A PROOF OF KOSNIOWSKI CONJECTURE

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ABSTRACT. Let $M$ be a unitary $S^1$-manifold with only isolated fixed points such that $M$ is not a boundary. We show that $4\chi(M) > \dim M$, where $\chi(M)$ is the Euler characteristic of $M$. This gives an affirmative answer of Kosniowski conjecture.

1. INTRODUCTION

A unitary manifold is a smooth closed manifold with a stably complex structure on its tangent bundle. With respect to smooth $S^1$-actions on unitary manifolds, there is a standing more than 40-year-old conjecture, posed by Kosniowski in his 1979’s paper [10, Conjecture A]. Specifically speaking, suppose that $M$ is a unitary $S^1$-manifold with isolated fixed points such that $M$ is not a boundary. Kosniowski conjectured that the number of fixed points is greater than $f(\dim M)$ where $f$ is some (linear) function. He further noted that the most likely function is $f(x) = \frac{x}{4}$. We know from [11 Chapter 1, (1.5)] that $\chi(M) = \chi(M^{S^1})$ where $\chi(\cdot)$ denotes the Euler characteristic and $M^{S^1}$ is the fixed point set. So $\chi(M)$ is equal to the number of fixed points. With this result together, Kosniowski conjecture is stated as follows.

Conjecture 1.1 (Kosniowski). Suppose that $M$ is a nonbounding unitary $S^1$-manifold with isolated fixed points. Then

$$4\chi(M) > \dim M.$$ 

Remark 1. It is well-known that if $M$ is a unitary $S^1$-manifold with isolated fixed points, then $\dim M$ must be even. Write $\dim M = 2n$, then Conjecture 1.1 becomes $2\chi(M) > n$.

Unitary manifolds form a class of nice behaved geometric objects, including complex manifolds, almost complex manifolds and symplectic manifolds etc. Thus, Kosniowski conjecture will probably play an important role on the various areas, such as topology, complex geometry, symplectic geometry, transformation groups and so on.

When $\chi(M) = 2$, in his thesis Kosniowski first proved the conjecture and determined that the possible value of $n$ is 1 or 3 (also see [9, 10]). Under the condition that the $S^1$-action on $M^{2n}$ can be extended to the $T^n$-action on $M^{2n}$, Lü–Tan in [14] proved the conjecture. Ma–Wen in [16] also discussed the case in which the $S^1$-action on $M^{2n}$ can be extended to the $T^{n-1}$-action on $M^{2n}$. When $\chi(M) = 3$ and $M$ is assumed to be almost complex, Jang showed in [7] that only $n = 2$ happens, so the conjecture holds in this case. In the setting of compact symplectic manifolds with symplectic circle actions fixing isolated points, Pelayo–Tolman showed

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in [19] that if the weights satisfy some subtle condition, then the action has at least \( n + 1 \) isolated fixed points. In the setting of almost complex manifolds with circle actions, Li–Liu showed in [13] that if an almost complex manifold \( M \) has some nonzero Chern number, then any \( S^1 \)-action on \( M \) must have at least \( n + 1 \) fixed points. All these provide the evidence to support the conjecture.

The main tool used here is the localization formula of rigid \( T_{x,y} \)-genus of \( M \) in terms of the fixed point data, essentially due to Krichever [11], which is a generalization of Atiyah–Hirzebruch formula for the \( \chi_y \)-genus of a complex \( S^1 \)-manifold ([2]). We will see that this formula plays an essential role on our work. We first read out a system of equations from the localization formula of the \( T_{x,y} \)-genus of \( M \). Then using Newton’s formula, we further change this system into a new system such that its coefficient matrix is a Vandemonde matrix, from which we induce some cyclic equalities if \( \chi(M) < \lfloor \frac{n}{2} \rfloor + 1 \), giving a contradiction. This establishes \( \chi(M) \geq \lfloor \frac{n}{2} \rfloor + 1 > \frac{n}{2} \), thus obtaining the affirmative answer of Kosniowski conjecture. The result is stated as follows.

**Theorem 1.2.** Suppose that \( M \) is a nonbounding unitary \( S^1 \)-manifold with isolated fixed points. Then \( 4 \chi(M) > \dim M \).

Making use of the localization formula of rigid \( L \)-genus, our approach is automatically applied to the case of oriented \( S^1 \)-manifolds, so we also have that

**Theorem 1.3.** Suppose that \( M \) is a nonbounding oriented closed smooth \( S^1 \)-manifold with isolated fixed points. Then \( 2 \chi(M) > \dim M \).

This note is organized as follows. In Section 2 we review the relative work with respect to genera and rigidity, especially for \( T_{x,y} \)-genus and its rigidity, and state a useful lemma. In Section 3 we give the proof of Theorem 1.2. Finally we carry out a simple investigation on the relation between the Euler characteristic and the top Chern number of a unitary \( S^1 \)-manifold in Section 4.

