 in modules. We also set

$$Q_s = \{ (x_1, f_1, \ldots, x_n, f_n) : x_i, f_i \in Q, i = 1, \ldots, n \} \cup \{ \text{range}(x_i) \cup \text{range}(f_i) < \text{range}(x_{i+1}) \cup \text{range}(f_{i+1}) : i < n \}.$$

We consider a coding function $\sigma$ (i.e. $\sigma$ is an injection) from $Q_s$ to the set $\{ 2j : j \in \mathbb{N} \}$ such that for every $\phi = (x_1, f_1, \ldots, x_n, f_n) \in Q_s$

(2.1) $\sigma(x_1, f_1, \ldots, x_{n-1}, f_{n-1}) < \sigma(x_1, f_1, \ldots, x_n, f_n)$

(2.2) $\max \{ \text{range}(x_n) \cup \text{range}(f_n) \} \leq m_{\sigma(\phi)}^4$

Although $x_i, f_i$ are elements of $c_{00}$ their role in the space $X_{ius}$ we shall define is quite different. Namely $x_i$ will be elements of the space itself and $f_i$ elements of its dual $X_{ius}^*$. For similar reasons we shall denote the standard basis of $c_{00}$ either by $(e_n)_n$ or $(e_n^*)_n$.

Definition 2.1. A sequence $\phi = (x_1, f_1, \ldots, x_{2k}, f_{2k}) \in Q_s$ is said to be a special sequence of length $2k$ provided that

(2.3) $x_1 = \frac{1}{n_{2j}} \sum_{l=1}^{n_{2j}} e_{1,l}, \quad f_1 = \frac{1}{m_{2j}} \sum_{l=1}^{n_{2j}} e_{1,l}^*, \quad \text{for some } j \in \mathbb{N}, \text{ such that } m_{2j}^{1/2} > 2k,$

where $(e_{1,l})_{l=1}^{n_{2j}}$ is a subset of the standard basis of $c_{00}$ of cardinality $n_{2j}$, and for every $1 \leq i < k$, setting $\phi_i = (x_1, f_1, \ldots, x_i, f_i)$

(2.4) $\|f_{2i}\| \leq \frac{1}{m_{\sigma(\phi_{2i-1})}}, \quad |f_{2i}(x_{2i})| \leq \frac{1}{m_{\sigma(\phi_{2i-1})}},$

(2.5) if $i < k$ then $x_{2i+1} = \frac{1}{n_{\sigma(\phi_{2i})}} \sum_{l=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,l}, \quad f_{2i+1} = \frac{1}{m_{\sigma(\phi_{2i})}} \sum_{l=1}^{n_{\sigma(\phi_{2i})}} e_{2i+1,l},$

where for every $i \geq 1$, $(e_{2i+1,l})_{l=1}^{n_{\sigma(\phi_{2i})}}$ is a subset of the standard basis of $c_{00}$ of cardinality $n_{\sigma(\phi_{2i})}$. 

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The norming set of the space $X_{ius}$.

The norming set $K$ will be equal to the union $\bigcup_{n=0}^{\infty} K_n$ and the sequence $(K_n)_n$ is increasing and inductively defined. The inductive definition of $K_n$ goes as follows:

We set
\[ K_n^0 = K_0 = \{ \pm e_n^* : n \in \mathbb{N} \} \quad \text{and} \quad K_n^j = \emptyset \text{ for } j = 1, 2, \ldots. \]
Assume that $K_{n-1} = \bigcup_j K_{n-1}^j$ has been defined. Then we set,
(a) for $j \in \mathbb{N}$
\[ K_{n+1}^{2j} = K_{n+1}^{2j} \cup \{ \frac{1}{m_{2j}} \sum_{i=1}^{d} f_i : d \leq n_{2j}, f_1 < \ldots < f_d, f_i \in K_{n-1} \}. \]
(b) For $j \in \mathbb{N}$ and every $\phi = (x_1, f_1, \ldots, x_{n_{2j+1}}, f_{n_{2j+1}})$ special sequence of length $n_{2j+1}$, (see Definition 2.1), such that $f_{2i} \in K_{n-1}^{\sigma(\phi_{2i-1})}$ for $i = 1, \ldots, n_{2j+1}/2$ (where $\phi_{2i-1} = (x_1, f_1, \ldots, x_{2i-1}, f_{2i-1})$) we define the set
\[ K_{n+1}^{2j+1} = \left\{ \frac{\pm 1}{m_{2j+1}} E(\lambda f_{2i} f_1 + f_{2i} + \ldots + \lambda f_{2j+1}, f_{n_{2j+1}} + f_{n_{2j+1}}) : \right\} \]
(2.6) $E$ interval of $\mathbb{N}$, supp$f_{2i} = \text{supp } f_{2i}$, $f_{2i} \in K_{n-1}^{\sigma(\phi_{2i-1})}$,
\[ |g(x_{2i})| \leq \frac{1}{m_{\sigma(\phi_{2i-1})}} \text{ for all } g \in K_{n-1}^{\sigma(\phi_{2i-1})} \]
\[ \lambda f_{2i} = f_{2i}(m_{\sigma(\phi_{2i-1})} x_{2i}) \text{ if } f_{2i}(x_{2i}) \neq 0, \quad \frac{\pm 1}{n_{2j+1}} \text{ otherwise}. \]

We define
\[ K_{n+1}^{2j+1} = \bigcup \{ K_{n+1}^{2j+1} : \phi \text{ is a special sequence of length } n_{2j+1} \} \bigcup K_{n-1}^{2j+1}, \]
and finally we set
\[ K_n = \bigcup_j K_n^j. \]

This completes the inductive definition of $K_n$ and we set,
\[ K = \bigcup_n K_n. \]

Let us observe that the set $K$ satisfies the following properties

(i) It is symmetric and for each $f \in K$, $\|f\|_{\infty} \leq 1$.
(ii) It is closed under interval projections (i.e. it is closed in the restriction of its elements on intervals).
(iii) It is closed under the $(A_{n_2}, \frac{1}{m_{n_2}})$ operations (i.e. for $f_1 < f_2 < \cdots < f_d$ in $K$
\[ \text{with } d \leq n_{2j} \text{ we have that } \frac{1}{m_{n_2}} \sum_{i=1}^{d} f_i \in K. \])
(iv) If $f \in K$ then either $f = \pm e_n^*$ or $f \in K_n^j$ for $n \geq 1, j \in \mathbb{N}$. In the later case we define the weight of $f$ as $w(f) = m_j$. Note that $w(f)$ is not necessarily unique.

The space $X_{ius}$ is the completion of the space $(c_{00}, \| \cdot \|_K)$ where
\[ \|x\|_K = \sup \{ \langle f, x \rangle : f \in K \}. \]

From the definition of the norming set $K$ it follows easily that $(e_n)_n$ is a bimonotone basis of $X_{ius}$. Also it is easy to see, using (iii), that the basis $(e_n)_n$ is boundedly complete.
Indeed, for \( x \in c_00 \) and \( E_1 < E_2 < \cdots < E_{n_2j} \) intervals of \( \mathbb{N} \) it follows from property (iii) of the norming set that,

\[
\|x\| \geq \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} \|E_i x\|. 
\]

Also from the choice of the sequences \((n_i)_i, (m_i)_i\), it follows that \( \frac{n_{2j}}{m_{2j}} \) increases to infinity.

To prove that the space \( X_{ius} \) is reflexive we need to show that the basis is shrinking. This requires some further work and we will present the argument later.

**Lemma 2.2.** Let \( \phi = (x_1, f_1, \ldots, x_{n_{2j}+1}, f_{n_{2j}+1}) \) be a special sequence of length \( n_{2j+1} \) such that:

(a) \( \{f_i : i = 1, \ldots, n_{2j+1}\} \subset K \) and for \( i \geq 2 \), \( w(f_i) = m_{\sigma(\phi_{i-1})} \).

(b) \( \text{For } 1 \leq i \leq n_{2j+1}/2, \|w(f_{2i})x_{2i}\| \leq 1. \)

Then there exists \( n \in \mathbb{N} \) such that \( K_{n,\phi}^{2j+1} \) is nonempty.

**Notation.** For every \( \phi \) special sequence of length \( n_{2j+1} \) such that \( K_{n,\phi}^{2j+1} \) is strictly singular operator. Moreover we keep the dependence only between \( f_{2i-1} \) and the family \( \{g \in K : w(g) = w(f_{2i}) \} \) to ensure that the space \( X_{ius} \) is unconditionally saturated.

**Definition 2.4** (The tree \( T_f \) of a functional \( f \in K \)). Let \( f \in K \). We call tree of \( f \) (or tree corresponding to the analysis of \( f \)) every finite family \( T_f = (f_\alpha)_{\alpha \in A} \) indexed by a finite tree \( A \) with a unique root \( 0 \in A \) such that the following conditions are satisfied:

1) \( f_0 = f \) and \( f_\alpha \in K \) for each \( \alpha \in A \).

2) If \( \alpha \in A \) is terminal node then \( f_\alpha \in K_0 \).

3) For every \( \alpha \in A \) which is not terminal, denoting by \( S_\alpha \) the set of the immediate successors of \( \alpha \), exclusively one of the following two holds:

(a) \( S_\alpha = \{\beta_1, \ldots, \beta_d\} \) with \( f_{\beta_1} < \cdots < f_{\beta_d} \) and there exists \( j \in \mathbb{N} \) such that \( d \leq n_{2j} \), and \( f_\alpha = \frac{1}{m_{2j}} \sum_{i=1}^{d} f_{\beta_i} \).

(b) There exists a special sequence \( \phi = (x_1, f_1, \ldots, x_{n_{2j}+1}, f_{n_{2j}+1}) \) of length \( n_{2j+1} \), an interval \( E \) and \( \varepsilon \in \{-1, 1\} \) such that \( f_\alpha = \frac{\varepsilon}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} E(\lambda f_{2i}, f_{2i-1} + f_{2i}) \in K_\phi \) and \( \{f_\beta : \beta \in S_\alpha\} = \{Ef_{2i-1} : Ef_{2i-1} \neq 0\} \cup \{Ef'_{2i} : Ef'_{2i} \neq 0\} \).

It follows from the inductive definition of \( K \) that every \( f \in K \) admits a tree, not necessarily unique.

3. THE SPACE \( X_{ius} \) IS UNCONDITIONALLY SATURATED

This section is devoted to show that the space \( X_{ius} \) is unconditionally saturated. We start with the following: We set

\[
\tilde{K} = \{\pm e_n : \frac{1}{m_{2j}} \sum_{i \in F} \pm e_i : \#F \leq n_{2j}, j \in \mathbb{N} \} \cup \{0\}. 
\]
Clearly $\tilde{K}$ is a subset of the norming set $K$ and it is easily checked that $\tilde{K}$ is a countable and compact set (in the pointwise topology). It is well known that the space $C(\tilde{K})$ is $c_0$-saturated. Observe also that $\| \cdot \|_{\tilde{K}} \leq \| \cdot \|_{X_{ius}}$ and hence the identity operator

$$I : (c_0, \| \cdot \|_{X_{ius}}) \to (c_0, \| \cdot \|_{\tilde{K}})$$

is bounded. Since the basis $(e_n)_n$ of $X_{ius}$ is boundedly complete, the space $X_{ius}$ does not contain $c_0$, therefore the operator $I$ is also strictly singular. These observations yield that every block subspace $Y$ of $X_{ius}$ contains a further block sequence $(y_n)$ such that $\|y_n\|_{X_{ius}} = 1$ and $\|y_n\|_{\tilde{K}} \underset{n \to \infty}{\longrightarrow} 0$. Our intention is to show the following:

**Proposition 3.1.** Let $(x_\ell)_\ell$ be a normalized block sequence in $X_{ius}$ such that $\|x_\ell\|_{\tilde{K}} \to 0$. Then there exists a subsequence $(x_\ell)_\ell \in M$ of $(x_\ell)$ which is an unconditional basic sequence.

The proof of this proposition requires certain steps and we attempt a sketch of the main ideas. First we assume, passing to a subsequence, that $\| \cdot \|_{\tilde{K}} \leq \| \cdot \|_{X_{ius}}$ and we claim that $(x_\ell)_\ell \in \mathbb{N}$ is an unconditional basic sequence. Indeed, consider a norm one combination $\sum_{\ell=1}^d b_\ell x_\ell$ and let $(\varepsilon_\ell)_{\ell=1}^d \in \{-1, 1\}^d$. We shall show that $\| \sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell \| > \frac{1}{4}$. Choose any $f \in K$ with $f(\sum_{\ell=1}^d b_\ell x_\ell) > \frac{3}{4}$ and we are seeking a $g \in K$ such that $g(\sum_{\ell=1}^d \varepsilon_\ell b_\ell x_\ell) \geq \frac{1}{4}$. To find such a $g$ a normal procedure is to consider a tree $(f_\alpha)_{\alpha \in A}$ of the functional $f$ and then inductively to produce a functional $g$ with a tree $(g_\alpha)_{\alpha \in A}$ such that

$$|f(x_\ell) - g(\varepsilon_\ell x_\ell)| < 2\sigma_\ell$$

which easily yields the desired result.

