Topological K-Theory for Hilbert Scheme Analogs

Ammar Husain

Abstract

In geometric representation theory, it is common to compute equivariant $K$ theory of schemes like $\text{Hilb}^n(\mathbb{A}^2)$ or $\text{Hilb}^n(X)$ for an ALE resolution $X \to \mathbb{A}^2/\Gamma$. If we abandon the algebraic nature and just look at this homotopically we see close relatives of $BS_n$ and $B(\Gamma \wr S_n)$. Therefore we compute the topological $K$ theory of these classifying spaces to fill in a small gap in the literature.

1 Introduction

In the seminal paper of Atiyah and Segal [1], they show that $K^\bullet(BG)$ for a finite group is isomorphic to the representation ring of $G$ completed at the augmentation ideal. Here we compute some examples. In particular at the sequences that come from Platonic groups, Weyl groups and groups associated to Hilbert schemes of du Val singularities. We can construct interesting generating functions in the cases when the groups come in countable families. In some of these cases this can be interpreted as replacing the usual algebraic $K^0$ and genuine equivariance with the much simpler Borel equivariant topological $K$ theory. This can be a very pale shadow of the Platonic ideal.

2 K Theory of discrete $BG$

2.1 Theorem ([2] Lück) For finite groups $G$\textsuperscript{1}

$$
K^0(BG) = \mathbb{Z} \times \prod_p (\mathbb{Z}_p)^{r(p,G)}
$$
$$
K^1(BG) = 0
$$

where $r(p,G)$ is the number of conjugacy classes $C$ such that $g \in C$ will have order $p^d$ for some $d \geq 1$. Similarly define $\tilde{r}(p,G) = r(p,G) + 1$ with $d = 0$ also allowed.

2.2 Definition (Rank Generating Functions) If there is a sequence of finite groups $G_n$ with $n \in \mathbb{N}$, define two generating functions as

$$
OGF(p,G*,x) = \sum r(p,G_n)x^n
$$
$$
\tilde{OGF}(p,G*,x) = \sum \tilde{r}(p,G_n)x^n
$$
$$
\tilde{OGF}(p,G*,x) = OGF(p,G*,x) + \frac{1}{1-x}
$$

\textsuperscript{1}Parenthesis are used to distinguish $p$-adic vs cyclic groups.
3 Platonic Groups

For finite subgroups $G \subset SL(2, \mathbb{C})$, there are few choices. In this section, we describe their $K^0(BG)$. The cohomologies of these groups are described in [3].

3.1 Lemma For the sequence of cyclic groups $Cyc_n = \mathbb{Z}_{n+1}$

$$O\tilde{G}F(p, Cyc, x) = \sum_{n=0}^{\infty} p^{\nu_p(n+1)} x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{|n+1|_p}$$

For the sequence of binary cyclic groups $BinCyc_n = \mathbb{Z}_{2(n+1)}$

$$O\tilde{G}F(p, BinCyc, x) = \sum_{n=0}^{\infty} p^{\nu_p(2(n+1))} x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{2(n+1)|_p}$$

Proof

$$\sum_n r(p, \mathbb{Z}_{n+1}) x^n = \sum_n \sum_{r=1}^{\nu_p(n+1)} \phi(p^r) \delta_{p^r | (n+1)} x^n$$

$$= \sum_n \sum_{r=1}^{\nu_p(n+1)} \phi(p^r) x^n$$

$$\sum d | n+1 \phi(d) = n+1$$

$$\sum_{n=0}^{\infty} r(p, G_n) x^n = \sum_{n=0}^{\infty} (p^{\nu_p(n+1)} - 1) x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{|n+1|_p - 1} x^n$$

where $\nu_p(n)$ is the p-adic valuation that indicates the highest power dividing $n$. $|n|_p = p^{-\nu_p(n)}$ is the p-adic norm.2.