2. **The \( T_{x,y} \)-Genus of Unitary \( S^1 \)-Manifolds**

First let us review the relative work with respect to genera and rigidity.

2.1. **Genera and rigidity.** In his seminar book [3], Hirzebruch introduced various multiplicative genera. A genus which is an invariant of unitary bordism classes can be defined by a homomorphism \( \phi : \Omega^U_\ast \to R \) where \( \Omega^U_\ast \) is the complex bordism ring and \( R \) is a commutative ring with unit. If we assume that \( R \) has no additive torsion, then \( \phi \) is completely determined by \( \phi \otimes \mathbb{Q} : \Omega^U_\ast \otimes \mathbb{Q} \to R \otimes \mathbb{Q} \). Using universal \( R \)-valued characteristic classes of special type, Hirzebruch gave the explicit description of \( \phi \otimes \mathbb{Q} \). For any multiplicative genus \( \phi \), there always exists a multiplicative sequence \( \{ K_i(c_1, \ldots, c_i) \} \) of polynomials given by a characteristic series such that \( \phi(M) = K_N(c_1, \ldots, c_N)[M] \) where \( c_i \) is the \( i \)-th Chern class of a unitary manifold \( M \) of \( \dim M = n \) and \( [M] \) is the fundamental homology class of \( M \).

It is well-known that a multiplicative genus \( \phi : \Omega^U_\ast \to R \) can induce an equivariant genus \( \phi^G : \Omega^{U,G}_\ast \to K(BG) \otimes R \), where \( \Omega^{U,G}_\ast \) is the ring of complex bordisms of unitary manifolds with actions of a connected compact Lie group \( G \). A multiplicative genus \( \phi \) is called rigid if \( \phi^G(M) = \phi(M) \) for any connected compact group \( G \), i.e., \( \phi(M) \in R \). The rigidity for \( S^1 \) implies the rigidity for \( G \) (also see
two orientations on the orientation of the unitary structure on the tangent bundle and some nonzero integer fixing only isolated points and established the localization formula of . Let the weight set of the complex and number-theoretic abstraction, -genus was further discussed in . In addition, Musin also showed in that if a genus is rigid, then it is -genus. In the view points of complex analysis and number-theoretic abstraction, -genus was further discussed in .

2.2. Localization formula of -genus. Throughout the following, assume that is a unitary manifold with an action of preserving the unitary structure and fixing only isolated points . Choose a fixed point the tangent space at is a complex -representation, so this forces dimension of to be even. Let then can be decomposed into the sum of irreducible -representations

where the action of on each is given by for and some nonzero integer . The collection , denoted by , gives the weight set of the complex -representation . The orientation of and the orientation of the unitary structure on the tangent bundle naturally induce two orientations on . Thus, the equivariant Euler class of the normal bundle to in is

where otherwise.

Let be the -genus of . Its -equivariant version is denoted by , corresponding to the characteristic series

As mentioned as before, the -genus of is rigid, so . Then the localization formula of is

Write . Then becomes

\begin{equation}
T_{x,y}^{S^1}(M) = \sum_{i=1}^{m} \varepsilon_i \prod_{j=1}^{n} \frac{x + yq^{w_{i,j}}}{1 - q^{w_{i,j}}} \in \mathbb{Z}[x, y][[q]]
\end{equation}
due to Krichever (also see [3, Theorem 9.4.8]). Let \( w^+_i \) (resp. \( w^-_i \)) denote the collection of all positive weights (resp. all negative weights) in \( w \) at the fixed point \( p_i \). By \(|w^+_i|\) we mean the number of weights in \( w^+_i \), so \(|w^+_i| + |w^-_i| = n\). Furthermore, as \( q \rightarrow 0 \), we have that

**Theorem 2.1** (Generalized Atiyah–Hirzebruch formula [11]).

\[
T_{x,y}(M) = T^{S^1}_{x,y}(M) = \sum_{i=1}^{m} \varepsilon_i \prod_{j=1}^{n} \frac{x + y q^{w^+_{i,j}}}{1 - q^{w^+_{i,j}}} = \sum_{i=1}^{m} \varepsilon_i x^{|w^+_i|} (-y)^{|w^-_i|} \in \mathbb{Z}[x, y].
\]

**Remark 2.** As \( q \rightarrow \infty \), there is also another expression of \( T_{x,y}(M) \)

\[
T_{x,y}(M) = \sum_{i=1}^{m} \varepsilon_i x^{|w^+_i|} (-y)^{|w^-_i|}.
\]

Clearly \( T_{x,y}(M) = T^{S^1}_{x,y}(M) \) does not depend on \( q \) and (2.1) is actually a Laurent polynomial in \( q \).

