Enumeration of curves with one singular point

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Abstract

In this paper we obtain an explicit formula for the number of curves in $\mathbb{P}^2$, of degree $d$, passing through $(d(d+3)/2-k)$ generic points and having a codimension $k$ singularity, where $k$ is at most 7. In the past, many of these numbers were computed using techniques from algebraic geometry. In this paper we use purely topological methods to count curves. Our main tool is a classical fact from differential topology: the number of zeros of a generic smooth section of a vector bundle $V$ over $M$, counted with a sign, is the Euler class of $V$ evaluated on the fundamental class of $M$.

Contents

1 Introduction 1
2 Overview 3
3 Algorithm 6
4 Summary of definitions and notation 7
5 Local structure of holomorphic sections 11
6 Transversality 18
7 Closure 35
8 Euler class 50
A Low degree checks 53
B Some details 54

1 Introduction

Enumerative geometry is a branch of mathematics concerned with the following question:

How many geometric objects are there which satisfy prescribed constraints?

A well known class of enumerative problems is that of singular curves in $\mathbb{P}^2$ (complex projective space) passing through the appropriate number of points. This question has been studied by algebraic geometers for a long time. However, in this paper we use purely topological methods to tackle this problem.
Let us denote the space of degree $d$-curves in $\mathbb{P}^2$ by $\mathcal{D}$. It follows that $\mathcal{D} \cong \mathbb{P}^{\delta_d}$, where $\delta_d = d(d+3)/2$. Let $\gamma_{p_2} \to \mathbb{P}^2$ be the tautological line bundle. A homogeneous degree $d$-polynomial $f$ (in 3 variables) induces a holomorphic section of the line bundle $\gamma_{p_2}^{*d} \to \mathbb{P}^2$. If $f$ is non-zero, then we will denote its equivalence class in $\mathcal{D}$ by $\tilde{f}$. Similarly, if $p$ is a non-zero vector in $\mathbb{C}^3$, we will denote its equivalence class in $\mathbb{P}^2$ by $\tilde{p}$.

**Definition 1.1.** Let $\tilde{f} \in \mathcal{D}$ and $\tilde{p} \in \mathbb{P}^2$. A point $\tilde{p} \in f^{-1}(0)$ is of singularity type $\mathcal{A}_k$, $\mathcal{D}_k$, $\mathcal{E}_6$, $\mathcal{E}_7$, $\mathcal{E}_8$ or $\mathcal{A}_8$ if there exists a coordinate system $(x,y): (\mathcal{U},\tilde{p}) \to (\mathbb{C}^2,0)$ such that $f^{-1}(0) \cap \mathcal{U}$ is given by

- $\mathcal{A}_k : y^2 + x^{k+1} = 0 \quad k \geq 0,
- \mathcal{D}_k : y^2 x + x^{k-1} = 0 \quad k \geq 4,
- \mathcal{E}_6 : y^3 + x^4 = 0,
- \mathcal{E}_7 : y^3 + yx^3 = 0,
- \mathcal{E}_8 : y^3 + x^5 = 0,
- \mathcal{A}_8 : x^4 + y^4 = 0.$

In more common terminology, $\tilde{p}$ is a smooth point of $f^{-1}(0)$ if it is a singularity of type $\mathcal{A}_0$; a simple node if its singularity type is $\mathcal{A}_1$; a cusp if its type is $\mathcal{A}_2$; a tacnode if its type is $\mathcal{A}_3$; a triple point if its type is $\mathcal{D}_1$; and a quadruple point if its type is $\mathcal{A}_8$.

We have several results (cf. Theorem 3.3-3.15, section 3) which can be summarized collectively as our main result. Although (3.3)-(3.15) may appear as equalities, the content of each of these equations is a theorem.

**MAIN THEOREM.** Let $\mathcal{X}_k$ be a singularity of type $\mathcal{A}_k$, $\mathcal{D}_k$ or $\mathcal{E}_k$. Denote $\mathcal{N}(\mathcal{X}_k,n)$ to be the number of degree $d$ curves in $\mathbb{P}^2$ that pass through $\delta_d - (k+n)$ generic points and have a singularity of type $\mathcal{X}_k$ at the intersection of $n$ generic lines.

(i) There is a formula for $\mathcal{N}(\mathcal{X}_k,n)$ if $k \leq 7$, provided $d \geq C_{\mathcal{X}_k}$ where

\[ C_{\mathcal{A}_k} = k + 1, \quad C_{\mathcal{D}_k} = k - 1, \quad C_{\mathcal{E}_6} = 4, \quad C_{\mathcal{E}_7} = 4. \]

(ii) There is an algorithm to explicitly compute these numbers.

The numbers $\mathcal{N}(\mathcal{X}_k,0)$ till $k \leq 7$ have also been computed by Maxim Kazarian [3] and Dimitry Kerner [4] using different methods. Our results for $n = 0$ agree with theirs. The bound $d \geq C_{\mathcal{X}_k}$ is imposed to ensure that the relevant bundle sections are transverse.\(^2\) The formulas for $\mathcal{N}(\mathcal{A}_1,n)$, $\mathcal{N}(\mathcal{A}_2,n)$ and $\mathcal{N}(\mathcal{A}_3,n)$ also appear in [7]. We extend the methods applied by the author to obtain the remaining formulas. This method carries over to the case of enumerating curves on a complex surface. With some further effort, the method can also be used to enumerate curves with more than one singular point. This will be pursued elsewhere.

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1. In this paper we will use the symbol $\tilde{A}$ to denote the equivalence class of $A$ instead of the standard $[A]$. This will make some of the calculations in section 7 easier to read.
2. However, this bound is not the optimal bound.
2 Overview

Our main tool will be the following well known fact from topology (cf. [2], Proposition 12.8).

**Theorem 2.1.** Let $V \to X$ be a vector bundle over a manifold $X$. Then the following are true:
(1) A generic smooth section $s : X \to V$ is transverse to the zero set.
(2) Furthermore, if $V$ and $X$ are oriented with $X$ compact then the zero set of such a section defines an integer homology class in $X$, whose Poincaré dual is the Euler class of $V$. In particular, if the rank of $V$ is same as the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is the Euler class of $V$, evaluated on the fundamental class of $X$, i.e.,
$$\pm |s^{-1}(0)| = \langle e(V), [X] \rangle.$$

**Remark 2.2.** Let $X$ be a compact, complex manifold, $V$ a holomorphic vector bundle and $s$ a holomorphic section that is transverse to the zero set. If the rank of $V$ is same as the dimension of $X$, then the signed cardinality of $s^{-1}(0)$ is same as its actual cardinality (provided $X$ and $V$ have their natural orientations).

However, for our purposes, the requirement that $X$ is a smooth manifold is too strong. We will typically be dealing with spaces that are smooth but have non-smooth closure. The following result is a stronger version of Theorem 2.1, that applies to singular spaces, provided the set of singular points is of real codimension two or more.

**Theorem 2.3.** Let $M \subset \mathbb{P}^N$ be a smooth, compact algebraic variety and $X \subset M$ a smooth subvariety, not necessarily closed. Let $V \to M$ be an oriented vector bundle, such that the rank of $V$ is same as the dimension of $X$. Then the following are true:
(1) The closure of $X$ is an algebraic variety and defines a homology class.
(2) The zero set of a generic smooth section $s : M \to V$ intersects $X$ transversely and does not intersect $\overline{X} - X$ anywhere.
(3) The number of zeros of such a section inside $X$, counted with signs, is the Euler class of $V$ evaluated on the homology class $[\overline{X}]$, i.e.,
$$\pm |s^{-1}(0) \cap \overline{X}| = \pm |s^{-1}(0) \cap X| = \langle e(V), [\overline{X}] \rangle.$$

**Remark 2.4.** All the subsequent statements we make are true provided $d$ is sufficiently large. The precise bound on $d$ is given in section 6.

We will now explain our strategy to compute $\mathcal{N}(X_k, n)$. Given a singularity $X_k$, let us also denote by $X_k$, the space of degree $d$-curves with a marked point $\tilde{p}$ such that the curve has a singularity of type $X_k$ at $\tilde{p}$, i.e.,
$$X_k := \{ (\tilde{f}, \tilde{p}) \in D \times \mathbb{P}^2 : \tilde{f} \text{ has a singularity of type } X_k \text{ at the point } \tilde{p} \}.$$

Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+n)}$ be $\delta_d-(k+n)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Define the following sets
$$H_i := \{ \tilde{f} \in D : f(p_i) = 0 \}, \quad H_i^* := \{ \tilde{f} \in D : f(p_i) = 0, \nabla f|_{p_i} \neq 0 \}$$
$$\hat{H}_i := H_i \times \mathbb{P}^2, \quad \hat{H}_i^* := H_i^* \times \mathbb{P}^2 \quad \text{and} \quad \hat{L}_i := D \times L_i. \quad (2.1)$$

By definition, our desired number $\mathcal{N}(X_k, n)$ is the cardinality of the set
$$\mathcal{N}(X_k, n) := |X_k \cap \hat{H}_1 \cap \ldots \cap \hat{H}_{\delta_d-(n+k)} \cap \hat{L}_1 \cap \ldots \cap \hat{L}_n|. \quad (2.2)$$
Step 1. If the degree $d$ is sufficiently large then the space $X_k$ is a smooth algebraic variety and its closure defines a homology class.

Lemma 2.5. (cf. section 6) The space $X_k$ is a smooth subvariety of $D \times \mathbb{P}^2$ of dimension $\delta_d - k$.

Step 2. If the points and lines are chosen generically, then the corresponding hyperplanes and lines defined in (2.1) will intersect our space $X_k$ transversely. Moreover, they won’t intersect any extra points in the closure.

Lemma 2.6. Let $\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_{\delta_d-(k+n)}$ be $\delta_d - (k+n)$ generic points in $\mathbb{P}^2$ and $L_1, L_2, \ldots, L_n$ be $n$ generic lines in $\mathbb{P}^2$. Let $\hat{H}_i, \hat{H}_i^*$ and $\hat{l}_i$ be as defined in (2.1). Then

$$X_k \cap \hat{H}_1 \cap \ldots \cap \hat{H}_{\delta_d-(k+n)} \cap \hat{l}_1 \cap \ldots \cap \hat{l}_n = X_k \cap \hat{H}_1^* \cap \ldots \cap \hat{H}_{\delta_d-(k+n)}^* \cap \hat{l}_1 \cap \ldots \cap \hat{l}_n$$

and every intersection is transverse.

Although we omit the details of the proof, this follows from an application of the families transversality theorem and Bertini’s theorem. The details of this proof can be found in [1].

Notation 2.7. Let $\gamma_D \to D$ and $\gamma_{\mathbb{P}^2} \to \mathbb{P}^2$ denote the tautological line bundles. If $c_1(V)$ denotes the first Chern class of a vector bundle then we set

$$y := c_1(\gamma_D^*) \in H^2(D; \mathbb{Z}), \quad a := c_1(\gamma_{\mathbb{P}^2}^*) \in H^2(\mathbb{P}^2; \mathbb{Z}).$$

As a consequence of Lemma 2.6 we obtain the following fact:

Lemma 2.8. The number $N(X_k, n)$ is given by $N(X_k, n) = \langle y^{\delta_d-(n+k)}a^n, [X_k] \rangle$.

Proof: This follows from Theorem 2.3 and Lemma 2.6. \hfill \Box

Remark 2.9. Here we are making an abuse of notation by referring to $y, a \in H^*(D \times \mathbb{P}^2; \mathbb{Z})$. The intended meaning is $\pi_D^*y$ and $\pi_{\mathbb{P}^2}^*a$, where $\pi_D, \pi_{\mathbb{P}^2} : D \times \mathbb{P}^2 \to D, \mathbb{P}^2$ are the projection maps. We will make a similar abuse of notation with vector bundles. Our intended meaning should be clear when we say, for instance, $\gamma_D^* \to D \times \mathbb{P}^2$.

The space $X_k$, unfortunately, is not easy to describe directly. Consequently, computing $N(X_k, n)$ directly is not a promising approach. Instead we will look at the space

$$\mathcal{P}X_k \subset D \times \mathbb{P}T\mathbb{P}^2.$$

This is the space of degree $d$-curves $\tilde{f}$, with a marked point $\tilde{p} \in \mathbb{P}^2$ and a marked direction $l_{\tilde{p}} \in \mathbb{P}T_{\tilde{p}}\mathbb{P}^2$, such that the curve $f$ has a singularity of type $X_k$ at $\tilde{p}$ and certain directional derivatives vanish along $l_{\tilde{p}}$, and certain other derivatives don’t vanish. To take a simple example, $\mathcal{P}A_2$ is the space of curves $f$ with a marked point $\tilde{p}$ and a marked direction $l_{\tilde{p}}$ such that $f$ has an $A_2$-node at $\tilde{p}$ and the Hessian is degenerate along $l_{\tilde{p}}$, but the third derivative along $l_{\tilde{p}}$ is non-zero. It turns out that this space is much easier to describe. The precise definition of the space $\mathcal{P}X_k$ is given in subsection 4.3.

Step 3. Since the space $\mathcal{P}X_k$ is described locally as the vanishing of certain sections that are transverse to the zero set they are smooth algebraic varieties.

Lemma 2.10. (cf. section 6) The space $\mathcal{P}X_k$ is a smooth subvariety of $D \times \mathbb{P}T\mathbb{P}^2$ of dimension $\delta_d - k$.  


Notation 2.11. Let $\tilde{\gamma} \to \mathbb{P}T\mathbb{P}^2$ be the tautological line bundle. The first Chern class of the dual will be denoted by $\lambda = c_1(\tilde{\gamma}^* ) \in H^2(\mathbb{P}T\mathbb{P}^2; \mathbb{Z})$.

Lemma 2.10 now motivates the following definition:

**Definition 2.12.** We define the number $N(\mathcal{P}\mathcal{X}_k, n, m)$ as

$$N(\mathcal{P}\mathcal{X}_k, n, m) := \langle y^{a - (k+n+m)} a^n \lambda^m, [\overline{\mathcal{P}\mathcal{X}_k}] \rangle. \quad (2.3)$$

The next Lemma relates the numbers $N(\mathcal{P}\mathcal{X}_k, n, m)$ and $N(\mathcal{X}_k, n)$.

**Lemma 2.13.** (cf. section B.1) The projection map $\pi : \mathcal{P}\mathcal{X}_k \to \mathcal{X}_k$ is one to one if $\mathcal{X}_k = \mathcal{A}_k, D_6, E_8$ or $E_8$ except for $\mathcal{X}_k = D_4$ when it is three to one. In particular,

$$N(\mathcal{X}_k, n) = N(\mathcal{P}\mathcal{X}_k, n, 0) \quad \text{if} \quad \mathcal{X}_k \not\subseteq D_4 \quad \text{and} \quad N(D_4, n) = \frac{N(\mathcal{P}D_4, n, 0)}{3}. \quad (2.4)$$

To summarize, the definition of $N(\mathcal{X}_k, n)$ is (2.2). Lemma 2.8 equates this number to a topological computation. We then introduce another number $N(\mathcal{P}\mathcal{X}_k, n, m)$ in definition 2.12 and relate it to $N(\mathcal{X}_k, n)$ in Lemma 2.13. In other words, we do not compute $N(\mathcal{X}_k, n)$ directly. We compute it indirectly by first computing $N(\mathcal{P}\mathcal{X}_k, n, m)$ and then using Lemma 2.13.

We give a brief idea of how to compute these numbers. Suppose we want to compute $N(\mathcal{P}\mathcal{X}_k, n, m)$. We first find some singularity $\mathcal{X}_l$ for which $N(\mathcal{P}\mathcal{X}_l, n, m)$ has been calculated and which contains $\mathcal{X}_k$ in its closure, i.e., we want $\mathcal{P}\mathcal{X}_k$ to be a subset of $\overline{\mathcal{P}\mathcal{X}_l}$. Usually, $l = k - 1$ but it is not necessary. Our next task is to describe the closure of $\mathcal{P}\mathcal{X}_l$ explicitly as

$$\overline{\mathcal{P}\mathcal{X}_l} = \mathcal{P}\mathcal{X}_l \cup \overline{\mathcal{P}\mathcal{X}_k} \cup \mathcal{B}. \quad (2.5)$$

Equivalently, we want an explicit description of the space $\mathcal{B}$. By definition 2.12 and Theorem 2.3

$$N(\mathcal{P}\mathcal{X}_k, n, m) := \langle e(\mathcal{W}_{n,m,k}), [\overline{\mathcal{P}\mathcal{X}_k}] \rangle = \pm |Q^{-1}(0) \cap \mathcal{P}\mathcal{X}_k|,$$

where

$$Q : D \times \mathbb{P}T\mathbb{P}^2 \to \mathcal{W}_{n,m,k} := \left( \bigoplus_{i=1}^{d-n+m+k} \gamma_D^* \right) \oplus \left( \bigoplus_{i=1}^{n} \gamma_D^* \right) \oplus \left( \bigoplus_{i=1}^{m} \tilde{\gamma}^* \right) \quad (2.6)$$

is a generic smooth section. We now have to construct a section $\Psi_{\mathcal{P}\mathcal{X}_k}$ of some vector bundle

$$\mathcal{V}_{\mathcal{P}\mathcal{X}_k} \to \overline{\mathcal{P}\mathcal{X}_l} = \mathcal{P}\mathcal{X}_l \cup \overline{\mathcal{P}\mathcal{X}_k} \cup \mathcal{B}$$

with the following properties: it should not vanish on $\mathcal{P}\mathcal{X}_l$ and it should vanish transversely on $\mathcal{P}\mathcal{X}_k$. In that case we are led to

$$\langle e(\mathcal{V}_{\mathcal{P}\mathcal{X}_k} \oplus \mathcal{W}_{n,m,k}), [\overline{\mathcal{P}\mathcal{X}_l}] \rangle = N(\mathcal{P}\mathcal{X}_k, n, m) + C(\mathcal{V}_{\mathcal{P}\mathcal{X}_k} \oplus Q),$$

where $C(\mathcal{V}_{\mathcal{X}_k} \oplus Q)$ is the contribution of the section $\mathcal{V}_{\mathcal{P}\mathcal{X}_k} \oplus Q$ to the Euler class from the points of $\mathcal{B}$. The left hand side is computable via splitting principle and the fact that $N(\mathcal{P}\mathcal{X}_l, n, m)$ is known. Therefore, once we know $C(\mathcal{V}_{\mathcal{X}_k} \oplus Q)$, we get a recursive formula for the number $N(\mathcal{P}\mathcal{X}_k, n, m)$ and iterate.

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The Euler class of this vector bundle is expressible in terms of the Euler classes of three canonical line bundles via the splitting principal.
Example 2.14. Suppose we wish to compute \( N(\mathcal{A}_5, n) \). This can be deduced from the knowledge of \( N(\mathcal{P}A_5, n,m) \). The obvious singularities which have \( \mathcal{A}_5 \)-nodes in its closure are \( \mathcal{A}_4 \)-nodes. In order to analyze the space \( \mathcal{P}A_4 \), we infer that (cf. Lemma 7.1, statement 10)

\[
\mathcal{P}A_4 = \mathcal{P}A_4 \cup \mathcal{P}A_5 \cup \mathcal{P}D_5.
\]

The corresponding line bundle \( \mathbb{L}_{\mathcal{P}A_5} \rightarrow \mathcal{P}A_4 \) with a section \( \Psi_{\mathcal{P}A_5} \) that does not vanish on \( \mathcal{P}A_4 \) and vanishes transversely on \( \mathcal{P}A_5 \) is defined in subsection 4.1. The verification of these properties of the section is proved in section 6 (Proposition 6.24). Finally, in Corollary 7.4 we show that if \( Q \) is a generic section of the vector bundle \( W_{n,m,5} \rightarrow D \times \mathbb{P}^2 \) then \( \Psi_{\mathcal{P}A_5} \oplus Q \) vanishes on all the points of \( \mathcal{P}D_5 \) with a multiplicity of 2. Hence, we conclude that

\[
\langle e(\mathbb{L}_{\mathcal{P}A_5} \oplus W_{n,m,5}), [\mathcal{P}A_4] \rangle = N(\mathcal{P}A_5, n,m) + 2N(\mathcal{P}D_5, n,m).
\] (2.7)

This gives us a recursive formula for \( N(\mathcal{P}A_5, n,m) \) in terms of \( N(\mathcal{P}A_4, n', m') \) and \( N(\mathcal{P}D_5, n,m) \) which is (3.7) in our algorithm.

Now we describe the basic organization of our paper. In section 3 we state the explicit algorithm to obtain the numbers \( N(\mathcal{X}_k, n) \) in our MAIN THEOREM in section 1. In section 4 we summarize all the spaces, vector bundles and sections of vector bundles we will encounter in the course of our computations. In section 5 we describe necessary and sufficient conditions for a curve \( f_1(0) \) to have a singularity of type \( \mathcal{X}_k \) at a point. In section 6 we describe the spaces \( \mathcal{P}\mathcal{X}_k \) as the vanishing of certain sections and the non-vanishing of certain other sections. Moreover, we show that these sections are transverse to the zero set. In section 7 we stratify the space \( \mathcal{P}\mathcal{X}_k \) as described in (2.5). Along the way we also compute the order to which a certain section vanishes around certain points (i.e., the contribution of the section to the Euler class of a bundle). Finally, using the splitting principal, in section 8 we compute the Euler class of the relevant bundles and obtain the recursive formula similar to (2.7) above.

3 Algorithm

We now give an algorithm to compute the numbers \( N(\mathcal{X}_k, n) \). We have implemented this algorithm in a Mathematica program to obtain the final answers. The program is available on our web page https://www.sites.google.com/site/ritwik371/home. We prove these formulas in section 8.

The base case for the recursion is:

\[
N(\mathcal{A}_1, n) = \begin{cases} 
3(d-1)^2, & \text{if } n = 0; \\
3(d-1), & \text{if } n = 1; \\
1, & \text{if } n = 2; \\
0, & \text{otherwise.}
\end{cases}
\] (3.1)

Next we will give an algorithm to compute \( N(\mathcal{P}\mathcal{X}_k, n,m) \). Using Lemma 2.13 we get our desired numbers \( N(\mathcal{X}_k, n) \). We note that using the ring structure of \( H^*(D \times \mathbb{P}^2; \mathbb{Z}) \), it is easy to see that for every singularity type \( \mathcal{X}_k \) we have

\[
N(\mathcal{P}\mathcal{X}_k, n,m) = -3N(\mathcal{P}\mathcal{X}_k, n+1,m-1) - 3N(\mathcal{P}\mathcal{X}_k, n+2,m-2) \quad \forall \ m \geq 2.
\] (3.2)
This follows from Lemma B.1 and B.2. Finally, we give recursive formulas for \( N(\mathcal{P}X_k, n, m) \):

\[
N(\mathcal{P}A_2, n, 0) = 2N(A_1, n) + 2(d - 3)N(A_1, n + 1) \\
N(\mathcal{P}A_2, n, 1) = N(A_1, n) + (2d - 9)N(A_1, n + 1) + (d^2 - 9d + 18)N(A_1, n + 2) \\
N(\mathcal{P}A_3, n, m) = N(\mathcal{P}A_2, n, m) + 3N(\mathcal{P}A_2, n, m + 1) + dN(\mathcal{P}A_2, n + 1, m) \\
N(\mathcal{P}A_4, n, m) = 2N(\mathcal{P}A_3, n, m) + 2N(\mathcal{P}A_3, n, m + 1) + (2d - 6)N(\mathcal{P}A_3, n + 1, m) \\
N(\mathcal{P}A_5, n, m) = 3N(\mathcal{P}A_4, n, m) + N(\mathcal{P}A_4, n, m + 1) + (3d - 12)N(\mathcal{P}A_4, n + 1, m) \\
\quad - 2N(\mathcal{P}D_5, n, m) \\
N(\mathcal{P}A_6, n, m) = 4N(\mathcal{P}A_5, n, m) + 0N(\mathcal{P}A_5, n, m + 1) + (4d - 18)N(\mathcal{P}A_5, n + 1, m) \\
\quad - 4N(\mathcal{P}D_6, n, m) - 3N(\mathcal{P}E_6, n, m) \\
N(\mathcal{P}A_7, n, 0) = 5N(\mathcal{P}A_6, n, 0) - N(\mathcal{P}A_6, n, 1) + (5d - 24)N(\mathcal{P}A_6, n + 1, 0) \\
\quad - 6N(\mathcal{P}D_7, n, 0) - 7N(\mathcal{P}E_7, n, 0) \\
N(\mathcal{P}D_4, n, m) = N(\mathcal{P}A_3, n, m) - 2N(\mathcal{P}A_3, n, m + 1) + (d - 6)N(\mathcal{P}A_3, n + 1, m) \\
N(\mathcal{P}D_5, n, m) = N(\mathcal{P}D_4, n, m) + N(\mathcal{P}D_4, n, m + 1) + (d - 3)N(\mathcal{P}D_4, n + 1, m) \\
N(\mathcal{P}D_6, n, m) = N(\mathcal{P}D_5, n, m) + 4N(\mathcal{P}D_5, n, m + 1) + dN(\mathcal{P}D_5, n + 1, m) \\
N(\mathcal{P}D_7, n, m) = 2N(\mathcal{P}D_6, n, m) + 4N(\mathcal{P}D_6, n, m + 1) + (2d - 6)N(\mathcal{P}D_6, n + 1, m) \\
N(\mathcal{P}E_6, n, m) = N(\mathcal{P}D_5, n, m) - N(\mathcal{P}D_5, n, m + 1) + (d - 6)N(\mathcal{P}D_5, n + 1, m) \\
N(\mathcal{P}E_7, n, m) = N(\mathcal{P}D_6, n, m) - N(\mathcal{P}D_6, n, m + 1) + (d - 6)N(\mathcal{P}D_6, n + 1, m)
\]

4 Summary of definitions and notation

4.1 The vector bundles involved

We now list down all the vector bundles that we will encounter. The first three of these have been defined in notations 2.7 and 2.11, (the tautological line bundles). Let \( \pi : D \times \mathbb{P}^2 \to D \times \mathbb{P}^2 \) be the projection map. We have the following bundles over \( D \times \mathbb{P}^2 \):

\[
\begin{align*}
\mathcal{L}_{A_0} & := \gamma_D \otimes \gamma_{y_2} \to D \times \mathbb{P}^2 \\
\mathcal{V}_{A_1} & := \gamma_D \otimes \gamma_{y_2} \otimes T^*\mathbb{P}^2 \to D \times \mathbb{P}^2 \\
\mathcal{L}_{A_2} & := (\gamma_D^* \otimes \gamma_{y_2}^* \otimes \Lambda^2 T^*\mathbb{P}^2)^2 \to D \times \mathbb{P}^2 \\
\mathcal{V}_{D_4} & := \gamma_D^* \otimes \gamma_{y_2}^* \otimes \text{Sym}^2(T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2) \to D \times \mathbb{P}^2 \\
\mathcal{V}_{E_8} & := \gamma_D^* \otimes \gamma_{y_2}^* \otimes \text{Sym}^3(T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2) \to D \times \mathbb{P}^2 
\end{align*}
\]

Associated to the map \( \pi \) there are pullback bundles

\[
\begin{align*}
\mathcal{L}_{\mathcal{L}_{A_0}} & := \pi^* \mathcal{L}_{A_0} \to D \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\mathcal{V}_{\mathcal{A}_1} & := \pi^* \mathcal{V}_{A_1} \to D \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\mathcal{V}_{\mathcal{D}_4} & := \pi^* \mathcal{V}_{D_4} \to D \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\mathcal{V}_{\mathcal{E}_8} & := \pi^* \mathcal{V}_{E_8} \to D \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\mathcal{V}_{\mathcal{P}A_2} & := \tilde{\gamma}^* \otimes \gamma_D \otimes \gamma_{y_2}^* \otimes \pi^* T^*\mathbb{P}^2 \to D \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\mathcal{V}_{\mathcal{P}D_5} & := \tilde{\gamma}^* \otimes \gamma_D^* \otimes \gamma_{y_2} \otimes \pi^* T^*\mathbb{P}^2 \to D \times \mathbb{P}^2 \times \mathbb{P}^2.
\end{align*}
\]
Finally, we have

\[
\begin{align*}
L_{\mathcal{P}_1} &:= (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \\
L_{\mathcal{P}_2} &:= \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \\
L_{\mathcal{P}_3} &:= \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \\
L_{\mathcal{P}_4} &:= \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \\
L_{\mathcal{P}_5} &:= \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \\
L_{\mathcal{P}_6} &:= \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2
\end{align*}
\]

\[k \geq 3 \quad L_{\mathcal{P}_6} := \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2 \]

\[k \geq 6 \quad L_{\mathcal{P}_6} := \gamma^s \otimes (TP^2/\gamma)^* s \otimes \gamma_p \otimes \gamma_s d \longrightarrow D \times PT^2.
\]

With the abuse of notation as explained in remark 2.9, the bundle \(TP^2/\gamma\) is the quotient of the bundles \(V\) and \(W\), where \(V\) is the pullback of the tangent bundle \(TP^2 \to \mathbb{P}^2\) via \(D \times PT^2 \longrightarrow D \times \mathbb{P}^2 \to \mathbb{P}^2\) and \(W\) is pullback of \(\tilde{\gamma} \to \mathbb{P}T^2\) via \(D \times PT^2 \to \mathbb{P}T^2\).

### 4.2 Sections of Vector Bundles

Let us define the notion of vertical derivatives.

