Conformal symmetry for black holes in four dimensions

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Abstract: We show that the asymptotic boundary conditions of general asymptotically flat black holes in four dimensions can be modified such that a conformal symmetry emerges. The black holes with the asymptotic geometry removed in this manner satisfy the equations of motion of minimal supergravity in five dimensions. We develop evidence that a two dimensional CFT dual of general black holes in four dimensions account for their black hole entropy.

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1 Introduction

For supersymmetric black holes it has long been understood how to account for the entropy of in terms of dual weakly coupled conformal field theories in two dimensions (some reviews are \[1–3\]). General arguments suggest that similar advances are precluded in settings that do not preserve supersymmetry even approximately. However, the general (non-supersymmetric) entropy formula takes a form that suggests a dual two dimensional conformal field theory even when supersymmetry is broken substantially \[4, 5\]. Our recent investigation of this situation in the setting of five dimensional black holes lead to a concrete procedure that might account for the black hole entropy of black holes even in situations far from the supersymmetric limit \[6\]. In our previous work we proposed that
the entropy of general black holes in five dimensions can be addressed by an analysis with the following components:

1. Establish that the causal structure and the thermodynamics of the black hole geometry is independent of a certain conformal factor. Accordingly this conformal factor can be interpreted as a specification of the environment of the black hole in a manner decoupled from its internal structure.

2. Show that the scalar wave equation exhibits \( SL(2, R) \times SL(2, R) \) symmetry for some specifications of the conformal factor. We refer to the geometry with conformal factor modified in this manner as the “subtracted” geometry. The subtracted geometry has the same near horizon properties as the original black hole but different asymptotics at large distances: it is not asymptotically flat.

3. Show that an auxiliary dimension can be introduced that lifts the subtracted geometry to one dimension higher such that both the separability and the \( SL(2, R) \times SL(2, R) \) symmetry of the scalar wave equation become manifest. In this setting the 2D conformal symmetry is linearly realized by representing the subtracted geometry as a U(1) coset.

In this paper we present computations that carry out these steps in the context of a large class of four dimensional black holes. This aspect of the present work is an adaptation to four dimensions of the analogous computations in five dimensions previously presented in [6]. This part of the paper generalizes the “hidden conformal symmetry” proposal [7] to include charges and sharpens it by providing a formulation directly in the geometry rather than relying on the wave equation. It also greatly sharpens our own suggestions [8] from over a decade ago.

These generalizations are interesting in their own right. However, in addition this paper also addresses some of aspects of our approach that present legitimate questions:

1. The subtracted black hole geometry does not generally satisfy the equations of motion since we simply change the geometry by declaration. We view this as acceptable, since the black hole “itself” cannot be in equilibrium unless it is surrounded by a “box” kept at the same temperature as the black hole. Such a box must be made from matter and it is usually not worthwhile to specify this matter explicitly. However, our construction is quite novel and its off-shell nature is one of its unusual features. In this paper we address this concern directly, by constructing in the non-rotating case the explicit matter such that the equations of motion are satisfied.

2. An appealing interpretation of the matter that supports the subtracted black hole solution in four dimensions is that it corresponds to minimal supergravity in five dimensions.

This paper is organized as follows. In section 2 we introduce the general black holes in four dimensions and analyze their causal structure and their thermodynamics. We
show that these features are independent of the conformal factor. In section 3 we analyze the scalar wave equation for general conformal factor. We determine the “subtracted” conformal factor such that the desirable features suggested by the standard conformal factor become exact. In section 4 we identify matter fields such that the black hole with subtracted conformal factor satisfies the equations of motion. In section 5 we exhibit the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ symmetry of the subtracted geometry explicitly, and we use an auxiliary five dimensional geometry to argue that the symmetry is enhanced to the conformal group in two dimensions. In section 6 we discuss hidden conformal symmetry from a 4D and a 5D point of view.

2 General black holes in four dimensions

In this section we introduce the black hole geometry in the fibered form we find useful. We review the causal structure of the black holes and derive their thermodynamics in a manner that exhibits independence of the conformal factor $\Delta_0$.

2.1 The black hole metric

The setting for our discussion is the rotating black hole solution of four dimensional string theory with four independent U(1) charges [5]. The asymptotic charges of the black hole are parametrized as:

$$
G_4 M = \frac{1}{4} m \sum_{I=0}^{3} \cosh 2\delta_I , \\
G_4 Q_I = \frac{1}{4} m \sinh 2\delta_I , \quad (I = 0, 1, 2, 3) \quad G_4 J = ma(\Pi_c - \Pi_s) ,
$$

where we employ the abbreviations

$$
\Pi_c \equiv \prod_{I=0}^{3} \cosh \delta_I , \quad \Pi_s \equiv \prod_{I=0}^{3} \sinh \delta_I . \quad (2.2)
$$

The parametric mass and angular momentum $m, a$ both have dimension of length.

We write the 4D metric as a fibration over a 3D base space

$$
ds_4^2 = -\Delta_0^{-1/2}(dt + A)^2 + \Delta_0^{1/2} \left( \frac{dr^2}{X} + d\theta^2 + \frac{X}{G} \sin^2 \theta d\phi^2 \right) ,
$$

where for the black holes we consider

$$
X = r^2 - 2mr + a^2 ,
$$

$$
G = r^2 - 2mr + a^2 \cos^2 \theta ,
$$

$$
A = \frac{2ma \sin^2 \theta}{G} [(\Pi_c - \Pi_s)r + 2m\Pi_s] d\phi ,
$$

$$
\Delta_0 = \prod_{I=0}^{3} (r + 2m \sinh^2 \delta_I) + 2a^2 \cos^2 \theta \left[ r^2 + mr \sum_{I=0}^{3} \sinh^2 \delta_I + 4m^2(\Pi_c - \Pi_s)\Pi_s \\
- 2m^2 \sum_{I<J<K} \sinh^2 \delta_I \sinh^2 \delta_J \sinh^2 \delta_K \right] + a^4 \cos^4 \theta . \quad (2.4)
$$
The fibered form (2.3) of the metric does not reduce to the one usually presented in textbooks for Kerr. However, the alternate form here simplifies manipulations significantly, especially when all the string theory charges are included.

The rather complicated conformal factor $\Delta_0$ simplifies in some special cases. The benchmark is the non-rotating case $a = 0$ where only the first term remains. However, the expression also simplifies with rotation when the four charges are equal in pairs

$$\Delta_0 = [(r + 2m \sin^2 \delta_1)(r + 2m \sin^2 \delta_2) + a^2 \cos^2 \theta]^2 \quad (2.5)$$

The generic case with rotation and four independent charges does not simplify.

2.2 Causal structure

It is instructive to analyze a general black hole geometry of the form (2.3) where $X$ is an arbitrary function of the radial variable $r$, the function $G$ is

$$G = X - a^2 \sin^2 \theta \quad (2.6)$$

and $\Delta_0, A_\phi$ are arbitrary functions of both $r$ and the polar angle $\theta$. We will return to the specific forms (2.4) later.

