Corrigendum

Corrigendum: Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order (2016 Inverse Problems 32 105009)

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The main result of [1] appears to be correct under slightly stronger assumptions. The correct version of theorem 1.1 in [1] is stated in theorem 0.1.

We now state the corrected result. Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with $C^\infty$ boundary. Consider the polyharmonic operator $(-\Delta)^m$, where $m \geq 1$ is an integer. The operator $(-\Delta)^m$ is positive and self-adjoint on $L^2(\Omega)$ with domain $H^{2m}(\Omega) \cap H^m_0(\Omega)$, where

$$
H^m_0(\Omega) = \{ u \in H^m(\Omega) : \gamma u = 0 \}.
$$

Herein what follows, $\gamma$ is the Dirichlet trace operator

$$
\gamma : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega), \quad \gamma u = (u|_{\partial\Omega}, \partial_\nu u|_{\partial\Omega}, \ldots, \partial_{\nu}^{m-1} u|_{\partial\Omega}),
$$

where $\nu$ is the unit outer normal to the boundary $\partial\Omega$, and $H^m(\Omega)$ and $H^m(\partial\Omega)$ are the standard $L^2$-based Sobolev spaces on $\Omega$ and its boundary $\partial\Omega$ for $s \in \mathbb{R}$.

Throughout the paper we shall assume that $n > m$.

Let $A \in W^{-\frac{m}{2},2}(\mathbb{R}^n, \mathbb{C}^n) \cap \mathcal{E}'(\Omega, \mathbb{C}^n)$ and $q \in W^{-\frac{m}{2},2}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega, \mathbb{C})$, where $\mathcal{E}'(\Omega) = \{ u \in \mathcal{D}'(\mathbb{R}^n) : \text{supp}(u) \subseteq \overline{\Omega} \}$ and $W^{s,p}(\mathbb{R}^n)$ is the standard $L^p$-based Sobolev space on $\mathbb{R}^n$, $s \in \mathbb{R}$ and $1 < p < \infty$, which is defined by the Bessel potential operator.

Now, consider the operator $D_A$ which is formally $A \cdot D$, where $D_j = -i\partial_{x_j}$, and the operator $m_q$ of multiplication by $q$. It is claimed in [1, appendix A] that the operators $D_A$ and $m_q$ are bounded on $H^m(\Omega)$ for $m < 2n$. This claim is true for $m_q$ but not for $D_A$. The claim is correct for the latter only if $m < n$. The reason is that the multiplication operation becomes bounded between appropriate Sobolev spaces only for $m < n$; see lines 11–14 in the proof.
of [1, proposition A.1] on page 16. Exactly the same mistake also appears in the following places: lines 11–14 on page 6, lines 2–4 from below on page 9 and lines 7–10 on page 10.

For $f = (f_0, \ldots, f_{m-1}) \in \prod_{j=0}^{m-1} H^{\alpha-j-1/2}(\partial \Omega)$, consider the Dirichlet problem

$$\begin{align*}
\mathcal{L}_{A,q} u &= 0 \quad \text{in } \Omega, \\
\gamma u &= f \quad \text{on } \partial \Omega.
\end{align*}$$

If $0$ is not in the spectrum of $\mathcal{L}_{A,q}$, the Dirichlet problem (0.1) has a unique solution $u \in H^m(\Omega)$. We define the Dirichlet-to-Neumann map formally as $\mathcal{N}_{A,q} := (\partial^m_{\nu} u|_{\partial \Omega}, \partial^{m+1}_{\nu} u|_{\partial \Omega}, \ldots, \partial^{2m-1}_{\nu} u|_{\partial \Omega})$ which we show to be well-defined and bounded operator from

$$\mathcal{N}_{A,q} : \prod_{j=0}^{m-1} H^{\alpha-j-1/2}(\partial \Omega) \to \prod_{j=0}^{m-1} H^\alpha-m+j+1/2(\partial \Omega).$$

The main result of [1] appears to be correct under slightly stronger assumptions on regularity of $q$.

**Theorem 0.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with $C^\infty$ boundary, and let $m \geq 2$ be an integer such that $n > m$. Suppose that $A_1, A_2 \in W^{-\frac{m-2}{2}, \infty}(\mathbb{R}^n, \mathbb{C}^n) \cap \mathcal{E}'(\Omega)$ and $q_1, q_2 \in W^{-\frac{m}{2}+\frac{\delta}{2}, \frac{\delta}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega)$ for some $0 < \delta < 1/2$, such that $0$ is not in the spectrums of $\mathcal{L}_{A_1,q_1}$ and $\mathcal{L}_{A_2,q_2}$. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then $A_1 = A_2$ and $q_1 = q_2$.

The proof of theorem 0.1 follows exactly the same approach as in [1] up to the last paragraph in section 3, except that throughout the paper the assumption $m < 2n$ must be replaced by $m < n$ as discussed above. To finish the proof, one should proceed as follows.

To show that $q_1 = q_2$, we substitute $A_1 = A_2$ and $a_1 = a_2 = 1$ into the identity [1, 3.10] and obtain

$$b_{q_1,q_2}^B (1 + h^{m/2} r_1, (1 + h^{m/2} r_2) e^{i\xi}) = 0.$$ 

From [1, (3.9)] we know that $||r||_{H^m(\partial \Omega)} = O(1)$ as $h \to 0$. We want to show that letting $h \to 0$, we get $\widehat{q_1}(\xi) - \widehat{q_2}(\xi) = 0$ for all $\xi \in \mathbb{R}^n$. We will only consider the term $b_{q_1,q_2}^B (h^{m/2} r_1, e^{i\xi})$. The justification for the other two terms follows similarly. Since $q_1, q_2 \in W^{-\frac{m}{2}+\frac{\delta}{2}, \frac{\delta}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega)$ for some $0 < \delta < 1/2$, we use [1, proposition 2.2] with $p = q = 2, s_1 = \frac{2}{\delta} - \delta, s_2 = \frac{2}{\delta}$ and $r = 2n/(2n - m)$ to get

$$|b_{q_1,q_2}^B (h^{m/2} r_1, e^{i\xi})| \leq Ch^{m/2} ||q_2 - q_1||_{W^{-\frac{m}{2}+\frac{\delta}{2}, \frac{\delta}{2}}(\mathbb{R}^n)} ||r_1 e^{i\xi}||_{W^{-\frac{m}{2}+\frac{\delta}{2}, \frac{\delta}{2}}(\mathbb{R}^n)}$$

$$\leq Ch^{m/2} ||q_2 - q_1||_{W^{-\frac{m}{2}+\frac{\delta}{2}, \frac{\delta}{2}}(\mathbb{R}^n)} ||e^{i\xi}||_{H^\alpha(\partial \Omega)} ||r_1||_{H^\alpha(\partial \Omega)} \leq O(h^{\delta}) ||r_1||_{H^\alpha(\partial \Omega)} \leq O(h^{\delta}).$$

This implies that $q_1 = q_2$ in $B$ completing the proof of theorem 0.1.

**Acknowledgments**

The author is very grateful to Karthik Iyer for pointing out these mistakes and for suggesting the ways to fix them.

**References**

[1] Assylbekov Y M 2016 Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order Inverse Problems 32 105009
Inverse problems for the perturbed polyharmonic operator with coefficients in Sobolev spaces with non-positive order

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Abstract
We show that the knowledge of the Dirichlet-to-Neumann map on the boundary of a bounded open set in \(\mathbb{R}^n\), \(n \geq 3\), for the perturbed polyharmonic operator \((-\Delta)^m + A \cdot D + q\), \(m \geq 2\), with \(2n > m\), \(A \in W^{-\frac{n-2}{2}, \frac{n-2}{2}}\) and \(q \in W^{-\frac{n-2}{2}, \frac{n-2}{2}}\), determines the potentials \(A\) and \(q\) in the set uniquely. The proof is based on a Carleman estimate with linear weights and with a gain of two derivatives and on the property of products of functions in Sobolev spaces.

Keywords: inverse problems, polyharmonic operators, Dirichlet-to-Neumann map

1. Introduction

Let \(\Omega \subset \mathbb{R}^n\), \(n \geq 3\), be a bounded open set with \(C^\infty\) boundary. Consider the polyharmonic operator \((-\Delta)^m\), where \(m \geq 1\) is an integer. The operator \((-\Delta)^m\) is positive and self-adjoint on \(L^2(\Omega)\) with domain \(H^m_0(\Omega) \cap H^m_0(\Omega)\), where

\[H^m_0(\Omega) = \{ u \in H^m(\Omega) : \gamma u = 0 \} .\]

This operator can be obtained as the Friedrichs extension starting from the space of test functions. This fact can be found, for example, in [10]. Here and in what follows, \(\gamma\) is the Dirichlet trace operator

\[\gamma : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega), \quad \gamma u = (u|_{\partial \Omega}, \partial_\nu u|_{\partial \Omega}, \ldots, \partial_\nu^{m-1} u|_{\partial \Omega}) ,\]

where \(\nu\) is the unit outer normal to the boundary \(\partial \Omega\), and \(H^s(\Omega)\) and \(H^s(\partial \Omega)\) are the standard \(L^2\)-based Sobolev spaces on \(\Omega\) and its boundary \(\partial \Omega\) for \(s \in \mathbb{R}\).

Throughout the paper we shall assume that \(2n > m\).
Let $A \in W^{2,\infty}(\mathbb{R}^n, \mathbb{C}^m) \cap C'(\overline{\Omega}, \mathbb{C}^n)$ and $q \in W^{2,\infty}(\mathbb{R}^n, \mathbb{C}) \cap C'(\overline{\Omega}, \mathbb{C})$, where $C'(\overline{\Omega}) = \{ u \in D'(\mathbb{R}^n) : \text{supp}(u) \subseteq \overline{\Omega} \}$ and $W^{s,p}(\mathbb{R}^n)$ is the standard $L^p$-based Sobolev space on $\mathbb{R}^n$, $s \in \mathbb{R}$ and $1 < p < \infty$, which is defined by the Bessel potential operator. Thus $W^{s,p}(\mathbb{R}^n)$ is the space of all distributions $u$ on $\mathbb{R}^n$ such that $J_s u \in L^p(\mathbb{R}^n)$, where $J_s$ is the operator defined as

$$u \mapsto ((1 + |\cdot|^2)^{-s/2} \hat{u}(\cdot))^\vee.$$ 

In the case of the $s \geq 0$ integer, $W^{s,p}(\mathbb{R}^n)$ coincides with the space of all functions whose all derivatives of order less or equal to $s$ is in $L^p(\mathbb{R}^n)$. The reader is referred to [33] for properties of these spaces.

Before stating the problem, we consider the bilinear forms $B_\lambda$ and $b_\eta$ on $H^m(\Omega)$ which are defined by

$$B_\lambda(u, v) = \langle A, \nabla D\hat{u} \rangle_{\mathbb{R}^n}, \quad b_\eta(u, v) = \langle q, \nabla \hat{v} \rangle_{\mathbb{R}^n}, \quad u, v \in H^m(\Omega),$$

where $(\cdot, \cdot)_{\mathbb{R}^n}$ is the distributional duality on $\mathbb{R}^n$, and $\hat{u}, \hat{v} \in H^m(\mathbb{R}^n)$ are extensions of $u$ and $v$, respectively. In appendix A, we show that these definitions are well-defined (i.e. independent of the choice of extensions $\hat{u}, \hat{v}$). Using a property on the multiplication of functions in Sobolev spaces, we show that the forms $B_\lambda$ and $b_\eta$ are bounded on $H^m(\Omega)$; see proposition A.2.

