An operator-theoretic approach to invariant integrals on quantum homogeneous \( \mathfrak{sl}_{n+1}(\mathbb{R}) \)-spaces

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Abstract

We present other examples illustrating the operator-theoretic approach to invariant integrals on quantum homogeneous spaces developed by Kürsten and the second author. The quantum spaces are chosen such that their coordinate algebras do not admit bounded Hilbert space representations and their self-adjoint generators have continuous spectrum. Operator algebras of trace class operators are associated to the coordinate algebras which allow interpretations as rapidly decreasing functions and as finite functions. The invariant integral is defined as a trace functional which generalizes the well-known quantum trace. We argue that previous algebraic methods would fail for these examples.

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1 Introduction

In a series of papers, Shklyarov, Sinel’shchikov and Vaksman studied non-commutative analogues of bounded symmetric domains of non-compact type [8–12]. The corresponding non-commutative algebras can be described by stating the commutation relations of their generators which are viewed as coordinate functions. As a first step toward the development of function theory and harmonic analysis, the authors defined covariant differential calculi and invariant integrals on these algebras. Naturally, because of the non-compactness, the invariant integral does not exist on polynomial functions in the coordinates. To circumvent the problem, the authors introduced algebras of finite functions on the quantum space. Basically, this was done by first considering a faithful Fock-type representation, where some distinguished self-adjoint operators have a discrete joint spectrum, and then adjoining functions with finite support (on the joint spectrum) to the (represented) algebra of coordinate functions [3, 8, 10, 11]. The algebra of finite functions can be equipped with a symmetry action of a quantum group,
and the invariant integral is defined as a generalization of the (well-known) quantum trace.

As is customary when defining quantum spaces, the approach of Shklyarov, Sinel’-shchikov and Vaksman is almost completely algebraic, using only a minimal amount of topology. On the other hand, Hilbert space representations of the coordinate algebras provide a systematic tool for exploring topological questions. This observation was the starting point for the operator-theoretic approach to invariant integrals proposed by Körsten and the second author [3]. The use of operator-theoretic methods has several advantages. For instance, it allows to adjoin a wider class of integrable functions to the coordinate algebra, one can prove density and continuity results, and it works for all Hilbert space representations in the same way. In particular, the operator-theoretic methods apply to algebras which to not admit bounded representations and where the “distinguished self-adjoint operators” do not have a discrete spectrum.

The objective of the present paper is to provide an example that can easily be treated by the operator-theoretic approach developed in [3] but for which purely algebraic methods seem to fail. In order to keep the exposure as close as possible to the previous paper [3], we take the same coordinate algebras and change only the involution and the values of the deformation parameter. The corresponding quantum spaces are known as real quantum hyperboloid and real \( q \)-Weyl algebra. The difference of the involution has profound consequences: First, there do not exist bounded Hilbert space \( * \)-representations, and second, the \( * \)-representations of the coordinate algebra are determined by self-adjoint operators with continuous spectrum [5, 6]. This impedes the description of integrable functions as finite functions of the self-adjoint generators since these operators are not of trace class and the generalized quantum trace will not exist on them. Nevertheless, we shall see that the methods from [3] can be applied even in this situation.

Let us briefly outline the main ideas of [3]. Suppose we are given a Hopf \( * \)-algebra \( \mathcal{U} \) acting on a \( * \)-algebra of coordinate functions \( \mathcal{X} \). It is natural to require that the action respects the Hopf \( * \)-structure of \( \mathcal{U} \) and the multiplicative structure of \( \mathcal{X} \). In other words, we assume that \( \mathcal{X} \) is a \( \mathcal{U} \)-module \( * \)-algebra. Let \( \pi : \mathcal{X} \to \mathcal{L}^+(\mathcal{D}) \) be a \( * \)-representation of \( \mathcal{X} \) into a \( * \)-algebra of closeable operators on a pre-Hilbert space \( \mathcal{D} \). Starting point of the operator-theoretic approach is an operator expansion of the action. This means that for each \( Z \in \mathcal{U} \) there exists a finite number of operators \( L_i, R_i \in \mathcal{L}^+(\mathcal{D}) \) such that

\[
\pi(Z \triangleright x) = \sum_i L_i \pi(x) R_i, \quad x \in \mathcal{X},
\]

where \( \triangleright \) denotes the (left) \( \mathcal{U} \)-action on \( \mathcal{X} \). Obviously, it suffices to know the operators \( L_i, R_i \) for a set of generators of \( \mathcal{U} \). The operator expansion allows us to extend the action to the \( * \)-algebra \( \mathcal{L}^+(\mathcal{D}) \) turning it into a \( \mathcal{U} \)-module \( * \)-algebra. Inside \( \mathcal{L}^+(\mathcal{D}) \), we find two \( \mathcal{U} \)-module \( * \)-subalgebra of particular interest. The first one is the algebra of finite rank operators and will be considered as the algebra of finite functions associated to \( \mathcal{X} \). The second one is an algebra of trace class operators which is stable under multiplication by operators from the operator expansion. This algebra will be viewed as an algebra of functions which vanish sufficiently rapidly at “infinity”. On both algebras, the invariant integral can be defined by a trace formula resembling the quantum trace of finite dimensional representations of \( \mathcal{U} \).
To make the paper more readable, we first discuss in Section 3 the lowest dimensional case of a $q$-Weyl algebra, namely the so-called real quantum hyperboloid. The general case will be treated in Section 4.

2 Preliminaries

Throughout the paper, $q$ is a complex number such that $|q| = 1$ and $q^4 \neq 1$. The letter $i$ stands for the imaginary unit, and we set $\lambda := q - q^{-1}$. Note that $\lambda \in i\mathbb{R}$.

Let $\mathcal{U}$ be a Hopf $*$-algebra with comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$. Adopting Sweedler’s notation, we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$ for $x \in \mathcal{U}$. An $*$-algebra $\mathcal{X}$ is called left $\mathcal{U}$-module $*$-algebra [2] if there is a left $\mathcal{U}$-action $\triangleright$ on $\mathcal{X}$ such that

\begin{equation}
 f \triangleright (xy) = (f \triangleright x)(f \triangleright y), \quad (f \triangleright x)^* = S(f)^* \triangleright x^*, \quad x, y \in \mathcal{X}, \quad f \in \mathcal{U}. \tag{2}
\end{equation}

For unital algebras, one additionally requires

\begin{equation}
 f \triangleright 1 = \varepsilon(f) 1, \quad f \in \mathcal{U}. \tag{3}
\end{equation}

By an invariant integral $h$ we mean a linear functional $h$ on $\mathcal{X}$ satisfying

\begin{equation}
 h(f \triangleright x) = \varepsilon(f) h(x), \quad x \in \mathcal{X}, \quad f \in \mathcal{U}. \tag{4}
\end{equation}

Synonymously, we refer to it as $\mathcal{U}$-invariant.

In this paper, the Hopf $*$-algebra under consideration will be a $q$-deformation of the universal enveloping algebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$ with non-compact real form. Let $n \in \mathbb{N}$. Recall that the Cartan matrix $(a_{ij})_{i,j=1}^n$ of $\mathfrak{sl}_{n+1}(\mathbb{C})$ is given by $a_{j,j+1} = a_{j+1,j} = -1$ for $j = 1, \ldots, n-1$, $a_{jj} = 2$ for $j = 1, \ldots, n$, and $a_{ij} = 0$ otherwise. The Hopf $*$-algebra $\mathcal{U}_{q}(\mathfrak{sl}_{n+1}(\mathbb{R}))$ is generated by $K_j, K_j^{-1}, E_j, F_j, j = 1, \ldots, n$, with relations [2]