Trajectories along the Killing direction parametrized by $t$ cease to be time-like at the static limit where

$$G = 0 \quad (2.7)$$

The volume inside this surface (but outside the event horizon) is the ergosphere.

In the ergosphere physical trajectories at fixed $r, \theta$ have a nontrivial component along the azimuthal angle $\phi$ because they would be spacelike if they were fully directed along $t$. However, all directions at fixed $r, \theta$ become space-like once the determinant in the $t - \phi$ plane

$$\det(t - \phi) = -X \sin^2 \theta \quad (2.8)$$

turns positive. This identifies the event horizon as the surface

$$X = 0 \quad (2.9)$$

The relation (2.6) ensures that the event horizon (2.9) is indeed inside the static limit (2.7), except at the poles $\sin \theta = 0$ where the two surfaces meet.

2.3 Black hole thermodynamics

It is worthwhile to analyze the thermodynamics of the black holes while remaining in the general setting with $X, \Delta_0, A_\phi$ unspecified and $G$ given by (2.6).

We first present the metric (2.3) as

$$ds^2 = \Delta_0^{1/2} \left( \frac{dr^2}{X} + X \frac{\sin^2 \theta}{G} d\phi^2 \right) + [\Delta_0^{1/2} d\theta^2 - \Delta_0^{-1/2} G(dt + A_\phi d\phi)^2] \quad (2.10)$$

and then focus on the region $X \sim 0$ near the event horizon (2.9). In this region $G \sim -a^2 \sin^2 \theta < 0$ so the geometry in the round bracket is independent of $\theta$. Moreover, this
part of the geometry has Lorentzian signature and can be interpreted as Rindler space.\footnote{We assume the horizon $X = 0$ is a simple pole in $X(r)$. A double pole corresponds to an extremal black holes which presents a challenge for a thermodynamic interpretation, as usual.} We present the acceleration of this Rindler space in terms of the Euclidean period

$$\beta_\phi = \frac{4\pi a}{(\partial_r X)_\text{hor}},$$

(2.11)

determined such that the “time” $\phi$ avoids a conical singularity.

The geometry of the black hole horizon is encoded in the square bracket of (2.10). Recalling again that $G \sim -a^2 \sin^2 \theta < 0$ it is recognized that the horizon has topology $S^2$ as expected. The determination of the Euclidean period (2.11) was carried out at any point on the event horizon so the geometry in the square bracket must be kept fixed as $\phi$ is periodically identified. This consideration determines the Euclidean periodicity of the asymptotic time $t$ as

$$\beta_H = -(A_\phi)_\text{hor} \beta_\phi .$$

(2.12)

We can interpret this formula in Lorentzian signature where it gives the rotational velocity

$$\Omega_H = \frac{\beta_\phi}{\beta_H} = -\frac{1}{(A_\phi)_\text{hor}} .$$

(2.13)

We will find it useful to introduce the reduced angular potential that has some of the overall factors removed

$$A_\text{red} = \frac{G}{a \sin^2 \theta} A_\phi \sim -a (A_\phi)_\text{hor} .$$

(2.14)

The explicit solution (2.4) gives a reduced angular potential $A_\text{red}$ that only depends on the radial coordinate. This in turn ensures that the rotational velocity (2.13) is independent of the position on the horizon, as it should be. This property motivates the use of the reduced angular potential $A_\text{red}$ without reference to the explicit solution.

With the notation (2.14) and the general result (2.11) we can rewrite the inverse temperature (2.12) as

$$\beta_H = \left( \frac{4\pi A_\text{red}}{\partial_r X} \right)_\text{hor} .$$

(2.15)

This expression is manifestly independent of the polar angle.

The black hole entropy is computed from the area of the event horizon. Reading the measure from the square bracket of (2.10) (at fixed asymptotic time $t$) we find

$$S = \frac{1}{4G_4} \int_{r=r_+} \sqrt{-G A_\phi^2 d\phi d\theta} = \frac{a}{4G_4} \int_{r=r_+} |A_\phi| \sin \theta d\phi d\theta = \frac{\pi (A_\text{red})_\text{hor}}{G_4} .$$

(2.16)

The integral was evaluated by exploiting the constancy of the rotational velocity (2.13) on the event horizon.

The formula (2.16) identifies the black hole entropy with the reduced potential $A_\text{red}$, up to a universal constant. This is surprising because the introduction of the potential $A_\text{red}$ in (2.14) relies on rotation of the black hole. The result nevertheless makes sense because the actual value or $A_\text{red}$ on the horizon remains finite in the limit of vanishing angular
momentum. More generally, our derivation of the inverse temperature (2.15) and the black hole entropy (2.16) relies on the rotation at intermediate steps but the final expressions are finite in the non-rotating limit and in agreement with those obtained using other methods. We will see angular momentum in a privileged role repeatedly in this work.

An important corollary to the expressions (2.11), (2.13), (2.15), (2.16) for the thermodynamic parameters is their independence of the conformal factor $\Delta_0$. We interpret this to mean that $\Delta_0$ characterizes the environment of the black hole rather than its internal structure.

2.4 Thermodynamics: explicit expressions

In the case of the explicit function $X$ given in (2.4), the event horizon is at the largest solution to the quadratic equation (2.9)

$$r_+ = m + \sqrt{m^2 - a^2}.$$  

(2.17)

For the solutions (2.4) the thermodynamic potential (2.11) becomes

$$\beta_H \Omega = \beta_\phi = \frac{2\pi a}{r_+ - m} = \frac{2\pi a}{\sqrt{m^2 - a^2}},$$

(2.18)

and the reduced potential (2.14) is

$$A_{\text{red}} = 2m[\Pi_c - \Pi_s]r + 2m\Pi_s].$$  

(2.19)

Then the inverse Hawking temperature (2.15) yields

$$\beta_H = \frac{4\pi m}{r_+ - m}[\Pi_c - \Pi_s]r + 2m\Pi_s] = 4\pi m \left( \frac{m}{\sqrt{m^2 - a^2}}(\Pi_c + \Pi_s) + (\Pi_c - \Pi_s) \right),$$

(2.20)

and the black hole entropy (2.16) becomes

$$S = \frac{2\pi m}{G_4}[(\Pi_c - \Pi_s)r_+ + 2m\Pi_s] = \frac{2\pi m}{G_4} \left( (\Pi_c + \Pi_s)m + (\Pi_c - \Pi_s)\sqrt{m^2 - a^2} \right).$$

(2.21)

The thermodynamic potentials (2.18), (2.20), and (2.21) agree with those found in [9] using conventional methods.

3 The subtracted geometry

The massless wave equation hints at a dual 2D CFT even for the general black holes we consider. This section discusses the appearance of hypergeometric structure, the precursor of SL(2, $R$) $\times$ SL(2, $R$) symmetry.

