Thermal convection in a linearly viscous fluid 
overlying a bidisperse porous medium

P. Dondl
Abteilung für Angewandte Mathematik
Albert-Ludwigs-Universität Freiburg
Hermann-Herder-Str. 10,
79104 Freiburg, Germany

and

B. Straughan
Department of Mathematics
University of Durham, Durham DH1 3LE, UK

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Abstract

A bidisperse porous medium is one with two porosity scales. There are the usual pores known as macro pores but also cracks or fissures in the skeleton which give rise to micro pores. In this article we develop and analyse a model for thermal convection where a layer of viscous incompressible fluid overlies a layer of bidisperse porous medium. Care has to be taken with the boundary conditions at the interface of the fluid and the porous material and this aspect is investigated. The situation is one in a layer which is heated from below and under appropriate conditions bimodal neutral curves are found. These depend on the ratio \( \hat{d} \) of the depth \( d \) of the fluid layer to the depth \( d_m \) of the porous layer. We show that there is a critical value of \( \hat{d} \) such that below this value convective motion initiates in the porous layer whereas for \( \hat{d} \) above this value the convective instability commences in the fluid layer.

1 Introduction

The problem of flow of a fluid overlying a porous medium saturated by the same fluid is one which has attracted the attention of many prominent scientists. A fundamental interface condition between the fluid and the porous medium was proposed by Beavers and Joseph \cite{beavers1967boundary}. The first analysis of thermal convection in the situation where a fluid overlies a saturated porous medium is due to
Nield [38] who successfully employed the Beavers - Joseph boundary condition to derive a satisfactory model. A surprising result for the same thermal convection problem was discovered by Chen and Chen [14] who showed that the ratio of fluid depth, $d$, to porous layer depth, $d_m$, defined by $\hat{d} = d/d_m$, is critical to determining the process for the onset of thermal convection in the two layer system. If $\hat{d}$ is below a critical value then convection commences in the porous layer whereas when $\hat{d}$ is above the critical value then convection commences in the fluid. This class of problem was further investigated both experimentally and theoretically by Chen and Chen [15, 16], Chen [13], McKay and Straughan [36], where the last mentioned article applies the theory to the problem of stone formation into regular patterns at the bottom of a shallow lake.

The subject of thermal convection or generally flow in a two layer system has been studied in much detail with a review of the early work and applications to various areas in industry or geophysics given in chapter 6 of Straughan [50]. The intense interest in this class of problem has been driven by the many applications to diverse areas such as heat pipe technology, renewable energy, see e.g. Straughan [50], contaminant dispersal in waterways, Hibi and Tomigashi [25], Hibi [24], or even blood flow in arteries and veins in the human body, see e.g. Sharma and Yadav [47], Tiwari et al. [52], Ponalagusamy and Manchi [42], Wajihah and Sankar [54]. Indeed, the last mentioned article involves 5 layer flow comprising Darcy media, Brinkman media, plasma, core flow, and plug flow.

There are many recent stability analyses of fluid flow in the two layer fluid - porous configuration, see e.g. Carr and Straughan [8], Chang et al. [11], Hill and Straughan [26, 28], Chang et al. [12], Samanta [45], Yin et al. [60], Tiberkin [56]. Particular analyses involving the nonlinear theory and bifurcations are given by Han et al. [23], McCurdy et al. [33], Lyu and Wang [34], and Hill and Straughan [27]. In addition, mathematical analysis of the structural stability of the two layer system has been thoroughly investigated, see e.g. Li et al. [31], Li et al. [32], Payne and Straughan [41].

In a separate development, there has been immense interest in thermal convection in a single layer of saturated porous material but when the porous skeleton is of double porosity type. By double porosity we mean that the solid skeleton contains pores of a visible size known as macro pores, but the skeleton itself contains cracks or fissures which give rise to much smaller micropores. Flow in such materials is additionally called bidisperse or bidispersive. The thermal convection problem in a bidisperse porous material was first developed by Nield and Kuznetsov [40] who allowed for different velocities, pressures and temperature fields in the macro and micro phases and a critical review of the topic is given by Gentile and Straughan [22]. There are many recent contributions driven by the need to understand bidispersive convection in real life applications, cf. Gentile and Straughan [22], Straughan [51], chapter 13. Analyses of bidispersive thermal convection in isotropic, anisotropic, vertical layer, inclined layer, and with rotation effects are given by Badday and Harfash [1], Capone et al. [6], Capone et al. [5], Capone and De Luca [3], Capone and Massa [4], Capone et al. [7], Chaloob et al. [9], Falsaperla et al. [18], Gentile and
 Straughan [20, 21], Saravanan and Vigneshwaran [46], Straughan [52, 54, 53], and the structural stability aspect of the system of equations is considered by Franchi et al. [19].

The object of the current work is to present a model for thermal convection in an incompressible viscous fluid when that fluid overlies a bidispersive porous medium saturated with the same fluid. We analyse the instability of thermal convection in this system and show there are definite relations between the onset of convective motion and the respective depths of the fluid and porous layers, and of the properties of the macro and micro pores. This is the first analysis we have seen of this problem and we believe it has much future application to diverse areas such as blood flow, heat transfer, and renewable energy.

2 Basic equations

We suppose a linearly viscous incompressible fluid is contained in the infinite layer \( \mathbb{R}^2 \times \{0 < z < d\} \) and below this is a bidisperse porous medium saturated with the same fluid and this occupies the infinite layer \( \mathbb{R}^2 \times \{-d_m < z < 0\} \), with gravity acting in the negative \( z \)-direction.

The equations for the fluid in the layer \( \mathbb{R}^2 \times \{0 < z < d\} \) are then, cf. Chandrasekhar [10],

\[
V_{i,t} + V_j V_{i,j} = -\frac{1}{\rho_0} p_{,i} + \nu \Delta V_i + \gamma g k T,
\]

\[V_{i,i} = 0,
\]

\[T_{,t} + V_i T_{,i} = \frac{k_f}{(\rho_0 c_p) f} \Delta T,
\]

where \( V_i(x, t) \) is the velocity field, \( T(x, t) \) is the temperature field, \( p(x, t) \) is the pressure field, \( x \) is the spatial point in the layer, and \( t \) is time. We use indicial notation throughout in conjunction with the Einstein summation convention, so that, for example,

\[ V_{i,i} \equiv \sum_{i=1}^{3} V_i T_{,i} = V_1 \frac{\partial T}{\partial x_1} + V_2 \frac{\partial T}{\partial x_2} + V_3 \frac{\partial T}{\partial x_3} \]

\[ \equiv U \frac{\partial T}{\partial x} + V \frac{\partial T}{\partial y} + W \frac{\partial T}{\partial z} ,
\]

where \( \mathbf{V} = (V_1, V_2, V_3) \equiv (U, V, W) \). In equations \( \gamma, g, \rho_0, k_f, \nu \) and \( c_p \) are the thermal expansion coefficient of the fluid, gravity, reference density, thermal conductivity of the fluid, kinematic viscosity of the fluid, and specific heat at constant pressure of the fluid. The vector \( \mathbf{k} = (0, 0, 1) \) and \( \Delta \) is the Laplacian

\[ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .
\]

