Cauchy filters from Pelant’s games

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Abstract

The language of finite games is used to rephrase Pelant’s proof of his result: The separable modification of the complete metric space $C([0, \omega_1])$ is not complete.

1 Introduction

The uniform space concepts not defined here may be found in Isbell [2].

For every uniform space $X$ there is another uniform space on the same set of points and compatible with the same topology, called here the separable modification of $X$ and denoted by $p_1X$. The uniformity of $p_1X$ is projectively generated by all uniformly continuous mappings from $X$ to separable metric spaces. Isbell’s notation for $p_1X$ is $X_{\aleph_1}$ and also $eX$ (pages 52 and 129 in [2], respectively).

What follows is a result of my attempt to understand Pelant’s proof [5,6] of his theorem:

There is a (not too large) complete metric space $X$ for which $p_1X$ is not complete.

The “not too large” qualification excludes examples such as the discrete space of measurable cardinality. In fact, in the spirit of [3] Thm 1.1, Pelant’s method lets us take $X = C([0, \omega_1])$, the Banach space of continuous real-valued functions on the compact space $[0, \omega_1]$ with the sup norm $\|\cdot\|$. In this paper I describe Pelant’s construction using level sets of functions in $C([0, \omega_1])$ and finite games instead of Pelant’s cornets and finite sequences of alternating quantifiers.

By virtue of Exercise 2(b) on page 52 in [2], every countable uniform cover of $X$ is a uniform cover of $p_1X$. Hence the set of all countable uniform covers of $X$ is a basis of uniform covers for $p_1X$. However, for the construction in the next section the property of being a point-finite uniform cover turns out to be more useful than being countable. The incompleteness of $p_1C([0, \omega_1])$ then follows by using a basis of point-finite covers. This is explained in section 3 along with several other consequences of the main theorem.

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2 The construction

As in the proof of Theorem 1.1 in [9], let

\[ M := \{ f \in C([0, \omega_1]) \mid f \text{ is monotone non-increasing, } f(0) = 1 \text{ and } f(\omega_1) = 0 \} . \]

With the subspace metric defined by the sup norm \(|-|\) on \(C([0, \omega_1])\), \(M\) is a complete metric space.

**Main Theorem.** Let \(U\) be a uniform structure on \(M\) such that

- the \(|-|\) topology on \(M\) is compatible with \(U\);
- the \(|-|\) uniformity on \(M\) is finer than \(U\); and
- \(U\) has a uniformity basis consisting of point-finite covers.

Then \(U\) is not complete.

The proof of the main theorem in this section is based on the proof of Th. 17 in [6] (pp. 58–60). It also incorporates elements of the proof of Th. 1.1 in [9]. It should be noted that [6, Th.17] deals with more general point characters of uniformities; the point-finite version that I prove here is a special case.

Write \(\omega^0 := \{1, 2, \ldots\}\). For every ordinal \(\beta < \omega_1\) let \(I_\beta: [0, \omega_1] \to \{0, 1\}\) be the characteristic function of the closed interval \([0, \beta]\). When \(n \in \omega^0\) and \(B = (\beta_0, \beta_1, \ldots, \beta_{n-1})\) is a finite sequence of ordinals \(< \omega_1\), let

\[ I_B := \max_{0 \leq k < n} \frac{(n-k)I_{\beta_k}}{n} \]

and note that \(I_B \in M\). When \(g \in M\) and \(\varepsilon > 0\), write

\[ \otimes[g, \varepsilon] := \{ f \in M \mid \|f - g\| \leq \varepsilon \} . \]

Covers of the form \(\{ \otimes[g, \varepsilon] \mid g \in M, \varepsilon > 0\}\), form a uniformity basis of the metric space \(M\). Hence any uniformity \(U\) satisfying the assumptions of the main theorem has a uniformity basis consisting of point-finite covers each of which is refined by the cover \(\{ \otimes[f, \varepsilon] \mid f \in M\}\) for some \(\varepsilon > 0\).

