SOME ERGODIC PROPERTIES OF INVERTIBLE CELLULAR AUTOMATA

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ABSTRACT. In this paper we consider invertible one-dimensional linear cellular automata (CA hereafter) defined on a finite alphabet of cardinality $p^k$, i.e., the maps $T_{f[l,r]} : \mathbb{Z}^n_{p^k} \rightarrow \mathbb{Z}^n_{p^k}$ which are given by $T_{f[l,r]}(x) = (y_n)_{n=−\infty}^{\infty}$, $y_n = f(x_{n+l}, \ldots, x_{n+r}) = \sum_{i=l}^{r} \lambda_i x_{n+i} (\mod p^k)$, $x = (x_n)_{n=−\infty}^{\infty} \in \mathbb{Z}^n_{p^k}$, and $f : \mathbb{Z}^r_{p^k}^{l+1} \rightarrow \mathbb{Z}^r_{p^k}$, over the ring $\mathbb{Z}_{p^k}$ ($k \geq 2$ and $p$ is a prime number), where $\gcd(p, \lambda_l) = 1$ and $p|\lambda_l$ for all $i \neq r$ (or $\gcd(p, \lambda_l) = 1$ and $p|\lambda_l$ for all $i \neq l$). Under some assumptions we prove that any right (left) permutative, invertible one-dimensional linear CA $T_{f[l,r]}$ and its inverse are strong mixing. We also prove that any right (left) permutative, invertible one-dimensional linear CA is Bernoulli automorphism without making use of the natural extension previously used in the literature.

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1. Introduction

Cellular automata (CA for brevity), first introduced by Ulam and von Neumann, has been systematically studied by Hedlund from purely mathematical point of view [10]. Hedlund’s paper started investigation of current problems in symbolic dynamics. CA have been widely investigated in a great number of disciplines (e.g., mathematics, physics, computer sciences, and etc.), the study of such dynamics from the point of view of the ergodic theory has received remarkable attention in the last few years ([2], [3], [5], [6], [7], [11], [13], [14], [18]). The dynamical behavior of $D$-dimensional linear CA (linear CA) over ring $\mathbb{Z}_m$ has been studied in [7]. In [19], Shereshevsky has studied ergodic properties of CA, he has also defined the inverse of a $D$-dimensional linear CA. Pivato [17] has characterized the invariant measures of bipermutative right-sided, nearest neighbor cellular automaton. Host et al. [11] have studied the role of uniform Bernoulli measure in the dynamics of cellular automata of algebraic origin.

Ito et al. [12] have characterized the invertible linear CA in terms of the coefficients of its local rule. Manzini and Margara [15] have obtained some necessary and sufficient conditions for a CA over $\mathbb{Z}_m$ to be invertible. They have given an explicit formula for the computation of the inverse of a $D$-dimensional linear CA. They have applied finite formal power series ($fps$ for brevity) to obtain the inverse of a $D$-dimensional linear CA. The technique of $fps$ is well known for the study of these problems (see [12] for details). In [2], the author has studied the topological entropy of an additive CA by using the $fps$.

It is well known that there are several notions of mixing (i.e. weak mixing, strong mixing, mildly mixing, harmonically mixing etc.) of measure-preserving transformations on probability space in ergodic theory. For example, recently, Pivato and Yassawi [18] developed broad sufficient conditions for convergence. They introduced the concepts of harmonic mixing for measures and diffusion for a linear CA. It is important to know how these notions are related with each other. In the last few decades, a lot of work is devoted to this subject (see, e.g. [14], [19] and [21]). Kleveland [14] has proved that if $r < 0$ or $l > 0$, then $T_{f[l,r]}$ is strong mixing, and some of these CA’s even are t-mixing. In [1], the author has studied some ergodic properties of 1-dimensional linear CA acting on the space of all doubly-infinite sequences taking values in ring $\mathbb{Z}_m$. 

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Although the 1-dimensional linear CA theory and the ergodic properties of this linear CA have developed somewhat independently, there are strong connections between ergodic theory and CA theory. For the definitions and some properties of 1-dimensional linear CA, we refer the reader to [10], [14] and [22] (see also [19] for details).

