Invariants of multidimensional time series based on their iterated-integral signature

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We introduce a novel class of features for multidimensional time series, that are invariant with respect to transformations of the ambient space. The general linear group, the group of rotations and the group of permutations of the axes are considered. The starting point for their construction is Chen’s iterated-integral signature.

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1 Introduction

The analysis of multidimensional time series is a standard problem in data science. Usually, as a first step, features of a time series must be extracted that are (in some sense) robust and that characterize the time series. In many applications the features should additionally be invariant to a particular group acting on the data. In character recognition on tablets for example, the angle from which the device is operated is usually not fixed. This leads to the requirement of rotation invariant features. In EEG analysis, invariants to the general linear group are beneficial [EMZMN2012]. In some applications the labelling of coordinates is arbitrary, for example if we have observations of some system from a set of random or unlabelled sensors. Here permutation invariants are called for.

As any time series in discrete time can, via linear interpolation, be thought of as a multidimensional curve, one is naturally lead to the search of invariants of curves. Invariant features, of (mostly) two-dimensional curves have been treated using various approaches. Among the techniques are Fourier series (of closed curves) [Gra1972, ZR1972, KG1982], wavelets [CK1986], curvature based methods [MM1986, COSTH1998] and integral invariants [MCHYS2006]. Closest to our setting is probably the work [FKK2010].

The usefulness of iterated integrals in data analysis has recently been realized, see for example [LLN2013, Gra2013, KSHGL2017, YLNSJC2017] and the introduction in [CK2016]. Let us demonstrate the appearance of iterated integrals on a very simple example. Let

\[ X : [0, T] \to \mathbb{R}^2 \]

be a smooth curve. Say, we are looking for a feature describing this curve that is unchanged if one is handed a rotated version of \( X \). Maybe the simplest one that one can come up with is the (squared) total displacement length \(|X_T - X_0|^2\). Now,

\[ |X_T - X_0|^2 = (X^1_T - X^1_0)^2 + (X^2_T - X^2_0)^2 \]

\[ = 2 \int_0^T (X^1_r - X^1_0) \dot{X}^1_r dr + 2 \int_0^T (X^2_r - X^2_0) \dot{X}^2_r dr \]

\[ = 2 \int_0^T \left( \int_0^r \dot{X}^1_u du \right) \dot{X}^1_r dr + 2 \int_0^T \left( \int_0^r \dot{X}^2_u du \right) \dot{X}^2_r dr \]

\[ = 2 \int_0^T \int_0^r dX^1_u dX^1_r + 2 \int_0^T \int_0^r dX^2_u dX^2_r \]

where we applied the fundamental theorem of calculus twice and introduced the notation \( dX^i_r = X^i_r dr \). We see that we have succeeded in expressing this simple invariant in terms of iterated integrals of \( X \); the collection of which is usually called its signature. The aim of this work can be summarized as describing all invariants that can be obtained in this way. It turns out, when formulated in the right way this search for invariants reduces to classical problems in invariant theory. In some sense this work can hence be seen as the application of classical results to the setting of invariant feature selection for time series. We also produce some mathematically new results. For example we exhibit, what seems for the first time, a basis for invariants with respect to the general linear group (Lemma 10), and state new geometric interpretations for certain terms of the signature (Section 3.4).

The paper is structured as follows. In the next section we introduce the signature of iterated integrals of a multidimensional curve, as well as some algebraic language to work with it. Based
on this signature, we present in Section 3 and Section 4 invariants to the general linear group and the special orthogonal group. Both are based on classical results in invariants theory. In the case of the general linear group we are able to present a linear basis of the invariants. For completeness, we present in Section 5 the invariants to permutations, which have been constructed in [NRRZ2005]. In Section 6 we show how to use all these invariants if an additional (time) coordinate is introduced.

For readers who want to use these invariants without having to go into the technical results, we propose the following route. The required notation is presented in the next section. The invariants are presented in Proposition 11, Proposition 31 and Proposition 39. Examples are given in Section 3.1 (in particular Remark 14), Example 34 and Example 40. All these invariants are also implemented in the software package D2018. For a python package for calculating the iterated-integrals signature we propose using the package iisignature, as described in [Rei2017].

2 The signature of iterated integrals

By a multidimensional curve $X$ we will denote a continuous mapping $X : [0, T] \to \mathbb{R}^d$ of bounded variation. The aim of this work is to find features (i.e. complex or real numbers) describing such a curve, that are invariant under the general linear group, the group of rotations and the group of permutations. Note, that in practical situation one is usually presented with a discrete sequence of data points in $\mathbb{R}^d$, a multidimensional time series. Such a time series can be easily transformed into a (piecewise) smooth curve by linear interpolation.

It was proven in [Che1957] (see [HJL2010] for a recent generalization) that a curve $X$ is almost completely characterized by the collection of its iterated integrals:

$$\int_0^T \int_0^{r_2} \ldots \int_0^{r_n} dX_{i_1}^{r_1} \ldots dX_{i_n}^{r_n}, \quad n \geq 1, \quad i_1, \ldots, i_n \in \{1, \ldots, d\}.$$ 

The collection of all these integrals is called the signature of $X$. In a first step, we can hence reduce the goal

Find functions $\Psi : \text{curves} \to \mathbb{R}$ that are invariant under the action of a group $G$.

to the goal

Find functions $\Psi : \text{signature of curves} \to \mathbb{R}$ that are invariant under the action of a group $G$.

By the Shuffle identity (Lemma 1), any polynomial function on the signature can be re-written as a linear function on the signature. Assuming that arbitrary functions are well-approximated by polynomial functions, we are lead to the final simplification, which is the goal of this paper

---

1 The reader might prefer to just think of a (piecewise) smooth curve. The standard example in applications will be the piecewise linear interpolation of $d$-dimensional data measured at discrete time points.

2 Since $X$ is of bounded variation the integrals are well-defined using classical Riemann-Stieltjes integration (see for example Chapter 6 in [Rud1964]). This can be pushed much further though. In fact the following considerations are purely algebraic and hence hold for any curve for which a sensible integration theory exists. A relevant example is Brownian motion which, although being almost surely nowhere differentiable, nonetheless admits a stochastic (Stratonovich) integral.
Find linear functions $\Psi : \text{signature of curves} \rightarrow \mathbb{R}$ that are invariant under the action of a group $G$.

### 2.1 Algebraic underpinning

Let us introduce some algebraic notation in order to work with the collection of iterated integrals. Denote by $T((\mathbb{R}^d))$ the space of formal power series in $d$ non-commuting variables $x_1, x_2, \ldots, x_d$. We can conveniently store all the iterated integrals of the curve $X$ in $T((\mathbb{R}^d))$, by defining the signature of $X$ to be

$$S(X)_{0,T} := \sum_{i_1, \ldots, i_n \in \{1, 2\}} x_1^{i_1} \cdots x_n^{i_n} \int_0^{r_1} \cdots \int_0^{r_n} dX_{i_1} \cdots dX_{i_n}.$$ 

Here the sum is taken over all $n \geq 0$ and all $i_1, \ldots, i_n \in \{1, 2\}$. For $n = 0$ the summand is, for algebraic reasons, taken to be the constant 1.

The algebraic dual of $T((\mathbb{R}^d))$ is $T(\mathbb{R}^d)$, the space of polynomials in $x_1, x_2, \ldots, x_d$. The dual pairing, denoted by $\langle \cdot, \cdot \rangle$ is defined by declaring all monomials to be orthonormal, so for example

$$\langle x_1 + 15 \cdot x_1 x_2 - 2 \cdot x_1 x_2 x_1, x_1 x_2 \rangle = 15.$$ 

Here we write the element of $T((\mathbb{R}^d))$ on the left and the element of $T(\mathbb{R}^d)$ on the right. We can “pick out” iterated integrals from the signature as follows

$$\langle S(X)_{0,T}, x_1^{i_1} \cdots x_n^{i_n} \rangle = \int_0^{r_1} \cdots \int_0^{r_n} dX_{i_1}^{r_1} \cdots dX_{i_n}^{r_n}.$$ 

The space $T((\mathbb{R}^d))$ becomes an algebra by extending the usual product of monomials, denoted $\cdot$, to the whole space by bilinearity. Note that $\cdot$ is non-commutative.

On $T(\mathbb{R}^d)$ we usually use the shuffle product $\shuffle$ which, on monomials, interleaves them in all order-preserving ways, so for example

$$x_1 \shuffle x_2 x_3 = x_1 x_2 x_3 + x_2 x_1 x_3 + x_2 x_3 x_1.$$ 

Note that $\shuffle$ is commutative.

Monomials, and hence homogeneous polynomials, have the usual concept of order or homogeneity. For $n \geq 0$ we denote the projection on polynomials of order $n$ by $\pi_n$, so for example

$$\pi_2(x_1 + 15 \cdot x_1 x_2 - 2 \cdot x_1 x_2 x_1) = 15 \cdot x_1 x_2.$$ 

See [Reu1993] for more background on these spaces.

