Quantum query complexity of symmetric oracle problems.

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December 27, 2018

Abstract

We study the query complexity of quantum learning problems in which the oracles form a group $G$ of unitary matrices. In the simplest case, one wishes to identify the oracle, and we find a description of the optimal success probability of a $t$-query quantum algorithm in terms of group characters. As an application, we show that $\Omega(n)$ queries are required to identify a random permutation in $S_n$. More generally, suppose $H$ is a fixed subgroup of the group $G$ of oracles, and given access to an oracle sampled uniformly from $G$, we want to learn which coset of $H$ the oracle belongs to. We call this problem coset identification and it generalizes a number of well-known quantum algorithms including the Bernstein-Vazirani problem, the van Dam problem and finite field polynomial interpolation. We provide character-theoretic formulas for the optimal success probability achieved by a $t$-query algorithm for this problem. One application involves the Heisenberg group and provides a family of problems depending on $n$ which require $n+1$ queries classically and only 1 query quantumly.

1 Introduction

An oracle problem is a learning task in which a learner tries to determine some information by asking certain questions to a teacher, called an oracle. In our setting the learner is a quantum computer and the oracle is an unknown unitary operator acting on some subsystem of the computer. The computer asks questions by preparing states, subjecting them to the oracle, measuring the results, and finally making a guess about the hidden information. How many queries to the oracle are needed by the computer to guess the correct answer with high probability?

This paper addresses the following oracle problem. Fix a finite group $G$ and a subgroup $H \leq G$. The elements of $G$ are encoded as unitary operators by some unitary representation $\pi: G \to U(V)$. Given oracle access to $\pi(a)$ (for some unknown $a \in G$) the learner must guess which coset of $H$ the element $a$ lies in.

We call this problem coset identification. This task includes as special cases univariate and multivariate polynomial interpolation over a finite field [6] [5], the group summation problem [13] [20] [2], symmetric oracle discrimination [4] and homomorphism evaluation [20]. In this paper prove that nonadaptive algorithms are optimal for this task (measuring optimality only by the query complexity of the algorithm). We provide tools to reduce the analysis of query complexity to purely character theoretic questions (which are themselves often combinatorial). In particular we derive a formula for the exact quantum query complexity for coset identification in terms of characters. In the case of symmetric oracle discrimination (which itself includes polynomial interpolation as a special case) we find the lower and upper bound for bounded error query complexity.

Another motivation for our work is the study of nonabelian oracles. Much is known about quantum speedups when the oracle is a standard Boolean oracle. Less is known about whether oracle problems with nonabelian symmetries can offer notable speedups. To that end we study the follow scenario: suppose a group $G$ acts by permutations on a finite set $\Omega$ (we call $\Omega$ a $G$-set). A learner is given access to a machine which takes an element $\omega \in \Omega$ and returns $a \cdot \omega$ for some hidden group element $a \in G$. With as few queries as possible the learner should guess the hidden element $a \in G$. The classical query complexity for this problem is a long-known invariant of $G$-sets called the base size. For instance, if $G$
is the full permutation group of $\Omega = \{1, \ldots, n\}$ then $n - 1$ queries are required classically to determine the hidden permutation. This problem is a special case of symmetric oracle discrimination and we can express the bounded error quantum query complexity of this purely in terms of the character of the $G$-set $\Omega$. For instance, we find that when $G$ is the full permutation group of $X = \{1, \ldots, n\}$ then $n - \sqrt{n} + n^{1/6}$ queries are necessary (and sufficient) to determine the hidden element. This meager improvement can be seen as an analogue of classical no-go lower bounds showing that quantum computation of total Boolean functions is hard (offers at best a polynomial speedup) \cite{2}.

This task can be further refined: fix a group $G$, a $G$-set $\Omega$, and a homomorphism $G \to X$ (where $X$ is some other group). The task is to determine $f(a)$ given access to a permutational black-box hiding $a$ through the action on $\Omega$. For instance, as above when $G = S_n$ (the symmetric group), $\Omega = \{1, \ldots, n\}$ its defining representation and $f$ the sign homomorphism, it requires $n - 1$ classical queries to determine $f(a)$. As a counterpoint to the harsh lower bound above we provide a family of examples for this task parametrized by $n$ in which the quantum query complexity is $1$ while the classical complexity is $O(n)$. The groups used are Heisenberg groups acting as small subgroups of the full permutation group. This example is a nonabelian analogue of the fact that good speedups can be found in computing partial Boolean functions \cite{3}.

The paper is organized as follows. In Sec. 2 we formalize coset identification in the context of quantum learning algorithms and review the notions of adaptive and nonadaptive learning. In Sec. 3 we prove that parallel queries suffice to produce an optimal algorithm for this task. Sec. 4 applies this theorem to symmetric oracle discrimination and addresses numerous example problems. In Sec. 5 we return to the general coset identification task and we prove the main theorem of this paper, Theorem \cite{5}, which is a formula for the success probability of an optimal $t$-query algorithm in terms of characters. We use this in Sec. 6 to compute the exact and bounded error query complexity of some special examples (including the Heisenberg group example).

## 2 Quantum learning from oracles.

A quantum or classical oracle problem is described by a set of hidden information $Y$, a function $f : Y \to X$ (the function to learn or compute), and a representation of $Y$ as operations on inputs of some kind (which determines the oracles). Classically such a representation consists of a set of inputs $\Omega$ and an assignment taking each $y \in Y$ to a permutation of $\Omega$, ie a map $\pi : Y \to \text{Sym}(\Omega)$. A classical oracle problem is specified by a tuple $(Y, \Omega, \pi, f)$. A classical computer has access to $\pi(y)$ for some unknown $y \in Y$ by spending one query to input $\alpha \in \Omega$ and learn $\pi(y) \cdot \alpha$. The goal is to determine $f(y)$ with a high degree of certainty \cite{4} with as few queries as possible. The quantum representation of oracles is described by a Hilbert space $V$ and an assignment taking each $y \in Y$ to a unitary operator of $V$, in other words a map $\pi : Y \to U(V)$. Thus a quantum oracle problem is specified by a tuple $(Y, V, \pi, f)$. The quantum computer spends one query to input a state $|\psi\rangle \in V$ to $\pi(y)|\psi\rangle$ to acquire the state $|\psi\rangle$; the goal is to produce a state and measurement scheme which outputs the value $f(y)$.

Any classical oracle problem $(Y, \Omega, \pi, f)$ determines a quantum oracle problem via linearization: oracles will act on the Hilbert space $\mathbb{C} \Omega$ (spanned by the orthonormal basis $\{|\omega\rangle \mid \omega \in \Omega\}$) by permutation matrices.

A symmetric oracle problem is an oracle problem in which the hidden information is a group $G$ (so we are replacing $Y$ with $G$) and the map $\pi$ is a homomorphism (to $\text{Sym}(\Omega)$ or $U(V)$). We often regard $V$ as a (left) $\mathbb{C}G$-module where $\mathbb{C}G$ is the group algebra of $G$ (spanned by an orthonormal basis sometimes written without kets as $\{g \mid g \in G\}$). In module notation we sometimes write $g \cdot v := \pi(g)(v)$ (for $g \in G, v \in V$) if the representation $\pi$ is understood from context. The quantum oracle arising from a symmetric classical problem is also symmetric.

\footnote{Throughout the paper we assume that the unknown information $a \in Y$ is sampled uniformly from $Y$.}
Of special interest to us is the case when the function \( f \) to be learned is compatible with the group structure \( G \). An instance of the coset identification problem is a symmetric oracle problem \((G, V, f)\) where the function \( f : G \to X \) is constant on left cosets of a subgroup \( H \leq G \) and distinct on distinct cosets. We also assume \( f \) is onto. The typical example is when \( X = \{gH \mid g \in G\} \) is the set of left cosets of \( H \) and \( f(g) = gH \). An equivalent formulation is to say that \( X \) is a transitive \( G \)-set and the map \( f : G \to X \) is a map of \((\text{left})\ G\)-sets (ie, \( f(gh) = gf(h) \) for all \( g, h \in G \)).

We examine bounded error and exact measures of query complexity. The exact (or zero error) query complexity of a learning problem is the minimum number of queries needed by an algorithm to compute \( f(y) \) with zero probability of error. The bounded error query complexity is the minimum number of queries needed by an algorithm to compute \( f(y) \) with probability \( \geq 2/3 \). The bounded error query complexity is often studied for a family of problems growing with a parameter \( n \) and so changing the constant 2/3 above to any number strictly greater than 1/2 will only change the query complexity by a constant factor mostly ignored in asymptotic analysis.

Broadly speaking, there are two qualitatively different approaches to solving an oracle problem. The first approach is to ask questions one at a time, carefully changing your questions as you receive more information. This is called using adaptive queries. The other approach is to prepare all your questions and ask them at once in one go (imagining the learner has access to multiple copies of the teacher). This is known as using non-adaptive, or parallel queries.

Classically the adaptive model is stronger than the nonadaptive model, since you can convert any non-adaptive algorithm into an adaptive one (by picking your questions in advance but asking them one at a time). This is well-known to be true also in the quantum setting. In the next section we will prove the converse for coset identification:

**Theorem 2.1.** Suppose \((G, V, f)\) describes an instance of coset identification. Then there exists a \( t \)-query quantum algorithm to determine \( f(a) \) with probability \( P \) if and only if there exists a \( t \)-query nonadaptive query algorithm which does the same.

This theorem is certainly not true for arbitrary learning problems: Grover’s algorithm provides an example in which any optimal algorithm must use adaptive queries [19]. To prove the theorem we must precisely state what adaptive and nonadaptive algorithms are.

### 2.1 Adaptive vs. nonadaptive: definitions

Recall that a quantum learning problem is described by a tuple \((Y, V, \pi, f : Y \to X)\) where \( Y \) indexes the set of hidden information, \( V \) is a finite dimensional Hilbert space, \( \pi : Y \to U(V) \) a representation of the unknown information by unitary operators, and \( f \) is the function to learn.

The standard model for an adaptive algorithm is as follows (see eg [2] Section 3.2): A t-query adaptive quantum algorithm for the quantum oracle problem \((Y, V, \pi, f : Y \to X)\) consists of a tuple \( A = (N, \psi, \{U_1, \ldots, U_t\}; \{E_x\}) \) where

- \( N \) is the dimension of the auxiliary workspace used in the computation
- \( |\psi\rangle \) is a unit vector in \( V \otimes \mathbb{C}^N \)
- \( \{U_1, \ldots, U_t\} \) is a set of unitary operators acting on \( V \otimes \mathbb{C}^N \)
- \( \{E_x\}_{x \in X} \) is a POVM with measurement outcomes indexed by \( X \).