In most of the cases, the choice for producing $g_\alpha$ from $f_\alpha$ is straightforward. Essentially there exists only one case where we need to be careful. That is when $f_\alpha \in K_\phi$ for some special sequence $\phi$. (i.e. $f_\alpha = E(\sum_{i=1}^{n_{2j+1}/2} \lambda f_{2i-1} f_{2i} + \cdots + \lambda f_{2n_{2j+1}} f_{2n_{2j+1} + 1} f_{2n_{2j+1} + 1} + f_{n_{2j+1}})$) and for some $i \leq n_{2j+1}/2$ and $\ell < d$ we have

$$\max \supp x_{\ell-1} < \min \supp (f_{2i-1}) \leq \max \supp x_\ell$$

$$\max \supp f_{2i} \geq \min \supp x_{\ell+1}.$$

In this case we produce $g_\alpha$ from $f_\alpha$ such that $g_\alpha \in K_\phi$. The form of $f_\alpha$ and hence $g_\alpha$ permits us to show that $|f_\alpha(x_\ell) - g_\alpha(\varepsilon_\ell x_\ell)| < 2\sigma_\ell$.

We pass now to present the proof and we start with the next notation and definitions.

**Notation.** Let $f \in K$ and $(f_\alpha)_{\alpha \in A}$ a tree of $f$. Then for every non terminal node $\alpha \in A$ we order the set $S_\alpha$ following the natural order of $\{\supp f_\beta\}_{\beta \in S_\alpha}$. Hence for $\beta \in S_\alpha$ we denote by $\beta^+$ the immediate successor of $\beta$ in the above order if such an object exists.

**Definition 3.2.** Let $f \in K$ and $(f_\alpha)_{\alpha \in A}$ be a tree of $f$. A couple of functionals $f_\alpha$, $f_\alpha^+$ is said to be a *depended couple with respect to* $f$, (w.r.t. $f$), if there exists $\beta \in A$ such that $\alpha, \alpha^+ \in S_\beta$, $f_\beta = \frac{\varepsilon}{m_{2j+1}} E(\sum_{i=1}^{n_{2j+1}/2} \lambda f_{2i} f_{2i+1, -1} + f_{2i+1}^\beta, f_{2i+1}^\beta, f_{2i+1}^\beta)$, $f_\alpha = Ef_{2i+1}^\beta$ and $f_{\alpha^+} = Ef_{2i}^\beta$ for some $i \leq n_{2j+1}/2$. 

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Lemma 3.6. Let \((x_k)_k\) be a normalized block sequence, \(f \in K\) and \(T_f = (f_\alpha)_{\alpha \in A}\) be a tree of \(f\). For \(k \in \mathbb{N}\), a couple of functionals \(f_\alpha, f_\alpha^+\) is said to be **depended couple with respect to \(f\) and \(x_k\)** (w.r.t.) if \(f_\alpha, f_\alpha^+\) is a depended couple w.r.t. \(f\) and moreover
\[
\max \supp x_{k-1} < \min \supp f_\alpha \leq \max \supp x_k
\]
and
\[
\max \supp f_\alpha^+ \geq \min \supp x_{k+1}.
\]
We also set
\[
F_{f,x_k} = \{ \alpha \in A : f_\alpha, f_\alpha^+ \text{ is a depended couple w.r.t. } f \text{ and } x_k \}.
\]
and
\[
F_f = \bigcup_k F_{f,x_k}.
\]

Remark 3.4. Let \((x_k)_k\) be a block sequence in \(X_{ius}\), \(f \in K\) and \((f_\alpha)_{\alpha \in A}\) be a tree of \(f\).

1. It is easy to see that for every \(k \in \mathbb{N}\) and every non terminal node \(\alpha \in A\) the set \(S_\alpha \cap F_{f,x_k}\) has at most one element.
2. As consequence of this, we obtain that for every \(k\) and \(\alpha_1, \alpha_2 \in F_{f,x_k}\) with \(\alpha_1 \neq \alpha_2\) we have that \(\alpha_1, \alpha_2\) are incomparable and \(|\alpha_1| \neq |\alpha_2|\), where we denote by \(|\alpha|\) the order of \(\alpha\) as a member of the finite tree \(A\).
3. It is also easy to see that for \(\alpha_1, \alpha_2 \in F_f\) with \(\alpha_1 \neq \alpha_2\), \(\alpha_1, \alpha_2\) are incomparable and hence \(\text{range}(f_{\alpha_1}) \cap \text{range}(f_{\alpha_2}) = \emptyset\).

Lemma 3.5. Let \((x_k)_k\) be a block sequence in \(X_{ius}\) such that \(\|x_k\|_K \leq \sigma_k\), \(f \in K\) and \((f_\alpha)_{\alpha \in A}\) be a tree of \(f\). We set \(y_k = x_k \mid_{\cup_{\alpha \in F_f} \text{supp}(f_\alpha)}\). Then we have that
\[
|f(y_k)| \leq 2\sigma_k.
\]

Proof. Let us first observe that for each \(q \in \mathbb{N}\) the set \(\{\text{range}(f_\alpha) : |\alpha| = q\}\) consists of pairwise disjoint sets. Therefore from the preceding remark we obtain that for each \(k\) and each \(q\) the set
\[
\{\alpha \in F_f : |\alpha| = q, \text{range}(f_\alpha) \cap \text{range}(x_k) \neq \emptyset\}
\]
contains at most two elements (one of them belongs to \(F_{f,x_k}\) and the other to \(F_{f,x_i}\) for some \(i \leq k - 1\)). Therefore
\[
|f(y_k)| \leq \sum_{\alpha \in F_f} \left( \prod_{0 \leq \gamma < \alpha} \frac{1}{w(f_\gamma)} \right) |f_\alpha(x_k)|
\]
\[
= \sum_{i} \sum_{\alpha \in F_f, |\alpha| = i} \left( \prod_{0 \leq \gamma < \alpha} \frac{1}{w(f_\gamma)} \right) |f_\alpha(x_k)| \leq 2\sigma_k \sum_{i} \frac{1}{m_i^1} \leq 2\sigma_k.
\]

The following lemma is the crucial step for the proof of the main result of this section.

Lemma 3.6. Let \((x_k)_k\) be a block sequence in \(X_{ius}\), \(f \in K\) and \((f_\alpha)_{\alpha \in A}\) be a tree of \(f\). For every \(k \in \mathbb{N}\) we set \(y_k = x_k \mid_{\cup_{\alpha \in F_f} \text{supp}(f_\alpha)}\). Then for every choice of signs \((\varepsilon_k)_k\) there exists a functional \(g \in K\) with a tree \((g_\alpha)_{\alpha \in A}\) such that
\begin{enumerate}
\item \(f(x_k - y_k) = g(\varepsilon_k(x_k - y_k))\)
\item For every \(\alpha \in A\), \(\text{supp}(f_\alpha) = \text{supp}(g_\alpha)\)
\item \(F_{f,x_k} = F_{g,x_k}\)
\end{enumerate}
for every \(k = 1, 2, \ldots\).
Proof. For the given tree $\{f_\alpha\}_{\alpha \in \mathcal{A}}$ of $f$, we define

$$D = \{ \beta \in \mathcal{A} : \text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset \text{ for at most one } k \}$$

and if $\beta \in S_\alpha$ then $\text{range}(f_\alpha) \cap \text{range}(x_i) \neq \emptyset$ for at least two $x_i$.

Let us observe that for every branch $b$ of $\mathcal{A}$, $b \cap D$ is a singleton. Furthermore, for $\beta \in D$ and $\gamma \in \mathcal{A}$ with $\beta \prec \gamma$ we have that $\gamma \notin \mathcal{F}_f$.

The definition of $(g_\alpha)_{\alpha \in \mathcal{A}}$ requires the following three steps.

**Step 1.** First we define the set $\{g_\beta : \beta \in D\}$ as follows.

(a) If $\beta \in D$ and there exists $\alpha \in \mathcal{A}$ with $\alpha \leq \beta$ and $f_\alpha, f_{\alpha^+}$ is a depended couple w.r.t. $f$ we set $g_\beta = f_\beta$.

(b) If $\beta \in D$ does not belong to the previous case and there exists a (unique) $k$ such that $\text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset$ then we set $g_\beta = \varepsilon_k f_\beta$.

(c) If $\beta \in D$ does not belong to case (a) and $\text{range}(f_\beta) \cap \text{range}(x_k) = \emptyset$ for all $k$ then we set $g_\beta = \varepsilon_k f_\beta$ where

$$k = \max \{l : \text{range}(x_l) < \text{range}(f_\beta)\}.$$ 

(We have assumed that $\min \text{range}(f_1) \leq \min \text{range}(f)$.)

Let us comment the case (a) in the above definition. First we observe that the unique $\alpha \in \mathcal{A}$ witnessing that $\beta$ belongs to the case (a) satisfies the following: either $\alpha = \beta$ or $|\alpha| = |\beta| - 1$. Moreover if this $\alpha$ does not belong to $\mathcal{F}_f$ then $\alpha = \beta$, $\alpha^+ \in D$. In this case, if we assume that there exists a (unique) $k$ such that $\text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset$ then $g_\alpha$ is defined by cases (b) or (c) and $g_{\alpha^+} = \varepsilon_k f_{\alpha^+}$ for the specific $k$. All these are straightforward consequences of the corresponding definitions.

**Step 2.** We set

$$D^+ = \{ \gamma \in \mathcal{A} : \text{ there exists } \beta \in D \text{ with } \beta \prec \gamma \}.$$ 

For $\gamma \in D^+$ we set $g_\gamma = \varepsilon_\beta f_\alpha$ where $\beta$ is the unique element of $D$ with $\beta \prec \gamma$ and $\varepsilon_\beta \in \{-1, 1\}$ is such that $g_\beta = \varepsilon_\beta f_\beta$.

Clearly for every $\beta \in D \cup D^+$, $(g_\gamma)_{\beta \leq \gamma}$ is a tree of the functional $g_\beta$. Furthermore for $\alpha \in D \cup D^+$ the following properties hold:

(1) $\text{supp}(f_\alpha) = \text{supp}(g_\alpha)$

(2) $w(f_\alpha) = w(g_\alpha)$

**Step 3.** We set

$$D^- = \{ \alpha \in \mathcal{A} : \text{ there exists } \beta \in D \text{ with } \alpha \prec \beta \}.$$ 

Observe that $\mathcal{A} = D \cup D^+ \cup D^-$ and using backward induction, for all $\alpha \in D^-$ we shall define $g_\alpha$ such that the above (1) and (2) hold and additionally the following two properties will be established.

(3) For $\alpha \in D^-$, $f_\alpha(x_k - y_k) = g_\alpha(\varepsilon_k(x_k - y_k))$ for all $k$.

(4) For $\alpha \in D^-$ and each $k$ we have that $\mathcal{F}_{f_\alpha, x_k} = \mathcal{F}_{g_\alpha, x_k}$.

Observe that for every $\alpha \in D^-$ we have that $f_\alpha \notin K_0$ and furthermore for every $\beta \in D$ $\mathcal{F}_{f_\beta} = \emptyset$.

We pass now to construct inductively $g_\alpha, \alpha \in D^-$ and to establish properties (1)–(4). Let assume that $\alpha \in D^-$ and for every $\beta \in S_\alpha$ either $\beta \in D$ or $g_\beta$ has been defined and properties (1)–(4) have been established. We consider the following three cases.

**Case 1.** $w(f_\alpha) = m_{2j}$ and $\alpha \in \mathcal{F}_f$.

That means that $f_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} f_\beta$ and each $f_\beta = \varepsilon_\ell^* f_{\ell}$ for some $\ell \in \mathbb{N}$. Then $S_\alpha \subset D$ and
from Step 1(a) we conclude that \( g_\beta = f_\beta \) for all \( \beta \in S_\alpha \). We set
\[
  g_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} g_\beta = f_\alpha.
\]
Furthermore for each \( k \) we have that \( \text{supp}(g_\alpha) \cap \text{supp}(x_k) \subset \text{supp}(y_k) \). Hence
\[
g_\alpha(\varepsilon_k(x_k - y_k)) = f_\alpha(x_k - y_k) = 0
\]
and also \( F_{g_\alpha} = F_{f_\alpha} = \emptyset \). Thus properties (3) and (4) hold while (1) and (2) are obvious.

