On Singer-Wermer conjecture in Fréchet algebras

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Abstract. We construct two Fréchet algebras admitting countably many mutually inequivalent Fréchet algebra topologies. The second example is a modification of the maiden (and first) example of a non-Banach Fréchet algebra with two inequivalent Fréchet algebra topologies, considered by Read to show that the famous Singer-Wermer conjecture does not hold in the Fréchet case. However the conjecture does hold by the second example which also admits countably many mutually equivalent Fréchet algebra topologies.
1 Introduction

Recently, the author posed a question about the existence of a Fréchet algebra with infinitely many inequivalent Fréchet algebra topologies [8, Question 4]. This question carries a lot of importance due to the following three reasons.

First, in automatic continuity theory, we would normally like to examine when and how the algebraic structure of the algebra $A$ determines the topological structure of $A$, in particular, the continuity aspect (and more particularly, the uniqueness of the topology of $A$; see [2, 7, 8, 9] for more details). So it is natural to expect that the non-uniqueness of the topology would reflect some properties of the algebraic structure of $A$.

In [4], Feldman constructed an example to show the failure of the Wederburn Theorem in the Banach algebra case. This algebra is $\ell_2 \oplus C$ as a vector space with the usual product in $\ell_2$ and trivial multiplication by the second summand, and has a norm in which $\ell_2$ is dense. In [10], Read constructed an example (see §2 below) to show the failure of the Singer-Wermer conjecture (the commutative case) in the Fréchet algebra case; this conjecture holds in the Banach algebra case (see [11]). Thus the situation on Fréchet algebras is markedly different from that on Banach algebras, and that a structure theory for Fréchet algebras behaves in a very distinctive
manner from Banach algebras; for example, the prestigious Michael problem is still remained unsolved since 1952.

Second, it is easy to give uncountably many inequivalent Fréchet space topologies to a very familiar Fréchet space, namely, spaces of holomorphic functions (see [12] for more details). However the same space is also a semisimple Fréchet algebra of power series, and so, it admits a unique Fréchet algebra topology (see [8] for more details). Thus it is expected that the case of having infinitely many inequivalent Fréchet algebra topologies by some Fréchet algebra should be rare, and the former situation should be common.

We note that in [6], Loy gave a method for constructing commutative Banach algebras with inequivalent complete norm topology. We exploit the idea of using the discontinuity of derivations to give a Fréchet algebra other inequivalent Fréchet algebra topologies. In view of a comment in the previous paragraph, it is interesting to recall an early exposition of Banach algebras of powers series and of their automorphisms and derivations, given by Grabiner in [5]. We remark that although the Fréchet algebra topology $\tau_0 + \tau_0$ (defined below) of $F_\infty \oplus F_\infty$ is not obtainable by our approach, other inequivalent Fréchet algebra topologies on $F_\infty \oplus F_\infty$ may be constructed. The purpose of this paper is to show that commutative Fréchet algebras $F_\infty$
and $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ admit countably many mutually inequivalent Fréchet algebra topologies.

Third, we do not know any non-trivial examples of (commutative) Banach algebras with infinitely many inequivalent complete algebra norms till this date (see Conclusions below). Of course, one may consider the trivial product on a Banach space with countably many inequivalent complete algebra norms to have such an example in the Banach algebra case.

2 First Example

In [10], Read showed that the algebra $\mathcal{F}_\infty = \mathcal{O}[X_0, X_1, \ldots]$ of all formal power series in infinitely many commuting indeterminates $X_0, X_1, \ldots$ has two inequivalent Fréchet algebra topologies $\tau_0$ and $\tau_c$. The natural derivation $\partial/\partial X_0$ is continuous with respect to the Fréchet algebra topology $\tau_c$. He also showed that the natural derivation $\partial/\partial X_0$ is discontinuous with respect to the Fréchet algebra topology $\tau_0$ and with the image the whole algebra, and that $X_0$ lies in the closure of the coefficient algebra $\mathcal{A}_0 = \mathcal{O}[X_1, X_2, \ldots]$. In fact, it is surprising to observe that for each $i \in \mathcal{N}$, the other natural derivation $\partial/\partial X_i$ is discontinuous with respect to the Fréchet algebra topol-
ogy $\tau_i$ (defined below), and that for each $i \in \mathbb{N}$, $X_i$ lies in the closure of the coefficient algebra $A_i = \mathcal{C}[[X_0, X_1, \ldots]]$ such that $X_i$ does not belong to $A_i$. Indeed, with respect to $\tau_i$ one has $X_n \to X_i$ as $n \to \infty$; so one has $X_n - X_i \to 0$ yet $\partial/\partial X_i(X_i - X_n) = 1$. Thus 1 is in the separating subspace for $\partial/\partial X_i$; since the subspace is an ideal, it is the whole algebra. This shows that for each $i \in \mathbb{N}$, $\partial/\partial X_i$ vanishes on a dense subset $A_i$ of $(\mathcal{F}_\infty, \tau_i)$. Not only this, but it is easy to see that for $i \neq j$, the Fréchet algebra topologies $\tau_i$ and $\tau_j$ on $\mathcal{F}_\infty$ are not mutually comparable, because the identity map is continuous in neither direction as $X_n \to X_i$ in $(\mathcal{F}_\infty, \tau_i)$ whereas $X_n \to X_j$ in $(\mathcal{F}_\infty, \tau_j)$. Hence $\mathcal{F}_\infty$ admits countably many mutually inequivalent Fréchet algebra topologies. Moreover, for $i \neq j$, the natural derivations $\partial/\partial X_i$ are continuous on $(\mathcal{F}_\infty, \tau_j)$ as $X_n \to X_j$ and $\partial/\partial X_i(X_j - X_n) = 0$.

