Finding Exact Minimal Polynomial by Approximations *

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Abstract

We present a new algorithm for reconstructing an exact algebraic number from its approximate value using an improved parameterized integer relation construction method. Our result is almost consistent with the existence of error controlling on obtaining an exact rational number from its approximation. The algorithm is applicable for finding exact minimal polynomial by its approximate root. This also enables us to provide an efficient method of converting the rational approximation representation to the minimal polynomial representation, and devise a simple algorithm to factor multivariate polynomials with rational coefficients. Compared with other methods, this method has the numerical computation advantage of high efficiency. The experimental results show that the method is more efficient than \texttt{identify} in \textit{Maple} 11 for obtaining an exact algebraic number from its approximation. In this paper, we completely implement how to obtain exact results by numerical approximate computations.

\textbf{keywords:} Algebraic number, Numerical approximate computation, Symbolic-numerical computation, Integer relation algorithm, Minimal polynomial.

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1 Introduction

Symbolic computations are principally exact and stable. However, they have the disadvantage of intermediate expression swell. Numerical approximate computations can solve large and complex problems fast, whereas only give approximate results. The growing demand for speed, accuracy and reliability in mathematical computing has accelerated the process of blurring the distinction between two areas of research that were previously quite separate. Therefore, algorithms that combine ideas from symbolic and numeric computations have been of increasing interest in recent two decades. Symbolic computations is for sake of speed by intermediate use of floating-point arithmetic. The work reported in [1, 2, 3, 4, 5, 6] studies recovery of approximate value from numerical intermediate results. A somewhat related topic are algorithms that obtain the exact factorization of an exact input polynomial by use of floating point arithmetic in a practically efficient technique [7, 8]. In the meantime, symbolic methods are applied in the field of numerical computations for ill-conditioned problems [9, 10, 11]. The main goal of hybrid symbolic-numeric computation is to extend the domain of efficiently solvable problems. However, there is a gap between approximate computations and exact results[12].

We consider the following question: Suppose we are given an approximate root of an unknown polynomial with integral coefficients and a bound on the degree and size of the coefficients of the polynomial. Is it possible to infer the polynomial and its exact root? The question was raised by Manuel Blum in Theoretical Cryptography, and Jingzhong Zhang in Automated Reasoning, respectively. Kannan et al answered the question in [13]. However, their technique is based on the Lenstra-Lenstra-Lovasz(LLL) lattice reduction algorithm, which is quite unstable in numerical computations. The function MinimalPolynomial in maple, which finds minimal polynomial for an approximate root, was implemented using the same technique. In this paper, we present a new algorithm for finding exact minimal polynomial and reconstructing the exact root by approximate value. Our algorithm is based on the improved parameterized integer relation construction algorithm, whose stability admits an efficient implementation with lower run times on average than the former algorithm, and can be used to prove that relation bounds obtained from computer runs using it are numerically accurate. The other function identify in maple, which finds a closed form for a decimal approximation of a number, was implemented using the integer relation construction algorithm. However, the choice of Digits of approximate value is fairly arbitrary [14]. In contrast, we fully analyze numerical behavior of an approximate to exact value and give how many Digits of approximate value, which can be obtained exact results. The work is regard as a further research in [15]. We solve the problem, which can be described as follows:

Given approximate value $\tilde{\alpha}$ at arbitrary accuracy of an unknown algebraic
number, and we also know the degree of the algebraic number \( n \) and an upper bound \( N \) of its height on minimal polynomial in advance. The problem will be solved in two steps: First, we discuss how much control error \( \varepsilon \) is, so that we can reconstruct the algebraic number \( \alpha \) from its approximation \( \tilde{\alpha} \) when it holds that \( |\alpha - \tilde{\alpha}| < \varepsilon \). Of course, \( \varepsilon \) is a function in \( n \) and \( N \). Second, we give an algorithm to compute the minimal polynomial of the algebraic number.

We are able to extend our results with the same methods to devise a simple polynomial-time algorithm to factor multivariate polynomials with rational coefficients, and provide a natural, efficient technique to the minimal polynomial representation.

