WEAK UNBOUNDED NORM TOPOLOGY AND DOUNFORD-PETTIS OPERATORS

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Abstract. In this paper, we study un-dual (in symbol, $E^o$) of Banach lattice $E$ and compare it with topological dual $E^*$. If $E^*$ has order continuous norm, then $E^* = E^o$. We introduce and study weakly unbounded norm topology ($wun$-topology) on Banach lattices and compare it with weak topology and $uaw$-topology. In the final, we introduce and study $wun$-Dunford-Pettis operators from a Banach lattice $E$ into Banach space $X$ and we investigate some of its properties and its relationships with others known operators.

1. Introduction

In [6], authors shows that $uo$-convergence need not be given by a topology, but $un$-convergence is topological. We will refer to this topology as $un$-topology. The smallest topology $\tau$ that each $un$-continuous functional $f : E \to \mathbb{R}$ is continuous with respect to that topology is called weakly unbounded norm topology (for short, $wun$-topology), and we denote it by $\tau_{wun}$. First we will ours motivate to write this article.

(1) We have defined $un$-continuous operators between two Banach lattices $E$ and $F$ in [11], and so we introduced the $un$-dual space for a Banach lattice $E$ and we study some of its properties.
(2) Such as the definition of weak topology for a normed space, we define $wun$-topology for a Banach lattice and compare it with $uaw$-topology which is introduced in [17]. We show that in general $un$-topology and $wun$-topology are different and by some conditions both topologies coincide.
(3) By studying of $uaw$-Dunford-Pettis operators in [15], it is interested to define a new generation of operators as $wun$-Dunford-Pettis operators. We study some of its properties and compare with other known classifications of operators.

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Here we bring some definitions of need.
Let $E$ be a vector lattice and $x \in E$. A net $(x_\alpha)_{\alpha \in A} \subseteq E$ is said to be order convergent to $x$ if there is a net $(z_\beta)_{\beta \in B}$ in $E$ such that $z_\beta \downarrow 0$ and for every $\beta \in B$, there exists $\alpha_0 \in A$ such that $|x_\alpha - x| \leq z_\beta$ whenever $\alpha \geq \alpha_0$. We denote this convergence by $x_\alpha \stackrel{o}{\to} x$ and write that $(x_\alpha)$ is $o$-convergent to $x$. In vector lattice $E$ we write $x_\alpha \stackrel{uo}{\to} x$ and say that $(x_\alpha)$ is $uo$-convergent to $x$ if $|x_\alpha - x| \wedge u \stackrel{\to}{\to} 0$ for every $u \in E^+$. In Banach lattice $E$ we write $x_\alpha \stackrel{un}{\to} x$ and say that $(x_\alpha)$ is $un$-convergent to $x$ if $|x_\alpha - x| \wedge \|\cdot\| \stackrel{\to}{\to} 0$ for every $u \in E^+$.

It was observed in [6] that $un$-convergence is topological. Let $x_0 \in E$ be arbitrary. For every $\epsilon > 0$ and non-zero $u \in X^+$, put $V_{\epsilon,u} = \{x \in X : ||x - x_0| \wedge u|| < \epsilon\}$.

The collection of all sets of this form is a base of $x_0$ neighbourhoods for a topology, and the convergence in this topology agrees with $un$-convergence. We will refer to this topology as $un$-topology.

Recall of [17], a net $(x_\alpha)$ in Banach lattice $E$ is unbounded absolutely weakly convergent ($uaw$-convergent) to $x$ if $(|x_\alpha - x| \wedge u)$ converges to zero weakly for every $u \in E^+$; we write $x_\alpha \stackrel{uaw}{\to} x$. It was observed in [17] that $uaw$-convergence is topological. By Theorem 4 of [17], if Banach lattice $E$ has order continuous norm, then $(x_\alpha) \subseteq E$ is $un$-null iff it is $uaw$-null in $E$.

Let $E$ be a vector lattice and $e \in E^+$. $e$ is weak unit, if band $B_e$ generated by $e$ is equal with $E$; equivalently, $x \wedge ne \uparrow x$ for every $x \in E^+$, and $e$ is strong unite when ideal $I_a$ generated by $e$ is equal $E$; equivalently, for every $x \geq 0$ there exists $n \in \mathbb{N}$ such that $x \leq ne$. A positive non-zero vector $a$ in a vector lattice $E$ is an atom if the ideal $I_a$ generated by $a$ coincides with span $a$. We say that $E$ is non-atomic if it has no atoms. We say that $E$ is atomic if $E$ is the band generated by all the atoms.

Let $E$ be a vector lattice, $E^\sim$ be the space of all order continuous functionals on $E$ and $(E^\sim)^\sim$ be the order bidual of $E$. Recall that a subset $A$ of $E$ is $b$-order bounded in $E$ if $A$ is order bounded in $(E^\sim)^\sim$. If each $b$-order bounded subset $A$ of vector lattice $E$ is order bounded, we say that $E$ has property (b). Let $X$, $Y$ be two Banach spaces, then the continuous operator $T : X \to Y$ is said to be:

- **Dounford-Pettis** whenever $(x_n) \subseteq X$ and $x_n \stackrel{w}{\to} 0$, then $T(x_n) \stackrel{\|\|}{\to} 0$. 
• **weakly compact** whenever $T$ carries the closed unit ball of $X$ to a relatively weakly compact subset of $Y$.

