THE BERNDT-BHARGAVA-GARVAN TRANSFER PRINCIPLE FOR SIGNATURE THREE

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Abstract. We offer a new proof for the Berndt-Bhargava-Garvan Transfer Principle that connects the signature-three elliptic theory of Ramanujan to the classical elliptic theory.

INTRODUCTION

In [1995] Berndt, Bhargava and Garvan introduce a Transfer Principle by means of which to pass between 'classical' elliptic function theory and elliptic function theory in signature three. Their transfer principle is a direct consequence of then-new transformation formulae that relate the hypergeometric functions $F_2 = F\left(\frac{1}{2}, \frac{1}{2}; 1; \bullet\right)$ and $F_3 = F\left(\frac{2}{3}, \frac{2}{3}; 1; \bullet\right)$: explicitly, if with $0 < p < 1$ we define

$$\alpha = \frac{p^3(2 + p)}{1 + 2p} \quad \text{and} \quad \beta = \frac{27}{4} \frac{p^2(1 + p)^2}{(1 + p + p^2)^3}$$

then

$$(1 + p + p^2) F_2(\alpha) = \sqrt{1 + 2p} F_3(\beta)$$

and

$$(1 + p + p^2) F_2(1 - \alpha) = \sqrt{3 + 6p} F_3(1 - \beta)$$

whence

$$\frac{F_2(1 - \alpha)}{F_2(\alpha)} = \sqrt{3} \frac{F_3(1 - \beta)}{F_3(\beta)}.$$ 

In [1995] these hypergeometric identities appear as Theorem 5.6, Corollary 5.7 and Corollary 5.8 respectively; the transfer principle itself appears as Theorem 5.9. The proofs in [1995] rest on calculations involving $q$-series, especially the cubic theta-functions $a(q), b(q)$ and $c(q)$ of the brothers Borwein.

Here, we present completely different proofs of these hypergeometric identities. Our proofs derive from the elliptic function $dn_3$ of Shen [2004]; they therefore serve to reinforce the place of $dn_3$ within the theory of elliptic functions in signature three. In Section 1 we review the construction of the elliptic function $dn_3$ and its coperiodic Weierstrass function $p$; further, we express the fundamental periods of the elliptic functions $dn_3$ and $p$ explicitly in terms of the 'signature-three' hypergeometric function $F\left(\frac{1}{3}, \frac{2}{3}; 1; \bullet\right)$. In Section 2 we review the classical Jacobian perspective on the Weierstrass function $p$; this facilitates explicit expressions for the fundamental periods of $dn_3$ in terms of the 'classical' hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}; 1; \bullet\right)$. In Section 3 we assemble the pieces to deduce the aforementioned hypergeometric identities.

1. The elliptic function $dn_3$

The function $dn_3$ is initially defined as a strictly-increasing function from $\mathbb{R}$ onto $\mathbb{R}$; this function is then seen to satisfy a differential equation whose solutions are known to be elliptic. Here, we outline the construction; for details, see [2004].
We begin by fixing $0 < \kappa < 1$ as modulus and \( \lambda = \sqrt{1 - \kappa^2} \) as complementary modulus. We define \( \delta_\kappa : \mathbb{R} \to \mathbb{R} \) to be the derivative of the function that is inverse to

\[
T \mapsto \int_0^T F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt.
\]

The function \( \delta_\kappa \) (easily) satisfies the initial condition \( \delta_\kappa(0) = 1 \) and (less easily) satisfies the differential equation

\[
9(\delta_\kappa')^2 = 4\left(1 - \delta_\kappa\right)\left(\delta_\kappa^3 + 3\delta_\kappa^2 - 4\lambda^2\right).
\]

The solution of this initial value problem is readily identified, as follows.