\begin{align*}
 K_i K_j &= K_j K_i, \quad K_j^{-1} K_j = K_j K_j^{-1} = 1, \quad K_i E_j = q^{a_{ij}} E_j K_i, \quad K_i F_j = q^{-a_{ij}} F_j K_i, \\
 E_i E_j - E_j E_i &= 0, \quad i \neq j \pm 1, \quad E_i^2 E_{j \pm 1} - (q + q^{-1}) E_j E_{j \pm 1} E_i + E_{j \pm 1} E_j^2 = 0, \\
 F_i F_j - F_j F_i &= 0, \quad i \neq j \pm 1, \quad F_i^2 F_{j \pm 1} - (q + q^{-1}) F_j F_{j \pm 1} F_i + F_{j \pm 1} F_j^2 = 0, \\
 E_i F_j - E_j F_i &= 0, \quad i \neq j, \quad E_j F_j - F_j E_j = \lambda^{-1} (K_j - K_j^{-1}), \quad j = 1, \ldots, n,
\end{align*}

comultiplication, counit and antipode given by

\begin{align*}
 \Delta(E_j) &= E_j \otimes 1 + K_j \otimes E_j, \quad \Delta(F_j) = F_j \otimes K_j^{-1} + 1 \otimes F_j, \quad \Delta(K_j) = K_j \otimes K_j, \\
 \varepsilon(K_j) &= \varepsilon(K_j^{-1}) = 1, \quad \varepsilon(E_j) = \varepsilon(F_j) = 0, \\
 S(K_j) &= K_j^{-1}, \quad S(E_j) = -K_j^{-1} E_j, \quad S(F_j) = -F_j K_j,
\end{align*}

and involution

\begin{align*}
 K_i^* &= K_i, \quad E_j^* = E_j, \quad F_j^* = F_j.
\end{align*}
If \( n = 1 \), we write \( K, K^{-1}, E, F \) rather than \( K_1, K_1^{-1}, E_1, F_1 \). These generators are hermitian and satisfy the relations

\[
KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F, \\
EF - FE = (K - K^{-1})/(q - q^{-1}).
\]

We turn now to operator-theoretic preliminaries. Let \( \mathcal{H} \) be a Hilbert space. For a closable densely defined operator \( T \) on \( \mathcal{H} \), we denote by \( D(T) \), \( \bar{T} \), \( T^* \) and \( |T| \) its domain, closure, adjoint and modulus, respectively. Given a dense linear subspace \( D \) of \( \mathcal{H} \), we set

\[
L^+(D) := \{ x \in \text{End}(D) \mid D \subset D(x^*), x^*D \subset D \}.
\]

Clearly, \( L^+(D) \) is a unital algebra of closeable operators. It becomes a \(*\)-algebra if we define the involution by \( x \mapsto x^+ := x^*\big|_D \). Since it should cause no confusion, we shall continue to write \( x^* \) in place of \( x^+\big|_D \). Unital \(*\)-subalgebras of \( L^+(D) \) are called \( O^*\)-algebras.

Given an \( O^*\)-algebra \( \mathfrak{A} \), set

\[
\mathbb{B}_1(\mathfrak{A}) := \{ t \in L^+(D) \mid \bar{t} \subset D, \bar{t}^* \subset D, \text{tr}(\bar{t}^*t) \text{ is a trace class for all } a, b \in \mathfrak{A} \}.
\]

It follows from [4, Lemma 5.1.4] that \( \mathbb{B}_1(\mathfrak{A}) \) is a \(*\)-subalgebra of \( L^+(D) \). Next, let

\[
\mathbb{F}(D) := \{ x \in L^+(D) \mid \bar{x} \text{ is bounded, } \dim(\bar{x}^*D) < \infty, \bar{x}D \subset D, \bar{D} \subset D \}.
\]

Note that each element \( A \in \mathbb{F}(D) \) can be written as \( A = \sum_{i=1}^n \alpha_i e_i \otimes f_i \), where \( n \in \mathbb{N}, \alpha_i \in C, f_i, e_i \in D \), and \( (e_i \otimes f_i)(x) := f_i(x)e_i \) for \( x \in D \). Obviously, \( \mathbb{F}(D) \subset \mathbb{B}_1(\mathfrak{A}) \) and \( 1 \notin \mathbb{B}_1(\mathfrak{A}) \) if \( \dim(\mathcal{H}) = \infty \).

By a \(*\)-representation \( \pi \) of a \(*\)-algebra \( \mathcal{X} \) on a domain \( D \), we mean a \(*\)-homomorphism \( \pi : \mathfrak{A} \to L^+(D) \). For notational simplicity, we usually suppress the representation and write \( x \) instead of \( \pi(x) \) when no confusion can arise.

## 3 Real quantum hyperboloid

The \(*\)-algebra \( A_q(1; \mathbb{R}) \) of coordinate functions on the real quantum hyperboloid is generated by two hermitian elements \( x \) and \( y \) fulfilling

\[
xy - q^2yx = 1 - q^2.
\]

Set

\[
Q := \lambda^{-1}(yx - xy) = q(1 - yx).
\]

Since \( \lambda \in i\mathbb{R} \), we have \( Q^* = Q \). The commutation relations of \( Q \) with \( x \) and \( y \) are given by

\[
Qy = q^2yQ, \quad Qx = q^{-2}xQ.
\]
Note that the two generators of the quantum disc [1] satisfy the same relation as $x$ and $y$, only the involution is different ($x^* = y$). In [12, Section 8], one can find an explicit construction of a $U_q(\mathfrak{su}_{1,1})$-action on the quantum disc. Replacing in [12] the involution on $v_\pm(0)$ by $v_\pm(0)^* = v_{\pm}(0)$ and performing the construction for $U_q(\mathfrak{sl}_2(\mathbb{R}))$ yields the following $U_q(\mathfrak{sl}_2(\mathbb{R}))$-action on $A_q(1; \mathbb{R})$:

$$K^\pm \triangleright y = q^{\pm 2} y, \quad E \triangleright y = i q y^2, \quad F \triangleright y = i,$$ \hspace{1cm} (10)

$$K^\pm \triangleright x = q^{\mp 2} x, \quad E \triangleright x = -i q^{-1}, \quad F \triangleright x = -i q^2 x^2. \hspace{1cm} (11)$$

By [12], this action turns $A_q(1; \mathbb{R})$ into a $U_q(\mathfrak{sl}_2(\mathbb{R}))$-module $*$-algebra.

The crucial step toward an invariant integration theory on the quantum hyperboloid is an operator expansion (1) of the action. This will be done in the next lemma.

**Lemma 1.** Assume that $\pi : A_q(1; \mathbb{R}) \to \mathcal{L}^+(D)$ is a $*$-representation of $A_q(1; \mathbb{R})$ such that $Q^{-1} \in \mathcal{L}^+(D)$. Set

$$A := -i \lambda^{-1} y, \quad B := -i \lambda^{-1} q^{-1} Q^{-1} x. \hspace{1cm} (12)$$

The formulas

$$K \triangleright f = QfQ^{-1}, \quad K^{-1} \triangleright f = Q^{-1} f Q,$$ \hspace{1cm} (13)

$$E \triangleright f = A f - QfQ^{-1} A,$$ \hspace{1cm} (14)

$$F \triangleright f = B f Q - q^2 f Q B$$ \hspace{1cm} (15)

applied to $f \in A_q(1; \mathbb{R})$ define an operator expansion of the action $\triangleright$ on $A_q(1; \mathbb{R})$. If $f$ is taken from $\mathcal{L}^+(D)$, then the same formulas turn the $O^*$-algebra $\mathcal{L}^+(D)$ into a $U_q(\mathfrak{sl}_2(\mathbb{R}))$-module $*$-algebra.

**Proof.** First we show that (13)–(15) define an action on $\mathcal{L}^+(D)$ which turns $\mathcal{L}^+(D)$ into a $U_q(\mathfrak{sl}_2(\mathbb{R}))$-module $*$-algebra. A straightforward calculation shows that

$$QA = q^2 A Q, \quad QB = q^{-2} B Q, \quad AB - BA = -\lambda^{-1} Q^{-1}. \hspace{1cm} (16)$$

That the action is well-defined can be proved by direct verification using (16). As a sample,

$$(EF - FE) \triangleright f = AB f Q + Q f B A - B A f Q - Q f A B$$

$$= (AB - BA) f Q - Q f (AB - BA)$$

$$= \lambda^{-1} (Q f Q^{-1} - Q^{-1} f Q) = \lambda^{-1} (K - K^{-1}) \triangleright f.$$

for all $f \in \mathcal{L}^+(D)$.

