A note on the size of $\mathcal{N}$-free families

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Abstract The $\mathcal{N}$ poset consists of four distinct sets $W, X, Y, Z$ such that $W \subseteq X, Y \subseteq X$, and $Y \subseteq Z$ where $W$ is not necessarily a subset of $Z$. A family $\mathcal{F}$, considered as a subposet of the $n$-dimensional Boolean lattice $\mathcal{B}_n$, is $\mathcal{N}$-free if it does not contain $\mathcal{N}$ as a subposet. Let $La(n, \mathcal{N})$ be the size of a largest $\mathcal{N}$-free family in $\mathcal{B}_n$. Katona and Tarján proved that $La(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k+1)$, where $k = \lfloor n/2 \rfloor$ and $A(n, 4, k+1)$ is the size of a single-error-correcting code with constant weight $k + 1$. In this note, we prove for $n$ even and $k = n/2$, $La(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k)$, which improves the bound on $La(n, \mathcal{N})$ in the second order term for some values of $n$ and should be an improvement for an infinite family of values of $n$, depending on the behavior of the function $A(n, 4, \cdot)$.

Keywords Forbidden subposets · Error-correcting codes · Extremal set theory

Mathematics Subject Classification 06A06

1 Introduction

The $n$-dimensional Boolean lattice, $\mathcal{B}_n$, denotes the partially ordered set (poset) $(2^{[n]}, \subseteq)$, where $[n] = \{1, \ldots, n\}$ and, for every finite set $S$, $2^S$ denotes the set of
For posets, $P = (P, \preceq)$ and $P' = (P', \preceq')$, we say $P'$ is a (weak) sub-poset of $P$ if there exists an injection $f : P' \to P$ that preserves the partial ordering. That is, whenever $u \preceq' v$ in $P'$, we have $f(u) \preceq f(v)$ in $P$. If $\mathcal{F}$ is a subposet of $\mathcal{B}_n$ such that $\mathcal{F}$ contains no subposet $P$, we say $\mathcal{F}$ is $P$-free.

$P$-free posets (or $P$-free families) have been extensively studied, beginning with Sperner’s theorem in 1928. Sperner [7] proved that the size of the largest antichain in $\mathcal{B}_n$ is $\left(\begin{array}{l}n \\ \lfloor n/2 \rfloor \end{array}\right)$. Erdős [2] generalized this result to chains. Katona and Tarján [6] addressed the problem of $\mathcal{V}$-free families and got an asymptotic result. Griggs and Katona [4] addressed $\mathcal{N}$-free families, obtaining Theorem 1.1 below. See Griggs and Li [5] for a survey of the progress on $P$-free families. Let $La(n, P)$ denote the size of the largest $P$-free family in $\mathcal{B}_n$.

The main result of this note is Theorem 1.4, in which, for some values of $n$, we improve the bounds on $La(n, \mathcal{N})$ in the second-order term. The poset $\mathcal{N}$ consists of four distinct sets $W, X, Y, Z$ such that $W \subset X$, $Y \subset X$, and $Y \subset Z$. However, $W$ is not necessarily a subset of $Z$. See Fig. 1. The earliest extremal result on $\mathcal{N}$-free families is Theorem 1.1.

**Theorem 1.1** (Griggs and Katona [4])

\[
\left(\begin{array}{l}n \\ \lfloor n/2 \rfloor \end{array}\right)\left(1 + \frac{1}{n} + \Omega\left(\frac{1}{n^2}\right)\right) \leq La(n, \mathcal{N}) \leq \left(\begin{array}{l}n \\ \lfloor n/2 \rfloor \end{array}\right)\left(1 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right).
\]

The construction for the lower bound of Theorem 1.1 comes directly from a previous result of Katona and Tarján [6] from 1983 on $\mathcal{V}$-free families. The poset $\mathcal{V}$ consists of three elements $X, Y, Z$ such that $Y \subset X$ and $Y \subset Z$. It is clear that $La(n, \mathcal{V}) \leq La(n, \mathcal{N})$ because any $\mathcal{V}$-free family is also $\mathcal{N}$-free.

To establish the lower bound, Katona and Tarján used a constant-weight code construction due to Graham and Sloane [3] from 1980. In the proof of Theorem 1.4, we obtain a lower bound that appears to be larger than the current known bound. However, whether it is an improvement depends on the behavior of some functions well-known in coding theory. In order to discuss our results we need some brief coding theory background.

### 1.1 Coding theory background

Let $A(n, 2\delta, k)$ denote the size of the largest family of $\{0, 1\}$-vectors of length $n$ such that each vector has exactly $k$ ones and the Hamming distance between any pair of distinct vectors is at least $2\delta$. This is the same as the size of the largest family of subsets
of \([n]\) such that each subset has size exactly \(k\) and the symmetric difference of any pair of distinct sets is at least \(2\delta\).

The quantity \(A(n, 2\delta, k)\) is important in the field of error-correcting codes. In fact, \(A(n, 4, k)\) computes the size of a single-error-correcting code with constant weight \(k\). Henceforth, we will use “SEC code” as shorthand for “single-error-correcting code.”