As mentioned above, every polynomial expression in terms of the signature can be re-written as a linear expression in (different) terms of the signature. This is the content of the following lemma, which is proven in [Ree1958].

---

3 In contrast to a power series, a polynomial only has finitely many terms.
Lemma 1 (Shuffle identity). Let \( X : [0, T] \to \mathbb{R}^d \) be a continuous curve of bounded variation, then for every \( a, b \in T(\mathbb{R}^d) \)

\[
\langle S(X)_{0,T}, a \rangle \langle S(X)_{0,T}, b \rangle = \langle S(X)_{0,T}, a \uplus b \rangle
\]

Remark 2. We have used this fact already in the introduction, where we confirmed by hand that

\[
\left( \langle S(X)_{0,T}, x_1 \rangle \right)^2 + \left( \langle S(X)_{0,T}, x_2 \rangle \right)^2 = 2 \langle S(X)_{0,T}, x_1 x_1 \rangle + 2 \langle S(X)_{0,T}, x_2 x_2 \rangle
\]

We will use the following fact repeatedly, which also explains the commonly used name tensor algebra for \( T(\mathbb{R}^d) \).

Lemma 3. The space of all multilinear maps on \( \mathbb{R}^d \times \ldots \times \mathbb{R}^d \) \((n\text{-times})\) is in a one-to-one correspondence with homogeneous polynomials of order \( n \) in the non-commuting variables \( x_1, \ldots, x_d \) by the following bijection

\[
\psi \mapsto \text{poly}(\psi) := \sum_{i_1, \ldots, i_n \in \{1, \ldots, d\}} \psi(e_{i_1}, e_{i_2}, \ldots, e_{i_n}) x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_n},
\]

with \( e_i \) the \( i \)-th canonical basis vector of \( \mathbb{R}^d \).

3 General linear group

Let

\[
GL(\mathbb{R}^d) = \{ A \in \mathbb{R}^{d \times d} : \det(A) \neq 0 \},
\]

be the general linear group of \( \mathbb{R}^d \).

Definition 4. For \( w \in \mathbb{N} \), we call \( \phi \in T(\mathbb{R}^d) \) a GL invariant of weight \( w \) if

\[
\langle S(AX)_{0,T}, \phi \rangle = (\det A)^w \langle S(X)_{0,T}, \phi \rangle
\]

for all \( A \in GL(\mathbb{R}^d) \).

Definition 5. Define a linear action of \( GL(\mathbb{R}^d) \) on \( T((\mathbb{R}^d)) \) and \( T(\mathbb{R}^d) \), by specifying on monomials

\[
Ax_{i_1} \cdots x_{i_n} := \sum_j (Ae_{i_1})_{j_1} x_{j_1} \cdots (Ae_{i_n})_{j_n} x_{j_n}
\]

\[
= \sum_j A_{j_1 i_1} \cdots A_{j_n i_n} x_{j_1} \cdots x_{j_n}.
\]

Lemma 6. For all \( A \in \mathbb{R}^{d \times d} \), \( X \) a curve,

\[
\langle S(AX)_{0,T}, \phi \rangle = \langle S(X)_{0,T}, A^\top \phi \rangle.
\]
Proof. It is enough to verify this on monomials $\phi = x_{\ell_1} \ldots x_{\ell_m}$. Then

$$\langle S(AX), \phi \rangle = \sum_j A_{\ell_1 j_1} \ldots A_{\ell_m j_m} \int dX^{j_1} \ldots dX^{j_m}$$

$$= \langle S(X), \sum_j A_{\ell_1 j_1} x_{j_1} \ldots A_{\ell_m j_m} x_{j_m} \rangle$$

$$= \langle S(X), A^\top \phi \rangle.$$ 

By Lemma 43 the set $\{ S(X)_0, T : X \text{ curve } \}$ spans $T(\mathbb{R}^d)$. Hence $\phi$ is a GL invariant of weight $w$ in the sense of Definition 4 if and only if for all $A \in GL(\mathbb{R}^d)$

$$A^\top \phi = (\det A)^w \phi.$$

Since the action respects homogeneity, we immediatly obtain that projection of invariants are invariants (take $B = (\det A)^{-w} A^\top$ in the following lemma):

Lemma 7. If $\phi \in T(\mathbb{R}^d)$ satisfies

$$B\phi = \phi,$$

for some $B \in GL(\mathbb{R}^d)$ then

$$B\pi_m \phi = \pi_m \phi,$$

for all $n \geq 1$.

Proof. By definition, the action of $GL$ on $T(\mathbb{R}^d)$ commutes with $\pi_n$. 

In order to apply classical results in invariant theory, we use the bijection $\text{poly}$ between multilinear functions and noncommuting polynomials, given in Lemma 3.

Lemma 8. For $\psi : (\mathbb{R}^d)^\times n \to \mathbb{R}$ multilinear and $A \in GL(\mathbb{R}^d)$,

$$\text{poly}[\psi(A \cdot)] = A^\top \text{poly}[\psi].$$

Proof.

$$\text{poly}[\psi(A \cdot)] = \sum_i \psi(Ae_{i_1} \ldots Ae_{i_n}) x_{i_1} \ldots x_{i_n}$$

$$= \sum_{i,j} A_{j_1 i_1} \ldots A_{j_n i_n} \psi(e_{j_1} \ldots e_{j_n}) x_{j_1} \ldots x_{j_n}$$

$$= \sum_j \psi(e_{j_1} \ldots e_{j_n}) A^\top x_{j_1} \ldots x_{j_n}$$

$$= A^\top B \psi.$$ 

\[\square\]
The simplest multilinear function
\[
\Psi : (\mathbb{R}^d)^n \to \mathbb{R},
\]
satisfying \(\Psi(Av_1, \ldots, Av_n) = \det(A)\Psi(v_1, \ldots, v_n)\) that one can maybe think of, is the determinant itself. That is, \(n = d\) and
\[
\Psi(v_1, \ldots, v_n) = \det[v_1v_2\ldots v_n],
\]
where \(v_1v_2\ldots v_n\) is the \(d \times d\) matrix with columns \(v_i\). Up to a scalar this is in fact the only one, and invariants of higher weight are built only using determinants as a building block.

The following result is classical. There seem to be at least three ways to prove it. Weyl [Wey1946] uses the “Capelli identities” and Dieudonné [DC1970, Section 2.5] uses the Young theory of the symmetric group. See [Gar1975] for a simple modern proof.

**Theorem 9.** Let \(V\) be a \(d\)-dimensional vector space. A multilinear map
\[
\psi : V \times \ldots \times V \to \mathbb{R}
\]
n times
satisfies
\[
\psi(Av_1, Av_2, \ldots, Av_n) = (\det A)^w \psi(v_1, v_2, \ldots, v_n)
\]
for all \(A \in \text{GL}(V)\) and \(v_1, \ldots, v_n \in V\) if and only if \(n = wd\) and \(\psi\) is the linear combination of functions of the form
\[
\det[v_{\sigma(1)}v_{\sigma(2)}\ldots v_{\sigma(d)}] \cdot \det[v_{\sigma(d+1)}v_{\sigma(d+2)}\ldots v_{\sigma(2d)}] \cdot \ldots \cdot \det[v_{\sigma((w-1)d+1)}v_{\sigma((w-1)d+2)}\ldots v_{\sigma(wd)}]
\]
where \(\sigma\) can be any permutation of the numbers \(1, \ldots, wd\).

To state the following theorem, we introduce the notion of Young diagrams, which play an important role in the representation theory of the symmetric group.

Let \(n \in \mathbb{N}\), and \(\lambda = (\lambda_1, \ldots, \lambda_r)\) be a partition, which we assume ordered as \(\lambda_1 \geq \lambda_2 \geq \ldots \lambda_r\).

We associate to it a **Young diagram**, which is a collection of \(n\) boxes, with left-justified rows. There are \(r\) rows, with \(\lambda_i\) boxes in the \(i\)-th row.

For example, the partition \((4, 2, 1)\) of 7 gives the Young diagram

```
  X
  X
  X
  X
  X
```

A **Young tableau** is obtained by filling these boxes with the numbers 1,..,\(n\). Continuing the example, the following is a Young tableau

```
  2 3 7 1
  5 4
  6
```
A Young tableau is **standard** if the values in every row are increasing (from left to right) and are increasing in every column (from top to bottom). The previous tableau was not standard; the following is.

\[
\begin{array}{ccc}
1 & 3 & 5 & 7 \\
5 & 4 & & \\
6 & & & \\
\end{array}
\]

To our surprise the following statement on a linear basis for the invariants of Theorem 9 cannot be found in the literature.