The algorithm uses \( t \)-queries to the oracle \( \pi(a) \) (with \( a \) sampled uniformly from \( Y \)) to produce the output state

\[
|\psi_A^A\rangle = U_t \pi(a) U_{t-1} \pi(a) \ldots \pi(a) U_1 \pi(a) |\psi\rangle
\]
upon which the algorithm executes the measurement described by \( \{ E_x \}_{x \in X} \).

In quantum circuit notation the preparation of the state \( |\psi^A_a\rangle \) reads

\[
|\psi\rangle \xrightarrow{\pi(a)} U_1 \xrightarrow{\pi(a)} \ldots \xrightarrow{\pi(a)} U_{t-1} \xrightarrow{\pi(a)} U_t |\psi\rangle = |\psi^A_a\rangle
\]

By contrast, an algorithm is nonadaptive if at any point during the algorithm, the input for some query does not depend on the results to any of the previous queries. Essentially this means that all the inputs are completely determined before the algorithm begins. Classically, \( t \) nonadaptive queries are identical to \( t \) simultaneous queries to \( t \) copies of an oracle. This motivates the following definition (cf. [14, Section 2]):

A \( t \)-query nonadaptive algorithm for the oracle problem \((Y, V, \pi, f)\) is a tuple \( A = (N, \psi, \{ E_x \})_{x \in X} \) where

- \( N \) is the dimension of the auxiliary register.
- \( |\psi\rangle \) is the input state, a unit vector of \( V^\otimes t \otimes \mathbb{C}^N \).
- \( \{ E_x \} \) is a POVM indexed by \( X \).

The algorithm operates on the input state to produce

\( |\psi^A_a\rangle = (\pi(a)^{\otimes t} \otimes I)|\psi\rangle \)

which is then measured using the POVM \( \{ E_x \} \). The next fact is very useful and follows immediately from definitions.

**Lemma 2.2.** A \( t \)-query nonadaptive algorithm for the problem \((Y, V, \pi, f)\) is the same as a single-query nonadaptive algorithm for the oracle problem \((Y, V^{\otimes t}, \pi^{\otimes t}, f)\).

In quantum circuit notation the nonadaptive preparation of the state \( |\psi^A_a\rangle \) is written

\[
|\psi\rangle \xrightarrow{\pi(a)} U_1 \xrightarrow{\pi(a)} \ldots \xrightarrow{\pi(a)} U_{t-1} \xrightarrow{\pi(a)} U_t |\psi\rangle = |\psi^A_a\rangle
\]

In either model, the algorithm \( A \) uses \( t \)-copies of the unitary \( \pi(a) \) to produce a state \( |\psi^A_a\rangle \). Using the POVM \( \{ E_x \} \) results in a measurement value \( x \in X \) with probability
\[ P(x \mid a) = \langle \psi^A_a | E_x | \psi^A_a \rangle \]

Since we assume the oracle is sampled uniformly from \( Y \), the probability that \( \mathcal{A} \) executes successfully is

\[ P_{\text{succ}}(\mathcal{A}) = \frac{1}{|Y|} \sum_{a \in Y} P(f(a) \mid a) = \frac{1}{|Y|} \sum_{a \in Y} \langle \psi^A_a | E_{f(a)} | \psi^A_a \rangle. \]

### 2.2 Symmetric oracle problems.

Suppose we have a symmetric oracle problem \((G, V, f)\). As mentioned in the introduction, since we are focusing on query complexity and not on issues of implementation, analysis of this problem depends only on the character \( \chi_V \) of \( \pi : G \to U(V) \). In fact, a little more is true. Let \( \text{Irr}(G) \) denote the set of irreducible characters of \( G \). Given a representation \( \pi : G \to U(V) \) define the set

\[ I(V) := \{ \chi \in \text{Irr}(G) \text{ appearing in the representation } V \} = \{ \chi \in \text{Irr}(G) \mid (\chi, \chi_V) > 0 \}. \]

Here we are using \((,\cdot)\) to denote the usual inner product of characters. If \( \chi \in \text{Irr}(G) \) and \((\chi, \chi_V) > 0\) we say that \( \chi \) appears in the representation \( V \).

#### Lemma 2.3. The optimal success probability of a \( t \)-query algorithm to solve a symmetric oracle problem \((G, V, f)\) depends only on \( I(V) \) and \( f \).

**Proof.** First, note that if \( U : V \to W \) is a Hilbert space isomorphism then we can define a new oracle problem \((G, W, U\pi U^{-1}, f)\) where the oracles now act on \( W \). Any \( t \)-query algorithm to solve the original problem can be “conjugated” by \( U \) (e.g., the input state \( |\psi\rangle \) becomes \( U|\psi\rangle \)) and the non-oracle unitaries and POVM are conjugated by \( U \) to produce a \( t \)-query algorithm for the new problem which succeeds with the same probability. Conversely any algorithm to solve the new problem can be conjugated by \( U^{-1} \) to solve the old problem with the same probability. Therefore oracle problems with isomorphic unitary representations of \( G \) will have the same \( t \)-query optimal success probability. In other words, only the character \( \chi_V \) is relevant.

Second, we claim that the multiplicities of irreducible characters in \( V \) is not important; only whether they appear in \( V \) or not. Indeed, adding a \( d \)-dimensional workspace to a computer’s original system \( V \) produces a new representation \( V \otimes \mathbb{C}^d \) of \( G \) with character \( d\chi_V \). Since we allow our algorithm to introduce any such workspace, we are in effect allowing it to increase the multiplicity of each character by a factor of \( d \). Note that this process will never produce irreps which did not appear in \( V \) to begin with. Hence the optimal success probability depends only on which irreps appear in \( V \), i.e., the set \( I(V) \).

**Remark.** We are usually indifferent to multiplicities of characters appearing in a given representation since we can always boost any non-zero multiplicities with the addition of a work space. What really matters is which irreps appear and which don’t.

#### Corollary 2.4. The optimal success probability of a \( t \)-query nonadaptive algorithm to solve a symmetric oracle problem \((G, V, f)\) depends only on \( I(V^\otimes t) \) and \( f \).

### 3 Parallel queries suffice.

Here we prove Theorem 2.1, namely that the optimal success probability for coset identification can be attained by a parallel (nonadaptive) algorithm. We prove this by showing that any \( t \)-query adaptive algorithm can be converted to a \( t \)-query nonadaptive algorithm without affecting the success probability. Another way to say this is that every \( t \)-query adaptive algorithm can be simulated by a \( t \)-query nonadaptive one. This technique is greatly inspired by the work Zhandry [20] who proves this result when \( G \) is abelian, and also bears resemblance to the lower bound technique of Childs, van Dam, Hung and
Shparlinski [8], where the special case of polynomial interpolation is addressed.

Let \( \pi : G \to U(V) \) be a unitary representation of \( G \). Let \( \mathbb{C}G \) denote the group algebra of \( G \). Each \( h \in G \) acts on \( \mathbb{C}G \) by left multiplication, an operator we denote \( L_h \). We will use the controlled multiplication operator (8) defined on \( V \otimes \mathbb{C}G \) by

\[
CM|v, g \rangle = \pi(g^{-1})|v \rangle \otimes |g\rangle.
\]

This defines a unitary operator and is a generalization of the standard CNOT gate (take \( G = \mathbb{Z}_2 \) and \( V = \mathbb{C}Z_2 \)). As such we draw it using circuit diagrams as

![Figure 1: Notation for the controlled multiplication gate CM.](image)

There are two \( G \)-actions on \( V \otimes \mathbb{C}G \) we use, one given by \( \pi(h) \otimes L_h \) and the other \( \text{id}_V \otimes L_h \). Our first observation is that \( CM \) intertwines these actions.

**Lemma 3.1.** The controlled multiplication operator satisfies

\[
CM|\pi(h)v \rangle \otimes |hg\rangle = |v, hg\rangle.
\]

The proof follows immediately from the definition of \( CM \). Note that \( CM \) is a \( \mathbb{C}G \)-module isomorphism \( V \otimes \mathbb{C}G \to \mathbb{1} \dim V \otimes \mathbb{C}G \) where \( \mathbb{1} \) denotes the trivial representation. In pictures the lemma reads

![Figure 2: Lemma 3.1 in pictures.](image)

The next property is crucial for our parallelization argument. Recall that if \( W \) is a \( \mathbb{C}G \)-module then \( I(W) \) denotes the set of irreducible characters of \( G \) which appear in \( W \). Since there are two actions on the space \( V \) we use the notation \( V_0 \) to denote the vector space \( V \) (forgetting both actions).

**Lemma 3.2.** Suppose \( W \) is a subrepresentation of \( \mathbb{C}G \). Then there is a subrepresentation \( Y \) of \( \mathbb{C}G \) such that the image of \( V_0 \otimes W \) under \( CM \) is contained in \( V_0 \otimes Y \) and \( Y \) satisfies \( I(Y) = I(V \otimes W) \). Here \( V \otimes W \) denotes the standard tensor product of \( \mathbb{C}G \)-modules.

**Proof.** By Lemma 2.3 \( CM \) is a \( \mathbb{C}G \)-module isomorphism \( V \otimes \mathbb{C}G \to \mathbb{1} \dim V \otimes \mathbb{C}G \) where \( V \) and \( \mathbb{1} \dim V \) denote the same vector space \( V_0 \) with the two different actions. Let \( Z \) denote the image of \( V_0 \otimes W \) under \( CM \). Then \( CM \) restricts to a \( \mathbb{C}G \)-module isomorphism \( V \otimes W \to Z \). Next let \( Y \) be the submodule of \( \mathbb{C}G \) which contains each irreducible of \( I(Z) \) with maximal multiplicity (so if \( \chi \) appears in \( Y \) then \( \chi \) appears with multiplicity \( \chi(e) \)). Now \( Z \cong V \otimes W \) as \( \mathbb{C}G \)-modules so in particular \( I(Z) = I(V \otimes W) \). Hence also \( I(Y) = I(V \otimes W) \).