3.2 Corollary (Product of Cyclic Groups) For $\mathbb{Z}_{n+1} \times \mathbb{Z}_{m+1}$ (which is realized in $GL(2, \mathbb{C})$ instead of $SL(2, \mathbb{C})$) we have

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2For $p = 2$, this is [4]
\[ OGF(p, \text{Cyc} \times \text{Cyc}, x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p^{\nu_p(n+1)} p^{\nu_p(m+1)} x^n y^m \]
\[ = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m \frac{1}{n+1 \mid_p m+1 \mid_p} \]

In particular if the sequence of groups \( \mathbb{Z}_{n+1}^2 \) we get

\[ OGF(p, \text{Cyc} \times \text{Cyc}, x) = \sum_{n=0}^{\infty} \frac{x^n}{n+1 \mid_p^2} \]

in analogy with the dilogarithm.

**3.3 Proposition (Dihedral)**

\[ OGF(2, \text{BinDih}_{\bullet+1}, x) = \sum_{n=0}^{\infty} (2 + \frac{1}{2 \mid 2n+2 \mid_2}) x^n \]
\[ = \frac{2}{1-x} + \frac{1}{2} OGF(p, \text{BinCyc}, x) \]
\[ OGF(p, \text{BinDih}_{\bullet+1}, x) = \sum_{n=0}^{\infty} 1 + \frac{1}{2 \mid 2n+2 \mid_p} (1 - 1) x^n \]
\[ = \frac{1}{2} \frac{1}{1-x} + \frac{1}{2} OGF(p, \text{BinCyc}, x) \]

**Proof** The conjugacy classes of the binary dihedral group are

- \( e \)
- \( a^n = x^2 \) which has order 2 contributes 1 to \( r(2, \text{BinDih}_n) \)
- \( a^m \simeq 2^{n-m} \) for \( m \neq n \). They have same order as \( m \) does in binary cyclic group \( \mathbb{Z}_{2n} \). Contribute half as much as in \( r(p, \text{BinCyc}_n) \)
- \( x \) order 4. Contribute 1 to \( r(2, \text{BinDih}_n) \)
- \( ax \) order 4. Contribute 1 to \( r(2, \text{BinDih}_n) \)
\[ r(2, \text{BinDih}_n) = 1 + 2 + \frac{1}{2}(r(2, \mathbb{Z}_{2n}) - 1) \]
\[ r(p \neq 2, \text{BinDih}_n) = \frac{1}{2}(\tilde{r}(p, \mathbb{Z}_{2n}) - 1) \]
\[ \tilde{r}(2, \text{BinDih}_n) = 2 + \frac{1}{2} \tilde{r}(2, \mathbb{Z}_{2n}) = 2 + \frac{1}{2} 2^{\nu_2(2n)} = 2 + 2^{\nu_2(2n) - 1} \]
\[ = 2 + \frac{1}{2} |2n|_2 \]
\[ \tilde{r}(p, \text{BinDih}_n) = 1 + \frac{1}{2}(\tilde{r}(p, \mathbb{Z}_{2n}) - 1) \]
\[ = 1 + \frac{1}{2} (p^\nu_p(2n) - 1) \]
\[ = 1 + \frac{1}{2} \left( \frac{1}{|2n|_p} - 1 \right) \]

The rest are the exceptional types which do not come in sequences. So we just list their \( r(p, G) \) for later use.

### 3.4 Lemma (Exceptional Platonic Groups)

\[ K^0(BA_4 = BT) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^1 \times \mathbb{Z}_{(3)}^2 \]
\[ K^0(BS_4 = BT_d) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^3 \times \mathbb{Z}_{(3)} \]
\[ K^0(BS_5) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^3 \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(5)} \]
\[ K^0(BA_5 = BI) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^1 \times \mathbb{Z}_{(3)} \times \mathbb{Z}_{(5)}^2 \]
\[ K^0(B\text{Bin}T = BSL(2, 3)) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^2 \times \mathbb{Z}_{(3)}^3 \]
\[ K^0(B\text{Bin}I = BSL(2, 5)) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^2 \times \mathbb{Z}_{(3)}^3 \times \mathbb{Z}_{(5)}^2 \]
\[ K^0(B\text{Bin}O) \cong \mathbb{Z} \times \mathbb{Z}_{(2)}^5 \times \mathbb{Z}_{(3)}^1 \]

For \( G \times \mathbb{Z}_2 \) or \( G \times \mathbb{Z}_4 \) with one of the \( G \) above, simply double/quadruple \( \tilde{r}(2, G) \) and leave the others the same. This takes care of all the exceptional finite subgroups of \( SO(3) \), \( O(3) \), \( Pin_{\pm}(3) \) and \( Spin(3) \).

