Twistor String Structure of the Kerr-Schild Geometry and Consistency of the Dirac-Kerr System. *

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Abstract

Kerr-Schild (KS) geometry of the rotating black-holes and spinning particles is based on the associated with Kerr theorem twistor structure which is defined by an analytic curve \( F(Z) = 0 \) in the projective twistor space \( Z \in \mathbb{CP}^3 \).

On the other hand, there is a complex Newman representation which describes the source of Kerr-Newman solution as a "particle" propagating along a complex world-line \( X(\tau) \in CM^4 \), and this world-line determines the parameters of the Kerr generating function \( F(Z) \). The complex world-line is really a world-sheet, \( \tau = (t + i\sigma) \), and the Kerr source may be considered as a complex Euclidean string extended in the imaginary time direction \( \sigma \). The Kerr twistor structure turns out to be adjoined to the Kerr complex string source, forming a natural twistor-string construction similar to the Nair-Witten twistor-string.

We show that twistor polarization of the Kerr-Newman solution may be matched with the massless solutions of the Dirac equation, providing consistency of the Dirac-Kerr model of spinning particle (electron). It allows us to extend the Nair-Witten concept on the scattering of the gauge amplitudes in twistor space to include massive KS particles.

1 Introduction

Kerr-Schild (KS) geometry is a background for the rotating black-hole (BH) solutions. On the other hand, the Kerr-Newman solution has gyromagnetic ratio \( q = 2 \), as that of the Dirac electron, and the KS geometry acquires central role as a model of spinning particle in gravity, in particular, as a Dirac-Kerr model of electron [1]. Consistency of Quantum theory with Gravity is one of the principal problems of modern physics, and in this paper we discuss a way to progress in this direction, showing that consistency of the Dirac-Kerr system for a massive particle (electron) may be provided by a twistor-string source.

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formed by the massless solutions of the Dirac equation aligned with the twistor
structure of the KS background. In many respects this twistor-string is similar
to the Nair-Witten twistor-string model [2], which allows us to join to the Nair-
Witten concept on the scattering of the gauge amplitudes in twistor space [3, 4],
extending this concept to include massive particles.

The KS solutions in 4D are algebraically special solutions of type D which
has two (doubled) Principal Null Congruences (PNC) corresponding to geodesic
lines of outgoing or ingoing photons. Tangent vectors to these congruences,
\( k_\mu(x)^\pm \), are null and determine two different coordinate system of the black-
hole solutions related to the ‘in’ or ‘out’ congruence. [5]. KS metrics have very
simple KS form

\[
g_{\mu\nu} = \eta_{\mu\nu} - 2H k_\mu k_\nu, \quad (1)
\]

where \( \eta_{\mu\nu} \) is metric of auxiliary Minkowski space-time, and the in (or out) null
vector field \( k^\mu(x), \ x \in M^4 \) determines symmetry of space-time, in particular,
direction of gravitational ‘dragging’. For rotating BH geometry the congruence
is twisting which causes the difficulties for its derivation and analysis. The
obtained exact solutions of the Einstein-Maxwell system of equations [6] indicate
that electromagnetic (em) field is not free with respect to the choice of the in
or out representation. The direction of congruence related with em field must
be the same as it is for metric, i.e. vector potential of em field \( A_\mu \) has to
satisfy the alignment condition , [7], \( A_\mu k^\mu = 0 \). Since the classical em-fields are
determined by the retarded potentials, the in-out symmetry of gravity turns
out to be broken and the solutions of the Einstein-Maxwell system have to be
based on the out-congruence, which has important physical consequences. [4]
The structure of Kerr congruence for the stationary Kerr-Newman solution at
rest is shown in Fig.1. It displays very specific twosheetedness of the KS space-

Figure 1: The Kerr singular ring and the Kerr congruence.

