Deformation of singular connections I: $G_2$–instantons with point singularities

Yuanqi Wang

Abstract

In dimension 7, we establish a Fredholm theory for a Dirac-type operator associated to a connection with point singularities. There are two applications. 1. over a closed 7-manifold, under some natural conditions, a $G_2$–instanton and its point singularities can still be "seen" when the $G_2$–structure is properly perturbed. 2. over a ball in $\mathbb{R}^7$, for any almost-Euclidean $G_2$–structure, there exists a $G_2$–monopole asymptotic to an arbitrary Hermitian Yang-Mills connection on $S^6$.

Contents

1 Introduction 2
2 Definitions and Setting 6
3 Local theory 12
  3.1 Separation of variable for the system in the model case. . . . . . . . 12
  3.2 Solutions to the ODEs on the Fourier-coefficients . . . . . . . . . . 16
  3.3 Local solutions for the model cone connection . . . . . . . . . . . . 22
4 Global Theory 27
  4.1 Global apriori estimate. . . . . . . . . . . . . . . . . . . . . . . . . 27
  4.2 Hybrid space and $C^0$–estimate. . . . . . . . . . . . . . . . . . . 30
  4.3 Compact imbedding . . . . . . . . . . . . . . . . . . . . . . . . . . 32
  4.4 Fredholm Theory . . . . . . . . . . . . . . . . . . . . . . . . . . . 33
5 Perturbation 37
6 Characterizing the cokernel 41
7 Appendix 44
  7.1 Appendix A: Weitzenböck formula in the model case. . . . . . . . . 44
  7.2 Appendix B: Proof of Lemma 3.1 . . . . . . . . . . . . . . . . . . . 46
  7.3 Appendix C: Fundamental facts on elliptic systems . . . . . . . . . 48
  7.4 Appendix D: Density and smooth convergence of Fourier Series . . . 54
  7.5 Appendix E: Various integral identities and proof of Proposition 3.23 54
1 Introduction

The celebrated work of Donaldson-Thomas [13] has inspired extensive studies on special holonomy. On a seven-dimensional manifold $M$ with a $G_2$--structure $\phi$, for a vector bundle $E \to M$, it’s suggested in [13] to consider the $G_2$--instanton equation of connections on $E$:

$$F_A \wedge \psi = 0,$$

where $\psi$ is the co-associative form uniquely determined by $\phi$, and $F_A$ is the curvature form of the connection $A$. As pointed out in [13], the $G_2$--instantons should form a basis for a Casson-Floer-type theory for 7--manifolds. By adding the torsion-free $G_2$--structures $\phi(\psi)$ as a ”parameter” in (1), a very natural moduli space is the set of solutions $(\phi, A)$ to (1) (modulo gauge and some other natural equivalence). According to the seminal paper of Tian [35], it’s expected that instantons with point singularities should appear in the natural compactification. Therefore, a fundamental step to understand this moduli space (and any related one) is to study the following question.

*Given a $G_2$--instanton with point singularities, can we still see the instanton and the singularity for nearby $G_2$--structures?*

The best story we can expect is that the singularity disappears by perturbing the $G_2$--structure. On mean curvature flows, by the work of Colding-Minicozzi [6], this indeed happens: any flow which develops singularity as ”shrinking donuts” (Angenent [2]) can be perturbed away. However, our following main result shows that this does not happen very often for $G_2$--instantons.

**Theorem 1.1.** Let $E$ be an admissible bundle defined away from finitely-many points $(O_j)$ on a 7--manifold with a $G_2$--structure. Suppose $E$ admits an admissible $G_2$--instanton with trivial co-kernel, then for any small enough admissible deformation of the $G_2$--structure, there exists a $G_2$--monopole with the same tangent connection at each $O_j$.

**Remark 1.2.** The above statement is not the most precise, but we hope it is easy to understand. The most precise version of Theorem [1.1] is Theorem 5.1, to which we strongly recommend the readers to pay attention. We will only prove Theorem 5.1.

**Remark 1.3.** In the rest of the article we call the points where $E$ is undefined singular points. Please notice that our definition also includes the case when the bundle is smooth across some of (or all) the singularities, and in this case we allow both singular and smooth connections. Nevertheless, we are more interested in the case when the bundle (connection)
is truly singular. When the bundle and connection are both smooth across a singular point, our **local right inverse** of the linearised operator is still different from the standard one (see [10], [23]).

**Remark 1.4.** When the $G_2$--form is co-closed, any $G_2$--monopole (140) on a closed manifold is an instanton. However, a locally defined one might not.

**Remark 1.5.** Theorem 1 of Yang [37] and Lemma [24] suggest that, via a bundle isomorphism and a smooth gauge away from the singularities, any $G_2$--instanton (on a singular bundle) with quadratic curvature blowing-up at each singular point can be reduced to the case in Theorem 5.1. The work of Tian [35] indicates that the tangent connections at the singularities are the cone connections (bundles) on $\mathbb{R}^7 \setminus O$, pulled back from a smooth Hermitian-Yang-Mills connection over $S^6$ (with respect to the standard nearly-Kähler structure) via the spherical projection (Remark 2.5). The work of Charbonneau-Harland [7] indicates that the deformation of these Hermitian connections can be identified with a subspace of the kernel of a Dirac operator.

**Remark 1.6.** We expect the co-kernel to be trivial for most singular instantons i.e. we have tranversality in most of the cases. This is reasonable at least when the inner product is unweighted: the instanton constructed by Walpuski [34] is rigid, and the self-adjointness implies the co-kernel is trivial.

The key to the deformation problem is a **Fredholm-theory** for the linearised operator (13). It is a Dirac operator i.e. the square of it is a Laplacian. In the model case, **though we can not do separation of variable to the deformation operator itself, we can do it for the Laplacian.** As the standard Laplacian in polar coordinate, we have a **polar coordinate formula for the Laplacian of any cone connection** (Lemma 3.1). Using the Galerkin method (see 7.1.2 in [15]), we can construct a **local inverse for the Laplacian**, and this gives a **local inverse for the deformation operator** between the desired weighted-Sobolev spaces.

To handle the non-linearity of the instanton equation (or to preserve the tangent cone), a theory of Sobolev-spaces is not sufficient. The spaces should satisfy some multiplicative properties. Therefore, it should be helpful to turn on the a priori Schauder-estimates of Douglis-Nirenberg (Theorem 1 in [14]). Nevertheless, the essentially difficulty is the $C^0$--estimate.

Our crucial observation is that the $W^{1,2}_{p,b}$-**estimate of sufficiently negative** $p$ **yields the $C^0$--estimate**. Moreover, to handle the non-linearity of (1), it suffices to consider a hybrid space consisting of a weighted-$C^{2,\alpha}$ space and the weighted Sobolev-space (with norm as the sum of the two). The global version of our main analytic theorem is:

**Theorem 1.7.** Let $E \to M$ be the same as in Theorem 5.1. Suppose $A$ is an admissible connection of order 4. Then for any $A$--generic negative $p$ (Definition 2.21) and $b \geq 0$, $L_A$ (see (13)) is $(p, b)$--Fredholm (Definition 2.18).
from $W^{1,2}_{p,b}$ to $L^2_{p,b}$ (weighted Sobolev-spaces in Definition 3.24). $L_A$ is also $(p,b)$-Fredholm from $H_{p,b}$ to $N_{p,b}$ (Hybrid spaces in Definition 4.7).

Remark 1.8. Our Fredholm-theory works for a much larger class of operators, as long as the model operator is cone-type and admits separation of variable with respect to some operator on the link. In particular, it works for the Laplace-type operators (Theorem 3.19). We only assume discreteness and some natural asymptotics of the spectrum of the operator on the link. When the eigenfunctions are explicitly known, one might have a summation formula for the heat kernel and Green’s function, thus more information can be extracted (Theorem 4.3 and 1.13 in [8]).

Remark 1.9. It does not follow directly from definition that the kernel of the formal adjoint is finite-dimensional, nor equal to the co-kernel. Nevertheless, a trick (in Lemma 6.3) ensures that we can decrease the blowing-up rate of the co-kernel a little bit with respect to the spectrum gaps. This does not only give us an interesting PDE-result, but also implies the co-kernel is precisely the kernel of the formal adjoint (Theorem 6.1).

Remark 1.10. We can’t have optimal Sobolev-estimates unless the weight is properly chosen. Roughly speaking, $A$-generic means the weight $p$ of our Fredholm-theory avoids some discrete values determined by the spectrum of the tangential operators. This phenomenon, in other settings, is well understood (see [10], [9], [24]).

Remark 1.11. Usually the weight in the Schauder-estimate is required to have non-negative power (Lemma 3 in [14]). Thus, to obtain Fredholmness for every negative $p$, we should use a different norm (see [11]) when the power is negative, and adopt a trick in [18] to avoid global interpolations (the use of (200)).

Remark 1.12. For Theorem 5.1 (Theorem 1.11), we only need the hybrid theory when $p \in (\frac{-5}{2}, \frac{-3}{2})$ and $b = 0$. The most important usage is to handle the first iteration (146). Nevertheless, a theory for all $p$ and $b$ is useful for other applications.

The local version of our deformation theory states as follows.

Theorem 1.13. In the setting of Remark 1.5, for any smooth $SO(m)$–bundle $E \to S^6$ equipped with a smooth Hermitian Yang-Mills connection $A_0$, there is a $\delta_0 > 0$, such that for any admissible $\delta_0$-deformation $(\phi, \psi)$ over $B_0(\frac{1}{4}) \subset \mathbb{R}^7$ of the Euclidean $G_2$–forms (167), there exists a $G_2$–monopole of $\psi$ (146) over $B_0(\frac{1}{4})$ tangent to $A_0$ at the origin $O$.

Remark 1.14. Currently there is only one Hermitian Yang-Mills connection known over $S^6$: the canonical connection (Example 2.2 in [36]). Theorem 1.13 produces concrete local examples of singular $G_2$–monopoles tangent to the canonical connection for almost Euclidean $G_2$–structures.
Historically, the Fredholm-problem of elliptic operators has been extensively studied. The most related work to the present article is done by Lockhart-McOwen [24]. They proved that, over non-compact manifolds, a large class of operators are Fredholm between proper weighted Sobolev-spaces. Melrose-Mendoza [26] also obtained similar results in the $W^{k,2}$-setting generalized to pseudo-differential operators. Our hybrid-spaces, though not the most general, are sufficient for this study and are specially designed for singular connections.

Very recently the author learned from Thomas Walpuski that, using cylindrical method for the deformation operator, and the theory in [24], he could also obtain a local inverse between weighted Schauder-spaces for cones. This local Schauder-estimate is well illustrated in Section 2.1 of [21]. The author also learned from Goncalo Oliveira that in [27], he obtained $G_2$-monopoles with different kind of singularities.

In the aspect of $G_2$-instantons or monopoles, related work are conducted by Walpuski [34], Sa Earp-Walpuski [31], and Oliveira [28, 27]. On monopoles in other settings, see the work of Foscolo [16] and Oliveira [28, 29]. In the metric setting, the most related research is done by Joyce [22] (ALE space), Degeratu-Mazzeo [9] (Quasi ALE space), Mazzeo [25] (Edge-operators), Donaldson [11] (conic Kähler), the author-Chen [8] (parabolic conic Kähler), Akutagawa-Carron-Mazzeo [1] (Yamabe problem on singular spaces). The author believes the above list is not complete, and refers the readers to the references therein.

Omitting a number of necessary intermediate results, the following diagram shows the important steps to prove the main theorems.

This article is organized as following: most of the notions and symbols are defined in Section 2. In Section 3.1 we do separation of variable, and reduce the "squared" model linearized equation to ODEs. In Section 3.2 we solve these ODEs. In Section 3.3 we establish the optimal local Sobolev-theory. In Section 4.1, 4.2, 4.4 we establish the global Fredholm theory of Sobolev and Hybrid spaces. In Section 5 we prove the main geometric theorems. In Section 6 we prove the PDE result and characterize the co-kernel.
Acknowledgements: The author would like to thank Professor Simon Donaldson for suggesting this problem to work on, and for numerous inspiring conversations. The author is grateful to Song Sun and Thomas Walpuski for many valuable discussions, and for careful reading of the previous versions of this article. The author is grateful to Alex Waldron, Lorenzo Foscolo, Gao Chen, and Professor Xianzhe Dai for valuable discussions.

2 Definitions and Setting

We work under the setting of Theorem 5.1. By a bundle, we mean an open cover and associated overlap functions. Two bundles with different overlap functions are considered to be different, even when they are isomorphic. The definitions in this section are all routine and natural, a reader familiar with related material such as [18], [14], [10] can skip this section and come back if necessary.

Definition 2.1. A smooth $SO(m)-$bundle $E \to M \setminus (\cup O_j)$ is said to be an admissable bundle if

- $E$ is defined by an admissible cover $U_{\rho_0}$ (Definition 2.2) for some $\rho_0 > 0$,
- for each singular point $O_j$, the overlap function between $V_{+,O_j}$ and $V_{-,O_j}$ does not depend on $r$ (see Remark 2.5) i.e. the overlap function is pulled back from the sphere.

Let $\Xi$ denote $\Omega^0(adE) \oplus \Omega^1(adE)$ (adE-valued 0–form and 1–form), and the corresponding bundle over $S^{n-1}$ as in Section 3.1. All the analysis in this article are on sections to $\Xi$, over $M$ or various domains. We omit $\Xi$ in the notations of the section spaces in Definition 2.19. All the definitions and discussions below apply to $\Xi$ as well. When $E$ is a complex bundle, we require it to be a $\mathbb{U}(m^2)$-bundle, and we still view it as a real bundle.

Definition 2.2. (Admissible open cover). Given an (reference) open cover of $M$ and a (reference) coordinate system, a refinement (with the same coordinate maps) denoted as $\mathbb{U}_{\tau_0} = \{B_l, B_{O_j}(V_{+,O_j}, V_{-,O_j}), l, j \text{ are integers with finite range}\}$ is called an $\tau_0-$admissible cover if the following conditions are satisfied.

1. Each $B_l$ is in the smooth part of $E$, the ball $100B_l$ (concentric and of radius 100 times larger) is still away from the singularities and is contained in a coordinate chart. This is different from saying that $B_l$ is a metric ball in the manifold. In this article, by abuse of notation, $B_l$ means both the ball in the chart and the open set in the manifold (it should be clear from the specific context which notion we mean).
2. Each $B_{O_j}$ is centred at a singular point of $E$ with radius $\tau_0$, and contain no other singular point. $O_j$ corresponds to the origin in the chart. $100B_{O_j}$ is still a ball in a coordinate chart and are disjoint from each other. Moreover, in this coordinate $\phi(O_j)$ is the standard $G_2$–form.

3. $\frac{B_j}{100}$ and $\frac{B_{O_j}}{100}$ still form a cover of $M$.

When $\tau_0$ small enough with respect to $M$ and $E$, this cover always exists if one adds enough balls of small radius.

The letter $O$ always means a singular point among the $O_j$‘s, and also the origin in the coordinate (by abuse of notations). We denote it as ”$B_O(\rho)$” when we want a ball with radius $\rho$. The symbols ”$B_O$” (”$B_{O_j}$”) without radius usually means one of balls in $\mathbb{U}_m$ defined above.

Let $M_\tau$ denote $M \setminus \cup_j B_{O_j}(\tau)$ (the part far away from the singularities).

**Remark 2.3.** In practice, we usually choose the coordinates as the normal coordinates of the underline Riemannian metric, though our definition allows any smooth coordinate. In [35], the existence of tangent cone connection (near the singularities) is proved in normal coordinates.

Let $B_O(1)$ denote the unit ball in $\mathbb{R}^n$ centred at the origin. Since $B_O(1)$ admits a natural smooth deformation retraction onto $S^{n-1} \times \{\frac{1}{2}\}$, the well known homotopy property (Theorem 6.8 and the last paragraph in page 58 of [4]) of bundles gives the following lemma.

**Lemma 2.4.** Any smooth $SO(m)$–bundle $\tilde{E} \to M \setminus (\cup_j O_j)$ defined by a locally finite cover is isomorphic to an admissible bundle $E$ in Definition 2.7. The isomorphism covers the identity map from $M \setminus (\cup_j O_j)$ to itself.

**Remark 2.5.** Near each $O_j$, for some $\tau_0 > 0$, the smooth isomorphism (away from $O_j$) is the one in Theorem 6.8 of [4], with respect to the natural homotopy deforming the identity map $id$ of $B_{O_j}(\tau_0)$ to the map $g \circ f$:

$$B_{O_j}(\tau_0) \simeq S^{n-1} \times (0, \tau_0) \xrightarrow{g} S^{n-1} \times (\frac{\tau_0}{2}) \xrightarrow{f} S^{n-1} \times (\frac{\tau_0}{4}),$$

where $g$ is the **spherical projection** $(x, t) \to (x, \frac{t}{2})$, and $f$ is the identity inclusion. Let $r$ (sometimes $r_x$) denote the Euclidean distance to the singular set $\{O_1, ..., O_m\}$ in the reference coordinate chart respectively.

**Remark 2.6.** Any bundle-valued $k$–form $\xi$ without $dr$– component (defined over $\mathbb{R}^n \setminus O$) can be viewed as a $r$–dependent bundle-valued $k$–form over $S^{n-1}(1)$. Let $|\xi|$ denote the usual Euclidean norm of $\xi$ (as a form over $\mathbb{R}^n \setminus O$)
, and $|\xi|_S$ denote the norm on the unit sphere with respect to the standard round metric (as a spherical form). The relation is

$$|\xi|^2 = \frac{1}{r^{2k}}|\xi|^2_S, \text{ for any } \xi. \tag{2}$$

**Definition 2.7.** (Admissible connections) Given a smooth bundle $E \to M$ with finite many singular points, and a smooth $G_2$–structure $(\phi, \psi)$ over $M$, a connection $A$ of $E$ is called an admissible connection of order $k_0$, if it satisfies the following conditions.

- $A$ is smooth away from the $O_j$’s.
- There exist a $\mu_1 > 0$, such that for any $O$ among the $O_j$’s, there is smooth connection $A_O$ on $E \to S^{n-1}$ such that the following holds in the reference coordinate chart.

$$\sum_{j=0}^{k_0} r^{j+1} |\nabla^j_{A_O}(A - A_O)| \leq C(-\log r)^{-\mu_1}, \tag{3}$$

where we view $A_O$ as the pulled-back connection over $\mathbb{R}^7 \setminus O$.

For the purpose of quantization, $A$ is also said to be of polynomial rate $\mu_1$ at $O_j$ (we omit the $O_j$ if the rate holds at every singular point).

Suppose for some constant $C$, $A$ satisfies (3) with right hand side replaced by $C r^{\mu_0}$ (at $O_j$); $\mu_0 > 0$, then $A$ is said to be of exponential rate $\mu_0$ at $O_j$.

**Remark 2.8.** When $A$ is admissible and satisfies the instanton equation away from the singularities, we call it an admissible instanton. In practice, the coordinate near the singularities are normal coordinate of $g_\phi$ (see Remark 2.3).

**Definition 2.9.** A connection $\underline{A}$ is satisfies Condition $\mathcal{S}_{A,p}$ if the following holds with respect to the reference instanton $A$.

- $\underline{A}$ is an admissible connection of order 3.
- $\underline{A}$ is close to $A$ in $H_p$ (Definition 4.17 and 2.19). Consequently,
- the tangent connections of $\underline{A}$ at each $O_j$ is the same as that of $A$;
- $\underline{A}$ is with the same polynomial rate as $A$ at each $O_j$. Moreover, if $A$ is with exponential rate $\mu_0 > 0$ at $O_j$, then $\underline{A}$ is with exponential rate $\min\{\mu_0, -\frac{3}{2} - p\}$ at the same point.

Near any singular point $O$ (among the $O_j$’s), the bundle $E$ is trivialized by 2 coordinate patches $U_+, U_-$ of $S^{n-1}$, then we choose the cover of $B_O$ as $V_+O(V_-O) = U_+(U_-) \times [0, \tau_0]$. In these coordinates, we can easily define the weighted Schauder norms for sections of $\Xi$ without involving any connection.
Definition 2.10. As in Definition 2.4 of [8], we don’t even need a connection to define the Schauder norms. Let $r_{x,y} = \min\{r_x, r_y\}$, $r_{x,y} = \max\{r_x, r_y\}$. Near a singular point $O$, let $\Gamma$ be a locally defined matrix-valued tensor in a coordinate chart of $\Xi$ (Definition 2.2), we define the following.

$$[\Gamma]^{(\mu, b)}_{\alpha, 2\Delta} = \begin{cases} \sup_{x,y \in \Xi} (-\log r_{x,y})^{\beta} (r_{x,y})^{-\alpha} \cdot |\Gamma(x) - \Gamma(y)|, & \text{when } \mu + \alpha \geq 0 \\ \sup_{x,y \in \Xi} (-\log r_{x,y})^{\beta} (r_{x,y})^{-\alpha} \cdot \frac{|\Gamma(x)| - |\Gamma(y)|}{|x-y|^{\alpha}}, & \text{when } \mu + \alpha < 0 \end{cases}$$

(4)

$$[\Gamma]_{0, 2\Delta} = \sup_{x \in \Xi} (-\log r_x)\cdot |\Gamma(x)|.$$  

(5)

The idea of (4) is to choose the weight function ”as small as possible”. Note that we allow $\mu$ to be any real number, while in Lemma 3 in [14], the power is required to be non-negative. We usually let $\Xi$ be $V_{+, O}$ ($V_{-, O}$) or a ball contained therein. We then define

$$|\xi|^{(\gamma, b)}_{2,\alpha, V_{+, O}} \triangleq |\nabla^2 \xi|^{(2+\gamma, b)}_{\alpha, V_{+, O}} + |\nabla \xi|^{(1+\gamma, b)}_{0, V_{+, O}} + |\xi|^{(\gamma, b)}_{0, V_{+, O}}.$$  

(6)

where the $\nabla$ is just the usual gradient in Euclidean coordinates. Moreover, by abuse of notation (which we adopt throughout this article in this case), the ”$\xi$” in (6) means the multi-matrix-valued function in $V_{+, O}$ representing $\xi$. $|\xi|^{(\gamma, b)}_{2,\alpha, V_{+, O}}$ is defined in the same way throughout this article, so does $|\xi|^{(\gamma, b)}_{2,\alpha, B}$ for any ball $B \subset V_{+, O}$ or $V_{-, O}$.

Definition 2.11. (Global Schauder norms) In the same context as Definition 2.2 let $\rho_0 > 0$ be independent of $A$ such that there exists a $\rho_0$–admissible cover $\mathcal{U}_{\rho_0}$. We define

$$|\xi|^{(\gamma, b)}_{2,\alpha, M, I} \triangleq \sup_{B_i \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, B_i} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{+, O_j}} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{-, O_j}}.$$  

(7)

The $|\xi|^{(\gamma, b)}_{2,\alpha, B_i}$ are the unweighted Schauder norms defined in (4.5),(4.6) in [18]. Actually we have 2 other ways to define the Schauder norms. One is by using the smaller cover:

$$|\xi|^{(\gamma, b)}_{2,\alpha, M, II} \triangleq \sup_{B_i \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, B_i} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{+, O_j}} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{-, O_j}}.$$  

(8)

The third definition is by using the naturally weighted Schauder norms in (4.17) of [18] (away from the singularity):

$$|\xi|^{(\gamma, b)}_{2,\alpha, M, III} \triangleq \sup_{B_i \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, B_i} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{+, O_j}} + \sup_{B_o_j \in \mathcal{U}_{\rho_0}} |\xi|^{(\gamma, b)}_{2,\alpha, V_{-, O_j}}.$$  

(9)

An easy but important lemma is
Lemma 2.12. The 3 norms in (7), (8), (9) are equivalent.

Proof. This is an easy exercise by definition. For the reader’s convenience, we still point out the crucial detail. Obviously norm I is stronger than norm III, and norm III is stronger than norm II. We only need to show norm II is stronger than norm I. This is because of the last item in Definition 2.2: $V_{+Oj} \setminus \frac{V_{+Oj}}{100}$ is covered by the $B_{100}'s$. Since the transition functions are smooth, then the Schauder norm of $\xi$ over $V_{+Oj} \setminus \frac{V_{+Oj}}{100}$ is controlled by the supreme of Schauder norms on the $B_{100}'s$.

The same holds for $V_{-Oj} \setminus \frac{V_{-Oj}}{100}$ and $B_i \setminus \frac{B_i}{100}$ away from the singularities. □

Definition 2.13. The weighted Schauder-space $C^{k,\alpha}_{(\gamma,b)}(M)$ consists of sections with the norm (7) being finite. This notation also applies to any domain.

