Permutation Invariant Algebras, a Fock Space Realization and the Calogero Model

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Abstract. We study permutation invariant oscillator algebras and their Fock space representations using three equivalent techniques, i.e. (i) a normally ordered expansion in creation and annihilation operators, (ii) the action of annihilation operators on monomial states in Fock space and (iii) Gram matrices of inner products in Fock space. We separately discuss permutation invariant algebras which possess hermitean number operators and permutation invariant algebras which possess non-hermitean number operators. The results of a general analysis are applied to the $S_M$-extended Heisenberg algebra, underlying the $M$-body Calogero model. Particular attention is devoted to the analysis of Gram matrices for the Calogero model. We discuss their structure, eigenvalues and eigenstates. We obtain a general condition for positivity of eigenvalues, meaning that all norms of states in Fock space are positive if this condition is satisfied. We find a universal critical point at which the reduction of the physical degrees of freedom occurs. We construct dual operators,

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leading to the ordinary Heisenberg algebra of free Bose oscillators. From the Fock-space point of view, we briefly discuss the existence of mapping from the Calogero oscillators to the free Bose oscillators and vice versa.
1 Introduction

The classical and quantum integrable model of \( M \) interacting particles on a line, introduced by Calogero [1], has been intensively studied during the past few years. The model and its generalizations [2] are connected with a number of physical problems, ranging from condensed matter physics [3] to gravity and black-hole physics [4]. The algebraic structure of the Calogero model and its successors, studied earlier using group theoretical methods [5], has recently been reconsidered by a number of authors in the framework of the \( S_M \)-extended Heisenberg algebra [6].

Apart from its particular realization, the \( S_M \)-extended Heisenberg algebra is basically a multi-mode oscillator algebra with permutation invariance. The general techniques for analysing such a class of oscillator algebras were developed earlier in a series of papers [7,8].

In the present paper we apply these techniques to the Calogero model in its operator formulation. We start our analysis with the two-body Calogero model. On the algebraic grounds, this model is described by a particular class of deformed single-mode oscillator algebras which were treated in a unified manner in [9].

In Section 2, we describe a single-mode algebra underlying the two-body Calogero model. We construct a mapping from this algebra to the ordinary Bose algebra. We also present the number operator \( N \) and the exchange operator \( K \) as (infinite) series in creation and annihilation operators. At the end of Section 2 we construct an algebra that is dual to the original algebra of the two-body Calogero model.

In Section 3 we discuss general multi-mode oscillator algebras with permutation invariance. We describe two distinct classes of these algebras: (i) algebras which possess well-defined hermitean number operators (i.e. transition number operators
$N_{ij}$, partial number operators $N_{ii} \equiv N_i$ and the total number operator $N \equiv \sum N_i$), and (ii) algebras which possess well-defined number operators but for which only the total number operator $N$ is hermitean. The analysis of these algebras is performed in three equivalent ways: using (i) a normally ordered expansion in creation and annihilation operators, (ii) the action of annihilation operators on monomial states in Fock space, and (iii) Gram matrices of scalar products in Fock space. We conclude Section 3 with a discussion of the general structure of transition number operators $N_{ij}$ and exchange operators $K_{ij}$.

The ideas developed in the preceding Sections are applied to the many-body Calogero model in Section 4. The algebra underlying the many-body Calogero model ( $S_M$-extended Heisenberg algebra) is discussed along the lines described in Section 3. Special attention is devoted to the analysis of Gram matrices and to the construction of number operators and exchange operators as an infinite series in creation and annihilation operators. We also find that the $S_M$-extended Heisenberg algebra can be defined as a generalized triple operator algebra. Generalizing the construction given in Section 2 to the multi-mode case, we define and investigate the structure of dual algebras. Section 4 ends with a short discussion of mappings from the Calogero algebra to the set of free bosonic oscillators. Section 5 is a short summary.

2 Two-body Calogero model and deformed single-mode oscillator algebras

Particular aspects of single-mode deformed oscillator algebras were studied by a number of authors, starting with seminal papers [10]. A unified view of deformed single-mode oscillator algebras was proposed in [9,11].
Basically, these algebras are generated by a set of generators $G$, involving annihilation ($a$) and creation ($a^\dagger$) operators, together with the well-defined number operator $N$:

$$G := \{1, a, a^\dagger, N\},$$

$$(a)^\dagger = a^\dagger \quad N = N^\dagger,$$

The following commutation relations hold:

$$[N, a] = -a, \quad [N, a^\dagger] = a^\dagger,$$

$$[N, a^\dagger a] = [N, aa^\dagger] = 0,$$

$$aa^\dagger - qa^\dagger a = G(N),$$

where $q \in \mathbb{R}$ and $G(N)$ is the hermitean, analytic function of the number operator.

Vacuum conditions are $a|0\rangle = 0$ and $N|0\rangle = 0$, with $\langle 0|0 \rangle = 1$. Since $[N, a^\dagger a] = [N, aa^\dagger] = 0$, we can write

$$a^\dagger a = \phi(N),$$

$$aa^\dagger = \phi(N + 1),$$

where $\phi(N) \geq 0$ is some function of the number operator. Several examples of algebras that belong to the class (1) and their corresponding functions $\phi(N)$ are given in [9].

Here we want to discuss a variant of the algebra (1), with $G(N) = 1 + 2\nu K$ and $q = 1$, namely

$$aa^\dagger - a^\dagger a = 1 + 2\nu K, \quad \nu \in \mathbb{R}$$

$$K = (-)^N \quad Ka = -aK.$$

In this equation $K$ is the exchange operator (see Sections 3 and 4) which here acts simply as a parity operator that separates the set of excited states $|n\rangle \propto a^{\dagger n}|0\rangle$ into even and odd subspaces. For $\nu > -\frac{1}{2}$, the algebra (3) possesses unitary infinite-dimensional representations. This algebra is known as the Calogero-Vasiliev algebra.
[6] (also termed the deformed Heisenberg algebra with reflection [12]) and provides an algebraic formulation of the two-particle Calogero model [1] described by the Hamiltonian ($x$ and $p$ are the relative coordinate and momentum, respectively)

$$2H = p^2 + x^2 + \frac{\nu(\nu - 1)}{x^2}K$$

which reduces to

$$2H = \{a, a^\dagger\}$$

after the identification

$$\sqrt{2}a = x + ip - \frac{\nu}{x}K$$

$$\sqrt{2}a^\dagger = x - ip + \frac{\nu}{x}K.$$  

*Remark 1.* Generalizations of the algebra (3) have been investigated in Refs. [13,14] and its connection to nonlinear parabosonic (parafermionic) supersymmetry have been described in [15].

As we have already described in [7,9], the analysis of the general (deformed) oscillator algebras could be carried out in three completely equivalent ways. One can express $aa^\dagger$ as a normally ordered expansion:

$$aa^\dagger = 1 + \sum_{k \geq 1} \alpha_k a^k a^\dagger.$$  

Alternatively, one can start the analysis by using the action of the annihilation operator on the states in Fock space:

$$aa^\dagger m|0\rangle = \phi(N + 1)a^{m-1}|0\rangle.$$  

The third way is to examine the vacuum matrix elements (Gram matrix)

$$A_{m,n} = \langle 0|a^m a^n|0\rangle = [\phi(n)]!\delta_{mn}. $$
These approaches are rather simple for the single-mode oscillators but becomes very powerful for the analysis of multimode (deformed) oscillator algebras (see Sections 3,4.).

Now, we show how the normally ordered expansion (6) works for the Calogero-Vasiliev algebra (3).

First, we calculate the function $\phi(N)$ and it reads

$$\phi(N) = N + \nu (1 + (-)^{1+N}).$$

(9)

Knowing $\phi(N)$, we recursively calculate the coefficients $\alpha_k$ (Eq.(6)) for the algebra (3):

$$\alpha_k = \frac{\phi(k+1) - 1 - \sum_{m=1}^{k-1} \alpha_m \phi(k) \cdots \phi(k+1-m)}{[\phi(k)]!}, \quad \forall \phi(k) \neq 0.$$

Similarly, we can expand the operators $K$ and $N$ in an infinite series in operators $a$ and $a^\dagger$, i.e.

$$K = 1 + \sum_{k \geq 1} \beta_k a^\dagger a^k.$$  

(10)

Using relations (3) and $Ka^\dagger n|0\rangle = (-)^n a^\dagger n|0\rangle$, one can recursively calculate the coefficients $\beta_k$ as

$$\beta_k = \frac{\left[(-)^k - 1\right]}{\phi(k)!} - \sum_{m=1}^{k-1} \beta_m \frac{1}{\phi(m-1)!}.$$

The expansion of the number operator $N$ reads

$$N = a^\dagger a + \sum_{n \geq 2} \gamma_n (a^\dagger)^n (a)^n,$$

(11)

where

$$\gamma_n = \frac{n - \sum_{k=1}^{n-1} \gamma_k \phi(n) \cdots \phi(n+1-k)}{[\phi(n)]!}.$$
Note that $\gamma_n = 0$ for $\phi(n) = 0$, $\phi(n - 1) \neq 0$.