For \(U \subseteq M\), \(f \in M\) and \(n \in \omega^0\), the finite game \(G(U, f, n)\) is played by two players Alice and Bob as follows: Alice moves first, and then the players alternate in choosing countable ordinals; each choice must be larger than or equal to the ordinals already chosen in previous moves. The game ends when the players have made \(n\) moves each.

Thus each run of the game is a sequence \((\alpha_0, \beta_0, \alpha_1, \beta_1, \ldots, \beta_{n-2}, \alpha_{n-1}, \beta_{n-1})\) of ordinals such that \(\alpha_0 \leq \beta_0 \leq \alpha_1 \leq \beta_1 \leq \ldots \leq \beta_{n-2} \leq \alpha_{n-1} \leq \beta_{n-1} < \omega_1\); say that Bob wins this run if \(\otimes[f \lor I_B, 1/n] \subseteq U\), where \(B\) is the sequence \((\beta_0, \beta_1, \ldots, \beta_{n-1})\) of Bob’s moves.

A game position is a prefix of a run of the game. A strategy for Bob is a mapping that takes any \(U\), \(f\), \(n\) and a game position ending with Alice’s move as inputs and produces a countable ordinal as output, to be used as Bob’s next move. Say that Bob wins the game \(G(U, f, n)\) if he has a winning strategy.

We also need the modified game \(G(U, f, n; k)\) for every \(1 \leq k \leq n\); it has the same rules as \(G(U, f, n)\) and in addition the first \(n-k\) moves by each player must be zero; that is, \(\alpha_i = \beta_i = 0\) for \(0 \leq i \leq n - k - 1\). Thus \(G(U, f, n; n) = G(U, f, n)\).

Another modified game is \(G(U, f, n; k, \alpha)\) for \(1 \leq k \leq n\) and \(\alpha < \omega_1\). It has the same rules as \(G(U, f, n; k)\) and in addition Alice must play \(\alpha_{n-k} = \alpha\).
For \( f \in M, n \in \omega^0, 1 \leq k \leq n \) and \( \alpha < \omega_1 \), write
\[
\mathcal{W}(f, n) := \{ U \subseteq M | \text{Bob wins } \mathcal{G}(U, f, n) \}
\]
\[
\mathcal{W}(f, n; k) := \{ U \subseteq M | \text{Bob wins } \mathcal{G}(U, f, n; k) \}
\]
\[
\mathcal{W}(f, n; k, \alpha) := \{ U \subseteq M | \text{Bob wins } \mathcal{G}(U, f, n; k, \alpha) \}
\]

In the proof of the main theorem, the following lemma is used to show that a certain \( U \)-Cauchy filter of subsets of \( M \) does not converge.

**Lemma 1.** If \( U \in \mathcal{W}(f, n) \) for some \( f \in M \) and some \( n \in \omega^0 \) then the \( \| \cdot \| \) diameter of \( U \) is 1.

**Proof.** Play the game \( \mathcal{G}(U, f, n) \) twice, both times with Bob using his winning strategy. In the first run Alice plays any legal moves. The run produces \( f \vee I_B \in U \), where \( B \) is the sequence of Bob’s moves. Since \( f \vee I_B \in C([0, \omega_1]) \) and \( f \vee I_B(\omega_1) = 0 \), there is \( \alpha < \omega_1 \) such that \( f \vee I_B(\alpha) = 0 \). Play the game \( \mathcal{G}(U, f, n) \) again; this time Alice’s first move is \( \alpha \) and she plays any legal moves after that. Let \( B' \) be the sequence of Bob’s moves in the second run. Then \( f \vee I_{B'} \in U \) and \( f \vee I_{B'}(\alpha) = 1 \). □

By the next lemma, every \( \mathcal{W}(f, n; k) \) generates a \( \sigma \)-filter of subsets of \( M \).