It remains to prove \( Z \subseteq V_0 \otimes Y \). Indeed, in the \( \mathbb{C}G \)-module \( \mathbb{1} \dim V \otimes \mathbb{C}G \) the subspace \( \mathbb{1} \dim V \otimes Y \) is the maximal subrepresentation containing only irreducibles in \( I(V \otimes W) \). As noted \( Z \) contains only irreducibles in \( I(V \otimes W) \) so therefore \( Z \subseteq \mathbb{1} \dim V \otimes Y \). The result follows since the underlying vector space of \( \mathbb{1} \dim V \) is \( V_0 \). \( \square \)
Now suppose \((G,V,f)\) is an instance of coset identification and \(A = (N,|\psi\rangle,\{U_1,...,U_t\},\{E_x\})\) is a \(t\)-query adaptive algorithm to evaluate the homomorphism \(f\). First, by replacing \(\pi\) with \(\pi \otimes I\) if necessary, we may assume that the algorithm does not use a workspace, that is \(N = 1\). We will describe a new adaptive algorithm \(A'\) which is a modification of \(A\) as follows. We introduce a new workspace which is a copy of \(C^G\). The new intermediate unitaries are \(U_1 \circ CM, U_2 \circ CM, \ldots, U_t \circ CM\). The input state is \(|\psi\rangle \otimes |\eta\rangle\) where \(\eta\) is the equal superposition state in \(C^G\). When the oracle is hiding the unitary \(\pi(a)\) this produces the following state:

\[
|\psi\rangle \pi(a) \quad U_1 \quad \pi(a) \quad U_{t-1} \quad \pi(a) \quad U_t
\]

Figure 3: Pre-measurement state for \(A'\).

Next measurement is performed: first the second register is measured in the standard basis of \(C^G\). Then the original POVM is applied to the first register. The result of these two measurements will be a pair \((g,x)\); the final output of the algorithm is \(gx\).

**Lemma 3.3.** The algorithm \(A'\) succeeds with the same success probability as \(A\).

**Lemma 3.4.** The algorithm \(A'\) can be simulated by a \(t\)-query parallel query algorithm.

**Proof of Theorem 2.1 from Lemmas 3.3 and 3.4** By the two lemmas, given any \(t\)-query adaptive algorithm \(A\) which solves coset identification with probability \(P\), there exists a \(t\)-query parallel query algorithm which succeeds with the same probability.

**Proof of Lemma 3.3.** Consider the pre-measurement state for \(A'\) given that the hidden group element is \(a \in G\). It can be written

\[
|\psi^A_a\rangle = \frac{1}{\sqrt{|G|}} \sum_{g \in G} |\psi^A_{g^{-1}a}\rangle \otimes |g\rangle.
\]

If the first measurement reads \(g\) then the state collapses to \(|\psi^A_{g^{-1}a}\rangle \otimes |g\rangle\). If the second measurement is now performed, the result will read \(f(g^{-1}a)\) with the same probability that the algorithm \(A\) would read this result given that the oracle was hiding \(g^{-1}a\). The algorithm then classically converts the result to \(gf(g^{-1}a)\) which is equal to \(f(a)\) since \(f\) is a left \(G\)-set map. So the following conditional probabilities are equal:

\[
P(A' \text{ outputs } f(a) \mid a \text{ is hidden }, \text{ first measurement result is } g) = P(A \text{ outputs } f(g^{-1}a) \mid g^{-1}a \text{ is hidden }).
\]

Denote these probabilities by \(P_{A'}(f(a) \mid a,g)\) and \(P_A(f(g^{-1}a) \mid g^{-1}a)\) respectively. Since the probability that the first measurement of \(A'\) reads \(g\) is \(1/|G|\) for all \(G\) and \(g\) is sampled independently of \(a\), we

Formally the algorithm \(A'\) is given by

\[
A' = (|G|,|\psi,\eta\rangle,\{U_1 \otimes I \circ CM,...,U_t \otimes I \circ CM\},\{E_x' = \sum_{g \in G} E_{g^{-1}a} \otimes |g\rangle\langle g|\}).
\]
compute the average case success probability by

\[ P_{\text{succ}}(A') = \frac{1}{|G|^2} \sum_{g \in G} \sum_{a \in G} P_A(f(a) \mid a, g) \]

\[ = \frac{1}{|G|^2} \sum_{g \in G} \sum_{a \in G} P_A(f(g^{-1}a) \mid g^{-1}a) \]

\[ = \frac{1}{|G|} \sum_{g \in G} P_{\text{succ}}(A) = P_{\text{succ}}(A). \]

Proof of Lemma 3.4. We rewrite the pre-measurement state of \( A' \) expressed by Figure 3 using Lemma 3.1. Denote the state that results when the hidden element is \( a \in G \) by \(|\psi_a\rangle\). We apply Lemma 3.1 diagrammatically from left to right:

|\psi_{a'}\rangle = |\psi\rangle \pi(a) U_1 \pi(a) \cdots U_{t-1} \pi(a) |\psi\rangle |\eta\rangle \eta \rangle \cdots \eta \rangle |\psi_{a'}\rangle = (I \otimes L_a) \circ \left( (U_t \otimes I) \circ CM \circ \cdots \circ (U_1 \otimes I) \circ CM \right) |\psi, \eta\rangle.

In the last step, in addition to applying Lemma 3.1 at the right of the diagram, we used the fact that \( L_{a^{-1}}|\eta\rangle = |\eta\rangle \) since \( \eta \) spans the trivial subspace of \( \mathbb{C}G \). In formulas we have

\[ |\psi_{a'}\rangle = (I \otimes L_a) \circ \left( (U_t \otimes I) \circ CM \circ \cdots \circ (U_1 \otimes I) \circ CM \right) |\psi, \eta\rangle. \]

Therefore we have converted this algorithm to a single-query algorithm using the oracle \( I \otimes L_a \) with initial state \( U|\psi, \eta\rangle \) where \( U = (U_t \otimes I) \circ CM \circ \cdots \circ (U_1 \otimes I) \circ CM \).

As in Lemma 3.2 let \( V_0 \) denote the underlying vector which hosts two \( G \) actions (the action given by \( \pi \) and the trivial action). Let \( \mathbb{C}|\eta\rangle \) denote the subspace spanned by \( |\eta\rangle \).

Claim. The image of \( V_0 \otimes \mathbb{C}|\eta\rangle \) under \( U \) is contained in \( V_0 \otimes Y \) where \( Y \subseteq \mathbb{C}G \) is a submodule satisfying \( I(Y) = I(V^\otimes) \).
This is readily proved by induction and Lemma 3.2. For instance, by Lemma 3.2 the image of $V_0 \otimes C|\eta\rangle$ under $CM$ is contained in $V_0 \otimes Y_1$ where $Y_1$ is a submodule with $I(Y_1) = I(V)$. The next part of $U$ is $U_1 \otimes I$ which sends $V_0 \otimes Y_1$ to itself. Now another $CM$ is applied and by Lemma 3.2 this sends $V_0 \otimes Y_1$ to $V_0 \otimes Y_2$ where $I(Y_2) = I(V \otimes Y_1) = I(V^{\otimes 2})$.

Having checked the claim, we see that the algorithm $A'$ may be simulated by a single query algorithm to the oracle $I \otimes L_a$ acting on the subspace $V_0 \otimes Y$. Since the irreducibles appearing in this action are $I(\dim V \otimes Y) = I(V^{\otimes t})$, this may be interpreted via Lemma 2.3 as a single-query algorithm using the representation $V^{\otimes t}$. By Lemma 2.2 this is the same as a $t$-query parallel algorithm using the representation $V$. This concludes the proof of Lemma 3.4.

Corollary 3.5. The optimal $t$-query success probability for an algorithm solving an instance of coset identification $(G, V, f)$ is equal to the optimal single-query success probability achievable solving the instance $(G, V^{\otimes t}, \pi^{\otimes t}, f)$.

4 Application to symmetric oracle identification.

Symmetric oracle discrimination is the following task: given oracle access to a symmetric oracle hiding a group element $a \in G$, determine $a$ exactly. This is the special case of coset identification in which $H = \{e\}$. Thus an instance of this problem is determined by a finite group $G$ and a (finite-dim) unitary representation $\pi : G \rightarrow U(V)$. The following theorem computes success probability of a single-query algorithm and is proved by Bucicovschi, Copeland, Meyer and Pommersheim:

**Theorem 4.1.** ([4], Theorem 1) Suppose $G$ is a finite group and $\pi : G \rightarrow U(V)$ a unitary representation of $G$. Then an optimal single-query algorithm to solve symmetric oracle discrimination succeeds with probability

$$P_{\text{opt}} = \frac{d_V}{|G|}$$

where

$$d_V = \sum_{\chi \in I(V)} |\chi(e)|^2.$$ 

The result of the previous section tells us that parallel algorithms are optimal for symmetric oracle discrimination.

**Theorem 4.2.** Suppose $G$ is a finite group and $\pi : G \rightarrow U(V)$ a unitary representation of $G$. Then an optimal $t$-query algorithm to solve symmetric oracle discrimination succeeds with probability

$$P_{\text{opt}} = \frac{d_{V^{\otimes t}}}{|G|}$$

where

$$d_{V^{\otimes t}} = \sum_{\chi \in I(V^{\otimes t})} |\chi(e)|^2.$$ 

**Proof.** Theorem 2.4 tells us that a $t$-query parallel algorithm achieves the optimal success probability. As noted this is equivalent to a single-query algorithm using the representation $\pi^{\otimes t} : G \rightarrow U(V^{\otimes t})$. Now apply Theorem 4.1.

To express the exact and bounded error query complexity of symmetric oracle discrimination we’re compelled to make the following definitions.

Let $V$ denote a $CG$-module. The quantum base size, denoted $\gamma(V)$, is the minimum $t$ for which every irrep of $G$ appears in $V^{\otimes t}$. If no such $t$ exists then $\gamma(V) = \infty$. The bounded error quantum base size,
denoted $\gamma^{\text{bdd}}(V)$ is the minimum $t$ for which
\[ \frac{1}{|G|} \sum_{\chi \in \text{I}(V^\otimes t)} \chi(e)^2 \geq 2/3. \]

If $(G, V)$ is a case of symmetric oracle discrimination then by Theorem 4.2, the number of queries needed to produce a probability 1 algorithm is $\gamma(V)$. That is, the exact quantum query complexity of the problem is equal to the quantum base size of $V$. Similarly the bounded error query complexity is $\gamma^{\text{bdd}}(V)$.