The first nontrivial value of \(\delta\) for \(A(n, 2\delta, k)\) is \(\delta = 2\). Graham and Sloane [3] give a lower bound construction for \(A(n, 4, k)\).

**Theorem 1.2** (Graham and Sloane [3]) \(A(n, 4, k) \geq n^{-1}\binom{n}{k}\).

### 1.2 Main result

Katona and Tarján [6] estimated the following lower bound for \(\mathcal{N}\)-free families.

**Theorem 1.3** Let \(k = \lfloor n/2 \rfloor\). Then,

\[
\mathcal{L}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k+1).
\]

The following theorem is our main result of the note.

**Theorem 1.4** Let \(n\) be even and let \(k = n/2\). Then,

\[
\mathcal{L}(n, \mathcal{N}) \geq \binom{n}{k} + A(n, 4, k).
\]

**Remark 1.5** This is potentially an improvement when \(n\) is even. We note that the same 3-level construction works for \(n\) odd and \(k = (n-1)/2\). This gives a family of size \(\binom{n}{k} + A(n, 4, k)\) nontrivially in three layers. However, since \(A(n, 4, k) = A(n, 4, k+1)\) in the odd case, this does not provide an improvement to the known bounds.

We believe that, for \(n \geq 6\), the quantity \(A(n, 4, k)\) is strictly unimodal as a function of \(k\) as long as \(3 \leq k \leq n - 3\). This strict unimodality has been established [1] for \(6 \leq n \leq 12\) and known bounds suggest that it is the case for larger values of \(n\) as well. If unimodality holds, then \(A(n, 4, k)\) would achieve its maximum uniquely at \(k = \lfloor n/2 \rfloor\) or \(k = \lceil n/2 \rceil\). Therefore, we expect (1) to also be a strict improvement over Theorem 1.3 in the case where \(n\) is even. However, to our knowledge, the unimodality of \(A(n, 4, k)\) has never been established and seems to be a highly nontrivial problem.

**Proof of Theorem 1.4** Given \(k = n/2\), let \(C\) be a constant weight SEC code of size \(A(n, 4, k)\). Define \(C_{\text{up}} = \{ c \cup \{i\} : c \in C, i \notin c \} \) and \(C_{\text{down}} = \{ c - \{i\} : c \in C, i \in c \}\). Let us list some important properties of \(C_{\text{up}} \cup C_{\text{down}}\).

**Claim** (i) Both \(C_{\text{up}}\) and \(C_{\text{down}}\) are SEC codes with constant weight \(k+1\) and \(k-1\), respectively.

(ii) If \(c'' \in C_{\text{up}}\) and \(c' \in C_{\text{down}}\), \(c' \not\subseteq c''\).
Proof (i) Let $c_1, c_2 \in C_{\text{up}}$. Then $|c_1 \triangle c_2| = |(c_1 - \{i\}) \triangle (c_2 - \{i\})| \geq 4$ since $(c_1 - \{i\}), (c_2 - \{i\}) \in C$ and their symmetric difference must be at least 4 in order for $C$ to be a 1-EC code. Thus, $C_{\text{up}}$ is a SEC code. By a similar argument, $C_{\text{down}}$ is a SEC code.

(ii) Let $c'' \in C_{\text{up}}, c' \in C_{\text{down}}$, and $c' \subseteq c''$. Then, $(c' \cup \{i\}), (c'' - \{i\}) \in C$. So, $|(c'' - \{i\}) \triangle (c' \cup \{i\})| \geq 4$. This implies that there are two members of $[n]$ that are in $(c' \cup \{i\}) - (c'' - \{i\})$. One is $i$ and the other is some $j \in c' - c''$, which contradicts the assumption that $c' \subseteq c''$. ■

In order to finish the proof, we just need to show that the family $\mathcal{F} = \binom{[n]}{k} \cup C_{\text{up}} \cup C_{\text{down}}$ is $N$-free.

To that end, suppose there is a subposet $N$ with elements $W, X, Y, Z$ where $W \subset X, Y \subset X$ and $Y \subset Z$ (see Fig. 1). Where is the element $X$?

We know that $X \notin C_{\text{down}}$ because it has to have elements below it and the elements of $C_{\text{down}}$ are all minimal in $\mathcal{F}$. We know that $X \notin \binom{[n]}{k}$ because that would force $W, Y \in C_{\text{down}}$ and, being subsets of $X$ would require $|W \triangle Y| = 2$, a contradiction to $C_{\text{down}}$ being a SEC code. Therefore, $X \in C_{\text{up}}$.

Now, where is $Y$? We know that $Y \notin C_{\text{up}}$ because $Y \subset X$. We know $Y \notin \binom{[n]}{k}$ because that would force $X, Z \in C_{\text{up}}$ and thus would force $|X \triangle Z| = 2$, this is a contradiction to the fact that $C_{\text{up}}$ is a SEC code. Therefore, $Y \in C_{\text{down}}$.

In order for the copy of $N$ to exist, $Y \subset X$, which implies $Y \subset X - \{i\}$ and so $|(Y \cup \{i\}) \triangle (X - \{i\})| = 2$. Recall, however, that $Y \cup \{i\}$ and $X - \{i\}$ are distinct members of $C$ and so have symmetric difference at least 4, a contradiction. □

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