1. Introduction

In this paper we study unique continuation properties of solutions of Schrödinger equations. In this class we shall include linear ones of the form

\[(1.1) \quad i\partial_t u + \Delta u = Vu, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R},\]

and non-linear ones of the type

\[(1.2) \quad i\partial_t u + \Delta u + F(u, \overline{u}) = 0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}.\]

Our goal is to obtain sufficient conditions on the behavior of the solution \(u\) at two different times \(t_0 = 0\) and \(t_1 = 1\) which guarantee that \(u \equiv 0\) is the unique solution of (1.1). In the case of the nonlinear equation (1.2) we are interested in deducing uniqueness of the solution from information on the difference of two possible solutions at two different times.

For dispersive models, and in particular for Schrödinger equations, these kind of uniqueness results have been obtained under the assumptions that the solutions coincide in a large sub-domain of \(\mathbb{R}^n\) at two different times.

For the 1-D cubic Schrödinger equation, i.e. \(F = \pm|u|^2u\), \(n = 1\) in (1.2), B.-Y. Zhang showed that if \(u = 0\) in \((-\infty, a) \times \{t_0, t_1\}\) (or in \((a, \infty) \times \{t_0, t_1\}\)) for some \(a \in \mathbb{R}\), then \(u \equiv 0\). His proof is based on inverse scattering theory (IST).

In [8] it was shown, under general assumptions on \(F\) in (1.2), that if \(u_1, u_2 \in C([0, 1] : H^s(\mathbb{R}^n)), s > n/2, s \geq 2\), such that

\[(1.3) \quad u_1(x, t) = u_2(x, t), \quad (x, t) \in \Gamma^c_{x_0} \times \{t_0, t_1\},\]

where \(\Gamma^c_{x_0}\) denotes the complement of a cone \(\Gamma_{x_0}\) with vertex in \(x_0 \in \mathbb{R}^n\) with opening \(< 180^\circ\), then \(u_1 \equiv u_2\). In [8] one of the key steps in the proof was a uniform exponential decay estimate in the time interval \([0, 1]\) obtained under the assumption that the corresponding solution has the same decay at times \(t_0 = 0, t_1 = 1\) (see Lemma 2.1 below). The proof of this estimate follows by combining energy estimates for the Fourier transform of the solution and its projection onto the positive and negative frequencies together with some classical estimates for pseudo-differential and singular integral operators. Roughly, this uniform estimate allows to extend (1.3) to the whole time interval \([0, 1]\). In this setting one can then use V. Isakov’s approach in [10] to obtain the desired result.
In [4]-[5], the results in [8] were extended to the case of semi-spaces, i.e. cones with opening $= 180^0$, to more general classes of potentials, and to less regular solutions.

Unique continuation of the kind described above has also been established in other dispersive equations. In particular, L. Robbiano [9] proved the following uniqueness result for the Korteweg-de Vries (KdV) equation. If two solutions $u_1, u_2$ of this equation, in an appropriate class, agree in a semi-line $(a, \infty)$ at time $t_0 = 0$ and for any $(x, t) \in (a, \infty) \times [0, 1]$ their difference $u_1 - u_2$ and its space derivatives up to order 2 are point-wise bounded by $ce^{-\alpha x}$, for some $\alpha > 9/4$, then $u_1 \equiv u_2$.

Similar uniqueness result for the KdV but under the assumption that one of the solutions is $u(x, t) \equiv 0$ and the other vanishes in a semi-line at the times $0, 1$ were previously proved in [12] by using the IST. This result was extended in [7] to any pair of solutions to the generalized KdV equation, which includes non-integrable models. More recently, based on the IST, S. Tarama [11] proved that if the initial data has an appropriate exponential decay for $x > 0$, then the corresponding solution of the KdV becomes analytic respect the $x$ variable for all $t > 0$. We notice that even in the KdV case neither of the results in [9] and [11] described above implies the other one, since in [9] the decay assumption is needed in a whole time interval $[0, 1]$, and the result in [11] does not apply to the difference of two arbitrary solutions of the KdV.

Our motivation came in part from the following result due to G. H. Hardy (see [10]) concerning the decay of a function and its Fourier transform. For $f : \mathbb{R} \to \mathbb{C}$ if $f(x) = O(e^{-\pi A x^2})$ and its Fourier transform $\hat{f}(\xi) = O(e^{-\pi B \xi^2})$ with $A, B > 0$ and $AB > 1$, then $f \equiv 0$. Also, if $A = B = 1$, then $f(x) = ce^{-\pi x^2}$.

This kind of uncertainty principle can be re-phased in terms of the solution, $v(x,t) = e^{it\partial_x^2/4\pi}v_0(x)$, of the free Schrödinger equation
\[
i \partial_t v + \frac{1}{4\pi} \partial_x^2 v = 0,
\]

since $e^{i\pi|x|^2}v(x,1) = e^{i\pi|x|^2}e^{it\partial_x^2/4\pi}v_0(x)$ is the Fourier transform of $e^{i\pi \xi^2}v_0(\xi)$.

Our main result concerning the equation (1.1) is the following.