To show that $\mathcal{L}^+(D)$ is a $U_q(\mathfrak{sl}_2(\mathbb{R}))$-module $*$-algebra, it suffices to verify (2) and (3) for the generators $E$, $F$, $K$ and $K^{-1}$. Observe that $K^{\pm 1} \triangleright 1 = 1 = \varepsilon(K^{\pm 1}) 1, \quad E \triangleright 1 = 0 = \varepsilon(E) 1$ and $F \triangleright 1 = 0 = \varepsilon(F) 1$, thus (3) holds. Let $f, g \in \mathcal{L}^+(D)$. Then

$$K^{\pm 1} \triangleright (f g) = Q^{\pm 1} f g Q^{\mp 1} = Q^{\pm 1} f Q^{\mp 1} Q^{\pm 1} g Q^{\mp 1} = (K^{\pm 1} \triangleright f) (K^{\pm 1} \triangleright g),$$

$$(E \triangleright f) g + (K \triangleright f) (E \triangleright g) = (A f - QfQ^{-1} A) g + QfQ^{-1} (A g - QgQ^{-1} A)$$

$$= A f g - Q f g Q^{-1} A = E \triangleright (f g),$$
and analogously \((F \triangleright f)(K^{-1} \triangleright g) + f(F \triangleright g) = F \triangleright (fg)\). As \(Q^* = Q\), we get
\[
(K^{\pm 1} \triangleright f)^* = (Q^{\mp 1}fQ^{\mp 1})^* = Q^{\mp 1}f^*Q^{\mp 1} = K^{\pm 1} \triangleright f^* = S(K^{\pm 1}) \triangleright f^*.
\]

Note that also \(A^* = A\) and \(B^* = B\) since \(\bar{q} = q^{-1}\) and \(i\lambda^{-1} \in \mathbb{R}\). Thus
\[
(E \triangleright f)^* = f^*A - AQ^{-1}f^*Q = -EK^{-1} \triangleright f^* = S(E)^* \triangleright f^*,
\]
\[
(F \triangleright f)^* = Qf^*B - q^{-2}BQf^* = Q(q^2f^*QB - Bf^*Q)Q^{-1} = -KF \triangleright f^* = S(F)^* \triangleright f^*.
\]

This proves (2). Therefore the \(\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))\)-action defined by (13)–(15) turns \(\mathcal{L}^+(D)\) into a \(\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))\)-module *-algebra.

To demonstrate that Equations (13)–(15) define an operator expansion of the action \(\triangleright\) on \(\mathcal{A}_q(1; \mathbb{R})\), it now suffices to verify it for the generators \(x\) and \(y\). The result follows then by applying the first relation of (2). Using (7), (9) and (12), we obtain
\[
K^{\pm 1} \triangleright y = Q^{\mp 1}yQ^{\mp 1} = q^{\mp 2}y, \quad K^{\pm 1} \triangleright x = Q^{\mp 1}xQ^{\mp 1} = q^{\mp 2}x,
\]
\[
E \triangleright y = Ay - QyQ^{-1}A = -i\lambda^{-1}(y^2 - q^2y^2) = iqy^2,
\]
\[
E \triangleright x = Ax - QxQ^{-1}A = -iq^{-2}\lambda^{-1}(q^2yx - xy) = -iq^{-1},
\]
and similarly \(F \triangleright y = i, F \triangleright x = -iq^2x^2\).

The aim of this section is to define an invariant integral on an appropriate class of operators. The problem arises because there does not exist a normalized invariant integral on \(\mathcal{A}_q(1; \mathbb{R})\). This can be seen as follows: If there were a \(\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))\)-invariant functional \(h\) on \(\mathcal{A}_q(1; \mathbb{R})\) satisfying \(h(1) = 1\), then
\[
1 = h(1) = -ih(F \triangleright y) = -i\varepsilon(F)h(y)
\]
would contradict \(\varepsilon(F) = 0\).

In [3], the quantum trace formula \(\text{tr}_q(X) := \text{tr}(XK^{-1})\) was generalized by replacing \(K^{-1}\) with the operator which realizes the operator expansion of \(K^{-1}\) on \(\mathcal{L}^+(D)\). The element \(X\) should belong to an algebra which has the property that the traces taken on the Hilbert space \(\mathcal{H} = \mathcal{D}\) exist. Two algebras with this property are described in Equations (5) and (6). The following proposition shows that the generalized quantum trace formula does define an invariant integral on these algebras.

**Proposition 2.** Suppose that \(\pi : \mathcal{A}_q(1; \mathbb{R}) \to \mathcal{L}^+(D)\) is a *-representation of \(\mathcal{A}_q(1; \mathbb{R})\) such that \(Q^{-1} \in \mathcal{L}^+(D)\). Let \(\mathfrak{A}\) be the \(O^*\)-algebra generated by the elements of \(\pi(\mathcal{A}_q(1; \mathbb{R}))\) and \(Q^{-1}\). Then the *-algebras \(\mathcal{B}_1(\mathfrak{A})\) and \(\mathcal{F}(D)\) defined in (5) and (6), respectively, are \(\mathcal{U}_q(\mathfrak{sl}_2(\mathbb{R}))\)-module *-algebras, where the action is given by (13)–(15). The linear functional
\[
h(g) := c \text{tr}(gQ^{-1}), \quad c \in \mathbb{R},
\]
defines an invariant integral on both \(\mathcal{B}_1(\mathfrak{A})\) and \(\mathcal{F}(D)\).
Proof. It is obvious from the definitions of \( B_1(\mathfrak{A}) \) and \( F(D) \) that both algebras are stable under the \( U_q(\mathfrak{sl}_2(\mathbb{R})) \)-action given by (13)–(15). Moreover, by Lemma 1, \( B_1(\mathfrak{A}) \) and \( F(D) \) are \( U_q(\mathfrak{sl}_2(\mathbb{R})) \)-module \(*\)-algebras.

Since the action is associative and \( \varepsilon \) is a homomorphism, it suffices to show the invariance for generators. Let \( g \in B_1(\mathfrak{A}) \). It follows from [4, Corollary 5.1.14] that \( \text{tr}(agb) = \text{tr}(gab) = \text{tr}(bag) \) for all \( a, b \in \mathfrak{A} \). Hence

\[
\begin{align*}
    h(K^{\pm 1} \triangleright g) &= h(Q^{\pm 1} g Q^{\mp 1} Q^{-1}) = \text{tr}(gQ^{-1}) = \varepsilon(K^{\pm 1}) h(g), \\
    h(E \triangleright g) &= h(A q^{1/2} - g q^{1/2} A^{-1} q^{1/2}) = \text{tr}(A g Q^{-1}) - \text{tr}(A g Q^{-1}) = 0 = \varepsilon(E) h(g), \\
    h(F \triangleright g) &= q \text{tr}(B g - q^2 g B Q^{-1}) = q \text{tr}(B g) - \text{tr}(B g) = 0 = \varepsilon(F) h(g),
\end{align*}
\]

where we used \( Q B = q^{-2} B Q \) in the last line. This proves the assertion for \( B_1(\mathfrak{A}) \).

Since \( F(D) \subset B_1(\mathfrak{A}) \), the same arguments apply to \( F(D) \).

Note that Proposition 2 was proved without referring explicitly to the \(*\)-representation \( \pi : A_q(1; \mathbb{R}) \rightarrow L^\infty(D) \). The only assumption on \( \pi \) was that \( Q^{-1} \in L^\infty(D) \). However, it is a priori not clear whether such a representation exists. That the answer to this question is affirmative was shown in [5]. For the convenience of the reader, we summarize the results from [5] (see also [6]).

