Cohomologies of Harmonic Bundles on Quasi-Compact Kähler manifolds

Jürgen Jost *, Yi-Hu Yang †‡ and Kang Zuo §

1 Introduction and the statements of the problems

Let $V$ be a complex flat vector bundle of rank $n$ on a complex manifold $M$; equivalently, this corresponds to a linear representation

$$\rho : \pi_1(M) \to GL(n, \mathbb{C}).$$

If $M$ is compact Kählerian and $\rho$ is semi-simple, that is that the Zariski closure of the image of $\rho$ in $GL(n, \mathbb{C})$ is semi-simple, by means of a result of Donaldson (in the case of Riemann surfaces, [5]) and Corlette (in the higher dimensional case, [4]), there exists a unique harmonic metric $u$ on $V$, equivalently, a $\rho$-equivariant harmonic map from the universal covering $\tilde{M}$ of $M$ into $\mathcal{P}_n := GL(n, \mathbb{C})/U(n)$– the set of positive definite Hermitian symmetric matrices of order $n$.

Let $D$ be the flat connection of $V$. Decompose $D = d' + d''$ into $(1, 0)$-part and $(0, 1)$-part. Suppose $\delta''$ and $\delta'$ are $(0, 1)$-type operator and $(1, 0)$-type operator respectively, s.t. $d' + \delta''$ and $\delta' + d''$ preserve the Hermitian metric $u$.

According to Simpson [25, 26], set

$$\partial = (d' + \delta')/2, \quad \overline{\partial} = (d'' + \delta'')/2,$$

$$\theta = (d' - \delta')/2, \quad \overline{\theta} = (d'' - \delta'').$$

It is easy to check that i) $\partial + \overline{\partial}$ preserves the metric $u$; ii) $< \theta x, y >_u = < x, \overline{\theta} y >_u$; iii) up to some constant, $\theta = (\partial u) u^{-1}$ and $\theta$ can be considered as a
1-form of type \((1,0)\) valued in \(\text{Hom}(\mathcal{V})\), which we call the \textit{Higgs field}.

Put

\[
D''_u = \overline{\partial} + \theta, \quad D'_u = \partial + \overline{\theta},
\]

\[
D^c_u = D''_u - D'_u.
\]

When \(M\) is one-dimensional, Hitchin [8] observed that the harmonicity of \(u\) gives rise to the structure of a Higgs bundle on \(\mathcal{V}\); namely, since \((D''_u)^2 = 0\), one can consider \(\mathcal{V}\) as a holomorphic vector bundle under the operator \(\overline{\partial}\), denoted by \(E\) later on, and hence \(\theta\) is a holomorphic 1-form valued in \(\text{Hom}(E)\). In the higher dimensional case, using Siu’s Bochner formula ([24], also see [21]), one can show that \(u\), as a map, is pluri-harmonic [4], equivalently, this is \((D''_u)^2 = 0\). So, in the higher dimensional case, one still has the above structure of a Higgs bundle with \(\theta \wedge \theta = 0\) (this is automatic in the case of Riemann surface). An equivalent but more geometric observation was actually discovered earlier by Jost-Yau [12], where from the harmonic map they derived a fibration which is holomorphic, namely \(\theta\) is holomorphic. Later on, we call \((E, \theta)\) with \(\theta \wedge \theta = 0\) the \textit{Higgs bundle} and \(D''_u\) the \textit{Higgs operator}, which is integrable.

**Definition.** A harmonic bundle on a complex manifold \(M\) is a complex flat vector bundle \(\mathcal{V}\) over \(M\) with a harmonic metric \(u\) satisfying the induced operator \(D''_u\) is integrable, so that one has an induced Higgs bundle \((E, \theta)\) as above.

For a harmonic bundle on a complex manifold, one can define various cohomologies.

Let us first give some formal definitions. Let \(\mathcal{V}\) be a harmonic bundle with a harmonic metric \(u\). By \(\mathcal{V}^D\) denote the locally constant sheaf of flat sections of \(\mathcal{V}\). This sheaf is resolved by de Rham complex of sheaves of \(C^\infty\) differential forms with coefficients in \(\mathcal{V}\):

\[
\mathcal{V}^D \to (\{A^p(\mathcal{V}), D\} = A^0(\mathcal{V}) \xrightarrow{D} A^1(\mathcal{V}) \xrightarrow{D} A^2(\mathcal{V}) \xrightarrow{D} \cdots) \quad (1)
\]

is a quasi isomorphism of complexes of sheaves. The sheaves of \(C^\infty\) forms are fine, so the sheaf cohomology \(H^*(M, \mathcal{V}^D)\) is naturally isomorphic to the cohomology of the complex of global sections

\[
\{A^p(\mathcal{V}), D\} = A^0(\mathcal{V}) \xrightarrow{D} A^1(\mathcal{V}) \xrightarrow{D} A^2(\mathcal{V}) \xrightarrow{D} \cdots \quad (2)
\]

Call the cohomology of the complex of global sections the \textit{de Rham cohomology}, denoted by \(H^*_{\text{DR}}(M, \mathcal{V})\).

For the corresponding Higgs bundle \((E, \theta)\), we also can define cohomologies as
follows. First, we have a complex of free sheaves of locally holomorphic forms valued in $E$, called the holomorphic Dolbeault complex:

\[ \{ \Omega; \theta \wedge \} = E \xrightarrow{\theta \wedge} \Omega^1(E) \xrightarrow{\theta \wedge} \Omega^2(E) \xrightarrow{\theta \wedge} \cdots \]  

(3)

The condition that $\theta \wedge \theta = 0$ insures that this is a complex. Then, define the Dolbeault cohomology with coefficients in $E$ to be the hypercohomology of the above complex

\[ H^\bullet_{Dol}(M, E) \]

denoted by $H^\bullet_{Dol}(M, E)$.

Furthermore, the complex of sheaves of local $C^\infty$ sections valued in $E$ with the differential $D''_u$

\[ \{ A^\bullet(E), D''_u \} = A^0(E) \xrightarrow{D''_u} A^1(E) \xrightarrow{D''_u} A^2(E) \xrightarrow{D''_u} \cdots \]  

(4)

gives a fine resolution of the holomorphic Dolbeault complex, so $H^\bullet_{Dol}(M, E)$ is naturally isomorphic to the cohomology of the complex of global sections $\{ A^\bullet(E), D''_u \}$. But, for later convenience, we call the cohomology of the complex of global sections the Higgs cohomology, denoted by $H^\bullet_{Higgs}(M, E)$; and the complex $\{ A^\bullet(E), D''_u \}$ the Higgs complex.