### 4 Weyl Groups

#### 4.1 Type A

For \( G = S_n \), \( \tilde{r}(p, S_n) \) is the number of partitions into powers of \( p \). In particular, \( \tilde{r}(2, S_n) \) is \([5]\) and \( r(3, S_n) \) is \([6]\). Note first two trivial groups \( S_0 \cong S_1 \cong \{e\} \).
4.1 Lemma

\[ OGF(p, A, x) = \sum \tilde{r}(p, n)x^n \]
\[ = \prod_{j \geq 0} \frac{1}{1 - x^{p^j}} \]

\[ OGF(p, A, x) = \sum r(p, n)x^n \]
\[ = \prod_{j \geq 0} \frac{1}{1 - x^{p^j}} \left( 1 - \frac{1}{1 - x} \right) \]
\[ = \frac{1}{1 - x} (\prod_{j \geq 1} \frac{1}{1 - x^{p^j}} - 1) \]

\[ g(p, A, x, z) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^{kp^j}}{k} \]

\[ g(p, A, x, 1) = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^{kp^j}}{k} = \log OGF(p, A, x) \]

\[ g(p, A, x^p, 1) = g(p, A, x, 1) - \log(1 - x) \]

Note that without the condition from \( p \), this would be related to the logarithm of the Dedekind \( \eta \) function.

**Proof**

\[ g(p, A, x, 1) = \log \sum \tilde{r}(p, n)x^n = \log \prod_{j \geq 0} \frac{1}{1 - x^{p^j}} \]
\[ = \sum_{j=0}^{\infty} \log \frac{1}{1 - x^{p^j}} \]
\[ = - \sum_{j=0}^{\infty} \log(1 - x^{p^j}) \]
\[ = - \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^{kp^j}}{k} \]
\[ = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^{kp^j}}{k} \]
\[
x \to x^p \implies \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^k p^j}{k} \to \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^{k+1} p^j}{k}
\]

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^k p^j}{k} = \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{x^k p^j}{k} - z^{-1} \sum_{k=1}^{\infty} \frac{x^k}{k}
\]

\[
g(p, A, x^p, z) = \frac{1}{z} (g(p, A, x, z) - \log(1 - x))
\]

\[
zg(p, A, x^p, z) = g(p, A, x, z) - \log(1 - x)
\]

4.2 Lemma When \(x\) approaches the root of unity \(x^l = 1\) and \((l, p) = 1\), there are divergences in \(g(p, A, x, z)\). You can begin to see this in figs. 1 and 2.

Proof

\[
g(p, A, x, z) = \sum_{j=0}^{\infty} z^j \sum_{k=1}^{l} \frac{x^k p^j}{k} \sum_{m=0}^{\infty} \frac{1}{k + ml}
\]

\[
\sum_{m=0}^{\infty} \frac{1}{k + ml} \to \infty
\]

We may play the usual games that we do with the generating function for all partitions such as finding asymptotics and Mellin transforms.

4.3 Lemma (Asymptotics of \(\tilde{r}(p, S_n)\) [7]) \(\log \tilde{r}(p, S_n) \approx \frac{\log^2 n}{n \log p}\). This slow growth rate relative to \(n!\) indicates \(OGF(p, A, x)\) would work well with Borel summation.