Before passing to the next case let us notice that there is no \( w \in F \) such that \( f_\alpha, f_{\alpha+} \) is a depended couple w.r.t. \( f \) and \( \alpha \notin F_f \). (See the comments after Step 1.)

\textbf{Case 2.} \( w(f_\alpha) = m_{2j} \) and \( \alpha \notin F_f \).

From the previous observation we obtain that \( \alpha \neq \beta \) for each \( \beta \in A \) with \( f_\beta, f_{\beta+} \) depended couple w.r.t. \( f \), and we set
\[
g_\alpha = \frac{1}{m_{2j}} \sum_{\beta \in S_\alpha} g_\beta.
\]
Our inductive assumptions yield properties (1) and (2). To establish property (3) let \( k \in \mathbb{N} \) and \( \beta \in D \cap S_\alpha \) be such that \( \text{range}(x_k) \cap \text{range}(f_\beta) \neq \emptyset \). Then \( g_\beta = \varepsilon_k f_\beta \) hence
\[
g_\beta(\varepsilon_k(x_k - y_k)) = \varepsilon_k g_\beta(x_k - y_k) = f_\beta(x_k - y_k).
\]
If \( \beta \in D^- \cap S_\alpha \) by the inductive assumption for each \( k \) we have
\[
g_\beta(\varepsilon_k(x_k - y_k)) = f_\beta(x_k - y_k).
\]
Therefore
\[
g_\alpha(\varepsilon_k(x_k - y_k)) = f_\alpha(x_k - y_k).
\]
Finally, for each \( k \)
\[
  F_{f_\alpha, x_k} = \bigcup_{\beta \in S_\alpha} F_{f_\beta, x_k} = \bigcup_{\beta \in S_\alpha \cap D^-} F_{f_\beta, x_k} = \bigcup_{\beta \in S_\alpha \cap D^-} F_{g_\beta, x_k} = F_{g_\alpha, x_k}
\]
which establishes property (4).

\textbf{Case 3.} \( f_\alpha = \frac{\varepsilon}{m_{2j+1}} E(f_{f_1} f_1^\alpha + f_2^\alpha + \ldots + \lambda f_{f_{n_{2j+1}}} f_{n_{2j+1}}^\alpha + f_{n_{2j+1}+1}^\alpha) \in K_\phi \) where
\[
  \{ f_\beta : \beta \in S_\alpha \} = \{ E f_1^\alpha : E f_1^\alpha \neq 0, 1 \leq i \leq n_{2j+1} \}, \varepsilon \in \{-1, 1\}, E \text{ is an interval and } \phi \text{ is a special sequence of length } n_{2j+1}.
\]
Let \( \phi = (z_1, f_1, \ldots, z_{n_{2j+1}}, f_{n_{2j+1}}) \). Without loss of generality we assume that \( E = \mathbb{N} \) and \( \varepsilon = 1 \). Let us observe that the definition of \( \{ g_\beta : \beta \in D \} \) and the inductive assumptions yield that for \( i \leq n_{2j+1}/2 \),
\begin{enumerate}
  \item \( f_{2i-1} = f_{2i-1}^\phi = g_{2i-1}^\phi \).
  \item \( w(f_{2i}) = w(f_{2i}^\phi) = w(g_{2i}^\phi) \).
  \item \( \text{supp}(f_{2i}) = \text{supp}(f_{2i}^\phi) = \text{supp}(g_{2i}^\phi) \).
\end{enumerate}
We define
\[
g_\alpha = \frac{1}{m_{2j+1}} \left( \lambda_{g_2} f_1 + g_1^\alpha + \lambda_{g_2} f_3 + g_3^\alpha + \cdots + \lambda_{g_{n_{2j+1}}} f_{n_{2j+1}} + g_{n_{2j+1}}^\alpha \right)
\]
where \( \{ g_\beta : \beta \in S_\alpha \} = \{ g_i^\alpha : 1 \leq i \leq n_{2j+1} \} \) while \( \lambda_{g_i} \) are defined as follows:
\begin{enumerate}
  \item \( g_{2i}^\phi(z_{2i}) \neq 0 \) then \( \lambda_{g_{2i}} = g_{2i}^\phi(m_{\sigma(2i-1)} z_{2i}) \).
  \item \( g_{2i}^\phi(z_{2i}) = 0 \) and \( f_{2i-1} = f_\beta \), there are two cases
    \begin{enumerate}
      \item If \( \beta \in F_f \) or \( \beta \notin F_f \) and \( \text{range}(f_\beta) \cap \text{range}(x_k) = \emptyset \) for all \( k \) we set \( \lambda_{g_{2i}} = \frac{1}{n_{2j+1}} \).
      \item If \( \beta \notin F_f \) and there exists (unique) \( k \) such that \( \text{range}(f_\beta) \cap \text{range}(x_k) \neq \emptyset \) then we set \( \lambda_{g_{2i}} = \varepsilon_k \lambda_{f_{2i}} \).
    \end{enumerate}
\end{enumerate}
Let us observe that in the case (6) b), as follows from the comments after Step 1, $g_{\beta^+} = \varepsilon_k f_{\beta^+}$ hence $f_{\beta^+} (z_{2i}) = 0$ if and only if $g_{\beta^+} (z_{2i}) = 0$.

From the above definition of $\lambda_{g_{2i}}$, $1 \leq i \leq n_{2j+1}/2$ and (i),(ii),(iii), we obtain that the functional $g_\alpha$ belongs to $K_\phi \subset K$.

Properties (1) and (2) are obvious for $g_\alpha$ and we check the rest. First we establish property (4).

Let $k$ be given. From Remark 3.3 (1) it follows that there exists at most one depended couple $f_{2i-1}^\alpha, f_{2i}^\alpha$ w.r.t. $f$ and $x_k$. Moreover if such a depended couple, $f_{2i-1}^\alpha, f_{2i}^\alpha$, exists then for every $i' \neq i$ it holds that $F_{f_{2i'},x_k} = \emptyset$. Therefore in this case we have that

$$\mathcal{F}_{f_{2i},x_k} = \mathcal{F}_{f_{2i},x_k} \cup \{\beta\}$$

where $f_{2i-1}^\alpha = f_{\beta}$. In the case that no such depended couple exists, it follows that $F_{f_{2i},x_k} \neq \emptyset$ for at most one $i$. This is a consequence of the definitions and the fact that the functionals $(f_i^\alpha)$ are successive. If such an $i$ exists then

$$\mathcal{F}_{f_{2i},x_k} = \mathcal{F}_{f_{2i},x_k}$$

The last alternative is that $F_{f_{2i},x_k} = \emptyset$. This description of $F_{f_{2i},x_k}$ and the inductive assumptions easily yield property (4). Namely, either $F_{g_\alpha,x_k} = F_{f_{2i},x_k} \cup \{\beta\}$ if (3.5) holds, $F_{g_\alpha,x_k} = F_{g_{2i},x_k}$ if (3.0) holds, or $F_{g_\alpha,x_k} = \emptyset$.

Finally we check property (3). Fix a number $k$ and $i \leq n_{2j+1}/2$. If $g_{2i}^\alpha = g_\beta$ and $\beta \in D^-$ the inductive assumption provides

$$g_{2i}^\alpha (\varepsilon_k (x_k - y_k)) = f_{2i}^\alpha (x_k - y_k).$$

If $\beta \in D$ and $\text{range}(f_{2i}^\alpha) \cap \text{range}(x_k) \neq \emptyset$ then $g_{2i}^\alpha = \varepsilon_k f_{2i}^\alpha$ which yields (3.7). Also if $\text{range}(f_{2i}^\alpha) \cap \text{range}(x_k) = \emptyset$ equality (3.7) trivially holds.

In the case $g_{2i-1}^\alpha = g_\beta$, $\beta \in S_\alpha$ we distinguish two subcases. First assume that $\beta \in \mathcal{F}_f$. Then $\text{supp}(g_{2i-1}^\alpha) = \text{supp}(f_{2i-1}^\alpha)$ and $\text{supp}(g_{2i-1}^\alpha) \cap \text{supp}(x_k - y_k) = \emptyset$ therefore

$$g_{2i-1}^\alpha (\varepsilon_k (x_k - y_k)) = 0 = f_{2i-1}^\alpha (x_k - y_k).$$

The second subcase is $\beta \notin \mathcal{F}_f$. As we have explained in the comments after Step 1 that means that either range$(f_{\beta}) \cap \text{range}(x_k) = \emptyset$, hence everything trivially holds, or $\beta, \beta^+ \in D$, $g_{\beta^+} = \varepsilon_k f_{\beta^+}$ and $\lambda_{g_{2i}^\alpha} = \varepsilon_k \lambda_{f_{2i}^\alpha}$. From these observations we conclude that

$$\lambda_{g_{2i}^\alpha} (g_{2i-1}^\alpha (\varepsilon_k (x_k - y_k))) = \lambda_{f_{2i}^\alpha} (f_{2i-1}^\alpha (x_k - y_k)).$$

All these derive the desired equality, namely

$$g_\alpha (\varepsilon_k (x_k - y_k)) = f_\alpha (x_k - y_k).$$

The inductive construction and the entire proof of the lemma is complete. \hspace{1cm} \Box

**Proof of Proposition 3.1.** Let $(\sigma_\ell)_{\ell}$ be a decreasing sequence of positive numbers such that $\sum_{\ell=1}^{d} \sigma_\ell \leq 1/8$. For each $\ell \in \mathbb{N}$ we select $k_\ell$ such that $\|x_{k_\ell}\|_{K} < \sigma_\ell$. For simplicity we assume that the entire sequence $(x_\ell)$ satisfies the above condition. Let $\sum_{\ell=1}^{d} b_\ell x_\ell$ be a finite linear combination which maximizes the norm of all vectors of the form $\sum_{\ell=1}^{d} c_\ell x_\ell$ with $|c_\ell| = |b_\ell|$. Assume furthermore that $\| \sum_{\ell=1}^{d} b_\ell x_\ell \| = 1$ and let $f \in K$ with $f(\sum_{\ell=1}^{d} b_\ell x_\ell) \geq 3/4$. Choose $\{\varepsilon_\ell\}_{\ell=1}^{d} \in \{-1, 1\}^{d}$ and consider the vector $\sum_{\ell=1}^{d} \varepsilon_\ell b_\ell x_\ell$. Lemma 3.0 yields that there exists $g \in K$ and that for each $\ell = 1, \ldots, d$, there exists a vector $y_\ell$ such that

$$g(\sum_{\ell=1}^{d} \varepsilon_\ell b_\ell (x_\ell - y_\ell)) = f(\sum_{\ell=1}^{d} b_\ell (x_\ell - y_\ell)).$$
Also Lemma 3.3 and Lemma 3.6 (2) and (3) yield that
\[ |g(y_\ell)| \leq 2\sigma_\ell \quad \text{and} \quad |f(y_\ell)| \leq 2\sigma_\ell \quad \text{for all} \quad \ell = 1, \ldots, d. \]

Hence
\[
\left\| \sum_{\ell=1}^{d} \varepsilon_{\ell} b_{\ell} x_{\ell} \right\| \geq |g(\sum_{\ell=1}^{d} \varepsilon_{\ell} b_{\ell} x_{\ell})| \geq |g(\sum_{\ell=1}^{d} \varepsilon_{\ell} b_{\ell} (x_{\ell} - y_{\ell}))| - \sum_{\ell=1}^{d} |g(y_\ell)|
\geq |f(\sum_{\ell=1}^{d} b_{\ell} x_{\ell})| - \sum_{\ell=1}^{d} |g(y_\ell)| - \sum_{\ell=1}^{d} |f(y_\ell)| \geq 3/4 - 2/4 = 1/4.
\]

This completes the proof of the proposition. \(\square\)

4. The space \(X_{ius}\) is indecomposable

In the last section we shall show that the space \(X_{ius}\) is indecomposable. This will be a consequence of a stronger result concerning the structure of the space \(\mathcal{B}(X_{ius})\) of the bounded linear operators acting on \(X_{ius}\). The proof adapts techniques related to H.I. spaces as they were presented in [AT]. Thus we will first consider the auxiliary space \(X_u\) and we will estimate the norm of certain averages of its basis. Next we will use the basic inequality to reduce upper estimation on certain averages to the previous results. Finally we shall compute the norms of linear combinations related to special sequences.

The auxiliary spaces \(X_u, X_{u,k}\)

We begin with the definition of the space \(X_u\) which will be used to provide us upper estimations for certain averages in the space \(X_{ius}\).

The space \(X_u\) is the mixed Tsirelson space \(T[(A_{4n_j}, \frac{1}{m_j})_{j=1}^{\infty}]\). The norming set \(W\) of \(X_u\) is defined in a similar manner as the set \(K\).

