Lightlike Hypersurfaces of an Indefinite Kaehler Manifold with an $(\ell, m)$-type Connection

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Abstract Jin [1] defined an $(\ell, m)$-type connection on semi-Riemannian manifolds. Semi-symmetric non-metric connection and non-metric $\phi$-symmetric connection are two important examples of this connection such that $(\ell, m) = (1, 0)$ and $(\ell, m) = (0, 1)$, respectively. In semi-Riemannian geometry, there are few literatures for the lightlike geometry, so we expose new theories for non-degenerate submanifolds in semi-Riemannian geometry. The goal of this paper is to study a characterization of a (Lie) recurrent lightlike hypersurface $M$ of an indefinite Kaehler manifold with an $(\ell, m)$-type connection when the characteristic vector field is tangent to $M$. In the special case that an indefinite Kaehler manifold of constant holomorphic sectional curvature is an indefinite complex space form, we investigate a lightlike hypersurface of an indefinite complex space form with an $(\ell, m)$-type connection when the characteristic vector field is tangent to $M$. Moreover, we show that the total space, the complex space form, is characterized by the screen conformal lightlike hypersurface with an $(\ell, m)$-type connection. With a semi-symmetric non-metric connection, we show that an indefinite complex space form is flat.

Keywords Compound Non-symmetric Non-metric Connection, Lightlike Hypersurface, Indefinite Kaehler Manifold, Indefinite Complex Space Form

\begin{equation}
\bar{T}(\bar{X}, \bar{Y}) = \ell \{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m \{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\},
\end{equation}

where $\ell$ and $m$ are two smooth functions on $\bar{M}$, $J$ is a tensor field of type $(1, 1)$ and $\theta$ is a 1-form associated with a smooth unit vector field $\zeta$, which is called the characteristic vector field of $M$, by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Throughout this paper, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the smooth vector fields on $\bar{M}$.

Two special cases are important for both the mathematical study and the applications to physics: (1) In case $(\ell, m) = (1, 0)$: This connection $\bar{\nabla}$ becomes a semi-symmetric non-metric connection. The notion of semi-symmetric non-metric connection was introduced by Ageshe-Chafle [2, 3] and later, studied by several authors [4]. (2) In case $(\ell, m) = (0, 1)$: $\bar{\nabla}$ becomes a non-metric $\phi$-symmetric connection such that $\phi(\bar{X}, \bar{Y}) = \bar{g}(J\bar{X}, \bar{Y})$. The notion of the non-metric $\phi$-symmetric connection was introduced by this author [5, 6].

Remark 1.1. Denote by $\bar{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. By directed calculations, we see that a linear connection $\nabla$ on $\bar{M}$ is an $(\ell, m)$-type connection if and only if $\nabla$ satisfies

\begin{equation}
\nabla_{\bar{X}}\bar{Y} = \bar{\nabla}_{\bar{X}}\bar{Y} + \theta(\bar{Y})\{\ell\bar{X} + mJ\bar{X}\}.
\end{equation}

The objective of study in this paper is lightlike hypersurfaces of an indefinite Kaehler manifold $M = (\bar{M}, \bar{g}, J)$ with an $(\ell, m)$-type connection subject to the conditions that (1) the tensor field $J$, defined by (1.1) and (1.2), is identical with the indefinite almost complex structure tensor $J$ of $\bar{M}$ and (2) the characteristic vector field $\zeta$ of $\bar{M}$ is tangent to $\bar{M}$. In this paper, we set $(\ell, m) \neq (0, 0)$ and we shall assume that $\zeta$ is unit spacelike, without loss of generality.