It may happen that one of these quantities is infinite. However when $V$ is a faithful representation then a classical result attributed to Brauer and Burnside ([11], Theorem 4.3) guarantees that every irrep of $G$ appears in one of the tensor powers $V^\otimes 0, V^\otimes 2, \ldots, V^\otimes m$ where $m$ is the number of distinct values of the character of $V$. If $V$ contains a copy of the trivial representation, then we can say that every irrep of $G$ is contained in some tensor power $V^\otimes t$ for some $t$. Hence in this case (with $V$ faithful and contains a copy of the trivial irrep) both $\gamma(V)$ and $\gamma^{\text{bdd}}(V)$ are finite.

In particular, this occurs whenever we “quantize” a classical symmetric oracle discrimination problem. This is the learning problem specified by a finite set $\Omega$ and a homomorphism $G \to \text{Sym}(\Omega)$. A query to an oracle hiding $a \in G$ consists of inputting $\omega \in \Omega$ and receiving $a \cdot \omega$. The learner must determine the hidden group element (or permutation) $a$. The quantized learning problem uses the homomorphism $G \to U(\mathbb{C} \Omega)$ sending elements of $G$ to permutation matrices. (Such a representation is called a permutation representation.) Then the quantized learning problem is faithful if the original problem is faithful and the $\mathbb{C}G$-module contains a copy of the trivial representation, namely span$\{\sum_{\omega \in \Omega} \langle \omega \rangle\}$.

This is precisely the situation we would like to study because we can compare the classical and quantum query complexity. Classically the exact and bounded error query complexities are equal, since if a classical algorithm does not use enough queries to identify the hidden permutation with certainty then it must make a guess between at least 2 equally likely permutations which behave the same on all the queries that were used, resulting in a success rate of at most 1/2.

**Example 4.3.**
- Suppose $\Omega = \{1, \ldots, n\}$ hosts the defining permutation representation of $G = S_n$. Then $n - 1$ queries are required to determine a hidden permutation $\sigma$.
- If we take the same action but restrict the group to $A_n \leq S_n$ then we need $n - 2$ queries to determine a hidden element $\sigma \in A_n$.
- Consider the action of the dihedral group $D_n$ on the set of vertices of an $n$-gon. Then 2 queries are required to determine a hidden group element.

In general the classical query complexity is a well-known invariant of a permutation group $G$ denoted $b(G)$ called the minimal base size or just base size of $G$ [12]. It may be defined to be the length of the smallest tuple $(\omega_1, \ldots, \omega_t) \subseteq \Omega^t$ with the property that $(g \cdot \omega_1, \ldots, g \cdot \omega_t) = (\omega_1, \ldots, \omega_t)$ if and only if $g = 1$. Thus the classical query complexity of symmetric oracle discrimination of $G \leq \text{Sym}(\Omega)$ is the base size of $G$ and the quantum exact (bounded error) query complexity is the (bounded error) quantum base size. We are naturally led to a broad group theoretic problem:

**Question.** What are the relationships between $b(G), \gamma(\mathbb{C} \Omega)$ and $\gamma^{\text{bdd}}(\mathbb{C} \Omega)$?

We are not aware of any direct comparison of these quantities in the group theory literature. Here we only compute the various quantities for some special cases. We saw earlier that $b(S_n) = n - 1$. We will prove

**Theorem 4.4.** Let $\gamma, \gamma^{\text{bdd}}$ denote the quantum base sizes for $S_n$ acting on $\{1, \ldots, n\}$. Then
1. $\gamma = n - 1$ are necessary for exact learning.

2. $\gamma^{\text{bld}} = n - 2\sqrt{n} + \Theta(n^{1/6})$ are necessary and sufficient to succeed with probability 2/3.

3. In fact, for any $\epsilon \in (0, 1)$, $n - 2\sqrt{n} + \Theta(n^{1/6})$ are necessary and sufficient to succeed with probability $1 - \epsilon$.

Proof. Recall that the irreducible characters of $S_n$ are parametrized by partitions of $n$ which can be written either as a sequence $[\lambda_1, \ldots, \lambda_t]$ or as a Young diagram with $t$ boxes (see the Appendix for the relevant information about the character theory of $S_n$). Let $V = \mathbb{C}[1, \ldots, n]$ denote the CG-module corresponding to the defining permutation representation of $S_n$. Then $V$ decomposes as a sum of two irreducibles:

$$V = V_{[n]} \oplus V_{[n-1,1]}.$$ 

We note that $V_{[n]}$ is the trivial representation. A well-known rule says that if $V_\lambda$ is a simple representation corresponding to the Young diagram $\lambda$ then the irreps appearing in $V \otimes V_\lambda$

$$I(V \otimes V_\lambda) = \{V_\mu \mid \mu \in \lambda^\pm\}.$$ 

where $\lambda^\pm$ is the set of Young diagrams obtained from $\lambda$ by adding then removing a box from lambda. In particular, this shows by induction that

$$I(V^{\otimes t}) = \{V_\mu \mid \mu \text{ has at least } n - t \text{ columns}\}.$$ 

We see that $n - 1$ queries are required until every irreducible is contained in $V^{\otimes t}$ (in particular, the sign representation corresponding to the partition $[1^n] = [1,1,\ldots,1]$ is not including in $V^{\otimes t}$ unless $t \geq n - 1$). This proves part (1) of the theorem.

To prove part (2) we must examine more closely the set $I_t = I(V^{\otimes t})$ consisting of all partitions with at least $n - i$ columns (ie $\lambda_i \geq n - i$). We are interested in the sum

$$d_t := d_{V^{\otimes t}} = \sum_{\chi \in I(V^{\otimes t})} \chi(e)^2.$$ 

By the RSK correspondence, this sum is equal to the number of sequences of the numbers $\{1, \ldots, n\}$ whose longest increasing subsequence is at least $n - t$ (see eg [16], Theorem 3.3.2). Now a deep result of Baik, Deift and Johannson [1] identifies the distribution of the $l_n$, the length of the longest increasing subsequence of a random permutation of $n$ elements, as the Tracy-Widom distribution (which also governs the largest eigenvalue of a random Hermitian matrix) of mean $2\sqrt{n}$ and standard deviation $n^{1/6}$.

In particular, Theorem 1.1 of [1] asserts that if $F(x)$ is the cumulative distribution function for the Tracy-Widom distribution, then

$$\lim_{n \to \infty} \text{Prob}\left(\frac{l_n - 2\sqrt{n}}{n^{1/6}} \leq x\right) = F(x)$$

Let $c$ be any real number. If we use $t = n - 2\sqrt{n} + cn^{1/6}$ queries, then our success probability will be

$$\text{Prob}(l_n \geq n - t) = 1 - \text{Prob}(l_n < 2\sqrt{n} - cn^{1/6}) = 1 - \text{Prob}\left(\frac{l_n - 2\sqrt{n}}{n^{1/6}} < -c\right) \rightarrow 1 - F(-c)$$

Thus for any $\epsilon \in (0, 1)$, if we wish to succeed with probability $1 - \epsilon$, it will be necessary and sufficient to use $t = n - 2\sqrt{n} + cn^{1/6}$ queries, where $c = -F^{-1}(\epsilon)$ (for $n$ sufficiently large). \hfill $\square$

Here is the analogous result for identifying an element of the alternating group.

**Theorem 4.5.** Consider the standard action of $A_n$ acting on $\{1, \ldots, n\}$. Then the quantum base sizes are given as follows.

1. $\gamma = n - \lceil \sqrt{n} \rceil$ are necessary for exact learning.

2. $\gamma^{\text{bld}} = n - 2\sqrt{n} + \Theta(n^{1/6})$ are necessary and sufficient to succeed with probability 2/3. In fact, for any $\epsilon \in (0, 1)$, $n - 2\sqrt{n} + \Theta(n^{1/6})$ are necessary and sufficient to succeed with probability $1 - \epsilon$. 

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Proof. Recall the following facts about the representation theory of $A_n$. The conjugate of a partition $\lambda$ is the partition $\lambda^\ast$ obtained by swapping the rows and columns of $\lambda$; in other words $\lambda^\ast = (\lambda_1, \lambda_2^\ast, \ldots)$ where $\lambda_i^\ast$ is the number of boxes in the $i$th column of $\lambda$. For each partition $\lambda$ of $n$ that is not self-conjugate, i.e., $\lambda \neq \lambda^\ast$, the restriction of $V_\lambda$ to $A_n$ is an irreducible representation $W_\lambda$ of $A_n$. Also, $W_\lambda = W_{\lambda^\ast}$. For self-conjugate $\lambda$, the representation $V_\lambda$ breaks up into two distinct irreducible representations $W_\lambda^+$ and $W_\lambda^\ast$ of equal dimension.

Recall from the previous proof that after $t$ queries, we get copies of all the $V_\lambda$ such that $\lambda_1 \geq n - t$. Observe that for any partition $\lambda$, we must have either $\lambda_1 \geq \lceil \sqrt{n} \rceil$ or $\lambda_1^\ast \geq \lceil \sqrt{n} \rceil$. (If both fail, the partition fits into a square of side length $\lceil \sqrt{n} \rceil - 1$, which contains fewer than $n$ boxes.) It follows that after $t = n - \lceil \sqrt{n} \rceil$ queries, for any $\lambda$, we have picked up a copy of $V_\lambda$ or $V_{\lambda^\ast}$. Hence we have every irreducible representation of $A_n$. Therefore, $n - \lceil \sqrt{n} \rceil$ queries suffice for exact learning. Showing that that fewer queries cannot suffice is similar. Here we make the observation that there exists a partition $\lambda$ such that $\lambda_1 < \lceil \sqrt{n} \rceil + 1$ and $\lambda_1^\ast < \lceil \sqrt{n} \rceil + 1$, since $n$ boxes can be packed into a square of side length $\lceil \sqrt{n} \rceil$. It follows that $t = n - \lceil \sqrt{n} \rceil - 1$ queries do not pick up the $V_\lambda$ or $V_{\lambda^\ast}$ for such $\lambda$. Thus, we do not get every irrep of $A_n$.

We now examine the bounded error case. For a positive integer $t$, let $p_t$ be the success probability of the optimal $t$-query algorithm for identifying a permutation of $S_n$ and let $q_t$ be the corresponding probability for $A_n$.

