Rayleigh waves in isotropic strongly elliptic thermoelastic materials with microtemperatures

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Abstract

This paper is concerned with the linear theory of thermoelasticity with microtemperatures, based on the entropy balance proposed by Green and Naghdi, which permits the transmission of heat as thermal waves of finite speed. We analyze the behavior of Rayleigh waves in an unbounded isotropic homogeneous strongly elliptic thermoelastic material with microtemperatures. The related solution of the Rayleigh surface wave problem is expressed as a linear combination of the elements of the bases of the kernels of appropriate matrices. The secular equation is established and afterwards an explicit form is written when some coupling constitutive coefficients vanish. Then, we solve numerically the secular equation by mean of a graphical method and by taking arbitrary data for strongly elliptic thermoelastic material.

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1 Introduction

There are many authors who study the materials having thermal variations at the microstructural level, such as viscous fluids, granular materials, composites and nanomaterials (e.g. [1-4] ans references). This is due to increasing interest in several class of nanomaterials used in the heat transfer industry, where the microtemperatures and microdeformations of the nanoparticles cannot be ignored. In future technologies studies related to propagation wave in the theory of thermoelastic materials with microtemperatures may be important.

On the other hand, during the past years several authors have studied the class of strongly elliptic materials. These materials are characterized by special properties, like negative Poisson’s ratio and negative stiffness (auxetic or antirubber materials). These particular structures (see for example [5]) expand laterally when stretched, in contrast to the behavior of ordinary materials. The ellipticity analysis is relevant in studying wave propagation [6] and has important applications in several contexts (e.g. [7-14]).
The propagation of thermoelastic waves has been discussed long ago by Lockett [15] and Lockett and Sneddon [16]. Chadwick [17] studied the coupled and modified character of the thermoelastic waves and noted that they are also damped. Later, Ivanov [18] used these results to discuss appropriate criteria for the behavior at infinity, in order to preserve the characteristic features of the Rayleigh waves known from the classical elasticity.

The effects of heat conduction upon the propagation of Rayleigh surface waves in a semi-infinite elastic solid has been studied by Chadwick and Windle [19] in isotropic thermoelastic bodies and by Chakraborty and Pal [20] and by Chadwick and Seet [21] for transversely isotropic materials. Further, Abouelregal [22] studied Rayleigh waves in a thermoelastic homogeneous solid half space in the context of a dual-phase-lag model. We have to point out that the wave motion in the form of acceleration waves and of shock waves is discussed in the recent book by Straughan [23] in an account of theories of heat conduction where the temperature may travel as a wave with finite speed. Further, the propagation of elastic waves and the propagation of Rayleigh surface waves has been studied in various contexts, in [24–33].

In [34], Ciarletta et al. study a homogeneous strongly elliptic thermoelastic body with microtemperatures following the theory of Iesan and Quintanilla [35]. In [35] it is presented a linearized theory based on the entropy balance proposed by Green and Naghdi [36]. Moreover, in [34] the authors show that there is neither dispersion nor attenuation in the wave propagation as a consequence of the entropy balance proposed by Green and Naghdi [36,37] and the strong ellipticity condition; this is in contrast to what we see in [38] where the theory of thermoelasticity with microtemperatures of Iesan and Quintanilla [39] is used. Further, the authors prove that only undamped plane harmonic waves exist for any direction of propagation. In the isotropic case the possible waves are undamped in time and there are three longitudinal and two transverse waves. We point out that all transverse waves have constant temperature.

In the present paper, the theory of thermoelasticity with microtemperatures (Iesan and Quintanilla [35], Ciarletta et al. [34]) is applied to the study of Rayleigh waves propagating at the thermally insulated stress-free surface of an isotropic, homogeneous strongly elliptic thermoelastic solid half-space with microtemperatures.

The layout of the paper is a follows. In Section 2, we state the set of basic equations describing the behavior of thermoelastic media with microtemperatures within the context of the theory developed in [35]. Further, we remark some of the results obtained in [34] as the conditions characterizing the strong ellipticity for isotropic materials.

