A birationality result for character varieties

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Abstract Let $M$ be an orientable, cusped hyperbolic 3–manifold of finite volume. We show that the restriction map $r: \mathcal{X}_0 \to \mathcal{X}(\partial M)$ from a Dehn surgery component in the $PSL_2(\mathbb{C})$–character variety of $M$ to the character variety of the boundary of $M$ is a birational isomorphism onto its image. This generalises a result by Nathan Dunfield. A key step in our proof is the exactness of Craig Hodgson’s volume differential on the eigenvalue variety.

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1 Introduction

Let $M$ be an orientable cusped complete hyperbolic 3–manifold of finite volume. There is a discrete and faithful representation of $\pi_1(M)$ into $PSL_2(\mathbb{C})$, and its character $\chi_0$ is known to be a smooth point of the $PSL_2(\mathbb{C})$–character variety $\mathcal{X}(M)$. The irreducible component $\mathcal{X}_0$ of $\mathcal{X}(M)$ containing $\chi_0$ is called a Dehn surgery component of $\mathcal{X}(M)$. The inclusion map $\partial M \to M$ induces a restriction map

$$r: \mathcal{X}(M) \to \mathcal{X}(\partial M).$$

It is shown in [4] that the restriction of $r$ to $\mathcal{X}_0$ is finite–to–one onto its image. In the case where $M$ has only got one cusp, Dunfield has shown in [4] that $r: \mathcal{X}_0 \to \mathcal{X}(\partial M)$ has degree one onto its image using Thurston’s Hyperbolic Dehn Surgery Theorem and a Volume Rigidity Theorem attributed to Gromov, Thurston and Goldman. This note generalises Dunfield’s result to manifolds with arbitrary number of cusps:

**Theorem 1** Let $M$ be an orientable, non–compact, complete hyperbolic 3–manifold of finite volume. Let $\mathcal{X}_0$ be a Dehn surgery component in the $PSL_2(\mathbb{C})$–character variety of $M$. Then the restriction map $r: \mathcal{X}_0 \to \mathcal{X}(\partial M)$ is a birational isomorphism onto its image.

**Corollary 2** Let $M$ be an orientable, non–compact, complete hyperbolic 3–manifold of finite volume with $h$ cusps. Let $\mathcal{X}_0$ be a Dehn surgery component in the $SL_2(\mathbb{C})$–character variety of $M$. Then the restriction map $r: \mathcal{X}_0 \to \mathcal{X}(\partial M)$ has degree at most $2^{-h}|H^1(M, \mathbb{Z}_2)|$ onto its image. In particular, if $H^1(M, \mathbb{Z}_2) \approx \mathbb{Z}_2^h$, then the map is a birational isomorphism.
The proof of Theorem 1 is a generalisation of Dunfield’s argument. Our new contributions are the construction of an explicit Zariski dense set in $\widetilde{X}_0$, and a proof of the fact that Hodgson’s volume differential [7] is an exact form on the eigenvalue variety [14]. The eigenvalue variety $\mathcal{E}(M)$ of a multi-cusped hyperbolic 3-manifold is a natural generalisation of the curve defined by the $A$–polynomial [2] for a once-cusped hyperbolic 3–manifold. Choose a basis $(M_i, L_i)$ for $H_1(T_i)$, where $T_i$ is a torus cross-section of the $i$–th cusp, and suppose that $M$ has $h$ cusps. Then the eigenvalue variety is the Zariski closure of the set of points $(m_1, l_1, \ldots, m_h, l_h)$ in $(\mathbb{C}\{0\})^{2h}$ with the property that there is a representation $\rho: \pi_1(M) \rightarrow SL_2(\mathbb{C})$ such that $\rho(M_i)$ and $\rho(L_i)$ have eigenvalues $m_i$ and $l_i$ respectively with respect to a common eigenvector. In this notation, Hodgson’s volume differential is the 1–form

$$\eta = -\frac{1}{2} \sum_{i=1}^{h} \left( \log |l_i| \, d \arg m_i - \log |m_i| \, d \arg l_i \right)$$

on the eigenvalue variety, where $M$ is oriented, each boundary component is given the induced orientation and $(M_i, L_i)$ is a left-handed basis with respect to this orientation.