Definition 2.14. For the local perturbation in Theorem 1.13, on $B_O(R)$, we need a Schauder space whose weights near $O$ and $\partial B_O(R)$ are different. To be precise, we define the space $C^{k,\alpha}_{(\gamma,t)}[B_O(R)]$ by the norm

$$|\xi|^{(\gamma),t}_{k,\alpha,B_O(R)} \triangleq \sum_{j=0}^{k} \sup_{x \in V_{+O}(R)} \min \{r_x^{\gamma+j}(R-r_x)^{t+j} \} |\nabla^j \xi|(x) + \sup_{x,y \in V_{+O}(R)} \min \{r_x^{\gamma+k+\alpha}, (R-r_x)^{t+k+\alpha} \} \frac{|\nabla \xi(x) - \nabla \xi(y)|}{|x-y|^\alpha} + \text{the same in } V_{-O}(R).$$

Definition 2.15. (Admissible $\delta_0$—deformations of $G_2$—structures) A $G_2$—structure $(\phi, \psi)$ is called an admissible $\delta_0$—deformation of $\phi$ if

- $\phi$ is smooth and $|\phi - \phi|_{C^0(M)} \leq \delta_0$. \hspace{1cm} (10)

where the $C^0(M)$—norm is defined by the base $G_2$—structure $\phi$;

- $\phi = \phi$ at each $O_j$.

Then we automatically have

$$|\phi - \phi|(x) \leq Cr_x \hspace{1cm} (11)$$

when $x$ is close to the singularities.

Since the $\phi$ determines a smooth metric $g_\phi$, and a smooth co-associative form $\psi$ (see [32] and [5]), we also obtain a small deformation of the base form $\psi$ such that

$$|\psi - \psi|_{C^0(M)} \leq C\delta_0. \hspace{1cm} (12)$$

Note that we don’t require $\phi$ to be closed, but when we want an instanton, we have to assume it’s co-closed i.e. $\psi$ is closed.
Definition 2.16. (General constants) The background data in this article is the dimension $n$ (in most cases it’s 7), the manifold $M$ and bundle $E \rightarrow (\Xi)$ with a fixed coordinate system, the $p, b$ in the weights, the reference $G_2$—structures $\phi$ and $\psi$, the tangent cone connections $A_O$ (and the bundle $E \rightarrow (\Xi)$ on the sphere), the Hölder-exponent $\alpha$, and the base connection $A$. Without further specification, the constants "$C$", $\delta_0$, $\epsilon_0$, $\mu_1$, $\vartheta_1$... in each estimate means a constant depending (at most) on the above data. We add sub-letters to the "$C$" when it depends on more data than the above, or when we want to emphasize the dependence on some specific factor. The "$C$’s" in different places might be different. The $\delta_0$, $\epsilon_0$, $\mu_1$, etc are usually small enough with respect to the above data. There are some auxiliary small numbers like $\epsilon$, $\delta$, which we usually let tend to 0.

When a bound depends only on the above data, we say it’s uniform.

Definition 2.17. (Special constants) For any $O$ among the singular points, we let $\bar{C}$ denote any constant depending only on the weights $p, b$, and the underlying cone connection $A_O$ (and the sub-symbol if there is any).

In particular, these $\bar{C}$’s do not depend on the radius of the underlying balls, so our requirements are fulfilled. They mainly appear in Section 3.2 and 3.3.

Definition 2.18. (($p, b$)—Fredholm operators and isomorphisms) In the space of $L^2_{loc}$—sections to $\Xi \rightarrow M \setminus \cup_j O_j$, consider the inner product given by the weighted space $L^2_{p, b}$. As in page 49 of [10], let $H$ and $N$ be Banach spaces of sections to the bundle $\Xi$, and $L$ is a bounded linear operator $H \rightarrow N$. $L$ is called a ($p, b$)—Fredholm operator if following conditions are satisfied.

- Both $H$ and $N$ are subspaces of $L^2_{p, b}$. Let $\perp$ be the orthogonal complement with respect the $L^2_{p, b}$—inner product.

- Image$L$ is closed in $N$. Both $Ker L$ and $coker L = N/\text{Image} L$ are finite dimensional.

- Coker$L$ is isomorphic to $\text{Image}^\perp L \cap N$, and under this isomorphism, $N$ admits a direct-sum decomposition

\[ N = \text{Image} L \oplus_{p, b} \text{coker} L, \]

where $\oplus_{p, b}$ is orthogonal with respect to $L^2_{p, b}$.

- $L : Ker^\perp L \cap H \rightarrow \text{Image} L$ is an isomorphism (under the norms of $H$ and $N$). The "isomorphism" means $L$ is bijective (restricted to the 2 closed subspaces), and both $L$ and $L^{-1}$ are bounded.

Definition 2.19. (Abbreviation of notations for the spaces of sections). When the log-power $b$ is equal to 0, we abbreviate all the notations $W_{p, b}^{1, 2}, L^2_{p, b}, H_{p, b}, N_{p, b}, J_{p, b}, Q_{p, b, A_O}, Q_{A, p, b},$ etc
as 
\[ W_p^{1,2}, L_p^2, H_p, N_p, J_p, Q_{p,AO}, Q_{A,p}, \text{etc.} \]

**Definition 2.20.** (Tensor products) The sign \( \otimes \) means a tensor product depending on (some of and at most) the reference \( G_2 \)-structure \( \phi, \psi \), the metric \( g_\psi \), the Euclidean metric in the coordinates, or some other \( G_2 \)-structure, manifold, or bundle. Thus the norms of these \( \otimes \)'s are bounded with respect to the above data. The \( \otimes \)'s in different places might be different. When we are considering some specific tensor product, we add sub-letter or symbol to the \( \otimes \) (like in Lemma 7.1 and proof of Proposition 3.3).

**Definition 2.21.** Given an admissible connection \( A \), let \( p \) be a real number. \( p \) is called \( A \)-generic if \( 1 - p \) and \( -p \) do not belong to the \( v \)-spectrum of any \( \Upsilon_{AO_j} \) (see (31) and Definition 3.5). This means neither \( 7.25 - (1 - p)^2 \) nor \( 7.25 - p^2 \) is an eigenvalue of any \( \Upsilon_{AO_j} \).

### 3 Local theory

#### 3.1 Separation of variable for the system in the model case.

By abuse of notation, we still let \( \Xi \) denote the space of sections to the bundle \( \Xi \) etc. By the monopole equation (140), the linearised operator with respect to \( \sigma \in \Omega^0_{\text{ad}E} \) and \( a \in \Omega^1_{\text{ad}E} \) (at \((0,0) \in \Omega^0_{\text{ad}E} \oplus \Omega^1_{\text{ad}E} = \Xi \) when \( \psi = \psi \)) is

\[
L_A[\begin{array}{c} \sigma \\ a \end{array}] = \begin{bmatrix} d^*_Aa \\ d_A\sigma + \star(d_Aa \wedge \psi) \end{bmatrix}
\]

where \( \psi \) is the base co-associative form in Theorem 5.1. Let \( L_{AO} \) denote the deformation operator of \( AO \) and Euclidean \( G_2 \)-structure. If the operator depends on any different \( G_2 \)-structure than the Euclidean one and \( \phi, \psi \), we add sub-symbol.

Thus \( L_{AO}^2 \) is still an operator from \( \Xi \) to itself. To achieve separation of variable for this operator, we should understand the bundle in another way.

Working in general dimension \( n \geq 4 \), given any \( \xi = \begin{bmatrix} \sigma \\ a \end{bmatrix} \in \Xi \), we write

\[
\sigma = \frac{\zeta}{r}, \quad a = a_r \frac{dr}{r} + a_s,
\]

where \( a_s \) does not have radial component, and \( a_r \) is a \( r \)-dependent section of \( \Omega^0_{E}(S^{n-1}) \). In another word, we want to view sections of \( \Xi \) as \( r \)-dependent sections of the bundle (over \( S^{n-1} \))

\[
\Xi = \Omega^0_{\text{ad}E}(S^{n-1}) \oplus \Omega^0_{\text{ad}E}(S^{n-1}) \oplus \Omega^1_{\text{ad}E}(S^{n-1}) \text{ under the basis in (14).}
\]

12
Let $\nabla_S$ denote the covariant derivative with respect to the connection $A_O$, viewed as a connection over $S^{n-1}$.

For the 0–form $\sigma$, the well known cone formula for the rough Laplacian reads as

$$-\nabla^* \nabla \sigma = \frac{\partial^2 \sigma}{\partial r^2} + \frac{n-1}{r} \frac{\partial \sigma}{\partial r} + \frac{\Delta_s \sigma}{r^2}. \tag{16}$$

Let $\zeta = r \sigma$, by Claim 3.11 we have

$$-r \nabla^* \nabla \left( \frac{\zeta}{r} \right) = \frac{\partial^2 \zeta}{\partial r^2} + \frac{n}{r} \frac{\partial \zeta}{\partial r} + \frac{(3-n) \zeta}{r^2} + \frac{\Delta_s \zeta}{r^2}, \tag{17}$$

where $\Delta_s$ is negative of the rough Laplacian of $A_O$ over $S^{n-1}$.

On 1–forms, we have the following polar coordinate formula.

**Lemma 3.1.** Suppose $A_O$ is a cone connection over $\mathbb{R}^n \setminus O$. Then

$$-\nabla^* \nabla a = \left( \frac{\partial^2 a_r}{\partial r^2} + \frac{n-3}{r} \frac{\partial a_r}{\partial r} - \frac{2(n-2)a_r}{r^2} + \frac{\Delta_s a_r + 2d_s^2 a_s}{r} \right) dr$$

$$+ \nabla_r (\nabla_r a_s) + \frac{n-1}{r} \frac{\Delta_s a_s + 2d_s^2 a_r}{r^2},$$

where $\Delta_s$ is the negative of the rough laplacian of $A_O$ on $S^{n-1}$.

The proof of Lemma 3.1 will be deferred to Section 7.2.

Next we return to dimension 7. By the formula in Lemma 3.1, $-L^2_{A_O}$ is the rough Laplacian of $A_O$ plus some algebraic operators, thus the polar coordinate formula naturally involves the $SU(3)$–structure of $S^6$ (see [17], [36]) for the formulas we need. Let $(\omega, \Omega)$ (as in [17]) be the standard $SU(3)$–structure over $S^6$, where $\omega$ is the standard Hermitian metric with respect to the almost complex structure, and $\Omega$ is the $(3,0)$–form. They satisfy

$$d\omega = 3\text{Re}\Omega, \ d\text{Im}\Omega = -2\omega^2. \tag{18}$$

Moreover, the standard $G_2$–forms can be written as

$$\phi_0 = r^2 dr \wedge \omega + r^3 \text{Re}\Omega, \ \psi_0 = -r^3 dr \wedge \text{Im}\Omega + \frac{r^4}{2} \omega^2. \tag{19}$$

A necessary algebraic definition is the following.

**Definition 3.2.** (Some specific tensor products) Let $\theta$ be a $p$–form and $\Theta$ be a $q$–form, both are possibly $adE$–valued.

Suppose $q > p$, then we define $\theta \cdot \Theta$ as the $q - p$ form

$$\theta \cdot \Theta(Y_1...Y_{q-p}) = \Sigma_{i_1,...,i_p} \theta(v_{i_1},...,v_{i_p}) \Theta(v_{i_1},...,v_{i_p},Y_1...Y_{q-p}),$$
where \( v_j \)'s form an orthogonal basis of the underlying metric.

Suppose \( q < p \), similar to the previous paragraph, we define \( \theta \circ \Theta \) as the \( p - q \) form \( \theta \circ \Theta(Y_1...Y_{p-q}) = \sum_{i_1,...,i_q} \theta(v_{i_1},...,v_{i_q}, Y_1...Y_{p-q}) \Theta(v_{i_1},...,v_{i_q}) \).

The order of multiplication is important, since they are matrix-valued.

The symbol "\( \otimes \)" means the tensor product

\[
F \otimes a = [F(e_i, e_j), a(e_i)]e^j,
\]
where \( F \) is an \( adE \)-valued 2-form, \( a \) is an 1-form, and the \( e^j \)'s form an orthonormal frame of the underline metric (which is the Euclidean metric in the model case).

The symbol "\( \otimes_S \)" means the \( \otimes \) over \( S^6 \) with respect to the standard round metric, so do the symbols \( \otimes_S \) and \( \otimes_{-S} \).

Routine computation gives the main result in this section.

**Proposition 3.3.** Under the basis in \([14]\), the equation

\[
-L^2_{AO}(\begin{array}{ccc} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & Id \end{array}) \begin{bmatrix} \zeta \\ a_r \\ a_s \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & Id \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix} \]  

(20)

is equivalent to

\[
\left\{ \begin{array}{l}
\frac{\partial^2 \zeta}{\partial r^2} + 4 \frac{\partial \zeta}{\partial r} - \frac{5 \zeta}{r^2} + \frac{\Theta_{AO,0}}{r^2} = f_0 \\
\frac{\partial^2 a_r}{\partial r^2} + 4 \frac{\partial a_r}{\partial r} - \frac{5 a_r}{r^2} + \frac{\Theta_{AO,1}}{r^2} = f_1 \\
\nabla_r(\nabla_r a_s) + \frac{6}{r} \nabla_r a_s - \frac{a_r}{r} + \frac{\Theta_{AO,2}}{r^2} = f_2
\end{array} \right.,
\]

(21)

where

\[
\begin{array}{c|c|c}
\Theta_{AO} & \zeta & \Theta_{AO,0} \\
& a_r & \Theta_{AO,1} \\
& a_s & \Theta_{AO,2} \\
\end{array}
\]

(22)

\[
= \begin{bmatrix}
\Delta_s \zeta + \zeta + [F_{AO}, a_s]_{\otimes_S} Re \Omega + [F_{AO}, a_r]_{\otimes_S} Im \Omega + [F_{AO}, a_s]_{\otimes_S} Im \Omega - [F_{AO}, a_r]_{\otimes_S} Re \Omega \\
\Delta_s a_r - 5a_r + 2d_s a_s + [F_{AO}, a_s]_{\otimes_S} Re \Omega + [F_{AO}, a_r]_{\otimes_S} Im \Omega - [F_{AO}, a_s]_{\otimes_S} Im \Omega \\
+ [F_{AO}, a_s]_{\otimes_S} Im \omega - [F_{AO}, a_r]_{\otimes_S} Re \omega
\end{bmatrix}
\]

The operator \( \Theta_{AO} \) is a smooth self-adjoint elliptic operator over \( \Xi \rightarrow S^6 \).

When \( A_O \) is a \( \psi_0 \)-instanton i.e. \( F_{AO} \wedge \psi_0 = 0 \) as a connection pulled back to \( \mathbb{R}^7 \setminus O \) (equivalent to that \( A_O \) is Hermitian Yang-Mills on \( S^6 \) \([37]\)), we have

\[
\begin{array}{c|c|c}
\Theta_{AO} & \zeta & \Delta_s \zeta + \zeta \\
& a_r & \Delta_s a_r - 5a_r + 2d_s a_s \\
& a_s & \Delta_s a_s + 2d_s a_r - 2F_{AO} \otimes_S a_s
\end{array}
\]

(23)
We only prove (22), equation (23) is a special case and is implied by Lemma 7.1 or by Lemma 2.4 in [36] (using (22)). It suffices to combine Lemma 3.1, Lemma 7.1 (and the proof of it), formulas (17), (18), (19). We say a few more for the readers’ convenience.

First, since $A_O$ is a cone connection, then

$$F_{A_O} \otimes a = F_{A_O} \otimes a_s = \frac{1}{r^2} F_{A_O} \otimes a_s.$$ (24)

Second, formula (19) directly implies

$$[F_{A_O}, a] \psi = \frac{1}{r^3} [F_{A_O}, a_s] \Im \Omega + \frac{1}{2r^2} [F_{A_O}, a_s] \Im \Omega + \frac{1}{2r^2} [F_{A_O}, a_s] \Im \Omega^2,$$ (25)

$$\ast([F_{A_O}, a] \wedge \psi) = [F_{A_O}, a] \varphi_0 = \frac{1}{r^2} [F_{A_O}, a_s] \Re \Omega + \frac{1}{r^3} [F_{A_O}, a_r] \Re \omega,$$ (26)

$$\ast([F_{A_O}, a] \wedge \psi) = [F_{A_O}, a] \varphi_0 = \frac{dr}{r^2} [F_{A_O}, a_s] \Re \omega + \frac{1}{r} [F_{A_O}, a_s] \Re \omega,$$ (27)

The proof of (22) is complete.

To show the self-adjointness of $\Upsilon_{A_O}$ as an operator over $S^6$, it suffices to note that $[F_{A_O}, a_s] \Re \omega$ is adjoint to $-[F_{A_O}, \zeta] \Im \Omega$, $[F_{A_O}, a_r] \Im \Omega$ is adjoint to $-d_a \sqrt{2}$, $d_s$ is adjoint to $2d_a \sqrt{2}$, and $\Re \Omega$ is adjoint to $-d_a \sqrt{2}$. Moreover, both $F_{A_O} \otimes a_s$ and $[F_{A_O}, a_s] \Re \omega$ are self-adjoint. A very important formula for verifying these relations is

Claim 3.4. For any $\text{ad}E$--valued $p$–form $a_1$, $\text{ad}E$--valued $q$–form $a_2$, and ordinary $(p+q)$–form $B$, we have

$$< a_1 \wedge B, a_2 > = (-1)^{pq} < a_1, a_2 \wedge B >.$$ (28)

The assumption that $\Xi \to S^6$ is a $SO(m)$–bundle implies $[F_A, \cdot]$ is anti-symmetric with respect to the inner product of the Lie-algebra of $so(m)$. □

We denote the eigenvalues of $\Upsilon_{A_O}$ as $\beta$, and the corresponding eigensection as $\Psi_{\beta}$ (there might be multiplicities) i.e

$$\Upsilon_{A_O} \Psi_{\beta} = \beta \Psi_{\beta}, \quad \Psi_{\beta} = (\phi_{1,\beta}, \phi_{0,\beta}, \phi_{1,\beta} \in \Omega^0_{\text{ad}E}(S^6), \phi_{2,\beta} \in \Omega^1_{\text{ad}E}(S^6).$$

We require $\Psi_{\beta}$ to be an orthonormal basis in $L^2_\Xi(S^6)$, which is the space of $L^2$--sections to $\Xi \to S^6$, with respect to the natural inner product of the direct sums in (15). By (21) and (182), the equation

$$-L_{A_O}^2 \xi = f$$ (29)
\[
\frac{d^2 \xi}{dr^2} + \frac{4 d \xi}{r \, dr} + \frac{(\beta - 5) \xi}{r^2} = f_\beta, \quad \xi = \Sigma_\beta \xi_\beta \Psi_\beta, \quad \text{where } f = \Sigma_\beta f_\beta \Psi_\beta. \quad (30)
\]

Let
\[
-v^2 = \beta - \frac{29}{4}. \quad (31)
\]

\(v\) is either a non-negative real number or a purely imaginary number. To reduce the above equation into the form we are most familiar with, we consider \(\xi_v = r^{\frac{2}{3}} \xi_\beta\), \(f_v = r^{\frac{2}{3}} f_\beta\), then (30) becomes
\[
\frac{d^2 \xi_v}{dr^2} + \frac{1}{r} \frac{d \xi_v}{dr} - \frac{v^2 \xi_v}{r^2} = f_v, \quad \text{where } \xi_v \text{ and } f_v \text{ only depend on } r. \quad (32)
\]

By abuse of notation, we shall study the ordinary differential equation
\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{d u}{dr} - \frac{v^2 u}{r^2} = f. \quad (33)
\]

**Definition 3.5.** \((v-\text{spectrum})\) Since the \(v\)'s are determined by \(\beta\) via (31), we call them \(v\)-spectrum of the tangential operators. By abuse of notation, we write \(\Psi_\beta\) as \(\Psi_v\), \(f_\beta\) as \(f_v\), \(\xi_\beta\) as \(\xi_v\) etc.

### 3.2 Solutions to the ODEs on the Fourier-coefficients

In this section, for any singular point \(O\) of the connection and bundle, we shall solve (29) locally for the cone connection \(A_O\). This is equivalent to solving the ODEs (33) of the Fourier-coefficients. By proving Theorem 3.6, we show the existence of good solutions to (33) with the correct and optimal \(L^2\)-estimates.

**Choosing different formulas for different spectrum, the solution for each \(v\) are given by** (35), (37), (41), (43). These solutions possess the properties for building up a deformation theory for singular connections.

We can not prove Theorem 3.6 only by "potential" estimates. To be precise, when \(v = 0\), Proposition 3.10, a result by potential estimate, is not the optimal estimate we want in Theorem 3.6. Nevertheless, the interesting thing is that the 2 terms in (43) actually enjoy some magic cancellation. This allows us to use integral identity to improve the non-optimal estimate in Proposition 3.10 to optimal estimate in Propositions 3.8. Since this technique involves integration by parts, we need to choose the solution properly with respect to the weight, so that the boundary terms in Lemma 7.16 vanish, so we have the identity (51).

This is similar to Theorem 9.9 of [18]: though the weak \(L^{2,1}\)-estimate can be done by Calderon-Zygmund potential estimate, the \(W^{2,2}\)-estimate still requires integration by parts.
**Theorem 3.6.** Suppose $f$ is supported in $(0, \frac{1}{100}]$ and vanishes near $r = 0$. Suppose $p < 0$ is $A_O$-generic and $b \geq 0$. Then for any $v$ among the $v$-spectrum of $\Upsilon_{A_O}$ (see Definition 3.5), there exists a solution $u$ to (33) with the following uniform estimate.

$$
\int_0^{\frac{1}{4}} u^2 (-\log r)^{2b} dr \leq \bar{C} \int_0^{\frac{1}{4}} f^2 r^{2p+1} (-\log r)^{2b} dr.
$$

(34)

$\bar{C}$ is as in Definition 2.17.

**Remark 3.7.** Suppose $p$ is not $A_O$-generic i.e. there is some $v$ such that $v = 1 - p$, there is a $f$ which violates the conclusion of Theorem 3.6. Namely, let $v = \frac{5}{2}$, $p = -\frac{3}{2}$, and $f = r^{\frac{5}{4}} (-\log r)^{-b-1-\epsilon}$, $b > 0$, $\frac{1}{100} > \epsilon > 0$, then $\int_0^{\frac{1}{4}} f^2 r^{-2} (-\log r)^{2b} dr < \infty$. However, there is no solution $u$ to (33) such that $\int_0^{\frac{1}{4}} u^2 r^{-6} (-\log r)^{2b} dr < \infty$. Using a limiting argument, this means we can’t find solution which satisfies the optimal bound in (34).

**Proof of Theorem 3.6.** It’s a combination of Proposition 3.8, 3.9, and 3.13 □

**Proposition 3.8.** Under the same conditions in Theorem 3.6, suppose $v > 0$, then there exists a solution $u$ to (33) such that

$$
\int_0^{\frac{1}{4}} u^2 r^{2p-3} (-\log r)^{2b} dr \leq \frac{C}{1 + |v|^3} \int_0^{\frac{1}{4}} f^2(x)x^{2p+1} (-\log x)^{2b} dx.
$$

**Proposition 3.9.** Under the same conditions in Theorem 3.6, suppose $v$ is purely imaginary, then there exists a solution $u$ to (33) such that

$$
\int_0^{\frac{1}{4}} u^2 r^{2p-3} (-\log r)^{2b} dr \leq C \int_0^{\frac{1}{4}} f^2(x)x^{2p+1} (-\log x)^{2b} dx.
$$

**Proposition 3.10.** Under the same conditions in Theorem 3.6, suppose $v = 0$, then there exists a solution $u$ to (33) such that

$$
\int_0^{\frac{1}{4}} u^2 r^{2p-3} (-\log r)^{2b-2} dr \leq \bar{C} \int_0^{\frac{1}{4}} f^2 x^{2p+1} (-\log x)^{2b} dx.
$$

**Proof of Proposition 3.8.** Case 1: $v > 1 - p$. We choose a solution to (33) as

$$
u = \frac{1}{2v} \left\{ r^v \int_r^{\frac{1}{4}} x^{-v+1} f(x) dx + r^{-v} \int_0^r x^{v+1} f(x) dx \right\}.
$$

(35)

We denote $u_I = \frac{r^v}{2v} \int_r^{\frac{1}{4}} x^{-v+1} f(x) dx$, $u_{II} = \frac{r^{-v}}{2v} \int_0^r x^{v+1} f(x) dx$. 