1 Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an $(\ell, m)$-type connection [1] if $\bar{\nabla}$ and its torsion tensor $\bar{T}$ satisfy

\begin{equation}
(\bar{\nabla}_{\bar{X}}\bar{g})(\bar{Y}, \bar{Z}) = -\ell\{\theta(\bar{Y})\bar{g}(\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(\bar{X}, \bar{Y})\} - m\{\theta(\bar{Y})\bar{g}(J\bar{X}, \bar{Z}) + \theta(\bar{Z})\bar{g}(J\bar{X}, \bar{Y})\},
\end{equation}

2 Preliminaries

Let $(\bar{M}, \bar{g}, J)$ be an indefinite Kaehler manifold equipped with an $(\ell, m)$-type connection $\bar{\nabla}$ and a Levi-Civita connec-
tion $\nabla$, where $\tilde{g}$ is a semi-Riemannian metric and $J$ is an indefinite almost complex structure such that

$$J^2 = -I, \quad \tilde{g}(J\tilde{X}, J\tilde{Y}) = \tilde{g}(\tilde{X}, \tilde{Y}), \quad (\nabla_X J)\tilde{Y} = 0.$$  \hspace{1cm} (2.1)

By direct calculation from (1.3) and (2.1), we see that

$$(\nabla_X J)\tilde{Y} = \ell(\theta(J\tilde{Y})\tilde{X} - \theta(\tilde{Y})J\tilde{X}) + m\{\theta(\tilde{Y})\tilde{X} + \theta(J\tilde{Y})J\tilde{X}\}. \hspace{1cm} (2.2)$$

Let $(M, g)$ be a lightlike hypersurface of $\tilde{M}$. Denote by $F(M)$ the algebra of smooth functions on $M$ and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle $E$ over $M$. Also denote by (2.1), the $i$-th equation of the three equations in (2.1). We use same notations for any others. It is known that the normal bundle $TM^\perp$ of $M$ is a subbundle of the tangent bundle $TM$, of rank 1. A complementary vector bundle $S(TM)$ of $TM^\perp$ in $TM$ is a non-degenerate distribution on $M$, which is called a screen distribution on $M$, such that

$$TM = TM^\perp \oplus_{\text{orth}} S(TM),$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. It is known [7] that, for any null section $\xi$ of $TM^\perp$ on a coordinate neighborhood $U \subset M$, there exists a unique null section $N$ of a unique vector bundle $tr(TM)$ in $S(TM)$ satisfying

$$\tilde{g}(\xi, N) = 1, \quad \tilde{g}(N, N) = \tilde{g}(\xi, X) = 0, \quad \forall X \in \Gamma(S(TM)).$$

We call $tr(TM)$ and $N$ the transversal vector bundle and the null transversal vector field of $M$ with respect to the screen distribution $S(TM)$, respectively. Then the tangent bundle $TM$ of $M$ is decomposed as follow:

$$TM = TM^\perp \oplus_{\text{orth}} S(TM). \hspace{1cm} (2.3)$$

In case the vector field $\xi$ is tangent to $M$. If $\xi$ belongs to $Rad(TM)$, then

$$\xi = a\xi, \quad 1 = \tilde{g}(\xi, \xi) = a^2 \tilde{g}(\xi, \xi) = 0, \quad a \in F(M).$$

It is a contradiction. Thus $\xi$ does not belong to $Rad(TM)$. This result enables one to choose a screen distribution $S(TM)$ which contains $\xi$. Thus we consider lightlike hypersurfaces $M$ of an indefinite Kaehler manifold $\tilde{M}$ with an $(\ell, m)$-type connection and a screen distribution $S(TM)$ which contains $\xi$.

Denote by $X, Y$ and $Z$ the smooth vector fields on $M$, unless otherwise specified. Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then the local Gauss and Weingartan formulee of $M$ and $S(TM)$ are given by

$$\nabla_X Y = \nabla_X Y + B(X, Y)N, \hspace{1cm} (2.4)$$

$$\nabla_X N = -A_\xi X + \tau(X)N; \hspace{1cm} (2.5)$$

$$\nabla_X PY = \nabla_X Y + C(X, PY)\xi, \hspace{1cm} (2.6)$$

$$\nabla_X \xi = -A_\xi X - \tau(X)\xi, \hspace{1cm} (2.7)$$

where $\nabla$ and $\nabla^*$ are the induced connections on $TM$ and $S(TM)$, respectively, $B$ and $C$ are the local second fundamental forms on $TM$ and $S(TM)$, respectively, $A_\xi$ and $A_\xi^*$ are the shape operators and $\tau$ is a 1-form.