For an isotropic bidisperse porous material we suppose the macro porosity is \( \phi \), the micro porosity is \( \epsilon \). If we denote \((U_f^l, p_f^l)\) to be the pore averaged...
velocity and pressure in the macropores and \((U_i^f, p_i^f)\) to be the pore averaged velocity and pressure in the micropores, then the governing equations of flow in the bidisperse porous medium may be written, cf. Gentile and Straughan [20], Straughan [52],

\[-\frac{\mu}{K_f} U_i^f - \zeta (U_i^f - U_i^p) - p_i^f + \rho_0 \gamma k_i g T^m = 0,\]

\[U_{i,i}^f = 0,\]

\[-\frac{\mu}{K_p} U_i^p - \zeta (U_i^p - U_i^f) - p_i^p + \rho_0 \gamma k_i g T^m = 0,\]

\[U_{i,i}^p = 0,\]

\[\rho_0 c_m T^m_{i,i} + \rho_0 c_f (U_i^f + U_i^p) T^m_{i,i} = k_m \Delta T^m,\]

where \(T^m(x, t)\) is the temperature field of the fluid in the bidisperse porous medium.

Equations (1) hold on the domain \(\{(x, y) \in \mathbb{R}^2 \times \{z \in (0, d)\} \times \{t > 0\}\) while (2) hold on the domain \(\{(x, y) \in \mathbb{R}^2 \times \{z \in (-d_m, 0)\} \times \{t > 0\}\). Equations (2) assume Darcy’s law holds and a Boussinesq approximation is employed. The variable \(\mu\) is the dynamic viscosity of the fluid, \(K_f\) and \(K_p\) are the macro and micro permeabilities, \(\zeta\) is an interaction coefficient which represents the momentum transfer between the macro and micro phases, and \((\rho_0 c)_m, k_m\) are given by

\[(\rho_0 c)_m = (1 - \phi)(1 - \epsilon)(\rho_0 c)_s + \phi (\rho_0 c)_f + \epsilon (1 - \phi)(\rho_0 c)_p,\]

and

\[k_m = (1 - \phi)(1 - \epsilon)k_s + \phi k_f + \epsilon (1 - \phi)k_p,\]

where \(s, f\) and \(p\) denote values in the solid skeleton, the fluid in the macropores, and the fluid in the micropores.

The boundary conditions at the top and bottom of the layer, \(z = d\) and \(z = -d_m\) are specified as follows

\[V_i = 0, \quad T = T_U, \quad \text{on} \quad z = d,\]

\[U_i^f = 0, \quad U_i^p = 0, \quad T^m = T_L, \quad \text{on} \quad z = -d_m,\]

where \(T_U, T_L\) are constants with \(T_L > T_U > 0\). Under these conditions (1) and (2) admit a steady conduction solution of form

\[\bar{V}_i = 0, \quad \bar{U}_i^f = 0, \quad \bar{U}_i^p = 0, \quad \bar{T} = T_0 - (T_0 - T_U) \frac{z}{d}, \quad 0 \leq z \leq d,\]

\[\bar{T}^m = T_0 - (T_L - T_0) \frac{z}{d_m}, \quad -d_m \leq z \leq 0,\]

where we have employed the fact that the temperature is continuous across the interface \(z = 0\). To determine the constant \(T_0\) we require the heat flux to be
continuous across the interface $z = 0$ and then

$$k_m \frac{dT_m}{dz} = k_f \frac{dT}{dz}, \quad \text{at } z = 0.$$ 

This yields $T_0$ as

$$T_0 = \frac{k_f T_U d_m + k_m T_L d}{d k_m + d_m k_f}. \quad (5)$$

3 Perturbation equations

As our goal is to study instability of the base solution (4) we now introduce perturbations to the variables $V_i, T, p_f, U_i^f, U_i^p, T^m, p^m$ as $u_i, \theta, \pi_f, u_i^f, u_i^p, \theta^m$ and $\pi^m$. We derive the perturbation equations for these variables from equations (1) and (2). However, it is convenient to present these in non-dimensional form with length scales $d, d_m$ in the fluid and bidisperse porous layers, and with corresponding velocity scales $U$ and $U_m$ for the base flow. The time scales are $T$ and $T^m$ where $T = d^2/\nu$, with $\nu = \mu/\rho_0$ and we pick $T = T^m$.

The pressure scale is $P = \mu U/d$. The temperature scales are

$$T^* = \frac{U(T_0 - T_U) d (\rho_0 c_p)_f}{k_f}$$

and

$$T^*_m = \frac{U_m (T_L - T_0) d_m (\rho_0 c_p)_f}{k_m}$$

and thus one finds

$$\frac{T^*_m}{T^*} = \left( \frac{k_m}{k_f} \right)^2 \hat{d}^2; \quad \frac{T_0 - T_U}{T_L - T_0} = \frac{k_m}{k_f} \hat{d}$$

where $\hat{d} = d/d_m$. It is convenient to introduce the notation

$$\hat{k} = \frac{k_f}{k_m}, \quad \hat{\kappa} = \frac{(\rho_0 c)_m}{(\rho_0 c)_f} \hat{k},$$

and to define the Prandtl number, $Pr$, and the porous Prandtl number, $Pr_m$, by

$$Pr = \frac{\nu}{\kappa} = \frac{\mu}{\rho_0} \frac{(\rho_0 c)_f}{k_f}, \quad Pr_m = \frac{\mu}{\rho_0} \frac{(\rho_0 c)_m}{k_m}$$

from which one may show $Pr_m = \hat{\kappa} Pr$.

The fluid and porous Rayleigh numbers $Ra$ and $Ra_m$ are defined by

$$Ra = \frac{\gamma g d^4 (T_0 - T_U)}{\kappa_f \nu d}, \quad (6)$$

where $\kappa_f = k_f/(\rho_0 c_p)_f$, and

$$Ra_m = \frac{\gamma g K_f (T_L - T_0)}{d_m^2} \left[ \frac{d_m}{(\rho_0 c)_f \nu} \right]. \quad (7)$$
from which one shows

\[ Ra = \left( \frac{\hat{d}}{k} \right)^2 \frac{Ra_m}{Da} \]

where \( Da \) is the Darcy number defined here as

\[ Da = \frac{K_f}{d^2}. \]

The relative permeability \( K_r \) is defined by

\[ K_r = \frac{K_f}{K_p}, \]

and another useful non-dimensional variable is \( \delta = \sqrt{K_p/d_m} \), from which we may see that \( Da = K_r \delta^2/d^2 \).

