The Jang equation, apparent horizons and the Penrose inequality

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Abstract
The Jang equation in the spherically symmetric case reduces to a first-order equation. This permits an easy analysis of the role apparent horizons play in the (non)existence of solutions. We demonstrate that the proposed derivation of the Penrose inequality based on the Jang equation cannot work in the spherically symmetric case. Thus it is fruitless to apply this method, as it stands, to the general case. We also show that those analytic criteria for the formation of horizons that are based on the use of the Jang equation are of limited validity for the proof of the trapped surface conjecture.

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1. Introduction

Jang introduced his eponymous equation in 1978 in the context of the initial value formulation of general relativity \cite{Jang}. The initial data for the Einstein equations consist of a quartet of objects \((g_{ij}, K^{ij}, \rho, J^i)\) which satisfy the constraints. Many of the difficult issues in this area are easier to express and solve if one knows that the scalar curvature of the metric \(g_{ij}\) is positive. Unfortunately, one has no guarantee that this is valid for a given initial data set.

The Jang equation is a nonlinear second-order elliptic equation for a scalar \(f\) which depends on \(g_{ij}\) and \(K^{ij}\). Given a metric \(g_{ij}\), and a solution \(f\), one constructs a related metric \(\tilde{g}_{ij} = g_{ij} + f_i f_j\). Jang showed that the scalar curvature of the new metric has nice positivity properties.

The first precise results about the Jang equation were derived by Schoen and Yau in their second article on the positivity of the total energy of the gravitational field \cite{SY1}. In this paper they proved that if asymptotically flat initial data do not contain any apparent horizon (either future or past) a regular solution must exist. The converse result, i.e., if a regular solution to the Jang equation does not exist, then the data must contain apparent horizons, played a key
role in their article ‘Existence of a black hole due to condensation of matter’ [3]. Given a
Riemannian metric with a nonzero-scalar curvature \((\mathcal{R})\), one can find a conformally related
manifold with zero-scalar curvature by solving the Lichnerowicz equation, \(8\nabla^2 \phi - \mathcal{R} \phi = 0\).
Schoen and Yau also showed (in [2]) that one could always solve the Lichnerowicz equation
on the ‘Jang transformed’ metric because of the positivity property discovered by Jang.

The Penrose inequality relates the area of the outermost apparent horizon to the ADM
mass. It has been suggested that the Jang equation, in a method described in section 4, can
be used to prove the Penrose inequality. We explicitly demonstrate, analysing spherically
symmetric initial data, that one cannot control simultaneously the mass of the three-manifold
and the area in the various steps of this construction. As a result, the Penrose inequality cannot
be derived by this method even in the spherically symmetric case, and therefore it could not
be effective in the general case. It remains an open question whether a radical alteration of the
method would give the desired result.

In the following sections we focus our attention on the Jang equation with spherically
symmetric data and seek a spherically symmetric solution. In this case, it reduces to a first-
order equation (essentially for the radial derivative of \(f\)) and it is much easier to determine
whether solutions do or do not exist. In the sections following section 4 we investigate the
validity of the Jang-equation-based methods for diagnosing the presence of trapped surfaces.
The net conclusion we draw from this analysis is that the Jang equation is not very useful as
either a predictor or finder of apparent horizons, and that it is not particularly suitable for the
proof of the trapped surface conjecture.

2. Spherically symmetric Jang equation

As described by Schoen and Yau, one starts with a pair \((g_{ij}, K_{ij})\). One extends the three-metric
to a four-metric (which is Riemannian, not pseudo-Riemannian) by adding trivially a fourth
coordinate (call it \(w\)) such that \(g_{ww} = +1\), \(g_{wi} = 0\), i.e.,

\[
g_{\mu\nu} = \begin{pmatrix} 1, & 0 \\ 0, & g_{ij} \end{pmatrix}.
\]

In this four-manifold we find a three-surface defined by \(w = f(x^i)\). The Jang equation is
that the mean curvature of this three-surface equals the trace of \(K_{ij}\), taken with respect to the
induced three-metric. This gives a second-order, nonlinear elliptic equation for the scalar
\(f\).