Combining with the same terms in \( \sum_{i=1}^{m} \varepsilon_i x^{|w^+_i|} (-y)^{|w^-_i|} \), we may write

\[
\sum_{i=1}^{m} \varepsilon_i x^{|w^+_i|} (-y)^{|w^-_i|} = \sum_{l=0}^{n} \chi^l(M) x^{n-l} y^l,
\]

where each \( \chi^l(M) \) is an integer. Compare with two sides of the following equation

\[
\sum_{i=1}^{m} \varepsilon_i \prod_{j=1}^{n} \frac{x + y q^{w^+_{i,j}}}{1 - q^{w^+_{i,j}}} = \sum_{l=0}^{n} \chi^l(M) x^{n-l} y^l,
\]

an easy observation gives the following formula

\[
(2.2) \quad \chi^l(M) = \sum_{i=1}^{m} \varepsilon_i \frac{\sigma_l(q^{w^+_{i,1}}, \ldots, q^{w^+_{i,n}})}{\prod_{j=1}^{n} (1 - q^{w^+_{i,j}})}
\]

for \( 0 \leq l \leq n \), where \( \sigma_l \) is the \( l \)-th elementary symmetric function. As far as the author knows, the formula (2.2) without signs \( \varepsilon_i \) first appeared in [8, (2.3) of Theorem, page 45] and was used very well in [9, page 172].

**Remark 3.** When \( x = 1 \), it is well-known that \( T_{1,y}(M) \) is exactly the Atiyah–Hirzebruch \( \chi_y \)-genus

\[
T_{1,y}(M) = \chi_y(M) = \sum_{l=0}^{n} \chi^l(M) y^l.
\]

Moreover,

1. The signature of \( M \) is \( \text{Sign}(M) = T_{1,1}(M) = \sum_{l=0}^{n} \chi^l(M) \);
2. The top Chern number of \( M \) is \( c_n[M] = T_{1,-1}(M) = \sum_{l=0}^{n} (-1)^l \chi^l(M) \);
3. The Todd-genus of \( M \) is \( \text{Todd}(M) = T_{1,0}(M) = \chi^0(M) \).

The following result can be found in [6] for a complex manifold and in [12] for an almost complex manifold. Actually it can be extended to the unitary situation automatically. Here a simple proof is included for a local completeness.
Lemma 2.2. For \( 0 \leq l \leq n \),

\[
\chi^l(M) = (-1)^n \chi^{n-l}(M).
\]

Proof. Using the rigidity of \( T_{x,y} \)-genus and replacing \( q \) by \( \frac{1}{q} \), the formula (2.2) becomes

\[
\chi^l(M) = \sum_{i=1}^{m} \varepsilon_i \frac{\sigma_i(q^{w_{i,1}}, \ldots, q^{w_{i,n}})}{\prod_{j=1}^{n} (1 - q^{w_{i,j}})} = \sum_{i=1}^{m} \varepsilon_i \frac{\sigma_i(q^{-w_{i,1}}, \ldots, q^{-w_{i,n}})}{\prod_{j=1}^{n} (1 - q^{-w_{i,j}})}
\]

\[
= \sum_{i=1}^{m} \varepsilon_i \frac{\sigma_i(q^{-w_{i,1}}, \ldots, q^{-w_{i,n}})}{(-1)^n (\prod_{j=1}^{n} q^{-w_{i,j}}) \prod_{j=1}^{n} (1 - q^{w_{i,j}})}
\]

\[
= (-1)^n \sum_{i=1}^{m} \varepsilon_i \frac{\sigma_{n-l}(q^{w_{i,1}}, \ldots, q^{w_{i,n}})}{\prod_{j=1}^{n} (1 - q^{w_{i,j}})}
\]

\[
= (-1)^n \chi^{n-l}(M)
\]

as desired. \( \square \)

Corollary 2.3. If \( n \) is odd, then \( \text{Sign}(M) = 0 \).

3. PROOF OF THEOREM 1.2

Let \( M^{2n} \) be a unitary manifold with an action of \( S^1 \) preserving the unitary structure and fixing only isolated points \( p_1, \ldots, p_m \). Suppose further that \( M^{2n} \) is not a boundary. Following the notations of Section 2, let \( f_i(q) = \frac{1}{\prod_{j=1}^{n} (1 - q^{w_{i,j}})} \) for \( 1 \leq i \leq m \).