A (formal) discussion tells us that if $M$ is compact Kähler, all the above cohomologies: the sheaf cohomology $H^\bullet(M, \mathcal{V}_D)$, the de Rham cohomology $H^\bullet_{DR}(M, \mathcal{V})$, the Dolbeault cohomology $H^\bullet_{Dol}(M, E)$, and the Higgs cohomology $H^\bullet_{Higgs}(M, E)$, are isomorphic. The above arguments tell us that one only needs to show that the de Rham cohomology $H^\bullet_{DR}(M, \mathcal{V})$ is isomorphic to the Higgs cohomology $H^\bullet_{Higgs}(M, E)$. This is just a consequence of the (local) Kähler identity for harmonic bundles and the theory of harmonic forms; for details, see the end of this section.

When $M$ is non-compact, e.g. $M = \overline{M} - D$, $D$ being a divisor of $\overline{M}$, the situation gets very complicated; one does not necessarily have the isomorphism between the de Rham cohomology $H^\bullet_{DR}(M, \mathcal{V})$ and the Dolbeault cohomology $H^\bullet_{Dol}(M, E)$. Roughly speaking, these cohomologies are too big for study and applications, so that we have to modify all the above four classes of cohomologies. A good method for modifying them are to consider certain sub-complexes by using certain growth condition and the corresponding cohomologies.

In order to modify these cohomologies, let us consider the following setup, which geometrically is the most interesting. Let $\overline{M}$ be a compact Kähler manifold with a Kähler metric $\omega_0$, $D = \sum_k D_k$ a normal crossing divisor, denote $\overline{M} - D$ by $M$ and the inclusion map $M \hookrightarrow \overline{M}$ by $j$. Call $M$ a quasi-compact Kähler manifold. One can equip $M$ with a complete Kähler metric $\omega$ which
is Poincaré-like near the divisor, namely, if $p \in D$ and $z = (z_1, z_2, \cdots, z_m)$ is a local coordinate at $p$ satisfying that $z(p) = (0, 0, \cdots, 0)$ and $\prod^{s'}_{1} z_k = 0$, $s' \leq s$, is the local minimal equation of $D$, then, near $p$

$$\omega = \sum_{k=1}^{s'} \frac{-1}{|z_k|^2} \log |z_k|^2 + \sum_{s'+1}^{m} dz_k \wedge d\bar{z}_k.$$ 

Now, suppose that $V$ is a harmonic bundle over $M$ with a harmonic metric $u$, $E, \theta$ the corresponding the Higgs bundle, the Higgs field. Using the metric $\omega$ on $M$ and the metric $u$, one can define (local) square-integrable forms valued in $V$ or $E$. Using (local) square-integrable forms, one then can define the corresponding sub-complexes on $M$ of (1) and (2) respectively as follows:

$$\{A^{1}_{(2)}(V), D\} = A^{1}_{(2)}(V) \xrightarrow{D} A^{2}_{(2)}(V) \xrightarrow{D} \cdots, \quad (5)$$

$$\{A^{*}_{(2)}(V), D\} = A^{0}_{(2)}(V) \xrightarrow{D} A^{1}_{(2)}(V) \xrightarrow{D} A^{2}_{(2)}(V) \xrightarrow{D} \cdots, \quad (6)$$

where the sheaves $A^{i}_{(2)}(V)$ are defined as follows:

for an open subset $U$ of $M$, $A^{i}_{(2)}(V)(U)$ is defined as the set of $V$-valued $i$-forms $\eta$ on $U \cap M$ with measurable coefficients and measurable exterior derivative $D\eta$, such that $\eta$ and $D\eta$ have finite $L^2$-norm; and $\{A^{*}_{(2)}(V), D\}$ is the complex of global sections of $\{A^{*}_{(2)}(V), D\}$.

Since the sheaves $A^{i}_{(2)}(V)$ are fine, the hypercohomology of the complex $\{A^{*}_{(2)}(V), D\}$ is isomorphic to the cohomology of the complex $\{A^{*}_{(2)}(V), D\}$. We call this cohomology the $L^2$-de Rham cohomology, denoted by $H_{DR, (2)}^*(M, V)$; correspondingly, the complex is called the $L^2$-de Rham complex.

On the other hand, as mentioned before, the sheaf cohomology $H^*(M, V^D)$ is too big in such a non-compact case, according to a suggestion of Deligne, one should consider the (middle) intersection cohomology $[1, 6, 7]$ with coefficients in the direct sheaf $j_*V$: $H^*_\text{int}(\overline{M}, j_*V)$. Then, one has the following

**Conjecture 1.** The intersection cohomology $H^*_\text{int}(\overline{M}, j_*V)$ is isomorphic to the $L^2$-de Rham cohomology $H_{DR, (2)}^*(M, V)$.

Now, we turn to the Dolbeault cohomology. In this case, we need to pay much more attention to the properties at the divisor of the representation $\rho$ of $\pi_1(M)$ and the asymptotic behavior of the harmonic metric $u$; but here, we will not involve in much more details, instead, we will give some direct statements; in the next two sections, these will be stated more clearly.
Similar to the $L^2$-de Rham complex, we define the $L^2$-holomorphic Dolbeault complex and the $L^2$-Higgs complex as follows:

\[ \{\Omega^*_{(2)}, \theta\wedge\} = E_{(2)} \xrightarrow{\theta\wedge} \Omega^1_{(2)}(E) \xrightarrow{\theta\wedge} \Omega^2_{(2)}(E) \xrightarrow{\theta\wedge} \cdots \quad (7) \]

\[ \{A^i_{(2)}(E), D^u_i\} = A^0_{(2)}(E) \xrightarrow{D'_u} A^1_{(2)}(E) \xrightarrow{D'_u} A^2_{(2)}(E) \xrightarrow{D'_u} \cdots \quad (8) \]

where the sheaves $A^i_{(2)}(E)$ are defined as follows:

for an open subset $U$ of $\mathcal{M}$, $A^i_{(2)}(E)(U)$ is defined as the set of $E$-valued $i$-forms $\eta$ on $U \cap M$ with measurable coefficients and measurable exterior derivative $D'_u\eta$, such that $\eta$ and $D'_u\eta$ have finite $L^2$-norm;

and the sheaves $\Omega^i_{(2)}(E)$ are defined as a sub-sheaves of $A^i_{(2)}(E)$ germs of which are local holomorphic forms.