3.1 Separability of the scalar wave equation

As in the previous section it is instructive to first consider a general metric of the form (2.3) where $X$, $G$, $\Delta_0$, $A_\phi$ are arbitrary functions, except for the rudimentary assumptions stated around (2.6). We will also assume that $A_{\text{red}}$ introduced in (2.14) depends on $r$ alone, as it does in our primary example.
The Laplacian derived by inverting the metric \((2.3)\) becomes

\[
\Delta_0^{-1/2} \left[ \partial_r X \partial_r \left( + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta - \frac{\Delta_0}{G} \frac{\partial^2}{\partial t^2} + \frac{G}{X \sin^2 \theta} (\partial_\phi - A_\phi \partial_t)^2 \right) \right].
\]

(3.1)

The last two terms generally mix \(r\) and \(\theta\) in a complicated way that obstructs separability. At this point we utilize \((2.6)\) for \(G\) and also introduce \(A_{\text{red}}\) from \((2.14)\). Then the Laplacian simplifies to

\[
\Delta_0^{-1/2} \left[ \partial_r X \partial_r - \frac{1}{X} (A_{\text{red}} \partial_t + a \partial_\phi)^2 + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 + \frac{A_{\text{red}}^2 - \Delta_0}{G} \frac{\partial^2}{\partial t^2} \right].
\]

(3.2)

In this equation it is just the last term that prevents separability, when disregarding (for now\(^2\)) the overall factor of \(\Delta_0^{-1/2}\).

In the actual geometry \((2.4)\) the reduced potential \(A_{\text{red}}\) is \((2.19)\) and the combination \(\Delta_0 - A_{\text{red}}^2\) contains a factor \(G\) that ensures factorization

\[
\frac{\Delta_0 - A_{\text{red}}^2}{G} = r^2 + 2mr \left( 1 + \sum_{I=0}^{3} s_I^2 \right) + 8m^2 (\Pi_c - \Pi_s) \Pi_s - 4m^2 \sum_{I<J<K} s_I^2 s_J^2 s_K^2 + a^2 \cos^2 \theta,
\]

(3.3)

where \(s_I^2 \equiv \sinh^2 \delta_i\). This expression implies separability of the Laplacian \((3.2)\) and so separability of the (massless) wave equation. The details of the expression still looks quite forbidding but this is primarily due to an intricate dependence on black hole parameters. The right hand side is in fact just a quadratic polynomial in \(r\), with the constant term depending quadratically on \(\cos \theta\).

### 3.2 The subtracted geometry

The differential equation \((3.2)\) with the effective potential \((3.3)\) has simplifying features beyond the separability. The radial equation has two regular singularities, at \(r = r_+\) and \(r = r_-\). In the neighbourhood of these singularities the equation can be solved by a power (the index) of \((r - r_{\pm})\), with corrections suppressed by integral powers of \((r - r_{\pm})\). The indices at these regular singularities are \(\frac{i\beta_{\pm}}{2\pi}\), i.e. essentially the outer and inner horizon temperatures. The radial equation has a third singularity at infinity, but this singularity is irregular. If it had been regular the radial equation would have been the hypergeometric equation, with its \(\text{SL}(2, \mathbb{R})\) symmetry permuting the three regular singularities. This situation is desirable because it would hint at an underlying conformal symmetry.

The irregular singularity at infinity is due to the asymptotic behavior \(\Delta_0 \sim r^4\) at large \(r\) which encodes the asymptotic flatness of spacetime. If instead \(\Delta_0 \rightarrow \Delta \sim r^2\) the singularity at infinity in the radial equation would be regular. For even more special warp factors with \(\Delta_0 \rightarrow \Delta \sim r\) at large \(r\) the radial equation maintains its hypergeometric character but, in addition, the angular equation simplifies to the familiar spherically symmetric form

\[
\left( \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right) \chi(\theta, \phi) = -l(l + 1) \chi(\theta, \phi).
\]

(3.4)

\(^2\)We will later identify a non-minimal coupling that ensures separability of massive scalars as well.
When $\Delta \sim r$ the geometry thus indicates an unbroken SU(2) R-symmetry. In this case the indices of the regular singularity at infinity are $(l, -l - 1)$.

It was shown in section 2 that both the causal structure and the thermodynamics of black holes is independent of the conformal factor $\Delta_0$. We interpret this as a demonstration that an alternate $\Delta_0 \to \Delta$ corresponds to a black hole with the same internal structure as the original one, but a black hole that finds itself in a different external environment.

We will focus on the warp factors $\Delta$ that preserve separability of the scalar wave equation and also analyticity in the coordinates. These technical assumptions identify a warp factor with the asymptotic behavior $\Delta \sim r$ uniquely as

$$\Delta_0 \to \Delta = (2m)^3 r (\Pi_c^2 - \Pi_s^2) + (2m)^4 \Pi_s^2 - (2m)^2 (\Pi_c - \Pi_s)^2 a^2 \cos^2 \theta .$$  (3.5)

In particular, the condition of separability determines the $\theta$-dependence by the requirement that $\Delta - A_{\text{red}}^2$ be factorizable by $G$:

$$\frac{\Delta - A_{\text{red}}^2}{G} = -4m^2 (\Pi_c - \Pi_s)^2 .$$  (3.6)

The condition of separability is powerful even in the nonrotating limit $a \to 0$. For example, the analysis of the Schwarzchild geometry geometry presented in [10] is not consistent with this criterion.

More general assignments of the conformal factor are possible if we allow asymptotic behavior $\Delta \sim r^2$ while maintaining separability. Such alternate conformal factors take the form $\Delta' = \text{const} \cdot G + \Delta$ where $\Delta$ is given in (3.6). The angular equation of such geometries generalize (3.4) in a manner that, for the purpose of the radial equation, can be absorbed in a renormalization of the separation constant $l \to l_{\text{eff}}$. The resulting effective angular momenta $l_{\text{eff}}$ generally differ from the non-negative integers assigned to $l$. Indeed, they do not even have to be real: complex $l_{\text{eff}}$ are interpreted physically in terms of superradiance, in the case of near extreme Kerr [11, 12]. In this paper we primarily analyze the minimally subtracted metric with conformal factor (3.4).

As we have explained, the motivation for changing the conformal factor $\Delta$ at will is the interpretation that such changes do nothing to alter the internal structure of the black hole, it merely changes the environment of the quantum black hole. This argument is not beyond reproach. It is therefore worth noting a less ambitious reasoning that motivates the same procedure: any setting where the scalar wave equation approximates the hypergeometric equation can be interpreted as a situation where the actual conformal factor $\Delta_0$ is well approximated by the $\Delta$ that gives the hypergeometric equation exactly. In any such setting we might as well consider the approximate geometry from the outset. The decoupling limit of the AdS/CFT correspondence is a special case of this reasoning, the near extreme limit of Kerr is another. In such settings the present point of view offers the advantage that we are not limited to the scalar wave equation. The geometry with enhanced symmetry allows the analysis of many other questions.

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Footnote: In the special case of a dilute gas (near-BPS) approximation where $\delta_I \gg 1$, $(I = 1, 2, 3)$, the exact warp factor $\Delta_0$ and the subtracted one $\Delta$ coincide, as expected.
3.3 Kerr

It is worthwhile making our considerations more explicit in the case of the pure Kerr black hole, without any charges. In this case the parametric mass and angular momenta are \( m = G_4 M \) and \( a = J/M \). The full conformal factor in (2.4) (or (2.5)) becomes

\[
\Delta_0 = (r^2 + a^2 \cos^2 \theta)^2 .
\] (3.7)

Separability for pure Kerr can be traced to factorization of the effective potential

\[
\frac{\Delta_0 - A_{\text{red}}^2}{G} = r^2 + 2mr + a^2 \cos^2 \theta ,
\] (3.8)

with \( A_{\text{red}} = 2mr \).