**Lemma 10.** A linear basis for the invariants Theorem 9 for \( n = wd \) is given by

\[
\{ v \mapsto \det[v_{C_1}] \ldots \det[v_{C_w}] \}
\]

where \( C_i \) are the columns of \( \Sigma \), and \( \Sigma \) ranges over all standard Young tableaus corresponding to the partition \( \lambda = (w, w, \ldots, w) \) of \( n \).

Here, for a sequence \( C = (c_1, \ldots, c_d) \), \( v_C \) denotes the matrix of column vectors \( v_{c_i} \), i.e.

\[
v_C = (v_{c_1}, \ldots, v_{c_d}).
\]

**Proof.** We will show that the invariants form an irreducible representation of \( S_n \), for which an explicit basis can be given.

We first sketch how the irreducible representations for \( S_n \) are constructed. Let us recall that a **tabloid** is an equivalence class of Young tableaux modulo permutations leaving the set of entries in each row invariant [Sag2013, Chapter 2]. For \( t \) a Young tableau denote \( \{t\} \) its tabloid, so for example

\[
\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}.
\]

The symmetric group \( S_n \) acts on Young tableaux as

\[
(\tau \cdot t)_{ij} := \tau(t_{ij}).
\]

For example

\[
(23) \cdot \begin{array}{cc}
2 & 4 \\
1 & 3 \\
\end{array} = \begin{array}{cc}
3 & 4 \\
1 & 2 \\
\end{array}
\]

In then acts on tabloids by \( \tau \cdot \{t\} := \{\tau \cdot t\} \). Define for a Young tableau \( t \)

\[
e_t := \sum_{\pi} \text{sign}(\pi)\pi\{t\},
\]
where the sum is over all \( \pi \in S_n \) that leave the set of values in each column invariant. For example with

\[
t = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

\[
e_t = \{t\} + (13)\{t\} + (24)\{t\} + (13)(24)\{t\}.
\]

Then

\[
\text{Irrep}_{(w,\ldots,w)} := \text{span}\{e_t : \text{Young tableau of shape } (w,\ldots,w)\}
\]

is an irreducible representation of \( S_n \) and

\[
\{e_t : \text{standard Young tableau of shape } (w,\ldots,w)\},
\]

forms a basis [Sag2013, Theorem 2.5.2]. This concludes the reminder on representation theory for \( S_n \).

We now show that the invariants are an irreducible representation isomorphic to \( \text{Irrep}_{(w,\ldots,w)} \).

First, \( S_n \) acts on \( n \)-multilinear functions by permutation of the arguments

\[
\tau \cdot \Psi(v_1,\ldots,v_n) := \Psi(v_{\tau(1)},\ldots,v_{\tau(n)}).
\]

This means, for tensors \( \Psi = w_1^* \otimes \ldots \otimes w_n^* \) (note the inverse)

\[
\tau \cdot w_1^* \otimes \ldots \otimes w_n^* = w_{\tau(1)}^* \otimes \ldots \otimes w_{\tau(n)}^*.
\]

Denote the space of \( GL \) invariants of weight \( w \) given in Theorem 9 by invariants. It is a sub-vectorspace of \( n \)-multilinear functions, \( n = wd \). Clearly \( S_n \cdot \text{invariants} = \text{invariants} \). It is in fact irreducible, since for any \( \sigma \in S_n \) the orbit of

\[
\text{det}[v_{\sigma(1)}\ldots v_{\sigma(d)}] \ldots \text{det}[v_{\sigma((w-1)d+1)}\ldots v_{\sigma(n)}]
\]

under \( S_n \) spans invariants.

For a tabloid of shape \((w,w,\ldots,w)\) define

\[
\iota(\{t\}) := e_{j_1}^* \otimes \ldots \otimes e_{j_n}^*,
\]

where

\[
j_\ell = i \quad \Leftrightarrow \quad \ell \in i\text{-th row of } \{t\}.
\]

For example

\[
\iota \left( \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \right) = e_1^* \otimes e_1^* \otimes e_2^* \otimes e_2^* \otimes e_1^* \otimes e_2^*.
\]
This is a homomorphism of $S_n$ representations. Indeed,

$$\iota(\tau \cdot \{t\}) = e_{j_1}^* \otimes \ldots \otimes e_{j_n}^*,$$

with

$$j_\ell = i \iff \ell \in \text{i-th row of } \tau \cdot \{t\}.$$

On the other hand

$$\tau \cdot \iota(\{t\}) = \tau \cdot e_{r_1}^* \otimes \ldots \otimes e_{r_n}^*$$

$$= e_{p_1}^* \otimes \ldots \otimes e_{p_n}^*,$$

with $p_\ell = i \iff \tau_{\tau^{-1}(\ell)} = i$

$$\iff \tau^{-1}(\ell) \in \text{i-th row of } \{t\}$$

$$\iff \ell \in \text{i-th row of } \tau \cdot \{t\}.$$

Hence $\iota(\tau \cdot \{t\}) = \tau \cdot \iota(\{t\})$, and $\iota$ is a homomorphism of $S_n$ representations. Hence, restricted to the irreducible representation $\text{Irrep}((w,\ldots,w))$, $\iota$ is either 0 or a bijection to its image [Sag2013, Theorem 1.6.5 (Schur’s lemma)]. We now show that its image is invariants and establish a basis for invariants at the same time.

Define the standard Young tableau of shape $(w,\ldots,w)$

$$t_{first} := \begin{array}{cccc}
1 & d + 1 & \ldots & \ldots \\
2 & d + 2 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
d & 2d & \ldots & n
\end{array}$$

Then for any (standard) Young tableau $t$ there exists unique $\sigma_t \in S_n$ such that

$$\sigma_t \cdot t_{first} = t.$$

Then

$$\iota(e_t) = \left(v \mapsto \det[v_{\sigma_t(1)} \ldots v_{\sigma_t(d)}] \cdot \ldots \cdot \det[v_{\sigma_t((w-1)d+1)} \ldots v_{\sigma_t(n)}]\right).$$

Indeed, since $\iota$ is a homomorphism of $S_n$ representation,

$$\iota(\sigma_t \cdot e_{t_{first}})(v_1, \ldots, v_n) = \iota(e_{t_{first}})(v_{\sigma_t(1)}, \ldots, v_{\sigma_t(n)})$$
It remains to check
\[ \iota(e_{t_{\text{first}}}) = \det[v_1..v_d]..\det[v_{(w-1)d+1}..v_n]. \]

In general, every \( \pi \in S^n \) that is column-preserving for \( t_{\text{first}} \) can be written as the product \( \pi_1 \cdot \cdots \cdot \pi_w \), with \( \pi_j \) ranging over the permutations of the entries of the \( j \)-th column \( t_{\text{first}} \). Then
\[
\iota(e_{t_{\text{first}}})(v_1,..,v_n) = \sum_{\pi} \text{sign } \pi \cdot \iota(\pi(\{t\}))(v_1,..,v_n)
= \sum_{\pi_j} \prod_j \text{sign } \pi_j \cdot \iota(\pi_1..\pi_w(\{t\}))(v_1,..,v_n)
= \sum_{\pi_j} \prod_j \text{sign } \pi_j \cdot e^*_{(\pi_1^{-1})_1} \otimes \cdots \otimes e^*_{(\pi_1^{-1})_d} \otimes e^*_{(\pi_2^{-1})_{(d+1)} \text{mod } d+1} \otimes \cdots
\]
\[
\cdots \otimes e^*_{(\pi^{-1}_{w-n})_{\text{mod } d+1}}(v_1,..,v_n)
= \det[v_1..v_d]..\det[v_{(w-1)d+1}..v_n],
\]
as desired.

Hence \( \iota(\text{Irrep}(w,..,w)) = \text{invariants} \) and
\[ \{ \iota(e_t) : t \text{ standard Young tableau of shape } (w,..,w) \}, \]
forms a basis for \text{invariants}, as claimed. \( \square \)

Applying Lemma 3 to Lemma 10 we get the invariants in \( T(\mathbb{R}^d) \).

**Proposition 11.** A linear basis for the space of GL invariants of order \( n = wd \) is given by
\[
\sum_{i_1,..,i_n \in \{1,..,d\}} g_{\Sigma}(i_1, i_2, \ldots, i_n)x_{i_1}x_{i_2} \ldots x_{i_n},
\]
where
\[
g_{\Sigma}(v) = \det[v_{C_1}]..\det[v_{C_w}],
\]
where \( C_i \) are the columns of \( \Sigma \), and \( \Sigma \) ranges over all standard Young tableaus corresponding to the partition \( \lambda = (w, w, \ldots, w) \) of \( n \).

**Remark 12.** By Lemma 7, for any invariant \( \phi \in T(\mathbb{R}^d) \) and \( n \geq 1 \) we have that \( \pi_n \phi \) is also invariant. Hence the previous theorem characterizes all invariants we are interested in (Definition 4), not just homogeneous ones.