17
There exists a $q$ such that $p > q > 1 - v$. By Hölder inequality,

$$u_i^2 \leq \frac{\tilde{C}_{1,2v}}{1 + |v|^2} \int_r^\frac{1}{2} x^{-2v+2}x^{-2q-1}(-\log r)^{-2b}dx \left[ \int_r^\frac{1}{2} f^2(x)x^{2q+1}(-\log r)^{-2b}dx \right]$$

$$\leq \frac{\tilde{C}}{(1 + |v|^2)|v|} r^{2v-2v-2q+2}(-\log r)^{-2b} \left[ \int_r^\frac{1}{2} f^2(x)x^{2q+1}(-\log r)^{-2b}dx \right], \quad q > 1 - v.$$ 

Thus

$$\int_0^\frac{1}{2} u_i^2 r^{2p-3}(-\log r)^{2b}dr \leq \frac{\tilde{C}}{(1 + |v|^2)^\frac{p}{2}} \int_0^\frac{1}{2} r^{2p-2q-1}dr \left[ \int_r^\frac{1}{2} f^2(x)x^{2q+1}(-\log x)^{2b}dx \right]$$

$$= \frac{\tilde{C}}{(1 + |v|^2)^\frac{p}{2}} \int_0^\frac{1}{2} f^2(x)x^{2q+1}(-\log x)^{2b}dx \int_0^x r^{2p-2q-1}dr, \quad p > q$$

$$= \frac{\tilde{C}}{(1 + |v|^2)^\frac{p}{2}} \int_0^\frac{1}{2} f^2(x)x^{2p+1}(-\log x)^{2b}dx, \quad p > q.$$ 

Since $p < 0$, we directly use (38) and (39) replacing the ”$q$” there by 0, ”$v$” by $-v$. Hence

$$\int_0^\frac{1}{2} u_i^2 r^{2p-3}(-\log r)^{2b}dr \leq \frac{\tilde{C}}{(1 + |v|^2)^\frac{p}{2}} \int_0^\frac{1}{2} f^2(x)x^{2p+1}(-\log x)^{2b}dx. \quad (36)$$

The proof is complete when $v > 1 - p$.

Case 2: $0 < v < 1 - p$. In this case we take the solution as

$$u = \frac{1}{-2v} \left\{ -r^v \int_0^r x^{-v+1}f(x)dx + r^{-v} \int_0^r x^{v+1}f(x)dx \right\}. \quad (37)$$

Let $u_i$ in this case be $\frac{r^v}{2v} \int_0^r x^{-v+1}f(x)dx$. Choose $q$ such that $p < q < 1 - v$, by Hölder inequality and Lemma 7.17, we calculate

$$u_i^2 \leq r^{2v} \left[ \int_0^r x^{-2v+2}x^{-2q-1}(-\log x)^{2b}dx \right] \left[ \int_0^r f^2(x)x^{2q+1}(-\log x)^{2b}dx \right]$$

$$\leq \tilde{C} r^{-2v+2}(-\log r)^{-2b} \left[ \int_0^r f^2(x)x^{2q+1}(-\log x)^{2b}dx \right], \quad q < 1 - v.$$ 

18
Thus
\[
\int_0^1 \frac{1}{2} u_1 r^{2p-3} (-\log r)^2 dr
\]
\[\leq \tilde{C} \int_0^1 \frac{1}{2} r^{2p-2q-1} dr \left[ \int_0^r f^2(x) x^{2q+1} (-\log x)^{2b} dx \right]
\]
\[= \tilde{C} \int_0^1 \frac{1}{2} f^2(x) x^{2q+1} (-\log x)^{2b} dx \int_x^1 \frac{1}{2} r^{2p-2q-1} dr, \ p < q
\]
\[= \tilde{C} \int_0^1 \frac{1}{2} f^2(x) x^{2q+1} (-\log x)^{2b} x^{2p-2q} dx, \ p < q.
\]
\[= \tilde{C} \int_0^1 \frac{1}{2} f^2(x) x^{2p+1} (-\log x)^{2b} dx.
\]

The estimate of the other term is similar. The proof of Proposition 3.8 is complete.

**Proof of Proposition 3.9**: In this case we write i.e \(-v^2 = \mu^2 > 0\), where \(\mu\) is real and positive. Hence equation (33) becomes
\[
\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} + \frac{\mu^2 u}{r^2} = f.
\]
\[
(40)
\]
Since the solutions to the homogeneous equation are \(\sin(\mu \log r)\), \(\cos(\mu \log r)\), then we choose a particular solution to (40) as
\[
u = \frac{\sin(\mu \log r)}{\mu} \int_0^r x \cos(\mu \log x) f(x) dx - \frac{\cos(\mu \log r)}{\mu} \int_0^r x \sin(\mu \log x) f(x) dx.
\]
\[
(41)
\]
Since the solutions to homogeneous equations are bounded, the estimate is easier than that of Proposition 3.8. Choosing \(q\) such that \(p < q < 0\), for the estimate of \(u_I = \frac{\sin(\mu \log r)}{\mu} \int_0^r x \cos(\mu \log x) f(x) dx\), we only need to use (38) and (39) by replacing the "\(r^m\)" there by \(\sin(\mu \log r)\), "\(x^{-m}\)" by \(\cos(\mu \log x)\). The estimate of the other term is similar.

**Proof of Proposition 3.10**: When \(v = 0\), the solutions to the homogeneous equation
\[
\frac{d^2 u}{dt^2} + \frac{1}{r} \frac{du}{dr} = 0
\]
\[
(42)
\]
are 1 and \(\log r\), then we choose a particular solution to
\[
\frac{d^2 u}{dt^2} + \frac{1}{r} \frac{du}{dr} = f \quad \text{as} \quad u = - \int_0^r x \log x f(x) dx + \log r \int_0^r x f(x) dx.
\]
\[
(43)
\]
Let \(u_{II} = \log r \int_0^r x f(x) dx\), we compute
\[
u^2 \leq (\log r)^2 (\int_0^r f^2 x dx) (\int_0^r x dx) = \frac{r^2 (\log r)^2}{2} \int_0^r f^2 x dx.
\]
\[
(44)
\]
19
Then \[
\int_{0}^{\frac{1}{2}} u_{12}^{2} r^{2p-3} (-\log r)^{2b-2} \, dr
\leq \int_{0}^{R} \int_{x}^{\frac{1}{2}} r^{2p-1} (-\log r)^{2b} \, dr \leq \tilde{C} \left( \int_{0}^{R} x^{2p} (-\log x)^{2b} \right)
\]
\[
= \tilde{C} \int_{0}^{R} f^{2} x^{2p+1} (-\log x)^{2b} \, dx.
\]
The estimate for the other term in (43) is similar.

Next we use integration by parts to improve Proposition 3.10 to 3.13. We verify the following identity, it doesn’t harm to have identities for every \( v \) instead of only for \( v = 0 \).

**Claim 3.11.** Suppose \( B, V^{2}, p \) are real numbers. Suppose \( u \) is a solution to the following differential equation

\[
\frac{d^{2}u}{dr^{2}} + \frac{B}{r} \frac{du}{dr} - \frac{V^{2}u}{r^{2}} = f,
\]
(45)

then the function \( \bar{u} = r^{p}u \) satisfies

\[
\frac{d^{2}\bar{u}}{dr^{2}} + \frac{(B - 2p)}{r} \frac{d\bar{u}}{dr} - \frac{[V^{2} + p(p - 1) + p(B - 2p)]}{r^{2}} \frac{\bar{u}}{r^{2}} = f r^{p}.
\]
(46)

In particular, when \( B = 1 \), we have

\[
\frac{d^{2}\bar{u}}{dr^{2}} + \frac{(1 - 2p)}{r} \frac{d\bar{u}}{dr} - \frac{[V^{2} - p^{2} - 2]}{r^{2}} \frac{\bar{u}}{r^{2}} = f r^{p}.
\]
(47)

Suppose \( u \) solves (33), consider \( \bar{u} \) as in Claim 3.11. Then \( u \) satisfies

\[
\frac{d^{2}\bar{u}}{dr^{2}} + \frac{k}{r} \frac{d\bar{u}}{dr} - \frac{a^{2}\bar{u}}{r^{2}} = \bar{f},
\]
(48)

where \( \bar{f} = r^{p}f \), \( k = 1 - 2p \), \( a^{2} = v^{2} - p^{2} \). Then

\[
k^{2} + 2a^{2} = 2v^{2} + 1 - 4p + 2p^{2}, \quad a^{4} - 2ka^{2} - 2a^{2} = (v^{2} - p^{2})(v^{2} - (p - 2)^{2}).
\]
(49)

Moreover, we consider the weight \( w_{0} \) as

\[
w_{0} = \left( \frac{1}{2} - r \right)^{10} (-\log r)^{2b}. \text{ Notice that } \frac{dw_{0}}{dr} \leq 0.
\]
(50)

Using \( \bar{u} = r^{p}u \) and integrating the square of (48), we directly verify
Lemma 3.12. Under the same condition on $f$ as in Theorem 3.6, suppose in each case we choose the solutions to (48) as in (33), (35), (41), (43). Then all the boundary terms in (212) are 0. Consequently,

$$\int_0^\frac{1}{2} f^2 r w_0 dr \quad \text{(51)}$$

$$= \int_0^\frac{1}{2} |\frac{d}{dr} \tilde{u}^2 r w_0| dr + (2v^2 + 1 - 4p + 2p^2) \int_0^\frac{1}{2} |\frac{d}{dr} \tilde{u} w_0| dr$$

$$+ (v^2 - p^2)(v^2 - (p - 2)^2) \int_0^\frac{1}{2} \tilde{u}^2 w_0 r^3 dr - (1 - 2p) \int_0^\frac{1}{2} |\frac{d}{dr} |\frac{d}{dr} w_0| dr$$

$$+ (3 - 2p)(v^2 - p^2) \int_0^\frac{1}{2} \tilde{u}^2 |\frac{d}{dr} w_0| dr - (v^2 - p^2) \int_0^\frac{1}{2} |\frac{d}{dr} w_0| dr.$$

Proposition 3.13. Under the same conditions in Theorem 3.6, suppose $v = 0$. Let $\tilde{u} = r^p u$, $u$ is the solution to (33) in (43), then

$$\int_0^\frac{1}{2} \tilde{u}^2 w_0 r^{-3} dr \leq C \int_0^\frac{1}{2} f^2 r w_0 dr. \quad \text{(52)}$$

Consequently, $u$ satisfies (34).

Proof of Proposition 3.13: By (50) and $p < 0$, the terms on the right hand side of (51) are all non-negative except the term $-(0 - p^2) \int_0^\frac{1}{2} \tilde{u}^2 w_0 r^3 dr$ (even this term is non-negative when $b \geq 1$, but we do not assume this in general). We compute

$$\frac{d^2 w_0}{dr^2} = \text{non-negative terms} + \frac{2b(2b - 1)}{r^2} (\frac{1}{2} - r)^{10} (-\log r)^{2b - 2}.$$ 

Hence (51) implies

$$\int_0^\frac{1}{2} \tilde{u}^2 r^{-3} w_0 dr \leq C \left\{ \int_0^\frac{1}{2} f^2 r w_0 dr + \int_0^\frac{1}{2} \tilde{u}^2 \left(\frac{1}{2} - r\right)^{10} (-\log r)^{2b - 2} dr \right\}$$

$$\leq C \left\{ \int_0^\frac{1}{2} f^2 r (-\log r)^{2b} dr + \int_0^\frac{1}{2} \tilde{u}^2 (-\log r)^{2b - 2} dr \right\}. \quad \text{(53)}$$

Proposition 3.10 implies $\int_0^\frac{1}{2} \tilde{u}^2 (-\log r)^{2b - 2} dr \leq C \int_0^\frac{1}{2} f^2 r (-\log r)^{2b} dr.$

The proof of (52) is complete by combining the above two inequalities. 

21
3.3 Local solutions for the model cone connection

Our goal in this section is to solve the following deformation equation in model case, and obtain optimal Sobolev-estimates.

\[ L_{A_0} \xi = f. \]  

(54)

**Definition 3.14.** Let \( w = r^{2p}(-\log r)^{2b}, \ p < 0. \) Let \( \mathcal{B} \) be an open set in \( \mathbb{R}^n \setminus O, \) and \( \xi \) be a smooth section of \( \Xi \) over \( \mathcal{B}, \) we define

\[
|\xi|_{W^{2,2}_{p,b,A_0}(\mathcal{B})}^2 \triangleq \int_\mathcal{B} |\nabla_{A_0} \xi|^2 w dV + \int_\mathcal{B} \frac{1}{r^2} |\nabla_{A_0} \xi|^2 w dV + \int_\mathcal{B} \frac{|\xi|^2}{r^4} w dV
\]

\[
|\xi|_{W^{1,2}_{p,b,A_0}(\mathcal{B})}^2 \triangleq \int_\mathcal{B} |\nabla_{A_0} \xi|^2 w dV + \int_\mathcal{B} \frac{|\xi|^2}{r^2} w dV; \ |\xi|_{L^2_{p,b}(\mathcal{B})}^2 \triangleq \int_\mathcal{B} |\xi|^2 w dV.
\]

Since \( A \) is admissible, the norm \( W^{2,2}_{p,b,A}(\mathcal{B}) \) (with connection \( A \) instead of \( A_0 \)) is equivalent to \( W^{2,2}_{p,b,A}(\mathcal{B}), \) etc. Thus, by deleting the symbol of the connections, we denote the spaces as \( W^{2,2}_{p,b}(\mathcal{B}), \)

\( W^{1,2}_{p,b}(\mathcal{B}), \)

\( L^2_{p,b}(\mathcal{B}). \)

When the bundle is trivial over \( B_0 \) and the connection is smooth across \( O, \) our \( W^{2,2}_{p,b} - \)norm is stronger than the usual \( W^{2,2} - \)norm weighted by \( w. \) It fits into our setting better, in the sense that the estimates in (58), Proposition 3.23, and Corollary 3.20 do not depend on the radius of the balls.

**Definition 3.15.** The space \( W^{2,2}_{p,b}(\mathcal{B}) \) is the completion of the section space

\[
\{ \xi | \xi \in C^2(\mathcal{B} \setminus O), |\xi|_{W^{2,2}_{p,b}(\mathcal{B})} < \infty \} \text{ under the } W^{2,2}_{p,b}(\mathcal{B}) - \text{norm.}
\]

The space \( W^{1,2}_{p,b}(\mathcal{B}) \) is the completion of the section space

\[
\{ f | f \in C^1(\mathcal{B} \setminus O), |f|_{W^{1,2}_{p,b}(\mathcal{B})} < \infty \} \text{ under the } W^{2,2}_{p,b}(\mathcal{B}) - \text{norm.}
\]

The space \( L^2_{p,b}(\mathcal{B}) \) is the completion of the following under the \( L^2_{p,b}(\mathcal{B}) - \text{norm.} \)

\[
\{ f | f \in C^\infty_c(\mathcal{B} \setminus O), \text{only finite terms in the series (30) are non-zero} \}.
\]

We define the space \( \mathcal{W}^{1,2}_{p,b} (\mathcal{W}^{2,2}_{p,b}, \mathcal{W}^{0,2}_{p,b}) \) as the following

\[
\mathcal{W}^{1,2}_{p,b}(B_0) = \{ \xi \in W^{1,2}_{p,b} \text{ in the coordinates } V_{+,O}, V_{-,O} ||\xi||_{W^{1,2}_{p,b}(B_0)} < \infty \}. \]  

(55)

**Lemma 3.16.** For any \( \rho > 0, \) \( L^2_{p,b}[B_0(\rho)] \) is the space of measurable functions on \( B_0(\rho) \) which are square integrable with respect to the weight \( w. \)

**Lemma 3.17.** \( W^{k,2}_{p,b}[B_0(\rho)] = \mathcal{W}^{k,2}_{p,b}[B_0(\rho)], \ k = 0, 1, 2. \)
Remark 3.18. The proofs of the above two lemmas are deferred to Section 7.4. Lemma 3.16 reduces Theorem 3.19 to a finite-dimensional problem. On Lemma 3.17, we expect that when the bundle $\Xi$ is trivial, $\mathcal{W}_{p,b}^{1,2}[B_O(\rho)]$ can even be approximated by sections that are smooth across the singularity. This is because $w$ and $r^{-2}w$ are $A_p$-weights (see Theorem 1 in [20]).

Our main result in this section is the following. The crucial observation is that the $L^2$–estimate (given by Theorem 3.6) and simple integration by parts yield the $W^{2,2}$–estimate.

Theorem 3.19. Suppose $p < 0$ is $A_O$–generic and $b \geq 0$. Then there is a bounded linear operator $Q_{p,b,A_O}$ from $L^2_{p,b}[B_O(\frac{1}{4})]$ to $W^{2,2}_{p,b}[B_O(\frac{1}{4})]$ such that

- $-L^2_{\lambda_O}Q_{p,b,A_O} = \text{Id}$ from $L^2_{p,b}[B_O(\frac{1}{4})]$ to itself,
- the bound on $Q_{p,b,A_O}$ is less than a $\bar{C}$ as in Definition 2.17.

Proof of Theorem 3.19: This is a direct application of Theorem 3.6. We first prove it assuming that $f$ satisfies the conditions in Definition 3.15 for $L^2_{p,b}[B_O(\frac{1}{4})]$. We write $f = \sum_{v<v_0} r^{-3}f_v \Psi_v$ for some $0 < v_0 < \infty$ (see Definition 3.5). For each $f_v$, we define $\xi_v$ as the solution to (32) in (35), (37), (41), (43), and let

$$Q_{p,b,A_O}f \triangleq \sum_v r^{-3}f_v \xi_v \Psi_v \triangleq \xi.$$ (56)

It suffices to bound the $L^2_{p-2,b}[B_O(\frac{1}{4})]$–norm of $\xi$. By Theorem 3.6, we find

$$\int_{B_O(\frac{1}{4})} \frac{|\xi|^2}{r^4} \, wdV = \sum_v \int_0^\frac{1}{4} \int_{S^6(1)} \xi_v r^{-9} |\Psi_v|^2 r^6 \theta d\theta dr = \sum_v \int_0^\frac{1}{4} |\xi_v|^2 r^{-3} wdr \leq \bar{C} \sum_v \int_0^\frac{1}{4} f_v^2 r \, wdr = \bar{C} \int_{B_O(\frac{1}{4})} |f|^2 \, w \, dV.$$ (57)

By Definition 3.15, (57), (58), and Proposition 3.23, the proof is complete when $f$ satisfies the a priori conditions in the first paragraph of this proof.

In the general case, for any $f \in L^2_{p,b}$, by Lemma 3.16 there exists a sequence $f_j \rightarrow f$ in $L^2_{p,b}$–topology, and $f_j$ satisfy the a priori conditions. We denote $\xi_j$ as $Q_{p,b,A_O}f_j$. By the a priori estimate proved (in the first step) and the linearity of $Q_{p,b,A_O}$, $\xi_j$ is a Cauchy-sequence in $W^{2,2}_{p,b}[B_O(\frac{1}{4})]$. By completeness, $\xi_j$ converges to $\xi_\infty$ in $W^{2,2}_{p,b}[B_O(\frac{1}{4})]$. Moreover, $\xi_\infty$ satisfies the estimate in Theorem 3.19. We thus define $Q_{p,b,A_O}f$ as $\xi_\infty$, the bounds in Theorem 3.19 implies that this definition does not depend on the approximation. The proof is complete.

Corollary 3.20. Let $p, b$ be as in Theorem 3.19. There exists a local right inverse $Q_{p,b,A_O}$ of $L_{A_O}$ with the following properties. For any $\tau \leq \frac{1}{10}$,
\( Q_{p,b,A_O} \) is bounded from \( L^2_{p,b}[B_O(\tau)] \) to \( W^{1,2}_{p,b}[B_O(\tau)] \). The bound is less than a \( \bar{C} \) as in Definition \[2.17\]. In particular, it does not depend on \( \tau \).

\( L_{A_O}Q_{p,b,A_O} = \text{Id} \) from \( L^2_{p,b}[B_O(\tau)] \) to itself.

**Proof of Corollary \[3.20\]**. By extending to vanish outside \( B_O(\tau) \), \( f \) can be viewed as a section in \( L^2_{p,b}[B_O(\frac{1}{4})] \). It suffices to take \( Q_{p,b,A_O} = -L_{A_O}Q_{p,b,A_O} \) and restrict it to \( B_O(\tau) \). Under this extension, \( Q_{p,b,A_O} \) does not depend on \( \tau \).

Next, we establish two crucial building-blocks of Theorem \[3.19\].

**Lemma 3.21.** Under the same conditions in Theorem \[3.19\], suppose \( f \) satisfies the a priori conditions in the first paragraph of proof of Theorem \[3.19\]. Let \( \xi = Q_{p,b,A_O}f \). Then for any \( \varrho \in (0, 1) \), the following bound holds.

\[
\int_{B_O(\frac{\varrho}{4})} \frac{|\nabla A_O\xi|^2}{r^2} \eta w dV \leq \bar{C}(\int_{B_O(\frac{\varrho}{4})} |\xi|^2 r^4 w dV + \int_{B_O(\frac{\varrho}{4})} |f|^2 w dV). \tag{58}
\]

\( \bar{C} \) is as in Definition \[2.17\], thus is independent of \( \varrho \).

**Remark 3.22.** The estimate is independent of \( \varrho \) because (58) is scaling-correct. This is important for (85). When \( f \) satisfies the a priori conditions, \( \Omega_{p,b,A_O}f \) is smooth away from \( O \), thus every term in the proofs of Theorem \[3.19\], Lemma \[3.21\], Proposition \[3.23\] makes sense.

**Proof.** Let \( \eta_\epsilon \) be the standard cut-off function (of the singular point \( O \)) which vanishes in \( B_O(\epsilon) \), and is identically 1 when \( r \geq 2\epsilon \). We have

\[
\epsilon |\nabla \eta_\epsilon| + \epsilon^2 |\nabla^2 \eta_\epsilon| < \bar{C}, \text{ when } \epsilon << \varrho. \tag{59}
\]

Let \( \chi \) be the standard cut-off function supported in \( B_O(\frac{\varrho}{4}) \) and identically 1 over \( B_O(\frac{\varrho}{15}) \), \( |\nabla \chi| \leq \frac{C}{\varrho} \). Lemma \[7.1\] implies

\[
\nabla_{A_O} \nabla_{A_O} \xi = -f + F_{A_O} \otimes \xi. \tag{60}
\]

We compute

\[
\int_{B_O(\frac{\varrho}{4})} \frac{|\nabla_{A_O} \xi|^2}{r^2} \eta_\chi^2 w dV \tag{61}
\]

\[
= \int_{B_O(\frac{\varrho}{4})} \frac{\langle \xi, \nabla_{A_O} \nabla_{A_O} \xi \rangle}{r^2} \eta_\chi^2 w^2 dV - \int_{B_O(\frac{\varrho}{4})} \langle \xi, [\nabla (\frac{\eta_\chi^2 w}{r^2})], \nabla_{A_O} \xi \rangle \rangle dV
\]

\[
= -\int_{B_O(\frac{\varrho}{4})} \frac{\langle \xi, f \rangle}{r^2} \eta_\chi^2 w dV + \int_{B_O(\frac{\varrho}{4})} \langle \xi, F_{A_O} \otimes \xi \rangle \rangle \frac{\eta_\chi^2 w^2 dV}{r^2}
\]

\[
- \int_{B_O(\frac{\varrho}{4})} \langle \xi, (\nabla \eta_\epsilon) \nabla_{A_O} \xi \rangle \leq \frac{2w}{r^2} dV - \int_{B_O(\frac{\varrho}{4})} \langle \xi, (\nabla^2 \frac{w^2}{r^2}) \nabla_{A_O} \xi \rangle \rangle \eta_\epsilon dV
\]

24
By Definition 3.15, the cheap estimate $|F_{A_0}| \leq \frac{C}{r}$, and Cauchy-Schwartz inequality, the first 2 terms on the most right hand side of (61) are bounded by the right hand side of (58), uniformly in $\epsilon$. Note
\[
|\nabla w| \leq \frac{Cw}{r} \Rightarrow |\nabla \left( \frac{\chi^2 w}{r^2} \right)| \leq \frac{\tilde{C}\chi^2 w}{r^2} + 2\chi|\nabla \chi| \frac{w}{r^2},
\] hence we obtain the following bound on the last term.
\[
\int_{B_{\Omega}(\frac{\xi}{r})} <\xi, (\nabla \frac{\chi^2 w}{r^2}) \cdot \nabla_{A_0} \xi > \eta dV
\]
\[
\leq \tilde{C} \int_{B_{\Omega}(\frac{\xi}{r})} \frac{\xi}{r^3} \eta \chi \frac{w}{r^2} dV + \tilde{C} \int_{B_{\Omega}(\frac{\xi}{r})} \xi \frac{|\nabla \chi| \eta}{r^2} dV
\]
\[
\leq \vartheta \int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\nabla_{A_0} \xi|^2}{r^2} \eta \chi^2 w dV + \tilde{C}_\vartheta \int_{B_{\Omega}(\frac{\xi}{r})} \frac{\xi}{r^4} \chi^2 w \eta dV
\]
\[
+ \tilde{C}_\vartheta \int_{B_{\Omega}(\frac{\xi}{r})} |\nabla \chi|^2 \frac{\xi}{r^2} \eta dV.
\]

On the last term above, we notice that in $B_{\Omega}(\frac{\xi}{r})$, $\frac{1}{r} \leq \frac{1}{\vartheta}$. By $|\nabla \chi| \leq \frac{C}{\vartheta}$, we obtain the following bound
\[
\tilde{C}_\vartheta \int_{B_{\Omega}(\frac{\xi}{r})} |\nabla \chi|^2 \frac{\xi}{r^2} \eta dV \leq \tilde{C}_\vartheta \int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\xi|^2}{r^4} \eta dV.
\]

Therefore (63) and (64) imply
\[
\int_{B_{\Omega}(\frac{\xi}{r})} <\xi, (\nabla \frac{\chi^2 w}{r^2}) \cdot \nabla_{A_0} \xi > \eta dV
\]
\[
\leq \vartheta \int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\nabla_{A_0} \xi|^2}{r^2} \eta \chi^2 w dV + \tilde{C}_\vartheta \int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\xi|^2}{r^4} \eta dV.
\]

Let $\vartheta = \frac{1}{10}$, plugging (65) in (61) and using the remark under (61), we find
\[
\int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\nabla_{A_0} \xi|^2}{r^2} \eta \chi^2 w dV
\]
\[
\leq \tilde{C} \left\{ \int_{B_{\Omega}(\frac{\xi}{r})} \frac{|\xi|^2}{r^4} \eta \chi^2 w dV + \int_{B_{\Omega}(\frac{\xi}{r})} |f|^2 \eta \chi^2 w dV + \int_{B_{\Omega}(\frac{\xi}{r})} <\xi, \nabla_{A_0, \nabla} \xi > \frac{\chi^2 w}{r^2} dV \right\}.
\]

It suffices to show $\Pi_1$ approaches 0 as $\epsilon \to 0$. The condition on $f$ implies $L^2_{A_0} \xi = 0$ in $B_{\Omega}(r_f)$, for some $r_f > 0$. Since $\xi = \Omega_{p,b,A_0} f \in L^2_{p-2, \Omega} B_{\Omega}(\frac{\xi}{r})$, Lemma 4.2 (with $\frac{p}{2}$ replaced by $p - 1$) gives
\[
|\xi| \leq \frac{C_f}{\vartheta^{\frac{p}{2}+p-\lambda}}, \quad |\nabla_{A_0} \xi| \leq \frac{C_f}{\vartheta^{\frac{p}{2}+p-\lambda}}, \quad \lambda > 0.
\]
Hence when $\epsilon$ goes to 0,
The proof of (58) is complete.