The geometric significance of this form is as follows. A representation $\rho: \pi_1(M) \rightarrow SL_2(\mathbb{C})$ determines a pseudo-hyperbolic structure on $M$ with holonomy $\rho$. Taking the volume of this structure (which is possibly zero or negative) gives a function on the $SL_2(\mathbb{C})$–character variety, $\text{Vol}_M: \widetilde{X}(M) \rightarrow \mathbb{R}$. Dunfield establishes fundamental results about this function, which carry over to our setting. These are recalled in Section 2. Work of Hodgson [7] and Neumann and Zagier [11] shows that $d\text{Vol}_M = -\frac{1}{2} \omega$ (see Section 4.5 of [2] and Section 3 below), where $\omega$ is the 1–form $\eta$ interpreted as a form on the complement of a suitable subvariety of $\widetilde{X}_0$. We prove in Section 4 that $\eta$ is also exact on the complement of a suitable subvariety of the eigenvalue variety. This is then used to show that the map $\tau$ on $PSL_2(\mathbb{C})$–characters has degree one by studying it on a suitable Zariski dense set of characters. This set is defined in Section 5. The proofs of Theorem 1 and Corollary 2 are put together in Section 6.

Dunfield applies his birationality theorem to settle a conjecture due to Boyer and Zhang concerning cyclic surgeries of certain hyperbolic knots. Theorem 1 can be used to understand Dehn surgery spaces of multi-cusped hyperbolic 3-manifolds; see Klaff [9] for an application.

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2 Volume of representations

We refer the reader to [3, 11] for standard facts about character varieties, and to [10] as a reference for algebraic geometry. This section summarises the material we need from Dunfield [4].

Let $M$ be a complete hyperbolic 3–manifold of finite volume. If $M$ is closed, then the volume of a representation $\rho: \pi_1(M) \rightarrow PSL_2(\mathbb{C})$ is well-defined by letting $\text{Vol}_M(\rho) = \int_F f_\rho^*(\text{vol}_{\mathbb{H}^3})$, where $f_\rho: \widetilde{M} \rightarrow \mathbb{H}^3$ is any smooth equivariant map, $\text{vol}_{\mathbb{H}^3}$ is the usual volume form on hyperbolic space and $F$ is any fundamental domain for $M$. The volume is an invariant of the conjugacy class of a representation, and Dunfield proves the following:
**Theorem 3** (Gromov-Thurston-Goldman, in Dunfield [4], Theorem 6.1) Let $M$ be a closed, hyperbolic 3–manifold of finite volume, and $\chi \in \chi(M)$. If $\rho$ is a representation with character $\chi$ and $\Vol_M(\rho) = \pm \Vol(M)$, then $\rho$ is discrete and faithful.

If $M$ is not closed, Dunfield defines the volume of a representation $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ with respect to a so-called pseudo-developing map $f_\rho : \hat{M} \to \mathbb{H}^3$. A pseudo-developing map is a $\rho$–equivariant map which satisfies two technical conditions which ensure that the integral defining the volume of $\rho$ is both finite and independent of the chosen pseudo-developing map. We recall the definition.