Define the hypercohomology of the complex $\{\Omega^*_{(2)}, \theta\wedge\}$ as the $L^2$-Dolbeault cohomology, denoted by $H^*_{\text{Dol},(2)}(M, E)$. Since the complex $\{A^i_{(2)}(E), D^u_i\}$ is a complex of fine sheaves, its hypercohomology is computed by the cohomology of the corresponding complex of global sections $\{A^i_{(2)}(E), D^u_i\}$; call this cohomology the $L^2$-Higgs cohomology, denoted by $H^*_{\text{Higgs},(2)}(M, E)$. Then, one has the following

**Conjecture 2.** The $L^2$-Dolbeault cohomology $H^*_{\text{Dol},(2)}(M, E)$ is isomorphic to the $L^2$-Higgs cohomology $H^*_{\text{Higgs},(2)}(M, E)$; equivalently, the complex $\{\Omega^*_{(2)}, \theta\wedge\}$ is quasi-isomorphic to the complex $\{A^i_{(2)}(E), D^u_i\}$.

As in the compact case, using the theory of harmonic forms, one can easily show that the $L^2$-de Rham cohomology $H^*_{\text{DR},(2)}(M, \nabla)$ is isomorphic to the $L^2$-Higgs cohomology $H^*_{\text{Higgs},(2)}(M, E)$. To this end, let us first recall the local Kähler identity for harmonic bundles.

Using the previous notation, one has the first order Kähler identities [26]

\[
(D'_u)^* = \sqrt{-1}\lambda(D''_u), \quad (D''_u)^* = -\sqrt{-1}\lambda(D'_u) \\
(D''_u)^* = -\sqrt{-1}\lambda(D), \quad (D)^* = \sqrt{-1}\lambda(D''_u),
\]

where $^*$ represents the adjoint of the respective operator by using the metrics $\omega$ and $u$. On the other hand, set

\[ \Delta = DD^* + D^*D, \quad \Delta'' = D''_u(D'_u)^* + (D''_u)^*D'_u. \]

Using the above first order identities, one then has

\[ \Delta = 2\Delta''. \]
This shows that spaces of $\Delta$-harmonic forms valued in the local system $\mathcal{V}$ can be identified with that of $\Delta''$-harmonic forms valued in the Higgs bundle $E$.

On the other hand, by $L^2$-theory, the $L^2$-de Rham cohomology $H^*_\text{DR},(2)(M,\mathcal{V})$ can be represented by $L^2$ $\Delta$-harmonic forms valued in $\mathcal{V}$ and the $L^2$-Higgs cohomology $H^*_\text{Higgs},(2)(M,E)$ can be represented by $L^2$ $\Delta''$-harmonic forms valued in $E$. So, $H^*_\text{DR},(2)(M,\mathcal{V}) \cong H^*_\text{Higgs},(2)(M,E)$.

**Remark 1** The intersection cohomology is a topological notion [6, 7]; but from the definition of the $L^2$-de Rham cohomology (and the $L^2$-Higgs cohomology), we know that it depends on the metrics $\omega$ and $u$, so that it is an analytical notion. From the next two sections, we will see that the $L^2$-Dolbeault cohomology is a much more algebraic notion, which is closely related to the properties at the divisor of the representation $\rho$. Thus, Conj. 1 and 2 establish the relations between these different notions. When the harmonic bundle comes from a variation of Hodge structure, these conjecture was first suggested by Deligne.

The conjectures were first proved by S. Zucker [29] when $M$ is a Riemann surface and the harmonic bundle comes from a variation of Hodge structure. It should be pointed out that in the Riemann surface’s case, the intersection cohomology $H^*_\text{int},(\bar{M},j_*\mathcal{V})$ is isomorphic to the sheaf cohomology $H^*(\bar{M},j_*\mathcal{V})$, but this is not in general valid in the higher dimensional case.

In the case that $M$ is a higher dimensional quasi-compact Kähler manifolds and $\mathcal{V}$ is a variation of Hodge structure, Conjecture 1 was proved independently by Cattani-Kaplan-Schmid [3] and Kashiwara-Kawai [16]. We also should mention the works of Looijenga [18] and Saper-Stern [22]; they showed independently that the $L^2$-cohomology is isomorphic to the intersection cohomology when the base manifold is an arithmetic variety but the representation is induced from a linear representation of the corresponding Lie group.

In the case that $M$ is a higher dimensional quasi-compact Kähler manifolds and $\mathcal{V}$ is a variation of Hodge structure, Conjecture 1 was proved independently by Cattani-Kaplan-Schmid [3] and Kashiwara-Kawai [16]. We also should mention the works of Looijenga [18] and Saper-Stern [22]; they showed independently that the $L^2$-cohomology is isomorphic to the intersection cohomology when the base manifold is an arithmetic variety but the representation is induced from a linear representation of the corresponding Lie group.

In the note, we survey our recent study about Conjecture 1 and 2 [9, 10]. In Section 2, we outline the work when the coefficient is a VHS; Section 3 outlines the work when the base manifold is a noncompact algebraic curve and the representation is unipotent. A natural problem is what the situation is when we don’t assume that the representation $\rho$ is unipotent.

## 2  Cohomologies valued in VHSs

In this section, we consider cohomologies with coefficients in a variation of Hodge structure and outline the results in [9].

Let $H_C = H_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$, $k$ a positive integer, and $S$ a real nondegenerate bilinear
form on $H_C$ satisfying $S(u, v) = (-1)^k S(v, u)$. A polarized Hodge structure of weight $k$ is a decomposition $H_C = \sum_{p+q=k} H^{p,q}$ satisfying $H^{p,q} = \overline{H^{q,p}}$ and

$$S(H^{p,q}, H^{r,s}) = 0, \quad \text{unless } p = s, q = r,$$

$$S(Cv, \overline{v}) > 0, \quad \text{if } v \in H^{p,q}, v \neq 0,$$

where $C$ is the Weil operator defined by $Cv = \sqrt{-1}^{p-q} v, v \in H^{p,q}$. Call $\dim_C H^{p,q}$ the Hodge number, denoted by $h^{p,q}$. Let $D$ be the classifying space of polarized Hodge structures of weight $k$, which is a homogeneous complex manifold. Denote by $G_{\mathbb{R}}$ the group of holomorphic automorphisms of $D$ (more precisely, the group preserving $S$ and $H_{\mathbb{R}}$), by $G_{\mathbb{C}}$ the complexification of $G_{\mathbb{R}}$; denote by $g_0$ (resp. $g$) the Lie algebra of $G_{\mathbb{R}}$ (resp. $G_{\mathbb{C}}$).