**Remark 13.** Note that each of these invariants \( \phi \) consists only of monomials that contain every variable \( x_1, \ldots, x_d \) at least once. This implies that \( \langle S(X)_{0,T}, \phi \rangle \) consists only of iterated integrals that contain every component \( X^1, \ldots, X^d \) of the curve at least once. Hence, if at least one of these components is constant, the whole expression will be zero.

Since \( \phi \) is invariant, this implies that \( \langle S(X)_{0,T}, \phi \rangle = 0 \) as soon as there is some coordinate transformation under which one component is constant, that is whenever the curve \( X \) stays in a hyperplane of dimension strictly less then \( d \).
3.1 Examples

We will use the following short notation:

\[ i_1 \ldots i_n := x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_n} \]

so, for example

\[ 1121 := x_1 x_1 x_2 x_1. \]

We present the invariants described in Section 2 for some special cases of \( d \) and \( w \).

We give the linear basis for certain dimensions \( d \) and orders \( w \), given in Lemma 10.

The case \( d = 2 \)

Level 2 \( (w = 1) \)

\[ 12 - 21 \]

Remark 14. Let us make clear that from the perspective of data analysis, the “invariant” of interest is really the action of this element in \( T(\mathbb{R}^d) \) on the signature of a curve.

In this example, the real number

\[ \langle S(X)_{0,T}, 12 - 21 \rangle = \int_0^T \int_{r_1}^{r_2} dX_1 dX_2^2 - \int_0^T \int_{r_1}^{r_2} dX_1^2 dX_2, \]

changes only by the determinant of \( A \in GL(\mathbb{R}^2) \) when calculating it for the transformed curve \( AX \),

\[ \langle S(AX)_{0,T}, 12 - 21 \rangle = \text{det}(A) \langle S(X)_{0,T}, 12 - 21 \rangle. \]

Level 4 \( (w = 2) \)

\[ 1212 - 1221 - 2112 + 2121 \]

\[ 1122 - 1221 - 2112 + 2211 \]

Remark 15. This is a linear basis of invariants in the fourth level. If one takes algebraic dependencies into consideration, the set of invariants becomes smaller. To be specific, assume that one already has knowledge of the invariant of level 2 (i.e. \( \langle S(X)_{0,T}, 12 - 21 \rangle \)). If, say in a machine learning application, the learning algorithm can deal sufficiently well with nonlinearities, one should not be required to provide additionally the square of this number. In other words \( |\langle S(X)_{0,T}, 12 - 21 \rangle|^2 \) can also be assumed to be “known”. But, by the shuffle identity (Lemma 1 in the Appendix), this can be written as

\[ \langle S(X)_{0,T}, 12 - 21 \rangle \cdot \langle S(X)_{0,T}, 12 - 21 \rangle = \langle S(X)_{0,T}, (12 - 21) \cup (12 - 21) \rangle \]

\[ = \langle S(X)_{0,T}, 4 \cdot 1122 - 4 \cdot 1221 - 4 \cdot 2112 + 4 \cdot 2211 \rangle. \]

Now, given \( \phi = 4 \cdot 1122 - 4 \cdot 1221 - 4 \cdot 2112 + 4 \cdot 2211 \) there is only one “new” independent invariant in the fourth level, namely \( 1212 - 1221 - 2112 + 2121 \).

A similar analysis can also be carried out for the following invariants, but we refrain from doing so, since it can be easily done with a computer algebra system.
Level 6 \((w = 3)\)

\[
\begin{align*}
121212 &− 121221 − 122112 + 122121 − 211212 + 211221 + 212112 − 212121 \\
112212 &− 112221 − 122112 + 122121 − 211212 + 211221 + 221112 − 221121 \\
121122 &− 121221 − 122112 + 122121 − 211212 + 211221 + 212112 − 212211 \\
112122 &− 112221 − 122112 + 122121 − 211212 + 211221 + 221112 − 221211 \\
111222 &− 112221 − 121212 + 122112 − 211212 + 212112 + 221112 − 222111 \\
\end{align*}
\]

The case \(d = 3\)

Level 3 \((w = 1)\)

\[
123 − 132 − 213 + 231 + 312 − 321
\]

Level 6 \((w = 2)\)

\[
\begin{align*}
123123 &− 312132 + 312312 + 213132 − 213231 − 213123 + 321213 − 312321 − 132231 − 132123 \\
&− 321231 + 321132 + 312321 + 132213 + 231231 + 321321 + 213321 + 123231 + 231123 − 312213 \\
&− 321123 − 231132 + 213213 + 132132 + 312231 − 213123 − 231321 − 132312 − 123213 − 321312 \\
&+ 312123 − 231213 + 231312 + 123231 + 123132 + 231321 + 213231 + 312213 + 123123 − 132123 \\
&− 213231 − 213321 + 213132 − 132231 + 122331 + 321321 + 312213 + 122132 − 212321 + 311322 + 133221 \\
&+ 132123 + 312231 + 233112 − 323112 − 231123 + 313212 − 133212 + 132312 + 131232 + 311223 \\
&− 232113 + 322113 − 123321 − 322131 + 123312 + 232131
\end{align*}
\]

Level 4 \((w = 1)\)

\[
\begin{align*}
1234 &− 1243 − 1324 + 1342 + 1423 − 1432 − 2134 + 2143 + 2314 − 2341 − 2413 + 2431 + 3124 \\
&− 3142 − 3214 + 3241 + 3412 − 3421 − 4123 + 4132 + 4213 − 4231 − 4312 + 4321
\end{align*}
\]

The case \(d = 4\)
3.2 The case \( d \) arbitrary, \( w = 1 \)

Whatever the dimension \( d \) of the curve’s ambient space, the space of invariants of weight 1 has dimension 1 and is spanned by

\[
\text{Inv}_d := \text{Inv}_d(x_1, \ldots, x_d) := \sum_{\sigma \in S_d} \text{sign}(\sigma) \ x_{\sigma(1)} \cdots x_{\sigma(d)} = \det \begin{pmatrix} x_1 & \cdots & x_d \\ \vdots & \ddots & \vdots \\ x_1 & \cdots & x_d \end{pmatrix}.
\]

Here, for a matrix \( C \) of non-commuting variables,

\[
\det C := \sum_{\tau} \text{sign} \tau \ \prod_i C_{i\tau(i)}.
\]

The following lemma tells us that \( \text{Inv}_d \), although a formerly on level \( d \), can be written in terms of expressions on level 1 and 2.

**Lemma 16.**

\[
\text{Inv}_d(x_1, \ldots, x_d) = \frac{1}{d} \sum_{j=1}^{d} (-1)^{j+1} x_j \sqcup \text{Inv}_{d-1}(x_1, \ldots, \widehat{x}_j, \ldots, x_d),
\]

where \( \widehat{x}_j \) denotes the omission of that argument.

**Proof.** We can express the determinant in terms of minors, with respect to any row (since the \( x_i \) are non-commuting, this does not work with columns!). So e.g. for \( d = 3 \)

\[
\text{Inv}_3 = x_1(x_2 x_3 - x_3 x_2) - x_2(x_1 x_3 - x_3 x_1) + x_3(x_1 x_2 - x_2 x_1)
\]

\[
= x_2 x_1 x_3 - x_3 x_1 x_2 - x_1 x_2 x_3 + x_3 x_2 x_1 + x_1 x_3 x_2 - x_2 x_3 x_1
\]

\[
= (x_2 x_3 - x_3 x_2)x_1 - (x_1 x_3 - x_3 x_1)x_2 + (x_1 x_2 - x_2 x_1)x_3
\]

\[
= \text{Insert-After}(x_1, r) \text{Inv}_2(x_2, x_3) - \text{Insert-After}(x_2, r) \text{Inv}_2(x_1, x_3)
\]

\[
+ \text{Insert-After}(x_3, r) \text{Inv}_2(x_1, x_2),
\]

for \( r = 0, 1, 2 \). Here, for a monomial of order \( n \geq r \),

\[
\text{Insert-After}(x_i, r) \phi,
\]

inserts the variable \( x_i \) after position \( r \), e.g.

\[
\text{Insert-After}(x_1, 1) \text{Inv}_2(x_2, x_3) = \text{Insert-After}(x_1, 1)\left(x_2 x_3 - x_3 x_2\right)
\]

\[
= x_2 x_1 x_3 - x_3 x_1 x_2.
\]

Summing up we see that

\[
3 \cdot \text{Inv}_3 = x_1 \sqcup \text{Inv}_2(x_2, x_3) - x_2 \sqcup \text{Inv}_2(x_1, x_3) + x_3 \sqcup \text{Inv}_2(x_1, x_2),
\]

\[
\text{Insert-After}(x_i, r) \phi,
\]
In general, for any \( i = 0, \ldots, d - 1 \),

\[
\text{Inv}_d = \sum_{r=1}^{d} (-1)^{j+1} \text{Insert-After}(x_j, r) \text{Inv}_{d-1}(x_1, \ldots, \hat{x}_j, \ldots, x_d),
\]

and summing over \( r \) gives

\[
\text{Inv}_d = \frac{1}{d} \sum_{j=1}^{d} (-1)^{j+1} x_j \text{Inv}_{d-1}(x_1, \ldots, \hat{x}_j, \ldots, x_d).
\]

\[\square\]

### 3.3 The case \( d = 2, \ w = 1 \)

**Geometric interpretation**

The invariant for \( d = 2, w = 1 \), namely \( \phi = x_1x_2 - x_2x_1 \) has a simple geometric interpretation: it picks out (two times\(^4\)) the area (signed, and with multiplicity) between the curve \( X \) and the cord spanned between its starting and endpoint. This follows from Green’s theorem \[^{[Rud1964]}\].