The rest of this paper is organized as follows. Section 2 illustrates the improved parameterized integer relation construction algorithm. In Section 3, we discuss how to recover a quadratic algebraic number and reconstruct minimal polynomial by approximation. Section 4 gives some experimental results. The final section concludes this paper.

2 Preliminaries

In this section, we first give some notations, and a brief introduction on integer relation problems. Then an improved parameterized integer relation construction algorithm is also reviewed.

2.1 Notations

Throughout this paper, \( \mathbb{Z} \) denotes the set of the integers, \( \mathbb{Q} \) the set of the rationals, \( \mathbb{R} \) the set of the reals, \( \mathbb{O}(\mathbb{R}^n) \) the corresponding system of ordinary integers, \( U(n-1, \mathbb{R}) \) the group of unitary matrices over \( \mathbb{R} \), \( GL(n, \mathbb{O}(\mathbb{R})) \) the group of unimodular matrices with entries in the integers, \( \text{col}_i \) \( B \) the i-th column of the matrix \( B \). The ring of polynomials with integral coefficients will be denoted \( \mathbb{Z}[X] \). The content of a polynomial \( p(X) \) in \( \mathbb{Z}[X] \) is the greatest common divisor of its coefficients. A polynomial in \( \mathbb{Z}[X] \) is primitive if its content is 1. A polynomial \( p(X) \) has degree \( d \) if \( p(X) = \sum_{i=0}^{d} p_i X^i \) with \( p_d \neq 0 \). We write \( \text{deg}(p) = d \). The length \( |p| \) of \( p(X) = \sum_{i=0}^{d} p_i X^i \) is the Euclidean length of the vector \( (p_0, p_1, \ldots, p_d) \); the height \( |p|_{\infty} \) of \( p(X) \) is the \( L_{\infty} \)-norm of the vector \( (p_0, p_1, \ldots, p_d) \), so \( |p|_{\infty} = \max_{0 \leq i \leq d} |p_i| \). An algebraic number is a root of a polynomial with integral coefficients. The minimal polynomial of an algebraic number \( \alpha \) is the irreducible polynomial in \( \mathbb{Z}[X] \) satisfied by \( \alpha \). The minimal polynomial is unique up to units in \( \mathbb{Z} \). The degree and height of an algebraic number are the degree and height, respectively, of its minimal polynomial.
2.2 Integer relation algorithm

There exists an integer relation amongst the numbers $x_1, x_2, \cdots, x_n$ if there are integers $a_1, a_2, \cdots, a_n$, not all zero, such that $\sum_{i=1}^{n} a_i x_i = 0$. For the vector $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T$, the nonzero vector $a \in \mathbb{Z}^n$ is an integer relation for $\mathbf{x}$ if $a \cdot \mathbf{x} = 0$. Here are some useful definitions and theorems [16, 17]:

Definition 1 ($M_\mathbf{x}$) Assume $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ has norm $|\mathbf{x}| = 1$. Define $\mathbf{x}^\perp$ to be the set of all vectors in $\mathbb{R}^n$ orthogonal to $\mathbf{x}$. Let $O(\mathbb{R}^n) \cap \mathbf{x}^\perp$ be the discrete lattice of integral relations for $\mathbf{x}$. Define $M_\mathbf{x} > 0$ to be the smallest norm of any relation for $\mathbf{x}$ in this lattice.

Definition 2 ($H_\mathbf{x}$) Assume $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$ has norm $|\mathbf{x}| = 1$. Furthermore, suppose that no coordinate entry of $\mathbf{x}$ is zero, i.e., $x_j \neq 0$ for $1 \leq j \leq n$ (otherwise $\mathbf{x}$ has an immediate and obvious integral relation). For $1 \leq j \leq n$ define the partial sums

$$s_j^2 = \sum_{j \leq k \leq n} x_k^2.$$

Given such a unit vector $\mathbf{x}$, define the $n \times (n-1)$ lower trapezoidal matrix $H_\mathbf{x} = (h_{i,j})$ by

$$h_{i,j} = \begin{cases} 0 & \text{if } 1 \leq i < j \leq n-1, \\ s_{i+1}/s_i & \text{if } 1 \leq i = j \leq n-1, \\ -x_i x_j/(s_j s_{j+1}) & \text{if } 1 \leq j < i \leq n. \end{cases}$$

Note that $h_{i,j}$ is scale invariant.