Let $V$ denote the $t$-fold tensor power of the defining representation of $S_n$. We can decompose $V$ as a direct sum of irreps of $S_n$ and if we know which $V_\lambda$ appear, we can determine which irreps of $A_n$ appear in $V$. In particular, each time we have a non-self-conjugate $\lambda$ such that $V_\lambda$ appears in $V$, we will have $W_\lambda$ appearing in $V$. Let’s consider the contribution of this appearance to the success probability $p_t$ and $q_t$, which is the square of the dimension divided by the order of the group. Since the dimension of $V_\lambda$ equals the dimension of $W_\lambda$, while the order of $S_n$ is twice the order of $A_n$, the contribution to $q_t$ is twice the contribution to $p_t$.

Now if $\lambda$ is self-conjugate then $V_\lambda$ decomposes into two irreps of $S_n$ of equal dimensions. The sum of the squares of these two irreps is thus one-half the square of the dimension of $V_\lambda$. Once we’ve divided by the sizes of the groups, we see that the contribution to $q_t$ is equal the contribution to $p_t$.

We have thus seen that for any $\lambda$ the contribution to $q_t$ is either $2$ or $3$ times the contribution to $p_t$. It follows that

$$p_t \leq q_t \leq 2p_t$$

Thus for $q_t \geq 2/3$ we must have $p_t \geq 1/3$, which as we showed in Theorem 4.4 requires $n - 2\sqrt{n} + \Theta(n^{1/6})$ queries. On the other hand, if we are given $n - 2\sqrt{n} + \Theta(n^{1/6})$ queries, we achieve $p_t \geq 2/3$, which forces $q_t \geq 2/3$.

The two theorems above show that there is very little speedup possible when trying to identify a permutation from the symmetric group or the alternating group. For the alternating group, one can at least get by with $\sqrt{n}$ fewer queries for exact quantum learning. Here there is an analogy to Van Dam’s problem of exactly learning the value of an $n$-long bitstring using queries to its bits [13]. Exact learning requires $n$ queries. However, if we are guaranteed in advance that the parity of the string is even, then only $\lceil n/2 \rceil$ queries are required for exact learning. To see this using the techniques of the current paper, we argue as follows. Let $G$ be the subgroup $\mathbb{Z}_2^n$ consisting of all strings of even parity. If we are allowed $t$ queries, then we can access those representations $\rho_x$ of $\mathbb{Z}_2^n$ corresponding to strings $x$ of Hamming weight less than or equal to $t$. If $\bar{x}$ is the bitwise complement of $x$, then $\rho_x$ and $\rho_{\bar{x}}$ take the same values on $G$. Now, for any string $x$, one of $x$ and $\bar{x}$ will have Hamming weight less than or equal to $\lceil n/2 \rceil$. Hence every representation of $G$ can be accessed by $\lceil n/2 \rceil$ queries to the oracle, and we will succeed with probability $1$.

5 Query complexity of coset identification.

In this section we derive a formula for the optimal success probability of a $t$-query algorithm to solve coset identification. In light of our previous result on parallelizability (Corollary 5.5), this boils down to finding a formula in the single-query case. This will directly generalize the single-query results of [7]
used in Section 4.

To state the result we fix some notation. Suppose \((G, V, f)\) is an instance of coset identification. Then \(f : G \rightarrow X\) is a \(G\)-set map. Let \(H\) denote the stabilizer of some \(x_0 \in X\). Given an \(H\)-representation \(W\) let \(W^\uparrow\) denote the induced representation of \(W\) (which is a representation of \(G\)). Likewise if \(W\) is a \(C_G\)-module then we denote by \(W^\downarrow\) the \(C_H\)-module obtained by restriction to \(H\). Recall that if \(V\) is a \(C_G\)-module then \(I(V)\) denotes the set of all irreducible characters of \(G\) appearing in \(V\). We sometimes use the notation \(I_G(V), I_H(V)\) to emphasize which group we are considering. Finally, given two representations \(A\) and \(B\) we let

\[
A_B := \text{the maximal subrepresentation of } A \text{ such that } I(A_B) \subseteq I(B).
\]

Thus \(A_B\) denotes the sum of all the isotypical components of \(A\) which correspond to an irreducible isotype appearing in \(B\). We will be interested in the quantities

\[
\frac{\dim A_B}{\dim A}
\]

which can be understood as the fraction of \(A\) which is shared with \(B\).

**Theorem 5.1.** An optimal single-query algorithm to solve the instance \((G, V, f)\) of coset identification succeeds with probability

\[
P_{\text{opt}} = \max_{Y \in \text{Irr}(H)} \frac{\dim (Y^\uparrow)_V}{\dim Y^\uparrow}.
\]

In words: to find the optimal success probability, you look at an irrep \(Y\) of \(H\) which appears in \(V^\downarrow\). Then you examine the fraction of \(Y^\uparrow\) which is shared with \(V\). Finally take the maximum over all irreps \(Y\) appearing in \(V^\downarrow\).

The next two sections are devoted to the proof of Theorem 5.1. First we prove the lower bound (i.e., existence of a state and measurement achieving the desired success probability) and then we prove the upper bound (optimality of that success probability).

### 5.1 The lower bound.

First we collect some facts about induced representations necessary for the proof. A fine treatment of the subject is contained in Serre’s book [17].

Suppose \(H\) is a subgroup of a finite group \(G\) and let \(Y\) denote a representation of \(H\). Note that \(C_G\) admits a right \(H\)-action. The representation of \(G\) induced from \(Y\) is

\[
Y^\uparrow = C_G \otimes_{C_H} Y.
\]

When \(H\) and \(G\) are understood we simply write \(Y^\uparrow\). Similarly if \(W\) is a representation of \(G\) then the restriction of \(W\) to \(H\) is denoted \(W^\downarrow\) or simply \(W^\downarrow\).

Frobenius reciprocity states that induction and restriction are adjoint functors. In terms of characters this means that if \(\psi\) is a character of \(H\) and \(\chi\) a character of \(G\) then

\[
(\psi^\uparrow, \chi) = (\psi, \chi^\downarrow).
\]

From the definition of induced representations, we can write

\[
Y^\uparrow = \bigoplus_t t \otimes Y
\]
where \( t \) ranges over a set of coset representatives for \( H \). Conversely, if a representation \( W \) of \( G \) contains an \( H \)-invariant subspace \( W_0 \) such that
\[
W = \bigoplus_t tW_0
\]
where \( t \) again ranges over a set of coset representatives for \( H \), then \( W \) is isomorphic to \( W_0 \) as \( G \)-representations.

In our situation all representations are unitary. In particular if \( Y \) is a unitary representation of \( H \) then \( Y^\dagger \) is equipped with the inner product determined by requiring the subspaces \( t \otimes Y \) to be pairwise orthogonal, and translating the inner product of \( Y \) to each subspace \( t \otimes Y \). With this inner product \( Y^\dagger \) is a unitary representation of \( G \). We will often denote the orthogonal projection onto \( e \otimes Y \) by \( E \). Then the orthogonal projection onto \( t \otimes Y \) is \( tEt^{-1} \), and we have \( \sum_t tEt^{-1} = \text{id}_{Y^\dagger} \).

If \( R \) is a ring, \( V \) an \( R \)-module and \( W \leq V \) a linear subspace, we let \( R \cdot W \) denote the submodule of \( V \) generated by \( W \) (ie the smallest submodule containing the subspace \( W \)). Similarly for \( r \in R \) we let \( r \cdot W \) denote the subspace \( \{rw: w \in W\} \).

**Proposition 5.2.** Suppose \( Y \) is an irreducible unitary representation of \( H \) (a subgroup of \( G \)). Also suppose \( V \) is a \( G \)-subrepresentation of \( Y^\dagger \). Let \( E \) denote orthogonal projection onto \( e \otimes Y \subset Y^\dagger \). Then there exists a unit vector \( \psi \in V \) such that
\[
\langle \psi | E \psi \rangle = \frac{\dim V}{\dim Y^\dagger}. 
\]

**Proof.** Let \( \Pi_Y \) denote orthogonal projection onto \( V \). Since \( Y \) is irreducible, \( \Pi_Y E \Pi_Y \) is a scalar multiple of a projection onto an \( H \)-invariant subspace of \( V \) which is either 0 or isomorphic to \( Y \). Let this subspace be \( Y' \), so we have
\[
\Pi_Y E \Pi_Y = \lambda \Pi_{Y'},
\]
for some non-zero scalar \( \lambda \in \mathbb{C} \). By multiplying on the left and right by \( \Pi_{Y'} \) we see that
\[
\Pi_{Y'} E \Pi_{Y'} = \lambda \Pi_{Y'} = \Pi_Y E \Pi_Y.
\]
Next, we claim that \( Y' \) is not zero (so it is in fact isomorphic to \( Y \) as an \( H \)-module). Indeed, we have
\[
\mathbb{C} G \cdot Y' = \sum_t t \cdot Y' = \sum_t \text{Im}(\Pi_{Y'} tEt^{-1} \Pi_{Y'}) \supset \text{Im}(\Pi_Y (\sum_t tEt^{-1}) \Pi_Y) = \text{Im}(\Pi_Y) = V
\]
where the sum is over a set of coset representatives of \( H \). This shows that \( \mathbb{C} G \cdot Y' \supset V \), and in fact \( \mathbb{C} G \cdot Y' = V \) since \( Y' \subset V \) and \( V \) is \( G \)-invariant. This allows us to find the scalar \( \lambda \) via
\[
\dim V = \text{Tr}(\sum_t \Pi_Y E_t \Pi_{Y'}) = \lambda \sum_t \text{Tr}(\Pi_{t \cdot Y'}) = \lambda |G| \dim Y = \lambda \dim Y^\dagger
\]
which yields \( \lambda = \frac{\dim V}{\dim Y^\dagger} \).