Then for \( 0 \leq l \leq n \), we rewrite the formula (2.2) into the following

(3.1)

\[
\chi^l(M) = \sum_{i=1}^{m} \varepsilon_i \sigma_i(q^{w_{i,1}}, \ldots, q^{w_{i,n}}) f_i(q).
\]

We know by Lemma 2.2 that \( \chi^l(M) = (-1)^n \chi^{n-l}(M) \). This implies that essentially there are \( \left\lfloor \frac{n}{2} \right\rfloor + 1 \) different formulae of the form (3.1), where

\[
\left\lfloor \frac{n}{2} \right\rfloor + 1 = \begin{cases} 
k + 1 & \text{if } n = 2k \\
1 + 1 & \text{if } n = 2k + 1.
\end{cases}
\]

Thus it suffices to consider \( k + 1 \) formulae of \( \chi^l(M), l = 0, 1, \ldots, k \). This gives the following system of \( k + 1 \) equations:

(3.2)

\[
\begin{align*}
\sum_{i=1}^{m} \varepsilon_i \sigma_0(q^{w_{i,1}}, \ldots, q^{w_{i,n}}) f_i(q) &= \chi^0(M) \\
\sum_{i=1}^{m} \varepsilon_i \sigma_1(q^{w_{i,1}}, \ldots, q^{w_{i,n}}) f_i(q) &= \chi^1(M) \\
& \vdots \\
\sum_{i=1}^{m} \varepsilon_i \sigma_k(q^{w_{i,1}}, \ldots, q^{w_{i,n}}) f_i(q) &= \chi^k(M).
\end{align*}
\]
Note that $\sigma_0(q^{w_1}, \ldots, q^{w_n}) = 1$ for all $i$. By the Newton’s formula (see [17] Problem 16-A)

$$s_l - \sigma_1 s_{l-1} + \cdots + (-1)^{l-1} \sigma_{l-1} s_1 + (-1)^l l \sigma_l = 0,$$

we can induce

$$l \sigma_l = (-1)^l \sigma_1^l + F_l$$

for $l \geq 2$, where $F_l$ is a symmetric polynomial. For example, when $l = 2$, we have $2\sigma_2 = \sigma_1^2 - s_2$ so $F_2 = -s_2$. When $l = 3$, we have

$$3\sigma_3 = (-1)^3 \sigma_1^3 + 3 \sigma_1 \sigma_2 + s_3$$

so $F_3 = 3 \sigma_1 \sigma_2 + s_3$. Then for $l \geq 2,$

$$l \chi^l(M) = \sum_{i=1}^{m} \varepsilon_i l \sigma_1(q^{w_{i,1}}, q^{w_{i,n}}) f_i(q)$$

$$= \sum_{i=1}^{m} \varepsilon_i (-1)^l \sigma_1^l(q^{w_{i,1}}, q^{w_{i,n}}) f_i(q) + \sum_{i=1}^{m} \varepsilon_i F_i(q) f_i(q)$$

where $F_i(q)$ is a certain function in $q$. Set $F_i(q) = \sum_{i=1}^{m} \varepsilon_i F_i(q) f_i(q)$. Then we can further change the system (3.2) of equations into

$$\begin{align*}
\sum_{i=1}^{m} \varepsilon_i f_i(q) &= \chi^0(M) \\
\sum_{i=1}^{m} \varepsilon_i \sigma_1(q^{w_{i,1}}, q^{w_{i,n}}) f_i(q) &= \chi^1(M) \\
\sum_{i=1}^{m} \varepsilon_i \sigma_1^2(q^{w_{i,1}}, q^{w_{i,n}}) f_i(q) &= 2 \chi^2(M) - F_2(q) \\
&\vdots \\
\sum_{i=1}^{m} \varepsilon_i \sigma_1^k(q^{w_{i,1}}, q^{w_{i,n}}) f_i(q) &= (-1)^k (k \chi^k(M) - F_k(q)).
\end{align*}$$

(3.3)

Obviously, we can regard

$$x(q) = (\varepsilon_1 f_1(q), \varepsilon_2 f_2(q), \ldots, \varepsilon_m f_m(q))^\top$$

as a nonzero solution of the equation system (3.3) with the following coefficient matrix

$$A(q) = \begin{pmatrix}
1 & \cdots & 1 \\
\sigma_1(q^{w_{1,1}}, q^{w_{1,n}}) & \cdots & \sigma_1(q^{w_{m,1}}, q^{w_{m,n}}) \\
\vdots & \vdots & \vdots \\
\sigma_1^k(q^{w_{1,1}}, q^{w_{1,n}}) & \cdots & \sigma_1^k(q^{w_{m,1}}, q^{w_{m,n}})
\end{pmatrix}.$$

For a convenience, we simply write

$$\sigma_{1,i} = \sigma_1(q^{w_{i,1}}, q^{w_{i,n}})$$

for $i = 1, \ldots, m$

and

$$b(q) = (b_0(q), b_1(q), b_2(q), \ldots, b_k(q))^\top$$

$$= (\chi^0(M), \chi^1(M), 2 \chi^2(M) - F_2(q), \ldots, (-1)^k (k \chi^k(M) - F_k(q)))^\top.$$
Then the system (3.3) can be written as the following simple form

\[(3.4) \quad A(q)x(q) = b(q).\]

Next, we are going to prove the following key lemma.