**Theorem 1.1.** Let $u \in C([0,1] : H^2(\mathbb{R}^n))$ be a strong solution of the equation (1.1) in the domain $(x, t) \in \mathbb{R}^n \times [0,1]$ with $V : \mathbb{R}^n \times [0,1] \to \mathbb{C}$, $V \in L^\infty(\mathbb{R}_x^n \times [0,1])$, and $\nabla_x V \in L^1_1([0,1] : L^\infty(\mathbb{R}^n))$. If there exist $\alpha > 2$ and $a > 0$ such that
\[
\text{(1.5)} \quad u_0 = u(\cdot, 0), \quad u_1 = u(\cdot, 1) \in H^1(e^{a|x|^\alpha} \, dx),
\]
and
\[
\text{(1.6)} \quad \lim_{r \to \infty} ||V||_{L^1 L^\infty([|x|>r])} = \lim_{r \to \infty} \int_0^1 \sup_{|x|>r} |V(x, t)| \, dt = 0,
\]
then $u \equiv 0$.

For the nonlinear equation (1.2) we shall prove,
Theorem 1.2. Let \( u_1, u_2 \in C([0,1] : H^k(\mathbb{R}^n)) \), \( k \in \mathbb{Z}^+ \), \( k > n/2 + 1 \) be strong solutions of the equation (1.2) in the domain \((x,t) \in \mathbb{R}^n \times [0,1]\), with \( F : \mathbb{C}^2 \to \mathbb{C} \), \( F \in C^k \) and \( F(0) = \partial_u F(0) = \partial_{\bar{u}} F(0) = 0 \).

If there exist \( \alpha > 2 \) and \( a > 0 \) such that

\[
\begin{align*}
(1.7) & \quad w_0 = u_1(\cdot, 0) - u_2(\cdot, 0), \quad w_1 = u_1(\cdot, 1) - u_2(\cdot, 1) \in H^1(e^{a|x|^\alpha}dx),
\end{align*}
\]

then \( u_1 \equiv u_2 \).

We shall say that \( f \in H^1(e^{a|x|^\alpha}dx) \) if \( f, \partial_{x_j} f \in L^2(e^{a|x|^\alpha}dx) \) for \( j = 1, \ldots, n \), i.e.

\[
\begin{align*}
(1.8) & \quad \int_{\mathbb{R}^n} |f(x)|^2 e^{a|x|^\alpha}dx + \sum_{j=1}^n \int_{\mathbb{R}^n} |\partial_{x_j} f(x)|^2 e^{a|x|^\alpha}dx < \infty.
\end{align*}
\]

Remarks

a) It will be clear from our proof below that Theorems 1.1 and 1.2 still hold assuming (1.5) and (1.7) respectively with \( \alpha = 2 \) and \( a \geq c_0 \), where

\[
\begin{align*}
c_0 &= c_0(\|u\|_{L^\infty H^2}; n; \|V\|_{L^\infty_{t,x}}; \|\nabla_x V\|_{L^1_t L^\infty_x}) > 0, \quad \text{in Theorem 1.1},
\end{align*}
\]

and

\[
\begin{align*}
c_0 &= c_0(\|u\|_{L^\infty H^2}; n; \|F\|_{C^k}) > 0, \quad \text{in Theorem 1.2}.
\end{align*}
\]

b) Theorems 1.1 and 1.2 can be seen as natural extensions to the Schrödinger equation of the results we recently obtained in [3] for the heat equation. In [3], the decay assumption was only assumed at time \( t = 1 \).

c) The method of proof of Theorems 1.1 and 1.2 follows a similar argument, which is based on two main steps. The first one is based on the exponential decay estimates obtained in [8]. These estimates are expressed in terms of the \( L^2 \)-norm with respect to the measure \( e^{\beta|x|}dx \) and involve bounds independent of \( \beta \). Here we shall use them to deduce similar ones but with higher order power in the exponent, i.e. with super-linear growth. The second main step is to establish asymptotic lower bounds for the \( L^2 \)-norm of the solution and its space gradient on the annulus domain \((x,t) \in \{R-1 < |x| < R\} \times [0,1]\).

The use of a lower bound was motivated by the recent work of J. Bourgain and C. E. Kenig [1] on a class of stationary Schrödinger operators (i.e. \(-\Delta + V(x)\)). There a key lower bound on the decay of the average of the solution over unit balls was deduced in terms of the size of their centers. In this second part we also follow some of the arguments found in [6].

d) In a forthcoming work we shall extend the results obtained here to the generalized KdV equation.

e) For the existence of solutions of the IVP associated to the equations (1.1) and (1.2) with data at \( t = 0 \) as in Theorem 1.1 and 1.2 we refer to [2] and references therein.

The rest of this paper is organized as follows. In section 2, we deduce the energy estimate with super-linear exponential growth in the interval \([0,1]\) under the assumption that a similar one holds for the solution at times \( t_0 = 0 \) and \( t_1 = 1 \). In section 3, we
shall obtain a lower bound for the $L^2$-norm of the solution and its gradient in the annular domain mentioned above. Finally, in section 4 we combine the results in the previous sections to prove Theorems 1.1 and 1.2.

2. WEIGHTED ENERGY ESTIMATES

In [8] the following exponential decay estimate was established.

**Lemma 2.1.** There exists $\epsilon > 0$ such that if $V : \mathbb{R}^n \times [0, 1] \to \mathbb{C}$ satisfies that
\[(2.1)\quad \|V\|_{L^1_t L^\infty_x} \leq \epsilon,\]
and $u \in C([0, 1] : L^2_x(\mathbb{R}^n))$ is a (strong) solution of the IVP
\[(2.2)\quad \begin{cases} \partial_t u + \Delta u = V u + H, & (x, t) \in \mathbb{R}^n \times [0, 1], \\ u(x, 0) = u_0(x). \end{cases}\]
with $H \in L^1_t ([0, 1] : L^2_x(e^{2\beta |x|} dx))$, and for some $\beta \in \mathbb{R}$
\[(2.3)\quad u_0, u_1 \equiv u(\cdot, 1) \in L^2(e^{2\beta |x|} dx),\]
then
\[(2.4)\quad \sup_{0 \leq t \leq 1} \|u(\cdot, t)\|_{L^2(e^{2\beta |x|} dx)} \leq c(\|u_0\|_{L^2(e^{2\beta |x|} dx)} + \|u_1\|_{L^2(e^{2\beta |x|} dx)} + \|H\|_{L^1_t L^2_x(e^{2\beta |x|} dx)}),\]
with $c$ independent of $\beta > 0$.