Finally, let \( \psi \) be any unit vector in \( Y' \). Then \( \frac{1}{|H|} \sum_{h \in H} h \psi h^* h^{-1} = \frac{1}{\dim Y} \Pi_{Y'} \psi \Pi_{Y'} \) since \( Y' \) is irreducible. Using this we compute
\[
\langle \psi | E \psi \rangle = \text{Tr}(\psi^* E) = \text{Tr}(\frac{1}{|H|} \sum_{h \in H} h \psi h^* h^{-1} E) = \frac{1}{\dim Y} \text{Tr}(\Pi_{Y'} E \Pi_{Y'}) = \frac{1}{\dim Y} \text{Tr}(\Pi_{Y'}) \lambda = \frac{\dim V}{\dim Y^\dagger}
\]
as needed. \( \square \)

**Proof of Theorem 5.1, lower bound.** Let \( Y \) be an irreducible constituent of \( V_\perp \) which maximizes the quantity
\[
\frac{\dim(Y^\dagger)_V}{\dim Y^\dagger}.
\]
Let \( V' \) denote the \( G \)-subrepresentation \((Y^\dagger)_V\) of \( Y^\dagger \) and let \( E \) denote the orthogonal projection onto the subspace \( e \otimes Y \subset Y^\dagger \). Then by Proposition 5.2 there exists a unit vector \( |\psi \rangle \in V' \) such that
\[
\langle \psi | E \psi \rangle = \frac{\dim V'}{\dim Y^\dagger} = \frac{\dim Y^\dagger}{\dim Y^\dagger}.
\]

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Our algorithm will use the input state $|\psi\rangle$ and the projective measurement $\{tEt^{-1}\}$, where $t$ ranges over a set of coset representatives for $H$ (so measuring outcome $t$ uniquely determines a coset of $H$). The measurement is used to distinguish the density operators $\rho_t = tpt^{-1}$ where $\rho = \frac{1}{|M|}\sum_{h\in H} h|\psi\rangle\langle h^{-1}|$. The success probability is

$$P_{\text{succ}} = \frac{1}{|G:H|} \sum_t \text{Tr}(\rho_t tEt^{-1}) = \text{Tr}(\rho E) = (|\psi\rangle E |\psi\rangle) = \frac{\dim Y_V^+}{\dim Y_V^+}.$$  

\[\Box\]

5.2 The upper bound.

In this section we prove the upper bound of Theorem 5.1 using a minimum-error quantum state discrimination approach \[10\]. We prove the theorem in two steps. The first is to show that an optimal measurement to distinguish the relevant density matrices forms a $G$-set under conjugation. The second step is to bound the optimal probability of successful discrimination given that the measurement is projective and supports an action of $G$.

The final stage of any quantum algorithm \[9\] consists in performing a measurement to distinguish some number of density matrices. With a given unitary representation $V$ of $G$ and a fixed $G$-set $X$ understood we say a set of operators $\{A_x\}_{x \in X}$ (on $V$) is orbital if $gA_xg^{-1} = A_{gx}$ for all $x$. The density matrices for a single query algorithm for coset identification is an orbital set.

Lemma 5.3. Suppose $\{\rho_x\}_{x \in X}$ is an orbital set of density matrices. Then there exists an optimal measurement to distinguish the states $\{\rho_x\}$ which is orbital.

Proof Eldar, Megretski, Verghese give the proof when $H = e$ \([9]$, Section 4.3) and it works in this setting as well. \[\Box\]

The next lemma is an equivariant version of the argument given by Chuang and Nielsen to show that arbitrary measurement operators can be simulated using projective measurements and ancilla spaces (see \[15\], Section 2.2.8).

Lemma 5.4. Suppose $\{E_x\}$ is an orbital POVM on the space $V$. Then there exists a unitary representation $W$ and a CG-module embedding

$$\iota : V \to W$$

together with a projective orbital measurement $\{E_x\}$ on $W$ such that for any state $|\psi\rangle$, the measurement statistics by measuring $|\psi\rangle$ with $\{E_x\}$ are identical to those given by the state $|\psi\rangle$ and measurement $\{E_x\}$.

Proof Let $W$ be the space $V \otimes \mathbb{C}G$ and fix a basepoint $x_0$ of the $G$-set $X$. Given a set of measurement operators $\mathcal{M} = \{M_x\}$ corresponding to the POVM $\{E_x\}$ let $C_{\mathcal{M}}$ be the controlled-$M$ operator acting on $W$ via

$$C_{\mathcal{M}}|\psi, g\rangle = \sqrt{|X|}M_{gx_0}|\psi\rangle \otimes |g\rangle.$$  

Note that $C_{\mathcal{M}}$ is a $\mathbb{C}G$-module endomorphism of $V \otimes \mathbb{C}G$, since

$$C_{\mathcal{M}}(h \cdot |\psi, g\rangle) = C_{\mathcal{M}}(hv, hg) = \sqrt{|X|}M_{hg}h|\psi\rangle \otimes |hg\rangle = \sqrt{|X|}hgM_{gx_0}^{-1}|\psi\rangle \otimes |hg\rangle = h \cdot C_{\mathcal{M}}|\psi, g\rangle.$$  

For the third equality we used the fact that $M$ is orbital, ie $M_{gx_0} = gM_{x_0}g^{-1}$. Now $C_{\mathcal{M}}$ is not necessarily invertible, but its kernel has zero intersection with the subspace $V \otimes |\eta\rangle$ where $|\eta\rangle \in \mathbb{C}G$ is the equal superposition state Even more, $C_{\mathcal{M}}$ preserves inner products on this subspace (which the factor of $\sqrt{|X|}$ accounts for). We take $\iota$ to be the inclusion of $V$ as $V \otimes |\eta\rangle$:

$$\iota|\psi\rangle = |\psi\rangle \otimes |\eta\rangle.$$  

\[3\]That is, a conventional quantum algorithm which performs only a single measurement at the end of the computation.
As noted, \( C_M : V \otimes |\eta) \to C_M(V \otimes |\eta) \) is an inner-product preserving \( \mathbb{C}G \)-module map. Hence there exists a unitary \( \mathbb{C}G \)-module endomorphism \( U \) which restricts to \( C_M \) on \( V \otimes |\eta) \). Finally we define the projective measurement \( \{ E_x \}_x \) by

\[
E_x = U^{-1} \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) U.
\]

Here \( I \) denotes the identity on \( V \). We check that this is an orbital measurement:

\[
hE_x h^{-1} = U^{-1} h \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) U^{-1} = U^{-1} \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) U = E_{hx}.
\]

Now suppose \( \ell|\psi) = |\psi) \otimes |\eta) \) is measured with the projective measurement \( \{ E_x \}_x \). Then the probability of reading outcome \( x \) is

\[
\langle \psi, \eta| E_x |\psi, \eta \rangle = \langle \psi, \eta| U^{-1} \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) U|\psi, \eta \rangle
\]

\[
= \langle C_M(|\psi, \eta))| \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) C_M|\psi, \eta \rangle
\]

\[
= \frac{|X|}{|G|} \left( \sum_{h \in G} M_{h, x_0}|\psi) \otimes |h) \right| \left( \sum_{g, g \cdot x_0 = x} I \otimes |g\rangle \langle g| \right) \left( \sum_{h' \in G} M_{h', x_0}|\psi) \otimes |h') \right)
\]

\[
= \frac{|X|}{|G|} \sum_{g, g \cdot x_0 = x} \langle M_{g, x_0}\psi| M_{g, x_0}\psi \rangle = \frac{|X|}{|G|} \sum_{g, g \cdot x_0 = x} \langle M_{x}\psi| M_{x}\psi \rangle
\]

\[
= \langle \psi| E_x |\psi \rangle.
\]

The first three equalities are definitions, the fourth expands the multiplication, the fifth is notational and the last follows since the number of \( g \) for which \( g \cdot x_0 = x \) is equal to \( |G|/|X| \) for all \( x \in X \) (since \( X \) is a transitive \( G \)-set). This proves the lemma. \( \square \)

**Lemma 5.5.** Suppose \( \{ E_x \}_{x \in X} \) is a projective orbital measurement on a \( \mathbb{C}G \)-module \( W \). Let \( W_x \) denote the image of \( E_x \). Let \( H \) denote the stabilizer of a point \( x_0 \in X \). Then \( W_{x_0} \) is an \( H \)-representation and \( W \cong W_{x_0}^\dagger \).

**Proof.** If \( \{ E_x \} \) is an orbital measurement then \( G \) acts on this set and the stabilizer of \( E_{x_0} \) is \( H \). Therefore \( hE_{x_0} h^{-1} = E_{x_0} \) for all \( h \in H \), ie \( E_{x_0} \) is a \( \mathbb{C}H \)-module homomorphism. Hence the image of \( E_x \) is invariant under \( H \).

Next, note that \( W_x \) is orthogonal to \( W_y \) for all \( x \neq y \) (this follows from the completeness relation). Furthermore, since \( E_{g \cdot x_0} = g E_{x_0} g^{-1} \), we have \( W_{g \cdot x_0} = g W \). Hence \( V = \bigoplus_{g \in G} g W_{x_0} \). By the characterization of induced representations discussed in Section 5, this shows \( V \cong W_{x_0}^\dagger \). \( \square \)

The lemmas above show that as long as we are willing to embed our original representation \( V \) into a larger representation \( W \), we may assume that \( W \) is induced from some representation \( Y \) of \( H \) and that the measurement operators are projections corresponding to the direct sum decomposition of \( W \) as an induced representation. In other words, the measurement operator corresponding to outcome \( x \in X \) is projection onto \( t \otimes Y \) where \( t \) is any element such that \( t \cdot x_0 = x \). The next lemma is the final key to unlocking the upper bound.