**Theorem 1.** The function \( \delta_\kappa \) satisfies

\[
\left(1 - \delta_\kappa\right)(\frac{1}{3} + p_\kappa) = \frac{4}{9} \kappa^2
\]

where \( p_\kappa = \wp(\bullet; g_2, g_3) \) is the Weierstrass function with invariants

\[
g_2 = \frac{4}{27}(9 - 8\kappa^2) = \frac{4}{27}(8\lambda^2 + 1)
\]

and

\[
g_3 = \frac{8}{729}(27 - 36\kappa^2 + 8\kappa^4) = \frac{8}{729}(8\lambda^4 + 20\lambda^2 - 1).
\]

**Proof.** See [2004]. \(\square\)

The elliptic function \( \text{dn}_3 \) of Shen is the elliptic extension of \( \delta_\kappa \) whose existence is guaranteed by this Theorem: thus,

\[
\text{dn}_3 = 1 - \frac{4}{9} \kappa^2 \frac{1}{3} + p_\kappa.
\]

The elliptic function \( \text{dn}_3 \) and the Weierstrass function \( p_\kappa \) are plainly coperiodic. We write \((2\omega_\kappa, 2\omega'_\kappa)\) for their fundamental pair of periods with \( \omega_\kappa \) and \(-i\omega'_\kappa \) strictly positive.

A virtue of this construction is that it provides immediate access to the real half-period \( \omega_\kappa \) in explicit hypergeometric terms.

**Theorem 2.** The real half-period \( \omega_\kappa \) of \( \text{dn}_3 \) and \( p_\kappa \) is given by

\[
\omega_\kappa = \frac{\pi}{2} F\left(\frac{1}{3}, \frac{2}{3}; 1; \kappa^2\right).
\]

**Proof.** From the definition of \( \delta_\kappa \), its least positive period is equal to twice the integral

\[
\int_0^{2\pi} F\left(\frac{1}{3}, \frac{2}{3}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt;
\]

this may be calculated by expanding the hypergeometric series and integrating termwise, with the result announced in the Theorem. \(\square\)

An explicit hypergeometric expression for the imaginary half-period \( \omega'_\kappa \) lies a little deeper. We bring it to light by applying to the Weierstrass function \( p_\kappa = \wp(\bullet; g_2, g_3) = \wp(\bullet; \omega_\kappa, \omega'_\kappa) \) the modular transformation according to which its imaginary period is divided by three: thus, we introduce the Weierstrass function

\[
g_\kappa = \wp(\bullet; \omega_\kappa, \frac{1}{3} \omega'_\kappa).
\]

**Theorem 3.** The Weierstrass functions \( g_\kappa \) and \( p_\lambda \) are related by

\[
g_\kappa(z) = -3 p_\lambda(\sqrt{3} i z).
\]
Proof. Let the Weierstrass function $q_\kappa$ have quadrinvariant $h_2$ and cubinvariant $h_3$: then

\[ h_2 = 120b^2 - 9g_2 \]

and

\[ h_3 = 280b^3 - 42bg_2 - 27g_3 \]

where $b$ is the value of $p_\kappa$ at $\frac{2}{3}\omega'_\kappa$; for this general consequence of Weierstrassian trimidiation, see Section 68 of [1973]. It is proved in [2004] Section 5 that $p_\kappa(\frac{2}{3}\omega'_\kappa)$ is precisely $-\frac{1}{3}$; using also $g_2$ and $g_3$ from Theorem 1 it follows that

\[ h_2 = \frac{4}{3}(1 + 8\kappa^2) \]

and

\[ h_3 = \frac{8}{27}(1 - 20\kappa^2 - 8\kappa^4). \]

A glance at Theorem 1 reveals the relationship between these invariants and those of $p_\lambda$: namely, $h_2$ is $9 = (\sqrt{3}i)^4$ times the quadrinvariant of $p_\lambda$, and $h_3$ is $-27 = (\sqrt{3}i)^6$ times the cubinvariant of $p_\lambda$. Finally, the homogeneity relation for $p$-functions completes the proof.

Here, notice both the switch to the complementary modulus and the quarter-rotation of the period lattice.

We are now prepared to identify the imaginary half-period $\omega'_\kappa$ in explicit hypergeometric terms.