In Section 3, we find the explicit solutions for surface waves propagation in a half space filled with an isotropic homogeneous strongly elliptic thermoelastic medium with microtemperatures. We prove that there is no dispersion, moreover we obtain that the solution of the Rayleigh surface wave problem is expressed as a linear combination of the elements of the (five) bases of the kernels of the appropriate matrices; these vectors are written in an explicit form. The secular equation is then established, then we solve numerically by mean of a grafical metod and by taking arbitrary data for strongly elliptic thermoelastic material.

Finally, in Section 4 all vectors of the bases of the considered kernels, and the corresponding secular equations, are established in three cases in which some of the coupling constitutive coefficients vanish.
2 Field equations

Let $\Omega$ be an unbounded region filled of a thermoelastic material with microstructure, as presented in [34,35]. For this type of bodies, the temperature $\theta^\prime$ at the point $X^\prime$ of the microelement $\omega$ is considered a linear function of the microcoordinates $X^\prime - X$, i.e. $\theta^\prime = \theta + T \cdot (X^\prime - X)$, where $X$ is the center of mass of $\omega$ in the reference configuration, $\theta$ and $T$ are the temperature and the microtemperature vector at $X$ and $\cdot$ is the euclidean scalar product. It is assumed that there exists a reference time $t_R$ such that

$$T(X,t_R) = T^R, \quad \theta(X,t_R) = \theta^R.$$  

In the following, a rectangular Cartesian coordinate system $Ox_k$, $k = 1, 2, 3$, is used. Letters in boldface, like $\mathbf{v}$, stand for tensors of order $p$, with components $v_{ij...s}$ ($p$ subscripts). Latin subscripts range over the integers $\{1,2,3\}$, Greek subscripts range over $\{1,2\}$ and the summation convention is employed. A superposed dot or a subscript preceded by a comma will mean partial derivative with respect to time or to the corresponding coordinate, respectively. Moreover, we suppress the dependence upon spatial and/or temporal variables when no confusion may occur. All involved functions are supposed to be sufficiently regular to ensure analysis to be valid.

In the context of the linear theory presented in [34,35] and in the absence of supply terms, the behavior of the isotropic homogeneous body possessing a center of symmetry is governed by the following equations

$$\begin{align*}
\mu u_{i,jj} + (\lambda + \mu) u_{j,ji} + \varepsilon_2 \tau_{i,jj} + (\varepsilon_1 + \varepsilon_2) \tau_{j,ji} - \beta \ddot{\chi},i &= \rho \ddot{u}_i, \\
\varepsilon_2 u_{i,jj} + (\varepsilon_1 + \varepsilon_2) u_{j,ji} + d_2 \tau_{i,jj} + (d_1 + d_3) \tau_{j,ji} - m \ddot{\chi},i &= b \ddot{r}_i, & \text{on } \Omega \times (t_R, \infty), \quad (1)
\end{align*}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ is the displacement vector field and $\mathbf{\tau} = (\tau_1, \tau_2, \tau_3)^T$ and $\chi$ are defined by

$$\begin{align*}
\tau_i &= \int_{t_R}^t [T_i(s) - T_i^R] \, ds, \\
\chi &= \int_{t_R}^t [\theta(s) - \theta^R] \, ds.
\end{align*}$$

Here $\rho$ is the reference mass density and $\lambda, \mu, \varepsilon_1, \varepsilon_2, d_1, d_2, d_3, \beta, m, k, a$ and $b$ are constitutive coefficients.

We remark that, for the considered theory, the coupling constitutive coefficients are $\varepsilon_1, \varepsilon_2, \beta$ and $m$.

We can see that the equations of the system (1) are uncoupled when

i) $\beta = 0, m \neq 0$ and $\varepsilon_1 = \varepsilon_2 = 0$ (Eq. (1)$_1$ reduces to the classical motion equation of elasticity);

ii) $\beta \neq 0, m = 0$ and $\varepsilon_1 = \varepsilon_2 = 0$ (Eq. (1)$_2$ involves only $\mathbf{\tau}$);

iii) $\beta = 0, m = 0, \varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$ (Eq. (1)$_3$ involves only $\chi$).

