TOPOLOGY, ENTROPY AND WITTEN INDEX
OF DILATON BLACK HOLES

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ABSTRACT

We have found that for extreme dilaton black holes an inner boundary must be introduced in addition to the outer boundary to give an integer value to the Euler number. The resulting manifolds have (if one identifies imaginary time) topology $S^1 \times R \times S^2$ and Euler number $\chi = 0$ in contrast to the non-extreme case with $\chi = 2$.

The entropy of extreme $U(1)$ dilaton black holes is already known to be zero. We include a review of some recent ideas due to Hawking on the Reissner-Nordström case. By regarding all extreme black holes as having an inner boundary, we conclude that the entropy of all extreme black holes, including $[U(1)]^2$ black holes, vanishes.

We discuss the relevance of this to the vanishing of quantum corrections and the idea that the functional integral for extreme holes gives a Witten Index. We have studied also the topology of “moduli space” of multi black holes. The quantum mechanics on black hole moduli spaces is expected to be supersymmetric despite the fact that they are not HyperKähler since the corresponding geometry has torsion unlike the BPS monopole case.

Finally, we describe the possibility of extreme black hole fission for states with an energy gap. The energy released, as a proportion of the initial rest mass, during the decay of an electro-magnetic black hole is 300 times greater than that released by the fission of an $^{235}U$ nucleus.

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1 Introduction

There has been a great deal of interest recently in the classical black hole solutions of theories with axion and dilaton fields and the behaviour of such black holes in the quantum theory. In particular there have been a number of new ideas about pair creation, entropy and the evaporation of extreme black holes. Our aim in this paper is to contribute to this discussion. Because some of these new ideas may appear at first sight to be puzzling, if not paradoxical, we would like, before passing on to the main business of the paper, to make some general, and hopefully clarifying, remarks about the significance of classical solutions and what role one assigns them in the quantum theory.

Traditionally one has thought of black holes, including primordial black holes, as having arisen from gravitational collapse. The classical solution, and its attendant classical spacetime is expected to provide a good description for holes rather larger than the Planck mass, but as the black hole losses mass by Hawking evaporation one expects that the classical picture should become less and less accurate. At this stage one might anticipate that processes that are classically forbidden, such as fission, might become important.

If the initial black hole has a charge which because either, (i) no ordinary particles carry it, or (ii) if they do they are so heavy as to suppress its creation by the field of the hole, then the Hawking evaporation is expected to lead to a limiting, near extreme state. Actually in case (i) it is difficult to see how the hole can have acquired the charge in the first place unless it was created in a pair creation process. This latter process is inherently quantum mechanical. One might be tempted to describe it using instanton methods and again a classical solution is involved. However it is not immediately clear that the appropriate boundary conditions at the horizons of these classical solutions are the same as those that apply to classical macroscopic black holes. This is particularly true in the extreme case. In the classical theory one never encounters an extreme black hole as such – one merely finds that under suitable circumstances that a sub-extreme hole may become more and more extreme, rather like a (massive) particle which may become more and more relativistic but which never actually attains the velocity of light. However, just as with massless particles, it is possible to contemplate a class of extreme black holes in their own right. Presumably such extreme black holes may only be produced by pair creation. If this is true then some properties of extreme holes might indeed be different from near extreme holes. Superficially this might give rise to the impression that some physical properties may change discontinuously as the extremal limit is approached. However such an impression would be illusory if the limit, as is suggested here, is never attained.

A second role for classical solutions in a field theory is as “solitons”. Solitons are typically topological non-trivial classical field configurations carrying a conserved charge – often a “central” charge in a supersymmetric theory – which is not carried by the “elementary” fields. There are many reasons for regarding extreme black holes as the solitons of supergravity and superstring theories. One often thinks of soliton as a weak coupling approximation to a single quantum state in the underlying Hilbert space of the theory. If extreme holes are solitons it seems reasonable to expect that:
1. Classical extreme hole solutions should have a different topology from non-extreme holes.

2. The classical entropy of the soliton solutions should vanish.

It is also not unreasonable to expect that soliton solutions might satisfy different boundary conditions from solutions representing non-extreme black holes formed during gravitational collapse, since solitons are usually only formed by pair creation.

A third role for classical solutions is as saddle points in the approximate evaluation of the functional integral giving some matrix element or giving a partition function. The classical solutions are typically “euclidean”, i.e. have signature ++++ and the appropriate boundary conditions then depend upon what matrix element or what partition function one is trying to evaluate. Thus if one wants to evaluate the thermodynamic properties of black holes one sums over bosonic fields which are periodic in imaginary time. Naturally the classical bosonic saddle point is periodic in imaginary time. This technique may be used to calculate the entropy of a system (containing black holes) at fixed temperature $T = \beta^{-1}$. In our opinion a great deal of wasted effort has gone into trying to interpret this equilibrium too literally in a purely classical way. Classically such an equilibrium is impossible. The classical solutions are saddle points used in the evaluation of the functional integral. They do not represent familiar astrophysical black holes. The classical saddle points are, for example, time symmetric and have both past and future horizons. Put more picturesquely they contain both black holes and white holes. One does not expect to encounter white holes in the classical theory.

In any event one finds in this way that non-extreme classical black holes contribute an amount $S$ of entropy given by

$$ S = \frac{1}{4} A, $$

(1)

where $A$ is the area of the event horizon. Quantum fluctuations will, in general, modify this formula. The classical saddle point giving this result has its mass and charge determined by the temperature and potential of the system.

To what extent one can trust eq. (1) for charged black holes? For the Schwarzschild black hole the thermodynamical approach obviously breaks down when the mass of a black hole becomes smaller than the Planck mass, and its temperature becomes greater than the Planck mass. For the charged black holes the temperature reaches some maximum value and then approaches zero in the extreme limit. Thus one might expect that the semiclassical description of the charged black holes to be quite reliable. However, a more detailed investigation of this question performed in [1, 2] casts doubt on this. The standard description breaks down if the emission of a single particle with the energy of the order of Hawking temperature would make the mass of the black hole smaller than its extreme value. Starting from this moment the thermodynamical approach becomes ambiguous; it does not allow one to calculate the entropy of the near extreme black holes in a reliable way, since one can no longer neglect the backreaction of created particles.

The situation with respect to extreme static holes is even more complicated. Firstly if the
coupling \( a \) to the dilaton is non-zero then the metric of \( U(1) \) black holes is singular on the horizon (at least in Einstein conformal gauge). Secondly the area of the event horizon of these black holes tends to zero as the extreme limit is approached. This presumably means that their contribution to the entropy vanishes. One of the purposes of the present paper is to check this point in detail. If the coupling to the dilaton vanishes, i.e. in the Einstein-Maxwell case, the situation is slightly different but the end result is similar. The evaluation of the action requires additional inner boundary terms [3]. These are chosen to make the extreme holes a *bona fide* solution of the classical variational problem. These boundary terms contribute to the classical action and hence to the classical entropy of the Reissner-Nordström black holes with the result that it vanishes [3], see also [4]:

\[
S = 0 .
\]

Of course, in general, thermal fluctuations and closed loops might be expected to give a non-vanishing contribution. The latter will in general diverge except in a well defined quantum theory such as superstring theory might be. One would not, however, expect the entire contribution to vanish because thermal fluctuations should contribute differently and moreover positively to the entropy, be they bosons or fermions.

In addition to the thermal or Gibbs partition function \( Z(\beta) = T r e^{-\beta H} \) in a supersymmetric theory, one may be interested in the Witten index \( Y(\beta) = T r (-)^F e^{-\beta H} \). This is given by a functional integral over boson and fermion fields both of which are periodic in imaginary time. Solitons and extreme holes holes should contribute to \( Y(\beta) \). Non-extreme holes cannot. This is because fermion fields must, for purely topological reasons, be anti-periodic on these backgrounds [5]. The point here is that for topology \( S^2 \times R^2 \) the spin structure is unique because the manifold is simply connected. Moreover this unique spin structure corresponds to spinors which are antiperiodic at infinity. For extreme holes we will argue that the topology is \( S^1 \times R \times S^2 \) which is not simply connected. Two spin structures are allowed, one of which is periodic at infinity.

Being a topological quantity, quantum and thermal fluctuations of fermions and bosons should cancel in the expression for \( Y(\beta) \). For this reason it may make more sense to evaluate \( Y(\beta) \) in the field theory limit of a putatively finite superstring theory. To do so one would need to consider boundary conditions on the extreme classical background in imaginary time which are quite different from what one uses for non-extreme holes. This is one of the principal points we wish to make in this paper.

We have given three different ways in which classical black hole solutions may be used in the quantum theory. No doubt the reader can think of others. Each forms the basis of a perturbative calculation of some process or physical property of a quantum black hole. In effect however nobody knows what a quantum black hole is really like, or indeed whether such a concept is well defined in the full quantum theory – not least, because we don’t yet have the full quantum theory, let alone the tools needed to analyse it. It may be that some special theories, heterotic string theory for example, exhibit an Olive-Montonen type duality which makes statements about solitons in the weak coupling limit exact statements about string states in the strong coupling limit. In that case the classical concept of an extreme black hole may carry over completely into
the quantum regime. Even then it is not obvious that the detailed properties will have much similarity with what we expect of macroscopic astrophysical black holes. In theories which do not admit such a duality one may have to recognize that the idea of a black hole is essentially a classical concept and that one has no right to be dogmatic ahead of time about what modification may be necessary in the quantum theory. Thus for example, the quasi-teleological properties ascribed in \[3\] to extreme black holes, while unfamiliar, may indeed be appropriate for particular calculations, for example of the Witten index. It is neither obvious nor excluded that they are appropriate for some sorts of scattering calculations. For that reason we have tried to explore some of the geometrical properties of extreme black holes and their possible physical consequences but any such consequences do not follow directly from the solutions themselves but rather, as we have tried to indicate above, from a whole set of physical assumptions about the physical role of the classical solutions in a particular calculation, subject to appropriate boundary conditions. Strictly speaking, statements such as “black holes have such and such properties” are meaningless without these implicit assumptions being carefully spelt out. In the classical theory considerable agreement exists about what the appropriate physical assumptions are. In the quantum theory no such agreement is even possible in principle at present because we lack such a theory. For that reason, and others, we have declined in this paper to speculate extensively about the possible consequences of our results for black hole evaporation and the so-called information paradox. However this process operates it is certainly a dynamical process and we feel that only a limited insight can be gained by examining the properties of purely static classical solutions. This is not the least so because the classical solutions may have past and future event horizons (or naked singularities). Classically these render the classical evolution ambiguous. Presumably this ambiguity would be resolved in a proper, dynamical quantum mechanical treatment. This we reserve for future study.

To summarize: axion-dilaton black holes are different from those of Schwarzschild, Reissner-Nordström and Kerr-Newman black holes. The presence of dilaton has a great affect on the geometry as well as the thermodynamical properties of these objects. Some properties of dilaton black holes are exactly the same as those of classical ones and some are not. For example, the entropy \(S\) calculated by the standard Euclidean methods is always equal to one quarter of the area of the horizon. However, Schwarzschild, and Kerr black holes never have vanishing surface area and hence entropy. By contrast some extreme dilaton black holes do have vanishing area of the horizon and vanishing entropy. The study of the topology and the entropy of the charged black holes and specifically of the charged extreme black holes is the purpose of this paper. In fact we shall only treat in detail static black holes and we have at present nothing new to say about the extreme Kerr solution.

The Euler number of the Schwarzschild and Kerr black holes equals 2 since the corresponding manifold is identified to be \(R^2 \times S^2\), see Fig. 1. The Gauss-Bonnet (G-B) theorem confirms this by representing this number as the volume G-B integral supplemented by the outer boundary term \([3, 7]\). One motivation for the present work was whether this going to be the same for all charged dilaton black holes and what happens in the extreme limit.

Because in the Einstein conformal gauge extreme holes may be singular at the horizon and so
standard definitions may break down, in what follows we are going to assume that by definition
the extreme solution is the one which allows one to identify the imaginary time coordinate $\tau = it$
with any period $\beta$. One is led to the following picture. For extreme black holes the topology of
Riemannian section is $S^1 \times R \times S^2$ and hence the Euler number should vanish, $\chi = 0$, see Fig. 2.

Thus our first aim is to understand better the topology and in addition to calculate the
Euler number of all charged axion-dilaton black holes using the explicit form of the metric in the
Euclidean space and the G-B theorem. The second reason why we are interested in Gauss-Bonnet
Lagrangian is related to the trace anomalies in gravity. The 4-dimensional trace anomaly contains
(in addition to other terms) the following expression \[ T(x) = g_{\mu\nu} < T^{\mu\nu} > = \frac{A}{32\pi^2} R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta}. \] (3)
The coefficient $A$ is known for all fields interacting with gravity. The integrated form of the
anomaly in the Euclidean space expresses the trace of the energy-momentum tensor through the
Euler number of a manifold
\[ \int d^4x \sqrt{-g} T(x) = A \chi. \] (4)
The knowledge of the local expression for the G-B term may be useful for the analysis of the way
in which the quantum anomaly affects the classical solutions. The third reason for our interest to
G-B term is the following. The above mentioned anomaly is actually related to the fact that in
topologically non-trivial backgrounds there exists a counterterm, proportional to the G-B action.
Therefore in general one may want to consider the classical gravitational action supplemented by
a G-B term with some arbitrary coefficient in front of it. This action will be able to absorb a
corresponding one-loop divergence. This in turn may affect any calculation of the action for some
configurations. Thus we would like to understand the value of the G-B action for the new family
dilaton black holes, including extreme ones. The presence of such term may affect different
configurations in a different way.

Our interest to the problem was greatly enhanced by the recent calculations, following an
earlier suggestion \[ 9, \] of the rate of the pair production of extreme black holes \[ 10, 11. \] This
process involves change in topology, and therefore the one-loop Gauss-Bonnet correction to the
classical Einstein-Maxwell-dilaton-axion action may become relevant. In addition, it was observed
in \[ 10, \] that the calculation of the instanton action related to the pair production of Reissner-
Nordström black holes has a strange feature: discontinuity at the limit to extreme. This leads to
a prediction that the rate of production of extreme black holes is suppressed comparatively to the
non-extreme ones by the amount equal to the entropy of the extreme Reissner-Nordström black
holes. The physical meaning of this discontinuity is the subject of a recent paper by Hawking,
Horowitz and Ross \[ 3. \]

\[ ^3 \text{This is in an analogy with the well known fact that the pure Yukawa coupling } g \bar{\psi} \psi \phi \text{ in quantum field theory}
\text{without } \lambda \phi^4 \text{ coupling is inconsistent. Only in presence of non-vanishing } \lambda \text{ the theory is consistent as a renormalizable quantum field theory. The actual value of both couplings is determined by the flow of renormalization group but it is inconsistent to take } \lambda = 0 \text{ from the beginning. Of course, gravity is not renormalizable but it still seems logical to proceed as in field theory when dealing with one-loop divergences.} \]
In addition the mere existence of instantons, describing pair production of non-extreme $U(1)$ black holes depends on the dilaton coupling; in fact they exist only for $0 \leq a < 1$ although the instantons for pair production of extreme black holes do exist for stringy case $a = 1$. The instantons describing pair production of $[U(1)]^2$ non-extreme as well as extreme black holes have been constructed by Ross [11]. The area of the horizon of these extreme black holes is not vanishing. We will show that the inner boundary at the horizon, which will be introduced for all extreme black holes, will change the value of their entropy as dramatic as for Reissner-Nordström black holes. Instead of being

\[ S_{[U(1)]^2} = \frac{A}{4} = \pi (M^2 - \Sigma^2), \]

where $\Sigma$ is the dilaton charge, the entropy will vanish. In the Reissner-Nordström case $\Sigma = 0$.