We set \(W_0 = \{ \pm e_n^* : n \in \mathbb{N} \} \cup \{0\}\), for \(j \in \mathbb{N}\) , \(W_0 = \cup_j W_0^j\). In the general inductive step we define
\[
W_{n}^j = W_{n-1}^j \cup \{ \frac{1}{m_j} \sum_{i=1}^{d} f_i : d \leq 4n_j, f_1 < \ldots < f_d \in W_{n-1} \}
\]
and \(W_n = \cup_j W_n^j\). Finally let \(W = \cup_n W_n\). The space \(X_u\) is the completion of \((c_{00}, \|\cdot\|_W)\) where
\[
\|x\|_W = \sup \{ \langle f, x \rangle : f \in W \}.
\]
It is clear that the norming set \(K\) of the space \(X_{ius}\) is a subset of the convex hull of \(W\). Hence we have that \(\|x\|_K \leq \|x\|_W\) for every \(x \in c_{00}\).

We also need the spaces \(X_{u,k} = T[(A_{4n_j}, \frac{1}{m_j})_{j=1}^{\infty}]\). The norm of such a space is denoted by \(\|\cdot\|_{u,k}\) and it is defined in a similar manner as the norm of \(X_u\). Namely we define \(W_{n}^j\), \(n \in \mathbb{N}\), \(1 \leq j \leq k\) as above and \(W_n^{(k)} = \bigcup_{j=1}^{k} W_n^j\). The norming set is \(W^{(k)} = \bigcup_{n=0}^{\infty} W_n^{(k)}\).

Spaces of this form have been studied in [BD] and it has been shown that such a space is either isomorphic to some \(\ell_p, 1 < p < \infty\), or to \(c_0\).

Before stating the next lemma we introduce some notations. For each \(k \in \mathbb{N}\) we set \(q_k = \frac{1}{\log 4n_k m_k}\) and \(p_k = \frac{1}{\log 4n_k m_k}\) its conjugate.

Lemma 4.1. For the sequences \((m_j), (n_j)\) used in the definition of \(X_{ius}\) and \(X_u, X_{u,k}\) the following hold:

(1) The sequence \((q_j)\) strictly increases to infinity.
(2) For $x = \sum a_{\ell} \in c_{00}$, $\|x\|_{u,k} \leq \|x\|_{p_k}$.

(3) $\left\| \frac{1}{n_{k+1}} \sum_{i=1}^{n_{k+1}} e_i \right\|_{p_k} \leq \frac{1}{m_{k+1}^3}$.

Proof. (1) Using that $m_j+1 = m_j^5$ and $n_j+1 = (4n_j)^{s_j}$ and the fact that $s_j$ increases to infinity we have that

$$q_{j+1} = \frac{1}{\log_{4n_j+1} m_{j+1}} = \frac{1}{\log_{4(4n_j)^{s_j}} m_j^5} > \frac{1}{s_j} \log_{4n_j} m_j = \frac{s_j}{5} q_j$$

hence $(q_j)_j$ strictly increases to infinity.

(2) We inductively show that for $f \in W_n^{(k)}$

$$|f(\sum a_{\ell} e_{\ell})| \leq \|\sum a_{\ell} e_{\ell}\|_{p_k}.$$  

For $n = 0$ it is trivial. The general inductive step goes as follows: for $f \in W_{n+1}^{(k)}$

$$f(\sum a_{\ell} e_{\ell}) = \frac{1}{m_j} \sum_{\ell=1}^{d} f_{\ell}(\sum a_{\ell} e_{\ell})$$

where $f_1 < f_2 < \cdots < f_d$, $d \leq 4n_j$ for some $j \leq k$. We set $E_i = \text{range}(f_i)$ and from our inductive assumption and Hölder inequality we obtain that

$$|f(\sum a_{\ell} e_{\ell})| \leq \frac{1}{m_j} \sum_{\ell \in E_i} a_{\ell} e_{\ell} \|p_k\|_{\ell} \leq \frac{d_{\ell}^2}{m_j} \left( \sum_{\ell \in E_i} \sum_{\ell \in E_i} a_{\ell} e_{\ell} \|p_k\|_{\ell}^2 \right)^{\frac{1}{2}}.$$  

Using that $p_k \leq p_j$ and $m_j = (4n_j)^{s_j}$ we obtain inequality (2).

(3)

$$\left\| \frac{1}{n_{k+1}} \sum_{i=1}^{n_{k+1}} e_i \right\|_{p_k} \leq \frac{1}{n_{k+1}^4} = \frac{1}{(4n_k)^{s_k}} = \frac{1}{m_k^3} \leq \frac{1}{m_{k+1}^3}.$$  

(Recall that $2^{s_k} \geq m_{k+1}^3$).

The tree $T_j$ of $f \in W$ is defined in a similar manner as for $f \in K$.

Lemma 4.2. Let $f \in W$ and $j \in \mathbb{N}$. Then

$$|f(\frac{1}{n_j} \sum_{i=1}^{n_j} e_{k_i})| \leq \begin{cases} \frac{2}{w(f) m_j}, & \text{if } w(f) < m_j \\ \frac{1}{w(f)}, & \text{if } w(f) \geq m_j. \end{cases}$$

If moreover we assume that there exists a tree $(f_\alpha)_{\alpha \in A}$ of $f$, such that $w(f_\alpha) \neq m_j$ for every $\alpha \in A$, we have that

$$|f(\frac{1}{n_j} \sum_{i=1}^{n_j} e_{k_i})| \leq \frac{2}{m_j^2}.$$  

In particular the upper estimations holds for every $f \in K$.

Proof. If $w(f) \geq m_j$ the estimation is an immediate consequence of the fact that $\|f\|_\infty \leq 1/w(f)$. Let $w(f) < m_j$ and $(f_\alpha)_{\alpha \in A}$ be a tree of $f$. We set

$$B = \{i : \text{there exists } \alpha \in A \text{ with } k_i \in \text{supp} f_\alpha \text{ and } w(f_\alpha) \geq m_j\}$$

Then we have that

$$|f(\frac{1}{n_j} \sum_{i \in B} e_{k_i})| \leq \frac{1}{w(f) m_j}.$$
To estimate \(|f(\frac{1}{n_j} \sum_{i \in B^c} e_{k_i})|\), we observe that \(f|_{\{k_i: i \in B^c\}} \in W^{(j-1)}\) (the norming set of \(X_{u,j-1}\)) hence Lemma 4.1 yields that

\[
|f(\frac{1}{n_j} \sum_{i \in B^c} e_{k_i})| \leq \frac{1}{m_j^3}.
\]

Combining (4.3) and (4.4) we obtain (4.1).

To see (4.2) we define the set

\(B = \{i : \text{there exists } \alpha \in A \text{ with } k_i \in \text{supp} f_\alpha \text{ and } w(f_\alpha) \geq m_{j+1}\}\)

and we conclude that

\[
|f(\frac{1}{n_j} \sum_{i \in B} e_{k_i})| \leq \frac{1}{m_{j+1}} < \frac{1}{m_j^3}.
\]

Furthermore from our assumption \(w(f_\alpha) \neq m_j\) for every \(\alpha \in A\) we conclude that

\(f|_{\{k_i: i \in B^c\}} \in W^{(j-1)}\). This yields that the corresponding of (4.3) remains valid and combining (4.4) and (4.5) we obtain (4.2).

**The basic inequality and its consequences**

Next we state and prove the basic inequality which is an adaptation of the corresponding result from [AT]. Actually the proof of the present statement is easier than the original one, due mainly to the low complexity of the family \(A_n\) (in [AT] are studied spaces defined with use of the Schreier families \((S_\xi)_{\xi<\omega_1}\) and also since the definition of the norming set \(K\) does not involve convex combinations. The role of this result is important since it includes most of the necessary computations (unconditional or conditional).

Recall that \(K\) and \(W\) denote the norming sets of \(X_{ius}\) and \(X_u\) respectively.

**Proposition 4.3. (Basic inequality)** Let \((x_k)\) be a block sequence in \(X_{ius}\), \((j_k)\) be a strictly increasing sequence of positive integers, \((b_k) \in c_{00}\), \(C \geq 1\) and \(\varepsilon > 0\) such that

a) \(\|x_k\| \leq C\) for every \(k\).

b) For every \(k \geq 1\), \(\#(\text{supp} x_k)|_{m_{j_k+1}} \leq \varepsilon\).

c) For every \(k \geq 1\), for all \(f \in K\) with \(w(f) < m_{j_k}\), we have that \(|f(x_k)| \leq \frac{C}{w(f)}\).

Then for every \(f \in K\) there exists \(g_1\) such that \(g_1 = h_1\) or \(g_1 = e_t^* + h_1\) where \(t \notin \text{supp} h_1, h_1 \in W, w(h_1) = w(f)\), and \(g_2 \in c_{00}\) with \(\|g_2\|_\infty \leq \varepsilon\) such that

\[
|f(\sum_{k \in E} b_k x_k)| \leq C(g_1 + g_2)(\sum |b_k| e_{k}),
\]

and \(\text{supp} g_1, \text{supp} g_2\) are contained in \(\{k : \text{supp}(f) \cap \text{range}(x_k) \neq \emptyset\}\).

d) If we additionally assume that for some \(j_0 \in \mathbb{N}\) we have that

\[
|f(\sum_{k \in E} b_k x_k)| \leq C(\max_{k \in E} |b_k| + \varepsilon \sum_{k \in E} |b_k|),
\]

for every interval \(E\) of positive integers and every \(f \in K\) with \(w(f) = m_{j_0}\), then \(h_1\) may be selected to have a tree \((h_\alpha)_{\alpha \in A_1}\) such that \(w(h_\alpha) \neq m_{j_0}\) for every \(\alpha \in A_1\).

Our intention is to apply the above inequality in order to obtain upper estimations for \(\ell_1\)-averages of rapidly increasing sequences. Observe that the above proposition reduces this problem to the estimations of the functionals \(g_1, g_2\) on a corresponding average of the basis in the space \(X_u\).

The proof in the general case, assuming only a), b), c), and in the special case, where additionally d) is assumed, is the same. We will make the proof only in the special case.
The proof in the general case arises by omitting any reference to the question whether a functional has weight $m_{j_0}$ or not. For the rest of the proof we assume that there exists $j_0 \in \mathbb{N}$ such that statement of Proposition is fulfilled.

**Proof of Proposition 4.3.** Let $f \in K$ and let $T_f = (f_\alpha)_{\alpha \in A}$ be a tree of $f$. For every $k$ such that $\text{supp}(f) \cap \text{range}(x_k) \neq \emptyset$ we define the set $A_k$ as follows:

$$A_k = \left\{ \alpha \in A : (i) \supp f_\alpha \cap \text{range}(x_k) = \supp(f) \cap \text{range}(x_k), \right.$$

$$\left. (ii) \text{ for all } \gamma < \alpha, \ w(\gamma) \neq m_{j_0}, \right.$$  

$$\left. (iii) \text{ there is no } \beta \in S_\alpha \text{ such that } \supp(f_\alpha) \cap \text{range}(x_k) = \supp(f_\beta) \cap \text{range}(x_k) \text{ if } w(\alpha) \neq m_{j_0} \right\}.$$  

From the definition, it follows easily that for every $k$ such that $\supp(f) \cap \text{range}(x_k) \neq \emptyset$ $A_k$ is a singleton.

We recursively define sets $(D_\alpha)_{\alpha \in A}$ as follows.

For every terminal node $\alpha$ of the tree we set $D_\alpha = \{ k : \alpha \in A_k \}$. For every non-terminal node $\alpha$ we define,

$$D_\alpha = \{ k : \alpha \in A_k \} \cup \bigcup_{\beta \in S_\alpha} D_\beta.$$

The following are easy consequences of the definition.

i) If $\beta < \alpha$, $D_\alpha \subset D_\beta$.

ii) If $w(\alpha) = m_{j_0}$, then $D_\beta = \emptyset$ for all $\beta \succ \alpha$.

iii) If $w(\alpha) \neq m_{j_0}$, then for every $\{ \{ k \} : k \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta \}$ is a family of successive subsets of $\mathbb{N}$.

iv) If $w(\alpha) \neq m_{j_0}$, for every $k \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta$ there exists $\beta \in S_\alpha$ such that $\min \supp x_k < \min \supp f_\beta \leq \max \supp x_k$ and for $k' \in D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta$ different form $k$ the corresponding $\beta'$ is different from $\beta$.

Inductively for every $\alpha \in A$ we define $g^1_\alpha$ and $g^2_\alpha$ such that

1. For every $\alpha \in A$, $\supp g^1_\alpha$ and $\supp g^2_\alpha \subset D_\alpha$.

2. If $w(\alpha) = m_{j_0}$, $g^1_\alpha = e^*_{k_\alpha}$, where $|b_{k_\alpha}| = \max_{k \in D_\alpha} |b_k|$ and $g^2_\alpha = \varepsilon \sum_{k \in D_\alpha} e^*_k$.