Using almost the same technique, we obtain

**Proposition 3.23.** Let \( p, b, f, \xi, \varrho \) be as in Lemma 3.21. Then

\[
\int_{B_O(\varrho)} |\nabla A_{\varrho} \xi|^2 w dV \leq \bar{C} \int_{B_O(\varrho)} \frac{|\xi|^2}{r^{2p}} - \log r \text{ } ^2 b dV + \bar{C} \int_{B_O(\varrho)} |f|^2 w dV.
\]

For the reader’s convenience, we still do the full proof of Proposition 3.23 in Section 7.5

**Definition 3.24.** Given an admissible connection \( A \), let \( \tau_0 \) be small enough. Let \( p < 0 \) be \( A \)-generic and \( b \geq 0 \). Let \( w_{p,b} \) be the smooth function such that for any singular point \( O_j \),

\[
w_{p,b} = \begin{cases} 
1 & \text{when } r \geq 2\tau_0, \\
\frac{1}{r^{2p}(- \log r)^{2b}} & \text{when } r \leq \tau_0,
\end{cases}
\]

(68)

\( r \) is the distance to \( O_j \) in local coordinates (by abuse of notation). Away from the coordinate neighbourhoods of the singular points, \( w_{p,b} \equiv 1 \).

Then we define the global \( L^2_{p,b} \) space as the completion of smooth functions (away from the singular points) under the norm \( | \cdot |_{L^2_{p,b}} \):

\[
|\xi|_{L^2_{p,b}} = \int_M |\xi|^2 w_{p,b} dV.
\]

(69)

We define the space \( W^{1,2}_{p,b} \) as the completion of smooth functions (away from the singular points) under the norm

\[
|\xi|_{W^{1,2}_{p,b}} = |\nabla A \xi|_{L^2_{p,b}} + |\xi|_{L^2_{p-1,b}}.
\]

(70)

**Convention of section-spaces:** the global norms over \( M \) are denoted just as \( W^{1,2}_{p,b} \) or \( L^2_{p,b} \) without any symbol on the domain, the local norms are usually with a symbol indicating the domain (c.f. Definition 3.15).

**Definition 3.25.** By abuse of notation, let \( dV \) denote all the volume forms of our integrations. The convention is: locally, it usually means the Euclidean volume form; globally, it usually means the volume form determined by \( \phi \).

Anyway, the \( dV \) in various cases are equivalent up to a constant depending on the (reference) \( G_2 \) structure.
4 Global Theory

By abuse of notation, from now on let \( f \) denote the image.

4.1 Global apriori estimate.

Definition 4.1. For any real number \( \tau \), let \( \vartheta_\tau \) denote the \( v \)–spectrum gap of \( \Upsilon'_{\text{AO}} \)s at \( \tau \) i.e the distance from \( \tau \) to the closest \( v \)–eigenvalue (of any \( \Upsilon_{\text{AO}} \)) other than \( \tau \) itself. When the gap \( > 1 \), we let \( \vartheta_\tau = 1 \).

The following bootstrapping lemma for the model operator is important especially for Theorem 4.5.

Lemma 4.2. Let \( \tau_0 \in (0, \frac{1}{10}) \), \( p < 0, b \geq 0 \). Suppose \( \xi \in L^2_{p-1,b}[B_O(2\tau_0)] \), and \( L^2_{\text{AO}} \xi = 0 \) in \( B_O(2\tau_0) \). Then \( \xi \) is actually in \( L^2_{p-1-\lambda,b}[B_O(2\tau_0)] \) for all \( 0 \leq \lambda < \vartheta_{\tau_0} \), and we have the following estimates.

\[
|\xi|_{L^2_{p-1-\lambda,b}[B_O(2\tau_0)]} \leq \frac{\bar{C}}{r_{\tau_0}} |\xi|_{L^2_{p,b}[B_O(2\tau_0)]}.
\]

(71)

\[
 r|\xi| + r^2 |\nabla \text{AO} \xi| + r^3 |\nabla^2 \text{AO} \xi| + r^4 |\nabla^3 \text{AO} \xi| \leq C_{\xi,\tau_0} r^{-\frac{3}{2}+p+\lambda} \text{ when } r < \tau_0.
\]

(72)

\( \bar{C} \) is independent of \( \tau_0 \).

Remark 4.3. We don’t need \( p \) to be \( \text{AO} \)–generic, since by condition (76), any “harmonic” section in \( W_{1,2}^{1,2} \) does not have non-trivial component on \( v = -p \).

Proof of Lemma 4.2: The idea of proof is to use Fourier-expansion to rule out some bad eigenvalues. We write

\[
\xi = r^{-\frac{3}{2}} U_v \Psi_v \text{ (see Definition 3.5).}
\]

(73)

Since \( \xi \) is harmonic, and \( U_v \) is the spherical inner product, by (21) (with right hand side as 0), we directly verify

\[
\frac{\partial^2 U_v}{\partial r^2} + \frac{1}{r} \frac{\partial U_v}{\partial r} - \frac{v^2 U_v}{r^2} = 0.
\]

(74)

This means

\[
U_v = \begin{cases} 
  c_{1,v} r^{-v} + c_{2,v} r^v; & v > 0, \\
  c_{1,v} + c_{2,v} \log r; & v = 0, \\
  c_{1,v} \sin(v \log r) + c_{2,v} \cos(v \log r), & v < 0.
\end{cases}
\]

(75)

c_{1,v}, c_{2,v} are constants. The condition \( \xi \in L^2_{p-1,b}[B_O(2\tau_0)] \) implies

\[
\int_0^{2\tau_0} U_v^2 r^{2p-1} (-\log r)^{2b} dr < \infty.
\]

(76)
Since $p < 0$, the terms $1, \log r, \sin(v \log r), \cos(v \log r), r^{-v}$ cannot appear. Moreover, $r^v$ cannot appear if $2v + 2p \leq 0$. Thus, only those $r^v$ with $v > -p$ will appear. Moreover, by the discreteness of the spectrum, only those $r^v$ with $v \geq -p + \vartheta_\pm$ could appear. In this case, we have a $v-$ independent $L^2_{p-1-\lambda, b}$-estimate by the $L^2_{p,b}$-norm.

$$
\int_0^{2\tau_0} |r^v|^2 r^{2p-2\lambda-1} (-\log r)^{2b} \, dr \leq \frac{(2\tau_0)^{2v+2p-2\lambda}(-\log 2\tau_0)^b}{2v + 2p - 2\lambda}.
$$

On the other hand,

$$
\int_0^{2\tau_0} |r^v|^2 r^{2p+1} (-\log r)^{2b} \, dr \geq (-\log 2\tau_0)^b \frac{(2\tau_0)^{2v+2p+2}}{2v + 2p + 2}.
$$

(77)

Then

$$
\int_0^{2\tau_0} |r^v|^2 r^{2p-2\lambda-1} (-\log r)^{2b} \, dr \leq \bar{C} \lambda \tau_0^{2-2\lambda} \int_0^{2\tau_0} |r^v|^2 r^{2p+1} (-\log r)^{2b} \, dr.
$$

(78)

Using (73) and (75), (78) is equivalent to (71). The estimate (72) is a direct consequence of (71) and Lemma 7.4 (with $k = 3$, $A = A_O$).

**Claim 4.4.** Under the same conditions in Lemma 4.2, we have

$$
|\xi|_{W^{1,2}_{p,b}|B_O(\tau_0)} \leq \frac{\bar{C}}{\tau_0} |\xi|_{L^2_{p,b}|B_O(2\tau_0)}; \quad \bar{C} \text{ is independent of } \tau_0.
$$

(79)

**Proof.** It suffices to show $\int_{B_O(\tau_0)} |\nabla A_O \xi|^2 \, wdV \leq \bar{C} \int_{B_O(2\tau_0)} \frac{|\xi|^2}{r^2} \, wdV$. It is a much easier version of Lemma 3.21; we only need to run the argument through with the measure $\eta_k \chi^2 \, wdV$ instead of $\frac{\eta_k \chi^2}{r^2} \, dV$.

Our main theorem in this section implies the image is closed.

**Theorem 4.5. (Global a priori estimate)** Suppose $\xi \in W^{1,2}_{p,b}$, then

$$
|\xi|_{W^{1,2}_{p,b}} \leq C(|L_A \xi|_{L^2_{p,b}} + |\xi|_{L^2_{p,b}}).
$$

**Proof.** The observation is that we can reduce the estimate for $L_A$ to estimate of the model operator. We only need to derive this estimate near the singularity. Away from the singularity it follows from the standard estimates, then we patch up the estimates in each piece.

For any $\epsilon_0$, when $\tau_0-$small enough, given $\xi \in W^{1,2}_{p,b}|B_O(2\tau_0)|$, by Corollary 3.20 there exists a $\eta$ such that

$$
L_{A_O} \eta = L_{A_O} \xi \quad \text{and} \quad |\eta|_{W^{1,2}_{p,b}(B_O(2\tau_0))} \leq \bar{C} |L_{A_O} \xi|_{L^2_{p,b}(B_O(2\tau_0))}; \quad \bar{C} \text{ is independent of } \epsilon_0 \text{ and } \tau_0.
$$

(80)

$$
\leq \bar{C} |(L_A - L_{A_O}) \xi|_{L^2_{p,b}(B_O(2\tau_0))} + \bar{C} |L_{A_O} \xi|_{L^2_{p,b}(B_O(2\tau_0))}
$$

$$
\leq \bar{C} |L_{A_O} \xi|_{L^2_{p,b}(B_O(2\tau_0))} + \epsilon_0 |\xi|_{W^{1,2}_{p,b}(B_O(2\tau_0))}.
$$

28
Then we estimate
\[
|\xi|_{W^{1,2}_{p,b}(B_0(\tau_0))} \leq |\xi - \eta|_{W^{1,2}_{p,b}(B_0(\tau_0))} + |\eta|_{W^{1,2}_{p,b}(B_0(\tau_0))} \\
\leq \tilde{C}|L_{A}\xi|_{L^2_{p,b}(B_0(2\tau_0))} + \epsilon_0|\xi|_{W^{1,2}_{p,b}(B_0(2\tau_0))} + |\xi - \eta|_{W^{1,2}_{p,b}(B_0(\tau_0))}.
\]
(81)
\[\xi - \eta\] satisfies \(L_{A\sigma}(\xi - \eta) = 0\) in \(B_0(2\tau_0)\). Then it’s smooth away from the singularity, and we have
\[ -L_{A\sigma}^2(\xi - \eta) = 0. \]
(82)

Thus \((81)\) and Claim \((4.3)\) (for \(\xi - \eta\)) yield
\[
|\xi|_{W^{1,2}_{p,b}(B_0(\tau_0))} \leq \tilde{C}|L_{A}\xi|_{L^2_{p,b}(B_0(2\tau_0))} + \epsilon_0|\xi|_{W^{1,2}_{p,b}(B_0(2\tau_0))} + \frac{\tilde{C}}{\tau_0} |\xi - \eta|_{L^2_{p,b}(B_0(2\tau_0))}. \]
(83)

Within \(B_0(2\tau_0)\), we have \(\frac{1}{\tau_0} \leq \frac{2}{\tau}\), then by definition and \((80)\), we obtain
\[
\frac{|\eta|_{L^2_{p,b}(B_0(2\tau_0))}}{\tau_0} \leq 4|\eta|_{L^2_{p,b}(B_0(2\tau_0))} \leq \tilde{C}|L_{A}\xi|_{L^2_{p,b}(B_0(2\tau_0))} + 4\epsilon_0|\xi|_{W^{1,2}_{p,b}(B_0(2\tau_0))} + \frac{\tilde{C}}{\tau_0} |\xi - \eta|_{L^2_{p,b}(B_0(2\tau_0))}, \]
(84)
where \(\tilde{C}\) does not depend on \(\epsilon_0\) or \(\tau_0\). Then \((79), (83),\) and \((81)\) imply
\[
|\xi|_{W^{1,2}_{p,b}(B_0(\tau_0))} \leq \tilde{C}|L_{A}\xi|_{L^2_{p,b}(B_0(2\tau_0))} + \tilde{C}\epsilon_0|\xi|_{W^{1,2}_{p,b}(B_0(2\tau_0))} + \frac{\tilde{C}}{\tau_0} |\xi - \eta|_{L^2_{p,b}(B_0(2\tau_0))}. \]
(85)

Away from the singularities we have
\[
|\xi|_{W^{1,2}_{p,b}(M_{\tilde{\omega}})} \leq C_{\epsilon_0,\tau_0}|L_{A}\xi|_{L^2_{p,b}(M_{\tilde{\omega}})} + C_{b,p,\tau_0}|\xi|_{L^2_{p,b}(M_{\tilde{\omega}})}. \]
(86)

Adding \((85)\) and \((86)\), we find
\[
|\xi|_{W^{1,2}_{p,b}} \leq C|L_{A}\xi|_{L^2_{p,b}} + \tilde{C}\epsilon_0|\xi|_{W^{1,2}_{p,b}} + C|\xi|_{L^2_{p,b}}. \]
(87)

Choosing the \(\epsilon_0\) (initially in \((80)\)) to be less than \(\frac{1}{2\tilde{C}}\) (we let \(\tau_0\) be small enough with respect to \(\epsilon_0\)), the proof of Theorem \((4.5)\) is complete. \(\square\)

**Theorem 4.6.** Let \(A,p,b\) be as in Theorem \((1.7)\) then \(Ker L_{A} \in W^{1,2}_{p,b}\) is finite-dimensional. Moreover, for any \(f \in Image L_{A}\), there exists a solution \(\xi \in Ker^{\perp} L_{A} \subset W^{1,2}_{p,b}\) such that \(\xi \in Ker^{\perp} L_{A} \subset W^{1,2}_{p,b}\) and \(|\xi|_{W^{1,2}_{p,b}} \leq C|f|_{L^2_{p,b}}\).

**Proof.** We first show that \(ker L_{A}\) is finite-dimensional. Were this not true, there exist countably many \(\xi_k\)’s in \(ker L_{A}\), such that for any \(k, \xi_k\) is not in the span of the preceding vectors. Then using the \(L^2_{p,b}\)-inner product, the Gram-Schmidt process produces an orthonormal sequence \(\hat{\xi}_k\) of sections in \(ker L_{A}\). On the other hand, by Theorem \((4.5)\), we have \(|\hat{\xi}_k|_{W^{1,2}_{p,b}} \leq C\), then Lemma \((4.10)\) implies \(\xi_k\) converges in \(L^2_{p,b}\) which contradicts the orthogonality.
Since $\text{Ker}L_A \subset W^{1,2}_{p,b}$ is now shown to be finite dimensional, we consider the projection of $\xi$ onto $\text{Ker}L_A$ (with respect to the $L^2_{p,b}$–inner product) as $\xi = \xi - \langle \xi, \text{Ker}L_A \rangle$, then $\xi \in \text{Ker}^\perp L_A$. (88)

To prove the estimate in Theorem 4.6, by Theorem 4.5, it suffices to show

$$|\xi|_{L^2_{p,b}} \leq C|f|_{L^2_{p,b}}.$$ (89)

Were (89) not true, there exists a sequence $\xi_i \in W^{1,2}_{p,b}$, $f_i = L_A \xi_i \in \text{Image}L_A \subset L^2_{p,b}$, such that

$$|\xi_i|_{L^2_{p,b}} = 1, \; \xi_i \in \text{ker}^\perp L_A, \; \text{but} \; |f_i|_{L^2_{p,b}} \rightarrow 0.$$ (90)

By Theorem 4.5 and Lemma 4.10 $\xi_i$ converges in $L^2_{p,b}$ to $\xi_\infty$. By the linearity of $L_A$, these in turn imply $\xi_i$ is a Cauchy-Sequence in $W^{1,2}_{p,b}$:

$$|\xi_i - \xi_j|_{W^{1,2}_{p,b}} \leq C|\xi_i - \xi_j|_{L^2_{p,b}} + C|f_i - f_j|_{L^2_{p,b}} \rightarrow 0.$$ (91)

Then $\xi_i$ converges to $\xi_\infty$ in $W^{1,2}_{p,b}$, $\xi_\infty \in W^{1,2}_{p,b}$, $|\xi_\infty|_{L^2_{p,b}} = 1$, and $L_A \xi_\infty = 0$. But $\xi_i \in \text{ker}^\perp L_A$ implies $\xi_\infty \perp \text{ker}L_A$. This is a contradiction. $\square$

### 4.2 Hybrid space and $C^0$–estimate.

Using an interpolation trick, the $C^0$–estimate is a direct corollary of the $W^{1,2}_{p,b}$-estimate. To see this, for any point $q$ close enough to a singular point $O$, $\xi \in W^{1,2}_{p,b} \subset L^2_{p-1,b}$ implies the average of $|\xi|$ over $B_q(\frac{r_q}{2})$ is bounded correctly i.e. by $r_q^{-\frac{2}{p}}(- \log r_q)^{-b}$. Since $\xi$ satisfies an elliptic equation, the interpolation trick gives the $C^0$–bound. For second order uniformly elliptic equations of divergence form, this can be done by the well known Nash-Moser iteration.

**Definition 4.7.** (Hybrid spaces) The hybrid spaces are defined as (their norms are defined in the parenthesis on the left hand side of “$< \infty$”)

- $H_{p,b} = \{\xi \mid C^{2,\alpha} \text{ away from } O \mid |\xi|_{W^{1,2}_{p,b}} + |\xi|_{L^2_{2,\alpha,M}^{(\frac{2}{p} + p)}} < \infty\}$ and

- $N_{p,b} = \{\xi \mid C^{1,\alpha} \text{ away from } O \mid |\xi|_{L^2_{p,b}} + |\xi|_{L^2_{1,\alpha,M}^{(\frac{2}{p} + p)}} < \infty\}$.

**Lemma 4.8.** Suppose $b \geq 0$, $p \leq -\frac{3}{2}$. Suppose $\xi_1, \; \xi_2 \in H_{p,b}$, then for any smooth tensor product $\otimes$, $|\xi_1 \otimes \xi_2| \in N_{p,b}$ and

$$|\xi_1 \otimes \xi_2|_{N_{p,b}} \leq C|\xi_1|_{H_{p,b}}|\xi_2|_{H_{p,b}}, \; C \text{ depends on the } C^2 – \text{norm of } \otimes.$$ (92)
This multiplicative property works for the quadratic non-linearity of (140).

**Proof.** By Definition 2.11, 2.10 and the conditions on $p, b$, we have $\xi_1 \otimes \xi_2 \in C^1_{\alpha, (2+p,b)}$ and

$$|\xi_1 \otimes \xi_2|_{C^1_{\alpha, (2+p,b)}} \leq C|\xi_1|_{C^1_{\alpha, (2+p,b)}}|\xi_2|_{C^1_{\alpha, (2+p,b)}} \leq C|\xi_1|_{H_{p,b}}|\xi_2|_{H_{p,b}}.$$  \hspace{1cm} (93)

The $L^2_{p,b}$-bound is estimated by making use of the $C^0$-norms:

$$\int_M |\xi_1 \otimes \xi_2|^2 w_{p,b}dV \quad p \leq \frac{3}{2}$$

$$\leq C|\xi_1|_{C^1_{\alpha, (2+p,b)}} \int_M \frac{|\xi_2|^2}{r^{5+2p(-\log r)^{2\beta}} w_{p,b}dV} \leq C|\xi_1|_{H_{p,b}} \int_M \frac{|\xi_2|^2}{r^2 w_{p,b}dV} \leq C|\xi_1|_{H_{p,b}}|\xi_2|_{H_{p,b}}.$$  \hspace{1cm} \Box

**Theorem 4.9.** Let $b \geq 0$, $0 < \alpha < 1$, and $\gamma$ be any real number. Suppose $f \in C^{(\gamma,b)}(M)$, and $\xi$ is $C^{2,\alpha}$ away from the singularities. Suppose $L_A \xi = f$ or $L_A^* \xi = f$, and $\xi \in L^2_{\gamma+b}$. Then $\xi$ satisfies

$$|\xi|_{L^2_{\gamma+b}} \leq C\{|f|_{(\gamma,b)} + |\xi|_{L^2_{\gamma+b}}\}. \hspace{1cm} (94)$$

Consequently,

$$|\xi|_{L^2_{\gamma+b}} \leq C\{|f|_{(\gamma,b)} + |\xi|_{L^2_{\gamma+b}}\}. \hspace{1cm} (95)$$

**Proof.** We only prove it for $L_A$, the proof for $L_A^*$ is the same.

Let $||.||_{k,\alpha,B}$ denote the weighted norm in (6.10) of [13] with respect to $B$.

Notice that this is **different** from $|.|_{k,\alpha,B}^y$ (Definition 2.13) on which the weight depends only on the distance to the singular point.