Notice that, knowing $\alpha_k$, $\beta_k$ and $\gamma_n$, we can obtain $\phi(n)$ from the same recurrent relation.

As we elaborated in [9], there exists a simple mapping of the general deformed algebra (1) to the ordinary Bose algebra $[b, b^\dagger] = 1$. The mapping is of the form

$$a = b \sqrt{\frac{a^3 a}{N}} \equiv b \sqrt{\frac{\phi(N)}{N}}.$$  \hspace{3cm} (12)

The inverse mapping exists if $\phi(N) \neq 0$, i.e. $\phi(N) > 0$, $\forall N$.

Using $\phi(N)$, Eq.(9), associated with the algebra (3), we obtain

$$a = \begin{cases} b & n = \text{even} \\ b \sqrt{\frac{N + 2n}{N}} & n = \text{odd}. \end{cases}$$

It is also possible to map the operators $a$ and $a^\dagger$ to the Bose operators $b$ and $b^\dagger$, using the expansion of the form

$$a = ( \sum_{k \geq 0} c_k b^k b^k ) \cdot b.$$  \hspace{3cm} (13)

Comparing Eq.(13) with Eq.(3), one can recursively calculate the coefficients $c_k$ as

$$c_k = \frac{1}{k!} \left[ \phi(k + 1) \right] + \sum_{m=1}^{k} (-)^k \binom{k}{m} \left[ \phi(k + 1 - m) \right].$$

For further purpose (see Section 4), it is convenient to define a new operator $\tilde{a}$ in a sense dual to $(a, a^\dagger)$, such that

$$[\tilde{a}, a^\dagger] = 1, \quad \tilde{a}|0\rangle = 0.$$  \hspace{3cm} (14)

The connection is

$$\tilde{a} = a \frac{N}{a^3 a} \equiv a \frac{N}{\phi(N)}, \quad \phi(N) > 0, \quad \forall N$$  \hspace{3cm} (15)

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where $N$ can be realized as in (11), or as

$$N = \frac{1}{2}\{a^\dagger, a\} - (\nu + \frac{1}{2}).$$

Using $\phi(N)$, Eq.(9), we find

$$\tilde{a} = \begin{cases} 
  a & n = \text{even} \\
  a_{N+2\nu}^N & n = \text{odd}.
\end{cases}$$

The construction of the mapping (15) is similar to that given in [16].

We also find

$$N = a^\dagger \tilde{a}, \quad N + 1 = \tilde{a} a^\dagger,$$

and

$$\tilde{a} a^m |0\rangle = n a^i(n-1) |0\rangle \quad \langle 0|\tilde{a}^m a^m |0\rangle = n! \delta_{mn}.$$ 

In the next section we turn to general multi-mode oscillator algebras with permutational invariance.

### 3 Intermezzo: Multi-mode oscillator algebras with permutation invariance

As we emphasized in [7,8], the analysis of the general multimode (deformed) oscillator algebras is more complicated and the three approaches, Eqs.(6-8), become more involved. Here we concentrate on permutational invariant multi-mode oscillator algebras. Invariance on the permutation group simplifies the analysis but one still has to distinguish between two cases. The first case are permutational invariant algebras with hermitean number operators. These algebras were analysed in [7,8]. In order to be a self-contained, in Subsection 3.1 we repeat the main points of this analysis.
The second case are permutation invariant algebras with non-hermitean number operators, not discussed previously. The analysis of these algebras is presented in Subsections 3.2 and 3.3.

### 3.1 Permutation invariant algebras with hermitean number operators

Let us consider a system of multi-mode oscillators described by $M$ pairs of creation and annihilation operators $a_i^\dagger$, $a_i$ ($i = 1, 2, ..., M$) hermitian conjugated to each other. We consider operator algebras which possess the well-defined transition number operators $N_{ij}$, the partial number operators $N_i \equiv N_{ii}$ and the total number operator $N = \sum N_i$. We also demand that the algebras be permutation invariant. In this subsection we suppose that all number operators are hermitean, i.e. $N_i^\dagger = N_i$, $N_i^\dagger = N_i$ and $N_{ij}^\dagger = N_{ji}$.

The relations involving the number operators and the operators $a_i^\dagger$, $a_i$ ($i = 1, 2, ..., M$) are

$$ [N_{ij}, a_k^\dagger] = \delta_{jk} a_i^\dagger, \quad [N_{ij}, a_k] = -\delta_{ik} a_j, \quad [N_{ij}, N_{kl}] = \delta_{jk} N_{il} - \delta_{il} N_{kj}. $$

$$ [N_i, a_j] = -\delta_{ij} a_i, \quad [N_i, a_j^\dagger] = \delta_{ij} a_i^\dagger, \quad [N_i, N_j] = 0 $$

$$ [N, a_k^\dagger] = a_k^\dagger, \quad [N, a_k] = -a_k. \quad (17) $$

In the associated Fock-like representation, let $|0\rangle$ denote the vacuum vector. Then, the scalar product is uniquely defined by $\langle 0|0 \rangle = 1$, and the vacuum conditions are $a_i|0\rangle = 0$, $a_i a_i^\dagger |0\rangle \neq 0$. A general $n$-particle state is a linear combination of monomial state vectors $(a_i^\dagger \cdots a_{i_n}^\dagger |0\rangle)$, $i_1, ..., i_n = 1, 2, ..., M$. The partial number operators $N_i$ are diagonal on the monomial states $(a_i^\dagger \cdots a_{i_n}^\dagger |0\rangle)$, with eigenvalues...
counting the number of operators $a_i^{\dagger}$ in the corresponding monomial state:

$$N_i(a_i^{\dagger} a_i^{\dagger} \cdots a_i^{\dagger} a_n^{\dagger} |0\rangle) = n_i(a_i^{\dagger} a_i^{\dagger} \cdots a_i^{\dagger} a_n^{\dagger} |0\rangle).$$

In Fock space considered, monomial states with different total number operator $N$ are orthogonal, as well as the states with the same $N$ but different partial number operators $N_i$. Two states are not orthogonal if they have the same partial number operators $N_1, N_2, \ldots N_M$.

The general multi-mode oscillator algebra with hermitean number operators and permutation invariance can be described in three ways [7]: (i) as a normally ordered expansion, (ii) as a set of relations in the Fock space and, (iii) using Gram matrix of scalar products in Fock space. These three approaches are independent but completely equivalent. Moreover, as we demonstrate in the rest of the paper, they can also be applied to the permutation invariant algebras with non-hermitean number operators. Which of the three approaches will be used, depends on the nature of the problem. In the concrete example of the Calogero model (Section 4) we are mainly interested in positivity of the norms in Fock space and in finding the critical points. Therefore, we shall analyse the problem using the Gram matrix.

Now, we shortly describe each of the three approaches.

In the first approach, the operator algebras are defined by a set of relations:

$$a_i a_j^{\dagger} \equiv \Gamma_{ij} (a^{\dagger}, a) =$$

$$= \delta_{ij} + C_{1,1} a_j^{\dagger} a_i + \sum_{n=1}^{\infty} \sum_{\pi, \sigma \in S_{n+1}} C_{\pi,\sigma} \sum_{k_1, \ldots, k_n=1}^{M} [\pi(j, k_1, \ldots, k_n)]^{\dagger} [\sigma(i, k_1, \ldots, k_n)], \quad (18)$$

where the operators $a_i$ are normalized in such a way that the coefficient of the $\delta_{ij}$ term is equal to 1. Several comments on the structure of the above expression are in order.
Permutation invariance guarantees that the coefficients in the expansion do not depend on concrete indices in normally ordered monomials, but only on certain linearly independent types of permutation invariant terms, schematically displayed above. The existence of the number operators $N_i$ implies that the annihilation and creation operators appearing in a monomial in the normally ordered expansion (18) have to appear in pairs, i.e. monomials are diagonal in the variables $k_1, ..., k_n$ (up to permutations) [8]. The symbol $[\sigma(i, k_1, ..., k_n)]$ denotes $a_{\sigma(i)}a_{\sigma(k_1)} \cdots a_{\sigma(k_n)} \equiv \sigma(a_ia_k \cdots a_{kn})$. Also, $C_{\pi,\sigma} = C_{\sigma,\pi}^*$, owing to the hermiticity of the operator product $a_ia_i^\dagger$. We consider only those relations in (18) that may allow the norm zero vectors, but do not allow the state vectors of negative norm in Fock space. The norm zero vectors imply relations between creation (annihilation) operators. Since these relations are consequences of Eq.(18), they need not be postulated independently. Also, there is no need to postulate relations $[ a_ia_j^\dagger ]$ separately, since they can differ from the relations $[ a_ia_j^\dagger |i=j\rangle ]$ only in the unique function $f(N, N_i)$. Finally, notice that, although there are infinitely many terms in the expansion, only finite terms are actually involved when acting on the finite monomial state in Fock space.