Definition 3 (Modified Hermite reduction) Let $H$ be a lower trapezoidal matrix, with $h_{i,j} = 0$ if $j > i$ and $h_{i,j} \neq 0$. Set $D = I_n$, define the matrix $D = (d_{i,j}) \in GL(n, O(\mathbb{R}))$ recursively as follows: For $i$ from 2 to $n$, and for $j$ from $i$-1 to 1 (step-1), set $q = \text{nint}(h_{i,j}/h_{j,j})$; then for $k$ from 1 to $j$ replace $h_{i,k}$ by $h_{i,k} - q h_{j,k}$, and for $k$ from 1 to $n$ replace $d_{i,k} - q d_{j,k}$, where the function nint denotes a nearest integer function, e.g., nint(t) = $\lceil t + 1/2 \rceil$.

Theorem 1 Let $\mathbf{x} \neq 0 \in \mathbb{R}^n$. Suppose that for any relation $m$ of $\mathbf{x}$ and for any matrix $A \in GL(n, O(\mathbb{R}))$ there exists a unitary matrix $Q \in U(n-1)$ such that $H = AH_\mathbf{x}Q$ is lower trapezoidal and all of the diagonal elements of $H$ satisfy $h_{j,j} \neq 0$. Then

$$\frac{1}{\max_{1 \leq j \leq n-1} |h_{j,j}|} = \min_{1 \leq j \leq n-1} \frac{1}{|h_{j,j}|} \leq |m|.$$
Remark 1 The inequality of Theorem 1 offers an increasing lower bound on the size of any possible relation. Theorem 1 can be used with any algorithm that produces $GL(n, \mathbb{O}(\mathbb{R}))$ matrices. Any $GL(n, \mathbb{O}(\mathbb{R}))$ matrix $A$ whatsoever can be put into Theorem 1.

Theorem 2 Assume real numbers, $n \geq 2$, $\tau > 1$, $\gamma > \sqrt{4/3}$, and that $0 \neq x \in \mathbb{R}^n$ has $\mathbb{O}(\mathbb{R})$ integer relations. Let $M_x$ be the least norm of relations for $x$. Then $PSLQ(\tau)$ will find some integer relation for $x$ in no more than

$$\left(\frac{n}{2}\right) \frac{\log(\gamma^{n-1}M_x)}{\log\tau}$$

iterations.

Theorem 3 Let $M_x$ be the smallest possible norm of any relation for $x$. Let $m$ be any relation found by $PSLQ(\tau)$. For all $\gamma > \sqrt{4/3}$ for real vectors $|m| \leq \gamma^{n-2}M_x$.

Remark 2 For $n=2$, Theorem 2 proves that any relation $0 \neq m \in \mathbb{O}(\mathbb{R}^2)$ found has norm $|m| = M_x$. In other words, $PSLQ(\tau)$ finds a shortest relation. For real numbers this corresponds to the case of the Euclidean algorithm.

Based on the theorems as above, and if there exists a known error controlling $\varepsilon$, then an algorithm for obtaining the integer relation was designed as follows:

Algorithm 1 Parameterized integer relation construction algorithm
Input: a vector $x$, and an error control $\varepsilon > 0$;
Output: an integer relation $m$.

Step 1: Set $i := 1$, $\tau := 2/\sqrt{3}$, and unitize the vector $x$ to $\tilde{x}$;

Step 2: Set $H_{\tilde{x}}$ by definition 3.