Let $E$ be a Banach lattice and $X$ Banach space, then the continuous operator $T : E \to X$ is said to be:

- **$M$-weakly compact** if $\lim \|Tx_n\| = 0$ holds for every norm bounded disjoint sequence $(x_n)$ of $E$.
- **order weakly compact** whenever $T[0,x]$ is relatively weakly compact subset of $X$ for each $x \in E^+$.
- **$b$-weakly compact** whenever for each $b$-order bounded subset $A$ of $E$, $T(A)$ is a relatively weakly compact. The class of $b$-weakly compact operators was firstly introduced by Alpay, Altin and Tonyali [2], the class of all $b$-weakly compact operators between $E$ and $X$ will be denoted by $W_b(E,X)$. One of the interesting properties of the class of $b$-weakly compact operators is that it satisfies the domination property. Some more investigations on $W_b(E,X)$ were done by [3, 4, 14].
- **$uaw$-Dunford-Pettis** if for every norm bounded sequence $(x_n)$ in $E$, $x_n \xrightarrow{un} 0$ in $E$ implies $\|Tx_n\| \to 0$ in $X$. These operators are introduced and examined in [15], the class of all $uaw$-Dunford-Pettis operators on $E$ will be denoted by $B_{UDP}(E)$. This is continued in [10]. Moreover if $F$ is a Banach lattice, a continuous operator $T : E \to F$ is said to be
- **$un$-continuous** if for every norm bounded and $un$-null net $(x_\alpha) \subseteq E$, $T(x_\alpha) \xrightarrow{un} 0$ in $F$. These operators are introduced and examined in [11].

Recall that a Banach lattice $E$ is said to have dual positive Schur property if every $w^*$-null positive sequence in $E^*$ is norm null.

Throughout this article $E$ and $X$ will be assumed to be Banach lattice and Banach space, respectively, and $(e_n)$ is the sequence of real numbers whose $n^{th}$ term is one and the rest are zero, i.e. $e_n := (0, 0, ..., 0, 1, 0, 0, ...)$ unless specified otherwise. For a normed space $X$, $A \subseteq X$ and $B \subseteq X^*$, $\sigma(A,B)$ is the smallest topology for $A$ such that each $f \in B$ is continuous on $A$ with respect to this topology.

### 2. Weakly unbounded norm topology

Let $E$ be a Banach lattice. A functional $f : E \to \mathbb{R}$ is *$un$-continuous*, if $x_\alpha \xrightarrow{un} 0$ implies $f(x_\alpha) \to 0$ for each norm bounded net $(x_\alpha) \subseteq E$. We denote the vector space of all *$un$-continuous* functionals on $E$ by
Let $E^\diamond$ be the second un-dual of Banach lattice $E$. It is clear that $E^\diamond$ is a subspace of $E^*$. The functional $f : \ell^1 \to \mathbb{R}$ defined by $f(x_1, x_2, x_3, ...) = \sum_{i=1}^{\infty} x_i$ is continuous but is not $un$-continuous. Therefore $E^\diamond \neq E^*$. Let $f \in E^\diamond$, we define $\|f\|_{E^\diamond} = \sup\{|f(x)| : x \in E, \|x\|_E \leq 1\}$. It is clear that $E^\diamond$ is a normed space.

**Theorem 2.1.** Let $E$ be a Banach lattice. Then we have the following assertions.

1. If $E^*$ has order continuous norm, then $E^* = E^\diamond$.
2. $E^\diamond$ is an ideal in $E^*$.
3. If $E$ is AM-space, then $E^\diamond$ is AL-space.

**Proof.**

1. Proof follows by Theorem 6.4 of [6].
2. Proof has similar argument of Proposition 5.3 of [9].
3. Let $E$ be an AM-space. Since $E^*$ is an AL-space, so $E^*$ has order continuous norm and therefore by part (1), we have $E^\diamond = E^*$ and it is an AL-space.

**Example 2.2.** Let $c$ be the sublattice of $\ell^\infty$ consisting of all convergent sequences. Since $c^* = \ell^1$ has order continuous norm, therefore by preceding theorem, we have $c^\diamond = c^* = \ell^1$.

Theorem 2.1 shows that $E^\diamond$ is a normed sublattice of a Banach lattice of $E$. Thus we define the second un-dual of Banach lattice $E$ which show by $E^{\diamond\diamond}$. $E^{\diamond\diamond}$ in general is not equal with topological second dual of $E$, $E^{**}$. Set $E = c$. It is obvious that $c^{\diamond\diamond} = c^\diamond = \ell^\infty \neq \ell^1 = c^{**}$ On the other hand $E^\diamond$ is neither norm closed nor order closed, since $E = \ell_2$, then $\ell_2^\diamond = c_{00}$.

**Proposition 2.3.** Let $E$ be a Banach lattice and $G$ be a sublattice of $E$ such that one of the following conditions hold.

1. $G$ is majorizing in $E$;
2. $G$ is norm dense in $E$;
3. $G$ is a projection band in $E$.

If $E^\diamond = E^*$, then $G^\diamond = G^*$.

**Proof.** We know that $G^\diamond \subseteq G^*$. Now assume that $f \in G^*$ and $(x_\alpha) \subseteq G$ is norm bounded and $un$-null in $G$. By Theorem 3.6 of [16], there exists $g \in E^*$ such that $f = g$ on $G$. Note that by assumption $g \in E^\diamond$. By Theorem 4.3 [9], $x_\alpha \xrightarrow{un} 0$ in $E$. We obvious that $(x_\alpha)$ is norm bounded in $E$. Therefore $f(x_\alpha) = g(x_\alpha) \xrightarrow{\|\|} 0$ in $\mathbb{R}$. It follows that $f \in G^\diamond$. ☐
Naturally, we can define weakly unbounded norm topology \((wun\text{-topology})\) as follows.