Another closely related issue is the possibility that at the last stages of the black hole evaporation, when black hole approaches its extreme limit, a black hole may continue evaporating by splitting into smaller extreme black holes carrying elementary electric and magnetic charges [2]. As was argued in [2], the probability of such process should be suppressed by the factor $\exp(\Delta S)$, where $\Delta S$ is the change of entropy. For large black holes the change of entropy $\Delta S$ is large and negative, which prevents their splitting, in accordance with the second law of black hole physics. Meanwhile, some extreme black holes studied in [2] have vanishing entropy, which allows them to split without violation of the second law of black hole physics. In this paper we will argue that in fact all extreme dilaton black holes have vanishing entropy. This makes the possibility of splitting of extreme black holes more plausible.

Finally the vanishing Euler number and the vanishing entropy seem an essential component of a new interpretation of the functional integral for extreme black holes in supersymmetric theories as being purely topological in nature and giving a Witten Index for supersymmetric soliton states. In fact independently of that interpretation one might argue that a zero is the only consistent value for the entropy of solitons.

Thus we have presented various reasons why the investigation of the topology of all known axion-dilaton black holes and of the behavior of the Gauss-Bonnet action is worthy of study.

The plan of the paper is the following. In Sec. 2 we will present a simple proof that the 4-dimensional G-B volume integral over the compact manifold without a boundary is a topological invariant. The next step is the calculation of the G-B Lagrangian for the static spherically symmetric non-extreme Euclidean axion-dilaton black holes. First we calculate the volume integral in Sec. 3. In Sec. 4 the general discussion of the boundary terms for the G-B action is presented, which is based on the work of Chern [12]. We discuss a special situation for manifolds with boundary, when the Euler number vanishes. In Sec. 5 we calculate the volume G-B integral over a compact manifold by putting the outer boundary at some radius $r_0$. The boundary terms at this boundary are calculated, which finally allows to verify the G-B theorem for all non-extreme black holes. The fact that the Hawking temperature is finite and non-vanishing plays an important role in providing a compactness of our manifold in the Euclidean time direction. Under specific assumptions about the area of integration in the Euclidean time we proceed with the corresponding
calculations for the extreme black holes in Sec. 6. We will come to the conclusion that without
placing an additional inner boundary we cannot have a consistent picture for extreme black holes.
The existence of this inner boundary affects the calculation of the entropy of extreme black holes,
which is presented in Sec. 7. In Sec. 8 we define the Witten Index of the fluctuations existing
in the extreme black hole background. The supersymmetric non-renormalization theorem which
was known before for the entropy of such black holes becomes relevant for the topological Witten
Index of the solitons. In Sec. 9 we prove that the Euler number of Euclidean multi black hole
solutions vanishes. The topology of the “moduli space” of multi black holes is studied both by
standard methods for the compact spaces as well as by investigation of the corresponding de
Rahm complex. In Sec. 10 an example of (4, 1) supersymmetric sigma model is given by the
target space of an uplifted dilatonic non-Abelian magnetic black hole. The non-HyperKählerian
properties of the manifold are due to the existence of the torsion in the uplifted geometry of the
magnetic black hole with the unbroken space-time supersymmetry. In Sec. 11 we have suggested
to consider an one-dimensional supersymmetric action in the black hole background to study
the supersymmetric quantum mechanics of the extreme black holes. Sec. 12 describes various
examples of splitting and fission of extreme black holes.

Some details of Gauss-Bonnet calculations can be found in Appendix. The figures illustrate
the topology of non-extreme and extreme black holes and their entropy and the topology of the
“relative moduli space” of two extreme black holes.

2 Topological invariance

We will start with a short reminder of the topological character of the G-B action [12, 7]. The
Euler number of a 4-manifold is the alternating sum of the Betti numbers,
\[
\chi[M] = \sum_{p=0}^{p=4} (-1)^p B_p .
\]
(6)
Gauss-Bonnet theorem relates the Euler number of the closed Riemannian manifold without a
boundary to the volume integral of the curvature of the four-dimensional metric,
\[
S_{GB} = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd} .
\]
(7)
In equation (7) the curvature 2-form is defined as
\[
R^a_{\;\;b} = d\omega^a_{\;b} + \omega^a_{\;c} \wedge \omega^c_{\;b} ,
\]
(8)
where \( \omega^a_{\;b} \) is a spin connection one-form. Consider some arbitrary variation of the metric, which
will result in some variation of the spin connection \( \delta \omega^a_{\;b} \). Under such variation the volume integral
in eq. (7) changes as follows:
\[
\delta S_{GB} = \frac{1}{32\pi^2} \int_M 2 \epsilon_{abcd} \delta R^{ab} \wedge R^{cd} ,
\]
(9)
where
\[ \delta R^a_b = d \delta \omega^a_b + \delta \omega^a_e \wedge \omega^e_b + \omega^a_e \wedge \delta \omega^e_b = D \delta \omega^e_b . \] (10)

We get
\[ \delta S_{GB} = \frac{1}{32 \pi^2} \int_M 2 \epsilon_{abcd} (D \delta \omega^{ab}) \wedge R^{cd} . \] (11)

Using the Bianchi identity for the Riemannian manifold
\[ DR^a_b = 0 \] (12)
and the covariant constancy of the tensor \( \epsilon_{abcd} \) in the form
\[ D \epsilon_{abcd} = 0 \] (13)
we can bring equation (11) to the form
\[ \delta S_{GB} = \frac{1}{16 \pi^2} \int_M D (\epsilon_{abcd} \delta \omega^{ab} \wedge R^{cd}) . \] (14)

Finally, we may notice that our covariant derivative \( D \) acts on an \( SO(4) \) invariant differential form and therefore we may replace it by the ordinary derivative \( d \), acting on this 3-form. Thus the volume part of the G-B action under the arbitrary change of the metric undergoes the following change:
\[ \delta S_{GB} = \frac{1}{16 \pi^2} \int_M d (\epsilon_{abcd} \delta \omega^{ab} \wedge R^{cd}) . \] (15)

We have shown that the variation of the Gauss-Bonnet form is an exact form. If the manifold has no boundary, the integral from the exact form vanishes according to the Stoke’s theorem. This proves the topological character of this expression, specifically its independence of the metric. If however the manifold does have a boundary, the volume integral (7) does depend on the metric since the variation equals to
\[ \delta S_{GB}^{volume} = \frac{1}{16 \pi^2} \int_{\partial M} (\epsilon_{abcd} \delta \omega^{ab} \wedge R^{cd}) . \] (16)

The explicit expressions for the boundary terms for the Euler number have been presented in [6] and [7] in a different form.

### 3 G-B volume integral

Now we can start calculating the G-B Lagrangian for axion-dilaton dilaton black holes [13, 2, 14]. The metric of spherically symmetric \( U(1) \) and \( U(1) \times U(1) \) black holes is given by
\[ ds^2 = -e^{2U}(r)dt^2 + e^{-2U}(r)dr^2 + R^2(r)d^2 \Omega . \] (17)
The G-B volume integral

\[ S_{volume}^{GB} = \frac{-1}{32\pi^2} \left[ \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd} \right] = \frac{1}{32\pi^2} \left[ \int d^4x \sqrt{-g} \left( R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right] \]  

(18)
can be calculated using the values of the Riemann tensor \( R_{\mu\nu\lambda\delta} \), Ricci tensor \( R_{\mu\nu} \) and Ricci scalar \( R \) for the metric of the form (17). The result of such calculation is the following, see Appendix:

\[ \sqrt{-g} \left( R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) = \frac{\partial}{\partial r} \left[ 4(e^{2U})' \left( 1 - (e^U R')^2 \right) \right] . \]  

(19)
The volume integral takes the form

\[ S_{volume}^{GB} = \frac{1}{32\pi^2} \left[ \int d^2\Omega \int dt \int dr \frac{\partial}{\partial r} \left[ 4(e^{2U})' \left( 1 - (e^U R')^2 \right) \right] \right] . \]  

(20)
It shows indeed that the G-B integrand is a total derivative. For this class of metrics we may perform the angular integration and we are left with

\[ S_{volume}^{GB} = \frac{-1}{2\pi} \left[ \int dt \int dr \frac{\partial}{\partial r} \left[ (e^{2U})' \left( 1 - (e^U R')^2 \right) \right] \right] . \]  

(21)
If we work in a Lorentzian spacetime with Minkowski signature there is nothing interesting to say about the integral. The integrand is time independent and the range of integration in \( t \) is infinite. One could identify in real time but this would introduce closed timelike curves and would also lead to singularities on the horizon. However if we use the Riemannian version of the metric,

\[ ds^2 = e^{2U}(r) dt^2 + e^{-2U}(r) dr^2 + R^2(r) d^2\Omega , \]  

(22)
where \( \tau \) is a periodic coordinate, the range of \( \tau \)-integration is constrained from 0 to \( \beta \) by the standard requirement [15]

\[ \beta = \frac{2\pi}{\kappa} . \]  

(23)
This range of integration is finite and not vanishing under the condition that the surface gravity, which is given by

\[ \kappa = \frac{1}{2} \left. \frac{\partial_r g_{tt}}{\sqrt{-g_{rr} g_{tt}}} \right|_{r=r_h} , \]  

(24)
is finite. For such class of metrics one can find a change of coordinates where the apparent singularity at the horizon at \( e^{2U}(r = r_h) = 0 \) is removed.

When the surface gravity does vanish, as for some extreme black holes with zero temperature, the range of integration in \( \tau \) is infinite. When the surface gravity is infinite, as for some extreme black holes with \( a > 1 \), the range of integration in \( \tau \) has to shrink to zero to give a finite result. We do not at present see how to give a straightforward meaning to the G-B integral for such solutions and will limit ourselves at present to the \( a \leq 1 \) solutions only.
Thus we consider a class of metric (22) and the Euclidean time range is $0 \leq \tau \leq \beta$, where

$$\beta = 4\pi \left((e^{2U})'|_{r=r_h}\right)^{-1} .$$

(25)

We will consider first the non-extreme black holes, when this range is finite, i.e. the temperature of the black hole $T = 2\pi \kappa$ is finite, and non-vanishing. There is a regular horizon, and we may integrate over time and get

$$S_{GB}^{volume} = -2 \left((e^{2U})'|_{r=r_h}\right)^{-1} \left[\int_{r_h}^{r_0} dr \frac{\partial}{\partial r} (e^{2U})' \left(1 - (e^{U} R')^2\right)\right].$$

(26)

We consider a compact manifold with $r_h \leq r \leq r_0$. The $r$-integration may also be performed and the result is

$$S_{GB}^{volume} = 2(e^{2U})'|_{r=r_h}^{-1} \left\{\left[(e^{2U})' \left(1 - (e^{U} R')^2\right)\right]_{r=r_h} - \left[(e^{2U})' \left(1 - (e^{U} R')^2\right)\right]_{r=r_0}\right\} .$$

(27)

Thus the volume G-B integral for the black holes with regular event horizon is

$$S_{GB}^{volume} = 2 \left(1 - (e^{U} R')^2\right)|_{r=r_h} - 2(e^{2U})'|_{r=r_h}^{-1} \left[(e^{2U})' \left(1 - (e^{U} R')^2\right)\right]_{r=r_0} .$$

(28)

The second term in this equation is vanishing in the limit when $r_0 \to \infty$ for black holes under consideration, since they are asymptotically flat spaces.

However, the G-B theorem applies only to compact manifolds perhaps with boundaries, which means that we cannot consider such limit. We have to comply with the conditions of the theorem and consider a manifold with a boundary at $r = r_0$.

4 Boundary corrections

To treat the G-B action with boundary terms it is convenient to use differential forms. Chern [12] has considered the Gauss-Bonnet form $\Omega$ of degree $n = 2p$,

$$\Omega = \frac{(-1)^p}{2^{2p} \pi^p p!} \varepsilon_{a_1 \ldots a_{2p}} R^{a_1 a_2} \ldots R^{a_{2p-1} a_{2p}} .$$

(29)

He has shown that this form, which is originally defined in a closed manifold $M^n$ of $n$ dimensions, can be defined in a manifold $M^{2n-1}$ of dimension $2n - 1$. This larger manifold is formed by the unit vectors of the original manifold. Chern has shown that $\Omega$ is equal to the exterior derivative of a differential form $\Pi$ of degree $n - 1$ in $M^{2n-1}$,

$$\Omega = -d\Pi .$$

(30)
The original integral over $M^n$ of the form $\Omega$ is equal to the same integral over a submanifold $V^n$ of $M^{2n-1}$, and, according to the Stoke’s theorem, to the integral over the boundary of $V^n$ of $\Pi$. Since the unit vectors have some isolated singular points, the boundary of $V^n$ corresponds to the singular points of the vector field defined in $M^n$. The integral of $\Pi$ over the boundary of $V^n$ is evaluated and proved to be equal to $\chi$. Thus according to Chern we get

$$S_{volume}^{GB} = \int_{M^n} \Omega = \int_{V^n} \Omega = \int_{\partial V} \Pi .$$  \hspace{1cm} (31)$$

For four-dimensional case the boundary terms have been given in explicit form using differential forms in [7]. Actually by comparing their boundary terms with the expression given by Chern [12] for $\Pi$ one may notice that EGH expression for the Euler number presented in eq. (32) can be rewritten as

$$S_{GB} = S_{volume}^{GB} + S_{boundary}^{GB} = \int_M \Omega - \int_{\partial M} \Pi = \int_{\partial V} \Pi - \int_{\partial M} \Pi .$$  \hspace{1cm} (32)$$

This expression provides an exact cancellation of the volume term by the boundary term under condition that the boundaries of the manifold $M^n$ and of a submanifold $V^n$ of the manifold $M^{2n-1}$ are the same.

For non-extreme black holes, which have regular horizon in Minkowski space and finite temperature, the Euler number equals 2, as explained in Sec. 2. Let us show how the G-B theorem reproduces this result. Equation (32) tells us that if the boundary of a submanifold $V^4$ is at $r = r_0$ as well as at $r = r_h$ we should get

$$S_{GB} = S_{volume}^{GB} + S_{boundary}^{GB} = \int_{r_0} \Pi - \int_{r_h} \Pi - \int_{r_0} \Pi = -\int_{r_h} \Pi .$$  \hspace{1cm} (33)$$

This is equivalent to the statement that the contribution from the volume integral at the outer boundary is exactly compensated by the boundary term, if the boundary term is correct. Basically it is a statement that there is no need to calculate it, one has to ignore the contribution from $r_0$ when calculating the volume integral, as if it would be legal to take the limit $r_0 \to \infty$.