Due to [7, Section 6.2], for a lightlike hypersurface $M$ of an indefinite Kaehler manifold $\tilde{M}$, $J(TM^\perp)$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1, such that $J(TM^\perp) \cap J(tr(TM)) = \{0\}$. It follow that $J(TM^\perp) \oplus J(tr(TM))$ is a subbundle of $S(TM)$, of rank 2. Thus there exist two non-degenerate almost complex distributions $D_o$ and $D$ on $M$ with respect to the indefinite almost complex structure $J$, i.e., $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^\perp) \oplus J(tr(TM)) \oplus_{\text{orth}} D_o,$$

$$D = \{TM^\perp \oplus_{\text{orth}} J(TM^\perp)\} \oplus_{\text{orth}} D_o.$$  \hspace{1cm} (2.8)

In this case, the decomposition of $TM$ is reduced to

$$TM = D \oplus J(tr(TM)). \hspace{1cm} (2.9)$$

Consider two null vector fields $U$ and $V$ and two 1-forms $u$ and $v$ such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \hspace{1cm} (2.9)$$

Denote by $S$ the projection morphism of $TM$ on $D$. Any vector field $X$ of $M$ is expressed as $X = SX + u(X)U$. Applying $J$ to this form, we have

$$JX = FX + u(X)N, \hspace{1cm} (2.10)$$

where $F$ is a tensor field of type $(1, 1)$ globally defined on $M$ by $F = J \circ S$. Applying $J$ to (2.10) and using (2.1) and (2.9), we have

$$F^2 X = -X + u(X)U. \hspace{1cm} (2.11)$$

As $u(U) = 1$ and $FU = 0$, the set $(F, u, U)$ defines an indefinite almost contact structure on $M$ and $F$ is called the structure tensor field of $M$.

3 \hspace{0.5cm} $(\ell, m)$-type connections

Using (1.1), (1.2), (2.4) and (2.10), we obtain

$$(\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y) - \ell(\theta(Y)g(X, Z) + \theta(Z)g(X, Y)) \hspace{1cm} (3.1)$$

$$-m\{\theta(Y)g(JX, Z) + \theta(Z)g(JX, Y)\},$$

$$T(X, Y) = \ell(\theta(Y)X - \theta(X)Y) + m\{\theta(Y)FX - \theta(X)FY\}, \hspace{1cm} (3.2)$$

$$B(X, Y) = B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}. \hspace{1cm} (3.3)$$

where $T$ is the torsion tensor with respect to $\nabla$ and $\eta$ is a 1-form such that $\eta(X) = \tilde{g}(X, N)$. From the fact that $B(X, Y) = \tilde{g}(\nabla_X Y, \xi)$, we know that $B$ is independent of the choice of the screen distribution $S(TM)$ and satisfy

$$B(X, \xi) = 0, \hspace{1cm} B(\xi, X) = 0. \hspace{1cm} (3.4)$$

The local second fundamental forms are related to their shape operators by

$$B(X, Y) = g(A_\xi^* X, Y) + mu(X)\theta(Y), \hspace{1cm} (3.5)$$

$$\tilde{g}(A_\xi^* X, N) = 0.$$

3.1 \hspace{0.5cm} local second fundamental forms on $TM$ and $S(TM)$,
Let \( M \) be a lightlike hypersurface of an indefinite Kaehler manifold \( \tilde{M} \) with an \((\ell, m)\)-type connection such that \( \zeta \) is tangent to \( M \). Then (1) \( A^\xi_\zeta \) is self-adjoint, and (2) \( B \) is symmetric on \( TM \) if and only if \( m = 0 \).

Proof. (1) From (3.3) and (3.5), we see that \( g(A^\zeta_\xi X, Y) = g(X, A^\xi_\zeta Y) \). Thus \( A^\xi_\zeta \) is self-adjoint. (2) If \( m = 0 \), then \( B \) is symmetric by (3.3). Conversely, if \( B \) is symmetric, then, taking \( X = V \) and \( Y = U \) to (3.3), we get \( m\theta(V) = 0 \). Also, taking \( X = \zeta \) and \( Y = U \) to (3.3) and using \( m\theta(V) = 0 \), we have \( m = 0 \). \( \square \)

4 Some results

Definition 1. The structure tensor field \( F \) of \( M \) is said to be Lie recurrent [8] if there exists a 1-form \( \varpi \) on \( M \) such that

\[
(\nabla_X F)Y = \varpi(X)FY.
\]

A lightlike hypersurface \( M \) of an indefinite Kaehler manifold \( \tilde{M} \) is said to be recurrent if it admits a recurrent structure tensor field \( F \).