**Proposition 5.6.** Suppose \( Y \) is an irreducible unitary representation of \( H \) (a subgroup of \( G \)). Next suppose \( V \) is a \( G \)-subrepresentation of \( Y^\dagger \). Let \( E \) denote orthogonal projection onto the subspace \( e \otimes Y \subset Y^\dagger \). Then for any unit vector \( \psi \in V \) we have

\[
\langle \psi| E \psi \rangle \leq \frac{\dim V}{\dim Y^\dagger}.
\]
Then \( \rho \) is an \( H \)-equivariant map \( Y' \to Y' \), so it is a scalar multiple of orthogonal projection onto \( Y' \). Taking traces we find

\[
\rho = \frac{1}{\dim Y} \Pi_{Y'}
\]

where we used that \( \dim Y' = \dim Y \). In particular, \( \rho \leq \frac{1}{\dim Y} \Pi_{Y'} \). Therefore

\[
\langle \psi|E\psi \rangle = \text{Tr}(\psi\psi^* E) = \text{Tr}(\rho E)
\]

\[
= \frac{1}{|G:H|} \sum_t \text{Tr}(t\rho t^{-1} tEt^{-1}).
\]

Here \( t \) ranges over a set of coset representatives for \( H \). Now since \( \rho \leq \frac{1}{\dim Y} \Pi_{Y'} \) we also have \( t\rho t^{-1} \leq \frac{1}{\dim Y} \Pi_{Y'} \) since \( \Pi_{Y'} \) is invariant under conjugation by \( t \). Hence the sum above is at most

\[
\frac{1}{|G:H| \dim Y} \sum_t \text{Tr}(\Pi_{Y'} tEt^{-1}).
\]

However, \( \sum_t tEt^{-1} \) is equal to the sum of the orthogonal projections onto the subspaces \( t \otimes Y \) which is the identity of \( Y^\uparrow \). Hence we obtain

\[
\langle \psi|E\psi \rangle \leq \frac{1}{|G:H| \dim Y} \dim V \text{Tr}(\Pi_{Y'}) = \frac{\dim V}{\dim W}.
\]

This proves the lemma when \( \psi \) belongs to an irreducible \( H \)-invariant subspace of \( V \). Next suppose that \( \psi = \lambda_1 \psi_1 + \cdots + \lambda_r \psi_r \) is a convex combination of orthogonal unit vectors such that each \( \psi_i \) belongs to an irreducible \( H \)-invariant subspace of \( V \). Then

\[
\langle \psi|E\psi \rangle = \sum_{i,j=1}^r \lambda_i \lambda_j \langle \psi_i|E\psi_j \rangle.
\]

Now as before we can write

\[
\langle \psi_i|E\psi_j \rangle = \frac{1}{|H|} \sum_{h \in H} \text{Tr}(h\psi_j \psi_i^* h^{-1} E).
\]

However the operator \( \sum_{h \in H} h\psi_j \psi_i^* h^{-1} \) is an \( H \)-module homomorphism between the irreducible \( H \)-invariant subspaces containing \( \psi_i \) and \( \psi_j \); thus it is a scalar times some fixed isomorphism. Taking traces we find that this scalar is 0 if \( i \neq j \). On the other hand we know that \( \langle \psi_i|E\psi_i \rangle \leq \frac{\dim V}{\dim W} \). This shows that

\[
\langle \psi|E\psi \rangle \leq \frac{\dim V}{\dim W} \sum_i \lambda_i^2 \langle \psi_i|E\psi_i \rangle \leq \frac{\dim V}{\dim W} \sum_i \lambda_i^2 \leq \frac{\dim V}{\dim W}.
\]

We are ready for

**Proof of Theorem 5.1** upper bound. Let \( (G, V, f) \) specify an instance of coset identification and let \( H \) denote the stabilizer of \( x_0 = f(e) \) (recall that the codomain of \( f \) is a \( G \)-set). Suppose an optimal single-query algorithm is given by an input state \( |\psi\rangle \in V \) (again we may assume there is no workspace by absorbing it into \( V \)) and POVM \( \{E_x\} \). By Lemmas 5.4 and 5.5, there is a representation \( Y \) of \( H \) and \( CG \)-submodule of \( Y^\uparrow \) isomorphic to \( V \) (which we identify with \( V \)) such that the success probability of our algorithm is equal to the success probability of an algorithm using input state \( |\psi\rangle \in V \subset Y^\uparrow \) and the projective measurement \( \{tEt^{-1}\}_t \), where \( E \) denotes orthogonal projection onto \( e \otimes Y \) and \( t \) ranges over a set of coset representatives for \( H \).

Now decompose \( Y \) into irreducible \( H \)-invariant orthogonal subspaces:

\[
Y = Y_1 \oplus \cdots \oplus Y_r.
\]
Then $Y^\dagger \cong \bigoplus_i Y_i^\dagger$ as CG-modules. Let $\Pi_i$ denote orthogonal projection onto $Y_i^\dagger$. Then $|\psi\rangle$ can be decomposed as a combination of orthogonal unit vectors

$$|\psi\rangle = \lambda_1|\psi_1\rangle + \cdots + \lambda_r|\psi_r\rangle$$

such that each $|\psi_i\rangle$ belongs to $Y_i^\dagger$. Even more is true: since $\lambda_i|\psi_i\rangle = \Pi_i|\psi\rangle$ and $\Pi_i$ is a CG-module map, we know $|\psi_i\rangle \in (Y_j^\dagger)_V$.

Note also that $E$ decomposes as $E = E_1 + \cdots + E_r$ where $E_i$ is orthogonal projection onto $e \otimes Y_i$.

We are ready to bound the success probability of the algorithm. Recall that we are using the Bratteli diagram/induction rules are conveniently described in a diagram.

A straightforward consequence is the following:

$$P_{\text{suc}} = \frac{1}{|G:H|} \sum_i \text{Tr}((t \rho)^{-1} E t) = \langle \psi | E \psi \rangle.$$ 

Now using the decomposition of $|\psi\rangle$ we have

$$\langle \psi | E \psi \rangle = \sum_{i=1}^r |\lambda_i|^2 \langle \psi_i | E_i \psi_i \rangle.$$ 

Now by Proposition 5.6 we have, for all $i$

$$\langle \psi_i | E_i \psi_i \rangle \leq \frac{\dim(Y_i^\dagger)_V}{\dim Y_i^\dagger}.$$ 

Therefore

$$P_{\text{suc}} \leq \sum_i |\lambda_i|^2 \frac{\dim(Y_i^\dagger)_V}{\dim Y_i^\dagger} \leq \max_{\chi \in \text{Irr}(H)} \frac{\dim Y_i^\dagger}{\dim Y_i^\dagger}. \quad \square$$

5.3 Query complexity.

We now know the success probability of an optimal single-query algorithm solving coset identification. As in Section 4, we combine this with the fact that an optimal $t$-query algorithm with access to the representation $V$ is the same as an optimal 1-query algorithm to $V^\otimes t$ to determine the optimal success probability for $t$-query algorithms:

**Corollary 5.7.** Let $(G, V, f)$ describe a case of coset identification. Then an optimal $t$-query algorithm succeeds with probability

$$P_{\text{opt}} = \max_{Y \in \text{Irr}(H)} \frac{\dim Y^\dagger_{V^\otimes t}}{\dim Y^\dagger}.$$ 

A straightforward consequence is the following:

**Theorem 5.8.** Let $(G, V, f)$ describe a case of coset identification. Then the zero-error quantum query complexity of the problem is the minimum $t$ for which there exists some $Y \in \text{Irr}(H)$ such that every irrep of $G$ appearing in $Y^\dagger$ also appears in $V$.

The bounded error quantum query complexity is the minimum $t$ for which

$$\max_{Y \in \text{Irr}(H)} \frac{\dim(Y^\dagger_{V^\otimes t})}{\dim Y^\dagger} \geq 2/3.$$ 

6 Examples of coset identification.

6.1 Identifying the coset of the Klein 4 group.

Here we present an easy demonstration of the machinery of the previous section. Consider the symmetric group on 4 letters $G = S_4$ with normal subgroup the Klein 4-group $H = \{e, (12)(34), (13)(24), (14)(23)\}$. Given access to the defining permutation representation $V$ of $S_4$ we would like to identify which coset of $H$ our permutation belongs to. Classically this requires 2 queries. To determine the quantum complexity we need to know the characters of $V$ and $S_4$. Of course $V$ is isomorphic to $Z_2 \times Z_2$ (say, using the generators $(12)(34)$ and $(13)(24)$) and has 4 characters labelled $\psi_{\alpha,\beta}$ with $\alpha, \beta \in \{0, 1\}$. The group $S_4$ has 5 characters parametrized by partitions of 4, denoted $\chi_{[4]}, \chi_{[3,1]}, \chi_{[2,2]}, \chi_{[2,1^2]}$ and $\chi_{[1^4]}$. The restriction/induction rules are conveniently described in a Bratteli diagram:
This indicates, for instance, that $\chi_{[2,1^2]}^\dagger = \psi_{0,0} + \psi_{0,1} + \psi_{1,0} + \psi_{1,1}$ and $\psi_{0,0}^\dagger = \chi_{[4]} + 2\chi_{[2^2]} + \chi_{[1^4]}$. Finally, we are given access to the defining permutation representation of $S_4$ which decomposes as $V = \chi_{[4]} + \chi_{[3,1]}$.

To find the optimal success probability of a single-query algorithm to determine which coset of $H$ a permutation belongs to, we examine the irreps of $H$ appearing in $V$. From the diagram we see that every irrep of $H$ appears in $V$, so we look at each one. First consider the trivial representation $\psi_{0,0}$. The only irrep of $S_4$ that appears in both $V$ and $\psi_{0,0}^\dagger$ is $\chi_{[4]}$, which contributes a one dimensional subspace to the 6 dimensional $\psi_{0,0}^\dagger$. Therefore using the irrep $\psi_{0,0}$ gives a success probability of $1/6$. Now consider $\psi_{0,1}$. In this case only $\chi_{[3,1]}$ appears in both $V$ and $\psi_{0,1}^\dagger$, and it contributes 3 dimensions to the 6 dimensional $\psi_{0,1}^\dagger$. Therefore the success probability using this irrep is $3/6 = 1/2$. The other characters $\psi_{1,0}$ and $\psi_{1,1}$ give the same ratio so the optimal success probability of a single-query quantum algorithm is $1/2$ (note a single-query classical algorithm can do no better than probability $1/6$).

That the optimal 2-query success probability is 1 can be verified using the fact that $V^\otimes 2$ contains a copy of every irrep of $S_4$ except the sign representation, and so using any of the irreps $\psi_{0,1}, \psi_{1,0}, \psi_{1,1}$ we can achieve probability $1$.

### 6.2 An action of the Heisenberg Group

We now consider a natural action of the Heisenberg group over a finite field for which the oracle identification problem achieves a significant quantum speedup over the best classical algorithm. For this action, we also show that a single query suffices to solve the coset identification problem, where the chosen subgroup $H$ is the center of the group.

Specifically, let $p$ be prime and let $n$ be a positive integer. Let $G = G(p, n)$ denote the Heisenberg group of all $(n + 2)$-by-$(n + 2)$ matrices with 1’s on the main diagonal and whose only other nonzero entries are in the first row and last column. Such matrices are in correspondence with triples $(x, y, z)$, with $x, y \in \mathbb{Z}_p^n$ and $z \in \mathbb{Z}_p$, where $(1, x, z)$ is the first row of the matrix and $(z, y, 1)$ is the last column of the matrix. Then $G(p, n)$ is a $p$-group of order $p^{2n+1}$.