In the second approach, in addition to the vacuum relations $a_i|0\rangle = 0$, $a_ia_j^\dagger|0\rangle = \delta_{ij}|0\rangle$, one can define the action of $a_i, i = 1, 2, ... M$, on monomial states $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle)$ through the relations

$$a_ia_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle = \sum_{k=1}^{n} \delta_{ik} \sum_{\sigma \in S_{n-1}} \phi_{\sigma}^k [\sigma(i_1, \hat{i}_k, \ldots, i_n)^\dagger |0\rangle, \quad (19)$$

where $\hat{i}_k$ denotes the omission of the creation operator (with index $i_k$) in all possible ways. (If the monomial state does not contain $a_i^\dagger$ at all, the RHS of (19) is equal to zero.) The sum is running over all linearly independent monomials and $\phi_{\sigma}^k$ are (complex) coefficients. The identity $\phi_{id}^1 = 1$ is implied by normalization in Eq. (18).
The coefficients $\phi^k_\sigma$ can be uniquely determined from $C_{\pi,\sigma}$ and vice versa. If we write the type of monomial state $(a_{i_1}^\dagger \cdots a_{i_n}^\dagger |0\rangle)$ as $1^{\nu_1}2^{\nu_2}\cdots M^{\nu_M}$, where $\nu_1, \nu_2, \ldots, \nu_M$ are multiplicities satisfying $\nu_i \geq 0$ and $\sum_{i=1}^M \nu_i = n$, we see that permutation invariance drastically reduces the number of independent terms in Eq.(19), i.e. from $M^n + 1$ (for the general algebra) to at most $(\sum_{k=1}^n k! n! (\nu_1! \cdots \nu_k!))$ independent terms (for the permutation invariant algebra).

**Example 1.** States for $n = 2$: hermitean $(N_{ij}, N_i)$

$$a_1(a_1^\dagger)^2|0\rangle$$

$$a_1a_1^\dagger a_2^\dagger|0\rangle, \quad a_2a_1^\dagger a_2^\dagger|0\rangle.$$

In the third approach, one defines the Gram matrix of scalar products

$$A_{i_n \cdots i_1; j_n \cdots j_1} = \langle 0| a_{i_n} \cdots a_{i_1} a_{j_1}^\dagger \cdots a_{j_n}^\dagger |0\rangle. \quad (20)$$

For the permutation invariant algebra, it is permutation invariant, meaning that the matrix element $\langle 0| a_{i_{\pi(n)}} \cdots a_{i_{\pi(1)}} a_{j_{\pi(1)}}^\dagger \cdots a_{j_{\pi(n)}}^\dagger |0\rangle$ does not depend on the permutation $\pi \in S_n$. The rank of the Gram matrix gives the number of linearly independent states in Fock space, which is positive definite if $A \geq 0$, i.e. if all eigenvalues are non-negative. The matrix and its rank depend only on the collection of multiplicities $\{\nu_1, \ldots, \nu_M\}$, which, written in descending order, give rise to a partition of $n$ [8]. The generic matrix (all indices different!) is of the type $n! \times n!$. It can be decomposed into terms of the right regular representation of the permutation group $S_n$ [8]. All other, non-generic matrices, are easily obtained from the generic matrix. Their order is $\frac{n!}{\nu_1! \cdots \nu_k!} \times \frac{n!}{\nu_1! \cdots \nu_k!}$, where $\sum \nu_k = n$.

We mention that the typical permutation invariant algebras with hermitean number operators are parastatistics/interpolation between parastatistics [17] and infinite
quon statistics [18]. Finally, we note that there is a simple way to unify a very large class of such permutation invariant algebras as a triple operator algebras [19]:

\[
[a_i, a_j^\dagger, a_k] = x\delta_{ij}a_k^\dagger + y\delta_{ik}a_j^\dagger + z\delta_{jk}a_i^\dagger,
\]

\[(x, y, z) \in \mathbb{R}
\] (21)

which can be rewritten as [20]

\[
a_i a_j^\dagger = qa_j^\dagger a_i + (1 + xN)\delta_{ij} + yN_{ij} + zN_{ji}.
\]

3.2 Permutation invariant algebras with non-hermitean number operators

There exist a large class of operator algebras which possess permutation invariance but for these \(N_i \neq N_i^\dagger\) and \(N_{ij} \neq N_{ji}\) (it still holds \(N = N^\dagger\)). The most important example is the many-body Calogero model[1,6]. Generally, in these algebras one can have \(a_i |0\rangle = 0\), but \(a_i a_j^\dagger |0\rangle \neq 0\) for \(i \neq j\). As a rule, a non-orthogonal monomial basis appears, i.e. two monomial states with the same total number operator \(N\) but different partial number operators \(N_i\) are not orthogonal.

The whole algebra can be obtained from one relation, e.g. \(a_1 a_2^\dagger\), and by successive application of permutation \(\pi \in S_M\), \(a_{\pi(1)} a_{\pi(2)} = \pi(a_1 a_2^\dagger)\). The general structure of the normally ordered expansion is

\[
a_i a_j^\dagger = c_0 + [a_j^\dagger a_i] + [a_j^\dagger B_{0,1} + B_{0,1}^\dagger a_i] + [B_{0,1}^\dagger B_{0,1}] + [B_{1,1}] + \cdots,
\]

(22)

where

\[
B_{m,n} = \sum_{k=1}^{M} (a_k^\dagger)^m a_k^n \quad B_{m,n}^\dagger = B_{n,m}.
\]

The above expansion displays the \(S_M\) symmetric structure as it contains the \(S_M\)-symmetric operators only. Notice that, although the RHS of Eq.(22) contains an
equal number of $a^\dagger$’s and $a$’s, they are not matched in pairs any longer (cf. Eq. (18)).

The general structure of the terms in the expansion of $a_i a_j^\dagger$ in (22) (as well as the structure of $N_{ij}$, $N_i$ and $K_{ij}$, see later) is

$$a^\dagger_m O a_n^\dagger, \quad m, n = i, j$$

where $( a_m, a_n )$ are $a_i$ or $a_j$ and $O$ is any normally ordered $S_M$-invariant polynomial in the operators $(a^\dagger, a)$. The indices $(i, j)$ appear explicitly in the expansion and all other indices are contained implicitly in the $S_M$-invariant operator $O$. Using this fact, one can uniquely define $[ a_i a_j^\dagger ]_{i=j}$ and it should coincide with $[ a_i a_i^\dagger ]$ up to the unique hermitean function of the operators $(N, N_i, N_i^\dagger, N_{ik}, ..)$. which are no longer diagonal on monomial states (and can change the eigenvalues of $N_k$ for fixed total $N$).

Going to the Fock-space description (19), we notice that, owing to the non-orthogonality of monomial states, there appear much more independent terms than in the orthogonal case (see Example 1).

**Example 2**: States for $n = 2$: non-hermitean $(N_{ij}, N_i)$

$$a_1 (a_1^\dagger)^2 |0\rangle, \quad a_2 (a_1^\dagger)^2 |0\rangle$$

$$a_1 a_1^\dagger a_2^\dagger |0\rangle, \quad a_2 a_1^\dagger a_2^\dagger |0\rangle, \quad a_3 a_1^\dagger a_2^\dagger |0\rangle.$$  

Generally, there are at most

$$\sum_{k=1}^{n} (k + 1) \frac{n!}{\nu_1!...\nu_k!}, \quad \sum_{i=1}^{k} \nu_i = n$$

independent terms.

The Gram matrices $A_{i_1...i_k:j_1...j_n}$ are hermitean, of the type $M^n$ and we require all
eigenvalues to be non-negative. The matrix elements of the particular Gram matrix are related by permutation symmetry. In the next section we study the structure of these matrices in more detail.

### 3.3 Operators $N_{ij}, N_i$ and $K_{ij}$

The transition number operators $N_{ij}$ can be expanded into an infinite ( normally ordered ) series in creation and annihilation operators. However, only finitely many terms are involved when $N_{ij}$ acts on a finite monomial state in Fock space.

We find that the general structure of $N_{ij}$ is (cf. (22))

$$N_{ij} = [a_i^\dagger a_j] + [a_i^\dagger B_{0,1} + B_{0,1}^\dagger a_j] + [B_{0,1}^\dagger B_{0,1}] + [B_{1,1}] + \cdots$$

(23)

The partial number operators $N_i$ are obtained from the above expression as $N_i = N_{ii}$.

Notice that $N_{ij}^\dagger \neq N_{ji}$.

The structure of the total number operator $N = \sum N_i$ is

$$N = [B_{0,1}^\dagger B_{0,1}] + [B_{1,1}] + [B_{0,2}^\dagger B_{0,2}] + [B_{0,2}^\dagger B_{0,1}^\dagger] + [B_{0,1}^\dagger B_{0,1}] + [B_{1,1}] + [B_{2,1} B_{0,1}] + [B_{0,1}^\dagger B_{1,2}] + [B_{2,2}] + \cdots$$

(24)

As we are interested only in the $S_M$-symmetric structure, the coefficients in the above expressions have been omitted. Notice that $N^\dagger = N$.