We shall feel free to use the terminology and conventions established there in [10]. Not only this, but our argument for this section is kept short because it uses the key ideas involved in defining the Fréchet algebra topology $\tau_0$ on $\mathcal{F}_\infty$. Let us therefore define ourselves bizarre topology $\tau_i$ on $\mathcal{F}_\infty$ for each $i \in \mathbb{N}$. For this, first, let us choose a linear functional $\psi : A(1) \to \mathcal{C}$ with the property that $\psi(X_0) = 0$ and $\psi = X_n^*$ on $A(1,0) = B(1,0)$ (the subspace of $A(1)$, consisting of all formal sums $\sum_{j=1}^{\infty} \lambda_j X_j$), i.e., the coordinate functional
such that $X_n^*(\sum_{j=1}^{\infty} \lambda_j X_j) = \lambda_n$ (and hence $\psi(X_n) = 1$ for all $n \in \mathbb{N}$). Note that this requires extending the sequence $(X_n), n \in \mathbb{N}_0,$ to a Hamel basis of $A^{(1)},$ and thus involves the axiom of choice. Then let $(\phi_n), n \in \mathbb{N}_0,$ be linear functionals on $A^{(1)}$ as follows: (a) For $n \in \mathbb{N}_0, n = i > 0, \phi_i = \psi,$ the “discontinuous” linear functional, defined above. (b) For $n \in \mathbb{N}_0, n \neq i, \phi_n = X_n^*,$ the coordinate functional such that $X_n^*(\sum_{j=0}^{\infty} \lambda_j X_j) = \lambda_n.$ For each $i > 0,$ let $\tau_i$ be the locally multiplicative convex topology on $\mathcal{B}$ (a non-commutative analogue of $\mathcal{F}_{\infty}$) of convergence in all the seminorms $\| \cdot \|_n^{(m)},$ where $\|a\|_n^{(m)} = |a^{(0)}| + \sum_{r=1}^{m} \sum_{i \in \{0,1,\ldots,n\}^r} |\otimes_{j=1}^r \phi_{ij}(a^{(r)})|$ for all $a \in \mathcal{B},$ for this sequence of linear functionals $\phi_n$ (for $i \neq j,$ the Fréchet algebra topologies $\tau_i$ and $\tau_j$ are mutually inequivalent, as described above). We remark that our discontinuous linear functional $\psi$ here is different from Read’s discontinuous linear functional (see [10, Definition 1.7]). In fact, the modification in the definition of $\psi$ is needed so as to change the position of $\phi_i = \psi$ in the sequence $(\phi_n)$ of linear functionals, in order to ultimately generate mutually inequivalent Fréchet algebra topologies $\tau_i$ on $\mathcal{F}_{\infty}.$

We claim that $\mathcal{B}$ is, in fact, complete under the topology $\tau_i$ for each $i > 0,$ so $(\mathcal{B}, \tau_i)$ is a Fréchet algebra. Since the averaging map $\alpha : \mathcal{B} \rightarrow \mathcal{B},$ the locally finite map such that $\alpha(X_{i_1} \otimes \ldots \otimes X_{i_n}) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{i_{\sigma(1)}} \otimes \ldots \otimes X_{i_{\sigma(n)}},$
is a $\tau_i$-continuous projection, the subspace $\mathcal{A} = \alpha(\mathcal{B}) = \ker(I - \alpha)$ is closed, so $(\mathcal{A}, \tau_i)$ is also a Fréchet algebra. Note that if we prove our claim we have our result; for since $\phi_i(X_n - X_i) = 0$ for $N > 0$ one sees that $\|X_n - X_i\|_{(m)} = 0$ for all $n, m$ with $n < N$. Hence, $X_n \to X_i$ in $\tau_i$. One may then use the derivation $\partial/\partial X_i$ as above, and the separating subspace is the whole algebra, be it $\mathcal{F}_\infty$ or $\mathcal{B}$. All that remains is to prove $\mathcal{B}$ complete.

First, we note that $(\mathcal{B}, \tau_c)$ is a Fréchet algebra; to define $\tau_c$, apply the method of Corollary 1.11 of [10] to the sequence of coordinate functionals $(X_n^*)$, obtaining algebra seminorms $|\cdot|_{(m)}$, $|\sum a_i X^\otimes i|_{(m)} = |a^{(0)}| + \sum_{r=1}^m \sum_{i \in \{0,1,\ldots,n\}^r} |a_i|$. To prove $\mathcal{B}$ complete, by using the seminorms $\|\cdot\|_{(m)}$, defined above, we show that the linear map $\Psi : \mathcal{B} \to \mathcal{B}$, defined by $\Psi(a) = a^{(0)} + \sum_{r=1}^\infty \sum_{i \in \mathbb{N}_0^r} X^\otimes i \otimes_{j=1}^r \phi_{ij}(a^{(r)})$, is bijective. (We note that, for each $i \in \mathbb{N}$, $\Psi : (\mathcal{B}, \tau_i) \to (\mathcal{B}, \tau_c)$ is continuous, because convergence under $\tau_i$ is precisely convergence of all the functionals $\otimes_{j=1}^r \phi_{ij}(a^{(r)})$ that are involved in $\Psi(a)$.) For each $r$, $\Psi$ maps $\mathcal{B}^{(r)}$ to $\mathcal{B}^{(r)}$, where $\mathcal{B}^{(r)}$ is the subspace of $n$-homogeneous formal power series $\sum_{i \in \mathbb{N}_0^n} b_{i}X^\otimes i$. It is therefore enough to show that $\Psi^{(r)} = \Psi|_{\mathcal{B}^{(r)}}$ is a bijection $\mathcal{B}^{(r)} \to \mathcal{B}^{(r)}$ for each $r$. Now, for $0 \leq k \leq r$, let $\mathcal{F}^{(r,k)}$ be the set of all $i = (i_1, i_2, \ldots, i_r) \in \mathbb{N}_0^r$ such that exactly $k$ of the $i_j$ are equal to zero and let $\mathcal{B}^{(r,k)}$ be the subspace consisting
of all power series of the form $\sum_{i \in F} a_i X^\otimes i$. Then $B^{(r)} = \oplus_{k=0}^r B^{(r,k)}$. Let $\beta_{r,k}$ be the projection onto $B^{(r,k)}$ parallel to the others. We claim that for each $k$, $\beta_{r,k}\Psi^{(r)}|_{B^{(r,k)}}$ is equal to the identity map on $B^{(r,k)}$; and we claim that $\Psi^{(r)}$ maps $B^{(r,k)}$ into $\oplus_{l=k}^r B^{(r,l)}$. Thus we shall show that the action of $\Psi^{(r)}$ is “lower triangular” and nonsingular, with respect to the decomposition $B^{(r)} = \oplus_{k=0}^r B^{(r,k)}$. Therefore, $\Psi^{(r)}$ - and hence $\Psi$ - is bijective.