The space $\mathbb{H}^3$ is the usual compactification of $\mathbb{H}^3$ obtained by adding the sphere at infinity. The space $\hat{M}$ is obtained from $M$ by adding countably many points as follows. The manifold $M$ is naturally the interior of a compact manifold $N$ with boundary. Choose a collar neighbourhood $T^2 \times [0, \infty]$ for each boundary component of $N$, where we assume that $T^2 \times \{\infty\} \subseteq \partial N$. Then $\hat{M}$ is the quotient space obtained by collapsing $T^2 \times \{\infty\}$ to a point. There is a natural inclusion $M \hookrightarrow \hat{M}$ and $\hat{M} \setminus M$ is a finite collection of points, one for each cusp of $M$. The construction for the universal cover is analogous. Lift the product structure at each boundary component of $N$ to $\hat{N}$. Each connected component of one of the collar neighbourhoods in $N$ is of the form $\mathbb{R}^2 \times [0, \infty]$, and $\hat{M}$ is obtained by collapsing each $\mathbb{R}^2 \times \{\infty\}$ to a point. Note that there is a natural map $\hat{M} \to \hat{M}$, and each $v \in \hat{M} \setminus \hat{M}$, has a neighbourhood of the form $N_v = (P_v \times [0, \infty)) \cup \{v\}$. The action of $\pi_1(M)$ by deck transformations extends naturally to $\hat{M}$ and preserves the chosen product structure of the cusps. With this notation, a $\rho$–equivariant map $f_\rho : \hat{M} \to \mathbb{H}^3$ is a pseudo-developing map if it satisfies the following two conditions:

1. $f_\rho(\hat{M}) \subseteq \mathbb{H}^3$ and $f_\rho(\hat{M} \setminus \hat{M}) \subseteq \partial \mathbb{H}^3$; and
2. for each $v \in \hat{M} \setminus \hat{M}$, $f_\rho$ maps each ray $\{p\} \times [0, \infty)$ in $N_v$ to a geodesic ray in $\mathbb{H}^3$ with ideal endpoint $f_\rho(v)$ and parameterises this ray by arc-length with respect to the cone parameter.

Given a pseudo-developing map $f_\rho : \hat{M} \to \mathbb{H}^3$, define $\Vol_M(\rho, f_\rho) = \int_F f_\rho^*(\text{vol}_{\mathbb{H}^3})$, where, as above, $F$ is a chosen fundamental domain. In fact, Dunfield takes the absolute value of this integral, but for our purposes, it will be more convenient to work with a signed volume that takes orientation into account. Also, Dunfield only considers hyperbolic 3–manifolds with one cusp. However, a careful examination of the material in Section 2.5 of [4] reveals that it applies to multi–cusped hyperbolic 3–manifolds. The following results from [4] are therefore at our disposal:

**Lemma 4** (Dunfield [4], Lemma 2.5.2) The function $\Vol_M : \chi_0 \to \mathbb{R}$ defined by taking $\Vol_M(\chi) = \Vol_M(\rho, f_\rho)$, where $\rho$ is any representation with character $\chi$ and $f_\rho$ is any pseudo-developing map for $\rho$, is well–defined.

**Lemma 5** (Dunfield [4], Lemma 2.5.4) Let $M$ be a complete cusped hyperbolic 3–manifold of finite volume, and let $N$ be a closed hyperbolic 3–manifold obtained by Dehn filling on $M$. Assume that $\rho$ is a representation of $\pi_1(M)$ which factors through a representation $\rho'$ of $\pi_1(N)$. Then $\Vol_N(\rho') = \Vol_M(\rho, f_\rho)$, where $f_\rho$ is any pseudo-developing map for $\rho$.

For a representation $\rho : \pi_1(M) \to SL_2(\mathbb{C})$, we define the volume of $\rho$ to be the volume of its composition with the quotient map $SL_2(\mathbb{C}) \to PSL_2(\mathbb{C})$.一事
3 Hodgson’s volume differential

For the material of this section, which is well-known to experts, we refer the reader to Hodgson [7], Neumann–Zagier [11] and Cooper-Culler-Gillet-Long-Shalen [2]. Complete details can be found in [8], and we will only give an overview.