The Kerr singular ring represents a branch line of space-time, showing
that congruence is propagating from negative sheet of metric onto positive one,
and the outgoing congruence is analytic extension of the ingoing one. The KS

1's may mistakenly be associated with white BH. The necessity of choice of out-congruence
is also argued in [8] from quantum point of view.
metrics have a rigid connection with metric of auxiliary Minkowski space-time \( \eta_{\mu\nu} \) and are practically unfastened from the position of horizon. KS approach allows one to obtain solutions which are form-invariant with respect to the position of horizons, and even to its presence, which allows one to consider the black-holes and spinning particles without horizon on the common grounds [9], as well as perform the analysis of deformations of the black-hole horizon by electromagnetic excitations [7, 10]. Besides, the twosheetedness of the Kerr space reproduces the properties of BH space-times which are desirable from quantum point of view [8].

2 The Kerr theorem

Kerr congruence forms a fiber bundle of the KS space-time which is determined by the Kerr theorem [11, 12, 13, 14]. The lightlike fibers of the Kerr congruence are real twistors (intersections of the complex conjugated null planes [12]). The Kerr theorem gives a rule to generate geodesic and shear-free (GSF) null congruences in Minkowski space-time \( M^4 \). Due to the specific form of the KS metrics, these congruences turn out to be also geodesic and shear-free in the curved KS space-times which justifies application of the KS formalism [6] to solutions of the Einstein-Maxwell field equations. The Kerr theorem states that any GSF congruence in \( M^4 \) is determined by some holomorphic generating function \( F(Z) \) of the projective twistor coordinates

\[
Z = (Y, \, \zeta - Yv, \, u + Y\bar{\zeta}),
\]

being an analytic solution of the equation \( F(Z) = 0 \) in projective twistor space, \( Z \in CP^3 \). The variable \( Y \) plays especial role, being projective spinor coordinate \( Y = \pi^2/\pi^1 \) and simultaneously the projective angular coordinate

\[
Y = e^{i\phi} \tan \frac{\theta}{2}.
\]

The dependence \( Y(x) \) is the output of the Kerr theorem which determines the Kerr congruence as a field of null directions

\[
k_{\mu}dx^\mu = P^{-1}(du + \bar{Y}d\zeta + Yd\bar{\zeta} - Y\bar{Y}dv).
\]

The null field \( k^\mu \) may also be expressed in spinor form

\[
k^\mu = \bar{\pi}\sigma^\mu \pi
\]

via the Pauli matrices \( \sigma^\mu \), or in the terms of spinor components [13, 4, 3]

\[
k_{a\dot{a}} = \sigma^\mu_{a\dot{a}}k_{\mu} = \pi_a\bar{\pi}_{\dot{a}}.
\]

Each real null ray represents a twistor which is fixed by projective twistor coordinates (2), or by homogenous coordinates

\[
Z^\alpha = \{\pi^a, \mu_{\dot{a}}\}.
\]

\[2\text{Twistor coordinates are defined via the null Cartesian coordinates } 2\frac{i}{2}z = x + iy, \quad 2\frac{i}{2}z = x - iy, \quad 2\frac{i}{2}u = z - t, \quad 2\frac{i}{2}v = z + t.\]
The spinor \( \pi^a \) determines the null direction \( k^\mu \), and spinor \( \mu_\lambda = x_\nu \sigma^\nu_\alpha \pi^\alpha \) fixes the position (equation) of a real null ray (or of the corresponding complex null plane). **Quadratic in \( Y \) functions** \( F(Y) \) determine two roots \( Y^\pm(x) \) of the equation \( F = 0 \) which corresponds to twosheetedness of the Kerr space-time. It was shown in seminal work [6] that any quadratic generating function \( F(Y) \) of the Kerr theorem determines a class of the exact solutions of the Einstein-Maxwell field equations. The case of quadratic in \( Y \) functions \( F(Y) = A(x)Y^2 + B(x)Y + C(x) = A(x)(Y - Y^+)(Y - Y^-) \) was studied in details [12, 15]. In fact, it describes the Kerr-Newman solution in a general position with arbitrary spin-orientation and Lorentz boost. This information is encoded in the coefficients \( A, B, C \), and the KS formalism allows one to write down the corresponding form of congruence and the exact solutions for metric and em field.