Let us assume that both the three-metric and the extrinsic curvature are spherically
symmetric. This means that we can write the metric as

\[
ds^2 = g_{rr} \, dr^2 + R^2(r) \, d\Omega^2
\]

where \(r\) is some radial coordinate and \(R\) is the areal (Schwarzschild) radius of the isometry
spheres. The spherically symmetric extrinsic curvature can be written as

\[
K_{ab} = n^a n^b K_I + (g^{ab} - n^a n^b) K_R
\]

where \(K_I\) and \(K_R\) are two scalars and \(n^a\) is the outward pointing unit normal to the surfaces of
constant \(R\).

We make the following coordinate transformation:

\[
\hat{w} = w - f(r) \\
\hat{r} = r \\
\hat{\theta} = \theta \\
\hat{\phi} = \phi
\]
where the $\bar{w} = 0$ surface is the slice we are interested in. The transformed metric becomes
\[ \bar{g}_{\mu \nu} = \begin{pmatrix} 1 & f' & 0 & 0 \\ f', & g_{rr} + f'^2 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ 0 & 0 & 0 & R^2 \sin^2 \theta \end{pmatrix} \]
and
\[ \bar{g}^{\mu \nu} = \begin{pmatrix} 1 + f'^2 g_{rr} & -f' g'' & 0 & 0 \\ -f' g'' & g'' & 0 & 0 \\ 0 & 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & 0 & \frac{1}{R^2 \sin^2 \theta} \end{pmatrix} \]
where the prime denotes derivative with respect to $r$.

The unit normal to the surface defined by $\bar{w} = w - f = 0$ is given by
\[ (\bar{n}^\mu) = \left(\frac{1 + f'^2 g_{rr}, -f' g''}{\sqrt{1 + f'^2 g_{rr}}}, 0, 0\right) \]
and the mean curvature of the slice is given by
\[ H = \bar{n}_\mu \bar{n}^{\mu} = \frac{\left(\frac{1 + f'^2 g_{rr}, -f' g''}{\sqrt{1 + f'^2 g_{rr}}}, 0, 0\right)}{\sqrt{1 + f'^2 g_{rr}}}, \frac{1}{r}. \]

The induced three-metric is given by
\[ \bar{g}^{ab} = \begin{pmatrix} \frac{1}{g_{rr}} & 0 & 0 \\ 0 & \frac{1}{R^2} & 0 \\ 0 & 0 & \frac{1}{R^2 \sin^2 \theta} \end{pmatrix} \]
and the relevant trace of $K$ is given by
\[ \bar{g}^{ab} K_{ab} = \frac{K_l}{1 + f'^2 g_{rr}} + 2 K_R \]
\[ = (K_l + 2 K_R) - K_l \frac{f'^2 g_{rr}}{1 + f'^2 g_{rr}} \]
\[ = \text{tr} K - K_l \frac{f'^2 g_{rr}}{1 + f'^2 g_{rr}}. \]

The spherically symmetric Jang equation is
\[ \frac{\sqrt{g_{rr}}}{R^2} \left( R^2 f' \frac{\sqrt{g_{rr}}}{\sqrt{1 + f'^2 g_{rr}}} \right) + \text{tr} K - K_l \frac{f'^2 g_{rr}}{1 + f'^2 g_{rr}} = 0. \]

This equation can be simplified by introducing
\[ k = \frac{f' \sqrt{g_{rr}}}{\sqrt{1 + f'^2 g_{rr}}} = \frac{f_j}{\sqrt{1 + f_j^2}} \Rightarrow f_j = \frac{k}{\sqrt{1 - k^2}}, \]
where $l$ is the unit proper distance in the radial direction, as a new variable. The Jang equation now becomes
\[ (R^2 k)_j + R^2 [\text{tr} K - K_l k^2] = 0. \]
The apparent horizon conditions are
\[ R_I \pm RK_R = 0 \]  
(6)
with the plus sign for a future apparent horizon and the minus for a past.