4.4 Lemma (Mellin Transform)

\[
F(p, A, s) \equiv \mathcal{M}(\log \tilde{OGF}(p, A, e^{-t}))
\]

\[
F(p, A, s) = \Gamma(s) \sum_{j \geq 0, k \geq 1} \frac{(kp^j)^{-s}}{k} = \Gamma(s) \frac{\zeta(s + 1)}{1 - p^{-s}}
\]

To recover the \(\tilde{OGF}\) and therefore the \(r(p, S_n)\), undo the Mellin transform and exponentiate.

Proof Let \(j(t) = \log \tilde{OGF}(p, A, e^{-t})\). This obeys a shift equation relating values at \(pt\) and \(t\).

\[
j(pt) = j(t) - \log(1 - e^{-t})
\]

\[
p^{-s} \mathcal{M}(j(t))(s) = \mathcal{M}(j(t))(s) - \mathcal{M}(\log(1 - e^{-t}))
\]

\[
(p^{-s} - 1) F(p, A, s) = -\mathcal{M}(\log(1 - e^{-t}))
\]
Figure 1: Type $A$ $g(p, A, x, 1)$ $p = 2$ with $j$ and $k$ sums cutoff at 20.

Figure 2: Type $A$ $g(p, A, x, 1)$ $p = 3$ with $j$ and $k$ sums cutoff at 20.

4.2 Type B/C

4.5 Proposition The conjugacy classes whose elements have order $p^r$ for some $r \geq 0$ are labelled by pairs of partitions such that the total number is $n$. Call those positive and negative cycles. If $p \neq 2$ the partition is entirely in positive cycles and has to be into $p^r$ parts. If $p = 2$ then there are both positive and negative cycles each with $2^r$ parts for possibly different $r$’s.

$$\tilde{r}(2, W_{B_n}) = \sum_{m=0}^{n} \tilde{r}(2, S_m)\tilde{r}(2, S_{n-m})$$

$$\tilde{r}(p, W_{B_n}) = \tilde{r}(p, S_n)$$

$$OGF(2, B, x) = OGF(2, A, x)OGF(2, A, x)$$

$$OGF(p \neq 2, B, x) = OGF(p \neq 2, A, x)$$

Proof For $B(\mathbb{Z}_2 \wr S_n)$, conjugacy classes are labelled by pairs of partitions [8]. Take the prime factorization of the order of the associated element. The negative cycles have a factor of 2 from the sign flip around the cycle. A negative cycle $(12 \cdots n)$ has order $2n$. This ensures that whenever $p \neq 2$, there can be no negative cycles. This reduces to type $A$. If $p = 2$, then we are simply taking the least common multiple of a bunch of powers of 2, so we just need to ensure the parts have length $2^r$. \hfill \square
4.3 Type D

4.6 Theorem

\[ \tilde{r}(2, W_{D_n}) = \sum_{m=0}^{n} \tilde{r}(2, S_m)\tilde{r}(2, S_{n-m}, \text{evenparts}) + \tilde{r}(2, S_n, \text{evenlengths}) \]

\[ \tilde{r}(p, W_{D_n}) = \tilde{r}(p, S_n) + \tilde{r}(p, S_n, \text{evenlengths}) \]

\[ OGF(2, D, x) = OGF(2, A, x)^2 \left( G(x, 1) + G(x, -1) \right) + OGF(2, A, x^2) \]

\[ = \frac{1}{2} OGF(2, A, x)^2 + \frac{1}{2} OGF(2, A, x)(1 - x) + OGF(2, A, x^2) \]

\[ OGF(p \neq 2, D, x) = OGF(p \neq 2, A, x) \]

\[ OGF(p \neq 2, D, x) = OGF(p \neq 2, A, x) \]

Proof For \( W_{D_n} \) the second partition must have an even number of parts. Also when all cycles are positive of even length, the same partition gives 2 separate conjugacy classes. For the positive cycles we get order as the least common multiple of the lengths. The negative partitions are different only in the first cycle as \((123\cdots n) \rightarrow (-n, -1, 2, 3\cdots n - 1)\). \( g^n(1\cdots n) = (12\cdots n) \) so has a power of 2 order if and only if the underlying partition did.