**Definition 4.1.** Let \(\pi : V \to M\) be a holomorphic vector bundle of rank \(k\) and \(s : M \to V\) be a holomorphic section. Suppose \(h : V|_U \to U \times \mathbb{C}^k\) is a holomorphic trivialization of \(V\) and \(\pi_1, \pi_2 : U \times \mathbb{C}^k \to U, \mathbb{C}^k\) the projection maps. Let

\[
\hat{s} := \pi_2 \circ h \circ s.
\]

For \(q \in U\), we define the vertical derivative of \(s\) to be the \(\mathbb{C}\)-linear map

\[
\nabla s|_q : T_q M \to V_q, \quad \nabla s|_q := (\pi_2 \circ h)|_{V_q}^{-1} \circ ds|_q,
\]

where \(V_q = \pi^{-1}(q)\), the fibre at \(q\). In particular, if \(v \in T_q M\) is given by a holomorphic map \(\gamma : B_\epsilon(0) \to M\) such that \(\gamma(0) = q\) and \(\frac{\partial \gamma}{\partial z}|_{z=0} = v\), then

\[
\nabla s|_q(v) := (\pi_2 \circ h)|_{V_q}^{-1} \circ \frac{\partial \hat{s}(\gamma(z))}{\partial z}|_{z=0}
\]

were \(B_\epsilon\) is an open \(\epsilon\)-ball in \(\mathbb{C}\) around the origin.\(^4\)

Finally, if \(v, w \in T_q M\) are tangent vectors such that there exists a family of complex curves \(\gamma : B_\epsilon \times B_\epsilon \to M\) such that

\[
\gamma(0, 0) = q, \quad \frac{\partial \gamma(x, y)}{\partial x}|_{(0, 0)} = v, \quad \frac{\partial \gamma(x, y)}{\partial y}|_{(0, 0)} = w
\]

then

\[
\nabla^{i+j} s|_q(v, \ldots, v, w, \ldots w) := (\pi_2 \circ h)|_{V_q}^{-1} \circ \left[ \frac{\partial^{i+j} \hat{s}(\gamma(x, y))}{\partial x^i \partial y^j} \right]|_{(0, 0)}.
\]

\(^4\)Not every tangent vector is given by a holomorphic map; however combined with the fact that \(\nabla s|_p\) is \(\mathbb{C}\)-linear, this definition determines \(\nabla s|_p\) completely.
Remark 4.2. In general the quantity in (4.2) is not well defined, i.e., it depends on the trivialization and the curve $\gamma$. Lemma 5.18 explains on what subspace this quantity is well defined.

Remark 4.3. The section $s : M \rightarrow V$ is transverse to the zero set if and only if the induced map

$$\tilde{s} := \tilde{s} \circ \varphi_d^{-1} : \mathbb{C}^n \rightarrow \mathbb{C}^k$$

is transverse to the zero set in the usual calculus sense, where $\varphi_U : U \rightarrow \mathbb{C}^n$ is a coordinate chart and $\tilde{s}$ is as defined in (4.1).

Let $f : \mathbb{P}^2 \rightarrow \gamma_{\mathbb{P}^2}^d$ be a section and $\tilde{p} \in \mathbb{P}^2$. We can think of $p$ as a non-zero vector in $\gamma_{\mathbb{P}^2}^d$ and $p^{\otimes d}$ a non-zero vector in $\gamma_{\mathbb{P}^2}^{d^2}$ 5. The quantity $\nabla f|_{\tilde{p}}$ acts on a vector in $\gamma_{\mathbb{P}^2}^d|_{\tilde{p}}$ and produces an element of $T_{\tilde{p}}^*\mathbb{P}^2$. Let us denote this quantity as $\nabla f|_p$, i.e.,

$$\nabla f|_p := (\nabla f|_{\tilde{p}})(p^{\otimes d}) \in T_{\tilde{p}}^*\mathbb{P}^2.$$  

(4.4)

Notice that $\nabla f|_{\tilde{p}}$ is an element of the fibre of $T_{\tilde{p}}^*\mathbb{P}^2 \otimes \gamma_{\mathbb{P}^2}^d$ at $\tilde{p}$ while $\nabla f|_p$ is an element of $T_{\tilde{p}}^*\mathbb{P}^2$.

Now observe that $\pi^*\mathbb{P}^2 \approx \gamma \oplus \pi^*\mathbb{P}^2/\tilde{\gamma} \rightarrow \mathbb{P}^2$, where $\pi : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is the projection map. Let us denote a vector in $\tilde{\gamma}$ by $v$ and a vector in $\pi^*\mathbb{P}^2/\tilde{\gamma}$ by $\tilde{w}$. Given $\tilde{f} \in \mathcal{D}$ and $\tilde{p} \in \mathbb{P}^2$, let

$$f_{ij} := \nabla^{ij} f|_p(v, \ldots, v, w, \ldots w).$$

(4.5)

Note that $f_{ij}$ is a number. Finally, since our sections are not defined on the whole space, we will use the notation $s : M \rightarrow V$ to indicate that $s$ is defined only on a subspace of $M$. With this terminology, we now explicitly define the sections that we will encounter in this paper.

$$\psi_{A_0} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{L}_{A_0}, \quad \{\psi_{A_0}(\tilde{f}, \tilde{p}))(f \otimes p^{\otimes d}) := f(p)$$

$$\psi_{A_1} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{V}_{A_1}, \quad \{\psi_{A_1}(\tilde{f}, \tilde{p}))(f \otimes p^{\otimes d}) := \nabla f|_p$$

$$\psi_{D_4} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{V}_{D_4}, \quad \{\psi_{D_4}(\tilde{f}, \tilde{p}))(f \otimes p^{\otimes d}) := \nabla^2 f|_p$$

$$\psi_{X_8} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{V}_{X_8}, \quad \{\psi_{X_8}(\tilde{f}, \tilde{p}))(f \otimes p^{\otimes d}) := \nabla^3 f|_p$$

$$\Psi_{A_0} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{L}_{A_0}, \quad \Psi_{A_0}(\tilde{f}, l_{\tilde{p}}) := \psi_{A_0}(\tilde{f}, \tilde{p}),$$

$$\Psi_{A_1} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{V}_{A_1}, \quad \Psi_{A_1}(\tilde{f}, l_{\tilde{p}}) := \psi_{A_1}(\tilde{f}, \tilde{p})$$

$$\Psi_{D_4} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{V}_{D_4}, \quad \Psi_{D_4}(\tilde{f}, l_{\tilde{p}}) := \psi_{D_4}(\tilde{f}, \tilde{p})$$

$$\Psi_{X_8} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{V}_{X_8}, \quad \Psi_{X_8}(\tilde{f}, l_{\tilde{p}}) := \psi_{X_8}(\tilde{f}, \tilde{p}).$$

We also have

$$\Psi_{P_{A_2}} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{V}_{P_{A_2}}, \quad \{\Psi_{P_{A_2}}(\tilde{f}, l_{\tilde{p}}))(f \otimes p^{\otimes d} \otimes v) := \nabla^2 f|_p(v, \cdot)$$

$$\Psi_{P_{D_5}} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{D_5}}, \quad \{\Psi_{P_{D_5}}(\tilde{f}, l_{\tilde{p}}))(f \otimes p^{\otimes d} \otimes v^{\otimes 2}) := \nabla^3 f|_p(v, \cdot)$$

$$\Psi_{P_{D_4}} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{D_4}}, \quad \{\Psi_{P_{D_4}}(\tilde{f}, l_{\tilde{p}}))(f \otimes p^{\otimes d} \otimes w^{\otimes 2}) := f_{02}$$

$$\Psi_{P_{X_8}} : \mathcal{D} \times \pi^*\mathbb{P}^2 \rightarrow \mathcal{L}_{P_{X_8}}, \quad \{\Psi_{P_{X_8}}(\tilde{f}, l_{\tilde{p}}))(f \otimes p^{\otimes d} \otimes w^{\otimes 3} \otimes w) := f_{03}$$

5Remember that $p$ is an element of $\mathbb{C}^3 - 0$ while $\tilde{p}$ is the corresponding equivalence class in $\mathbb{P}^2$.
We also have $\Psi_J : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathbb{L}_J$ given by
\[
\{\Psi_J(f, l_p)\}(f \otimes p^{\otimes d} \otimes v^{\otimes g} \otimes w^{\otimes 3}) := \left( -\frac{f_3^3}{8} + \frac{3f_{22}f_{31}f_{40}}{16} - \frac{f_{13}f_{30}^2}{16} \right).
\] (4.6)

When $k \geq 3$ we have $\Psi_{P\mathcal{A}_k} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathbb{L}_{P\mathcal{A}_k}$ given by
\[
\{\Psi_{P\mathcal{A}_k}(f, l_p)\}(f \otimes p^{\otimes d} \otimes v^{\otimes k} \otimes w^{\otimes(2k-6)}) := f_{02}^{k-3} \mathcal{A}_k^f.
\]

Similarly, when $k = 6, 7$ we have $\Psi_{P\mathcal{D}_k} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathbb{L}_{P\mathcal{D}_k}$ given by
\[
\{\Psi_{P\mathcal{D}_k}(f, l_p)\}(f \otimes p^{\otimes d} \otimes v^{\otimes(2k-8)} \otimes w^{\otimes(2k-12)}) := f_{02}^{k-6} \mathcal{D}_k^f.
\]

The section $\Psi_{P\mathcal{D}_k}$ is given by $f_{12}^3 \mathcal{D}_8^f$. The expressions for $\mathcal{A}_k^f$ (resp. $\mathcal{D}_k^f$) are given below explicitly in (4.7) (resp. (4.8)), till $k = 7$ (resp. till $k = 8$). The algorithm to obtain $\mathcal{A}_k^f$ (resp. $\mathcal{D}_k^f$) for any $k$ is given in Lemma 5.5 (resp. Lemma 5.11) and (5.8) (resp. (5.16)).

Here is an explicit formula for $\mathcal{A}_k^f$ till $k = 7$.
\[
\begin{align*}
\mathcal{A}_3^f &= f_{30}, \\
\mathcal{A}_4^f &= f_{40} - \frac{3f_{21}^3}{f_{02}}, \\
\mathcal{A}_5^f &= f_{50} - \frac{10f_{12}f_{31}f_{41}}{f_{02}} + \frac{15f_{12}f_{31}^2}{f_{02}^2} \\
\mathcal{A}_6^f &= f_{60} - \frac{15f_{21}f_{31}f_{41}}{f_{02}} - \frac{10f_{21}^3}{f_{02}} + \frac{60f_{12}f_{31}f_{41}}{f_{02}} + \frac{45f_{21}^2f_{31}}{f_{02}^2} - \frac{15f_{12}f_{31}^3}{f_{02}^2} - \frac{90f_{12}^2f_{31}^2}{f_{02}^2} \\
\mathcal{A}_7^f &= f_{70} - \frac{21f_{21}f_{31}}{f_{02}} - \frac{35f_{21}^3}{f_{02}} + \frac{105f_{12}f_{31}f_{41}}{f_{02}} + \frac{105f_{21}^2f_{31}}{f_{02}^2} + \frac{70f_{12}f_{31}^3}{f_{02}^2} + \frac{210f_{12}f_{31}^2f_{41}}{f_{02}^2} \\
&- \frac{105f_{03}f_{21}f_{31}^2}{f_{02}^2} - \frac{420f_{12}f_{31}f_{41}}{f_{02}^2} - \frac{630f_{12}^2f_{31}f_{41}}{f_{02}^2} + \frac{105f_{13}f_{31}^2}{f_{02}^2} + \frac{315f_{03}f_{12}f_{31}^2}{f_{02}^2} + \frac{630f_{12}^2f_{31}^2}{f_{02}^2}.
\end{align*}
\] (4.7)

Here is an explicit formula for $\mathcal{D}_k^f$ till $k = 8$.
\[
\begin{align*}
\mathcal{D}_6^f &= f_{40}, \\
\mathcal{D}_7^f &= f_{50} - \frac{5f_{31}^3}{3f_{12}}, \\
\mathcal{D}_8^f &= f_{60} + \frac{5f_{03}f_{31}f_{50}}{3f_{12}} - \frac{5f_{31}f_{41}}{f_{12}} - \frac{10f_{03}f_{31}^3}{3f_{12}^2} + \frac{5f_{22}f_{31}^2}{f_{12}^2}.
\end{align*}
\] (4.8)

4.3 The spaces involved.

We begin by explaining a terminology. If $l_p \in \mathbb{P}T_p\mathbb{P}^2$, then we say that $v \in l_p$ if $v$ is a tangent vector in $T_p\mathbb{P}^2$ and lies over the fibre of $l_p$. We now define the spaces that we will encounter.

\[
\begin{align*}
\mathcal{X}_k &:= \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p} \} \\
\mathcal{X}_k &:= \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{X}_k \text{ at } \tilde{p} \} = \pi^{-1}(\mathcal{X}_k) \\
\text{if } k > 1 & \quad \mathcal{P}\mathcal{A}_k := \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{A}_k \text{ at } \tilde{p}, \nabla^2 f|_p(v, \cdot) = 0 \text{ if } v \in l_p \} \\
\mathcal{P}\mathcal{D}_4 &:= \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{D}_4 \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, v) = 0 \text{ if } v \in l_p \} \\
\text{if } k > 4 & \quad \mathcal{P}\mathcal{D}_k := \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{D}_k \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, \cdot) = 0 \text{ if } v \in l_p \} \\
\text{if } k = 6, 7 \text{ or } 8 & \quad \mathcal{P}\mathcal{E}_k := \{(f, \tilde{p}) \in \mathcal{D} \times \mathbb{P}^2 : f \text{ has a singularity of type } \mathcal{E}_k \text{ at } \tilde{p}, \nabla^3 f|_p(v, v, \cdot) = 0 \text{ if } v \in l_p \}
\end{align*}
\]
We also need the definitions for a few other spaces which will make our computations convenient.

\begin{align*}
\mathcal{D}_1^h & := \{(\tilde{f}, \tilde{l}_p) \in \mathcal{D} \times \mathbb{P}^2 \mathbb{T}^2 : f(p) = 0, \nabla f|_{l_p} = 0, \nabla^2 f|_{l_p}(v, \cdot) \neq 0, \forall \ v \neq 0 \in l_p \} \\
\mathcal{D}_4^4 & := \{(\tilde{f}, \tilde{l}_p) \in \mathcal{D} \times \mathbb{P}^2 \mathbb{T}^2 : f(p) = 0, \nabla f|_{l_p} = 0, \nabla^2 f|_{l_p} \equiv 0, \nabla^3 f|_{l_p}(v, v, v) \neq 0, \forall \ v \neq 0 \in l_p \} \\
\mathcal{X}_8^\# & := \{(\tilde{f}, \tilde{l}_p) \in \mathcal{D} \times \mathbb{P}^2 \mathbb{T}^2 : f(p) = 0, \nabla f|_{l_p} = 0, \nabla^2 f|_{l_p} \equiv 0, \nabla^3 f|_{l_p} = 0, \\
& \quad \nabla^4 f|_{l_p}(v, v, v) \neq 0 \forall \ v \neq 0 \in l_p \} \\
\mathcal{X}_8^{\#}\# & := \{(\tilde{f}, \tilde{l}_p) \in \mathcal{D} \times \mathbb{P}^2 \mathbb{T}^2 : (\tilde{f}, \tilde{l}_p) \in \mathcal{X}_8^\#, \Psi_{\mathcal{I}}(\tilde{f}, \tilde{l}_p) \neq 0, \text{where } \Psi_{\mathcal{I}} \text{ is defined in (4.6)} \}.
\end{align*}

\section{5 Local structure of holomorphic sections}

We give a necessary and sufficient criterion for a curve $f^{-1}(0)$ to have a singularity of type $X_k$ at the point $\tilde{p}$. Let $\rho = \rho(x, y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ and $i, j$ be non-negative integers. We define

$$\rho_{ij} := \frac{\partial^{i+j} \rho}{\partial^i x \partial^j y}|_{(x,y)=(0,0)}.$$ 

\textbf{Lemma 5.1.} Let $\rho = \rho(x, y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $\rho_{00} = 0$ and $\nabla \rho|_{(0,0)} \neq 0$. Then there exists a coordinate chart $(u, v)$ centered at the origin so that $\rho(u, v) = v^2 + u$.

\textbf{Proof:} Follows immediately by considering the Taylor expansion of $\rho$. \hfill \Box

\textbf{Corollary 5.2.} A curve $\rho^{-1}(0)$ has an $A_0$-node at the origin if and only if it satisfies the hypothesis of Lemma 5.1.

\textbf{Lemma 5.3.} Let $\rho = \rho(x, y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00}, \nabla \rho|_{(0,0)} = 0$ and $\nabla^2 \rho|_{(0,0)}$ is non-degenerate. Then there exists a coordinate chart $(u, v)$ centered at the origin so that $\rho(u, v) = v^2 + u^2$.

\textbf{Proof:} This is the Morse Lemma, which again follows by considering the Taylor expansion of $\rho$. \hfill \Box

\textbf{Corollary 5.4.} A curve $\rho^{-1}(0)$ has an $A_1$-node if and only if it satisfies the hypothesis of Lemma 5.3.

\textbf{Lemma 5.5.} Let $\rho = \rho(r, s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho(0,0), \nabla \rho|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 f(w, \cdot) = 0$, i.e., the Hessian is degenerate. Let $x = w_1 r + w_2 s, y = -w_2 r + w_1 s$ and $\rho_{ij}$ be the partial derivatives with respect to the new variables $x$ and $y$. If $\rho_{02} \neq 0$, there exists a coordinate chart $(u, v)$ centered around the origin in $\mathbb{C}^2$ such that

$$\rho = \begin{cases} v^2, & \text{or} \\ v^2 + u^{k+1}, & \text{for some } k \geq 2. \end{cases}$$

(5.1)

\textbf{Remark 5.6.} In terms of the new coordinates we have $\rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = 0$ and $\rho_{02} \neq 0$. Here $\partial_x + 0\partial_y = (1,0)$ is the distinguished direction along which the Hessian is degenerate.
Proof: Let the Taylor expansion of \( \rho \) in the new coordinates be given by

\[
\rho(x, y) = A_0(x) + A_1(x)y + A_2(x)y^2 + \ldots.
\]

By our assumption on \( \rho \), \( A_2(0) \neq 0 \). We claim that there exists a holomorphic function \( B(x) \) such that after we make a change of coordinates \( y = y_1 + B(x) \), the function \( \rho \) is given by

\[
\rho = \hat{A}_0(x) + \hat{A}_2(x)y_1^2 + \hat{A}_3(x)y_1^3 + \ldots
\]

for some \( \hat{A}_k(x) \) (i.e., \( \hat{A}_1(x) \equiv 0 \)). To see this, we note that this is possible if \( B(x) \) satisfies the identity

\[
A_1(x) + 2A_2(x)B + 3A_3(x)B^2 + \ldots \equiv 0.
\]  

(5.2)

Since \( A_2(0) \neq 0 \), \( B(x) \) exists by the Implicit Function Theorem. Therefore, we can compute \( \hat{A}_0(x) \). Hence,

\[
\rho = v^2 + \frac{A_3}{3!}x^3 + \frac{A_4}{4!}x^4 + \ldots, \quad \text{where} \quad v = \sqrt{(\hat{A}_2 + \hat{A}_3y_1 + \ldots)y_1},
\]

(5.3)

satisfies (5.1).

Following the above procedure we find \( A_i^\rho \) for \( i = 3, \ldots, 7 \). In particular,

\[
A_3^\rho = \rho_{30}, \quad A_4^\rho = \rho_{40} - \frac{3\rho_{21}^2}{\rho_{02}}, \quad A_5^\rho = \rho_{50} - \frac{10\rho_{21}\rho_{31}}{\rho_{02}} + \frac{15\rho_{12}\rho_{21}^2}{\rho_{02}^2}.
\]  

(5.4)

Corollary 5.7. Let the hypothesis be as in Lemma 5.5. The curve \( \rho^{-1}(0) \) has an \( A_k \)-node (for \( k \geq 2 \)) at the origin if and only if \( \rho_{02} \neq 0 \) and the directional derivatives \( A_i^\rho \) obtained in (5.3) are zero for all \( i \leq k \) and \( A_{k+1}^\rho \neq 0 \). Furthermore, if \( \tau \) is any holomorphic function that does not vanish at the origin, then

\[
A_{k+1}^\tau = \tau_{00} A_{k+1}^\rho \quad \text{and} \quad (\tau\rho)^{k-3}_{02} A_{k+1}^\tau = \tau_{00}^k \rho_{02}^k A_{k+1}^\rho.
\]

(5.5)

Finally, if \( A_i^\rho = 0 \) for \( i \leq k \) then the quantity \( A_{k+1}^\rho \) is invariant under

\[
x \rightarrow x + T_1(x, y), \quad y \rightarrow y + T_2(x, y)
\]

(5.6)

\[
y \rightarrow y + x, \quad x \rightarrow x
\]

(5.7)

where \( T_1 \) and \( T_2 \) are holomorphic functions that vanish at the origin and whose derivative also vanish at the origin, i.e.,

\[
T_i(0, 0) = 0, \quad \nabla T_i(0, 0) = 0, \quad i = 1, 2.
\]

Proof: The first assertion follows immediately from (5.3). To prove (5.5), note that by (5.3)

\[
A_{k+1} = \frac{\partial^{k+1} \rho(x, y)}{\partial x^{k+1}} \bigg|_{(0, 0)} \Rightarrow A_{k+1}^\tau = \frac{\partial^{k+1} \tau \rho(x, y)}{\partial x^{k+1}} \bigg|_{(0, 0)} = \tau_{00} A_{k+1}^\rho
\]

which follows from the fact that \( A_i^\rho = 0 \) for all \( i \leq k \). The second equation follows similarly by observing that \( (\tau\rho)_{02} = \tau_{00} \rho_{02} \). We have omitted here the proofs of (5.6) and (5.7). The details of the proof can be found in [1]. 

\[
\square
\]
Remark 5.8. The quantity $p_{02}^{-1} A_k^0$ is defined even when $p_{02} = 0$. These quantities induce sections $\Psi_{PA_k}$ of the line bundles $\mathcal{L}_{PA_k} \rightarrow D \times \mathbb{P}T^2$ of subsection 4.1. The induced section is defined to be

$$\{ \Psi_{PA_k}(f, l_p) \} = f \otimes p^{\otimes d} \otimes v^{\otimes k} \otimes w^{\otimes (2k-6)} := f_{02}^{-1} A_k^0,$$

where $A_k^0$ is the number we get by replacing $\rho_{ij}$ by $f_{ij}$ in $A_k^0$ ($f_{ij}$ is defined in (5.4)). Note that (5.5) and (5.6) imply that this section is well defined restricted to $\Psi_{PA_{k-1}}^{-1}(0)$, i.e., it is independent of the trivialization and independent of the curve chosen. This is easily seen by unwinding definition 4.1. The details of this can be found in [1]. Finally, note that (5.7) implies that the section $\Psi_{PA_k}$ is well defined on the quotient space $\pi^*(\mathbb{P}T^2)/\gamma$, since the quantity in (5.8) is invariant under $w \rightarrow w + v$.

Next we analyze singularities when the Hessian is identically zero.

Lemma 5.9. Let $\rho = \rho(x, y)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00}, \nabla \rho|_{(0,0)}, \nabla^2 \rho|_{(0,0)} = 0$ and there does not exist a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 \rho(w, w, \cdot) = 0$. Then, there exists a coordinate chart $(u, v)$ centered at the origin so that $\rho(u, v) = u^3 + v^3$.

Proof: The Taylor expansion of $\rho$ is given by

$$\rho = \frac{\rho_{00}}{6} x^3 + \frac{\rho_{11}}{2} x^2 y + \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \ldots$$

We make an observation from multi linear algebra that if $\nabla^3 \rho$ is non degenerate, then the cubic term in the Taylor expansion has no repeated factors (this analogous to the similar fact that if $\nabla^2 \rho$ is non degenerate, then the quadratic term in the Taylor expansion is not a perfect square). Hence, we can make a linear change of coordinates so that $\rho$ is given by

$$\rho = x_1^3 + y_1^3 + \eta x_1^2 y_1^2 + x_1^3 h_1 + y_1^3 h_2$$

where $h_1$ and $h_2$ are holomorphic functions of $x_1$ and $y_1$ vanishing at the origin and $\eta$ is some number. We now make a change of coordinate $x_1 = x_2 + A y_1^2$ to get rid of the coefficient of $x_2^3 y_1^2$. Equating coefficients we get $3A + \eta = 0$. Hence,

$$\rho = x_2^3 + y_1^3 + x_2^3 h_3 + y_1^3 h_4,$$

where $h_3$ and $h_4$ are holomorphic functions of $x_2$ and $y_1$ that vanish at the origin. This is equivalent to $u^3 + v^3$ after a change of coordinates.\]

Corollary 5.10. A curve $\rho^{-1}(0)$ has a $D_4$-node if and only if it satisfies the hypothesis of Lemma 5.9.

Lemma 5.11. Let $\rho = \rho(r, s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00}, \nabla \rho|_{(0,0)}, \nabla^2 \rho|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^2 \rho(w, w, \cdot) = 0$. Let $x = w_1 r + w_2 s$, $y = -w_2 r + w_1 s$ and $\rho_{ij}$ be the partial derivatives with respect to the new variables $x$ and $y$. If $\rho_{12} \neq 0$, there exists a coordinate chart $(u, v)$ centered around the origin in $\mathbb{C}^2$ such that

$$\rho(u, v) \equiv \begin{cases} v^2 u & \text{or} \\ v^2 u + u^{k-1} & \text{for some } k \geq 5. \end{cases}$$
Note that in terms of the new coordinates we have

\[ \rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = \rho_{02} = \rho_{30} = \rho_{21} = 0, \rho_{12} \neq 0. \]

**Proof:** In terms of the new coordinates \( x \) and \( y \), The Taylor expansion of \( \rho \) is given by

\[ \rho = \frac{\rho_{12}}{2} xy^2 + \frac{\rho_{03}}{6} y^3 + \frac{\rho_{40}}{24} x^4 + \ldots \]

We claim that there exists a holomorphic function \( G(y) \), such that after making a change of coordinate \( x = x_1 + G(y) \), the function is given by \( \rho = x_1 g(x_1, y) \), i.e., we can kill off all powers of \( y \). Assuming such a \( G(y) \) exists, we can now apply the argument as in Lemma 5.5. Let

\[ g(x_1, y) = A_0(x_1) + A_1(x_1)y + A_2(x_1)y^2 + \ldots. \]

We can make a change of coordinate \( y = y_1 + B(x_1) \) so that

\[ g = \hat{A}_0(x_1) + A_2(x_1)y_1^2 + \ldots \]

i.e., \( \hat{A}_1(x_1) \equiv 0 \). This is possible since \( A_2(0) \neq 0 \). That gives us

\[ \rho = x_1(\hat{A}_0(x_1) + \hat{A}_2(x_1)y_1^2 + \hat{A}_3(x_1)y_1^3 + \ldots) \]

If we set

\[ \hat{A}_0(x_1) = \frac{D^\rho_0}{4!} x_1^3 + \frac{D^\rho_4}{5!} x_1^4 + \ldots \]  \( (5.9) \)

then \( \rho \) is given by

\[ \rho = x_1(y_1^2 + \frac{D^\rho_0}{4!} x_1^3 + \frac{D^\rho_4}{5!} x_1^4 + \ldots) \]

If \( \hat{A}_0(x_1) \equiv 0 \) then \( \rho = x_1y_2^2 \) is of the intended form. Otherwise let \( k \) be the smallest integer such that \( D^\rho_{k+1} \neq 0 \). Let

\[ x_2 = \frac{1}{k-1} \sqrt{\frac{D^\rho_{k+1}}{(k-1)!} x_1^{k-1} + \frac{D^\rho_{k+2}}{k!} x_1^k + \ldots} \]  \( (5.10) \)

with \( x_1 = Cx_2 + O(x_2^2) \) and \( C = ((k-1)!/D^\rho_{k+1})^{k-1} \). In these new coordinates, \( \rho \) is given by

\[ \rho = (Cx_2 + x_2^2 h)y_2^2 + x_2^{k-1} \]

for some holomorphic function \( h(x_2, y_2) \). Now define \( y_3 = y_2\sqrt{C + x_2 h} \). Therefore,

\[ \rho = y_3^2 x_2 + x_2^{k-1} \]  \( (5.11) \)

to get the intended form as in \( (5.11) \).

It remains to show that \( G(y) \) exists. By our assumption on \( \rho \), we know that \( \rho_{30} = \rho_{21} = 0 \) and \( \rho_{12} \neq 0 \). Hence, the Taylor expansion of \( \rho \) is given by

\[ \rho(x, y) = P_{12}(x, y)xy^2 + P_{03}(y)y^3 + P_{40}(x, y)x^4 + P_{31}(x, y)x^3y \]  \( (5.12) \)

\(^6\)Notice that \( A_2(0) = \frac{\rho_{12}}{2} \) is non-zero by hypothesis.
for some holomorphic functions $P_{ij}$ with $P_{12}(0,0) \neq 0$. Recall that we want $x = x_1 + G(y)$ so that $\rho = x_1 g(x_1, y)$, i.e., the coefficients of $y^n$ are killed for all $n$. This is equivalent to saying that we want to find a $G$ such that $\rho(G(y), y) = 0$.7 Plugging in $x = x_1 + yH(y)$ in (5.12) we get that

$$0 = \rho(yH(y), y) = P_{12}(yH(y), y)y^3H(y) + P_{03}(y)y^3 + P_{40}(yH(y), y)y^4H(y)^4 + P_{31}(yH(y), y)y^3H(y).$$

This implies that

$$P_{12}(yH(y), y)H(y) + P_{03}(y) + P_{40}(yH(y), y)yH(y)^4 + P_{31}(yH(y), y)H(y) = 0$$

By the implicit function theorem, $H(y)$ exists since $P_{12}(0,0) \neq 0$, whence $G(y) = yH(y)$ exists. □

In practice we first find $H(y)$ as a power series and then find $A(x_1)$ as a power series. That ultimately gives us $D_f^0$. Following the above procedure, we prove (4.8).