In [19], Shereshevsky shown that if \( f \) is right (left) permutative and \( 0 \leq l < r \) (resp. \( l < r \leq 0 \)), then the natural extension of the dynamical system \((\mathbb{Z}_m^\mathbb{Z}, B, \mu, T_{f[l,r]} \)) is a Bernoulli automorphism and also he proved that if \( r < 0 \) or \( l > 0 \) and \( T_{f[l,r]} \) is surjective, then the natural extension of the dynamical system \((\mathbb{Z}_m^\mathbb{Z}, B, \mu, T_{f[l,r]} \)) is a \( K \)-automorphism. Later, in [20], Sherehevsky has also shown that if \( f \) is left (right) permutative and \( l \neq 0 \) \( (r \neq 0, \) then the natural extension of \( T_{f[l,r]} \) is a \( K \)-automorphism. In [19, 20], Sherehevsky has used the natural extension so as to convert a noninvertible CA into an invertible dynamical system. Kleveland [14] proved that any bipermutative CA is a Bernoulli system with respect to the uniform measure. In general, the technical definitions of Bernoulli and Kolmogorov automorphisms in ergodic theory are studied for invertible transformations (see [8, 19, 20, 14, 21] for details). Thus, in this paper, we shall restrict our attention to certain invertible 1-dimensional linear CA over the ring \( \mathbb{Z}_m \) \( (m \geq 2) \).

In this paper we are only interested in invertible 1-dimensional linear CA and some of their ergodic properties. We consider invertible 1-dimensional linear CA defined on a finite alphabet of cardinality \( p^k \) where \( p \) is prime number and \( k \geq 2 \) is a positive integer. Without loss of generality, we focus on \( k = 2 \).

One of the interesting parts of this paper is the idea of using the simple characterization of the invertibility of an invertible 1-dimensional linear CA to prove some strong ergodic properties. Therefore, we think that our results will also give a possibility of proving certain ergodic properties for a complete formal classification of invertible multi-dimensional CA defined on alphabets of composite cardinality.

Under some assumptions we prove that any right (left) permutative, invertible one-dimensional linear CA \( T_{f[l,r]} \) and its inverse are strong mixing. We also prove that any right (left) permutative, invertible 1-dimensional linear CA is a Bernoulli automorphism without making use of the natural extension previously used in the literature \([8, 19, 20]\).

The rest of this paper is organized as follows: In Section 2, we give basic definitions and notations. In Section 3, we study the invertibility and permutativity of 1-dimensional linear CA. In Section 4 we investigate the strong mixing property of this invertible 1-dimensional linear CA and its inverse. In Section 5, we study the Bernoulli automorphism. In Section 6, we conclude by pointing some further problems.

2. Preliminaries

Let \( \mathbb{Z}_m = \{0, 1, \ldots, m-1\} \) \( (m \geq 2) \) be the ring of the integers modulo \( m \) and \( \mathbb{Z}_m^\mathbb{Z} \) be the space of all doubly-infinite sequences \( x = (x_n)_{n=-\infty}^{\infty} \in \mathbb{Z}_m^\mathbb{Z} \) and \( x_n \in \mathbb{Z}_m \). A CA can be defined as a continuous homomorphism over \( \mathbb{Z}_m^\mathbb{Z} \) relative to the product topology. The shift \( \sigma : \mathbb{Z}_m^\mathbb{Z} \rightarrow \mathbb{Z}_m^\mathbb{Z} \) defined by \((\sigma x)_i = x_{i+1}\) is a homeomorphism of compact metric space \( \mathbb{Z}_m^\mathbb{Z} \).