By Lemma 7.4 and multiplication of weight, in $B \triangleq B_q(\frac{x}{100})$, we have

$$|\xi|_{L^2_{\gamma+b}} \leq C|L_A \xi|_{L^2_{\gamma+b}}.$$  \hspace{1cm} (96)

It suffices to prove that (97) holds for any $\mu < \frac{1}{10}$.

$$|\xi|_{L^2_{\gamma+b}} \leq \mu|\nabla \xi|_{L^2_{\gamma+b}} + C\mu \frac{r^\gamma}{(-\log r_q)^{\gamma}} |\xi|_{L^2_{\gamma+b}(B)}. \hspace{1cm} (97)$$

Assuming (97), we go on to prove Theorem 4.9. By (96), we obtain

$$|\xi|_{L^2_{\gamma+b}} \leq C|L_A \xi|_{L^2_{\gamma+b}} + \mu C\nabla \xi|_{L^2_{\gamma+b}} + C\mu \frac{r^\gamma}{(-\log r_q)^{\gamma}} |\xi|_{L^2_{\gamma+b}(B)}. \hspace{1cm} (98)$$
Let $\mu C < \frac{1}{10}$, then $|\xi|_{1,\alpha,B}^\frac{2}{3} \leq C |f|_{\alpha,B}^\frac{2}{3} + C_\mu \frac{r_q^{\frac{2}{3} - \gamma}}{(- \log r_q)^{b}} |\xi|_{L^2_{p^{-\frac{2}{3} + \gamma}}(B)}$. (99)

In particular, on the $C^0$-norm, we have

$$r_q^{\frac{2}{3}} |\xi|(q) \leq C |f|_{\alpha,B}^\frac{2}{3} + C_\mu \frac{r_q^{\frac{2}{3} - \gamma}}{(- \log r_q)^{b}} |\xi|_{L^2_{p^{-\frac{2}{3} + \gamma}}(B)}.$$ (100)

By definition we have $r_q^{\frac{2}{3}} |f|_{\alpha,B} \leq C \frac{r_q^{\frac{2}{3} - \gamma}}{(- \log r_q)^{b}} |f|_{\alpha,B}^{(\gamma,b)}$. (101)

Then $|\xi|(q) \leq C \frac{r_q^{\frac{2}{3} - \gamma}}{(- \log r_q)^{b}} [|f|_{\alpha,B}^{(\gamma,b)} + |\xi|_{L^2_{p^{-\frac{2}{3} + \gamma}}(B)}]$. (102)

Since $q$ is an arbitrary point near the singularity, and $|f|_{\alpha,B}^{(\gamma,b)} \leq |f|_{\alpha,M}^{(\gamma,b)}$, the proof in general is the same, except that we have to spell out the following:

$$\sup_{y \in B_{\alpha,M}(x)} |\nabla \xi|(y) + \frac{C_\mu r_q^{\frac{2}{3} - \gamma}}{(- \log r_q)^{b}} |\xi|_{L^2_{p^{-\frac{2}{3} + \gamma}}(B)}.$$ (103)

Replacing "$2\mu$" by $\mu$, by definition, we deduce (92). Then (95) is a Corollary of Proposition 4.3 and (94).

4.3 Compact imbedding

**Lemma 4.10.** Suppose $p_1 - 1 < p_2$, or $p_1 - 1 = p_2$ and $b_1 > b_2$. Then for any ball $B$ such that $\partial B$ does not intersect the singularities, the imbedding $W^{1,2}_{p_1,b_1}(B) \to L^2_{p_2,b_2}(B)$ is compact.

Consequently, the imbedding $W^{1,2}_{p_1,b_1} \to L^2_{p_2,b_2}$ (of global spaces) is compact.

**Proof.** It suffices to assume $B$ is centred at a singular point $O$, and does not contain any other singular point. We only prove the case when $p_1 - 1 = p_2$ and $b_1 = b_2 + 1$. The proof in general is the same, except that we have to spell out
more notations. By definition, the imbedding from $W^{1,2}_{p_1,b_1}(B)$ to $L^2_{p_1-1,b_1}(B)$ is bounded. For any concentric and smaller ball $B(R)$, we have

$$
\int_{B(R)} |u|^2 w_{p_2,b_2} dx \leq \frac{1}{(-\log R)^2} \int_{B(R)} |u|^2 w_{p_1-1,b_1} dx.
$$

(103)

Then suppose $|u|_{W^{1,2}_{p_1,b_1}(B)} \leq C_1$, we can choose $R_m$ depending on $b$ and $C_1$ such that

$$
\left( \int_{B(R_m)} |u|^2 w_{p_2,b_2} dx \right)^{\frac{1}{2}} \leq \frac{1}{m}.
$$

(104)

Given a sequence $u_i$ such that $|u_i|_{W^{1,2}_{p_1,b_1}(B)} \leq C$, using compactness of the imbedding $W^{1,2}(B \setminus B(R_m)) \rightarrow L^2(B \setminus B(R_m))$, we obtain a Cauchy-subsequence $u_{m,j}$ in $L^2_{p_2,b_2}(B \setminus B(R_m))$ and

$$
|u_{m,j}|_{L^2_{p_2,b_2}(B(R_m))} \leq \frac{1}{m}.
$$

This means for any $m$ and the $u_{m,j}$, there is a $N_m$ such that $j, l > N_m$ implies

$$
|u_{m,j} - u_{m,l}|_{L^2_{p_2,b_2}(B)} < \frac{4}{m}.
$$

(105)

When $b > a$, we choose $u_{b,j}$ as a subsequence of $u_{a,j}$. Then we can write $u_{aa}$ as $u_{m,j_a}$, $u_{bb}$ as $u_{m,j_b}$. Since they are subsequence of $u_{m,j}$, then $j_a > a$, $j_b > b$. Consider the diagonal sequence $u_{aa}$. By the discussion above, for any $m$, when $a, b > m + N_m$, we have

$$
|u_{aa} - u_{bb}|_{L^2_{p_2,b_2}(B)} < \frac{4}{m}.
$$

(106)

Thus the diagonal sequence $u_{aa}$ is a Cauchy-Sequence in $L^2_{p_2,b_2}(B)$.

\[\square\]

### 4.4 Fredholm Theory

Using the local inverse in Corollary 3.20 and the above compact Sobolev-imbedding, it’s almost standard to prove $L_A$ is Fredholm.

We consider the linearized equation $L_A \xi = f$ on a small ball.

\[\text{Proposition 4.11. Let } A, p, b \text{ be as in Theorem 1.7. There is a } \tau_0 > 0 \text{ such that for any singular point } O, \text{ there exists a bounded linear map } Q_{A,p,b} \text{ from } L^2_{p,b}[B_O(\tau_0)] \rightarrow W^{1,2}_{p,b}[B_O(\tau_0)] \text{ with the following properties.}\]

- $L_A Q_{A,p,b} = \text{Id}$ from $L^2_{p,b}[B_O(\tau_0)]$ to itself.
- The bound on $Q_{A,p,b}$ is less than a $\bar{C}$ as in Definition 2.17.\]

33
Consequently, equation (107) admits a solution $\xi$ such that
\[
|\xi|_{W^{1,2}_{p,b}[B_O(\tau_0)]]} \leq \tilde{C}|f|_{L_{p,b}^2[B_O(\tau_0)]}. \tag{108}
\]
In particular, $\tilde{C}$ does not depend on $\tau_0$.

Remark 4.12. Note that for $W^{1,2}_{p,b}$-estimate, we don’t have to shrink domain. This is similar to the case of standard Laplace equation (Theorem 9.9 in [18]).

Proof. The proof is similar to that of Theorem 5.2 in [18]. Equation (107) is equivalent to
\[
\xi = Q_{p,b,A_O}f + Q_{p,b,A_O}P_{A_O,A}\xi \triangleq \Box\xi, \quad P_{A_O,A} = L_{A_O} - L_A, \tag{109}
\]
where $Q_{p,b,A_O}$ is the one in Corollary 3.20 ($\tau = \tau_0$).

By Definition 2.7 and Lemma 7.11 for any $\epsilon_0$, when $\tau_0$ is small enough with respect to $\epsilon_0$ and $\psi$, we obtain the following for any $\xi_1, \xi_2 \in W^{1,2}_{p,b}[B_O(\tau_0)]$.
\[
|P_{A_O,A}(\xi_1 - \xi_2)|_{L_{p,b}^2[B_O(\tau_0)]} \leq \tilde{C}\epsilon_0|\xi_1 - \xi_2|_{W^{1,2}_{p,b}[B_O(\tau_0)]}. \tag{110}
\]
By the optimal $W^{1,2}_{p,b}$-estimate of $Q_{p,b,A_O}$, we obtain
\[
|\Box(\xi_1 - \xi_2)|_{W^{1,2}_{p,b}[B_O(\tau_0)]} \leq \tilde{C}\epsilon_0|\xi_1 - \xi_2|_{W^{1,2}_{p,b}[B_O(\tau_0)]}. \tag{111}
\]
Let $\tilde{C}\epsilon_0 < \frac{1}{4}$, $\Box$ is a contract mapping from $W^{1,2}_{p,b}[B_O(\tau_0)]$ to itself, then there must be a unique fixed point $\xi$ which solves (107). We define $Q_{A,p,b}f$ as this $\xi$, the uniqueness also implies $Q_{A,p,b}$ is linear. The condition $\tilde{C}\epsilon_0 < \frac{1}{4}$, (109), (110), and Corollary 3.20 imply
\[
|\xi|_{W^{1,2}_{p,b}[B_O(\tau_0)]} \leq |Q_{p,b,A_O}f|_{W^{1,2}_{p,b}} + \frac{1}{4}|\xi|_{W^{1,2}_{p,b}[B_O(\tau_0)]}. \tag{112}
\]
Therefore (108) is true. $\square$

The following Lemma is well known.

Lemma 4.13. (Lemma 1.3.1 of [19], Theorem 4.3 of [23]). Under the same assumptions in Proposition 4.11 on $A$, for the $\tau_0$ obtained there, the $\tau_0$ - admissible cover $U_{\tau_0}$ satisfies the following property. For any ball $B_l \in U_{\tau_0}$ away from the singularity, there exists a local parametrix $Q_l : L^2(B_l) \rightarrow W^{1,2}(B_l)$ such that $L_AQ_l = Id + K_l$, $K_l$ is compact ($L^2(B_l) \rightarrow L^2(B_l)$).

Proof. For the reader’s convenience we mention a little bit: any section $\xi \in L^2(B_l)$ can be extended as 0 outside $B_l$, thus gives a section in $H^0 (L^2)$ defined in page 6 of [19]. Choosing the ”$\phi$” in (a) of Lemma 1.3.1 in [19] to be the standard cutoff function which is identically 1 in $B_l$ but vanishes outside $2B_l$, Lemma 1.3.1 in [19] says $\phi(L_AQ_l - Id)$ is an infinitely-smoothing operator (defined in first line of page 12 in [19]). Then by restricting $\phi(L_AQ_l - Id)\xi$ onto $B_l$, the proof is complete. $\square$
**Theorem 4.14.** Let $A, p, b$ be as in Theorem 1.7, there is a bounded linear operator $Q : (L^2_{p,b} \to W^{1,2}_{p,b})$ such that $K_{p,b} \triangleq L_A Q - \text{id} : (L^2_{p,b} \to L^2_{p,b})$ is compact.

**Proof.** We consider the $\tau_0$-admissible cover $U_{\tau_0}$ in Definition 2.2, for the $\tau_0$ in Proposition 4.11. In the $B_{O_j}$’s (near the singular points), we use the right inverse $Q_j$ constructed in Proposition 4.11. In the $B_l$’s (away from the singular points), we use the $Q_l$’s in Lemma 4.13. Then let

$$Q(f) = \sum_j \varphi_j Q_j(\chi_j f) + \sum_l \varphi_l Q_l(\chi_l f), \quad (113)$$

where $\varphi_j, \varphi_l$’s are the partition of unity of $U_{\tau_0}$ (with co-centric balls with radius $\frac{1}{5}$ of the original one), and $\chi_j (\chi_l)$ are smooth functions such that

$$\chi_j (\chi_l) = \begin{cases} 
1, & \text{over supp} \varphi_j \subset B_{O_j} (\text{supp} \varphi_l \subset B_l), \\
0, & \text{outside } B_{O_j} (B_l). 
\end{cases}$$

Then $\varphi_j \chi_j = \varphi_j (\varphi_l \chi_l = \varphi_l)$.

For any smooth function $\varphi$ and section $\xi = \begin{bmatrix} \sigma \\ a \end{bmatrix}$, we calculate

$$L_A (\varphi \xi) = \varphi L_A \xi + G(d \varphi, \xi), \quad (114)$$

$$G(d \varphi, \xi) = \begin{bmatrix} 0 & - \ast (d \varphi \wedge a) \\
d \varphi \wedge \sigma & \ast (d \varphi \wedge a \wedge \psi) \end{bmatrix}$$

is an algebraic operator. (115)

Thus $L_A Q \xi = f + \sum_j G[d \varphi_j, Q_j(\chi_j f)] + \sum_l G[d \varphi_l, Q_l(\chi_l f)] + \sum_l \varphi_l K_l(\chi_l f). \quad (116)$

Since each $\varphi_j$ is smooth, Corollary 3.20 yields

$$|\sum_j G[d \varphi_j, Q_j(\chi_j f)]|_{W^{1,2}_{p,b}} \leq C|f|_{L^2_{p,b}}, \quad (117)$$

Lemma 4.13 implies $|\sum_l \varphi_l K_l(\chi_l f)|_{L^2_{p,b}} \leq C|f|_{L^2_{p,b}}$. Let

$$K_{p,b}(f) = \sum_j G[d \varphi_j, Q_j(\chi_j f)] + \sum_l G[d \varphi_l, Q_l(\chi_l f)] + \sum_l \varphi_l K_l(\chi_l f), \quad (118)$$

we obtain $|K_{p,b} f|_{W^{1,2}_{p,b}} \leq C|f|_{L^2_{p,b}}.$

By Lemma 4.10, $K_{p,b}$ is compact from $L^2_{p,b}$ to itself. \hfill \Box

**Proof of Theorem 1.7.** By Theorem 4.14 using the argument in page 50 of [10], and in the proof of Theorem 4 in [15], the proof of the Sobolev-theory part of Theorem 1.7 is complete.

The Hybrid part in Theorem 1.7 is a direct corollary of the crucial $C^0$-estimate in Theorem 4.9. Suppose $\bar{f}$ is in cokernel, (158) yields $L_A^* \bar{f} = 0$ away from

35
from the singularities. Then Theorem 4.9 for \( L^*_A (\gamma = \frac{9}{2} + p) \), with ”f” being 0 and ”ξ” there being \( \bar{f} \) yields
\[
|\bar{f}|_{C^{1,\alpha}_{(\frac{9}{2} + p),b}(M)} \leq C|\bar{f}|_{L^2_{p,b}}.
\] (119)
This means \( \text{Coker} L_A \subset N_{p,b} \) (Definition 4.7).

Since \( \text{Ima} g\bar{A}|_{W^{1,2}_{p,b}} \) is closed in \( L^2_{p,b} \), for any \( f \in N_{p,b} \), we have a resolution into parallel and perpendicular components with respect to \( \text{Image} L_A|_{W^{1,2}_{p,b}} \):
\[
f = f^\parallel + f^\perp, \ f^\perp \in \text{Coker} L_A.
\] (120)
The estimate (119) means \( f^\perp \in N_{p,b} \) and
\[
|f^\perp|_{C^{1,\alpha}_{(\frac{7}{2} + p),b}} \leq C|f^\perp|_{L^2_{p,b}} \leq C|f|_{L^2_{p,b}} \leq C|f|_{N_{p,b}}.
\] (121)
The decomposition (120) implies \( f^\parallel \in N_{p,b} \). Thus we have proved

**Lemma 4.15.** Suppose \( f \in N_{p,b} \), then
\[
|f^\parallel|_{N_{p,b}} \leq C|f|_{N_{p,b}}, \ |f^\perp|_{N_{p,b}} \leq C|f|_{N_{p,b}}.
\] (122)

Lemma 4.15 and Theorem 4.6 yields a solution to \( L_A \xi = f^\parallel \) which is orthogonal to the kernel and
\[
|\xi|_{L^2_{p-1,b}} \leq |\xi|_{W^{1,2}_{p,b}} \leq C|f|_{L^2_{p,b}}.
\] (123)
Theorem 4.9 (\( \gamma = \frac{7}{2} + p \)) and (123) gives
\[
|\xi|_{L^2_{(\frac{7}{2} + p),b}} \leq C|f|_{N_{p,b}}. \text{ Therefore } |\xi|_{H_{p,b}} \leq C|f|_{N_{p,b}}.
\] (124)
The proof of the Hybrid-spaces part of Theorem 1.7 is complete.

**Remark 4.16.** (The pre-image space \( J_{p,b} \)) We define
\[
J_{p,b} = \{ \xi \in H_{p,b} | \xi \perp \text{ker} L_A \}.
\] (125)
Theorem 1.7 says \( L_A \) is an isomorphism from \( J_{p,b} \) to \( \text{Image} L_A|_{H_{p,b}} \subset N_{p,b} \). Denoting \( |\xi|_{H_{p,b}} \) as \( |\xi|_{J_{p,b}} \) when \( \xi \in J_{p,b} \), we rewrite (124) as
\[
|L_A^{-1}f|_{J_{p,b}} \leq C|f|_{N_{p,b}}, \text{ for any } f \in \text{Image} L_A|_{H_{p,b}}.
\] (126)
5 Perturbation

In this section we don’t need the log-weight, thus we conform to the abbreviation convention in Definition 2.19 i.e. there will be no ”b” in the symbols of the function spaces. We prove the most precise version of Theorem 1.1.

Theorem 5.1. Let $M$ be a 7−manifold with a smooth $G_2$−structure $(\phi, \psi)$, and $E \to M$ be an admissible $SO(m)$−bundle defined away from finitely-many points (Definition 2.7). Suppose $A$ is an admissible $\psi$−instanton of order 4 (Definition 2.7 and (1)). Then for any $A$−generic $p \in (-\frac{5}{2}, -\frac{3}{2})$ (Definition 2.21), there exists a $\delta_0 > 0$ with the following property.

Suppose $coker L_A = \{0\}$ (Theorem 6.1), for any admissible $\delta_0$−deformation $(\phi, \psi)$ of $(\phi, \psi)$ (Definition 2.15), there exists a $\psi$−$G_2$ monopole $(A, \sigma)$ such that $A$ satisfies Condition $\textcircled{S}_{A, p}$ (Definition 2.9). In particular, the tangent connection of $A$ at each singular point is the same as that of $A$. When $\psi$ is closed, $A$ is a $\psi$−instanton.

Proposition 5.2. Under the same conditions as in Theorem 5.1, for any $\lambda_1 > 0$ and $\epsilon_1$, there is a $\delta_0$ with the following property. For any admissible $\delta_0$−deformation $\psi$ of $\psi$, we have

$$|\star \psi (F_A \wedge \psi)|_{C^{1,0}_{(1+\lambda_1)}(M)} < \epsilon_1, \text{ for any } 0 < \alpha \leq 1. \quad (127)$$

Thus for any $\lambda_2 > 0$ and $\epsilon_1$, the following is true when $\delta_0$ is small enough.

$$|\star \psi (F_A \wedge \psi)|_{N_{-\frac{5}{2} + \lambda_2}} < \epsilon_1. \quad (128)$$

Proof. The essential issue can be elaborated by the $C^0$−estimate. The instanton condition (1) implies

$$F_A \wedge \psi = F_A \wedge (\psi - \psi). \quad (129)$$

Let $\rho_0 > 0$ be small enough such that $B_O(\rho_0)$ is within the coordinate chart near $O$. When $r \leq \rho_0$, the admissible condition implies $\psi - \psi = [\psi - \psi(O)] + [\psi(O) - \psi]$. Hence $|\psi - \psi| < Cr$. Adjust $\rho_0$ such that $C\rho_0^{\lambda_1} < \frac{\epsilon_1}{r^{1+\lambda_1}}$, we find

$$|F_A \wedge (\psi - \psi)| \leq \frac{C}{r} \leq \frac{C\rho_0^{\lambda_1}}{r^{1+\lambda_1}} < \frac{\epsilon_1}{r^{1+\lambda_1}}. \quad (130)$$

When $r \geq \rho_0$, still by the condition in Proposition 5.2, we have

$$|F_A \wedge (\psi - \psi)| \leq C\delta_0\rho_0^{-2} = C\delta_0(\frac{\epsilon_1}{2})^{-\frac{2}{r^{1+\lambda_1}}} < \epsilon_1, \text{ when } \delta_0 \text{ is small enough.} \quad (131)$$

Thus we obtain the $C^0$−bound

$$|F_A \wedge \psi|_{C^{0}_{(1+\lambda_1)}(M)} \approx |\star \psi (F_A \wedge \psi)|_{C^{0}_{(1+\lambda_1)}(M)} < C\epsilon_1. \quad (132)$$
where \( \approx \) means equivalent up to a constant in the sense of Definition 2.16.

The bounds on \( |\nabla_A \star \psi (F_A \wedge \psi)|_{C^{(2+\lambda_1)}(M)} \) and \( |\nabla_A^2 \star \psi (F_A \wedge \psi)|_{C^{(3+\lambda_1)}(M)} \) are similar. For the reader's convenience, we still do the gradient bound.

\[
\nabla_{A,\psi} \star \psi (F_A \wedge \psi) = \nabla_{A,\psi} \star \psi (F_A \wedge [\psi - \psi]) \tag{133}
\]

For the first term, by (132) and that \(|\psi|_{C^0(M)} \leq C\), we have

\[
||\nabla_{\psi}(\star \psi)(F_A \wedge [\psi - \psi])|_{C^{(2+\lambda_1)}(M)} \leq ||\nabla_{\psi}(\star \psi)(F_A \wedge [\psi - \psi])|_{C^{(1+\lambda_1)}(M)} \leq C\epsilon_1. \tag{134}
\]

For the second term, we have the following cheap estimate

\[
|\star \psi (F_A \wedge \nabla_{\psi}[\psi - \psi])| \leq \frac{C\delta_0}{r^2}. \tag{135}
\]

By exactly the same trick (relaxing the weight a little bit) from (130) to (132), we obtain for any \( \lambda_1 > 0 \)

\[
|\star \psi (F_A \wedge \nabla_{\psi}[\psi - \psi])|_{C^{(2+\lambda_1)}(M)} < \epsilon_1 \text{ when } \delta_0 \text{ is small enough.} \tag{136}
\]

In the same way it follows that the \( C^{(2+\lambda_1)}(M) \)-norm of the third term is less than \( \epsilon_1 \) (using \(|\nabla_{A,\psi}F_A| \leq \frac{C}{r}\)). Then we obtain

\[
|\nabla_{A,\psi} \star \psi (F_A \wedge \psi)|_{C^{(2+\lambda_1)}(M)} < C\epsilon_1 \text{ when } \delta_0 \text{ is small enough.} \tag{137}
\]

The proof of the Hessian estimate \( |\nabla_{A,\psi}^2 \star \psi (F_A \wedge \psi)|_{C^{(3+\lambda_1)}(M)} < C\epsilon_1 \) (138) is absolutely the same, except that we have one more negative power on \( r \).

By replacing "\( C\epsilon_1 \)" by \( \epsilon_1 \) (since \( \epsilon_1 \) is arbitrary and \( C \) does not depend on it), the estimates (132), (137), (138) amount to

\[
|\star \psi (F_A \wedge \psi)|_{C^{(3+\lambda_1)}(M)} < \epsilon_1. \tag{139}
\]

By Lemma 7.6, the proof of (127) is complete. Using Definition 4.7, the proof of (128) is done by letting \( \lambda_1 = \frac{A}{2} \) in (127) and \( \delta_0 \) small enough. \( \Box \)

The monopole equation with respect to \( \psi \) is

\[
\star \psi (F_{A+a} \wedge \psi) + d_{A+a}\sigma = 0, \text{ with gauge fixing equation } d_{A+a}^*\sigma = 0 \tag{140}
\]

It's equivalent to

\[
\star \psi (d_{A+a} \wedge \psi) + \frac{1}{2} \star \psi ([a, \sigma] \wedge \psi) + d_{A}\sigma + [a, \sigma] + \star \psi (F_A \wedge \psi) = 0 \tag{141}
\]

with gauge fixing. Equation (140) and (141) can be written as

\[
L_{A,\psi} [\sigma] = \left[ -\frac{1}{2} \star \psi ([a, \sigma] \wedge \psi) - [a, \sigma] - \star \psi (F_A \wedge \psi) \right] \tag{142}
\]

38
Lemma 5.3. Under the same conditions as in Theorem 5.1, $L_{A,\psi}$ is an isomorphism from $J_p$ to $N_p$, and the bounds (on itself and the inverse of it) are uniform for all admissible $\delta_0$–deformations of $\psi$, when $\delta_0$ is small enough with respect to the data in Definition 2.16.

Proof. The proof is exactly as that of Proposition 4.11, for the reader’s convenience we include the crucial detail.

For any $f \in N_p$, the equation $L_{A,\psi}\begin{bmatrix} \sigma \\ a \end{bmatrix} = f$ is equivalent to

$$\begin{bmatrix} \sigma \\ a \end{bmatrix} = L^{-1}_A (L_A - L_{A,\psi}) \begin{bmatrix} \sigma \\ a \end{bmatrix} + L^{-1}_A f. \tag{144}$$

The right hand side of (144) is a contract mapping in terms of $\begin{bmatrix} \sigma \\ a \end{bmatrix}$ from $J_p$ to itself, thus iteration implies (143) can be uniquely solved in $J_p$. The bound on $L^{-1}_A$ follows from the last part in the proof of Proposition 4.11.