\( \tilde{r}(2, S_{n-m}, \text{evenparts}) \) counts partitions of \( n - m \) into powers of 2 length parts and there must be an even number of them. Keep track of the number of parts by \( u \); so we only wish to take the sum of only even powers of \( u \).

\[ G(x, u) = \prod_{j \geq 0} \frac{1}{1 - ux^{2^j}} \]

\[ \sum \tilde{r}(2, S_n, \text{evenparts})x^n = \frac{1}{2}(G(x, 1) + G(x, -1)) \]

\[ G(x, -1) = 1 - x \]

\[ \sum \tilde{r}(2, S_n, \text{evenparts})x^n = \frac{1}{2} G(x, 1) + \frac{1}{2} (1 - x) \]

That means that \( \tilde{r}(2, S_n, \text{evenparts}) = \frac{1}{2} \tilde{r}(2, S_n) \) for all \( n \geq 2 \).

\( \tilde{r}(2, S_n, \text{evenlengths}) \) counts partitions of \( n \) in powers of 2 but none can be \( 2^0 \). If \( n \) is odd, then it is 0. This means we can halve all the parts and remove the condition of even lengths.

\[ \tilde{r}(2, S_n, \text{evenlengths}) = \tilde{r}(2, S_{n/2}) \]

\[ \sum \tilde{r}(2, S_n, \text{evenlengths})x^n = OGF(2, A, x^2) \]

\[ \square \]

In contrast, \( \tilde{r}(p \neq 2, S_n, \text{evenlengths}) = 0 \) because then \( p^r \) are all odd.
4.4 Exceptionals

4.7 Lemma (Exceptional Examples) The exceptional classes finish off the possible Weyl groups and they can be read from [9, 10].

- \( W_{D_4} \) \( r(2) = 10 \) \( r(3) = 1 \)
- \( W_{F_4} \) \( r(2) = 13 \) \( r(3) = 3 \)
- \( W_{G_2} \) \( r(2) = 3 \) \( r(3) = 1 \)
- \( W_{E_6} \) \( r(2) = 9 \) \( r(3) = 4 \) \( r(5) = 1 \)
- \( W_{E_7} \) \( r(2) = 23 \) \( r(3) = 4 \) \( r(5) = 1 \) \( r(7) = 1 \)
- \( W_{E_8} \) \( r(2) = 31 \) \( r(3) = 6 \) \( r(5) = 2 \) \( r(7) = 1 \)

4.8 Lemma \((H_{3/4})\) If we relax the condition to finite Coxeter groups from Weyl groups, the symmetries of the dodecahedron and 600-cell are also allowed.

- \( H_3 \) \( r(2) = 3 \) \( r(3) = 1 \) \( r(5) = 2 \)
- \( H_4 \) \( r(2) = 6 \) \( r(3) = 2 \) \( r(5) = 5 \)

Proof This can be read off from Sage:

\[
\begin{align*}
W &= \text{ReflectionGroup}([\text{'H',3}]); W \\
\text{CW} &= W.\text{conjugacy_classes_representatives}(); \\
\text{orders} &= [\text{CW}[i].\text{order()} \text{ for } i \text{ in range}(0, \text{len(CW))}]; \text{orders}
\end{align*}
\]

5 Analogy with Hilbert Schemes

5.1 Definition \((\text{Hilb}^n \mathbb{C}^2)\) We may resolve \( \mathbb{C}^n/S_n \) by taking ideals of length \( n \) in \( \mathbb{C}[x, y] \). Maximal ideals like \((x-a, y-b)\) will be points and if there are \( n \) disjoint points we get an ideal of length \( n \). Similarly define \( \text{Hilb}^n \) for other surfaces resolving \( \mathbb{C}^2/G \).

If we didn’t work algebraically, but topologically instead we would see a contractible space quotiented by \( S_n \). The action isn’t free which weakens the analogy. Repeat the same process for \( \mathbb{C}^2/G \) with \( G \) a binary Platonic group indexed by an ADE Lie algebra. In the purely homotopic world this resembles \( B(G\wr S_n) \) as a proxy for \( \text{Hilb}^n(\mathbb{C}^2/G) \). We cannot accommodate \( T \) equivariance because that escapes the world of quotients by finite groups. It can only be approximated by \( \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \) for which one may refer to (3.2).