**Lemma 2.** Let \( f \in M, n \in \omega^0, 1 \leq k \leq n \), and \( U_j \in \mathcal{W}(f, n; k) \) for \( j \in \omega \). Then
\[
\bigcap_{j \in \omega} U_j \in \mathcal{W}(f, 2n; 2k).
\]

**Proof.** Write \( U := \bigcap_{j \in \omega} U_j \). For every \( j \in \omega \) Bob has a winning strategy \( S_j \) for the game \( \mathcal{G}(U_j, f, n; k) \). Bob’s winning strategy for \( \mathcal{G}(U, f, 2n; 2k) \) is the following: In the game position \( \alpha_0 \leq \beta_0 \leq \ldots \leq \beta_{i-1} \leq \alpha_i \) where \( i \leq 2n-1 \) is even, Bob’s next move is \( \beta_i := \sup_{j \in \omega} \beta_{j,i} \), where \( \beta_{j,i} \) is chosen by Bob’s strategy \( S_j \) in the position
\[
\alpha_0 \leq \beta_0 \leq \alpha_2 \leq \ldots \leq \alpha_{i-2} \leq \beta_{i-2} \leq \alpha_i
\]

of the game \( \mathcal{G}(U_j, f, n; k) \). In the game position \( \alpha_0 \leq \beta_0 \leq \ldots \leq \beta_{i-1} \leq \alpha_i \) where \( i \leq 2n-1 \) is odd, Bob’s next move is \( \beta_i := \alpha_i \).

Let \( B = \langle \beta_0, \beta_1, \ldots, \beta_{2n-1} \rangle \) be the resulting sequence of Bob’s moves. For every \( j \in \omega \) let \( B_j \) be the sequence \( \langle \beta_{j,0}, \beta_{j,2}, \ldots, \beta_{j,2n-2} \rangle \) of the choices made by strategy \( S_j \) in the positions \( \alpha_0 \leq \beta_0 \leq \alpha_2 \leq \beta_2 \leq \ldots \leq \alpha_{i-2} \leq \beta_{i-2} \leq \alpha_i \) with \( i \) even. Then \( \star [f \vee I_{B_j}, 1/n] \subseteq U_j \) for every \( j \) because \( S_j \) is a winning strategy for Bob in \( \mathcal{G}(U_j, f, n; k) \). But \( \| (f \vee I_B) - (f \vee I_{B_j}) \| \leq 1/2n \) for every \( j \) and therefore \( \star [f \vee I_B, 1/2n] \subseteq U \). □

**Corollary 3.** Let \( f \in M \) and \( n \in \omega^0 \). Then \( \mathcal{W}(f, n) \subseteq \mathcal{W}(f, 2n) \).

**Corollary 4.** Let \( \mathcal{P} \) be a point-finite cover of \( M, f \in M, n \in \omega^0 \) and \( 1 \leq k \leq n \). Then the set \( \mathcal{P} \cap \mathcal{W}(f, n; k) \) is finite.

**Proof.** Assume to the contrary that there is a countable infinite set \( \mathcal{N} \subseteq \mathcal{P} \cap \mathcal{W}(f, n; k) \). Then \( \bigcap \mathcal{N} \in \mathcal{W}(f, 2n; 2k) \) by Lemma 2. Hence \( \bigcap \mathcal{N} \neq \emptyset \), which contradicts \( \mathcal{P} \) being point-finite. □

Our next goal is to prove that the finite set \( \mathcal{P} \cap \mathcal{W}(f, n; k) \) is not empty. This is the crucial step in the proof of the main theorem.
Lemma 5. Let $P$ be a point-finite cover of $M$, and let $n \in \omega^\omega$ be such that $P$ is refined by the cover \{\@g, 2/\$n \mid g \in M\}. Then $P \cap W(f, n; k) \neq \emptyset$ for every $f \in M$ and every $1 \leq k \leq n$.