It is not immediately clear what role apparent horizons play in the disappearance of regular solutions to equation (5). What is obvious, however, is that the Jang equation does not distinguish between future and past horizons. If one reverses the sign of the extrinsic curvature in the initial data, one will get the same solution, \( k \), for equation (5) but with the sign reversed. It is clear from equation (4) that \( f_I \rightarrow \pm \infty \) as \( k \rightarrow \pm 1 \). Therefore, the blowup occurs when the derivative of the solution goes to infinity and this is insensitive to the sign of the extrinsic curvature. On the other hand, if \( k \) rises above +1, then from equation (9) one can see that there must be future trapped surfaces. Similarly, if \( k \) decreases below −1, then there exist past trapped surfaces.

The great advantage of this analysis is that we have reduced the Jang equation to a fairly simple first-order equation and that one can now identify quite easily when the solution becomes unphysical. We confirm the Schoen–Yau result that blowup can only occur when the initial data have apparent horizons. In sections 5 and 6 we will apply this equation in a number of simple situations. It will be shown, by considering special families of initial data, that blowup can occur far beyond the point where the first apparent horizon appears. We also construct families where, despite the existence of apparent horizons, no blowup ever occurs.

3. Uniqueness and apparent horizons

We can rewrite equation (5) in the following way:
\[ R^2 k_I + 2 R[R_J k + RK_R] + R^2 K_I (1 - k^2) = 0. \]  
(7)
Let us assume that there exists a solution to this equation for data without either a future or past apparent horizon. Further assume that at some point \( |k| < 1 \). For example, at a regular centre one needs \( k = 0 \) at \( R = 0 \) (otherwise, since \( k \) is the derivative of \( f \), there would be a conical singularity at the origin). We can show, by method of contradiction, that \(-1 < k < +1\). We assume that \( k \) starts off less than 1 and rises up through +1. At \( k = 1 \), equation (7) becomes
\[ R^2 k_I + 2 R [R_J + RK_R] = 0. \]  
(8)
Since no future apparent horizons exist, we have \( R_J + RK_R > 0 \). Therefore, from equation (8), one gets \( k_J < 0 \) which contradicts the fact that it must rise up through \( k = 1 \).

Alternatively, one could have \( k \) dropping down through −1. At \( k = -1 \), equation (7) becomes
\[ R^2 k_I + 2 R [-R_J + RK_R] = 0. \]  
(9)
Since we assume no past apparent horizons, \( R_J - RK_R > 0 \). Therefore from equation (9) follows \( k_J > 0 \) which contradicts the fact that it must drop down through \( k = -1 \). Hence, if a solution exists, it must lie between −1 and +1. In turn, this means that \( f_I \) is well defined and it can be integrated to find \( f \) itself.

Let us consider a ‘critical’ initial data set on the boundary between ‘good’ and ‘bad’ solutions to the Jang equation. This will be a set where the maximum value of \( k \) equals +1 (or the minimum equals −1). Therefore, we have a point in this critical data set where \( k = \pm 1 \) and \( k_I = 0 \). If this is substituted into equation (7) one finds that the point where the equation breaks down must satisfy
\[ \pm R_J + RK_R = 0, \]  
(10)
i.e., at an apparent horizon. The reader should be warned that this is a very special case. While the Jang equation ceases to have regular solutions only when there are apparent horizons, the points where the gradient of \( f \) goes infinite will almost always not coincide with apparent horizons.