### 3 Complex string as source of Kerr geometry

Structure of the Kerr-Newman solution admits a complex interpretation (suggested by Newman [19]) as being generated by a source propagating along a complex world-line \( X^\mu(\tau) \in CM^4 \), and parameters of the quadratic in \( Y \) function \( F(Y) \) may be easily determined from parameters of the world-line [12, 15]. Since a complex world-line \( X^\mu(\tau) \) in \( CM^4 \) is parametrized by complex time \( \tau = t + i\sigma \), it is really a complex world-sheet with target space \( CM^4 \), [17]. Thereby, the complex Kerr-Newman source is equivalent to some complex string. The KS twistor structure may be described by a complex retarded-time construction, in which twistors of the Kerr congruence are the real sections of the complex light cones emanated from the complex world-line. In this way the Kerr-Newman solution may be generalized to solutions of a broken \( N = 2 \) supergravity [18]. Projection of the real KS twistors onto complex world line selects on the world-sheet a strip \( Im \tau \in [-a, a] \), forming an open Euclidean string extended along the complex time direction \( \sigma = Im\tau \). The parameter \( \sigma \) is linked with angular directions of twistor lines, \( \sigma = a \cos \theta \) \({}^4\). Condition of the existence of real slice for the complex twistors determines the end points of the string \( \tau = t \pm i\sigma \) [17]. In the same time, consistency of boundary conditions demands orientifolding this string, by doubling of the world-sheet and turning it into a closed but folded string [17] [2]. The joined to the end points of this string twistors form two especial null rays emanated in the North and South directions, \( k_N = \overline{\pi}_N \sigma^\nu \pi_N \) and \( k_S = \overline{\pi}_S \sigma^\nu \pi_S \). [1]. They play peculiar role in the KS geometry, controlling parameters of the function \( F \), and therefore, the twistorial structure in whole. The corresponding end points of complex string may be marked by quark indices \( \Psi^\alpha_\overline{N}(S) \). It was obtained in [12, 2] that the null directions \( k_N \) and \( k_S \) may be set in the one-to-one correspondence with solutions of the Dirac equation \( \Psi = (\phi, \chi^\alpha) \). Putting \( \pi_N = \phi \) and \( \pi_S = \chi \), one sees that the Dirac wave function \( \Psi \) manages the position, orientation and boost of the Kerr source. It gives a combined Dirac-Kerr model of spinning particle, in

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\(^{3}\text{Newman suggested this construction for } CM^4, [10], \text{ while the application to curved spacetimes demands the KS representation for metric [12, 2].}\)

\(^{4}\text{In fact, each point of the complex string has adjoined complex twistor parameter } Y \text{ with the additional freedom of the rotations } e^{i\phi} \text{ around z-axis. Up to this } U(1) \text{ symmetry, each point of the complex Kerr string has an adjoined real twistor line of the Kerr congruence, forming the Kerr twistor-string structure [2].}\)
which twistor structure is controlled by the solutions of the Dirac equation [1]. Plane waves of the Dirac wave function propagate along the null directions \( k_N \) and \( k_S \). The essential drawback of this model is that the Dirac equation and its wave functions are considered in \( M^4 \), instead of the consistent treatment on the KS background. However, the exact solutions of the massive Dirac equation on the Kerr background are unknown, and moreover, there are evidences that they cannot be consistent with KS structure in principle. Note also that the plane waves are inconsistent with the Kerr background and there is a related problem with consistency of the conventional Fourier transform on the twosheeted Kerr background.