It is proved in [34] that a thermoelastic material with microtemperatures is strongly elliptic if and only if

$$\rho > 0, \quad a > 0, \quad b > 0, \quad k > 0,$$

(2)
\[ \lambda + 2\mu > 0, \quad \mu > 0, \quad (\varepsilon_1 + 2\varepsilon_2)^2 < (\lambda + 2\mu)d, \quad \varepsilon_2^2 < \mu d, \] 

where \( d = d_1 + d_2 + d_3 \). Consequently, it is also \( d > 0, \ d_2 > 0 \).

In what follows, it is useful to introduce the following polynomials

\[ q_2(t) = t^2 - a_2 t + a_0, \quad q_3(t) = t^3 - b_4 t^2 + b_2 t - b_0, \quad t \in \mathbb{C} \]  

with

\[
\begin{align*}
    a_2 &= \frac{\mu}{\rho} + \frac{d_2}{b}, \quad a_0 = \frac{\mu d_2 - \varepsilon_2^2}{\rho b}, \\
    b_4 &= \left( \frac{\lambda + 2\mu}{\rho} + \frac{d}{b} \right) + \frac{1}{a} \left( \frac{m_2^2}{b} + \frac{\beta^2}{\rho} \right) + \frac{k}{a}, \\
    b_2 &= \frac{1}{\rho ab} \left\{ (ad + m^2) \left[ (\lambda + 2\mu)d - (\varepsilon_1 + 2\varepsilon_2)^2 \right] + [d\beta - (\varepsilon_1 + 2\varepsilon_2)m]^2 \right\} + \frac{k}{a} \left( \frac{\lambda + 2\mu}{\rho} + \frac{d}{b} \right), \\
    b_0 &= \frac{k}{\rho ab} \left[ (\lambda + 2\mu)d - (\varepsilon_1 + 2\varepsilon_2)^2 \right].
\end{align*}
\]

Under the restrictions imposed by the strong ellipticity conditions (2), (3), we can easily prove that

\[ a_2 > 0, \quad a_0 > 0, \quad b_4 > 0, \quad b_2 > 0, \quad b_0 > 0, \]

so that if the polynomials have a real root, it must be positive. Moreover, it is proved in [34] that if the constitutive coefficients satisfy the conditions (2) and (3), then the equation

\[ q_2(t)q_3(t) = 0 \]

has only real (and positive) solutions. In particular, we have that the roots of \( q_2(t) \) are

\[ t_{1,2} = \frac{\mu b + \rho d_2 \pm \sqrt{(\mu b - \rho d_2)^2 + 4\rho b\varepsilon_2^2}}{2\rho b}. \]  

On the other hand, in [34] it is shown that the cubic equation \( q_3(t) = 0 \) has three different solutions if

\[ h_0^2 < \frac{4}{27} h_1^3, \]  

with \( h_0 = -(2b_4^3 - 9b_2b_4 + 27b_0)/27 \), and \( h_1 = (b_4^2 - 3b_2)/3 \), and these roots are

\[ t_k = \frac{b_4}{3} + 2\sqrt{\frac{h_1}{3}} \cos \left[ \frac{1}{3} \arccos \left( \frac{-3h_0}{2h_1} \sqrt{\frac{3}{h_1}} \right) - \frac{2\pi}{3} (k + 1) \right] \quad k = 3, 4, 5. \]  

In the following, we assume that the constitutive coefficients are such that Eqs. (2), (3) and (6) are satisfied.
3 Rayleigh surface waves

In what follows, $\Omega$ is a half-space made of an isotropic homogeneous strongly elliptic thermoelastic material with microtemperatures such that the coupling constitutive coefficients are non zero ($m \neq 0$, $\beta \neq 0$ and $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$). This half-space is characterized by $x_2 \geq 0$. We study the propagation of a surface wave in the $x_1$-direction and with attenuation in the $x_2$-direction. The surface $x_2 = 0$ is assumed stress free and thermally insulated. To this end, we seek for solutions of the system (1) in the form

$$u_\alpha = u_\alpha (x_1 - vt, x_2), \quad u_3 = 0,$$
$$\tau_\alpha = \tau_\alpha (x_1 - vt, x_2), \quad \tau_3 = 0, \quad \chi = \chi (x_1 - vt, x_2), \quad \text{with } v \in \mathbb{C},$$