![Figure 1: A curve \( X = (X^1, X^2) \) is shown, with shaded area given by \( \frac{1}{2}(S(X)_{0,T}, x_1x_2 - x_2x_1) = \frac{1}{2} \int_0^T \int_0^{r_2} dX_1^1 dX_2 - \frac{1}{2} \int_0^T \int_0^{r_2} dX_2^1 dX_1^2 \).](image)

**Connection to correlation**

Assume that \( X \) is a continuous curve, piecewise linear between some time point \( t_i, \ i = 0, \ldots, n \)\(^5\).

\[^{4}\] The prefactor \( 1/2 \) is irrelevant, so we will speak of \( \phi \) and also of \( \frac{1}{2} \phi \) as picking out the area.

\[^{5}\] The standard example is a curve that is discretely observed at times \( t_i \) and linearly interpolated in between.
The area is then explicitly calculated as
\[
\int_0^T \int_0^r dX_u^1 dX_r^2 - \int_0^T \int_0^r dX_u^2 dX_r^1 = \int_0^T (X_r^1 - X_0^1) dX_r^2 - \int_0^T (X_r^2 - X_0^2) dX_r^1 = 1/2 \sum_{i=0}^{n-1} (X_{t_{i+1}}^1 - X_{t_i}^1 + X_{t_i}^1 - X_{t_0}^1) (X_{t_{i+1}}^2 - X_{t_i}^2) - 1/2 \sum_{i=0}^{n-1} (X_{t_{i+1}}^2 - X_{t_i}^2 + X_{t_i}^2 - X_{t_0}^2) (X_{t_{i+1}}^1 - X_{t_i}^1)
\]
\[
= \sum_{i=0}^{n-1} X_{t_i}^1 \left[ X_{t_{i+1}}^2 - X_{t_0}^2 \right] - \sum_{i=0}^{n-1} \left[ X_{t_{i+1}}^1 - X_{t_i}^1 \right] X_{t_i}^2 = \text{Corr}(X^2 - X_0^2, X^1)_1 - \text{Corr}(X^1 - X_0^1, X^2)_1
\]

Here, for two vectors \(a, b\) of length \(n\)

\[
\text{Corr}(a, b)_1 := \sum_{i=0}^{n-1} a_{i+1} b_i,
\]

the lag-one cross-correlation, which is a commonly used feature in data analysis.

In particular, if the curve starts at 0, we have

\[
\int_0^T \int_0^r dX_u^1 dX_r^2 - \int_0^T \int_0^r dX_u^2 dX_r^1 = \text{Corr}(X^2, X^1)_1 - \text{Corr}(X^1, X^2)_1,
\]

which is an antisymmetrized version of the lag-one cross-correlation.

**Remark 17.** Note that it is immediate that the antisymmetrized version of the lag \(\tau\) cross-correlation, \(\tau \geq 2\) are also invariants of the curve. Where they can be found in the signature \(S(X)\) is unknown to us.

### 3.4 The case \(d = 3, w = 1\)

**Geometric interpretation**

In this case, we have a geometric interpretation of the single invariant. Given a curve \(X\), let \(V\) be a sixth of the invariant we have identified. Then

\[
V = \frac{1}{6} \langle S(X)_{0,T}, 123 - 132 - 213 + 231 + 312 - 321 \rangle = \frac{1}{6} \langle S(X)_{0,T}, (12 - 21) \uplus 3 + (23 - 32) \uplus 1 + (31 - 13) \uplus 2 \rangle
\]

\[
= \frac{1}{6} \left[ \langle S(X)_{0,T}, 12 - 21 \rangle \langle S(X)_{0,T}, 3 \rangle + \langle S(X)_{0,T}, 23 - 32 \rangle \langle S(X)_{0,T}, 1 \rangle + \langle S(X)_{0,T}, 31 - 13 \rangle \langle S(X)_{0,T}, 2 \rangle \right],
\]
which is a special case of Lemma 16. We have a clear geometric interpretation in terms of areas and increments.

If the curve is closed, then $X_{0,T}^1 = X_{0,T}^2 = X_{0,T}^3 = 0$, so $V$ is clearly 0. If the curve lies in a plane, by Remark 13 $V$ is also 0.

### 3.4.1 A tetrahedron

If $a, b, c$ and $d$ are points in $\mathbb{R}^3$, then we say the **signed volume** of the tetrahedron with corners $a, b, c$ and $d$ (in order) is

$$\frac{1}{6}[(b-a) \times (c-a) \cdot (d-a)] = \frac{1}{6} \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \end{bmatrix}.$$  

Its absolute value is the volume of the tetrahedron.

**Proposition 18.** A linearly interpolated curve through four points has $V$ equal to the signed volume of the tetrahedron on them, and thus $|V|$ is equal to the volume of its convex hull.

**Proof.** Without loss of generality we set $a = 0$. If the points are coplanar, then both $V$ and the volumes are 0. We are concerned with the case where the points are in general position. Consider the curve to be made of three line segments, between the points 0, $b$, $c$ and $d$. Then the convex hull is a tetrahedron, whose volume is a third of the volume of the parallelepiped whose sides are the three line segments. Without loss of generality, 0, $b$ and $c$ (which form the base triangle, say) lie in the $x_1x_2$ plane. $V$ and the volumes are unchanged under a shear which fixes the $x_1x_2$ plane and the base triangle but sends $d$ to the $x_3$ axis. Then

$$V = \frac{1}{6} \langle S(X)_{0,T}, 12 - 21 \rangle \langle S(X)_{0,T}, 3 \rangle$$

$$= \frac{1}{3} \frac{1}{2} \langle S(X)_{0,T}, 12 - 21 \rangle \langle S(X)_{0,T}, 3 \rangle$$

$$= \frac{1}{3} \text{(signed area of base triangle)} \cdot \text{(perpendicular height of } d \text{ from base triangle)}$$

$$= \text{signed volume of tetrahedron}$$

Consider a tetrahedron whose first point is 0 and last point is $(0, 0, 1)$ on the $x_3$-axis. We know $V$ of the curve through the points is its signed volume, and we know

$$V = \frac{1}{6} \langle S(X)_{0,T}, 12 - 21 \rangle \langle S(X)_{0,T}, 3 \rangle = \frac{1}{6} \langle S(X)_{0,T}, 12 - 21 \rangle$$

This gives a slightly non-obvious formula for the signed volume of a tetrahedron as follows. Look along the edge of the tetrahedron from the beginning to the end. The signed volume is $\frac{1}{3}$ of the length of the edge times the signed area of the triangle that you see.

**Proposition 19.** The signed volume of the tetrahedron between the four points $(0, 0, 0), (p_1, p_2, p_3), (q_1, q_2, q_3), (0, 0, r_3)$ is $\frac{r_3}{3}$ times the signed area of the triangle between $(0, 0), (p_1, p_2)$ and $(q_1, q_2)$.  

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3.4.2 Two tetrahedra

Let \( p_1 \ldots p_5 \) be five points in general position. Consider the linearly interpolated curve through the points in order, made of 4 line segments. Pick axes so that \( p_1 \) is at the origin and \( p_5 \) is on the \( x_3 \) axis.

**Proposition 20.** \( V \) is the sum of the signed volumes of the tetrahedra with corners \( p_1, p_3, p_4, p_5 \) and \( p_1, p_2, p_3, p_5 \).

**Proof.** The two tetrahedra both have \( p_1 p_5 \) as the edge from beginning to end. The sum of the signed volumes of the two tetrahedra is, by Proposition 19, \( \frac{1}{3} \) times the distance \( p_1 p_5 \) times the sum of the signed areas, projected onto the \( x_1 x_2 \) plane, of the disjoint triangles \( p_1 p_2 p_3 \) and \( p_1 p_3 p_4 \). That projected area is exactly the signed area of the curve \( p_1 p_2 p_3 p_4 \) in the \( x_1 x_2 \) plane. Thus the sum of the signed volumes is \( \frac{1}{6} \langle S(X)_{0,T}, 12 - 21 \rangle \langle S(X)_{0,T}, 3 \rangle \), which is \( V \).