Nakajima takes \( K_T(\text{Hilb}_n(\mathbb{C}^2/G)) \) in a procedure that gives the \( U_q(\hat{g}) \) for the corresponding Lie algebra [11]. For \( G = e \) the trivial group, this constructs a q-boson algebra. \( G = \mathbb{Z}_2 \) like the type B/C Weyl groups above corresponds to \( A_1 \) or \( U_q(\mathfrak{s}l_2) \).

5.2 Lemma Conjugacy classes in \( G \wr S_n \) are given by a partition of \( n \) and a labelling of each part with a conjugacy class of \( G \). We can decorate all with the same conjugacy class of \( G \) and get an analog of the decomposition into q-boson algebras.

\[\text{This isn’t exceptional in some ways but very exceptional in others.}\]
5.3 Corollary

\[ OGf(p, G \wr A, x) = \tilde{OGF}(p, A, x)^{\tilde{r}(p, G)} \]

This allows us to recover \( K^0(\mathbb{B}(G; S_n)) \) by looking at the appropriate coefficient of these functions for each \( p \). We can also easily let \( G \) vary if need be.

**Proof** First we give a partition of \( n \) and to have \( p \) power order it must be into \( p \) power parts. The conjugacy classes coloring the parts also have to be \( p \) power to maintain this condition. That gives \( \tilde{r}(p, G) \) choices for the colors of the parts. This procedure is in bijection with counting the number of ways of dividing \( n \) up into \( \tilde{r}(p, G) \) parts and then each one of those gets the structure of a partition. The values of \( \tilde{r}(p, G) \) of the binary Platonic groups were already given in section 3. Together on the generating function this amounts to taking the \( \tilde{r}(p, G) \)th power. \( \square \)

5.4 Lemma

\[
\begin{align*}
F(p, G \wr A, s) & \equiv M(\log \tilde{OGF}(p, G \wr A, e^{-t})) \\
F(p, G \wr A, s) & = \tilde{r}(p, G) \Gamma(s) \frac{\zeta(s+1)}{1 - p^{-s}} \\
\sum y^m F(p, G_m \wr A, s) & = \tilde{OGF}(p, G, y) \Gamma(s) \frac{\zeta(s+1)}{1 - p^{-s}}
\end{align*}
\]

**Proof**

\[
\begin{align*}
F(p, G \wr A, s) & \equiv M(\log \tilde{OGF}(p, e^{-t})) \\
& = \tilde{r}(p, G) M(\log \tilde{OGF}(p, A, e^{-t})) = \tilde{r}(p, G) F(p, A, s) \\
F(p, A, s) & = \Gamma(s) \sum_{j \geq 0, k \geq 1} \frac{(kp^j)^{-s}}{k} = \Gamma(s) \frac{\zeta(s+1)}{1 - p^{-s}} \\
F(p, G \wr A, s) & = \tilde{r}(p, G) \Gamma(s) \frac{\zeta(s+1)}{1 - p^{-s}}
\end{align*}
\]

\( \square \)

6 Conclusion

We have applied the theorem of Lück to compute \( K^0(BG) \) for cases of finite groups that are relevant to Platonic solids, Weyl groups and Hilbert schemes. When the groups come in a natural countable family, we may form generating functions akin to \( \eta(q) \) and partitions.

These were example computations that leave many questions raised. These are driven by understanding the relations between the algebraic and topological K-theories of \( \mathbb{C}^2/G \). This goes into what is lost and what is kept by the comparison map [12]. There is also the distinction between genuine equivariant K theory and the Borel equivariant K theory that we have considered here. In addition, conjugacy classes in Weyl groups are related to nilpotent coadjoint orbits/W-algebras via [13, 14] so we can translate the prime power conditions there too.
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