**Definition 2.4.** The smallest topology \(\tau\) that each \(f \in E^\circ\) is continuous with respect to that topology is called weakly unbounded norm topology (for short, \(wun\text{-topology}\)), and we denote by \(\tau_{wun}\). In the other words,

\[
\tau_{wun} := \bigcap \{\tau : \text{each } f \in E^\diamond \text{ is } \tau\text{-continuous}\}.
\]

For every \(\epsilon > 0\) and each \(f \in E^\diamond\), put

\[
V_{\epsilon,f} = \{x \in X : |f(x)| < \epsilon\}.
\]

It easily follows from Definition 2.4.2 of [12] that the collection of all \(V_{\epsilon,f}\) is a subbasis for \(wun\)-topology at zero.

Let \(x, y \in V_{\epsilon,f}\) and \(0 \leq \lambda \leq 1\). We have

\[
|f(\lambda x + (1-\lambda)y)| = \lambda |f(x)| + (1-\lambda) |f(y)| \leq \lambda \epsilon + (1-\lambda) \epsilon = \epsilon.
\]

Therefore \(\lambda x + (1-\lambda)y \in V_{\epsilon,f}\).

Hence \(\tau_{wun}\) is a locally convex in Banach lattice \(E\).

Let \(E\) be a Banach lattice. It is obvious that for each norm bounded net \((x_{\alpha}) \subseteq E\), \(x_{\alpha} \overset{wun}{\longrightarrow} 0\) if and only if for each \(f \in E^\circ\), \(f(x_{\alpha}) \overset{0}{\longrightarrow} 0\) in \(\mathbb{R}\).

It is obvious that every \(un\)-null net is \(wun\)-null in a Banach lattice, but in general the converse not holds. The following example shows that in general both topologies \(un\) and \(wun\)-topology are not the same. On the other hand by Proposition 3.5 of [9], every norm bounded and disjoint net in order continuous Banach lattice \(E\) is \(wun\)-null. Thus if we set \(E = \ell^1\), then \((e_n)\) is \(wun\)-null in \(\ell^1\).

**Example 2.5.** Consider the sequence \((e_n)\) in the sublattice \(c\) of \(\ell^\infty\). By Example 2.2, \(c^\circ = \ell^1\). For each \(f = (x_1, x_2, \ldots, x_n, \ldots) \in c^\circ\), \(f(e_n) = x_n \overset{\|\|}{\longrightarrow} 0\), therefore \(e_n \overset{wun}{\longrightarrow} 0\) in \(c\), but \((e_n)\) is not \(un\)-null in \(c\). Consider \(u = (1, 1, 1, \ldots) \in c^+\). We have \(\|e_n \wedge u\| = \|e_n\| = 1 \nless 0\).

The following facts are in \(wun\)-topology that will be used throughout the paper.

**Lemma 2.6.** Let \(E\) be a Banach lattice and \((x_{\alpha}) \subseteq E\), then

1. \(x_{\alpha} \overset{wun}{\longrightarrow} x\) iff \((x_{\alpha} - x) \overset{wun}{\longrightarrow} 0\);
2. \(wun\)-limits are unique;
3. If \(x_{\alpha} \overset{wun}{\longrightarrow} x\) and \(y_{\beta} \overset{wun}{\longrightarrow} y\), then \(ax_{\alpha} + by_{\beta} \overset{wun}{\longrightarrow} ax + by\), for any scalars \(a, b\);
4. If \(x_{\alpha} \overset{wun}{\longrightarrow} x\), then \(y_{\beta} \overset{wun}{\longrightarrow} x\), for every subnet \((y_{\beta})\) of \((x_{\alpha})\).

**Proof.**

(1) The proof is clear.
On the other hand, (E and (x bounded and disjoint in ℓ-uaw-topology (in short uaw-topology. Note that wun-is not weak convergent to zero in ℓ-topology in Banach lattice (n∞, x ∈ E. When E has strong unit then by Theorem 2.3 of [9], E∞ = E* and therefore, τw = τwun in E. □

Remark 2.7. Since wun-limits are unique, therefore for each x ∈ E, {x} is wun-closed. By condition 2 of Lemma 2.6, τwun is a vector topology on E, and (E, τwun) is a topological vector space. By Theorem 1.12 of [16], τwun is a Hausdorff topology.

Note that wun-topology is different with weak topology (in short w-topology). Consider the sequence (e_n) in ℓ^1. Since (e_n) is norm bounded and disjoint in ℓ^1, then by Example 2.5(1), e_n wun→ 0 in ℓ^1. On the other hand, (e_n) is not w-null in ℓ^1. Therefore τw ≠ τwun. Since E∞ is a subspace of E*, w-topology is weaker then wun-topology. Thus every w-null net is wun-null for each Banach lattice E.

Proposition 2.8. Let E be a Banach lattice. If E has strong unit, then w-topology and wun-topology coincide.

Proof. Since E∞ ⊆ E*, follows that wun-topology is weaker than w-topology in Banach lattice E. When E has strong unit then by Theorem 2.3 of [9], E∞ = E* and therefore, τw = τwun in E. □

O. Zabti in [17] has been introduced unbounded absolutely weakly topology (in shorn uaw-topology) and investigated some of its properties. Note that wun-topology is different with uaw-topology. By Lemma 2 of [17], the sequence (e_n) ⊆ ℓ^∞ is uaw-null in ℓ^∞, but (e_n) is not weak convergent to zero in ℓ^∞, and so by Proposition 2.8, it is not wun-null in ℓ^∞.