so that the system (1) becomes

$$[t_{1i} - \rho v^2 u_{i,1}]_{,1} + [t_{2i}]_{,2} = 0,$$
$$[\Lambda_{1i} - v^2 b \tau_{i,1}]_{,1} + [\Lambda_{2i}]_{,2} = 0,$$
$$[S_1 - v^2 a \chi_{,1}]_{,1} + [S_2]_{,2} = 0,$$

where

$$t_{ai} = \lambda \delta_{ia} u_{\beta, \beta} + \mu (u_{a, \beta} + u_{i, \alpha}) + \varepsilon_1 \delta_{ia} \tau_{\beta, \beta} + \varepsilon_2 (\tau_{a, \beta} + \tau_{i, \alpha}) + \nu \beta \delta_{ia} \chi_1,$$
$$\Lambda_{ai} = \varepsilon_1 \delta_{ia} u_{\beta, \beta} + \varepsilon_2 (u_{a, \beta} + u_{i, \alpha}) + d_1 \delta_{ia} \tau_{\beta, \beta} + d_2 \tau_{a, \alpha} + d_3 \tau_{i, i} + \nu m \delta_{ia} \chi_1,$$
$$S_\alpha = v \beta u_{a,1} + v m \tau_{a,1} + k \chi_{,a}. $$

Let be

$$\mathcal{U} = (u_1, u_2, \tau_1, \tau_2, \chi)^T, \quad \mathcal{T}_\alpha = (t_{a1}, t_{a2}, \Lambda_{a1}, \Lambda_{a2}, S_\alpha)^T.$$ 

Since we study the wave with attenuation in the direction $x_2$ and the surface $x_2 = 0$ is stress free and is thermally insulated, we have the following asymptotic conditions

$$\lim_{x_2 \to +\infty} \mathcal{U} (x_1, x_2, t) = 0, \quad \lim_{x_2 \to +\infty} \mathcal{T}_\alpha (x_1, x_2, t) = 0, \quad \forall x_1 \in \mathbb{R}, \; t \geq 0,$$

and the boundary conditions

$$\mathcal{T}_2 (x_1, 0, t) = 0, \quad \forall x_1 \in \mathbb{R}, \; t \geq 0.$$ 

Now, we seek for solutions $\mathcal{U}$ of the above problem in the following exponential form

$$\mathcal{U} = \tilde{\mathcal{U}} e^{i \lambda (x_1 - vt + px_2)} \quad \text{with } \tilde{\mathcal{U}} = (U_1, U_2, A_1, A_2, B)^T.$$ 

Here, $\mathcal{U} = (U_1, U_2, 0)^T$ and $\Lambda = (A_1, A_2, 0)^T$ are complex constant vectors and $B$ is a complex constant with $|\mathcal{U}| \neq 0$ or $|\Lambda| \neq 0$ or $B \neq 0$. Further, $v$ is such that

$$v = v_R - vi \quad \text{with } v_R \geq 0, \; v_I \geq 0,$$

and $p$ is such that

$$p = \alpha + \beta i \quad \text{with } \beta > 0,$$

in order to satisfy the asymptotic conditions (12). In particular, the real part of $v$ gives the wave speed and the imaginary part gives the rate of damping in time.
Substituting the exponential form \((14)\) into Eqs. \((9), (10)\) we arrive to a homogeneous linear algebraic system

\[
\mathcal{D}_p \tilde{U} = 0,
\]

where the matrix \(\mathcal{D}_p\) is defined as

\[
\mathcal{D}_p = p^2 Q_1 + p Q_2 + R
\]

with

\[
Q_1 = \begin{pmatrix}
\mu & 0 & \varepsilon_2 & 0 & 0 \\
0 & \lambda + 2\mu & 0 & \varepsilon_1 + 2\varepsilon_2 & 0 \\
\varepsilon_2 & 0 & d_2 & 0 & 0 \\
0 & \varepsilon_1 + 2\varepsilon_2 & 0 & d & 0 \\
0 & 0 & 0 & 0 & k
\end{pmatrix},
\]

\[
Q_2 = \begin{pmatrix}
0 & \lambda + \mu & 0 & \varepsilon_1 + \varepsilon_2 & 0 \\
\lambda + \mu & 0 & \varepsilon_1 + \varepsilon_2 & 0 & v\beta \\
0 & \varepsilon_1 + \varepsilon_2 & 0 & d_1 + d_3 & 0 \\
\varepsilon_1 + \varepsilon_2 & 0 & d_1 + d_3 & 0 & mv \\
0 & v\beta & 0 & mv & 0
\end{pmatrix},
\]