We consider the usual action of $G(p, n)$ on the set $X = \mathbb{Z}_p^{n+2}$, considered as column vectors, by matrix-vector multiplication. The corresponding classical oracle identification problem turns out to have complexity $b(G) = n + 1$. To see this note that $y$ and $z$ can be determined by the single query $(0, \ldots, 0, 1)$. Further queries give affine conditions on $x$, and it requires at least $n$ of these to determine the value of $x$.

In contrast to the $n + 1$ queries needed to solve this question classically, we now show that a single quantum query suffices to solve the problem with high probability, and that two queries suffice to solve the problem with certainty.

**Theorem 6.1.** Let $G(p, n)$ denote the Heisenberg group defined above acting by multiplication on the set of column vectors $X = \mathbb{Z}_p^{n+2}$. Then an optimal single query quantum algorithm solves the oracle identification problem with probability

$$P_{\text{opt}} = 1 - \frac{1}{p} + \frac{2}{p^{n+1}} - \frac{1}{p^{2n+1}}.$$  

Furthermore, two queries suffices to solve the oracle identification problem with probability 1.
We will prove this theorem shortly. Before doing so, let us consider a related coset identification problem. Let \( H \lt G(p, n) \) be the subgroup in which \( x = y = 0 \). Then \( H \) is a subgroup of order \( p \), and in fact \( H \) is the center of \( G(p, n) \). The coset identification problem with respect to this subgroup \( H \) asks us to determine the values of \( x \) and \( y \). In the classical case, \( n + 1 \) queries are again required. However this time, a single quantum query solves the coset identification problem with certainty.

**Theorem 6.2.** Let \( G = G(p, n) \) denote the Heisenberg group acting by multiplication on the set of column vectors \( X = \mathbb{Z}_p^{n+2} \). Let \( H \) be center of \( G \), the set of all matrices in \( G \) for which \( x = y = 0 \). Then this coset identification problem can be solved with a single quantum query with probability 1.

In order to prove these theorems, we must understand the representation theory of \( G = G(p, n) \), which we now describe briefly. The group \( G \) has \( p^{2n} \) one-dimensional irreducible representations and \( p - 1 \) irreducible representations of dimension \( p^n \). The one-dimensional representations will be denoted \( \chi_{\alpha,\beta} \), indexed by tuples \( \alpha, \beta \in \mathbb{Z}_p^n \). We identify these representations with their characters which are given by the formula

\[
\chi_{\alpha,\beta}(x, y, z) = \omega^{\alpha x + \beta y},
\]

with \( \omega \) denoting a primitive \( p \)-th root of unity.

The \( p^n \) dimensional representations denoted \( \rho_c \), with \( c \in \mathbb{Z}_p, c \neq 0 \) are described as follows. Let \( U \) be the vector space of all complex-valued functions on \( (\mathbb{Z}_p)^n \). Fix \( c \in \mathbb{Z}_p \) with \( c \neq 0 \). Then there is an irreducible representation \( \rho_c \) of \( G \) on \( U \) given by \( [\rho_c(x, y, z)f](w) = \omega^{c(y w + z)} f(w + x) \).

The character of this representation is given by \( \theta_c(x, y, z) = \begin{cases} p^n \omega^{cz} & \text{if } x = y = 0, \\ 0 & \text{otherwise.} \end{cases} \)

In order to understand the query complexity of the oracle identification problem we must decompose the representation \( V = \mathbb{C}^X \) into irreducible representations. Since this representation comes from a permutation representation of \( G \), each character value \( \chi_V(x, y, z) \) is simply the number of fixed points of the matrix \( A = (x, y, z) \). This number of fixed points is determined by the rank of the matrix \( A' = A - I \).

If \( (x, y, z) = 0 \), then \( A' \) has rank 0, and if \( x \) and \( y \) are both nonzero, then \( A' \) has rank 2. In all other cases \( A' \) has rank 1. We thus obtain the following character values of our given permutation representation \( V \):

\[
\chi_V(x, y, z) = \begin{cases} p^{n+2} & \text{if } (x, y, z) = (0, 0, 0) \\ p^n & \text{if } x \neq 0 \text{ and } y \neq 0 \\ p^{n+1} & \text{otherwise.} \end{cases}
\]

To find the number of copies of the trivial representation \( \chi_{0,0} \) appearing in \( \chi_V \), we simply average these values and obtain \( \langle \chi_V, \chi_{0,0} \rangle = p^n + 2(p - 1) \).

Now let \( \phi \) be any nontrivial irreducible character of \( G \). We compute the number \( \langle \chi_V, \phi \rangle \) of copies of \( \phi \) appearing in \( \chi_V \) as follows

\[
\langle \chi_V, \phi \rangle = \frac{1}{|G|} \sum_{(x, y, z) \in G} \chi_V(x, y, z) \phi(x, y, z) = \frac{1}{|G|} \sum_{(x, y, z) \in G} (\chi_V(x, y, z) - p^n) \phi(x, y, z) \\
= \frac{1}{|G|} \sum_{(x, y, z)'} (p^{n+1} - p^n) \phi(x, y, z) + (p^{n+2} - p^n) \phi(0, 0, 0) \\
= \frac{p - 1}{p^{n+1}} \left[ \sum_{(x, y, z)'} \phi(x, y, z) + (p + 1) \phi(0, 0, 0) \right]
\]

where \( (x, y, z)' \) indicates a sum over those \( (x, y, z) \) such that \( x = 0 \) or \( y = 0 \), but \( (x, y, z) \neq (0, 0, 0) \).

Taking \( \phi = \theta_c \) in this formula, we conclude that \( V \) contains \( p - 1 \) copies of \( \rho_c \). Taking \( \phi = \chi_{\alpha,\beta} \), we get

\[
\langle \chi_V, \chi_{\alpha,\beta} \rangle = \begin{cases} p - 1 & \text{if } \alpha = 0 \text{ or } \beta = 0, \text{ but not both} \\ 0 & \text{if } \alpha \neq 0 \text{ and } \beta \neq 0. \end{cases}
\]
We conclude that our $V$ contains copies of all irreducible representations of $G$ except the $\chi_{\alpha,\beta}$ for which both $\alpha$ and $\beta$ are nonzero. The optimal single-query quantum success probability is thus given by

$$P_{\text{opt}} = \frac{1}{|G|} \left( |G| - \sum_{(\alpha,\beta) \neq (0,0)} 1 \right) = 1 - \frac{1}{p} + \frac{2}{pn+1} - \frac{1}{p2n+1},$$

as claimed.

If two queries are allowed, we have access to the representation $V \otimes V$. Noting that $\chi_{\alpha,\beta} = \chi_{\alpha,0} \otimes \chi_{0,\beta}$, it follows that $V \otimes V$ contains every irreducible representation of $G$. Hence, there is a probability 1 algorithm with two quantum queries.

Finally, we turn our attention to the coset identification problem for the subgroup $H = \{(0,0,z) | z \in \mathbb{Z}_n \}$. To see that there is a probability one algorithm, note that any of the nontrivial characters of $H$ induces up to $p^n$ times one of the $\rho_c$. Since $\rho_c$ is contained in $V$, it follows that the coset identification problem can be solved with one query.

\[ \square \]

### 6.3 Guessing the sign of a permutation.

Suppose there is an unknown permutation $g \in G = S_n$ for some $n \geq 2$. We wish to learn the sign of $g$ using queries to the standard action of $S_n$ on $\{1,...,n\}$. This is an instance of the hidden coset problem where $H = A_n$. Classically, $n-1$ queries are necessary to determine the sign of $g$. In fact, any fewer queries and we do not learn anything about the sign. Quantumly, we have

**Theorem 6.3.** Let $n \geq 2$ and consider the standard action of $S_n$ on $\{1,...,n\}$. Consider the hidden coset problem for the subgroup $H = A_n$. That is we wish to determine the sign of a hidden permutation. For exact learning, $t = \left\lceil \frac{n}{2} \right\rceil$ quantum queries suffice. With any smaller number of quantum queries, one cannot do any better than random guessing ($p = 1/2$).

**Proof.** For facts and notation about representations of $S_n$ and $A_n$, we refer the reader to the proofs of Theorems 4.4 and 4.5.

Let $V$ be the defining representation of $S_n$, and suppose we use $t$ queries so that we have access to $V' = V^\otimes t$. Suppose $\lambda$ is a non-self-conjugate partition such that $V'$ contains $V_{\lambda}$. Letting $Y = W_{\lambda}$, we see that $Y^\dagger$ consists of one copy of $V_{\lambda}$ and one copy of $V_{\lambda^\ast}$. Hence the quotient of dimensions

$$\frac{\dim (Y^\dagger)_{V'}}{\dim Y^\dagger}$$

equals 1 if $V'$ contains both $V_{\lambda}$ and $V_{\lambda^\ast}$, and $\frac{1}{2}$ if $V'$ contains $V_{\lambda}$ but not $V_{\lambda^\ast}$. Now consider a self-conjugate partition $\lambda$ contained in $V'$. In this case, if we take $Y = W_{\lambda}^\dagger$, then $Y^\dagger$ is $V_{\lambda}$. Hence in this case the quotient of dimensions is 1.

We thus wish to find the smallest $t$ such that $V^\otimes t$ contains both $V_{\lambda}$ and $V_{\lambda^\ast}$ for some partition $\lambda$ (including the possibility that $\lambda$ is self-conjugate). For such $t$, we will have a $t$-query probability 1 algorithm and for fewer queries we cannot do better than probability 1/2, which is random guessing.

For even $n$, the value $t = n/2$ produces the partition $\lambda = (n/2 + 1, 1, \ldots, 1)$ (with $n/2 - 1$ 1’s) and its conjugate $\lambda^\ast = (n/2, 1, \ldots, 1)$ (with $n/2$ 1’s). For odd $n$, the value $t = \frac{n-1}{2}$ produces the self-conjugate partition $\left( \frac{n+1}{2}, 1, \ldots, 1 \right)$ (with $\frac{n-1}{2}$ 1’s). In either case $t = \left\lceil \frac{n}{2} \right\rceil$ gives a probability 1 success, and fewer queries give success probability 1/2.

\[ \square \]

**Acknowledgements.** We would like to thank Hanspeter Kraft, David Meyer and Marino Romero for helpful communications.

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