The conformal factor (3.5) in the subtracted geometry simplifies to

\[
\Delta_0 \rightarrow \Delta = 4m^2(2mr - a^2 \cos^2 \theta) .
\] (3.9)

The subtracted geometry still has \( A_{\text{red}} = 2mr \) and separability now relies on the factorization

\[
\frac{\Delta - A_{\text{red}}^2}{G} = -4m^2 .
\] (3.10)

The additional hypergeometric structure is due to the improved asymptotic behavior of the effective potential.

The special case of near-extreme Kerr can be usefully analyzed by focussing on the near horizon region isolated by the NHEK limit [13]. The NHEK scaling limit tunes the parameters of the black hole so \( \sqrt{m^2 - a^2} = \epsilon \lambda \rightarrow 0 \) as \( \lambda \rightarrow 0 \) while simultaneously focussing on the near horizon region where \( r - m = \lambda U \rightarrow 0 \). In the NHEK limit the black hole reduces to a warped AdS\(_3\) geometry with radius \( \ell^2 = 2m^2 \) and warp factor \( \Omega = \frac{1}{2}(1 + \cos^2 \theta) \).

Applying the NHEK limit directly on the conformal factor (3.7) we recover the NHEK warp factor

\[
\sqrt{\Delta_0} \rightarrow \ell \frac{1}{2}(1 + \cos^2 \theta) ,
\] (3.11)

If we instead apply the NHEK limit on the subtracted conformal factor (3.9) we find the warp factor

\[
\sqrt{\Delta} \rightarrow \ell \sqrt{1 + \sin^2 \theta} .
\] (3.12)

It is clear from this comparison that our subtraction procedure differs from the NHEK limit even for the rapidly spinning black holes where both analyses apply. In either approach the scalar field equation is in hypergeometric form and it is possible that this means there are two valid CFT descriptions for rapidly spinning black holes. It would of course be interesting to find a relation between these descriptions. This in no way presents a contradiction: both of these schemes involve a scalar field equation of a hypergeometric form and it is possible that this means there are two valid CFT descriptions for rapidly spinning black holes. It would of course be interesting to find a relation between these descriptions.
3.4 Asymptotic behavior of the subtracted geometry

The asymptotic behavior of the subtracted geometry is

\[ ds^2_4 \sim -\Delta^{-1/2} r^2 dt^2 + \Delta^{1/2} \frac{1}{r^2} (dr^2 + r^2 d\Omega^2_2) , \]

where \( \Delta \sim \ell^3 r \) with \( \ell^3 = 2m^3 (\Pi_c^2 - \Pi_s^2) \). We can introduce \( R = 4\ell^3 r^{1/4} \) and write this asymptotic behavior as

\[ ds^2_4 \sim -\left( \frac{R}{4\ell} \right)^6 dt^2 + dR^2 + \left( \frac{R}{4\ell} \right)^2 d\Omega^2_2 . \]

In this form the asymptotic geometry has an obvious scaling symmetry \( ds^2_4 \rightarrow \lambda^2 ds^2_4 \) that is implemented by taking \( R \rightarrow \lambda R \) and \( t \rightarrow \lambda^{-2} t \). The nonstandard scaling of time is reminiscent of the Lifshitz symmetry that has recently been developed for applications of holography to condensed matter systems (some representative works are [14–16]). It would be interesting to develop the asymptotic symmetry of the present context in more detail.

4 The matter supporting the geometry

The black hole metric with subtracted conformal factor does not a priori satisfy the equations of motion. In this section we explicitly identify a matter configuration that supports the geometry.

4.1 Physical matter

Our reasoning up to this stage has been to freely modify the conformal factor \( \Delta_0 \rightarrow \Delta \) as needed in order that the scalar wave equation exhibits enhanced symmetries, without the black hole thermodynamics and causal structure having been modified. This lead us to the specific assignment (3.5) for the conformal factor, while keeping the remaining parts of the solution (2.4) intact.

The subtracted geometry with conformal factor \( \Delta \) does not satisfy the equations of motion with the matter that was specified before the subtraction procedure. In particular, the Kerr geometry with subtracted conformal factor is not a vacuum solution. However, we can form a genuine solution by specifying appropriate matter that supports the solution.

To assess the situation we focus for now on the non-rotating solutions. The Einstein tensor for the geometries (2.4) with arbitrary conformal factor \( \Delta = e^{-4U} \) then becomes

\[ G_{\phi\phi} = G_{\theta\theta} = -G_{r r} = e^{2U} \left( r \partial_r U + 1 \right) \left( (r - 2m) \partial_r U + 1 \right) , \]
\[ G_{i i} = G_{\theta\theta} + 2e^{2U} r(r - 2m) \left( \partial_r U - (\partial_r U)^2 \right) . \]

The hatted coordinates refer to the standard orthonormal frame. We wish to find matter with an energy-momentum tensor that equates this Einstein tensor.

It turns out that for the static case it is sufficient to consider the STU Lagrangian in the absence of pseudoscalars:

\[ \mathcal{L} = -\frac{1}{16\pi G_4} \left( R - \frac{1}{2} \sum_{i=1}^3 \nabla_\mu \eta_i \nabla^\mu \eta_i - \frac{1}{4} e^{-q_1 - q_2 - q_3} F_{0\mu\nu} F^{0\mu\nu} - \frac{1}{4} e^{-q_1 - q_2 - q_3} \sum_{i=1}^3 e^{2q_i} F_{i\mu\nu} F^{i\mu\nu} \right) . \]
In the case of a spherically symmetric configuration (without magnetic fields) the energy
momentum becomes
\[
8\pi G_4 T^{\phi\phi} = 8\pi G_4 T_{\theta\theta} = -8\pi G_4 T_{rr} ,
\]
\[
8\pi G_4 T^{\phi\phi} = \frac{1}{4} X e^{2U} \sum_{i=1}^{3} \left( \partial_{\eta} \eta_i \right)^2 - \frac{1}{4} e^{-\eta_m - \eta_2 - \eta_3} \left( (F_{ri})^2 + \sum_{i=1}^{3} e^{2\eta_i} (F_{ri})^2 \right) ,
\]
\[
8\pi G_4 T_{rr} = \frac{1}{4} X e^{2U} \sum_{i=1}^{3} \left( \partial_{\eta} \eta_i \right)^2 + \frac{1}{4} e^{-\eta_m - \eta_2 - \eta_3} \left( (F_{ri})^2 + \sum_{i=1}^{3} e^{2\eta_i} (F_{ri})^2 \right) .
\]
(4.3)

As a check on the equations we may compute the Einstein tensor (4.1) for the standard
conformal factor \( \Delta_0 = e^{-4U} = \prod_{i=0}^{3} h_i \) with \( h_I = r + 2m \sinh^2 \delta_I \) and verify that the result
agrees with the energy-momentum tensor (4.3) for the matter
\[
e^{-\eta_i} = h_i \sqrt{\frac{h_0}{h_1 h_2 h_3}} , \quad i = 1, 2, 3 ,
\]
\[
A_I^I = \frac{2m \sinh \delta_I \cosh \delta_I}{h_I} , \quad I = 0, 1, 2, 3 .
\]
(4.4)

This is the matter that supports the solution with standard conformal factor.