Remark 2.9. Note that if Banach lattice E has order continuous norm and (x_α) ⊆ E is norm bounded and uaw-null, then by Proposition 5 of [17], x_α wun→ 0 in E. If E is an AM-space with strong unit and (x_n) ⊆ E with x_n wun→ 0, then x_n w→ 0 and by Exercise 5 of page 355 of [1], |x_n| w→ 0. It follows that x_n uaw→ 0 in E.
Proposition 2.10. If a Banach lattice $E$ is atomic with order continuous norm, then $(E, \tau_{un})^* = E^\circ$.

Proof. It follows from Theorem 5.2 of [9] that un-topology is locally convex. Therefore, $E^\circ$ separates the points of $E$. Thus, by Theorem 3.10 of [16] we have $(E, \tau_{un})^* = E^\circ$. \qed

Let $G$ be a sublattice of $E$ and $(x_\alpha) \subseteq G$. By Theorem 3.6 of [16], $x_\alpha \wto 0$ in $G$ iff $x_\alpha \to 0$ in $E$. The situation is different for wun-convergence.

Example 2.11. (1) Consider the sequence $(e_n)$ of $\ell^1$. It is wun-null in $\ell^1$ while is not wun-null in $\ell^\infty$.

(2) Note that $\ell^1$ is an order continuous Banach lattice with a weak unit $e$. It is known that $\ell^1$ can be represented as an order and norm dense ideal in $L_1(\mu)$ for some finite measures $\mu$. Consider the sequence $(x_n) = (\frac{1}{2}(e_1 - e_n)) \subseteq \ell^1$. Since $L_1(\mu)$ has order continuous norm and is non-atomic, therefore by Corollary 5.4 of [9], $x_n \un 0$ in $L_1(\mu)$. On the other hand $(x_n)$ is not wun-null in $\ell^1$. Since $(x_n)$ is un-convergent to $\frac{1}{2}e_1$ in $\ell^1$ and therefore is wun-convergent to $\frac{1}{2}e_1$ in $\ell^1$.

Let $G$ be a sublattice of a Banach lattice $E$. In the following, we bring the conditions that if a net $(x_\alpha) \subseteq G$ is wun-null in $G$, then is wun-null in $E$ and vice versa.

Theorem 2.12. Let $G$ be a sublattice of Banach lattice $E$ and $(x_\alpha) \subseteq G$.

(1) If $E^* = E^\circ$ and $x_\alpha \un 0$ in $E$, then $x_\alpha \un 0$ in $G$.

(2) If $G^* = G^\circ$ and $x_\alpha \un 0$ in $G$, then $x_\alpha \un 0$ in $E$.

Proof. (1) Let $(x_\alpha) \subseteq G$ and $x_\alpha \un 0$ in $E$. By $E^* = E^\circ$, $x_\alpha \to 0$ in $E$. By Theorem 3.6 of [16], $x_\alpha \to 0$ in $G$ and therefore $x_\alpha \un 0$ in $G$.

(2) If $x_\alpha \un 0$ in $G$, then $x_\alpha \to 0$ in $G$ and therefore $x_\alpha \to 0$ in $E$. So $x_\alpha \un 0$ in $E$. \qed

Corollary 2.13. Let $G$ be a sublattice of Banach lattice $E$ and $(x_\alpha) \subseteq G$. If $E^* = E^\circ$ and $G$ is a majorizing or norm dense or band projection in $E$, then $x_\alpha \un 0$ in $E$ iff $x_\alpha \un 0$ in $G$. 
Let $T : E \to F$ be a continuous operator between two Banach lattices. Then $T$ has a un-adjoint if there exists the unique operator $T^\circ : F^\circ \to E^\circ$ satisfying
\[ < T^\circ y^\circ, x > = < y^\circ, Tx > = y^\circ(Tx), \quad \forall y^\circ \in F^\circ, \forall x \in E. \]
It easily follows from $F^\circ \subseteq F^*$ that $T^\circ = T^*|_{F^\circ}$.

**Theorem 2.14.** Let $T : E \to F$ be an operator between two Banach lattices. If $T$ is un-continuous, then $T$ has un-adjoint.

**Proof.** Assume that $T$ is un-continuous. It is enough to prove that $T^\circ(F^\circ) \subseteq E^\circ$. Let $y^\circ \in F^\circ$ and $(x_\alpha)$ be norm bounded and un-null net in $E$. Since $T$ is un-continuous, we have $(Tx_\alpha)$ is norm bounded and un-null in $F$. As $y^\circ$ is un-continuous, $T^\circ y^\circ(x_\alpha) = y^\circ(Tx_\alpha) \overset{\text{un}}{\to} 0$. Thus, $T^\circ y^\circ \in E^\circ$.

**Example 2.15.** The operator $T : C[0,1] \to c_0$, given by
\[ T(f) = (\int_0^1 f(x) \sin x dx, \int_0^1 f(x) \sin 2xdx, ...) \]
is a un-continuous. By Theorem 2.14, $T$ has un-adjoint. We have $T^\circ : c_0^\circ \to (C[0,1])^\circ$, given by
\[ (T^\circ(x_1, x_2, ...), f) = \langle (x_1, x_2, ...), Tf \rangle = \sum_{n=1}^{\infty} x_n \int_0^1 f(x) \sin nx \, dx, \]
for all $(x_1, x_2, ...) \in c_0^\circ$ and $f \in C[0,1]$.

Now, assume that $Q_E$ be a natural mapping from $E$ into $E^{**}$ where $\langle x', Q_E(x) \rangle = \langle x, x' \rangle = x'(x)$ for all $x \in E$ and $x' \in E^*$. Since $E^{**}$ is a subspace of $E^{**}$, we have the following lemma.