**Lemma 3.1.** It is impossible that \(k + 1 > m.\)

**Proof.** Suppose that \(k + 1 > m.\) Let us look at the system (3.4). Since \(M^{2n}\) has been assumed to be not a boundary, without any loss of generality, we may assume that all \(\sigma_{1,i}, i = 1, \ldots, m\) are mutually distinct. In fact, if there are some two \(\sigma_{1,i}\) and \(\sigma_{1,i'}\) that are both equal, then their weight sets are the same, too. When \(\varepsilon_{i_1} \neq \varepsilon_{i_2},\) we can do a simple surgery to delete two fixed points \(p_{i_1}\) and \(p_{i_2}.\) This will unchange the \(S^1\)-manifold \(M\) up to unitary bordism. When \(\varepsilon_{i_1} = \varepsilon_{i_2},\) we can move those terms corresponding to the \(i_2\)-th column of \(A(q)\) in the left side of the system (3.4) into its right side. Whichever of above cases happens, we can always modify the system (3.4) into a new system, satisfying that all \(\sigma_{1,j}\)'s appearing in the new coefficient matrix are distinct. Actually this will reduce the number of polynomials in \(x(q)\) to be smaller, but this smaller number can not become zero since \(M^{2n}\) is nonbounding. So the above distinct assumption for all \(\sigma_{1,j}\)'s will not produce any essential influence on our further argument.

Since \(k + 1 > m,\) we have \(k + 1 \geq m + 1.\) Without any loss of generality, we may assume that \(k = m,\) so the system (3.4) is formed by \(m + 1\) equations. Then the system (3.4) can produce \(m + 1\) systems, each of which consists of \(m\) equations, stated as follows:

\[(3.5) \quad V_l(q)x(q) = b_l(q)\]

for \(l = 0, 1, \ldots, m,\) where

\[b_l(q) = (b_0(q), b_1(q), \ldots, b_{l-1}(q), \hat{b}_l(q), b_{l+1}(q), \ldots, b_m(q))^\top\]

means that the \((l + 1)\)-th element is deleted in \(b(q),\) and

\[V_l(q) = \begin{pmatrix} 1 & \cdots & 1 & \cdots & 1 \\ \sigma_{1,1} & \cdots & \sigma_{1,i} & \cdots & \sigma_{1,m} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \sigma_{i,1}^{l-1} & \cdots & \sigma_{i,1}^{l-1} & \cdots & \sigma_{i,m}^{l-1} \\ \sigma_{i+1,1} & \cdots & \sigma_{i+1,1} & \cdots & \sigma_{i+1,m} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ \sigma_{m,1}^m & \cdots & \sigma_{m,1}^m & \cdots & \sigma_{m,m}^m \end{pmatrix}\]

is a Vandermonde matrix obtained by deleting the \((l + 1)\)-th row of \(A(q).\) It is easy to see that \(b_0(q), b_1(q), \ldots, b_m(q)\) are linearly dependent.

Now suppose that

\[(3.6) \quad u_0(q)b_0(q) + u_1(q)b_1(q) + \cdots + u_m(q)b_m(q) = 0\]
where all \( u_l(q) \)'s are functions in \( q \). More explicitly, (3.6) can be written as

\[
\begin{align*}
\begin{cases}
  u_0(q)b_1(q) + (u_1(q) + \cdots + u_m(q))b_0(q) = 0 \\
  (u_0(q) + u_1(q))b_2(q) + (u_2(q) + \cdots + u_m(q))b_1(q) = 0 \\
  \vdots \\
  (u_0(q) + \cdots + u_{m-2}(q))b_{m-1}(q) + (u_{m-1}(q) + u_m(q))b_{m-2}(q) = 0 \\
  (u_0(q) + \cdots + u_{m-2}(q) + u_{m-1}(q))b_m(q) + u_m(q)b_{m-1}(q) = 0.
\end{cases}
\end{align*}
\]

(3.7)

Our next task is to determine the precise expressions of all functions \( u_l(q) \)'s. We proceed our argument as follows.