Step 3: Produce matrix $D \in GL(n, \mathbb{O}(\mathbb{R}))$ using modified Hermite Reduction;

Step 4: Set $\tilde{x} := \tilde{x} \cdot D^{-1}$, $H := D \cdot H$, $A := D \cdot A$, $B := B \cdot D^{-1}$.
   case 1: if $\tilde{x}_j = 0$, then $m := \text{col}_j B$, goto Step 11;
   case 2: if $h_{i,i} < \varepsilon$, then $m := \text{col}_{n-1} B$, goto Step 11;

Step 5: $i := i + 1$;

Step 6: Choose an integer $r$, such that $\tau^r|h_{r,r}| \geq \tau^j|h_{i,i}|$, for all $1 \leq j \leq n - 1;$

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Step 7: Define $\alpha := h_{r,r}$, $\beta := h_{r+1,r}$, $\lambda := h_{r+1,r+1}$, $\sigma := \sqrt{\beta^2 + \lambda^2}$;

Step 8: Change $h_r$ to $h_{r+1}$, and define the permutation matrix $R$;

Step 9: Set $\bar{x} := \bar{x} \cdot R$, $H := R \cdot H$, $A := R \cdot A$, $B := B \cdot R$, if $i=n-1$, then goto Step 4;

Step 10: Define $Q := (q_{i,j}) \in U(n-1, R)$, $H := H \cdot Q$, goto Step 4;

Step 11: return $m$.

By algorithm we can find the integer relation $m$ of the vector $x = (1, \tilde{\alpha}, \tilde{\alpha}^2, \cdots, \tilde{\alpha}^n)$. So, we get a nonzero polynomial of degree $n$, i.e.,

$$G(x) = m \cdot (1, x, x^2, \cdots, x^n)^T.$$ \hspace{1cm} (1)

Our main task is to show that polynomial is uniquely determined under assumptions, and discuss the controlling error $\varepsilon$ in algorithm in the next section.

3 Reconstructing minimal polynomial from its approximation

In this section, we will solve such a problem: For a given floating number $\tilde{\alpha}$, which is an approximation of unknown algebraic number, how do we obtain the exact value? Without loss of generality, we first consider the recovering quadratic algebraic number from its approximate value, and then generalize the results to the case of algebraic number of high degree. At first, we have some lemmas as follows:

**Lemma 1** Let $f = \sum_{i=0}^{n} a_i x^i \in \mathbb{Z}[x]$ be a polynomial of degree $n > 0$, and let $\varepsilon = \max_{1 \leq i \leq n} |\alpha^i - \tilde{\alpha}_i|$ for the rest of this paper, where $\alpha_i$ for $1 \leq i \leq n$ are the rational approximations to the powers $\alpha^i$ of algebraic number $\alpha$, and $\tilde{\alpha}_0 = 1$. Then

$$|f(\alpha) - f(\tilde{\alpha})| \leq \varepsilon \cdot n \cdot |f|_{\infty}.$$ \hspace{1cm} (2)

**Proof:** Since $f(\alpha) - f(\tilde{\alpha}) = \sum_{i=0}^{n} a_i (\alpha^i - \tilde{\alpha}_i)$, we get $|f(\alpha) - f(\tilde{\alpha})| = |\sum_{i=1}^{n} a_i (\alpha^i - \tilde{\alpha}_i)|$, and then

$$|\sum_{i=1}^{n} a_i (\alpha^i - \tilde{\alpha}_i)| \leq \sum_{i=1}^{n} |a_i| \cdot |(\alpha^i - \tilde{\alpha}_i)| \leq \sum_{i=1}^{n} |a_i| \cdot \varepsilon \leq n \cdot |f|_{\infty} \cdot \varepsilon.$$ 

So, the lemma is finished.
Lemma 2 Let \( h \) and \( g \) be nonzero polynomials in \( \mathbb{Z}[x] \) of degree \( n \) and \( m \), respectively, and let \( \alpha \in \mathbb{R} \) be a zero of \( h \) with \( |\alpha| \leq 1 \). If \( h \) is irreducible and \( g(\alpha) \neq 0 \), then
\[
|g(\alpha)| \geq n^{-1} \cdot |h|^{-m} \cdot |g|^{1-n}.
\]
\[ (3) \]
Proof: See Proposition(1.6) of [13]. If \( |\alpha| > 1 \), a simple transform of it does.