One can consider a family of polarized Hodge structures parameterized by a complex manifold $M$; this is the notion of (rational) variation of polarized Hodge structures (in brief, VHS). For the precise definition, one can refer to e.g. [23], p. 220. As in the introduction, we always assume that $M = \overline{M} - D$ is a quasi-compact Kähler manifold with a Poincaré-like metric $\omega$, $D$ being a normal crossing divisor, $j : M \to \overline{M}$ the inclusion map.

Let $\{M, H_Z \subset H_{\mathbb{C}}, \{F^p\}_{p=0}^{k}, \nabla = \nabla^{1,0} + \nabla^{0,1}, S\}$ be a VHS, where $H_{\mathbb{C}}$ is a flat bundle under a flat connection $\nabla$. $H_Z$ is a flat lattice, $\{F^p\}_{p=0}^{k}$ is a Hodge filtration, and $S$ is a polarization. The polarization $S$ defines a Hermitian metric

$$h(\cdot, \cdot) = S(C\cdot, \overline{\cdot}),$$

called Hodge metric, where the bar is the conjugation with respect to $H_Z$. By $\| \cdot \|$ denote the corresponding norm. Take the successive quotients $F^p/F^{p-1}$, called Hodge bundles, denoted by $E^p$. Accordingly, we have the induced $\mathcal{O}$-linear map $\theta^p : E^p \to E^{p-1}$ of $\nabla^{1,0}$. Set $E = \bigoplus E^p$ and $\theta = \bigoplus \theta^p$. Since $(\nabla^{1,0})^2 = 0, \theta \wedge \theta = 0$. So, the pair $(E, \theta)$ is a Higgs bundle. One can also assign to the variation a ($p$-equivariant) period mapping $\tilde{\phi} : \tilde{M} \to D$ from the universal covering $\tilde{M}$ of $M$ to $D$, which is holomorphic; here, $\rho$ is the induced linear representation of $\pi_1(M)$ by $\nabla$.

Using the Poincaré-like metric $\omega$ on $M$ and the Hodge metric $h$ on $H_{\mathbb{C}}$, we define $\Omega'(H_{\mathbb{C}})_{(2)}$ to be the sheaf of germs of local $L^2$ holomorphic $r$-forms valued in $j_*H_{\mathbb{C}}$ on $\overline{M}$, where $j$ is the inclusion map of $M$ into $\overline{M}$. According to the Hodge filtration $\{F^p\}$ of the variation, one can then construct a filtration of $\Omega'(H_{\mathbb{C}})_{(2)}$, denoted by $F^p\Omega'(H_{\mathbb{C}})_{(2)}$, which is the sheaf of germs of local $L^2$-holomorphic $r$-forms on $\overline{M}$ with values in $j_*F^{p-r}$. On the other hand, the quotient bundle $E^{p-r} = F^{p-r}/F^{p-r+1}$ inherits a quotient norm from $F^{p-r}$; using this quotient norm, we define $(\Omega'^r \otimes E^{p-r})_{(2)}$ as the sheaves of germs of local $L^2$-holomorphic forms on $\overline{M}$ with values in $j_*E^{p-r}$, which, as seen in
the following proposition 1, is actually \((j_!E^p \otimes \Omega^r_M(\log D))_{(2)}\), the sheaf of germs of local \(L^2\)-sections in \(j_!E^p \otimes \Omega^r_M(\log D)\). Then, one can show the following

**Theorem 1** The projection \(F^p \rightarrow E^p\) induced by \(p \geq r\) induces a natural projection

\[ F^p \Omega^r(\mathbb{H}_C)_{(2)} \rightarrow (\Omega^r \otimes E^{p-r})_{(2)}; \]

and the sequence

\[ 0 \rightarrow F^{p+1} \Omega^r(\mathbb{H}_C)_{(2)} \hookrightarrow F^p \Omega^r(\mathbb{H}_C)_{(2)} \rightarrow (\Omega^r \otimes E^{p-r})_{(2)} \rightarrow 0 \]

is exact, where \(\hookrightarrow\) is the inclusion map.

**Remark 2** Theorem 1 was first proved by Zucker in the case of Riemann surfaces [29].

Since the following two results are local, w.l.o.g., we can assume \(M = (\bigtriangleup^*)^m\), \(\bigtriangleup^*\) being the punctured disk. \(\pi_1(M)\) is then a free abelian group generated by \(m\) elements \(\sigma_1, \sigma_2, \ldots, \sigma_m\), \(\sigma_i\) corresponding to the counter-clockwise path around 0 of the \(i\)-th component of \((\bigtriangleup^*)^m\). Denote by \(\gamma_i\) the image of \(\sigma_i\) under \(\rho\), which is possibly trivial and (if nontrivial) referred to as the \(i\)-th monodromy transformation of the variation. By a result of Borel [23], up to some finite lifting, throughout this paper we assume that each \(\gamma_i\) is unipotent. Set \(N_i = \log \gamma_i\) with the elements on the diagonal being zero.

**Corollary 1** \(\theta\) and hence \(\nabla^{1,0}\) have the following asymptotic behavior at \((0, 0, \ldots, 0)\)

\[ \sum_{i=1}^{n} N_i \frac{dz_i}{z_i}; \]

and are \(L^2\)-bounded.

For later applications, one needs some explicit expressions of the sheaves \(\Omega^r(\mathbb{H}_C)_{(2)}\). This is done by the following proposition 1, which says that the sheaf \(\Omega^r(\mathbb{H}_C)_{(2)}\) can be defined algebraically, just using the logarithmic monodromies \(N_1, \ldots, N_m\) and the corresponding weight filtrations, and lies in \(j_!H^c \otimes \Omega^r_M(\log D), r = 1, \ldots, m\). As a consequence of this fact together with the asymptotic behavior of \(\theta\) and the theorem 1, we obtain the \(L^2\) holomorphic Dolbeault complex on \(\overline{M}\)

\[(*)\quad E_{(2)} \overset{\theta}{\rightarrow} (E \otimes \Omega^1_M(\log D))_{(2)} \overset{\theta}{\rightarrow} (E \otimes \Omega^2_M(\log D))_{(2)} \overset{\theta}{\rightarrow} \cdots , \]

which is furthermore independent of the Poincaré-like metric and the Hodge metric. The key point of the proof is the norm estimates [2] of the Hodge metric near the singularity.
Proposition 1 The sheaves $\Omega^0(\mathcal{H}_C)(2), \Omega^1(\mathcal{H}_C)(2), \cdots, \Omega^n(\mathcal{H}_C)(2)$ are determined by the monodromies $N_1, \cdots, N_n$ and can be expressed in terms of the weight filtrations $\{W^j\}, j = 1, \cdots, n$ on the domain $D_\epsilon = \{(t_1, \cdots, t_n) \mid \log |t_1| > \epsilon, \cdots, \log |t_n| > \epsilon, -\log |t_n| > \epsilon\}, \epsilon > 0$. More precisely, if considering the case of $\dim \mathcal{M} = 2$, one has the following explicit formulae,