**Proposition 21.** Assume that, projected onto the \( x_1 x_2 \) plane, the points \( p_1 \ldots p_5 \) form a convex quadrilateral. Assume, further, that the points \( p_2 \ldots p_5 \) form a convex spherical quadrilateral when enlarged onto the unit sphere centred at \( p_1 \) and same for the points \( p_1 \ldots p_4 \) around \( p_5 \). Then the convex hull of \( p_1 \ldots p_5 \) is the union of the tetrahedron with corners \( p_1, p_3, p_4, p_5 \) and the one with corners \( p_1, p_2, p_3, p_5 \). These two tetrahedra have the same sign of signed volume, and they have disjoint interiors.

**Proof.** Because of the convex spherical quadrilaterals, the following edges must be on the boundary of the convex hull: \( p_1 p_2, p_1 p_3, p_1 p_4, p_1 p_5, p_2 p_5, p_3 p_5, p_4 p_5 \). The following edges must be on the boundary because of any of the convexity assumptions: \( p_2 p_3, p_3 p_4 \).

Thus every point is a vertex of the convex hull. The convex hull is therefore made of two tetrahedra which share a triangular face. All the edges of the two tetrahedra \( p_1, p_3, p_4, p_5 \) and \( p_1, p_2, p_3, p_5 \) are on the boundary of the convex hull, so they must be the two.

In the plane, the two triples \( p_1 p_2 p_3 \) and \( p_1 p_3 p_4 \) form two triangles making up a convex quadrilateral. They must either both be specified clockwise or both anticlockwise. It follows that the signs of the signed volumes of the tetrahedra agree.

**Proposition 22.** Under the assumptions of Proposition 21, the curve has the property that \( |V| \) is the volume of the convex hull.

**Proof.** By Proposition 20, \( V \) is the sum of the signed volumes of the two tetrahedra, which by Proposition 21 make up the convex hull. They have the same sign, so \( |V| \) is the hull’s volume.

3.4.3 More tetrahedra

The same argument will apply to more than five points \( p_1 \ldots p_n \). Let \( \gamma \) be the linearly interpolated curve through the points (which is defined up to reparametrisation). Let \( V_\gamma \) denote the value of \( V \) corresponding to the curve \( \gamma \).

**Proposition 23.** \( V_\gamma \) is the sum of the signed volumes of the tetrahedra \( (p_1, p_2, p_3, p_n), (p_1, p_3, p_4, p_n), \ldots, (p_1 p_{n-2} p_{n-1} p_n) \).
Proposition 24. If the points are in general position, are a convex polygon when projected onto the $x_1x_2$ plane and with $p_2 \ldots p_n$ convex when viewed from $p_1$ and $p_1 \ldots p_{n-1}$ convex when viewed from $p_n$, then the linearly interpolated curve through the points has the property that $|V_\gamma|$ is the volume of its convex hull.

3.4.4 A curve

Definition 25. Let $\gamma : [0, T] \to \mathbb{R}^3$ be any curve. Define its signed volume to be the following limit, if it exists

$$\text{Signed-Volume}(\gamma) := \lim_{|\pi| \to 0} V_\gamma^\pi$$

Here $\gamma^\pi$ is the piecewise linear curve obtained from $\gamma$ and the partition $\pi$ of $[0, T]$.

Theorem 26. Let $\gamma : [0, T] \to \mathbb{R}^3$ a continuous curve of bounded variation. Then its signed volume exists and

$$\text{Signed-Volume}(\gamma) = V_\gamma$$

Proof. Fix some sequence $\{\pi^n\}_{n \in \mathbb{N}}$, of partitions of $[0, T]$ with $|\pi^n| \to 0$. and interpolate $\gamma$ linearly along each $\pi^n$ to obtain a sequence of linearly interpolated curves $\gamma^n$. Then

$$\text{Signed-Volume}(\gamma^n) = V_{\gamma^n}.$$  

By stability of the signature we get convergence

$$V_{\gamma^n} \to V_\gamma,$$

and this is independent of the particular sequence $\pi^n$ chosen. \qed

The previous theorem is almost a tautology, but one corollary that then does not use signed volume is

Corollary 27. Let $\gamma : [0, T] \to \mathbb{R}^3$ be of bounded variation such that when collapsed perpendicular to the line through its endpoints, it forms a strictly convex closed curve, and when all but each end is projected onto a sphere around that end it forms a strictly convex curve. Then the signed volume equals the volume of the convex hull and we have

$$V(X) = \text{Volume}(\text{Convex-Hull}(X_{[0,T]})).$$

This follows because any discretization will satisfy Proposition 24. An example is a piece of helix of up to a single revolution.
4 Rotations

Let

\[ SO(\mathbb{R}^d) = \{ A \in GL(\mathbb{R}^d) : AA^\top = \text{id}, \det(A) = 1 \} , \]

be the group of rotations of \( \mathbb{R}^d \).

**Definition 28.** We call \( \phi \in T(\mathbb{R}^d) \) an **SO invariant** if

\[ \langle S(X)_{0,T}, \phi \rangle = \langle S(AX)_{0,T}, \phi \rangle \]

for all \( A \in SO(\mathbb{R}^d) \) and all curves \( X \).

Alternatively, as explained in Section \[3\]
\[ A^\top \phi = \phi, \]
for all \( A \in SO(\mathbb{R}^d) \), where the action on \( T(\mathbb{R}^d) \) was given in Definition \[5\].

Since \( \det(X) = 1 \), any \( GL \) invariant of weight \( w \geq 1 \) (Section \[3\]) is automatically an \( SO \) invariant. But there are \( SO \) invariants that are not \( GL \) invariants (of any weight), for example, for \( d = 2 \),
\[ \phi := x_1x_1 + x_2x_2. \]

**Theorem 29 (\[Wey1946\], Theorem 2.9.A).** Let \( V \) be a \( d \)-dimensional vector space with inner product \( \langle \cdot, \cdot \rangle \). A multilinear map

\[ \psi : V \times \ldots \times V \to \mathbb{R} \]

\[ n \text{ times} \]

satisfies

\[ \psi(Av_1, Av_2, \ldots, Av_n) = \psi(v_1, v_2, \ldots, v_n) \]

for all \( A \in SO(V) \) and \( v_1, \ldots, v_n \in V \) if and only if \( \psi \) is the linear combination of functions of the form

\[ \prod_{\{i,j\} \in \mathcal{P}} \langle v_i, v_j \rangle \prod_{\{i_1, \ldots, i_d\} \in \mathcal{P}} \det[v_{i_1}, \ldots, v_{i_d}], \]

where \( \mathcal{P} \) is a partition of \( \{1, \ldots, n\} \) into sets of size 2 and sets of size \( d \). In particular \( n = a \cdot 2 + b \cdot d \) for some \( a, b \in \mathbb{N} \). The order of the vectors in \( \det[..] \) does not matter, since it only changes the overall sign.

**Proof.** By \[Wey1946\], Theorem 2.9.A or \[Pro2006\], p.390 every polynomial

\[ f : (\mathbb{R}^d)^m \to \mathbb{R}, \]

that is invariant under \( SO(\mathbb{R}^d) \) can be written as a polynomial in terms of the scalar products

\[ \langle v_i, v_j \rangle, \quad i, j \in \{1, \ldots, m\}, \]
and the determinants
\[ \det[v_{i_1} \ldots v_{i_d}], \quad i_1, \ldots, i_d \in \{1, \ldots, m\}. \]

Now a multilinear map
\[ \psi : V^n \to \mathbb{R}, \]
is a polynomial on \( V^n \), viz
\[ \psi(v_1, \ldots, v_n) = \sum_{i=(i_1, \ldots, i_n)} a_i \langle e_{i_1}, v_1 \rangle \cdots \langle e_{i_n}, v_n \rangle \]
\[ a_i := \psi(e_{i_1}, \ldots, e_{i_n}). \]

Hence
\[ \psi(v_1, \ldots, v_n) = \Psi \left( \langle v_i, v_j \rangle : i, j \in \{1, \ldots, m\}; \det[v_{i_1} \ldots v_{i_d}] : i_1, \ldots, i_d \in \{1, \ldots, m\} \right), \]
for some polynomial \( \Psi \). Now the fact that \( \psi \) is multilinear forces \( \Psi \) to be of the form claimed. \( \square \)

**Remark 30.** For any vectors \( v^{(i)}, w^{(j)} \),
\[ \det[v^{(1)} \ldots v^{(d)}] \det[w^{(1)} \ldots w^{(d)}] = \det \begin{pmatrix} \langle v^{(1)}, w^{(1)} \rangle & \langle v^{(1)}, w^{(2)} \rangle & \cdots & \langle v^{(1)}, w^{(d)} \rangle \\ \langle v^{(2)}, w^{(1)} \rangle & \langle v^{(2)}, w^{(2)} \rangle & \cdots & \langle v^{(2)}, w^{(d)} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v^{(d)}, w^{(1)} \rangle & \langle v^{(d)}, w^{(2)} \rangle & \cdots & \langle v^{(d)}, w^{(d)} \rangle \end{pmatrix}. \]
Indeed, both sides coincide for \( v^{(i)} = w^{(i)} = e_i \) and both change sign under permutation of two \( v \)'s or two \( w \)'s. Hence they coincide for any \( v^{(i)}, w^{(i)} \in \{e_1, \ldots, e_d\} \). Both are multilinear, so the coincide for all inputs.