In view of definition of $\Psi$ above, $\beta_{r,k}\Psi^{(r)}|_{B^{(r,k)}}$ is equal to the identity map if and only if we have

\[ \otimes_{i=1}^r \phi_{l_i}(a) = a_1 \ldots \ldots (1) \]

for each $l = (l_1, \ldots, l_r) \in \mathcal{F}^{(r,k)}$ and $a \in B^{(r,k)}$; and $\Psi^{(r)}$ maps $B^{(r,k)}$ into $\oplus_{l=k}^r B^{(r,l)}$ if and only if

\[ \otimes_{i=1}^r \phi_{l_i}(a) = 0 \ldots \ldots (2) \]

for all $m < k$, $l = (l_1, \ldots, l_r) \in \mathcal{F}^{(r,m)}$ and $a \in B^{(r,k)}$. Let us prove (1) and (2) together, by induction on $r$.

When $r = 1$, statement (1) demands that $\phi_{l_1}(a) = a_{l_1}$ if either $l_1 = 0$ and $a \in B^{(1,1)}$, or $l_1 > 0$ and $a \in B^{(1,0)}$. For $l_1 > 0$, we have two cases: (i) if $l_1 \neq i$, the functional $\phi_{l_1} = X_{l_1}^*$ anyway, so the assertion is true; and (ii) if $l_1 = i$, $\phi_i = \psi = X_{l_1}^*$ on $B^{(1,0)}$, so the assertion is true. But when $l_1 = 0,$
we note that $B^{(1,1)}$ consists solely of multiples $a_0X_0$; since $\phi_0 = X_0^*$, the assertion is true for $l_1 = 0$ also. Statement (2) demands that $\phi_{l_1}(\mathbf{a}) = 0$ if $l_1 > 0$ and $\mathbf{a} \in B^{(1,1)}$. Again we have two cases: (i) if $l_1 \neq i$, the functional $\phi_{l_1} = X_{l_1}^*$, so $X_{l_1}^*(a_0X_0) = 0$; and (ii) if $l_1 = i$, $\phi_i = \psi$, so $\psi(a_0X_0) = 0$. So the assertion is correct.

When $r > 1$, we proceed by induction. First we establish statements (1) and (2) when $r = 2$. In this case, from Definition 1.9 of \cite{10}, we have the “tensor product by rows” $\phi_{l_1} \otimes \phi_{l_2} : B^{(2)} \rightarrow \mathcal{C}$ by

$$\phi_{l_1} \otimes \phi_{l_2}(\mathbf{a}) = \phi_{l_1}(\sum_{j=0}^{\infty} X_j \phi_{l_2}(P_j a)) ........(3)$$

To prove statement (1), consider the case when $\mathbf{l} = (l_1, l_2) \in F^{(2,k)}$ and $\mathbf{a} \in B^{(2,k)}$. Note that $P_j a$ lies in $B^{(1,1)}$ if $j > 0$, but in $B^{(1,0)}$ if $j = 0$, because the division on the left by $X_0$ removes a factor $X_0$ from each monomial $X_j$. If $l_1 = 0$, then $(l_2) \in F^{(1,0)}$ (i.e., $l_2 > 0$), so $\phi_{l_2}(\mathbf{a}) = 0$ when $\mathbf{a} \in B^{(1,1)}$ (here, if $l_2 = i$, then $\phi_i = \psi$, so $\psi(a_0X_0) = 0$; otherwise, $\phi_{l_2} = X_{l_2}^*$ and $X_{l_2}^*(a_0X_0) = 0$), but will send $\mathbf{b} \in B^{(1,0)}$ to $b_{l_2}$ (here, $\mathbf{b} = \sum_{j=1}^{\infty} b_j X_j$ and if $l_2 = i$, then $\phi_i = \psi = X_i^*$ on $B^{(1,0)}$; otherwise, $\phi_{l_2} = X_{l_2}^*$ and $\phi_{l_2}(\sum_{j=1}^{\infty} b_j X_j) = b_{l_2}$). Accordingly, when $l_1 = 0$, statement (3) is equal to $\phi_0(X_0(P_0 a)_{l_2}) = (P_0 a)_{l_2} = a_{l_1, l_2}$. If on the other hand, $l_1 > 0$, then we have two cases: (i) if $l_1 = i$, then $\phi_{l_1} = \phi_i$. So, the statement (3) is
equal to $\phi_i(\sum_{j=0}^{\infty} X_j \phi_{l_2}(P_j a)) = \phi_i(X_0 \phi_{l_2}(P_0 a)) + \phi_i(\sum_{j=1}^{\infty} X_j \phi_{l_2}(P_j a)) = \phi_i(\sum_{j=1}^{\infty} X_j \phi_{l_2}(P_j a)) = \phi_{l_2}(P_i a) = \phi_0(a_0 X_0) = \phi_i = \psi = X_i^*$ on $B^{(1,0)}$.

In this case, $l_2 \in \mathcal{F}^{(1,1)}$ and $P_i a \in B^{(1,1)}$. So, the statement (3) is equal to $(P_i a)_{l_2} = a_{l_1, l_2}$ as before. (ii) if $l_1 \neq i$, then $\phi_{l_1} = X_{l_1}^*$, so the statement (3) is equal to $\phi_{l_2}(P_i a)$. In this case, $(l_2) \in \mathcal{F}^{(1,k)}$ and $P_{l_1} a \in B^{(1,k)}$ so the statement (3) is equal to $(P_{l_1} a)_{l_2} = a_{l_1, l_2}$ as before.