Corollary 1 Let \( h \) and \( g \) be nonzero polynomials in \( \mathbb{Z}[x] \) of degrees \( n \) and \( m \), respectively, and let \( \alpha \in \mathbb{R} \) be a zero of \( h \) with \( |\alpha| \leq 1 \). If \( h \) is irreducible and \( g(\alpha) \neq 0 \), then
\[
|g(\alpha)| \geq n^{-1} \cdot (n + 1)^{\frac{-n}{2}} \cdot (m + 1)^{\frac{-m}{2}} \cdot |h|^{-m} \cdot |g|^{1-n}.
\]
\[ (4) \]
Proof: First notice that \(|f|^2 \leq (n + 1) \cdot |f|_\infty^2\) holds for any polynomial \( f \) of degree at most \( n > 0 \), so \(|f| \leq \sqrt{n + 1} \cdot |f|_\infty\). From Lemma 2 we get
\[
|g(\alpha)| \geq n^{-1} \cdot (n + 1)^{\frac{-n}{2}} \cdot (m + 1)^{\frac{-m}{2}} \cdot |h|^{-m} \cdot |g|^{1-n}.
\]
So, the corollary is finished.

Theorem 4 Let an approximate value \( \tilde{\alpha} \) belong to an unknown algebraic number \( \alpha \) of degree \( n > 0 \). Assume that the existence of the polynomial \( G(x) = \sum_{i=0}^{n} a_i x^i \), where \( a_n \neq 0 \). Suppose \( n \) and upper bound \( N \) on the degree and height of minimal polynomial \( g(x) \) on the algebraic number \( \alpha \) are known, respectively. If
\[
|G(\tilde{\alpha})| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n} - n \cdot \varepsilon \cdot |G|_{\infty},
\]
then
\[
|G(\alpha)| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n}.
\]
Proof: From lemma [11] we notice that \(|G(\alpha) - G(\tilde{\alpha})| \leq \varepsilon \cdot n \cdot |G|_{\infty} \), and
\[
|G(\alpha) - G(\tilde{\alpha})| \geq |G(\alpha)| - |G(\tilde{\alpha})|, \text{ so } |G(\alpha)| \leq |G(\tilde{\alpha})| + n \cdot \varepsilon \cdot |G|_{\infty}.
\]
From the assumption of the theorem, since
\[
|G(\tilde{\alpha})| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n} - n \cdot \varepsilon \cdot |G|_{\infty}.
\]
\[ (5) \]
So, the theory is finished.

Corollary 2 If \(|G(\alpha)| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n} \), where \( G(x) \) is constructed by the parameterized integer relation construction algorithm as above,
the upper bound \( N \) on the height of its minimal polynomial \( g(x) \) on an algebraic number \( \alpha \) are known. Then

\[
G(\alpha) = 0, \tag{6}
\]

and the primitive part of polynomial \( G(x) \) is the minimal polynomial of algebraic number \( \alpha \).

**Proof:** Proof is given by contradiction. According to Lemma\(^2\) suppose on the contrary that \( G(\alpha) \neq 0 \), then

\[
|G(\alpha)| \geq n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n}.
\]

From theory\(^4\) we get

\[
|G(\alpha)| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n}.
\]

So, \( G(\alpha) = 0 \). Since algebraic number \( \alpha \) is degree \( n > 0 \), then the primitive polynomial of \( G(x) \) denotes \( pp(G(x)) \), hence \( pp(G(x)) \) is just irreducible and equal to \( g(x) \). Of course, it is unique.

So, the corollary is finished.

### 3.1 Recovering quadratic algebraic number from approximate value

For simplicity, we discuss how to obtain quadratic algebraic number from its approximation using integer relation algorithm. Let \( \tilde{\alpha} \) be the approximate value, considering the vector \( v = (1, \tilde{\alpha}, \tilde{\alpha}^2) \). Our goal is to find a vector \( w \) which has all integer entries such that the dot product of \( v \) and \( w \) is less than a lower bound, which is obtained and we are able to get the size of the neighborhood is \( 1/(12\sqrt{3}N^4) \) from theorem\(^4\) The following theorem answers the basic questions of this approach.