We shall use Lemma 2.1 to deduce further weighted inequalities.

**Corollary 2.1.** If in addition to the hypothesis in Lemma 2.1 one has that for some $\beta > 0$
\[(2.5)\quad u_0, u_1 \in L^2(e^{2\beta |x|} dx),\]
and $H \in L^1_t ([0, 1] : L^2_x(e^{2\beta |x|} dx))$, then
\[(2.6)\quad \sup_{0 \leq t \leq 1} \|u(\cdot, t)\|_{L^2(e^{\pm 2\beta |x|} dx)} \leq c(\|u_0\|_{L^2(e^{2\beta |x|} dx)} + \|u_1\|_{L^2(e^{2\beta |x|} dx)} + \|H\|_{L^1_t L^2_x(e^{2\beta |x|} dx)}),\]
with $c$ independent of $\beta > 0$.

**Proof.** From (2.4) it follows that for any $\beta > 0$
\[(2.7)\quad \sup_{0 \leq t \leq 1} \|u(\cdot, t)\|_{L^2(e^{\pm 2\beta |x|} dx)} \leq c(\|u_0\|_{L^2(e^{2\beta |x|} dx)} + \|u_1\|_{L^2(e^{2\beta |x|} dx)} + \|H\|_{L^1_t L^2_x(e^{2\beta |x|} dx)}) \equiv \Phi,\]
for any $j = 1, \ldots, n$. Hence, for any $t \in [0, 1]$ one has that

$$\|u(\cdot, t)\|_{L^2(e^{2\beta|x|}/\sqrt{n} \, dx)} \leq \sum_{j=1}^{n} \left( \int_{\mathbb{R}^n} |u(x, t)|^2 e^{2\beta|x|} \, dx \right)^{1/2} \leq 2n \Phi.$$  

Taking the supremum on $t \in [0, 1]$ we get the desired inequality (2.6).

**Corollary 2.2.** If in addition to the hypothesis in Lemma 2.1 one assumes that for some $a > 0$ and $\alpha > 1$

$$u_0, u_1 \in L^2(e^{a|x|\alpha} \, dx),$$

and $H \in L^1([0, 1] : L^2(e^{a|x|\alpha} \, dx))$, with $u \in C([0, 1] : H^1(\mathbb{R}^n))$, then there exists $c_a > 0$ such that

$$\sup_{0 \leq t \leq 1} \int_{|x| \geq c_a} |u(x, t)|^2 e^{a|x|^\alpha/(10\sqrt{n})^\alpha} \, dx$$

$$\leq c(\|u_0\|_{L^2(e^{a|x|\alpha} \, dx)}^2 + \|u_1\|_{L^2(e^{a|x|\alpha} \, dx)}^2) + c \int_0^1 \int_{\mathbb{R}^n} |H(x, t)|^2 e^{a|x|\alpha} \, dx \, dt + c \sum_{l=0}^{1} \int_0^1 \int_{\mathbb{R}^n} |\partial^l_r u(x, t)|^2 \, dx \, dt.$$

The constant $c$ in (2.10) can be taken uniform in a set of $a$’s bounded from below away from zero.

**Proof.** We multiply the equation in (2.2) by $\eta_R(x) = \eta(x/R)$ with $\eta \in C^\infty$ non-decreasing radial function such that $\eta(x) = 0$ if $|x| < 1$ and $\eta(x) = 1$ if $|x| > 2$.

Using the notation $u_R(x, t) = u(x, t) \eta_R(x)$ we get the new equation

$$i \partial_t u_R + \Delta u_R = V u_R + \tilde{H}_R,$$

with

$$\tilde{H}_R = H \eta_R - 2 \partial_r \eta_R \partial_r u - (\partial^2_r \eta_R + \frac{n-1}{r} \partial_r \eta_R) u$$

Applying Corollary 2.1 (estimate (2.6)) it follows that

$$\int_{|x| > 2R} |u(x, t)|^2 e^{2\beta|x|}/\sqrt{n} \, dx \leq c \sum_{j=0}^{1} \int_{|x| > R} |u_j(x)|^2 e^{2\beta|x|} \, dx$$

$$+ c \int_0^1 \int_{|x| > R} |H(x, t)|^2 e^{2\beta|x|} \, dx \, dt$$

$$+ c \int_0^1 \int_{R<|x|<2R} \left( |\partial_r u|^2 \frac{1}{R^2} + \frac{|u|^2}{R^4} \right) e^{2\beta|x|} \, dx \, dt.$$
Since
\[ \int_0^1 \int_{R < |x| < 2R} \left( \frac{\partial_r u}{R^2} + \frac{|u|^2}{R^4} \right) e^{2\beta |x|} \, dx \, dt \]
(2.14)
multiplied by \( e^{-4\beta R} \) one has that
\[ A_1 \equiv e^{-4\beta R} \int_{|x| > 2R} |u(x, t)|^2 e^{2\beta |x|/\sqrt{n}} \, dx \]
\[ \leq c e^{-4\beta R} \sum_{j=0}^1 \int_{|x| > R} |u_j(x)|^2 e^{2\beta |x|} \, dx \]
(2.15)
\[ + c e^{-4\beta R} \int_0^1 \int_{|x| > R} |H(x, t)|^2 e^{2\beta |x|} \, dx \, dt \]
\[ + c \int_0^1 \int_{R < |x| < 2R} \left( \frac{\partial_r u}{R^2} + \frac{|u|^2}{R^4} \right) \, dx \, dt \equiv B_1 + B_2 + B_3. \]