First we introduce some notations. Write \( q = e^{i\varphi} \) with \( |\varphi| < \pi \) and \( s(\varphi) \) for the sign of \( \varphi \). Consider the Pauli matrices

\[
\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Let \( \mathcal{K} \) be a Hilbert space. The Pauli matrices act on \( \mathcal{K} \oplus \mathcal{K} \) in an obvious way. We denote by \( T \) and \( P \) the multiplication operator by the variable \( t \) and the differential operator \( i\frac{\partial}{\partial t} \) acting on \( \mathcal{L}^2(\mathbb{R}) \), respectively. An operator \( \omega \) on \( \mathcal{K} \) is called a symmetry if \( \omega \) is unitary and self-adjoint.

For the definition of an integrable (well-behaved) \(*\)-representation, we refer the reader to [5] (see also [7]). By [5, Theorem 3.7], each integrable \(*\)-representation of \( A_q(1; \mathbb{R}) \) such that \( x \) and \( y \) are self-adjoint operators and that \( \ker Q = \{ 0 \} \) is unitarily equivalent to a representation which is given by one of the following models.

(i): \( y = e^T \otimes \omega \), \( x = ((-1)^k q e^{2(i\varphi-k\pi)} P + 1) e^{-T} \otimes \omega \)
\( \text{on } \mathcal{H} = \mathcal{L}^2(\mathbb{R}) \otimes \mathcal{K} \), where \( \omega \) is a symmetry on \( \mathcal{K} \) and \( k \in \{ 0, s(\varphi) \} \).

(ii): \( y = e^T \otimes \sigma_1 \), \( x = q e^{(2\varphi-s(\varphi)\pi)} P e^{-T} \otimes \sigma_0 \sigma_1 + e^{-T} \otimes \sigma_1 \)
\( \text{on } \mathcal{H} = \mathcal{L}^2(\mathbb{R}) \otimes (\mathcal{K} \oplus \mathcal{K}) \).

The operator \( Q \) and its inverse are given by

(i): \( Q = (-1)^{k+1} e^{2(i\varphi-k\pi)} P \otimes 1 \), \( Q^{-1} = (-1)^{k+1} e^{-2(i\varphi-k\pi)} P \otimes 1 \),

(ii): \( Q = -e^{(2\varphi-s(\varphi)\pi)} P \otimes \sigma_0 \), \( Q^{-1} = -e^{-2(\varphi-s(\varphi)\pi) P} \otimes \sigma_0 \).
There exists a dense linear space \( D \subset \mathcal{H} \) such that \( D \) is an invariant core for each of the self-adjoint operators \( x, y, Q \) and \( Q^{-1} \) and the commutation relation of these operators are pointwise satisfied on \( D \). For instance, set

\[
\mathcal{F} := \text{Lin}\{e^{-a^2 + \gamma t}; \; \epsilon > 0, \; \gamma \in \mathbb{C}\},
\]

and take \( D = \mathcal{F} \otimes K \) and \( D = \mathcal{F} \otimes (K \oplus K) \) for representations of type (I) and (II), respectively, (see [6, Proposition 2]). This proves, in particular, the existence of representations which satisfy the assumptions of Proposition 2.

Motivated by a similar result in [3], we view \( B_q(1; \mathbb{R}) \) as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at “infinity” and \( \mathcal{F}(D) \) as the infinitely differentiable functions with compact support.

Note that the representation theory of \( A_q(1; \mathbb{R}) \) is much more subtle in comparison with the quantum disc treated in [3]. Whereas for the quantum disc one can define algebras of functions which vanish sufficiently rapidly at “infinity” and an invariant integral on purely algebraic level (see [3, Lemma 3.4] and [3, Proposition 3.3]), the same method does not apply to \( A_q(1; \mathbb{R}) \). For instance, since the self-adjoint operator \( Q \) has continuous spectrum, there does not exist a non-zero continuous function \( \psi \) on \( \sigma(Q) = \mathbb{R} \) such that \( \psi(Q)Q^{-1} \) is of trace class. Furthermore, apart from the trivial representation \( x = \alpha, \; y = \alpha^{-1} \), where \( \alpha \in \mathbb{R} \setminus \{0\} \), the algebra \( A_q(1; \mathbb{R}) \) does not admit other (irreducible) bounded \(*\)-representations. Nevertheless, we succeeded in establishing an invariant integration theory on \( A_q(1; \mathbb{R}) \).

4 Real \( q \)-Weyl algebra

Recall that \( |q| = 1, \; q^4 \neq 1 \). Let \( n \in \mathbb{N} \). The \(*\)-algebra \( A_q(n; \mathbb{R}) \) with hermitian generators \( x_1, \ldots, x_n, y_1, \ldots, y_n \) and relations

\[
\begin{align*}
  y_k y_l &= q y_l y_k, \quad k < l, \quad (20) \\
  x_k x_l &= q^{-1} x_l x_k, \quad k < l, \quad (21) \\
  x_l y_k &= q y_k x_l, \quad k \neq l, \quad (22) \\
  x_k y_k &= q^2 y_k x_k - (1 - q^2) \sum_{j=k+1}^{n} q^{j-k} y_j x_j + (1 - q^2) q^{n-k}, \quad k < n, \quad (23) \\
  x_n y_n &= q^2 y_n x_n + (1 - q^2) \quad (24)
\end{align*}
\]

is called real \( q \)-Weyl algebra [6].

Define the hermitian elements

\[
Q_j = \lambda^{-1}(y_j x_j - x_j y_j), \quad j \leq n, \quad Q_{n+1} = 1. \quad (25)
\]

A straightforward calculation shows that

\[
\begin{align*}
  Q_k y_j &= y_j Q_k, \quad j < k, \quad Q_k y_j = q^2 y_j Q_k, \quad j \geq k, \quad (26) \\
  Q_k x_j &= x_j Q_k, \quad j < k, \quad Q_k x_j = q^{-2} x_j Q_k, \quad j \geq k. \quad (27)
\end{align*}
\]

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This immediately implies

\[ Q_k Q_l = Q_l Q_k, \quad \text{for all } k, l \leq n + 1. \]  

(28)

From the definition of \( Q_k \) and (23), we obtain

\[ Q_k = \lambda^{-1}(y_k x_k - x_k y_k) = \lambda^{-1}(1 - q^2)(y_k x_k + \sum_{j=0}^{n} q^{j-k} y_j x_j - q^{n-k}) \]

\[ = q(q^{n-k} - \sum_{j=0}^{n} q^{j-k} y_j x_j), \]

so that

\[ q^{-1}Q_k = q^{-n-k} - \sum_{j=0}^{n} q^{j-k} y_j x_j = -y_k x_k + q(q^{n-(k+1)} - \sum_{j=0}^{n} q^{j-(k+1)} y_j x_j) \]

\[ = -y_k x_k + Q_{k+1}. \]

Hence

\[ y_k x_k = (Q_{k+1} - q^{-1}Q_k), \quad x_k y_k = (Q_{k+1} - qQ_k), \]  

(29)

where the second equation follows from the first one by taking adjoints. Equation (29) also holds for \( k = n \). Furthermore, from (29),

\[ x_k y_k - q^2 y_k x_k = (1 - q^2)Q_{k+1}. \]  

(30)

As in Section 3, write \( q = e^{i\varphi} \) with \( |\varphi| < \pi \) and set \( q_0 := e^{i\varphi/2} \). Consider the \( \mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{R})) \)-action on \( \mathcal{A}_q(n; \mathbb{R}) \) defined by

\[ j < n : \quad E_j \triangleright y_k = 0, \quad k \neq j + 1, \quad E_j \triangleright y_{j+1} = -i q_0^{-1} y_j, \quad E_j \triangleright x_k = 0, \quad k \neq j, \quad E_j \triangleright x_{j+1} = i q_0^{-1} x_j, \quad (31) \]

\[ F_j \triangleright y_k = 0, \quad k \neq j, \quad F_j \triangleright y_j = i q_0 y_{j+1}, \quad F_j \triangleright x_k = 0, \quad k \neq j + 1, \quad F_j \triangleright x_{j+1} = -i q_0 x_j, \quad (32) \]

\[ K_j \triangleright y_k = y_k, \quad k \neq j, j + 1, \quad K_j \triangleright y_{j+1} = q y_j, \quad K_j \triangleright y_j = q^{-1} y_{j+1}, \quad (33) \]

\[ K_j \triangleright x_k = x_k, \quad k \neq j, j + 1, \quad K_j \triangleright x_{j+1} = q^{-1} x_j, \quad K_j \triangleright x_j = q x_{j+1}, \quad (34) \]

\[ j = n : \quad E_n \triangleright y_k = i q y_n y_k, \quad k < n, \quad E_n \triangleright y_n = i q y_n^2, \quad (35) \]

\[ E_n \triangleright x_k = 0, \quad k < n, \quad E_n \triangleright x_n = -i q^{-1}, \quad (36) \]

\[ F_n \triangleright y_k = 0, \quad k < n, \quad F_n \triangleright y_n = i, \quad (37) \]

\[ F_n \triangleright x_k = -i q^2 x_k x_n, \quad k < n, \quad F_n \triangleright x_n = -i q^2 x_n^2, \quad (38) \]

\[ K_n \triangleright y_k = q y_k, \quad k < n, \quad K_n \triangleright y_n = q^2 y_n, \quad (39) \]

\[ K_n \triangleright x_k = q^{-1} x_k, \quad k < n, \quad K_n \triangleright x_n = q^{-2} x_n, \quad (40) \]