For the statement (2), consider $a \in B^{(2,k)}$ as before, but now use sequences $1 \in \mathcal{F}^{(r,m)}$ for some $m < k$. If $l_1 = 0$, then $l_2 \in \mathcal{F}^{(1,0)}$; since each $P_j a$ is either in $B^{(1,1)}$ or $B^{(1,0)}$, $\phi_{l_2}(P_j a) = 0$ for all $P_j a$ (even when $l_2 = i$). If $l_1 > 0$, then we have two cases: (i) if $l_1 = i$, then $\phi_{l_1} = \phi_i = \psi$, so $\phi_i(\sum_{j=0}^{\infty} X_j \phi_{l_2}(P_j a)) = \phi_{l_2}(P_i a)$. Since $l_2 \in \mathcal{F}^{(1,m)}$ and $P_{l_1} a \in B^{(1,1)}$, $\phi_{l_2}(P_i a) = 0$. (ii) if $l_1 \neq i$, then $\phi_{l_1} = X_{l_1}^*$, so the statement (3) is equal to $\phi_{l_2}(P_{l_1} a)$. Since $(l_2) \in \mathcal{F}^{(1,m)}$ and $P_{l_1} a \in B^{(1,k)}$ so this is zero also. Thus the statement (1) and (2) are proved for $r = 2$.

Now one establishes statements (1) and (2) by induction when $r > 2$. For this, follow Read’s argument in our case.

**Theorem 2.1** Let $i \in \mathcal{I}^+$ be fixed. $(B, \tau_i)$ is complete with respect to the seminorms $\| \cdot \|^{(m)}_n$. The derivation $\partial/\partial X_i : (B, \tau_i) \to (B, \tau_i)$ is discontinuous, and its separating subspace is all of $B$. The derivation $\partial/\partial X_i : (A, \tau_i) \to$
\((A, \tau_i)\) is also discontinuous, and its separating subspace is all of \(A\).

**Proof.** Follow the proof of [10, Theorem 2.5] in this case and the result follows.  \(\Box\)

### 3 Second Example

We now show that the algebra \(F_\infty \oplus F_\infty\) has countably many mutually inequivalent Fréchet algebra topologies. In fact, the algebra \(F_\infty\) has been completed by the adjunction of a radical so that \(F_\infty \oplus F_\infty\) has Fréchet algebra topologies in which \(F_\infty\) is dense in those topologies.

For us, let \(A = F_\infty\) be a commutative (Fréchet) algebra and \(M = F_\infty\) a commutative (Fréchet) \(A\)-module. Let \(H^1(F_\infty, F_\infty)\) denote the first algebraic cohomology group, \(H^1_C(F_\infty, F_\infty)\) the first continuous cohomology group, where the cochains are required to be bounded. Thus with the usual conventions \(H^1(F_\infty, F_\infty)\) is the space of derivations of \(F_\infty\) into itself; \(H^1_C(F_\infty, F_\infty)\) the space of continuous derivations of \(F_\infty\) into itself with the Fréchet algebra topology \(\tau_c\) on \(F_\infty\).

Let \(F_\infty\) be a commutative Fréchet algebra with Fréchet algebra topology \(\tau_i\) generated by the sequence \((p_{k,i}')\) for each \(i \in \mathbb{Z}^+\), by §2 and \(\tau_c\) generated
by the sequence \((p_k)\). Let \(\mathcal{A}\) denote the vector space direct sum \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\) with product

\[(x, m)(y, n) = (xy, x \cdot n + y \cdot m)\]

and seminorms

\[q_k(x, m) = p_k(x) + p_k(m),\]

generating the Fréchet algebra topology \(\tau_c + \tau_c\), and seminorms

\[q'_k(x, m) = p'_k(x) + p'_k(m),\]

generating the Fréchet algebra topology \(\tau'_c + \tau'_c\) for each \(i \in \mathbb{Z}^+\). For each \(i \in \mathbb{Z}^+, \partial_i = \partial/\partial X_i \in H^1(\mathcal{F}_\infty, \mathcal{F}_\infty)\), the functional

\[q_{k, \partial_i} : (x, m) \rightarrow p_k(x) + p_k(\partial_i(x) - m)\]

is defined on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\) and is, in fact, easily seen to be a submultiplicative seminorm thereon; for each \(i \in \mathbb{Z}^+\), the sequence \((q_{k, \partial_i})\) generates the Fréchet algebra topology \(\tau_{\partial_i}\) equivalent to the Fréchet algebra topology \(\tau_c + \tau_c\) (see Theorem 3.2 below). For each \(i \in \mathbb{Z}^+, \partial_i = \partial/\partial X_i \in H^1(\mathcal{F}_\infty, \mathcal{F}_\infty)\), the functional

\[q'_{k, \partial_i} : (x, m) \rightarrow p'_k(x) + p'_k(\partial_i(x) - m)\]

is defined on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\) and is, in fact, easily seen to be a submultiplicative seminorm thereon; for each \(i \in \mathbb{Z}^+\), the sequence \((q'_{k, \partial_i})\) generates
the Fréchet algebra topology $\tau_{\partial_i}$ inequivalent to the Fréchet algebra topology $\tau_i + \tau_i$ (see Theorem 3.2 below). For each $i \in \mathbb{Z}^+$, $\partial_i = \partial/\partial X_i$, the map

$$\theta_{\partial_i} : (x, m) \to (x, \partial_i(x) - m)$$

is an isometric isomorphism of $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ under $(q_{k, \partial_i})$ (respectively, $(q'_{k, \partial_i})$) into $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_c)$ (respectively, $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$) and so extends uniquely to a map of the completion $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty)_{\partial_i}$ of $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ into $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_c)$ (respectively, $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$). In particular, if $\iota : x \to (x, 0)$ is the natural embedding of $\mathcal{F}_\infty$ into $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, then, for each $i \in \mathbb{Z}^+$, $q''_{k, \partial_i} : x \to q_{k, \partial_i}(\iota(x))$ (respectively, $q'''_{k, \partial_i} : x \to q'_{k, \partial_i}(\iota(x))$) is a seminorm on $\mathcal{F}_\infty$ and $\theta_{\partial_i} \circ \iota$ extends to an isometric isomorphism of $(\mathcal{F}_\infty)_{\partial_i}$, the completion of $\mathcal{F}_\infty$ under $(q_{k, \partial_i})$ (respectively, $(q'''_{k, \partial_i})$), with $\overline{\text{Gr}(\partial_i)}$, the closure (respectively, in $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$ and in $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$) of the graph of $\partial_i$.