Next, we fix \( 2\beta = bR^{\alpha-1} \), with \( b = b(\alpha) > 0 \) to be determined, integrate the inequality (2.15) in \( R \) in the interval \([1, \infty)\), and consider the resulting terms separately. First, for the term coming from \( B_1 \) using Fubini’s theorem we write
\[ \int_1^\infty e^{-2bR^\alpha} \sum_{j=0}^1 \int_{|x| > 1} |u_j(x)|^2 e^{2\beta |x|} \, dx \, dR \]
(2.16)
\[ = \sum_{j=0}^1 \int_{|x| > 1} \left( \int_1^r e^{-2bR^\alpha + bR^{\alpha-1} r} \, dR \right) |u_j(x)|^2 \, dx, \]
where \( r = |x| \). To deduce an upper bound for this expression we see that \( \varphi(R) = bR^{\alpha-1}(r - 2R) \) has its maximum at \( R_M = (\alpha - 1)r/2\alpha < r/2 \), hence
\[ \int_1^r e^{-2bR^\alpha + bR^{\alpha-1} r} \, dR \leq r e^{\varphi(R_M)} \]
(2.17)
\[ = r e^{b(\alpha-1)^{\alpha-1} r/(2^{\alpha-1}\alpha^{\alpha-1})} = r e^{b_\alpha r^{\alpha}} = |x| e^{b_\alpha |x|^\alpha}, \]
i.e. \( b_\alpha = b(\alpha-1)^{\alpha-1}/(2^{\alpha-1}\alpha) \). This estimated inserted in (2.16) yields the bound
\[ \sum_{j=0}^1 \int_{\mathbb{R}^n} |u_j(x)|^2 e^{b_\alpha |x|^\alpha} \, |x| \, dx. \]
(2.18)
A similar argument provides the following upper bound for the term coming from $B_2$ in (2.15)

\[ \int_0^1 \int_{\mathbb{R}^n} |H(x, t)|^2 e^{b_\alpha |x|^\alpha} |x| \, dx \, dt. \tag{2.19} \]

Next, we shall deduce a lower bound for the term arising from $A_1$. Using again Fubini’s theorem this can be written as

\[ \int_1^\infty e^{-2bR^\alpha} \int_0^1 \int_{|x|>2R} |u(x, t)|^2 e^{2\beta |x|/\sqrt{n}} \, dx \, dR \tag{2.20} \]

\[ = \int_2^\infty \int_{S^{n-1}} \left( \int_1^{r/2} e^{-2bR^\alpha + bR_0^{\alpha-1} r/\sqrt{n}} \, dR \right) |u(x, t)|^2 \, r^{n-1} \, dS \, dr. \]

Since $\eta(R) = -2bR^\alpha + bR_0^{\alpha-1} r/\sqrt{n}$ has its maximum at $\bar{R}_M = (\alpha-1) r/2 \alpha \sqrt{n} < r/2 \sqrt{n} < r/2$, we take $R_0 = (\alpha-1) r/10\alpha \sqrt{n}$, $r > c_\alpha$ and $c_\alpha > 10\alpha \sqrt{n}/(\alpha-1) > 2$ to bound from below the integral inside (2.20) as

\[ \int_1^{r/2} e^{-2bR^\alpha + bR_0^{\alpha-1} r/\sqrt{n}} \, dR > \int_{\bar{R}_M}^{2\bar{R}_M} e^{bR_0^{\alpha-1} (r/\sqrt{n})} \, dR \tag{2.21} \]

\[ \geq \frac{2\alpha - 1}{5} \frac{r}{\alpha} e^{b(\alpha-1)\alpha-1/(10\alpha-1)\alpha \sqrt{n}^\alpha} \]

\[ = \frac{2\alpha - 1}{5} \frac{r}{\alpha} e^{b_\alpha r^\alpha/(5\alpha-1)\alpha \sqrt{n}^\alpha}. \]

The term obtained from $B_3$ in (2.15) can be handled as

\[ \int_1^\infty \int_0^1 \int_{R<|x|<2R} \left( \frac{|\partial_r u|^2}{R^2} + \frac{|u|^2}{R^4} \right) \, dx \, dt \, dR \tag{2.22} \]

\[ c \leq \int_0^1 \int_{|x| \geq 1} (|\partial_r u|^2 + |u|^2) \left( \int_{|x|/2} |\partial_r u(x, t)|^2 \, dx \right) \, dt \]

\[ \leq c \int_0^1 \int_{\mathbb{R}^n} (|u(x, t)|^2 + |\partial_r u(x, t)|^2) \, dx \, dt. \]

Thus, collecting this information and those in the estimates (2.14)-(2.22), fixing $b$ such that $b_\alpha + \epsilon = b(\alpha-1)\alpha-1/(2\alpha-1)\alpha^\alpha + \epsilon = a$, with $\epsilon > 0$ small enough we obtain the desired estimate (2.10).