Theorem 4.1. There exist no recurrent lightlike hypersurface of an indefinite Kaehler manifold with an \((\ell, m)\)-type connection such that \( \zeta \) is tangent to \( M \).

Proof. From the above definition and (3.11), we get

\[
\varpi(X)FY = u(Y)A_N X - B(X, Y)U + \ell \theta(FY)X - \theta(FY)X + m\theta(Y)X + \theta(FY)X,
\]

Taking the scalar product with \( N \) to this and using (3.6), we have

\[
\varpi(Y)V = \{\ell\eta(Y) + m\nu(Y)\}\theta(FY) - \{\ell\nu(Y) - m\eta(Y)\}\theta(Y).
\]

Replacing \( Y \) by \( \xi \) to this and using the facts that \( F\xi = -V \) and \( \theta(\xi) = 0 \), we obtain \( \{\ell\eta(Y) + m\nu(Y)\}\theta(V) = 0 \). It follows that

\[
\ell\theta(V) = 0, \quad m\theta(V) = 0.
\]

Replacing \( Y \) by \( V \) to (4.1) and using the last equation, we obtain \( \varpi = 0 \).

Replacing \( X \) by \( \xi \) and \( V \) to (4.1) with \( \varpi = 0 \) by turns, we obtain

\[
m\theta(X) = -\ell\theta(FX), \quad \ell\theta(X) = m\theta(FX).
\]

As \((\ell, m) \neq (0, 0)\), from the last two equations we obtain \((\ell^2 + m^2)\theta(X) = 0 \). Taking \( X = \zeta \) to this, we get \( \ell^2 + m^2 = 0 \). It follows that \( \ell = 0 \) and \( m = 0 \). It is a contradiction to \((\ell, m) \neq (0, 0)\). Thus we have our theorem. \( \square \)

Definition 2. The structure tensor field \( F \) of \( M \) is said to be Lie recurrent [8] if there exists a 1-form \( \vartheta \) on \( M \) such that

\[
(\mathcal{L}_\vartheta F)Y = \vartheta(X)FY,
\]

where \( \mathcal{L}_\vartheta \) denotes the Lie derivative on \( M \) with respect to \( X \). \( F \) is called Lie parallel if \( \vartheta = 0 \). A lightlike hypersurface \( M \) of an indefinite Kaehler manifold \( \tilde{M} \) is called Lie recurrent if it admits a Lie recurrent structure tensor field \( F \).

Theorem 4.2. Let \( M \) be a Lie recurrent lightlike hypersurface of an indefinite Kaehler manifold \( \tilde{M} \) with an \((\ell, m)\)-type connection such that \( \zeta \) is tangent to \( M \). Then we have the following three assertions:

(1) the structure tensor field \( F \) is Lie parallel,

(2) the 1-form \( \tau \) satisfies \( \tau = 0 \), and

(3) \( A^\xi_\zeta \) and \( A_N \) satisfy \( A^\xi_\zeta U = A^\xi_N V = 0 \) and \( A_N V = 0 \).

Proof. (1) Using the above definition, (2.10), (11), (3.2) and (3.11), we get

\[
\vartheta(X)FY = -\nabla_{FY}X + F\nabla_YX + u(Y)A_N X - B(X, Y)U - \{B(X, Y) - m\theta(Y)u(X)\}U.
\]

Taking \( Y = \xi \) to (4.2) and using (3.4), and the fact that \( F\xi = -V \), we have

\[
\vartheta(X)V = \nabla_YX + F\nabla_X\xi.
\]

Taking the scalar product with \( V \) to (4.3) and using \( g(FX, V) = 0 \), we have

\[
u(\nabla_YX) = g(\nabla_YX, V) = 0.
\]

Replacing \( Y \) by \( V \) to (4.2) and using the fact that \( FV = \xi \), we have

\[
\vartheta(X)\xi = -\nabla_{\xi}X + F\nabla_X\xi - \{B(X, V) - m\theta(V)u(X)\}U.
\]

Applying \( F \) to this equation and using (2.11) and (4.4), we obtain

\[
\vartheta(X)V = \nabla_YX + F\nabla_X\xi.
\]
Comparing this equation with (4.3), we get $\partial = 0$. Thus $F$ is Lie parallel.