\[
R = \begin{pmatrix}
\lambda + 2\mu - \rho v^2 & 0 & \varepsilon_1 + 2\varepsilon_2 & 0 & v\beta \\
0 & \mu - v^2\rho & 0 & \varepsilon_2 & 0 \\
\varepsilon_1 + 2\varepsilon_2 & 0 & d_0 - bv^2 & 0 & mv \\
0 & \varepsilon_2 & 0 & d_0 - bv^2 & 0 \\
v\beta & 0 & mv & 0 & k - av^2
\end{pmatrix},
\]

in other words we should have \(\tilde{U} \in \ker \mathcal{D}_p\). Since \(\tilde{U}\) is a non-trivial solution of the homogeneous algebraic system \((17)\), then we obtain the following propagation condition

\[
\det \mathcal{D}_p = 0.
\]

We observe that \(\det \mathcal{D}_p\) is a homogeneous polynomial with respect to \(v^2\) and \(p^2 + 1\); consequently, \(v = 0\) if and only if \(p^2 + 1 = 0\). Now, if we introduce

\[
t = \frac{v^2}{p^2 + 1} \quad \text{with} \quad v \neq 0,
\]

we can rewrite the condition \((19)\) as

\[
q_2(t)q_3(t) = 0,
\]

where \(q_2\) and \(q_3\) are defined in \((1)\).

The propagation condition \((21)\) holds if and only if \(t\) is a root of \(q_2\) or \(q_3\). We suppose that \(q_2\) and \(q_3\) do not have a common root. The (real and positive) solutions of Eq. \((21)\) are expressed in Eqs. \((5)\) and \((7)\).

The wave-number \(\kappa\) does not appear in Eq. \((19)\), so that the phase velocity \(v\) cannot depend on \(\kappa\) and therefore there is no dispersion.

Let be \(t_k\) a solution of Eq. \((21)\) \((k = 1, \ldots, 5)\) and be \(p_k = \alpha_k + \beta_k i\) the value of \(p\) corresponding to \(t_k\) through Eq. \((20)\) and satisfying Eq. \((16)\). Then, Eqs. \((15)\) and \((20)\) imply

\[
t_k \left[\alpha_k^2 - \beta_k^2 + 1\right] = v_R^2 - v_I^2, \quad t_k \alpha_k \beta_k = - v_R v_I, \quad \forall k.
\]