By Lemma 5.3, (142) is equivalent to the following equation

$$\begin{bmatrix} \sigma \\ a \end{bmatrix} = L^{-1}_{A,\psi} P \begin{bmatrix} \sigma \\ a \end{bmatrix}, \tag{145}$$

where $P \begin{bmatrix} \sigma \\ a \end{bmatrix}$ means the right hand side of (142).

The first iteration is $L^{-1}_{A,\psi} P \begin{bmatrix} 0 \\ 0 \end{bmatrix} = L^{-1}_{A,\psi} \begin{bmatrix} 0 \\ - *_{\psi} (F_A \wedge \psi) \end{bmatrix}. \tag{146}$

For any $p$ as in Theorem 5.1, there is a $\lambda_2$ such that $p > -\frac{5}{2} + \lambda_2$, thus Proposition 5.2 implies the following bound on the first iteration

$$|L^{-1}_{A,\psi} P \begin{bmatrix} 0 \\ 0 \end{bmatrix}|_{J_p} \leq C |*_{\psi} (F_A \wedge \psi)|_{N_p} < C\epsilon_1 \text{ when } \delta_0 \text{ is small enough.} \tag{147}$$

Proof of Theorem 5.1: To solve the taming-pair equation (140), it suffices to show that $L^{-1}_{A,\psi} P$ is a contract mapping when restricted to a small enough neighbourhood of 0 in $J_p$, then iteration as in section 3 of [8] implies the existence of a unique solution close enough to 0.

Since $p < -\frac{5}{2}$, this is an easy consequence of the multiplicative property of $J_p (H_p), N_p$ in Lemma 4.8, and that the 2 terms

$$-\frac{1}{2} *_{\psi} ([a,a] \wedge \psi) \text{ and } -[a,\sigma] \tag{148}$$

39
are quadratic. For the reader’s convenience, we include the crucial detail.

We compute

$$P \left[ \frac{\sigma_1}{a_1} \right] - P \left[ \frac{\sigma_2}{a_2} \right]$$

(149)

$$= -\frac{1}{2} \{ \ast \varphi \left( [a_1 - a_2, a_1] \wedge \psi \right) + \frac{1}{2} \ast \varphi \left( [a_2, a_2 - a_1] \wedge \psi \right) \} - \{ [a_1 - a_2, \sigma_1] + [a_2, \sigma_2 - \sigma_1] \}$$

Then Lemma 4.8 implies

$$|P \left[ \frac{\sigma_1}{a_1} \right] - P \left[ \frac{\sigma_2}{a_2} \right]|_{N_p} \leq C \left| \frac{\sigma_1}{a_1} - \frac{\sigma_2}{a_2} \right|_{J_p}$$

(150)

Thus, letting $\epsilon_1$ small enough with respect to the "$C$" above and the "$C$" in (126), the condition (\left| \frac{\sigma_1}{a_1} + \frac{\sigma_2}{a_2} \right|_{J_p} < \epsilon_1 \) implies

$$|L_{A,\varphi}^{-1} P \left[ \frac{\sigma_1}{a_1} \right] - L_{A,\varphi}^{-1} P \left[ \frac{\sigma_2}{a_2} \right]|_{J_p} \leq \frac{1}{2} \left| \frac{\sigma_1}{a_1} - \frac{\sigma_2}{a_2} \right|_{J_p}$$

(151)

The proof of the contract mapping is complete.

Denote $A + a$ as $A_1$. When $\psi$ is closed, note that by applying $d_{A_1} \varphi$ to (141) away from the singularity, we obtain $d_{A_1} \varphi dA_1 \sigma = 0$. Then we choose the cut-off function $\eta$ as in (59), and calculate

$$0 = \int_M < d_{A_1} \varphi \eta \sigma, \sigma > dV = \int_M < d_{A_1} \sigma, (d\eta_\epsilon) \wedge \sigma > + \int_M \eta_\epsilon |d_{A_1} \sigma|^2 dV.$$

(152)

$\sigma \in J_p$ implies $|d_{A_1} \sigma| \leq \frac{C_\sigma}{r^{\frac{1}{2} + p}}$, $|\sigma| \leq \frac{C_\sigma}{r^{\frac{1}{2} + p}}$. Then

$$| \int_M < d_{A_1} \sigma, (d\eta_\epsilon) \wedge \sigma | \leq C_\sigma \int_{B(\epsilon) - B(2\epsilon)} \frac{1}{r^{\frac{1}{2} + p}} \frac{1}{r^{\frac{1}{2} + p}} \leq C \epsilon^{-2p} \to 0 \text{ as } \epsilon \to 0.$$  

(153)

Let $\epsilon \to 0$ in (152), monotone convergence theorem and (153) implies $\int_M |d_{A_1} \sigma|^2 dV = 0$. This means $d_{A_1} \sigma = 0$, and thus (141) says $A_1$ is a $\psi$-instanton.

**Proof of Theorem 1.13:** Using Corollary 3.20, this is much easier than Theorem 5.1. The crucial trick is to cut off the nonlinear term and error in (140) [(142)] i.e. let $\xi$ denote $\left[ \frac{a}{\sigma} \right]$, we should consider the equation

$$L_{A_0,\varphi} \xi = \chi \left[ \xi \otimes \xi - \ast \varphi (F_{A_0} \wedge \psi) \right], \xi \otimes \xi = -\frac{1}{2} \ast \varphi \left( [a, a] \wedge \psi \right) = [a, \sigma],$$

(154)
\( \chi \) is the standard cut-off function \( \equiv 1 \) in \( B_O(\frac{1}{4}) \) and \( \equiv 0 \) outside \( B_O(\frac{5}{16}) \). Using (102) with \( \gamma = \frac{7}{2} + p \) (for any \( p \) as in Theorem 5.1), and proof of Theorem 1 in [14] [in \( B_O(\frac{7}{16}) \) and near \( \partial B_O(\frac{7}{16}) \)], we obtain

\[
|\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{L^2(\frac{7}{16})} \leq C\{[|L_{A_O}\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{1,\alpha,B_O(\frac{5}{16})} + |\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{L^{p-1}(\frac{7}{16})}] \} \quad (155)
\]

We define

- \( \tilde{H}_p = \{ \xi \text{ is } C^{2,\alpha} \text{ away from } O | |\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{W^{1,2}(B_O(\frac{7}{16}))} + |\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{1,\alpha,B_O(\frac{7}{16})} < \infty \} \) and
- \( \tilde{N}_p = \{ \xi \text{ is } C^{1,\alpha} \text{ away from } O | |\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{L^{p}(B_O(\frac{7}{16}))} + |\xi|^{(\frac{7}{2} + p)\frac{2}{7}}_{1,\alpha,B_O(\frac{7}{16})} < \infty \} \).

Using Corollary 5.20 and (155), the \( L_{A,O,\psi_0} : \tilde{H}_p \rightarrow \tilde{N}_p \) is inverted by \( Q_{p,A_O} \) which is linear and bounded by a \( C \) as in Definition 2.17. Moreover, the advantage of cutting-off the monopole equation is

**Claim 5.4.** We have \( |\chi f|_{C^{1,\alpha}_{(\frac{7}{2} + p)\frac{2}{7}}[B_O(\frac{7}{16})]} \leq C|f|_{C^{1,\alpha}_{(\frac{7}{2} + p)\frac{2}{7}}[B_O(\frac{7}{16})]} \); Consequently,

\[
|\chi_1 \otimes \chi_2|_{\tilde{N}_p} \leq C|\chi_1|_{\tilde{H}_p}|\chi_2|_{\tilde{H}_p}, \quad \text{where } \otimes \text{ and } \chi \text{ are as in (154)}.
\]

The proof of the above is similar to Lemma 4.8, the cutoff function \( \chi \) plays the key role. **Claim 5.4** means we can avoid the boundary estimate near \( \partial B, B \) as in Theorem 1.13. With the help of Proposition 5.2 by going through the proof of Lemma 5.3, Theorem 5.1, and

- replacing the \( A \) and \( \psi \) (Proposition 5.2) by \( A_O \) and \( \psi_0 \),
- replacing the \( L_{A,O,\psi} \), \( L_A \) in Lemma 5.3 by \( L_{A,O,\psi} \), \( L_{A,O} \) respectively,
- replacing the \( L_{A,O\psi}^{-1} \) in proof of Theorem 5.1 by \( L_{A,O,\psi}^{-1} \),
- replacing the \( J_p, N_p \) in proof of Theorem 5.1 by \( \tilde{H}_p, \tilde{N}_p \) respectively,

we obtain a solution \( \xi \) to (154) in \( \tilde{H}_p \), for any \( p \) as in Theorem 5.1. Since \( \chi \equiv 1 \) in \( B_O(\frac{1}{4}) \), \( A_O + a \) solves the monopole-equation therein [see (142) and the discussion above it]. The proof of Theorem 1.13 is complete. \( \square \)

6 Characterizing the cokernel

The formal adjoint of \( L_A \) is

\[
L^*_{A} f = L_A f + G(\frac{d\omega_{p,b}}{\omega_{p,b}}, f) \text{ defined away from the singularities.} \quad (156)
\]
The cokernel is defined as $\text{Image}^\perp L_A$ i.e.
\[
coker L_A \triangleq \{ f \in L^2_{p,b} \mid \int_M < L_A \xi, f > w_{p,b} dV = 0 \text{ for all } \xi \in W^{1,2}_{p,b} \}. \quad (157)
\]
Taking the test sections $\xi$ in (157) as smooth sections supported away from the singularities, using Theorem 4.9 (for $\gamma = \frac{9}{2} + p$), we deduce
\[
coker L_A \subset \{ f \in N_{p,b} \mid L^*_A f = 0 \text{ away from the singularities} \}. \quad (158)
\]
However, a priori, there is no guarantee that the right hand side of (158) is finite-dimensional. Fortunately, when $p$ is $A$–generic, the ”blowing-up” rate of elements in cokernel can be improved as follows.

**Theorem 6.1.** Let $A, p, b$ be as in Theorem 1.7. For all $0 < \mu < \vartheta_{1-p},$
\[
coker L_A = \{ f \in C^{1,\alpha}_{\left(\frac{9}{2}+p-\mu\right)}(M) \mid L^*_A f = 0 \text{ away from the singularities} \}. \quad (159)
\]
Moreover, $|f|_{C^{1,\alpha}_{\left(\frac{9}{2}+p-\mu\right)}(M)} \leq C|f|_{L^2_{p,b}}$ for any $f \in \text{ker}(\text{Id} - K^*_{p,b})$. In particular, if $f$ satisfies the two conditions in the parentheses of (159) for some $\mu > 0$, then $f$ satisfies them for all $\mu < \vartheta_{1-p}$.

**Proof.** By Theorem 5 of Appendix D.5 in [15], $\text{ker}(\text{Id} - K^*_{p,b}) \subset L^2_{p,b}$. Lemma 6.3 implies any $f \in \text{ker}(\text{Id} - K^*_{p,b})$ is actually in $L^2_{p-\mu,b}$ with uniform bound in terms of the $L^2_{p,b}$–norm of $f$. Then Theorem 6.1 is a direct corollary of (158), Lemma 6.3, 6.4, and Theorem 4.9 (for $L^*_A$ by taking $\gamma = \frac{9}{2} + p - \mu$).

The crucial observation is that in the setting of Theorem 6.1 we have
\[
Q_{\omega,j,p,b} f = Q_{\omega,j,p+\mu,b} f \text{ for any } j. \quad (160)
\]
The reason is that the $v$–spectrum of the tangential operators are fixed. Thus, if $v > (<) 1 - p$, the same holds with $p$ with replaced by $p + \mu$ or $p - \mu$. Hence the solution formulas (in (35), (37), (41), and (43)) do not change.

**Proof.** It’s directly implies by the Lemmas 6.2, 6.3 and 6.4.

**Lemma 6.2.** Let $A, p, b, \mu$ be as in Theorem 6.1, then $L^2_{p,b} \subset L^2_{p+\mu,b}$ is an invariant subspace of $K_{p+\mu,b}$, and $K_{p+\mu,b} = K_{p,b}$ when restricted to $L^2_{p,b}$. Consequently, for any $f \in L^2_{p,b}$, we have
\[
|K_{p,b} f|_{L^2_{p,b}} \leq C_{\mu} |f|_{L^2_{p+\mu,b}}. \quad (161)
\]
Proof. By (118), we only need to show the parametrices $Q_j$ near the singularities satisfies the property asserted. The parametrices away from the singularity do not depend on the weight chosen.

In the setting of (110) and (111), let $Q_{j,p,b}$ denote $Q_{A,p,b}$ near $O_j$. By (143) and uniqueness of fixed point of contract mappings, we find $Q_{j,p,b}f = Q_{j,p+\mu,b}f$ when $f \in C^\infty_c([B_{O_j} \setminus O_j])$. Since $Q_{j,p,b}$ ($Q_{j,p+\mu,b}$) is bounded from $L^2_{p,b}(B_{O_j})$ ($L^2_{p+\mu,b}(B_{O_j})$) to $W^{1,2}_{p,b}(B_{O_j})$ ($W^{1,2}_{p+\mu,b}(B_{O_j})$), their extensions (as in proof of Theorem 3.19) agree on $L^2_{p,b}(B_{O_j}) \subset L^2_{p+\mu,b}(B_{O_j})$. Hence
\[
|Q_{j,p,b}f|_{L^2_{p,b}(B_{O_j})} \leq |f|_{L^2_{p+\mu,b}(B_{O_j})} \text{ when } f \in L^2_{p,b}(B_{O_j}).
\]
(162)

\[\square\]

Lemma 6.3. Let $A,p,b,\mu$ be as in Theorem 6.1, then $K_{p,b}^*$ is a bounded operator from $L^2_{p,b}$ to $L^2_{p-\mu,b}$. The bound is uniform as in Definition 2.10.

Proof. Assuming $\xi \in L^2_{p,b}$ vanishes near the singularities (thus $\xi \in L^2_{p,b}$ for any $p$), by Lemma 6.2 we find
\[
\int_M <K_{p,b}(\frac{\xi}{\mu}), f > w_{p,b}dV \leq C|K_{p,b}(\frac{\xi}{\mu})|_{L^2_{p,b}}|f|_{L^2_{p,b}} \leq C|\frac{\xi}{\mu}|_{L^2_{p+\mu,b}}|f|_{L^2_{p,b}}.
\]
(163)

On the other hand, let $\eta_\epsilon$ be the cutoff function of the singular points satisfying condition (59). Let $\xi = \eta_\epsilon^2 \frac{K_{p,b}^* \xi}{\mu}$, we obtain
\[
\int_M <K_{p,b}(\frac{\xi}{\mu}), f > w_{p,b}dV = \int_M \eta_\epsilon^2 |K_{p,b}^2 f|^2 \frac{w_{p,b}}{\mu}dV \geq C|\eta_\epsilon K_{p,b}^* f|_{L^2_{p-\mu,b}}^2.
\]
Hence (163) and (164) imply
\[
|\eta_\epsilon K_{p,b}^* f|_{L^2_{p-\mu,b}}^2 \leq C|\eta_\epsilon^2 K_{p,b}^* f|_{L^2_{p-\mu,b}} |f|_{L^2_{p,b}} \leq C|\eta_\epsilon K_{p,b}^* f|_{L^2_{p-\mu,b}} |f|_{L^2_{p,b}}.
\]
(165)

Then $|\eta_\epsilon K_{p,b}^* f|_{L^2_{p-\mu,b}} \leq C|f|_{L^2_{p,b}}$. Let $\epsilon \to 0$, by the monotone convergence theorem of Lebesgue measure, the proof of Lemma 6.3 is complete. \[\square\]

Lemma 6.4. Suppose $L_A^* f = 0$ away from the singular points and $f \in C^{1,\alpha}_{\frac{1}{2}+\mu}(M)$ for some $\mu > 0$. Then $f \in \text{coker}L_A$.

Proof. With the same $\eta_\epsilon$ as in Lemma 6.3, we compute
\[
\int_M <L_A \xi, f > w_{p,b}dV = \lim_{\epsilon \to 0} \int_M <L_A \xi, \eta_\epsilon f > w_{p,b}dV = \lim_{\epsilon \to 0} \int_M \eta_\epsilon G(d\eta_\epsilon, f) > w_{p,b}dV \triangleq \lim_{\epsilon \to 0} \Pi_\epsilon.
\]
43
By (59), (156), (157), and Hölder inequality, since $\mu > 0$, we obtain
\[
\Pi_\epsilon \leq C|\xi|_{L^2_{p-1,b}}\left(\int_M |G(d\eta_\epsilon, f)|^2 r^2 w_{p,b} dV\right)^{\frac{1}{2}}
\]
\[
\leq C\Sigma_j |\xi|_{L^2_{p-1,b}} \left(\int_{B_O(2\epsilon)-B_O(\epsilon)} |f|^2 w_{p,b} r^{2\mu} dV\right)^{\frac{1}{2}}
\]
\[
\leq C\epsilon^{\mu}|\xi|_{L^2_{p-1,b}} |f|_{L^2_{p-\mu,b}} \to 0 \text{ as } \epsilon \to 0.
\]

The proof is complete. \qed

7 Appendix

7.1 Appendix A: Weitzenböck formula in the model case.

Suppose $p + q \leq n$, $\Phi$ is a $p$–form, $\nu$ is a $q$–form, and both are possibly $adE$–valued. We have
\[
\star(\Phi \wedge \nu) = (\star \Phi)\wedge \nu = (-1)^{pq}\Phi \wedge (\star \nu).
\] (166)

Let the $e_i$’s be the standard coordinate vectors in $\mathbb{R}^7$. The standard (Euclidean) $G_2$–structure over $\mathbb{R}^7 \setminus O$ is
\[
\phi_0 = e^{123} - e^{145} - e^{167} + e^{246} - e^{257} - e^{347} - e^{356}
\] (167)
\[
\psi_0 = -e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567}
\]

Notice the $e_i$’s here are not the same as the ones in Section 7.2. We abuse notations on frames in different sections.

Lemma 7.1. Suppose $A_O$ is a cone connection over $E \to \mathbb{R}^7 \setminus O$. Let $L_{A_O}$ be the deformation operator of $A_O$ with respect to $\phi_0, \psi_0$ (see (13)), and $[\sigma_a]$ be as in Section 3.4. Then
\[
L_{A_O}[\sigma_a] = \left[ \nabla^* \nabla \sigma - \star([F_{A_O}, \sigma] \wedge \psi_0) + \nabla^* \nabla a + F_{A_O} \otimes a - [F_{A_O}, a] \wedge \psi_0 \right].
\] (168)

Suppose $A_O$ is a $G_2$–instanton with respect to the standard (Euclidean) $G_2$–structures i.e. $\star(F_{A_O} \wedge \phi_0) = -F_{A_O}$ or $(F_{A_O} \wedge \psi_0 = 0)$, then
\[
L_{A_O}[\sigma_a] = \left[ \nabla^* \nabla \sigma + 2F_{A_O} \otimes a \right].
\] (169)
Proof. All the forms in this proof can possibly be $ad E$–valued.

By \((166)\) and \((13)\), we have $L_{A_0}[\sigma_a] = \left[ \frac{d^*a}{d\sigma + da \lrcorner \phi_0} \right]$. Hence

$$L_{A_0}^2[\sigma_a] = \left[ \frac{d^*d\sigma - \ast([F_{A_0}, a] \lrcorner \psi_0)}{d\sigma + da \lrcorner \phi_0} \right] \ast([F_{A_0}, \sigma] \lrcorner \psi_0) + dd^*a + \{d[da \lrcorner \phi_0] \lrcorner \phi_0\}. \quad (170)$$

We first prove the general formula \((168)\). For any 1-form $B = B_i e^i$, we compute

$$(dB \lrcorner \phi_0)(e_1) = -d_6 B_7 + d_7 B_6 - d_4 B_5 + d_5 B_4 - d_3 B_2 + d_2 B_3 \quad (171)$$

Let $b$ be a 2–form, write $b = \sum_{i<j} b_{ij} e^i \wedge e^j$. Let $B = b \lrcorner \phi_0$, then

\begin{align*}
B_7 &= -b_{16} + b_{25} - b_{34}, & B_6 &= -b_{24} + b_{17} - b_{35}, & B_5 &= -b_{27} - b_{14} + b_{36}, \\
B_4 &= b_{15} + b_{37} + b_{26}, & B_3 &= -b_{47} - b_{56} + b_{12}, & B_2 &= -b_{13} - b_{46} + b_{57}.
\end{align*}

Then
$$\{[d(b \lrcorner \phi_0)] \lrcorner \phi_0\}(e_1) = -\sum_{i<j} d_i b_{ij} - <db, e^{625}> + <db, e^{634}> + <db, e^{427}> + <db, e^{537}> = d^*b(e_1) - (db \lrcorner \psi_0)(e_1).$$

Therefore, by computing the component on $e_2, \ldots, e_7$ similarly, we arrive at the following intermediate result.

**Claim 7.2.** For any $ad E$-valued 2–form $b$, the following formula holds.

$$[d(b \lrcorner \phi_0)] \lrcorner \phi_0 = d^*b - db \lrcorner \psi_0. \quad (173)$$

Let $b = da$, we obtain $[d(da \lrcorner \phi_0)] \lrcorner \phi_0 = d^*da - [F_{A_0}, a] \lrcorner \psi_0$. Using the Bochner-identity $d^*da + dd^*a = \nabla^*\nabla a + F_{A_0} \boxtimes a$, we find

$$dd^*a + \{d[da \lrcorner \phi_0] \lrcorner \phi_0\} = \nabla^*\nabla a + F_{A_0} \boxtimes a - [F_{A_0}, a] \lrcorner \psi_0. \quad (174)$$

The proof of \((168)\) is complete.

Next, suppose $A_0$ is a $G_2$–instanton with respect to $\phi_0$, we prove \((169)\). Notice in this case we automatically have $\ast([F_{A_0}, a] \lrcorner \psi_0) = 0$ and $\ast([F_{A_0}, \sigma] \lrcorner \psi_0) = 0$. We compute

$$-[F_{A_0}, a] \lrcorner \psi_0 = -\ast(\phi_0 \lrcorner [F_{A_0}, a]) = \ast\{\phi_0 \lrcorner [F_{A_0}, a] \} + \ast\{\phi_0 \lrcorner a \wedge [F_{A_0}, a]\}. \quad (175)$$

The instanton equation implies

$$\ast\{\phi_0 \wedge F_{A_0} \wedge a \} = -\ast(\phi_0 \wedge F_{A_0}) \lrcorner a = F_{A_0} \lrcorner a, \quad (176)$$

and

$$\ast\{\phi_0 \wedge a \wedge F_{A_0} \} = -\ast\{a \wedge \phi_0 \wedge F_{A_0} \} = a \lrcorner \ast(\phi_0 \wedge F_{A_0}) = -a \lrcorner F_{A_0}. \quad (177)$$

Then we get

$$-[F_{A_0}, a] \lrcorner \phi_0 = F_{A_0} \lrcorner a - a \lrcorner F_{A_0} = \sum_{i,j} [F_{ij}, a_i] e^j \triangleq F_{A_0} \boxtimes a. \quad (178)$$

The proof of \((169)\) and Lemma \(7.1\) is complete. 

45
7.2 Appendix B: Proof of Lemma 3.1

The polar coordinate formula for 1–forms is more involved. We need some preliminary identities. The Euclidean metric is equal to \( dr^2 + r^2 g_{S^{n-1}} \). For any point \((p, r) \in \mathbb{R}^7 \setminus O\), we choose \( e_1, ..., e_{n-1} \) as an orthonormal frame on \( S^{n-1} \) near \( p \). Furthermore, we require \( e_1, ..., e_{n-1} \) to be the geodesic coordinates on the sphere at \( p \).

As vector fields defined under polar coordinate near \( p \times (0, 1) \), for any \( r \), let \( \nabla^S \) denote the covariant derivative induced on \( S^{n-1}(r) \). \( \nabla^S \) is just the Levi-Civita connection of the induced metric. Since (the metric on) \( S^{n-1}(r) \) differs from the unit sphere by a constant rescaling, and \( e'_i \)s are the geodesic coordinates at \( p \), then

\[
\nabla^S e_j e_i = 0 \text{ along } p \times (0, 1), \text{ for all } i, j.
\]

Notice the \( e_i \)'s here are not the same as the ones in Section 7.1. We abuse these notations.