4 Consistency of the Dirac-Kerr system

Consistent solutions of the Dirac equation may easily be obtained for the massless fields which are aligned with the Kerr congruence. For the aligned to PNC solutions, the Dirac spinor \( \Psi_D^\dagger = (\phi, \chi) = (A, B, C, D) \) has to satisfy the relation \( \Psi_D k_\mu \gamma^\mu = 0 \), which yields, \([18, 19]\), \( A = D = 0 \), and the functions \( B \) and \( C \) take the form \([18]\, \text{App. B}\)

\[
B = f_B(\bar{Y}, \bar{\tau})/\bar{\tilde{r}}, \quad C = f_C(Y, \tau)/\tilde{r}, \quad \tilde{r} = r + ia \cos \theta, \tag{8}
\]

where \( f_B \) and \( f_C \) are arbitrary analytic functions of the complex angular variable \( Y \) and the retarded-time \( \tau = t - \tilde{r} \), obeying the relations \( \tau_2 = \tau_4 = 0 \), and \( Y_2 = Y_4 = 0 \). Due to this analyticity, the wave solutions have poles at some values of \( Y \), or a series of such poles \( f_B = \sum_i a_i/(Y - Y_i) \), leading to singular beams along some of twistor lines.\(^5\) It turns out that the treatment of the massless solutions resolves four problems: 1) existence of the exact self-consistent solutions, 2) the Dirac plane waves on the KS background turn into the lightlike beams concentrating near singular twistor lines with directions \( k_N \) and \( k_S \), 3) there appears natural relation to the massless core of superstring theory \([20]\) and to the massless Nair-Witten twistor models for scattering, \([4,3]\), 4) the Dirac massless solutions form the currents and lightlike momenta \( p(Y) \) distributed over sphere, \( Y \in S^2 \), and the total nonzero mass appears after averaging over sphere. The real source of the Kerr-Newman solution is a disk, D2-brane boundary (base) of a “holographic” twistor bundle. The total mass has a few contributions \([9]\), including the source itself and the mass-energy of the fields distributed over null rays. It depends on \( Y \in S^2 \) which may be factorized into angular part \( e^{i\phi} \), and parameter \( \sigma = a \cos \theta = a \frac{1 + Y}{1 - Y} \) along the complex Kerr string. Orientifolding is expected to be equivalent to normal ordering

\[
p^\mu = \langle :\bar{\Psi}\gamma^\mu\Psi: \rangle = \int_{S^2} \mu(Y) dY d\bar{Y} \langle :\bar{\Psi}\gamma^\mu\Psi: \rangle. \tag{9}\]

In the rest frame

\[
m = \langle p^0(Y) \rangle = \int_{S^2} \mu(Y) dY d\bar{Y} p^0(Y). \tag{10}\]

\(^5\)It is valid for em-field too. \([10]\)}
Therefore, mechanism of the origin of mass from the massless solutions is similar to that of the dual (super)string models and corresponds to an initial Wheeler’s idea of a geon \[21\] (“mass without mass” obtained from the averaged em field + gravity), as well as to the initial Ramond idea on averaging of the local string structure in the dual string models \[22\]. The corresponding wave solutions on the Kerr background have singular beams which take asymptotically the form of the exact singular pp-wave solutions by A. Peres \[23\].