Since all roots \(t_k\) are positive, we arrive to

\[
\alpha_k \beta_k \leq 0 \quad \Rightarrow \quad \alpha_k \leq 0.
\]
Consequently, we can remark by using Eqs. (22) that
\[ v_R v_I = 0 \quad \Rightarrow \quad \alpha_k = 0 \quad \text{or} \quad \beta_k = 0; \]
in particular, we obtain from (22)
\[ v = v_R : p_k = \sqrt{1 - \frac{v_R^2}{t_k}} \quad \text{if} \quad v_R < \min_{k \in \{1, \ldots, 5\}} \sqrt{t_k}, \quad \text{and} \quad p_k = \alpha_k \quad \text{if} \quad v_R \geq \sqrt{t_k} \quad \text{for some} \quad k \quad \text{but it is not compatible with the condition (16)}; \]
\[ v = v_I : p_k = \beta_k i \quad \forall k. \]
The solution (14) of the problem corresponding to \( p_k \ (k = 1, \ldots, 5) \) is
\[ \mathcal{U}^{(k)} = \mathcal{U}^{(k)} e^{ix(x_1 - vt + p_k x_2)} = \mathcal{U}^{(k)} e^{-x/\beta_k x_2} e^{ix(x_1 - vt - \alpha_k x_2)}, \quad \text{if} \quad k = 1, 2, \]
where
\[ \mathcal{U}^{(k)} = \begin{pmatrix} U_1^{(k)}, U_2^{(k)}, A_1^{(k)}, A_2^{(k)}, B^{(k)} \end{pmatrix}^T \in \ker \mathcal{D}_{p_k}. \quad \text{(25)} \]
We calculate the solutions of the corresponding homogeneous linear system (17) and we arrive to
\[ \mathcal{U}^{(k)} = \begin{pmatrix} \Gamma_k, p_k \Gamma_k, \Lambda_k, p_k \Lambda_k, \frac{v}{m \beta t_k} [\Gamma_k \Lambda_k - (\varepsilon_1 + 2 \varepsilon_2) (\beta \Gamma_k + m \Lambda_k)] \end{pmatrix}^T, \quad \text{if} \quad k = 3, 4, 5, \]
with
\[ \Phi_k = \frac{b}{\varepsilon_2} \left( t_k - \frac{d_2}{b} \right), \]
\[ \Gamma_k = b \beta \left( t_k - \frac{d_2}{b} \right) + m (\varepsilon_1 + 2 \varepsilon_2), \quad \Lambda_k = \rho m \left( t_k - \frac{\lambda + 2 \mu}{\rho} \right) + \beta (\varepsilon_1 + 2 \varepsilon_2). \]
Let be \( \mathbf{n}^{(k)} = (1, p_k, 0)^T \). It is then obvious that:
\[ k = 1, 2: \quad \mathcal{U}^{(k)} \text{ are such that } \mathcal{U}^{(k)} = (-p_k \Phi_k, \Phi_k, -p_k, 1, 0)^T \text{ and } \mathbf{A}^{(k)} = (-p_k, 1, 0) \text{ are orthogonal to } \mathbf{n}^{(k)}; \]
\[ k = 3, 4, 5: \quad \mathcal{U}^{(k)} \text{ are such that } \mathcal{U}^{(k)} = (\Gamma_k, p_k \Gamma_k, 0)^T \text{ and } \mathbf{A}^{(k)} = (\Lambda_k, p_k \Lambda_k, 0) \text{ are parallel to } \mathbf{n}^{(k)}. \]
The more general solution \( \mathcal{U} \) of our problem is given by a linear combination of the \( \mathcal{U}^{(k)} \)
\[ \mathcal{U}(x_1, x_2, t) = \sum_{k=1}^{5} \gamma_k \mathcal{U}^{(k)} e^{ix(x_1 - vt + p_k x_2)}, \quad \text{(26)} \]
where \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_5)^T \) is a non-zero constant vector. Substituting the expression (26) into Eq. (10), we arrive to
\[ \mathcal{T}_2(x_1, x_2, t) = i \chi \sum_{k=1}^{5} \gamma_k \mathbf{s}_{p_k} \mathcal{U}^{(k)} e^{ix(x_1 - vt + p_k x_2)}, \quad \text{(27)} \]
with

\[ S_{p_k} = \begin{pmatrix} \mu p_k & \mu & p_k \varepsilon_2 & \varepsilon_2 & 0 \\ \lambda & (\lambda + 2\mu)p_k & \varepsilon_1 & p_k (\varepsilon_1 + 2\varepsilon_2) & v\beta \\ p_k \varepsilon_2 & \varepsilon_1 & d_2 p_k & d_3 & 0 \\ \varepsilon_2 & p_k (\varepsilon_1 + 2\varepsilon_2) & d_1 & d_2 p_k & m v \\ 0 & v\beta & 0 & m v & k p_k \end{pmatrix}. \]

With the aid of the boundary conditions (13), Eq. (27) leads to

\[ T_2(x_1, 0, t) = i \pi \sum_{k=1}^{5} \gamma_k S_{p_k} \tilde{U}^{(k)} e^{i\kappa(x_1 - vt)} = 0, \quad \forall x_1 \in \mathbb{R}, \forall t \geq 0, \]

and, equivalently,

\[ \mathcal{A} \gamma = 0 \quad (28) \]

where \( \mathcal{A} = \|a_{hk}\| \) with \((a_{1k}, a_{2k}, \ldots, a_{5k})^T = S_{p_k} \tilde{U}^{(k)}\). A non trivial solution \( \gamma \) of Eq. (28) exists if and only if

\[ \det \mathcal{A} = 0, \quad (29) \]

which represents the secular equation for the complex parameter \( v \). We have to select the solutions of the secular equation (29) satisfying the conditions (15).