3. If $w(\alpha) \neq m_{j_0}$, $g^1_\alpha = h_\alpha$ or $g^1_\alpha = e^*_{k_\alpha} + h_\alpha$ where $k_\alpha \notin \text{supp} h_\alpha$, $h_\alpha \in W$ and $w(h_\alpha) = w(f_\alpha)$.

4. For every $\alpha \in A$ the following inequality holds

$$|a(\sum_{k \in D_\alpha} b_k x_k)| \leq C(\sum_{k \in D_\alpha} |b_k| |e_k|.)$$

For every terminal node we set $g^1_\alpha = g^2_\alpha = 0$ if $D_\alpha = \emptyset$, otherwise we set $g_\alpha = e^*_{k_\alpha}$ if $D_\alpha = \{ k \}$ and $g^2_\alpha = 0$. Assume that we have defined the functionals $g^1_\beta$ and $g^2_\beta$, satisfying (1) - (4), for every $\beta \in A$ with $|\beta| = k$, and let $\alpha \in A$ with $|\alpha| = k - 1$. If $D_\alpha = \emptyset$ we set $g^1_\alpha = g^2_\alpha = 0$. Let $D_\alpha \neq \emptyset$. We distinguish two cases.

**Case 1.** $w(\alpha) = m_{j_0}$.

Let $T_\alpha = D_\alpha \setminus \bigcup_{\beta \in S_\alpha} D_\beta = \{ k : \alpha \in A_k \}$. We set $T^2_\alpha = \{ k \in T_\alpha : m_{j_{k+1}} \leq m_j \}$ and $T^1_\alpha = T_\alpha \setminus T^2_\alpha$. In the pointwise estimations we shall make below, we shall discard the coefficient $\lambda_{f_\alpha}$, which appears in the definition of the special functionals, since $|\lambda_{f_\alpha}| \leq 1$.

From condition b) in the statement, it follows that for each $k \in T^2_\alpha$

$$|a(x_k)| \leq \#(\supp x_k)\|a\|_\infty \leq \#(\supp x_k) \frac{1}{m_j} \leq \varepsilon \leq C\varepsilon.$$  

(4.8)
We define
\[ g^2_\alpha = \varepsilon \sum_{k \in T^2_\alpha} e^*_k + \sum_{\beta \in S_\alpha} g^2_\beta. \]
We observe that \( \|g^2_\alpha\|_{\infty} \leq \varepsilon \), and that \( |f_\alpha(x_k)| \leq C\varepsilon = Cg^2_\alpha(e_k) \), for every \( k \in T^2_\alpha \).

Let \( T^1_\alpha = \{k_1 < k_2 < \ldots < k_l\} \). By the definition of \( T^1_\alpha \) we have that \( m_j < m_{j+1} < \ldots < m_{j+k_i} \). Thus condition c) in the statement implies that
\[ (4.9) \quad |f_\alpha(x_k)| \leq \frac{C}{m_j} = \frac{1}{m_j} e^*_{k_i}(Ck_i), \quad \text{for every } 2 \leq i \leq l. \]

We set
\[ g^1_\alpha = e^*_{k_i} + \frac{1}{m_j} \left( \sum_{i=2}^{l} e^*_k + \sum_{\beta \in S_\alpha} g^1_\beta \right). \]
(The term \( e^*_{k_1} \) does not appear if \( w(f_\alpha) < m_{j_2} \) for every \( k \in T_\alpha \)). We have to show that \( h_\alpha = \frac{1}{m_j}(\sum_{i=2}^{l} e^*_k + \sum_{\beta \in S_\alpha} g^1_\beta) \in W \). From the inductive hypothesis, we have that \( g^1_\beta = h_\beta \) or \( g^1_\beta = e^*_{k_i} + h_\beta \), \( h_\beta \in W \), for every \( \beta \in S_\alpha \). For \( \beta \in S_\alpha \), such that \( g^1_\beta = e^*_{k_i} + h_\beta \), let \( E^1_\beta = \{n \in \mathbb{N} : n < k_3 \} \) and \( E^2_\beta = \{n \in \mathbb{N} : n > k_3 \} \). We set \( h_\beta^1 = E^1_\beta h_\beta \), \( h_\beta^2 = E^2_\beta h_\beta \). For every \( \beta \), such that \( g^1_\beta = e^*_{k_i} + h_\beta \), the functionals \( h^1_\beta, e^*_{k_i}, h^2_\beta \) are successive belonging to \( W \), and for \( \beta \neq \beta' \in S_\alpha \) the corresponding functionals have disjoint range, since \( \text{supp} g^1_\beta \) is an interval, remark (iii) after the definition of \( D_\alpha \).

From the remark iv) after the definition of \( D_\alpha \) we have that \( \#T^1_\alpha \leq n_j \). It follows that
\[ \#(\{e^*_{k_i}, 2 \leq i \leq l\} \cup \{e^*_{k_i}, h^1_\beta, h^2_\beta : \beta \in S_\alpha, g^1_\beta = e^*_{k_i} + h_\beta\} \cup \{h_\beta : \beta \in S_\alpha, g^1_\beta = h_\beta\}) \leq 4n_j. \]
Therefore \( h_\alpha = \frac{1}{m_j}(\sum_{i=2}^{l} e^*_k + \sum_{\beta \in S_\alpha} g^1_\beta) \in W \). It remains to show property 4). By (4.9) we have that
\[ |f_\alpha(x_k)| \leq \|x_k\| \leq Ce^*_{k_i}(e_k) = g^1_\alpha(Ce_k). \]

We also have that
\[ |f_\alpha(\sum_{k \in \cup_{j \in S_\alpha} D_\beta} b_k x_k)| \leq \sum_{\beta \in S_\alpha} |f_\alpha(\sum_{k \in D_\beta} b_k x_k)| \]
\[ \leq \frac{1}{m_j} \sum_{\beta \in S_\alpha} |f_\beta(\sum_{k \in D_\beta} b_k x_k)| \]
\[ \leq \frac{1}{m_j} \sum_{\beta \in S_\alpha} (g^1_\beta + g^2_\beta)(C \sum_{k \in D_\beta} |b_k| e_k) \]
\[ \leq (g^1_\alpha + g^2_\alpha)(C \sum_{k \in D_\alpha} |b_k| e_k). \]

Case 2. \( w(f_\alpha) = m_{j_0} \). In this case we have that \( D_\alpha \) is an interval of the positive integers and \( D_\gamma = \emptyset \), for every \( \gamma > \alpha \). Let \( k_\alpha \) such that \( b_{k_\alpha} = \max_{k \in D_\alpha} |b_k| \). We set
\[ g^1_\alpha = e^*_{k_\alpha} \text{ and } g^2_\alpha = \varepsilon \sum_{k \in D_\alpha} e^*_k. \]

Then we have that
\[ |f_\alpha(\sum_{k \in D_\alpha} b_k x_k)| \leq C(\max_{k \in D_\alpha} |b_k| + \varepsilon \sum_{k \in D_\alpha} |b_k|) = (g^1_\alpha + g^2_\alpha)(C \sum_{k \in D_\alpha} |b_k| e_k). \]
\[ \square \]
**Definition 4.4.** Let \( k \in \mathbb{N} \). A vector \( x \in c_{00} \) is said to be a \( C - \ell^k_1 \) average if there exists \( x_1 < \ldots < x_k \), \( \|x_i\| \leq C \|x\| \) and \( x = \frac{1}{k} \sum_{i=1}^{k} x_i \). Moreover, if \( \|x\| = 1 \) then \( x \) is called a normalized \( C - \ell^k_1 \) average.

**Lemma 4.5.** Let \( j \geq 1 \), \( x \) be an \( C - \ell^1_1 \)-average. Then for every \( n \leq n_{j-1} \) and every \( E_1 < \ldots < E_n \), we have that

\[
\sum_{i=1}^{n} \|E_i x\| \leq C (1 + \frac{2n}{n_j}) < \frac{3}{2} C.
\]

We refer to [S], (or [GM], Lemma 4), for a proof.

**Proposition 4.6.** For every normalized block sequence \((y_i)\) and every \( k \geq m_2 \) there exists a linear combination of \((y_i)\) which is a normalized \( 2 - \ell^k_1 \) average.

**Proof.** Given \( k \geq m_2 \) there exists \( j \in \mathbb{N} \) such that \( m_{2j-1} < k \leq m_{2j+1} \). Recall that \( m_{2j+2} = (4n_{2j+1})^{s_{2j+1}} \) and \( m_{2j+2} < 2^{s_{2j+1}} \). Hence setting \( s = s_{2j+1} \) we have that \( k^s \leq m_{2j+2} \) and \( 2^{-s} < \frac{1}{m_{2j+2}} \). Observe that

\[
\| \sum_{i=1}^{k^s} y_i \| \geq \frac{k^s}{m_{2j+2}}.
\]

Assuming that there is no normalized \( 2 - \ell^1 \) average in \((y_i : i \leq k^s)\) and following the proof of Lemma 3 in [GM] we obtain that

\[
\| \sum_{i=1}^{k^s} y_i \| < k^s \cdot 2^{-s}.
\]

Since \( 2^{-s} < \frac{1}{m_{2j+2}} \), (4.10) and (4.11) derive a contradiction. \( \square \)

**Definition 4.7.** A block sequence \((x_k)\) in \( X_{ius} \) is said to be a \((C, \varepsilon)\)-rapidly increasing sequence (R.I.S.), if there exists a strictly increasing sequence \((j_k)\) of positive integers such that

a) \( \|x_k\| \leq C \).

b) \( \#(\text{range}(x_k)) \frac{1}{m_{j_k+1}} < \varepsilon \).

c) For every \( k = 1, 2, \ldots \) and every \( f \in K \) with \( w(f) < m_{j_k} \) we have that \( |f(x_k)| \leq \frac{C}{w(f)} \).

**Remark 4.8.** Let \((x_k)\) be a block sequence in \( X_{ius} \) such that each \( x_k \) is a normalized \( \frac{4C}{3} - \ell^1_{1/k} \) average and let \( \varepsilon > 0 \) be such that for each \( k \), \( \#(\text{range}(x_k)) \frac{1}{m_{j_k+1}} < \varepsilon \). Then Lemma 4.5 yields that condition (c) in the above definition is also satisfied hence \((x_k)\) is a \((C, \varepsilon)\) R.I.S. In this case we shall call \((x_k)\) as a \((C, \varepsilon)\) R.I.S. of \( \ell_1 \) averages. Let also observe that Proposition 4.6 ensures that for each block sequence \((y_i)\) and every \( \varepsilon > 0 \) there exists \((x_k)\) which is a \((3, \varepsilon)\) R.I.S. of \( \ell_1 \) averages.

**Proposition 4.9.** Let \((x_k)_{k=1}^{n_j} \) be a \((C, \varepsilon)\)-R.I.S such that \( \varepsilon \leq \frac{1}{n_j} \). Then

1) For every \( f \in K \)

\[
|f(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k)| \leq \begin{cases} 
\frac{3C}{m_j w(f)}, & \text{if } w(f) < m_j \\
\frac{2C}{w(f)} + \frac{3C}{n_j}, & \text{if } w(f) \geq m_j.
\end{cases}
\]

In particular \( \| \frac{1}{n_j} \sum_{k=1}^{n_j} x_k \| \leq \frac{2C}{m_j} \).
2) If for \( j_0 = j \) the assumption d) of the basic inequality is fulfilled (Proposition 4.3), for a linear combination \( \frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i \), where \(|b_i| \leq 1\), then

\[
\left\| \frac{1}{n_j} \sum_{i=1}^{n_j} b_i x_i \right\| \leq \frac{4C}{m_j^2}.
\]

3) If \((x_i)^{n_j}_{i=1}\) is a \((3, \varepsilon)\) rapidly increasing sequence of \(\ell_1\) averages then

\[
\frac{1}{m_{2j}} \leq \left\| \frac{1}{n_{2j}} \sum_{i=1}^{n_{2j}} x_i \right\| \leq \frac{6}{m_{2j}}.
\]

**Proof.** The proof of 1) is an application of the basic inequality and Lemma 4.2. Indeed for \( f \in K \), the basic inequality yields that there exist \( h_1 \in W \) with \( w(f) = w(h_1) \), \( t \in \mathbb{N} \) with \( t \notin \text{supp} h_1 \), and \( h_2 \in c_{00} \) with \( \|h_2\|_\infty \leq \varepsilon \), such that

\[
|f(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k)| \leq (e^*_t + h_1 + h_2)C(\frac{1}{n_j} \sum_{k=1}^{n_j} e_k).
\]

Using Lemma 4.2 and the fact that \( \varepsilon \leq \frac{1}{n_j} \), we obtain

\[
|f(\frac{1}{n_j} \sum_{k=1}^{n_j} x_k)| \leq \begin{cases} \frac{C}{n_j^2} + \frac{2C}{w(f)m_j} + C\varepsilon & \text{if } w(f) < m_j \\ \frac{C}{n_j^2} + \frac{2C}{w(f)m_j} + C\varepsilon & \text{if } w(f) \geq m_j. \end{cases}
\]

To prove 2) we observe that the basic inequality yields the existence of \( h_1, h_2 \) such that \( h_1 \) has a tree \((h_\alpha)_{\alpha \in A} \) such that \( w(h_\alpha) \neq m_j \) for every \( \alpha \in A \) and \( \|h_2\|_\infty \leq \varepsilon \). This and Lemma 4.2 yield that

\[
|f(\frac{1}{n_j} \sum_{k=1}^{n_j} b_k x_k)| \leq (e^*_t + h_1 + h_2)C(\frac{1}{n_j} \sum_{k=1}^{n_j} e_k) \leq \frac{C}{n_j} + \frac{2C}{m_j^2} + C\varepsilon \leq \frac{4C}{m_j^2}.
\]

The upper estimation in 3) follows from 1) for \( C = 3 \). For the lower estimation in 3), for every \( i \leq n_{2j} \), we choose a functional \( f_i \) belonging to the pointwise closure of \( K \) such that \( f_i(x_i) = 1 \) and \( \text{range}(f_i) \subset \text{range}(x_i) \). Then it is easy to see that the functional \( f = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} f_i \) belongs to the same set and provides the required result.