Alternatively rather than using the papers by Chern [12] we could use the G-B theorem as given in [7], where the boundary terms are written explicitly:

$$S_{bound}^{EGH} = -\frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2\theta^{ab} \wedge R^{cd} - \frac{4}{3} \theta^{ab} \wedge \theta^a_c e^c \wedge \theta^b_e) .$$  \hspace{1cm} (34)$$

In equation (34) $\theta^{ab}$ is the second fundamental form of the boundary. Let us specify all the above-mentioned forms for the axion-dilaton black holes under consideration. Our metric (22) corresponds to the following vierbein forms

$$e^0 = e^U(r)dt, \quad e^1 = e^{-U}(r)dr, \quad e^2 = R(r)d\theta, \quad e^3 = R(r)\sin \theta d\phi .$$  \hspace{1cm} (35)$$

The spin-connections are

$$\omega^{01} = 1/2 (e^{2U})' dt, \quad \omega^{21} = e^U R'd\theta, \quad \omega^{31} = e^U R' \sin \theta d\phi, \quad \omega^{32} = \cos \theta d\phi .$$  \hspace{1cm} (36)$$
The curvature 2-forms is given by \( R = d\omega + \omega \land \omega \), see Appendix.

The volume G-B integral for this curvature becomes

\[
S_{\text{GB}}^{\text{volume}} = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^{ab} \land R^{cd} = \frac{1}{4\pi^2} \int_V d(\omega^{01} \land R^{23}),
\]

where

\[
R^{23} = d\omega^{23} + \omega^{21} \land \omega^{13} = \sin \theta \left( 1 - (e^U R')^2 \right) d\theta d\phi.
\]

Using Stoke’s theorem we get

\[
S_{\text{GB}}^{\text{volume}} = \frac{1}{4\pi^2} \int_{\partial V} \omega^{01} \land R^{23}.
\]

Our assumption about the range of integration in Euclidean time, given in eq. (25) in terms of the integral of the one-form is reduced to

\[
\frac{1}{2\pi} \int \omega^{01}|_h = \frac{1}{2\pi} \int_0^\beta d\tau \omega^{01}|_h = \int_0^\beta d\tau T = 1,
\]

where \( T \) is the black hole temperature. Integrating out the angular variables and considering the boundary \( \partial V \) to be at \( r_0 \) and at \( r_h \) we have as before

\[
S_{\text{GB}}^{\text{volume}} = 2 \left( 1 - (e^U R')^2 \right)|_{r_h} - 2(e^{2U})'|_{r=r_h}^{-1} [(e^{2U})' \left( 1 - (e^U R')^2 \right)]_{r=r_0}.
\]

To specify the boundary corrections we may define the metric at the boundary \( r = r_0 \) as

\[
d s_0^2 = e^{2U}(r_0) dt^2 + e^{-2U}(r_0) dr^2 + R^2(r_0) d^2 \Omega.
\]

The non-vanishing spin connections of this metric are \( (\omega^{32})_0 = \cos \theta d\phi \). Hence the second fundamental form \( \theta^{ab} = \omega^{ab} - (\omega^{ab})_0 \) at the boundary \( r = r_0 \) is

\[
\theta^{01} = 1/2(e^U)' dt, \quad \theta^{21} = e^U R' d\theta, \quad \theta^{31} = e^U R' \sin \theta d\phi, \quad \theta^{32} = 0.
\]

The calculation of the EGH boundary term

\[
S_{\text{EGH}}^{\text{boundary}} = -\frac{1}{32\pi^2} \int_{\partial M} \epsilon_{abcd} (2\theta^{ab} \land R^{cd} - \frac{4}{3} \theta^{ab} \land \theta^a \epsilon \land \theta^{eb})\]

is straightforward. We find that the part of the first term in eq. (14) which does not contain \( \theta^{01} \) is completely cancelled by the second term. The part which does contain \( \theta^{01} \) is

\[
S_{\text{EGH}}^{\text{boundary}} = -\frac{1}{4\pi^2} \int_{\partial M} \theta^{01} \land R^{23} = -\frac{1}{4\pi^2} \int_{\partial M} \omega^{01} \land R^{23}.
\]

At \( r_0 \) this expression coincides with the contribution from the outer boundary of the volume integral (with the opposite sign) and we get

\[
S_{\text{GB}} = S_{\text{GB}}^{\text{volume}} + S_{\text{EGH}}^{\text{boundary}} = \frac{1}{4\pi^2} \left( \int_{\partial V} - \int_{\partial M} \right) \omega^{01} \land R^{23} = -\frac{1}{4\pi^2} \left( \int_{\partial M} \omega^{01} \land R^{23} \right)_{r=r_h}.
\]
It is easy to recognize the Chern 3-form
\[ \Pi = \frac{1}{4\pi^2} \omega^{01} \wedge R^{23} \] (47)
and the fact that the non-vanishing contribution is coming only from the inner boundary, when calculating the volume integral. Before we will use the specific form of different black hole metrics we may summarize the result as follows.

We have shown that for a 4-dimensional compact manifold with an outer boundary the G-B action is given by the integral of the Chern 3-form \( \Pi \) at the horizon of the black hole.

Thus for our class of solutions we get the following action
\[ S_{GB} = \int \Pi|_{r_h} . \] (48)
If one takes into account eq. (40), which is equivalent to the condition that the black hole has a finite and non-vanishing temperature, one gets
\[ S_{GB} = \int \Pi|_{r_h} = 2 \left( 1 - (e^U R')^2 \right) |_{r_h} . \] (49)

**5 G-B theorem for the non-extreme black holes**

To verify the relation between the Euler number of the non-extreme black holes and the G-B action presented above we would like to consider different known static axion-dilaton black holes.

i) Non-extreme \( U(1) \) dilaton black holes. The metric is given by eq. (17), where
\[ e^{2U} = (1 - \frac{r_+}{r})(1 - \frac{r_-}{r}) \frac{1-a^2}{1+a^2}, \quad R = r \left( 1 - \frac{r_-}{r} \right) \frac{1-a^2}{1+a^2}, \] (50)
where the mass and the charge of the black hole are
\[ M = \frac{r_+}{2} + \left( \frac{1-a^2}{1+a^2} \right) \frac{r_-}{2}, \quad Q^2 = \frac{r_+ r_-}{1+a^2} . \] (51)
Non-extreme black holes have \( r_+ > r_- \) and the event horizon is at \( r_+ = r_h \). For all of those black holes at the horizon \( e^U|_{r_h} = 0 \) and \( R'(r)|_{r_h} \) is non-singular. Therefore the G-B action as given in eq. (17) becomes equal to 2 since
\[ (e^U R')^2 |_{r=r_h} = 0 . \] (52)

ii) Non-extreme \( U(1) \times U(1) \) axion-dilaton black holes. The metric is
\[ e^{2U} = \frac{(r - r_-)(r - r_+)}{R^2}, \quad R^2 = r^2 - \Upsilon^2 . \] (53)
The mass and the electric and the magnetic charges of the black hole are related to \( r_+, r_- \) and to the axion-dilaton charge is \( \Upsilon \).

As long as the black hole is non-extreme, we can verify that the combination given in eq. (52) vanishes, and again the G-B action (49) equals 2.

Thus, as expected, for non-extreme black holes the G-B theorem with the action given in eq. (49) leads to the correct geometrical answer.

## 6 G-B action for extreme black holes

Extreme black holes have either vanishing temperature (surface gravity at the horizon) or finite temperature. We have agreed not to consider solutions with formally infinite temperature. Consider first extreme black holes with vanishing temperature (surface gravity).

\[
(e^{2U})'|_{r=r_h} = 0. \tag{54}
\]

Such black holes usually have a regular horizon but the range of the \( \tau \) integration is infinite. Again we cannot apply the G-B theorem since the space is not compact in \( \tau \)-direction. One way to fix this problem would be to consider the volume integral to be extended only to \( r = r_h + \epsilon \) simultaneously with restricting the range of \( \tau \)-integration by \( 0 \leq \tau \leq \beta \). Even if ignored the issue of compactness and tried to calculate the G-B volume integral we would find that we are in a situation where the product of \( \beta \times (e^{2U})'|_{r=r_h} = \infty \times 0 \) is not well defined and some additional assumption is required. One simple procedure which we begin by tentatively trying out and then rejecting is to assume that, as before, the product is

\[
\beta (e^{2U})'|_{r=r_h+\epsilon} = 4\pi. \tag{55}
\]

This is in agreement with the prescription above which makes the range of the \( \tau \)-integration finite. If such restriction has been imposed, which may be taken as our definition of the limit to extreme, we still have to examine the behavior of the term \( (e^UR^2)_{extr} |_{r=r_h} \) if one first takes the limit to extreme and then afterwards takes the limit of this expression as we approach the horizon.

### Examples

1) Extreme Reissner-Nordström black holes.

\[
e^{2U} = (1 - \frac{r_+}{r})^2, \quad R = r, \tag{56}
\]

and the mass and the charge of the black hole are

\[
M = Q = r_+ . \tag{57}
\]
The function $R(r)$ is just $r$ and therefore
\[
\left( (e^UR')^2 \right)_{\text{extr}} \big|_{r=r^+} = 0,
\] (58)
and G-B action (48) equals 2 under the condition (55).

ii) Extreme $U(1) \times U(1)$ black holes. Here again eq. (58) is valid and we see that for the extreme $U(1) \times U(1)$ axion-dilaton black holes the G-B action (48) with eq. (55) imposed equals 2.

iii) Extreme $U(1)$ dilaton black holes with $a \leq 1$.
\[
e^{2U} = \left( 1 - \frac{r_+}{r} \right)^2, \quad R = r \left( 1 - \frac{r_+}{r} \right)^2,
\] (59)
where the mass and the charge of the black hole are
\[
M = \frac{r_+}{1 + a^2}, \quad Q^2 = \frac{(r_+)^2}{1 + a^2}.
\] (60)

The temperature of black holes with $a < 1$ vanishes. Therefore we have to repeat the discussion of the problem in the beginning of this section with the same proposal to keep condition given in eq. (55). We assume again that the integration over $\tau$ is from 0 to $\beta$, and $\beta$ is defined in eq. (55). The basic difference between these solutions and the one discussed above is the fact that the function $R'(r)$ at the horizon is singular. The function $e^U$ still vanishes at the horizon, by the definition of the horizon. However, the product $e^U R'$ does not vanish at the horizon anymore. The G-B action (48) becomes
\[
S_{GB} = 2 - \left( (e^UR')^2 \right)_{\text{extr}} \big|_{r=r_h} = 2 - \left( \frac{a^2}{1 + a^2} \right)^2.
\] (61)
This situation is obviously non-satisfactory. The G-B volume integral with the boundary term at outer infinity, given in eq. (48), is neither an integer nor a topological invariant. Indeed, we can see a strong dependence on the metric through the dilaton coupling $a$. One might try to blame our prescription for dealing with $\tau$-integration for these extreme solutions. However, if we consider the stringy case with $a = 1$ we will find that the problem is not due to this prescription. The reason for that is the following. The surface gravity of $a = 1$ extreme $U(1)$ dilaton black hole is finite. Therefore one does not encounter a problem since there exists a natural choice of $\tau$-integration in this case, which is the same as in all non-extreme cases with finite temperature. The resulting G-B action (48) is still not satisfactory,
\[
S_{GB} = 2 - \left( (e^UR')^2 \right)_{\text{extr}} \big|_{r=r_h} = 2 - \left( \frac{a^2}{1 + a^2} \right)^2 \right|_{a=1} = \frac{3}{2}.
\] (62)

\footnote{The only exception is the Reissner-Nordström extreme black hole with $a = 0$.}
To clarify the situation we consider the domain of integration to be such that \( r \) does not reach the horizon. We also add the boundary terms at the horizon.

The boundary near the horizon \( r = r_h + \epsilon \) will be defined by the metric in eq. (42) with \( r_0 = r_h + \epsilon \). In the end of calculations we will send \( \epsilon \) to zero. It is clear that we will have a complete cancellation of the volume term by the boundary term if we use in the boundary term the \( \Pi \)-form of Chern as suggested in \([7]\),

\[
S_{\text{GB}} = S_{\text{GB}}^{\text{volume}} + S_{\text{GB}}^{\text{boundary}} = \int_{\partial V} \Pi - \int_{\partial M} \Pi = 0 , \tag{63}
\]

since now

\[
\partial V = \partial M . \tag{64}
\]

Thus we have found that by adding the inner boundary we get complete agreement with Gauss-Bonnet theorem and that for all extreme black holes with singular horizons \( \chi = 0 \). We conclude that the extreme black holes are very different from the near extreme ones despite the fact that the extreme black holes have a small difference in the metric as compared to the non-extreme ones when the mass is only slightly greater than the extreme value.

Since we were forced to make this crucial step for all \( 0 < a \leq 1 \), it seems natural to look back and to revise our treatment of \( a = 0 \) extreme Reissner-Nordström black holes as well as \( U(1) \times U(1) \) black holes. Indeed, if we stress that by definition any extreme black hole is the one which allows to identify the imaginary time coordinate \( \tau = it \) with any period \( \beta \), we may well relax the assumption about the area of integration in \( \tau \)-direction, given in eq. (55). Under such condition we could get any result from the volume integral supplemented by the outer boundary contribution. The only way to get a unique result is to add the inner boundary. As we have explained above, this will lead unambiguously to \( \chi = 0 \) for the extreme Reissner-Nordström black holes as well as extreme \( U(1) \times U(1) \) black holes.

The calculations given above are entirely consistent with the following geometric picture. In all extreme cases the Killing vector field \( \frac{\partial}{\partial \tau} \) vanishes at no finite point in the manifold. The horizon is at infinite distance. By contrast in the non-extreme case the horizon is at a finite distance and must be included to complete the manifold. However if one does so one now has an entire 2-sphere of fixed points at the horizon. The different values for the Euler number now follow from the general fixed point theorems described in \([6]\). The difference is precisely that between a “cigar” and a “pipette” and is illustrated in Figs. 1 and 2.

7 Entropy of extreme black holes

If we accept the point of view, presented above, about the existence of the inner boundary for all extreme black holes, we have to revise the calculation \([16]\) of the on-shell action and of the
entropy for such black holes, as was first been emphasized by S. W. Hawking in lectures at the
Newton Institute describing a preliminary version of the results reported in [3]. A remarkable
property of \( U(1) \) dilaton black holes with \( 0 < a \leq 1 \) is the following. The boundary conditions
are not relevant for these calculations and the change in the boundary conditions does not affect
the result. The semiclassical on-shell action as well as the entropy vanish for all these solutions
no matter whether we introduce the inner boundary or not. To explain this we will repeat the
calculation of the Lagrangian, for example for extreme dilaton electric black holes.

Our starting point for the calculation of the on-shell action will be the following Lagrangian

\[
I = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( \mathcal{L}_{\text{grav}} + \mathcal{L}_{\text{dil}} + \mathcal{L}_{\text{gauge}} \right). \tag{65}
\]

The gravitational part of the action has a Landau-Lifshitz form

\[
\sqrt{-g} \mathcal{L}_{\text{grav}} = -\sqrt{-g} R + \partial_\mu \sqrt{-g} \omega^\mu, \tag{66}
\]

where the vector \( \omega^\mu \) in the total derivative term in the gravitational Lagrangian (\( K \)-term) is

\[
\omega^\mu = g^{\lambda\rho} \Gamma^\mu_{\lambda\rho} - g^{\lambda\mu} \Gamma^\nu_{\nu\lambda}. \tag{67}
\]

The \( K \)-term removes the second derivatives of the metric from the Lagrangian. The gravitational
part of the action is given by eq. (66), and the vector \( \omega^\mu \) in the total derivative term in the
Lagrangian is given by eq.(67). Eq.(67) can be also given in the form

\[
\omega^\mu = -\frac{1}{\sqrt{-g}} \partial_\lambda \left( \sqrt{-g} g^{\lambda\mu} \right) - g^{\lambda\mu} \left( \partial_\lambda \ln \sqrt{-g} \right). \tag{68}
\]

For these calculations there will be no need to rewrite the volume integral for the total derivative
part in the Lagrangian (second term in eq. (66)) as a surface integral (\( K \)-term). Also we will not
transform the gauge part of the action to a surface integral. It will be sufficient to keep all terms
in a volume integral in what follows.