**Theorem 5** Let \( \tilde{\alpha} \) be an approximate value belonging to an unknown quadratic algebraic number \( \alpha \), if

\[
\varepsilon = |\alpha - \tilde{\alpha}| < 1/(12\sqrt{3}N^4), \tag{7}
\]

where \( N \) is the upper bound on the height of its minimal polynomial. Then \( G(\alpha) = 0 \), and the primitive part of \( G(x) \) is its minimal polynomial, where \( G(x) = \sum_{i=0}^{2} a_i x^i \) is constructed using integer relation algorithm as above.

**Proof:** From theorem\(^4\) and corollary\(^2\) if and only if

\[
|G(\alpha)| < n^{-1} \cdot (n + 1)^{-n + \frac{1}{2}} \cdot |G|_{\infty}^{-n} \cdot N^{1-n},
\]
Therefore, $G(\alpha) = 0$. Under the assumption of the theorem we get $n = 2$ and $|G(\tilde{\alpha})| > 0$, hence it is obvious that inequality (5) holds, i.e.,

$$0 < n^{-1} \cdot (n + 1)^{-\frac{1}{2}} \cdot |G|^{-n} \cdot N^{1-n} - n \cdot \varepsilon \cdot |G|_{\infty}. \quad (8)$$

So, solving inequality (8) yields

$$\varepsilon < 1/(12\sqrt{3}|G|_{\infty}^3N).$$

From theorem 3 and $\alpha$ is a quadratic algebraic number, so $|G|_{\infty}$ is just equal to $N$.

So, the theory is finished.

This theorem leads to the following algorithm for recovering the quadratic algebraic number of $\tilde{\alpha}$:

**Algorithm 2** Recovering quadratic algebraic number algorithm

Input: an floating number $(\tilde{\alpha}, N)$ belonging to an unknown quadratic algebraic number $\alpha$, i.e., satisfying (7).

Output: an quadratic algebraic number $\alpha$.

Step 1: Construct the vector $v$;

Step 2: Compute $\varepsilon$ satisfying (4);

Step 3: Call algorithm 1 to find an integer relation $w$ for $v$;

Step 4: Obtain $w(x)$ the corresponding polynomial;

Step 5: Let $g(x)$ be the primitive part of $w(x)$;

Step 6: Solve the equation $g(x) = 0$ and choose the corresponding algebraic number to $\alpha$;

Step 7: return $\alpha$.

**Theorem 6** Algorithm 2 works correctly as specified and uses $O(\log N)$ binary bit operations, where $N$ is the upper bound of height on its minimal polynomial.

**Proof**: Correctness follows from theorem 5. The cost of the algorithm is $O(\log N)$ binary bit operations obviously.
3.2 Obtaining minimal polynomial of high degree

If $\alpha$ is a real number, then by definition $\alpha$ is algebraic exactly if, for some $n$, the vector

$$ (1, \alpha, \alpha^2, \cdots, \alpha^n) \quad (9) $$

has an integer relation. The integer coefficient polynomial of lowest degree, whose root $\alpha$ is, is determined uniquely up to a constant multiple; it is called the \textit{minimal polynomial} for $\alpha$. Integer relation algorithm can be employed to search for minimal polynomial in a straightforward way by simply feeding them the vector \((9)\) as their input. Let $\tilde{\alpha}$ be an approximate value belonging to an unknown algebraic number $\alpha$, considering the vector $v = (1, \tilde{\alpha}, \tilde{\alpha}^2, \cdots, \tilde{\alpha}^n)$, how to obtain the exact minimal polynomial from its approximate value? We have the same technique answer to the question from the following theorem.

\textbf{Theorem 7} Let $\tilde{\alpha}$ be an approximate value belonging to an unknown algebraic number $\alpha$ of degree $n > 0$, if

$$ \varepsilon = |\alpha - \tilde{\alpha}| < 1/(n^2(n + 1)^{n - \frac{1}{2}} N^{2n}), \quad (10) $$

where $N$ is the upper bound on the height of its minimal polynomial. Then $G(\alpha) = 0$, and the primitive part of $G(x)$ is its minimal polynomial, where $G(x) = \sum_{i=0}^{n} a_i x^i$ is constructed using the parameterized integer relation construction algorithm as above.

\textbf{Proof:} The proof can be given similarly to that in theorem 5.