\( T : \mathbb{Z}_m^\mathbb{Z} \rightarrow \mathbb{Z}_m^\mathbb{Z} \) is defined by \((T x)_i = f(x_{i+1}, \ldots, x_{i+r})\), where \( f : \mathbb{Z}_m^{r+1} \rightarrow \mathbb{Z}_m \) is a given local rule or map. It is well known that \( T \) commutes with \( \sigma \). Martin et al. [16] have defined a local rule \( f \) to be linear if it can be written as

\[
(2.1) \quad f(x_1, \ldots, x_r) = \sum_{i=1}^{r} \lambda_i x_i \pmod{m},
\]

where at least one among \( \lambda_1, \ldots, \lambda_r \) is nonzero, \( \pmod{m} \). We consider 1-dimensional linear CA \( T_{f[l,r]} \) determined by the rule \( f \):

\[
(2.2) \quad (T_{f[l,r]} x) = (y_n)_{n=-\infty}^{\infty}, y_n = f(x_{n+1}, \ldots, x_{n+r}) = \sum_{i=l}^{r} \lambda_i x_{n+i} \pmod{m},
\]
where $\lambda_l, \ldots, \lambda_r \in \mathbb{Z}_m$.

We are going to use the notation $T_{[l,r]}^f$ for linear CA-map defined in (2.2) to emphasize the local rule $f$ and the numbers $l$ and $r$. The $fps$ associated with $f$ given in (2.1) is defined as $F(X) = \sum_{i=l}^{r} \lambda_i X^{-i}$ (see [7] and [15] for details).

3. Invertible 1-dimensional linear CA and permutativity

In this section, we study the invertibility of a 1-dimensional linear CA generated by a linear local rule with respect to modulo $m$ ($m \geq 2$) and we investigate the relation between the invertibility of the CA generated by a linear local rule $f$ and the permutativity of the local rule $f$.

The notion of permutative CA was first introduced by Hedlund in [10]. If the linear local rule $f : \mathbb{Z}_m^{n-l+1} \to \mathbb{Z}_m$ is given in (2.1), then it is permutative in the $j$th variable if and only if $gcd(\lambda_j, m) = 1$, where $gcd$ denotes the greatest common divisor. A local rule $f$ is said to be right (respectively, left) permutative, if $gcd(\lambda_j, m) = 1$ (respectively, $gcd(\lambda_j, m) = 1$). It is said that $f$ is bipermutative if it is both left and right permutative.

**Example 3.1.** Consider the local rule $f : \mathbb{Z}_3^3 \to \mathbb{Z}_3$ given by $f(x_{-1}, x_0, x_1) = (2x_{-1} + 2x_0 + x_1)(mod\ 3)$, then $f$ is both left and right permutative, that is, bipermutative.

From [19], it is clear that if the local rule $f : \mathbb{Z}_m^n \to \mathbb{Z}_m$ is right (left) permutative, then so is its $n$th iteration $f^n : \mathbb{Z}_m^{n(n-1)+1} \to \mathbb{Z}_m$ for every integer $n \geq 1$. Also, in [19], Shereshevsky has stated that the $n$th iteration $T_{[l,r]}^n f$ of CA $T_{[l,r]} f$ generated by the additive local rule $f$ coincides with the CA $T_{[l,r]} f^n$.

Ito et al. [12] characterize the invertible linear CA in terms of the coefficients of the local rule. They have shown that if $T_{[l,r]} : \mathbb{Z}_m^2 \to \mathbb{Z}_m^2$ is the linear CA given by $f$:

$$T_{[l,r]}(x)(n) = \sum_{j=l}^{r} \lambda_j x(n+j) \ (mod\ m),$$

then $T_{[l,r]}$ is invertible if and only if for each prime factor $p$ of $m$ there exists a unique coefficient $\lambda_i$ ($l \leq j \leq r$) such that $gcd(p, \lambda_j) = 1$, that is, $p | \lambda_i$ and $gcd(p, \lambda_j) = 1$ for all $i \neq j$.

In this way, if $m = p^k$ with $p$ prime, then $T_{[l,r]}$ is invertible and left permutative (respectively, invertible and right permutative) if and only if $gcd(p, \lambda_i) = 1$ and $p | \lambda_i$ for $i \neq l$ (respectively, $gcd(p, \lambda_r) = 1$ and $p | \lambda_i$ for $i \neq r$).

In this paper, we only consider the following results originally stated for higher dimensions in [15] for 1-dimensional linear CA.

It is clear that for $m = p^k$ we can state that $fps F$ is invertible if and only if there exists a unique coefficient $\lambda_j$ such that $gcd(\lambda_j, p) = 1$. So, $fps F$ can be written as follows:

$$(3.1) \quad F(X) = \lambda_j X^{-j} + pH(X),$$

where $gcd(\lambda_j, p) = 1$ and $H(X) = \sum_{i=l, i \neq j}^{r} \lambda_i X^{-i}$.