The vector fields \( \partial \partial r, e_i r = 1, ..., e_{n-1} r \) form an orthonormal basis for the Euclidean metric over \( \mathbb{R}^7 \setminus O \).

**Lemma 7.3.** In a neighbourhood of \( p \times (0, 1) \) in \( \mathbb{R}^n \setminus O \), we have

\[
\nabla \frac{\partial}{\partial r} e_i = \frac{e_i}{r}; \quad \nabla \frac{\partial}{\partial r} e^i = -\frac{e^i}{r}; \quad \nabla e_i dr = re^i. \tag{180}
\]

\[
\nabla e_i e^k = \nabla^S e_i e^k - \delta_{ik} \frac{dr}{r}; \quad \nabla e_i e_j = \nabla^S e_i e_j - \delta_{ij} r \frac{\partial}{\partial r}. \tag{181}
\]

Hence we compute for any \( \phi \in \Omega^1_\Xi(S^{n-1}) \) that

\[
\nabla \frac{\partial}{\partial r} \phi = \nabla \frac{\partial}{\partial r} (\phi e^i) = -\frac{\phi}{r}; \quad \nabla \frac{\partial}{\partial r} \nabla \frac{\partial}{\partial r} \phi = \frac{2\phi}{r^2}. \tag{182}
\]

**Proof of Lemma 3.1.** We first observe that, for any bundle-valued form \( b \),

\[
-\nabla^* \nabla b = \sum_{k=1}^n \nabla^2 b(v_k, v_k), \quad (v_k) \text{ is an orthonormal basis.} \tag{183}
\]

This definition does not depend on the orthonormal basis chosen, then we can use \( \frac{\partial}{\partial r}, \frac{e_1}{r}, ..., \frac{e_{n-1}}{r} \) to obtain

\[
-\nabla^* \nabla b = \nabla^2 b(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}) + \frac{1}{r^2} \sum_{i=1}^{n-1} \nabla^2 b(e_i, e_i) \tag{184}
\]

\[
= \nabla \frac{\partial}{\partial r} \nabla \frac{\partial}{\partial r} b + \frac{1}{r^2} \sum_{i=1}^{n-1} \nabla_i \nabla_i b - \frac{1}{r^2} \nabla (\nabla \frac{\partial}{\partial r} e_i) b
\]

\[
= \nabla \frac{\partial}{\partial r} \nabla \frac{\partial}{\partial r} b + \frac{1}{r^2} \sum_{i=1}^{n-1} \nabla_i \nabla_i b - \frac{1}{r^2} \nabla (\nabla \frac{\partial}{\partial r} e_i) b + \frac{n - 1}{r} \nabla \frac{\partial}{\partial r} b.
\]

46
The $\nabla_i \nabla_i b$ should be understood as $\nabla_i (\nabla_i b)$.

Part I: we compute the rough laplacian of the radial part $a_r \frac{dr}{r}$. By (184),

$$-\nabla^i \nabla_i (a_r \frac{dr}{r}) = \nabla \varphi \nabla \varphi (a_r \frac{dr}{r}) + \frac{1}{r^2} \Sigma_{i=1}^{n-1} \nabla_i \nabla_i (a_r \frac{dr}{r}) - \frac{1}{r^2} \nabla (\Sigma_i \nabla_i e_i) (a_r \frac{dr}{r}) + \frac{n-1}{r} \nabla \varphi (a_r \frac{dr}{r}).$$

Term-wise computation gives

$$\nabla \varphi \nabla \varphi (a_r \frac{dr}{r}) = (\nabla \varphi a_r) \frac{dr}{r} - a_r \frac{dr}{r^2},$$

$$\nabla \varphi \nabla \varphi (a_r \frac{dr}{r}) = (\nabla \varphi \nabla \varphi a_r) \frac{dr}{r} - 2(\nabla \varphi a_r) \frac{dr}{r^2} + \frac{2}{r^3} a_r dr.$$

For the hardest term $\Sigma_{i=1}^{n-1} \nabla_i \nabla_i (a_r \frac{dr}{r})$, fix $i$, we compute

$$\nabla_i \nabla_i (a_r \frac{dr}{r}) = (\nabla_i \nabla_i a_r) \frac{dr}{r} + 2(\nabla_i a_r) (\nabla_i \frac{dr}{r}) + a_r \nabla_i \nabla_i \frac{dr}{r}. (187)$$

Using

$$\nabla_i \frac{dr}{r} = e_i, \quad \nabla_i \nabla_i \frac{dr}{r} = -\frac{dr}{r} + \nabla_i^s e_i, \quad \text{and} \quad \Sigma_{i=1}^{n-1} \nabla_i \nabla_i a_r = \Delta_S a_r + \nabla_S^i \nabla_i a_r,$$

we obtain (by summing up $i$ in (187))

$$\Sigma_{i=1}^{n-1} \nabla_i \nabla_i (a_r \frac{dr}{r}) = (\Delta_S a_r) \frac{dr}{r} + (\nabla_S^i \nabla_i a_r) \frac{dr}{r} + 2d_S a_r - (n-1) a_r \frac{dr}{r} + a_r (\Sigma_i \nabla_i^s e_i)$$

$$\text{(189)}$$

By (179), (180), (185), and (189), along $p \times (0,1)$, we have

$$-\nabla^i \nabla_i (a_r \frac{dr}{r})$$

$$= (\nabla \varphi \nabla \varphi a_r) \frac{dr}{r} + \frac{n-3}{r} (\nabla \varphi a_r) \frac{dr}{r} + \frac{1}{r^2} (\Delta_S a_r - (2n-4) a_r) \frac{dr}{r} + \frac{2}{r^3} d_S a_r. \text{(190)}$$

Since (190) is independent of the coordinate chosen, and $p$ is arbitrary, then it holds everywhere on $\mathbb{R}^7 \setminus O$.

Part II: we compute the rough laplacian of the radial part $a_s$ (which does not have radial component). First we have

$$-\nabla^i \nabla_i a_s = \nabla \varphi \nabla \varphi a_s + \frac{1}{r^2} \Sigma_{i=1}^{n-1} \nabla_i \nabla_i a_s - \frac{1}{r^2} \nabla (\Sigma_i \nabla_i e_i) a_s + \frac{n-1}{r} \nabla \varphi a_s.$$

In this case, we only have to compute the crucial term $\nabla_i \nabla_i a_s$. We write

$$a_s = \Sigma_{i=1}^{n-1} a_i,$$

then

$$\nabla_i a_s = (\nabla_i a_k) e_k + a_k \nabla_i e_k = \nabla_i^s a_s + a_k (\nabla_i e_k - \nabla_i^s e_k) = \nabla_i^s a_s - a_s (e_i) \frac{dr}{r}.$$ 

\[ \text{In this case}, \ we \ only \ have \ to \ compute \ the \ crucial \ term \ \nabla_i \nabla_i a_s. \ \text{We \ write} \ a_s = \Sigma_{i=1}^{n-1} a_i, \ \text{then} \]
\[ \nabla_i a_s = (\nabla_i a_k) e_k + a_k \nabla_i e_k = \nabla_i^s a_s + a_k (\nabla_i e_k - \nabla_i^s e_k) = \nabla_i^s a_s - a_s (e_i) \frac{dr}{r}. \]
To compute $\nabla_i \nabla_i a_s$, it suffices to compute $\nabla_i \nabla_i^S a_s$ and $\nabla_i [a_s(e_i) \frac{dr}{r}]$.

$$\Sigma_i \nabla_i \nabla_i^S a_s = \Sigma_i \nabla_i \nabla_i^S a_s - \Sigma_i (\nabla_i^S a_s)(e_i) \frac{dr}{r} = \Delta_S a_s + \nabla_i^S a_s + d^* a_s \frac{dr}{r}. \quad (192)$$

On the other hand

$$\Sigma_i \nabla_i [a_s(e_i) \frac{dr}{r}] = \Sigma_i \{ [ \nabla_i^S a_s(e_i)] \frac{dr}{r} + a_s (\nabla_i^S e_i) \frac{dr}{r} \} = -(d^* a_s) \frac{dr}{r} + a_s (\nabla_i^S e_i) \frac{dr}{r} + a_s. \quad (193)$$

Then $\Sigma_i \nabla_i \nabla_i a_s = \Delta_S a_s + 2 (d^* a_s) \frac{dr}{r} + \nabla_i^S a_s - a_s (\nabla_i^S e_i) \frac{dr}{r} - a_s. \quad (194)$

By $(181)$, $(194)$, and $(179)$, the following holds along $p \times (0,1)$

$$-\nabla^* \nabla a_s = \nabla \frac{\sigma}{\sigma} \nabla \frac{\sigma}{\sigma} a_s + \frac{n-1}{r} \nabla \frac{\sigma}{\sigma} a_s + \frac{1}{r^2} (\Delta_S a_s - a_s) + (2d^* a_s) \frac{dr}{r^3}. \quad (195)$$

It does not depend on the coordinates chosen, thus holds everywhere. The proof of Lemma 3.1 is completed by combining $(190)$ and $(195)$. $\square$

### 7.3 Appendix C: Fundamental facts on elliptic systems

In this section, we work under the same conditions as in Theorem 1.7. For any $q$ near a singular point $O$, denote

$$B = B_q(\frac{r_q}{100}), \quad (196)$$

then $B$ lies in one coordinate sector. Let $L_\psi$ be the local elliptic operator with $A = 0$ in $B$ i.e. $L_\psi [ \sigma \begin{array}{c} \sigma \\ a \end{array} ] = \begin{array}{c} d^* a \\ d\sigma + da, \phi \end{array}$. For any locally defined $G_2$—structure $(\phi, \psi)$, we have the following weighted estimate due to Nirenberg-Douglis [14].

$$|\xi|_{2,\alpha,B}^* \leq C |L_\psi \xi|_{1,\alpha,B} + C |\xi|_{0,B}. \quad (197)$$

where ”$C$” depends at most on the $C^5$—norm and the non-degeneracy of $\phi$.

An easy but important building-block is

**Lemma 7.4.** Suppose $\xi \in C^{k,\alpha}(B)$, the following estimate holds.

$$|\xi|_{k,\alpha,B}^* \leq C |L_\alpha \xi|_{k-1,\alpha,B} + C |\xi|_{0,B}; \quad k = 2, 3, 4. \quad (198)$$

The estimate holds the same with $L_\alpha$ replaced by $L_\alpha^*$. 58
Proof. We only prove it for \( L_A \) when \( k = 2 \). On \( L_A^* \), note that Definition 3.24 implies \( \left| dw_{\alpha,B} \right| \leq \frac{C}{r} \), thus we verify that \( \left| dw_{\alpha,B} \right|_{2,0,B} \leq \left| dw_{\alpha,B} \right|_{2,0,B} < C \). Then formula (156) implies that the proof for \( L_A \) directly carries over to \( L_A^* \).

The admissible conditions implies \( |A|^{[1]}_{2,0,B} \leq C \), hence \( |A|^{[1]}_{1,0,B} \leq C \). Using

\[
|L_A - L_\psi| = |A \otimes_{p_q} \xi| \text{ and splitting of weight, we have }
\]

\[
|A \otimes_{q} \xi|^{[1]}_{1,0,B} \leq |A|^{[1]}_{1,0,B} |\xi|^{*}_{1,0,B} \leq C |\xi|^{*}_{1,0,B}.
\]

Thus \( |L_A - L_\psi| \xi|^{[1]}_{1,0,B} \leq C |\xi|^{*}_{1,0,B} \), and this implies

\[
|\xi|^{*}_{2,0,B} \leq C (|L_A \xi|^{[1]}_{1,0,B} + |\xi|_{0,B}) + |L_A - L_\psi| \xi|^{[1]}_{1,0,B} \leq C (|L_A \xi|^{[1]}_{1,0,B} + |\xi|^{*}_{1,0,B})
\]

By Lemma 6.32 in [18] (standard interpolation), for any \( \mu \in (0, \frac{1}{100}) \), we have

\[
|\xi|^{*}_{1,0,B} \leq \mu \|\nabla^2 \xi|^{[2]}_{0,B} + C_\mu |\xi|_{0,B}.
\]

(199)

Then (198) and (199) imply Lemma 7.3.

Proposition 7.5. Let \( \gamma \) be any real number, suppose \( \xi \in C^{2,\alpha} \) away from the singularity, \( L_A \xi \in C^{1,\alpha}_{(\gamma,b)}(M) \), \( \xi \in C^{0}_{(\gamma-1,b)}(M) \). Then \( \xi \in C^{1,\alpha}_{(\gamma-1,b)}(M) \), and

\[
|\xi|^{(\gamma-1,b)}_{2,0,M} \leq C |L_A \xi|^{(\gamma,b)}_{1,0,M} + C |\xi|^{(\gamma-1,b)}_{0,0,M}.
\]

Proof. We only consider the case when \( 1 + \gamma + \alpha \geq 0 \). By our choice of \( x \) and \( y \) (the paragraph above (201)), the proof is similar when it’s negative. Without loss of generality, for any singular point \( O \) we only consider \( x \in B_O(\rho_0) \) and \( B_{\frac{r_0}{1000}} \). For any \( y \in B_{\frac{r_0}{1000}} \), the distance from \( y \) to \( \partial B_{\frac{r_0}{1000}} \) is less than \( r_y \), and we have \( 2r_y > r_x > \frac{r_y}{2} \), \( 10(-\log r_y) > -\log r_x > \frac{\log r_y}{10} \). Thus \( r_{x,y} \simeq r_y \simeq r_x \simeq r_{x,y} \). Hence, using the proof of 4.20 in Theorem 4.8 of [18] and Lemma 7.3 multiplying both sides of the estimate in Lemma 7.3 by \( r_y^{-1}(-\log r_y)^b \), we obtain the following lower order estimate

\[
|\xi|^{(\gamma-1,b)}_{2,0,M} \leq C |L_A \xi|^{(\gamma,b)}_{1,0,M} + C |\xi|^{(\gamma-1,b)}_{0,0,M}.
\]

(200)

The term left to estimate is the following one with highest order.

\[
Q_{x,y} = (-\log r_{x,y})^b y^{1+\gamma+b} \frac{\nabla^2 \xi(x) - \nabla^2 \xi(y)}{|x-y|^\alpha}.
\]

This can be done in a standard way. We can assume \( r_x \geq r_y \), since otherwise we only need to interchange them. We note that this choice is different from the paragraph above (6.15) in [18]. Suppose \( y \in B_{\frac{r_y}{2000}} \), the proof of (200) implies one more conclusion:

\[
Q_{x,y} \leq C |L_A \xi|^{(\gamma,b)}_{1,0,M} + C |\xi|^{(\gamma-1,b)}_{0,0,M}.
\]

(201)
When \( y \notin B_x \left( \frac{r_0}{200} \right) \) (we only need to consider \( y \) in \( B_\nu(r_0) \)), we find

\[
Q_{x,y} \leq (\log r_x)^b r_x^{1+\gamma|\nabla^2 \xi(x)|} + (\log r_y)^b r_y^{1+\gamma|\nabla^2 \xi(y)|} \leq C|\xi|^{(\gamma-1,b)}_{2,0,M}.
\]

The proof of Proposition 7.5 is complete.

Working on each coordinate patch separately, by the proof of Lemma 6.32 in [18] with slight modification (on log weight as Proposition 7.5), we obtain

Lemma 7.6. (Interpolation) For any \( \mu < \frac{1}{9000}, b \geq 0, 0 < \alpha < 1, \) and real numbers \( k, \) there exists a constant \( C_{\mu,k,\alpha,b} \) with the following property. For any section \( \xi \) and non-negative integer \( j, \) the following interpolation holds.

\[
|\nabla^j \xi|^{(k,b)}_{\alpha,M} \leq \mu |\nabla^{j+1} \xi|^{(k+1,b)}_{0,M} + C_{\mu,k,\alpha,b} |\nabla^j \xi|^{(k,b)}_{0,M}, \tag{203}
\]

where \( \nabla^j \xi \) is viewed as a combination of locally defined matrix-valued tensors in each chart of \( U_{\rho_0}. \)

Lemma 7.7. Suppose \( B \) is a ball such that \( 2B \) is contained in a single coordinate neighbourhood away from the singularity. Suppose \( \xi \in W^{1,2}(2B) \) and \( L_A\xi \in C^{1,\alpha}(2B) \) (in the sense of strong solution). Then \( u \in C^{2,\alpha}(B). \)

Remark 7.8. We believe Lemma 7.7 is in literature. Since the author cannot find an exact reference, we still give a proof for the readers’ convenience.

Proof. We apply \( -L_A \) to the equation again to obtain \( -L_A^2 \xi = -L_A f \in C^{\alpha}(2B). \) From the proof of Lemma 7.1, we see that the difference of the Weitzenböck formula between the model case and the general case is some lower order term (concerning at most first derivative of \( \xi \) in local coordinates). Then

\[
\Delta_{g_{\phi}} \xi = \nabla \xi \otimes T_{\phi,\psi,A,1} + \xi \otimes T_{\phi,\psi,A,0} - L_A f, \tag{204}
\]

where the \( T \)'s are tensors depending algebraically on \( \phi, \psi, A, \) and their derivatives. The \( T \)'s might be only locally defined, but this is sufficient.

The important point is that \( \Delta_{g_{\phi}} \xi \) means the metric Laplacian of each entry of \( \xi \) in local coordinates, thus \( (204) \) is actually a bunch of scalar equations. Then Lemma 7.7 follows by applying Lemma 7.9 successively.

Lemma 7.9. (18) Under the same conditions on \( B \) in Lemma 7.7, suppose \( \xi \in W^{1,2}(2B) \) is a weak solution to

\[
\Delta_{g_{\phi}} \xi = h \text{ in } 2B. \tag{205}
\]

Suppose \( h \in L^p(2B), p \geq 2, \) then \( \xi \in L^{2p}(2B). \) Suppose \( h \in C^{\alpha}(2B), 0 < \alpha < 1, \) then \( \xi \in C^{2,\alpha}(2B). \)
Proof. It suffices to construct local solutions (with optimal regularity) to (205). Viewing (205) as a bunch of scalar equations, let the boundary value over $\partial B$ be 0, when $h \in L^p(2B)$ and $p \geq 2$, Theorem 9.15 in [18] implies that (205) admits a solution $\xi \in L^{2,p}(B)$ (in $B$). When $h \in C^\infty(2B)$, Theorem 6.14 in [18] gives a solution $\xi \in C^{2,\alpha}(B)$ to (205). In both cases, $\Delta_g(\xi - \bar{\xi}) = 0$, therefore $\xi - \bar{\xi}$ is smooth by Lemma 7.10. Then $\bar{\xi} = (\xi - \bar{\xi}) + \bar{\xi} \in L^{2,p}(B)$ or $C^{2,\alpha}(B)$, when $h \in L^p(B)$ or $C^\infty(B)$, respectively.

Lemma 7.10. (see Gilkey [13]) Suppose $f \in L^2_{p,b}$ belongs to the cokernel of $L_A$ in distribution sense (see [177]). Then $f$ is smooth away from the singular points. $\xi \in \ker L_A \subset W^{1,2}_{p,b}$ also implies $\xi$ is smooth away from the singularities.

Proof. It’s an easy exercise on pseudo-differential operators. We only need to show the conditions imply $L_A^* f = 0$ where we view $L_A^* f$ as an element in $H^{-1}$ (see Lemma 1.2.1 in [19]). Thus Lemma 1.3.1 of [19] is directly applicable.

To achieve this, for any ball $B$ such that $100B$ is still away from the singularities, we choose $\eta$ as the standard cutoff function which vanishes outside $2B$ and is identically 1 in $B$. We also choose $\chi$ as the standard cutoff function which vanishes outside $3B$ and is identically 1 in $2B$. By Lemma 1.2.1 and 1.1.6 in [19], using a limiting argument with respect the smoothing of $\chi f$, we obtain $\eta L_A^* (\chi f) = 0$ as an element in $H^{-1}$. Since $L_A^*$ is elliptic, we conclude by Lemma 1.3.1 of [19] that $f$ is smooth in $B$.

Lemma 7.11. Suppose $\tau_0 \leq \delta$, and $\psi_1, \psi_2$ are two $G_2$—structures over $B_0(2\tau_0)$, $|\psi_1 - \psi_2| < \delta$. Suppose $A_1, A_2$ are 2 connections over $B_0(\tau_0)$, $|A_1 - A_2| < \frac{\delta}{r}$.

Then $|L_{A_1, \psi_1} - L_{A_2, \psi_2}| \leq C\delta|\nabla_{A_2} \xi| + \frac{C\delta|\xi|}{r}$ in $B_0(\tau_0)$.

(206)

Remark 7.12. The $C$ depends on $C^2$—norms of $\psi_1, \psi_2$ and $C^0$—norm of $rA_1$ in local coordinates. The $\nabla_{A_2}$ is the covariant derivative with respect to the Euclidean metric in the coordinate. The estimate (206) still holds for $\nabla_{A_1}$ and with respect to any smooth metric.

Proof. In [13], we only estimate the difference from $d^*_A a$, the errors from the other terms are similar. By $d_A^* = -\star d_A \star$, we note $\star_{\psi_1} d_{A_1} \star_{\psi_2} \star_{\psi_2} d_{A_2} \star_{\psi_2} = (\star_{\psi_1} \star_{\psi_2}) d_{A_1} \star_{\psi_1} + \star_{\psi_2} d_{A_2} (\star_{\psi_1} \star_{\psi_2}) + \star_{\psi_2} d_{A_2} (\star_{\psi_1} \star_{\psi_2})$. We only estimate the last term $\star_{\psi_2} d_{A_2} (\star_{\psi_1} \star_{\psi_2})$, the estimate of the other terms are similar and easier. Using $d_{A_2} [(\star_{\psi_1} \star_{\psi_2}) a] = (\star_{\psi_1} \star_{\psi_2}) \otimes \nabla_{A_2} a + [\nabla(\star_{\psi_1} \star_{\psi_2})] \otimes a$, we get $|L_{A_1, \psi_1} - L_{A_2, \psi_2}| \leq C\delta|\nabla_{A_2} \xi| + C|\xi|$. Then we obtain (206) when $r < \tau_0$.

7.4 Appendix D: Density and smooth convergence of Fourier Series

Lemma 7.13. Let $S$ be a closed Riemannian manifold (of any dimension), and $\Xi \to S$ be a smooth $SO(m)$—vector bundle with an inner product. Suppose
\(A_S\) is a smooth connection on \(\Xi\), and \(\Delta_{A_S} = \nabla^*_{A_S} \nabla_{A_S} + \delta\) is a self-adjoint Laplacian-type operator acting on sections of \(\Xi\), where \(\nabla^*_{A_S} \nabla_{A_S}\) is the rough Laplacian of \(A_S\) and the Riemannian metric, \(\delta\) is a smooth algebraic operator (which does not concern any covariant derivative). Let \(\beta\) be the real eigenvalues of \(\Delta_{A_S}\) repeated according to their multiplicities, and \(\Psi_\beta\) be the corresponding orthonormal basis in \(L^2(S)\). Then for any smooth section \(f\) to \(\Xi\), the Fourier-series \(\sum_\beta \int \beta \Psi_\beta\) (of \(f\)) converges to \(f\) in the \(C^\infty\)-topology.

Moreover, the speed of convergence only depends on the \(C^\infty\)-norm of \(f\), i.e., there exists a integer \(\tau > 0\) depending only on \(\Delta_{A_S}\), such that for any \(\epsilon > 0\) and integer \(s \geq 0\), there exists a \(k\) depending only on \(|f|_{W^{2r+2,2}(S)}\), \(\epsilon\), and \(\Delta_{A_S}\), such that \(|f - \sum_{\beta<k} \int \beta \Psi_\beta|_{W^{2r,2}(S)} < \epsilon\).

Corollary 7.14. In the setting of Lemma 7.16 and Theorem 7.19, let \(f \in C_c^\infty[B_0(\rho) \setminus O]\), then the Fourier-series in (30) converges in \(C^0[B_0(\rho)]\) to \(f\).

Proof of Lemma 7.13. Since \(\Delta_{A_S}\) is bounded from below, by considering \(\Delta_{A_S} + a_0I\) for some big enough \(a_0\), we can assume all the \(\beta\)'s are larger than 1000.