The desired Einstein tensor (4.1) satisfies \( G_{\phi\phi} = G_{\theta\theta} = -G_{rr} \) even for an arbitrary
conformal factor. Comparing with the energy momentum tensor (4.3) for the STU-model
we see that a combination of scalars and vectors will be appropriate matter also in the
general case.

The specific conformal factor (3.5) that we focus on corresponds to
\[
U = -\frac{1}{4} \ln \left( (2m)^3 (r (\Pi^2_e - \Pi^2_s) + 2m \Pi^2_s) \right) .
\]
(5.5)

Evaluating the right hand side of (4.1) for this \( U \) we determine the scalar and vector matter
needed to support the subtracted solution as:
\[
8\pi G_4 T^{\text{scalar}}_{\tilde{r} \tilde{r}} = \frac{1}{2} (G_{\tilde{r} \tilde{r}} - G_{\tilde{\theta} \tilde{\theta}}) = \frac{3}{16} X e^{2U} \frac{(\Pi^2_e - \Pi^2_s)^2}{(r (\Pi^2_e - \Pi^2_s) + 2m \Pi^2_s)^2} ,
\]
\[
8\pi G_4 T^{\text{vector}}_{\tilde{r} \tilde{r}} = \frac{1}{2} (G_{\tilde{r} \tilde{r}} + G_{\tilde{\theta} \tilde{\theta}}) = e^{2U} \left[ \frac{3}{4} + \frac{(2m)^2 \Pi^2_e \Pi^2_s}{4(r (\Pi^2_e - \Pi^2_s) + 2m \Pi^2_s)^2} \right] .
\]
(4.6)

At this point comparison with the scalar and vector terms in (4.3) gives simple ordinary
differential equations for the matter.

The solution for the matter is not unique. For example, a duality transformation will
leave the Einstein geometry invariant but change the matter. We will construct just the
simplest solution and note just one obvious ambiguity. Equating the first line in (4.6) with
the scalar term in (4.3) we find (for some choice of integration constant):
\[
\eta_i = -\frac{1}{2} \ln \left( (2m)^3 r (\Pi^2_e - \Pi^2_s) + \Pi^2_s \right) , \quad i = 1, 2, 3 .
\]
(4.7)
\[
F_{ri} = e^{-\frac{1}{2} \eta_i + U} = 1 , \quad i = 1, 2, 3 ,
\]
\[
F_{rr} = \Pi_e \Pi_s ,
\]
(4.8)
\[
F_{\tilde{r} \tilde{r}} = \frac{4m^2 (r (\Pi^2_e - \Pi^2_s) + 2m \Pi^2_s)^2} .
\]
A different integration constant in (4.7) would rescale $F_{\mu t}^i$ by some factor $e^{-\frac{i}{2}\delta\eta}$ (with $\delta\eta$ some constant) and simultaneously rescale $F_{\mu i}^0$ by $e^{\frac{3}{2}\delta\eta}$. For example, the addition of $2\ln 2m$ to the right hand side of (4.7) makes the scalar field $\eta_i$ dimensionless and gives $F_{\mu i}^i$ and $F_{\mu 0}^0$ the same dimension (of inverse length). The choice made in (4.7) avoids the introduction of an arbitrary scale (like $2m$) and will be convenient later.

The scalar and vector fields (4.7), (4.8) were constructed such that the Einstein equations are satisfied. It remains to verify that the matter field equations of motion are also obeyed. This is a straightforward exercise.

The three-fold symmetry of the supporting matter means the subtracted solution does not distinguish between the three charges. One subtle dependence nevertheless remains: the classical solution is written in terms of “dressed” charges, the product of the quantum charges and various moduli. The position in moduli space may therefore specify the charge assignment implicitly. This is the mechanism that applies in the extremal case where the attractor mechanism forces the “dressed” charges to be identical.

### 4.2 Discussion of matter

Any geometry is a solution to Einstein’s equations, if the energy momentum tensor is chosen as the Einstein tensor of the geometry. The nontrivial question is always whether such matter is physical. The standard criterion is to ask whether the matter specified by the Einstein tensor satisfies suitable energy conditions. In the present situation all of the standard energy conditions — dominant, strong, weak, and null — are in fact satisfied (at least when there is no rotation). The black holes with subtracted conformal factor are therefore physical.

However, it is significant that we have gone beyond these criteria, by finding explicit matter. Indeed, we have shown that the subtracted geometries are solutions to the same theory as the original asymptotically flat black holes. This is explicit evidence that the black holes with subtracted conformal factor are physical. But it is presumably also useful for analyzing the physics of these solutions.

The realization of the subtracted geometries as solutions to the same theory as the original black holes suggests that there is a more direct relation between these geometries. We expect that it can be obtained by a solution generating technique on the original black hole, at least for the STU-model. Actually, in the Schwarzschild case the subtracted geometry emerges as a specific Harrison transformation within Dilaton-Maxwell-Einstein gravity, somewhat akin to transformations considered within Maxwell-Einstein gravity in [10]. The identification of a transformation relating the true matter to the auxiliary matter (4.7), (4.8) that we have identified in the non-rotating case could also serve as a practical strategy for generalizations to the cases with rotation. It would be interesting to compare the result of such considerations with the matter inferred from another method in the next section.

### 5 A five dimensional interpretation of the subtracted geometry

In this section we realize separability of the scalar wave equation geometrically, by introducing an auxiliary dimension. The construction identifies a locally AdS$_3$ geometry that
accounts for the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ symmetry of the hypergeometric radial equation. It also gives a simplified representation of the matter that supports the subtracted geometry.

5.1 Simplified derivation of the scalar wave equation

A useful initial goal is to seek a geometric interpretation of separability by attempting to construct a factorized spacetime directly. One of the ways that the geometry (2.3) couples the angular and radial coordinates is through the function $G$. It is therefore advantageous to expand the geometry such that the spurious poles at $G = 0$ cancel explicitly:

$$\Delta^{-1/2} ds^2_4 = - \frac{G}{\Delta} \left( dt + \frac{a \sin^2 \theta}{G} A_{\text{red}} d\phi \right)^2 + \frac{X \sin^2 \theta}{G} d\phi^2 + \frac{dr^2}{X} + d\theta^2$$

$$= \frac{1}{4m^2(\Pi_c - \Pi_s)^2} dt^2 - \frac{1}{4m^2(\Pi_c - \Pi_s)^2} \Delta \left[ A_{\text{red}} dt + 4m^2(\Pi_c - \Pi_s)^2 a \sin^2 \theta d\phi \right]^2 + \frac{dr^2}{X} + d\theta^2 + \sin^2 \theta d\phi^2. \quad (5.1)$$