The dilaton part of the Lagrangian is

\[
\sqrt{-g} \mathcal{L}_{\text{dil}} = 2\sqrt{-g} \partial^\mu \phi \cdot \partial_\mu \phi. \tag{69}
\]

The gauge part of the Lagrangian for the purely electric solution is

\[
\sqrt{-g} \mathcal{L}_{\text{gauge}} = \sqrt{-g} \mathcal{L}_{\text{electr}} = -\sqrt{-g} e^{-2\phi} F^{\mu\nu} F_{\mu\nu}. \tag{70}
\]

The maximally supersymmetric purely electric extreme black holes are described by the following
metric [4]

\[
ds^2 = e^{2U} dt^2 - e^{-2U} d\vec{x}^2. \tag{71}
\]

Before using the field equations, let us calculate the total derivative term in the gravitational part
of the Lagrangian for the ansatz (71). We find using eq. (68) that

\[
\partial_\mu \left( \sqrt{-g} \omega^\mu \right) = -2 \partial_i \partial_i U. \tag{72}
\]
The total gravitational part of the Lagrangian becomes
\[ \sqrt{-g} L_{\text{grav}} = -\sqrt{-g} R - 2 \partial_i \partial_i U. \] (73)

At this stage we may start taking the equations of motion into account. The dilaton for maximally supersymmetric extreme black holes is related to the metric as follows:
\[ \phi = U. \] (74)

The first equation of motion which will be used to calculate the on-shell Lagrangian is the one which relates the scalar curvature to the dilaton contribution,
\[ R - 2 \partial^\mu \phi \cdot \partial_\mu \phi = 0. \] (75)

It follows that, on-shell,
\[ \sqrt{-g} (L_{\text{grav}} + L_{\text{dil}}) = -2 \partial_i \partial_i U. \] (76)

To treat the gauge action we have to use another equation of motion,
\[ \nabla^2 \phi - \frac{1}{2} e^{-2\phi} F^2 = 0. \] (77)

For the electric solution with \( U = \phi \) it leads to
\[ \sqrt{-g} L_{\text{electr}} = 2 \partial_i \partial_i \phi, \] (78)

and the total on-shell Lagrangian becomes
\[ \sqrt{-g} \mathcal{L} = -2 \partial_i \partial_i U + 2 \partial_i \partial_i \phi = 0. \] (79)

The same procedure allows to establish that the on-shell Lagrangian for all \( 0 < a \leq 1 \) black holes vanishes, as well as for pure magnetic ones with \( a = 1 \). This is the only known to us configuration, where the calculation of the entropy of extreme solutions coincides with the limit from non-extreme ones. This can be easily seen in Fig. 3 where the \( a = 1 \) pure electric or pure magnetic solutions are in the corners. The Lagrangian and therefore the action vanishes for \( PQ = 0 \) dilaton black holes. In other words, we get the same result by evaluating the action directly as we get by taking the \( PQ = 0 \) limit of the expression for black holes with regular horizon.

Note that in the process of calculation of the on-shell Lagrangian for maximally supersymmetric extreme dilatonic black holes, we never faced the problem of going to Euclidean signature, choosing a proper gauge for the vector potentials, and thinking about boundary surfaces (the horizon versus infinity). All of those problems, however, arise for the non-maximally supersymmetric \( U(1) \times U(1) \) dilaton black holes (Reissner-Nordström black holes can be viewed as a particular case of \( U(1) \times U(1) \) dilaton black holes [4]). The corresponding configurations on the Fig. 3 correspond to the sides of the diamond.

In [16], the generalization of the Gibbons-Hawking method of the calculation of the Euclidean action for dilaton black holes was presented. One starts with non-extreme black holes characterized by some finite temperature and surface gravity, and performs the calculation of the on-shell
action in Euclidean signature by compactifying the Euclidean time coordinate. It turns out that one can express the total action as a surface integral and evaluate the contribution from the extrinsic curvature and the gauge terms from the outer boundary. As a final step, the extremal limit was considered, and the result was:

\[ S_{\chi=2} = \frac{1}{4} A = \pi(M^2 - \Sigma^2) = \frac{1}{2} \pi |z_1^2 - z_2^2| = 2\pi |PQ| , \]

(80)

where \( z_1, z_2 \) are the central charges of extreme black holes defined in [2] and \( P \) and \( Q \) are the electric and magnetic charges of the 2 \( U(1) \) fields. One can see this on the border of the shaded region on Fig. 3. This corresponds to the topology of the near extreme Euclidean black hole, presented in Fig. 1. If however we would like to associate with the extreme black holes the topology, presented in Fig. 2, this will affect our calculation of the Euclidean action for all extreme configurations which are not maximally supersymmetric, for which \( |z_1^2 - z_2^2| \neq 0 \).

The idea is that if we regard the spacetime as having an inner boundary the action will have to be augmented by an extra boundary term and this will affect the calculation of the action and hence the entropy [3]. If we add an inner boundary for these configurations, we must subtract the value of \( S \) which comes from the outer boundary. This can be easily understood by looking at the calculation of the entropy in the Sec. 2 of [16]. The result is that now entropy vanishes,

\[ S_{\chi=0} = \frac{1}{4} \pi |z_1^2 - z_2^2| - \frac{1}{2} \pi |z_1^2 - z_2^2| = 0 . \]

(81)

This is essentially the generalization of the result found originally by Hawking for \( \Sigma = 0 \) and described in [3] and confirmed by Teitelboim [4]. We find this conclusion particularly satisfying because one does not expect a soliton state to have non-vanishing entropy. It also fits in with our interpretation of the functional integral about these backgrounds as giving a Witten Index.

### 8 The Witten index

In this section we shall discuss the relevance of the previous results to calculations of the Witten index for supergravity theories. To begin with recall that for any supersymmetric theory one may define the Witten index \( W(\beta) \) by

\[ W(\beta) = Tr_{\mathcal{H}} e^{-\beta H} (-1)^F , \]

(82)

where \( H \) is the Hamiltonian, \( F \) the fermion number operator and the trace is taken over the suitable Hilbert space of theory. Note that if the factor \((-1)^F\) is replaced by 1 we obtain the Gibbs partition function \( Z(\beta) \) at the temperature \( T = \beta^{-1} \).

If one represents \( W(\beta) \) by a formal path integral then both the bosonic and fermionic fields should be taken to be periodic with period \( \beta \), in contrast to the Gibbs partition function for which the bosonic fields are periodic but the fermionic fields are antiperiodic. Formally at least, if the Hilbert space \( \mathcal{H} \) consists of states which are paired by a supersymmetry operator \( Q \) then
$W(\beta)$ should be independent of $\beta$ and its numerical value should be strictly zero. If one adds the vacuum state which is invariant under all supersymmetry operations $Q$ then the value of $W(\beta)$ will be unity.

The case we shall mainly be interested in is when $\mathcal{H}$ consists of a short supermultiplet of soliton states together with their quantum fluctuations. The lowest, purely bosonic, soliton state is taken to be an extreme black hole. For our purposes an classical extreme black hole solution may be taken to be one which allows one to identify the imaginary time coordinate $\tau = it$ with any period $\beta$. In the semiclassical expansion, which corresponds to large $\beta$, one may attempt to evaluate $W(\beta)$ by a functional integral over matter fields, including the metric $g_{\mu\nu}$, which are periodic in imaginary time with period $\beta$. The dominant contribution might be anticipated to arise from a classical extreme black hole solution whose mass is $M$. The main result from the old and more recent discussions of extreme black holes is that, even for the case of extreme black holes in Einstein-Maxwell theory, the classical euclidean action $I$ works out to be given by

$$I = M\beta ,$$

where $M$ is the total mass. To obtain this result one must, as we have seen, pay careful attention to boundary condition and boundary terms. In particular one must treat the extreme black holes as being essentially different from non-extreme black holes. Thus one should not evaluate the action $I$ by taking a limit. If one does so one misses an essential inner boundary term not present in the non-extreme case. We refer the reader to [3], [4] for more details. It follows from (83) that the classical entropy $S$ of extreme black holes vanishes,

$$S = 0 .$$

This result has already been obtained for extreme $U(1)$dilaton black holes [4]. It is also true that for extreme black holes, the topology of the Riemannian section is $S^1 \times R \times S^2$ and hence that the Euler characteristic $\chi$ vanishes.

$$\chi = 0 .$$

Note that, as we have seen, if one wishes to evaluate $\chi$ using the Gauss-Bonnet theorem one must pay special attention to boundary terms. In particular there is an inner boundary contribution in the extreme case which is absent in the non-extreme case.

Extreme black holes are known to be invariant under global supersymmetry transformations, generated by Killing spinor fields $\epsilon(x)$ which tend to constant at infinity

$$\lim_{|x| \to \infty} \epsilon(x) \to \epsilon .$$

In a theory with $N$ supersymmetries, $N'$ of which are unbroken the soliton states will fall into short supermultiplets of dimension $2^{2(N-N')}$. For example, in $N = 2$ supergravity there are $2N = 4$ possible supersymmetry generators. $2N'$ of them (corresponding to the rest frame, say) leave the solution invariant and $2(N - N') = 2$ generate a supermultiplet of 4 states with spins $(0^+, +1/2, -1/2, 0^-)$. Each state in the multiplet has, at the tree level, the same mass. Because
the entropy $S = 0$ no further degeneracy is introduced by tree level thermodynamic effects such as are encountered for non-extreme black holes.

We turn now to the calculations of fluctuations around the classical soliton backgrounds. To calculate such fluctuations, for example in perturbation theory using functional determinants, one must adopt suitable boundary conditions. As we indicated above the appropriate choice is for the fermions to be periodic in imaginary time with period $\beta$. In fact, it seems rather plausible, though we have not calculated this directly, that if we adopt anti-periodic boundary conditions for the fermions, the one-loop terms would diverge on the horizon because of the well-known Tolman redshifting in a background gravitational field with $g_{00} \neq$ constant.

In fact rather than treat the fluctuations in components it is more efficient to treat the fluctuations using superfield techniques as has recently been demonstrated by one of us \cite{[17]}, \cite{[2]} and by C. Hull \cite{[18]}. In that work the arguments were presented that at higher than one-loop order no appropriate supersymmetric counterterms exists consistent with the background supersymmetry. The basic idea is that the existence of Killing vectors allows to choose a coordinate system where the solution is independent on some bosonic coordinate of the space. The presence of a Killing spinor in turn implies the possibility to choose a coordinate system in superspace such that the solution is independent of some direction in the anticommuting coordinates of the superspace. Actually the argument is valid not only for the counterterms but for any corrections to the effective action. This improvement is based on the fact that the manifestly supersymmetric Feynman graph rules\footnote{For $N = 4$ theory for which manifestly supersymmetric rules are not known, one may use the Feynman rules which have only $N = 2$ supersymmetry manifest. It is remarkable that for the black holes with some unbroken $N = 4$ supersymmetry, the corresponding manifestly realized $N = 2$ supersymmetry also has some unbroken part.} always lead to the expressions local in fermionic variables. Therefore, the total expressions, describing the Feynman supergraphs become local in anticommuting variables after some intermediate integrations. This property is sufficient to prove the vanishing of the superspace integrals over the whole superspace, when the integrand is independent on one or more of the fermionic coordinates of the superspace. This result follows directly from Berezin rules of integration over anticommuting variables. One has to take into account that the arguments, given above are formal. They correspond to the statement, which follows from the formal Ward Identities and are valid in any order of perturbation theory. It is well known from experience with non-abelian gauge theories, that the analysis of such formal Ward Identities is necessary but not sufficient condition for absence of quantum corrections. The first place where we encounter the problem with the arguments, given above is related to one-loop corrections. The so-called conformal anomaly is expressed in terms of the integral not over the total superspace, but as the integral over the chiral superspace only\footnote{In addition, other supersymmetric expressions may exists, which are presented by integrals not over the total superspace but only part of it. Such terms require special considerations, since they may or may not include the integration over the Killing directions. Hopefully, there is a finite number of such terms as different from the infinite number of generic corrections to the effective action, described by the full superspace integrals. We are grateful to K. Stelle for this observation.}

We may study the subtleties in supersymmetric non-renormalization theorems related to the
one-loop conformal anomalies. For $N = 4$ supersymmetric theories such anomalies are absent in some versions of the theory. For $N = 2$ they exist. The corresponding integral over the chiral superspace does not automatically vanishes for the extreme Reissner-Nordström configuration. However, the part of the anomaly, which is local in $x$-variables is given by the Euler characteristic $\chi$ of the background solution. In particular, the only one-loop counterterm in all pure supergravities is proportional to the Euler characteristic $\chi$ and the coefficient is related to the anomaly. For a general solution of the Einstein-Maxwell equations the divergent counterterm does exist, since both the anomaly coefficient is non-vanishing and the Euler characteristic $\chi$ is non-vanishing. The vanishing of the Euler characteristic for extreme black hole solutions is thus a necessary condition for the consistency of the picture we are proposing and we find it very satisfying that this condition is indeed met.

The result is that the contribution to $W(\beta)$ from the spin zero soliton state is given, to all orders in perturbation theory, by

$$W(\beta) = \exp(-M\beta) .$$

(87)

Note in particular the argument outlined above indicates that the maximal central charge condition, valid for $N = 2$ supersymmetry,

$$M^2 = \frac{\mathcal{P}^2}{\kappa^2} ,$$

(88)

where $\kappa^2 = 4\pi G$, $P$ is the magnetic charge and $G$ is Newton’s constant is maintained in all orders by virtue of the fact that neither the mass nor the magnetic charge receive quantum corrections. Summing over the 4 soliton states we than obtain the answer

$$W(\beta) = 0 .$$

(89)

The argument sketched above applies to the Witten index in the case that $\mathcal{H}$ is the one-soliton Hilbert Space. In the next section we will attempt to extend it to the case of more than one soliton.

9 Multi-Black Holes

In addition to classical solutions representing single isolated black holes there exist multi-black hole solutions representing an arbitrary number, $k$ say, of black holes admitting Killing spinors \cite{19}. In what follows we shall, for completeness describe the solutions and their moduli spaces for arbitrary values of the dilaton coupling constant $a$ despite the fact that only the values $a^2 = 0, \frac{1}{3}, 1$ and 3 have yet been identified as arising from a supergravity theory. The spacetime metric is

$$ds^2 = -V_{\frac{1-a^2}}^2 dt^2 + V_{\frac{1+a^2}}^2 d\mathbf{x}^2 ,$$

(90)

where $V = V(\mathbf{x})$ is given by

$$V = 1 + \sum_{i=1}^{j=k} \frac{\mu_i}{|\mathbf{x} - \mathbf{x}_i|} .$$

(91)
The points $x_i$ correspond to extreme horizons.