**Lemma 2.16.** Let $E$ be a Banach lattice. Then $Q_E(E) \subseteq E^{**}$.

**Proof.** Let $(x_\alpha') \subseteq E^\circ$ be a norm bounded and un-null net in $E^\circ$. By Theorem 2.1 we know that $E^\circ$ is an ideal in $E^*$, and so $x_\alpha' \wedge y' \in E^\circ$ for all $y' \in E^*$. It follows that $x_\alpha' \wedge (x_\alpha' \wedge y') \overset{\|\|}{\to} 0$ in $E^\circ$. Thus for each $y' \in E^*$, we have $x_\alpha' \wedge y' \overset{\|\|}{\to} 0$ in $E^\circ$.

Let $x \in E$. If $x = 0$, $Q_E(0) \in E^{\infty}$, and proof holds. Now assume that $x \neq 0$. Then there exists $y' \in E^*$ such that $y'(x) = 1$. Then we have $(y')^+(x) \geq 1$. Since $x_\alpha'(x) \wedge (y')^+(x) \overset{\|\|}{\to} 0$ in $E^\circ$, follows that $x_\alpha'(x) \overset{\|\|}{\to} 0$ in $E^\circ$, and so $Q_E(x)(x_\alpha') \to 0$. It follows that $Q_E(x) \in E^{\infty}$ and proof follows. \hfill $\Box$
Now the Lemma 2.16 make motivation to us for definition a new
topology for $E^\circ$, that is, the smallest topology on $E^\circ$ such that each
$Q_E(x)$ is continuous with respect to it where $x \in E$. This topology is
called weak* unbounded topology (for short $w^*\text{-}\text{un}$-topology). In this
way, $x_\alpha \xrightarrow{w^*\text{-}\text{un}} 0$ if and only if $x'(x) \rightarrow 0$ for all $x \in E$. It is clear that the
$w^*\text{-}\text{un}$-topology in $E^\circ$ is a subtopology of $w^*$-topology in $E^*$, and $w^*\text{-}\text{un}$-
topology is a subset of $w\text{-}\text{un}$-topology in $E^\circ$, that is, $\sigma(E^\circ, Q_E(E)) \subseteq 
\sigma(E^\circ, E^{**}) \subseteq \sigma(E^\circ, (E^\circ)^*) \subseteq \sigma(E^\circ, E^{**})$.

**Theorem 2.17.** Let $E$ be a Banach lattice. Then $B_{E^\circ} = \{x' \in 
E^\circ : \|x\| \leq 1\}$ is $w^*\text{-}\text{un}$-compact.

*Proof.* It is obviously that $A \subseteq E^\circ$ is $w^*\text{-}\text{un}$-closed in $E^\circ$ if and only if
there exists $B \subseteq E^\circ$ which is $w^*$-closed in $E^*$ and $A = B \cap E^\circ$. Since
$B_{E^\circ} = B_{E^*} \cap E^\circ$ and $B_{E^*}$ is $w^*$-compact, proof follows. \hfill \Box

In the following we have some facts for $w\text{-}\text{un}$-topology and $w^*\text{-}\text{un}$-
topology in $E^\circ$ which theirs proofs has similar arguments such as clas-
sical studying for $w^*$ and $w$-topologies in $E^*$.

**Corollary 2.18.** Suppose that $E$ and $F$ are Banach lattices. Then we
have the following assertions.

1. If $T \in B(E, F)$, then $T^\circ$ is $w^*\text{-}\text{un}$-continuous. Conversely,
   if $S$ is a $w^*\text{-}\text{un}$-continuous linear operator from $F^\circ$ into
   $E^\circ$, then there is a $T$ in $B(E, F)$ such that $T^\circ = S$.
2. If $T \in B(E, F)$, then $T^\circ Q_E(F) \subseteq Q_F(F)$ and $Q_F^{-1} T^\circ Q_E = T$.
3. A bounded linear operator from a Banach lattice into a Banach
   lattice is $w\text{-}\text{un}$-compact if and only if its adjoint is $w\text{-}\text{un}$-compact.
4. Suppose that $T \in B(E, F)$ and $Q_F$ is the natural map from $F$
   into $F^{\circ\circ}$. Then $T$ is $w\text{-}\text{un}$-compact if and only if $T^\circ(E^{\circ\circ}) \subseteq 
   Q_F(F)$.

**Definition 2.19.** Let $E$ and $F$ be two Banach lattices. A continuous
operator $T : E \rightarrow F$ is said to be, weak unbounded norm continuous
(or, $w\text{-}\text{un}$-continuous for short), if $x_\alpha \xrightarrow{w\text{-}\text{un}} 0$ in $E$ implies
$Tx_\alpha \xrightarrow{w\text{-}\text{un}} 0$ in $F$ for each norm bounded net $(x_\alpha) \subseteq E$. The collection of all $w\text{-}\text{un}$-
continuous operators from $E$ to $F$, will be denoted by $L_{w\text{-}\text{un}}(E, F)$.

**Example 2.20.** (1) Let $G$ be a majorizing or norm dense or band
projection of $\ell^\infty$. Then each continuous operator from $G$ to $\ell^\infty$
is $w\text{-}\text{un}$-continuous.

(2) Consider the functional $f : \ell^\infty \rightarrow \mathbb{R}$ defined with
$f(x_1, x_2, ...) = \lim_{n \rightarrow \infty} x_n$. 

Since \( f \) is positive, \( f \) is continuous. Now if \( (x_n) \subseteq \ell^\infty \) is norm bounded and \( x_n \xrightarrow{un} 0 \) then \( x_n \xrightarrow{w} 0 \) and therefore \( f(x_n) \xrightarrow{w} 0 \).

Hence \( f(x_n) \xrightarrow{un} 0 \) in \( \mathbb{R} \).