Consider the operator $D_\lambda$, which is formally $A \cdot D$, where $D_j = -i \partial_j$, and the operator $m_\eta$ of multiplication by $\eta$. To be precise, for $u \in H^m(\Omega)$, $D_\lambda(u)$ and $m_\eta(u)$ are defined as

$$\langle D_\lambda(u), \psi \rangle_{\Omega} = B_\lambda(u, \psi) \quad \text{and} \quad \langle m_\eta(u), \psi \rangle_{\Omega} = b_\eta(u, \psi), \quad \psi \in C_0^\infty(\Omega).$$

Then the operators $D_\lambda$ and $m_\eta$ are bounded $H^m(\Omega) \rightarrow H^m(\Omega)$ (see corollary A.3), and hence, standard arguments show that the operator

$$\mathcal{L}_{A,\eta} = (-\Delta)^m + D_\lambda + m_\eta : H^m_0(\Omega) \rightarrow H^{-m}(\Omega) = (H^m_0(\Omega))',$$

is Fredholm operator with zero index; see appendix B. For $f = (f_0, \ldots, f_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega)$, consider the Dirichlet problem

$$\mathcal{L}_{A,\eta} u = 0 \quad \text{in} \quad \Omega, \quad \gamma u = f \quad \text{on} \quad \partial \Omega. \quad (1.1)$$

If $0$ is not in the spectrum of $\mathcal{L}_{A,\eta}$, the Dirichlet problem (1.1) has a unique solution $u \in H^m(\Omega)$. We define the Dirichlet-to-Neumann map $N_{A,\eta}$ as follows

$$\langle N_{A,\eta} f, \eta \rangle_{\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, D^\alpha \eta)_{L^2(\Omega)} + B_\lambda(u, \eta) + b_\eta(u, \eta), \quad (1.2)$$

where $h = (h_0, \ldots, h_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega)$, and $\eta \in H^m(\Omega)$ is an extension of $h$, that is $\gamma \eta = h$. It is shown in appendix B that $N_{A,\eta}$ is a well-defined (i.e. independent of the choice of $\eta$) bounded operator

$$N_{A,\eta} : \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \rightarrow \left( \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \right)' = \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial \Omega).$$

The inverse boundary problem for the perturbed polyharmonic operator $\mathcal{L}_{A,\eta}$ is to determine $A$ and $\eta$ in $\Omega$ from the knowledge of the Dirichlet-to-Neumann map $N_{A,\eta}$.

When $m = 1$ the operator $L_{A,\eta}$ is the first order perturbation of the Laplacian and $N_{A,\eta}$ is formally given by $N_{A,\eta} f = (\Delta u + i (A \cdot \nu) u)_{|\partial \Omega}$, where $u$ is an $H^m(\Omega)$ solution to the equation $L_{A,\eta} u = 0$. It was shown in [29] that in this case there is an obstruction to
uniqueness in this problem given by the following gauge equivalence of the set of the Cauchy data: if $\psi \in W^{1,\infty}$ in a neighborhood of $\Omega$ and $\psi|_{\partial \Omega} = 0$, then $C_{A,q} = C_{A + \nabla \psi, q}$; see also [19, lemma 3.1]. Hence, given $C_{A,q}$ we may only hope to recover the magnetic field $dA$ and electric potential $q$. Here and in what follows the magnetic field $dA$ is defined by

$$
dA = \sum_{1 \leq j,k \leq n} (\partial_j A_k - \partial_k A_j) dx_j \wedge dx_k.
$$

Due to the lack smoothness of $A$, this definition is in the sense of distributions.

Starting with the paper of Sun [29], inverse boundary value problems for the magnetic Schrödinger operators have been extensively studied. It was shown in [29] that the hope mentioned above is justified provided that $A \in W^{2,\infty}$, $q \in L^\infty$ and $dA$ satisfies a smallness condition. The smallness condition was removed in [22] for $C^\infty$ magnetic and electric potentials, and also for compactly supported $C^2$ magnetic potentials and essentially bounded electric potentials. The regularity assumption on magnetic potentials were subsequently weakened to $C^1$ in [32], and then to Dini continuous in [27]. All these results were obtained under the assumption that zero is not a Dirichlet eigenvalue for the magnetic Schrödinger operator in $\Omega$. There are two best result by now. One is due to Krupchyk and Uhlmann [19], where they prove uniqueness under the assumption that magnetic and electric potentials are of class $L^\infty$. Another is due to Haberman [12] in dimension $n = 3$, where the uniqueness is shown for the case when $q \in W^{-1,3}$ and $A \in W^{5,3}$ for some $s > 0$ with certain smallness condition.

It was shown in [17] that the obstruction to uniqueness coming from the gauge equivalence when $m = 1$ can be eliminated by considering operators of higher order. More precisely, they show that for $m \geq 2$ the set of Cauchy data $C_{A,q}$ determines $A$ and $q$ uniquely provided that $A \in W^{1,\infty}(\Omega, C^0) \cap \mathcal{E}'(\Omega, C^0)$ and $q \in L^\infty(\Omega)$. They also show that the uniqueness result holds without the assumption $A = 0$ on $\partial \Omega$ but for $C^\infty$ magnetic and electric potentials. This is also true for $A \in W^{1,\infty}(\Omega, C^0)$ and $q \in L^\infty(\Omega)$ when the boundary of the domain $\Omega$ is connected.

The purpose of this paper is to relax the regularity assumption on $A$ from $W^{1,\infty}$ to $W^{\frac{2n}{n+2},\frac{n}{n+2}}$ class and $q$ from $L^\infty$ to $W^{-\frac{2n}{n+2},\frac{n}{n+2}}$ for the perturbed polyharmonic operator $L_{A,q}$ with $m \geq 2$. Therefore, throughout the paper we assume that $m \geq 2$. Our main result is as follows.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set with $C^\infty$ boundary, and let $m \geq 2$ be an integer such that $2n > m$. Suppose that $A_1, A_2 \in W^{-\frac{2n}{n+2},\frac{n}{n+2}}(\mathbb{R}^n, C^0) \cap \mathcal{E}'(\Omega, C^0)$ and $q_1, q_2 \in W^{-\frac{2n}{n+2},\frac{n}{n+2}}(\mathbb{R}^n, C) \cap \mathcal{E}'(\Omega, C)$. Let $\mathcal{N}_{A_1,q_1} \neq \mathcal{N}_{A_2,q_2}$, then $A_1 = A_2$ and $q_1 = q_2$.

The assumption $2n > m$ is related to the dual space of $W^{n,\frac{n}{n+2}}$ and the estimate on products of functions in different Sobolev spaces. It seems to the author that the techniques of the present paper can be adopted to the case $2n < m$ by changing regularity assumptions. We hope to consider this problem in future work. It would be also interesting to study this problem on the borderline $2n = m$.

The key ingredient in the proof of theorem 1.1 is a construction of complex geometric optics solutions for the operator $A \in W^{-\frac{2n}{n+2},\frac{n}{n+2}}(\mathbb{R}^n, C^0) \cap \mathcal{E}'(\Omega, C^0)$ and $q \in W^{-\frac{2n}{n+2},\frac{n}{n+2}}(\mathbb{R}^n, C) \cap \mathcal{E}'(\Omega, C)$. For this, we use the method of Carleman estimates which is based on the corresponding Carleman estimate for the Laplacian, with a gain of two derivatives, due to Salo and Tzou [28]. Another important tool in our proof is the property of products of functions in Sobolev spaces [26]. This was used in the paper of Brown and Torres
The idea of constructing such solutions for the Schrödinger operator goes back to the fundamental paper due to Sylvester and Uhlmann [30]. Such solutions were first introduced in [6] in the setting of quantum inverse scattering problem.

The inverse boundary value problem of the recovery of a zeroth order perturbation of the biharmonic operator, that is when \( m = 2 \), has been studied by Isakov [15], where uniqueness result was obtained, similarly to the case of the Schrödinger operator. In [14], the uniqueness result was extended to \( q \in L^{n/2}(\Omega) \), \( n > 4 \) by Ikehata. These results were extended for zeroth order perturbation of the polyharmonic operator with \( q \in L^{n/2m} \), \( n > 2m \) by Krupchyk and Uhlmann [18]. In the case \( m = 1 \), that is for zeroth order perturbation of the Schrödinger operator, global uniqueness result was established by Lavine and Nachman [20] for \( q \in L^{n/2}(\Omega) \), following an earlier result of Chanillo [5] for \( q \in L^{n/2+\varepsilon}(\Omega) \), \( \varepsilon > 0 \) and Novikov [23] for \( q \in L^{\infty}(\Omega) \).

Higher ordered polyharmonic operators occur in the areas of physics and geometry such as the study of the Kirchhoff plate equation in the theory of elasticity, and the study of the Paneitz–Branson operator in conformal geometry; for more details see monograph [8].

We would like to remark that the problem considered in this paper can be considered as a generalization of the Calderón’s inverse conductivity problem [3], known also as electrical impedance tomography, for which the reduction of regularity have been studied extensively.

In the fundamental paper by Sylvester and Uhlmann [30] it was shown that \( C^2 \) conductivities can be uniquely determined from boundary measurements. The regularity assumptions were weakened to \( C^{1/2+\varepsilon} \) conductivities by Brown [1], and corresponding result for \( C^{1/2} \) conductivities was obtained by Paivarinta et al [24]. Uniqueness result for \( C^{1/2+\varepsilon} \) conormal conductivities was shown by Greenleaf et al [9]. There is a recent work by Haberman and Tataru [13] which gives a uniqueness result for Calderón’s problem with \( C^1 \) conductivities and with Lipschitz continuous conductivities, which are close to the identity in a suitable sense. Very recent work of Caro and Rogers [4] shows that Lipschitz conductivities can be determined from the Dirichlet-to-Neumann map. Finally, Haberman [11] gives uniqueness results for conductivities with unbounded gradient. In particular, uniqueness for conductivities in \( W^{1,n}(\Omega) \) with \( n = 3, 4 \) is obtained.

The structure of the paper is as follows. Section 2 is devoted to the construction of complex geometric optics solutions for the perturbed polyharmonic operator \( \mathcal{L}_{\alpha,q} \) with \( A \in W^{n/2+\varepsilon}(\mathbb{R}^n, C^n) \cap \mathcal{E}'(\Omega, C^n) \) and \( q \in W^{-n/2+\varepsilon}(\mathbb{R}^n, C) \cap \mathcal{E}'(\Omega, C) \). Then the proof of theorem 1.1 is given in section 3. In appendix A, we study mapping properties of the operators \( D_{\alpha} \) and \( m_{\gamma} \). Finally, appendix B is devoted to the well-posedness of the Dirichlet problem for \( \mathcal{L}_{\alpha,q} \) with \( A \in W^{-n/2+\varepsilon}(\mathbb{R}^n, C^n) \cap \mathcal{E}'(\Omega, C^n) \) and \( q \in W^{-n/2+\varepsilon}(\mathbb{R}^n, C) \cap \mathcal{E}'(\Omega, C) \), \( m \geq 2 \). When constructing such solutions, we shall first derive Carleman estimates for the operator \( \mathcal{L}_{\alpha,q} \).

2. Carleman estimates and complex geometric optics solutions

In this section we construct the complex geometric optics solutions for the equation \( \mathcal{L}_{\alpha,q}u = 0 \) in \( \Omega \) with

\[
A \in W^{-n/2+\varepsilon}(\mathbb{R}^n, C^n) \cap \mathcal{E}'(\Omega, C^n)
\]

and \( q \in W^{-n/2+\varepsilon}(\mathbb{R}^n, C) \cap \mathcal{E}'(\Omega, C) \), \( m \geq 2 \). When constructing such solutions, we shall first derive Carleman estimates for the operator \( \mathcal{L}_{\alpha,q} \).