Let $i \in \mathbb{Z}^+$ be fixed. Now if $\partial_i$ is continuous, then $(q_{k, \partial_i})$ is equivalent to $(q_k)$ on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ (see Theorem 3.2 below) and $(q'''_{k, \partial_i})$ is equivalent to $(p_k)$ on $\mathcal{F}_\infty$. Thus $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty)_{\partial_i} = (\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_c)$ and $(\mathcal{F}_\infty)_{\partial_i} = (\mathcal{F}_\infty, \tau_c)$.

This is the case when $\mathcal{F}_\infty$ is a Fréchet algebra under the Fréchet algebra topology $\tau_c$. In the discontinuous case $q'''_{k, \partial_i}$ is a discontinuous seminorm on $(\mathcal{F}_\infty, \tau_i)$ and $\iota$ is a discontinuous isomorphism. We study this case in detail. First, by Theorem 3.2 below, $(q'_{k, \partial_i})$ is not equivalent to $(q'_{k, \iota})$ on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$.  

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Suppose now that $\partial_i$ is discontinuous. This is the case when $\mathcal{F}_\infty$ is a Fréchet algebra under the Fréchet algebra topology $\tau_i$, by §2. Then we have

$$\text{Gr}(\partial_i)^{\tau_i} \cap 0 \oplus \mathcal{F}_\infty \neq \{0\}.$$ 

Thus if $\mathcal{F}_\infty$ is a Fréchet algebra with $H^1(\mathcal{F}_\infty, \mathcal{F}_\infty) \neq H^1_c(\mathcal{F}_\infty, \mathcal{F}_\infty)$, then $\mathcal{F}_\infty$ has a completion with a non-trivial nil ideal, that is, the completion $(\mathcal{F}_\infty)_{\partial_i}$ under $(q''_{k,\partial_i})$ is, in fact, $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$ due to the following theorem. We remark that if $A$ is a semisimple Fréchet algebra, then $H^1(A, A) = H^1_c(A, A)$ (see [4] for details), and in particular, if $A$ is a pro-$C^*$-algebras, then $H^1(A, A) = 0$. However we do not know an example of a non-semisimple, non-Banach Fréchet algebra such that $H^1(A, M) = 0$ for any $A$-module $M$ (in particular, such that $H^1(A, A) = 0$); the Singer-Wermer conjecture holds for commutative Banach algebras. Banach algebras with $H^1(A, M) = H^1_c(A, M)$ are discussed in [1].

**Theorem 3.1** Let $i \in \Z^+$ be fixed. Let $(\mathcal{F}_\infty, \tau_i)$ be a commutative Fréchet algebra, $\partial_i = \partial/\partial X_i$ a non-zero derivation of $\mathcal{F}_\infty$ into itself. Then the algebra $(\mathcal{F}_\infty)_{\partial_i}$ admits a Fréchet algebra topology $\tau'_{\partial_i}$, generated by $(q'_{k,\partial_i})$, and which is inequivalent to the Fréchet algebra topology $\tau_i + \tau_i$, generated by $(q'_{k,i})$. In particular, the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ admits countably many Fréchet algebra topologies inequivalent to the Fréchet algebra topology $\tau_i + \tau_i$. 

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Proof. First, by §2, we note that, for each $i \in \mathbb{Z}^+$, $\partial_i$ vanishes on a dense subset $\mathcal{A}_i$ (which is the coefficient algebra of $\mathcal{F}_\infty$); we also note that this is possible due to the unique property of the Fréchet algebra topology $\tau_i$ as explained in §2. Now follow the proof of [6, Theorem 1] for this case. □

Remarks A. We note that $(\mathcal{F}_\infty, \tau_i)$ is dense in $(\overline{\mathcal{F}_\infty})_{\partial_i} = (\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_i)$, and $\text{Rad}(\mathcal{F}_\infty \oplus \mathcal{F}_\infty) = \mathcal{F}_\infty^* \oplus \mathcal{F}_\infty$.

B. Theorem 3.1 gives countably many mutually inequivalent Fréchet algebra topologies on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$; for mutual inequivalence, see Theorem 3.5 below. In fact, we have a more stronger result than Theorem 3.1 as follows. We note that, for each $i \in \mathbb{Z}^+$, the derivation $\partial_i$ on $\mathcal{F}_\infty$ induces a natural derivation $D_i : (a, x) \mapsto (0, \partial_i(a))$ on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$.

**Theorem 3.2** The algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ is a Fréchet algebra with respect to the sequences $(q_k)$, $(q'_{k,i})$, $(q_{k,\partial_i})$ and $(q'_{k,\partial_i})$. The Fréchet algebra topologies $\tau_{\partial_i}$ generated by the sequences $(q_k, \partial_i)$ are equivalent to the Fréchet algebra topology $\tau_c + \tau_c$ if and only if for each $i \in \mathbb{Z}^+$, $\partial_i$ is continuous on $(\mathcal{F}_\infty, \tau_c)$ if and only if for each $i \in \mathbb{Z}^+$, the natural derivation $D_i$ is continuous on $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_c)$. Let $i \in \mathbb{Z}^+$ be fixed. Then the Fréchet algebra topology $\tau_{\partial_i}'$ generated by the sequence $(q'_{k,\partial_i})$ is not equivalent to the Fréchet algebra topology $\tau_i + \tau_i$ if and only if $\partial_i$ is discontinuous on $(\mathcal{F}_\infty, \tau_i)$ if and
only if the natural derivation $D_i$ is discontinuous on $(F_{\infty} \oplus F_{\infty}, \tau_i + \tau_i)$.