Now, we want to investigate, from a numerical point of view, the secular equation (29) with respect to the complex parameter \( v \). To this aim, we will take arbitrary values for the relevant constitutive parameters, compatible with restrictions (2), (3). In particular, we look for a numerical solution of Eq. (29).

We note that this relation contains the unknown \( v \) both explicitly and implicitly through \( p_k, k = 1, \ldots, 5 \), that should be taken as the solutions of relation (19). The solution of this system of two nonlinear equations is not easy, and we take another approach. We define

\[ \mathcal{F}(\text{Re}(v), \text{Im}(v)) = \ln |\det \mathcal{A}|. \]

The presence of the logarithm is convenient because the function has a wide range of variability. We can now make a graphics of the function \( \mathcal{F} \) looking for a minimum. In Figure 1 we show the graphics that we have obtained, where it is possible to see the presence of a minimum around \( v = 0.62 - 0.08i \).

### 4 Special cases

In this section, we consider the class of isotropic strongly elliptic thermoelastic media with microtemperatures when some of the coupling coefficients vanish; in particular, we consider the following cases:

i) \( \beta = 0, m \neq 0 \) and \( \varepsilon_1 = \varepsilon_2 = 0 \):

The propagation condition (19) is rewritten as Eq. (21), where \( q_2 \) and \( q_3 \) reduce to

\[ q_2 = \left( t - \frac{\mu}{\rho} \right) \left( t - \frac{d_2}{b} \right), \quad q_3 = \left( t - \frac{\lambda + 2\mu}{\rho} \right) \left( t^2 - \frac{m^2 + ad + bk}{ab} t + \frac{kd}{ab} \right), \quad (30) \]
so that

\[ t_1 = \frac{\mu}{\rho}, \quad t_2 = \frac{d_2}{b}, \quad t_3 = \frac{\lambda + 2\mu}{\rho}, \quad t_{4,5} = \frac{m^2 + ad + bk \pm \sqrt{m^4 + (ad - bk)^2 + 2m^2(ad + bk)}}{2ab}. \]  

Let be \( p_k \) the values of \( p \) corresponding, through (20), to the roots \( t_k \) defined in (31) and (32). The kernels, associated with \( \mathcal{D}_{p_k} \), are spanned by

\[ \tilde{U}^{(1)} = (-p_1, 1, 0, 0, 0)^T, \]
\[ \tilde{U}^{(2)} = (0, 0, -p_2, 1, 0)^T, \]
\[ \tilde{U}^{(3)} = (1, p_3, 0, 0, 0)^T, \]
\[ \tilde{U}^{(4)} = (0, 0, \Pi_4, p_4\Pi_4, mv(bt_4 - d_2))^T, \]
\[ \tilde{U}^{(5)} = (0, 0, \Pi_5, p_5\Pi_5, mv(bt_5 - d_2))^T, \]

where

\[ \Pi_k = m^2t_k + (at_k - k)(d_1 + d_3), \quad k = 4, 5. \]

The secular equation (29) reduces to

\[
v \left[ 4\mu^2p_1p_2 + (\rho v^2 - 2\mu)^2 \right] \left\{ bp_4 \left[ bv^2 - (d_2 + d_3) \right] \left[ (k - at_5) \left[ bv^2 - (d_2 + d_3) \right] + m^2v^2 \right] + p_5 \left[ p_3p_4(d_2 + d_3)^2 \left( -2abt_5 + ad + bk \right) + a(d - bt_5) \left[ bv^2 - (d_2 + d_3) \right]^2 + m^2(d_2 + d_3) \left[ (p_3p_4 + 1)(d_2 + d_3) - bv^2 \right] \right\} = 0.
\]
ii) $\beta \neq 0$, $m = 0$ and $\varepsilon_1 = \varepsilon_2 = 0$:

The propagation condition (19) leads to Eq. (21) where $q_2$ is defined by Eq. (30) and $q_3$ reduces to

$$q_3(t) = (bt - d) \left( t^2 - \frac{a(\lambda + 2\mu) + \beta^2 + k\rho}{a\rho} t + \frac{(\lambda + 2\mu)k}{a\rho} \right).$$