We need the following Theorem to concentrate on the problem of inverting a finite $fps$, associated to 1-dimensional linear CA, over a prime power.

**Theorem 3.2.** ([15], Theorem 3.2) Let $F(X)$ denote an invertible finite $fps$ over $\mathbb{Z}_{p^k}$, and let $\lambda_j$ and $H$ be defined as in (3.1). Let $\lambda_j^{-1}$ be such that $\lambda_j^{-1} \lambda_j \equiv 1 \ (mod\ p)$. Then, the inverse of $fps F$
Thus, we can find the inverse of
\[ \tilde{\gamma} \]
where
\[ \tilde{\gamma} = \gamma_{j-1} X^{-j}(1 + p\tilde{H}(X) + p^2 \tilde{H}(X)^2 + \ldots + p^{k-1} \tilde{H}(X)^{k-1}), \]
where \( \tilde{H}(X) = -\gamma_{j}^{-1} X^{-j} H(X) \).

It is clear that if \( F(X) \) is given as the equation (3.1), then the local rule \( f \) associated with \( F(X) \) can be written as follows;
\[ f(x_1, \ldots, x_r) = \lambda_j x_j + p \sum_{i=1, i \neq j}^r \lambda_i x_i (\text{mod } p^k). \]

**Example 3.3.** Let \( f(x_1, x_2, x_3) = 2x_1 + 2x_2 + x_3 (\text{mod } 2^2) \). Then it is easy to see that \( f \) is right permutative and is not left permutative. The finite \( fps \) \( F \) associated with the local rule \( f \) is
\[ F(X) = 2X^{-1} + 2X^{-2} + X^{-3} \]
\[ = X^{-3}[1 + (2X^1 + 2X^2)]. \]
Thus, we can find the inverse of \( F \) as follows:
\[ G(X) = X^3[1 - 2(X^1 + X^2)]. \]
The local rule \( g \) related to the finite \( fps \) \( G(X) \) is
\[ g(x_{-5}, x_{-4}, x_{-3}) = 2x_{-5} + 2x_{-4} + x_{-3} (\text{mod } 2^2). \]
It is clear that \( g \) is not left permutative. Thus, we conclude that if \( f \) is left (respectively, right) permutative and invertible, then its inverse \( g \) is also left (respectively, right) permutative.

In Section 4 and 5, we will need some of the identities that will appear in the proof of Proposition 3.4. Hence, we will include the proof of Proposition 3.4.

**Proposition 3.4.** Given 1-dimensional linear CA
\[ (T_f[t,r])_n = f(x_{n+t}, \ldots, x_{n+r}) = \sum_{i=1}^r \lambda_i x_{n+i} \pmod{p^k}, \]
where \( p \) is a prime number. If \( \gcd(p, \lambda_i) = 1 \) and \( p|\lambda_i \) for all \( i \neq r \) (respectively, \( \gcd(p, \lambda_i) = 1 \) and \( p|\lambda_i \) for all \( i \neq l \)), then \( f \) is right (respectively, left) permutative and \( T_f[t,r] \) is invertible.

**Proof.** For brevity without loss of generality we focus on \( k = 2 \). For \( k > 2 \), similarly the proof can be satisfied.

Let us consider the following local rule
\[ f(x_1, \ldots, x_r) = p \sum_{i=1}^{r-1} \beta_i x_i + \lambda_r x_r \pmod{p^2}, \]
where \( \gcd(p, \lambda_r) = 1 \) and \( \beta_i \in \mathbb{Z}_{p^2} \). From Theorem 3.3, it is clear that the inverse of the local rule \( f \) is as follows:
\[ g(x_{-(2r-1)}, \ldots, x_{-r}) = -p \sum_{i=-r}^r \gamma_i x_i + \lambda_r x_{-r} \pmod{p^2}, \]
where \( \gamma_i \in \mathbb{Z}_{p^2} \). Thus, if 1-dimensional CA generated by the local rule \( f \) is \( T_f[t,r] \), then the inverse of \( T_f[t,r] \) is \( T_f^{-1}[t,r] = T_g[-(2r-1),-r] \), generated by the local rule \( g \).