Instead of proving that these formulas define a \( \mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{R})) \)-action on \( \mathcal{A}_q(n; \mathbb{R}) \), we give an operator expansion of the action showing thus its existence since the algebra admits faithful \*-representations. More explicitly, if \( \pi : \mathcal{A}_q(n; \mathbb{R}) \to \mathcal{L}^+(D) \) is a faithful \*-representation and \( \pi(\mathcal{A}_q(n; \mathbb{R})) \) a \( \mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{R})) \)-module \*-algebra with action \( \triangleright \), then setting \( X \triangleright f := \pi^{-1}(X \triangleright \pi(f)) \), \( f \in \mathcal{A}_q(n; \mathbb{R}) \), \( X \in \mathcal{U}_q(\mathfrak{sl}_{n+1}(\mathbb{R})) \), defines
Proposition 3. where \( k \) satisfying for all sense well-behaved. Integrable representations of \( L^+ (D) \).

Remind that we only consider “integrable” representations which are in a certain sense well-behaved. Integrable representations of \( \mathcal{A}_q (n; \mathbb{R}) \) were defined and classified in [6]. Before describing an operator expansion of the action, we state the results from [6]. This allows us to perform any algebraic manipulation in \( L^+ (D) \) and to verify directly if certain assumptions on the operators are satisfied.

We shall adopt the notational conventions of Section 3. In particular, \( T \) and \( P \) denote the multiplication operator by the variable \( t \) and the differential operator \( i \frac{d}{dz} \) acting on \( L^2 (\mathbb{R}) \), respectively. Set \( \mathcal{H} = L^2 (\mathbb{R})^n \otimes \mathcal{K} \), where \( \mathcal{K} \) is a Hilbert space. Consider the following series of Hilbert space operators on \( \mathcal{H} \).

\[
(I) : \quad y_l = (\bigotimes_{j=1}^{n-l} e^{(q-k_{n+j-1})P}) \otimes e^T \otimes 1 \otimes \cdots \otimes 1 \otimes \omega_l,
\]
\[
x_l = (\bigotimes_{j=1}^{n-l} (-1)(k_{n+j-1}+1)e^{(-q-k_{n+j-1})P}) \otimes (\bigotimes_{j=1}^l \frac{1}{e_T} + (q-k_l)\pi P + 1) e^{-T} \otimes 1 \otimes \cdots \otimes 1 \otimes \omega_l
\]
for all \( l = 1, \ldots, n \), where \( k_j \in \{0, s(q)\} \) and the operators \( \omega_j \) are symmetries on \( \mathcal{K} \) satisfying \( \omega_j \omega_l = (\omega_1)^{k_l} q l \omega_j \) for \( j > l \).

\[
(II) : \quad y_l = (\bigotimes_{j=1}^{n-l} e^{(q-k_{n+j-1})P}) \otimes e^T \otimes 1 \otimes \cdots \otimes 1 \otimes \omega_l,
\]
\[
x_l = (\bigotimes_{j=1}^{n-l} (-1)(k_{n+j-1}+1)e^{(-q-k_{n+j-1})P}) \otimes (\bigotimes_{j=1}^l \frac{1}{e_T} + (q-k_l)\pi P + 1) e^{-T} \otimes 1 \otimes \cdots \otimes 1 \otimes \omega_l
\]
for \( l = 2, \ldots, n \), and

\[
y_1 = (\bigotimes_{j=1}^{n-1} e^{(q-k_{n+j-1})P}) \otimes e^T \otimes \omega_1,
\]
\[
x_1 = (\bigotimes_{j=1}^{n-1} (-1)(k_{n+j-1}+1)e^{(-q-k_{n+j-1})P}) \otimes (\bigotimes_{j=1}^l \frac{1}{e_T} + (q-k_l)\pi P + 1) e^{-T} \otimes \omega_0 \omega_1
\]
where \( k_j \in \{0, s(q)\}, j = 2, \ldots, n \), and the operators \( \omega_j \) are symmetries on \( \mathcal{K} \) satisfying \( \omega_j \omega_l = (\omega_1)^{k_l} q l \omega_j \) for \( j > l \), \( \omega_j \omega_1 = \omega_0 \omega_j \) for \( j \geq 2 \), and \( \omega_1 \omega_1 = -\omega_0 \omega_1 \).

The proof of the following facts can be found in [6].

**Proposition 3.**

i Both families of operators \( x_1, \ldots, x_n, y_1, \ldots, y_n \) define an integrable *-representation of \( \mathcal{A}_q (n; \mathbb{R}) \).

ii Any integrable *-representation of \( \mathcal{A}_q (n; \mathbb{R}) \) such that \( \ker Q_j = \{0\} \) for all \( j = 1, \ldots, n \) is unitarily equivalent to one of the above form.

iii There exists a dense domain \( D \) of \( \mathcal{H} \) such that \( D \) is an invariant core for each of the self-adjoint operators \( x_j, \ y_j, \ Q_j, \ j = 1, \ldots, n \), and the commutation relations in \( \mathcal{A}_q (n; \mathbb{R}) \) are pointwise fulfilled on \( D \), for instance, \( D := F \otimes \mathcal{K} \) satisfies these conditions, where \( F \) is defined as in (19).
iv With $D$ given as in (iii), the *-representation of $\mathcal{A}_q(n; \mathbb{R})$ into $\mathcal{L}^+(D)$ is faithful.

In the remainder of this section we shall exclusively work with representations of the series $(I)$ and assume $k_n = \ldots = k_1 = 0$. It follows from the above formulas that in this case the operators $Q_l$, $Q_l^{-1}$, $|Q_l|^{1/2}$ and $|Q_l|^{-1/2}$ are given by

$$Q_l = (-1)^{n-l+1}(\otimes e^{2\varphi_P})|Q_l|^{1/2} \otimes 1 \cdots \otimes 1, \quad Q_l^{-1} = (-1)^{n-l+1}(\otimes e^{-2\varphi_P})|Q_l|^{-1/2} \otimes 1 \cdots \otimes 1,$$

$$|Q_l|^{1/2} = (\otimes e^{\varphi_P})|Q_l| \otimes 1 \cdots \otimes 1, \quad |Q_l|^{-1/2} = (\otimes e^{-\varphi_P})|Q_l|^{-1} \otimes 1 \cdots \otimes 1.$$  \hspace{1cm} (43)

By a slight abuse of notation, we denote operators and their restrictions to a dense domain $D$ by the same symbol.