(2) Taking $X = U$ to $\nabla_V X + F\nabla_\xi X = 0$ and using (2.11) and (3.9), we get

$$F(A_{\xi} V) + \tau(V)U - A_{\xi} \xi + u(A_{\xi} \xi)U = 0.$$  

Taking the scalar product with $N$ to this equation, we obtain

$$g(A_{\xi} V, U) = 0. \tag*{(4.5)}$$

Replacing $X$ by $V$ to (4.2) and using (2.11), (3.3), (3.5) and (3.10), we get

$$\tau \xi = -(\bar{\nabla}_Y \theta)(\bar{Z}) \{f\bar{Y} + m\bar{Y}\} \tag*{(5.2)}$$

Taking the scalar product with $U$ to (4.6) and using (3.5) and (4.5), we have

$$B(X, U) - m\theta(U)u(X) = \tau(FX). \tag*{(4.7)}$$

From this equation and (3.8), we see that

$$u(A_{\xi} X) = \tau(FX). \tag*{(4.8)}$$

Replacing $X$ by $U$ to (4.2) and using (2.11), (3.3) (3.8) and (3.9), we get

$$u(Y)A_{\xi} X - F(A_{\xi} FY) - A_{\xi} Y - \tau(FY)U = 0. \tag*{(4.9)}$$

Taking the scalar product with $V$ to (4.9) and using (4.8), we get $\tau(FY) = 0$.

(3) As $\tau = 0$, from (4.7) we have $B(X, U) = m\theta(U)u(X)$. Thus

$$B(U, X) = m\theta(X). \tag*{(11.1)}$$

Taking $X = U$ to (3.5) and using (4.11), we get $g(A_{\xi} U, X) = 0$. Using this and the fact that $S(TM)$ is non-degenerate, we have $A_{\xi}^2 U = 0$. Replacing $X$ by $\xi$ to (4.3) and using (2.7), (3.7) and the fact that $\tau = 0$, we obtain $A_{\xi}^2 V = 0$. On the other hand, replacing $Y$ by $U$ to (4.6) and using (4.10), we get $A_{\xi} V = A_{\xi}^2 U$. Thus we see that $A_{\xi}^2 V = 0$.

where $\tilde{R}$ is the curvature tensor of the Levi-Civita connection $\tilde{\nabla}$ on $\tilde{M}$.

Denote by $R$ the curvature tensors of the $(\ell, m)$-type connection $\nabla$ on $M$. By directed calculations from (1.2) and (1.3), we see that

$$R(X, Y)Z = R(X, Y)Z + B(X, Z)A_{\bar{N}} Y - B(Y, Z)A_{\bar{N}} X + \{[\nabla_X C](Y, PZ) - ([\nabla_Y C](X, PZ) + \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)$$

$$- \ell[\theta(X)C(Y, PZ) - \theta(Y)C(X, PZ)] - m[\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)]\} \xi, \tag*{(5.4)}$$

respectively. Comparing the tangential and transversal components of the left and right terms of (5.2) and using (5.1), (5.3) and (5.3), we obtain

$$R(X, Y)Z = B(X, Z)A_{\bar{N}} X - B(Y, Z)A_{\bar{N}} Y + \{[\nabla_X \theta](Y, Z)\{fY + mF\}$$

$$- \{[\nabla_Y \theta](Z)\{fX + mFX\} + \theta(Z)\{\bar{Y}X - (\bar{Y}X)\} + (\bar{X}m)FY - (\bar{Y}m)FX\} \tag*{(5.5)}$$

Taking the scalar product with $N$ to (5.2) such that $\tilde{Z} = PZ$ and substituting (5.1), (5.3) and (5.4) into the resulting equa-