Consider lifts \( \tilde{\chi}_0 \) of \( \chi_0 \) and \( \tilde{\mathcal{X}}_0 \) of \( \mathcal{X}_0 \) to the \( SL_2(\mathbb{C}) \)–character variety. Also denote \( \tilde{\rho}_0 \) a lift of \( \tilde{\chi}_0 \) to the \( SL_2(\mathbb{C}) \)–representation variety. We may choose a fundamental domain \( D_0 \) for the action of \( \tilde{\rho}_0 \) consisting of a union of ideal hyperbolic polyhedra (see Epstein and Penner [5]). The ideal vertices of the polyhedra correspond to fixed points of peripheral groups. Given a smooth 1–parameter family of representations \( \tilde{\rho}_t \) sufficiently close to \( \tilde{\rho}_0 \), we obtain a smooth 1–parameter family of fundamental domains \( D_t \) obtained by small deformations of \( D_0 \), because all peripheral groups have images not contained in \( \{ \pm E \} \) near \( \tilde{\rho}_0 \) and so the fixed point sets vary smoothly. Writing \( \tilde{\chi}_t \) for the character of \( \tilde{\rho}_t \), we have \( \operatorname{Vol}_M(\tilde{\chi}_t) = \operatorname{Vol}_M(\tilde{\rho}_t) = \operatorname{Vol}(D_t) \), noting that there is a natural definition of a pseudo-developing map at the complete structure, and that this can be deformed for nearby representations using \( D_t \).

Choose a basis \( \{ \mathcal{M}_i, \mathcal{L}_i \} \) for \( H_1(T_i) \), where \( T_i \) is a torus cross-section of the \( i \)–th cusp, and we use the same orientation conventions as in the introduction. The eigenvalues \( m_i(t) \) and \( l_i(t) \) associated to a common eigenvector of \( \tilde{\rho}_t(\mathcal{M}_i) \) and \( \tilde{\rho}_t(\mathcal{L}_i) \) vary smoothly with \( t \). Hodgson [7] computes the derivative of volume using the Milnor-Schl"afli formula for the derivative of volume of a smooth 1–parameter family of hyperbolic polyhedra with ideal vertices, obtaining:

\[
\frac{d}{dt} \operatorname{Vol}_M(\tilde{\chi}_t) = -\frac{1}{2} \sum_{i=1}^h \left( \log |l_i(t)| \frac{d}{dt} \arg m_i(t) - \log |m_i(t)| \frac{d}{dt} \arg l_i(t) \right).
\]

As discussed in [7, 2], the above application of the Schl"afli formula can be modified to show that the above formula holds at each point of the character variety (not just on \( \mathcal{X}_0 \)), except possibly at the points where a peripheral subgroup is contained in \( \{ \pm E \} \). The key idea is to allow both ideal and finite vertices, and negatively oriented or degenerate polyhedra (whose volume is negative or zero respectively). The main issue at \( \{ \pm E \} \) is that the fixed point set may not converge, and so the above argument of deforming polyhedra with ideal vertices may not apply. To avoid this situation, let

\[
V = \tilde{\mathcal{X}}_0 \cap \bigcup_{i=1,\ldots,h} \{ \chi(\mathcal{M}_i)^2 = \chi(\mathcal{L}_i)^2 = 4 \}.
\]

Then \( V \) is a proper subvariety of \( \tilde{\mathcal{X}}_0 \), and the function \( \operatorname{Vol}_M : \tilde{\mathcal{X}}_0 \setminus V \to \mathbb{R} \) is smooth.

4 Exactness of the volume differential

Given \( \gamma \in \pi_1(M) \), there is a rational function \( I_\gamma : \tilde{\mathcal{X}}_0 \to \mathbb{C} \) defined by \( \tilde{\chi} \to \tilde{\chi}(\gamma) \). A well-known application of Thurston’s Hyperbolic Dehn Surgery Theorem shows that \( \tilde{\mathcal{X}}_0 \) is a smooth point of \( \tilde{X}(M) \), and that the map \( I : \tilde{\mathcal{X}}_0 \to \mathbb{C}^h \) defined by \( I(\tilde{\chi}) = (I_{\mathcal{M}_1}, \ldots, I_{\mathcal{M}_h}) \) maps a small open neighbourhood of \( \tilde{\mathcal{X}}_0 \) analytically to an open neighbourhood of \( (e_2, \ldots, e_2) \), where \( I_{\mathcal{M}_i}(\tilde{\chi}_0) = e_2 \).