**Proposition 4.10.** The space \( X_{ius} \) is reflexive.

**Proof.** As we have already explained after the definition of the norming set \( K \), the basis is boundedly complete. Therefore, to show that the space \( X_{ius} \) is reflexive we need to prove that the basis is shrinking.

Assume on the contrary. Namely there exists \( x^* = w^* - \sum_{n=1}^{\infty} b_n e^*_n \) and \( x^* \neq <e^*_t> \).

Then there exists \( \varepsilon > 0 \) and successive intervals \((E_k)_k\) such that \( \|E_k x^*\| > \varepsilon \). Choose \((x_k)_k \) in \( X_{ius} \) such that \( \text{supp}(x_k) \subset E_k \), \( \|x_k\| = 1 \) and \( x^*(x_k) > \varepsilon \). It follows that every convex combination \( \sum a_k x_k \) satisfies

\[
\|\sum a_k x_k\| > \varepsilon.
\]

Next for \( j \) sufficiently large such that \( \frac{4}{m_{2j}} < \varepsilon \) we define \( y_1, y_2, \ldots, y_{n_{2j}} \) a \( (\frac{2}{\varepsilon}, \frac{1}{n_{2j}}) \) R.I.S. of \( \ell_1 \) averages and each \( y_i \) is some average of \((x_k)_k\). Proposition 1.19(1) yields that

\[
\left\| \frac{1}{n_{2j}} (y_1 + y_2 + \cdots + y_{n_{2j}}) \right\| \leq \frac{4}{m_{2j}} \varepsilon < \varepsilon.
\]

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Clearly (4.17) contradicts (4.16) and the basis is shrinking. □

The structure of $\mathcal{B}(X_{i\mathfrak{u}s})$

**Definition 4.11.** A sequence $\chi = (x_1, f_1, x_2, f_2, \ldots, x_{n_2+1}, f_{n_2+1})$ is said to be a depended sequence of length $n_{2j+1}$ if the following conditions are fulfilled

(i) There exists $\phi = (x_1, f_1, y_2, f_2, \ldots, x_{2i-1}, f_{2i-1}, y_{2i}, f_{2i}, \ldots, y_{2i+1}, f_{2i+1})$ such that $\supp(y_{2i}) = \supp(x_{2i})$ and $\|y_{2i} - x_{2i}\| \leq \frac{1}{n_{2i}}$

where for $1 \leq i < n_{2j+1}$, $f_{i+1} = \sigma(\phi_i)$.

(ii) For $i \leq n_{2j+1}/2$ we have that

$$x_{2i} = \frac{c_{2i}}{n_{2i}} \sum_{l=1}^{n_{2i}} x_{2l}$$

where $(x_{2l}^2)$ is a $(3, \frac{1}{n_{2i}})$ R.I.S. of $\ell^1$ averages, $c_{2i} \in (0, 1)$.

(iii) $f_{2i}(x_{2i}) \geq \frac{1}{12n_{2i}}$.

The following is a consequence of the previous results, and we sketch the proof of it.

**Lemma 4.12.** Let $(y_k)_k$ be a normalized block sequence in $X_{i\mathfrak{u}s}$ and $(e_n)_{n \in M}$ be a subsequence of its basis. Then for all $j \in \mathbb{N}$ there exists a depended sequence

$$\chi = (x_1, f_1, x_2, f_2, \ldots, x_{n_{2j+1}}, f_{n_{2j+1}})$$

of length $n_{2j+1}$ such that for each $i \leq n_{2j+1}/2$, $x_{2i-1} \in \langle e_{n} \rangle_M$ and $x_{2i} \in \langle y_k \rangle_k$.

**Proof.** Let $j_1 \in \mathbb{N}$, $j_1$ even such that $m_{j_1}^2 > n_{2j+1}$. We set

$$x_1 = \frac{1}{n_{j_1}} \sum_{i=1}^{n_{j_1}} e_{1,i} \quad \text{and} \quad f_1 = \frac{1}{m_{j_1}} \sum_{i=1}^{n_{j_1}} e_{1,i}^*,$$

such that $x_1 \in \langle e_{n} \rangle_M$. Let $j_2 = \sigma(x_1, f_1)$. Using Proposition 4.6, we choose an $(3, \frac{1}{n_{j_2}})$ R.I.S. $(x_{2l}^2)_{l=1}^{n_{j_2}} \in \langle y_k \rangle_k$ such that $x_1 < x_{2l}^2$ for every $l \leq n_{j_2}$. Next we choose for every $l \leq n_{j_2}$ a functional $f_{2l}^2 \in K$ such that $f_{2l}^2(x_{2l}^2) \geq \frac{2}{3} \|x_{2l}^2\|_2 \geq \frac{2}{3}$ and $\text{range}(f_{2l}^2) \subseteq \text{range}(x_{2l}^2)$. We set

$$f_2 = \frac{1}{m_{j_2}} \sum_{l=1}^{n_{j_2}} f_{2l}^2$$

and

$$x_2 = \frac{c_2}{n_{j_2}} \sum_{l=1}^{n_{j_2}} x_{2l}^2 \quad \text{where} \quad c_2 = \frac{1}{6}(1 - \frac{m_{j_2}}{n_{j_2}}).$$

From Proposition 4.9, it follows that $\|x_2\| \leq \left(\frac{1}{m_{j_2}} - \frac{1}{n_{j_2}}\right)$. We also have that

$$f_2(x_2) \geq \frac{1}{m_{j_2}} \frac{c_2}{n_{j_2}} \sum_{l=1}^{n_{j_2}} f_{2l}^2(x_{2l}^2) \geq \frac{2}{3} \frac{c_2}{m_{j_2}} \geq \frac{1}{12m_{j_2}}.$$

We choose $y_2 \in Q$, that is $y_2$ is a finite sequence with rational coordinates, such that $\|y_2 - x_2\| \leq \frac{1}{n_{j_2}}$ and $\supp(y_2) = \supp(x_2)$. It follows that $\|y_2\| \leq \frac{1}{m_{j_2}}$ and therefore $(x_1, f_1, y_2, f_2)$ is a special sequence of length 2.

We set $j_3 = \sigma(x_1, f_1, y_2, f_2)$ and we choose

$$x_3 = \frac{1}{n_{j_3}} \sum_{l=1}^{n_{j_3}} e_{3,l} \quad \text{and} \quad f_3 = \frac{1}{m_{j_3}} \sum_{l=1}^{n_{j_3}} e_{3,l}^*.$$
such that range($y_2$) $\cup$ range($f_2$) $<$ range($x_3$) and $x_3 \in (e_n)_M$. Next we choose $x_4, f_4$ and $y_4$ as in the second step, and it is clear that the procedure goes through up to the choice of $x_{n_2j+1}, f_{n_2j+1}$ and $y_{n_2j+1}$.

\textbf{Remark 4.13.} a) Let us observe that the proof of Lemma 4.12 yields that if $\chi = (x_1, f_1, x_2, f_2, \ldots, x_{n_2j+1}, f_{n_2j+1})$ is a depended sequence, then for every $i \leq n_2j+1/2$ it holds that $x_{2i} = \frac{c_{2i}}{n_{2j+1}} \sum_{l=1}^{n_{2j+1}} x_l^2$, where $(x_{2i})_l$ is a $R.I.S.$, $j_{2i} = \sigma(\phi_{2i-1})$ and $c_{2i} \leq \frac{1}{\epsilon}$. It follows from Proposition 4.9 that 

$$\|m_{j_2}, x_{2i}\| \leq 1, \text{ and also if } f \in K \text{ and } w(f) < m_{j_2}, \text{ then, } f(m_{j_2}, x_{2i}) \leq \frac{2}{w(f)}.$$ 

b) Definition 4.11 essentially describes that a depended sequence is a small perturbation of a special sequence. Its necessity occurs from the restriction in the definition of the special sequence $\phi = (x_1, f_1, \ldots, x_{n_2j+1}, f_{n_2j+1})$ that each $x_i \in Q$ (i.e. $x_i(n)$ is a rational number) not permitting to find such elements $x_i$ in every block subspace.

Next we state the basic estimations of averages related to depended sequences.

\textbf{Lemma 4.14.} Let $\chi = (x_1, f_1, x_2, f_2, \ldots, x_{n_2j+1}, f_{n_2j+1})$ be a depended sequence of length $n_2j+1$. Then the following inequality holds:

$$\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_j, x_i \right\| \leq \frac{8}{m_{2j+1}^2}$$

where $m_j, = w(f_i)$.

\textbf{Lemma 4.15.} Let $\phi = (x_1, f_1, \ldots, x_{n_2j+1}, f_{n_2j+1})$ be a special sequence. For every $i \leq n_2j+1/2$, let $\sigma(x_1, f_1, \ldots, x_{2i-1}, f_{2i-1}) = j_{2i}$ and let $y_{2i} = \frac{m_{j_2}}{n_{j_2}} \sum_{l=1}^{n_{j_2}} e_{2i}$ be such that

$$\text{supp}(f_{2i}) \cap \text{supp}(y_{2i}) = 0 \text{ and } \text{supp}(f_{2i-1}) < \text{supp}(y_{2i}) < \text{supp}(f_{2i+1}).$$

Then it holds that

$$\left\| \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right\| \leq \frac{8}{m_{2j+1}^2}.$$

These two lemmas are the key ingredients for proving the main results for the structure of $X_{in}$ and $B(X_{in})$. We proceed with the proof of the main results and we will provide the proof of the two lemmas at the end.