5 Indefinite complex space forms

Definition 3. An indefinite complex space form $\bar{M}(c)$ is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature $c$ such that

$$\tilde{R}(\bar{X}, \bar{Y})\bar{Z} = \frac{c}{4} \{g(\bar{Y}, \bar{Z})\bar{X} - g(\bar{X}, \bar{Z})\bar{Y} + g(\bar{Y}, \bar{Z})\bar{J}\bar{X} - g(\bar{J}\bar{X}, \bar{Z})\bar{J}\bar{Y}$$

$$+ 2g(\bar{X}, \bar{J}\bar{Y})\bar{Z}\}, \tag*{(5.1)}$$

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tion and using (3.6)₂, we get
\[
(\nabla_X C)(Y, PZ) = \nabla_Y C(X, PZ) - \tau(X) + \ell \theta(X) C(Y, PZ) + \{\tau(Y) + \ell \theta(Y)\} C(X, PZ) - m(\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)) - (\nabla_X \theta)(PZ)\{\ell v(Y) + mv(Y)\} + (\nabla_Y \theta)(PZ)\{\ell v(X) + mv(X)\} - \theta(PZ)\{(X\ell)\eta(Y) - (Y\ell)\eta(X) + (Xm)v(Y) - (Ym)v(X)\} = \frac{c}{4}(\eta(X)g(Y, PZ) - \eta(Y)g(X, PZ) + v(X)g(FY, PZ) - v(Y)g(FX, PZ) + 2v(PZ)g(X, JY)].
\]

(5.7)

**Definition 4.** A screen distribution \( S(TM) \) is called totally geodesic \([7]\) in \( M \) if \( C = 0 \) on a cooerinate neighborhood \( U \).

**Theorem 5.1.** Let \( M \) be a lightlike hypersurface of an indefinite complex space form \( \tilde{M}(c) \) with an \( (\ell, m) \)-type connection such that \( \zeta \) is tangent to \( M \). If one of the following four conditions is satisfied:

1. \( M \) Lie recurrent,
2. \( U \) is parallel with respect to \( \nabla \),
3. \( V \) is parallel with respect to \( \nabla \),
4. \( S(TM) \) is totally geodesic in \( M \),

then \( c = 0 \) and \( M(c) \) is flat.

**Proof.** (1) Applying \( \nabla_X \) to \( \theta(\zeta) = 0 \) and using (2.7), (3.4) and (3.5), we get
\[
(\nabla_X \theta)(\zeta) = \theta(A_2^2 X).
\]

(5.8)

Replacing \( Z \) by \( \xi \) to (5.5) and using (3.4) and (5.8), we have
\[
R(X, Y)\xi = \frac{c}{4}\{u(Y)FX - u(X)FY - 2\bar{g}(X, JY)V + \theta(A_2^2 X)\{\ell Y + mFY\} - \theta(A_2^2 Y)\{\ell X + mFX\}.\]

In general, using the Gauss-Weingarten formulae (2.6) and (2.7) for \( S(TM) \), we obtain the Codazzi equation for \( S(TM) \) such that
\[
R(X, Y)\xi = -\nabla_X^c(A_2^2 Y) + \nabla_Y^c(A_2^2 X) + A_2^2 [X, Y] - \tau(X)A_2^2 Y + \tau(Y)A_2^2 X + \{C(Y, A_2^2 X) - C(X, A_2^2 Y) - 2\delta r(X, Y)\}\xi.
\]

If \( M \) Lie recurrent, then we have \( \tau = 0 \) and \( A_2^2 U = A_2^2 V = 0 \) by Theorem 4.2. Comparing the radical components of the last two equations, we obtain
\[
C(Y, A_2^2 X) - C(X, A_2^2 Y) = \frac{c}{4}\{u(Y)v(X) - u(X)v(Y)\} + \theta(A_2^2 X)\{\ell v(Y) + mv(Y)\} - \theta(A_2^2 Y)\{\ell v(X) + mv(X)\},
\]
since \( \tau = 0 \). Taking \( X = V \) and \( Y = U \) to this equation and using the fact that \( A_2^2 U = A_2^2 V = 0 \), we obtain \( c = 0 \).