**Corollary 5.12.** Let the hypothesis be as in Lemma 5.11. Then the curve $\rho^{-1}(0)$ has a $D_k$-node if and only if the directional derivatives $D_f^0$ obtained in (5.9) are zero for all $i \leq k$ and $D^\rho_{k+1} \neq 0$. Furthermore, if $\tau$ is a holomorphic function that does not vanish at the origin, then

$$D_{k+1}^\rho = \tau_0 D_{k+1}^\rho \quad \text{and} \quad (\tau \rho)^{k-6} D_{k+1}^\rho = \tau_0^{k-5} \rho^{k-6} D_{k+1}^\rho. \quad (5.13)$$

Finally, if $D_i^\rho = 0$ for $i \leq k$ then the quantity $D^\rho_{k+1}$ is invariant under

$$x \longrightarrow x + T_1(x, y), \quad y \longrightarrow y + T_2(x, y) \quad (5.14)$$

$$x \longrightarrow x, \quad y \longrightarrow y + x, \quad (5.15)$$

where $T_1$ and $T_2$ are holomorphic functions that vanish at the origin and whose derivative also vanish at the origin, i.e.,

$$T_i(0,0) = 0, \quad \nabla T_i(0,0) = 0, \quad \text{where} \quad i = 1,2.$$

**Proof:** The first assertion follows immediately from (5.11). To prove the second assertion, note that by (5.9)

$$D^\rho_{k+1} = \frac{\partial^{k+1} \rho(x_1, y_2)}{\partial x_1^{k+1}} \bigg|_{(0,0)} \implies D^\tau_{k+1} = \frac{\partial^{k+1} \tau \rho(x_1, y_2)}{\partial x_1^{k+1}} \bigg|_{(0,0)} = \tau_0 D^\rho_{k+1}$$

which follows from the product rule and the fact that $D_i^\rho = 0$ for all $i \leq k$. The second equation follows similarly by observing that $(\tau \rho)_1 = \tau_0 \rho_1$. We have omitted here the proofs of (5.14) and (5.15). The details of the proof can be found in [1]. □

**Remark 5.13.** Similar to remark 5.8, the quantities $\rho_{12}^{k-6} D_k^\rho$ \footnote{Note that $\rho(G(y), y)$ gives us the part of the Taylor expansion of $\rho(x_1 + G(y), y)$ that only depends on $y$. We desire that part to be zero, which is equivalent to killing off all the $y^n$ terms in the expansion of $\rho(x_1 + G(y), y)$.} induce sections of the line bundle $\mathbb{L}_{P\mathcal{D}_k} \longrightarrow \mathcal{D} \times \mathbb{P} \mathbb{P}^2$ given by

$$\{ \Psi_{P\mathcal{D}_k}(f, l_p) \} (f \otimes p^{\otimes d} \otimes v^{\otimes(2k-8)} \otimes w^{\otimes(2k-12)}) := f^{k-3} D_k^f. \quad (5.16)$$

Equations (5.13) and (5.14) imply that this section is well defined restricted to $\Psi_{P\mathcal{D}_{k-1}}^{-1}(0)$. Equation (5.15) implies that the section $\Psi_{P\mathcal{D}_k}$ is well defined on the quotient space.
Next we analyze the singularities $\mathcal{E}_6$ and $\mathcal{E}_7$.

**Lemma 5.14.** Let $\rho = \rho(r, s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00} = \nabla \rho|_{(0,0)} = \nabla^2 \rho|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^3 \rho(w, w, \cdot) = 0$. Let $x = w_1 r + w_2 s$, $y = -w_2 r + w_1 s$ and $\rho_{ij}$ be partial derivatives with respect to the new coordinates, $x$ and $y$. If $\rho_{12} = 0$ and $\rho_{03} \neq 0, \rho_{40} \neq 0$, there exists a coordinate chart $(u, v)$ centered at the origin so that

$$\rho(u, v) = v^3 + u^4.$$  \hfill (5.17)

Note that in terms of the new coordinates $x$ and $y$, we get that

$$\rho_{00} = \rho_{10} = \rho_{01} = \rho_{20} = \rho_{11} = \rho_{02} = \rho_{21} = \rho_{12} = 0, \rho_{03} \neq 0, \rho_{40} \neq 0.$$  

**Proof:** The Taylor expansion of $\rho$ is given by,

$$\rho(x, y) = P_{03}(x, y)y^3 + P_{40}(x, y)x^4 + \kappa_1 x^3 y + \kappa_2 x^2 y^2 + \kappa_3 x^3 y^2$$  \hfill (5.18)

in terms of the new variables $x$ and $y$, for some constants $\kappa_1, \kappa_2, \kappa_3$ and holomorphic functions $P_{03}$ and $P_{40}$ on a neighborhood of the origin in $\mathbb{C}^2$ such that $P_{03}(0, 0), P_{40}(0, 0) \neq 0$. We claim that there exists constants $\eta_1, \eta_2$ and $\eta_3$, such that if we make the substitution

$$x = \dot{x} + \eta_1 \dot{y}, \quad y = \dot{y} + \eta_2 x^2 + \eta_3 x^3$$

then $\rho$ is given by

$$\rho = \hat{P}_{03}(\dot{x}, \dot{y})\dot{y}^3 + \hat{P}_{40}(\dot{x}, \dot{y})\dot{x}^4$$  \hfill (5.19)

such that $\hat{P}_{03}(0, 0), \hat{P}_{40}(0, 0) \neq 0$. To see this, we note that this is possible if the following equations are satisfied:

$$\frac{\rho_{40}}{6} \eta_1 + \kappa_1 = 0, \quad \frac{\rho_{03}}{2} \eta_2 + \frac{\rho_{40}}{4} \eta_1^2 + 3\kappa_1 \eta_1 + \kappa_2 = 0,$$

$$\frac{\rho_{03}}{2} \eta_3 + \frac{\rho_{13}}{2} \eta_2 + \frac{\rho_{50}}{12} \eta_1^2 + \frac{\rho_{41}}{6} \eta_1 + 3\kappa_1 \eta_1^2 \eta_2 + 4\kappa_2 \eta_1 \eta_2 + \kappa_3 = 0.$$  

Solutions to $\eta_1, \eta_2, \eta_3$ exist, since $\rho_{40} \neq 0$ and $\rho_{03} \neq 0$. It is easy to see that (5.19) is equivalent to (5.17) after a change of coordinates, since $\hat{P}_{03}(0, 0), \hat{P}_{40}(0, 0) \neq 0$. \hfill \Box

**Corollary 5.15.** A curve $\rho^{-1}(0)$ has an $\mathcal{E}_6$-node if and only if it satisfies the hypothesis of Lemma 5.14.

**Lemma 5.16.** Let $\rho = \rho(r, s)$ be a holomorphic function defined on a neighbourhood of the origin in $\mathbb{C}$ such that $\rho_{00}, \nabla \rho|_{(0,0)}, \nabla^2 \rho|_{(0,0)} = 0$ and there exists a non-zero vector $w = (w_1, w_2)$ such that at the origin $\nabla^3 \rho(w, w, \cdot) = 0$. Let $x = w_1 r + w_2 s$, $y = -w_2 r + w_1 s$. Let $\rho_{ij}$ be the partial derivatives with respect to the new variables $x$ and $y$. If $\rho_{12} = \rho_{40} = 0$ and $\rho_{03} \neq 0, \rho_{31} \neq 0$, then there exists a coordinate chart $(u, v)$ centered at the origin so that

$$\rho(u, v) = v^3 + u^3 v.$$  \hfill (5.20)
In terms of the new coordinates \( x \) and \( y \), we get that
\[
\rho_{00} = \rho_{10} = \rho_{01} = \rho_{11} = \rho_{02} = \rho_{20} = \rho_{12} = \rho_{40} = 0,\ 
\rho_{03} \neq 0, \rho_{31} \neq 0.
\]

**Proof:** The Taylor expansion of \( \rho \) is given by,
\[
\rho(x, y) = P_{03}(x, y) y^3 + P_{31}(x, y) x^3 y + \kappa x^2 y^2 + P_{50}(x) x^5
\]
in terms of the new coordinates \( x \) and \( y \), for constant \( \kappa \) and holomorphic functions \( P_{03}, P_{31}, \) and \( P_{50} \) such that \( P_{03}(0,0), P_{31}(0,0) \neq 0 \). We claim that there exists a holomorphic function \( B(\hat{x}) \) and constant \( \eta_1 \) such that if we make the substitution
\[
x = \hat{x} + \eta_1 \hat{y}, \quad y = \hat{y} + B(\hat{x}) \hat{x}^2
\]
then \( \rho \) is given by
\[
\rho = \hat{P}_{03}(\hat{x}, \hat{y}) \hat{y}^3 + \hat{P}_{31}(\hat{x}, \hat{y}) \hat{x}^3 \hat{y}
\]
To see this, note that this is possible if the following equations are satisfied:
\[
P_{31}(\hat{x}, B(\hat{x}) \hat{x}^2) B(\hat{x}) + P_{03}(\hat{x}, B(\hat{x}) \hat{x}^2) \hat{x} B(\hat{x})^3 + k \hat{x} B(\hat{x})^2 + P_{50}(\hat{x}) = 0,
\]
\[
\frac{\rho_{31}}{2} \eta_1 + \frac{\rho_{03}}{2} B(0) + \kappa = 0.
\]
A solution to \( B(\hat{x}) \) exists since \( \rho_{31} \neq 0 \). We see that
\[
B(0) = -\frac{P_{50}(0)}{P_{31}(0,0)} = -\frac{\rho_{50}}{20 \rho_{31}}
\]
which implies the existence of \( \eta_1 \). It is easy to see that (5.22) is equivalent to (5.20) after a change of coordinates, since \( \hat{P}_{03}(0,0), \hat{P}_{31}(0,0) \neq 0 \).

**Corollary 5.17.** A curve \( \rho^{-1}(0) \) has an \( \mathcal{E}_7 \)-node if and only if it satisfies the hypothesis of Lemma 5.16.

Let us now summarize certain facts about sections of vector bundles, involving the vertical derivative.

**Lemma 5.18.** Let \( L \rightarrow M \) a complex line bundle over a two dimensional complex manifold \( M \), \( s : M \rightarrow L \) a holomorphic section and \( q \in M \) a point in \( M \). Let \( v, w \in T_q M \) be two tangent vectors at the point \( q \). Then the following are true:

1. If \( s(q) = 0 \) and \( \nabla^i s|_q = 0 \) for all \( i < k \) then \( \nabla^k s|_q \) is well defined. Furthermore, for any tangent vectors \( v, w \in T_q M \) and non-negative integers \( i \) and \( j \) such that \( i + j = k \), the quantity
\[
\mathbf{s}_{ij} := \nabla^{i+j} s|_q(v, \cdot \cdot \cdot, v, w, \cdot \cdot \cdot, w)
\]
(5.23)
is also well defined.

2. If \( s(q) = 0, \nabla s|_q = 0 \) and \( \nabla^2 s|_q(v, \cdot) = 0 \) then \( \nabla^3 s|_q(v, v, v) \) is also well defined.
3. Let $A^i_k$ be the corresponding sections induced from the quantities $A^\alpha_k$ obtained in (5.3) by replacing $\rho_{ij}$ with $\delta_{ij}$. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_{x(v, \cdot)} = 0$ and $\delta_{i}^{-3}A^i_k = 0$ for all $i < k$, then $\delta_{02}^{-3}A^i_k$ is well defined. Furthermore, the quantity $\delta_{02}^{-3}A^i_k$ is invariant under $w \to w + v$.

4. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$ and $\nabla^3 s|_{q(v, v, \cdot)} = 0$, then $\nabla^4 s|_{x(v, v, v, v)}$ is well defined.

5. Let $D^i_k$ be the corresponding sections induced from the quantities $D^\alpha_k$ obtained in (5.9) by replacing $\rho_{ij}$ with $\delta_{ij}$. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_{q(v, v, \cdot)} = 0$, $\nabla^4 s|_{q(v, v, v, v)} = 0$ and $\delta_{i}^{-3}D^i_k = 0$ for all $i < k \leq 8$, then $\delta_{12}^{-3}D^i_k$ (resp. $\delta_{12}^{-3}D^i_k$ if $k = 8$) is well defined. Furthermore, the quantity $\delta_{12}^{-3}D^i_k$ is invariant under $w \to w + v$.

6. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$ and $\nabla^3 s|_{q(v, v, \cdot)} = 0$, then $\nabla^3 s|_{q(v, w, w)}$ is well defined. Furthermore, the quantity $\nabla^3 s|_{q(v, w, w)}$ is invariant under $w \to w + v$.

7. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_{q(v, v, \cdot)} = 0$, and $\nabla^3 s|_{q(v, w, w)} = 0$, then $\nabla^4 s|_{q(v, v, v)}$ is well defined.

8. If $s(q) = 0$, $\nabla s|_q = 0$, $\nabla^2 s|_q = 0$, $\nabla^3 s|_{q(v, v, \cdot)} = 0$, $\nabla^3 s|_{q(v, w, w)} = 0$ and $\nabla^4 s|_{q(v, v, v)} = 0$, then $\nabla^4 s|_{q(v, v, v, w)}$ is well defined. Furthermore, the quantity $\nabla^3 s|_{q(v, v, v, w)}$ is invariant under $w \to w + v$.

**Proof:** We omit the details of the proof here; the details can be found in [1]. These facts follow in a straightforward way by unwinding definition 4.1. As explained in remarks 5.8 and 5.13, Corollary 5.7 and 5.12 imply Lemma 5.18, statement 3 and 5, respectively. \(\square\)

**Remark 5.19.** Let us mention a pedantic point about our notation. The $\ast$ introduced in the notation of (5.23) might seem strange to the reader. We have done that to be consistent with (4.5). According to our notation, if $f : \mathbb{P}^2 \to \mathbb{P}^d$ is a section and $\bar{p} \in \mathbb{P}^2$ then

$$f_{ij} := \nabla^i j f|_{\bar{p}}(v, \cdots, v, w, \cdots w) \in \gamma_{p^d}$$

$$\ast$$

$$f_{ij} := \{\nabla^i j f|_{\bar{p}}(v, \cdots, v, w, \cdots w)\} (p^d) = \nabla^i j f|_{\bar{p}}(v, \cdots, v, w, \cdots w) \in \mathbb{C}.$$  

Since we encounter the second quantity more in our computations, we have denoted that as $f_{ij}$. Notice that if $\tilde{f}_{ij}$ is well defined, then so is $f_{ij}$.

6 Transversality

In this section we give an explicit description of the spaces $A_0$, $A_1$ and $\mathcal{P}X_k$ in terms of vanishing and non-vanishing of bundle sections. Before proceeding, let us state three important Lemmas. We will then show that these spaces satisfy the hypothesis of one of these three Lemmas.

**Lemma 6.1.** Let $M$ be a smooth manifold and $\{S_i\}_{i=0}^k$ be a family of subspaces defined as:

$$S_i := \{p \in M : \zeta_0(p) = 0, \ldots, \zeta_i(p) = 0, \zeta_{i+1}(p) \neq 0\} \quad \forall \quad 0 \leq i \leq k \quad (6.1)$$

where $\zeta_i : M \to V_i$ are sections of vector bundles only defined on the subspace

$$\{p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0\}.$$
If the section 
\[ \zeta_i : \{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \} \to V_i \]
is transverse to the zero set for \(0 \leq i \leq k + 1\) then
\[ \overline{S}_{k-1} = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0 \} = S_{k-1} \cup \overline{S}_k. \] (6.2)

In particular, \(\overline{S}_{k-1}\) is a smooth manifold.

**Lemma 6.2.** Let \(M\) be a smooth manifold and \(\{S_i\}_{i=-1}^k\) be a family of subspaces defined as follows:
\[ S_{-1} = \{ p \in M : \zeta_0(p) \neq 0, \varphi(p) \neq 0 \}, \]
\[ S_i = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_i(p) = 0, \zeta_{i+1}(p) \neq 0, \varphi(p) \neq 0 \} \quad \forall \ 0 \leq i \leq k \]
where \(\zeta_i : M \to V_i\) are sections of vector bundles only defined on the subspace
\[ \{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \} \]
and \(\varphi : M \to W\) is defined. Suppose that
\[ \zeta_i : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{i-1}(p) = 0, \varphi(p) \neq 0 \} \to V_i \]
is transverse to the zero set for \(0 \leq i \leq k + 1\). Then
\[ \overline{S}_{k-1} = S_{k-1} \cup \overline{S}_k \cup B \quad \text{where} \quad B := \{ p \in \overline{S}_{k-1} : \varphi(p) = 0 \}. \] (6.3)
Furthermore, if \(\varphi : M \to W\) is transverse to the zero set, then
\[ M = S_{-1} \cup S_0 \cup B' \quad \text{where} \quad B' := \{ p \in M : \varphi(p) = 0 \}. \] (6.4)

**Lemma 6.3.** Let \(M\) be a smooth manifold and let \(S_0 \subset M\) be a subspace defined by
\[ S_0 = \{ p \in M : \zeta_0(p) = 0, \zeta_1(p) \neq 0, \varphi(p) \neq 0 \}. \]
Here \(\zeta_i : M \to V_i\) are sections of vector bundles only defined on the subspace
\[ \{ p \in M : \zeta_0(p), \ldots, \zeta_{i-1}(p) = 0 \} \]
and \(\varphi : M \to W\) is defined. If the sections
\[ \zeta_0 : M \to V_0, \quad \varphi : \zeta_0^{-1}(0) \to W, \quad \zeta_1 : \zeta_0^{-1}(0) - \varphi^{-1}(0) \to V_1 \]
are transverse to the zero set, then
\[ \overline{S}_0 = \{ p \in M : \zeta_0(p) = 0 \}. \] (6.5)

In particular, \(\overline{S}_0\) is a smooth manifold.

**Proofs of Lemma 6.1, 6.2 and 6.3:** See Appendix B.1.

We now give an explicit description of our spaces as algebraic varieties.
Proposition 6.4. The space $A_0$ can be described as
\[ A_0 = \{(\hat{f}, \tilde{p}) \in D \times \mathbb{P}^2 : \psi_{A_0}(\hat{f}, \tilde{p}) = 0, \, \psi_{A_1}(\hat{f}, \tilde{p}) \neq 0 \}. \] (6.6)
Furthermore, the sections of the vector bundles
\[ \psi_{A_0} : D \times \mathbb{P}^2 \to L_{A_0}, \, \psi_{A_1} : \psi_{A_0}^{-1}(0) \to V_{A_1} \]
are transverse to the zero set. In particular, $A_0$ is a smooth manifold of dimension $\delta_d + 1$.

Corollary 6.5. The space $\overline{A}_0$ is a smooth manifold of dimension $\delta_d + 1$ that can be described as
\[ \overline{A}_0 = \{(\hat{f}, \tilde{p}) \in D \times \mathbb{P}^2 : \psi_{A_0}(\hat{f}, \tilde{p}) = 0 \}. \]

Proposition 6.6. The space $A_1$ can be described as
\[ A_1 = \{(\hat{f}, \tilde{p}) \in \overline{A}_0 : \psi_{A_1}(\hat{f}, \tilde{p}) = 0, \, \psi_{A_2}(\hat{f}, \tilde{p}) \neq 0, \, \psi_{D_4}(\hat{f}, \tilde{p}) \neq 0 \}. \] (6.7)
Furthermore, the sections of the vector bundles,
\[ \psi_{A_1} : \overline{A}_0 \to V_{A_1}, \, \psi_{D_4} : \psi_{A_1}^{-1}(0) \to V_{D_4}, \, \psi_{A_2} : \psi_{A_1}^{-1}(0) - \psi_{D_4}^{-1}(0) \to L_{A_2} \]
are transverse to the zero set if $d \geq 2^9$. In particular, $A_1$ is a smooth manifold of dimension $\delta_d - 1$ if $d \geq 1$.

Corollary 6.7. The space $\overline{A}_1$ is a smooth manifold of dimension $\delta_d - 1$ that can be described as
\[ \overline{A}_1 = \{(\hat{f}, \tilde{p}) \in \overline{A}_0 : \psi_{A_1}(\hat{f}, \tilde{p}) = 0 \}, \quad \text{provided } d \geq 2. \]

Corollary 6.8. The spaces $A_2$ and $D_4$ are smooth manifolds, of dimension $\delta_d - 2$ and $\delta_d - 4$ respectively, if $d \geq 2$.

Proposition 6.9. The space $\hat{A}_0$ can be described as
\[ \hat{A}_0 = \{(\hat{f}, l_\tilde{p}) \in D \times \mathbb{P}^2 \mathbb{P}^2 : \Psi_{A_0}(\hat{f}, l_\tilde{p}) = 0, \, \Psi_{A_1}(\hat{f}, l_\tilde{p}) \neq 0 \}. \]
Furthermore, the sections of the vector bundles
\[ \Psi_{\hat{A}_0} : D \times \mathbb{P}^2 \to L_{\hat{A}_0}, \, \Psi_{\hat{A}_1} : \Psi_{A_0}^{-1}(0) \to V_{A_1} \]
are transverse to the zero set.

Corollary 6.10. The space $\overline{A}_0$ is a smooth manifold of dimension $\delta_d + 2$ that can be described as
\[ \overline{A}_0 = \{(\hat{f}, l_\tilde{p}) \in D \times \mathbb{P}^2 \mathbb{P}^2 : \Psi_{\hat{A}_0}(\hat{f}, l_\tilde{p}) = 0 \}. \]

Proposition 6.11. The space $\hat{A}_1$ can be described as
\[ \hat{A}_1 = \{(\hat{f}, l_\tilde{p}) \in \overline{A}_0 : \Psi_{\hat{A}_1}(\hat{f}, l_\tilde{p}) = 0, \, \Psi_{\hat{A}_2}(\hat{f}, l_\tilde{p}) \neq 0, \, \Psi_{D_4}(\hat{f}, l_\tilde{p}) \neq 0 \}. \]
Furthermore, the sections of the vector bundles
\[ \Psi_{\hat{A}_1} : \overline{A}_0 \to V_{\hat{A}_1}, \, \Psi_{D_4} : \Psi_{\hat{A}_1}^{-1}(0) \to V_{D_4}, \, \Psi_{\hat{A}_2} : \Psi_{\hat{A}_1}^{-1}(0) - \Psi_{D_4}^{-1}(0) \to L_{\hat{A}_2} \]
are transverse to the zero set if $d \geq 2^{10}$.

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\(^9\)The section $\psi_{D_4}(\hat{f}, \tilde{p})$ is non-zero is vacuously true if $\psi_{A_1}(\hat{f}, \tilde{p}) \neq 0$. We have stated the proposition in this way so that it is clear that our spaces satisfy the hypothesis of Lemma 6.3.

\(^{10}\)Again, $\Psi_{D_4}(\hat{f}, l_\tilde{p}) \neq 0$ is vacuously true if $\Psi_{\hat{A}_2}(\hat{f}, l_\tilde{p}) \neq 0$. 

20
Proposition 6.13. The space $\hat{A}_1^\#$ can be described as
\[ \hat{A}_1^\# = \{ (\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{P}A_2}(\tilde{f}, l_\tilde{p}) \neq 0 \}. \] (6.8)
Furthermore, the sections of the vector bundles
\[ \Psi_{\hat{A}_0} : \mathcal{D} \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{\hat{A}_0}, \quad \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}^{-1}(0) \rightarrow \mathbb{V}_{\hat{A}_1}, \quad \Psi_{\mathcal{P}A_2} : \Psi_{\mathcal{P}A_2}^{-1}(0) \rightarrow \mathbb{V}_{\mathcal{P}A_2} \]
are transverse to the zero set, provided $d \geq 2$.

Corollary 6.14. The space $\overline{A}_1^\#$ is a smooth manifold of dimension $\delta_d$ that can be described as
\[ \overline{A}_1^\# = \{ (\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{A}}_0 : \Psi_{\overline{A}_1}(\tilde{f}, l_\tilde{p}) = 0 \}, \text{ provided } d \geq 2. \]

Proposition 6.15. The space $\mathcal{P}A_2$ can be described as
\[ \mathcal{P}A_2 = \{ (\tilde{f}, l_\tilde{p}) \in \overline{A}_1^\# : \Psi_{\mathcal{P}A_2}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{P}A_3}(\tilde{f}, l_\tilde{p}) \neq 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_\tilde{p}) \neq 0 \}. \] (6.9)
Furthermore, the sections of the vector bundles
\[ \Psi_{\mathcal{P}A_2} : \overline{A}_1^\# \rightarrow \mathbb{V}_{\mathcal{P}A_2}, \quad \Psi_{\mathcal{P}A_3} : \Psi_{\mathcal{P}A_2}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}A_3}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}A_2}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_4} \]
are transverse to the zero set, provided $d \geq 3$.

Corollary 6.16. The space $\overline{A}_2$ is a manifold of dimension $\delta_d - 2$ and can be described as
\[ \overline{A}_2 = \{ (\tilde{f}, l_\tilde{p}) \in \overline{A}_1^\# : \Psi_{\mathcal{P}A_2}(\tilde{f}, l_\tilde{p}) = 0 \}, \text{ provided } d \geq 3. \]

Proposition 6.17. The space $\hat{D}_4^\#$ can be described as
\[ \hat{D}_4^\# = \{ (\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{D}_4}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{P}A_3}(\tilde{f}, l_\tilde{p}) \neq 0 \}. \] (6.10)
Furthermore, the sections of the vector bundles
\[ \Psi_{\hat{A}_0} : \mathcal{D} \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{\hat{A}_0}, \quad \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}^{-1}(0) \rightarrow \mathbb{V}_{\hat{A}_1}, \quad \Psi_{\hat{D}_4} : \Psi_{\hat{A}_1}^{-1}(0) \rightarrow \mathbb{V}_{\hat{D}_4}, \quad \Psi_{\mathcal{P}A_3} : \Psi_{\hat{D}_4}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}A_3} \]
are transverse to the zero set, provided $d \geq 3$.

Corollary 6.18. The space $\overline{D}_4^\#$ is a manifold of dimension $\delta_d - 4$ and can be described as
\[ \overline{D}_4^\# = \{ (\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\overline{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{D}_4}(\tilde{f}, l_\tilde{p}) = 0 \}, \text{ provided } d \geq 3. \]

Proposition 6.19. The space $\mathcal{P}D_4$ can be described as
\[ \mathcal{P}D_4 = \{ (\tilde{f}, l_\tilde{p}) \in \overline{A}_2 : \Psi_{\mathcal{P}A_3}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{P}D_4}(\tilde{f}, l_\tilde{p}) \neq 0 \}. \] (6.11)
Furthermore, the sections of the vector bundles
\[ \Psi_{\mathcal{P}A_3} : \overline{A}_2 \rightarrow \mathbb{L}_{\mathcal{P}A_3}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}A_3}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_4}, \quad \Psi_{\mathcal{P}D_4} : \Psi_{\mathcal{P}D_4}^{-1}(0) \rightarrow \mathbb{L}_{\mathcal{P}D_5} \]
are transverse to the zero set, provided $d \geq 3$. 

21
Corollary 6.20. The space $\overline{PD}_4$ is a manifold of dimension $\delta_d - 3$ and can be described as
\[ \overline{PD}_4 = \{(\tilde{f}, l_p) \in \overline{PA}_2 : \Psi_{PA_3}(\tilde{f}, l_p) = 0, \ \Psi_{PD_4}(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 3. \]

Proposition 6.21. The space $PA_3$ can be described as
\[ PA_3 = \{(\tilde{f}, l_p) \in \overline{PA}_2 : \Psi_{PA_3}(\tilde{f}, l_p) = 0, \ \Psi_{PA_4}(\tilde{f}, l_p) \neq 0, \ \Psi_{PD_4}(\tilde{f}, l_p) \neq 0\}. \quad (6.12) \]
Furthermore, the sections of the vector bundles
\[ \Psi_{PA_3} : \overline{PA}_2 \to \mathbb{L}_{PA_3}, \quad \Psi_{PD_4} : \Psi^{-1}_{PA_3}(0) \to \mathbb{L}_{PD_4}, \quad \Psi_{PA_4} : \Psi^{-1}_{PA_3}(0) - \Psi^{-1}_{PD_4}(0) \to \mathbb{L}_{PA_4} \]
are transverse to the zero set provided $d \geq 4$.

Corollary 6.22. The space $\overline{PA}_3$ is a manifold of dimension $\delta_d - 3$ and can be described as
\[ \overline{PA}_3 = \{(\tilde{f}, l_p) \in \overline{PA}_2 : \Psi_{PA_3}(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 4. \]

Corollary 6.23. The space $A_3$ is a smooth manifold of dimension $\delta_d - 3$, if $d \geq 3$.

Proposition 6.24. If $k > 3$, the space $PA_k$ can be described as
\[ PA_k = \{(\tilde{f}, l_p) \in \overline{PA}_3 : \Psi_{PA_i}(\tilde{f}, l_p) = 0 \text{ for } 4 \leq j \leq k, \ \Psi_{PA_{k+1}}(\tilde{f}, l_p) \neq 0, \ \Psi_{PD_4}(\tilde{f}, l_p) \neq 0\}. \quad (6.13) \]
Furthermore, the sections of the vector bundles $\Psi_{PA_i} : \Psi^{-1}_{PA_{i-1}}(0) - \Psi^{-1}_{PD_4}(0) \to \mathbb{L}_{PA_i}$ are transverse to the zero set for all $4 \leq i \leq k + 1$, provided $d \geq k + 1$.

Corollary 6.25. The space $A_k$ is a smooth manifold of dimension $\delta_d - k$, if $d \geq k$.