Let us assume that a regular solution exists. We can show that this solution is unique. Assume that equation (5) possesses two regular solutions, \( k_1 \) and \( k_2 \), that coincide at an initial point \( l_0 \) (possibly \( l_0 = 0 \)). Define \( Y \equiv R^2 (k_1 - k_2) \); one easily finds that

\[
Y(l) = -(2K_R - \text{tr} K)(k_1 + k_2)Y.
\]

One has, after a little calculation and a Gronwall-type argument, that

\[
|Y(l)| \leq |Y(l_0)| \exp \left( \int_{l_0}^l d(l)(2K_R - \text{tr} K)(k_1 + k_2) \right).
\]

We know that \( (k_1 + k_2) \) is finite and we will assume that the extrinsic curvature within the integrand is regular enough to ensure that the integral converges. Since \( Y(l_0) = 0 \), one immediately sees that \( Y \) is identically zero and \( k_1 = k_2 \). Note that this uniqueness result holds without any assumption about trapped surfaces.

4. The Jang equation and the Penrose inequality

Jang quite rightly pointed out that if his equation was sensitive to the existence of apparent horizons (he specifically expected that regular solutions could be absent in such a case), then it may be useful in proving the Penrose inequality [1]. The Penrose inequality is a statement relating the asymptotic mass of a Cauchy hypersurface and the apparent horizon area on this hypersurface. It has been proven only in special cases—in the ‘Riemannian case’ [4, 5] and in spherically symmetric spacetimes [6]. The existing proofs (or scenarios for the proofs [7–9]; see also references therein) are heavily based on the fact that the Hawking mass of an apparent horizon itself satisfies the Penrose inequality and that is monotonic in a very special class of foliations (the inverse mean curvature foliations).

It has been widely suggested that a three-step approach might allow one to prove the Penrose inequality in the general case. Let us assume that one is given an asymptotically flat initial data set with an outermost apparent horizon. One first solves the Jang equation in the region between the outermost horizon and infinity. Since this region has no horizons, there must be a regular solution to the Jang equation. The second step is to perform the Jang transformation and obtain a metric whose scalar curvature has a positivity property—assuming in addition the dominant energy condition for material fields—that guarantees that the Lichnerowicz equation has a solution (see [2]). This allows us to conformally transform to a manifold with zero-scalar curvature. Then—provided that the original apparent horizon transforms into an outermost minimal surface—one may be able to use the inverse mean curvature flow argument of Huisken and Ilmanen [4] and show that the Penrose inequality holds. We argue that this procedure cannot be implemented even in the spherical case and therefore there is no hope of doing so in the general, nonspherical, case.

Let us have a spherical set of initial data with suitable decay at infinity and identify the outermost apparent horizon. For concreteness, let us assume it is a future apparent horizon. Therefore, on this surface we have

\[
R_{ij} + R K_R = 0, \quad R_{ij} - R K_R \geq 0.
\]

Adding the two conditions, we immediately get \( R_{ij} \geq 0 \). This means that the mean curvature of the outermost apparent horizon is nonnegative. The inverse mean curvature flow argument of Huisken and Ilmanen [4] requires that the starting surface has zero mean curvature. Thus, this condition has to be maintained under whatever transformations are made to the manifold. We end up comparing a final inner area to a final asymptotic mass. Since one really wants to
compare the initial area to the initial mass, we do not want to make any changes which either reduce the inner area or increase the mass. The mean curvature of every spherical surface outside the outermost horizon is positive in spherically symmetric geometries. A minimal surface with $K_R = 0$ is an apparent horizon, while a minimal surface with $K_R \neq 0$ is either past or future trapped so must have an apparent horizon outside it.

We first need to solve the Jang equation, equation (7), i.e.,

$$R^2k_{,l} + 2R[R_{,l} + RK_R] + R^2K_l(1 - k^2) = 0,$$

(13) between the horizon and infinity. The only uncertainty is in the choice of boundary conditions at the horizon. These will be dictated by the demand that the method of Huisken and Ilmanen works, and that it works in turn only if the outermost horizon actually is a minimal surface. It happens that only $k = 1$ and $k = -1$ do the trick; starting from a three-manifold with an apparent horizon in the first step, one ends with a three-manifold having a minimal surface in the third step.