(2) If \( U \) is parallel with respect to \( \nabla \), then, taking the scalar product with \( U \) and \( U \) to (3.9) such that \( \nabla_X U = 0 \) by turns, we obtain
\[
\theta(U)\{\ell v(X) - m\eta(X)\} = 0, \quad \tau(X) + \ell \theta(U)u(X) = 0,
\]
respectively. From these two equations, we obtain
\[
\ell \theta(U) = 0, \quad m\theta(U) = 0, \quad \tau = 0.
\]

(5.9)

Applying \( \nabla_X \) to (5.9)₁,₂ and using (2.4) and the fact that \( \nabla_X U = 0 \), we get
\[
(X\ell)\theta(U) + \ell(\nabla_X \theta)(U) = 0, \quad (Xm)\theta(U) + m(\nabla_X \theta)(U) = 0.
\]

(5.10)

Also, taking the scalar product with \( \eta \) to (3.9): \( F(A_2^2 X, Y) = 0 \) and using (3.6), (5.9)₁,₂ and the fact that \( \nabla_X U = 0 \), we obtain
\[
C(X, U) = 0, \quad (\nabla_X C)(Y, U) = 0.
\]

(5.11)

Taking \( PZ = U \) to (5.7) and using (5.10)₁,₂ and (5.11)₁,₂, we obtain \( c = 0 \).

(3) If \( V \) is parallel with respect to \( \nabla \), then, taking the scalar product with \( V \) and \( U \) to (3.10) such that \( \nabla_X V = 0 \) by turns, we have
\[
\ell \theta(V)u(X) = 0, \quad \tau(X) = \theta(V)\{\ell v(X) - m\eta(X)\},
\]
respectively. From these two equations, we obtain
\[
\ell \theta(V) = 0, \quad \tau(X) = -m\theta(V)\eta(X).
\]

(5.12)

Applying \( \nabla_X \) to (5.12)₁ and using (2.4) and the fact that \( \nabla_X V = 0 \), we have
\[
(X\ell)\theta(V) + \ell(\nabla_X \theta)(V) = 0.
\]

(5.13)

Using (5.12), the equation (3.10) is reduced to
\[
F(A_2^2 X) + m\theta(V)\{FX + \eta(X)V\} = 0.
\]

Taking the scalar product with \( N \) to this equation and using (3.5), (3.8), (5.12)₁ and the fact that \( \nabla_X V = 0 \), we obtain
\[
C(X, V) = 0, \quad (\nabla_X C)(Y, U) = 0.
\]

Taking \( PZ = V \) to (5.7) and using (5.13) and the last two equations, we get
\[
- \{(Xm)\theta(V) + m(\nabla_X \theta)(V)\}v(Y) + \{(Ym)\theta(V) + m(\nabla_Y \theta)(V)\}v(X) = \frac{c}{4}\{(\eta(X)u(Y) - \eta(Y)u(X) + 2\bar{g}(X, JY)\}.
\]

Taking \( X = \xi \) and \( Y = U \) to this equation, we have \( c = 0 \).

(4) Taking \( X = \xi \) and \( X = V \) to (3.8) by turns and using (3.4)₂, we have
\[
\ell \theta(V) = 0, \quad B(V, U) = -m\theta(V), \quad B(U, V) = 0.
\]

(5.14)
As \( \theta(J\zeta) = 0 \) and \( \theta(N) = 0 \), we have \( \theta(F\zeta) = 0 \) due to (2.10). Also we have \( v(F\zeta) = 0 \) and \( u(F\zeta) = 0 \). Taking \( X = U \) and \( Y = F\zeta \) to (3.3), we obtain

\[
B(U, F\zeta) = B(F\zeta, U).
\]

Replacing \( X \) by \( F\zeta \) to (3.8) and using (5.14), we have

\[
B(F\zeta, U) = 0. \quad \text{(5.15)}
\]

Applying \( \nabla_U \) by \( \ell(\theta)(V) = 0 \) and using (3.10), (5.14) and (5.15), we get

\[
(U\ell)(\theta)(V) + \ell(\nabla_U \theta)(V) = 0. \quad \text{(5.16)}
\]

As \( S(TM) \) is totally geodesic in \( (M, \ell) \) is reduced to

\[
- \langle \nabla_X Y, g (PZ) \{ \ell \eta(Y) + m v(Y) \} \rangle + \langle \nabla_Y \theta(PZ) \{ \ell \eta(X) + m v(X) \} + \theta(PZ) \{ Y \ell - \eta(Y) \} \rangle + \eta(Y) g(X, PZ)
\]