\textbf{Proposition 4.16.} Let $M \in \mathbb{N}$ and let $(y_k)_k$ be a normalized block sequence. Then we have that

$$\text{dist}(S_{(e_n)_M}, S_{(y_k)_k}) = 0.$$ 

\textbf{Proof.} For a given $\epsilon > 0$ we choose $j \in \mathbb{N}$ such that $\frac{8}{m_{2j+1}^2} < \epsilon$. From Lemma 4.12 there exists a depended sequence $\chi = (x_1, f_1, \ldots, x_{n_2j+1}, f_{n_2j+1})$ such that $x_{2i-1} \in (e_n)_M$, $x_{2i} \in (y_k)_k$ for every $i \leq n_2j+1/2$. Set

$$e = \frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_{j_{2i-1}}, x_{2i-1} \text{ and } y = \frac{m_{2j+1}}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_{j_{2i}}, y_{2i}.$$ 

We have that $e \in (e_n : n \in M)$ and $y \in (y_i : i \in M)$. From Lemma 4.14 we have that $\|e - y\| \leq \frac{8}{m_{2j+1}^2}$. To obtain a lower estimation of the norm of $e$ and $y$ we consider the functional $f = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \lambda_{2i}, f_{2i-1} + f_{2i}$ where $\lambda_{2i} = f_{2i}(m_{j_{2i}}, y_{2i})$ and
\[ \phi = (x_1, f_1, y_2, f_2, \ldots, y_{n_{2j+1}}, f_{n_{2j+1}}) \] is the special sequence associated to the depended sequence \( \chi \). From the definition of the depended sequence, \( f_{2i}(m_{2j}, x_{2i}) \geq \frac{1}{12} \), and \( \|x_{2i} - y_{2i}\| \leq \frac{1}{n_{2j}} \) for every \( i \leq n_{2j+1}/2 \). It follows that

\[ \lambda_{f_{2i}} = f(m_{2j}, y_{2i}) \geq f(m_{2j}, x_{2i}) - m_{2j}, \|x_{2i} - y_{2i}\| \geq \frac{1}{12} - \frac{1}{m_{2j}} > \frac{1}{24}. \]

Therefore

\[ \|e\| \geq f(e) = \frac{m_{2j+1}}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} \lambda_{f_{2i}}(m_{2j}, x_{2i-1}) \geq \frac{1}{48}, \]

and

\[ \|y\| \geq f(y) = \frac{m_{2j+1}}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} f_{2i}(m_{2j}, x_{2i}) \geq \frac{1}{24}. \]

These lower estimations and the fact that \( \|e - y\| \leq \frac{8}{m_{2j+1}} \) easily yields the desired result. \( \square \)

**Lemma 4.17.** Let \( T : X \rightarrow X \) be a bounded operator. Then

\[ \lim_{n} \text{dist}(T e_n, R e_n) = 0. \]

**Proof.** Without loss of generality we may assume that \( \|T\| = 1 \). Since \( (e_n) \) is weakly null, by a small perturbation of \( T \) we may assume that \( T(e_n) \) is a finite block, \( T(e_n) \in Q \) and \( \min \text{supp}(T(e_n)) = \infty \). Let \( I(e_n) \) be the smallest interval containing \( \text{supp}(T(e_n)) \). Passing to a subsequence \( (e_n)_{n \in M} \), we may assume that \( I(e_n) < I(e_m) \) for every \( n, m \in M \) with \( n < m \).

If the result is not true, we may assume, on passing to a further subsequence, that there exists \( \delta > 0 \) such that

\[ \text{dist}(T e_n, R e_n) > 2\delta \] for every \( n \in M \).

It follows that \( \|P_{n-1} T e_n\| > \delta \) or \( \|(I - P_n) T e_n\| > \delta \). Therefore for every \( n \in M \) we can choose \( x_n^* \in K \) such that

\[ x_n^*(T e_n) \geq \delta, \quad \text{range}(x_n^*) \cap \text{range}(e_n) = \emptyset, \quad \text{and} \quad \text{range}(x_n^*) \subset I(e_n). \]

Since \( T \) is bounded, for every \( j \in N \) we have that

\[ \|T(\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} e_i)\| \leq \|T\| \|\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} e_i\| = \frac{1}{m_{2j}}. \]

Also for every \( j \in N \) and \( k_1 < k_2 < \cdots < k_{n_{2j}} \) in \( M \), the functional \( h_{2j} = \frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} x_{k_i}^* \) is in \( K \) and

\[ \|T(\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} e_i)\| = \|\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} T e_i\| \geq h_{2j}(\frac{1}{m_{2j}} \sum_{i=1}^{n_{2j}} T e_i) \geq \frac{\delta}{m_{2j}}. \]
We consider now a special sequence $\phi = (x_1, f_1, \ldots, x_{n_2+1}, f_{n_2+1})$ which is defined as follows: for every $i \geq 0$,

$$x_{2i+1} = \frac{1}{n_2} \sum_{j=1}^{n_2(e_{2i})} e_{2i+1,j}, \quad f_{2i+1} = \frac{1}{m_2} \sum_{j=1}^{m_2(e_{2i})} e_{2i+1,j}$$

$$x_{2i} = \frac{1}{n_2} \sum_{j=1}^{n_2(e_{2i-1})} T e_{2i,j}, \quad f_{2i} = \frac{1}{m_2} \sum_{j=1}^{m_2(e_{2i-1})} T x_{2i,j}$$

where $e_{i,\ell} \in \{e_n : n \in M\}$, $x_{s,i,j}$, $T e_{2i,j}$ and $I(e_{i,\ell}) < I(e_{s,j})$ if either $i < s$ or $i = s$ and $\ell < j$. This is possible by our assumption $I(e_n) < I(e_m)$ for $n, m \in M$ with $n < m$. Observe that $f_{2i}(m_2(e_{2i-1})x_{2i}) \geq \delta$ and also that $\text{range}(f_\ell) \cap \text{range}(x_{2i}) = \emptyset$ for every $\ell \neq 2i$. Consider now the following vector:

$$x = \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_2 \sum_{j=1}^{m_2(e_{2i-1})} e_{2i,j}.$$

Then

$$T(x) = \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} m_2 \sum_{j=1}^{m_2(e_{2i-1})} x_{2i,j}.$$

and

$$\|Tx\| \geq \frac{1}{m_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} (\lambda f_{2i} f_{2i-1} + f_{2i})Tx \geq \frac{\delta}{2m_{2j+1}}.$$

On the other hand, if $y_{2i} = \frac{m_2(e_{2i-1})}{n_2(e_{2i-1})} \sum_{j=1}^{n_2(e_{2i-1})} e_{2i,j}$, then we have that $\text{supp}(y_{2i}) \cap \text{supp}f_{2i} = \emptyset$ and $x_{2i-1} < y_{2i} < x_{2i+1}$ for every $i \leq n_{2j+1}/2$, and therefore by Lemma 4.15 we have that

$$\|x\| = \|x_{2i+1}\| \leq \frac{8}{m_{2j+1}}.$$

It follows that $\|T\| \geq \frac{\delta}{m_{2j+1}}$, a contradiction for $j$ sufficiently large. \hfill \Box

**Proposition 4.18.** Let $T : X_{ius} \to X_{ius}$ be a bounded operator. Then there exists $\lambda \in \mathbb{R}$ such that $T - \lambda I$ is strictly singular.

**Proof.** By Lemma 4.17 there exists $\lambda \in \mathbb{R}$ and $M \in [N]$ such that $\lim_{n \in M} \|T e_n - \lambda e_n\| = 0$. Let $\varepsilon > 0$. Passing to a further subsequence $(e_{n_k})_k$, we may assume that $\|T e_{n_k} - \lambda e_{n_k}\| \leq \varepsilon 2^{-k}$ for every $k \in N$. It follows that the restriction of $T - \lambda I$ to $[e_{n_k}, k \in N]$ is of norm less than $\varepsilon$. By Proposition 4.16 it follows that $T - \lambda I$ is strictly singular. \hfill \Box

The following two corollaries are consequences of Proposition 4.18 (see [GM]).

**Corollary 4.19.** There does not exist a non trivial projection $P : X_{ius} \to X_{ius}$.

**Corollary 4.20.** The space $X_{ius}$ is not isomorphic to any proper subspace of it.

It remains to prove lemmas 4.14 and 4.15. We start with the following.
Lemma 4.21. Let \( j \in \mathbb{N} \), \( n_{2j+1} < m_j < m_{j_2} < \ldots < m_{j_2r} \) be such that \( 2r \leq n_{2j+1} < m_j^{1/2} \). Let also \( j_0 \in \mathbb{N} \) be such that \( m_{j_0} \neq m_j \), for every \( i = 1, \ldots, 2r \) and \( m_j^{1/2} > n_{2j+1} \). Then if \( h_1 < \ldots < h_{2r} \in K \) are such that \( w(h_i) = m_j \), for every \( i = 1, \ldots, 2r \), then

\[
|\sum_{k=1}^{r} (\lambda_{2k-1} h_{2k-1} + h_{2k}) \left( \frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} c_{k_l} \right)| < \frac{1}{n_{2j+1}} \text{,}
\]

for every choice of real numbers \( (\lambda_{2k-1})_{k=1}^{r} \) with \( |\lambda_{2k-1}| \leq 1 \) for every \( k \leq r \).

b) If \( (x_i)_{i=1}^{n_{j_0}} \) is a \( (3, \frac{1}{n_{j_0}}) \)-R.I.S of \( E_{1} \) averages, then

\[
|\sum_{k=1}^{r} (\lambda_{2k-1} h_{2k-1} + h_{2k}) \left( \frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_{i_l} \right)| < \frac{1}{n_{2j+1}} \text{,}
\]

for every choice of real numbers \( (\lambda_{2k-1})_{k=1}^{r} \) with \( |\lambda_{2k-1}| \leq 1 \) for every \( k \leq r \).

Proof. We shall give the proof of b) and we shall indicate the minor changes for the proof of a).

From the estimations on the R.I.S, Proposition 4.9 for every \( k \leq 2r \) we have that

\[
|h_k\left( \frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_{i_l} \right)| \leq \begin{cases} \frac{9}{w(h_k)}, & \text{if } w(h_k) < m_{j_0} \\ \frac{3}{m_r} + \frac{6}{n_{j_0}}, & \text{if } w(h_k) = m_r > m_{j_0} \end{cases} 
\]

Using that \( m_{j+1} = m_j^{1/2} \) for every \( j \) and \( |\lambda_{2k-1}| \leq 1 \) for every \( k \leq r \) and \( 4.24 \), we get that

\[
|\sum_{k=1}^{r} (\lambda_{2k-1} h_{2k-1} + h_{2k}) \left( \frac{m_{j_0}}{n_{j_0}} \sum_{l=1}^{n_{j_0}} x_{i_l} \right)| \leq \sum_{k:w(h_k) \leq m_{j_0}} \frac{9}{w(h_k)} + \sum_{r > j_0} \frac{3}{m_r} + \frac{12r}{n_{j_0}} \leq \frac{10}{w(h_1)} + \frac{4}{m_r^2} + \frac{12r}{n_{j_0}} < \frac{1}{n_{2j+1}} \text{.}
\]

For the proof of a) using Lemma 4.2 for the estimations on the basis we get the corresponding inequality to \( 4.24 \), from which follows inequality \( 4.22 \). \( \square \)

Proof of Lemma 4.14. Let \( \chi = (x_1, f_1, \ldots, x_{n_{2j+1}}, f_{n_{2j+1}}) \) be a depended sequence and \( \phi = (y_1, f_1, y_2, f_2, \ldots, y_{n_{2j+1}}, f_{n_{2j+1}}) \) the special sequence associated to \( \chi \). In the rest of the proof we shall assume that \( \chi = \phi \). The general proof follows by slight and obvious modifications of the present proof. Hence we assume that \( \phi = (x_1, f_1, \ldots, x_{n_{2j+1}}, f_{n_{2j+1}}) \).

From Lemma 4.2 and Remark 4.13 it follows that the sequence \( (m_j, x_i)_{i=1}^{n_{j_0}+1} \) satisfies assumptions a), c) of the basic inequality for \( C = 2 \). Furthermore the properties of the function \( \sigma \) yield that assumption b) is also satisfied for \( \varepsilon = 1/n_{2j+1} \).

The rest of the proof is devoted to establish that the sequence \( (m_j, x_i) \); satisfies the crucial condition d) for \( m_{j_0} = m_{2j+1} \) and \( (h_i) = \left( \frac{(-1)^{i}+1}{n_{2j+1}} \right) \). Let

First we consider \( f \in K_\phi \). Then \( f \) is of the form

\[
f = E\left( \frac{\varepsilon}{m_{2j+1}^{1/2}} (\lambda f_1 + \lambda f_2 + \ldots + \lambda f_{n_{2j+1}} f_{n_{2j+1}+1} + f_{n_{2j+1}+1}) \right),
\]

where \( \varepsilon \in \{-1, 1\} \) and \( E \) an interval of \( \mathbb{N} \). Let us recall that \( w(f'_1) = w(f_2) \) and \( \supp(f'_1) = \supp(f_2) \) and therefore \( \text{range}(f'_1) \cap \text{range}(x_k) = \emptyset \) for every \( k \neq 2i \). Let

\[
i_0 = \min\{i \leq n_{2j+1}/2 : \supp(f) \cap (\text{range}(x_{2i-1}) \cup \text{range}(x_{2i})) = \emptyset \}.
\]

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For every \( i \) follows that, for every \( i \)

\[
|f(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1}m_j, x_i)| = |E \frac{1}{m_{2j+1}} \sum_{k=1}^{n_{2j+1}/2} (\lambda_f' f_{2k-1} + f_{2k}) (\sum_{i=1}^{n_{2j+1}} (-1)^{i+1}m_j, x_i)| \leq
\]

\[\tag{4.25}
\frac{1}{m_{2j+1}} |\lambda_f' \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1}) - \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1})|.
\]

\[\tag{4.26}
+ \frac{1}{m_{2j+1}} \sum_{i=0}^{n_{2j+1}/2} (\lambda_f' f_{2i-1} - f_{2i} (m_{j_{2i+1}}, x_{2i+1})).
\]