Proof. In this proof letters $\lambda$ and $\beta$, with or without subscripts, stand for countable ordinals. For $f \in M$, $1 \leq k \leq n$ and $\alpha$, define the function $f^{k+1} \alpha \in M$ by $f^{k+1} \alpha := f \lor (k I_\alpha/n)$.

The proof proceeds by induction on $k$, starting with $k = 1$. In the game $G(f, n; 1)$ all except the last move by each player are $0$. Hence

\[ W(f, n; 1) = \{U \subseteq M \mid \forall \beta \exists \alpha : \exists [f^{k+1} \alpha, 1/n] \subseteq U\}. \]

Since $P$ is refined by \{\@g, 2/\$n \mid g \in M\}, there is $U_0 \in P$ for which \$g, 2/\$n \subseteq U_0$. Since \$g^{k+1} \alpha, 1/n \subseteq \circ [f^{k+1} \alpha, 1/n]$ for every $\beta$, it follows that Bob wins $G(U_0, f, n; 1)$ with any strategy. Hence $U_0 \in P \cap W(f, n; 1)$. That concludes the proof for $k = 1$.

For the induction step, take $k \leq n - 1$ and assume the conclusion of the lemma holds for every $f \in M$. Take any $f \in M$.

Claim: For every $\alpha$ there exists $U \in P$ for which Bob wins $G(U, f, n; k + 1, \alpha)$.

By the induction assumption with $f^{k+1} \alpha$ in place of $f$ we have $P \cap W(f^{k+1} \alpha, n; k) \neq \emptyset$, hence there is $U \in P$ and a winning strategy $S$ for Bob in the game $G(U, f^{k+1} \alpha, n; k)$. Bob’s winning strategy for $G(U, f, n; k + 1, \alpha)$ is to choose $\beta_{n-k-1} = \alpha$ and then follow strategy $S$ in the subsequent moves. That proves the claim.

Now observe that for $\alpha \leq \alpha'$ we have $W(f, n; k + 1, \alpha) \supseteq W(f, n; k + 1, \alpha')$. Since the sets $P \cap W(f, n; k + 1, \alpha)$ are finite by Corollary 4 there is $\alpha_0$ such that

\[ P \cap W(f, n; k + 1, \alpha_0) \subseteq P \cap W(f, n; k + 1, \alpha) \]

for every $\alpha$. We have $P \cap W(f, n; k + 1, \alpha_0) \neq \emptyset$ by the claim, and Bob wins $G(U, f, n; k + 1)$ for every $U \in P \cap W(f, n; k + 1, \alpha_0)$. That completes the induction step. \(\Box\)

Lemma 6. Let $U$ be a uniform structure on the set $M$ such that the metric uniformity of $M$ is finer than $U$ and $U$ has a uniformity basis consisting of point-finite covers. Let $f \in M$. Then there is a filter $F$ of subsets of $M$ such that

- $F$ is $U$-Cauchy; and
- for every $U \in F$ there is $n \in \omega$ for which Bob wins $G(U, f, 2^n)$.

Proof. For every $U$-uniform cover $P$ there is $r(P) \in \omega$ such that $P$ is refined by \{\@g, 2^{1-n} \mid g \in M\} for every $n \geq r(P)$. Define

\[ F_0 := \left\{ \bigcap (P \cap W(f, 2^n)) \mid P \text{ is a point-finite } U\text{-uniform cover and } n \geq r(P) \right\}. \]

If $F_0$ has the finite intersection property then clearly the filter $F$ generated by $F_0$ is $U$-Cauchy.

Take any finite subset \{A_0, A_1, \ldots, A_j\} of $F_0$ and let $A := \bigcap_{i=0}^j A_i$. There are point-finite $U$-uniform covers $P_i$ and $n_i$ such that $n_i \geq r(P_i)$ and $A_i = \bigcap (P_i \cap W(f, 2^{n_i}))$ for $0 \leq i \leq j$. Write $n := \max_i n_i$. The sets $P_i \cap W(f, 2^{n_i})$ are finite by Corollary 4 hence

\[ A = \bigcap_{0 \leq i < j} \bigcap (P_i \cap W(f, 2^{n_i})) \in W(f, 2^{n+1}) \]
by Lemma 2 and Corollary 3. Thus $F_0$ has the finite intersection property and the filter generated by $F_0$ has the second property in the lemma.