Proposition 6.26. The space $PD_5$ can be described as
\[ PD_5 = \{(\tilde{f}, l_p) \in \overline{PD}_4 : \Psi_{PD_5}^L(\tilde{f}, l_p) = 0, \ \Psi_{PD_6}(\tilde{f}, l_p) \neq 0, \ \Psi_{PE_6}(\tilde{f}, l_p) \neq 0\} \quad (6.14) \]
Furthermore, the sections of the vector bundles
\[ \Psi_{PD_5} : \overline{PD}_4 \to \mathbb{L}_{PD_5}, \quad \Psi_{PD_6} : \Psi^{-1}_{PD_5}(0) \to \mathbb{L}_{PD_6}, \quad \Psi_{PE_6} : \Psi^{-1}_{PD_5}(0) \to \mathbb{L}_{PE_6} \]
are transverse to the zero set, provided $d \geq 4$.

Corollary 6.27. The space $\overline{PD}_5$ is a manifold of dimension $\delta_d - 5$ and can be described as
\[ \overline{PD}_5 = \{(\tilde{f}, l_p) \in \overline{PD}_4 : \Psi_{PD_5}^L(\tilde{f}, l_p) = 0\}, \text{ provided } d \geq 4. \]

Corollary 6.28. The space $D_5$ is a smooth manifold of dimension $\delta_d - 5$, if $d \geq 3$.

Proposition 6.29. The space $PE_6$ can be described as
\[ PE_6 = \{(\tilde{f}, l_p) \in \overline{PD}_5 : \Psi_{PE_6}(\tilde{f}, l_p) = 0, \ \Psi_{PE_7}(\tilde{f}, l_p) \neq 0, \ \Psi_{PE_8}(\tilde{f}, l_p) \neq 0\} \quad (6.15) \]
Furthermore, the sections of the vector bundles
\[ \Psi_{PE_6} : \overline{PD}_5 \to \mathbb{L}_{PE_6}, \quad \Psi_{PE_7} : \Psi_{PE_6}^L(0) \to \mathbb{L}_{PE_7}, \quad \Psi_{PE_8} : \Psi_{PE_6}^L(0) \to \mathbb{L}_{PE_8} \]
are transverse to the zero set, provided $d \geq 4$. 

22
Corollary 6.30. The space $\overline{PE}_6$ is a manifold of dimension $\delta_d - 6$ and can be described as

$$\overline{PE}_6 = \{(\tilde{f}, l_\bar{p}) \in \overline{PD}_5 : \Psi_{PD_6}(\tilde{f}, l_\bar{p}) = 0\}, \text{ provided } d \geq 4.$$  

Corollary 6.31. The space $E_6$ is a manifold of dimension $\delta_d - 6$, if $d \geq 3$.

Proposition 6.32. The space $PD_6$ can be described as

$$PD_6 = \{(\tilde{f}, l_\bar{p}) \in \overline{PD}_5 : \Psi_{PD_6}(\tilde{f}, l_\bar{p}) = 0, \Psi_{PD_7}(\tilde{f}, l) \neq 0, \Psi_{PD_6}(\tilde{f}, l_\bar{p}) \neq 0\}. \quad (6.16)$$

Furthermore, the sections of the vector bundles

$$\Psi_{PD_6} : \overline{PD}_5 \rightarrow L_{PD_6}, \quad \Psi_{PD_6} : \Psi_{PD_6}^{-1}(0) \rightarrow L_{PD_6}, \quad \Psi_{PD_7} : \Psi_{PD_6}^{-1}(0) - \Psi_{PD_6}^{-1}(0) \rightarrow L_{PD_7} \quad (6.17)$$

are transverse to the zero set, provided $d \geq 5$.

Corollary 6.33. The space $\overline{PD}_6$ is a manifold of dimension $\delta_d - 6$ and can be described as

$$\overline{PD}_6 = \{(\tilde{f}, l_\bar{p}) \in \overline{PD}_5 : \Psi_{PD_6}(\tilde{f}, l_\bar{p}) = 0\}, \text{ provided } d \geq 5.$$  

Corollary 6.34. The space $D_6$ is a manifold of dimension $\delta_d - 6$, if $d \geq 4$.

Proposition 6.35. If $k > 6$, then the space $PD_k$ can be described as

$$PD_k = \{(\tilde{f}, l_\bar{p}) \in \overline{PD}_6 : \Psi_{PD_j}(\tilde{f}, l_\bar{p}) = 0 \text{ if } 7 \leq j \leq k, \Psi_{PD_{k+1}}(\tilde{f}, l_\bar{p}) \neq 0, \Psi_{PD_6}(\tilde{f}, l_\bar{p}) \neq 0\}. \quad (6.18)$$

Furthermore, the section of the vector bundle $\Psi_{PD_i} : \Psi_{PD_{i-1}}^{-1}(0) - \Psi_{PD_6}^{-1}(0) \rightarrow L_{PD_i}$ is transverse to the zero set for all $7 \leq i \leq k + 1$ provided $d > k - 2$.

Corollary 6.36. The space $D_k$ is a manifold of dimension $\delta_d - k$, if $d \geq k - 2$.

Proposition 6.37. The space $PE_7$ can be described as

$$PE_7 = \{(\tilde{f}, l_\bar{p}) \in \overline{PE}_6 : \Psi_{PE_6}(\tilde{f}, l_\bar{p}) = 0, \Psi_{PE_6}(\tilde{f}, l_\bar{p}) \neq 0, \Psi_{PE_6}(\tilde{f}, l_\bar{p}) \neq 0\}. \quad (6.19)$$

Furthermore, the sections of the vector bundles

$$\Psi_{PE_6} : \overline{PE}_6 \rightarrow L_{PE_7}, \quad \Psi_{PE_6} : \Psi_{PE_6}^{-1}(0) \rightarrow L_{PE_8}, \quad \Psi_{PE_6} : \Psi_{PE_6}^{-1}(0) \rightarrow L_{PE_8}$$

are transverse to the zero set, provided $d \geq 4$.

Corollary 6.38. The space $\overline{PE}_7$ is a manifold of dimension $\delta_d - 6$ and can be described as

$$\overline{PE}_7 = \{(\tilde{f}, l_\bar{p}) \in \overline{PE}_6 : \Psi_{PE_7}(\tilde{f}, l_\bar{p}) = 0\}, \text{ provided } d \geq 4.$$  

Corollary 6.39. The space $E_7$ is a manifold of dimension $\delta_d - 7$, if $d \geq 4$.

Proposition 6.40. The space $PE_7$ can also be described as

$$PE_7 = \{(\tilde{f}, l_\bar{p}) \in \overline{PD}_6 : \Psi_{PE_6}(\tilde{f}, l_\bar{p}) = 0, \Psi_{PE_6}(\tilde{f}, l_\bar{p}) \neq 0, \Psi_{PE_6}(\tilde{f}, l_\bar{p}) \neq 0\}. \quad (6.20)$$

Furthermore, the sections of the vector bundles

$$\Psi_{PE_6} : \overline{PD}_6 \rightarrow L_{PE_6}, \quad \Psi_{PE_6} : \Psi_{PE_6}^{-1}(0) \rightarrow L_{PE_8}, \quad \Psi_{PE_6} : \Psi_{PE_6}^{-1}(0) \rightarrow L_{PE_8}$$

are transverse to the zero set, provided $d \geq 4$.  

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23
Corollary 6.41. The space $\mathcal{PE}_7$ is a manifold of dimension $\delta_d - 6$ and can also be described as
\[ \mathcal{PE}_7 = \{(\tilde{f}, l_\tilde{p}) \in \mathbb{P}D_6 : \Psi_{\mathcal{PE}_6}(\tilde{f}, l_\tilde{p}) = 0\}, \text{ provided } d \geq 4. \]

Proposition 6.42. The spaces $\hat{X}_8^\#$ and $\hat{X}_8^{\#}$ can be described as
\[ \hat{X}_8^\# = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{D}_4}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\hat{X}_8}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\mathcal{PE}_7}(\tilde{f}, l_\tilde{p}) = 0\} \]
\[ \hat{X}_8^{\#} = \{(\tilde{f}, l_\tilde{p}) \in \hat{X}_8^\# : \Psi_{\hat{J}}(\tilde{f}, l_\tilde{p}) \neq 0\}. \quad (6.21) \]

Furthermore, the sections of the vector bundle
\[ \Psi_{\hat{A}_0} : \mathcal{D} \times \mathbb{P}T^2 \rightarrow \mathbb{L}_{\hat{A}_0}, \quad \Psi_{\hat{A}_1} : \Psi_{\hat{A}_0}(0) \rightarrow \mathbb{V}_{\hat{A}_1}, \quad \Psi_{\hat{D}_4} : \Psi_{\hat{A}_1}(0) \rightarrow \mathbb{V}_{\hat{D}_4}, \quad \Psi_{\hat{X}_8} : \Psi_{\hat{D}_4}(0) \rightarrow \mathbb{V}_{\hat{X}_8}, \quad \Psi_{\mathcal{PE}_7} : \Psi_{\hat{X}_8}(0) \rightarrow \mathbb{V}_{\mathcal{PE}_7}, \quad \Psi_{\hat{J}} : \Psi_{\hat{X}_8}(0) - \Psi_{\mathcal{PE}_7}(0) \rightarrow \mathbb{L}_{\hat{J}} \]
are transverse to the zero set, provided $d \geq 4$.

Corollary 6.43. The spaces $\overrightarrow{X}_8^\#$ and $\overrightarrow{X}_8^{\#}$ are equal to each other. They are manifolds of dimension $\delta_d - 7$ and can be described as
\[ \overrightarrow{X}_8^\# = \overrightarrow{X}_8^{\#} = \{(\hat{f}, l_{\hat{p}}) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\hat{A}_0}(\hat{f}, l_{\hat{p}}) = 0, \Psi_{\hat{A}_1}(\hat{f}, l_{\hat{p}}) = 0, \Psi_{\hat{D}_4}(\hat{f}, l_{\hat{p}}) = 0, \Psi_{\hat{X}_8}(\hat{f}, l_{\hat{p}}) = 0\} \]
provided $d \geq 4$.

6.1 Proofs of Propositions

Let $\mathcal{F} \cong \mathbb{C}^{\delta_d + 1}$ denote the space of homogeneous polynomials of degree $d$. Let $\mathcal{F}^*$ be the subspace of non-zero polynomials. This can also be thought of as the space of polynomials in two variables of degree at most $d$. If $\mathcal{V} \rightarrow M$ is any vector bundle then a section
\[ \psi : \mathcal{D} \times M \rightarrow \pi^*_\mathcal{D} \gamma^*_\mathcal{D} \otimes \pi^*_{\mathcal{M}} V \]
induces a section
\[ \dot{\psi} : \mathcal{F}^* \times M \rightarrow \pi^*_{\mathcal{M}} V \quad \text{given by} \quad \dot{\psi}(f, x) := \{\psi(\tilde{f}, x)\}(f). \quad (6.22) \]
We note that $\psi$ is transverse to zero at $(\tilde{f}, x)$ if and only if $\dot{\psi}$ is transverse to zero at $(f, x)$.

Proof of Proposition 6.4: Equation (6.6) follows from Corollary 5.2. We will now show transversality. We will use the setup of definition 6.1, remark 6.3 and (6.22). Suppose $f, \tilde{p}) \in \psi_{\hat{A}_0}^{-1}(0)$. Choose homogeneous coordinates $[X : Y : Z]$ on $\mathbb{P}^2$ so that $\tilde{p} = [0 : 0 : 1]$ and let
\[ \mathcal{U} := \{[X : Y : Z] \in \mathbb{P}^2 : Z \neq 0\}, \varphi_{\mathcal{U}} : \mathcal{U} \rightarrow \mathbb{C}^2, \varphi_{\mathcal{U}}([X : Y : Z]) = (X/Z, Y/Z). \]
Let us also denote $x := X/Z$ and $y := Y/Z$. The section
\[ \psi_{\hat{A}_0} : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{L}_{\hat{A}_0} := \pi^*_\mathcal{D} \gamma^*_\mathcal{D} \otimes \pi^*_{\mathbb{P}^2} \gamma^*_\mathbb{P}^2 \]
induces a section \( \tilde{\psi}_{A_0} : \mathcal{F}^* \times \mathbb{P}^2 \rightarrow \pi_{\mathbb{P}^2}^* \gamma_{\mathbb{P}^2} \) as given in (6.22). We shall denote the bundle by \( \gamma_{\mathbb{P}^2} \).

We now observe that with respect to the standard trivialization of \( \gamma_{\mathbb{P}^2} \rightarrow \mathcal{F}^* \times \mathbb{P}^2 \), the induced map \( \tilde{\psi}_{A_0} : \mathcal{F}^* \times C^2 \rightarrow C \) of the section \( \tilde{\psi}_{A_0} \) (cf. (4.3), remark 4.3) is given by

\[
\tilde{\psi}_{A_0}(f, x, y) = f(x, y) := f_{00} + f_{10}x + f_{01}y + \frac{f_{20}}{2} x^2 + f_{11}xy + \frac{f_{02}}{2} y^2 + \cdots.
\]

By remark 4.3, it suffices to show that this induced map \( \tilde{\psi}_{A_0} \) is transverse to zero at \((f, 0, 0)\). Since the Jacobian matrix of this map at \((f, 0, 0)\) is

\[
d\tilde{\psi}_{A_0}|_{(f,0,0)} = \begin{pmatrix} 1 & 0 & 0 & \cdots \end{pmatrix},
\]

where the first column is partial derivative with respect to \( f_{00} \), transversality follows. Next we will prove that \( \psi_{A_1} \) restricted to \( \psi_{A_0}^{-1}(0) \) is transverse to zero. By Lemma 5.18, statement 1 we conclude that if \( \psi_{A_0}(\tilde{f}, \tilde{p}) = 0 \), then \( \psi_{A_1}(\tilde{f}, \tilde{p}) \) is well defined. Let

\[
(\tilde{f}, \tilde{p}) \in \psi_{A_1}^{-1}(0) \subset \psi_{A_0}(0).
\]

With respect to the standard trivialization of \( \pi_D^* \gamma_D^* \otimes \mathcal{V}_{A_1} \rightarrow \psi_{A_0}^{-1}(0) \), the induced map of the section \( \tilde{\psi}_{A_1} \) (cf. definition 4.1, (4.1)) is given by

\[
\tilde{\psi}_{A_1} : (\mathcal{F}^* \times C^2) \cap \psi_{A_0}^{-1}(0) \rightarrow C^2, \quad \tilde{\psi}_{A_1}(f, x, y) = (f_x(x, y), f_y(x, y)).
\]

Since the function \( \tilde{\psi}_{A_0} \) is transverse to the zero set at \((f, 0, 0)\), showing that \( \tilde{\psi}_{A_1} \) is transverse to zero at \((f, 0, 0)\) is equivalent to showing that the map

\[
\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1} : \mathcal{F}^* \times C^2 \rightarrow C^3, \quad \tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1}(f, x, y) = (f(x, y), f_x(x, y), f_y(x, y))
\]

is transverse to zero. Since \( f(x, y) = f_{00} + f_{10}x + f_{01}y + \cdots \), the Jacobian at \((f, 0, 0)\) is

\[
d(\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1})|_{(f,0,0)} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \end{pmatrix},
\]

where the first three columns are partial derivatives with respect to \( f_{00}, f_{10} \) and \( f_{01} \). \( \square \)

**Proof of Corollary 6.5:** This follows immediately from Lemma 6.1 and Proposition 6.4. \( \square \)

**Proof of Proposition 6.6:** Equation (6.7) follows from Corollary 5.4 and Corollary 6.5. We have already shown the transversality of \( \psi_{A_1} \) in the proof of Proposition 6.4. We will now show the transversality of \( \psi_{D_1} \) and \( \psi_{A_2} \). Let us start with \( \psi_{D_1} \). By Lemma 5.18, statement 1 we conclude that \( \psi_{D_1} \) is well defined restricted to \( \psi_{A_1}^{-1}(0) \). As in the proof of Proposition 6.4, proving transversality is equivalent to showing that the map

\[
\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1} \oplus \tilde{\psi}_{D_1} : \mathcal{F}^* \times C^2 \rightarrow C^6, \quad (f, x, y) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy})
\]

is transverse to zero at the point \((f, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0)\) is a \( 6 \times (\delta_d + 3) \) matrix which has full rank if \( d \geq 2 \); this follows, for instance, if the first six columns of the matrix are partial derivatives with respect to \( f_{00}, f_{10}, f_{01}, f_{20}, f_{11} \) and \( f_{02} \).
Next let us show transversality of the section $\psi_{A_2}$. The section is well defined on $\Psi_{A_1}^{-1}(0)$ by Lemma 5.18, statement 1. As before, proving transversality is equivalent to showing that the map

$$\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1} \oplus \tilde{\psi}_{A_2} : \mathcal{F}^* \times \mathbb{C}^2 \longrightarrow \mathbb{C}^4, \quad (f, x, y) \mapsto (f(x, y), \ f_x, \ f_y, \ f_{xx}f_{yy} - f_{xy}^2)$$

is transverse to zero at the point $(f, 0, 0)$. The Jacobian matrix of this map at $(f, 0, 0)$ is a $4 \times (\delta_d + 3)$ matrix which has full rank if $d \geq 2$ and $\psi_{D_1}(\bar{f}, \bar{p}) \neq 0$: take the first four columns of the matrix as the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, and $f_{20}$, $f_{20}$ or $f_{11}$, depending on whether $f_{02}$, $f_{20}$ or $f_{11}$ is non-zero. One of them is guaranteed to be non-zero if $\psi_{D_1}(\bar{f}, \bar{p}) \neq 0$.

**Proof of Corollary 6.7:** This follows immediately from Lemma 6.3 and Proposition 6.6.

**Proof of Corollary 6.8:** This follows from Proposition 6.6, Corollary 5.7 and Corollary 5.10.

**Proof of Proposition 6.9:** This is identical to the proof of Proposition 6.4.

**Proof of Corollary 6.10:** This follows immediately from Lemma 6.1 and Proposition 6.9.

**Proof of Proposition 6.11:** This is identical to the proof of Proposition 6.6.

**Proof of Corollary 6.12:** This follows immediately from Lemma 6.3 and Proposition 6.11.

**Proof of Proposition 6.13:** Equation (6.8) is the definition of $\hat{A}^\#$, so there is nothing to prove. To show transversality we continue with the setup of the proof of Proposition 6.4, but choose coordinate chart so that

$$\tilde{\mathcal{U}} := \{(a \partial_x, \ b \partial_y) \in \mathbb{P}TP^2| \mathcal{U} : a \neq 0\}, \ \varphi_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \longrightarrow \mathbb{C}^3, \ \varphi_{\tilde{\mathcal{U}}}(a \partial_x, \ b \partial_y) = (x, y, \eta),$$

where $\eta := b/a$. By Lemma 5.18, statement 1 we conclude that $\Psi_{P,A_2}(\bar{f}, \bar{p})$ is well defined restricted to $\Psi_{A_1}^{-1}(0)$. With respect to the standard trivialization, the induced map $\Psi_{P,A_2}$ restricted to $\Psi_{A_1}^{-1}(0)$ is given by

$$\tilde{\Psi}_{P,A_2} : (\mathcal{F}^* \times \mathbb{C}^3) \cap \tilde{\Psi}_{A_1}^{-1}(0) \longrightarrow \mathbb{C}^2, \quad f, x, y, \eta \mapsto (f_{xx} + \eta f_{xy}, \ f_{xy} + \eta f_{yy}).$$

As before, this is equivalent to showing that the map

$$\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{P,A_2} : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^5, \quad (f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy})$$

is transverse to zero at $(f, 0, 0, 0)$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $5 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 2$. This is easy to see if the first five columns of the matrix are partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, and $f_{20}$ and $f_{11}$.

**Proof of Corollary 6.14:** This follows immediately from Lemma 6.1 and Proposition 6.13.

**Proof of Proposition 6.15:** Equation (6.9) follows from Lemma 5.5 and Corollary 5.7. To see this, we observe that Corollary 5.7 (for $k = 2$) gives us a necessary and sufficient condition for a curve $\rho^{-1}(0)$ to have an $A_2$-node. In terms of bundle sections, this is equivalent to the statement that the
space \( \mathcal{P}A_2 \) can be described as
\[
\mathcal{P}A_2 = \{ (f, l_p) \in \mathcal{D} \times \mathbb{RP}^2 : \Psi_{\mathcal{A}_0}(f, l_p) = 0, \Psi_{\mathcal{A}_1}(f, l_p) = 0, \Psi_{\mathcal{P}A_2}(f, l_p) = 0, \Psi_{\mathcal{P}A_3}(f, l_p) = 0, \Psi_{\mathcal{P}D_4}(f, l_p) \neq 0, \Psi_{\mathcal{P}D_4}(f, l_p) \neq 0 \}.
\]

The desired equation (6.9) now follows from Corollary 6.10 and 6.14.

We will now show transversality, having already proved it for the section \( \Psi_{\mathcal{P}A_2} \) in the proof of Proposition 6.13. Let us start with \( \Psi_{\mathcal{P}A_3} \). By Lemma 5.18, statement 2 we conclude that restricted to \( \Psi_{\mathcal{P}A_2}^{-1}(0) \), the section \( \Psi_{\mathcal{P}A_3} \) is well defined. As before, showing that the section \( \Psi_{\mathcal{P}A_3} \) is transverse to the zero set is equivalent to showing that the map
\[
\tilde{\Psi}_{\mathcal{A}_0} \oplus \tilde{\Psi}_{\mathcal{A}_1} \oplus \tilde{\Psi}_{\mathcal{P}A_2} \oplus \tilde{\Psi}_{\mathcal{P}A_3} : \mathcal{F}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^6,
\]
\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy})
\]
is transverse to zero, where
\[
\hat{x} := x + \eta y \quad \text{and} \quad f_{xxy} := (\partial_x + \eta \partial_y)^3 f(x, y).
\]
The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(6 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\). This follows by looking at the first six columns of the matrix which are partial derivatives with respect to \(f_0, f_1, f_0, f_2, f_0, f_1\) and \(f_3\). Next let us show the transversality of \( \Psi_{\mathcal{P}D_4} \). Note that \( \Psi_{\mathcal{P}D_4} \) is well defined restricted to \( \Psi_{\mathcal{P}A_2}^{-1}(0) \) since \( \nabla^2 f|_p \) is well defined (cf. Lemma 5.18, statement 1). Showing that the section \( \Psi_{\mathcal{P}D_4} \) is transverse to the zero set is equivalent to showing that the map
\[
\tilde{\Psi}_{\mathcal{A}_0} \oplus \tilde{\Psi}_{\mathcal{A}_1} \oplus \tilde{\Psi}_{\mathcal{P}A_2} \oplus \tilde{\Psi}_{\mathcal{P}D_4} : \mathcal{F}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^6,
\]
\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{y})
\]
is transverse to zero at \((f, 0, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(6 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 2\); look at the first six columns which are partial derivatives with respect to \(f_0, f_1, f_2, f_0, f_1\) and \(f_2\).

**Proof of Corollary 6.16:** This follows immediately from Lemma 6.3 and Proposition 6.15. \(\square\)

**Proof of Proposition 6.17:** Equation (6.10) is the definition of \( \mathcal{D}^2_f \). Towards showing transversality, note that the transversality of the sections \( \mathcal{A}_0 \) and \( \mathcal{A}_1 \) in the sections is taken care of in the proof of Proposition 6.13. Moreover, proving the transversality of \( \mathcal{D}^2_4 \) is identical to the proof of transversality of \( \mathcal{D}_4 \) in Proposition 6.6. We will now show the transversality of the section \( \mathcal{P}A_3 \) restricted to \( \Psi_{\mathcal{D}^4_4}^{-1}(0) \). Then it is easy to see that \( \Psi_{\mathcal{P}A_2}(f, l_p) = 0 \). Hence, \( \Psi_{\mathcal{P}A_3} \) is well defined (cf. Lemma 5.18, statement 2). As before, showing that the section \( \Psi_{\mathcal{P}A_3} \) is transverse to the zero set is equivalent to showing that the map
\[
\tilde{\Psi}_{\mathcal{A}_0} \oplus \tilde{\Psi}_{\mathcal{A}_1} \oplus \tilde{\Psi}_{\mathcal{D}^4_4} \oplus \tilde{\Psi}_{\mathcal{P}A_3} : \mathcal{F}^3 \times \mathbb{C}^3 \longrightarrow \mathbb{C}^7,
\]
\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxy})
\]
is transverse to zero and \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(7 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); choose the first seven columns as the partial derivatives with respect to \(f_0, f_1, f_2, f_0, f_1, f_2 \) and \(f_3\). \(\square\)
Proof of Corollary 6.18: This follows immediately from Lemma 6.1 and Proposition 6.17.

Proof of Proposition 6.19: Proving (6.11) requires some care. Unravelling the definition of $\mathcal{P}\mathcal{D}_4$ and using Corollary 5.10 we get

$$\mathcal{P}\mathcal{D}_4 = \{(\tilde{f}, l_p) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_p) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{D}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0 \}.$$  

It is evident via linear algebra that

$$\Psi_{\mathcal{D}_4}(\tilde{f}, l_p) = 0, \quad \Psi_{\mathcal{P}A_3}(\tilde{f}, l_p) = 0 \Rightarrow \Psi_{\mathcal{P}A_2}(\tilde{f}, l_p) = 0, \quad \Psi_{\mathcal{P}A_4}(\tilde{f}, l_p) = 0, \quad \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}, l_p) = 0. \quad (6.24)$$

Using equations (6.23) and (6.24), we get that

$$\mathcal{P}\mathcal{D}_4 = \{(\tilde{f}, l_p) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_p) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_2}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}A_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{P}\mathcal{D}_4}(\tilde{f}, l_p) = 0 \}.$$  

Hence, using (6.25), Corollary 6.16, 6.14 and 6.10 we obtain (6.11). Next we prove transversality of the bundle sections. We have already proved the transversality of the sections $\Psi_{\mathcal{P}A_3}$ in Proposition 6.15. We will now show the transversality of the sections $\Psi_{\mathcal{P}\mathcal{D}_4}$ and $\Psi_{\mathcal{P}\mathcal{D}_3}$. Since $\nabla f|_p$ is restricted to $\Psi_{\mathcal{P}A_3}(0)$, we conclude that $\Psi_{\mathcal{P}\mathcal{D}_4}$ is well defined by Lemma 5.18, statement 1. As before, showing that the section $\Psi_{\mathcal{P}\mathcal{D}_4}$ is transverse to the zero set is equivalent to showing that the map

$$\tilde{\Psi}_{\tilde{A}_0} \oplus \tilde{\Psi}_{\tilde{A}_1} \oplus \tilde{\Psi}_{\mathcal{P}A_2} \oplus \tilde{\Psi}_{\mathcal{P}A_4} \oplus \tilde{\Psi}_{\mathcal{P}\mathcal{D}_4} : \mathcal{F}^* \times \mathbb{C}^3 \to \mathbb{C}^7,$$

is transverse to zero at $(f,0,0,0,0)$, where $\tilde{x} = x + \eta y$. The Jacobian matrix of this map at $(f,0,0,0,0)$ is a $7 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 3$; take the first seven columns as the partial derivatives with respect to $f_00, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}$ and $f_{02}$.

Next let us show that $\Psi_{\mathcal{P}\mathcal{D}_3}$ is transverse. Note that restricted to $\Psi_{\mathcal{P}\mathcal{D}_4}(0)$, the section $\Psi_{\mathcal{P}\mathcal{D}_3}$ is well defined; when restricted to $\Psi_{\mathcal{P}\mathcal{D}_4}(0)$, we infer that $f(p) = 0$, $\nabla f|_p = 0$ and $\nabla^2 f|_p = 0$. Hence, by Lemma 5.18, statement 1, the quantity $\nabla^3 f|_p$ is well defined. Consequently, the section $\Psi_{\mathcal{P}\mathcal{D}_3}$ is also well defined. Showing that the section $\Psi_{\mathcal{P}\mathcal{D}_3}$ is transverse to the zero set is equivalent to showing that the map

$$\tilde{\Psi}_{\tilde{A}_0} \oplus \tilde{\Psi}_{\tilde{A}_1} \oplus \tilde{\Psi}_{\mathcal{P}A_2} \oplus \tilde{\Psi}_{\mathcal{P}A_4} \oplus \tilde{\Psi}_{\mathcal{P}\mathcal{D}_4} \oplus \tilde{\Psi}_{\mathcal{P}\mathcal{D}_3} : \mathcal{F}^* \times \mathbb{C}^3 \to \mathbb{C}^8,$$

is transverse to zero at $(f,0,0,0,0)$, where $\tilde{x} = x + \eta y$. The Jacobian matrix of this map at $(f,0,0,0,0)$ is an $8 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 3$; take the first eight columns as the partial derivatives with respect to $f_00, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}$ and $f_{21}$.

Proof of Corollary 6.20: This follows immediately from Lemma 6.1 and Proposition 6.19.

Proof of Proposition 6.21: By Corollary 5.7 (for $k = 3$), we get that

$$\mathcal{P}A_3 = \{(f, l_p) \in \mathcal{D} \times \mathbb{P}^2 : \Psi_{\tilde{A}_0}(f, l_p) = 0, \Psi_{\tilde{A}_1}(f, l_p) = 0, \Psi_{\mathcal{P}A_2}(f, l_p) = 0, \Psi_{\mathcal{P}A_3}(f, l_p) = 0, \Psi_{\mathcal{P}\mathcal{D}_4}(f, l_p) = 0 \}.$$  

(6.26)
Equation (6.12) now follows from (6.26) and Corollary 6.16, 6.14 and 6.10.