The exchange operators $K_{ij}$, $i, j = 1, 2, ...M$, generate the symmetric group $S_M$.

They are defined as follows:

$$K_{ij} = K_{ji}, \quad (K_{ij})^2 = 1, \quad K_{ij}^\dagger = K_{ij},$$

$$K_{ij} K_{jl} = K_{jl} K_{il} = K_{il} K_{ij}, \quad i \neq j, \quad i \neq l, \quad j \neq l.$$
The representation of the symmetric group $S_M$ exists on every deformed algebra of $a_i$ and $a_i^\dagger$, $(i = 1, ..., M)$, in the sense

$$K_{ij}a_j = a_iK_{ij}, \quad K_{ij}a_k = a_kK_{ij},$$

$$K_{ij}a_j^\dagger = a_i^\dagger K_{ij}, \quad K_{ij}a_k^\dagger = a_k^\dagger K_{ij},$$

for $k \neq i$ and $k \neq j$.

The vacuum condition is $K_{ij}|0\rangle = \pm|0\rangle$, and we choose $K_{ij}|0\rangle = +|0\rangle$.

**Remark 2.** Note that the following change of the definitions in (26):

$$K_{ij}a_j = -a_iK_{ij}, \quad K_{ij}a_k = +a_kK_{ij},$$

(and similarly for $a_i^\dagger$), directly leads to contradiction since, from Eqs.(25,26), we obtain two apparently different results:

$$K_{ij}K_{jk}a_k = a_iK_{ij}K_{jk}$$

$$K_{jk}K_{ki}a_k = -a_iK_{jk}K_{ki}.$$  

The important fact is that if the algebra of the oscillators is permutation invariant, then the $K_{ij}$ operators can be expressed similarly as $N_i$, $N_i$, $N_{ij}$, namely as an infinite series expansion in creation and annihilation operators. If the algebra is not permutation invariant, such a representation of exchange operators may not exist. We point out the difference between the operators $N_{ij}$ and $K_{ij}$. The exchange operators $K_{ij}$ act "globally" (and simultaneously) on the right on any monomial in $a_j^\dagger$ and $a$, interchanging indices $i$ and $j$ and keeping all other indices at rest. Transition number operators $N_{ij}$ act "locally", turning only one $a_j^\dagger$ into $a_i^\dagger$, at the same place.
where \(a_j^\dagger\) is sitting. The action of \(N_{ij}\) can be repeated at most \(n_j\) times, where \(n_j\) counts the number of \(a_j^\dagger\)'s in the monomial. If there is only one \(a_j^\dagger\), then we have

\[
N_{ij}(\cdots a_j^\dagger \cdots)|0\rangle = (\cdots a_j^\dagger \cdots)|0\rangle.
\]

Let \([a_i^\dagger, a_j^\dagger] = 0\) for \(i \neq j\). We symbolically denote the eigenstate of \(N_i\) as

\[
(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle.
\]

Then,

\[
N_i(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle = n_i(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle
\]

\[
N_j(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle = n_j(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle
\]

\[
N_{ij}(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle = n_j(\cdots a_i^{n_i+1} \cdots a_j^{n_j-1} \cdots)|0\rangle
\]

\[
N_{nj}(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle = n_j!(\cdots a_i^{n_i+n_j} \cdots a_j^{n_j} \cdots)|0\rangle
\]

\[
N_{ij}N_{ji}(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle = (n_i + n_j)!(\cdots a_i^{n_i} \cdots a_j^{n_j} \cdots)|0\rangle
\]

We also obtain

\[
K_{ij} = \frac{1}{(n_i + n_j)!}(N_{ji})^{n_i}(N_{ij})^{n_j} = \frac{1}{(n_i + n_j)!}(N_{ij})^{n_j}(N_{ji})^{n_i},
\]

or alternatively

\[
K_{ij} = \begin{cases} 
(N_{ij} \frac{1}{N_{ji}})^{n_j-n_i} & \text{if } n_i < n_j \\
1 & \text{if } n_i = n_j \\
(N_{ji} \frac{1}{N_{ij}})^{n_i-n_j} & \text{if } n_i > n_j 
\end{cases}
\]

where \(n_i\) and \(n_j\) again denote, respectively, the number of \(a_i^\dagger\) and \(a_j^\dagger\) in the monomial \((\cdots a_i^\dagger \cdots a_j^\dagger)|0\rangle\). If \([a_i^\dagger, a_j^\dagger] \neq 0\) for \(i \neq j\), there is generally no such a simple relation between \(K_{ij}\) and \(N_{ij}\).

Below we give two examples of \(K_{ij}\) operators for permutation invariant algebras with hermitean number operators. An example of \(K_{ij}\) operators for permutation invariant algebra with non-hermitean number operators is given in the next section.
**Example 3.** Heisenberg algebra of Bose oscillators $b_i, b_i^\dagger, i = 1, ..M$.

Algebra:

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i^\dagger, b_j^\dagger] = [b_i, b_j] = 0.$$ 

Number operators:

$$N_{ij} = b_i^\dagger b_j, \quad N_i = b_i^\dagger b_i, \quad N = \sum_{i=1}^{M} b_i^\dagger b_i.$$ 

Exchange operators:

$$K_{ij} = :e^{-\left(b_i^\dagger - b_j^\dagger\right)\left(b_i - b_j\right)}: = \sum_{k=0}^{\infty} \frac{(-)^k}{k!} \left(b_i^\dagger - b_j^\dagger\right)^k \left(b_i - b_j\right)^k.$$ 

**Example 4.** Clifford algebra of Fermi oscillators $f_i, f_i^\dagger, i = 1, ..M$.

Algebra:

$$\{f_i, f_j^\dagger\} = \delta_{ij}, \quad \{f_i^\dagger, f_j^\dagger\} = \{f_i, f_j\} = 0.$$ 

Number operators:

$$N_{ij} = f_i^\dagger f_j, \quad N_i = f_i^\dagger f_i, \quad N = \sum_{i=1}^{M} f_i^\dagger f_i.$$ 

Exchange operators:

$$K_{ij} = :e^{-\left(f_i^\dagger - f_j^\dagger\right)\left(f_i - f_j\right)}: = 1 - \left(f_i^\dagger - f_j^\dagger\right)\left(f_i - f_j\right).$$

### 4 Application: The M-body Calogero model

The results of a general analysis of permutation invariant multi-mode oscillator algebras with non-hermitean number operators (Subsections 3.2 and 3.3) will be now applied to the M-body Calogero model. We also generalize a several concepts (e.g. an infinite series expansion of $a_i a_j^\dagger, N, N_i, N_{ij}, K_{ij}$ and dual algebra) introduced
in Section 2. We find, as particularly interesting, the structure and the eigensystem of the Gram matrices. The analysis of the Gram matrices enable us to locate the universal critical point of the M-body Calogero model at $\nu = -\frac{1}{M}$.

### 4.1 The M-body Calogero model and the multi-mode oscillator algebras

The M-body Calogero model, describing M identical bosons on the line, is defined by the following Hamiltonian [1]:

$$H = -\frac{1}{2} \sum_{i=1}^{M} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{M} x_i^2 + \frac{\nu(\nu - 1)}{2} \sum_{i \neq j}^{M} \frac{1}{(x_i - x_j)^2}.$$  \hspace{1cm} (28)

For simplicity, we have set $\hbar$, the mass of particles and the frequency of harmonic oscillators equal to one. The dimensionless constant $\nu$ is the coupling constant (and/or the statistical parameter) and $M$ is the number of particles. The Hamiltonian (28) can be factorized by creation and annihilation operators of the $S_M$-extended Heisenberg algebra [6].

Let us introduce the following analogs of creation and annihilation operators [6]:

$$a_i^\dagger = \frac{1}{\sqrt{2}} (-D_i + x_i), \quad a_i = \frac{1}{\sqrt{2}} (D_i + x_i),$$

where

$$D_i = \partial_i + \nu \sum_{j, j \neq i}^{M} \frac{1}{x_i - x_j} (1 - K_{ij})$$

are Dunkl derivatives and $K_{ij}$ are exchange operators (Eqs.(25-26)) generating the symmetric group $S_M$. One can easily check that the commutators of creation and annihilation operators are

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad \hspace{1cm} (29)$$

$$[a_i, a_j^\dagger] = A_{ij} = \left( 1 + \nu \sum_{k=1}^{M} K_{ik} \right) \delta_{ij} - \nu K_{ij}. \hspace{1cm} (29)$$

20
The action of \( K_{ij} \) on \( a_i^\dagger \) and \( a_i \) is given in Eq.(26).

Remark 3. The following definitions

\[
K_{ij} f_j = -f_i K_{ij},
K_{ij} f_k = -f_k K_{ij},
\]

are also consistent and one can study the algebra defined by the anticommutator \( \{a_i, a_j^\dagger\} \):

\[
\{a_i, a_j^\dagger\} = A_{ij} = \left(1 + \nu \sum_{k=1}^{N} K_{ik}\right) \delta_{ij} - \nu K_{ij} \quad \forall i, j.
\]

Note that \( \{a_i, a_j\} \neq 0 \), if \( \nu \neq 0 \). Moreover, one can show that the general form of the algebra, namely \( [a_i, a_j^\dagger]_q = A_{ij}(\nu), |q| \leq 1 \), with the condition \( K_{ij} a_j = a_i K_{ij} \), Eq.(26), is equivalent to the algebra \( [a_i, a_j^\dagger]_q = A_{ij}(-\nu) \), with the conditions \( K_{ij} a_j = -a_i K_{ij} \) and \( K_{ij} a_k = -a_k K_{ij} \).