Just as in the case of a single extreme black hole in the case of the multi black holes the Euler characteristic of the corresponding Euclidean solutions vanishes. The easiest way to see this is to use the fact that the Killing vector field $\frac{\partial}{\partial \tau}$ where $\tau = it$ has no fixed points. Then the result follows by Hopf’s theorem about everywhere non-vanishing vector fields. On a manifold with boundary (which may have more than one connected component) Hopf’s theorem states that the sum of the indices of a vector field which is either transverse to the boundary or every-where parallel to the boundary equals to Euler number of the manifold. In our case we consider the four-dimensional sub-manifold $N$ of the Euclidean multi-black hole spacetime (with $\tau$ identified with some arbitrary period) for which $0 < \epsilon_1 \leq V^{-1} \leq \epsilon_2 < 1$ for some $\epsilon_1$ and $\epsilon_2$. If $\epsilon_2$ is close to 1 then the set of points $V^{-1} = \epsilon_2$ may be regarded as the component of the boundary near infinity. If $\epsilon_1$ is very small then the set of points for which $V^{-1} = \epsilon_1$ gives us an inner boundary with $k$ connected components. Since the length of the vector field $\frac{\partial}{\partial \tau}$ is $\sqrt{\epsilon_1}$ it is everywhere non-vanishing in the sub-manifold $N$. Thus the indices vanish. The vector field also clearly lies in the three-dimensional submanifolds $V = \text{constant}$. Thus it is parallel to the boundary.

Of course we could also apply the Gauss-Bonnet theorem with boundary as we did earlier for the case of one black hole but the calculation would be more involved because of the more complicated form of the metric and the additional internal boundaries, one for each horizon.

Thus just as in the case of a single extreme black hole the counterterms coming from the Gauss-Bonnet action must vanish on these backgrounds. The entropy is also clearly zero. This is completely consistent with our idea that the functional integral corresponds to the Witten index. Now since the Witten index is purely topological it is unchanged by changes of the temperature parameter $\beta$ it should be possible to evaluate it for very large $\beta$, i.e. at very low energies. However the low energy quantum behaviour of solitons is believed to be governed by quantum mechanics on their “moduli space” or space of zero-modes. The Witten index should then be related to a topological index of the relevant quantum mechanics. Since we have seen that the Witten index vanishes, partly because of the vanishing of the Euler number, it would seem that the relevant index on the moduli space must also vanish. In what follows we shall investigate to what extent this is true.

To begin with consider the geometry and topology of the moduli spaces. If the black holes are identical we have $\mu_1 = \mu_2 = \ldots = \mu_k = \mu$. For fixed $\mu$ the solutions then depend on $3k$ parameters or “moduli ”. The space of parameters is just the “moduli space” $\mathcal{M}_k$. In the present case the moduli space is given by the $k$ position vectors $x_i$ subject to the condition that no two coincide. Moreover since the holes are identical, permutations of the $x_i$’s give the same configuration. Thus the moduli space corresponds to the configuration space $C_k(R^3)$ of $k$ unordered distinct points in $R^3$, i.e.

$$\mathcal{M}_k \equiv C_k(R^3) \equiv \left( (R)^3 - \Delta \right) / S_k .$$

(92)

where $\Delta$ is the “diagonal set ” for which at least two of the $x_i$’s coincide and $S_k$ is the permutation group on $k$ letters.
In what follows we shall be more interested in the “relative moduli space” $\mathcal{M}_k^{\text{rel}} = C_k(R^3)/R^3$. This is obtained by identifying configurations which differ by an overall translation. Thus

$$\mathcal{M}_k \equiv \mathcal{M}_k^{\text{rel}} \times R^3 .$$

(93)

In the simplest non-trivial case, $k = 2$, it is easy to see that the relative moduli space is given by the non-zero position vector

$$\mathbf{r} = x_2 - x_1$$

(94)

with $\mathbf{r}$ and $-\mathbf{r}$ identified. Thus topologically

$$\mathcal{M}_2^{\text{rel}} = R \times \mathbb{RP}^2 ,$$

(95)

where $\mathbb{RP}^2$ is the 2-sphere with opposite points identified. This is a non-orientable manifold with Euler characteristic equal to one. In fact this is general, i.e.

$$\chi(\mathcal{M}_k^{\text{rel}}) = 1 .$$

(96)

To see that this is true we use the Poincaré polynomial $P(t, \tilde{\mathcal{M}}_k)$ of $\tilde{\mathcal{M}}_k$, the universal covering space of $\mathcal{M}_k$. This is the space obtained if we do not identify the points, corresponding to $\tilde{C}(R^3)_k$. We have, by definition, for any manifold

$$P(t, \mathcal{M}) = \Sigma_{p=0}^{\dim \mathcal{M}} t^p B_p(\mathcal{M}) ,$$

(97)

where the Betti numbers $B_p(\mathcal{M}) = \dim H^p(\mathcal{M}, R)$ of the $p$’th cohomology group of $\mathcal{M}$. Clearly

$$\chi(\mathcal{M}) = P(-1, \mathcal{M})$$

(98)

For $C_k(R^n)$, $n \geq 3$ we have

$$P(t) = (1 + t^2)(1 + 2t^2) \ldots (1 + (k - 1)t^2) ,$$

(99)

whence

$$\chi(\tilde{C}(R^n)_k) = k! .$$

(100)

Now, by considering a simplicial decomposition of a manifold $\mathcal{M}$ and lifting it to any covering space $\tilde{\mathcal{M}}$ one sees that

$$\chi(\mathcal{M}) = \chi(\tilde{\mathcal{M}})/|\Gamma| ,$$

(101)

where $|\Gamma|$ is the order of the group $\Gamma$ of “deck transformations” such that $\mathcal{M} \equiv \tilde{\mathcal{M}}/\Gamma$. In the present case $\Gamma = S_k$ the permutation group on $k$ letters and therefore because the order of the permutation group $S_k$ is $k!$ it follows that

$$\chi(C(R^n)_k) = 1 .$$

(102)

The moduli carries a natural metric obtained by restricting the kinetic energy functional to the finite dimensional submanifold of the configuration space of static solutions. Unlike the topology
the metric depends on the dilaton coupling constant $a$. Consider, for simplicity, the simplest case $k = 2$. The metric is given by

$$ds^2 = \gamma(r)dr^2$$

where

$$\gamma(r) = 1 - (3 - a^2)\frac{M}{r} + 4(1 + M\frac{1 + a^2}{2r})^{\frac{3-a^2}{1+a^2}} - 4,$$

and $r = |r|$. Thus $\tilde{M}_2^{rel}$ is asymptotically flat (as $r \to \infty$). If $a^2 \leq \frac{1}{3}$ this is joined by a throat to an asymptotically conical region (as $r \to 0$). If $a^2 = \frac{1}{3}$ there is no conical region, just an infinitely long throat or “drain”. If $a^2 \geq \frac{1}{3}$ but $a^2 \leq 3$, including the string case $a^2 = 1$ there is a conical singularity at finite distance at $r = 0$. If $a^2 = 3$ the metric is flat. The qualitative form of the relative moduli space in the case $a^2 < \frac{1}{3}$ is illustrated in figure 4 as a surface of revolution obtained by suppressing one of the angular coordinates. Note that the metric has been calculated assuming that the only fields contributing to the velocity dependent forces are gravity, the vector particle and the dilaton. Interestingly the critical value of the dilaton coupling constant, $a^2 = \frac{1}{3}$ corresponds to the reduction of 5-dimensional Einstein-Maxwell theory to 4-dimensions. It has recently been shown that the solutions in 5-dimensions, which represent extreme black strings are complete and every where non-singular [24]. These 5-dimensional Einstein-Maxwell solutions may also be regarded as solutions of 5-dimensional supergravity theory.

Quantum mechanics on these moduli spaces, but not with reference to supersymmetry or a Witten index has been considered by a number of authors [23]. Because they are not Kähler as one expects if they have $N = 2$ supersymmetry let alone HyperKähler which one expects if they have $N = 4$ supersymmetry it is not obvious how to extend these calculations to the supersymmetric case. By contrast the moduli spaces of BPS monopoles are HyperKähler [26] and this plays an important role in Sen’s work on S-duality [27]. The fact that the moduli space of extreme Reissner-Nordstrom black holes and of Kaluza-Klein monopoles is not HyperKähler has been a long-standing puzzle. It may signal a breakdown of supersymmetry. It may equally well indicate the need to consider some alternative form of supersymmetric quantum mechanics on the moduli space. A suggestion of this kind has been attributed to Witten in [28]. In the case of $a^2 = 3$ Paul Townsend has pointed out that one may regard the multi-solitons as being solutions of $N = 8$ supergravity with half the maximum number of supersymmetries. It is then reasonable to expect a moduli space with $N = 8$ supersymmetries to be not just HyperKähler but flat. This may also explain why the neutral fivebranes have a flat moduli space [29].

In the light of the proposal of this paper that it is the Witten index which is important it seems natural to consider a topological theory such as the de Rham complex. Since the moduli space is non-compact however there are apparently, in contrast to the familiar case of a closed manifold, few general results relating the existence of harmonic forms to the topology of the manifold. For that reason we will try to construct them explicitly. To that end we consider the simplest case $k = 2$ and to look for square-integrable harmonic forms. We may trivially factor out the dependence on the centre of mass coordinate so we consider $\mathcal{M}_2^{rel}$. It seems reasonable to restrict attention to spherically symmetric harmonic forms.
The one-form
\[ \omega_1 = \frac{1}{r^2} \, \frac{dr}{\sqrt{\gamma}} \] (105)
and its Hodge dual
\[ \omega_2 = * \omega_1 = \sin \theta d\theta d\phi \] (106)
are both closed, co-closed and hence harmonic. They will be square integrable with respect to the volume form of the metric \( \gamma dr^2 \) provided
\[ \int_0^\infty \frac{\gamma^{-\frac{1}{2}}}{r^2} \, dr < \infty . \] (107)
This condition will hold as long as
\[ a^2 < \frac{1}{3} . \] (108)
It is a theorem that a square integrable harmonic form on a geodesically complete Riemannian manifold must be both closed and co-closed [20]. Thus we need not have checked the closure and co-closure if this condition holds.

Under inversion on the 2-sphere \( \{ r, \theta, \phi \} \rightarrow \{ r, \pi - \theta, \phi + \pi \} \), which corresponds physically to interchanging the two black holes, we have
\[ \omega_1 \rightarrow \omega_1 \] (109)
but
\[ \omega_2 \rightarrow -\omega_2 . \] (110)

Thus only \( \omega_1 \) remains well defined on the identified space \( \mathcal{M}_2 \). Now
\[ \omega_1 = d\omega_0 \] (111)
where the harmonic function or zero-form \( \omega_0 \) is given by
\[ \omega_0 = \int \frac{dr}{\sqrt{\gamma}} . \] (112)
The harmonic function \( \omega_0 \) is not square integrable. It is however the limit of a square integrable one-form. Thus from the point of view of \( L^2 \) cohomology it is trivial. It is clear that \( \omega_2 \) is not the exterior derivative of a smooth one-form. This indicates that the square integrable or \( L^2 \) cohomology of \( \{ \mathcal{M}_2, ds^2 \} \) the covering space equipped with its metric \( ds^2 \) is, in the case that \( a^2 < \frac{1}{3} \), given entirely by forms on the 2-sphere. If we pass to the identified space \( \mathcal{M}_2 \) we find that the two form \( \omega_2 \) does not descend because it is odd under the antipodal map.

It would seem to follow from this that if we regard the black holes as indistinguishable bosons then the \( L^2 \) cohomology of the moduli space is trivial in the case \( a^2 < \frac{1}{3} \). Therefore if for example in the case of Einstein-Maxwell theory (i.e. \( a^2 = 0 \)) our identification of the Witten index with the index of the de Rham complex is correct then the Witten index should vanish both for single extreme holes as well as for multi-extreme-black holes. The case \( a^2 = 3 \) is also cohomologically trivial. The case \( a^2 = 1 \) is is unclear because of the conical singularity on the moduli space.
10 Supersymmetric Sigma Model with Black Hole Target Space

As the first step in direction to investigate the supersymmetric quantum mechanics on moduli space of extreme dilaton black holes we will study here the supersymmetric sigma models in the black hole target space. Some four-dimensional extreme $a = 1$ dilaton black holes with electromagnetic fields [13] have been recently reinterpreted as solutions with unbroken supersymmetry of the effective action of heterotic string theory in critical dimension. Generic space-time supersymmetric dilaton black hole is related to a supersymmetric sigma model with torsion when the fundamental dilaton is not constant. Indeed, we are interested in space-time supersymmetric solutions of heterotic string theory in ten-dimensional target space, which contains in particular the dilatino $\lambda$ transformation rule. The supersymmetry rules are (our notation for this section are given in [30]).

\begin{align}
\delta \psi_\mu &= (\partial_\mu - \frac{1}{4} \Omega_\mu^{ab} \gamma_{ab}) \epsilon, \\
\delta \lambda &= (\gamma^\mu \partial_\mu \phi + \frac{1}{4} H_{\mu\nu\rho} \gamma^{\mu\nu\rho}) \epsilon, \\
\delta \chi &= -\frac{1}{4} F_{\mu\nu} \gamma^{\mu\nu} \epsilon.
\end{align}

One can see from eq. (114) that bosonic configuration with vanishing dilatino can have also vanishing supersymmetry variations of dilatino $\delta \lambda(\epsilon) = 0, \epsilon \neq 0$. However, in presence of dilaton, depending on three-dimensional space coordinate $\vec{x}$ one has to have a non-vanishing torsion $H$. Thus, as long as we are considering dilaton supersymmetric black holes with non-trivial dilaton field, we have geometries with torsion from the point of view of supersymmetric sigma model. It is interesting that if we look on solution with constant dilaton, we may afford to have unbroken space-time supersymmetry without torsion. For example one may choose the metric to be self-dual stringy ALE solutions [31]. This solution indeed forms a target space of the supersymmetric sigma model with HyperKähler geometry. Stringy multi-monopole solutions [32] which are T-dual to a special class of ALE solutions, formed by multi-center metrics of Gibbons-Hawking type, do have a non-trivial dilaton as well as non-vanishing torsion and therefore the geometry is not HyperKähler. Still multi-monopoles provide a target space for a sigma model with world-sheet (4, 4) supersymmetry [33] which is non-HyperKähler target space.

Magnetic black holes were uplifted in [34], [30]. The details of supersymmetric uplifting are described in [35], [36]. When supersymmetric solution of four-dimensional $N = 4$ supergravity is reinterpreted as a solution of a heterotic string effective action with unbroken supersymmetry it always relates the four-dimensional vector field of the charged $a = 1$ black hole to a non-diagonal component of the metric, which is equal to a 2-form field $B_{\mu4}$. The supersymmetric sigma model of such theory always has torsion and the geometry cannot be HyperKähler! In

\footnote{One can avoid torsion for the dilaton depending on some null coordinate $\gamma^u \partial_u \phi \epsilon = 0$, with Killing spinors satisfying $\gamma^u \epsilon = 0$, like in the case of gravitational waves.}
some cases T-duality transformation can be performed on the target space as well as on world-

sheet variables which is consistent with supersymmetry and removes torsion. However, generic

space-time supersymmetric dilaton $a = 1$ black hole before any duality transformation is related
to a supersymmetric sigma model with torsion when the fundamental dilaton is not constant.
More details about supersymmetry versus T-duality can be found in [36].

The $(4,0)$ supersymmetric sigma model for magnetic uplifted dilaton black hole [13] was disc-

cussed in [34] and presented explicitly for the throat limit. We will present here the $(4,1)$ super-
symmetric sigma model for magnetic non-Abelian uplifted dilaton black hole without considering
the throat limit.