Combining with (3.5) and (3.6), we have that

\[
(\sum_{l=0}^{m} u_l(q)V_l(q))x(q) = 0
\]

whose coefficient matrix \( \sum_{l=0}^{m} u_l(q)V_l(q) \) is explicitly written as follows:

\[
\begin{pmatrix}
  \sigma_{1,1}u_0(q) + \sum_{j=1}^{m} u_j(q) & \sigma_{1,2}u_0(q) + \sum_{j=1}^{m} u_j(q) & \cdots & \sigma_{1,m}u_0(q) + \sum_{j=1}^{m} u_j(q) \\
  \sigma_{1,1}^{2} \sum_{j=2}^{m} u_j(q) + \sigma_{1,1} \sum_{j=2}^{m} u_j(q) & \sigma_{1,2}^{2} \sum_{j=2}^{m} u_j(q) + \sigma_{1,2} \sum_{j=2}^{m} u_j(q) & \cdots & \sigma_{1,m}^{2} \sum_{j=2}^{m} u_j(q) + \sigma_{1,m} \sum_{j=2}^{m} u_j(q) \\
  \vdots & \vdots & \ddots & \vdots \\
  \sigma_{1,1}^{m-1} \sum_{j=2}^{m} u_j(q) + \sigma_{1,2}^{m-1} \sum_{j=2}^{m} u_j(q) & \cdots & \sigma_{1,m}^{m-1} \sum_{j=2}^{m} u_j(q) + \sigma_{1,m}^{m-1} \sum_{j=2}^{m} u_j(q) \\
  \sigma_{1,1}^{m} \sum_{j=2}^{m} u_j(q) + \sigma_{1,2}^{m} \sum_{j=2}^{m} u_j(q) & \cdots & \sigma_{1,m}^{m} \sum_{j=2}^{m} u_j(q) + \sigma_{1,m}^{m} \sum_{j=2}^{m} u_j(q)
\end{pmatrix}
\]

In order to give the precise expressions of all functions \( u_l(q) \)'s, we consider \( m \) diagonal elements of the matrix \( \sum_{l=0}^{m} u_l(q)V_l(q) \), which are expressed in \( u_l(q) \). We let these \( m \) expressions in \( u_l(q) \) become zero, so that we obtain the following system of \( m \) equations

\[
(3.9) \quad W(q)u(q) = 0
\]

where \( u(q) = (u_0(q), u_1(q), \ldots, u_m(q))^{\top} \) and the coefficient matrix

\[
W(q) = \begin{pmatrix}
  \sigma_{1,1} & 1 & 1 & \cdots & \cdots & 1 & 1 \\
  \sigma_{1,2} & \sigma_{1,2} & 1 & \cdots & \cdots & 1 & 1 \\
  \sigma_{1,3} & \sigma_{1,3} & \sigma_{1,3} & \cdots & \cdots & 1 & 1 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
  \sigma_{1,m-1} & \cdots & \cdots & \sigma_{1,m-1} & \sigma_{1,m-1} & 1 & 1 \\
  \sigma_{1,m} & \cdots & \cdots & \sigma_{1,m} & \sigma_{1,m} & \sigma_{1,m} & \sigma_{1,m}
\end{pmatrix}
\]

8
By a direct calculation, we obtain that

\[
\begin{align*}
    u_1(q) &= \frac{\sigma_{1,1}-\sigma_{1,2}}{\sigma_{1,2}-1} u_0(q) \\
    u_1(q) + u_2(q) &= \frac{\sigma_{1,1}-\sigma_{1,2}}{\sigma_{1,2}-1} u_0(q) \\
    \vdots & \quad \vdots \\
    u_1(q) + \cdots + u_{m-1}(q) &= \frac{\sigma_{1,1}-\sigma_{1,m}}{\sigma_{1,m}-1} u_0(q) \\
    u_1(q) + \cdots + u_{m-1}(q) + u_m(q) &= -\sigma_{1,1} u_0(q).
\end{align*}
\] (3.10)

Taking \(u_0(q) = 1\), from (3.10) we obtain the precise expressions of all \(u_i(q)\) in (3.6), which are stated as follows:

\[
\begin{align*}
    u_0(q) &= 1 \\
    u_1(q) &= \frac{\sigma_{1,1}-\sigma_{1,2}}{\sigma_{1,2}-1} \\
    u_2(q) &= \frac{\sigma_{1,1}-\sigma_{1,3}}{\sigma_{1,3}-1} - \frac{\sigma_{1,1}-\sigma_{1,2}}{\sigma_{1,2}-1} = \frac{(\sigma_{1,1}-1)(\sigma_{1,2}-\sigma_{1,3})}{(\sigma_{1,2}-1)(\sigma_{1,3}-1)} \\
    \vdots & \quad \vdots \\
    u_{m-1}(q) &= \frac{\sigma_{1,1}-\sigma_{1,m}}{\sigma_{1,m}-1} - \frac{\sigma_{1,1}-\sigma_{1,m-1}}{\sigma_{1,m-1}-1} = \frac{(\sigma_{1,1}-1)(\sigma_{1,m-1}-\sigma_{1,m})}{(\sigma_{1,m-1}-1)(\sigma_{1,m}-1)} \\
    u_m(q) &= \frac{\sigma_{1,1}-\sigma_{1,m}}{\sigma_{1,m}-1} - \sigma_{1,1} = \frac{\sigma_{1,m}(1-\sigma_{1,1})}{\sigma_{1,m}-1}.
\end{align*}
\] (3.11)