See, for instance Thurston [13] (Section 5.8), Neumann–Zagier [11] or Porti [12] (Corollaries 3.27 and 3.28).
This implies that there is a simply connected neighbourhood $V \subset \tilde{X}_0$ of $\tilde{\chi}_0$ with the property that the volume of any character $\tilde{\chi} \in V$ is independent of the path of integration chosen from $\tilde{\chi}_0$ to $\tilde{\chi}$ inside $V$ and, moreover, that the volume of $\tilde{\chi}$ only depends on its image under the restriction map $r: \tilde{X}_0 \to \tilde{X}(\partial M)$, because the latter can be imbedded with the coordinates $(I_{M_1}, I_{L_1}, I_{M_2}, \ldots, I_{M_h}, I_{L_h}, I_{M_h, L_h})$. Note that this observation is not valid globally; for instance, at the character of the discrete and faithful representation, we have $r(\chi_0) = r(\tilde{\chi}_0)$ but $\text{Vol}_M(\tilde{\chi}_0) = -\text{Vol}_M(\chi_0) = -\text{Vol}(M) = \text{Vol}_M(\chi_0)$. We will now show that points with this property are exceptional: the peripheral traces do determine the volume in the complement of a proper subvariety.

**Proposition 6** Denote $\mathfrak{C}_0$ the component of $\mathfrak{C}(M)$ corresponding to $\tilde{X}_0$. Then there is a proper subvariety $U$ of $\mathfrak{C}_0$ such that the 1-form $\eta$ is exact on $\mathfrak{C}_0 \setminus U$.

Moreover, there is a proper subvariety $V' \subset \tilde{X}_0$ containing $V$ with the property that the restriction $\text{Vol}_M: \tilde{X}_0 \setminus V' \to \mathbb{R}$ factors through a real valued function on $r(\tilde{X}_0 \setminus V')$. In particular, if $\chi_1, \chi_2 \in \tilde{X}_0 \setminus V'$ and $r(\chi_1) = r(\chi_2)$, then $\text{Vol}_M(\chi_1) = \text{Vol}_M(\chi_2)$.

**Proof** The quotient map $p: \mathfrak{C}_0 \to \tilde{X}(\partial M)$ is division by $\Gamma = \mathbb{Z}^2_2$, where the $h$ $\mathbb{Z}_2$-factors are generated by $(m_i, l_i) \to (m_i^{-1}, l_i^{-1})$ for $1 \leq i \leq h$. The union of the fixed point sets is contained in the proper subvariety

$$U = \mathfrak{C}_0 \cap \bigcup_{i=1,\ldots,h} \{m_i^2 = l_i^2 = 1\}.$$ 

Notice that $U$ corresponds to the subvariety $V$ of $\tilde{X}_0$ defined in the previous section.

Denote $p(\mathfrak{C}_0) = Y_0 \subset \tilde{X}(\partial M)$. Then $p^*: H^1(Y_0 \setminus p(U), \mathbb{R}) \to H^1(\mathfrak{C}_0 \setminus U, \mathbb{R})$ is injective with image $H^1(\mathfrak{C}_0 \setminus U, \mathbb{R})^\Gamma$ (see, for instance, Hatcher [5], Proposition 3G.1). The form $\eta$ is invariant under the action of $\Gamma$, and hence $[\eta] \in H^1(\mathfrak{C}_0 \setminus U, \mathbb{R})^\Gamma$. Whence there is a unique class $c \in H^1(Y_0 \setminus p(U), \mathbb{R})$ that maps to $[\eta]$.

The map $r: \tilde{X}_0 \to Y_0$ has finite degree [14], whence it is a branched cover and the branch set is contained in a proper subvariety of $\tilde{X}_0$. Denote $V'$ the union of this subvariety with $V$, and note that $V$ is invariant under the covering transformations, so that $r: \tilde{X}_0 \setminus V' \to Y_0$ is a finite cover onto its image. Let $W' = r(V')$. Since subvarieties have real co-dimension at least two, the restriction $r: \tilde{X}_0 \setminus V' \to Y_0 \setminus W'$ is a finite cover of connected topological spaces, and so

$$r^*: H^1(Y_0 \setminus W', \mathbb{R}) \to H^1(\tilde{X}_0 \setminus V', \mathbb{R})$$

is an isomorphism. The definitions of $\eta$, $r$ and $p$ as well as Hodgson’s formula for the volume differential imply that $d\text{Vol}_M \in r^*(c)$. Whence $r^*(c) = [d\text{Vol}_M] = 0$. Since $r^*$ is an isomorphism, we have $c = 0$, and so $[\eta] = p^*(c) = 0$. This completes the proof of the exactness statement.