□

A similar argument provides the following result.
Corollary 2.3. If in addition to the hypothesis in Lemma 2.1 one assumes that for some \( a \in \mathbb{R} \) and \( \alpha > 1 \)

\[
(2.23) \quad u_0, u_1 \in L^2(e^{ax_1|x_1|^{a-1}} \, dx),
\]

and \( H \in L^1([0, 1] : L^2(e^{ax_1|x_1|^{a-1}} \, dx)) \), with \( u \in C([0, 1] : H^1(\mathbb{R}^n)) \), then there exists \( c_\alpha \) such that

\[
\sup_{0 \leq t \leq 1} \int_{|x_1| \geq c_\alpha} |u(x, t)|^2 e^{ax_1|x_1|^{a-1}/(10)^a} \, dx
\]

\[
\leq c(\|u_0\|_{L^2(e^{ax_1|x_1|^{a-1}} \, dx)}^2 + \|u_1\|_{L^2(e^{ax_1|x_1|^{a-1}} \, dx)}^2)
\]

\[
+ c \int_0^1 \int_{\mathbb{R}^n} |H(x, t)|^2 e^{ax_1|x_1|^{a-1}} \, dx \, dt
\]

\[
+ c \sum_{i=0}^{1} \int_0^1 \int_{\mathbb{R}^n} |\partial_{x_1} u(x, t)|^2 \, dx \, dt.
\]

In Corollaries 2.2 and 2.3 it suffices to assume that \( u, \partial_r u \) and \( u, \partial_{x_1} u \) belong to \( C([0, 1] : L^2(\mathbb{R}^n)) \), respectively.

Also, the results in this section extend to equations of the form

\[
(2.25) \quad i\partial_t u + \Delta u = V_1 u + V_2 \bar{u} + H,
\]

with the potentials \( V_j(x, t) \), \( j = 1, 2 \) satisfying the assumption (2.1).

3. LOWER BOUNDS ESTIMATES

This section is concerned with lower bounds for the \( L^2 \)-norm of the solution of the equations (1.1) and (1.2) and its gradient in the domain \( \{R - 1 < |x| < R \} \times [0, 1] \).

Lemma 3.1. Assume that \( R > 0 \) and \( \varphi : [0, 1] \rightarrow \mathbb{R} \) is a smooth function. Then, there exists \( c = c(n, \|\varphi\|_\infty + \|\varphi''\|_\infty) > 0 \) such that, the inequality

\[
(3.1) \quad \frac{\alpha^{3/2}}{R^2} \|e^{(\alpha^1 \varphi + \varphi(t))e_1^2} \|_{L^2(dx \, dt)} \leq c \|e^{(\alpha^1 \varphi + \varphi(t))e_1^2}(i\partial_t + \Delta)g\|_{L^2(dx \, dt)}
\]

holds, when \( \alpha \geq cR^2 \) and \( g \in C_c^\infty(\mathbb{R}^{n+1}) \) has its support contained in the set

\[
\{ (x, t) : |\frac{x}{R} + \varphi(t)e_1| \geq 1 \}.
\]

Proof. We shall follow the arguments in [6] and [3]. Let \( f = e^{(\alpha^1 \varphi + \varphi(t))e_1^2} g \). Then,

\[
(3.2) \quad e^{(\alpha^1 \varphi + \varphi(t))e_1^2}(i\partial_t + \Delta)g = S_\alpha f - 4\alpha A_\alpha f,
\]
where

\[ S_\alpha = i \partial_t + \Delta + \frac{4\alpha^2}{R^2} \mid \frac{x}{R} + \varphi e_1 \mid^2, \]

\[ A_\alpha = \frac{1}{R} \left( \frac{x}{R} + \varphi e_1 \right) \cdot \nabla + \frac{n}{2R^2} + \frac{i \varphi'}{2} \left( \frac{x}{R} + \varphi \right). \]

Thus,

\[ (3.3) \quad A_\alpha^* = -A_\alpha, \quad S_\alpha^* = S_\alpha, \]

and

\[ \| e^{\alpha | \frac{x}{R} + \varphi e_1 |^2} (i \partial_t + \Delta)g \|_2^2 = \langle S_\alpha f - 4\alpha A_\alpha f, S_\alpha f - 4\alpha A_\alpha f \rangle \]
\[ \geq -4\alpha \langle (S_\alpha A_\alpha - A_\alpha S_\alpha) f, f \rangle = -4\alpha \langle [S_\alpha, A_\alpha] f, f \rangle. \]

A calculation shows that

\[ [S_\alpha, A_\alpha] = \frac{2}{R^2} \Delta - \frac{4\alpha^2}{R^2} \mid \frac{x}{R} + \varphi e_1 \mid^2 - \frac{1}{2} \left( \frac{x}{R} + \varphi \right) \varphi'' + \varphi'^2 + \frac{2i\varphi'}{R} \partial_{x_1} \]

and

\[ \| e^{\alpha | \frac{x}{R} + \varphi e_1 |^2} (i \partial_t + \Delta)g \|_2^2 \geq \frac{16\alpha^3}{R^4} \int \mid \frac{x}{R} + \varphi e_1 \mid^2 |f|^2 dxdt + \frac{8\alpha}{R^2} \int |\nabla f|^2 dxdt \]
\[ + 2\alpha \int [(\frac{x}{R} + \varphi) \varphi'' + \varphi'^2] |f|^2 dxdt - \frac{8\alpha i}{R} \int \varphi' \partial_{x_1} f \bar{f} dxdt. \]