$$\Omega^0(\mathcal{H}_C)(2) = t_1t_2\mathcal{H}_C + t_1 \bigcup_{l_2 - l_1 \leq 0} \mathcal{W}_{l_1}W_{l_2} + \bigcup_{l_1 \leq 0} \mathcal{W}_{l_1l_2},$$

$$\Omega^1(\mathcal{H}_C)(2) = \frac{dt_1}{t_1} \otimes (t_1t_2\mathcal{H}_C + t_1 \bigcup_{l_2 - l_1 \leq 0} \mathcal{W}_{l_1}W_{l_2} + \bigcup_{l_1 \leq 0} \mathcal{W}_{l_1l_2}) + + \frac{dt_2}{t_2} \otimes (t_1t_2\mathcal{H}_C + t_1 \bigcup_{l_2 \leq l_1 - 2} \mathcal{W}_{l_1}W_{l_2} + \bigcup_{l_1 \leq 0} \mathcal{W}_{l_1l_2}),$$

$$\Omega^2(\mathcal{H}_C)(2) = \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \otimes (t_1t_2\mathcal{H}_C + t_1 \bigcup_{l_2 \leq l_1 - 2} \mathcal{W}_{l_1}W_{l_2} + \bigcup_{l_1 \leq 0} \mathcal{W}_{l_1l_2}),$$

where $\mathcal{W}_{l_1l_2} = \mathcal{W}_{l_1}^1 \cap \mathcal{W}_{l_2}^2$ and $\nu \in \mathcal{W}_{l_1}^1 \cap \mathcal{W}_{l_2}^2$, $l_2 - l_1 \leq 0$, implies that $\nu$ has nontrivial projections to both $Gr_{l_1}W_{l_1}$ and $Gr_{l_2}W_{l_2}$.

In the above, we carefully analyze the $L^2$ holomorphic Dolbeault complex $\{(\mathcal{E} \otimes \Omega^r_M(\log D))(2), \theta\}$ of a VHS on $\overline{M}$. For uniformity of notations, we from now on denote $(\mathcal{E}^p \otimes \Omega^r_M(\log D))(2)$ by $Gr^p_F\Omega^r(\mathcal{H}_C)(2)$; it is easy to see that it is actually a piece of $(\mathcal{E} \otimes \Omega^r_M(\log D))(2)$.

By the infinitesimal period relation of $\nabla^{1,0}$ and the $L^2$-boundedness of $\theta$, we have

$$\theta(Gr^p_F\Omega^r(\mathcal{H}_C)(2)) \subset Gr^p_F\Omega^{r+1}(\mathcal{H}_C)(2).$$

(More precisely, we should restrict to a piece of $\theta$.) Thus, we obtain a holomorphic Dolbeault subcomplex of $\{(\mathcal{E} \otimes \Omega^r_M(\log D))(2), \theta\}$ on $\overline{M}$

$$\{Gr^p_F\Omega(\mathcal{H}_C)(2), \theta\}.$$

Let $F^pA^k(\mathcal{H}_C)(2) = \oplus_{r+s=k}(A^r \otimes \mathcal{E}^s \otimes \Omega^r_M(\log D))(2)$ and $D''_1 = \overline{\nabla^{1,0}}$ for $p \geq 0$. Here, $A^{r,s}$ is the sheaf of germs of local forms of type $(r, s)$ (not necessarily smooth) on $\overline{M}$ and $(A^r \otimes \mathcal{E}^s \otimes \Omega^r_M(\log D))(2)$ is the sheaf of germs of local $L^2\mathcal{E}^s$-valued forms $\phi$ of type $(r, s)$ on $\overline{M}$ for which $\overline{\nabla} \phi$ are $L^2$ in the weak sense, where the action of $\overline{\nabla}$ is defined as follows: Let $\phi$ be a form of type $(r, s)$ and $v$ a holomorphic section of $\mathcal{E}^p$, then $\overline{\nabla}(\phi \otimes v) = \overline{\nabla} \phi \otimes v$.

By the boundedness of $\nabla^{1,0}$, $D''_1(F^pA^k(\mathcal{H}_C)(2)) \subset F^pA^{k+1}(\mathcal{H}_C)(2)$. Using the Hodge filtration $\{F^p\}$, take the successive quotients

$$[Gr^p_FA^k(\mathcal{H}_C)(2)](2) := F^pA^k(\mathcal{H}_C)(2)/F^{p+1}A^k(\mathcal{H}_C)(2),$$
which, by the theorem 1 (which is also true for the differentiable case by the proof), can be identified with \( \oplus r+s=k(A^r,s \otimes E^{p-r})\). Here, \((A^r,s \otimes E^{p-r})\) is the sheaf on \(\overline{M}\) of germs of local \(L^2\) valued forms \(\phi\) of type \((r,s)\) for which \(\overline{\partial} \phi\) are in the weak sense. Denote the induced map of \(D''_p\) by \(D''\), which is actually \(\overline{\partial} + \theta\) and satisfies \((D'') = 0\). We now obtain a complex of fine sheaves on \(M\)

\[
\{[\text{Gr}^p_F A^r(\mathbb{H}_C)](2), D''\},
\]

for \(p \geq 0\) and the holomorphic Dolbeault subcomplex \(\{\text{Gr}^p_F \Omega(\mathbb{H}_C)(2), \theta\}\) is its subcomplex. Similar to the complex \(\{\text{Gr}^p_F \Omega(\mathbb{H}_C)(2), \theta\}, ([\text{Gr}^p_F A(\mathbb{H}_C)](2), D'')\) can also be considered as a piece of a larger complex of fine sheaves: Denote by \([\text{Gr}^p_F A^k(\mathbb{H}_C)](2)\) the sheaf of germs of local \(L^2\) \(k\)-forms \(\phi\) (not necessarily smooth) with values in \(E\) on \(\overline{M}\), for which \(\overline{\partial} \phi\) are also in the weak sense. It is clear that \(\{[\text{Gr}^p_F A^r(\mathbb{H}_C)](2), D''\}\) is a complex of fine sheaves and \(\{[\text{Gr}^p_F A^r(\mathbb{H}_C)](2), D''\}\) is a piece of it. Then we can show the following

**Theorem 2** The holomorphic Dolbeault complex \(\{\text{Gr}^p_F \Omega(\mathbb{H}_C)(2), \theta\}\) is quasi-isomorphic to the complex \(\{[\text{Gr}^p_F A(\mathbb{H}_C)](2), D''\}\) under the inclusion map for \(p \geq 0\).