Taking \( Y = U \) and \( PZ = V \) to this equation by turns and using (5.14), we obtain \( c = 0 \)

**Definition 5.** A lightlike hypersurface \( M \) is said to be screen conformal \( [9] \) if there exists a non-vanishing smooth function \( \varphi \) on \( U \) such that

\[
C(X, PY) = \varphi B(X, PY). \quad \text{(5.17)}
\]

**Theorem 5.2.** Let \( M \) be a screen conformal lightlike hypersurface of an indefinite complex space form \( M(c) \) with an \((\ell, m)\)-type connection such that \( \zeta \) is tangent to \( M \). Then the vector field \( \mu \), defined by \( \mu = U - \varphi V \), is an eigenvector of \( A^*_\zeta \) corresponding to the eigenvalue \( -m \theta(V) \). If \( m \theta(V) = 0 \), then \( c = 0 \).

**Proof.** Taking \( Y = \xi \) to (5.17) and using (3.4), we get

\[
C(\xi, PZ) = 0. \quad \text{Thus} \quad C(\xi, V) = 0. \quad \text{Taking} \quad X = \xi \text{to (3.8) and using} \ C(\xi, V) = 0, \text{we have}
\]

\[
\ell(\theta)(V) = 0. \quad \text{(5.18)}
\]

Applying \( \nabla_X \) to \( C(Y, PZ) = \varphi B(Y, PZ) \), we have

\[
(\nabla_X C)(Y, PZ) = (X\varphi) B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).
\]

Substituting this equation into (5.7) and using (5.6), we obtain

\[
\{ X\varphi - 2\varphi \tau(Y) \} B(Y, PZ) - \{ Y\varphi - 2\varphi \tau(Y) \} B(X, PZ)
\]

Replacing \( X \) by \( \xi \) to this equation and using (3.4) and (3.8), we have

\[
\{ \xi \varphi - 2\varphi \tau(\xi) \} B(Y, PZ) + \ell(\nabla_X \theta)(PZ) \{ \ell \eta(Y) + m g(Y, \mu) \}
\]

Taking \( Y = \xi \) to (3.8) and using (3.10), we have

\[
B(X, \mu) = m \{ \theta(U) u(X) - \theta(V) v(X) \}. \quad \text{(5.21)}
\]

As \( \theta(J\zeta) = 0 \) and \( \theta(N) = 0 \), we have \( \theta(F\zeta) = 0 \). Also we have \( v(F\zeta) = 0 \) and \( u(F\zeta) = 0 \). Replacing \( Y \) by \( \mu \) to (3.3) and using (5.21), we obtain

\[
B(\mu, X) = m \{ \theta(X) - \theta(V) g(X, \mu) \}. \quad \text{(5.22)}
\]

Taking \( X = V \) and \( X = F\zeta \) to (5.22) by turns and using (5.20), we obtain

\[
B(\mu, V) = B(\mu, F\zeta) = 0, \quad (\mu \ell)(\theta)(V) + \ell(\nabla_u \theta)(V) = 0. \quad \text{(5.23)}
\]

Replacing \( X \) by \( \mu \) to (3.5) and using (5.22), we have

\[
\varphi \{ m(\nabla_\xi \theta)(V) + \theta(V) \xi m \} = \frac{3}{c}. \quad \text{(5.24)}
\]

Thus \( \mu \) is an eigenvector of \( A^*_\zeta \) corresponding to the eigenvalue \( -m \theta(V) \). If \( m \theta(V) = 0 \), then, applying \( \nabla_\xi \) to this and using (3.7) and (3.10), we have

\[
(\xi m) \theta(V) + m(\nabla_\xi \theta)(V) = 0.
\]

From this equation and (5.24), we obtain \( c = 0 \).

**Corollary 5.3.** Let \( M \) be a lightlike hypersurface of an indefinite complex space form \( M(c) \) with a semi-symmetric non-metric connection. If \( M \) is screen conformal, then \( A^*_\zeta \mu = 0 \) and \( c = 0 \).
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