We start by recalling the Carleman estimate for the semiclassical Laplace operator \(-h^2\Delta\) with a gain of two derivatives, established in [28]. Let \( \Omega \) be an open set in \( \mathbb{R}^d \) such that \( \Omega \subset \subset \hat{\Omega} \) and let \( \varphi \in C^\infty(\hat{\Omega}, \mathbb{R}) \). Consider the conjugated operator

\[
\mathcal{L}_{\alpha,q}u = 0
\]
\[ P_\varphi = e^{\varphi/h}(-h^2\Delta)e^{-\varphi/h}, \]

and its semiclassical principal symbol
\[ p_\varphi(x, \xi) = \xi^2 + 2i\nabla \varphi \cdot \xi - |\nabla \varphi|^2, \quad x \in \tilde{\Omega}, \quad \xi \in \mathbb{R}^n. \]

Following [16], we say that \( \varphi \) is a limiting Carleman weight for \(-h^2\Delta \) in \( \tilde{\Omega} \), if \( \nabla \varphi \approx 0 \) in \( \tilde{\Omega} \) and the Poisson bracket of Re \( p_\varphi \) and Im \( p_\varphi \) satisfies
\[ \{ \text{Re} p_\varphi, \text{Im} p_\varphi \}(x, \xi) = 0 \quad \text{when} \quad p_\varphi(x, \xi) = 0, \quad (x, \xi) \in \tilde{\Omega} \times \mathbb{R}^n. \]

In this paper we shall consider only the linear Carleman weights \( \varphi(x) = \alpha \cdot x \), \( \alpha \in \mathbb{R}^n, |\alpha| = 1 \).

In what follows we consider the semiclassical norm on the standard Sobolev space \( H^s(\mathbb{R}^n) \), \( s \in \mathbb{R} \),
\[ \|u\|_{H^s_0(\mathbb{R}^n)} = \|\langle hD\rangle^s u\|_{L^2(\mathbb{R}^n)}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2}. \]

Our starting point is the following Carleman estimate for the semiclassical Laplace operator \(-h^2\Delta \) with a gain of two derivatives, which is due to Salo and Tzou [28].

**Proposition 2.1.** Let \( \varphi \) be a limiting Carleman weight for \(-h^2\Delta \) in \( \tilde{\Omega} \), and let \( \varphi_\varepsilon = \varphi + \frac{h}{2\varepsilon} \varphi^2 \). Then for \( 0 < h \ll \varepsilon \ll 1 \) and \( s \in \mathbb{R} \), we have
\[ \frac{h}{\varepsilon^2} \|u\|_{H^s_0(\mathbb{R}^n)} \lesssim C \|e^{\varphi/h}(-h^2\Delta)e^{-\varphi/h}u\|_{H^s_0(\mathbb{R}^n)}, \quad C > 0, \]
for all \( u \in C_0^\infty(\Omega) \).

Next, we state theorem on products of functions in Sobolev spaces. This result is well-known, see theorem 2 in [26, section 4.4.4].

**Proposition 2.2.** Let \( 1 < p, q < \infty \) and \( 0 < s_1 \leq s_2 < n \min(1/p, 1/q) \). Then \( W^{s_1, p}(\mathbb{R}^n) \cap W^{s_2, q}(\mathbb{R}^n) \subseteq W^{s, t}(\mathbb{R}^n) \) where \( 1/t = 1/p + 1/q - s_2/n \).

Now we shall derive Carleman estimate for the perturbed operator \( \mathcal{L}_{A, q} \) with \( A \in W^{-s_2, \infty}(\mathbb{R}^n, \mathbb{C}^n) \cap \mathcal{E}(\tilde{\Omega}, \mathbb{C}^n) \) and \( q \in W^{-s_2, \infty}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}(\tilde{\Omega}, \mathbb{C}) \). To that end we shall iterate \( m \) times inequality in proposition 2.1 and use it with \( s = -m \), and with fixed \( \varepsilon > 0 \) being sufficiently small, that is independent of \( h \). We have the following result.

**Proposition 2.3.** Let \( \varphi \) be a limiting Carleman weight for \(-h^2\Delta \) in \( \tilde{\Omega} \), and suppose that \( A \in W^{-s_2, \infty}(\mathbb{R}^n, \mathbb{C}^n) \cap \mathcal{E}(\tilde{\Omega}, \mathbb{C}^n) \) and \( q \in W^{-s_2, \infty}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}(\tilde{\Omega}, \mathbb{C}) \). Then for \( 0 < h \ll \varepsilon \ll 1 \), we have
\[ \|u\|_{H^s_0(\mathbb{R}^n)} \lesssim \frac{1}{h^m} \|e^{\varphi/h}(h^{2m}\mathcal{L}_{A, q})e^{-\varphi/h}u\|_{H^s_0(\mathbb{R}^n)}, \quad (2.1) \]
for all \( u \in C_0^\infty(\Omega) \).

**Proof.** Iterating the Carleman estimate in proposition 2.1 \( m \) times, \( m \geq 2 \), we get the following Carleman estimate for the polyharmonic operator,
\[ \frac{h^m}{\varepsilon^{m/2}} \|u\|_{H^s_0(\mathbb{R}^n)} \lesssim C \|e^{\varphi/h}(-h^2\Delta)^m e^{-\varphi/h}u\|_{H^s_0(\mathbb{R}^n)}, \]
for all \( u \in C_0^\infty(\Omega) \), \( s \in \mathbb{R} \) and \( 0 < h \ll \epsilon \ll 1 \). We shall use this estimate with \( s = -m \), and with fixed \( \epsilon > 0 \) being sufficiently small but independent of \( h \):

\[
\frac{h^m}{\epsilon^{m/2}} \| u \|_{H^m_{\epsilon^{-m}}(\mathbb{R}))} \leq C \| e^{\epsilon^j}(1-h^2\Delta)^me^{\epsilon^j}u \|_{H^m_{\epsilon^{-m}}(\mathbb{R}))},
\]

(2.2)

for all \( u \in C_0^\infty(\Omega) \) and \( 0 < h \ll \epsilon \ll 1 \).

In order to prove the proposition it will be convenient to use the following characterization of the semiclassical norm in the Sobolev space \( H^{-m}(\mathbb{R}^n) \)

\[
\| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)} = \sup_{0 \neq \psi \in C_0^\infty(\mathbb{R}^n)} \frac{\langle \psi, \psi \rangle_{\mathbb{R}^n}}{\| \psi \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)}},
\]

(2.3)

where \( \langle \cdot, \cdot \rangle_{\mathbb{R}^n} \) is the distribution duality on \( \mathbb{R}^n \).

Let \( \varphi = h + \frac{\epsilon}{\epsilon^2}2 \) be the convexified weight with \( \epsilon > 0 \) such that \( 0 < h \ll \epsilon \ll 1 \), and let \( u \in C_0^\infty(\Omega) \). Then for all \( 0 = \psi \in C_0^\infty(\mathbb{R}^n) \), by duality of the spaces \( W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n, \mathbb{C}) \) and \( W^{\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n, \mathbb{C}) \) (this is the reason for our assumption that \( 2n > m \)) and by proposition 2.2, we have

\[
| \langle e^{\epsilon^j}h^m, D_\varphi(e^{-\epsilon^j}u), \psi \rangle_{\mathbb{R}^n} | \leq C h^{2m} \| q \|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)} \| u \|_{W^{\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)}
\]

\[
\leq C h^{2m} \| q \|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)} \| u \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)} \| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)}
\]

\[
\leq C h^{2m} \| q \|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)} \| u \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)} \| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)}.
\]

Therefore, by (2.3), we obtain

\[
\| e^{\epsilon^j}h^m, D_\varphi(e^{-\epsilon^j}u) \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)} \leq C h^{2m} \| q \|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)} \| u \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)}.
\]

(2.4)

For all \( 0 = \psi \in C_0^\infty(\Omega) \), we can show

\[
| \langle e^{\epsilon^j}h^m, D_\varphi(e^{-\epsilon^j}u), \psi \rangle_{\mathbb{R}^n} | = \| (h^{2m-1}A, e^{\epsilon^j}h^m, \psi)|u(1 + h^2\varphi)D\varphi + hD\varphi(\psi) + hD\varphi(\psi)\|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)}.
\]

In the last step we used duality between \( W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n, \mathbb{C}) \) and \( W^{\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n, \mathbb{C}) \). In the case \( m \geq 3 \), we use proposition 2.2 with \( p = q = 2 \), \( s_1 = \frac{m-2}{2} \) and \( s_2 = \frac{m}{2} \), to get

\[
\| -u(1 + h^2\varphi)D\varphi(\psi) + hD\varphi(\psi) \|_{W^{-\frac{m}{2}, \frac{m}{2}}(\mathbb{R}^n)} \leq C \| -u(1 + h^2\varphi)D\varphi + hD\varphi(\psi) \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)} \| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)}
\]

\[
\leq C \| -u(1 + h^2\varphi)D\varphi + hD\varphi(\psi) \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)} \| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)}
\]

\[
\leq C \| -u(1 + h^2\varphi)D\varphi + hD\varphi(\psi) \|_{H^m_{\epsilon^{-m}}(\mathbb{R}^n)} \| \psi \|_{H^{-m}_{\epsilon^{-m}}(\mathbb{R}^n)}
\]

for some constant \( C > 0 \) depending only on \( \varphi \). When \( m = 2 \), we use Hölder’s inequality and Sobolev embedding \( H^1(\mathbb{R}^n) \subset L^{2n}(\mathbb{R}^n) \) (see [31, chapter 13, proposition 6.4]), and obtain
\[ \|u(1 + h \varphi \varepsilon)D\varphi \psi + hD\psi\|_{L^{\infty}_{h}(\mathbb{R}^n)} \leq C\|u(1 + h \varphi \varepsilon)D\varphi + hD\psi\|_{L^{2}_{h}(\mathbb{R}^n)} \|\psi\|_{L^{\infty}_{h}(\mathbb{R}^n)} \]
\[ \leq C\|u\|_{H^{m}_{h}(\mathbb{R}^n)} \|\psi\|_{H^{m}_{h}(\mathbb{R}^n)} \]
\[ \leq \frac{C}{h}\|u\|_{H^{m}_{h}(\mathbb{R}^n)} \|\psi\|_{H^{m}_{h}(\mathbb{R}^n)}, \]

for some constant \( C > 0 \) depending only on \( \varphi \). Therefore, for \( m \geq 2 \), we get

\[ \|e^{i\beta/\hbar}h^{-2m}D\psi(\cdot, \psi)\|_{H^{m}_{h}(\mathbb{R}^n)} \leq Ch^{m}\|A\|_{W^{\infty,-2}_{\hbar}(\mathbb{R}^n)} \|\psi\|_{H^{m}_{h}(\mathbb{R}^n)}. \]

Hence, by (2.3), we obtain

\[ \|e^{i\beta/\hbar}h^{-2m}D\psi(\cdot, \psi)\|_{H^{m}_{h}(\mathbb{R}^n)} \leq Ch^{m}\|A\|_{W^{\infty,-2}_{\hbar}(\mathbb{R}^n)} \|\psi\|_{H^{m}_{h}(\mathbb{R}^n)}. \]

Combining these estimates with (2.2) and (2.4) we get that for small enough \( h > 0 \)

\[ \|\psi\|_{H^{m}_{h}(\mathbb{R}^n)} \leq \frac{1}{h^{m}}\|e^{i\beta/\hbar}(h^{-2m}L_{A,h})e^{-i\beta/\hbar}\|_{H^{m}_{h}(\mathbb{R}^n)}. \]

Using that

\[ e^{-i\beta/\hbar} = e^{-i\beta/\hbar}e^{-i\beta/\hbar} \]

we obtain (2.1). \( \square \)

Let \( \varphi \in C_{0}^{\infty}(\Omega, \mathbb{R}) \) be a limiting Carleman weight for \(-h^2\Delta\). Set

\[ L_{\varphi} = e^{i\beta/\hbar}(h^{-2m}L_{A,h})e^{-i\beta/\hbar}. \]

Then by proposition A.4 we have

\[ \langle L_{\varphi}u, v \rangle_{\Omega} = \langle u, L_{\varphi}^{*}v \rangle_{\Omega}, \]

where \( L_{\varphi}^{*} = e^{-i\beta/\hbar}(h^{2}L_{A,h}^{*})e^{i\beta/\hbar} \) is the formal adjoint of \( L_{\varphi} \), and \( \langle \cdot, \cdot \rangle_{\Omega} \) is the distribution duality on \( \Omega \). We have that

\[ L_{\varphi}^{*} : C_{0}^{\infty}(\Omega) \rightarrow H^{-m}(\mathbb{R}^n) \cap E'(\Omega) \]

is bounded. Therefore, the estimate (2.1) holds for \( L_{\varphi}^{*} \), since \(-\varphi \) is a limiting Carleman weight as well.