Let us first see what happens when we pick $k = 1$. At the horizon $R_j + RK_R = 0$ so that $k_j = 0$. Differentiating the Jang equation, and using these conditions yields $R^2k_{,ll} + 2R[R_{,l} + RK_R]_{,l} = 0$. Since the surface is the outermost horizon we get $[R_{,l} + RK_R]_{,l} > 0$ so $k_{,ll} < 0$. Thus, the function decreases below the critical value of $k = 1$ as one moves out and since there are no further horizons we get a regular solution which asymptotes to zero.

Now make the Jang transformation. The only metric component that changes is

$$\tilde{g}_{rr} = g_{rr} + f^2 = g_{rr}(1 + f_j^2) = \frac{g_{rr}}{1 - k^2}.$$

Since $k = 1$ and $k_j = 0$ at the horizon we have $1 - k^2 \approx A l^2$ near the horizon for some constant $A$. This means that the proper distance in the transformed metric from points `near' the horizon to the horizon itself becomes infinite. However, the area of the spheres does not change because the angular metric components are unaffected. This means that the manifold `near' the horizon gets transformed into an infinitely long cylinder whose cross-section asymptotes to the original area of the horizon and the three-scalar curvature along the cylinder is a positive constant $(3) \tilde{R} \sim 1/R_{H}^2$, $R_H$ is the areal radius of the apparent horizon). Now, however, when solving the Lichnerowicz equation on this manifold we find that the conformal factor cannot go to 1 at both ends since it behaves as $\exp(-Cl)$ near the horizon. Such behaviour was first observed by Schoen and Yau in [2]. This means that, while a manifold with zero-scalar curvature can be constructed, we have no ‘inner’ minimal surface whose area approximates the original area of the horizon.

The second choice is to pick $k = -1$ at the horizon. If it is a future horizon we have $k_j > 0$ and there is a regular solution to the Jang equation. In addition, near the horizon $1 - k^2 \approx B l$. Now the radial metric, after the Jang transformation, blows up at the horizon but does so in such a way that the proper distance remains finite. Further, the inner surface becomes a minimal surface with the same original area because the mean curvature scales to zero. While the area of the nearby surfaces is unchanged, the proper distance between them becomes large. Again, when solving the Lichnerowicz equation with $\phi = 1$ at both ends we expect $\phi_j$ to be negative at the inner boundary so in this case the mean curvature of nearby surfaces will become negative. Indeed, the Jang-transformed scalar curvature satisfies a positivity property of the form $(3) \tilde{R} \geq 2A_l A' + 2\nabla^2 A'$ with $(A') = (kk' + k(1 - k^2) K_l, 0, 0)$. This is sufficient to show that the manifold can be conformally transformed to one with zero-scalar curvature, i.e., we can solve $8\nabla^2 \phi - (3) \tilde{R} \phi = 0, \phi > 0$ with $\phi = 1$ at both ends. A solution can be expected rather as in the case $(3) \tilde{R} \geq 0$, i.e., with an interior minimum. This means that $\phi$ decreases at the inner minimal surface. The mean curvature of this conformally
transformed surface becomes negative and a new minimal surface appears somewhere outside it. Its area can be expected to be different from the area of the original apparent horizon; it can be smaller than the original area, if \( \bar{R} \geq 0 \), but since we do not actually know the sign of the scalar curvature, there is no simple way of establishing which area is bigger.

Thus in both cases, \( k = \pm 1 \), there is no simple possibility for exerting the necessary control over the asymptotic mass and the area of outermost minimal surfaces. We do not exclude the possibility that the Jang equation can be useful in order to establish the validity of the Penrose inequality, but that would require a significant alteration and extension of the procedure.