Proof. Certainly $(F_{\infty} \oplus F_{\infty}, (q_k))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty} \oplus F_{\infty}, (q'_{k,i}))$) is a Fréchet algebra and $q_{k,\partial_i}$ (respectively, $q'_{k,\partial_i}$) is a seminorm on $F_{\infty} \oplus F_{\infty}$ for each $k \in \mathbb{N}$. For $(a, x), (b, y) \in F_{\infty} \oplus F_{\infty}$, we have

$$q_{k,\partial_i}((a, x)(b, y)) = p_k(ab) + p_k(a \cdot (\partial_i(b) - y) + b \cdot (\partial_i(a) - x)) \leq (p_k(a) + p_k(\partial_i(a) - x))(p_k(b) + p_k(\partial_i(b) - y)) = q_{k,\partial_i}((a, x))q_{k,\partial_i}((b, y)),$$

and so $q_{k,\partial_i}$ (respectively, $q'_{k,\partial_i}$) is a submultiplicative seminorm on $F_{\infty} \oplus F_{\infty}$ for each $k \in \mathbb{N}$. We now show that $(F_{\infty} \oplus F_{\infty}, (q_{k,\partial_i}))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty} \oplus F_{\infty}, (q'_{k,i}))$) is a Fréchet algebra. Let $((a_n, x_n))$ be a Cauchy sequence in $(F_{\infty} \oplus F_{\infty}, (q_{k,\partial_i}))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty} \oplus F_{\infty}, (q'_{k,i}))$). Then $(a_n)$ and $(\partial_i(a_n) - x_n)$ are Cauchy sequences in $(F_{\infty}, (p_k))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty}, (p'_{k,i}))$). Since $F_{\infty}$ is a Fréchet space, there exists $a \in F_{\infty}$ and $x \in F_{\infty}$ such that $a_n \to a$ and $\partial_i(a_n) - x_n \to x$. Then $((a_n, x_n)) \to ((a, \partial_i(a) - x))$ in $(F_{\infty} \oplus F_{\infty}, (q_{k,\partial_i}))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty} \oplus F_{\infty}, (q'_{k,i}))$) and so $(F_{\infty} \oplus F_{\infty}, (q_{k,\partial_i}))$ (respectively, for each $i \in \mathbb{Z}^+$, $(F_{\infty} \oplus F_{\infty}, (q'_{k,i}))$) is a Fréchet algebra.

Suppose that $\partial_i$ is continuous. Then, for each $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and a constant $c_m > 0$ such that

$$q_{m,\partial_i}((a, x)) \leq p_m(a) + c_m p_{n(m)}(a) + p_m(x) \leq (1 + c_m)q_{n(m)}((a, x)),$$
and so the two topologies are equivalent, by the open mapping theorem for Fréchet spaces.

Conversely, suppose that the two topologies are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. Then, for each $m \in \mathbb{N}$, there exists $n(m) \in \mathbb{N}$ and a constant $c_m > 0$ such that $q_{m,\partial_i}((a, x)) \leq c_m q_n((a, x)) ((a, x) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty)$. Hence $p_{m,\partial_i}(a) \leq q_{m,\partial_i}((a, 0)) \leq c_m q_{n(m)}((a, 0)) = c_m p_{n(m)}(a) (a \in \mathcal{F}_\infty)$, and so, $\partial_i$ is continuous on $(\mathcal{F}_\infty, \tau_c)$.

The first half of the proof of the second statement has already been discussed in proof of Theorem 3.1. Suppose that for each $i \in \mathbb{Z}^+$, the Fréchet algebra topology $\tau_{\partial_i}^\prime$ is not equivalent to the Fréchet algebra topology $\tau_i + \tau_i$ on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. Then, for all $n \in \mathbb{N}$ and a constant $c > 0$, there exists $m \in \mathbb{N}$ such that $q_{m,\partial_i}^\prime((a, x)) > c q_{n,i}((a, x)) ((a, x) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty)$. Hence $p_{m,i}(\partial_i(a)) \geq q_{m,\partial_i}^\prime((0, a)) > c q_{n,i}^\prime((0, a)) = c p_{n,i}^\prime(a) (a \in \mathcal{F}_\infty)$, and so, $\partial_i$ is discontinuous on $(\mathcal{F}_\infty, \tau_i)$.

Remarks 1. For each $i \in \mathbb{Z}^+$, the natural derivation $D_i$ does not vanish on a dense subset $\mathcal{F}_\infty$ of the algebra $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$; however, it does vanish on the coefficient algebra $\mathcal{A}_i$ since the restriction of the natural derivation $D_i$ on $\mathcal{F}_\infty$ is $\partial_i$.

2. It is easy to see that for each $i \in \mathbb{Z}^+$, the Fréchet algebra topologies $\tau_c + \tau_c$
and $\tau_i + \tau_i$ are inequivalent, since if they are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, then since $\mathcal{F}_\infty$ can be identified with the closed subalgebra $\mathcal{F}_\infty \oplus 0$, we see that for each $i \in \mathbb{Z}^+$, the Fréchet algebra topologies $\tau_c$ and $\tau_i$ are equivalent, a contradiction. For the same reason, for $i \in \mathbb{Z}^+$, the Fréchet algebra topologies $\tau_i + \tau_i$ are mutually inequivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. Similarly, for $i \in \mathbb{Z}^+$, the Fréchet algebra topologies $\tau_c + \tau_c$ and $\tau_{\partial_i}$ are inequivalent, since if they are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, then, by Theorem 3.2 and by the transitivity property of the relation “equivalence of topologies”, the Fréchet algebra topologies $\tau_{\partial_i}$ and $\tau'_{\partial_i}$ are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, a contradiction to the remark, given after Theorem 3.5 below. Also, for each $i \in \mathbb{Z}^+$, the Fréchet algebra topologies $\tau_i + \tau_i$ and $\tau_{\partial_i}$ are inequivalent, since if they are equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, then, by Theorem 3.2 and by the transitivity property of the relation “equivalence of topologies”, the Fréchet algebra topologies $\tau_i + \tau_i$ and $\tau_c + \tau_c$ are equivalent, a contradiction to the fact given above.

3. From the first half of Theorem 3.2 we have the following

**Corollary 3.3** The Fréchet algebra topologies $\tau_{\partial_i}$ are mutually equivalent on the algebra $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$.