Theorem 2 then implies Conjecture 2 in the case of variation of Hodge structure.

### 3 Cohomologies valued in harmonic bundles on non-compact curves

In this section, we consider cohomologies valued in harmonic bundles on non-compact curves. In order to define the \(L^2\)-cohomology, we need a suitable harmonic metric, which reflects the geometry of the representation \(\rho\). Throughout this section, we assume that \(\rho\) is unipotent near the divisor.

#### 3.1 Harmonic metrics in the case of noncompact curves

From now on, we assume that \(M\) is a noncompact curve, i.e. a compact Riemannian surface deleting finitely many points, and change the symbol \(M\) into \(S\), the compactification of which is denoted by \(\overline{S}\); \(S = \overline{S} \setminus \{p_1, \ldots, p_s\}\) and \(j: S \to \overline{S}\) is the inclusion map. Under such a case together with the assumption that the representation \(\rho\) is unipotent, one can easily get a \(\rho\)-equivariant harmonic map of finite energy and its behavior near the punctures \(\{p_1, \ldots, p_s\}\). It is worth pointing out that in [13], the authors considered these problems without the restriction of dimension. Since there the target manifolds are very general, the construction of the initial metrics are getting very complicated. In the present case, since the representation is linear, we will give an explicit construction though the basic idea is the same as [13]; more importantly, we
can see explicitly why the initial map is of finite energy, and hence the analysis of \cite{13} is applicable. The idea of the construction for the initial map will be used in the most general case, where representations need not be unipotent and also the energy of maps need not be finite. Here, we should also point out that the construction in \cite{14} is also a very special one (cf. \cite{11}); in \cite{27, 28}, we consider some constructions with some artificial singularities. As before, we take the Poincaré-like metric on the base manifold $S$, and by $t$ denote local Euclidean complex coordinate.

Let $\rho: \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple linear representation, and restrict $\rho$ to a neighborhood of $p_i$, say a small punctured disk $\Delta^*$ around $p_i$, which we call the boundary representation of $\rho$, denoted by $\rho_i$. Call the representation $\rho$ unipotent if each $\rho_i(\gamma)$ is a unipotent matrix, where $\gamma$ is a circle around $p_i$. A result of Borel tells us that this is the case for VHS up to a suitable lifting (cf. e.g. \cite{23}, Lemma 4.5). Take the logarithm of $\rho_i(\gamma)$, denoted by $N$, which is upper-triangle and all the diagonal entries of which are 0 under a suitable basis.

We now proceed to construct an initial metric on $L_\rho$, equivalently, an initial $\rho$-equivariant map from $\tilde{S}$ to $GL(n, \mathbb{C})/U(n)$. To this end, let us first give some preliminary. Let $P_n$ be the set of all positive definite hermitian symmetric matrices of order $n$. $GL(n, \mathbb{C})$ acts transitively on $P_n$ by $g \circ H := gH^t\bar{g}$, $H \in P_n$, $g \in GL(n, \mathbb{C})$. Obviously, the action has the isotropic subgroup $U(n)$ at the identity $I_n$. Thus $P_n$ can be identified with the coset space $GL(n, \mathbb{C})/U(n)$, and can be uniquely endowed an invariant metric\footnote{In terms of matrices, such an invariant metric can be defined as follows. At the identity $I_n$, the tangent elements just are hermitian matrices; let $A, B$ be such matrices, then the Riemannian inner product $\langle A, B \rangle_{p_n}$ is defined by $tr(AB)$. In general, let $H \in P_n$, $A, B$ two tangent elements at $H$, then the Riemannian inner product $\langle A, B \rangle_{p_n}$ is defined by $tr(H^{-1}AH^{-1}B)$.} up to some constants. In particular, under such a metric, the geodesics through the identity $I_n$ are of the form $\exp(tA)$, $t \in \mathbb{R}$, $A$ being a hermitian matrix.

Let the Jordan normal form of $N$ have $p$ Jordan blocks, denoted by $N_j, 1 \leq j \leq p$. By the Jacobson-Morosov theorem, one can expand each block $N_j$ into an $sl_2$-triple $\{Y_j, N_j, N_j^-\}$, i.e., $[Y_j, N_j] = 2N_j$, $[Y_j, N_j^-] = -2N_j^-$ and $[N_j, N_j^-] = Y_j$; a theorem of Kostant tells us that such a triple, up to conjugations, is unique (cf. e.g. \cite{17}). Take the Euclidean coordinate $t$ on $\Delta^*$ with $t(p_i) = 0$ and $t = re^{\sqrt{-1}x}$; also take the universal covering

$$H_\alpha = \{z = x + \sqrt{-1}y \mid x \in \mathbb{R}, \ y > -\log \alpha\}$$

of $\Delta^*$ with $y = -\log r$, for a positive number $\alpha < 1$. Corresponding to a (once and for all) fixed flat sections basis of $L_\rho$, we construct the required Hermitian
metric of $L_\rho$ on $\Delta^*$ as

$$h_i = \begin{pmatrix} M_1 & 0 \\ \vdots & \ddots \\ 0 & M_p \end{pmatrix},$$

where $M_j = \exp(x N_j) \circ \exp((\frac{1}{2} \log |\log r|) Y_j)$. In the following, $h_i$ is considered as both a metric and the above matrix under the fixed basis. Clearly, $h_i$, as a map from $H_\alpha$ to $\mathcal{P}_n$, is $\rho_i$-equivariant when changing $\log r$ into $y$. Geometrically, this is a geodesic $\rho_i$-equivariant embedding of $\Delta^*$ into $\mathcal{P}_n$ which maps the puncture to the infinity of $\mathcal{P}_n$, as can be explicitly seen when one embeds geodesically the upper-half plane into the real hyperbolic 3-space $H^3$ by using the matrix model of $H^3$ (for more details, cf. [11]). We also remark that each $h_i$ is actually harmonic. Now one can easily extend the metrics $\{h_i\}$ to a global metric on $L_\rho$, denoted by $h_0$. Naturally, the corresponding map is $\rho$-equivariant.