Hence, using the hypothesis on the support on \( g \) and the Cauchy-Schwarz inequality, the absolute value of the last two terms in (3.4) can be bounded by a fraction of the first two terms on the right hand side of (3.4), when \( \alpha \geq cR^2 \) for some large \( c \) depending on \( \| \varphi' \|_\infty + \| \varphi'' \|_\infty \). This yields (3.1) and Lemma 3.1.

\[ \square \]

**Theorem 3.1.** Let \( u \in C([0,1]: H^1(\mathbb{R}^n)) \) be a strong solution of

\[ (3.5) \quad i \partial_t u + \Delta u + Vu = 0, \quad t \in [0,1], \quad x \in \mathbb{R}^n. \]

If

\[ (3.6) \quad \int_0^1 \int_{\mathbb{R}^n} (|u|^2 + |\nabla_x u|^2)(x,t)dxdt \leq A^2, \]

\[ (3.7) \quad \int_{1/2}^{1/2+1/8} \int_{|x|<1} |u|^2(x,t)dxdt \geq 1, \]

and

\[ (3.8) \quad \| V \|_{L^\infty(\mathbb{R}^n \times [0,1])} \leq L. \]
then there exists $R_0 = R_0(n, A, L) > 0$ and a constant $c = c(n)$ such that for $R \geq R_0$ it follows that

$$\delta(R) \equiv \left( \int_0^1 \int_{R-1 < |x| < R} \left( |u|^2 + |\nabla_x u|^2 \right)(x, t) dx dt \right)^{1/2} \geq ce^{-\frac{cR^2}{2}}.$$  

Proof. Let $\theta_R$, $\theta \in C_c^\infty(\mathbb{R}^n)$ and $\varphi \in C^\infty([0, 1])$ be functions verifying $\theta_R(x) = 1$ if $|x| \leq R - 1$, $\theta_R(x) = 0$ if $|x| > R$, $\theta(x) = 1$ if $|x| \leq 1$, $\theta(x) = 0$ if $|x| \geq 2$, $0 \leq \varphi \leq 3$, $\varphi = 3$ in the interval $[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$ and $\varphi = 0$ in $[0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$.

With this choice of $\varphi$, we will apply the Lemma 3.1 to the function

$$g(x, t) = \theta_R(x) \theta(\frac{x}{R} + \varphi(t)e_1)u(x, t).$$

Observe that $g$ has compact support on $\mathbb{R}^n \times (0, 1)$ and satisfies the hypothesis in Lemma 3.1, $g = u$ in $B_{R-1} \times \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right]$, where $|\frac{x}{R} + \varphi(t)e_1| \geq 3 - 1 = 2$.

$$(i\partial_t + \Delta + V)g = \theta(\frac{x}{R} + \varphi e_1)(2\nabla \theta_R(x) \cdot \nabla u + u\Delta \theta_R(x)) + \theta_R(x) \left( 2R^{-1}\nabla \theta(\frac{x}{R} + \varphi e_1) \cdot \nabla u + R^{-2}u\Delta \theta(\frac{x}{R} + \varphi e_1) + i\varphi' \partial_x \theta(\frac{x}{R} + \varphi e_1)u \right),$$

and that the first and second terms on the right hand side of (3.10) are supported respectively in $B_R \setminus B_{R-1} \times [0, 1]$, where $|\frac{x}{R} + \varphi e_1| \leq 4$, and in $\{(x, t) : 1 \leq |\frac{x}{R} + \varphi e_1| \leq 2\}$. Thus,

$$\|e^{\alpha(\frac{x}{R} + \varphi(t)e_1)^2}g\|_{L^2(dx dt)} \geq e^{4\alpha} \|u\|_{L^2(B_1 \times \left[\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}\right])} \geq e^{\frac{4\alpha}{2}},$$

and combining (3.11) and (3.8) with $\alpha \geq cR^2$

$$\frac{\alpha^{3/2}}{cR^2} \left\| e^{\alpha(\frac{x}{R} + \varphi(t)e_1)^2}g \right\|_{L^2(dx dt)} \leq L \left\| e^{\alpha(\frac{x}{R} + \varphi(t)e_1)^2}g \right\|_{L^2(dx dt)} + e^{16\alpha} \delta(R) + e^{4\alpha} A.$$

If we choose $\alpha = cR^2$, it is possible to hide the first term on the right hand side of (3.12) in the left hand side of the same inequality, when $R \geq R_0(L)$. This and (3.11) imply that

$$R \leq c \left( e^{8cR^2} \delta(R) + A \right),$$

when $R \geq R_0(L)$, which implies the Theorem 3.1 when $R \geq R_0(L, A)$.

4. PROOF OF THEOREMS 1.1 AND 1.2

In this section we shall combine the results in Sections 2 and 3 to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1