It was explained in [30] that the magnetic black hole solution has to be supplemented by the

non-Abelian vector field to avoid Lorentz and supersymmetry anomaly. The uplifted magnetic

black hole geometry corresponds to a decomposition of the manifold $M^{1,9} \rightarrow M^{1,5} \times M^4$ and
the tangent space $SO(1,9) \rightarrow SO(1,5) \times SO(4)$. The curved manifold is four-dimensional $M^4$,
it has Euclidean signature and it has a self-dual curvature for a torsionful connection. The
six-dimensional manifold $M^{1,5}$ is flat. Four-dimensional manifolds are known to have interesting
properties when considered as target space for supersymmetric sigma models. The supersymmetric
sigma models $(p,q)$ where $p$ is the number of left and $q$ is the number of right supersymmetries,
are described in detail in [37] for $p,q \leq 2$. An extensive list of references can also be found there.
In addition to that for our purpose it will be sufficient to use the description of supersymmetric
sigma models with $p$ or $q$ equal 4 as given by Hull and Witten [38] and Howe and Papadopoulous
[40]. Our method of investigation of the world-sheet supersymmetry of the sigma model in the
black hole target space will be very close to the method which was applied by Callan, Harvey
and Strominger [33] for the analysis of the world-sheet supersymmetry $(4,4)$ of the sigma model
in the target space of the fivebrane. In what follows we are going to show that the non-Abelian
magnetic black hole target space provides $(4,1)$ supersymmetry of the sigma model. Note that
we are suggesting a world-sheet description of the magnetic dilaton black holes which differs from
the one used in [39]. The main difference is in the role which the the vector field plays. For us the
four-dimensional abelian $U(1)$ vector field becomes a non-diagonal component of the metric as
well as of the two-form field in a compactified dimension $x^4$. In addition we have a non-Abelian
$SU(2)$ vector which will appear in the interaction with the left-handed world-sheet fermions.

The generic $(4,1)$ supersymmetric sigma model [40] with manifestly realized $(1,1)$ supersym-

metry in the four-dimensional Euclidean space is given by

$$ I_{(4,1)} = \int d^2z d^2\theta (G_{\mu\nu} + B_{\mu\nu}) D_+ X^\mu D_- X^\nu , \quad (116) $$

where the unconstrained $(1,1)$ superfield is given by

$$ X^\mu(x^\mu, \theta^+, \theta^-) = x^\mu(z) + \theta^+ \lambda_+^a(z) e_a^\mu(x) - \theta^- \lambda_-^a(z) e_a^\mu(x) - \theta^+ \theta^- F^\mu(z) . \quad (117) $$

Action (116) has additional 3 supersymmetries for some special backgrounds $G_{\mu\nu}(X), B_{\mu\nu}(X)$
which admit covariantly constant complex structures $J_{\tau}^{\mu\nu} = e_a^\mu J_{\tau}^a e_b^\nu$ with special properties.
Let us show that the non-Abelian uplifted magnetic black hole \[^{[30]}\]
\[
\begin{align*}
\text{d}s^2_{(10)} &= \text{d}t^2 - e^{4\phi} \text{d}x^2 - (\text{d}x^4 + V_i \text{d}x^i)^2 - \text{d}x^4 \text{d}x^4, \\
B_{(10)} &= \frac{1}{2} B_{\mu\nu} \text{d}x^\mu \wedge \text{d}x^\nu = -V_i \text{d}x^i \wedge \text{d}x^4,
\end{align*}
\]
\[
\begin{align*}
\partial_i \partial_i e^{2\phi} &= 0, \quad e^{2\phi} = 1 + \sum_s \frac{2M_s}{|\vec{x} - \vec{x}_s|}, \quad 2\partial_i V_{ij} = \pm \varepsilon_{ikl} \partial_k e^{2\phi} \\
A^i_k = \Omega_{k-}^{i m} &= 2\delta_{k [i} \partial_{m]} e^{-2\phi} \quad A_4^{ij} = \Omega_{4-}^{ij} = \pm \varepsilon_{ijk} \partial_k e^{-2\phi}.
\end{align*}
\]

has all the corresponding properties. This solution has flat six dimensions and non-flat four-dimensional Euclidean space \((\vec{x}, x^4)\). The vierbein basis is \(e^j_i = e^{2\phi} \delta^j_i, \quad e^4_4 = V, \quad e^4_i = 0\). Thus, one can use the general form of the \((1, 1)\) supersymmetric action \((116)\) with the non-trivial four-dimensional part of the metric and 2-form field as given in \((118)\) and \(\mu, \nu; \alpha, \beta = 1, 2, 3, 4\).

\[
G_{ij}(X) = -e^{4\phi(X)} \delta_{ij} - V_i(X)V_j(X), \quad G_{4i}(X) = B_{4i}(X) = -V_i(X), \quad G_{44} = -1 \quad \text{(119)}
\]

where the functions \(\phi\) and \(V_i\) of the \((1, 1)\) superfield \(X\) are the same as the corresponding functions of \(x\) in \((118)\). Let us remind the reader that the non-Abelian vector field was obtained via spin embedding into the gauge group and therefore it is a quantity derived from the given values of the metric (vierbein) and a 2-form field. Also the second torsionful connection as well as both curvatures are derived quantities: if one has the metric and the 2-form field, one can derive other characteristics of the configuration. In the sigma model the same property is seen as follows. If one would like to perform the integration over the world-sheet fermionic coordinates one would recover the supersymmetric action, in which however \((1, 1)\) supersymmetry is not realized manifestly anymore. The action in addition to the first term, depending on ordinary \(x\), will contain terms with covariant derivatives acting on right-moving fermions \(\lambda_+^a\) and left moving fermions \(\lambda_-^a\) as well as the term quartic in fermions. In the covariant derivatives of the right(left)-moving fermions one has to use the torsionful spin-connection \(\Omega^{\mu+}_{ab}(\Omega^{\mu-}_{ab})\). The quartic term contains the torsionful curvature which has the exchange properties for the positive and negative torsionful spin connections under condition \(dH = 0\).

The origin of additional 3 world-sheet supersymmetries is related directly to the fact that our black hole has space-time unbroken supersymmetries \([30]\). Eqs. \((113)\) have a zero mode with
\[
(1 \pm \gamma_5)\epsilon_\pm \equiv (1 \pm \gamma_1 \gamma_2 \gamma_3 \gamma_4)\epsilon_\pm = 0. \quad \text{(120)}
\]

For one choice of the sign in solution one gets a constant spinors with positive chirality \(\epsilon_+\), for the other choice one gets a constant negative chirality spinor \(\epsilon_-\).

The difference between black hole target space and that of the fivebrane \([33]\) at this stage is the following. For the fivebrane one of the torsionful spin connection is self-dual and the other one is anti-self-dual. This means that for one choice of sign in the solution one has simultaneously
\[
(\partial_\mu - \frac{1}{4} \Omega^{ab}_{\mu+} \gamma_{ab})\epsilon_+ = 0, \quad (\partial_\mu - \frac{1}{4} \Omega^{ab}_{\mu-} \gamma_{ab})\epsilon_- = 0. \quad \text{(121)}
\]
Therefore one promotes \((1, 1)\) supersymmetric sigma model to the \((4, 4)\) supersymmetric one. The procedure is described in \([33]\). One constructs 3 self-dual complex structures out of space time Killing spinors of one type of chirality and the 3 anti-self-dual complex forms out of the spinors of the opposite chirality. For the black hole target space we have only \((\partial_\mu - \frac{1}{4}\Omega_{\mu+}^a \gamma_{ab})\epsilon_+ = 0\) from the gravitino part of unbroken space-time supersymmetry and therefore the self-dual properties of \(\Omega_+\) spin connection.

\[
\Omega_{+i}^{jk} = 2\Omega_{+i}^{jl} = -2\epsilon_{ijkl}\partial_k e^{-2\phi}.
\] (122)

The other property of the fivebranes \((\partial_\mu - \frac{1}{4}\Omega_{\mu-}^a \gamma_{ab})\epsilon_- = 0\) is not valid for the black hole \((118)\). The \(\Omega_-\) spin connection is presented in eq. \((118)\) and it is clear that it is not (anti)self-dual. Thus we can construct only one type of 3 complex structures, which allows to promote the world-sheet supersymmetry to \((4, 1)\) only. The reason for that is the following. One of the properties of the 3 complex structures \(J_{\mu r}^a\) is that they have to be covariantly constant with respect to either \((\partial_\mu - \frac{1}{4}\Omega_{\mu+}^a \gamma_{ab})\) for promoting to \((4, 1)\) or with respect to \((\partial_\mu - \frac{1}{4}\Omega_{\mu-}^a \gamma_{ab})\) to promote supersymmetry to \((1, 4)\). In the black hole case only the first possibility is available. Thus we conclude that the supersymmetric sigma model action in the black hole target space has \((4, 1)\) world-sheet supersymmetry\(^8\). The black hole provides an excellent example of a constrained extended \((4, 1)\) superfield \(X_{\mu}(x^\mu, \theta^+_{\bar{a}}, \theta^-_{\bar{r}}, \bar{\theta}^+_{\bar{a}}, \bar{\theta}^-_{\bar{r}}), r = 1, 2, 3,\) satisfying the following constraint \([40]\)

\[
D_{r+}X^\mu = J^\mu_{r \nu}D_{\bar{0}+}X^\nu.
\] (123)

Three complex structures \(J_{\mu r}^a\) are constructed from the Killing spinors of unbroken space-time supersymmetry of the black hole configuration. They satisfy all conditions required by Howe and Papadopolous \([40]\) for the \((4, 1)\) supersymmetric sigma model. In particular, the complex structures obey the quaternion algebra, they are covariantly constant with respect to torsion-full \(\Omega_+\)-connection and the holonomy group is \(Sp(1)\). However, the manifold is obviously not HyperKähler since there is a torsion \(H_{3ij} = \pm \frac{1}{3}\epsilon_{ijk}\partial_k e^{2\phi}\).

The relation of supersymmetric sigma model in the target space given by uplifted magnetic black hole manifold to the geometry of the moduli space still has to be understood. However, we believe that the sigma model presented above clearly indicates that there is no reason for the \(a = 1\) dilaton black holes to rely on HyperKähler manifold for \(N = 4\) supersymmetric quantum mechanics. Rather one may expect the manifold of the type described above where extended supersymmetry of quantum fluctuation exists due to the existence of unbroken space-time supersymmetry on manifold with torsion.

\(^8\)Our analysis does not exclude that there are more supersymmetries which may still be hidden, however \((4, 1)\) is certainly available.
11 Towards $N = 4$ supersymmetric Quantum Mechanics of Black Holes

To confirm the conclusion of the previous section we may dimensionally reduce the supersymmetric sigma model on the world sheet to the one-dimensional theory on the world line. It is useful for this purpose to perform the integration over the fermionic coordinates in (116) and eliminate auxiliary fields. This will also permit us to compare our dimensionally reduced action on black hole manifolds with the supersymmetric quantum mechanics of monopoles in $N = 4$ Yang-Mills theory [41]. One gets the following Lagrangian

$$\mathcal{L}_2 = \left( G_{\mu \nu} + B_{\mu \nu} \right) \partial_\tau x^\mu \partial_\tau x^\nu + i \lambda_+^a (\nabla_\tau (\pm) \lambda_+^a) - i \lambda_-^a (\nabla_\tau (\pm) \lambda_-^a)$$

- $\frac{1}{2} R^{(+)}_{ab,cd} \lambda_+^a \lambda_+^b \lambda_-^c \lambda_-^d$. \hspace{1cm} (124)

The right(left)-handed fermions $\lambda_+^a (\lambda_-^a)$ have covariant derivatives with respect to torsionful spin connections $\Omega_+ (\Omega_-)$. The torsionful curvatures $R_\pm = d \Omega_\pm + \Omega_\pm \wedge \Omega_\pm$ have the exchange properties

$$R_{ab,cd}^{(+)} = R_{cd,ab}^{(-)}.$$ \hspace{1cm} (125)

Those exchange properties generalize the symmetry properties of the torsionless Riemann tensor $R_{ab,cd} = R_{cd,ab}$. For our non-Abelian black hole the torsionful curvatures are given either by the Yang-Mills field strength or by the gravitational torsionful curvature, depending on the position of the indices. One is expressed through another by eq. (125). Because of that one could rewrite the last term in (124) as

$$\frac{1}{2} R^{(+)}_{ab,cd} \lambda_+^a \lambda_+^b \lambda_-^c \lambda_-^d = \frac{1}{4} R^{(+)}_{ab,cd} \lambda_+^a \lambda_+^b \lambda_-^c \lambda_-^d + \frac{1}{4} R^{(-)}_{ab,cd} \lambda_-^a \lambda_-^b \lambda_+^c \lambda_+^d.$$ \hspace{1cm} (126)

We would like to stress that for the black holes $R_{ab,cd}^{(+)} \neq R_{cd,ab}^{(+)}$ as well as $R_{ab,cd}^{(-)} \neq R_{cd,ab}^{(-)}$ which is another manifestation of the fact that we deal with manifolds with torsion.

Lagrangian (124) has (4, 1) supersymmetry. Four right-handed ones relate $x$ with $\lambda_+$ and one left-handed relates $x$ with $\lambda_-$, as explained above in terms of the action depending on superfields. Lagrangian (124) is invariant up to terms which are total derivatives in $z$ or $\bar{z}$. Therefore if we will perform dimensional reduction of the Lagrangian (124) by requiring $x, \lambda_+, \lambda_-$ to be independent on $\sigma$ and dependent on $\tau$ we will keep all abovementioned supersymmetries of the Lagrangian preserved. The reduced lagrangian is

$$\mathcal{L}_{bh}^{QM} = \frac{1}{2} G_{\mu \nu} \partial_\tau x^\mu \partial_\tau x^\nu + \frac{i}{\sqrt{2}} \lambda_+^a (\nabla_\tau (\pm) \lambda_+^a) - \frac{i}{\sqrt{2}} \lambda_-^a (\nabla_\tau (\pm) \lambda_-^a)$$

- $\frac{1}{4} R^{(+)}_{ab,cd} \lambda_+^a \lambda_+^b \lambda_-^c \lambda_-^d - \frac{1}{4} R^{(-)}_{ab,cd} \lambda_-^a \lambda_-^b \lambda_+^c \lambda_+^d$. \hspace{1cm} (127)

The torsion is gone from the bosonic part of the action, since the term $B_{\mu \nu} \partial_\tau x^\mu \partial_\tau x^\nu$ is not available anymore. However, the fermionic part of one-dimensional action (127) keeps track of torsion, it
remains in both covariant derivatives as well as in the fact that there are two different curvature
tensors. Covariant derivatives on fermions are
\[
(\nabla_{\tau}^{(+)} \lambda^+)^a \equiv \partial_{\tau} \lambda^a_+ + \partial_{\tau} x^\mu \Omega_{+\mu}^{\ ab} \lambda_{+b}
\]
\[
(\nabla_{\tau}^{(-)} \lambda^-)^a \equiv \partial_{\tau} \lambda^-_a + \partial_{\tau} x^\mu \Omega_{-\mu}^{\ ab} \lambda_{-b}.
\]
(128)

In notation of [30] we have \(\Omega_{\pm} = \omega \mp \frac{3}{2} H\), where \(\omega\) is metric compatible spin connection and the
torsion for the black hole manifold is given by the 3-form
\[
H = \pm dx^4 \wedge dx^i \wedge dx^j \epsilon_{ijk} \partial_k e^{2\phi}.
\]
(129)
which is closed since the dilaton \(e^{2\phi}\) satisfies harmonic equation of motion \(\partial_k \partial^k e^{2\phi} = 0\)
\[
dH = \mp dx^4 \wedge dx^i \wedge dx^j \wedge dx^l \epsilon_{ijk} \partial_l \partial_k e^{2\phi} = 0 .
\]
(130)
The last condition \(dH = 0\) is necessary for the proof of the exchange properties of curvatures
(125) and for the proof of supersymmetry of the action (127).