Towards showing transversality of the bundle sections, note that the sections $\Psi_{PA_3}$ and $\Psi_{PD_4}$ are already transverse (cf. Proposition 6.15 and 6.19). We now prove transversality of $\Psi_{PA_4}$. By Lemma 5.18, statement 3 we conclude that $\Psi_{PA_4}$ is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

$$
\bar{\Psi}_{A_0} \oplus \bar{\Psi}_{A_1} \oplus \bar{\Psi}_{PA_2} \oplus \bar{\Psi}_{PA_3} \oplus \bar{\Psi}_{PA_4} : F^* \times C^3 \rightarrow C^7,
$$

$$(f, x, y, \eta) \mapsto \left( f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{\bar{\eta} \bar{x} \bar{y}}, f_{yy} A_4^{f(\bar{x}, \bar{y})} \right) \quad (6.27)$$

is transverse to zero at $(f, 0, 0, 0)$, where $\bar{x} = x + \eta y$. From (5.4) it follows that

$$f_{yy} A_4^{f(\bar{x}, \bar{y})} = f_{yy} f_{\bar{x} \bar{x} \bar{y}} - 3 f_{\bar{x} \bar{y}}^2.$$

Hence, the Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $7 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 4$ and $f_{02} \neq 0$; take the first seven columns as the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$, $f_{30}$ and $f_{40}$. Notice that the condition $\Psi_{PD_4}(\bar{f}, \bar{l}_p) \neq 0$ (which is equivalent to $f_{02} \neq 0$) is necessary to conclude that the Jacobian matrix has full rank.

**Proof of Corollary 6.22:** This follows immediately from Lemma 6.3 and Proposition 6.21.

**Proof of Corollary 6.23:** We observe that $A_3 = \pi(PA_3)$, where $\pi : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathcal{D} \times \mathbb{P}^2$ is the projection map. Let us continue with the setup of Proposition 6.21 and consider (6.27). It suffices to show that the zero set of the map

$$F^* \times C^2 \rightarrow C^5, \quad (f, x, y) \mapsto \left( f(x, y), f_x, f_y, f_{xx} f_{yy} - f_{xy}^2, (\partial_x - \frac{f_{xy}}{f_{yy}} \partial_y)^3 f(x, y) \right)$$

is smooth submanifold of $F^* \times C^2$ at $(f, 0, 0)$. The Jacobian of this map at $(f, 0, 0)$ is a $5 \times (\delta_d + 3)$ matrix which has full rank if $d \geq 3$; take the first five columns as the partial derivatives with respect to $f_{00}$, $f_{10}$, $f_{01}$, $f_{20}$ and $f_{30}$.

**Proof of Proposition 6.24:** By Corollary 5.7, we get that

$$\mathcal{P} A_k = \{ (\bar{f}, \bar{l}_p) \in \mathcal{D} \times \mathbb{P}^2 : \bar{\Psi}_{A_0}(\bar{f}, \bar{l}_p) = 0, \bar{\Psi}_{A_1}(\bar{f}, \bar{l}_p) = 0, \bar{\Psi}_{PA_2}(\bar{f}, \bar{l}_p) = 0, \ldots, \bar{\Psi}_{PA_k}(\bar{f}, \bar{l}_p) = 0, \bar{\Psi}_{PA_{k+1}}(\bar{f}, \bar{l}_p) \neq 0, \bar{\Psi}_{PD_4}(\bar{f}, \bar{l}_p) \neq 0 \}. \quad (6.28)$$

Equation (6.13) now follows from (6.28) and Corollary 6.16, 6.14 and 6.10.

Towards proving transversality of the bundle section $\Psi_{PA_i}$, note that by Lemma 5.18 and statement 3 this section is well defined. Showing that the section $\Psi_{PA_i}$ is transverse to the zero set is equivalent to showing that the map

$$\bar{\Psi}_{A_0} \oplus \bar{\Psi}_{A_1} \oplus \bar{\Psi}_{PA_2} \oplus \bar{\Psi}_{PA_3} \oplus \ldots \bar{\Psi}_{PA_i} : F^* \times C^3 \rightarrow C^{i+3},
$$

$$(f, x, y, \eta) \mapsto \left( f, f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{\bar{x} \bar{x} \bar{y}}, f_{yy} A_{4}^{f(\bar{x}, \bar{y})}, f_{yy}^{2} A_{5}^{f(\bar{x}, \bar{y})}, \ldots, f_{yy}^{i-3} A_{i}^{f(\bar{x}, \bar{y})} \right) \quad (6.29)$$

is transverse to zero at $(f, 0, 0, 0)$, where $\bar{x} = x + \eta y$. Recall that $A_i^f$ are defined in (5.3) (implicitly) and in (5.4) (explicitly till $i = 7$). Hence, the Jacobian matrix of this map at $(f, 0, 0, 0)$ is an
\((i + 3) \times (\delta_d + 4)\) matrix which has full rank if \(d \geq i\) and \(f_{02} \neq 0\); take the first \((i + 3)\) columns of the matrix as partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}, \ldots, f_{10}\). Notice that the condition \(\Psi_{PD_4}(\tilde{f}, l_p) \neq 0\) (which is equivalent to \(f_{02} \neq 0\)) is necessary to conclude that the Jacobian matrix has full rank. \(\square\)

**Proof of Corollary 6.25:** This is similar to the proof of Corollary 6.23. We observe that \(A_k = \pi(\mathcal{P}A_k)\), where \(\pi : D \times \mathbb{P}T^2 \rightarrow D \times \mathbb{P}^2\) is the projection map. It suffices to show that the zero set of the map \(\mathcal{F}^* \times C^2 \rightarrow C^{k+2}\) given by

\[
(f, x, y) \mapsto \left( f(x, y), f_x, f_y, f_{xx} f_{yy} - f_{xy}^2, f_{yy} A_4 f^{(x,y)}, f_{yy} A_5 f^{(x,y)}, \ldots, f^{k-3} A_k f^{(x,y)} \right),
\]

where \(\tilde{x} = x \frac{f_{xx} f_{yy} - f_{xy}^2}{f_{yy}}\), is smooth submanifold of \(\mathcal{F}^* \times C^{k+2}\) at \((f, 0, 0)\). The Jacobian of this map at \((f, 0, 0)\) is a \((k + 2) \times (\delta_d + 3)\) matrix which has full rank if \(d \geq k\); take the first \((k + 2)\) columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}, \ldots, f_{10}\). Notice that we require \(f_{02} \neq 0\) to conclude that the matrix has full rank. \(\square\)

**Proof of Proposition 6.26:** Proving (6.14) requires some care. Unravelling the definition of \(\mathcal{P}D_5\) and using Corollary 5.12 for \(k = 5\) we see that

\[
\mathcal{P}D_5 := \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{A_2}(\tilde{f}, l_p) = 0, \Psi_{PD_4}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0\}.
\]  

(6.30)

Standard linear algebra implies that

\[
\Psi_{PD_4}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_5}(\tilde{f}, l_p) = 0
\]

\[
\iff \Psi_{PD_4}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_5}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_5}(\tilde{f}, l_p) = 0, \quad \Psi_{PD_5}(\tilde{f}, l_p) = 0.
\]  

(6.31)

Hence, using (6.30) and (6.31), we get that

\[
\mathcal{P}D_5 = \{(\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{A_2}(\tilde{f}, l_p) = 0, \Psi_{A_3}(\tilde{f}, l_p) = 0, \Psi_{PD_4}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0\}.
\]  

(6.32)

Equation (6.14) now follows from (6.32), Corollary 6.20, 6.16, 6.14 and 6.10.

We have already proved the transversality of the section \(\Psi_{PD_5}^{L}\) in Proposition 6.19. We will now show the transversality of the sections \(\Psi_{PD_6}\) and \(\Psi_{PD_5}\). By Lemma 5.18, statement 4 and 6, these two sections are well defined. Showing that the section \(\Psi_{PD_6}\) is transverse to the zero set is equivalent to showing that the map

\[
\Psi_{A_0} \oplus \Psi_{A_1} \oplus \Psi_{PD_2} \oplus \Psi_{PD_4} \oplus \Psi_{PD_5} \oplus \Psi_{PD_6} : \mathcal{F}^* \times C^3 \rightarrow C^0,
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{yy} + \eta f_{yy}, f_{yy}, f_{yy}, f_{yy}, f_{yy})
\]

is transverse to zero at \((f, 0, 0, 0)\), where \(\tilde{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(9 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 4\); take the first 9 columns as the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{40}, \ldots, f_{10}\).

Showing that the section \(\Psi_{PD_6}\) is transverse to the zero set is equivalent to showing that the map

\[
\Psi_{A_0} \oplus \Psi_{A_1} \oplus \Psi_{PD_2} \oplus \Psi_{PD_4} \oplus \Psi_{PD_5} \oplus \Psi_{PD_6} : \mathcal{F}^* \times C^3 \rightarrow C^0,
\]

30
is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(9 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 4\); take the first 9 columns as the partial derivatives with respect to \(f_0, f_{10}, f_{11}, f_{20}, f_{21}, f_{30}, f_{31}, f_{40}, f_{41}\) and \(f_{12}\). 

**Proof of Corollary 6.27**: This follows immediately from Lemma 6.3 and Proposition 6.26.

**Proof of Corollary 6.28**: This basically follows from the setup of Proposition 6.26. The proof is similar to the proof of Corollary 6.23 and 6.25.

**Proof of Proposition 6.29**: Proving (6.15) requires some care. Unravelling the definition of \(\mathcal{P}E_6\) and using Corollary 5.15 we conclude that

\[
\mathcal{P}E_6 = \{(\tilde{f}, l_p) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_p) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_A}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_D}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_E}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_X}(\tilde{f}, l_p) = 0 \}. \tag{6.33}
\]

Using (6.33) and (6.31) we get that

\[
\mathcal{P}E_6 = \{(\tilde{f}, l_p) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{\tilde{A}_0}(\tilde{f}, l_p) = 0, \Psi_{\tilde{A}_1}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_A}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_D}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_E}(\tilde{f}, l_p) = 0, \Psi_{\tilde{P}_X}(\tilde{f}, l_p) = 0 \}. \tag{6.34}
\]

Equation (6.15) now follows from (6.34) and Corollary 6.27, 6.20, 6.16, 6.14 and 6.10.

We have already proved the transversality of the sections \(\Psi_{\mathcal{P}E_6}\) in Proposition 6.26. We will now show the transversalit of the sections \(\Psi_{\mathcal{P}E_7}\) and \(\Psi_{\mathcal{P}X_6}\). Let us start with \(\Psi_{\mathcal{P}E_7}\). By Lemma 5.18, statement 7, the section is well defined. Showing that the section \(\Psi_{\mathcal{P}E_7}\) is transverse to the zero set is equivalent to showing that the map

\[
(\tilde{f}, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\tilde{x}}, f_{y\eta}, f_{\tilde{x}x}, f_{\tilde{x}y})
\]

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(10 \times (\delta_d + 3)\) matrix which has full rank if \(d \geq 4\); take the first 10 columns as the partial derivatives with respect to \(f_0, f_{10}, f_{11}, f_{20}, f_{21}, f_{30}, f_{31}, f_{40}, f_{41}\) and \(f_{12}\).

Observe that restricted to \(\Psi_{\mathcal{P}E_6}^{-1}(0)\), \(\nabla^2 f|_p = 0\), whence \(\Psi_{\mathcal{P}X_6}\) is well defined by Lemma 5.18, statement 1. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
(\tilde{f}, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{x\tilde{x}}, f_{y\eta}, f_{\tilde{x}x}, f_{\tilde{x}y})
\]

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \(10 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 3\); take the first 10 columns as the partial derivatives with respect to \(f_0, f_{10}, f_{11}, f_{20}, f_{21}, f_{30}, f_{31}, f_{40}, f_{41}\) and \(f_{12}\).

**Proof of Corollary 6.30**: This follows immediately from Lemma 6.3 and Proposition 6.29.
Proof of Corollary 6.31: This is identical to the setup of Proposition 6.29. The proof is similar to the proof of Corollary 6.23 and 6.25. □

Proof of Proposition 6.32: Proving (6.16) requires some care. We begin by observing that Corollary 5.12 (for k = 6) implies that

\[
\mathcal{PD}_6 = \{(f, l_p) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{A_0}(\bar{f}, \bar{l}_p) = 0, \Psi_{A_1}(\bar{f}, \bar{l}_p) = 0, \Psi_{D_1}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_6}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_6}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_7}(\bar{f}, \bar{l}_p) \neq 0, \Psi_{PD_8}(\bar{f}, \bar{l}_p) \neq 0\}. \quad (6.35)
\]

Using (6.35) and (6.31) we conclude that

\[
\mathcal{PD}_6 = \{(f, l_p) \in \mathcal{D} \times \mathbb{P}T^2 : \Psi_{A_0}(\bar{f}, \bar{l}_p) = 0, \Psi_{A_1}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_6}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_6}(\bar{f}, \bar{l}_p) = 0, \Psi_{PD_7}(\bar{f}, \bar{l}_p) \neq 0, \Psi_{PD_8}(\bar{f}, \bar{l}_p) \neq 0\}. \quad (6.36)
\]

Equation (6.16) now follows from (6.36) and Corollary 6.27, 6.20, 6.16, 6.14 and 6.10.

Towards proving transversality, note that we have already shown the transversality of $\Psi_{PD_6}$ in Proposition 6.26. We now prove transversality of $\Psi_{PD_6}$ and $\Psi_{PD_7}$. Let us start with $\Psi_{PD_6}$. Note that restricted to $\Psi_{PD_6}^{-1}(0)$, the quantity $\nabla^2 f_p$ vanishes, whence the section $\Psi_{PD_6}$ is well defined (cf. Lemma 5.18, statement 1). Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{PD_6} : F^* \times \mathbb{C}^3 \to \mathbb{C}^{10},
\]

is transverse to zero at $(f, 0, 0, 0)$, where $\hat{x} = x + \eta y$. The Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $10 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 4$ and $f_{02} \neq 0$. This follows by taking the first ten columns to be partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{30}$ and $f_{12}$.

Observe that $\Psi_{PD_7}$ is well defined by Lemma 5.18, statement 5. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{PD_6} : F^* \times \mathbb{C}^3 \to \mathbb{C}^{10},
\]

is transverse to zero at $(f, 0, 0, 0)$, where $\hat{x} = x + \eta y$. From (4.8) we see that

\[
f_{\tilde{x}yy} D_7^{(\hat{x}, \hat{y})} = \tilde{x}yy f_{\tilde{x}yyyy} - \frac{5 f_{\tilde{x}yyyy}^2}{3}.
\]

Hence, the Jacobian matrix of this map at $(f, 0, 0, 0)$ is a $10 \times (\delta_d + 4)$ matrix which has full rank if $d \geq 5$ and $f_{12} \neq 0$. This is evident by taking the first ten columns to be partial derivatives with respect to $f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{30}$ and $f_{50}$. Notice that the condition $f_{12} \neq 0$ is necessary to conclude that the matrix has full rank. □

Proof of Corollary 6.33: This follows immediately from Lemma 6.3 and Proposition 6.32. □

Proof of Corollary 6.34: This basically follows from the setup of Proposition 6.32. The proof is similar to the proof of Corollary 6.23 and 6.25. □
Proof of Proposition 6.35: Proving (6.18) requires some care. Notice that Corollary 5.12 implies that

$$\mathcal{P}D_k = \{ (\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{D_4}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \ldots, \Psi_{PD_k}(\tilde{f}, l_p) = 0, \Psi_{PD_{k+1}}(\tilde{f}, l_p) \neq 0, \Psi_{PE_6}(\tilde{f}, l_p) \neq 0 \}. \quad (6.37)$$

Equation (6.18) now follows from (6.37), (6.31) and Corollary 6.33, 6.27, 6.20, 6.16, 6.14 and 6.10.

We now prove transversality of the bundle sections \(\Psi_{PD_k}\). By Lemma 5.18, statement 5 these sections are well defined. Showing that these sections are transverse to the zero set is equivalent to showing that the map

$$\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{PA_2} \oplus \tilde{\Psi}_{PA_3} \oplus \tilde{\Psi}_{PD_4} \oplus \tilde{\Psi}_{PD_5} \oplus \tilde{\Psi}_{PD_6} \oplus \ldots \tilde{\Psi}_{PD_k} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{i+3},$$

$$(f, x, y, \eta) \mapsto \left( f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxy}, f_{xyy} \right)$$

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). Recall that \(D_i^f\) is defined in (5.10) implicitly and in (4.8) explicitly till \(i = 8\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \((i + 3) \times (\delta_d + 4)\) matrix which has full rank if \(d \geq i - 2\) and \(f_{12} \neq 0\). This follows by choosing the first \((i + 3)\) columns to be the partial derivatives with respect to \(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{40}, f_{03}, f_{50}\), and \(f_{11-2.0}\). Notice that the condition \(\Psi_{PE_6}(\tilde{f}, l_p) \neq 0\) (which is equivalent to \(f_{12} \neq 0\)) is necessary to conclude that the Jacobian matrix has full rank.

\[\square\]

Proof of Corollary 6.36: This basically follows from the setup of Proposition 6.35. The proof is similar to the proof of Corollary 6.23 and 6.25.

\[\square\]

Proof of Proposition 6.37: Proving (6.19) requires some care. Unravelling the definition of \(\mathcal{P}E_{\tau}\) in conjunction with Corollary 5.17 we gather

$$\mathcal{P}E_{\tau} = \{ (\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{D_4}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \ldots, \Psi_{PD_{k+1}}(\tilde{f}, l_p) = 0, \Psi_{PE_6}(\tilde{f}, l_p) \neq 0, \Psi_{PD_6}(\tilde{f}, l_p) \neq 0 \}. \quad (6.38)$$

Hence, using (6.38) and (6.31), we get that

$$\mathcal{P}E_{\tau} = \{ (\tilde{f}, l_p) \in D \times \mathbb{P}T^2 : \Psi_{A_0}(\tilde{f}, l_p) = 0, \Psi_{A_1}(\tilde{f}, l_p) = 0, \Psi_{PA_2}(\tilde{f}, l_p) = 0, \Psi_{PA_3}(\tilde{f}, l_p) = 0, \Psi_{PD_4}(\tilde{f}, l_p) = 0, \Psi_{PD_5}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \Psi_{PD_6}(\tilde{f}, l_p) = 0, \Psi_{PD_{k+1}}(\tilde{f}, l_p) = 0, \Psi_{PE_6}(\tilde{f}, l_p) \neq 0, \Psi_{PD_6}(\tilde{f}, l_p) \neq 0 \}. \quad (6.39)$$

Equation (6.19) now follows from equation (6.39) and Corollary 6.30, 6.27, 6.20, 6.16, 6.14 and 6.10.

We have already proved transversality of \(\Psi_{PE_{\tau}}\) in Proposition 6.29. We now prove transversality of \(\Psi_{PE_{\tau}}\) and \(\Psi_{PD_6}\). Let us start with \(\Psi_{PE_{\tau}}\). By Lemma 5.18, statement 8 the section is well defined. Showing that this section is transverse to the zero set, is equivalent to showing that the map

$$\tilde{\Psi}_{A_0} \oplus \tilde{\Psi}_{A_1} \oplus \tilde{\Psi}_{PA_2} \oplus \tilde{\Psi}_{PA_3} \oplus \tilde{\Psi}_{PD_4} \oplus \tilde{\Psi}_{PD_5} \oplus \tilde{\Psi}_{PD_6} \oplus \tilde{\Psi}_{PD_{k+1}} \oplus \tilde{\Psi}_{PE_{\tau}} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{i+3},$$

$$(f, x, y, \eta) \mapsto \left( f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxy}, f_{xyy}, f_{xxx}, f_{xxy}, f_{xyy} \right)$$

is transverse to zero at \((f, 0, 0, 0)\), where \(\hat{x} = x + \eta y\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \(11 \times (\delta_d + 4)\) matrix which has full rank if \(d \geq 4\) and \(f_{12} \neq 0\). This follows by taking the first
eleven columns as the partial derivatives with respect to \( f_{00}, f_{10}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{12}, f_{40} \) and \( f_{31} \).

To show that \( \Psi_{P\mathcal{X}_S} \) is transverse to the zero set, we observe that restricted to \( \Psi_{P\mathcal{E}_S}^{-1}(0) \), the quantity \( \nabla^2 f|_p \) is identically zero (via linear algebra). Hence, the section is well defined by Lemma 5.18, statement 1. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\bar{\Psi}_A_0 \oplus \bar{\Psi}_A_1 \oplus \bar{\Psi}_P A_2 \oplus \bar{\Psi}_P A_3 \oplus \bar{\Psi}_P D_4 \oplus \bar{\Psi}_P D_5 \oplus \bar{\Psi}_P E_6 \oplus \bar{\Psi}_P E_7 \oplus \bar{\Psi}_P \mathcal{X}_S : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx} + \eta f_{xy}, f_{xy} + \eta f_{yy}, f_{xxy}, f_{xyy}, f_{xxxy}, f_{xxyy}, f_{xyyy})
\]
is transverse to zero at \((f, 0, 0, 0)\), where \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \( 11 \times (\delta_d + 4) \) matrix which has full rank if \( d \geq 4 \) and \( f_{12} \neq 0 \); take the first eleven columns as the partial derivatives with respect to \( f_{00}, f_{10}, f_{20}, f_{11}, f_{30}, f_{02}, f_{21}, f_{12}, f_{40} \), and \( f_{31} \).

**Proof of Corollary 6.38:** This follows immediately from Lemma 6.3 and Proposition 6.37.

**Proof of Corollary 6.39:** This basically follows from the setup of Proposition 6.37. The proof is similar to the proof of Corollary 6.23 and 6.25.

**Proof of Proposition 6.40:** Observe that \( \Psi_{P\mathcal{D}_6} = \Psi_{P\mathcal{E}_7} \). Equation (6.20) follows from Proposition 6.37, Corollary 6.33 and 6.30. Transversality of \( \Psi_{P\mathcal{E}_7} \) has been proven in Proposition 6.32 (there it was denoted as \( \Psi_{P\mathcal{D}_6} \)). Proving that the sections \( \Psi_{P\mathcal{E}_8} \) and \( \Psi_{P\mathcal{X}_S} \) are transverse to the zero set is almost identical to the proof in Proposition 6.37.

**Proof of Corollary 6.41:** This follows immediately from Lemma 6.3 and Proposition 6.40.

**Proof of Proposition 6.42:** Equation (6.21) is the definition(s) of the spaces \( \mathcal{X}_S^\# \) and \( \mathcal{X}_S^{\#\#} \) so there is nothing to prove. We have already proved the transversality of the sections \( \bar{\Psi}_A_0, \bar{\Psi}_A_1 \) and \( \bar{\Psi}_D_4 \) in Proposition 6.17. We will now show the transversality of the sections \( \bar{\Psi}_A_0, \bar{\Psi}_E_6, \) and \( \bar{\Psi}_E_7 \).

Let us start with \( \bar{\Psi}_X_S \). Observe that by Lemma 5.18, statement 1 this section is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\bar{\Psi}_A_0 \oplus \bar{\Psi}_A_1 \oplus \bar{\Psi}_D_4 \oplus \bar{\Psi}_X_S : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{10},
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xyy}, f_{xyy}, f_{xyy})
\]
is transverse to zero at \((f, 0, 0, 0)\). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is a \( 10 \times (\delta_d + 4) \) matrix which has full rank if \( d \geq 3 \); take the first ten columns as the partial derivatives with respect to \( f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}, f_{31} \).

By Lemma 5.18, statement 1 the section \( \Psi_{P\mathcal{E}_7} \) is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\bar{\Psi}_A_0 \oplus \bar{\Psi}_A_1 \oplus \bar{\Psi}_D_4 \oplus \bar{\Psi}_X_S \oplus \bar{\Psi}_{P\mathcal{E}_7} : \mathcal{F}^* \times \mathbb{C}^3 \rightarrow \mathbb{C}^{11},
\]

\[
(f, x, y, \eta) \mapsto (f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xyy}, f_{xyy}, f_{xyy}, f_{xyy}, f_{xyy})
\]
is transverse to zero at \((f, 0, 0, 0)\), where \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \( 11 \times (\delta_d + 4) \) matrix which has full rank if \( d \geq 4 \); take the eleven columns as the partial derivatives.
with respect to \( f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}, f_{03} \) and \( f_{40} \).

Finally, observe that by Lemma 5.18, statement 1 the section \( \Psi_J \) is well defined. Showing that this section is transverse to the zero set is equivalent to showing that the map

\[
\Psi_{\hat{A}_0} \oplus \Psi_{\hat{A}_1} \oplus \Psi_{\hat{D}_4} \oplus \Psi_{\hat{X}_8} \oplus \Psi_J : \mathcal{F}^* \times \mathbb{C}^3 \longrightarrow \mathbb{C}^{11},
\]

\[
(f, x, y, \eta) \mapsto \left(f(x, y), f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xyy}, f_{yy}, f_{xyy}, f_{yyy}, \right)
\]

is transverse to zero at \((f, 0, 0, 0)\), where \( \hat{x} = x + \eta y \). The Jacobian matrix of this map at \((f, 0, 0, 0)\) is an \( 11 \times (\delta_d + 4) \) matrix which has full rank if \( d \geq 4 \) and \( f_{40} \neq 0 \). This follows by taking the first eleven columns to be the partial derivatives with respect to \( f_{00}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}, f_{03} \) and \( f_{13} \). Notice that the condition \( \Psi_{\mathcal{P}E_7}(\hat{f}, l_p) \neq 0 \) (which is equivalent to \( f_{40} \neq 0 \)) is necessary for the matrix to have full rank. \( \square \)

**Proof of Proposition 6.43:** The result for \( \overline{X}_8^\# \) follows from Lemma 6.1 and Proposition 6.42 while that for \( \overline{X}_8^{\#3} \) follows from Lemma 6.3 and Proposition 6.42. \( \square \)

### 7 Closure

We compute the closure of the various spaces; this is the main result(s) of this section.

**Lemma 7.1.** Let \( X_k \) be a singularity of type \( A_k, D_k, E_k \) or \( X_8 \). Then the closures are given by:

1. \( \overline{\mathcal{A}}_0 = \mathcal{A}_0 \cup \mathcal{A}_1 \) if \( d \geq 2 \).
2. \( \overline{A}_1 = \mathcal{A}_1^\# \cup \mathcal{P}A_2 \) if \( d \geq 3 \).
3. \( \overline{D}_4^\# = \mathcal{D}_4^\# \cup \mathcal{P}D_4 \) if \( d \geq 3 \).
4. \( \overline{P}D_4 = \mathcal{P}D_4 \cup \mathcal{P}D_5 \) if \( d \geq 4 \).
5. \( \overline{P}E_6 = \mathcal{P}E_6 \cup \overline{P}E_7 \cup \overline{X}_8^\# \) if \( d \geq 4 \).
6. \( \overline{P}D_5 = \mathcal{P}D_5 \cup \mathcal{P}D_6 \cup \overline{P}E_6 \) if \( d \geq 4 \).
7. \( \overline{P}D_6 = \mathcal{P}D_6 \cup \mathcal{P}D_7 \cup \overline{P}E_7 \) if \( d \geq 5 \).
8. \( \overline{P}A_2 = \mathcal{P}A_2 \cup \overline{P}A_3 \cup \overline{D}_4^\# \) if \( d \geq 4 \).
9. \( \overline{P}A_3 = \mathcal{P}A_3 \cup \mathcal{P}A_4 \cup \overline{P}D_4 \) if \( d \geq 5 \).
10. \( \overline{P}A_4 = \mathcal{P}A_4 \cup \mathcal{P}A_5 \cup \overline{P}D_5 \) if \( d \geq 6 \).
11. \( \overline{P}A_5 = \mathcal{P}A_5 \cup \mathcal{P}A_6 \cup \overline{P}D_6 \cup \overline{P}E_6 \) if \( d \geq 7 \).
12. \( \overline{P}A_6 = \mathcal{P}A_6 \cup \overline{P}A_7 \cup \overline{P}D_7 \cup \overline{P}E_7 \cup \overline{X}_8^{\#3} \) if \( d \geq 8 \).
Proof of Lemma 7.1 (1): Follows from Corollary 6.5, Proposition 6.4, Corollary 6.7.

Proof of Lemma 7.1 (2): Follows from Corollary 6.12, 6.14, Proposition 6.13, Corollary 6.16.

Proof of Lemma 7.1 (3): Follows from Corollary 6.18, Proposition 6.17, Corollary 6.20, 6.16, 6.14 and 6.10.

Proof of Lemma 7.1 (4): Follows from Corollary 6.20, Proposition 6.19, Corollary 6.27.

Proof of Lemma 7.1 (5): Corollary 6.43 implies that

$$\overline{X}_8^\# = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}\mathbb{T}^2 : \Psi_{\overline{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_3}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_5}(\tilde{f}, l_\tilde{p}) = 0\}.$$ 

Standard linear algebra implies that

$$\overline{X}_8^\# = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{D} \times \mathbb{P}\mathbb{T}^2 : \Psi_{\overline{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_2}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_3}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_4}(\tilde{f}, l_\tilde{p}) = 0, \Psi_{\overline{A}_5}(\tilde{f}, l_\tilde{p}) = 0\}.$$ 

By Corollary 6.30, 6.27, 6.20, 6.16, 6.14, 6.10, we conclude that

$$\overline{X}_8^\# = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{P}\mathbb{E}_6 : \Psi_{\overline{A}_6}(\tilde{f}, l_\tilde{p}) = 0\}.$$ 

Lemma 7.1, statement 5 now follows from (7.1), Corollary 6.38, Proposition 6.29 and Corollary 6.30.

Proof of Lemma 7.1 (6): Follows from Corollary 6.27, Proposition 6.26, Corollary 6.33 and 6.30.

Proof of Lemma 7.1 (7): Follows from Proposition 6.35. Since the section

$$\Psi_{\mathbb{E}_6} : \mathcal{P}\mathbb{D}_6 \longrightarrow \mathbb{L}\mathbb{E}_6$$

is transverse to the zero set (as proved in Proposition 6.32), the hypothesis of the second part of Lemma 6.2 is satisfied. By Proposition 6.32 and Corollary 6.33 we conclude that

$$\mathcal{P}\mathbb{D}_6 = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{P}\mathbb{D}_6 : \Psi_{\mathbb{D}_7}(\tilde{f}, l_\tilde{p}) \neq 0, \Psi_{\mathbb{E}_6}(\tilde{f}, l_\tilde{p}) \neq 0\}.$$ 

Corollary 6.41 now proves our claim.