After performing a similarity transformation on the Hamiltonian \( H \),

\[
(\prod_{i<j}^{M} |x_i - x_j|^{-\nu}) \; H \; (\prod_{i<j}^{M} |x_i - x_j|^\nu),
\]

we obtain the reduced Hamiltonian \( H' \) which, when restricted to the space of symmetric functions, takes the following simple form:

\[
H' = \frac{1}{2} \sum_{i=1}^{M} \{a_i, a_i^\dagger\} = \sum_{i=1}^{M} a_i^\dagger a_i + E_0, \quad (30)
\]

\[
[H', a_i] = -a_i, \quad [H', a_i^\dagger] = a_i^\dagger \quad [H', \{a_i, a_j^\dagger\}] = 0.
\]

Notice that one can define the general Hamiltonian \( \bar{H} \) as

\[
\bar{H} = \frac{1}{2} \sum_{i=1}^{M} \{a_i, a_i^\dagger\} = N + E_0,
\]

which is not restricted to the symmetric states only, i.e it acts on the states in the whole Fock space. In the space of symmetric states, it coincides with \( H' \), \( H' = \bar{H} \).

The ground-state energy is \( E_0 = \frac{M}{2}(1 + \nu(M - 1)) \). The Fock-space representation
is defined by $a_i |0\rangle = 0, \forall i$ and $K_{ij} |0\rangle = \epsilon |0\rangle, \forall (i,j)$. As $\epsilon^2 = 1$, we fix $\epsilon$ to $+1$. The Fock space is spanned by monomials $(a_i^{\dagger n_1} \cdots a_M^{\dagger n_M} |0\rangle)$. In the following we analyse the full Fock space of states since (i) we want to apply ideas from Section 3 to the $S_M$ - extended Heisenberg algebra (29) and (ii) we want to obtain the positivity of physical states as a consequence of positivity of states in the complete Fock space. Physical states are symmetric (antisymmetric) states for the bosonic (fermionic) systems. The algebraic analysis of the physical space of symmetric states for a two- and a three-body Calogero model (28) is given in [21]. A general approach to the algebra of observables and dynamical symmetry algebra for the M-body Calogero model was proposed in [22].

We point out that the algebra (29) can be defined in a new way, without exchange operators $K_{ij}$. The construction relies on the generalization of the triple operator algebras (21) [19,20]. The only difference is that now the number operators $N_i$ and $N_{ij}$ are not hermitean.

Eliminating $\nu K_{ij} = [a_i, a_j^\dagger]$ for $i \neq j$, we find

$$[a_i, B_{0,1}^\dagger] = 1, \quad \forall i$$

$$a_i[a_i, a_j^\dagger] = [a_i, a_j^\dagger]a_j, \quad \forall (i,j) \ i \neq j$$

$$a_i[a_j, a_i^\dagger] = [a_j, a_i^\dagger]a_j, \quad \forall (i,j), \ i \neq j$$

$$a_k[a_i, a_j^\dagger] = [a_i, a_j^\dagger]a_k, \quad \forall (i,j,k) \ i \neq j \neq k \neq i. \quad (31)$$

(It is understood that the hermitean counterparts of these relations also hold.) Notice that the single-mode algebra (3) is a true triple operator algebra (21) since it can be rewritten as $[[a, a^\dagger], a^\dagger] = 2a^\dagger$ and $[[a, a^\dagger], a] = -2a$.

The algebra (31) is still a permutation invariant algebra, but the indices on the LHS
and RHS are not the same (cf. the second and third relations in Eq.(31)). The last relation in (31) can be written as

\[
[a_k, [a_i, a_j^\dagger]] = 0, \quad \forall (i, j, k), \ i \neq j \neq k \neq i
\]

\[
[a_i, a_j^\dagger] = [a_j, a_i^\dagger], \quad \forall (i, j).
\]

For the triple operator algebras, the Fock-space representation is defined by the two generalized vacuum conditions (one can notice the similarity with Green’s paras-statistics [17]):

\[
a_i |0\rangle = 0, \quad \forall i
\]

\[
a_i a_j^\dagger |0\rangle = -\nu |0\rangle, \quad \forall (i, j), \ i \neq j
\] (32)

Under these conditions, the Fock representation is uniquely determined and is equivalent to the first construction, Eq.(29). However, this algebra does not depend on \(\nu\) and \(K_{ij}\), i.e. in the formulation of the triple operator algebras, the interaction parameter \(\nu\) enters only through the vacuum condition (32).

We also obtain the consistency conditions in the Fock representation, namely

\[
([a_i, a_j^\dagger])^2 = \nu^2, \quad [a_k, ([a_i, a_j^\dagger])^2] = 0,
\]

\[
[a_i, a_j^\dagger][a_j, a_k^\dagger] = [a_j, a_k^\dagger][a_i, a_j^\dagger] = [a_i, a_k^\dagger][a_i, a_j^\dagger].
\] (33)

There is a simple generalization of the triple operator algebra to include fermions (i.e. anticommutators) and quons (i.e. q-commutators).

### 4.2 Gram matrices for the Calogero model

In the rest of this section, we discuss the Calogero model in the framework of the three approaches proposed in Section 3. The special attention is devoted to the
Gram matrix approach.

In the first approach we express $a_i a_j^\dagger$ as a normally ordered expansion:

$$a_i a_j^\dagger = -\nu K_{ij} + a_j^\dagger a_i \quad \forall (i, j), \ i \neq j$$

$$a_i a_i^\dagger = 1 + a_i^\dagger a_i + \nu \sum_{l,l \neq i} K_{il}.$$  \hspace{1cm} (34)

This is obviously a permutation invariant algebra. We can write it in the form of Eq.(22) if we know the expansion of $K_{ij}$ in terms of $a_i$ and $a_j^\dagger$ (and vice versa). Later we shall give such a construction.

In the second approach we have to know the action of $a_i$ on the monomial states $(a_1^{i_1} \cdots a_M^{i_M} \langle 0 \rangle)$ in the Fock space.

For the one - particle states $(a_i^\dagger \langle 0 \rangle)$ , using $a_i \langle 0 \rangle = 0, \forall i$, we find

$$a_i a_j^\dagger \langle 0 \rangle = \begin{cases} -\nu \langle 0 \rangle & \text{if } i \neq j \\ (1 + (M - 1))\langle 0 \rangle & \forall i = j \end{cases}$$

For the two - particle states $(a_i^\dagger a_j^\dagger \langle 0 \rangle)$ we find:

$$a_i a_j^\dagger a_j^\dagger \langle 0 \rangle = [A_{ij_1} a_j^\dagger + a_j^\dagger A_{ij_2}]\langle 0 \rangle.$$  

There are only four types of relations, owing to $[a_j^\dagger, a_j^\dagger] = 0$:

$$\begin{aligned} a_1 (a_1^\dagger)^2 \langle 0 \rangle &= (\nu (M - 2) + 2) a_1^\dagger \langle 0 \rangle + \nu B_{0,1}^\dagger \langle 0 \rangle, \\ a_1 (a_2^\dagger)^2 \langle 0 \rangle &= -\nu (a_1^\dagger + a_2^\dagger)\langle 0 \rangle, \\ a_1 a_1^\dagger a_2^\dagger \langle 0 \rangle &= (\nu (M - 2) + 1) a_2^\dagger \langle 0 \rangle, \\ a_1 a_2^\dagger a_3^\dagger \langle 0 \rangle &= -\nu (a_3^\dagger + a_2^\dagger)\langle 0 \rangle. \end{aligned}$$