With the above non-dimensionalization one may show that the linearized fluid perturbation equations have form

\[
\begin{align*}
    u_{i,t} &= -\pi_{i,j} + \Delta u_i + Rak_i \theta, \\
    u_{i,i} &= 0, \\
    Pr \theta_{t,ij} &= w + \Delta \theta
\end{align*}
\]  

where \( w = u_3 \) and these equations hold on \( \mathbb{R}^2 \times \{ z \in (0,1) \} \times \{ t > 0 \} \) while the linearized bidispersive porous media equations have form

\[
\begin{align*}
    -u_{i,j}^f - \xi(u_{i,j}^f - u_{i,j}^p) - \pi_{i,j}^f + Ra_m k_i \theta_m &= 0, \\
    u_{i,i}^f &= 0, \\
    -K_r u_{i,j}^p - \xi(u_{i,j}^p - u_{i,j}^f) - \pi_{i,j}^p + Ra_m k_i \theta_m &= 0, \\
    u_{i,i}^p &= 0, \\
    \frac{Pr_m}{d^2} \theta^m_{t,ij} &= w^f + w^p + \Delta \theta^m
\end{align*}
\]

where \( w^f = u_{3,j}^f, w^p = u_{3,j}^p \) and equations (9) hold on \( \mathbb{R}^2 \times \{ z_m \in (-1,0) \} \times \{ t > 0 \} \). The non-dimensional momentum transfer interaction coefficient \( \xi \) is defined by \( \xi = \zeta K_f/\mu \).

Equations (8) and (9) when reduced to the thermal convection instability problem essentially represent a twelfth order system of equations. We thus require twelve boundary conditions. In this work we employ a normal mode instability analysis and it is sufficient to require the following conditions

\[ w = w' = \theta = 0, \quad \text{on } z = 1, \]

where \( w' = \partial w/\partial z \). This corresponds to a fixed upper surface. Also,

\[ w^f = w^p = \theta^m = 0, \quad \text{on } z = -1. \]

The remaining six boundary conditions come from considerations on the interface \( z = 0 \). At the microscopic level the velocity \( W \) in the fluid should be continuous across the interface with the actual velocity \( W^f \) in a macro pore and
with the actual velocity \( W_p \) in a micro pore. In dimensional form this requires at \( z = 0 \),

\[
W = \frac{W_f}{\phi}, \quad \text{and} \quad W = \frac{W_p}{\epsilon(1 - \phi)},
\]

where \( w_f \) and \( w_p \) are the pore averaged values. Likewise the dimensional temperatures are continuous so

\[
\theta = \theta_m, \quad \text{at} \quad z = 0.
\]

In addition the normal component of heat flux \( q \cdot n \) is continuous across \( z = 0 \). Two further conditions are needed and these arise by requiring continuity of normal stress, and by appealing to a combination of appropriate forms of the experimentally verified Beavers and Joseph \cite{2} condition. Details of these boundary conditions are amplified below.

4 Instability analysis

The next step is to remove the pressure terms \( \pi, \pi_f, \) and \( \pi_p \) from (8) and (9) and this we do by taking curl curl of equations (8) and (9) retaining the third component. This results in the system of equations

\[
\begin{align*}
\sigma \Delta w &= \Delta^2 w + Ra \Delta^* \theta, \\
\sigma Pr \theta &= w + \Delta \theta,
\end{align*}
\]

and

\[
\begin{align*}
(1 + \xi) \Delta w_f - \xi \Delta w_p - Ra^m \Delta^* \theta^m &= 0, \\
(K_r + \xi) \Delta w_p - \xi \Delta w_f - Ra^m \Delta^* \theta^m &= 0, \\
\frac{\sigma_m Pr_m}{\partial z^2} \theta^m &= w^f + w^p + \Delta \theta^m,
\end{align*}
\]

where we have represented time by \( e^{\sigma t} \) in the fluid equations and by \( \exp(\sigma_m t) \) in the bidispersive porous equations. The symbol \( \Delta^* \) is the horizontal Laplacian. Equations (10) hold on \( \mathbb{R}^2 \times (0,1) \) while (11) hold on \( \mathbb{R}^2 \times (-1,0) \).

We next solve (11) in terms of \( \Delta w_f \) and \( \Delta w_p \). We represent \( w \) and \( \theta \) as \( w = W(z)h(x,y), \theta = \Theta(z)h(x,y) \), and \( w_f = W_f(z)h_m(x,y), \quad w_p = W_p(z)h_m(x,y), \quad \theta_m = \Theta_m(z)h_m(x,y) \), where \( h \) and \( h_m \) are plan forms which tile the plane, cf. Chandrasekhar \cite{10}, pp. 43-52, and are typical of the hexagonal convection cell forms found in real life. The functions \( h \) and \( h_m \) satisfy the relations \( \Delta^* h = -a^2 h \) and \( \Delta^* h_m = -a_m^2 h_m \), for wavenumbers \( a \) and \( a_m \). We reduce (10) to two second order equations by setting \( \Delta w = \chi \), and then we arrive at the following coupled system of equations to solve for the growth rate (eigenvalues) \( \sigma, \sigma_m \),

\[
\begin{align*}
(D^2 - a^2)W - \chi &= 0, \\
(D^2 - a^2)\chi - Ra a^2 \Theta &= \sigma \chi, \\
(D^2 - a^2)\Theta + W &= Pr \sigma \Theta,
\end{align*}
\]

7
on $z \in (0, 1)$, where $D = d/dz$, and

$$\begin{align*}
(D^2 - a_m^2)W^f + Ra_m a_m^2 (K_r + 2\zeta) (K_r + \zeta + \xi K_r) \Theta_m &= 0, \\
(D^2 - a_m^2)W^p + Ra_m a_m^2 (1 + 2\zeta) (K_r + \zeta + \xi K_r) \Theta_m &= 0, \\
(D^2 - a_m^2)\Theta_m + W^f + W^p &= \frac{Pr_m}{d^2} \sigma_m \Theta_m,
\end{align*}$$

(13)
on $z_m \in (-1, 0)$.

The non-dimensional boundary conditions are

$$\begin{align*}
W &= W' = \Theta = 0, & z &= 1, \\
W^f &= W^p = \Theta^m = 0, & z &= -1,
\end{align*}$$

(14)
together with the non-dimensional interface conditions

$$\begin{align*}
W &= \frac{W^p}{\epsilon(1 - \phi)}, & W &= \frac{W^f}{\phi}, & z &= 0, \\
\theta_m &= \theta \left(\frac{k_m}{k_f}\right)^2 \hat{d}, & \frac{d\theta_m}{dz_m} &= \frac{d\theta}{dz} \frac{k_m}{k_f}, & z &= 0,
\end{align*}$$

(15)
where the latter two arise due to continuity of temperature, and continuity of heat flux.

For the remaining interface conditions we argue as follows.