Similarly, let us consider the following local rule
\[ f(x_{-l}, \ldots, x_{-r}) = \lambda_{-l} x_{-l} + p \sum_{i=-l+1}^{-r} a_i x_i \pmod{p^k}, \]
where \( \gcd(p, \lambda_i) = 1, \) for brevity again without loss of generality we focus on \( k = 2. \) So we can obtain the finite \( \text{fps} \ F \) associated with \( f \) as follows:

\[
F(X) = \lambda_{-l}X^l(1 + p\lambda_{-l}^{-1}\sum_{i=-l+1}^{r} a_iX^{-i-l}).
\]

From Theorem 3.3, the inverse of \( F \) is obtained as

\[
G(X) = \lambda_{-l}X^{-l}(1 - p\lambda_{-l}^{-1}\sum_{i=-l+1}^{r} a_iX^{-i-l}).
\]

Therefore, the inverse of local rule \( f \) is obtained as follows:

\[
g(x_l, \ldots, x_{2(l-r)}) = (\lambda_{-l}^{-1}x_l - p\sum_{i=-l+1}^{r} (\lambda_{-l}^{-1})^2a_ix_{2l+i}) \pmod{p^2}.
\]

\[\square\]

4. **The Ergodic Properties of Invertible 1-Dimensional Linear CA**

In this section, we study some ergodic properties of the invertible 1-dimensional linear CA. To be specific, Coven and Paul [9] showed that any surjective CA preserves the uniform measure. This result is restated by Shereshevsky [19] and reproved by Kleveland [14]. Next, Kleveland [14] established that any one-dimensional left- or right-permutative CA is mixing.

**Definition 4.1.** Let \( T : X \to X \) be a measure-preserving transformation on a probability space \((X, \mathcal{B}, \mu)\); then \( T \) is called **strong mixing** if for any \( A, B \in \mathcal{B} \) it satisfies the following equation:

\[
\lim_{n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).
\]

In order to prove the main results of this section and Theorem 5.4, we consider the \( \sigma \)-algebra \( \mathcal{B} \) of Borelian sets of \( \mathbb{Z}_m^n \) and the uniform Bernoulli probability measure \( \mu : \mathcal{B} \to [0,1] \), which is defined in the cylinders \( C = \{ x \in \mathbb{Z}_m^n : x_n = j_0, \ldots, x_{n+a} = j_s \} \) as \( \mu(C) = m^{-(s+1)} \).

We can easily verify that the Bernoulli measure is defined as follows:

\[
\mu(a[j_0, j_1, \ldots, j_s]_{s+a}) = \mu(\{ x \in \mathbb{Z}_m^n : x_n = j_0, \ldots, x_{n+a} = j_s \}) = p_{(j_0)}p_{(j_1)}\cdots p_{(j_s)},
\]

where \((p(0), p(1), \ldots, p(m-1))\) is a probability vector. If for all \( i, j \in \mathbb{Z}_m \) equality \( p(i) = p(j) \) holds then \( \mu \) is called uniform Bernoulli measure. It is a general fact that every surjective CA preserves the uniform Bernoulli measure (see [10] [14] [19] for details).

**Proposition 4.2.** Let \( T_{f[l,r]} \) be an invertible 1-dimensional linear CA over \( \mathbb{Z}_p^k \) and \( \gcd(p, \lambda_r) = 1, \) \( p|\lambda_i \) for all \( i \neq r. \) Then, \( T_{f[l,r]} \) is uniform Bernoulli measure-preserving transformation.