For the three - particle states $(a_i^\dagger a_j^\dagger a_k^\dagger \langle 0 \rangle)$ we find

$$a_i a_j^\dagger a_j^\dagger a_k^\dagger \langle 0 \rangle = [A_{ij_1} a_j^\dagger a_j^\dagger + a_j^\dagger A_{ij_2} a_j^\dagger + a_j^\dagger a_j^\dagger A_{ij_3}]\langle 0 \rangle.$$
There are seven types of relations:

\[ a_1(a_1^\dagger)^3|0\rangle = (\nu(M-3) + 3)a_1^{i2}|0\rangle + \nu(a_1^\dagger B_{0,1}^1 + B_{0,2}^1)|0\rangle, \]
\[ a_1(a_2^\dagger)^3|0\rangle = -\nu(a_2^\dagger a_1^\dagger + a_2^{i2} + a_1^{i2})|0\rangle, \]
\[ a_1(a_1^\dagger)^2a_2^\dagger|0\rangle = (\nu(M-2) + 2)a_1^\dagger a_2^\dagger|0\rangle - \nu a_2^{i2}|0\rangle + \nu a_2^\dagger B_{0,1}^1|0\rangle, \]
\[ a_1a_1^\dagger(a_2^\dagger)^2|0\rangle = (\nu(M-2) + 1)a_2^{i2}|0\rangle - \nu a_1^\dagger a_2^\dagger|0\rangle, \]
\[ a_1a_1^\dagger a_2^\dagger a_3^\dagger|0\rangle = (\nu(M-3) + 1)a_2^{i2}a_3^\dagger|0\rangle, \]
\[ a_1(a_2^\dagger)^2a_3^\dagger|0\rangle = -\nu(a_2^\dagger a_3^\dagger + a_2^{i2} + a_1^\dagger a_3^\dagger)|0\rangle, \]
\[ a_1a_2^\dagger a_3^\dagger a_4^\dagger|0\rangle = -\nu(a_2^\dagger a_3^\dagger + a_2^{i2}a_4^\dagger + a_3^\dagger a_4^\dagger)|0\rangle. \]

It is a simple task to generalize these equations to an arbitrary multiparticle state, i.e. \(((a_1^\dagger)^{n_1} \cdots (a_M^\dagger)^{n_M})|0\rangle\):

\[
a_1(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2} \cdots (a_M^\dagger)^{n_M}|0\rangle \equiv a_1|n_1; n_2; \cdots n_M\rangle =
\]
\[
n_1|n_1 - 1; n_2; n_3\rangle + \nu \text{ sgn}(n_1 - n_2) \sum_{k=1}^{|n_1 - n_2|} |\text{min}(n_1, n_2) + k - 1; \text{max}(n_1, n_2) - k; n_3\rangle +
\]
\[
+ \nu \text{ sgn}(n_1 - n_3) \sum_{k=1}^{|n_1 - n_3|} |\text{min}(n_1, n_3) + k - 1; n_2; \text{max}(n_1, n_3) - k\rangle + \cdots
\]
\[
+ \nu \text{ sgn}(n_1 - n_M) \sum_{k=1}^{|n_1 - n_M|} |\text{min}(n_1, n_M) + k - 1; n_2; \cdots; \text{max}(n_1, n_M) - k\rangle \quad (35)
\]

We use these formulas in the constructing Gram matrices for different \(M\) (the third approach).

Now we easily obtain the structure of the matrix elements of Gram matrices
\[
\langle 0|a_{i_n^\dagger} \cdots a_{i_j^\dagger} a_{j_1^\dagger} \cdots a_{j_n^\dagger}|0\rangle. \]
We explicitly give several examples (for \(M = 2, 3\)) in
the Appendix and below we discuss eigenvalues and eigenvectors of Gram matrices corresponding to one- and two-particle states for any \( M \).

(1) One-particle states \( (a_i^\dagger |0\rangle, i = 1, 2, \ldots M) \):
The matrix of one-particle states is of order \( M \) and has only two distinct entries: \(-\nu\) and \( 1 + \nu(M - 1) \). Its eigenvalues and typical eigenvectors are

| Eigenvalue | Degeneracy | Eigenvector | Comments |
|------------|------------|-------------|----------|
| 1          | 1          | \( B_{0,1}^i |0\rangle \) |          |
| \( 1 + M\nu \) | \( M - 1 \) | \( (a_1^\dagger - a_i^\dagger) |0\rangle \) | \( i \neq 1 \) |

The positivity condition implies that all eigenvalues should be positive, meaning \( 1 + M\nu > 0 \) or \( \nu > -1/M \).

(2) Two-particle states \( (a_i^\dagger a_j^\dagger |0\rangle, (i, j) = 1, 2, \ldots M) \):
The matrix of two-particle states is of order \( M^2 \) and has four distinct entries of the form

\[
\langle 0 | a_i^2 (a_j^\dagger)^2 |0\rangle = a \\
\langle 0 | a_1^2 (a_2^\dagger)^2 |0\rangle = \langle 0 | a_1^2 a_2^\dagger a_3 |0\rangle = \langle 0 | a_1 a_2 a_3 |0\rangle = b \\
\langle 0 | a_1^2 a_1^\dagger a_2^\dagger |0\rangle = \langle 0 | a_1 a_2 a_3^\dagger |0\rangle = c \\
\langle 0 | a_1 a_2 a_3 a_4 |0\rangle = d
\]
where \( a = [1 + \nu(M - 1)][2 + \nu(M - 1)] - \nu^2(N - 1) \), \( b = -\nu - \nu^2(M - 2) \), \( c = 2\nu^2 \)
and \( d = [1 + \nu(M - 1)][2 + \nu(M - 2)] \).

Its eigenvalues and typical eigenvectors are
| Eigenvalue | Degeneracy | Eigenvector | Comments |
|------------|------------|-------------|----------|
| 0          | $M(M-1)/2$ | $[a_i^\dagger, a_j^\dagger] |0\rangle$ | $\forall (i,j), i \neq j$ |
| $2(1 + M\nu)$ | $M$ | $\{a_i^\dagger, B_{0,1}^\dagger \} - 2B_{0,2}^\dagger |0\rangle$ | $\forall i, M \geq 2$ |
| $(1 + M\nu)(2 + M\nu)$ | $M-1$ | $\{(a_i^\dagger - a_j^\dagger), B_{0,1}^\dagger \} - M(a_i^{12} - a_j^{12}) |0\rangle$ | $i \neq 1; M \geq 2$ |
| $2(1 + M\nu)(1 + \nu(M-1))$ | $M(M-3)/2$ | $\{(a_i^\dagger - a_j^\dagger), (a_k^\dagger - a_l^\dagger) \} |0\rangle$ | $(i \neq j \neq k \neq l)$ |

Here, $B_{0,1} = \sum_i a_i$ and $B_{0,2} = \sum_i a_i^2$. Note that null-eigenstates are identically equal to zero owing to the commutation relation $[a_i^\dagger, a_j^\dagger] \equiv 0$, which is satisfied $\forall (i,j)$.

The positivity condition implies again that all non-zero eigenvalues are positive, which is satisfied if $1 + M\nu > 0$, i.e. $\nu > -1/M$.

One can show that the same condition for positivity of eigenvalues for three and more particle states also holds. There is a universal critical point, $\nu = -1/M$, at which all matrix elements of an arbitrary multi-state Gram matrix are equal to $\left(\frac{k!}{M^k}\right)$, where $k$ denotes a $k$-particle state. This can be proved by induction and here we sketch the proof.

The generic Gram matrix is of type $(M^k \times M^k)$. At the critical point $\nu = -1/M$ we find

$$A_{ij} \equiv \langle 0 | a_i a_j^\dagger | 0 \rangle = \frac{1}{M},$$

$$A_{i_1 i_2 i_3 i_4} \equiv \langle 0 | a_{i_1} a_{i_2} a_{i_3} a_{i_4}^\dagger | 0 \rangle =$$

$$= A_{i_1 j_1} \langle 0 | a_j a_{j_2}^\dagger | 0 \rangle + A_{i_1 j_2} \langle 0 | a_j a_{j_1}^\dagger | 0 \rangle = \frac{2}{M^2},$$

$$\vdots$$

$$A_{i_1 \cdots i_k j_1 \cdots j_k} \equiv \langle 0 | a_{i_1} a_{i_2} a_{i_3} \cdots a_{i_k} a_{j_1} a_{j_2} \cdots a_{j_k}^\dagger | 0 \rangle =$$

$$= A_{i_1 j_1} \langle 0 | a_{i_2} a_{i_3} \cdots a_{i_k} a_{j_2} a_{j_3} \cdots a_{j_k}^\dagger | 0 \rangle + A_{i_1 j_2} \langle 0 | a_{i_1} a_{i_2} a_{i_3} a_{i_4} a_{i_5} a_{j_3} \cdots a_{j_k}^\dagger | 0 \rangle + \cdots$$
\[ + A_{i_1 j_k} \langle 0 | a_{i_k} \cdots a_{i_2} a_{j_1}^\dagger \cdots a_{j_{k-1}}^\dagger | 0 \rangle = \]
\[ = k \frac{1}{M} \langle 0 | a_{i_k} \cdots a_{i_2} a_{j_2}^\dagger \cdots a_{j_{k-1}}^\dagger | 0 \rangle = \frac{k!}{M^k}. \]

The rank of this matrix is one. There is only one state, namely \( B_{0,1}^{t_k} | 0 \rangle \), which corresponds to the center of mass and has positive norm. The corresponding eigenvalue is \( \frac{k!}{M^k} M^k = k! \). All other eigenstates are null-states with zero norm. Diagonal matrix elements are larger than \( \frac{k!}{M^k} \) if \( \nu > -1/M \) and the positivity conditions for eigenvalues are satisfied.