We used (2.6) relating $G$ to $X$ and (3.6) relating $\Delta$ and $A_{\text{red}}$. We can disentangle radial and polar variables further by writing the conformal factor (3.5) as $\Delta = \rho + \gamma$ where

$$\rho = A_{\text{red}}^2 - 4m^2(\Pi_c - \Pi_s)^2 X = 8m^3 \left[ r(\Pi_c^2 - \Pi_s^2) + 2m \Pi^2 - \frac{a^2}{2m} (\Pi_c - \Pi_s)^2 \right],$$

$$\gamma = 4m^2(\Pi_c - \Pi_s)^2 a^2 \sin^2 \theta. \quad (5.2)$$

The subtracted metric (5.1) now simplifies to

$$ds^2_4 = \Delta^{1/2} \left( - \frac{X}{\rho} dt^2 + \frac{dr^2}{X} + d\theta^2 \right) + \Delta^{-1/2} \rho \sin^2 \theta \left( d\phi - \frac{a A_{\text{red}}}{\rho} dt \right)^2. \quad (5.3)$$

We used the identity

$$\frac{\rho \gamma}{\rho + \gamma} \left( \frac{p}{\gamma} - \frac{q}{\rho} \right)^2 = \frac{p^2}{\gamma^2} + \frac{q^2}{\rho^2} - \frac{(p + q)^2}{\gamma + \rho}, \quad (5.4)$$

with the identifications

$$p = 4m^2(\Pi_c - \Pi_s)^2 a \sin^2 \theta d\phi,$$

$$q = A_{\text{red}} dt. \quad (5.5)$$

The metric in the form (5.3) almost decouple the angular and radial dependence. For example, we can use this expression to separate variables in the Laplacian quite easily:

$$\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu\nu} \right) \partial_{\nu}$$

$$= \Delta^{-1/2} \left[ \frac{\rho}{X} \left( \partial_t + \frac{a A_{\text{red}}}{\rho} \partial_\phi \right)^2 + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{\gamma + \rho}{\rho \sin^2 \theta} \partial_\phi^2 \right]$$

$$= \Delta^{-1/2} \left[ \frac{1}{X} \left( \rho \partial_t^2 + 2a A_{\text{red}} \partial_\phi \partial_t + a^2 \partial_\phi^2 \right) + X \partial_r^2 + \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2 \right]. \quad (5.6)$$

The recovery of the correct scalar wave equation gives a check on our algebra.
5.2 A five dimensional lift

It is instructive to reconsider separability from a five dimensional point of view. The last form of the metric in (5.1) is a good starting point. In that expression the second term is quite awkward but it can be presented as $-\Delta B^2$

$$B = \frac{p + q}{2m(\Pi_c - \Pi_s)\Delta} = \frac{(\Pi_c - \Pi_s)r + 2m\Pi_s}{(\Pi_c - \Pi_s)\Delta} dt + \frac{2m(\Pi_c - \Pi_s)a \sin^2 \theta d\phi}{\Delta}.$$  (5.7)

It is natural to cancel this term by introducing an auxiliary coordinate $\alpha$ and so consider the five dimensional auxiliary metric

$$ds^2_5 = \Delta(d\alpha + B)^2 + \Delta^{-1/2}ds^2_4 = -\frac{X}{\rho} dt^2 + \frac{dr^2}{X} + \rho \left( d\alpha + \frac{A_{red}}{2m(\Pi_c - \Pi_s)\rho} dt \right)^2 + d\theta^2 + \sin^2 \theta (d\phi + 2ma(\Pi_c - \Pi_s) d\alpha)^2.$$  (5.8)

This geometry is locally AdS$_3 \times S^2$. The sphere is fibered over the AdS$_3$ base by the shifted angle $\phi' = \phi + 2ma(\Pi_c - \Pi_s)\alpha = \phi + 2G_4 J\alpha$. This does not prevent the product form from making separability explicit. Additionally, the AdS$_3$ accounts for the hypergeometric form of the scalar wave equation. Furthermore, this five dimensional lift has the same geometry as the one obtained in the dilute gas approximation $\delta_I \gg 1$ ($I = 1, 2, 3$) [17].

The auxiliary five dimensional geometry (5.8) was introduced as a dimensionless geometry, without a specific scale. Therefore the radius of the sphere $\ell_S = 1$ is a pure number. Similarly the scale $\ell_A = 2$ of the AdS$_3$ factor is a pure number. A related issue is that the auxiliary coordinate $\alpha$ has dimension length$^{-2}$. Assuming that $\alpha$ is periodic with periodicity $2\pi R_\alpha$, the radius $R_\alpha$ will have dimension of length$^{-2}$ as well. It is preferable to keep these awkward assignments of dimensions rather than introducing a specific scale that would in any case be arbitrary.

An additional benefit of the five dimensional representation of the black hole is that it provides a geometrical interpretation of the matter supporting the subtracted solution, previously introduced for the non-rotating case in (4.7), (4.8). To see this we compute the electric field strength for the gauge field $B$ (5.7) in the non-rotating case

$$F^B_{tr} = \frac{\Pi_c \Pi_s}{4m^2((\Pi_c^2 - \Pi_s^2)r + 2m\Pi_s^2)^2}.$$  (5.9)

This expression is identical to $F^0_{tr}$ given in (4.8). We can therefore identify the gauge field in 4D with the graviphoton gauge field (5.7), at least in the non-rotating case. The scalar field (4.7) introduced directly in four dimensions can similarly be identified with the dilaton determined through

$$e^{-2\Phi_4} = R_\alpha \sqrt{\Delta}.$$  (5.10)

The precise identification is $2\Phi_4 = \eta_i$ ($i = 1, 2, 3$). The remaining gauge fields $F^i_{tr}$ introduced in (4.8) are constants that can be identified with a constant field strength in five dimensions. Thus the overall representation is the one where the subtracted black hole geometry is a solution to minimal supergravity in five dimensions.
5.3 The effective BTZ black hole

It is worthwhile to rewrite the five dimensional auxiliary metric (5.8) explicitly as BTZ×$S^2$ with the BTZ black hole presented in the standard form

\[
ds_{\text{BTZ}}^2 = -\frac{(\rho_3^2 - \rho_+^2)(\rho_3^2 - \rho_-^2)}{\ell_A^2 \rho_3^2} \, dt_3^2 + \frac{\ell_A^2 \rho_3^2}{(\rho_3^2 - \rho_+^2)(\rho_3^2 - \rho_-^2)} \, d\rho_3^2 + \rho_3^2 \left( d\phi_3 + \frac{\rho_+ \rho_-}{\ell_A \rho_3^2} \, dt_3 \right)^2 .
\] (5.11)

The BTZ coordinates are identified as

\[
\rho_3^2 = (2mR_\alpha)^2[2mr(\Pi_c^2 - \Pi_s^2) + (2m)^2\Pi_s^2 - a^2(\Pi_c - \Pi_s)^2],
\]

\[
t_3 = \frac{\ell_A}{R_\alpha(2m)^3(\Pi_c^2 - \Pi_s^2)^2} t,
\]

\[
\phi_3 = \frac{\alpha}{R_\alpha} + \frac{t_3}{\ell_A} .
\] (5.12)