To construct the complex geometric optics solutions for the operator \( L_{A,h} \), we need to convert the Carleman estimate (2.1) for \( L_{\varphi}^{*} \) into the following solvability result. The proof is essentially well-known, and we include it here for the convenience of the reader. In what follows, we shall write

\[ \|u\|_{H^{m}_{h}(\Omega)} = \sum_{|\alpha| \leq m} \|h^{2m}u\|_{L^{1}_{h}(\Omega)}, \]

\[ \|v\|_{H^{m}_{h}(\Omega)} = \sup_{0 \neq f \in H^{m}_{h}(\Omega)} \frac{\langle v, \phi \rangle_{\Omega}}{\|\phi\|_{H^{m}_{h}(\Omega)}} = \sup_{0 \neq f \in H^{m}_{h}(\Omega)} \frac{\|\phi\|_{H^{m}_{h}(\Omega)}}{\|\phi\|_{H^{m}_{h}(\Omega)}}. \]
Proposition 2.4. Let
\[ A \in W^{-\frac{n-1}{2}}(\mathbb{R}^n, \mathbb{C}^n) \cap E'(\Omega, \mathbb{C}^n) \]
and
\[ q \in W^{-\frac{n}{2}}(\mathbb{R}^n, \mathbb{C}) \cap E'(\Omega, \mathbb{C}) \]
and let \( \varphi \) be a limiting Carleman weight for the semiclassical Laplacian on \( \tilde{\Omega} \). If \( h > 0 \) is small enough, then for any \( v \in H^{-m}(\Omega) \), there is a solution \( u \in H^m(\Omega) \) of the equation
\[ e^{\varphi/h}(h^{2m}L_{\lambda,q})e^{-\varphi/h}u = v \quad \text{in} \quad \Omega, \]
which satisfies
\[ \|u\|_{H^m(\Omega)} \lesssim \frac{1}{h^m} \|v\|_{H^{-m}(\Omega)}. \]

Proof. Let \( v \in H^{-m}(\Omega) \) and let us consider the following complex linear functional
\[ L : L^\infty_{\varphi}(\Omega) \to \mathbb{C}, \quad L^\infty_{\varphi,w}w \mapsto (w, \varphi)_{\Omega}. \]
By the Carleman estimate (2.1) for \( L^\infty_{\varphi} \), the map \( L \) is well-defined. Let \( w \in C^\infty(\Omega) \). Then we have
\[ |L(L^\infty_{\varphi}w)| = |(w, \varphi)_{\Omega}| \leq \|w\|_{H^m(\mathbb{R}^n)} \|v\|_{H^{-m}(\Omega)} \leq \frac{C}{h^m} \|v\|_{H^{-m}(\Omega)} \|L^\infty_{\varphi}w\|_{H^m(\mathbb{R}^n)}. \]
By the Hahn–Banach theorem, we may extend \( L \) to a linear continuous functional \( \tilde{L} \) on \( H^{-m}(\mathbb{R}^n) \), without increasing its norm. By the Riesz representation theorem, there exists \( u \in H^m(\mathbb{R}^n) \) such that for all \( \psi \in H^{-m}(\mathbb{R}^n) \),
\[ \tilde{L}(\psi) = \langle \psi, \pi \rangle_{\mathbb{R}^n}, \quad \text{and} \quad \|u\|_{H^m(\mathbb{R}^n)} \leq \frac{C}{h^m} \|v\|_{H^{-m}(\Omega)}. \]
Let us now show that \( L^\infty_{\varphi}u = v \) in \( \Omega \). To that end, let \( w \in C^\infty(\Omega) \). Then
\[ \langle L^\infty_{\varphi}u, \varphi \rangle_{\Omega} = \langle u, L^\infty_{\varphi}w \rangle_{\mathbb{R}^n} = \tilde{L}(L^\infty_{\varphi}w) = (w, \varphi)_{\Omega} = (v, \varphi)_{\Omega}. \]
The proof is complete. \( \square \)

Our next goal is to construct the complex geometric optics solutions for the equation
\[ L_{\lambda,q}u = 0 \quad \text{in} \quad \Omega \quad \text{with} \quad A \in W^{-\frac{n-1}{2}}(\mathbb{R}^n, \mathbb{C}^n) \cap E'(\Omega, \mathbb{C}^n) \quad \text{and} \quad q \in W^{-\frac{n}{2}}(\mathbb{R}^n, \mathbb{C}) \cap E'(\Omega, \mathbb{C}) \]
using the solvability result proposition 2.4. Complex geometric optics solutions are the solutions of the following form
\[ u(x, \zeta; h) = e^{i\zeta x + h^{m/2}r(\zeta, \zeta; h)}, \quad \zeta \in \mathbb{C}^n \quad \text{such that} \quad \zeta \cdot \zeta = 0, \quad |\zeta| \sim 1, \quad a \in C^\infty(\Omega) \text{ is an amplitude,} \quad r \text{ is a correction term,} \quad \text{and} \quad h > 0 \text{ is a small parameter.} \]
Let us conjugate \( h^{2m}L_{\lambda,q} \) by \( e^{i\zeta/h} \). We have
\[ e^{i\zeta/h}h^{2m}L_{\lambda,q}e^{-i\zeta/h} = (-h^2\Delta a - 2i\zeta \cdot \nabla a) + h^{2m}D_a + h^{2m-1}m_a + h^{2m}m_{\lambda,q}. \]
as follows
\[ e^{\text{i}z\xi}h^{2m}L_{A,q}e^{\text{i}z\xi} = (-h^2\Delta - 2i\xi \cdot h\nabla - 2i\xi \cdot h\nabla)^m + h^{2m}D_a + h^{2m-1}m_A(\xi_0 + \xi) + h^{2m}m_a. \]

Then (2.5) is a solution to \( L_{A,q}u = 0 \) if and only if
\[ e^{\text{i}z\xi}h^{2m}L_{A,q}(e^{\text{i}z\xi}h^{m/2}r) = -e^{\text{i}z\xi}h^{2m}L_{A,q}(e^{\text{i}z\xi}a) \]
\[ = -\sum_{k=0}^{m} \frac{m!}{k!(m-k)!}(-h^2\Delta - 2i\xi \cdot h\nabla)^m a - (2i\xi \cdot h\nabla)^m (-2i\xi_0 \cdot h\nabla)a \]
\[ - h^{2m}D_a - h^{2m-1}m_A(\xi_0 + \xi)(a) - h^{2m}m_a(a) \quad \text{in} \; \Omega. \] (2.7)

If \( a \in C^\infty(\Omega) \) satisfies
\[ (\xi_0 \cdot \nabla)^ka = 0 \quad \text{in} \; \Omega \]
for some \( k_0 \geq 1 \) integer, then, using the fact that \( \xi_0 = O(h) \), one can show that the lowest order of \( h \) on the right-hand side of (2.7) is \( k_0 - 1 + 2(m - k_0 + 1) = 2m - k_0 + 1 \). In order to get
\[ \|e^{\text{i}z\xi}h^{2m}L_{A,q}(e^{\text{i}z\xi}a)\|_{H^m(\Omega)} \leq O(h^{m+m/2}), \]
we should choose \( k_0 \) satisfying \( 2m - k_0 + 1 \geq m + m/2 \) and hence \( k_0 \leq (m + 2)/2 \). Since \( m \geq 2 \), we should choose \( a \in C^\infty(\Omega) \), satisfying the following transport equation,
\[ (\xi_0 \cdot \nabla)^ka = 0 \quad \text{in} \; \Omega. \] (2.8)

The choice of such \( a \) is clearly possible. Having chosen the amplitude \( a \) in this way, we obtain the following equation for \( r \),
\[ e^{\text{i}z\xi}h^{2m}L_{A,q}(e^{\text{i}z\xi}h^{m/2}r) = -e^{\text{i}z\xi}h^{2m}L_{A,q}(e^{\text{i}z\xi}a) \]
\[ = -(-h^2\Delta - 2i\xi \cdot h\nabla)^m a - (2i\xi \cdot h\nabla)^m (-2i\xi_0 \cdot h\nabla)a \]
\[ - h^{2m}D_a - h^{2m-1}m_A(\xi_0 + \xi)(a) - h^{2m}m_a(a) = g \quad \text{in} \; \Omega. \] (2.9)

Notice that \( g \) belongs to \( H^m(\Omega) \) and we would like to estimate \( \|g\|_{H^m(\Omega)} \). To that end, we let \( \psi \in C^\infty_0(\Omega) \) such that \( \psi = 0 \). Then using the fact that \( \xi_0 = O(h) \), we get by the Cauchy-Schwarz inequality,
\[ |((-h^2\Delta - 2i\xi \cdot h\nabla)^m a, \psi)|_{L^2(\Omega)} | + |((-h^2\Delta - 2i\xi \cdot h\nabla)^m (-2i\xi_0 \cdot h\nabla)a, \psi)|_{L^2(\Omega)} | \leq O(h^{2m-1}) \|\psi\|_{L^2(\Omega)} \leq O(h^{2m-1}) \|\psi\|_{H^m(\Omega)}. \] (2.10)

In the case \( m \geq 3 \), we use proposition 2.2 with \( p = q = 2 \), \( s_1 = \frac{m-2}{2} \) and \( s_2 = \frac{m}{2} \), to get
\[ |(h^{2m-1}m_A(\xi_0 + \xi)(a), \psi)|_{L^2(\Omega)} \leq Ch^{2m-1}\|A\|_{W^{-\frac{3}{2},\frac{3}{2}}(\mathbb{R}^r)} \|A\|_{W^{-\frac{3}{2},\frac{3}{2}}(\mathbb{R}^r)} \|\psi\|_{H^m(\Omega)} \leq O(h^{2m-1}) \|\psi\|_{H^m(\Omega)} \leq O(h^{2m-1}) \|\psi\|_{H^m(\Omega)}. \]
\[ \leq O(h^{2m}) \|\psi\|_{H^m(\Omega)}. \]
When $m = 2$, we use Hölder’s inequality, and obtain
\[
|h^{m}m_{A(\zeta_{0} + \zeta_{1})}(a), \psi|_{\Omega} | \leq C h^{3} \| A \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \| a \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \\
\leq O(h^{3}) \| a \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \| \varphi \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \\
\leq O(h^{3}) \| \varphi \|_{L^{\frac{1}{\alpha}}(\Omega)} \leq O(h^{3}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)}.
\]
Therefore, for $m \geq 2$, we get
\[
|h^{m-1}m_{A(\zeta_{0} + \zeta_{1})}(a), \psi|_{\Omega} | \leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)}.
\]
Similarly, in the case $m \geq 3$, we use proposition 2.2 with $p = q = 2, s_{1} = \frac{m - 2}{2}$ and $s_{2} = \frac{m}{2}$, to get
\[
|h^{m}D_{a}(a), \psi|_{\Omega} | \leq C h^{3} \| A \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \| \varphi \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \\
\leq O(h^{3}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)} \leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)} \\
\leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)},
\]
when $m = 2$, we again use Hölder’s inequality and Sobolev embeddings $H^{1}(\mathbb{R}^{n}) \subset L^{\frac{2n}{n-2}}(\mathbb{R}^{n})$ (see [31, chapter 13, proposition 6.4]), and obtain
\[
|h^{m}D_{a}(a), \psi|_{\Omega} | \leq h^{3} \| A \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \| \varphi \|_{L^{\frac{2}{3}}(\mathbb{R}^{n})} \\
\leq O(h^{3}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)} \leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)}.
\]