If $u \not\equiv 0$, we can assume, after a possible translation, dilation, and multiplication by a constant, that $u = u(x, t)$ satisfies the hypothesis of Theorem 3.1. Therefore, there exist $R_0 = R_0(n, A, L) > 0$ and $c = c(n)$ such that for $R \geq R_0$

$$\delta(R) = \left( \int_0^1 \int_{R-1 < |x| < R} \left( |u|^2 + |\nabla_x u|^2 \right)(x, t) dx dt \right)^{1/2} \geq ce^{-cR^2}.$$
Next, we take \( \phi_R \in C^\infty(\mathbb{R}^n) \) radial with \( \phi_R(x) = 0 \) if \(|x| < R-1\), \( \phi_R(x) = 1 \) if \(|x| > R\), and \( \partial_r \phi_R(r) \geq 0 \), to get from (1.1) the equations

\[
i \partial_t(u\phi_R) + \Delta(u\phi_R) = V_R(u\phi_R) + 2\nabla x u \cdot \nabla x \phi_R + \Delta \phi_R u = V_R(u\phi_R) + H,
\]

and for \( j = 1, \ldots, n \)

\[
i \partial_t(\partial_{x_j}(u\phi_R)) + \Delta(\partial_{x_j}(u\phi_R)) = V_R(\partial_{x_j}(u\phi_R)) + \partial_{x_j}V_R(u\phi_R) + 2\nabla x \partial_{x_j} \phi_R \cdot \nabla x u + 2\nabla x \phi_R \cdot \nabla x \partial_{x_j} u + \Delta \phi_R \partial_{x_j} u - \Delta \partial_{x_j} \phi_R u = V_R(\partial_{x_j}(u\phi_R)) + \hat{H}_j,
\]

where \( V_R(x,t) = \phi_{R-1}(x) V(x,t) \).

In the following inequalites \( c_0 \) will denote a constant independent of \( R \) which may change from line to line. Now applying Corollary 2.2 to the equation (4.2) we obtain, for \( R \) large enough depending on \( \alpha \), that for \( a \) large enough depending on \( \alpha \),

\[
\sup_{0 \leq t \leq 1} \int_{|x| > R} |u(x,t)|^2 e^{a|x|^\alpha/(10\sqrt{n})} \, dx \\
\leq \sup_{0 \leq t \leq 1} \int_{|x| \geq c_0} |(u\phi_R)(x,t)|^2 e^{a|x|^\alpha/(10\sqrt{n})} \, dx \\
\leq c \sum_{j=0}^1 \|u_j\|_{L^2(e^{a|x|^\alpha} \, dx)}^2 + c \int_0^1 \int_{R-1 < |x| < R} (|u|^2 + |\nabla x u|^2) e^{a|x|^\alpha} \, dx \, dt \\
+ c \int_0^1 \int_{\mathbb{R}^n} |\partial_{x_j}^\alpha(u\phi_R)|^2 \, dx \, dt \leq c_0 + c_0 e^{aR^\alpha}.
\]

Corollary 2.2 (see remark at the end of section 2) with \( a/(10\sqrt{n})^\alpha \) instead of \( a \), the equation (4.3) and the previous estimate (4.4) leads to

\[
\sup_{0 \leq t \leq 1} \int_{|x| > R} |\nabla x u(x,t)|^2 e^{a|x|^\alpha/(10\sqrt{n})} \, dx \\
\leq \sup_{0 \leq t \leq 1} \int_{|x| \geq c_0} |\nabla x (u\phi_R)(x,t)|^2 e^{a|x|^\alpha/(10\sqrt{n})} \, dx \\
\leq c_0 + c \int_0^1 \int_{R-1 < |x| < R} (|u|^2 + |\nabla x u|^2 + |\nabla^2 x u|^2) e^{a|x|^\alpha} \, dx \, dt \\
+ c \int_0^1 \int_{\mathbb{R}^n} |u\phi_R \partial_{x_j} V_R|^2 e^{aR^\alpha/(10\sqrt{n})} \, dx \, dt \\
\leq c_0 + c_0 e^{aR^\alpha}.
\]
Combining (4.4)-(4.5) one sees that

\[ \sup_{0 \leq t \leq 1} \int_{|x| > R} (|u|^2 + |\nabla_x u|^2)(x, t)e^{a|x|^\alpha/(10\sqrt{t})}dx \leq c_0 + c_0e^{aR^\alpha}. \]  

(4.6)

From (4.6) and (4.1) we conclude for any \( \mu > 1 \) with \( \mu R - 1 > R \) that

\[ c_0e^{c(\mu R)^2}e^{a(\mu R - 1)^\alpha/(10\sqrt{t})} \leq c_0\delta(\mu R)e^{a(\mu R - 1)^\alpha/(10\sqrt{t})^{2\alpha}} \leq c_0 + c_0e^{aR^\alpha}. \]  

(4.7)

Finally, since \( \alpha > 2 \) taking \( \mu \) sufficiently large in (4.7) we get a contradiction. Hence, \( u \equiv 0 \).

**Proof of Theorem 1.2**

We consider the difference of the two solutions

\[ w(x, t) = u_1(x, t) - u_2(x, t), \]  

(4.8)

which satisfies the equation

\[ i\partial_tw + \Delta w = \frac{F(u_1, \bar{u}_1) - F(u_2, \bar{u}_2)}{u_1 - u_2}w. \]  

(4.9)

Also, its \( \partial_{x_j} \)-derivative, \( j = 1, \ldots, n \), solves

\[ i\partial_tw + \Delta w = \partial_u F(u_1, \bar{u}_1)\partial_{x_j}w + \partial_\bar{u} F(u_1, \bar{u}_1)\partial_{x_j}\bar{w} \]

\[ + \left( \frac{\partial_u F(u_1, \bar{u}_1) - \partial_u F(u_2, \bar{u}_2)}{u_1 - u_2} \right) \partial_{x_j}u_2w \]

\[ + \left( \frac{\partial_\bar{u} F(u_1, \bar{u}_1) - \partial_\bar{u} F(u_2, \bar{u}_2)}{\bar{u}_1 - \bar{u}_2} \right) \partial_{x_j}\bar{u}_2\bar{w} \]

\[ = \partial_u F(u_1, \bar{u}_1)\partial_{x_j}w + \partial_\bar{u} F(u_1, \bar{u}_1)\partial_{x_j}\bar{w} + H_j, \]  