It is easiest to appreciate the theorem by seeing how it justifies the following algorithm for obtaining minimal polynomials from its approximation:

\textbf{Algorithm 3} Obtaining minimal polynomial algorithm

\textit{Input:} an floating number $(\tilde{\alpha}, n, N)$ belong to an unknown algebraic number $\alpha$, i.e., satisfying (10).

\textit{Output:} $g(x)$, the minimal polynomial of $\alpha$.

\begin{enumerate}
  \item \textit{Step 1:} Construct the vector $v$;
  \item \textit{Step 2:} Compute $\varepsilon$ satisfying (10);
  \item \textit{Step 3:} Call algorithm [7] to find an integer relation $w$ for $v$;
  \item \textit{Step 4:} Obtain $w(x)$ the corresponding polynomial;
  \item \textit{Step 5:} Let $g(x)$ be the primitive part of $w(x)$;
  \item \textit{Step 6:} return $g(x)$.
\end{enumerate}
Theorem 8 Algorithm 3 works correctly as specified and uses $O(n(\log n + \log N))$ binary bit operations, where $n$ and $N$ are the degree and height of its minimal polynomial, respectively.

Proof: Correctness follows from theorem 7. The cost of the algorithm is $O(n(\log n + \log N))$ binary bit operations obviously.

The method of obtaining minimal polynomials from an approximate value can be extended to the set of complex numbers and many applications in computer algebra and science.

This yields a simple factorization algorithm for multivariate polynomials with rational coefficients: We can reduce a multivariate polynomial to a bivariate polynomial using the Hilbert irreducibility theorem, the basic idea was described in [5], and then convert a bivariate polynomial to a univariate polynomial by substituting a transcendental number in [18] or an algebraic number of high degree for a variate in [8]. It can find the bivariate polynomial’s factors, from which the factors of the original multivariate polynomial can be recovered using Hensel lifting. After this substitution we can get an approximate root of the univariate polynomial and use our algorithm to find the irreducible polynomial satisfied by the exact root, which must then be a factor of the given polynomial. This is repeated until the factors are found.

The other yields an efficient method of converting the rational approximation representation to the minimal polynomial representation. The traditional representation of algebraic numbers is by their minimal polynomials [19, 20, 21, 22]. We now propose an efficient method to the minimal polynomial representation, which only needs an approximate value, degree and height of its minimal polynomial, i.e., an ordered triple $< \tilde{\alpha}, n, N >$ instead of an algebraic number $\alpha$, where $\tilde{\alpha}$ is its approximate value, and $n$ and $N$ are the degree and height of its minimal polynomial, respectively, denotes $< \alpha >=< \tilde{\alpha}, n, N >$. It is not hard to see the computations in the representation can be changed to computations in the other without loss of efficiency, the rational approximation method is closer to the intuitive notion of computation.

4 Experimental Results

Our algorithms are implemented in Maple. The following examples run in the platform of Maple 11 and PIV3.0G,512M RAW. The first three examples illuminate how to obtain exact quadratic algebraic number and minimal polynomials. Example 4 tests our algorithm for factoring primitive polynomials with integral coefficients.

Example 1 Let $\alpha$ be an unknown quadratic algebraic number. We only
know an upper bound of height on its minimal polynomial $N = 47$. According to theorem 5 compute quadratic algebraic number $\alpha$ as follows: First obtain control error $\varepsilon = 1/(12 \sqrt{3} * N^4) = 1/(1807729447692 * \sqrt{3}) \approx 1.0 \times 10^{-8}$. And then assume that we use some numerical method to get an approximation $\hat{\alpha} = 11.937253933$, such that $|\alpha - \hat{\alpha}| < \varepsilon$. Calling algorithm 2 yields as follows: Its minimal polynomial is $g(x) = x^2 - 8 \times x - 47$. So, we can obtain the corresponding quadratic algebraic number $\alpha = 4 + 3\sqrt{7}$.

Remark 3 The function identify in maple 11 needs Digits=13, whereas our algorithm only needs 9 digits.