**Proof.** Let \( T_{f[l,r]} \) be an invertible 1-dimensional CA and consider a cylinder set

\[
C = \{ j_0, j_1, \ldots, j_s \}_{s+a} = \{ x \in \mathbb{Z}_p^k : x_n^{(0)} = j_0, \ldots, x_{n+a}^{(0)} = j_s \}
\]

Then the first preimage of \( C \) under \( T_{f[l,r]} \) is as follows:

\[
T_{f[l,r]}^{-1}(C) = T_{f[l,r]}^{-1}(\{ x \in \mathbb{Z}_p^k : x_n^{(0)} = j_0, \ldots, x_{n+a}^{(0)} = j_s \}) = \bigcup_{i_0, i_1, \ldots, i_{(r-l)+s}} (a+l)[i_0, i_1, \ldots, i_{(r-l)+s}],
\]

where \( x_n^{(0)} = \sum_{i=l}^r a_i x_{a+i}^{(1)} \pmod{p^k} \) and \( x_{n+a}^{(0)} = \sum_{i=l}^r a_i x_{r+a+i}^{(1)} \pmod{p^k}. \) It is clear that

\[
(a+l)[i_0, i_1, \ldots, i_{(r-l)+s}] = \{ x \in \mathbb{Z}_p^k : x_{a+i}^{(1)} = i_0, \ldots, x_{a+s+r}^{(1)} = i_{r-l+s} \}.
\]
Then we have
\[
\mu(C) = \mu\left( \{ x \in \mathbb{Z}^k_{p^k} : x_a^{(0)} = j_0, \ldots, x_{a+s}^{(0)} = j_s \} \right)
\]
\[
= \mu(T_{f[l,r]}^{-1}) \left( \{ x \in \mathbb{Z}^k_{p^k} : x_a^{(0)} = j_0, \ldots, x_{a+s}^{(0)} = j_s \} \right)
\]
\[
= \mu\left( \bigcup_{i_0, i_1, \ldots, (r-l)+s} (a+t)i_0, i_1, \ldots, i(r-l)+a+s+r \right)
\]
\[
= \left( p^k \right)^{(s+1)}.
\]

In [14], for a CA \( T_{f[l,r]} \) Kleveland has proved that if \( r < 0 \) or \( l > 0 \), then \( T_{f[l,r]} \) is strong mixing. He has also proved that if \( f \) is permutative in \( x_l \) and \( l < 0 \) or if \( f \) is permutative in \( x_r \) and \( r > 0 \), then \( T_{f[l,r]} \) is strong mixing (see [14] for details).

Under some conditions, now by using a simple characterization of invertibility we are going to prove that the invertible linear CA \( T_{f[l,r]} \) and its inverse are strong mixing.

**Theorem 4.3.** Let \( T_{f[l,r]} \) be an invertible 1-dimensional linear CA over \( \mathbb{Z}_{p^2} \) and \( \gcd(p, \lambda_i) = 1 \) and \( p|\lambda_i \) for all \( i \neq r \), \( l > 0 \). Then both \( T_{f[l,r]} \) and \( T_{g[-(2r-l), -r]} \) are strong mixing, where \( g \) is the local rule given in (3.5).

**Proof.** Let us firstly consider right permutative local rule \( f \) defined as follows:
\[
f(x_l, \ldots, x_r) = \sum_{i=l}^{r} \lambda_i x_i (\text{mod } p^2),
\]
where \( \lambda_i \in \mathbb{Z}_{p^2} \) and \( 0 < l < r \).

Let \( A = a \left[ i_0, \ldots, i_k \right]_{k+a} \) and \( B = b \left[ j_0^{(0)}, \ldots, j_k^{(0)} \right]_{l+b} \) be two cylinder sets. Then we can observe that
\[
A \cap T_{f[l,r]}^{-n} B = \bigcup_{x_{k+1}, \ldots, x_{n-1}} \bigcup_{j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}} (a[i_0, \ldots, i_k, x_{k+1}, \ldots, x_{n-1}, j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}], c),
\]
where \( f(j_{il}^{(n)}, \ldots, j_{ir+t}^{(n)}) = \sum u=l \lambda_u j_{il+u} (\text{mod } p^2) = j_i^{(n-1)} \) and \( c = a + b + k + t + nr \). Thus, for all \( nl > k \) we have
\[
\mu(A \cap T_{f[l,r]}^{-n} B) = \\
= \mu\left( \bigcup_{x_{k+1}, \ldots, x_{n-1}} \bigcup_{j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}} (a[i_0, \ldots, i_k, x_{k+1}, \ldots, x_{n-1}, j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}], c) \right)
\]
\[
= \mu(A) \left( \sum_{x_{k+1}, \ldots, x_{n-1}} p(x_{k+1}) \cdots p(x_{n-1}) \sum_{j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}} p(j_{nl}^{(n)}) \cdots p(j_{nr+t}^{(n)}) \right)
\]
\[
= \mu(A) \sum_{j_{nl}^{(n)}, \ldots, j_{nr+t}^{(n)}} p(j_{nl}^{(n)}) \cdots p(j_{nr+t}^{(n)})
\]
\[
= \mu(A) \mu(B).
\]