5 Solutions with singular Beams as analogs of the plane and spherical waves

Note, that in general the conventional electromagnetic (em) plane waves are not consistent solutions in a curved space-time. The closest gravitational analogs of the em plane waves are the so called “exact plane wave” solutions which are singular at infinity \[24, 25\]. More general are the plane-fronted wave solutions (pp-waves) which have the unique symmetry along a covariantly constant null direction \(k^\mu\) (one can set \(k^\mu = du, \ u = z - t\) and, therefore, belong to the KS class \(g_{\mu\nu} = \eta_{\mu\nu} + H(u)k_\mu k_\nu\). They may be regular at infinity, but should have one or more poles in the finite region of the orthogonal to \(k^\mu\) plane \(x + iy\). These poles determine the position of singular beams. Finally, there is a third type of singularity at the front of solution, at a fixed value of the retarded time \(u\). \[24, 25\]. Similar, the conventional spherical (or even the ellipsoidal) harmonics cannot be exact em solution for the curved Kerr background. So, in principle, the conventional Fourier analysis cannot be used on the KS space-times. The KS solutions with singular em beams aligned with a constant null vector may be exact solutions to the coupled system (em + gravity) and be considered as analogs of the exact plane waves. Similar, the solutions with singular beams aligned with one or many rays of the Kerr congruence turn out to be analogs of the conventional smooth harmonics on the KS space-time. They form overfilled system of coherent states, and appear in many problems related with the exact solutions on the KS background \[18, 27, 10\]. Such beams lead to some unexpected physical consequences. In particular, the beam-like solutions have very strong back reaction on metric and break the topological structure of the black-hole horizons, leading to its topological instability with respect to electromagnetic excitations \[27, 10\]. The KS approach gives as a broad class of the exact em solutions with beams which are consistent with the KS gravity and aligned with twistor structure.

6 Multi-particle KS solutions and Quantum Gravity

There is a very broad class of exact em solutions with beams coupled with KS gravity. They are based on the Kerr theorem related with the Kerr generating functions \(F(Y)\) of higher degrees in \(Y\). \[26\]. The corresponding KS congruences form multisheeted Riemann surfaces, and the resulting space-times turn out to be multisheeted too, which generalizes the known twosheetedness of the Kerr geometry.
If we have a system of $k$ particles with known parameters $q_i$, one can form the function $F$ as a product of the $k$ given blocks $F_i(Y) = F(Y|q_i)$ with a known dependence $F(Y|q_i)$,

$$F(Y) = \prod_{i=1}^{k} F_i(Y). \quad (11)$$

The solution of the equation $F = 0$ acquires $2k$ roots $Y_i^{\pm}$, leading to $2k$-sheeted twistor space.

Figure 2: The lightlike interaction in Multi-sheeted twist or space via a common twistor line connecting the out-sheet of one particle to the in-sheet of another.

The twistorial structure on the $i$-th ($+$) or $(-)$ sheet is determined by the equation $F_i = 0$ and does not depend on the other functions $F_j, j \neq i$. Therefore, the particle $i$ does not feel the twistorial structures of other particles. Similar, singular sources of the $k$ Kerr’s spinning particles are determined by equations $F = 0, dy F = 0$ which acquires the form

$$\prod_{l=1}^{k} F_l = 0, \quad \sum_{i=1}^{k} \prod_{l \neq i}^{k} F_l dy F_i = 0 \quad (12)$$
and splits into \( k \) independent relations

\[
F_i = 0, \quad \prod_{i \neq i}^k F_i dY F_i = 0, \tag{13}
\]

showing that \( i \)-th particle does not feel also the singular sources of other particles. The space-time splits on the independent twistorial sheets, and therefore, the twistor structure related to the \( i \)-th particle plays the role of its “internal space”. It looks wonderful. However, it is a natural generalization of the well known twosheetedness of the Kerr space-time which remains one of the mysteries of the Kerr solution for the very long time. In spite of independence of the twistor structures positioned on the different sheets, there is an interaction between them via the singular lightlike beams (pp-strings) which appear on the common twistor lines connecting the different sheets of the particles \[26\] and play the role of propagators in the Nair-Witten concept on scattering amplitudes in twistor space \[4\]. Multisheetedness of the KS spacetimes cannot be interpreted in the frame of classical gravity, however it has far-reaching consequences for quantum gravity \[8\], in which metric is considered as operator \( \hat{g}_{\mu\nu} \) leading to an “effective geometry”,

\[
\langle \text{out} | \hat{g}_{\mu\nu} | \text{in} \rangle = g_{\mu\nu} \langle \text{out} | \text{in} \rangle. \tag{14}
\]