**Remark 4.** The critical point is universal in the sense that all algebras of the form \( [a_i, a_j^\dagger]_q = A_{ij}(\nu), \ |q| \leq 1 \) have the same critical point \( \nu = -1/M \). The case \( q = -1 \), for which the algebra takes a fermionic form, is of special interest:

\[ \{ f_i, f_j^\dagger \} = A_{ij}, \quad \{ F, F^\dagger \} = M; \]
\[ \{ f_i, F^\dagger \} = 1, \quad F^2 = F^{\dagger 2} = 0. \]

Here, \( F = \sum f_i \). It follows that the one-particle Gram matrix for \( q = -1 \) is the same as for \( q = 1 \):

\[
\begin{pmatrix}
1 + \nu(M - 1) & -\nu & \cdots & -\nu \\
-\nu & 1 + \nu(M - 1) & \cdots & -\nu \\
\cdots & \cdots & \cdots & \cdots \\
-\nu & -\nu & \cdots & 1 + \nu(M - 1)
\end{pmatrix}
\]

The matrices for two- and many-particle cases depend on \( q \) and will be treated in separate paper. It appears that \( \nu = -1/M \) could be interpreted as a physically interesting point [23]. At this point, the Fock space reduces to the Fock space of a single harmonic oscillator, corresponding to the centre-of-mass coordinate.
4.3 Operators \( N_{ij}, N_i, N \) and \( K_{ij} \)

Now, we proceed to the construction of \( N_{ij}, N_i, N \) and \( K_{ij} \) operators. This construction can be performed for any \( M \) but, for simplicity, we present the results for the first non-trivial case \( M = 3 \). All these constructions exist only if the positivity condition \( \nu > -1/M \), is satisfied. For \( M = 3, \nu > -1/3 \). The construction starts with expanding the corresponding operator in a series in \( a_i \) and \( a_i^\dagger \); for example, (indices are omitted for brevity)

\[
K_{ij} = c_0 + \sum c_1 a_i^\dagger a + \sum c_2 a_i^\dagger a a + \cdots \tag{36}
\]

Using the definitions (25,26), we act with (36) on the vacuum (which gives \( c_0 = 1 \)), then on the one-particle state, the two-particle state, etc... In this way, we obtain linear recursive relations which are easily solved. The result for \( K_{12} \) and \( M = 3 \) is

\[
K_{12} = 1 - \frac{1}{(1 + 3\nu)}b_{12}^\dagger b_{12} + \frac{1}{2(1 + 3\nu)^2}b_{12}^2 b_{12} - \frac{\nu}{2(1 + 3\nu)^2(2 + 3\nu)}b_{12}^\dagger b_{123} b_{12} + \cdots, \tag{37}
\]

where \( b_{12} = a_1 - a_2 \) and \( b_{123} = a_1 + a_2 - 2a_3 \). One gets \( K_{13} \) and \( K_{23} \) from \( K_{12} \) using permutation invariance. Knowing \( K_{ij} \), one can find a normally ordered expansion \( a_i a_j^\dagger \).

Similarly, one finds \( M = 3 \)

\[
N_1 = \frac{1}{(1 + 3\nu)}a_1^\dagger a_1 + \frac{\nu}{(1 + 3\nu)}a_1^\dagger B_{0,1} - \frac{\nu}{4(1 + 3\nu)(2 + 3\nu)}a_1^\dagger b_{231} b_{23} - \frac{\nu(1 + \nu)}{4(1 + 3\nu)^2(2 + 3\nu)}a_1^\dagger b_{231} b_{231} - \frac{\nu}{2(1 + 3\nu)^2(2 + 3\nu)}a_1^\dagger b_{23} b_{231} + \cdots. \tag{38}
\]

Here, \( b_{23} = a_2 - a_3 \) and \( b_{231} = a_2 + a_3 - 2a_1 \). \( N_{12} \) is easy to obtain from the above formula. Note that \( N_1^\dagger \neq N_1 \). Similarly, \( N_{12}^\dagger \neq N_{21} \). However, the total number
operator $N$ is hermitean, $N^\dagger = N$ and for $M = 3$:

$$
N = \frac{1}{(1+3\nu)} \sum_{i=1}^{3} a_i^\dagger a_i + \frac{\nu}{(1+3\nu)} (\sum_{i=1}^{3} a_i^\dagger)(\sum_{i=1}^{3} a_i) + \\
+ \frac{\nu}{(1+3\nu)^2(2+3\nu)} \sum_{i<j}^{3} (a_i^\dagger - a_j^\dagger)^2(a_i - a_j)^2 + \\
+ \frac{2\nu^2}{(1+3\nu)^2(2+3\nu)} [\sum_{i=1}^{3} a_i^\dagger a_i - \sum_{i<j}^{3} a_i^\dagger a_j][\sum_{i=1}^{3} a_i - \sum_{i<j}^{3} a_i a_j] \equiv \\
\equiv \frac{1}{(1+3\nu)} B_{1,1} + \frac{\nu}{(1+3\nu)} B_{0,1}^\dagger B_{0,1} + \\
+ \frac{\nu}{(1+3\nu)^2(2+3\nu)} [2\nu^2 B_{0,2}^\dagger - \frac{1}{2} B_{0,1}^2][\frac{3}{2} B_{0,2} - \frac{1}{2} B_{0,1}^2] + \\
+ 3B_{2,2} + B_{0,2}^\dagger B_{0,2} - 2(B_{2,1}B_{0,1} + h.c.) + 2\sum_{i=1}^{3} a_i^\dagger B_{1,1}a_i \quad (39)
$$

The result is consistent with the general expression $\bar{H} - E_0 = \frac{1}{2} \sum_i \{a_i, a_i^\dagger\} - E_0 = N$. In the limit $\nu \to 0$, we reproduce the standard result $N = \sum_i a_i^\dagger a_i$. Although the above expressions seem to be divergent at the critical point, it appears that for $\nu = -1/3$, the degrees of freedom, the Fock space and the related algebra are substantially reduced [23] and the above expansions are completely regular, giving $N = \frac{1}{3} B_{0,1}^\dagger B_{0,1}$ at $\nu = -1/3$.

### 4.4 Dual operators $\tilde{a}_i$ and dual algebra $\tilde{A}$

Owing to the fact that $[a_i, a_j] = 0$ and $[a_i^\dagger, a_j^\dagger] = 0, \forall (i,j)$, we can define the operators $\tilde{a}_i, i = 1, 2,...M, \nu > \frac{1}{M},$ such that

$$
\tilde{a}_i(a_{i_1}^\dagger \cdots a_{i_m}^\dagger |0\rangle) = \sum_{\alpha=1}^{m} \delta_{\alpha i} a_{i_1}^\dagger \cdots a_{i_m}^\dagger a_{i_m} \cdots a_{i_1}^\dagger |0\rangle,
$$

\(\tilde{a}_i |0\rangle = 0\);
where the hat denotes omission of the corresponding operator.

The sum on the RHS contains $m_i$ terms. We find that:

$$\tilde{a}_i (a_{i_1}^{m_1} \cdots a_{i_{M_i}}^{m_{M_i}} |0\rangle) = m_i (a_{i_1}^{m_1} \cdots a_{i_{M_i}-1}^{m_{M_i}} \cdots a_{i_M}^{m_M} |0\rangle)$$

and

$$[\tilde{a}_i, a_j^\dagger] = \delta_{ij}, \quad [\tilde{a}_i, \tilde{a}_j] = 0, \quad \forall (i, j). \quad (41)$$

These relations are satisfied on all monomial states in Fock space. If we define a dual Fock space, spanned by monomials $(\langle 0|\tilde{a}_{i_1} \cdots \tilde{a}_{i_M})$, we obtain the following relation, as a consequence of Eq.(41):

$$\langle 0|\tilde{a}_{i_1}^{\tilde{m}_1} \cdots \tilde{a}_{i_M}^{\tilde{m}_M} a_{i_1}^{\dagger m_1} \cdots a_{i_M}^{\dagger m_M} |0\rangle = \prod_{k=1}^{M} n_k! \delta_{m_k \tilde{m}_k}$$

We call the operators $\tilde{a}_i$ the bosonic duals of operators $a_i^\dagger$.

The transition number operators $N_{ij}$, the partial number operators $N_i$ and the total number operator $N$ can now be expressed as:

$$N_{ij} = a_i^\dagger \tilde{a}_j, \quad \forall (i, j),$$

$$N_i = a_i^\dagger \tilde{a}_i, \quad \forall i,$$

$$N = \sum_{i=1}^{M} a_i^\dagger \tilde{a}_i \quad \forall i.$$

From the expression for $N_{ij}$ we obtain $\tilde{a}_j$ and vice versa. Symbolically,

$$\tilde{a}_j = a_j + \sum_{k \geq 1} a_i^k a_i^{k+1}.$$
where \( b_{kji} = (a_k - a_i) + (a_j - a_i) \) and \( b_{kj} = (a_k - a_j) \). Hence, we obtain new families of commuting operators \( \tilde{a}_i(\nu), \ i = 1, 2...M \) and \( \nu > -1/M \). They satisfy a new commutation relation:

\[
[\tilde{a}_i(\nu), \tilde{a}^\dagger_j(\nu)] = \tilde{A}_{ij}(\nu)
\]

and we call it dual algebra \( \tilde{A}(\nu) \) to the algebra of Eq.(29). Of course, \([\tilde{a}_i(\mu), \tilde{a}_j(\nu)] \neq 0 \) for \( \nu \neq \mu \) (as \([a_i(\mu), a_j(\nu)] \neq 0 \) for \( \nu \neq \mu \)).