The transformation gives the identifications $\ell_A = 2$ and

\[
\rho_\pm = 2mR_\alpha[m(\Pi_c + \Pi_s) \pm \sqrt{m^2 - a^2(\Pi_c - \Pi_s)}] .
\] (5.13)

These assignments are equivalent to the physical BTZ parameters

\[
M_3 = \frac{\rho_+^2 + \rho_-^2}{8G_3 \ell_A^2} = \frac{m^2R_\alpha^2}{4G_3} (2m^2(\Pi_c^2 + \Pi_s^2) - a^2(\Pi_c - \Pi_s)^2) ,
\]

\[
J_3 = \frac{\rho_+ \rho_-}{4G_3 \ell_A} = \frac{m^2R_\alpha^2}{2G_3} (4m^2\Pi_c\Pi_s + a^2(\Pi_c - \Pi_s)^2) .
\] (5.14)

The effective Newton’s constant in three dimensions is determined in terms of the Newton’s constant in four dimensions by comparing the reduction from five dimensions on a sphere with radius $\ell_S = 1$ to the reduction on a circle with radius $R_\alpha$:

\[
\frac{1}{G_3} = \frac{4\pi \ell_S^2}{G_5} = \frac{4\pi \ell_A^2}{2\pi R_\alpha G_4} = \frac{2}{R_\alpha G_4} .
\] (5.15)

This in turn gives the Brown-Henneaux central charge of the effective AdS$_3$ with radius $\ell_A = 2$:

\[
c = \frac{3\ell_A}{2G_3} = \frac{6}{R_\alpha G_4} .
\] (5.16)

Recall that $R_\alpha$ has dimension of inverse (length)$^2$ so this expression is dimensionless. We are also interested in the effective conformal weights

\[
h_+ = \frac{M_3 \ell_A + J_3}{2} = \frac{m^4R_\alpha}{G_4}(\Pi_c + \Pi_s)^2 ,
\]

\[
h_- = \frac{M_3 \ell_A - J_3}{2} = \frac{m^2(m^2 - a^2)R_\alpha}{G_4}(\Pi_c - \Pi_s)^2 .
\] (5.17)

Again these expressions are dimensionless because $R_\alpha$ has dimension of (length)$^{-2}$.
The conformal dimensions are generally complicated functions of all physical charges (with implicit dependence on moduli), black hole mass, and black hole angular momentum. However, the combination of charges

\[ I_4 = \frac{4m^4 \Pi_c \Pi_s}{G_4^2}, \quad (5.18) \]

is independent of moduli and dependent only on the quantized charges. It is normalized to be an integer. It follows that the effective 3D angular momentum simplifies as

\[ J_3 = h_+ - h_- = \frac{1}{k} (I_4 + J^2). \quad (5.19) \]

We use the notation \( k = c/6 \) where \( c \) is given in (5.16). This is consistent with expectations from (generalized) level matching.

The entropy computed from (5.17) by using Cardy’s formula gives

\[ S = 2\pi \left( \sqrt{\frac{ch_+}{6}} + \sqrt{\frac{ch_-}{6}} \right) 
= \frac{2\pi m}{G_4} \left( (\Pi_c + \Pi_s)m + (\Pi_c - \Pi_s)\sqrt{m^2 - a^2} \right). \quad (5.20) \]

This agrees with the entropy (2.21) of the original four dimensional black hole, as it should.

The agreement for the entropy is not impressive in and by itself. In fact, it follows automatically from the local \( \text{AdS}_3 \) structure (for review see [2]). In order for a counting to be claimed we must specify the scale \( R_\alpha \) which is arbitrary for now. Additionally, we must ascertain that there really is a physical conformal symmetry for which Cardy’s formula (5.20) performs asymptotic state counting. These are the issues we address in the next section.

6 Hidden conformal symmetry

There are several promising routes from the facts we have presented to a useful underlying 2D conformal symmetry. In this section we discuss a 4D interpretation (inspired by Kerr/CFT) and a five dimensional interpretation (generalizing AdS/CFT correspondence).

6.1 2D conformal symmetry from 4D

The subtracted geometry with conformal factor (3.5) has \( \text{SL}(2, R) \times \text{SL}(2, R) \) symmetry. Accordingly, we can represent the scalar Laplacian in the two forms

\[ \ell^2 \nabla^2 = \mathcal{R}_1^2 + \mathcal{R}_2^2 - \mathcal{R}_3^2, \]
\[ = \mathcal{L}_1^2 + \mathcal{L}_2^2 - \mathcal{L}_3^2, \quad (6.1) \]

where the linear differential operators \( \mathcal{R}_i \) \((i = 1, 2, 3)\) and \( \mathcal{L}_i \) \((i = 1, 2, 3)\) commute with each other and satisfy \( \text{SL}(2, R) \) algebras

\[ [\mathcal{R}_i, \mathcal{R}_j] = 2i\epsilon_{ijk}(-)^{\delta_{k3}} \mathcal{R}_k; \quad [\mathcal{L}_i, \mathcal{L}_j] = 2i\epsilon_{ijk}(-)^{\delta_{k3}} \mathcal{L}_k. \quad (6.2) \]
We can construct the differential operators explicitly by comparing with global AdS$_3$

\[ ds^2_3 = \ell^2 (dp^2 - \cosh^2 \rho d\tau^2 + \sinh^2 \rho d\sigma^2) . \]  

(6.3)

In this standardized setting the Laplacian takes the form (6.1) with the SL(2, R) × SL(2, R) generators

\[ \mathcal{R}_\pm = \mathcal{R}_1 \pm i \mathcal{R}_2 = e^{\pm i(\tau + \sigma)} (\mp i \partial_\rho + \tanh \rho \partial_\tau + \coth \rho \partial_\sigma) , \]

\[ \mathcal{R}_3 = \partial_\tau + \partial_\sigma . \]  

(6.4)

and \( \mathcal{L}_i \) determined by taking \( \tau \rightarrow -\tau \) in these expressions. Comparing the Laplacian in global AdS$_3$

\[ \ell^2 \nabla^2 = \frac{1}{\sinh 2\rho} \partial_\rho \sinh 2\rho \partial_\rho - \frac{1}{\cosh^2 \rho} \partial_\tau^2 + \frac{1}{\sinh^2 \rho} \partial_\sigma^2 , \]  

(6.5)

and the Laplacian (3.2) we find the identifications

\[ \sinh^2 \rho = \frac{r - r_+}{r_+ - r_-} , \]

\[ \sigma - \tau = -\frac{2\pi i}{\beta_L} \left( t - \frac{\beta_R}{\beta_H \Omega_H} \phi \right) , \]

\[ \sigma + \tau = -\frac{2\pi i}{\beta_H \Omega_H} \phi . \]  

(6.6)

These identifications map the SL(2, R) × SL(2, R) generators in global AdS$_3$ (6.4) to the ones adapted to the subtracted black hole background. It is important that the resulting SL(2, R) generators are canonically normalized due to the nonabelian nature of SL(2, R). In particular the normalized Cartan generators become

\[ \pi \mathcal{R}_3 = \beta_R i \partial_t + \beta_H \Omega_H i \partial_\phi , \]

\[ \pi \mathcal{L}_3 = \beta_L i \partial_t . \]  

(6.7)

It may be useful to introduce yet another set of coordinates

\[ t^- = \frac{i}{2} (\sigma - \tau) , \]

\[ t^+ = \frac{i}{2} (\sigma + \tau) . \]  

(6.8)

such that the generators (6.7) are represented canonically as \( \mathcal{R}_3 = i \partial_{t^+} , \mathcal{L}_3 = i \partial_{t^-} \). These coordinates generalize the preferred coordinates introduced in [7].