Since the subvariety $U$ has real co-dimension at least 2, $[\eta] = 0$ implies that there is a function $\text{Vol}_\mathfrak{C}: \mathfrak{C}_0 \setminus U \to \mathbb{R}$ with $d\text{Vol}_\mathfrak{C} = \eta$, and which we normalise so that for some $x_1 \in \mathfrak{C}_0$ with $r(x_1) \in Y_0 \setminus W'$, we have $\text{Vol}_\mathfrak{C}(x_1) = \text{Vol}_M(\chi)$, where $\chi$ satisfies $p(x_1) = r(\chi)$. Since $\eta$ is invariant under $\Gamma$, there is a function $\text{Vol}_\mathfrak{C}: Y_0 \setminus W' \to \mathbb{R}$ such that $\text{Vol}_\mathfrak{C}: \mathfrak{C}_0 \setminus U \to \mathbb{R}$ factors through $\text{Vol}_\mathfrak{C}$.

The last claim follows if we show that $\text{Vol}_M: \tilde{X}_0 \setminus V' \to \mathbb{R}$ also factors through $\text{Vol}_\mathfrak{C}$. This is done using the following construction from [14]. If $\tilde{X}(M)$ is a variety in $\mathfrak{C}^m$, we now define the variety
\( \tilde{X}_E(M) \) in \( \mathfrak{C}^m \times (\mathfrak{C} \setminus \{0\})^{2h} \) by adding \([m_i^{\pm 1}, l_i^{\pm 1}, \ldots, m_h^{\pm 1}, l_h^{\pm 1}] \) to the coordinate ring of \( \tilde{X}(M) \) and adding the following generators to the ideal defining \( \tilde{X}(M) \):

\[
\begin{align*}
I_{M_i} &= m_i + m_i^{-1}, \\
I_{L_i} &= l_i + l_i^{-1}, \\
I_{M_i L_i} &= m_i l_i + m_i^{-1} l_i^{-1},
\end{align*}
\]

for \( i = 1, \ldots, h \), noting that the left-hand sides are polynomials in the coordinates of \( \tilde{X}(M) \). The projections \( p_2 : \tilde{X}_E(M) \to \tilde{X}(M) \) and \( r_E : \tilde{X}_E(M) \to \mathfrak{E}(M) \) are dominating maps. Every point in \( \tilde{X}_E(M) \) is of the form \((\chi, x)\), where \( \chi \in \tilde{X}(M) \) and \( x \in \mathfrak{E}(M) \). We define \( \text{Vol} : \tilde{X}_E(M) \to \mathbb{R} \) by \( \text{Vol}(\chi, x) = \text{Vol}_M(\chi) \). In these coordinates, Hodgson’s work shows \( \text{d} \text{Vol} = r_E^* \eta \) in the complement of the set of characters sending a peripheral subgroup to \( \eta \in \mathfrak{E}(M) \). Moreover, there is a unique discrete and faithful character \( \chi_0 \in \mathfrak{X}_0(M) \) which corresponds to the complete hyperbolic structure on \( M \) with the chosen orientation, and a neighbourhood \( U(\chi_0) \) such that if \( \kappa \in N \) and \( \chi_\kappa \) is the character of the holonomy of \( M_\kappa \), then \( \chi_\kappa \in U(\chi_0) \).