**Remark 2.21.** Note that the operator \( T : \ell^1 \rightarrow \ell^\infty \) defined by

\[
T(x_1, x_2, \ldots) = \left(\sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots\right)
\]

is continuous, while is not \( w_{un} \)-continuous.

**Theorem 2.22.** A functional \( f : E \rightarrow \mathbb{R} \) is \( un \)-continuous if and only if is \( w_{un} \)-continuous.

**Proof.** Let \( f \) is \( un \)-continuous and \( (x_\alpha) \subseteq E \) is norm bounded with \( x_\alpha \xrightarrow{un} 0 \) in \( E \). Therefore for each \( x^* \in E^* \), we have \( x^* (x_\alpha) \xrightarrow{\parallel \cdot \parallel} 0 \) in \( \mathbb{R} \). Since \( f \) is \( un \)-continuous, therefore by Theorem 2.14, \( f \) has \( un \)-adjoint. Hence \( f^* (\mathbb{R}^*) \subseteq E^* \). Therefore for all \( y^* \in E^* \), we have

\[
y^* (f(x_\alpha)) = f^* y^* (x_\alpha) \xrightarrow{\parallel \cdot \parallel} 0 \text{ in } \mathbb{R}.
\]

Hence \( f(x_\alpha) \xrightarrow{un} 0 \) in \( \mathbb{R} \).

Conversely, let \( (x_\alpha) \subseteq E \) is norm bounded and \( x_\alpha \xrightarrow{un} 0 \) in \( E \). It is clear that \( x_\alpha \xrightarrow{un} 0 \) in \( E \) and therefore \( f(x_\alpha) \xrightarrow{un} 0 \) in \( \mathbb{R} \). So \( f(x_\alpha) \xrightarrow{\parallel \cdot \parallel} 0 \) in \( \mathbb{R} \). \( \square \)

**Corollary 2.23.** Let \( E \) and \( F \) be two Banach lattices. Similar to Therem 2.22, if operator \( T : E \rightarrow F \) is \( un \)-continuous, then is \( w_{un} \)-continuous.

**Remark 2.24.** Note that, if Banach lattice \( E \) is an atomic \( KB \)-space, then by Theorem 7.5 of [9], \( B_E \) is \( un \)-compact. Since \( \tau_{un} \subseteq \tau_{w_{un}} \), therefore \( B_E \) is \( w_{un} \)-compact.

**Theorem 2.25.** Let \( E \) be an atomic Banach lattice with order continuous norm. If \( A \subseteq E \) is a convex set, then \( w_{un} \)-closure of \( A \) is the same as its \( un \)-closure, that is; \( \overline{A}_{w_{un}} = \overline{A}_{un} \).

**Proof.** Since \( \tau_{w_{un}} \subseteq \tau_{un} \), \( \overline{A}_{un} \subseteq \overline{A}_{w_{un}} \). On the other hand, by Theorem 5.2 of [9], \( un \)-topology is locally convex, hence if \( x \notin \overline{A}_{un} \) then by Theorem 3.13 [5] there exists some \( f \in E^* \), \( \alpha \in \mathbb{R} \) and an \( \epsilon > 0 \) such that

\[
f(a) \leq \alpha < \alpha + \epsilon < f(x),
\]

for all \( a \in \overline{A}_{un} \). Therefore, \( \overline{A}_{un} \subseteq B = \{ y : f(y) \leq \alpha \} \). By Proposition 2.10 \( f \) is \( w_{un} \)-continuous, thus \( B \) is \( w_{un} \)-closed. Hence \( \overline{A}_{w_{un}} \subseteq B \). Therefore, \( x \notin \overline{A}_{w_{un}} \). Consequently, \( \overline{A}_{un} = \overline{A}_{w_{un}} \). \( \square \)
3. wun-Dunford-Pettis operators

A continuous operator $T$ from Banach lattice $E$ into Banach space $X$ is a wun-Dunford-Pettis whenever $x_n \overset{wun}{\rightarrow} 0$ in $E$ implies $Tx_n \overset{\|\|}{\rightarrow} 0$ in $X$ for each norm bounded sequence $(x_n) \subseteq E$.

Example 3.1. Operator $T: C[0, 1] \rightarrow \ell^1$, given by

$$T(f) = \left( \frac{\int_0^1 f(x) \sin x \, dx}{1^2}, \frac{\int_0^1 f(x) \sin 2x \, dx}{2^2}, \ldots \right)$$

is a wun-Dunford-Pettis operator. Let $(f_n) \subseteq C[0, 1]$ is norm bounded and $f_n \overset{wun}{\rightarrow} 0$. Since $(C[0,1])^\ast = (C[0,1])^\circ$, so $f_n \overset{\ast}{\rightarrow} 0$ in $C[0,1]$. By continuity of $T$, we have $T(f_n) \overset{\ast}{\rightarrow} 0$ in $\ell^1$ and by Schur property of $\ell^1$, $T(f_n) \overset{\|\|}{\rightarrow} 0$ in $\ell^1$.

Remark 3.2. Let $E$ and $F$ be two Banach lattices and $X$ be a Banach space. If $T: E \rightarrow F$ and $S: F \rightarrow X$ are wun-Dunford-Pettis, then $ST$ is wun-Dunford-Pettis.

It is clear that if $T$ is wun-Dunford-Pettis, then it is Dunford-Pettis and $\sigma$-un-continuous. The identity operator $I: \ell^1 \rightarrow \ell^1$ is a $\sigma$-un-continuous, but it is not wun-Dunford-Pettis operator.

Here we give an example to illustrate the difference between Dunford-Pettis and wun-Dunford-Pettis operators.