Therefore, for $m \geq 2$, we get
\[
|h^{m-1}D_{a}(a), \psi|_{\Omega} | \leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)}.
\]
Finally, using proposition 2.2, we show that
\[
|h^{m}m_{q}(a), \psi|_{\Omega} | \leq h^{2m} \| q \|_{W^{\frac{2}{3}, \frac{3}{2}}(\mathbb{R}^{n})} \| a \|_{W^{\frac{2}{3}, \frac{3}{2}}(\mathbb{R}^{n})} \\
\leq O(h^{2m}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)} \leq O(h^{2}) \| \varphi \|_{H^{1}_{\alpha}(\Omega)}.
\]
Thus, combining this together with the estimates (2.10)–(2.12) in (2.9), and using (2.3) and $m \geq 2$, we can conclude that
\[
\| g \|_{H^{1}_{\alpha}(\Omega)} \leq O(h^{2}) \leq O(h^{m+\frac{m}{2}}).
\]

Thanks to this and proposition 2.4, for $h > 0$ small enough, there exists a solution $r \in H^{m}(\Omega)$ of (2.9) such that
\[
\| h^{m/2} r \|_{H^{1}_{\alpha}(\Omega)} \lesssim \frac{1}{h^{m}} \| \varphi \|_{H^{\alpha}(\Omega)} \| h^{2m} \mathcal{L}_{A, \varphi}(h^{2m} a) \|_{L^{\frac{2}{3}}(\Omega)} = \frac{1}{h^{m}} \| g \|_{H^{\alpha}(\Omega)} \lesssim h^{m/2}.
\]
Therefore, $\| r \|_{H^{1}_{\alpha}(\Omega)} = O(1)$. The discussion of this section can be summarized in the following proposition.

**Proposition 2.5.** Let $\Omega \subset \mathbb{R}^{n}, n \geq 3$, be a bounded open set with $C^{\infty}$ boundary, and let $m \geq 2$ be an integer such that $2n > m$. Suppose that $A \in W^{\frac{2}{3}, \frac{3}{2}}(\mathbb{R}^{n}, \mathbb{C}^{n}) \cap \mathcal{E}(\Omega)$ and $q \in W^{\frac{2}{3}, \frac{3}{2}}(\mathbb{R}^{n}, \mathbb{C}) \cap \mathcal{E}(\Omega)$, and let $\zeta \in \mathbb{C}^{n}$ be such that $\zeta \cdot \zeta = 0, \zeta = \zeta_{0} + \zeta_{1}$ with $\zeta_{0}$ being independent of $h > 0, |\text{Re} \zeta_{0}| = |\text{Im} \zeta_{0}| = 1$, and $\zeta_{1} = O(h)$ as $h \to 0$. Then for all $h > 0$ small enough, there exists a solution $u(x, \zeta; h) \in H^{m}(\Omega)$ to the equation $\mathcal{L}_{A, \varphi} u = 0$ in $\Omega$, of the form
where the function \( a(\cdot, \zeta) \in C^\infty(\Omega) \) satisfies (2.8) and the remainder term \( r \) is such that \( \|r\|_{\mathcal{H}_0^2(\Omega)} = \mathcal{O}(1) \) as \( h \to 0 \).

3. Proof of theorem 1.1

The first ingredient in the proof of theorem 1.1 is a standard reduction to a larger domain; see [30]. For the proof we follow [19, proposition 3.2] and [27, lemma 4.2].

Proposition 3.1. Let \( \Omega, \Omega' \subset \mathbb{R}^n \) be two bounded opens sets such that \( \Omega \subset \subset \Omega' \) and \( \partial \Omega \) being \( C^\infty \). Let \( A_1, A_2 \in W^{\cdot, \frac{n-1}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega) \) and \( q_1, q_2 \in W^{\cdot, \frac{n-1}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega) \). If \( \mathcal{N}_{A_{1, q_1}} = \mathcal{N}_{A_{1, q_1}} \), then \( \mathcal{N}_{A_{1, q_1}} = \mathcal{N}_{A_{1, q_1}} \), where \( \mathcal{N}_{A_{1, q_1}} \) denotes the set of the Dirichlet-to-Neumann map for \( \mathcal{L}_{A_{1, q_1}} \) in \( \Omega' \), \( j = 1, 2 \).

Proof. Let \( f' \in \prod_{j=0}^{n-1} H^{m-1/2}(\partial \Omega') \) and let \( u'_1 \in H^m(\Omega') \) be a unique solution to \( \mathcal{L}_{A_{1, q_1}} u'_1 = 0 \) in \( \Omega' \) with \( \gamma u'_1 = f' \) on \( \partial \Omega' \), where \( \gamma \) denotes the Dirichlet trace on \( \partial \Omega' \). Let \( u_1 = u'_1 \mid \Omega \in H^m(\Omega) \) and \( f = \gamma u_1 \). Since \( \mathcal{N}_{A_{1, q_1}} = \mathcal{N}_{A_{1, q_1}} \), we can guarantee the existence of \( u_2 \in H^m(\Omega) \) satisfying \( \mathcal{L}_{A_{1, q_1}} u_2 = 0 \) and \( \gamma u_2 = f \). In particular \( \varphi := u_2 - u_1 \in H^m_0(\Omega) \subset H^m(\Omega) \). We define

\[ u'_2 = u'_1 + \varphi \in H^m(\Omega'), \]  

(3.1)

so that we have \( u'_2 = u_2 \) in \( \Omega \). It follows that \( \gamma' u'_2 = \gamma' u'_1 = f' \) on \( \partial \Omega' \).

Next, we show that \( u'_2 \) satisfies \( \mathcal{L}_{A_{1, q_1}} u'_2 = 0 \) in \( \Omega' \). For this, let \( \psi \in C^\infty(\Omega') \). Then we have

\[ \langle \mathcal{L}_{A_{1, q_1}} u'_2, \psi \rangle_{\Omega'} = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u'_2, D^\alpha \psi \rangle_{L^2(\Omega')} + \langle D_{A_1}(u'_2), \psi \rangle_{\Omega'} + \langle m_{\alpha}(u'_1), \psi \rangle_{\Omega'}. \]

Since \( A_2 = 0 \) and \( q_2 = 0 \) outside of \( \Omega \), by (3.1), with \( \varphi \in H^m_0(\Omega) \), we can rewrite the above equality as

\[ \langle \mathcal{L}_{A_{1, q_1}} u'_2, \psi \rangle_{\Omega'} = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u'_1, D^\alpha \psi \rangle_{L^2(\Omega')} + \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha \varphi, D^\alpha (\psi\mid \Omega) \rangle_{L^2(\Omega')} \]

\[ + B_{A_2}(u'_2, \psi_{\mid \Omega'}) + b_{q_2}(u'_2, \psi_{\mid \Omega'}) \]

\[ = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u_1, D^\alpha \psi \rangle_{L^2(\Omega')} - \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u, D^\alpha (\psi\mid \Omega) \rangle_{L^2(\Omega')} \]

\[ + \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u_2, D^\alpha (\psi\mid \Omega) \rangle_{L^2(\Omega')} + B_{A_2}(u_2, \psi_{\mid \Omega'}) \]

\[ + b_{q_2}(u_2, \psi_{\mid \Omega'}). \]

Note that

\[ \langle \mathcal{N}_{A_{2, q_2}} f, \gamma (\psi\mid \Omega) \rangle_{\Omega} = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \langle D^\alpha u_2, D^\alpha (\psi\mid \Omega) \rangle_{L^2(\Omega')} \]

\[ + B_{A_2}(u_2, \psi_{\mid \Omega'}) + b_{q_2}(u_2, \psi_{\mid \Omega'}). \]
Therefore, we have

\[
\langle \mathcal{L}_{A_2, q_2} u_2', \psi \rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \langle D^\alpha u_1', \nabla^\alpha \psi \rangle_{L^2(\Omega')} = \mathcal{A} + \mathcal{N}_{A_2, q_2} f, \psi \rangle_{\partial \Omega'}. \]

Since \( \mathcal{N}_{A_1, q_1} = \mathcal{N}_{A_2, q_2} \) and since \( \langle \mathcal{N}_{A_1, q_1} f, \gamma(\psi\vert_{\partial \Omega}) \rangle = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \langle D^\alpha u_1, \nabla^\alpha (\psi\vert_{\partial \Omega}) \rangle_{L^2(\Omega)} + B_{A_1}(u_1, \psi\vert_{\partial \Omega}) + b_{q_1}(u_1, \psi\vert_{\partial \Omega}), \)

we come to

\[
\langle \mathcal{L}_{A_2, q_2} u_2', \psi \rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \langle D^\alpha u_1', \nabla^\alpha \psi \rangle_{L^2(\Omega')} + B_{A_1}(u_1, \psi\vert_{\partial \Omega}) + b_{q_1}(u_1, \psi\vert_{\partial \Omega}).
\]

Using that \( A_1 = 0 \) and \( q_1 = 0 \) outside \( \Omega \), we obtain

\[
\langle \mathcal{L}_{A_2, q_2} u_2', \psi \rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \langle D^\alpha u_1', \nabla^\alpha \psi \rangle_{L^2(\Omega')} + \langle D_{A_1}(u_1)', \psi \rangle_{\Omega} + \langle m_{q_1}(u_1)', \psi \rangle_{\Omega}
\]

\[
= \langle \mathcal{L}_{A_2, q_2} u_1, \psi \rangle_{\Omega'} = 0.
\]

This shows that \( \mathcal{L}_{A_2, q_2} u_2' = 0 \) in \( \Omega' \).

Using the analogous arguments one can show that \( \mathcal{N}_{A_1, q_1} f' = \mathcal{N}_{A_2, q_2} f' \) on \( \partial \Omega' \), which finishes the proof.

The second ingredient is the derivation of the following integral identity based on the assumption that \( \mathcal{N}_{A_1, q_1} = \mathcal{N}_{A_2, q_2} \).

**Proposition 3.2.** Let \( \Omega \subset \mathbb{R}^n, n \geqslant 3 \), be a bounded open set with \( C^\infty \) boundary. Assume that \( A_1, A_2 \in W^{\frac{n}{2}, \infty}((\mathbb{R}^n, C) \cap E'(\Omega)) \) and \( q_1, q_2 \in W^{\frac{n}{2}, \infty}(\mathbb{R}^n, C) \cap E'(\Omega) \). If \( \mathcal{N}_{A_1, q_1} = \mathcal{N}_{A_2, q_2} \), then the following integral identity holds

\[
0 = \mathcal{B}_{A_2} - \mathcal{B}_{A_1}(u_1, \bar{\alpha}_2) + \mathcal{B}_{A_2} - \mathcal{B}_{A_1}(u_1, \bar{\alpha}_2)
\]

\[
= \langle D_{A_2} - D_{A_1}(u_1), \bar{\alpha}_2 \rangle_{\Omega} + \langle m_{q_1}(u_1), \bar{\alpha}_2 \rangle_{\Omega}, \tag{3.2}
\]

for any \( u_1, u_2 \in H^m(\Omega) \) satisfying \( \mathcal{L}_{A_2, q_2} u_1 = 0 \) in \( \Omega \) and \( \mathcal{L}_{A_2, q_2} u_2 = 0 \) in \( \Omega \), respectively.

Recall that \( \mathcal{L}_{A, q}^* = \mathcal{L}_{\pi, \pi': D, \pi} \) is the formal adjoint of \( \mathcal{L}_{A, q} \).