In terms of the fluid and pore averaged velocities at the microscopic level, the Beavers and Joseph [2] condition may be applied separately to the macro and micro components to yield in dimensionless form

$$\frac{1}{d^2} \frac{\partial u_\beta}{\partial z} = \frac{\alpha_1}{\sqrt{K_f}} (u_\beta - u_\beta^f), \quad \beta = 1, 2,$$

(16)
and

$$\frac{1}{d^2} \frac{\partial u_\beta}{\partial z} = \frac{\alpha_2}{\sqrt{K_p}} (u_\beta - u_\beta^p), \quad \beta = 1, 2,$$

(17)
where $\alpha_1$ and $\alpha_2$ are experimentally determined constants. For a single porosity material Beavers and Joseph [2] write that “$\alpha$ is a dimensionless quantity depending on the material parameters which characterize the structure of permeable material within the boundary region”. Our first approach is to take an averaged value of each of the conditions (16) and (17) and propose at the interface $z = 0$,

$$\frac{\partial u_\beta}{\partial z} = (A_1 + A_2)u_\beta - A_1 u_\beta^f - A_2 u_\beta^p, \quad \beta = 1, 2,$$

(18)
where

$$A_1 = \frac{\alpha_1 d}{2 \sqrt{K_f}}, \quad A_2 = \frac{\alpha_2 d}{2 \sqrt{K_p}}.$$
We next differentiate (18) and employ the incompressibility conditions to derive the interface condition

$$w_{zz} = (A_1 + A_2)w_z - A_1 \hat{w}_m^f - A_2 \hat{w}_m^p$$

(20)

where the derivatives are with respect to $z$, $z_m$ non-dimensional, $0 \leq z \leq 1$, $-1 \leq z_m \leq 0$.

We also consider an alternative procedure whereby we combine the Beavers and Joseph [2] conditions (16) and (17) in a manner which reflects the macro and micro contributions. Thus, define the constants $c_1$ and $c_2$ by

$$c_1 = \frac{\phi}{\phi + \epsilon(1 - \phi)}, \quad c_2 = \frac{\epsilon(1 - \phi)}{\phi + \epsilon(1 - \phi)}.$$

We now produce an equation of form (18) but where now

$$A_1 = c_1 \alpha_1 d / \sqrt{K_f}, \quad A_2 = c_2 \alpha_2 d / \sqrt{K_p}.$$  

(21)

We thus derive an equation of form (20) but with $A_1$ and $A_2$ as given here. The differences in employing (20) and the pore weighted version just discussed are considered in the numerical results section. For ease in understanding the numerical results where the effect of parameter variation upon the solution is considered we specifically write the two versions of (20) here as, equal splitting,

$$w_{zz} = \left( \frac{\alpha_1 d}{2 \sqrt{K_f}} + \frac{\alpha_2 d}{2 \sqrt{K_p}} \right) w_z - \frac{\alpha_1 d}{2 \sqrt{K_f}} \hat{w}_m^f - \frac{\alpha_2 d}{2 \sqrt{K_p}} \hat{w}_m^p$$

(22)

and pore weighted

$$w_{zz} = \left( \frac{c_1 \alpha_1 d}{\sqrt{K_f}} + \frac{c_2 \alpha_2 d}{\sqrt{K_p}} \right) w_z - \frac{c_1 \alpha_1 d}{\sqrt{K_f}} \hat{w}_m^f - \frac{c_2 \alpha_2 d}{\sqrt{K_p}} \hat{w}_m^p.$$  

(23)

At the microscopic level, continuity of normal stress at the interface $z = 0$ requires

$$\pi^f = \pi - 2\mu w_z, \quad \text{for a macropore},$$

and

$$\pi^p = \pi - 2\mu w_z, \quad \text{for a micropore}.$$  

Our first continuity of normal stress interface condition is

$$\frac{\pi^f + \pi^p}{2} = \pi - 2\mu w_z, \quad \text{on } z = 0.$$  

(24)

Equation (24) is differentiated with respect to $x_\alpha$, $\alpha = 1, 2$, and one then employs the differential equations (3)1,2 and (9)1−4 in the forms

$$\pi_{,z} = \Delta u_\alpha - \sigma u_\alpha, \quad u_{,\alpha, z} + w_{,z} = 0,$$

$$\pi^f_{,z} = -u^f_{,\alpha} - \xi (u^f_{,\alpha} - u^p_{,\alpha}), \quad u^f_{,\alpha, z} + w^f_{,z} = 0,$$

$$\pi^p_{,z} = -K_p u^p_{,\alpha} - \xi (u^p_{,\alpha} - u^f_{,\alpha}), \quad u^p_{,\alpha, z} + w^p_{,z} = 0.$$  

(25)
to eliminate the pressure terms. The differentiated form of (24) is rewritten using (25) as
\[
\frac{1}{2} \left( -w_f^\alpha - \xi[u_f^\alpha - u_p^\alpha] - K_r u_p^\alpha - \xi[u_p^\alpha - u_f^\alpha] \right) = \Delta u_\alpha - \sigma u_\alpha - 2\mu w_{z\alpha},
\]
where \(\alpha = 1, 2\). Note that in this case the \(\xi\) terms disappear. Now differentiate (26) for \(\alpha = 1\) with respect to \(x\) and for \(\alpha = 2\) with respect to \(y\). This yields after summation to
\[
\frac{1}{2} \left( -w_{\alpha,\alpha} - K_r u_{\alpha,\alpha} \right) = \Delta u_{\alpha,\alpha} - \sigma u_{\alpha,\alpha} - 2\mu w_{z\alpha\alpha}.
\]
Then use the incompressibility conditions to find
\[
\frac{1}{2} \left( w_f^{z\alpha} + K_r w_p^{z\alpha} \right) = -\Delta w_{\alpha} + \sigma w_{\alpha} - 2\mu \Delta^* w_{\alpha}.
\]
Equation (25) allows one to determine the following non-dimensional interface condition
\[
\frac{1}{2} \left( D_m w_f^{\alpha} + K_r D_m w_p^{\alpha} \right) = \frac{Da}{d} (\sigma D w - D^3 w - 3\Delta^* D w),
\]
where \(D = d/dz, z \in (0,1), D_m = d/dz_m, z_m \in (-1,0)\).

Alternatively, one may employ a weighted form of (24) where we write
\[
\frac{1}{2} \left( w_f^f + K_r w_p^p \right) = \pi - 2\mu w_z.
\]
This then leads to a weighted form of (28). In deriving the weighted form the interaction terms involving \(\xi\) do not vanish. The precise forms are given in the next section.

Thus, the complete set of boundary conditions are (14), (15), together with (20) and (28), or the weighted equivalents of these latter two equations.