Example 2 For obtaining exact minimal polynomials from approximate root $\hat{\alpha}$, we only know degree $n = 3$ and height $N = 17$ of its minimal polynomial. According to theorem 4 just as do in Example 1: First get the error $\varepsilon = 1/(n^2(n+1)^{n-4} * N^{2n}) = 1/(6951619872) \approx 1.4 \times 10^{-10}$. Assume that we use some numerical method to get an approximation $\hat{\alpha} = 16.808034642702$, such that $|\alpha - \hat{\alpha}| < \varepsilon$. Calling algorithm 3 yields as follows: Its minimal polynomial is $g(x) = x^3 - 17 \times x^2 + 4 \times x - 13$.

Example 3 Let a known floating number $\tilde{\alpha}$ belonging to some algebraic number $\alpha$ of degree $n = 4$, where $\tilde{\alpha} = 3.14626436994198$, we also know an upper bound of height $N = 10$ on its minimal polynomial. According to theorem 4 we can get the error $\varepsilon = 1/(n^2(n+1)^{n-4} * N^{2n}) = 1/(4^2 \times 5^7 \times 10^8) \approx 2.2 \times 10^{-12}$. Calling algorithm 4 if only the floating number $\tilde{\alpha}$, such that $|\alpha - \tilde{\alpha}| < \varepsilon$, then we can get its minimal polynomial $g(x) = x^4 - 10 \times x^2 + 1$. So, the exact algebraic number $\alpha$ is able to denote $< \alpha >=< 3.14626436994198, 4, 10 >$, i.e., $< \sqrt{2} + \sqrt{3} >=< 3.14626436994198, 4, 10 >$.

Example 4 This example is an application in factoring primitive polynomials over integral coefficients. For the convenience of display in the paper, we choose a very simple polynomial as follows:

$$p = 3x^9 - 9x^8 + 3x^7 + 6x^5 - 27x^4 + 21x^3 + 30x^2 - 21x + 3$$

We want to factor the polynomial $p$ via reconstruction of minimal polynomials over the integers. First, we transform $p$ to a primitive polynomial as follows:

$$p = x^9 - 3x^8 + x^7 + 2x^5 - 9x^4 + 7x^3 + 10x^2 - 7x + 1,$$

We see the upper bound of coefficients on polynomial $p$ is 10, which has relation with an upper bound of coefficients of the factors on the primitive polynomial $p$ by Landau-Mignotte bound [23]. Taking $N = 5$, $n = 2$ yields $\varepsilon = 1/(2^2 * (2 + 1)^2 - \sqrt{5}) \approx 8.0 \times 10^{-5}$. Then we compute the approximate root on $x$. With Maple we get via $[\text{solve}(p = 0, x)]$:

$$S = [2.618033989, 1.250523220, -0.9223475138, 0.3819660113, 0.2192284350]$$
According to theorem \[ \text{7} \] let \( \tilde{\alpha} = 2.618033989 \) be an approximate value belonging to some quadratic algebraic number \( \alpha \), calling algorithm \[ \text{3} \] yields as follows:

\[
p_1 = x^2 - 3 \cdot x + 1.
\]

And then we use the polynomial division to get

\[
p_2 = x^7 + 2 \cdot x^3 - 3 \cdot x^2 - 4 \cdot x + 1.
\]

Based on the Eisenstein’s Criterion \[ \text{24} \], the \( p_2 \) is irreducible in \( \mathbb{Z}[X] \). So, the \( p_1 \) and \( p_2 \) are the factors of primitive polynomial \( p \).

5 Conclusion

In this paper, we have presented a new method for obtaining exact results by numerical approximate computations. The key technique of our method is based on an improved parameterized integer relation construction algorithm, which is able to find an exact relation by the accuracy control \( \varepsilon \) in formula \( \text{10} \) is an exponential function in degree and height of its minimal polynomial. The result is almost consistent with the existence of error controlling on obtaining an exact rational number from its approximation in \[ \text{15} \]. Using our algorithm, we have succeed in factoring multivariate polynomials with rational coefficients and providing an efficient method of converting the rational approximation representation to the minimal polynomial representation. Our method can be applied in many aspects, such as proving inequality statements and equality statements, and computing resultants, etc.. Thus we can take fully advantage of approximate methods to solve larger scale symbolic computation problems.

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