**Proof.** Since \( u_2 \in H^m(\Omega) \) satisfies \( \mathcal{L}_{A_2, q_2} - D_{A_2} \bar{\alpha}_2 = 0 \), the following

\[
0 = \langle \mathcal{L}_{A_2, q_2} - D_{A_2} \bar{\alpha}_2, \psi \rangle_{\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \langle D^\alpha u_2, \nabla^\alpha \psi \rangle_{L^2(\Omega)} - \langle D_{A_1}(\bar{\alpha}_2), \psi \rangle_{\Omega} + \langle m_{q_2}(u_2), \psi \rangle_{\Omega}, \tag{3.3}
\]

holds for every \( \psi \in C_0^\infty(\Omega) \). Density and continuity imply that (3.3) holds also for all \( \psi \in H_0^m(\Omega) \).

The hypothesis that \( \mathcal{N}_{A_1, q_1} = \mathcal{N}_{A_2, q_2} \) implies the existence of \( v_2 \in H^m(\Omega) \) satisfying \( \mathcal{L}_{A_2, q_2} v_2 = 0 \) in \( \Omega \) such
Then \( \psi = u_1 - v_2 \in H^m_0(\Omega) \). Hence, applying (3.3) we obtain
\[
0 = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \left( D^\alpha (u_1 - v_2), (u_1 - v_2) \right)_{\Omega} + \langle m_{q_1 - D A_q}(\pi_2), (u_1 - v_2) \rangle_{\Omega}.
\]
(3.4)

The equality \( \langle N_{A_{q_1}} \gamma_1, \gamma \bar{\pi}_2 \rangle_{\partial \Omega} = \langle N_{A_{q_2}} \gamma_2, \gamma \bar{\pi}_2 \rangle_{\partial \Omega} \) together with the definition (1.2) gives
\[
0 = \sum_{|\alpha| = m} \frac{m!}{\alpha!} \left( D^\alpha (u_1 - v_2), D^\alpha \pi_2 \right)_{\Omega} - (B_{A_q}(u_1, \pi_2) - B_{A_q}(v_2, \pi_2))
\]
\[
+ b_{q_1}(u_1, \pi_2) - b_{q_2}(v_2, \pi_2).
\]
Combining this with (3.4) and using proposition A.4, we derive the integral identity (3.5) as desired.

Let \( A_1, A_2 \in W^{-\frac{m+1}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}(\overline{\Omega}) \) and \( q_1, q_2 \in W^{\frac{3}{2}-\frac{m+1}{2}}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}(\overline{\Omega}) \) as in the statement of theorem 1.1. In order to show that \( A_1 = A_2 \), we will need to use the Poincaré lemma for currents [25]. We need this reduction to apply the Poincaré lemma for currents, which requires the domain to be simply connected. Therefore, we reduce the problem to a larger simply connected domain. In particular, to a ball.

Let \( B \) be an open ball in \( \mathbb{R}^n \) such that \( \Omega \subset B \). According to proposition 3.1, we know that \( N_{A_{q_1}} = N_{A_{q_2}} \), where \( N_{A_{q_j}} \) denotes the Dirichlet-to-Neumann map for \( L_{A_{q_j}} \) in \( B \), \( j = 1, 2 \). Now, by proposition 3.2, the following integral identity holds
\[
B_{A_{q_1}}(u_1, \pi_2) + B_{q_2}(u_1, \pi_2) = 0
\]
(3.5)
for any \( u_1, u_2 \in H^m(B) \) satisfying \( L_{A_{q_1}} u_1 = 0 \) in \( B \) and \( L_{A_{q_2}} u_2 = 0 \) in \( B \), respectively. Here and in what follows, by \( B_{A_{q_1}} \) and \( B_{q_2} \) we denote the bilinear forms corresponding to \( A_1 - A_2 \) and \( q_1 - q_2 \), respectively, defined (by means of (A.1)) in the ball \( B \).

The main idea of the proof of theorem 1.1 is to use the integral identity (3.5) with \( u_1, u_2 \in H^m(B) \) being complex geometric optics solutions for the equations \( L_{A_{q_1}} u_1 = 0 \) in \( B \) and \( L_{A_{q_2}} u_2 = 0 \) in \( B \), respectively. In order to construct these solutions, consider \( \xi, \mu_1, \mu_2 \in \mathbb{R}^n \) such that \( |\mu_1| = |\mu_2| = 1 \) and \( \mu_1 \cdot \mu_2 = \xi \cdot \mu_1 = \xi \cdot \mu_2 = 0 \). For \( h > 0 \), we set
\[
\zeta_j = \frac{h \xi}{2} + \sqrt{1 - h^2 |\xi|^2} \mu_1 + i \mu_2, \quad \zeta_2 = -\frac{h \xi}{2} + \sqrt{1 - h^2 |\xi|^2} \mu_1 - i \mu_2.
\]
So we have \( \zeta_j, \zeta_j = 0, j = 1, 2 \), and \( \zeta_1 - \zeta_2 = h \xi \).

By proposition 2.5, for all \( h > 0 \) small enough, there are solutions \( u_1(\cdot, \zeta_1; h) \) and \( u_2(\cdot, \zeta_2; h) \) in \( H^m(B) \) to the equations
\[
L_{A_{q_1}} u_1 = 0 \text{ in } B \quad \text{and} \quad L_{A_{q_2}}^* u_2 = 0 \text{ in } B,
\]
respectively, of the form
\[
u_1(x, \zeta_1; h) = e^{ih \zeta_1}(a_1(\mu_1 + i \mu_2) + h^{m/2} r_1(x, \zeta_1; h))
\]
\[
u_2(x, \zeta_2; h) = e^{ih \zeta_2}(a_2(\mu_1 - i \mu_2) + h^{m/2} r_2(x, \zeta_2; h)),
\]
(3.6)
where the amplitudes \( a_1(\mu_1 + i \mu_2), a_2(\mu_1 - i \mu_2) \in C^\infty(B) \) satisfy the transport equations
and
\[(\mu_1 + i\mu_2) \cdot \nabla^2 a_1(x, \mu_1 + i\mu_2) = 0, \quad \text{in} \quad B, \quad (3.7)\]
and
\[(\mu_1 - i\mu_2) \cdot \nabla^2 a_2(x, \mu_1 - i\mu_2) = 0, \quad \text{in} \quad B, \quad (3.8)\]
and the remainder terms \(r_1(\cdot, \zeta_1; h)\) and \(r_2(\cdot, \zeta_2; h)\) satisfy
\[
\|r_j\|_{H^m(B)} = O(1), \quad j = 1, 2. \quad (3.9)
\]
We substitute \(u_1\) and \(u_2\) given by (3.6) into (3.5), and get
\[
0 = \frac{1}{h} b^B_{\mu_1 + i\mu_2} (A_1 - A_2) (a_1 + h^{m/2} r_1, e^{i\xi}(\sigma_2 + h^{m/2} r_2)) \\
+ B^B_{\mu_1 - i\mu_2} (a_1 + h^{m/2} r_1, e^{i\xi}(\sigma_2 + h^{m/2} r_2)) \\
+ b^B_{\mu_2 - \mu_1} (a_1 + h^{m/2} r_1, e^{i\xi}(\sigma_2 + h^{m/2} r_2)). \quad (3.10)
\]
Multiplying this by \(h\) and letting \(h \to 0\), we obtain that
\[
b^B_{(\mu_1 + i\mu_2)} (A_1 - A_2) (a_1, e^{i\xi} \sigma_2) = 0. \quad (3.11)
\]
Here we have used (3.9), proposition A.2 and the fact that \(a_1, a_2 \in C^\infty(B)\) to conclude that
\[
|b^B_{\mu_1 + i\mu_2} (A_1 - A_2) (a_1 + h^{m/2} r_1, e^{i\xi}(\sigma_2 + h^{m/2} r_2))| \\
\leq C |A_1 - A_2|_W \cdot \|a_1 + h^{m/2} r_1\|_{H^m(B)} \|\sigma_2 + h^{m/2} r_2\|_{H^m(B)} \\
\leq C (\|a_1\|_{H^m(B)} + \|r_1\|_{H^m(B)})(\|\sigma_2\|_{H^m(B)} + \|r_2\|_{H^m(B)}) \leq O(1).
\]
and
\[
|b^B_{\mu_2 - \mu_1} (a_1 + h r_1, e^{i\xi}(\sigma_2 + h r_2))| \\
\leq C |q_1 - q_2|_W \cdot \|a_1 + h^{m/2} r_1\|_{H^m(B)} \|\sigma_2 + h^{m/2} r_2\|_{H^m(B)} \\
\leq C (\|a_1\|_{H^m(B)} + \|r_1\|_{H^m(B)})(\|\sigma_2\|_{H^m(B)} + \|r_2\|_{H^m(B)}) \leq O(1).
\]
Substituting \(a_1 = a_2 = 1\) in (3.11), we obtain
\[
((\mu_1 + i\mu_2) \cdot (A_1 - A_2), e^{i\xi})_B = 0. \quad (3.12)
\]
This implies that
\[
((\mu_1 + i\mu_2) \cdot (\tilde{A}_1(\xi) - \tilde{A}_2(\xi)) = 0, \quad \text{for all} \quad \mu, \xi \in \mathbb{R}^n, \quad \mu \cdot \xi = 0, \quad (3.13)
\]
where \(\tilde{A}_j\) stands for the Fourier transform of \(A_j, j = 1, 2\). It follows from (3.13) that
\[
\partial_{\xi_j}(\tilde{A}_{1,k} - \tilde{A}_{2,k}) - \partial_{\xi_k}(\tilde{A}_{1,j} - \tilde{A}_{2,j}) = 0 \quad \text{in} \quad \Omega, \quad 1 \leq j, k \leq n, \quad (3.14)
\]
in the sense of distributions. Indeed, for each \(\xi = (\xi_1, \ldots, \xi_n)\) and for \(j = k, 1 \leq j, k \leq n\), consider the vector \(\mu = \mu(\xi, j, k)\) such that \(\mu_j = -\xi_j, \mu_k = \xi_k\) and all other components are equal to zero. Therefore, \(\mu\) satisfy \(\mu \cdot \xi = 0\). Hence, using (3.13) we obtain
\[
\xi_j \cdot (\tilde{A}_{1,k}(\xi) - \tilde{A}_{2,k}(\xi)) - \xi_k \cdot (\tilde{A}_{1,j}(\xi) - \tilde{A}_{2,j}(\xi)) = 0,
\]
which proves (3.14) in the sense of distributions.

Our goal is to show that \(A_1 = A_2\). Considering \(A_1 - A_2\) as a 1-current and using the Poincaré lemma for currents, we conclude that there is \(\psi \in D'(\mathbb{R}^n)\) such that \(d\psi = A_1 - A_2 \in W^{-\frac{m}{2} - 1} - \hat{\mathcal{E}}(\mathbb{R}^n, C^\infty) \cap \mathcal{E}'(\mathbb{R}^n)\); see [25]. Note that \(\psi\) is a constant, say
c ∈ C, outside B since A₁ − A₂ = 0 in \( \mathbb{R}^n \setminus B \) (and also near \( \partial B \)). Considering \( \psi = c \) instead of \( \psi \), we may and shall assume that \( \psi ∈ \mathcal{E}(B, C) \).

To show that \( A₁ = A₂ \), consider (3.11) with \( a₂(\cdot, μ₁ − iμ₂) = 1 \) and \( a₁(\cdot, μ₁ + iμ₂) \) satisfying

\[
((μ₁ + iμ₂) \cdot \nabla)̂a₁(x, μ₁ + iμ₂) = 1 \text{ in } B. \tag{3.15}
\]

The latter choice is possible thanks to (3.7), (3.8) and the assumption that \( m ≥ 2 \). The equation (3.15) is just an inhomogeneous \( \bar{\partial} \)-equation and one can solve it by setting

\[
a₁(x, μ₁ + iμ₂) = \frac{1}{2π} \int_{\mathbb{R}^2} \frac{χ(x - x₁μ₁ - x₂μ₂)}{y₁ + iy₂} \, dy₁ \, dy₂,
\]

where \( χ ∈ C_0^∞(\mathbb{R}^n) \) is such that \( χ ≡ 1 \) near \( \bar{B} \); see [27, lemma 4.6].