**Corollary 4.** Suppose that the operators $x_j$, $y_j$, $j = 1, \ldots, n$, are given by the formulas of the series $(I)$ and assume that $k_n = \ldots = k_1 = 0$. Then there exists a dense domain $D$ of $\mathcal{H}$ such that $D$ is an invariant core for the self-adjoint operators $x_j$, $y_j$, $Q_j$, $j = 1, \ldots, n$, the commutation relations in $\mathcal{A}_q(n; \mathbb{R})$ are pointwise fulfilled on $D$, the *-representation $\pi : \mathcal{A}_q(n; \mathbb{R}) \to \mathcal{L}^+(D)$ is faithful, and $|Q_j|^{-1/2} \in \mathcal{L}^+(D)$. The operators $x_j$, $y_j$ and $|Q_j|^{1/2}$ satisfy the following commutation relations.

$$|Q_k|^{1/2} y_j = y_j |Q_k|^{1/2}, \ j < k, \quad |Q_k|^{1/2} y_j = q y_j |Q_k|^{1/2}, \ j \geq k,$$

$$|Q_k|^{1/2} x_j = x_j |Q_k|^{1/2}, \ j < k, \quad |Q_k|^{1/2} x_j = q^{-1} x_j |Q_k|^{1/2}, \ j \geq k.$$  \hspace{1cm} (44)

**Proof.** The assertions concerning $x_j$, $y_j$ and $Q_j$ are just a repetition of Proposition 3. Recall from Section 3 that $\mathcal{F} := \text{Lin}\{e^{-\epsilon t^2+\gamma t}; \epsilon > 0, \gamma \in \mathbb{C}\}$ and $q = e^{i\varphi}$ with $|\varphi| < \pi$. The operators $e^{\alpha T}$ and $e^{\beta P}$, $\alpha, \beta \in \mathbb{R}$, act on $e^{-\epsilon t^2+\gamma t} \in \mathcal{F}$ by (see [5, Lemma 1.1])

$$e^{\alpha T}(e^{-\epsilon t^2+\gamma t}) = e^{-\epsilon t^2+(\gamma+\alpha)t}, \quad e^{\beta P}(e^{-\epsilon t^2+\gamma t}) = e^{-e(t+i\beta)^2+\gamma(t+i\beta)}.
$$

Obviously, $\mathcal{F}$ is invariant under the action of $e^{\alpha T}$ and $e^{\beta P}$. Hence $D := \mathcal{F} \otimes \mathcal{K}$ satisfies the conditions of the corollary. On $\mathcal{F}$, the operators $e^{\alpha T}$ and $e^{\beta P}$ obey the commutation relation

$$e^{\beta P} e^{\alpha T} = e^{i\beta \alpha} e^{\alpha T} e^{\beta P}.$$  \hspace{1cm} (45)

Now (44) is easily proved by inserting the expressions of $y_j$, $x_j$, $|Q_j|^{1/2}$ and applying (45) since $e^{i\varphi} = q$.

**Lemma 5.** Suppose we are given an integrable *-representation of $\mathcal{A}_q(n; \mathbb{R})$ such that the operators $x_j$, $y_j$ and the domain $D$ satisfy the conditions of Corollary 4. Set $q_0 := e^{i\varphi}/2$, where $|\varphi| < \pi$ and $q = e^{i\varphi}$. Define

$$\rho_k = |Q_k|^{1/2} |Q_{k+1}|^{-1} |Q_{k+2}|^{1/2}, \quad k < n, \quad \rho_n = |Q_n|^{1/2} |Q_n|^{1/2};$$

$$A_k = i \lambda^{-1} y_0^{-1} q^{-1} Q_{k+1}^{-1} x_{k+1} y_k, \quad k < n, \quad A_n = -i \lambda^{-1} y_n,$$

$$B_k = -i \lambda^{-1} q_0 \rho_k^{-1} Q_{k+1}^{-1} x_{k+1} y_k, \quad k < n, \quad B_n = -i \lambda^{-1} q^{-1} \rho_n^{-1} x_n.$$  \hspace{1cm} (46-48)


Then the operators $\rho_k$, $A_k$, $B_k$ are hermitian and they obey the following commutation relations:

\begin{align}
\rho_i \rho_j &= \rho_j \rho_i, \quad \rho_j^{-1} \rho_j = 1,
\rho_i A_j &= q^{\alpha_{ij}} A_j \rho_i, \quad \rho_i B_j = q^{-\alpha_{ij}} B_j \rho_i, \quad (49)
\end{align}

\begin{align}
A_i A_j - A_j A_i &= 0, \quad i \neq j \pm 1, \quad A_i^2 A_{j \pm 1} - (q + q^{-1}) A_j A_{j \pm 1} A_{j \pm 1} A_j^2 &= 0, \quad (50)
B_i B_j - B_j B_i &= 0, \quad i \neq j \pm 1, \quad B_i^2 B_{j \pm 1} - (q + q^{-1}) B_{j \pm 1} B_j B_{j \pm 1} B_j^2 &= 0, \quad (51)
A_i B_j - B_j A_i &= 0, \quad i \neq j, \quad A_j B_j - B_j A_j = \lambda^{-1} (\rho_j - \rho_j^{-1}), \quad j < n, \quad (52)
A_n B_n - B_n A_n &= -\lambda^{-1} \rho_n^{-1}, \quad (53)
\end{align}

where $(\alpha_{ij})^n_{i,j=1}$ denotes the Cartan matrix of $\text{sl}(n+1, \mathbb{C})$.

**Proof.** Clearly, the operators $\rho_k$ are hermitian. It follows from (44) that

\begin{align}
\rho_j y_k &= y_k \rho_j, \quad k \neq j, j + 1, \quad \rho_j y_j = q y_j \rho_j, \quad \rho_j y_{j+1} = q^{-1} y_{j+1} \rho_j, \quad (54)
\rho_j x_k &= x_k \rho_j, \quad k \neq j, j + 1, \quad \rho_j x_j &= q^{-1} x_j \rho_j, \quad \rho_j x_{j+1} = q x_{j+1} \rho_j, \quad (55)
\rho_n y_k &= q y_k \rho_n, \quad \rho_n x_k = q^{-1} x_k \rho_n, \quad k < n, \quad \rho_n y_n = q^2 y_n \rho_n, \quad \rho_n x_n = q^{-2} x_n \rho_n. \quad (56)
\end{align}

Observe that $i \lambda \in \mathbb{R}$ and $q = q_0^2$. Thus, by (22), (26), (27) and the preceding, we have:

\begin{align}
k \neq n : \quad A_k^* &= i \lambda^{-1} q_0 q y_k x_k y_{k+1} Q_{k+1}^1 = i \lambda^{-1} q_0 q^{-2} Q_{k+1}^1 y_k x_k = A_k,
B_k^* &= -i \lambda^{-1} q_0 q y_k x_k y_{k+1} Q_{k+1}^1 \rho_k^{-1} = -i \lambda^{-1} q_0 q^{-1} Q_{k+1}^1 y_k x_k = B_k, \quad (57)
k = n : \quad A_n^* &= -i \lambda^{-1} y_n = A_n, \quad B_n^* = -i \lambda^{-1} q x_n \rho_n^{-1} = -i \lambda^{-1} q^{-1} \rho_n^{-1} x_n = B_n. \quad (58)
\end{align}

Equation (49) is easily shown using (28) and (54)–(56). The first relations of (50)–(52) follow by straightforward computations using the commutation rules in $A_q(n; \mathbb{R})$ and (54)–(56). Let $l < n$. Since $(-1)^{n-l+1} (-1)^{n-(l+2)+1} = 1$, we find from Equations (43) and (46) that $\rho_l^{-1} Q_l Q_{l+1}^1 Q_{l+2} = \rho_l$. Thus

\begin{align}
A_k B_k - B_k A_k &= \lambda^{-2} \rho_k^{-1} Q_{k+1}^2 (x_{l+2} y_{l+2} x_{l+1} y_l - y_{l+1} x_{l+1} y_{l+1} y_l),
&= \lambda^{-2} \rho_k^{-1} Q_{k+1}^2 [(Q_{l+2} - q Q_{l+1})(Q_{l+1} - q^{-1} Q_l) - (Q_{l+2} - q^{-1} Q_{l+1})(Q_{l+1} - q Q_l)],
&= \lambda^{-2} \rho_k^{-1} Q_{k+1}^2 [(q - q^{-1}) Q_{l+2} Q_l - (q - q^{-1}) Q_{l+1}^2],
&= \lambda^{-1} (\rho_l - \rho_l^{-1}), \quad (59)
\end{align}

where we applied (29) in the second equality. The proof of (53) is similar and easier.