**Theorem 4.**

\[ \omega'_\kappa = i\frac{\sqrt{3}}{2}\pi F\left(\frac{1}{3}, \frac{1}{3}, 1; 1 - \kappa^2\right). \]

*Proof.* Upon comparing the fundamental half-periods of the Weierstrass functions involved, we see at once that Theorem 3 implies the relation

\[ \omega'_\kappa = i\sqrt{3}\omega_\lambda; \]

the present Theorem therefore follows from Theorem 2 in view of the fact that $\lambda^2 = 1 - \kappa^2$. □

2. The Jacobian reformulation

Here, we derive alternative explicit expressions for the fundamental periods of $dn_3$ in terms of the hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}, 1; \bullet\right)$ that is associated to the classical theory of Jacobian elliptic functions.

Recall (from Chapter XXII of [1927] for instance) the fact that if $p$ is a Weierstrass function with real midpoint values $e_1 > e_2 > e_3$ then

\[ p(z) = e_3 + \frac{e_1 - e_3}{sn^2(z (e_1 - e_3)^{1/2})} \]

where $sn = sn(\bullet, k)$ is the Jacobian elliptic function with modulus $k$ given by

\[ k^2 = \frac{e_2 - e_3}{e_1 - e_3}. \]

Recall also that $sn$ has fundamental pair of periods $(4K, 2iK')$ where

\[ K = \frac{1}{2}\pi F\left(\frac{1}{2}, 1; 1; k^2\right) \]

and

\[ K' = \frac{1}{2}\pi F\left(\frac{1}{2}, 1; 1; 1 - k^2\right); \]

recall further that addition of $2K$ to the argument of $sn$ effects merely a reversal of sign, so that $sn^2$ has $(2K, 2iK')$ as a fundamental pair of periods. It follows that the Weierstrass function $p$ has as fundamental pair of half-periods

\[ \omega = \frac{K}{(e_1 - e_3)^{1/2}} \quad \text{and} \quad \omega' = i\frac{K'}{(e_1 - e_3)^{1/2}}. \]
We now elaborate upon these facts as they apply to the Weierstrass function \( p_\kappa \) that is coperiodic with \( \text{dn}_3 \). Let the (acute) modular angle \( \theta \) be defined by
\[
\kappa = \sin \theta
\]
and introduce the abbreviations
\[
s = \sin \frac{1}{3} \theta \quad \text{and} \quad c = \cos \frac{1}{3} \theta
\]
so that trigonometric trimidiation yields
\[
\kappa = s(3 - 4s^2).
\]
Factorizing the right-hand side of the differential equation
\[
(p_\kappa')^2 = 4(p_\kappa)^4 - g_2 p_\kappa - g_3
\]
with \( g_2 \) and \( g_3 \) as in Theorem 1 reveals that the midpoint values of \( p_\kappa \) are given by
\[
e_1 = \frac{1}{9}(8s^4 + 12s^2 - 3 + 8\sqrt{3} s^3 c)
\]
\[
e_2 = \frac{1}{9}(-8s^4 + 12s^2 - 3 - 8\sqrt{3} s^3 c)
\]
In terms of the foregoing choices of notation, these deliberations have the following outcome.

**Theorem 5.** The fundamental half-periods of \( p_\kappa \) and \( \text{dn}_3 \) are given by
\[
 r \omega_\kappa = \frac{1}{2} \pi F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)
\]
and
\[
 r \omega'_\kappa = i \frac{1}{4} \pi F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - k^2\right)
\]
where \( r > 0 \) is given by
\[
r^2 = \frac{1}{3\sqrt{3}}(8s^3 c + \sqrt{3}(8s^4 - 12s^2 + 3))
\]
and \( k > 0 \) is given by
\[
k^2 = \frac{16s^3 c}{8s^3 c + \sqrt{3}(8s^4 - 12s^2 + 3)}.
\]

**Proof.** First substitute the midpoint values into \( r = (e_1 - e_3)^{1/2} \) and \( k^2 = (e_2 - e_3)/(e_1 - e_3) \); then invoke the relationship between Weierstrassian periods and Jacobian periods that was recalled above.

\[\square\]

3. The hypergeometric identities

We are now in a position to address the hypergeometric identities to which we alluded in our Introduction. In fact, these hypergeometric identities will follow at once from from a direct comparison of the formulae in Theorem 2 and Theorem 4 with the formulae in Theorem 5 once we introduce the parameter \( p \) from [1995].