Since all \(\sigma_{1,i}\)'s are distinct, it is easy to see that all \(u_i(q)\)'s are nonzero. Putting (3.11) into (3.7), we conclude that

\[
\begin{align*}
    b_1(q) &= \sigma_{1,1} b_0(q) \\
    b_2(q) &= \sigma_{1,2} b_1(q) \\
    \vdots & \quad \vdots \\
    b_{m-1}(q) &= \sigma_{1,m-1} b_{m-2}(q) \\
    b_m(q) &= \sigma_{1,m} b_{m-1}(q)
\end{align*}
\] (3.12)

which gives cyclic equalities among \(b_0(q), b_1(q), \ldots, b_m(q)\). This implies that all \(b_0(q), b_1(q), \ldots, b_m(q)\) must be nonzero. In fact, if one of them is zero, then the cyclic equalities in (3.12) will induce that they are all zero. Moreover, we can yield from the system (3.4) that \(x(q) = 0\), which is impossible.

Now, from the equation system (3.12) we easily read out a contradiction equation

\[b_1(q) = \sigma_{1,1} b_0(q)\]

since \(b_0(q) = \chi^0(M)\) and \(b_1(q) = \chi^1(M)\) are only nonzero integers.

It remains to prove that the equation system (3.9) established beforehand and all expressions of \(u_i(q)\) in (3.11) are compatible with the system (3.8).

First we check the first equation in the system (3.8), which has a little bit difference from the general case. Since we have assumed that \(\sigma_{1,1} u_0(q) + \sum_{j=1}^{m} u_j(q) = 0\), it suffices to check that

\[
\sum_{i=2}^{m} (\sigma_{1,i} u_0(q) + \sum_{j=1}^{m} u_j(q)) \varepsilon_i f_i(q)
\]
must be zero. Using the equations in (3.10), (3.11) and (3.12), we directly calculate as follows:
\[
\sum_{i=2}^{m} (\sigma_{i}u_{0}(q) + \sum_{j=1}^{m} u_{j}(q))\varepsilon_{i}f_{i}(q)
\]
\[
= (\sigma_{1,2} - \sigma_{1,1})\varepsilon_{2}f_{2}(q) + \cdots + (\sigma_{1,m} - \sigma_{1,1})\varepsilon_{m}f_{m}(q)
\]
\[
= \sigma_{1,2}\varepsilon_{2}f_{2}(q) + \cdots + \sigma_{1,m}\varepsilon_{m}f_{m}(q) - \sigma_{1,1}(\varepsilon_{2}f_{2}(q) + \cdots + \varepsilon_{m}f_{m}(q))
\]
\[
= \sigma_{1,2}\varepsilon_{2}f_{2}(q) + \cdots + \sigma_{1,m}\varepsilon_{m}f_{m}(q) - \sigma_{1,1}(b_{0}(q) - \varepsilon_{1}f_{1}(q))
\]
\[
= \sigma_{1,1}\varepsilon_{1}f_{1}(q) + \sigma_{1,2}\varepsilon_{2}f_{2}(q) + \cdots + \sigma_{1,m}\varepsilon_{m}f_{m}(q) - \sigma_{1,1}b_{0}(q)
\]
\[
= b_{1}(q) - \sigma_{1,1}b_{0}(q)
\]
\[
= 0
\]
as desired.

Next let us check the \(l\)-th equation in the system (3.8), where \(l \geq 2\). We need to show that
\[
\sum_{i \neq l} \left( \sigma_{i,l} \sum_{j=0}^{l-1} u_{j}(q) + \sigma_{i,l}^{-1} \sum_{j=l}^{m} u_{j}(q) \right)\varepsilon_{i}f_{i}(q)
\]
is equal to zero. From the equations in (3.11), we see easily that
\[
\sum_{j=0}^{l-1} u_{j}(q) = 1 + \frac{\sigma_{1,1} - \sigma_{1,l}}{\sigma_{1,l} - 1} = \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1}
\]
and
\[
\sum_{j=l}^{m} u_{j}(q) = -\sigma_{1,l} - \frac{\sigma_{1,1} - \sigma_{1,l}}{\sigma_{1,l} - 1} = -\frac{\sigma_{1,l}(\sigma_{1,1} - 1)}{\sigma_{1,l} - 1}.
\]
So
\[
\sum_{i \neq l} \left( \sigma_{i,l} \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} - \sigma_{i,l}^{-1} \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} \right)\varepsilon_{i}f_{i}(q)
\]
\[
= \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} \left( \sigma_{1,1}\varepsilon_{1}f_{1}(q) + \cdots + \sigma_{1,l-1}\varepsilon_{l-1}f_{l-1}(q) + \sigma_{1,l+1}\varepsilon_{l+1}f_{l+1}(q) + \cdots + \sigma_{1,m}\varepsilon_{m}f_{m}(q) \right)
\]
\[
- \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} \left( \sigma_{1,1}^{-1}\varepsilon_{1}f_{1}(q) + \cdots + \sigma_{1,l-1}^{-1}\varepsilon_{l-1}f_{l-1}(q) + \sigma_{1,l+1}^{-1}\varepsilon_{l+1}f_{l+1}(q) + \cdots 
\]
\[
+ \sigma_{1,m}^{-1}\varepsilon_{m}f_{m}(q) \right)
\]
\[
= \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} (b_{l}(q) - \sigma_{l,l}^{-1}\varepsilon_{l}f_{l}(q)) - \frac{\sigma_{l,l}(\sigma_{1,1} - 1)}{\sigma_{1,l} - 1} (b_{l-1}(q) - \sigma_{l,l}^{-1}\varepsilon_{l}f_{l}(q))
\]
\[
= \frac{\sigma_{1,1} - 1}{\sigma_{1,l} - 1} (b_{l}(q) - \sigma_{1,l}b_{l-1}(q))
\]
\[
= 0
\]
as desired.