Thus we have got an one-dimensional supersymmetric action which has at least \(N = 4\) super-
symmetry or more and codifies nicely all information about the uplifted non-Abelian magnetic \(a = 1\) extreme dilaton black hole (118) whose manifold is not HyperKähler. This action cer-
tainly represents some possibility to work out the supersymmetric quantum mechanics related to
the black hole. The quantization of this one-dimensional action most likely will not lead to any
problems and we hope to perform it and report about the results elsewhere.

It is rather instructive to compare our action (127) with the one, associated with the the
supersymmetric quantum mechanics of monopoles in \(N = 4\) Yang-Mills theory in the form given
in [41].
\[
\mathcal{L}_{\text{mon}}^{QM} = \frac{1}{2} G_{\mu \nu} \partial_{\tau} x^\mu \partial_{\tau} x^\nu - \frac{i}{2} \bar{\lambda}^a \gamma_0 (\nabla_{\tau} \lambda)^a - \frac{1}{2} R_{ab,cd} \bar{\lambda}^a \lambda^b \lambda^c \lambda^d.
\]
(131)
This one-dimensional supersymmetric action is a corollary of the analysis of the boson and fermion
zero modes in the monopole moduli space and it is supposed to describe the supersymmetric quantum
mechanics of the monopoles in their moduli space. The metric here is HyperKähler, the
covariant derivative of fermions is metric compatible and there is one torsionless Riemann tensor
in the four-fermion coupling.

In our case the one-dimensional supersymmetric action (127) is available and looks rather
unique from the point of view of the possibilities to have any \(N = 4\) supersymmetric quantum
mechanics, related to the black hole geometry (118). The relation of this action to the moduli
space approach as well as the quantization of this theory remain to be investigated.
In this section we explore approach to extreme black holes as a special quantum system. Having revised our view on the entropy of all extreme black holes we are in a better position to discuss and develop the idea, suggested by Linde and described in the Appendix of [2] about the splitting of the extreme black holes. It was suggested there that the probability of this process is suppressed by the factor $e^{\Delta S}$, where $\Delta S$ is the change of the total entropy of the system of black holes.

The simplest example of splitting concerns $U(1) a = 1$ purely magnetic (or purely electric) extreme black holes. They have vanishing entropy (and vanishing area of horizon). Therefore they presumably can split (or diffuse quantum mechanically, without any energy release) into smaller purely magnetic or electric black holes, since $\Delta S = 0$ in this case.

A more complicated example considered in [16] was the splitting of the extreme $U(1) \times U(1)$ electric-magnetic black hole into a purely magnetic and a purely electric one. At the time that this example was studied it was qualified as follows: such bifurcation is forbidden classically but could in principle occur in a quantum-mechanical process. The reason for this is that classically, if cosmic censorship holds, and appropriate energy conditions then a theorem of Hawking states that black holes cannot bifurcate and their area cannot decrease. Setting aside the purely classical concept of cosmic censorship, if one identifies one quarter of the area of the event horizon with entropy then the fact that the entropy of extreme $U(1) \times U(1)$ electric-magnetic black hole was previously considered to be $S = 2\pi|PQ|$ as explained above and that the entropy of the products of the splitting was vanishing according to the same formula, it would also follow that splitting was forbidden thermodynamically. It seems to us that while one might be prepared to give up cosmic censorship in the quantum or near quantum regime, and while one can certainly not expect the energy conditions to hold in all circumstances in the quantum regime one would expect the results of thermodynamics to remain true. Now that we have come to conclusion that the entropy of all extreme black holes vanishes, the splitting of black holes discussed above starts with zero entropy state and ends with zero entropy state. Such processes can be viewed as decays of nuclei or elementary particles. We will present some details of this splitting here since this is the simplest one and we will later compare it with the black hole splitting in the theory with dilaton coupling $\sqrt{3}$ where the process of splitting one black hole into two will be accompanied by the release of energy.

Thus we consider first the decay of one extreme $a = 1$ $U(1) \times U(1)$ electric-magnetic black hole into pure electric $U(1)$ and pure magnetic $U(1)$. It can be described by

$$\begin{align*}
(P, Q) &\rightarrow (P, 0) + (0, Q) .
\end{align*}$$

(132)

The extreme electric-magnetic black hole has the following relations between parameters:

$$\begin{align*}
M_{\text{initial}} &= \frac{|P| + |Q|}{\sqrt{2}} , \\
\Sigma &= \frac{|P| - |Q|}{\sqrt{2}} ,
\end{align*}$$

(133)
which gives

\[ M^2 + \Sigma^2 - P^2 - Q^2 = 0 . \]  

(134)

For simplicity consider the situation with both charges positive and \( P \geq Q \). For this case the solution above has one quarter of supersymmetries of \( N = 4 \) supergravity unbroken, which corresponds to \( \epsilon^3_4 \) and \( \epsilon^+_4 \), see ref. [2] for details. The parameters of the daughter black holes are related to those of the parent as

\[
M_{\text{final}} = M_1 + M_2, \quad M_1 = \Sigma_1 = \frac{|P|}{\sqrt{2}}, \\
\Sigma = \Sigma_1 + \Sigma_2, \quad M_2 = -\Sigma_2 = \frac{|Q|}{\sqrt{2}}.
\]  

(135)

and

\[
M_1^2 + \Sigma_1^2 - P^2 = 0, \\
M_2^2 + \Sigma_2^2 - Q^2 = 0.
\]  

(136)

For positive charges this means that the magnetic solution by itself has one half of unbroken supersymmetries \( \epsilon^3_4, \epsilon^+_4 \) and \( \epsilon^{-}_4, \epsilon^{-}_4 \). The electric solution by itself has one half of unbroken supersymmetries \( \epsilon^3_4, \epsilon^+_4 \) and \( \epsilon^{-}_4, \epsilon^{+}_4 \). However, everywhere in space the total configuration has the same unbroken supersymmetry as the parent solution, i.e. one quarter of \( N = 4 \) unbroken supersymmetries given by \( \epsilon^+_4, \epsilon^+_4 \).

These black holes are in equilibrium with each other, since the attractive force between them vanishes due to supersymmetry [2]. Indeed, let us consider Newtonian, Coulomb and dilatonic forces. The force between two distant objects of masses and charges \((M_1, Q_1, P_1, \Sigma_1)\) and \((M_2, Q_2, P_2, \Sigma_2)\) is

\[
F_{12} = -\frac{M_1 M_2}{r_{12}^2} + \frac{Q_1 Q_2}{r_{12}^2} + \frac{P_1 P_2}{r_{12}^2} - \frac{\Sigma_1 \Sigma_2}{r_{12}^2}.
\]  

(137)

The dilatonic force is attractive for charges of the same sign and repulsive for charges of opposite sign. Using the relations (133) for the masses and dilaton charges in terms of the magnetic and electric charges \( P_1 = P, P_2 = 0, Q_1 = 0, Q_2 = Q \), we see that \( F_{12} \) vanishes.

There is no release of energy during this splitting since \( M_{\text{initial}} - M_{\text{final}} = 0 \) but it is also not suppressed since there is no change in entropy, according to our new picture.

Now consider the new case, where indeed we deal with fission. Consider the theory

\[
S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R + 2\partial^\mu \phi \cdot \partial_\mu \phi - e^{-2a\phi} F_{\mu\nu} F^{\mu\nu} \right).
\]  

(138)

with arbitrary dilaton coupling \( a \). The usual extreme Reissner-Nordström black hole with \( |P| = |Q| \) and vanishing dilaton solves equation of motion of this theory (138) for arbitrary \( a \). For \( a = 0 \)
this solution has one half of unbroken supersymmetry when embedded into $N = 2$ supergravity \cite{12}. For $a = 1$ an extreme Reissner-Nordström black hole with $|P| = |Q|$ and vanishing dilaton solves the equations of motion of the theory with two vector fields and has one quarter of unbroken supersymmetries when embedded into $N = 4$ supergravity \cite{2}. The solution which was called a $[U(1)]^2$ black holes (to stress that one vector is electric and another one is magnetic) also solves the equation of motion for the axion field, which shows that the axion is constant. However, it is actually a solution of the system of fields with only one vector field, if we do not care about having $F \ast F = 0$. Special case of $\sqrt{2}|Q| = \sqrt{2}|P| = M$ and $\Sigma = 0$ just means that $F^2 = 0$ and $\phi = 0$.

The nice property of the solution where the dilaton is vanishing in this non-trivial way, is that it solves equations of motion of the theory (138) for arbitrary values of $a$. The dilaton eq.

$$\nabla^2 \phi - \frac{1}{2} a F^2 = 0$$

(139)

is solved for $\phi = \text{const}$ since $F^2 = 0$. The gravitational and vector equations are not affected by the value of $a$ since $e^{-2a\phi} = 1$. Thus we have established that our no dilaton $\sqrt{2}|Q| = \sqrt{2}|P| = M$ Einstein-Maxwell dyon solves the eqs. of motion of the bosonic Lagrangian (138). Now let us focus on the $a = \sqrt{3}$ case. This can be embedded into dimensionally reduced 5d supergravity with the dilaton being $g_{55}$. The solutions with unbroken supersymmetry of this theory have to saturate the following bound \cite{19}

$$M \geq \frac{1}{\sqrt{1 + a^2}} \sqrt{Q^2 + P^2} = \frac{1}{2} \sqrt{Q^2 + P^2}.$$  (140)

Our Einstein-Maxwell dyon with $\sqrt{2}|Q| = \sqrt{2}|P| = M$ does not saturate the bound. Indeed for $|Q| = |P| = \text{the bound is}$

$$M \geq \frac{1}{\sqrt{2}} |Q|,$$

(141)

whereas the extreme Einstein-Maxwell dyon state of the same theory has the mass $M_{\text{dyon}} = \sqrt{2}|Q|$. Thus there is a gap between the supersymmetric state with $M_{\text{susy}} = \frac{1}{2} \sqrt{Q^2 + P^2}$ and $M_{\text{dyon}} = \sqrt{Q^2 + P^2}$. We can use the fact that the extreme Einstein-Maxwell dyon state has the energy higher than the supersymmetric ground state of this theory.

We consider the decay of one extreme Einstein-Maxwell $|Q| = |P|$ dyon into the supersymmetric monopole and supersymmetric electropole of $a = \sqrt{3}$ theory.

It can be described by

$$(P, Q) \rightarrow (P, 0) + (0, Q).$$  (142)

The extreme electric-magnetic black hole has the following relations between parameters:

$$M_{\text{initial}} = \frac{|P| + |Q|}{\sqrt{2}} = \sqrt{2}|Q|,$$

$$\Sigma = \frac{|P| - |Q|}{\sqrt{2}} = 0,$$

(143)
which gives
\[ M^2 + \Sigma^2 - P^2 - Q^2 = 0. \] (144)

The parameters of the daughter black holes are related to those of the parent as
\[ M_{\text{final}} = M_1 + M_2 = |Q|, \quad M_1 = \frac{1}{\sqrt{3}} \Sigma_1 = \frac{|P|}{2}, \]
\[ \Sigma = \Sigma_1 + \Sigma_2 = 0, \quad M_2 = -\frac{1}{\sqrt{3}} \Sigma_2 = \frac{|Q|}{2}. \] (145)

and
\[ M_1^2 + \Sigma_1^2 - P^2 = 0, \]
\[ M_2^2 + \Sigma_2^2 - Q^2 = 0. \] (146)

The magnetic black hole by itself and the electric one by itself each have unbroken supersymmetry. However the configuration with both monopole and electropole most likely does not have common Killing spinors. This has to be investigated. If indeed this turns out to be the case, it would be very close to what we have observed during the decay of the \( a = 1 \) dyon. Monopole far away from electropole has an increased number of unbroken supersymmetries and the same for the electropole. However the additional Killing spinors for monopole are different from those for the electropole. Therefore the total number of unbroken supersymmetries of the configuration after decay remains the same as before decay.

The gap in energy between the initial 1-black-hole state and final 2-black-hole state is
\[ M_{\text{initial}} - M_{\text{final}} = (\sqrt{2} - 1)|Q|. \] (147)

The efficiency of the energy release is
\[ \frac{M_{\text{initial}} - M_{\text{final}}}{M_{\text{initial}}} = (1 - \frac{1}{\sqrt{2}}) \sim 0.3. \] (148)

The black holes which are the product of decay are not in an equilibrium with each other, since the attractive gravitational force between them is overcompensated by the repelling dilaton force. Consider again eq. (149) for \( Q_1 = 0 \) and \( P_2 = 0 \).
\[ F_{12} = -\frac{M_1 M_2}{r_{12}^2} - \frac{\Sigma_1 \Sigma_2}{r_{12}^2}. \] (149)

Using the relations for the masses and dilaton charges in terms of the magnetic and electric charges we see that
\[ F_{12} = -\frac{M_1 M_2}{r_{12}^2} + a^2 \frac{M_1 M_2}{r_{12}^2} = \frac{Q^2}{2r_{12}^2}. \] (150)
The monopole and electropole repel strongly in this theory, since the monopole and electropole have dilaton charges of the opposite sign. Interesting enough, the original state in this example was such that there was no dilaton anywhere. After the decay the monopole fragment carries away a dilaton charge and the electropole fragment carries away an exactly the same value but the opposite sign dilaton charge. The total process looks like the inverse annihilation of the dilaton charge.

$$\Sigma_{\text{monopole}} = -\Sigma_{\text{electropole}} = \frac{\sqrt{3}|P|}{2} = \frac{\sqrt{3}|Q|}{2}. \quad (151)$$

It is energetically advantageous to create opposite dilaton charges in this theory and to separate simultaneously the electric solution from the magnetic one. In our next example we will consider the state before fission which has an arbitrary dilaton charge. Still we will see that the release of energy during the fission proceeds by the fragmentation of the original state with dilaton charge imbalance:

$$\Sigma_{\text{monopole}} = -\Sigma_{\text{electropole}} + \frac{\sqrt{3}|P|}{2} - \frac{\sqrt{3}|Q|}{2}. \quad (152)$$

The fission described above actually will proceed starting with more general extreme electromagnetic solution of $a = \sqrt{3}$ theory as quoted in [45]. The extreme solution in isotropic coordinates is given in terms of two independent parameters, electric and magnetic charges, $Q$ and $P$ or in terms of their special combinations $\alpha, \beta$.

$$ds^2 = -e^{2U}dt^2 + e^{-2U}d\bar{x}^2, \quad (153)$$

where the metric and the scalar field are given by

$$e^{-4U} = CD, \quad \exp\left(-\frac{4\phi}{\sqrt{3}}\right) = \frac{D}{C}. \quad (154)$$

The simplest form of the vector field is given in terms of the magnetic potential and dual magnetic potential $\tilde{A}_\phi$.

$$A_\phi = P \cos \theta, \quad \tilde{A}_\phi = Q \cos \theta. \quad (155)$$

The vector field part of the action, as usual for dilaton black holes can be written as $F^* \tilde{F}$. Equation of motion require $d\tilde{F} = 0$ and the existence of the dual magnetic potential.