If we consider the Kerr-Newman solution for an isolated BH or spinning particle, we obtain the smooth space-time apart from the Kerr singular ring. However, the exact solution for the Kerr-Newman source surrounded by remote massive or massless particles will contain series of singular beams along the twistor lines of Kerr congruence connecting the Kerr source with these remote particles. Analysis of the action of such beams on the BH horizon, given in \[10\], showed that the beams have very strong back reaction on metric and even the very weak beams pierce the horizon changing its topology. The beams, caused by external particles \[26\], as well as the beams caused by the em zero point field, \[7, 27\], lead to a fine-grained topological fluctuations of horizon. The exact wave solutions on the KS background show that the beams are coherent and freely overpass the BH horizon \[28\]. It revokes the loss of information inside of BH.

Considering metric as an operator, one has to give an operator meaning to the Kerr theorem. Immediate way is to consider the complex position of the Kerr source \( X^\mu(\tau) \) and its velocity \( \dot{X}^\mu(\tau) \) as coordinates of a quantum oscillator in spirit of the original Ramond paper \[22\]. It leads to the operator meanings of the the equation \( F = 0 \) via coefficients of the function \( F(Y) \), and suggests an operator approach to the Kerr-Schild equations \[6\].

7 KS twistor structure and the Nair twistor WZW model

Nair considers \( S^2 \) as a momentum space of the massless gauge bosons and constructs a classical WZW model over the sphere \( S^2 \) which is parametrized by the spinor coordinates \( \pi^a \), \[3\]. This construction is close to our field of null directions \( k^\mu(\pi^a) \) in \( M^4 \), which is parametrized by \( Y \in S^2, \ Y = \pi^2/\pi^1 \). The
suggested by Nair and Witten treatment of the scattering amplitudes in twistor space \( CP^3 \) assumes that the time evolution occurs in \( M^4 \) along twistor null lines \( Z = \text{const.} \), while \( S^2 \) has only topological duty, playing the role of 2D boundary (base) of the “holographic” twistor bundle of \( M^4 \). In the KS twistor structure this boundary is the real disk-like source of the Kerr solution, forming the area for complex fields on \( S^2 \) and the currents. Introducing two-dimensional spinor fields \( \Psi(\pi) \) on the sphere, Nair considers correlation function

\[
\langle \Psi_r(\pi) \Psi_s(\pi') \rangle = \frac{\delta_{rs}}{\pi \pi'} \sim \frac{\delta_{rs}}{Y - Y'}
\]

(15)

and the local currents \( J^a = \bar{\Psi} t^a \Psi \), showing that OPE takes in homogenous spinor coordinates the usual form of the corresponding Kac-Moody algebra of the WZW model\[29\] with central extension \( k = 1 \). In this case the complex-analytic structure of WZW model is lifted from \( S^2 \) to twistor space \( CP^3 \) over \( M^4 \). Twistor string structure of the KS geometry has much in common with the Nair WZW model and Witten’s twistor string B-model, as well as many specific features. However, in all the cases twistor approach displays the principal advantage, providing a natural extension of CFT from \( S^2 \) to \( M^4 \). Since the massless KS twistor space allows one to describe massive spinning particles, the Nair-Witten treatment of the scattering amplitudes of the gauge fields may be extended in the KS geometry to describe scattering of the gauge fields on the massive black-holes and spinning particles.

8 Conclusion

We have showed that consistency of the Dirac-Kerr model for a massive spinning particle in gravity (in particular electron) has to be based on the massless solutions of the Dirac equation aligned with the KS twistor background. It provides a natural twistor string structure of the KS geometry and generalization of the Nair-Witten approach to scattering of the gauge amplitudes in twistor space to include massive particles formed from the KS twistor-string bundle.

The considered here aligned solutions of the massless Dirac equation were extracted from a fermionic part of the super-Kerr-Newman solutions to broken \( N = 2 \) supergravity \[15\], which rises interests in the corresponding solutions to supergravity. Self-consistency of the considered aligned solutions shows that the KS twistor structure gives a clue to unification of quantum theory and gravity.

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