5. The existence problem in various gauges

As was mentioned in the introduction, one use of the Jang equation was in the Schoen and Yau article [3]. It is difficult to find initial data which satisfy all the conditions laid down in the Schoen and Yau paper. Essentially, the only configuration we have found is where one considers a spherical piece of a flat Friedmann universe glued to some asymptotically flat data. The blowup of the solution can be observed in this situation.

Let us consider a flat spherical region of radius \( R_0 \), where the extrinsic curvature is pure constant trace, i.e., \( K_{ab} = \frac{K_0}{3} \delta_{ab} \). We seek a solution of equation (5) with the standard boundary condition of \( k = 0 \) at \( R = 0 \). There is a natural scaling in the problem so we can choose \( K_0 = -3 \). This will give a positive solution and blowup occurs if and only if \( k \geq 1 \).

Hence, the equation we deal with is

\[
(R^2 k)_{,R} - R^2 [3 - k^2] = 0.
\]

An upper bound can be found by setting \( k^2 = 0 \) and, since we are only interested in the region where \( k \leq 1 \), a lower bound is obtained by setting \( k = 1 \). Therefore, \( \frac{3}{2} R \leq k \leq R \).

This means that for some value of \( R \) which lies in the range \((1, 3/2)\), \( k \) will pass through 1. Numerically, this has been found to occur at \( R = 1.29 \) [10]. We have no contradiction to the Schoen–Yau result because the first apparent horizon is at \( R = 1 \). This calculation tells us that for any flat Friedmann sphere with \( |K_0| = 3 \) and whose radius exceeds \( R = 1.29 \), independent of how it is extended into the asymptotically flat region, the Jang equation has no regular solution.

If we consider theorem 2 of the Schoen–Yau black-hole paper [3], and apply their criterion for the absence of solutions to the Jang equation to this model, we find that they demand that no solution can exist if \( R_0 \geq \sqrt{2\pi} \). Again there is no contradiction, but note that it is more than a factor of 3 bigger than 1.29.

Another situation where the spherical Jang equation can be easily analysed is on the Painlevé–Gullstrand [11] slice of the Schwarzschild solution. This is the Schwarzschild slice where the spatial metric is flat and the extrinsic curvature satisfies

\[
K_i = \sqrt{\frac{m}{2 R^3}}, \quad K_R = -\sqrt{\frac{2m}{R^3}}.
\]

The Jang equation now becomes

\[
(R^2 k)_{,R} - \sqrt{\frac{mR^2}{2}} (3 + k^2) = 0.
\]

The solution of this equation at large \( R \) must go as

\[
k \approx \sqrt{\frac{2m}{R}}.
\]
From equation (4) it follows that

\[ f_R = \frac{k}{1 - k^2} \approx \sqrt{\frac{2m}{R}}. \]

This can be integrated to give

\[ f \approx 2\sqrt{2mR}. \]

We are interested in solutions that go to zero (or some constant value) at infinity. This solution does not satisfy this. Any data which asymptote to this slice will have the same nonexistence property.

Schoen and Yau demand that \( \text{tr} K \approx 1/R^3 \). This would immediately exclude the Painlevé–Gullstrand data. The \( 1/R^3 \) condition seems unnecessarily strong, it may well be that \( \text{tr} K \approx 1/R^{2+\epsilon} \) would suffice. The one thing that is clear is that some falloff condition is required.

The Jang equation may not possess a regular solution if the data contain an apparent horizon. It is easy to show that the converse cannot hold. Consider any moment of time symmetry data with a minimal surface, i.e., something like the moment of time symmetry slice of the Schwarzschild solution, glued to some smooth interior. Then the source term in the Jang equation vanishes and \( f \equiv 0 \) is obviously a regular solution.