Together with all arguments as above, we complete the proof. \(\square\)
Proof of Theorem 1.2. By Lemma 3.1, we must have
\[ m \geq k + 1 = \left\lceil \frac{n}{2} \right\rceil + 1 > \frac{n}{2} \]
as desired. This completes the proof of Theorem 1.2. □

Corollary 3.2. The coefficient matrix of the system (3.2) has rank \( k + 1 \).

4. An Observation

Suppose that \( M^{2n} \) is a unitary \( S^1 \)-manifold fixing isolated points, where \( M^{2n} \) is not necessarily assumed to be nonbounding.

First assume that \( w_{i,1}, \ldots, w_{i,|w_i^-|} \) are all negative weights in the weight set \( w_i \).

Now, using the same way as in [9, (*), page 172], we change the expression of \( \chi^l(M) \) into

\[
\chi^l(M) = \sum_{i=1}^{m} \varepsilon_i (-1)^{|w_i^-|} \frac{\sigma_l(q^{w_{i,1}}, \ldots, q^{w_{i,n}})}{\prod_{j \leq |w_i^-|} (1 - q^{-w_{i,j}}) \prod_{j > |w_i^-|} (1 - q^{w_{i,j}})}. \tag{4.1}
\]

Since \( \chi^l(M) \) is an integer, to calculate \( \chi^l(M) \) is enough to determine the constant term in the right side of (4.1). Obviously, only those terms with \( |w_i^-| = l \) in the right side of (4.1) can produce the constant term. An easy argument shows that

\[
\chi^l(M) = (-1)^l(n_i^+ - n_i^-)
\]

where \( n_i^+ \) (resp. \( n_i^- \)) denotes the number of those fixed points \( p_i \) with \( |w_i^-| = l \) and \( \varepsilon_i = +1 \) (resp. \( \varepsilon_i = -1 \)). Then the top Chern number

\[
c_n[M] = T_{1,-1}(M) = \sum_{l=0}^{n} (-1)^l \chi^l(M) = \sum_{l=0}^{n} (n_i^+ - n_i^-).
\]

On the other hand, since the Euler characteristic \( \chi(M) \) is exactly equal to the number of all fixed points, we have

\[
\chi(M) = \sum_{l=0}^{n} (n_i^+ + n_i^-).
\]

Furthermore, we conclude that

\[
\chi(M) = c_n[M] + 2 \sum_{l=0}^{n} n_i^-
\]

and

\[
\chi(M) + c_n[M] = 2 \sum_{l=0}^{n} n_i^+.
\]

As in the case of almost complex, we see that all signs \( \varepsilon_i \) of fixed points are positive if and only if \( \chi(M) = c_n[M] \). A classical result of Thomas [20] tells us that \( \chi(M^{2n}) = c_n[M] \) is sufficient for a unitary manifold to be almost complex. Whether \( \chi(M^{2n}) = c_n[M] \) holds or not completely does depend upon the choices of all signs \( \varepsilon_i \) of fixed points. It seems to be not easy to determine when \( \varepsilon_i \) is chosen as \( +1 \) or \( -1 \). Both (4.2) and (4.3) only tell us that \( \chi(M) \equiv c_n[M] \mod 2 \). As a result, it is stated as follows.
Corollary 4.1. Let $M$ be a unitary $S^1$-manifold fixing isolated points. Then $\chi(M)$ and $c_n[M]$ have the same parity.

As a consequence, it is not difficult to see that when $\chi(M) = 2$, both (4.2) and (4.3) force all $n_i$ to be zero, so $\chi(M) = c_n[M]$ and then $M$ is almost complex, as seen in [9, 10].

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