To verify the second equations of (50) and (51), we first observe that

\begin{align}
A_{k-1} A_k &= q A_k A_{k-1} + \lambda^{-1} q^{-1} Q_k^1 x_k y_{k-1}, \quad k < n, \quad (57)
B_{k-1} B_k &= q^{-1} B_k B_{k-1} - \lambda^{-1} \rho_k^{-1} Q_k^1 y_k x_{k-1}, \quad k < n, \quad (58)
A_{n-1} A_n &= q A_n A_{n-1} - \lambda^{-1} q_0 Q_n^1 y_{n-1}, \quad (59)
B_{n-1} B_n &= q^{-1} B_n B_{n-1} - \lambda^{-1} q_0 q^{-1} \rho_n^{-1} Q_n^1 x_{n-1}. \quad (60)
\end{align}

To see this, consider

\begin{align}
\rho_k^{-1} Q_k^{-1} x_k y_{k-1} x_k &= \rho_k^{-1} Q_k^{-1} x_k y_{k-1} x_k \rho_k^{-1} Q_k^{-1} x_k, \\
&= \rho_k^{-1} Q_k^{-1} y_{k+1} (q^2 x_k y_k - q^{-2}(1 - q^2) Q_{k+1}) \rho_k^{-1} Q_k^{-1} x_k, \\
&= q^{-1} Q_k^{-1} y_{k+1} Q_k^{-1} x_k + \lambda q^{-1} Q_k^{-1} y_{k+1} x_k, \quad (59)
\end{align}
where the second equality was obtained by inserting Equation (30). Multiplying both
sides by \((-i\lambda^{-1}q_0)^2\) gives (58) since \((-i\lambda^{-1}q_0)^2\lambda q^{-1} = -\lambda^{-1}\). Equations (57), (59)
and (60) are proved similarly.

Next we claim that

$$A_k Q_k^{-1} x_{k+1} y_{k-1} = q Q_k^{-1} x_{k+1} y_{k-1} A_k,$$

(61)

$$A_{k-1} Q_k^{-1} x_{k+1} y_{k-1} = q^{-1} Q_k^{-1} x_{k+1} y_{k-1} A_{k-1},$$

(62)

$$B_k \rho_k^{-1} \rho_k^{-1} Q_k^{-1} y_{k+1} x_{k-1} = q^{-1} \rho_k^{-1} \rho_k^{-1} Q_k^{-1} y_{k+1} x_{k-1} B_k,$$

(63)

$$B_{k-1} \rho_k^{-1} \rho_k^{-1} Q_k^{-1} y_{k+1} x_{k-1} = q^{-1} \rho_k^{-1} \rho_k^{-1} Q_k^{-1} y_{k+1} x_{k-1} B_{k-1},$$

(64)

$$A_n Q_n^{-1} y_{n-1} = q Q_n^{-1} y_{n-1} A_n,$$

(65)

$$A_{n-1} Q_n^{-1} y_{n-1} = q Q_n^{-1} y_{n-1} A_{n-1},$$

(66)

$$B_n \rho_n^{-1} \rho_n^{-1} Q_n^{-1} x_{n-1} = q^{-1} \rho_n^{-1} \rho_n^{-1} Q_n^{-1} x_{n-1} B_n,$$ (67)

$$B_{n-1} \rho_n^{-1} \rho_n^{-1} Q_n^{-1} x_{n-1} = q^{-1} \rho_n^{-1} \rho_n^{-1} Q_n^{-1} x_{n-1} B_{n-1}.$$ (68)

All these equations are easily shown by straightforward calculations. As a sample,

$$Q_k^{-1} x_{k+1} y_k Q_k^{-1} x_{k+1} y_{k-1} = Q_k^{-1} x_{k+1} y_k x_{k+1} y_{k-1} = q Q_k^{-1} x_{k+1} y_k x_{k+1} y_{k-1}$$

implies (61).

Now, the second equations of (50) and (51) follow readily from (57)–(60) and (61)–
(68). For example, if \(k < n\), then computing (57) \(A_k - q^{-1} A_k\) of (57) and applying (61)
gives (50) with the plus sign, and \(A_{k-1} \cdot (57) - q^{-1}(57) \cdot A_{k-1}\) together with (62) gives
(50) with the minus sign. By the same method one proves the remaining relations.

We are now in a position to present the operator expansion of the action announced
in the beginning of this section.

**Lemma 6.** Suppose we are given an integrable \(*\)-representation of \(A_q(n; \mathbb{R})\) such that
the operators \(x_j, y_j\) and the domain \(D\) satisfy the conditions of Corollary 4. With the
operators \(\rho_k, A_k\) and \(B_k\) defined in Lemma 5, set

$$K_j \triangleright f = \rho_j f \rho_j^{-1}, \quad K_j^{-1} \triangleright f = \rho_j^{-1} f \rho_j,$$

(69)

$$E_j \triangleright f = A_j f - \rho_j f \rho_j^{-1} A_j,$$

(70)

$$F_j \triangleright f = B_j f \rho_j - q^2 f \rho_j B_j$$

(71)

for \(j = 1, \ldots, n\). Then Equations (69)–(71) define a \(U_q(sl_{n+1}(\mathbb{R}))\)-action \(\triangleright\) on \(L^+(D)\)
turning \(L^+(D)\) into a \(U_q(sl_{n+1}(\mathbb{R}))\)-module \(*\)-algebra. Its restriction to \(A_q(n; \mathbb{R})\),
considered as subalgebra of \(L^+(D)\), is given by the formulas (31)–(42).

**Proof.** The operators \(\rho_j, A_j, B_j\) satisfy the same commutation relations as the corre-
sponding operators in [3, Lemmas 4.1 and 4.2] (with \(\epsilon_1 = \ldots = \epsilon_n = 1\) in [3, Equation
(64)]). Since only these relations are needed in order to verify that (69)–(71) define a
\(U_q(sl_{n+1})\)-action on \(L^+(D)\), the proof of this fact runs completely analogous to that
of [3, Lemmas 4.2]. By the same argument, the action satisfies (3) and the first equation.
of (2) since these relations are independent of the involution. It remains to verify that \( \triangleright \) is consistent with the second equation of (2). This follows immediately from the proof of Lemma 1 by replacing \( Q, A \) and \( B \) with \( \rho_j, A_j \) and \( B_j \), respectively.

Applying (44), one easily proves by direct calculations that (69) yields the action of \( K_i^{\pm 1} \) on \( x_k \) and \( y_k \), \( j, k = 1, \ldots, n \).

Next, using the commutation rules of \( x_i, y_i, Q_i, A_i \) and \( \rho_i \), we obtain for \( j < n \)

\[
E_j \triangleright x_k = i\lambda^{-1}q_0^{-1}q^{-1}(Q_{j+1}^{-1}x_{j+1}y_jx_k - \rho_jx_k\rho_j^{-1}Q_{j+1}^{-1}x_{j+1}y_j)
= i\lambda^{-1}q_0^{-1}q^{-1}(Q_{j+1}^{-1}x_{j+1}y_jx_k - Q_{j+1}^{-1}x_{j+1}y_jx_k) = 0, \quad k \neq j,
\]

and, similarly, \( E_j \triangleright y_k = 0 \) if \( k \neq j + 1 \). Further, again for \( j < n \),

\[
E_j \triangleright x_j = i\lambda^{-1}q_0^{-1}q^{-1}(Q_{j+1}^{-1}x_{j+1}y_jx_j - \rho_jx_j\rho_j^{-1}Q_{j+1}^{-1}x_{j+1}y_j)
= i\lambda^{-1}q_0^{-1}q^{-1}Q_{j+1}^{-1}(q^2y_jx_j - x_jy_j)x_{j+1} = iq_0^{-1}x_{j+1},
\]

\[
E_j \triangleright y_{j+1} = i\lambda^{-1}q_0^{-1}q^{-1}(Q_{j+1}^{-1}x_{j+1}y_jy_{j+1} - \rho_jy_{j+1}\rho_j^{-1}Q_{j+1}^{-1}x_{j+1}y_j)
= i\lambda^{-1}q_0^{-1}Q_{j+1}^{-1}(x_{j+1}y_{j+1} - y_{j+1}x_{j+1})y_j = -iq_0^{-1}y_j,
\]

where we used (25) and (30). If \( k \neq n \), Equations (20), (22) and (56) give

\[
E_n \triangleright x_k = -i\lambda^{-1}(y_nx_k - \rho_nx_k\rho_n^{-1}y_n) = -i\lambda^{-1}(y_nx_k - y_nx_k) = 0,
\]

\[
E_n \triangleright y_k = -i\lambda^{-1}(y_ny_k - \rho_ny_k\rho_n^{-1}y_n) = -i\lambda^{-1}(y_ny_k - q^2y_ny_k) = iq_ny_ny_k.
\]

The action of \( E_n \) on \( x_n \) and \( y_n \) can be computed by replacing in the proof of Lemma 1 \( x, y, \rho \) and \( A \) with \( x_n, y_n, \rho_n \) and \( A_n \), respectively.