We now want to show that each $h_i$, and hence $h_0$, is of finite energy. For simplicity, we may assume here that the Jordan normal form of $N$ has only one Jordan block. Let $\{Y, N, N^-\}$ be the corresponding $\mathfrak{sl}_2$-triple. The semisimple element $Y$ can actually be described as follows. Canonically, $\mathbb{C}^n$ has a filtration

$$0 \subset W_{-(n-1)} \subset W_{-(n-3)} \subset \cdots \subset W_{-3} \subset W_{-1} = \mathbb{C}^n,$$

satisfying that $N(W_i) \subset W_{i-2}$, $Y$ preserves each $W_i$, and all the quotients $W_i/W_{i-2}$ are 1-dimensional. Then the (induced) action of $Y$ on $W_i/W_{i-2}$ is multiplying by $i$. Actually, one can also choose a basis $\{e_{-(n-1)}, e_{-(n-3)}, \ldots, e_{-3}, e_{-1}\}$ of $\mathbb{C}^n$, which is compatible with the above filtration (i.e., $Ne_j = e_{j-2}$ and $\{e_j\}_{j \leq i}$ generates $W_i$) and satisfies $Ye_i = ie_i \cdot \exp((\frac{1}{2} \log |\log r|) Y)$, under the above basis, can then be written as

$$\begin{pmatrix} |\log r|^{-(n-1)/2} & 0 & \cdots & 0 & 0 \\ 0 & |\log r|^{-(n-3)/2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & |\log r|^{-3/2} & 0 \\ 0 & 0 & \cdots & 0 & |\log r|^{-1/2} \end{pmatrix}.$$  \hspace{1cm} (11)

Then, using the invariant metric of $\mathcal{P}_n$, a simple computation shows that the energy of $h_i$ satisfies

$$E(h_i) = \int_{\Delta^*} |h_i^{-1} dh_i|^2 \leq C \int_0^\alpha |\log r|^{-2} r^{-1} dr < \infty.$$
As mentioned before, since the initial map $h_0$ is of finite energy, the analysis of [13] works; in particular, one has the following

**Proposition 2** Let $\rho : \pi_1(S) \to GL(n, \mathbb{C})$ be a semisimple representation all the boundary representations of which are unipotent, $h_0$ be the $\rho$–equivariant map (initial metric) constructed above. Then there exists a $\rho$-equivariant harmonic map (harmonic metric) of finite energy

$$h : \tilde{S} \to GL(n, \mathbb{C})/U(n),$$

which has the same asymptotic behavior as $h_0$ near the punctures, where $\tilde{S}$ is the universal covering of $S$; moreover the norm of the derivative $dh$ of $h$ satisfies, when going down to $S$ and measured near the divisor with respect to the Poincaré-like metric and the standard Riemannian symmetric metric on $GL(n, \mathbb{C})/U(n)$,

$$|dh|^2 \leq C|\log r|^2$$

for some constant $C > 0$, where $r$ is the radial Euclidean coordinate of $\Delta^*$. 

**Remark 3** From (11) and the related argument there, one has some explicit asymptotic estimates of the harmonic metric $h$, which are the same as the case of VHS (cf. [22]).

### 3.2 The $L^2$-cohomology: The $L^2$-Poincaré Lemma

For the direct image sheaf $j_\ast L_\rho$ of the local system $L_\rho$ on $\mathfrak{S}$, one has the Čech cohomology $H^\ast(\mathfrak{S}, j_\ast L_\rho)$; on the other hand, using the Poincaré-like metric $\omega$ on $S$ and the harmonic metric $h$ on $L_\rho$, one can define a complex $\{A^i_\ast(L_\rho), D\}$ of sheaves over $\mathfrak{S}$ as in the introduction and then $L^2$-de Rham cohomology $H^\ast(\{\Gamma(A^i_\ast(L_\rho)), D\})$, denoted by $H^\ast_{DR,(2)}(\mathfrak{S}, L_\rho)$.

**Theorem 3** There exists a natural identification

$$H^\ast(\mathfrak{S}, j_\ast L_\rho) \cong H^\ast_{DR,(2)}(\mathfrak{S}, L_\rho).$$

This is Conjecture 1 in the present situation.

Canonically, the proof of Theorem 1 is reduced to prove

**Theorem 4** *(The $L^2$-Poincaré lemma)* The complex $\{A^i_\ast(L_\rho), D\}$ is a resolution of $j_\ast L_\rho$. This is equivalent to saying that

1) $j_\ast L_\rho = \{\eta \in A^0_\ast(L_\rho) \mid D\eta = 0\}$;

2) the differential $D$ satisfies the Poincaré lemma, i.e., if an $i$-form $\eta$ in $A^i_\ast(L_\rho)$ is $D$-closed, then there exists an $i - 1$-form $\sigma \in A^{i-1}_\ast(L_\rho)$ satisfying $D\sigma = \eta$, for $i = 1, 2$. 

13
Remark 4 In the case of higher dimension, one needs to consider the intersection cohomology $H^*_\text{int}(\overline{S}, j_*L_\rho)$ instead of the Čech cohomology $H^*(\overline{S}, j_*L_\rho)$.

3.3 The $L^2$-Higgs cohomology: The $L^2$-$\overline{\partial}$-Poincaré Lemma

As seen in the introduction, the harmonic metric $h$ on $L_\rho$ induces the structure of a Higgs bundle on $L_\rho$: $(E, D'' = \overline{\partial} + \theta)$, satisfying $D = D' + D''$ with $D' = \partial + \overline{\theta}$; moreover, the Hermitian connection $\overline{\partial} + \theta$ w.r.t. the metric $h$ has bounded curvature under the metric $h$ and the Poincaré-like metric so that $E$ can be analytically extended to $\overline{S}$, denoted by $j_*E$ as usual, which is especially coherent. Furthermore, $\theta$ has a log-singularity, i.e. $\theta \sim \frac{dt}{t^N}$. It is especially worth pointing out that by an argument of Simpson (cf. [25]), the residue $N$ of $\theta$ here coincides with the logarithmic monodromy $N$ in the local system $L_\rho$; so although under different bundle structures, we have the same weight filtration under certain suitable identification. Throughout this subsection, we consider the Higgs bundle $(E, D'' = \overline{\partial} + \theta)$ together with the harmonic metric $h$, satisfying that the meromorphic sections of $E$ at the punctures have the same estimates as the case of VHS [23], just forgetting that it comes from the local system corresponding to the representation $\rho$.

As seen in the introduction, using the Poincaré-like metric $\omega$ on $S$ and the harmonic metric $h$ on $(E, D'')$, one can define a complex $\{A_{(2)}(E), D''\}$ of fine sheaves on $S$ and hence the corresponding cohomology $H^*(\{\Gamma(A_{(2)}(E)), D''\})$ of the complex of global sections—the $L^2$-Higgs cohomology of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$, denoted by $H^*_{\text{Higgs},(2)}(\overline{S}, E)$.