5 Numerical method

We solve equations (12) and (13) by a Chebyshev tau method, cf. Dongarra et al. [17], coupled with the QZ algorithm for a generalized matrix eigenvalue problem, cf. Moler and Stewart [37]. Equations (12) are transformed into the Chebyshev domain \((-1,1)\) and equations (13) are likewise transformed into the same domain with the interface now being \(z = -1\). The variables \(W, \chi, \Theta, W^f, W^p\) and \(\Theta^m\) are written as finite series of Chebyshev polynomials, e.g.
\[
W = \sum_{i=0}^{N} W_i T_i(z),
\]
for Fourier coefficients \(W_i\). This yields a block matrix generalized eigenvalue problem of form \(Ax = \sigma Bx\) for \(6N \times 6N\) matrices \(A, B\) with \(B\) singular, where
\[
x = (W, \chi, \Theta, W^f, W^p, \Theta^m).
\]
\[ \tilde{W} = (W_0, \ldots, W_N), \ldots, \tilde{\Theta} = (\Theta_0^m, \ldots, \Theta_N^m). \]

The boundary conditions are likewise expanded in Chebyshev polynomials and added as rows of the matrices \( A \) and \( B \) via a similar procedure to that explained in Dongarra et al. [17].

The complete set of boundary conditions on \( z = 1 \) or \( z = -1 \) are now

\[ W = 0, \quad DW = 0, \quad \Theta = 0, \quad W^f = 0, \quad W^p = 0, \quad \Theta_m = 0, \quad z = 1, \]
\[ W = \frac{W_f}{\phi}, \quad W = \frac{W_p}{\epsilon(1 - \phi)}, \quad z = -1, \]
\[ \Theta_m = \Theta \frac{a^2}{k^2}, \quad D_m \Theta_m + \frac{d}{k} D\Theta = 0, \quad z = -1, \]
\[ A + a^2 W - 2(A_1 + A_2) DW - 2A_2 \frac{d}{k} D_m W^p - 2A_1 \frac{d}{k} D_m W^f = 0, \quad z = -1, \]

and

\[ 2DA - 4a^2 DW - \frac{d}{D_a} D_m W^f - \frac{K_r d}{D_a} D_m W^p = 2\sigma DW, \quad z = -1. \]

This is for the case (24) and (20) with \( A_1 \) and \( A_2 \) given by (19). When the weighted version of the Beavers - Joseph and continuity of normal stress conditions are employed then (29) holds but with \( A_1 \) and \( A_2 \) given by (21). However, in the weighted case (30) should be replaced by

\[ 2DA - 4a^2 DW - \frac{2d}{D_a} [c_1 + \xi(c_1 - c_2)] D_m W^f - \frac{2d}{D_a} [K_r c_2 + \xi(c_2 - c_1)] D_m W^p = 2\sigma DW, \]

where it is to be observed that the interaction terms involving \( \xi \) are present.

## 6 Numerical results

This section reports on numerical results for the critical Rayleigh number and critical wavenumber for the equations for thermal convection in a linearly viscous fluid overlying a bidisperse porous material. We choose parameter values appropriate to water being the working fluid and a bidisperse porous material being based upon a glass bead skeleton.

As this is the first calculation for this problem we restrict attention to \( \alpha_1 = \alpha_2 = \alpha \) for the Beavers - Joseph constant. Since there are many parameters we believe this is justified. For numerical values of the many parameters we refer to Gentile and Straughan [22] who employed tabulated experimental values of
Beavers and Joseph [2] reported values of $\alpha$ for a single porosity material in the range 0.1 to 4. These values are for a granular aloxite material and for man made porous foams. In this work we concentrate on more granular materials and choose $\alpha$ values close to 0.1. The values reported in Gentile and Straughan [22] suggest we take relative permeabilities of $K_r = 25, 151.7, 263.16$.

We here choose to investigate the behaviour of the critical $Ra_m$ and $a_m$ values upon $K_r, \alpha, \xi, \phi, \epsilon$, and upon the equal splitting interface conditions of Beavers and Joseph and continuity of normal stress together with the analogous pore weighted interface conditions.

For core values using glass beads and water the thermal conductivities, densities and specific heats are taken from the internet version of Engineering Toolbox to yield $k_m, k_f, (\rho c)_m$ and $(\rho c)_f$. We also employ the parameter ranges in Gentile and Straughan [22] to find values for $\xi, K_r$ and $\delta$. In this way we obtain $Pr = 6, Pr_m = 0.75828, Da = 0.161278 \times 10^{-2}, \delta = 0.3279 \times 10^{-2}, \hat{k} = 0.16736, \phi = 0.3, \epsilon = 0.3, \alpha = 0.1, \xi = 0.02987, \hat{\kappa} = 0.12638, K_r = 25$, although specific values will be varied at appropriate points in our discussion. The nature of the onset of convective motion in all cases depends on $\hat{d}$, the depth of fluid layer to depth of porous layer. There is a critical value of $\hat{d}$ such that when $\hat{d}$ is below this value then convective motion commences in the porous layer whereas such motion is initiated in the fluid layer when $\hat{d}$ is above the critical value. The critical $\hat{d}$ value also depends strongly on the other parameters in the problem and this variation is examined in detail here.

The bimodal behaviour of the neutral curves is displayed in figures 1 and 2. Figure 1 shows that when $\hat{d} = 0.12$ the lowest minimum for $Ra_m$ is at $a_m = 2.1$ and this corresponds to convection initiating in the porous layer, whereas when $\hat{d} = 0.13$ the lowest minimum for $Ra_m$ is at $a_m = 17.0$ and this corresponds to convective motion initiating in the fluid layer. From table 4 we see that the critical value for $\hat{d}$ corresponding to figure 1 is when $\hat{d}_{crit} \in (0.1260, 0.1261)$. Observe that from table 4 the wavenumber jumps from $a_m = 2.1$ to $a_m = 17.7$ as the initiation of convection switches from the porous layer to the fluid layer. Since $a_m$ is inversely proportional to the aspect ratio of the convection cell this means that for a non-dimensional depth of porous layer of 1 the width of the cell changes from $2\pi/2.1$ to $2\pi/17.7$, i.e. from 2.99 to 0.355, or from wide cells in the porous layer to cells 8.42 times smaller in the fluid layer. Of course, the aspect ratio also depends on other parameters in the bidisperse porous medium. Figure 2 also displays a switch from convection in the porous layer to convection in the fluid layer, but now when $\hat{d}$ is fixed. In this case the switch of convection is due to the macro porosity changing.
It is noticeable that the minima associated to the fluid are widely separated in figure 1 whereas in 2 it is the minima associated to the porous medium which display the greater variation. Since figure 2 is displaying changes in the porosity the larger variation in the “porous minimum” is to be expected.

Table 1 displays how the critical values of the porous Rayleigh number $R_{am}$ and the critical values of the porous wavenumber $a_m$ change as $K_r$ is varied. Since we are effectively fixing $K_f$ in our computations changing $K_r$ corresponds to changing $1/K^p$. As $K_r$ increases in table 1 from values of 1.5 through to 263.16 we see that $R_{am}$ increases, both for the porous and fluid minima. This corresponds to the layered system becoming less easy to convect as $\Delta T = T_L - T_U$ increases. Since $K_r$ increasing corresponds to $K^p$ decreasing this means it is more difficult for the fluid to move in the micro pores and so we expect the system to be more stable. It is worth observing that in all our computations we have found that at criticality the growth rate $\sigma$ is real.