Proof of Lemma 7.1 (8): Follows from Corollary 6.16, Proposition 6.15, Corollary 6.22, 6.18, 6.14 and 6.10.

Proof of Lemma 7.1 (9): Follows from the second part of Lemma 6.2 and Proposition 6.24. Since the section $$\Psi_{\mathbb{D}_4} : \mathcal{P}\mathbb{A}_3 \longrightarrow \mathbb{L}_{\mathbb{D}_4}$$ is transverse to the zero set (cf. Proposition 6.21), the hypothesis of the second part of Lemma 6.2 is satisfied. Corollary 6.20 and 6.22 now imply that

$$\mathcal{P}\mathbb{D}_4 = \{(\tilde{f}, l_\tilde{p}) \in \mathcal{P}\mathbb{A}_3 : \Psi_{\mathbb{D}_4}(\tilde{f}, l_\tilde{p}) = 0\}.$$ 

(7.2)
By Proposition 6.21 and Corollary 6.22 we get that
\[ \mathcal{P}A_3 = \{(f, l_p) \in \mathcal{PA}_3 : \Psi_{\mathcal{PA}_4}(f, l_p) \neq 0, \ \Psi_{\mathcal{PD}_4}(f, l_p) \neq 0\}. \]
The second part of Lemma 6.2 now proves our claim. \[\square\]

Proof of Lemma 7.1 (10): By Lemma 6.2 applied to
\[ M = \overline{\mathcal{PA}_3}, \ \zeta_0 = \Psi_{\mathcal{PA}_4}, \ \zeta_1 = \Psi_{\mathcal{PA}_5}, \ \zeta_2 = \Psi_{\mathcal{PA}_6}, \ \varphi = \Psi_{\mathcal{PD}_4}, \]
and Proposition 6.24, it suffices to show that
\[ \{(f, l_p) \in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(f, l_p) = 0\} = \overline{\mathcal{PD}_5}. \quad (7.3) \]
Let us show that the left hand side of (7.3) is a subset of the right hand side. We claim that
\[ \overline{\mathcal{PA}_4} \cap \overline{\mathcal{PD}_4} = \emptyset. \quad (7.4) \]
To see this, first we observe that if \((f, l_p) \in \overline{\mathcal{PD}_4}\) then \(\Psi_{\mathcal{PD}_4}(f, l_p) = 0\) and \(\Psi_{\mathcal{PD}_5}(f, l_p) \neq 0\). Therefore,
\[ \Psi_{\mathcal{PA}_4}(f, l_p) = \Psi_{\mathcal{PD}_4}(f, l_p)\Psi_{\mathcal{PD}_5}(f, l_p) - 3\Psi_{\mathcal{PD}_4}(f, l_p)^2 = -3\Psi_{\mathcal{PD}_5}(f, l_p)^2 \neq 0. \]
This implies that if \((\tilde{f}(t), l_p(t))\) lies in a small neighborhood of \((f, l_p)\) then \(\Psi_{\mathcal{PA}_4}((\tilde{f}(t), l_p(t))) \neq 0\), proving (7.4). By Lemma 7.1, statement 9 we have \(\overline{\mathcal{PA}_4} \subset \overline{\mathcal{PA}_3}\). Therefore,
\[ \{(f, l_p) \in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(f, l_p) = 0\} \subset \{(f, l_p) \in \overline{\mathcal{PA}_3} : \Psi_{\mathcal{PD}_4}(f, l_p) = 0\}. \]
The right hand side above equals
\[ \mathcal{PD}_4 \cup \overline{\mathcal{PD}_5} \]
by (7.2) and Lemma 7.1, statement 4. Hence,
\[ \{(f, l_p) \in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(f, l_p) = 0\} \subset \overline{\mathcal{PD}_5} \]
by (7.4). This proves that the left hand side of (7.3) is a subset of the right hand side. For the converse note since \(\overline{\mathcal{PA}_4}\) is a closed set, it suffices to show that
\[ \{(\tilde{f}, l_p) \in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_p) = 0\} \supset \mathcal{PD}_5. \quad (7.5) \]
We will simultaneously prove statement (7.5) and also prove the following statement
\[ \overline{\mathcal{PA}_5} \cap \mathcal{PD}_5 = \emptyset. \quad (7.6) \]
Since \(\mathcal{PD}_5\) and \(\mathcal{PA}_5\) are both subsets of \(\overline{\mathcal{PA}_3}\), we can consider closures inside \(\overline{\mathcal{PA}_3}\).

Claim 7.2. Let \((\tilde{f}, l_p) \in \mathcal{PD}_5\). Then there exists a solution \((\tilde{f}(t), l_p(t)) \in \overline{\mathcal{PA}_3}\) near \((f, l_p)\) to the set of equations
\[ \Psi_{\mathcal{PD}_4}(\tilde{f}(t), l_p(t)) \neq 0, \ \Psi_{\mathcal{PA}_4}(\tilde{f}(t), l_p(t)) = 0. \quad (7.7) \]
Moreover, whenever such a solution \((\tilde{f}(t), l_p(t))\) is sufficiently close to \((f, l_p)\) it lies in \(\mathcal{PA}_4\), i.e., \(\Psi_{\mathcal{PA}_4}(\tilde{f}(t), l_p(t)) \neq 0\). In particular \((\tilde{f}(t), l_p(t))\) does not lie in \(\mathcal{PA}_5\).
It is easy to see that claim 7.2 proves statements (7.5) and (7.6) simultaneously.

**Proof:** Let \( v \in \tilde{\gamma}, \tilde{w} \in \pi^*T\mathbb{P}^2/\tilde{\gamma} \) and \( f_{ij} \) be as defined in (4.5), subsection 4.2. Equation (7.7) is a functional equation since the quantities \( \Psi_{\mathcal{P}_D}(\tilde{f}(t), l_p(t)) \) and \( \Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t)) \) are functionals, i.e., they act on vectors \( v \) and \( \tilde{w} \) and produce a number. We will first solve the corresponding equation

\[
\{\Psi_{\mathcal{P}_D}(\tilde{f}(t), l_p(t))\}(p \otimes f^{\otimes d} \otimes \tilde{w}^{\otimes 2}) = f_{02}(t) \neq 0
\]

\[
\{\Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t))\}(p \otimes f^{\otimes d} \otimes v^{\otimes 2} \otimes \tilde{w}) = f_{02}(t)A^4_f(t) = 0. \tag{7.8}
\]

In (7.8) equality holds as numbers. It is easy to see that the only solutions to (7.8) are of the form

\[
f_{21}(t) = u, \quad f_{02}(t) = \frac{3u^2}{f_{40}(t)}. \tag{7.9}
\]

Equation (7.9) implies that the only solutions to the functional equation (7.7) is of the form

\[
\Psi_{\mathcal{P}_D}^4(\tilde{f}(t), l_p(t)) = t, \quad \Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t)) = \frac{3t^2}{\Psi_{\mathcal{P}_D}(\tilde{f}(t), l_p(t))}. \tag{7.10}
\]

where equality holds as functionals.

**Remark 7.3.** To avoid confusion, let us explain our notation carefully. The notation \( (\tilde{f}(t), l_p(t)) \) is simply used to indicate that \( (\tilde{f}(t), l_p(t)) \) is some point sufficiently close to \( (\tilde{f}, l_p) \). We denote the functional \( \Psi_{\mathcal{P}_D}(\tilde{f}(t), l_p(t)) \) by the letter \( t \). Hence, \( t \) is close to the zero functional, since \( \Psi_{\mathcal{P}_D}(\tilde{f}, l_p) = 0 \). Next, we denote the number

\[
\{\Psi_{\mathcal{P}_D}^4(\tilde{f}(t), l_p(t))\}(v^{\otimes 2} \otimes \tilde{w})
\]

by the symbol \( f_{21}(t) \). This number is close to the number zero. We also need to denote this number with some symbol. We decided to use the symbol \( u \). Hence, we have this seemingly awkward equation \( f_{21}(t) = u \).

To summarize, \( t \) is functional, while \( u \) is a number. Now comes a crucial observation: the sections

\[
\Psi_{\mathcal{P}_D} : \mathcal{P}_A \longrightarrow \mathcal{L}_{\mathcal{P}_D} \quad \text{and} \quad \Psi_{\mathcal{P}_D}^{-1} : \mathcal{L}_{\mathcal{P}_D} \longrightarrow \mathcal{P}_D
\]

are transverse to the zero set (cf. Proposition 6.22). Since \( \Psi_{\mathcal{P}_D}^{-1}(0) \) is a smooth manifold, we can extend the section \( \Psi_{\mathcal{P}_D}^4 \) outside a small neighborhood of \( \Psi_{\mathcal{P}_D}^{-1}(0) \) using the exponential map (recall that \( \Psi_{\mathcal{P}_D}^{-1} \) is well defined only on \( \Psi_{\mathcal{P}_D}^{-1}(0) \)). Therefore, there exists a solution \( (\tilde{f}(t), l_p(t)) \) close to \( (\tilde{f}, l_p) \) to (7.10). This proves our first assertion. Now we need to show that any such solution satisfies the condition \( \Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t)) \neq 0 \) if \( t \) is sufficiently small. Observe that

\[
f_{02}(t)^2A^4_f(t) = 15f_{12}(t)u^2 + O(u^3) \quad \text{using (7.9)},
\]

\[
\Rightarrow \quad \Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t)) = 15\Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t))t^2 + O(t^3) \tag{7.11}
\]

Since \( (\tilde{f}, l_p) \in \mathcal{P}_D \), we get that \( \Psi_{\mathcal{P}_A}(\tilde{f}, l_p) \neq 0 \) (cf. Proposition 6.26). Hence, by (7.11), if \( t \) is sufficiently small then \( \Psi_{\mathcal{P}_A}(\tilde{f}(t), l_p(t)) \neq 0 \). This proves claim 7.2. \( \square \)

Before proving the next Lemma, we prove a corollary which will be used in the proof of (3.7). Since this corollary follows immediately from the previous discussion, we prove it here.
Corollary 7.4. Let $\mathcal{W} \rightarrow \mathcal{D} \times \mathbb{P}\mathbb{TP}^2$ be a vector bundle such that the rank of $\mathcal{W}$ is same as the dimension of $\mathcal{PD}_5$ and $\mathcal{Q} : \mathcal{D} \times \mathbb{P}\mathbb{TP}^2 \rightarrow \mathcal{W}$ a generic smooth section. Suppose $(\tilde{f}, l_{\tilde{p}}) \in \mathcal{PD}_5 \cap \mathcal{Q}^{-1}(0)$. Then the section

$$\Psi_{\mathcal{PA}_5} \oplus \mathcal{Q} : \overline{\mathcal{PA}}_1 \rightarrow \mathbb{L}_{\mathcal{PA}_5} \oplus \mathcal{W}$$

vanishes around $(\tilde{f}, l_{\tilde{p}})$ with a multiplicity of 2.

Proof: Since the section $\mathcal{Q}$ is generic, $\mathcal{Q}^{-1}(0)$ intersects $\mathcal{PD}_5$ transversely. Since the rank of $\mathcal{W}$ is equal to the dimension of $\mathcal{PD}_5$ there exists a unique solution $(\tilde{f}(t), l_{\tilde{p}}(t)) \in \overline{\mathcal{PA}}_3$ near $(\tilde{f}, l_{\tilde{p}})$ to the set of equations

$$\Psi_{\mathcal{PD}_5}(\tilde{f}(t), l_{\tilde{p}}(t)) = t,$$

$$\Psi_{\mathcal{PD}_4}(\tilde{f}(t), l_{\tilde{p}}(t)) = \frac{3t^2}{\Psi_{\mathcal{PD}_6}(\tilde{f}(t), l_{\tilde{p}}(t))},$$

$$\mathcal{Q}(\tilde{f}(t), l_{\tilde{p}}(t)) = 0.$$  

The claim follows from (7.10) combined with the added condition $\mathcal{Q}(\tilde{f}(t), l_{\tilde{p}}(t)) = 0$. Equation (7.11) now proves our claim, since $\Psi_{\mathcal{PD}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0$. \[\square\]

Remark 7.5. This idea is due to Aleksey Zinger - the crucial observation that we can use the transversality of the bundle sections to describe the neighborhood of a point.

Proof of Lemma 7.1 (11): By Lemma 6.2 applied to $M = \overline{\mathcal{PA}}_3$, $\zeta_0 = \Psi_{\mathcal{PA}_4}$, $\zeta_1 = \Psi_{\mathcal{PA}_5}$, $\zeta_2 = \Psi_{\mathcal{PA}_6}$, $\zeta_3 = \Psi_{\mathcal{PA}_7}$, $\varphi = \Psi_{\mathcal{PD}_4}$, and Proposition 6.24, it suffices to show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0\} = \overline{\mathcal{PD}}_6 \cup \overline{\mathcal{PE}}_6.$$  

(7.12)

We will do this in two steps. We will show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\} = \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PD}}_6 : \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\}.$$  

(7.13)

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) = 0\} = \overline{\mathcal{PE}}_6.$$  

(7.14)

It follows that (7.13) and (7.14) imply (7.12). To see this, note that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PD}}_6 : \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) = 0\} \subset \overline{\mathcal{PE}}_6 \quad \text{using Corollary 6.33 and 6.30.}$$  

(7.15)

It is now easy to see that (7.15), (7.13) and (7.14) imply (7.12).

We will now start with the proof of (7.13). We will show that the left hand side of (7.13) is a subset of the right hand side. This follows from (7.6). By Lemma 7.1, statement 10 we know that $\overline{\mathcal{PA}}_5 \subset \overline{\mathcal{PA}}_4$. Therefore, in conjunction with (7.3), we gather that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0\} \subset \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_4 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0\} = \overline{\mathcal{PD}}_5.$$  

The right hand side above equals $\mathcal{PD}_5 \cup \overline{\mathcal{PD}}_6 \cup \overline{\mathcal{PE}}_6$ by Lemma 7.1, statement 6. Hence, by (7.6)

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\} \subset \{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PD}}_6 : \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\}.$$  

Now we will show the converse. Since $\overline{\mathcal{PA}}_5$ is a closed space, it suffices to show that

$$\{(\tilde{f}, l_{\tilde{p}}) \in \overline{\mathcal{PA}}_5 : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_{\tilde{p}}) = 0, \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\} \subset \{(\tilde{f}, l_{\tilde{p}}) \in \mathcal{PD}_6 : \ \Psi_{\mathcal{PE}_6}(\tilde{f}, l_{\tilde{p}}) \neq 0\}.$$  

(7.16)

As before, we will simultaneously prove (7.16) and also prove that

$$\overline{\mathcal{PA}}_6 \cap \mathcal{PD}_6 = \emptyset.$$  

(7.17)
Claim 7.6. Let \((\tilde{f}, \tilde{l}_p) \in \mathcal{PD}_6\). Then there exists a solution \((\tilde{f}(t), \tilde{l}_p(t)) \in \overline{\mathcal{PA}_3}\) near \((\tilde{f}, \tilde{l}_p)\) to the set of equations

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0, \quad \Psi_{\mathcal{PA}_4}(\tilde{f}(t), \tilde{l}_p(t)) = 0, \quad \Psi_{\mathcal{PA}_5}(\tilde{f}(t), \tilde{l}_p(t)) = 0. \tag{7.18}
\]

Moreover, whenever such a solution \((\tilde{f}(t), \tilde{l}_p(t))\) is sufficiently close to \((\tilde{f}, \tilde{l}_p)\) it lies in \(\mathcal{PA}_5\), i.e., \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0\). In particular, \((\tilde{f}(t), \tilde{l}_p(t))\) does not lie in \(\mathcal{PA}_6\).

It is clear that claim 7.6 proves (7.16) and (7.17) simultaneously.

Proof: As before, we will first solve the equation

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_{4}^{f(t)} = 0, \quad f_{02}(t)^2A_{5}^{f(t)} = 0. \tag{7.19}
\]

The only solutions to (7.19) are of the form

\[
f_{02}(t) = u \quad \Rightarrow \quad f_{21}(t) = \left( \frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D_{7}^{f(t)}}}{15f_{12}(t)} \right) u \quad \Rightarrow \quad f_{40}(t) = 3 \left( \frac{5f_{31}(t) \pm \sqrt{-15f_{12}(t)D_{7}^{f(t)}}}{15f_{12}(t)} \right)^2 u. \tag{7.20}
\]

The second equation comes from solving a quadratic arising from \(f_{02}(t)^2A_{5}^{f(t)} = 0\) while the third is from solving \(f_{02}(t)A_{4}^{f(t)} = 0\) and using \(f_{21}\) from the second equation. Since \((\tilde{f}, \tilde{l}_p) \in \mathcal{PD}_6\) we know that \(f_{12} \neq 0\) and \(D_{7}^{f} \neq 0\). Hence, there are always two solutions for \((f_{21}(t), f_{40}(t))\). Equation (7.20) implies that the only solutions to the functional equation (7.18) is of the form

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t), \tilde{l}_p(t)) = t \\
\Psi_{\mathcal{PD}_5}(\tilde{f}(t), \tilde{l}_p(t)) = \left( \frac{5\Psi_{\mathcal{PE}_5}(\tilde{f}(t), \tilde{l}_p(t)) \pm \sqrt{-15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), \tilde{l}_p(t))\Psi_{\mathcal{PD}_7}(\tilde{f}(t), \tilde{l}_p(t))}}{15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), \tilde{l}_p(t))} \right) t \quad \Rightarrow \quad \Psi_{\mathcal{PD}_6}(\tilde{f}(t), \tilde{l}_p(t)) = 3 \left( \frac{5\Psi_{\mathcal{PE}_5}(\tilde{f}(t), \tilde{l}_p(t)) \pm \sqrt{-15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), \tilde{l}_p(t))\Psi_{\mathcal{PD}_7}(\tilde{f}(t), \tilde{l}_p(t))}}{15\Psi_{\mathcal{PE}_6}(\tilde{f}(t), \tilde{l}_p(t))} \right)^2 t. \tag{7.21}
\]

where equality holds as functionals. Since the sections

\[
\Psi_{\mathcal{PD}_4} : \overline{\mathcal{PA}_3} \longrightarrow \mathbb{L}_{\mathcal{PD}_4}, \quad \Psi_{\mathcal{PD}_5}^{-1} : \Psi_{\mathcal{PE}_5}(0) \longrightarrow \mathbb{L}_{\mathcal{PD}_5} \quad \text{and} \quad \Psi_{\mathcal{PD}_6} : \Psi_{\mathcal{PE}_6}(0) \longrightarrow \mathbb{L}_{\mathcal{PD}_6}
\]

are transverse to the zero set (as proved in Proposition 6.19 and 6.26), there exists a solution \((\tilde{f}(t), \tilde{l}_p(t))\) close to \((\tilde{f}, \tilde{l}_p)\) to (7.21). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0\) if \(t\) is sufficiently small. To prove that we observe

\[
f_{02}(t)^3A_{3}^{f(t)} = \frac{D_{7}^{f(t)}}{f_{12}(t)}u^2 + O(u^3) \quad \Rightarrow \quad \Psi_{\mathcal{PA}_6}(\tilde{f}(t), \tilde{l}_p(t)) = \frac{\Psi_{\mathcal{PD}_7}(\tilde{f}(t), \tilde{l}_p(t))}{\Psi_{\mathcal{PE}_6}(\tilde{f}(t), \tilde{l}_p(t))}t^2 + O(t^3) \tag{7.22}
\]
Since \((\tilde{f},\tilde{l}_p)\in \mathcal{PD}_6\), we get that \(\Psi_{\mathcal{PD}_6}(\tilde{f},\tilde{l}_p),\Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p)\neq 0\) (see Proposition 6.32). Hence, (7.22) implies that if \(t\) is sufficiently small \(\Psi_{\mathcal{PA}_5}(\tilde{f}(t),\tilde{l}_p(t)) \neq 0\), which proves claim 7.6.

Before proving (7.14), we prove a corollary which will be used in the proof of (3.8).

**Corollary 7.7.** Let \(\mathbb{W} \rightarrow \mathcal{D} \times \mathbb{P}^2\) be a vector bundle such that the rank of \(\mathbb{W}\) is same as the dimension of \(\mathcal{PD}_6\). Let \(Q : \mathcal{D} \times \mathbb{P}^2 \rightarrow \mathbb{W}\) be a generic smooth section. Suppose \((\tilde{f},\tilde{l}_p)\in \mathcal{PD}_6 \cap Q^{-1}(0)\). Then the section

\[
\Psi_{\mathcal{PA}_5} \oplus Q : \overline{\mathcal{A}_5} \rightarrow L_{\mathcal{PA}_5} \oplus \mathbb{W}
\]

vanishes around \((\tilde{f},\tilde{l}_p)\) with a multiplicity of 4.

**Proof:** As before, in the proof of claim 7.4, this follows from (7.22) and the fact that \(Q^{-1}(0)\) intersects \(\mathcal{PD}_6\) transversely. Each branch of \(\sqrt{f_1^2D_1^f}\) contributes with a multiplicity of 2. Hence, the total multiplicity is 4. \(\Box\)

Next we will prove (7.14). We will show that the left hand side is a subset of the right hand side. Note that \(\overline{\mathcal{PA}_5} \subset \overline{\mathcal{PA}_4}\) is implied by Lemma 7.1, statement 10, whence

\[
\{(\tilde{f},\tilde{l}_p) \in \overline{\mathcal{PA}_5} : \Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\} \subset \{(\tilde{f},\tilde{l}_p) \in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\}.
\]

The right hand side above can be simplified (7.3) and Corollary 6.30 as follows:

\[
\{(\tilde{f},\tilde{l}_p)\in \overline{\mathcal{PA}_4} : \Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\} = \{(\tilde{f},\tilde{l}_p) \in \overline{\mathcal{PD}_5} : \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\} = \overline{\mathcal{PE}_6}.
\]

This implies

\[
\{(\tilde{f},\tilde{l}_p) \in \overline{\mathcal{PA}_5} : \Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\} \subset \overline{\mathcal{PE}_6}.
\]

Now we prove the converse. Since \(\overline{\mathcal{PA}_5}\) is a closed set, it suffices to show that

\[
\{(\tilde{f},\tilde{l}_p) \in \overline{\mathcal{PA}_5} : \Psi_{\mathcal{PD}_4}(\tilde{f},\tilde{l}_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f},\tilde{l}_p) = 0\} \subset \mathcal{PE}_6. \tag{7.23}
\]

As before, we will simultaneously prove (7.23) and also prove the following:

\[
\overline{\mathcal{PA}_6} \cap \mathcal{PE}_6 = \emptyset. \tag{7.24}
\]

**Claim 7.8.** Let \((\tilde{f},\tilde{l}_p)\in \mathcal{PE}_6\). Then there exists a solution \((\tilde{f}(t),\tilde{l}_p(t))\in \overline{\mathcal{PA}_3}\) near \((\tilde{f},\tilde{l}_p)\) to the set of equations

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t),\tilde{l}_p(t)) \neq 0, \quad \Psi_{\mathcal{PA}_4}(\tilde{f}(t),\tilde{l}_p(t)) = 0, \quad \Psi_{\mathcal{PA}_5}(\tilde{f}(t),\tilde{l}_p(t)) = 0. \tag{7.25}
\]

Moreover, whenever such a solution \((\tilde{f}(t),\tilde{l}_p(t))\) is sufficiently close to \((\tilde{f},\tilde{l}_p)\), it lies in \(\mathcal{PA}_5\), i.e., \(\Psi_{\mathcal{PA}_6}(\tilde{f}(t),\tilde{l}_p(t)) \neq 0\). In particular \((\tilde{f}(t),\tilde{l}_p(t))\) does not lie in \(\mathcal{PA}_6\).

Note that claim 7.8 proves (7.23) and (7.24) simultaneously.

**Proof:** As before, we will first solve the equations

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_4^{(t)} = 0, \quad f_{02}(t)^2 A_5^{(t)} = 0. \tag{7.26}
\]
It is easy to see that the only solutions to (7.26) are of the form
\[ f_{21}(t) = u, \quad f_{02} = \frac{3u^2}{f_{40}(t)}, \quad \text{and} \quad f_{12} = \frac{2f_{31}(t)}{f_{40}(t)} u - \frac{3f_{50}(t)}{5f_{40}(t)^2} u^2. \] (7.27)

Note that since \((\tilde{f}, l_p) \in \mathcal{P}E_6\) we get that \(f_{40} \neq 0\). Equation (7.27) implies that the only solutions to the functional equation (7.25) is of the form
\[ \Psi_{PD_6}(\tilde{f}(t), l_p(t)) = t \]
\[ \Psi_{PD_4}(\tilde{f}(t), l_p(t)) = \frac{3t^2}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))} \]
\[ \Psi_{PE_6}(\tilde{f}(t), l_p(t)) = \frac{2\Psi_{PE_6}(\tilde{f}(t), l_p(t))}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))} t + O(t^2) \] (7.28)

where equality holds as functionals. Since the sections
\[ \Psi_{PD_4} : \overline{\mathcal{A}_3} \rightarrow \mathbb{L}_{PD_4}, \quad \Psi_{PD_6}^{-1} : \mathbb{L}_{PD_6} \rightarrow \mathbb{L}_{PD_5} \quad \text{and} \quad \Psi_{PE_6} : \Psi_{PD_6}^{-1} \rightarrow \mathbb{L}_{PE_6} \]
are transverse to the zero set (as proved in Proposition 6.19 and 6.26), there exists a solution \((\tilde{f}(t), l_p(t))\) close to \((\tilde{f}, l_p)\) to (7.28). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{PA_6}(\tilde{f}(t), l_p(t)) \neq 0\) if \(t\) is sufficiently small. To prove that we observe
\[ f_{02}(t)^3 A_6^{f(t)} = -15f_{02}(t)u^3 + O(u^4) \quad \text{using (7.27).} \]
\[ \Rightarrow \quad \Psi_{PA_6}(\tilde{f}(t), l_p(t)) = -15\Psi_{PA_6}(\tilde{f}(t), l_p(t))t^3 + O(t^4) \] (7.29)

Since \((\tilde{f}, l_p) \in \mathcal{P}E_6\), we get that \(\Psi_{PA_6}(\tilde{f}, l_p) \neq 0\) (see Proposition 6.29). Hence, (7.29) implies that if \(t\) is sufficiently small \(\Psi_{PA_6}(\tilde{f}(t), l_p(t)) \neq 0\) which proves claim 7.8.

We now prove a corollary which will be used in proving (3.8).

**Corollary 7.9.** Let \(\mathbb{W} \rightarrow D \times \mathbb{P}T\mathbb{P}^2\) be a vector bundle such that the rank of \(\mathbb{W}\) is same as the dimension of \(\mathcal{P}E_6\). Let \(\mathbb{Q} : D \times \mathbb{P}T\mathbb{P}^2 \rightarrow \mathbb{W}\) be a generic smooth section. Suppose \((\tilde{f}, l_p) \in \mathcal{P}E_6 \cap \mathbb{Q}^{-1}(0)\). Then the section
\[ \Psi_{PA_6} \oplus \mathbb{Q} : \overline{\mathcal{A}_5} \rightarrow \mathbb{L}_{PA_6} \oplus \mathbb{W} \]
vanishes around \((\tilde{f}, l_p)\) with a multiplicity of 3.

**Proof:** Follows from the fact that \(\mathbb{Q}^{-1}(0)\) intersects \(\mathcal{P}E_6\) transversely and (7.29).

**Proof of Lemma 7.1 (12):** By Lemma 6.2 applied to
\[ M = \overline{\mathcal{A}_3}, \quad \zeta_0 = \Psi_{PA_4}, \quad \zeta_1 = \Psi_{PA_5}, \quad \zeta_2 = \Psi_{PA_6}, \quad \zeta_3 = \Psi_{PA_7}, \quad \zeta_4 = \Psi_{PA_8}, \quad \varphi = \Psi_{PD_4}, \]
and Proposition 6.24, it suffices to show that
\[ \{ (\tilde{f}, l_p) \in \overline{\mathcal{A}_6} : \Psi_{PD_4}(\tilde{f}, l_p) = 0 \} = \overline{\mathcal{D}_7} \cup \overline{\mathcal{E}_7} \cup \overline{\mathcal{A}_8^{\#3}}. \] (7.30)
We will do this in three steps. We will show that

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) \neq 0\} \equiv \{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}D_7} : \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) \neq 0\}
\]  \hspace{1cm} (7.31)

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{\mathcal{X}_6}(\tilde{f}, l_\tilde{p}) \neq 0\} \equiv \{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}E_7} : \Psi_{\mathcal{X}_6}(\tilde{f}, l_\tilde{p}) \neq 0\}
\]  \hspace{1cm} (7.32)

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{\mathcal{X}_6}(\tilde{f}, l_\tilde{p}) = 0\} \equiv \overline{X}_8^{\#}\]
\]  \hspace{1cm} (7.33)

Observe that (7.31), (7.32) and (7.33) prove (7.30). To see this, note that

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}E_7} : \Psi_{\mathcal{X}_6}(\tilde{f}, l_\tilde{p}) = 0\} \subset \overline{X}_8^{\#}
\]
\]  \hspace{1cm} (7.34)

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}D_7} : \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) = 0\} \subset \overline{\mathcal{E}_7} \cup \overline{X}_8^{\#}.
\]
\]  \hspace{1cm} (7.35)

Equation (7.34) follows from Corollary 6.38 and 6.43. Equation (7.35) follows from Proposition 6.35 and Corollary 6.38 and 6.43. It is now easy to see that (7.31), (7.32) and (7.33) combined with (7.34) and (7.35) prove (7.30).