**Proof.** Clearly, by Theorem 3.2 for each $i \in \mathbb{Z}^+$, the Fréchet algebra
topology \( \tau_{\partial_i} \) generated by the sequence \((q_k, \partial_i)\) is equivalent to the Fréchet algebra topology \( \tau_c + \tau_c \), since the derivations \( \partial_i \) is continuous on \((\mathcal{F}_\infty, \tau_c)\). Now, the Fréchet algebra topology \( \tau_{\partial_i} \) is equivalent to \( \tau_{\partial_j} \) by the transitivity property of the relation “equivalence of topologies”.

In fact, we can sharpen the above corollary as follows.

**Corollary 3.4** Let \( i \) and \( j \) be in \( \mathbb{N}^+ \) such that \( i \neq j \). The Fréchet algebra topology \( \tau_{\partial_i} \) is equivalent to the Fréchet algebra topology \( \tau_{\partial_j} \) on the algebra \( \mathcal{F}_\infty \oplus \mathcal{F}_\infty \) if and only if the natural derivation \( D_i \) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j})\) and the natural derivation \( D_j \) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})\).

**Proof.** Suppose that for \( i \neq j \), the natural derivation \( D_i \) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j})\) and the natural derivation \( D_j \) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})\). Then, for each \( m \in \mathbb{N} \) and \((a, x) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty\), there exist \( n(m) \in \mathbb{N} \) and constants \( c_m, c'_m > 0 \) such that

\[
q_{m, \partial_i}((a, x)) \leq p_m(a) + p_m(\partial_i(a)) + p_m(x) \leq p_m(a) + c_m p_n(m)(a) + p_m(x) \leq (1 + c_m) q_n(m)((a, x)) \leq (1 + c_m) c'_m q_{n(m), \partial_j}((a, x)),
\]

and so, the two topologies \( \tau_{\partial_i} \) and \( \tau_{\partial_j} \) are equivalent on the algebra \( \mathcal{F}_\infty \oplus \mathcal{F}_\infty \), by the open mapping theorem for Fréchet spaces.

Conversely, suppose that for \( i \neq j \), the two topologies \( \tau_{\partial_i} \) and \( \tau_{\partial_j} \) are equivalent on the algebra \( \mathcal{F}_\infty \oplus \mathcal{F}_\infty \). Then, for each \( m \in \mathbb{N} \) and \((a, x) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty\), there exist \( n(m), l(m) \in \mathbb{N} \) and constants \( c_m, c'_m > 0 \) such that
\[ q_{m,\partial_i}(a,x) \leq c_m q_{n(m),\partial_j}(a,x). \] Hence

\[ q_{m,\partial_i}(D_j(a,x)) \leq c_m q_{n(m),\partial_j}(D_j(a,x)) \leq c_m' q_{l(m)}(D_j(a,x)) \]

since the sequences \((q_k,\partial)_i\) and \((q_k)\) are equivalent and

\[ c_m' q_{l(m)}(D_j(a,x)) \leq c_m q_{l(m)}(a,x) \leq c_m' q_{l(m),\partial_i}(a,x) \]

since \(D_j\) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_d + \tau_c)\). Thus, for \(i \neq j\), the natural derivation \(D_i\) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j})\) and the natural derivation \(D_j\) is continuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})\).

\[ \square \]

**Theorem 3.5** The Fréchet algebra topologies \(\tau_{\partial_i}'\) are mutually inequivalent on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\) if and only if for each \(i, j \in \mathbb{Z}^+\) such that \(i \neq j\), the natural derivation \(D_i\) is discontinuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j}')\).

**Proof.** Let \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j}')\) be a commutative Fréchet algebra and let \(D_i\) be the discontinuous natural derivation on \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\), induced by \(\partial_i\). Then, following the arguments of Theorem 1 of [5],

\[ S = \{ m = (m_1, m_2) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty : (0, m) \in \overline{\text{Gr}(D_i)_{\partial_j} + \tau_{\partial_j}} \}, \]

so that \(S\) is certainly contained in \(\overline{D_i(\mathcal{F}_\infty \oplus \mathcal{F}_\infty)_{\partial_j}'} \). Now a diagonal argument shows that \(S\) is closed. Then, following the arguments of Theorem 1
of \[6\], we have \( F_\infty \oplus F_\infty \oplus S = \text{Gr}(D_i) \tau'_\partial + \tau'_\partial \) and \( \theta_{D_i} \) is an automorphism of \( F_\infty \oplus F_\infty \oplus S \). Since \( D_i \) is discontinuous so is \( \theta_{D_i} \) and so the topologies \( \tau'_\partial + \tau'_\partial \) and \( \tau'_{D_i} \) (defined analogously as \( \tau'_\partial \)) are not equivalent on \( F_\infty \oplus F_\infty \oplus S \). It is easy to see that the topologies \( \tau'_\partial \) and \( \tau''_{D_i} \) (defined analogously as \( \tau''_{\partial} \)) are equivalent on \( F_\infty \oplus F_\infty \) as \( \tau'_\partial \leq \tau''_{D_i} \) on \( F_\infty \oplus F_\infty \) and the open mapping theorem for Fréchet spaces. So, if the topologies \( \tau'_\partial \) and \( \tau'_\partial \) are equivalent on \( F_\infty \oplus F_\infty \), then the topologies \( \tau'_\partial + \tau'_\partial \) and \( \tau'_{D_i} \) are equivalent on \( F_\infty \oplus F_\infty \oplus S \), a contradiction. Hence the Fréchet algebra topologies \( \tau'_\partial \) are mutually inequivalent.