From table 1 the porous wavenumber $a_m$ increases by a factor of approximately 2.5 over the range of variation of $K_r$ but the wavenumber corresponding to the fluid decreases from 24.4 or 26.1 to 9.3 or 8.3, respectively, depending on the value of $\hat{d}$. Thus, at the onset of convection when the motion is initiated in the porous layer increasing $K_r$ decreases the cell aspect ratio whereas the opposite is true when the convective motion is initiated in the fluid layer. The quantitative value of this effect does demonstrate that the presence of the bidisperse layer is significant in both cell aspect ratio and whether convection will occur.

Table 2 displays how $R_{am}$ and $a_m$ change at criticality with variation in the Beavers - Joseph interface parameter. We find that increasing $\alpha$ decreases the value of $R_{am}$ and also that of $a_m$. Since the coefficient $\alpha$ multiplies $u_\alpha - u^p_L$ increasing its value has the effect of increasing the shear flow term $\partial u_\alpha/\partial z$ which is making the system more stable and increasing the convection cell aspect ratio.

Table 3 displays the effect of changing the momentum transfer coefficient $\xi$ upon $R_{am}$ and $a_m$. It is seen that a relative change of over 4 in $\xi$ reduces both $R_{am}$ and $a_m$, although the change is relatively small.

Table 4 demonstrates how the critical values of $R_{am}$ and $a_m$ are affected by changing the macro porosity $\phi$ and the micro porosity $\epsilon$. For a fixed value of $\phi$ increasing $\epsilon$ leads to an increase in $R_{am}$ and $a_m$ for the convection initiation in the porous medium, while the values associated to initiation in the fluid part of the layer show little variation. This indicates that increasing $\epsilon$ for fixed $\phi$ helps to stabilize the fluid layer. This could be an important factor in any application which requires the fluid/porous layer to not convect.

Increasing $\phi$ for fixed $\epsilon$ decreases both $R_{am}$ and $a_m$ as might be expected since a larger macro porosity should mean convective motion is easier to commence. However, care must be taken in interpretation. The Rayleigh number $R_{am}$ as defined in (7) may be rewritten as

$$R_{am} = \frac{\gamma g(T_L - T_U)d_m d^2}{\kappa_m \nu} \frac{Da}{(1 + \hat{d}/\hat{k})} \frac{(\rho c)_f}{(\rho c)_m}.$$ (32)
Clearly $Ra_m$ has the correct structure but since $Da = K_f/d^2$ there is a direct dependence on the macro permeability $K_f$ and it is widely believed that $K_f$ depends strongly on $\phi$. For example, for glass spheres Chen [13] uses the Carmen-Kozeny relation

$$K_f = \frac{d_s^2}{172.8} \frac{\phi^3}{(1-\phi)^2}$$

where $d_s$ is the diameter of the glass beads. It is clear that $Ra_m$ in (32) would then depend strongly on $\phi$ and in any application this would need to be taken into account.

We have calculated the necessary values of $a_m$ even though they are not displayed in table 4. For the first set of values where $\phi = 0.5$ and $\epsilon = 0.2, \cdots, 0.6$, convection always initiates in the porous zone, and $a_m$ takes values 1.4, 1.8, 2.0, 2.1 and 2.2, respectively, while $a_m^{(2)}$ has values 18.6 and 18.7 four times. This means lowering the micro porosity increases the width of the convection cells. It is useful to compare this with the case of a single porosity material as studied in Straughan [49]. Figure 7 and table VII of that work are appropriate as $\phi = 0.5$ there. However, it is difficult to compare directly since the meaning of terms is different, for example, here $\delta = \sqrt{K_p/d_m}$ whereas in Straughan [49], $\delta = \sqrt{K/d_m}$ with $K$ being the permeability in the single porosity medium. Nevertheless, the case in Straughan [49] demonstrates convection initiates strongly in the porous region. The single porosity case is formally achieved for equations (8) and (9) by taking $\xi = 0$ and $K_r = 1$, although this is a formal calculation since $\epsilon \to 0$ needs a careful treatment of interface conditions. Although if we take $\xi = 0$ and $K_r = 1$ and compute $a_m, Ra_m, a_m^{(2)}$ and $Ra_m^{(2)}$ we find for $d = 0.08, a_m = 1.0, Ra_m = 3.183, a_m^{(2)} = 27.6, Ra_m^{(2)} = 5.124$ whereas when $d = 0.09, a_m = 0.7, Ra_m = 2.695, a_m^{(2)} = 23.6, Ra_m^{(2)} = 3.080$. If these values are compared with corresponding values in table 4 for $K_r = 1.5$ and table 4 then we see that as $\epsilon$ decreases $Ra_m$ decreases and the layer system becomes less stable. This is important because for some applications one requires the layer to not convect and so the presence of a bidisperse porous medium is highly beneficial. For example, in experiments with a solar pond Wang et al. [58] found that the presence of a porous layer stabilizes the pond and increases efficiency, but, additionally the presence of a bidisperse porous material further stabilizes and increases efficiency. Of course, our model does not describe a solar pond where one must also account for a salt distribution and different boundary conditions due to solar radiation heating. However, we do believe it is very interesting that our model does predict the presence of a bidisperse porous layer could be very beneficial in applications.

Finally tables 5 and 6 show the variation in the values of $Ra_m$ depending on whether the Beavers-Joseph and continuity of normal stress conditions employ the equally split (ES) value of 0.5 or the pore weighted (PW) values. For the $K_r$ and $\alpha$ values selected here the variation is relatively small, but not totally insignificant. It would be useful to have experimental results to compare with in order to assess which class of interface conditions is preferable.
7 Conclusions

We have formulated equations for thermal convection in a fluid layer which overlies a layer of bidisperse (or double porosity) porous medium saturated by the same fluid. The macro and micro porosities \( \phi \) and \( \epsilon \) and the analogous macro and micro permeabilities \( K_f \) and \( K_p \) are explicitly included and play a major role in the model along with the depth of the fluid layer \( d \) and the depth of the porous layer \( d_m \). The interface conditions between the fluid and porous medium are very important and we have adopted two approaches. One is to adopt an equal weighting to both the macro and micro phases while the other approach employs a pore weighted average reflecting the relative porosities \( \phi \) and \( \epsilon \).

We suggest a variant of the Beavers and Joseph [2] interface condition which holds when the porous medium is one of bidispersive type. In general, this allows for two Beavers - Joseph interface coefficients \( \alpha_1 \) and \( \alpha_2 \) corresponding to the macro and micro phases. In our numerical results we assume \( \alpha_1 = \alpha_2 \) although in specific applications \( \alpha_1 \) may need to be different from \( \alpha_2 \) and our model allows for this.