Let us prove (7.31). To see why the left hand side is a subset of the right hand side, recall (7.17). We also recall a result:

\[
\overline{X}_8^{\#} = \{(\tilde{f}, l_\tilde{p}) \in D \times \mathbb{R}\mathbb{T}^2 : \Psi_{\mathcal{A}_0}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{\mathcal{A}_1}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{\mathcal{D}_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{\mathcal{X}_6}(\tilde{f}, l_\tilde{p}) = 0\}.
\]

Now observe that \(\overline{\mathcal{P}A_6} \subset \overline{\mathcal{P}A_5}\) is implied by Lemma 7.1, statement 11, whence

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0\} \subset \{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_5} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0\}.
\]

By (7.12), the right hand side above equals \(\overline{\mathcal{P}D_6} \cup \overline{\mathcal{E}_6}\). But by Lemma 7.1, statement 7 and statement 5, we get

\[
\overline{\mathcal{P}D_6} \cup \overline{\mathcal{E}_6} = \mathcal{P}D_6 \cup \mathcal{P}E_6 \cup \overline{\mathcal{P}D_7} \cup \overline{\mathcal{P}E_7} \cup \overline{X}_8^{\#}.
\]

This implies, by (7.17) and (7.24), that

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) \neq 0\} \subset \overline{\mathcal{P}D_7}.
\]

Hence, the left hand side of (7.31) is a subset of its right hand side.

Next let us show that the right hand side of (7.31) is a subset of its left hand side. Since \(\overline{\mathcal{P}A_6}\) is a closed set, it suffices to show that

\[
\{(\tilde{f}, l_\tilde{p}) \in \overline{\mathcal{P}A_6} : \Psi_{PD_4}(\tilde{f}, l_\tilde{p}) = 0, \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) \neq 0\} \supset \{(\tilde{f}, l_\tilde{p}) \in \mathcal{P}D_7 : \ \Psi_{PE_6}(\tilde{f}, l_\tilde{p}) \neq 0\}.
\]
\]  \hspace{1cm} (7.36)

We will simultaneously prove (7.36) and also prove the following:

\[
\overline{\mathcal{P}A_7} \cap \mathcal{P}D_7 = \emptyset.
\]
\]  \hspace{1cm} (7.37)

**Claim 7.10.** Let \((\tilde{f}, l_\tilde{p}) \in \mathcal{P}D_7\). Then there exists a solution \((\tilde{f}(t), l_\tilde{p}(t))\) in \(\overline{\mathcal{P}A_3}\) near \((\tilde{f}, l_\tilde{p})\) to the set of equations

\[
\Psi_{PD_4}(\tilde{f}(t), l_\tilde{p}(t)) \neq 0, \Psi_{\mathcal{A}_4}(\tilde{f}(t), l_\tilde{p}(t)) = 0, \Psi_{\mathcal{D}_4}(\tilde{f}(t), l_\tilde{p}(t)) = 0, \Psi_{\mathcal{A}_6}(\tilde{f}(t), l_\tilde{p}(t)) = 0.
\]
\]  \hspace{1cm} (7.38)

Moreover, whenever such a solution \((\tilde{f}(t), l_\tilde{p}(t))\) is sufficiently close to \((\tilde{f}, l_\tilde{p})\) it lies in \(\mathcal{P}A_6\), i.e., \(\Psi_{\mathcal{A}_4}(\tilde{f}(t), l_\tilde{p}(t)) \neq 0\). In particular \((\tilde{f}(t), l_\tilde{p}(t))\) does not lie in \(\overline{\mathcal{P}A_7}\).
Note that claim 7.10 proves (7.36) and (7.37) simultaneously.

**Proof:** As before, we will first solve the equation

\[ f_{02}(t) \neq 0, \quad f_{02}(t)A_4^{f(t)} = 0, \quad f_{02}(t)^2 A_5^{f(t)} = 0 \quad \text{and} \quad f_{02}(t)^3 A_6^{f(t)} = 0. \quad (7.39) \]

We claim that the *only* solutions to (7.39) that go to zero as \( f_{02}(t) \) goes to zero are of the form

\[ f_{02}(t) = u^2 + O(u^4) \quad (7.40) \]

\[ f_{21}(t) = \frac{f_{31}(t)}{3f_{12}(t)} u^2 + \sqrt{\beta(t)} u^3 + O(u^4) \quad (7.41) \]

\[ f_{40}(t) = \frac{f_{31}(t)^2}{3f_{12}(t)^2} u^2 + O(u^3) \]

\[ \left( f_{50}(t) - \frac{5f_{31}(t)^2}{3f_{12}(t)} \right) = -15f_{12}(t)\beta(t)u^2 + O(u^3) \]

where \( \beta(t) = -\frac{f_{03}(t)f_{31}(t)^3}{162f_{12}(t)^4} + \frac{f_{22}(t)f_{31}(t)^2}{18f_{12}(t)^3} - \frac{f_{41}(t)f_{31}(t)}{18f_{12}(t)^2} + \frac{f_{60}(t)}{90f_{12}(t)} \quad (7.42) \)

for just *one* choice of a branch of \( \sqrt{\beta(t)} \). We will see shortly that \( \beta(t) \neq 0 \). The value for \( f_{40} \) can be calculated using \( f_{21}, f_{02} \) and \( f_{02}(t)A_4^{f(t)} = 0 \) while the next equation follows by using the first three equations and \( f_{02}(t)^2 A_5^{f(t)} = 0 \). Let us now explain how we obtain (7.40) and (7.41). The equation \( f_{02}(t)^3 A_6^{f(t)} = 0 \) is a cubic equation in \( f_{21}(t) \), i.e., it is of the form

\[ A_3(f_{02}(t))f_{21}(t)^3 + A_2(f_{02}(t))f_{21}(t)^2 + A_1(f_{02}(t))f_{21}(t) + A_0(f_{02}(t)) = 0. \]

Since \( f_{12}(t) \neq 0 \) it follows that as \( f_{02}(t) \) goes to zero \( A_2 \) remains non zero. Hence, there exists a unique holomorphic function \( P(f_{02}(t)) \), of \( f_{02}(t) \) (close to the zero function), such that if we make a change of variables

\[ f_{21}(t) = H + P(f_{02}(t)) \]

then our cubic equation becomes

\[ \hat{A}_3(f_{02}(t))H^3 + \hat{A}_2(f_{02}(t))H^2 + \hat{A}_0(f_{02}(t)) = 0. \]

The argument is same as in Lemma 5.5, where we show the existence of \( B(x) \) (it is simply an application of the Implicit Function Theorem). Since \( \hat{A}_2(0) \neq 0 \), we can divide out by \( \hat{A}_2(f_{20}(t)) \) and get

\[ \hat{A}_3(f_{02}(t))H^3 + H^2 + \hat{A}_0(f_{02}(t)) = 0. \quad (7.43) \]

By a simple calculation, it is easy to see that

\[ \hat{A}_0(f_{02}(t)) = -\beta(t)f_{02}(t)^3 + O(f_{02}(t)^4). \]

Assuming \( \beta(t) \neq 0 \) we can make a change of variables

\[ \hat{f}_{02} = f_{02}(t) \left( -\beta(t)f_{02}(t)^2 \right)^{\frac{3}{2}} , \quad \hat{H} = H(1 + \hat{A}_3(f_{02}(t))H)^{\frac{1}{2}}. \]

\[^{11}\text{In other words, choosing the other branch of the square root does not give us any extra solutions.}\]
Our cubic equation (7.43) now becomes

\[ \hat{H}^2 - \beta(t)f_{02}^3 = 0. \] (7.44)

Now, it is easy to see that the only small solutions to (7.44) are of the form

\[ \hat{H} = \sqrt{\beta(t)}u^3, \quad \hat{f}_{02} = u^2 \]

for just one choice of \( \sqrt{\beta(t)} \). In other words, by choosing just one branch of \( \sqrt{\beta(t)} \), we get all the possible small solutions of (7.44). By inverting the change of coordinates, \((H, f_{02}) \rightarrow (\hat{H}, \hat{f}_{02})\), we conclude that the only small solutions to (7.43) are of the form

\[ H = \sqrt{\beta(t)}u^3 + O(u^4), \quad f_{02}(t) = u^2 + O(u^4). \]

(Note that the transformation \((H, f_{02}) \rightarrow (\hat{H}, \hat{f}_{02})\) is identity to first order, i.e. the Jacobian matrix of this transformation at the origin is the identity matrix.) It follows that

\[ P(f_{02}(t)) = -\frac{f_{31}(t)}{3f_{12}(t)}f_{02}(t) + O(f_{02}(t)^2) = -\frac{f_{31}(t)}{3f_{12}(t)}u^2 + O(u^4). \]

This gives us (7.41) and (7.40). It remains to show that \( \beta(t) \neq 0 \). To see this, note that

\[ \beta(t) = \frac{D_{5}^{f}(t)}{90f_{12}(t)} - \frac{f_{30}(t)f_{31}(t)D_{7}^{f}(t)}{54f_{12}(t)^{3}} \] (7.45)

Since \((\hat{f}, \hat{p}_p) \in PD_7, D_{7}^{f} = 0 \) and \( D_{8}^{f} \neq 0 \). Therefore, by (7.45) \( \beta(t) \neq 0 \) for small \( t \).

Equation (7.42) combined with (7.45) imply that the only solutions to the functional equation (7.38) are of the form

\[
\begin{align*}
\Psi_{PD_4}(\hat{f}(t), l_p(t)) &= t^2 + O(t^4) \\
\Psi_{PD_5}(\hat{f}(t), l_p(t)) &= \frac{\Psi_{PE_6}(\hat{f}(t), l_p(t))}{3\Psi_{PE_6}(\hat{f}(t), l_p(t))}t^3 + O(t^4) \\
\Psi_{PD_6}(\hat{f}(t), l_p(t)) &= \frac{\Psi_{PE_6}(\hat{f}(t), l_p(t))^2}{3\Psi_{PE_6}(\hat{f}(t), l_p(t))}t^2 + O(t^3) \\
\Psi_{PD_7}(\hat{f}(t), l_p(t)) &= -15\Psi_{PE_6}(\hat{f}(t), l_p(t))B(\hat{f}(t), l_p(t))t^2 + O(t^3),
\end{align*}
\]

where

\[ B(\hat{f}(t), l_p(t)) = \frac{\Psi_{PD_6}(\hat{f}(t), l_p(t))}{90\Psi_{PE_6}(\hat{f}(t), l_p(t))} - \frac{\Psi_{PD_4}(\hat{f}(t), l_p(t))\Psi_{PE_6}(\hat{f}(t), l_p(t))\Psi_{PD_4}(\hat{f}(t), l_p(t))}{54\Psi_{PE_6}(\hat{f}(t), l_p(t))^3} \]

and equality holds as functionals. Since the sections

\[ \Psi_{PD_4} : \overline{AP_3} \rightarrow \mathbb{L}_{PD_4}, \quad \Psi_{PD_5}^L : \Psi_{PD_4}^{-1}(0) \rightarrow \mathbb{L}_{PD_5}, \quad \Psi_{PD_6} : \Psi_{PD_5}^{-1}(0) \rightarrow \mathbb{L}_{PD_6}, \quad \Psi_{PD_7} : \Psi_{PD_6}^{-1}(0) - \Psi_{PE_6}^{-1}(0) \rightarrow \mathbb{L}_{PD_7} \]

are transverse to the zero set (as proved in Proposition 6.19, 6.26 and 6.35 ), there exists a solution \((\hat{f}(t), l_p(t))\) close to \((\hat{f}, \hat{p}_p)\) to (7.21). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{PA_4}(\hat{f}(t), l_p(t)) \neq 0 \) if \( t \) is sufficiently small. To prove that we observe

\[ f_{02}^4 A_7^{f(t)} = 630f_{12}(t)^2\beta(t)^2u^6 + O(u^7) \quad \text{using (7.42)} \]

\[ \implies \Psi_{PA_4}(\hat{f}(t), l_p(t)) = 630\Psi_{PE_6}(\hat{f}(t), l_p(t))B(\hat{f}(t), l_p(t))t^6 + O(t^7) \] (7.47)
Since \((\tilde{f}, l_p) \in \mathcal{PD}_7\), we get that \(B(\tilde{f}(t), l_p(t))^2 \neq 0\) and \(\Psi_{\mathcal{PE}}(\tilde{f}(t), \tilde{p}(t)) \neq 0\). Hence, \((7.47)\) implies that if \(t\) is sufficiently small then \(\Psi_{\mathcal{PA}_7}(\tilde{f}(t), l_p(t)) \neq 0\), since all the sections are continuous. This proves claim 7.10.

Before proving (7.32), let us prove a corollary which will be used in the proof of (3.9).

**Corollary 7.11.** Let \(\mathbb{W} \to \mathcal{D} \times \mathbb{PTP}^2\) be a vector bundle such that the rank of \(\mathbb{W}\) is same as the dimension of \(\mathcal{PD}_7\). Let \(\mathcal{Q} : \mathcal{D} \times \mathbb{PTP}^2 \to \mathbb{W}\) be a generic smooth section. Suppose \((\tilde{f}, l_p) \in \mathcal{PD}_7 \cap \mathcal{Q}^{-1}(0)\). Then the section

\[
\Psi_{\mathcal{PA}_7} \oplus \mathcal{Q} : \overline{\mathcal{PA}_6} \to L_{\mathcal{PA}_7} \oplus \mathbb{W}
\]

vanishes around \((\tilde{f}, l_p)\) with a multiplicity of 6.

**Proof:** Follows from the fact that \(\mathcal{Q}^{-1}(0)\) intersects \(\mathcal{PD}_7\) transversely and \((7.47)\). \(\square\)

Next we will prove (7.32). To show that the left hand side is a subset of the right hand side note that \(\overline{\mathcal{PA}_6} \subset \overline{\mathcal{PA}_5}\) by Lemma 7.1, statement 11. Consequently,

\[
\{(\tilde{f}, l_p) : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PX}_8}(\tilde{f}, l_p) \neq 0\}
\]

is contained in

\[
\{(\tilde{f}, l_p) : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PX}_8}(\tilde{f}, l_p) \neq 0\}.
\]

But the quantity above, by (7.14), equals

\[
\{(\tilde{f}, l_p) \in \overline{\mathcal{PE}_6} : \Psi_{\mathcal{PX}_8}(\tilde{f}, l_p) \neq 0\} \subset \overline{\mathcal{PE}_6} \cup \overline{\mathcal{PE}_7} \cup \overline{\mathcal{PX}_8},
\]

where the last inclusion follows from Lemma 7.1, statement 5. Therefore, the left hand side of (7.32) is a subset of its right hand side, using (7.24).

For the converse, since \(\overline{\mathcal{PA}_6}\) is a closed set, it suffices to show that

\[
\{(\tilde{f}, l_p) \in \overline{\mathcal{PA}_6} : \Psi_{\mathcal{PD}_4}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PE}_6}(\tilde{f}, l_p) = 0, \Psi_{\mathcal{PX}_8}(\tilde{f}, l_p) \neq 0\} \supset \{(\tilde{f}, l_p) \in \mathcal{PE}_7 : \Psi_{\mathcal{PX}_8}(\tilde{f}, l_p) \neq 0\}. \tag{7.48}
\]

We will simultaneously prove this statement and also prove

\[
\overline{\mathcal{PA}_7} \cap \mathcal{PE}_7 = \varnothing. \tag{7.49}
\]

**Claim 7.12.** Let \((\tilde{f}, l_p) \in \mathcal{PE}_7\). Then there exists a solution \((\tilde{f}(t), l_p(t)) \in \overline{\mathcal{PA}_3}\) near \((\tilde{f}, l_p)\) to the set of equations

\[
\Psi_{\mathcal{PD}_4}(\tilde{f}(t), l_p(t)) \neq 0, \Psi_{\mathcal{PA}_4}(\tilde{f}(t), l_p(t)) = 0, \Psi_{\mathcal{PA}_5}(\tilde{f}(t), l_p(t)) = 0, \Psi_{\mathcal{PA}_6}(\tilde{f}(t), l_p(t)) = 0. \tag{7.50}
\]

Moreover, whenever such a solution \((\tilde{f}(t), l_p(t))\) is sufficiently close to \((\tilde{f}, l_p)\) it lies in \(\mathcal{PA}_6\), i.e., \(\Psi_{\mathcal{PA}_7}(\tilde{f}(t), l_p(t)) \neq 0\). In particular \((\tilde{f}(t), l_p(t))\) does not lie in \(\mathcal{PA}_7\).

Note that claim 7.12 proves (7.48) and (7.49) simultaneously.

**Proof:** As before, we will first solve the equation

\[
f_{02}(t) \neq 0, \quad f_{02}(t)A_4^{(t)} = 0, \quad f_{02}(t)^2A_5^{(t)} = 0, \quad f_{02}(t)^3A_6^{(t)} = 0. \tag{7.51}
\]
The only solutions to (7.51), that converge to zero as \( f_{02}(t) \) and \( f_{12}(t) \) go to zero are

\[
\begin{align*}
  f_{12}(t) &= u \\
  f_{21}(t) &= -\frac{3}{2f_{03}(t)}u^2 + O(u^3) \\
  f_{02}(t) &= -\frac{9}{4f_{31}(t)f_{03}(t)}u^3 + O(u^4) \\
  f_{40}(t) &= -\frac{3f_{31}(t)}{f_{03}(t)}u + O(u^2)
\end{align*}
\]

(7.52)

We use the first two equations and \( f_{02}(t)^3A_6^f(t) = 0 \) to obtain the third equation; the first three equations and \( f_{02}(t)A_4^f(t) = 0 \) to obtain the last equation. Let us now explain how we obtained the second equation. Using \( f_{02}(t)^2A_5^f(t) = 0 \) we can solve for \( \frac{f_{02}}{f_{21}} \) and get

\[
\frac{f_{02}(t)}{f_{21}(t)} = \frac{10f_{31}(t) - \sqrt{100f_{31}(t)^2 - 60f_{50}(t)u}}{2f_{50}(t)} = \frac{3}{2f_{31}(t)}u + O(u^2)
\]

(7.53)

It is easy to see that we never really used the fact that \( f_{50}(t) \neq 0 \); the equality of the first and last term remains valid even when \( f_{50} = 0 \). However, we do need to justify why we did not choose the other branch of the square root. We will explain that shortly. Plugging in the value of \( f_{02} \) from (7.53) in equation \( f_{02}(t)^3A_6^f(t) = 0 \) and by using the Implicit Function Theorem, we get the expression for \( f_{21}(t) \) in (7.52). And now using the value of \( f_{21}(t) \) and (7.53) we get the expression for \( f_{02}(t) \) in (7.52).

It remains to show that why we did not chose the other branch of the square root. It is easy to see that if we chose the other branch, it would imply that as \( f_{02}(t) \) and \( f_{21}(t) \) go to zero, the ratio \( L_t := \frac{f_{21}(t)}{f_{02}(t)} \) tends to a finite number \( L \), since \( f_{31} \neq 0 \). Using \( f_{03}(t)^3A_6^f(t) = 0 \) we can solve for \( f_{31}(t) \) as a quadratic equation and get that

\[
f_{31}(t) = \frac{30L_tf_{12}(t) \pm \sqrt{10} \sqrt{-15L_t^3f_{02}(t)f_{03}(t) + 45L_t^2f_{02}(t)f_{22}(t) - 15L_tf_{02}(t)f_{41}(t) + f_{02}(t)f_{60}(t)}}{10}.
\]

It is now easy to see that \( f_{31}(t) \) tends to zero as \( f_{12}(t) \) and \( f_{02}(t) \) tend to zero. This gives us a contradiction, since \( f_{31} \neq 0 \).

Since \((\hat{f}, \hat{p}) \in \mathcal{PE}_7\) we get that \( f_{03}, f_{31} \neq 0 \). Equation (7.52) implies that the only solutions to the functional equation (7.50) are of the form

\[
\begin{align*}
  \Psi_{\mathcal{PE}_6}(\hat{f}, \hat{p})(t) &= t \\
  \Psi_{\mathcal{PD}_6}^i(\hat{f}, \hat{p})(t) &= -\frac{3}{2\Psi_{\mathcal{P}\mathcal{X}_6}(\hat{f}, \hat{p})(t)}t^2 + O(t^3) \\
  \Psi_{\mathcal{PD}_4}(\hat{f}, \hat{p})(t) &= -\frac{9}{4\Psi_{\mathcal{P}\mathcal{E}_6}(\hat{f}, \hat{p})(t)\Psi_{\mathcal{P}\mathcal{X}_6}(\hat{f}, \hat{p})(t)}t^3 + O(t^4) \\
  \Psi_{\mathcal{PE}_7}(\hat{f}, \hat{p})(t) &= -\frac{3\Psi_{\mathcal{P}\mathcal{E}_6}(\hat{f}, \hat{p})(t)}{\Psi_{\mathcal{P}\mathcal{X}_6}(\hat{f}, \hat{p})(t)}t^4 + O(t^2)
\end{align*}
\]

(7.54)

where equality holds as functionals. Since the sections

\[
\begin{align*}
  \Psi_{\mathcal{PD}_4} : \mathcal{PD}_3 &\rightarrow \mathbb{L}_{\mathcal{PD}_4}, & \Psi_{\mathcal{PD}_6}^i : \Psi_{\mathcal{PD}_4}^{-1}(0) &\rightarrow \mathbb{L}_{\mathcal{PD}_5}, \\
  \Psi_{\mathcal{PE}_6} : \Psi_{\mathcal{PD}_5}^{-1}(0) &\rightarrow \mathbb{L}_{\mathcal{PE}_6}, & \Psi_{\mathcal{PE}_7} : \Psi_{\mathcal{PE}_6}^{-1}(0) &\rightarrow \mathbb{L}_{\mathcal{PE}_7}
\end{align*}
\]

47
are transverse to the zero set (as proved in Proposition 6.19, 6.26 and 6.29), there exists a solution \((\tilde{f}(t), \tilde{l}_p(t))\) close to \((f, l_p)\) for (7.54). This proves our first assertion. Next we need to show that any such solution satisfies the condition \(\Psi_{\mathcal{P}_A'}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0\) if \(t\) is sufficiently small. To prove that we observe
\[
\begin{align*}
f_0^4 A_T^4(t) &= -\frac{2835}{16 f_0^3(t)} u^7 + O(u^8) \quad \text{using (7.52)} \\
\implies \Psi_{\mathcal{P}_A'}(\tilde{f}(t), \tilde{l}_p(t)) &= -\frac{2835}{16 \Psi_{\mathcal{P}_A'}(f(t), l_p(t))} t^7 + O(t^8)
\end{align*}
\]
Since \((\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{E}_7\), we get that \(\Psi_{\mathcal{P}_A'}(\tilde{f}, \tilde{l}_p) \neq 0\). Hence, (7.55) implies that if \(t\) is sufficiently small \(\Psi_{\mathcal{P}_A'}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0\), since all the sections are continuous. This proves claim 7.12.

Before proving (7.33), let us prove a corollary which will be used in the proof of (3.9).

**Corollary 7.13.** Let \(W \longrightarrow D \times \mathbb{P}^2\) be a vector bundle such that the rank of \(W\) is same as the dimension of \(\mathcal{P}\mathcal{E}_7\). Let \(Q : D \times \mathbb{P}^2 \longrightarrow W\) be a generic smooth section. Suppose \((\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{E}_7 \cap Q^{-1}(0)\). Then the section
\[
\Psi_{\mathcal{P}_A'} + Q : \mathcal{P}\mathcal{A}_6 \longrightarrow L_{\mathcal{P}_A'} + W
\]
vanishes around \((\tilde{f}, \tilde{l}_p)\) with a multiplicity of 7.

**Proof:** Follows from the fact that \(Q^{-1}(0)\) intersects \(\mathcal{P}\mathcal{E}_7\) transversely and (7.55). 

Finally, we will prove (7.33). Let us show that the left hand side is contained in the right hand side. Note that \(\mathcal{P}\mathcal{A}_6 \subset \mathcal{P}\mathcal{A}_5\) by Lemma 7.1, statement 11. Therefore,
\[
\{(\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{A}_6 : \Psi_{\mathcal{P}_D}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_E}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_X}(\tilde{f}, \tilde{l}_p) = 0\}
\]
is contained in
\[
\{(\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{A}_5 : \Psi_{\mathcal{P}_D}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_E}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_X}(\tilde{f}, \tilde{l}_p) = 0\}.
\]
The last quantity above equals, due to (7.14), and (7.1) and Corollary 6.43
\[
\text{LHS} = \{(\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{E}_6 : \Psi_{\mathcal{P}_X}(\tilde{f}, \tilde{l}_p) = 0\} = \hat{\lambda}_8^{\#b}.
\]
This implies that the left hand side of (7.33) is a subset of its right hand side.

Next let us show that the right hand side of (7.33) is a subset of its left hand side. Since \(\mathcal{P}\mathcal{A}_6\) is a closed set, it suffices to show that
\[
\{(\tilde{f}, \tilde{l}_p) \in \mathcal{P}\mathcal{A}_6 : \Psi_{\mathcal{P}_D}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_E}(\tilde{f}, \tilde{l}_p) = 0, \Psi_{\mathcal{P}_X}(\tilde{f}, \tilde{l}_p) = 0\} \supset \hat{\lambda}_8^{\#b}
\]
We will simultaneously prove this statement and also prove
\[
\mathcal{P}\mathcal{A}_7 \cap \hat{\lambda}_8^{\#b} = \emptyset
\]

**Claim 7.14.** Let \((\tilde{f}, \tilde{l}_p) \in \hat{\lambda}_8^{\#b}\). Then there exists a solution \((\tilde{f}(t), \tilde{l}_p(t)) \in \mathcal{P}\mathcal{A}_6\) near \((\tilde{f}, \tilde{l}_p)\) to the set of equations
\[
\Psi_{\mathcal{P}_D}(f(t), l_p(t)) \neq 0, \Psi_{\mathcal{P}_A}(f(t), l_p(t)) = 0, \Psi_{\mathcal{P}_A'}(f(t), l_p(t)) = 0, \Psi_{\mathcal{P}_A}(f(t), l_p(t)) = 0.
\]
Moreover, whenever such a solution \((\tilde{f}(t), l_p(t))\) is sufficiently close to \((\tilde{f}, \tilde{l}_p)\) it lies in \(\mathcal{P}\mathcal{A}_6\), i.e., \(\Psi_{\mathcal{P}_A'}(\tilde{f}(t), \tilde{l}_p(t)) \neq 0\). In particular, \((f(t), l_p(t))\) does not lie in \(\mathcal{P}\mathcal{A}_7\).
Notice that claim 7.14 proves (7.56) and (7.57) simultaneously.

**Proof:** As before, we will first solve the equation

\[
 f_{02}(t) 
eq 0, \quad f_{02}(t)A_4^{f(t)} = 0, \quad f_{02}(t)^2 A_5^{f(t)} = 0, \quad f_{02}(t)^3 A_6^{f(t)} = 0. \tag{7.59}
\]

The only solutions to (7.59) that converge to zero as \( f_{02}(t), f_{12}(t) \) and \( f_{03} \) go to zero are

\[
 f_{21}(t) = u \\
 f_{02}(t) = \frac{3u^2}{f_{40}} \quad \text{using} \quad f_{02}(t)A_4^{f(t)} = 0 \\
 f_{12}(t) = \frac{2f_{31}}{f_{40}} u - \frac{3f_{50}}{5f_{40}^2} u^2 \quad \text{using} \quad f_{02}(t)^2 A_5^{f(t)} = 0 \\
 f_{03}(t) = \left( -\frac{6f_{31}}{f_{40}^2} + \frac{9f_{22}}{f_{40}} \right) u + \left( -\frac{9f_{41}}{f_{40}^2} + \frac{36f_{31}f_{50}}{5f_{40}^3} \right) u^2 \\
 + \left( -\frac{54f_{50}^2}{25f_{40}^4} + \frac{9f_{60}}{5f_{40}^3} \right) u^3 \quad \text{using} \quad f_{02}(t)^3 A_6^{f(t)} = 0. \tag{7.60}
\]

Note that since \( (\tilde{f}, l) \in \tilde{X}_s \) we get that \( f_{40} \neq 0 \). Equation (7.60) implies that the only solutions to the functional equation (7.58) are of the form

\[
 \Psi_{PD_4}(\tilde{f}(t), l_p(t)) = t \\
 \Psi_{PD_4}(\tilde{f}(t), l_p(t)) = \frac{3}{\Psi_{PD_4}(\tilde{f}(t), l_p(t))} t^2 \\
 \Psi_{PE_6}(\tilde{f}(t), l_p(t)) = \frac{2\Psi_{PE_6}(\tilde{f}(t), l_p(t))}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))} t + O(t^2) \\
 \Psi_{PX_s}(\tilde{f}(t), l_p(t)) = \left( -\frac{6\Psi_{PE_6}(\tilde{f}(t), l_p(t))}{\Psi_{PD_6}(\tilde{f}(t), l_p(t))} + \frac{9\varphi(\tilde{f}(t), l_p(t))}{\Psi_{PE_7}(\tilde{f}(t), l_p(t))} \right) t + O(t^2). \tag{7.61}
\]

The functional \( \varphi \) is given by

\[
 \{ \varphi(\tilde{f}(t), l_p(t)) \} (f \otimes P^{\otimes d} \otimes v^{\otimes 2} \otimes \hat{w}^{\otimes 2}) := f_{22},
\]

where notations are as defined in subsection 4.2. Equality holds here as functionals. Since the sections

\[
 \Psi_{PD_4} : \overline{PA}_3 \longrightarrow \mathbb{L}_{PD_4}, \quad \Psi_{PD_4}^{-1} : \mathbb{L}_{PD_4}(0) \longrightarrow \mathbb{L}_{PD_4}, \\
 \Psi_{PE_6} : \overline{PE}_6(0) \longrightarrow \mathbb{L}_{PE_6}, \quad \Psi_{PX_s} : \overline{PX}_s(0) \longrightarrow \mathbb{L}_{PX_s}
\]

are transverse to the zero set (as proved in Proposition 6.19, 6.26 and 6.29), there exists a solution \((\tilde{f}(t), l_p(t))\) close to \((\tilde{f}, l)\) to (7.61). This proves our first assertion. Next we need to show that any such solution satisfies the condition \( \Psi_{PA_r}(\tilde{f}(t), l_p(t)) \neq 0 \) if \( t \) is small. Observe

\[
 f_{02}(t)^4 A_7^{f(t)} = \left( -\frac{f_{31}(t)^3}{8f_{40}(t)^3} + \frac{3f_{22}(t)f_{31}(t)}{16f_{40}(t)^2} - \frac{f_{13}(t)}{16f_{40}(t)} \right) u^5 + O(u^6)
\]

using (7.60)

\[
 \implies \Psi_{PA_r}(\tilde{f}(t), l_p(t)) = \frac{\Psi_{PJ}(\tilde{f}(t), l_p(t))}{\Psi_{PE_7}(\tilde{f}(t), l_p(t))^3} t^5 + O(t^6) \tag{7.62}
\]
Since \((\hat{f}, l_{\hat{p}}) \in \mathcal{X}'_{\delta,0}\), we get that \(\Psi_{P,A_f}(\hat{f}, l_{\hat{p}}) \neq 0\) and \(\Psi_{P,E}(\hat{f}, l_{\hat{p}}) \neq 0\). Hence, (7.62) implies that if \(t\) is small \(\Psi_{P,A_f}(\hat{f}(t), l_{\hat{p}}(t)) \neq 0\), which proves claim 7.14. This finishes the proof of Lemma 7.1, statement 12.