(4.10)

The potential

\[ V_1(x, t) = \frac{F(u_1, \bar{u}_1) - F(u_2, \bar{u}_2)}{u_1 - u_2}. \]  

(4.11)

in the equation (4.9) satisfies the hypothesis of Theorem 3.1, therefore

\[ \delta(R) = (\int_0^1 \int_{R-1 < |x| < R} (|w|^2 + |\nabla_x w|^2)(x, t)dxdt)^{1/2} \geq ce^{-cR^{2}}. \]  

(4.12)

Next, we shall follow the argument in (4.2)-(4.5). Thus, we multiply the equations in (4.9)-(4.10) by \( \phi_R(x) \) defined before (4.2) and observe that the potentials \( V_1(x, t) \) in (4.11) and

\[ V_2(x, t) = \partial_u F(u_1, \bar{u}_1), \quad V_3(x, t) = \partial_\bar{u} F(u_1, \bar{u}_1), \]  

(4.13)

satisfy that

\[ \|\phi_{R-1}V_j\|_{L^1_xL^\infty_x} \to 0, \quad \text{as} \quad R \uparrow \infty, \quad j = 1, 2, 3. \]  

(4.14)
In particular, we can apply Corollary 2.2 (see the remark at the end of Section 2) to the equation (4.9) and argue as in (4.4) to get that for $R$ sufficiently large

\begin{equation}
\sup_{0 \leq t \leq 1} \int_{|x|>R} |w(x,t)|^2 e^{a|x|^\alpha/(10\sqrt{n})^\alpha} \, dx \leq c_0 e^{aR^\alpha}
\end{equation}

Also we see that the terms

\begin{equation}
\frac{\partial_j F(u_1, \bar{u}_1) - \partial_j F(u_2, \bar{u}_2)}{u_1 - u_2} \partial_x u_2 \, w,
\end{equation}

and

\begin{equation}
\frac{\partial_j F(u_1, \bar{u}_1) - \partial_j F(u_2, \bar{u}_2)}{\bar{u}_1 - \bar{u}_2} \partial_x \bar{u}_2 \, \bar{w},
\end{equation}

in the equation (4.10) belong to $L^\infty_t([0,1]: L^\infty_x(\mathbb{R}^n))$, and contain a factor already estimated in (4.15). Also, the additional terms coming from the commutator between multiplication by $\phi_R$ and the Laplacian $\Delta$ are similar to those considered in (4.2)-(4.3), and their bounds are analogous to those described in (4.4)-(4.5). Hence, the reminding of the proof of Theorem 1.2 follows the argument given in (4.2)-(4.7) in the proof of Theorem 1.1, therefore it will be omitted.

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References

[1] Bourgain, J., and Kenig, C. E., On localization in the continuous Anderson-Bernoulli model in higher dimensions, to appear in Invent. Math.
[2] Cazenave, T., An introduction to nonlinear Schrödinger equations, Textos de Métodos Matemáticos 22, Universidade Federal de Rio de Janeiro (1989)
[3] Escauriaza, L., Kenig, C. E., Ponce, G., and Vega, L., Decay at infinity of caloric functions within characteristic hyperplanes, pre-print
[4] Ionescu, I. D., and Kenig, C. E., $L^p$ Carleman inequalities and uniqueness of solutions of nonlinear Schrödinger equations, to appear in Acta Math
[5] Ionescu, I. D., and Kenig, C. E., Uniqueness properties of solutions of Schrödinger equations, to appear in J. Funct. Anal.
[6] Isakov, V., Carleman type estimates in anisotropic case and applications J. Diff. Eqs. 105 (1993), 217-238
[7] Kenig, C. E., Ponce, G., and Vega, L., On unique continuation of solutions to the generalized KdV equation, Math. Res. Letters 10 (2003), 833-846
[8] Kenig, C. E., Ponce, G., and Vega, L. On the support of solutions of nonlinear Schrödinger equations, Comm. Pure Appl. Math. 60 (2002), 1247-1262
[9] Robbiano, L., Unicité forte á l’infini pour KdV, Control Opt. and Cal. Var. 8 (2002), 933-939
[10] Stein, E. M., and Shakarchi. R., Complex Analysis, Princeton Lectures in Analysis, Princeton University Press, (2003)

[11] Tarama, S., Analytic solutions of the Korteweg-de Vries equation, J. Math. Kyoto Univ. 44 (2004), 1-32

[12] Zhang, B.-Y., Unique continuation for the Korteweg-de Vries equation, SIAM J. Math. Anal. 32 (1992), 55-71

[13] Zhang, B.-Y., Unique continuation for the nonlinear Schrödinger equation, Proc. Roy. Soc. Edinburgh Sect. A 127 (1997), 191-205

Luis Escauriaza  
Departamento de Matematicas  
Universidad del Pais Vasco  
Apartado 644  
48080 Bilbao  
Spain  
E-mail: mtpeszul@lg.ehu.es

Carlos E. Kenig  
Department of Mathematics  
University of Chicago  
Chicago, Il. 60637  
USA  
E-mail: cek@math.uchicago.edu

Gustavo Ponce  
Department of Mathematics  
University of California  
Santa Barbara, CA 93106  
USA  
E-mail: ponce@math.ucsb.edu

Luis Vega  
Departamento de Matematicas  
Universidad del Pais Vasco  
Apartado 644  
48080 Bilbao  
Spain  
E-mail: mtpvegol@lg.ehu.es