Our numerical results show that the bidisperse porous medium is very different from the single porous medium case in that the coefficients of the double porosity material do have a strong effect on convective instability. Since the two layer convection problem with a fluid overlying a bidisperse porous medium does have serious application to renewable energy generation, see e.g. Wang et al. [58, 59], we believe the current work is very useful. Of course, two layer convection with a single porosity medium already has many parameters, see e.g. Chen [13], Chen and Chen [14, 15, 16], Straughan [48, 49], while convection in a single bidisperse layer likewise involves many parameters. The combined problem is necessarily complicated and, therefore, involves a lot of parameters. Future work will apply this theory to specific renewable energy situations.

Conflict of interest statement. This work does not have any conflicts of interest.

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| $K_r$ | $d$ | $a_m$ | $Ra_m$ | $a_m^{(2)}$ | $Ra_m^{(2)}$ |
|------|-----|-------|--------|-------------|-------------|
| 1.5  | 0.08| 1.2   | 4.221  | 28.1        | 7.755       |
| 1.5  | 0.09| 1.2   | 3.986  | 24.4        | 4.765       |
| 1.5  | 0.1  | 1.0   | 3.963  | 21.2        | 3.014       |
| 5    | 0.1  | 1.6   | 9.030  | 22.5        | 10.621      |
| 5    | 0.11 | 1.6   | 8.746  | 20.1        | 7.172       |
| 10   | 0.11 | 1.8   | 13.901 | 20.4        | 14.604      |
| 10   | 0.12 | 1.8   | 13.515 | 18.4        | 10.193      |
| 20   | 0.12 | 2.0   | 19.589 | 18.7        | 20.734      |
| 20   | 0.13 | 2.0   | 19.128 | 17.0        | 14.852      |
| 25   | 0.12 | 2.1   | 21.474 | 18.7        | 26.015      |
| 25   | 0.126 | 2.1  | 21.178 | 17.7        | 21.276      |
| 25   | 0.1261 | 2.1 | 21.173 | 17.7       | 21.206      |
| 25   | 0.13  | 2.1   | 20.991 | 17.0        | 18.671      |
| 151.7 | 0.17 | 2.5   | 28.572 | 12.2        | 36.445      |
| 151.7 | 0.18 | 2.5   | 28.147 | 11.2        | 27.910      |
| 263.16 | 0.20 | 2.6   | 28.423 | 9.3         | 29.038      |
| 263.16 | 0.21 | 2.7   | 27.243 | 8.3         | 22.277      |

Table 1: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, $Ra_m, a_m$, and the second minimum, $Ra_m^{(2)}, a_m^{(2)}$, for indicated values of $K_r$. Here, $Pr = 6, Pr_m = 0.75828, \delta = 0.003279, k = 0.16736, \phi = 0.3, \epsilon = 0.3, \alpha = 0.1, \xi = 0.02987, \tilde{\kappa} = 0.12638$. The $\hat{d}$ values are shown in the table.

| $\alpha$ | $d$ | $a_m$ | $Ra_m$ | $a_m^{(2)}$ | $Ra_m^{(2)}$ |
|----------|-----|-------|--------|-------------|-------------|
| 0.1      | 0.12 | 2.1   | 21.474 | 18.7        | 26.014      |
| 0.105    | 0.12 | 2.1   | 21.355 | 18.6        | 25.630      |
| 0.11     | 0.12 | 2.0   | 21.201 | 18.4        | 25.224      |
| 0.115    | 0.12 | 2.0   | 20.971 | 18.2        | 24.793      |
| 0.12     | 0.12 | 1.9   | 20.621 | 18.0        | 24.333      |
| 0.125    | 0.12 | 1.6   | 19.944 | 17.7        | 23.842      |
| 0.125    | 0.125 | 1.5  | 19.787 | 16.7        | 19.887      |
| 0.125    | 0.126 | 1.5  | 19.750 | 16.5        | 19.184      |
| 0.1      | 0.13 | 2.1   | 20.991 | 17.0        | 18.670      |
| 0.11     | 0.13 | 2.0   | 20.868 | 16.6        | 17.939      |
| 0.12     | 0.13 | 1.8   | 20.492 | 16.1        | 17.097      |

Table 2: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, $Ra_m, a_m$, and the second minimum, $Ra_m^{(2)}, a_m^{(2)}$, for indicated values of the Beavers - Joseph parameter $\alpha$. Here, $Pr = 6, Pr_m = 0.75828, \delta = 0.003279, k = 0.16736, \phi = 0.3, \epsilon = 0.3, \alpha = 0.1, \xi = 0.02987, \tilde{\kappa} = 0.12638, K_r = 25$. The $\hat{d}$ values are shown in the table.
Table 3: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, $Ra_m$, $a_m$, and the second minimum, $Ra_m^{(2)}$, $a_m^{(2)}$, for indicated values of the non-dimensional momentum transfer coefficient $\xi$. Here, $Pr = 6$, $Pr_m = 0.75828$, $\delta = 0.003279$, $\hat{k} = 0.16736$, $\phi = 0.3$, $\epsilon = 0.3$, $\hat{\kappa} = 0.12638$, $K_r = 25$, $\alpha = 0.1$. The $\hat{d}$ values are shown in the table.
| \( \phi \) | \( \epsilon \) | \( d \) | \( a_m \) | \( Ra_m \) | \( a_m^{(2)} \) | \( Ra_m^{(2)} \) |
|---|---|---|---|---|---|---|
| 0.5 | 0.2 | 0.12 | 1.4 | 12.310 | 18.6 | 24.010 |
| 0.5 | 0.3 | 0.12 | 1.8 | 17.210 | 18.7 | 25.239 |
| 0.5 | 0.4 | 0.12 | 2.0 | 19.782 | 18.7 | 25.994 |
| 0.5 | 0.5 | 0.12 | 2.1 | 21.394 | 18.7 | 26.503 |
| 0.5 | 0.6 | 0.12 | 2.2 | 22.495 | 18.7 | 26.871 |
| 0.2 | 0.5 | 0.12 | 2.4 | 25.195 | 18.7 | 27.311 |
| 0.3 | 0.5 | 0.12 | 2.3 | 24.232 | 18.7 | 27.104 |
| 0.4 | 0.5 | 0.12 | 2.2 | 23.008 | 18.7 | 26.843 |
| 0.6 | 0.5 | 0.12 | 2.0 | 19.197 | 18.7 | 26.043 |
| 0.3 | 0.3 | 0.12 | 2.1 | 21.474 | 18.7 | 26.015 |
| 0.3 | 0.3 | 0.1260 | 2.1 | 21.178 | 17.7 | 21.276 |
| 0.3 | 0.3 | 0.1261 | 2.1 | 21.173 | 17.7 | 21.206 |
| 0.3 | 0.3 | 0.13 | 2.1 | 20.991 | 17.0 | 18.671 |
| 0.5 | 0.2 | 0.14 | 1.3 | 11.692 | 15.6 | 12.909 |
| 0.5 | 0.2 | 0.1436 | 1.3 | 11.595 | 15.1 | 11.604 |
| 0.5 | 0.2 | 0.1437 | 1.3 | 11.593 | 15.1 | 11.570 |
| 0.5 | 0.2 | 0.15 | 1.3 | 11.439 | 14.2 | 9.630 |
| 0.3 | 0.5 | 0.12 | 2.3 | 24.232 | 18.7 | 27.104 |
| 0.3 | 0.5 | 0.1234 | 2.3 | 24.054 | 18.1 | 24.097 |
| 0.3 | 0.5 | 0.1235 | 2.3 | 24.049 | 18.1 | 24.014 |
| 0.3 | 0.5 | 0.13 | 2.3 | 23.703 | 17.0 | 19.296 |