Let \( M \) be a complete hyperbolic 3–manifold with \( h \) cusps and a chosen orientation. Choose a basis \( \{M_i, L_i\} \) for \( H_1(T_i) \), where \( T_i \) is a torus cross section of the \( i \)–th cusp. Denote by \( M_\kappa \) the oriented 3–manifold obtained by Dehn surgery on \( M \) with coefficient \( \kappa = (p_1, q_1; \ldots; p_h, q_h) \), where \((p_i, q_i)\) is either a co–prime pair of integers or \( \infty \). Thurston showed that \( M_\kappa \) has a complete hyperbolic structure for all \( \kappa \) in a neighbourhood \( N = \left\{ (\infty; \ldots; \infty) \right\} \) in \( S^2 \times \ldots \times S^2 \), and that \( \lim_{\kappa \to \infty} \text{Vol}(M_\kappa) = \text{Vol}(M) \).

\[ L \subset \{ \chi_\kappa \ | \ \kappa \in S' \}. \]

We claim that \( W \) is a Zariski dense subset of \( \mathfrak{X}_0 \). Choose a lift \( \tilde{\chi}_0 \) of \( \chi_0 \) in the \( \text{SL}_2(\mathfrak{C}) \)–character variety and corresponding lifts \( \tilde{\mathfrak{X}}_0 \) of \( \mathfrak{X}_0 \), \( \tilde{U} \) of \( U(\chi_0) \) and \( \tilde{W} \subset \tilde{U} \) of \( W \). Since the quotient map \( \tilde{\mathfrak{X}} \to \mathfrak{X} \) is finite–to–one, it suffices to show that \( \tilde{W} \) is a Zariski dense subset of \( \tilde{\mathfrak{X}}_0 \).

Given \( \gamma \in \pi_1(M) \), there is a rational function \( I_\gamma : \tilde{\mathfrak{X}}_0 \to \mathfrak{C}^h \) defined by \( \chi \mapsto \chi(\gamma) \). Thurston shows in [13], Section 5.8, that the map \( I : \tilde{\mathfrak{X}}_0 \to \mathfrak{C}^h \) defined by \( I(\chi_\rho) = (\gamma_1, \ldots, \gamma_h) \) maps a small open neighbourhood of \( \chi_0 \) to an open neighbourhood of \( (\epsilon_1, 2, ..., \epsilon_h, 2) \), where \( \gamma_i(\tilde{\chi}_0) = \epsilon_i \cdot 2 \) and \( \gamma_i \) is a fixed primitive element of \( H_1(T_i) \). This is only possible if the functions \( I_{\gamma_i} \) are algebraically independent over \( \mathfrak{C} \) as elements of \( \mathfrak{C}[\tilde{\mathfrak{X}}_0] \).

Assume that \( \tilde{W} \) is not Zariski dense in \( \tilde{\mathfrak{X}}_0 \). We proceed by complete induction on the number of cusps. If \( h = 1 \), then \( \tilde{W} \) is a finite collection of points. This is not possible since \( \tilde{W} \) contains the
holonomy characters of infinitely many pairwise non–isometric closed hyperbolic manifolds because their volumes tend to the volume of $M$.

So assume that $\tilde{W}$ is a Zariski dense subset of $\tilde{X}_0(N)$ whenever $N$ has fewer than $h$ cusps, and that the hypothesis fails for $M$, which has $h$ cusps. Thus, $\tilde{W}$ is contained in a finite union of proper subvarieties $\cup U_i \subset \tilde{X}_0$ where $\dim U_i \leq h - 1$. Performing surgery on one cusp, say the first, gives an infinite family of complete hyperbolic $(h - 1)$–cusped manifolds $M_j = M(1, p_j; \infty, \ldots; \infty)$. For each $M_j$, surgeries on the remaining cusps with resulting coefficients contained in $S'$ give a Zariski dense set in $X_0(M_j)$ by the hypothesis; whence each $X_0(M_j)$ must equal some $U_i$, and in particular, infinitely many of these Dehn surgery components are identical. By passing to a subsequence and renumbering, we may assume that $U_1 = X_0(M_j)$ for each $M_j$. The discrete and faithful character of $M_j$ is contained in a finite set determined by the intersection of $U_1$ with the hypersurfaces $I'_{\gamma_1} = 4$, $i = 2, \ldots, h$. Thus, infinitely many of the holonomy representations are conjugate, which is again not possible since the surgery coefficients tend to $(\infty; \ldots; \infty)$, and hence $\lim \Vol(M_j) = \Vol(M)$. This proves that $\tilde{W}$ is a Zariski dense subset of $\tilde{X}_0(M)$.