To estimate the sum in (4.25) and (4.26), we partition the set \( \{i_0, \ldots, n_{2j+1}/2\} \) into two sets \( A \) and \( B \), where \( A = \{i : f'_{2i}(x_{2i}) \neq 0\} \) and \( B \) is its complement. For every \( i \in A \), \( i > i_0 \), using that \( \lambda_f' f_{2i} = f'_{2i}(m_{j_{2i}}, x_{2i}) \), we have that

\[\tag{4.27}
\lambda_f' f_{2i-1} - f_{2i} (m_{j_{2i}}, x_{2i}) = f'_{2i}(m_{j_{2i}}, x_{2i}) - f'_{2i}(m_{j_{2i}}, x_{2i}) = 0.
\]

For every \( i \in B \) we have that \( f'_{2i}(x_{2i}) = 0 \), and therefore, \( |\lambda_f' f_{2i}| = \frac{1}{n_{2j+1}} \), see (2.6). It follows that, for every \( i \in B, i > i_0 \)

\[\tag{4.28}
|\lambda_f' f_{2i-1} - f_{2i} (m_{j_{2i}}, x_{2i})| = |\lambda_f' f_{2i}| = \frac{1}{n_{2j+1}^2}.
\]

For the sum \( |\lambda_f' \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1}) - \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1})| \) distinguishing whether or not \( E\sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1}) = 0 \) and whether \( i_0 \in A \) or \( i_0 \in B \), it follows easily using the previous arguments that

\[\tag{4.29}
|\lambda_f' \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1}) - \sum_{i=0}^{n_{2j+1}/2} (m_{j_{2i+1}} - x_{2i+1})| \leq 1
\]

Summing up (4.27)–(4.29) we have that

\[\tag{4.30}
|f(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1}m_j, x_i)| \leq \frac{1}{m_{2j+1}} (\frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}^2}) < \frac{1}{n_{2j+1}^3}.
\]

Consider now a special sequence \( \psi = (y_1, g_1, y_2, g_2, \ldots, y_{n_{2j+1}}, g_{n_{2j+1}}) \). Let \( i_1 = \min\{i \in \{1, \ldots, n_{2j+1}\} : y_i \neq x_i \text{ or } g_i \neq f_i\} \), and \( k_0 \in \mathbb{N} \) such that \( i_1 = 2k_0 - 1 \) or \( 2k_0 \).

Consider a functional \( g \in K_\psi \) which is defined from this special sequence. Then we have that

\[g = \frac{1}{m_{2j+1}} (\lambda g_1 + g_2' + \ldots + \lambda g_{n_{2j+1} - 1} + g_{n_{2j+1}}'),\]

where \( E \) is an interval of \( \mathbb{N} \) and \( w(g_{2i}) = w(g_{2i}) \) for every \( i \leq n_{2j+1}/2 \). Observe that \( \text{range}(x_i) \cap \text{range}(g_k) = \emptyset \) for every \( i \geq i_1 \) and every \( k < i_1 \). Let

\[i_0 = \min\{i \leq n_{2j+1}/2 : \text{supp}(g) \cap (\text{range}(x_{2i-1}) \cup \text{range}(x_{2i})) \neq \emptyset\}.
\]
Let \( i_0 < k_0 \). Then
\[
|g\left(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_j x_i\right)| \leq \\
\frac{1}{m_{2j+1}} \left| E\lambda g_{2i_0} g_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E g'_{2i_0}(m_{j_{2i_0}} x_{2i_0}) \right| \\
+ \frac{1}{m_{2j+1}} \sum_{k=0}^{k_0-1} \left| \sum_{i=i_0+1}^{k} (\lambda g'_{2i} g_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - g'_{2i}(m_{j_{2i}} x_{2i})) \right| \\
+ \frac{1}{m_{2j+1}} \sum_{k=0}^{k_0-1} \left| \sum_{i=i_0+1}^{k} (\lambda g'_{2i} g_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - g'_{2i}(m_{j_{2i}} x_{2i})) \right|, 
\]
where the sum in (4.32) makes sense when \( i_0 < k_0 - 1 \). If \( i_0 \geq k_0 \) we get that
\[
|g\left(\sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_j x_i\right)| \leq \frac{1}{m_{2j+1}} \left| E \sum_{k \geq i_0} (\lambda g'_{2k} g_{2k-1} + g'_{2k})(\sum_{i \geq i_0} m_{j_{2i-1}} x_{2i-1} - m_{j_{2i}} x_{2i}) \right|.
\]

The proof of the upper estimation for the two cases is almost identical, so we shall give the proof in the case \( i_0 < k_0 \).

As in the previous case, for the sum in (4.31), (4.32) we have that
\[
|E\lambda g_{2i_0} g_{2i_0-1}(m_{j_{2i_0-1}} x_{2i_0-1}) - E g'_{2i_0}(m_{j_{2i_0}} x_{2i_0})| + \\
\sum_{k=0}^{k_0-1} \left| \sum_{i=i_0+1}^{k} (\lambda g'_{2i} g_{2i-1}(m_{j_{2i-1}} x_{2i-1}) - g'_{2i}(m_{j_{2i}} x_{2i})) \right| \leq 2.
\]

To estimate the sum in (4.33), first we observe that from the injectivity of \( \sigma \) it follows that there exists at most one \( k \geq i_1 \) such that
\[
w(g_k) \in \{m_j : i_1 \leq i \leq n_{2j+1}\}.
\]

Let \( 2i-1 \geq i_1 \) be such that \( m_{j_{2i-1}} \neq w(g_k) \) for every \( k \geq i_1 \). Then functionals \( g_{2k-1}, g'_{2k} \), \( k \geq k_0 \) satisfy the assumptions of Lemma 4.21 and therefore we get that
\[
\left| \sum_{k \geq k_0} (\lambda g'_{2k} g_{2k-1} + g'_{2k})(m_{j_{2k}} x_{2k-1}) \right| \leq \frac{1}{n_{2j+1}}.
\]

Also for every \( 2i \geq i_1 \) such that \( m_{j_{2i}} \neq w(g_k) \) for every \( k \geq i_1 \), the functionals \( g_{2k-1}, g'_{2k}, \) \( k \geq k_0 \) satisfy the assumptions of Lemma 4.21 and therefore we get that
\[
\left| \sum_{k \geq k_0} (\lambda g'_{2k} g_{2k-1} + g'_{2k})(m_{j_{2k}} x_{2k}) \right| \leq \frac{1}{n_{2j+1}}.
\]

For the unique \( i \geq i_1 \), such that there exists \( k \geq i_1 \) and \( w(g_k) = m_{j_i} \), if such an \( i \) exists, we have that, using Lemma 4.21
\[
\left| \sum_{k \geq k_0} (\lambda g'_{2k} g_{2k-1} + g'_{2k})(m_{j_i} x_{i}) \right| \leq 1 + \frac{1}{n_{2j+1}}.
\]

Now we distinguish if \( i_1 = 2k_0 - 1 \) or \( i_1 = 2k_0 \). If \( i_1 = 2k_0 - 1 \), we have that \( \text{range}(g_k) \cap \text{range}(x_i) = \emptyset \) for every \( k < 2k_0 - 1 \) and every \( i \geq 2k_0 - 1 \), and from (4.35)-(4.37) we get
that
\[
| \sum_{k \geq k_0} (\lambda g_{2k} g_{2k-1} + g'_{2k}) \left( \frac{1}{n_{2j+1}} \sum_{i=2k_0-1}^{n_{2j+1}} (-1)^{i+1} m_j, x_i \right) | \\
\leq \frac{1}{n_{2j+1}} \left( \frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \right) \leq \frac{3}{n_{2j+1}}.
\]

(4.38)

If \( i_1 = 2k_0 \) then we have that \( \text{range}(x_{2k_0-1}) \cap \text{range}(g_k) = \emptyset \) for every \( k \geq 2k_0 \) and \( k < 2k_0 - 1 \), and from (4.36)-(4.37) we get that
\[
| \sum_{k \geq k_0} (\lambda g_{2k} g_{2k-1} + g'_{2k}) \left( \frac{1}{n_{2j+1}} \sum_{i=2k_0-1}^{n_{2j+1}} (-1)^{i+1} m_j, x_i \right) | \\
\leq \frac{1}{n_{2j+1}} \left( \frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \right) \leq \frac{3}{n_{2j+1}}.
\]

(4.39)

From (4.34), (4.38) and (4.39) we get that
\[
| \sum_{i=1}^{n_{2j+1}} (-1)^{i+1} m_j, x_i | \leq \frac{1}{n_{2j+1}} \left( \frac{2}{n_{2j+1}} + \frac{4}{n_{2j+1}} \right) < \frac{1}{n_{2j+1}}.
\]

(4.40)

The inequalities (4.30) and (4.40) yield that indeed condition d) is satisfied for \( \varepsilon = 1/n_{2j+1} \). Proposition 4.9 (2) derives the desired result and the proof is complete.

Proof of Lemma 4.15. To prove this we shall follow similar arguments as in the proof of Lemma 4.14. We shall establish conditions a), b), c) and d) of the basic inequality, for \( C = 2 \), \( \varepsilon = 1/n_{2j+1} \) and \( m_{j_0} = m_{2j+1} \). Lemma 4.2 yields that the sequence \( (y_{2j}) \), satisfies the assumptions a) and c) of the basic inequality for \( C = 2 \). Furthermore the properties of the function \( \sigma \) yield that assumption b) is also satisfied for \( \varepsilon = 1/n_{2j+1} \).

To establish condition d) we shall show that for every \( f \in K \) with \( w(f) = m_{2j+1} \), it holds that
\[
| f(1/n_{2j+1} \sum_{i=1}^{n_{2j+1}} y_{2i}) | \leq \frac{1}{n_{2j+1}} \left( \frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \right) < \frac{1}{n_{2j+1}}.
\]

First let us observe that for every \( f \in K_\phi \), \( f = E_{m_{2j+1}} \sum_{k=1}^{n_{2j+1}/2} (\lambda g_{2k} f_{2k-1} + f'_{2k}) \) it holds that \( f(1/n_{2j+1} \sum_{i=1}^{n_{2j+1}/2} y_{2i}) = 0 \). This is due to \( \text{supp} f'_{2i} = \text{supp} f_{2i} \) and \( \text{supp} (f_{2i-1}) < y_{2i} < \text{supp} (f_{2i+1}) \) for every \( i \leq n_{2j+1}/2 \).

Let \( \phi = (z_1, g_1, z_2, g_2, \ldots, z_{n_{2j+1}}, g_{n_{2j+1}}) \) be a special sequence of length \( n_{2j+1} \) and let
\[
f = E_{m_{2j+1}} \sum_{k=1}^{n_{2j+1}/2} (\lambda g_{2k} g_{2k-1} + g'_{2k}) \text{ belonging to } K_\phi \).
\]

Without loss of generality we may assume that \( E = \mathbb{N} \). Let \( i_1 = \min \{ i \leq n_{2j+1} : z_i \neq x_i \text{ or } f_i \neq g_i \} \), and \( k_0 \in \mathbb{N} \) such that \( i_1 = 2k_0 - 1 \) or \( i_1 = 2k_0 \). Observe that \( \text{range}(g_k) \cap \text{range}(y_{2i}) = \emptyset \) for every \( k < i_1 \) and every \( 2i \geq i_1 \).
From the injectivity of $\sigma$, it follows that there exists at most one $k \geq i_1$ such that $$w(g_k) \in \{m_{j_i} : i_1 \leq i \leq n_{2j+1}\}.$$ Let $2i \geq i_1$ such that $w(g_k) \neq m_{j_2}$ for all $k \geq i_1$. Then the functionals $g'_{2k-1}, g'_{2k}, k \geq k_0$ satisfy the assumptions of Lemma 4.21(a), and therefore it follows that

$$\left| \sum_{k \geq k_0} \lambda g'_{2k-1} + g'_{2k} \right| < \frac{1}{n_{2j+1}}.$$

(4.41)

For the unique $2i \geq i_1$ such that there exists $k \geq i_1$ with $w(g_k) = m_{j_2}$, if such $2i$ exists, we have that

$$\left| \sum_{k \geq k_0} \lambda g'_{2k-1} + g'_{2k} \right| < 1 + \frac{1}{n_{2j+1}}.$$

(4.42)

Summing up (4.41)-(4.42) we get that

$$\left| f \left( \frac{1}{n_{2j+1}} \sum_{i=1}^{n_{2j+1}/2} y_{2i} \right) \right| \leq \frac{1}{m_{2j+1}} \left( \frac{1}{n_{2j+1}} + \frac{1}{n_{2j+1}} \right) < \frac{1}{n_{2j+1}}.$$

(4.43)

Inequality (4.43) implies that condition d) of the basic inequality is fulfilled, and Proposition 4.9 yields the desired result.

\[ \square \]

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