From (3.11), we have

\[
h^B_{(μ₁ + iμ₂) \nabla} (a₁, e^{ivξ}) = 0.
\]

Using the fact that \( μ₁ ⋅ ξ = μ₂ ⋅ ξ = 0 \), we obtain

\[
0 = -h^B_{(μ₁ + iμ₂) \nabla} (a₁, e^{ivξ}) = -((μ₁ + iμ₂) ⋅ \nabla)\hat{ψ}, e^{ivξ}a₁)_{B}
\]

\[
= (\hat{ψ}, e^{ivξ}(μ₁ + iμ₂) ⋅ \nabla a₁)_{B} = (\hat{ψ}, e^{ivξ})_{B}.
\]

This gives \( \hat{ψ} = 0 \), and hence we have \( ψ = 0 \) in \( B \), which completes the proof of \( A₁ = A₂ \).

To show that \( q₁ = q₂ \), we substitute \( A₁ = A₂ \) and \( a₁ = a₂ = 1 \) into the identity (3.10) and obtain

\[
h^B_{q₁ - a₁} (1 + h^{m/2}τ₁, (1 + h^{m/2}τ₂)e^{ivξ}) = 0.
\]

Letting \( h → 0^+ \), we get \( \hat{q}_1(ξ) = \hat{q}_2(ξ) = 0 \) for all \( ξ ∈ \mathbb{R}^n \). This implies that \( q₁ = q₂ \) in \( B \) completing the proof of theorem 1.1.

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Appendix A. Mapping properties of \( DA \) and \( mq \)

Let \( Ω ⊂ \mathbb{R}^n, n ≥ 3 \), be a bounded open set with \( C^∞ \) boundary, and \( m ≥ 2 \) be an integer such that \( 2n > m \). Let \( A ∈ W^{-\frac{m}{2}, \frac{m}{2}}(Ω, C^n) \cap \mathcal{E}(Ω, C^n) \) and \( q ∈ W^{−\frac{m}{2}, \frac{m}{2}}(Ω, C) \cap \mathcal{E}(Ω, C) \), where \( W^{s,p}(Ω) \) is the standard \( L^p \)-based Sobolev space on \( \mathbb{R}^n, s ∈ \mathbb{R} \) and \( 1 < p < ∞ \). The reader is referred to [33] for properties of these spaces.

We start with considering the bilinear forms \( B^{m}_{A} \) and \( B^{m}_{q} \) on \( H^m(\mathbb{R}^n) \) which are defined by

\[
B^{m}_{A}(u, v) = ⟨A, vDu⟩_{\mathbb{R}^n}, \quad B^{m}_{q}(u, v) = ⟨q, uv⟩_{\mathbb{R}^n}, \quad u, v ∈ H^m(\mathbb{R}^n).
\]

The following result shows that the forms \( B^{m}_{A} \) and \( B^{m}_{q} \) are bounded on \( H^m(\mathbb{R}^n) \). The proof is based on a property on multiplication of functions in Sobolev spaces given in proposition 2.2.
Proposition A.1. The bilinear forms $B_{\tilde{A}}^\omega$ and $b_q^\omega$ on $H^m(\mathbb{R}^n)$ are bounded and satisfy
\[
|b_q^\omega(u, v)| \leq C\|q\|_{W^{\omega, p(\mathbb{R}^n)}} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}
\]
and
\[
|B_{\tilde{A}}^\omega(u, v)| \leq C\|A\|_{W^{\omega, p(\mathbb{R}^n)}} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}
\]
for all $u, v \in H^m(\mathbb{R}^n)$.

Proof. First, we give the proof for the form $b_q^\omega$. Using the duality between $W^{\omega, p(\mathbb{R}^n)}$ and $W^{\omega, p(\mathbb{R}^n)}$, and using proposition 2.2, we conclude that for all $u, v \in H^m(\mathbb{R}^n)$
\[
|b_q^\omega(u, v)| = |\langle q, uv \rangle_{\mathbb{R}^n}| \leq \|q\|_{W^{\omega, p(\mathbb{R}^n)}} \|uv\|_{W^{\omega, p(\mathbb{R}^n)}}
\]
\[
\leq C\|q\|_{W^{\omega, p(\mathbb{R}^n)}} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}.
\]
Now, we give the proof for the form $B_{\tilde{A}}^\omega$. In the case $m \geq 3$, using the duality between $W^{\omega, p(\mathbb{R}^n)}$ and $W^{\omega, p(\mathbb{R}^n)}$, and using proposition 2.2 with $p = q = 2, s_1 = \frac{m}{2}$ and $s_2 = \frac{m}{2}$, we conclude that
\[
|B_{\tilde{A}}^\omega(u, v)| = |\langle A, vDu \rangle_{\mathbb{R}^n}| \leq \|A\|_{W^{\omega, p(\mathbb{R}^n)}} \|vDu\|_{W^{\omega, p(\mathbb{R}^n)}}
\]
\[
\leq C\|A\|_{W^{\omega, p(\mathbb{R}^n)}} \|Du\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}
\]
\[
\leq C\|A\|_{W^{\omega, p(\mathbb{R}^n)}} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}.
\]
When $m = 2$, we use Hölder’s inequality and Sobolev embedding $H^2(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n)$ (see [31, chapter 13, proposition 6.4]), and obtain
\[
|B_{\tilde{A}}^\omega(u, v)| = |\langle A, vDu \rangle_{\mathbb{R}^n}| \leq \|A\|_{L^4(\mathbb{R}^n)} \|vDu\|_{L^{\frac{4}{3}}(\mathbb{R}^n)}
\]
\[
\leq \|A\|_{L^4(\mathbb{R}^n)} \|Du\|_{L^2(\mathbb{R}^n)} \|v\|_{L^{\frac{4}{3}}(\mathbb{R}^n)}
\]
\[
\leq C\|A\|_{L^4(\mathbb{R}^n)} \|Du\|_{L^2(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}
\]
\[
\leq C\|A\|_{L^4(\mathbb{R}^n)} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}.
\]
Therefore
\[
|B_{\tilde{A}}^\omega(u, v)| \leq C\|A\|_{W^{\omega, p(\mathbb{R}^n)}} \|u\|_{H^p(\mathbb{R}^n)} \|v\|_{H^p(\mathbb{R}^n)}
\]
for all $u, v \in H^m(\mathbb{R}^n)$. \qed

The bilinear forms $B_{\tilde{A}}$ and $b_q$ on $H^m(\Omega)$, which were defined in the introduction, can be rewritten as
\[
B_{\tilde{A}}(u, v) = B_{\tilde{A}}^\omega(\tilde{u}, \tilde{v}), \quad b_q(u, v) = b_q^\omega(\tilde{u}, \tilde{v}), \quad u, v \in H^m(\Omega)
\]
(A.1)
where $\tilde{u}, \tilde{v} \in H^m(\mathbb{R}^n)$ are extensions of $u$ and $v$, respectively. First, we show that these definitions are well-defined, i.e. independent of the choice of extensions $\tilde{u}, \tilde{v}$. Indeed, let $u_1, u_2 \in H^m(\mathbb{R}^n)$ be such that $u_1 = u_2 = u$ in $\Omega$, and let $v_1, v_2 \in H^m(\mathbb{R}^n)$ be such that $v_1 = v_2 = v$ in $\Omega$. Then we need to show that
\[
B_{\tilde{A}}^\omega(u_1 - u_2, v_1 - v_2) = 0 \quad \text{and} \quad b_q^\omega(u_1 - u_2, v_1 - v_2) = 0.
\]
(A.2)
Since \( A \) and \( q \) are supported in \( \Omega \), for any \( \phi, \psi \in \mathcal{S}(\mathbb{R}^n) \) with \( \phi = \psi = 0 \) in \( \Omega \), we have
\[
B_A^\mathbb{R}^n (\phi, \psi) = \langle A, \psi D\phi \rangle_{\mathbb{R}^n} = 0 \quad \text{and} \quad b_q^\mathbb{R}^n (\phi, \psi) = \langle q, \psi \psi \rangle_{\mathbb{R}^n} = 0.
\]
Since \( \mathcal{S}(\mathbb{R}^n) \) is dense in \( H^m(\mathbb{R}^n) \) and \( B_A^\mathbb{R}^n \) and \( b_q^\mathbb{R}^n \) are continuous bilinear forms (by proposition A.1), we get (A.2).

The next result shows that the bilinear forms \( B_A \) and \( b_q \) are bounded on \( H^m(\Omega) \). This is a consequence of proposition A.1.

**Proposition A.2.** The bilinear forms \( B_A \) and \( b_q \) on \( H^m(\Omega) \) are bounded and satisfy
\[
|b_q(u, v)| \leq C||q||_W \sup_{\tau} ||u||_{H^m(\Omega)} ||v||_{H^m(\Omega)}
\]
and
\[
|B_A(u, v)| \leq C||A||_W \sup_{\tau} ||u||_{H^m(\Omega)} ||v||_{H^m(\Omega)}
\]
for all \( u, v \in H^m(\Omega) \).

**Proof.** Let \( u, v \in H^m(\Omega) \) and let \( \tilde{\Omega} \) be a bounded open neighborhood of \( \Omega \). Then there is a bounded linear map \( E : H^m(\Omega) \rightarrow H^m_0(\tilde{\Omega}) \) such that \( E|_\Omega = \text{Id} \); see [7, theorem 6.44]. Then according to estimates proven in proposition A.1, we obtain
\[
|b_q(u, v)| = |b_q^E(E(u), E(v))| \leq C||q||_W \sup_{\tau} ||E(u)||_{H^m(\tilde{\Omega})} ||E(v)||_{H^m(\tilde{\Omega})}
\]
and
\[
|B_A(u, v)| = |B_A^E(E(u), E(v))| \leq C||A||_W \sup_{\tau} ||E(u)||_{H^m(\tilde{\Omega})} ||E(v)||_{H^m(\tilde{\Omega})}.
\]
These estimates finish the proof. \( \square \)

Now, for \( u \in H^m(\Omega) \), we define \( D_A(u) \) and \( m_q(u) \), for any \( v \in H^m_0(\Omega) \) by
\[
\langle D_A(u), v \rangle_{\Omega} = B_A(u, v) \quad \text{and} \quad \langle m_q(u), v \rangle_{\Omega} = b_q(u, v).
\]

The following result, which is an immediate corollary of proposition A.2, implies that \( D_A, m_q \) are bounded operators from \( H^m(\Omega) \) to \( H^{-m} (\Omega) \). The norm on \( H^{-m}(\Omega) \) is the usual dual norm given by
\[
||y||_{H^{-m}(\Omega)} = \sup_{\phi \in H^m(\Omega)} \frac{\langle y, \phi \rangle_{\Omega}}{||\phi||_{H^m(\Omega)}}.
\]

**Corollary A.3.** The operators \( B_A \) and \( b_q \) are bounded from \( H^m(\Omega) \) to \( H^{-m}(\Omega) \) and satisfy
\[
||m_q(u)||_{H^{-m}(\Omega)} \leq C||q||_W \sup_{\tau} ||u||_{H^m(\Omega)}
\]
and
\[
||D_A(u)||_{H^{-m}(\Omega)} \leq C||A||_W \sup_{\tau} ||u||_{H^m(\Omega)}
\]
for all \( u \in H^m(\Omega) \).

Finally, we record and give the proof of the following useful identities.