It is a central issue already at the level of the SL(2, R) × SL(2, R) symmetry that the generators (6.4) are globally ill-defined [7]: the azimuthal angle is identified as \( \phi \equiv \phi + 2\pi \) and the generators \( \mathcal{R}_\pm \) and \( \mathcal{L}_\pm \) transform under this identification. It is natural to interpret this ambiguity in terms of a thermal CFT which, because it is defined on a torus, obeys the equivalences \( \mathcal{R}_\pm \equiv \mathcal{R}_\pm e^{-4\pi^2 T_{CFT}^\pm} , \mathcal{L}_\pm \equiv \mathcal{L}_\pm e^{-4\pi^2 T_{CFT}^\pm} \). This interpretation determines

\footnote{The notation \( w^\pm \sim e^{\mp i z^2} \) was used in [7]. The utility of the \( \tau, \sigma \) coordinates was noted already in [18].}
the relative normalization of the (dimensionful) physical temperatures \( T_{R,L} = \beta_{R,L}^{-1} \) and the (dimensionless) CFT temperatures as

\[
T_{L,R}^{\text{CFT}} = T_{L,R} \cdot \beta_{R} R_{H} \Omega_{H} .
\]  

(6.9)

To put these values in perspective it is interesting to assume that the CFT accounts for the black hole entropy (2.21) by satisfying the Cardy formula in the canonical ensemble

\[
S = \frac{\pi^2}{3} \left( c_{L} T_{L}^{\text{CFT}} + c_{R} T_{R}^{\text{CFT}} \right) .
\]  

(6.10)

The central charges inferred from this assumption are

\[
c_{L} = c_{R} = 12 \cdot \frac{1}{4 \pi^2} \frac{S_{L,R}}{T_{L,R}} \frac{\beta_{H} \Omega_{H}}{\beta_{R}} = 12J .
\]  

(6.11)

It is non-trivial that this procedure gives the same central charge in the L and R sectors (equivalent coincidences were noted in [19, 20]).

The result (6.11) and the procedure leading to it is a generalization of the hidden conformal symmetry approach [7] to the setting with general charges (see also [21–26]). A central weakness of the approach is the assumption that the \( SL(2, R) \times SL(2, R) \) symmetry is enhanced to a Virasoro symmetry (squared) and the assumption that Cardy’s formula applies. It is not presently known how to justify these assumptions. It is nevertheless interesting that the computation suggests a master CFT with the central charge (6.10). This value is familiar from the Kerr/CFT correspondence [27]; but the considerations here suggest (following [7]) that this one theory accounts for the entropy of black holes far from extremality.

Our extension to the setting with arbitrary charges creates a tension between this optimistic interpretation of the Kerr/CFT correspondence and the standard description (such as [28]) that applies near the BPS limit. Such “large charge” descriptions invoke CFT’s with a central charge that depends on spacetime charges and in many cases these CFT’s describes black holes with a range of the angular momenta. It would be interesting to delineate the range of applicability of these disparate descriptions.

6.2 2D conformal symmetry from 5D

Our embedding of the subtracted geometry into five dimensions suggests a different approach to the apparent conformal symmetry: the local \( \text{AdS}_3 \times S^2 \) invites reference to standard AdS/CFT correspondence [29] or, more precisely, the Brown-Henneaux result [30]. Concretely, we can map the BTZ black hole (5.11) into global \( \text{AdS}_3 \) (6.3) through the identifications

\[
t^\pm = \frac{i}{2} (\sigma \pm \tau) = \frac{r_+ \pm r_-}{2 \ell_A} z^\pm ,
\]  

(6.12)

where, according to the embedding (5.12),

\[
z^- = \phi_3 - \frac{t_3}{\ell_A} = \frac{\alpha}{R_{\alpha}} ,
\]

\[
z^+ = \phi_3 + \frac{t_3}{\ell_A} = \frac{\alpha}{R_{\alpha}} + \frac{2\ell}{R_{\alpha}(2m)^2(\Pi_{\ell}^2 - \Pi_{s}^2)} t .
\]  

(6.13)
An obvious advantage of this approach is that the SL(2,R)’s do in fact extend to full Virasoro’s: diffeomorphisms that act on $z^\pm$ (while preserving asymptotic AdS$_3$) form a Virasoro algebra in AdS$_3$. A key issue then becomes the value of the central charge, given previously in (5.16). It is determined entirely by $R_\alpha$, the periodicity of the auxiliary dimension. This parameter can be inferred from the fibration of the sphere $S^2$ over BTZ: according to (5.8) it is only the combination $\alpha + 2G_4 J\phi$ that enters so the azimuthal shift symmetry $\phi \rightarrow \phi + 2\pi$ is equivalent to $R_\alpha = 2G_4 J$. The central charge (5.16) then returns to the Kerr value $c = 12J$.

Although the 5D interpretation thus appears to have the same central charge as the 4D interpretation, the states in the Virasoro representation are quite different: in 5D the states generally depend on the coordinate $\alpha$. An added value of the 5D representation is that it realizes modular invariance (and spectral flow symmetry) in a simple manner. This ensures that conformal symmetry acts on a sufficient number of conformal primaries that the black hole entropy is accounted for, rather than just on the AdS$_3$ vacuum. It would be interesting to understand the relations between the 4D and 5D interpretations.

There is another interesting periodicity that is determined by the set-up. Recall that the azimuthal angle $\phi$ plays the role of time near the horizon and regularity of the Euclidean geometry fixes its imaginary periodicity as (2.11). This periodicity is computed with a combination of $\phi$ and the asymptotic time $t$ fixed, and this in turn determines the imaginary periodicity of $t$ as (2.12). These shifts are with $\alpha$ fixed but (5.8) show that they are equivalent to the imaginary periodicity

$$\beta_\alpha = \frac{\beta_\phi}{2ma(\Pi_c - \Pi_s)} = \frac{2\pi}{2m(\Pi_c - \Pi_s)\sqrt{m^2 - a^2}}.$$ (6.14)

This quantity can be interpreted as usual as the chemical potential for excitations with momentum along the auxiliary direction $\alpha$. It can be expressed geometrically as

$$\frac{1}{\beta_\alpha} = \frac{A_+ - A_-}{16\pi^2},$$ (6.15)

where $A_\pm$ are the areas of the outer and inner horizon. The geometric nature of this formula suggests a robust significance of this potential. We defer further exploration of its origin to future work.

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