To introduce this parameter, we begin by noting that the rule
\[
(0, 1) \to (0, 1) : p \mapsto \sqrt{\frac{3p^2}{1 + p + p^2}}
\]
defines a bijection; we may coordinate this with the bijection
\[
(0, \frac{1}{3} \pi) \to (0, 1) : \theta \mapsto 2s = 2\sin \frac{1}{3} \theta
\]
to obtain a parametrization according to which \( s = \sin \frac{1}{3} \theta \) is given by
\[
s = \frac{\sqrt{3}}{2} \frac{p}{(1 + p + p^2)^{1/2}}
\]
and the complementary $c = \cos \frac{1}{3} \theta$ is given by
\[ c = \frac{1}{2} \frac{2 + p}{(1 + p + p^2)^{1/2}} \]
while inversely
\[ p = 2 \frac{s^2 + \sqrt{3}sc}{3 - 4s^2}. \]
In terms of this parametrization, the original modulus $\kappa = \sin \theta$ of $\operatorname{dn}_3$ is given by
\[ \kappa^2 = s^2 (3 - 4s^2)^2 = \frac{27}{4} \frac{p^2 (1 + p)^2}{(1 + p + p^2)^3}. \]
Now return to the context of Theorem 5. A moderate amount of calculation confirms that
\[ 8s^4 - 12s^2 + 3 = \frac{3}{2(1 + p + p^2)^2} (2 + 4p - 2p^3 - p^4) \]
and
\[ s^3 c = \frac{3\sqrt{3}}{16} \frac{p^3 (2 + p)}{(1 + p + p^2)^2}, \]
whence (after further calculation) the spread of the midpoint values is given by
\[ r^2 = e_1 - e_3 = \frac{1 + 2p}{(1 + p + p^2)^2} \]
and the Jacobian modulus is given by
\[ k^2 = \frac{e_2 - e_3}{e_1 - e_3} = \frac{p^3 (2 + p)}{1 + 2p}. \]
As in the Introduction, for typographical convenience we shall adopt the abbreviations
\[ F_2 = F(\frac{1}{3}; \frac{1}{2}; 1; \bullet) \]
and
\[ F_3 = F(\frac{1}{3}; \frac{2}{3}; 1; \bullet) \]
in each of the following three Theorems. We shall also adopt from [1995] the abbreviations
\[ \alpha = \frac{p^3 (2 + p)}{1 + 2p} \]
and
\[ \beta = \frac{27}{4} \frac{p^2 (1 + p)^2}{(1 + p + p^2)^3}. \]
Notice that $\alpha$ is precisely $k^2$ and $\beta$ is none other than $\kappa^2$.

The following result is [1995] Theorem 5.6.

**Theorem 6.** If $0 < p < 1$ then
\[ (1 + p + p^2) F_2(\alpha) = \sqrt{1 + 2p} F_3(\beta) \]

**Proof.** Simply compare the formula for $\omega_\kappa$ in Theorem 2 with the formula for $\omega_\kappa$ in Theorem 5, taking note of how the original modulus $\kappa$, the Jacobian modulus $k$ and the spread $r^2 = e_1 - e_3$ depend on the parameter $p$ as displayed leading up to the present Theorem. □

The following result is [1995] Corollary 5.7.
Theorem 7. If $0 < p < 1$ then
\[
(1 + p + p^2) F_2(1 - \alpha) = \sqrt{3 + 6p} F_3(1 - \beta)
\]

Proof. Simply compare the formula for $\omega'_\kappa$ in Theorem 4 with the formula for $\omega'_\kappa$ in Theorem 5 again taking note of how $\kappa$, $k$ and $r^2$ depend on the parameter $p$. \qed

The following result is [1995] Corollary 5.8 (mildly rewritten).

Theorem 8. If $0 < p < 1$ then
\[
\frac{F_2(1 - \alpha)}{F_2(\alpha)} = \sqrt{3} \frac{F_3(1 - \beta)}{F_3(\beta)}.
\]

Proof. An immediate consequence of Theorem 6 and Theorem 7 \qed

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