Example 3.3. The operator $T: \ell^1 \rightarrow \ell^\infty$ defined by

$$T(x_1, x_2, \ldots) = \left( \sum_{i=1}^{\infty} x_i, \sum_{i=1}^{\infty} x_i, \ldots \right)$$

is a Dunford-Pettis operator ($\ell^1$ has Schur property and $T$ is a continuous operator). Consider $(e_n) \subseteq \ell^1$. $e_n \overset{wun}{\rightarrow} 0$ in $\ell^1$. We have $T(e_n) = (1, 1, 1, \ldots)$, therefore $(Te_n)$ is not convergent to zero in norm topology. Thus $T$ is not wun-Dunford-Pettis operator.

Remark 3.4. It is clear that if $E^\ast = E^\circ$, then operator $T: E \rightarrow X$ is Dunford-Pettis iff it is wun-Dunford-Pettis.

Proposition 3.5. A linear operator from a Banach lattice into a Banach lattice is wun-Dunford-Pettis if and only if it is wun-norm sequentially continuous.

Proof. The forward implication is an easy consequence of the wun-wun continuity of wun-Dunford-Pettis operators along with the fact that a subset of a Banach lattice consisting of the terms and limit of a wun-convergent sequence is wun-compact. The converse follows
directly from the fact that \textit{wun}-compact subsets of a normed space are \textit{wun}-sequentially compact. \hfill \square

**Proposition 3.6.** If each Dunford-Pettis operator \( T : E \to F \) between two Banach lattices is \textit{wun}-Dunford-Pettis, then the norm of \( E^* \) is order continuous or \( F = \{0\} \).

**Proof.** The proof has the similar argument of Theorem 3.1 of [10]. \hfill \square

**Theorem 3.7.** Let \( F \neq \{0\} \) be a reflexive Banach lattice. The zero operator is the only \textit{wun}-Dunford-Pettis positive operator \( T : \ell^1 \to F \).

**Proof.** Let \( T : \ell^1 \to F \) be a positive operator. Since \( F \) is reflexive, then by Theorem 5.29 of [1], \( T \) is a weakly compact operator. By Theorem 5.85 of [1], \( \ell^1 \) has Dunford-Pettis property. Therefore by Theorem 5.82 of [1], \( T \) is Dunford-Pettis. Since the norm of \((\ell^1)^*\) is not order continuous and \( F \neq \{0\} \), so by Proposition 3.6, \( T \) is not \textit{wun}-Dunford-Pettis. \hfill \square

**Remark 3.8.** It is known that every compact operator between Banach lattices is Dunford-Pettis. In the case of a \textit{wun}-Dunford-Pettis operator, the situation is different. The Example 3.3 is compact while it is not \textit{wun}-Dunford-Pettis.

Here we give an example that it illustrate \textit{uaw}-Dunford-Pettis operators differ from \textit{wun}-Dunford-Pettis operators.

**Example 3.9.** For each continuous operator \( T : C[0,1] \to c_0 \), the adjoint operator \( T^* : \ell^1 \to (C[0,1])^* \) is a \textit{uaw}-Dunford-Pettis. Indeed let \((x_n) \subseteq \ell^1 \) be norm bounded and \( x_n \xrightarrow{\text{uaw}} 0 \). By Proposition 5 of [17], \( x_n \xrightarrow{w^*} 0 \) in \( \ell^1 \) and therefore \( T^*(x_n) \xrightarrow{w^*} 0 \) in \((C[0,1])^*\). Since \( C[0,1] \) has dual positive Schur property, so we have \( T^*(x_n) \xrightarrow{\|\cdot\|} 0 \) in \((C[0,1])^*\). Note that for each continuous operator \( T : C[0,1] \to c_0 \), the adjoint operator \( T^* : \ell^1 \to (C[0,1])^* \) is Dunford-Pettis (we know that \( T^* \) is continuous and \( \ell^1 \) has Schur property). Since \((\ell^1)^* = \ell^\infty \) does not has order continuous norm and \((C[0,1])^* \neq 0\), therefore by Proposition 3.6, there exists some \( T : C[0,1] \to c_0 \) such that \( T^* \) is not \textit{wun}-Dunford-Pettis.

**Remark 3.10.** If \( T : E \to X \) is a \textit{wun}-Dunford-Pettis where \( E \) has order continuous norm, then \( T \) is a \( M \)-weakly compact and therefore by Theorem 1 of [15], it is a \textit{uaw}-Dunford-Pettis.

**Theorem 3.11.** Let \( T : E \to X \) be an operator from AM-space \( E \) to Banach space \( X \). Then the following assertions are equivalent:

1. \( T \) is \( M \)-weakly compact.
(2) \( T \) is weakly compact.
(3) \( T \) is Dunford-Pettis.
(4) \( T \) is wun-Dunford-Pettis.
(5) \( T \) is uaw-Dunford-Pettis.
(6) \( T \) is \( b \)-weakly compact.
(7) Moreover if \( E \) has property \((b)\), \( T \) is order weakly compact.

**Proof.**
(1) ⇔ (2) By Theorem 5.62 of [1], the proof is complete.
(2) ⇒ (3) By Theorems 5.85 and 5.82 of [1], \( T \) is a Dunford-Pettis operator.
(3) ⇒ (1) Since \( E \) is an \( AM \)-space, then by Theorem 4.23 of [1], \( E^* \) is an \( AL \)-space and therefore \( E^* \) has order continuous norm. By Theorem 3.7.10 of [13], \( T \) is \( M \)-weakly compact.
(3) ⇔ (4) Since \( E^* = E^\circ \), therefore \( x_n \overset{w}{\rightarrow} 0 \) in \( E \) iff \( x_n \overset{wun}{\rightarrow} 0 \) in \( E \). Hence the proof is clear.
(3) ⇔ (5) Follows from Corollary 3.7 of [10].
(2) ⇒ (6) Since each \( b \)-order bounded set in Banach lattice is norm bounded, hence the proof is clear.
(6) ⇒ (2) Let \( B_E \) be a closed unit ball of \( E \). Since \( E \) has strong unit, so \( B_E \) is an order interval. Therefore \( (TB_E) \) is a relatively weakly compact subset of \( X \).
(6) ⇔ (7) By property \((b)\) of \( E \), \( A \subseteq E \) is order bounded if and only if it is \( b \)-order bounded, hence the proof is clear.