The preceding shows that the action of \( E_i, i = 1, \ldots, n \), on the generators of \( A_q(n; \mathbb{R}) \) is given by Equations (31), (32), (37) and (38). The analogous statement for \( F_i, i = 1, \ldots, n \), is proved similarly.

Recall that a generalization of the quantum trace formula [2, Section 7.1.6] was obtained in [3] by introducing the operator \( \Gamma := \prod_{i=1}^n \rho_i^{-i(n-i+1)} \). This definition redefines the definition of the distinguished element \( K_0 := \prod_{i=1}^n A_i^{i(n-i+1)} \in U_q(\mathfrak{sl}_{n+1}) \) satisfying \( XK_0 = K_0X \) for all \( X \in U_q(\mathfrak{sl}_{n+1}) \). Moreover, \( K_0 \) appears in the quantum trace as a density operator. This analogy will be used in the following proposition to define an invariant integral. Note that

\[
\Gamma = |Q_1|^{-n}|Q_2| \cdots |Q_n|, \quad n > 1, \quad \Gamma = |Q_1|^{-1}, \quad n = 1,
\]

exactly as in [3, Equation (71)].

**Proposition 7.** Suppose we are given a \(*\)-representation of \( A_q(n; \mathbb{R}) \) into \( \mathcal{L}^+ (D) \) such that the operators \( x_j, y_j, j = 1, \ldots, n \) are given by the formulas of the series (I) with \( k_n = \ldots = k_1 = 0 \). Assume that \( D \) is of the form described in Corollary 4. Let \( \mathfrak{A} \) be the \( O^* \)-algebra generated by the elements of \( A_q(n; \mathbb{R}) \cup \{|Q_k|^{1/2}, |Q_k|^{-1/2}\}_{k=1}^n \). Then the \(*\)-algebras \( \mathbb{F}(D) \) and \( \mathbb{B}_1 (\mathfrak{A}) \) defined in Equations (6) and (5), respectively, are \( U_q(\mathfrak{sl}_{n+1}(\mathbb{R})) \)-module \(*\)-algebras, where the action is given by (69)–(71). The linear functional

\[
h(f) := c \text{tr}(f \Gamma), \quad c \in \mathbb{R}, \quad (72)
\]

defines an invariant integral on both \( \mathbb{F}(D) \) and \( \mathbb{B}_1 (\mathfrak{A}) \).
Proof. Since the operators $\rho_k, A_k, B_k$ from Lemma 6 and $\Gamma$ satisfy the same commutation relations as the corresponding operators in [3] (with $\epsilon_1 = \ldots = \epsilon_n = 1$ in [3, Equation (64)]), and since only these relations are needed in the proof of [3, Proposition 4.2], the proof of Proposition 7 is literally the same.

Observe that $z_n, z_n^*, K_n^{\pm 1}, E_n$ and $F_n$ satisfy the relations of $A_q(1; \mathbb{R})$. In particular, Equation (17) applies, telling us that we are dealing with a non-compact quantum space. Again, $B_1(A)$ is considered as the algebra of infinitely differentiable functions which vanish sufficiently rapidly at “infinity” and $F(D)$ as the infinitely differentiable functions with compact support.

Let us finally remark that, for $n > 1$, the operators $K_n^{\pm 1}, E_j$ and $F_j$, $j = 1, \ldots, n-1$, generate the Hopf $*$-algebra $U_q(sl_n(\mathbb{R}))$, and (31)–(36) define a $U_q(sl_n(\mathbb{R}))$-action on $A_q(n; \mathbb{R})$ such that $A_q(n; \mathbb{R})$ becomes a $U_q(sl_n(\mathbb{R}))$-module $*$-algebra. This action on $A_q(n; \mathbb{R})$ is well-known because it can be obtained from a $O(SL_q(n, \mathbb{R}))$-coaction and a dual pairing of $U_q(sl_n(\mathbb{R}))$ and $O(SL_q(n, \mathbb{R}))$ (see Sections 1.3.5, 9.3.3 and 9.3.4 in [2]). Now Proposition 7 asserts that we can develop a $U_q(sl_n(\mathbb{R}))$-invariant integration theory on $A_q(n; \mathbb{R})$, that is, the $*$-algebras $F(D)$ and $B_1(A)$ are $U_q(sl_n(\mathbb{R}))$-module $*$-algebras and Equation (72) defines a $U_q(sl_n(\mathbb{R}))$-invariant functional on both algebras. Note furthermore that, under the assumptions of Lemma 5, we obtain a $*$-representation $\pi : U_q(sl_n(\mathbb{R})) \to L^+(D)$ by assigning $\pi(K_j) = \rho_j$, $\pi(E_j) = A_j$ and $\pi(F_j) = B_j$ for $j = 1, \ldots, n-1$.

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References

[1] Klimec, S. and A. Lesniewski: A two-parameter quantum deformation of the unit disc. J. Funct. Anal. 155 (1993), 1–23.
[2] Klimyk, A.U. and K. Schmüdgen: Quantum Groups and Their Representations. Springer-Verlag, Berlin, 1997.
[3] Kürsten, K.-D. and E. Wagner: An operator-theoretic approach to invariant integrals on quantum homogeneous $SU_{n,1}$-spaces. Publ. Res. Inst. Math. Sci. 43 (2007), 1–37.
[4] Schmüdgen, K.: Unbounded operator algebras and representation theory. Birkhäuser, Basel, 1990.
[5] Schmüdgen, K.: Integrable Operator Representations of $\mathbb{R}^2_q, X_{q,7}$ and $SL_q(2, \mathbb{R})$. Commun. Math. Phys. 159 (1994), 217–237.
[6] Schmüdgen, K.: Operator representations of the real twisted canonical commutation relations. J. Math. Phys. 35 (1994), 3211–3229.
[7] Schmüdgen, K.: On well-behaved unbounded representations of $*$-algebras. J. Operator Theory 48 (2002), 487–502.

[8] Shklyarow, D. L., S. D. Sinel’shchikov and L. L. Vaksman: Integral representations of functions in the quantum disk. I. (Russian) Mat. Fiz. Anal. Geom. 4 (1997), 286–308. On Function Theory in Quantum Disc: Integral Representations. E-print, arXiv: math.QA/9808015.

[9] Shklyarow, D. L., S. D. Sinel’shchikov and L. L. Vaksman: On Function Theory in Quantum Disc: Covariance. E-print, arXiv:math/9808037

[10] Shklyarow, D. L., S. D. Sinel’shchikov and L. L. Vaksman: Quantum Matrix Balls: Differential and Integral Calculi. E-print, arXiv:math.QA/9905035.

[11] Shklyarow, D. L., S. D. Sinel’shchikov and L. L. Vaksman: $q$-analogues of some bounded symmetric domains. Czech. J. Phys. 50 (2000), 175–180.

[12] Sinel’shchikov, S. D. and L. L. Vaksman: On $q$-analogues of Bounded Symmetric Domains and Dolbeault Complexes. Math. Phys. Anal. Geom. 1 (1998), 75–100.