Due to \( -r < 0 \), the Theorem 6.2 in [14] is satisfied. Thus, the linear CA \( T_{g[-(2r-l), -r]} \) is strong mixing.

It is well known that if the measure-preserving transformation of probability space is strong mixing, then it is both weak mixing and ergodic.
5. Bernoulli automorphism

As mentioned in introduction, our goal in this section is to show that certain invertible 1-dimensional linear CA is Bernoulli automorphism, which is strong ergodic property, without making use of the natural extension which Shereshevsky [8, 19, 20] employs in his proofs.

Definition 5.1. The partitions $\xi = \{C_i\}$ and $\eta = \{D_j\}$ of the measure space $(\mathbb{Z}_m^n, \mathcal{B}, \mu)$ are called $\varepsilon$-independent ($\varepsilon \geq 0$), if

$$\sum_{i, j} |\mu(C_i \cap D_j) - \mu(C_i)\mu(D_j)| \leq \varepsilon.$$ 

The partitions are independent, if they are 0-independent. A partition $\xi$, that is, an automorphism of the measure space $(\mathbb{Z}_m^n, \mathcal{B}, \mu)$, is weakly Bernoulli for $T$, that is, an automorphism of the measure space $(\mathbb{Z}_m^n, \mathcal{B}, \mu)$, if all its shifts are pairwise independent. A partition $\xi = \{C_i\}$ is weakly Bernoulli for $T$, that is, an automorphism of the measure space $(\mathbb{Z}_m^n, \mathcal{B}, \mu)$, if all its shifts are pairwise independent. A partition $\xi = \{C_i\}$ is weakly Bernoulli for $T$, if for every $\varepsilon > 0$ there exists an integer $N > 0$ such that the partitions $\bigvee_{k=0}^{n} T^k\xi$ and $\bigvee_{k=0}^{N+n} T^k\xi$ are $\varepsilon$-independent for all $n \geq 0$.

The automorphism $(\mathbb{Z}_m^n, \mathcal{B}, \mu, T_{f[l,r]})$ is Bernoulli if and only if it has a generator $\xi$ which is (weakly) Bernoulli for $T$.

In this section, our purpose is to prove that the certain invertible 1-dimensional CA $T_{f[l,r]}$ is Bernoulli automorphism.

Lemma 5.2. If the local rules $f$ and $g$ is defined as (3.4) and (3.5), respectively, then partitions $\bigvee_{k=0}^{n} T_{g[-(2r-l),-r]}^k\xi(-i, i)$ and $\bigvee_{k=0}^{n} T_{f[l,r]}^{-k}\xi(-i, i)$ are $\varepsilon$-independent.

Proof. Follow the definitions.

Lemma 5.3. If the local rules $f$ and $g$ is defined as (3.4) and (3.5), respectively, then the partition $\xi(-i, i)$ is weakly Bernoulli for an invertible 1-dimensional linear CA $T_{f[l,r]}$.

Proof. Let us consider the partition $\xi(-i, i) = \bigvee_{u=-i}^{i} \sigma^{-u}\xi$, where $\sigma$ is the shift transformation and $\xi$ is the zero-time partition of $\mathbb{Z}_p^n$, that is, $\xi = \{a[j] : j \in \mathbb{Z}_p^n\}$, then we have

$$\bigvee_{k=-n}^{0} T_{f[l,r]}^k\xi(-i, i) = \bigvee_{k=0}^{n} T_{f[l,r]}^{-k}\xi(-i, i) \leq \xi(-i, i) \vee \xi(l-i, r+i) \vee \ldots \vee \xi(nl-i, nr+i).$$

(5.1)