**Proposition A.4.** For any \( u, v \in H^m(\Omega) \), the forms \( B_A \) and \( b_q \) satisfy the following identities

\[
B_A(u, v) = -B_A(v, u) = b_{DA}(u, v) \quad \text{and} \quad b_q(u, v) = b_q(v, u).
\]

**Proof.** According to the definition (A.1) and density of \( \mathcal{S}(\mathbb{R}^n) \) in \( H^m(\mathbb{R}^n) \), it is sufficient to prove for the case \( \Omega \subset \mathbb{R}^n \). This follows by straightforward computations

\[
b_q^{\mathbb{R}^n}(u, v) = \langle q, uv \rangle_{\mathbb{R}^n} = \langle m_q(v), u \rangle_{\mathbb{R}^n},
\]

and using product rule

\[
B_A^{\mathbb{R}^n}(u, v) = \langle A, vDu \rangle_{\mathbb{R}^n} = -\langle A, uDv \rangle_{\mathbb{R}^n} + \langle A, D(uv) \rangle_{\mathbb{R}^n}
\]

\[
= -B_A^{\mathbb{R}^n}(v, u) - \langle D \cdot A, uv \rangle_{\mathbb{R}^n}
\]

\[
= -B_A^{\mathbb{R}^n}(v, u) - b_{DA}(u, v).
\]

The proof is thus finished. \( \square \)

**Appendix B. Well-posedness and Dirichlet-to-Neumann map**

Let \( \Omega \subset \mathbb{R}^n, \ n \geq 3, \) be a bounded open set with \( C^\infty \) boundary, and let \( A \in W^{-\frac{m}{2}, 2}(\mathbb{R}^n, \mathbb{C}^n) \cap \mathcal{E}'(\Omega, \mathbb{C}^n) \) and \( q \in W^{-\frac{m}{2}, 2}(\mathbb{R}^n, \mathbb{C}) \cap \mathcal{E}'(\Omega, \mathbb{C}) \), where \( n > m \).

For \( f = (f_0, \ldots, f_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \), consider the Dirichlet problem

\[
\mathcal{L}_{A, q} u = 0 \quad \text{in} \quad \Omega, \quad \gamma u = f \quad \text{on} \quad \partial \Omega. \quad (B.1)
\]

Here, by \( \gamma \) we denote the Dirichlet trace operator, given by

\[
\gamma : H^m(\Omega) \rightarrow \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega), \quad \gamma u = (u|_{\partial \Omega}, \partial_\nu u|_{\partial \Omega}, \ldots, \partial_{\nu}^{m-1} u|_{\partial \Omega}),
\]

which is bounded and surjective, see [10, theorem 9.5, page 226].

First aim of this appendix is to use the standard variational arguments to show the well-posedness of the problem (B.1). First, consider the following inhomogeneous problem

\[
\mathcal{L}_{A, q} u = F \quad \text{in} \quad \Omega, \quad \gamma u = 0 \quad \text{on} \quad \partial \Omega, \quad (B.2)
\]

with \( u \in H^m(\Omega) \).

To define a sesquilinear form \( a \), associated to the problem (B.2), for \( u, v \in C_0^\infty(\Omega) \), we integrate by parts and get

\[
\langle \mathcal{L}_{A, q} u, v \rangle_{\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, \bar{D}^\alpha v)_{L^2(\Omega)} + \langle D_A(u), \bar{v} \rangle_{\Omega} + \langle m_q(u), \bar{v} \rangle_{\Omega} = a(u, v).
\]
Therefore, $a$ is defined on $H^m(\Omega)$ by

$$a(u, v) := \sum_{|\alpha|=m} m! \left( D^\alpha u, D^\alpha v \right)_{L^2(\Omega)} + B_k(u, \tau) + b_q(u, \tau), \quad u, v \in H^m(\Omega).$$

Note that this is not a unique way to define a sesquilinear form associated to the problem (B.2).

Now, we show that the sesquilinear form $a$ can be extended to a bounded form on $H^m_0(\Omega)$. Using duality and proposition A.1, for $u, v \in H^m_0(\Omega)$, we obtain

$$|a(u, v)| \leq \sum_{|\alpha|=m} m! \|D^\alpha u\|_{L^2(\Omega)} \|D^\alpha v\|_{L^2(\Omega)} + \left( |A|_{W^{-\frac{m}{2}}(\mathbb{R}^d)} + |q|_{W^{-\frac{m}{2}}(\mathbb{R}^d)} \right) \|u\|_{H^m(\Omega)} \|v\|_{H^m(\Omega)}$$

$$\leq C \|u\|_{H^m(\Omega)} \|v\|_{H^m(\Omega)}.$$  \hspace{1cm} (B.3)

Thus, the sesquilinear form $a$ is a bounded form on $H^m_0(\Omega)$.

Applying Poincaré’s inequality, we have

$$\|u\|_{H^m(\Omega)} \leq C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)}, \quad u \in H^m_0(\Omega).$$  \hspace{1cm} (B.4)

Write $q = q^0 + (q - q^0)$ with $q^0 \in L^\infty(\Omega, \mathbb{C})$ and $\|q - q^0\|_{W^{-\frac{m}{2}}(\mathbb{R}^d)}$ small enough, and write $A = \tilde{A} + (A - \tilde{A})$ with $\tilde{A} \in L^\infty(\Omega, \mathbb{C})^n$ and $\|A - \tilde{A}\|_{W^{-\frac{m}{2}}(\mathbb{R}^d)}$ small enough. Using (B.4) and proposition A.2, for $\varepsilon > 0$, we obtain that

$$\text{Re } a(u, u) \geq \sum_{|\alpha|=m} m! \|D^\alpha u\|_{L^2(\Omega)}^2 - |B_k(u, u)| - |b_q(u, u)|$$

$$\geq C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)}^2 - |B_k(u, u)| - |b_q(u, u)|$$

$$\geq C \sum_{|\alpha|=m} \|D^\alpha u\|_{L^2(\Omega)}^2 - |A^\dagger|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} - \|q^\dagger\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2$$

$$- C|A - \tilde{A}|_{W^{-\frac{m}{2}}(\mathbb{R}^d)} \|u\|_{H^m(\Omega)}^2 - C\|q - q^0\|_{W^{-\frac{m}{2}}(\mathbb{R}^d)} \|u\|_{H^m(\Omega)}^2$$

$$\geq C \|u\|_{H^m(\Omega)}^2 - |A^\dagger|_{L^\infty(\Omega)} \|Du\|_{L^2(\Omega)}^2 - |A^\dagger|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}^2$$

$$- C\|q - q^0\|_{W^{-\frac{m}{2}}(\mathbb{R}^d)} \|u\|_{H^m(\Omega)}^2,$$ \hspace{1cm} $C, C' > 0, \quad u \in H^m_0(\Omega).$

Taking $\varepsilon > 0$ sufficiently small, we get

$$\text{Re } a(u, u) \geq C \|u\|_{H^m(\Omega)}^2 - C_0 \|u\|_{L^2(\Omega)}^2, \quad C, C_0 > 0, \quad u \in H^m_0(\Omega).$$

Therefore, the form $a$ is coercive on $H^m_0(\Omega)$. As the inclusion map $H^m_0(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the operator

$$L_{A, q} = (-\Delta)^m + D_k + m_q : H^m_0(\Omega) \rightarrow H^{-m}(\Omega) = (H^m_0(\Omega))^\dagger,$$

is Fredholm operator with zero index; see [21, theorem 2.34].

Positivity of the operator $L_{A, q} + C_0 : H^m_0(\Omega) \rightarrow H^{-m}(\Omega)$ and an application of the Lax–Milgram lemma implies that $L_{A, q} + C_0$ has a bounded inverse. By compact Sobolev embedding $H^m_0(\Omega) \hookrightarrow H^{-m}(\Omega)$ and the Fredholm theorem, the equation (B.2) has a unique solution $u \in H^m_0(\Omega)$ for any $F \in H^{-m}(\Omega)$ if one is outside a countable set of eigenvalues.
Now, consider the Dirichlet problem
\[ \mathcal{L}_{A,q}u = 0 \quad \text{in} \quad \Omega, \]
\[ \gamma u = f \quad \text{on} \quad \partial \Omega, \quad (B.5) \]
with \( f = (f_0, \ldots, f_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega). \) We assume that 0 is not in the spectrum of \( \mathcal{L}_{A,q} : H_0^m(\Omega) \to H^{-m}(\Omega). \) By [10, theorem 9.5, page 226], there is \( w \in H^m(\Omega) \) such that \( \gamma w = f. \) According to corollary A.3, we have \( \mathcal{L}_{A,q}w \in H^{-m}(\Omega). \) Therefore, \( u = v + w, \) with \( v \in H_0^m(\Omega) \) being the unique solution of the equation \( \mathcal{L}_{A,q}v = -\mathcal{L}_{A,q}w \in H^{-m}(\Omega), \) is the unique solution of the Dirichlet problem \((B.5).\)

Under the assumption that 0 is not in the spectrum of \( \mathcal{L}_{A,q}, \) the Dirichlet-to-Neumann map \( N_{A,q} \) is defined as follows: let \( f, h \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega). \) Then we set
\[ \langle N_{A,q}f, h \rangle_{(\partial \Omega)} = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, D^\alpha \nu_h)_{L^2(\Omega)} + B_h(u, \nu_h) + b_h(u, \nu_h), \quad (B.6) \]
where \( u \) is the unique solution of the Dirichlet problem \((B.5)\) and \( v_h \in H^m(\Omega) \) is an extension of \( h, \) that is \( \gamma v_h = h. \) In this appendix we show that \( N_{A,q} \) is a well-defined (i.e. independent of the choice of the v_h) bounded operator
\[ N_{A,q} : \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \to \left( \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \right) \to \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial \Omega). \]

Let us first show that the definition \((B.6)\) of \( N_{A,q}f \) is independent of the choice of an extension \( v_h \) of \( h. \) For this, let \( v_{h,1}, v_{h,2} \in H^m(\Omega) \) be such that \( \gamma v_{h,1} = \gamma v_{h,2} = h. \) Note that \( v_{h,1} - v_{h,2} \in H_0^m(\Omega). \) Then we have to show that
\[ \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, D^\alpha (v_{h,1} - v_{h,2}))_{L^2(\Omega)} + \langle D_h(u), (v_{h,1} - v_{h,2}) \rangle_{(\Omega)} + \langle m_q(u), (v_{h,1} - v_{h,2}) \rangle_{(\Omega)} = 0. \quad (B.7) \]
For any \( w \in C_0^\infty(\Omega) \) and for \( u \in H^m(\Omega) \) solution of the Dirichlet problem \((B.5),\) we have
\[ 0 = \langle \mathcal{L}_{A,q}u, w \rangle = \sum_{|\alpha|=m} \frac{m!}{\alpha!} (D^\alpha u, D^\alpha w)_{L^2(\Omega)} + \langle D_h(u), w \rangle_{(\Omega)} + \langle m_q(u), w \rangle_{(\Omega)}. \]
Density of \( C_0^\infty(\Omega) \) in \( H^m_0(\Omega) \) and continuity of the form on \( H^m_0(\Omega) \) give \((B.7).\)

Now we show that \( N_{A,q}f \) is a well-defined element of \( \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial \Omega). \) From \((B.3),\) it follows that
\[ |N_{A,q}u, \tilde{F}|_{(\partial \Omega)} \leq C \|u\|_{H^m(\Omega)} \|v_h\|_{H^m(\Omega)} \leq C \|u\| \prod_{j=0}^{m-1} H^{m-j-1/2}(\Omega), \]
where
\[ \|h\| \prod_{j=0}^{m-1} H^{m-j-1/2}(\Omega) = (\|h_0\|_{H^{m-1/2}(\Omega)} + \cdots + \|h_{m-1}\|_{H^{m-1/2}(\Omega)})^{1/2} \]
is the product norm on the space \( \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega). \) Here we have used the fact that the extension operator \( \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \ni h \mapsto v_h \in H^m(\Omega) \) is bounded, again see [10, theorem 9.5, page 226]. Hence, we have that \( N_{A,q}f \) belongs to \( (\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega))^\prime = \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial \Omega). \) The proof given above also shows that
\[ N_{A,q} : \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \to \left( \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial \Omega) \right) '= \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial \Omega) \]

is bounded.

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