The definition and structure of the algebra dual to a general algebra of \( a_i \) and \( a_i^\dagger \) operators is an interesting problem. Its physical meaning is connected with the construction of new models which are dual to the initial one.

### 4.5 Mapping to free Bose oscillators

It was found that the M-body Calogero model in the harmonic potential (28) could be mapped to M ordinary free Bose oscillators [24]. The mapping was performed in the coordinate space (not in the Fock space) and no restriction on \( \nu \) was found or discussed. Since the whole Fock space (spanned by the monomials \( a_{i_1}^{\dagger m_1} \cdots a_{i_M}^{\dagger m_M} |0\rangle \)) for the M-body Calogero model with \( \nu > -1/M \), is isomorphic to the Fock space of M free Bose oscillators with \( \nu = 0 \), we conclude that there must exist a regular mapping from \((a_i, a_i^\dagger)\) to \((b_i, b_i^\dagger)\) and vice versa. To ensure the existence of the mapping \( a = \Psi(b, b^\dagger) \), the following relations have to be satisfied.

\[
[a_i^\dagger, a_i^\dagger] = [a_i, a_j] = 0,
\]

\[
[N_i, a_j^\dagger] = \delta_{ij} a_i^\dagger, \quad \forall(i, j).
\]

(42)

The sufficient condition for the existence of the inverse real mapping \( b = \Psi^{-1}(a, a^\dagger) \) is \( \nu > -1/M \).
Our results on mappings can be generalized in the following way. If two algebras of operators, e.g. \((a_i, a_i^\dagger)\) and \((b_i, b_i^\dagger)\), have completely isomorphic Fock spaces (i.e. the same structure of all Gram matrices), then there exists a regular, real mapping from \(a_i\) to \(b_i\) and vice versa. If one Fock space is isomorphic with a subspace of the second Fock space, then there exists the mapping \(a = \Psi(b, b^\dagger)\), but there is no inverse mapping. The construction of the mapping for Calogero operators \(a_i, i = 1, \ldots, M\), (Eq.(29)), will be considered in a separate publication.

5 Conclusion

In conclusion, we want to point out the main results of the paper. In section 2 we have applied the general results of Ref.[9] to the Calogero-Vasiliev single-mode oscillator algebra (3), underlying the two-body Calogero model. We have expressed the number operator \(N\) (11) and the exchange operator \(K\) (10) as an infinite series in creation and annihilation operators and have recursively calculated the coefficients of expansion. We have found a mapping (13) from Calogero-Vasiliev oscillators \((a, a^\dagger)\) to the Bose oscillators \((b, b^\dagger)\). The mapping has the form of an infinite series in \((b, b^\dagger)\). Finally, we have defined a new operators \((\tilde{a}, \tilde{a}^\dagger)\) which are dual to the operators \((a, a^\dagger)\) in the sense that \([\tilde{a}, a^\dagger] = 1\). We have found a connection between the operators \(a, \tilde{a}\) and \(b\), Eqs.(12,15,16).

Section 3 is devoted to the generalization of the single-mode oscillator algebras to the multi-mode case. We have discussed two distinct classes of multi-mode oscillator algebras: (i) permutation invariant algebras with hermitean number operators and (ii) permutation invariant algebras with non-hermitean number operators. The class (ii) has not been discussed previously and in Subsection 3.2 several new results
for those algebras were given. Both classes have been treated in three completely equivalent ways, first proposed in [7]. In the analysis, we have used concepts of a normally ordered expansion (18), the action of annihilation operators on the states in Fock space (19) and the notion of Gram matrices of scalar products in Fock space (20). We have found the general structure of the number operators, Eqs.(23,24), and the exchange operators, Eqs.(25)-(27). The results of this section have been applied in Section 4 to the $S_M$-extended Heisenberg algebra (29), underlying the M-body Calogero model. While the previous analyses of this algebra were performed mainly on the symmetric (or antisymmetric) subspace of the whole Fock space [6], here we have analysed the whole Fock space of states. Our main results are the following.

We have rewritten the $S_M$-extended Heisenberg algebra in the form of the (generalized) triple operator algebra (31). This is a generalization of the known result for the single-mode case (3). Then, we have found the action of annihilation operators on the monomials in Fock space (35). Using this, we have calculated one- and two-particle Gram matrices and discussed their structure and eigensystem. We have found that there exists a universal critical point in Fock space, given by $\nu = -1/M$, and all states in Fock space have positive norms for $\nu > -1/M$. As we have made a comment in Remark 4, the same critical point exists for a large class of $S_M$-extended Heisenberg algebras. Then, we have proceeded with the construction of number operators and exchange operators. We have given explicit examples of the structure of these operators in the case of $M = 3$, Eqs.(37)-(39). Generalizing the construction of the dual algebra from Section 2, we have defined a dual multi-mode algebra in terms of the operators $(\tilde{a}_i, \tilde{a}_i^\dagger)$ (40,41). With this operators, we have been able to write the number operators in a compact form. Finally, we have shortly discussed the existence of mapping from the $S_M$-extended Heisenberg algebra (29) to Bose
oscillators.
We note that the Calogero model has been related [25] to $q$-deformed quantum mechanics [26]. The ideas presented here may help in elucidating the connection between algebraic structures arising from the deformation of the phase space of ordinary quantum mechanics and Calogero-type models.

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A Appendix

Here, we give two examples of complete Gram matrices for $M = 2$ and $M = 3$ oscillators and two-particle states.

**Example A.1** The Gram matrix for $M = 2$. The matrix is written in the basis \( \{ a_1^\dagger |0\rangle, a_2^\dagger |0\rangle, a_1^\dagger a_2^\dagger |0\rangle, a_2^\dagger a_1^\dagger |0\rangle \} \).

\[
\begin{pmatrix}
    a & b & b & b \\
    b & a & b & b \\
    b & b & d & d \\
    b & b & d & d \\
\end{pmatrix},
\]

where $a = 2 + 3\nu$, $b = -\nu$ and $d = 1 + \nu$.

**Example A.2** The Gram matrix for $M = 3$. The matrix is written in the basis \( \{ a_1^\dagger |0\rangle, a_2^\dagger |0\rangle, a_3^\dagger |0\rangle, a_1^\dagger a_2^\dagger |0\rangle, a_1^\dagger a_3^\dagger |0\rangle, a_2^\dagger a_3^\dagger |0\rangle, a_2^\dagger a_1^\dagger |0\rangle, a_3^\dagger a_1^\dagger |0\rangle, a_3^\dagger a_2^\dagger |0\rangle \} \).

\[
\begin{pmatrix}
    a & b & b & b & c & b & b & c \\
    b & a & b & b & c & b & b & c \\
    b & b & a & c & b & b & c & b \\
    b & b & a & c & b & b & c & b \\
    b & c & b & d & b & d & b & b \\
    b & c & b & d & b & d & b & b \\
    c & b & b & d & b & b & d & b \\
    c & b & b & d & b & b & d & b \\
\end{pmatrix},
\]

where $a = 2 + 6\nu + 2\nu^2$, $b = -\nu - \nu^2$, $c = 2\nu^2$ and $d = 1 + 3\nu + 2\nu^2$.

It is straightforward to write two-particle Gram matrices for any $M$. Non-zero
matrix elements are of the type ($i, j, k, l) = 1, 2, \ldots M$)

$$\langle 0 | a_i^2 a_i^{12} | 0 \rangle \equiv a = [1 + \nu(M - 1)][2 + \nu(M - 1)] - \nu^2(M - 1),$$

$$\langle 0 | a_i^2 a_j^{12} | 0 \rangle = \langle 0 | a_i a_j a_i^{12} | 0 \rangle = \langle 0 | a_i a_j a_k a_i^{12} | 0 \rangle =$$

$$= \langle 0 | a_i a_j a_i^\dagger a_k^\dagger | 0 \rangle \equiv b = -\nu - \nu^2(M - 2),$$

$$\langle 0 | a_i a_j a_k^{12} | 0 \rangle = \langle 0 | a_i a_j a_k a_i^\dagger a_j^\dagger | 0 \rangle \equiv c = 2\nu^2,$$

$$\langle 0 | a_i a_j a_i^\dagger a_j^\dagger | 0 \rangle \equiv d = [1 + \nu(M - 1)][1 + \nu(M - 2)].$$

It is understood that different indices are not equal.
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