In the spherically symmetric case consider some compact distribution of matter which is instantaneously at rest. Equation (5) reduces to \((R^2k)_t = 0\). Therefore, there is a family of solutions \( k = D/R^2 \), where \( D \) is any constant. The only solution which is regular at \( R = 0 \) is the \( D = 0 \) one. Hence, the \( f = 0 \) regular solution is unique in the case of spherically symmetric moment-of-time-symmetry data.

Both the existence and uniqueness results extend to spherical maximal slices. The Jang equation in this case is

\[ (R^2k)_t - R^2K_tk^2 = 0. \] (14)

Obviously, \( k = 0 \) (and thus \( f = \) constant) is a solution. This result also holds in the nonspherical case. The only inhomogeneous term in the second-order Jang equation is \( \text{tr} K \). Therefore, \( f = \) constant is a solution in the maximal case irrespective of whether the data contain apparent horizons or not.

In the spherical maximal case we can give a simple proof that \( k = 0 \) is the only solution. Again, this is a proof by contradiction. Let us assume that a nontrivial solution to equation (14) exists. This solution must vanish at the origin and let us assume that it is positive, at least initially. In terms of the proper distance coordinates, let us choose an \( l_1 \) which satisfies two conditions. We want that \( R^2k \) at \( l_1 \) is the maximum over the interval \([0, l_1]\) and also that

\[ \int_0^{l_1} |K_t|k \, dl < 1. \]

Let us integrate equation (14) over the interval \([0, l_1]\) to get

\[ R^2k|_{l_1} = \int R^2K_tk^2 \, dl \leq \max |R^2k| \int |K_t|k \, dl. \]

This is a contradiction because \( R^2k|_{l_1} = \max |R^2k| \). Therefore \( l_1 \) cannot exist and so also a nontrivial \( k \). The only assumptions needed are that \( K_t \) is finite and that the proper distance coordinates are well behaved.
6. Tr $K$ and (non)existence of solutions

It was demonstrated in the previous section that the Jang equation can always be solved for spherical maximal data. Therefore, it is interesting to investigate the role that $\text{tr } K$ plays in allowing/preventing the existence of regular solutions. An easy place to analyse this is in the ‘transverse’ gauge, where one assumes that $K_i \equiv 0$. In this case, equation (5) reduces to

$$(R^2 k)_{,j} + R^2 \text{tr } K = 0. \quad (15)$$

This can be integrated out to some radius $R = R_1$ from the centre to give

$$R^2 k(R_1) = - \int R^2 \text{tr } K \, dl. \quad (16)$$

This now gives

$$k(R_1) = - \frac{\int \text{tr } K \, dv}{A}, \quad (17)$$

where the numerator is the proper volume integral over the sphere inside $R = R_1$ and $A = 4\pi R_1^2$ is the area of the surface of the sphere. Hence, if $|\int \text{tr } K \, dv| \geq A$ for any sphere, then we will not have a regular solution while if $|\int \text{tr } K \, dv| < A$ for all spheres we have no blowup.

The Hamiltonian constraint of general relativity can be used to show that this condition, i.e.,

$$\frac{|\int \text{tr } K \, dv|}{A} \geq 1 \quad (18)$$

can only hold if a significant amount of matter is contained within the sphere. The Hamiltonian constraint is

$$(^3)R - K^{ij}K_{ij} + \text{tr } K^2 = 16\pi \rho. \quad (19)$$

In the gauge where $K_i \equiv 0$ this reduces to

$$(^3)R + \frac{1}{2} \text{tr } K^2 = 16\pi \rho. \quad (20)$$

If the scalar curvature is nonnegative this gives $\text{tr } K^2 \leq 32\pi \rho$.

Returning to the condition (18), we can use the Schwarz inequality to get

$$1 \leq \frac{|\int \text{tr } K \, dv|}{A} \leq \sqrt{\frac{\int \text{tr } K^2 \, dv V^2}{A}} \leq \sqrt{\frac{32\pi \rho \, dv V^2}{A}}. \quad (21)$$

Therefore, if the Jang equation does not have a regular solution, there must be a sphere such that $M > A^2 / 32\pi V$, where $M$ is the matter content.