In Claim 7.15 we note that \(\Delta_{A_S}^l \sum_{\beta<\kappa} \int \beta \Psi_\beta\) is the Fourier partial sum of \(\Delta_{A_S}^l f\). Let \(F_k\) denote \(\sum_{\beta \geq k} \int \beta \Psi_\beta\), and \(k = 100 + \sup_{0 \leq l \leq s} k_l\), the standard \(W^{2,2}\)-estimate for \(\Delta_{A_S}\) and Claim 7.15 imply
\[
|\Delta_{A_S}^{s-k} F_{k,m}|_{W^{2,2}(S)} \leq C|\Delta_{A_S}^s F_{k,m}|_{L^2(S)} + C|\Delta_{A_S}^{s-k} F_{k,m}|_{L^2(S)} \leq C\epsilon \tag{207}
\]
uniformly in \(m\). Let \(m \to \infty\), we find \(\Delta_{A_S}^k f \in W^{2,2}(S)\) and
\[
|\Delta_{A_S}^{s-k} F_k|_{W^{2,2}(S)} \leq C|\Delta_{A_S}^s F_k|_{L^2(S)} + C|\Delta_{A_S}^{s-k} F_k|_{L^2(S)} \leq C\epsilon. \tag{208}
\]
By induction, using Theorem 5.2 in [23], by similar estimates as (207) and (208), we obtain \(|F_k|_{W^{2,2}(S)} \leq C\epsilon\). Replacing \(C\epsilon\) by \(\epsilon\), the proof of Lemma 7.13 is complete by assuming the following.

Claim 7.15. For any \(\epsilon > 0\), integer \(l \geq 0\), there exists a \(k_l\) depending only on \(|f|_{W^{2r+2,2}(S)}\), \(\epsilon\), \(l\), \(\Delta_{A_S}\), such that \(|\Delta_{A_S}^l f - \Delta_{A_S}^l (\sum_{\beta<\kappa} \int \beta \Psi_\beta)|_{L^2(S)} < \epsilon\).

The proof of the Claim is by the asymptotic property of zeta-functions. For any positive integer \(t\), using
\[
\int \beta = \int \beta < \int \beta \Psi_\beta = \int \beta < \int \beta \Delta_{A_S} \Psi_\beta = \int \beta < \int \beta \Delta_{A_S} \Psi_\beta, \tag{209}
\]
we get \(|\int \beta| < C|\Delta_{A_S}^l|_{W^{2,2}(S)}\). Then \(|\int \beta F_{k_l}|_{L^2(S)} \leq C|\int \beta|_{W^{2,2}(S)}|\sum_{\beta \geq k_l} \int \beta \Psi_\beta|\). The sum \(\sum_{\beta \geq k_l} \int \beta \Psi_\beta\) is part of the zeta-function of \(\Delta_{A_S}\). There exists a large enough \(\tau\) with respect to the data in Lemma 7.13 such that \(\sum_{\beta \geq k_l} \frac{1}{\beta^r}\) converges to an analytic function of \(t - l \geq \tau\). By Corollary 2.43 in [3], or Lemma 1.10.1 in [19], we can take \(\tau = \frac{\dim S^2}{2} + 2\). Nevertheless, we don’t need \(\tau\) to be explicit.
Let $t = r + l$, and $k_l$ be large enough with respect to $\epsilon$ and the zeta function of $\Delta_{A_S}$, the proof of Claim 7.15 is complete. 

**Proof of Corollary 7.14** The condition $f \in C^\infty_c[B_O(\rho) \setminus O]$ implies that, by viewing $f$ as an $r$–dependent smooth section, the Sobolev norms of $\hat{f}$ are uniformly bounded in $r$ i.e $|f(r, \cdot)|_{W^{2,2}(S^{n-1})} \leq C_{f,t}$. Moreover, $f$ (and its Fourier-coefficients) vanishes when $r$ is small enough. Then for any $\epsilon$, let $s = \frac{n-1}{4} + 10$, by Sobolev imbedding, there exists a $k$ as in Lemma 7.13 such that the estimate

$$|f(r, \cdot) - \Sigma_{\beta<k} f_{\beta}(r) \Psi_{\beta}|_{C^0(S^{n-1})} \leq \tilde{C} f(r, \cdot) - \Sigma_{\beta<k} f_{\beta}(r) \Psi_{\beta}|_{W^{2,2}(S^{n-1})} < \epsilon$$

holds uniformly in $r$. The proof of Corollary 7.14 is complete. 

**Proof of Lemma 3.16** Without loss of generality, we assume $\rho = 1$. Dropping the last condition in Definition 3.15, we first show that $C^\infty_c[B_O(1) \setminus O]$ is dense in $L^2_{p,b}[B_O(1)]$. We assume $f$ satisfies the condition after the "which" in Lemma 3.16. Under Local coordinate, $\hat{f}$ is a matrix-valued function. For any $\epsilon > 0$, by absolute continuity of Lebesgue integration (Theorem 4.12 in [38]), for any small enough $h > 0$, we can decompose $f = f_0 + f_+$: $f_0$ is supported in $V_{+O} \setminus V_{+h}$ and $\int_{V_{+O} \setminus V_{+h}} |f_0|^2 wdV < (\frac{\epsilon}{2})^2$, $f_+$ is supported in $V_{+h}$, where $V_{+h}$ is the set of points with distance to $\partial V_{+O}$ greater than $h$.

Since $f_+$ is supported away from the singular point, using Lemma 7.2 in [18], we can find $f_+$ such that $|f_+ - f_1|_{L^2_{p,b}(V_{+O})} < \frac{\epsilon}{2}$, $\text{supp} f_+ \subset V_{+h}$. Then

$$|\hat{f} - f_1|_{L^2_{p,b}(V_{+O})} \leq |\hat{f} - f_0|_{L^2_{p,b}(V_{+O})} + |f_0 - f_1|_{L^2_{p,b}(V_{+O})} < \epsilon.$$ 

Let $\hat{f}_+$ be the same approximation in $V_{-O}$. We denote the partition of unity over $S^{n-1}$ subordinate to $U_+, U_-$ as $\eta_+, \eta_-$, and pull them back to $\mathbb{R}^7 \setminus O$. Let $f_+ = \eta_+ f_+ + \eta_- f_-$, we obtain

$$|f - f_1|_{L^2_{p,b}[B_O(1)]} \leq |\eta_+ f - \eta_+ f_+|_{L^2_{p,b}[V_{+O}]} + |\eta_- f - \eta_- f_-|_{L^2_{p,b}[V_{-O}]} < 2\epsilon. \quad (210)$$

Viewing $f_+$ as a $r$–dependent smooth section of $\Xi \to S^6(1)$, Corollary 7.14 implies that the series $\Sigma_{\nu_\mu} f_{\nu_\mu}(r)\Psi_\nu$ (see (39)) converges to $f_+$ in $C^0[B_O(1)]$, and the following holds for some large enough $v_0 > 0$.

$$\int_{B_O(1)} |f_+ - f_{[v_0],r}|^2 wdV < \epsilon^2, \quad \text{where } f_{[v_0],r} \triangleq \Sigma_{\nu<v_0} f_{\nu_\mu}(r)\Psi_\nu. \quad (211)$$

Inequalities (210) and (211) imply $|f_{[v_0],r} - f_+|_{L^2_{p,b}[B_O(1)]} < 3\epsilon$. 

The following proof does not depend on Corollary 7.14.
**Proof of Lemma 3.17:** We only consider the case $k = 1$, and assume $\rho = 1$. The assertion $W_{p,b}^{1,2}[B_0(1)] \subset W_{p,b}^{1,2}[O(1)]$ is an easy exercise using monotone convergence theorem and Theorem 7.4 in [18] away from the singularity.

The assertion $W_{p,b}^{1,2}[B_0(1)] \subset W_{p,b}^{1,2}[O(1)]$ can be approximated by smooth sections defined in $B$, as in (43), let $\text{Lemma 7.16}$. Under the conditions as in Proposition 3.10 and 3.13, let $\text{Proof of Lemma 3.17}$.

7.5 Appendix E: Various integral identities and proof of Proposition 3.23

**Lemma 7.16.** Under the conditions as in Proposition 3.10 and 3.13, let $u$ be as in (43), let $\bar{u}$ and $\bar{f}$ be as in Claim 3.11 and (48), we have

\[
\int_0^{\frac{1}{2}} \bar{f}^2 r w_0 dr = \int_0^{\frac{1}{2}} \frac{d^2 \bar{u}}{dr^2} \bar{u}^2 r w_0 dr + k^2 \int_0^{\frac{1}{2}} \frac{d \bar{u}}{dr} \bar{u}^2 r^2 w_0 dr + a^4 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr
\]

**Proof of Lemma 7.16:** Integrating the square of both hand sides of (48) over $(0, \frac{1}{2})$ with respect to $r w_0$, we obtain

\[
\int_0^{\frac{1}{2}} \bar{f}^2 r w_0 dr = \int_0^{\frac{1}{2}} \frac{d^2 \bar{u}}{dr^2} \bar{u}^2 r w_0 dr + k^2 \int_0^{\frac{1}{2}} \frac{d \bar{u}}{dr} \bar{u}^2 r^2 w_0 dr + a^4 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr + 2k \int_0^{\frac{1}{2}} \frac{d^2 \bar{u}}{dr^2} \frac{\bar{u}}{r^2} w_0 dr - 2a^2 \int_0^{\frac{1}{2}} \frac{d \bar{u}}{dr} \frac{\bar{u}}{r^2} w_0 dr - 2a^2 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr - 2ka^2 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr.
\]

Integration by parts (of $\frac{d}{dr}$) gives

\[
2k \int_0^{\frac{1}{2}} \frac{d^2 \bar{u}}{dr^2} \frac{\bar{u}}{r^2} w_0 dr = -k \int_0^{\frac{1}{2}} \frac{d \bar{u}}{dr} \bar{u}^2 w_0 dr + k \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr \tag{214}
\]

\[
-2ka^2 \int_0^{\frac{1}{2}} \frac{d \bar{u}}{dr} \frac{\bar{u}}{r^2} w_0 dr = -2ka^2 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr + ka^2 \int_0^{\frac{1}{2}} \frac{\bar{u}^2 w_0}{r^3} dr - ka^2 \frac{\bar{u}^2 w_0}{r^3} \bigg|_0^{\frac{1}{2}}.
\]

54
Proof. This is an absolutely easy practice in calculus. We only prove (216) we have more terms of the same nature. We compute
\[ -2a^2 \int_0^1 \frac{d^2 \bar{u}}{dr^2} \frac{\bar{u}}{r} w_0 dr \]
\[ = -2a^2 \frac{du}{dr} \frac{u}{r} w_0 \bigg|_0^1 - 2a^2 \int_0^1 \frac{\bar{u}^2 w_0}{r^3} dr + 2a^2 \int_0^1 \frac{\bar{u}^2 dw_0}{r^3} dr - a^2 \bar{u}^2 w_0 \bigg|_0^1. \]
\[ + a^2 \int_0^1 \frac{d^2 w_0}{dr^2} \bigg|_0^1 + 2a^2 \int_0^1 \frac{du}{dr} \frac{\bar{u}}{r} w_0 dr - a^2 \int_0^1 \frac{\bar{u}^2 dw_0}{r^3} dr. \]

Plugging the above in (213), the proof of Lemma 7.16 is complete. \[ \square \]

**Lemma 7.17.** Let \( b \geq 0 \). For any \( k \), there exists a constant \( C_{k,b} \) which depends only on the positive lower bound of \( |k - 1| \) (not on the upper bound), with the following properties. Suppose \( k < 1 \), then
\[ \int_0^r x^{-k}(-\log x)^{2b} dx \leq \frac{C_{k,b}}{1-k} x^{1-k}(-\log r)^{2b}, \quad \text{for all } r \in [0, \frac{1}{2}], \quad (216) \]
Suppose \( k > 1 \), then
\[ \int_r^\infty x^{-k}(-\log x)^{2b} dx \leq \frac{C_{k,b}}{k-1} x^{1-k}(-\log r)^{2b}, \quad \text{for all } r \in [0, \frac{1}{2}]. \quad (217) \]

**Proof.** This is an absolutely easy practice in calculus. We only prove (216) assuming \( b < \frac{1}{2} \). The general case and proof of (217) are similar except that we have more terms of the same nature. We compute
\[ \int_0^r x^{-k}(-\log x)^{2b} dx = \frac{x^{1-k}(-\log x)^{2b}}{1-k} \bigg|_0^r + \frac{2b}{k-1} \int_0^r x^{-k}(-\log x)^{2b-1} dx. \]

Since \( 2b - 1 < 0 \), we have \( \int_0^r x^{-k}(-\log x)^{2b-1} dx \leq C \int_0^r x^{-k} dx = \frac{C x^{1-k}}{1-k} \). Thus the right hand side of (218) is bounded by \( \frac{C x^{1-k}(-\log r)^{2b}}{1-k} \). \[ \square \]

**Proof of Proposition 3.23** Let \( \eta \) be as in (59), let \( d_j \) denote \( \nabla A_0 \frac{\partial}{\partial x_j} \) (under Euclidean metric and coordinate), we compute for any \( \eta > 0 \) that
\[ \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} |\nabla A_0 \nabla A_0 \xi|^2 \eta \chi^2 w dV = \Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} d_j d_j \xi, d_i d_i \xi > \eta \chi^2 w dV \quad (219) \]
\[ = -\Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} < d_j \xi, d_j d_i \xi > \eta \chi^2 w dV - \Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} < d_j \xi, d_i d_i \xi > [\nabla_j (\eta \chi^2 w)] dV \]
\[ = -\Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} < d_j \xi, d_j d_i \xi > \eta \chi^2 w dV + \Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} < d_j \xi, [F_{ij}, d_i] > \eta \chi^2 w dV \]
\[ - \Sigma_{j,i} \int_{B_{\overline{\Omega}}(\frac{\eta}{\epsilon})} < d_j \xi, d_i d_i \xi > [\nabla_j (\eta \chi^2 w)] dV \]

55
Then we distribute all derivatives like $\nabla (\eta w)$. By the method in (68), Lemma 4.2 and the proof of (68), all the integrals containing $\nabla \eta$ tend to 0 as $\epsilon \to 0$, thus the equality between top and bottom of (219) gives

$$\int_{\overline{B}_0(\frac{3}{4})} |\nabla^2_{\substack{A_0}} \xi|^2 \chi^2 \, dV$$

$$= \int_{\overline{B}_0(\frac{3}{4})} |\nabla^*_{\substack{A_0}} \nabla_{\substack{A_0}} \xi|^2 \chi^2 \, dV - \sum_{j,i} \int_{\overline{B}_0(\frac{3}{4})} [F_{ij}, \xi] \, d\xi > \chi^2 \, dV$$

$$- \sum_{j,i} \int_{\overline{B}_0(\frac{3}{4})} [F_{ij}, \xi] \, d\xi > \chi^2 \, dV$$

$$+ \sum_{j,i} \int_{\overline{B}_0(\frac{3}{4})} [F_{ij}, \xi] \, d\xi > \chi^2 \, dV.$$  

Using (62), (60), and (220), by the proof of (63) and (65), we deduce

$$\int_{\overline{B}_0(\frac{3}{4})} |\nabla^2_{\substack{A_0}} \xi|^2 \chi^2 \, dV$$

$$\leq \tilde{C}_\delta \int_{\overline{B}_0(\frac{3}{4})} \frac{|\nabla^2_{\substack{A_0}} \xi|^2}{r^2} \chi^2 \, dV + \tilde{C}_\delta \int_{\overline{B}_0(\frac{3}{4})} |\nabla^2 \chi|^2 |\nabla_{\substack{A_0}} \xi|^2 \chi \, dV$$

$$+ \tilde{C}_\delta \int_{\overline{B}_0(\frac{3}{4})} \frac{|\xi|^2}{r^4} \chi^2 \, dV + \tilde{C} \int_{\overline{B}_0(\frac{3}{4})} |\chi|^2 \, dV + \vartheta \int_{\overline{B}_0(\frac{3}{4})} \frac{|\nabla^2_{\substack{A_0}} \xi|^2}{r^2} \chi^2 \, dV.$$  

Let $\chi$ be the standard cutoff function which is identically 1 over $B_0(\frac{3}{4})$ and vanishes outside $B_0(\frac{\theta}{15})$, the proof of (63) implies

$$\tilde{C}_\delta \int_{\overline{B}_0(\frac{3}{4})} |\nabla^2 \chi|^2 |\nabla_{\substack{A_0}} \xi|^2 \chi \, dV \leq \tilde{C}_\delta \int_{\overline{B}_0(\frac{3}{4})} \frac{|\nabla^2_{\substack{A_0}} \xi|^2}{r^2} \chi \, dV.$$  

Let $\vartheta = \frac{1}{20}$, combining (221) and (222), the proof is complete. □
# 8 Notation and Subject Index

The locations in the column of "definition" includes the nearby material.

| Subject or Notation | definition |
|---------------------|------------|
| $A$-generic         | Def 2.21   |
| admissible connections | Def 2.7   |
| $H_{p,b}, H_p, N_{p,b}, N_p$ | Def 4.4, Def 2.19 |
| $C^{k,\alpha}_{\gamma,b}, C^{k,\alpha}_{\gamma} \mid \cdot \mid_{k,\alpha}^{(\gamma)}$ | Def 2.13, Def 2.11, Def 2.19 |
| $\mid \cdot \mid_{k,\alpha,B}^{(\gamma)} \mid \cdot \mid_{k,\alpha}$ | Proof of Theorem 4.9 (9) |
| $\mathbb{U}, \mathbb{U}_0$-admissible cover, $\mathbb{U}_{p_0}$ | Def 2.2, Def 2.11 |
| $\text{condition } \mathfrak{S}_{A,p}$ | Def 2.9 |
| admissible $\delta_0$-deformation of the $G_2$-structure, $\phi, \psi, \phi_0, \psi_0$ | Def 2.15, (167) |
| $W^{2,2}_{p,b}, W^{1,2}_{p,b}, \mathbb{W}_{p,b}^2, L^2_{p,b}, L^2_p$ | Def 3.24, Def 3.15, Def 2.19 |
| $\otimes, \bigotimes, \otimes$ | Def 3.2, Def 2.20 |
| $C$ | Def 2.17, Def 2.20 |
| $L_A, L_{A^O}, L_{A^O,\psi}, L_A^*$ | Corollary 3.20, Def 2.19 |
| $J_{p,b}, J_p$ | Remark 4.16, Def 2.19 |
| $G(\cdot, \cdot)$ | (115) |
| $v, v-$spectrum, $\beta, \Psi_\beta, \Psi_v$ | Def 3.5 |
| $K_{p,b}$ | Theorem 4.14 |
| $\Xi$ | Def 2.1 |
| $\perp, \parallel$ | Def 2.18 |
| $O, O_j, V_{O,+}, V_{O,-}, U_+, U_-$ | Def 2.2 |
| $\Upsilon_{A^O}, \Upsilon_{A^{O_j}}$ | Proposition 3.3 |
| $\Delta_s$ | (17) |
| $w, w_{p,b}$ | Def 3.15, Def 3.24 |
| $dV$ | Def 3.25 |
| $\vartheta_{-p}, \vartheta_{1-p}$ | Def 4.1 |
| $\text{coker } L_A$ | (157) |
| $d_j, d_i$ | (219) |
| $r, r_z, r_{x,y}, r_{x,y}$ | Remark 2.5, Def 2.10 |
| $(p,b)$-Fredholm | Def 2.18 |

## References

[1] K. Akutagawa, G. Carron, R. Mazzeo. *The Yamabe problem on stratified spaces*. arXiv:1210.8054. To appear in Geometric and Functional Analysis.

[2] S.B. Angenent. *Shrinking doughnuts*. Nonlinear Diffusion Equations and their Equilibrium States (Gregynog, 1989).
N. Berline, E. Getzler, M. Vergne. *Heat Kernels and Dirac Operators*. Grundlehren Text Editions. 2004.

R. Bott, L.W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. 1982.

R.L. Bryant. *Some remarks on $G_2$–structures*. Proceedings of 12th Gökova Geometry-Topology Conference. 75-109.

T.H. Colding, W.P. Minicozzi II. *Generic mean curvature ow I; generic singularities*. Annals of Mathematics 175 (2012), 755-833.

B. Charbonneau, D. Harland. *Deformations of nearly Kähler instantons*. arXiv:1510.07720.

X.X. Chen, Y.Q. Wang. *Bessel functions, heat kernel and the Conical Kähler-Ricci flow*. Journal of Functional Analysis. Volume 269, Issue 2 .

A. Degeratu, R. Mazzeo. *Fredholm theory for elliptic operators on quasi-asymptotically conical spaces*. arXiv:1406.3465.

S.K. Donaldson. *Floer homology groups in Yang-Mills theory*. Cambridge Tracts in Mathematics. 147.

S.K. Donaldson. *Kähler metrics with cone singularities along a divisor*. In: Essays on Mathematics and its applications (P.M. Pardalos et al., Eds.), Springer, 2012, pp. 49-79.

S.K. Donaldson, E. Segal. *Gauge Theory in higher dimensions, II*. from: Geometry of special holonomy and related topics, (NC Leung, ST Yau, editors), Surv. Differ. Geom. 16, International Press (2011) 141.

S.K. Donaldson, R.P. Thomas. *Gauge Theory in Higher Dimensions*. from: The Geometric Universe, (SA Huggett, L J Mason, KP Tod, S Tsou, NMJ伍德豪斯, editors), Oxford Univ. Press (1998) 3147.

A. Douglis, L. Nirenberg. *Interior estimates for elliptic systems of partial differential equations*. Communications on Pure and Applied Mathematics. Volume 8, Issue 4, pages 503-538, November 1955.

L.C. Evans. *Partial differential equations*. Graduate Studies in Mathematics, Vol 19. AMS.

L. Foscolo. *Deformation theory of periodic monopoles (with singularities)*. arXiv:1411.6946.

L. Foscolo, M. Haskins. *New $G_2$–holonomy cones and exotic nearly Kähler structure on $S^6$ and $S^3 \times S^3$*. arxiv1501.07838.

D. Gilbarg, N.S. Trudinger. *Elliptic Partial Differential Equations of Second Order*. Springer.
[19] P.B. Gilkey. Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Library of Congress Catalog Card Number 84-061166. ISBN 0-914098-20-9.

[20] V. Goldshtein, A. Ukhlov. Weighted Sobolev spaces and embedding theorems. Trans. Amer. Math. Soc., 361 (2009), 3829-3850.

[21] M. Haskins, H.J. Hein, J. Nordström. Asymptotically cylindrical Calabi-Yau manifolds. arXiv:1212.6929. To appear in J. Diff. Geom.

[22] D.D. Joyce. Compact manifolds with special holonomy. Oxford Mathematical Monographs. Oxford University Press. 2000.

[23] H.B. Lawson, M.L. Michelson. Spin Geometry. Princeton mathematical series: 38.

[24] R.B. Lockhart, R.C. McOwen. Elliptic differential operators on noncompact manifolds. Annali della Scuola Normale Superiore di Pisa - Classe di Scienze (1985) Volume: 12, Issue: 3, page 409-447.

[25] R. Mazzeo. Elliptic theory of differential edge operators I. Comm. Partial Differential Equations 16 (1991), no.10, 1615-1664.

[26] R.B. Melrose, G. Mendoza. Elliptic operators of totally characteristic type. MSRI, Berkeley, CA June 1983 MSRI 047-83.

[27] G. Oliveira. \(G_2\)-monopoles with singularities (examples). Unpublished work.

[28] G. Oliveira. Monopoles on the Bryant-Salamon \(G_2\)-manifolds. Journal of Geometry and Physics, vol. 86 (2014), pp. 599-632, ISSN 0393-0440.

[29] G. Oliveira. Calabi-Yau Monopoles for the Stenzel Metric. To appear in Communications in Mathematical Physics.

[30] G. Oliveira. Monopoles on 3 dimensional AC manifolds. arXiv:1412.2252.

[31] H. Sa Earp, T. Walpuski. \(G_2\)-instantons over twisted connected sums. Geometry Topology 19 (2015) 1263-1285.

[32] D.A. Salamon, T. Walpuski. Notes on the octonians. arXiv:1005.2820.

[33] J. Song, X. Wang. The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality. arXiv1207.4839. To appear in Geometry and Topology.

[34] T. Walpuski. \(G_2\)-instantons on generalised Kummer constructions. Geometry and Topology 17 (2013). 2345-2388.

[35] G. Tian. Gauge theory and calibrated geometry. Ann. Math. 151. 193-268 (2000).

[36] F. Xu. On instantons on nearly \(\text{Kähler}\) 6-manifolds. Asian. J. Math. 2009 International Press Vol. 13, No. 4, pp. 535-568, December 2009.
[37] B.Z. Yang. The uniqueness of tangent cones for Yang-Mills connections with isolated singularities. Advances in Mathematics (180), 648-691. 2003.

[38] M.Q. Zhou. Theory of Real Functions (Mandarin Chinese). Peking University Press; 2nd edition (1991).

Yuanqi Wang, Department of Mathematics, Stony Brook University, Stony Brook, NY, USA; ywang@scgp.stonybrook.edu.