In particular, the roots $t_1, t_2$ are defined by (31) and the other roots are

$$t_3 = \frac{d}{b}, \quad t_{4,5} = \frac{a(\lambda + 2\mu) + \beta^2 + k\rho \pm \sqrt{\beta^4 + [a(\lambda + 2\mu) - \rho k]^2 + 2\beta^2 [a(\lambda + 2\mu) + \rho k]}}{2a\rho}.$$

Consequently, we obtain

$$\mathbf{\tilde{U}}^{(1)} = (-p_1, 1, 0, 0, 0)^T,$$
$$\mathbf{\tilde{U}}^{(2)} = (0, 0, -p_2, 1, 0)^T,$$
$$\mathbf{\tilde{U}}^{(3)} = (0, 0, 1, p_3, 0)^T,$$
$$\mathbf{\tilde{U}}^{(4)} = (\Omega_4, p_4\Omega_4, 0, 0, \beta \nu(\rho t_4 - \mu))^T,$$
$$\mathbf{\tilde{U}}^{(5)} = (\Omega_5, p_5\Omega_5, 0, 0, \beta \nu(\rho t_5 - \mu))^T,$$

where

$$\Omega_k = \beta^2 t_k + (at_k - k)(\lambda + \mu), \quad k = 4, 5.$$

We can calculate the secular equation (29) and we obtain

$$v \left[ [bv^2 - (d_2 + d_3)]^2 + p_2 p_3 (d_2 + d_3)^2 \right] \left\{ p_4 \rho (\rho v^2 - 2\mu) \left[ \beta v^2 - (k - at_5) (2\mu - \rho v^2) \right] + p_5 \left[ 4\mu^2 p_1 p_4 (a(\lambda + 2\mu - 2\rho t_5) + k\rho) + a (\rho v^2 - 2\mu)^2 (\lambda + 2\mu - \rho t_5) + 2\beta^2 \mu (2\mu + 2\mu p_1 p_4 - \rho v^2) \right] \right\} = 0.$$

iii) $\beta = 0$, $m = 0$, $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$:

The propagation condition (19) leads to Eq. (21), where $q_2$ is defined by Eq. (30) and $q_3$ reduces to

$$q_3(t) = \left( t - \frac{k}{a} \right) \left( t^2 - \frac{b(\lambda + 2\mu)}{\rho b} t + \frac{(\lambda + 2\mu)d - (\varepsilon_1 + 2\varepsilon_2)^2}{\rho b} \right).$$

In particular, the roots $t_1, t_2$ are defined by (5) and other three roots are

$$t_3 = \frac{k}{a}, \quad t_{4,5} = \frac{b(\lambda + 2\mu) + d\rho \pm \sqrt{[b(\lambda + 2\mu) - \rho d]^2 + 4\rho b(\varepsilon_1 + 2\varepsilon_2)^2}}{2\rho b}.$$

The vectors of the bases of the kernels of $\mathcal{D}_{p_k}$ are defined by

$$\mathbf{\tilde{U}}^{(1)} = (\varepsilon_1 + 2\varepsilon_2, (\varepsilon_1 + 2\varepsilon_2)p_1, \Psi_1, p_2\Psi_1, 0)^T,$$
$$\mathbf{\tilde{U}}^{(2)} = (\varepsilon_1 + 2\varepsilon_2, (\varepsilon_1 + 2\varepsilon_2)p_2, \Psi_2, p_2\Psi_2, 0)^T,$$
$$\mathbf{\tilde{U}}^{(3)} = (-\varepsilon_2 p_3, \varepsilon_2, p_3\hat{\Psi}_3, \hat{\Psi}_3, 0)^T,$$
$$\mathbf{\tilde{U}}^{(4)} = (-\varepsilon_2 p_4, \varepsilon_2, p_4\hat{\Psi}_4, \hat{\Psi}_4, 0)^T,$$
$$\mathbf{\tilde{U}}^{(5)} = (0, 0, 0, 0, \varepsilon_2)^T.$$

10
where

\[ \Psi_k = \rho t_k - (\lambda + 2\mu), \quad k = 1, 2 \]

\[ \hat{\Psi}_k = \rho t_k - \mu, \quad k = 3, 4 \]

It is possible to calculate the secular equation \([29]\), but the obtained formula is too long to be reported here.

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