In section 5 we discussed the special case where the extrinsic curvature was pure constant trace and found that if the region was large enough, regular solutions do not exist. This result can be generalized to the situation where the extrinsic curvature is pure trace, but not necessarily constant, i.e., $K_i = K_R$. Equation (5) becomes

$$(R^2 k)_{,j} + R^2 \text{tr } K \left[1 - \frac{k^2}{3}\right] = 0. \quad (22)$$

Let us integrate this out to some radius $R_1$ and get

$$k(R_1) = - \frac{\int \text{tr } K \left[1 - \frac{k^2}{3}\right] \, dv}{A}. \quad (23)$$
Let us assume there exists a regular solution, i.e., \(|k| < 1\) and that \(\mathrm{tr} \ K\) has a fixed sign, say, positive. One then gets

\[
1 > \frac{\int |K|}{A} \left[ 1 - \frac{k^2}{4} \right] \mathrm{d}v > \frac{2 \int \mathrm{tr} K \mathrm{d}v}{3A}.
\]  

(24)

Therefore, in a region which satisfies

\[
\frac{\int \mathrm{tr} K \mathrm{d}v}{A} \geq \frac{3}{2},
\]  

(25)

a regular solution to the Jang equation cannot exist. In the flat Friedmann model we analysed in section 5, we showed that a lower bound for this constant was 1.29. This 3/2 may well be sharp. As above, this inequality can be related to the requirement that the sphere contain a large amount of matter. A generalization of this approach to nonspherical cases can be found in [22].

7. Concluding remarks

The Penrose inequality, \(A \leq 16\pi m^2\), where \(A\) is the area of the outermost future apparent horizon (with a nonnegative optical scalar \(R - KK\) outward from the horizon) and \(m\) is the ADM mass, is a consequence of cosmic censorship. Much evidence for its correctness exists: it is true for spherically symmetric data [17]; it is true in the moment-of-time-symmetry case [4]; no numerical counterexample exists in the general case. We believe it to be valid in general. However, the Jang-equation-based scenario cannot be used to prove it, since it does not work in the spherically symmetric case (where the Penrose inequality is correct). A new idea is needed.

Another use of the Jang equation is in attempts to give a precise statement of the trapped surface conjecture [14]—that the compression of matter leads to the formation of an apparent horizon, as in the Schoen–Yau trapped surface article [3]. This was also problematic. This equation is—as laid out in preceding sections—rather insensitive to the existence of apparent horizons. To begin with, it does not distinguish between future and past marginal surfaces. This means that in order to obtain statements about the existence of apparent horizons, one needs to control the sign of the quantity \(R - KK\). This was done in a recent analysis of this problem [13]. Moreover, when gauge conditions are imposed which set the trace of the extrinsic curvature small enough—such as the maximal slicing condition—then the Jang equation does not ‘see’ any horizons. Clearly, the diagnostic power of the Jang equation with regard to trapped surfaces is limited.

Having said that, it is necessary to point out that a demonstration of the trapped surface conjecture remains elusive. It has been well established in the spherically symmetric case [12, 15–18] in spacetimes sliced with the use of ‘reasonable’ gauge conditions, such as maximal, constant mean curvature, flat or polar slicings. The results concerning nonspherical situations remain patchy ([19–21] and references therein) and apply mostly to moment of time symmetry initial data. The conjecture itself seems to be self-evident. It is therefore surprising that it is so difficult to formulate a clear quantitative description of this initial phase of gravitational collapse. Seen from this perspective, the approach based on the Jang equation [3, 13] is valuable in working in the ‘large tr \(K\)’ sector of initial data, where it yields a quantitative description of a valid and interesting physical problem.
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