We can then reformulate the theorem: \( \psi \) is invariant if and only if \( \psi \) is the linear combination of functions of the form
\[ \prod_{\{i, j\} \in \mathcal{P}} \langle v_i, v_j \rangle \prod_{\{i_1, \ldots, i_d\} \in \mathcal{P}} \det[v_{i_1} \ldots v_{i_d}], \]
where \( \mathcal{P} \) is a partition of \( \{1, \ldots, n\} \) into sets of size 2 and at most one set of size \( d \).

In the language of invariants for \( T(\mathbb{R}^d) \) we have

**Proposition 31.** The \( \text{SO} \) invariants of homogeneity \( n \) are spanned by
\[ \text{poly}(\Psi), \]
where \( \Psi \) ranges over the invariants of the previous remark and \text{poly} is given in Lemma 3.

In the case \( d = 2 \), here is another way to arrive at a full set of invariants, even giving an explicit basis. Taking inspiration from [Flu2000], which concerns rotation invariants of images, we work in the complex vector space \( T(\mathbb{C}^2) \).
Theorem 32. Define

\[ z_1 = x_1 + ix_2 \]
\[ z_2 = x_1 - ix_2. \]

The space of SO invariants on level \( n \) in \( T(\mathbb{C}^2) \) is spanned freely by

\[ z = z_{j_1} \cdots z_{j_n} \quad \text{with} \quad \#\{ r : j_r = 1 \} = \#\{ r : j_r = 2 \}. \]

The space of SO invariants on level \( n \) in \( T(\mathbb{R}^2) \) is spanned freely by

\[ \text{Re}[z], \text{Im}[z] \quad \text{with} \quad \#\{ r : j_r = 1 \} = \#\{ r : j_r = 2 \} \text{ and } z_1 = 1. \]

Remark 33. In particular for \( d = 2 \) and \( n \) even, the dimension of rotation invariants on level \( n \) in \( T(\mathbb{R}^2) \) is equal to \( \binom{n}{n/2} \).

Proof. 1. \( z \) is invariant
Let

\[ A_\theta := \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \]

Then (recall Definition 5)

\[ A_\theta^\top z_1 = A_\theta^\top (x_1 + ix_2) = \cos(\theta)x_1 + \sin(\theta)x_2 + i(\sin(\theta)x_1 + \cos(\theta)x_2) = e^{-i\theta}z_1 \]
\[ A_\theta^\top z_2 = e^{i\theta}z_2. \]

Hence

\[ A_\theta^\top z_{j_1} \cdots z_{j_n} = z_{j_1} \cdots z_{j_n} \forall \theta \quad \text{if and only} \quad \#\{ r : j_r = 1 \} = \#\{ r : j_r = 2 \}. \]

2. They form a basis
Now \( x_{j_1} \cdots x_{j_n} : j_i \in \{1, 2\} \) is a basis of \( \pi_n T(\mathbb{C}^2) \) with respect to \( \mathbb{C} \). Hence \( z_{j_1} \cdots z_{j_n} \) is (the map \( (x_1, x_2) \mapsto (z_1, z_2) \) is invertible). By Step 1 we have hence exhibited a basis (with respect to \( \mathbb{C} \)) for all invariants in \( \pi_n T(\mathbb{C}^2) \).

3. Real invariants
The space of SO invariants on level \( n \) in \( T(\mathbb{C}^2) \) is spanned freely by the set of

\[ z_{j_1} \cdots z_{j_n} \quad \text{with} \quad \#\{ r : j_r = 1 \} = \#\{ r : j_r = 2 \}. \]

Adding and subtracting the elements with \( j_1 = 2 \) from the elements with \( j_1 = 1 \), we get that the space of SO invariants on level \( n \) in \( T(\mathbb{C}^2) \) is spanned freely by the set of

\[ (z_{j_1} \cdots z_{j_n} + z_{3-j_1} \cdots z_{3-j_n}) \quad \text{and} \quad (z_{j_1} \cdots z_{j_n} - z_{3-j_1} \cdots z_{3-j_n}) \]
\[ \text{with} \quad \#\{ r : j_r = 1 \} = \#\{ r : j_r = 2 \} \text{ and } j_1 = 1. \]
Because $z_{3-j_1} \cdots z_{3-j_n}$ is the complex conjugate of $z_{j_1} \cdots z_{j_n}$, this means that the space of $SO$ invariants on level $n$ in $T(C^2)$ is spanned freely by the set of

$$\text{Re}(z_{j_1} \cdots z_{j_n}) \quad \text{and} \quad \text{Im}(z_{j_1} \cdots z_{j_n})$$

with $\#\{r : j_r = 1\} = \#\{r : j_r = 2\}$ and $j_1 = 1$.

This is an expression for a basis of the SO invariants in terms of real combinations of basis elements of the tensor space. They thus form a basis for the SO invariants for the free real vector space on the same set, namely $\pi_n T(R^2)$.

Example 34. We give maximally linearly independent subsets of the invariants given in Proposition 31, for certain dimensions $d$ and orders $w$.

Consider $d = 2$

Order 2

\begin{align*}
11 + 22 \\
-12 + 21
\end{align*}

Order 4

\begin{align*}
1111 - 1122 + 1212 + 1221 + 2112 + 2121 - 2211 + 2222 \\
-1112 - 1121 + 1211 - 1222 + 2111 - 2122 + 2212 + 2221 \\
1111 + 1122 - 1212 + 1221 + 2112 - 2121 + 2211 + 2222 \\
-1112 + 1121 - 1211 - 1222 + 2111 + 2122 - 2212 + 2221 \\
1111 + 1122 + 1212 - 1221 + 2112 + 2121 + 2211 + 2222 \\
1112 - 1121 - 1211 - 1222 + 2111 + 2122 + 2212 + 2221
\end{align*}

Consider $d = 3$

Order 3

\begin{align*}
123 - 132 + 312 - 321 + 231 - 213
\end{align*}

Consider $d = 4$

Order 2

\begin{align*}
11 + 22 + 33 + 44.
\end{align*}
5 Permutations

Denote by $S_d$ the group of permutations of $[d] := \{1, .., d\}$.

**Lemma 35.** For $\sigma \in S_d$, define $M(\sigma) \in GL(\mathbb{R}^d)$ as

$$M(\sigma)_{ij} = 1 \quad \text{if} \quad i = \sigma(j).$$

Then $M : S_d \to GL(\mathbb{R}^d)$ is a group homomorphism and moreover $M(\sigma^{-1}) = M(\sigma)^\top$.

**Proof.** Regarding the first point, for $i = \{1, .., d\}$,

$$M(\sigma)M(\tau)e_i = M(\sigma)e_{\tau(i)} = e_{\sigma(\tau(i))} = M(\sigma \tau)e_i.$$  

Regarding the last point

$$M_{ij} = 1 \quad \text{if} \quad i = \sigma^{-1}(j)$$

$\iff$

$$M_{ij} = 1 \quad \text{if} \quad \sigma(i) = j$$

$\iff$

$$M_{ij} = 1 \quad \text{if} \quad j = \sigma(i).$$

$S_d$ then acts on $T((\mathbb{R}^d))$ and $T(\mathbb{R}^d)$ via Definition 5. Explicitly,

$$\sigma \cdot x_{i_1}..x_{i_n} = x_{\sigma(i_1)}..x_{\sigma(i_n)}.$$

**Definition 36.** We call $\phi \in T(\mathbb{R}^d)$ a **permutation invariant** if

$$\langle S(M(\sigma)X)_{0,T}, \phi \rangle = \langle S(X)_{0,T}, \phi \rangle$$

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for all $\sigma \in S_d$ and all curve $X$.

Alternatively, as explained in Section [3]

$$M(\sigma)\phi = \phi,$$

for all $\sigma \in S_d$. Equivalently,

$$M(\sigma)\phi = \phi,$$

for all $\sigma \in S_d$.