The following lemma is a key.

**Lemma 1** $\theta$ is an $L^2$-bounded operator.

**Proof.** The proof uses the norm estimates of $h$ in the sense of Higgs bundles.

Based on this lemma, as seen in the introduction, one has a sub-complex of $\{A_{(2)}(E), D''\}$—the $L^2$-holomorphic Dolbeault complex $\{\Omega_{(2)}(E), \theta\}$ and the corresponding hypercohomology $H^*(\{\Omega_{(2)}(E), \theta\})$, the $L^2$-Dolbeault cohomology of $\overline{S}$ valued in the Higgs bundle $(E, D = \overline{\partial} + \theta)$, denoted by $H^*_{\text{Dol},(2)}(\overline{S}, E)$.

Then, we have

**Theorem 5** ($L^2$-$\overline{\partial}$-Poincaré lemma) The inclusion

$$i : \{\Omega_{(2)}(E), \theta\} \hookrightarrow \{A_{(2)}(E), D''\}$$

is a quasi-isomorphism; and hence one has

$$H^*_{\text{Higgs},(2)}(\overline{S}, E) \cong H^*_{\text{Dol},(2)}(\overline{S}, E).$$
The proof of the theorem is reduced to show the following lemma, which was first proved by Zucker in the case of VHS (cf. [29], Proposition 6.4).

**Lemma 2** Let $V$ be a holomorphic line bundle on $\Delta^*$ with generating section $\sigma$, and with a Hermitian metric satisfying

$$||\sigma||^2 \sim |\log r|^k, \quad k \in \mathbb{Z}, \quad k \neq 1.$$

Then for every germ of an $L^2(0,1)$-form $\phi = f d\tau \otimes \sigma$ at the puncture, there exists an $L^2$ section $u \otimes \sigma$ with $\partial u = f d\tau$.

**References**

[1] J. Cheeger, M. Goresky, & R. MacPherson, *L^2*-cohomology and intersection homology of singular algebraic varieties. Seminar on differential geometry, ed. by S.-Y. Yau, Ann. Math. Studies, 102 (1982), 303-340.

[2] E. Cattani, A. Kaplan & W. Schmid, *Degeneration of Hodge structures*. Ann. Math., 38, (1986), 457-535.

[3] E. Cattani, A. Kaplan, & W. Schmid, *L^2 and intersection cohomologies for a polarizable variation of Hodge structure*. Inventiones Math., 87 (1987), 217-252.

[4] K. Corlette, *Flat G-bundles with canonical metrics*. J. Differential Geometry, 28 (1988), 361-382.

[5] S. K. Donaldson, *Twisted harmonic maps and self-duality equations*, Proc. London Math.Soc., 55 (1987) 127-131.

[6] M. Goresky & R. MacPherson, *Intersection homology theory*. Topology, 19 (1980), 135-162.

[7] M. Goresky & R. MacPherson, *Intersection homology, II*. Inventiones Math., 71 (1983), 77-129.

[8] N. Hitchin, *The self-duality equations on a Riemann surface*. Proc.London Math.Soc., 76 (1987), 59-126.

[9] J. Jost, Y.-H. Yang, & K. Zuo, *Chomologies of polarized variation of Hodge structure over quasi-compact Kähler manifolds*. J. Algebraic Geometry, 16 (2007), 401-434.

[10] J. Jost, Y.-H. Yang, & K. Zuo, *Cohomologies of unipotent harmonic bundles on noncompact curves*. Crelle’s Journal, 609, 2007.
[11] J. Jost, Y.-H. Yang, & K. Zuo, Linear representations of $\pi_1$ of quasi-projective varieties, constructions of harmonic maps, and parabolic structures. Preprint, 2004.

[12] J. Jost & S.-T. Yau, Harmonic mappings and Kähler manifolds. Math. Ann., 262 (1983), 145-166.

[13] J. Jost & K. Zuo, Harmonic maps and $SL(n, \mathbb{C})$-representations of $\pi_1$ of quasi-projective manifolds. J. Algebraic Geometry, 5 (1996), 77-106.

[14] J. Jost & K. Zuo, Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasi-projective manifolds. J. Differential Geometry, 47 (1997), 469-503.

[15] M. Kashiwara, The asymptotic behavior of a variation of polarized Hodge structure. Publ. Res. Inst. Math. Sci. 21 (1985), 853–875.

[16] M. Kashiwara & T. Kawai, The Poincaré lemma for variations of polarized Hodge structure. Publ. Res. Inst. Math. Sci. 23 (1987), 345–407.

[17] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. American Journal of Mathematics, 81, (1959) 973-1032.

[18] E. Looijenga, $L^2$-cohomology of locally symmetric varieties. Compositio Math., 67(1988), 3-20.

[19] V. B. Mehta & C. S. Seshadri, Moduli of vector bundles on curves with parabolic structures. Math. Ann., 248 (1980), 205-239.

[20] A. Miller, S. Müller-Stach, Watermann, Y.-H. Yang, & K. Zuo, Chow-Künneth decomposition for universal families over Picard modular surfaces. Algebraic Cycles and Motives, edited by J. Nagel and C. Peters, London Mathematical Society, Lecture Notes Series 344, 241-276.

[21] J. Sampson, Applications of harmonic maps to Kähler geometry. Contemporary Math., 49 (1986), 125-134.

[22] L. Saper & M. Stern, $L^2$-cohomology of arithmetic varieties. Ann. Math., 132(1989), 1-69.

[23] W. Schmid, Variation of Hodge structure: The singularities of the period mapping. Inventiones Math., 22 (1973), 211-319.

[24] Y.-T. Siu, The complex-analcity of harmonic maps and the strong rigidity of compact Kähler manifolds. Ann. Math., 112 (1980), 73-111.

[25] C. Simpson, Harmonic bundles on noncompact curves. J. Amer. Math. Soc., 3 (1990),
[26] ———, Higgs bundles and local system. Publ. Math. IHES, 75 (1992), 5-95.

[27] Y.-H. Yang, Meromorphic differentials with twisted coefficients on compact Riemann surfaces. To appear in "Calculus of Variations and PDE, 2008"; arXiv:math.DG/0703542.

[28] ———, Harmonic bundles with prescribed singularities over compact Kähler manifolds, in preparation.

[29] S. Zucker, Hodge theory with degenerating coefficients: $L^2$ cohomology in the Poincaré metric. Ann. Math., 109(1979), 415-476.