Functions $C, D$ are given by

$$D = \left(1 + \frac{\beta}{r}\right)^2 + \frac{\beta^2(\beta + \alpha)}{r^2(\beta - \alpha)}, \quad (156)$$

and

$$C = \left(1 - \frac{\alpha}{r}\right)^2 - \frac{\alpha^2(\beta + \alpha)}{r^2(\beta - \alpha)}. \quad (157)$$

\[^9\text{In theories where the dilaton is massive the phenomenon of repelling of black holes after the splitting was observed in [44].}\]
The mass and the scalar charge are given by

\[ M = \frac{1}{4}(\beta - \alpha) , \quad \Sigma = \frac{\sqrt{3}}{4}(\alpha + \beta) . \] (158)

The electric and magnetic charges are

\[ Q^2 = \frac{\beta^3}{4(\beta - \alpha)} , \quad P^2 = \frac{\alpha^3}{4(\alpha - \beta)} . \] (159)

Thus

\[ M^2 + \Sigma^2 - P^2 - Q^2 = 0 . \] (160)

Solution is manifestly invariant under duality transformation

\[ P \to Q , \quad \phi \to -\phi , \quad \Sigma \to -\Sigma , \quad \alpha \to -\beta . \] (161)

The mass of the extreme solutions is then given in terms of electric and magnetic charges by

\[ M = \frac{1}{2}(P^2 + Q^2)^{\frac{1}{2}} . \] (162)

The electric charge of this solution does not have to be equal to the magnetic one. There is also a non-constant dilaton now. Only if we use a particular case of this solution with \(|P| = |Q|\) will we get the previously discussed case of the Einstein-Maxwell dyon with

\[ M = \frac{1}{2}(P^2 + Q^2)^{\frac{1}{2}} = \sqrt{2}|Q| . \] (163)

This generic extreme solution is not supersymmetric unless either \(P = 0\) or \(Q = 0\). It will also undergo an explosive fission into monopole \(M_1 = \frac{1}{\sqrt{3}} \Sigma_1 = \frac{|P|}{2}\) and electropole with \(M_2 = -\frac{1}{\sqrt{3}} \Sigma_2 = \frac{|Q|}{2}\). This time the repelling force will be

\[ F_{12} = -\frac{M_1 M_2}{r_{12}^2} + a^2 \frac{M_1 M_2}{r_{12}^2} = \frac{|PQ|}{2r_{12}^2} . \] (164)

The existence of this solution gives a richer picture of fission. The initial state has arbitrary relation between the electric and magnetic charge and a non-vanishing dilaton. The products of decay are monopole and electropoles with some magnetic and electric charges which also are not equal to each other.

One may compare the general case \(P^2 = -\left(\frac{2}{3}\right)^3 Q^2\) with the one discussed above. The gap between \((P,Q)\) state and \((P,0), (0, Q)\) state is

\[ M_{\text{initial}} - M_{\text{final}} = \frac{1}{2}(P^2 + Q^2)^{\frac{1}{2}} - \frac{1}{2}(|P| + |Q|) . \] (165)
The efficiency of the energy release

\[
\frac{M_{\text{initial}} - M_{\text{final}}}{M_{\text{initial}}} = 1 - \frac{|P| + |Q|}{(P^2 + Q^2)^{\frac{1}{2}}}
\]

(166)

is a function of \((\frac{\alpha}{\beta})\). The maximum is reached when the solution becomes a familiar Reissner-Nordström one at \(\alpha = -\beta\). The maximum was found before to be about 0.3.

One can consider the situation with the electric charge which is proportional to some elementary electric charge \(e\) times an integer and the magnetic charge such that the Dirac quantization condition is imposed.

\[
Q = en, \quad P = \frac{m}{2e}, \quad PQ = \frac{nm}{2},
\]

(167)

where \(n, m\) are some integers. The spectrum of excited states of the \((P, Q)\) black hole will look like

\[
M(m, n) = \frac{1}{2} \left( \left( \frac{m}{2e} \right)^{\frac{2}{3}} + (en)^{\frac{2}{3}} \right)^{\frac{4}{7}}.
\]

(168)

Any such state with arbitrary integers \((m, n)\) will eventually decay into the black holes with charges \((P = \frac{m}{2e}, 0)\), and \((0, Q = en)\) with the release of energy, described above. A process of further splitting without the release of energy might then continue until \(m\) elementary monopoles \((P = \frac{1}{2e}, 0)\) and \(n\) elementary electropoles \((0, Q = e)\) are produced. The system \((168)\) has many discrete levels and transitions between levels may also be possible, mostly with release of energy.

One can as well consider the situation that from the beginning we have a \([U(1)]^2\) black hole as in [2]. The electric charge belongs to one \(U(1)\) group and the magnetic one to the other. In such case the quantization of charges may be different.

The splitting of atomic nucleus resulting in the release of large amounts of energy is called fission. In what follows we are going to estimate the amount of energy released during the black hole decay. Suppose that the black hole has a Planckian mass \(M_p \sim 1.2 \times 10^{19} \text{ GeV} \sim 2 \times 10^{-5} \text{ g}\). In the maximum efficiency case described above the energy released during the black hole fission is about \(6 \times 10^{15} \text{ erg} \sim 120 \text{ kg of TNT}\). To compare the efficiency of black hole fission with that of Uranium, suppose that we have not just one Planckian scale black hole of \(2 \times 10^{-5}\) g but many black holes, with a total mass 1 kg. The fission of one kilogram of extreme black holes will result in energy release of about 6 megatons of TNT, to be compared with 20 kilotons of TNT during the explosion of one kilogram of \(^{235}\text{U}\). The gravitational black hole fission bomb can be 300 times more efficient than the atomic one!

### 13 Conclusion

We have performed the calculation of the Euler number of the axion-dilaton black holes. For all non-extreme solutions, which (in Minkowski space) have regular horizons, the calculation of
the Euclidean Gauss-Bonnet volume integral with the outer boundary terms always results in the value \( \chi = 2 \) as expected from the fact that the topology of such manifolds is \( R^2 \times S^2 \). The corresponding configurations are presented in Fig. 1.

Extreme dilaton \( U(1) \) black holes do not have regular horizons, at least according to the canonical choice of spacetime metric. Our strategy has been firstly to calculate the Euclidean Gauss-Bonnet volume integral using the condition \( \beta \kappa = 2\pi \). Note that the temperature of these black holes \( T = \frac{\kappa}{2\pi} = \frac{1}{4\pi r^2} \left( \frac{r_+ - r_-}{r_+} \right)^{1+a^2} \) is zero for \( a < 1 \) or finite for \( a = 1 \).

Despite the fact that the horizon of extreme solutions is singular, the volume integral with the outer boundary term turned out to be finite, but the result was \( 2 - \left( \frac{a^2}{1+a^2} \right)^2 \). It is neither an integer nor a topological invariant, since there is an obvious dependence on the metric. It was natural therefore to remove the singular horizon and to add an inner boundary near the horizon. We have found that if we use the correct boundary term as given in [7], the volume term is completely cancelled by the boundary term. This may be interpreted as discontinuity in the value of the Euler number for extreme (\( \chi = 0 \), see Fig. 2) and non-extreme (\( \chi = 2 \), see Fig. 1) black holes.

Extreme Reissner-Nordstrom and \( U(1) \times U(1) \) black holes have regular horizons. The calculation of the Euclidean Gauss-Bonnet volume integral with the outer boundary terms results in the value \( \chi = 2 \). One can obtain this result by imposing the condition that the integration over the Euclidean time is performed in the region from zero till \( \beta \) where the combination \( \beta \kappa \) is held fixed and equal to \( 2\pi \). We should stress, however, that this calculation has an intrinsic ambiguity, related to the fact that the temperature of such black holes vanishes, \( T = 0 \). Therefore the volume integral is not well defined. Imposing the inner boundary for extreme black holes seems to be the only consistent way to be in agreement with the general definition that extreme black holes are such as to allow one to identify the imaginary time coordinate \( \tau = it \) with any period \( \beta \).

The study of the topological invariants of the extreme black holes performed in this paper, shows the crucial importance of the boundary corrections to the Euler number of such geometries. We believe that at present when the creation of such black holes in some background vector fields seems to be possible [9, 10], it is useful to understand the topological properties of these new black holes in addition to their thermodynamical properties.

As mentioned above, the topology of extreme Reissner-Nordstrom black holes as well as their entropy was studied recently [3, 4]. Discussions of this issue at the Durham conference “Quantum Concepts of Space and Time” have shown that there is a complete agreement about the existence of the inner boundary for all extreme black holes, which leads to the vanishing Euler number as well as the entropy.

A careful analysis of the boundary conditions for the extreme black holes in the path integral used in this paper in the context of Gauss-Bonnet theorem has been recently developed further. We have come to the conclusion that the supersymmetric non-renormalization theorem for extreme black holes with unbroken supersymmetries implies the absence of quantum corrections to a
properly defined Witten Index. It is essential for this interpretation that fermion fields are taken to be periodic in imaginary time. This is only possible for extreme holes. For non-extreme holes the fermion fields must be taken to be anti-periodic in imaginary time, consistent with the functional integral giving a thermodynamic partition function.

In an attempt to study the possible approaches to supersymmetric quantum mechanics of the fluctuations in the extreme black hole background we have studied a supersymmetric sigma model in the black hole target space. The non-K"ahlerian character of the corresponding geometry was explained by the existence of the torsion.

The idea of possible splitting and joining of extreme black holes suggested in \cite{2} acquires an additional support with the new understanding that the entropy of all (and not only $U(1)$ dilaton extreme black holes) vanishes. The process of splitting and joining of various extreme black holes is not forbidden anymore by the second law of black hole physics. A discussion of this appears in \cite{3}. We have studied some examples of fission of extreme dilaton black holes which are energetically advantageous. The existence of a gap between the states in a specific model indicates that fission might be possible with an explosive release of energy.

To conclude, there have been several developments recently in the theory of charged black holes. A new picture seems to be emerging: the relation between extreme and non-extreme black holes resembles the one between the massless $m=0$ and massive $m>0$ non-Abelian gauge fields. Even a very light massive vector field has 3 degrees of freedom and the corresponding field theory is non-renormalizable. The massless field had 2 degrees of freedom and the corresponding field theory is renormalizable. The limit $m \to 0$ does not coincide with the massless theory $m=0$. Extreme black holes with the mass strictly equal to the central charge $M=|Z|$ are very different from the non-extreme ones with $M>|Z|$ and their limit $M \to |Z|$ is in some important respects very different from the extreme case.

As with massless Yang-Mills fields, extreme black holes seem to have features which open some hope to make progress in understanding the underlying quantum theory.

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Appendix

We have started the calculation of the G-B Lagrangian

\[
L_{GB} = R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2
\]

for the class of metrics \(ds^2 = e^{2U}(r) d\tau^2 + e^{-2U}(r) dr^2 + R^2(r) d^2\Omega\) by using Mathematica. The Lagrangian was obtained in the form

\[
L_{GB} = \frac{4}{R^2} \left( [(e^{2U})' R']^2 - (e^{2U})'' (e^{2U})' (e^{2U})'' + 2 (e^{2U}) (e^{2U})' R' (r) R'' \right).
\]

In particular for extreme dilaton black holes with arbitrary coupling \(a\) presented in eq. (59) the G-B Lagrangian was obtained in the following form:

\[
\sqrt{g} L_{GB} = -\frac{2 r_+ (1 - \frac{r_+}{r})^{-1+\frac{a^2}{1+a^2}}}{(1 + a^2)^2 r^2}
\]

\[
+ \frac{2 r_+ (1 - \frac{r_+}{r})^{-1+\frac{4a^2}{1+a^2}} \left( (1 - \frac{r_+}{r})^{\frac{a^2}{1+a^2}} + a^2 r_+ (1 - \frac{r_+}{r})^{-1+\frac{4a^2}{1+a^2}} \right)^2}{(1 + a^2)^2 r^2},
\]

which can be simplified to the following form:

\[
\sqrt{g} L_{GB} = -\frac{2 r_+ (1 - \frac{r_+}{r})^{-1+\frac{a^2}{1+a^2}}}{(1 + a^2)^2 r^2} \left( 1 - \left( 1 - \frac{r_+}{r} + \frac{a^2 r_+}{(1 + a^2) r} \right)^2 \right).
\]

However, it was easy to observe looking on eq. (170) that \(\sqrt{g} L_{GB}\) can be rewritten in the form

\[
\sqrt{g} L_{GB} = -4 \frac{\partial}{\partial r} \left( (e^{2U})' \left( 1 - (e^U R')^2 \right) \right),
\]

since we expected to find some total derivative. In this form we have presented it in eq. (19) of the paper.

Let us show how this expression can be used for Schwarzschild black hole where \(e^{2U} = 1 - 2m/r\) and \(R(r) = r\):

\[
\chi = \frac{1}{32\pi^2} \left[ \int d^4 x \sqrt{-g} \left( R_{\mu\nu\lambda\delta} R^{\mu\nu\lambda\delta} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right] + S_{\text{bound}}
\]

\[
= -\frac{4\pi}{32\pi^2} \int_0^{8\pi m} d\tau \int_{2m}^{r_0} dr \frac{\partial}{\partial r} \frac{2m}{r^2} (1 - (1 - 2m/r)) - \frac{16m^3}{r_0^3} = 2.
\]
The simple form of the G-B Lagrangian (19) was confirmed when we have used the vierbein formulation and differential forms, as explained in Sec. 5. For completeness we present here all the non-vanishing curvature forms:

\[ R^{01} = \frac{1}{2} (e^U)^n d\tau , \quad R^{02} = -\frac{1}{2} (e^{2U})' e^U R' d\tau d\theta , \]

\[ R^{03} = -\frac{1}{2} (e^{2U})' e^U R' \sin \theta dr d\phi , \quad R^{31} = \frac{1}{2} (e^U R')' \sin \theta dr d\phi , \]

\[ R^{32} = -\sin \theta \left(1 - (e^U R')^2\right) d\theta d\phi , \quad R^{21} = \frac{1}{2} (e^U R')' dr d\theta . \] (175)
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Figure 1: The geometry of the $r - \tau$ space for non-extreme black holes. The circles are lines of constant $r$. The shaded region has an outer boundary but no inner boundary. The vector field $\frac{\partial}{\partial \tau}$ has a fixed point set at the horizon. The space has the topology $\mathbb{R}^2$. Thus $\chi = 1$. 
Figure 2: The geometry of the $r - \tau$ space for the extreme black holes. The circles are lines of constant $r$. The shaded region has an outer and inner boundary. The vector field $\frac{\partial}{\partial \tau}$ has no fixed points. The surface has an infinitely long spine, and has topology $S^1 \times R \sim R^2 - \{0\}$. Thus $\chi = 0$. 
Figure 3: The entropy $S$ of the dilaton black holes as the function of the electric $Q$ and magnetic $P$ charges. The shaded region presents the entropy of non-extreme and near extreme $[U(1)]^2$ black holes, whose topology is shown on Fig. 1. The bold line shows the vanishing entropy of extreme black holes, whose topology is shown on Fig. 2.
Figure 4: The relative moduli space for two extreme holes with $0 \leq a < \frac{1}{3}$. What is illustrated is the covering space $\tilde{\mathcal{M}}_2^{\text{rel}}$. Each circle represents a 2-sphere. To obtain $\mathcal{M}_2^{\text{rel}}$ one must identify antipodal points on these spheres. The identified space is non-orientable and has $\chi = 1$. 
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