**Remark 37.** An $SO$ invariant is a permutation invariant, if we restrict to even permutations.

We follow [NRRZ2005, Section 3]. To a monomial $x_{i_1} \cdots x_{i_n}$,

we associate the following set partition of $[n] := \{1,\ldots,n\}$

$$\nabla(x_{i_1} \cdots x_{i_n}) := \{\{\ell : i_\ell = p\} : p \in [d]\} \setminus \{\emptyset\}.$$ 

**Example 38.** Let $d = 3$, then

$$\nabla(x_2x_3x_2x_2x_1) = \{1,3,4\}, \{2\}, \{5\}.$$ 

Note that for every permutation $\sigma \in S_d$,

$$\nabla(x_{i_1} \cdots x_{i_n}) = \nabla(x_{\sigma(i_1)} \cdots x_{\sigma(i_n)}).$$

(3)

**Proposition 39** ([NRRZ2005, Section 3]). Define

$$M_A := \sum_{i: \nabla(x_{i_1} \cdots x_{i_n})=A} x_{i_1} \cdots x_{i_n}.$$ 

Then $\{M_A : A \text{ is set partition of } [n]\}$ is a linear basis for the space of permutation invariants of homogeneity $n$.

**Proof.** By (3), each $M_A$ is permutation invariant.

For $A, A'$ distinct set partitions of $[n]$ the monomials in $M_A$ and the monomials in $M_{A'}$ do not overlap. Hence the proposed basis is linearly independent.

Now, if $\phi$ is permutation invariant and if for some $i, i'$, $\nabla(x_{i_1} \cdots x_{i_n}) = \nabla(x_{i'_1} \cdots x_{i'_n})$ then the coefficient of $x_i$ and $x_{i'}$ must coincide. Hence the proposed basis spans invariants of homogeneity $n$.

**Example 40.** Consider $d = 3$

Order $n = 1$

$$1 + 2 + 3$$
Order $n = 2$

$$33 + 22 + 11$$
$$32 + 31 + 23 + 21 + 13 + 12$$

Order $n = 3$

$$333 + 222 + 111$$
$$332 + 331 + 223 + 221 + 113 + 112$$
$$323 + 313 + 232 + 212 + 131 + 121$$
$$322 + 311 + 233 + 211 + 133 + 122$$
$$321 + 312 + 231 + 213 + 132 + 123$$

6 An additional (time) coordinate

Assume now that $X = (X^0, X^1, \ldots, X^d) : [0, T] \rightarrow \mathbb{R}^{1+d}$. Here $X^0$ plays a special role, in that we assume that it is not affected by the space transformations under consideration. Often $X^0(t) = t$ is just a component keeping track of time.

Consider $GL$ invariants for the moment.

**Definition 41.** Let

$$\tilde{GL}(\mathbb{R}^d) := \{ A \in GL(\mathbb{R}^{1+d}) : Ae_0 = e_0, A^{-1}e_0 = e_0 \},$$

the space of invertible maps of $\mathbb{R}^{1+d}$ leaving the first direction unchanged. We call $\phi \in T(\mathbb{R}^{1+d})$ a $\tilde{GL}$ invariant of weight $w$ if

$$A^\top \phi = (\det A)^w \phi,$$

for all $A \in \tilde{GL}(\mathbb{R}^d)$.

Consider the $GL(\mathbb{R}^2)$ invariant of weight 1

$$x_1x_2 - x_2x_1.$$

Since elements of $\tilde{GL}(\mathbb{R}^2)$ leave the variable $x_0$ unchanged, a straightforward way to produce $\tilde{GL}$ invariants presents itself: insert $x_0$ at the same position in every monomial. For example

$$x_1x_0x_2 - x_2x_0x_1$$

is a $\tilde{GL}(\mathbb{R}^2)$ invariant of weight 1. We now formalize this idea and show that we get every $\tilde{GL}$ invariant this way.
Define the linear map $\text{Remove}$ of "removing instances of } x_0 \text{" on monomials, as \\
\text{Remove } x_{i_1}..x_{i_m} := \prod_{\ell : i_\ell \neq 0} x_{i_\ell}, \\
\text{so for example } \\
\text{Remove } x_0x_1x_0x_3 = x_1x_3 \\
\text{Remove } x_0x_0 = 1.

Define for $U \subset [m]$ and $i = (i_1,..,i_m)$ \\
i|_U = (i_\ell : \ell = 1,..,m; \ell \in U).

Define the linear map of restriction to $U$ on polynomials of order $m$ by defining on monomials \\
x_i|_U := x_i|_U \\
so for example \\
x_{i_1}x_{i_2}x_{i_3}|_{\{1,3\}} = x_{i_1}x_{i_3}.

For $z = (z_1,..,z_{m+1}) \in \mathbb{N}^{m+1}$ denote by $\text{Insert}_z$ the linear operator on polynomials of order $m$ by defining it on monomials as follows. For a monomial $x_{i_1}..x_{i_m}$ of order $m$, $\text{Insert}_z$ inserts $z_1$ occurrences of $x_0$ before $x_{i_1}$, $z_2$ occurrences of $x_0$ before $x_{i_2}$, $..$, $z_m$ occurrences of $x_0$ before $x_{i_m}$ and $z_{m+1}$ occurrences of $x_0$ after $x_{i_m}$. For example \\
\text{Insert}_{(2,1,4)}x_1x_2 = x_0x_0x_1x_0x_1x_0x_0x_0x_0.

**Theorem 42.** The space of $\tilde{GL}$ invariant of weight $w$, homogeneous of degree $m$ is spanned by the polynomials \\
\text{Insert}_z\psi, \\
with $0 \leq n \leq m$, $\psi$ is an $GL$ invariant of weight $w$ and homogeneity $n$ and $z \in \mathbb{N}^{n+1}$ such that \\
$\sum_{\ell} z_\ell = m - n.$

**Proof.** Let $n, \psi, z$ be as in the statement, then $\text{Insert}_z\psi$ is $\tilde{GL}$ invariant of weight $w$. Indeed: for $\tilde{A} = \text{diag}(1, A) \in \tilde{GL}(\mathbb{R}^d)$, with $A \in GL(\mathbb{R}^d), \\
\tilde{A} \text{Insert}_z\psi = \text{Insert}_z A\psi = (\det A)^w \text{Insert}_z\psi.$

Let now $\phi$ be of order $m$ be $\tilde{GL}$ invariant modulo time of weight $w$. Define for $U \subset [m]$ \\
$\phi^U := \sum_{i : i_\ell = 0, \ell \in U; i_j \neq 0, j \notin U} (\phi, x_i)x_i,$
which collects all monomials having $x_0$ exactly at the positions in $U$. Then

$$\phi = \sum_{U \subset [m]} \phi^U.$$

Now, since $\phi$ is $\tilde{GL}$ invariant of weight $w$ and since $\tilde{GL}$ leaves

$$\text{span}\{x_i : i_\ell = 0, \ell \in U; i_j \neq 0, j \notin U\}$$

invariant, we get that $\phi^U$ is $\tilde{GL}$ invariant of weight $w$. Clearly, there is $0 \leq n \leq m$ and $i \in \mathbb{N}^{n+1}$ such that

$$\text{Insert } \phi^U = \phi^U.$$

Lastly, $\text{Remove } \phi^U$ is $GL$ invariant, since for $\tilde{A} = \text{diag}(1, A) \in \tilde{GL}(\mathbb{R}^d)$, with $A \in GL(\mathbb{R}^d),

$$A \text{ Remove } \phi^U = \text{Remove } \tilde{A}\phi^U = (\text{det } \tilde{A})^w \text{Remove } \phi^U = (\text{det } A)^w \text{Remove } \phi^U.$$

Hence every invariant is in the span of the set given in the statement.

The corresponding statements for rotations and permutations are completely analogous, so we omit stating them.

7 Appendix

Lemma 43. For $n \geq 1$

$$\text{span}\{\pi_n S(X)_{0,T} : X \text{ curve } \} = \pi_n T((\mathbb{R}^d)). \quad (4)$$

Proof. It is clear by definition that the left hand side of (4) is included in $\pi_n T((\mathbb{R}^d))$. We show the other direction and use ideas of [CF2010, Proposition 4]. Let $x_{i_1} \cdots x_{i_n} \in \pi_n T((\mathbb{R}^d))$ be given. Let $X$ be the piecewise linear path, that results from the concatenation of the vectors $t_1 e_{i_1}, t_2 e_{i_2}$ up to $t_n e_{i_n}$, where $e_i, i = 1, \ldots, d$ is the standard basis of $\mathbb{R}^d$. Its signature is given by (see for example [PV2010, Chapter 6])

$$S(X)_{0,1} = \exp(t_n x_{i_n}) \cdot \cdots \cdot \exp(t_1 x_{i_1}) =: \phi(t_1, \ldots, t_n),$$

where the exponential function is defined by its power series. Then

$$\frac{d}{dt_n} \cdots \frac{d}{dt_1} \phi(0, \ldots, 0) = x_{i_n} \cdot \cdots \cdot x_{i_1}.$$

Combining this with the fact that left hand side of (4) is a closed set we get that

$$x_{i_n} \cdot \cdots \cdot x_{i_1} \in \text{span}\{\pi_n (S(X)_{0,1}) : X \text{ curve } \}.$$

These elements span $\pi_n T((\mathbb{R}^d))$, which finishes the proof.
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