**Proof of the main theorem.** Take any $f \in M$ (for example $f = I_0$). Let $F$ be a filter with the two properties in Lemma 6. By Lemma 1, $F$ does not converge in the topology of $M$.

3 Corollaries

Following Pelant [6], I have written the main theorem in a form that not only applies to the space $p_1M$, but is also useful for questions about point-finite covers.

3.1 Completeness of the separable modification

All that is now needed to prove that the space $p_1C([0,\omega_1])$ is not complete is the following result of Vidossich [12].

**Lemma 7.** Let $X$ be any uniform space. The uniformity $p_1X$ has a basis consisting of countable point-finite covers.

Combining the main theorem in section 2 with Lemma 7 we obtain:

**Theorem 8.** The uniform spaces $p_1M$ and $p_1C([0,\omega_1])$ are not complete.

Pelant [5] constructed a complete metric space $X$ for which $p_1X$ is not complete. Although using a different terminology, the proof of Theorem 8 here is a modification of Pelant’s construction in [5] and [6]. The fact that we can take $X = C([0,\omega_1])$ is not surprising in view of [9 Thm 1.1].

The theorem has an interesting application in the theory of **uniform measures** [3]. For any uniform space $X$, let $U_b(X)$ be the space of bounded real-valued functions on $X$. Let $M_u(X)$ be the space of uniform measures; that is, the linear functionals on $U_b(X)$ that are continuous on every bounded uniformly equicontinuous subset of $U_b(X)$ in the $X$-pointwise topology.

In their work on topological centres, Ferri and Neufang [1] defined also the space of linear functionals $U_b(X)$ that are **sequentially** continuous on bounded uniformly equicontinuous subsets of $U_b(X)$. In section 8.1 of [3], where $M_{u\sigma}(X)$ denotes the space of such functionals, I prove that $M_{u\sigma}(X) = M_u(p_1X)$. The following corollary is then an immediate consequence of Theorem 8 using the relationship between the space $M_u(X)$ and the completion of $X$ (section 6.5 in [3]).

**Corollary 9.** $M_{u\sigma}(M) \neq M_u(M)$ and $M_{u\sigma}(C([0,\omega_1])) \neq M_u(C([0,\omega_1]))$.  

I don’t know any simple proof of Corollary 9, or in fact any other construction, without using measurable cardinals, of a uniform space $X$ such that $M_{u\sigma}(X) \neq M_u(X)$.

3.2 Point-finite refinements of uniform covers

Let $X$ be a metric space. Since $X$ is paracompact, every open cover of $X$ is refined by an open locally finite cover. Thus it is natural to ask, as Stone [11] and then Isbell [2] p.144] did: Is every uniform cover of $X$ refined by a uniformly locally finite uniform cover? By [2 VIII.3], an equivalent question is: Is every uniform cover of $X$ refined by a point-finite uniform cover?
Pelant [4,7] and Ščepin [10] constructed metric spaces $X$ for which the answer is negative. In a later paper, Pelant, Holický and Kalenda [9] prove that the answer is no for $X = C([0, \omega_1])$; this immediately follows also from the main theorem in section [2].

Let $X$ be a Banach space with the metrizable uniformity defined by its norm. Pelant [8] proved that $X$ has a uniformity basis consisting of point-finite covers if and only if $X$ is uniformly homeomorphic to a subset of $c_0(\Gamma)$ for some index set $\Gamma$. Thus we get Theorem 1.1 in [9]: The Banach space $C([0, \omega_1])$ is not uniformly homeomorphic to a subset of $c_0(\Gamma)$ for any $\Gamma$. This and related results are discussed in more detail in [9].

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