8 Euler class

Finally, we are ready to prove the recursive formulas stated in section 3. The notations are as in section 4 and notations 2.7 and 2.11.

**Proof of Equation (3.1):** Let \(Q : D \times \mathbb{P}^2 \rightarrow W\) be a generic smooth section to

\[
W := \left( \bigoplus_{i=1}^{\delta_d-(n+1)} \gamma_{D}^{i} \right) \oplus \left( \bigoplus_{i=1}^{n} \gamma_{y_{2}}^{i} \right) \rightarrow D \times \mathbb{P}^2.
\]

By Lemma 2.8 and Theorem 2.3 we conclude

\[
\mathcal{N}(A_1, n) = \langle e(W), [\overline{A_1}] \rangle = \pm |A_1 \cap Q^{-1}(0)|.
\]

By Lemma 7.1, statement 1, \(\overline{A}_0 = A_0 \cup \overline{A}_1\). The section \(\psi_{A_1} : \overline{A}_0 \rightarrow \nu_{A_1}\) vanishes on \(A_1\) transversely and doesn’t vanish on \(A_0\) (cf. Proposition 6.4). Therefore, the zeros of the section

\[
\psi_{A_1} \oplus Q : \overline{A}_0 \rightarrow \nu_{A_1} \oplus W
\]
counted with a sign is our desired number, whence

\[
\mathcal{N}(A_1, n) = \langle e(\nu_{A_1})e(W), [\overline{A}_0] \rangle = \langle \text{PD}[\overline{A}_0]e(\nu_{A_1})e(W), [D \times \mathbb{P}^2] \rangle.
\]

Now Proposition 6.5, 6.4 and Theorem 2.1 imply that the Poincaré dual \(\text{PD}[\overline{A}_0]\) of \(\overline{A}_0\) is the Euler class \(e(L_{\overline{A}_0})\). We may now use the splitting principle and Lemma B.1 to conclude that

\[
\mathcal{N}(A_1, n) = \langle (y + da)((y + da)^2 - 3a(y + da) + 3a^2)y^{\delta_d-(n+1)}a^n, [D \times \mathbb{P}^2] \rangle.
\]

Equation (3.1) now follows. \(\square\)

**Proof of Equation (3.3) and (3.4):** Let \(W_{n,m,2}\) and \(Q\) be as in (2.6) with \(k = 2\). By definition, \(\mathcal{N}(P,A_2, n, m)\) is the signed cardinality of the intersection of \(P,A_2\) with \(Q^{-1}(0)\). By Lemma 7.1, statement refA1cl we gather that

\[
\overline{A}_1 = \overline{A}'_1 = \hat{A}'_1 \cup P,A_2.
\]

By Proposition 6.13, the section \(\Psi_{P,A_2} : \overline{A}_1 \rightarrow \nu_{P,A_2}\) vanishes on \(P,A_2\) transversely and by definition it doesn’t vanish on \(\hat{A}'_1\). Hence, the zeros of the section

\[
\Psi_{P,A_2} \oplus Q : \overline{A}_1 \rightarrow \nu_{P,A_2} \oplus W_{n,m,2},
\]
counted with a sign, is our desired number. Via the splitting principle and Lemma B.1 we have

\[
\mathcal{N}(P,A_2, n, m) = \langle e(\nu_{P,A_2})e(W_{n,m,2}), [\overline{A}_1] \rangle
\]

\[
= \langle ((\lambda + y + da)^2 - 3a(\lambda + y + da) + 3a^2)y^{\delta_d-(n+m+2)}a^n\lambda^m, [\overline{A}_1] \rangle.
\]

50
Next we use the fact that
\[ \langle \pi^*(y^d-a_{n_1}) \lambda, \overline{A_1} \rangle = \langle y^d-a_{n_1}, [\overline{A_1}] \rangle \quad \text{and} \quad \langle \pi^*(y^d-a_{n_1}) \lambda, \overline{A_1} \rangle = 0 \]
for all \( n_1 \). This follows from Lemma B.2 and the fact that \( \overline{A_1} \) is a smooth manifold (cf. Corollary 6.7). Finally, using the ring structure of \( H^*(D \times \mathbb{P}^2; \mathbb{Z}) \) (cf. Lemma B.2), we obtain equations (3.3) and (3.4). Here \( \pi : D \times \mathbb{P}^2 \to D \times \mathbb{P}^2 \) is the projection map.

Proof of Equation (3.5): Let \( \mathbb{W}_{n,m,3} \) and \( Q \) be as in (2.6) with \( k = 3 \). By Lemma 7.1, statement 8 we have
\[ \overline{PA}_2 = \mathcal{L}_{PA_2} \cup \overline{PA}_3 \cup \mathcal{D}_4^\# . \]
The section \( \Psi_{PA_3} : \overline{PA}_2 \to \mathcal{L}_{PA_3} \) doesn’t vanish on \( \mathcal{L}_{PA_2} \) and vanishes transversely on \( \overline{PA}_3 \). Furthermore, it does not vanish on any point of \( \mathcal{D}_4^\# \) (by definition). Hence, the zeros of the section
\[ \Psi_{PA_3} \oplus Q : \overline{PA}_2 \to \mathcal{L}_{PA_3} \oplus \mathbb{W}_{n,m,3} \]
counted with a sign is \( \mathcal{N}(\mathcal{L}_{PA_3}, n, m) \). A similar computation using the product formula for the first Chern class of a product of line bundles, proves the equation.

Proof of Equation (3.6): Let \( \mathbb{W}_{n,m,4} \) and \( Q \) be as in (2.6) with \( k = 4 \). By Lemma 7.1, statement 9 we have that
\[ \overline{PA}_3 = \mathcal{L}_{PA_3} \cup \overline{PA}_4 \cup \overline{PD}_4 . \]
The section \( \Psi_{PA_4} : \overline{PA}_3 \to \mathcal{L}_{PA_4} \) doesn’t vanish on \( \mathcal{L}_{PA_3} \) and vanishes transversely on \( \overline{PA}_4 \) (cf. Proposition 6.24). Furthermore, it does not vanish on any point of \( \overline{PD}_4 \). Hence, the zeros of the section
\[ \Psi_{PA_4} \oplus Q : \overline{PA}_3 \to \mathcal{L}_{PA_4} \oplus \mathbb{W}_{n,m,4} \]
counted with a sign is \( \mathcal{N}(\mathcal{L}_{PA_4}, n, m) \), which proves the equation.

Proof of Equation (3.7): Let \( \mathbb{W}_{n,m,5} \) and \( Q \) be as in (2.6) with \( k = 5 \). By Lemma 7.1, statement 10 we have that
\[ \overline{PA}_4 = \mathcal{L}_{PA_4} \cup \overline{PA}_5 \cup \overline{PD}_5 . \]
The section \( \Psi_{PA_5} : \overline{PA}_4 \to \mathcal{L}_{PA_5} \) doesn’t vanish on \( \mathcal{L}_{PA_4} \) and vanishes transversely on \( \overline{PA}_5 \) (see Proposition 6.24). Furthermore, the section
\[ \Psi_{PA_5} \oplus Q : \overline{PA}_4 \to \mathcal{L}_{PA_5} \oplus \mathbb{W}_{n,m,5} \]
vanishes on \( \overline{PD}_5 \) with a multiplicity of 2 (cf. Corollary 7.4). Hence,
\[ \langle e(\mathcal{L}_{PA_5}) e(\mathbb{W}_{n,m,5}), [\overline{PA}_4] \rangle = \mathcal{N}(\mathcal{L}_{PA_5}, n, m) + 2\mathcal{N}(\overline{PD}_5, n, m) \]
completing the proof.

Proof of Equation (3.8): Let \( \mathbb{W}_{n,m,6} \) and \( Q \) be as in (2.6) with \( k = 6 \). By Lemma 7.1, statement 11 we have that
\[ \overline{PA}_5 = \mathcal{L}_{PA_5} \cup \overline{PA}_6 \cup \overline{PD}_6 \cup \overline{PE}_6 . \]
The section \( \Psi_{PA_6} : \overline{PA_6} \to \mathbb{L}_{PA_6} \) doesn’t vanish on \( PA_5 \) and vanishes transversely on \( PA_6 \). Furthermore, the section 
\[
\Psi_{PA_6} \oplus Q : \overline{PA_6} \to \mathbb{L}_{PA_6} \oplus W_{n,m,6}
\]
vanishes on \( PD_6 \) and \( PE_6 \) with a multiplicity of 3 and 4 respectively (cf. Corollary 7.7 and 7.9).

**Proof of Equation (3.9):** Let \( W_{n,0,7} \) and \( Q \) be as in (2.6) with \( m = 0 \) and \( k = 7 \). By Lemma 7.1, statement 11 we have that 
\[
\overline{PA_6} = PA_6 \cup PA_7 \cup PD_7 \cup PE_7 \cup \hat{X}_8^{\#}.
\]
The section \( \Psi_{PA_7} : \overline{PA_7} \to \mathbb{L}_{PA_7} \) doesn’t vanish on \( PA_6 \) and vanishes transversely on \( PA_7 \). Furthermore, the section 
\[
\Psi_{PA_7} \oplus Q : \overline{PA_7} \to \mathbb{L}_{PA_7} \oplus W_{n,m,7}
\]
vansishes on \( PD_7 \) and \( PE_7 \) with a multiplicity of 6 and 7 respectively (cf. Corollary 7.7 and 7.9). Let us assume the section vanishes with a multiplicity of \( \eta \) on \( \hat{X}_8^{\#} \). Hence,
\[
\langle e(\mathbb{L}_{PA_7})e(W_{n,0,7}), [\overline{PA_6}] \rangle = N(PA_7, n, 0) + 6N(PD_7, n, 0) + 7N(PE_7, n, 0) + \eta \langle e(W_{n,0,7}), [\hat{X}_8^{\#}] \rangle
\]
It is easy to see that \( \langle e(W_{n,0,7}), [\hat{X}_8^{\#}] \rangle = 0 \), which proves the equation.

**Proof of Equation (3.10):** Let \( W_{n,m,4} \) and \( Q \) be as in (2.6) with \( k = 4 \). By Lemma 7.1, statement 9 we have 
\[
\overline{PA_3} = PA_3 \cup PA_4 \cup PD_4.
\]
The section \( \Psi_{PD_4} : \overline{PA_4} \to \mathbb{L}_{PD_4} \) doesn’t vanish on \( PA_3 \) and vanishes transversely on \( PD_4 \). Furthermore, this section does not vanish on any point of \( PA_4 \). Hence, the zeros of the section 
\[
\Psi_{PD_4} \oplus Q : \overline{PA_4} \to \mathbb{L}_{PD_4} \oplus W_{n,m,4}
\]
counted with a sign is \( N(PD_4, n, m) \) which proves the equation.

**Proof of Equation (3.11):** Let \( W_{n,m,5} \) and \( Q \) be as in (2.6) with \( k = 5 \). By Lemma 7.1, statement 4 we have that 
\[
\overline{PD_4} = PD_4 \cup PD_5.
\]
The section \( \Psi_{PD_5} : \overline{PD_5} \to \mathbb{L}_{PD_5} \) doesn’t vanish on \( PD_4 \) and vanishes transversely on \( PD_5 \). Hence, the zeros of the section 
\[
\Psi_{PD_5} \oplus Q : \overline{PD_5} \to \mathbb{L}_{PD_5} \oplus W_{n,m,5}
\]
counted with a sign is \( N(PD_5, n, m) \), which proves the equation.

**Proof of Equation (3.12):** Let \( W_{n,m,6} \) and \( Q \) be as in (2.6) with \( k = 6 \). By Lemma 7.1, statement 6 we have 
\[
\overline{PD_5} = PD_5 \cup PD_6 \cup PE_6.
\]
The section \( \Psi_{PD_6} : \overline{PD_6} \to \mathbb{L}_{PD_6} \) doesn’t vanish on \( PD_5 \) and vanishes transversely on \( PD_6 \). Furthermore, it does not vanish on any point of \( PE_6 \). Hence, the zeros of the section 
\[
\Psi_{PD_6} \oplus Q : \overline{PD_6} \to \mathbb{L}_{PD_6} \oplus W_{n,m,6}
\]
counted with a sign is $N(\mathcal{P}D_6, n, m)$, which proves the equation.

**Proof of Equation (3.13):** Let $\mathcal{Q}_{n,m,7}$ and $\mathcal{Q}$ be as in (2.6) with $k = 7$. By Lemma 7.1, statement 7 we have that

$$\mathcal{P}D_6 = \mathcal{P}D_6 \cup \mathcal{P}D_7 \cup \mathcal{P}E_7.$$  

The section $\Psi_{\mathcal{P}D_7} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}D_7}$ doesn’t vanish on $\mathcal{P}D_6$ and vanishes transversely on $\mathcal{P}D_7$. Furthermore, it does not vanish on any point of $\mathcal{P}E_7$. Hence, the zeros of the section

$$\Psi_{\mathcal{P}D_7} \oplus \mathcal{Q} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}D_7} \oplus \mathcal{Q}_{n,m,7}$$  

counted with a sign is $N(\mathcal{P}D_7, n, m)$, which proves the equation.

**Proof of Equation (3.14):** Let $\mathcal{Q}_{n,m,6}$ and $\mathcal{Q}$ be as in (2.6) with $k = 6$. By Lemma 7.1, statement 6 we have that

$$\mathcal{P}D_6 = \mathcal{P}D_6 \cup \mathcal{P}D_6 \cup \mathcal{P}E_6.$$  

The section $\Psi_{\mathcal{P}E_6} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}E_6}$ doesn’t vanish on $\mathcal{P}D_6$ and vanishes transversely on $\mathcal{P}E_6$. Hence, the zeros of the section

$$\Psi_{\mathcal{P}E_6} \oplus \mathcal{Q} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}E_6} \oplus \mathcal{Q}_{n,m,6}$$  

counted with a sign is $N(\mathcal{P}E_6, n, m)$, which proves the equation.

**Proof of Equation (3.15):** Let $\mathcal{Q}_{n,m,7}$ and $\mathcal{Q}$ be as in (2.6) with $k = 7$. By Lemma 7.1, statement 7 we have that

$$\mathcal{P}D_6 = \mathcal{P}D_6 \cup \mathcal{P}E_7 \cup \mathcal{P}D_7.$$  

The section $\Psi_{\mathcal{P}E_6} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}E_6}$ doesn’t vanish on $\mathcal{P}D_7$ and vanishes transversely on $\mathcal{P}E_7$ (cf. Proposition 6.40). Hence, the zeros of the section

$$\Psi_{\mathcal{P}E_6} \oplus \mathcal{Q} : \mathcal{P}D_6 \to \mathbb{L}_{\mathcal{P}E_6} \oplus \mathcal{Q}_{n,m,7}$$  

counted with a sign is $N(\mathcal{P}E_7, n, m)$, which proves the equation.

### A Low degree checks

**Verification of the number $N(A_1, 0) = 3(d - 1)^2$:**

- $d = 1$: There are no nodal lines.
- $d = 2$: The number of line pairs that pass through 4 general points is $\frac{1}{2}{4 \choose 2} = 6$.
- $d = 3$: The number of nodal cubics passing through 8 general points are the rational cubics passing through these points; this number 12 can also be computed through Kontsevich’s recursion formula.

**Verification of the number $N(A_1, 1) = 3(d - 1)$:**

- $d = 1$: There are no nodal lines.
- $d = 2$: The number of line pairs that pass through 3 points and meet on a line is $\frac{3}{2} = 3$.

**Verification of the number $N(A_2, 0) = 12(d - 1)(d - 2)$:**

- $d = 1$: There are no lines with a cusp.
- $d = 2$: The only way a conic can have a cusp is if its a double line. There are no double lines through
three generic points.
d = 4 : The number of quartics with a cusp is 72. This is same as the number of genus two curves
with a cusp and equals 72 (cf. [6], pp. 19).

Verification of the number \( N(PA_3, n, m) \):
d = 3 : The number \( N(PA_3, n, m) \) can be verified by direct geometric means for all values of \( n \) and
\( m \) in the case of cubics (cf. [1]).

Verification of the number \( N(A_4, 0) = 60(d - 3)(3d - 5) \):
d = 3 : There are no cubics with an \( A_4 \)-node.

Verification of the number \( N(D_4, 0) = 15(d - 2)^2 \):
d = 2 : There are no conics with a \( D_4 \)-node.
d = 3 : The only way a cubic can have a \( D_4 \)-node is, if it breaks into three distinct lines intersecting
at a common point. The number of such configurations passing through 5 points is \( \frac{1}{7} \times \binom{5}{2} \times \binom{3}{2} = 15 \).

Verification of the number \( N(D_4, 1) = 6(d - 2) \):
d = 2 : There are no conics with a \( D_4 \)-node on a line.
d = 3 : The number of triple lines, having a common point at a given line and passing through four
points is \( \binom{4}{2} = 6 \).

Verification of the number \( N(PD_4, n, 1) \):
d = 3 : The number \( N(PD_4, n, 1) \) can be verified by direct geometric means for all values of \( n \) in the
case of cubics (cf. [1]).

Verification of the number \( N(PD_6, n, m) \):
d = 4 : The number \( N(PD_6, n, m) \) can be verified by direct geometric means for all values of \( n \) and
\( m \) in the case of quartics (cf. [1]).

Verification of the number \( N(E_6, 0) = 21(d - 3)(4d - 9) \):
d = 3 : There are no cubics with an \( E_6 \)-node.
d = 4 : An \( E_6 \)-node contributes three to the genus of a curve. Since a smooth quartic has genus
three, the quartics with an \( E_6 \)-node have genus zero. The number of such quartics through 8 points
is 147 (cf. [8], pp. 24).

B Some details

B.1 Proof of some lemmas

Proof of Lemma 2.13: Let \( X_k = A_k \). If \( \tilde{f}, \tilde{l}_{\tilde{p}} \in PA_k \) then the projection map has to be one to
one, because otherwise the kernel of the Hessian would have two linearly independent vectors (which
would imply it is identically zero). Similar argument holds if \( X_k = D_k \) for \( k \geq 5 \) or \( E_6 \), \( E_7 \), or \( E_8 \).
Next let \( X_k = D_4 \). If \( \tilde{f}, \tilde{l}_{\tilde{p}} \in D_4 \) then there exists three distinct directions \( l_{\tilde{p}} \) along which the third
derivative vanishes. Hence, the projection map is three to one. Finally, since the projection map is
orientation preserving, on the level of homology

\[
\pi_*([\overline{P X_k}]) = [X_k] \in H_*(D \times \mathbb P^2; \mathbb Z) \text{ if } X_k \neq D_4 \ \text{ and } \ \pi_*([\overline{PD_4}]) = 3[D_4] \in H_*(D \times \mathbb P^2; \mathbb Z). \quad (B.1)
\]
This proves (2.4) (using definition 2.12).

Proof of Lemma 6.1: It is evident that the left hand side of (6.1) is a subset of its right hand side, since all the sections are continuous. For the converse, we will show that if \( p \) belongs to the right hand side, then there exists a sequence \( p_n \) in \( S_{k-1} \) that converges to \( p \). Since the section

\[ \zeta_k : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0 \} \rightarrow V_k \]

is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations

\[ \zeta_0(p_t) = 0, \ldots, \zeta_{k-1}(p_t) = 0, \quad \zeta_k(p_t) = t \]

if \( t \) is sufficiently small. By definition, \( p_t \) belongs to \( S_{k-1} \) if \( t \neq 0 \). This gives us a sequence that lies in \( S_{k-1} \) and converges to \( p \). To finish the proof, it suffices to prove (6.2) by showing that

\[ S_k = \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0, \quad \zeta_k(p) = 0 \} \]

This follows from an identical argument as before, using transversality of the section

\[ \zeta_{k+1} : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_k(p) = 0 \} \rightarrow V_{k+1}. \]

Proof of Lemma 6.2: We will show that the left hand side of (6.3) is a subset of its right hand side. We may assume that \( p \in \overline{S}_{k-1} - S_{k-1} \) and \( \varphi(p) \neq 0 \). We claim that \( p \in \overline{S}_k \). In other words, we need to show that there exists a sequence in \( S_k \) that converges to \( p \). Since the section

\[ \zeta_{k+1} : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_k(p) = 0, \quad \varphi(p) \neq 0 \} \rightarrow V_{k+1} \]

is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations

\[ \zeta_0(p_t) = 0, \ldots, \zeta_k(p_t) = 0, \quad \zeta_{k+1}(p_t) = t \]

if \( t \) is sufficiently small. By definition, \( p_t \) belongs to \( S_k \) if \( t \neq 0 \). This gives us a sequence that lies in \( S_k \) and converges to \( p \). This proves that the left hand side of (6.3) is a subset of its right hand side; if \( p \in \overline{S}_{k-1} - S_{k-1} \) and \( \varphi(p) = 0 \), then \( p \in B \). Next, let us show that the right hand side of (6.3) is a subset of its left hand side. Since \( B \subseteq \overline{S}_{k-1} \), it suffices to show that \( \overline{S}_k \subseteq \overline{S}_{k-1} \). We need to show that if \( p \in \overline{S}_k \), then there exists a sequence in \( S_{k-1} \) that converges to \( p \). Since by hypothesis \( p \in \overline{S}_k \), there exists a sequence \( p_n \in S_k \) that converges to \( p \), i.e., \( p_n \) satisfies the equations:

\[ \zeta_0(p_n) = 0, \ldots, \zeta_{k-1}(p_n) = 0, \quad \zeta_k(p_n) = 0, \quad \zeta_{k+1}(p_n) \neq 0, \quad \varphi(p_n) \neq 0. \]

However, since the section

\[ \zeta_k : \{ p \in M : \zeta_0(p) = 0, \ldots, \zeta_{k-1}(p) = 0, \quad \varphi(p) \neq 0 \} \rightarrow V_k \]

is transverse to the zero set, we can conclude that there exists a solution \( p_{n,t} \) near \( p_n \) for the set of equations:

\[ \zeta_0(p_{n,t}) = 0, \ldots, \zeta_{k-1}(p_{n,t}) = 0, \quad \zeta_k(p_{n,t}) = t. \]

Moreover, since \( \varphi(p_n) \neq 0 \) and \( \varphi \) is continuous, we get that \( \varphi(p_{n,t}) \neq 0 \) if \( t \) is sufficiently small. Hence, \( p_{n,t} \) lies in \( S_{k-1} \). This gives us a sequence in \( S_{k-1} \) that converges to \( p \), which proves (6.3). Finally, we will prove (6.4). Let us prove that

\[ \overline{S}_0 \supset \{ p \in M : \zeta_0(p) = 0, \quad \varphi(p) \neq 0 \}. \]
In other words, we need to show that if \( \zeta_0(p) = 0 \) and \( \varphi(p) \neq 0 \), then there exists a sequence in \( S_0 \) that converges to \( p \). Since the section

\[
\zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \to V_1
\]
is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations

\[
\zeta_0(p_t) = 0, \quad \zeta_1(p_t) = t.
\]

Since \( \varphi \) is continuous and \( \varphi(p) \neq 0 \), we get that \( \varphi(p_t) \neq 0 \) if \( t \) is sufficiently small. This gives us the desired sequence in \( S_0 \). Now using (B.2) and the definition of \( S_{-1} \) and \( B \), we get that

\[
M = \{ p \in M : \zeta_0(p) \neq 0, \varphi(p) \neq 0 \} \cup \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \cup \{ p \in M : \varphi(p) = 0 \}
\]
is contained in \( S_{-1} \cup S_0 \cup B' \). The reverse inclusion is vacuous.

**Proof of Lemma 6.3:** Observe that the left hand side of (6.5) is a subset of its right hand side. For the converse, assume that \( p \) belongs to the right hand side. We need to show that there exists a sequence in \( S_0 \) that converges to \( p \). Let us assume that \( \varphi(p) \neq 0 \). Since the section

\[
\zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \to V_1
\]
is transverse to the zero set, there exists a solution \( p_t \) near \( p \) to the set of equations \( \zeta_0(p_t) = 0 \) and \( \zeta_1(p_t) = t \). Moreover, since \( \varphi(p) \neq 0 \), we conclude that \( \varphi(p_t) \neq 0 \) if \( t \) is small. Hence, \( p_t \in S_0 \) for all small \( t \) which gives us the desired sequence. Next let us assume that \( \varphi(p) = 0 \). Since the section

\[
\varphi : \{ p \in M : \zeta_0(p) = 0 \} \to W
\]
is transverse to the zero set, there exists a solution \( p_{t_1} \) near \( p \) to the set of equations

\[
\zeta_0(p_{t_1}) = 0, \quad \varphi(p_{t_1}) = t_1.
\]

Since \( \varphi(p_{t_1}) \neq 0 \), there exists a solution \( p_{t_1,t_2} \) near \( p_{t_1} \) to the set of equations \( \zeta_0(p_{t_1,t_2}) = 0 \) and \( \zeta_1(p_{t_1,t_2}) = t_2 \) because the section

\[
\zeta_1 : \{ p \in M : \zeta_0(p) = 0, \varphi(p) \neq 0 \} \to V_1
\]
is transverse to the zero set. Moreover, since \( \varphi(p_{t_1}) \neq 0 \), we infer that \( \varphi(p_{t_1,t_2}) \neq 0 \) if \( t_2 \) is sufficiently small. Hence, \( p_{t_1,t_2} \) lies in \( S_0 \). This gives us the desired sequence which proves Lemma 6.3.

**B.2 Chern classes and projectivized bundle**

**Lemma B.1.** ([5], Theorem 14.10) Let \( \mathbb{P}^n \) be the \( n \)-dimensional complex projective space and \( \gamma \to \mathbb{P}^n \) the tautological line bundle. Then the total Chern class \( c(T\mathbb{P}^n) \) is given by \( c(T\mathbb{P}^n) = (1 + c_1(\gamma^*))^{n+1} \).

**Lemma B.2.** ([2], pp. 270) Let \( V \to M \) be a complex vector bundle, of rank \( k \), over a smooth manifold \( M \) and \( \pi : \mathbb{P}V \to M \) the projectivization of \( V \). Let \( \tilde{\gamma} \to \mathbb{P}V \) be the tautological line bundle over \( \mathbb{P}V \) and \( \lambda = c_1(\tilde{\gamma}^*) \). There is a linear isomorphism

\[
H^*(\mathbb{P}V; \mathbb{Z}) \cong H^*(M; \mathbb{Z}) \otimes H^*(\mathbb{P}^{k-1}; \mathbb{Z}) \quad (B.3)
\]
and an isomorphism of rings

\[ H^*(\mathbb{P}V; \mathbb{Z}) \cong \frac{H^*(M; \mathbb{Z})[\lambda]}{\langle \lambda^k + \lambda^{k-1}\pi^*c_1(V) + \lambda^{k-2}\pi^*c_2(V) + \ldots + \pi^*c_k(V) \rangle}. \quad (B.4) \]

In particular, if \( \omega \in H^*(M; \mathbb{Z}) \) is a top cohomology class then

\[ \langle \pi^*(\omega)\lambda^{k-1}, [\mathbb{P}V] \rangle = \langle \omega, [M] \rangle, \]

i.e., \( \lambda^{k-1} \) is a cohomology extension of the fibre.

References

[1] S. Basu and R. Mukherjee, Enumeration of curves with one singular point: Further details. available at https://www.sites.google.com/site/ritwik371/home.

[2] R. Bott and L. W. Tu, Differential forms in algebraic topology, vol. 82 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1982.

[3] M. È. Kazaryan, Multisingularities, cobordisms, and enumerative geometry, Uspekhi Mat. Nauk, 58 (2003), pp. 29–88.

[4] D. Kerner, Enumeration of singular algebraic curves, Israel J. Math., 155 (2006), pp. 1–56.

[5] J. W. Milnor and J. D. Stasheff, Characteristic classes, Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.

[6] R. Vakil, Enumerative geometry of plane curves of low genus. available at http://arxiv.org/abs/math/9803007.

[7] A. Zinger, Counting plane rational curves: old and new approaches. available at http://arxiv.org/abs/math/0507105.

[8] A. Zinger, Enumeration of genus-three plane curves with a fixed complex structure, J. Algebraic Geom., 14 (2005), pp. 35–81.