Similarly one gets

$$\bigvee_{k=0}^{N+n} T_{f[l,r]}^k\xi(-i, i) = \bigvee_{k=0}^{n} (T^{-1})_{f[l,r]}^{-(k+N)}\xi(-i, i) \leq \xi(-2r-l)N-i, -rN+i) \vee \ldots \vee \xi(-(2r-l)(N+1)-i, -r(N+1)+i) \vee \ldots \vee \xi(-(2r-l)(N+n)-i, -r(N+n)+i).$$

(5.2)

Thus, from (5.1) and (5.2) it is clear that $\bigvee_{k=0}^{n} T_{f[l,r]}^{-k}\xi(-i, i) \leq \xi(-i, nr+i)$.
and
\[ \bigvee_{k=N}^{N+n} T_{f[l,r]}^k(\xi(-i, i) \leq \xi(-(2r-l)(N+n) - i, -rN+i). \]

For any \( n > 0 \) we have
\[ \sum_{a=1}^{p^{2n+4i+2}} \sum_{b=1}^{p^{2((N+n)(2r-l) - Nr+2i+1)}} |\mu(C_a \cap C_b) - \mu(C_a)\mu(C_b)| < \varepsilon, \]
where \( C_a \in \xi(-i, nr+i) \) and \( C_b \in \xi(-(2r-l)(N+n) - i, -rN+i) \). Thus, for every \( i \geq 0 \) the partition \( \xi(-i, i) = \bigvee_{u=-i}^{i} \sigma^{-u}\xi \) is weakly Bernoulli for the automorphism \( T_{f[l,r]} \).

**Theorem 5.4.** If the conditions of Lemma 5.2 are satisfied, then the dynamical system \((\mathbb{Z}_{p^2}, \mathcal{B}, \mu, T_{f[l,r]})\) is a Bernoulli automorphism.

**Proof.** From Lemma 5.3, it is clear that the partition \( \xi(-i, i) = \bigvee_{u=-i}^{i} \sigma^{-u}\xi \) is a generator for \( T_{f[l,r]} \), that is \( \bigvee_{n=-\infty}^{\infty} T_{f[l,r]}^n(\xi(-i, i) = \varepsilon \) (see [19]). Thus, \( T_{f[l,r]} \) is a Bernoulli automorphism.

**Remark 5.5.** Similarly, one can prove that for \( k > 2 \) the dynamical system \((\mathbb{Z}_{p^2}, \mathcal{B}, \mu, T_{f[l,r]})\) defined by the local rule in (5.4) is a Bernoulli automorphism. Also one can prove that if the local rule \( f \) is defined as
\[ f(x_l, \ldots, x_r) = (p_1 \cdot p_2 \ldots p_h \sum_{i=l}^{r-1} \lambda_ix_i + \lambda_rx_r)(\text{mod} \ m), \]
where \( m = p_1^{k_1} p_2^{k_2} \ldots p_h^{k_h} \) and for all \( i \in [l, r] \cap \mathbb{Z}, \lambda_i \in \mathbb{Z}_m \) and \( \gcd(\lambda_r, m) = 1 \), then the invertible 1-dimensional CA \( T_{f[l,r]} \) generated by the local rule \( f \) is a Bernoulli automorphism.

6. Conclusions

In this paper, our main results are Theorem 4.3 and Theorem 5.4.

- Theorem 4.3 states that any right (left) permutable, the invertible linear CA \( T_{f[l,r]} \) and its inverse are strong mixing.
- Theorem 5.4 states that under the conditions of Lemma 5.2 the invertible 1-dimensional linear CA \( T_{f[l,r]} \) is Bernoulli automorphism.

One of the interesting parts of this paper is the idea of using the simple characterization of the invertibility of an invertible 1-dimensional linear CA to prove some strong ergodic properties without making use of the natural extension. This method provides considerable technical simplifications. Therefore, we think that our results will also give a possibility of proving certain ergodic properties for a complete formal classification of invertible multi-dimensional CA defined on alphabets of composite cardinality (or the other finite rings). In [4], Akin and Şiap have investigated CAs over the Galois rings. Thus, similar computations and explorations of CA’s over different rings remain to be of interest.

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