Table 4: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, \( Ra_m, a_m \), and the second minimum, \( Ra_m^{(2)}, a_m^{(2)} \), for indicated values of the macro porosity \( \phi \) and micro porosity \( \epsilon \). \( Pr = 6, Pr_m = 0.75828, \delta = 0.003279, \hat{k} = 0.16736, \xi = 0.02987, \hat{\kappa} = 0.12638, K_r = 25, \alpha = 0.1 \). The \( d \) values are shown in the table.
Table 5: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, \( R_{m} \), and the second minimum, \( R_{m}^{(2)} \), for a comparison between the equally split (ES) and pore weighted (PW) versions of the Beavers-Joseph interface condition, and the continuity of normal stress at the interface. Here \( Pr = 6 \), \( Pr_m = 0.75828 \), \( \delta = 0.003279 \), \( \hat{k} = 0.16736 \), \( \phi = 0.3 \), \( \epsilon = 0.3 \), \( \hat{\kappa} = 0.12638 \), \( \xi = 0.02987 \), \( \alpha = 0.1 \). The \( \hat{d} \), \( K_r \) values are shown in the table.

| \( K_r \) | \( \hat{d} \) | \( R_{m}(ES) \) | \( R_{m}(PW) \) | \( R_{m}^{(2)}(ES) \) | \( R_{m}^{(2)}(PW) \) |
|---|---|---|---|---|---|
| 20 | 0.12 | 19.589 | 18.199 | 20.734 | 20.997 |
| 20 | 0.13 | 19.128 | 17.666 | 14.852 | 15.216 |
| 25 | 0.12 | 21.474 | 20.132 | 26.015 | 26.310 |
| 25 | 0.13 | 20.991 | 19.539 | 18.671 | 19.103 |
| 151.7 | 0.17 | 28.572 | 28.048 | 36.445 | 39.233 |
| 151.7 | 0.18 | 28.147 | 27.632 | 27.910 | 30.664 |
| 151.7 | 0.19 | 27.113 | 24.152 | 22.277 | 26.821 |
| 263.16 | 0.19 | 29.013 | 28.340 | 29.038 | 33.579 |
| 263.16 | 0.20 | 28.423 | 27.790 | 22.277 | 26.821 |
| 263.16 | 0.21 | 27.243 | 26.790 | 22.277 | 26.821 |

Table 6: The minimum values of the porous Rayleigh number and corresponding wavenumber for the first minimum, \( R_{m} \), and the second minimum, \( R_{m}^{(2)} \), for a comparison between the equally split (ES) and pore weighted (PW) versions of the Beavers-Joseph interface condition, and the continuity of normal stress at the interface. Here \( Pr = 6 \), \( Pr_m = 0.75828 \), \( \delta = 0.003279 \), \( \hat{k} = 0.16736 \), \( \phi = 0.3 \), \( \epsilon = 0.3 \), \( \hat{\kappa} = 0.12638 \), \( \xi = 0.02987 \), \( \alpha = 0.1 \). The \( \hat{d} \), \( \alpha \) values are shown in the table.

| \( \alpha \) | \( \hat{d} \) | \( R_{m}(ES) \) | \( R_{m}(PW) \) | \( R_{m}^{(2)}(ES) \) | \( R_{m}^{(2)}(PW) \) |
|---|---|---|---|---|---|
| 0.1 | 0.12 | 21.474 | 20.132 | 26.014 | 26.310 |
| 0.105 | 0.12 | 21.355 | 20.018 | 25.630 | 26.034 |
| 0.11 | 0.12 | 21.201 | 19.867 | 25.224 | 25.745 |
| 0.115 | 0.12 | 20.971 | 19.681 | 24.793 | 25.442 |
| 0.12 | 0.12 | 20.621 | 19.442 | 24.333 | 25.123 |
| 0.125 | 0.12 | 19.944 | 19.115 | 23.842 | 24.788 |
| 0.13 | 0.12 | 18.608 | 17.659 | 24.434 | 24.059 |
| 0.135 | 0.12 | 18.352 | 17.352 | 24.434 | 24.059 |
| 0.1 | 0.13 | 20.991 | 19.539 | 18.670 | 19.103 |
| 0.1 | 0.13 | 20.868 | 19.372 | 17.939 | 18.584 |
| 0.12 | 0.13 | 20.492 | 19.075 | 17.097 | 18.005 |
| 0.13 | 0.13 | 18.352 | 17.352 | 17.352 |
Figure 1: Graph of $Ra_m$ vs. $a_m$. Here, $\delta = 0.003279$, $\hat{k} = 0.16736$, $\phi = 0.3$, $\epsilon = 0.3$, $\alpha = 0.1$, $\xi = 0.02987$, $\hat{\kappa} = 0.12638$, $Pr = 6$, $K_r = 25$. The minimum values on the $\hat{d} = 0.12$ curve are at $a_m = 2.1$, $Ra_m = 21.474$ and $a_m = 18.7$, $Ra_m = 26.015$, on the $\hat{d} = 0.13$ curve they are $a_m = 2.1$, $Ra_m = 20.991$ and $a_m = 17.0$, $Ra_m = 18.671$. The instability when $\hat{d} = 0.12$ initiates in the porous medium, whereas when $\hat{d} = 0.13$ it initiates in the fluid.
Figure 2: Graph of $Ra_m$ vs. $a_m$. Here, $\delta = 0.003279$, $\dot{k} = 0.16736$, $\dot{d} = 0.13$, $\epsilon = 0.2$, $\alpha = 0.1$, $\xi = 0.02987$, $\dot{k} = 0.12638$, $Pr = 6$, $K_r = 25$. The minimum values on the $\phi = 0.2$ curve are at $a_m = 2.0$, $Ra_m = 19.683$ and $a_m = 17.1$, $Ra_m = 18.191$, on the $\phi = 0.5$ curve they are $a_m = 1.4$, $Ra_m = 11.973$ and $a_m = 17.0$, $Ra_m = 17.486$. The instability when $\phi = 0.5$ initiates in the porous medium, whereas when $\phi = 0.2$ it initiates in the fluid.