Let \( S, T : E \rightarrow F \) be two positive operators satisfying \( 0 \leq S \leq T \) with \( T \) is wun-Dunford-Pettis, it is clear that \( S \) is wun-Dunford-Pettis.

We give an example that an operator \( T \) is wun-Dunforde-Pettis while its adjoint is not wun-Dunford-Pettis and vic versa. In the following with under certain conditions, an operator \( T \) is wun-Dunford-Pettis iff \( T^* \) is wun-Dunford-Pettis.

**Example 3.12.**
(1) Consider the operator \( T : C[0,1] \rightarrow c_0 \), given by

\[
T(f) = \left( \int_0^1 f(x) \sin x \, dx, \int_0^1 f(x) \sin 2x \, dx, \ldots \right).
\]

If \( (f_n) \subseteq C[0,1] \) is norm bounded and \( f_n \overset{wun}{\rightarrow} 0 \) then \( f_n \overset{w}{\rightarrow} 0 \). We have \( \|Tf_n\| = \sup_{m \geq 1} \int_0^1 f_n(t) \sin mt \, dt \leq \int_0^1 |f_n(t)| \, dt \rightarrow 0 \). Hence \( T \) is a wun-Dunforde-Pettise. Similar to Example 3 of [15], adjoint of \( T \), \( T^* \) is not wun-Dunford-Pettis.
The functional $f : \ell^1 \to \mathbb{R}$ defined by

$$f(x_1, x_2, ...) = \sum_{i=1}^{n} x_i$$

is not wun-Doufnord-Pettis, but $f^*$ is wun-Dounford-Pettis.

**Theorem 3.13.** Let $E$ and $F$ be two Banach lattices such that $E$ and $F^*$ have strong unit. Then $T : E \to F$ is wun-Doufnord-Pettis iff $T^*$ is wun-Dounford-Pettis.

**Proof.**

(1) Let $T : E \to F$ be a wun-Dounford-Pettis. Since $E$ has strong unit, then by Theorem 3.11, $T$ is b-weakly compact operator. because $E$ has strong unit therefore it ia an AM-space. By Theorem 4.23 of [1], $E^*$ is an AL-space. Each AL-space is a KB-space. Therefore $E^*$ is a KB-space. Hence by Theorem 3.1 of [4], $T^*$ is b-weakly compact. Since $F^*$ has strong unit, hence by Theorem 3.11, $T^*$ is wun-Doufnord-Pettis.

(2) The Proof has similar argument of (1) and by Theorem 3.5 of [4], proof follows.

**Theorem 3.14.** Let $F$ be a Banach lattice. If for each arbitrary Banach lattice $E$, operator $T : E \to F$ is wun-Doufnord-Pettis, then

1. $F$ is KB-space.
2. $T$ is b-weakly compact.

**Proof.**

(1) Let $c_0$ be embeddable in $F$ and $T : c_0 \to F$ be this embedding. Then there exist two positive constants $K$ and $M$ satisfying

$$K\|x_n\| \leq \|T(x_n)\| \leq M\|x_n\| \text{ for all } (x_n) \subseteq c_0.$$

Consider the sequence $(e_n) \subseteq c_0. e_n \xrightarrow{\text{wun}} 0$ and norm bounded in $c_0$ but $\|T(e_n)\| \geq K\|e_n\| = K > 0$ which contradicts with assumption. Therefore $c_0$ is not embeddable in $F$. Hence by Theorem 4.61 of [1], $F$ is a KB-space.

(2) By past part we have $F$ is KB-space. Since $c_0$ is not embeddable in $F$, then by Theorem 4.63 of [1], there exist a KB-space $H$, a lattice homomorphism $Q : E \to H$ and a continuous operator $S : H \to F$ such that $T = SQ$. Let $(x_n)$ be a b-order bounded disjoint sequence in $E$. It is clear that $(Q(x_n))$ is also b-order bounded and disjoint sequence in $H$. By Lemma 2.1 of [4], $Q(x_n) \xrightarrow{\text{w}} 0$ in $H$. Thererore $T(x_n) = SQ(x_n) \xrightarrow{\text{w}} 0$. So $T$ is b-weakly compact.
Remark 3.15. Note that if $F$ is $KB$-space, then every operator $T$ from a Banach lattice $E$ into $F$, in general, is not $wun$-Dounford-Pettis. By Example 3.12 there exists adjoint operator $T^*$ from $\ell^1$ into $KB$-space $(C[0, 1])^*$ such that it is not $wun$-Dounford-Pettis.

Remark 3.16. Let $E$ and $F$ be two Banach lattices. If an operator $T : E \to F$ is $b$-weakly compact, in general, $T$ is not $wun$-Dounford-Pettis necessarily. We know that $\ell^1$ is $KB$-space. Therefore by Theorem 4.61 of [1], $c_0$ is not embeddable in $\ell^1$. By Proposition 2.2 of [4], for any Banach lattice $E$, each operator from $E$ into $\ell^1$ is $b$-weakly compact. On the other hand, the identity operator $I : \ell^1 \to \ell^1$ is not $wun$-Dounford-Pettis.

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