Conversely, we observe that \( D_i \) is discontinuous on \( (F_\infty \oplus F_\infty, \tau'_\partial) \). For this, let \( ((a_n, x_n)) \) be a sequence in \( (F_\infty \oplus F_\infty, \tau'_\partial) \) such that it converges to \((0, 0)\) in \( F_\infty \oplus F_\infty \). So, \( q'_{k,\partial_j}((a_n, x_n)) \to 0 \). Now we see that \( D_i((a_n, x_n)) = (0, \partial_i(a_n)) \) does not converge to \((0, 0)\). Since \( q'_{k,\partial_j}((0, \partial_i(a_n))) = p'_{k,j}((\partial_i(a_n))) \), we show that \( p'_{k,j}((\partial_i(a_n))) \) does not converge to 0. For this, suppose that for \( i \neq j \), the Fréchet algebra topologies \( \tau'_\partial \) and \( \tau'_\partial \) are not equivalent. Then, for all \( n \in \mathbb{N} \) and \( c > 0 \), there exists some \( m \in \mathbb{N} \) such that

\[
q'_{m,\partial_i}((a, x)) > cq'_{n,\partial_j}((a, x)) \quad ((a, x) \in F_\infty \oplus F_\infty).
\]
Hence, for \((a, x) \in \mathcal{F}_\infty \oplus \mathcal{F}_\infty\),
\[q_{m, j}(D_i(a, x)) = q'_{m, j}((0, \partial_i(a))) = p'_{m, j}(\partial_i(a)) > c q'_{n, j}((0, \partial_i(a))).\]

Since \(q'_{n, j}((0, \partial_i(a))) = c p'_{n, i}(\partial_i(a))\) and \(\partial_i\) is discontinuous on \((\mathcal{F}_\infty, \tau_i)\), we see that \(p'_{k, j}(\partial_i(a_n))\) does not converge to 0 whenever \(a_n\) converges to 0. So \(D_i\) is discontinuous on \((\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau'_{\partial_j})\). Hence the result follows. \(\square\)

We remark that for each \(i \in \mathbb{Z}^+\), the Fréchet algebra topologies \(\tau_{\partial_i}\) and \(\tau'_{\partial_i}\) are mutually inequivalent on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\), because if they are equivalent, then, by Corollary 3.3 and by the transitivity property of the relation “equivalence of topologies”, we see that for \(i \neq j\), the Fréchet algebra topologies \(\tau_{\partial_i}\) and \(\tau'_{\partial_j}\) are equivalent on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\), and so, for \(i, j \in \mathbb{Z}^+\) with \(i \neq j\), the Fréchet algebra topologies \(\tau'_{\partial_i}\) and \(\tau'_{\partial_j}\) are equivalent on the algebra \(\mathcal{F}_\infty \oplus \mathcal{F}_\infty\), a contradiction.

We give the following extension of Theorem 3.1 in the Fréchet case.

**Theorem 3.6** Let \(A\) be a commutative Fréchet algebra, \((D_i)\) a sequence of non-zero derivations of \(A\) into a commutative Fréchet \(A\)-module \(M\). If \(D_i\) vanishes on dense subset of \(A\) for each \(i \in \mathbb{Z}^+\), then the algebra \(\overline{A}_{D_i}\) admits a Fréchet algebra topology \(\tau'_{D_i}\), generated by \((q'_{k, D_i})\) (defined analogously as \((q'_k, \partial_i)\)), and is inequivalent to the Fréchet algebra topology, generated by \((q'_{k, i})\) (defined analogously). \(\square\)
4 Conclusions

4.1. The above examples confirm the impression that derivations can behave quite badly on Fréchet algebras, which impression is initially given simply by the fact that $\partial_i = \partial/\partial X_i$ map outside the radical, on any formal power series algebra $\mathcal{A}_i[[X_i]]$, and so the Singer-Wermer conjecture does not hold in this case. Also, it does not hold for the derivation $f \mapsto f'$ on the algebra $\text{Hol}(U), U$ a domain in $\mathcal{C}$. However this conjecture does hold for the derivations $D_i$ on the Fréchet algebra $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})$ as $\text{Rad}(\mathcal{F}_\infty \oplus \mathcal{F}_\infty) = \mathcal{F}_\infty^* \oplus \mathcal{F}_\infty$ and the image of $D_i$ is $0 \oplus \mathcal{F}_\infty$. Moreover, it holds for pro-$C^*$-algebras (and, in particular, for $C(X), X$ a hemicompact $k$-space). So, it would be interesting to see which classes of Fréchet algebras satisfy this conjecture among all Fréchet algebras. Also, it is of interest to see which classes of Fréchet algebras satisfy $H^1(A, A) = 0$.

4.2. Moreover, it would also be interesting to construct examples of Fréchet algebras with uncountably many inequivalent Fréchet algebra topologies. The author believes that the answer of this question is in the affirmative, but has been unable to settle the question in general. For example, in the Banach case, one may modify the Feldman’s example to have an example of a Banach algebra with uncountably many inequivalent complete algebra norms.
by using distinct (discontinuous) linear functionals on a Banach algebra.

References

[1] W. G. Bade and P. C. Curtis, Jr., *The continuity of derivations of Banach algebras*, J. Funct. Anal. 16 (1974), 372-387.

[2] R. L. Carpenter, *Uniqueness of topology for commutative semisimple F-algebras*, Proc. Amer. Math. Soc. 29 (1971), 113-117.

[3] R. L. Carpenter, *Continuity of derivations on F-algebras*, Amer. J. Math. 93 (1971), 500-502.

[4] C. Feldman, *The Wedderburn principal theorem in Banach algebras*, Proc. Amer. Math. Soc. 2 (1951), 771-777.

[5] S. Grabiner, *Derivations and automorphisms of Banach algebras of power series*, Mem. Amer. Math. Soc. 146 (1974), 1-124.

[6] R. J. Loy, *Commutative Banach algebras with non-unique complete norm topology*, Bull. Austral. Math. Soc. 10 (1974), 409-420.

[7] S. R. Patel, *Fréchet algebras, formal power series, and automatic continuity*, Studia Math. 187 (2008), 125-136.
S. R. Patel, *Uniqueness of the Fréchet algebra topology on certain Fréchet algebras*, Studia Math. 234 (2016), 31-47.

S. R. Patel, *Fréchet algebras, formal power series, and analytic structure*, J. Math. Anal. Appl. 394 (2012), 468-474.

C. J. Read, *Derivations with large separating subspace*, Proc. Amer. Math. Soc. 130 (2002), 3671-3677.

M. P. Thomas, *The image of a derivation is contained in the radical*, Anna. of Math. 128 (1988), 435-460.

D. Vogt, *Non-natural topologies on spaces of holomorphic functions*, Ann. Polon. Math. 108 (2013), 215-217.

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