The quotient map $q : \tilde{X}_0 \to X_0$ and the restriction $r : X_0 \to \mathfrak{X}(\partial M)$ are finite–to–one, and therefore the set $r(W)$ is a Zariski dense subset of $r(X_0)$. Whence the set $Z = r(W) \setminus p(U)$, where $U$ is the subvariety from Proposition 6 is also Zariski dense. Moreover, $Z$ has the property that if $z \in Z$, then there is a character $\chi \in r^{-1}(z)$ which is the character of a holonomy of a closed hyperbolic manifold obtained by Dehn filling on $M$. It is unique (up to complex conjugation) by Mostow rigidity.

6 Proofs

Proof of Theorem 1 Assume that $M$ has $h$ cusps. It suffices to show that each point in the set $Z \subset r(X_0)$ of the previous section has one preimage. By construction, for each $z \in Z$, there is a closed hyperbolic 3–manifold $N = M(\gamma_1, \ldots, \gamma_h)$ obtained by Dehn filling on $M$ and a preimage $\chi \in r^{-1}(z)$ such that $\chi$ is the character of a holonomy for the complete hyperbolic structure on $N$. In particular, we have $\Vol_M(\chi) = \Vol_N(\chi) = \pm \Vol(N)$ by Lemma 5 and Theorem 3.

Now assume that $\chi' \in r^{-1}(z)$ is another preimage. Proposition 6 yields $\Vol_M(\chi) = \Vol_M(\chi')$. We claim that $\chi'$ also factors through $\pi_1(N)$. The characters of peripheral elements are completely determined by $z$. Thus, each peripheral subgroup has a non–trivial rotation in its image and the curves $\gamma_i$ are represented by parabolics. Since a peripheral subgroup is abelian, the images of the $\gamma_i$ must be trivial, and hence $\chi'$ factors through $\pi_1(N)$. Lemma 5 may now be applied:

\[ \Vol_N(\chi') = \Vol_M(\chi') = \Vol_M(\chi) = \Vol_N(\chi) = \pm \Vol(N). \]

Thus, $\chi'$ is a discrete and faithful character corresponding to the complete structure on $N$, and hence by Mostow Rigidity either $\chi' = \chi$ or $\chi' = \overline{\chi}$. Since complex conjugation reverses orientation and hence changes the sign of the volume function, we have $\Vol_N(\chi') = -\Vol_N(\chi)$ in the second case, which implies an offending statement: $\Vol(N) = 0$.

Proof of Corollary 1 The proof of Dunfield, Corollary 3.2, applies almost verbatim. A representation $\rho : \pi_1(M) \to \text{PSL}_2(\mathbb{C})$ has $|H^1(M, \mathbb{Z}_2)|$ pairwise distinct lifts to $\text{SL}_2(\mathbb{C})$. Now $H_1(\partial M, \mathbb{Z}_2) \cong \mathbb{Z}_2^{2h}$ and $\text{im}(H_1(\partial M, \mathbb{Z}_2) \to H_1(M, \mathbb{Z}_2)) \cong \mathbb{Z}_2^h$. So by duality, if $|H^1(M, \mathbb{Z}_2)| = 2^{h+k}$, then the image of these $2^{h+k}$ representations consists (generically) of exactly $2^k$ points in $\mathfrak{X}(\partial M)$. The genericity hypothesis applies to $\rho_0$. Since distinct lifts of $\rho_0$ may be on distinct components of the $\text{SL}_2(\mathbb{C})$–character variety, the degree of $r : \tilde{X}_0 \to \tilde{X}(\partial M)$ is at most $2^k$. 

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